Stability and instability in the Stefan problem with surface tension

by

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Vitæ

Mahir Hadžić was born in Zvornik, Bosnia-Herzegovina, on August 24, 1982 . He obtained his Sc.M. degree in Mathematics from University of Vienna, Austria in May 2005. In the fall of 2005, he started a Ph.D. program in the Division of Applied Mathematics at Brown University, under the supervision of Professor Yan Guo. While at Brown he was supported by research assistantships and a teaching fellowship. He won a Brown University Sigma Xi award in May 2010.

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Dedicated to my parents Almasa and Huso Hadžić

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by Mahir Hadžić, Ph.D., Brown University, May 2010

We develop a high-order nonlinear energy method to study the stability of steady states of the Stefan problem with surface tension. There are two prominent classes of steady states: flat planes and round spheres.

In the case of steady planes, we prove that the equilibria are always stable, asymptotically converging to a nearby flat hypersurface, in arbitrary dimesnions. Our proof relies on an energy method along the fixed domain.

In the case of steady spheres, we establish sharp nonlinear stability and instability criterion in arbitrary dimensions. Our nonlinear stability proof relies on an energy method along the moving domain, and the discovery of a new "momentum conservation law". Our nonlinear instability proof relies on a variational framework which leads to the sharp growth rate estimate for the linearized problem, as well as a bootstrap framework to overcome the nonlinear perturbation with severe high-order derivatives. The instability result is surprising, as it stands in stark contrast to the known stability results for the related Mullins-Sekerka problem.

CHAPTER 1

Introduction

Stefan problem is one of the best known two phase free boundary problems. It has been studied in various mathematical and physics literature for over hundred and fifty years, owing its importance not just to the physical models it describes, but also to its phenomenological character: it is a simple model that "exhibits some attractive mathematical features, which also underlie more sophisticated models", as phrased in the Introduction to [**53**]. The main focus of this thesis is the Stefan problem *with surface tension*; it accounts for a range of phenomena such as undercooling, superheating, phase nucleation (i.e. spontaneous creation of phases), and crystal growth.

1. Stefan problem with surface tension and related models

The unknowns are the temperature functions $u^{\pm}$ of the different phases and the *free* (or *moving*) boundary Γ between them. While the temperature satisfies the heat conduction equation in the phases, the problem is *nonlinear* since the boundary is *a-priori unknown* and its geometry is nonlinearly coupled to the temperature functions $u^{\pm}$. Let $\Omega \subset \mathbb{R}^n$ contain the two phases $\Omega^+(t)$ and $\Omega^-(t)$, i.e. $\bar{\Omega^-} = \Omega \setminus \Omega^-(t)$. If $u^{\pm}:\mathbb{R}_+ \times \Omega^{\pm} \to \mathbb{R}$ and $\Gamma(t) = \partial\Omega^+(t) \cap \partial\Omega^-(t)$ are the unknowns, then the Stefan problem with surface tension reads as follows:

$$u_t^{\pm} - \Delta u^{\pm} = 0 \quad \text{in } \Omega^{\pm} \tag{1.1}$$

$$u^+ = u^- = \sigma\kappa \quad \text{on } \Gamma(t) \tag{1.2}$$

$$[\partial_n u]_-^+ = V, \tag{1.3}$$

FIGURE 1. Jožef Stefan

where κ and V are the mean curvature and normal velocity of the hypersurface Γ respectively. $\sigma>0$ is a constant called *coefficient of surface tension*. The jump of normal derivatives across Γ is denoted by $[\partial_n u]^+_-$ i.e.:

$$[\partial_n u]^+_- := \partial_n u^+ - \partial_n u^-.$$

Note that the boundary condition (1.2) implies that we can define a *continuous* function $u:\mathbb{R}_+\times\Omega\to\mathbb{R}$ such that u agrees with $u^\pm$ on $\Omega^\pm$ respectively. On the other hand, boundary condition (1.3) implies a *discontinuity* in the normal derivatives of u at the boundary Γ. Stefan problem was originally introduced under the assumption that phase transition occurs at a constant temperature. In other words, surface tension is neglected and by setting $\sigma=0$ in (1.2), new Dirichlet boundary condition reads

$$u^+ = u^- = 0 \quad \text{on} \quad \Gamma(t). \tag{1.4}$$

This model was used by the Slovenian physicist and poet Jožef Stefan in 1889 to describe melting of polar ice caps (testifying to the importance Stefan recognized in this model), cf. [**50**]. It is usually called the *classical Stefan problem*. From the mathematical point of view, it is very important to note that (1.4) allows us to use maximum principle techniques and with suitable boundary conditions characterize the phase by the sign of the temperature function u. This insight was used to develop various notions of weak solutions to the classical Stefan problem and establish their regularity.

Adding surface tension to the problem is by no means a “small modification”. Physically, we justify the presence of surface tension by adding a requirement that there must exist a surface energy contribution proportional to the surface area of Γ. It helps stabilize the system by counteracting the release of latent heat across the moving boundary that drives it. Condition (1.2) is referred to as the *Gibbs-Thomson correction*. We lose the maximum principle, and thus locally around the boundary we may encounter an *undercooled* liquid ($u<0$ in part of the liquid region) or a *superheated* solid ($u>0$ in part of the solid region). These kinds of phenomena are well observed in practice and it is in fact possible to cool water down to $-42°$C and keep it in a liquid state at the same time.

We quote here a paragraph from the introduction to the seminal work of ALMGREN and WANG [**2**]: “When the first author began studying models for crystal growth, it was not clear to him that several common assumptions about the way in which crystals may freeze were simultaneously mathematically compatible. It seemed a stringent condition to impose that crystals and temperature should interact in such a way that interface surface tension should exactly balance temperature. One implication of such a requirement is that a temperature maximum principle cannot hold for heat flow. Indeed, on the boundary of a small shrinking crystal with high curvatures, temperatures would have to be low and become even lower as the crystal melts. Similarly, on the boundary of a small shrinking hole in a crystal with large curvatures of the opposite sign, temperatures would have to be high and become even higher as the hole freezes.” This should caution the reader that it is not a-priori clear how surface tension “stabilizes” the system. Indeed, in their paper, the authors prove a global existence result (without uniqueness) for the weakly formulated two-phase Stefan problem, as the curvature condition (1.2) allows for a suitable weak formulation in the language of geometric measure theory. A similar deep result around the same time was proven by LUCKHAUS [**41**] (with no uniqueness).

In the strong formulation we demand that the solution be smooth enough so that the equations (1.1) - (1.3) hold in the classical sense. As a consequence, we expect singularity formation for general initial data as suggested by numerical and experimental work, cf. R. Almgren [**1**], Visintin [**53**] and references therein, respectively. Yet, we should expect well-posedness locally-in-time and try to understand global behavior in regimes close to the equilibrium. This thesis focuses on the strong formulation of the problem and surface tension will help augment the energy with a positive definite coercive term, that will prove necessary to conduct various energy estimates.

Important variants of both the classical and surface tension case arise when the heat equation (1.1) is replaced by

$$\Delta u^{\pm} = 0 \quad \text{in} \quad \Omega^{\pm}(t). \tag{1.5}$$

Without surface tension, the problem (1.5), (1.4), and (1.3) is called the two-phase *quasi-stationary Stefan problem* or the two-phase *Hele-Shaw* problem. When surface tension is included, it is typically referred to as *Mullins-Sekerka* or two phase Hele-Shaw problem with surface tension. In certain regimes, these problems can be used as an approximation to the Stefan problem. In the two dimensional setting, Hele-Shaw type problems arise as equations of motion of a fluid compressed between two thin plates.

2. Main results

We shall note that there are two prominent classes of time-independent solutions (equilibria) to the Stefan problem with surface tension: *flat hypersurfaces* and *round spheres*. Namely, it is easy to check that

$$(\bar{u}, \bar{\Gamma}) \equiv (0, \{x_n = c\}) \quad \text{or} \quad (\bar{u}, \bar{\Gamma}) \equiv (\frac{n-1}{R}, S_R(\mathbf{x}))$$

satisfy the equations (1.1) - (1.3). Here $c \in \mathbb{R}$ is a constant and $S_R(\mathbf{x})$ stands for a sphere of radius $R > 0$ centered at $\mathbf{x} \in \mathbb{R}^n$. A natural important question is that of dynamical stability of the above steady states. To answer it, one needs to develop an appropriate functional analytic framework, within which a notion of *dynamical* *perturbation* can be formulated, as well as an accompanying well-posedness theory in these spaces. As it turns out, choice of these spaces is dictated by the natural energy norms of parabolic Sobolev type. In the case that a particular equilibrium is stable, we wish to prove a global-in-time well-posedness result together with a rate of decay estimate towards the equilibrium. In the case an equilibrium is dynamically unstable, we require a refined understanding of the underlying linear instability so to characterize *morphological changes* occurring in the system.

2.1. The flat case. Chapter 2 concerns the stability of the flat steady states. We choose to work in a periodic box (with respect to the first $(n-1)$ directions). We set $\Omega = \mathbb{T}^{n-1} \times [-1,1]$ and on the 'top' and 'bottom' boundaries $\mathbb{T}^{n-1} \times \{x_n = \pm 1\}$, we impose the homogeneous Neumann condition, $\partial_n u^{\pm} = 0$. Denote the 1-parameter family of steady state solutions by $(\bar{u}, \bar{\Gamma})_c \equiv (0, \{x_n = c\})$. We then loosely state the following stability theorem:

THEOREM 2.1 (Loose statement of Theorem 1.1, Chapter 2). *The steady state $(\bar{u}, \bar{\Gamma})_c$ is dynamically stable: for any sufficiently small perturbation (u_0, Γ_0) of $(\bar{u}, \bar{\Gamma})_c$, the corresponding initial value problem has a unique global-in-time classical solution. Furthermore, this solution converges exponentially fast to a 'nearby' steady state $(\bar{u}, \bar{\Gamma})_{c^*}$ for some constant c^* close to c.*

Theorem 2.1 is the *first global existence and uniqueness result* for classical solutions to the two-phase Stefan problem with surface tension, established by HADŽIĆ & GUO [**31**].

2.2. The round case. We are interested in the $(n+1)$-parameter family of solutions: $(\bar{u}, \bar{\Gamma})_{\mathbf{x},R} \equiv (n-1/R, S_R(\mathbf{x}))$, where the radius R and the n coordinates determining $\mathbf{x}$ account for $(n+1)$ degrees of freedom in the choice of the steady state. Furthermore, we fix the outer domain $\Omega = B_{R_*}(0)$ to be a ball of fixed radius $R_* > 1$ and we impose the homogeneous Neumann boundary condition on $\partial\Omega$, namely $u_n = 0$. Surprisingly, this steady state may be unstable, depending on the size of the radius R_* of the outer domain Ω. This is in *stark contrast* to the stability result for the related Hele-Shaw model with surface tension [**12**] and [**16**] and also to our result for the flat case. We *characterize* the stable/unstable behavior by introducing the parameter

$$\zeta := \frac{1}{|\Omega|} - \frac{n-1}{|\mathbb{S}^{n-1}|} = \frac{1}{R_*^n \omega_n} - \frac{n-1}{n\omega_n R^{n-1}},$$

where ω_n stands for the volume of the unit ball in $\mathbb{R}^n$. If $\zeta > 0$ (i.e. $R_* < \left(\frac{nR^{n-1}}{n-1}\right)^{1/n}$), then the following stability statement holds:

THEOREM 2.2 (Loose statement of Theorems 1.1 and 1.2 in Chapter 3). *Assume $\zeta > 0$. The steady state $(\bar{u}, \bar{\Gamma})$ is dynamically stable: for any sufficiently small perturbation (u_0, Γ_0) of $(\bar{u}, \bar{\Gamma})$, the corresponding initial value problem has a unique global-in-time classical solution. Furthermore, this solution converges exponentially fast to a 'nearby' steady state $(\tilde{u}, \tilde{\bar{\Gamma}})_{\tilde{\mathbf{x}},R}$ for some constant R close to 1 and $\tilde{\mathbf{x}}$ close to $\mathbf{x}$. These 'nearby' steady states are completely determined by the initial datum.*

If, on the other hand, $\zeta < 0$ (i.e. $R_* < \left(\frac{nR^{n-1}}{n-1}\right)^{1/n}$), we prove that the linearized Stefan problem has one strictly positive eigenvalue λ, which indicates an instability. We turn this into a nonlinear instability result:

THEOREM 2.3 (Loose statement of Theorem 1.3 in Chapter 3). *Assume $\zeta < 0$. Consider a 1-parameter family of generic initial data $(u_0, f_0)_\delta$, δ-distance away from the steady state $(n-1/R, S_R(\mathbf{x}))$. Then the associated solution $(u^\delta(t), f^\delta(t))$ is dynamically unstable: there is a positive constant K independent of δ, such that for $0 < t \le T^\delta \approx \log\frac{1}{\delta}$, we have $||u^\delta(t) - 1, f^\delta(t)||_{L^2} \ge K$.*

The instability of $(\bar{u}, \bar{\Gamma}) \equiv (n-1, \mathbb{S}^{n-1})$ does not contain morphological changes as we show that the unstable eigenvector is radially symmetric. This implies shrinking or spreading along radial directions only.

Nonlinear stability and instability statements above have been proved in two-dimensional setting by HADŽIĆ & GUO [**32**] and in arbitrary dimensions by HADŽIĆ [**33**], content of which basically comprises Chapter 3 of the thesis.

In Chapter 4 we observe that the above characterization of stability through the sign of ζ confirms the so-called "Poincaré turning point principle". It states that the stability of a family of steady states is transformed to instability as the associated "binding energy" passes through a critical point. This is made precise in Chapter 4 and further comments are made regarding the aspects of nucleation phenomena that Stefan model tries to capture.

Our results are a continuation of the program of developing a robust energy method to investigate and characterize morphological stabilities/instabilities in the setting of free boundary problems in partial differential equations.

3. Previous work

It has been known that the classical Stefan problem admits unique global weak solutions in several dimensions as shown in the works of Friedman [**23, 24**] and Kamenomostskaya [**37**]. Under suitable restrictions, regularity of weak solutions for the one-phase problem has been established by Friedman & Kinderlehrer[**25**], Caffarelli [**8, 9**], Kinderlehrer & Nirenberg [**38, 39**]. Regularity results for the two-phase problem were obtained by DiBenedetto [**19**], Caffarelli & Evans [**10**], among others. Existence of regular viscosity solutions was shown in a series of important papers by Athanasopoulos, Caffarelli & Salsa [**4, 5, 6, 7**]. For a comprehensive overview we recommend the monograph of Caffarelli & Salsa [**11**].

As to the classical solutions, Hanzawa [**35**] and Meirmanov [**43**] prove the existence of solutions, but without well-posedness, as the regularity is lost with respect to the initial smoothness assumptions. This gap is bridged and local well-posedness is established in a forthcoming work by Hadžić & Shkoller [**34**].

As to the Mullins-Sekerka problem (aka Hele-Shaw problem with surface tension), global existence and stability for the two-phase problem close to a sphere in two dimensions has been obtained by Constantin & Pugh [**16**] and Chen [**12**], and in arbitrary dimensions by Escher & Simonett [**22**]. Global stability for the one-phase quasi-stationary Stefan problem is established by FRIEDMAN & REITICH [**27**]. Local-in-time solutions in parabolic Hölder spaces in arbitrary dimensions are established by CHEN, HONG & YI [**13**].

As to the Stefan problem with surface tension, global weak existence theory (without uniqueness) is developed by ALMGREN & WANG [**2**], LUCKHAUS [**41**] and there is an important subsequent improvement of Luckhaus' result by RÖGER [**48**]. The authors employ a time-discretized scheme, updated through a sequence of variational problems, and rely on geometric measure theory methods to pass to the limit. More intuitively speaking, there is a gradient flow structure in the problem that allows this procedure to work. FRIEDMAN & REITICH [**26**] consider the Stefan problem with small surface tension i.e. $\sigma \ll 1$ in (1.2). The local existence of classical solutions for the Stefan problem is studied by RADKEVICH [**47**]. ESCHER, PRÜSS & SIMONETT [**20**] prove a local existence and uniqueness result in suitable Besov spaces, relying on the L^p-regularity theory. They work under the assumption that Γ is a graph. Linear stability and instability results for steady spheres are obtained by PRÜSS & SIMONETT [**46**].

If kinetic undercooling is included in the model, namely if the Gibbs-Thomson condition (1.2) is replaced by

$$u^+ = u^- = \sigma\kappa + V, \tag{3.6}$$

then the problem (1.1), (3.6) and (1.3) is called Stefan problem with surface tension and kinetic undercooling. Local existence is shown by CHEN & REITICH [**15**], RADKEVICH [**47**]. SONER [**49**] proves global existence of weak solutions by passing to a limit in the related phase field model.

The list of above references is only the tip of the iceberg, as the Stefan problem has been studied in a variety of mathematical literature over the past century. For a comprehensive overview of mathematical references to various versions of the Stefan model we refer the reader to the monographs of VISINITIN [**52, 53**].

4. Brief overview of ideas and methods

In the following we present an informal description of basic ideas and tools used to prove the main theorems.

4.1. Leitmotiv: Nonlinear energy method. Nonlinear energy method is a robust general idea used to prove global-in-time existence of solutions. It consists of two main steps:

- identifying the natural energy structure: *instant energy* $\mathcal{E}$ and *dissipation* $\mathcal{D}$.
- proving an inequality of the type:

$$\sup_{0\le s\le t}\mathcal{E}(s)+\int_0^t \mathcal{D}(s)\,ds\le \mathcal{E}(0)+C\sup_{0\le s\le t}\sqrt{\mathcal{E}(s)}\int_0^t\mathcal{D}(s)\,ds. \tag{4.7}$$

Note that under the smallness assumption on $\sup_{0\le s\le t}\mathcal{E}(s)$, we may absorb the rightmost term in (4.7) into the left-hand side, to obtain an a-priori estimate

$$\sup_{0\le s\le t}\mathcal{E}(s)+\frac{1}{2}\int_0^t \mathcal{D}(s)\,ds\le \mathcal{E}(0),$$

thus recovering the smallness assumption given $\mathcal{E}(0)$ is assumed small. The argument above is obviously circular, yet in applications it is usually implemented via an iteration scheme, allowing us to prove an estimate of the form:

$$\sup_{0\le s\le t}\mathcal{E}^{m+1}(s)+\int_0^t \mathcal{D}^{m+1}(s)\,ds\le \mathcal{E}(0)+C\sup_{0\le s\le t}\sqrt{\mathcal{E}^m(s)}\int_0^t\mathcal{D}^{m+1}(s)\,ds.$$

4.2. Conservation Laws. While the above is indeed the main idea luring in the background of the proof of Theorem 2.1 (i.e. Theorem 1.1, Chapter 2), it is certainly insufficient to prove the stability theorem in the round case i.e. Theorem 2.2 (i.e. Theorems 1.1 and 1.2, Chapter 3). The reason is that the natural energy is **not** positive definite in the round case, due to the presence of $(n+1)$ neutral modes in the linearized problem. Thus, a major new input is the discovery of conservation laws, that allow us to control the neutral directions, as explained in Introduction to Chapter 3. As a consequence, more than merely a stability statement is proved. Namely, the conserved quantities allow us to determine the exact asymptotic state of the solution as well as the exponential decay rate, as formulated in Theorem 1.2.

4.3. Instability result. In the instability regime (i.e. $\zeta < 0$, round case) linear problem possesses one positive eigenvalue and the challenge is to turn this *linear* instability into a *rigorous nonlinear* instability proof. This is accomplished by using a profound general technique developed by Guo & Strauss [**30**], that requires precise understanding of growth rates for the linear problem. Thus, a non-standard variational problem is devised to deal with this issue, as well as an appropriate functional analytic framework to address the notion of perturbation, dictated by the variational structure of the eigenvalue problem.

4.4. Some technical aspects. Constructing a solution to free boundary problems is a non-trivial task in general, even after the a-priori estimates are available. For instance, recent progress in the well-posedness theory for incompressible and compressible Euler equations with free boundary illustrates this point quite vividly. Among others, the reader may consult the works of Coutand & Shkoller [**17, 18**] or Jang & Masmoudi [**36**].

A common technical ingredient in proofs of Theorems 2.1 and 2.2 is the construction scheme for the solution. We choose to implement an interation scheme: we update the $(m+1)$-st step by solving a linear problem with boundary conditions from the m-th step. In doing so, we destroy the natural energy estimate, but we compensate for this by adding artificial viscosity of fourth order to (1.3). Details are described in Section 3 of Chapter 2 and Subsection 3.1 of Chapter 3. There is however a major difference between our analysis of the flat and the round case. In Chapter 2 we choose to facilitate our estimates by fixing the domain via the so-called Hanzawa transform. In Chapter 3 however, fixing the domain would amount to use of polar coordinates, creating an artificial singularity at 0. We choose to work on a moving domain, making the energy contribution from the bulk transparent and compactly expressed, but turning the extraction of energy on the moving boundary technically more complicated, yet *intrinsically geometric* and thus more natural. A new structure is recognized, via the so-called *reduction* procedure as described in Subsection 3.2.3, Chapter 3.

CHAPTER 2

The flat case

1. Introduction

We shall work in $\Omega = \mathbb{T}^{n-1} \times [-1,1]$ where $\mathbb{T}^{n-1}$ stands for an $(n-1)$-dimensional torus. Let us assume that the moving interface $\Gamma(t)$ is a graph given by $x_n = \rho(t,x')$. Here $\rho: [0,T] \times \mathbb{T}^{n-1} \to \mathbb{R}$ is some smooth function such that $\bigcup_{0 \le t \le T} \Gamma(t) \subset \Omega$ and $T > 0$. Define the liquid/solid phase $\Omega^{\pm}(t)$ by setting $\Omega^{\pm}(t) = \left\{(x', x_n) \in \Omega \,\middle|\, x_n \gtrless \rho(t,x')\right\}$. We note that $\Omega = \Omega^+(t) \cup \Omega^-(t)$. In order to formulate the problem we first specify the initial conditions. Let $\Gamma_0 = \text{graph}(\rho_0)$ be the initial position of the free boundary and $v_0 : \Omega \to \mathbb{R}$ be the initial temperature. The unknowns are the interface $\left\{\Gamma(t); t \ge 0\right\}$ and the temperature function $v : [0,T] \times \Omega \to \mathbb{R}$. We denote the normal velocity of Γ by V and normalize it to be positive if Γ is locally expanding $\Omega^+(t)$. The mean curvature of $\Gamma(t)$ is given by

$$\kappa(t) = \nabla \cdot \Big(\frac{\nabla \rho(t)}{\sqrt{1 + |\nabla \rho(t)|^2}}\Big).$$

The Stefan problem with surface tension is now given by:

$$\partial_t v - \Delta v = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.8}$$

$$v = \kappa(t) \qquad \text{on } \Gamma(t),\ t \ge 0, \tag{1.9}$$

$$\partial_n v = 0 \qquad \text{on } \mathbb{T}^{n-1} \times \{x_n = \pm 1\} \tag{1.10}$$

$$V = [\partial_\nu v]^+_- \qquad \text{on } \Gamma(t),\ t > 0, \tag{1.11}$$

$$v(0,\cdot) = v_0 \qquad \text{in } \Omega, \tag{1.12}$$

$$\Gamma(0) = \Gamma_0. \tag{1.13}$$

Given v we write v^+ and v^- for the restriction of v to $\Omega^+(t)$ and $\Omega^-(t)$, respectively. With this notation $[\partial_\nu v]^+_-$ stands for the jump of the normal derivatives across the interface $\Gamma(t)$, namely

$$[\partial_\nu v]_-^+ := \partial_\nu v^+ - \partial_\nu v^-,$$

where ν stands for the unit normal on the hypersurface $\Gamma(t)$ with respect to $\Omega^-(t)$. If we replace the boundary condition (1.9) with

$$v = 0 \qquad \text{on} \quad \Gamma(t),\;\; t \geq 0, \tag{1.14}$$

then we are referring to the *classical Stefan problem*.

The difficulties in dealing with the existence of solutions of the problem (1.8) - (1.13) arise from the nonlinear coupling between the temperature v and the boundary ρ. This connection is expressed through the boundary conditions (1.9) and (1.11). The equation (1.11) is a Neumann-type boundary condition for v. It is hyperbolic in nature as opposed to the parabolic diffusion process in the regions Ω^+ and Ω^-.

From the technical point of view the first major obstacle for the analysis is the moving boundary. To deal with this issue we shall first transform the problem to the fixed domain by applying the so-called Hanzawa transform. To this end let us fix a small positive constant $\alpha < \frac{1}{3}$ and choose a cut-off function $\phi \in C^\infty(\mathbb{R})$ with $\mathrm{Im}(\phi) \subset [0,1]$ and

$$\phi(z) = \begin{cases} 1, & |z| \leq \alpha, \\ 0, & |z| > 1 - \alpha \end{cases} \qquad ||\phi'||_{L^\infty(\mathbb{R})} < C.$$

Define now a diffeomorphism

$$\Theta(t, x', x_n) = (t, x', x_n + \phi(x_n)\rho(x', t))$$

and the function $u(t,x',x_n) = v(\Theta(t,x',x_n))$. Observe that $u(t,x',0) = v(t,x',\rho(x',t))$ and at the outer boundaries $\partial\Omega^\pm := \mathbb{T}^{n-1} \times \{x_n = \pm 1\}$, we have $v|_{\partial\Omega^\pm} = u|_{\partial\Omega^\pm}$. This is the version of the transform first introduced by Hanzawa (cf. [**35**]). In the new coordinates the heat operator $\partial_t - \Delta$ transforms into a more complicated operator whose coefficients depend on the interface function ρ and the cut-off function ϕ. Following the calculations from [**20**], we find that the Laplace operator Δ in the new coordinates takes the form

$$\Delta_\Theta u = \Delta_{x'} u + a_\rho u_{nn} - \mathcal{B}\cdot\nabla_{x'} u_n - d_\rho u_n,$$

where

$$a_\rho := \frac{1+|\phi\nabla\rho|^2}{(1+\phi'\rho)^2}, \quad \mathcal{B} := \frac{2\phi}{1+\phi'\rho}\nabla\rho, \tag{1.15}$$

$$d_\rho := \frac{\phi\Delta\rho}{1+\phi'\rho} - \frac{(\phi^2)'|\nabla\rho|^2}{(1+\phi'\rho)^2} + \frac{\phi''\rho(1+|\phi\nabla\rho|^2)}{(1+\phi'\rho)^3}. \tag{1.16}$$

Furthermore, the operator ∂_t in the new coordinates reads

$$(\partial_t)_\Theta u = \partial_t u + e_\rho u_n$$

where

$$e_\rho := -\frac{\phi\rho_t}{1+\phi'\rho}. \tag{1.17}$$

Note that RHS of (1.9) remains unchanged in the new coordinates. In order to transform the boundary condition (1.11) into the new coordinates we first observe that

$$(\partial_i)_\Theta u = \partial_i u - \frac{\phi\partial_i\rho}{1+\phi'\rho}\partial_n u,\ 1\le i\le n-1, \qquad (\partial_n)_\Theta u = \frac{1}{1+\phi'\rho}\partial_n u.$$

We thus conclude that at the boundary $\mathbb{T}^{n-1}\times\{x_n=0\}$:

$$\nabla v|_\Gamma = \nabla_\Theta u|_{x_n=0} = (\nabla_{x'}u,0) - \frac{\partial_n}{1+\phi'\rho}(\phi\nabla\rho,1) = (\nabla_{x'}u,0) - \partial_n u(\nabla\rho,1).$$

The outward unit normal is given by $\nu(x',t) = \frac{(-\nabla\rho(x',t),1)}{\sqrt{1+|\nabla\rho|^2}}$. Thus the normal velocity V takes the form $V = \frac{-\partial_t\rho}{\sqrt{1+|\nabla\rho|^2}}$. Using the above expressions we derive the formula for $[\partial_n v]_-^+|_\Gamma$. Namely on $\mathbb{T}^{n-1}\times\{x_n=0\}$ we have

$$[\partial_n v]_-^+|_\Gamma = [(\nabla_{x'}u,0) - \partial_n u(\nabla\rho,1)]_-^+ \cdot \frac{(-\nabla\rho(x',t),1)}{\sqrt{1+|\nabla\rho|^2}} = [\partial_n u]_-^+\sqrt{1+|\nabla\rho|^2}.$$

It is thus easy to see that the equation (1.11) transforms into

$$\partial_t\rho = (1+|\nabla\rho|^2)[\partial_n u]_+^-.$$

For the sake of notational simplicity we also set

$$\langle\rho\rangle := \sqrt{1+|\nabla\rho|^2}.$$

The Stefan problem (1.8) - (1.13) now takes the following form:

$$u_t - \Delta_{x'} u - a_\rho u_{nn} + \mathcal{B} \cdot \nabla_{x'} u_n + c_\rho u_n = 0 \tag{1.18}$$

$$u = \kappa \qquad \text{on } \mathbb{T}^{n-1} \times \{x_n = 0\} \tag{1.19}$$

$$\partial_n u = 0 \qquad \text{on } \mathbb{T}^{n-1} \times \{x_n = \pm 1\} \tag{1.20}$$

$$u(0,x) = u_0(x) \qquad x \in \Omega \tag{1.21}$$

$$\rho(0,x') = \rho_0(x') \qquad x' \in \mathbb{T}^{n-1} \tag{1.22}$$

$$\rho_t = \langle\rho\rangle^2 [u_n]_+^- \qquad \text{on } \mathbb{T}^{n-1} \times \{x_n = 0\}, \tag{1.23}$$

where we set $c_\rho := d_\rho + e_\rho$ with d_ρ and e_ρ given by (1.16) and (1.17) respectively. Recall that a_ρ and $\mathcal{B}$ are given by (1.15). In order to deal with the hyperbolic equation (1.23) we introduce the regularization of the jump relation (1.23):

$$\rho_t + \epsilon \Delta^2 \rho_t = \langle\rho\rangle^2 [u_n]_+^- \qquad \text{on } \mathbb{T}^{n-1} \times \{x_n = 0\}. \tag{1.24}$$

We shall refer to the problem of finding the solution to (1.18)-(1.22) and (1.24) as to the regularized Stefan problem.

Notation. For notational simplicity we define for any multi-index $\mu = (\mu_1, \dots, \mu_{n-1})$ and $s \in \mathbb{N}$

$$\partial_s^\mu = \partial_t^s \partial_{x_1}^{\mu_1} \dots \partial_{x_{n-1}}^{\mu_{n-1}}. \tag{1.25}$$

Note that the operator ∂_s^μ acts only in directions tangential to the boundary $\mathbb{T}^{n-1} \times \{x_n = 0\}$. The Latin letters are always used to refer to the differentiation with respect to the time variable t and Greek letters to refer to the differentiation with respect to the first $n-1$ spatial variables $x_1, \dots, x_{n-1}$. If each component of μ' is not greater then that of μ and $s' \le s$, we write $(\mu', s') \le (\mu, s)$. We write $(\mu', s') < (\mu, s)$ if $(\mu', s') \le (\mu, s)$ and $|\mu'| < |\mu|$ or $s' < s$. We also denote $C_{s'}^{\mu'} = \binom{(\mu,s)}{(\mu',s')}$. For given functions $\omega : \mathbb{T}^{n-1} \to \mathbb{R}$ and $\mathcal{U} : \Omega \to \mathbb{R}$, we denote $\omega_i = \partial_{x_i} \omega$, $i = 1, \dots, n-1$ and $\mathcal{U}_n = \partial_{x_n} \mathcal{U}$. With $x = (x_1, \dots, x_n)$ and $x' = (x_1, \dots, x_{n-1})$ we set

$$\nabla_{x'} \mathcal{U} = (\partial_{x_1} \mathcal{U}, \dots, \partial_{x_{n-1}} \mathcal{U}), \quad |\nabla_{x'}^2 \mathcal{U}|^2 = \sum_{i,j=1}^{n-1} (\partial_{x_i x_j} \mathcal{U})^2, \quad \Delta_{x'} \mathcal{U} = \sum_{i=1}^{n-1} \partial_{x_i x_i} \mathcal{U}.$$

We additionally denote $\Omega_f^{\pm} = \mathbb{T}^{n-1} \times [0, \pm 1]$. For any $t \in]0, \infty]$ introduce

$$C^\infty(]0,t[;\Omega_f) := \Big\{ u :]0,t[\times \Omega \to \mathbb{R};\ u\mathbf{1}_{\Omega_f^{\pm}} \in C^\infty(]0,t[\times \Omega_f^{\pm}) \Big\},$$

and analogously define $C^\infty(\Omega_f)$ by surpressing the time dependence. For a given $l \in \mathbb{N}$ set

$$H^l(\Omega_f) = \Big\{ u : \Omega \to \mathbb{R};\ u\mathbf{1}_{\Omega_f^{\pm}} \in H^l(\Omega_f^{\pm}) \Big\},$$

where H^l stands for the standard $W^{l,2}$ Sobolev space. The Einstein summation convention is used throughout the chapter when dealing with repeated indices. The letter C will stand for a generic constant that may change from line to line.

We define the following high-order energy norms:

$$\begin{aligned} \mathcal{E}(\mathcal{U},\omega;\psi)(t) := & \sum_{|\mu|+2s\le 2k} \int_\Omega \Big\{ \frac{1}{2}(\partial_s^\mu \mathcal{U}(t))^2 + |\partial_s^\mu \nabla_{x'}\mathcal{U}(t)|^2 + a_{\psi(t)}(\partial_s^\mu \mathcal{U}_n(t))^2 \Big\} + \\ & + \sum_{|\mu|+2s\le 2k} \int_{\mathbb{T}^{n-1}} \Big\{ \frac{1}{2} |\partial_s^\mu \nabla \omega(t)|^2 \langle \psi(t) \rangle^{-1} + I_{\psi(t)}(\nabla^2 \partial_s^\mu \omega, \nabla^2 \partial_s^\mu \omega)(t) \Big\}, \end{aligned} \tag{1.26}$$

$$\begin{aligned} \mathcal{D}(\mathcal{U},\omega;\psi)(t) := & \sum_{|\mu|+2s\le 2k} \int_\Omega \Big\{ (\partial_s^\mu \mathcal{U}_t(t))^2 + |\partial_s^\mu \nabla_{x'}\mathcal{U}(t)|^2 + a_{\psi(t)}(\partial_s^\mu \mathcal{U}_n(t))^2 + \\ & |\nabla_{x'}^2 \partial_s^\mu \mathcal{U}(t)|^2 + 2a_{\psi(t)}|\partial_s^\mu \nabla_{x'} \mathcal{U}_n(t)|^2 + \big(a_{\psi(t)}\partial_s^\mu \mathcal{U}_{nn}(t)\big)^2 \Big\} \\ & + 2\sum_{|\mu|+2s\le 2k} \int_{\mathbb{T}^{n-1}} |\partial_s^\mu \nabla \omega_t(t)|^2 \langle \psi(t) \rangle^{-1}, \end{aligned} \tag{1.27}$$

where for given functions $\omega, \psi : \mathbb{T}^{n-1} \to \mathbb{R}$, we define

$$I_\psi(\nabla^2\omega, \nabla^2\omega) := |\nabla^2 \omega|^2 \langle \psi \rangle^{-1} - \sum_{k=1}^{n-1} \big(\nabla \omega_k \cdot \nabla \psi\big)^2 \langle \psi \rangle^{-3}. \tag{1.28}$$

Recall that $a_{\psi(t)}$ is given by (1.15). It is crucial to observe that I_ψ is a positive definite bilinear form. Namely,

(1.29)
$$\begin{aligned} I_\psi(\nabla^2\omega, \nabla^2\omega) = & \int_{\mathbb{T}^{n-1}} \Big\{ |\nabla^2\omega|^2 \langle \psi \rangle^{-3} + \Big(|\nabla^2 \omega|^2 |\nabla \psi|^2 - \sum_{k=1}^{n-1} \big(\nabla\omega_k \cdot \nabla \psi\big)^2 \Big) \langle \psi \rangle^{-3} \Big\} \\ & \ge \int_{\mathbb{T}^{n-1}} |\nabla^2 \omega|^2 \langle \psi \rangle^{-3}. \end{aligned}$$

Note that we have used the Cauchy-Schwarz inequality in the last estimate. The above norms are the weighted versions of parabolic Sobolev norms given by

$$||(\mathcal{U},\omega)||_{\mathcal{E}} := \sum_{|\mu|+2s\leq 2k} \Big\{ ||\partial_s^\mu \mathcal{U}||^2_{H^1(\Omega)} + ||\partial_s^\mu \nabla\omega||^2_{H^1} \Big\} \tag{1.30}$$

and

$$||(\mathcal{U},\omega)||_{\mathcal{D}} := \sum_{|\mu|+2s\leq 2k} \Big\{ ||\partial_s^\mu \mathcal{U}_t||^2_{L^2(\Omega)} + ||\partial_s^\mu \nabla\mathcal{U}||^2_{H^1(\Omega)} + ||\partial_s^\mu \nabla\omega_t||^2_2 \Big\}. \tag{1.31}$$

Given $||\psi||_\infty$ small enough so that $(1+\phi'\psi)^2 \geq \delta > 0$ and $||\nabla\psi||_\infty$ bounded, we conclude that there exists $C>0$ so that

$$\frac{1}{C}||(\mathcal{U},\omega)||_{\mathcal{E}} \leq \mathcal{E} \leq C||(\mathcal{U},\omega)||_{\mathcal{E}}, \quad \frac{1}{C}||(\mathcal{U},\omega)||_{\mathcal{D}} \leq \mathcal{D} \leq C||(\mathcal{U},\omega)||_{\mathcal{D}}.$$

For any $t \in]0,\infty]$ define the function spaces

$$\mathcal{S}_u(0,t) := \Big\{ \bigcap_{l=0}^{k} W^{l,\infty}(]0,t[;H^{2k+1-2l}(\Omega_f)) \Big\} \cap \Big\{ \bigcap_{l=0}^{k+1} H^l(]0,t[;H^{2k+2-2l}(\Omega_f)) \Big\}$$

and

$$\mathcal{S}_\rho(0,t) := \Big\{ \rho;\, \rho \in \bigcap_{l=0}^{k} W^{l,\infty}(]0,t[;H^{2k+2-2l}(\mathbb{T}^{n-1})),\ \rho_t \in \bigcap_{l=0}^{k} H^l(]0,t[;H^{2k+1-2l}(\mathbb{T}^{n-1})) \Big\}.$$

The energy space $\mathcal{S}(0,t)$ is defined by $\mathcal{S}(0,t) := \mathcal{S}_u(0,t) \times \mathcal{S}_\rho(0,t)$. The *instant energy* $\mathcal{E}$ and the *dissipation* $\mathcal{D}$ are respectively given by

$$\mathcal{E} \equiv \mathcal{E}(u,\rho) := \mathcal{E}(u,\rho;\rho), \tag{1.32}$$

$$\mathcal{D} \equiv \mathcal{D}(u,\rho) := \mathcal{D}(u,\rho;\rho). \tag{1.33}$$

In the rest of the chapter we shall always assume $k \geq n$, where n is the dimension of the space the domain Ω belongs to. Observe that the stationary solutions to the Stefan problem (1.18) - (1.23) are given by $(u,\rho) \equiv (0,\bar{\rho})$, where $\bar{\rho} \in \mathbb{R}$ is a given constant. Note that $\mathcal{E}(u,\rho-\bar{\rho}) = \mathcal{E}(u,\rho)$. The main result of the chapter is the following theorem.

THEOREM 1.1. *Assume that* $u(0,\cdot)=u_0\in H^{2k+1}(\Omega_f)$, $\rho(0,\cdot)=\rho_0\in H^{2k+2}(\mathbb{T}^{n-1})$, $u_0\big|_{\mathbb{T}^{n-1}}=\kappa(0)$ *and* $\partial_n u_0\big|_{\mathbb{T}^{n-1}\times\{\pm1\}}=0$. *Then there exists a sufficiently small constant* $M^*>0$ *such that if*

$$\mathcal{E}(u_0,\rho_0)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0-\int_\Omega u_0(1+\phi'\rho_0)\Big|\le M^*,$$

then there exists a unique solution to the Stefan problem (1.18) - (1.23) $(u,\rho)\in\mathcal{S}(0,\infty)$, *satisfying the global bound*

$$\mathcal{E}(u,\rho)(t)+\frac{1}{2}\int_s^t\mathcal{D}(u,\rho)(\tau)\,d\tau\le\mathcal{E}(u,\rho)(s),\quad t\ge s\ge 0. \tag{1.34}$$

Moreover, given the stationary solution $(u,\rho)\equiv(0,\bar\rho)$, *such that*

$$\int_{\mathbb{T}^{n-1}}\bar\rho=\int_{\mathbb{T}^{n-1}}\rho_0-\int_\Omega u_0(1+\phi'\rho_0),$$

then for such small initial datum there exist constants $K_1,K_2>0$ *such that*

$$\mathcal{E}(u,\rho)(t)+||\rho(t)-\bar\rho||_2^2\le K_1e^{-K_2t}\quad\textit{for all}\quad t\ge 0.$$

Remarks. By choosing k large enough, we can make the solution as smooth as desired in a classical sense. Furthermore, the temporal derivatives in the expression $\mathcal{E}(u_0,\rho_0)$ are naturally defined through the equations (1.18) and (1.23). The compatibility conditions $u(0,\cdot)\Big|_{\mathbb{T}^{n-1}}=\kappa(0)$ and $\partial_n u_0\big|_{\mathbb{T}^{n-1}\times\{\pm1\}}=0$ ensure that the solution is continuous at $t=0$.

The proof of theorem will strongly rely on careful examination of the regularized Stefan problem (1.18) - (1.22) and (1.24). For this purpose we introduce the appropriate energy norms incorporating the additional viscosity coefficient ϵ.

$$\begin{aligned}\mathcal{E}_\epsilon(\mathcal{U},\omega;\psi):=\mathcal{E}(\mathcal{U},\omega;\psi)+\epsilon\sum_{|\mu|+2s\le 2k}\int_{\mathbb{T}^{n-1}}\Big\{&|\partial_s^\mu\Delta\nabla\omega|^2\langle\psi\rangle^{-1}\\ &+I_\psi(\nabla^2\partial_s^\mu\Delta\omega,\nabla^2\partial_s^\mu\Delta\omega)\Big\}\end{aligned} \tag{1.35}$$

$$\mathcal{D}_\epsilon(\mathcal{U},\omega;\psi):=\mathcal{D}(\mathcal{U},\omega;\psi)+2\epsilon\sum_{|\mu|+2s\le 2k}\int_{\mathbb{T}^{n-1}}|\nabla^2\partial_s^\mu\Delta\omega|^2\langle\psi\rangle^{-1}. \tag{1.36}$$

The above norms are the weighted versions of parabolic Sobolev norms given by

$$||(\mathcal{U},\omega)||_{\mathcal{E}_\epsilon}:=||(\mathcal{U},\omega)||_{\mathcal{E}}+\epsilon||\partial_s^\mu\Delta\nabla\omega||_{H^1}^2 \tag{1.37}$$

and

$$||(\mathcal{U},\omega)||_{\mathcal{D}_\epsilon} := ||(\mathcal{U},\omega)||_{\mathcal{D}} + \epsilon||\partial_s^\mu \Delta\nabla\omega_t||_2^2. \tag{1.38}$$

Given $||\psi||_\infty$ small enough so that $(1+\phi'\psi)^2 \geq \delta > 0$ and $||\nabla\psi||_\infty$ bounded, we conclude that there exists $C>0$ so that

$$\frac{1}{C}||(\mathcal{U},\omega)||_{\mathcal{E}_\epsilon} \leq \mathcal{E}_\epsilon \leq C||(\mathcal{U},\omega)||_{\mathcal{E}_\epsilon}, \quad \frac{1}{C}||(\mathcal{U},\omega)||_{\mathcal{D}_\epsilon} \leq \mathcal{D}_\epsilon \leq C||(\mathcal{U},\omega)||_{\mathcal{D}_\epsilon}.$$

In this sense the above norms are equivalent and this observation will be often implicitly used throughout the chapter. For any $t \in]0,\infty]$ define the function space

$$\mathcal{S}_\rho^\epsilon(0,t) := \Big\{\rho;\ \rho \in \bigcap_{l=0}^{k} W^{l,\infty}(]0,t[;H^{2k+4-2l}(\mathbb{T}^{n-1})),\ \rho_t \in \bigcap_{l=0}^{k} H^l(]0,t[;H^{2k+3-2l}(\mathbb{T}^{n-1}))\Big\}.$$

The energy space $\mathcal{S}_\epsilon(0,t)$ is defined by $\mathcal{S}_\epsilon(0,t) := \mathcal{S}_u(0,t) \times \mathcal{S}_\rho^\epsilon(0,t)$. The major part of the analysis will be concerned with proving the following result, which states that the regularized Stefan problem has unique global solutions with small initial data - independent of ϵ.

Theorem 1.2. *Assume that $u^\epsilon(0,\cdot) = u_0^\epsilon \in H^{2k+1}(\Omega_f)$, $\rho^\epsilon(0,\cdot) = \rho_0^\epsilon \in H^{2k+4}(\mathbb{T}^{n-1})$, $u_0^\epsilon\Big|_{\mathbb{T}^{n-1}} = \nabla\cdot\frac{\nabla\rho_0^\epsilon}{\langle\rho_0^\epsilon\rangle}$ and $\partial_n u_0^\epsilon|_{\mathbb{T}^{n-1}\times\{\pm1\}} = 0$. There exists a sufficiently small constant $M>0$ independent of ϵ, such that the following statement holds: if the inequality*

$$\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon) + \Big|\int_{\mathbb{T}^{n-1}} \rho_0^\epsilon - \int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big| \leq M$$

holds, then there exists a unique global solution to the regularized Stefan problem (1.18)-(1.22) and (1.24) $(u^\epsilon,\rho^\epsilon) \in \mathcal{S}_\epsilon(0,\infty)$, satisfying the bound

$$\mathcal{E}_\epsilon(u^\epsilon,\rho^\epsilon)(t) + \frac{1}{2}\int_0^t \mathcal{D}_\epsilon(u^\epsilon,\rho^\epsilon)(\tau)\,d\tau \leq \mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon), \quad t \geq 0. \tag{1.39}$$

Remark. The temporal derivatives in the expression $\mathcal{E}(u_0,\rho_0)$ are naturally defined through the equations (1.18) and (1.24).

The Stefan problem has been studied in a variety of mathematical literature over the past century (see for instance [**53**]). It has been known that classical Stefan problem admits unique global weak solutions in several dimensions ([**23**], [**24**] and [**37**]). The references to the regularity of weak solutions of the two-phase classical

Stefan are, among others, [**5**], [**6**], [**10**], [**19**]. Local classical solutions are established in [**35**] and [**43**].

If the diffusion equation (1.8) is replaced by the elliptic equation $\Delta v = 0$, then the resulting problem is called the quasi-stationary Stefan problem with surface tension. Global existence for the two-phase quasi-stationary Stefan problem close to a sphere in two dimensions has been obtained in [**12**] and [**16**], and in arbitrary dimensions in [**22**]. Global stability for the one-phase quasi-stationary Stefan problem is established in [**27**]. Local-in-time solutions in parabolic Hölder spaces in arbitrary dimensions are established in [**13**].

As to the Stefan problem with surface tension, global weak existence theory (without uniqueness) is analyzed in [**41**] and [**48**]. In [**26**] the authors consider the Stefan problem with small surface tension i.e. $\sigma \ll 1$ if (1.9) is substituted by $v = \sigma\kappa(t)$. The local existence result for the Stefan problem is studied in [**47**]. In [**20**] the authors prove a local existence and uniqueness result in suitable Besov spaces, relying on the L^p-regularity theory. Linear stability and instability results for spheres (which are equilibria for the Stefan problem with surface tension) are contained in [**46**]. Some of the references for the Stefan problem with surface tension and kinetic undercooling effects are [**15**], [**47**], [**49**], [**52**].

We establish a *global-in-time* existence, uniqueness and exponential decay of classical solutions to the Stefan problem with surface tension near a flat steady state (Theorem 1.1). The major difficulty consists of proving Theorem 1.2 which establishes the existence and uniqueness result for the regularized Stefan problem with the energy estimate

$$\mathcal{E}_\epsilon(t) + \int_0^t \mathcal{D}_\epsilon(\tau)\,d\tau \le \mathcal{E}_\epsilon(0) + C\int_0^t \sqrt{\mathcal{E}_\epsilon(\tau)}\mathcal{D}_\epsilon(\tau)\,d\tau, \tag{1.40}$$

where C does not depend on ϵ. Combined with the smallness assumption on the instant energy $\mathcal{E}_\epsilon$ the estimate (1.40) gives (1.39) and the global-in-time existence. For a fixed ϵ we first construct local-in-time solution for the regularized Stefan problem (1.18)- (1.22) and (1.24). The crux of our method is the use of high order energy estimates, for the differential operator ∂_s^μ acts only in tangential directions with respect to the boundary $\mathbb{T}^{n-1}\times\{x_n=0\}$. This is very convenient when deriving the energy identities because the Neumann boundary operator commutes with ∂_s^μ. The diffusion equation (1.18) is then used to control high order derivatives of u with respect to the normal direction x_n, as it is presented in Lemmas 3.5 and 3.6. We set up an iteration scheme, which generates a sequence of iterates $\{(u^m,\rho^m)\}_{m\in\mathbb{N}}$. Such iteration is well defined, but it breaks the natural energy setting due to lack of exact cancellations in the presence of the cross-terms. With fixed ϵ, we crucially use the regularization to prove that $\{(u^m,\rho^m)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in the energy space. As $m\to\infty$ the unpleasant cross-terms disappear and we recover (1.40) in the limit. We conclude the proof of Theorem 1.1 by letting $\epsilon\to 0$.

The chapter is organized as follows: In Section 2 we derive general energy identities for a model Stefan problem. In Section 3 the iteration scheme for proving the local existence is set up and the actual energy identities are derived, based on Section 2. Furthermore, some basic estimates are established, which are then used in Section 4 to prove the crucial energy estimates. Section 5 is entirely devoted to the proof of the local-in-time existence nd uniqueness. The main results, Theorems 1.1 and 1.2 are proved in Section 6.

2. Energy identities

Let $I=[0,q]$ for some $0<q\le\infty$. The derivation of the energy identities crucially depends on the following model problem:

$$\mathcal{U}_t-\Delta_{x'}\mathcal{U}-\epsilon\mathcal{U}_{nn}=f \quad \text{on } \Omega\times I \tag{2.41}$$

$$\mathcal{U}=\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}+G \quad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\}\times I \tag{2.42}$$

$$\partial_n\mathcal{U}=0 \quad \text{on } \mathbb{T}^{n-1}\times\{x_n=\pm1\}\times I \tag{2.43}$$

$$[\mathcal{U}_n]_+^-=\left(\omega_t+\epsilon\Delta^2\omega_t\right)\langle\psi\rangle^{-2}+h \quad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\}\times I \tag{2.44}$$

We shall denote

$$g:=-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}+G.$$

For most of the identities we shall derive, only the leading term $\Delta\chi\langle\psi\rangle^{-1}$ in (2.42) will be relevant. We can thus write the equation (2.42) in the alternative form

$$\mathcal{U} = \Delta\chi\langle\psi\rangle^{-1} + g \quad \text{on } \mathbb{T}^{n-1}\times\{x_n = 0\}\times I. \tag{2.45}$$

We define the energies $\bar{\mathcal{E}}_\epsilon$ and $\bar{\mathcal{D}}_\epsilon$ (for the model problem) by setting $k=0$ in the definitions (1.35) and (1.36) of $\mathcal{E}_\epsilon$ and $\mathcal{D}_\epsilon$, respectively.

LEMMA 2.1. *Let each of the functions $\mathcal{U}$, ω, χ and ψ be five times continuously differentiable with respect to the space variable and each of its spatial partial derivatives of order ≤ 5 once continuously differentiable with respect to the time variable. The following identity holds:*

$$\frac{d}{dt}\bar{\mathcal{E}}_\epsilon(\mathcal{U},\omega;\psi) + \bar{\mathcal{D}}_\epsilon(\mathcal{U},\omega;\psi) = \int_\Omega \{P+R\} - \int_{\mathbb{T}^{n-1}}\{Q+S+T\}, \tag{2.46}$$

where

$$P \equiv P(\mathcal{U},\psi,f) := f\mathcal{U} - (\mathfrak{e})_n\mathcal{U}_n\mathcal{U}, \tag{2.47}$$

$$\begin{aligned} R \equiv R(\mathcal{U},\psi,f) &:= f^2 + \mathcal{U}_n^2(\mathfrak{e})_t - 2\mathcal{U}_t\mathcal{U}_n(\mathfrak{e})_t \\ &-2\nabla_{x'}\mathcal{U}_n\cdot\nabla_{x'}(\mathfrak{e})\mathcal{U}_n + 2\Delta_{x'}\mathcal{U}\mathcal{U}_n(\mathfrak{e})_n, \end{aligned} \tag{2.48}$$

(2.49)
$$\begin{aligned} Q \equiv Q(\chi,\omega,\psi,g,h) &= \nabla\omega_t\cdot\big(\nabla\chi - \nabla\omega\big)\langle\psi\rangle^{-1} + \epsilon\Delta\nabla\omega_t\cdot\big(\Delta\nabla\chi - \Delta\nabla\omega\big)\langle\psi\rangle^{-1} \\ &-\frac{1}{2}\Big\{|\nabla\omega|^2\langle\psi\rangle_t^{-1} + \epsilon|\Delta\nabla\omega|^2\langle\psi\rangle_t^{-1}\Big\} + \omega_t\nabla\chi\cdot\nabla(\langle\psi\rangle^{-1}) \\ &+\epsilon\Delta\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\chi - (\omega_t + \epsilon\Delta^2\omega_t)g - \langle\psi\rangle^2 h\mathcal{U}|_{\mathbb{T}^{n-1}}, \end{aligned}$$

(2.50)
$$\begin{aligned} S \equiv S(\chi,\omega,\psi,g,h) &:= 2\nabla\omega_t\cdot\big(\nabla\chi_t - \nabla\omega_t\big)\langle\psi\rangle^{-1} + 2\epsilon\Delta\nabla\omega_t\cdot\big(\Delta\nabla\chi_t - \Delta\nabla\omega_t\big)\langle\psi\rangle^{-1} \\ &+2\nabla\chi_t\cdot\nabla(\langle\psi\rangle^{-1})\omega_t + 2\epsilon\Delta\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\chi_t - 2\big(\omega_t + \epsilon\Delta^2\omega_t\big)\big(\Delta\chi\langle\psi\rangle_t^{-1} + g_t\big) \\ &-2h\langle\psi\rangle^2\mathcal{U}_t|_{\mathbb{T}^{n-1}}, \end{aligned}$$

$$\begin{aligned} T \equiv T(\chi,\omega,\psi,G,h) &= A(\chi,\omega,\psi,g,h) + B(\chi,\omega,\psi,g,h) + 2\Delta G\big(\omega_t + \epsilon\Delta^2\omega_t\big) \\ &+2\Delta\mathcal{U}|_{\mathbb{T}^{n-1}}h\langle\psi\rangle^2. \end{aligned} \tag{2.51}$$

Here, the functions A and B are given by:

(2.52)

$$A := 2\Delta\omega_t\big(\Delta\chi - \Delta\omega\big)\langle\psi\rangle^{-1} - 2\Delta\omega_t\psi_i\psi_j(\chi_{ij} - \omega_{ij})\langle\psi\rangle^{-3} - |\nabla^2\omega|^2\langle\psi\rangle_t^{-1}$$
$$+2\omega_{it}\nabla\omega_i\cdot\nabla(\langle\psi\rangle^{-1}) - 2\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\omega + \omega_{jk}\omega_{ik}\psi_j\psi_{it}\langle\psi\rangle^{-3} + \big(\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle_t^{-3}$$
$$-2\omega_{jk}\Big\{\big[\psi_i\psi_j\omega_{kt}\langle\psi\rangle^{-3}\big]_i - \psi_i\psi_j\omega_{ikt}\langle\psi\rangle^{-3}\Big\} + 2\omega_{kt}\Big\{\big[\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}\big]_k - \psi_i\psi_j\omega_{ijk}\langle\psi\rangle^{-3}\Big\}$$

and

$$\begin{aligned}
B :=\ & 2\epsilon\Delta^2\omega_t\big(\Delta^2\chi - \Delta^2\omega\big)\langle\psi\rangle^{-1} - 2\epsilon\Delta^2\omega_t\psi_i\psi_j(\Delta\chi_{ij} - \Delta\omega_{ij})\langle\psi\rangle^{-3} \\
& -\epsilon|\nabla^2\Delta\omega|^2\langle\psi\rangle_t^{-1} + 2\epsilon\Delta\omega_{it}\nabla\Delta\omega_i\cdot\nabla(\langle\psi\rangle^{-1}) \\
& -2\epsilon\Delta\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta^2\omega + \epsilon\Delta\omega_{jk}\Delta\omega_{ik}\psi_j\psi_{it}\langle\psi\rangle^{-3} + \epsilon\big(\Delta\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle_t^{-3} \\
& -2\epsilon\Delta\omega_{jk}\Big\{\big[\psi_i\psi_j\Delta\omega_{kt}\langle\psi\rangle^{-3}\big]_i - \psi_i\psi_j\Delta\omega_{ikt}\langle\psi\rangle^{-3}\Big\} \\
& +2\epsilon\Delta\omega_{kt}\Big\{\big[\psi_i\psi_j\Delta\omega_{ij}\langle\psi\rangle^{-3}\big]_k - \psi_i\psi_j\Delta\omega_{ijk}\langle\psi\rangle^{-3}\Big\} \\
& +2\epsilon\Big\{\Delta\big(\Delta\chi\langle\psi\rangle^{-1} - \psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big) - \big(\Delta^2\chi\langle\psi\rangle^{-1} - \psi_i\psi_j\Delta\chi_{ij}\langle\psi\rangle^{-3}\big)\Big\}\Delta^2\omega_t.
\end{aligned} \tag{2.53}$$

Proof. We start by multiplying the equation (2.41) by $\mathcal{U}$ and integrating over Ω. By a direct computation,

$$\frac{1}{2}\partial_t\int_\Omega \mathcal{U}^2 + \int_\Omega\Big\{|\nabla_{x'}\mathcal{U}|^2 + \mathfrak{e}(\mathcal{U}_n)^2\Big\} - \int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U} = \int_\Omega P(\mathcal{U},\psi,f), \tag{2.54}$$

where $P(\mathcal{U},\psi,f)$ is given by (2.47). Using the boundary conditions (2.45) and (2.44), we obtain

$$\begin{aligned}
-\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U} = & -\big\{\omega_t\Delta\chi\langle\psi\rangle^{-1} + \epsilon\Delta^2\omega_t\Delta\chi\langle\psi\rangle^{-1}\big\} \\
& -\langle\psi\rangle^2 h\big[\Delta\chi\langle\psi\rangle^{-1} + g\big] - (\omega_t + \epsilon\Delta^2\omega_t)g.
\end{aligned} \tag{2.55}$$

Integrating by parts in the first term on the RHS of (2.55) above, we arrive at

$$\begin{aligned}
& -\textstyle\int_{\mathbb{T}^{n-1}}\Big\{\omega_t\Delta\chi\langle\psi\rangle^{-1} + \epsilon\Delta^2\omega_t\Delta\chi\langle\psi\rangle^{-1}\Big\} = \\
& \textstyle\int_{\mathbb{T}^{n-1}}\nabla(\omega_t\langle\psi\rangle^{-1})\cdot\nabla\chi + \epsilon\Delta\nabla\omega_t\cdot\nabla(\Delta\chi\langle\psi\rangle^{-1}).
\end{aligned} \tag{2.56}$$

By the product rule, the integrand on the RHS of (2.56) can be written as

$$\nabla\omega_t\langle\psi\rangle^{-1}\cdot\nabla\chi + \epsilon\Delta\nabla\omega_t\cdot\Delta\nabla\chi\langle\psi\rangle^{-1} + \omega_t\nabla(\langle\psi\rangle^{-1})\cdot\nabla\chi + \epsilon\Delta\omega_t\nabla(\langle\psi\rangle^{-1})\cdot\Delta\nabla\chi.$$

In each of the terms $\nabla\omega_t \langle\psi\rangle^{-1}\cdot\nabla\chi$ and $\Delta\nabla\omega_t\cdot\Delta\nabla\chi\langle\psi\rangle^{-1}$ we set $\chi=\omega+(\chi-\omega)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t\Big\{|\nabla\omega|^2\langle\psi\rangle^{-1}+\epsilon|\Delta\nabla\omega|^2\langle\psi\rangle^{-1}\Big\}+\nabla\omega_t\cdot\big(\nabla\chi-\nabla\omega\big)\langle\psi\rangle^{-1}+\\
&+\epsilon\Delta\nabla\omega_t\cdot\big(\Delta\nabla\chi-\Delta\nabla\omega\big)\langle\psi\rangle^{-1}-\frac{1}{2}\Big\{|\nabla\omega|^2\langle\psi\rangle_t^{-1}+\epsilon|\Delta\nabla\omega|^2\langle\psi\rangle_t^{-1}\Big\}+\\
&+\omega_t\nabla\chi\cdot\nabla(\langle\psi\rangle)^{-1}+\epsilon\Delta\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\chi
\end{aligned}\tag{2.57}$$

Thus plugging (2.57) into (2.56) yields

$$-\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}=\frac{1}{2}\partial_t\int_{\mathbb{T}^{n-1}}\Big\{|\nabla\omega|^2\langle\psi\rangle^{-1}+\epsilon|\Delta\nabla\omega|^2\langle\psi\rangle^{-1}\Big\}+\int_{\mathbb{T}^{n-1}}Q(\chi,\omega,\psi,g,h),\tag{2.58}$$

where $Q(\chi,\omega,\psi,g,h)$ is given by (2.49). To complete the derivation of (2.46), we take the square of the equation (2.41) and integrate over Ω:

$$\begin{aligned}
&\partial_t\Big\{\int_\Omega|\nabla_{x'}\mathcal{U}|^2+\mathfrak{e}\mathcal{U}_n^2\Big\}+\int_\Omega\Big\{\mathcal{U}_t^2+|\nabla_{x'}^2\mathcal{U}|^2+2a_\psi|\nabla_{x'}\mathcal{U}_n|^2+\mathfrak{e}^2\mathcal{U}_{nn}^2\Big\}\\
&-2\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}_t+2\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\Delta_{x'}\mathcal{U}=\int_\Omega R(\mathcal{U},\psi,f),
\end{aligned}\tag{2.59}$$

where $R(\mathcal{U},\psi,f)$ is given by (2.48). The goal is to evaluate the two integrals over $\mathbb{T}^{n-1}$ on LHS of (2.59) using the boundary conditions (2.45) and (2.44). We first treat the integral $\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}_t$. Integrating by parts in the leading order term, we obtain

$$\begin{aligned}
-2\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]_+^-\mathcal{U}_t=2\int_{\mathbb{T}^{n-1}}&\Big\{|\nabla\omega_t|^2\langle\psi\rangle^{-1}+\epsilon|\Delta\nabla\omega_t|^2\langle\psi\rangle^{-1}\Big\}\\
+\int_{\mathbb{T}^{n-1}}&S(\chi,\omega,\psi,g,h),
\end{aligned}\tag{2.60}$$

where $S(\chi,\omega,\psi,g,h)$ is given by (2.50). Note that the expression (2.50) is obtained similarly to (2.57) by setting $\chi=\omega+(\chi-\omega)$ in the leading order terms.

The second integral over $\mathbb{T}^{n-1}$ in the identity (2.59) is more delicate. We shall make use of the boundary conditions (2.42) and (2.44) to evaluate it. The relation (2.42) is used is to exploit the full algebraic structure of the curvature-type term $\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}$, which is important in the energy estimates later on. We have:

$$\begin{aligned}
&\langle\psi\rangle^2[\mathcal{U}_n]_+^-\Delta_{x'}\mathcal{U}=\Delta\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)(\omega_t+\epsilon\Delta^2\omega_t)\\
&+\Delta G\big(\omega_t+\epsilon\Delta^2\omega_t\big)+\langle\psi\rangle^2h\Delta_{x'}\mathcal{U}.
\end{aligned}\tag{2.61}$$

Observe that

(2.62)
$$
\begin{aligned}
&2\int_{\mathbb{T}^{n-1}}\Delta\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\omega_t=2\int_{\mathbb{T}^{n-1}}\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\Delta\omega_t=\\
&\partial_t\int_{\mathbb{T}^{n-1}}|\nabla^2\omega|^2\langle\psi\rangle^{-1}-\int_{\mathbb{T}^{n-1}}|\nabla^2\omega|^2\langle\psi\rangle_t^{-1}+2\int_{\mathbb{T}^{n-1}}\omega_{it}\nabla\omega_i\cdot\nabla(\langle\psi\rangle^{-1})-\\
&2\int_{\mathbb{T}^{n-1}}\nabla\omega_t\cdot\nabla(\langle\psi\rangle^{-1})\Delta\omega+2\int_{\mathbb{T}^{n-1}}\Delta\omega_t\big(\Delta\chi-\Delta\omega\big)\langle\psi\rangle^{-1}-\\
&2\int_{\mathbb{T}^{n-1}}\omega_{kkt}\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}-2\int_{\mathbb{T}^{n-1}}\Delta\omega_t\big(\psi_i\psi_j(\chi_{ij}-\omega_{ij})\langle\psi\rangle^{-3}\big).
\end{aligned}
$$

Here, just like in (2.57) we substituted $\chi=\omega+(\chi-\omega)$ in the leading order terms. Note that we have repeatedly used integration by parts. Integrating by parts twice, we obtain

$$
\begin{aligned}
&-2\int_{\mathbb{T}^{n-1}}\omega_{kkt}\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}=2\int_{\mathbb{T}^{n-1}}\omega_{kt}\psi_i\psi_j\omega_{ijk}\langle\psi\rangle^{-3}+\\
&2\int_{\mathbb{T}^{n-1}}\omega_{kt}\Big\{\big[\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}\big]_k-\psi_i\psi_j\omega_{ijk}\langle\psi\rangle^{-3}\Big\}=\\
&-2\int_{\mathbb{T}^{n-1}}\omega_{ikt}\psi_i\psi_j\omega_{jk}\langle\psi\rangle^{-3}-2\int_{\mathbb{T}^{n-1}}\omega_{jk}\Big\{\big[\psi_i\psi_j\omega_{kt}\langle\psi\rangle^{-3}\big]_i-\psi_i\psi_j\omega_{ikt}\langle\psi\rangle^{-3}\Big\}\\
&+2\int_{\mathbb{T}^{n-1}}\omega_{kt}\Big\{\big[\psi_i\psi_j\omega_{ij}\langle\psi\rangle^{-3}\big]_k-\psi_i\psi_j\omega_{ijk}\langle\psi\rangle^{-3}\Big\}.
\end{aligned}
\tag{2.63}
$$

We now single out the t-derivative in the first term on RHS of (2.63) to obtain

$$
\begin{aligned}
-2\omega_{ikt}\psi_i\psi_j\omega_{jk}\langle\psi\rangle^{-3}=&-\partial_t\Big\{\sum_{k=1}^{n-1}\big(\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle^{-3}\Big\}+\omega_{jk}\omega_{ik}\psi_j\psi_{it}\langle\psi\rangle^{-3}\\
&+\big(\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle_t^{-3}.
\end{aligned}
\tag{2.64}
$$

We combine the identities (2.62), (2.63) and (2.64) to conclude

$$
\begin{aligned}
&2\int_{\mathbb{T}^{n-1}}\Delta\big(\Delta\chi\langle\psi\rangle^{-1}-\psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\omega_t=\\
&\partial_t\int_{\mathbb{T}^{n-1}}|\nabla^2\omega|^2\langle\psi\rangle^{-1}-\partial_t\Big\{\sum_{k=1}^{n-1}\int_{\mathbb{T}^{n-1}}\big(\nabla\omega_k\cdot\nabla\psi\big)^2\langle\psi\rangle^{-3}\Big\}+\int_{\mathbb{T}^{n-1}}A\\
&=\partial_t\int_{\mathbb{T}^{n-1}}I_\psi(\nabla^2\omega,\nabla^2\omega)+\int_{\mathbb{T}^{n-1}}A,
\end{aligned}
$$

where I_ψ and A are given by (1.28) and (2.52) respectively.

In the ϵ-dependent part on RHS of (2.61) we set $\omega^* = \Delta\omega$ and $\chi^* = \Delta\chi$. We can write

$$\Delta\big(\Delta\chi\langle\psi\rangle^{-1} - \psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\Delta^2\omega_t = \big(\Delta\chi^*\langle\psi\rangle^{-1} - \psi_i\psi_j\chi^*_{ij}\psi^{-3}\big)\Delta\omega^*_t$$
$$+\Big\{\Delta\big(\chi^*\langle\psi\rangle^{-1} - \psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big) - \big(\Delta\chi^*\langle\psi\rangle^{-1} - \psi_i\psi_j\chi^*_{ij}\langle\psi\rangle^{-3}\big)\Big\}\Delta\omega^*_t.$$

We may now apply the same computation as in (2.62) to conclude

$$2\epsilon\int_{\mathbb{T}^{n-1}}\Delta\big(\Delta\chi\langle\psi\rangle^{-1} - \psi_i\psi_j\chi_{ij}\langle\psi\rangle^{-3}\big)\Delta^2\omega_t = \partial_t\int_{\mathbb{T}^{n-1}}\epsilon I_\psi(\nabla^2\Delta\omega, \nabla^2\Delta\omega) + \int_{\mathbb{T}^{n-1}} B,$$

where B is given by (2.53). We combine the above identities to write the final form of the second integral over $\mathbb{T}^{n-1}$ in the identity (2.59):

$$\begin{aligned}2\int_{\mathbb{T}^{n-1}}\langle\psi\rangle^2[\mathcal{U}_n]^-_+\Delta_{x'}\mathcal{U} = \partial_t\Big\{\int_{\mathbb{T}^{n-1}} I_\psi(\nabla^2\omega, \nabla^2\omega) + \epsilon I_\psi(\nabla^2\Delta\omega, \nabla^2\Delta\omega)\Big\}\\ + \int_{\mathbb{T}^{n-1}} T(\chi, \omega, \psi, G, h),\end{aligned} \tag{2.65}$$

where T is given by (2.51). By summing the identities (2.54) and (2.59), plugging (2.58) in (2.54) and (2.60) and (2.65) into (2.59) and collecting terms, we conclude the proof of the lemma. $\square$

3. Iteration scheme and the basic estimates

We shall set up an iterative scheme in order to solve the regularized Stefan problem locally-in-time. Let $(u^m, \rho^m) \in \big\{C^\infty(]0,\infty[\times\Omega_f) \times C^\infty(]0,\infty[\times\mathbb{T}^{n-1})\big\} \cap \mathcal{S}_\epsilon(0,\infty)$ be given and let the initial data satisfy $u^{m+1}(0,x) = u^\epsilon_0(x) \in C^\infty(\Omega_f) \cap H^{2k+1}(\Omega_f)$. We solve the following problem:

$$u^{m+1}_t - \Delta_{x'}u^{m+1} - a_{\rho^m}u^{m+1}_{nn} + B_{\rho^m}\cdot\nabla_{x'}u^{m+1}_n + c_{\rho^m}u^{m+1}_n = 0 \tag{3.66}$$

$$u^{m+1} = \kappa^m \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n = 0\} \tag{3.67}$$

$$\partial_n u^{m+1} = 0 \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n = \pm 1\} \tag{3.68}$$

Here

$$\kappa^m := \nabla\cdot\Big(\frac{\nabla\rho^m}{\langle\rho^m\rangle}\Big). \tag{3.69}$$

Having obtained u^{m+1}, we solve the equation

$$\rho_t^{m+1} + \epsilon\Delta^2\rho_t^{m+1} = \langle\rho^m\rangle^2\,[u_n^{m+1}]_+^- \qquad \text{on } \mathbb{T}^{n-1}\times\{x_n=0\} \tag{3.70}$$

for ρ^{m+1}, with the given initial condition $\rho^{m+1}(0,x') = \rho_0^\epsilon(x') \in C^\infty(\mathbb{T}^{n-1}) \cap H^{2k+4}(\mathbb{T}^{n-1})$. We assume that the compatibility conditions hold: $u_0^\epsilon\big|_{\mathbb{T}^{n-1}} = \nabla\cdot\frac{\nabla\rho_0^\epsilon}{\langle\rho_0^\epsilon\rangle}$ and $\partial_n u_0^\epsilon = 0$ on $\mathbb{T}^{n-1}\times\{x_n=\pm1\}$. The solution (u^{m+1},ρ^{m+1}) to the problem (3.66) - (3.68) and (3.70) exists and belongs to $\{C^\infty(]0,\infty[\times\Omega_f)\times C^\infty(]0,\infty[\times\mathbb{T}^{n-1})\}\cap \mathcal{S}_\epsilon(0,\infty)$ (see Chapter 4 of [**40**]). We aim for proving the convergence of the sequence (u^m,ρ^m) to the solution of the regularized Stefan problem in the energy space. Applying the tangential differential operator ∂_s^μ (recall (1.25)) to the equations (3.66), (3.67) and (3.70), we obtain

$$\partial_s^\mu u_t^{m+1} - \partial_s^\mu\Delta_{x'}u^{m+1} - a_{\rho^m}\partial_s^\mu u_{nn}^{m+1} = f_{\mu,s}^m$$

$$\partial_s^\mu u^{m+1} = \partial_s^\mu\kappa^m = \Delta\partial_s^\mu\rho^m\langle\rho^m\rangle^{-1} + g_{\mu,s}^m = \Delta\partial_s^\mu\rho^m\langle\rho^m\rangle^{-1} - \rho_i^m\rho_j^m\Delta\partial_s^\mu\rho_{ij}^m\langle\rho^m\rangle^{-1} + G_{\mu,s}^m$$

$$[\partial_s^\mu u_n^{m+1}]_+^-\langle\rho^m\rangle^2 = \partial_s^\mu\rho_t^{m+1} + \epsilon\Delta^2\rho_t^{m+1} + h_{\mu,s}^m\langle\rho^m\rangle^2,$$

where

$$f_{\mu,s}^m = \Big\{\partial_s^\mu\Big(a_{\rho^m}u_{nn}^{m+1}\Big) - a_{\rho^m}\partial_s^\mu u_{nn}^{m+1}\Big\} - \partial_s^\mu\Big(B_{\rho^m}\cdot\nabla_{x'}u_n^{m+1}\Big) - \partial_s^\mu\Big(c_{\rho^m}u_n^{m+1}\Big), \tag{3.71}$$

$$g_{\mu,s}^m = \sum_{\substack{|\mu'|+s'\\<|\mu|+s}} C_{s'}^{\mu'}\partial_{s'}^{\mu'}\Delta\rho^m\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-1}) + \partial_s^\mu(\nabla\rho^m\cdot\nabla(\langle\rho^m\rangle^{-1})), \tag{3.72}$$

(3.73)
$$G_{\mu,s}^m = \sum_{\substack{|\mu'|+s'\\<|\mu|+s}} C_{s'}^{\mu'}\partial_{s'}^{\mu'}\Delta\rho^m\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-1}) - \big\{\partial_s^\mu\big(\rho_i^m\rho_j^m\rho_{ij}^m\langle\rho^m\rangle^{-1}\big) - \rho_i^m\rho_j^m\partial_s^\mu\rho_{ij}^m\langle\rho^m\rangle^{-1}\big\},$$

$$h_{\mu,s}^m = \sum_{\substack{|\mu'|+s'\\<|\mu|+s}} C_{s'}^{\mu'}\partial_{s'}^{\mu'}(\rho_t^{m+1} + \epsilon\Delta^2\rho_t^{m+1})\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-2}). \tag{3.74}$$

For any $l\in\mathbb{N}$ let us define

$$\mathcal{E}^l := \mathcal{E}_\epsilon(u^l,\rho^l;\rho^{l-1}), \quad \mathcal{D}^l := \mathcal{D}_\epsilon(u^l,\rho^l;\rho^{l-1}), \tag{3.75}$$

where $\mathcal{E}_\epsilon$ and $\mathcal{D}_\epsilon$ are defined by (1.35) and (1.36) respectively. Setting $\mathcal{U}=\partial_s^\mu u^{m+1}$, $\omega=\partial_s^\mu \rho^{m+1}$, $\chi=\partial_s^\mu\rho^m$, $\psi=\rho^m$, $f=f_{\mu,s}^m$, $g=g_{\mu,s}^m$, $G=G_{\mu,s}^m$ and $h=h_{\mu,s}^m$, the identity (2.46) implies

$$\frac{d}{dt}\mathcal{E}^{m+1}(t)+\mathcal{D}^{m+1}(t)=\int_\Omega\{P^m+R^m\}-\int_{\mathbb{T}^{n-1}}\{Q^m+S^m+T^m\}. \tag{3.76}$$

Here $P^m=\sum_{|\mu|+2s\le 2k}P_{\mu,s}^m$ and R^m, Q^m, S^m and T^m are defined analogously, whereby

$$P_{\mu,s}^m=P(\partial_s^\mu u^{m+1},\rho^m,f_{\mu,s}^m),\quad Q_{\mu,s}^m=Q(\partial_s^\mu\rho^m,\partial_s^\mu\rho^{m+1},\rho^m,g_{\mu,s}^m,h_{\mu,s}^m) \tag{3.77}$$

and

$$\begin{aligned} &R_{\mu,s}^m:=R(\partial_s^\mu u^{m+1},\rho^m,f_{\mu,s}^m),\quad S_{\mu,s}^m:=S(\partial_s^\mu\rho^m,\partial_s^\mu\rho^{m+1},\rho^m,g_{\mu,s}^m,h_{\mu,s}^m),\\ &T_{\mu,s}^m:=T(\partial_s^\mu\rho^m,\partial_s^\mu\rho^{m+1},\rho^m,g_{\mu,s}^m,h_{\mu,s}^m). \end{aligned} \tag{3.78}$$

The inequality (1.29) implies that the instant energy $\mathcal{E}^l$ is positive definite. In order to estimate P^m, R^m, Q^m, S^m and T^m we first need to establish some basic auxiliary estimates.

Lemma 3.1. *The following identity holds*

$$\partial_t\Big\{\int_\Omega u^{m+1}(1+\phi'\rho^m)\Big\}=\partial_t\Big\{\int_{\mathbb{T}^{n-1}}\rho^{m+1}\Big\}. \tag{3.79}$$

Proof. We multiply the equation (3.66) with $(1+\phi'\rho^m)$ and integrate over Ω. We thus obtain

$$\begin{aligned} &\int_\Omega(1+\phi'\rho^m)(u_t^{m+1}-\Delta_{x'}u^{m+1})-\int_\Omega\frac{1+|\phi\nabla\rho^m|^2}{1+\phi'\rho^m}u_{nn}^{m+1}+2\int_\Omega\phi\nabla\rho^m\cdot\nabla_{x'}u_n^{m+1}\\ &+\int_\Omega\phi\Delta\rho^m u_n^{m+1}-\int_\Omega\Big(\frac{(\phi^2)'|\nabla\rho^m|^2}{1+\phi'\rho^m}-\frac{\phi''\rho^m(1+|\phi\nabla\rho^m|^2)}{(1+\phi'\rho^m)^2}\Big)u_n^{m+1}-\int_\Omega\phi\rho_t^m u_n^{m+1}=0. \end{aligned}$$

Integrating by parts we have $\int_\Omega-\phi\rho_t^m u_n^{m+1}=\int_\Omega\phi'\rho_t^m u^{m+1}$. Using this identity, we obtain

$$\int_\Omega(1+\phi'\rho^m)u_t^{m+1}+\int_\Omega(1+\phi'\rho^m)e_\rho u_n^{m+1}=\partial_t\Big\{\int_\Omega(1+\phi'\rho^m)u^{m+1}\Big\}.$$

Observe that the integration by parts implies

$$\int_\Omega-(1+\phi'\rho^m)\Delta_{x'}u^{m+1}=\int_\Omega\phi'\nabla_{x'}\rho^m\cdot\nabla_{x'}u^{m+1}$$

and $\int_\Omega \phi\Delta\rho^m u_n^{m+1} = -\int_\Omega \phi\nabla\rho^m\cdot\nabla_{x'}u_n^{m+1}$. Thus

$$-\int_\Omega (1+\phi'\rho^m)\Delta_{x'}u^{m+1} + \int_\Omega 2\phi\nabla\rho^m\cdot\nabla_{x'}u_n^{m+1} + \int_\Omega \phi\Delta\rho^m u_n^{m+1} = 0$$

Note further that

$$\partial_n\Big(\frac{1+|\phi\nabla\rho^m|^2}{1+\phi'\rho^m}\Big) = \frac{(\phi^2)'|\nabla\rho^m|^2}{1+\phi'\rho^m} - \frac{\phi''\rho^m(1+|\phi\nabla\rho^m|^2)}{(1+\phi'\rho^m)^2}$$

Using integration by parts again we have

$$-\int_\Omega \frac{1+|\phi\nabla\rho^m|^2}{1+\phi'\rho^m}u_{nn}^{m+1} - \int_\Omega\Big(\frac{(\phi^2)'|\nabla\rho^m|^2}{1+\phi'\rho^m} - \frac{\phi''\rho^m(1+|\phi\nabla\rho^m|^2)}{(1+\phi'\rho^m)^2}\Big)u_n^{m+1} =$$
$$-\int_{\mathbb{T}^{n-1}}\langle\rho^m\rangle^2[u_n^{m+1}]_+^- = -\partial_t\Big\{\int_{\mathbb{T}^{n-1}}\rho^{m+1}\Big\}.$$

This finishes the proof of the lemma. □

The importance of this identity is reflected in the fact that it allows to control terms with purely temporal derivatives of ρ^{m+1}:

Lemma 3.2. *There exist positive constants K and $\theta_0<1$ such that for any $\theta\le\theta_0$ such that if*

$$\Big|\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon - \int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big| < \theta,$$

$$\mathcal{E}^m\le\theta, \quad ||\nabla\rho^{m-1}||_\infty\le 1 \quad \textit{and} \quad \sum_{p=0}^k||\partial_p\rho^m||_2 \le K\sqrt{\mathcal{E}^m}+\theta,$$

then $||\nabla\rho^m||_\infty\le 1$ *and* $\sum_{p=0}^k||\partial_p\rho^{m+1}||_2\le K\sqrt{\mathcal{E}^{m+1}}+\theta$.

