Multiple Dirichlet Series
Associated to Prehomogeneous Vector Spaces and the Relation with GL₃(ℤ) Eisenstein Series

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Li-Mei began her graduate studies in the fall of 2008 at Brown University. She received her Master of Science in 2010 and started her dissertation under the direction of Jeffrey Hoffstein. With her collaborators, Thomas Hulse, Eren Mehmet Kiral and Chan Ieong Kuan, Li-Mei published a paper, “The Sign of Fourier Coefficients of Half-integral Weight Cusp Forms” in the International Journal of Number Theory. Also as a graduate student, Li-Mei was first a teaching assistant, then a teaching fellow. She participated in programs run by the Sheridan Center for Teaching and Learning and served as the graduate student liaison to the Sheridan Center for the mathematics department for two years. In the fall of 2012, Li-Mei received a departmental award for excellent teaching. She graduates from Brown University with her Ph.D. in mathematics in May 2013.
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CHAPTER 1

INTRODUCTION

In [9], Katok and Sarnak show that for a GL$_2$ Maass form $\varphi$,

$$
\sum_{z \in \Lambda_d} \varphi(z) = \sum_{\text{Shim}(F_j) = \varphi} \rho_j(1)\rho_j(d),
$$

where the sum on the left side is over Heegner points of discriminant $d$, the sum on the right side is over all half-integral weight forms that lift to $\varphi$ under the Shimura correspondence described in [14], and $\rho_j(n)$ denotes the $n$-th Fourier coefficient of $F_j$.

Katok and Sarnak’s result is a generalization of a well-known result for GL$_2$ Eisenstein series. Setting $E(z, s)$ to be the real analytic GL$_2$ Eisenstein series, the following is true.

**Proposition 1.1.** If $d$ a fundamental discriminant,

$$
\int d^{s/2} \sum_{z \in \Lambda_d} E(z, s) = \zeta(s)L(s, \chi_d) = \zeta_{\mathbb{Q}(\sqrt{d})}(s),
$$

where $\chi_d$ is the quadratic character corresponding to the field extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$.

This result for GL$_2$ Eisenstein series is extended by Chinta and Offen in [3]. In this paper, they compute a particular orthogonal period of the GL$_3$ Eisenstein series...
and show that a special value of the Eisenstein series is equal to a linear combination of products of double Dirichlet series.

In fact, all these results are special cases of Jacquet’s conjecture, stated in [7], which relates integrals of $G = \text{GL}_r$ cusp forms over orthogonal subgroups of $G$ to Whittaker coefficients on the double cover of $G$ via a relative trace formula. In this dissertation, I generalize the results of Chinta and Offen for $\text{GL}_3$ Eisenstein series, providing further evidence for Jacquet’s conjecture. In particular, in [3] Chinta and Offen compute the case for the discriminant $d = 1$, and in this thesis, I obtain the result for all negative $d$.

The method I use involves looking at multiple Dirichlet series associated to prehomogeneous vector spaces. A prehomogeneous vector space is pair $(G,V)$ of a complex algebraic group $G$ and a complex vector space $V$ such that $G$ acts on $V$ with an open dense orbit. A key fact is that the algebra of relative invariants of such an action is finitely generated. For example, one could take the space of binary quadratic forms with the action of $\text{GL}_2(\mathbb{C})$. The algebra of relative invariants is then just generated by the usual discriminant $b^2 - 4ac$.

M. Sato and Shintani, in [13], define zeta functions associated to prehomogeneous vector spaces with a single relative invariant. They prove some basic results, for example showing that these zeta functions have an analytic continuation and satisfy a functional equation. The study of multiple Dirichlet series coming from prehomogeneous vector spaces was originated by F. Sato in [10], [12], [11].

The multiple Dirichlet series I construct come from the action of a non-reductive group and have multiple relative invariants rather than a single invariant. This construction gives a function of several, rather than just one, complex variables.
This thesis has three parts:

- **Background**: I record some facts about the Hurwitz zeta function and binary quadratic forms. None of these results are new.

- **The GL$_2$ case**: Although the result in this part is well-known, the proof given is (as far as I know) new. This part is included, in part, to give the idea for the GL$_3$ case, which is similar, but slightly more complicated.

- **The GL$_3$ case**: This section contains my original results, using methods similar to the previous section and is broken into multiple chapters. The first chapter in this section constructs the multiple Dirichlet series and relates it to the GL$_3$ Eisenstein series. The second rewrites the series using genus theory and recovers products of double Dirichlet series. The third chapter investigates the genus theory in more depth, explicitly computing the necessary coefficients. Finally, in the last chapter, the main result is achieved by combining all these results.
CHAPTER 2
BACKGROUND

§ 2.1 The Hurwitz Zeta Function

It will be necessary to use some basic facts about Hurwitz zeta functions when manipulating the multiple Dirichlet series attached to the space of ternary quadratic forms. Therefore, we give a brief introduction to the theory of Hurwitz zeta functions.

For $0 < a \leq 1$, the Hurwitz zeta function is defined to be

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}.$$ 

Like the Riemann zeta function, which the Hurwitz zeta function specializes to when $a = 1$, this converges absolutely for $\Re s = \sigma > 1$ and has an analytic continuation to the complex plane with simple pole at $s = 1$, as shown in [1].

The Hurwitz zeta function is important because we can express it as a combination of Dirichlet $L$-series.

Proposition 2.1. If $k$ and $m$ are relatively prime integers, then

$$\zeta(w, \frac{k}{m}) = \frac{m^w}{\varphi(m)} \sum_{\chi \mod m} \bar{\chi}(k)L(w, \chi).$$

Here, the sum is over all characters $\chi$, primitive and imprimitive, mod $m$. 

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Proof. This proposition follows from the orthogonality of characters, that is, the fact that

\[
\sum_{\chi \mod m} \chi(n)\bar{\chi}(k) = \begin{cases} 
\varphi(m) & \text{if } n \equiv k \mod m \\
0 & \text{otherwise}
\end{cases}.
\]

Then we have

\[
\frac{m^w}{\varphi(m)} \sum_{\chi \mod m} \bar{\chi}(k)L(w,\chi) = \frac{m^w}{\varphi(m)} \sum_{\chi \mod m} \sum_{n>0} \chi(n)\bar{\chi}(k) \frac{1}{n^w} \\
= m^w \sum_{n\equiv k \mod m \atop n>0} \frac{1}{n^w} \\
= \sum_{n\equiv k \mod m \atop n>0} \frac{1}{(n/m)^w} \\
= \sum_{\ell=0}^{\infty} \frac{1}{(\ell + k/m)^w} \\
= \zeta \left( w,\frac{k}{m} \right),
\]

completing the proof. \hfill \Box

§ 2.2 Binary Quadratic Forms

To show that the multiple Dirichlet series associated to the space of ternary quadratic forms is equal to a product of double Dirichlet series, we will need some standard facts about binary quadratic forms.

Let \( X_2 \) be the space of binary quadratic forms. The group \( \text{SL}_2(\mathbb{Z}) \) acts on \( X_2 \) by linear substitution. Namely,

\[
\gamma \circ Q(x, y) = Q(\gamma(x, y)).
\]
For a fixed discriminant $D$, there are finitely many $\text{SL}_2(\mathbb{Z})$-orbits. When choosing a representative from each orbit we have the following proposition.

**Proposition 2.2.** A representative $Q(x, y) = ax^2 + bxy + cy^2$ for an $\text{SL}_2(\mathbb{Z})$ orbit in $X_2$ can be chosen such that $(a, b) = 1$.

This statement can be found in the exercises of [4] with some hints at the proof.
CHAPTER 3

THE CASE OF THE GL₂ EISENSTEIN SERIES

Although the following result is well-known, we present a proof of it here to give a simple example of the methods we will use in the case of GL₃.

§ 3.1 Preliminaries

Let \( E(z, s) \) denote the real analytic Eisenstein series for GL₂. Namely, setting

\[
\Gamma = \text{SL}_2(\mathbb{Z}) \quad \text{and} \quad \Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \bigg| n \in \mathbb{Z} \right\}
\]

we have

\[
E(z, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \Im(\gamma z)^s.
\]

Next, let \( E^*(z, s) \) be the half-integral weight Eisenstein series for GL₂ studied by Goldfeld and Hoffstein in [6]. To define \( E^*(z, s) \), let

\[
\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \bigg| c \equiv 0 \pmod{4} \right\}.
\]

For \( \gamma \in \Gamma_0(4) \), define \( j_\gamma(z) \) to be the factor of automorphy of the theta series:

\[
j_\gamma(z) = \begin{pmatrix} c \\ d \end{pmatrix} \varepsilon_d^{-1}(cz + d)^{\frac{1}{2}}
\]
where \( (\frac{c}{d}) \) is the extension of the Jacobi symbol described by Shimura in [14] and
\[
\varepsilon_d = \begin{cases} 
1 & \text{if } d \equiv 1 \mod 4 \\
i & \text{if } d \equiv 3 \mod 4
\end{cases}.
\]

Then we recall the well-known fact mentioned in the introduction, Proposition 1.1, which says that a sum of special values of the Eisenstein series is equal to the product of a zeta function and a quadratic Dirichlet L-function.

In [6], it is shown that the \( d \)-th Fourier coefficient of \( E^*(z, s) \) is essentially \( L(s, \chi_d) \), so the proposition can be rewritten as

**Proposition 3.1.** Let \( d < 0 \). Then
\[
\sum_{z \in \Lambda_d} E(z, s) = \rho(1)\rho(d)
\]
where \( \rho(n) \) is the \( n \)-th Fourier coefficient of \( E^*(z, s) \).

§ 3.2 Proof

The idea of the proof in this case is to look at the space \( X^+ \) of positive definite integral binary quadratic forms. To a binary quadratic form \( Q(x, y) = ax^2+bxy+cy^2 \), we associate the matrix
\[
Q = \begin{pmatrix} 
2a & b \\
b & 2c
\end{pmatrix}
\]
so that (abusing notation), \( Q(x, y) = \frac{1}{2}(x, y)Q(x, y)^t \).

Usually, we think of \( \Gamma = \text{SL}_2(\mathbb{Z}) \) acting on the space of binary quadratic forms. In this case, the space of invariants of the action is one-dimensional. Namely, it is
generated by the discriminant, $b^2 - 4ac$. The key for this proof, though, is to only consider the action of $\Gamma_\infty$ on $X^+$. As usual, the action is by linear substitution of variables, so that

$$\gamma \circ Q(x,y) = Q(\gamma(x,y)).$$

In terms of matrices, this means that

$$\gamma \circ Q = \gamma^t Q \gamma.$$

With this action, the space of invariants is two-dimensional. The discriminant is still invariant, of course, but so is $a = Q(1,0)$. For clarity, set

$$r_1(Q) = \text{disc } Q = b^2 - 4ac$$
$$r_2(Q) = Q(1,0) = a$$

to be the two generating invariants.

We form the double Dirichlet series associated to the space $X^+$ with the action of $\Gamma_\infty$ by

$$Z(s,w) = \sum_{Q \in \Gamma_\infty \setminus X^+} \frac{1}{|r_1(Q)|^w |r_2(Q)|^s}.$$

By rewriting this sum in two different ways, we will obtain, on the one hand, a sum of values of the Eisenstein series, and on the other, the quadratic Dirichlet $L$-series. The first way involves writing the quotient $\Gamma_\infty \setminus X^+$ explicitly and using Hurwitz zeta functions. The second way splits the sum up further.

