Stratified and steady periodic water waves

by

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Dedicated to G. B. Walsh

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Abstract of "Stratified and steady periodic water waves"

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This thesis considers two-dimensional stratified water waves propagating under the force of gravity over an impermeable flat bed and with a free surface. In the absence of surface tension, it is proved that there exists of a global continuum of classical solutions that are periodic and traveling. These waves, moreover, can exhibit large density variation, speed, and amplitude. When the motion is assumed to be driven by capillarity on the surface and a gravitational force acting on the body of the fluid, it is shown that there exists global continua of such solutions. In both regimes, this is accomplished by first constructing a 1-parameter family of laminar flow solutions, then applying bifurcation theory methods to obtain local curves of small amplitude solutions branching from the laminar curve at an eigenvalue of the linearized problem. Each solution curve is then continued globally by means of a degree theoretic argument in the spirit of Rabinowitz. We also provide an alternate global bifurcation theorem via the analytic continuation method of Dancer.

Finally, we consider the question of symmetry for two-dimensional stably stratified steady periodic gravity water waves with surface profiles monotonic between crests and troughs. We provide sufficient conditions under which such waves are necessarily symmetric. We do this by first exploiting some elliptic structure in the governing equations to show that, in certain size regimes, a maximum principle holds. This then forms the basis for a method of moving planes argument.

CHAPTER 1

Background

1.1. Equations of motion

In this section we introduce the fundamental equations of motion governing stratified fluids. The main purpose here is twofold. First, we wish to develop a suitable physical intuition to motivate the mathematics done in the later chapters. Second, we hope to convince the reader of the physical validity of our analysis, and inform him of its limitations. The latter point is not to be overlooked; all of the mathematical results we shall derive are contingent upon some basic physical assumptions on the underlying system.

That said, this is a mathematical text and so we do wish to not overemphasize non-mathematical topics. The derivations of the governing equations are therefore left quite brief, and some topics are, in the interests of maintaing a clean exposition, excised entirely. In this section we shall also, as a matter of necessity, be forced to temporarily put aside some mathematical concerns, such as regularity of the quantities under consideration, and the well-posedness of the derived models. These issues shall be treated with care in the remainder of the thesis.

1.1.1. The Cauchy Equation. Consider a collection of a single species of particles inhabiting a smooth region $\Omega = \Omega(t) \subset \mathbb{R}^n$ (for the purposes of this section, we shall always take $n = 2$ or 3). Then, if $\rho = \rho(t, \mathbf{x}) > 0$ denotes the density of the particles at each point $\mathbf{x} \in \Omega(t)$, the total mass $m = m(t)$ is given by

$$m(t) := \int_{\Omega(t)} \rho(t, \mathbf{x}) \, d\mathbf{x}.$$

Conservation of mass can then be formulated as follows:

$$\begin{aligned} 0 = m'(t) &= \int_{\Omega(t)} \partial_t \rho \, d\mathbf{x} + \int_{\partial\Omega(t)} \rho \mathcal{V} \, dS \\ &= \int_{\Omega(t)} \partial_t \rho \, d\mathbf{x} + \int_{\partial\Omega(t)} \rho \mathbf{u} \cdot \mathbf{n} \, ds. \end{aligned}$$

Here $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ denotes the velocity of the particle at $\mathbf{x}$, while $\mathcal{V}$ is the normal velocity, $\mathbf{n}$ the outward unit normal, and dS the surface measure on $\partial\Omega$. The equality of the last two lines follows from the observation that, for the particles to remain within the domain Ω — which they must by definition — the outward normal velocity $\mathbf{u} \cdot \mathbf{n}$ needs to coincide with the normal velocity of the boundary.

Assuming suitable regularity, the divergence theorem can be applied to the boundary integral in the last line above, which yields

$$\int_{\Omega} \left[\partial_t \rho + \nabla \cdot (\rho \mathbf{u})\right] \, d\mathbf{x} = 0. \tag{1.1.1}$$

Observe that the choice of Ω was arbitrary, and thus (1.1.1) must hold for any collection of particles. We are therefore justified in concluding that the integrand above must vanish point-wise, that is,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \text{in } \Omega. \tag{1.1.2}$$

Equation (1.1.2) is referred to as the *continuity equation* or the *mass equation*, and is, in some sense, the most fundamental conservation law governing the fluids we study: mass conservation shall be assumed, in precisely the form above, in all the physical regimes we consider. This is in marked contrast to the equations governing the conservation of fluid particle momentum, which we discuss next.

Again, let $\Omega = \Omega(t) \subset \mathbb{R}^n$ be a smooth region consisting of a collection of particles. For the purposes of exposition, we make the assumption that the motion of the fluid is well behaved on some time interval $[0, T)$. Denote

$$\Omega_T := \bigcup_{t \in [0,T)} \{t\} \times \Omega(t), \qquad \partial\Omega_T := \bigcup_{t \in [0,T)} \{t\} \times \partial\Omega(t).$$

We may then define the Lagrangian flow map, or particle trajectory, $\mathbf{X} = \mathbf{X}(t, \mathbf{x}) : \Omega_T \to \mathbb{R}^n$, to be the solution of the following differential equation:

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{u}(t, \mathbf{X}), & \text{in } \Omega_T, \\ \mathcal{V}(\partial\Omega) = \mathbf{u} \cdot \mathbf{n}, & \text{on } \partial\Omega_T, \\ \mathbf{X}|_{t=0} = \boldsymbol{\iota}_{\Omega_0}, & \Omega|_{t=0} = \Omega_0, \end{cases} \tag{1.1.3}$$

where $\boldsymbol{\iota}_{\Omega_0}$ is identity function on the initial domain Ω_0 (which we take to be given.) Intuitively, $\mathbf{X}(t, \mathbf{x})$ gives the location of the fluid particle that began at $\mathbf{x}$ at time $t = 0$. It follows that the change in momentum for that particle is:

$$\begin{aligned}\partial_t[\rho(t,\mathbf{X})\mathbf{u}(t,\mathbf{X})] &= [(\partial_t\rho)(t,\mathbf{X}) + \dot{\mathbf{X}}\cdot(\nabla\rho)(t,\mathbf{X})]\mathbf{u}(t,\mathbf{X}) \\ &\quad + \rho(t,\mathbf{X})[(\partial_t\mathbf{u})(t,\mathbf{X}) + \dot{\mathbf{X}}\cdot(\nabla\mathbf{u})(t,\mathbf{X})] \\ &= [(\partial_t + \mathbf{u}\cdot\nabla)\rho\mathbf{u}](t,\mathbf{X}),\end{aligned} \tag{1.1.4}$$

where $\mathbf{u}\cdot\nabla := \sum_i u_i\partial_{x_i}$. The operator appearing in the parenthesis on the right-hand side of (1.1.4) is referred to as the *material* or *advective derivative* associated with the flow. From the derivation above, it is apparent that physically it indicates the change of a quantity associated with a particle, as that particle is evolved by the flow. Typically the material derivative is written D, or D/Dt, but, in order to avoid confusion with the Jacobian, we instead use the convention

$$\mathcal{D} := \partial_t + \mathbf{u}\cdot\nabla.$$

Newtonian mechanics tells us that the material derivative of the momentum $\rho\mathbf{u}$ should balance the total force acting on the particle. These forces can be divided into two categories: body forces, which act on the particle's center of mass, and stress forces which act on the surface of the particle. In all of the scenarios that we shall consider in this thesis, the only body force will be gravity. We therefore introduce the vector $\mathbf{g} := g\mathbf{e}_n$, where g is a given, strictly positive quantity (the gravitational constant), and $\mathbf{e}_n$ is the vector $(0,0,1)$ for $n=3$, and $(0,1)$ for $n=2$. The body force shall be taken to be $-\rho\mathbf{g}$ (the sign reflecting the choice of $\mathbf{e}_n$ as pointing 'upwards').

Understanding the stress, $\mathbf{t}$, shall require some additional concepts from continuum mechanics. As stress is a purely surface phenomenon, we infer that $\mathbf{t}$ should depend on the outward unit normal vector, denote it $\mathbf{n}$. Indeed, since the particular form of the dependence should be independent of the geometry of the surface (for the infinitesimal reasoning to have any hope of proving correct), we can discern that the dependence is linearly through a symmetric second-order tensor T:

$$\mathbf{t} = \mathsf{T}\cdot\mathbf{n}.$$

This fact is derived in introductory texts of continuum mechanics (cf., e.g., [**52**]). The stress acts on the surface, whereas the material derivative of the momentum, and the body force, are all defined in the interior of the fluid domain. Assuming some minimal regularity, we therefore use the divergence theorem to conclude that the point-wise stress force contribution is given by $\nabla \cdot \mathbf{T}$.

Collecting these observations, we arrive finally at the Cauchy equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\partial_t + \mathbf{u} \cdot \nabla)(\rho \mathbf{u}) = \nabla \cdot \mathbf{T} - \rho \mathbf{g}, \end{cases} \quad \text{in } \Omega_T. \tag{1.1.5}$$

This is not yet a complete account of the motion, since boundary conditions are not specified. We forego this discussion until section 1.1.4.

Nothing we have done up to this point is specific to fluids, and consequently (1.1.5) is applicable to a wide range of continuous media. Unsurprisingly, therefore, there is significant advantage to specializing further. This process shall lead us to the Navier–Stokes, and finally the incompressible Euler equations, which shall monopolize our attention in the bulk of this work.

1.1.2. The incompressibility condition. One of the first simplifying assumptions we shall make is that of incompressibility, which is an almost universal practice in the literature of water waves. However, an almost equally prevalent assumption is that of irrotationality, which we are quite pointedly avoiding (see section 1.2.2). With that in mind, some discussion of the appropriateness of the incompressible regime is merited.

From a physical standpoint, incompressibility is the requirement that the total mass of the fluid is conserved (as opposed to the mass of each individual fluid particle.) Informally, this behavior is anticipated when working with liquids, but not, e.g., for gasses, in which the separation of the particles allows for contraction or expansion. A more carefully dimensional analysis reveals that the relevant physical parameter to consider is the Mach number, which is defined as the ratio

$$\mathrm{Ma} := \frac{U_{\mathrm{char}}}{U_{\mathrm{sound}}},$$

where U_{char} is the characteristic velocity, and U_{sound} is the speed of sound in the medium. Supposing that the fluid consists of particles of a single species (we remark on the general case at the end of section 1.1.3),then the density should depend functionally on the pressure and temperature. That is, the thermodynamic laws of state suggest

$$\rho = \varrho(P, T), \tag{1.1.6}$$

for some ϱ, where P is the pressure and T is the temperature. To simplify our discussion, let us treat the simpler case where $\varrho = \varrho(P)$. Taking the material derivative of the state equation (1.1.6), we obtain

$$\mathcal{D}\rho = \frac{d\varrho}{dP}\mathcal{D}\varrho. \tag{1.1.7}$$

If we also require the flow to be isentropic, then the quantity $d\varrho/dP$ is on the order of the speed of sound in the medium. A straightforward non-dimensionalization of (1.1.7) gives the following:

$$\hat{\mathcal{D}}\hat{\rho} = \mathrm{Ma}^2 \widehat{\frac{d\varrho}{dP}} \hat{\mathcal{D}}\hat{P},$$

where the $\hat{\cdot}$ denotes a non-dimensional quantity (cf. [**56**]). Thus, when $\mathrm{Ma} \ll 1$, we may assume that

$$\mathcal{D}\rho = \mathcal{D}\varrho = 0. \tag{1.1.8}$$

The above reasoning is not dependent on the flow being isothermal; one need only complete a similar, though slightly more elaborate, non-dimensionalization to deduce (1.1.9) in the more general case. We also remark that the formal correctness of (1.1.9) is dependent only on the order of Ma. In principle, then, one can encounter compressible flows in water and incompressible flows in air, so long as U_{char} is of the appropriate magnitude. In other words, incompressibility is *not* a material property and a careful distinction must be made between incompressible flows and incompressible fluids.

Let us now parse out the mathematical consequences of the incompressibility condition. First, observe that (1.1.8) in conjunction with the continuity equation (1.1.2) implies immediately that $\nabla \cdot \mathbf{u} = 0$. This is the definition of incompressibility one often encounters in the mathematical literature, and it is the one we shall adopt:

Definition 1.1.1. A flow $\mathbf{u}$ is said to be incompressible provided that

$$\nabla \cdot \mathbf{u} \equiv 0, \qquad \text{in } \Omega_T. \tag{1.1.9}$$

Equivalently, if $\mathbf{X}$ is the Lagrangian flow map associated to $\mathbf{u}$, incompressibility is said to hold when exponentiation along $\mathbf{X}$ is volume preserving, i.e.

$$\det D\mathbf{X}(t, \cdot) \equiv 1, \qquad \text{in } \Omega_T. \tag{1.1.10}$$

The equivalence of the two statements is a familiar fact from differential geometry, which we now discuss. In the interests of brevity, we take $n = 2$. These arguments can be easily augmented to work in arbitrary dimensions, modulo some additional geometric machinery (cf., e.g., [**24**]). Let $\Omega(t) \subset \mathbb{R}^2$ be the domain occupied by the fluid at time $t \geq 0$. For the flow to preserve total volume, we clearly need

$$|\Omega(t)| := \int_{\Omega(t)} 1\, d\mathbf{x} = \int_{\Omega(0)} 1\, d\mathbf{x} = |\Omega(0)|, \qquad \text{for all } t \geq 0.$$

To understand the above statement more concretely, it is useful to temporarily adopt a Lagrangian perspective. Let $\mathbf{u} = (u, v) \in \mathbb{R}^2$ denote the velocity field. We define the Lagrangian flow map $\mathbf{X}$ as in (1.1.3). To ascertain the total volume of the fluid, we need only investigate $\mathbf{X}$, since $\Omega(t) = \mathbf{X}(t, \Omega(0))$. In other words, if we informally assume that $\mathbf{X}(t, \cdot)$ is a diffeomorphism (which is valid for t sufficiently small) then

$$|\Omega(t)| = \int_{\Omega(0)} \det D\mathbf{X}(t, \mathbf{x})\, d\mathbf{x},$$

where D denotes the Jacobian in the spatial variables. The flow is therefore incompressible (in the sense that Lagrangian flow map is volume preserving) whenever

$$\det D\mathbf{X}(t, \cdot) \equiv 1, \qquad \text{in } \Omega(0), \text{ for all } t \geq 0,$$

Clearly this holds for $t = 0$ (since $\mathbf{X}|_{t=0}$ is the identity map on $\Omega(0)$). Moreover, we compute

$$\begin{aligned}
\partial_t \det D\mathbf{X} &= \partial_t \left((\partial_x \mathbf{X}_1)(\partial_y \mathbf{X}_2) - (\partial_x \mathbf{X}_2)(\partial_y \mathbf{X}_1)\right) \\
&= (\mathbf{u}_x(\mathbf{X}) \cdot \nabla \mathbf{X}_1)\partial_y \mathbf{X}_2 + (\mathbf{u}_y(\mathbf{X}) \cdot \nabla \mathbf{X}_2)\partial_x \mathbf{X}_1 \\
&\qquad - (\mathbf{u}_x(\mathbf{X}) \cdot \nabla \mathbf{X}_2)\partial_y \mathbf{X}_1 - (\mathbf{u}_y(\mathbf{X}) \cdot \nabla \mathbf{X}_1)\partial_x \mathbf{X}_2 \\
&= (u_x(\mathbf{X}) + v_y(\mathbf{X}))((\partial_x \mathbf{X}_1)(\partial_y \mathbf{X}_2) - (\partial_x \mathbf{X}_2)(\partial_y \mathbf{X}_1)) \\
&= (\nabla \cdot \mathbf{u})(\mathbf{X})(\det D\mathbf{X}).
\end{aligned}$$

Hence, volume is preserved when $\nabla \cdot \mathbf{u} = 0$.

1.1.3. The Navier–Stokes and Euler equations. Viewed in comparison with (1.1.6), equation (1.1.9) elicits an alternate interpretation of incompressibility, namely that it is precisely the physical regime in which the density is decoupled from the other thermodynamic quantities. That is, the thermodynamic equation of state (1.1.6) tying the density to the pressure and temperature is replaced by (1.1.8), which is a condition relating the density only to the velocity. From a mathematical viewpoint, this is particularly attractive because it enables us to describe the evolution of the fluid using only the mass and momentum equations.

Supposing the flow is incompressible, we now hope to take the general equations of motion (1.1.5) and produce a more convenient theory that is specific to fluids. This entails introducing *constitutive laws*, which are assumptions on the structure of T that are, essentially, phenomenologically tailored to the desired medium. With that in mind, we make the following definition: a fluid is said to be *Newtonian* provided that its stress tensor T is isotropic and depends linearly on the strain. For Newtownian fluids that are also incompressible, the stress can be written in component form as follows

$$\mathsf{T}_{ij} = -P\delta_{ij} + \mu\partial_{x_i}u_j, \qquad \text{for } i,\, j = 1, \ldots, n,$$

where P is the pressure (a scalar quantity), δ_{ij} is the Kronecker delta, and μ is a material constant known as the *viscosity* (cf., e.g., [**56**]). The Cauchy equation (1.1.5) and the incompressibility condition thus become

$$\begin{cases} \rho(\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P + \mu\Delta\mathbf{u} - \rho\mathbf{g}, \\ (\partial_t + \mathbf{u}\cdot\nabla)\rho = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \qquad \text{in } \Omega_T. \tag{1.1.11}$$

Equation (1.1.11) is referred to as the Navier–Stokes equation and is the basic model of incompressible fluids. Due to its importance, the Navier–Stokes equation has attracted a phenomenal body of literature, though some important open questions persist (most famously, global well-posedness.) In this thesis, we are primarily interested in understanding the behavior of waves in water, where the viscosity μ is extremely small. We will therefore work from a different governing system, the (incompressible) Euler equations:

$$\begin{cases} \rho(\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P - \mathbf{g}\rho, \\ (\partial_t + \mathbf{u}\cdot\nabla)\rho = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \qquad \text{in } \Omega_T. \tag{1.1.12}$$

Clearly, one obtains (1.1.12) from (1.1.11) by formally taking $\mu = 0$. To be more precise, for a flow with characteristic velocity U_{char}, density ρ_{char}, and length L_{char}, define the *Reynolds* *number* Re by

$$\mathrm{Re} := \frac{L_{\mathrm{char}}\, \rho_{\mathrm{char}}\, U_{\mathrm{char}}}{\mu}.$$

Then a non-dimensionalization of the momentum equation in (1.1.11) reveals that

$$\hat{\rho}(\partial_t + \hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} = -\nabla \hat{P} + \frac{1}{\mathrm{Re}} \Delta \hat{\mathbf{u}} - \hat{\rho}\mathbf{g}.$$

Here $\hat{\cdot}$ denotes a non-dimensional quantity. Thus the Euler equations are the formal limit of the Navier-Stokes equations as the Reynolds number approaches infinity. Since $\mu \ll 1$ in water, the Euler equation is a suitable model for flows where the L_{char}, ρ_{char}, and U_{char} are not extremely large. There is one substantial caveat to this, which involves the behavior near solid boundaries; see the discussion in section 1.1.4. In the mathematical literature, confirming that solutions to Navier–Stokes converge (in some sense) to solutions of Euler as $\mathrm{Re} \to \infty$ is called the *vanishing viscosity limit* problem, and has been studied by a number of authors (cf., e.g., [**35, 36**]).

Remark. The above derivations are valid for fluids where the particles are of a single species, but some authors suggest that a different argument is required for fluids where the particles may be of varying species. The distinction between the multi-species and single species cases does not particularly concern us in this work, however, since they both result in the same governing equation (1.1.12). The only difference between the two is that, for heterogeneous fluids, the validity of the incomperssible and inviscid regime is justified not by the magnitudes of Re and Ma, but instead on an assumption of *non-diffusivity.* Physically, this is the requirement that the diffusivity coefficient of the particles must be vanishingly small, i.e. the time scale for diffusion is substantially longer than that of the flow (cf., e.g., [**4**]).

1.1.4. Conditions on the boundary. Up to this point we have formulated the governing equations for the particles in the interior of a fluid domain $\Omega = \Omega(t)$; in this section we consider the complement $[\Omega(t)]^c$ and the boundary $\partial\Omega(t)$.

First let us consider the simplest case where a solid barrier confines the fluid to a fixed domain. Suppose that there is a portion of the boundary $\Gamma^{(1)} \subset \partial\Omega$ that is immovable and impermeable. Since the fluid particles cannot penetrate the boundary, we require

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad \text{on } \Gamma^{(1)} \text{ for all } t \geq 0, \tag{1.1.13}$$

where $\mathbf{n} : \Gamma^{(1)} \to \mathbb{R}^n$ is the outward unit normal. For flows governed by Navier–Stokes, we need an additional requirement because of the second-order operator in the momentum equation. One candidate is the so-called *no-slip* condition: for every tangent vector $\mathbf{s}$,

$$\mathbf{u} \cdot \mathbf{s} = 0, \qquad \text{on } \Gamma^{(1)} \text{ for all } t \geq 0. \tag{1.1.14}$$

For the Euler equation, which are only first-order, we cannot impose (1.1.14) without overdetermining the system. We say, therefore, that slip is allowed on the boundary for inviscid flows. This constitutes a significant inconsistency between the two regimes. For example, suppose that Ω is the half-space $\mathbb{R}^n_+$, and that the boundary $\Gamma^{(1)}$ is a plate that is being pulled tangentially to the flow. An inviscid fluid will experience no shear force from the boundary, and is thus completely oblivious to its motion. In contrast, (1.1.14) will produce vorticity near the boundary that will be diffused by the viscosity throughout the domain. A less contrived scenario that is of particular interest is that where a wind blowing over a quiescent body of water. The Navier–Stokes equation predicts that the wind will produce a shear force on the surface generating wave motion, while the Euler equation is incapable of modeling this phenomenon.

Next consider the situation where the exterior of the fluid is occupied not by a solid boundary, but by a second, distinct fluid. The boundary portion $\Gamma^{(2)}(t) \subset \partial\Omega(t) \setminus \Gamma^{(1)}$ then represents a *material interface* delineating the regions in space inhabited by each species of fluid. In order for the interface to be maintained (i.e., for the fluids to immiscible), it is necessary for

$$\mathcal{V} = \mathbf{u} \cdot \mathbf{n}, \qquad \text{on } \Gamma^{(2)} \text{ for all } t \geq 0, \tag{1.1.15}$$

where $\mathcal{V} : \Gamma^{(2)}_T \to \mathbb{R}^n$ denotes the normal velocity of the boundary, and $\mathbf{n} : \Gamma^{(2)}_T \to \mathbb{R}^n$ is the outward unit normal. Equation (1.1.15) is known as the *kinematic condition.* Observe that the kinematic condition actually encompasses the impermeability condition if we mandate *a priori* that $\mathcal{V}(\Gamma^{(1)}) = 0$.

When $\partial\Omega(t)$ is locally a graph in a neighborhood of each boundary point, condition (1.1.15) can be stated more explicitly. For simplicity let $n = 2$ and suppose that for some smooth $\eta = \eta(t, x)$,

$$\Gamma^{(2)}(t) = \{(x, y) \in \mathbb{R}^2 : y = \eta(t, x),\ x \in \mathbb{R}\}. \tag{1.1.16}$$

Then, on $\Gamma^{(2)}(t)$,

$$\mathbf{n} = \frac{(-\eta_x,\, 1)}{\sqrt{1+\eta_x^2}}, \qquad \mathbf{u}\cdot\mathbf{n} = \frac{-u\eta_x + v}{\sqrt{1+\eta_x^2}}. \tag{1.1.17}$$

To compute $\mathcal{V}$, note that the speed of the particle at $(x, \eta(t,x))$ is just $(0, \eta_t(t,x))$. Hence the normal velocity is

$$\mathcal{V} = (0, \eta_t)\cdot\mathbf{n} = \frac{\eta_t}{\sqrt{1+\eta_x^2}}.$$

Combining the above expression with (1.1.17), (1.1.15) becomes

$$\eta_t + u\eta_x = v, \qquad \text{on } \Gamma^{(2)} \text{ for all } t \geq 0. \tag{1.1.18}$$

It is easy to see that in three-dimensions the analogous condition is

$$\eta_t + u\eta_x + v\eta_y = w, \qquad \text{on } \Gamma^{(2)} \text{ for all } t \geq 0, \tag{1.1.19}$$

where here $\eta = \eta(t,x,y)$. As a consequence of (1.1.13) and (1.1.15), we note that particles on the boundary must remain on the boundary as the flow evolves. In other words, recalling the definition of the Lagrangian map $\mathbf{X}$ in (1.1.3), we have

$$\partial\Omega(t) = \mathbf{X}(t)(\partial\Omega(0)).$$

Note that in (1.1.16) we have made an implicit assumption that $\Gamma^{(2)}$ is unbounded in x, and admits a global parameterization $x \mapsto (x,\, \eta(t,x))$. Clearly neither of these restrictions are necessary: (1.1.18)–(1.1.19) will hold for each local parameterization.

In addition to (1.1.15), there must be some balancing of the pressure forces on the material interface to prevent the fluids from mixing. The simplest scenario will be that the pressures are exactly equal, which results in the *dynamic boundary condition*:

$$P = P_{\text{ext}}, \qquad \text{on } \Gamma^{(2)} \text{ for all } t \geq 0, \tag{1.1.20}$$

where P_{ext} is the pressure of the fluid in $[\Omega(t)]^c$, which we shall take to be a given quantity. The dynamic condition proves to be a good physical model when the characteristic length scale of the boundary is of moderate size. However, at very small length scales, surface tension will begin to assert itself, and we find instead that

$$P = P_{\text{ext}} - \sigma\kappa, \qquad \text{on } \Gamma^{(2)} \text{ for all } t \geq 0, \tag{1.1.21}$$

where $\sigma > 0$ is a material constant (the coefficient of surface tension), and κ is the mean curvature of $\Gamma^{(2)}$.

1.1.5. Steady flows. Steady flows are an important special class of solutions to the water wave problem. A wave is called *steady* or *traveling* provided that, when it is regarded in a coordinate frame moving with a judiciously chosen fixed velocity, it appears independent of time. In general, this is thought to be possible due to a dispersive character in the physics. Steady waves are extremely common in nature, occurring in all manner or media, but there is an especially strong historical connection to water waves. Indeed, the entire field of study was initiated by Russell [**61**] who by chance observed a wave of permanent configuration traveling along the Glasgow–Edinburgh canal in 1844. The beginnings of a mathematical theory soon followed (with Cauchy and Laplace lending early and important contributions), but a complete understanding of the phenomenon proved remarkably resistant to these and later efforts. It is only quite recently that many of the fundamental questions have been given satisfactory answers, and a great many remain open.

Mathematically, to say a wave is traveling means that the vector field has the ansatz $\mathbf{u}(t,\mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x} - t\mathbf{c})$, for some profile $\hat{\mathbf{u}}$, wave speed $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^n$, and likewise for the pressure, and density. It follows that

$$\partial_t \mathbf{u} = (-\mathbf{c}\cdot\nabla)\hat{\mathbf{u}}, \qquad \partial_t \rho = -\mathbf{c}\cdot\nabla\hat{\rho}.$$

Notice that here, for these equalities to make sense, we must further require that the domain $\Omega(t)$ is invariant under the transformation $\mathbf{x} \mapsto \mathbf{x} - \mathbf{c}t$. In particular, we need $\Omega(t) = \Omega(0)$, meaning that the domain becomes fixed in the moving coordinates.

Abusing notation, we identify $\mathbf{u}$ and $\hat{\mathbf{u}}$ as well as $\Omega(t)$ and $\Omega(0)$. The Euler equations (1.1.12) then become the steady system

$$\begin{cases} \rho(\mathbf{u}-\mathbf{c})\cdot\nabla\mathbf{u} = -\nabla P - \mathbf{g}\rho, \\ (\mathbf{u}-\mathbf{c})\cdot\nabla\rho = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \qquad \text{in } \Omega. \tag{1.1.22}$$

Let us consider the boundary conditions as in section 1.1.4. If there is a portion of $\partial\Omega$, denote it $\Gamma^{(1)}$, that corresponds to an impermeable barrier, so that (1.1.13) applies, then we clearly must have that $\mathbf{c}\cdot\mathbf{n} = 0$, whence

$$(\mathbf{u}-\mathbf{c})\cdot\mathbf{n} = 0, \qquad \text{on } \Gamma^{(1)}. \tag{1.1.23}$$

Now suppose that there is a component $\Gamma^{(2)} \subset \partial\Omega \setminus \Gamma^{(1)}$ on which the boundary is free. For simplicity, say that we have a graph representation as in (1.1.16). Note that this means that $c_n = 0$. Since the boundary must reflect the traveling character of the flow, we conclude that $\boldsymbol{\eta}(t, \mathbf{x}) = \hat{\boldsymbol{\eta}}(\mathbf{x} - t\mathbf{c})$, for some profile $\hat{\boldsymbol{\eta}}$, which we henceforth write $\boldsymbol{\eta}$. We find then, that

$$(\mathbf{u} - \mathbf{c}) \cdot \nabla\boldsymbol{\eta} = \mathbf{u}_n, \qquad \text{on } \Gamma^{(2)}. \tag{1.1.24}$$

As the dynamic boundary conditions (1.1.20)–(1.1.21) involve no time derivatives, their representation in the moving frame is unchanged. Observing (1.1.22)–(1.1.24), it is clear that the relative vector field $\mathbf{u}_{\mathrm{rel}} := \mathbf{u} - \mathbf{c}$ satisfies the standard Euler equations with no time dependence, i.e., it is a steady solutions in the traditional sense.

1.2. Stratified flows

The equations we obtained in section 1.1 describe the motion of incompressible, inviscid flows with possibly heterogeneous density. These differ from the more familiar constant density equations in two respects: the continuity equation (1.1.2) is satisfied nontrivially, and the momentum equation has non-constant coefficients. It is not immediately obvious what, if any, effect this will have on the existence theory and qualitative property of the solutions. In this section, we explore some of the archetypical features of stratified flows, particularly those that are steady. Though in the chapters that follow we shall exclusively be considering the two-dimensional case, some of the more distinctive features of stratified flows are to be found only in higher dimensions, so we shall refrain from specializing until section 1.2.3.

1.2.1. Gravity and stratification. Consider the scenario where $g = 0$ in the Euler equations (1.1.12), which models flows where inertia (represented by the material derivative) far outweigh the gravitational force. Since gravity only acts in a single direction (the $\mathbf{e}_n$ direction) and occurs as a lower-order term, naively we might expect it to have little mathematical importance. In fact, this proves to be far from true. Following Yih [**80**], we introduce the associated vector field $\hat{\mathbf{u}} := \sqrt{\rho}\mathbf{u}$, and the associated material derivative $\hat{\mathcal{D}} := \partial_t + \hat{\mathbf{u}} \cdot \nabla$. We shall assume that the flow is steady for some wave speed $\mathbf{c} = (c_1, c_2, 0)$.

From the continuity equation (1.1.2), we find

$$\hat{\mathcal{D}}(\rho^{-1/2}) = \rho^{1/2}(\mathbf{u}-\mathbf{c})\cdot\nabla(\rho^{-1/2}) = -\frac{1}{2}\rho^{-1}(\mathbf{u}-\mathbf{c})\cdot\nabla\rho = 0.$$

The steady Euler equation (1.1.22) then implies

$$\hat{\mathcal{D}}\hat{\mathbf{u}} = (\hat{\mathcal{D}}\rho^{-1/2})\rho\mathbf{u} + \mathcal{D}(\rho\mathbf{u}) = -\nabla P. \tag{1.2.1}$$

In other words, (1.2.1) states that $\hat{\mathbf{u}}$ satisfies the *constant density* Euler equation for steady motion in the absence of gravity. Moreover, to each solution of the homogeneous problem, and each choice of density function, we have a corresponding solution to the heterogenous problem. Without the addition of the gravitational force term $\rho\mathbf{g}$, stratified flows are of no real mathematical interest.

The physical consequences of density variation are loosely divided into two categories, *inertial* and *gravitational*. The former refers to the effect on the flow due to the inhomogeneity of the fluid particle inertia, while the latter relates to those arising from the differing levels of gravitational force felt by the particles. At an equally heuristic level, we can view the transformation $\mathbf{u} \mapsto \hat{\mathbf{u}}$ as quotienting out the inertial effects so that, if there is no gravity, we truly have a trivial reduction to the constant density setting. We shall make use of this intuition extensively later.

In flows where the inertial forces are relatively small, the gravitational effects become very pronounced. Consider a steady stratified fluid where the motion is vanishingly weak. That is, suppose that $\mathbf{u}$ and $\mathbf{c}$ have characteristic velocity $U_{\text{char}} = O(\epsilon)$, where $\epsilon \ll 1$. Without making any further length scale prescriptions, we investigate what occurs qualitatively when $\epsilon \to 0$. Since $\nabla\mathbf{u}$ has order $U_{\text{char}}/L_{\text{char}}$, for characteristic length $L_{\text{char}} = O(1)$, we have that

$$\rho(\mathbf{u}-\mathbf{c})\cdot\nabla\mathbf{u} = O(\epsilon^2).$$

Inserting this into (1.1.22), we infer that

$$\nabla P + \rho\mathbf{g} = -\rho(\mathbf{u}-\mathbf{c})\cdot\nabla\mathbf{u} = O(\epsilon^2). \tag{1.2.2}$$

Cross differentiating the equations for u and w, leads to the following identity:

$$\partial_z(\rho(\mathbf{u}-\mathbf{c})\cdot\nabla u) = -P_{xz} = \partial_x(\rho(\mathbf{u}-\mathbf{c})\cdot\nabla w + g\rho). \tag{1.2.3}$$

Doing likewise with the equations for v and w, we see

$$\partial_z(\rho(\mathbf{u}-\mathbf{c})\cdot\nabla v) = -P_{yz} = \partial_y(\rho(\mathbf{u}-\mathbf{c})\cdot\nabla w + g\rho). \tag{1.2.4}$$

Note that, by (1.2.2), P_{xz} and P_{yz} are $O(\epsilon^2)$. We do not wish to make smallness assumptions on the stratification, so *a priori* we must take $\rho_z = O(1)$, and hence (1.2.3) and (1.2.4) imply

$$\rho_x = O(\epsilon^2), \qquad \rho_y = O(\epsilon^2). \tag{1.2.5}$$

This last equation expresses the fact that, since the z-direction is distinguished by the presence of the gravitational force, the stratification will likewise have an expected direction. In fact, by the continuity equation we have an even stronger statement:

$$w\rho_z = -(u-c_1)\rho_x - (v-c_2)\rho_y = O(\epsilon^3).$$

Since $\rho_z = O(1)$, for the above relation to hold, we need

$$w = O(\epsilon^3). \tag{1.2.6}$$

That is, the vertical velocity of the fluid is necessarily two orders of magnitude smaller than the two horizontal velocities. Consequently, the flow will take on a distinctly two-dimensional structure. Incompressibility becomes

$$u_x + v_y = O(\epsilon^3), \tag{1.2.7}$$

and the momentum equations (in the two horizontal directions) reduce to the following:

$$\begin{cases} \rho(u-c_1)u_x + \rho(v-c_2)u_y + P_x = O(\epsilon^4), \\ \rho(u-c_1)v_x + \rho(v-c_2)v_y + P_y = O(\epsilon^4). \end{cases} \tag{1.2.8}$$

Together (1.2.7)–(1.2.8) show that, up to $O(\epsilon^3)$, $(u,\, v)$ satisfies the two-dimensional steady incompressible Euler equations on each horizontally oriented plane $z = z_0$. Indeed, in light of (1.2.5), we have (locally) that

$$\rho(x,y,z) = \varrho(z) + O(\epsilon^2). \tag{1.2.9}$$

Replacing ρ in (1.2.8) with ϱ, it follows that, on each such plane, the rescaled horizontal velocities satisfy the *constant density* Euler equations. Qualitatively, then, the fluid is organized as a vertical layering of two-dimensional and constant density flows. Since there is no viscosity, the strata communicate with each only through the pressure and density (which both remain functions of z) via the hydrostatic relation:

$$P_z + g\rho = O(\epsilon^4).$$

This last equation is found from the w-momentum equation.

The argument above, which is due originally to Yih (cf. [**79**]) suggests the existence of a striking phenomenon known as *blocking*. Suppose that, apart from the consideration of the last several paragraphs, it is known *a priori* that the motion is horizontally two-dimensional. This is usually justified on the basis of some symmetry in the domain, e.g., if the fluid is confined to a channel $\Omega = \{(x, y, z) : (y, z) \in [-1, 1] \times [-1, 1]\}$, we expect that motion on any plane parallel to the vertical walls will be identical. Then, without loss of generality we may assume that

$$u_y = w_y = v = 0,$$

and thus, by the incompressibility condition (1.2.7),

$$u_x = O(\epsilon^3). \tag{1.2.10}$$

Up to third-order in ϵ, therefore, the horizontal velocity is simply constant. Accordingly, any barrier that is submerged within the flow orthogonal to the horizontal motion will result in stagnant regions. To see this, observe that equation (1.2.6) prevents the fluid from bypassing the obstacle vertically, while the impermeability condition (1.1.13) forces u_x (the normal velocity) to vanish on the surface of the object. From (1.2.10) we conclude that $u \equiv 0$ upstream and downstream on each horizontal plane that meets the barrier. Similarly, if we were to tow the object through the fluid, then the implication of (1.2.10) is that the fluid downstream must adhere to the object, while the fluid upstream is pushed as if it were a solid body.

Of course, as with any inviscid flow, this analysis is only correct modulo some caveats about the behavior near the boundary. Nonetheless, the general predictions of blocking have been verified experimentally by many authors (cf. [**23, 46, 81**]). In particular, it has been studied in connection with the problem of slow, atmospheric flow over mountainous topography. A more rigorous mathematical justification — e.g. in the same vein as the vanishing viscosity limit problem — is of considerable interest mathematically, but seems to be entirely open.

1.2.2. Vorticity in stratified flows. For a flow $\mathbf{u}$, the *vorticity* is defined as $\boldsymbol{\omega} := \nabla \times \mathbf{u}$. In the case of two-dimensional motion, $\boldsymbol{\omega}$ will have only one nontrivial component, and so it is customary to treat the vorticity as the scalar quantity $\omega := v_x - u_y$, where $\mathbf{u} = (u, v)$. The vast majority of the literature on water waves is concerned with the case of so-called *irrotational* flow, where it is assumed that $\boldsymbol{\omega} \equiv 0$. Irrotationality is a physically valid assumption in some situations — see THEOREM 1.2.1 and the surrounding discussion — but perhaps the most compelling reason for working with irrotational flows is simple mathematical convenience. Note that, if $\nabla \times \mathbf{u} \equiv 0$, then there exists a velocity potential ϕ satisfying $\nabla\phi = \mathbf{u}$. The incompressibility condition (1.1.9) implies further that $\Delta\phi = 0$ in Ω, i.e. ϕ is harmonic. The problem of determining the velocity is thus reduced to identifying the behavior on the boundary, since there is a robust and extensive solution theory for the Laplace equation. For a fixed boundary (or an unbounded domain) the problem of existence becomes trivial. The free boundary problem remains difficult, but the decoupling of the interior and boundary behavior affords the use of machinery from complex analysis and harmonic analysis that would otherwise be inaccessible.

In this thesis, however, we shall allow the flows to be rotational for reasons that we now endeavor to explain. With that in mind, we begin by deriving an equation governing the dynamics of the vorticity. We shall do this in two-dimensions, since the three dimensional case differs only in technical complexity.

Starting from the Euler equations (1.1.12) and cross differentiating, we find

$$\rho(u_{yt} + u_y u_x + u u_{xy} + v_y u_y + v u_{yy}) + \rho_y(u_t + u u_x + v u_y) = -P_{xy},$$

and

$$\rho(v_{xt} + u_x v_x + u v_{xx} + v_x v_y + v_{yx}) + \rho_x(v_t + u v_x + v v_y) = -P_{xy} + g\rho_x.$$

Taking the difference and using incompressibility to eliminate several terms, these two equations reduce to the following

$$\rho^2(\omega_t + \mathbf{u} \cdot \nabla\omega) = \rho_x P_y - \rho_y P_x,$$

or, equivalently,

$$\omega_t + \mathbf{u} \cdot \nabla\omega = \frac{1}{\rho^2}\left(\nabla\rho \cdot \nabla^\perp P\right). \tag{1.2.11}$$

The left-hand side above is nothing but the material derivative of the vorticity, so that (1.2.11) describes the advection of vorticity by the flow. In particular, we see that vorticity is always generated by the component of density gradient that is orthogonal to the pressure gradient. This fact generalizes to three-dimensions, where we find

$$\boldsymbol{\omega}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \frac{1}{\rho^2}\left(\nabla\rho \times \nabla P\right). \tag{1.2.12}$$

Observe that, even absent density variation, there is a significant difference between (1.2.11) and (1.2.12): the presence of the *vortex stretching term* $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$. The dynamics of the vorticity is therefore highly dimensionally dependent.

In light of (1.2.11)–(1.2.12), we see that one cannot expect the flow to be irrotational (even with irrotational initial data), unless it is know *a priori* that the gradients of the density and pressure are parallel. Indeed, $\nabla\rho \times \nabla P$ is referred to as the *baroclinic vector* and plays a destabilizing role in the vorticity dynamics, similar to the vortex stretching term in the homogeneous theory. Flows in which the baroclinic vector vanishes identically are called *barotropic*; surprisingly, they are the expected behavior in a variety of physical situations. Observe that, if $\nabla\rho \times \nabla P = 0$, then the level sets of the density (the *isopycnic surfaces*) and those of the pressure (*isobars*) coincide, meaning that there exists a functional relationship between ρ and P. We may then let ϱ be given so that $\rho = \varrho(P)$. Then the continuity equation becomes

$$\varrho'(P)\left(\partial_t + \mathbf{u}\cdot\nabla\right)P = 0, \tag{1.2.13}$$

and thus $\mathcal{D}P = 0$ in regions of the fluid domain where $\varrho' \neq 0$. This is, in fact, quite surprising. Naively we expect fluid particles to move from areas of higher pressure to areas of lower pressure, i.e. in the direction of the pressure gradient. But (1.2.13) shows that the opposite must be true in stratified barotropic flows: since the motion is along isobars, the velocity must be orthogonal to the pressure gradient. We remark that this situation also occurs in the geostrophic approximation in geophysical fluid dynamics, where it is the byproduct of rotational forcing (cf. [**57**]).

Recall that in section 1.2.1 it was argued that slow moving steady stratified flows are composed of two-dimensional infinitesimal homogeneous flows layered vertically (i.e. in the direction of the gravitational force.) Consequently, each layer of the fluid will be irrotational in the two-dimensional sense, so long as that is the case in the far field. While the argument used to arrive at this conclusion was highly specialized to the weak flow regime, by considering the baroclinic vector and the circulation, a similar insight into the structure of general stratified flows can be obtained. The *circulation* of the flow around any smooth, simple curve $\mathcal{C}(t) \subset \Omega(t)$ is defined to be the quantity

$$\Gamma(t) := \int_{\mathcal{C}(t)} u\,dx + v\,dy + w\,dz = \int_{\mathcal{S}(t)} \boldsymbol{\omega} \cdot \mathbf{n}\,dS, \tag{1.2.14}$$

where $\mathcal{S}(t)$ is a surface bounded by $\mathcal{C}(t)$, dS denotes the surface measure on $\mathcal{S}$, and $\mathbf{n}$ the outer unit normal. The circulation, as can be seen from the definition above, describes roughly the vorticity per unit area. Unlike the vorticity, however, the circulation is actually conserved by the flow in a number of cases. For example, the classical Kelvin–Helmholtz theorem in fluid mechanics states that, for an inviscid fluid, the following identity holds:

Theorem 1.2.1 (Kelvin–Helmholtz, cf. [**57**]). *Consider an inviscid, incompressible flow (defined locally in time) in the domain $\Omega(t) \subset \mathbb{R}^n$. Let $\mathcal{C}_0$ be any smooth closed curve in $\Omega(0)$, and let $\mathcal{C} = \mathcal{C}(t)$ be image of $\mathcal{C}_0$ under the Lagrangian flow map. Suppose that, for some $T > 0$, $\mathcal{C}(t)$ is smooth, simple and closed on $[0,T)$, and that $\mathbf{u}$ is barotropic along $\mathcal{C}$. Then the circulation $\Gamma(t)$ around $\mathcal{C}(t)$ satisfies $\mathcal{D}\Gamma(t) \equiv 0$ on $[0,T)$.*

One important consequence is that, if the fluid is initially at rest (hence $\Gamma(0) = 0$), then the circulation around any barotropic curve remains zero even as the curve is evolved by the flow. This is the justification for the physicality of constant density irrotational flows: if a fluid is homogeneous (hence barotropic) and started from rest, it remains irrotational.

To understand how the situation changes for baroclinic flows, we appeal to a generalization of Theorem 1.2.1 due to Bjerknes:

Theorem 1.2.2 (Bjerknes, [**6**]). *Let the flow and fluid domain be defined as before. Let $\mathcal{C}_0 \subset \Omega(0)$ be be any smooth, simple curve bounding the surface $\mathcal{S}_0$. Let $\mathcal{C} = \mathcal{C}(t)$ and $\mathcal{S} = \mathcal{S}(t)$ denote the images of $\mathcal{C}_0$ and $\mathcal{S}_0$ under the Lagrangian flow map, respectively. Suppose that, for some $T > 0$, $\mathcal{C}(t)$ is smooth, simple and closed, while $\mathcal{S}(t)$ is simply connected on the time interval $[0,T)$. Then*

$$\mathcal{D}\Gamma(t) = -\int_{\mathcal{S}(t)} \frac{1}{\rho^2}\,(\nabla\rho \times \nabla P) \cdot \mathbf{n}\,dS, \qquad \text{for all } t \in [0,T). \tag{1.2.15}$$

In particular, $\mathcal{D}\Gamma(t) \equiv 0$ along $\mathcal{C}$ on $[0,T)$, if $\mathcal{C}_0$ (and hence $\mathcal{C}$) lies within a surface of constant density or pressure.

When the fluid is started from rest, we now have that $\Gamma \equiv 0$ on any isopycnic surface. Comparing (1.2.15) to the second expression for the circulation in (1.2.14), we infer that, for an inviscid, incompressible fluid started from rest, the component of $\boldsymbol{\omega}$ orthogonal to any isopycnic surface vanishes. This property is referred to as *layerwise irrotationality*. Note that, if the isopyncic surfaces are planes of the form $z = z_0$, then the normal component of $\boldsymbol{\omega}$ is merely the two-dimensional vorticity ω. In other words, as in the case of weak, steady flows, in any flow started from rest (that is incompressible and inviscid) we observe a vertical layering of isopycnic surfaces. Once again, this conforms our expectation that stratification generates vorticity: in contrast to the homogeneous case, a heterogeneous fluid started from rest will exhibit *vortex sheets*, the appearance of vorticity arising from the tangential shear of isopycnic layers slipping along one another at varying speeds.

To summarize, in this section we have seen that irrotational flows can occur for stratified fluids, e.g. if the flow is barotropic and started from rest. There are many physically relevant problems in which this assumption is valid, but it is clearly far from universal. In light of (1.2.11)–(1.2.12), moreover, we see that if vorticity does appear (e.g. due to external forcing), it will not be attenuated but rather intensified by the heterogenity in the density. In everything that follows, therefore, we shall work in the rotational regime.

1.2.3. Steady stratified flows in two-dimensions. We previously introduced the concept of a steady flow (or traveling wave) in section 1.1.5. In the present section, we discuss some mathematical machinery that is useful for the two-dimensional case.

For a steady incompressible flow two-dimensional flow $\mathbf{u} = (u - c, v)$, there exists a (relative) stream function ψ satisfying $\nabla^{\perp}\psi = \mathbf{u}$, or more explicitly:

$$\psi_x = -v, \qquad \psi_y = u - c.$$

The definition of ψ is completed by prescribing its value on the boundary, on which ψ is known to be constant. To see this, note that on a solid boundary portion (one where the impermeability condition (1.1.23) holds), then $\nabla\psi$ and the outward unit normal $\mathbf{n}$ must be parallel, since they are each orthogonal to $\mathbf{u}$. On portions of the boundary that are free and parameterized by $x \mapsto (x, \eta(x))$, the kinematic condition (1.1.24) becomes

$$\psi_y \eta_x = -\psi_x,$$

whence ψ is constant along the free surface, as claimed. The level sets of ψ, referred to as the *streamlines*, encode a great deal of information about the flow, since $\nabla\psi$ is orthogonal to the velocity field at each point. For instance, the continuity equation implies that the streamlines are isopycnic surfaces.

Were it not for the presence of the density term ρ, the steady Euler equations could be entirely rewritten in terms of ψ. In fact, we later show that there is a semilinear scalar formulation in this case (see (1.2.22)). It is therefore natural to consider generalizations to the stream function that incorporate the effects of density stratification. This was essentially the insight that drove the works of Dubreil–Jacotin [**26**], and later that of Long [**45**], who a considered a function ψ_{DJ} such that $\nabla^{\perp}\psi_{\mathrm{DJ}} = \rho\mathbf{u}$. We shall, however, be using a further refinement of this approach due to Yih (cf. [**80**]) motivated by our discussion in section 1.2.1. There we argued that the transformation $\mathbf{u} \mapsto \sqrt{\rho}\mathbf{u}$ removed the inertial effects of stratification, leaving only the effects of gravitation. With that in mind, we introduce ψ_{YIH} satisfying $\nabla^{\perp}\psi_{\mathrm{YIH}} = \sqrt{\rho}\mathbf{u}$, which we call the (relative) *pseudo-stream function*. Since there will be no further need for the standard stream function, nor that of Dubreil-Jacotin–Long, in what follows we simply write ψ for ψ_{YIH}.

It is often convenient to express the steady Euler equations in terms of ψ. Directly from the definitions, we find that the momentum equation is equivalent to

$$\begin{cases} \psi_y\psi_{xy} - \psi_x\psi_{yy} = -P_x, \\ -\psi_y\psi_{xx} + \psi_x\psi_{xy} = -P_y - \rho g, \end{cases} \tag{1.2.16}$$

and mass conservation can be expressed via the following identity

$$\nabla^{\perp}\psi \cdot \nabla\rho = 0. \tag{1.2.17}$$

An important result due to Bernoulli is that the quantity

$$E := P + \frac{\rho}{2}\left((u-c)^2 + v^2\right) + g\rho y = P + \frac{1}{2}|\nabla\psi|^2 + g\rho y,$$

is constant along streamlines. E is called the *total hydraulic head*. Some authors prefer to define the total head as E/ρ, in which case it clearly corresponds to an energy density. From a mathematical perspective, though, the definition above is more convenient, even if its physical meaning is slightly more obscure.

Let us now prove that E is indeed a function of ψ. First we compute

$$\begin{aligned}
\nabla E &= \nabla P + (\psi_x\psi_{xx} + \psi_y\psi_{xy},\, \psi_x\psi_{xy} + \psi_y\psi_{yy}) \\
&\qquad + gy\nabla\rho + \rho\mathbf{g} \qquad (1.2.18)\\
&= (\psi_x\psi_{yy} - \psi_y\psi_{xy},\, \psi_y\psi_{xx} - \psi_x\psi_{xy}) \\
&\qquad + (\psi_x\psi_{xx} + \psi_y\psi_{xy},\, \psi_x\psi_{xy} + \psi_y\psi_{yy}) \\
&\qquad\quad + gy\nabla\rho \qquad (1.2.19)\\
&= (\Delta\psi)\nabla\psi + gy\nabla\rho. \qquad (1.2.20)
\end{aligned}$$

Here line (1.2.19) is found from (1.2.18) by using (1.2.16) to express $\nabla P + \rho\mathbf{g}$ in terms of ψ. Now, recalling the equivalent expression for conservation of mass (1.2.17), we have from (1.2.20) that

$$\nabla^\perp\psi \cdot \nabla E = (\Delta\psi)(\nabla^\perp\psi \cdot \nabla\psi) + gy(\nabla^\perp\psi \cdot \nabla\rho) = 0,$$

which proves the claim.

The conservation of E along streamlines is an extremely useful fact; it is one of the only tools we possess for relating the pressure, kinetic energy, and potential energy. It will also provide us with a scalar formulation for the system (1.2.16). Note that

$$\nabla E = \frac{dE}{d\psi}\nabla\psi, \qquad \text{and} \qquad \nabla\rho = \frac{d\rho}{d\psi}\nabla\psi.$$

Hence, on the condition that $\nabla\psi$ is nowhere vanishing, we obtain from (1.2.20) the following:

$$\frac{dE}{d\psi} = \Delta\psi + gy\frac{d\rho}{d\psi}. \qquad (1.2.21)$$

This rather remarkable identity is called Yih's equation, or the Dubreil-Jacotin–Long–Yih equation, in recognition of the fact that Dubreil-Jacotin and Long each derived a similar scalar equation earlier. As noted earlier, those earlier works were done in terms ψ_{DJ}, so that the resulting equation was less desirable in form than (1.2.21).

To compare this to the homogeneous case, let us suppose that ρ is a constant. Equation (1.2.11) put into the steady frame implies that $\nabla^\perp\psi \cdot \nabla\omega = 0$, that is, ω can be thought of as a function of ψ. We may therefore let γ be defined by $\omega = \gamma(\psi)$. If Γ is an antiderivative of γ (with resect to ψ), then

$$\nabla\Gamma = \gamma\nabla\psi = (-\psi_{xx} - \psi_{yy})\,\nabla\psi.$$

Inserting this expression into (1.2.21), we see that

$$\Delta\psi = \frac{dE}{d\psi} = -\gamma(\psi). \tag{1.2.22}$$

This is the well-known semilinear formulation for the constant density case. Yih's equation is essentially a generalization of (1.2.22) to the heterogeneous setting, and will serve as the basis for our analysis in the coming chapters.

1.3. Overview of thesis

With the background established, we conclude this chapter with a summary of our results, and a discussion of the structure of the thesis. Each of the remaining chapters began as a journal article; chapters 2 and 4 have each been accepted for publication (cf. [**76**] and [**74**], respectively), while chapter 3 has been submitted. As such, they are written to be more-or-less self-contained, though both the latter chapters build upon chapter 2. This results in a certain degree of redundancy in the exposition, which we view as preferable to having a unified document that cannot be approached except in its entirety.

We begin, in chapter 2, by building an existence theory for steady stratified gravity waves in water. In particular, we are considering the case where the flow is periodic in the direction of propagation, and the boundary consists of a free surface above and an impermeable barrier below. We will prove the existence of a global continuum of these waves, under certain assumptions on the given physical quantities. This work generalizes a recent contribution of Constantin and Strauss (cf. [**15**]) to the stratified regime, which results in some nontrivial technical obstacles. The basic machinery is that of (degree theoretic) global bifurcation theory.