Proof. Observe first that the assumption on ρ^{m-1} implies that $\langle\rho^{m-1}\rangle\le\sqrt{2}$. Using the Sobolev inequality, we obtain

$$||\nabla\rho^m||_\infty \le C_*||\nabla\rho^m||_{H^{(n+1)/2}} \le C_*\sqrt{2}\mathcal{E}^m \le C_*\sqrt{2}\theta.$$

Thus, choosing $\theta_0\le\frac{1}{C_*\sqrt{2}}$ guarantees $||\nabla\rho^m||_\infty\le 1$. By (3.79) we have

$$\int_{\mathbb{T}^{n-1}}\partial_s\rho_t^{m+1} = \int_\Omega \partial_s u_t^{m+1} + \int_\Omega \phi'\sum_{p=0}^{s+1}\binom{s+1}{p}\partial_p u^{m+1}\partial_{s+1-p}\rho^m.$$

Also, $\int_{\mathbb{T}^{n-1}}\rho^{m+1} = \int_\Omega (u^{m+1} + \phi' u^{m+1}\rho^m) + \int_{\mathbb{T}^{n-1}}\rho_0^\epsilon - \int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)$. Thus, for $0 \le s \le k-1$, using the Cauchy-Schwarz inequality, definition (3.75) of $\mathcal{E}^{m+1}$ and the main assumption in the statement of the lemma, we obtain

$$\Big|\int_{\mathbb{T}^{n-1}} \partial_s \rho_t^{m+1}\Big| \le ||\partial_{s+1}u^{m+1}||_2 + C\sum_{p=1}^{s+1} ||\partial_p u^{m+1}||_{L^2(\Omega)} ||\partial_{s+1-p}\rho^m||_2$$
$$\le \sqrt{\mathcal{E}^{m+1}} + C\sqrt{\mathcal{E}^{m+1}}\sum_{p=0}^{k}||\partial_p\rho^m||_2.$$

Furthermore,

$$\Big|\int_{\mathbb{T}^{n-1}} \rho^{m+1}\Big| \le C||u^{m+1}||_2 + C||u^{m+1}||_{L^2(\Omega)}||\rho^m||_2 + \Big|\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon - \int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big|$$
$$\le C\sqrt{\mathcal{E}^{m+1}} + C\sqrt{\mathcal{E}^{m+1}}\sum_{p=0}^{k}||\partial_p\rho^m||_2 + \theta.$$

By the Poincaré inequality we get

$$\sum_{p=0}^{k}||\partial_p\rho^{m+1}||_2 \le C\sum_{p=0}^{k}||\partial_p\nabla\rho^{m+1}||_2 + C\sum_{p=0}^{k}||\int_{\mathbb{T}^{n-1}}\partial_p\rho^{m+1}||_2. \tag{3.80}$$

The first term on the right-hand side is estimated by $C\sqrt{\mathcal{E}^{m+1}}$, by the definition (3.75) of $\mathcal{E}^{m+1}$. By the previous two inequalities and the assumptions of the lemma, we can estimate the second sum on RHS of (3.80) by $C\sqrt{\mathcal{E}^{m+1}} + C\sqrt{\mathcal{E}^{m+1}}\big(K\sqrt{\mathcal{E}^m}+\theta\big)+\theta$. Keeping in mind that $\mathcal{E}^m \le \theta$, we choose $1 \le K$ large enough and $\theta < \frac{1}{C_*\sqrt{2}}$ small enough so that

$$\sum_{p=0}^{k}||\partial_p\rho^{m+1}||_2 \le K\sqrt{\mathcal{E}^{m+1}}+\theta.$$

□

In the following we shall work under the standing assumption

$$\mathcal{E}^m \le \theta, \quad ||\nabla\rho^{m-1}||_\infty \le 1 \quad \sum_{p=0}^{k}||\partial_p\rho^m||_2 \le K\sqrt{\mathcal{E}^m}+\theta, \tag{3.81}$$

with $\theta \le \theta_0$ where θ_0 is given as in Lemma 3.2.

LEMMA 3.3. *Let $\xi \in C^\infty(J,\mathbb{R})$ and $J \subseteq \mathbb{R}$ an interval such that every derivative of ξ is uniformly bounded on J.*

(a) *Let (μ,r) be a pair of indices such that $|\mu|+2r\le 2k$ and $(|\mu|,r)\neq(0,0)$. Then there exists a positive constant C such that*

$$||\partial_r^\mu\big[\xi(|\nabla\rho^m|^2)\big]||_{H^1}\le C\sqrt{\mathcal{E}^m} \tag{3.82}$$

and

$$||\partial_r^\mu\big[\xi(|\nabla\rho^m|^2)\big]||_{H^3}\le \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}. \tag{3.83}$$

(b) *There exists a positive constant C such that*

$$||\partial_s^\mu\nabla\rho^m||_{H^2}\le C\sqrt{\mathcal{D}^{m+1}}, \tag{3.84}$$

where $|\mu|+2s\le 2k$. Furthermore,

$$||\partial_{s+1}^\mu\xi(|\nabla\rho^m|^2)||_2\le C\sqrt{\mathcal{D}^m} \tag{3.85}$$

for all (μ,s) satisfying $|\mu|+2s\le 2k$.

(c) *For any pair of indices (μ,s) such that $|\mu|+2s\le 2k$ there exists a positive constant C and a small parameter λ such that*

$$||\partial_s^\mu\rho_t^{m+1}||_2^2+2\epsilon||\partial_s^\mu\Delta\rho_t^{m+1}||_2^2+\epsilon^2||\partial_s^\mu\Delta^2\rho_t^{m+1}||_2^2\le\frac{C}{\lambda}\mathcal{E}^{m+1}+C\lambda\mathcal{D}^{m+1}. \tag{3.86}$$

Proof. *Part (a):* Let $\alpha=\mu+\tau$ for any given multi-index of length $n-1$ satisfying $|\tau|\le 1$. Let first $r=0$. By assumption $|\alpha|\ge 1$. Using Moser's inequality (cf. [**29**], Lemma 5) and Leibniz' rule (cf. [**29**], Lemma 4), we have

$$\begin{aligned}||\partial^\alpha\big[\xi(|\nabla\rho^m|^2)\big]||_2&\le C\max_{1\le d\le|\alpha|}\Big(||\xi^{(d)}(|\nabla\rho^m|^2))||_\infty||\nabla\rho^m||_\infty^{|\alpha|-1}\Big)||\partial^\alpha(|\nabla\rho^m|^2)||_2\\&\le C||\partial^\alpha(|\nabla\rho^m|^2)||_2\le C||\partial^\alpha\nabla\rho^m||_2||\nabla\rho^m||_\infty\le C\sqrt{\mathcal{E}^m}.\end{aligned}$$

Let $r\ge 1$.

$$\begin{aligned}\partial_r^\alpha\big[\xi(|\nabla\rho^m|^2)\big]&=\partial^\alpha\Big\{\sum_{d=1}^r\sum_{\substack{s_1+\dots+s_d=r\\ s_i>0}}C_d\xi^{(d)}(|\nabla\rho^m|^2)\partial_{s_1}(|\nabla\rho^m|^2)\dots\partial_{s_d}(|\nabla\rho^m|^2)\Big\}\\&=\sum_{d=1}^r\sum_{\substack{s_1+\dots+s_d=r\\ s_i>0}}\sum_{\substack{\gamma_1+\dots+\gamma_{d+1}\\=\alpha}}C_dC_{\gamma_1\dots\gamma_{d+1}}\partial_{s_1}^{\gamma_1}(|\nabla\rho^m|^2)\dots\partial_{s_d}^{\gamma_d}(|\nabla\rho^m|^2)\partial^{\gamma_{d+1}}\big[\xi^{(d)}(|\nabla\rho^m|^2)\big].\end{aligned}$$

For any $i=1,\dots,d$, we have $\partial_{s_i}^{\gamma_i}(|\nabla\rho^m|^2)=\sum_{l=0}^{s_i}\sum_{\delta\le\gamma_i}C_{l,\delta}\partial_l^\delta\nabla\rho^m\partial_{s_i-l}^{\gamma_i-\delta}\nabla\rho^m$. Thus if $|\gamma_i|+2s_i\le k+1$, then by the Sobolev inequality

$$||\partial_l^\delta\nabla\rho^m||_\infty\le||\partial_l^\delta\nabla\rho^m||_{H^{\frac{n+1}{2}}}\le C\sqrt{\mathcal{E}^m},$$

and analogously $||\partial_{s_i-l}^{\gamma_i-\delta}\nabla\rho^m||_\infty\le C\sqrt{\mathcal{E}^m}$, implying $||\partial_{s_i}^{\gamma_i}(|\nabla\rho^m|^2)||_\infty\le C\mathcal{E}^m$. If $|\gamma_{d+1}|\le k+1$, we use the Sobolev and Moser's inequality to conclude

$$||\partial^{\gamma_{d+1}}\left[\xi^{(d)}(|\nabla\rho^m|^2)\right]||_\infty\le C\sqrt{\mathcal{E}^m}.$$

If there exists $1\le j\le d$ such that $|\gamma_j|+2s_j>k+1$, then $|\gamma_i|+2s_i\le k+1$ for $1\le i\le d$, $i\neq j$ and additionally, $|\gamma_{d+1}|\le k+1$. Thus we can estimate the term containing γ_j in superscript in L^2-norm and the remaining terms in L^∞-norm. If on the other hand $|\gamma_i|+2s_i\le k+1$ for every $1\le i\le d$, we estimate the term $\partial^{\gamma_{d+1}}\left[\xi^{(d)}(|\nabla\rho^m|^2)\right]$ in L^2-norm and the remaining terms in L^∞-norm. We conclude that

$$||\partial_r^\alpha\left[\xi(|\nabla\rho^m|^2)\right]||_2\le C\sqrt{\mathcal{E}^m},$$

for the specified range of α-s and r-s. The inequality (3.83)is proved similarly.

Part (b): By (3.67), $\Delta\rho^m=u^{m+1}\langle\rho^m\rangle+\rho_i^m\rho_j^m\rho_{ij}^m\langle\rho^m\rangle^{-2}$. Let $\gamma=\mu+\tau$ where $|\tau|=1$. Applying ∂_s^γ to the above identity, we get

$$\begin{aligned}\partial_s^\gamma\Delta\rho^m=&\sum_{\gamma',s'}C_{s'}^{\gamma'}\partial_{s'}^{\gamma'}(u^{m+1})\partial_{s-s'}^{\gamma-\gamma'}(\langle\rho^m\rangle)+\\&\sum_{\substack{\sum(\gamma_l+s_l)\\=\gamma+s}}C_{s_1,s_2,s_3,s_4}^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\partial_{s_1}^{\gamma_1}\rho_i^m\partial_{s_2}^{\gamma_2}\rho_j^m\partial_{s_3}^{\gamma_3}\rho_{ij}^m\partial_{s_4}^{\gamma_4}(\langle\rho^m\rangle^{-2})\end{aligned}$$

Observe that $\int_{\mathbb{T}^{n-1}}u^{m+1}=0$ since $u^{m+1}=\kappa^m=\nabla\cdot\left(\nabla\rho^m\langle\rho^m\rangle^{-1}\right)$ on $\mathbb{T}^{n-1}$. Let us fix $(\gamma',s')\le(\gamma,s)$. If $|\gamma'|>0$ note that $\int_\Omega\partial_{s'}^{\gamma'}u^{m+1}=0$. We use the trace inequality and then the Poincaré inequality on Ω to deduce

$$||\partial_{s'}^{\gamma'}u^{m+1}||_{L^2(\mathbb{T}^{n-1})}\le C||\partial_{s'}^{\gamma'}u^{m+1}||_{H^1(\Omega)}\le C||\partial_{s'}^{\gamma'}\nabla u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{D}^{m+1}}.$$

If $|\gamma'|=0$ by the Poincaré inequality and the trace inequality:

$$\begin{aligned}||\partial_{s'}u^{m+1}||_{L^2(\mathbb{T}^{n-1})}&\le C||\partial_{s'}\nabla_{x'}u^{m+1}||_{L^2(\mathbb{T}^{n-1})}\\&\le C||\partial_{s'}\nabla_{x'}u^{m+1}||_{H^1(\Omega)}\le C\sqrt{\mathcal{D}^{m+1}}.\end{aligned}$$

By part (a), $||\partial_{s-s'}^{\gamma-\gamma'}(\langle\rho^m\rangle)||_2 \leq C + C\sqrt{\mathcal{E}^m}$. Furthermore, if for some $1 \leq l \leq 4$ we have $|\gamma_l| + 2s_l \geq k$, we estimate the term containing γ_l in superscript in L^2-norm and the remaining terms in L^∞-norm. Using part (a) we deduce

$$||\partial_{s_1}^{\gamma_1}\rho_i^m \partial_{s_2}^{\gamma_2}\rho_j^m \partial_{s_3}^{\gamma_3}\rho_{ij}^m \partial_{s_4}^{\gamma_4}(\langle\rho^m\rangle^{-2})||_2 \leq C(\mathcal{E}^m)^{3/2} \sum_{|\mu|+2s\leq 2k} ||\partial_s^\mu \nabla\rho^m||_{H^2}.$$

Thus, summing over all pairs (μ, s) yields

$$\sum_{\substack{|\mu|+2s\leq 2k\\ \gamma=\mu+\tau, |\tau|=1}} ||\partial_s^\gamma \Delta\rho^m||_2 \leq (C + C\sqrt{\mathcal{E}^m})\sqrt{\mathcal{D}^{m+1}} + C(\mathcal{E}^m)^{3/2} \sum_{|\mu|+2s\leq 2k} ||\partial_s^\mu \nabla\rho^m||_{H^2}. \tag{3.87}$$

Since $\int_{\mathbb{T}^{n-1}} \nabla\rho^m = 0$ and $|\gamma| \geq 1$ elliptic regularity implies

$$\sum_{|\mu|+2s\leq 2k} ||\partial_s^\mu \nabla\rho^m||_{H^2} \leq C \sum_{\substack{|\mu|+2s\leq 2k\\ \gamma=\mu+\tau, |\tau|=1}} ||\partial_s^\gamma \Delta\rho^m||_2. \tag{3.88}$$

Combining (3.87), (3.88) and choosing $\mathcal{E}^m$ sufficiently small we deduce the claim.

Part (c): We apply the differential operator ∂_s^μ to the 'jump relation' (3.70) and take squares on both sides to obtain

$$||\partial_s^\mu \rho_t^{m+1}||_2^2 + 2\epsilon||\partial_s^\mu \Delta\rho_t^{m+1}||_2^2 + \epsilon^2||\partial_s^\mu \Delta^2\rho_t^{m+1}||_2^2 = ||\partial_s^\mu\big(\langle\rho^m\rangle^2 [u_n^{m+1}]_+^-\big)||_2^2.$$

Next

$$\begin{aligned}
||\partial_s^\mu\Big(\langle\rho^m\rangle^2 [u_n^{m+1}]_+^-\Big)||_2 &= ||\sum_{\mu',s'} C_{s'}^{\mu'} \partial_{s'}^{\mu'}(\langle\rho^m\rangle^2)\partial_{s-s'}^{\mu-\mu'}[u_n^{m+1}]_+^-||_2\\
&\leq C \sum_{\substack{|\mu'|+2s'\\ \leq k}} ||\partial_{s'}^{\mu'}(\langle\rho^m\rangle^2)||_\infty ||\partial_{s-s'}^{\mu-\mu'}[u_n^{m+1}]_+^-||_2 +\\
&\quad + C \sum_{\substack{|\mu'|+2s'\\ > k}} ||\partial_{s'}^{\mu'}(\langle\rho^m\rangle^2)||_2 ||\partial_{s-s'}^{\mu-\mu'}[u_n^{m+1}]_+^-||_{L^\infty(\mathbb{T}^{n-1})}\\
&\leq (C\sqrt{\mathcal{E}^m}+1)\Big(\frac{C}{\lambda}\mathcal{E}^{m+1} + C\lambda\mathcal{D}^{m+1}\Big) \leq \frac{C}{\lambda}\mathcal{E}^{m+1} + C\lambda\mathcal{D}^{m+1}.
\end{aligned}$$

Here we assume $\mathcal{E}^m \leq 1$. Terms involving L^∞-norm are first estimated by the Sobolev inequality. Then we use the standard trace inequality $||v||_{L^2(\partial\Omega)} \leq \frac{C}{\lambda}||v||_{L^2(\Omega)} + \lambda||\nabla v||_{L^2(\Omega)}$ to bound the terms involving $[u_n^{m+1}]_+^-$. Observe that the same proof is easily adapted to yield the bound

$$||\partial_s \rho_t^{m+1}||_2 \leq C\sqrt{\mathcal{D}^{m+1}}, \qquad \text{for} \quad s \leq k. \tag{3.89}$$

□

Remark. The estimate (3.86) will play a crucial role in the energy estimates for the problem of local existence. For any pair (μ, s), $|\mu| + 2s \leq 2k$ and $|\mu| \geq 1$, the elliptic regularity and the estimate (3.86) imply

$$||\nabla^4 \partial_s^{\mu-1} \nabla \rho_t^{m+1}||_2 \leq \frac{C}{\lambda \epsilon^2} \sqrt{\mathcal{E}^{m+1}} + \lambda \sqrt{\mathcal{D}^{m+1}}. \tag{3.90}$$

Lemma 3.4. *There exists a positive constant C such that*

(a) *For $r \geq 0$, $|\mu| + 2s \leq 2k$ and $(|\mu|, s) \neq (0,0)$*

$$||\partial_s^\mu \partial_{n^r}(a_{\rho^m})||_{L^2(\Omega)} \leq C\sqrt{\mathcal{E}^m} \tag{3.91}$$

(b) *For $r \geq 0$ and $|\mu| + 2s \leq 2k$*

$$||\partial_s^\mu \partial_{n^r}(B_{\rho^m})||_{L^2(\Omega)} \leq C\sqrt{\mathcal{E}^m} \tag{3.92}$$

(c) *For $r \geq 0$ and $|\mu| + 2s \leq 2k$*

$$||\partial_s^\mu \partial_{n^r}(c_{\rho^m})||_{L^2(\Omega)} \leq C\sqrt{\mathcal{E}^m} + C\sqrt{\mathcal{D}^m}. \tag{3.93}$$

Furthermore, for $r \geq 0$, $|\mu| + 2s \leq 2k - 1$

$$||\partial_s^\mu \partial_{n^r}(c_{\rho^m})||_{L^2(\Omega)} \leq C\sqrt{\mathcal{E}^m}. \tag{3.94}$$

Proof. *Part (a):* We note:

$$\begin{aligned}
\partial_s^\mu \partial_{n^r}(a_{\rho^m}) &= \partial_s^\mu \partial_{n^r}\big((1 + |\phi \nabla \rho^m|^2)(1 + \phi' \rho^m)^{-2}\big) \\
&= \partial_s^\mu \Big(\sum_{p=0}^{r} \partial_{n^p} F_1(|\phi \nabla \rho^m|) \partial_{n^{r-p}} F_2(\phi' \rho^m) \Big),
\end{aligned}$$

where $F_1(x) := 1 + x^2$ and $F_2(x) := (1+x)^{-2}$. Observe that

$$\partial_{n^{r-p}} F_2(\phi' \rho^m) = \sum_{d=1}^{r-p} \sum_{\substack{s_1 + \dots s_d \\ = r-p}} C_d F_2^{(d)}(\phi' \rho^m)(\rho^m)^d \phi^{(s_1)} \dots \phi^{(s_d)},$$

and thus

$$
\begin{aligned}
||\partial_s^\mu \partial_{n^{r-p}} F_2(\phi'\rho^m)||_{L^2(\Omega)} &\leq C\sum_{d=1}^{r-p} ||\partial_s^\mu \left[F_2^{(d)}(\phi'\rho^m)(\rho^m)^d\right]||_{L^2(\Omega)} \\
&\leq C\sum_{d=1}^{r-p}\sum_{\mu',s'} C_{s'}^{\mu'} ||\partial_{s'}^{\mu'}\left[[F_2^{(d)}(\phi'\rho^m)\right]\partial_{s-s'}^{\mu-\mu'}\left[G_d(\rho^m)\right]||_{L^2(\Omega)},
\end{aligned}
$$

where $G_d(x) := x^d$. By Moser's inequality, for any pair of indices (ν,q) such that $|\nu|+2q\leq 2k$, $(|\nu|,q)\neq(0,0)$ we have $||\partial_q^\nu\left[[F_2^{(d)}(\phi'\rho^m)\right]||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}$ if $|\nu|>0$, and $||\partial_q^\nu\left[[F_2^{(d)}(\phi'\rho^m)\right]||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}+\theta$ if $|\nu|=0$. We have used Lemma 3.2. Similarly, if $|\nu|+2q\leq 2k$, we have $||\partial_q^\nu\left[G_d(\rho^m)\right]||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}$ when $|\nu|>0$, and $||\partial_q^\nu\left[G_d(\rho^m)\right]||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}+\theta$ when $|\nu|=0$. Thus

$$
||\partial_{s'}^{\mu'}\left[[F_2^{(d)}(\phi'\rho^m)\right]\partial_{s-s'}^{\mu-\mu'}\left[G_d(\rho^m)\right]||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}, \tag{3.95}
$$

where we hit the terms with lower order derivatives with L^∞-norms, depending on whether $|\mu'|+2s'\leq k$ or $|\mu'|+2s'\geq k$. Additionally, we use the assumption that $\theta<1$ and $\mathcal{E}^m<1$. This implies $||\partial_s^\mu\partial_{n^{r-p}}F_2(\phi'\rho^m)||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}$ for $0\leq p\leq r$ and $|\mu|+2s\leq 2k$. In the same way we prove $||\partial_s^\mu\partial_{n^p}F_1(|\phi\nabla\rho^m|)||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}$ for $0\leq p\leq r$, $|\mu|+2s\leq 2k$ and $(|\mu|,s)\neq(0,0)$. Using the same idea of estimating lower order terms with L^∞-norms as in the proof of (3.95), we conclude the proof of part (a). The proof of part (b) follows in a completely analogous way.

Part (c): To prove part (c), recall that $c_{\rho^m}=d_{\rho^m}+e_{\rho^m}$, where d_{ρ^m} and e_{ρ^m} are given by

$$
d_{\rho^m} := \frac{\phi\Delta\rho^m}{1+\phi'\rho^m} - \frac{(\phi^2)'|\nabla\rho^m|^2}{(1+\phi'\rho^m)^2} + \frac{\phi''\rho^m(1+|\phi\nabla\rho^m|^2)}{(1+\phi'\rho^m)^3}, \quad e_{\rho^m} := \frac{\phi\rho_t^m}{1+\phi'\rho^m}.
$$

The analogous proof as in the part (a) implies that $||\partial_s^\mu\partial_{n^r}(d_{\rho^m})||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}$. Furthermore, since $||\partial_s^\mu\rho_t^m||\leq C\sqrt{\mathcal{D}^m}$ if $|\mu|+2s\leq 2k$, we use the same method as in part (a) to prove $||\partial_s^\mu\partial_{n^r}(e_{\rho^m})||_{L^2(\Omega)}\leq C\sqrt{\mathcal{D}^m}$. We thus conclude (3.93). From the definition of $\mathcal{E}^m$ (cf. (3.75)) and the assumption of Lemma 3.2, we have $||\partial_s^\mu\rho_t^m||\leq C\sqrt{\mathcal{E}^m}$ for $|\mu|+2s\leq 2k-1$, $|\mu|>0$; also $||\partial_s\rho_t^m||\leq C\sqrt{\mathcal{E}^m}+\theta$ for all $s\leq k-1$. Now we use the same method as in the proof of part (a) to deduce (3.94). $\square$

LEMMA 3.5. *For all pairs of indices* (μ,s) *such that* $|\mu|+2s+r\le k+\frac{n}{2}+3$, *the inequality* $||\partial_s^\mu\partial_{n^r}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}$ *holds.*

Proof. We prove the claim by induction in r. In case $r=1$ the claim is obvious from the definition of $\mathcal{E}^m$. Let the claim be true for all $r\le\vartheta$ for some $1<\vartheta<k+n/2+3$. We have to prove the claim for $r=\vartheta+1$, i.e.

$$||\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}, \tag{3.96}$$

if $|\mu|+2s\le k+\frac{n}{2}-\vartheta+2$. Let (μ,s) be such a pair of indices. Then

$$\begin{aligned}
&\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}=\partial_s^\mu\partial_{n^{\vartheta-1}}u_{nn}^{m+1}\\
&=\partial_s^\mu\partial_{n^{\vartheta-1}}\Big\{\big(u_t^{m+1}-\Delta_{x'}u^{m+1}+B_{\rho^m}\cdot\nabla_{x'}u_n^{m+1}+c_{\rho^m}u_n^{m+1}\big)a_{\rho^m}^{-1}\Big\}\\
&=\sum_{\mu',s'}\sum_{w=0}^{\vartheta-1}C(\mu',s',w)\Big\{\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}+\\
&+\sum_{\mu'',s''}\sum_{p=0}^{w}C(\mu'',s'',p)\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}+\\
&+\sum_{\mu'',s''}\sum_{p=0}^{\vartheta-1}C(\mu'',s'',p)\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}\Big\}\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big).
\end{aligned} \tag{3.97}$$

Observe that $||\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}$ and $||\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}$ for any triple of indices $(\mu',s',w)\le(\mu,s,\vartheta-1)$ by inductive hypothesis. Further, by Lemma 3.4 and the Sobolev inequality $||\partial_{s''}^{\mu''}\partial_{n^p}B_{\rho^m}||_{L^\infty(\Omega)}\le C\sqrt{\mathcal{E}^m}$, $||\partial_{s''}^{\mu''}\partial_{n^p}c_{\rho^m}||_{L^\infty(\Omega)}\le C\sqrt{\mathcal{E}^m}$ and $||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^\infty(\Omega)}\le C\sqrt{\mathcal{E}^m}+C$, for $(\mu'',s'',p)\le(\mu',s',w)\le(\mu,s,\vartheta-1)$. Applying these estimates to the above identity and using the inductive assumption, we obtain

$$||\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{E}^{m+1}}+C\mathcal{E}^m\sqrt{\mathcal{E}^{m+1}}\le C\sqrt{\mathcal{E}^{m+1}},$$

where we recall the smallness assumption on $\mathcal{E}^m$, specially $\mathcal{E}^m\le 1$. This finishes the proof of the lemma. □

LEMMA 3.6. *If* $|\mu|+r+2s\le 2k+1$,

$$||\partial_s^\mu\partial_{n^r}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}}. \tag{3.98}$$

Proof. We prove the claim by induction in r. In case $r=1$ the claim is obvious from the definition of $\mathcal{E}^m$. Let the claim be true for all $r\le\vartheta$ for some $1<\vartheta<2k+1$. We have to prove the claim for $r=\vartheta+1$, i.e.

$$||\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}||_{L^2(\Omega)}\le C\sqrt{\mathcal{E}^{m+1}},$$

if $|\mu|+2s+\vartheta\le 2k$. Let (μ,s) be such a pair of indices. Then

$$\begin{aligned}
&\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}=\partial_s^\mu\partial_{n^{\vartheta-1}}u_{nn}^{m+1}\\
&=\partial_s^\mu\partial_{n^{\vartheta-1}}\Big\{\big(u_t^{m+1}-\Delta_{x'}u^{m+1}+B_{\rho^m}\cdot\nabla_{x'}u_n^{m+1}+c_{\rho^m}u_n^{m+1}\big)a_{\rho^m}^{-1}\Big\}\\
&=\sum_{\mu',s'}\sum_{w=0}^{\vartheta-1}C(\mu',s',w)\Big\{\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}+\\
&+\sum_{\mu'',s''}\sum_{p=0}^{w}C(\mu'',s'',p)\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)\cdot\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}+\\
&+\sum_{\mu'',s''}\sum_{p=0}^{w}C(\mu'',s'',p)\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}\Big\}\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big).
\end{aligned}\tag{3.99}$$

We analyze separately the case when $|\mu'|+w+2s'\le k$ and $|\mu'|+w+2s'\ge k$. **Case 1.** In the case $|\mu'|+w+2s'\ge k$ note that

$$|\mu-\mu'|+2(s-s')+\vartheta-w\le 2k-k=k.$$

By the Sobolev inequality and Lemma 3.4, we have

$$||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^\infty(\Omega)}\le C\sqrt{\mathcal{E}^m}.$$

If $(\mu'',s'',p)\le(\mu',s',w)$, $|\mu''|+p+2s''\le k$ and

$$||\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)||_{L^\infty(\Omega)}\le||\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)||_{H^{n/2+1}(\Omega)}\le C\sqrt{\mathcal{E}^m},$$

where we have used the Sobolev inequality and Lemma 3.4 respectively. Similarly $||\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)||_{L^\infty(\Omega)}\le C\sqrt{\mathcal{E}^m}$. This implies

$$\begin{aligned}
&\int_\Omega|\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)|^2|(\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1})|^2(\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big))^2\\
&\le C||\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)||_{L^\infty(\Omega)}^2||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}||_{L^2(\Omega)}^2\times\\
&\quad\times||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^\infty(\Omega)}^2\le C(\mathcal{E}^m)^2\mathcal{E}^{m+1},
\end{aligned}$$

where we have used the inductive hypothesis to deduce

$$||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}||_{L^2(\Omega)}^2 \leq C\mathcal{E}^{m+1}.$$

Analogously

$$\int_\Omega (\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big))^2(\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1})^2(\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big))^2$$
$$\leq C(\mathcal{E}^m)^2\mathcal{E}^{m+1}$$

If on the other hand $|\mu''|+p+2s''>k$, we use the Sobolev inequality and Lemma 3.5 to get

$$||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}||_{L^\infty(\Omega)} \leq ||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}||_{H^{n/2+1}(\Omega)} \leq C\sqrt{\mathcal{E}^{m+1}},$$

where we note that

$$|\mu'-\mu''|+(w-p+1)+n/2+1+2(s'-s'')\leq k+n/2+1,$$

so that Lemma 3.5 is applicable. In analogous fashion it follows

$$||\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}||_{L^\infty(\Omega)} \leq C\sqrt{\mathcal{E}^{m+1}}.$$

We also note that

$$\int_\Omega \Big(\partial_{s'}^{\mu'}\partial_{n^w}\partial u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}\Big)^2(\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big))^2$$
$$\leq ||\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}||_{L^2(\Omega)}^2||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^\infty(\Omega)}^2$$
$$\leq C\mathcal{E}^{m+1}\mathcal{E}^m \leq C\mathcal{E}^{m+1}.$$

Observe that we have used the inductive hypothesis in the last inequality above. This completes the first case.

Case 2. In the case $|\mu'|+w+2s'\leq k$, by the Sobolev inequality and Lemmas 3.4 and 3.5,

$$||\partial_{s''}^{\mu''}\partial_{n^p}\big(B_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}\nabla_{x'}u^{m+1}||_{L^\infty(\Omega)} \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{E}^{m+1}},$$

and $||\partial_{s''}^{\mu''}\partial_{n^p}\big(c_{\rho^m}\big)\partial_{s'-s''}^{\mu'-\mu''}\partial_{n^{w-p+1}}u^{m+1}||_{L^\infty(\Omega)}\leq C\sqrt{\mathcal{E}^{m+1}}$ for $(\mu'',s'',p)\leq(\mu',s',w)$. By Lemma 3.4, $||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^m}+C$. We also note

$$\begin{aligned}&\int_\Omega\Big(\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}\Big)^2(\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big))^2\\&\leq||\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}||_{L^\infty(\Omega)}^2||\partial_{s-s'}^{\mu-\mu'}\partial_{n^{\vartheta-1-w}}\big(a_{\rho^m}^{-1}\big)||_{L^2(\Omega)}^2\\&\leq C\mathcal{E}^{m+1}\mathcal{E}^m+C\mathcal{E}^{m+1}\leq C\mathcal{E}^{m+1}.\end{aligned}$$

By the Sobolev inequality and Lemma 3.5,

$$||\partial_{s'}^{\mu'}\partial_{n^w}u_t^{m+1}-\partial_{s'}^{\mu'}\partial_{n^w}\Delta_{x'}u^{m+1}||_{L^\infty(\Omega)}^2\leq C\mathcal{E}^{m+1}.$$

We combine the above estimates to conclude $||\partial_s^\mu\partial_{n^{\vartheta+1}}u^{m+1}||_{L^2(\Omega)}\leq C\sqrt{\mathcal{E}^{m+1}}$ and this completes the second case and finishes the proof of the lemma. □

4. Energy estimates

Recalling the assumptions on (u^m,ρ^m) and the initial data before the formulation of the problem (3.66) - (3.68) and (3.70), we state the main result of this section:

LEMMA 4.1. *Let K and $\theta\leq\theta_0$ be given as in Lemma 3.2. There exists $0<L\leq\theta$ and $\mathcal{T}^\epsilon$ such that if*

$$\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon-\int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big|\leq\frac{L}{2}$$

and for some $m\in\mathbb{N}$

$$\sup_{0\leq t\leq\mathcal{T}^\epsilon}\mathcal{E}^m(t)+\int_0^{\mathcal{T}^\epsilon}\mathcal{D}^m(\tau)\,d\tau\leq L,$$

$$||\nabla\rho^{m-1}||_\infty\leq 1,\quad\sum_{p=0}^k||\partial_p\rho^m||_2\leq K\sqrt{\mathcal{E}^m}+\theta,$$

then

$$\sup_{0\leq t\leq\mathcal{T}^\epsilon}\mathcal{E}^{m+1}(t)+\int_0^{\mathcal{T}^\epsilon}\mathcal{D}^{m+1}(\tau)\,d\tau\leq L.$$

Proof. With the preparation from Chapter 2, we are ready to estimate RHS of (3.76) term by term. Note that the assumptions of Lemma 3.2 are fulfilled and we are thus able to use Lemmas 3.3 - 3.6 in the forthcoming estimates. Let (μ,s) be an arbitrary, but fixed pair of indices satisfying $|\mu|+2s\leq 2k$.

Term $\int_\Omega P^m_{\mu,s}$: Recall that $P^m_{\mu,s}=P(\partial^\mu_s u^{m+1},\rho^m,f^m)$ is given by (3.77), where P is given by (2.47) and f^m by (3.71). Thus, combining (2.47) and (3.71) we can estimate the first term on RHS of $\int_\Omega P^m_{\mu,s}$:

$$
\begin{aligned}
&\Big|\int_\Omega \Big\{\partial^\mu_s(a_{\rho^m}u^{m+1}_{nn})-a_{\rho^m}\partial^\mu_s u^{m+1}_{nn}\Big\}\partial^\mu_s u^{m+1}\Big|\\
&\leq C\sum_{(|\mu'|,s')\neq(0,0)}\Big|\int_\Omega \partial^{\mu'}_{s'}a_{\rho^m}\partial^{\mu-\mu'}_{s-s'}u^{m+1}_{nn}\partial^\mu_s u^{m+1}\Big| = C\sum_{\substack{|\mu'|+2s'\leq k\\ (|\mu'|,s')\neq(0,0)}} + C\sum_{|\mu'|+2s'>k}\\
&\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{D}^{m+1}}\leq C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}.
\end{aligned}
\tag{4.100}
$$

In the first sum, observe that $||\partial^{\mu'}_{s'}a_{\rho^m}||_{L^\infty(\Omega)}\leq C\sqrt{\mathcal{E}^m}$ for $|\mu'|+2s'\leq k$, by the Sobolev inequality and Lemma 3.3. In the second sum observe that $||\partial^{\mu-\mu'}_{s-s'}u^{m+1}_{nn}||_{L^\infty(\Omega)}\leq C\sqrt{\mathcal{D}^{m+1}}$, by the Sobolev inequality and Lemma 3.6. By glancing at (2.47) and (3.71) the second term in the expression $\int_\Omega P^m_{\mu,s}$ is given by $\int_\Omega \partial^\mu_s(B_{\rho^m}\cdot\nabla_{x'}u^{m+1}_n)\partial^\mu_s u^{m+1}$. It is estimated in a completely analogous way and is bounded by $C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}$ again. By (3.77), the third term on RHS (2.47) renders the third term in $\int_\Omega P^m_{\mu,s}$. We have

$$
\begin{aligned}
&\Big|\int_\Omega \partial^\mu_s\big(c_{\rho^m}u^{m+1}_n\big)\partial^\mu_s u^{m+1}\Big|\leq C\sum_{\mu',s'}\Big|\int_\Omega \partial^{\mu'}_{s'}c_{\rho^m}\partial^{\mu-\mu'}_{s-s'}u^{m+1}_n\partial^\mu_s u^{m+1}\Big|\\
&= C\sum_{|\mu'|+2s'\leq k} + C\sum_{|\mu'|+2s'>k}\\
&\leq C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}+C\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^{m+1}}.
\end{aligned}
\tag{4.101}
$$

In the first sum we estimate $||\partial^\mu_s c_{\rho^m}||_\infty$ like above (since $|\mu'|+2s'\leq k$). In the second sum, for $|\mu'|+2s'\geq k$, $||\partial^{\mu-\mu'}_{s-s'}u^{m+1}_n||_{L^\infty(\Omega)}\leq C\sqrt{\mathcal{E}^{m+1}}$ by the Sobolev inequality and Lemma 3.6. By Lemma 3.3, $||\partial^{\mu'}_{s'}c_{\rho^m}||_{L^2}\leq C\sqrt{\mathcal{D}^m}$. Finally, the fourth term of $\int_\Omega P^m_{\mu,s}$ (again use (2.47) to identify the fourth term and the equations (3.77) and (3.71) to plug in the appropriate values), is estimated by using the Cauchy-Schwarz inequality

$$
\begin{aligned}
&\Big|\int_\Omega (a_{\rho^m})_n\partial^\mu_s u^{m+1}_n\partial^\mu_s u^{m+1}\Big|\leq ||(a_{\rho^m})_n||_{L^\infty(\Omega)}||\partial^\mu_s u^{m+1}_n||_{L^2(\Omega)}||\partial^\mu_s u^{m+1}||_{L^2(\Omega)}\\
&\leq C\sqrt{D^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^{m+1}}.
\end{aligned}
\tag{4.102}
$$

Term $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$: We now proceed with the estimates for the expression $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$, where $Q^m_{\mu,s}$ is given by (3.77), where Q is defined by (2.49), $g^m_{\mu,s}$ is given by (3.72) and $h^m_{\mu,s}$ is given by (3.74). The first two terms of $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ are the cross-terms and they deserve special attention. For any $\eta > 0$,

(4.103)
$$\Big|\int_{\mathbb{T}^{n-1}} \partial^\mu_s \nabla \rho^{m+1}_t \big\{\partial^\mu_s \nabla \rho^m \langle \rho^m\rangle^{-1} - \partial^\mu_s \nabla \rho^{m+1} \langle \rho^m\rangle^{-1}\big\}\Big|$$
$$\leq \eta ||\partial^\mu_s \nabla \rho^{m+1}_t||_2^2 + \frac{C}{\eta}\Big(||\partial^\mu_s \nabla \rho^m||_2^2 + ||\partial^\mu_s \nabla \rho^{m+1}||_2^2\Big) \leq \eta \mathcal{D}^{m+1} + \frac{C}{\eta}(\mathcal{E}^m + \mathcal{E}^{m+1}).$$

$$\begin{aligned}
&\Big|\epsilon \int_{\mathbb{T}^{n-1}} \partial^\mu_s \nabla \Delta \rho^{m+1}_t \cdot \Big\{\Delta \partial^\mu_s \nabla \rho^m \langle \rho^m\rangle^{-1} - \Delta \partial^\mu_s \nabla \rho^{m+1} \langle \rho^m\rangle^{-1}\Big\}\Big| \\
&\leq \eta \epsilon ||\partial^\mu_s \Delta \nabla \rho^{m+1}_t||_2^2 + \frac{\epsilon C}{\eta}\Big(||\partial^\mu_s \Delta \nabla \rho^m||_2^2 + ||\partial^\mu_s \Delta \nabla \rho^{m+1}||_2^2\Big) \\
&\leq \eta \mathcal{D}^{m+1} + \frac{C}{\eta}(\mathcal{E}^m + \mathcal{E}^{m+1}).
\end{aligned} \tag{4.104}$$

Observe that the constant C does not depend on ϵ. The third term in $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ is given by (2.49) and (3.77). Note that $\partial_t\big(\langle \rho^m\rangle^{-1}\big) = \frac{2\nabla \rho^m \nabla \rho^m_t}{\langle \rho^m\rangle^3}$. By Lemma 3.3, we conclude $||\langle \rho^m\rangle^{-1}_t||_\infty \leq C\mathcal{D}^m$. We then obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{T}^{n-1}} \Big\{|\partial^\mu_s \nabla \rho^{m+1}|^2 \langle \rho^m\rangle^{-1}_t + \epsilon |\partial^\mu_s \Delta \nabla \rho^{m+1}|^2 \langle \rho^m\rangle^{-1}_t\Big\}\Big| \\
&\leq C\mathcal{D}^m\big(||\partial^\mu_s \nabla \rho^{m+1}_t||_2^2 + \epsilon ||\partial^\mu_s \Delta \nabla \rho^{m+1}||_2^2\big) \leq C\mathcal{E}^{m+1}\mathcal{D}^m.
\end{aligned} \tag{4.105}$$

To estimate the fourth term of $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ (which is obtained as the fourth term of (2.49) together with the definition (3.77)), we use the Cauchy-Schwarz inequality to get

$$\begin{aligned}
&\Big|\int_{\mathbb{T}^{n-1}} \partial^\mu_s \rho^{m+1}_t \partial^\mu_s \nabla \rho^m \cdot \nabla(\langle \rho^m\rangle^{-1})\Big| \leq ||\partial^\mu_s \rho^{m+1}_t||_2 ||\partial^\mu_s \nabla \rho^m||_2 ||\nabla(\langle \rho^m\rangle^{-1})||_\infty \leq \\
&C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m} \leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}).
\end{aligned} \tag{4.106}$$

Analogously, the fifth term in $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ is estimated as follows:

$$\begin{aligned}
&\epsilon\Big|\int_{\mathbb{T}^{n-1}} \Delta \nabla \partial^\mu_s \rho^{m+1}_t \cdot \nabla(\langle \rho^m\rangle^{-1}) \Delta \partial^\mu_s \rho^m\Big| \leq \\
&||\sqrt{\epsilon}\Delta \nabla \partial^\mu_s \rho^{m+1}_t||_2 ||\nabla(\langle \rho^m\rangle^{-1})||_\infty ||\sqrt{\epsilon}\Delta \partial^\mu_s \rho^m||_2 \leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{D}^m}\sqrt{\mathcal{E}^m} \\
&\leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}).
\end{aligned} \tag{4.107}$$

We first note that the sixth term of (2.49) contains g. Note that in $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}\; g = g^m_{\mu,s}$, where $g^m_{\mu,s}$ is defined by (3.72). We shall first estimate $||g^m_{\mu,s}||_2$ and $||\sqrt{\epsilon}\nabla g^m_{\mu,s}||_2$ and then use the Cauchy-Schwarz inequality. For any $|\mu'|+s'<|\mu|+s$, we have

$$||\partial^{\mu'}_{s'}\Delta\rho^m\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-1})||_2+||\partial^{\mu}_{s}(\nabla\rho^m\cdot\nabla(\langle\rho^m\rangle^{-1}))||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}.$$

The inequality follows by estimating the term with smaller order space-derivatives in L^∞-norm, which can then be estimated by the Sobolev inequality and Lemma 3.3. Similarly, recalling (3.75):

$$||\sqrt{\epsilon}\nabla\big(\partial^{\mu'}_{s'}\Delta\rho^m\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-1})\big)||_2+||\sqrt{\epsilon}\nabla\big(\partial^{\mu}_{s}(\nabla\rho^m\cdot\nabla(\langle\rho^m\rangle^{-1}))\big)||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}.$$

Therefore $||g^m_{\mu,s}||_2+||\sqrt{\epsilon}\nabla g^m_{\mu,s}||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$ and we can bound the sixth term in $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ by

$$\begin{aligned}\Big|\int_{\mathbb{T}^{n-1}}\big(\partial^\mu_s\rho_t^{m+1}+\epsilon\Delta^2\rho_t^{m+1}\big)g^m_{\mu,s}\Big|&\leq||\partial^\mu_s\rho_t^{m+1}||_2||g^m_{\mu,s}||_2\\ +||\sqrt{\epsilon}\Delta\nabla\rho_t^{m+1}||_2||\sqrt{\epsilon}\nabla g^m_{\mu,s}||_2&\leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}\leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}).\end{aligned}\tag{4.108}$$

The last term in $\int_{\mathbb{T}^{n-1}} Q^m_{\mu,s}$ is extracted from the last term of (2.49), which contains h. By (3.77), $h=h^m_{\mu,s}$ where $h^m_{\mu,s}$ is given by (3.74). For the notational simplicity, we set $h^m_{\mu,s}=:h_1+\epsilon h_2$, where

$$h_1:=\sum_{\substack{|\mu'|+s'\\<|\mu|+s}}C^{\mu'}_{s'}\partial^{\mu'}_{s'}\rho_t^{m+1}\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2}),\; h_2:=\sum_{\substack{|\mu'|+s'\\<|\mu|+s}}C^{\mu'}_{s'}\partial^{\mu'}_{s'}\Delta^2\rho_t^{m+1}\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2}).\tag{4.109}$$

Note that for $|\mu'|+s'<|\mu|+s$, we have

$$\begin{aligned}&||\langle\rho^m\rangle^2\partial^{\mu'}_{s'}\rho_t^{m+1}\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_2\\ &\leq||\langle\rho^m\rangle^2||_\infty||\partial^{\mu'}_{s'}\rho_t^{m+1}\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}.\end{aligned}\tag{4.110}$$

Here, if $|\mu'|+2s'\leq k$ then $||\partial^{\mu'}_{s'}\rho_t^{m+1}||_\infty\leq C\sqrt{\mathcal{E}^{m+1}}$ and if $|\mu'|+2s'\geq k$ then $||\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_\infty\leq C\sqrt{\mathcal{E}^m}$. We use the Sobolev inequality and Lemma 3.3 to conclude the estimate. This implies $||h_1||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}$. Note that in similar fashion, for $|\mu'|+s'<|\mu|+s$,

$$||\sqrt{\epsilon}\partial^{\mu'}_{s'}\Delta^2\rho_t^{m+1}\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2})||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}.$$

In other words $||\sqrt{\epsilon}h_2||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}$. From the proof of part (b) of Lemma 3.3, we deduce $||\partial_s^\mu \kappa^m||_2 \leq C\sqrt{\mathcal{D}^{m+1}}$ and also $||\sqrt{\epsilon}\partial_s^\mu \kappa^m||_2 \leq C\sqrt{\mathcal{D}^{m+1}}$ (recall here (3.69)). Thus the last term of $\int_{\mathbb{T}^{n-1}} Q_{\mu,s}^m$ is bounded by

$$\begin{aligned}\Big|\int_{\mathbb{T}^{n-1}} \langle\rho^m\rangle^2 h_{\mu,s}^m \partial_s^\mu \kappa^m\Big| &\leq C||h_1||_2||\partial_s^\mu\kappa^m||_2 + C||\sqrt{\epsilon}h_2||_2||\sqrt{\epsilon}\partial_s^\mu\kappa^m||_2 \\ &\leq C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}.\end{aligned} \tag{4.111}$$

Term $\int_\Omega R_{\mu,s}^m$: Note that $R_{\mu,s}^m$ is given by (3.78) where $f_{\mu,s}^m$, is defined by (3.71). Our first task is to estimate the first term of $\int_\Omega R_{\mu,s}^m$, namely $\int_\Omega (f_{\mu,s}^m)^2$. Observe that

$$\begin{aligned}\int_\Omega (f_{\mu,s}^m)^2 \leq\, & C\sum_{\substack{(\mu',s')\\ \neq(0,0)}} \int_\Omega (\partial_{s'}^{\mu'} a_{\rho^m})^2 (\partial_{s-s'}^{\mu-\mu'} u_{nn}^{m+1})^2 \\ &+ C\sum_{(\mu',s')}\int_\Omega |\partial_{s'}^{\mu'} B_{\rho^m}|^2 |\partial_{s-s'}^{\mu-\mu'}\nabla_{x'} u_n^{m+1}|^2 + C\sum_{\mu',s'}\int_\Omega (\partial_{s'}^{\mu'} c_{\rho^m})^2(\partial_{s-s'}^{\mu-\mu'} u_n^{m+1})^2\end{aligned} \tag{4.112}$$

If $|\mu'|+2s' \leq k$, by Lemma 3.4

$$||\partial_{s'}^{\mu'} a_{\rho^m}||_{L^\infty(\Omega)} + ||\partial_{s'}^{\mu'} B_{\rho^m}||_{L^\infty(\Omega)} + ||\partial_{s'}^{\mu'} c_{\rho^m}||_{L^\infty(\Omega)} \leq C\sqrt{\mathcal{E}^m}.$$

Thus for $|\mu'|+2s' \leq k$ and $(\mu',s') \neq (0,0)$, RHS of (4.112) is bounded by $C\mathcal{E}^m\mathcal{D}^{m+1}$. If $|\mu'|+2s' > k$, then $|\mu-\mu'|+2(s-s') \leq k-1$ and from Lemma 3.6

$$||\partial_{s-s'}^{\mu-\mu'} u_{nn}^{m+1}||_{L^\infty(\Omega)} + ||\partial_{s-s'}^{\mu-\mu'}\nabla_{x'} u_n^{m+1}||_{L^\infty(\Omega)} + ||\partial_{s-s'}^{\mu-\mu'} u_n^{m+1}||_{L^\infty(\Omega)} \leq C\sqrt{\mathcal{E}^{m+1}}.$$

In addition to this, for such (μ',s'), we use Lemma 3.4 to conclude

$$||\partial_{s'}^{\mu'} a_{\rho^m}||_{L^2(\Omega)} + ||\partial_{s'}^{\mu'} B_{\rho^m}||_{L^2(\Omega)} + ||\partial_{s'}^{\mu'} c_{\rho^m}||_{L^2(\Omega)} \leq C\sqrt{\mathcal{D}^m}.$$

Thus, for every $\lambda > 0$

$$\int_\Omega (\partial_{s'}^{\mu'} a_{\rho^m})^2 (\partial_{s-s'}^{\mu-\mu'} u_{nn}^{m+1})^2 \leq ||\partial_{s'}^{\mu'} a_{\rho^m}||_{L^2(\Omega)}^2 ||\partial_{s-s'}^{\mu-\mu'} u_{nn}^{m+1}||_{L^\infty(\Omega)}^2 \leq CD^m\mathcal{E}^{m+1}.$$

Analogously,

$$\int_\Omega \left\{|\partial_{s'}^{\mu'} B_{\rho^m}|^2 |\partial_{s-s'}^{\mu-\mu'}\nabla_{x'} u_n^{m+1}|^2 + (\partial_{s'}^{\mu'} c_{\rho^m})^2 (\partial_{s-s'}^{\mu-\mu'} u_n^{m+1})^2\right\} \leq CD^m\mathcal{E}^{m+1},$$

for all $(\mu',s') \leq (\mu,s)$ satisfying $|\mu'|+2s' > k$. Combining the estimates for $|\mu'|+2s' \leq k$ and $|\mu'|+2s' > k$, we obtain

$$\int_\Omega (f_{\mu,s}^m)^2 \leq C\mathcal{E}^m\mathcal{D}^{m+1} + CD^m\mathcal{E}^{m+1} \tag{4.113}$$

The second term of the expression $\int_\Omega R^m_{\mu,s}$ (the second term on RHS of (2.48) and (3.78)) is bounded by:

$$\Big|\int_\Omega (\partial^\mu_s u_n^{m+1})^2 (a_{\rho^m})_t\Big| \le ||(a_{\rho^m})_t||_{L^\infty(\Omega)} ||\partial^\mu_s u_n^{m+1}||^2_{L^2(\Omega)} \le C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1} \tag{4.114}$$

and similarly the third term $\Big|\int_\Omega \partial^\mu_s u_t^{m+1} \partial^\mu_s u_n^{m+1} (a_{\rho^m})_n\Big| \le C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}$. Note further that for the fourth term in $\int_\Omega R^m_{\mu,s}$ (use (2.48) and (3.78)), by Lemma 3.3,

$$\begin{aligned}&\Big|\int_\Omega \nabla_{x'}\partial^\mu_s u_n^{m+1}\cdot\nabla_{x'}a_{\rho^m}\partial^\mu_s u_n^{m+1}\Big|\\ &\le C||\nabla_{x'}\partial^\mu_s u_n^{m+1}||_{L^2(\Omega)}||\nabla_{x'}a_{\rho^m}||_{L^\infty(\Omega)}||\partial^\mu_s u_n^{m+1}||_{L^2(\Omega)} \le C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}.\end{aligned} \tag{4.115}$$

By Lemma 3.3, the last term in $\int_\Omega R^m_{\mu,s}$ (last term on RHS of (2.48) and (3.78)) is bounded by:

$$\begin{aligned}&\Big|\int_\Omega \Delta_{x'}\partial^\mu_s u^{m+1}(a_{\rho^m})_n\partial^\mu_s u_n^{m+1}\Big|\\ &\le ||\Delta_{x'}\partial^\mu_s u^{m+1}||_{L^2(\Omega)}||(a_{\rho^m})_n||_{L^\infty(\Omega)}||\partial^\mu_s u_n^{m+1}||_{L^2(\Omega)} \le C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}.\end{aligned} \tag{4.116}$$

Term $\int_{\mathbb{T}^{n-1}} S^m_{\mu,s}$: Note that $S^m_{\mu,s}$ is given by (3.78) where S is given by (2.50), $g^m_{\mu,s}$ by (3.72) and $h^m_{\mu,s}$ by (3.74). The first two terms on RHS of (2.50) are the cross-terms and in order to estimate them we shall exploit the part (c) of Lemma 3.3. It turns out that the constants on the right-hand side will depend on ϵ. Note that for any $\eta,\lambda>0$

$$\begin{aligned}&\Big|\int_{\mathbb{T}^{n-1}} \partial^\mu_s\nabla\rho_t^{m+1}\Big(\partial^\mu_s\nabla\rho_t^m\langle\rho^m\rangle^{-1} - \partial^\mu_s\nabla\rho_t^{m+1}\langle\rho^m\rangle^{-1}\Big)\Big|\\ &\le \frac{C}{\eta}||\partial^\mu_s\nabla\rho_t^{m+1}||_2^2 + \eta||\partial^\mu_s\nabla\rho_t^m||_2^2 \le \frac{C}{\eta\lambda\epsilon^4}\mathcal{E}^{m+1} + \frac{C\lambda}{\eta}\mathcal{D}^{m+1} + \eta\mathcal{D}^m.\end{aligned} \tag{4.117}$$

In the last estimate we have used the estimate (3.90). Similarly,

$$\begin{aligned}&\Big|\epsilon\int_{\mathbb{T}^{n-1}} \partial^\mu_s\Delta\nabla\rho_t^{m+1}\Big(\partial^\mu_s\Delta\nabla\rho_t^m\langle\rho^m\rangle^{-1} - \partial^\mu_s\Delta\nabla\rho_t^{m+1}\langle\rho^m\rangle^{-1}\Big)\Big|\\ &\le \frac{\epsilon C}{\eta}||\partial^\mu_s\Delta\nabla\rho_t^{m+1}||_2^2 + \eta\epsilon||\partial^\mu_s\Delta\nabla\rho_t^m||_2^2 \le \frac{C}{\eta\lambda\epsilon^3}\mathcal{E}^{m+1} + \frac{C\epsilon\lambda}{\eta}\mathcal{D}^{m+1} + \eta\mathcal{D}^m.\end{aligned} \tag{4.118}$$

The third term of $\int_{\mathbb{T}^{n-1}} S^m_{\mu,s}$ is given by the third term on RHS of (2.50) and (3.78):

$$
\begin{aligned}
(4.119)\qquad &\Big|\int_{\mathbb{T}^{n-1}} \partial^\mu_s \rho^{m+1}_t \partial^\mu_s \nabla \rho^m_t \cdot \nabla(\langle\rho^m\rangle^{-1})\Big| \leq \\
&||\partial^\mu_s \rho^{m+1}_t||_2 ||\partial^\mu_s \nabla \rho^m_t||_2 ||\nabla(\langle\rho^m\rangle^{-1})||_\infty \leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{D}^m}\sqrt{\mathcal{E}^m} \\
&C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}).
\end{aligned}
$$

Similarly, the fourth term in $\int_{\mathbb{T}^{n-1}} S^m_{\mu,s}$ (given by the fourth term on RHS of (2.50) and (3.78)) is bounded by:

$$
\begin{aligned}
(4.120)\qquad &\epsilon\Big|\int_{\mathbb{T}^{n-1}} \Delta\partial^\mu_s \rho^m_t \Delta\nabla\partial^\mu_s \rho^{m+1}_t \cdot \nabla(\langle\rho^m\rangle^{-1})\Big| \leq \\
&||\sqrt{\epsilon}\Delta\partial^\mu_s \rho^m_t||_2 ||\sqrt{\epsilon}\Delta\nabla\partial^\mu_s \rho^{m+1}_t||_2 ||\nabla(\langle\rho^m\rangle^{-1})||_\infty \leq C\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m} \\
&\leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}).
\end{aligned}
$$

Note that the fifth term in $\int_{\mathbb{T}^{n-1}} S^m_{\mu,s}$, by (2.50) and (3.78)) involves the function $g^m_{\mu,s}$, where $g^m_{\mu,s}$ is given by (3.72). The crucial step in estimating this term in $\int_{\mathbb{T}^{n-1}} S^m$, is to observe that $||(g^m_{\mu,s})_t||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$. This is proved by first differentiating $g^m_{\mu,s}$ with respect to t, and then in each product estimating the terms with lower order space derivatives in L^∞-norm and the other one in L^2-norm. The same method applies to show $||\sqrt{\epsilon}\nabla(g^m_{\mu,s})_t||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$. Also observe that

$$
||\Delta\partial^\mu_s \rho^m \langle\rho^m\rangle^{-1}_t||_2 \leq ||\Delta\partial^\mu_s \rho^m||_2 ||\langle\rho^m\rangle^{-1}_t||_\infty \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}.
$$

Analogous proof shows that

$$
||\sqrt{\epsilon}\nabla\big(\Delta\partial^\mu_s \rho^m \langle\rho^m\rangle^{-1}_t\big)||_2 \leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}.
$$

Using the above inequalitites and the Cauchy-Schwarz inequality we establish

$$
\begin{aligned}
(4.121)\qquad &\Big|\int_{\mathbb{T}^{n-1}} \partial^\mu_s \rho^{m+1}_t \Big(\Delta\partial^\mu_s \rho^m \langle\rho^m\rangle^{-1}_t + (g^m_{\mu,s})_t\Big)\Big| \leq ||\partial^\mu_s \rho^{m+1}_t||_2 ||\Delta\partial^\mu_s \rho^m \langle\rho^m\rangle^{-1}_t + (g^m_{\mu,s})_t||_2 \\
&\leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m} \leq C\sqrt{\mathcal{E}^m}\big(\mathcal{D}^m + \mathcal{D}^{m+1}\big).
\end{aligned}
$$

Integrating by parts and using the analogous argument as in (4.121) we get

(4.122)
$$\epsilon\Big|\int_{\mathbb{T}^{n-1}}\partial_s^\mu\Delta^2\rho_t^{m+1}\Big(\Delta\partial_s^\mu\rho^m\langle\rho^m\rangle_t^{-1}+(g_{\mu,s}^m)_t\Big)\Big|=\epsilon\Big|\int_{\mathbb{T}^{n-1}}\partial_s^\mu\Delta\nabla\rho_t^{m+1}\cdot\nabla\Big(\Delta\partial_s^\mu\rho^m\langle\rho^m\rangle_t^{-1}+(g_{\mu,s}^m)_t\Big)\Big|$$
$$\leq||\sqrt{\epsilon}\partial_s^\mu\Delta\nabla\rho_t^{m+1}||_2||\sqrt{\epsilon}\nabla\Big(\Delta\partial_s^\mu\rho^m\langle\rho^m\rangle_t^{-1}+(g_{\mu,s}^m)_t\Big)||_2$$
$$\leq C\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}\leq C\sqrt{\mathcal{E}^m}\big(\mathcal{D}^m+\mathcal{D}^{m+1}\big)$$

The sixth and the last term in $\int_{\mathbb{T}^{n-1}}S^m$ (given through the last term on RHS of (2.50) and (3.78)) involves the function $h_{\mu,s}^m$, where $h_{\mu,s}^m$ is given by (3.74). Since $\partial_s^\mu\kappa_t^m=\nabla\cdot\partial_s^\mu\big(\nabla\rho^m\langle\rho^m\rangle_t^{-1}\big)$, we can integrate by parts to obtain

$$\int_{\mathbb{T}^{n-1}}\langle\rho^m\rangle^2h_{\mu,s}^m\partial_s^\mu\kappa_t^m=-\int_{\mathbb{T}^{n-1}}\nabla\big(\langle\rho^m\rangle^2h_{\mu,s}^m\big)\cdot\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)_t.$$

We split $h_{\mu,s}^m=h_1+\epsilon h_2$ as in (4.109). For $|\mu'|+s'<|\mu|+s$

$$\begin{aligned}&||\nabla\big(\langle\rho^m\rangle^2h_1\big)||_2\leq||\nabla(\langle\rho^m\rangle^2)||_\infty||\partial_{s'}^{\mu'}\rho_t^{m+1}\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-2})||_2+\\ &+||\langle\rho^m\rangle^2||_\infty\Big(||\partial_{s'}^{\mu'}\nabla\rho_t^{m+1}\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-2})||_2+||\partial_{s'}^{\mu'}\rho_t^{m+1}\partial_{s-s'}^{\mu-\mu'}\nabla(\langle\rho^m\rangle^{-2})||_2\Big)\\ &\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}+C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}.\end{aligned}\tag{4.123}$$

Here we have used L^2-L^∞ estimated by separating the cases $|\mu'|+2s'\leq k$ and $|\mu'|+2s'>k$. Since

$$||\sqrt{\epsilon}\nabla\big(\langle\rho^m\rangle^2\partial_{s'}^{\mu'}\Delta^2\rho_t^{m+1}\partial_{s-s'}^{\mu-\mu'}(\langle\rho^m\rangle^{-2})\big)||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}},$$

we deduce

$$||\sqrt{\epsilon}\nabla(\langle\rho^m\rangle^2h_2)||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}},\tag{4.124}$$

where h_2 is given by (4.109). Similarly, $||\partial_{s+1}^\mu\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$ and also $||\sqrt{\epsilon}\partial_{s+1}^\mu\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$.

In summary,

$$\begin{aligned}&\Big|\int_{\mathbb{T}^{n-1}}\langle\rho^m\rangle^2h_{\mu,s}^m\partial_s^\mu\kappa_t^m\Big|\leq||\nabla\big(\langle\rho^m\rangle^2h_1\big)||_2||\partial_s^\mu\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)||_2+\\ &+||\sqrt{\epsilon}\big(\langle\rho^m\rangle^2h_2\big)||_2||\sqrt{\epsilon}\partial_s^\mu\big(\nabla\rho^m\langle\rho^m\rangle^{-1}\big)||_2\\ &\leq C\mathcal{E}^m\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}}\leq C\mathcal{E}^m\mathcal{D}^m+C\mathcal{E}^m\mathcal{D}^{m+1}.\end{aligned}\tag{4.125}$$

Term $\int_{\mathbb{T}^{n-1}}T_{\mu,s}^m$: Recall that $T_{\mu,s}^m$ is defined by (3.78) where T is given by (2.51), $G_{\mu,s}^m$ by (3.73) and $h_{\mu,s}^m$ by (3.74). In particular the term A- the first term on RHS of (2.51) is given by (2.52). The first two terms of the expression A are the cross-terms. Using integration by parts, we obtain

$$
\begin{aligned}
&\Big|\int_{\mathbb{T}^{n-1}} \partial_s^\mu \Delta\rho_t^{m+1}\left(\partial_s^\mu\Delta\rho^m-\partial_s^\mu\Delta\rho^{m+1}\right)\langle\rho^m\rangle^{-1}\Big|=\\
&\Big|\int_{\mathbb{T}^{n-1}} \partial_s^\mu \nabla\rho_t^{m+1}\cdot\nabla\Big(\left(\partial_s^\mu\Delta\rho^m-\partial_s^\mu\Delta\rho^{m+1}\right)\langle\rho^m\rangle^{-1}\Big)\Big|\\
&\le\lambda||\partial_s^\mu\nabla\rho_t^{m+1}||_2^2+\frac{C}{\lambda}||\nabla\left(\partial_s^\mu\Delta\rho^m\langle\rho^m\rangle^{-1}\right)||_2^2+\\
&\frac{C}{\lambda}||\nabla\left(\partial_s^\mu\Delta\rho^{m+1}\langle\rho^m\rangle^{-1}\right)||_2^2\le\lambda\mathcal{D}^{m+1}+\frac{C}{\epsilon\lambda}(\mathcal{E}^m+\mathcal{E}^{m+1}),
\end{aligned}
\tag{4.126}
$$

where we note that by the definition of $\mathcal{E}^m$, $||\partial_s^\mu(\nabla\rho^m\langle\rho^m\rangle^{-1})||_{H^3}\le\frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}$ for $|\mu|+2s\le 2k$. Similarly, for the second cross term:

$$
\begin{aligned}
&\Big|\int_{\mathbb{T}^{n-1}} \partial_s^\mu \Delta\rho_t^{m+1}\Big(\rho_i^m\rho_j^m\left(\partial_s^\mu\rho_{ij}^m-\partial_s^\mu\rho_{ij}^{m+1}\right)\Big)\langle\rho^m\rangle^{-3}\Big|\\
&\qquad\le\lambda\mathcal{D}^{m+1}+\frac{C}{\epsilon\lambda}(\mathcal{E}^m+\mathcal{E}^{m+1}).
\end{aligned}
\tag{4.127}
$$

The proof of (4.127) relies on the same idea as above; we first integrate by parts and then establish the estimate

$$
||\nabla\Big(\rho_i^m\rho_j^m\left(\partial_s^\mu\rho_{ij}^m-\partial_s^\mu\rho_{ij}^{m+1}\right)\langle\rho^m\rangle^{-3}\Big)||_2^2\le\frac{C}{\epsilon}(\mathcal{E}^m+\mathcal{E}^{m+1}).
$$

By $A-(\text{crossterms})$ we denote the sum of all the remaining terms in the expression A (recall (2.52) and the fact that $\psi=\rho^m$ in our case). Terms of the form $\langle\rho^m\rangle_i^{-1}$, $\langle\rho^m\rangle_t^{-1}$, $\langle\rho^m\rangle_t^{-3}$, ρ_i^m and ρ_{ij}^m for $1\le i,j\le n-1$ are bounded in L^∞-norm by $C\sqrt{\mathcal{E}^m}$, by Lemma 3.3. Terms of the form $\langle\rho^m\rangle^{-3}$ are estimated by 1 in L^∞-norm. Note that in the last two terms in the expression A (2.52) the leading order derivatives cancel out after the the product rule has been applied within the parentheses. Using these observations and applying the Cauchy-Schwarz inequality, we conclude

$$
|A-(\text{crossterms})|\le C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}).
\tag{4.128}
$$

Recall now that the term B (the second expression on RHS of (2.51)) is given by (2.53). The first two terms in the expression B are again the cross-terms. By part (c) of Lemma 3.3, we obtain

$$\begin{aligned}
&\left|\epsilon\int_{\mathbb{T}^{n-1}}\partial_s^\mu\Delta^2\rho_t^{m+1}\big(\partial_s^\mu\Delta^2\rho^m-\partial_s^\mu\Delta^2\rho^{m+1}\big)\left<\rho^m\right>^{-1}\right|\leq\\
&\epsilon^2||\partial_s^\mu\Delta^2\rho_t^{m+1}||_2^2+C||\partial_s^\mu\Delta^2\rho^m||_2^2+C||\partial_s^\mu\Delta^2\rho^{m+1}||_2^2\leq\\
&\leq\frac{C}{\lambda}\mathcal{E}^{m+1}+(\lambda+C\mathcal{E}^m)\mathcal{D}^{m+1}+\frac{C}{\epsilon}\big(\mathcal{E}^m+\mathcal{E}^{m+1}\big).
\end{aligned}\tag{4.129}$$

Analogously, we establish

$$\begin{aligned}
&\left|\epsilon\int_{\mathbb{T}^{n-1}}\partial_s^\mu\Delta^2\rho_t^{m+1}\Big(\rho_i^m\rho_j^m\big(\partial_s^\mu\Delta\rho_{ij}^m-\partial_s^\mu\Delta\rho_{ij}^{m+1}\big)\Big)\left<\rho^m\right>^{-3}\right|\leq\\
&\frac{C}{\lambda}\mathcal{E}^{m+1}+(\lambda+C\mathcal{E}^m)\mathcal{D}^{m+1}+\frac{C}{\epsilon}\big(\mathcal{E}^m+\mathcal{E}^{m+1}\big).
\end{aligned}\tag{4.130}$$

We denote the sum of the remaining terms in the expression B by $B-$(crossterms). The same idea as in the estimates for $|A-$(crossterms)$|$ works. It is important to note that we have canceling of the highest order derivatives within the parentheses in the last three expressions on RHS of (2.53). In addition to that, we factorize $\epsilon=\sqrt{\epsilon}\times\sqrt{\epsilon}$. By the Cauchy-Schwarz inequality,

$$|B-(\text{crossterms})|\leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}).\tag{4.131}$$

The third term of $\int_{\mathbb{T}^{n-1}}T_{\mu,s}^m$ is given by the third term on RHS of (2.51) together with (3.78). Recall that $G_{\mu,s}^m$ is given by (3.73). In order to estimate it, we first integrate by parts.

$$\begin{aligned}
&\int_{\mathbb{T}^{n-1}}\Delta G_{\mu,s}^m\big(\partial_s^\mu\rho_t^{m+1}+\epsilon\partial_s^\mu\Delta^2\rho_t^{m+1}\big)=\\
&\quad-\int_{\mathbb{T}^{n-1}}\nabla G_{\mu,s}^m\cdot\partial_s^\mu\nabla\rho_t^{m+1}-\epsilon\int_{\mathbb{T}^{n-1}}\Delta\nabla G_{\mu,s}^m\cdot\partial_s^\mu\Delta\nabla\rho_t^{m+1}.
\end{aligned}$$

The crucial observation is $||\nabla G_{\mu,s}^m||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$, $||\sqrt{\epsilon}\Delta\nabla G_{\mu,s}^m||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}$. Both inequalities follow in the standard way, by using $L^\infty-L^2$ type estimates and (3.75). The third term of $\int_{\mathbb{T}^{n-1}}T_{\mu,s}^m$ is then bounded by:

$$\begin{aligned}
&\left|\int_{\mathbb{T}^{n-1}}\Delta G_{\mu,s}^m\big(\partial_s^\mu\rho_t^{m+1}+\epsilon\partial_s^\mu\Delta^2\rho_t^{m+1}\big)\right|\leq||\nabla G_{\mu,s}^m||_2||\partial_s^\mu\nabla\rho_t^{m+1}||_2+\\
&+||\sqrt{\epsilon}\Delta\nabla G_{\mu,s}^m||_2||\sqrt{\epsilon}\partial_s^\mu\Delta\nabla\rho_t^{m+1}||_2\leq C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}}\leq C\sqrt{\mathcal{E}^m}(\mathcal{D}^m+\mathcal{D}^{m+1}).
\end{aligned}\tag{4.132}$$

We integrate by parts in the fourth and the last term of $\int_{\mathbb{T}^{n-1}} T^m_{\mu,s}$ (given by the last term on RHS of (2.51) and (3.78)). Recall $\Delta\mathcal{U}|_{\mathbb{T}^{n-1}} = \Delta\partial_s^\mu u^{m+1}|_{\mathbb{T}^{n-1}} = \Delta\partial_s^\mu \kappa^m$.

$$\int_{\mathbb{T}^{n-1}} \langle\rho^m\rangle^2 h^m_{\mu,s} \Delta\partial_s^\mu \kappa^m =$$
$$-\int_{\mathbb{T}^{n-1}} \nabla(\langle\rho^m\rangle^2 h_1)\cdot\partial_s^\mu\nabla\kappa^m - \epsilon\int_{\mathbb{T}^{n-1}} \nabla(\langle\rho^m\rangle^2 h_2)\cdot\partial_s^\mu\nabla\kappa^m.$$

By (4.109), we set $h^m_{\mu,s} = h_1 + \epsilon h_2$. By the trace inequality,

$$||\nabla\partial_s^\mu\kappa^m||_2 = ||\partial_s^\mu\nabla_{x'}u^{m+1}||_{L^2(\mathbb{T}^{n-1})} \le C||\partial_s^\mu\nabla u^{m+1}||_{H^1(\Omega)} \le C\sqrt{\mathcal{D}^{m+1}}.$$

By (4.123), we obtain

$$\left|\int_{\mathbb{T}^{n-1}} \nabla(\langle\rho^m\rangle^2 h_1)\cdot\partial_s^\mu\nabla\kappa^m\right| \le ||\nabla(\langle\rho^m\rangle^2 h_1)||_2||\partial_s^\mu\nabla\kappa^m||_2 \le C\sqrt{\mathcal{E}^m}\mathcal{D}^{m+1}. \tag{4.133}$$

On the other hand, by Lemma 3.3 and $L^\infty - L^2$ type estimates, $||\sqrt{\epsilon}\partial_s^\mu\nabla\kappa^m||_2 \le C\sqrt{\mathcal{D}^m}$. By (4.124),

$$\begin{aligned}\left|\epsilon\int_{\mathbb{T}^{n-1}} \nabla(\langle\rho^m\rangle^2 h_2)\cdot\partial_s^\mu\nabla\kappa^m\right| &\le ||\sqrt{\epsilon}\nabla(\langle\rho^m\rangle^2 h_2)||_2||\sqrt{\epsilon}\partial_s^\mu\nabla\kappa^m||_2\\ &\le C\sqrt{\mathcal{E}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{D}^m} \le C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}),\end{aligned} \tag{4.134}$$

where we recall (4.109) again. From the estimates (4.133) and (4.134), the last term in $\int_{\mathbb{T}^{n-1}} T^m_{\mu,s}$ is bounded by

$$\left|\int_{\mathbb{T}^{n-1}} \langle\rho^m\rangle^2 h^m_{\mu,s}\Delta\partial_s^\mu\kappa^m\right| \le C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}). \tag{4.135}$$

Using the identity (3.76) and summing the estimates (4.100) - (4.122) and (4.125) - (4.135) to get a bound on the right-hand side of the identity (3.76), we arrive at

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}^{m+1}(t) + \mathcal{D}^{m+1}(t) \le C\sqrt{\mathcal{E}^m}(\mathcal{D}^m + \mathcal{D}^{m+1}) + C\sqrt{\mathcal{D}^m}\sqrt{\mathcal{D}^{m+1}}\sqrt{\mathcal{E}^{m+1}}\\ &+C(\eta+\lambda)\mathcal{D}^{m+1} + \frac{C}{\eta}(\mathcal{E}^m + \mathcal{E}^{m+1}) + \frac{C}{\lambda}\mathcal{E}^{m+1} + \eta\mathcal{D}^m\\ &+\frac{C}{\eta\lambda\epsilon^4}\mathcal{E}^{m+1} + \frac{C\lambda\epsilon}{\eta}\mathcal{D}^{m+1} + \frac{C}{\lambda\epsilon}(\mathcal{E}^m + \mathcal{E}^{m+1}) + C\mathcal{E}^{m+1}\mathcal{D}^m.\end{aligned} \tag{4.136}$$

Now integrate in time over the interval $[0,t]$ to get

(4.137)

$$\mathcal{E}^{m+1}(t)+\int_0^t \mathcal{D}^{m+1}(\tau)\,d\tau \le \mathcal{E}^{m+1}(0)+C\sup_{s\le t}\sqrt{\mathcal{E}^m(s)}\Big(\int_0^t \mathcal{D}^m(\tau)\,d\tau+\int_0^t \mathcal{D}^{m+1}(\tau)\,d\tau\Big)+$$
$$+C\sup_{s\le t}\sqrt{\mathcal{E}^{m+1}(s)}\int_0^t \sqrt{\mathcal{D}^m(s)}\sqrt{\mathcal{D}^{m+1}(s)}\,ds+C(\eta+\lambda)\int_0^t \mathcal{D}^{m+1}(\tau)\,d\tau$$
$$+Ct\big(\frac{1}{\eta}+\frac{1}{\lambda}\big)\sup_{s\le t}\mathcal{E}^{m+1}(s)+\frac{Ct}{\eta}\sup_{s\le t}\mathcal{E}^m(s)+\frac{Ct}{\eta\lambda\epsilon^4}\sup_{s\le t}\mathcal{E}^{m+1}(s)+$$
$$+\frac{C\lambda\epsilon}{\eta}\int_0^t \mathcal{D}^{m+1}(\tau)\,d\tau+\eta\int_0^t \mathcal{D}^m(\tau)\,d\tau+\frac{Ct}{\lambda\epsilon}\Big(\sup_{s\le t}\mathcal{E}^m(s)+\sup_{s\le t}\mathcal{E}^{m+1}(s)\Big)+$$
$$+\sup_{s\le t}\mathcal{E}^{m+1}(s)\int_0^t \mathcal{D}^m(\tau)\,d\tau.$$

Note that by Cauchy-Schwarz inequality we have

(4.138)

$$\sup_{s\le t}\sqrt{\mathcal{E}^{m+1}(s)}\int_0^t \sqrt{\mathcal{D}^m(s)}\sqrt{\mathcal{D}^{m+1}(s)}\,ds \le \sup_{s\le t}\sqrt{\mathcal{E}^{m+1}(s)}\sqrt{\int_0^t \mathcal{D}^m(s)\,ds}\sqrt{\int_0^t \mathcal{D}^{m+1}(s)\,ds}$$
$$\le \Big(\int_0^t \mathcal{D}^m(\tau)\,d\tau\Big)^{1/2}\big(\sup_{s\le t}\mathcal{E}^{m+1}(s)+\int_0^t \mathcal{D}^{m+1}(\tau)\,d\tau\big).$$

By assumption $\sup_{s\le t}\mathcal{E}^m(s)+\int_0^t \mathcal{D}^m(\tau)\,d\tau\le L$, and thus from (4.137) and (4.138), for any $t'\le t$:

$$\mathcal{E}^{m+1}(t')+\int_0^{t'} \mathcal{D}^{m+1}(\tau)\,d\tau \le \frac{L}{2}+L^{3/2}+\frac{Ct}{\eta}L+\eta L+\frac{CtL}{\lambda\epsilon}+$$
$$\big(\sup_{s\le t}\mathcal{E}^{m+1}(s)+\int_0^t \mathcal{D}^{m+1}(\tau)\,d\tau\big)\Big\{L^{1/2}+C(\eta+\lambda)+Ct(\frac{1}{\eta}+\frac{1}{\lambda})+\frac{Ct}{\eta\lambda\epsilon^4}+\frac{C\lambda\epsilon}{\eta}+\frac{Ct}{\lambda\epsilon}+L\Big\}.$$

Since the above inequality holds for any $t'\le t$, we obtain

$$\big(\sup_{s\le t}\mathcal{E}^{m+1}(s)+\int_0^t \mathcal{D}^{m+1}(\tau)\,d\tau\big)\Big\{1-\Big(L^{1/2}+C(\eta+\lambda)+Ct(\frac{1}{\eta}+\frac{1}{\lambda})+\frac{Ct}{\eta\lambda\epsilon^4}+\frac{C\lambda\epsilon}{\eta}+\frac{Ct}{\lambda\epsilon}+L\Big)\Big\}$$
$$\le \frac{L}{2}+CL^{3/2}+\frac{Ct}{\eta}L+\eta L+\frac{CtL}{\lambda\epsilon}.$$

We first choose η small and then λ small so that $C(\lambda+\eta)+\frac{C\lambda\epsilon}{\eta}$ is small. Further, we choose t (t depends on ϵ) and L small so that

$$L^{1/2}+C(\eta+\lambda)+Ct(\frac{1}{\eta}+\frac{1}{\lambda})+\frac{Ct}{\eta\lambda\epsilon^4}+\frac{C\lambda\epsilon}{\eta}+\frac{Ct}{\lambda\epsilon}+L<\frac{1}{3},$$

and

$$\frac{L}{2}+C(L)^{3/2}+\frac{Ct}{\eta}L+\eta L+\frac{CtL}{\lambda\epsilon}\le\frac{2}{3}L,\quad L\le\theta.$$

With such a choice of L and $t =: \mathcal{T}^\epsilon$, we obtain $\sup_{s\le \mathcal{T}^\epsilon} \mathcal{E}^{m+1}(s) + \int_0^{\mathcal{T}^\epsilon} \mathcal{D}^{m+1}(\tau)\,d\tau \le L$ and this finishes the proof of Lemma 4.1. $\square$

5. Regularized Stefan problem.

The principal goal of this section is the following local existence theorem:

THEOREM 5.1. *Assume that* $u^\epsilon(0,\cdot) = u_0^\epsilon \in H^{2k+1}(\Omega_f)$, $\rho^\epsilon(0,\cdot) = \rho_0^\epsilon \in H^{2k+4}(\mathbb{T}^{n-1})$, $u_0^\epsilon\Big|_{\mathbb{T}^{n-1}} = \nabla\cdot \frac{\nabla\rho_0^\epsilon}{\langle \rho_0^\epsilon\rangle}$ *and* $\partial_n u_0^\epsilon\big|_{\mathbb{T}^{n-1}\times\{\pm1\}} = 0$. *For any sufficiently small* $L>0$ *there exists* $t^\epsilon > 0$ *depending on* L *and* ϵ *such that if*

$$\mathcal{E}_\epsilon(u_0^\epsilon, \rho_0^\epsilon; \rho_0^\epsilon) + \Big|\int_{\mathbb{T}^{n-1}} \rho_0^\epsilon - \int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\Big| \le \frac{L}{2}$$

then there exists a unique solution $(u^\epsilon,\rho^\epsilon)\in \mathcal{S}_\epsilon(0,t^\epsilon)$ *to the regularized Stefan problem (1.18)-(1.22) and (1.24) defined on the time interval* $[0,t^\epsilon[$, *satisfying the bound*

$$\sup_{0\le t\le t^\epsilon} \mathcal{E}_\epsilon(u^\epsilon,\rho^\epsilon;\rho^\epsilon)(t) + \int_0^{t^\epsilon} \mathcal{D}_\epsilon(u^\epsilon,\rho^\epsilon;\rho^\epsilon)(\tau)\,d\tau \le L$$

and $\mathcal{E}_\epsilon(u^\epsilon,\rho^\epsilon)(\cdot)$ *is continuous on* $[0,t^\epsilon[$.