To get the quotient explicitly, we have the following lemma.
Lemma 3.1. A set of representatives for the quotient $\Gamma_\infty \setminus X^+$ is

$$\left\{ \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \bigg| a > 0 \text{ and } 0 \leq b \leq 2a - 1 \text{ and } c > \frac{b^2}{4a}, \ c \in \mathbb{Z} \right\}$$

Proof. We compute

$$(1 \ 0) \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} (1 \ 0) = \begin{pmatrix} 2a & 2a + b \\ b + c & b + 2c \end{pmatrix}.$$ 

Therefore, we see that $Q_1 \sim Q_2$ modulo the action of $\Gamma_\infty$ if $b_1 \equiv b_2 \pmod{2a}$, and a condition on the set of representatives of $\Gamma_\infty \setminus X^+$ is that $0 \leq b \leq 2a - 1$. Since the form is positive definite, $a > 0$ and also $4ac - b^2 > 0$. Therefore, $c$ ranges freely over the integers greater than $\frac{b^2}{4a}$. So a set of representatives for $\Gamma_\infty \setminus X^+$ has $a > 0$, $0 \leq b \leq 2a - 1$ and $c > \frac{b^2}{4a}$, $c \in \mathbb{Z}$ as desired. $\blacksquare$

Therefore, we can rewrite our double Dirichlet series as

$$Z(s, w) = \sum_{a > 0} \sum_{0 \leq b < 2a - 1} \frac{1}{as} \frac{1}{|b^2 - 4ac|^w}$$

$$= \sum_{a > 0} \frac{1}{a^s} \sum_{0 \leq b < 2a - 1} \frac{1}{|b^2 - 4ac|^w}$$

Focusing on the inner sum, we see that

$$\sum_{0 \leq b < 2a - 1} \frac{1}{|b^2 - 4ac|^w} = \frac{1}{(4a)^w} \sum_{0 \leq b < 2a - 1} \frac{1}{|\frac{b^2}{4a} - c|^w}$$

$$= \frac{1}{(4a)^w} \sum_{0 \leq b < 2a - 1} \zeta(w, \alpha)$$

where $\alpha$ is $\frac{-b^2}{4a} \pmod{1}$.
Using Proposition 2.1, we can rewrite this sum of Hurwitz zeta functions in terms of $L$-series, but we need to be careful to ensure that $\alpha$ is a fraction expressed in lowest terms.

Let $d = (b^2, 4a)$. Then $d$ can be expressed as $d = d_1^2 d_0$, where $d_0$ is square-free. Since $d \mid b^2$, in fact the values of $b$ with $(b^2, 4a) = d_1^2 d_0$ and $b \mod 2a$ will be the $kd_1 d_0$ with $k$ between $0$ and $\lfloor 2a/d_1 d_0 \rfloor$ inclusive such that $(k, 4a/d_1) = 1$. With this notation, and with the understanding that $\zeta(w, a)$ might mean $\zeta(w, a \mod 1)$, our sum becomes

$$
\frac{1}{(4a)^w} \sum_{0 \leq b \leq 2a-1 \atop (b^2, 4a)=d} \zeta(w, \frac{b^2}{4a/d}) = \frac{1}{(4a)^w} \sum_{d_1 \mid 4a \atop d = d_1^2 d_0} \sum_{k=0}^{\lfloor 2a/d_1 d_0 \rfloor} \sum_{(k, 4a/d_1) = 1} \zeta(w, \frac{-k^2 d_0}{4a/d_1^2 d_0})
$$

$$
= \frac{1}{(4a)^w} \sum_{d_1 \mid 4a \atop d = d_1^2 d_0} \sum_{k=0}^{\lfloor 2a/d_1 d_0 \rfloor} \frac{(4a/d)^w}{\varphi(4a/d)} \sum_{\chi \mod 4a/d} \bar{\chi}(-k^2 d_0) L(w, \chi)
$$

$$
= \sum_{d \mid 4a \atop d = d_1^2 d_0} \frac{1}{d \varphi(4a/d)} \sum_{\chi \mod 4a/d} \bar{\chi}(-d_0) \left( \sum_{k=0}^{\lfloor 2a/d_1 d_0 \rfloor} \bar{\chi}(k^2) \right) L(w, \chi).
$$

But since the character sum in parentheses is just

$$
\sum_{k=0}^{\lfloor 2a/d_1 d_0 \rfloor} \bar{\chi}(k^2) = \begin{cases} 
\frac{d_1}{2} \varphi(4a/d) & \text{if } \chi \text{ trivial or quadratic} \\
0 & \text{otherwise}
\end{cases}
$$
this sum is in fact
\[
\sum_{d \mid 4a} \sum_{\chi \mod 4a/d} \bar{\chi}(-d_0) L(w, \chi).
\]

With this, we can rewrite the original double Dirichlet series \(Z(s, w)\) as
\[
Z(s, w) = \sum_{a > 0} \frac{1}{a^s} \sum_{d \mid 4a} \sum_{\chi \mod 4a/d} \bar{\chi}(-d_0) L(w, \chi).
\]

Carefully exchanging the order of the first two sums, we get
\[
Z(s, w) = \sum_{d \text{ odd}} \frac{d_1}{2d^w} \sum_{k=1}^{\infty} \frac{1}{(dk)^s} \sum_{\chi \mod 4k} \bar{\chi}(-d_0) L(w, \chi)
\]
\[
+ \sum_{2 \mid d} \frac{d_1}{2d^w} \sum_{k=1}^{\infty} \frac{1}{(dk/2)^s} \sum_{\chi \mod 2k} \bar{\chi}(-d_0) L(w, \chi)
\]
\[
+ \sum_{4 \mid d} \frac{d_1}{2d^w} \sum_{k=1}^{\infty} \frac{1}{(dk/4)^s} \sum_{\chi \mod k} \bar{\chi}(-d_0) L(w, \chi).
\]

Note that the sums over characters include all characters, primitive and imprimitive. We can change these sums to include only primitive quadratic characters by summing over square-free, odd divisors of \(k\) (or \(2k\) or \(4k\)). Modulo a square-free odd number \(m_0\), there is a unique primitive quadratic character, which we will call \(\chi_{m_0}\). Of course, there are also primitive quadratic characters mod 4 and 8 to be added in if 4 or 8 divide \(k\), \(2k\) or \(4k\). Modulo 4, there is a unique primitive quadratic character \(\chi_4\) and modulo 8 there are two primitive quadratic characters, \(\chi_8\) and \(\chi'_8\). With this notation, we rewrite the series as
\[ Z(s, w) = \sum_{d \text{ odd}} \frac{d_1}{2d^{w+s}} \sum_{k=1}^{\infty} \frac{1}{k^s} \left( \bar{\chi}_4(-d_0)P(k, \chi_4, w)L(w, \chi_4) + g_d(1, k, w) \right) \]
\[ + \sum_{m_0|2k} \bar{\chi}_{m_0}(-d_0)P(\frac{2k}{m_0}, \chi_{m_0}, w)L(w, \chi_{m_0}) \]
\[ + \sum_{2|d} \frac{d_1}{2d^{w+s}} \sum_{k=1}^{\infty} \frac{2^s}{k^s} \left( f_d(1, k, w) + g_d(2, k, w) \right) \]
\[ + \sum_{m_0|2k} \bar{\chi}_{m_0}(-d_0)P(\frac{2k}{m_0}, \chi_{m_0}, w)L(w, \chi_{m_0}) \]
\[ + \sum_{4|d} \frac{d_1}{2d^{w+s}} \sum_{k=1}^{\infty} \frac{4^s}{k^s} \left( f_d(2, k, w) + g_d(3, k, w) \right) \]
\[ + \sum_{m_0|k} \bar{\chi}_{m_0}(-d_0)P(\frac{k}{m_0}, \chi_{m_0}, w)L(w, \chi_{m_0}) \]

where we define the functions \( f_d(\ell, k, w) \) and \( g_d(\ell, k, w) \) to be

\[ f_d(\ell, k, w) = \begin{cases} 
\bar{\chi}_4(-d_0)P(\frac{k}{2^\ell}, \chi_4, w)L(w, \chi_4) & \text{if } 2^\ell | k \\
0 & \text{otherwise}
\end{cases} \]  

(3.1)

\[ g_d(\ell, k, w) = \begin{cases} 
P(\frac{k}{2^\ell}, \chi_8, w)\bar{\chi}_8(-d_0)L(w, \chi_8) + P(\frac{k}{2^\ell}, \chi'_8, w)\bar{\chi}'_8(-d_0)L(w, \chi'_8) & \text{if } 2^\ell | k \\
0 & \text{otherwise}
\end{cases} \]  

(3.2)

and the functions \( P(n, w) \) are correction polynomials

\[ P(n, \chi, w) = \prod_{p|n} \left( 1 - \frac{\chi(p)}{p^w} \right). \]
On the other hand, we can rewrite $Z(s, w)$ in terms of the Eisenstein series. We first rewrite $Z(s, w)$ by splitting the sum further, as follows.

$$
\sum_{Q \in \Gamma_\infty \setminus \mathbb{X}} \frac{1}{|r_1(Q)|^w |r_2(Q)|^s} = \sum_{Q \in \Gamma \setminus \mathbb{X}} \frac{1}{|\text{disc } Q|^w} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \frac{1}{|\gamma \circ Q(1, 0)|^s}
$$

We can compute $\gamma \circ Q(1, 0)$ to find that if

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
$$

then

$$\gamma \circ Q(1, 0) = A^2a + ACb + C^2c.
$$

Note, however, that this is equal to

$$\gamma \circ Q(1, 0) = a|Cz + A|^2
$$

where

$$z = \frac{b + \sqrt{b^2 - 4ac}}{2a}.
$$

Then, setting $z$ as above, the inner sum becomes

$$
\sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \frac{1}{|\gamma \circ Q(1, 0)|^s} = \frac{1}{a^s} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \frac{1}{|Bz + A|^{2s}}
$$

$$= \frac{1}{a^{s}y^s} E(z, s)
$$

$$= \frac{2^s}{d^{s/2}} E(z, s),
$$
Therefore, we see that

\[ Z(s, w) = \sum_{Q \in \Gamma \setminus X^+} \frac{2^s}{|b^2 - 4ac|^{w + s/2}} E(z, s). \]

But fixing the discriminant of \( Q \), we can rewrite the sum as

\[ Z(s, w) = 2^s \sum_{d < 0} \frac{1}{|d|^{w + s/2}} \sum_{z \in \Lambda_d} E(z, s) \]

where \( \Lambda_d \) is the set of Heegner points of discriminant \( d \).

With these two representations of our original double Dirichlet series \( Z(s, w) \), we recover the original result. Restricting to the case of \( d \) an odd fundamental discriminant, that is, \( d \) is square-free and \( 1 \mod 4 \), we can compare \( d^{-w} \) coefficients. On the one hand, we have

\[ \frac{2^s}{d^{s/2}} \sum_{z \in \Lambda_d} E(z, s), \]

namely the sum of special values of the Eisenstein series.

On the other hand, if we ignore the problems at 2, we can compute the \( d^{-w} \) coefficient from the first representation of \( Z(s, w) \). The first thing we have to do is group all the terms with \( w \)'s in them. Ignoring the problems at the prime 2 and assuming all numbers are “nice” (that is, quadratic reciprocity works perfectly),
So restricting to a particular $d$, we get, up to some factors, that the $d^{-w}$ coefficient is $\zeta(s)L(s, \chi_d)$, as desired.