Chapter 3 concerns the situation when the effects of surface tension on the boundary are taken into account (i.e., where the dynamic condition on the boundary is (1.1.21) rather than (1.1.20)). We show that, under similar conditions to those in chapter 2, there exist global continua of steady solutions. Here, though, the local behavior is considerably more interesting: when the coefficient of surface tension lies in a certain set of values (determined by the other physical parameters), the bifurcation that occurs is non-simple, resulting in multiple continua. We study this scenario in some detail. Additionally, we use an alternative global bifurcation theory, due to Dancer, to prove that the continua are, in fact, smooth. A more detailed account of the mathematical history is given later, but we mention that these results generalize (in a number of ways) the work of Wahlén on the homogeneous case.

Finally, in chapter 4, we prove some results on the symmetry of steady stratified water waves. In the previous chapters, the solutions that we find are symmetric by construction, at least locally. However, it has recently been shown that, for waves of constant density, symmetry always follows if the surface profile is assumed to be monotone (cf. [**14, 13**]). We show that this remains true for heterogeneous flows, given some additional smallness assumptions.

CHAPTER 2

Stratified and steady periodic gravity waves

2.1. Introduction

Stratification is a common feature of ocean waves, where the presence of salinity, in concert with external gravitational force, can produce substantial heterogeneity in the fluid. The pronounced effects that may accompany even a moderate density variation have earned stratified flows a great deal of scholarly attention, particularly in the realm of geophysical fluid dynamics. In this chapter, we develop a theory for stratified gravity waves that are traveling and periodic.

If we imagine a wave on the open ocean, past experience suggests that it may be *regular* in the following sense. First, it is essentially two-dimensional. That is, the motion will be identical along any line that runs parallel to a crest. Second, if we regard the wave in a coordinate system moving with some constant speed, it appears steady. Finally, the profile is periodic in the direction of motion, descending monotonically from a single crest to a single trough once per period.

We now formulate governing equations for waves of this form. Fix a Cartesian coordinate system so that the x-axis points in the direction of propagation, and the y-axis is vertical. We assume that the floor of the ocean is flat and occurs at $y = -d$. Let $y = \eta(x,t)$ be the free surface at the interface between the atmosphere and the fluid. We shall normalize η by choosing the axes so that the free surface is oscillating around the line $y = 0$. As usual we let $u = u(x,y,t)$ and $v = v(x,y,t)$ denote the horizontal and vertical velocities, respectively. Let $\rho = \rho(x,y,t) > 0$ be the density.

Incompressibility of the fluid is represented mathematically by the requirement that the vector field (u,v) be divergence free for all time:

$$u_x + v_y = 0. \tag{2.1.1}$$

Taking the fluid to be inviscid, conservation of mass implies that the density of a fluid particle remains constant as it follows the flow. This is expressed by the continuity equation

$$\rho_t + u\rho_x + v\rho_y = 0. \tag{2.1.2}$$

Next, the conservation of momentum is described by Euler's equations

$$\begin{cases} u_t + uu_x + vu_y = -\frac{P_x}{\rho}, \\ v_t + uv_x + vv_y = -\frac{P_y}{\rho} - g, \end{cases} \tag{2.1.3}$$

where $P = P(x, y, t)$ denotes the pressure and g is the gravitational constant. Here, of course, we assume that the only external force acting on the fluid is gravity.

On the free surface, we must ensure that the pressure of the fluid matches the atmospheric pressure of the air above, that we shall denote P_{atm}. Thus,

$$P = P_{\text{atm}} \qquad \text{on } y = \eta(x, t). \tag{2.1.4}$$

The corresponding boundary condition for the vector field is motivated by the fact that fluid particles that reside on the free surface continue to do so as the flow develops. This observation is manifested in the kinematic condition

$$v = \eta_t + u\eta_x \qquad \text{on } y = \eta(x, t). \tag{2.1.5}$$

Since we cannot have any fluid moving normal to the flat bed occurring at $y = -d$, we require

$$v = 0 \qquad \text{on } y = -d. \tag{2.1.6}$$

Note that there is no accompanying condition on u because in the inviscid case we allow for slip, that is, nonzero horizontal velocity along solid boundaries.

We seek traveling periodic wave solutions (u, v, ρ, P, η) to (2.1.1)–(2.1.6). More precisely, we take this to mean that, for fixed $c > 0$, the solution appears steady in time and periodic in the x-direction when observed in a frame that moves with constant speed c to the right. The vector field will thus take the form $u = u(x - ct, y)$, $v = v(x - ct, y)$, where each of these is L-periodic in the first coordinate. Likewise for the scalar quantities $\rho(x, y, t) = \rho(x - ct, y)$, $P = P(x - ct, y)$, and $\eta = \eta(x - ct)$, again with L-periodicity in the first coordinate. We therefore take moving coordinates

$$(x - ct, y) \mapsto (x, y),$$

which eliminate time dependency from the problem. In the moving frame, (2.1.1)–(2.1.3) become

$$\begin{cases} u_x + v_y = 0, \\ (u-c)\rho_x + v\rho_y = 0, \\ (u-c)u_x + vu_y = -\frac{P_x}{\rho}, \\ (u-c)v_x + vv_y = -\frac{P_y}{\rho} - g, \end{cases} \tag{2.1.7}$$

throughout the fluid domain. Meanwhile, the reformulated boundary conditions are

$$\begin{cases} v = (u-c)\eta_x & \text{on } y = \eta(x), \\ v = 0 & \text{on } y = -d, \\ P = P_{\text{atm}} & \text{on } y = \eta(x), \end{cases} \tag{2.1.8}$$

where u, v, ρ, P are taken to be functions of x and y, η is a function of x, and all of them are L-periodic in x.

In the event that $u = c$ somewhere in the fluid we say that *stagnation* has occurred, as in the moving frame the fluid appears to be stationary at that point. In order to avoid roll-up and other instability phenomena, we shall restrict our attention to the case where $u < c$ throughout.

Recall that we have chosen our axes so that η oscillates around the line $y = 0$. Mathematically this equates to

$$⨍_0^L \eta(x)dx = 0. \tag{2.1.9}$$

In effect, this couples the depth to the problem, so we need not treat d as a free parameter. The trade-off, as we shall see, is that this normalization will have significant technical consequences later.

An often indispensable tool in the study of incompressible fluids is the stream function—especially in the case of two-dimensional flow in bounded domains. This is a function whose curl is the vector field (u, v), and thus, the gradient is orthogonal to the field at each point in the fluid. For a homogeneous fluid, a standard method is to then reformulate the problem for the stream function instead of the flow. What recommends this approach is the fact that the boundaries of the domain will be level sets of the stream function. Therefore, by adopting streamline coordinates, we can fix the domain and eliminate the free surface problem.

However, for a heterogeneous fluid this generally proves insufficient, as it does not sufficiently capture the effects of stratification. Instead, we employ a more generalized object whose use was pioneered by Yih and Long, among others (cf. [**82, 45**], for example). Observe that by conservation of mass and incompressibility, ρ is transported, and the vector field is divergence free. Therefore we may introduce a (relative) *pseudostream function* $\psi = \psi(x,y)$, defined uniquely up to a constant by

$$\psi_x = -\sqrt{\rho}v, \qquad \psi_y = \sqrt{\rho}(u-c).$$

Here we have the addition of a ρ term to the typical definition of the stream function for an incompressible fluid. This neatly captures the inertial effects of the heterogeneity of the flow (see the treatment in [**82**], for example). The particular choice of $\sqrt{\rho}$ is merely to simplify algebraically what follows.

It is a straightforward calculation to check that ψ is, indeed, a (relative) stream function in the usual sense, i.e., its gradient is orthogonal to the vector field in the moving frame at each point in the fluid domain:

$$(u-c)\psi_x + v\psi_y = 0.$$

Moreover, (2.1.8) implies that the free surface and flat bed are each level sets of ψ. For definiteness we choose $\psi \equiv 0$ on the free boundary, so that $\psi \equiv p_0$ on $y = -d$, where p_0 is the quantity

$$p_0 := \int_{-d}^{\eta(x)} \sqrt{\rho(x,y)}\,[u(x,y)-c]\,dy. \tag{2.1.10}$$

To see that this is well defined, that is, the integral on the right-hand side is independent of x, we calculate

$$\begin{aligned}\frac{dp_0}{dx} &= \eta_x\left(\sqrt{\rho(x,\eta(x))}\,[u(x,\eta(x))-c]\right) + \int_{-d}^{\eta(x)} \partial_x\left(\sqrt{\rho(x,y)}\,[u(x,y)-c]\right)dy \\ &= \sqrt{\rho(x,\eta(x))}v(x,\eta(x)) - \int_{-d}^{\eta(x)} \partial_y\left(\sqrt{\rho}v\right)dy = 0.\end{aligned}$$

We shall call p_0 the (relative) *pseudovolumetric mass flux*; it represents the amount of fluid flowing through a vertical line extending from the bed to the free surface and with respect to the transformed vector field $(\sqrt{\rho}(u-c), \sqrt{\rho}v)$. The level sets of ψ will be called the *streamlines* of the flow.

Since ρ is transported, it must be constant on the streamlines, and hence, we may think of it as a function of ψ. Abusing notation we may let $\rho : [p_0, 0] \to \mathbb{R}^+$ be given such that

$$\rho(x, y) = \rho(-\psi(x, y)) \tag{2.1.11}$$

throughout the fluid. The reason for taking $-\psi$ here will become apparent in the next section. When there is risk of confusion, we shall refer to the ρ occurring on the right-hand side above as the *streamline density function*. We shall focus our attention on the case where the density is nondecreasing as depth increases. This is entirely reasonable from a physical standpoint. Indeed, hydrodynamic stability requires that the depth be monotonically increasing with depth, making this assumption standard in the literature. The level set $-\psi = p_0$ corresponds to the flat bed, and the set where $-\psi = 0$ corresponds to the free surface. Therefore, we require that the streamline density function is nonincreasing as a function of $-\psi$.

By Bernoulli's theorem the quantity

$$E := P + \frac{\rho}{2}\left((u-c)^2 + v^2\right) + g\rho y$$

is constant along streamlines, that is,

$$(u - c)E_x + vE_y = 0.$$

This can be verified directly from (2.1.7). In the case of an inviscid fluid, E represents the energy of the fluid particle at (x, y). The first term on the right-hand side, P, gives the energy due to internal pressure. The second and third terms combined describe the kinetic energy, while the last is gravitational potential energy. When evaluated on the free surface, E is usually referred to as the *hydraulic* or *total head* of the fluid.

Under the assumption that $u < c$ throughout the fluid and given the fact that E is constant along streamlines, there exists a function $\beta : [0, |p_0|] \to \mathbb{R}$ such that

$$\frac{dE}{d\psi}(x, y) = \beta(\psi(x, y)). \tag{2.1.12}$$

For want of a better name we shall refer to β as the *Bernoulli function* corresponding to the flow. Physically it describes the variation of specific energy as a function of the streamlines. Define

$$B(p) := \int_0^p \beta(-s)ds$$

for $p_0 \leq p \leq 0$, and let B have minimum value $B_{\min}$.

Let us briefly outline a few notational conventions. Let $\overline{D_\eta}$ denote the closure of the fluid domain

$$D_\eta := \left\{(x,y) \in \mathbb{R}^2 : -d < y < \eta(x)\right\}.$$

For any integer $m \geq 1$ and $\alpha \in (0,1)$, we say that a bounded domain $D \subset \mathbb{R}^2$ is $C^{m+\alpha}$ provided that each point in the boundary denoted ∂D is locally the graph of a $C^{m+\alpha}$ function. Furthermore, for fixed $m \geq 1$ and $\alpha \in (0,1)$, we define the space $C^{m+\alpha}_{\text{per}}(\overline{D})$ to consist of those functions $f : \overline{D} \to \mathbb{R}$ with Hölder continuous derivatives (of exponent α) up to order m and that are L-periodic in the x-variable. Similarly, we shall take $C^m_{\text{per}}(\overline{D})$ to be the space of m-times continuously differentiable functions which are L-periodic in x.

Our main result is the following.

Theorem 2.1.1. *Fix a wave speed $c > 0$, wavelength $L > 0$, and a relative mass flux $p_0 < 0$. Fix any $\alpha \in (0,1)$, and let the functions $\beta \in C^{1+\alpha}([0,|p_0|])$ and $\rho \in C^{1+\alpha}([p_0,0])$ be given such that the* (L-B) *condition holds (see Definition* 2.3.1 *for a precise statement). Also we assume the streamline density function ρ is nonincreasing.*

Consider traveling solutions to the stratified water wave problem (2.1.1)–(2.1.6) *of speed c, relative pseudomass flux p_0, Bernoulli function β, and streamline density function ρ such that $u < c$ throughout the fluid. There exists a connected set $\mathcal{C}$ of solutions (u,v,ρ,η) in the space $C^{2+\alpha}_{per}(\overline{D_\eta}) \times C^{2+\alpha}_{per}(\overline{D_\eta}) \times C^{2+\alpha}_{per}(\overline{D_\eta}) \times C^{3+\alpha}_{per}(\mathbb{R})$ with the following properties.*

(i) *$\mathcal{C}$ contains a laminar flow (with a flat surface $\eta \equiv 0$ and all streamlines parallel to the bed).*

(ii) *Along some sequence $(u_n, v_n, \rho_n, \eta_n) \in \mathcal{C}$, either $\max_{\overline{D_{\eta_n}}} u_n \uparrow c$, $\min_{\overline{D_{\eta_n}}} u_n \downarrow -\infty$, or $\mathcal{C}$ contains more than one distinct laminar solution.*

Furthermore, each nonlaminar flow $(u,v,\rho,\eta) \in \mathcal{C}$ is regular in the sense that

(iii) *u, v, ρ, and η each have period L in x;*

- *within each period the wave profile η has a single crest and trough; say, the crest occurs at $x = 0$;*
- *u, ρ, and η are symmetric, v antisymmetric across the line $x = 0$;*

(iv) *a water particle located at (x,y), with $0 < x < \frac{L}{2}$ and $y > -d$ has positive vertical velocity $v > 0$; and*

(v) $\eta'(x) < 0$ *on* $(0, \frac{L}{2})$.

A few remarks. The (L-B) condition above arises in the local theory as a means of guaranteeing the existence of a minimizer to a particular Sturm–Liouville problem. In Lemma 2.3.3 we will show that a sufficient condition for (L-B) to hold is the following:

$$\begin{aligned} g\rho(0)p_0^2 \quad > \quad & \int_{p_0}^0 \left\{ \frac{4\pi^2}{L^2}(2B(p) - 2B_{\min} + 2\epsilon_0)^{3/2} \right. \\ & \left. + (p-p_0)^2((2B(p) - 2B_{\min} + 2\epsilon_0)^{1/2} + g\rho'(p)) \right\} dp, \end{aligned} \tag{2.1.13}$$

where

$$\epsilon_0 := \max\left\{ \left(2g\|\rho'\|_\infty p_0^2 e^{|p_0|}\right)^{2/3}, \ \left(2g\|\rho'\|_\infty\right)^2, \ \left(4\|\rho'\|_\infty\right)^2, \ (8g|p_0|\rho(0))^{2/3} \right\}. \tag{2.1.14}$$

Though this is far from necessary, it has the advantage of being entirely explicit.

In order to better situate this result in the larger context of geophysical fluid dynamics, let us give a quick comment on length-scales. As is evident from Theorem 2.1.1, our method allows us to treat mathematically waves with arbitrary speed and period. From a physical standpoint, however, we caution that our model ceases to be valid in certain scales. First, to ensure that the flow is incompressible, the Mach number must be far less than one. Also, in order to justify working within the inviscid regime, we must assume that the speed of the wave is substantially faster than the time scale for diffusion in the fluid (a situation typical of salinity in ocean wave; cf. [**4**] for a more careful derivation). Finally, to safely neglect Coriolis effects in (2.1.3), we need the rate of ambient rotation Ω to be of larger order than the ratio c/L.

Finally, we note that stagnation will occur along some sequence in $\mathcal{C}$ unless serious pathologies occur. We shall prove in section 2.7 that if there is some $\{u_n, v_n, \eta_n\}$ in $\mathcal{C}$, with

$$\liminf_{n\to\infty} \underline{\min}_{D_{\eta_n}} u_n \downarrow -\infty, \qquad \liminf_{n\to\infty} \underline{\min}_{D_{\eta_n}} (c - u_n) > 0,$$

then $\eta_n \to 0$ uniformly. That is, the fluid domain is pinching off to nothing, while a current of arbitrarily fast leftward-moving particles develops.

Now let us discuss the history of this problem. Physically, one of the most distinctive features of stratified flows is their ability to exhibit so-called internal waves. These are flows in which the motion is essentially driven by a density gradient in the fluid rather than gravitational force. Qualitatively, this can result in some highly counterintuitive behavior.

For example, in the ocean one sometimes encounters "dead water," a phenomenon where a stratified wave is moving quite rapidly near the floor, but relatively slowly near the surface. A ship encountering such a wave would observe quiescent water but experience a large amount of drag (cf., e.g., [**19**] for a lengthy discussion). Given such phenomena, previous mathematical investigations of heterogeneous fluids have largely focused on the study of these internal waves, where the effects of density variation are most pronounced. The typical setup is to consider a two-dimension stratified fluid (or several layered homogeneous fluids) confined between two impermeable horizontal boundaries. Some substantial efforts in this genre include [**3, 4, 40, 68**], among many others.

The free surface problem has mainly been studied in the context of solitary waves, that is, waves that limit to a constant height at $\pm\infty$. A scalar governing equation for the pseudostream function was developed by Long [**45**] in 1953 and later improved by Yih (cf. [**82**] and the references therein) that greatly aided these efforts. The standard approach, for solitary waves, became to take the Long–Yih equation; assuming a given upstream density profile, one then works downstream to deduce properties of the wave. This is done, for example, in [**7, 58, 42**], and [**69**].

On the other hand, traveling periodic stratified waves have not received nearly as much treatment in the literature. The first substantial results are due to Dubreil-Jacotin in 1937 [**26**], generalizing her work on the homogeneous case in [**25**]. Remarkably, she had, even at that early date, already derived Long's equation (in fact, Long's equation is referred to as the Dubreil-Jacotin–Long equation in some circles). Dubreil-Jacotin was able to analyze the fixed boundary problem, i.e., existence of traveling periodic stratified waves between two horizontal plates. Linearizing around these solutions, she gave a system of integral equations governing small amplitude, traveling stratified gravity waves, from which she was able to produce a class of solutions of that type under certain assumptions. Yanowitch [**78**] later obtained similar results by using a variational argument with an alternate governing equation due to Love, again assuming small amplitude. In 1953, Ter-Krikorov [**65**], by means of an asymptotic argument and Yih's equation, proved the existence of long, stationary stratified waves. This was in answer to a large body of results on the long wave problem that assumed the flow was homogeneous and potential.

In this thesis we take a new approach rooted in the work of Constantin and Strauss on the homogeneous case (cf. [**15, 16**] and others). In 2004, these authors were able to prove the existence of a large class of rotational (homogeneous) gravity waves that were regular in the sense we discussed earlier. This was done under very weak assumptions relating the volumetric mass flux to the strength of the vorticity [**15**]. By generalizing the methods of Constantin and Strauss to the heterogeneous case, we shall inherit many of the benefits of that paper. For instance, there will be no mathematical restrictions placed upon the wave speed or wavelength. Moreover, as the hypotheses of Theorem 2.1.1 make clear, we need only some inequality between the density variation, specific energy, and volumetric mass flux in order to conclude existence of a global continuum of solutions.

To emphasize the effects of heterogeneity, we hew closely to the organizational structure of [**15**]. We begin, in section 2.2, by using ψ to change variables and thereby fix the domain. We proceed to derive a scalar, nonlinear boundary value problem that describes the height above the flat bed in the new coordinates. As a consequence of (2.1.9), the reformulated problem takes the form of a quasi-linear elliptic differential operator added to an integro-differential operator. It is this second term that shall require the utmost care; we will show it completely accounts for the effects of the stratification. Given the long history of integral equations in the study of heterogeneous waves, perhaps the presence of the integro-differential operator here can be viewed as natural. We emphasize, though, that one of the strengths of our method is that the governing equation is still "more-or-less" elliptic (in a sense that will become clear later) and thus, substantially easier to treat.

In section 2.3, we prove the existence of a 1-parameter family of laminar flows. For these solutions, the height equation reduces to an ordinary differential equation. However, because of the stratification term, this takes the form of a nonlinear boundary value problem with operator coefficients. Analyzing the linearized problem along the curve of laminar flows, we use a result of Crandall and Rabinowitz to prove that there exists a simple generalized eigenvalue from which bifurcates a local curve of small amplitude solutions. As one might expect given the volume of work on small amplitude stratified waves, this proves substantially more difficult than the corresponding results for the homogeneous case in [**15**].

The analysis of section 2.4 continues the local curve to a global bifurcation curve using a degree theoretic argument. We are thereby able to seamlessly treat both small and large amplitude solutions. However, this require strong a priori estimates in order to guarantee that the operator has the necessary compactness for it to be admissible in the sense of degree theory. This will be achieved by first "freezing" the stratification term, then applying Schauder theory to the resulting uniformly elliptic operator. Finally, we estimate the frozen operator in order to return to the full problem. The main result of this section is a global bifurcation theorem in the form of an alternative, following in the footsteps of Rabinowitz [**60**].

We devote section 2.5 to investigating the nodal properties along the global curve. It is shown that $\mathcal{C}$ can only return to the curve of laminar flows in a certain interval of parameter values.

In section 2.6 we provide uniform bounds in the $C^{3+\alpha}$-norm along the global curve. These, with the results of section 2.5, allow us to prove that Theorem 2.1.1 follows from the global bifurcation theorem. This is done in section 2.7.

2.2. Reformulation of the problem

The main goal of this section is to reformulate problem (2.1.7)–(2.1.8) so that the fluid domain $\overline{D_\eta}$ is transformed into a fixed domain, whose closure we shall denote $\overline{R}$. As usual for two-dimensional incompressible flow, the main tool here will be the pseudostream function ψ introduced in the previous section. Recall that we have $\psi \equiv 0$ on the free surface, $\psi \equiv -p_0$ on the flat bed, where p_0 is the relative pseudomass flux and ψ is defined uniquely by the requirement that

$$\psi_x = -\sqrt{\rho}v, \qquad \psi_y = \sqrt{\rho}(u-c). \tag{2.2.1}$$

In light of (2.2.1) and (2.1.1)–(2.1.2), the governing equations inside the fluid become

$$\begin{cases} \psi_y\psi_{xy} - \psi_x\psi_{yy} = -P_x \\ -\psi_y\psi_{xx} + \psi_x\psi_{xy} = -P_y - \rho g \end{cases} \qquad \text{in } \overline{D_\eta}, \tag{2.2.2}$$

whereas, the boundary conditions (2.1.8) are

$$\begin{cases} \psi_x = -\psi_y\eta_x & \text{on } y = \eta(x), \\ P = P_{\text{atm}} & \text{on } y = \eta(x), \\ \psi_x = 0 & \text{on } y = -d. \end{cases} \tag{2.2.3}$$

Recall we have that the quantity

$$E = P + \frac{\rho}{2}\left((u-c)^2 + v^2\right) + g\rho y \tag{2.2.4}$$

is constant along streamlines. In particular then, evaluating the above equation on the free surface $\psi \equiv 0$, we find

$$|\nabla\psi|^2 + 2g\rho(x,\eta(x))\,(\eta(x)+d) = Q \qquad \text{on } y = \eta(x), \tag{2.2.5}$$

where the constant $Q := (2E|_\eta - P_{\text{atm}} + g\rho|_\eta d)$. Note that this Q gives roughly the energy density along the free surface of the fluid. Critically, however, we shall see that as a consequence of our normalization of η, d is not a parameter for the problem. On the contrary, in all but the most trivial cases, d varies along $\mathcal{C}$. In our analysis, therefore, we shall, instead, be viewing Q as parameterizing the continuum.

We now wish to find from (2.1.7) and (2.2.4) a scalar PDE satisfied by ψ. In the case where the fluid is homogeneous, it is easy to show that this will take the form of a semilinear equation:

$$\Delta\psi = \gamma(\psi) \tag{2.2.6}$$

for some function $\gamma : [0, |p_0|] \to \mathbb{R}$. It is not hard to show that, in this scenario, γ will describe the change in vorticity as a function of the streamlines. When one allows for density variation, however, one expects there must be an additional term accounting for the gravitational effects of the stratification.

One can prove that the corresponding relation in the heterogeneous case takes the form

$$\frac{dE}{d\psi} = \Delta\psi + gy\frac{d\rho}{d\psi}.$$

This is known as Yih's equation or the Yih–Long equation. In this work, it shall serve as our governing equation for ψ. Recall that in the previous section we introduced the Bernoulli function β and streamline density function ρ. Rewriting the above expression we arrive at an equation of roughly the same form as (2.2.6):

$$\beta(\psi) = \Delta\psi - gy\rho'(-\psi). \tag{2.2.7}$$

Were it not for the presence of y on the right-hand side, this would reduce to the homogeneous case—a fact that agrees with our intuitive notion that density stratification should reintroduce the depth into the problem as a serious consideration. We remark that comparing this to (2.2.6) we see that the final term on the right accounts precisely for the gravitational effects of stratification, the inertial effects having been captured in the choice of the ψ.

With (2.2.7) in hand, we now make a change of variables to eliminate the free boundary. The new coordinates we will denote (q, p), where

$$q = x, \qquad p = -\psi(x, y).$$

This scheme is sometimes referred to as *semi-Lagrangian* coordinates, in recognition of the fact that we are working, in some sense, halfway between streamline coordinates and the usual Lagrangian system (cf. [**69**]).

By means of a scaling argument, we may take $L := 2\pi$. Then, under the transformation

$$(x, y) \mapsto (q, p),$$

the closed fluid domain $\overline{D_\eta}$ is mapped to the rectangle

$$\overline{R} := \left\{(q, p) \in \mathbb{R}^2 : 0 \le q \le 2\pi,\ p_0 \le p \le 0\right\}.$$

The purpose of the minus sign is simply to flip the rectangle so that the free surface will correspond to the top of R, while the flat bed will mapped to the bottom. Given this, it will be convenient to put

$$T := \{(q, p) \in R : p = 0\}, \qquad B := \{(q, p) \in R : p = p_0\}.$$

Note that, in light of (2.1.11) and (2.1.12), we have that

$$\beta = \beta(-p), \qquad \rho = \rho(p).$$

Moreover, the assumption that the streamline density function is nonincreasing becomes

$$\rho_p \le 0. \tag{2.2.8}$$

Next, following the ideas of Dubreil-Jacotin (cf. [**25, 26**]), define

$$h(q, p) := y + d, \tag{2.2.9}$$

which gives the height above the flat bottom on the streamline corresponding to p and at $x = q$. We calculate

$$\psi_y = -\frac{1}{h_p}, \qquad \psi_x = \frac{h_q}{h_p}. \tag{2.2.10}$$

Note that this implies $h_p > 0$, because we have stipulated that $u < c$ throughout the fluid. The change of variables then gives

$$\begin{aligned} \partial_x &= \partial_q - \tfrac{h_q}{h_p}\partial_p, \\ \partial_y &= h_p^{-1}\partial_p, \\ \partial_p &= h_p\partial_y, \\ \partial_q &= \partial_x + h_q\partial_y. \end{aligned} \tag{2.2.11}$$

Using these expressions we can solve for u and v in (2.2.10) to obtain

$$u = c - \frac{1}{\sqrt{\rho}h_p}, \qquad v = -\frac{h_q}{\sqrt{\rho}h_p}. \tag{2.2.12}$$

In order to formulate (2.2.7) in terms of h, we observe that

$$\begin{aligned} \Delta\psi &= \partial_x^2\psi + \partial_y^2\psi \\ &= -\partial_x(\sqrt{\rho}v) + \partial_y(\sqrt{\rho}(u-c)) \\ &= \partial_x\left(\frac{h_q}{h_p}\right) + \partial_y\left(-h_p^{-1}\right) \\ &= \left(\frac{-h_q}{h_p}\right)\partial_p\left(\frac{h_q}{h_p}\right) + \partial_q\left(\frac{h_q}{h_p}\right) + h_p^{-1}\partial_p\left(-h_p^{-1}\right) \\ &= \left(\frac{-h_q}{h_p}\right)\left(\frac{h_ph_{pq} - h_qh_{pp}}{h_p^2}\right) + \frac{h_ph_{qq} - h_qh_{pq}}{h_p^2} + \frac{h_{pp}}{h_p^3}. \end{aligned}$$

Hence, Yih's equation (2.2.7) becomes the following:

$$-h_p^3\beta(-p) = \left(1 + h_q^2\right)h_{pp} + h_{qq}h_p^2 - 2h_qh_ph_{pq} - g(h-d)h_p^3\rho_p, \tag{2.2.13}$$

where we have used (2.2.9) to write $y = h - d$. Recall, however, that we have normalized η so that it has mean zero. Taking the mean of (2.2.9) along T, we obtain

$$d = d(h) = \fint_0^{2\pi} h(q,0)dq. \tag{2.2.14}$$

That is, the average depth d must be viewed as a linear operator acting on h. Namely, it is the average value of h over T. Where there is no risk of confusion, we shall suppress this dependency and simply write d. As we shall see, the addition of this integral term to the governing equations will be the single most significant departure from the homogeneous case, both technically and qualitatively. In recognition of this fact, we shall refer to the mapping $h \mapsto -g(h-d(h))h_p^3\rho_p$ as the *stratification operator*. Equation (2.2.13) will then take the form of a quasi-linear elliptic operator added to the stratification operator.

Next consider the boundaries of the transformed domain. On the bed we must have by the definition of h that

$$h \equiv 0 \qquad \text{on } B. \tag{2.2.15}$$

If we were to reformulate the problem in terms of y not h, then the integral term would disappear in (2.2.13), but the boundary condition on B would become $y \equiv -d$. We see then that, in the presence of density variation, the depth cannot simply be eliminated from the problem.

Throughout the fluid we have by (2.2.1) and (2.2.12):

$$\psi_x^2 = h_q^2 h_p^{-2}, \qquad \psi_y^2 = h_p^{-2}.$$

Given this, the definition of Q given in (2.2.5) becomes the requirement

$$1 + h_q^2 + h_p^2\left(2g\rho h - Q\right) = 0 \qquad \text{on } T. \tag{2.2.16}$$

Altogether, then, combining (2.2.13), (2.2.15), and (2.2.16) we have that the fully reformulated problem is the following. Find $(h,\ Q) \in C^{2+\alpha}_{\text{per}}(\overline{R}) \times \mathbb{R}$ satisfying

$$\begin{cases} \left(1+h_q^2\right) h_{pp} + h_{qq}h_p^2 - 2h_q h_p h_{pq} & \\ \qquad -g(h-d(h))h_p^3\rho_p = -h_p^3\beta(-p), & p_0 < p < 0, \\ 1 + h_q^2 + h_p^2(2g\rho h - Q) = 0, & p = 0, \\ h = 0, & p = p_0, \end{cases} \tag{2.2.17}$$

and $h_p > 0$. Here $\rho \in C^{1+\alpha}([p_0, 0]; \mathbb{R}^+)$ and $\beta \in C^{1+\alpha}([0, |p_0|]; \mathbb{R})$ are given function with $\rho_p \leq 0$.

We now prove the equivalence of the height equation problem to the original Euler equation formulation. Because the differential equations relating ψ and h, (2.2.2)–(2.2.3), do not include ρ, this result follows more or less from the methods of the constant density case in [**15**]. Nonetheless, for clarity we recapitulate that argument here.

Lemma 2.2.1. *Problem* (2.2.17) *is equivalent to problem* (2.1.1)–(2.1.6).

Proof. The preceding development shows that (2.1.1)–(2.1.6) implies (2.2.17) for some ρ, β, and Q. It remains only to prove the converse. Fix p_0, and let h be a solution to (2.2.17) of class $C^{2+\alpha}_{\mathrm{per}}(\overline{R})$ corresponding to a given value of Q and streamline density function ρ, Bernoulli function β, with $h_p > 0$ in $\overline{R}$.

Define the $C^{1+\alpha}_{\mathrm{per}}(\overline{R})$ functions

$$F(q,p) := \frac{1}{h_p(q,p)}, \qquad G(q,p) := -\frac{h_q(q,p)}{h_p(q,p)}. \tag{2.2.18}$$

Then, as $h_{pq} = h_{qp}$,

$$F_q + F_p G - G_p F = -h_{pq}h_p^{-2} + h_q h_{pp} h_p^{-3} + h_{pq}h_p^{-2} - h_{pp}h_q h_p^{-3} = 0 \tag{2.2.19}$$

throughout $\overline{R}$. Also, we note that the free surface of the flow is given by $\eta(x) = h(x,0) - d(h)$.

Our first task is to recover the pseudostream function ψ. Fix $x_0 \in \mathbb{R}$, and let ψ denote the solution of the ODE

$$\psi_y(x_0, y) = -F(x_0, -\psi(x_0, y)), \tag{2.2.20}$$

along with the initial condition $\psi(x_0, \eta(x_0)) = 0$. By assumption, h_p is bounded strictly away from zero on $\overline{R}$. We may let $\delta > 0$ be given such that $F \geq \delta$. Thus $\psi(x_0, \cdot)$ is increasing at a rate greater than δ as y decreases. Thus (2.2.20) is solvable until $\psi(x_0, y) = -p_0$ for some y. Then for each $x \in \mathbb{R}$, we may define $\psi(x,y)$ on some interval $[y(x), \eta(x)]$, with $y < \eta$. By uniqueness of solutions to (2.2.20) and the periodicity of h, it follows that ψ is periodic in x within its domain of definition.

We claim that $y(x) = -d(h)$ for all $x \in \mathbb{R}$. To prove this, let $H(x,y) := -G(x, -\psi(x,y))$ for $x \in \mathbb{R}$, $y \in [y(x), \eta(x)]$. Then by (2.2.19) and (2.2.20),

$$H_y(x,y) = G_p \psi_y = -G_p F = -F_q - F_p G = -F_q + F_p H.$$

On the other hand, by C^1-dependence of ψ on the parameter x_0, we may differentiate to find

$$(\psi_x)_y = (\psi_y)_x = -F_q + F_p \psi_x.$$

Thus ψ_x and H satisfy the same ODE. We have $\psi(x, \eta(x)) = 0$ by definition. Differentiating this relation yields

$$\psi_x = -\psi_y \eta' = F(x, -\psi(x, \eta(x)))\eta'.$$

Likewise, by definition of η we have

$$\eta'(x) = h_q,$$

and thus

$$H(x,\eta(x)) = -G(x,-\psi(x,\eta(x))) = F(x,-\psi(x,\eta(x)))\eta'(x)$$

by (2.2.18). We have shown that H and ψ_x have identical initial data at $(x,\eta(x))$; hence by uniqueness we conclude

$$\psi_x(x,y) = H(x,y) = -G(x,-\psi(x,y)) \qquad \text{for all } x \in \mathbb{R},\ y \in [y(x),\eta(x)]. \tag{2.2.21}$$

Now, C^1-dependence of y on x enables us to differentiate the relation $\psi(x,y(x)) = -p_0$,

$$\psi_x(x,y(x)) + \psi_y(x,y(x))\frac{dy}{dx} = 0 \qquad \text{for all } x \in \mathbb{R}.$$

By (2.2.20) and (2.2.21) this implies

$$G(x,-p_0) - F(x,-p_0)\frac{dy}{dx} = 0 \qquad \text{for all } x \in \mathbb{R}.$$

But as we have seen, $h_q = \eta'$, so that $h_q(q,p_0) = 0$ and therefore $G(\cdot,-p_0) \equiv 0$. Also, we have that $F \geq \delta > 0$, so we may conclude that $y(x) \equiv y_0$, a constant.

Finally, we must show that $y_0 = p_0$. To do so we observe that $h(0,p_0) = 0$, $h(0,0) = \eta(0) + d(h)$. Then by (2.2.20) evaluated at $x_0 = 0$, we have

$$\begin{aligned}
\eta(0) + d = h(0,0) = \int_{p_0}^{0} h_p(0,p)dp &= \int_{p_0}^{0} \frac{dp}{F(0,p)} \\
&= \int_{y_0}^{\eta(0)} \frac{-d\psi(0,y)}{F(0,-\psi(0,y))} \\
&= \int_{y_0}^{\eta(0)} dy \\
&= \eta(0) - y_0.
\end{aligned}$$

Thus $y_0 = -d$.

It remains now to show that ψ constructed above constitutes a pseudostream function for a solution to (2.1.1)–(2.1.6). We have already proved that $\psi = p_0$ on $y = -d$, and $\psi = 0$ on $y = \eta(x)$. Also,

$$F^2 + G^2 = \psi_y^2 + \psi_x^2 = |\nabla\psi|^2 \qquad \text{in } \overline{R} \tag{2.2.22}$$

by (2.2.18) and (2.2.21). On the other hand, from the boundary condition at $p = 0$ in (2.2.16), we find $F^2 + G^2 = Q - 2g\rho h$ on the free surface. But as $q = x$ and $h = y + d$, we may combine this with (2.2.22) to conclude that $|\nabla\psi|^2 + 2g\rho(y + d) = 0$ on $y = \eta(x)$.

Finally, differentiating (2.2.20) and (2.2.21) and summing the two yields

$$\begin{aligned}
\Delta\psi &= F_p\frac{d\psi}{dy} + G_p\frac{d\psi}{dx} - G_q \\
&= -F_pF - G_pG - G_q \\
&= h_{pp}h_p^{-3} + h_{pp}h_q^2h_p^{-3} - h_{pq}h_qh_p^{-2} + h_{qq}h_p^{-1} - h_qh_{pq}h_p^{-2}.
\end{aligned}$$

In light of (2.3), this becomes

$$\Delta\psi = -\beta(\psi) + gy\rho'(-\psi) \qquad \text{in } \overline{R}. \tag{2.2.23}$$

We now define u and v through (2.2.12), and, by abuse of notation, take $\rho(x,y) = \rho(-\psi(x,y))$. Then, recalling the moving frame transformation and the definition of the pseudostream function, we have by (2.2.23) and the arguments of the preceding paragraph that $(u,v,\rho,\eta) \in C^{1+\alpha}_{\text{per}}(\overline{D_\eta}) \times C^{1+\alpha}_{\text{per}}(\overline{D_\eta}) \times C^{1+\alpha}_{\text{per}}(\overline{D_\eta}) \times C^{2+\alpha}_{\text{per}}(\overline{D_\eta})$ solves the original problem (2.1.1)–(2.1.6). ☐

2.3. Local bifurcation

The goal of this section is to prove the existence of a local curve of small-amplitude solutions to (2.1.1)–(2.1.6). The main product of our efforts will be the following theorem.

THEOREM 2.3.1 (local bifurcation). *Let $c > 0$, $p_0 < 0$, $\alpha \in (0,1)$, and $C^{1+\alpha}$ functions β and ρ defined on $[0, |p_0|]$, $[p_0, 0]$, respectively, be given. Consider the traveling solutions of speed c with relative mass flux p_0 of the water wave problem* (2.1.1)–(2.1.6) *with Bernoulli function β and streamline density function ρ such that $u < c$ throughout the fluid. If β and ρ satisfy* (L-B) *(see Definition* 2.3.1*) and $\rho_p \leq 0$, then there exists a C^1 curve $\mathcal{C}_{loc}$ of small-amplitude solutions $(u,v,\rho,\eta) \in C^{2+\alpha}_{per}(\overline{D_\eta}) \times C^{2+\alpha}_{per}(\overline{D_\eta}) \times C^{1+\alpha}_{per}(\overline{D_\eta}) \times C^{3+\alpha}_{per}(\overline{D_\eta})$. The solution curve $\mathcal{C}_{loc}$ contains precisely one laminar flow (with $\eta \equiv 0$).*

2.3.1. Overview. Following the ideas set forth in the previous section, we shall work with the equivalent height equation formulation (2.2.17). First we shall prove the existence of a 1-parameter family $\mathcal{T}$ of laminar solutions. We then show the existence of a local curve of nonlaminar solutions bifurcating from a simple eigenvalue of the linearized problem along

$\mathcal{T}$. The result will then follow from an application of the local bifurcation theory of Crandall and Rabinowitz.

In several ways, it will be this section where the stratification term will be problematic. This will be immediately apparent in the next lemma, where its presence will make unattainable an explicit formula for the laminar solutions. The consequences of this, at least from a technical standpoint, will cascade through the subsequent development, as we will be forced to make perturbation arguments along a curve of solutions we can only hope to represent implicitly. Though these concerns are purely mathematical in nature, they are not without physical analogue. Indeed, as we remarked earlier, one of the striking features of stratified flows is their propensity to exhibit significant internal waves—even in a small amplitude regime.

2.3.2. Laminar flow. Consider laminar flow solutions to the height equation (2.2.17). By laminar we mean parallel shear flows where the free surface is flat. Any such solution must then take the form $H = H(p)$, with $\eta \equiv 0$. The height equation then reduces to an ODE with operator coefficients:

$$H_{pp} - g(H - d(H))H_p^3\rho_p = -H_p^3\beta(-p), \qquad p_0 < p < 0 \tag{2.3.1}$$

along with with the boundary conditions

$$H = 0 \qquad \text{on } p = p_0 \tag{2.3.2}$$

and

$$1 + H_p^2(2g\rho H - Q) = 0 \qquad \text{on } p = 0. \tag{2.3.3}$$

Owing mainly to the presence of the stratification term, (2.3.1)–(2.3.2) constitutes a nonlinear, nonautonomous ODE boundary value problem which we cannot solve explicitly. Our approach is motivated by the observation that if we can replace β and ρ with functions of y, the resulting autonomous ODE becomes tractable. Indeed, any solution to the height equation should satisfy $H_p > 0$; hence the vertical variable y is a strictly increasing function of p, and so inverting the two is a valid change of variables. This is an adaptation of a technique commonly seen in the analysis of solitary waves in a channel, where it can be convenient to invert p and y on the profile at $\pm\infty$ (cf. [**4, 42, 68**] and many others). As in the first section, define

$$B(p) := \int_0^p \beta(-s)ds, \qquad B_{\min} := \min_{p\in[p_0,0]} B(p) \leq 0.$$

LEMMA 2.3.1 (laminar flow). *Suppose that the streamline density function ρ satisfies* (2.2.8)*. Then there exists a* 1*-parameter family of solutions $H(\cdot;\lambda)$ to the laminar flow equation* (2.3.1)–(2.3.3) *with $H_p > 0$, where $0 \leq -2B_{\min} < \lambda < Q$.*

PROOF. In accordance with (2.2.9), let $Y := H - d$ denote the vertical variable. By specifying the free surface $\eta \equiv 0$, we force $Y(0) = 0$. Thus $H(0) = d$, and so by (2.3.3),

$$1 + Y_p^2(2g\rho d - Q) = 0 \qquad \text{on } p = 0. \tag{2.3.4}$$

It therefore suffices to find Y satisfying

$$Y_{pp} + (\beta(-p) - gY\rho_p)\, Y_p^3 = 0, \qquad p_0 < p < 0 \tag{2.3.5}$$

along with the boundary conditions

$$Y = 0 \qquad \text{on } p = 0, \tag{2.3.6}$$

$$Y = -d \qquad \text{on } p = p_0, \tag{2.3.7}$$

where d (viewed as a positive real number) and Y additionally satisfy (2.3.4) for some choice of Q.

We now change variables. Put $s := Y(p)$. Then from (2.3.5) we calculate

$$\begin{aligned} Y_{pp}Y_p^{-3} &= \left(-\frac{dB}{ds} + gY\frac{d\rho}{ds}\right) Y_p \\ &= \left(-\frac{dB}{ds} + gY\frac{d\rho}{ds}\right) \frac{ds}{dp}. \end{aligned}$$

Therefore we may rewrite (2.3.5) as

$$\left(-\frac{1}{2}Y_p^{-2}\right)_p + (F(Y))_p = 0, \qquad p_0 < p < 0, \tag{2.3.8}$$

where

$$F'(Y) = \left(\frac{dB}{ds} - gY\frac{d\rho}{ds}\right)\bigg|_{s=Y} = \left(\frac{dB}{ds} - gY\frac{d\rho}{dp}\right)\bigg|_{Y=s},$$

or equivalently,

$$\frac{dF}{ds} = \left(\frac{dB}{dp} - gs\frac{d\rho}{ds}\right)\frac{dp}{ds}. \tag{2.3.9}$$

For definiteness, we take $F(0) = 0$, so that integrating we find

$$F(Y) = B(p) + \int_Y^0 gr\frac{d\rho}{ds}(r)dr \tag{2.3.10}$$

and

$$F(Y) = B(p) + \int_p^0 gY(r)\frac{d\rho}{dp}(r)dr. \tag{2.3.11}$$

Note that by (2.2.8), the integrand on the right-hand side is nonnegative; hence $F > B_{\min}$.

Returning to (2.3.8) we integrate once to obtain

$$Y_p = \frac{1}{\sqrt{\lambda + 2F(Y)}}, \tag{2.3.12}$$

where λ is a constant of integration, with $\lambda > -2B_{\min}$. Note that, by (2.3.10), F is actually an unknown. Using the fact that $F(0) = 0$, (2.3.12) and (2.3.4) together imply

$$d = \frac{Q - \lambda}{2g\rho(0)}. \tag{2.3.13}$$

Inverting (2.3.12) and combining it with (2.3.9) we arrive at a first order system for p and F:

$$\begin{cases} \frac{dp}{ds} = \sqrt{\lambda + 2F(s)}, \\ \frac{dF}{ds} = (B'(p) - gs\rho'(p))\sqrt{\lambda + 2F(s)}, \end{cases} \tag{2.3.14}$$

where by abuse of notation ρ and B here have been extended continuously so that their domain encompasses the entire real line. We have that both Y and p vanish when $s = 0$. Locally, then, we can solve this initial value problem. Let $(F, p) = (F(s;\lambda), p(s;\lambda))$ be such a solution for a fixed choice of $\lambda > -2B_{\min}$, and let the maximum domain of definition be $(s_-(\lambda), s_+(\lambda))$. We must show that p attains the value p_0 somewhere in $(s_-, 0)$.

Elementary theory of ordinary differential equations tells us that, if the domain of definition is bounded from below, then the solution must blow-up as we approach the boundary. More precisely, we must have $|F|^2 + |p|^2 \to \infty$ as $s \to s_-$, provided that $s_- > -\infty$. First consider the F equation in (2.3.14). Rearranging terms and integrating in s yields

$$F = \left(\int_0^s (B'(p) - gr\rho'(p))\, dr\right)^2 - \lambda,$$

which implies

$$|F(s;\lambda)| \leq |\lambda| + \left(\|\beta\|_\infty |s| + gs^2\|\rho'\|_\infty\right)^2.$$

It follows that F is bounded for finite s. Next, recall that we have chosen $\lambda > -2B_{\min}$; whence,

$$\lambda + 2F > \lambda + 2B_{\min} > 0.$$

Putting $\epsilon(\lambda) := \lambda + 2B_{\min}$, by (2.3.14) we have

$$\frac{dp}{ds} \geq \sqrt{\epsilon} > 0. \tag{2.3.15}$$

But this uniform lower bound on $\frac{dp}{ds}$ implies that if $s_- = -\infty$, then $p \to -\infty$ as $s \to s_-$. On the other hand, if $s_- > -\infty$, then the arguments of the previous paragraph guarantee that $|F|$ is bounded on $(s_-, 0]$, so again we conclude $p \to -\infty$ as $s \to s_-$. In either event, we may choose $d = d(\lambda) > 0$ to be the unique number satisfying $p(-d) = p_0$. Inverting p (which is valid by (2.3.15)) we get a function $Y = Y(p; s)$ satisfying (2.3.5)–(2.3.7). Finally, let $Q = Q(\lambda)$ be defined by the relation (2.3.13):

$$Q(\lambda) = \lambda + 2g\rho(0)d(\lambda). \tag{2.3.16}$$

Thus for each $\lambda > -2B_{\min}$, $H(p;\lambda) := Y(p;\lambda) + d(\lambda)$ is a solution the laminar flow equations with $Q = Q(\lambda)$. $\square$

2.3.3. Eigenvalue problem. Up to this point we have produced a family $\mathcal{T}$ of laminar solutions to the height equation parametrized by the variable λ, which is drawn from a suitable range determined a priori by the given function β and the value of p_0. We will now linearize the full height equation around H for fixed λ by evaluating the Fréchet derivative. Ultimately we show that the linearized problem has a simple eigenvalue at some λ^*.

Fix $\epsilon > 0$, and let $h(q,p) = H(p) + \epsilon m(q,p)$. Then from (2.2.17)

$$\begin{aligned}\left(1+\epsilon^2 m_q^2\right)(H_{pp} + \epsilon m_{pp}) + \epsilon m_{qq}(H_p + \epsilon m_p)^2 - 2\epsilon^2 m_q m_{qp}(H_p + \epsilon m_p)& \\ - g(H + \epsilon m - \ d(H) - \epsilon d(m))(H_p + \epsilon m_p)^3 \rho_p& \\ = -(H_p + \epsilon m_p)^3 \beta(-p)&\end{aligned}$$

for $p_0 < p < 0$. Differentiating in ϵ and evaluating at $\epsilon = 0$ yields the following linearized equation:

$$m_{pp} + m_{qq}H_p^2 - g\rho_p\left(3m_p(H - d(H))H_p^2 + (m - d(m))H_p^3\right) = -3H_p^2 m_p \beta(-p) \tag{2.3.17}$$

for $p_0 < p < 0$.

Applying the same procedure to the two boundary conditions we find

$$2g\rho m H_p^2 + 2(2g\rho H - Q)m_p H_p = 0 \qquad \text{on } p = 0$$

and

$$m = 0 \qquad \text{on } p = p_0.$$

We may simplify the nonlinear condition on T by noting that for all λ,

$$H(0) = \frac{Q-\lambda}{2g\rho(0)} \Longrightarrow 2g\rho(0)H(0) - Q = -\lambda. \tag{2.3.18}$$

Also, from (2.2.10),

$$H_p^2 = Y_p^2 = \frac{1}{\lambda + 2F(Y)} = \lambda^{-1} \qquad \text{on } p = 0. \tag{2.3.19}$$

Combining these we find that the the linearized problem is the following: for fixed $\lambda > -2B_{\min}$, find m that is 2π-periodic in q satisfying

$$\begin{cases} m_{pp} + m_{qq}H_p^2 - g\rho_p\big(3m_p(H - d(H))H_p^2 + (m - d(m))H_p^3\big) & \\ \qquad = -\, 3H_p^2 m_p \beta(-p), & p_0 < p < 0, \\ m = 0, & p = p_0, \\ g\rho m = \lambda^{3/2} m_p, & p = 0. \end{cases} \tag{2.3.20}$$

Observe that the only effect of the variable density on the boundary is in the addition of the constant $\rho(0)$ on $p = 0$. This similarity with the constant density case will allow many of the same calculations to push through virtually unaltered. On the other hand, the stratification term in (2.3.17) will present a significant technical barrier as it introduces both the nonlocal operator d and a zeroth order term.

From this section onward we shall consider only λ in the range $\lambda \geq -2B_{\min} + \epsilon_0$, where ϵ_0 is as in (2.1.14). As will be made clear later, the purpose of slightly decreasing our range of admissible λ is essentially to ensure that H_p is bounded uniformly away from zero for λ small.

In order for bifurcation to occur, some control over the relative sizes of ρ, B, β, and p_0 is necessary—even in the homogeneous case. With this in mind, we make the following definition.

Definition 2.3.1. We say the pseudovolumetric mass flux p_0, streamline density function ρ, and Bernoulli function β collectively satisfy the local bifurcation condition provided

$$\text{(L-B)}\qquad \inf_{\lambda\geq -2B_{\min}+\epsilon_0}\ \inf_{\phi\in\mathscr{S}} \left\{ \frac{-g\rho(0)\phi(0)^2 + \int_{p_0}^0 H_p^{-3}(p;\lambda)\phi_p(p)^2\,dp}{\int_{p_0}^0 \phi^2(p)\left(H_p^{-1}(p;\lambda)+g\rho_p(p)\right)dp} \right\} < -1,$$

where $\mathscr{S} := \{\phi \in H^1((p_0,0)) : \phi \not\equiv 0,\ \phi(p_0) = 0\}$ and $H(\cdot;\lambda)$ is the solution to (2.3.1)–(2.3.3) given by Lemma 2.3.1.

We shall restrict our attention to case where (L-B) is satisfied. The next lemma gives the precise motivation for this choice.

Lemma 2.3.2 (eigenvalue problem). *Assuming* (2.2.8) *and* (L-B), *there exists* $\lambda^* > -2B_{\min} + \epsilon_0$ *and a solution* $m(q,p) \not\equiv 0$ *of* (2.3.20) *that is even and* 2π*-periodic in* q.

Proof. We seek special solutions of the form $m(q,p) = M(p)\cos(kq)$ for some $k \in \mathbb{Z}^\times$, as we require 2π-periodicity in q. Note that this implies

$$d(m) = M(0)\fint_0^{2\pi} \cos(kq)\,dq = 0.$$

Also, since solutions of this form will necessarily satisfy the ODE

$$m_{qq} = -k^2 m,$$

we may rewrite the interior equation of (2.3.20) in self-adjoint form:

$$\left\{H_p^{-3}m_p\right\}_p + \left\{\left(H_p^{-1} + g\rho_p k^{-2}\right)m_q\right\}_q = 0.$$

In turn, this requires that M satisfy

$$\left\{H_p^{-3}M_p\right\}_p = \left(k^2H_p^{-1} + g\rho_p\right)M, \qquad p_0 < p < 0. \tag{2.3.21}$$

Any value of k will suffice, but for simplicity, we focus on finding a solution where $k = 1$. To do so we approach (2.3.21) as a Sturm–Liouville problem. Note that, as a consequence of (2.1.14), the term in parenthesis above is nonnegative for $k = 1$.

In that connection we consider the minimization problem

$$\mu = \mu(\lambda) = \inf_{\phi\in\mathscr{S}} \mathcal{R}(\phi;\lambda), \tag{2.3.22}$$

where the Rayleigh quotient $\mathcal{R}$ is

$$\mathcal{R}(\phi;\lambda) := \frac{-g\rho(0)\phi(0)^2 + \int_{p_0}^0 H_p(p;\lambda)^{-3}\phi_p(p)^2\,dp}{\int_{p_0}^0 \phi(p)^2\left(H_p(p;\lambda)^{-1} + g\rho_p(p)\right)dp} \tag{2.3.23}$$

and $\mathscr{S} := \{\phi \in H^1((p_0, 0)) : \phi \not\equiv 0,\ \phi(p_0) = 0\}$. For each λ, the function M that attains the minimum $\mu(\lambda)$ will satisfy the boundary conditions of the linearized problem (2.3.20) as well as the ODE:

$$\left\{H_p^{-3} M_p\right\}_p = -\mu(\lambda)\left(H_p^{-1} + g\rho_p\right) M.$$

The task is to find $\lambda^* > -2B_{\min} + \epsilon_0$ such that $\mu(\lambda^*) = -1$. The corresponding M will be precisely the special solution we seek.