Remark. Note that the constant L is independent of ϵ.

Proof. Convergence. Combining Lemmas 3.2 and 4.1, we obtain a uniform-in-m bound on the sequence $\{(u^m,\rho^m)\}_m$. Our goal is to show that $\{(u^m,\rho^m)\}_m$ is a Cauchy sequence in the energy space. For any $l\in\mathbb{N}$ let $v^{l+1} := u^{l+1}-u^l$ and $\sigma^{l+1} = \rho^{l+1}-\rho^l$. By subtracting two consecutive equations in the iteration process, we obtain

$$v_t^{m+1} - \Delta_{x'} v^{m+1} - a_{\rho^m} v_{nn}^{m+1} = f_m^\circ, \tag{5.139}$$

$$\begin{aligned} v^{m+1} &= \Delta\sigma^m \langle\rho^m\rangle^{-1} + g_m^\circ \\ &= \Delta\sigma^m\langle\rho^m\rangle^{-1} - \rho_i^m\rho_j^m\sigma_{ij}^m\langle\rho^m\rangle^{-1} + G_m^\circ \quad \text{on} \quad \mathbb{T}^{n-1}\times\{x_n=0\}, \end{aligned} \tag{5.140}$$

$$\partial_n v^{m+1} = 0 \qquad \text{on} \quad \mathbb{T}^{n-1}\times\big\{x_n=\pm1\big\}, \tag{5.141}$$

$$[v_n^{m+1}]_+^- = \big(\sigma_t^{m+1} + \epsilon\Delta^2\sigma_t^{m+1}\big)\langle\rho^m\rangle^{-2} + h_m^\circ \quad \text{on} \quad \mathbb{T}^{n-1}\times\{x_n=0\}. \tag{5.142}$$

Here

$$f^\circ_m = -B_{\rho^m}\cdot\nabla_{x'}v_n^{m+1} - c_{\rho^m}v_n^{m+1} + u^m_{nn}(a_{\rho^m}-a_{\rho^{m-1}})$$
$$-\nabla_{x'}u^m_n\cdot\big(B_{\rho^m}-B_{\rho^{m-1}}\big) - u^m_n(c_{\rho^m}-c_{\rho^{m-1}}), \tag{5.143}$$

$$G^\circ_m = \Delta\rho^{m-1}\big(\langle\rho^m\rangle^{-1} - \langle\rho^{m-1}\rangle\big) + \rho^{m-1}_{ij}\big(\sigma^m_i\rho^m_j\langle\rho^m\rangle^{-1} + \rho^{m-1}_i\sigma^m_j\langle\rho^m\rangle^{-1} +$$
$$\rho^{m-1}_i\rho^{m-1}_j(\langle\rho^m\rangle^{-1} - \langle\rho^{m-1}\rangle)\big),$$

$$g^\circ_m = -\rho^m_i\rho^m_j\sigma^m_{ij}\langle\rho^m\rangle^{-1} + G^\circ_m,$$

$$h^\circ_m = -\big(|\nabla\rho^m|^2 - |\nabla\rho^{m-1}|^2\big)[u^m_n]^-_+\langle\rho^m\rangle^{-2}. \tag{5.144}$$

After applying the differential operator ∂^μ_s to the equations (5.139), (5.140) and (5.142) and singling out the leading-order terms we arrive at:

$$\partial^\mu_s v^{m+1}_t - \Delta_{x'}v^{m+1} - a_{\rho^m}v^{m+1}_{nn} = f'_m, \tag{5.145}$$

$$\partial^\mu_s v^{m+1} = \partial^\mu_s\Delta\sigma^m\langle\rho^m\rangle^{-1} + g'_m = \partial^\mu_s\Delta\sigma^m\langle\rho^m\rangle^{-1} - \rho^m_i\rho^m_j\partial^\mu_s\sigma^m_{ij}\langle\rho^m\rangle^{-1} + G'_m, \tag{5.146}$$

$$[\partial^\mu_s v^{m+1}_n]^-_+ = \big(\partial^\mu_s\sigma^{m+1}_t + \epsilon\partial^\mu_s\Delta^2\sigma^{m+1}_t\big)\langle\rho^m\rangle^{-2} + h'_m, \tag{5.147}$$

where

$$f'_m = \partial^\mu_s f^\circ_m + \big(\partial^\mu_s(a_{\rho^m}\partial^\mu_s v^{m+1}_{nn}) - a_{\rho^m}\partial^\mu_s v^{m+1}_{nn}\big), \tag{5.148}$$

$$g'_m = \partial^\mu_s g^\circ_m + \sum_{\substack{|\mu'|+s'\\<|\mu|+s}}\partial^{\mu'}_{s'}\Delta\sigma^m\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-1}), \tag{5.149}$$

$$G'_m = \partial^\mu_s G^\circ_m + \sum_{\substack{|\mu'|+s'\\<|\mu|+s}}\partial^{\mu'}_{s'}\Delta\sigma^m\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-1}) - \Big(\partial^\mu_s\big(\rho^m_i\rho^m_j\sigma^m_{ij}\langle\rho^m\rangle^{-1}\big) - \rho^m_i\rho^m_j\partial^\mu_s\sigma^m_{ij}\langle\rho^m\rangle^{-1}\Big), \tag{5.150}$$

$$h'_m = \partial^\mu_s h^\circ_m + \sum_{\substack{|\mu'|+s'\\<|\mu|+s}}\partial^{\mu'}_{s'}\big(\sigma^{m+1}_t + \epsilon\Delta^2\sigma^{m+1}_t\big)\partial^{\mu-\mu'}_{s-s'}(\langle\rho^m\rangle^{-2}). \tag{5.151}$$

As a next step, we use the identities from Chapter 2 to obtain the energy identities for the problem (5.145) - (5.147). Respecting the notations of Chapter 2 we set for any $l\in\mathbb{N}$

$$f=f_l',\ g=g_l',\ G=G_l',\ h=h_l',\ \mathcal{U}=\partial_s^\mu v^{l+1},\ \omega=\partial_s^\mu\sigma^{l+1},\ \chi=\partial_s^\mu\sigma^l \text{ and } \psi=\rho^l. \tag{5.152}$$

Additionally, we introduce the notations

$$e^l:=\mathcal{E}_\epsilon(v^l,\sigma^l;\rho^{l-1}),\quad d^l:=\mathcal{D}_\epsilon(v^l,\sigma^l;\rho^{l-1}),$$

where $\mathcal{E}_\epsilon$ and $\mathcal{D}_\epsilon$ are defined by (1.35) and (1.36) respectively. Using (2.46) and (5.152), we arrive at

$$\frac{d}{dt}e^{m+1}+d^{m+1}=\int_\Omega\{p^m+r^m\}-\int_{\mathbb{T}^{n-1}}\{q^m+s^m+t^m\}, \tag{5.153}$$

where

(5.154)

$$p^m:=\sum_{|\mu|+2s\le 2k}P(\partial_s^\mu v^{m+1},\rho^m,f_m'),\quad r^m:=\sum_{|\mu|+2s\le 2k}R(\partial_s^\mu v^{m+1},\rho^m,f_m')$$

$$q^m:=\sum_{|\mu|+2s\le 2k}Q(\partial_s^\mu\sigma^m,\partial_s^\mu\sigma^{m+1},\rho^m,g_m',h_m'),\quad s^m:=\sum_{|\mu|+2s\le 2k}S(\partial_s^\mu\sigma^m,\partial_s^\mu\sigma^{m+1},\rho^m,g_m',h_m'),$$

$$t^m:=\sum_{|\mu|+2s\le 2k}T(\partial_s^\mu\sigma^m,\partial_s^\mu\sigma^{m+1},\rho^m,g_m',h_m').$$

Here P, Q, R, S and T are defined by (2.47), (2.49), (2.48), (2.50) and (2.51) respectively. Our aim is to prove that for suitably small $t\le t(\epsilon)$ there exists a $\Lambda<1$ such that

$$e^{m+1}(t)+\int_0^t d^{m+1}(\tau)\,d\tau\le\Lambda\big(e^m(t)+\int_0^t d^m(\tau)\,d\tau\big).$$

We shall accomplish this by estimating the terms p^m, r^m, q^m, s^m and t^m on RHS of the identity (5.153). These estimates will be largely analogous to the estimates from Chapter 4. However, due to the formally new terms $\partial_s^\mu f_m^\circ$, $\partial_s^\mu g_m^\circ$, $\partial_s^\mu G_m^\circ$, $\partial_s^\mu h_m^\circ$ appearing in the definitions (5.148), (5.149), (5.150) and (5.151) of f_m', g_m', G_m' and h_m' respectively, we need to make several preparatory steps. First, for any $l\in\mathbb{N}$

$$||a_{\rho^m}-a_{\rho^{m-1}}||_{H^l(\Omega)}+||B_{\rho^m}-B_{\rho^{m-1}}||_{H^l(\Omega)}\le C||\sigma^m||_{H^{l+1}(\Omega)}, \tag{5.155}$$

and

$$||c_{\rho^m}-c_{\rho^{m-1}}||_{H^l(\Omega)}\leq C||\sigma_t^m||_{H^l}+C||\sigma^m||_{H^{l+2}(\Omega)}. \tag{5.156}$$

Note that $\sup_{0\leq t\leq t^\epsilon}\mathcal{E}^m(t)+\int_0^{t^\epsilon}\mathcal{D}^m(\tau)\,d\tau\leq L$. In particular, for $|\mu|+2s\leq 2k$

$$||\partial_s^\mu u^m||_{H^1(\Omega)}\,,||\partial_s^\mu a_{\rho^m}||_{L^2(\Omega)}\,,||\partial_s^\mu B_{\rho^m}||_{L^2(\Omega)}\leq C\sqrt{L}. \tag{5.157}$$

Furthermore, $||\partial_s^\mu c_{\rho^m}||_{L^2(\Omega)}\leq C\sqrt{L}$ for $|\mu|+2s\leq 2k-1$. We can now use the Sobolev inequality to bound the lower order derivatives of u^m, a_{ρ^m}, B_{ρ^m} and c_{ρ^m} in L^∞-norm by $C\sqrt{L}$. The major step is to provide the analogoues of part (c) of Lemma 3.3 for the function σ^{m+1} instead of ρ^{m+1} and Lemma 3.5 for the function v^{m+1} instead of u^{m+1}. By the boundary condition (5.147) and the proof of Lemma 3.3, part (c), we deduce

(5.158)

$$||\partial_s^\mu\sigma_t^{m+1}||_2^2+2\epsilon||\partial_s^\mu\Delta\sigma_t^{m+1}||_2^2+\epsilon^2||\partial_s^\mu\Delta^2\sigma_t^{m+1}||_2^2\leq\lambda d^{m+1}+\frac{C}{\lambda}e^{m+1}+e^m(\mathcal{E}^m+\mathcal{D}^m)$$

and

$$||\partial_s^\mu\sigma_t^{m+1}||_2^2+2\epsilon||\partial_s^\mu\Delta\sigma_t^{m+1}||_2^2+\epsilon^2||\partial_s^\mu\Delta^2\sigma_t^{m+1}||_2^2\leq Cd^{m+1}+e^m\mathcal{D}^m. \tag{5.159}$$

As in Lemma 3.5,

$$||\partial_s^\mu\partial_{n^r}v^{m+1}||_{L^2(\Omega)}^2\leq Ce^{m+1}+Ce^m. \tag{5.160}$$

The proof of (5.160) is completely analogous to the proof of Lemma 3.5 whereby, due to the addition of the formally new term $\partial_s^\mu f_m^\circ$ in the definition of f_m' we need to exploit the relations (5.155) and (5.156), which are responsible for the occurence of the term Ce^m on RHS of (5.160). We proceed fully analogously to the energy estimates in Chapter 4 to estimate the right-hand side of the energy identity (5.153). The terms involving $\partial_s^\mu f_m^\circ$ and $\partial_s^\mu h_m^\circ$ require an additional care. In the estimate for the term $\int_\Omega p^m=\int_\Omega f_m'\partial_s^\mu v^{m+1}$, where f_m' is given by (5.148), we single out the term $\int_\Omega\partial_s^\mu f_m^\circ\partial_s^\mu v^{m+1}$. Here f_m° is given by (5.143). Writing

$$\partial_s^\mu(c_{\rho^m}v_n^{m+1})=\sum_{\mu',s'}C_{s'}^{\mu'}\partial_{s'}^{\mu'}c_{\rho^m}\partial_{s-s'}^{\mu-\mu'}v_n^{m+1}=\sum_{\mu'+2s'\leq k}+\sum_{\mu'+2s'>k} \tag{5.161}$$

we estimate the lower-order terms in both sums in L^∞-norm and the higher-order terms in L^2-norm. By the Cauchy-Schwarz inequality and (5.160),

$$|\int_\Omega \partial_s^\mu (c_{\rho^m} v_n^{m+1}) \partial_s^\mu v^{m+1}| \le C\sqrt{\mathcal{D}^m}\sqrt{e^{m+1}}(\sqrt{e^m}+\sqrt{e^{m+1}}). \tag{5.162}$$

Similarly, $|\int_\Omega \partial_s^\mu (B_{\rho^m}\cdot\nabla_{x'} v_n^{m+1})\partial_s^\mu v^{m+1}| \le C\sqrt{\mathcal{E}^m}(e^{m+1}+d^{m+1})$. In analogous fashion, we find

$$\begin{aligned}&\Big|\int_\Omega \partial_s^\mu\big(u_{nn}^m(a_{\rho^m}-a_{\rho^{m-1}})-\nabla_{x'}u_n^m\cdot\big(B_{\rho^m}-B_{\rho^{m-1}}\big)-u_n^m(c_{\rho^m}-c_{\rho^{m-1}})\big)\partial_s^\mu v^{m+1}\Big|\\ &\quad\le C\sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{e^{m+1}}+C\sqrt{\mathcal{E}^m}\sqrt{d^m}\sqrt{e^{m+1}},\end{aligned} \tag{5.163}$$

where we rely on (5.155) and (5.156). The term r^m is of the form $r^m=(f_m')^2+(\text{rest})$, ($r^m$ is defined in (5.154)). The formally new term to estimate has the form $\int_\Omega(\partial_s^\mu f_m^\circ)^2$, where f_m° is given by (5.143). By the same splitting idea as in (5.161) and the estimate (5.160),

$$|\int_\Omega(\partial_s^\mu f_m^\circ)^2| \le C\mathcal{E}^m(e^m+e^{m+1}+d^{m+1})+C\mathcal{D}^m e^{m+1}+C\mathcal{D}^m e^m+C\mathcal{E}^m d^m.$$

Recall now that s^m and t^m are defined by (5.154). The last term in each of the expressions (2.50) and (2.51) has the form $\langle\rho^m\rangle^2 h_m' v_t^{m+1}|_{\mathbb{T}^{n-1}}$ and $\langle\rho^m\rangle^2 h_m'\Delta_{x'}v^{m+1}|_{\mathbb{T}^{n-1}}$, respectively. By (5.151), the formally new term (with respect to $h_{\mu,s}^m$ defined by (3.74)) is $\partial_s^\mu h_m^\circ$. By (5.144), we conclude

$$\partial_s^\mu h_m^\circ = -\sum_{\mu',s'} C_{s'}^{\mu'}\partial_{s'}^{\mu'}([u_n^m]_+^-)\partial_{s-s'}^{\mu-\mu'}\big((|\nabla\rho^m|^2-|\nabla\rho^{m-1}|^2)\langle\rho^m\rangle^{-2}\big) = \sum_{\mu'+2s'\le k}+\sum_{\mu'+2s'>k}$$

Hitting the lower order terms with L^∞-norm and the higher order terms with L^2-norm and using the trace inequality to estimate $||\partial_{s'}^{\mu'}[u_n^m]_+^-||_2$, we obtain

$$||\partial_s^\mu h_m^\circ||_2 \le (\lambda\sqrt{\mathcal{D}^m}+\frac{C}{\lambda}\sqrt{\mathcal{E}^m})\sqrt{e^m}, \qquad ||\partial_s^\mu h_m^\circ||_2\le C\sqrt{\mathcal{D}^m}\sqrt{e^m}. \tag{5.164}$$

Note that $v_t^{m+1}|_{\mathbb{T}^{n-1}}=(\kappa_{\rho^m}-\kappa_{\rho^{m-1}})_t$. It is easy to check that

$$||\partial_s^\mu v_t^{m+1}|_{\mathbb{T}^{n-1}}||_2 \le C(||\nabla^2\sigma_t^m||_2+||\nabla\sigma_t^m||_2+||\nabla^2\sigma^m||_2+||\nabla\sigma^m||_2).$$

By the definitions of d^{m+1} and e^{m+1}, $||\partial_s^\mu v_t^{m+1}|_{\mathbb{T}^{n-1}}||_2 \leq \frac{C}{\sqrt{\epsilon}}\sqrt{d^m} + C\sqrt{e^m}$. Combining this with (5.164) and the Cauchy-Schwarz inequality, we obtain

$$\Big|\int_{\mathbb{T}^{n-1}} \langle \rho^m \rangle^2 \partial_s^\mu h_m^\circ v_t^{m+1}|_{\mathbb{T}^{n-1}}\Big| \leq (\lambda\sqrt{\mathcal{D}^m} + \frac{C}{\lambda}\sqrt{\mathcal{E}^m})\sqrt{e^m}(\frac{C}{\sqrt{\epsilon}}\sqrt{d^m} + C\sqrt{e^m}).$$

Choosing $\lambda = \sqrt{\epsilon}$ we arrive at

(5.165)
$$\Big|\int_{\mathbb{T}^{n-1}} \langle \rho^m \rangle^2 \partial_s^\mu h_m^\circ v_t^{m+1}|_{\mathbb{T}^{n-1}}\Big| \leq C\sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{d^m} + C\sqrt{\epsilon}\sqrt{\mathcal{D}^m}e^m + \frac{C}{\epsilon}\sqrt{\mathcal{E}^m}\sqrt{e^m}\sqrt{d^m} + \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}e^m.$$

We want to estimate the time integral of the right-hand side of the inequality (5.165). For any $t \leq t^\epsilon, \lambda > 0$, we have

$$\int_0^t \sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{d^m}\, d\tau \leq \lambda \int_0^t d^m(\tau)\, d\tau + \frac{C}{\lambda}\sup_{0\leq s\leq t} e^m(s)\int_0^t \mathcal{D}^m(\tau)\, d\tau. \tag{5.166}$$

Similarly,

(5.167)
$$\int_0^t \sqrt{\mathcal{D}^m}e^m\, d\tau \leq \int_0^t e^m + \sup_{0\leq s\leq t} e^m(s)\int_0^t \mathcal{D}^m(\tau)\, d\tau \leq t\sup_{0\leq s\leq t} e^m(s) + L\sup_{0\leq s\leq t} e^m(s).$$

Since $\mathcal{E}^m \leq L$, for any $\lambda > 0$, we obtain

$$\int_0^t \frac{C}{\epsilon}\sqrt{\mathcal{E}^m}\sqrt{e^m}\sqrt{d^m}\, d\tau \leq \lambda\int_0^t d^m(\tau)\, d\tau + \frac{C}{\lambda\epsilon^2}Lt\sup_{0\leq s\leq t} e^m(s), \tag{5.168}$$

and similarly,

$$\int_0^t \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}e^m\, d\tau \leq \frac{C}{\sqrt{\epsilon}}t\sqrt{L}\sup_{0\leq s\leq t} e^m(s). \tag{5.169}$$

Noting that $||\Delta_{x'}v^{m+1}|_{\mathbb{T}^{n-1}}||_2 \leq \frac{C}{\sqrt{\epsilon}}\sqrt{e^m}$, we can use the Cauchy-Schwarz inequality and the estimate (5.164) to get

$$\Big|\int_{\mathbb{T}^{n-1}} \langle \rho^m \rangle^2 \partial_s^\mu h_m^\circ \Delta_{x'}v^{m+1}|_{\mathbb{T}^{n-1}}\Big| \leq \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{D}^m}e^m. \tag{5.170}$$

We get

$$\int_0^t \frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{D}^m}e^m\, d\tau \leq \frac{C}{\epsilon}t\sup_{0\leq s\leq t} e^m(s) + L\sup_{0\leq s\leq t} e^m(s). \tag{5.171}$$

In analogy to the estimate (4.136) together with (5.162), (5.163), (5.165) and (5.170), we obtain

$$\frac{d}{dt}e^{m+1}+d^{m+1}\le Ce^m(\frac{1}{\eta}+\frac{C}{\lambda\epsilon})+\eta d^m+Ce^{m+1}(\frac{1}{\eta}+\frac{1}{\lambda}+\frac{C}{\eta\lambda\epsilon^4}+\frac{C}{\lambda\epsilon})+Cd^{m+1}(\sqrt{L}+\eta+\lambda+\frac{C\lambda\epsilon}{\eta})$$
$$+C\sqrt{\mathcal{D}^m}\sqrt{d^{m+1}}\sqrt{e^{m+1}}+C\mathcal{D}^m e^{m+1}+C\sqrt{\mathcal{D}^{m+1}}\sqrt{e^{m+1}}(\sqrt{e^m}+\sqrt{e^{m+1}})+C\sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{e^{m+1}}$$
$$+C\sqrt{\mathcal{E}^m}\sqrt{d^m}\sqrt{e^{m+1}}+C\sqrt{\mathcal{D}^m}\sqrt{e^m}\sqrt{d^m}+C\sqrt{\epsilon}\sqrt{\mathcal{D}^m}e^m+\frac{C}{\epsilon}\sqrt{\mathcal{E}^m}\sqrt{e^m}\sqrt{d^m}$$
$$+\frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{E}^m}e^m+\frac{C}{\sqrt{\epsilon}}\sqrt{\mathcal{D}^m}e^m.$$

Integrating the above inequality over $[0,t]$ and proceeding as in (4.137), using (5.166), (5.167), (5.168), (5.169), (5.171), we conclude

$$\begin{aligned}
&e^{m+1}(t)+\int_0^t d^{m+1}(\tau)\,d\tau\le Ct(\frac{1}{\eta}+\frac{C}{\lambda\epsilon})\sup_{0\le s\le t}e^m(s)+\eta\int_0^t d^m(\tau)\,d\tau+\\
&Ct(\frac{1}{\eta}+\frac{1}{\lambda}+\frac{C}{\eta\lambda\epsilon^4}+\frac{C}{\lambda\epsilon})\sup_{0\le s\le t}e^{m+1}(s)+C(\sqrt{L}+\eta+\lambda+\frac{C\lambda\epsilon}{\eta})\int_0^t d^{m+1}(\tau)\,d\tau\\
&+C\lambda\big(\int_0^t d^m(\tau)\,d\tau+\int_0^t d^{m+1}(\tau)\,d\tau\big)+\\
&(\frac{CL}{\lambda}+L+t+\frac{CL}{\lambda\epsilon^2}t+\frac{C\sqrt{L}}{\sqrt{\epsilon}}t+\frac{C}{\epsilon}t)\sup_{0\le s\le t}e^m(s)+C(t+L+\sqrt{L})\sup_{0\le s\le t}e^{m+1}(s)
\end{aligned}\tag{5.172}$$

We choose η, λ, L and $t=:t^\epsilon\le\mathcal{T}^\epsilon$ small, so that $\eta+C\lambda<\chi<\frac{1}{5}$

$$Ct(\frac{1}{\eta}+\frac{C}{\lambda\epsilon})+\frac{CL}{\lambda}+L+t^\epsilon+\frac{CL}{\lambda\epsilon^2}t^\epsilon+\frac{C\sqrt{L}}{\sqrt{\epsilon}}t^\epsilon+\frac{C}{\epsilon}t^\epsilon<\chi<\frac{1}{5},$$

$$Ct^\epsilon(\frac{1}{\eta}+\frac{1}{\lambda}+\frac{C}{\eta\lambda\epsilon^4}+\frac{C}{\lambda\epsilon})+C(t^\epsilon+L+\sqrt{L})<\chi<\frac{1}{5}$$

and $C(\sqrt{L}+\eta+\lambda+\frac{C\lambda\epsilon}{\eta})<\chi<\frac{1}{5}$. Taking supremum over $[0,t^\epsilon]$, we arrive at

$$\sup_{0\le\tau\le t^\epsilon}e^{m+1}(\tau)+\int_0^{t^\epsilon}d^{m+1}(s)\,ds\le\chi\Big\{\sup_{0\le\tau\le t^\epsilon}e^m(\tau)+\int_0^{t^\epsilon}d^m(s)\,ds\Big\}+$$
$$+\chi\Big\{\sup_{0\le\tau\le t^\epsilon}e^{m+1}(\tau)+\int_0^{t^\epsilon}d^{m+1}(s)\Big\}ds.$$

Therefore,

$$\sup_{0\le\tau\le t^\epsilon}e^{m+1}(\tau)+\int_0^{t^\epsilon}d^{m+1}(s)\,ds\le\frac{\chi}{1-\chi}\Big\{\sup_{0\le\tau\le t^\epsilon}e^m(\tau)+\int_0^{t^\epsilon}d^m(s)\,ds\Big\}.\tag{5.173}$$

We observe now that the conservation law (3.79) and the fact that $\sigma^{m+1}(0)= v^{m+1}(0)=0$ imply

$$\int_{\mathbb{T}^{n-1}} \sigma^{m+1} = \int_\Omega v^{m+1}(1+\phi'\rho^m)+\int_\Omega \phi' u^m\sigma^m.$$

By the Poincaré inequality, previous identity and the uniform bounds on ρ^m and u^m we get

(5.174)

$$||\sigma^{m+1}||_2^2 \le C||\nabla\sigma^{m+1}||_2^2+C||\int_{\mathbb{T}^{n-1}}\sigma^{m+1}||_2^2 \le C||\nabla\sigma^{m+1}||_2^2+C||v^{m+1}||_2^2+C||u^m||^2_{L^2(\Omega)}||\sigma^m||_2^2$$
$$\le Ce^{m+1}+C\mathcal{E}^m||\sigma^m||_2^2 \le Ce^{m+1}+CL||\sigma^m||_2^2.$$

With L and χ so small that $CL+2C\frac{\chi}{1-\chi}=:\Lambda<1$, we obtain by (5.173),

$$\sup_{0\le\tau\le t^\epsilon}||\sigma^{m+1}(\tau)||_2^2 \le C\frac{\chi}{1-\chi}\Big\{\sup_{0\le\tau\le t^\epsilon} e^m(\tau)+\int_0^{t^\epsilon} d^m(s)\,ds\Big\}+$$
$$+CL\sup_{0\le\tau\le t^\epsilon}||\sigma^m(\tau)||_2^2 \le \frac{\Lambda}{2}\Big\{\sup_{0\le\tau\le t^\epsilon} e^m(\tau)+\int_0^{t^\epsilon} d^m(s)\,ds\Big\}+\Lambda\sup_{0\le\tau\le t^\epsilon}||\sigma^m(\tau)||_2^2.$$

Adding $\big\{\sup_{0\le\tau\le t^\epsilon} e^{m+1}(\tau)+\int_0^{t^\epsilon} d^{m+1}(s)\,ds\big\}$ to the both sides of the above inequality and using (5.173) again we get

$$\sup_{0\le\tau\le t^\epsilon}\Big\{e^{m+1}(\tau)+\int_0^{t^\epsilon} d^{m+1}(s)\,ds+||\sigma^{m+1}(\tau)||_2^2\Big\}\le$$
$$C\frac{\chi}{1-\chi}\Big\{\sup_{0\le\tau\le t^\epsilon} e^m(\tau)+\int_0^{t^\epsilon} d^m(s)\,ds\Big\}+\frac{\Lambda}{2}\Big\{\sup_{0\le\tau\le t^\epsilon} e^m(\tau)+\int_0^{t^\epsilon} d^m(s)\,ds\Big\}+$$
$$\sup_{0\le\tau\le t^\epsilon}\Lambda||\sigma^m(\tau)||_2^2 \le \Lambda\sup_{0\le\tau\le t^\epsilon}\Big\{e^m(\tau)+\int_0^{t^\epsilon} d^m(s)\,ds+||\sigma^m(\tau)||_2^2\Big\}.$$

Define the Banach spaces

$$X:=\Big\{(v,\sigma)\Big|\ ||(v,\sigma)||_{\mathcal{E}_\epsilon}+||\sigma||_2^2<\infty\Big\},\quad Y:=\Big\{(v,\sigma)\Big|\ ||(v,\sigma)||_{\mathcal{D}_\epsilon}<\infty\Big\},$$

where $||(\cdot,\cdot)||_{\mathcal{E}_\epsilon}$ and $||(\cdot,\cdot)||_{\mathcal{D}_\epsilon}$ are defined by (1.37) and (1.38) respectively. Let $Z:= L^\infty\big([0,t^\epsilon];X\big)\cap L^2\big([0,t^\epsilon];Y\big)$, equipped with the natural norm $||(v,\sigma)||_Z:=||(v,\sigma)||_{\mathcal{E}_\epsilon}+ ||\sigma||_2^2+||(v,\sigma)||_{\mathcal{D}_\epsilon}$. Since Λ can be chosen arbitrarily small, we have proven that the sequence $\big((v^l,\sigma^l)\big)_{l\in\mathbb{N}}$ satisfies $||(v^{m+1},\sigma^{m+1})||_Z\le\Lambda'||(v^m,\sigma^m)||_Z$, for some $\Lambda'<1$. This implies that $\big((u^l,\rho^l)\big)_{l\in\mathbb{N}}$ is a Cauchy sequence and converges strongly in Z. Thus the *whole* sequence $\big((u^l,\rho^l)\big)_{l\in\mathbb{N}}$ converges to a solution $(u^\epsilon,\rho^\epsilon)$ of the regularized Stefan problem in the original energy space. In addition to this, passing to a limit in (3.79), we obtain the conservation law

$$\partial_t\Big\{\int_\Omega u(1+\phi'\rho)\Big\} = \partial_t\Big\{\int_{\mathbb{T}^{n-1}}\rho\Big\}. \tag{5.175}$$

Uniqueness. We want to prove uniqueness in the class of functions (u,ρ) satisfying $\sup_{0\le t<t^\epsilon}\mathcal{E}_\epsilon(u,\rho)(t)+\int_0^{t^\epsilon}\mathcal{D}_\epsilon(\tau)\,d\tau\le L$, where L may be chosen smaller if necessary. Let us assume that there exists another solution (v,σ) satisfying the same initial conditions $(v(x,0),\sigma(x',0))=(u_0(x),\rho_0(x'))$ and the bound $\sup_{0\le t<t^\epsilon}\mathcal{E}_\epsilon(v,\sigma)(t)+\int_0^{t^\epsilon}\mathcal{D}_\epsilon(v,\sigma)(\tau)\,d\tau\le L$. After subtracting them and setting $w:=u-v$, $\tau:=\rho-\sigma$, we obtain

$$w_t-\Delta_{x'}w-a_\rho w_{nn}=f^*, \tag{5.176}$$

$$w=\Delta\tau\left\langle\rho\right\rangle^{-1}+g^*=\Delta\tau\left\langle\rho\right\rangle^{-1}-\rho_i\rho_j\tau_{ij}\left\langle\rho\right\rangle^{-1}+G^*\quad\text{on}\quad\mathbb{T}^{n-1}\times\{x_n=0\}, \tag{5.177}$$

$$[w_n]_+^-=\big(\tau_t+\epsilon\Delta^2\tau_t\big)\left\langle\rho\right\rangle^{-2}+h^*\quad\text{on}\quad\mathbb{T}^{n-1}\times\{x_n=0\}, \tag{5.178}$$

where

$$\begin{aligned}
f^*&=-\mathcal{B}\cdot\nabla_{x'}w_n-c_\rho w_n+(a_\rho-a_\sigma)v_{nn}+(\mathcal{B}-B_\sigma)\nabla_{x'}v_n+(c_\sigma-c_\rho)v_n,\\
G^*&=\Delta\sigma\big(\left\langle\rho\right\rangle^{-1}-\left\langle\sigma\right\rangle^{-1}\big)+\sigma_{ij}\big(\tau_i\rho_j\left\langle\rho\right\rangle^{-1}+\sigma_i\tau_j\left\langle\rho\right\rangle^{-1}+\sigma_i\sigma_j\big(\left\langle\rho\right\rangle^{-1}-\left\langle\sigma\right\rangle^{-1}\big)\big),\\
g^*&=-\rho_i\rho_j\tau_{ij}\left\langle\rho\right\rangle^{-1}+G^*\\
h^*&=-(|\nabla\rho|^2-|\nabla\sigma|^2)[v_n]_+^-\left\langle\rho\right\rangle^{-2}.
\end{aligned}$$

We use Chapter 2 to derive the accompanying energy identities. To this end we set $\mathcal{U}=w$, $\psi=\rho$, $\omega=\chi=\tau$, $f=f^*$, $g=g^*$, $G=G^*$ and $h=h^*$. Here ρ takes the role of ρ^{m+1} and σ the role of ρ^m and additionally, the cross-terms vanish since $\omega=\chi=\tau$. With $k\ge n$ sufficiently large, the regularity assumptions of Lemma 2.1 are fulfilled. We are thus naturally led to the following energy quantities:

$$\mathcal{E}^*:=\tilde{\mathcal{E}}_\epsilon(w,\tau;\rho),\quad \mathcal{D}^*:=\tilde{\mathcal{D}}_\epsilon(w,\tau;\rho),$$

where $\tilde{\mathcal{E}}_\epsilon$ and $\tilde{\mathcal{D}}_\epsilon$ are defined to be the RHS of (1.35) and (1.36) with $k=0$, respectively. In addition to this we define $P^*=P(w,\rho,f^*)$ and analogously Q^*, R^*, S^* and T^*. Using the identity (2.46), we obtain

$$\frac{d}{dt}\mathcal{E}^*+\mathcal{D}^*=\int_\Omega\Big\{P^*+R^*\Big\}-\int_{\mathbb{T}^{n-1}}\Big\{Q^*+S^*+T^*\Big\}. \tag{5.179}$$

Our goal at this stage is to prove the inequality of the form

$$\frac{d}{dt}\mathcal{E}^*(t)+\mathcal{D}^*(t)\le C\mathcal{E}^*(t)+C\sqrt{L}\mathcal{D}^*(t), \tag{5.180}$$

which would enable us to absorb the multiple of $\mathcal{D}^*$ on the right-hand side into the left-hand side and then use the Gronwall's inequality to conclude that $\mathcal{E}^*(t)=0$ for any $t\ge 0$. It is essential that the constant C in the above estimate does not depend on ϵ so that the smallness bound on L remains independent of ϵ. That the identity (5.180) indeed holds, follows analogously to the energy estimates from the Chapter 4 applied to the right-hand side of (5.179). Here we strongly exploit the unfirom bounds on $\mathcal{E}_\epsilon(u,\rho)$ and $\mathcal{E}_\epsilon(v,\sigma)$. In particular we know that

$$||(a_\rho)_t||_\infty,\ ||\nabla_{x'}(a_\rho)||_\infty,\ ||(a_\rho)_n||_\infty,\ ||\mathcal{B}||_\infty,\ ||c_\rho||_\infty,\ ||B_\sigma||_\infty,\ ||c_\sigma||_\infty\le C\sqrt{L},$$

$$||v_n||_{L^\infty(\Omega)},\ ||\nabla_{x'}v_n||_{L^\infty(\Omega)},\ ||v_{nn}||_{L^\infty(\Omega)}\le C\sqrt{L}$$

A major difference from the existence part of the proof is the absence of cross-terms in the energy identities (since $\omega=\chi$ in the notation of Chapter 2). In addition to that, we work in a lower order energy space and we can thus use the above uniform estimates to bound the term $[v_n]_+^-$ by $C\sqrt{L}$ in L^∞-norm. This observation is crucial when estimating h^*. Knowing that the ϵ-dependence comes only from the estimates of the cross-terms (cf. (4.117), (4.118), (4.126), (4.127), (4.129) and (4.130)), we conclude that the constants on the right-hand side of (5.180) *do not* depend on ϵ. Choosing L suitably small (5.180) implies $\frac{d}{dt}\mathcal{E}^*\le C\mathcal{E}^*$ implying $\mathcal{E}^*(t)\le C\int_0^t\mathcal{E}^*(s)\,ds$, since $\mathcal{E}^*(0)=0$. By Gronwall's inequality, we conclude $\mathcal{E}^*(t)=0$. In addition to this the conservation law (5.175) gives $||\tau||_2^2\le C\mathcal{E}^*$. This estimate follows in the same way as (5.174). Thus $(u,\rho)=(v,\sigma)$. This finishes the proof of the uniqueness claim.

Continuity. Integrating the identity (3.76) over the time interval $[s,t]$, we obtain

(5.181)
$$\mathcal{E}^{m+1}(t)-\mathcal{E}^{m+1}(s)+\int_s^t\mathcal{D}^{m+1}(s)=\int_s^t\int_\Omega P^m+R^m-\int_s^t\int_{\mathbb{T}^{n-1}}Q^m+S^m+T^m.$$

However, since $(u^l,\rho^l)\to(u,\rho)$ strongly in the energy space, we may pass to the limit in (5.181) to conclude

$$\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)+\int_s^t\mathcal{D}_\epsilon(\tau)\,d\tau=\int_s^t\int_\Omega\bar{P}+\bar{R}-\int_s^t\int_{\mathbb{T}^{n-1}}\bar{Q}+\bar{S}+\bar{T}. \tag{5.182}$$

Here $\bar{P}=\sum_{|\mu|+2s\le 2k}P(\partial_s^\mu u,\rho,f_{\mu,s})$, where $f_{\mu,s}$ is defined by dropping the index m in the definition (3.71) of $f^m_{\mu,s}$. The terms $\bar{Q}$, $\bar{R}$, $\bar{S}$ and $\bar{T}$ are defined analogously. We claim that

$$\Big|\int_s^t\int_\Omega\bar{P}+\bar{R}-\int_s^t\int_{\mathbb{T}^{n-1}}\bar{Q}+\bar{S}+\bar{T}\Big|\le C\int_s^t\sqrt{\mathcal{E}_\epsilon(\tau)}\mathcal{D}_\epsilon(\tau)\,d\tau. \tag{5.183}$$

The inequality follows easily from the energy estimates in Chapter 4. We observe that the estimates involving ϵ on the right-hand side, are used only when estimating the cross-terms (cf. (4.117), (4.118), (4.126), (4.127), (4.129) and (4.130)). However, the cross-terms vanish as m goes to ∞ (since $\chi=\omega=\partial_s^\mu\rho$). As a result, we obtain the estimate (5.183) with the constant C on the right-hand side which does not depend on ϵ. Using (5.182) and (5.183), we obtain

$$\Big|\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)+\int_s^t\mathcal{D}_\epsilon(\tau)\,d\tau\Big|\le C\int_s^t\sqrt{\mathcal{E}_\epsilon(\tau)}\mathcal{D}_\epsilon(\tau)\,d\tau. \tag{5.184}$$

In addition to that, for any $0\le s<t\le t^\epsilon$ we have

$$\big|\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)\big|\le C\Big|\int_s^t\mathcal{D}_\epsilon(\tau)\,d\tau\Big|\Big(1+\sup_{0\le s\le t^\epsilon}\sqrt{\mathcal{E}_\epsilon(s)}\Big)\longrightarrow 0\quad\text{as}\quad s\to t,$$

since $\sup_{0\le s\le t^\epsilon}\sqrt{\mathcal{E}_\epsilon(s)}\le\sqrt{L}$. This finishes the proof of Theorem 5.1. □

6. Global stability

Proof of Theorem 1.2: We exploit the estimate (5.184) to prove the theorem. We shall abbreviate $(\mathcal{E},\mathcal{D})(u,\rho;\rho)(t)=:(\mathcal{E},\mathcal{D})(t)$ and $(\mathcal{E}_\epsilon,\mathcal{D}_\epsilon)(u^\epsilon,\rho^\epsilon;\rho^\epsilon)(t)=:(\mathcal{E}_\epsilon,\mathcal{D}_\epsilon)(t)$.

Existence. Let $M\le L/2$ where L is given in Lemma 4.1. Let $(u^\epsilon,\rho^\epsilon)$ be the associated solution to the regularized Stefan problem given by Theorem 5.1. Let us extend the solution to the maximal time interval $[0,\bar{t}^\epsilon[$, with $t^\epsilon\le\bar{t}^\epsilon$. Set

$$\mathcal{T}:=\sup_t\Big\{0<t\le\bar{t}^\epsilon:\sup_{0\le s\le t}\mathcal{E}_\epsilon(u^\epsilon,\rho^\epsilon)(s)+\int_0^t\mathcal{D}_\epsilon(u^\epsilon,\rho^\epsilon)(\tau)\,d\tau\le 2M\Big\}.$$

Theorem 5.1 guarantees $\mathcal{T}\ge t^\epsilon>0$. For any $t<\mathcal{T}$, the estimate (5.184) with $s=0$ implies

$$\mathcal{E}_\epsilon(t)+\int_0^t\mathcal{D}_\epsilon(\tau)\,d\tau\le\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon)+C\sup_{0\le s\le\mathcal{T}}\mathcal{E}_\epsilon^{\frac12}(s)\int_0^t\mathcal{D}_\epsilon(\tau)\,d\tau, \tag{6.185}$$

and thus

$$\sup_{0\le t\le\mathcal{T}}\mathcal{E}_\epsilon(t)+\int_0^{\mathcal{T}}\mathcal{D}_\epsilon(\tau)\,d\tau\le M+C\sqrt{2M}\int_0^{\mathcal{T}}\mathcal{D}_\epsilon(\tau)\,d\tau. \tag{6.186}$$

Choose $M<\min\{\frac{1}{32C^2},L/2\}=:M_1$. Inequality (6.186) implies

$$\sup_{0\le t\le\mathcal{T}}\mathcal{E}_\epsilon(t)+\int_0^{\mathcal{T}}\mathcal{D}_\epsilon(\tau)\,d\tau\le\frac43M<2M,$$

which would contradict the choice of $\mathcal{T}$ in case $\mathcal{T}$ were finite. Thus $\mathcal{T}=\infty$ (which implies $\bar{t}^\epsilon=\infty$) and the estimate (1.39) follows easily from (6.185) and the above choice of M. This proves the theorem.

Uniqueness. We want to prove uniqueness in the class of functions (u,ρ) satisfying $\sup_{0\le t<\infty}\mathcal{E}_\epsilon(u,\rho)(t)+\int_0^t\mathcal{D}_\epsilon(u,\rho)(\tau)\,d\tau\le 2M$, where M may be chosen smaller if necessary. It is done in exactly the same way as the uniqueness proof in Theorem 5.1.

□

Proof of Theorem 1.1: *Claim 1*: Let $K\le M$ be any positive number, where M is given by Theorem 1.2. If the initial data (u_0,ρ_0) satisfy

$$\mathcal{E}(u_0,\rho_0)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0-\int_\Omega u_0(1+\phi'\rho_0)\Big|\le\frac{K}{3},$$

then there exists a unique global solution to the Stefan problem (1.18) - (1.23). Moreover, we obtain the global bound $\sup_{0\le t<\infty}\mathcal{E}(t)+\int_0^t\mathcal{D}(\tau)\,d\tau\le K$.

Proof of Claim 1. Let $\{(u^\epsilon,\rho^\epsilon)\}_\epsilon$ be a family of solutions of the regularized Stefan problem with the initial condition $(u^\epsilon(x,0),\rho^\epsilon(x',0))=(u_0^\epsilon,\rho_0^\epsilon)$ satisfying the regularity assumptions of Theorem 1.2 and chosen so that

$$(u_0^\epsilon,\rho_0^\epsilon)\to(u_0,\rho_0),\quad \mathcal{E}(u_0^\epsilon,\rho_0^\epsilon;\rho_0^\epsilon)\to\mathcal{E}(u_0,\rho_0;\rho_0)\quad\text{as}\quad \epsilon\to 0$$

and

$$\sum_{|\mu|+2s\leq 2k}\epsilon\int_{\mathbb{T}^{n-1}}|\nabla^2\partial_s^\mu\Delta\rho_0^\epsilon|\leq\sqrt{\epsilon}.$$

Thus for ϵ small, we have $\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon)+\left|\int_{\mathbb{T}^{n-1}}\rho_0^\epsilon-\int_\Omega u_0^\epsilon(1+\phi'\rho_0^\epsilon)\right|\leq\frac{K}{2}$. Theorem 1.2 guarantees global existence of the solution $(u^\epsilon,\rho^\epsilon)$ and also gives the estimate $\sup_{0\leq t<\infty}\mathcal{E}_\epsilon(t)+\int_0^\infty\mathcal{D}_\epsilon(\tau)\,d\tau\leq K$. Since $\mathcal{E}\leq\mathcal{E}_\epsilon$ and $\mathcal{D}\leq\mathcal{D}_\epsilon$, we obtain $\sup_{0\leq t<\infty}\mathcal{E}(u^\epsilon,\rho^\epsilon)(t)+\int_0^\infty\mathcal{D}(u^\epsilon,\rho^\epsilon)(\tau)\,d\tau\leq K$. Passing to the limit as $\epsilon\to 0$, we obtain the solution (u,ρ) to the original Stefan problem (1.18) - (1.23). The uniqueness claim follows by setting $\epsilon=0$ in the proof of the uniqueness statement of Theorem 1.2. This finishes the proof of *Claim 1*. In the same way as we derived the inequality (6.185), we deduce for any $T>0$:

$$\sup_{0\leq t\leq T}\mathcal{E}_\epsilon(t)+\int_0^T\mathcal{D}_\epsilon(\tau)\,d\tau\leq\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon)+C\sqrt{K}\int_0^T\mathcal{D}_\epsilon(\tau)\,d\tau. \tag{6.187}$$

If we choose $K<\min\{\frac{1}{(2C)^2},M\}=:M_1$, absorb the right-most term into the left-hand side and drop the supremum sign, we obtain for any $t>0$

$$\mathcal{E}_\epsilon(t)+\frac{1}{2}\int_0^t\mathcal{D}_\epsilon(s)\,ds\leq\mathcal{E}_\epsilon(u_0^\epsilon,\rho_0^\epsilon).$$

We let $\epsilon\to 0$ and by lower semicontinuity and the assumptions on initial data, we obtain

$$\mathcal{E}(t)+\frac{1}{2}\int_0^t\mathcal{D}(s)\,ds\leq\mathcal{E}(u_0,\rho_0). \tag{6.188}$$

In addition to this, we obtain the conservation law

$$\partial_t\Big\{\int_{\mathbb{T}^{n-1}}\rho\Big\}=\partial_t\Big\{\int_\Omega u(1+\phi'\rho)\Big\}. \tag{6.189}$$

Let us set $M^*:=\frac{M_1}{12}$, where M_1 is defined in the line after (6.187), and assume $\mathcal{E}(u_0,\rho_0)+\left|\int_{\mathbb{T}^{n-1}}\rho_0-\int_\Omega u_0(1+\phi'\rho_0)\right|\leq M^*$. *Claim 1* guarantees the global existence of the solution (u,ρ) and also gives the global bound $\sup_{0\leq t<\infty}\mathcal{E}(t)+\int_0^\infty\mathcal{D}(\tau)\,d\tau\leq\frac{M_1}{4}$.

In order to prove (1.34), we first fix any $s>0$. The idea is to solve the Stefan problem with the new initial data $(u^1(x,0),\rho^1(x',0))=(u(x,s),\rho(x',s))$. The problem allows for unique solutions by *Claim 1*, since

$$\mathcal{E}(u_0^1,\rho_0^1)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0^1-\int_\Omega u_0^1(1+\phi'\rho_0^1)\Big| = \mathcal{E}(u,\rho)(s)+\Big|\int_{\mathbb{T}^{n-1}}\rho_0-\int_\Omega u_0(1+\phi'\rho_0)\Big|$$
$$\leq \frac{M_1}{4}+\frac{M_1}{12}=\frac{M_1}{3}.$$

In addition to this we have the global bound $\sup_{0\leq t<\infty}\mathcal{E}(u^1,\rho^1)(t)+\int_0^\infty\mathcal{D}(u^1,\rho^1)(\tau)\,d\tau\leq M_1$ (again by *Claim 1*). We are thus in the uniqueness regime and we conclude $(u^1,\rho^1)(t)=(u,\rho)(t+s)$ for any $t\geq 0$. We may now use the estimate (6.188) to obtain (1.34).

The second main ingredient in proving the decay is to control the instant energy in terms of the dissipation, i.e. to prove that there exists a constant $C>0$ such that $\mathcal{E}(t)\leq C\mathcal{D}(t)$. We know that for $|\mu|+2s\leq 2k$

$$||\partial_s^\mu\nabla\rho||_{H^1}\leq C\sqrt{\mathcal{D}}. \tag{6.190}$$

Thus, the only non-trivial term left to estimate is $||u||_{L^2(\Omega)}$.

Claim 2: There exists a constant $C>0$ such that $||u||_{L^2(\Omega)}\leq C\sqrt{\mathcal{D}}$.

Proof of Claim 2. Let $x\in\Omega$ and $x'\in\mathbb{T}^{n-1}$ be arbitrarily chosen. By the mean value theorem

$$u(x)=u(x')+\int_0^1\nabla u(tx+(1-t)x')\cdot(x-x')\,dt.$$

Note that $\int_{\mathbb{T}^{n-1}}u(x')\,dx'=0$ because $u=\nabla\cdot\left(\frac{\nabla\rho}{\langle\rho\rangle}\right)$ on $\mathbb{T}^{n-1}$. We thus integrate with respect to x' over $\mathbb{T}^{n-1}$ and then with respect to x over Ω to obtain

$$|\mathbb{T}^{n-1}|\int_\Omega u(x)\,dx=\int_\Omega\int_{\mathbb{T}^{n-1}}\int_0^1\nabla u(tx+(1-t)x')(x-x')\,dtdx'dx.$$

Therefore

$$\Big|\int_\Omega u(x)\,dx\Big|\leq\frac{1}{|\mathbb{T}^{n-1}|}\max_{\substack{x\in\Omega\\ x'\in\mathbb{T}^{n-1}}}|x-x'||\Omega||\mathbb{T}^{n-1}|||\nabla u||_{L^\infty(\Omega)}$$
$$\leq C||\nabla u||_{L^\infty(\Omega)}\leq C||\nabla u||_{H^{\frac{n}{2}+1}(\Omega)}\leq C\sqrt{\mathcal{D}}.$$

By the Poincaré inequality,

$$||u||_{L^2(\Omega)} \leq C|| \int_\Omega u||_{L^2(\Omega)} + C||\nabla u||_{L^2(\Omega)} \leq C\sqrt{\mathcal{D}}.$$

This finishes the proof of *Claim 2*. As explained above, *Claim 2* and the estimate (6.190) together, imply that there exists $C>0$ such that

$$\mathcal{E} \leq C\mathcal{D}. \tag{6.191}$$

Plugging (6.191) into (1.34) yields for any $s>0$ and some constant $\alpha>0$:

$$\mathcal{E}(t) + \alpha \int_s^t \mathcal{E}(\tau)\, d\tau \leq \mathcal{E}(s). \tag{6.192}$$

As in [**42**], p. 135, define a function $V(s) := \int_s^\infty \mathcal{E}(\tau)\, d\tau$. From (6.192), $\alpha V(s) \leq \mathcal{E}(s)$,

$$V'(s) = -\mathcal{E}(s) \leq -\alpha V(s)$$

and thus $V(s) \leq V(0)e^{-s\alpha}$. We integrate (6.192) with respect to s over the time interval $[t/2, t]$ to get

$$\mathcal{E}(t)\frac{t}{2} \leq V(\frac{t}{2}).$$

Thus $\mathcal{E}(t) \leq \frac{2C}{t} e^{-\frac{\alpha}{2}t}$. There exist $k_1, K_2 > 0$ such that for any $t>0$

$$\mathcal{E}(t) \leq k_1 e^{-K_2 t}. \tag{6.193}$$

Integrating the conservation law (6.189) implies $\int_{\mathbb{T}^{n-1}} (\rho(x',t) - \bar{\rho})\, dx' = \int_\Omega u(1+\phi'\rho)$. By an argument analogous to (5.174), $||\rho(t) - \bar{\rho}||_2^2 \leq C\mathcal{E}(u,\rho)(t)$. Combining this inequality with (6.193), we conclude

$$\mathcal{E}(u,\rho)(t) + ||\rho(t) - \bar{\rho}||_2^2 \leq K_1 e^{-K_2 t}$$

for some new constant $K_1 > 0$, K_2 as in (6.193) and for all $t \geq 0$. This finishes the proof of the theorem. $\square$

CHAPTER 3

The round case

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ denote a domain that contains a liquid and a solid separated by an interface Γ. The solid phase $\Omega^-(t)$ is defined as a region encircled by $\Gamma(t)$ and the liquid phase is given by $\Omega^+(t) := \Omega \setminus \overline{\Omega^-}$. The unknowns are the location of the interface $\left\{\Gamma(t); t \geq 0\right\}$ and the temperature function $v : [0,T] \times \Omega \to \mathbb{R}$. Let Γ_0 be the initial position of the free boundary and $v_0 : \Omega \to \mathbb{R}$ be the initial temperature. We denote the normal velocity of Γ by V and normalize it to be *positive if* Γ *is locally expanding* $\Omega^+(t)$. Furthermore, we denote the mean curvature of Γ by κ. With these notations, (v, Γ) satisfies the following free boundary value problem:

$$\partial_t v - \Delta v = 0 \quad \text{in} \quad \Omega \setminus \Gamma. \tag{1.194}$$

$$v = \sigma\kappa \quad \text{on} \quad \Gamma, \tag{1.195}$$

$$V = [v_n]_-^+ \quad \text{on} \quad \Gamma, \tag{1.196}$$

$$v_n = 0 \quad \text{on} \quad \partial\Omega, \tag{1.197}$$

$$v(0, \cdot) = v_0; \quad \Gamma(0) = \Gamma_0 \tag{1.198}$$

where $\sigma > 0$ is the surface tension coefficient. Given v, we write v^+ and v^- for the restriction of v to $\Omega^+(t)$ and $\Omega^-(t)$, respectively. With this notation $[v_n]_-^+$ stands for the jump of the normal derivatives across the interface $\Gamma(t)$, namely $[v_n]_-^+ := v_n^+ - v_n^-$, where n stands for the unit normal on the hypersurface $\Gamma(t)$ with respect to $\Omega^-(t)$.

Steady states and the perturbation (u, f)**.** It is well known (cf. [**46**]) that there are steady spheres for the problem (1.194) - (1.198) and they are parametrized by its radius $\bar{R}$ and the center $\mathbf{x}_0 \in \mathbb{R}^n$, such that the ball $B_{\bar{R}}(\mathbf{x}_0)$ with radius $\bar{R}$ centered at $\mathbf{x}_0$ belongs to Ω. We are thus speaking of an $(n+1)$-parameter family of steady state solutions. To each such $(n+1)$-tuple, we associate a time-independent solution given by $(\bar{v},\bar{\Gamma}) \equiv (\sigma(n-1)/\bar{R}, S_{\bar{R}}(\mathbf{x}_0))$, where $S_{\bar{R}}(\mathbf{x}_0) = \partial B_{\bar{R}}(\mathbf{x}_0)$.

Coordinates. We shall assume that

$$\Omega := B_{R_*}(0) \subset \mathbb{R}^n$$

is a ball of radius $R_* > 1$. We shall parametrize Γ as a 'graph' over the unit sphere $\mathbb{S}^{n-1}$ centered at $\mathbf{x}_0$ with $|\mathbf{x}_0| + 1 < R_*$:

$$\Gamma(t) = \left\{ \mathbf{x} \in \Omega \,\middle|\, \mathbf{x} = R(t,\xi)\xi + \mathbf{x}_0 \right\}, \tag{1.199}$$

where $\xi \in \mathbb{S}^{n-1}$. Here $R : [0,\infty[\times \mathbb{S}^{n-1} \to \mathbb{R}$ is a sufficiently smooth function such that $\bigcup_{0\le t\le T} \Gamma(t) \subset \Omega$, $T > 0$ and additionally $\Gamma_0 = \left\{ \mathbf{x} \in \Omega \,\middle|\, \mathbf{x} = R_0(\xi)\xi + \mathbf{x}_0 \right\}$ for some $R_0 > 0$. We introduce a map $\phi : [0,\infty[\times \mathbb{S}^{n-1} \to [0,\infty[\times \Gamma$,

$$\phi(t,\xi) := R(t,\xi)\xi + \mathbf{x}_0. \tag{1.200}$$

In the following we summarize some basic facts from hypersurface geometry and introduce the necessary notation. Detailed proofs are found in the appendix. The map ϕ induces a metric $\mathbf{g}$ on Γ, whose line element equals $R^{n-2}|g|\,d\xi$, where in local coordinates $|g| = \sqrt{R^2 + |\nabla_g R|^2}$ (see Appendix, Lemma 0.11). Here ∇_g stands for the Riemmanian gradient with respect to the standard metric g on the unit sphere $\mathbb{S}^{n-1}$. We can also express the unit normal n on Γ in local coordinates. Prescribe a local system of coordinates $\{t_1, \ldots, t_{n-1}\}$. The Riemmanian metric g on $\mathcal{S}$ is given by $g_{\alpha\beta} = \xi_{,\alpha} \cdot \xi_{,\beta}$. The unit normal n on Γ is given by

$$n = \frac{R\xi - g^{\alpha\beta} R_{,\alpha} \xi_{,\beta}}{\sqrt{R^2 + |\nabla_g R|^2}}.$$

The mean curvature κ and the normal velocity V in local coordinates read (see Appendix, Lemma 0.10 and formula (0.583)):

$$H(t,\xi) := \kappa \circ \phi = \frac{n-1}{|g|} - \frac{1}{R} \nabla_g \cdot \frac{\nabla_g R}{|g|} \quad \text{and} \quad V \circ \phi = \phi_t n = -\frac{R_t R}{|g|}. \tag{1.201}$$

In the rest of the chapter we shall normalize $\sigma=1$ and analyze the stability of the steady state $(\bar{v},\bar{\Gamma})\equiv(n-1,S_1(\mathbf{0}))=(n-1,\mathbb{S}^{n-1})$. Hence, it is natural to introduce the perturbation $f:[0,\infty[\times\mathbb{S}^{n-1}\to\mathbb{R}$ from the steady sphere $\mathbb{S}^{n-1}$ by setting $f:=R-1$. Similarly we set $u:=v-(n-1)$. Linearizing the curvature H and $|g|$ around $R=1$, we obtain

$$H(f)=n-1-f-\Delta_g f+N(f);\quad |g|=1+f+\frac{|\nabla_g f|^2}{2}+\Psi(f) \tag{1.202}$$

where Δ_g stands for the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ with respect to the metric g (see (0.588)); $N(f)$ denotes the nonlinear remainder in the expansion of $H(f)$ and $\Psi(f)=O(|f|^3+|\nabla_g f|^3)$. If we set $u_0=v_0-(n-1)$ and $f_0=R_0-1$, we can formulate the problem in terms of the perturbation (u,f):

$$\partial_t u-\Delta u=0 \quad \text{in} \quad \Omega\setminus\Gamma. \tag{1.203}$$

$$u=\kappa-(n-1) \quad \text{on} \quad \Gamma, \tag{1.204}$$

$$V=[u_n]_-^+ \quad \text{on} \quad \Gamma, \tag{1.205}$$

$$u_n=0 \quad \text{on} \quad \partial\Omega, \tag{1.206}$$

$$u(0,\cdot)=u_0;\quad f(0)=f_0 \tag{1.207}$$

Motivation and linear version of main results. To motivate our results let us first investigate the eigenvalue problem for the linearized Stefan problem:

$$\lambda v-\Delta v=0 \quad \text{on } \Omega\setminus\mathbb{S}^{n-1}, \tag{1.208}$$

$$v=-(n-1)f-\Delta_g f \quad \text{on } \mathbb{S}^{n-1}, \tag{1.209}$$

$$[v_n]_-^+=-\lambda f \quad \text{on } \mathbb{S}^{n-1}, \tag{1.210}$$

$$v_n=0 \quad \text{on } \partial\Omega;\ [v]_-^+=0 \quad \text{on } \mathbb{S}^{n-1}, \tag{1.211}$$

with $\lambda \neq 0$. Observe that RHS of the equation (1.209) simply is the linearization around $(n-1)$ of the curvature operator H. Note that by integrating (1.208) over $\Omega \setminus \mathbb{S}^{n-1}$ and using the boundary condition (1.210), we obtain the first important identity $\lambda\Big(\int_\Omega v + \int_{\mathbb{S}^{n-1}} f\Big) = 0$. Multiplying (1.208) by v, integrating over $\Omega \setminus \mathbb{S}^{n-1}$ and using the boundary conditions (1.209) and (1.210), we are led to the identity

$$\lambda\Big(\int_\Omega v^2 + \int_{\mathbb{S}^{n-1}} \big\{|\nabla_g f|^2 - (n-1)f^2\big\}\Big) + \int_\Omega |\nabla v|^2 = 0. \tag{1.212}$$

Setting $\mathcal{I}(v,f) := \int_\Omega v^2 + \int_{\mathbb{S}^{n-1}} \big\{|\nabla_g f|^2 - (n-1)f^2\big\}$ and denoting $\mathbf{P}v = v - \frac{1}{|\Omega|}\int_\Omega v$, $\mathbb{P}f = f - \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f$, we note that

$$\begin{aligned}
\mathcal{I}(v,f) &= \frac{1}{|\Omega|}\Big(\int_\Omega v\Big)^2 + \int_\Omega |\mathbf{P}v|^2 + \int_{\mathbb{S}^{n-1}} \big\{|\nabla_g f|^2 - (\mathbb{P}f)^2\big\} - \frac{n-1}{|\mathbb{S}^{n-1}|}\Big(\int_{\mathbb{S}^{n-1}} f\Big)^2 \\
&= \Big(\frac{1}{|\Omega|} - \frac{n-1}{|\mathbb{S}^{n-1}|}\Big)\Big(\int_\Omega v\Big)^2 + \tilde{\mathcal{I}}(v,f),
\end{aligned}$$

where we note that $\tilde{\mathcal{I}}(v,f) = \int_\Omega |\mathbf{P}v|^2 + \int_{\mathbb{S}^{n-1}} \big\{|\nabla_g f|^2 - (\mathbb{P}f)^2\big\} \geq 0$ by Wirtinger's inequality (cf. (1.216)) and $\int_\Omega v = -\int_{\mathbb{S}^{n-1}} f$. Hence, if

$$\zeta := \frac{1}{|\Omega|} - \frac{n-1}{|\mathbb{S}^{n-1}|} > 0,$$

then we easily see that for $\lambda > 0$, equation (1.212) can not have non-trivial solutions and thus, all the eigenvalues are non-positive. On the other hand, if $\zeta < 0$, one can show that there exists a pair (v,f) such that $\mathcal{I}(v,f) < 0$. One can formulate a variational problem so that there exists $\lambda_0 > 0$

$$-\frac{1}{\lambda_0} = \min\Big\{\frac{\mathcal{I}(v,f)}{\int_\Omega |\nabla v|^2}\Big|(v,f) \in \mathcal{C}\Big\},$$

where the constraint set $\mathcal{C}$ is given by

$$\begin{aligned}
\mathcal{C} := \Big\{(v,f)\Big| v \in H^1(\Omega \setminus \mathbb{S}^{n-1}),\ f \in H^{5/2}(\mathcal{S}),\ v = -(n-1)f - \Delta_g f \text{ on } \mathbb{S}^{n-1}, \\
\int_{\mathbb{S}^{n-1}} f s_{1,i} = 0,\ \int_\Omega |\nabla v|^2 \neq 0\Big\};
\end{aligned}$$

(the spherical harmonics $s_{1,i}$ of order 1 are defined below, in the "Notation and spherical harmonics decomposition" paragraph). The minimizer (v_0, z_0) is the eigenvector associated to λ_0 and leads to the fastest growing mode $e^{\lambda t}(v_0, z_0)$ for the linear Stefan problem. For a complete analysis of the eigenvalue problem via a different method, see [**46**].

The above analysis suggests that in the *stable case* $\zeta > 0$, we should expect the solutions to exist globally close to the steady sphere $\mathcal{S}$. Contrary to that, in the *unstable case* $\zeta < 0$, an instability should develop.

Notation and spherical harmonics decomposition. Throughout the rest of the chapter, we shall denote the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ by $\mathcal{S}$. Boldface letters are reserved for vectors in $\mathbb{R}^n$. The normal velocity of a moving surface Σ is denoted by V_Σ and the index Σ is dropped when it is clear what surface we are referring to. We also set $\Omega_\Sigma := \Omega \setminus \Sigma$.

Furthermore, we will typically leave out the domain when referring to a Sobolev norm of functions $g : \mathcal{S} \to \mathbb{R}$, i.e. $||g||_{H^s} := ||g||_{H^s(\mathcal{S})}$. For a given set $A \subset \mathbb{R}^n$, we denote the closure of A by $\mathrm{cl}(A)$. For any $t \in]0, \infty]$ introduce

$$C^\infty([0,t[;\Omega^\pm) := \Big\{ u : [0,t] \times \Omega \to \mathbb{R};\ u\mathbf{1}_{\mathrm{cl}(\Omega^\pm)} \in C^\infty([0,t[\times \mathrm{cl}(\Omega^\pm)) \Big\},$$

and analogously define $C^\infty(\Omega^\pm)$ by suppressing the time dependence. For any $k \in \mathbb{N} \cup \{0\}$ set

$$\begin{aligned} H^k(\Omega^\pm) &= \Big\{ u : \Omega \to \mathbb{R};\ u\mathbf{1}_{\mathrm{cl}(\Omega^\pm)} \in H^k(\mathrm{cl}(\Omega^\pm)) \Big\}; \\ ||u||_{H^k(\Omega^\pm)} &:= ||u\mathbf{1}_{\mathrm{cl}(\Omega^+)}||_{H^k(\mathrm{cl}(\Omega^+))} + ||u\mathbf{1}_{\mathrm{cl}(\Omega^-)}||_{H^k(\mathrm{cl}(\Omega^-))}, \end{aligned} \tag{1.213}$$

where H^k stands for the standard $W^{k,2}$ Sobolev space.

For a given function $f \in L^2(\mathcal{S})$, we introduce the spherical harmonics decomposition for f, thus replacing the Fourier decomposition used in [**32**]. Following [**44**], for any $k \in \mathbb{N}_0$ let H_k stand for the set of homogeneous polynomials of degree k in n variables. We define the set of spherical harmonics S^k to consist of the restrictions of these polynomials onto the unit sphere:

$$S_k = \Big\{ s : \mathcal{S} \to \mathbb{R};\ s = p|_{\mathcal{S}},\ p \in H_k \Big\}.$$

The set S_k is a finite dimensional subspace of $L^2(\mathcal{S})$ of dimension $N(k)$ and is spanned by an orthonormal basis $(s_{k,j})_{j=1}^{N(k)}$. It is well known that $N(0) = 1$ and $N(1) = n$.

By [**44**], Lemma 2, we know that the spherical harmonics $s_{k,i}$ form a complete orthonormal basis $\mathcal{X}$ of $L^2(\mathcal{S})$ given by

$$\mathcal{X} = \bigcup_{k=0}^{\infty} \bigcup_{i=1}^{N(k)} \{s_{k,i}\}.$$

For any $k \in \mathbb{N}_0$, $i \in \mathbb{N}$ and a given function $g \in L^2(\mathcal{S})$, we set $g_{k,i} := \int_{\mathcal{S}} g s_{k,i} \, d\xi$. We further denote

$$\mathbb{P}_k f = \sum_{i=1}^{N(k)} g_{k,i} s_{k,i}. \tag{1.214}$$

Hence, $g = \sum_{k=0}^{\infty} \mathbb{P}_k g$. Note that $\mathbb{P}_0 g = \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} g \, d\xi$. For any $k \geq 0$, we define the projection onto the "higher modes" $\mathbb{P}_{k+}$ by

$$\mathbb{P}_{k+} g := \sum_{j=k}^{\infty} \mathbb{P}_j g. \tag{1.215}$$

For the sake of simplicity, we also denote $\mathbb{P} f := \mathbb{P}_{1+} f = f - \mathbb{P}_0 f$. If a function w is prescribed on Ω then $\mathbf{P}_0 w := \frac{1}{|\Omega|} \int_\Omega w$ and $\mathbf{P} w := w - \mathbf{P}_0 w$. An important observation that will be used several times in our analysis is the Wirtinger's inequality: there is a positive constant C_W such that for any $f \in H^1(\mathcal{S})$ the following inequality holds:

$$\int_{\mathcal{S}} \left\{ |\nabla_g f|^2 - (n-1) |\mathbb{P} f|^2 \right\} \geq C_W ||\nabla_g \mathbb{P}_{2+} f||_{L^2}^2. \tag{1.216}$$

The proof is straightforward and follows by decomposing f into spherical harmonics and using the property $\Delta_g s_{k,i} + (n-1) k^2 s_{k,i} = 0$.

Conservation laws. Assume for the moment u and f are classical solutions to (1.203) - (1.205). The first 'conservation-of-mass' law arises by simply integrating (1.203) over Ω_Γ, using the Stokes' formula and the boundary conditions:

$$\partial_t \int_\Omega u + \partial_t \int_{\mathcal{S}} \frac{(1+f)^n}{n} = 0. \tag{1.217}$$

The second law states the fundamental energy identity; it arises by multiplying (1.194) by $(1+u)$, integrating over Ω and making use of the fact that $\int_\Gamma V \kappa \, dS = \partial_t |\Gamma(t)|$ (where $|\Gamma(t)| = \int_{\mathcal{S}} |g| R^{n-2} \, d\xi$ stands for the surface volume of $\Gamma(t)$):

$$\frac{1}{2} \partial_t \int_\Omega (1+u)^2 + \partial_t \int_{\mathcal{S}} (1+f)^{n-2} \sqrt{(1+f)^2 + |\nabla_g f|^2} + \int_\Omega |\nabla u|^2 = 0. \tag{1.218}$$

In addition to these two well-known identities, we discover a new 'conservation-of-momentum' law. It arises by multiplying the equation (1.203) by the harmonic functions $p_i(x)=x_i+\frac{R_*^n x_i}{|x|^n}$. After integrating over Ω_Γ, using the Green's identity and the boundary conditions, we arrive at:

$$\partial_t\int_\Omega u p_i=\partial_t\int_{\mathcal{S}}\big\{F_i(R,\xi)-F_i(1,\xi)\big\}. \tag{1.219}$$

Here

$$F_i(R,\xi):=\Big(\frac{n-1}{n+1}R^{n+1}+R_*^n R\Big)s_{1,i}(\xi)\qquad i=1,\dots n. \tag{1.220}$$

The details of the proof are presented in Lemma 3.16. Let us denote

$$m_0=\int_\Omega u_0+\int_{\mathcal{S}}\{\sum_{k=1}^n\frac{1}{n}\binom{n}{k}f_0^k\}, \tag{1.221}$$

$$m_i:=\int_\Omega u_0 p_i-\int_{\mathcal{S}}\big\{F_i(R(0),\xi)-F_i(1,\xi)\big\},\quad i=1,\dots,n. \tag{1.222}$$

The conservation laws above play an important role in our proof of the stability.

1.1. Main theorems in the stable case. The natural 0-th order energy is defined by

(1.223)

$$\mathcal{E}_{(0)}(u,f):=\frac{1}{2}\int_\Omega u^2+\int_\Omega u+\int_{\mathcal{S}}\Big(R^{n-2}\sqrt{(1+f)^2+|\nabla_g f|^2}-1\Big);\quad \mathcal{D}_{(0)}(u,f):=\int_\Omega|\nabla u|^2.$$

Let $l\geq n/2$. By $\mathcal{E}_{(+)}$, we denote the energy quantity involving high-order time derivatives:

(1.224)

$$\mathcal{E}_{(+)}(\mathcal{U},\omega):=\frac{1}{2}\sum_{k=1}^{l-1}\int_\Omega\mathcal{U}_{t^k}^2+\frac{1}{2}\sum_{k=0}^{l-1}\int_\Omega|\nabla\mathcal{U}_{t^k}|^2+\frac{1}{2}\sum_{k=1}^{l-1}\int_{\mathcal{S}}\Big\{|\nabla_g\omega_{t^k}|^2-(n-1)\omega_{t^k}^2\Big\}.$$

Similarly, $\mathcal{D}_{(+)}$ is defined through:

$$\mathcal{D}_{(+)}(\mathcal{U},\omega):=\sum_{k=1}^{l}\int_\Omega\mathcal{U}_{t^k}^2+\sum_{k=1}^{l-1}\int_\Omega|\nabla\mathcal{U}_{t^k}|^2+\sum_{k=1}^{l}\int_{\mathcal{S}}\Big\{|\nabla_g\omega_{t^k}|^2-(n-1)\omega_{t^k}^2\Big\}. \tag{1.225}$$

Summing (1.223), (1.224) and (1.225), we introduce the *temporal energy* and the *temporal dissipation*:

$$\mathcal{E}(\mathcal{U},\omega):=\mathcal{E}_{(0)}(\mathcal{U},\omega)+\mathcal{E}_{(+)}(\mathcal{U},\omega), \tag{1.226}$$

$$\mathcal{D}(\mathcal{U},\omega):=\mathcal{D}_{(0)}(\mathcal{U},\omega)+\mathcal{D}_{(+)}(\mathcal{U},\omega). \tag{1.227}$$

Remark. It is not clear that $\mathcal{E}_{(0)}$ is positive definite from the definition above. Similarly, due to the presence of terms $-(n-1)\int_{\mathcal{S}}\omega_{tk}^2$, it also not clear that $\mathcal{E}_{(+)}$ and $\mathcal{D}_{(+)}$ are coercive. However, under appropriate smallness assumptions, the positivity will become apparent later.

In addition to the (temporal) energy $\mathcal{E}$ and (temporal) dissipation $\mathcal{D}$, we introduce the higher order space-time energy quantities $\mathfrak{E}$ and $\mathfrak{D}$ that incorporate mixed space-time derivatives. For any $0<\nu\leq 1$, we set:

$$\mathfrak{E}^{\nu}(u,f):=\sum_{q=0}^{l-1}\nu^q\mathfrak{E}_q(u,f);\quad \mathfrak{D}^{\nu}(u,f):=\sum_{q=0}^{l-1}\nu^q\mathfrak{D}_q(u,f). \tag{1.228}$$

Before we define $\mathfrak{E}_q(u,f)$ and $\mathfrak{D}_q(u,f)$, we introduce the functions A_μ^ξ, rather technical in nature, but arise naturally when identifying the leading order in the space-time energies. For a given multi-index $|\xi|=2q$, for any $|\mu|=2q$, let A_μ^ξ be functions depending smoothly on f and $\nabla_g f$ with the additional property

$$A_\mu^\xi=\begin{cases} O(f,\nabla_g f) & \mu\neq\xi,\\ 1+O(f,\nabla_g f), & \mu=\xi\end{cases} \tag{1.229}$$

Exact definition of A_μ^ξ and more detailed description are given in (3.414) and the paragraph thereafter. Property (1.229) is all we need to establish coercivity of the energy: for $0\leq q\leq l-2$

$$\begin{aligned}\mathfrak{E}_q(u,f):=\frac{1}{2}||\nabla u_{t^{l-1-q}}||^2_{H^{2q}(\Omega^\pm)}+2^{2q}\sum_{|\xi|=2q}\sum_{i=1}^n\int_{\mathcal{S}}\Big\{\Big|\nabla_g\sum_{|\mu|=2q+1}A_\mu^{\xi_i}\partial_g^\mu f_{t^{l-1-q}}\Big|^2\\ -(n-1)\Big|\sum_{|\mu|=2q+1}A_\mu^{\xi_i}\partial_g^\mu f_{t^{l-1-q}}\Big|^2\Big\};\end{aligned} \tag{1.230}$$

$$\mathfrak{E}_{l-1}(u,f):=\frac{1}{2}||\nabla u||^2_{H^{2(l-1)}(\Omega^\pm)}. \tag{1.231}$$

Here $\xi_i := \xi + e_i$, where e_i stands for the i-th standard unit vector in $\mathbb{R}^n$. Observe that $\mathfrak{E}_q$ is positive definite under the smallness assumption on f and $\nabla_g f$, as it follows from the form (1.229) of coefficients A_μ^ξ. Similarly, for any $0 \le q \le l-1$

$$\begin{aligned}\mathfrak{D}_q(u,f) := ||\nabla u_{t^{l-1-q}}||^2_{H^{2q+1}(\Omega^\pm)} + 2^{2q-1} \sum_{|\xi|=2q} \int_{\mathcal{S}} \Big\{ \Big| \nabla_g \sum_{|\mu|=2q} A_\mu^\xi \partial_g^\mu f_{t^{l-q}} \Big|^2 \\ -(n-1) \Big| \sum_{|\mu|=2q} A_\mu^\xi \partial_g^\mu f_{t^{l-q}} \Big|^2 \Big\} \ge 0,\end{aligned} \tag{1.232}$$

by Wirtinger's inequality (cf. (1.216)). Finally, for any $0 < \alpha$, $0 < \nu \le 1$ we define the *total energy* and *total dissipation* as:

$$E_{\alpha,\nu} := \mathcal{E} + \alpha \mathfrak{E}^\nu; \quad D_{\alpha,\nu} := \mathcal{D} + \alpha \mathfrak{D}^\nu.$$

THEOREM 1.1. *Assume $\zeta > 0$. Then there exist $0 < \alpha, \nu < 1$, a constant C_* and a sufficiently small constant $M^* > 0$ such that if*

$$E_{\alpha,\nu}(u_0, f_0) + C_* \sum_{i=0}^n m_i^2 \le M^*,$$

then there exists a unique solution to the Stefan problem (1.203) - (1.207) (u,f), satisfying the global bound

$$E_{\alpha,\nu}(u,f)(t) + C_* \sum_{i=0}^n m_i^2 + \frac{1}{2} \int_s^t D_{\alpha,\nu}(u,f)(\tau)\, d\tau \le E_{\alpha,\nu}(u,f)(s) + C_* \sum_{i=0}^n m_i^2, \quad t \ge s \ge 0. \tag{1.233}$$

The second theorem improves the stability to an asymptotic stability statement, and states that the solution converges to a 'nearby' steady sphere:

THEOREM 1.2. *Under the assumptions from the previous theorem, there exists $\bar{\mathbf{x}}$ close to $\bar{\mathbf{x}}_0$ and $\bar{R}$ close to 1 such that the solution (v,Γ) to the Stefan problem converges exponentially fast to the steady state $(1/\bar{R}, S_{\bar{R}}(\bar{\mathbf{x}}_0)$.*

Remark. The analogous statement holds for the steady states $(v,\Gamma) \equiv (\sigma/\bar{R}, S_{\bar{R}}(\bar{\mathbf{x}}_0)$ of the original Stefan problem (1.194) - (1.196). Here, the stability condition reads $\zeta(\sigma, \bar{R}) := \frac{1}{\sigma|\Omega|} - \frac{n-1}{|S_{\bar{R}}|\bar{R}^2} > 0$, where $|S_{\bar{R}}|$ is the surface area of the sphere of radius $\bar{R}$.

Remark. The role of the weight ν in the definition (1.228) of $\mathfrak{E}^\nu$ and $\mathfrak{D}^\nu$ is technical in nature. It is introduced naturally in order to close the energy estimates in Subsection 3.2.4.

1.2. Main theorem in the unstable case. As it is shown in Subsection 4.2, with the help of an appropriate change of variables, the Stefan problem can be formulated as a problem on a fixed domain $\Omega_{S_1(\bar{\mathbf{x}}_0)}$ for the unknowns $w:\Omega_{S_1(\bar{\mathbf{x}}_0)}\to\mathbb{R}$ and $f:\mathbb{S}^1\to\mathbb{R}$ and it takes the form

$$\partial_t(w,f)=\mathcal{L}(w,f)+U(w,f) \tag{1.234}$$

where the linear operator $\mathcal{L}$ and the nonlinearity U are defined explicitly by (4.546) and (4.568) respectively. In order to formulate the instability statement, we need to specify the notion of the initial perturbation. Lemma 4.3 based on [**46**], under the instability assumption $\zeta<0$, shows that the linearized operator for the Stefan problem possesses countably many real eigenvalues $\lambda_0,\lambda_1,\lambda_2,\dots$ with finite multiplicity. The leading eigenvalue λ_0 is positive and simple, $\lambda_1=0$ and all the remaining eigenvalues are strictly negative. If $m(i)$ stands for the multiplicity of the eigenvalue λ_i, we associate the eigenvectors $e_{i,k}$ to the eigenvalue λ_i, where $1\le k\le m(i)$. The set $\cup_{\substack{i=0\\ i\neq 1}}^{\infty}\cup_{k=1}^{m(i)}\left\{e_{i,k}\right\}$ forms an orthonormal basis with respect to the inner product $\langle\cdot,\cdot\rangle_I$ defined by (4.553). We refer to

$$(w_0,f_0)=\sum_{i=0}^{\infty}\sum_{k=1}^{m(i)}c_{i,k}e_{i,k};\quad c_{0,1}\neq 0 \tag{1.235}$$

as the *generic* profile if there exists a non-trivial contribution along the direction of the growing eigenvector (v_0,z_0), i.e. $c_{0,1}\neq 0$. In the following theorem we assert the local existence for small initial data and nonlinear instability of the steady spheres.