Putting this all together, we recover the known result, that a sum of special values of the Eisenstein series is (up to some factors) equal to the product of the first and $d$-th Fourier coefficients of the half-integral weight Eisenstein series.
CHAPTER 4

THE MULTIPLE DIRICHLET SERIES AS A SUM OF EISENSTEIN SERIES VALUES

In this chapter, we will construct the multiple Dirichlet series associated to the space of ternary quadratic forms and prove its relationship with the $SL_3(\mathbb{Z})$ Eisenstein series.

§ 4.1 The $SL_3(\mathbb{Z})$ Eisenstein Series

The $SL_3(\mathbb{Z})$ Eisenstein Series is described in detail in [2]. For completeness, we record the pertinent definitions and theorems here.

Let $\tau \in \mathcal{H} = G/ZK$, where $G = GL_3(\mathbb{R})$, $K = O(3)$ and $Z$ is the set of scalar matrices. Then $\tau$ will be given in coordinates by

$$\tau = \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(4.1)

Also, let $\Gamma$ denote $SL_3(\mathbb{Z})$ and let $P$ be the minimal parabolic subgroup in $SL_3(\mathbb{Z})$, namely

$$P = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$
Then the $SL_3(\mathbb{Z})$ Eisenstein series is defined to be

$$E(\nu_1, \nu_2, \tau) = \sum_{g \in \mathcal{P} \setminus \Gamma} I_{(\nu_1, \nu_2)}(g \cdot \tau),$$

where

$$I_{(\nu_1, \nu_2)}(\tau) = y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2}. $$

Setting $x_1 x_2 = x_3 + x_4$, the Eisenstein series can be written explicitly as

$$E(\nu_1, \nu_2, \tau) = I_{(\nu_1, \nu_2)}(\tau) \times \sum_{(A_1,B_1,C_1)\neq (0,0,0)} \frac{1}{(A_2,B_2,C_2)\neq 0, A_1 C_2 + B_1 B_2 + C_1 A_2 = 0} \left( (A_1 x_3 + B_1 x_1 + C_1)^2 + (A_1 x_2 + B_1)^2 y_1^2 + A_1^2 y_1^2 y_2^2 \right)^{-3\nu_1/2} \left( (A_2 x_4 - B_2 x_2 + C_2)^2 + (A_2 x_1 - B_2)^2 y_2^2 + A_2^2 y_1^2 y_2^2 \right)^{-3\nu_2/2}. $$

(4.2)

The $A$'s, $B$'s and $C$'s are the invariants of an orbit of $G_\infty \setminus G$. For $g = (a_{ij}) \in G$, the invariants are:

$$
\begin{align*}
A_1 &= -a_{31} & A_2 &= a_{22} a_{31} - a_{21} a_{32} \\
B_1 &= -a_{32} & B_2 &= a_{21} a_{33} - a_{23} a_{31} \\
C_1 &= -a_{33} & C_2 &= a_{23} a_{32} - a_{22} a_{33}.
\end{align*}
$$

(4.3)

§ 4.2 The Multiple Dirichlet Series for the space of Ternary Quadratic Forms

We want to construct a multiple Dirichlet from the space of ternary quadratic forms which can be interpreted as both a sum of values of the $SL_3(\mathbb{Z})$ Eisenstein series and as a product of double Dirichlet series. To do this, let $X_3$ be the space
of ternary quadratic forms, and let $X_3^+$ be the space of positive definite ternary quadratic forms. The minimal parabolic subgroup in SL$_3(\mathbb{Z})$ $P$ acts on $X_3^+$ by linear substitution, and the action has three relative invariants. Namely the invariants are

$$r_1(T) = r(s^2 - 4pq) + (pu^2 - stu + qt^2) = \text{disc}(T)$$
$$r_2(T) = s^2 - 4pq = \text{disc}(T_1)$$
$$r_3(T) = p$$

when $T = \left( \begin{array}{ccc} 2p & s & t \\ s & 2q & u \\ t & u & 2r \end{array} \right)$ and $T_1 = \left( \begin{array}{ccc} 2p & s \\ s & 2q \end{array} \right)$.

Then we form the multiple Dirichlet series by

$$Z(s_1, s_2, s_3) = \sum_{T \in P \setminus X_3^+(\mathbb{Z})} \frac{1}{|r_1(T)|^{s_1}|r_2(T)|^{s_2}|r_3(T)|^{s_3}}.$$  \hfill (4.4)

In this chapter, we will split up the sum further to show that this series can be interpreted as a sum of special values of the minimal parabolic Eisenstein series. The first step is as follows.

$$Z(s_1, s_2, s_3) = \sum_{T \in P \setminus X_3^+(\mathbb{Z})} \frac{1}{|r_1(T)|^{s_1}|r_2(T)|^{s_2}|r_3(T)|^{s_3}}$$
$$= \sum_{T \in \Gamma \setminus X_3^+(\mathbb{Z})} \sum_{\gamma \in P \setminus \Gamma} \frac{1}{|r_1(\gamma \circ T)|^{s_1}|r_2(\gamma \circ T)|^{s_2}|r_3(\gamma \circ T)|^{s_3}}$$
$$= \sum_{T \in \Gamma \setminus X_3^+(\mathbb{Z})} \frac{1}{\text{disc}(T)^{s_1}} \sum_{\gamma \in P \setminus \Gamma} \frac{1}{|r_2(\gamma \circ T)|^{s_2}|r_3(\gamma \circ T)|^{s_3}}.$$
For now, we will define the Eisenstein series on quadratic forms to be exactly

\[ E(s_2s_3, Q) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} |r_2(\gamma \circ Q)|^{-s_2} |r_3(\gamma \circ Q)|^{-s_3}, \]

so

\[ Z(s_1, s_2, s_3) = \sum_{T \in \Gamma \setminus X_2^+(\mathbb{Z})} \frac{1}{\text{disc}(T)^{s_1}} E(s_2, s_3, T). \quad (4.5) \]

The next section is devoted to showing how the Eisenstein series defined in Section 4.1 is related to this new Eisenstein series. The two are, in fact, very closely related.

§ 4.3 Relating the Two Eisenstein Series

In this section, we will prove the following proposition, giving the relationship between the two Eisenstein series defined previously in this chapter.

**Proposition 4.1.** Let \( T \) be the binary quadratic form given by \( 2\tau T \), where \( \tau \) is as shown in (4.1) and \( I_{\nu_1, \nu_2} \), \( r_2 \) and \( r_3 \) are as defined above. Then

\[ E(\nu_1, \nu_2, \tau) = \text{disc}(\tau)^{(\nu_1 + 2\nu_2)/2} \sum_{P \setminus \Gamma} r_2(\gamma \circ T)^{-3\nu_2/2} r_3(\gamma \circ T)^{-3\nu_1/2}. \]

**Proof.** First, recall from equation 4.2 that the Eisenstein series is a sum over \( P \setminus \Gamma \) of terms involving the invariants \( A_1, B_1, C_1, A_2, B_2, \) and \( C_2 \). These terms are the product of powers of

\[ f_1(\tau) := (A_1x_3 + B_1x_1 + C_1)^2 + (A_1x_2 + B_1)^2 y_1^2 + A_1^2 y_1^2 y_2^2 \]

and

\[ f_2(\tau) := (A_2x_4 - B_2x_2 + C_2)^2 + (A_2x_1 - B_2)^2 y_2^2 + A_2^2 y_1^2 y_2^2. \]
Focusing on $f_1(\tau)$, we can expand and rewrite it to get

\[
  f_1(\tau) = (A_1 x_3 + B_1 x_1 + C_1)^2 + (A_1 x_2 + B_1)^2 y_1^2 + A_1^2 y_1^2 y_2^2
  = A_1^2 (x_3^2 + x_2^2 y_1^2 + y_1^2 y_2^2) + B_1^2 (x_1^2 + y_1^2) + C_1^2 + 2 A_1 B_1 (x_1 x_3 + x_2 y_1^2)
  + 2 A_1 C_1 x_3 + 2 B_1 C_1 x_1
  = T(A_1, B_1, C_1)
\]

where $T$ is given by

\[
  2 \tau \tau^T = 2 \cdot \begin{pmatrix}
    y_1^2 y_2^2 + x_2^2 y_1^2 + x_3^2 & x_1 x_3 + x_2 y_1^2 & x_3 \\
    x_1 x_3 + x_2 y_1^2 & x_1^2 + y_1^2 & x_1 \\
    x_3 & x_1 & 1
  \end{pmatrix}.
\]

Note that this seemingly extraneous factor of 2 comes from the fact that we are normalizing our forms so that (again abusing notation),

\[
  Q(a, b, c) = \frac{1}{2} (a, b, c) Q(a, b, c)^T.
\]

Now looking at $f_2(\tau)$ and rewriting it, we get

\[
  f_2(\tau) = (A_2 x_4 - B_2 x_2 + C_2)^2 + (A_2 x_1 - B_2)^2 y_2^2 + A_2^2 y_1^2 y_2^2
  = A_2^2 (y_1^2 y_2^2 + x_4^2 + x_2^2 y_2^2) + B_2^2 (x_2^2 + y_2^2) + C_2^2 - 2 A_2 B_2 (x_2 x_4 + x_1 y_2^2)
  + 2 A_2 C_2 x_4 - 2 B_2 C_2 x_2
  = \frac{1}{y_1^2} T'(C_2, B_2, A_2)
\]

where $T'$ is the adjoint matrix of $T$.

This means that, if $\gamma$ has top row $(A_1, B_1, C_1)$ and 2-by-2 minors $C_2$, $B_2$ and $A_2$,
then we can express \( f_1(\tau) \) and \( f_2(\tau) \) as

\[
f_1(\tau) = T((1,0,0)\gamma) = \gamma \circ T(1,0,0) = r_3(\gamma \circ T)
\]
\[
f_2(\tau) = \frac{1}{y^1_1} T'((0,0,1)\gamma') = \frac{1}{y^1_1} (\gamma \circ T)'(0,0,1) = \frac{1}{y^1_1} r_2(\gamma \circ T)
\]

where \( \gamma' \) denotes the adjoint matrix for \( \gamma \). Therefore, we see that in fact the functions \( f_1 \) and \( f_2 \) are closely related to the invariants \( r_2 \) and \( r_3 \), and that

\[
E(\nu_1, \nu_2, \tau) = y^1_1^{3\nu_2} I_{\nu_1, \nu_2}(\tau) \sum_{P \backslash \Gamma} r_2(\gamma \circ T)^{-3\nu_2/2} r_3(\gamma \circ T)^{-3\nu_1/2}
\]
\[
= y_1^{2\nu_1+4\nu_2} y^1_2^{\nu_1+2\nu_2} \sum_{P \backslash \Gamma} r_2(\gamma \circ T)^{-3\nu_2/2} r_3(\gamma \circ T)^{-3\nu_1/2}
\]
\[
= \text{disc}(\tau)^{(\nu_1+2\nu_2)/2} \sum_{P \backslash \Gamma} r_2(\gamma \circ T)^{-3\nu_2/2} r_3(\gamma \circ T)^{-3\nu_1/2}
\]

as desired.
Chapter 5

The Multiple Dirichlet Series as a Product of Double Dirichlet Series

The goal of this chapter is to show that the multiple Dirichlet series \( Z(s_1, s_2, s_3) \) is a Dirichlet series in the variable \( s_3 \) whose coefficients are (as functions of \( s_1 \) and \( s_2 \)) linear combinations of products of double Dirichlet series. To do this, we will make use of genus theory and the results of [3].