By the continuity of μ and in light of the (L-B), it suffices to prove that $\mu(\lambda) \geq -1$ for λ sufficiently large. By (2.2.8) we have that $\rho_p \leq 0$, and we found in the previous section that this induces a uniform lower bound on $F(\cdot;\lambda)$. That is, for any $\lambda > -2B_{\min} + \epsilon_0$,

$$\min_p F(Y(p;\lambda);\lambda) \geq B_{\min}.$$

We may therefore choose λ such that

$$\lambda > g\rho(0) + g\|\rho_p\|_\infty^2 + g^{3/2}\sqrt{\rho(0)}\|\rho_p\|_\infty - 2B_{\min}$$

and conclude

$$H_p^{-1} = (\lambda + 2F(Y(p)))^{1/2} \geq (\lambda + 2B_{\min})^{1/2} \geq g\|\rho_p\|_\infty + \sqrt{g\rho(0)}.$$

Then, again using the fact that $\rho_p \leq 0$, we obtain

$$H_p^{-1} + g\rho_p \geq \sqrt{g\rho(0)}.$$

Now let $w \in \mathscr{S}$ be given and fix a $\lambda > -2B_{\min} + \epsilon_0$. Then

$$\begin{aligned}
\int_{p_0}^0 \left(H_p^{-3} w_p^2 + \left(H_p^{-1} + g\rho_p\right) w^2\right) dp \quad &\geq \quad \sqrt{g\rho(0)} \int_{p_0}^0 \left(w^2 + g\rho(0) w_p^2\right) dp \\
&\geq \quad 2g\rho(0) \int_{p_0}^0 w w_p dp \\
&= \quad g\rho(0) w(0)^2.
\end{aligned}$$

Thus $\mathcal{R}(w) \geq -1$. As this holds for arbitrary $w \in \mathscr{S}$ and all admissible λ, we conclude $\mu(\lambda) \geq -1$. □

As we indicated in the first section, there is an explicit condition (2.1.13) on p_0, ρ, and β that implies (L-B). Moreover, for any choice of β, ρ, taking p_0 sufficiently small (and restricting β and ρ to the decreased domain), will be enough to guarantee that this size condition will hold. Though the size condition is not necessary, it is still general enough to allow for a great variety of flows.

LEMMA 2.3.3 (sufficiency of size condition). *If p_0, ρ, and β satisfy the size condition* (2.1.13), *then they satisfy* (L-B).

PROOF. We must show that for some $\lambda \geq -2B_{\min} + \epsilon_0$, we have $\mu(\lambda) < -1$ in the sense of (2.3.22). In (2.3.15) we showed that $\frac{dp}{ds}$ was bounded below uniformly in p. Therefore,

$$-p_0 = -p(-d) = \int_{-d}^{0} \frac{dp}{ds} ds \geq \sqrt{\epsilon_0} d.$$

If $\epsilon_0 = 0$, then we are in the constant density case, and this lemma has already been proved in [**15**]. Otherwise,

$$d \leq -\frac{p_0}{\sqrt{\epsilon_0}}. \tag{2.3.24}$$

As $Y_p = H_p$, we see by (2.3.12)

$$\begin{aligned} H_p^{-1}(p;\lambda) &= \left(\lambda + 2B(p) + 2\int_p^0 g\rho_p(r) Y_\lambda(r) dr\right)^{1/2} \\ &\leq (\lambda + 2B(p) + 2gd(\lambda)|p_0|\|\rho_p\|_\infty)^{1/2} \\ &\leq (\lambda + 2B(p) + \epsilon_0)^{1/2}, \end{aligned}$$

where the last line comes from (2.3.24) and the definition of ϵ_0. Pairing this with the size condition (2.1.13), we have that for λ sufficiently near $-2B_{\min} + \epsilon_0$,

$$\begin{aligned} \int_{p_0}^0 \left(H_p^{-3} + (p-p_0)^2 \left(H_p^{-1} + g\rho_p\right)\right) dp &< \int_{p_0}^0 \big((\lambda + 2B(p) + \epsilon_0)^{3/2} \\ &\qquad + (p-p_0)^2((\lambda + 2B(p) + \epsilon_0)^{1/2} + g\rho_p)\big) dp \\ &< g\rho(0)p_0^2. \end{aligned}$$

Now take $w = p - p_0 \in \mathscr{S}$, and let λ be such that the above inequality holds. Then,

$$\mathcal{R}(w;\lambda) = \frac{-g\rho(0)p_0^2 + \int_{p_0}^0 H_p^{-3} dp}{\int_{p_0}^0 (p-p_0)^2 (H_p^{-1} + g\rho_p) dp} < -1.$$

It follows that $\mu(\lambda) = \inf_{\mathscr{S}} R(\cdot;\lambda) < -1$, as desired. Hence the local bifurcation condition holds. $\square$

For convenience we denote $G(p;\lambda) := 2F(Y(p;\lambda);\lambda)$. Thus, for instance, we have $H_p(p) = (\lambda + G(p;\lambda))^{-1/2}$. The great difficulty that arises from the stratification term in (2.3.1) in large part stems from the fact that G depends on λ. Given that, before proceeding further, we set ourselves to a technical task: We must determining precisely how the $\lambda + G(\cdot;\lambda)$ term varies along $\mathcal{T}$.

For the next several proofs we will denote differentiation with respect to λ by a dot.

Lemma 2.3.4 (sign of $1+\dot{G}$). *For $\lambda \geq -2B_{\min} + \epsilon_0$, $1+\dot{G}$ is strictly positive. In fact, we have the following bound: $-1/2 \leq \dot{G} \leq 0$.*

Proof. By rewriting (2.3.11), we have the following expression for G:

$$G(p;\lambda) = 2B(p) + 2\int_p^0 gY(r;\lambda)\rho_p(r)dr. \tag{2.3.25}$$

Since B is independent of λ, and, indeed, the λ of G derives entirely from that of Y, we differentiate to find

$$1+\dot{G}(p;\lambda) = 1 + 2\int_p^0 g\dot{Y}(r;\lambda)\rho_p(r)dr. \tag{2.3.26}$$

But note that from (3.6) we can calculate

$$\dot{Y}_p = -\frac{1}{2}(\lambda + G(p))^{-3/2}\big(1+\dot{G}\big) \Longrightarrow \dot{Y}(p) = \frac{1}{2}\int_p^0 (\lambda+G(r))^{-3/2}\big(1+\dot{G}(r)\big)dr; \tag{2.3.27}$$

hence

$$1+\dot{G}(p) = 1 + \int_p^0\int_r^0 g\rho'(r)(\lambda+G(s))^{-3/2}\big(1+\dot{G}(s)\big)dsdr.$$

Now we reverse the order of integration to find

$$\begin{aligned} 1+\dot{G}(p) &= 1 + \int_p^0\int_p^s g\rho_p(r)(\lambda+G(s))^{-3/2}\big(1+\dot{G}(s)\big)drds \\ &= 1 + \int_p^0 k(p,s)\big(1+\dot{G}(s)\big)ds, \end{aligned} \tag{2.3.28}$$

where $k(p,s) := \int_p^s g\rho_p(r)(\lambda+G(s))^{-3/2}dr$. This is a linear Volterra integral equation of the second type, albeit a simple one. As k is a continuous map on $\Delta := \{(p,s) : p_0 \leq p \leq s \leq 0\}$ we have that

$$1+\dot{G}(p) = 1 + \int_p^0 \tilde{k}(p,s)ds, \tag{2.3.29}$$

where

$$k_1(p,s) := k(p,s), \qquad k_m(p,s) := \int_p^s k_{m-1}(p,r)k(r,s)dr,\ m \geq 2$$

and the resolvent kernel $\tilde{k}$ is given by

$$\tilde{k}(p,s) := \sum_{m=1}^{\infty} k_m(p,s).$$

The fact that the series converges comes from the following estimate:

$$|k_m(p,s)| \leq \left(\sup_{(p,s)\in\Delta} |k(p,s)| \right) \frac{(s-p)^{m-1}}{(m-1)!}.$$

From this it is clear that controlling the quantity in parentheses allows us to control the size $\tilde{k}$ and thereby the sign of $1+\dot{G}$. However, as we are taking take $\lambda > -2B_{\min} + \epsilon_0$, we have

$$\sup_{(p,s)\in\Delta} |k(p,s)| \leq g|p_0|\epsilon_0^{-3/2} \|\rho_p\|_\infty,$$

since $G \geq 2B_{\min}$ implies $\lambda + G \geq \epsilon_0$. Inserting this into the definition of $\tilde{k}$ we find

$$\sup_{(p,s)\in\Delta} \left|\tilde{k}(p,s)\right| \leq g|p_0|\epsilon_0^{-3/2} \|\rho_p\|_\infty e^{s-p} \leq |p_0|\epsilon_0^{-3/2} \|\rho_p\|_\infty e^{|p_0|}.$$

Integrating this expression and applying the definition of ϵ_0,

$$1+\dot{G}(p) \geq 1 - p_0^2\epsilon_0^{-3/2}\|\rho_p\|_\infty e^{|p_0|} = \frac{1}{2}.$$

Thus we have shown that $1+\dot{G}$ is strictly positive, as desired.

Finally, to derive the upper bound we merely reexamine (2.3.27). That is, from (2.3.27) we see that $\dot{Y} > 0$, as we have shown $1+\dot{G} > 0$. But then (2.3.26) implies $\dot{G} \leq 0$. □

With the sign of $1+\dot{G}$ established, we are now in a position to better understand the relation between Q and λ set down by (2.3.16).

Corollary 2.3.1 (convexity of Q). *For all $\lambda > -2B_{\min} + \epsilon_0$, we have that Q is a convex function of λ with a unique minimum λ_0.*

Proof. First recall that by the definition, $Q(\lambda) = \lambda - 2g\rho(0)Y(p_0;\lambda)$, where $Y(\cdot;\lambda)$ is the corresponding solution to the laminar flow. Differentiating twice in λ, we obtain

$$-\frac{\ddot{Q}(\lambda)}{2g\rho(0)} = \ddot{Y}(p_0;\lambda).$$

It follows that to prove the corollary it suffices to show $\ddot{Y}(p_0;\lambda) < 0$ for all $\lambda > -2B_{\min} + \epsilon_0$.

To do so we first note that by differentiating (2.3.26) by λ,

$$\ddot{G}(p;\lambda) = 2\int_p^0 g\ddot{Y}(r;\lambda)\rho'(r)dr. \tag{2.3.30}$$

On the other hand, differentiating (2.3.27) we find

$$\ddot{Y}(p;\lambda) = \frac{1}{2}\int_p^0 \frac{\ddot{G}(r;\lambda)}{(\lambda+G(r;\lambda))^{3/2}}dr - \frac{3}{4}\int_p^0 \frac{(1+\dot{G}(r;\lambda))^2}{(\lambda+G(r;\lambda))^{5/2}}dr. \tag{2.3.31}$$

Substituting (2.3.31) into (2.3.30) and exchanging the order of integration we find that for each fixed λ, $\ddot{Y}(\cdot;\lambda)$ solves the integral equation

$$\ddot{Y}(p) = \ell(p) + \int_p^0 j(p,r)\ddot{Y}(r)dr, \qquad p\in[p_0,0], \tag{2.3.32}$$

where

$$j(p,r) := \int_p^r \frac{g\rho'(r)}{(\lambda+G(s))^{3/2}}ds, \qquad \ell(p) := -\frac{3}{2}\int_p^0 \frac{(1+\dot{G}(r))^2}{(\lambda+G(r))^{5/2}}dr.$$

Observe that $\ell\leq 0$ and $\|\ell\|_\infty = -\ell(p_0)$. Now this is a Volterra integral equation with solution given by

$$\ddot{Y}(p) = \ell(p) + \int_p^0 \tilde{j}(p,r)\ell(r)dr,$$

where the resolvent kernel $\tilde{j}$ is given in the same fashion as the previous lemma. From this representation formula we immediately derive the estimate

$$\ddot{Y}(p_0;\lambda) \leq \left(1-|p_0|\|\tilde{j}\|_\infty\right)\ell(p_0). \tag{2.3.33}$$

As $\ell(p_0)<0$, we are done if we can show that the quantity in parenthesis is positive. Unsurprisingly, this is again a consequence of the definition of ϵ_0 in (2.1.13). Again let Δ denote the triangular region $\Delta := \{(p,r) : p_0\leq p\leq r\leq 0\}$. Observe that

$$\sup_{(p,r)\in\Delta}|j(p,r)| \leq \epsilon_0^{-3/2}g\|\rho_p\|_\infty|p_0|,$$

and thus

$$\sup_{(p,r)\in\Delta}\left|\tilde{j}(p,r)\right| \leq |p_0|\epsilon_0^{-3/2}g\|\rho_p\|_\infty e^{|p_0|}.$$

But by the definition of ϵ_0, this becomes

$$\sup_{(p,r)\in\Delta}\left|\tilde{j}(p,r)\right| \leq \frac{1}{2|p_0|}.$$

Inserting this last expression into (2.3.33) proves convexity of Q (and the concavity of Y).

To show the existence of a global minimum we note that

$$\dot{Q}(\lambda) = 1 - 2g\rho(0)\dot{Y}(p_0;\lambda),$$

thus $\dot{Q} > 0$ for λ sufficiently large, since we have shown $\ddot{Y}(p_0;\lambda) < 0$ for all λ. $\square$

We now return to the linearized problem. With the previous lemma in hand we are prepared to prove the monotonicity of μ in some neighborhood of λ^*. This will in turn give the uniqueness of λ^*. For convenience, we denote $a = a(p;\lambda) := H_p(p;\lambda)^{-1}$.

LEMMA 2.3.5 (monotonicity of μ). *μ is a strictly increasing function of λ whenever $\mu(\lambda) < 0$.*

PROOF. For each $w \in \mathscr{S}$ and each λ, define $L_\lambda w := -\{H_p^{-3} w_p\}_p$. For each λ, let $w(p) = w(p;\lambda)$ be the solution to the eigenvalue problem

$$L_\lambda w = \mu\left(H_p^{-1} + g\rho_p\right) w, \qquad w(p_0) = 0, \qquad w_p(0) = \lambda^{-3/2} g\rho(0) w(0),$$

where $\mu = \mu(\lambda)$ is the minimum eigenvalue. As in the previous lemma, denote by a dot differentiation with respect to λ, and for notational convenience, we drop the explicit dependence on λ. Thus, $\dot{a} = \frac{1+\dot{G}}{2a}$. Likewise,

$$\frac{\partial}{\partial\lambda} Lw = -\left\{\frac{3}{2} a\big(1+\dot{G}\big) w_p + a^3 \dot{w}_p\right\}_p. \tag{2.3.34}$$

Hence

$$L\dot{w} - \frac{3}{2}\left\{a\big(1+\dot{G}\big)w_p\right\}_p = \frac{\partial}{\partial\lambda} Lw = \dot{\mu}(a+g\rho_p)w + \mu\left(\frac{1+\dot{G}}{2a}\right) w + \mu(a+g\rho_p)\dot{w}.$$

We also have from the boundary conditions satisfied by w that $\dot{w}(p_0) = 0$ and

$$\dot{w}_p(0) = -\frac{3}{2}\lambda^{-5/2} g\rho(0)w(0) + \lambda^{-3/2} g\rho(0)\dot{w}(0).$$

Multiplying (2.3.34) by $\dot{w}$ and integrating yields

$$(\dot{w}, Lw) = \mu(\dot{w}, (a+g\rho_p)w).$$

Likewise, multiplying (2.3.34) by w and integrating,

$$\begin{aligned}(L\dot{w}, w) - \frac{3}{2}\left(\left\{a\big(1+\dot{G}\big)w_p\right\}_p, w\right) = (\dot{\mu}(a+g\rho_p)w, w) + \left(\mu\left(\frac{1+\dot{G}}{2a}\right) w, w\right) \\ + (\mu(a+g\rho_p)\dot{w}, w).\end{aligned}$$

We may integrate by parts to evaluate the second term on the left-hand side in the equation directly above:

$$-\frac{3}{2}\left(\left\{a\big(1+\dot{G}\big)w_p\right\}_p, w\right) = \frac{3}{2}\left(a\big(1+\dot{G}\big)w_p, w_p\right) - \frac{3}{2} a\big(1+\dot{G}\big)w_p w\Big|^0.$$

On the other hand,

$$
\begin{aligned}
(\dot{w}, Lw) - (L\dot{w}, w) &= \int_{p_0}^{p} \left(-\dot{w}\{a^3 w_p\}_p + w\{a^3 \dot{w}_p\}_p\right) dp \\
&= \left(\dot{w}_p a^3 w - w_p a^3 \dot{w}\right) \Big|^0 .
\end{aligned}
$$

Adding the last three equations gives

$$
\begin{aligned}
\frac{3}{2}\left(a\big(1+\dot{G}\big)w_p, w_p\right) - \frac{3}{2}a\big(1+\dot{G}\big)w_p w\Big|^0 &= (\dot{\mu}(a+g\rho_p)w, w) \\
&\quad + \left(\mu\left(\frac{1+\dot{G}}{2a}\right)w, w\right) + (\dot{w}_p w - w_p \dot{w})\, a^3\Big|^0 .
\end{aligned}
$$

Now as $G(0) = 0$ for all λ, $\dot{G}(0) = 0$. Thus the boundary conditions evaluated at $p = 0$ are the following:

$$
\begin{aligned}
a^3(\dot{w}_p w - w_p \dot{w}) + \frac{3}{2}w_p w &= \lambda^{3/2}\left(-\frac{3}{2}\lambda^{-5/2} g\rho w^2 + \lambda^{-3/2} g\rho\dot{w}w\right) \\
&\quad -\lambda^{3/2}\left(\lambda^{-3/2} g\rho\dot{w}w\right) + \frac{3}{2}\lambda^{1/2}\left(\lambda^{-3/2} g w^2\right) \\
&= 0.
\end{aligned}
$$

Thus dropping the boundary terms, we have

$$
\frac{3}{2}\left(a\big(1+\dot{G}\big)w_p, w_p\right) = (\dot{\mu}(a+g\rho_p)w, w) + \left(\mu\left(\frac{1+\dot{G}}{2a}\right)w, w\right).
$$

But, a (and $a + g\rho_p$) are strictly positive for admissible λ. Hence, when $\mu < 0$, we must have $\dot{\mu} > 0$ as claimed. This completes the lemma. □

Applying these lemmas, we have, ultimately, the following result.

Lemma 2.3.6 (location of λ^*). *Under the hypotheses of the previous lemmas, the solution λ^* of $\mu(\lambda^*) = -1$ (whose existence is given by Lemma 2.3.2) is unique. Moreover, $\lambda^* < \lambda_0$.*

Proof. By continuity of μ and the preceding lemmas, there exists some λ^*, with $\mu(\lambda^*) = -1$. Moreover, since μ is monotonically increasing when $\mu(\lambda)$ is in any sufficiently small neighborhood of -1, this λ^* must be unique.

We now prove that the size conditions given are sufficient to ensure that $\lambda_0 \neq \lambda_*$. If $\lambda_0 < -2B_{\min} + \epsilon_0$, then we are done. Likewise, if $\epsilon_0 = 0$, then we are in the constant density case, and this result is already known. So we shall assume $\lambda_0 \geq \epsilon_0 > 0$.

Fixing $\lambda = \lambda_0$, let $\phi \in \mathscr{S}$ be given by

$$\phi(p) := \int_{p_0}^{p} \frac{1+\dot{G}(r;\lambda_0)}{(\lambda_0 + G(r;\lambda_0))^{3/2}} dr, \qquad p_0 \le p \le 0.$$

Then, using (2.3.27) we estimate

$$\int_{p_0}^{0} Y_p(p)^{-3}\phi_p(p)^2 dp = \int_{p_0}^{0} \frac{\big(1+\dot{G}(p)\big)^2}{(\lambda_0 + G(p))^{3/2}} dp \le 2\dot{Y}(p_0).$$

Now,

$$\phi(0)^2 = \left(\int_{p_0}^{0} \frac{1+\dot{G}(p)}{(\lambda_0 + G(p))^{3/2}} dp \right)^2 = 4\dot{Y}(p_0)^2.$$

But, for λ_0, we know from (2.3.16) that $\dot{Y}(p_0) = \frac{1}{2g\rho(0)}$. Combining these estimates we see that the numerator of the Rayleigh quotient $\mathcal{R}(\cdot;\lambda_0)$ is dominated by

$$-g\rho(0)\phi(0)^2 + \int_{p_0}^{0} Y_p(p)^{-3}\phi_p(p)^2 dp \le -\frac{4\dot{Y}(p_0)}{2\dot{Y}(p_0)} + 2\dot{Y}(p_0) = 0.$$

Thus $\mu(\lambda_0) \le 0$.

Now let $\phi \in \mathcal{S}$ minimize $\mathcal{R}(\phi;\lambda_0)$. Then multiplying the equation satisfied by ϕ and integrating by parts we find that, for $p_0 < p < 0$,

$$a^3\phi\phi_p = -\mu(\lambda_0)\int_{p_0}^{p} (a + g\rho_p)\phi^2 dr + \int_{p_0}^{p} a^3\phi_p^2 dr.$$

Therefore, since we have already shown that $-\mu(\lambda_0) \le 0$, we see that $a^3\phi\phi_p$ is a positive and increasing function of p.

If we, instead, multiply the equation satisfied by ϕ by $\phi(1+\dot{G})^{-1}$ and integrate, we arrive at the following identity:

$$-g\rho(0)\phi(0)^2 + \int_{p_0}^{0} \frac{a^3}{1+\dot{G}}\phi_p^2 dp - \int_{p_0}^{0} \frac{a^3\dot{G}_p}{\big(1+\dot{G}\big)^2}\phi\phi_p dp = \mu(\lambda_0)\int_{p_0}^{0} \frac{a+g\rho_p}{1+\dot{G}}\phi^2 dp.$$

Note that we have used the fact that $\dot{G}(0) = 0$. To show that $\mu(\lambda_0) > -1$, therefore, we need to prove that

$$-g\rho(0)\phi(0)^2 + \int_{p_0}^{0} \frac{a^3}{1+\dot{G}}\phi_p^2 dp - \int_{p_0}^{0} \frac{a^3\dot{G}_p}{\big(1+\dot{G}\big)^2}\phi\phi_p dp + \int_{p_0}^{0} \frac{a+g\rho_p}{1+\dot{G}}\phi^2 dp > 0. \tag{2.3.35}$$

First we observe that, for any $\phi \in \mathscr{S}$,

$$\begin{aligned}
\phi(0)^2 &= \left(\int_{p_0}^0 \phi_p(p)dp\right)^2 \\
&\leq \left(\int_{p_0}^0 \frac{(\lambda_0 + G(p))^{3/2}}{1+\dot{G}(p)}\phi_p(p)^2 dp\right)\left(\int_{p_0}^0 \frac{1+\dot{G}(p)}{(\lambda_0+G(p))^{3/2}}dp\right) \\
&= \frac{1}{g\rho(0)}\int_{p_0}^0 \frac{(\lambda_0+G(p))^{3/2}}{1+\dot{G}(p)}\phi_p(p)^2 dp \\
&= \frac{1}{g\rho(0)}\int_{p_0}^0 \frac{a^3}{1+\dot{G}}\phi_p^2 dp.
\end{aligned}$$

Thus the first two terms in (2.3.35) give a nonnegative contribution.

Note, however, that the third term is nonpositive, since $\dot{G}_p \geq 0$. As we have seen, $a^3\phi\phi_p$ is increasing; therefore, we can bound the third term from below by

$$\begin{aligned}
-\int_{p_0}^0 \frac{a^3\dot{G}_p}{\left(1+\dot{G}\right)^2}\phi\phi_p dp &\geq -a(0)^3\phi(0)\phi_p(0)\int_{p_0}^0 \frac{\dot{G}_p}{\left(1+\dot{G}\right)^2}dp \\
&\geq -4g\rho(0)\phi(0)^2|p_0|\left\|\dot{G}_p\right\|_\infty \qquad (2.3.36) \\
&\geq -4g\|\rho_p\|_\infty|p_0|\phi(0)^2. \qquad (2.3.37)
\end{aligned}$$

On the other hand, integrating the fourth term on the left-hand side of (2.3.35) by parts gives

$$\int_{p_0}^0 \frac{a+g\rho_p}{1+\dot{G}}\phi^2 dp = A(0)\phi(0)^2 - 2\int_{p_0}^0 \frac{A}{1+\dot{G}}\phi\phi_p dp + \int_{p_0}^0 \frac{A\dot{G}_p}{\left(1+\dot{G}\right)^2}\phi^2 dp,$$

where $A(p) := \int_{p_0}^p (a(r;\lambda_0) + g\rho_p(r))dr$. Note that the final term on the right-hand side above is nonnegative.

The next task is to estimate these quantities. In that regard we note

$$-2\int_{p_0}^0 \frac{a^{-3}A}{1+\dot{G}}dp \geq -4|p_0|A(0)\epsilon_0^{-3/2}. \qquad (2.3.38)$$

Then, again exploiting the fact that $a^3\phi\phi_p$ is increasing, we use (2.3.37)–(2.3.38) to find

$$\begin{aligned}
&-\int_{p_0}^0 \frac{a^3\dot{G}_p}{\left(1+\dot{G}\right)^2}\phi\phi_p dp + \int_{p_0}^0 \frac{a+g\rho_p}{1+\dot{G}}\phi^2 dp \\
&\qquad\geq \left(|p_0|^{-1}A(0) - 4\|\rho_p\| - 4A(0)\epsilon_0^{-3/2}g\rho(0)\right)|p_0|\phi(0)^2.
\end{aligned}$$

So long as the quantity in parenthesis on the right-hand side above is nonnegative, we have $\mu(\lambda_0) > -1$. But, since

$$A(0) > \left(\epsilon_0^{1/2} - g\|\rho_p\|_\infty\right)|p_0| > \frac{1}{2}|p_0|\epsilon_0^{1/2},$$

we have that

$$|p_0|^{-1}A(0) - \|\rho_p\|_\infty - 4A(0)\epsilon_0^{-3/2}g\rho(0) > \frac{1}{2}|p_0|^{-1}A(0) - \|\rho_p\|_\infty > 0,$$

where the last two inequalities follow from the definition of ϵ_0 in (2.1.14). □

2.3.4. Proof of local bifurcation. All that is left for us now is to verify the hypotheses of the Crandall–Rabinowitz bifurcation theorem presented in [**18**]. As in the previous sections let the transformed fluid domain be

$$R := \{(q,p) : 0 < q < 2\pi,\ p_0 < p < 0\},$$

with boundaries

$$T := \{(q,p) \in R : p = 0\}, \qquad B := \{(q,p) \in R : p = p_0\},$$

and define

$$X := \left\{h \in C^{3+\alpha}_{\mathrm{per}}\left(\overline{R}\right) : h = 0 \text{ on } B\right\}, \qquad Y = Y_1 \times Y_2 := C^{1+\alpha}_{\mathrm{per}}\left(\overline{R}\right) \times C^{2+\alpha}_{\mathrm{per}}(T).$$

Let $h(q,p) := H(p) + w(q,p)$. Then by the full height equation, w must satisfy the following PDE:

$$\begin{aligned} &\left(1 + w_{qq}^2\right)(H_{pp} + w_{pp}) + w_{qq}(H_p + w_p)^2 - 2w_q w_{pq}(H_p + w_p) \\ &-g(H + w - d(H) - d(w))(H_p + w_p)^3\rho_p + (H_p + w_p)^3\beta(-p) = 0 \qquad \text{in } R, \end{aligned} \tag{2.3.39}$$

$$1 + w_q^2 + (H_p + w_p)^2(2g\rho(H + w) - Q) = 0 \qquad \text{on } T, \tag{2.3.40}$$

together with periodicity in q and vanishing on B.

Conforming to the framework of [**18**], we introduce a nonlinear operator

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : (-2B_{\min} + \epsilon_0, \infty) \times X \to Y$$

defined, for $w \in X$, $\lambda > -2B_{\min} + \epsilon_0$, by

$$\begin{aligned} \mathcal{F}_1(\lambda, w) \quad &:= \quad \left(1 + w_{qq}^2\right)(H_{pp} + w_{pp}) + w_{qq}(H_p + w_p)^2 - 2w_q w_{pq}(H_p + w_p) \\ &\qquad -g(H + w - d(H) - d(w))(H_p + w_p)^3\rho_p + (H_p + w_p)^3\beta(-p), \end{aligned} \tag{2.3.41}$$

$$\mathcal{F}_2(\lambda, w) \quad := \quad 1 + w_q^2 + (H_p + w_p)^2(2g\rho(H + w) - Q). \tag{2.3.42}$$

Note that by definition of the laminar solution $H(\cdot;\lambda)$, we have $\mathcal{F}(\lambda,0)\equiv 0$. For future reference, we evaluate the Fréchet derivatives $\mathcal{F}_{1w}$, $\mathcal{F}_{2w}$ at $w=0$:

$$\begin{aligned}\mathcal{F}_{1w} &= \partial_p^2 + H_p^2\partial_q^2 + 3H_p^2\beta(-p)\partial_p \\ &\quad -3g(H-d(H))H_p^2\rho_p\partial_p - gH_p^3\rho_p(1-d),\end{aligned} \tag{2.3.43}$$

$$\begin{aligned}\mathcal{F}_{2w} &= \left(2g\rho H_p^2 + 2H_p(2g\rho H - Q)\partial_p\right)\big|_T \\ &= \left(2g\rho\lambda^{-1} - 2\lambda^{1/2}\partial_p\right)\Big|_T.\end{aligned} \tag{2.3.44}$$

In order to show that the eigenvalue at λ^* is simple we must characterize the null space and range of the linearized operator $\mathcal{F}_w(\lambda^*,0)$. This is accomplished in the following two lemmas.

LEMMA 2.3.7 (null space). *The null space of $\mathcal{F}_w(\lambda^*,0)$ is one-dimensional.*

PROOF. Fix $\lambda=\lambda^*$. We have, by our work in Lemma 2.3.2, that $M(p)\cos q\in\mathcal{N}(\mathcal{F}_w(\lambda^*,0))$, where $\mathcal{N}(\cdot)$ denotes the null space. We need only prove uniqueness to complete the lemma.

Let m in the null space be given. Then, since m is even and 2π-periodic in q, we may expand

$$m(q,p)=\sum_{k=0}^{\infty}\cos(kq)m_k(p).$$

The fact that $\mathcal{F}_w(\lambda^*,0)m=0$ then implies

$$\sum_{k=0}^{\infty}\left(\partial_p^2 + H_p^2\partial_q^2 + 3H_p^2\beta(-p)\partial_p - 3g(H-d(H))H_p^2\rho_p\partial_p - gH_p^3\rho_p(1-d)\right)$$
$$\cos(kq)m_k(p)=0$$

in R, and

$$\sum_{k=0}^{\infty}\left(\left(2g\rho H_p^2 + 2H_p(2g\rho H-Q)\partial_p\right)\cos(kq)m_k(p)\right)\big|_T=0.$$

We conclude that each term in the series above must vanish. But, note that

$$d(m_k(p)\cos(kq)) = m_k(0)\fint_0^{2\pi}\cos(kq)dq=0;$$

hence, for each $k>0$,

$$\left\{-g\rho_p + H_p^{-1}k^2 + \partial_p\left(H_p^{-3}\partial_p\right)\right\}m_k=0, \qquad p_0<p<0.$$

From the second series and the boundary conditions satisfied by m, we have that for $k \geq 0$,

$$\partial_p m_k - g\rho\lambda^{-3/2} m_k = 0 \qquad \text{on } p = 0,$$

along with

$$m_k = 0 \qquad \text{on } p = p_0.$$

For $k = 1$, then, m must be a constant multiple of M, as both satisfy the identical ODE boundary value problem. Moreover, if $k \geq 2$, we see that $m_k \equiv 0$. Otherwise, the Rayleigh quotient in the minimization problem will be $-k^2 < -1$, which contradicts the choice of λ^*.

The difficulty arises in the case $k = 0$, since the d-term persists. Using the fact $d(m_0) = m_0(0)$, we have that m_0 satisfies

$$\left\{-g\rho_p + \partial_p \left(H_p^{-3}\partial_p\right)\right\} m_0 = -g\rho_p m_0(0), \qquad p_0 < p < 0,$$

along with the same boundary conditions as before on $p = 0$ and $p = p_0$. From this we wish to conclude that $m_0 \equiv 0$. To see why this must be the case we rewrite it in the form

$$\left(a^3 m_0'\right)' = g\rho'(m_0 - m_0(0)) = g\rho' \int_0^p m_0'(r)dr,$$

where $p \in (p_0, 0)$ and prime denotes differentiation with respect to p. Integrating this equation and using the boundary condition at $p = 0$ we find

$$\begin{aligned}
a^3 m_0' &= g\rho(0)m_0(0) + \int_0^p g\rho'(r) \int_0^r m_0'(s)dsdr \\
&= g\rho(0)m_0(0) + \int_0^p \left(\int_s^p g\rho'(r)dr\right) m_0'(s)ds \\
&= g\rho(0)m_0(0) + \int_0^p k(p,s)a(s)^3 m_0'(s)ds,
\end{aligned}$$

where the kernel k is given by

$$k(p,s) := \int_s^p g\rho'(r)a(s)^{-3}dr, \qquad p_0 \leq p < s \leq 0.$$

This is nothing but a Volterra-integral equation of the form

$$\phi(t) = C + \int_0^t k(t,s)\phi(s)ds, \qquad p_0 \leq t < s \leq 0, \tag{2.3.45}$$

with $\phi(0) = C$. As everything here is smooth and we are working on a compact interval, standard theory of integral equations gives a unique solution for $\phi = \phi(p; C)$. Fix a value of C. Then multiplying (2.3.45) by a^{-3} and integrating we find

$$\begin{aligned}\int_0^{p_0} \phi(t;C)a(t)^{-3}dt &= C\int_0^{p_0} a(t)^{-3}dt + \int_0^{p_0} a(t)^{-3}\int_0^t k(t,s)\phi(s;C)dsdt \\ &= C\int_0^{p_0} a(t)^{-3}dt + \int_0^{p_0} K(t)\phi(t;C)dt,\end{aligned}$$

where

$$K(t) := \int_t^{p_0} k(s,t)a(s)^{-3}ds.$$

Now we note that we must have that $a^3 m_0' = \phi(\cdot; m_0(0))$, where ϕ is as above. Then, by the linearity of $\phi(\cdot;C)$ in C, we see that

$$a^3 m_0'(p) = g\rho(0)m_0(0)\phi(p;1), \qquad p_0 < p < 0.$$

But observe that $a(p)^{-3} = (\lambda+G(p))^{-3/2}$ and recalling our argument of the previous section, we have that $\phi(p;1) = 1+\dot{G}(p)$, as the kernels of the integral equations (2.3.28) and (2.3.45) (scaled so that $C=1$) are identical. Hence, integrating the above equation we find

$$m_0(0) = g\rho(0)m_0(0)\int_{p_0}^0 \frac{1+\dot{G}(p)}{(\lambda+G(p))^{3/2}}dp = 2\dot{Y}(0).$$

Assuming $m_0(0) \neq 0$, we arrive at a contradiction, since by the above identity

$$\dot{Q}(\lambda) = 1 - 2g\rho(0)\dot{Y}(0) = 0.$$

We conclude that $m_0(0) \neq 0$ only when $\lambda = \lambda_0$. However, Lemma 2.3.6 assures us that $\lambda_* < \lambda_0$, so this is impossible.

We have, therefore, shown that $m_0(0)$ must be 0. This is an immediate consequence of the fact that $m(0) = m'(0) = 0$ while m satisfies a second order, linear ODE.

In summary, we have showed that all the modes $k \neq 1$ vanish, thus $m(q,p) = m_1(p)\cos q$. By Lemma 2.3.2, see that m_1 is a constant multiple of M. Thus the null space is one-diminsional with generator $M(p)\cos q$. □

Lemma 2.3.8 (range). *The pair* $(\mathcal{A},\mathcal{B})$ *belongs to the range of the linear operator* $\mathcal{F}_w(\lambda^*,0)$ *iff it satisfies the orthogonality condition*

$$\iint_R \mathcal{A}a^3\phi^* dqdp + \frac{1}{2}\int_T \mathcal{B}a^2\phi^* = 0,$$

where ϕ^* *generates the nulls space* $\mathcal{F}_w(\lambda^*,0)$.

PROOF. Suppose first that $(\mathcal{A},\mathcal{B}) \in Y$ is in the range of $\mathcal{F}_w(\lambda^*,0)$. Then there exists some $v \in X$ such that $\mathcal{A} = \mathcal{F}_{1w}(\lambda^*,0)v$, and $\mathcal{B} = \mathcal{F}_{2w}(\lambda^*,0)v$. It follows that

$$\begin{aligned}
\iint_R \mathcal{A}a^3\phi^* dqdq &= \iint_R \left(\left(a^3 v_p\right)_p + av_{qq} - a^3 g\rho_p v\right)\phi^* dqdp \\
&= \iint_R \left(\left(a^3 \phi^*_p\right)_p + a\phi^*_{qq} - a^3 g\rho_p \phi^*\right) v dqdp \\
&\qquad + \int_T \left(a^3 v_p \phi^* - a^3 v\phi^*_p\right) dq \\
&= \int_T \left(a^3 v_p \phi^* - a^3 v\phi^*_p\right) dq,
\end{aligned}$$

as on B, both v and ϕ^* vanish. Now, on T we have $a^3 = \lambda^{3/2}$. Moreover, the ODE satisfied by ϕ^* gives $g\rho\phi^* = \lambda^{3/2}\phi^*_p$ on T. Thus,

$$\begin{aligned}
\frac{1}{2}\int_T \mathcal{B}a^2\phi^* dq &= \int_T \left(g\rho v - \lambda^{3/2} v_p\right)\phi^* dq \\
&= \int_T \left(\lambda^{3/2}\phi^*_p v - \lambda^{3/2} v_p \phi^*\right) dq.
\end{aligned}$$

Combining this expression with the last gives necessity of the orthogonality condition.

To demonstrate sufficiency we let any pair $(\mathcal{A},\mathcal{B}) \in Y$ be given satisfying the orthogonality condition. We must show that there exists a solution $u \in X$ to the following:

$$\begin{cases} -g\rho_p(u - d(u)) + a^{-2}u_{qq} + a^{-3}\left(a^3 u_p\right)_p = \mathcal{A} & \text{in } R, \\ 2\left(g\rho a^{-2}u - au_p\right) = \mathcal{B} & \text{on } T, \\ u = 0 & \text{on } B. \end{cases} \tag{2.3.46}$$

Our methodology here will be to freeze the operator d, replacing it with a fixed real number. The resulting problem will be an elliptic PDE, so that we may use Schauder theory to extract solutions.

We shall do this successively in several stages. First, consider the problem

$$\begin{cases} -\epsilon v^{(\epsilon)} + -g\rho_p v^{(\epsilon)} + a^{-2}v^{(\epsilon)}_{qq} + a^{-3}\left(a^3 v^{(\epsilon)}_p\right)_p = \mathcal{A} & \text{in } R, \\ 2\left(g\rho a^{-2}v^{(\epsilon)} - av^{(\epsilon)}_p\right) = \mathcal{B} & \text{on } T, \\ v^{(\epsilon)} = 0 & \text{on } B, \end{cases} \tag{2.3.47}$$

where $v^{(\epsilon)}$ is periodic in q. By standard elliptic theory, for a sequence of $\epsilon > 0$ going to zero, problem (2.3.47) will have a unique solution.

We claim, moreover, that these solutions are bounded in $C^{1+\alpha}_{\text{per}}(\overline{R})$. By contradiction, suppose otherwise. Then there is some sequence of $\epsilon \to 0$ such that $\|v^{(\epsilon)}\|_{C^{1+\alpha}(\overline{R})} \to \infty$.

For each ϵ, put $u^{(\epsilon)} := v^{(\epsilon)}/\|v^{(\epsilon)}\|_{C^{1+\alpha}(\overline{R})}$. Thus, $u^{(\epsilon)}$ has unit norm for all ϵ. Given that $v^{(\epsilon)}$ solves (2.3.47), we have, additionally,

$$-g\rho_p u^{(\epsilon)} + a^{-2}u^{(\epsilon)}_{qq} + a^{-3}\left(a^3 u^{(\epsilon)}_p\right)_p \longrightarrow 0 \qquad \text{in } C^{1+\alpha}_{\text{per}}\left(\overline{R}\right)$$

and

$$g\rho a^{-2}u^{(\epsilon)} - au^{(\epsilon)}_p \longrightarrow 0 \qquad \text{in } C^{1+\alpha}_{\text{per}}(T).$$

Observe that (2.3.47) is uniformly elliptic. Applying Schauder estimates ensures that the sequence $\{u^{(\epsilon)}\}$ is uniformly bounded in $C^{2+\alpha}_{\text{per}}(\overline{R})$. So by compactness, we have a subsequence converging strongly to some $u \in C^2_{\text{per}}(\overline{R})$. By an argument identical to the previous lemma, we can show that u is in the null space of $\mathcal{F}_w(\lambda^*, 0)$ and hence, a multiple of ϕ^*. This follows from the fact that problem immediately above is $\mathcal{F}_{1w}$ without the stratification term. As we have seen, when we expand any element in the null space, this term will drop out in all modes $k \geq 1$. Similarly, for the $k = 0$ term, we may simply bypass the integral equation argument (as we are, so-to-speak, already given that the right-hand side of the interior equation is zero) and use the same Rayleigh quotient manipulations to conclude $u_0 \equiv 0$. So, indeed, u is a constant multiple of ϕ^*.

But by definition of $v^{(\epsilon)}$, we have

$$\iint_R \mathcal{A}a^3\phi^* dqdp = \iint_R a^3\phi^*\left(-\epsilon v^{(\epsilon)} + \mathcal{F}_{1w}(\lambda^*, 0)v^{(\epsilon)}\right) dqdp.$$

Applying the exact same manipulations on the second term in the integrand above as we did in proving necessity yields

$$0 = -\epsilon\iint_R a^3 v^{(\epsilon)}\phi^* dqdp = \iint_R a^3 u^{(\epsilon)}\phi^* dqdp \Longrightarrow 0 = \iint_R a^3 u\phi^* dqdp.$$

However, we have found that u is in the linear span of ϕ^*, so the equation above implies $u \equiv 0$. This contradicts the fact that $\|u\|_{C^{1+\alpha}(\overline{R})} = 1$. Hence, $\{v^{(\epsilon)}\}$ is bounded.

It now follows that $\{v^{(\epsilon)}\}$ has a strongly convergent subsequence in $C^1_{\text{per}}(\overline{R})$. The limit, denote it v, will satisfy (2.3.47) with $\epsilon = 0$ in the sense of distributions. But, again appealing to standard elliptic regularity theory, this implies that $v \in X$.

We have thus shown that for each $(\mathcal{A}, \mathcal{B})$ satisfying the orthogonality condition, we may find a unique solution in X to the following problem:

$$\begin{cases} -g\rho_p u + a^{-2}u_{qq} + a^{-3}\left(a^3 u_p\right)_p = \mathcal{A} & \text{in } R, \\ 2\left(g\rho a^{-2}u - au_p\right) = \mathcal{B} & \text{on } T, \\ u = 0 & \text{on } B. \end{cases} \tag{2.3.48}$$

Now to return to (2.3.46) we make the following observation: if the pair $(\mathcal{A}, \mathcal{B})$ satisfies the orthogonality condition, then so does $(\mathcal{A} - f(p), \mathcal{B})$, where f is any smooth function. To see why this is the case recall that we have proved that ϕ^* is of the form $\phi^*(q,p) = M(p)\cos q$. Therefore,

$$\begin{aligned} \iint_R (\mathcal{A} - f)a^3\phi^* dqdp &= \iint_R \mathcal{A}a^3\phi^* dqdp - \int_{p_0}^0 f(p)a(p)^3 M(p)dp \int_0^{2\pi} \cos q dq \\ &= \iint_R \mathcal{A}a^3\phi^* dqdp \\ &= \frac{1}{2}\int_T \mathcal{B}a^2\phi^*, \end{aligned}$$

so, indeed, the orthogonality condition holds for $(\mathcal{A} - f(p), \mathcal{B})$.

Fix $\sigma \in \mathbb{R}$. In light of our previous observation and our proof of the uniqueness and existence of solutions to (2.3.48) in X, it follows that there exists a unique solution $u^{(\sigma)}$ of

$$\begin{cases} -g\rho_p u^{(\sigma)} + a^{-2}u_{qq}^{(\sigma)} + a^{-3}\left(a^3 u_p^{(\sigma)}\right)_p = \mathcal{A} - g\rho_p\sigma & \text{in } R, \\ 2\left(g\rho a^{-2}u^{(\sigma)} - au_p^{(\sigma)}\right) = \mathcal{B} & \text{on } T, \\ u^{(\sigma)} = 0 & \text{on } B. \end{cases} \tag{2.3.49}$$

We may, therefore, define a mapping $\Psi : \mathbb{R} \to \mathbb{R}$ by $\Psi(\sigma) := d(u^{(\sigma)})$, where $u^{(\sigma)}$ solves (2.3.49). Suppose that Ψ has a fixed point σ. Then we may simply substitute $d(u^{(\sigma)})$ for σ in (2.3.49) to find that $u^{(\sigma)}$ is the sought-after solution to (2.3.46). Thus, it suffices to prove that Ψ has a (unique) fixed point.

Let σ, τ be given with $\sigma \neq \tau$, and let $u^{(\sigma)}$, $u^{(\tau)}$ solve (2.3.49) for σ, τ, respectively. Then subtracting the equations satisfied by $u^{(\sigma)}$ and $u^{(\tau)}$ and dividing by $\sigma - \tau$, we arrive at a solution to the following problem:

$$\begin{cases} -g\rho_p v + a^{-2}v_{qq} + a^{-3}\left(a^3 v_p\right)_p = -g\rho_p & \text{in } R, \\ 2\left(g\rho a^{-2}v - av_p\right) = 0 & \text{on } T, \\ v = 0 & \text{on } B. \end{cases} \tag{2.3.50}$$

By applying the Schauder estimates for Dirchlet and oblique boundary conditions, moreover, we have that the solution v of this equation is in X.

By contradiction, assume that $d(v) = 1$. Then, by virtue of the fact v solves (2.3.50), we have that $v \in \mathcal{N}(\mathcal{F}_w(\lambda^*, 0))$. It follows from our previous lemma that v is a constant multiple of ϕ^*. But this is a contradiction, since

$$d(\phi^*) = M(0) \fint_0^{2\pi} \cos q dq = 0.$$

Thus we are assured $d(v) \neq 1$.

Fix $\sigma,\ h \in \mathbb{R}$. Examining (2.3.49), we readily see

$$\frac{\Psi(\sigma + h) - \Psi(\sigma)}{h} = \fint_T \left(\frac{u^{(\sigma+h)} - u^{(\sigma)}}{h} \right) dq = d(v).$$

Hence, Ψ is differentiable everywhere, and $\Psi' \equiv d(v)$. Therefore, the function $\sigma \mapsto \Psi(\sigma) - \sigma$ is monotonic with derivative uniformly equal to $d(v) - 1 \neq 0$. So for some σ^*, we have $0 = \psi(\sigma^*) - \sigma^*$. This proves the existence of a fixed point of Ψ and thereby, a solution to (2.3.46). The lemma follows. $\square$

Finally, we must ensure that the so-called transversality or crossing condition holds.

LEMMA 2.3.9 (technical condition). *If ϕ^* generates the null space of $\mathcal{F}(\lambda^*, 0)$, then $\mathcal{F}_{w\lambda}(\lambda^*, 0)\phi^* \notin \mathcal{R}(\mathcal{F}_w(\lambda^*, 0))$.*

PROOF. First we calculate the mixed Fréchet derivatives of $\mathcal{F}$ at $\lambda = \lambda^*$, $w = 0$:

$$\begin{aligned} \mathcal{F}_{1\lambda w}(\lambda^*, 0) &= -\big(1 + \dot{G}\big)a^{-4}\partial_q^2 - 3\big(1 + \dot{G}\big)a^{-4}\beta(-p)\partial_p - 3g\dot{Y}a^{-2}\rho_p\partial_p \\ &\quad + 3gY\big(1 + \dot{G}\big)a^{-4}\rho_p\partial_p + \frac{3}{2}g\big(1 + \dot{G}\big)a^{-5}\rho_p(1 - d), \\ \mathcal{F}_{2\lambda w}(\lambda^*, 0) &= \Big(-2g\rho\lambda^{-2} - \lambda^{-1/2}\partial_p\Big)\Big|_T. \end{aligned}$$

By the previous lemma, it suffices to show that the pair $(\mathcal{F}_{1\lambda w}(\lambda^*, 0)\phi^*, \mathcal{F}_{2\lambda w}(\lambda^*, 0)\phi^*)$ does not satisfy the orthogonality condition. Equivalently, if we put

$$\begin{aligned} \Xi := &\iint_R \Big(\big(1 + \dot{G}\big)a^{-1}(\phi^*)^2 - 3\big(1 + \dot{G}\big)a^{-1}\beta(-p)\phi^*\phi_p^* \\ &- 3g\dot{Y}a\rho_p\phi^*\phi_p^* + 3gY\big(1 + \dot{G}\big)a^{-1}\rho_p\phi^*\phi_p^* + \frac{3}{2}g\big(1 + \dot{G}\big)a^{-2}\rho_p(\phi^*)^2 \Big) dqdp \\ &+ \int_T \left(-2g\rho a^{-2}(\phi^*)^2 - \frac{1}{2}a\phi^*\phi_p^* \right) dq, \end{aligned}$$

our lemma will be proven if we can show $\Xi \neq 0$ (again, because $d(\phi^*) = 0$). We will demonstrate that, in fact, $\Xi < 0$. To keep our notation concise, let $\Xi = \Xi_1 + \cdots + \Xi_7$, where Ξ_i denotes the ith term in the sum above. It is clear a priori that $\Xi_1 > 0$, while $\Xi_5, \Xi_6 < 0$.

Consider the boundary terms Ξ_6 and Ξ_7. On T, we have $a = \lambda^{1/2}$, and ϕ^* satisfies $g\rho\phi^* = \lambda^{3/2}\phi_p^*$. Therefore,

$$\Xi_7 = -\frac{1}{2}\int_T \lambda^{1/2}\phi^*\phi_p^* dq = -\frac{1}{2}\int_T \lambda^{-1} g\rho(\phi^*)^2 dq = \frac{1}{4}\Xi_6. \tag{2.3.51}$$

Now, by the ODE satisfied by ϕ^* in R, we have

$$(a + g\rho_p)\phi^* = \left(a^3\phi_p^*\right)_p = 3a(\beta(-p) - gY\rho_p)\phi_p^* + a^3\phi_{pp};$$

whence,

$$3a^{-1}gY\rho_p\phi_p^* = a\phi_{pp}^* - (g\rho_p + a)a^{-2}\phi^* + 3a^{-1}\beta(-p)\phi_p^*.$$

Substituting this expression into Ξ_4 yields the following:

$$\begin{aligned}
\Xi_4 &= \iint_R 3gY\big(1+\dot{G}\big)a^{-1}\rho_p\phi^*\phi_p^* dqdp \\
&= \iint_R \Big(a\big(1+\dot{G}\big)\phi^*\phi_{pp}^* - (g\rho_p + a)a^{-2}\big(1+\dot{G}\big)(\phi^*)^2\Big)\, dqdp - \Xi_2 \\
&= \iint_R \Big(a\big(1+\dot{G}\big)\phi^*\phi_{pp}^* - g\rho_p a^{-2}\big(1+\dot{G}\big)(\phi^*)^2\Big)\, dqdp - \Xi_1 - \Xi_2.
\end{aligned} \tag{2.3.52}$$

Next, consider the quantity

$$\begin{aligned}
\Xi_2 + \Xi_3 + \Xi_4 &= \iint_R \Big(a\big(1+\dot{G}\big)\phi^*\phi_{pp}^* - g\rho_p a^{-2}\big(1+\dot{G}\big)(\phi^*)^2 \\
&\qquad - 3g\dot{Y}a\rho_p\phi^*\phi_p^*\Big)dqdp - \Xi_1.
\end{aligned} \tag{2.3.53}$$

Calculating that $\dot{G}_p = -2g\dot{Y}\rho_p$ and $a_p = a^{-1}(\beta(-p) - gY\rho_p)$, we integrate the first term by parts to find

$$\begin{aligned}
\Xi_2 + \Xi_3 + \Xi_4 &= \iint_R \Big(\Big(a^{-1}gY\big(1+\dot{G}\big)\rho_p - ga\dot{Y}\rho_p\Big)\,\phi^*\phi_p^* - a^{-1}\beta(-p)(1+\dot{G})\phi^*\phi_p^* \\
&\qquad - a\big(1+\dot{G}\big)(\phi_p^*)^2 - ga^{-2}\big(1+\dot{G}\big)\rho_p(\phi^*)^2\Big)dqdp \\
&\quad + \int_T a\big(1+\dot{G}\big)\phi^*\phi_p^* dq - \Xi_1 \\
&= \iint_R \Big(-ga^{-2}\big(1+\dot{G}\big)\rho_p(\phi^*)^2 - a\big(1+\dot{G}\big)(\phi_p^*)^2\Big)\, dqdp \\
&\quad + \frac{1}{3}\left(\Xi_2 + \Xi_3 + \Xi_4\right) - \Xi_1 - 2\Xi_7,
\end{aligned} \tag{2.3.54}$$

where we have used the familiar fact that $\dot{G}(0) = 0$.

Now, simply combining identities (2.3.51), (2.3.52), (2.3.53), and (2.3.54) gives

$$\begin{aligned}
\Xi &= \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 + \frac{5}{4}\Xi_6 \\
&= \Xi_1 + \frac{3}{2}\left(\iint_R \left(-ga^{-2}\big(1+\dot{G}\big)\rho_p(\phi^*)^2 - a\big(1+\dot{G}\big)(\phi_p^*)^2\right) dqdp - \Xi_1 - 2\Xi_7\right) \\
&\quad + \Xi_5 + \frac{5}{4}\Xi_6 \\
&= -\frac{1}{2}\Xi_1 - \frac{3}{2}\iint_R a\big(1+\dot{G}\big)(\phi_p^*)^2 dqdp + \frac{1}{2}\Xi_6.
\end{aligned}$$

Thus $\Xi < 0$, and the claim is proven. □

We are now ready to prove the local bifurcation theorem.