THEOREM 1.3. *For any smooth generic profile such that $||w_0||^2_{L^2(\Omega)}+||f_0||^2_{L^2}=1$, there exists a sufficiently small number $\theta_0>0$, such that for any $\delta>0$, there exists a unique solution (w^δ,f^δ) to (1.234) with initial conditions $\left\{(\delta w_0,\delta f_0)\right\}_\delta$ which on the time interval $[0,T^\delta:=\frac{1}{\lambda_0}\log\frac{\theta_0}{\delta}]$ satisfies*

$$||(w^\delta,f^\delta)(t)-c_{0,1}\delta e^{\lambda_0 t}(v_0,z_0)||_{L^2}\le C\delta^2e^{2\lambda_0 t}+C\delta,$$

where C is independent of δ. Furthermore, there exists a constant $K_0>0$ independent of δ, such that

$$||w^\delta(T^\delta)||^2_{L^2(\Omega)}+||f^\delta(T^\delta)||^2_{L^2}\geq K_0. \tag{1.236}$$

Remark. Our theorem shows that the nonlinear unstable dynamics is characterized by the growing mode $e^{\lambda_0 t}(v_0,z_0)$ over the time scale $[0,T^\delta]$. If we perturb away from $S_1(\mathbf{0})=\mathcal{S}$ we show that v_0 is spherically symmetric ((v_0,z_0) is the eigenvector associated to λ_0), hence our instability does NOT contain morphological changes, but only spreading or shrinking along the radial direction.

1.3. Strategy of the proofs and basic ideas. There are several major mathematical novelties in this work, with respect to the results of Chapter 2, i.e. [**31**]. Let us first note that just from the definition (1.223) of $\mathcal{E}_{(0)}$ it is not even clear that this quantity is positive definite. An important step is to Taylor-expand the term $(1+f)^{n-2}\sqrt{(1+f)^2+|\nabla_g f|^2}-1$. We obtain

$$\mathcal{E}_{(0)}=\frac{1}{2}\int_\Omega u^2+\int_\Omega u+\int_{\mathcal{S}} f+\frac{1}{2}\int_{\mathcal{S}}|\nabla_g f|^2+\int_{\mathcal{S}} O(|f|^3+|\nabla_g f|^3)$$

To extract a positive quadratic contribution, we exploit the conservation-of-mass law (1.217) and note that $\int_\Omega u+\int_{\mathcal{S}} f=-\frac{n-1}{2}\int_{\mathcal{S}} f^2+O(f^3)$. Using this observation, by a calculation similar to the one in the equation following (1.212), we can conclude

$$\mathcal{E}_{(0)}=\frac{1}{2}\int_\Omega|\mathbf{P}u|^2+\frac{1}{2}\zeta\big(\int_\Omega u\big)^2+\frac{1}{2}\int_{\mathcal{S}}\big\{|\nabla_g f|^2-(n-1)|\mathbb{P}f|^2\big\}+O(|f|^3+|\nabla_g f|^3)$$

Note that $\int_{\mathcal{S}}\big\{|\nabla_g f|^2-(n-1)|\mathbb{P}f|^2\big\}\geq 0$ by the Wirtinger's inequality. Hence, if $\zeta>0$ it does seem that $\mathcal{E}_{(0)}$ is nonnegative, modulo the third order remainder. Unfortunately, note that the L^2-norm of the first mode $\mathbb{P}_1 f$ (recall (1.215)) is certainly not controlled by the expression $\int_{\mathcal{S}}\big\{|\nabla_g f|^2-(n-1)|\mathbb{P}f|^2\big\}=\int_{\mathcal{S}}\big\{|\nabla_g \mathbb{P}_{2+}f|^2-(n-1)|\mathbb{P}_{2+}f|^2\big\}$. The analysis of the first modes turns out to be impossible without the new conservation-of-momentum law (1.219). More specifically, the problem lies in ensuring the positivity of $\mathcal{E}_{(0)}$ due to the non-trivial presence of the first modes in the third order remainder. The key to circumvent this severe difficulty is to exploit (1.219), so to write:

$$((n-1)+R_*^n)\int_{\mathcal{S}} f\, s_{1,i} = \int_{\Omega} u p_i + \int_{\mathcal{S}} O(|f|^2+|\nabla_g f|^2), \quad i=1,\dots,n.$$

and thus, we can bound the L^2-norm of $\mathbb{P}_1 f$ by the L^2-norms of u and $(\mathbb{P}_0+\mathbb{P}_{2+})f$, which are in turn bounded by the energy $\mathcal{E}_{(0)}$. See Lemma 3.17.

Energy. Unlike in Chapter 2 or in our previous work [**31**], the higher-order energy estimate is considerably more delicate due to the geometric structure of our problem. In order to preserve the boundary condition, one needs to take tangential derivatives. Unfortunately, usual spatial derivatives ∂_{x^i} could lead to linear growth in time when we interchange from one local chart to another, which presents a difficulty in constructing the global solution. Instead, we divide our main energy estimate in two steps. We note that ∂_t is the only globally defined vector field which is 'almost' tangential with respect to $\Gamma(t)$, so we start by deriving the estimate for the temporal energy, defined as in (1.226) and (1.227):

$$\sup_{0\le s\le t} \mathcal{E}(s) + \int_0^t \mathcal{D}(s) \le \mathcal{E}(0) + C \sup_{0\le s\le t}(\sqrt{\mathcal{E}}+\sqrt{\mathfrak{E}})\int_0^t \mathcal{D}(s)\,ds. \tag{1.237}$$

Instead of deriving the similar energy estimates for the other tangential derivatives, we make use of the temporal energy estimate (1.237) and the heat equation in the bulk to bound $\mathfrak{E}$ by $\mathcal{E}$ and $\mathcal{D}$. We refer to this as the *reduction* step. The estimate we obtain for the space-time energy is the following:

$$\sup_{0\le s\le t} \mathfrak{E}^\nu(s) + \int_0^t \mathfrak{D}^\nu(s) \le \mathfrak{E}^\nu(0) + C \sup_{0\le s\le t}\sqrt{\mathfrak{E}^\nu}\int_0^t \mathfrak{D}^\nu(s)\,ds + C^*\int_0^t \mathcal{D}(s)\,ds. \tag{1.238}$$

Based on the previous two estimates, we are able to obtain the a-priori bound for the solution of the original Stefan problem

$$\sup_{0\le s\le t} E_{\alpha,\nu}(s) + \frac{1}{2}\int_0^t D_{\alpha,\nu}(s)\,ds \le E_{\alpha,\nu}(0),$$

which is the central estimate that will allow us to extend the solution globally. We chose to define our energy on a moving domain, with the aim of preserving the transparency of the geometric structure of the problem. We prove the temporal energy estimates in Subsection 3.2.2 and with the space-time energy estimates in Subsection 3.2.4.

Regularization and the iterative procedure. In order to establish the local existence for the Stefan problem, we set-up an iteration scheme, which generates a sequence of iterates $\{(u^m, f^m)\}_{m\in\mathbb{N}}$. As in Chapter 2 or [**31**], such an iteration is well defined, but it breaks the natural energy setting due to the lack of exact cancellations in the presence of cross-terms. We design the elliptic regularization

$$-\frac{f_t R}{|g|} - \epsilon\frac{\Delta_g^2 f_t}{|g|R^{n-2}} = [u_n]_-^+ \circ \phi.$$

to overcome this difficulty. For a fixed ϵ, we use it to prove that $\{(u^m, f^m)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in the energy space. However, note that the expression $u^{m+1} - u^m$ does not a-priori make sense, since the iterates u^m and u^{m+1} are defined on different domains Ω^{m-1} and Ω^m, respectively. Thus, we transform the regularized Stefan problem to a problem on a fixed domain by applying a suitable change of variables and then prove the Cauchy property.

Stability for $\zeta > 0$. After establishing the existence of the global solution under the positivity assumption on ζ, we further prove the exponential decay of the solution to a nearby steady circle which is uniquely determined via conservation laws (1.217) and (1.219). Just like in [**31**], we have to prove that the energy $E_{\alpha,\nu}$ is bounded by the dissipation $D_{\alpha,\nu}$. However, the situation is considerably more complicated in comparison to the Stefan problem close to flat steady surfaces. The reason is that the Dirichlet condition (1.195) can not be directly used to estimate the L^2-norm of f against the L^2-norm of ∇u, since $\int_\Gamma(\kappa-1)\neq 0$. Again, we crucially exploit the conservation laws (1.217) and (1.219) together with the condition $\zeta > 0$, to prove this estimate. This is carried out in Subsection 3.4.

Instability for $\zeta < 0$. The passage from the linear to nonlinear instability is a delicate problem in general. In Section 4 we start by proving the local existence under the assumption $\zeta < 0$ (Theorem 4.1), analogously to the stable case. However, to prove the instability, we resort to a bootstrap technique as developed by Guo &

Strauss in [**30**]. The main idea is to show that on the time scale of instability, some stronger norm $|||\cdot|||$ (given by the high-order energy) is actually controlled by the weaker norm $||\cdot||$ (which is simply L^2-norm in our case). The details of this general procedure are described in Lemma 4.9. However, in order to apply Lemma 4.9, a detailed understanding of the linear problem and the growing eigenmode is necessary. In Subsection 4.1, we provide a variational characterization of the positive eigenvalue (Lemma 4.5) and establish sharp growth estimate for the linear problem in suitable norms (Lemma 4.8). To finally prove Theorem 1.3 (Subsection 4.2), we have to reformulate the Stefan problem so that the Duhamel principle can be applied. Again, we transform the problem to a problem on the fixed domain with the severe nonlinearity $N(f)$ coming from the curvature term, contained in the Dirichlet boundary condition. To ensure the linearity of the boundary condition, we further modify the unknowns and as a result obtain the problem (4.564) - (4.567), to which we apply Lemma 4.9.

2. Energy identities

2.1. Regularization. In order to prove Theorem 1.1, we shall regularize the Stefan problem by adding viscosity to the 'jump' relation (1.196). Namely, for $\epsilon > 0$, we consider the following equation:

$$V + \epsilon\Lambda = [u_n]_-^+ \quad \text{on } \Gamma, \tag{2.239}$$

where $\Lambda \circ \phi = -\frac{\Delta_g f_t}{R^{n-2}|g|}$. Thus, expressed as an equation on $\mathcal{S}$, the equation (2.239) reads

$$-\frac{f_t R}{|g|} - \epsilon \frac{\Delta_g^2 f_t}{R^{n-2}|g|} = [u_n]_-^+ \circ \phi,$$

where ϕ is given by (1.200). We shall refer to the problem (1.194), (1.195) and (2.239) as the *regularized Stefan problem*. In the rest of this subsection we define a number of energy quantities that will be necessary for our subsequent energy estimates.

The stable case. We introduce the additional ϵ-dependent energy terms that incorporate the viscosity:

(2.240)
$$\mathcal{E}_\epsilon(\mathcal{U},\omega) := \mathcal{E}(\mathcal{U},\omega) + \hat{\mathcal{E}}^\epsilon(\omega); \quad \hat{\mathcal{E}}^\epsilon(\omega) := \frac{1}{2}\sum_{k=0}^{l-1}\epsilon\int_{\mathcal{S}}\big\{|\nabla_g^3\omega_{t^k}|^2 - (n-1)|\nabla_g^2\omega_{t^k}|^2\big\},$$

(2.241)
$$\mathcal{D}_\epsilon(\mathcal{U},\omega) := \mathcal{D}(\mathcal{U},\omega) + \hat{\mathcal{D}}^\epsilon(\omega); \quad \hat{\mathcal{D}}^\epsilon(\omega) := \sum_{k=0}^{l-1}\epsilon\int_{\mathcal{S}}\big\{|\nabla_g^3\omega_{t^{k+1}}|^2 - (n-1)|\nabla_g^2\omega_{t^{k+1}}|^2\big\}.$$

Accordingly, with regard to the higher-order space-time norms $\mathfrak{E}$ and $\mathfrak{D}$, for any $0 \le q \le l-1$ we define:

(2.242)
$$\hat{\mathfrak{E}}_q^\epsilon(f) := \epsilon 2^{2q}\sum_{|\xi|=2p}\sum_{i=1}^{n}\int_{\mathcal{S}}\big\{|\nabla_g^3\sum_{|\mu|=2q+1}A_\mu^{\xi_i}\partial_g^\mu f_{t^c}|^2 - (n-1)|\nabla_g^2\sum_{|\mu|=2q+1}A_\mu^{\xi_i}\partial_g^\mu f_{t^c}|^2\big\},$$

(2.243)
$$\hat{\mathfrak{D}}_q^\epsilon(f) := \epsilon 2^{2q-1}\int_0^t\int_{\mathcal{S}}\sum_{|\xi|=2q}\Big\{|\nabla_g^3\sum_{|\mu|=2q}A_\mu^{\xi}\partial_g^\mu f_{t^{c+1}}|^2 - (n-1)|\nabla_g^2\sum_{|\mu|=2q}A_\mu^{\xi}\partial_g^\mu f_{t^{c+1}}|^2\Big\}.$$

Here we recall again the form (1.229) of the coefficients A_μ^ξ. We define the total higher order energies in the natural way:

$$\hat{\mathfrak{E}}^{\epsilon,\nu} = \sum_{q=0}^{l-1}\nu^q\hat{\mathfrak{E}}_q^\epsilon;\ \mathfrak{E}_\epsilon^\nu = \mathfrak{E}^\nu + \hat{\mathfrak{E}}^{\epsilon,\nu}; \quad \hat{\mathfrak{D}}^{\epsilon,\nu} = \sum_{q=0}^{l-1}\nu^q\hat{\mathfrak{D}}_q^\epsilon;\ \mathfrak{D}_\epsilon^\nu = \mathfrak{D}^\nu + \hat{\mathfrak{D}}^{\epsilon,\nu}. \tag{2.244}$$

Again, if $\nu = 1$, we simply drop the index 1 in the notation introduced in (2.244).

The unstable case. In order to prove the local existence theorem in the case $\zeta < 0$, we introduce the appropriate energy quantities. Using (1.215), we define the energy $\mathcal{M}$ by

(2.245)
$$\mathcal{M}(\mathcal{U},\omega) := \frac{1}{2}\sum_{k=0}^{l-1}\int_\Omega \mathcal{U}_{t^k}^2 + \frac{1}{2}\sum_{k=0}^{l-1}\int_\Omega|\nabla\mathcal{U}_{t^k}|^2 + \frac{1}{2}\sum_{k=0}^{l-1}\int_{\mathcal{S}}\big\{|\nabla_g\omega_{t^k}|^2 - (n-1)|\mathbb{P}_{2+}\omega_{t^k}|^2\big\};$$

and the dissipation $\mathcal{N}$ is defined through:

$$\mathcal{N}(\mathcal{U},\omega) := \sum_{k=1}^{l}\int_\Omega \mathcal{U}_{t^k}^2 + \sum_{k=0}^{l-1}\int_\Omega|\nabla\mathcal{U}_{t^k}|^2 + \sum_{k=1}^{l}\int_{\mathcal{S}}\big\{|\nabla_g\omega_{t^k}|^2 - |\mathbb{P}_{2+}\omega_{t^k}|^2\big\}. \tag{2.246}$$

The quantities $\mathcal{M}$ and $\mathcal{N}$ are the analogues of $\mathcal{E}$ and $\mathcal{D}$ defined by (1.226) and (1.227). In analogy to (1.228), for any $0<\nu\leq 1$ we define the space-time energy quantities:

(2.247)
$$\mathfrak{M}^{\nu}=\sum_{q=0}^{l-1}\nu^q\mathfrak{M}_q,\quad \mathfrak{N}^{\nu}=\sum_{q=0}^{l-1}\nu^q\mathfrak{N}_q$$

where, for any $0\leq q\leq l-1$, we define

(2.248)
$$\mathfrak{M}_q(u,f):=||\nabla u_{t^{l-1-q}}||^2_{H^{2q}(\Omega)}+2^{2q}\sum_{|\xi|=2q}\sum_{i=1}^{n}\int_{\mathcal{S}}\Big\{\Big|\nabla_g\sum_{|\mu|=2q}A_{\mu}^{\xi_i}\partial_g^{\mu}f_{t^{l-1-q}}\Big|^2$$
$$-(n-1)\Big|\mathbb{P}_{2+}\sum_{|\mu|=2q}A_{\mu}^{\xi_i}\partial_g^{\mu}f_{t^{l-1-q}}\Big|^2\Big\};$$
$$\mathfrak{N}_q(u,f):=||\nabla u_{t^{l-1-q}}||^2_{H^{2q+1}(\Omega)}+2^{2q-1}\sum_{|\xi|=2q}\int_{\mathcal{S}}\Big\{\Big|\nabla_g\sum_{|\mu|=2q}A_{\mu}^{\xi}\partial_g^{\mu}f_{t^{l-q}}\Big|^2-(n-1)\Big|\sum_{|\mu|=2q}A_{\mu}^{\xi}\partial_g^{\mu}f_{t^{l-q}}\Big|^2\Big\}.$$

Finally, for any $0<\beta$, $0<\nu\leq 1$, the *total energy* and the *total dissipation* for the unstable case are defined by

$$M_{\beta,\nu}:=\mathcal{M}+\beta\mathfrak{M}^{\nu};\quad N_{\beta,\nu}:=\mathcal{N}+\beta\mathfrak{N}^{\nu}.$$

As to the energy quantities that arise from the regularization (2.239), analogously to (2.240) - (2.244), we define

(2.249)
$$\mathcal{M}_{\epsilon}(\mathcal{U},\omega):=\mathcal{M}(\mathcal{U},\omega)+\hat{\mathcal{M}}^{\epsilon}(\omega);\ \hat{\mathcal{M}}^{\epsilon}(\omega):=\frac{1}{2}\sum_{k=0}^{l-1}\int_{\mathcal{S}}\{|\nabla_g^3\omega_{t^k}|^2-(n-1)|\mathbb{P}_{2+}\nabla_g^2\omega_{t^k}|^2\}$$

and

(2.250)
$$\mathcal{N}_{\epsilon}(\mathcal{U},\omega):=\mathcal{N}(\mathcal{U},\omega)+\hat{\mathcal{N}}^{\epsilon}(\omega);\ \hat{\mathcal{N}}^{\epsilon}(\omega):=\sum_{k=0}^{l-1}\int_{\mathcal{S}}\{|\nabla_g^3\omega_{t^{k+1}}|^2-(n-1)|\mathbb{P}_{2+}\nabla_g^2\omega_{t^{k+1}}|^2\}.$$

Furthermore, for any $0\leq q\leq l-1$ we define:

(2.251)
$$\hat{\mathfrak{M}}_q^{\epsilon}(f):=\epsilon 2^{2q}\sum_{|\xi|=2q}\sum_{i=1}^{n}\int_{\mathcal{S}}\Big\{\Big|\nabla_g^3\sum_{|\mu|=2q}A_{\mu}^{\xi_i}\partial_g^{\mu}f_{t^{l-1-q}}\Big|^2-(n-1)\Big|\mathbb{P}_{2+}\nabla_g^2\sum_{|\mu|=2q}A_{\mu}^{\xi_i}\partial_g^{\mu}f_{t^{l-1-q}}\Big|^2\Big\};$$
$$\hat{\mathfrak{N}}_q^{\epsilon}(f):=\epsilon 2^{2q-1}2^{2q}\sum_{|\xi|=2q}\int_{\mathcal{S}}\Big\{\Big|\nabla_g^3\sum_{|\mu|=2q}A_{\mu}^{\xi}\partial_g^{\mu}f_{t^{l-q}}\Big|^2-(n-1)\Big|\mathbb{P}_{2+}\nabla_g^2\sum_{|\mu|=2q}A_{\mu}^{\xi_i}\partial_g^{\mu}f_{t^{l-q}}\Big|^2\Big\},$$

and hence the total higher order energies are given by

$$\hat{\mathfrak{M}}^{\epsilon,\nu}=\sum_{q=0}^{l-1}\nu^q\hat{\mathfrak{M}}_q^{\epsilon};\ \mathfrak{M}_{\epsilon}^{\nu}=\mathfrak{M}^{\nu}+\hat{\mathfrak{M}}^{\epsilon,\nu};\quad \hat{\mathfrak{N}}^{\epsilon,\nu}=\sum_{q=0}^{l-1}\nu^q\hat{\mathfrak{N}}_q^{\epsilon};\ \mathfrak{N}_{\epsilon}^{\nu}=\mathfrak{N}^{\nu}+\hat{\mathfrak{N}}^{\epsilon,\nu}. \tag{2.252}$$

2.2. Trace. Let γ stand for the trace operator on the curve Γ. By ∂_t^* we shall denote the time differentiation operator along the hypersurface Γ defined by

$$\partial_t^* u\circ\phi=\partial_t(u\circ\phi). \tag{2.253}$$

With these notations, applying the chain rule in the above formula, we arrive at the following decomposition:

$$\partial_t^* u=\gamma[u_t]+Vu_n+V_{\|}\cdot\nabla_{\mathbf{g}}u, \tag{2.254}$$

where V is the normal velocity, $V_{\|}\circ\phi=R_t\xi$ is the tangential velocity and $\nabla_{\mathbf{g}}u$ is the tangential gradient of u with respect to the metric $\mathbf{g}$ on Γ. Iterating the formula (2.254) k times and using the product rule, we arrive at the following decomposition formula for the operator ∂_{t^k}:

$$\gamma[u_{t^k}]=\partial_{t^k}^* u-\big[\gamma,\partial_{t^k}\big](u), \tag{2.255}$$

where the commutator $\big[\gamma,\partial_{t^k}\big](u)$ is defined through

$$\big[\gamma,\partial_{t^k}\big](u)\circ\phi:=\sum_{p=0}^{k-1}\sum_{q+r=k-p}C^{p,q,r}\partial_{t^{k-1-p}}((V\circ\phi)^q R_t^r)\xi^{i_1}\dots\xi^{i_r}\partial_g^{i_1}\dots\partial_g^{i_r}\partial_n^q u_{t^p}\circ\phi, \tag{2.256}$$

and $C^{p,q,r}$ are some positive real constants. The following trace lemma will be of crucial importance for estimating terms appearing in (2.256):

LEMMA 2.1. *Let $v\in H^{k+1}(\Omega)$, $f\in H^k(\mathcal{S})$ and $q,r\in\mathbb{N}$ such that $r+q=k$. Let μ be a multi-index such that $|\mu|=r$ Then the following inequality holds:*

$$||\partial_{\mathbf{g}}^{\mu}\partial_n^q v||_{L^2(\Gamma)}\le C\big(1+||f||_{H^k}\big)||v||_{H^{k+1}(\Omega)}.$$

Proof. Using the formulas $\partial_n=n_k\partial_{x^k}$ and $\partial_{\mathbf{g}}^i=\partial_{x^i}-n_in_k\partial_{x^k}$, we may write

$$\partial_{\mathbf{g}}^{\mu}\partial_n^q v=\sum_{j=1}^{r}F_j(n)Q_{k+1-j}(u).$$

Here $F_j(n)$ is a function involving at most $j-1$ tangential derivatives of n and thus at most j derivatives of R, since $n=1+O(\nabla_g R)$. On the other hand $Q_m(u)$ consists of terms containing m Cartesian derivatives of u. The idea is to use the Sobolev estimates: if $j\leq\frac{k+1}{2}$, we estimate $||F_j||_{L^\infty}\leq C||F_j||_{H^{n/2}}$ by the Sobolev inequality, whereas Q_{k+j-1} is estimated in L^2-norm. We then use the trace inequality, to conclude $||Q_{k+j-1}||_{L^2}\leq C||u||_{H^{k+j}(\Omega)}$. In the case $j>\frac{k+1}{2}$, we estimate F_j in L^2-norm and Q_{k+j-1} in L^∞-norm together with the Sobolev and trace inequalities. Combining the two cases the claim of the lemma easily follows. Note that we only use the trace inequality for the Cartesian derivatives on Γ. Observe that in the inequality $||v||_{L^2(\Gamma)}\leq C||v||_{H^1(\Omega)}$, the constant C depends only on the C^1-norm of f. However, for k sufficiently large, the C^1-norm of f is bounded by the H^{k+1}-norm of f. $\square$

2.3. Identities. Let $I=[0,q]$ for some $0<q\leq\infty$. The derivation of the energy identities crucially depends on the following model problem:

$$\mathcal{U}_t-\Delta\mathcal{U}=\mathcal{F}\quad\text{on }\Omega\times I \tag{2.257}$$

$$\mathcal{U}^\pm=\bar{\mathcal{U}}+\tilde{\mathcal{U}}^\pm\quad\text{on }\Gamma, \tag{2.258}$$

$$\bar{\mathcal{U}}\circ\phi=-\frac{(n-1)\chi}{F}-\frac{\Delta_g\chi}{\psi|g|F}+G(\psi,\chi)\quad\text{on }\mathcal{S}, \tag{2.259}$$

$$[\partial_\nu\mathcal{U}]^+_-=\bar{\mathcal{V}}+\tilde{\mathcal{V}}\quad\text{on }\Gamma, \tag{2.260}$$

$$\bar{\mathcal{V}}\circ\phi=-\omega_t\frac{\psi}{|g|F}-\epsilon\frac{\Delta_g^2\omega_t}{|g|F}+h(\psi,\omega)\quad\text{on }\mathcal{S}. \tag{2.261}$$

We should think of $\tilde{\mathcal{U}}^\pm$, $\tilde{\mathcal{V}}$, $G(\psi,\chi)$ and $h(\psi,\chi)$ as a certain kind of 'error terms' that will not contribute to the extraction of the energy. We first define the stable case energies $\bar{\mathcal{E}}_\epsilon$ and $\bar{\mathcal{D}}_\epsilon$ (for the model problem) through:

$$\bar{\mathcal{E}}_\epsilon(\mathcal{U},\omega):=\frac{1}{2}\int_\Omega\left\{\mathcal{U}^2+|\nabla\mathcal{U}|^2\right\}+\frac{1}{2}\int_\mathcal{S}\left\{|\nabla_g\omega|^2-(n-1)\omega^2\right\}+\frac{1}{2}\epsilon\int_\mathcal{S}\left\{|\nabla_g^3\omega|^2-(n-1)|\nabla_g^2\omega|^2\right\} \tag{2.262}$$

and

$$\bar{\mathcal{D}}_\epsilon(\mathcal{U},\omega):=\int_\Omega\Big\{\mathcal{U}_t^2+|\nabla\mathcal{U}|^2\Big\}+\int_{\mathcal{S}}\Big\{|\nabla_g\omega_t|^2-(n-1)\omega_t^2\Big\}+\epsilon\int_{\mathcal{S}}\Big\{|\nabla_g^3\omega_t|^2-(n-1)|\nabla_g^2\omega_t|^2\Big\}. \tag{2.263}$$

As to the unstable case, we define the energies $\bar{\mathcal{M}}_\epsilon$ and $\bar{\mathcal{N}}_\epsilon$ by

$$\begin{aligned}\bar{\mathcal{M}}_\epsilon(\mathcal{U},\omega):=&\frac{1}{2}\int_\Omega\Big\{\mathcal{U}^2+|\nabla\mathcal{U}|^2\Big\}+\frac{1}{2}\int_{\mathcal{S}}\big\{|\nabla_g\omega|^2-(n-1)|\mathbb{P}_{2+}\omega|^2\big\}\\&+\frac{1}{2}\epsilon\int_{\mathcal{S}}\big\{|\nabla_g^3\omega|^2-(n-1)|\mathbb{P}_{2+}\nabla_g^2\omega|^2\big\}.\end{aligned} \tag{2.264}$$

$$\begin{aligned}\bar{\mathcal{N}}_\epsilon(\mathcal{U},\omega):=&\int_\Omega\Big\{\mathcal{U}_t^2+|\nabla\mathcal{U}|^2\Big\}+\int_{\mathcal{S}}\big\{|\nabla_g\omega_t|^2-(n-1)|\mathbb{P}_{2+}\omega_t|^2\big\}\\&+\epsilon\int_{\mathcal{S}}\big\{|\nabla_g^3\omega_t|^2-(n-1)|\mathbb{P}_{2+}\nabla_g^2\omega_t|^2\big\}.\end{aligned} \tag{2.265}$$

Remark. Note that the first modes of ω are included in the energy quantities $\bar{\mathcal{M}}_\epsilon$ and $\bar{\mathcal{N}}_\epsilon$.

Lemma 2.2. *The following identities hold:*

(a)

$$\bar{\mathcal{E}}_\epsilon(t)+\int_0^t\bar{\mathcal{D}}_\epsilon(s)\,ds=\bar{\mathcal{E}}_\epsilon(0)+\int_\Omega(\mathcal{U}+\mathcal{U}_t)\mathcal{F}+\int_0^t\int_\Gamma\Big\{O+P\Big\}+\int_0^t\int_{\mathcal{S}}\Big\{Q+S\Big\}, \tag{2.266}$$

(b)

$$\bar{\mathcal{M}}_\epsilon(t)+\int_0^t\bar{\mathcal{N}}_\epsilon(s)\,ds=\bar{\mathcal{M}}_\epsilon(0)+\int_\Omega(\mathcal{U}+\mathcal{U}_t)\mathcal{F}+\int_0^t\int_\Gamma\Big\{O^u+P^u\Big\}+\int_0^t\int_{\mathcal{S}}\Big\{Q^u+S^u\Big\}, \tag{2.267}$$

where (recalling (1.201))

$$O=O^u=O(\mathcal{U}):=\tilde{\mathcal{U}}^\pm\mathcal{U}_n^\pm+\bar{\mathcal{U}}\tilde{\mathcal{V}}+V(\mathcal{U}^\pm)^2, \tag{2.268}$$

$$P=P^u=P(\mathcal{U}):=\big(\partial_t^*\tilde{\mathcal{U}}^\pm-[\gamma,\partial_t]\mathcal{U}^\pm\big)\mathcal{U}_n^\pm+\partial_t^*\bar{\mathcal{U}}\tilde{\mathcal{V}}+V|\nabla\mathcal{U}^\pm|^2, \tag{2.269}$$

(2.270)

$$Q = Q(\chi,\omega,\psi,F) = -(n-1)\omega_t(\omega-\chi) + \nabla_g\omega_t\cdot\nabla_g(\omega-\chi) - \epsilon(n-1)\Delta_g\omega_t\Delta_g(\omega-\chi)$$
$$+\epsilon\Delta_g\nabla_g\omega_t\cdot\Delta_g\nabla_g(\omega-\chi) + (n-1)\chi\omega_t(\frac{\psi^{n-1}}{F^2}-1) + \Delta_g\chi\omega_t(\frac{\psi^{n-2}}{|g|F^2}-1)$$
$$+\epsilon(n-1)\chi\Delta_g^2\omega_t(\frac{\psi^{n-2}}{F^2}-1) + \epsilon\Delta_g\chi\Delta_g^2\omega_t(\frac{\psi^{n-3}}{|g|F^2}-1) + h\bar{\mathcal{U}}\circ\phi\psi^{n-2}|g| - G\big(\omega_t\frac{\psi}{|g|F}$$
$$+\epsilon\frac{\Delta_g^2\omega_t}{|g|F}\big)\psi^{n-2}|g|; \quad Q^u = Q + (\mathbb{P}_0+\mathbb{P}_1)\omega(\mathbb{P}_0+\mathbb{P}_1)\omega_t + \epsilon\mathbb{P}_1\omega_{\theta^2}\mathbb{P}_1\omega_{\theta^2 t}.$$

(2.271)

$$S = S(\chi_t,\omega_t,\psi,F) := -(n-1)\omega_t(\omega_t-\chi_t) + \nabla_g\omega_t\cdot\nabla_g(\omega_t-\chi_t) - \epsilon(n-1)\Delta_g\omega_t\Delta_g(\omega_t-\chi_t)$$
$$+\epsilon\Delta_g\nabla_g\omega_t\cdot\Delta_g\nabla_g(\omega_t-\chi_t) + (n-1)\chi_t\omega_t(\frac{\psi^{n-1}}{F^2}-1) + \Delta_g\chi_t\omega_t(\frac{\psi^{n-2}}{|g|F^2}-1)$$
$$+\epsilon(n-1)\chi_t\Delta_g^2\omega_t(\frac{\psi^{n-2}}{F^2}-1) + \epsilon\Delta_g\chi_t\Delta_g^2\omega_t(\frac{\psi^{n-3}}{|g|F^2}-1) - \big(-(n-1)\chi(\frac{1}{F})_t - \Delta_g\chi\big(\frac{1}{\psi|g|F}\big)_t$$
$$+G_t\big)\bar{\mathcal{V}}\circ\phi\psi^{n-2}|g| + \big(\frac{\chi_t}{F^2} + \frac{\Delta_g\chi_t}{\psi|g|F^2}\big)h\psi^{n-2}|g|; \quad S^u = S + |(\mathbb{P}_0+\mathbb{P}_1)\omega_t|^2 + \epsilon|\mathbb{P}_1\omega_{\theta^2 t}|^2.$$

Proof. Clearly, part (b) follows easily from part (a). Hence, in the rest of the proof we only focus on the identity in the part (a) of Lemma 2.2. The first energy identity arises by multiplying (2.257) by $\mathcal{U}$ and integrating over Ω. We obtain:

$$\frac{1}{2}\partial_t\int_\Omega \mathcal{U}^2 - \int_\Gamma V(\mathcal{U}^\pm)^2 + \int_\Omega |\nabla\mathcal{U}|^2 - \int_\Gamma \mathcal{U}^\pm\mathcal{U}_n^\pm = \int_\Omega \mathcal{U}\mathcal{F}.$$

Using (2.258) and (2.260):

$$-\int_\Gamma \mathcal{U}^\pm\mathcal{U}_n^\pm = -\int_\Gamma \bar{\mathcal{U}}\bar{\mathcal{V}} - \int_\Gamma (\tilde{\mathcal{U}}^\pm\mathcal{U}_n^\pm + \bar{\mathcal{U}}\tilde{\mathcal{V}}) \tag{2.272}$$

We shall evaluate $-\int_\Gamma \bar{\mathcal{U}}\bar{\mathcal{V}}$ by transforming it to the integral over $\mathcal{S}$ using (2.259) and (2.261):

$$\begin{aligned}-\int_\Gamma \bar{\mathcal{U}}\bar{\mathcal{V}} = &-\int_{\mathcal{S}} \big(\frac{(n-1)\chi}{F} + \frac{\Delta_g\chi}{\psi|g|F}\big)\big(\omega_t\frac{\psi}{|g|F} + \epsilon\frac{\Delta_g^2\omega_t}{|g|F}\big)\psi^{n-2}|g| \\ &-\int_{\mathcal{S}} h\bar{\mathcal{U}}\circ\phi\psi^{n-2}|g| - \int_{\mathcal{S}} G\big(\omega_t\frac{\psi}{|g|F} + \epsilon\frac{\Delta_g^2\omega_t}{|g|F}\big)\psi^{n-2}|g|.\end{aligned} \tag{2.273}$$

We proceed toward the extraction of the energy contribution from the first integral on RHS of (2.273). Note that

(2.274)

$$-\int_{\mathcal{S}}\Big(\frac{(n-1)\chi}{F}+\frac{\Delta_g\chi}{\psi|g|F}\Big)\big(\omega_t\frac{\psi}{|g|F}+\epsilon\frac{\Delta_g^2\omega_t}{|g|F}\big)\psi^{n-2}|g| = -\int_{\mathcal{S}}(n-1)\chi\omega_t\frac{\psi^{n-1}}{F^2}-\int_{\mathcal{S}}\frac{\Delta_g\chi\omega_t\psi^{n-2}}{|g|F^2}$$
$$-\epsilon\int_{\mathcal{S}}(n-1)\frac{\chi\Delta_g^2\omega_t\psi^{n-2}}{F^2}-\epsilon\int_{\mathcal{S}}\frac{\Delta_g\chi\Delta_g^2\omega_t\psi^{n-3}}{|g|F^2}$$
$$=\frac{1}{2}\partial_t\int_{\mathcal{S}}\big\{|\nabla_g\omega|^2-(n-1)\omega^2\big\}+\frac{1}{2}\epsilon\partial_t\int_{\mathcal{S}}\big\{|\nabla_g^3\omega|^2-(n-1)|\nabla_g^2\omega|^2\big\}$$
$$+(n-1)\int_{\mathcal{S}}\omega_t(\omega-\chi)-\int_{\mathcal{S}}\nabla_g\omega_t\cdot\nabla_g(\omega-\chi)+\epsilon(n-1)\int_{\mathcal{S}}\Delta_g\omega_t\Delta_g(\omega-\chi)$$
$$-\epsilon\int_{\mathcal{S}}\Delta_g\nabla_g\omega_t\cdot\Delta_g\nabla_g(\omega-\chi)-(n-1)\int_{\mathcal{S}}\chi\omega_t\big(\frac{\psi^{n-1}}{F^2}-1\big)-\int_{\mathcal{S}}\Delta_g\chi\omega_t\big(\frac{\psi^{n-2}}{|g|F^2}-1\big)$$
$$-\epsilon(n-1)\int_{\mathcal{S}}\chi\Delta_g^2\omega_t\big(\frac{\psi^{n-2}}{F^2}-1\big)-\epsilon\int_{\mathcal{S}}\Delta_g\chi\Delta_g^2\omega_t\big(\frac{\psi^{n-3}}{|g|F^2}-1\big).$$

The identities (2.272) - (2.274) give the formulas (2.268) and (2.270) for O and Q, respectively. The second energy identity arises by multiplying (2.257) by $\mathcal{U}_t$ and integrating over Ω. We obtain:

$$\int_\Omega\mathcal{U}_t^2+\frac{1}{2}\partial_t\int_\Omega|\nabla\mathcal{U}|^2-\int_\Gamma V|\nabla\mathcal{U}^\pm|^2-\int_\Gamma\mathcal{U}_t^\pm\mathcal{U}_n^\pm=\int_\Omega\mathcal{U}_t\mathcal{F}.$$

Using (2.258) and (2.260)

$$\begin{aligned}-\int_\Gamma\mathcal{U}_t^\pm\mathcal{U}_n^\pm &= -\int_\Gamma(\partial_t^*\bar{\mathcal{U}}+\partial_t^*\tilde{\mathcal{U}}^\pm-[\gamma,\partial_t]\mathcal{U}^\pm)\mathcal{U}_n^\pm\\ &= -\int_\Gamma\partial_t^*\bar{\mathcal{U}}\bar{\mathcal{V}}-\int_\Gamma\Big\{(\partial_t^*\tilde{\mathcal{U}}^\pm-[\gamma,\partial_t]\mathcal{U}^\pm)\mathcal{U}_n^\pm+\partial_t^*\bar{\mathcal{U}}\tilde{\mathcal{V}}.\Big\}\end{aligned}\tag{2.275}$$

We shall evaluate $-\int_\Gamma\partial_t^*\bar{\mathcal{U}}\bar{\mathcal{V}}$ by transforming it to the integral over $\mathbb{S}^1$ using (2.259) and (2.261). By (2.259)

$$\partial_t(\bar{\mathcal{U}}\circ\phi)=-(n-1)\frac{\chi_t}{F}-\frac{\Delta_g\chi_t}{\psi|g|F}-(n-1)\chi\big(\frac{1}{F}\big)_t-\Delta_g\chi\big(\frac{1}{\psi|g|F}\big)_t+G_t$$

Thus, we have

(2.276)

$$
\begin{aligned}
&-\int_\Gamma \partial_t^* \bar{\mathcal{U}}\bar{\mathcal{V}} = -\int_{\mathcal{S}} \partial_t(\bar{\mathcal{U}}\circ\phi)\bar{\mathcal{V}}\circ\phi\psi^{n-2}|g| = -\int_{\mathcal{S}} \big(-(n-1)\frac{\chi_t}{F} - \frac{\Delta_g\chi_t}{\psi|g|F} - (n-1)\chi(\frac{1}{F})_t \\
&\qquad -\Delta_g\chi\big(\frac{1}{\psi|g|F}\big)_t + G_t\big) \times \big(-\frac{\omega_t\psi}{|g|F} - \epsilon\frac{\Delta_g^2\omega_t}{|g|F} + h\big)\psi^{n-2}|g| \\
&= -(n-1)\int_{\mathcal{S}} \chi_t\omega_t\frac{\psi^{n-1}}{F^2} - \int_{\mathcal{S}} \frac{\Delta_g\chi_t\psi^{n-2}}{|g|F^2} - \epsilon(n-1)\int_{\mathcal{S}} \frac{\chi_t\Delta_g^2\omega_t\psi^{n-2}}{F^2} - \epsilon\int_{\mathcal{S}} \frac{\Delta_g\chi_t\Delta_g^2\omega_t\psi^{n-3}}{|g|F^2} \\
&+ \int_{\mathcal{S}} \big(-(n-1)\chi(\frac{1}{F})_t - \Delta_g\chi\big(\frac{1}{\psi|g|F}\big)_t + G_t\big)\bar{\mathcal{V}}\circ\phi\psi^{n-2}|g| - \int_{\mathcal{S}} \big(\frac{\chi_t}{F} + \frac{\Delta_g\chi_t}{\psi|g|F}\big)h\psi^{n-2}|g| \\
&= \int_{\mathcal{S}} \big\{|\nabla_g\omega_t|^2 - (n-1)\omega_t^2\big\} + \epsilon\int_{\mathcal{S}} \big\{|\nabla_g^3\omega_t|^2 - (n-1)|\nabla_g^2\omega_t|^2\big\} \\
&+ (n-1)\int_{\mathcal{S}} \omega_t(\omega_t - \chi_t) - \int_{\mathcal{S}} \nabla_g\omega_t\cdot\nabla_g(\omega_t - \chi_t) + \epsilon(n-1)\int_{\mathcal{S}} \Delta_g\omega_t\Delta_g(\omega_t - \chi_t) \\
&- \epsilon\int_{\mathcal{S}} \Delta_g\nabla_g\omega_t\cdot\Delta_g\nabla_g(\omega_t - \chi_t) - (n-1)\int_{\mathcal{S}} \chi_t\omega_t\big(\frac{\psi^{n-1}}{F^2} - 1\big) - \int_{\mathcal{S}} \Delta_g\chi_t\omega_t\big(\frac{\psi^{n-2}}{|g|F^2} - 1\big) \\
&- \epsilon(n-1)\int_{\mathcal{S}} \chi_t\Delta_g^2\omega_t\big(\frac{\psi^{n-2}}{F^2} - 1\big) - \epsilon\int_{\mathcal{S}} \Delta_g\chi_t\Delta_g^2\omega_t\big(\frac{\psi^{n-3}}{|g|F^2} - 1\big) \\
&+ \int_{\mathcal{S}} \big(-(n-1)\chi(\frac{1}{F})_t - \Delta_g\chi\big(\frac{1}{\psi|g|F}\big)_t + G_t\big)\bar{\mathcal{V}}\circ\phi\psi^{n-2}|g| - \int_{\mathcal{S}} \big(\frac{\chi_t}{F^2} + \frac{\Delta_g\chi_t}{\psi|g|F^2}\big)h\psi^{n-2}|g|
\end{aligned}
$$

The identities (2.275) and (2.276) imply the formulas (2.269) and (2.271) for P and S respectively.

3. Stable case $\zeta > 0$

3.1. Iteration scheme. In this section we set up an iteration scheme for proving the local existence of solutions to the Stefan problem (1.203) - (1.207). We want to find a sequence of iterates $(u^k, \Gamma^k)_{k\in\mathbb{N}}$, where each curve Γ^k is expressed as a graph over $\mathcal{S}$ via the mapping $\phi^k : [0,\infty[\times\mathbb{S}^1 \to \Gamma^k$, $\phi^k(t,\xi) = R^k(t,\xi)\xi$.

The iteration is defined as follows: given (u^m, R^m) for some $m\in\mathbb{N}$ and smooth initial data $u^{m+1}(0,\cdot) = u_0(\cdot)$, find u^{m+1} by solving the equation

$$\partial_t u^{m+1} - \Delta u^{m+1} = 0, \quad \text{on} \quad \Omega_{\Gamma^m}, \tag{3.277}$$

with the given Dirichlet boundary condition

$$u^{m+1} = \kappa^m \quad \text{on} \quad \Gamma^m. \tag{3.278}$$

After we have obtained u^{m+1}, for given smooth initial data $R^{m+1}(0,\cdot) = R_0(\cdot)$, find R^{m+1} by solving the following equation on Γ^m:

$$V^{m+1} + \epsilon \Lambda^{m+1} = [\partial_n u^{m+1}]_-^+, \tag{3.279}$$

where $V^{m+1} : \Gamma^m \to \mathbb{R}$ is given by $V^{m+1} \circ \phi^m = -\frac{f_t^{m+1} R^m}{|g^m|}$ and $\Lambda^{m+1} : \Gamma^m \to \mathbb{R}$ is defined through $\Lambda^{m+1} \circ \phi^m = -\frac{\Delta_g^2 f_t^{m+1}}{|g^m|(R^m)^{n-2}}$. Here $|g^m| = \sqrt{(R^m)^2 + |\nabla_g R^m|^2}$. The relation (3.279) can be expressed as an equation on $\mathcal{S}$:

$$-\frac{f_t^{m+1} R^m}{|g^m|} - \epsilon \frac{\Delta_g^2 f_t^{m+1}}{|g^m|(R^m)^{n-2}} = [\partial_n u^{m+1}]_-^+ \circ \phi^m \quad \text{on} \quad \mathcal{S}, \tag{3.280}$$

Our goal is to prove that the sequence of iterates (u^m, f^m) converges to the solution of the regularized Stefan problem. Applying the temporal differentiation operator ∂_{t^k} $(0 \le k \le l-1)$ to the equations (3.277), (3.278) and (3.280), we obtain

$$u_{t^{k+1}}^{m+1} - \Delta u_{t^k}^{m+1} = 0 \quad \text{on} \quad \Omega^m, \tag{3.281}$$

$$u_{t^k}^{m+1} = \bar{u}_k^m + \tilde{u}_k^m \quad \text{on} \quad \Gamma^m, \tag{3.282}$$

where

$$\bar{u}_k^m = \partial_{t^k}^* \kappa^m \tag{3.283}$$

and thus, (3.283) can be rewritten as

$$\bar{u}_k^m \circ \phi^m = -(n-1) f_{t^k}^m - \frac{\Delta_g f_{t^k}^m}{R^m |g^m|} + G_k^m \quad \text{on} \quad \mathcal{S}, \tag{3.284}$$

$$[\partial_n u_{t^k}^{m+1}]_-^+ = \bar{v}_k^m + \tilde{v}_k^m \quad \text{on} \quad \Gamma^m, \tag{3.285}$$

where

$$\bar{v}_k^m = \partial_{t^k}^* (V^{m+1} + \epsilon \Lambda^{m+1}) \tag{3.286}$$

We can thus rewrite

$$\bar{v}_k^m \circ \phi^m = -\frac{f_{t^{k+1}}^{m+1} R^m}{|g^m|} - \epsilon \frac{\Delta_g^2 f_{t^{k+1}}^{m+1}}{|g^m|(R^m)^{n-2}} + h_k^m \quad \text{on} \quad \mathcal{S}, \tag{3.287}$$

where

$$\tilde{u}_k^m = -[\gamma^m, \partial_{t^k}]u^{m+1}, \tag{3.288}$$

$$\tilde{v}_k^m = -[\gamma^m, \partial_{t^k}][\partial_n u^{m+1}]_-^+, \tag{3.289}$$

$$G_k^m = -\sum_{q=0}^{k-1} \Delta_g f_{t^q}^m \Big(\frac{1}{R^m|g^m|}\Big)_{t^{k-q}} + \partial_{t^k} N^*(f^m). \tag{3.290}$$

(3.291)

$$h_k^m = -h_{k,1}^m - \epsilon h_{k,2}^m; \quad h_{k,1}^m = -\sum_{q=0}^{k-1} f_{t^{q+1}}^{m+1} \partial_{t^{k-q}} \Big(\frac{R^m}{|g^m|}\Big),\ h_{k,2}^m = -\sum_{q=0}^{k-1} \Delta_g^2 f_{t^{q+1}}^{m+1} \Big(\frac{1}{|g^m|(R^m)^{n-2}}\Big)_{t^{k-q}}.$$

In (3.290) above, $N^*(f)$ is the quadratic remainder in the decomposition $H(f) = n-1-(n-1)f - \frac{\Delta_g f}{R|g|} + N^*(f)$, i.e. $N^*(f) = \frac{n-1}{|g|} - (n-1)(1-f) - \nabla_g(\frac{1}{|g|}) \cdot \frac{\nabla_g f}{R}$. As a first step toward the introduction of the energy norm for the iterates, we state the version of the conservation law (1.217) which takes into account the iteration.

Lemma 3.1. *The following conservation law holds:*

$$\int_\Omega u_t^{m+1} + \int_{\mathcal{S}} f_t^{m+1} = -\int_{\mathcal{S}} f_t^{m+1}((R^m)^{n-1} - 1), \tag{3.292}$$

Proof. Upon integrating (3.277) over Ω^m and using integration by parts we arrive at

$$\begin{aligned}\partial_t \int_\Omega u^{m+1} &= -\int_{\Gamma^m} [\partial_n u^{m+1}]_-^+ = -\int_{\mathcal{S}} \Big(\frac{R^m f_t^{m+1}}{|g^m|} + \epsilon \frac{\Delta_g^2 f_t^{m+1}}{(R^m)^{n-2}|g^m|}\Big)(R^m)^{n-2}|g^m| \\ &= -\int_{\mathcal{S}} f_t^{m+1} - \int_{\mathcal{S}} ((R^m)^{n-1} - 1) f_t^{m+1},\end{aligned}$$

which implies the claim. □

From (3.292), we immediately see

$$\partial_t \int_\Omega u^{m+1} + \partial_t \int_{\mathcal{S}} \frac{1}{n}(R^{m+1})^n = \int_{\mathcal{S}} ((R^{m+1})^{n-1} - (R^m)^{n-1}) f_t^{m+1}. \tag{3.293}$$

For any $k \in \mathbb{N}$ define:

$$\mathcal{E}_{(+)}^k := \mathcal{E}_{\epsilon,+}(u^k, f^k), \qquad \mathcal{D}_{(+)}^k := \mathcal{D}_{\epsilon,+}(u^k, f^k), \tag{3.294}$$

where

$$\mathcal{E}_{\epsilon,+}(\mathcal{U},\omega)=\mathcal{E}_{(+)}(\mathcal{U},\omega)+\frac{1}{2}\sum_{k=1}^{l-1}\epsilon\int_{\mathcal{S}}\left\{|\nabla_g^3\omega_{t^k}|^2-(n-1)|\nabla_g^2\omega_{t^k}|^2\right\};$$
$$\mathcal{D}_{\epsilon,+}(\mathcal{U},\omega)=\mathcal{D}_{(+)}(\mathcal{U},\omega)+\sum_{k=1}^{l-1}\epsilon\int_{\mathcal{S}}\left\{|\nabla_g^3\omega_{t^{k+1}}|^2-(n-1)|\nabla_g^2\omega_{t^{k+1}}|^2\right\}.$$

Setting

$$\begin{aligned}\mathcal{U}=\partial_{t^k}u^{m+1},\ \omega=R_{t^k}^{m+1},\ \chi=R_{t^k}^m,\ \psi=R^m,\ \bar{\mathcal{U}}=\bar{u}_k^m,\ \tilde{\mathcal{U}}=\tilde{u}_k^m,\ \bar{\mathcal{V}}=\bar{V}_k^m,\\ \tilde{\mathcal{V}}=\tilde{v}_k^m,\ G=G_k^m,\ h=h_k^m,\ F\equiv 1\ ,\ \mathcal{F}\equiv 0\end{aligned} \tag{3.295}$$

the identity (2.266) implies:

$$\mathcal{E}_{(+)}^{m+1}(t)+\int_0^t\mathcal{D}^{m+1}(s)\,ds=\mathcal{E}_{\epsilon,+}(0)+\int_0^t\int_{\Gamma^m}\Big\{O^m+P^m\Big\}+\int_0^t\int_{\mathcal{S}}\Big\{Q^m+S^m\Big\}. \tag{3.296}$$

Here $O^m:=\sum_{k=1}^{l-1}O_k^m$, $P^m=\sum_{k=0}^{l-1}P_k^m$, $Q^m=\sum_{k=1}^{l-1}Q_k^m$ and $S^m=\sum_{k=0}^{l-1}S_k^m$, whereby

$$O_k^m:=O(\partial_{t^k}u^{m+1}),\qquad P_k^m:=P(\partial_{t^k}u^{m+1}) \tag{3.297}$$

and

$$Q_k^m:=Q(R_{t^k}^m,R_{t^k}^{m+1},R^m),\qquad S_k^m:=S(R_{t^k}^m,R_{t^k}^{m+1},R^m). \tag{3.298}$$

On the level of iterates, for any $j\in\mathbb{N}$ we define the 0-th order energy by

$$\begin{aligned}\mathcal{E}_{(0)}^j=&\frac{1}{2}\int_{\Omega^{j-1}}(\mathbf{P}u^j)^2+\frac{\zeta}{2}\Big(\int_{\Omega^{j-1}}u^j\Big)^2+\frac{1}{2}\int_{\mathcal{S}}\left\{|\nabla_g f^j|^2-(n-1)|\mathbb{P}f^j|^2\right\}\\&+\frac{1}{2}\epsilon\int_{\mathcal{S}}\Big\{|\nabla_g^3 f^j|^2-(n-1)|\nabla_g^2 f^j|^2\Big\},\end{aligned}$$

$$\mathcal{D}_{(0)}^j=\int_{\Omega^{j-1}}|\nabla u^j|^2.$$

Remark. Note that $\mathcal{E}^j$ is slightly different from $\mathcal{E}_{(0)}$.

3.1.1. *The* 0*-th order energy identity.* To derive the 0-th order energy identity we multiply the equation (3.277) by u^{m+1} and integrate over Ω^m. We obtain

$$\frac{1}{2}\partial_t\int_{\Omega^m}(u^{m+1})^2+(n-1)\int_{\Omega^m}u_t^{m+1}+\int_{\mathcal{S}}H^m f_t^{m+1}(R^m)^{n-1}-\epsilon\int_{\Gamma^m}\kappa^m\Lambda^{m+1}+\int_{\Omega^m}|\nabla u^{m+1}|^2=0. \tag{3.299}$$

We apply the ϵ-dependent part of the formulas (2.273) and (2.274) to evaluate $-\epsilon\int_{\Gamma^m}\kappa^m\Lambda^{m+1}$. With $\chi=f^m$, $\omega=f^{m+1}$, $F\equiv 1$, $\psi=R^m$, $g=g^m$, $G=N^*(f^m)$ and $h\equiv 0$, we obtain

$$
\begin{aligned}
-\epsilon\int_{\Gamma^m}\kappa^m\Lambda^{m+1} &= -\epsilon\int_{\mathcal{S}} H^m\frac{\Delta_g^2 f_t^{m+1}}{(R^m)^{n-2}|g^m|}(R^m)^{n-2}|g^m| \\
&= \frac{1}{2}\partial_t\epsilon\int_{\mathcal{S}}\Big\{|\nabla_g^3 f^k|^2-(n-1)|\nabla_g^2 f^k|^2\Big\}-\int_{\mathcal{S}}A^m,
\end{aligned}
\tag{3.300}
$$

where

$$
\begin{aligned}
A^m =& -\epsilon\Delta_g f_t^{m+1}\Delta_g(f^{m+1}-f^m)+\epsilon\Delta_g\nabla_g f_t^{m+1}\cdot\Delta_g\nabla_g(f^{m+1}-f^m) \\
&+\epsilon\Delta_g f^m\Delta_g^2 f_t^{m+1}\Big(\frac{1}{R^m|g^m|}-1\Big)+\epsilon N^*(f^m)\Delta_g^2 f_t^{m+1};
\end{aligned}
\tag{3.301}
$$

the quadratic remainder term $N^*(f)$ is defined in the line after (3.291). We expand

$$
H^m(R^m)^{n-1}=n-1+(n-1)(n-2)f^m-\Delta_g f^m+N_1(f^m),
$$

where $N_1(h)=O(|h|^2+|\nabla_g h|^2+|\nabla_g^2 h|^2)$. Hence, using (3.293), we have

$$
\begin{aligned}
&(n-1)\int_{\Omega^m}u_t^{m+1}+\int_{\mathcal{S}}H^m f_t^{m+1}(R^m)^{n-1}=-(n-1)\int_{\mathcal{S}}f_t^{m+1}-(n-1)^2\int_{\mathcal{S}}f_t^{m+1}f^m \\
&-\sum_{k=2}^{n-1}\int_{\mathcal{S}}\binom{n-1}{k}(f^m)^k f_t^{m+1}+\int_{\mathcal{S}}\{(n-1)+(n-1)(n-2)f^m-\Delta_g f^m+N_1(f^m)\}f_t^{m+1} \\
&=\int_{\mathcal{S}}\{-(n-1)f^m f_t^{m+1}+\nabla_g f^m\cdot\nabla_g f_t^{m+1}\}+\int_{\mathcal{S}}\{N_1(f^m)-\sum_{k=2}^{n-1}\int_{\mathcal{S}}\binom{n-1}{k}(f^m)^k\}f_t^{m+1}
\end{aligned}
$$

Using the previous identity, we obtain from (3.299):

$$
\begin{aligned}
&\frac{1}{2}\partial_t\{\int_{\Omega^m}(u^{m+1})^2-(n-1)\int_{\mathbb{S}^1}(f^{m+1})^2\Big\}+\frac{1}{2}\partial_t\int_{\mathcal{S}}|\nabla_g f^{m+1}|^2 \\
&+\frac{1}{2}\partial_t\epsilon\int_{\mathcal{S}}\Big\{|\nabla_g^3 f^k|^2-(n-1)|\nabla_g^2 f^k|^2\Big\}+\int_{\Omega^m}|\nabla u^{m+1}|^2 \\
&\quad=\int_{\mathcal{S}}\{-(n-1)(f^{m+1}-f^m)f_t^{m+1}+\nabla_g(f^{m+1}-f^m)\cdot\nabla_g f_t^{m+1}\} \\
&\quad-\int_{\mathcal{S}}\{N_1(f^m)-\sum_{k=2}^{n-1}\int_{\mathcal{S}}\binom{n-1}{k}(f^m)^k\}f_t^{m+1}+\int_{\mathcal{S}}A^m.
\end{aligned}
\tag{3.302}
$$

We now make the fundamental calculation:

(3.303)
$$\int_{\Omega^m}(u^{m+1})^2-(n-1)\int_{\mathcal{S}}(f^{m+1})^2=\int_{\Omega^m}(\mathbf{P}u^{m+1})^2\,dx+|\Omega^m|(\mathbf{P}_0u^{m+1})^2$$
$$-(n-1)\Big(\int_{\mathcal{S}}(\mathbb{P}f^{m+1})^2+|\mathcal{S}|(\mathbb{P}_0f^{m+1})^2\Big)=\int_{\Omega^m}(\mathbf{P}u^{m+1})^2\,dx-(n-1)\int_{\mathcal{S}}(\mathbb{P}f^{m+1})^2$$
$$+\Big(\frac{1}{|\Omega^m|}-\frac{n-1}{|\mathcal{S}|}\Big)\Big(\int_{\Omega^m}u^{m+1}\Big)^2+\frac{n-1}{|\mathcal{S}|}\Big(\Big(\int_{\Omega^m}u^{m+1}\Big)^2-\Big(\int_{\mathcal{S}}f^{m+1}\Big)^2\Big).$$

Using (3.302) and (3.303), we arrive at

$$\partial_t\mathcal{E}_0^{m+1}+\mathcal{D}_{(0)}^{m+1}=A(u^{m+1},f^m,f^{m+1}), \tag{3.304}$$

where

(3.305)
$$A(u^{m+1},f^m,f^{m+1}):=-\frac{n-1}{2|\mathcal{S}|}\partial_t\Big\{\Big(\int_{\Omega^m}u^{m+1}\Big)^2-\Big(\int_{\mathcal{S}}f^{m+1}\Big)^2\Big\}+\int_{\mathcal{S}}\Big\{-(n-1)(f^{m+1}-f^m)f_t^{m+1}$$
$$+\nabla_g(f^{m+1}-f^m)\cdot\nabla_g f_t^{m+1}\Big\}+\Big\{\Big\{N_1(f^m)-\sum_{k=2}^{n-1}\int_{\mathcal{S}}\binom{n-1}{k}(f^m)^k\Big\}f_t^{m+1}\Big\}+\int_{\mathcal{S}}A^m.$$

Upon integrating (3.304) in time and summing it with (3.296), we arrive at the fundamental energy identity:

$$\begin{aligned}\mathcal{E}_\epsilon^{m+1}(t)+\int_0^t\mathcal{D}_\epsilon^{m+1}(\tau)\,d\tau&=\mathcal{E}_\epsilon(0)+\int_0^t A(u^{m+1},f^m,f^{m+1})\\&+\int_0^t\int_{\Gamma^m}\Big\{O^m+P^m\Big\}+\int_0^t\int_{\mathcal{S}}\Big\{Q^m+S^m\Big\}.\end{aligned} \tag{3.306}$$

3.2. Energy estimates. In this section we prove the basic a-priori estimates for the regularized Stefan problem. In order to bound the higher-order space-time energy, we first introduce the necessary notations. We denote $\mathfrak{E}_q^{j,\nu}:=\mathfrak{E}_q^{\nu}(u^j,f^j)$, $\hat{\mathfrak{E}}_q^{j,\nu}:=\hat{\mathfrak{E}}_q^{\epsilon,\nu}(u^j,f^j)$ and $\mathfrak{E}_{\epsilon,q}^j:=\mathfrak{E}_q^j+\hat{\mathfrak{E}}_q^j$ for any $j\in\mathbb{N}$, where in the definitions (2.248) and (**??**) Ω is substituted by Ω^j. Analogously we define $\mathfrak{D}_q^{j,\nu}$, $\hat{\mathfrak{D}}_q^{j,\nu}$ and $\mathfrak{D}_{\epsilon,q}^j$. For any $j\in\mathbb{N}$ we finally set $\mathfrak{E}^{j,\nu}=\sum_{q=0}^{l-1}\nu^q\mathfrak{E}_q^j$ and we define $\hat{\mathfrak{E}}^{j,\nu}$ analogously. Summing the ϵ-independent energy $\mathfrak{E}^j$ and the ϵ-dependent energy $\hat{\mathfrak{E}}^j$, we define $\mathfrak{E}_\epsilon^{j,\nu}:=\mathfrak{E}^{j,\nu}+\hat{\mathfrak{E}}^{j,\nu}$. In a fully analogous fashion, we introduce $\mathfrak{D}^{j,\nu}$, $\hat{\mathfrak{D}}^{j,\nu}$ and $\mathfrak{D}_\epsilon^{j,\nu}$. We follow the notational convention of dropping the superscript ν when $\nu=1$. Let us set

$$E_{\alpha,\nu}^j=\mathcal{E}_\epsilon^{j,\nu}+\alpha\mathfrak{E}_\epsilon^{j,\nu};\quad D_{\alpha,\nu}^j=\mathcal{D}_\epsilon^{j,\nu}+\alpha\mathfrak{D}_\epsilon^{j,\nu}.$$

LEMMA 3.2. *Let $\zeta := \frac{1}{|\Omega|} - \frac{n-1}{|\mathcal{S}|} > 0$. Then there exist positive constants α, ν, C_a, C_A, θ_0 and t^ϵ, such that for any $\theta, \theta_1 < \theta_0$, $t \le t^\epsilon$ such that if $E_{\alpha,\nu}(0) < \theta/2$ and*

$$\sum_{k=1}^{l-1} ||\mathbb{P}_0 f^m_{t^k}||_{L^2} + \sum_{k=1}^{l-1} ||\mathbb{P}_1 f^m_{t^k}||^2_{L^2} \le C_a \mathcal{E}^m, \quad \sum_{k=1}^{l} ||\mathbb{P}_0 f^m_{t^k}||_{L^2} + \sum_{k=1}^{l} ||\mathbb{P}_1 f^m_{t^k}||^2_{L^2} \le C_a \mathcal{D}^m,$$

$$E^m_{\alpha,\nu}(t) + \int_0^t D^m_{\alpha,\nu}(\tau)\, d\tau \le \theta, \quad \sup_{0\le s\le t} ||f^m||^2_{H^{2l}} \le C_A(\theta_1 + \int_0^t \mathfrak{D}^{m,\nu})$$

then

(3.307)

$$\sum_{k=1}^{l-1} ||\mathbb{P}_0 f^{m+1}_{t^k}||_{L^2} + \sum_{k=1}^{l-1} ||\mathbb{P}_1 f^{m+1}_{t^k}||^2_{L^2} \le C_a \mathcal{E}^{m+1}, \quad \sum_{k=1}^{l} ||\mathbb{P}_0 f^{m+1}_{t^k}||_{L^2} + \sum_{k=1}^{l} ||\mathbb{P}_1 f^{m+1}_{t^k}||^2_{L^2} \le C_a \mathcal{D}^{m+1},$$

(3.308)

$$E^{m+1}_{\alpha,\nu}(t) + \int_0^t D^{m+1}_{\alpha,\nu}(\tau)\, d\tau \le \theta \quad \textit{and} \quad \sup_{0\le s\le t} ||f^{m+1}||^2_{H^{2l}} \le C_A(\theta_1 + \frac{t}{\epsilon^2}\int_0^t \mathfrak{D}^{m+1,\nu}).$$

Remark. For any $j \in \mathbb{N}$, any Sobolev exponent $k \in \mathbb{N} \cup \{0\}$ and a function $v : \Omega^j \to \mathbb{R}$, we shall denote the Sobolev norm $||v||_{H^k(\Omega^{j,\pm})}$ simply by $||v||_{H^k(\Omega^j)}$ (recall that $||\cdot||_{H^k(\Omega^\pm)}$ is defined in (1.213)).

3.2.1. *Proof of (3.307) (zero-th and the first modes) and second claim of (3.308).* Let p_i be a homogeneous polynomials of degree 1 uniquely associated to the spherical harmonic s_i, $i = 1, \ldots, n$. In other words: $s_i = \mathrm{Tr}|_{\mathbb{S}^1}(p_i)$. Multiply (3.277) by $p_i \circ \mathcal{S}^m$ and integrate over Ω. Here $\mathcal{S}^m : \Omega \to \Omega$ is a diffeomorphism with the property $\mathcal{S}^m|_{\Gamma^m} = (\phi^m)^{-1}$.

$$\int_{\mathcal{S}} (R^m)^{n-1} f^{m+1}_t s_i + \epsilon \Delta^2_{\mathcal{B}} f^{m+1}_t s_i \, d\xi = \int_\Omega u^{m+1}_t p_i \circ \mathcal{S}^m + \int_\Omega \nabla u^{m+1} \cdot \nabla\big(p_i \circ \mathcal{S}^m\big).$$

Hence

(3.309)

$$(1+\epsilon)\int_{\mathcal{S}} f^{m+1}_t s_i \, d\xi = -\int_{\mathcal{S}} \big((R^m)^{n-1} - 1\big) f^{m+1}_t s_i \, d\theta + \int_\Omega u^{m+1}_t p_i \circ \mathcal{S}^m + \int_\Omega \nabla u^{m+1} \cdot \nabla\big(p_i \circ \mathcal{S}^m\big).$$

Let $1 \le k \le l$ and apply the differential operator $\partial_{t^{k-1}}$ to both sides of (3.309). Applying the Leibniz product rule and the Cauchy-Schwarz inequality, we arrive at:

(3.310)

$$\Big|\int_{\mathcal{S}} f^{m+1}_{t^k} s_i \, d\xi\Big| \le C \sum_{j=0}^{k-1} \big\{ ||u^{m+1}_{t^{j+1}}||_{L^2(\Omega^m)} + ||\nabla u^{m+1}_{t^j}||_{L^2(\Omega^m)} \big\} + C \sum_{j'=0}^{k-1} ||f^m_{t^{j'}}||_{L^2} \sum_{j=1}^{k} ||f^{m+1}_{t^j}||_{L^2}.$$

Observe that, due to the assumed bound on $\int_0^t D^m_{\alpha,\nu}$, we have $\sup_{0\le s\le t}||f^m||^2_{H^{2l}} \le C_A(\theta+\frac{\theta}{\alpha\nu^{l-1}})$. Hence,

(3.311)
$$\sum_{j'=0}^{k-1}||f^m_{t^{j'}}||_{L^2} \le ||(\mathbb{P}_0+\mathbb{P}_1)f^m||_{L^2}+\sum_{j=1}^{k-1}||(\mathbb{P}_0+\mathbb{P}_1)f^m_{t^j}||_{L^2}+(\mathcal{E}^m)^{1/2}$$
$$\le CC_A^{1/2}\Big(\big(\frac{\theta}{\alpha\nu^{l-1}}\big)^{1/2}+\theta^{1/2}\Big)+C_a^{1/2}\theta^{1/2}+\theta^{1/2}=\Big(C\big(\frac{C_A}{\alpha\nu^{l-1}}\big)^{1/2}+1+C_a^{1/2}\Big)\theta^{1/2}=:\mu.$$

The estimates (3.310) and (3.311) give

(3.312)
$$\Big|\int_{\mathbb{S}^1} f^{m+1}_{t^k}s_i\,d\xi\Big| \le C\sum_{j=0}^{k-1}\big\{||u^{m+1}_{t^{j+1}}||_{L^2(\Omega^m)}+||\nabla u^{m+1}_{t^j}||_{L^2(\Omega^m)}\big\}+C\mu\sum_{j=1}^{k}||f^{m+1}_{t^j}||_{L^2}.$$

We next use the conservation law (3.292). Namely, applying the differential operator $\partial_{t^{k-1}}$ to (3.292), we have

$$\partial_{t^{k-1}}\Big(\int_\Omega u^{m+1}_t\Big)+\int_{\mathcal{S}} f^{m+1}_{t^k}=-\int_{\mathcal{S}}\partial_{t^{k-1}}\big(f^{m+1}_t((R^m)^{n-1}-1)\big). \tag{3.313}$$

To extract the highest order contribution from the first term on LHS of (3.313) we observe the following differentiation rule for a function $\mathcal{U}$ defined on some moving domain $\Omega(t)$:

$$\partial_t\int_{\Omega(t)}\mathcal{U}=\int_{\Omega(t)}\mathcal{U}_t+\int_{\partial\Omega(t)}V\mathcal{U}, \tag{3.314}$$

where V stands for the normal velocity of the moving boundary $\partial\Omega(t)$. Using (3.314) $(k-1)$ times consecutively, it is easy to prove the formula

(3.315)
$$\partial_{t^{k-1}}\Big(\int_{\Omega^m}u^{m+1}_t\Big)=\int_\Omega u^{m+1}_{t^k}+\sum_{q=1}^{k-1}\partial_{t^{k-1-q}}\Big(\int_{\Gamma^m}V_{\Gamma^m}[u^{m+1}_{t^q}]^+_-\Big)=\int_\Omega u^{m+1}_{t^k}+B_k(V_{\Gamma^m},u^{m+1}),$$

where $B_k(V_{\Gamma^m},u^{m+1}):=\sum_{q=1}^{k-1}\partial_{t^{k-1-q}}\Big(\int_{\Gamma^m}V_{\Gamma^m}[u^{m+1}_{t^q}]^+_-\Big)$. Hence, from (3.313) and (3.315), we obtain

$$\int_\Omega u^{m+1}_{t^k}+\int_{\mathcal{S}} f^{m+1}_{t^k}=-\int_{\mathcal{S}}\partial_{t^{k-1}}\big(f^{m+1}_t((R^m)^{n-1}-1)-B_k(V_{\Gamma^m},u^{m+1})=-B^m_k, \tag{3.316}$$

where $B_k^m := \int_{\mathcal{S}} \partial_{t^{k-1}}\big(f_t^{m+1}((R^m)^{n-1}-1)+B_k(V_{\Gamma^m},u^{m+1})$. Hence, using (3.316), we obtain

$$
\begin{aligned}
&\Big(\int_\Omega u_{t^k}^{m+1}\Big)^2-\Big(\int_{\mathcal{S}} f_{t^k}^{m+1}\Big)^2=\Big(\int_\Omega u_{t^k}^{m+1}-\int_{\mathcal{S}} f_{t^k}^{m+1}\Big)\Big(\int_\Omega u_{t^k}^{m+1}+\int_{\mathcal{S}} f_{t^k}^{m+1}\Big)\\
&=-B_k^m\Big(2\int_\Omega u_{t^k}^{m+1}+B_k^m\Big)=-2B_k^m\int_\Omega u_{t^k}^{m+1}-(B_k^m)^2.
\end{aligned}
\tag{3.317}
$$

By the product rule and the Cauchy-Schwarz inequality:

$$
\sum_{k=1}^{l}\Big|\int_{\mathcal{S}}\partial_{t^{k-1}}\big(f_t^{m+1}((R^m)^{n-1}-1)\Big|\le C\sum_{k=1}^{l}\sum_{j'=0}^{k-1}||f_{t^{j'}}^m||_{L^2}\sum_{j=1}^{k}||f_{t^j}^{m+1}||_{L^2}\le C\mu\sum_{j=1}^{l}||f_{t^j}^{m+1}||_{L^2}.
\tag{3.318}
$$

In an analogous fashion we show

$$
|B_k|\le C\sum_{p=1}^{l-1}||f_{t^p}^m||_{L^2}\sum_{j=1}^{k-1}||u_{t^j}^{m+1}||_{H^1(\Omega^m)}\le C\mu\sum_{j=1}^{k-1}||u_{t^j}^{m+1}||_{H^1(\Omega^m)}.
\tag{3.319}
$$

Furthermore, from (3.316) we obtain for any $1\le k\le l$

$$
||\mathbb{P}_0 f_{t^k}^{m+1}||_{L^2}\le C||u_{t^k}^{m+1}||_{L^2(\Omega)}+C\Big|\int_{\mathcal{S}}\partial_{t^{k-1}}\big(f_t^{m+1}((R^m)^{n-1}-1)+B_k\Big|.
\tag{3.320}
$$

Now we may use (3.318), (3.319) and the definition of B_j^m to conclude

$$
\sum_{k=1}^{l}|B_k^m|\le\sum_{k=1}^{l}\Big|\int_{\mathcal{S}}\partial_{t^{k-1}}\big(f_t^{m+1}((R^m)^{n-1}-1)\Big|+\sum_{k=1}^{l}|B_k|\le C\mu\sum_{k=1}^{l}||f_{t^k}^{m+1}||_{L^2}+C\mu\sum_{k=1}^{l-1}||u_{t^k}^{m+1}||_{H^1(\Omega^m)}.
$$

Using this observation together with (3.317), we conclude that

$$
\begin{aligned}
&\sum_{k=1}^{l}\Big|\Big(\int_\Omega u_{t^k}^{m+1}\Big)^2-\Big(\int_{\mathcal{S}} f_{t^k}^{m+1}\Big)^2\Big|\le\sum_{k=1}^{l}\Big\{\lambda||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+C(1+\frac{1}{\lambda})|B_k^m|^2\Big\}\le\lambda\sum_{k=1}^{l}||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2\\
&+C\mu^2\Big(\sum_{k=1}^{l}||f_{t^k}^{m+1}||_{L^2}^2+\sum_{k=1}^{l-1}||u_{t^k}^{m+1}||_{H^1(\Omega^m)}^2\Big).
\end{aligned}
\tag{3.321}
$$

Analogously we prove $(l\to l-1)$

$$
\sum_{k=1}^{l-1}\Big|\Big(\int_\Omega u_{t^k}^{m+1}\Big)^2-\Big(\int_{\mathcal{S}} f_{t^k}^{m+1}\Big)^2\Big|\le C(\lambda+\nu^2)\sum_{k=1}^{l-1}\Big\{||u_{t^k}^{m+1}||_{H^1(\Omega^m)}^2+||f_{t^k}^{m+1}||_{L^2}^2\Big\}.
\tag{3.322}
$$

Note that (3.318), (3.319) and (3.320) in particular imply

(3.323)
$$\sum_{j=1}^{l}||\mathbb{P}_0 f_{t^j}^{m+1}||_{L^2}^2 \leq C\sum_{j=1}^{l}\Big\{||u_{t^j}^{m+1}||_{L^2(\Omega^m)}^2+||\mathbb{P}_1 f_{t^j}^{m+1}||_{L^2}^2+||\mathbb{P}_{2+}f_{t^j}^{m+1}||_{L^2}^2\Big\}+C\sum_{j=1}^{l-1}||u_{t^j}^{m+1}||_{H^1(\Omega^m)}^2.$$

Namely, using the smallness of θ and ν, together with (3.323), we obtain from (3.310):

(3.324)
$$\sum_{k=1}^{l}||\mathbb{P}_1 f_{t^k}^{m+1}||_{L^2(\mathbb{S}^1)}^2 \leq C\sum_{k=1}^{l}\Big\{||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2+||\nabla u_{t^{k-1}}^{m+1}||_{L^2(\Omega^m)}^2+\mu||\mathbb{P}_{2+}f_{t^k}^{m+1}||_{L^2}^2\Big\}.$$

Positive definiteness. In the following we wish to prove that $\mathcal{E}^k$ is a positive definite quantity. Note that for any $k\geq 1$:

(3.325)
$$\begin{aligned}
&\int_{\Omega^m}(u_{t^k}^{m+1})^2-(n-1)\int_{\mathcal{S}}(f_{t^k}^{m+1})^2\\
&=\int_{\Omega}(\mathbf{P}u_{t^k}^{m+1})^2\,dx+|\Omega|(\mathbf{P}_0 u_{t^k}^{m+1})^2-(n-1)\int_{\mathcal{S}}(\mathbb{P}f_{t^k}^{m+1})^2-(n-1)|\mathcal{S}|(\mathbb{P}_0 f_{t^k}^{m+1})^2\\
&=\int_{\Omega}(\mathbf{P}u_{t^k}^{m+1})^2\,dx-(n-1)\int_{\mathcal{S}}(\mathbb{P}f_{t^k}^{m+1})^2+\zeta\Big(\int_{\Omega}u_{t^k}^{m+1}\Big)^2+\frac{n-1}{|\mathcal{S}|}\Big(\Big(\int_{\Omega}u_{t^k}^{m+1}\Big)^2-\Big(\int_{\mathcal{S}}f_{t^k}^{m+1}\Big)^2\Big),
\end{aligned}$$

where, recall, $\zeta=\frac{1}{|\Omega|}-\frac{n-1}{|\mathcal{S}|}$. A fundamental observation is that the Wirtinger inequality implies:

$$\int_{\mathbb{S}^1}|\omega_{\theta t^k}|^2-(n-1)\int_{\mathcal{S}}|\mathbb{P}\omega_{t^k}|^2\geq C_W||\mathbb{P}_{2+}\omega_{\theta t^k}||_{L^2(\mathcal{S})}^2.$$

Even though this would suffice to establish positivity of the energies $\mathcal{E}^k$, $\mathcal{D}^k$, it features only the second and higher-order modes of ω on the right-hand side. However, the inequality (3.310) gives the obvious bound on the L^2-norm of the first modes of $f_{t^k}^{m+1}$. Let $\eta>0$ be a small number to be specified later. Using the definition (3.294) of $\mathcal{E}^m$, $\mathcal{D}^m$, and combining (3.325), (3.321) and (3.324), we have

$$\begin{aligned}
(1+\eta)\mathcal{D}^{m+1}\geq\sum_{k=1}^{l}\Big\{&(1+\eta)\Big[\int_{\Omega}(\mathbf{P}u_{t^k}^{m+1})^2+\int_{\Omega}|\nabla u_{t^{k-1}}^{m+1}|^2+\zeta\Big(\int_{\Omega}u_{t^k}^{m+1}\Big)^2+\epsilon C_W||\mathbb{P}_{2+}f_{\theta^3 t^k}^{m+1}||_{L^2}^2\\
&-C\lambda||u_{t^k}^{m+1}||_{L^2(\Omega^m)}^2-C\mu^2\Big(||u_{t^k}^{m+1}||_{L^2(\Omega)}^2+||\nabla u_{t^{k-1}}^{m+1}||_{L^2}^2+||\mathbb{P}_1 f_{t^k}^{m+1}||_{L^2}^2+||\mathbb{P}_{2+}f_{t^k}^{m+1}||_{L^2}^2\Big)\Big]\\
&+C_W||\mathbb{P}_{2+}f_{\theta t^k}^{m+1}||_{L^2}^2+\eta||f_{\theta t^k}^{m+1}||_{L^2}^2-\eta C\sum_{j=1}^{k}\Big\{||u_{t^j}^{m+1}||_{L^2(\Omega^m)}^2+||\nabla u_{t^{j-1}}^{m+1}||_{L^2(\Omega^m)}^2+\nu||\mathbb{P}_{2+}f_{t^j}^{m+1}||_{L^2}^2\Big\}.
\end{aligned}$$

After regrouping, we obtain

$$(1+\eta)\mathcal{D}^{m+1}+C\sum_{k=1}^{l}\Big\{\mu^2\Big(||u^{m+1}_{t^k}||^2_{L^2(\Omega^m)}+||\nabla u^{m+1}_{t^{k-1}}||^2_{L^2(\Omega^m)}+||\mathbb{P}_1 f^{m+1}_{t^k}||^2_{L^2}+||\mathbb{P}_{2+}f^{m+1}_{t^k}||^2_{L^2}\Big)$$
$$+C\lambda\sum_{k=1}^{l}||u^{m+1}_{t^k}||^2_{L^2(\Omega^m)}+\eta\sum_{j=1}^{k}\Big\{||u^{m+1}_{t^j}||^2_{L^2(\Omega^m)}+||\nabla u^{m+1}_{t^{j-1}}||^2_{L^2(\Omega^m)}+\mu||\mathbb{P}_{2+}f^{m+1}_{t^j}||^2_{L^2}\Big\}\Big\}$$
$$\geq C'\sum_{k=1}^{l}\Big\{\epsilon C_W||\mathbb{P}_{2+}f^{m+1}_{\theta^3 t^k}||^2_{L^2}+||u^{m+1}_{t^k}||^2_{L^2(\Omega^m)}+||\nabla u^{m+1}_{t^{k-1}}||^2_{L^2(\Omega^m)}+\eta||f^{m+1}_{\theta t^k}||^2_{L^2}+C_W||\mathbb{P}_{2+}f^{m+1}_{\theta t^k}||^2_{L^2}\Big\}.$$

First choosing η and then λ, θ, μ and ϵ small, we obtain

$$\mathcal{D}^{m+1}\geq C''\sum_{k=1}^{l}\Big\{||u^{m+1}_{t^k}||^2_{L^2(\Omega)}+||\nabla u^{m+1}_{t^{k-1}}||^2_{L^2(\Omega)}+||f^{m+1}_{\theta t^k}||^2_{L^2}+\epsilon||\mathbb{P}_{2+}f^{m+1}_{\theta^3 t^k}||^2_{L^2}\Big\} \tag{3.326}$$

Let us denote

$$\mathcal{D}_{\|}(u,f):=\sum_{k=1}^{l}\Big\{||u_{t^k}||^2_{L^2(\Omega^m)}+||\nabla u_{t^{k-1}}||^2_{L^2(\Omega^m)}+||f_{\theta t^k}||^2_{L^2}+\epsilon||f_{\theta^3 t^k}||^2_{L^2}\Big\}$$

Using (3.326) together with (3.323) and the definition (3.294) of $\mathcal{D}^j$, we conclude that there exists a constant $C>0$ such that

$$\frac{1}{C}\mathcal{D}_{\|}(u^{m+1},f^{m+1})\leq\mathcal{D}^{m+1}\leq C\mathcal{D}_{\|}(u^{m+1},f^{m+1}). \tag{3.327}$$

Similarly we introduce

$$\mathcal{E}_{\|}(u,f):=\sum_{k=0}^{l-1}\Big\{||u_{t^k}||^2_{L^2(\Omega^m)}+||\nabla u_{t^k}||^2_{L^2(\Omega^m)}\Big\}+||\mathbb{P}_0 f||^2_{L^2}+||\mathbb{P}_2 f||^2_{L^2}+\sum_{k=1}^{l-1}\Big\{||f_{\theta t^k}||^2_{L^2}+\epsilon||f_{\theta^3 t^k}||^2_{L^2}\Big\} \tag{3.328}$$

A proof analogous to the proof of (3.326), which uses (3.322) instead of (3.321) leads to

$$\mathcal{E}^{m+1}\geq\mathcal{E}^{m+1}_{(0)}+C_3\sum_{k=1}^{l-1}\Big\{||u^{m+1}_{t^k}||^2_{L^2(\Omega^m)}+||\nabla u^{m+1}_{t^k}||^2_{L^2(\Omega^m)}+||f^{m+1}_{\theta t^k}||^2_{L^2}+\epsilon||\mathbb{P}_{2+}f^{m+1}_{\theta^3 t^k}||^2_{L^2}.\Big\} \tag{3.329}$$

Like above we conclude that there exists a constant $C>0$ such that

$$\frac{1}{C}\mathcal{E}_{\|}(u^{m+1},f^{m+1})\leq\mathcal{E}^{m+1}\leq C\mathcal{E}_{\|}(u^{m+1},f^{m+1}). \tag{3.330}$$

With θ small enough and C_a larger if necessary, from (3.323) and (3.324) and bounds (3.326) and (3.329) we immediately deduce the second and the first claim of (3.307) respectively.