§ 5.1 Explicit Coset Representatives

We begin by writing down explicit coset representatives for the quotient \( P \backslash X_3^+ \).

Given a ternary quadratic form, we can represent it by a symmetric matrix

\[
T = \begin{pmatrix}
2p & s & t \\
2s & 2q & u \\
t & u & 2r
\end{pmatrix}
\]

Then we can describe the coset representatives with restrictions on \( p, q, r, s, t \) and \( u \) as follows:

**Proposition 5.1.** A set of coset representatives for \( B(3) \backslash X_3^+ \) is

\[
\left\{ T = \begin{pmatrix}
2p & s & t \\
2s & 2q & u \\
t & u & 2r
\end{pmatrix} \middle| p > 0, \ 0 \leq s \leq 2p - 1, \ q \in \mathbb{Z}, \ q > \frac{s^2}{4p}, \ r \in \mathbb{Z}, \ r > \frac{ps^2 - stu + qt^2}{s^2 - 4pq}, \ (t, u) \in \mathbb{Z}^2/Q\mathbb{Z}^2 \right\}
\]

where \( Q \) is the matrix \( \begin{pmatrix} 2p & s \\ s & 2p \end{pmatrix} \).
Proof. Considering how $B(3)$ acts on $X_3^+$, we compute

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
2p & s & t \\
s & 2q & u \\
t & u & 2r \\
\end{pmatrix}
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
2p & 2pa+s & 2pb+sc+t \\
2pa+s & 2pa^2+2as+2q & 2abp+bs+acs+2cq+at+u \\
2pb+sc+t & 2abp+acs+at+bs+2cq+u & 2pb^2+2bcs+bt+2qc^2+2cu+b+2r \\
\end{pmatrix}
\]

So we see that $T_1 \sim T_2$ if $s_1 \equiv s_2 \mod 2p$. But also, note that if $T_1 \sim T_2$, then

\[
\begin{pmatrix}
t_1 \\
u_1 \\
\end{pmatrix}
= \begin{pmatrix}
t_2 \\
u_2 \\
\end{pmatrix}
+ \begin{pmatrix}
2p & s & t \\
s & 2q & c \\
\end{pmatrix}
\begin{pmatrix}
b \\
0 \\
\end{pmatrix}
.
\]

Now, if $T_1 \sim T_2$ and $s_1 = s_2$, then $a$ is zero since $s_1 = 2p_1a + s_2$. So we see that $T_1 \sim T_2$ implies that $(t_1, u_1)$ and $(t_2, u_2)$ lie in the same coset of $\mathbb{Z}^2/Q\mathbb{Z}^2$, where $Q = \begin{pmatrix} 2p & s \\ s & 2p \end{pmatrix}$.

Therefore, we see that

\[
\left\{ T = \begin{pmatrix} 2p & s & t \\ s & 2q & u \\ t & u & 2r \end{pmatrix} \mid p > 0, \ 0 \leq s \leq 2p - 1, \ q \in \mathbb{Z}, \ q > \frac{s^2}{4p} \right\}.
\]

\[
\left\{ r \in \mathbb{Z}, \ r > \frac{ps^2-stu+at^2}{s^2-4pq}, \ (t, u) \in \mathbb{Z}^2/Q\mathbb{Z}^2 \right\}
\]

is a set of coset representatives for the quotient $B(3) \backslash X_3^+$.

\qed
§ 5.2 Rewriting the Sum with Genus Theory

Using the result of the previous section, we can rewrite the multiple Dirichlet series as

\[
Z(s_1, s_2, s_3) = \sum_{p > 0} \frac{1}{|p|^{s_3}} \sum_{q > s^2/4p} \frac{1}{|s^2 - 4pq|^{s_2}} \sum_{r > (pu^2 - stu + qt^2)/(s^2 - 4pq)} \frac{1}{|r(s^2 - 4pq) + (pu^2 - stu + qt^2)|^{s_1}}.
\]

Notice, however, that the conditions on the first two sums are exactly the same as the bounds we had in the \(GL_2\) case. Therefore, we can express \(Z(s_1, s_2, s_3)\) as

\[
Z(s_1, s_2, s_3) = \sum_{d < 0} \frac{1}{|d|^{s_2}} \sum_{Q \in \Gamma \setminus \Gamma_1} \frac{1}{|Q(1, 0)|^{s_3}} \sum_{r > Q(t,u) \in \mathbb{Z}^2 / \mathbb{Q} \mathbb{Z}^2} \frac{1}{|rd + Q(t,u)|^{s_1}}.
\]

Defining

\[
\zeta_Q(s) = \sum_{\gamma \in \Gamma \setminus \Gamma} \frac{1}{|\gamma \circ Q(1, 0)|^s},
\]

and noting that a translation by \(\gamma \in \Gamma\) will not change the innermost sum, we can rewrite the series as

\[
Z(s_1, s_2, s_3) = \sum_{d < 0} \frac{1}{|d|^{s_2}} \sum_{Q \in \Gamma \setminus \Gamma_1} \frac{1}{\text{disc } Q = d} \sum_{r > Q(t,u) \in \mathbb{Z}^2 / \mathbb{Q} \mathbb{Z}^2} \zeta_Q(s_3) \frac{1}{|rd + Q'(t,u)|^{s_1}}.
\]
Now the innermost sum is just some Dirichlet series, and we can express it as

$$\sum_{\substack{rd+Q'(t,u)<0 \\ (t,u)\in\mathbb{Z}^2/Q\mathbb{Z}^2}} \frac{1}{|rd+Q'(t,u)|^{s_1}} = \sum_{k<0} \frac{A_k(Q)}{|k|^{s_1}},$$

where

$$A_k(Q) = \# \left\{ (t,u) \in \mathbb{Z}^2/Q\mathbb{Z}^2 \mid Q'(t,u) \equiv k \mod d \right\}. \quad (5.1)$$

Now, $A_k(Q)$ depends only on the genus class of $Q$ (modulo $d$) since forms in the same genus class are precisely those which represent the same congruence classes modulo $d$. Therefore, we can break the sum over $\text{SL}_2(\mathbb{Z})$ classes of binary quadratic forms into a sum over the genus classes. Denote that two forms are in the same genus class by $Q_1 \sim Q_2$. This gives

$$Z(s_1, s_2, s_3) = \sum_{d<0} \frac{1}{|d|^{s_2}} \sum_{[Q_i]} \sum_{\text{genus classes mod } d} \sum_{k<0} \frac{A_k(Q_i)}{|k|^{s_1}} \sum_{Q \sim Q_i} \zeta_Q(s_3). \quad (5.2)$$

§ 5.3 The Sum of $\zeta_Q(s)$ Over a Genus Class

The innermost sum of (5.2) is computed in [3] in Proposition 4.1. Namely, Chinta and Offen show that

$$\sum_{Q \sim Q_i} \zeta_Q(s) = \#O_d^{\times} \sum_{2^\omega(d)-1} \chi(Q_i) L_{O_d}(s, \chi) \quad (5.3)$$

where $\chi(d)$ denotes the group of genus class characters of the group of $\text{SL}_2(\mathbb{Z})$ equivalence classes of primitive integral binary quadratic forms $\text{Cl}(d)$, $O_d$ is the
order $\mathbb{Z}[d + \sqrt{d}/2]$, and $L_{\mathcal{O}_d}(s, \chi)$ is defined to be

$$L_{\mathcal{O}_d}(s, \chi) = \sum_{a} \frac{\chi(a)}{N(a)s}.$$ 

In the definition of $L_{\mathcal{O}_d}(s, \chi)$, the sum runs over all invertible ideals of $\mathcal{O}_d$.

Following [3], we can define the genus class characters as follows. First, we define $\chi^{(p)}$ on $\text{Cl}(d)$ by

$$\chi^{(p)}(Q) = \begin{cases} 
\chi_{p'}(a) & \text{if } (p, a) = 1 \\
\chi_{p'}(c) & \text{if } (p, c) = 1 
\end{cases}$$

where $p' = (-1)^{(p-1)/2}p$. Of course, if $Q$ is primitive, then one of the two conditions will be satisfied. Now, writing $d = d_1d_2$ where $d_1$ is an even fundamental discriminant and $d_2$ is an odd discriminant, we can set $d_0$ to be $d_1$ times each distinct prime factor of $d_2$. Then we can define the genus class character $\chi_{e', e''}$, where $e'_1e''_2 = d_0$, as follows:

$$\chi_{e'_1, e''_2} = \prod_{p | e_1} \chi^{(p)}.$$ 

As noted by Chinta and Offen, as $e_1$ ranges over positive odd squarefree divisors of $d$, $\chi_{e_1, e_2}$ ranges over all the genus characters exactly once if $d$ is even and exactly twice if $d$ is odd. Therefore, we can rewrite the innermost sum of (5.2) (that is, (5.3)) as

$$\sum_{Q \sim Q_i} \zeta_Q(s) = \frac{\#\mathcal{O}_d^\times}{2^{\omega(d)-1}} \sum_{e_1|d \text{ odd, squarefree}} \chi_{e'_1, e''_2}(Q_i) L_{\mathcal{O}_d}(s, \chi_{e'_1, e''_2}).$$

Now we can use the results of Kaneko in [8] and Chinta and Offen in [3] to express this L-function in terms of Dirichlet L-functions. Namely, in Proposition 4.2 of [3],
it is shown that for fundamental discriminants \( e_1 \) and \( e_2 \) with \( d = e_1 e_2 f^2 \), that

\[
L_{\mathcal{O}_d}(s, \chi_{e_1, e_2}) = \prod_{p^k \parallel f} P_k(p^{-s}, \chi_{e_1(p), \chi_{e_2(p)}})
\]

(5.4)

where \( P_k(p^{-s}, \chi_{e_1(p), \chi_{e_2(p)}}) \) is the correction polynomial defined by the generating series

\[
F(u, X; \alpha, \beta) = \sum_{k \geq 0} P_k(u, \alpha, \beta) X^k = \frac{(1 - \alpha u X)(1 - \beta u X)}{(1 - X)(1 - pu^2 X)}.
\]

We see, then, that we get products of Dirichlet L-functions coming into the series.

In particular,

\[
Z(s_1, s_2, s_3) = \sum_{d < 0} \frac{1}{|d|^{s_2}} \sum_{[Q_i]} \sum_{k < 0} \frac{A_k(Q_i)}{|k|^{s_1}} \cdot \frac{\#\mathcal{O}_d^\times}{2\omega(d-1)} \cdot \prod_{p \parallel f} P_i(p^{-s_3}, \chi_{e_1(p), \chi_{e_2(p)}}).
\]

(5.5)

In the next chapter, we will carefully compute the values of the coefficients \( A_k(Q_i) \) so that this expression for \( Z(s_1, s_2, s_3) \) becomes explicit.
The goal of this chapter is to compute the coefficients $A_k(Q)$. That is, we need to count

$$A_k(Q) = \# \{(t, u) \in \mathbb{Z}^2 / Q\mathbb{Z}^2 | Q'(t, u) \equiv k \mod d\}$$

where $d = \text{disc } Q$.

We first show that it is enough to consider just the principal forms of discriminant $d$, then compute the coefficients for principal forms.

§ 6.1 Cosetting to Freedom

First define $S_Q$ to be the multi-set

$$S_Q = \{Q'(t, u) | (t, u) \in \mathbb{Z}^2 / Q\mathbb{Z}^2\}.$$

That is, consider the values of $Q'(t, u)$ and retain the multiplicities. Then computing $A_k(Q)$ is the same as counting the multiplicity of $k$ in $S_Q$. Now we prove the following lemma, showing that we need only compute $A_k(Q)$ in the case where $Q$ is a principal form.