Proof of Theorem 2.3.1. By definition of the laminar solutions $H(\cdot;\lambda)$, we have $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda \geq -2B_{\min} + \epsilon_0$. Moreover, by the regularity assumptions on ρ and β as well as the definition of X, we see that $\mathcal{F}_\lambda$, $\mathcal{F}_w$, $\mathcal{F}_{\lambda w}$, and $\mathcal{F}_{ww}$ exist and are continuous. Applying Lemma 2.3.7 and Lemma 2.3.8, moreover, we see that $\mathcal{N}(F_w(\lambda^*, 0))$ and $Y \setminus \mathcal{R}(\mathcal{F}_w(\lambda^*, 0)$ are both one-dimensional with the former generated by ϕ^*. Finally, Lemma 2.3.9 shows $\mathcal{F}_{w\lambda}(\lambda^*, 0)\phi^*$ is not contained in $\mathcal{R}(\mathcal{F}_w(\lambda^*, 0))$.

Thus we have satisfied all the hypotheses of the Crandall–Rabinowitz theorem on bifurcation from a simple eigenvalue (cf. [**18**, Theorem 2]). This allows us to conclude that there exists a C^1 local bifurcation curve

$$\mathcal{C}_0' = \{\lambda(s), w(s) : |s| < \epsilon\},$$

for $\epsilon > 0$ sufficiently small, such that $(\lambda(0), w(0)) = (\lambda^*, 0)$ and

$$\{(\lambda, w) \in \mathcal{U} : w \not\equiv 0,\ \mathcal{F}(\lambda, w) = 0\} = \mathcal{C}_0',$$

where $\mathcal{U}$ is some neighborhood of $(\lambda^*, 0) \in [-2B_{\min} + \epsilon_0, \infty) \times X$. Moreover, for $|s| < \epsilon$,

$$w(s) = s\phi^* + o(s) \qquad \text{in } X. \tag{2.3.55}$$

Furthermore, since $h = H + w$ and $H_p > 0$ in $\overline{R}$, we may restrict our attention to a smaller C^1-curve $\mathcal{C}_{\text{loc}}' \subset \mathcal{C}_0'$ containing $(\lambda^*, 0)$ and along which $h_p > 0$ in $\overline{R}$. Then by Lemma 2.2.1, we have the existence of a C^1-curve $\mathcal{C}_{\text{loc}}$ of solutions to (2.1.1)–(2.1.6).

2.4. Global bifurcation

We now prove that we can continue the curve C'_{loc} whose existence was established in Theorem 2.3.1. Continuing with our notation from the previous section, denote

$$X := \left\{ h \in C^{3+\alpha}_{\text{per}}\left(\overline{R}\right) : h = 0 \text{ on } B \right\}, \qquad Y = Y_1 \times Y_2 := C^{1+\alpha}_{\text{per}}\left(\overline{R}\right) \times C^{2+\alpha}_{\text{per}}(T),$$

where the subscript "per" indicates 2π-periodicity and evenness in q, R is the rectangle $(0, 2\pi) \times (p_0, 0)$, and $T = (0, 2\pi) \times \{p = 0\}$. Next define

$$\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : \mathbb{R} \times X \to Y$$

by

$$\mathcal{G}_1(h) := \left(1 + h_q^2\right) h_{pp} + h_{qq} h_p^2 - 2h_q h_p h_{pq} - g(h - d(h)) h_p^3 \rho_p + h_p^3 \beta(-p), \tag{2.4.1}$$

$$\mathcal{G}_2(Q, h) := \left(1 + h_q^2 + h_p^2 (2g\rho h - Q)\right)\Big|_{p=0}. \tag{2.4.2}$$

Recall that here $d : X \to \mathbb{R}$ denotes the linear operator mapping an element of X to its average value on T. We remark, also, that the laminar flow solutions $(H(\lambda), \lambda)$ found in Lemma 2.3.1 satisfy $\mathcal{G}(H(\lambda), Q(\lambda)) = 0$ for all $\lambda \geq -2B_{\min} + \epsilon_0$.

In order for our arguments to have any traction, we will need to make strong use of a priori estimates. It will often prove convenient to consider uniformly elliptic differential operators that approximate $\mathcal{G}$ or more specifically, $\mathcal{G}_1$. For any $\sigma \in \mathbb{R}$, we therefore define $\mathcal{G}^{(\sigma)} : \mathbb{R} \times X \to Y$ by

$$\mathcal{G}_1^{(\sigma)}(h) := \left(1 + h_q^2\right) h_{pp} + h_{qq} h_p^2 - 2h_q h_p h_{pq} - g(h - \sigma) h_p^3 \rho_p + h_p^3 \beta(-p), \tag{2.4.3}$$

$$\mathcal{G}_2^{(\sigma)}(Q, h) := \mathcal{G}_2(Q, h) = \left(1 + h_q^2 + h_p^2 (2g\rho h - Q)\right)\Big|_{p=0}. \tag{2.4.4}$$

That is, we replace the d-term of $\mathcal{G}_1$ with the real number σ. As will be seen later, in the function space we shall work in, this defines a uniformly elliptic differential operator for each $\sigma \in \mathbb{R}$. This is essential, for it allows us to exploit Schauder theory in order to prove the compactness properties we need.

Let $\delta > 0$ be given. In order to ensure the uniformity of the ellipticity of $\mathcal{G}_1^{(\sigma)}$ and obliqueness of $\mathcal{G}_2^{(\sigma)}$ we will work in the set

$$\mathcal{O}_\delta := \left\{ (Q, h) \in \mathbb{R} \times X : h_p > \delta \text{ in } \overline{R},\ h < \frac{Q - \delta}{2g\rho} \text{ on } T \right\}.$$

Likewise, we put

$$S_\delta := \text{closure in } \mathbb{R} \times X \text{ of } \{(Q,h) \in \mathcal{O}_\delta : \mathcal{G}(Q,h) = 0,\ h_q \not\equiv 0\},$$

and let C'_δ be the component of S_δ that contains the point (Q^*, H^*), where $Q^* := Q(\lambda^*)$, $H^* := H(\cdot;\lambda^*)$. Thus C'_δ contains the local curve C'_{loc}.

For later reference, we compute the Fréchet derivatives of $\mathcal{G}$ and $\mathcal{G}^{(\sigma)}$:

$$\begin{aligned}
\mathcal{G}_{1h}(h) &= 2h_q h_{pp}\partial_q + \left(1 + h_q^2\right)\partial_p^2 + 2h_p h_{qq}\partial_p + h_p^2\partial_q^2 \\
&\quad - 2(h_{pq}h_q\partial_p + h_{pq}h_p\partial_q + h_p h_q \partial_p\partial_q) \\
&\quad - 3g\rho_p(h - d(h))h_p^2\partial_p - g\rho_p h_p^3(1-d) + 3h_p^2\beta(-p)\partial_p, && (2.4.5)
\end{aligned}$$

$$\mathcal{G}_{2h}(Q,h) = \left(2h_q\partial_q + 2g\rho h_p^2 + 2h_p(2g\rho h - Q)\partial_p\right)\big|_T, \qquad (2.4.6)$$

$$\begin{aligned}
\mathcal{G}^{(\sigma)}_{1h}(h) &= \mathcal{G}_{1h}(h) + 3g(d(h) - \sigma)h_p^3\rho_p\partial_p + g\rho_p h_p^3 d \\
&= 2h_q h_{pp}\partial_q + \left(1 + h_q^2\right)\partial_p^2 + 2h_p h_{qq}\partial_p + h_p^2\partial_q^2 \\
&\quad - 2(h_{pq}h_q\partial_p + h_{pq}h_p\partial_q + h_p h_q \partial_p\partial_q) \\
&\quad - 3g\rho_p(h - \sigma)h_p^2\partial_p - g\rho_p h_p^3 + 3h_p^2\beta(-p)\partial_p, && (2.4.7)
\end{aligned}$$

$$\mathcal{G}^{(\sigma)}_{2h}(Q,h) = \mathcal{G}_{2h}(Q,h). \qquad (2.4.8)$$

In the spirit of Rabinowitz [**60**], we shall apply a degree theoretic argument to get a global continuation theorem in the form of an alternative result.

THEOREM 2.4.1 (global bifurcation). *Let $\delta > 0$ be given. One of the following must hold:*

(i) $\mathcal{C}'_\delta$ *is unbounded in* $\mathbb{R} \times X$.

(ii) $\mathcal{C}'_\delta$ *contains another trivial point* $(Q(\lambda), H(\lambda)) \in \mathcal{T}$, *with* $\lambda \neq \lambda^*$.

(iii) $\mathcal{C}'_\delta$ *contains a point* $(Q,h) \in \partial\mathcal{O}_\delta$.

Observe, however, that $\mathcal{G}$ is not a compact perturbation of identity, which rules out using the classical Leray–Schauder degree (see, e.g., [**27**]). In light of the nonlinear boundary operator $\mathcal{G}_2$, we, instead, employ a variant degree theory developed by Healey and Simpson (cf. [**32**]). In order to do so, we must first establish two lemmas on the topological properties of the map $\mathcal{G}$. The structure of the arguments in both cases will be to begin by using elliptic estimates on $\mathcal{G}^{(\sigma)}$ (which is uniformly elliptic in $\mathcal{O}_\delta$). Then, taking advantage of the fact that the operator d can be easily estimated in X, we reinsert the stratification term and are able to make conclusions about $\mathcal{G}$.

We emphasize that these lemmas are key. Once $\mathcal{G}$ has been shown to be admissible in the sense of Healey–Simpson degree (cf. Definition 4.10 of [**32**]), the proof of Theorem 4.1 is identical to that of the homogeneous case in [**15**]. Indeed, the technical obstacles presented by the stratification term are completely confined to the following three proofs.

Lemma 2.4.1 (proper map). *Suppose K is a compact subset of Y and D is a closed, bounded set in $\overline{\mathcal{O}_\delta}$, then $\mathcal{G}^{-1}(K) \cap D$ is compact in $\mathbb{R} \times X$.*

Proof. First we prove that for each Q, $\sigma \in \mathbb{R}$, the operator $h \mapsto \mathcal{G}^{(\sigma)}(Q,h)$ is uniformly elliptic and oblique in $\overline{\mathcal{O}_\delta}$. The former follows from the fact that the coefficients of the higher order terms in $\mathcal{G}_{1h}$ satisfy

$$4(1+h_q^2)h_p^2 - 4h_q^2h_p^2 = 4h_p^2 > \delta^2.$$

Notice that the bound here does not in any way depend on σ or Q. Similarly, we have that the boundary operator $h \mapsto \mathcal{G}_2^{(\sigma)}(h)$ is uniformly oblique as the coefficient of h_p satisfies

$$|(2g\rho h - Q)h_p| \geq \delta^2 \qquad \text{on } T$$

for $(Q,h) \in \overline{\mathcal{O}_\delta}$.

Let $\{(f_j, g_j)\}$ be a convergent sequence in $Y = Y_1 \times Y_2$ and assume that for each $j \geq 1$, $(f_j, g_j) = \mathcal{G}(Q_j, h_j)$ for some $(Q_j, h_j) \in \overline{\mathcal{O}_\delta}$, with $\{h_j\}$ bounded in $C^{3+\alpha}_{\mathrm{per}}(\overline{R})$, Q_j bounded in $\mathbb{R}$. We wish to show that there exists a subsequence of $\{(Q_j, h_j)\}$ convergent in $\mathbb{R} \times X$.

Towards that end we denote $\theta_j := \partial_q h_j$ for $j \geq 1$. Then, differentiating the relation between (Q_j, h_j) and (f_j, g_j), we find by (2.4.1),

$$\begin{aligned}
\partial_q f_j &= \partial_q \mathcal{G}_1(h_j) \\
&= \left(1 + (\partial_q h_j)^2\right)\partial_p^2\theta_j + \left(\partial_q\left(1 + (\partial_q h_j)^2\right)\right)\partial_p^2 h_j + (\partial_p h_j)^2\partial_q^2\theta_j + \left(\partial_q^2 h_j\right)\partial_q(\partial_p h_j)^2 \\
&\quad - 2(\partial_q h_j)(\partial_p h_j)\partial_p\partial_q\theta_j - 2\left(\partial_q^2 h_j\right)(\partial_p h_j)(\partial_p\partial_q h_j) - 2(\partial_q h_j)(\partial_p\partial_q h_j)^2 \\
&\quad - g(\partial_p h_j)^3\rho_p\theta_j - g(h_j - d(h_j))\partial_q(\partial_p h_j)^3 + \partial_q(\partial_p h_j)^3\beta(-p) \quad \text{in } R \text{ for all } j \geq 1.
\end{aligned}$$

Grouping terms above we may rewrite this in the form

$$\begin{aligned}&\left(1+(\partial_q h_j)^2\right)\partial_p^2\theta_j+(\partial_p h_j)^2\partial_q^2\theta_j-2(\partial_q h_j)(\partial_p h_j)\partial_q\partial_p\theta_j\\&\quad=\partial_q f_j+F\left(\partial_q h_j,\partial_p\partial_q h_j,\partial_q^2 h_j,\partial_p h_j,\partial_p^2 h_j,d(h_j),\rho_p\right)\qquad\text{in } R,\end{aligned}\tag{2.4.9}$$

where F is the cubic polynomial dictated by the previous expression. Notice that we have picked up a dependence on $d(h_j)$ and ρ_p that was not there in the constant density case presented in [**15**]. This will not be an issue, however, as these terms can still be bounded in X.

We may likewise differentiate the equation on T to discover that, for each $j \geq 1$,

$$\begin{aligned}\partial_q g_j &= \partial_q \mathcal{G}_2(Q_j, h_j)\\&= 2(\partial_q h_j)\partial_q\theta_j+2(\partial_p h_j)(2g\rho h_j-Q_j)\partial_p\theta_j+(\partial_p h_j)^2(2g\rho\partial_q h_j)\qquad\text{on } T.\end{aligned}$$

Equivalently,

$$2(\partial_q h_j)\partial_q\theta_j+2(\partial_p h_j)(2g\rho h_j-Q_j)\partial_p\theta_j=\partial_q g_j+G(\partial_q h_j,\partial_p h_j,\rho)\qquad\text{on } T \text{ for all } j\geq 1,\tag{2.4.10}$$

where G is the quadratic polynomial determined by the previous equation. Finally, we observe that

$$\theta_j = 0 \qquad \text{on } B.\tag{2.4.11}$$

Since $\{(f_j, g_j)\}$ is convergent in $C^{1+\alpha}_{\text{per}}(\overline{R}) \times C^{2+\alpha}_{\text{per}}(T)$, we have that the sequences $\{\partial_q f_j\}$, $\{\partial_q g_j\}$ are Cauchy in $C^{\alpha}_{\text{per}}(\overline{R})$ and $C^{1+\alpha}_{\text{per}}(T)$, respectively. Moreover, by the boundedness of $\{h_j\}$ in $C^{3+\alpha}_{\text{per}}(\overline{R})$ (and thereby the boundedness of $d(h_j)$ in this space), we have that F is uniformly bounded in $C^{1+\alpha}_{\text{per}}(\overline{R})$ and G is uniformly bounded in $C^{2+\alpha}_{\text{per}}(T)$. It follows that each of these, viewed as a sequence in j, is Cauchy. Then, by the compactness of the embeddings, the right-hand sides of (2.4.9) and (2.4.10) are precompact in $C^{\alpha}_{\text{per}}(\overline{R})$ and $C^{1+\alpha}_{\text{per}}(T)$, respectively. Possibly passing to a subsequence, we may take both to be convergent in these spaces.

We now consider differences $\theta_j - \theta_k$ for $j, k \geq 1$. By (2.4.9) we have

$$(1+(\partial_q h_j)^2)\partial_p^2(\theta_j-\theta_k)+(\partial_p h_j)^2\partial_q^2(\theta_j-\theta_k)-2(\partial_q h_j)(\partial_p h_j)\partial_q\partial_p(\theta_j-\theta_k)=F_{jk}\quad\text{in } R,$$

where by our arguments in the previous paragraph we know $F_{jk} \to 0$ in $C^\alpha(\overline{R})$. Similarly, from (2.4.11), we have that $\theta_j - \theta_k$ vanishes on the bottom, and on the top (2.4.10) tells us

$$2(\partial_q h_j)\partial_q(\theta_j - \theta_k) + 2(\partial_p h_j)(2g\rho h_j - Q_j)\partial_p(\theta_j - \theta_k) = G_{jk} \qquad \text{on } T.$$

Here $G_{jk} \to 0$ in $C^{1+\alpha}(T)$, again by the considerations of the preceding paragraph. We now apply the mixed-boundary condition Schauder estimates to the differences $\theta_j - \theta_k$ to deduce that $\|\theta_j - \theta_k\|_{C^{2+\alpha}(\overline{R})} \to 0$ as $j, k \to \infty$. We have shown, therefore, that all third derivatives of h_j are Cauchy, except possibly for $\partial_p^3 h_j$. To demonstrate the same holds for $\{\partial_p^3 h_j\}$, we use the PDE to express $\partial_p^2 h_j$ in terms of the derivatives of order less than or equal to two:

$$\begin{aligned}\partial_p^2 h_j \quad &= \quad \left(1 + (\partial_q h_j)^2\right)^{-1} \Big(f_j - \partial_q^2 h_j (\partial_p h_j)^2 + 2(\partial_q h_j)(\partial_p h_j)(\partial_p \partial_q h_j) \\ &\qquad + g(h_j - \sigma)(\partial_p h_j)^3 \rho_p + (\partial_p h_j)^3 \beta(-p) \Big) \qquad \text{in } R,\ j \geq 1.\end{aligned}$$

But we have seen that the right-hand side is Cauchy in $C^{1+\alpha}(\overline{R})$; hence $\{\partial_p^3 h_j\}$ is also Cauchy in $C^\alpha_{\text{per}}(\overline{R})$. We conclude that the original sequence $\{h_j\}$ had a convergent subsequence in $C^{2+\alpha}_{\text{per}}(\overline{R})$, and hence $\{(Q_j, h_j)\}$ had a convergent subsequence in $\mathbb{R} \times X$. □

Lemma 2.4.2 (Fredholm map). *For each $(Q, h) \in \mathcal{O}_\delta$, the linearized operator $\mathcal{G}_h(Q, h)$ is a Fredholm map of index 0 from X to Y.*

Proof. In the previous lemma we established that $\mathcal{G}^{(\sigma)}$ was uniformly elliptic and oblique for all $(Q, h) \in \mathcal{O}_\delta$ and for all $\sigma \in \mathbb{R}$. We remark also that these bounds are independent of σ.

Now let $\psi \in C^{3+\alpha}_{\text{per}}(\mathbb{R})$ be given and fix $\sigma, Q \in \mathbb{R}$, $h \in X$. Put

$$\phi^{(i)} := \partial_q \left(\mathcal{G}^{(\sigma)}_{ih}(Q, h)[\psi] \right) - \left(\partial_q \mathcal{G}^{(\sigma)}_{ih}(Q, h) \right) [\psi] \qquad \text{for } i = 1, 2.$$

Here, by $(\partial_q \mathcal{G}_{ih}^{(\sigma)}(Q,h))[\psi]$ we mean differentiating the coefficients of $\mathcal{G}_{ih}^{(\sigma)}$ in q, then applying the resulting operator to ψ. Then $\partial_q \psi$ satisfies

$$\begin{cases} \mathcal{G}_{1h}^{(\sigma)}(h)\partial_q\psi = \phi^{(1)} & \text{on } R, \\ \\ \mathcal{G}_{2h}^{(\sigma)}(Q,h)\partial_q\psi = \phi^{(2)} & \text{on } T, \\ \\ \partial_q\psi = 0 & \text{on } B, \end{cases}$$

which is a uniformly elliptic PDE with an oblique boundary condition. The classical Schauder estimates ensure the existence of a constant $C > 0$, independent of ψ, such that

$$\begin{aligned} C\|\partial_q\psi\|_{C^{2+\alpha}(\overline{R})} &\leq \|\partial_q\psi\|_{C^\alpha(\overline{R})} + \left\|\phi^{(1)}\right\|_{C^\alpha(\overline{R})} + \left\|\phi^{(2)}\right\|_{C^{1+\alpha}(T)} \\ &\leq \|\psi\|_{C^{2+\alpha}(\overline{R})} + \left\|\partial_q\left(\mathcal{G}_{1h}^{(\sigma)}(h)\psi\right)\right\|_{C^\alpha(\overline{R})} \\ &\qquad + \left\|\partial_q\left(\mathcal{G}_{2h}^{(\sigma)}(Q,h)\psi\right)\right\|_{C^{1+\alpha}(T)}. \end{aligned}$$

On the other hand, we may express $\partial_p^2\psi$ via the partial differential equation to arrive at an estimate for $\partial_p^3\psi$ of the same type. Combining we get

$$C\|\psi\|_{C^{3+\alpha}(\overline{R})} \leq \|\psi\|_{C^{2+\alpha}(\overline{R})} + \left\|\partial_q\phi^{(1)}\right\|_{C^\alpha(\overline{R})} + \left\|\partial_q\phi^{(2)}\right\|_{C^{1+\alpha}(T)}.$$

If we now apply the classical Schauder estimates to ψ, we find that for some $C > 0$, independent of ψ,

$$C\|\psi\|_{C^{2+\alpha}(\overline{R})} \leq \|\psi\|_{C^\alpha(\overline{R})} + \left\|\partial_q\left(\mathcal{G}_{1h}^{(\sigma)}(h)\psi\right)\right\|_{C^\alpha(\overline{R})} + \left\|\partial_q\left(\mathcal{G}_{2h}^{(\sigma)}(Q,h)\psi\right)\right\|_{C^{1+\alpha}(T)}.$$

Together with the previous line, this implies that there exists a constant $C = C(\sigma) > 0$ such that, for all $\psi \in C^{3+\alpha}(\overline{R})$ even and periodic in q that vanish on B,

$$C(\sigma)\|\psi\|_X \leq \|\psi\|_{C^\alpha(\overline{R})} + \left\|\mathcal{G}_{1h}^{(\sigma)}(h)\psi\right\|_{Y_1} + \left\|\mathcal{G}_{2h}^{(\sigma)}(Q,h)\psi\right\|_{Y_2}. \tag{2.4.12}$$

Of course, this is not quite the estimate we are after, since there is a lingering dependence on σ. In order to remedy this we make the following elementary, but very useful, observation: for any ψ as above, we have

$$|d(\psi)| \leq \fint_T |\psi| dq \leq \|\psi\|_{C^\alpha(\overline{R})}.$$

We are, therefore, able to estimate terms involving d easily in spaces of Hölder continuous functions. In particular,

$$\begin{aligned} \left\| \mathcal{G}_{1h}(h)\psi - \mathcal{G}_{1h}^{(\sigma)}(h)\psi \right\|_{C^\alpha(\overline{R})} &= \left\| 3g(d(h)-\sigma)h_p^3\rho_p\psi_p + g\rho_p h_p^3 d(\psi)\right\|_{C^\alpha(\overline{R})} \\ &\leq C\|\psi\|_{C^{1+\alpha}(\overline{R})}, \end{aligned} \tag{2.4.13}$$

where the constant C above depends only on ρ_p, $\|h\|_X$ and our choice of σ. Likewise, an identical argument gives the estimates

$$\left\| \partial_i \mathcal{G}_{1h}(h)\psi - \partial_i \mathcal{G}_{1h}^{(\sigma)}(h)\psi \right\|_{C^\alpha(\overline{R})} \leq C\|\psi\|_{C^{2+\alpha}(\overline{R})}, \qquad i = p,\ q,$$

where C depends again on ρ_p, $\|h\|_X$ and the choice of σ. Of course, we do not need to make such arguments to estimate the $\mathcal{G}_{2h}^{(\sigma)}(h)$ term, as it is identical to $\mathcal{G}_{2h}(h)$.

Combining these observations with our first estimate (2.4.12), we find that, for some $C > 0$ and all $\psi \in X$,

$$C\|\psi\|_X \leq \|\psi\|_{C^{2+\alpha}(\overline{R})} + \|\mathcal{G}_{1h}(h)\psi\|_{Y_1} + \|\mathcal{G}_{2h}(Q,h)\psi\|_{Y_2}. \tag{2.4.14}$$

The key point here is that $X = C^{3+\alpha}_{\mathrm{per}}(\overline{R})$ is compactly embedded in the spaces who appear on the right-hand side of (2.4.14). To show that the null space of $\mathcal{G}_h(Q,h)$ is finite dimensional, for example, it suffices to show that the unit ball is compact. But for any sequence $\{\psi_n\} \subset \{\psi \in \mathcal{N}(\mathcal{G}_h(Q,h)) : \|\psi\|_{C^{3+\alpha}(\overline{R})} \leq 1\}$, we have that, by the compactness of the embedding, there exists a convergent subsequence $\{\psi_{n_k}\}$ in $C^{2+\alpha}(\overline{R})$. This subsequence is Cauchy in $C^{2+\alpha}(\overline{R})$, so applying (2.4.14), we see it is also Cauchy in X. Finally, by completeness, $\{\psi_{n_k}\}$ is convergent in X. It follows that the unit ball in the null space is compact; hence the null space is finite dimensional.

The proof that the range is closed is, likewise, standard. Again, the argument hinges on the compact embedding of $C^{3+\alpha}(\overline{R}) \subset\subset C^{2+\alpha}(\overline{R})$. For brevity, we omit the details.

Now, the Fredholm index has a discrete range; hence by the connectedness of $\mathcal{O}_\delta$ it must be constant on this set. But $\mathcal{G}_h(Q^*, H^*)$, which we denoted $\mathcal{F}_w(\lambda^*, 0)$ in the previous section, was shown to have a one-dimensional null space in Lemma 2.3.7. Moreover, by the orthogonality condition of Lemma 2.3.8, its range has codimension one. Since $(Q^*, H^*) \in \mathcal{O}_\delta$ by construction, the Fredholm index is uniformly 0 along the continuum $\mathcal{C}'_\delta$. $\square$

Finally, we prove a result characterizing the spectrum of the operator $\mathcal{G}$.

Lemma 2.4.3 (spectral properties). (i) *for all* $\delta > 0$*, there exists* $c_1, c_2 > 0$ *such that for all* $(Q, h) \in \mathcal{O}_\delta$*, with* $|Q| + \|h\|_X \leq M$*, for all* $\psi \in X$*, and for all real* $\mu \geq c_2$*, we have*

$$c_1\|\psi\|_X \leq \mu^{\frac{\alpha}{2}}\|(A-\mu)\psi\|_{Y_1} + \mu^{\frac{1+\alpha}{2}}\|B\psi\|_{Y_2},$$

where $A = A(Q, h) = \mathcal{G}_{1h}(h)$ *and* $B = B(Q, h) = \mathcal{G}_{2h}(Q, h)$*.*

(ii) *Define the spectrum* $\Sigma = \Sigma(Q, h)$ *by*

$$\Sigma(Q, h) := \{\mu \in \mathbb{C} : A - \mu \text{ is not isomorphic from } \{\psi \in X : B\psi = 0 \text{ on } T\} \text{ onto } Y_1\}.$$

Then Σ *consists entirely of the eigenvalues of finite multiplicity with no finite accumulation points. Furthermore, there is a neighborhood* $\mathcal{N}$ *of* $[0, +\infty)$ *in the complex plane such that* $\Sigma(\lambda, w) \cap \mathcal{N}$ *is a finite set.*

(iii) *For all* $(Q, h) \in \mathcal{O}_\delta$*, the boundary operator* $\mathcal{G}_{2h}(Q, h)$ *from* $X \to Y_2$ *is onto.*

Proof. In this proof we follow the well-tread path laid forth by Agmon in [**1**]. The only novelty here is in (i), where we must do a little work before we can apply the elliptic estimates. The remaining parts are completely standard; their proofs rely only on (i), Lemma 2.4.1, and Lemma 2.4.2 (cf. [**1**], [**15**], or [**32**], for example). For that reason, we omit them here and devote our attention to (i).

Fix $\sigma \in \mathbb{R}$, and let $\psi \in X$ and $\mu \in \mathbb{C}$ be given. Put $\theta := \arg\mu$, and suppose for some $\epsilon > 0$, $|\theta| < \pi/2 + \epsilon$. Consider the operator $D^{(\sigma)} := A^{(\sigma)} + e^{i\theta}\partial_t^2$ on $\mathbb{R} \times R$, where $A^{(\sigma)} := \mathcal{G}_{1h}^{(\sigma)}(h)$. Then, $D^{(\sigma)}$ is elliptic for each σ, and the boundary condition on the top is oblique; hence the complementing condition holds. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a cutoff function supported compactly in the interval $I := (-1, 1)$. Put

$$e(t) := e^{i|\mu|^{1/2}t}\zeta(t), \qquad \phi(t, q, p) := e(t)\psi(q, p).$$

We apply the Schauder estimates in $\mathbb{R}^3$ with the boundary operator B to $\phi(t, q, p)$ to deduce

$$C\|\phi\|_{C^{3+\alpha}(I\times R)} \leq \|\phi\|_{C^\alpha(I\times R)} + \|D^{(\sigma)}\phi\|_{C^{1+\alpha}(I\times R)} + \|B\phi\|_{C^{2+\alpha}(I\times T)} \tag{2.4.15}$$

for some constant $C > 0$ independent of ψ.

A quick calculation readily confirms the existence of $C,\ C' > 0$, depending only on ζ, with

$$C\mu^{\frac{\alpha}{2}} \leq \|e\|_{C^\alpha(\mathbb{R})} \leq C'\mu^{\frac{\alpha}{2}}, \qquad C\mu^{\frac{1+\alpha}{2}} \leq \|e\|_{C^{1+\alpha}(\mathbb{R})} \leq C'\mu^{\frac{1+\alpha}{2}}. \tag{2.4.16}$$

Using this, we can unpack (2.4.15) to derive the following set of estimates:

$$\begin{aligned}\left\|D^{(\sigma)}\phi\right\|_{C^{1+\alpha}(I\times R)} &= \Big\|e(t)\mathcal{G}^{(\sigma)}_{1h}(h)[\psi] - \mu e(t)\psi \\ &\quad + \left(|\mu|^{1/2}\zeta' + \zeta''\right)e^{i\theta}e^{i|\mu|^{1/2}t}\psi\Big\|_{C^{1+\alpha}(I\times R)} \\ &\leq C|\mu|^{(1+\alpha)/2}\Big(\left\|\left(\mathcal{G}_{1h}(h)-\mu\right)\psi\right\|_{C^{1+\alpha}(R)} \\ &\quad + \left\|\left(G_{1h}(h) - \mathcal{G}^{(\sigma)}_{1h}(h)\right)\psi\right\|_{C^{1+\alpha}(R)} + \|\psi\|_{C^{1+\alpha}(R)}\Big)\end{aligned}$$

for $|\mu|$ sufficiently large. We can estimate the second term in parenthesis by appealing to (2.4.13) in the previous lemma, yielding

$$\left\|D^{(\sigma)}\phi\right\|_{C^{1+\alpha}(I\times R)} \leq C|\mu|^{\frac{1+\alpha}{2}}\left(\|(\mathcal{G}_{1h}(h)-\mu)\psi\|_{C^{1+\alpha}(R)} + \|\psi\|_{C^{2+\alpha}(R)}\right) \tag{2.4.17}$$

for $|\mu|$ sufficiently large.

Similarly, analyzing the boundary terms we find for some $C > 0$, independent of ψ, and $|\mu|$ sufficiently large,

$$\begin{aligned}\|B\phi\|_{C^{2+\alpha}(I\times T)} &= \|e(t)\mathcal{G}_{2h}(Q,h)[\psi]\|_{C^{2+\alpha}(I\times T)} \\ &\leq C|\mu|^{\frac{2+\alpha}{2}}\|\psi\|_{C^{2+\alpha}(T)}.\end{aligned} \tag{2.4.18}$$

On the other hand, using (2.4.16) and the fact $C^{3+\alpha} \subset\subset C^{k+\alpha}$, for $k < 3$, we can show that for some $C > 0$, independent of ψ, and for all $|\mu|$ sufficiently large,

$$\|\phi\|_{C^{3+\alpha}(\mathbb{R}^3)} \geq C\mu^{\frac{1}{2}}\|\psi\|_{C^{3+\alpha}(R)}. \tag{2.4.19}$$

Now, inserting (2.4.17), (2.4.18), and (2.4.19) into (2.4.15), we arrive at the inequality in (i). □

Let $\mathcal{W}$ be an open bounded subset of X, and fix $Q \in \mathbb{R}$. By virtue of Lemmas 2.4.1–2.4.3, we conclude that $\mathcal{G}(Q,\cdot) : \overline{\mathcal{W}} \to Y$ is admissible in the sense of [**32**]. This enables us to use the generalization of the Leray–Schauder degree introduced in that paper, which we now briefly recapitulate.

Let $y \in Y \setminus \mathcal{G}(Q, \partial\mathcal{W})$ be a regular value of $\mathcal{G}(Q,\cdot)$. We define the (Healey–Simpson) degree of $\mathcal{G}(Q,\cdot)$ at y with respect to $\mathcal{W}$ by

$$\deg\left(\mathcal{G}(Q,\cdot), \mathcal{W}, y\right) := \sum_{w\in\mathcal{G}^{-1}(\{y\})} (-1)^{\nu(w)},$$

where $\nu(w)$ is the number of positive real eigenvalues counted by multiplicity of $\mathcal{G}_{1w}(Q, w)$, subject to the boundary condition $\mathcal{G}_{2w}(Q, w) = 0$ on T. By Lemma 2.4.3, $\nu(w)$ is finite. Likewise, the properness of $\mathcal{G}(Q, \cdot)$ established in Lemma 2.4.1 implies $\mathcal{G}^{-1}(\{y\}) \cap \mathcal{W}$ is a finite set. The degree is therefore well defined at proper values. In the usual way, this definition is extended to critical values via the Sard–Smale theorem (this is valid by Lemma 2.4.2).

Our interest in degree theory stems from the fact that the degree is invariant under any homotopy that respects the boundaries of $\mathcal{W}$ in the following sense. Let $\mathcal{U} \subset [0, 1] \times \mathcal{W}$ be open. Define $\mathcal{U}_t := \{w \in \mathcal{W} : (t, w) \in \mathcal{U}\}$ and likewise $\partial\mathcal{U}_t := \{w \in \mathcal{W} : (t, w) \in \partial\mathcal{U}\}$.

Lemma 2.4.4 (homotopy invariance). *The degree is invariant under admissible homotopies. That is, suppose $\mathcal{U} \subset [0, 1] \times \mathcal{W}$ is open. If $\mathcal{H} \in C^2(\overline{\mathcal{U}}; Y)$ is proper and, for each $t \in [0, 1]$, $\mathcal{H}(t, \cdot)$ is admissible (i.e., satisfies the conclusions of Lemmas* 2.4.1–2.4.3*), then we say $\mathcal{H}$ is an admissible homotopy and have*

$$\deg\ (\mathcal{H}(0, \cdot), \mathcal{U}_0, y) = \deg\ (\mathcal{H}(1, \cdot), \mathcal{U}_1, y)$$

provided $y \notin \mathcal{H}(t, \partial\mathcal{U}_t)$ for all $t \in [0, 1]$.

From our previous analysis it follows that $\mathcal{G}(Q_1, \cdot) \mapsto \mathcal{G}(Q_2, \cdot)$ is an admissible homotopy. Thus, in light of Lemma 2.4.4, the degree will remain constant as we move along the continuum. Assuming that all three alternatives of Theorem 2.4.1 fail, we use this feature to generate a contradiction. At this stage, we have reestablished all the relevant properties of $\mathcal{G}$ that were true in the constant density case. Repeating (verbatim) the proof in [**15**] we obtain Theorem 2.4.1.

2.5. Nodal pattern

We now seek to investigate the second possibility of the global bifurcation theorem, namely, that the continuum arcs back and intersects $\mathcal{T}$ at some point aside from λ^*. This will be done by assuming there exists some other laminar solution on $\mathcal{C}'_\delta$ and analyzing the nodal pattern to show it to be, in fact, equal to λ^*. In particular, we will be concerned with the vanishing of h_q. We therefore work in the set $\Omega := (0, \pi) \times (p_0, 0) \subset R$. Denote

$$\begin{aligned} \partial\Omega_t &:= \{(q, 0) : q \in (0, \pi)\}, \\ \partial\Omega_b &:= \{(q, p_0) : q \in (0, \pi)\}, \\ \partial\Omega_r &:= \{(\pi, p) : p \in (p_0, 0)\}, \\ \partial\Omega_l &:= \{(0, p) : p \in (p_0, 0)\}. \end{aligned}$$

It follows that $h = 0$ on $\partial\Omega_b$ for all $(Q, h) \in \mathcal{C}'_\delta$, and for $h \in X$, periodicity and evenness in q yield $h_q = 0$ on $\partial\Omega_r \cup \partial\Omega_l$. In the following lemmas we will attempt to prove that for $h \in \mathcal{C}'_\delta \setminus \{(Q^*, H^*)\}$, we have, additionally,

$$\begin{cases} h_q < 0 & \text{in } \Omega \cup \partial\Omega_t, \\ h_{qp} < 0 & \text{on } \partial\Omega_b, \\ h_{qq} < 0 & \text{on } \partial\Omega_l, \\ h_{qq} > 0 & \text{on } \partial\Omega_r, \end{cases} \tag{2.5.1}$$

and on the bottom corners of Ω:

$$h_{qqp}(0, p_0) < 0, \qquad h_{qqp}(\pi, p_0) > 0, \tag{2.5.2}$$

at the top corners, we have

$$h_{qq}(\pi, 0) > 0, \qquad h_{qq}(0, 0) < 0. \tag{2.5.3}$$

These inequalities define an open set in X.

Our first result is the following.

Lemma 2.5.1. *Properties* (2.5.1)–(2.5.3) *hold in a small neighborhood of* (Q^*, H^*) *in* $\mathbb{R} \times C^{3+\alpha}_{per}(\bar{\Omega})$ *along the bifurcation curve* $\mathcal{C}'_{loc} \setminus \{(Q^*, H^*)\}$ *originating from the point* (Q^*, H^*).

The proof of this lemma does not make any special reference to the form of the operator $\mathcal{G}$, only the basic properties of the eigenfunction w^* and the local bifurcation curve. Not surprisingly, therefore, it follows with virtually no modification from the homogeneous case treated in [**15**]. We therefore omit it and concentrate on the remaining lemmas, which will require more finesse.

Differentiating the relation $\mathcal{G}_1^{(\sigma)}(h) = 0$ by q, for any choice of $\sigma \in \mathbb{R}$, yields $\mathcal{G}_{1h}^{(\sigma)}(h)[h_q] = 0$ for $(Q,h) \in \mathcal{C}'_\delta$, where we recall that

$$\begin{aligned}\mathcal{G}_{1h}^{(\sigma)}(h)[\phi] &= 2h_q h_{pp}\phi_q + \left(1+h_q^2\right)\phi_{pp} + 2h_p h_{qq}\phi_p + h_p^2\phi_{qq} \\ &\quad - 2(h_{pq}h_q\phi_p + h_q h_p \phi_q + h_p h_q \phi_{pq}) \\ &\quad - 3g\rho_p(h-\sigma)h_p^2\phi_p - g\rho_p h_p^3\phi + 3h_p^2\beta(-p)\phi_p.\end{aligned}$$

Likewise, differentiating the boundary relation we find that $\mathcal{G}_{2h}(Q,h)[h_q] = 0$, for $(Q,h) \in \mathcal{C}'_\delta$, where

$$\mathcal{G}_{2h}(Q,h)[\phi] := 2h_q\phi_q + 2g\rho h_p^2\phi + 2h_p(2g\rho h - Q)\phi_p.$$

An important observation is that $\mathcal{G}_{1h}^{(\sigma)}$—though elliptic for given $\sigma \in \mathbb{R}$ and $h \in \mathcal{O}_\delta$—has a zeroth order term. We have dictated that $\rho_p \leq 0$, so that the sign will go precisely "the wrong way," in the sense that the maximum principle will not hold, in general. The argument will be saved by the observation that the ϕ under consideration is known a priori to be nonpositive. If we denote

$$\mathcal{H}^{(\sigma)}(h) := \mathcal{G}_{1h}^{(\sigma)}(h) + g\rho_p h_p^3,$$

then $\mathcal{H}^{(\sigma)}(h)$ is uniformly elliptic with no zeroth order terms. Moreover, every nonpositive ϕ that is a solution relative to $\mathcal{G}_{1h}^{(\sigma)}(h)$ is a subsolution relative to $\mathcal{H}^{(\sigma)}(h)$.

LEMMA 2.5.2. *Properties* (2.5.1)–(2.5.3) *hold along* $\mathcal{C}'_\delta \setminus \{(Q^*, H^*)\}$ *unless there exists* $\lambda \neq \lambda^*$, *with* $(Q(\lambda), H(\lambda)) \in \mathcal{C}'_\delta$.

PROOF. By the local bifurcation theorem we know that in a sufficiently small neighborhood of (Q^*, H^*), the bifurcation curve $\mathcal{C}'_\delta$ consists entirely of the curve C'_{loc}. Suppose by contradiction that the statement of the lemma is false. As the previous lemma implies that (2.5.1)–(2.5.3) will hold near (Q^*, H^*) in $\mathbb{R} \times C^{3+\alpha}(\bar{\Omega})$, we have the existence of some $(Q,h) \in \mathcal{C}'_\delta$, with $h_q \not\equiv 0$ where at least one of the properties fail, although they hold on a sequence $(Q_j, h_j) \to (Q,h)$. Continuity then implies that $h_q \leq 0$ on $\bar{\Omega}$, as this holds for each h_j. Since $\mathcal{C}'_\delta \subset \mathcal{O}_\delta$, we have that $\mathcal{G}_{1h}^{(\sigma)}(Q,h)$ is uniformly elliptic, where we take $\sigma := d(h)$.

We wish to apply and exploit Hopf's lemma to conclude the last three inequalities of (2.5.1) hold. To do so, we first note that, as $h_q \leq 0$ in Ω and $h_q \equiv 0$ on $\partial\Omega_l \cup \partial\Omega_r \cup \partial\Omega_b$, we have $\sup_{\overline{\Omega}} h_q = 0$. But then, by our comments above, we see that $\phi := h_q$ is a subsolution relative to the linear elliptic operator $\mathcal{H}^{(\sigma)}(h)$. Applying Hopf's lemma to $\mathcal{H}^{(\sigma)}(h)$, we conclude

$$\left.\frac{\partial \phi}{\partial \nu}\right|_{(q_0,p_0)} > 0,$$

where (q_0, p_0) is any point where ϕ vanishes and ν is the outward unit normal at (q_0, p_0) (cf., for example, Theorem 2.15 of [**28**]). In particular, since $\partial_\nu = -\partial_q$ on $\partial\Omega_l$ and $h_q \equiv 0$ on this set, we have $h_{qq} < 0$ on $\partial\Omega_l$. An identical argument shows $h_{qq} > 0$ on $\partial\Omega_r$ and $h_{qp} < 0$ on $\partial\Omega_b$.

To establish the strict inequality $h_q < 0$ on $\partial\Omega_t$, we appeal to the nonlinear boundary condition (2.4.2). We already have that $h_q \leq 0$ in this region by continuity. Suppose that for some $q_0 \in (0, \pi)$, we have $h_q(q_0, 0) = 0$. Letting $\phi = h_q$ we use our expression for $\mathcal{G}_{2h}(Q, h)[\phi]$. Furthermore,

$$2h_p(2g\rho h - Q)h_{qp} = 0 \qquad \text{at } (q_0, 0).$$

Again, by Hopf's lemma, we have that $h_{qp}(q_0, 0) > 0$. Moreover, as $(Q, h) \in \mathcal{O}_\delta$, we are guaranteed $h_p \geq \delta > 0$. As $\rho > 0$, this implies $2g\rho h = Q$ on T. But then, the nonlinear boundary condition for the original problem $\mathcal{G}(Q, h) = 0$ would imply $1 + h_q^2 = 0$, which is a contradiction. Hence we have the strict inequality, and therefore (2.5.1) holds for h.

In order to produce a contradiction, therefore, all that remains is to verify the corner properties (2.5.2) and (2.5.3). By continuity we have that $h_{qqp}(\pi, p_0) \geq 0$. Suppose that it vanishes. We have that $h(q, p_0) = 0$ for all q; hence $h_q(\pi, p_0) = h_{qq}(\pi, p_0) = h_{qqq}(\pi, p_0) = \cdots = 0$. Also, by evenness and periodicity in q we have $h_q(\pi, p) = 0$ for all p. Hence $h_{qp}(\pi, p) = h_{qpp}(\pi, p) = \cdots = 0$ for all p. But then if $\nu = \nu_q \hat{q} + \nu_p \hat{p}$, where $\hat{q} = (1, 0)$, $\hat{p} = (0, 1)$, is any vector exiting Ω at (π, p_0), then

$$\frac{\partial h_q}{\partial \nu} = \nu_q h_{qq} + \nu_p h_{pp} = 0 \qquad \text{at } (\pi, p_0)$$

and

$$\frac{\partial^2 h_q}{\partial \nu^2} = 0 \qquad \text{at } (\pi, p_0)$$

by the previous analysis. This violates Serrin's edge lemma producing a contradiction (see, for example, Theorem E.8 of [**28**]). An identical argument applied to $(0, p_0)$ proves that (2.5.2) holds.

For (2.5.3) we consider the point (0,0). By evenness we know that $h_q(0,p) = 0$ for all $p \in (p_0, 0)$. Thus $h_{qp}(0,p) = 0$ for all such p. By continuity, moreover, we are guaranteed that $h_{qq}(0,0) \leq 0$. By contradiction suppose that $h_{qq}(0,0) = 0$. Then differentiating the relation $\mathcal{G}_{2h}(Q,h)[h] = 0$ twice yields

$$h_q h_{qq} + g\rho h_q h_p^2 + h_p h_{pq}(2g\rho h - Q) = 0 \qquad \text{on } \partial\Omega_t$$

and

$$\begin{aligned} h_{qq}^2 + h_q h_{qq} + g\rho h_{qq} h_p^2 + 2g\rho h_q h_{pq} + h_{pq}^2(2g\rho h - Q) & \\ + h_p h_{pqq}(2g\rho h - Q) + 2g\rho h_q h_p h_{pq} &= 0 \qquad \text{on } \partial\Omega_t. \end{aligned}$$

Evaluating both of these at $(0,0)$ and taking our assumption into account, we have all terms involving h_q, h_{qq}, and h_{pq} drop out. Hence

$$h_p h_{pqq}(2g\rho h - Q) = 0 \qquad \text{at } (0,0).$$

But we already have seen that $2g\rho h - Q$ cannot be zero, as this leads to a violation of the of the nonlinear boundary condition. Moreover, h_p is bounded uniformly away from 0 in O_δ, so we must conclude that $h_{pqq}(0,0) = 0$. As before this violates Serrin's edge lemma, as any vector ν leaving Ω through $(0,0)$ can be written as $\nu = \nu_p \hat{p} + \nu_q \hat{q}$, and therefore

$$\frac{\partial h_q}{\partial \nu} = \nu_q h_{qq} + \nu_p h_{pq} = 0 \qquad \text{at } (0,0)$$

and

$$\frac{\partial^2 h_q}{\partial \nu^2} = 0 \qquad \text{at } (0,0),$$

since $h_{pqq}(0,0) = 0$. This proves that (2.5.3) holds for h; hence we have a contradiction establishing the lemma. □

Combining the two previous lemmas we see that the nodal properties (2.5.1)–(2.5.3) will hold along the continuum $\mathcal{C}'_\delta$ unless it intersects the laminar curve somewhere other than λ^*. We now conclude this section by showing this cannot occur for λ to the right of $-2B_{\min} + \epsilon_0$.

Lemma 2.5.3. *If a trivial solution $(Q(\lambda), H(\lambda)) \in \mathcal{C}'_\delta$, with $\lambda > -2B_{\min} + \epsilon_0$, then $\lambda = \lambda^*$.*

Proof. Let $(Q(\lambda), H(\lambda)) \in \mathcal{C}'_\delta$ be given. Clearly if $\lambda = \lambda^*$, we are done, so suppose that this is not the case. As we know locally to (Q^*, H^*) that $\mathcal{C}'_\delta = \mathcal{C}'_{\text{loc}}$, we may assume without loss that there exists a curve $\mathcal{C}'_\delta(\lambda, \lambda^*) \subset \mathcal{C}'_\delta$ originating at (Q^*, H^*) and terminating at (Q, H) that does not intersect $\mathcal{T}$ except at the points λ, λ^*. Let $\{(Q(\lambda_n), h_n)\}$ be a sequence in $\mathcal{C}'_\delta(\lambda, \lambda^*) \setminus \mathcal{T}$ such that $(\lambda_n, h_n) \to (\lambda, H(\lambda))$ in $\mathbb{R} \times C^{3+\alpha}(\bar{\Omega})$. Thus $\partial_q h_n \not\equiv 0$, and $\mathcal{G}(Q, h_n) = 0$ for all $n \geq 1$. We claim that $\partial_q h_n < 0$ for $n \geq 1$. By the first lemma of this section this holds true for all h in some sufficiently small neighborhood of (Q^*, H^*) in X. The arguments of the previous lemma, however, can be directly applied to show that the nodal properties hold along $\mathcal{C}'_\delta(\lambda, \lambda^*) \setminus \{(Q(\lambda), H(\lambda))\}$. That is, if we assume that for some $h \in \mathcal{C}'_\delta(\lambda, \lambda^*)$, (2.5.1)–(2.5.3) fail, then we may again choose a sequence of functions on $\mathcal{C}'_\delta(\lambda, \lambda^*)$ approaching h where the properties hold. The previous lemma can then be applied without further modification to produce a contradiction. This proves the claim. Thus, $\partial_q h_n < 0$ for $n \geq 1$.

We now differentiate the relation $\mathcal{G}(Q_n, h_n)[h_n] = 0$ with respect to q to find that, for each $n \geq 1$,

$$\begin{aligned}
0 &= \partial_q \mathcal{G}_1(h_n) h_n \\
&= \left(1 + (\partial_q h_n)^2\right) \partial_p^2 \partial_q h_n + 2(\partial_q h_n)\left(\partial_q^2 h_n\right)\left(\partial_p^2 h_n\right) + \left(\partial_q^3 h_n\right)(\partial_p h_n)^2 \\
&\quad + 2\left(\partial_q^2 h_n\right)(\partial_p h_n)(\partial_p \partial_q h_n) - 2\left(\partial_q^2 h_n\right)(\partial_p h_n)(\partial_p \partial_q h_n) - 2(\partial_q h_n)(\partial_p \partial_q h_n)^2 \\
&\quad - 2(\partial_q h_n)(\partial_p h_n)\left(\partial_p \partial_q^2 h_n\right) - 3g(h_n - d(h_n))(\partial_p h_n)^2(\partial_q \partial_p h_n)\rho_p \\
&\quad - g(\partial_q h_n)(\partial_p h_n)^3 \rho_p + 3(\partial_p h_n)^2(\partial_q \partial_p h_n)\beta(-p) \\
&= \mathcal{G}_{1h}^{(\sigma_n)}(h_n)[v_n],
\end{aligned}$$

where $v_n := \|\partial_q h_n\|^{-1}_{C^{2+\alpha}(\bar{\Omega})} \partial_q h_n$ and $\sigma_n := d(h_n)$. Likewise, we may differentiate the boundary relation to conclude that $\mathcal{G}_{2h}^{(\sigma_n)}(Q_n, h_n)[v_n] = \mathcal{G}_{2h}(Q_n, h_n)[v_n] = 0$ for all $n \geq 1$.

For brevity, denote $(Q, H) := (Q(\lambda), H(\lambda))$ and $\sigma := d(H)$. We observe that since $h_n \to H$ in $C^{3+\alpha}(\bar{\Omega})$, we must have $\sigma_n \to \sigma$. Combining this with the convergence of $h_n \to H$ in the $C^{3+\alpha}$-norm, we see $\{\mathcal{G}_{1h}^{(\sigma_n)}(h_n)\}$ is a Cauchy sequence in $\mathcal{L}(C^{2+\alpha}(\bar{\Omega}), C^\alpha(\bar{\Omega}))$.

Then for $i, j \geq 1$, we have

$$\left\| \left(\mathcal{G}_{1h}^{(\sigma_i)}(h_i) - \mathcal{G}_{1h}^{(\sigma_j)}(h_j) \right) v_j \right\|_{C^\alpha(\bar{\Omega})} \leq \left\| \mathcal{G}_{1h}^{(\sigma_i)}(h_i) - \mathcal{G}_{1h}^{(\sigma_j)}(h_j) \right\|_{\mathcal{L}\left(C^{2+\alpha}(\bar{\Omega}), C^\alpha(\bar{\Omega})\right)} \to 0 \qquad \text{as } i, j \to \infty,$$

since $\|v_j\|_{C^{2+\alpha}(\bar{\Omega})} = 1$. Likewise, $\{\mathcal{G}_{2h}^{(\sigma_n)}(Q_n, h_n)\}$ is a Cauchy sequence in $\mathcal{L}(C^{2+\alpha}(T), C^{1+\alpha}(T))$, so that

$$\|(\mathcal{G}_{2h}^{(\sigma_i)}(Q_i, h_i) - \mathcal{G}_{2h}^{(\sigma_j)}(Q_j, h_j))v_j\|_{C^{1+\alpha}(T)} \to 0 \qquad \text{as } i, j \to \infty.$$

By linearity we have

$$\mathcal{G}_h^{(\sigma_i)}(Q_i, h_i)[v_i - v_j] + \left(\mathcal{G}_h^{(\sigma_i)}(Q_i, h_i) - \mathcal{G}_h^{(\sigma_j)}(Q_j, h_j) \right) [v_j] = 0 \qquad \text{for all } i, g \geq 1.$$

Notice that for each $i, j \geq 1$, we have that $\mathcal{G}_h^{(\sigma_i)}(Q_i, h_i)$ is uniformly elliptic; hence we apply the Schauder estimates to the above equation and conclude that there exists a generic constant $C > 0$ independent of i, j satisfying

$$C\|v_i - v_j\|_{C^{2+\alpha}(\bar{\Omega})} \leq \|v_i - v_j\|_{C^0(\bar{\Omega})} + \left\| \left(\mathcal{G}_h^{(\sigma_i)}(Q_i, h_i) - \mathcal{G}_h^{(\sigma_j)}(Q_j, h_j) \right) v_j \right\|_{C^\alpha(\bar{\Omega}) \times C^{1+\alpha}(T)}.$$

Our previous estimates imply that the last term on the right-hand side vanishes as $i, j \to \infty$. On the other hand, as the $\{v_j\}$ are uniformly bounded in $C^{2+\alpha}(\bar{\Omega})$, the compact embedding of $C^{2+\alpha}(\bar{\Omega}) \subset\subset C^0(\bar{\Omega})$ guarantees there is subsequence convergent in $C^0(\bar{\Omega})$. But then the inequality above implies this subsequence converges in $C^{2+\alpha}(\bar{\Omega})$ as well. By abuse of notation we identity the subsequence with $\{v_j\}$ itself. We note also that as $d(v_j) = d(\partial_q h_j) = 0$, for all $j \geq 1$, that the limit must also have mean zero. Then we may let m be given, with $\partial_q m \in C^{2+\alpha}(\bar{\Omega})$ and $v_j \to \partial_q m$ in this space. It follows that $\|\partial_q m\|_{C^{2+\alpha}(\bar{\Omega})} = 1$, and

$$\mathcal{G}_h^{(\sigma)}(Q, H)[\partial_q m] = \mathcal{G}_h(Q, H)[\partial_q m] = 0,$$

where the first equality follows from the fact that $\partial_q m$ has mean zero so the d term in $\mathcal{G}_h$ vanishes.