Proof of the second claim of (3.308). Note that the equation (3.280) can be rewritten in the form:

$$f_t^{m+1}+\epsilon\Delta_{\mathcal{B}}f_t^{m+1}=|g^m|(R^m)^{n-2}[u_n^{m+1}]_-^+ - f_t^{m+1}\big((R^m)^{n-1}-1\big). \tag{3.331}$$

Let α be a multi-index and $|\alpha|\leq 2l-4$. Apply the differential operator ∂_g^α to the equation (3.331), take the square, integrate over $\mathcal{S}$ and then integrate in time. We readily obtain

$$\begin{aligned}&\int_0^t\Big(||\partial_g^\alpha f_t^{m+1}||_{L^2}^2+2\epsilon||\nabla_g^2\partial_g^\alpha f_t^{m+1}||_{L^2}^2+\epsilon^2||\nabla_g^4\partial_g^\alpha f_t^{m+1}||_{L^2}^2\Big)\\&\leq 2\int_0^t\int_{\mathcal{S}}\big[\partial_g^\alpha\big((R^m)^{n-2}|g^m|[u_n^{m+1}]_-^+\big)\big]^2+2\int_0^t\int_{\mathcal{S}}\big[\partial_g^\alpha f_t^{m+1}\big((R^m)^{n-1}-1\big)\big]^2\end{aligned}$$

Using the Leibniz rule, L^2-L^∞ type estimates and the trace inequality, we easily conclude that there exists a constant C_1, such that

$$\epsilon^2||\nabla_g^4\partial_g^\alpha f_t^{m+1}||_{L^2}^2\leq C_1\sup_{0\leq s\leq t}||f^m||_{H^{2l-4}}^2\int_0^t\mathfrak{D}^{m+1}. \tag{3.332}$$

On the other hand, since $\partial_t||\partial_g^\alpha f^{m+1}||_{L^2}\leq||\partial_g^\alpha f_t^{m+1}||_{L^2}$, using the smallness assumptions in Lemma 3.2, we easily deduce

$$\epsilon^2||\nabla_g^4\partial_g^\alpha f^{m+1}(t)||_{L^2}^2\leq\epsilon^2||\nabla_g^4\partial_g^\alpha f^{m+1}(0)||_{L^2}^2+C_1C_A(\theta_1+\theta)t\int_0^t\mathfrak{D}^{m+1}.$$

Thus, using the above for any $0\leq|\alpha|\leq 2l-4$, assuming that $||f_0||_{H^{2l}}^2\leq K\theta_1$ for some constant $K>0$ and choosing θ,θ_1 such that $(\theta+\theta_1)C_1, K\theta_1\leq 1$ suitably small, we obtain

$$||f^{m+1}||_{H^{2l}}^2\leq C_A\big(\theta_1+\frac{t}{\epsilon^2}\int_0^t\mathfrak{D}^{m+1}\big) \tag{3.333}$$

and this proves the second claim of (3.308). If we apply the operator ∂_g^α with $|\alpha|\leq 2l-2$ to the equation (3.331), in the same way as above we get (with no ϵ!)

$$||f^{m+1}||_{H^{2l-2}}^2\leq C_A\big(\theta_1+t\int_0^t\mathfrak{D}^{m+1}\big) \tag{3.334}$$

Note that the same proof also gives the bound

$$||f^{m+1}||^2_{H^{2l-2}} \leq C_A\big(\theta_1 + t^2 \sup_{0\leq s\leq t} \mathfrak{E}^{m+1}(s)\big). \tag{3.335}$$

3.2.2. *Temporal energy estimates.* We start by an auxiliary lemma that will be subsequently used in the energy estimates below.

LEMMA 3.3. *The following inequalities hold:*

(a) *For any* $0 \leq k \leq l-1$ *the following inequalities hold:*

$$\int_0^t ||[\gamma^m, \partial_{t^{k+1}}]u^{m+1}||^2_{L^2(\Gamma^m)} \leq C \sup_{0\leq s\leq t} \sqrt{\mathfrak{E}^m(s)} \int_0^t \mathfrak{D}^{m+1} + \int_0^t \mathfrak{D}^m \sup_{0\leq s\leq t} \mathfrak{E}^{m+1}(s)$$

$$\int_0^t ||[\gamma^m, \partial_{t^k}]u^{m+1}||^2_{H^1(\Gamma^m)}, \int_0^t ||[\gamma^m, \partial_{t^k}]u_n^{m+1}||^2_{L^2(\Gamma^m)} \leq C \sup_{0\leq s\leq t} \sqrt{\mathfrak{E}^m(s)} \int_0^t \mathfrak{D}^{m+1}.$$

(b) *For any* $0 \leq k \leq l-1$*, the following inequality holds (*∂_t^* *is defined in (2.253)):*

$$\int_0^t ||\partial^*_{t^k} V^{m+1}||^2_{L^2(\Gamma^m)} \leq C \sup_{0\leq s\leq t} (1 + \sqrt{\mathfrak{E}^m(s)}) \int_0^t \mathcal{D}^{m+1}.$$

(c) *Let* $\xi \in C^\infty(J,\mathbb{R})$ *and* $J \subseteq \mathbb{R}$ *an interval such that every derivative of* ξ *is uniformly bounded on* J*. Let* $0 < r < l$*. Then there exists a positive constant* C *such that*

$$||\partial_{t^r}\left[\xi((R^m)^2 + (f^m_\theta)^2)\right]||_{H^1} \leq C\big(\sqrt{\mathcal{E}^m} + \sqrt{\mathfrak{E}^m}\big) \tag{3.336}$$

and

$$||\partial_{t^r}\left[\xi((R^m)^2 + (f^m_\theta)^2)\right]||_{H^3} \leq \frac{C}{\sqrt{\epsilon}}\big(\sqrt{\mathcal{E}^m} + \sqrt{\mathfrak{E}^m}\big). \tag{3.337}$$

(d) *For any* $k = 1,\ldots,l$*, the following inequality holds:*

$$||f^{m+1}_{t^{k+1}}||^2_{L^2} + 2\epsilon||\nabla^2_g f^{m+1}_{t^{k+1}}||^2_{L^2} + \epsilon^2||\nabla^4_g f^{m+1}_{t^{k+1}}||^2_{L^2} \leq \bar{\eta}\mathfrak{D}^{m+1} + \frac{Ct}{\bar{\eta}}\mathcal{E}^{m+1}$$

Recall the formula (2.256):

$$\left[\gamma, \partial_{t^k}\right](u) \circ \phi := \sum_{p=0}^{k-1} \sum_{q+r=k-p} C^{p,q,r} \partial_{t^{k-1-p}}((V \circ \phi^m)^q R^r_t) \xi^{i_1} \ldots \xi^{i_r} \partial_g^{i_1} \ldots \partial_g^{i_r} \partial_n^q u_{t^p} \circ \phi^m.$$

Note that for any $1 \leq p \leq l-1$ $||\partial_{t^{l-1-p}}((V \circ \phi^m)^q R^r_t)||_{L^2} \leq C\sqrt{\mathfrak{E}^m}$ and for any $q, r \in \mathbb{N}_0$ such that $q + r = l - p$, we have

$$||\partial_{s^r}\partial_{n^q} u^{m+1}_{t^p}||_{L^\infty(\Gamma^m)} \leq C||\nabla u^{m+1}_{t^p}||_{H^{l-p}(\Gamma^m)} \leq C\sqrt{\mathfrak{D}^{m+1}},$$

where we use the trace inequality and the fact that $l-p+1\leq 2(l-1-p)+2$ (observe (1.232)). If $p=0$, then $||\partial_{t^{l-1-p}}((V\circ\phi^m)^q R_t^r)||_{L^2}\leq C\mathfrak{E}^m\sqrt{\mathfrak{D}^m}$ and from the trace inequality, (1.230) and the fact that $q+r=l-p$:

$$||\partial_{s^r}\partial_{n^q}u^{m+1}||_{L^\infty(\Gamma^m)}\leq C||u^{m+1}||_{H^{l+n/2+1}(\Gamma^m)}\leq C\sqrt{\mathfrak{E}^{m+1}},$$

where we note that $||u^{m+1}||_{H^{l+n/2+1}(\Gamma^m)}\leq C\sqrt{\mathfrak{E}^{m+1}}$ if $n/2\leq l$ (observe (1.230)). Combining the above inequalities, we deduce

$$\int_0^t ||[\gamma^m,\partial_{t^{k+1}}]u^{m+1}||^2_{L^2(\Gamma^m)}\leq C\sup_{0\leq s\leq t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathfrak{D}^{m+1}+\int_0^t\mathfrak{D}^m\sup_{0\leq s\leq t}\mathfrak{E}^{m+1}(s)$$

and this proves the first claim of part (a) of Lemma 3.3. The other two claims are proven similarly. As to the part (b), fix $0\leq k\leq l-1$. Then

$$\int_0^t ||\partial_{t^k}\big(V^{m+1}\circ\phi^m\big)||^2_{L^2}\leq C\sum_{q=0}^k\int_0^t ||f^{m+1}_{t^{q+1}}\big(\frac{R^m}{|g^m|}\big)_{t^{k-q}}||^2_{L^2}$$
$$\leq C\sum_{q=0}^k\sup_{0\leq s\leq t}||\big(\frac{R^m}{|g^m|}\big)_{t^{k-q}}||^2_{L^\infty}\int_0^t ||f^{m+1}_{t^{q+1}}||^2_{L^2}\leq C(1+\sup_{0\leq s\leq t}\sqrt{\mathfrak{E}^m(s)})\int_0^t\mathcal{D}^{m+1}$$

and this finishes the proof of part (b). Note that we used the Leibniz rule in the first inequality, and then standard L^∞-L^2 type inequality, combined with the Sobolev inequality. Part (c) follows using similar techniques and the definition (1.228) of $\mathfrak{E}$ and $\mathfrak{D}$. This proof is in fact analogous to the proof of part (a) of Lemma 3.3 in [**31**]. Part (d) follows by applying ∂_{t^k} to (3.331), taking squares and integrating. On the right hand side we apply the trace estimate and interpolation inequality to conclude.

$\square$

Our goal is to estimate RHS of the energy identity (3.374). The following lemma renders the desired inequalities:

LEMMA 3.4. *The following inequalities hold:*

(a)

$$\Big|\int_0^t A(u^{m+1},f^m,f^{m+1})\Big|\leq\lambda\big(\int_0^t\hat{\mathcal{D}}^{m+1}+\int_0^t\mathcal{D}^{m+1}\big)+\frac{C_0t}{\lambda}m_0^2$$
$$+\frac{C_0C_At}{\lambda}\big(\theta_1+\int_0^t\mathfrak{D}^m+\int_0^t\mathfrak{D}^{m+1}+\sup_{0\leq s\leq t}(\mathfrak{E}^{m+1}(s))^2\big)$$

(b)

$$
\begin{aligned}
&\Big|\int_0^t\int_{\Gamma^m}\{O^m+P^m\}+\int_0^t\int_{\mathcal{S}}\{Q^m+S^m\}\Big|\\
&\le C\sup_{0\le s\le t}\big(\sqrt{\mathfrak{E}^m}(s)+\sqrt{\hat{\mathfrak{E}}^m(s)}+\sqrt{\mathcal{E}^m(s)}\big)\Big(\int_0^t\mathfrak{D}^{m+1}+\int_0^t\hat{\mathfrak{D}}^{m+1}+\int_0^t\mathcal{D}^m_\epsilon+\int_0^t\mathcal{D}^{m+1}_\epsilon\Big)\\
&+\lambda\Big(\int_0^t\mathcal{D}^m+\int_0^t\mathcal{D}^{m+1}\Big)+\frac{C}{\lambda}\Big(\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathfrak{D}^{m+1}+\sup_{0\le s\le t}\mathfrak{E}^{m+1}(s)\int_0^t\mathfrak{D}^m\\
&+\sup_{0\le s\le t}||\mathbb{P}_1f^m(s)||^2_{L^2}\int_0^t\mathfrak{D}^m\Big)+C\sup_{0\le s\le t}||f^m(s)||_{H^l}\Big(\int_0^t\mathcal{D}^m_\epsilon+\int_0^t\mathcal{D}^{m+1}_\epsilon\Big)+^1
\end{aligned}
$$

Proof. Part (a). Recall that the expression A is defined by (3.305). In order to deal with the first term on RHS of (3.305), we first use the conservation law (3.292) to expand that expression:

(3.338)

$$
\begin{aligned}
&\frac{1}{2}\partial_t\Big\{\big(\int_{\Omega^m}u^{m+1}\big)^2-\big(\int_{\mathbb{S}^1}f^{m+1}\big)^2\Big\}=\big(\int_{\Omega^m}u_t^{m+1}+\int_{\mathbb{S}^1}f_t^{m+1}\big)\big(\int_{\Omega^m}u^{m+1}-\int_{\mathbb{S}^1}f^{m+1}\big)\\
&+\big(\int_{\Omega^m}u^{m+1}+\int_{\mathbb{S}^1}f^{m+1}\big)\big(\int_{\Omega^m}u_t^{m+1}-\int_{\mathbb{S}^1}f_t^{m+1}\big)=-\int_{\mathcal{S}}\big((R^m)^{n-1}-1\big)f_t^{m+1}\times\big(\int_{\Omega^m}u^{m+1}-\int_{\mathbb{S}^1}f^{m+1}\big)\\
&+\Big(m_0-\int_{\mathcal{S}}\sum_{k=2}^n\frac{1}{n}\binom{n}{k}(f^{m+1})^k+\int_0^t\int_{\mathcal{S}}f_t^{m+1}\big((R^{m+1})^{n-1}-(R^m)^{n-1}\big)\Big)\Big(\int_{\Omega^m}u_t^{m+1}-\int_{\mathbb{S}^1}f_t^{m+1}\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\Big|\int_0^t\partial_t\Big\{\big(\int_{\Omega^m}u^{m+1}\big)^2-\big(\int_{\mathbb{S}^1}f^{m+1}\big)^2\Big\}\Big|\le\lambda\int_0^t\int_{\mathbb{S}^1}(f_t^{m+1})^2+\lambda\int_0^t\int_{\Omega^m}(u_t^{m+1})^2\\
&+\frac{C_0t}{\lambda}\sup_{0\le s\le t}\big(m_0^2+||u^{m+1}||^2_{L^2(\Omega)}+||f_t^{m+1}||^2_{L^2}+\sum_{k=1}^n\int_{\mathcal{S}}\{(f^m)^{2k}+(f^{m+1})^{2k}\}\big).
\end{aligned}
\tag{3.339}
$$

We estimate the higher powers of f^m and f^{m+1} via the relation [2]:

$$
\begin{aligned}
\int_{\mathcal{S}}(f^{m+1})^{2k}&\le C||f^{m+1}||^{2k}_{L^\infty}\le C||f^{m+1}||^{2k}_{H^{n/2}}\\
&\le C_A^k\big(\theta_1+t^2\sup_{0\le s\le t}\mathfrak{E}^{m+1}(s)\big)^k\le CC_A^k\big(\theta_1+\sup_{0\le s\le t}\mathfrak{E}^{m+1}(s)^k\big),
\end{aligned}
$$

where we generously applied the formalism $\theta^m=\theta$ for a small constant θ and exploited the assumption $t<1$. Similarly, $\int_{\mathcal{S}}(f^m)^{2k}\le CC_A^k\big(\theta_1+\sup_{0\le s\le t}\mathfrak{E}^m(s)^k\big)$. We finally

[2] fill in the reference

conclude

$$
\begin{aligned}
(3.340)\qquad & \Big|\int_0^t \partial_t\Big\{\big(\int_{\Omega^m} u^{m+1}\big)^2-\big(\int_{\mathbb{S}^1} f^{m+1}\big)^2\Big\}\Big| \\
&\quad \le \lambda\int_0^t \mathcal{D}^{m+1}+\frac{C_0 t}{\lambda}\Big(CC_A^k\big(\theta_1+\sum_{k=1}^n \sup_{0\le s\le t}\big\{\mathfrak{E}^m(s)^k+\mathfrak{E}^{m+1}(s)^k\big\}\big)\Big)
\end{aligned}
$$

The second integral on RHS of (3.305) is a cross-term and is easily estimated as follows:

$$
\begin{aligned}
(3.341)\qquad & \Big|\int_0^t\int_{\mathcal{S}}\Big\{-(n-1)(f^{m+1}-f^m)f_t^{m+1}+\nabla_g(f^{m+1}-f^m)\cdot\nabla_g f_t^{m+1}\Big\}\Big| \\
&=\Big|\int_0^t\int_{\mathcal{S}}\Big\{-((n-1)f^{m+1}+\Delta_g f^{m+1})f_t^{m+1}+((n-1)f^m+\Delta_g f^m)f_t^{m+1}\Big\}\Big| \\
&\le\lambda\int_0^t ||f_t^{m+1}||_{H^1}^2+\frac{C_0}{\lambda}\int_0^t\{||(I-\mathbb{P}_1)f^m||_{H^1}^2+||(I-\mathbb{P}_1)f^{m+1}||_{H^1}^2\} \\
&\le\lambda\int_0^t\mathcal{D}^{m+1}+\frac{C_0 t}{\lambda}C_A\big(2\theta_1+\int_0^t\mathfrak{D}^m+\int_0^t\mathfrak{D}^{m+1}\big),
\end{aligned}
$$

where we used the second claim of[3] and the assumption that $t<1$. As to the third integral on RHS of (3.305), from the assumptions of Lemma 3.2, we have

$$
\begin{aligned}
(3.342)\qquad & \int_0^t\int_{\mathcal{S}}\{N_1(f^m)-\sum_{k=2}^{n-1}\int_{\mathcal{S}}\binom{n-1}{k}(f^m)^k\}f_t^{m+1}\le\lambda\int_0^t\int_{\mathcal{S}}(f_t^{m+1})^2 \\
&+\frac{C_0 t}{\lambda}\sup_{0\le s\le t}\{||N_1(f^m)||_{L^2}^2+\sum_{k=2}^{n-1}||(f^m)^k||_{L^2}^2\}\le\lambda\int_0^t\mathcal{D}^{m+1}+\frac{C_0C_At}{\lambda}\big(\theta_1+\int_0^t\mathfrak{D}^m\big)^2.
\end{aligned}
$$

The last on RHS of (3.305) is $\int_{\mathcal{S}}A_m$, which is given by (3.301). The first two terms on RHS of (3.301) are the cross-terms and are estimated analogously to (3.341):

$$
\begin{aligned}
(3.343)\qquad & \Big|\int_0^t\int_{\mathcal{S}}-\epsilon\Delta_g f_t^{m+1}\Delta_g(f^{m+1}-f^m)+\epsilon\Delta_g\nabla_g f_t^{m+1}\cdot\Delta_g\nabla_g(f^{m+1}-f^m)\Big|\le\lambda\epsilon\int_0^t||f_{\theta^3 t}^{m+1}||_{L^2}^2 \\
&+\frac{C_0}{\lambda}\big(\epsilon\int_0^t||(I-\mathbb{P}_1)f^m||_{H^3}^2+\epsilon\int_0^t||(I-\mathbb{P}_1)f^{m+1}||_{H^3}^2\big)\le\lambda\Big(\int_0^t\hat{\mathcal{D}}^{m+1}+\epsilon\int_0^t\mathcal{D}^{m+1}\Big) \\
&+\frac{C_0C_At}{\lambda}\big(\theta_1+\int_0^t\mathfrak{D}^m+\int_0^t\mathfrak{D}^{m+1}\big).
\end{aligned}
$$

[3]Fill in the reference

As to the third term on RHS of (3.301), we start by integrating by parts:

(3.344)

$$\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g f^m\Delta_g^2 f_t^{m+1}\big(\frac{1}{R^m|g^m|}-1\big)\Big|\leq\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g\nabla_g f^m\cdot\Delta_g\nabla_g f_t^{m+1}\big(\frac{1}{R^m|g^m|}-1\big)\Big|\\
&+\Big|\int_0^t\int_{\mathcal{S}}\Delta_g f^m\Delta_g\nabla_g f_t^{m+1}\nabla_g\big(\frac{1}{R^m|g^m|}\big)\Big|\leq\lambda\epsilon\int_0^t||\Delta_g\nabla_g f_t^{m+1}||_{L^2}^2\\
&+\frac{C_0t}{\lambda}\epsilon\sup_{0\leq s\leq t}\big(||\frac{1}{R^m|g^m|}-1||_{L^\infty}^2||\nabla_g^3 f^m||_{L^2}^2\big)\\
&+\lambda\epsilon\int_0^t||\nabla_g^3 f_t^{m+1}||_{L^2}^2+\frac{C_0t}{\lambda}\epsilon\sup_{0\leq s\leq t}\big(||\nabla_g\big(\frac{1}{R^m|g^m|}\big)||_{L^\infty}^2||\nabla_g^2 f^m||_{L^2}^2\big)\\
&\leq\lambda\big(\int_0^t\hat{\mathcal{D}}^{m+1}+\epsilon\int_0^t\mathcal{D}^{m+1}\big)+\frac{C_0t}{\lambda}\sup_{0\leq s\leq t}\big(||\mathbb{P}f^m||_{H^{n/2+1}}^2\epsilon||\mathbb{P}\nabla_g^2 f^m||_{H^1}^2\big)\\
&\leq\lambda\big(\int_0^t\hat{\mathcal{D}}^{m+1}+\epsilon\int_0^t\mathcal{D}^{m+1}\big)+\frac{C_0t}{\lambda}C_A\big(\theta_1+\int_0^t\mathfrak{D}^m\big),
\end{aligned}$$

where we recall the definitions (2.241) and (1.227) of $\hat{\mathcal{D}}^\epsilon$ and $\mathcal{D}$ respectively. In the last estimate we used the smallness assumption on $||f^m||_{H^{2l}}$ of Lemma 3.2. Finally, the last term on RHS of (3.301) can be estimated analogously to (3.344) and we obtain:

(3.345)

$$\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}N^*(f^m)\Delta_g^2 f_t^{m+1}\Big|\leq\lambda\int_0^t\mathcal{D}^{m+1}+\frac{C_0t}{\lambda}\sup_{0\leq s\leq t}||f^m||_{H^{n/2+1}}C_A\big(\theta_1+\int_0^t\mathfrak{D}^m\big).$$

Summing the estimates (3.340) - (3.345), we get the claim of the part (a) of Lemma 3.4.

Proof of part (b). Part (b) follows by systematically applying the energy estimates, similarly to the part (a). These estimates are analogous to the ones from Section 3.2.3 in [**32**].

Estimating $\int_0^t\int_{\Gamma^m}O_k^m$. Recall that O_k^m is defined by (3.297) and (2.268):

$$O_k^m=-[\gamma^m,\partial_{t^k}]u^{m+1}[\partial_n u_{t^k}^{m+1}]_-^+-\partial_{t^k}^*u^{m+1}[\gamma^m,\partial_{t^k}][u_n^{m+1}]_-^++V_{\Gamma^m}(u_{t^k}^{m+1})^2. \tag{3.346}$$

In order to estimate the first term on RHS of (3.346), we apply the relation (2.255) to write

$$[\partial_n u_{t^k}^{m+1}]_-^+=\partial_{t^k}^*[u_n^{m+1}]_-^+-[\gamma^m,\partial_{t^k}][u_n^{m+1}]_-^+.$$

We next note that from (2.244)

(3.347)
$$\Big|\int_0^t\int_{\Gamma^m}[\gamma^m,\partial_{t^k}]u^{m+1}[\gamma^m,\partial_{t^k}][u_n^{m+1}]_-^+\Big|\le\int_0^t\|[\gamma^m,\partial_{t^k}]u^{m+1}\|_{L^2(\Gamma^m)}\|[\gamma^m,\partial_{t^k}][u_n^{m+1}]_-^+\|_{L^2(\Gamma^m)}$$
$$\le\int_0^t\|[\gamma^m,\partial_{t^k}]u^{m+1}\|^2_{L^2(\Gamma^m)}+\int_0^t\|[\gamma^m,\partial_{t^k}][u_n^{m+1}]_-^+\|^2_{L^2(\Gamma^m)}\le C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathfrak{D}^{m+1},^4$$

Furthermore, using (3.279) yields:

(3.348)
$$\Big|\int_0^t\int_{\Gamma^m}[\gamma^m,\partial_{t^k}]u^{m+1}\partial^*_{t^k}[u_n^{m+1}]_-^+\Big|=\Big|\int_0^t\int_{\Gamma^m}[\gamma^m,\partial_{t^k}]u^{m+1}\partial^*_{t^k}\big(V^{m+1}+\epsilon\Lambda^{m+1}\big)\Big|$$
$$\le\frac{C}{\lambda}\int_0^t\|[\gamma^m,\partial_{t^k}]u^{m+1}\|^2_{L^2(\Gamma^m)}+\lambda\int_0^t\|\partial^*_{t^k}V^{m+1}\|^2_{L^2(\Gamma^m)}$$
$$+\epsilon\Big|\int_0^t\int_{\mathcal{S}}[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m\partial_{t^k}\big(\frac{\Delta_g^2 f_t^{m+1}}{|g^m|}(R^m)^{n-2}\big)|g^m|\Big|,$$

where Λ^{m+1} is defined in the line after (3.279). The first two terms on right-most side above are easily estimated by

(3.349)
$$\frac{C}{\lambda}\int_0^t\|[\gamma^m,\partial_{t^k}]u^{m+1}\|^2_{L^2(\Gamma^m)}+\lambda\int_0^t\|\partial^*_{t^k}V^{m+1}\|^2_{L^2(\Gamma^m)}\le C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathfrak{D}^{m+1}.$$

The last term on right-most side of (3.348) is estimated by righting out $\partial_{t^k}\big(\frac{\Delta_g^2 f_t^{m+1}}{|g^m|}(R^m)^{n-2}\big)=\frac{\Delta_g^2 f_{t^{k+1}}^{m+1}}{|g^m|(R^m)^{n-2}}+h^m_{k,2}$ where $h^m_{k,2}$ is given by (3.291). Integrating by parts and using the standard energy estimates, we conclude

(3.350)
$$\epsilon\Big|\int_0^t\int_{\mathbb{S}^1}[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m h^m_{k,2}|g^m|\Big|\le C\sup_{0\le s\le t}\mathfrak{E}^m\int_0^t\mathcal{D}^{m+1}+\int_0^t\|[\gamma^m,\partial_{t^k}]u^{m+1}\|^2_{H^1(\Gamma^m)}$$
$$\le C\sup_{0\le s\le t}\sqrt{\hat{\mathfrak{E}}^m(s)}\int_0^t\hat{\mathcal{D}}^{m+1}+C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathcal{D}^{m+1}.$$

Putting (3.347) - (3.350) together, we immediately obtain:

$$\begin{aligned}\Big|\int_0^t\int_{\Gamma^m}[\gamma^m,\partial_{t^k}]u^{m+1}\partial^*_{t^k}[u_n^{m+1}]_-^+\Big|&\le C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathfrak{D}^{m+1}+C\epsilon\lambda\int_0^t\hat{\mathcal{D}}^{m+1}\\&+C\epsilon^2\lambda\int_0^t\mathcal{D}^{m+1}+C\sup_{0\le s\le t}\sqrt{\hat{\mathfrak{E}}^m(s)}\int_0^t\hat{\mathcal{D}}^{m+1}+C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathcal{D}^{m+1}.\end{aligned}\tag{3.351}$$

Estimating $\int_0^t \int_{\Gamma^m} P_k^m$. Recall that P_k^m is defined by (3.297) and (2.269):

(3.352)
$$P_k^m = \{\partial_t([\gamma^m, \partial_{t^k}]u^{m+1,\pm}) - [\gamma^m, \partial_t]u_{t^k}^{m+1,\pm}\}\partial_n u_{t^k}^{m+1,\pm} + \partial_{t^{k+1}}^* u^{m+1}[\gamma^m, \partial_{t^k}][u_n^{m+1}]_-^+ \\ + V_{\Gamma^m}|\nabla u^{m+1,\pm}|^2.$$

It is, however, easy to see that the expression in the first parenthesis above simplifies, i.e.,

$$\partial_t([\gamma^m, \partial_{t^k}]u^{m+1}) - [\gamma^m, \partial_t]u_{t^k}^{m+1} = [\gamma^m, \partial_{t^{k+1}}]u^{m+1}.$$

The first contribution on RHS of (3.352) is thus easily estimated: Note that

(3.353)
$$\Big|\int_0^t \int_{\Gamma^m} [\gamma^m, \partial_{t^{k+1}}]u^{m+1}\partial_n u_{t^k}^{m+1,\pm}\Big| \le \int_0^t \frac{C}{\lambda}||[\gamma^m, \partial_{t^{k+1}}]u^{m+1}||^2_{L^2(\Gamma^m)} + \lambda \int_0^t ||\partial_n u_{t^k}^{m+1,\pm}||^2_{L^2(\Gamma^m)} \\ \le \frac{C}{\lambda} \sup_{0\le s\le t} \sqrt{\mathfrak{E}^m(s)} \int_0^t \mathfrak{D}^{m+1} + \frac{C}{\lambda} \sup_{0\le s\le t} \mathfrak{E}^{m+1} \int_0^t \mathcal{D}^m + C\lambda \int_0^t \mathfrak{D}^{m+1},$$

where we used part (a) of Lemma 3.3 and the trace inequality in the second inequality. To estimate the second contribution on RHS of (3.352), we first recall that $\partial_{t^{k+1}}^* u^{m+1} \circ \phi^m = -f_{t^{k+1}}^m - \frac{\Delta_g f_{t^{k+1}}^m}{R^m|g^m|} + G_{k+1}^m$ (G_{k+1}^m defined by (3.290)). Using integration by parts, Young's inequality, trace inequality, Lemma 3.3 and standard energy estimates, we obtain

(3.354)
$$\Big|\int_0^t \int_{\Gamma^m} \partial_n u_{t^k}^{m+1,\pm} + \partial_{t^{k+1}}^* u^{m+1}[\gamma^m, \partial_{t^k}][u_n^{m+1}]_-^+\Big| \le \lambda \int_0^t \mathcal{D}^m + \frac{C}{\lambda} \sup_{0\le s\le t} \sqrt{\mathfrak{E}^m(s)} \int_0^t \mathcal{D}^{m+1} \\ + C \sup_{0\le s\le t} \sqrt{\mathfrak{E}^m(s)} \int_0^t \mathcal{D}^{m+1} + C \sup_{0\le s\le t} \big(||f^m(s)||_{H^{n/2+2}} + \sqrt{\mathcal{E}^m(s)}\big) \int_0^t \mathcal{D}^m$$

The last term on RHS of (3.352) is quite standard to estimate, and we obtain:

(3.355)
$$\int_0^t \int_{\Gamma^m} V_{\Gamma^m}|\nabla u_{t^k}^{m+1,\pm}|^2 \le C \sup_{0\le s\le t} ||V_{\Gamma^m}||_{L^\infty(\Gamma^m)} \int_0^t ||u_{t^k}^{m+1}||^2_{H^2(\Omega^m)} \\ \le C \sup_{0\le s\le t} \sqrt{\mathfrak{E}^m(s)} \int_0^t \mathfrak{D}^{m+1}.$$

Estimating $\int_0^t \int_{\mathcal{S}} Q_k^m$**.** Recall that Q_k^m is defined by (3.298), where Q is given by (2.270) with (3.295). The first four terms on RHS of (2.270) are the cross-terms, first two of which is estimated as follows:

(3.356)
$$\begin{aligned}
&\Big|\int_0^t\int_{\mathcal{S}} f^{m+1}_{t^{k+1}}(f^{m+1}_{t^k}-f^m_{t^k})\Big|+\Big|\int_0^t\int_{\mathcal{S}}\nabla_g f^{m+1}_{t^{k+1}}\cdot\nabla_g(f^{m+1}_{t^k}-f^m_{t^k})\Big|\\
&\le\lambda\int_0^t ||f^m_{t^k}||^2_{H^1}+\frac{\bar{C}}{\lambda}\int_0^t\big(||f^{m+1}_{t^{k+1}}||^2_{H^1}+||f^{m+1}_{t^k}||^2_{H^1}\big)\\
&\le\lambda\int_0^t\mathcal{D}^m+\lambda t\sup_{0\le s\le t}||f^m(s)||^2_{L^2}+\frac{\bar{C}}{\lambda}\big(t\sup_{0\le s\le t}||f^{m+1}(s)||^2_{H^1}+\eta\int_0^t\mathfrak{D}^{m+1}+\frac{Ct}{\epsilon\eta}\sup_{0\le s\le t}\mathcal{E}^{m+1}(s)\big)\\
&\le\lambda\int_0^t\mathcal{D}^m+tC_A\big(1+\frac{\bar{C}}{\lambda}\big)\sup_{0\le s\le t}\big(\theta_1^2+\mathfrak{E}^{m+1}(s)+\mathfrak{E}^m(s)\big)+\frac{\bar{C}\eta}{\lambda}\int_0^t\mathfrak{D}^{m+1}+\frac{\bar{C}t}{\epsilon\eta\lambda}\sup_{0\le s\le t}\mathcal{E}^{m+1}(s),
\end{aligned}$$

where we used part (d) of Lemma 3.3 in the second estimate (with $\bar{\eta}=\epsilon\eta$) and the inequality (3.335) in the last line to estimate $\sup_{0\le s\le t}\big(||f^m(s)||_{H^j},||f^{m+1}(s)||_{H^j}\big)$. Similarly we estimate the third and the fourth cross-term to obtain:

(3.357)
$$\begin{aligned}
&\Big|\epsilon\int_0^t\int_{\mathcal{S}}\Delta_{\mathcal{B}}f^{m+1}_{t^{k+1}}\Delta_{\mathcal{B}}(f^{m+1}_{t^k}-f^m_{t^k})\Big|+\Big|\epsilon\int_0^t\int_{\mathcal{S}}\Delta_{\mathcal{B}}\nabla_g f^{m+1}_{t^{k+1}}\cdot\Delta_{\mathcal{B}}\nabla_g(f^{m+1}_{t^k}-f^m_{t^k})\Big|\\
&\le\lambda\int_0^t\mathcal{D}^m+\frac{\bar{C}C_At}{\lambda}\sup_{0\le s\le t}\big(\theta_1^2+\mathfrak{E}^{m+1}(s)+\mathfrak{E}^m(s)\big)+\frac{\eta}{\lambda}\int_0^t\mathfrak{D}^{m+1}+\frac{\bar{C}t}{\epsilon^2\eta}\sup_{0\le s\le t}\mathcal{E}^{m+1}(s),
\end{aligned}$$

where we use part (d) of Lemma 3.3 with $\bar{\eta}=\eta\epsilon^2$.The fifth, sixth, seventh and eighth term are estimated in a standard way, via integration by parts and $L^\infty-L^2-L^2$ type estimates, to obatin (recall $F\equiv 1$):

$$\begin{aligned}
&\int_0^t\big|f^m_{t^k}f^{m+1}_{t^{k+1}}((R^m)^{n-1}-1)\big|+\int_0^t\big|\Delta_g f^m_{t^k}f^{m+1}_{t^{k+1}}(\frac{(R^m)^{n-2}}{|g^m|}-1)\big|\\
&+\epsilon\int_0^t\big|f^m_{t^k}\Delta_g^2f^{m+1}_{t^{k+1}}((R^m)^{n-2}-1)\big|+\epsilon\int_0^t\big|\Delta_g f^m_{t^k}\Delta_g^2f^{m+1}_{t^{k+1}}(\frac{(R^m)^{n-3}}{|g^m|}-1)\big|\\
&\le C\sup_{0\le s\le t}\big(||f^m(s)||_{H^{n/2+2}}+\sqrt{\mathfrak{E}^m(s)}\big(\int_0^t\mathcal{D}^m+\int_0^t\mathcal{D}^{m+1}\big).
\end{aligned}\tag{3.358}$$

Note that next-to-last term of $\int_0^t\int_{\mathbb{S}^1}Q^m_k$, given by (3.298) and (2.270) (with (3.295) and (3.283)), involves the term $h^m_k=h^m_{k,1}+\epsilon h^m_{k,2}$, given by (3.291). By the trace decomposition,$\bar{u}^m_k=\partial^*_{t^k}u^{m+1}=u^{m+1}_{t^k}-[\gamma^m,\partial_{t^k}]u^{m+1}$. Using this, the usual $L^\infty-L^2-L^2$ type estimates and part (a) of Lemma 3.3, we establish

(3.359)
$$\Big|\int_0^t\int_{\mathcal{S}}h^m_{k,1}\bar{u}^m_k\circ\phi^m(R^m)^{n-2}|g^m|\Big|\le C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathcal{D}^{m+1}+C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t\mathfrak{D}^{m+1},$$

In the same way

$$
\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathcal{S}} h_{k,2}^m \bar{u}_k^m \circ \phi^m (R^m)^{n-2}|g^m|\Big| \\
&\le C\epsilon\int_0^t ||h_{k,2}^m||_{L^2}||u_{t^k}^{m+1}||_{L^2(\Gamma^m)} + \epsilon\Big|\int_0^t\int_{\mathcal{S}} h_{k,2}^m[\gamma^m,\partial_{t^k}]u^{m+1}\circ\phi^m (R^m)^{n-2}|g^m|\Big| \\
&\le C\sup_{0\le s\le t}\sqrt{\hat{\mathfrak{E}}^m(s)}\int_0^t \hat{\mathcal{D}}^{m+1} + C\sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\int_0^t \mathcal{D}^{m+1}.
\end{aligned}
\tag{3.360}
$$

Combining (3.359) and (3.360), we obtain

$$
\Big|\int_0^t\int_{\mathcal{S}} h_k^m \bar{u}_k^m \circ\phi^m (R^m)^{n-2}|g^m|\Big| \le C\big(\sup_{0\le s\le t}\sqrt{\mathfrak{E}_\epsilon^m(s)} + \sup_{0\le s\le t}\sqrt{\mathcal{E}^m(s)}\big)\Big(\int_0^t \mathcal{D}^{m+1} + \int_0^t \mathfrak{D}^{m+1}\Big).
\tag{3.361}
$$

The last term of $\int_0^t\int_{\mathbb{S}^1} Q_k^m$, given by (3.298) and (2.270), involves the term G_k^m, which is defined by (3.290) with (3.295). The following estimate holds:

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathcal{S}} G_k^m\big(R^m f_{t^{k+1}}^{m+1} + \epsilon\Delta_g^2 f_{t^{k+1}}^{m+1}\big)(R^m)^{n-2}\Big| \\
&\le C\big(\sup_{0\le s\le t}||f^m||_{H^{n/2+3}} + \sup_{0\le s\le t}\sqrt{\mathfrak{E}^m(s)}\big)\Big(\int_0^t \mathcal{D}_\epsilon^m + \int_0^t \mathcal{D}_\epsilon^{m+1}\Big).
\end{aligned}
\tag{3.362}
$$

The proof of (3.362) is analogous to its 2-dimensional analog from [**32**], Subsection 3.2.1. Again, it is rather lengthy and technical, relying on the standard energy estimates.

Estimating $\int_0^t\int_{\mathcal{S}} S_k^m$. Recall that S_k^m is given by (3.298) and (2.271) (with (3.295)). The first four terms on RHS of (2.271) are the cross-terms:

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathcal{S}} f_{t^{k+1}}^{m+1}(f_{t^{k+1}}^{m+1} - f_{t^{k+1}}^m)\Big| \le \lambda\int_0^t ||f_{t^{k+1}}^m||_{L^2}^2 + \frac{\bar{C}}{\lambda}\int_0^t ||f_{t^{k+1}}^{m+1}||_{L^2}^2 \\
&\le \lambda\int_0^t \mathcal{D}^m + \frac{\bar{C}}{\lambda}\int_0^t \big(\eta||u_{t^k}^{m+1}||_{H^2(\Omega^m)}^2 + \frac{C}{\eta}||u_{t^k}^{m+1}||_{H^1(\Omega^m)}^2\big) \\
&\le \lambda\int_0^t \mathcal{D}^m + \frac{\bar{C}\eta}{\lambda}\int_0^t \mathfrak{D}^{m+1} + \frac{\bar{C}}{\eta\lambda}t\sup_{0\le s\le t}\mathcal{E}^{m+1},
\end{aligned}
\tag{3.363}
$$

where we used part (d) of Lemma 3.3 and the trace inequality in the last inequality. Similarly, for the second term, we have

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathcal{S}} \nabla_g(f_{t^{k+1}}^m - f_{t^{k+1}}^{m+1})\cdot\nabla_g f_{t^{k+1}}^{m+1}\Big| \le \eta\int_0^t\int_{\mathcal{S}} |\nabla_g f_{t^{k+1}}^m|^2 + \frac{\bar{C}}{\eta}\int_0^t\int_{\mathcal{S}} |\nabla_g f_{t^{k+1}}^{m+1}|^2 \\
&\le \eta\int_0^t \mathcal{D}^m + \frac{\bar{C}\eta}{\lambda\epsilon}\int_0^t\int_{\Omega^m} |\nabla u_{t^k}^{m+1}|^2 + \frac{\bar{C}}{\eta}\lambda\int_0^t\int_{\Omega^m} |\nabla^2 u_{t^k}^{m+1}|^2.
\end{aligned}
\tag{3.364}
$$

As to the third cross-term on RHS of (2.271), we have

(3.365)

$$\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g f^{m+1}_{t^{k+1}}\Delta_g(f^{m+1}_{t^{k+1}}-f^m_{t^{k+1}})\Big|\leq\lambda\epsilon\int_0^t||\nabla_g^2 f^m_{t^{k+1}}||^2_{L^2}+\frac{\bar{C}\epsilon}{\lambda}\int_0^t||\nabla^2_g f^{m+1}_{t^{k+1}}||^2_{L^2}\\
&\leq\lambda\int_0^t\hat{\mathcal{D}}^m+\frac{\bar{C}}{\epsilon\lambda}\big(\eta||u^{m+1}_{t^k}||^2_{H^2(\Omega^m)}+\frac{C}{\eta}||u^{m+1}_{t^k}||^2_{H^1(\Omega^m)}\big)\\
&\leq\lambda\int_0^t\hat{\mathcal{D}}^m+\frac{\bar{C}\eta}{\epsilon\lambda}\int_0^t\mathfrak{D}^{m+1}+\frac{\bar{C}}{\epsilon\eta\lambda}t\sup_{0\leq s\leq t}\mathcal{E}^{m+1}
\end{aligned}$$

and an analogous estimate for the fourth (cross-)term in (2.271) gives

(3.366)

$$\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g\nabla_g f^{m+1}_{t^{k+1}}\cdot\Delta_g\nabla_g(f^{m+1}_{t^{k+1}}-f^m_{t^{k+1}})\Big|\leq\lambda\int_0^t\hat{\mathcal{D}}^m+\frac{\bar{C}\eta}{\epsilon\lambda}\int_0^t\mathfrak{D}^{m+1}+\frac{\bar{C}}{\epsilon\eta\lambda}t\sup_{0\leq s\leq t}\mathcal{E}^{m+1}.$$

In analogy to the estimate (3.358), we obtain the estimate on the fifth, sixth, seventh and eighth term on RHS of (2.271) (keep in mind that $F\equiv 1$):

(3.367)

$$\begin{aligned}
&\int_0^t\int_{\mathcal{S}}\big|f^m_{t^{k+1}}f^{m+1}_{t^{k+1}}((R^m)^{n-1}-1)\big|+\int_0^t\int_{\mathcal{S}}\Big|\Delta_g f^m_{t^{k+1}}f^{m+1}_{t^{k+1}}\big(\frac{(R^m)^{n-2}}{|g^m|}-1\big)\Big|\\
&+\epsilon\int_0^t\int_{\mathcal{S}}\Big|f^m_{t^{k+1}}\Delta^2_g f^{m+1}_{t^{k+1}}\big((R^m)^{n-2}|g^m|-1\big)\Big|+\epsilon\int_0^t\int_{\mathcal{S}}\Big|\Delta_g f^m_{t^{k+1}}\Delta^2_g f^{m+1}_{t^{k+1}}\big(\frac{(R^m)^{n-3}}{|g^m|}-1\big)\Big|\\
&\leq C\sup_{0\leq s\leq t}||f^m(s)||_{H^n/2+3}\big(\int_0^t\mathcal{D}^{m+1}_\epsilon+\int_0^t\mathcal{D}^m_\epsilon\big).
\end{aligned}$$

The next-to-last term on RHS of $\int_0^t\int_{\mathcal{S}}S^m_k$ involves the expression G^m_k which is given by (3.290) and hence

$$(G^m_k)_t=-\sum_{q=0}^{k-1}\Delta_g f^m_{t^{q+1}}\big(\frac{1}{R^m|g^m|}\big)_{t^{k-q}}-\sum_{q=0}^{k-1}\Delta_g f^m_{t^q}\big(\frac{1}{R^m|g^m|}\big)_{t^{k-q+1}}+\partial_{t^{k+1}}N^*(f^m)$$

Separating the cases $q\leq k/2$ and $q>k/2$ and taking L^2 and L^∞-norms accordingly, we conclude

$$\int_0^t||(G^m_k)_t||^2_{L^2}\leq C(||\mathbb{P}_1 f^m||^2_{L^2}+\sup_{0\leq s\leq t}\mathfrak{E}^m(s))\int_0^t\mathfrak{D}^m. \tag{3.368}$$

With (3.368) in hand, by standard energy estimates we obtain

(3.369)

$$\begin{aligned}
&\int_0^t\int_{\mathcal{S}}\Big(\Delta_g f^m_{t^k}\big(\frac{1}{R^m|g^m|}\big)_t+(G^m_k)_t\Big)\partial^*_{t^k}u^{m+1}\circ\phi^m|g^m|\leq\lambda\int_0^t\mathcal{D}^{m+1}+\lambda\mathfrak{E}^m\int_0^t\mathfrak{D}^{m+1}\\
&+\frac{C}{\lambda}(||\mathbb{P}_1 f^m||^2_{L^2}+\sup_{0\leq s\leq t}\sqrt{\mathfrak{E}^m(s)})\int_0^t\mathfrak{D}^m,
\end{aligned}$$

where we used the trace inequality, part (a) of Lemma 3.3 and the estimate (3.368).

Summing the inequalities (3.351), (3.353) - (3.355), (3.356) - (3.358), (3.361), (3.362), (3.363) - (3.367) and (3.369), we finish the proof of part (b) of Lemma 3.4. $\square$

Let us introduce a small parameter θ_2 such that

$$\sup_{0\leq s\leq t}\big(\mathcal{E}_\epsilon^m(s)+\mathfrak{E}_\epsilon^m(s)\big)+\int_0^t\left\{\mathcal{D}_\epsilon^m(s)+\mathfrak{D}_\epsilon^m(s)\right\}ds\leq\theta_2. \tag{3.370}$$

We shall later express the desired constant θ_0 (from the statement of Lemma 3.2) with the help of θ_2. From the identity (3.306) and Lemma 3.4, we arrive at:

(3.371)

$$\begin{aligned}
&\mathcal{E}_\epsilon^{m+1}+\int_0^t\mathcal{D}_\epsilon^{m+1}\leq\mathcal{E}_\epsilon(0)+\frac{C_0t}{\lambda}\big(m_0^2+\mathcal{E}^m+\mathcal{E}^{m+1}+(\mathfrak{E}^{m+1})^2+C_A(\theta_1+\int_0^t\mathfrak{D}^m+\int_0^t\mathfrak{D}^{m+1})\big)\\
&+\frac{\bar{C}C_At}{\lambda}\sup_{0\leq s\leq t}\big(\theta_1+\mathfrak{E}^{m+1}(s)+\mathfrak{E}^m(s)\big)+\frac{\bar{C}t}{\epsilon^2\eta\lambda}\mathcal{E}_\epsilon^{m+1}+\frac{\bar{C}\eta}{\epsilon\lambda}\int_0^t\mathfrak{D}^{m+1}\\
&+C\big(\sup_{0\leq s\leq t}\big(\sqrt{\mathfrak{E}^m(s)}+\sqrt{\hat{\mathfrak{E}}^m(s)}+||f^m(s)||_{H^{n/2+3}}\big)+\lambda\big)\int_0^t\big(\mathfrak{D}^m+\mathfrak{D}^{m+1}\big)+C\int_0^t\mathcal{D}^m\sup_{0\leq s\leq t}\mathfrak{E}^{m+1}(s)\\
&\leq\mathcal{E}_\epsilon(0)+\frac{C_0t}{\lambda}(m_0^2+\theta_1+\theta_2)+\frac{\bar{C}t}{\lambda}\big(\theta_1+\theta_2\big)+C\big(\lambda+\theta_1+\theta_2\big)\theta_2+\frac{C_0t}{\lambda}\big(\mathcal{E}^{m+1}+(\mathfrak{E}^{m+1})^2\big)\\
&+\frac{\bar{C}t}{\epsilon^2\eta\lambda}\mathcal{E}_\epsilon^{m+1}+\big(C(\lambda+\theta_1^{1/2}+\theta_2^{1/2})+\frac{C_0t}{\lambda}+\frac{\bar{C}\eta}{\epsilon\lambda}\big)\int_0^t\mathfrak{D}^{m+1}+\frac{\bar{C}t}{\lambda}\sup_{0\leq s\leq t}\mathfrak{E}^{m+1}(s)+C\theta_2\sup_{0\leq s\leq t}\mathfrak{E}^{m+1},
\end{aligned}$$

where we used the smallness assumption above. Let us set

$$\mathcal{K}(\theta_2,t):=\frac{C_0t}{\lambda}(m_0^2+\theta_1+\theta_2)+\frac{\bar{C}t}{\lambda}(\theta_1+\theta_2)+C\big(\lambda+\theta_1+\theta_2\big)\theta_2, \tag{3.372}$$

which shortens the notation for the constant term appearing on RHS of (3.371).

3.2.3. *Reduction and representations of $\int_\Gamma\Delta_{\mathcal{B}}vv_n$ and $\int_\Gamma v_tv_n$.* Just to provide intuition, assume for the moment that (u,f) is the solution to the Stefan problem (1.203) - (1.205). The energies $\mathfrak{E}$ and $\mathfrak{D}$ arise by applying the space-time differential operators to the heat equation (1.194). For instance, after applying $\partial_x\partial_t$ to the equation (1.194), multiplying by $-\Delta\partial_x\partial_tu$ and integrating over Ω, we run into the trouble of extracting the boundary energy contribution. The difficulty is contained in the restriction of the spatial differentiation operator ∂_{x^i} to the moving surface Γ. Namely $\partial_{x^i}=\tau_i\partial_s+n_i\partial_n$ on Γ, whereby $|n_i|$ is of order 1 and hence the normal component of ∂_{x^i} can not be assumed small. Thus, the restriction of the operator ∂^μ to Γ inherits non-trivial high order normal derivatives of u. This presents a difficulty in extracting the energy contribution, which is overcome through the *reduction* procedure explained in this subsection. The main idea is to recover the spatial derivatives via the equation in the bulk and the established temporal energy estimate in Subsection 3.2.2. We first decompose the Laplace operator on Γ as $\Delta u|_\Gamma = \Delta_{\mathcal{B}} u + \kappa u_n + u_{nn}$. Here $\Delta_{\mathcal{B}}$ stands for the Laplace-Beltrami operator on Γ (for the definition see (0.588) and the sentence preceeding it). Since $u_t = \Delta u$ in the bulk, we can represent u_{nn} in terms of the u_t (which is controlled by the temporal energy) and $\Delta_{\mathcal{B}} u$ (which helps us extract the boundary energy contribution). We now rigorously justify the above heuristics.

In order to facilitate our arguments, we shall denote the multi-index μ of order $2p$, by an ordered $2p$-tuple $\mu = (i_1, i_2, \dots, i_{2p})$, where $i_k \in \{1, \dots, n\}$ for $k = 1, \dots 2p$. Let $c = l - p - 1$ and set $v = \partial^\mu u^{m+1}_{tc} = \partial_{x^{i_1}} \partial_{x^{i_2}}, \dots \partial_{x^{i_{2p}}} \partial_t^c u^{m+1}$. Choose a multi-index μ such that $|\mu| = 2p$. Applying the differential operator $\partial^\mu \partial_{t^{l-(p+1)}}$ to the equation (1.194) and setting $v = \partial^\mu \partial_{t^{l-(p+1)}} u$, we obtain

$$\partial_t v - \Delta v = 0.$$

Multiplying the above equation by $-\Delta v$ and integrating over Ω^m, we obtain $\int_{\Omega^m} (\Delta v)^2 = \int_{\Omega^m} \partial_t v \Delta v$. On the other hand, using the Stokes theorem, we obtain

$$\int_{\Omega^m} (\Delta v)^2 = \int_{\Omega^m} \nabla^2 v : \nabla^2 v - \int_{\Gamma^m \cup \partial\Omega} v_k v_{kl} n_l + \int_{\Gamma^m \cup \partial\Omega} \Delta v v_n. \tag{3.373}$$

Using the basic decomposition formula $\partial_{x^i} = \partial^i_{\mathbf{g}} + n_i \partial_n$ (cf. (0.584) and (0.585)).

$$\begin{aligned}
&\int_{\Gamma^m \cup \partial\Omega} v_k v_{lk} n_l = \int_{\Gamma^m \cup \partial\Omega} \partial^k_{\mathbf{g}} v v_{lk} n_l + \int_{\Gamma^m \cup \partial\Omega} v_n n_k v_{kl} n_l = \int_{\Gamma^m \cup \partial\Omega} \partial^k_{\mathbf{g}} v \big[\partial^k_{\mathbf{g}}(v_l) n_l\big] + \int_{\Gamma^m \cup \partial\Omega} v_n v_{nn} \\
&= \int_{\Gamma^m \cup \partial\Omega} \partial^k_{\mathbf{g}} v \big[\partial^k_{\mathbf{g}}(v_l n_l) - \partial^l_{\mathbf{g}} v \partial^k_{\mathbf{g}}(n_l)\big] + \int_{\Gamma^m \cup \partial\Omega} v_n \big(\Delta v - \kappa v_n - \Delta_{\mathcal{B}} v\big) \\
&= \textstyle\int_{\Gamma^m \cup \partial\Omega} \Big(\partial^k_{\mathbf{g}} v \partial^k_{\mathbf{g}} v_n - v_n \Delta_{\mathcal{B}} v - \partial^k_{\mathbf{g}} v \partial^l_{\mathbf{g}} v \partial^k_{\mathbf{g}}(n_l) - \kappa (v_n)^2\Big) + \int_{\Gamma^m \cup \partial\Omega} v_n \Delta v \\
&= -2 \int_{\Gamma^m \cup \partial\Omega} v_n \Delta_{\mathcal{B}} v - \int_{\Gamma^m \cup \partial\Omega} \partial^k_{\mathbf{g}} v \partial^l_{\mathbf{g}} v \partial^k_{\mathbf{g}}(n_l) - \int_{\Gamma^m \cup \partial\Omega} \kappa (v_n)^2 + \int_{\Gamma^m \cup \partial\Omega} v_n \Delta v.
\end{aligned}$$

Note that in the fourth equality we used the fact that $v_l \partial^k_{\mathbf{g}}(n_l) = \partial^l_{\mathbf{g}} v \partial^k_{\mathbf{g}}(n_l)$. This holds since

$$v_l \partial^k_{\mathbf{g}}(n_l) = (\partial^l_{\mathbf{g}} v + n_l v_n) \partial^k_{\mathbf{g}}(n_l) = \partial^l_{\mathbf{g}} v \partial^k_{\mathbf{g}}(n_l) + \frac{1}{2} v_n \partial^k_{\mathbf{g}}(|n|^2) = \partial^l_{\mathbf{g}} v \partial^k_{\mathbf{g}}(n_l).$$

In the last equality above, we used the integration-by-parts formula. Further note that

$$-\int_{\Omega^m} v_t \Delta v = \frac{1}{2}\partial_t \int_{\Omega^m} |\nabla v|^2 + \frac{1}{2}\int_{\Gamma^m} V_{\Gamma^m} |\nabla v|^2 - \int_{\Gamma^m} v_t v_n.$$

Hence, going back to (3.373) and the remark before, we obtain

$$\begin{aligned}&\int_{\Omega^m} \nabla^2 v : \nabla^2 v + \frac{1}{2}\partial_t \int_{\Omega^m} |\nabla v|^2 + \int_{\Gamma^m \cup \partial\Omega} \kappa v_n^2 \\ &= \int_{\Gamma^m \cup \partial\Omega} v_t v_n - 2\int_{\Gamma^m \cup \partial\Omega} \Delta_{\mathcal{B}} v v_n - \int_{\Gamma^m \cup \partial\Omega} \partial^i_{\mathbf{g}} v \partial^j_{\mathbf{g}} v \partial^i_{\mathbf{g}}(n_j).\end{aligned}$$

Upon integrating in time over the interval $[0,t]$, we arrive at the fundamental identity

(3.374)

$$\begin{aligned}&\frac{1}{2}\int_{\Omega^m} |\nabla v(t)|^2 + \int_0^t \int_{\Omega^m} |\nabla^2 v|^2 + \int_0^t \int_{\Gamma^m \cup \partial\Omega} \kappa v_n^2 = \frac{1}{2}\int_{\Omega^m} |\nabla v(0)|^2 - 2\int_0^t \int_{\Gamma^m \cup \partial\Omega} \Delta_{\mathcal{B}} v v_n \\ &\quad + \int_0^t \int_{\Gamma^m \cup \partial\Omega} v_t v_n + \int_0^t \int_{\Gamma^m} \partial^i_{\mathbf{g}} v \partial^j_{\mathbf{g}} v \partial^i_{\mathbf{g}}(n_j) - \frac{1}{2}\int_0^t \int_{\Gamma^m} V_{\Gamma^m} |\nabla v|^2\end{aligned}$$

Let $\mathcal{C}$ denote either Γ^m or $\partial\Omega$. Recall again that for $i=1,\dots,n$, the operator ∂_{x^i} on $\mathcal{C}$ can be written as $\partial_{x^i} = \partial^i_{\mathbf{g}} + n_i \partial_n$. Hence, for a given multi-index $\mu = (i_1,\dots,i_{2p})$ of order $2p$, we have

(3.375)

$$\partial^\mu u_{t^c} = (\partial^{i_1}_{\mathbf{g}} + n_{i_1}\partial_n)\dots(\partial^{i_{2p}}_{\mathbf{g}} + n_{i_{2p}}\partial_n) u_{t^c} = \sum_{k=0}^{2p} \sum_{\substack{J \subset \{i_1,\dots,i_{2p}\} \\ |J|=k}} \prod_{j \in J} n_j \partial^{J^{co}}_{\mathbf{g}} \partial_{n^k} u_{t^c} + G_{\alpha,\beta}(g) u_{t^c},$$

where $G_{\alpha,\beta}(g)$ is of the form

$$G_{\alpha,\beta}(g) = \sum_{|\alpha|+\beta<2p} \partial^\alpha_{\mathbf{g}} \partial^\beta_n.$$

As to the first term of on RHS of (3.375), we carry out the crucial technical step, that we will refer to as the *reduction procedure*. Given $b \geq 2$ and $c \in \mathbb{N}$ we note the following:

$$\partial^b_n u_{t^c} = \partial^{b-2}_n \big(\Delta u_{t^c} - \kappa \partial_n u_{t^c} - \Delta_{\mathcal{B}} u_{t^c}\big) = \partial^{b-2}_n u_{t^{c+1}} - \Delta_{\mathcal{B}} \partial^{b-2}_n u_{t^c} - \kappa \partial^{b-1}_n u_{t^c}. \tag{3.376}$$

Using (3.376) carefully, by induction we deduce for b even:

$$\partial_{n^b} u_{t^c} = \sum_{j=0}^{b/2} (-1)^m \binom{b/2}{m} \Delta^m_{\mathcal{B}} u_{t^{c+b/2-m}} + \sum_{|\alpha|+\beta+2\gamma<b} M^b_{\alpha,\beta,\gamma}(\kappa) \partial^\alpha_{\mathbf{g}} \partial^\beta_n u_{t^{c+\gamma}}; \tag{3.377}$$

and for b odd:

$$\partial_{n^b} u_{t^c} = \sum_{m=0}^{\lfloor b/2\rfloor} (-1)^m \binom{\lfloor b/2\rfloor}{m} \Delta_{\mathcal{B}}^m \partial_n u_{t^{c+\lfloor b/2\rfloor - m}} + \sum_{|\alpha|+\beta+2\gamma<b} M^b_{\alpha,\beta,\gamma}(\kappa) \partial^\alpha_{\mathbf{g}} \partial^\beta_n u_{t^{c+\gamma}}.^5 \tag{3.378}$$

Using the relations (3.377) and (3.378), we may write for k even

(3.379)

$$\partial^{J^{co}}_{\mathbf{g}} \partial_{n^k} u_{t^c} = \sum_{m=0}^{k/2} (-1)^m \binom{k/2}{m} \partial^{J^{co}}_{\mathbf{g}} \Delta_{\mathcal{B}}^m u_{t^{c+b/2-m}} + \sum_{|\alpha|+\beta+2\gamma<2p} M^{1,k}_{\alpha,\beta,\gamma}(g) \partial^\alpha_{\mathbf{g}} \partial^\beta_n u_{t^{c+\gamma}};$$

and for k odd:

(3.380)

$$\partial^{J^{co}}_{\mathbf{g}} \partial_{n^k} u_{t^c} = \sum_{m=0}^{\lfloor k/2\rfloor} (-1)^m \binom{\lfloor k/2\rfloor}{m} \partial^{J^{co}}_{\mathbf{g}} \Delta_{\mathcal{B}}^m \partial_n u_{t^{c+\lfloor b/2\rfloor - m}} + \sum_{|\alpha|+\beta+2\gamma<2p} M^{1,k}_{\alpha,\beta,\gamma}(g) \partial^\alpha_{\mathbf{g}} \partial^\beta_n u_{t^{c+\gamma}}.^6$$

Hence, using (3.375), (3.379) and (3.380), we obtain:

(3.381)

$$\begin{aligned}
\partial^\mu u_{t^c} &= \sum_{k\,\text{even}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{k/2} (-1)^m \binom{k/2}{m} \partial^{J^{co}}_{\mathbf{g}} \Delta_{\mathcal{B}}^m u_{t^{c+k/2-m}} \\
&+ \sum_{k\,\text{odd}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{\lfloor k/2\rfloor} (-1)^j \binom{\lfloor k/2\rfloor}{m} \partial^{J^{co}}_{\mathbf{g}} \Delta_{\mathcal{B}}^m \partial_n u_{t^{c+\lfloor k/2\rfloor - m}} + \sum_{|\alpha|+\beta+2\gamma<2p} F_{\alpha,\beta,\gamma}(g) \partial^\alpha_{\mathbf{g}} \partial^\beta_n u_{t^{c+\gamma}},
\end{aligned}$$

where

$$F_{\alpha,\beta,\gamma} = \sum_{k=0}^{2p} \sum_{\substack{J\subset\{i_1,\dots,i_{2p}\}\\|J|=k}} \prod_{j\in J} n_j\, M^{1,k}_{\alpha,\beta,\gamma}\ (\gamma>0); \quad F_{\alpha,\beta,0} = \sum_{k=0}^{2p} \sum_{\substack{J\subset\{i_1,\dots,i_{2p}\}\\|J|=k}} \prod_{j\in J} n_j\, M^{1,k}_{\alpha,\beta,\gamma} + G_{\alpha,\beta}(g).$$

With the use of formula (3.381) we write out the expression for $\Delta_{\mathcal{B}} v$ on Γ^m:

$$\begin{aligned}
\Delta_{\mathcal{B}} \partial^\mu u_{t^c} &= \sum_{k\,\text{even}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{k/2} (-1)^m \binom{k/2}{m} \partial^{J^{co}}_{\mathbf{g}} \Delta_{\mathcal{B}}^{m+1} u_{t^{c+k/2-m}} \\
&+ \sum_{k\,\text{odd}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{\lfloor k/2\rfloor} (-1)^m \binom{\lfloor k/2\rfloor}{m} \partial^{J^{co}}_{\mathbf{g}} \Delta_{\mathcal{B}}^{m+1} \partial_n u_{t^{c+\lfloor k/2\rfloor - m}} + \tilde{F}_\mu \\
&=: \sum_{k\,\text{even}} \sum_{|J|=k} \sum_{m=0}^{k/2} a_{m,J} \prod_{j\in J} n_j + \sum_{k\,\text{odd}} \sum_{|J|=k} \sum_{m=0}^{\lfloor k/2\rfloor} b_{m,J} \prod_{j\in J} n_j + \tilde{F}_\mu,
\end{aligned} \tag{3.382}$$

[5] Explain $M^b_{\alpha,\beta,\gamma}$.

where we define $\tilde{F}_\mu := \Delta_{\mathcal{B}}\big\{\sum_{|\alpha|+\beta+2\gamma<2p} F_{\alpha,\beta,\gamma}(g)\partial_{\mathbf{g}}^{\alpha}\partial_n^{\beta} u_{t^{c+\gamma}}\big\}$, and by product rule, we can write it in the form

$$\tilde{F}_\mu = \sum_{|\alpha|+\beta+2\gamma<2p+2} \tilde{F}_{\alpha,\beta,\gamma}(g)\partial_{\mathbf{g}}^{\alpha}\partial_n^{\beta} u_{t^{c+\gamma}}.$$

Furthermore,

(3.383)

$$a_{m,J} = (-1)^m \binom{|J|/2}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m+1} u_{t^{c+|J|/2-m}}, \quad b_{m,J} = (-1)^m \binom{\lfloor |J|/2 \rfloor}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m+1} \partial_n u_{t^{c+\lfloor |J|/2\rfloor -m}}.$$

For v_n on Γ^m we obtain:

(3.384)

$$\begin{aligned}
\partial_n \partial^{\mu} u_{t^c} &= \sum_{k\,\text{even}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{k/2} (-1)^m \binom{k/2}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m} \partial_n u_{t^{c+k/2-m}} \\
&+ \sum_{k\,\text{odd}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{\lfloor k/2\rfloor} (-1)^m \binom{\lfloor k/2\rfloor}{m} \big\{ \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m} u_{t^{c+\lfloor k/2\rfloor-m+1}} - \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m+1} u_{t^{c+\lfloor k/2\rfloor -m}} \big\} + \bar{F}_\mu \\
&=: \sum_{k\,\text{even}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{k/2} c_{m,J} + \sum_{k\,\text{odd}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{\lfloor k/2\rfloor} (d_{m,J} - e_{m,J}) + \bar{F}_\mu,
\end{aligned}$$

where

(3.385)

$$\begin{aligned}
&c_{m,J} = (-1)^m \binom{|J|/2}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m} \partial_n u_{t^{c+|J|/2-m}}, \quad d_{m,J} = (-1)^m \binom{\lfloor |J|/2\rfloor}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m} u_{t^{c+\lfloor |J|/2\rfloor -m+1}}, \\
&e_{m,J} = (-1)^m \binom{\lfloor |J|/2\rfloor}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m+1} u_{t^{c+\lfloor |J|/2\rfloor -m}}.
\end{aligned}$$

Furthermore, $\bar{F}_\mu$ is defined by $\bar{F}_\mu = \sum_{|\alpha|+\beta+2\gamma<2p+1} \bar{F}_{\alpha,\beta,\gamma}(g)\partial_{\mathbf{g}}^{\alpha}\partial_n^{\beta} u_{t^{c+\gamma}}$. Similarly, using (3.381), we obtain an expression for v_t on Γ^m:

(3.386)

$$\begin{aligned}
\partial^{\mu} u_{t^{c+1}} &= \sum_{k\,\text{even}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{k/2} (-1)^m \binom{k/2}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m} u_{t^{c+k/2-m+1}} \\
&+ \sum_{k\,\text{odd}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{\lfloor k/2\rfloor} (-1)^m \binom{\lfloor k/2\rfloor}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^{m} \partial_n u_{t^{c+\lfloor k/2\rfloor -m+1}} + \sum_{\alpha+\beta+2\gamma<2p} F_{\alpha,\beta,\gamma}(g)\partial_{\mathbf{g}}^{\alpha}\partial_n^{\beta} u_{t^{c+\gamma+1}} \\
&=: \sum_{k\,\text{even}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{k/2} A_{m,J} + \sum_{k\,\text{odd}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{\lfloor k/2\rfloor} B_{m,J} + \hat{F}_\mu,
\end{aligned}$$

where

(3.387)
$$A_{m,J} = (-1)^m \binom{|J|/2}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^m u_{t^{c+|J|/2-m+1}}, \quad B_{m,J} = (-1)^m \binom{\lfloor |J|/2 \rfloor}{m} \partial_{\mathbf{g}}^{J^{co}} \Delta_{\mathcal{B}}^m \partial_n u_{t^{c+\lfloor |J|/2 \rfloor - m+1}}$$

and $\hat{F}_\mu$ is of the form $\hat{F}_\mu = \sum_{|\alpha|+\beta+2\gamma<2p} F_{\alpha,\beta,\gamma}(g) \partial_{\mathbf{g}}^\alpha \partial_n^\beta u_{t^{c+\gamma+1}}$. With formulas (3.382), (3.384) and (3.386) in hand, we can evaluate the the integrals $\int_{\Gamma^m} \Delta_{\mathcal{B}} v v_n$ and $\int_{\Gamma^m} v_t v_n$.

Evaluating the expression $\int_{\Gamma^m} \Delta_{\mathcal{B}} v v_n$. The goal is to arrive at the identity (3.394), which helps us extract the boundary energy contribution and isolate the 'error' terms. Using (3.382) and (3.384), we may write

$$\begin{aligned}
&\sum_{|\mu|=2p} \int_{\mathcal{C}} \Delta_{\mathcal{B}} v v_n = \sum_{|\mu|=2p} \int_{\mathcal{C}} \Delta_{\mathcal{B}} \partial^\mu u_{t^c} \partial_n \partial^\mu u_{t^c} \\
&= \sum_{\substack{i_r=1,\\ r=1,\dots,2p}}^{n} \int_{\mathcal{C}} \Big(\sum_{k \text{ even}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{k/2} a_{m,J} + \sum_{k \text{ odd}} \sum_{|\bar{J}|=k} \prod_{j \in \bar{J}} n_j \sum_{m=0}^{\lfloor k/2 \rfloor} b_{m,\bar{J}} \Big) \times \\
&\times \Big(\sum_{k \text{ even}} \sum_{|J|=k} \prod_{j\in J} n_j \sum_{m=0}^{k/2} c_{m,J} + \sum_{k \text{ odd}} \sum_{|\bar{J}|=k} \prod_{j \in \bar{J}} n_j \sum_{m=0}^{\lfloor k/2 \rfloor} (d_{m,\bar{J}} - e_{m,\bar{J}}) \Big) \\
&+ \sum_{|\mu|=2p} \int_{\mathcal{C}} \Delta_{\mathcal{B}} v \bar{F}_\mu + \sum_{|\mu|=2p} \int_{\mathcal{C}} v_n \tilde{F}_\mu - \sum_{|\mu|=2p} \int_{\mathcal{C}} \tilde{F}_\mu \bar{F}_\mu.
\end{aligned} \tag{3.388}$$

Let us simplify the first integral on RHS of (3.388). We observe that if J and $\bar{J}$ are two subsets of the index set $I = \{i_1, \dots, i_{2p}\}$ such that $J \neq \bar{J}$, then there obviously exists at least one index i_δ, such that $i_\delta \in J \cap \bar{J}^c$ or $i_\delta \in \bar{J} \cap J^c$. Without loss of generality assume $i_\delta \in J \cap \bar{J}^c$. In this case, we note immediately

$$\sum_{i_\delta=1}^{n} \prod_{j\in J} n_j \partial_{\mathbf{g}}^{J^{co}} \prod_{r \in \bar{J}} n_r \partial_{\mathbf{g}}^{\bar{J}^{co}} = \sum_{i_\delta=1}^{n} n_{i_\delta} \partial_{\mathbf{g}}^{i_\delta} \prod_{j \in J \setminus i_\delta} n_j \partial_{\mathbf{g}}^{J^{co}} \prod_{r\in \bar{J}} n_r \partial_{\mathbf{g}}^{\bar{J}^{co} \setminus i_\delta} = 0,$$

since $\sum_{i_\delta=1}^n n_{i_\delta} \partial_{\mathbf{g}}^{i_\delta} = n \cdot \nabla_{\mathbf{g}} = 0$. Because of this, the first term on RHS of (3.388) reads:

(3.389)
$$\begin{aligned}
&\int_{\mathcal{C}} \sum_{\substack{i_r=1\\ r\in\{1,\dots,2p\}}}^{n} \sum_{k \text{ even}} \sum_{\substack{|J|=k\\ J\subset I}} \prod_{j\in J} n_j^2 \sum_{m=0}^{k/2} a_{m,J} \sum_{m=0}^{k/2} c_{m,J} + \int_{\mathcal{C}} \sum_{\substack{i_r=1\\ r\in\{1,\dots,2p\}}}^{n} \sum_{k \text{ odd}} \sum_{\substack{|J|=k\\ J\subset I}} \prod_{j \in J} n_j^2 \sum_{m=0}^{\lfloor k/2 \rfloor} b_{m,J} \sum_{m=0}^{\lfloor k/2 \rfloor} (d_{m,J} - e_{m,J}) \\
&= \sum_{k \text{ even}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{|J|=k} \sum_{m=0}^{k/2} a_{m,J} \sum_{m=0}^{k/2} c_{m,J} + \sum_{k \text{ odd}} \binom{2p}{k} \int_{\mathcal{C}} \sum_{|J|=k} \sum_{m=0}^{\lfloor k/2 \rfloor} b_{m,J} \sum_{m=0}^{\lfloor k/2 \rfloor} (d_{m,J} - e_{m,J})
\end{aligned}$$

where we used $\sum_{j=1}^2 n_j^2=1$. In the first term on the RHS of (3.389), we single out the $\int_{\mathcal{C}} a_{k/2,J}c_{k/2,J}$ contribution, as we shall use it to extract the leading order energy contribution (see (3.383) and (3.385)):

$$
\begin{aligned}
&\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\sum_{m=0}^{k/2}a_{m,J}\sum_{m=0}^{k/2}c_{m,J}=\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}a_{k/2,J}c_{k/2,J}\\
&+\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\big(-a_{k/2,J}c_{k/2,J}+\sum_{m=0}^{k/2}a_{m,J}\sum_{m=0}^{k/2}c_{m,J}\big)\\
&=\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|\xi|=2p-k}\partial_{\mathbf{g}}^{\xi}\Delta_{\mathcal{B}}^{k/2+1}u_{t^c}\partial_{\mathbf{g}}^{\xi}\Delta_{\mathcal{B}}^{k/2}\partial_n u_{t^c}\\
&\qquad+\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\big(-a_{k/2,J}c_{k/2,J}+\sum_{m=0}^{k/2}a_{m,J}\sum_{m=0}^{k/2}c_{m,J}\big)\\
&=-\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|\xi|=2p}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}\\
&\qquad+\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\big(-a_{k/2,J}c_{k/2,J}+\sum_{m=0}^{k/2}a_{m,J}\sum_{m=0}^{k/2}c_{m,J}\big).
\end{aligned}\tag{3.390}
$$

Similarly, (see (3.383) and (3.385))

(3.391)

$$
\begin{aligned}
&\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\sum_{m=0}^{\lfloor k/2\rfloor}b_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}(d_{m,J}-e_{m,J})=-\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}b_{\lfloor k/2\rfloor,J}e_{\lfloor k/2\rfloor,J}\\
&+\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\sum_{m=0}^{\lfloor k/2\rfloor}b_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}d_{m,J}-\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\big(-b_{\lfloor k/2\rfloor,J}e_{\lfloor k/2\rfloor,J}+\sum_{m=0}^{\lfloor k/2\rfloor}b_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}e_{m,J}\big)\\
&=-\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|\xi|=2p}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}+\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\sum_{m=0}^{\lfloor k/2\rfloor}b_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}d_{m,J}\\
&-\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\big(-b_{\lfloor k/2\rfloor,J}e_{\lfloor k/2\rfloor,J}+\sum_{m=0}^{\lfloor k/2\rfloor}b_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}e_{m,J}\big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|\xi|=2p}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}-\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|\xi|=2p}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}\\
&=-2^{2p}\int_{\mathcal{C}}\sum_{|\xi|=2p}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c},
\end{aligned}
$$

where we used the binomial expansion formula. Hence, using this observation, we conclude from (3.388) - (3.391)

$$\sum_{|\mu|=2p}\int_{\mathcal{C}}\Delta_{\mathcal{B}}v v_n = -2^{2p}\int_{\mathcal{C}}\sum_{|\xi|=2p}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c} + \int_{\mathcal{C}} I_p + \sum_{|\mu|=2p}\int_{\mathcal{C}}\Delta_{\mathcal{B}}v\bar{F}_\mu + \sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\tilde{F}_\mu - \sum_{|\mu|=2p}\tilde{F}_\mu\bar{F}_\mu, \tag{3.392}$$

where

$$I_p := \sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\Big(-a_{k/2,J}c_{k/2,J} + \sum_{m=0}^{k/2}a_{m,J}\sum_{m=0}^{k/2}c_{m,J}\Big) + \sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\sum_{m=0}^{\lfloor k/2\rfloor}b_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}d_{m,J} - \sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\Big(-b_{\lfloor k/2\rfloor,J}e_{\lfloor k/2\rfloor,J} + \sum_{m=0}^{\lfloor k/2\rfloor}b_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}e_{m,J}\Big). \tag{3.393}$$

We use the trace formula (2.255) to further simplify the first integral on RHS of (3.392):

$$\begin{aligned}&\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c} = \int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\big(\partial_{t^c}^{*}u - [\gamma^m,\partial_{t^c}]u\big)\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\big(\partial_{t^c}^{*}u_n - [\gamma^m,\partial_{t^c}]u_n\big)\\ &= \int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_{t^c}^{*}u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_{t^c}^{*}u_n - \int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c} - \int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u_n\\ &+ \int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u_n.\end{aligned}$$

Plugging the above in (3.392), we obtain the following important formula:

$$\begin{aligned}&\sum_{|\mu|=2p}\int_{\mathcal{C}}\Delta_{\mathcal{B}}v v_n = -2^{2p}\sum_{|\xi|=2p}\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_{t^c}^{*}u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_{t^c}^{*}u_n + 2^{2p}\sum_{|\xi|=2p}\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}\\ &+2^{2p}\sum_{|\xi|=2p}\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u_n - 2^{2p}\sum_{|\xi|=2p}\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u_n\\ &+\int_{\mathcal{C}}I_p + \sum_{|\mu|=2p}\int_{\mathcal{C}}\Delta_{\mathcal{B}}v\bar{F}_\mu + \sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\tilde{F}_\mu - \sum_{|\mu|=2p}\tilde{F}_\mu\bar{F}_\mu.\end{aligned} \tag{3.394}$$

Evaluating the expression $\int_{\Gamma^m} v_t v_n$. Our goal is to prove the formula (3.398). Using the representations (3.384) and (3.386) of v_n and v_t respectively, just like in (3.388), (3.389) and (3.390), we arrive at:

(3.395)

$$\sum_{|\mu|=2p}\int_{\mathcal{C}} v_t v_n = \sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\sum_{m=0}^{k/2}A_{m,J}\sum_{m=0}^{k/2}c_{m,J}+\sum_{k\,\text{odd}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\sum_{m=0}^{\lfloor k/2\rfloor}B_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}(d_{m,J}-e_{m,J})$$
$$+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar{F}_\mu+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu$$
$$=\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}A_{k/2,J}c_{k/2,J}+\sum_{k\,\text{even}}\binom{2p}{k}\int_{\mathcal{C}}\sum_{|J|=k}\Big(-A_{k/2,J}c_{k/2,J}+\sum_{m=0}^{k/2}A_{m,J}\sum_{m=0}^{k/2}c_{m,J}\Big)$$
$$+\sum_{k\,\text{odd}}\binom{2p}{k}\sum_{|J|=k}\int_{\mathcal{C}}\sum_{m=0}^{\lfloor k/2\rfloor}B_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}(d_{m,J}-e_{m,J})+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar{F}_\mu+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu.$$

We only concentrate on the case when k is even, whereas the odd k-s are treated as an error term. Note that

$$\sum_{k\,\text{even}}\binom{2p}{k}\sum_{|J|=k}\int_{\Gamma^m}A_{k/2,J}c_{k/2,J}=2^{2p-1}\sum_{|\xi|=2p}\int_{\mathcal{C}}\partial^\xi_{\mathbf{g}}u^{m+1}_{t^{c+1}}\partial^\xi_{\mathbf{g}}\partial_n u^{m+1}_{t^c}. \tag{3.396}$$

Let us abbreviate

$$J_p:=\sum_{k\,\text{even}}\binom{2p}{k}\sum_{|J|=k}\Big(-A_{k/2,J}c_{k/2,J}+\sum_{m=0}^{k/2}A_{m,J}\sum_{m=0}^{k/2}c_{m,J}\Big)$$
$$+\sum_{k\,\text{odd}}\binom{2p}{k}\sum_{|J|=k}\sum_{m=0}^{\lfloor k/2\rfloor}B_{m,J}\sum_{m=0}^{\lfloor k/2\rfloor}(d_{m,J}-e_{m,J}).$$

Using (3.395) and (3.396), we arrive at the following identity:

(3.397)

$$\sum_{|\mu|=2p}\int_{\mathcal{C}}v_tv_n=2^{2p-1}\sum_{|\xi|=2p}\int_{\mathcal{C}}\partial^\xi_{\mathbf{g}}u^{m+1}_{t^{c+1}}\partial^\xi_{\mathbf{g}}\partial_n u^{m+1}_{t^c}+\int_{\mathcal{C}}J_p+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar{F}_\mu+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu.$$

Just like in (3.394), we use the trace formula (2.255) to simplify the first integral on RHS of (3.397) and we obtain, in analogy to (3.394), an important identity:

(3.398)

$$\sum_{|\mu|=2p}\int_{\mathcal{C}}v_tv_n=2^{2p-1}\sum_{|\xi|=2p}\int_{\mathcal{C}}\partial^\xi_{\mathbf{g}}\partial^*_{t^{c+1}}u^{m+1}\partial^\xi_{\mathbf{g}}\partial^*_{t^c}u^{m+1}_n-2^{2p-1}\sum_{|\xi|=2p}\int_{\mathcal{C}}\partial^\xi_{\mathbf{g}}[\gamma^m,\partial_{t^{c+1}}]u^{m+1}\partial^\xi_{\mathbf{g}}\partial_n u^{m+1}_{t^c}$$
$$-2^{2p-1}\sum_{|\xi|=2p}\int_{\mathcal{C}}\partial^\xi_{\mathbf{g}}u^{m+1}_{t^{c+1}}\partial^\xi_{\mathbf{g}}[\gamma^m,\partial_{t^c}]u^{m+1}_n+2^{2p-1}\sum_{|\xi|=2p}\int_{\mathcal{C}}\partial^\xi_{\mathbf{g}}[\gamma^m,\partial_{t^{c+1}}]u^{m+1}\partial^\xi_{\mathbf{g}}[\gamma^m,\partial_{t^c}]u^{m+1}_n$$
$$+\int_{\mathcal{C}}J_p+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_t\bar{F}_\mu+\sum_{|\mu|=2p}\int_{\mathcal{C}}v_n\hat{F}_\mu-\sum_{|\mu|=2p}\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu.$$

3.2.4. *Space-time energy estimates.* We proceed to estimate RHS of (3.394), term by term. In the next two lemmas, we collect the estimates on the last seven integrals ("error terms") on RHS of (3.394) and RHS of (3.398) respectively:

LEMMA 3.5. *The following inequality holds:*

$$
\begin{aligned}
&\sum_{|\xi|=2p}\Big\{\Big|\int_0^t\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}\Big|+\Big|\int_0^t\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u_n\Big|\\
&+\Big|\int_0^t\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u_n\Big|\Big\}+\Big|\int_0^t\int_{\mathcal{C}}I_p\Big|+\sum_{|\mu|=2p}\Big\{\Big|\int_0^t\int_{\mathcal{C}}\Delta_{\mathcal{B}}v\bar{F}_\mu\Big|\\
&+\Big|\int_0^t\int_{\mathcal{C}}v_n\tilde{F}_\mu\Big|+\Big|\int_0^t\int_{\mathcal{C}}\tilde{F}_\mu\bar{F}_\mu\Big|\Big\}\le\eta\int_0^t||v||^2_{H^2(\Omega^m)}+\frac{C\lambda}{\eta}\sum_{q=0}^{p}\int_0^t\mathfrak{D}_q^{m+1}+\frac{C}{\lambda\eta}\int_0^t\mathcal{D}^{m+1}\\
&+\lambda\int_0^t\mathfrak{D}_p^{m+1}+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}_q^{m+1}+\eta\int_0^t\mathfrak{D}_p^{m+1}+\frac{C}{\eta}\sup_{0\le s\le t}\mathfrak{E}^m(s)\sum_{q=p+1}^{l-1}\mathfrak{D}_q^{m+1}.
\end{aligned}
$$

LEMMA 3.6. *The following inequality holds:*

$$
\begin{aligned}
&\sum_{|\xi|=2p}\Big\{\Big|\int_0^t\int_{\mathcal{C}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^{c+1}}]u^{m+1}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}^{m+1}\Big|+\Big|\int_0^t\int_{\mathcal{C}}\partial_{\mathbf{g}}^{\xi}u_{t^{c+1}}^{m+1}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u_n^{m+1}\Big|\\
&+\Big|\int_0^t\int_{\mathcal{C}}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^{c+1}}]u^{m+1}\partial_{\mathbf{g}}^{\xi}[\gamma^m,\partial_{t^c}]u_n^{m+1}\Big|\Big\}+\Big|\int_0^t\int_{\mathcal{C}}J_p\Big|+\sum_{|\mu|=2p}\Big\{\Big|\int_0^t\int_{\mathcal{C}}v_t\bar{F}_\mu\Big|\\
&+\Big|\int_0^t\int_{\mathcal{C}}v_n\hat{F}_\mu\Big|+\Big|\int_0^t\int_{\mathcal{C}}\hat{F}_\mu\bar{F}_\mu\Big|\Big\}\le\eta\int_0^t||v||^2_{H^2(\Omega^m)}+\frac{C\lambda}{\eta}\sum_{q=0}^{p}\int_0^t\mathfrak{D}_q^{m+1}+\frac{C}{\lambda\eta}\int_0^t\mathcal{D}^{m+1}\\
&+\lambda\int_0^t\mathfrak{D}_p^{m+1}+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}_q^{m+1}+\eta\int_0^t\mathfrak{D}_p^{m+1}+\frac{C}{\eta}\sup_{0\le s\le t}\mathfrak{E}^m(s)\sum_{q=p+1}^{l-1}\mathfrak{D}_q^{m+1}.
\end{aligned}
$$

Proofs of Lemmas 3.5 and 3.6. The proof of the second lemma is analogous to the proof of the first lemma. To prove Lemma 3.5, we start by estimating the last three terms on LHS. Recall the definition of $\bar{F}_\mu$ in the line after (3.385) and the formula $\bar{F}_\mu=\sum_{|\alpha|+\beta+2\gamma<2p+1}\bar{F}_{\alpha,\beta,\gamma}(g)\partial_{\mathbf{g}}^{\alpha}\partial_n^{\beta}u_{t^{c+\gamma}}$. Let us, thus, fix a triple (α,β,γ)

satisfying $\alpha+\beta+2\gamma\leq 2p$.