**Lemma 6.1.** Let $Q_0$ be the principal form with discriminant $d$, and let $Q$ be a
primitive binary quadratic form with the same discriminant. Then there exists \( \alpha \) in \( \mathbb{Z}/|d|\mathbb{Z}^\times \) such that

\[
S_Q = \alpha S_{Q_0} = \{ \alpha Q'_0(t, u) \mid (t, u) \in \mathbb{Z}^2/Q_0\mathbb{Z}^2 \},
\]

retaining multiplicities as before.

**Proof.** We consider two cases. The first is when \( d \equiv 0 \mod 4 \) and the second is when \( d \equiv 1 \mod 4 \).

**Case 1:** \( d \equiv 0 \mod 4 \). In this case, let \( n = -\frac{d}{4} \). Then we have

\[
Q_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2n \end{pmatrix}, \quad Q'_0 = \begin{pmatrix} 2n & 0 \\ 0 & 2 \end{pmatrix}.
\]

Now suppose we have another binary quadratic form \( Q \) with discriminant \( d \), and \( Q \) and \( Q' \) are given by

\[
Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad Q' = \begin{pmatrix} 2c & -b \\ -b & 2a \end{pmatrix}.
\]

Since \( Q \) and \( Q' \) are primitive, either \( a \) or \( c \) is relatively prime to \( d \). Assume that \( (a, d) = 1 \). The case of \( (c, d) = 1 \) follows from the same argument.

Then note that \( Q'(t, u) = ct^2 - btu + au^2 \), so

\[
aQ'(t, u) = ac t^2 - ab tu + a^2 u^2
\]
\[
= nt^2 + \left( au - \frac{b}{2} t \right)^2
\]
\[
= Q'_0 \left( t, au - \frac{b}{2} t \right).
\]
Of course, since \( d \equiv 0 \mod 4 \), \( b \) must be even, and \( \frac{b}{2} \) is an integer. So we see that

\[
S_Q = \{ a^{-1}Q'_0(t, au - bt/2) \mid (t, u) \in \mathbb{Z}^2 / Q\mathbb{Z}^2 \}.
\]

Now, if we could show that

\[
\{ Q'_0(t, au - bt/2) \mid (t, u) \in \mathbb{Z}^2 / Q\mathbb{Z}^2 \} = \{ Q'_0(t, u) \mid (t, u) \in \mathbb{Z}^2 / Q_0\mathbb{Z}^2 \}
\]

we’d be done.

Consider

\[
T = \left\{ (t, au - bt/2) \mid (t, u) \in \mathbb{Z}^2 / \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \mathbb{Z}^2 \right\}.
\]

Then \( T \) can be rewritten as

\[
T = \left\{ (t, v) \in \begin{pmatrix} 1 & 0 \\ -b/2 & a \end{pmatrix} \mathbb{Z}^2 / \begin{pmatrix} 2a & b \\ 0 & 2n \end{pmatrix} \mathbb{Z}^2 \right\}.
\]

But now, we will see that a set of representatives for \( T \) is also a set of representatives for \( \mathbb{Z}^2 / Q_0\mathbb{Z}^2 \), so

\[
\{ Q'_0(t, au - bt/2) \mid (t, u) \in \mathbb{Z}^2 / Q\mathbb{Z}^2 \} = \{ Q'_0(t, u) \mid (t, u) \in \mathbb{Z}^2 / Q_0\mathbb{Z}^2 \}.
\]

If \((t_1, v_1) \sim (t_2, v_2) \mod \begin{pmatrix} 2a & b \\ 0 & 2n \end{pmatrix}\), where \((t_i, v_i) \in \begin{pmatrix} 1 & 0 \\ -b/2 & a \end{pmatrix} \mathbb{Z}^2\), then

\[
(t_1 - t_2, v_1 - v_2) = \begin{pmatrix} 2a & b \\ 0 & 2n \end{pmatrix} (x, y)^t.
\]
for some $(x, y) \in \mathbb{Z}^2$. Then notice that

\[
\begin{pmatrix} 2a & b \\ 0 & 2n \end{pmatrix} (x, y)^t = \begin{pmatrix} 2 & 0 \\ 0 & 2n \end{pmatrix} \begin{pmatrix} a & b/2 \\ 0 & 1 \end{pmatrix} (x, y)^t,
\]

so in fact, $(t_1, v_1) \sim (t_2, v_2) \mod Q_0$.

On the other hand, if $(t_1, v_1) \sim (t_2, v_2) \mod Q_0$ for $(t_i, v_i) \in \begin{pmatrix} 1 & 0 \\ -b/2 & a \end{pmatrix} \mathbb{Z}^2$, then, letting $t_1 - t_2 = t$ and $v_1 - v_2 = v$,

\[
(t, v) = \begin{pmatrix} 2 & 0 \\ 0 & 2n \end{pmatrix} (x, y)^t
\]

for some $(x, y) \in \mathbb{Z}^2$. But then

\[
\begin{pmatrix} 2 & 0 \\ 0 & 2n \end{pmatrix} (x, y)^t = \begin{pmatrix} 2 & b \\ 0 & 2n \end{pmatrix} \begin{pmatrix} 1/a & -b/2a \\ 0 & 1 \end{pmatrix} (x, y)^t.
\]

But also, since $(t_i, v_i) \in \begin{pmatrix} 1 & 0 \\ -b/2 & a \end{pmatrix} \mathbb{Z}^2$,

\[
(t, v) = \begin{pmatrix} 1 & 0 \\ -b/2 & a \end{pmatrix} (x', y')^t
\]

for some $(x', y') \in \mathbb{Z}^2$, and in particular, $t = x' = 2x$ and $v = -bx'/2 + ay' = -bx + ay' = 2ny$. So

\[
2ny + bx = ay' = (2ac - b^2/2)y + bx
\]

and $bx - b^2/2y \equiv 0 \mod a$. But since $(b, a) = 1$, we have

\[
a|(x - by/2),
\]
implying
\[
\left( \begin{array}{cc} 1/a & -b/2a \\ 0 & 1 \end{array} \right) (x, y)^t \in \mathbb{Z}^2.
\]

Thus, \((t_1, v_1) \sim (t_2, v_2) \mod \left( \begin{array}{cc} 2a & b \\ 0 & 2n \end{array} \right)\). This tells us that a set of representatives for \(T\) is a set of representatives for \(\mathbb{Z}^2/Q_0 \mathbb{Z}^2\), and vice versa, giving us the desired result.

**Case 2:** \(d = 1 \mod 4\). In this case, we set \(n = \frac{1-d}{4}\). Then

\[
Q_0 = \begin{pmatrix} 2 & 1 \\ 1 & 2n \end{pmatrix} \quad Q'_0 = \begin{pmatrix} 2n & -1 \\ -1 & 2 \end{pmatrix}.
\]

Now suppose we have another binary quadratic form \(Q\) of discriminant \(d\), and \(Q\) and \(Q'\) are given by

\[
Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \quad Q' = \begin{pmatrix} 2c & -b \\ -b & 2a \end{pmatrix}.
\]

Again, since \(Q\) and \(Q'\) are primitive, either \(a\) or \(c\) is relatively prime to \(d\). Assume that \((a, d) = 1\). The case of \((c, d) = 1\) follows from the same argument.

Then note that, since \(Q'(t, u) = ct^2 - btu + au^2\),

\[
aQ'(t, u) = act^2 - abtu + a^2u^2
\]

\[
= nt^2 - t \left( au + \frac{1-b}{2}t \right) + \left( au + \frac{1-b}{2}t \right)^2
\]

\[
= Q_0' \left( t, au + \frac{1-b}{2}t \right).
\]
Of course, since $d = b^2 - 4ac \equiv 1 \mod 4$, $\frac{1-b}{2} \in \mathbb{Z}$. Therefore, we see that

$$S_Q = \{a^{-1}Q_0(t, au + (1-b)t/2)| (t, u) \in \mathbb{Z}^2/Q\mathbb{Z}^2\}.$$ 

As in the previous case, we’ll consider the quotient we’re summing over, and show that a set of representatives for this quotient is a set of representatives for $\mathbb{Z}^2/Q_0\mathbb{Z}^2$. We let

$$T = \left\{(t, au + \frac{1-b}{2}) | (t, u) \in \mathbb{Z}^2/Q\mathbb{Z}^2\right\},$$

and see that

$$T = \left\{(t, v) \in \left(\frac{1}{1-b} \begin{array}{c} 0 \\ a \end{array} \right)\mathbb{Z}^2 / \left(\frac{2a}{a} \begin{array}{c} b \\ a \end{array} \frac{b-1}{2} + 2n\right)\mathbb{Z}^2\right\}.$$

So we now want $(t_1, v_1) \sim (t_2, v_2) \mod \left(\frac{2a}{a} \begin{array}{c} b \\ a \end{array} \frac{b-1}{2} + 2n\right)$ if and only if $(t_1, v_1) \sim (t_2, v_2) \mod \left(\begin{array}{c} 2 \\ 1 \end{array} \right) \frac{1}{2n}$ for $(t_i, v_i) \in \left(\frac{1}{1-b} \begin{array}{c} 0 \\ a \end{array} \right)\mathbb{Z}^2$.

First suppose $(t_1, v_1) \sim (t_2, v_2) \mod \left(\frac{2a}{a} \begin{array}{c} b \\ a \end{array} \frac{b-1}{2} + 2n\right)$. Then, letting $t = t_1 - t_2$ and $v = v_1 - v_2$, we know that for some $(x, y)$ in $\mathbb{Z}^2$,

$$(t, v)^t = \left(\frac{2a}{a} \begin{array}{c} b \\ a \end{array} \frac{b-1}{2} + 2n\right) (x, y)^t$$

$$= \left(\begin{array}{c} 2 \\ 1 \end{array} \right) \frac{1}{2n} \left(\begin{array}{c} a \\ 0 \end{array} \right) \frac{b-1}{4} (x, y)^t,$$

showing that $(t_1, v_1) \sim (t_2, v_2) \mod \left(\begin{array}{c} 2 \\ 1 \end{array} \right) \frac{1}{2n}$.

On the other hand, if $(t_1, v_1) \sim (t_2, v_2) \mod \left(\begin{array}{c} 2 \\ 1 \end{array} \right) \frac{1}{2n}$, with $(t_i, v_i) \in \left(\frac{1}{1-b} \begin{array}{c} 0 \\ a \end{array} \right)\mathbb{Z}^2$,
we have, for some \((x, y) \in \mathbb{Z}^2\),

\[
(t, v)^t = \begin{pmatrix} 2 & 1 \\ 1 & 2n \end{pmatrix} (x, y)^t \\
= \begin{pmatrix} 2a & b \\ a & \frac{b-1}{2} + 2n \end{pmatrix} \begin{pmatrix} \frac{1}{a} & \frac{1-b}{2a} \\ 0 & \frac{1}{a} \end{pmatrix} (x, y)^t.
\]

We need only determine, now, that \(\begin{pmatrix} \frac{1}{a} & \frac{1-b}{2a} \\ 0 & \frac{1}{a} \end{pmatrix} (x, y)^t\) is in \(\mathbb{Z}^2\), or in other words, that \(a\) divides \(x + \frac{1-b}{2} y\). Since \((t_i, v_i) \in \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & a \end{pmatrix} \mathbb{Z}^2\),

\[
(t, v) = (2x + y, x + 2ny) = (x', (1-b)x'/2 + ay')
\]

for some \(x'\) and \(y'\) in \(\mathbb{Z}\). Using the system of equations obtained from this relation, we find that

\[
x + \frac{1-b}{2} y = \frac{1}{4n-1} \left( \left( 2n + \frac{b-1}{2} \right) x' - ay' + \left( \frac{-b + b^2}{2} \right) x' + (a - ab)y' \right) \\
= \frac{1}{4n-1} \left( \left( 2n + \frac{b^2 - 1}{2} \right) x' - aby' \right) \\
= \frac{a}{4n-1} (2cx' - by').
\]

However, since \((a,b) = 1\) and \(4n - 1 = -d = 4ac - b^2\), this implies that \((a, 4n - 1) = 1\). Therefore, \(a\) divides \(x + \frac{1-b}{2} y\) as desired.