As we have argued before, evenness and periodicity in q together imply that $\partial_q h_j$ vanishes on $\partial\Omega_r \cup \partial\Omega_l$ for $j \geq 1$. Moreover, as $h_j \equiv 0$ on $\partial\Omega_b$, we have that $v_j \equiv 0$ on $\partial\Omega_b \cup \partial\Omega_l \cup \partial\Omega_r$. We have, additionally, that $\partial_q h_j < 0$ in Ω for each $j \geq 1$. Together these imply that $\partial_q m \equiv 0$ on $\partial\Omega_b \cup \partial\Omega_l \cup \partial\Omega_r$ and $\partial_q m \leq 0$ in Ω. But $\partial_q m$ has unit norm and satisfies the uniformly elliptic equation $\mathcal{G}_h^{(\sigma)}[\partial_q m] = 0$; hence we may apply the maximum principle to conclude $\partial_q m < 0$ in Ω.

Mirroring our derivation of the eigenfunction at (Q^*, H^*), we expand $\partial_q m$ in a sine series

$$\partial_q m(q,p) = \sum_{k=0}^{\infty} f_k(p) \sin kq, \qquad q,p \in R.$$

As $\mathcal{G}_h(Q,H)[\partial_q m] = 0$, each mode in the series expansion must also satisfy this relation. Evaluating $\mathcal{G}_h(Q,H)$ we may drop all terms involving q derivatives to find

$$\begin{aligned}
\mathcal{G}_{1h}(Q,H) &= \partial_p^2 + H_p^2 \partial_q^2 - 3g\rho_p (H - d(H)) H_p^2 \partial_p - g\rho_p H_p^3 + 3H_p^2 \beta(-p)\partial_p, \\
\mathcal{G}_{2h}(Q,H) &= \left(2g\rho H_p^2 + 2H_p(2g\rho H - Q)\partial_p\right)\big|_T = \left(g\lambda^{-1} - \lambda^{1/2}\partial_p\right)\Big|_T.
\end{aligned}$$

In particular, for $k = 1$ find

$$0 = (\partial_p^2 + H_p^2 \partial_q^2 - 3g\rho_p (H - d(H)) H_p^2 \partial_p - g\rho_p H_p^3 + 3H_p^2 \beta(-p)\partial_p) f_1 \qquad \text{for all } p \in (p_0, 0),$$

or in self-adjoint form:

$$\partial_p \left(H_p^{-3} \partial_p f_1\right) = \left(H_p^{-1} + g\rho_p\right) f_1 \qquad \text{for all } p \in (p_0, 0).$$

Similarly, applying $\mathcal{G}_{2h}(Q,H)$ we find

$$f_1(p_0) = 0, \qquad f_1'(0) = g\rho(0)\lambda^{-3/2} f_1(0).$$

Returning to our original notation, this implies

$$-1 = \mathcal{R}(f_1; \lambda) \geq \mu(\lambda),$$

where we recall $\mathcal{R}$ is the Rayleigh quotient defined in (2.3.23). If $\mu(\lambda) < -1$, then -1 is not the minimum eigenvalue, so f_1 is an excited state of the above Sturm–Liouville problem. We conclude by standard Sturm–Liouville theory that f_1 vanishes at some point in $(p_0, 0)$. But we know from explicit computation that

$$f_1(p) = \frac{2}{\pi} \int_0^{\pi} (\partial_q m)(q,p) \sin q dq < 0, \qquad p \in (p_0, 0)$$

as $\partial_q m < 0$. This contradicts the existence of nodes, so we must have, instead, that $\mu(\lambda) = -1 = \mu(\lambda^*)$. By Lemma 2.3.6 it follows that $\lambda = \lambda^*$ as desired. $\square$

2.6. Uniform regularity

In the previous section we considered the second alternative of Theorem 2.4.1. Now we wish to delve further into the first. We make the central aim of this section the establishment of uniform bounds in the Hölder norm along the continuum $\mathcal{C}'_\delta$. The ultimate product of our efforts will be the following theorem.

THEOREM 2.6.1 (uniform regularity). *Let $\delta > 0$ be given. If $\sup_{(h,Q)\in\mathcal{C}'_\delta} \sup_{\overline{R}} h_p$ and $\sup_{(h,Q)\in\mathcal{C}'_\delta} |Q|$ are finite, then $\sup_{(h,Q)\in\mathcal{C}'_\delta} \|h\|_{C^{3+\alpha}(\overline{R})}$ is finite.*

To treat the second derivatives we shall employ a suite of a priori estimates for nonlinear elliptic equations with oblique boundary conditions due to Lieberman and Trudinger (cf. [**44**]). We shall not, however, need these results in their full generality, so, instead, we streamline them into a single statement.

First note that due to the periodicity in q, the domain R can be considered as embedded on a torus, allowing us to ignore the seeming lack of smoothness at the corner points $q = 0,\ 2\pi$. Next, in keeping with Lieberman and Trudinger's notational framework we consider a differential operator $F = F(z, \xi, r) \in C^2(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}, \mathbb{R})$ and boundary operator $G = G(z, \xi) \in C^2(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$. Here $\mathbb{S}$ denotes the space of 2×2 real symmetric matrices, D denotes the gradient operator, and D^2 the Hessian. Both of these are taken as acting on the space of smooth real-valued functions on $\overline{R}$ which are 2π-periodic in the first variable. As usual, we say that F is elliptic at $(h, \xi, r) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}$ provided that the matrix $F_r := [\partial F/\partial r_{ij}]_{1\le i,j\le 2}$ is positive definite at that point. Moreover, if Λ_1 and Λ_2 denote the minimum and maximum eigenvalue of F_r, respectively, then F is said to be uniformly elliptic provided that the ratio Λ_2/Λ_1 is bounded. The boundary operator G is said to be oblique at a point $(q, 0)$ if the normal derivative $\chi := G_\xi \cdot (0, -1)$ is positive at $(q, 0)$ for all $(h, \xi) \in \mathbb{R} \times \mathbb{R}^2$.

Assume that F and G are given as above and consider the corresponding nonlinear elliptic boundary value problem:

$$\begin{cases} F\left(h, Dh, D^2h\right) = 0 & \text{in } R, \\ G(h, Dh) = 0 & \text{on } p = 0, \\ h = 0 & \text{on } p = p_0. \end{cases} \tag{2.6.1}$$

THEOREM 2.6.2 (Lieberman-Trudinger, [**44**]). *Let $h \in C^2(\overline{R})$ be a solution to* (2.6.1) *that is 2π-periodic in the first variable and for some constant $K > 0$ satisfies $|h| + |Dh| \leq K$ in $\overline{R}$. Suppose further that for some positive constant M, the functions $F = F(z, \xi, r)$ and $G = G(z, \xi)$ satisfy the following structural conditions:*

$$\Lambda_2 \leq M\Lambda_1, \tag{2.6.2}$$

$$|F|,\ |F_\xi|,\ |F_z| \leq M\Lambda_1(|r| + 1), \tag{2.6.3}$$

$$F_{rr} \leq 0, \tag{2.6.4}$$

$$|G|, |G_z|, |G_\xi|, |G_{zz}|, |G_{z\xi}|, |G_{\xi\xi}| \leq M\chi \tag{2.6.5}$$

for all $(z, \xi, r) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}$ such that $|z| + |\xi| \leq K$. Then there are positive constants $\mu = \mu(M)$ and $C = C(K, M)$ such that $h \in C^{2+\mu}_{per}(\overline{R})$ and

$$\|h\|_{C^{2+\mu}_{per}(\overline{R})} \leq C.$$

This is a nontransparent restatement of the original result presented in [**44**], so we take a brief aside to outline its development. The general idea will be to incrementally estimate the Hölder norms, beginning at $C^{0+\mu}$ and working upwards to $C^{2+\mu}$. For clarity, all citations from [**44**] are marked with the abbreviation L-T.

PROOF. Let $F \in C^2(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}, \mathbb{R})$ and $G \in C^2(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ be given and consider the boundary value problem (2.6.1). First we wish to show that assuming (2.6.2)–(2.6.5), F and G meet the hypotheses of L-T Theorem 2.1, namely, structural requirements (L-T F1), (L-T F2), and (L-T G2). But observe that conditions (2.6.2) and (2.6.3) imply (L-T F1) and (L-T F2), respectively, where we are taking μ_0 to be a constant. Moreover, (2.6.5) ensures that condition (L-T G2) holds, again taking μ_0 to be a constant function. So, indeed, the hypotheses of L-T Theorem 2.1 hold. Applying the theorem, we have for any $C^1(\overline{R}) \cap C^2(R)$ solution h, with $|h| < M$, there exists positive constants $\mu = \mu(n, M, \mu_0)$ and $C = C(n, M, \mu_0, R)$ with (2.6.2)

$$[h]_{\mu;R} \leq C.$$

Here $[\cdot]_{\mu;R}$ denotes the Hölder seminorm. If we slightly abuse notation and drop the dependencies on n and R, it follows immediately that there exists a constant $C = C(K, M)$ such that

$$\|h\|_{C^{\mu}_{\mathrm{per}}(R)} \leq C.$$

Next we wish to show that structural requirements (L-T 4.1)–(L-T 4.4) are a proper subset of (2.6.2)–(2.6.5). Clearly, (2.6.2) is equivalent to (L-T 4.1). Also, taking $\theta = 0$ and noting that $F_x = 0$, we see that (2.6.4) gives both (L-T 4.2) and (L-T 4.3). Finally, (2.6.5) along with the obliqueness of G implies (L-T 4.4), with an appropriate choice of μ_1. This verifies all of the hypotheses of L-T Theorem 4.1; hence for some $\mu' > 0$ and $C' > 0$, both depending on M and K, we have

$$[Dh]_{\mu',R} \leq C'.$$

By abuse of notation we can therefore choose μ and C positive constants such that

$$\|h\|_{C^{1+\mu}_{\mathrm{per}}(R)} \leq C.$$

Again we note that these constants depend only on K and M.

For the higher derivatives we shall need to make use of L-T Theorem 5.4 and L-T Theorem 6.2. Conditions (L-T 5.1) and (L-T 5.2) coincide exactly with conditions (L-T 4.1) and (L-T 4.2), respectively. Similarly, condition (L-T 5.4) follows from the positive definiteness of F and (L-T 5.22) from (2.6.5). We have already seen that (L-T 4.3) holds, so applying the second statement of L-T Theorem 5.4, there exists a positive constant—which we again denote C—depending only on K and M with

$$\sup_R |D^2 h| \leq C.$$

Lastly, in order to apply L-T Theorem 6.2 and get an estimate of the full $C^{2+\mu}$-norm we need only verify that structural conditions (L-T 6.1), (L-T 6.2), (L-T 6.3), and (L-T 6.4)' hold. But (L-T 5.1) and (L-T 4.3) together imply (L-T 6.1), and (L-T 6.4)' clearly follows from (L-T 5.22). Positive definiteness of F, moreover, gives (L-T 6.2) for a sufficiently large $\bar{\mu}_2$. As this implies we have a bound on the C^2-norm of $\Lambda_1^{-1}F$, (L-T 6.3) also holds for large enough μ_3. Applying L-T Theorem 6.2, then, and combining with the previous estimates yields the theorem. □

That settled, we return to the proof of the uniform regularity theorem.

Proof of Theorem 2.6.1. We begin by proving the $L^\infty(\overline{R})$ norm of h is uniformly bounded along the continuum. To do so we note that by (2.2.9)

$$0 \le h(q,p) = y + d \le \eta(0) + d,$$

and thus the definition of p_0 in (2.1.10) yields

$$\eta(0) + d \le \frac{|p_0|}{\inf_{\overline{D_\eta}}(\sqrt{\rho}(c-u))} = |p_0| \sup_{\overline{R}} h_p,$$

where the last equality follows from (2.2.10). Thus assuming the hypothesis of the theorem, that is, that h_p is uniformly bounded in L^∞ along the continuum, then the above inequalities together give the boundedness of the L^∞-norm of h.

Next to bound h_q we fix Q and differentiate in q the full height equation (2.2.17) to find

$$\begin{aligned} -3h_{pq}h_p^2\beta(-p) &= \left(1+h_q^2\right)h_{ppq} + 2h_q h_{qq} h_{pp} + 2h_{qq}h_{pq}h_p + h_{qqq}h_p^2 \\ &\quad - 2h_{qq}h_p h_{pq} - 2h_q h_{pq}^2 - 2h_q h_p h_{pqq} \\ &\quad - g h_q h_p^3 \rho_p - 3g(h-d)h_{pq}h_p^2\rho_p \qquad \text{in } R, \end{aligned} \tag{2.6.6}$$

$$2h_q h_{qq} + 2h_{pq}h_p(2g\rho h - Q) + 2g\rho h_q h_p^2 = 0 \qquad \text{on } p = 0, \tag{2.6.7}$$

$$h_q = 0 \qquad \text{on } p = p_0. \tag{2.6.8}$$

Consider now the function $k(q,p) := h(q,p) - d + \epsilon q e^{np}$, where $\epsilon > 0$ and $n \in \mathbb{N}$ will be specified later. Inserting $k - \epsilon q e^{np}$ in place of $h - d$ in (2.6.6) we have

$$\begin{aligned} O\left(\epsilon^2\right) = &- 6\epsilon q n e^{np} k_{pq} k_p \beta(-p) + 3k_{pq}k_p^2\beta(-p) - 3\epsilon q n e^{np} k_p^2 \beta(-p) + 2k_q k_{qq} k_{pp} \\ &- 2\epsilon n^2 q e^{np} k_q k_{qq} - \epsilon e^{np} k_{pp} k_{qq} + \left(1 + k_q^2\right) k_{ppq} - 2\epsilon e^{np} k_q k_{ppq} \\ &- \left(1+k_q^2\right)\epsilon n^2 e^{np} - 2k_q k_{pq}^2 + 2\epsilon e^{np} k_{pq}^2 + 4\epsilon n e^{np} k_{pq} k_q - 2k_p k_q k_{pqq} \\ &+ 2\epsilon e^{np} k_p k_{pqq} + 2\epsilon n q e^{np} k_q k_{pqq} + k_p^2 k_{qqq} - 2\epsilon n q e^{np} k_p k_{qqq} - g k_q k_p^3 \rho_p \\ &+ 3g\epsilon q n e^{np} k_q k_p^2 \rho_p + g\epsilon e^{np} k_p^3 \rho_p - 3g k k_{pq} k_p^2 \rho_p + 6g\epsilon q n e^{np} k k_p k_{pq} \rho_p \\ &+ 3g\epsilon q e^{np} k_{pq} k_p^2 \rho_p + 3g\epsilon n e^{np} k k_p^2 \rho_p \qquad \text{in } R. \end{aligned} \tag{2.6.9}$$

At an interior maximum for k_q in R we would necessarily have

$$k_{pq} = k_{qq} = 0, \qquad k_{qpp} \le 0, \qquad k_{qqq} \le 0, \qquad k_{qpp}k_{qqq} \ge k_{qqp}^2. \tag{2.6.10}$$

Thus evaluating (2.6.9) at such a maximum point gives the following simplified equation:

$$\begin{aligned} O(\epsilon^2) &= k_{ppq} + k_{ppq}(k_q - \epsilon e^{np})^2 - 2k_{pqq}(k_q - \epsilon e^{np})(k_p - \epsilon nqe^{np}) \\ &\quad + k_{qqq}(k_p - \epsilon nqe^{np})^2 - gk_q\left[k_p^3\rho_p - 3\epsilon qne^{np}k_p^2\rho_p\right] \\ &\quad - \epsilon e^{np}\left[n^2 + n^2k_q^2 + 3\beta(-p)nk_p^2 - gk_p^3\rho_p - 3gnkk_p^2\rho_p\right]. \end{aligned} \tag{2.6.11}$$

Likewise, inserting k into (2.6.8) we find $k_q = \epsilon e^{np_0}$ on $p = p_0$. In particular, this implies that at the interior maximum, k_q is positive. We may therefore choose $n \in \mathbb{N}$ to ensure that the quantities in square brackets above are positive. In fact, it suffices merely to have $n^2 + 3\beta(-p)nk_p^2 > 0$ and $3\epsilon qne^{np} - k_p > 0$ in R to achieve this, taking into account the signs of ρ_p, k, k_p, and k_q. But combining this fact with the signs implied by (2.6.10) yields a contradiction, since for ϵ small we must have that the right-hand side of (2.6.11) is negative. We conclude k_q does not have an interior maximum. On the top we have from (2.6.7)

$$1 + (k_q - \epsilon)^2 + (k_p - \epsilon qn)^2\left(2g\rho(k + d - \epsilon q) - Q\right) = 0, \qquad p = 0.$$

By assumption, $h_p^2 = (k_p - \epsilon nqe^{np})^2$ is uniformly bounded along the continuum, and by our previous argument $h - d = k - \epsilon qe^{np}$ is as well. But then, as the maximum of k_q occurs on the boundary, and we have shown that on the boundary k_q is controlled by these quantities, we must have that the maximum of k_q is uniformly bounded along $\mathcal{C}'_\delta$. Clearly, the same must hold for h_q. We can repeat the argument but, instead, taking the auxiliary function $k = h - \epsilon qe^{np}$, which will show that the minimum of h_q is likewise bounded.

For the second derivatives we fix $\sigma \in \mathbb{R}$ and apply Theorem 2.6.2 to $\mathcal{G}^{(\sigma)}$. To do this we set

$$\begin{aligned} F(z,\xi,r) &:= \left(1 + \xi_1^2\right)r_{22} + r_{11}\xi_2^2 - 2\xi_1\xi_2 r_{12} - g(z - \sigma)\xi_2^3\rho_p + \xi_2^3\beta(-p), \\ G(z,\xi) &:= 1 + \xi_1^2 + \xi_2^2(2g\rho z - Q) \qquad \text{on } p = 0, \end{aligned}$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $r = (r_{11}, r_{12}, r_{22}) \in \mathbb{R}^3$. By using cutoff functions, Theorem 2.6.2 is applicable in a subset in which solutions exists a priori and for each choice of σ. In particular, $\xi_2 \geq \delta$ implies (2.6.2), (2.6.3) is immediate, and as $F_{rr} = 0$, (2.6.4) holds. Finally, as the only modification to G that occurs when considering the variable density case is the addition of the constant $\rho(0)$, a trivial modification of the constant density argument in [**15**] shows that (2.6.5) holds. It follows that for each $\sigma \in \mathbb{R}$, Theorem 2.6.2 applies to the solution of $\mathcal{G}^{(\sigma)}[h] = 0$. Of course, this introduces the possibility that the constant C will depend on σ. However, every $h \in \mathcal{C}'_\delta$ is such a solution for $\sigma = d(h)$, and recall that we have $\sigma = d(h) \leq \sup_R |h|$. Thus we may select K independently of σ. Moreover, since in the definition of F, σ does not occur in any of the r_{ij} terms, it does not affect Λ_1 and Λ_2. Indeed, the characteristic polynomial satisfied by the eigenvalues is

$$\Lambda_i^2 - \left(1 + \xi_1^2 + \xi_2^2\right)\Lambda_i - \left(1 + \xi_1^2\right)\xi_2^2 - 4\xi_1^2\xi_2^2 = 0, \qquad i = 1, 2,$$

which does not depend on σ. This is an obvious consequence of the fact that the effects of density variation in the height equation are consigned to the lower order terms. Finally, the term containing σ in F can be easily estimated in terms of K. This fact, along with the uniform boundedness of the L^∞-norms of h and Dh along the continuum, allows us to conclude that there exists $C > 0$ such that

$$\sup_{h \in \mathcal{C}'_\delta} \|h\|_{C^{2+\mu}_{\mathrm{per}}(R)} \leq C.$$

Finally, we establish third derivative bounds. Denoting $\theta = h_q$, we can differentiate the height equation in q with fixed Q to find

$$\tag{2.6.12} \begin{cases} \left(1 + h_q^2\right)\theta_{pp} - 2h_p h_q \theta_{pq} + h_p^2 \theta_{qq} & \\ \qquad = f(h, h_p, h_q, h_{pp}, h_{pq}, h_{qq}) & \text{in } R, \\ h_q \theta_q + g\rho h_p^2 \theta + (2g\sigma\kappa[h] - Q)h_p\theta_p = 0 & \text{on } p = 0, \\ \theta = 0 & \text{on } p = p_0, \end{cases}$$

with $C^{1+\mu}_{\mathrm{per}}(\overline{R})$ coefficients and right-hand side f a $C^{\mu}_{\mathrm{per}}(\overline{R})$ function. Since we have already proved h is bounded uniformly in $C^{2+\mu}_{\mathrm{per}}(\overline{R})$, it follows that the right-hand side of (2.6.12) is bounded uniformly in $C^{\mu}_{\mathrm{per}}(\overline{R})$ along the continuum. As we have seen (in the proof of Lemma 2.4.1, for example), so long as h_p is bounded uniformly away from 0 along the continuum, the problem above will be uniformly elliptic with a uniformly oblique boundary condition. We may therefore apply Schauder theory to obtain uniform a priori estimates of θ in $C^{2+\mu}_{\mathrm{per}}(\overline{R})$ over all of $\mathcal{C}'_\delta$.

It remains only to bound h_{ppp} in $C^{\mu}_{\mathrm{per}}(\overline{R})$. By means of the height equation (2.2.13), we may express h_{pp} in terms of the lower order derivatives of h:

$$h_{pp} = -\left(1 + h_q^2\right)^{-1}\left(h_p^3\beta(-p) + h_{qq}h_p^2 - 2h_q h_p h_{pq} - g(h - d(h))h_p^3\rho_p\right).$$

But the right-hand side above is in $C^{1+\mu}_{\text{per}}(\overline{R})$ by the arguments of the previous paragraphs. Thus h_{ppp} in $C^{\mu}_{\text{per}}(\overline{R})$ as desired.

To transition back to the original Hölder exponent α, we merely note that if $h \in C^{3+\mu}_{\text{per}}(\overline{R})$, then it is certainly in $C^{2+\alpha}_{\text{per}}(\overline{R})$. The arguments we have used to derive the third derivative bounds in no way relied on the particular value of μ, so running through them with α, instead, we find $h \in C^{3+\alpha}_{\text{per}}(\overline{R})$.

2.7. Main result

With the regularity established in the previous section, we are now prepared to begin the task of proving the main theorem, Theorem 2.1.1. Theorem 2.4.1 gave us three possibilities for the continuation of the local bifurcation curve. Then, Theorem 6.1 of the previous section allows us to conclude that if $\max_{\overline{R}} \partial_p h$ and Q remain bounded along the continuum, then h remains bounded in X. We now unpack the remaining alternatives and consider their significance in the context of the original problem (2.1.7)–(2.1.8).

Proof of Theorem 2.1.1. Fix $\delta > 0$. If $\mathcal{C}'_\delta$ is unbounded in $\mathbb{R} \times X$, then at least one of the following must occur:

- there exists a sequence $(Q_n, h_n) \in \mathcal{C}'_\delta$, with $\lim_{n\to\infty} Q_n = \infty$;
- there exists a sequence $(Q_n, h_n) \in \mathcal{C}'_\delta$, with $\lim_{n\to\infty} \max_{\overline{R}} \partial_p h_n = \infty$; or,
- $\mathcal{C}'_\delta$ contains another trivial point $(Q(\lambda), H(\lambda)) \in \mathcal{T}$, $-2B_{\min} < \lambda \le -2B_{\min} + \epsilon_0 < \lambda^*$,

where the second alternative follows from Theorem 2.6.1 and the third from the lemmas of section 2.5. If $\mathcal{C}'_\delta$ contains a point of $\partial\mathcal{O}_\delta$, then either

- there exists a $(Q, h) \in \mathcal{C}'_\delta$, with $\partial_p h = \delta$ somewhere in $\overline{R}$; or
- there exists a $(Q, h) \in \mathcal{C}'_\delta$, with $h = \frac{Q-\delta}{2g\rho}$ somewhere on T.

Thus Theorem 2.4.1 tell us that at least one of the five alternatives above must occur. Consider the first alternative. By the definition of p_0 in (2.1.10) we can estimate

$$\inf_{y\in[-d,\eta(x)]} \sqrt{\rho(x,y)}(c - u(x,y)) \le \frac{p_0}{\eta(x) + d} \le \sup_{y\in[-d,\eta(x)]} \sqrt{\rho(x,y)}(c - u(x,y))$$

and thus appealing to (2.2.5) we deduce

$$
\begin{aligned}
Q &= \rho(0,\eta(0))(c-u(0,\eta(0)))^2 + 2g\rho(0,\eta(0))(\eta(0)+d) \\
&\le \sup_{\overline{D_\eta}} \rho(c-u)^2 + \frac{2g\rho(0,\eta(0))p_0}{\inf_{\overline{D_\eta}} \sqrt{\rho}(c-u)}.
\end{aligned}
\tag{2.7.1}
$$

If for some $\delta > 0$ the first alternative holds, we conclude from (2.7.1) that for the corresponding sequence $(u_n, v_n, \rho_n, \eta_n)$ of solutions to (2.1.7)–(2.1.8), either $\sup_{\overline{D_{\eta_n}}} \rho_n(c-u_n) \to \infty$ or $\inf_{\overline{D_{\eta_n}}} \rho_n(c-u_n) \to 0$.

Similarly, if the second alternative holds, then simply by rearranging the change of variables in (2.2.12) we have

$$
\partial_p h_n = \frac{1}{\sqrt{\rho_n}(c-u_n)};
$$

hence $\max_{\overline{R}} \partial_p h_n \to \infty$ iff $\inf_{\overline{D_{\eta_n}}} \rho_n(c-u_n) \to 0$.

Next suppose that for some decreasing sequence $\delta_n \to 0$ the fourth alternative holds. Again appealing to (2.2.12) we see immediately that the corresponding sequence $(u_n, v_n, \rho_n, \eta_n)$ must satisfy $\sup_{\overline{D_{\eta_n}}} \rho_n(c-u_n) \to \infty$ as $n \to \infty$.

Finally, if we have a sequence $\delta_n \to 0$ for which the fifth alternative holds, then there exists a sequence $\{(Q_n, h_n)\}$, with $(Q_n, h_n) \in \mathcal{C}'_{\delta_n}$ for each $n \ge 1$ and $\sup_T (2g\rho h_n - Q_n) \to 0$. We have shown previously that $\partial_q h$ vanishes on $\partial\Omega_l$. Thus if we evaluate the boundary condition at the crest $(0,0)$, we find

$$
\frac{1}{(\partial_p h_n(0,0))^2} = Q_n - 2g\rho_n(0)h_n(0,0) \le Q_n - 2g\rho_n(0)h_n(q,0) \qquad \text{for all } q \in [0,2\pi].
$$

Therefore $\partial_p h_n(0,0) \to \infty$ as $n \to \infty$. In particular, we have $\max_{\overline{R}} \partial_p h_n \to \infty$ so that the fourth alternative implies the second.

By definition the family of continua $\mathcal{C}'_\delta$ indexed by $\delta > 0$ is increasing as δ decreases. We may therefore define $\mathcal{C}' := \bigcup_{\delta>0} \mathcal{C}'_\delta$ to be the maximal continuum. Then by the considerations of the section 2.2 and, in particular, Lemma 2.1, there exists a connected set $\mathcal{C}$ of solutions to (2.1.1)–(2.1.6) corresponding to $\mathcal{C}'$. The arguments of the preceding paragraphs, moreover, imply that either $\sup_{\overline{D_{\eta_n}}} u_n \to c$, $\inf_{\overline{D_{\eta_n}}} u_n \to -\infty$ along some sequence in $\mathcal{C}$ or $\mathcal{C}$ contains more than one distinct laminar flow. This completes the proof.

Remark. In the homogeneous case it was proved in [**15**] that if $\inf_{\overline{D}_{\eta_n}} u_n \to -\infty$ along some sequence in the continuum, then necessarily $\sup_{\overline{D}_{\eta_n}} u_n \to c$ along some sequence. In other words, there are waves on $\mathcal{C}$ whose speed is arbitrarily close to the speed of the wave profile.

We conjecture that the same holds true in the heterogeneous case. Suppose, on the contrary, that for some sequence $\inf_{\overline{D_{\eta n}}} u_n \to -\infty$, but u_n remains uniformly bounded away from c on $\mathcal{C}$. Let (Q_n, h_n) be the corresponding sequence of solutions to the height equation. Then we can show that $\sup_{\overline{D_{\eta n}}} \partial_p h_n$, $\sup_{\overline{D_{\eta n}}} h \to 0$. In particular, this implies $\|\eta_n\|_\infty$, and $d(h_n) \to 0$ in the limit. As the fluid domain is bounded between these two values, this entails that, moving along the continuum, we have regions of the fluid moving arbitrarily fast to left, while simultaneously the fluid is vertically pinching off. This is a consequence of the fact that if $u_n \to -\infty$, then in order to keep a constant pseudomass flux p_0, the domain must be vanishing. Given the pathology of the scenario, we strongly believe it does not occur, though we have been unable to rule it out.

CHAPTER 3

Steady stratified periodic gravity waves with surface tension

3.1. Introduction

Initiated by the breakthrough work of Constantin and Strauss (cf. [**15**]), there has recently been a great deal of interest in the theory of traveling water waves with vorticity (see, e.g., [**70, 71, 72, 73, 76**].) Historically, of course, most investigations of water waves have focused on the irrotational case, which has several extremely convenient mathematical features. While appropriate in a wide variety of scenarios, however, the assumption of zero vorticity is far from universally valid. For instance, ocean waves typically have a rotational layer near the surface due to the effects of wind shear (cf. [**50**]).

In fact, the wind is known to itself generate steady waves. When it first begins to blow over a body of still water, the wind produces a fine, hexagonal meshwork of wavelets whose motion is governed entirely by capillarity (i.e. surface tension.) If the breeze is maintained, then these waves will grow larger, eventually evolving into capillary-gravity waves and finally into gravity waves. Over the course of this progression, the waves develop a remarkable robustness: while pure capillary waves will quickly dissipate in the absence of external forcing, capillary-gravity and gravity waves can propagate long distances unaided (cf. [**39**]).

Surface tension's central role in the formation of wind-driven waves argues strongly for the mathematical consideration of steady, capillary-gravity waves with vorticity, particularly since wind-waves are one of the most important classes of rotational water waves. The goal of the present chapter, therefore, is to develop a global theory for traveling, periodic, water waves with surface tension building upon the local theory derived by Wahlén (cf. [**70, 72, 73**]).

Briefly, our new results come in five parts. The first two of these relate to the phenomenon of double (local) bifurcation for capillary-gravity waves. It is well known in the theory of irrotational capillary-gravity waves that, when the coefficient of surface tension resides in a distinguished set, the eigenvalue of the linearized problem at the point of bifurcation will be of multiplicity two and is otherwise simple. This distinctive feature was first observed in the irrotational case by Wilton (cf. [**77**]) and has been the subject of numerous papers since. Wahlén observed the presence of double bifurcation points in the rotational case and proved the existence of local bifurcation curves in certain regimes (cf. [**70**]). These results, however, are incomplete in the sense that they do not necessarily describe the entire solution set near the bifurcation point. The first of our contributions is to describe in detail the local bifurcation diagram obtained in [**70**] at double bifurcation points. In particular we confirm the existence of curves of mixed solutions conjectured by Wahlén (see the discussion following Theorem 3.1.1). Secondly, using the bifurcation theory for analytic operators pioneered by Dancer, we extend the local curves to obtain global continua of solutions. The application of these techniques to steady rotational waves is in and of itself novel and pays some additional dividends. Most notably, it leads to our third result, which is proving that the solution continua are path-connected, a fact that was unknown even for the constant density, gravity wave case originally considered in [**15**]. Fourthly, for the case of simple local bifurcation, we also use the more standard degree theoretic method to obtain a global bifurcation theorem. Finally, employing the machinery developed by the author in [**76**] (recorded in chapter 2), all of these results are obtained for fluids with general (stable) stratification, which is entirely new, even for the local bifurcation theory. This improvement is quite important from a physical standpoint, since stratification phenomenon is both very common in ocean waves (due to salinity and temperature variations), and exerts a potentially significant effect on the dynamics of the flow (see, e.g., [**82**], or the discussion in chapter 2.)

That said, let us now recall the setup for traveling, stratified, capillary-gravity waves. If we imagine a wave on the open ocean, past experience suggests that it may be *regular* in the following sense. First, it is essentially two-dimensional. That is, the motion will be identical along any line that runs parallel to a crest. Second, if we regard the wave in a coordinate system moving with some constant speed, it appears steady. Finally, the profile is periodic in the direction of motion.

We now formulate governing equations for waves of this form. Fix a Cartesian coordinate system so that the x-axis points in the direction of propagation, and the y-axis is vertical.

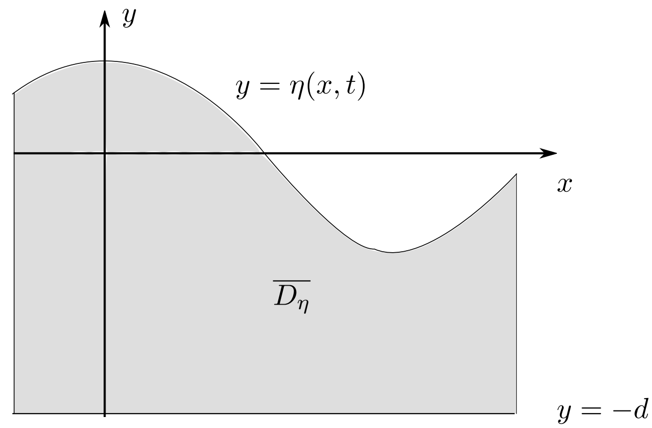

FIGURE 1. The fluid domain $\overline{D_\eta}$. The wave is assumed to propagate to the left with speed c.

We assume that the floor of the ocean is flat and occurs at $y = -d$. Let $y = \eta(x,t)$ be the free surface at the interface between the atmosphere and the fluid. We shall normalize η by choosing the axes so that the free surface is oscillating around the line $y = 0$. As usual we let $u = u(x,y,t)$ and $v = v(x,y,t)$ denote the horizontal and vertical velocities, respectively. Let $\rho = \rho(x,y,t) > 0$ be the density.

Incompressibility of the fluid is represented mathematically by the requirement that the vector field (u,v) be divergence free for all time:

$$u_x + v_y = 0. \tag{3.1.1}$$

Taking the fluid to be inviscid, conservation of mass implies that the density of a fluid particle remains constant as it follows the flow. This is expressed by the continuity equation

$$\rho_t + u\rho_x + v\rho_y = 0. \tag{3.1.2}$$

Next, the conservation of momentum is described by Euler's equations,

$$\begin{cases} \rho u_t + \rho u u_x + \rho v u_y &= -P_x, \\ \rho v_t + \rho u v_x + \rho v v_y &= -P_y - g\rho, \end{cases} \tag{3.1.3}$$

where $P = P(x,y,t)$ denotes the pressure and g is the gravitational constant. Here, of course, we assume that the only external force acting on the fluid is gravity.

For capillary waves, we formulate the boundary condition on the free surface as folows. At each point we say that the pressure of the fluid deviates from the atmospheric pressure of the air above, P_{atm}, in proportion to the (mean) curvature at that point. This results in the dynamic boundary condition

$$P = P_{\text{atm}} - \sigma \frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}, \qquad \text{on } y = \eta(x,t), \tag{3.1.4}$$

where $\sigma > 0$ is the coefficient of surface tension, a material property that we take to be given and constant.

The corresponding boundary condition for the velocity is motivated by the fact that fluid particles that reside on the free surface continue to do so as the flow develops. This observation is manifested in the kinematic condition:

$$v = \eta_t + u\eta_x, \qquad \text{on } y = \eta(x,t). \tag{3.1.5}$$

Since we cannot have any fluid moving normal to the flat bed occurring at $y = -d$, we require

$$v = 0, \qquad \text{on } y = -d. \tag{3.1.6}$$

Note that there is no accompanying condition on u because in the inviscid case we allow for slip, that is, nonzero horizontal velocity along solid boundaries.

We seek traveling periodic wave solutions (u, v, ρ, P, η) to (3.1.1)-(3.1.6). More precisely, we take this to mean that, for fixed $c > 0$, the solution appears steady in time and periodic in the x-direction when observed in a frame that moves with constant speed c to the right. The vector field will thus take the form $u = u(x - ct, y)$, $v = v(x - ct, y)$, where each of these is L-periodic in the first coordinate. Likewise for the scalar quantities: $\rho(x, y, t) = \rho(x-ct, y)$, $P = P(x - ct, y)$, and $\eta = \eta(x - ct)$, again with L-periodicity in the first coordinate. We therefore take moving coordinates

$$(x - ct, y) \mapsto (x, y),$$

which eliminates time dependency from the problem. In the moving frame (3.1.1)-(3.1.3) become

$$\begin{cases} u_x + v_y & = & 0, \\ (u-c)\rho_x + v\rho_y & = & 0, \\ \rho(u-c)u_x + \rho v u_y & = & -P_x, \\ \rho(u-c)v_x + \rho v v_y & = & -P_y - g\rho, \end{cases} \tag{3.1.7}$$

throughout the fluid domain. Meanwhile, the reformulated boundary conditions are

$$
\left\{
\begin{array}{rcll}
v & = & (u-c)\eta_x, & \text{on } y = \eta(x), \\
v & = & 0, & \text{on } y = -d, \\
P & = & P_{\text{atm}} - \sigma\eta_{xx}\left(1+\eta_x^2\right)^{-3/2}, & \text{on } y = \eta(x),
\end{array}
\right.
\tag{3.1.8}
$$

where u, v, ρ, P are taken to be functions of x and y, η is a function of x, and all of them are L-periodic in x.

In the event that $u = c$ somewhere in the fluid we say that *stagnation* has occurred. This is a slight abuse of terminology, since in our definition we require only that the horizontal velocity vanishes. In order to avoid roll-up and other instability phenomena, we shall restrict our attention to the case where $u < c$ throughout.

Recall that we have chosen our axes so that η oscillates around the line $y = 0$. Mathematically this equates to

$$
\fint_0^L \eta(x)dx = 0.
\tag{3.1.9}
$$

In effect, this couples the depth to the problem, so we need not treat d as a free parameter. The trade-off, as we shall see, is that this normalization will have significant technical consequences later.

Observe that by conservation of mass and incompressibility, ρ is transported and the vector field is divergence free. Therefore we may introduce a (relative) *pseudo-stream function* $\psi = \psi(x, y)$, defined uniquely up to a constant by:

$$
\psi_x = -\sqrt{\rho}v, \qquad \psi_y = \sqrt{\rho}(u-c).
$$

Here we have the addition of a factor involving ρ to the typical definition of the stream function for an incompressible fluid. This neatly captures the inertial effects of the heterogeneity of the flow (see the treatment in [**82**], for example). The particular choice of $\sqrt{\rho}$ is merely to simplify algebraically what follows.

It is a straightforward calculation to check that ψ is indeed a (relative) stream function in the usual sense, i.e. its gradient is orthogonal to the vector field in the moving frame at each point in the fluid domain:

$$
(u-c)\psi_x + v\psi_y = 0.
$$

Moreover, (3.1.8) implies that the free surface and flat bed are each level sets of ψ. For definiteness we choose $\psi \equiv 0$ on the free boundary, so that $\psi \equiv -p_0$ on $y = -d$, where p_0 is the *(relative) pseudo-voumetric mass flux*:

$$p_0 := \int_{-d}^{\eta(x)} \sqrt{\rho(x,y)}\,[u(x,y) - c]\,dy. \tag{3.1.10}$$

It is an easy calculation to check that p_0 is well-defined (see discussion following (2.1.10)). Physically, p_0 describes the rate of fluid moving through any vertical line in the fluid domain (and with respect to the transformed vector field $\sqrt{\rho}(u - c, v)$.)

Since ρ is transported, it must be constant on the streamlines and hence, we may think of it as a function of ψ. Abusing notation we may let $\rho : [p_0, 0] \to \mathbb{R}^+$ be given such that

$$\rho(x,y) = \rho(-\psi(x,y)) \tag{3.1.11}$$

throughout the fluid. When there is risk of confusion, we shall refer to the ρ occurring on the right-hand side above as the *streamline density function*. We shall focus our attention on the case where the density is nondecreasing as depth increases. This is entirely reasonable from a physical standpoint. Indeed, hydrodynamic stability requires that the depth be monotonically increasing with depth, so that fluids with this type of density distribution are referred to in the literature as *stably stratified*. The level set $-\psi = p_0$ corresponds to the flat bed, and the set where $-\psi = 0$ corresponds to the free surface. Therefore, we require that the streamline density function is nonincreasing as a function of $-\psi$. Note that this does not preclude the case of constant density.

By Bernoulli's theorem (for gravity waves) the quantity

$$E := P + \frac{\rho}{2}\left((u - c)^2 + v^2\right) + g\rho y,$$

is constant along streamlines, that is

$$(u - c)E_x + vE_y = 0.$$

This can be verified directly from (3.1.7). In the case of an inviscid fluid, E represents the energy of the fluid particle at (x, y). The first term on the right-hand side, P, gives the energy due to internal pressure. The second and third terms combined describe the kinetic energy, while the last is gravitational potential energy. In particular, evaluating the above equation on the level set $\psi = 0$, we find

$$|\nabla\psi|^2 + 2g\rho(x,\eta(x))\,(\eta(x)+d) = Q, \qquad \text{on } y = \eta(x) \tag{3.1.12}$$

where the constant $Q := 2(E|_\eta - P_{\mathrm{atm}} + g\rho|_\eta d)$ gives roughly the energy density along the free surface of the fluid. We shall view Q as parameterizing the continuum of solutions.

Under the assumption that $u < c$ throughout the fluid, and given the fact that E is constant along streamlines, there exists a function $\beta : [0, |p_0|] \to \mathbb{R}$ such that

$$\frac{dE}{d\psi}(x,y) = -\beta(\psi(x,y)). \tag{3.1.13}$$

For want of a better name we shall refer to β as the *Bernoulli function* corresponding to the flow. Physically it describes the variation of specific energy as a function of the streamlines. Define

$$B(p) := \int_0^p \beta(-s)ds$$

for $p_0 \leq p \leq 0$ and let B have minimum value $B_{\min}$.

We now briefly outline a few notational conventions. Let $\overline{D_\eta}$ denote the closure of the fluid domain

$$D_\eta := \left\{(x,y) \in \mathbb{R}^2 : -d < y < \eta(x)\right\}.$$

For any integer $m \geq 1$ and $\alpha \in (0,1)$, we say that a bounded domain $D \subset \mathbb{R}^2$ is $C^{m+\alpha}$ provided that at each point in the boundary, denoted ∂D, is locally the graph of a $C^{m+\alpha}$ function. That is, a function with Hölder continuous derivatives (of exponent α) and up to order m. Furthermore, for fixed $m \geq 1$ and $\alpha \in (0,1)$ we define the space $C^{m+\alpha}_{\mathrm{per}}(\overline{D})$ to consist of those functions $f : \overline{D} \to \mathbb{R}$ with Hölder continuous derivatives (of exponent α) up to order m and that are L-periodic in the first coordinate. Similarly, we shall take $C^m_{\mathrm{per}}(\overline{D})$ to be the space of m-times continuously differentiable functions which are L-periodic in the first coordinate. In this work, we shall seek solutions to the water wave problem (u, v, ρ, η, Q) in the space

$$\mathscr{S} := C^{2+\alpha}_{\mathrm{per}}(\overline{D_\eta}) \times C^{2+\alpha}_{\mathrm{per}}(\overline{D_\eta}) \times C^{2+\alpha}_{\mathrm{per}}(\overline{D_\eta}) \times C^{3+\alpha}_{\mathrm{per}}(\mathbb{R}) \times \mathbb{R}. \tag{3.1.14}$$

Occasionally, for brevity, we shall abuse notation and write $(u, v, \rho, \eta) \in \mathscr{S}$, or simply $(u, v, \eta) \in \mathscr{S}$, by which we mean $(u, v, \rho, \eta, Q) \in \mathscr{S}$, where ρ is found from the (given) streamline density function using (3.1.11) and Q is calculated via (3.1.12).

Finally, in describing the structure of the continua of solutions, we shall make use of the following definition.

DEFINITION 3.1.1. A subset $\mathcal{K} \subset \mathscr{S}$ is said to admit a *global, locally injective, continuous parameterization* provided that $\mathcal{K}$ can be described as the image of a locally injective $\kappa \in C([0,\infty);\mathscr{S})$. Furthermore, for $m \geq 1$, we say that $\mathcal{K}$ has a *locally injective, C^m-reparameterization (resp. analytic reparameterization)* if, for each $s_* \in (0,\infty)$, there exists a continuous $\tau : (-1,1) \to \mathbb{R}^+$ such that $\tau(0) = s_*$ and the map $s \mapsto \kappa(\tau(s))$ is injective and of class C^m (resp. analytic).

There are several key features distinguishing the sets described above from the standard C^m-curves of differential geometry. Observe that, while κ may be locally injective, DEFINITION 3.1.1 does not require it to be *globally* injective, i.e., the curve $\mathcal{K}$ may self-intersect. This scenario is not permissible for C^m-curves since in the neighborhood of a point of self-intersection $\mathcal{K}$ fails to be a one-dimensional submanifold of $\mathscr{S}$. Also, it is important to note that we do not require κ' to be non-vanishing. As a consequence, the global chart may not be of class C^m (resp. analytic), even though at each parameter value we can reparameterize to obtain that level of smoothness.

Our notation established, we may now state our first theorem for stratified capillary-gravity waves.

THEOREM 3.1.1. (Global Continuum from Simple Bifurcation) *Fix a wave speed $c > 0$, wavelength $L > 0$, relative mass flux $p_0 < 0$ and coefficient of surface tension $\sigma > 0$. Fix any $\alpha \in (0,1)$, and let the functions $\beta \in C^{3+\alpha}([0,|p_0|])$ and $\rho \in C^{1+\alpha}([p_0,0];\mathbb{R}^+)$ be given such that the* (L-B) *condition holds (see* DEFINITION 3.3.1*). Also we assume the streamline density function ρ is nonincreasing. There exists a set $\Sigma_1 \subset (0,\infty)$ such that the following is true. Σ_1 contains a half-line, (σ_c,∞), and, if $\sigma \in \Sigma_1$, then we have:*

(a) *There exists a connected set $\mathcal{C} \subset \mathscr{S}$ of solutions (u,v,ρ,η,Q) of the traveling, stratified, water wave problem with surface tension* (3.1.1)–(3.1.6)*. The set $\mathcal{C}$, moreover, has the following properties.*

- *$\mathcal{C}$ contains a laminar flow (with a flat surface $\eta \equiv 0$ and all streamlines parallel to the bed); denote it by (U_*, Q_*).*

- *Either* (i) $\mathcal{C}$ *contains more than one laminar flow, or it is unbounded in the sense that, along some sequence* $\{(u_n, v_n, \rho_n, \eta_n)\} \subset \mathcal{C}$, *we have*

$$\text{(ii)}\ \max u_n \uparrow c \qquad \textit{or} \qquad \text{(iii)}\ \min u_n \downarrow -\infty.$$

(b) *Furthermore, there exists a path-connected subset* $\mathcal{K} \subset \mathcal{C}$ *such that*

- $\mathcal{K}$ *admits a global, locally injective, continuous parameterization with a locally injective* C^1 *reparameterization.*
- *Either* $\mathcal{K}$ *is a closed loop, or it unbounded in the sense that alternative* (a)(ii) *or* (a)(iii) *must occur along* $\mathcal{K}$.

(c) *All flows* $(u, v, \rho, \eta) \in \mathcal{C}$ *are regular in the sense that:*

- u, v, ρ *and* η *each have period* L *in* x;
- u, ρ *and* η *are symmetric,* v *antisymmetric across the line* $x = 0$.

Remark. The Local Bifurcation Condition (L-B) essentially states that there exists an eigenvalue of the linearized problem with lowest mode, and is both necessary and sufficient for our result to hold. Still, in Lemma 3.3.3, we derive an explicit sufficient condition that implies (L-B):

$$(3.1.15)\qquad \begin{aligned} (g\rho(0)+\sigma)p_0^2 > \int_{p_0}^{0} \bigg\{ & \frac{4\pi^2}{L^2}\left(2B(p) - 2B_{\min} + 2\epsilon_0\right)^{3/2} \\ & + (p-p_0)^2 \Big(\left(2B(p) - 2B_{\min} + 2\epsilon_0\right)^{1/2} + g\rho'(p) \Big) \bigg\} dp, \end{aligned}$$

where ϵ_0 is defined by

$$(3.1.16)\qquad \epsilon_0^{3/2} := \max\Big\{ 2g\|\rho'\|_{L^\infty} p_0^2 e^{|p_0|},\ (2g\|\rho'\|_{L^\infty})^3,\ (4\|\rho'\|_{L^\infty})^3,\ 8g|p_0|\rho(0) \Big\}.$$

This is most certainly not optimal. Indeed, obtaining a complete characterization of the set of data (p_0, ρ, β) for which (L-B) holds is an important and largely open problem, even in the special case when $\rho \equiv 1$ and $\sigma = 0$. It is worth mentioning that (3.1.15) can be satisfied by assuming $|p_0|$ and $\|\rho'\|_{L^\infty}$ are sufficiently small, or σ is large. From this it is easy to see that the set $\Sigma := \{\sigma > 0 : \text{(L-B) holds}\}$ contains a neighborhood of $+\infty$.

We note also that much is known about the structure of the continuum $\mathcal{C}$ near the point of bifurcation (U_*, Q_*). There exists a neighborhood $\mathcal{U}$ of (U_*, Q_*) in $\mathscr{S}$ in which $\mathcal{C} \cap \mathcal{U} = \mathcal{K} \cap \mathcal{U}$ comprises all solutions to (3.1.1)–(3.1.6), and (U_*, Q_*) is the only laminar solution in $\mathcal{U}$. Moreover, all nonlaminar flows in $\mathcal{U} \cap \mathcal{C}$ exhibit the following monotonicity properties:

- within each period the wave profile η has a single crest and trough; say the crest occurs at $x = 0$;
- a water particle located at (x, y) with $0 < x < L/2$ and $y > -d$ has positive vertical velocity $v > 0$.

For the proof of these results, see THEOREM 3.3.1.

Finally, let us note that in certain special cases it can be shown that Σ_1 is a countable set, see PROPOSITION 3.9.1.

As we shall see, THEOREM 3.1.1 is obtained by means of a bifurcation argument with the family of laminar flows taking on the role of the trivial solutions. When $\sigma \in \Sigma_1$ (for example, if σ is sufficiently large) we will show that local bifurcation occurs at a simple (generalized) eigenvalue.

On the other hand, for $\sigma \notin \Sigma_1$, the eigenvalue will be not be simple. In order to discuss this scenario more carefully, we must first make a few definitions. Let D be a smooth domain in $\mathbb{R}^2$ and let Banach spaces $X, Y \subset C^{m+\alpha}_{\text{per}}(\overline{D})$ be given for some $m \geq 1$, $\alpha \in (0,1)$. For a smooth map $\mathcal{F} : \mathbb{R} \times X \to Y$ with $\mathcal{F}(\cdot, 0) \equiv 0$, the generic bifurcation problem is to find all nontrivial solutions (λ, w) of

$$\mathcal{F}(\lambda, w) = 0, \tag{3.1.17}$$

in a neighborhood of a given solution $(\lambda_*, 0)$. Suppose that the null-space of $\mathcal{F}_w(\lambda_*, 0)$ is two-dimensional, spanned by $\{\phi_1, \phi_2\}$, and $\phi_i = \phi_i(x, y)$ is of minimal period L/n_i in the first coordinate for $i = 1, 2$, where n_1 and n_2 are distinct natural numbers. Assume further that there exists a (continuously) parameterized curve of solutions,

$$\mathcal{C}_{\text{loc}} := \{(\lambda(s), w(s)) : s \in (-\epsilon, \epsilon)\} \subset \mathbb{R} \times X,$$

to (3.1.17) with $(\lambda(0), w(0)) = (\lambda_*, 0)$. We shall refer to these solutions as *pure* provided that

$$w(s) = s\phi_i + o(s) \qquad \text{in } X,$$

for either $i = 1$ or $i = 2$ but not both. That is, pure solutions are those that locally lie near the interior of the span of either ϕ_1 or ϕ_2. Conversely, the elements of $\mathcal{C}_{\text{loc}}$ are called *mixed* *solutions* if there exists $\xi_1, \xi_2 \in \mathbb{R}^\times$ such that

$$w(s) = s\,(\xi_1\phi_1 + \xi_2\phi_2) + o(s) \qquad \text{in } X.$$

Clearly, mixed solutions locally lie near the interior of the span of ϕ_1 and ϕ_2.

The relevance of these concepts to the case at hand stems from the fact that, for $\sigma \in \mathbb{R}^+ \setminus \Sigma_1$, local bifurcation occurs at eigenvalues of multiplicity two (see COROLLARY 3.3.1.) That is, the null space of the linearized operator is spanned by two functions, $\{\phi_1, \phi_2\}$, of minimal period $L/n_1, L/n_2$ in x, respectively. This feature of the eigenvalue problem is well-known for homogeneous capillary-gravity waves and has been very well analyzed in the irrotational case (cf. [**34, 53, 66**] for analytic results and, e.g. [**54, 62, 64**] and the references contained therein for some numerical results.) For general vorticity, this scenario was analyzed by Wahlén [**70**]. Briefly, his argument was the following. So long as there is no resonance, i.e. n_2 is not an integer multiple of n_1, then the dimension of the null space can be reduced to one by considering the restriction to the subspaces of L/n_1- or L/n_2-periodic solutions. Applying the Crandall-Rabinowitz arguments developed for the simple eigenvalue case, one can prove the existence of two C^1-curves bifurcating from the family of laminar flows, say $\mathcal{C}_{1,\text{loc}}$ and $\mathcal{C}_{2,\text{loc}}$. These consist of small amplitude solutions of minimal period L/n_1 and L/n_2, respectively. In fact, $\mathcal{C}_{i,\text{loc}}$ are both curves of pure solutions in the sense outlined above.

The price payed for the relative ease of this approach is that, in restricting the solutions space, we are likely missing the mixed solutions. To see why this may be the case, note that a linear combination $\xi_1\phi_1 + \xi_2\phi_2$ may have *minimal* period L. Such solutions would therefore not be included in either $\mathcal{C}_{1,\text{loc}}$ or $\mathcal{C}_{2,\text{loc}}$.