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathcal{C}}\Delta_{\mathcal{B}}\bar{F}_{\alpha,\beta,\gamma}(g)\partial_g^\alpha\partial_n^\beta u_{t^{c+\gamma}}\Big|\leq\int_0^t||\nabla_{\mathbf{g}}v||_{H^{1/2}(\mathcal{C})}||\bar{F}_{\alpha,\beta,\gamma}(g)\partial_{\mathbf{g}}^\alpha\partial_n^\beta u^{m+1}_{t^{c+\gamma}}||_{H^{1/2}(\mathcal{C})}\\
&\leq\eta\int_0^t||\nabla_{\mathbf{g}}v||^2_{H^{1/2}(\mathcal{C})}+\frac{C}{\eta}\int_0^t||\bar{F}_{\alpha,\beta,\gamma}(g)\partial_{\mathbf{g}}^\alpha\partial_n^\beta u^{m+1}_{t^{c+\gamma}}||^2_{H^{1/2}(\mathcal{C})}\\
&\leq C\eta\int_0^t||\nabla v||^2_{H^1(\Omega^m)}+\frac{C}{\eta}||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{H^{|\alpha|+\beta}(\Omega^m)}\\
&\leq C\eta\int_0^t||\nabla v||^2_{H^1(\Omega^m)}+\frac{C\lambda}{\eta}||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{H^{|\alpha|+\beta+1}(\Omega^m)}+\frac{C}{\lambda\eta}\int_0^t||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{L^2(\Omega^m)}\\
&\leq C\eta\int_0^t||\nabla v||^2_{H^1(\Omega^m)}+\frac{C\lambda}{\eta}\sum_{q=0}^p\int_0^t\mathfrak{D}_q^{m+1}+\frac{C}{\lambda\eta}\int_0^t\mathcal{D}^{m+1};
\end{aligned}\tag{3.399}$$

in the first inequality we use the $H^{1/2}$-version of the Cauchy-Schwarz inequality and in the second inequality we use the Young's inequality. In the third estimate we apply the trace inequality and in the fourth estimate we use the interpolation inequality between Sobolev spaces. In order for the fifth inequality to be valid, we have to make sure that (recall the definition (1.232))

$$||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{H^{|\alpha|+\beta+1}(\Omega^m)}\leq C\sum_{q=0}^p\int_0^t\mathfrak{D}_q^{m+1}\tag{3.400}$$

for our choice of the triple (α,β,γ). Note however, that $c+\gamma=l-1-(p-\gamma)$ and since $|\alpha|+\beta+2\gamma\leq 2p$, we immediately conclude that $|\alpha|+\beta+1\leq 2(p-\gamma)+1$. This implies that $||\nabla u^{m+1}_{t^{c+\gamma}}||^2_{H^{|\alpha|+\beta+1}(\Omega^m)}\leq C\mathfrak{D}^{m+1}_{p-\gamma}$. Since $0\leq\gamma\leq p$, the inequality (3.400) follows. Thus, from (3.399) we infer

$$\Big|\sum_{|\mu|=2p}\int_0^t\int_{\Gamma^m}\int_{\Gamma^m}\Delta_{\mathcal{B}}v\bar{F}_\mu\Big|\leq\eta\int_0^t||v||^2_{H^2(\Omega^m)}+\frac{C\lambda}{\eta}\sum_{q=0}^p\int_0^t\mathfrak{D}_q^{m+1}+\frac{C}{\lambda\eta}\int_0^t\mathcal{D}^{m+1}.\tag{3.401}$$

Recall that $\tilde{F}_\mu=\sum_{|\alpha|+\beta+2\gamma<2p+2}\tilde{F}_{\alpha,\beta,\gamma}(g)\partial_{\mathbf{g}}^\alpha\partial_n^\beta u_{t^{c+\gamma}}$. By virtue of the trace inequality, it is easy to see from the previous formula that

$$||\tilde{F}_\mu||^2_{L^2(\mathcal{C})}\leq C\sum_{q=0}^p\int_0^t\mathfrak{D}_q^{m+1}.\tag{3.402}$$

The proof of this estimate is analogous to the proof of (3.400). Therefore

$$\begin{aligned}\Big|\int_0^t\int_{\mathcal{C}} v_n\tilde{F}_\mu\Big| &\le \eta\int_0^t ||\tilde{F}_\mu||^2_{L^2(\mathcal{C})} + \frac{C}{\eta}\int_0^t ||v_n||_{L^2(\mathcal{C})} \\ &\le C\eta\sum_{q=0}^{p}\int_0^t \mathfrak{D}_q^{m+1} + \frac{C\lambda}{\eta}\int_0^t ||\nabla^2 v||^2_{L^2(\Omega^m)} + \frac{C}{\lambda\eta}\int_0^t \mathcal{D}^{m+1};\end{aligned} \tag{3.403}$$

where we used the estimate (3.402) in the second inequality and the trace inequality combined with the interpolation inequality between Sobolev spaces. Using the Cauchy-Schwarz inequality, the estimate (3.402) and the trace inequality, it is easy to obtain (recall that $\bar{F}_\mu$ is defined in the line after (3.385)):

$$\Big|\int_0^t\int_{\mathcal{C}} \bar{F}_\mu\tilde{F}_\mu\Big| \le \lambda\sum_{q=0}^{p}\int_0^t \mathfrak{D}_q^{m+1} + \frac{C}{\lambda}\int_0^t \mathcal{D}^{m+1}. \tag{3.404}$$

Estimating term $\int_0^t\int_{\mathcal{C}} I_p$. Recall the definition (3.393) of I_p and (3.383) and (3.385). Note that for $\mathcal{C}=d_\rho\Omega$, $\int_{\mathcal{C}} I_p=0$, because in any of the products of the form $a_m c_j$, $b_m d_j$ and $b_m e_j$ at least one term has the form $\partial_{\mathbf{g}}^a\partial_n u_{t^b}$ and is thus equal to 0 on $d_\rho\Omega$. Let us now assume that $\mathcal{C}=\Gamma^m$. Let $0\le k\le 2p$ be an even number and fix $0\le m,j\le k/2$ such that m or j is strictly smaller that $k/2$. Assume without loss of generality that $j<k/2$. Let the multi-index α satisfy $|\alpha|=p-k/2+m$ and let $b:=p-k/2+j$. Note that $|\alpha|\le p$ and $b\le p-1$. With these notations the expressions for a_m and c_j are of the form $a_m=C\partial_{\mathbf{g}}^{2\alpha+2}u^{\pm}_{t^{l-1-|\alpha|}}$ and $c_j=C\partial_{\mathbf{g}}^{2b}\partial_n u^{\pm}_{t^{l-1-b}}$, where C is some positive constant. We can thus estimate

$$\begin{aligned}&\Big|\int_0^t\int_{\Gamma^m} a_m c_j\Big| = C\Big|\int_0^t\int_{\Gamma^m} \partial_{\mathbf{g}}^{2\alpha+2}u^{\pm}_{t^{l-1-|\alpha|}}\partial_{\mathbf{g}}^{2\beta}\partial_n u^{\pm}_{t^{l-1-b}}\Big| \le C||\partial_{\mathbf{g}}^{2\alpha+1}u^{\pm}_{t^{l-1-|\alpha|}}||_{H^{1/2}}||\partial_{\mathbf{g}}^{2b}\partial_n u^{\pm}_{t^{l-1-b}}||_{H^{1/2}} \\ &\le \lambda\int_0^t ||\nabla u^{m+1}_{t^{l-1-|\alpha|}}||^2_{H^{2|\alpha|+1}(\Omega^m)} + \frac{C}{\lambda}\int_0^t ||\nabla u_{t^{l-1-b}}||^2_{H^{2b+1}(\Omega^m)} \le \lambda\int_0^t \mathfrak{D}^{m+1}_{|\alpha|} + \frac{C}{\lambda}\int_0^t \mathfrak{D}_b^{m+1} \\ &\le \lambda\int_0^t \mathfrak{D}_p^{m+1} + \frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t \mathfrak{D}_q^{m+1}.\end{aligned} \tag{3.405}$$

In the first inequality we used the $H^{1/2}$ version of the Cauchy-Schwarz inequality. In the last inequality we made use of the fact that $\beta\le p-1$. In the case that k is odd, in analogous manner we can conclude

$$\Big|\int_0^t \int_{\Gamma^m} b_m d_j \Big| \leq \lambda \int_0^t \mathfrak{D}_p^{m+1} + \frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t \mathfrak{D}_q^{m+1}, \tag{3.406}$$

for a pair of indices $0 \leq m, j \leq \lfloor \frac{k}{2} \rfloor$, such that m or j is strictly smaller than $\lfloor \frac{k}{2} \rfloor$. As to the expressions of the form $\int_0^t \int_{\Gamma^m} b_m e_j$, we have to allow for the possibility that both m and j may be equal to $\lfloor \frac{k}{2} \rfloor$. Let α be a multi-index staisfying $|\alpha| = p - \lfloor \frac{k}{2} \rfloor + m$ and let $b := p - \lfloor \frac{k}{2} \rfloor + j$. Note that $|\alpha|, b \leq p$. With these notations the expressions for b_m and e_j are of the form $b_m = C\partial_{\mathbf{g}}^{2\alpha+1} u^{\pm}_{t^{l-1-|\alpha|}}$, and $c_j = C\partial_{\mathbf{g}}^{2(b-1)+1}\partial_n u^{\pm}_{t^{l-1-(b-1)}}$, where C is some positive constant. Hence, using the same idea as in (3.405), we deduce

$$\Big|\int_0^t \int_{\Gamma^m} b_m e_j \Big| \leq \lambda \int_0^t \mathfrak{D}_{|\alpha|}^{m+1} + \frac{C}{\lambda}\int_0^t \mathfrak{D}_{b-1}^{m+1} \leq \lambda \int_0^t \mathfrak{D}_p^{m+1} + \frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t \mathfrak{D}_q^{m+1}. \tag{3.407}$$

Here, we make use of the fact that $b-1 \leq p-1$. Combining (3.405) - (3.407), we conclude

$$\Big|\int_0^t \int_{\mathcal{C}} I_p \Big| \leq \lambda \int_0^t \mathfrak{D}_p^{m+1} + \frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t \mathfrak{D}_q^{m+1}. \tag{3.408}$$

We now proceed to estimate the first three terms on RHS in the claim of Lemma 3.5. From the definition (2.255) of $[\gamma, \partial_{t^c}]$, we realize that if $\mathcal{C} = d_\rho \Omega$ then $\int_{\mathcal{C}} \nabla_{\mathbf{g}} \partial_{\mathbf{g}}^{\xi} [\gamma, \partial_{t^c}] u^{m+1} \cdot \nabla_{\mathbf{g}} \partial_{\mathbf{g}}^{\xi} \partial_n u_{t^c}^{m+1} = 0$. Hence, the expression we want to estimate is given by:

$$\sum_{d=0}^{c-1}\sum_{q+r=c-d} C_{d,q,r}\int_0^t\int_{\Gamma^m} \nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\big(\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}\circ\phi^m(R_t^m)^r)\theta_1^{i_1}\dots\xi_r^{i_r}\partial_g^{i_1}\dots\partial_g^{i_r}\partial_{n^q}u_{t^d}^{m+1}\big)\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}^{m+1}.$$

Let us fix some $0 \leq d \leq c-1 = l-p-2$. If α is a multi-index satisfying $|\alpha| = r$ and $P_r(\xi)$ a polynomial of order r in the components of ξ, then a generic summand above is estimated as follows:

$$\begin{aligned}&\Big|\int_0^t\int_{\Gamma^m} \nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\big(\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}\circ\phi^m(R_t^m)^r)P_r(\xi)\partial_{\mathbf{g}}^{\alpha}\partial_{n^q}u_{t^d}^{m+1}\big)\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}^{m+1}\Big|\\ &\leq \eta\int_0^t ||\partial_{\mathbf{g}}^{\xi}\partial_n u_{t^c}^{m+1}||^2_{H^{1/2}(\Gamma^m)} + \frac{C}{\eta}\int_0^t ||\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}\circ\phi^m(R_t^m)^r)\partial_{\mathbf{g}}^{\alpha}\partial_{n^q}u_{t^d}^{m+1}||^2_{H^{2p+3/2}(\Gamma^m)},\end{aligned} \tag{3.409}$$

where we used the $H^{1/2}$-version of the Cauchy-Schwarz inequality and the Young's inequality. Note that, by the trace inequality, we can estimate

$$\int_0^t ||\partial_{\mathbf{g}^\xi}\partial_n u_{t^c}^{m+1}||^2_{H^{1/2}(\Gamma^m)} \leq C\int_0^t ||\nabla u_{t^c}^{m+1}||^2_{H^{2p+1}(\Omega^m)} \leq C\int_0^t \mathfrak{D}_p^{m+1}.$$

Observe that

$$||\partial_{\mathbf{g}}^\alpha \partial_{n^q} u_{t^d}^{m+1}||^2_{H^{2p+3/2}(\Gamma^m)} \leq C||\nabla u_{t^d}^{m+1}||^2_{H^{2p+1+q+r}(\Omega^m)} \leq C\int_0^t \mathfrak{D}^{m+1}_{l-1-d} \leq C\sum_{q=p+1}^{l-1}\int_0^t \mathfrak{D}_q^{m+1},$$

where the trace trace inequality and the definition (1.232) of $\mathfrak{D}_q$ are used. By taking L^2 and L^∞ norms respectively, trace inequality and previous estimate, we obtain (recall (1.228))

(3.410)
$$\int_0^t ||\partial^*_{t^{c-1-d}}(V^q_{\Gamma^m}\circ\phi^m(R_t^m)^r)\partial_{\mathbf{g}}^\alpha\partial_{n^q}u_{t^d}^{m+1}||^2_{H^{2p+3/2}(\Gamma^m)} \leq C\sup_{0\leq s\leq t}\sqrt{\mathfrak{E}^m(s)}\sum_{q=p+1}^{l-1}\int_0^t \mathfrak{D}_q^{m+1}.$$

Using the inequalities between (3.409) and (3.410), we conclude

(3.411)
$$\Big|\int_{\mathcal{C}} \nabla_{\mathbf{g}}\partial_{\mathbf{g}}^\xi[\gamma,\partial_{t^c}]u^{m+1}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^\xi\partial_n u_{t^c}^{m+1}\Big| \leq \eta C\int_0^t \mathfrak{D}_p^{m+1} + \frac{C}{\eta}\sup_{0\leq s\leq t}\sqrt{\mathfrak{E}^m(s)}\sum_{q=p+1}^{l-1}\int_0^t \mathfrak{D}_q^{m+1}$$

In a fully analogous manner we can prove

$$\begin{aligned}&\Big|\int_0^t\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^\xi u_{t^c}\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^\xi[\gamma^m,\partial_{t^c}]u_n\Big| + \Big|\int_0^t\int_{\mathcal{C}}\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^\xi[\gamma^m,\partial_{t^c}]u\cdot\nabla_{\mathbf{g}}\partial_{\mathbf{g}}^\xi[\gamma^m,\partial_{t^c}]u_n\Big| \\ &\leq \eta C\int_0^t \mathfrak{D}_p^{m+1} + \frac{C}{\eta}\sup_{0\leq s\leq t}\sqrt{\mathfrak{E}^m(s)}\sum_{q=p+1}^{l-1}\int_0^t \mathfrak{D}_q^{m+1}.\end{aligned} \tag{3.412}$$

□

Evaluating $\sum_{|\xi|=2p}\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}\partial^*_{t^c}u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}\partial^*_{t^c}u_n^{m+1}$ via boundary conditions. In order to extract the coercive energy part in this integral, we have to evaluate the pull-back of the operator $\partial_{\mathbf{g}^m}$ on $\mathcal{S}$ with respect to $\phi^m:\mathcal{S}\to\Gamma^m$. For any $1\leq i\leq n$, the following formula holds

$$\phi^\#\partial^i_{\mathbf{g}^m} = \frac{1}{R^m}\partial_g^i + \sum_{j=1}^n c_j^i(f^m)\partial_g^j. \tag{3.413}$$

Here, for any $1 \le i,j \le n$, c^i_j is a smooth, uniformly bounded function in a given local chart, and is a polynomial expression in f^m, $|g^m|-1$ and $\partial^k_g f^m$, $k=1,\dots,n$. Thus, given a multi-index $|\xi|=2p$, we have

$$\partial^\xi_{\mathbf{g}^m} = \sum_{|\mu|=2p} A^\xi_\mu \partial^\mu_g + \sum_{|\mu|\le 2p-1} B^\xi_\mu \partial^\mu_g. \tag{3.414}$$

For $\mu \ne \xi$, A^ξ_μ are polynomials in f^m, $|g^m|-1$ and $\partial^k_g f^m$, $k=1,\dots,n$. If $\mu=\xi$ then $A^\xi_\xi = \frac{1}{(R^m)^{|\xi|}} + \text{polynomial}(f^m, |g^m|-1, \partial^k_g f^m)$. The functions B^ξ_μ are polynomials in terms of the form $\partial^\nu_g f^m$ with $|\nu| \le 2p+1-|\mu|$. Let e_i stand for the i-th unit vector. If ξ is a given multi-index, we denote

$$\xi_i := \xi + e_i. \tag{3.415}$$

Hence

$$\int_{\Gamma^m} \partial^\xi_{\mathbf{g}} \nabla_{\mathbf{g}} \partial^*_{t^c} u^{m+1} \cdot \partial^\xi_{\mathbf{g}} \nabla_{\mathbf{g}} \partial^*_{t^c} [u^{m+1}_n] = \int_{\mathcal{S}} \Big(\sum_{|\mu|=2p+1} A^{\xi_i}_\mu \partial^\mu_g + \sum_{|\mu|\le 2p} B^{\xi_i}_\mu \partial^\mu_g \Big) H^m_{t^c} \times$$
$$\times \Big(\sum_{|\mu|=2p+1} A^{\xi_i}_\mu \partial^\mu_g + \sum_{|\mu|\le 2p} B^{\xi_i}_\mu \partial^\mu_g \Big) \Big(\frac{f^{m+1}_t R^m}{|g^m|} + \epsilon \frac{\Delta^2_g f^{m+1}_t}{|g^m|(R^m)^{n-2}} \Big) (R^m)^{n-2} |g^m|$$

Recall the formula $H^m = (n-1) - (n-1) f^m - \frac{\Delta_g f^m}{R^m |g^m|} + N^*(f^m)$. Hence

$$H^m_{t^c} = -(n-1) f^m_{t^c} - \frac{\Delta_g f^m_{t^c}}{R^m |g^m|} - G^m_c,$$

where G^m_c is defined by (3.290). Thus

$$\begin{aligned} \sum_{|\mu|=2p+1} A^{\xi_i}_\mu \partial^\mu_g H^m_{t^c} &= -(n-1) \sum_{|\mu|=2p+1} A^{\xi_i}_\mu \partial^\mu_g f^m_{t^c} - \sum_{|\mu|=2p+1} A^{\xi_i}_\mu \frac{\partial^\mu_g \Delta_g f^m_{t^c}}{R^m |g^m|} \\ &- \sum_{|\mu|=2p+1} A^{\xi_i}_\mu \Big[\partial^\mu_g \frac{\Delta_g f^m}{R^m |g^m|} - \frac{\partial^\mu_g \Delta_g f^m}{R^m |g^m|} \Big] - \sum_{|\mu|=2p+1} A^{\xi_i}_\mu \partial^\mu_g G^m_c \\ &= -(n-1) \sum_{|\mu|=2p+1} A^{\xi_i}_\mu \partial^\mu_g f^m_{t^c} - \frac{\Delta_g \big(\sum_\mu A^{\xi_i}_\mu \partial^\mu_g f^m_{t^c} \big)}{R^m |g^m|} + G^{\xi_i, c}, \end{aligned} \tag{3.416}$$

where

(3.417)
$$G^{\xi_i,c} = \frac{\big[\Delta_g \big(\sum_\mu A^{\xi_i}_\mu \partial^\mu_g f^m_{t^c} \big) - \sum_\mu A^{\xi_i}_\mu \partial^\mu_g \Delta_g f^m_{t^c} \big]}{R^m |g^m|} - \sum_{|\mu|=2p+1} A^{\xi_i}_\mu \Big[\partial^\mu_g \frac{\Delta_g f^m_{t^c}}{R^m |g^m|} - \frac{\partial^\mu_g \Delta_g f^m_{t^c}}{R^m |g^m|} \Big]$$

Also note that

$$-\partial_{t^c}\Big(\frac{f_t^{m+1}R^m}{|g^m|}+\epsilon\frac{\Delta_g^2 f_t^{m+1}}{|g^m|(R^m)^{n-2}}\Big)=-\frac{f_{t^{c+1}}^{m+1}R^m}{|g^m|}-\epsilon\frac{\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}+h_c^m,$$

where h_c^m is defined by (3.291). Thus

(3.418)

$$\begin{aligned}&\sum_{|\mu|=2p+1}A_\mu^{\xi_i}\partial_g^\mu\big(V^{m+1}\circ\phi^m+\epsilon\Lambda^{m+1}\circ\phi^m\big)=-\sum_\mu A_\mu^{\xi_i}\partial_g^\mu\Big(\frac{f_{t^{c+1}}^{m+1}R^m}{|g^m|}+\epsilon\frac{\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}-h_c^m\Big)\\&=-\sum_\mu A_\mu^{\xi_i}\Big(\frac{\partial_g^\mu f_{t^{c+1}}^{m+1}R^m}{|g^m|}+\epsilon\frac{\partial_g^\mu\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}\Big)+\sum_\mu A_\mu^{\xi_i}\partial_g^\mu h_c^m\\&-\sum_\mu A_\mu^{\xi_i}\Big[\partial_g^\mu\Big(\frac{f_{t^{c+1}}^{m+1}R^m}{|g^m|}-\frac{\partial_g^\mu f_{t^{c+1}}^{m+1}R^m}{|g^m|}\Big)+\epsilon\partial_g^\mu\Big(\frac{\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}\Big)-\epsilon\frac{\partial_g^\mu\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}\Big]\\&=-\frac{\big(\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}\big)_t R^m}{|g^m|}-\epsilon\frac{\Delta_g^2\big(\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}\big)_t}{|g^m|(R^m)^{n-2}}+h^{\xi_i,c+1},\end{aligned}$$

where

(3.419)

$$\begin{aligned}&h^{\xi_i,c+1}:=\sum_\mu\frac{(A_\mu^{\xi_i})_t\partial_g^\mu f_{t^c}^{m+1}}{|g^m|}+\epsilon\sum_\mu\frac{\Delta_g^2\big(A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}\big)_t-A_\mu^{\xi_i}\partial_g^\mu\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}\\&-\sum_\mu A_\mu^{\xi_i}\Big[\partial_g^\mu\Big(\frac{f_{t^{c+1}}^{m+1}R^m}{|g^m|}\Big)-\frac{\partial_g^\mu f_{t^{c+1}}^{m+1}R^m}{|g^m|}+\epsilon\partial_g^\mu\Big(\frac{\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}\Big)-\epsilon\frac{\partial_g^\mu\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}\Big]+\sum_\mu A_\mu^{\xi_i}\partial_g^\mu h_c^m.\end{aligned}$$

Using formulas (3.416) and (3.418) and setting $\chi=\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^m$, $\omega=\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}$, we obtain

(3.420)

$$\begin{aligned}&\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial_g^\xi\nabla_g\partial_{t^c}^*u^{m+1}\cdot\partial_g^\xi\nabla_g\partial_{t^c}^*[u_n^{m+1}]=\sum_{|\xi|=2p}\sum_{i=1}^n\int_0^t\int_{\mathcal S}\Big(-(n-1)\chi-\frac{\Delta_g\chi}{R^m|g^m|}+G^{\xi_i,c}\Big)\\&\times\Big(-\frac{\omega_t R^m}{|g^m|}-\epsilon\frac{\Delta_g^2\omega_t}{|g^m|(R^m)^{n-2}}+h^{\xi_i,c+1}\Big)|g^m|(R^m)^{n-2}.\end{aligned}$$

Now we may apply the identities (2.273) and (2.274) to the integral on the right-most side of (3.420). With χ and ω already chosen, we set $\psi=R^m$, $g=g^m$, $G=G^{\xi_i,c}$ and $h=h^{\xi_i,c+1}$. Thus, from (2.273) and (2.274), we obtain:

(3.421)

$$\begin{aligned}&\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial_{\mathbf g}^\xi\nabla_{\mathbf g}\partial_{t^c}^*u^{m+1}\cdot\partial_{\mathbf g}^\xi\nabla_{\mathbf g}\partial_{t^c}^*[u_n^{m+1}]+\frac12\sum_{|\xi|=2p}\sum_{i=1}^n\int_{\mathcal S}\{|\nabla_g\sum_\mu A_\mu^{\xi_i}\partial_g^\mu|^2-(n-1)|\sum_\mu A_\mu^{\xi_i}\partial_g^\mu|^2\}\Big|_0^t\\&+\frac{\epsilon}{2}\sum_{|\xi|=2p}\sum_{i=1}^n\int_{\mathcal S}\{|\nabla_g^3\sum_\mu A_\mu^{\xi_i}\partial_g^\mu|^2-(n-1)|\nabla_g^2\sum_\mu A_\mu^{\xi_i}\partial_g^\mu|^2\}\Big|_0^t=\sum_{|\xi|=2p}\sum_{i=1}^n\int_0^t\int_{\mathcal S}Q(\chi,\omega,R^m),\end{aligned}$$

where Q is defined by (2.270). In view of the dependence of χ and ω on the index c (i.e. p since, $c=l-1-p$) and the iteration step m, we introduce the notation $\tilde{Q}^m_c = Q(\chi,\omega,R^m)$, with χ and ω as specified before (3.420). The energy estimate for the "error term" $\int_0^t \int_{\mathcal{S}} \tilde{Q}^m_c$ is the content of the following lemma and Lemma 3.9:

LEMMA 3.7. *For $1 \le c \le l-1$ the following inequality holds:*

$$\Big|\int_0^t \int_{\mathcal{S}} \tilde{Q}^m_c\Big| \le \lambda \int_0^t \mathfrak{D}^{m+1}_p + \frac{\tilde{C}\lambda t}{\epsilon^4} \sup_{0\le s\le t} \mathfrak{E}^{m+1}(s) + \frac{\tilde{C}t}{\lambda} \sup_{0\le s\le t} \big(\mathfrak{E}^{m+1}_p(s) + \mathfrak{E}^m_p(s) + \hat{\mathfrak{E}}^{m+1}_p + \hat{\mathfrak{E}}^m_p\big)$$
$$+ \frac{\tilde{C}C_A t\epsilon}{\lambda}\big(\theta + \int_0^t \mathfrak{D}^m\big) + \frac{\tilde{C}C_A t\epsilon}{\lambda}\big(\theta_1 + t\int_0^t \mathfrak{D}^{m+1}\big) + C \sup_{0\le s\le t} ||f^m(s)||_{H^l}\big(\int_0^t \mathfrak{D}^m_p + \int_0^t \mathfrak{D}^{m+1}_p\big)$$
$$+ C \sup_{0\le s\le t} (\mathfrak{E}^m(s))^{1/2} \int_0^t \mathfrak{D}^{m+1}_{p+1} + C\epsilon \sup_{0\le s\le t} ||f^m(s)||_{H^l} \mathcal{E}^m + Ct\int_0^t \hat{\mathfrak{D}}^{m+1}_p + C \sup_{0\le s\le t} ||f^m(s)||_{H^l} \hat{\mathfrak{E}}^m_p$$
$$+ C \sup_{0\le s\le t} \big(||f^m(s)||_{H^l} + (\mathfrak{E}^m(s))^{1/2}\big)\big(\int_0^t \mathfrak{D}^m + \int_0^t \hat{\mathfrak{D}}^m + \int_0^t \mathfrak{D}^{m+1}_p + \int_0^t \hat{\mathfrak{D}}^{m+1}_p\big).$$

Before we prove Lemma 3.7, we state the following trace-type lemma that will be repeatedly used in the forthcoming estimates:

LEMMA 3.8. *For any pair of indices (μ,c), $|\mu|=p$, $c=l-1-p$ we have*

$$\epsilon^2 \int_0^t ||\nabla^3_g \partial^\mu_g f^{m+1}_{t^{c+1}}||^2_{L^2} \le \eta \int_0^t \mathfrak{D}^{m+1}_p + \frac{C}{\eta}\int_0^t \mathfrak{E}^{m+1}_p.$$

As a consequence, we obtain

$$\epsilon^2 \int_0^t ||\nabla^3_g \partial^\mu_g f^{m+1}_{t^{c+1}}||^2_{L^2} \le \eta \int_0^t \mathfrak{D}^{m+1}_p + \frac{Ct}{\eta} \sup_{0\le s\le t} \mathcal{E}^{m+1}(s).$$

Proof. The first estimate follows by applying the operator ∂^μ_c to the regularized jump equation (3.280). We then take squares, integrate and use the trace inequality. The second estimate follows after using the interpolation on the term $\mathfrak{E}^{m+1}_p$ (recall (1.230)). $\square$

Proof of Lemma 3.7. Recall that $\tilde{Q}^m_c = Q(\chi,\omega,R^m)$, with χ and ω as specified in the line before (3.420) and Q is defined by (2.270). For the sake of notational conciseness, we denote

$$\chi^c(=\chi) := \sum_\mu A^{\xi_i}_\mu \partial^\mu_g f^m_{t^c}; \quad \omega^c(=\omega) := \sum_\mu A^{\xi_i}_\mu \partial^\mu_g f^{m+1}_{t^c}. \tag{3.422}$$

The first four terms on RHS of (2.270) are cross-terms. The generic constant used to estimate these cross-terms will always be denoted by $\tilde{C}$, as it will be important later to keep track of this distinction. For the first term on RHS of (2.270), we have

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathcal{S}}\omega^c\big(\omega^c-\chi^c\big)\Big|\leq\lambda\int_0^t||\omega_t^c||^2+\frac{\tilde{C}}{\lambda}\int_0^t\big(||\omega^c||_{L^2}^2+||\chi^c||_{L^2}^2\big)\\
&\leq\lambda\int_0^t||\omega_t^c||^2+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(||\omega^c(s)||_{L^2}^2+||\chi^c(s)||_{L^2}^2\big)\\
&\leq\lambda\int_0^t\mathfrak{D}_p^{m+1}+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\mathfrak{E}_p^{m+1}(s)+\mathfrak{E}_p^m(s)\big).
\end{aligned}\tag{3.423}$$

As to the second term on RHS of (2.270), we have

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathcal{S}}\nabla_g\omega_t^c\nabla_g\big(\omega^c-\chi^c\big)\Big|\\
&\leq\lambda\int_0^t||\nabla_g^{2p+2}f_{c+1}^{m+1}||_{L^2}^2+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(||\nabla_g^{2p+2}f_{t^c}^{m+1}||_{L^2}^2+||\nabla_g^{2p+2}f_{t^c}^m||_{L^2}^2\big)\\
&\leq\lambda\int_0^t\mathfrak{D}_p^{m+1}+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1}+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\mathfrak{E}_p^{m+1}+\mathfrak{E}_p^m\big),
\end{aligned}\tag{3.424}$$

where we argued analogously to the estimate (3.423) and in addition to that, in the last estimate we used Lemma 3.8 with $\eta=\epsilon^2$. The third and the fourth cross-terms are estimated in a fully analogous way, making use of Lemma 3.8 (with $\eta=\epsilon^2$) and the trace inequality:

$$\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g\omega_t^c\Delta_g(\omega^c-\chi^c)\Big|+\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g\nabla_g\omega_t^c\cdot\Delta_g\nabla_g(\omega^c-\chi^c)\Big|\\
&\quad\leq\lambda\int_0^t\mathfrak{D}_p^{m+1}+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{E}^{m+1}+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\mathfrak{E}_p^{m+1}+\mathfrak{E}_p^m\big).
\end{aligned}\tag{3.425}$$

By standard $L^\infty-L^2-L^2$ type estimates it is easy to estimate the fifth and the sixth term:

$$\Big|\int_0^t\int_{\mathcal{S}}\chi^c\omega_t^c((R^m)^{n-1}-1)\Big|+\Big|\int_0^t\int_{\mathcal{S}}\Delta_g\chi^c\omega_t^c(\frac{(R^m)^{n-2}}{|g^m|}-1)\Big|\leq C||f^m||_{H^{n/2+3}}\big(\int_0^t\mathfrak{D}_p^m+\int_0^t\mathfrak{D}_p^{m+1}\big).\tag{3.426}$$

Analogously, the seventh and the eighth term are estimated as follows:

$$\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathcal{S}}\chi^c\Delta_g^2\omega_t^c((R^m)^{n-2}-1)\Big|+\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g\chi^c\Delta_g^2\omega_t^c(\frac{(R^m)^{n-3}}{|g^m|}-1)\Big|\\
&\leq C||f^m||_{H^{n/2+3}}\big(\int_0^t\hat{\mathfrak{D}}_p^m+\int_0^t\hat{\mathfrak{D}}_p^{m+1}\big).
\end{aligned}$$

The last two terms in the expression for $\int_0^t\int_{\mathbb{S}^1}\tilde{Q}_c^m$ involve the expressions $G^{\xi_i,c}$ and $h^{\xi_i,c+1}$, defined by (3.417) and (3.419) respectively. The following inequality holds

(3.427)

$$\Big|\int_0^t\int_{\mathcal{S}}\Big(-(n-1)\chi^c-\frac{\Delta_g\chi^c}{R^m|g^m|}\Big)h^{\xi_i,c+1}\Big|+\Big|\int_0^t\int_{\mathcal{S}}\Big(\frac{\omega_t^cR^m}{|g^m|}+\epsilon\frac{\Delta_g^2\omega_t^c}{|g^m|(R^m)^{n-2}}\Big)G^{\xi_i,c}\Big|$$
$$+\Big|\int_0^t\int_{\mathcal{S}}h^{\xi_i,c+1}G^{\xi_i,c}\Big|\leq C(\sqrt{\mathfrak{E}^m}+||f^m||_{H^l})\Big(\int_0^t\mathfrak{D}^m+\int_0^t\hat{\mathfrak{D}}^m\Big)$$
$$+C(\sqrt{\mathfrak{E}^m}+||f^m||_{H^l})\Big(\int_0^t\mathfrak{D}^{m+1}+\int_0^t\hat{\mathfrak{D}}^{m+1}\Big).$$

These estimates follow by a routine use of L^2 and L^∞ estimates, together with the Cauchy-Schwarz inequality. □

In the case $c=0$ we have $p=l-1$ and LHS of the identity (3.421) takes the form

$$\sum_{|\xi|=2l-2}\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}[u_n^{m+1}]=\sum_{|\xi|=2l-2}\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}\big(V^{m+1}+\epsilon\Lambda^{m+1}\big).$$

We now make an exception and treat $\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}V^{m+1}$ as an "error term" and we use the boundary conditions for u^{m+1} only to extract the energy contribution from $\epsilon\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}\Lambda^{m+1}$. The corresponding energy identity thus reads

(3.428)

$$\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}[u_n^{m+1}]+\frac{\epsilon}{2}\sum_{|\xi|=2p}\sum_{i=1}^n\int_{\mathcal{S}}\{|\nabla_g^3\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f^{m+1}|^2$$
$$-(n-1)|\nabla_g^2\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f^{m+1}|^2\}\Big|_0^t=-\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}V^{m+1}+\int_{\mathcal{S}}Q_0^\epsilon(\chi,\omega,R^m),$$

where Q_0^ϵ is defined to be the ϵ-dependent part of the definition (2.270) of Q. We can now state the lemma that provides the energy estimate for RHS of (3.428):

Lemma 3.9. *The following inequality holds*[7]*:*

$$\Big|\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}V^{m+1}\Big|+\Big|\int_0^t\int_{\mathcal{S}}Q_0^\epsilon(\chi,\omega,R^m)\Big|\leq\lambda\int_0^t\mathfrak{D}_{l-1}^{m+1}+\frac{C}{\eta\lambda}\int_0^t\mathcal{D}^{m+1}$$
$$+\frac{C\eta}{\lambda}\int_0^t\mathfrak{D}_{l-1}^{m+1}+C\epsilon||\mathbb{P}_1f^m||_{L^2}^4+C||\mathbb{P}_1f^m||_{L^2}^2\hat{\mathfrak{E}}^m+C\sup_{0\leq s\leq t}||f^m(s)||_{H^l}\mathfrak{E}_p^m+Ct\int_0^t\hat{\mathfrak{D}}_p^{m+1}$$
$$+Ct\sup_{0\leq s\leq t}(\mathfrak{E}^m(s))^{1/2}\big(\hat{\mathfrak{E}}^m+\hat{\mathfrak{E}}_p^{m+1}\big).$$

[7]Change the estimate accordingly.

Proof. In order to estimate the first integral on right-most side of (3.428), we shall not evaluate it using the boundary conditions for u^{m+1}. Instead, we proceed in the following way (note that $c=0$ implies $p=l-1$):

$$
\begin{aligned}
&\Big|\int_0^t\int_{\Gamma^m}\partial^\xi_{\mathbf{g}}\nabla_{\mathbf{g}}u^{m+1}\cdot\partial^\xi_{\mathbf{g}}\nabla_{\mathbf{g}}V^{m+1}\Big|\leq\lambda\int_0^t||(\partial^\sharp_{\mathbf{g}})^\xi\nabla^\sharp_{\mathbf{g}}(\frac{f_t^{m+1}R^m}{|g^m|})||^2_{L^2}\\
&+\frac{C}{\lambda}\big(\frac{C}{\eta}\int_0^t||\nabla u^{m+1}||^2_{L^2(\Omega^m)}+\eta\int_0^t||\nabla u^{m+1}||^2_{H^{2l-1}(\Omega^m)}\big)\\
&\leq\lambda\int_0^t\mathfrak{D}^{m+1}_{l-1}+\frac{C}{\eta\lambda}\int_0^t\mathcal{D}^{m+1}+\frac{C\eta}{\lambda}\int_0^t\mathfrak{D}^{m+1}_{l-1},
\end{aligned}\tag{3.429}
$$

where the trace inequality and the interpolation are used in the first inequality (note that the potentially large constant $\frac{C}{\eta\lambda}$ stands in front of the *temporal* dissipation term). To estimate $\big|\int_0^t\int_{\mathcal{S}}Q_0^\epsilon(\chi^0,\omega^0,R^m)\big|$ first recall the definition (3.422) of ω^0, χ^0 and note that from the definition of Q_0^ϵ above and (2.270), we have

$$
\begin{aligned}
&Q_0^\epsilon(\chi^0,\omega^0),R^m)=-\epsilon(n-1)\Delta_g\omega^0_t\Delta_g(\omega^0-\chi^0)+\epsilon\Delta_g\nabla_g\omega^0_t\cdot\Delta_g\nabla_g(\omega^0-\chi^0)\\
&+\epsilon(n-1)\chi^0\Delta_g^2\omega^0_t((R^m)^{n-2}-1)+\epsilon\Delta_g\chi^0\Delta_g^2\omega^0_t(\frac{(R^m)^{n-3}}{|g|}-1)\\
&-\epsilon h_\epsilon^{\xi_i,1}\bar{\mathcal{U}}\circ\phi^m(R^m)^{n-2}|g|-\epsilon G^{\xi_i,0}\frac{\Delta_g^2\omega^0_t}{|g|}(R^m)^{n-2}|g|,
\end{aligned}\tag{3.430}
$$

where $h_\epsilon^{\xi_i,1}$ is the ϵ-dependent part of $h^{\xi_i,1}$ defined by (3.419) and $G^{\xi_i,0}$ is defined by (3.417). The cross-terms above are estimated in the following way:

$$
\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g\omega^0_t\Delta_g(\omega^0-\chi^0)\Big|+\epsilon\Big|\int_0^t\int_{\mathcal{S}}\Delta_g\nabla_g\omega^0_t\cdot\Delta_g\nabla_g(\omega^0-\chi^0)\Big|\\
&\leq\lambda\epsilon\int_0^t||\omega^0_t||^2_{H^3}+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(||\omega^0||^2_{H^3}+||\chi^0||^2_{H^3}\big)\\
&\leq\lambda\int_0^t\hat{\mathcal{D}}_{l-1}+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big(\hat{\mathfrak{E}}^m+\hat{\mathfrak{E}}^{m+1}+||\mathbb{P}_1f^m||^2_{L^2}+||\mathbb{P}_1f^{m+1}||^2_{L^2}\big)\\
&\leq\lambda\int_0^t\hat{\mathcal{D}}_{l-1}+\frac{\tilde{C}t}{\lambda}\sup_{0\leq s\leq t}\big\{\hat{\mathfrak{E}}^m+\hat{\mathfrak{E}}^{m+1}+C_A(\theta_1+\int_0^t\mathfrak{D}^m+\frac{t}{\epsilon^2}\int_0^t\mathfrak{D}^{m+1})\big\}
\end{aligned}\tag{3.431}
$$

The third and the fourth term in (3.430) are estimated as follows:

(3.432)

$$
\begin{aligned}
&\epsilon\Big|\int_0^t\int_{\mathcal{S}}\chi^0\Delta_g^2\omega^0_t((R^m)^{n-2}-1)+\Delta_g\chi^0\Delta_g^2\omega^0_t(\frac{(R^m)^{n-3}}{|g|}-1)\Big|\leq C\epsilon\sup_{0\leq s\leq t}||f^m||_{H^l}\int_0^t||\chi^0||_{H^3}||\omega^0||_{H^3}\\
&\leq C\sup_{0\leq s\leq t}||f^m||_{H^l}\int_0^t\big(\hat{\mathfrak{E}}^m+\hat{\mathfrak{D}}^{m+1}\big)\leq Ct\sup_{0\leq s\leq t}||f^m||_{H^l}\hat{\mathfrak{E}}^m+C\sup_{0\leq s\leq t}||f^m||_{H^l}\int_0^t\mathfrak{D}^{m+1}.
\end{aligned}
$$

As to the last two terms of (3.430), again by similar estimates we obtain:

$$\begin{aligned}\epsilon\Big|\int_0^t\int_{\mathcal{S}} h_\epsilon^{\xi_i,1}\bar{\mathcal{U}}\circ\phi^m(R^m)^{n-2}|g|+G^{\xi_i,0}\frac{\Delta_g^2\omega_t^0}{|g|}(R^m)^{n-2}|g|\Big| \\ \le Ct\sqrt{\mathfrak{E}^m}\hat{\mathfrak{E}}^m+Ct\sqrt{\mathfrak{E}^m}\hat{\mathfrak{E}}^{m+1}+C\sqrt{\mathfrak{E}^m}\int_0^t\mathfrak{D}^{m+1}.\end{aligned} \tag{3.433}$$

□

Evaluating $\sum_{|\xi|=2p}\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\partial_{t^{c+1}}^* u^{m+1}\partial_{\mathbf{g}}^\xi\partial_{t^c}^*[u_n^{m+1}]$ via boundary conditions. We now extract the coercive quadratic contribution from the first term on RHS of (3.398). Note that

$$\sum_\mu A_\mu^\xi\partial_g^\mu H_{t^{c+1}}^m=-(n-1)\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^m-\frac{\Delta_g\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^m}{|g^m|R^m}+G^{\xi,c+1} \tag{3.434}$$

where $G^{\xi,c+1}$ is given by analogy to the formula (3.417) for $G^{\xi_i,c}$. Note that

$$\begin{aligned}&\sum_{|\mu|=2p+1}A_\mu^\xi\partial_g^\mu\big(V^{m+1}\circ\phi^m+\epsilon\Lambda^{m+1}\circ\phi^m\big)=-\sum_\mu A_\mu^\xi\partial_g^\mu\Big(\frac{f_{t^{c+1}}^{m+1}R^m}{|g^m|}+\epsilon\frac{\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}-h_c^m\Big)\\ &=-\frac{\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^{m+1}R^m}{|g^m|}-\epsilon\frac{\Delta_g^2\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}+h^{\xi,c+1},\end{aligned} \tag{3.435}$$

where

$$h^{\xi,c+1}=-\sum_\mu A_\mu^\xi\Big[\partial_g^\mu\Big(\frac{f_{t^{c+1}}^{m+1}R^m}{|g^m|}\Big)-\frac{\partial_g^\mu f_{t^{c+1}}^{m+1}R^m}{|g^m|}+\epsilon\partial_g^\mu\Big(\frac{\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}\Big)-\epsilon\frac{\partial_g^\mu\Delta_g^2 f_{t^{c+1}}^{m+1}}{|g^m|(R^m)^{n-2}}\Big]+\sum_\mu A_\mu^\xi\partial_g^\mu h_c^m.$$

Using the formulas (3.434) and (3.435) and shortening $\chi'=\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^m$, $\omega'=\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^{m+1}$, we can write

$$\begin{aligned}&\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\partial_{t^{c+1}}^* u^{m+1}\partial_{\mathbf{g}}^\xi\partial_{t^c}^*[u_n^{m+1}]=\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial_g^\xi\partial_{t^{c+1}}^*\kappa^m\partial_g^\xi\partial_{t^c}^*\big(V^{m+1}+\epsilon\Lambda^{m+1}\big)\\ &\sum_\xi\int_0^t\int_{\mathcal{S}}\Big\{(n-1)\chi'+\frac{\Delta_g\chi'}{|g^m|R^m}-G^{\xi,c+1}\Big\}\Big\{\frac{\omega'R^m}{|g^m|}+\epsilon\frac{\Delta_g^2\omega'}{|g^m|(R^m)^{n-2}}-h^{\xi,c+1}\Big\}|g^m|(R^m)^{n-2}.\end{aligned} \tag{3.436}$$

We now apply the identity (2.276) to evaluate the integral on the right-most side of (3.436). With $\chi=\chi'$ and $\omega=\omega'$ already specified, we set $\psi=R^m$, $g=g^m$, $G=G^{\xi,c+1}$

and $h=h^{\xi,c+1}$. Thus, using (3.436), we obtain

(3.437)
$$\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial^\xi_{\mathbf{g}}\partial^*_{t^{c+1}}u^{m+1}\partial^\xi_{\mathbf{g}}\partial^*_{t^c}[u^{m+1}_n]-\int_0^t\int_{\mathcal{S}}\sum_\xi\{|\nabla_g\sum_\mu A^\xi_\mu\partial^\mu_g f^{m+1}_{t^{c+1}}|^2-(n-1)|\sum_\mu A^\xi_\mu\partial^\mu_g f^{m+1}_{t^{c+1}}|^2\}$$
$$-\epsilon\int_0^t\int_{\mathcal{S}}\sum_\xi\{|\nabla^3_g\sum_\mu A^\xi_\mu\partial^\mu_g f^{m+1}_{t^{c+1}}|^2-(n-1)|\nabla^2_g\sum_\mu A^\xi_\mu\partial^\mu_g f^{m+1}_{t^{c+1}}|^2\}=\int_0^t\int_{\mathcal{S}}\sum_\xi S(\chi',\omega',R^m),$$

where S is given by (2.271). In view of the dependence of χ' and ω' on the index c and the iteration step m, we introduce the notation $\tilde{S}^m_c=S(\chi',\omega',R^m)$. The energy estimate for the "error term" $\int_0^t\int_{\mathcal{S}}\tilde{S}^m_c$ is the content of the following lemma:

Lemma 3.10. *The following inequality holds:*

$$\Big|\int_0^t\int_{\mathcal{S}}\tilde{S}^m_c\Big|\le\lambda\int_0^t\mathfrak{D}^m_p+\frac{\tilde{C}\eta}{\lambda}\int_0^t\mathfrak{D}^{m+1}_p+\frac{\tilde{C}t}{\eta\lambda\epsilon}\mathfrak{E}^{m+1}_p+C\sup_{0\le s\le t}\big(||f^m(s)||_{H^l}+(\mathfrak{E}^m(s))^{1/2}\big)\times$$
$$\times\Big(\int_0^t\mathfrak{D}^m+\int_0^t\hat{\mathfrak{D}}^m+\int_0^t\mathfrak{D}^{m+1}_p+\int_0^t\hat{\mathfrak{D}}^{m+1}_p\Big).$$

Proof. The proof of the lemma is analogous to the proof of Lemma 3.7 and follows by a routine application of energy estimates. □

3.2.5. *Proof of Lemma 3.2.* From (3.370) and the bound on $||f^m||_{H^{2l}}$ assumed in Lemma 3.2, we conclude $||f^m||^2_{H^{2l}}\le C_A(\theta_1+\theta_2)$. Using this bound (and suppressing the index A in C_A) we combine Lemmas 3.5 - 3.10 to conclude that for any $0\le p\le l-1$, $0\le t\le T\le 1$

(3.438)
$$\mathfrak{E}^{m+1}_{\epsilon,p}+\int_0^t\mathfrak{D}^{m+1}_{\epsilon,p}\le\mathfrak{E}^{m+1}_{\epsilon,p}(0)+C(\eta+\lambda)\int_0^t\mathfrak{D}^{m+1}_p+\frac{C}{\lambda}\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}^{m+1}_q+\frac{C}{\eta\lambda}\int_0^t\mathcal{D}^{m+1}$$
$$+C\sqrt{\theta_2}\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}^{m+1}_q+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+\frac{\tilde{C}t}{\lambda}\big((1+\frac{t}{\epsilon^2})\sup_{0\le s\le t}\mathfrak{E}^{m+1}_{\epsilon,p}(s)+\theta_1+\theta_2\big)$$
$$+C(\sqrt{\theta_1}+\sqrt{\theta_2}+\lambda)\int_0^t\mathfrak{D}^{m+1}_p+\hat{C}\sup_{0\le s\le t}\big(\sqrt{\mathfrak{E}^m(s)}+||f^m(s)||_{H^4}+\lambda+t\big)t$$
$$\times\big\{\hat{\mathfrak{E}}^{m+1}(s)+\hat{\mathfrak{E}}^m(s)\big\}+\hat{C}\epsilon||f^m||^4_{L^2},$$

where we employ the convention $\sum_a^b\cdots=0$ if $a,b\in\mathbb{N}$ and $a>b$ and $\mathfrak{D}^{m+1}_l:=0$. We choose λ, η small, and allow the constants C to become larger (depending on the choice of λ and η). We conclude that there exist constants $\tilde{C}_1$, $\tilde{C}_2$ and C_2 such that for any $0 \le p \le l-1$:

(3.439)

$$\begin{aligned}&\big(1-\tilde{C}_1 t-\frac{\tilde{C}_1 t}{\epsilon^2}\big)\big(\mathfrak{E}_p^{m+1}+\int_0^t \mathfrak{D}_p^{m+1}\big)+\big(1-\tilde{C}_2 t-\hat{C}_2 t\big)\big(\hat{\mathfrak{E}}_p^{m+1}+\int_0^t \hat{\mathfrak{D}}_p^{m+1}\big)\\&\qquad\le \mathfrak{E}_{\epsilon,p}(0)+\mathcal{J}+\nu^l\sum_{q=p+1}^{l-1}\int_0^t \mathfrak{D}_q^{m+1}+C\sum_{q=0}^{p-1}\int_0^t \mathfrak{D}_q^{m+1}+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+C^*\int_0^t \mathcal{D}^{m+1},\end{aligned}$$

whereby $\nu > 0$ is a small, but fixed number and

$$\mathcal{J}:=\frac{\tilde{C}t}{\lambda}(\theta_1+\theta_2)+\hat{C}\sup_{0\le s\le t}\big(\sqrt{\mathfrak{E}^m(s)}+||f^m(s)||_{H^4}+\lambda+t\big)t\big\{\hat{\mathfrak{E}}^{m+1}(s)+\hat{\mathfrak{E}}^m(s)\big\}+\hat{C}\epsilon||f^m||_{L^2}^4.$$

Furthermore, let $D_1 := 1-\tilde{C}_1 t-\frac{\tilde{C}_1 t}{\epsilon}$, $D_2 := 1-\tilde{C}_2 t-C_2 t$. For each $0\le p\le l-1$, multiply (3.439) by ν^p and sum over all $p=0,1,\dots,l-1$. We obtain

$$\begin{aligned}&D_1\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_p^{m+1}+D_1\sum_{p=0}^{l-1}\nu^p\int_0^t \mathfrak{D}_p^{m+1}+D_2\sum_{p=0}^{l-1}\nu^p\big(\hat{\mathfrak{E}}_p^{m+1}+\int_0^t\hat{\mathfrak{D}}_p^{m+1}\big)\le\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0)+\sum_{p=0}^{l-1}\nu^p\mathcal{J}\\&+\nu^l\sum_{p=0}^{l-1}\nu^p\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}_q^{m+1}+C\sum_{p=0}^{l-1}\nu^p\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}_q^{m+1}+\sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+C^*\int_0^t\mathcal{D}^{m+1}\Big).\end{aligned}$$

From here we infer

$$\begin{aligned}&D_1\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_p^{m+1}+\big(D_1-\sum_{q=1}^{l-1}\nu^q\big)\int_0^t\mathfrak{D}_0^{m+1}+\sum_{p=1}^{l-1}\big(D_1\nu^p-\nu^l\sum_{q=0}^{p-1}\nu^q-\sum_{q=p+1}^{l-1}\nu^q\big)\int_0^t\mathfrak{D}_p^{m+1}\\&+D_2\sum_{p=0}^{l-1}\nu^p\big(\hat{\mathfrak{E}}_p^{m+1}+\int_0^t\hat{\mathfrak{D}}_p^{m+1}\big)\le\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0)+\sum_{p=0}^{l-1}\nu^p\mathcal{J}\\&\quad+\sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+C^*\int_0^t\mathcal{D}^{m+1}\Big)\end{aligned}$$

Let us set $d_0 = D_1-\sum_{q=1}^{l-1}\nu^q$ and for $1\le p\le l-1$ set $d_p = D_1\nu^p-\nu^l\sum_{q=0}^{p-1}\nu^q-\sum_{q=p+1}^{l-1}\nu^q$. From the previous inequality we immediately obtain

(3.440)

$$\begin{aligned}&D_1\mathfrak{E}^{m+1,\nu}+\sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{D}_p^{m+1}+D_2\hat{\mathfrak{E}}^{m+1,\nu}+D_2\int_0^t\hat{\mathfrak{D}}^{m+1,\nu}\\&\qquad\le\delta^\nu+\mathcal{J}^\nu+\sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+C^*\int_0^t\mathcal{D}^{m+1}\Big),\end{aligned}$$

where $\delta^\nu := \sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0)$ and $\mathcal{J}^\nu := \sum_{p=0}^{l-1}\nu^p\mathcal{J}$. Furthermore, let $D_1 := 1-\tilde{C}_1 t-\frac{\tilde{C}_1 t}{\epsilon}$, $D_2 := 1-\tilde{C}_2 t-C_2 t$. For each $0\le p\le l-1$, multiply (3.439) by ν^p and sum over all

$p=0,1,\ldots,l-1$. We obtain

$$
\begin{aligned}
&D_1\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_p^{m+1}+D_1\sum_{p=0}^{l-1}\nu^p\int_0^t\mathfrak{D}_p^{m+1}+D_2\sum_{p=0}^{l-1}\nu^p\big(\hat{\mathfrak{E}}_p^{m+1}+\int_0^t\hat{\mathfrak{D}}_p^{m+1}\big)\le\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0)+\sum_{p=0}^{l-1}\nu^p\mathcal{J}\\
&+\nu^l\sum_{p=0}^{l-1}\nu^p\sum_{q=p+1}^{l-1}\int_0^t\mathfrak{D}_q^{m+1}+C\sum_{p=0}^{l-1}\nu^p\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}_q^{m+1}+\sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+C^*\int_0^t\mathcal{D}^{m+1}\Big).
\end{aligned}
$$

From here we infer

$$
\begin{aligned}
&D_1\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_p^{m+1}+\big(D_1-\sum_{q=1}^{l-1}\nu^q\big)\int_0^t\mathfrak{D}_0^{m+1}+\sum_{p=1}^{l-1}\big(D_1\nu^p-\nu^l\sum_{q=0}^{p-1}\nu^q-\sum_{q=p+1}^{l-1}\nu^q\big)\int_0^t\mathfrak{D}_p^{m+1}\\
&+D_2\sum_{p=0}^{l-1}\nu^p\big(\hat{\mathfrak{E}}_p^{m+1}+\int_0^t\hat{\mathfrak{D}}_p^{m+1}\big)\\
&\qquad\le\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0)+\sum_{p=0}^{l-1}\nu^p\mathcal{J}+\sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+C^*\int_0^t\mathcal{D}^{m+1}\Big)
\end{aligned}
$$

Let us set $d_0=D_1-\sum_{q=1}^{l-1}\nu^q$ and for $1\le p\le l-1$ set $d_p=D_1\nu^p-\nu^l\sum_{q=0}^{p-1}\nu^q-\sum_{q=p+1}^{l-1}\nu^q$. From the previous inequality we immediately obtain

$$
\begin{aligned}
&D_1\mathfrak{E}^{m+1,\nu}+\sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{D}_p^{m+1}+D_2\hat{\mathfrak{E}}^{m+1,\nu}+D_2\int_0^t\hat{\mathfrak{D}}^{m+1,\nu}\\
&\qquad\le\delta^\nu+\mathcal{J}^\nu+\sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+C^*\int_0^t\mathcal{D}^{m+1}\Big),
\end{aligned}
\tag{3.441}
$$

where $\delta^\nu:=\sum_{p=0}^{l-1}\nu^p\mathfrak{E}_{\epsilon,p}(0)$ and $\mathcal{J}^\nu:=\sum_{p=0}^{l-1}\nu^p\mathcal{J}^p$. Let $0<\alpha\le1$ be a real number to be specified later. Multiplying the inequality (3.441) by α and adding it to (3.371), we arrive at

(3.442)

$$
\begin{aligned}
&\mathcal{E}_\epsilon^{m+1}+\int_0^t\mathcal{D}_\epsilon^{m+1}+\alpha\Big(D_1\sup_{0\le s\le t}\mathfrak{E}^{m+1,\nu}+\sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{D}_p^{m+1}\Big)+\alpha D_2\big(\sup_{0\le s\le t}\hat{\mathfrak{E}}^{m+1,\nu}+\int_0^t\hat{\mathfrak{D}}^{m+1,\nu}\big)\\
&\le\mathcal{E}_\epsilon(0)+\mathcal{K}(\theta_2,t)+\frac{C_0t}{\lambda}\big(\mathcal{E}^{m+1}+(\mathfrak{E}^{m+1})^2\big)+\frac{\bar{C}t}{\epsilon^2\eta\lambda}\mathcal{E}_\epsilon^{m+1}+C\big(\lambda+\theta_1^{1/2}+\theta_2^{1/2}+\frac{C_0t}{\lambda}+\frac{\bar{C}\eta}{\epsilon\lambda}\big)\int_0^t\mathfrak{D}^{m+1}\\
&+\big(\frac{\bar{C}t}{\lambda}+C\theta_2\big)\sup_{0\le s\le t}\mathfrak{E}^{m+1}+\alpha\Big(\delta^\nu+\mathcal{J}^\nu+\sum_{p=0}^{l-1}\nu^p\Big(\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\le s\le t}\mathcal{E}^{m+1}+C^*\int_0^t\mathcal{D}^{m+1}\Big)\Big)
\end{aligned}
$$

Absorbing the factors of $\mathcal{E}_\epsilon^{m+1}$, $\int_0^t \mathcal{D}_\epsilon^{m+1}$ and $\mathfrak{E}_\epsilon^{m+1} + \int_0^t \mathfrak{D}_\epsilon^{m+1}$ from RHS into LHS and recalling that $\mathfrak{E}^\nu + \int_0^t \mathfrak{D}^\nu \le \mathfrak{E} + \int_0^t \mathfrak{D} \le \nu^{-(l-1)}\big(\mathfrak{E}^\nu + \int_0^t \mathfrak{D}^\nu\big)$, we obtain

(3.443)

$$
\begin{aligned}
&\big(1 - \frac{C_0 t}{\lambda} - \frac{\bar{C} t}{\epsilon^2 \eta \lambda} - \frac{\alpha \tilde{C} \lambda t}{\epsilon^4} \sum_{p=0}^{l-1} \nu^p\big) \mathcal{E}_\epsilon^{m+1} + (1 - \alpha C^* \sum_{p=0}^{l-1} \nu^p) \int_0^t \mathcal{D}_\epsilon^{m+1} \\
&+ (\alpha D_1 - (\frac{\bar{C} t}{\lambda} + C\theta_2)\nu^{-(l-1)}) \sup_{0 \le s \le t} \mathfrak{E}^{m+1,\nu} \\
&+ \sum_{p=0}^{l-1} \Big(\alpha d_p - C\lambda - C\big(\lambda + \theta_1^{1/2} + \theta_2^{1/2} + \frac{C_0 t}{\lambda} + \frac{\bar{C}\eta}{\epsilon\lambda}\big)\Big) \int_0^t \mathfrak{D}_p^{m+1} + \alpha D_2 \big(\sup_{0 \le s \le t} \hat{\mathfrak{E}}^{m+1,\nu} + \int_0^t \hat{\mathfrak{D}}^{m+1,\nu}\big) \\
&\le \mathcal{E}_\epsilon(0) + \mathcal{K}(\theta_2, t) + \alpha \delta^\nu + \alpha \mathcal{J}^\nu + \frac{C_0 t}{\lambda} \nu^{-2(l-1)} (\mathfrak{E}^{m+1,\nu})^2
\end{aligned}
$$

Now set $\alpha = \min(\frac{\theta_1}{\theta_2}, \frac{1}{20 C^* \sum_{p=0}^{l-1} \nu^p}, 1)$ and choose λ, μ, θ_1, θ_2, ν and then $t = t^\epsilon$ so that

$$
\begin{aligned}
&\frac{C_0 t}{\lambda} + \frac{\bar{C} t}{\epsilon^2 \eta \lambda} + \frac{\alpha \tilde{C} \lambda t}{\epsilon^4} \sum_{p=0}^{l-1} \nu^p < 1/10;\ \alpha D_1 - (\frac{\bar{C} t}{\lambda} + C\theta_2)\nu^{-(l-1)}) \ge \frac{9}{10}\alpha; \\
&\alpha d_p - C\lambda - C\big(\lambda + \theta_1^{1/2} + \theta_2^{1/2} + \frac{C_0 t}{\lambda} + \frac{\bar{C}\eta}{\epsilon\lambda}\big) > \frac{9}{10} \alpha \nu^p; \\
&\mathcal{K}(\theta_2, t) < \frac{\theta_1}{10};\ \alpha \mathcal{J}^\nu < \frac{\theta_1}{10};\ \frac{C_0 t}{\lambda} \nu^{-2(l-1)} < \frac{1}{10} \alpha^2.
\end{aligned}
$$

Let $\theta_0 = \theta_2 \nu^{l-1}$ and assume $\mathcal{E}_\epsilon(0) + \alpha\delta^\nu \le \theta_0/2$ (and hence $\mathcal{E}_\epsilon(0) + \alpha\delta^\nu \le \theta_2/2$), we obtain from (3.443)

$$
\sup_{0 \le s \le t} \mathcal{E}_\epsilon^{m+1} + \int_0^t \mathcal{D}_\epsilon^{m+1} + \alpha\big(\sup_{0 \le s \le t} \mathfrak{E}_\epsilon^{m+1,\nu} + \int_0^t \mathfrak{D}_\epsilon^{m+1,\nu}\big) \le \frac{7}{9}\theta_2 + \frac{1}{9}\big(\alpha \sup_{0 \le s \le t} (\mathfrak{E}_\epsilon^{m+1,\nu}\big)^2.
$$

From the previous inequality we conclude that there exists a (possibly smaller) t^ϵ such that the first claim of (3.308) holds. This finishes the proof of Lemma 3.2.

3.3. Regularized Stefan problem. The principal goal of this section is the following local existence theorem:

THEOREM 3.11. *Let $\zeta > 0$. For any sufficiently small $L > 0$ there exists $T > 0$ depending on L and independent of ϵ and constants $0 < \alpha, \nu \le 1$ such that if $E_{\alpha,\nu}(u_0^\epsilon, f_0^\epsilon) \le \frac{L}{2}$ then there exists a unique solution (u^ϵ, f^ϵ) to the regularized Stefan problem defined on the time interval $[0,T[$, with $E_{\alpha,\nu}(u^\epsilon, f^\epsilon)(\cdot)$ continuous on $[0,T[$ and satisfying the bound*

$$
\sup_{0 \le t \le T} E_{\alpha,\nu}(u^\epsilon, f^\epsilon)(t) + \int_0^T D_{\alpha,\nu}(u^\epsilon, f^\epsilon)(\tau)\, d\tau \le L.
$$

3.3.1. *Local existence for the regularized Stefan problem.* Having obtained the uniform bounds on the sequence (u^m, R^m), our goal in this section is to prove that this sequence is actually Cauchy in the energy space. In order to do that, we need to compare two consecutive iterates in the energy norm. However, te expression $\mathcal{E}_\epsilon(u^{m+1}-u^m, R^{m+1}-R^m)$, does not a-priori make sense, because the functions u^{m+1} and u^m are defined on *different* domains Ω^m and Ω^{m-1}, respectively. To overcome this difficulty, we shall pull-back the function u^m from Ω^{m-1} onto $\Omega_{\mathcal{S}}$, where we recall $\Omega_{\mathcal{S}} = \Omega \setminus \mathcal{S}$. To apply this change of variables, we first define the map $\bar{\mathbf{x}} = \Theta^m(t,\mathbf{x})$, where $\Theta^m(t,\mathbf{x}) = \pi^m(t,\mathbf{x})\mathbf{x}$ and π^m is a smooth scalar-valued function with the property

$$\pi^m(t,x) = \begin{cases} 1/R^{m-1}(\frac{\mathbf{x}}{|\mathbf{x}|}), & |\mathbf{x}|-1 \le d, \\ 1, & |\mathbf{x}|-1 \ge 2d \end{cases} \quad d \quad \text{is small.} \tag{3.444}$$

For any $\mathbf{x} \in \Gamma^{m-1}$, we may write $\mathbf{x} = R^{m-1}(\xi \frac{\xi}{|\xi|}) = R^{m-1}(\xi)$, by the definition of Γ^{m-1}. Hence

$$\Theta^m(t,\mathbf{x}) = \pi^m(t, R^{m-1}(\xi)\frac{\mathbf{x}}{|\mathbf{x}|})R^{m-1}(\xi)\frac{\mathbf{x}}{|\mathbf{x}|} = \frac{1}{R^{m-1}(\xi)}R^{m-1}(\xi)\frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{|\mathbf{x}|} \in \mathcal{S}.$$

Thus π^m does the job for us: $\pi^m : \Omega^{m-1} \to \Omega_{\mathcal{S}}$. Note that this map is natural from the point of view of our *geometric* assumption that the evolving interfaces Γ^m are, in fact, 'graphs' over the unit sphere $\mathcal{S}$.