Hence, we arrive at the desired result. Namely, in both cases,

\[
S_Q = aS_{Q_0}.
\]
where in fact,

\[ \alpha = \begin{cases} 
  a^{-1} & \text{if } (a, d) = 1 \\
  c^{-1} & \text{if } (c, d) = 1 
\end{cases} \]

\[ \Box \]

§ 6.2 Computing $A_k(Q)$ for Principal Forms

As a consequence of the previous section, we only need to compute $A_k(Q)$ for $Q$ a principal form of discriminant $d$. As before, we split into two cases: $d \equiv 0 \mod 4$ and $d \equiv 1 \mod 4$.

6.2.1 The Case $d \equiv 0 \mod 4$

This is the more complicated of the two cases because of the factors of 2. In this case, we have the following proposition.

**Proposition 6.1.** Let $Q$ be the principal binary quadratic form with discriminant $d = -4n$, and let $\omega(a)$ denote the number of distinct odd prime factors of $a$. Suppose $(d, k) = m$ where $m = m_0m_1^2$ with $m_0$ squarefree, and let $m_0 = 2^\delta p_1 \ldots p_\ell$ (where $\delta = 0$ or 1) and $m_1 = 2^{2\alpha_0}q_1^{2\alpha_1} \ldots q_r^{2\alpha_r}$. Let $d' = d/m$ and $k' = k/m$. Assume that
\(m_0^{-1}k'\) is a square modulo every odd prime power of \(d'\). Then

\[
A_k(Q) = \begin{cases} 
    m_12^{\omega(d/m)+\gamma} & \text{if } d'm_0 \equiv 3 \mod 4 \\
    & \text{or } 4 \mid d' \text{ and } m_0 \cdot \frac{d'}{4} \equiv m_0k' \equiv 1 \mod 4 \\
    & \text{or } 2 \nmid m_0 \text{ and } 32 \mid d' \\
    m_12^{\omega(d/m)+\gamma-1} & \text{if } 2\|d' \text{ and } 2 \nmid m_0 \\
    & \text{or } 2 \nmid d' \text{ and } 2 \mid m_0 \\
    & \text{or } m_0d' \equiv 1 \mod 4 \\
    & \text{or } 4 \mid d' \text{ and } 2 \mid m_0 \text{ and } (m_0/2)^{-1}(k' - \frac{d'}{4}) \equiv 2 \mod 8 \\
    & \text{or } 4 \mid d' \text{ and } m_0 \cdot \frac{d'}{4} \equiv 3 \mod 4 \\
    & \text{or } 8\|d' \text{ and } m_0^{-1}k' \equiv 1 \mod 8 \\
    & \text{or } 8\|d' \text{ and } m_0^{-1}k' \equiv 3 \mod 8 \text{ and } 2m_0^{-1} \cdot \frac{d'}{8} \equiv 2 \mod 8 \\
    & \text{or } 8\|d' \text{ and } m_0^{-1}k' \equiv 7 \mod 8 \text{ and } 2m_0^{-1} \cdot \frac{d'}{8} \equiv 6 \mod 8 \\
    & \text{or } 16\|d' \text{ and } m_0^{-1}k' \equiv 1 \mod 8 \\
    & \text{or } 16\|d' \text{ and } m_0^{-1}k' \equiv 5 \mod 8 \\
    0 & \text{otherwise},
\end{cases}
\]

where

\[
\gamma = \begin{cases} 
    0 & \text{if } 2 \nmid d' \text{ or } 2\|d' \\
    1 & \text{if } 4\|d' \\
    2 & \text{if } 8 \mid d'
\end{cases}.
\]

In the course of the proof, it will also become clear that if \(m_0^{-1}k'\) is not a square modulo every prime power divisor of \(d'\), there are zero solutions and \(A_k(Q) = 0\).
Proof. When $Q$ is the principal form of discriminant $d = -4n$, then

$$Q(x, y) = x^2 + ny^2$$

and

$$Q'(x, y) = nx^2 + y^2.$$  

Also, $\mathbb{Z}^2/Q\mathbb{Z}^2$ has representatives

$$\mathbb{Z}^2/Q\mathbb{Z}^2 = \{(t, u)|0 \leq t \leq 1 \text{ and } u \leq 0 \leq 2n - 1\}.$$  

Therefore, the set $S_Q$ is

$$S_Q = \{u^2|0 \leq u \leq 2n - 1\} \cup \{u^2 + n|0 \leq u \leq 2n - 1\},$$

and we have to count the solutions to $u^2 = k$ and $u^2 + n = k$.

First consider the equation

$$u^2 \equiv k \mod d.$$  

Since $(d, k) = m = m_0m_1$, then $u^2$ must also be a multiple of $m$ and in particular, $u = m_0m_1x$ where $x$ ranges from 0 to $(d'/2)m_1$ where $d' = d/m$. Our equation then becomes

$$m_0^2m_1^2x^2 \equiv m_0m_1^2k' \mod m_0m_1^2d'$$

which we can reduce to

$$m_0x^2 \equiv k' \mod d'$$

Notice that $(d', k') = 1$ since we factored out the greatest common divisor of $d$ and
Also, for the equation to have solutions, we must have \((m_0, d') = 1\). If not, that is, if \(p\) divides \(m_0\) and \(d'\), then \(p\) would also divide \(k'\). Therefore, we can invert \(m_0\) modulo \(d'\) and solve

\[
x^2 \equiv m_0^{-1} k' \mod d'.
\]

Now we use the Chinese Remainder Theorem to say that if this equation has solutions, there are \(2^{\omega(d') + \gamma}\) of them modulo \(d'\) where \(\omega(d')\) is the number of distinct odd prime factors of \(d'\) and

\[
\gamma = \begin{cases} 
0 & \text{if } 2 \nmid d' \text{ or } 2\|d' \\
1 & \text{if } 4\|d' \\
2 & \text{if } 8 \mid d'
\end{cases}
\]

However, remember that \(x\) is ranging from 0 to \(m_1[d'/2]\), so we actually have \(m_12^{\omega(d') + \gamma - 1}\) solutions to this equation. Also, note that there are solutions when \((m_0, d') = 1\) and \(m_0^{-1} k'\) is a square modulo every prime power dividing \(d'\).

With this in mind, we turn our attention to the number of solutions to \(u^2 + n \equiv k \mod d\). Recall the notation \((d, k) = m_0m_1^2 = 2^\delta p_1 \ldots p_l 2^{2\alpha_0} q_1^{2\alpha_1} \ldots q_r^{2\alpha_r}\). Then a solution to our equation must have \(u = 2^{\delta + \alpha_0 - \varepsilon} p_1 \ldots p_l q_1^{\alpha_1} \ldots q_r^{\alpha_r} x\) where \(\varepsilon = 0\) or 1. In particular,

\[
\varepsilon = \begin{cases} 
1 & \text{if } \delta = 0 \text{ and } d' \text{ odd} \\
0 & \text{otherwise}
\end{cases}
\]
Using that a solution \( u \) must have this form, we rewrite the equation in question as

\[
2^{2\delta+2\alpha_0-2\varepsilon} p_1 \cdots p_\ell q_1^{2\alpha_1} \cdots q_r^{2\alpha_r} x^2 - \frac{2^\delta p_1 \cdots p_\ell q_1^{2\alpha_1} \cdots q_r^{2\alpha_r} d'}{4} \equiv mk' \mod md'
\]

and further reduce to

\[
2^\delta - 2\varepsilon + 2 p_1 \cdots p_\ell x^2 + d' \equiv 4k' \mod 4d'.
\]

At this point, there are six cases, coming from the two possibilities for \( \delta \) (that is, whether \( m_0 \) is even or odd), and from three possibilities for the two-divisibility of \( d' \) (namely, \( d' \) odd, \( 2 \| d' \) and \( 4 \| d' \)). However, only three of these cases have solutions.

First, if \( 2 \| d' \), then \( \varepsilon = 0 \) and we have

\[
2^\delta + 2 p_1 \cdots p_\ell x^2 + d' \equiv 4k' \mod 4d'.
\]

But if \( 2 \| d' \), then this cannot have solutions. However, we need to look at the number of solutions to this equation in combination with the solutions to \( u^2 \equiv k \mod d \).

For this particular configuration of \( k \) and \( d \), if \( m_0 \) is even, then there are still no solutions since \((m_0, d') = 1\) is a necessary condition for solutions to exist. However, if \( m_0 \) is odd, there are solutions to \( u^2 \equiv k \mod d \) assuming \( m_0^{-1} k' = (p_1 \cdots p_\ell)^{-1} k' \) is a square modulo each prime power divisor of \( d' \).

Next, if \( d' \) is odd and \( m_0 \) is even (i.e. \( \delta = 1 \)), then \( \varepsilon = 0 \) and we have

\[
8 p_1 \cdots p_\ell x^2 + d' \equiv 4k' \mod 4d'.
\]

But \( d' \) odd implies that this has no solutions either. Again looking at the corresponding equation \( u^2 \equiv k \mod d \) in this case, we see that there solutions if
\(m_0^{-1}k' = (2p_1 \ldots p_\ell)^{-1}k'\) is a square modulo each prime power divisor of \(d'\).

In the remaining cases, we reduce to three different equations, each which must be considered separately.

Case 1: \(m_0\) odd, \(2 \nmid d'\). In this case, \(\varepsilon = 1\) and our equation looks like

\[p_1 \ldots p_\ell x^2 + d' \equiv 4k' \mod 4d',\]

and \(x\) ranges from \(0\) to \([4d' \cdot m_1^{-1}]\).

As before, for there to be solutions, it must be the case that \(m_0 = p_1 \ldots p_\ell\) and \(4d'\) are relatively prime. If not, then \(d'\) and \(k'\) would share a common factor \(p_i\). Therefore, we can invert \(m_0\) modulo \(4d'\) and rewrite the equation as

\[x^2 + m_0^{-1}d' \equiv 4m_0^{-1}k' \mod 4d'.\]

Now we can use the Chinese Remainder Theorem and work modulo each prime power dividing \(4d'\) to count solutions. Modulo an odd prime power \(\pi^\beta\) which divides \(4d'\), we get the equation

\[x^2 \equiv 4m_0^{-1}k' \mod \pi^\beta,\]

which has two solutions modulo \(\pi^\beta\) if there are any solutions. Then the only thing to check is what happens modulo \(4\). Modulo \(4\), we get

\[x^2 + m_0^{-1}d' \equiv 0 \mod 4.\]

Then there are two solutions (modulo 4) when \(m_0^{-1}d' \equiv 3 \mod 4\), and there are no
solutions if $m_0^{-1}d' \equiv 1 \mod 4$.