Worse still, when resonance does occur (i.e. $n_2/n_1 \in \mathbb{N}$), this type of argument offers even less information. Since restriction to the L/n_1-periodic solutions no longer renders the null space one-dimensional (as L/n_2–periodic solutions are now also L/n_1–periodic), the only conclusion we can draw via Crandall-Rabinowitz is that there is at least one local curve of pure L/n_2-minimal periodic solutions. In the language we have been using thus far, this means that we can prove the existence of $\mathcal{C}_{2,\text{loc}}$, but not $\mathcal{C}_{1,\text{loc}}$. This, of course, says nothing about the existence (or nonexistence) of curves of mixed solutions.

In the present chapter we devote some effort to advancing the study of the bifurcation near double eigenvalues. Rather than using Crandall-Rabinowitz, we instead carry out our analysis via the full Lyapunov-Schmidt procedure. Owing to a great deal of degeneracy in the problem, a complete account is, unfortunately, beyond the scope of this thesis. Nonetheless, we are able to obtain some detailed results on the structure of the full solution set.

THEOREM 3.1.2. (Local Continua from Double Bifurcation) *Let the wave speed $c > 0$, the relative mass flux $p_0 < 0$, surface tension constant σ, streamline density function $\rho \in C^{2+\alpha}([0, |p_0|])$ and Bernoulli function $\beta \in C^{1+\alpha}([0, |p_0|])$, $0 < \alpha < 1$ be given. Suppose further that p_0, ρ, β and σ collectively satisfy* (L-B).

In contrast to THEOREM 3.1.1*, suppose $\sigma \notin \Sigma_1$ but non-degeneracy conditions* (3.4.1) *and* (3.4.2) *hold. Then there exists four C^1-curves, $\{\mathcal{C}_{\mathrm{i,loc}}\}_{i=1}^4$, of small-amplitude, traveling wave solutions of* (3.1.1)-(3.1.6) *in the space $\mathscr{S}$ with the following properties.*

- *Each $(u, v, \rho, \eta, Q) \in \bigcup_i \mathcal{C}_{\mathrm{i,loc}}$ has speed c, relative mass flux p_0 and satisfies $u < c$ throughout the fluid.*
- *For $i = 1, 2$, each $(u, v, \rho, \eta, Q) \in \mathcal{C}_{\mathrm{i,loc}}$, $i = 1, 2$, has minimal period $2\pi/n_i$ in x.*
- *For $i = 1, 2, 3, 4$, each of the curves $\{\mathcal{C}_{\mathrm{i,loc}}\}$ contain precisely one laminar flow, while for each nontrivial solution $(u, v, \rho, \eta, Q) \in \mathcal{C}_{i,loc}$, with $i = 1, 2$, we have*
  - *(i) the functions u and η are symmetric around the line $x = 0$ while v is anti-symmetric;*
  - *(ii) the function η has precisely one maximum (crest) and one minimum (trough) per minimal period; and*
  - *(iii) the wave profile is strictly monotone between crest and trough.*

REMARK. Referring back to the preceding discussion, the first two curves, $\mathcal{C}_{1,\mathrm{loc}}$ and $\mathcal{C}_{2,\mathrm{loc}}$ are the pure solutions, while the remaining two are the mixed solutions.

The nondegeneracy conditions (3.4.1)–(3.4.2) ensure that, upon applying the Lyapunov-Schmidt reduction, the solutions of the bifurcation equation are described totally by the expansion to cubic order. It is possible to omit condition (3.4.2) but, in this case, rather than having the complete solution set near (U_*, Q_*), we can only guarantee the existence of $\mathcal{K}_1$. Condition (3.4.1) can be similarly relaxed (see the remarks in section 3.4). More detailed information on the local structure of the bifurcation diagram is outlined in LEMMA 3.4.1–LEMMA 3.4.5 and THEOREM 3.4.1.

Once the local bifurcation picture has been resolved, we are able to continue the curves $\{\mathcal{C}_{i,\mathrm{loc}}\}_{i=1}^4$ to produce global continua of solutions. All told, this gives the last of our main results:

THEOREM 3.1.3. (Global Continua from Double Bifurcation) *Assume the hypotheses of* THEOREM 3.1.2. *There exist global continua of solutions extending $\{\mathcal{C}_{i,\mathrm{loc}}\}_{i=1}^4$ in the sense that the following statements hold.*

(a) *There exists a path-connected set $\mathcal{K} \subset \mathscr{S}$ of solutions (u, v, ρ, η, Q) to* (3.1.1)–(3.1.6). *The set $\mathcal{K}$ has the following structural properties.*

- *$\mathcal{K}$ contains a laminar flow, (U_*, Q_*).*
- *There exists four path-connected sets $\{\mathcal{K}_i\}_{i=1}^4$ such that $\mathcal{K} = \bigcup_i \mathcal{K}_i$ and $(U_*, Q_*) \in \bigcap_i \mathcal{K}_i$. In fact, there exists a neighborhood $\mathcal{U}$ of (U_*, Q_*) in $\mathscr{S}$ such that*

$$\mathcal{C}_{i,\mathrm{loc}} = \mathcal{K}_i \cap \mathcal{U}, \qquad \textit{for } i = 1, \ldots, 4.$$

- *Each $\mathcal{K}_i$ admits a global, locally injective, continuous parameterization with a locally injective C^1-reparameterization.*
- *For each $i = 1, \ldots, 4$, we have the following alternatives: Either* (i) *$\mathcal{K}_i$ is a closed loop or, along some sequence $\{(u_n, v_n, \rho_n, \eta_n, Q_n)\} \subset \mathcal{K}_i$,*

$$\text{(ii)}\ \max u_n \uparrow c \qquad \textit{or} \qquad \text{(iii)}\ \min u_n \downarrow -\infty.$$

(b) *All flows $(u, v, \rho, \eta) \in \mathcal{K}$ are regular in the sense that:*

- *u, v, ρ and η each have period L in x;*
- *u, ρ and η are symmetric, v antisymmetric across the line $x = 0$.*

Steady capillary-gravity waves, as we remarked earlier, are an important, classical object of study in fluid mechanics and geophysical fluid dynamics. Unsurprisingly, therefore, they have been the subject of a truly voluminous quantity of research, far more than we can hope to completely summarize here. The vast majority of the work on steady, capillary-gravity waves considers the irrotational regime. There are physical justifications for this choice, but perhaps the most compelling reason is mathematical: when there is no vorticity, ψ is a harmonic function. All that remains is to determine the free surface η. As the vector field has zero curl, moreover, we have the existence of a velocity potential ϕ, which enables the use of conformal maps and other powerful techniques from complex analysis. For instance, there are a number of equivalent reformulations of (3.1.1)–(3.1.7) as a scalar equation for θ, the slope of η (cf. [**43, 51**] for the classical case where $\sigma = 0$, and, e.g. [**53**] for the generalization to $\sigma > 0$.)

Of course, with the introduction of vorticity, the usefulness of these tools is greatly diminished. For that reason, the structure of our arguments in this chapter follows the general program for steady, rotational waves established in [**15, 71, 70, 76**]. We begin, in section 3.2, by reformulating the problem in terms of scalar quantities and in a semi-Lagrangian coordinate system. As a consequence, the transformed domain will be fixed, and the system of equations (3.1.1)-(3.1.6) is replaced by a non-linear, integro-differential, boundary value problem for a scalar function, h, that describes the height above the flat bed. The interior equation satisfied by h has the following form:

$$\mathcal{L}(D^2h, Dh) + \mathcal{I}(Dh, h) = 0,$$

where D^2h denotes the Hessian of h, Dh the gradient, $\mathcal{L}$ is an elliptic operator and $\mathcal{I}$ is an integro-differential operator. As in chapter 2, we will show that $\mathcal{I}$ totally accounts for the presence of stratification (indeed, if ρ is constant, then $\mathcal{I} \equiv 0$). On the other hand, the technical challenges due to surface tension present themselves, naturally, on the boundary. The reformulation of the dynamic condition (in particular, the mean curvature term in (3.1.4)) results in a non-degenerate Venttsel-type condition on the image of the free surface.

In section 3.3, we prove the existence of a 1-parameter family, $\mathcal{T}$, of laminar flows. Since these flows are, by definition, functions of the streamlines, the integro-differential equation reduces to an ODE with operator coefficients (the operator coming from $\mathcal{I}$). We proceed to show that, if $\sigma \in \Sigma_1$ — e.g., it is sufficiently large — then there exists a point on $\mathcal{T}$ at which the problem has a simple (generalized) eigenvalue. Using the theory of Crandall-Rabinowitz, we prove that there is a branch of small-amplitude, non-laminar solutions bifurcating from $\mathcal{T}$ at that point. This is, essentially, a generalization of [**70**] to the stratified case.

Next, in section 3.4, we consider the case when $\sigma \notin \Sigma_1$. As mentioned above, this means that the bifurcation from $\mathcal{T}$ is not simple. In fact, we show that, for all but at most one value of σ, the bifurcation is precisely double, i.e. the null space of the linearized operator has algebraic multiplicity two. Using the classical method of Lyapunov-Schmidt, we give several results on the bifurcation diagram for such σ. This is done by explicit computation of the coefficients in the bifurcation equation. In particular, we show that, under the non-degeneracy condition, there are four C^1-curves of small-amplitude, non-laminar solutions emanating from the bifurcation point, as described in THEOREM 3.1.2. Let us note that this result, though preliminary in the sense that it requires some additional non-degeneracy assumption, is new even in the case of constant density.

The remainder of the chapter is devoted to the global analysis. In section 3.5, we use a degree theoretic argument to prove an alternative theorem in the style of Rabinowitz [**60**]. This allows us to continue globally the curve found via simple local bifurcation in section 3.3. Of course, this involves first proving some strong compactness properties about the reformulated problem — a task made difficult both by the nonlocal operator term in the interior equation, and the nonstandard, second-order boundary conditions.

To continue the local curves obtained in the case of double bifurcation, we employ a global bifurcation theory due to Dancer (cf. [**11, 20, 21, 22**]) that exploits the analyticity of the integro-differential operator in the reformulated problem. As we make clear, this method enjoys several advantages with respect to the more standard degree theoretic approach of section 3.5. First, it enables us to prove an alternative theorem that continues each of the local curves found in section 3.4. Since the crossing at the bifurcation point is no longer odd, this would be non-trivial with a degree theoretic argument. Second, we are able to prove higher regularity of the global continuum. In fact, we show that the continuum $\mathcal{K}$ of solutions to the reformulated problem is a curve with a locally injective, *analytic* reparameterization.

In section 3.7, we derive uniform bounds on the $C^{3+\alpha}$-norm along the continuum $\mathcal{C}$ in terms of the first-order derivatives, the mean curvature on the surface and the energy Q. In section 3.8, these bounds, along with the alternative theorems of sections 3.5 and 3.6, are combined to produce THEOREM 3.1.1 and THEOREM 3.1.3. Finally, section 3.9 analyzes some important special cases: homogeneous, irrotational waves and homogeneous rotational waves.

3.2. Reformulation of the problem

In this section we set ourselves to the task of finding an equivalent formulation of (3.1.7)–(3.1.8) that eliminates the free boundary problem. We shall accomplish this by making a change of variables to transform the fluid domain $\overline{D_\eta}$ in a fixed domain, whose closure we shall denote $\overline{R}$. As usual for incompressible problems, this will be done by means of the pseudo-stream function ψ and will come at the cost of some additional nonlinearity in the governing equation.

To see why this is a natural choice of coordinates, recall that we have $\psi \equiv 0$ on the free surface, $\psi \equiv -p_0$ on the flat bed, where p_0 is the relative pseudo-mass flux and ψ is defined uniquely by the requirement that

$$\psi_x = -\sqrt{\rho}v, \qquad \psi_y = \sqrt{\rho}(u-c). \tag{3.2.1}$$

In light of (3.2.1) and (3.1.1)-(3.1.2), the governing equations inside the fluid become

$$\begin{cases} \psi_y\psi_{xy} - \psi_x\psi_{yy} & = & -P_x \\ -\psi_y\psi_{xx} + \psi_x\psi_{xy} & = & -P_y - \rho g \end{cases} \qquad \text{in } \overline{D_\eta}, \tag{3.2.2}$$

whereas the boundary conditions (3.1.8) are

$$\begin{cases} \psi_x & = & -\psi_y\eta_x, & \text{on } y = \eta(x), \\ P & = & P_{\text{atm}} - \sigma\eta_{xx}\left(1+\eta_x^2\right)^{-3/2}, & \text{on } y = \eta(x), \\ \psi_x & = & 0, & \text{on } y = -d. \end{cases} \tag{3.2.3}$$

Also, by Bernoulli's theorem, the quantity

$$E = P + \frac{\rho}{2}\left((u-c)^2 + v^2\right) + g\rho y, \tag{3.2.4}$$

is constant along streamlines. In particular, then, evaluating the above equation on the free surface we find

$$|\nabla\psi|^2 + 2g\rho(x,\eta(x))\left(\eta(x)+d\right) = Q, \qquad \text{on } y = \eta(x), \tag{3.2.5}$$

where the constant $Q := 2(E|_\eta - P_{\text{atm}} + g\rho|_\eta d)$. Note that this Q gives roughly the energy density along the free surface of the fluid. Critically, however, we shall see that as a consequence of our normalization of η, d is not a parameter for the problem. On the contrary, in all but the most trivial cases, d varies along $\mathcal{C}$. In our analysis, therefore, we shall instead be viewing Q as parameterizing the continuum.

With that in mind, we begin by deriving a scalar PDE satisfied by ψ. In the case where the fluid is homogeneous, it is not hard to check that

$$-\Delta\psi = \omega, \tag{3.2.6}$$

where $\omega := v_x - u_y$ is the vorticity. Moreover, again exploiting the homogeneity, one can readily verify that ω is transported, i.e. there exists some *vorticity function*, $\gamma : [0, |p_0|] \to \mathbb{R}$, such that $\omega = \gamma(\psi)$. It is therefore possible to view (3.2.6) as a semilinear elliptic equation, where γ is prescribed. This is the approach taken, for instance, by Constantin and Strauss ([**15, 16**]) in the (homogeneous) gravity wave case, and Wahlén ([**71, 70**]) for (homogeneous) capillary and (homogeneous) capillary-gravity waves.

In the presence of stratification, however, we cannot generally expect the vorticity to be a function of ψ. Instead, we must look back to the Bernoulli theorem to find an appropriate quantity conserved along streamlines. In particular, one can prove that

$$\frac{dE}{d\psi} = \Delta\psi + gy\frac{d\rho}{d\psi}.$$

Recall that in the previous section we introduced the Bernoulli function β and streamline density function ρ. Rewriting the above expression we arrive at an equation of roughly the same form as (3.2.6):

$$-\beta(\psi) = \Delta\psi - gy\rho'(-\psi). \tag{3.2.7}$$

This is known as Yih's equation or the Yih–Long equation (see, e.g., [**82**]). In this work, it shall serve as our governing equation for ψ. Were it not for the presence of y on the right-hand side, this would reduce to the homogeneous case — a fact that agrees with our intuitive notion that density stratification should reintroduce the depth into the problem as a serious consideration. One can easily check, moreover, that, if $\rho \equiv 1$, then β and γ coincide exactly.

With (3.2.7) in hand, we now make a change of variables to eliminate the free boundary. The new coordinates we will denote (q, p) where

$$q = x, \qquad p = -\psi(x, y).$$

This scheme is sometimes referred to as *semi-Lagrangian* coordinates, in recognition of the fact that we are working, in some sense, halfway between Lagrangian (or streamline) coordinates and the usual Eulerian system (cf. [**69**]).

By means of a scaling argument, we may take $L := 2\pi$. Then, under the transformation

$$(x, y) \mapsto (q, p),$$

the closed fluid domain $\overline{D_\eta}$ is mapped to the rectangle

$$\overline{R} := \{(q,p) \in \mathbb{R}^2 : 0 \le q \le 2\pi,\ p_0 \le p \le 0\}.$$

Given this, it will be convenient to put

$$T := \{(q,p) \in R : p = 0\}, \qquad B := \{(q,p) \in R : p = p_0\}.$$

Note that, in light of (3.1.11) and (3.1.13), we have that

$$\beta = \beta(-p), \qquad \rho = \rho(p).$$

Moreover, the assumption that the streamline density function is nonincreasing becomes

$$\rho_p \le 0. \tag{3.2.8}$$

Next, following the ideas of Dubreil–Jacotin (cf. [**25, 26**]), define

$$h(q,p) := y + d \tag{3.2.9}$$

which gives the height above the flat bottom on the streamline corresponding to p and at $x = q$. We calculate:

$$\psi_y = -\frac{1}{h_p}, \qquad \psi_x = \frac{h_q}{h_p}. \tag{3.2.10}$$

Note that this implies $h_p > 0$, because we have stipulated that $u < c$ throughout the fluid. The change of variables then gives

$$u = c - \frac{1}{\sqrt{\rho} h_p}, \qquad v = -\frac{h_q}{\sqrt{\rho} h_p}. \tag{3.2.11}$$

Performing the Dubreil–Jacotin transformation on (3.2.7), we find that, in the interior of the fluid,

$$(1 + h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_q h_p h_{pq} - g(h - d(h))\rho_p h_p^3 = -h_p^3\beta(-p),$$

where we have used (3.2.9) to write $y = h - d$. Recall, however, that we have normalized η so that it has mean zero. Taking the mean of (3.2.9) along T, we obtain

$$d = d(h) = \fint_0^{2\pi} h(q,0)\,dq. \tag{3.2.12}$$

That is, the average depth d must be viewed as a linear operator acting on h. Namely, it is the average value of h over T. Where there is no risk of confusion, we shall suppress this dependency and simply write d. As we shall see, the addition of this integral term to the governing equations will be the single most significant departure from the homogeneous case, both technically and qualitatively. In recognition of this fact, we shall refer to the mapping $h \mapsto -g(h-d(h))h_p^3\rho_p$ as the *stratification operator*. See chapter 2 for details.

On the bottom, we have by construction that $h \equiv 0$. The more interesting boundary condition is found on the top of R, which corresponds to the free surface in the original formulation. There, we have from (3.2.3) that

$$1 + h_q^2 + h_p^2(2\sigma\kappa[h] + 2g\rho h - Q) = 0, \qquad p = 0,$$

where the curvature, κ, is defined by

$$\kappa[h] := -\frac{h_{qq}}{\left(1+h_q^2\right)^{3/2}}.$$

This calculation is carried out in [**71**], for example.

Combining these observations together we see that the completely reformulated problem is the following. For a given Bernoulli function β, find $(h, Q) \in C^{3+\alpha}_{\mathrm{per}}(\overline{R}) \times \mathbb{R}$ with $h_p > 0$ and satisfying the height equation

$$\text{(3.2.13)} \qquad \begin{cases} (1+h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_qh_ph_{pq} & \\ \qquad -g(h-d(h))\rho_p h_p^3 = -h_p^3\beta(-p), & p_0 < p < 0, \\ & \\ 1 + h_q^2 + h_p^2(2\sigma\kappa[h] + 2g\rho h - Q) = 0, & p = 0, \\ h = 0, & p = p_0. \end{cases}$$

LEMMA 3.2.1. (Equivalency) *Problem* (3.2.13) *is equivalent to problem* (3.1.1)-(3.1.6)

PROOF. This is proven for $\sigma = 0$ in LEMMA 2.2.1. A straightforward modification of that argument — which is itself a simple generalization of a similar result in [**15**] — proves the lemma. For brevity, we omit the details here. The reader is also referred to the discussion in Remark 3.8. ☐

3.3. Local bifurcation at simple eigenvalues

3.3.1. Overview. The goal of this section is to prove the existence of local curves of small amplitude solutions to (3.1.7)–(3.1.8). In continuation from the previous section, we shall make use of the equivalent formulation (3.2.13) to prove the existence of a 1-parameter family of laminar flow solutions, $\mathcal{T}$. These will serve as the trivial solution curve for the standard Crandall-Rabinowitz machinery. Linearizing the problem around the laminar flows, we are thereby able to prove the existence of a local curve of non-laminar solutions bifurcating from $\mathcal{T}$ at a simple eigenvalue under certain size conditions. The analysis culminates in the following result.

THEOREM 3.3.1. (Local Bifurcation at Simple Eigenvalues) *Fix $\alpha \in (0,1)$ and let the wave speed $c > 0$, the relative mass flux $p_0 < 0$, surface tension constant σ, streamline density function $\rho \in C^{2+\alpha}([p_0, 0]; \mathbb{R}^+)$ and Bernoulli function $\beta \in C^{1+\alpha}([0, |p_0|])$ be given. Suppose further that p_0, ρ, β and σ collectively satisfy* (L-B). *If $\sigma \in \Sigma_1$ (as defined in* (3.3.6)*), then the following holds.*

There exists a C^1-curve $\mathcal{C}_{loc} \subset \mathscr{S}$ of small amplitude, traveling wave solutions (u, v, ρ, η) of (3.1.1)-(3.1.6) *with period 2π, speed c and relative mass flux p_0, satisfying $u < c$ throughout the fluid. The curve $\mathcal{C}_{\rm loc}$ has the following properties.*

- *(i) $\mathcal{C}_{loc}$ contains precisely one laminar flow. denote it (Q_*, U_*).*
- *(ii) $\mathcal{C}_{loc} \setminus \{(Q_*, U_*)\}$ comprises all nonlaminar solutions in a sufficiently small neighborhood of (Q_*, U_*) in $\mathbb{R} \times C^{2+\alpha}_{\rm per}(\overline{D_\eta})$.*
- *(iii) For each nontrivial solution $(u, v, \rho, \eta) \in \mathcal{C}_{loc}$,*
  - *the functions u, ρ and η are symmetric around the line $x = 0$ while v is antisymmetric;*
  - *the function η has precisely one maximum (crest) and one minimum (trough) per period;*
  - *the wave profile is strictly monotone between crest and trough.*

3.3.2. Eigenvalue problem. We begin by establishing the existence of a 1-parameter family of laminar flows. By laminar flow we mean that the streamlines (which, we recall, include the free surface) are parallel to the bed. Waves of this type will accordingly have the ansatz $h = H(p)$, which reduces the height equation (3.2.13) to the following (operator coefficient) ODE boundary value problem:

$$
\begin{cases}
H_{pp} - gH_p^3(H - d(H))\rho_p = -H_p^3\beta(-p), & p_0 < p < 0, \\
1 + H_p^2(2g\rho H - Q) = 0, & p = 0, \\
H = 0, & p = p_0.
\end{cases}
\tag{3.3.1}
$$

The basic existence theory for laminar solutions comes in the form of the following lemma.

Lemma 3.3.1. (Laminar Flow) *Suppose that the streamline density function ρ satisfies $\rho_p \leq 0$. Then there exists a 1-parameter family of solutions $(H(\cdot;\lambda), Q(\lambda))$ to the laminar flow equation* (3.3.1) *with $H_p > 0$, where $0 \leq -2B_{\min} < \lambda < Q$.*

Proof. Because the curvature of the laminar flows is zero, (3.3.1) is identical to the problem satisfied by laminar stratified gravity waves in the absence of surface tension. The lemma therefore follows from Lemma 2.3.1. □

Remark. For later reference, we note it was also shown in the previous chapter that

$$H(0) = \frac{Q-\lambda}{2g\rho(0)}, \qquad H_p(0) = \lambda^{-1/2}.$$

In contrast to the theory for homogeneous gravity and capillary-gravity waves, there is no explicit formula for these laminar flows. However, H_p satisfies the following implicit equation

$$H_p(p;\lambda) = \frac{1}{\sqrt{\lambda + G(p;\lambda)}}, \qquad p_0 < p < 0,$$

where

$$G(p;\lambda) := 2B(p) + 2\int_p^0 g\left(H(r;\lambda) - d(H(r;\lambda))\right)\rho_p(r)dr, \qquad p_0 < p < 0.$$

It will occasionally be convenient to make use of the shorthand

$$a(\cdot;\lambda) := H_p(\cdot;\lambda)^{-1}, \qquad Y(\cdot;\lambda) := H(\cdot;\lambda) - d(H).$$

Finally, we shall denote $\dot{H} := \partial H/\partial\lambda$, $\dot{Y} := \partial Y/\partial\lambda$, $\dot{G} := \partial G/\partial\lambda$ and $\dot{Q} := dQ/d\lambda$. In fact, we have the following useful identity

$$\dot{Y}(p) = \frac{1}{2}\int_{p_0}^0 \frac{1+\dot{G}(r)}{(\lambda+G(r))^{3/2}}dr = \frac{1}{2}\int_p^0 \left(1+\dot{G}(r)\right)a(r)^{-3}dr, \qquad p_0 \leq p \leq 0. \tag{3.3.2}$$

This is found by differentiating the relation $Y_p = (\lambda+G)^{1/2}$ in λ, followed by integrating in p.

We now linearize the full height equation (3.2.13) along the curve of laminar solutions, which we shall denote $\mathcal{T}$. As in chapter 2, when there is heterogeneity in the fluid, it will be necessary to ensure that λ is bounded strictly away from $-2B_{\min}$. Recall that we have defined

$$\epsilon_0^{3/2} := \max\left\{2g\|\rho'\|_{L^\infty}p_0^2 e^{|p_0|},\ (2g\|\rho'\|_{L^\infty})^3,\ (4\|\rho'\|_{L^\infty})^3,\ 8g|p_0|\rho(0)\right\}.$$

Note that having $\epsilon_0^{3/2}$ greater than the second quantity on the right-hand side above implies, for $\lambda > -2B_{\min} + \epsilon_0$,

$$a + g\rho_p \geq \epsilon_0^{1/2} - g\|\rho_p\|_{L^\infty} \geq \epsilon_0/2 > 0$$

if $\rho_p \not\equiv 0$. The same holds true, of course, when $\rho_p \equiv 0$, since in this case the left-hand side reduces to a, which is strictly positive when $\lambda > -2B_{\min}$. It was shown in Lemma 2.3.4 that, when $\epsilon_0^{3/2} > 2g\|\rho_p\|_{L^\infty}p_0^2 e^{|p_0|}$, then $-1/2 \leq \dot{G}(\cdot;\lambda) \leq 0$. In particular, this implies that, $a + g\rho_p > 0$, and $1 + \dot{G} > 1/2 > 0$. Both of these facts are critical for our analysis and so from this section onward we shall always be assuming that $\lambda > -2B_{\min} + \epsilon_0$. The remaining two quantities on the right-hand side above are, in fact, not strictly necessary. In chapter 2 they were used to avoid a certain pathology that might occur if the point of bifurcation λ_* occurred to the right of a particular value λ_0. That will not be an issue for the present chapter. Nonetheless, for simplicity of exposition, we keep the same definition of ϵ_0.

Now, fixing any such λ, we calculate that the linearized problem about $H(\cdot;\lambda)$ is:

$$(3.3.3)\qquad \begin{cases} m_{pp} + m_{qq}H_p^2 - g\rho_p(3m_p(H - d(H))H_p^2 + (m - d(m))H_p^3), & \\ \qquad = -3H_p^2 m_p \beta(-p) & p_0 < p < 0, \\ m = 0, & p = p_0, \\ g\rho m - \sigma m_{qq} - \lambda^{3/2} m_p = 0, & p = 0. \end{cases}$$

Since we seek solutions that are 2π-periodic and even in q, it is natural to search for m with the ansatz $m(q,p) = M(p)\cos(nq)$, for some $n \geq 0$. Inserting this into (3.3.3) we see immediately that M must satisfy the following:

$$(3.3.4)\qquad \begin{cases} m_{pp} - n^2 M H_p^2 - g\rho_p(3M_p(H - d(H))H_p^2 + (M - \delta_{n0}M(0))H_p^3) & \\ \qquad = -3H_p^2 M_p \beta(-p), & p_0 < p < 0, \\ M = 0, & p = p_0, \\ g\rho M + n^2\sigma M - \lambda^{3/2} M_p = 0 & p = 0, \end{cases}$$

where $\delta_{n0} = 1$ if $n = 0$ and vanishes otherwise. This is a simple consequence of the fact that

$$d(m) = \fint_0^{2\pi} m(q,0)dq = M(0)\fint_0^{2\pi} \cos(nq)dq = \delta_{n0}M(0).$$

Using the equation satisfied by H, (3.3.1), we may put (3.3.4) into self-adjoint form:

$$\text{(3.3.5)} \qquad \begin{cases} \{H_p^{-3}M_p\}_p = (n^2H_p^{-1} + g\rho_p)M - g\rho_p\delta_{n0}M(0), & p_0 < p < 0, \\ M = 0, & p = p_0, \\ \lambda^{3/2}M_p = (n^2\sigma + g\rho)M, & p = 0. \end{cases}$$

Solving the above problem for any value of n will produce a 2π-periodic solution to the linearized problem, but, as we are primarily interested in solutions of minimal period 2π, we shall focus on the case where $n = 1$.

With that in mind, we make the following definition.

$$\text{(3.3.6)} \qquad \begin{aligned} \Sigma_1 := \{\sigma \in \mathbb{R}^+ \colon & \text{there exists a unique } \lambda > -2B_{\min} + \epsilon_0 \text{ such that} \\ & \text{problem (3.3.5) has a solution } (M,n) \text{ if and only if } n = 1\}, \end{aligned}$$

$$\text{(3.3.7)} \qquad \begin{aligned} \Sigma_2 := \{\sigma \in \mathbb{R}^+ \colon & \text{there exists a } \lambda > -2B_{\min} + \epsilon_0 \text{ such that} \\ & \text{problem (3.3.5) has precisely two linearly independent} \\ & \text{nontrivial solutions } (M_1, n_1) \text{ and } (M_2, n_2), \\ & \text{where } n_1 = 1,\ n_2 \geq 2\}. \end{aligned}$$

and

$$\text{(3.3.8)} \qquad \begin{aligned} \Sigma_3 := \{\sigma \in \mathbb{R}^+ : & \text{ there exists a } \lambda > -2B_{\min} + \epsilon_0 \text{ such that} \\ & \text{problem (3.3.5) has at least two linearly independent solutions} \\ & (M_1, n_1) \text{ and } (M_2, n_2) \text{ where } n_1 = 1,\ n_2 = 0\} \end{aligned}$$

For fixed p_0, ρ and β, we shall prove that the set Σ_1 consists of those values of σ for which simple bifurcation occurs at some point along $\mathcal{T}$. The curve of nonlaminar solutions will, locally, be of the form $(Q(\lambda), H(p; Q(\lambda)) + \epsilon M(p)\cos q + O(\epsilon^2))$, and hence of minimal period 2π. On the other hand, if σ lies in the set Σ_2, then we expect double bifurcation to occur and there will be several curves of 2π-periodic solutions that, nevertheless, have a minimal period strictly less than 2π. The last set, Σ_3, consists of those values of σ where there is a zero eigenvalue of the linearized problem. We shall prove that Σ_3 is either a singleton or the empty set (see Lemma 3.3.7). Furthermore, in Corollary 3.3.1, we show that every σ for which (L-B) holds is in precisely one of Σ_1, Σ_2 and Σ_3. We strongly suspect that Σ_1 is generic. In fact, in section 3.9 we provide an elementary proof of the (previously known) result that, for irrotational, constant density waves, Σ_2 is countable.

This section is concerned with the simple bifurcation case, so our first task is to produce sufficient conditions under which $\sigma \in \Sigma_1$. In order to apply the linear theory developed in chapter 2, we introduce a modified density function, $\tilde{\rho}$, defined by

$$\tilde{\rho} := \rho + \frac{\sigma}{g}.$$

It follows that, for each $\lambda \geq -2B_{\min} + \epsilon_0$, $(H(\cdot;\lambda), \tilde{Q}(\lambda))$ is a laminar solution with streamline density function $\tilde{\rho}$, where

$$\begin{aligned} \tilde{Q}(\lambda) &:= 2\sigma H(0;\lambda) + Q(\lambda) \\ &= \sigma \left(\frac{\left(\sigma^{-1} g\rho(0) + 1\right) Q(\lambda) - \lambda}{g\rho(0)} \right) = \frac{g\tilde{\rho}(0) Q(\lambda) - \sigma\lambda}{g\rho(0)}. \end{aligned} \tag{3.3.9}$$

It is important to note that $\tilde{Q}$ inherits all of the important properties of Q. These we summarize in the following lemma.

LEMMA 3.3.2. *Let $\lambda > -2B_{min} + \epsilon_0$ be given and let $\tilde{Q}$ be defined as in* (3.3.9). *Then $\tilde{Q}$ is strictly convex as a function of λ with a unique minimum occurring at $\tilde{\lambda}_0$. Moreover, if the unique minimum of Q occurs at λ_0, then $\lambda_0 < \tilde{\lambda}_0$.*

PROOF. We know by COROLLARY 2.3.1 that Q is a strictly convex function of λ. Differentiating (3.3.9) twice in λ confirms that the same is true of $\tilde{Q}$. To see $\tilde{Q}$ attains its minimum, note that evaluating (3.3.9) at $\lambda = \lambda_0$, we have $d\tilde{Q}/d\lambda < 0$. But, since $dQ/d\lambda \nearrow 1$ as $\lambda \to \infty$, while $g\tilde{\rho} > \sigma$, taking λ sufficiently large we must have $d\tilde{Q}/d\lambda > 0$. Hence, there exists some $\tilde{\lambda}_0 > \lambda_0$ where $d\tilde{Q}/d\lambda = 0$. ☐

Now, since we are searching for a solution to (3.3.5) with $n^2 = 1$, we can simply replace ρ with $\tilde{\rho}$ to rewrite the problem as something reminiscent of the stratified gravity wave case. In particular, we see that it is equivalent to finding a solution M of

$$\begin{cases} H_p\{H_p^{-3} M_p\}_p - g\tilde{\rho}_p H_p M = M, & p_0 < p < 0 \\ M = 0, & p = p_0 \\ \lambda^{3/2} M_p = g\tilde{\rho} M, & p = 0. \end{cases} \tag{3.3.10}$$

We have therefore reduced the problem to that considered in chapter 2. Recalling the methodology of that chapter, let us now consider the minimization problem:

$$\mu = \mu(\lambda) = \inf_{\phi \in \mathcal{S}} \mathcal{R}(\phi; \lambda) \tag{3.3.11}$$

where $\mathcal{S} := \{\phi \in H^1((p_0, 0)) \colon \phi \not\equiv 0,\ \phi(p_0) = 0\}$ and the Rayleigh quotient $\mathcal{R}$ is given by

$$\mathcal{R}(\phi; \lambda) := \frac{-g\tilde{\rho}(0)\phi(0)^2 + \int_{p_0}^0 H_p(p;\lambda)^{-3}\phi_p(p)^2 dp}{\int_{p_0}^0 \phi(p)^2 \left(H_p(p;\lambda)^{-1} + g\tilde{\rho}_p(p)\right) dp}. \tag{3.3.12}$$

By the standard variational theory for Sturm-Liouville problems, the existence of a nontrivial solution to (3.3.10) for some $\lambda_* > -2B_{\min} + \epsilon_0$ is equivalent to the statement $\mu(\lambda_*) = -1$. Given that fact, we make the following definition.

Definition 3.3.1. We say the pseudo-volumetric mass flux p_0, coefficient of surface tension σ, streamline density function ρ and Bernoulli function β collectively satisfy the *capillary-gravity local bifurcation condition* provided that

(L-B) there exists $\lambda > -2B_{\min} + \epsilon_0$ and $M \not\equiv 0$ solving the linearized problem (3.3.10).

Equivalently,

$$\inf_{\lambda \geq -2B_{\min}+\epsilon_0} \mu(\lambda) \leq -1.$$

This last statement can also be written

$$\inf_{\lambda \geq -2B_{\min}+\epsilon_0} \inf_{\phi \in \mathcal{S}} \left\{ \frac{-g\rho(0)\phi(0)^2 - \sigma\phi(0)^2 + \int_{p_0}^0 H_p(p;\lambda)^{-3}\phi_p(p)^2 dp}{\int_{p_0}^0 \phi(p)^2 \left(H_p(p;\lambda)^{-1} + g\rho_p(p)\right) dp} \right\} < -1. \tag{3.3.13}$$

where $\mathcal{S} := \{\phi \in H^1((p_0, 0)) : \phi \not\equiv 0,\ \phi(p_0) = 0\}$ and $H(\cdot;\lambda)$ is the laminar flow solution given by Lemma 3.3.1.

(L-B) is both necessary and sufficient, but clearly suffers from being highly non-explicit. However, as was alluded to in the remarks following Theorem 3.1.1, if we are willing to sacrifice necessity, there is an entirely explicit sufficient condition under which the capillary-gravity local bifurcation holds, namely (3.1.15).

Lemma 3.3.3. (Sufficiency of Size Condition) *If p_0, ρ, β and σ satisfy the size condition* (3.1.15), *then they satisfy* (L-B).

Proof. This lemma follows from Lemma 2.3.3 with $\tilde{\rho}$ in place of ρ. $\square$

We also remark that from (3.3.13) it is evident that, if (L-B) holds for $(p_0, \sigma_1, \rho, \beta)$, then it hold for $(p_0, \sigma_2, \rho, \beta)$ whenever $\sigma_2 \geq \sigma_1$. In particular, this implies that if the Local Bifurcation Condition is satisfied for the problem without surface tension (i.e., with $\sigma = 0$), then it is satisfied for all values of $\sigma > 0$ as well.

For future reference we now quote another result from chapter 2.

LEMMA 3.3.4. (Monotonicity) *μ is a strictly increasing function of λ when $\mu(\lambda) < 0$.*

This result implies, for instance, that any $\lambda > -2B_{\min}+\epsilon_0$ satisfying $\mu(\lambda) = -1$ is necessarily unique. The next lemma gives the existence of such a λ.

LEMMA 3.3.5. (Eigenvalue Problem) *Assuming* (L-B)*, there exists $\lambda_* > -2B_{\min} + \epsilon_0$ and a nontrivial solution $M = M(p)$ to* (3.3.5) *with $n = 1$.*

PROOF. In the previous paragraphs we have shown that the producing such an M is equivalent to finding λ_* such that $\mu(\lambda_*) = -1$. Condition (L-B) ensures that $\mu(\lambda) < -1$, for some $\lambda > -2B_{\min} + \epsilon_0$, so it only remains to show that $\mu(\lambda) \geq -1$ for λ sufficiently large. With that in mind, let λ be given such that

$$\lambda > -2B_{\min} + \left(g\|\rho_p\|_{L^\infty} + \sqrt{g\rho(0) + \sigma}\right)^2.$$

Recalling that $G(\cdot;\lambda) \geq 2B_{\min}$, we find that, for any such λ,

$$H_p^{-1} \geq H_p^{-1} + g\rho_p = (\lambda + G)^{1/2} + g\rho_p \geq (\lambda + 2B_{\min})^{1/2} - g\|\rho_p\|_{L^\infty} \geq \sqrt{g\rho(0) + \sigma}.$$

Now let $\phi \in \mathcal{S}$ be given and take λ as above. Then

$$\begin{aligned}\int_{p_0}^0 \left(H_p^{-3}\phi_p^2 + (H_p^{-1} + g\rho_p)\phi^2\right) dp &\geq \sqrt{g\rho(0) + \sigma}\int_{p_0}^0 \left((g\rho(0) + \sigma)\phi_p^2 + \phi^2\right) dp \\ &\geq 2\left(g\rho(0) + \sigma\right)\int_{p_0}^0 \phi\phi_p dp = \left(g\rho(0) + \sigma\right)\phi(0)^2.\end{aligned}$$

From the last inequality we conclude $\mathcal{R}(\phi) \geq -1$. Since the selection of ϕ above was arbitrary, this proves $\mu(\lambda) \geq -1$, which completes the lemma. □

From Remark 3.3.2, we have that

$$\dot{Q} = 1 - 2g\rho(0)\dot{Y}(p_0).$$

As we stated in the proof of LEMMA 3.3.2, the right-hand side above vanishes for $\lambda = \lambda_0$, and approaches 1 monotonically from below as $\lambda \to \infty$. Therefore, for some λ_c, we have

$$4\dot{Y}(p_0;\lambda) < \frac{1}{g\rho(0) + g\|\rho_p\|_{L^\infty}|p_0|}, \qquad \text{for all } \lambda > \lambda_c. \tag{3.3.14}$$

As $2g\rho(0)\dot{Y}(p_0;\lambda_0) = 1$, and Q obtains its unique minimum at λ_0, we must have, in particular, that λ_c occurs to the right of λ_0. The significance of this quantity will be made clear later. Briefly, the point is that, when $\lambda_* > \lambda_c$, we will show that the null space of the linearized problem is one-dimensional, whereas in general it may be two-dimensional. In fact, when ρ is taken to be a constant, one can instead take $\lambda_c = \lambda_0$ (cf. [**70**]), in which case it satisfies the implicit relation

$$\int_{p_0}^{0} H_p(p;\lambda_c)^3 dp = \frac{1}{g\rho(0)}. \tag{3.3.15}$$

Since $H_p(\cdot;\lambda)$ is strictly monotonic as a function of λ, (3.3.15) serves as an equivalent definition of λ_c when ρ is constant.

Next, put

$$\sigma_c := (g\rho(0))^2 \int_{p_0}^{0} \left(H_p(p;\lambda_c)^{-1} + g\rho_p(p)\right)\left(\int_{p_0}^{p} H_p(s;\lambda_c)^3 ds\right)^2 dp. \tag{3.3.16}$$

The importance of this quantity is found in the following lemma.

LEMMA 3.3.6. *Suppose that p_0, σ, ρ, and β are given satisfying* (L-B) *(or size condition* (3.1.15)*). Then, for a unique λ_*, there exists a solution of the form $m = M(p)\cos q$ to* (3.3.3)*. Moreover, if we assume that*

$$\sigma \geq \sigma_c, \tag{3.3.17}$$

then $\lambda_c < \lambda_*$. (Note that this is an explicit statement as neither H nor ρ depend on σ.)

PROOF. Writing (L-B) in terms of $\tilde{\rho}$ we find that p_0, $\tilde{\rho}$ and β satisfy the Local Bifurcation Condition of chapter 2. Thus applying the lemmas of section 2.3 gives the existence of a solution M of (3.3.10) for a unique λ_*. As we have commented before, this is equivalent to solving (3.3.5), so $m(q,p) := M(p)\cos q$ is indeed the solution to (3.3.3) we seek.

Next, we the minimization problem (3.3.11). For each λ, the function M that attains the minimum, $\mu(\lambda)$, will satisfy the boundary conditions of the linearized problem (3.3.10) as well as the ODE:

$$\{H_p^{-3}M_p\}_p = -\mu(\lambda)(H_p^{-1} + g\rho_p)M.$$

By construction, therefore, $\mu(\lambda_*) = -1$. Fix $\lambda = \lambda_c$, and let $\phi \in \mathcal{S}$ be defined as

$$\phi(p) = \int_{p_0}^{p} a(s)^{-3} ds, \qquad p_0 < p < 0.$$

Note that ϕ is therefore the solution of the following boundary value problem:

$$(a^3\phi_p)_p = 0, \ \text{ for } p_0 < p < 0, \qquad \phi(p_0) = 0, \qquad \lambda^{3/2}\phi_p(0) = g\rho(0)\phi(0).$$

Evaluating $\mathcal{R}(\phi;\lambda)$ yields

$$\begin{aligned}
\mathcal{R}(\phi;\lambda_c) &= \frac{-g\tilde{\rho}(0)\phi(0)^2 + \int_{p_0}^0 a(p)^3\phi_p(p)^2 dp}{\int_{p_0}^0 \phi(p)^2\left(a(p) + g\tilde{\rho}_p(p)\right)dp} \\
&= \frac{-\sigma\phi(0)^2}{\int_{p_0}^0 \phi(p)^2\left(a(p;\lambda) + g\rho_p(p)\right)dp} \\
&= -\sigma\sigma_c^{-1}.
\end{aligned}$$

For $\sigma \geq \sigma_c$, the above identity implies $\mu(\lambda_c) \leq -1$. By the monotonicity of μ, we conclude that λ_c lies to the left of λ_* as desired. ☐

We end this subsection with a lemma characterizing the set of σ for which zero is an eigenvalue of the linearized problem.

Lemma 3.3.7. (Zero Eigenvalue) *Fix p_0, ρ and β. There exists at most one value of σ for which both* (L-B) *holds and there exists a nontrivial* (M, n) *solving* (3.3.5) *with $n = 0$ and $\lambda = \lambda_*$. In particular, Σ_3 is either a singleton or the empty set.*

Proof. Let σ_0 be given as in the statement of the lemma. Then by (3.3.5) with $n = 0$, there exists M solving

$$(a^3M_p)_p - g\rho_p(M - M(0)) = 0, \qquad M(p_0) = 0, \qquad \lambda^{3/2}M_p(0) = g\rho(0)M(0),$$

for $\lambda = \lambda_*$. Note that all the effects of surface tension have been removed from the equation, as σ occurs only as a coefficient of n in (3.3.5). Indeed, this problem is identical to that studied in chapter 2, where it was shown that a nontrivial solution exists if and only if $\lambda_* = \lambda_0$.

In order to prove the lemma, therefore, we must show that there is at most one value of σ for which λ_* can coincide with λ_0. To make the dependence of λ_* on σ explicit, define

$$\mathcal{R}(\phi;\lambda,\sigma) := \frac{-g\rho(0)\phi(0)^2 - \sigma\phi(0)^2 + \int_{p_0}^0 a^3\phi_p^2 dp}{\int_{p_0}^0 \phi^2\left(a + g\rho_p\right)}dp,$$

for $\phi \in \mathcal{S}$. Then, for each σ for which (L-B) is satisfied, we may let $\lambda_* = \lambda_*(\sigma)$ to be the smallest value of $\lambda > -2B_{\min} + \epsilon_0$ such that

$$\inf_{\phi \in \mathcal{S}} \mathcal{R}(\phi; \lambda_*(\sigma), \sigma) = -1.$$

Clearly, λ_* is continuous in σ on its domain.

Now let σ_1, σ_2 be given with $\sigma_1 < \sigma_2$, (L-B) hold for σ_1 (hence also for σ_2.) By the definition of $\lambda_*(\cdot)$, there exists $\phi_1, \phi_2 \in \mathcal{S}$ such that

$$\mathcal{R}(\phi_1; \lambda_*(\sigma_1), \sigma_1) = -1 = \mathcal{R}(\phi_2; \lambda_*(\sigma_2), \sigma_2).$$

But, recalling the meaning of $\mathcal{R}(\cdot; \cdot, \cdot)$, we have

$$\begin{aligned}
\mathcal{R}(\phi_2; \lambda_*(\sigma_2), \sigma_2) &\leq \mathcal{R}(\phi_1; \lambda_*(\sigma_2), \sigma_2) \\
&= \mathcal{R}(\phi_1; \lambda_*(\sigma_2), \sigma_1) + \frac{(\sigma_1 - \sigma_2)\, \phi_1(0)^2}{\int_{p_0}^0 \phi_1^2 \,(a + g\rho_p)\, dp} \\
&< \mathcal{R}(\phi_1; \lambda_*(\sigma_2), \sigma_1).
\end{aligned}$$

The strict inequality derives from the fact that $\phi_1(0) \neq 0$, in light of the equation (3.3.5) satisfied by ϕ_1. We have therefore shown $\mathcal{R}(\phi_1; \lambda_*(\sigma_2), \sigma_1) > -1$. By the strict monotonicity of μ, we conclude that $\lambda_*(\sigma_2)$ lies to the right of $\lambda_*(\sigma_1)$. It follows that λ_* is a strictly increasing function of σ. Hence there is at most one value of σ for which $\lambda_*(\sigma) = \lambda_0$. $\square$

3.3.3. Proof of local bifurcation from simple eigenvalues. In short, the previous section assures us that, if (L-B) and size condition (3.3.17) each hold, then, at some $\lambda_* > -2B_{\min} + \epsilon_0$, there exists a solution of the linearized problem (3.3.3). In order to infer the existence of curve of non-laminar flows local to this point, we shall make use of the classical theorem of Crandall and Rabinowitz. In this section and onward, we shall use $\mathcal{N}(\mathcal{L})$ to denote the null space of a linear operator $\mathcal{L}$ and $\mathcal{R}(\mathcal{L})$ to denote its range.

Theorem 3.3.2. (Crandall-Rabinowitz, [**18**]) *Let X and Y be Banach spaces, $I \subset \mathbb{R}$ an open interval with $\lambda_* \in I$. Suppose that $\mathcal{F} : I \times X \to Y$ is a continuous map with the following properties:*

(i) $\mathcal{F}(\lambda, 0) = 0$, for all $\lambda \in I$;

(ii) $D_1\mathcal{F}$, $D_2\mathcal{F}$ and $D_1D_2\mathcal{F}$ exist and are continuous, where D_i denotes the Fréchet derivative with respect to the i-th coordinate;

(iii) $D_2\mathcal{F}(\lambda_*, 0)$ *is a Fredholm operator of index 0, in particular, the null space is one-dimensional and spanned by some element* w_**;*

(iv) $D_1 D_2\mathcal{F}(\lambda_*, 0)w_* \notin \mathcal{R}(D_2\mathcal{F}(\lambda_*, 0))$.

Then there exists a continuous local bifurcation curve $\{(\lambda(s), w(s)) \in \mathbb{R} \times X : |s| < \epsilon\}$ *with* $\epsilon > 0$ *sufficiently small such that* $(\lambda(0), w(0)) = (\lambda_*, w_*)$*, and*

$$\{(\lambda, w) \in \mathcal{U} : w \neq 0, \mathcal{F}(\lambda, w) = 0\} = \{(\lambda(s), w(s)) \in \mathbb{R} \times X : |s| < \epsilon\}$$

for some neighborhood $\mathcal{U}$ *of* $(\lambda_*, 0)$ *in* $\mathbb{R} \times X$*. Moreover, we have*

$$w(s) = sw_* + o(s), \qquad \text{in } X,\ |s| < \epsilon.$$

If $D_2^2\mathcal{F}$ *exists and is continuous, then the curve is of class* C^1.

All that is left for us now is to verify the hypotheses of the preceding theorem. As in the previous sections let the transformed fluid domain be

$$R := \{(q, p) : 0 < q < 2\pi,\ p_0 < p < 0\},$$

with boundaries

$$T := \{(q, p) \in R : p = 0\}, \qquad B := \{(q, p) \in R : p = p_0\},$$

and define

$$X := \{h \in C^{3+\alpha}_{\text{per}}(\overline{R}) : h = 0 \text{ on } B\}, \qquad Y = Y_1 \times Y_2 := C^{1+\alpha}_{\text{per}}(\overline{R}) \times C^{1+\alpha}_{\text{per}}(T).$$

Let $h(q, p) := H(p) + w(q, p)$. Then by the full height equation, w must satisfy the following PDE

$$\begin{aligned} &(1 + w_q^2)(H_{pp} + w_{pp}) + w_{qq}(H_p + w_p)^2 - 2w_q w_{pq}(H_p + w_p) \\ &-g(H + w - d(H) - d(w))(H_p + w_p)^3\rho_p + (H_p + w_p)^3\beta(-p) = 0 \qquad \text{in } R, \end{aligned} \tag{3.3.18}$$

$$1 + w_q^2 + (H_p + w_p)^2(2\sigma\kappa[w] + 2g\rho(H + w) - Q) = 0 \qquad \text{on } T, \tag{3.3.19}$$

together with periodicity in q and vanishing on B.

Conforming to the framework of [**18**], we introduce a nonlinear operator

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : (-2B_{\min} + \epsilon_0, \infty) \times X \to Y$$

defined, for $w \in X$, $\lambda > -2B_{\min} + \epsilon_0$, by

$$\begin{aligned}\mathcal{F}_1(\lambda, w) &:= (1+w_q^2)(H_{pp}+w_{pp}) + w_{qq}(H_p+w_p)^2 - 2w_q w_{pq}(H_p+w_p) \\ &\quad - g(H+w-d(H)-d(w))(H_p+w_p)^3\rho_p + (H_p+w_p)^3\beta(-p)\end{aligned} \tag{3.3.20}$$

$$\mathcal{F}_2(\lambda, w) := 1 + w_q^2 + (H_p+w_p)^2(2\sigma\kappa[w] + 2g\rho(H+w) - Q). \tag{3.3.21}$$

Note that by definition of the laminar solution $H(\cdot;\lambda)$, we have $\mathcal{F}(\lambda, 0) \equiv 0$. For future reference, we evaluate the Fréchet derivatives $\mathcal{F}_{1w}$, $\mathcal{F}_{2w}$ at $w = 0$:

$$\mathcal{F}_{1w} = \partial_p^2 + H_p^2\partial_q^2 + 3H_p^2\beta(-p)\partial_p - 3g(H-d(H))H_p^2\rho_p\partial_p - gH_p^3\rho_p(1-d) \tag{3.3.22}$$

$$\begin{aligned}\mathcal{F}_{2w} &= \left(2g\rho H_p^2 + 2H_p(-2\sigma\partial_q^2 + 2g\rho H - Q)\partial_p\right)\Big|_T \\ &= \left(2g\rho\lambda^{-1} - 2\lambda^{1/2}\partial_p - 2\sigma\lambda^{-1}\partial_q^2\right)\Big|_T.\end{aligned} \tag{3.3.23}$$

In order to show that the eigenvalue at λ_* is simple we must characterize the null space and range of the linearized operator $\mathcal{F}_w(\lambda_*, 0)$. This is accomplished in the following three lemmas

THEOREM 3.3.3. (Null Space) *For $\lambda_* \neq \lambda_0$, the null space of $\mathcal{F}_w(\lambda_*, 0)$ is at most two-dimensional. In particular, it is one-dimensional provided $\lambda_* > \lambda_c$. Zero is an eigenvalue if and only if $\lambda_* = \lambda_0$.*

PROOF. In the previous section we showed that $M(p)\cos q \in \mathcal{N}(\mathcal{F}_w(\lambda_*, 0))$, hence the null space is at least one-dimensional. Using a generalization of the arguments in [**70**], we now demonstrate that, when $\lambda_* \neq \lambda_0$, then $\mathcal{N}(\mathcal{F}_w(\lambda_*, 0))$ can be at most two-dimensional. Furthermore, when $\lambda_* > \lambda_c$ it is at most one-dimensional.

Let $m \in \mathcal{N}(\mathcal{F}_w(\lambda_*, 0))$ be given. Then, by the evenness of m in q, we may decompose it via a cosine series:

$$m(q,p) = \sum_{n=0}^{\infty} m_n(p)\cos(nq).$$

By the definition of λ_*, we have that m_1 is in the span of M, the minimizer of the Rayleigh quotient $\mathcal{R}$. Because m is in the null space of the linearized operator, each m_{n_i} must satisfy (3.3.4) for $n = n_i$.

The case when $m_0 \not\equiv 0$ was dealt with in LEMMA 3.3.7. Recall that it was argued that this occurs precisely when $\lambda_* = \lambda_0$.