The inverse map $x(\bar{x})$ to the above change of variables (3.444) is given by

$$x = \rho^m(\bar{x})\bar{x}.$$

To convince ourselves that the function $\rho^m : \Omega_{\mathcal{S}} \to \mathbb{R}$ is well defined, we observe that ρ^m has to satisfy the relation

$$\pi^m(t, \rho^m(t,\bar{x})\bar{x})\rho^m(\bar{x}) - 1 = 0.$$

The existence of such a ρ^m can be established by the implicit function theorem, applied to the equation $F(t,s,\bar{x}) = 0$, where $F : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is given by

$F(t,s,v)=\pi^m(t,sv)s-1$. Namely, $F_s(t,s,v)=(\nabla\pi^m\cdot v)s+\pi^m>0$ since $\pi^m\approx 1$ and $\nabla\pi^m=O(\nabla_g f^{m-1})$ is small. We define $\bar{u}^m:\Omega_{\mathcal{S}}\to\mathbb{R}$ by setting

$$\bar{u}^m(\bar{x}):=u^m(t,\rho^m(t,x)). \tag{3.445}$$

The heat operator $\partial_t-\Delta$ on the domain Ω^{m-1} will transform into a more complicated operator in the new coordinates as stated in the following lemma:

LEMMA 3.12. *The push forward of the operator $\partial_t-\Delta$ with respect to the map $\Theta^m:\Omega^{m-1}\to\mathcal{S}$ reads:*

$$(\partial_t-\Delta)^{\#}\bar{u}=\bar{u}_t-\Delta_{\bar{x}}\bar{u}-a^m_{ij}\bar{u}_{\bar{x}^i\bar{x}^i}-b^m_i\bar{u}_{\bar{x}^i},$$

where

$$a^m_{ij}:=((\pi^m)^2-1)\delta_{ij}+2\pi^m x^j\pi^m_{x^i}+x^ix^j|\nabla\pi^m|^2,\quad b^m_i:=2\pi^m_{x^i}+\Delta\pi^m x^i-\pi^m_t x^i. \tag{3.446}$$

Furthermore,

$$[\bar{u}^{m+1}_n]^+_-=\frac{(R^m)^2}{|g^m|}\left([u^{m+1}_n]^+_-\right)\circ\phi^m \tag{3.447}$$

Note that (3.444) implies

$$\bar{x}_{x^i}=\pi^m e_i+\pi^m_{x^i}x. \tag{3.448}$$

From $\pi^m(x)=1/\rho^m(\bar{x})$, we obtain $\pi^m_{x^i}=-(\nabla\rho^m\cdot\bar{x}_{x^i})/(\rho^m)^2$, which in turn, combined with (3.448), after an elementary calculation implies

$$\pi^m_{x^i}=-\frac{\rho^m_{\bar{x}^i}}{(\rho^m)^2\Psi}, \tag{3.449}$$

After further differentiating (3.449) with repspect to x^i and using the relation (3.448), we arrive at

$$\Delta\pi^m=\frac{-\Delta\rho^m}{(\rho^m)^3\Psi}+\frac{2|\nabla\rho^m|^2\Psi-2|\nabla\rho^m|^2\nabla\rho^m\cdot\bar{x}+\rho^m|\nabla\rho^m|^2\Psi^{-1}\bar{x}^j\bar{x}^k\rho^m_{\bar{x}^j\bar{x}^k}}{(\rho^m)^4\Psi^2}.$$

where $\Psi=\rho^m-\nabla\rho^m\cdot\bar{x}$. Similarly it is not hard to see that $\pi^m_t=-\rho^m_t/(\rho^m)^2$. In order to evaluate the Laplacian in new coordinates, by (3.445) we first write

$$u(t,x)=\bar{u}^m(t,\pi^m(t,x)x). \tag{3.450}$$

Applying Δ_x to the left-hand side, we obtain

(3.451) $$\Delta u(t,x) = (\pi^m)^2 \Delta_{\bar{x}} \bar{u}^m + \big(2\pi^m x^j \pi^m_{x^i} + x^i x^j |\nabla \pi^m|^2\big) \bar{u}^m_{\bar{x}^i \bar{x}^j} + \big(2\pi^m_{x^i} + \Delta \pi^m x^i\big) \bar{u}^m_{x^i}$$

Applying ∂_t to the left hand side of (3.450), we obtain

(3.452) $$u_t(t,x) = \bar{u}^m_t + \pi^m_t \nabla_{\bar{x}} \bar{u}^m \cdot x.$$

Since $(\partial_t - \Delta) u^m = 0$, we conclude from (3.451) and (3.452)

$$(\partial_t - \Delta_{\bar{x}}) \bar{u}^m = a_{ij} \bar{u}^m_{\bar{x}^i \bar{x}^i} + b_i \bar{u}^m_{\bar{x}^i},$$

where a^m_{ij} and b^m_i are given by (3.446). Furthermore, using (3.449), it is easy to see that

$$a_{ij} = \frac{1-(\rho^m)^2}{(\rho^m)^2} \delta_{ij} - 2\frac{\rho^m_{\bar{x}^i} \bar{x}^j}{(\rho^m)^2 \Psi} + \frac{\bar{x}^i \bar{x}^j |\nabla \rho^m|^2}{(\rho^m)^2 \Psi^2}.$$

We now turn to the proof of (3.447). Note that

$$u^{m+1}_{x^i} = \nabla_{\bar{x}} \bar{u}^{m+1} \bar{x}_{x^i} = \nabla_{\bar{x}} \bar{u}^{m+1} \big(\pi^{m+1} e_i + \pi^{m+1}_{x^i} x\big) = \nabla_{\bar{x}} \bar{u}^{m+1} \big(\frac{e_i}{\rho^{m+1}} - \frac{\rho^{m+1}_{\bar{x}^i}}{(\rho^{m+1})^2 \Psi^{m+1}} \rho^{m+1} \bar{x}\big).$$

From here, we infer the formula

$$\nabla u^{m+1}(x) = \frac{1}{\rho^{m+1}} \nabla \bar{u}^{m+1}(\bar{x}) - \frac{\nabla \rho^{m+1}(\bar{x})}{\rho^{m+1}(\bar{x}) \Psi^{m+1}(\bar{x})} \bar{x} \cdot \nabla \bar{u}^{m+1}(\bar{x}).$$

From the above formula, we obtain

(3.453) $$[\nabla u^{m+1}]^+_- \circ \phi^m(\xi) = \frac{1}{\rho^{m+1}(\xi)} [\nabla \bar{u}^{m+1}]^+_-(\xi) - \frac{\nabla_g \rho^{m+1}(\xi)}{\rho^{m+1}(\xi) \Psi^{m+1}(\xi)} \xi \cdot [\nabla \bar{u}^{m+1}]^+_-(\xi)$$

Note that

$$\xi \cdot [\nabla \bar{u}^{m+1}]^+_-(\xi) = \xi \cdot \big([\nabla_g \bar{u}^{m+1}]^+_- + [\bar{u}^{m+1}_n]^+_- n_{\mathcal{S}}\big) = [\bar{u}^{m+1}_n]^+_-,$$

since $[\nabla_g \bar{u}^{m+1}]^+_- = 0$ (due to the fact that $u^{m+1,+}|_{\mathcal{S}} = u^{m+1,-}|_{\mathcal{S}}$) and $\xi \cdot n_{\mathcal{S}} = |n_{\mathcal{S}}|^2 = 1$. Observe further that $\rho^{m+1}(\xi) = R^m(\xi)$. Furthermore, since $\nabla \rho^{m+1}(\xi) \cdot \xi = \nabla_g \rho^{m+1} \cdot \xi = 0$, we also have $\Psi^{m+1}(\xi) = \rho^{m+1}(\xi) = R^m(\xi)$. Form these observations and from (3.453) we obtain the formula

$$[\nabla u^{m+1}]^+_- \circ \phi^m = \frac{1}{\rho^{m+1}} [\bar{u}^{m+1}_n]^+_- n_{\mathcal{S}} - \frac{\nabla_g \rho^{m+1}}{\rho^{m+1} \Psi^{m+1}} [\bar{u}^{m+1}_n]^+_- = \frac{1}{R^m} [\bar{u}^{m+1}_n]^+_- n_{\mathcal{S}} - \frac{\nabla_g R^m}{(R^m)^2} [\bar{u}^{m+1}_n]^+_-.$$

It is straightforward to see that $n_{\Gamma^m} \circ \phi^m = \frac{R^m}{|g^m|} n_{\mathcal{S}} - \frac{\nabla_g f^m}{|g^m|}$, where $n_{\mathcal{S}}$ stands for the unit normal on $\mathcal{S}$. Hence

$$\Big([\nabla u^{m+1}]_-^+ \cdot n_{\Gamma^m}\Big) \circ \phi^m = \Big(\frac{1}{R^m}[\bar{u}_n^{m+1}]_-^+ n_{\mathcal{S}} - \frac{\nabla_g f^m}{(R^m)^2}[\bar{u}_n^{m+1}]_-^+\Big) \cdot \Big(\frac{R^m}{|g^m|} n_{\mathcal{S}} - \frac{\nabla_g f^m}{|g^m|}\Big)$$
$$= \frac{1}{|g^m|}[\bar{u}_n^{m+1}]_-^+ + \frac{|\nabla_g f^m|^2}{|g^m|(R^m)^2}[\bar{u}_n^{m+1}]_-^+ = \frac{|g^m|}{(R^m)^2}[\bar{u}_n^{m+1}]_-^+$$

and this proves (3.447). $\square$

Since $\Big([\nabla u^{m+1}]_-^+ \cdot n_{\Gamma^m}\Big) \circ \phi^m = \frac{-f_t^{m+1} R^m}{|g^m|} + \epsilon \frac{-\Delta_g f_t^{m+1}}{(R^m)^{n-2}|g^m|}$, from (3.447), we immediately obtain

$$[\bar{u}_n^{m+1}]_-^+ = -\frac{f_t^{m+1} R^m}{|g^m|} - \epsilon \frac{\Delta_g^2 f_t^{m+1}}{(R^m)^{n-2}|g^m|} - \Xi(f^m), \tag{3.454}$$

where

$$\Xi(f^m) \equiv \Xi^m := \big(1 - \frac{|g^m|}{(R^m)^2}\big)[\bar{u}_n^{m+1}]_-^+.$$

For any $k \in \mathbb{N}$, let $v^{k+1} := \bar{u}^{k+1} - \bar{u}^k$ and $\sigma^{k+1} = f^{k+1} - f^k$. By subtracting two consecutive equations in the iteration process, we obtain:

$$v_t^{m+1} - \Delta v^{m+1} = f_m^\circ \quad \text{on} \quad \Omega^m \tag{3.455}$$

$$v^{m+1} = -\sigma^m - \frac{\Delta_g \sigma^m}{R^m |g^m|} + G_m^\circ \quad \text{on} \quad \mathcal{S}, \tag{3.456}$$

$$v_n^{m+1} = 0 \quad \text{on} \quad \partial\Omega, \tag{3.457}$$

$$[v_n^{m+1}]_-^+ = -\frac{\sigma_t^{m+1} R^m}{|g^m|} - \epsilon \frac{\Delta_g^2 \sigma_t^{m+1}}{(R^m)^{n-2}|g^m|} + h_m^\circ \quad \text{on} \quad \mathcal{S}. \tag{3.458}$$

Here

$$f_m^\circ = a_{ij}^{m+1} \bar{u}_{ij}^{m+1} + b_i^{m+1} \bar{u}_i^{m+1} - a_{ij}^m \bar{u}_{ij}^m + b_i^m \bar{u}_i^m,$$

$$G_m^\circ = N^*(f^m) - N^*(f^{m-1}),$$

$$h_m^\circ = -f_t^m\big(\frac{R^m}{|g^m|} - \frac{R^{m-1}}{|g^{m-1}|}\big) - \epsilon \Delta_g^2 f_t^m\big(\frac{1}{(R^m)^{n-2}|g^m|} - \frac{1}{(R^{m-1})^{n-2}|g^{m-1}|}\big) - \Xi^m + \Xi^{m-1}.$$

Here, the quadratic nonlinearity $N^*(\cdot)$ is defined in the line after (3.291). Upon applying the differential operator ∂_{t^k} to the equations (3.455), (3.456) and (3.458) and singling out the leading order terms, we arrive at:

$$v^{m+1}_{t^{k+1}} - \Delta v^{m+1}_{t^k} = f'_m, \tag{3.459}$$

$$v^{m+1}_{t^k} = -\sigma^m_{t^k} - \frac{\Delta_g \sigma^m_{t^k}}{R^m|g^m|} + G'_m \quad \text{on} \quad \mathcal{S}, \tag{3.460}$$

$$[\partial_n v^{m+1}_{t^k}]^+_- = -\sigma^{m+1}_{t^{k+1}} \frac{R^m}{|g^m|} - \epsilon \frac{\Delta_g^2 \sigma^{m+1}_{t^{k+1}}}{(R^m)^{n-2}|g^m|} + h'_m \quad \text{on} \quad \mathcal{S}, \tag{3.461}$$

where

$$f'_m = \partial_{t^k} f^\circ_m \tag{3.462}$$

$$G'_m = -\sum_{q=0}^{k-1} \Delta_g \sigma^m_{t^q} \big(\frac{1}{R^m|g^m|}\big)_{t^{k-q}} + \partial_{t^k} G^\circ_m. \tag{3.463}$$

$$h'_m = -\sum_{q=0}^{k-1} \sigma^{m+1}_{t^{q+1}} \big(\frac{R^m}{|g^m|}\big)_{t^{k-q}} - \epsilon \sum_{q=0}^{k-1} \Delta_g^2 \sigma^{m+1}_{t^{q+1}} \big(\frac{1}{(R^m)^{n-2}|g^m|}\big)_{t^{k-q}} + \partial_{t^k} h^\circ_m. \tag{3.464}$$

In order to obtain the energy identities for the problem (3.459) - (3.461), we may use the identities from Section 2. Respecting the notations from (2.257) - (2.261) from Section 2, for any $j \in \mathbb{N}$ we set

$$\mathcal{F} = f'_j, F \equiv 1, G = g'_j, h = h'_j, \mathcal{U} = v^{j+1}_{t^k}, \omega = \sigma^{j+1}_{t^k}, \chi = \sigma^j_{t^k}, \text{ and } \psi = R^j. \tag{3.465}$$

Note that $\Gamma = \mathcal{S}$ and hence, since the domain is not time-dependent, we have $V_\Gamma = 0$ and the terms $\tilde{\mathcal{U}}^\pm$ and $\tilde{\mathcal{V}}$ in equations (2.258) and (2.260) are simply equal to 0. In other words, the expressions $O(\mathcal{U})$ and $P(\mathcal{U})$ defined in (2.268) and (2.269) are equal to 0. We define the energy:

$$\bar{e}^j := \sum_{k=0}^{l-1} \bar{\mathcal{E}}_\epsilon(v^j, \sigma^j), \quad \bar{d}^j := \sum_{k=0}^{l-1} \bar{\mathcal{D}}_\epsilon(v^j, \sigma^j),$$

where $\bar{\mathcal{E}}_\epsilon$ and $\bar{\mathcal{D}}_\epsilon$ are defined by (2.262) and (2.263) respectively. Furthermore, using (2.266) and (3.465) and keeping in mind that $O=P=0$, we arrive at

$$\bar{e}^{m+1}(t)+\int_0^t \bar{d}^{m+1}(s)\,ds=\int_0^t\int_{\mathbb{S}^1}\{q^m+s^m\}, \tag{3.466}$$

where $q^m=\sum_{k=0}^{l-1}Q(\sigma^m_{t^k},\sigma^{m+1}_{t^k},R^m)$, $s^m=\sum_{k=0}^{l-1}S(\sigma^m_{t^k},\sigma^{m+1}_{t^k},R^m)$. Here, Q and S are defined by (2.270) and (2.271) respectively. Just like in Section 3.2, for any $0<\nu\le 1$, we introduce the higher order energy:

$$\mathfrak{e}^{j,\nu}:=\mathfrak{E}^\nu(v^j,\sigma^j),\ \hat{\mathfrak{e}}^{j,\nu}:=\hat{\mathfrak{E}}^{\epsilon,\nu}(v^j,\sigma^j),\quad \mathfrak{d}^{j,\nu}:=\mathfrak{D}^\nu(v^j,\sigma^j),\ \hat{\mathfrak{d}}^{j,\nu}:=\hat{\mathfrak{D}}^{\epsilon,\nu}(v^j,\sigma^j); \tag{3.467}$$

here $j\in\mathbb{N}$, $0\le k\le l-1$. Like in the previous sections, we drop the index ν if $\nu=1$. Before we explain the energy estimates, we first examine the form of f°_m, G°_m and h°_m featuring RHS of equations (3.455), (3.456) and (3.458) respectively. Note that f°_m can be rewritten in the form

$$f^\circ_m=(a^{m+1}_{ij}-a^m_{ij})\bar{u}^{m+1}_{ij}+a^m_{ij}(\bar{u}^{m+1}_{ij}-\bar{u}^m_{ij})+(b^{m+1}_i-b^m_i)\bar{u}^{m+1}_i+b^m_i(\bar{u}^{m+1}_i-\bar{u}^m_i).$$

We observe that $a^{m+1}_{ij}-a^m_{ij}$ can be written as $A_{ij}(\sigma^m,\nabla_g\sigma^m)$ for some smooth function with bounded derivatives A_{ij}. Similarly we can write $b^{m+1}_i-b^m_i$ in the form $B_i(\sigma^m,\nabla_g\sigma^m,\nabla^2_g\sigma^m,\sigma^m_t)$, for some smooth function B_i. In other words f°_m takes the form

$$f^\circ_m=A_{ij}(\sigma^m,\nabla_g\sigma^m)\bar{u}^{m+1}_{ij}+a^m_{ij}v^{m+1}_{ij}+B_i(\sigma^m,\nabla_g\sigma^m,\nabla^2_g\sigma^m,\sigma^m_t)\bar{u}^{m+1}_i+b^m_iv^{m+1}_i$$

In order to write h°_m in a similar form, note that

$$\begin{aligned}-\Xi^m+\Xi^{m-1}&=-\big(1-\frac{|g^m|}{(R^m)^2}\big)[\bar{u}^{m+1}_n]^+_-+\big(1-\frac{|g^{m-1}|}{(R^{m-1})^2}\big)[\bar{u}^m_n]^+_-\\&=-[v^{m+1}_n]^+_-\big(1-\frac{|g^m|}{(R^m)^2}\big)+[\bar{u}^m_n]^+_-\big(\frac{|g^m|}{(R^m)^2}-\frac{|g^{m-1}|}{(R^{m-1})^2}\big)\end{aligned}$$

and hence

$$\begin{aligned}h^\circ_m=&-f^m_t\big(\frac{R^m}{|g^m|}-\frac{R^{m-1}}{|g^{m-1}|}\big)-\epsilon\Delta^2_gf^m_t\big(\frac{1}{(R^m)^{n-2}|g^m|}-\frac{1}{(R^{m-1})^{n-2}|g^{m-1}|}\big)\\&-[v^{m+1}_n]^+_-\big(1-\frac{|g^m|}{(R^m)^2}\big)+[\bar{u}^m_n]^+_-\big(\frac{|g^m|}{(R^m)^2}-\frac{|g^{m-1}|}{(R^{m-1})^2}\big).\end{aligned} \tag{3.468}$$

Similar structure is also shared by G°_m. The conclusion is that f°_m, G°_m and h°_m have a quadratic form, always containing one term that can be bounded by the norms of $v^{j+1}-v^j$ and $\sigma^{j+1}-\sigma^j$, and the other term bounded by the norms of $\bar{u}^j$ and f^j (and hence a-priori bounded due to Lemma 3.2). In order to bound the first modes of $\sigma^{m+1}_{t^k}$, we shall use relation (3.458). If we rewrite (3.458) in the form

$$\sigma^{m+1}_t + \epsilon\Delta^2_g\sigma^{m+1}_t = -|g^m|(R^m)^{n-2}[v^{m+1}_n]^+_- + |g^m|(R^m)^{n-2}h^\circ_m - ((R^m)^{n-1}-1)\sigma^{m+1}_t,$$

upon applying ∂_{t^q}, multiplying the whole equation by $\sigma^{m+1}_{t^{q+1}}$ and integrating over $\mathcal{S}$, we obtain

(3.469)

$$\begin{aligned}&||\sigma^{m+1}_{t^{q+1}}||^2_{L^2} + \epsilon||\nabla^2_g\sigma^{m+1}_{t^{q+1}}||^2_{L^2} \leq \Big(||(|g^m|(R^m)^{n-2}[v^{m+1}_n]\circ\phi^m)_{t^q}||_{L^2} + ||(|g^m|(R^m)^{n-2}h^\circ_m)_{t^q}||_{L^2}\\ &+||(((R^m)^{n-1}-1)\sigma^{m+1}_t)_{t^q}||_{L^2}\Big)||\sigma^{m+1}_{t^{q+1}}||_{L^2} \leq \frac{5}{4}\Big(||(|g^m|(R^m)^{n-2}[v^{m+1}_n]\circ\phi^m)_{t^q}||^2_{L^2}\\ &+|||g^m|(R^m)^{n-2}h^\circ_m||^2_{L^2} + ||(((R^m)^{n-1}-1)\sigma^{m+1}_t)_{t^q}||^2_{L^2}\Big) + \frac{1}{5}||\sigma^{m+1}_{t^{q+1}}||^2_{L^2}.\end{aligned}$$

We note that $||(|g^m|(R^m)^{n-2}[v^{m+1}_n]\circ\phi^m)_{t^q}||^2_{L^2} \leq \frac{C}{\lambda}e^{m+1} + \lambda||\nabla v^{m+1}||^2_{H^1(\Omega)}$. It is further easy to see that $||(|g^m|(R^m)^{n-2}h^\circ_m)_{t^q}||^2_{L^2} + ||(((R^m)^{n-1}-1)\sigma^{m+1}_t)_{t^q}||^2_{L^2} \leq C\theta\sum_{k=0}^{l-1}\big(||\sigma^m_{t^k}||^2_{L^2} + ||\sigma^{m+1}_{t^{k+1}}||^2_{L^2}\big)$. Thus, using (3.469), we obtain the bound

$$\begin{aligned}&\sum_{k=0}^{l-1}\Big\{\sup_{0\leq s\leq t}||\mathbb{P}_1\sigma^{m+1}_{t^k}||^2_{L^2} + \int_0^t ||\mathbb{P}_1\sigma^{m+1}_{t^{k+1}}||^2_{L^2}\Big\} \leq \frac{C}{\lambda}t\sup_{0\leq s\leq t}\bar{e}^{m+1} + \lambda\int_0^t \mathfrak{d}^{m+1}\\ &+C\theta\sum_{k=0}^{l-1}\Big\{\sup_{0\leq s\leq t}||\sigma^m_{t^k}||^2_{L^2} + \int_0^t ||\sigma^m_{t^{k+1}}||^2_{L^2}\Big\}.\end{aligned} \tag{3.470}$$

Motivated by (3.470), we introduce the following temporal energy quantities:

$$e^j := \bar{e}^j + \sum_{k=0}^{l-1}||\mathbb{P}_1\sigma^j_{t^k}||^2_{L^2}; \quad d^j := \bar{d}^j + \sum_{k=0}^{l-1}||\mathbb{P}_1\sigma^j_{t^{k+1}}||^2_{L^2}. \tag{3.471}$$

Remark. Observe that the energy norm e^j is not equivalent to $\mathcal{E}^j$, because the first modes of σ^j are also controlled by $\mathcal{E}^j$; in other words $e^j \geq C_E||\mathbb{P}_1\sigma^j||^2_{L^2}$ for some constant C_E. An analogous remark holds for d^j.

The next lemma is the analogue of Lemma 3.2, where we collect preparatory statements, necessary to establish (3.473). Note that in the following the constant C_ϵ depends on ϵ and we shall make use of the ϵ-regularization (3.280).

LEMMA 3.13. (a) *For any $j\in\mathbb{N}$, the following conservation law holds:*

$$\partial_t\Big\{\int_{\Omega^j} v^{j+1}+\int_{\mathcal{S}}\sigma^{j+1}\Big\}=-\int_{\mathcal{S}}((R^j)^{n-1}-1)\sigma_t^{j+1}+\int_{\mathcal{S}}\{h_j^\circ+f_j^\circ\}.$$

(b) *The energy norms e^j and d^j are equivalent to $\mathcal{E}_\parallel+||\mathbb{P}_1\sigma||_{L^2}^2$ and $\mathcal{D}_\parallel$ respectively.*

(c) *There exist constants K^* and $0<\nu^*\leq 1$ such that for any $j\in\mathbb{N}$ the following inequality for space-time energies holds:*

$$\sup_{0\leq s\leq t}\mathfrak{e}^{j+1,\nu^*}+\int_0^t\mathfrak{d}^{j+1,\nu^*}\leq C_\epsilon(\theta+t)\big(\sup_{0\leq s\leq t}\mathfrak{e}^{j,\nu^*}+\int_0^t\mathfrak{d}^{j,\nu^*}\big)+K^*\int_0^t d^{j+1}(\tau)\,d\tau. \tag{3.472}$$

(d) *For a possibly smaller t^ϵ there exists a C such that for any $t\leq t^\epsilon$ the following inequality for the temporal energies holds:*

$$\sup_{0\leq s\leq t}e^{m+1}(s)+\int_0^t d^{m+1}(\tau)\,d\tau\leq C_\epsilon(\theta+t)\Big(\sup_{0\leq s\leq t}e^m(t)+\int_0^t d^m(\tau)\,d\tau\Big). \tag{3.473}$$

Using (3.473) together with part (c) of Lemma 3.13 yields a contraction bound on the total energy:

$$\begin{aligned}
&\sup_{0\leq s\leq t}e^{m+1}(s)+\int_0^t d^{m+1}(\tau)\,d\tau+\sup_{0\leq s\leq t}\mathfrak{e}^{m+1,\nu^*}+\int_0^t\mathfrak{d}^{m+1,\nu^*}\\
&\leq\Lambda\Big(\sup_{0\leq s\leq t}e^m(t)+\int_0^t d^m(\tau)\,d\tau\Big)+C(\theta+t)\Big(\sup_{0\leq s\leq t}\mathfrak{e}^{m,\nu^*}+\int_0^t\mathfrak{d}^{m,\nu^*}\Big)\\
&\quad+K^*\Lambda\Big(\sup_{0\leq s\leq t}e^m(t)+\int_0^t d^m(\tau)\,d\tau\Big)\\
&\leq\Lambda^*\Big(\sup_{0\leq s\leq t}e^m(s)+\int_0^t d^m(\tau)\,d\tau+\sup_{0\leq s\leq t}\mathfrak{e}^{m,\nu^*}+\int_0^t\mathfrak{d}^{m,\nu^*}\Big),
\end{aligned} \tag{3.474}$$

where $\Lambda^*:=\Lambda(1+K^*)+C(\theta+t)$. Note that $\Lambda^*<1$, if we choose θ small enough and possibly a smaller t^ϵ so that $\Lambda<\frac{1-C(\theta+t)}{1+K^*}$. Having proved (3.474), we conclude that the sequence $(v^k,\sigma^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the energy space defined for some t^ϵ small enough. Passing to the limit we recover the solution to the regularized Stefan problem on the time interval $[0,t^\epsilon[$.

3.3.2. *Uniqueness.* By the use of Gronwall's lemma, we prove uniqueness of the solutions to the regularized Stefan problem:

Lemma 3.14. *In the class of functions u,f satisfying* $\sup_{0\le s\le t^\epsilon} E_{\alpha,\nu}(u,f)+\int_0^{t^\epsilon} D_{\alpha,\nu}(u,f)(s)\,ds\le L$ *(where L may be chosen smaller if necessary), the solution (u,f) to the regularized Stefan problem is unique.*

Proof. The proof is rather straightforward and follows by the usual energy estimates. The difference is that we do not have to differentiate higher in the energy space. For completeness, we outline the argument of the proof. Let us assume that there exists another solution $(\tilde u,\tilde f)$ satisfying the same initial condition $(\tilde u(x,0),\tilde f(\theta,0))=(u_0(x),f_0(\theta))$ and the same smallness bound $\sup_{0\le s\le t^\epsilon} E_{\alpha,\nu}(\tilde u,\tilde f)+\int_0^{t^\epsilon} D_{\alpha,\nu}(\tilde u,\tilde f)(s)\,ds\le L$. We shall formulate the Stefan problem on the fixed domain by pulling both functions u and $\tilde u$ onto the fixed domain $\Omega_{S_1(\mathbf{x}_0)}$. Without loss of generality we assume $\Omega_{S_1(\mathbf{x}_0)}=\Omega_{\mathcal{S}}$. To this end, we shall apply the change of variables described in Subsection 3.3.1. We define $\Theta(t,x):=\pi(t,x)x$ and $\tilde\Theta(t,x):=\tilde\pi(t,x)x$, where π and $\tilde\pi$ are defined as in (3.444) by substituting R^{m-1} with $(1+f)$ and $(1+\tilde f)$ respectively. We set $U:=u\circ\Theta$ and $\tilde U=\tilde u\circ\tilde\Theta$. The corresponding coefficients a_{ij}, b_i, $\tilde a_{ij}$, $\tilde b_i$ are defined analogously to a_{ij}^m, b_i^m (given by (3.446)). Subtracting the two sets of equations (that we get for (U,f) and $(\tilde U,\tilde f)$), we obtain a new problem for $(V,h):=(U-\tilde U,f-\tilde f)$ $\big($note that $\tilde R=1+\tilde f$ and $|\tilde g|=\sqrt{\tilde R^2+\tilde f_\theta^2}\big)$:

$$V_t-\Delta V=A^\circ, \tag{3.475}$$

$$V=-h-\frac{\Delta_g h}{R|g|}+B^\circ, \tag{3.476}$$

$$[V_n]_-^+=-\frac{h_tR}{|g|}-\epsilon\frac{\Delta_g^2 h_t}{R^{n-2}|g|}+C^\circ \text{ on } \mathcal{S}, \tag{3.477}$$

where

$$A^\circ=(a_{ij}-\tilde a_{ij})U_{ij}+\tilde a_{ij}V_{ij}+(b_i-\tilde b_i)U_i+\tilde b_iV_i,$$

$$B^\circ=N^*(f)-N^*(\tilde f)+\Delta_g\tilde f\Big(\frac{1}{R|g|}-\frac{1}{\tilde R|\tilde g|}\Big),$$

$$C^\circ=-\tilde f_t\Big(\frac{R}{|g|}-\frac{\tilde R}{|\tilde g|}\Big)-\epsilon\Delta_g^2\tilde f_t\Big(\frac{1}{(R)^{n-2}|g|}-\frac{1}{(\tilde R)^{n-2}|\tilde g|}\Big)-\Xi(f)+\Xi(\tilde f).$$

Here $N^*(f)$ is defined in line after (3.291) and $\Xi(f)$ and $\Xi(\tilde{f})$ are defined by replacing f^m by f and $\tilde{f}$ in the definition of $\Xi(f^m)$ in the line after (3.454), respectively. We use the notation from (2.257) - (2.261) and Lemma 2.2 from Section 2 to derive the accompanying energy identities. To this end, we set

$$\mathcal{F}=A^\circ, F\equiv 1, G=B^\circ, h=C^\circ, \mathcal{U}=V, \omega=\chi=h \text{ and } \psi=R.$$

Same remark as in the line after (3.465) applies here; since $\Gamma=\mathcal{S}$ is time-independent, we conclude that the expressions $O(\mathcal{U})$ and $P(\mathcal{U})$ defined in (2.268) and (2.269) are equal to 0. In formal comparison to the problem (3.455) - (3.458), f takes the role of f^{m+1}, and $\tilde{f}$ the role of f^m. Additionally, the cross-terms vanish since $\omega=\chi=h$. We are naturally led to the following energy quantities:

$$\mathcal{E}^* := \tilde{\mathcal{E}}_\epsilon(V,h), \quad \mathcal{D}^* := \tilde{\mathcal{D}}_\epsilon(V,h),$$

where $\tilde{\mathcal{E}}_\epsilon$ and $\tilde{\mathcal{D}}_\epsilon$ are defined by (2.262) and (2.263), respectively. In addition to this, we define $Q^*=Q(h,h,R)$ and analogously S^*. Using the identity (2.266), we obtain

$$\mathcal{E}^*(t)+\int_0^t \mathcal{D}^*(s)\,ds=\int_0^t\int_{\Omega_{\mathcal{S}}}\big(V+V_t\big)A+\int_0^t\int_{\mathcal{S}}\big\{Q^*+S^*\big\}. \tag{3.478}$$

In addition to the above energies $\mathcal{E}^*$ and $\mathcal{D}^*$, we define

$$\begin{aligned}\mathfrak{E}^* &= ||\nabla V||^2_{L^2(\Omega)}+\int_{\mathcal{S}}\Big\{|\nabla_g^2 h|^2-(n-1)|\nabla_g h|^2\Big\}-\int_{\mathcal{S}}|\mathbb{P}_{2+}\nabla_g^2 h_{\theta^2}|^2\big(\frac{1}{|g|^3}-1\big)\\ &\quad+\epsilon\int_{\mathbb{S}^1}\Big\{|h_{\theta^4}|^2-|h_{\theta^3}|^2\Big\}-\epsilon\int_{\mathbb{S}^1}|\mathbb{P}_{2+}h_{\theta^4}|^2\big(\frac{1}{R|g|^3}-1\big)\end{aligned}$$

$$\mathfrak{D}^* = ||\nabla V||^2_{H^1(\Omega)}+\int_{\mathbb{S}^1}\Big\{|h_{\theta t}|^2-|h_t|^2\Big\}+\epsilon\int_{\mathbb{S}^1}\Big\{|f_{\theta^3 t}|^2-|f_{\theta^2 t}|^2\Big\}$$

Note that the quantities $\mathfrak{E}^*$ and $\mathfrak{D}^*$ are identical to $\mathfrak{E}$ and $\mathfrak{D}$ in the case when $l=1$. Our goal is to prove that for some $0<\alpha^*\le 1$ the following inequality holds

$$\begin{aligned}&\mathcal{E}^*(t)+\alpha^*\mathfrak{E}^*(t)+\int_0^t\big\{\mathcal{D}^*(s)+\alpha^*\mathfrak{D}^*(s)\big\}\,ds\\ &\quad\le C\int_0^t\big\{\mathcal{E}^*(s)+\mathfrak{E}^*(s)\big\}\,ds+C(\sqrt{L}+\alpha^*)\int_0^t\mathcal{D}^*(s)\,ds,\end{aligned} \tag{3.479}$$

which would enable us to absorb the multiple of $\int_0^t\mathcal{D}^*$ on the right-hand side into the left-hand side and then use the Gronwall's inequality to conclude that $\mathcal{E}^*(t)+\mathfrak{E}^*(t)=0$

for any $t \geq 0$. It is essential that the constant C in the above estimate does not depend on ϵ so that the smallness bound on L remains independent of ϵ. That the identity (3.479) indeed holds, follows analogously to the energy estimates from Subsections 3.2.4 and 3.2.2 applied to the right-hand side of (3.478). Here we strongly exploit the uniform bounds on $E_{\alpha,\nu}(u,f)$ and $E_{\alpha,\nu}(\tilde{u},\tilde{f})$. A major difference from the existence part of the proof is the absence of cross-terms in the energy identities (since $\omega = \chi$ in the notation of Section 2). In addition to that, we work in a lower order energy space ($l=1$) and we can thus use the above uniform estimates to bound the term $[V_n]_-^+$ by $C\sqrt{L}$ in L^∞-norm. Knowing that the ϵ-dependence comes only from the estimates of the cross-terms we conclude that the constants on the right-hand side of (3.479) *do not* depend on ϵ. Choosing L and α^* suitably small (3.479) implies $\mathcal{E}^*(t) + \mathfrak{E}^*(t) \leq C \int_0^t \left\{\mathcal{E}^*(s) + \mathfrak{E}^*(s)\right\} ds$, which, by Gronwall's inequality leads to $\mathcal{E}^*(t) + \mathfrak{E}^*(t) = 0$. This finishes the proof of the uniqueness statement of Theorem 3.11.

$\square$

3.3.3. *Continuity for the regularized Stefan problem.* The continuity statement is important because it will allow us to extend the solution beyond the the time interval $[0, t^\epsilon[$.

LEMMA 3.15. *The total energy $E^\epsilon_{\alpha,\nu}(u,f)(t) + \int_0^t D^\epsilon_{\alpha,\nu}(u,f)(s)\,ds$ is continuous as a function of t on $[0,t^\epsilon[$.*

Sketch of the proof. Integrating over $[s,t]$ instead of $[0,t]$ while arriving at the identity (3.306) in Section 3.1, we obtain

$$\begin{aligned} &\mathcal{E}_\epsilon^{m+1}(t) - \mathcal{E}_\epsilon^{m+1}(s) + \int_s^t \mathcal{D}_\epsilon^{m+1}(\tau)\,d\tau \\ &\quad = \int_s^t A(u^{m+1}, f^m, f^{m+1}) + \int_s^t \int_{\Gamma^m} \Big\{O^m + P^m\Big\} + \int_s^t \int_{\mathcal{S}} \Big\{Q^m + S^m\Big\}. \end{aligned} \tag{3.480}$$

We claim that we may pass to the limit as $m \to \infty$ in each of the terms in (3.480) to obtain:

$$\mathcal{E}_\epsilon(t) - \mathcal{E}_\epsilon(s) + \int_s^t \mathcal{D}_\epsilon(\tau)\,d\tau = \int_s^t \bar{A} + \int_s^t \int_\Gamma \Big\{\bar{O} + \bar{P}\Big\} + \int_s^t \int_{\mathcal{S}} \Big\{\bar{Q} + \bar{S}\Big\}, \tag{3.481}$$

where

$$\bar{A}=A(u,f,f),\ \bar{O}=\sum_{k=1}^{l-1}O(u_{t^k}),\ \bar{P}=\sum_{k=0}^{l-1}P(u_{t^k}),\ \bar{Q}=\sum_{k=1}^{l-1}Q(u_{t^k},f_{t^k},R),\ \bar{S}=\sum_{k=0}^{l-1}S(u_{t^k},f_{t^k},R).$$

The passage to the limit is justified for the following reason: we may apply the differential operator ∂_{t^k} to the equality (3.450) to express the derivatives of u^{m+1} in terms of derivatives of $\bar{u}^{m+1}$ and f^m. This enables us to express the L^2 norms of $\partial_{t^k}u^{m+1}$ on Ω^m in terms of the space-time norms of $\bar{u}^{m+1}$ and f^m on $\Omega_{\mathcal{S}}$ and $\mathcal{S}$, respectively. Hence, using the strong convergence of the sequence (u^j,f^j), we may pass to the limit to conclude (3.481). In order to bound RHS of (3.481), by the same argument, we pass to the limit as $m\to\infty$ in the estimate (3.371), where we keep in mind that we work on the time interval $[s,t]$. Note that all the estimates coming from the cross-terms will vanish, and those are precisely the ones involving the constant $\bar{C}$. As a consequence we obtain

(3.482)
$$\begin{aligned}&\Big|\int_s^t\bar{A}+\int_s^t\int_\Gamma\bar{O}+\bar{P}-\int_s^t\int_{\mathcal{S}}\bar{Q}+\bar{S}\Big|\le CC_0(t-s)\sup_{s\le\tau\le t}\big(\mathcal{E}(\tau)+\mathfrak{E}(\tau)+\theta_1+\int_s^t\mathfrak{D}\big)\\&+C(\lambda+\sup_{0\le\tau\le t}||\mathbb{P}_1f||_{L^2}^2+C\sqrt{\mathfrak{E}})\int_s^t\mathcal{D}_\epsilon+C\big(\sqrt{\mathfrak{E}}+\lambda+(\mathcal{E}_{(0)})^{1/2}\big)\int_0^t\mathfrak{D}_\epsilon\end{aligned}$$

Using (3.481), (3.482) and the triangle inequality, we obtain for any $0\le s<t\le t^\epsilon$:

$$\begin{aligned}&\big|\mathcal{E}_\epsilon(t)-\mathcal{E}_\epsilon(s)\big|\le C\Big|\int_s^t\mathcal{D}_\epsilon(\tau)\,d\tau\Big|\Big(1+C(\lambda+\sup_{0\le\tau\le t}||\mathbb{P}_1f||_{L^2}^2+\sqrt{\mathfrak{E}})\Big)\\&+CC_0(t-s)\sup_{s\le\tau\le t}\big(\mathcal{E}(\tau)+\mathfrak{E}(\tau)+\theta_1+\int_s^t\mathfrak{D}\big)+C\big(\sqrt{\mathfrak{E}}+\lambda+(\mathcal{E}_{(0)})^{1/2}\big)\int_s^t\mathfrak{D}_\epsilon\longrightarrow0\quad\text{as}\quad s\to t,\end{aligned}$$

since $\sup_{0\le\tau\le t^\epsilon}\mathcal{E}_\epsilon(\tau)$ and $\mathfrak{E}_\epsilon$ are bounded on $[0,t^\epsilon[$. This proves the continuity of $\mathcal{E}_\epsilon$ on the small time interval $[0,t^\epsilon[$. In a completely analogous fashion, by passing to the limit in (3.438), it follows that $\mathfrak{E}_\epsilon$ is continuous on the interval $[0,t^\epsilon[$. □

3.3.4. *Momentum conservation and smallness of the first modes.* Let us for the moment be a little bit more general and assume that Γ is parametrized as a "graph" over the unit sphere centered at $\mathbf{x}_0$, i.e. $\Gamma=\{\mathbf{x}\in\mathbb{R}^n;\,\mathbf{x}=R\xi+\mathbf{x}_0,\ \xi\in\mathcal{S}\}$. For $i=1,\dots,n$ define

$$F_i^{\mathbf{x}_0}(R,\xi)=\int^R r^{n-1}\big((n-1)[r\xi^i+\mathbf{x}_0^i]+R_*^n\frac{r\xi^i+\mathbf{x}_0^i}{|r\xi+\mathbf{x}_0|^n}\big)\,dr\tag{3.483}$$

Note that the above definition agrees with the definition (1.222) when $\mathbf{x}_0=\mathbf{0}$.

LEMMA 3.16. *Let* $p_i(x)=(n-1)x_i+\frac{R_*^n x_i}{|x|^n}$. *Let* s_i *be the spherical harmonic of order* 1 *associated with the homogeneous polynomial* x_i, *i.e.* $s_i=x_i|_{\mathcal{S}}$. *Then, on the time interval of existence of the solution to the Stefan problem, the following conservation law holds:*

$$\partial_t\int_\Omega up_i=\partial_t\int_{\mathcal{S}}F_i^{x_0}(R,\xi)\,d\xi+\epsilon\int_{\mathcal{S}}\Delta_g^2 f_t\big((n-1)R\xi^i+R_*^n\frac{R\xi^i+\mathbf{x}_0^i}{|R\xi+\mathbf{x}_0|^n}\big)\,d\xi. \tag{3.484}$$

Proof. Note that p_i is harmonic away from the origin and enjoys the property

$$\partial_n p_i\Big|_{\partial\Omega}=0. \tag{3.485}$$

Let further $\Omega_\delta:=\Omega\setminus B_\delta(0)$. Multiply (1.194) by p_i and integrate over Ω_δ. Using the Green's second identity, harmonicity of p_i and (3.485), we obtain

$$\int_{\Omega_+}u_t^+p_a-\int_\Gamma\partial_n u^+p_a\,dS+\int_\Gamma u^+\partial_n p_a=0,$$

$$\int_{\Omega_-}u_t^-p_a+\int_\Gamma\partial_n u^-p_a\,dS-\int_\Gamma u^-\partial_n p_a-\int_{S_\delta(0)}u_n p_a+\int_{S_\delta(0)}u\partial_n p_a=0.$$

Summing the above equations and using the jump condition (1.196), we obtain

$$\partial_t\int_{\Omega_\delta}up_a=\int_\Gamma Vp_a\,dS+\int_{S_\delta(0)}u_n p_a-\int_{S_\delta(0)}u\partial_n p_a.$$

Note that $\partial_n p_i\Big|_{S_\delta(0)}=s_i-(n-1)\frac{R_*^n}{\delta^n}s_i$. Hence

$$\begin{aligned}\int_{S_\delta(0)}u\partial_n p_i\,dS&=\int_{\mathcal{S}}u(\delta,\xi)\big(s_i-(n-1)\frac{R_*^n}{\delta^n}s_i\big)\delta^{n-1}\,d\xi\\&=\int_{\mathcal{S}}u(\delta,\xi)\delta^{n-1}s_i\,d\xi-(n-1)R_*^n\int_{\mathcal{S}}u(\delta,\xi)\frac{s_i}{\delta}\,d\xi.\end{aligned}$$

The first integral on right-most side converges to 0 as $\delta\to 0$. As to the second integral, observe that

$$\int_{\mathcal{S}}u(\delta,\xi)\frac{s_i}{\delta}\,d\xi=\int_{\mathcal{S}}\frac{u(\delta,\xi)-u(0,0)}{\delta}s_i\,d\xi\to\int_{\mathcal{S}}u_r(0,0)s_i\,d\xi=0\quad\text{as}\quad\delta\to 0.$$

Similarly $\int_{S_\delta(0)}u_n p_i\to 0$ as $\delta\to 0$. Finally, passing to the limit as $\delta\to 0$, we get

$$\partial_t\int_\Omega up_i=\int_\Gamma Vp_i\,dS, \tag{3.486}$$

where the integral $\int_\Omega up_i$ is interpreted in the sense of the existing limit as $\delta\to 0$, i.e. $\int_\Omega up_i = \lim_{\delta\to 0}\int_{\Omega_\delta} up_i$ (keep in mind that $p_i \approx \frac{1}{r^{n-1}}$ at the origin and this singularity is integrable). Thus

$$\begin{aligned}\int_\Gamma Vp_i &= \int_{\mathcal{S}}\big(\frac{f_tR}{|g|}+\epsilon\frac{\Delta_g^2 f_t}{|g|R^{n-2}}\big)\big((n-1)[R\xi^i+\mathbf{x}_0^i]+R_*^n\frac{R\xi^i+\mathbf{x}_0^i}{|R\xi+\mathbf{x}_0|^n}\big)s_iR^{n-2}|g|\,d\xi\\ &= \partial_t\int_{\mathcal{S}}F_i^{\mathbf{x}_0}(R,\xi)\,d\xi+\epsilon\int_{\mathcal{S}}\Delta_g^2 f_t\big((n-1)R\xi^i+R_*^n\frac{R\xi^i+\mathbf{x}_0^i}{|R\xi+\mathbf{x}_0|^n}\big),\end{aligned}$$

as claimed. $\square$

Remark. One could conceivably try to obtain a hierarchy of conservation laws for higher momenta, by multiplying the equation (1.203) by appropriate functions that restrict to a combination of spherical harmonics of k-th degree on $\mathcal{S}$. However, this strategy fails to render new conservation laws, since the newly constructed test functions p contain a non-integrable singularity.

Remark. Observe that we obtain exactly the conservation law (1.220) if we set $\mathbf{x}_0=\mathbf{0}$ and $\epsilon=0$.

We shall exploit the conservation laws given by (1.217) and Lemma 3.16 to prove that the natural energy $\mathcal{E}_{(0)}$ (given by (1.223)) is in fact positive and controls all the modes of f. To this end, the following lemma is necessary:

LEMMA 3.17. *Assume $\mathbf{x}_0=\mathbf{0}$. There exists a constant C such that the following inequalities hold:*

(3.487)
$$\sum_{i=1}^n|f_{1,i}|\le C\Big(|m_0|+\sum_{i=1}^n|m_i|+C||\nabla u||_{L^2(\Omega^\pm)}+C\sum_{k>n}|a_k|+C\epsilon\sqrt{t}\big(\int_0^t||f_t||_{L^2}^2\big)^{1/2}\Big)$$

(3.488)
$$|\mathbb{P}_0 f|\le C\Big(|m_0|+\sum_{i=1}^n|m_i|+||u||_{L^2(\Omega^\pm)}+||\nabla u||_{L^2(\Omega^\pm)}+\sum_{k>n}|a_k|+\epsilon t\int_0^t||f_t||_{L^2}^2\Big).$$

Proof. Recall that $m_i=\int_\Omega u_0p_i-\int_{\mathcal{S}}\big\{F_i(R_0,\xi)-F_i(1,\xi)\big\}\,d\xi$ for $i=1,\dots,n$ and m_0 is given by (1.221). Observe that

$$\int_{\mathcal{S}}F_i(1+f,\theta)-F_i(1,\theta)\,d\theta=\int_{\mathcal{S}}\partial_fF_i(1,\theta)f\,d\theta+\int_{\mathcal{S}}O(f^2). \tag{3.489}$$

From the definition (1.220) of F_i we have $\partial_f F_i(1,\theta) = \big((n-1)+R_*^n\big)s_{1,i}$. Using the decomposition $f=\sum_{k=0}^{\infty}\sum_{i=1}^{N(k)} f_{k,i}s_{k,i}$, we conclude

$$\sum_{j=1}^{n} f_{1,j}\int_{\mathcal{S}}\big((n-1)+R_*^n\big)s_{1,i}s_{1,j} = \int_{\mathcal{S}}\Big(F_i(1+f,\xi)-F_i(1,\xi)\Big)\,d\xi - \int_{\mathcal{S}} O(f^2)$$
$$-\int_{\mathcal{S}}\partial_f F_i(1,\xi)\big(f_{0,1}+\sum_{k\geq 2} f_{k,i}s_{k,i}\big).$$

From the orthonormality property of the basis $\{s_i\}_{i\in\mathbb{N}_0}$ and the fact that

$$\int_{\mathcal{S}}\Big(F_i(1+f,\xi)-F_i(1,\xi)\Big)\,d\xi = \int_{\Omega} up_i - m_i + \epsilon\int_0^t\int_{\mathcal{S}} G_i(R,\xi)\,d\xi,$$

we conclude for any $i=1,\ldots,n$

(3.490)
$$\big((n-1)+R_*^n\big)f_{1,i} = -m_i + \int_{\Omega} up_i - \int_{\mathcal{S}} O(f^2) - \int_{\mathcal{S}}\partial_f F_i(1,\xi)\big(f_{0,1}+\sum_{k\geq 2} f_{k,i}s_{k,i}\big) + \epsilon\int_0^t\int_{\mathcal{S}} G_i(R,\xi)\,d\xi.$$

Note that for $i=1,\ldots,n$, we have

$$\Big|\int_{\mathcal{S}} O(f^2)\Big| + \Big|\int_{\mathcal{S}}\partial_f F_i(1,\theta)(f_{0,1}+\sum_{k\geq 2} f_{k,i}s_{k,i}\Big| \leq C(\sum_{k\geq 0} f_{k,i}^2 + |f_{0,1}| + \sum_{k\geq 2}|f_{k,i}|)$$
$$\leq C\sqrt{M}\sum_{j=1}^{n}|a_{1,j}| + C(1+\sqrt{M})\sum_{k=0,k\geq 2}|a_{k,i}|.$$

In order to bound the second term on RHS of (3.490), we use esentially the polar coordinates:

$$\int_{\Omega} up_i = \int_0^{R_*}\int_{\mathcal{S}} u(r,\xi)\Big((n-1)r+\frac{R_*^n}{r^{n-1}}\Big)r^{n-1}s_{1,i}\,d\xi\,dr = \int_0^{R_*}\int_{\mathcal{S}} u(r,\xi)\Big(r^n+R_*^n\Big)s_{1,i}\,d\xi\,dr$$
$$= -\frac{1}{n-1}\int_0^{R_*}\int_{\mathcal{S}} u(r,\xi)\Big(r^n+R_*^n\Big)\Delta_g s_{1,i}\,d\xi\,dr = \frac{1}{n-1}\int_0^{R_*}\int_{\mathcal{S}}\Big(r^n+R_*^n\Big)\nabla_g u(r,\xi)\cdot\nabla_g s_{1,i}\,d\xi\,dr$$
$$\leq \frac{R_*^n+R_*^n}{n-1}\int_0^{R_*}||\nabla_g u(r)||_{L^2(\mathcal{S})}||\nabla_g s_{1,i}||_{L^2(\mathcal{S})} \leq C||\nabla u||_{L^2(\Omega)}.$$

Using the integration by parts, boundedness of $||f||_{H^l}$ on $[0,T]$ and the Cauchy-Schwarz inequality, it is easy to see that

$$\epsilon\Big|\int_0^t\int_{\mathcal{S}} G_i(R,\xi)\,d\xi\Big| \leq C\epsilon\sqrt{t}\Big(\int_0^t ||f_t||_{L^2}\Big)^{1/2}.$$

Using (3.490) and the previous three inequalities, together with the smallness of M, we conclude that

$$\sum_{i=1}^{n}|f_{1,i}|\leq C\sum_{i=1}^{n}|m_i|+C||\nabla u||_{L^2(\Omega^\pm)}+C|f_{0,1}|+C\sum_{k\geq 2}|f_{k,i}|+C\epsilon\sqrt{t}\Big(\int_0^t||f_t||_{L^2}^2\Big)^{1/2}. \tag{3.491}$$

□

We use Lemma 3.17 to prove a necessary smallness estimate on $||f||_{H^{2l}}$.

LEMMA 3.18. *There exists a constant C such that the following inequality holds:*

$$||f||_{H^{2l}}^2\leq C(m_0^2+\sum_{i=1}^{n}m_i^2)+CM+C\epsilon tM. \tag{3.492}$$

Proof. The boundary condition (1.204) can be rewritten in the following way:

$$-\Delta_g f-(n-1)f=u\circ\phi-N(f), \tag{3.493}$$

where $N(f)$ is the quadratic nonlinearity defined by (1.202). Let $|\xi|=2l-1$ be a given multi-index. Apply ∂_g^ξ to both sides of (3.493), multiply by $\partial_g^\xi f$ and integrate over $\mathcal{S}$ to obtain

$$\int_{\mathcal{S}}\{|\nabla_g\partial_g^\xi f|^2-(n-1)(\partial_g^\xi f)^2\}=\int_{\mathcal{S}}\partial_g^\xi(u\circ\phi)\partial^\xi f+\int_{\mathcal{S}}\partial_g^\xi(N(f))\partial_g^\xi f. \tag{3.494}$$

From (3.494), using the Leibniz rule, the Cauchy-Schwarz inequality, the trace inequality and the smallness of $||f||_{H^{2l}}$, we can conclude (see (1.215))

$$||\mathbb{P}_{2+}\nabla_g\partial_g^\xi f||_{L^2}^2\leq C||u||_{H^{2l-1}(\Omega^\pm)}^2+C||\mathbb{P}_1 f||_{L^2}^2.$$

By the Sobolev inequality, we conclude

$$||\mathbb{P}_{2+}f||_{H^{2l}}^2\leq C\mathfrak{E}+C||\mathbb{P}_1 f||_{L^2}^2\leq C\mathfrak{E}+C\mathcal{E}+C\sum_{i=0}^{n}m_i^2+CM||\mathbb{P}_{2+}f||_{L^2}^2+C\epsilon t\int_0^t||f_t||_{L^2}^2,$$

where we used the estimate (3.487) in the second inequality. This estimate implies

$$||\mathbb{P}_{2+}f||_{H^{2l}}^2\leq C(m_0^2+m_a^2+m_b^2)+C(\mathfrak{E}+\mathcal{E})+C\epsilon t\int_0^t||f_t||_{L^2}^2. \tag{3.495}$$

From (3.495), (3.487) and (3.488), we easily obtain

$$
\begin{aligned}
||f||^2_{H^{2l}} &\le C(m_0^2+m_a^2+m_b^2)+C(\mathfrak{E}+\mathcal{E})\\
&\le C(m_0^2+m_a^2+m_b^2)+C(M/(\alpha\nu^{l-1})+M)+C\epsilon tM\\
&\le C(m_0^2+m_a^2+m_b^2)+CM+C\epsilon tM.
\end{aligned}
$$

as claimed in the lemma. $\square$

Estimates for $\mathcal{E}_{(0)}$ revisited. Let $M\le L/2$, where L is given by Theorem 3.11 and let (u,f) be the unique solution of the regularized Stefan problem with the initial conditions satisfying $E_\alpha(u_0,f_0)\le M$. Now recall the definition (1.223) and the expansion (1.202) $|g|=1+f+|\nabla_g f|^2/2+\Psi(f)$, where $\Psi(f)=O(|f|^3+|\nabla_g f|^3)$. This implies $R^{n-2}|g|-1=(n-1)f+|\nabla_g f|^2/2+\Psi_1(f)$, where $\Psi_1(f)=O(|f|^3+|\nabla_g f|^3)$. We have

$$
\begin{aligned}
\mathcal{E}_{(0)}(u,f) &= \frac{1}{2}\int_\Omega u^2+\int_\Omega u+\int_{\mathcal{S}}\left(R^{n-2}|g|-1\right)=\frac{1}{2}\int_\Omega u^2+\int_\Omega u+(n-1)\int_{\mathcal{S}}f+\int_{\mathcal{S}}\frac{|\nabla_g f|^2}{2}+\int_{\mathcal{S}}\Psi_1(f)\\
&=\frac{1}{2}\int_\Omega u^2-\frac{n-1}{2}\int_{\mathcal{S}}f^2+\int_{\mathcal{S}}\frac{|\nabla_g f|^2}{2}+\int_{\mathcal{S}}\Psi_1(f)+O(f^3)=\frac{1}{2}\int_\Omega(\mathbf{P}u)^2\,dx-\frac{n-1}{2}\int_{\mathcal{S}}(\mathbb{P}f)^2\\
&\quad+\Big(\frac{1}{2|\Omega|}-\frac{n-1}{2|\mathcal{S}|}\Big)\Big(\int_\Omega u\Big)^2+\frac{1}{2|\mathcal{S}|}\Big(\Big(\int_\Omega u\Big)^2-\Big(\int_{\mathcal{S}}f\Big)^2\Big)+\int_{\mathcal{S}}|\nabla_g f|^2+\int_{\mathcal{S}}\Psi_1(f)+O(f^3)\\
&=\frac{1}{2}\int_\Omega(\mathbf{P}u)^2\,dx+\frac{\zeta}{2}\Big(\int_\Omega u\Big)^2+\frac{1}{2}\int_{\mathcal{S}}\{|\nabla_g f|^2-(n-1)\mathbb{P}f^2\}+\frac{1}{2|\mathcal{S}|}\Big(\Big(\int_\Omega u\Big)^2-\Big(\int_{\mathcal{S}}f\Big)^2\Big)\\
&\quad+\int_{\mathcal{S}}\Psi_1(f)+O(f^3).
\end{aligned}
$$

Note that, again due to (1.217),

$$
\Big(\int_\Omega u\Big)^2-\Big(\int_{\mathcal{S}}f\Big)^2=\Big(m_0-\sum_{k=2}^n\frac{1}{n}\binom{n}{k}\int_{\mathcal{S}}f^k\Big)\Big(\int_\Omega u-\int_{\mathcal{S}}f\Big),\tag{3.496}
$$

and it is hence of third order. Using the estimates (3.487), (3.488) and the equality (3.496), it is easy to see that

$$
\begin{aligned}
\Big|\frac{1}{2|\mathcal{S}|}\Big(\Big(\int_\Omega u\Big)^2-\Big(\int_{\mathcal{S}}f\Big)^2\Big)+\int_{\mathcal{S}}\Psi_1(f)+O(f^3)\Big| &\le \frac{C}{\lambda}m_0^2+\lambda\Big(\int_\Omega u\Big)^2+\lambda a_0^2\\
&\quad+C\sqrt{M}\Big(\sum_{i=0}^n m_i^2+||u||^2_{L^2(\Omega^\pm)}+||\nabla u||^2_{L^2(\Omega)}+\int_{\mathcal{S}}|\mathbb{P}_{2+}f|^2+\epsilon t\int_0^t||f_t||^2_{L^2}\Big).
\end{aligned}
$$

The last inequality implies that there exists a constant C_* such that

(3.497)
$$\mathcal{E}_{(0)} \geq C_*\big(||u||^2_{L^2(\Omega^\pm)} + \int_{\mathcal{S}} |\mathbb{P}_{2+}f|^2\big) - C_*\sum_{i=0}^n m_i^2 - C_*\sqrt{M}||\nabla u||^2_{L^2(\Omega^\pm)} - C_*\sqrt{M}\epsilon t\int_0^t ||f_t||^2_{L^2}.$$

Thus, for small M this implies that $\tilde{\mathcal{E}}_{(0)} + C_*\sum_{i=0}^n m_i^2 + \mathcal{E}_{(1)} + \int_0^t \mathcal{D}_{(1)}$ is a positive definite quantity. After summing the zero-th and the first level energy identities and adding $C_*\sum_{i=0}^n m_i^2$ to both sides, we obtain

(3.498)
$$\begin{aligned}&C_*\sum_{i=0}^n m_i^2 + \mathcal{E}_{(0)}(t) + \hat{\mathcal{E}}^\epsilon_{(0)}(t) + ||\nabla u(t)||^2_{L^2(\Omega^\pm)} + \int_0^t ||\nabla u||^2_{L^2(\Omega^\pm)} + \int_0^t ||u_t||^2_{L^2(\Omega^\pm)}\\ &+\int_0^t\int_{\mathcal{S}}\{|\nabla_g f_t|^2 - (n-1)f_t^2\} = C_*\sum_{i=0}^n m_i^2 + \mathcal{E}_{(0)}(0) + \hat{\mathcal{E}}^\epsilon_{(0)}(0) + ||\nabla u_0||^2_{L^2(\Omega^\pm)}\\ &+\epsilon\int_{\mathcal{S}}\big(\Delta_g f + N^*(f)\big)\Delta_g^2 f_t\big(\frac{1}{R|g|} - 1\big) + \int_0^t\int_\Gamma\Big\{O_1 + P_1\Big\} + \int_0^t\int_{\mathcal{S}}\Big\{Q_1 + S_1\Big\},\end{aligned}$$

whereby O_1, P_1, Q_1 and S_1 are defined by dropping the index m in the definitions (3.297) and (3.298). Estimates analogous to (3.344) and (3.345) imply the bound on the ϵ-dependent terms on RHS of (3.498).

$$\begin{aligned}&\Big|\epsilon\int_{\mathcal{S}}\big(\Delta_g f + N^*(f)\big)\Delta_g^2 f_t\big(\frac{1}{R|g|} - 1\big)\Big| \leq C_0\big(\lambda\int_0^t \mathcal{D}^\epsilon + \lambda\epsilon\int_0^t \mathcal{D}\big)\\ &+\frac{C_0 t}{\lambda}\sup_{0\leq s\leq t}\big(||f||^2_{H^l}\hat{\mathcal{E}}^\epsilon_{(0)}\big) + C_0 t\epsilon\sup_{0\leq s\leq t}\big(||f||^2_{H^l}||\mathbb{P}_1 f||^2_{L^2}\big).\end{aligned} \tag{3.499}$$

3.3.5. *Uniform interval of existence: $t^\epsilon \geq T$.* Our objective is to prove that the solution (u^ϵ, f^ϵ) exists on a time interval independent of ϵ.

LEMMA 3.19. *There exists a $T > 0$ independent of ϵ such that the solution (u^ϵ, f^ϵ) exists on the time interval $[0,T]$ and $E_{\alpha,\nu}$ is continuous on $[0,T[$.*

Proof. As in Subsection 3.3.3, we can pass to the limit as $m \to \infty$ in the estimate (3.438). In doing so, we use the bound (3.492) to ensure the smallness of $||f^m||_{H^l}$. Namely,

$$||f^m||_{H^l} \leq ||f||_{H^l} + ||f^m - f||_{H^l} \leq \frac{3}{2}||f||_{H^l} \leq C\sum_{i=0}^n m_i^2 + CM + C\epsilon t M, \tag{3.500}$$

for m large enough. Note that all the estimates involving the constant $\tilde{C}$ simply vanish (as they are used for cross-terms only), and for any $0\leq p\leq l-1$, we recover:

$$\begin{aligned}\sup_{0\leq s\leq t}\mathfrak{E}_p+\int_0^t\mathfrak{D}_p+(1-C_2t)\big(\sup_{0\leq s\leq t}\hat{\mathfrak{E}}_p^{\epsilon}+\int_0^t\hat{\mathfrak{D}}_p\big)\\ \leq\mathfrak{E}_{\epsilon,p}(0)+\nu^l\int_0^t\mathfrak{D}_{p+1}+C\sum_{q=0}^{p-1}\int_0^t\mathfrak{D}_q+\epsilon\mathcal{J}_1+C^*\int_0^t\mathcal{D},\end{aligned}\tag{3.501}$$

where $\mathcal{J}_1:=C||f||_{L^2}^4$. Multiplying (3.501) by ν^p and summing over all $p=0,1,\ldots,l-1$, just like in the derivation of (3.441), we can infer

$$\sup_{0\leq s\leq t}\mathfrak{E}^{\nu}+\sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{D}_p+(1-C_2t)\big(\sup_{0\leq s\leq t}\hat{\mathfrak{E}}^{\epsilon,\nu}+\int_0^t\hat{\mathfrak{D}}^{\epsilon,\nu}\big)\leq\mathfrak{E}_{\epsilon}^{\nu}(0)+\epsilon\sum_{q=0}^{l-1}\nu^q\mathcal{J}_1+\sum_{q=0}^{l-1}\nu^qC^*\int_0^t\mathcal{D},\tag{3.502}$$

whereby we observe that none of the constants in the inequality (3.502) contains ϵ in the denominator. Secondly, we pass to the limit as $m\to\infty$ in the estimate (3.371). Thereby we use the identity (3.498) together with the energy estimate (3.499). Note that all the terms containing the constant $\bar{C}$ vanish and we arrive at:

$$\begin{aligned}&\sup_{0\leq s\leq t}\mathcal{E}(s)+C_*\sum_{i=0}^{n}m_i^2+\sup_{0\leq s\leq t}\hat{\mathcal{E}}^{\epsilon}(s)+\int_0^t\mathcal{D}_{\epsilon}\\ &\quad\leq\mathcal{E}_{\epsilon}(0)+C_*\textstyle\sum_{i=0}^n m_i^2+\frac{C_0t}{\lambda}\sup_{0\leq s\leq t}\hat{\mathcal{E}}_{(0)}^{\epsilon}(s)+C_0t\epsilon\sup_{0\leq s\leq t}\big(||f||_{H^{n/2+3}}^2||\mathbb{P}_1f||_{L^2}^2\big)\\ &\qquad+C(\lambda+||f||_{H^{n/2+3}}+\sqrt{M})\int_0^t\mathcal{D}_{\epsilon}+C(\lambda+||f||_{H^{n/2+3}}+\sqrt{M})\int_0^t\mathfrak{D}\end{aligned}\tag{3.503}$$

Multiplying (3.502) by α and summing it with (3.503), we obtain

$$\begin{aligned}&\sup_{0\leq s\leq t}\mathcal{E}(s)+C_*\sum_{i=0}^{n}m_i^2+\alpha\sup_{0\leq s\leq t}\mathfrak{E}^{\nu}(s)+(1-C(\lambda+||f||_{H^4}+\sqrt{M})-\alpha\sum_{q=0}^{l-1}\nu^qC^*)\int_0^t\mathcal{D}\\ &+\sum_{p=0}^{l-1}(\alpha d_p-C(\lambda+||f||_{H^4}+\sqrt{M}))\int_0^t\mathfrak{D}_p\\ &+(1-C_0t)\sup_{0\leq s\leq t}\hat{\mathcal{E}}^{\epsilon}(s)+(1-C(\lambda+||f||_{H^4}+\sqrt{M}))\int_0^t\hat{\mathcal{D}}^{\epsilon}+\alpha(1-C_2t)\big(\sup_{0\leq s\leq t}\hat{\mathfrak{E}}^{\epsilon,\nu}+\int_0^t\hat{\mathfrak{D}}^{\epsilon,\nu}\big)\\ &\quad\leq\mathcal{E}_{\epsilon}(0)+C_*\sum_{i=0}^{n}m_i^2+\alpha\mathfrak{E}_{\epsilon}^{\nu}(0)+\epsilon\mathcal{J}_2,\end{aligned}\tag{3.504}$$

where $\mathcal{J}_2:=\alpha\sum_{q=0}^{l-1}\nu^q\mathcal{J}_1+C_0t\sup_{0\leq s\leq t}\big(||f||_{H^{n/2+3}}^2||\mathbb{P}_1f||_{L^2}^2\big)$. Using the bound (3.500) to ensure the smallness of $||f^m||_{H^l}$ and choosing λ and M sufficiently small, we obtain the following energy bound for any $t \le t^\epsilon$ from (3.504):

$$\begin{aligned}
&\sup_{0\le s\le t}\mathcal{E}(s)+C_*\sum_{i=0}^{n}m_i^2+\alpha\sup_{0\le s\le t}\mathfrak{E}^\nu(s)+\big(\frac{3}{4}-\epsilon CTM\big)\big(\int_0^t\mathcal{D}+\alpha\int_0^t\mathfrak{D}^\nu\big)\\
&+(1-C_0t)\sup_{0\le s\le t}\hat{\mathcal{E}}^\epsilon(s)+\frac{3}{4}\int_0^t\hat{\mathcal{D}}^\epsilon+\alpha(1-C_2t)\big(\sup_{0\le s\le t}\hat{\mathfrak{E}}^{\epsilon,\nu}+\int_0^t\hat{\mathfrak{D}}^{\epsilon,\nu}\big)\\
&\le\mathcal{E}(0)+C_*\sum_{i=0}^{n}m_i^2+\hat{\mathcal{E}}^\epsilon(0)+\alpha\mathfrak{E}^\nu(0)+\alpha\hat{\mathfrak{E}}^{\epsilon,\nu}(0)+\epsilon\mathcal{J}_2,
\end{aligned}\tag{3.505}$$

where $\mathcal{J}_2:=\alpha\sum_{q=0}^{l-1}\nu^q\mathcal{J}_1+C_0t\sup_{0\le s\le t}\big(||f||^2_{H^{n/2+3}}||\mathbb{P}_1f||^2_{L^2}\big)$. Now extend $[0,t^\epsilon[$ to a maximal interval $[0,\bar{t}^\epsilon[$ such that $E_{\alpha,\nu}(s)+\int_0^s D_{\alpha,\nu}$ is continuous on $[0,\bar{t}^\epsilon[$ and the inequality (3.505) holds for any $0\le t\le\bar{t}^\epsilon$. Let $L\le\theta$, where θ is given by Lemma 3.2. Set

$$\mathcal{T}:=\sup_t\Big\{0<t<\bar{t}^\epsilon:\sup_{0\le s\le t}E_{\alpha,\nu}(u^\epsilon,f^\epsilon)(s)+C_*\sum_{i=0}^{n}m_i^2+\int_0^t D_{\alpha,\nu}(u^\epsilon,f^\epsilon)(\tau)\,d\tau\le L\Big\}.$$

Note that $\mathcal{T}\ge t^\epsilon>0$. Furthermore, let us choose $\epsilon_0,\theta>0,T$ small enough such that $C_0T,C_2T<\frac{3}{8}$, and $\epsilon_0CTM<\frac{1}{8}$. Note that $\mathcal{E}_\epsilon(0)+C_*\sum_{i=0}^{n}m_i^2+\alpha\mathfrak{E}^\nu(0)\le L/2$. From (3.505), we conclude that for any $t\le T$ and any $\epsilon\le\epsilon_0$ the following inequality holds:

$$E_{\alpha,\nu}+C_*\sum_{i=0}^{n}m_i^2+\int_0^t D_{\alpha,\nu}\le\frac{4}{5}L<L\tag{3.506}$$

However, if $\bar{t}^\epsilon<T$, then $\mathcal{T}<T$ and the inequality (3.506) holds for any $t\le\mathcal{T}$. This contradicts the definition of $\mathcal{T}$ together with the continuity of $\mathcal{E}_\epsilon$ and hence $\bar{t}^\epsilon\ge T$.

□

3.4. Proofs of Stability Theorems.

Proof of Theorem 1.1. Since the inequality (3.505) holds on a time interval $[0,T]$, where T does not depend on ϵ, we may pass to the limit as $\epsilon\to0$. The sequence (u^ϵ,f^ϵ) converges to the solution (u,f) of the original Stefan problem. In addition to that $\epsilon CTM\to0$ and $\epsilon\mathcal{J}_2\to0$ as $\epsilon\to0$, since $\mathcal{J}_2$ is bounded. Due to the choice of $(u_0^\epsilon,f_0^\epsilon)$, we have $\hat{\mathcal{E}}^\epsilon(0)$, $\hat{\mathfrak{E}}^{\epsilon,\nu}(0)\to0$, as $\epsilon\to0$. Together with the weak lower semicontinuity we obtain the energy bound:

$$\sup_{0\le s\le t}\Big(\mathcal{E}(s)+C_*\sum_{i=0}^{n}m_i^2+\alpha\mathfrak{E}^\nu(s)\Big)+\frac{1}{2}\Big(\int_0^t\mathcal{D}(\tau)\,d\tau+\alpha\int_0^t\mathfrak{D}^\nu(\tau)\,d\tau\Big)$$
$$\le\mathcal{E}(0)+C_*\sum_{i=0}^{n}m_i^2+\alpha\mathfrak{E}^\nu(0), \tag{3.507}$$

for any $0\le t\le T$, where $T=\min(\frac{1}{2C_0},\frac{1}{2C_2})$. The uniqueness proof is analogous to the proof of uniqueness in Theorem 3.11, where we formally set $\epsilon=0$. Given the initial data (u_0,f_0) such that the smallness assumptions of Theorem 1.1 are satisfied, we can construct a solution on time interval $[0,T[$. From (3.507), at $t=T/2$, we can conclude $\mathcal{E}(T/2)+\sum_{i=1}^n m_i^2+\alpha\mathfrak{E}(T/2)\le\mathcal{E}(0)+C_*\sum_{i=0}^n m_i^2+\alpha\mathfrak{E}(0)$. The idea is to solve the Stefan problem with new initial data $(u^1(x,0),f^1(\theta,0)=(u(x,\frac{T}{2}),f(\theta,\frac{T}{2}))$. The problem allows for a unique solution that exists at least on a time interval of length $[0,T[$. Due to the uniqueness we have $(u^1(x,t),f^1(\theta,t))=(u(x,t+\frac{T}{2}),f(\theta,t+\frac{T}{2}))$ and we have thus extended the solution (u,f) to the interval $[0,\frac{3T}{2}[$. Iterating this procedure, we conclude that the solution exists for all $t\ge 0$. Furthermore we conclude that (3.507) holds for any $t>0$. In order to prove (1.233), we apply the same idea. Let first $M^*\le\frac{M}{4}$ and assume $E_{\alpha,\nu}(u_0,f_0)+C_*\sum_{i=0}^n m_i^2\le M^*$. By the above considerations there exists a global solution (u,f) with the initial value (u_0,f_0) and it satisfies the global bound $\sup_{0\le s\le\infty}E_{\alpha,\nu}(s)+C_*\sum_{i=0}^n m_i^2+\int_0^\infty D_{\alpha,\nu}(\tau)\,d\tau\le\frac{M}{2}$. Fix any $s>0$. We solve the Stefan problem with the new initial data $(u^1(x,0),f^1(\theta,0))=(u(x,s),f(\theta,s))$. The problem allows for the unique solution since

$$E_{\alpha,\nu}(u_0^1,f_0^1)+C_*\sum_{i=0}^{n}m_i^2=E_{\alpha,\nu}(u,f)(s)+C_*\sum_{i=0}^{n}m_i^2\le\frac{M}{2}.$$

In addition to this we have the global bound $\sup_{0\le t<\infty}E_{\alpha,\nu}(u^1,f^1)(t)+C_*\sum_{i=0}^n m_i^2+\int_0^\infty D_{\alpha,\nu}(u^1,f^1)(\tau)\,d\tau\le M$ (again by the same consideration as above). We are thus in the uniqueness regime and we conclude $(u^1,f^1)(t)=(u,f)(t+s)$ for any $t\ge 0$. We may now use the estimate (3.507) to obtain (1.233).

Proof of Theorem 1.2. Our first objective is to find the candidate point $\mathbf{x}_0$ close to $\mathbf{0}$ and the limiting radius $\bar{R}$, so that the solution $(v(t),\Gamma(t))$ would converge to $(1/\bar{R},S_{\bar{R}}(\mathbf{x}_0)$ based on the 'mass'- and 'momentum conservation'. To this end, for any $\mathbf{x}_0$ recall the definition (3.483):

$$F_i^{\mathbf{x}_0}(R,\xi)=\int^R r^{n-1}\big((n-1)[r\xi^i+\mathbf{x}_0^i]+R_*^n\frac{r\xi^i+\mathbf{x}_0^i}{|r\xi+\mathbf{x}_0|^n}\big)\,dr$$

Choose $\bar{R}$ (close to 1) to be the solution of $\int_\Omega \frac{n-1}{\bar{R}}+\frac{1}{n}\int_{\mathcal{S}}\bar{R}^n=\int_\Omega((n-1)+u_0)+\frac{1}{n}\int_{\mathcal{S}}(1+f_0)^n$.

LEMMA 3.20. *There exists a point $\mathbf{x}_0$ close to $\mathbf{0}$ such that*

$$\int_{\mathcal{S}} F_i^{\mathbf{x}_0}(\bar{R},\xi)\,d\xi=m_i+\int_{\mathcal{S}}F_i(1,\xi)\,d\xi; \tag{3.508}$$

Sketch of the proof. The solvability of the above system of equations follows easily from the implicit function theorem. □

Having found the point $\mathbf{x}_0$ and the radius $\bar{R}$, we parametrize Γ as a "graph" over the sphere of radius $\bar{R}$ centered at $\mathbf{x}_0$, $S_{\bar{R}}(\mathbf{x}_0)$. We set $\Gamma=\{\mathbf{x};\ \mathbf{x}=\tilde{R}\tilde{\xi}+\mathbf{x}_0,\ \tilde{\xi}\in\mathcal{S}\}$, $\tilde{R}=1+\tilde{f}$ and $v=\frac{n-1}{\bar{R}}+\tilde{u}$. With respect to the new parametrization of Γ, the conservation laws (1.217) and (1.219) take the form

$$\partial_t\int_\Omega\tilde{u}+\partial_t\int_{\mathcal{S}}\big\{\bar{R}^{n-1}\tilde{f}+\frac{1}{n}\binom{n}{k}\sum_{k=2}^n\bar{R}^{n-k}\tilde{f}^k\big\}=0;\quad \partial_t\int_\Omega\tilde{u}p_i+\partial_t\int_{\mathcal{S}}F_i^{\mathbf{x}_0}(\tilde{R},\tilde{\xi})\,d\tilde{\xi}=0. \tag{3.509}$$

After integrating the above conservation laws in time, we note that our choice of the center $\mathbf{x}_0$ and the radius $\bar{R}$ of the shifted sphere, together with the definitions (1.221) and (1.222) of m_0 and m_i, implies

$$\int_\Omega\tilde{u}+\int_{\mathcal{S}}\big\{\bar{R}^{n-1}\tilde{f}+\frac{1}{n}\binom{n}{k}\sum_{k=2}^n\bar{R}^{n-k}\tilde{f}^k\big\}=0;\quad \int_\Omega\tilde{u}p_i+\int_{\mathcal{S}}F_i^{\mathbf{x}_0}(\tilde{R},\tilde{\xi})\,d\tilde{\xi}=0. \tag{3.510}$$

Let us introduce the map $\tilde{\phi}:[0,\infty[\times\mathcal{S}\to[0,\infty[\times\Gamma$,

$$\tilde{\phi}(t,\xi):=\tilde{R}(t,\tilde{\xi})+\mathbf{x}_0.$$

With respect to the parametrization $\tilde{\phi}$, we redefine the 0-th order energy $\mathcal{E}_{(0)}$:

$$\mathcal{E}_{(0)}(\tilde{u},\tilde{f})=\frac{1}{2}\int_\Omega\big\{\big(\frac{n-1}{\bar{R}}+\tilde{u}\big)^2-\frac{(n-1)^2}{\bar{R}^2}\big\}+\int_{\mathcal{S}}\big\{\sqrt{(\bar{R}+\tilde{f})^2+|\nabla_g\tilde{f}|^2}-\bar{R}\big\}.$$

The only difference from the definition (1.223) is in subtracting $(n-1)^2/\bar{R}^2$ instead of $(n-1)^2$ in the first integral, and $\bar{R}$ instead of 1 in the second integral above. The higher order energies $\mathcal{E}_{(+)}$ and $\mathcal{D}_{(+)}$ are defined precisely as before.

LEMMA 3.21. *The temporal energy $\mathcal{E}(\tilde{u},\tilde{f})$ is positive definite.*

Proof. Using the new conservation laws (3.510), same idea as in Lemma 3.17, we can bound the first and zero-th order momenta of $\tilde{f}$:

(3.511)

$$|\mathbb{P}_1\tilde{f}|\leq C\big(||\nabla\tilde{u}||_{L^2(\Omega^\pm)}+||\mathbb{P}_{2+}\tilde{f}||_{L^2(\mathcal{S})}\big);\ |\mathbb{P}_0\tilde{f}|\leq C\big(||\tilde{u}||_{L^2(\Omega^\pm)}+||\nabla\tilde{u}||_{L^2(\Omega^\pm)}+||\mathbb{P}_{2+}\tilde{f}||_{L^2(\mathcal{S})}\big)$$

With the above bounds we obtain an estimate analogous to (3.497):

$$\mathcal{E}_{(0)}(\tilde{u},\tilde{f})\geq C_1\big(||\tilde{u}||^2_{L^2(\Omega^\pm)}+\int_{\mathcal{S}}|\mathbb{P}_{2+}\tilde{f}|^2\big)-C_1\sqrt{M}||\nabla\tilde{u}||^2_{L^2(\Omega^\pm)},$$

where from we easily infer the claim of the lemma. □

The following lemma states the analogue of the stability estimate (1.233), where the energy quantities are expressed as functions of $\tilde{u}$ and $\langle f\rangle$.

LEMMA 3.22. *The following inequality holds:*

$$E_{\alpha,\nu}(\tilde{u},\tilde{f})(t)+\frac{1}{2}\int_s^t D_{\alpha,\nu}(\tilde{u},\tilde{f})(\tau)\,d\tau\leq E_{\alpha,\nu}(\tilde{u},\tilde{f})(s),\quad t\geq s\geq 0.$$

Proof. The proof of this lemma is completely analogous to the proof of (1.233). Formally, the only difference is the absence of the term $C_*\sum_{i=0}^n m_i^2$ in the statement of the inequality. The reason for this is Lemma 3.21, which (again, due to our definition of $\tilde{u}$ and $\tilde{f}$) enables us to conclude positivity of $\mathcal{E}(\tilde{u},\tilde{f})$ without adding any constant to it. □

From now on, we shall abuse the notation and denote $\tilde{u}$, $\tilde{R}$, $\tilde{f}$, $\tilde{\theta}$ and $\tilde{\phi}$ respectively by u, R, f, θ and ϕ. The main ingredient in proving the exponential decay is to control the instant energy in terms of the dissipation, as stated in the following lemma:

LEMMA 3.23. *There exists a constant $C>0$ such that $E_{\alpha,\nu}\leq CD_{\alpha,\nu}$.*

Proof. Note that the only term, which a-priori can not be bounded by a constant multiple of D, is precisely $\mathcal{E}_{(0)}$. Recall the expansion (1.202):

$$u\circ\phi = -(n-1)f - \Delta_g f + N(f).$$

Fix a spherical harmonic $s_{k,i}$ of degree k, where $k \geq 2$ and $1 \leq i \leq N(k)$ (recall the notation introduced in the introduction). Multiply both sides of the above relation by $s_{k,i}$ and integrate over $\mathcal{S}$. Observing that $\int_{\mathcal{S}}(-(n-1)f - \Delta_g f)s_{k,i} = (n-1)(k^2-1)f_{k,i}$, we get

$$f_{k,i} = \frac{1}{(n-1)(k^2-1)}\int_{\mathcal{S}} u\circ\phi\, s_{k,i} - \frac{1}{(n-1)(k^2-1)}\int_{\mathcal{S}} N(f)\, s_{k,i}. \tag{3.512}$$

Note that

$$\Big|\int_{\mathcal{S}} u\circ\phi\, s_{k,i}\Big| = \Big|\int_{\mathcal{S}}\big(u\circ\phi - \frac{1}{|\mathcal{S}|}\int_{\mathcal{S}} u\circ\phi\big)\, s_{k,i}\Big| \leq C||u\circ\phi - \frac{1}{|\mathcal{S}|}\int_{\mathcal{S}} u\circ\phi||_{L^2} \leq C||\nabla u||_{L^2(\Omega^\pm)},$$

where we used the Sobolev inequality in the last estimate above. The last inequality, together with (3.512), immediately implies

$$|f_{k,i}| \leq C\sqrt{\mathcal{D}} + C\Big|\int_{\mathcal{S}} N(f)\, s_{k,i}\, d\xi\Big|,\ k \geq 2 \tag{3.513}$$

In order to estimate $\mathbb{P}_0 f$ (recall (1.214)), we first invoke the decomposition

$$u\circ\phi R^{n-2}|g| = -(n-1)f - \Delta_g f + q(f), \tag{3.514}$$

where q stands for the nonlinear remainder with leading quadratic order term. Assuming without loss of generality that $\bar{R} = 1$ in the last relation in (3.510) and integrating (3.514) over $\mathcal{S}$, we find

$$\int_\Gamma u = -(n-1)\int_{\mathcal{S}} f + \int_{\mathcal{S}} q(f). \tag{3.515}$$

Multiplying the conservation law $\int_\Omega u + \int_{\mathcal{S}} f + \sum_{k=2}^n \binom{n}{k}\int_{\mathcal{S}} f^k/n\} = 0$ by $\frac{1}{|\Omega|}$ and (3.515) by $\frac{1}{|\Gamma|}$ and subtracting the two equations, we obtain

(3.516)

$$\big(\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma|}\big)\int_{\mathcal{S}} f = -\big(\frac{1}{|\Omega|}\int_\Omega u - \frac{1}{|\Gamma|}\int_\Gamma u\big) - \frac{1}{n|\Omega|}\sum_{k=2}^n \binom{n}{k}\int_{\mathcal{S}} f^k + \frac{1}{|\Gamma|}\int_{|\mathcal{S}|} q(f).$$

Note that $\zeta - \big(\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma|}\big) = \frac{n-1}{|\Gamma|} - \frac{n-1}{|\mathbb{S}^1|}$ and hence $|\zeta - \big(\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma|}\big)| \le C(M^*)^{1/2}$, which, for M^* small enough implies that $|\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma|}| \ge \zeta/2 > 0$. Hence, upon dividing (3.516) by $K_1 := \frac{1}{|\mathcal{S}|}\big(\frac{1}{|\Omega|} - \frac{n-1}{|\Gamma|}\big)$, we conclude

$$\mathbb{P}_0 f = -\frac{1}{K_1}\Big(\frac{1}{|\Omega|}\int_\Omega u - \frac{1}{|\Gamma|}\int_\Gamma u\Big) - \frac{1}{nK_1|\Omega|}\sum_{k=2}^{n}\binom{n}{k}\int_{\mathcal{S}} f^k + \frac{1}{K_1|\Gamma|}\int_{|\mathcal{S}|} q(f). \tag{3.517}$$

From the mean value theorem, we deduce

$$\Big|\frac{1}{|\Omega|}\int_\Omega u - \frac{1}{|\Gamma|}\int_\Gamma u\Big| \le C||\nabla u||_{L^2(\Omega^\pm)}.$$

Thus, from (3.517) and the previous inequality, we deduce

$$|\mathbb{P}_0 f| \le C||\nabla u||_{L^2(\Omega^\pm)} + C||f||_{L^2}^2 \tag{3.518}$$

Summing (3.518), (3.511) and (3.513), we obtain

$$\begin{aligned} |\mathbb{P}_0 f| + \sum_{k=1}^{\infty}|\mathbb{P}_k f| &\le C||\nabla u||_{L^2(\Omega^\pm)} + C||f||_{L^2}^2 + C\sum_{k=2}^{\infty}\sum_i \Big\{\Big|\int_{\mathcal{S}} N(f) s_{k,i}\Big| \\ &\le C||\nabla u||_{L^2(\Omega^\pm)} + C||f||_{L^2}^2 + C||N(f)||_{L^2} \le C||\nabla u||_{L^2(\Omega^\pm)} + C||f||_{L^2}^2. \end{aligned} \tag{3.519}$$

Using the smallness of $||f||_{L^2}$, from (3.519) we obtain

$$\sum_{k=0}^{\infty}|\mathbb{P}_k f| \le C||\nabla u||_{L^2(\Omega^\pm)} \le C\sqrt{\mathcal{D}}. \tag{3.520}$$

Finally, from the previous inequality and the conservation law (1.217), we immediately deduce

$$\Big|\int_\Omega u\Big| \le C\sqrt{\mathcal{D}}. \tag{3.521}$$

The inequalities (3.520) and (3.521) imply the bound $\mathcal{E}_{(0)} \le C\mathcal{D}$, which in turn implies the estimate claimed in the lemma. □

Using the same argument as in Chapter 2 to prove decay, we use the inequality (1.233) and Lemma 3.23 to conclude that there exist constants $k_1, k_2 > 0$ such that

$$E_{\alpha,\nu}(u,f)(t) \le k_1 e^{-k_2 t}, \ t \ge 0.$$

This finishes the proof of Theorem 1.2.