Then, in total, if there are solutions to $u^2 + n \mod d$, there are $2^{\omega d'+1}$ solutions modulo $4d'$. However, since $x$ ranges up to $[4d' \cdot \frac{m_1}{4}]$, we actually have $m_1 2^{\omega(d')-1}$ solutions to $u^2 + n \equiv k \mod d$.

Thinking about the corresponding equation $u^2 \equiv k \mod d$, we see that there are solutions to this equation exactly when $m_0^{-1}k'$ is a square modulo each prime power divisor of $d'$, which is a necessary condition for $u^2 + n \equiv k \mod d$ to have solutions. Therefore, in this case, when $A_k(Q)$ is nonzero, it is equal to

$$A_k(Q) = \begin{cases} m_1 2^{\omega(d') + \gamma} & \text{if } m_0d' \equiv 3 \mod 4 \\ m_1 2^{\omega(d') + \gamma - 1} & \text{if } m_0d' \equiv 1 \mod 4 \end{cases}$$

where $\gamma$ is defined as before (and is zero in this case).

Case 2: $m_0$ odd, $4 \mid d'$. In this case, $\varepsilon = 0$ and our equation looks like

$$p_1 \ldots p_\ell x^2 + d'' \equiv 4k' \mod d',$$

where $d'' = d'/4$ and $x$ ranges from 0 to $[d' \cdot \frac{m_1}{2}]$.

Again, $m_0 = p_1 \ldots p_\ell$ and $d'$ must be relatively prime for there to be solutions, so we may invert $m_0$ to get

$$x^2 + m_0^{-1}d'' \equiv 4m_0^{-1}k' \mod d'.$$

Modulo an odd prime power divisor of $d'$, there are solutions if $m_0^{-1}k'$ is a square. If there are solutions modulo this odd prime power, there are exactly two solutions.
Therefore, we have remaining only the prime 2.

Looking modulo $4 \cdot 2^{\beta_0}$, where $2^{\beta_0} \parallel d''$, we get the equation

$$x^2 + m_0^{-1} 2^{\beta_0} d_0 \equiv m_0^{-1} k' \pmod{4 \cdot 2^{\beta_0}}$$

where $d_0 = d'' / 2^{\beta_0}$.

Now if $\beta_0 = 0$, we are just looking modulo 4, and our equation really looks like

$$x^2 + m_0^{-1} d_0 \equiv m_0^{-1} k' \pmod{4}.$$

This equation has either zero or two solutions, depending on whether $m_0^{-1} d'' = m_0^{-1} d_0$ and $m_0^{-1} k'$ are equivalent modulo 4 or not. If both are 1 or 3 mod 4, then there are two solutions to this equation. However, if they are different, there are no solutions.

Using the Chinese Remainder theorem, we see that if there are solutions to $u^2 + n \pmod{d}$, there are $2^{\omega(d') + \gamma}$ solutions modulo $d'$. However, since $x$ ranges up to $\lceil d' \cdot \frac{m_1}{2} \rceil$, we actually have $m_1 2^{\omega(d') + \gamma - 1}$ solutions to $u^2 + n \equiv k \pmod{d}$.

Looking at the corresponding equation $u^2 \equiv k \pmod{d}$, it is a necessary condition that $m_0^{-1} k'$ be a square modulo each odd prime power of $d'$ for there to be solutions.

There are solutions modulo 4 if $m_0^{-1} k' \equiv 1 \pmod{4}$, and there are no solutions if $m_0^{-1} k' \equiv 3 \pmod{4}$. Therefore, in this subcase, when $A_k(Q)$ is nonzero, we get

$$A_k(Q) = \begin{cases} 
   m_1 2^{\omega(d') + \gamma} & \text{if } m_0 d'' \equiv m_0 k' \equiv 1 \pmod{4} \\
   m_1 2^{\omega(d') + \gamma - 1} & \text{if } m_0 d'' \equiv m_0 k' \equiv 3 \pmod{4} \\
   m_1 2^{\omega(d') + \gamma - 1} & \text{if } m_0 d'' \equiv 3 \pmod{4} \text{ and } m_0 k' \equiv 1 \pmod{4}.
\end{cases}$$
If $\beta_0 = 1$, then we look modulo 8, and our equation looks like

$$x^2 + 2m_0^{-1}d_0 \equiv m_0^{-1}k' \mod 8.$$ 

This equation has either zero or four solutions modulo 8. The possibilities are $2m_0^{-1}d_0 \equiv 2, 6 \mod 8$ and $m_0^{-1}k' \equiv 1, 3, 5, 7 \mod 8$. The only combinations that will yield solutions are the case where $2m_0^{-1}d_0 \equiv 2 \mod 8$ and $m_0^{-1}k' \equiv 3 \mod 8$, and the case where $2m_0^{-1}d_0 \equiv 6 \mod 8$ and $m_0^{-1}k' \equiv 7 \mod 8$.

Using the Chinese Remainder theorem, we see that if there are solutions to $u^2 + n \mod d$, there are $2^{\omega(d')} + \gamma$ solutions modulo $d'$. However, since $x$ ranges up to $[d' \cdot \frac{m_1}{2}]$, we actually have $m_1 2^{\omega(d') + \gamma - 1}$ solutions to $u^2 + n \equiv k \mod d$.

Considering the corresponding equation $u^2 \equiv k \mod d$, we have the same necessary condition for solutions as before, namely that $m_0^{-1}k'$ be a square modulo every prime power divisor of $d'$. We also need $m_0^{-1}k' \equiv 1 \mod 8$, which tells us that we either have solutions to $u^2 + n \equiv k$ or $u^2 \equiv k$, but never both. Therefore, in this subcase, when $A_k(Q)$ is nonzero, we get

$$A_k(Q) = m_1 2^{\omega(d') + \gamma - 1}.$$ 

The case where $\beta_0 = 2$ is similar. The only difference is that, modulo 8, the equation looks like

$$x^2 + (4 \mod 8) \equiv (1, 3, 5, 7) \mod 8.$$ 

where the expression in the parentheses on the left side is $4m_0^{-1}d_0$ and the expression on the right is still $m_0^{-1}k'$. 
Again, to have solutions to $u^2 \equiv k$, we need $m_0^{-1}k' \equiv 1 \mod 8$, but to have solutions to $u^2 + n \equiv k$, we need $m_0^{-1}k' \equiv 5 \mod 8$. These are again mutually exclusive, and we see that, as before,

$$A_k(Q) = m_12^{\omega(d') + \gamma - 1}$$

when $A_k(Q)$ is nonzero.

Finally, if $\beta_0 \geq 3$, we have twice as many solutions as in the previous two subcases. This is because, modulo 8, our equation looks like

$$x^2 \equiv m_0^{-1}k' \mod 8$$

which has 4 solutions if $m_0^{-1}k' \equiv 1 \mod 8$ and 0 otherwise. The conditions $u^2 + n \equiv k$ to have solutions is the same as the condition for $u^2 \equiv k$ to have solutions, so in this subcase, we get that, when $A_k(Q)$ is nonzero, it is

$$A_k(Q) = m_12^{\omega(d') + \gamma}.$$

**Case 3: $m_0$ even, $4 \mid d'$.** In this case, $\varepsilon = 0$ and our equation looks like

$$2p_1 \ldots p_kx^2 + d'' \equiv k' \mod d',$$

where $d'' = d'/4$ and $x$ ranges from 0 to $\lfloor d' \cdot m_3^{-1} \rfloor$.

Note that, since $k'$ and $d'$ must be relatively prime, $k'$ is odd. This tells us that $d''$ must be odd, or there are no solutions.
Modulo any odd prime power $\pi^\beta$ dividing $d'$, we need to solve

$$m_0 x^2 \equiv k' \mod \pi^\beta$$

but $m_0 = 2p_1 \ldots p_\ell$ is relatively prime to $\pi^\beta$ since $m_0$ and $d'$ can only have powers of 2 as common factors. Therefore, we can invert $m_0$ modulo $\pi^\beta$ and solve

$$x^2 \equiv m_0^{-1}k' \mod \pi^\beta.$$  

Then if $m_0^{-1}k'$ is a square, there are two solutions modulo $\pi^\beta$.

Now we check modulo 8. We know $d''$ and $k'$ are odd, and again, our equation looks like

$$2p_1 \ldots p_\ell x^2 + d'' \equiv k'.$$

For there to be solutions, we need $p^{-1}(k' - d'') \equiv 2 \mod 8$, where $p = p_1 \ldots p_\ell$, so that $p^{-1} \cdot \frac{k' - d''}{2} \equiv 1 \mod 4$. If there are solutions, then there are two solutions modulo 4 (and 4 is the largest power of 2 dividing $d'$).

If there are solutions to $u^2 + n \equiv k$ in this case, then, there are $2^{\omega d' + \gamma}$ of them. Since $x$ ranges up to $\lfloor d' \cdot \frac{m_1}{2} \rfloor$, though, we actually have $m_1 2^{\omega (d') + \gamma - 1}$ solutions.

Notice that if $k'$ is odd, then there can’t be solutions to $u^2 \equiv k \mod d$, since our equation would reduce to $m_0 x^2 \equiv k' \mod 4d''$ and $m_0$ is even in this case. Therefore, in this case, when $A_k(Q)$ is non-zero, it is

$$A_k(Q) = m_1 2^{\omega (d') + \gamma - 1}.$$
Having considered all the cases, we see that we arrive at the desired result. \qed

6.2.2 The Case $d \equiv 1 \mod 4$

The case where $d \equiv 1 \mod 4$ is considerably simpler. In particular, we have the following.

**Proposition 6.2.** Let $Q$ be the principal binary quadratic form of discriminant $d = 1 - 4n$, and let $\omega(a)$ denote the number of distinct odd prime factors of $a$ as before. Suppose $(d, k) = m$ where $m = m_0m_1^2$ with $m_0$ squarefree. Then

$$A_k(Q) = \begin{cases} m_12^{\omega(d/m)} & \text{if } Q \text{ represents } k \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** If $Q$ is a principal form with discriminant $d$ where $d \equiv 1 \mod 4$, then, setting $n = (1 - d)/4$, we have $Q(x, y) = x^2 + xy + ny^2$.

If $Q(x, y) = x^2 + xy + ny^2$, then it is given by the matrix

$$Q = \begin{pmatrix} 2 & 1 \\ 1 & 2n \end{pmatrix},$$

so $\mathbb{Z}^2/Q\mathbb{Z}^2$ has representatives

$$\mathbb{Z}^2/Q\mathbb{Z}^2 = \{(1, u)|1 \leq u \leq 2n - 1\} \cup \{(2, u)|2 \leq u \leq 2n\} \cup \{(0, 0)\}.$$  

Also,

$$Q'(x, y) = nx^2 - xy + y^2.$$
Now, we will use the facts that

\[ n - u + u^2 \equiv (2n - u)^2 \mod |d| \]

and

\[ 4n - 2u + u^2 \equiv (u - 1)^2 \mod |d|. \]

These congruences come from

\[ n - u + u^2 - (2n - u)^2 = n - 4n^2 - u + 4nu \equiv (n - u)(1 - 4n) \equiv 0 \mod |d| \]

and

\[ 4n - 2u + u^2 - (u - 1)^2 = 4n - 1 \equiv 0 \mod |d|, \]

where we are using the fact that \(4n \equiv 1 \mod |d|\).