Let us now consider the case $\lambda_* \neq \lambda_0$. Suppose that n_1, n_2 and n_3 are distinct positive integers such that $1 \in \{n_1, n_2, n_3\}$ and $m_{n_1}, m_{n_2}, m_{n_3} \not\equiv 0$. We shall prove that (i) for $\lambda > \lambda_c$, each m_{n_i} is in the span of m_1, and (ii) at most two of $\{m_{n_1}, m_{n_2}, m_{n_3}\}$ are linearly independent. For simplicity let us denote $m_{n_i} := \phi_i$ for $i = 1, 2, 3$.

As $n_i \geq 1$, from (3.3.5) we conclude that the functions ϕ_i solve the following ODE:

$$-(a^3\phi_i')' + g\rho'\phi_i = -n_i^2 a\phi, \qquad p_0 < p < 0,$$

along with the boundary conditions

$$\phi_i(p_0) = 0, \qquad \lambda^{3/2}\phi_i'(0) = (n_i^2\sigma + g\rho(0))\phi(0), \qquad i = 1, 2, 3.$$

Multiplying the first equation by ϕ_j and integrating, we obtain the following identity: for any $p_0 \leq p \leq 0$,

$$(a^3\phi_i\phi_j')\Big|^p = \int_{p_0}^p a^3\phi_i'\phi_j' dr + \int_{p_0}^p (n_i^2 a + g\rho')\phi_i\phi_j dr > 0. \tag{3.3.24}$$

When $p = 0$ we find from the boundary conditions that

$$\begin{aligned} -n_i^2\left(\int_{p_0}^0 a\phi_i\phi_j dp - \sigma\phi_i(0)\phi_j(0)\right) &= -g\rho(0)\phi_i(0)\phi_j(0) + \int_{p_0}^0 a^3\phi_i'\phi_j' dp \\ &\quad + \int_{p_0}^0 g\rho'\phi_i\phi_j dp. \end{aligned} \tag{3.3.25}$$

In particular, when $i = j$ we get the useful identity:

$$-n_i^2\left(\int_{p_0}^0 a\phi_i^2 dp - \sigma\phi_i(0)^2\right) = -g\rho(0)\phi_i(0)^2 + \int_{p_0}^0 a^3(\phi_i')^2 dp + \int_{p_0}^0 g\rho'\phi_i^2 dp. \tag{3.3.26}$$

Note that the quantity in parenthesis on the left-hand side need not have a sign. In some sense, this is what is responsible for the appearance of double bifurcation.

Importing some notation from [**70**], let us define the space $\mathbb{H} := L^2([p_0, 0]) \times \mathbb{C}$ and endow it with additional structure by prescribing the indefinite inner product

$$[\tilde{u}_1, \tilde{u}_2]_{\mathbb{H}} := \int_{p_0}^0 a u_1 \overline{u_2} dp - \sigma b_1 \overline{b_2}, \qquad \text{for each } \tilde{u}_i = (u_i, b_i) \in \mathbb{H},\ i = 1, 2.$$

Note that $[\cdot,\cdot]_{\mathbb{H}}$ consists of an infinite-dimensional positive definite term and a one-dimensional negative definite term. As a trivial consequence, we have that any maximal negative definite subspace of $\mathbb{H}$ (with respect to $[\cdot,\cdot]_{\mathbb{H}}$) is one-dimensional.

We also observe that, taking (3.3.25) and exchanging the roles of ϕ_i and ϕ_j, reveals

$$-n_i^2\left(\int_{p_0}^0 a\phi_i\phi_j dp - \sigma\phi_i(0)\phi_j(0)\right) = -n_j^2\left(\int_{p_0}^0 a\phi_i\phi_j dp - \sigma\phi_i(0)\phi_j(0)\right).$$

Recalling the definition of $[\cdot,\cdot]_{\mathbb{H}}$, this last identity implies immediately that,

$$[\tilde{\phi}_i, \tilde{\phi}_j]_{\mathbb{H}} = 0, \qquad \text{for } i,j = 1,2,3, i \neq j,$$

where $\tilde{\phi}_i := (\phi_i, \phi_i(0))$. In other words, the $\tilde{\phi}_i$ are mutually orthogonal with respect to this indefinite inner product.

Let $D \subset \mathbb{H}$ be the dense set $D := \{(u,b) \in \mathbb{H} : u \in H^2([p_0, 0]), u(p_0) = 0, b = u(0)\}$. In particular, we have by the boundary conditions imposed in (3.3.4) that $\tilde{\phi}_i$ is an element of D.

Now let $\lambda > \lambda_c$ be given. Then, for any $\tilde{u} \in D \setminus \{0\}$, we estimate

$$\begin{aligned}
|b|^2 = |u(0)|^2 &= \left| \int_{p_0}^0 u'(p)dp \right|^2 \\
&\leq \int_{p_0}^0 \frac{1+\dot{G}(p;\lambda)}{a(p;\lambda)^3} dp \int_{p_0}^0 \frac{a(p;\lambda)^3}{1+\dot{G}(p;\lambda)} |u'(p)|^2 dp \\
&< 4\dot{Y}(p_0) \int_{p_0}^0 a(p;\lambda)^3 u'(p)^2 dp.
\end{aligned}$$

The last inequality come from the estimate $1 + \dot{G} \geq 1/2$ and the identity (3.3.2) evaluated at $p = p_0$. Since $\lambda_* > \lambda_c$, it follows from above that

$$|b|^2 < \frac{1}{g\rho(0) + g\|\rho_p\|_{L^\infty}|p_0|} \int_{p_0}^0 a(p;\lambda)^3 u'(p)^2 dp. \tag{3.3.27}$$

Using (3.3.27) with $\tilde{u} = \tilde{\phi}_i$, we find

$$\int_{p_0}^0 a^3 (\phi_i')^2 dp - g\rho(0)\phi_i(0)^2 - g\|\rho'\|_{L^\infty}|p_0| > 0.$$

From (3.3.24) with $i = j$, however, we conclude that $\phi_i(\cdot)^2$ is a strictly increasing function attaining its maximum value at p = 0. It follows that

$$0 < -g\rho(0)\phi_i(0)^2 + \int_{p_0}^0 a^3(\phi_i')^2 dp + \int_{p_0}^0 g\rho'\phi_i^2 dp = -n_i^2 [\tilde{\phi}_i, \tilde{\phi}_i]_{\mathbb{H}}.$$

Here the last equality is simply a restatement of (3.3.26). But now we have shown that each nontrivial $\tilde{\phi}_i$ is negative definite with respect to $[\cdot,\cdot]_{\mathbb{H}}$. Since they are all mutually orthogonal, this is impossible unless they lie within the same one-dimensional subspace of $\mathbb{H}$. Thus, for $\lambda > \lambda_c$, $\mathcal{N}(\mathcal{F}_w(\lambda, 0))$ is given by the linear span of m_1.

For $\lambda \leq \lambda_c$, the quantity appearing on the right-hand side of (3.3.26) may be negative and so we must approach the analysis with greater care. Let us consider the indefinite inner product

$$[\tilde{u}_1,\tilde{u}_2]_D := \int_{p_0}^0 a^3 u_1' \overline{u_2'} dp + \int_{p_0}^0 g\rho' u_1 \overline{u_2} dp - g\rho(0) u_1(0)\overline{u_2(0)}, \qquad \text{for every } \tilde{u}_1,\tilde{u}_2 \in D.$$

We claim that every subspace of D that is maximal and negative definite with respect to $[\cdot,\cdot]_D$ must be one-dimensional. Since there is an infinite dimensional negative semi-definite term in the definition of $[\cdot,\cdot]_D$, proving this claim is not automatic.

Let $\tilde{u} \in D$ be given. Then we estimate

$$\begin{aligned}
\int_{p_0}^0 g\rho'(p) u(p)^2 dp &= \int_{p_0}^0 g\rho'(p) \left(\int_{p_0}^p u'(r) dr \right)^2 dp \\
&\geq \int_{p_0}^0 g\rho'(p)(p-p_0) \int_{p_0}^p u'(r)^2 dr dp \\
&= \int_{p_0}^0 u'(r)^2 \int_r^0 g\rho'(p)(p-p_0) dp dr \\
&\geq -\frac{1}{2} g \|\rho'\|_{L^\infty} p_0^2 \int_{p_0}^0 u'(p)^2 dp.
\end{aligned}$$

Recall, however, that by the definition of ϵ_0,

$$a^3 \geq \epsilon_0^{3/2} \geq 2g\|\rho'\|_{L^\infty} p_0^2 e^{|p_0|} > 2g\|\rho'\|_{L^\infty} p_0^2.$$

With this inequality we are more-or-less done, for combining it with the last inequality we get

$$[\tilde{u},\tilde{u}]_D \geq \frac{1}{2}\int_{p_0}^0 a^3 (u')^2 dp - g\rho(0) u(0)^2.$$

The right-hand side above clearly defines an indefinite inner product on D for which each maximal negative semi-definite subspace is of dimension one. The same must therefore be true of $(D, [\cdot,\cdot]_D)$. This proves the claim.

Now, using the notation we have developed, (3.3.25) can be rewritten:

$$-n_i^2 [\tilde{\phi}_i, \tilde{\phi}_j]_{\mathbb{H}} = [\tilde{\phi}_i, \tilde{\phi}_j]_D, \qquad i,j = 1,2,3. \tag{3.3.28}$$

Since the $\tilde{\phi}_i$ are orthogonal with respect to $[\cdot,\cdot]_{\mathbb{H}}$, (3.3.28) says they must also be orthogonal with respect to $[\cdot,\cdot]_D$. Appealing to the claim, we conclude that at most one of them is negative semi-definite with respect to $[\cdot,\cdot]_D$. Without loss of generality, we may therefore assume that $[\tilde{\phi}_2,\tilde{\phi}_2]_D,\ [\tilde{\phi}_3,\tilde{\phi}_3]_D > 0$. But then (3.3.28) implies $\tilde{\phi}_2$ and $\tilde{\phi}_3$ are each negative definite with respect to $[\cdot,\cdot]_{\mathbb{H}}$, which is impossible unless they are in the same one-dimensional subspace. For general $\lambda \neq \lambda_0$, therefore, we have proved $\mathcal{N}(\mathcal{F}_w(\lambda, 0))$ is at most two-dimensional. $\square$

REMARK. From the arguments above it is clear that, when $\lambda_* = \lambda_0$, the dimension of $\mathcal{N}(\mathcal{F}_w(\lambda_*, 0))$ is at most three. This is simply a consequence of the fact that there can be at most two nonzero eigenvalues — one whose eigenfunction is negative semi-definite with respect to $[\cdot,\cdot]_{\mathbb{H}}$ and one whose eigenfunction is positive definite. When ρ is a constant, one can leverage this observation to prove that the null space will be at most two-dimensional. Unfortunately, for general ρ, ϕ_0, the eigenfunction for $n = 0$, need not be orthogonal to the other eigenfunctions (with respect to $[\cdot,\cdot]_{\mathbb{H}}$.)

The previous lemma motivates the choice of notation in (3.3.6)–(3.3.8): Σ_1 consists of those values of σ for which simple bifurcation occurs; Σ_2 consists of those for which we have double bifurcation; and Σ_3 (if it is nonempty) contains the unique value of σ for which zero is an eigenvalue and we cannot rule out triple bifurcation. The next corollary makes this statement more precise.

COROLLARY 3.3.1. *Fix p_0, ρ and β. Denote*

$$\Sigma := \{\sigma \in \mathbb{R}^+ : (p_0, \sigma, \rho, \beta) \textit{ collectively satisfy } \text{(L-B)}\}.$$

Then

$$(\sigma_c, \infty] \subset \Sigma_1 \qquad \textit{and} \qquad \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 = \Sigma.$$

PROOF. THEOREM 3.3.3 implies that, if $\lambda_* \neq \lambda_0$, then $\mathcal{N}(\mathcal{F}_w(\lambda_*, 0))$ is either one-dimensional or two-dimensional. First consider the one-dimensional case, i.e. assume there exists $\phi \in X$ generating the null space. Since ϕ is even in q, we may write it as a cosine series:

$$\phi(q,p) = \sum_{n=0}^{\infty} M_n(p) \cos(nq).$$

Observe that, because $\mathcal{F}_w(\lambda_*, 0)\phi = 0$, we have additionally

$$\mathcal{F}_w(\lambda_*, 0)\left(M_n(p)\cos(nq)\right) = 0, \qquad \text{for all } n \geq 0.$$

The local bifurcation condition guarantees that $M_1 \not\equiv 0$. But, since $\mathcal{N}(\mathcal{F}_w(\lambda_*, 0))$ is one-dimensional, it follows that $M_n \equiv 0$, for $n \neq 1$. Hence there is a unique solution (n_1, M_1) to (3.3.5) with $\lambda = \lambda_*$, $n_1 = 1$. We infer that, if $\sigma > \sigma_c$, then $\sigma \in \Sigma_1$.

In particular, note that by LEMMA 3.3.6, when $\sigma > \sigma_c$, then the corresponding value of λ_* will lie to the right of λ_c and thus the null space is one-dimensional. In other words, $(\sigma_c, \infty] \subset \Sigma_1$.

Suppose now that $\lambda_* \neq \lambda_0$ and the null space of $\mathcal{F}_w(\lambda_*, 0)$ is two-dimensional with generators ϕ_1 and ϕ_2. Then, decomposing each of these into cosine series we find that

$$\phi_1(q,p) = M_1(p)\cos(n_1 q), \qquad \phi_2(q,p) = M_2(p)\cos(n_2 q),$$

where (n_1, M_1) and (n_2, M_2) are distinct solutions to (3.3.5) with $\lambda = \lambda_*$, $n_1 = 1$ and $n_2 \neq 1$. In fact, by LEMMA 3.3.7, we actually have $n_2 \geq 2$. By definition this means $\sigma \in \Sigma_2$.

Finally, if $\lambda_* = \lambda_0$, then in view of LEMMA 3.3.7, we have that $\sigma \in \Sigma_3$ — indeed, it must be the case that $\Sigma_3 = \{\sigma\}$. Since these possibilities are exhaustive, we conclude $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. □

LEMMA 3.3.8. (Local Fredholm Map) *For each λ, the map $\mathcal{F}_w(\lambda, 0) : X \to Y$ is Fredholm with index 0.*

PROOF. This follows trivially from combining LEMMA 3.5 of [**70**] and LEMMA 2.4.2. □

LEMMA 3.3.9. (Range) *The pair $(\mathcal{A}, \mathcal{B})$ belongs to the range of the linearized operator $\mathcal{F}_w(\lambda_*, 0)$ if and only if it satisfies the orthogonality condition:*

$$\iint_R \mathcal{A} a^3 \phi^* dq dp + \frac{1}{2}\int_T \mathcal{B} a^2 \phi^* dq = 0,$$

where ϕ^* generates the null space of $\mathcal{F}_w(\lambda_*, 0)$.

PROOF. Suppose first that $(\mathcal{A}, \mathcal{B}) \in Y$ is in the range of $\mathcal{F}_w(\lambda_*, 0)$. Then there exists some $v \in X$ such that $\mathcal{A} = \mathcal{F}_{1w}(\lambda_*, 0)v$, and $\mathcal{B} = \mathcal{F}_{2w}(\lambda_*, 0)v$. It follows that

$$\begin{aligned}
\iint_R \mathcal{A} a^3 \phi^* dq dq &= \iint_R \big((a^3 v_p)_p + a v_{qq} - a^3 g \rho_p v\big)\phi^* dq dp \\
&= \iint_R \big((a^3 \phi^*_p)_p + a\phi^*_{qq} - a^3 g \rho_p \phi^*\big) v dq dp + \int_T (a^3 v_p \phi^* - a^3 v \phi^*_p) dq \\
&= \int_T (a^3 v_p \phi^* - a^3 v \phi^*_p) dq,
\end{aligned}$$

as on B, both v and ϕ^* vanish. Now, on T we have $a^3 = \lambda^{3/2}$. Moreover, from the definition of ϕ^* we have on T that $g\rho\phi^* - \sigma\phi^*_{qq} = \lambda^{3/2}\phi^*_p$. Thus,

$$\begin{aligned}
\frac{1}{2}\int_T \mathcal{B}a^2\phi^* dq &= \int_T \big(g\rho\lambda^{-1}v - \lambda^{1/2}v_p - \sigma\lambda^{-1}v_{qq}\big)\lambda\phi^* dq \\
&= \int_T \big((g\rho\phi^* - \sigma\phi^*_{qq})v - \lambda^{3/2}\phi^* v_p\big)dq \\
&= \int_T \big(\lambda^{3/2}\phi^*_p v - \lambda^{3/2}\phi^* v_p\big)dq.
\end{aligned}$$

Combining this expression with the last gives necessity of the orthogonality condition.

Sufficiency follows in the same way as Lemma 3.6 of [**70**]. The main point of that argument is that, once we have established $\mathcal{F}_w(\lambda, 0)$ is Fredholm of index 0, then dimension counting and necessity imply sufficiency. For brevity we omit the details. □

Finally, we must ensure that the so-called transversality or crossing condition holds.

Lemma 3.3.10. (Transversality Condition) *Suppose that ϕ^* generates the null space of $\mathcal{F}(\lambda_*, 0)$. Then $\mathcal{F}_{w\lambda}(\lambda_*, 0)\phi^* \notin \mathcal{R}(\mathcal{F}_w(\lambda_*, 0))$.*

Proof. First we calculate the mixed Frécfhet derivatives of $\mathcal{F}$ at $(\lambda_*, 0)$:

$$\begin{aligned}
\mathcal{F}_{1\lambda w}(\lambda_*, 0) &= -(1+\dot{G})a^{-4}\partial_q^2 - 3(1+\dot{G})a^{-4}\beta(-p)\partial_p - 3g\dot{Y}a^{-2}\rho_p\partial_p \\
&\qquad + 3gY(1+\dot{G})a^{-4}\rho_p\partial_p + \frac{3}{2}g(1+\dot{G})a^{-5}\rho_p(1-d) \\
\mathcal{F}_{2\lambda w}(\lambda_*, 0) &= \bigg(-2g\rho\lambda^{-2} - \lambda^{-1/2}\partial_p + 2\sigma\lambda^{-2}\partial_q^2\bigg)\bigg|_T.
\end{aligned}$$

By the previous lemma, it suffices to show that the pair $(\mathcal{F}_{1\lambda w}(\lambda_*, 0)\phi^*, \mathcal{F}_{2\lambda w}(\lambda_*, 0)\phi^*)$ does not satisfy the orthogonality condition. Equivalently, if we put

$$\begin{aligned}
\Xi := \iint_R \bigg(&(1+\dot{G})a^{-1}(\phi^*)^2 - 3(1+\dot{G})a^{-1}\beta(-p)\phi^*\phi^*_p \\
&- 3g\dot{Y}a\rho_p\phi^*\phi^*_p + 3gY(1+\dot{G})a^{-1}\rho_p\phi^*\phi^*_p + \frac{3}{2}g(1+\dot{G})a^{-2}\rho_p(\phi^*)^2\bigg)dqdp \\
&+ \int_T \bigg(-g\rho a^{-2}(\phi^*)^2 - \frac{1}{2}a\phi^*\phi^*_p + \sigma a^{-2}\phi^*_{qq}\phi^*\bigg)dq,
\end{aligned}$$

our lemma will be proven if we can show $\Xi \neq 0$ (again, because $d(\phi^*) = 0$). We will demonstrate that, in fact, $\Xi < 0$. To keep our notation concise, let $\Xi = \Xi_1 + \ldots + \Xi_8$, where Ξ_i denotes the i-th term in the sum above. Note that the only additional term from the

Ξ considered in Lemma 2.3.9 is Ξ_8. By simply integrating by parts once, and noting the boundary terms vanish by periodicity, we find

$$\Xi_8 = -\sigma\lambda_*^{-1}\int_T \phi_q^2 dq \leq 0.$$

The lemma follows. □

Combining these lemmas, we now prove the main theorem for simple eigenvalues.

Proof of Theorem 3.3.1. That $\mathcal{F}$ satisfies conditions (i)-(iii) of Theorem 3.3.2 is elementary. Likewise, $\mathcal{F}_{ww}(\lambda_*, 0)$ exists and is continuous. Finally, conditions (iii)-(iv) follow from Lemma 3.3.8 and Lemma 3.3.10, respectively.

We are therefore able to conclude the existence of a C^1-curve $\mathcal{C}'_{\text{loc}}$ of non-laminar solutions to the height equation bifurcating from $\mathcal{T}$ at $(H(\cdot;\lambda_*), Q(\lambda_*))$. In a neighborhood $\mathcal{U}$ of the bifurcation point, moreover, this curve and $\mathcal{T}$ together comprise the complete solution set and each $h \in \mathcal{C}'_{\text{loc}} \cap \mathcal{U}$ can be written

$$h(q,p) = H(p;\lambda_*) + \epsilon M(p)\cos q + o(\epsilon), \qquad \text{in } X. \tag{3.3.29}$$

Since $H_p(\cdot;\lambda) > 0$, (3.3.29) implies that, possibly by shrinking $\mathcal{U}$, we may assume that $h_p > 0$ on $\mathcal{C}'_{\text{loc}}$. Then, in light of Lemma 3.2.1, there is a corresponding C^1-curve $\mathcal{C}_{\text{loc}}$ of solutions to the original problem, (3.1.1)–(3.1.8). Since $\mathcal{C}'_{\text{loc}} \cap \mathcal{T} = \{(\lambda_*, 0)\}$, we are guaranteed that $\mathcal{C}_{\text{loc}}$ contains only one laminar solution: that corresponding to $H(\cdot;\lambda_*)$. Finally, part (iii) of the Theorem statement follows from (3.3.29), possibly by further restricting to a subneighborhood of (H_*, Q_*) in $\mathcal{U}$. □

3.4. Local bifurcation from double eigenvalues

If we do not assume that $\sigma \in \Sigma_1$, then we cannot rule out the possibility that null space of $\mathcal{F}_w(\lambda_*, 0)$ is two-dimensional (or, indeed, that zero is an eigenvalue.) Were this to be the case, the Crandall-Rabinowitz arguments of the previous section break down and we are forced to revert to a more general methodology. To successfully prosecute this program requires that we impose some additional nondegeneracy conditions. Suppose that $\sigma \in \Sigma_2 \cup \Sigma_3$. We may therefore let $n_2 \neq 1$ be given so that there exists a nontrivial solution to (3.3.5) with $\lambda = \lambda_*$, $n = n_2$. We require

$$n_2 \geq 3 \tag{3.4.1}$$

and

(3.4.2)

$$0 \neq \Theta_{1111}\Theta_{2222} \neq \Theta_{1122}\Theta_{2211}, \qquad \Theta_{1111}\Psi_{22} \neq \Theta_{2211}\Psi_{11} \qquad \Theta_{2222}\Psi_{11} \neq \Theta_{1122}\Psi_{22},$$

where Θ_{iijj}, Ψ_{ii} are given by (3.4.25)–(3.4.26) and (3.4.14), respectively, for $i, j = 1, 2$. In particular, we note that (3.4.1) implies $\sigma \notin \Sigma_3$. The purpose of dictating $n_2 \neq 0$ is merely to avoid considering the zero eigenvalue case, which require some additional work. In fact, Wahlén was able to give a fairly complete analysis of the local bifurcation when this does occur, though the argument is quite subtle (cf. [**70**]). The reason for avoiding $n_2 = 2$ is less obvious, but results from certain symmetries in the underlying problem that shall become clear later (in particular, see LEMMA 3.4.2.)

Under these assumptions, the eventual product of our efforts in this section will be the proof of THEOREM 3.1.2. We shall accomplish this in several stages, according to the Lyapunov-Schmidt procedure. The techniques here are not new, though, as we shall see, there is a certain degree of degeneracy in the problem that will require some care.

Step I: Reduction to finite-dimensional problem. We begin by recalling a weighted inner-product on Y introduced in the previous section. For all pairs $(\mathcal{A}_i, \mathcal{B}_i) \in Y$, $i = 1, 2$, let

$$\begin{aligned} ((\mathcal{A}_1, \mathcal{B}_1), (\mathcal{A}_2, \mathcal{B}_2))_Y := & \iint_R a^3(\cdot; \lambda_*)\mathcal{A}_1(\cdot)\mathcal{A}_2(\cdot)\,dq\,dp \\ & + \frac{1}{2}\int_T a^2(\cdot; \lambda_*)\mathcal{B}_1(\cdot)\mathcal{B}_2(\cdot)\,dq. \end{aligned} \tag{3.4.3}$$

Denote $X_A := \mathcal{N}(\mathcal{F}_w(\lambda_*, 0))$. Then, as we have seen in THEOREM 3.3.3, X_A will be at most dimension two. Suppose that it is spanned by ϕ_1 and ϕ_2, where $\phi_i(q,p) = M_i(p)\cos n_i q$, $i = 1, 2$. Of course, the most interesting case given our previous analysis, is when $n_1 = 1$. But there is little to gain at this stage in making such a restriction.

It follows directly from LEMMA 3.3.9 that the range of $\mathcal{F}_w(\lambda_*, 0)$ is given by the orthogonal complement (with respect to $(\cdot,\cdot)_Y$) of the linear span of $\{\phi_1, \phi_2\}$. So, if we now define

$$X_B := \left\{h \in X : ((h, h|_T), (\phi_i, \phi_i|_T))_Y = 0,\ i = 1, 2\right\},$$

$$Y_A := \left\{(\mathcal{A}, \mathcal{B}) \in Y : \mathcal{A}|_T = \mathcal{B},\ (\mathcal{A}, \mathcal{B}) \in \operatorname{span}\{(\phi_1, \phi_1|_T), (\phi_2, \phi_2|_T)\}\right\},$$

$$Y_B := \left\{(\mathcal{A}, \mathcal{B}) \in Y : \mathcal{A}|_T = \mathcal{B},\ ((\mathcal{A}, \mathcal{B}), (\phi_i, \phi_i|_T))_Y = 0,\ i = 1, 2\right\},$$

then $X = X_A \oplus X_B$, $Y = Y_A \oplus Y_B$. Finally, let $A_X : X \to X_A$, $B_X : X \to X_B$ denote the projections onto X_A and X_B, respectively, and similarly for $A_Y : Y \to Y_A$, $B_Y : Y \to Y_B$.

As always in bifurcation theory, we are trying to find all solutions to the problem

$$\mathcal{F}(\lambda, w) = 0, \tag{3.4.4}$$

in a neighborhood of $(\lambda_*, 0) \in \mathbb{R} \times X$. Applying the projections above, we see that this is equivalent to solving

$$A_Y \mathcal{F}(\lambda, A_X w + B_X w) = 0, \tag{3.4.5}$$

$$B_Y \mathcal{F}(\lambda, A_X w + B_X w) = 0, \tag{3.4.6}$$

locally near $(\lambda_*, 0)$.

Consider (3.4.6). Merely by unravelling the definitions above, we have

$$A_Y \mathcal{F}_w(\lambda_*, 0) B_X \equiv 0,$$

and therefore, for any $w \in X$,

$$B_Y \mathcal{F}_w(\lambda_*, 0) w - \mathcal{F}_w(\lambda_*, 0) B_X w = B_Y \mathcal{F}_w(\lambda_*, 0)(1 - B_X) w = B_Y \mathcal{F}_w(\lambda_*, 0) A_X w = 0.$$

This proves that $\mathcal{F}_w(\lambda_*, 0)$ commutes with projections onto the infinite dimension spaces in the sense that

$$\mathcal{F}_w(\lambda_*, 0) B_X = B_Y \mathcal{F}_w(\lambda_*, 0). \tag{3.4.7}$$

Now we note that, by construction, $\mathcal{F}_w(\lambda_*, 0) B_X$ is an isomorphism from X onto $Y_B = \mathcal{R}(\mathcal{F}_w(\lambda_*, 0))$. Thus, in view of (3.4.7), the Implicit Function Theorem allows us to solve for $B_X w$ as a function of $A_X w$ and λ near $(\lambda_*, 0)$. In particular, if we write $A_X w = \xi_1 \phi_1 + \xi_2 \phi_2$, for some $\xi_1, \xi_2 \in \mathbb{R}$, then there exists a neighborhood $U \subset \mathbb{R}^2 \times \mathbb{R}$ of $(0, 0, \lambda_*)$ and a smooth $\zeta : U \to Y_B$ such that, in a sufficiently small neighorhood of $(\lambda_*, 0)$ in $\mathbb{R} \times X$, all solutions of (3.4.6) are given by $\{(\lambda, \xi_1 \phi_1 + \xi_2 \phi_2 + \zeta(\xi_1, \xi_2; \lambda)) : (\xi_1, \xi_2, \lambda) \in U\}$. Moreover, it is easy to see that $\zeta(\cdot, \cdot; \lambda) = O(|\xi|^2)$, in this neighborhood.

Taking these solutions and inserting them into (3.4.5), we infer that (3.4.4) reduces to the following finite-dimensional problem: Find all $(\xi_1, \xi_2) \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$ in a sufficiently small neighborhood U of $(0, 0, \lambda_*)$ such that

$$\mathfrak{B}(\xi_1, \xi_2; \lambda) = 0, \tag{3.4.8}$$

where the bifurcation function $\mathfrak{B} : U \to \mathbb{R}^2$ is defined by

$$\mathfrak{B}_i(\xi_1, \xi_2; \lambda) := (\phi_i, \mathcal{F}(\lambda, \xi_1\phi_1 + \xi_2\phi_2 + \zeta(\xi_1, \xi_2; \lambda)))_Y\,, \qquad i = 1, 2. \tag{3.4.9}$$

To simplify this expression, we repeatedly Taylor expand $\mathcal{F}$. In doing so, we shall let

$$\langle \mathcal{F}_{ww}(\lambda, 0)\phi_i, \phi_j\rangle \in \mathcal{L}(X, Y)$$

denote the second Fréchet derivative of $\mathcal{F}$ in the ϕ_i then ϕ_j direction, $i, j = 1, 2$. With this notation, expanding (3.4.9) around $(\lambda_*, 0)$ yields, for $i, j, k, \ell = 1, 2$,

$$\mathfrak{B}_i = (\lambda - \lambda_*)\xi_j\Psi_{ij} + \xi_j\xi_k\Phi_{ijk} + \xi_j\xi_k\xi_\ell\Theta_{ijk\ell} + O(|\lambda - \lambda_*|^2|\xi| + |\lambda - \lambda_*||\xi|^4 + |\xi|^5), \tag{3.4.10}$$

where we are using summation convention and

$$\Psi_{ij} := \frac{1}{2}\left(\phi_i,\ \mathcal{F}_{\lambda w}(\lambda_*, 0)\phi_j\right)_Y\,, \qquad i, j = 1, 2, \tag{3.4.11}$$

$$\Phi_{ijk} := \frac{1}{2}\left(\phi_i,\ \langle \mathcal{F}_{ww}(\lambda_*, 0)\phi_j, \phi_k\rangle\right)_Y\,, \qquad i, j, k = 1, 2. \tag{3.4.12}$$

$$\Theta_{ijk\ell} := \frac{1}{6}\left(\phi_i,\ \langle\langle \mathcal{F}_{www}(\lambda_*, 0)\phi_j, \phi_k\rangle\,, \phi_\ell\rangle\right)_Y\,, \qquad i, j, k, \ell = 1, 2. \tag{3.4.13}$$

Step II: Computation of coefficients. Clearly, before we can begin to characterize the solutions of (3.4.8), we must first determine Ψ_{ij}, Φ_{ijk} and $\Theta_{ijk\ell}$ more precisely. As one might expect, this will be an elementary but involved computation. What will simplify things immensely, however, is the special structure of ϕ_1 and ϕ_2. That is, we know $\phi_i(q, p) = M_i(p)\cos(n_i q)$, where M_i is the solution of (3.3.5) with $n = n_i$, for $i = 1, 2$. This will aid us in evaluating the integrals in $(\cdot, \cdot)_Y$.

Also, observe that multiplying (3.3.5) by M_i and integrating by parts one obtains, for $p \in (p_0, 0]$, $i = 1, 2$,

$$M_i M_i' = a^{-3}\int_{p_0}^{p} a^3 (M_i')^2 dr + a^{-3}\int_{p_0}^{p} (a n_i^2 + g\rho_p) M_i^2 dr.$$

As the right-hand side above is strictly positive, we surmise that M_i and M_i' do not vanish on $(p_0, 0)$ and are of the same sign. By linearity of the equation, we may simply take them both to be positive.

REMARK. In general, for a Sturm-Liouville problem one has that the only eigenfunction without nodes is the ground state. We have argued, however, that M_i, M_i' are both nonvanishing for $i = 1, 2$. This seeming contradiction is precisely due to the presence of the surface tension term on the boundary, which alters the orthogonality relation between the eigenfunctions.

LEMMA 3.4.1. (Linear Terms) *The matrix* Ψ_{ij} *defined by* (3.4.11) *is diagonal and negative definite.*

PROOF. We shall start by explicitly computing Ψ_{ij}. Using the expressions for $\mathcal{F}_{\lambda w}(\lambda_*, 0)$ found in the previous section, we see that, for $i, j = 1, 2$,

(3.4.14)

$$\begin{aligned}\Psi_{ij} = \Bigg[\int_{p_0}^{0} n_j^2 a^{-1}(1+\dot{G})M_i M_j dp - 3\int_{p_0}^{0}(1+\dot{G})a^{-1}\beta M_i M_j' dp - 3g\int_{p_0}^{0}\dot{Y}a\rho_p M_i M_j' dp \\ + 3g\int_{p_0}^{0} Y(1+\dot{G})a^{-1}\rho_p M_i M_j' dp + \frac{3}{2}g\int_{p_0}^{0}(1+\dot{G})a^{-2}\rho_p M_i M_j \\ - \frac{1}{2}\left(2g\rho\lambda^{-1}M_i M_j + \lambda^{1/2}M_i M_j' + 2\sigma\lambda^{-1}n_j^2\right)\Big|_T\Bigg]\int_0^{2\pi}\cos(n_i q)\cos(n_j q)dq.\end{aligned}$$

Thus, $\Psi_{ij} = 0$ for $i \neq j$. Arguing as in LEMMA 3.3.10, moreover, we can show that $\Psi_{ii} < 0$, $i = 1, 2$. $\square$

Let us now attempt to characterize Φ_{ijk}. Unfortunately, the conclusions we will be able to draw will be far less satisfying than for the previous result.

LEMMA 3.4.2. (Quadratic Terms) *Let* Φ_{ijk} *be defined as in* (3.4.12) *then, if* $n_2 \neq 2n_1$, *we have* $\Phi_{ijk} = 0$, *for* $i, j, k = 1, 2$. *On the other hand, if* $n_2 = 2n_1$, *then necessarily,* $\Phi_{111}, \Phi_{222}, \Phi_{221}, \Phi_{212}, \Phi_{122} = 0$. *In general,* Φ_{112} *and* Φ_{122} *(* $= \Phi_{211}$*) are given by equations* (3.4.19) *and* (3.4.18), *respectively.*

PROOF. Recycling notation slightly, for the purpose of the next computation we let ϕ and ψ be given elements of X with $d(\phi) = d(\psi) = 0$. This restriction is natural as we shall eventually be applying this to ϕ_1, ϕ_2, each of which have this property. Then,

$$
\begin{aligned}
\langle \mathcal{F}_{1ww}(\lambda, w)\phi, \psi\rangle \quad = \quad & 2w_q\psi_q\phi_{pp} + 2(H_{pp} + w_{pp})\phi_q\psi_q + 2w_q\psi_{pp}\phi_q \\
& +2(H_p + w_p)\psi_p\phi_{qq} + 2(H_p + w_p)\psi_{qq}\phi_p + 2w_{qq}\psi_p\phi_p \\
& -2w_q\psi_{pq}\phi_p - 2w_{qp}\psi_q\phi_p - 2(H_p + w_p)\psi_q\phi_{pq} - 2w_q\psi_p\phi_{pq} \\
& -2w_{pq}\psi_p\phi_q - 2(H_p + w_p)\psi_{pq}\phi_q - 3g\rho_p(H_p + w_p)^2\psi_p\phi \\
& -3g\rho_p(H_p + w_p)^2\psi\phi_p + 6(H_p + w_p)\psi_p\phi_p\beta(-p) \\
& -6g\rho_p(H_p + w_p)(H + w - d(H) - d(w))\psi_p\phi_p,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \mathcal{F}_{2ww}(\lambda, w)\phi, \psi\rangle \quad = \quad & 2\phi_q\psi_q + 2\phi_p\psi_p\left(2\sigma\kappa[w] + 2g\rho(H + w) - Q\right) \\
& +2(H_p + w_p)\psi_p\left(2\sigma\kappa'[w]\phi + 2g\rho\phi\right) \\
& +2(H_p + w_p)\phi_p\left(2\sigma\kappa'[w]\psi + 2g\rho\psi\right) \\
& +2\sigma(H_p + w_p)^2\langle\kappa''[w]\phi, \psi\rangle,
\end{aligned}
$$

where

$$
\langle\kappa''[w]\phi, \psi\rangle = \frac{3(\phi_{qq}\psi_q + \phi_q\psi_{qq})}{(1 + w_q^2)^{5/2}} - \frac{15w_q w_{qq}}{(1 + w_q^2)^{7/2}}\phi_q\psi_q.
$$

Evaluating each of these at $(\lambda_*, 0)$, we arrive at the following

$$
\begin{aligned}
\langle \mathcal{F}_{1ww}(\lambda_*, 0)\phi, \psi\rangle = {} & 2H_{pp}\phi_q\psi_q + 2H_p\psi_p\phi_{qq} + 2H_p\psi_{qq}\phi_p - 2H_p\psi_q\phi_{pq} \\
& - 2H_p\psi_{qp}\phi_q - 3g\rho_p H_p^2\psi_p\phi - 3g\rho_p H_p^2\psi\phi_p \\
& - 6g\rho_p H_p(H - d(H))\psi_p\phi_p + 6H_p\psi_p\phi_p\beta(-p)
\end{aligned}
\tag{3.4.15}
$$

$$
\begin{aligned}
\langle \mathcal{F}_{2ww}(\lambda_*, 0)\phi, \psi\rangle = {} & 2\phi_q\psi_q + 2\phi_p\psi_p(2g\rho H - Q) + 2H_p\psi_p(2g\rho\phi - 2\sigma\phi_{qq}) \\
& + 2H_p\phi_p(2g\rho\psi - 2\sigma\psi_{qq}) + 6\sigma H_p^2(\phi_{qq}\psi_q + \phi_q\psi_{qq}).
\end{aligned}
\tag{3.4.16}
$$

Note that, as the above expressions make clear, the regularity of $\mathcal{F}_{ww}$ guarantees that $\langle \mathcal{F}_{ww}(\lambda, 0)\phi, \psi\rangle = \langle \mathcal{F}_{ww}(\lambda, 0)\psi, \phi\rangle$. So, recalling the definition of Φ_{ijk} in (3.4.12), we see immediately that $\Phi_{ijk} = \Phi_{ikj}$, for $i, j, k = 1, 2$.

To keep things manageable we will evaluate the inner-product in (3.4.12) in two steps. First we compute weighted the L^2 inner-product over R:

$$
\begin{aligned}
\left(a^3\phi_i, \langle \mathcal{F}_{1ww}(\lambda_*, 0)\phi_j, \phi_k\rangle\right)_{L^2} = C^{(1)}_{ijk}\bigg[& n_j n_k \left(a^3 H_{pp} M_i, M_j M_k\right)_{L^2} \\
& - 2n_j n_k \left(a^2 M_i, (M_j M_k)'\right)_{L^2}\bigg] \\
- C^{(2)}_{ijk}\bigg[& 2n_j^2 \left(a^2 M_i, M_j M_k'\right)_{L^2} + 2n_k^2 \left(a^2 M_i, M_j' M_k\right)_{L^2} \\
& + 3g \left(a\rho_p M_i, (M_j M_k)'\right)_{L^2} - 6\left(a^2 \beta M_i, M_j' M_k'\right)_{L^2} \\
& + 6g\left(a^2 \rho_p (H - d(H)) M_i, M_j' M_k'\right)_{L^2}\bigg],
\end{aligned}
$$

where, for $i, j, k = 1, 2$, we put

$$
\begin{aligned}
C^{(1)}_{ijk} &:= \int_0^{2\pi} \cos(n_i q)\sin(n_j q)\sin(n_k q)\,dq, \\
C^{(2)}_{ijk} &:= \int_0^{2\pi} \cos(n_i q)\cos(n_j q)\cos(n_k q)\,dq.
\end{aligned}
$$

Similarly, evaluating the weighted L^2 inner-product on T yields,

$$
\begin{aligned}
\frac{1}{2}\left(a^2\phi_i, \langle \mathcal{F}_{2ww}(\lambda_*, 0)\phi_j, \phi_k\rangle\right)_{L^2(T)} = C^{(3)}_{ijk} & \left[n_j n_k \lambda_* M_i M_j M_k + 3n_j^2 n_k \sigma M_i M_j M_k\right] \\
+ C^{(2)}_{ijk}\bigg[& \lambda_*^{1/2}(2g\rho + 2\sigma n_k^2) M_i M_j' M_k \\
& + \lambda_*^{1/2}(2g\rho + 2\sigma n_j^2) M_i M_j M_k' \\
& - \lambda_*^2 M_i M_j' M_k'\bigg] \\
+ 3C^{(4)}_{ijk} & \sigma n_j n_k^2 M_i M_j M_k,
\end{aligned}
$$

where

$$
\begin{aligned}
C^{(3)}_{ijk} &:= \int_0^{2\pi} \cos(n_i q)\cos(n_j q)\sin(n_k q)\,dq, \\
C^{(4)}_{ijk} &:= \int_0^{2\pi} \cos(n_i q)\sin(n_j q)\cos(n_k q)\,dq.
\end{aligned}
$$

On both the boundary and interior, therefore, we will have massive cancellation due to the integrals of the various trigonometric terms. For instance, when $i = j = k$, all of these integrals vanish, which implies $\Phi_{111} = \Phi_{222} = 0$. More strikingly, unless $n_2 = 2n_1$, *every* term will vanish and $\Phi_{ijk} = 0$, for $i, j, k = 1, 2$. Suppose that we do have this relationship between n_2 and n_1, then, after the dust clears, we are left finally with the following:

$$\Phi_{111},\ \Phi_{222},\ \Phi_{221},\ \Phi_{212},\ \Phi_{122} = 0, \tag{3.4.17}$$

$$\begin{aligned}\frac{2}{\pi}\Phi_{211} = &-n_1^2\left(a^3 H_{pp} M_2, M_1^2\right)_{L^2} - 3g\left(a\rho_p M_2, (M_1^2)'\right)_{L^2} \\ &- 6g\left(a^2\rho_p(H - d(H))M_2, (M_1')^2\right)_{L^2} + 6\left(a^2\beta M_2, (M_1')^2\right)_{L^2} \\ &+ \left(4\lambda_*^{1/2}(2g\rho + 2\sigma n_1^2)M_2 M_1 M_1' - \lambda_* n_1^2 M_2 M_1^2 - \lambda_*^2 M_2 (M_1')^2\right)\Big|_T,\end{aligned} \tag{3.4.18}$$

and

$$\begin{aligned}\frac{2}{\pi}\Phi_{112},\ \frac{2}{\pi}\Phi_{121} = &\ n_1 n_2\left(a^3 H_{pp} M_2, M_1^2\right)_{L^2} - 2n_1^2\left(a^2 M_1^2, M_2'\right)_{L^2} - n_2^2\left(a^2 M_2, (M_1^2)'\right)_{L^2} \\ &- 3g\left(a\rho_p M_1, (M_1 M_2)'\right)_{L^2} - 3g\left(a^2\rho_p(H - d(H))M_2', (M_1^2)'\right)_{L^2} \\ &+ 3\left(a^2\beta M_2', (M_1^2)'\right)_{L^2} - 2n_1 n_2\left(a^2 M_1, (M_1 M_2)'\right)_{L^2} \\ &+ \Big(\lambda_* n_1 n_2 M_1^2 M_2 + \lambda_*^{1/2}(2g\rho + 2\sigma n_2^2)M_1 M_1' M_2 \\ &+ \lambda_*^{1/2}(2g\rho + 2\sigma n_1^2)M_1^2 M_2' - \lambda_*^2 M_1 M_1' M_2'\Big)\Big|_T.\end{aligned} \tag{3.4.19}$$

This completes the lemma. ∎

Given the high degree of degeneracy at the quadratic level, in order to characterize the complete solution set of the bifurcation equation we forced to consider the cubic terms. These we compute in the next lemma.

LEMMA 3.4.3. (Cubic Terms) *Let $\Theta_{ijk\ell}$ be given as in* (3.4.13)*. Then*

$$\Theta_{1112},\ \Theta_{1121},\ \Theta_{1211},\ \Theta_{2111},\ \Theta_{2221},\ \Theta_{2212},\ \Theta_{2122},\ \Theta_{1222} = 0.$$

The remaining coefficients are generally nonvanishing; Θ_{iiii} is given by equation (3.4.25)*, while Θ_{iijj} is given by* (3.4.26)*, for $i, j = 1, 2$ and $i \neq j$.*

PROOF. As in the previous lemma, we begin with an explicit calculation. Differentiating once more in w the expressions for $\mathcal{F}_{ww}(\lambda, w)$ found in LEMMA 3.4.2, we find, for $\phi, \psi, \theta \in X$

with $d(\phi) = d(\psi) = d(\theta) = 0$,

$$\begin{aligned}
\langle\langle \mathcal{F}_{1www}(\lambda, w)\phi, \psi\rangle, \theta\rangle &= 2\phi_{pp}\psi_q\theta_q + 2\phi_q\psi_q\theta_{pp} + 2\phi_q\psi_{pp}\theta_q + 2\phi_{qq}\psi_p\theta_p \\
&\quad +2\phi_p\psi_{qq}\theta_p + 2\phi_p\psi_p\theta_{qq} - 2\phi_{pq}\psi_p\theta_q - 2\phi_{pq}\psi_q\theta_p \\
&\quad -2\phi_p\psi_{pq}\theta_q - 2\phi_q - 2\phi_p\psi_{pq}\theta_q - 2\phi_p\psi_q\theta_{pq} - 2\phi_q\psi_p\theta_{pq} \\
&\quad -6g\rho_p(H_p + w_p)(\phi_p\psi_p\theta + \phi_p\psi\theta_p + \phi\psi_p\theta_p) \\
&\quad -6g\rho_p(H + w - d(H) - d(w))\phi_p\psi_p\theta_p + 6\beta(-p)\phi_p\psi_p\theta_p,
\end{aligned}$$

and, on the boundary,

$$\begin{aligned}
\langle\langle \mathcal{F}_{2www}(\lambda, w)\phi, \psi\rangle, \theta\rangle &= 2\phi_p\psi_p(2\sigma\kappa'[w]\theta + 2g\rho\theta) + 2\psi_p\theta_p(2\sigma\kappa'[w]\phi + 2g\rho\phi) \\
&\quad +2\phi_p\theta_p(2\sigma\kappa'[w]\psi + 2g\rho\psi) + 4\sigma(H_p + w_p)\phi_p\langle\kappa''[w]\theta, \psi\rangle \\
&\quad +4\sigma(H_p + w_p)\psi_p\langle\kappa''[w]\theta, \phi\rangle + 4\sigma(H_p + w_p)\theta_p\langle\kappa''[w]\phi, \psi\rangle \\
&\quad +2\sigma(H_p + w_p)^2\langle\langle\kappa'''[w]\phi, \psi\rangle, \theta\rangle,
\end{aligned}$$

where

$$\langle\langle\kappa'''[w]\phi, \psi\rangle, \theta\rangle = -15\frac{(w_q\phi_q\psi_q\theta_q)_q}{(1 + w_q^2)^{7/2}} + 105\frac{w_q^2 w_{qq}\phi_q\psi_q\theta_q}{(1 + w_q^2)^{9/2}}.$$

Note that this last line implies $\kappa'''[0] = 0$.

Evaluating each of these at $(\lambda_*, 0)$, we compute

$$\begin{aligned}
\langle\langle \mathcal{F}_{1www}(\lambda_*, 0)\phi, \psi\rangle, \theta\rangle = 2\phi_{pp}\psi_q\theta_q + 2\phi_q\psi_q\theta_{pp} + 2\phi_q\psi_{pp}\theta_q + 2\phi_{qq}\psi_p\theta_p \\
+ 2\phi_p\psi_{qq}\theta_p + 2\phi_p\psi_p\theta_{qq} - 2\phi_{pq}\psi_p\theta_q - 2\phi_{pq}\psi_q\theta_p \\
- 2\phi_p\psi_{pq}\theta_q - 2\phi_q - 2\phi_p\psi_{pq}\theta_q - 2\phi_p\psi_q\theta_{pq} \\
- 6g\rho_p H_p(\phi_p\psi_p\theta + \phi_p\psi\theta_p + \phi\psi_p\theta_p) \\
- 6g\rho_p(H - d(H))\phi_p\psi_p\theta_p \\
+ 6\beta(-p)\phi_p\psi_p\theta_p - 2\phi_q\psi_p\theta_{pq},
\end{aligned} \tag{3.4.20}$$

$$
\begin{aligned}
\langle\langle \mathcal{F}_{2www}(\lambda_*,0)\phi,\psi\rangle,\theta\rangle = {} & 2\phi_p\psi_p(-2\sigma\theta_{qq}+2g\rho\theta) + 2\psi_p\theta_p(-2\sigma\phi_{qq}+2g\rho\phi) \\
& + 2\phi_p\theta_p(-2\sigma\psi_{qq}+2g\rho\psi) \\
& + 12\sigma H_p\phi_p(\psi_{qq}\theta_q+\psi_q\theta_{qq}) \\
& + 12\sigma H_p\psi_p(\phi_{qq}\theta_q+\phi_q\theta_{qq}) \\
& + 12\sigma H_p\theta_p(\phi_{qq}\psi_q+\phi_q\psi_{qq}).
\end{aligned} \tag{3.4.21}
$$

We remark again that, owing to the regularity of $\mathcal{F}$ — and as (3.4.20)-(3.4.21) make explicit — we have $\Theta_{ijk\ell} = \Theta_{i\varpi(j)\varpi(k)\varpi(\ell)}$, where ϖ is any permutation on $\{1,2\}$, and $i,j,k,\ell = 1,2$.

To compute the inner product $(\cdot,\cdot)_Y$, we first calculate that, by (3.4.20),

$$
\begin{aligned}
\frac{1}{2}\left(a^3\phi_i, \langle\langle \mathcal{F}_{1www}(\lambda_*,0)\phi_j,\phi_k\rangle,\phi_\ell\rangle\right)_{L^2(R)} = {} & n_k n_\ell C^{(1)}_{ijkl}\Big[\left(a^3 M_i M_j'', M_k M_\ell\right)_{L^2} \\
& \qquad - \left(a^3 M_i M_j', (M_k M_\ell)'\right)_{L^2}\Big] \\
& + n_j n_\ell C^{(2)}_{ijk\ell}\Big[\left(a^3 M_i M_k'', M_j, M_\ell\right)_{L^2} \\
& \qquad - \left(a^3 M_i M_k', (M_j, M_\ell)'\right)_{L^2}\Big] \\
& + n_j n_k C^{(3)}_{ijk\ell}\Big[\left(a^3 M_i M_\ell'', M_j M_k\right)_{L^2} \\
& \qquad - \left(a^3 M_i M_\ell', (M_j M_k)'\right)_{L^2}\Big] \\
& - C^{(4)}_{ijk\ell}\Big[n_j^2\left(a^3 M_i M_j, M_k' M_\ell'\right)_{L^2} \\
& \qquad + n_k^2\left(a^3 M_i M_j', M_k M_\ell'\right)_{L^2} \\
& \qquad + n_\ell^2\left(a^3 M_i M_j', M_k' M_\ell\right)_{L^2} \\
& \qquad + 3g\left(\rho_p a^2 M_i M_j', (M_k M_\ell)'\right)_{L^2} \\
& \qquad + 3g\left(\rho_p a^2 M_i M_j, M_k' M_\ell'\right)_{L^2} \\
& \qquad + 3\left(a^6 H_{pp} M_i M_j', M_k' M_\ell'\right)_{L^2}\Big],
\end{aligned} \tag{3.4.22}
$$

where the constants above are defined by

$$
\begin{cases}
C^{(1)}_{ijk\ell} := \int_0^{2\pi} \cos(n_i q)\cos(n_j q)\sin(n_k q)\sin(n_\ell q)dq, \\
C^{(2)}_{ijk\ell} := \int_0^{2\pi} \cos(n_i q)\sin(n_j q)\cos(n_k q)\sin(n_\ell q)dq, \\
C^{(3)}_{ijk\ell} := \int_0^{2\pi} \cos(n_i q)\sin(n_j q)\sin(n_k q)\cos(n_\ell q)dq, \\
C^{(4)}_{ijk\ell} := \int_0^{2\pi} \cos(n_i q)\cos(n_j q)\cos(n_k q)\cos(n_\ell q)dq.
\end{cases}
\tag{3.4.23}
$$

Likewise, on the top we find

$$
\frac{1}{12}\left(a^2\phi_i, \langle\langle \mathcal{F}_{2www}(\lambda_*, 0)\phi_j, \phi_k\rangle, \phi_\ell\rangle\right)_{L^2(T)} = C^{(4)}_{ijk\ell}\left(a^2 M_i M_j' M_k' M_\ell'\right)\Big|_T .
\tag{3.4.24}
$$

Here we have used the boundary condition on T to simplify.