4. The unstable case $\zeta<0$

4.1. Local existence and linear instability. The instability norms $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ (defined in (2.264) and (2.265)) differ from their stable counterparts $\bar{\mathcal{E}}$ and $\bar{\mathcal{D}}$ basically by an addition of L^2-norms of ω and ω_t, respectively. Schematically

$$\bar{\mathcal{M}}(\mathcal{U},\omega)\approx\bar{\mathcal{E}}(\mathcal{U},\omega)+||\omega||_{L^2}^2;\quad \bar{\mathcal{N}}(\mathcal{U},\omega)\approx\bar{\mathcal{D}}(\mathcal{U},\omega)+||\omega_t||_{L^2}^2.$$

It is simple to see that our method used in the stable case to construct local solutions, is rather insensitive to the above (small) alteration in the choice of norms:

THEOREM 4.1. *Let $\zeta<0$. For any sufficiently small $L>0$ there exists $\tau>0$ depending on L and constants $0<\beta,\nu\le1$ such that if*

$$M^{\epsilon}_{\beta,\nu}(u_0^{\epsilon},f_0^{\epsilon})\le\frac{L}{2}$$

then there exists a unique solution $(u^{\epsilon},f^{\epsilon})$ to the regularized Stefan problem defined on the time interval $[0,\tau[$, satisfying the bound

$$\sup_{0\le t\le\tau}M^{\epsilon}_{\beta,\nu}(u^{\epsilon},f^{\epsilon}f)(t)+\int_0^{\tau}N^{\epsilon}_{\beta,\nu}(u^{\epsilon},f^{\epsilon})(s)\,ds\le L$$

and $M^{\epsilon}_{\beta,\nu}(u^{\epsilon},f^{\epsilon})(\cdot)$ is continuous on $[0,\tau[$.

Before we prove the theorem, we need to establish a-priori estimates in analogy to Lemma 3.2. We denote $\mathfrak{M}_q^j:=\mathfrak{M}_q(u^j,f^j)$, $\hat{\mathfrak{M}}_q^j:=\hat{\mathfrak{M}}_q^{\epsilon}(u^j,f^j)$, for any $j\in\mathbb{N}$, whereby in the definition (2.248), Ω is substituted by Ω^j. Analogously we define $\mathfrak{N}_q^j$ and $\hat{\mathfrak{N}}_q^j$. For any $j\in\mathbb{N}$ we finally set $\mathfrak{M}^j=\sum_{q=1}^k\mathfrak{M}_q^j$ and we define $\hat{\mathfrak{M}}^j$ analogously. Summing the ϵ-independent and the ϵ-dependent energies $\mathfrak{M}^j$ and $\hat{\mathfrak{M}}^j$, we introduce $\mathfrak{M}_{\epsilon}^j:=\mathfrak{M}^j+\hat{\mathfrak{M}}^j$. In a fully analogous fashion, we introduce $\mathfrak{N}^j$, $\hat{\mathfrak{N}}^j$ and $\mathfrak{N}_{\epsilon}^j$. Let us introduce

$$M_{\beta}^j=\mathcal{M}_{\epsilon}^j+\beta\mathfrak{M}_{\epsilon}^j;\quad N_{\beta}^j=\mathcal{N}_{\epsilon}^j+\beta\mathfrak{N}_{\epsilon}^j.$$

LEMMA 4.2. *Let $\zeta<0$. Then there exist positive constants β, C_B, θ_0 and τ^{ϵ}, such that for any $\theta,\theta_1<\theta_0$, $t\le\tau^{\epsilon}$ such that if $M_{\beta}(0)<\theta/2$ and*

$$M_{\beta}^m(t)+\int_0^t N_{\beta}^m(\tau)\,d\tau\le\theta,\quad \sup_{0\le s\le t}||f^m||_{L^2}^2\le C_B(\theta_1+\int_0^t\mathfrak{N}^m)$$

then

$$M_\beta^{m+1}(t)+\int_0^t N_\beta^{m+1}(\tau)\,d\tau \le \theta \quad \textit{and} \quad \sup_{0\le s\le t}||f^{m+1}||^2_{L^2}\le C_B(\theta_1+t\int_0^t \mathfrak{N}^{m+1}). \tag{4.522}$$

Remark. Note that the norm $||\nabla_g f||^2_{H^{2l-1}}$ is already contained in the definitions of $\mathcal{M}$ and $\mathfrak{M}$ and hence, at variance with the formulation of Lemma 3.2, the second claim of Lemma 4.2 is only concerned with the L^2-norm of f^{m+1}.

Remark. We claim that for any $0\le k\le l-1$ the following inequality holds:

$$\int_0^t\Big(||f^{m+1}_{t^{k+1}}||^2_{L^2}+2\epsilon||\nabla_g^2 f^{m+1}_{t^{k+1}}||^2_{L^2}+\epsilon^2||\nabla_g^4 f^{m+1}_{t^{k+1}}||^2_{L^2}\Big)\le \frac{C}{\eta}\int_0^t||u^{m+1}||^2_{L^2(\Omega^m)}+\eta\int_0^t\mathfrak{N}^{m+1}_{l-1-k}. \tag{4.523}$$

This claim follows by first applying the differential operator ∂_{t^k} to the regularized jump equation (3.280), taking the squares and integrating over $\mathcal{S}$. By the standard trace inequality and the assumptions of smallness on $\mathcal{M}^m_\epsilon$ and $\mathfrak{M}^m_\epsilon$, it is straightforward to get

$$||\partial_{t^k}([u_n^{m+1}]^+_-\circ\phi^m)||_{L^2}\le \eta\sum_{q=0}^k||\nabla^2 u_{t^q}||^2_{L^2(\Omega^m)}+\frac{C}{\eta}\sum_{q=0}^k||\nabla u^{m+1}_{t^q}||^2_{L^2(\Omega^m)}.$$

We now exploit the fact that $u_t^{m+1}=\Delta u^{m+1}$ to conclude that $\sum_{q=0}^k||\nabla u^{m+1}_{t^q}||^2_{L^2(\Omega^m)}\le C||\nabla u^{m+1}||^2_{H^{2k}(\Omega^m)}$. Using the standard interpolation inequality and the definition of $\mathfrak{N}^j$, we obtain (4.523).

Proof of Lemma 4.2. The second claim of (4.522) is proved in the same way as the inequality (3.333) is deduced from the jump relation (3.331). The proof of the first claim of the lemma is analogous to the proof of the first claim of (3.308) in Lemma 3.2. The energy identities from Section 2 are nearly the same, except for the additional quadratic terms that appear in the definitions (2.270) and (2.271) of Q^u and S^u, respectively. We first analyze the temporal energies: the identity (2.267) implies

$$\mathcal{M}_\epsilon(s)+\int_0^t\mathcal{N}_\epsilon(s)\,ds=\mathcal{M}_\epsilon(0)+\int_0^t\int_{\Gamma^m}\big\{O^{m,u}+P^{m,u}\big\}+\int_0^t\int_{\mathbb{S}^1}\big\{Q^{m,u}+S^{m,u}\big\}. \tag{4.524}$$

Here $O^{m,u} := \sum_{k=0}^{l-1} O_k^{m,u}$, $P^{m,u} = \sum_{k=0}^{l-1} P_k^{m,u}$, $Q^{m,u} = \sum_{k=0}^{l-1} Q_k^{m,u}$ and $S^{m,u} = \sum_{k=0}^{l-1} S_k^{m,u}$, whereby

$$O_k^{m,u} := O^u(\partial_{t^k} u^{m+1}), \qquad P_k^{m,u} := P^u(\partial_{t^k} u^{m+1}) \tag{4.525}$$

$$Q_k^{m,u} := Q^u(R^m_{t^k}, R^{m+1}_{t^k}, R^m), \qquad S_k^{m,u} := S^u(R^m_{t^k}, R^{m+1}_{t^k}, R^m). \tag{4.526}$$

Estimating RHS of (4.524) is analogous to the estimates from Subsection 3.2.2. Namley, $O^{m,u} = O^m$, $P^{m,u} = P^m$ and

$$\begin{aligned} Q^{m,u} &= Q^m + \sum_{k=0}^{l-1} \int_0^t \int_{\mathcal{S}} \big\{ (\mathbb{P}_0 + \mathbb{P}_1) f_{t^k}^{m+1} (\mathbb{P}_0 + \mathbb{P}_1) f_{t^{k+1}}^{m+1} + \epsilon \mathbb{P}_1 \Delta_{\mathcal{B}} f_{t^k}^{m+1} \mathbb{P}_1 \Delta_{\mathcal{B}} f_{t^{k+1}}^{m+1} \big\}; \\ S^{m,u} &= S^m + \sum_{k=0}^{l-1} \int_0^t \int_{\mathcal{S}} \big\{ ((\mathbb{P}_0 + \mathbb{P}_1) f_{t^{k+1}}^{m+1})^2 + \epsilon |\mathbb{P}_1 \nabla_g^2 f_{t^{k+1}}^{m+1}|^2 \big\}. \end{aligned} \tag{4.527}$$

We estimate the second term on RHS of the first equation in (4.527) in the following manner:

$$\begin{aligned} &\Big| \sum_{k=0}^{l-1} \int_0^t \int_{\mathcal{S}} \big\{ (\mathbb{P}_0 + \mathbb{P}_1) f_{t^k}^{m+1} (\mathbb{P}_0 + \mathbb{P}_1) f_{t^{k+1}}^{m+1} + \epsilon \mathbb{P}_1 \Delta_{\mathcal{B}} f_{t^k}^{m+1} \mathbb{P}_1 \Delta_{\mathcal{B}} f_{t^{k+1}}^{m+1} \big\} \Big| \\ &\leq \sum_{k=0}^{l-1} \Big\{ \lambda \big\{ \int_0^t ||f_{t^{k+1}}^{m+1}||_{L^2}^2 + \epsilon ||\nabla_g^2 f_{t^{k+1}}^{m+1}||_{L^2}^2 \big\} + \frac{C}{\lambda} \int_0^t \big\{ ||f_{t^k}^{m+1}||_{L^2}^2 + \epsilon ||\nabla_g^2 f_{t^k}^{m+1}||_{L^2}^2 \big\} \Big\} \\ &\leq \lambda \int_0^t \mathcal{N}_\epsilon^{m+1} + \eta \int_0^t \mathfrak{N}^{m+1} + \frac{C}{\eta} \int_0^t ||u^{m+1}||_{L^2(\Omega^m)}^2 + \frac{CC_B}{\lambda} (\theta_1 + t \int_0^t \mathfrak{N}^{m+1}) + \frac{C}{\lambda} \int_0^t \hat{\mathcal{M}}^{m+1}, \end{aligned} \tag{4.528}$$

where, in the second inequality we used the estimate (4.523) and the second inequality in (4.522) to bound $\int_0^t \{ ||f^{m+1}||_{L^2}^2 + \epsilon ||\nabla_g^2 f^{m+1}||_{L^2}^2 \}$ by the last two terms on the rightmost side of (4.528). The second term on RHS of the second equation in (4.527) is again estimated using the estimate (4.523):

$$\Big| \sum_{k=0}^{l-1} \int_0^t \int_{\mathcal{S}} \big\{ ((\mathbb{P}_0 + \mathbb{P}_1) f_{t^{k+1}}^{m+1})^2 + \epsilon |\mathbb{P}_1 \nabla_g^2 f_{t^{k+1}}^{m+1}|^2 \big\} \Big| \leq \eta \int_0^t \mathfrak{N}_\epsilon^{m+1} + \frac{C}{\eta} \int_0^t ||u^{m+1}||_{L^2(\Omega^m)}^2. \tag{4.529}$$

Using the identity (4.524) and the estimates analogous to part (b) of Lemma 3.4, together with the additional estimates (4.528) and (4.529), we arrive at the following inequality:

(4.530)

$$\mathcal{M}_\epsilon^{m+1}+\int_0^t\mathcal{N}_\epsilon^{m+1}\leq\mathcal{M}_\epsilon(0)+\frac{\tilde{C}t}{\lambda}\big(\mathcal{M}^m+\mathcal{M}^{m+1}\big)+\big(\frac{\bar{C}t}{\eta\lambda}+\frac{\bar{C}t}{\epsilon\eta\lambda}\big)\mathcal{M}^{m+1}$$
$$+C\big(\lambda+\sup_{0\leq s\leq t}||f^m||_{H^{n/2+3}}+\sqrt{\mathfrak{M}^m}\big)\big(\int_0^t\mathcal{N}_\epsilon^m+\int_0^t\mathcal{N}_\epsilon^{m+1}\big)+\frac{CC_B}{\lambda}(\theta_1+t\int_0^t\mathfrak{N}^{m+1})+\frac{C}{\lambda}\int_0^t\hat{\mathcal{M}}^{m+1}$$
$$+\big(C\lambda+\frac{\bar{C}\eta}{\lambda}+\frac{\bar{C}\eta}{\epsilon\lambda}+C\sqrt{\mathcal{M}^m}\big)\int_0^t\mathfrak{N}^{m+1}+C\int_0^t\mathcal{N}^m\sup_{0\leq s\leq t}\mathfrak{M}^{m+1}(s)+\frac{C}{\lambda}\int_0^t||u^{m+1}||^2_{L^2(\Omega^m)}.$$

We now turn our attention to the estimates for the space-time energies $\mathfrak{M}^j$ and $\mathfrak{N}^j$. Note that the estimates (3.403), (3.409), (3.412) carry over to the unstable case, where the energy terms $\mathfrak{D}^m$, $\mathfrak{D}^{m+1}$, $\mathcal{D}^{m+1}$ are replaced by $\mathfrak{N}^m$, $\mathfrak{N}^{m+1}$ and $\mathcal{N}^{m+1}$, respectively. The identity (3.421) reads

(4.531)

$$\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}\partial_{t^c}^*u^{m+1}\cdot\partial_{\mathbf{g}}^\xi\nabla_{\mathbf{g}}\partial_{t^c}^*[u_n^{m+1}]$$
$$+\frac{1}{2}\sum_{|\xi|=2p}\sum_{i=1}^n\int_{\mathcal{S}}\{|\nabla_g\sum_\mu A_\mu^{\xi_i}\partial_g^\mu|^2-(n-1)|\mathbb{P}_{2+}\sum_\mu A_\mu^{\xi_i}\partial_g^\mu|^2\}\Big|_0^t$$
$$+\frac{\epsilon}{2}\sum_{|\xi|=2p}\sum_{i=1}^n\int_{\mathcal{S}}\{|\nabla_g^3\sum_\mu A_\mu^{\xi_i}\partial_g^\mu|^2-(n-1)|\mathbb{P}_{2+}\nabla_g^2\sum_\mu A_\mu^{\xi_i}\partial_g^\mu|^2\}\Big|_0^t=\sum_{|\xi|=2p}\sum_{i=1}^n\int_0^t\int_{\mathcal{S}}Q^u(\chi,\omega,R^m),$$

where Q^u is defined by (2.270) and $\chi=\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^m$, $\omega=\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}$. For the sake of simplicity, let us denote $\tilde{Q}_c^{m,u}:=Q^u(\chi,\omega,R^m)$ and note that

$$\tilde{Q}_c^{m,u}=\tilde{Q}_c^m+\int_0^t\int_{\mathcal{S}}\mathbb{P}_1\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}\mathbb{P}_1\partial_t\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}$$
$$+\epsilon\int_0^t\int_{\mathcal{S}}\mathbb{P}_1\Delta_{\mathcal{B}}\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}\mathbb{P}_1\Delta_{\mathcal{B}}\partial_t\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}.$$

The estimates analogous to Lemma 3.7 imply:

(4.532)

$$\Big|\tilde{Q}_{2p+1,c}^m\Big|\leq C\big(t+\sqrt{\mathfrak{M}^m}+\lambda+\sup_{0\leq s\leq t}||f^m||_{L^2}\big)\Big(\int_0^t\mathfrak{N}_p^m+\int_0^t\mathfrak{N}_p^{m+1}\Big)+\big(\frac{\tilde{C}t}{\lambda}+\frac{\tilde{C}t}{\lambda\epsilon^4}\big)\sup_{0\leq s\leq t}\big(\mathfrak{M}_p^m+\mathfrak{M}_p^{m+1}\big).$$

Furthermore,

(4.533)
$$\Big|\int_0^t\int_{\mathcal{S}}\mathbb{P}_1\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}\mathbb{P}_1\partial_t\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}+\epsilon\int_0^t\int_{\mathcal{S}}\mathbb{P}_1\Delta_{\mathcal{B}}\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}\mathbb{P}_1\Delta_{\mathcal{B}}\partial_t\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}\Big|$$
$$\le\lambda\int_0^t||\mathbb{P}_1\partial_t\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}||_{L^2}^2+\frac{C}{\lambda}\int_0^t||\mathbb{P}_1\sum_\mu A_\mu^{\xi_i}\partial_g^\mu f_{t^c}^{m+1}||_{L^2}^2$$
$$\le C\lambda\int_0^t\big(\mathcal{N}^{m+1}+\mathfrak{N}_p^{m+1}\big)+\frac{C}{\lambda}\sup_{0\le s\le t}\sqrt{\mathfrak{M}^m(s)}\int_0^t\big(\eta\mathfrak{N}_p^{m+1}+\frac{C}{\eta}||u^{m+1}||_{L^2(\Omega^m)}^2\big)+\frac{C}{\lambda}\int_0^t||f^{m+1}||_{L^2}^2,$$

where we used the estimate (4.523) in the third inequality above. Observe that the identity (3.437) reads

(4.534)
$$\sum_{|\xi|=2p}\int_0^t\int_{\Gamma^m}\partial_{\mathbf{g}}^\xi\partial_{t^{c+1}}^* u^{m+1}\partial_{\mathbf{g}}^\xi\partial_{t^c}^*[u_n^{m+1}]$$
$$-\int_0^t\int_{\mathcal{S}}\sum_\xi\{|\nabla_g\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^{m+1}|^2-(n-1)|\mathbb{P}_{2+}\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^{m+1}|^2\}$$
$$-\epsilon\int_0^t\int_{\mathcal{S}}\sum_\xi\{|\nabla_g^3\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^{m+1}|^2-(n-1)|\mathbb{P}_{2+}\nabla_g^2\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^{m+1}|^2\}=\int_0^t\int_{\mathcal{S}}\sum_\xi S(\chi',\omega',R^m),$$

where S^u is defined by (2.271) and $\chi'=\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^m$, $\omega'=\sum_\mu A_\mu^\xi\partial_g^\mu f_{t^{c+1}}^{m+1}$. For the sake of simplicity, let us denote $\tilde{S}_c^{m,u}:=S^u(\chi,\omega,R^m)$ and note that

$$\tilde{S}_c^{m,u}=\tilde{S}_c^m+\int_0^t\int_{\mathcal{S}}|\mathbb{P}_1\omega'|^2+\epsilon\int_0^t\int_{\mathcal{S}}|\nabla_g^2\omega'|^2$$

The estimates analogous to Lemma 3.9 imply the bound on $\tilde{S}_c^m$:

$$|\tilde{S}_c^m|\le C\big(\sqrt{\mathfrak{M}^m}+\lambda+\sup_{0\le s\le t}||f^m||_{L^2}\big)\Big(\int_0^t\big(\mathfrak{N}_p^m+\hat{\mathfrak{N}}_p^m\big)+\int_0^t\big(\mathfrak{N}_p^{m+1}+\hat{\mathfrak{N}}_p^{m+1}\big)\Big)$$
$$+\frac{\tilde{C}\eta}{\lambda}\int_0^t\mathfrak{N}_p^{m+1}+\frac{\tilde{C}t}{\eta\lambda\epsilon}\mathfrak{M}_p^{m+1}. \tag{4.535}$$

Furthermore, analogously to (4.533), using the estimate (4.523) we obtain

(4.536)
$$\Big|\int_0^t\int_{\mathcal{S}}|\mathbb{P}_1\omega'|^2+\epsilon\int_0^t\int_{\mathcal{S}}|\nabla_g^2\omega'|^2\Big|\le\lambda\int_0^t\mathcal{N}^{m+1}+\frac{C\eta}{\lambda}\int_0^t\mathfrak{N}_p^{m+1}+\frac{C}{\eta\lambda}\int_0^t||u^{m+1}||_{L^2(\Omega^m)}^2.$$

From the argument in the paragraph preceding (4.531) and the estimates (4.532) - (4.536), we obtain the energy inequality analogous to (3.439):

(4.537)
$$\begin{aligned}&\big(1-\tilde{C}_1 t-\frac{\tilde{C}_1 t}{\epsilon}\big)\big(\mathfrak{M}_p^{m+1}+\int_0^t \mathfrak{N}_p^{m+1}\big)+\big(1-\tilde{C}_2 t-C_2 t\big)\big(\hat{\mathfrak{M}}_p^{m+1}+\int_0^t \hat{\mathfrak{N}}_p^{m+1}\big)\\&\quad\leq \mathfrak{M}_{\epsilon,p}(0)+\nu^l\int_0^t \mathfrak{N}_{p+1}^{m+1}+C\sum_{q=0}^{p-1}\int_0^t \mathfrak{N}_q^{m+1}+\mathcal{K}^p+\frac{\tilde{C}\lambda t}{\epsilon^4}\sup_{0\leq s\leq t}\mathcal{M}^{m+1}+C^*\int_0^t \mathcal{N}^{m+1},\end{aligned}$$

where ν is a small parameter and $\mathcal{K}^p$ is given by:

$$\mathcal{K}^p=C\big(t+\sqrt{\mathfrak{M}^m}+\lambda+\sup_{0\leq s\leq t}||f^m||_{L^2}\big)\int_0^t\big(\mathfrak{N}_p^m+\hat{\mathfrak{N}}_p^m\big)+\big(\frac{\tilde{C}t}{\lambda}+\frac{\tilde{C}t}{\lambda\epsilon^4}\big)\sup_{0\leq s\leq t}\mathfrak{M}_p^m.$$

Using the inequalities (4.530) and (4.537) in the same way as it is done in Subsection 3.2.5, we conclude the proof of Lemma 4.2. □

Proof of Theorem 4.1. The proof of the local existence on an ϵ-dependent time interval $[0,\tau^\epsilon[$ together with the proof of continuity is fully analogous to the proofs in Subsections 3.3.1, 3.3.2 and 3.3.3. In order to prove that the solution exists on a short time interval independent of ϵ, we proceed analogously to the Subsection 3.3.5. We first pass to the limit as $m\to\infty$ in the inequality (4.537). All the terms involving $\tilde{C}$, $\bar{C}$ in the estimates vanish in the limit, because they are used to bound the cross-terms. Thus, in turn we obtain an estimate with constants that do not contain ϵ in the denominator:

(4.538)
$$\begin{aligned}&\mathfrak{M}_p+\int_0^t \mathfrak{N}_p+\big(1-C_2 t\big)\big(\hat{\mathfrak{M}}_p^\epsilon+\int_0^t \hat{\mathfrak{N}}_p^\epsilon\big)\\&\quad\leq \mathfrak{M}_p(0)+\nu^l\int_0^t \mathfrak{N}_{p+1}+C\sum_{q=0}^{p-1}\int_0^t \mathfrak{N}_q+\mathcal{K}_1+C^*\int_0^t \mathcal{N}+\frac{C}{\eta\lambda}\int_0^t ||(u,f)||_{L^2}^2,\end{aligned}$$

where $||(u,f)||_{L^2}^2:=||u||_{L^2(\Omega)}^2+||f||_{L^2}^2$ and $\mathcal{K}_1=C\big(t+\sqrt{\mathfrak{M}}+\lambda+\sup_{0\leq s\leq t}||f||_{L^2}\big)\int_0^t\big(\mathfrak{N}_p+\hat{\mathfrak{N}}_p^\epsilon\big)$. Note that the key novelty with respect to the energy estimates in the stable case, is the presence of the (potentially) large multiple of the L^2-norm of (u,f) on RHS of (4.538). Multiplying (4.538) by ν^p and summing over $p=0,\dots,l-1$, just like in (3.502), we obtain

$$\begin{aligned}&\sup_{0\le s\le t}\mathfrak{M}^{\nu}+\sum_{p=0}^{l-1}d_p\int_0^t\mathfrak{N}_p+(1-C_2t)\big(\sup_{0\le s\le t}\hat{\mathfrak{M}}^{\epsilon,\nu}+\int_0^t\hat{\mathfrak{N}}^{\epsilon,\nu}\big)\\&\le\mathfrak{M}^{\nu}_{\epsilon}(0)+\epsilon\sum_{q=0}^{l-1}\nu^q\mathcal{K}_1+\sum_{q=0}^{l-1}\nu^qC^*\int_0^t\mathcal{N}+\frac{C\sum_{q=0}^{l-1}\nu^q}{\eta\lambda}\int_0^t||(u,f)||_{L^2},\end{aligned}\tag{4.539}$$

We now pass to the limit as $m\to\infty$ in (4.530). Note that the term $\frac{CC_B}{\lambda}(\theta_1+t\int_0^t\mathfrak{N}^{m+1})$ used to estimate $\int_0^t||f^{m+1}||^2_{L^2}$ can, in the limit, be replaced by the bound $\int_0^t||f||^2_{L^2}\le C\int_0^t||(u,f)||^2_{L^2}$. We obtain

$$\begin{aligned}&\mathcal{M}_\epsilon+\int_0^t\mathcal{N}_\epsilon\le\mathcal{M}_\epsilon(0)+C\big(\lambda+\sup_{0\le s\le t}||f||_{H^{n/2+3}}+\sqrt{\mathfrak{M}}\big)\int_0^t\mathcal{N}_\epsilon\\&+C\big(\lambda+\sqrt{\mathcal{M}}\big)\int_0^t\mathfrak{N}+C\int_0^t\mathcal{N}\sup_{0\le s\le t}\mathfrak{M}(s)+\frac{C}{\lambda}\int_0^t||(u,f)||^2_{L^2}.\end{aligned}\tag{4.540}$$

Combining the estimates (4.539) and (4.540) in the same way as the estimates (3.502) and (3.503) are used in Subsection 3.3.5, we conclude that there exists a $\tau>0$ such that $\tau^\epsilon\ge\tau$ for any ϵ. $\square$

Linear instability. In order to find the growing mode λ_0 of the Stefan problem with surface tension, we turn our attention to the associated eigenvalue problem:

$$\lambda v-\Delta v=0\quad\text{on }\Omega\setminus\mathcal{S},\tag{4.541}$$

$$v=-(n-1)f-\Delta_g f\quad\text{on }\mathcal{S},\tag{4.542}$$

$$[v_n]^+_-=-\lambda f\quad\text{on }\mathcal{S},\tag{4.543}$$

$$v_n=0\quad\text{on }\partial\Omega;\ [v]^+_-=0\quad\text{on }\mathcal{S}.\tag{4.544}$$

For any $k\in\mathbb{N}\cup\{0\}$ set

$$H^k(\Omega^\pm)=\Big\{u:\Omega\to\mathbb{R};\ u\mathbf{1}_{\mathrm{cl}(\Omega^\pm)}\in H^k(\mathrm{cl}(\Omega^\pm))\Big\},\tag{4.545}$$

where $\Omega^+:=\Omega\setminus\mathrm{cl}(B_1(\mathbf{0}))$ and $\Omega^-:=B_1(\mathbf{0})$. We define the linear operator $\mathcal{L}$

$$\mathcal{L}(v,f):=(\Delta v,-[\partial_n v]^+_-),\tag{4.546}$$

where the domain of $\mathcal{L}$ is given by

(4.547)
$$D(\mathcal{L}) := \Big\{(v,f)\in H^2(\Omega^\pm)\times H^{7/2}(\mathcal{S});\ v_n=0 \text{ on } \partial\Omega;\ [v]_-^+=0 \text{ on } \mathcal{S};\ v=-(n-1)f-\Delta_g f \text{ on } \mathcal{S}\Big\}.$$

The following lemma states a number of important properties of the operator $\mathcal{L}$ and the proof can be found in [**46**]:

LEMMA 4.3. *There are countably many eigenvalues $\lambda_0, \lambda_1, \lambda_2 \dots$, they are all of finite multiplicity and the associated eigenvectors are smooth. There is exactly one simple positive eigenvalue λ_0. Furthermore, $\lambda_1 = 0$ and its eigenspace has dimension three. All other eigenvalues are negative.*

An important step in obtaining the sharp linear growth estimates is to provide the *variational characterization* of the positive eigenvalue λ_0. To do so, we identify the natural symmetric product structure dictated by the linear problem; we define a bilinear operator $\langle\cdot,\cdot\rangle$ on $D(\mathcal{L})\times D(\mathcal{L})$. Let (v,f), $(w,g)\in D(\mathcal{L})$. Set

$$\langle (v,f),(w,h)\rangle := \int_\Omega vw - \int_{\mathcal{S}} f((n-1)I+\Delta_g)h = \int_\Omega vw + \int_{\mathcal{S}} (\nabla_g f\cdot\nabla_g h - (n-1)fh)$$

LEMMA 4.4. *The operator $\mathcal{L}$ is a symmetric operator on $D(\mathcal{L})$, with respect to $\langle\cdot,\cdot\rangle$. Furthermore, the eigenvectors associated to any two different eigenvalues are orthogonal with respect to $\langle\cdot,\cdot\rangle$.*

Proof. A straightforward computation yields:

$$\begin{aligned}
&\langle \mathcal{L}(v,f),(w,h)\rangle = \langle (\Delta v, -[\partial_n v]_-^+),(w,h)\rangle = \int_\Omega \Delta v w + \int_{\mathcal{S}} [\partial_n v]_-^+((n-1)I+\Delta_g)h \\
&= \int_\Omega \Delta w v + \int_{\mathcal{S}} [\partial_n v]_-^+ w - \int_{\mathcal{S}} [\partial_n w]_-^+ v - \int_{\mathcal{S}} [v_n]_-^+ w = \int_\Omega \Delta w v - \int_{\mathcal{S}} [\partial_n w]_-^+ v \\
&= \int_\Omega \Delta w v - \int_{\mathcal{S}} -[\partial_n w]_-^+((n-1)I+\Delta_g)f = \langle (v,f),(\Delta w, -[\partial_n w]_-^+)\rangle = \langle (v,f),\mathcal{L}(w,h)\rangle.
\end{aligned}$$

Note that in the second equality, we used the Green's identity to rewrite $\int_\Omega \Delta v w$ and we also used the boundary condition $w = -((n-1)I+\Delta_g)h$ to rewrite $\int_{\mathcal{S}}[\partial_n v]_-^+((n-1)I+\Delta_g)h$. Note also that the third equality implies

$$\langle \mathcal{L}(v,f),(w,h)\rangle = -\int_\Omega \nabla v\cdot\nabla w. \tag{4.548}$$

If λ and μ are two different eigenvalues (w.l.o.g. $\mu\neq 0$), with associated eigenvectors (v,f) and (w,h) respectively, then

$$\langle \mathcal{L}(v,f),(w,h)\rangle = \lambda\langle (v,f),(w,h)\rangle = \frac{\lambda}{\mu}\langle (v,f),\mathcal{L}(w,h)\rangle$$

and hence, due to the symmetry of $\langle\cdot,\cdot\rangle$, we have $\langle \mathcal{L}(v,f),(w,h)\rangle = 0$. $\square$

Variational principle. Before stating the variational principle, natural to the linearized Stefan problem, we introduce some further notation. To every eigenvalue λ_i with multiplicity $m(i)$, we associate the eigenvectors $e_{i,1},\dots,e_{i,m(i)}$, where we denote $e_{i,k}=(v_{i,k},z_{i,k})$ for $1\le k\le m(i)$. The null-space $N_{\mathcal{L}}$ of the operator $\mathcal{L}$ is spanned by $n+1$ eigenvectors $e_{1,1}=(1,-1)$, $e_{1,i}=(0,s_i)$ $(i=2,\dots,n+1)$, associated to the eigenvalue $\lambda_1=0$. Let us define the functional space D_1 by:

(4.549)

$$D_1 := \Big\{(v,f)\in H^1(\Omega^{\pm})\times H^{5/2}(\mathcal{S});\ v_n=0 \text{ on } \partial\Omega;\ [v]_-^+=0 \text{ on } \mathcal{S};\ v=-(n-1)f-\Delta_g f \text{ on } \mathcal{S}\Big\}$$

By $N_{\mathcal{L}}^{\perp}\subset D_1$ we denote the orthogonal complement of $N_{\mathcal{L}}$ in D_1 with respect to $\langle\cdot,\cdot\rangle$. In other words,

$$N_{\mathcal{L}}^{\perp} = \Big\{(v,f)\in D_1 \big|\ \int_\Omega v + \int_{\mathcal{S}} f = 0,\ \mathbb{P}_1 f = 0\Big\}.$$

The following lemma is analogous to its two dimensional version stated in Lemmas 4.5 and 4.6 in [**32**]. We obtain the following result:

Lemma 4.5. *The following variational characterization of the positive eigenvalue λ_0 holds:*

$$-\frac{1}{\lambda_0} = \min_{(v,f)\in N_{\mathcal{L}}^{\perp}} \frac{\langle (v,f),(v,f)\rangle}{\int_\Omega |\nabla v|^2}.$$

Furthermore, if we denote the associated eigenvector by $e_{0,1}=(v_{0,1},z_{0,1})$, then $v_{0,1}$ is spherically symmetric.

Proof. For $(v,f)\in N_{\mathcal{L}}^{\perp}$ let us set

$$\mathcal{I}(v,f) := \frac{\langle (v,f),(v,f)\rangle}{\int_\Omega |\nabla v|^2}. \tag{4.550}$$

Let $\iota = \inf_{(v,f)\in N_{\mathcal{L}}^{\perp}} \mathcal{I}(v,f)$.

Boundedness from below: let $(v,f)\in N_{\mathcal{L}}^{\perp}$. Since $\int_\Omega v + \int_{\mathcal{S}} f = 0$ and $\int_{\mathcal{S}} v = -(n-$

$1)\int_{\mathcal{S}} f$, we deduce that $\int_{\mathcal{S}} v=(n-1)\int_\Omega v$. By the mean value theorem $\left|\frac{1}{|\Omega|}\int_\Omega v-\frac{1}{|\mathcal{S}|}\int_{\mathcal{S}} v\right|\leq C||\nabla v||_{L^2(\Omega^\pm)}$, and thus $|\zeta|\left|\int_\Omega v\right|\leq C||\nabla v||_{L^2(\Omega^\pm)}$, where $||\cdot||_{L^2(\Omega^\pm)}$ is defined by (4.545). Since $|\zeta|>0$, we conclude

$$\langle (v,f),(v,f)\rangle=\int_\Omega |\mathbf{P}v|^2+\int_{\mathcal{S}}|\nabla_g f|^2-(n-1)|\mathbb{P}f|^2)+\zeta\Big(\int_\Omega v\Big)^2\geq \zeta\Big(\int_\Omega v\Big)^2\geq -\frac{C}{\zeta}||\nabla v||^2_{L^2(\Omega^\pm)},$$

and hence $\iota>-\infty$.

Negativity: $\iota<0$. Let

$$v_{d,\epsilon}(r)=\begin{cases} d \quad 0\leq r\leq 1-\epsilon \text{ and } 1+\epsilon\leq r\leq R_*,\\ d+\frac{1-d}{\epsilon}(r-(1-\epsilon)), \quad 1-\epsilon\leq r\leq 1,\\ v_{d,\epsilon}(2-r), \quad 1\leq r\leq 1+\epsilon.\end{cases}$$

Note that $v_{d,\epsilon}(1)=1$ and we define $f_{d,\epsilon}=-1$. Hence $\mathbb{P}_1 f_{d,\epsilon}=0$ for any d,ϵ. For a fixed d note that $\int_\Omega v_{d,\epsilon}=\omega_n R_*^n d+O(\epsilon)$ and thus the condition $\int_\Omega v_{d,\epsilon}+\int_{\mathcal{S}} f_{d,\epsilon}=0$ reads $\omega_n R_*^n d-n\omega_n=O(\epsilon)$ and hence $d=n/R_*^n+O(\epsilon)$. On the other hand, it is easy to see that $\int_\Omega |\mathbf{P}v_{d,\epsilon}|^2=O(\epsilon)$. Note however

$$\langle (v_{d,\epsilon},f_{d,\epsilon}),(v_{d,\epsilon},f_{d,\epsilon})\rangle=\zeta\Big(\int_\Omega v_{d,\epsilon}\Big)^2+\int_\Omega |\mathbf{P}v_{d,\epsilon}|^2=\zeta\omega_n^2R_*^{2n}d^2+O(\epsilon)<0$$

for small ϵ. To conclude that $\iota<0$, we slightly perturb the function $v_{d,\epsilon}$ to a smooth function v_{neg} so that the $\int_\Omega|\nabla v_{\text{neg}}|^2$ is well defined.

Existence of the minimizer. Let (v_n,f_n) be an infimizing sequence for the above variational problem. Without loss of generality assume $\int_\Omega |\nabla v_n|^2=1$ (otherwise rescale). As in the proof of boundedness,

$$|\int_\Omega v_n|^2\leq \frac{C}{\zeta}||\nabla v_n||_{L^2(\Omega^\pm)} \tag{4.551}$$

and hence from here and the Poincaré inequality $||v_n||_{L^2(\Omega^\pm)}\leq C||\nabla v_n||_{L^2(\Omega^\pm)}=C$. Due to (4.551), $\mathbb{P}_1 f_n=0$ and the condition $v|\mathcal{S}=-(n-1)f-\Delta_g f$, we deduce that there exists a positive constant C such that $||f_n||_{H^2}\leq C$ for any n. By Banach-Alaoglu, we deduce that there exist weak limits v^* and f^* of $\{v_n\}$ and $\{f_n\}$ in $H^1(\Omega)$ and $H^2(\mathcal{S})$ respectively. Hence $\langle (v_n,f_n),(v_n,f_n)\rangle\to\langle (v^*,f^*),(v^*,f^*)\rangle$ as $n\to\infty$. Furthermore, by weak lower semi-continuity of $||\cdot||_{H^1}$, we deduce

$1 = \liminf_{n\to\infty} ||\nabla v_n||_{L^2(\Omega^\pm)} \geq ||\nabla v^*||_{L^2(\Omega^\pm)}$. Since $\mathcal{I}(v_n, f_n) < 0$ for n large enough, we deduce

$$\iota = \liminf_{n\to\infty} \mathcal{I}(v_n, f_n) \geq \liminf_{n\to\infty} \frac{\langle (v_n, f_n), (v_n, f_n)\rangle}{||\nabla v^*||^2_{L^2(\Omega^\pm)}} = \mathcal{I}(v^*, f^*)$$

and hence (v^*, f^*) is a minimizer. Note that $||\nabla v^*||_{L^2(\Omega^\pm)} = 1$. Otherwise assume $||\nabla v^*||_{L^2(\Omega^\pm)} < 1$. Then $\langle (v^*, f^*), (v^*, f^*)\rangle = \iota ||\nabla v^*||^2_{L^2(\Omega^\pm)} > \iota = \liminf_{n\to\infty} \langle (v_n, f_n), (v_n, f_n)\rangle = \langle (v^*, f^*), (v^*, f^*)\rangle$, which is a contradiction (note that we used the fact $\iota < 0$).

Euler-Lagrange equation. Setting $\iota = -\frac{1}{\lambda_0}$, the Euler-Lagrange equation for the variational problem of minimizing $\mathcal{I}$ over $N_{\mathcal{L}}^\perp$ allows to conclude that the minimizer (v^*, f^*) is the weak solution of the eigenvalue problem $\mathcal{L}(v, f) = \lambda_0(v, f)$, $(v, f) \in D_1$. In other words, for any test function $\varphi \in C^\infty(\Omega^\pm)$:

$$\int_\Omega \left\{\lambda_0 v^* \varphi - \nabla v^* \cdot \nabla \varphi\right\} - \lambda_0 \int_{\mathbb{S}^1} f\varphi = 0. \tag{4.552}$$

By standard elliptic regularity, it is standard to see that $v^* \in H^2(\Omega^\pm)$ Hence λ_0 is a positive eigenvalue for the linearized Stefan problem, with an associated eigenvector (v^*, f^*). Note that $v^*|_{\mathcal{S}} \in H^{3/2}(\mathcal{S})$ and the Dirichlet boundary condition $v^*|_{\mathcal{S}} = -(n-1)f^* - \Delta_g f^*$ implies $f^* \in H^{7/2}(\mathcal{S})$. By a classical bootstrap argument, we conclude that $(v^*, f^*) \in C^\infty(\Omega^\pm) \times C^\infty(\mathcal{S})$. □

Sharp linear growth. With the help of Lemma 4.5, we can turn $N_{\mathcal{L}}^\perp$ into a normed space. Fix some $I > 0$ and introduce a new norm $||\cdot||_I$ through the following scalar product $\langle \cdot, \cdot, \rangle_I$:

$$\langle (v,f),(w,h)\rangle_I = \langle (v,f),(w,h)\rangle + (\frac{1}{\lambda_0} + I) \int_\Omega \nabla v \cdot \nabla w, \quad (v,f),\ (w,h) \in N_{\mathcal{L}}^\perp. \tag{4.553}$$

From Lemma 4.5, we immedaitely see that $\langle (v,f),(v,f)\rangle_I \geq I ||\nabla v||^2_{L^2(\Omega^\pm)} \geq 0$. Furthermore, if $\langle (v,f),(v,f)\rangle_I = 0$, then $||\nabla v||_{L^2(\Omega^\pm)} = 0$ and hence $v \equiv$ const, which implies $(v,f) = \sum_{i=1}^{n+1} k_i e_{1,i}$, for some reals k_i, $i = 1, \ldots, n+1$. Since $(v,f) \in N_{\mathcal{L}}^\perp$, it follows $(v,f) = (0,0)$. Thus, $\langle \cdot, \cdot \rangle_I$ is an inner product on $N_{\mathcal{L}}^\perp$ and it defines the norm $||\cdot||_I := \sqrt{\langle \cdot, \cdot \rangle_I}$. Let us first show that this norm controls the L^2-norm on $N_{\mathcal{L}}^\perp$:

LEMMA 4.6. *There exists a constant* $C>0$ *such that for any* $(v,f)\in N_{\mathcal{L}}^{\perp}$, $||v||_{L^2(\Omega)}+||f||_{L^2}\leq C||(v,f)||_I$.

Proof. Let $(v,f)\in N_{\mathcal{L}}^{\perp}$. Using the mean value theorem, we conclude $|\zeta|\left|\int_\Omega v\right|\leq C||\nabla v||_{L^2(\Omega^\pm)}$. From this and the Poincaré inequality, we deduce $||v||_{L^2(\Omega)}\leq C||\nabla v||_{L^2(\Omega^\pm)}$. Furthermore, from $v|\mathcal{S}=-(n-1)f-\Delta_g f$, we obtain $\int_{\mathcal{S}}|\nabla_g f|^2-(n-1)|\mathbb{P}f|^2\leq C||v||_{L^2(\mathcal{S})}+C|\int_{\mathcal{S}}f|^2$. Since $\mathbb{P}_1 f=0$ and $|\int_{\mathcal{S}}f|=|\int_{\mathcal{S}}v|\leq C||v||_{L^2(\mathcal{S})}$, from the trace inequality we deduce

$$||f||_{L^2}^2\leq\int_{\mathcal{S}}|\nabla_g f|^2-(n-1)|\mathbb{P}f|^2+\frac{1}{|\mathcal{S}|}\Big(\int_{\mathcal{S}}f\Big)^2\leq C||v||_{L^2(\mathcal{S})}^2\leq C||v||_{H^1(\Omega^\pm)}^2\leq C||\nabla v||_{L^2(\Omega^\pm)}^2.$$

Since $||(v,f)||_I^2\geq I||\nabla v||_{L^2(\Omega^\pm)}^2$, we conclude the claim of the lemma. □

Using the symmetry of the operator $\mathcal{L}$ with respect to the bilinear structure $\langle\cdot,\cdot\rangle$ and the variational characterization of Lemma 4.5, we can work a bit harder to obtain the following lemma:

LEMMA 4.7. *The set* $\mathcal{B}=\cup_{i=0}^{\infty}\cup_{k\leq m(i)}\{e_{i,k}\}$ *forms an orthonormal basis of* $N_{\mathcal{L}}^{\perp}$ *with respect to* $\langle\cdot,\cdot\rangle_I$.

The natural decomposition into spaces $N_{\mathcal{L}}$ and $N_{\mathcal{L}}^{\perp}$ dictates the "right" choice of norms, with respect to which we shall measure the linear growth rate. For a given $y\in D_1$, let $y=y_{N_{\mathcal{L}}}+y^{\perp}$, where $y_{N_{\mathcal{L}}}$ is the projection onto $N_{\mathcal{L}}$ and $y^{\perp}$ projection onto $N_{\mathcal{L}}^{\perp}$, with respect to $\langle\cdot,\cdot\rangle$. We define

$$||y||^2:=||y_{N_{\mathcal{L}}}||_{L^2(\Omega)\times L^2(\mathcal{S})}^2+||y^{\perp}||_I^2. \tag{4.554}$$

Due to Lemma 4.6, we immediately see that

$$||y||_{L^2(\Omega)\times L^2(\mathbb{S}^1)}\leq C||y||. \tag{4.555}$$

LEMMA 4.8. *Let* $(w_0,f_0)=\sum_{i=0}^{\infty}\sum_{1\leq k\leq m(i)}a_{i,k}e_{i,k}$. *Then there exists a constant* $C_{\mathcal{L}}$, *such that*

$$||e^{\mathcal{L}t}(w_0,f_0)||\leq C_{\mathcal{L}}e^{\lambda_0 t}||(w_0,f_0)||.$$

Proof. Note that

$$
\begin{aligned}
||e^{\mathcal{L}t}(w_0,f_0)||^2 &= ||\sum_{1\le k\le n+1} a_{1,k}e_{1,k} + \sum_{i\ge 0, i\ne 1} e^{\lambda_i t}\sum_{1\le k\le m(i)} a_{i_k}e_{i,k}||^2 \\
&= ||\sum_{1\le k\le n+1} a_{1,k}e_{1,k}||^2_{L^2(\Omega)\times L^2(\mathcal{S})} + e^{2\lambda_0 t}|a_{0,1}|^2 + \sum_{i\ge 2} e^{2\lambda_i t}\sum_{1\le k\le m(i)} |a_{i,k}|^2 \\
&\le Ce^{2\lambda_0 t}\sum_{i,k}|a_{i,k}|^2,
\end{aligned}
$$

and the claim follows. □

4.2. Proof of Theorem 1.3. The abstract formulation. We shall use the bootstrap method developed in [**30**] to prove the instability of the stationary surfaces in the case $\zeta < 0$. The following lemma is a simple modification of Lemma 1 in [**29**], which suffices to establish the *nonlinear* instability of steady spheres for the Stefan problem.

LEMMA 4.9. *Assume that X is a Banach space with the associated norm $||\cdot||_X$ and $\mathcal{L}: D\subset X\to X$ is a linear operator. Assume a norm $||\cdot||$ with the property $||\cdot||_X \le C||\cdot||$ and such that $e^{t\mathcal{L}}$ generates a strongly continuous semigroup satisfying*

$$||e^{t\mathcal{L}}||_{X\to X} \le C_{\mathcal{L}} e^{\lambda_0 t} \tag{4.556}$$

for some $C_{\mathcal{L}}$ and $\lambda_0 > 0$ ($||\cdot||_{X\to X}$ stands for the operator norm). Assume a nonlinear operator $U: D\to X$, a norm $|||\cdot|||$ on D and a constant C_U, such that

$$||U(y)|| \le C_U|||y|||^2 \tag{4.557}$$

for all $y\in D$ and $|||y|||<\infty$. Assume that there exists a small constant σ such that for any solution $y(t)$ to the equation $y' = \mathcal{L}y + U(y)$, with $|||y(t)|||\le\sigma$, there exist constants C_1 and C_2 and a small constant μ, such that the following energy estimate holds:

$$|||y(t)|||^2 \le C_1|||y_0|||^2 + C_2\int_0^t ||y(s)||^2\,ds. \tag{4.558}$$

Consider a family of initial data $y^\delta(0)=\delta y_0$ with $||y_0||=1$ and $|||y_0|||<\infty$ and let θ_0 be a sufficiently small (fixed) number. Then there exists some constant $C>0$ such that if

$$0\le t\le T:=\frac{1}{\lambda}\log\frac{\theta_0}{\delta},$$

we have

$$\left|\left|y(t)-\delta e^{\mathcal{L}t}y_0\right|\right|\le C\Big(|||y_0|||^2+1\Big)\delta^2e^{2\lambda_0 t}.$$

In particular, if there exists a constant C_p such that $||\delta e^{\mathcal{L}t}y_0||\ge C_p\delta e^{\lambda_0 t}$, then there exists en escape time $T^\delta\le T$ such that

$$\left|\left|y(T^\delta)\right|\right|\ge K_0>0, \tag{4.559}$$

where K_0 depends explicitly on $C_{\mathcal{L}}$, C_U, C_2, C_p, y_0, λ_0 and is independent of δ.

Proof of nonlinear instability. Let $M^{**}\le\frac{L}{2}$, where L is given by Theorem 4.1 and let (u,f) be the unique solution of the regularized Stefan problem with the initial data satisfying $E^\epsilon_{\beta,\nu}(u^\epsilon_0,f^\epsilon_0)\le M^{**}$. In analogy to the derivation of (3.505), we can choose λ, M^{**}, ν small, use the smallness of $||f||_{H^l}$ and pass to the limit as $\epsilon\to 0$ to arrive at

$$\sup_{0\le s\le\tau}M_{\beta,\nu}(u,f)(s)+\frac{1}{2}\int_0^\tau N_{\beta,\nu}(u,f)(s)\,ds\le M_{\beta,\nu}(u_0,f_0)+C\int_0^\tau||(u,f)||^2_{L^2}\,ds \tag{4.560}$$

as stated in Theorem 1.3. To apply Lemma 4.9 and finish the proof of Theorem 1.3, we have to formulate the Stefan problem on a fixed domain. To this end we apply the change of variables described in Subsection 3.3.1: we set $\bar{x}=\pi(t,x)x$, where π is defined by dropping the super-indices in the definition (3.444). Setting $v:=u\circ\pi$ leads to an equation for the function $v:\Omega_{\mathcal{S}}\to\mathbb{R}$:

$$v_t-\Delta v=a_{ij}u_{x^ix^j}+b_iu_{x^i}, \tag{4.561}$$

where a_{ij} and b_i are defined by dropping the index m in (3.446). From (3.447), we deduce

$$[v_n]^+_-=-\frac{f_tR^3}{|g|^2}\text{ on }\mathbb{S}^1. \tag{4.562}$$

The Dirichlet boundary condition simply reads:

$$v=-(n-1)f-\Delta_g f+N(f) \text{ on } \mathcal{S}. \tag{4.563}$$

In order to set the problem (4.561) - (4.563) in the framework of Lemma 4.9, we have to consider the new unknown $w(x):=v(x)-N(f(\frac{x}{|x|}))$. This way we insure that the nonlinear boundary condition (4.563) transforms into a linear condition for w. We obtain

$$w_t-\Delta w=\mathcal{R}(w,f), \quad \text{in } \Omega_{\mathcal{S}}, \tag{4.564}$$

$$w=-(n-1)f-\Delta_g f \quad \text{on } \mathcal{S}, \tag{4.565}$$

$$[w_n]_-^+=-\frac{f_tR^3}{|g|^2} \text{ on } \mathcal{S}, \tag{4.566}$$

$$w_n=0 \quad \text{on } \partial\Omega. \tag{4.567}$$

Here

$$\begin{aligned}\mathcal{R}(w,f) &= a_{ij}v_{x^ix^j}+b_iv_{x^i}-(\partial_t-\Delta)N(f(\frac{\cdot}{|\cdot|}))\\ &= a_{ij}\big(w+N(f(\frac{\cdot}{|\cdot|}))\big)_{x^ix^j}+b_i\big(w+N(f(\frac{\cdot}{|\cdot|}))\big)_{x^i}-(\partial_t-\Delta)N(f(\frac{\cdot}{|\cdot|})).\end{aligned}$$

To apply Lemma 4.9 set $X:=L^2(\Omega)\times L^2(\mathcal{S})$ and $D=D_1$ as defined in (4.549). The norm $||\cdot||$ is defined by (4.554) and we also define

$$|||(w,f)|||^2:=||w||^2_{H^{2l-1}(\Omega^{\pm})}+||f_\theta||^2_{H^{2l-1}}$$

Observe that if $(w,f)\in X$ is a solution to the Stefan problem (4.564) - (4.567), then the norm $|||\cdot|||$ is equivavlent to the norms defined by $M_{\beta,\nu}$. This follows easily from the boundedness of the Jacobian of the coordinate transformation $\bar{x}=\pi(t,x)x$ and the fact that the time derivatives of w and f can be recursively expressed in terms of purely spatial derivatives via the relations $w_t=\Delta w+F$ and $f_t=-\frac{|g|^2}{R^3}[w_n]_-^+$. From this remark, the bound (4.555) and the inequality (4.560), we immediately conclude that there exist constants C_1 and C_2, such that the inequality (4.558) is satisfied. Furthermore, we define

$$U(w,f) = (\mathcal{R}(w,f), -(\frac{|g|^2}{R^3} - 1)[w_n]_-^+). \tag{4.568}$$

Note that the time derivative of f that appears in the definition of $\mathcal{R}(w,f)$ can be alternatively expressed as a purely spatial differential operator via (4.566). With this notation, the problem (4.564) - (4.567) takes the form $\partial_t(w,f) = \mathcal{L}(w,f) + U(w,f)$. It is straightforward to verify $||U(w,f)|| \leq C|||(w,f)|||^2$ and thus, the assumption (4.557) from Lemma 4.9 is satisfied whereas Lemma 4.8 guarantees (4.556). We may apply Lemma 4.9 to conclude that (1.236) holds and this finishes the proof of Theorem 1.3.

□

CHAPTER 4

Stefan problem in physical units, onset of nucleation and further comments

Following Chapter 4 in [**53**], we formulate the Stefan problem with relevant physical coefficients:

(0.569) $$c_i\partial_t v^i - k_i\Delta v^i = 0 \quad \text{in} \quad \Omega\setminus\Gamma,\ i=+,-.$$

(0.570) $$v = \sigma\kappa \quad \text{on} \quad \Gamma,$$

(0.571) $$lV = [kv_n]_-^+ \quad \text{on} \quad \Gamma,$$

(0.572) $$v_n = 0 \quad \text{on} \quad \partial\Omega,$$

(0.573) $$v(0,\cdot) = v_0; \quad \Gamma(0) = \Gamma_0,$$

where $[kv_n]_-^+ := k_+v_n^+ - k_-v_n^-$. In the first equation above, $c_i > 0$ stands for the heat capacity per unit volume (which is defined to be the heat necessary to increase the temperature of a unit volume by one degree). The coefficient of thermal conductivity is denoted by k_i and l stands for the density of latent heat (which corresponds to the heat exchanged by a phase transition of a unit volume). The surface tension coefficient is denoted by σ as before. A measure, naturally associated with the problem is given by $dc := \sum_{i=\pm} c_i 1_{\Omega^i}\,dx$. Linearizing the problem (0.569) - (0.573) around a steady state $(v,\Gamma) \equiv (\frac{\sigma}{\bar{R}}, S_{\bar{R}}(\mathbf{0}))$, we can conduct the same analysis as we did for the problem (1.208) - (1.211) to conclude that the stability is contingent upon the sign of the coefficient

$$\zeta(\bar{R}) := \frac{1}{\int_\Omega dc} - \frac{\sigma(n-1)}{l|S_{\bar{R}}|\bar{R}^2}.$$

Just like before, if $\zeta > 0$ the problem is linearly stable and if $\zeta < 0$ it is linearly unstable. This result is also derived in [**46**]. Our analysis of the full nonlinear problem carries over and one can accordingly state corresponding nonlinear statements.

Binding energy interpretation. Recall the conservation law (1.217). In its geometrically invariant formulation, (1.217) reads

$$\partial_t \int_\Omega u + \partial_t |\Omega^-| = 0$$

This motivates the definition of a potential functional U:

$$U(\bar{u}, \bar{\Gamma}) = U(\bar{R}) := \int_\Omega \bar{u} + l|\Omega^-|.$$

A simple calculation leads to

$$U(\bar{R}) = \frac{\sigma(n-1)\omega_n R_*^n}{\bar{R}} + l\omega_n \bar{R}^n.$$

It is of interest to understand the critical points of U. Note that the equation $\partial_{\bar{R}} U = 0$ is equivalent to the equation

$$-\frac{\sigma(n-1)\omega_n R_*^n}{\bar{R}^2} + nl\omega_n \bar{R}^{n-1} = 0.$$

Hence, the solution obviously satisfies $\zeta(\bar{R}) = 0$. In fact, the stability condition $\zeta > 0$ is equivalent to the assumption $\partial_{\bar{R}} U > 0$ and similarly, the instability condition is characterized by $\partial_{\bar{R}} U < 0$. This is an elementary observation with a rather remarkable interpretation. Namely, we may term the energy U the binding energy, as it is in accordance with the so-called "Poincaré turning point principle", which states that the steady states exchange stability to instability as we cross the critical point of the binding energy. Intuitively speaking, system struggles more and more to keep itself stable as $\bar{R}$ approaches the critical argument value. As soon as it is pass that point, system is unstable and the binding energy decreases as the parts of the system are no longer held together. Surprisingly, a similar situation is encountered in the work on stability of relativistic galaxies, where the "Poincaré turning point" principle has been verified so far only numerically as in the work of ANDRÉASSON & REIN [**3**]. Full stability in that context is a major open problem.

There are two "criticality regimes" that we want to explore a bit further now. One of them arises in the context of the "nucleation radius" i.e., when the size of the whole domain collapses to the size of the inner phase, and another one comes about in the context of "small spheres" when we take the domain to be the whole space, i.e. $\Omega=\mathbb{R}^n$ and impose suitable boundary condition for the temperature at spatial infinity.

Critical radius. To illustrate our first point, let us work under the assumption $c_+ = c_- = c$ and let us further assume $n=3$. To be able to apply the stability Theorem 1.1, we need to ensure that $\zeta = \frac{3}{4\pi c R_*^3} - \frac{2\sigma}{4\pi l \bar{R}^4} > 0$, which is equivalent to $\bar{R}^4 > \frac{2\sigma c}{3l} R_*^3$. On the other hand, the steady state must be contained in the domain $\Omega = B_{R_*}(0)$ i.e. $R_* > \bar{R}$. Combining the two inequalities we conclude that R_* has to satisfy

$$R_* > R_{cr}, \quad R_{cr} := \frac{2\sigma c}{3l}.$$

For any R_* smaller than the critical radius R_{cr}, the stability theorem simply does not apply. It now seems a fundamental question to attach an appropriate physical meaning to this critical size. As a first step we compute the size of R_{cr}. From the equation (3.1) in [**53**], we see that for water at room pressure at about 0°C, the surface coefficient $\sigma \sim 1.2\times 10^{-5}\mathrm{cm\,K}$ and the latent heat $l \sim 80\,\mathrm{cal\,cm^{-3}}$. The heat capacity c is basically the definition of the energy unit cal and hence $c \sim 1\frac{\mathrm{cal}}{\mathrm{K cm^3}}$. Plugging these numbers into the definition of R_{cr}, we obtain

$$R_{cr} \sim 10^{-7}\mathrm{cm}.$$

This is indeed the critical nucleation scale as observed in various physics literature (cf. [**53**], [**12**]).

Small Spheres. The second important situation that we want to mention is the case $\Omega = \mathbb{R}^n$. In this setting we replace the Neumann boundary condition (0.572) by the following condition: for a given constant v_∞ we demand

$$v(x,t) \to v_\infty \quad \text{as} \quad |x| \to \infty. \tag{0.574}$$

Observe that the steady states form an n-parameter family of solutions. The n degrees of freedom are contained in the choice of the center $\mathbf{x}_0$ of the steady sphere, whereas due to the boundary condition (0.574), the temperature $\bar{v} = v_\infty$ is fixed, and thus the radius $\bar{R}=(n-1)/v_\infty$ is determined. Hence, all steady states have the form $(\bar{v},\bar{\Gamma})\equiv(v_\infty, S_{1/v_\infty}(\mathbf{x_0}))$ for some $\mathbf{x}_0\in\mathbb{R}^n$. Note that this problem, interestingly enough, has an n-dimensional null space; this is to be contrasted with $n+1$-dimensionality of the null space when the domain is bounded. The existence of n neutral modes is a reflection of the n momentum conservation laws. Namely, for $i=1,\dots,n$, define $p_i^\infty(x)=\frac{x_i}{|x|^2}$. Then, just like in the proof of (1.219), we deduce

$$\partial_t\int_\Omega u p_i^\infty\, dc+\partial_t\int_{\mathcal{S}} R s_{1,i}\, d\xi=0.$$

Since the "mass" $\int_\Omega v_\infty\, dc+\int_{\mathcal{S}}\frac{R^n}{n}$ is evidently infinite, the "mass conservation" is not a constraint and unlike the bounded domain case, it cannot be responsible for changes in the radius of the steady state. Heuristically, it is clear that any alteration of the radius of the inner domain is negligible with respect to the size of the infinite domain $\mathbb{R}^n$, which is why we are referring to "small spheres".

The problem (0.569) - (0.571), (0.574) and (0.573) is manifestly linearly unstable: since $\int_\Omega dc=\infty$, from the above definition of ζ we conclude $\zeta<0$. Formally, one can see this by linearizing the problem around the steady state $(\bar{v},\bar{\Gamma})\equiv(v_\infty, S_{1/v_\infty}(\mathbf{0}))$ and adapting the variational analysis of Section 4. This linear instability has been observed in a 2-dimensional case in [**14**]. The methods of Section 4 extend in a straightforward way to a full nonlinear instability statement. More importantly, we obtain that the growing mode is spherically symmetric, which indicates that spherical shape remains "stable" despite the dynamical instability that takes place. This observation is also made in [**14**], and the authors contrast this phenomenon with the spherical shape "instability" of radially symmetric solutions with an interface of *large* radius, as studied in [**45**].

APPENDIX A

Basic hypersurface geometry and important formulas

Recall that $\mathcal{S}=\{\xi\in\mathbb{R}^n,\, |\xi|=1\}$ is the unit sphere in $\mathbb{R}^n$. We can 'parametrize' the moving surface Γ by setting

$$\Gamma=\{\mathbf{x}\in\mathbb{R}^n;\, \mathbf{x}=R(\xi,t)\xi,\, \xi\in\mathcal{S}\},$$

where $R:\mathcal{S}\times[0,\infty[\to\mathbb{R}$ is a real valued function. Let $x\in\Gamma$ and prescribe a local system of coordinates $\{t_1,\dots,t_{n-1}\}$ and a Riemmanian metric g on $\mathcal{S}$ given by $g_{\alpha\beta}=\xi_{,\alpha}\cdot\xi_{,\beta}$. Observe that

$$x_{,\alpha}=R_{,\alpha}\xi+R\xi_{,\alpha.}$$

Then the metric $\mathbf{g}$ induced on Γ is given by

$$\mathbf{g}_{\alpha\beta}=x_{,\alpha}\cdot x_{,\beta}=R_{,\alpha}R_{,\beta}+R^2g_{\alpha\beta}. \tag{0.575}$$

The unit normal on Γ, n is given by

$$n=\frac{R\xi-g^{\alpha\beta}R_{,\alpha}\xi_{,\beta}}{\sqrt{R^2+|\nabla_gR|^2}}. \tag{0.576}$$

Here, $\nabla_g=(\partial_g^1,\dots,\partial_g^n)$ denotes the Riemmanian gradient associated with the metric g on $\mathcal{S}$ and in local coordinates it is given by the expression

$$\partial_g^i=g^{\alpha\beta}\xi^i_{,\beta}\partial_{t^\alpha},\quad i=1,\dots,n\quad\text{and}\quad 1\le\alpha,\beta\le n-1.$$

The mean curvature H of Γ is defined to be the metric contraction of the second fundamental form. The second fundamental form $\mathbf{h}_{\alpha\beta}$ is given by

$$\mathbf{h}_{\alpha\beta}=x_{,\alpha}\cdot n_{,\beta}. \tag{0.577}$$

The mean curvature is then given by $H=\mathbf{g}^{\alpha\beta}\mathbf{h}_{\beta\alpha}$. Using the above formulae, we arrive at the following lemma:

Lemma 0.10. *The mean curvature is given by the formula*

$$H(\xi,t)=\frac{n-1}{\sqrt{R^2+|\partial_g R|^2}}-\frac{1}{R}\nabla_g\cdot\frac{\nabla_g R}{\sqrt{R^2+|\nabla_g R|^2}}. \tag{0.578}$$

Proof. Let us evaluate the second term on the right-hand side of (0.578). Recall that $\nabla_g\cdot X=\frac{1}{|g|}\partial_\alpha(|g|X^\alpha)$. Hence, after using the product rule we obtain

$$\begin{aligned}\nabla_g\cdot\frac{\nabla_g R}{\sqrt{R^2+|\nabla_g R|^2}} &= \frac{1}{|g|}\partial_\alpha\Big(\frac{|g|g^{\alpha\beta}R_{,\beta}}{\sqrt{R^2+|\nabla_g R|^2}}\Big)\\ &= \frac{\partial_\alpha|g|}{|g|}\frac{g^{\alpha\beta}R_{,\beta}}{\sqrt{R^2+|\nabla_g R|^2}}+\frac{g^{\alpha\beta}_{,\beta}R_{,\beta}}{\sqrt{R^2+|\nabla_g R|^2}}+\frac{g^{\alpha\beta}_{,\alpha,\beta}}{\sqrt{R^2+|\nabla_g R|^2}}\\ &\quad -\frac{Rg^{\alpha\beta}R_{,\beta}R_{,\alpha}+g^{\alpha\beta}R_{,\beta}g^{\gamma\delta}_{,\alpha}R_{,\gamma}R_{,\delta}+g^{\gamma\delta}(R_{,\gamma}R_{,\delta})_{,\alpha}g^{\alpha\beta}R_\beta}{(\sqrt{R^2+|\nabla_g R|^2})^3}\\ &=: (*),\end{aligned}$$

We fix a point $\xi\in\mathcal{S}$ and choose to work with normal coordinates at that point. This is not a restriction, because the result will be independent of the coordinate choice. Normal coordinates have the property that $g_{\alpha\beta}(\xi)=g^{\alpha\beta}(\xi)=\delta_{\alpha\beta}$. In addition to that we know that $(g_{\alpha\beta})_{,\gamma}(\xi)=g^{\alpha\beta}_{,\gamma}(\xi)=0$ for any $1\le\alpha,\beta,\gamma\le n-1$. Note that $\partial_\alpha|g|=\frac{1}{2}|g|g^{\beta\gamma}(g_{\beta\gamma})_{,\alpha}$ and hence it vanishes at ξ by our choice of coordinates. The expression (*) further simplifies to

$$\nabla_g\cdot\frac{\nabla_g R}{\sqrt{R^2+|\nabla_g R|^2}}=\frac{\Delta R(R^2+|\nabla R|^2)-R|\nabla R|^2-\sum_{\alpha,\beta}R_{,\alpha}R_{,\beta}R_{,\alpha,\beta}}{(\sqrt{R^2+|\nabla_g R|^2})^3} \tag{0.579}$$

On the other hand, since $x_{,\alpha}\cdot\bar n=0$ we have from (0.577):

$$\begin{aligned}\mathbf{h}_{\alpha\beta}&=x_{,\alpha}\cdot\bar n_{,\beta}=x_{,\alpha}\cdot\frac{\big(R\xi-g^{\gamma\epsilon}R_{,\gamma}\xi_{,\epsilon}\big)_{,\beta}}{\sqrt{R^2+|\nabla_g R|^2}}=x_{,\alpha}\cdot\frac{x_{,\beta}-g^{\gamma\epsilon}_{,\beta}R_{,\gamma}\xi_{,\epsilon}-g^{\gamma\epsilon}R_{,\gamma,\beta}\xi_{,\epsilon}-g^{\gamma\epsilon}R_{,\gamma}\xi_{,\epsilon,\beta}}{\sqrt{R^2+|\nabla_g R|^2}}\\ &=\frac{1}{\sqrt{R^2+|\nabla_g R|^2}}\Big(\mathbf{g}_{\alpha\beta}-g^{\gamma\epsilon}_{,\beta}g_{\epsilon\alpha}RR_{,\gamma}-g^{\gamma\epsilon}R_{,\gamma,\beta}Rg_{\epsilon\alpha}-g^{\gamma\epsilon}R_{,\gamma}\xi_{,\epsilon,\beta}\cdot x_{,\alpha}\Big)=:(**),\end{aligned}$$

where we used $x_{,\alpha}=R\xi_{,\alpha}+R_{,\alpha}\xi$. Note that

$$\xi_{,\epsilon,\beta}\cdot x_{,\alpha}=\xi_{,\epsilon,\beta}\cdot(R\xi_{,\alpha}+R_{,\alpha}\xi)=R_{,\alpha}\xi_{,\epsilon,\beta}\cdot\xi+R\xi_{,\epsilon,\beta}\xi_{,\alpha}.$$

However, since $\partial_\beta(\xi_{,\epsilon}\cdot\xi)=0$, it follows that $\xi_{,\epsilon,\beta}\cdot\xi=-\xi_{,\epsilon}\cdot\xi_{,\beta}=g_{\epsilon\beta}$. Hence, from (**) we get

$$\mathbf{h}_{\alpha\beta}=\frac{\mathbf{g}_{\alpha\beta}-g^{\gamma\epsilon}_{,\beta}g_{\epsilon\alpha}RR_{,\gamma}-g^{\gamma\epsilon}R_{,\gamma,\beta}Rg_{\epsilon\alpha}+g^{\gamma\epsilon}g_{\epsilon\beta}R_{,\gamma}R_{,\alpha}-g^{\gamma\epsilon}R_{,\gamma}R\xi_{,\epsilon,\beta}\xi_{,\alpha}}{\sqrt{R^2+|\nabla_g R|^2}} \tag{0.580}$$

Before we compute the curvature, observe that

$$\mathbf{g}^{\alpha\beta}g^{\gamma\epsilon}R_{,\gamma}R\xi_{,\epsilon,\beta}\xi_{,\alpha}=RR_{,\epsilon}\mathbf{g}^{\alpha\beta}\xi_{,\epsilon,\beta}\xi_{,\alpha}=\frac{1}{2}RR_{,\epsilon}\mathbf{g}^{\alpha\beta}(\xi_{,\alpha}\cdot\xi_{,\beta})_{,\epsilon}=\frac{1}{2}RR_{,\epsilon}\mathbf{g}^{\alpha\beta}(g_{\alpha\beta})_{,\epsilon}=0, \tag{0.581}$$

where the very last equality holds only at ξ, by our choice of coordinates. Note that, at the point $R(\xi)\xi$

$$\mathbf{g}^{\alpha\beta}=\frac{1}{R^2(R^2+|\nabla R|^2)}\big((R^2+|\nabla R|^2)\delta^{\alpha\beta}-R_{,\alpha}R_{,\beta}\big) \tag{0.582}$$

This is easy to check by keeping in mind that at $R(\xi)\xi$, $\mathbf{g}_{\alpha\beta}=R^2\delta_{\alpha\beta}+R_{,\alpha}R_{,\beta}$. Using the normal coordinates, (0.580), (0.581) and (0.582), we obtain

$$\begin{aligned}
&\sqrt{R^2+|\nabla_g R|^2}\,\mathbf{g}^{\alpha\beta}\mathbf{h}_{\beta\alpha}=\mathbf{g}^{\alpha\beta}\mathbf{g}_{\alpha\beta}-\mathbf{g}^{\alpha\beta}(RR_{,\alpha,\beta}-R_{,\alpha}R_{,\beta})=n-1-\mathbf{g}^{\alpha\beta}\big(RR_{,\alpha,\beta}-R_{,\alpha}R_{,\beta}\big)\\
&=n-1-\frac{\big((R^2+|\nabla R|^2)\delta^{\alpha\beta}-R_{,\alpha}R_{,\beta}\big)(RR_{,\alpha,\beta}-R_{,\alpha}R_{,\beta})}{R^2(R^2+|\nabla R|^2)}\\
&=n-1-\frac{R(R^2+|\nabla R|^2)\Delta R-(R^2+|\nabla R|^2)|\nabla R|^2-\sum RR_{,\alpha}R_{,\beta}R_{,\alpha,\beta}+|\nabla R|^4}{R^2(R^2+|\nabla R|^2)},
\end{aligned}$$

where we used the fact that $\delta^{\alpha\beta}R_{,\alpha,\beta}=\Delta R$, $\delta^{\alpha\beta}R_{,\alpha}R_{,\beta}=|\nabla R|^2$ and $\sum_{\alpha,\beta}R_{,\alpha}R_{,\beta}R_{,\alpha}R_{,\beta}=|\nabla R|^4$. Hence by this identity,

$$\begin{aligned}
H=\mathbf{g}^{\alpha\beta}\mathbf{h}_{\beta\alpha}&=\frac{n-1}{\sqrt{R^2+|\nabla_g R|^2}}-\frac{(R^2+|\nabla R|^2)\Delta R-R|\nabla R|^2-\sum_{\alpha,\beta}R_{,\alpha}R_{,\beta}R_{,\alpha,\beta}}{R(R^2+|\nabla R|^2)^{3/2}}\\
&=\frac{n-1}{\sqrt{R^2+|\partial_g R|^2}}-\frac{1}{R}\nabla_g\cdot\frac{\nabla_g R}{\sqrt{R^2+|\nabla_g R|^2}},
\end{aligned}$$

where in the last equality we use the identity (0.579). □

Let $x(\xi,t)=R(\xi,t)\xi$, $\xi\in\mathcal{S}$. The normal velocity V is given by $V=x_t\cdot n=R_t\xi\cdot n$. Observing that for any $\alpha\in\{1,\dots,n-1\}$ $\xi_\alpha\cdot\xi=0$ we obtain the expression

$$V(\xi,t)=\frac{RR_t}{\sqrt{R^2+|\nabla_g R|^2}}. \tag{0.583}$$

Furthermore, let $u:\Omega\to\mathbb{R}$ be a given real function. Then for any $x\in\Gamma$ we have

$$\nabla u(x)=\nabla_{\mathbf{g}}u(x)+n\partial_n u(x)=(\partial^1_{\mathbf{g}},\dots,\partial^n_{\mathbf{g}})u(x)+n\partial_n u(x), \tag{0.584}$$

where $\nabla_{\mathbf{g}}=(\partial^1_{\mathbf{g}},\dots,\partial^n_{\mathbf{g}})$ is the Riemmanian gradient of the manifold Γ given in local coordinates by

$$\partial^i_{\mathbf{g}}=\mathbf{g}^{\alpha\beta}x^i_{,\beta}\partial_{t^\alpha},\quad i=1,\dots,n\quad\text{and}\quad 1\le\alpha,\beta\le n-1. \tag{0.585}$$

In fact, formula (0.584) can be used as an alternative definition of the Riemmanian gradient, as it is done in [**28**], Ch. 16.

Remark. To see this, let $M\subset\mathbb{R}^n$ be a smooth hypersurface equipped with a local coordinate system $\eta:U\subset\mathbb{R}^{n-1}\to M$. Let $v:\mathbb{R}^n\to\mathbb{R}$ be a smooth function. The induced metric $\tilde{g}$ is given by $\tilde{g}_{\alpha\beta}=\eta_{,\alpha}\cdot\eta_{,\beta}$. Then, by the chain rule $\partial_\alpha v(\eta(t^1,\dots,t^{n-1}))=\partial_{x^i}v\eta^i_{,\alpha}$. On the other hand

$$\partial_{x^i}v\eta^i_{,\alpha}=\left(\nabla^i_{\tilde{g}}v+\tilde{n}^i\partial_n v\right)\eta^i_{,\alpha}=\partial^i_{\tilde{g}}v\eta^i_{,\alpha}.$$

We also know that $\partial^i_{\tilde{g}}v$ uniquely decomposes into a linear combination of vector fields ∂_α, $1\le\alpha\le n-1$, because $\partial_{\tilde{g}}v$ is a projection of ∇v onto the tangent space of M. Hence, we can write $\partial^i_{\tilde{g}}=\tau^\gamma_i\partial_\gamma$. Note that $\partial_\gamma=\partial^i_{\tilde{g}}\eta^i_{,\gamma}=\tau^\gamma_i\eta^i_{,\gamma}\partial_\gamma$. Hence, we want $\tau^\gamma_i\eta^i_{,\gamma}=\delta^\gamma_\alpha$, which, after inverting, implies $\tau^\gamma_i=\tilde{g}^{\epsilon\gamma}\eta^i_{,\epsilon}$. This proves the formula (0.585).

Note that $\nabla_{\mathbf{g}}f\cdot\nabla_{\mathbf{g}}g=\mathbf{g}^{\alpha\beta}f_{,\alpha}g_{,\beta}$. An important formula relates the volume elements $dvol_{\mathbf{g}}$ and $dvol_g$ of the manifolds Γ and $\mathcal{S}$ respectively. We formulate it in the following lemma:

Lemma 0.11.

$$dvol_{\mathbf{g}}=R^{n-2}\sqrt{R^2+|\nabla_g R|^2}\,dvol_g.$$

Proof. Recall that $d\mathrm{vol}_h=\sqrt{\det h}\,dt^1\wedge\cdots\wedge dt^{n-1}$, where h is an arbitrary Riemannian metric and $(t^1,\dots,t^{n-1})$ a choice of local coordinates. Thus, in order to prove (0.11) we need to prove that

$$\frac{\det\mathbf{g}}{\det g}=R^{2n-2}(R^2+|\nabla_g R|^2). \tag{0.586}$$

In order to do so, we first establish that $\frac{\det \mathbf{g}}{\det g}$ is coordinate independent and then prove (0.586), by a particular choice of cooridnates. Assume that we apply the coordinate change $\xi \to \xi'$. It is well known that in that case

$$\det(g') = \Big[\frac{\partial(t^1,\dots,t^{n-1})}{\partial(t^{1'},\dots,t^{(n-1)'})}\Big]^2 \det(g). \tag{0.587}$$

However, due to (0.575), $\mathbf{g}$ as a function of ξ is also a metric on $\mathcal{S}$ and hence, obeys the same coordinate transformation rule (0.587) as g. Hence, $\frac{\det \mathbf{g}'}{\det(g')} = \frac{\det \mathbf{g}}{\det(g)}$. Now choose the coordinate fields $\partial_{t^1},\dots,\partial_{t^{n-1}}$ such that at a fixed point ξ, $g_{\alpha\beta} = \delta_{\alpha\beta}$. Then orient the fields in such a way that $R_{,\alpha} = 0$ for all $\alpha \in \{2,\dots,n-1\}$. Then, at ξ, $\det(g) = 1$ and by (0.575) $\det(\mathbf{g}) = R^{2n-2}(R^2 + R_{,1}^2) = R^{2n-2}(R^2 + |\nabla_g R|^2)$, as desired. $\square$

Using the previous lemma, we prove a well-known formula for the first variation of a moving hypersurface:

Lemma 0.12. *Let $S(\Gamma) := \int_\Gamma dvol_{\mathbf{g}}$ be the volume of the hypersurface Γ. Then*

$$\frac{d}{dt} S(\Gamma) = \int_\Gamma H(x) V(x)\, dvol_{\mathbf{g}}(x).$$

Proof. Observe that

$$\begin{aligned}
\int_\Gamma H(x)V(x)\, d\mathrm{vol}_{\mathbf{g}}(x) &= \int_{\mathcal{S}} \Big(\frac{n-1}{\sqrt{R^2+|\nabla_g R|^2}} - \frac{1}{R}\nabla_g \cdot \frac{\nabla_g R}{\sqrt{R^2+|\nabla_g R|^2}}\Big) \frac{R R_t}{\sqrt{R^2+|\nabla_g R|^2}} \\
&\qquad \times R^{n-2}\sqrt{R^2+|\nabla_g R|^2}\, dS(\xi) \\
&= \int_{\mathcal{S}} \frac{(n-1)R^{n-1}R_t}{\sqrt{R^2+|\nabla_g R|^2}} - R^{n-2}R_t \nabla_g \cdot \frac{\nabla_g R}{\sqrt{R^2+|\nabla_g R|^2}}\, dS(\xi) \\
&= \int_{\mathcal{S}} \frac{(n-1)R^{n-1}R_t}{\sqrt{R^2+|\nabla_g R|^2}}\, dS(\xi) + \int_{\mathcal{S}} R^{n-2}\frac{\nabla_g R}{\sqrt{R^2+|\nabla_g R|^2}} \cdot \nabla_g R_t \\
&\qquad + \nabla_g(R^{n-2}) \frac{\nabla_g R}{\sqrt{R^2+|\nabla_g R|^2}} R_t\, dS(\xi) \\
&= (n-2)\int_{\mathcal{S}} R^{n-3}R_t \frac{R^2+|\nabla_g R|^2}{\sqrt{R^2+|\nabla_g R|^2}} + R^{n-2}\frac{R R_t + \nabla_g R\cdot \nabla_g R_t}{\sqrt{R^2+|\nabla_g R|^2}}\, dS(\xi) \\
&= \int_{\mathcal{S}} \partial_t(R^{n-2})\sqrt{R^2+|\nabla_g R|^2} + R^{n-2}\partial_t\Big(\sqrt{R^2+|\nabla_g R|^2}\Big)\, dS(\xi) \\
&= \partial_t \int_{\mathcal{S}} R^{n-2}\sqrt{R^2+|\nabla_g R|^2}\, dS(\xi) = \partial_t \int_\Gamma dS(\Gamma) = \frac{d}{dt}S(\Gamma),
\end{aligned}$$

where we used the integration by parts in the fourth identity. $\square$

Remark. Integration by parts. On a given Riemmanian manifold (M,h), with a metric h, the following integration by parts rule holds: given the C_0^2-functions $f,g:M\to\mathbb{R}$, we have

$$\int_M f\Delta_{\mathcal{B}}g\,d\mathrm{vol}_h=\int_M g\Delta_{\mathcal{B}}f\,d\mathrm{vol}_h=\int_M -\nabla_h f\cdot\nabla_h g\,d\mathrm{vol}_h.$$

Here, $\Delta_{\mathcal{B}}$ stands for the Laplace-Beltrami operator on M and it is defined by $\Delta_{\mathcal{B}}f:=\nabla_h\cdot\nabla_h f$. The following important relation holds:

$$\Delta_{\mathcal{B}}u=\sum_{i=1}^{n}\nabla_h^i\nabla_h^i u \tag{0.588}$$

More generally, by [**51**](Ch. 2, Prop. 2.2), for a given vector field X on M, we have

$$\int_M f\nabla_h\cdot X\,d\mathrm{vol}_h=-\int_M \nabla_h f\cdot X\,d\mathrm{vol}_h.$$

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