Therefore, the set \( S_Q \) is

\[ S_Q = \{0\} \cup \{(2n - u)^2|1 \leq u \leq 2n - 1\} \cup \{(u - 1)^2|2 \leq u \leq 2n\} \]

\[ = \{0\} \cup \{u^2|1 \leq u \leq 4n - 1\}, \]

and computing \( A_k(Q) \) amounts to counting the solutions to \( u^2 \equiv k \mod |d| \), which was done for the previous case. \( \square \)
CHAPTER 7

THE MAIN RESULT

With the results of the previous two chapters, we will now show that special values of the minimal parabolic $GL_3$ Eisenstein series are equal to linear combinations of products of double Dirichlet series.

§ 7.1 Tying Up the Loose Ends into Beautiful Bows

Recall that on the one hand, from (4.5),

$$Z(s_1, s_2, s_3) = \sum_{T \in \Gamma \setminus \mathcal{X}_3(2)} \frac{1}{|\text{disc}(T)|^{s_1}} E(s_2, s_3, T),$$

and on the other hand, from (5.5) that

$$Z(s_1, s_2, s_3) = \sum_{d < 0} \frac{1}{|d|^{s_2}} \sum_{[Q_i]} \sum_{\genus \text{ classes mod } d} \sum_{k < 0} \frac{A_k(Q_i)}{|k|^{s_1}} \cdot \frac{\#O_d^\times}{2^{\omega(d) - 1}}$$

$$\times \sum_{e_1 | d, \text{ odd, squarefree}} \chi_{e_1 e_2'}(Q_i) L(s_3, \chi_{e_1}) L(s_3, \chi_{e_2}) \prod_{p \parallel f} P_i(p^{-s_3}, \chi_{e_1}(p), \chi_{e_2}(p)).$$

By equating $|k|^{-s_1}$ coefficients, we will get the result.
From (4.5), we see that the $|k|^{-s_1}$ coefficient is

$$
\sum_{T \in \Gamma \setminus \chi_2^+ \atop \text{disc } T = k} E(s_2, s_3, T).
$$

(7.1)

From (5.5), the $|k|^{-s_1}$ coefficient is

$$
\sum_{d < 0} \frac{1}{|d|^{s_2}} \sum_{\left[ Q_i \right] \text{ genus classes } \mod d} \frac{A_k(Q_i)}{2^{\omega(d) - 1}} \sum_{\chi \in \chi(d)} \chi(Q_i) L_{\mathcal{O}_d}(s_3, \chi)
$$

$$
= \sum_{d < 0} \frac{1}{|d|^{s_2}} \sum_{\left[ Q_i \right] \text{ genus classes } \mod d} \frac{A_k(Q_i)}{2^{\omega(d) - 1}} \sum_{\substack{e_1 | d \atop \text{odd, sq.free}}} \chi_{e_1, e_2}(Q_i) L_{\mathcal{O}_d}(s_3, \chi_{e_1, e_2}).
$$

(7.2)

Using the results of the previous chapter, we can replace the $A_k(Q)$ by an explicit value. Note that if $k$ and $d$ are relatively prime, then exactly one genus class will represent $k$. However, if $d$ and $k$ share a factor, then there might be multiple genus classes representing $k$. In particular, if $d$ and $k$ are not relatively prime, but the principal genus class does represent $k$, then using the coset lemma, we see that there are $m_1$ other classes that represent $k$ as well, where $m = (k, d)$. Also, if $k$ is represented by some class and $k$ is not relatively prime to $d$, we can use the ideas from the coset lemma to show that the principal class represents $k$.

In particular, we start by splitting the sum into a sum over $d \equiv 1 \mod 4$ and a sum over $d \equiv 0 \mod 4$. Set

$$
S_1 = \sum_{d < 0} \frac{1}{|d|^{s_2}} \sum_{\left[ Q_i \right] \text{ genus classes } \mod d} \frac{A_k(Q_i)}{2^{\omega(d) - 1}} \sum_{\substack{e_1 | d \atop \text{odd, sq.free}}} \chi_{e_1, e_2}(Q_i) L_{\mathcal{O}_d}(s_3, \chi_{e_1, e_2})
$$
and
\[ S_2 = \sum_{d < 0 \atop d \equiv 0 \mod 4} \frac{1}{|d|^{s_2}} \sum_{[Q_i] \text{ genus classes mod } d} \frac{A_k(Q_i)}{2^{\omega(d) - 1}} \sum_{e_1 \mid d \text{ odd, sq.free}} \chi_{e_1' e_2'}(Q_i) L_{\mathcal{O}_d}(s_3, \chi_{e_1' e_2'}). \]

We consider each sum separately. Writing \( d = e_1 e_2 f^2 \) and considering \( m \mid k \) (i.e. the candidates to be the greatest common divisor of \( k \) and \( d \)), and adding back in the integers \( d \) which are 3 mod 4, we get that \( S_1 \) is
\[ S_1 = \sum_{m \mid k} \sum_{\mu_0 \mid m_0} \sum_{e_1 = \mu_0 e_1 e_{12}} \sum_{m_1 \mid m_2} \frac{1 + \chi_{-4}(e_1 e_2)}{(e_1 e_2)^{s_2}} \sum_{f = \mu_1 \varphi} \frac{2^{\omega(d') - \omega(d)}}{|\mu_1 \varphi|^{2s_2}} \chi_{e_1' e_2'}(k') L_{\mathcal{O}_d}(s_3, \chi_{e_1' e_2'}) \]
\[ = \sum_{m \mid k} m_1 2^{\omega(m_1)} \sum_{e_1, e_2} \frac{1}{(e_1 e_2)^{s_2}} \sum_{f > 0, \text{ odd}} \frac{1}{f^{2s_2}} \chi_{k'}(e_1) L_{\mathcal{O}_d}(s_3, \chi_{e_1' e_2'}) \]
\[ + \sum_{m \mid k} m_1 2^{\omega(m_1)} \sum_{e_1, e_2} \frac{\chi_{-4}(e_1 e_2)}{(e_1 e_2)^{s_2}} \sum_{f > 0, \text{ odd}} \frac{1}{f^{2s_2}} \chi_{k'}(e_1) L_{\mathcal{O}_d}(s_3, \chi_{e_1' e_2'}), \]
where the parameters of the sums have the additional conditions that the \( e_i \) are odd and squarefree with \( \varepsilon_i \) relatively prime to \( k' \) and to each other. As before, \( d' \) denotes \( d/m \) and \( k' \) denotes \( k/m \), and \( m = m_0 m_1^2 \) with \( m_0 \) squarefree.

As in [3], we can consider the innermost sum (over \( f \)) and rewrite it using (5.4).
This gives

\[
S_1 = \sum_{m \mid k} m 2^{\omega(m)} \sum_{e_1, e_2} \chi_k'(e_1) \chi_k''(e_2) L(s_1, \chi_{e_1}) L(s_3, \chi_{e_2}) \prod_{p \mid k'} \sum_{i=0}^{\infty} \frac{P_i(p^{-s_1}, \chi_{e_1}(p), \chi_{e_2}(p))}{p^{2is_2}}
\]

\[+ \sum_{m \mid k} m 2^{\omega(m)} \sum_{e_1, e_2} \chi_{-4}(e_1 e_2) \chi_k'(e_1) \chi_k''(e_2) L(s_1, \chi_{e_1}) L(s_3, \chi_{e_2}) \prod_{p \mid k'} \sum_{i=0}^{\infty} \frac{P_i(p^{-s_1}, \chi_{e_1}(p), \chi_{e_2}(p))}{p^{2is_2}}
\]

\[
= \sum_{m \mid k} m 2^{\omega(m)} \sum_{e_1, e_2} \chi_k'(e_1) \chi_k''(e_2) L(s_1, \chi_{e_1}) L(s_3, \chi_{e_2}) \times \frac{\zeta_{2k'}(2s_2) \zeta_{2k'}(2s_2 + 2s_3 - 1)}{L_{2k'}(s_1 + 2s_2, \chi_{e_1}) L_{2k'}(s_3 + 2s_2, \chi_{e_2})} + \sum_{m \mid k} m 2^{\omega(m)} \sum_{e_1, e_2} \chi_{-4}(e_1 e_2) \chi_k'(e_1) \chi_k''(e_2) L(s_1, \chi_{e_1}) L(s_3, \chi_{e_2}) \times \frac{\zeta_{2k'}(2s_2) \zeta_{2k'}(2s_2 + 2s_3 - 1)}{L_{2k'}(s_1 + 2s_2, \chi_{e_1}) L_{2k'}(s_3 + 2s_2, \chi_{e_2})},
\]

where \( \zeta_n \) and \( L_n \) denote zeta functions and L-functions with the Euler factor at primes dividing \( n \) removed. Then we see that \( S_1 \) can be expressed as a linear combination of products:

\[
S_1 = \sum_{\psi=1, \chi_{-4}} \sum_{m \mid k} m 2^{\omega(m)} \zeta_{2k'}(2s_2) \zeta_{2k'}(2s_2 + 2s_3 - 1) \times \left( \sum_{e_1} L(s_3, \chi_{e_1}) \psi \chi_k'(e_1) e_1^{s_2} L_{2k'}(s_3 + 2s_2, \chi_{e_1}) \right) \left( \sum_{e_2} L(s_3, \chi_{e_2}) \psi(e_2) e_2^{s_2} L_{2k'}(s_3 + 2s_2, \chi_{e_2}) \right).
\]

The expressions in parentheses are double Dirichlet series that come up in the Fourier Whittaker expansion of the GL3 Eisenstein series.
The computation of $S_2$ is similar, and we get that $S_2$ is

$$S_2 = \sum_{m|k} \frac{m_1 2^{\omega(m_1)}}{4^{s_2}} \zeta_{k'}(2s_2)\zeta_{k'}(2s_3 + 2s_2 - 1)$$

$$\times \left( \sum_{e_1} \frac{L(s_3, \chi_{e_1}) \chi_{-4}(e_1)}{e_1^{s_2} L_{k'}(s_3 + 2s_2, \chi_{e_1})} \right) \left( \sum_{e_2} \frac{L(s_3, \chi_{-4e_2})}{e_2^{s_2} L_{k'}(s_3 + 2s_2, \chi_{-4e_2})} \right).$$

Therefore, we see that the $k^{-s_1}$ coefficient in the series $Z(s_1, s_2, s_3)$ is both equal to a sum of special values of the minimal parabolic $GL_3$ Eisenstein series, and a linear combination of double Dirichlet series that arise as Fourier coefficients of the Eisenstein series on the double cover of $GL_3$. 
CHAPTER 8

CONCLUSION

§ 8.1 Future Research

A natural next step is to prove the analogous results for the maximal parabolic $GL_3$ Eisenstein series described, for example, in [5]. The same methods used in this dissertation can be used to attack this problem, and in fact, this case is expected to be easier. The only differences will be that instead of acting by the minimal parabolic subgroup $P$, we will use the action of the maximal parabolic subgroup, and that in this case there will be two relative invariants rather than three. In fact, I have some preliminary results in this direction, and I hope to finish this problem soon.

Once I have obtained the results for the maximal parabolic $GL_3$ Eisenstein series, I plan to work on the maximal parabolic $GL_3$ Eisenstein series induced from a $GL_2$ cusp form. This should involve introducing some numerator other than 1 in the definition of the multiple Dirichlet series. Proving results in this case would be very exciting because they would suggest how to induct up from $GL_3$ to higher $GL_n$.

It seems that similar methods could be applied to the general $GL_n$ case. However, it may also be that the genus theory necessary becomes significantly more complicated. Still, I remain optimistic.
BIBLIOGRAPHY