As we saw in the analysis of the quadratic terms, the symmetries of the underlying equation will be expressed through the quantities in (3.4.23). In particular, we can check that, for i, j, k, ℓ with an odd number of ones and twos, all of the coefficients vanish. Hence $\Theta_{ijk\ell} = 0$ in this case. If there are an even number, then by the permutation property observed above, it suffices to consider two types of indices: (i) Θ_{iiii} and (ii) Θ_{iijj}, for $i, j = 1, 2$ with $i \neq j$. A truly elementary calculation tells us

$$
C^{(1)}_{iiii} = \frac{\pi}{4},\ C^{(1)}_{iijj} = \frac{\pi}{2},\ C^{(4)}_{iiii} = \frac{3\pi}{4},\ C^{(4)}_{iijj} = \frac{\pi}{2},
$$

while $C^{(2)}_{iiii}, C^{(2)}_{iijj}, C^{(3)}_{iiii}, C^{(3)}_{iijj} = 0$, for $i, j = 1, 2$ with $i \neq j$. Using the information we obtained from (3.4.22)–(3.4.24) we find that, for such i, j,

$$
\begin{aligned}
\frac{1}{\pi}\Theta_{iiii} = \frac{1}{12} n_i^2 &\left[\left(a^3 M_i^3, M_i''\right)_{L^2} - 2\left(a^3 M_i^2, (M_i')^2\right)_{L^2}\right] \\
&- \frac{3}{4}\Big[n_i^2\left(a^3 M_i^2, (M_i')^2\right)_{L^2} + 3g\left(\rho_p a^2 M_i^2, (M_i')^2\right)_{L^2} \\
&\quad + \left(a^6 H_{pp} M_i, (M_i')^3\right)_{L^2}\Big] + \frac{3}{4}\left(a^2 M_i\left(M_i'\right)^3\right)\Big|_T
\end{aligned}
\tag{3.4.25}
$$

and

$$
\begin{aligned}
\frac{1}{\pi}\Theta_{iijj} &= \frac{1}{6}n_j^2\left[\left(a^3M_iM_i'', M_j^2\right)_{L^2} - 2\left(a^3M_iM_i', M_jM_j'\right)_{L^2}\right] \\
&\quad - \frac{1}{6}\Bigg[n_i^2\left(a^3M_i^2, (M_j')^2\right)_{L^2} + 2n_j^2\left(a^3M_iM_i', M_jM_j'\right)_{L^2} \\
&\qquad + 6g\left(\rho_p a^2M_iM_i', M_jM_j'\right)_{L^2} + 3g\left(\rho_p a^2M_i^2, (M_j')^2\right)_{L^2} \\
&\qquad\quad + 3\left(a^6H_{pp}M_iM_i', (M_j')^2\right)_{L^2}\Bigg] + \frac{1}{2}\left(a^2M_iM_i'(M_j')^2\right)\Big|_T .
\end{aligned} \tag{3.4.26}
$$

This completes the proof. □

Step III: Control of higher order terms. The next step is to use the scaling in (3.4.10) to show that all solutions (locally) must lie in a particular subregion of U. In the case of mixed solutions, this effort is complicated by the fact that Φ_{ijk} is degenerate in the ξ_1-direction. Consequently, in order to apply the standard theory we must restrict ourselves to working in sets that lie in complements of cusps surrounding the ξ_1-axis.

Lemma 3.4.4. (Scaling of Solutions) (a) *Suppose that*

$$
\Phi_{112}, \Phi_{121}, \Phi_{211} \neq 0. \tag{3.4.27}
$$

For each $C > 0$, there is a constant $K > 0$ and open subregion $V \subset U$ with $(0, 0, \lambda_*) \in V$ such that, if $\mathfrak{B}(\xi, \lambda) = 0$ for some $(\xi, \lambda) \in V \setminus \{(\xi, \lambda) : |\xi_1| < C|\xi_2|\}$, then (ξ, λ) satisfies the scaling relation $|\xi| \leq K|\lambda - \lambda_*|$.

(b) *Suppose instead that*

$$
\Phi_{112}, \Phi_{121}, \Phi_{211} = 0, \qquad 0 \neq \Theta_{1111}\Theta_{2222} \neq \Theta_{1122}\Theta_{2211}. \tag{3.4.28}
$$

Then there exists a constant $K > 0$ and and open subset V of U with $(0, 0, \lambda_*) \in V$ such that, if $\mathfrak{B}(\xi, \lambda) = 0$ for some $(\xi, \lambda) \in V$, then $|\xi| \leq K|\lambda - \lambda_*|^{1/2}$.

Proof. First consider the situation in part (a). By contradiction suppose that, for some $C > 0$, there exists a sequence $\{(\xi^{(n)}, \lambda_n)\} \subset V \setminus \{(\xi, \lambda) : |\xi_1| < C|\xi_2|\}$ such that $\lambda_n \searrow \lambda_*$, $\xi^{(n)} \to 0$ as $n \to \infty$, and

$$
\frac{|\xi^{(n)}|}{|\lambda_n - \lambda_*|} \to \infty, \qquad \text{as } n \to \infty.
$$

We may therefore write $\lambda_n - \lambda_* = \rho_n|\xi^{(n)}|$, where $\rho_n \to 0$ as $n \to \infty$. Then the equation for $\mathfrak{B}_1$ in (3.4.10) becomes, for all $n \geq 1$,

$$0 = \xi_1^{(n)} \left|\xi^{(n)}\right| \rho_n \Psi_{11} + 2\xi_1^{(n)}\xi_2^{(n)}\Phi_{112} + O\left(\left|\xi^{(n)}\right|^3\right).$$

Diving through by $|\xi^{(n)}|^2$ and taking $n \to \infty$, each term above but the second clearly vanishes. So, in order to produce a contradiction we will prove that the second term is bounded uniformly away from zero. By construction we know $|\xi_1^{(n)}| > C|\xi_2^{(n)}|$, for $n \geq 1$. Suppose that there exists a $C' > 0$ such that $C'|\xi_1^{(n)}|^2 < |\xi_2^{(n)}|^2$. Then,

$$\left|\frac{\xi_1^{(n)}\xi_2^{(n)}}{|\xi^{(n)}|^2}\Phi_{112}\right| \geq \frac{C}{1+C'}|\Phi_{112}| > 0.$$

Thus, we are we done if we can produce such a constant C'.

Suppose no such C' exists. Then, possibly by passing to a subsequence, it must be the case that $|\xi_2^{(n)}|/|\xi_1^{(n)}| \to 0$ as $n \to \infty$. By the equation for $\mathfrak{B}_2$ we have, $\forall n \geq 1$,

$$0 = \xi_1^{(n)} \left|\xi^{(n)}\right| \rho_n \Psi_{22} + (\xi_1^{(n)})^2\Phi_{211} + O\left(\left|\xi^{(n)}\right|^3\right).$$

As before, we may divide through by $|\xi^{(n)}|^2$ and take $n \to \infty$ to find that the first term and the higher order terms vanish in the limit. But,

$$\lim_{n\to\infty}\left|\frac{|\xi_1^{(n)}|^2}{|\xi^{(n)}|^2}\Phi_{211}\right| = |\Phi_{211}| > 0,$$

so we have a contradiction. This completes part (a).

To prove (b), we shall employ a similar blow-up argument. Now, however, as the coefficients $\Theta_{ijk\ell}$ are assumed to be non-degenerate, we will not be restricted to working in the complements of cusps.

By contradiction suppose that there exists a sequence $\{(\xi^{(n)}, \lambda_n)\} \subset U$ with $\lambda_n \searrow \lambda_*$, $\xi^{(n)} \to 0$, and

$$\frac{\left|\xi^{(n)}\right|}{|\lambda_n - \lambda_*|^{1/2}} \to \infty, \qquad \text{as } n \to \infty.$$

We may therefore write $\lambda_n - \lambda_* = \rho_n|\xi^{(n)}|^2$, where $\rho_n \to 0$ as $n \to \infty$. Then the bifurcation equation becomes, for $n \geq 1$, $i = 1, 2$,

$$0 = \xi_i^{(n)}\rho_n \left|\xi^{(n)}\right|^2 \Psi_{ii} + \xi_j\xi_k\xi_\ell\Theta_{ijk\ell} + O(|\xi^{(n)}|^4), \tag{3.4.29}$$

Dividing through by $|\xi^{(n)}|^3$ and taking $n \to \infty$ we see that the first term and the higher order terms vanish in the limit. On the other hand, by hypothesis we have, for $i = 1, 2$,

$$\left| \frac{\xi_j \xi_k \xi_\ell}{\left|\xi^{(n)}\right|^3} \Theta_{ijk\ell} \right| \geq \inf_{|\xi|=1} |\xi_j \xi_k \xi_\ell \Theta_{ijk\ell}| > 0.$$

Hence the third term is bounded above and strictly away from zero. Having arrived at a contradiction, we conclude the lemma holds. $\square$

Remark. Recall that, when we prove Theorem 3.1.2, we will be assuming (3.4.1)–(3.4.2) with $n_1 = 1$, which together imply (3.4.28). Hence, we expect the conclusions of part (b) to hold.

Fix $C > 0$ and define the set W as follows

$$W := \begin{cases} V \setminus \{(\xi,\lambda) \in U : |\xi_1| < C|\xi_2|\}, & \text{if (3.4.27) holds,} \\ V, & \text{if (3.4.28) holds,} \end{cases} \tag{3.4.30}$$

where V is as in Lemma 3.4.4. Let $W^\pm \subset W$ denote the sets $\{(\xi,\lambda) \in W : \pm(\lambda - \lambda_*) > 0\}$. Then, working in $W^\pm$, we are justified in letting $\epsilon = \epsilon(\xi,\lambda) > 0$, $\theta = \theta(\xi,\lambda) \in \mathbb{R}^2$ be defined by the relations

$$\xi = \pm\epsilon\theta, \qquad \epsilon = \begin{cases} |\lambda - \lambda_*|, & \text{if (3.4.27) holds,} \\ |\lambda - \lambda_*|^{1/2}, & \text{if (3.4.28) holds.} \end{cases}$$

Note that Lemma 3.4.4 guarantees $\|\theta\|_{L^\infty(V)}$ is bounded. Rewriting (3.4.10) in terms of ϵ and θ we obtain the following system of equations for (θ, ϵ):

$$0 = \widetilde{\mathfrak{B}}_i^\pm(\theta,\epsilon) := \begin{cases} \pm\theta_j \Psi_{ij} + \theta_j\theta_k \Phi_{ijk} + O(\epsilon), & \text{if (3.4.27) holds,} \\ \pm\theta_j \Psi_{ij} + \theta_j\theta_k\theta_\ell \Theta_{ijk\ell} + O(\epsilon), & \text{if (3.4.28) holds,} \end{cases} \tag{3.4.31}$$

for $i, j, k, \ell = 1, 2$.

We wish to say that solutions to (3.4.31) with $\epsilon = 0$ are, in some sense, the only solutions to (3.4.10) in the set W. To reach this conclusion requires taming the higher order terms, represented by the $O(\epsilon)$ in (3.4.31). We shall do this by proving the following lemma.

Lemma 3.4.5. (Regular Value) *Let $W \subset U$ be as in* (3.4.30). *Then zero is a regular value of $\theta \mapsto \widetilde{\mathfrak{B}}^\pm(\theta, 0)$ in W, provided that either (a)* (3.4.27) *holds, or (b)* (3.4.28) *holds and we have, additionally, that*

$$\Theta_{1111}\Psi_{22} \neq \Theta_{2211}\Psi_{11}, \qquad \Theta_{2222}\Psi_{11} \neq \Theta_{1122}\Psi_{22}. \tag{3.4.32}$$

Proof. Assume first that hypothesis (a) holds and say that, for some $\theta \neq 0$ in $W^{\pm}$, we have $\widetilde{\mathfrak{B}}^{\pm}(\theta, 0) = 0$. Then, using our knowledge of the specific forms of Φ_{ijk} and Ψ_{ij} garnered from Lemma 3.4.2–Lemma 3.4.3, we see that, from the equation for $\widetilde{\mathfrak{B}}^{\pm}_1$,

$$\pm\Psi_{11} + 2\theta_2\Phi_{112} = 0.$$

Here we have used the fact that $\theta_1 \neq 0$, since $\theta \in W$ and is nonzero. But, the Jacobian of $\theta \mapsto \widetilde{\mathfrak{B}}^{\pm}(\theta, 0)$ is easily computed to be

$$\det \begin{pmatrix} \pm\Psi_{11} + 2\theta_2\Phi_{112} & 2\theta_1\Phi_{112} \\ 2\theta_1\Phi_{211} & \pm\Psi_{22} \end{pmatrix} = \pm(\pm\Psi_{11} + 2\theta_2\Phi_{112})\Psi_{22} - 4\theta_1^2\Phi_{112}\Phi_{211}. \tag{3.4.33}$$

We have already shown that the first term on the right-hand side vanishes, and, by hypothesis, the second term is strictly nonzero. On the other hand, in the case where $\theta = 0$, (3.4.33) shows us the Jacobian is given by $\Psi_{11}\Psi_{22} > 0$. Hence zero is a regular value in W.

Next consider what happens when the hypotheses of (b) are satisfied. Then, using the information on the cubic terms obtained in Lemma 3.4.3, we see that

$$\widetilde{\mathfrak{B}}^{\pm}(\theta, 0) = \begin{pmatrix} \pm\theta_1\Psi_{11} + \theta_1\theta_2^2\Theta_{1122} + \theta_1^3\Theta_{1111} \\ \pm\theta_2\Psi_{22} + \theta_1^2\theta_2\Theta_{2211} + \theta_2^3\Theta_{2222} \end{pmatrix}.$$

Taking $\theta_1 = 0$ we have automatically that $\widetilde{\mathfrak{B}}^{\pm}_1 = 0$. To get nontrivial solutions, therefore, requires that θ_2 satisfies

$$\pm\Psi_{22} + \theta_2^2\Theta_{2222} = 0. \tag{3.4.34}$$

Note this implies that, depending on the sign of Θ_{2222}, either $\widetilde{\mathfrak{B}}^{+}(0, \theta_2, 0) = 0$ has two solutions and $\widetilde{\mathfrak{B}}^{-}(0, \theta_2, 0) = 0$ has none, or the reverse is true. In any event, evaluating the Jacobian of $\widetilde{\mathfrak{B}}^{\pm}$ at these points yields

$$\det \begin{pmatrix} \pm\Psi_{11} + \theta_2^2\Theta_{1122} & 0 \\ 0 & \pm\Psi_{22} + 3\theta_2^2\Theta_{2222} \end{pmatrix} = 2\theta_2^2\Theta_{2222}(\pm\Psi_{11} + \theta_2^2\Theta_{1122}).$$

Thus the Jacobian vanishes if and only if $\pm\Psi_{11} + \theta_2^2\Theta_{1122} = 0$. Of course, if $\Theta_{1122} = 0$, this never occurs and we are done. Otherwise, combining this with (3.4.34) leads to

$$\mp\frac{\Psi_{11}}{\Theta_{1122}} = \theta_2^2 = \mp\frac{\Psi_{22}}{\Theta_{2222}},$$

which contradicts (3.4.32). (Note that $\Theta_{1111}, \Theta_{2222} \neq 0$ by the nondegeneracy condition (3.4.28).) Thus all zeros of $\theta \mapsto \widetilde{\mathfrak{B}}^{\pm}(\theta, 0)$ in $W^{\pm}$ with $\theta_1 = 0$ are simple. Due to the symmetry of $\widetilde{\mathfrak{B}}$, we can simply reverse the roles of one and two above to get a similar result for the case when $\theta_2 = 0$, $\theta_1 \neq 0$.

Finally let us consider solutions for which neither θ_1 nor θ_2 vanish. Then simplifying we find

$$0 = \begin{pmatrix} \pm\Psi_{11} + \theta_2^2\Theta_{1122} + \theta_1^2\Theta_{1111} \\ \pm\Psi_{22} + \theta_1^2\Theta_{2211} + \theta_2^2\Theta_{2222} \end{pmatrix}, \tag{3.4.35}$$

so that the Jacobian evaluated at such points comes to

$$\begin{aligned} \det\left(\frac{\partial\widetilde{\mathfrak{B}}^{\pm}}{\partial\theta}\right)\bigg|_{(\theta,0)} &= (\pm\Psi_{11} + \theta_2^2\Theta_{1122} + 3\theta_1^2\Theta_{1111})(\pm\Psi_{22} + \theta_1^2\Theta_{2211} + 3\theta_2^1\Theta_{2222}) \\ &\quad - 4\theta_1^2\theta_2^2\Theta_{1122}\Theta_{2211} \\ &= 4\theta_1^2\theta_2^2\left(\Theta_{1111}\Theta_{2222} - \Theta_{1122}\Theta_{2211}\right). \end{aligned} \tag{3.4.36}$$

But this quantity is nonvanishing for $\theta \neq 0$ by (3.4.28). This completes the lemma. ☐

Examining the proof of Lemma 3.4.5, we can count the number of nontrivial roots of the map $\theta \mapsto \widetilde{\mathfrak{B}}^{\pm}(\theta, 0)$ in W under various assumptions. First suppose hypothesis (b) (i.e., (3.4.28) and (3.4.32) hold.) Then, by (3.4.34), taking $\theta_1 = 0$ and $\Phi_{2222} > 0$, there are no nontrivial solutions to $\widetilde{\mathfrak{B}}^{-}(0, \theta_2, 0) = 0$, and two nontrivial solutions to $\widetilde{\mathfrak{B}}^{+}(0, \theta_2, 0) = 0$. If $\Phi_{2222} < 0$, then there are no nontrivial solutions to $\widetilde{\mathfrak{B}}^{+}(0, \theta_2, 0) = 0$ and two for $\widetilde{\mathfrak{B}}^{-}(0, \theta_2, 0) = 0$. The same holds true when we take $\theta_2 = 0$ and search for nontrivial solutions. Thus there are a total of four possible nontrivial solutions where precisely one of θ_1 and θ_2 is zero.

On the other hand, (3.4.35) shows us that if neither θ_1 nor θ_2 vanish, then (θ_1^2, θ_2^2) solve the linear system

$$A\begin{pmatrix} \theta_1^2 \\ \theta_2^2 \end{pmatrix} = \mp\begin{pmatrix} \Psi_{11} \\ \Psi_{22} \end{pmatrix}, \qquad \text{where } A := \begin{pmatrix} \Theta_{1111} & \Theta_{1122} \\ \Theta_{2211} & \Theta_{2222} \end{pmatrix}.$$

Note that A is invertible by assumption (3.4.28). Therefore, one of $\widetilde{\mathfrak{B}}^{+}, \widetilde{\mathfrak{B}}^{-}$ has no such solutions with $\theta_1, \theta_2 \neq 0$, while the other has four. In total, we find that there are eight nontrivial solutions of $\widetilde{\mathfrak{B}}^{\pm}(\theta,0)=0$ in W:

$$\left(\theta_1=0,\ \theta_2=\pm|\Psi_{22}/\Theta_{2222}|^{1/2}\right), \qquad \left(\theta_1=\pm|\Psi_{11}/\Theta_{1111}|^{1/2},\ \theta_2=0\right),$$
(3.4.37)
$$\left(\theta_1=\pm|A|^{-1}|\Theta_{2222}\Psi_{11}-\Theta_{2211}\Psi_{22}|^{1/2},\ \theta_2=\pm|A|^{-1}|\Theta_{1111}\Psi_{22}-\Theta_{1122}\Psi_{11}|^{1/2}\right).$$

Suppose now that the situation in (a) occurs, that is, $\Phi_{112},\Phi_{121},\Phi_{211}\neq 0$. Then if we let $\theta_1=0$, we find that θ_2 must be zero as well (this follows both from the bifurcation equation and, at a more basic level, from the definition of W.) Similarly, there are no nontrivial solutions with $\theta_2=0$. However, if we suppose that $\theta_1,\theta_2\neq 0$, then we can solve $\widetilde{\mathfrak{B}}_1^{\pm}(\theta,\epsilon)$ for θ_2 to find

$$\theta_2=\mp\frac{\Psi_{11}}{2\Phi_{112}},$$

hence, $\widetilde{\mathfrak{B}}_2^{\pm}(\theta,0)=0$ if and only if

$$\theta_1^2=\mp\frac{\theta_2\Psi_{22}}{\Phi_{211}}=\frac{\Psi_{11}\Psi_{22}}{2\Phi_{112}\Phi_{221}}.$$

Since Ψ_{11} and Ψ_{22} are of the same sign, we see that the above equation has two solutions or zero, depending on whether or not Φ_{112} and Φ_{211} have the same sign. If $\Phi_{112}\Phi_{211}<0$, therefore, we find that there are no nontrivial solutions to the bifurcation equation, while if $\Phi_{112}\Phi_{211}>0$ then there are two nontrivial roots of $\widetilde{\mathfrak{B}}^{\pm}(\theta,0)=0$:

$$\left(\theta_1=\sqrt{\frac{\Psi_{11}\Psi_{22}}{2\Phi_{112}\Phi_{221}}},\ \theta_2=\mp\frac{\Psi_{11}}{2\Phi_{112}}\right), \qquad \left(\theta_1=-\sqrt{\frac{\Psi_{11}\Psi_{22}}{2\Phi_{112}\Phi_{221}}},\ \theta_2=\mp\frac{\Psi_{11}}{2\Phi_{112}}\right).$$

Putting these observations together with Lemma 3.4.4 and Lemma 3.4.5, we obtain the following result characterizing the solution set of the bifurcation equation under various assumptions on $\Phi_{ijk},\Theta_{ijk\ell}$.

Theorem 3.4.1. (Bifurcation structure at double eigenvalues) *Suppose that at $\lambda_*\geq -2B_{min}+\epsilon_0$ the null space of $\mathcal{F}_w(\lambda_*,0)$ is spanned by $\phi_i=M_i(p)\cos(n_iq)$, $i=1,2$, where $0<n_1<n_2$. Let Φ_{ij}, Θ_{ijk} be defined as above, for $i,j,k=1,2$.*

(a) If Φ_{ij} and $\Theta_{ijk\ell}$ satisfy (3.4.28) and (3.4.32), then there exist four C^1-curves, $\{\mathcal{C}'_{\mathrm{loc},i}\}_{i=1}^4$, of non-laminar solutions of the height equation bifurcating from $\mathcal{T}$ at λ_*. In a sufficiently small neighborhood of $(Q(\lambda_*),H_*)$ in X, moreover, these constitute all nontrivial solutions. Two of these curves consist of pure solutions — one of minimal period $2\pi/n_1$, one of minimal period $2\pi/n_2$ — and two are of mixed solutions.

(b) If instead we have that (3.4.27) *holds, then there exist a C^1-curve of minimal period $2\pi/n_2$ solutions to the height equation bifurcating from $\mathcal{T}$ at λ_*. If additionally we have $\Phi_{112}\Phi_{211} > 0$, then there are two C^1-curves of mixed-type, non-laminar solutions bifurcating from $\mathcal{T}$ at λ_*. In a sufficiently small neighborhood of the kind described in* Lemma 3.4.4*, these comprise all nontrivial solutions.*

Proof. Consider the scenario in (a). As we have seen in the preceding lemmas, this means that the bifurcation equation is equivalent to solving (3.4.31) near $(\theta = 0, \epsilon = 0)$. In light of Lemma 3.4.5, moreover, the Implicit Function theorem tells us that to each solution of $\widetilde{\mathfrak{B}}^{\pm}(\theta, 0) = 0$, there corresponds a curve of solutions to the bifurcation equation. The arguments of the preceding paragraphs tell us that there are precisely four of these, two of mixed type (i.e. with $\theta_1, \theta_2 \neq 0$) and two of pure type — one with $(\theta_1 = 0, \theta_2 \neq 0)$ and one with $(\theta_1 \neq 0, \theta_2 = 0)$. The union of these curves gives the complete solution set of the bifurcation equation in set in a sufficiently small neighborhood of $(\lambda_*, 0, 0)$.

Next let us look at the situation in (b). Fix $C > 0$ and let W be defined as before. Working within W we know that there are no pure solutions, but there are two of mixed type if and only if Φ_{112} and Φ_{211} share the same sign. If this does occur, then as we argued in the previous paragraph, Lemma 3.4.5 and the Implicit Function theorem together give the existence of three curves of mixed solutions.

However, these clearly do not give the entire solution set— Indeed, we know there must be a curve of purely $2\pi/n_2$-periodic solutions. This follows, as in [**70**], by noting that restricting the solution space X to $2\pi/n_2$-periodic solutions reduces to one the dimension of the null space of $\mathcal{F}_w(\lambda_*, 0)$. Hence the existence of such a curve follows from a routine application of Crandall-Rabinowitz. Note that this would correspond to taking $\xi_1 = 0$, and thus lies outside of W no matter the choice of C. □

We are finally ready to assemble the results of the previous steps into a proof of the main theorem. This amounts to nothing more than using the sufficient conditions derived in Lemma 3.4.2 and Lemma 3.4.3 in conjunction with Theorem 3.4.1.

Proof of Theorem 3.1.2. Recall that, by hypothesis, we have (3.4.1) and (3.4.2). Since we are taking $n_1 = 1$ and $n_2 \geq 3$, by Lemma 3.4.1 this in turn implies (3.4.28) and (3.4.32) each hold. Applying part (a) of Theorem 3.4.1, therefore, we conclude that

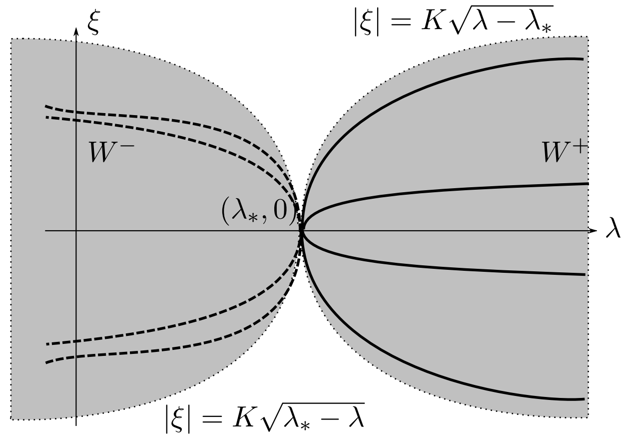

FIGURE 2. A typical bifurcation diagram. In this case we have $\Phi_{112}, \Phi_{221} = 0$, $\Theta_{1111}, \Theta_{2222} > 0$ and $\Theta_{1111}\Theta_{2222} < \Theta_{1122}\Theta_{2211}$. The two solid line pitchforks on the right consist of pure solutions, while the two dashed pitchforks on the left are mixed solution curves. The shaded regions are W^+ and W^-.

there exists four C^1-curves of solutions, $\{\mathcal{C}'_{i,\text{loc}}\}_{i=1}^4$, of the reformulated problem bifurcating from $\mathcal{T}$ at (Q_*, H_*). Likewise, by LEMMA 3.2.1, there exists four C^1-curves, $\{\mathcal{C}_{i,\text{loc}}\}_{i=1}^4$, of solutions to the original problem (3.1.1)–(3.1.7). All that remains to prove are the various nodal properties of these solutions.

Property (i) follows from the symmetries of the height equation, as we have argued before. Now, without loss of generality, we may suppose that each $(Q, h) \in \mathcal{C}'_{i,\text{loc}}$ can be written

$$h(q,p) = H(p;\lambda) + \epsilon\phi_i(p)\cos(n_i q) + o(\epsilon), \qquad \text{in X},$$

for $i = 1, 2$. In other words, we are choosing the labeling so that $\mathcal{C}'_{1,\text{loc}}$ and $\mathcal{C}'_{2,\text{loc}}$ are the curves of pure solutions, while $\mathcal{C}'_{3,\text{loc}}$ and $\mathcal{C}'_{4,\text{loc}}$ are the mixed solutions. By taking ϵ sufficiently small (or equivalently, by restricting ourselves to a sufficiently small neighborhood of the bifurcation point), we have that for $(Q, h) \in \mathcal{C}'_{i,\text{loc}}$, h has minimal period $2\pi/n_i$ in q. Moreover, since

$$h_q(q,p) = -\epsilon n_i \phi_p(p)\sin(n_i q) + o(\epsilon), \qquad \text{in } X,$$

we see that, again for $\epsilon > 0$ sufficiently small, $h_q(q,p)$ has the opposite sign of $\sin(n_i q)$ for each $(q,p) \in \overline{R}$. But, recall that $h_q = v/(c-u)$, which describes the slope of the vector field in the Eulerian formulation. It follows that, if we restrict $\mathcal{C}_{i,\text{loc}}$ to be solutions with $\epsilon > 0$

and sufficiently small, then the wave profile, indeed all of the streamlines above the bed, are monotonic between crestline and troughline. This implies properties (ii)–(iii). $\square$

REMARK. Elements of the mixed solution curves $\mathcal{C}'_{3,\mathrm{loc}}$ and $\mathcal{C}'_{4,\mathrm{loc}}$ can likewise be characterized

$$h(q,p) = H(p;\lambda) + \epsilon\left(\xi_1\phi_1(p)\cos q + \xi_2\phi_2(p)\cos\left(n_2 q\right)\right) + o(\epsilon), \qquad \text{in } X,$$

where $\xi_1, \xi_2 \in \mathbb{R}^\times$ are found from (3.4.37).

3.5. Global bifurcation from simple eigenvalues

The next step in our program is to continue the local curves of THEOREM 3.3.1 and THEOREM 3.1.2 globally. In this section we shall address the former (i.e., the simple bifurcation case), while the latter will be the subject of section 3.6.

As in previous section, we denote

$$X := \{h \in C^{3+\alpha}_{\mathrm{per}}(\overline{R}) : h = 0 \text{ on } B\}, \qquad Y = Y_1 \times Y_2 := C^{1+\alpha}_{\mathrm{per}}(\overline{R}) \times C^{1+\alpha}_{\mathrm{per}}(T),$$

where the subscript "per" indicates 2π-periodicity and evenness in q, R is the rectangle $(0, 2\pi) \times (p_0, 0)$ and $T = [0, 2\pi] \times \{p = 0\}$. In this section, we let the nonlinear operator $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : \mathbb{R} \times X \to Y$ be given by

$$\mathcal{G}_1(h) := (1 + h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_q h_p h_{pq} - g(h - d(h))\rho_p h_p^3 + h_p^3\beta(-p) \tag{3.5.1}$$

$$\mathcal{G}_2(Q, h) := \left(1 + h_q^2 + h_p^2(2\sigma\kappa[h] + 2g\rho h - Q)\right)\Big|_{p=0}. \tag{3.5.2}$$

Recall that κ denotes the mean curvature:

$$\kappa[h] := -\frac{h_{qq}}{\left(1 + h_q^2\right)^{3/2}}.$$

Note that, by construction, the laminar flow solutions $H(\cdot; Q)$ found in LEMMA 3.3.1 satisfy

$$\mathcal{G}(H(\cdot;\lambda), Q(\lambda))) = 0, \qquad \text{for all } \lambda \geq -2B_{\min} + \epsilon_0.$$

The main engine for our continuation argument will be Kielhöfer degree theory, a generalization of the classical theory of Leray-Schauder (cf. [**37**]). The basic outline for this type of argument is by now well established (see, e.g., [**15, 32, 60, 76**]). Fixing $\delta > 0$, put

$$\mathcal{O}_\delta := \left\{(Q, h) \in \mathbb{R} \times X : h_p > \delta \text{ in } \overline{R},\ Q - 2\sigma\kappa[h] - 2g\rho h > \delta\ \right\}.$$

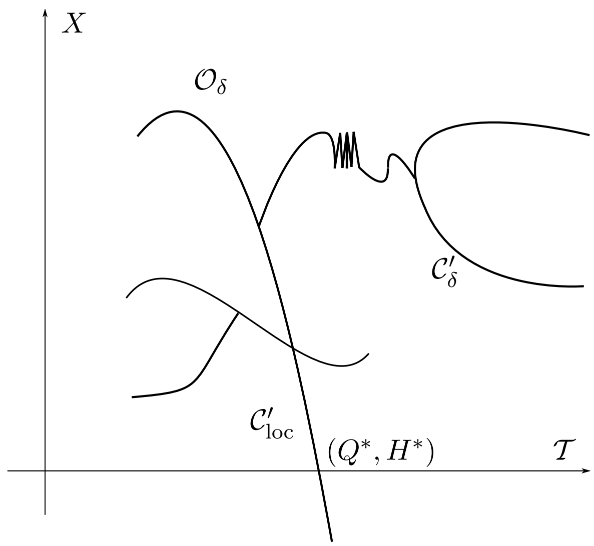

FIGURE 3. Global bifurcation diagram

Likewise, denote

$$S_\delta := \text{closure in } \mathbb{R} \times X \text{ of } \{(Q,h) \in \mathcal{O}_\delta : \mathcal{G}(Q,h) = 0, \; h_q \not\equiv 0\},$$

and let C'_δ be the component of S_δ that contains the point (Q_*, H_*), where $Q_* := Q(\lambda_*)$, $H_* := H(\cdot;\lambda_*)$ is the point where Crandall-Rabinowitz bifurcation occurs. We note that this is valid since the laminar flows all have zero curvature, hence $(Q^*, H^*) \in \cap_\delta \mathcal{O}_\delta$. Thus C'_δ contains the local curve C'_{loc}. A typical bifurcation diagram for this scenario is shown in Figure 3. Note that, the continuum $\mathcal{C}'_\delta$ is connected, but need not be path-connected in general. We shall discuss this important fact more in the next section.

The main result of this section is the following.

THEOREM 3.5.1. (Global Bifurcation from Simple Eigenvalues) *Let $\delta > 0$ be given. One of the following alternatives must hold.*

(i) $\mathcal{C}'_\delta$ is unbounded in $\mathbb{R} \times X$.

(ii) $\mathcal{C}'_\delta$ contains another trivial point $(Q,H) \in \mathcal{T}$, with $Q \neq Q^*$.

(iii) $\mathcal{C}'_\delta$ contains a point $(Q,h) \in \partial\mathcal{O}_\delta$.

The majority of the work in proving THEOREM 3.5.1 is showing the admissibility of $\mathcal{G}$ in the sense of Kielhöfer degree. This will, of course, hinge on being able to derive suitable *a priori* estimates for the linearized operators — a task that is difficult in two respects. First, the addition of the nonlocal operator d to the interior equation renders (at least initially) the usual elliptic regularity results inapplicable. The method for resolving this issue, first presented [**76**], is to "freeze" d to a constant, apply regularity theory to the resulting elliptic problem, then argue back to estimates on the unfrozen problem.

This leads us to the second difficulty: because of the higher order (tangential) derivatives on T, even the frozen elliptic problem lies outside the domain of standard Schauder theory. Though slightly more obscure, this type of boundary condition is usually referred to as *Venttsel-type*, and has received some attention in the literature (see, e.g. [**47, 48**]). In the interest of self-containedness, we present here one of the more basic linear result, as that is all that we shall require in this section. This is done by fleshing out a remark in [**41**] on the solvability of such problems in the constant coefficient case.

Lemma 3.5.1. *Fix $n \geq 2$ and let L be a second-order, linear, uniformly elliptic differential operator on $\mathbb{R}^n_+$,*

$$L := \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b_i(x)\partial_{x_i} + c(x), \qquad x \in \mathbb{R}^n_+;$$

M a homogeneous, second-order, linear, uniformly elliptic differential operator on $\partial\mathbb{R}^n_+$,

$$M := \sum_{i,j=1}^{n-1} \mathfrak{a}_{ij}(x')\partial_{x_i}\partial_{x_j}, \qquad x' \in \partial\mathbb{R}^n_+;$$

and N a first-order, linear, oblique differential operator on $\partial\mathbb{R}^n_+$,

$$N := \sum_{i=1}^{n} \mathfrak{b}_i(x')\partial_{x_i}, \qquad x' \in \partial\mathbb{R}^n_+,\ |\mathfrak{b}_n| \geq \beta > 0.$$

Suppose that

$$\|a_{ij}\|_{C^{2+\alpha}(\mathbb{R}^n_+)},\ \|b_i,\ c\|_{C^{\alpha}(\mathbb{R}^n_+)},\ \|\mathfrak{a}_{ij},\ \mathfrak{b}_i\|_{C^{\alpha}(\partial\mathbb{R}^n_+)} < K$$

and let $\kappa_L,\ \kappa_M$ denote the ellipticity constants of L and M respectively. Then there exists $C > 0$ (depending on K, β, n, κ_L and κ_M) such that, for all $u \in C^{2+\alpha}(\overline{\mathbb{R}^n_+})$,

$$\|u\|_{C^{2+\alpha}(\overline{\mathbb{R}^n_+})} \leq C\left(\|u\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|Lu\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|(M+N)u\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\right). \tag{3.5.3}$$

Proof. First note that, without loss of generality, we may take $b_i, c \equiv 0$. To see this, first suppose we had the lemma for homogeneous L. Then, for general L, we can treat the lower-order terms as a right-hand side and apply (3.5.3) to find,

$$\|u\|_{C^{2+\alpha}(\overline{\mathbb{R}^n_+})} \leq C\Big(\|u\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|Lu\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \sum_i \|b_i\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})}\|u\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})}$$
$$+ \|c\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})}\|u\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|(M+N)u\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\Big).$$

Next, interpolate the $C^{1+\alpha}$-norm on the right-hand side between C^{α} and $C^{2+\alpha}$ and absorb the latter term into the left-hand side to get

$$\begin{aligned}\|u\|_{C^{2+\alpha}(\overline{\mathbb{R}^n_+})} &\leq \frac{1}{2}\|u\|_{C^{2+\alpha}(\overline{\mathbb{R}^n_+})} + C\left(\|u\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|Lu\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|(M+N)u\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\right). \\ &\leq C\left(\|u\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|Lu\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|(M+N)u\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\right),\end{aligned}$$

where, as required, $C = C(K, \beta, n, \kappa_L, \kappa_M)$. So, indeed, it suffices to assume that L is homogeneous.

Now let us introduce a few notational conventions. Let the Poisson map, $\mathcal{P} : C^{\alpha}(\mathbb{R}^n_+) \times C^{2+\alpha}(\partial\mathbb{R}^n_+) \to C^{2+\alpha}(\mathbb{R}^n_+)$, be defined by $\mathcal{P}(f,g) := u$, where u is the unique solution of the Dirichlet problem

$$(1-L)u = f, \text{ in } \mathbb{R}^n_+, \qquad u(x',0) = g(x'), \ \forall x' \in \partial\mathbb{R}^+_n.$$

By standard Schauder theory this operator is well-defined, bounded and linear in each component. We may therefore define a Dirichlet-to-Neumann operator, $\mathcal{N}$, by

$$\mathcal{N}(f,g) := (\partial_{x_n}\mathcal{P}(f,g))\Big|_{\partial\mathbb{R}^n_+}.$$

Claim. There exists a positive constant $C = C(K, n, \kappa_L)$ such that, for any $(f,g) \in C^{\alpha}(\overline{\mathbb{R}^n_+}) \times C^{1+\alpha}(\partial\mathbb{R}^n_+)$,

$$\|\mathcal{N}(f,g)\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} \leq C\left(\|f\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|g\|_{C^{1+\alpha}(\partial\mathbb{R}^n_+)}\right).$$

Proof of Claim. Let $g \in C^{2+\alpha}(\partial\mathbb{R}^n_+) = C^{2+\alpha}(\mathbb{R}^{n-1})$ and $f \in C^{\alpha}(\overline{\mathbb{R}^n_+})$ be given. By linearity of L, $\mathcal{N}(f,g) = \mathcal{N}(f,0) + \mathcal{N}(0,g)$. The first term, however, can be easily controlled since $\mathcal{P}(f,0)$ satisfies the classical Schauder estimate for vanishing boundary data:

$$\|\mathcal{P}(f,0)\|_{C^{2+\alpha}(\overline{\mathbb{R}^n_+})} \leq C\|f\|_{C^{\alpha}(\overline{\mathbb{R}^+_n})}, \tag{3.5.4}$$

where $C = C(\|a_{ij}\|_{C^\alpha}, \kappa_L, n) > 0$. Therefore,

$$\begin{aligned}
\|\mathcal{N}(f,g)\|_{C^\alpha(\partial\mathbb{R}^n_+)} &\le \|\mathcal{N}(0,g)\|_{C^\alpha(\partial\mathbb{R}^n_+)} + \|\mathcal{N}(f,0)\|_{C^\alpha(\partial\mathbb{R}^n_+)} \\
&\le \|\mathcal{N}(0,g)\|_{C^\alpha(\partial\mathbb{R}^n_+)} + \|\mathcal{P}(f,0)\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})} \\
&\le C\left(\|\mathcal{N}(0,g)\|_{C^\alpha(\partial\mathbb{R}^n_+)} + \|f\|_{C^\alpha(\overline{\mathbb{R}^n_+})}\right).
\end{aligned}$$

Our main task is thus to estimate $\mathcal{N}(0,g)$. Denote by Δ' the Laplacian operator on $\mathbb{R}^{n-1}$. Suppose at first that the coefficients of L are constant. The advantage of this is that it implies $1-L$ and $1-\Delta'$ commute, from which we obtain the following identity

$$\mathcal{P}\left(0, \left(1-\Delta'\right)g\right) = \left(1-\Delta'\right)\mathcal{P}(0,g).$$

Hence,

$$\begin{aligned}
\left\|\mathcal{P}\left(0,\left(1-\Delta'\right)g\right)\right\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})} &\le \|1-\Delta'\|_{\mathcal{L}(C^{3+\alpha}(\overline{\mathbb{R}^n_+}),C^{1+\alpha}(\overline{\mathbb{R}^n_+}))}\|\mathcal{P}(0,g)\|_{C^{3+\alpha}(\overline{\mathbb{R}^n_+})} \\
&\le C\|g\|_{C^{3+\alpha}(\partial\mathbb{R}^n_+)} \\
&\le C\|(1-\Delta')g\|_{C^{1+\alpha}(\partial\mathbb{R}^n_+)}, \qquad (3.5.5)
\end{aligned}$$

where the constant $C = C(\|a_{ij}\|_{C^\alpha}, \kappa_L, n) > 0$.

Standard elliptic theory tells us that $1-\Delta'$ is a bounded linear operator mapping the space $C^{2+\alpha}(\mathbb{R}^{n-1})$ surjectively to $C^\alpha(\mathbb{R}^{n-1})$. Then, replacing g by $(1-\Delta')^{-1}g$ in (3.5.5), we get

$$\|\mathcal{P}(0,g)\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})} \le C\|g\|_{C^{1+\alpha}(\partial\mathbb{R}^n_+)}. \qquad (3.5.6)$$

The claim is an immediate consequence of this inequality, since

$$\|\mathcal{N}(0,g)\|_{C^\alpha(\partial\mathbb{R}^n_+)} \le \|\mathcal{P}(0,g)\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})} \le C\|g\|_{C^{1+\alpha}(\partial\mathbb{R}^n_+)}.$$

Combing this with (3.5.4) we have the desired result in the constant coefficient case.

In order to complete the proof of the claim, therefore, we need only to reprove (3.5.6) for variable coefficients. Naturally, the idea will be to estimate the commutator $[1-L, 1-\Delta']$ effectively.

Put $u := \mathcal{P}(0,(1-\Delta')g)$, $v := \mathcal{P}(0,g)$. That is, unraveling notation, take u and v to satisfy:

$$(1-L)u = 0, \text{ in } \mathbb{R}^n_+, \qquad u = (1-\Delta')g, \text{ on } \partial\mathbb{R}^n_+, \qquad (3.5.7)$$

$$(3.5.8) \qquad (1-L)v = 0, \text{ in } \mathbb{R}^n_+, \qquad v = g, \text{ on } \partial\mathbb{R}^n_+.$$

Put $w := (1-\Delta')v$. Then, from (3.5.8), we have that w satisfies

$$(3.5.9) \qquad (1-L)w = [1-L, 1-\Delta']v, \text{ in } \mathbb{R}^n_+, \qquad w = (1-\Delta')g, \text{ on } \partial\mathbb{R}^n_+.$$

Here we have let $[\cdot,\cdot]$ denote the commutator. Subtracting (3.5.7) from (3.5.9) leads to

$$(1-L)(w-u) = [1-L, 1-\Delta']v, \text{ in } \mathbb{R}^n_+, \qquad w-u = 0, \text{ on } \partial\mathbb{R}^n_+.$$

Applying the Schauder estimates to this last equation we find

$$(3.5.10) \qquad \|w-u\|_{C^{2+\alpha}(\overline{\mathbb{R}^n_+})} \le C\|[1-L, 1-\Delta']v\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})}.$$

Here $C = C(\|a_{ij}\|_{C^\alpha}, \kappa_L, n) > 0$. But note that $[1-L, 1-\Delta']$ is a third-order operator whose coefficients involves up to second-order derivatives of a_{ij}. We therefore obtain from estimate (3.5.10) that

(3.5.11)

$$\|u-(1-\Delta')v\|_{C^{2+\alpha}(\overline{\mathbb{R}^n_+})} \le C\|v\|_{C^{3+\alpha}(\overline{\mathbb{R}^n_+})} \le C\|g\|_{C^{3+\alpha}(\partial\mathbb{R}^n_+)} \le C\|(1-\Delta')g\|_{C^{1+\alpha}(\partial\mathbb{R}^n_+)},$$

where the constant $C = C(K, \kappa_L, n) > 0$.

Now, rephrasing this in the earlier notation, we find

$$\begin{aligned} \left\|\mathcal{P}(0,(1-\Delta')g)\right\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})} \le{}& \left\|(1-\Delta')\mathcal{P}(0,g)\right\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})} \\ &+ \left\|\mathcal{P}(0,(1-\Delta')g) - (1-\Delta')\mathcal{P}(0,g)\right\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})}. \end{aligned}$$

The first term on the right-hand side can be estimated as in (3.5.5), while the second term is bounded from above by (3.5.11). Equivalently, if we replace g by $(1-\Delta')^{-1}g$, it follows that

$$\|\mathcal{N}(0,g)\|_{C^{\alpha}(\partial\mathbb{R}^n_+)} \le \|\mathcal{P}(0,g)\|_{C^{1+\alpha}(\overline{\mathbb{R}^n_+})} \le C\|g\|_{C^{1+\alpha}(\partial\mathbb{R}^n_+},$$

where $C = C(K, \kappa_L, n) > 0$. This proves the claim. $\square$

Let $u \in C^{2+\alpha}(\overline{\mathbb{R}^n_+})$ be given. For notational convenience, we let h denotes the trace of u on $\partial\mathbb{R}^n_+$. The standard Schauder estimates for the Dirichlet problem are

$$(3.5.12) \qquad \|u\|_{C^{2+\alpha}(\overline{\mathbb{R}^n_+})} \le C\left(\|u\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|h\|_{C^{2+\alpha}(\partial\mathbb{R}^n_+)} + \|Lu\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})}\right),$$

where the constant $C = C(\|a_{ij}\|_{C^\alpha}, \kappa_L, n)$.

Let $(f,g) \in C^{2+\alpha}(\overline{\mathbb{R}^n_+}) \times C^{\alpha}(\partial\mathbb{R}^n_+)$ be given, and suppose

$$(1-L)u = f, \text{ in } \mathbb{R}^n_+, \qquad (M+N)u = g, \text{ on } \partial\mathbb{R}^n_+.$$

We can express the second equation above entirely as a problem on $\partial\mathbb{R}^n_+$ by means of the operator $\mathcal{N}(f,\cdot)$:

$$\left(\sum_{i,j=1}^{n-1} \mathfrak{a}_{ij}\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n-1} \mathfrak{b}_i \partial_{x_i}\right) h = -\mathfrak{b}_n \mathcal{N}(f,h) + g, \qquad \text{on } \partial\mathbb{R}^n_+. \tag{3.5.13}$$

Suppose that $n \geq 3$. Applying the Schauder estimates on $\mathbb{R}^{n-1}$ to this equation yields

$$\|h\|_{C^{2+\alpha}(\partial\mathbb{R}^n_+)} \leq C\left(\|h\|_{C^{\alpha}(\partial\mathbb{R}^n_+)} + \|\mathcal{N}(f,h)\|_{C^{\alpha}(\partial\mathbb{R}^n_+)} + \|g\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\right), \tag{3.5.14}$$

where $C = C(K, \kappa_M, \beta, n) > 0$. Applying the claim to the $\mathcal{N}$-term above and interpolating the resulting $C^{1+\alpha}$-norm between C^{α} and $C^{2+\alpha}$ we obtain

$$\begin{aligned}
\|h\|_{C^{2+\alpha}(\partial\mathbb{R}^n_+)} &\leq C\left(\|h\|_{C^{\alpha}(\partial\mathbb{R}^n_+)} + \|f\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|h\|_{C^{1+\alpha}(\partial\mathbb{R}^n_+)} + \|g\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\right) \\
&\leq \frac{1}{2}\|h\|_{C^{2+\alpha}(\partial\mathbb{R}^n_+)} + C\left(\|h\|_{C^{\alpha}(\partial\mathbb{R}^n_+)} + \|f\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|g\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\right) \\
&\leq C\left(\|h\|_{C^{\alpha}(\partial\mathbb{R}^n_+)} + \|f\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|g\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\right). \qquad (3.5.15)
\end{aligned}$$

Here $C = C(K, \kappa_L, \kappa_M, \beta, n)$.

Inserting (3.5.15) into (3.5.12) — keeping in mind that $u = h$ on $\partial\mathbb{R}^n_+$ and $\|f\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} \leq \|u\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|Lu\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})}$ — we arrive finally at the following:

$$\|u\|_{C^{2+\alpha}(\mathbb{R}^n_+)} \leq C\left(\|u\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|Lu\|_{C^{\alpha}(\overline{\mathbb{R}^n_+})} + \|(M+N)u\|_{C^{\alpha}(\partial\mathbb{R}^n_+)}\right), \tag{3.5.16}$$

where $C = C(K, \kappa_L, \kappa_M, \beta, n)$. This completes the lemma for $n > 2$.

For the case $n = 2$ we introduce an artificial variable, t, to move up a dimension. Notice that the only place where the previous argument fails is in deriving (3.5.14), as Schauder theory would require the boundary to be at least two-dimensional. Let h, f, g and $\mathcal{N}(f,h)$ be as above and let their extensions $(\bar{h}, \bar{g}, \overline{\mathcal{N}(f,h)}) \in C^{2+\alpha}(\mathbb{R}^2) \times C^{\alpha}(\mathbb{R}^2) \times C^{1+\alpha}(\mathbb{R}^2)$ be defined as follows

$$\bar{h}(t,x) := h(x), \qquad \bar{g}(t,x) := g(x), \qquad \overline{\mathcal{N}(f,h)}(t,x) := \mathcal{N}(f,h)(x), \qquad \forall (t,x) \in \mathbb{R}^2.$$

Of course, for any $k \geq 0$, the $C^{k+\alpha}(\mathbb{R}^2)$-norm of the extensions $(\bar{h}, \bar{g}, \overline{\mathcal{N}})$ coincides with the $C^{k+\alpha}(\mathbb{R})$-norm of the original original functions $(h, g, \mathcal{N})$.

Define operators $\overline{M}$, $\overline{N}$ by

$$\overline{M} := M + \partial_t^2, \qquad \overline{N} = N + \partial_t.$$

Then $(M+N)g = h$ iff $(\overline{M}+\overline{N})\bar{g} = \bar{h}$, so from (3.5.13) we obtain

$$\left(\mathfrak{a}_{11}\partial_x^2 + \partial_t^2 + \mathfrak{b}_1\partial_x + \partial_t\right)\bar{h} = -\mathfrak{b}_2\overline{\mathcal{N}(f,h)} + \bar{g}, \qquad \text{on } \mathbb{R}^2.$$

As we are now in the proper dimension, the usual elliptic estimates yield (3.5.14) with $\bar{h}$, $\bar{g}$ and $\overline{\mathcal{N}}$ in place of h, g and $\mathcal{N}$, respectively. Then, using the fact that the extensions have the same relevant norms as the original functions, we may argue as before to get (3.5.15) and thereby reestablish (3.5.16) for $n = 2$. $\square$

Remark. A standard cut-off function argument generalizes Lemma 3.5.1 to the case at hand, i.e. where we have $\Omega = R$, is bounded with periodic boundary conditions on the lateral sides and Dirichlet data on the top.

Since we now have adequate *a priori* estimates for the frozen problem, we fix any $\tau \in \mathbb{R}$ and define $\mathcal{G}^{(\tau)} : \mathbb{R} \times X \to Y$ by

$$\mathcal{G}_1^{(\tau)}(h) := (1+h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_qh_ph_{pq} - g(h-\tau)h_p^3\rho_p + h_p^3\beta(-p) \tag{3.5.17}$$

$$\mathcal{G}_2^{(\tau)}(Q,h) := \mathcal{G}_2(Q,h) = \left(1 + h_q^2 + h_p^2(2\sigma\kappa[h] + 2g\rho h - Q)\right)\bigg|_{p=0}. \tag{3.5.18}$$

We will show that, for $h \in \mathcal{O}_\delta$, $\mathcal{G}_1^{(\tau)}$ is a uniformly elliptic, quasilinear differential operator, while $\mathcal{G}_2^{(\tau)}$ is a non-degenerate, uniformly oblique Venttsel-type boundary operator. For later reference we now compute the Fréchet derivatives of $\mathcal{G}$ and $\mathcal{G}^{(\tau)}$:

$$\begin{aligned}\mathcal{G}_{1h}(h) = {} & 2h_qh_{pp}\partial_q + (1+h_q^2)\partial_p^2 + 2h_ph_{qq}\partial_p + h_p^2\partial_q^2 \\ & - 2(h_{pq}h_q\partial_p + h_{pq}h_p\partial_q + h_ph_q\partial_p\partial_q) \\ & - 3g\rho_p(h-d(h))h_p^2\partial_p - g\rho_ph_p^3(1-d) + 3h_p^2\beta(-p)\partial_p\end{aligned} \tag{3.5.19}$$

$$\mathcal{G}_{2h}(Q,h) = \left(2h_q\partial_q + 2h_p(2\sigma\kappa[h] + 2g\rho h - Q)\partial_p + 2g\rho h_p^2 + 2\sigma h_p^2\kappa'[h]\right)\bigg|_T \tag{3.5.20}$$

$$
\begin{aligned}
\mathcal{G}_{1h}^{(\tau)}(h) &= \mathcal{G}_{1h}(h) + 3g(d(h)-\sigma)h_p^3\rho_p\partial_p + g\rho_p h_p^3 d \\
&= 2h_q h_{pp}\partial_q + (1+h_q^2)\partial_p^2 + 2h_p h_{qq}\partial_p + h_p^2\partial_q^2 \\
&\quad - 2(h_{pq}h_q\partial_p + h_{pq}h_p\partial_q + h_p h_q \partial_p\partial_q) \\
&\quad - 3g\rho_p(h-\tau)h_p^2\partial_p - g\rho_p h_p^3 + 3h_p^2\beta(-p)\partial_p
\end{aligned} \tag{3.5.21}
$$

$$
\mathcal{G}_{2h}^{(\tau)}(Q,h) = \mathcal{G}_{2h}(Q,h), \tag{3.5.22}
$$

where

$$
\kappa'[h] := 3h_{qq}\left(1+h_q^2\right)^{-5/2}\partial_q - \left(1+h_q^2\right)^{-3/2}\partial_q^2.
$$

Following section 2.4, we now establish the key admissibility lemmas.

Lemma 3.5.2. (Proper Map) *Suppose K is a compact subset of Y and D is a closed, bounded set in $\overline{\mathcal{O}_\delta}$, then $\mathcal{G}^{-1}(K)\cap D$ is compact in $\mathbb{R}\times X$.*

Proof. Let $\{(f_j,g_j)\}$ be a convergent sequence in $Y = Y_1\times Y_2$ and assume that for each $j\geq 1$, $(f_j,g_j) = \mathcal{G}(Q_j,h_j)$ for some $(Q_j,h_j)\in\overline{\mathcal{O}_\delta}$ with $\{h_j\}$ bounded in $C^{3+\alpha}_{\mathrm{per}}(\overline{R})$, Q_j bounded in $\mathbb{R}$. We wish to show that there exists a subsequence of $\{(Q_j,h_j)\}$ convergent in $\mathbb{R}\times X$.

Denote $\theta_j := \partial_q h_j$, for $j\geq 1$. Then, differentiating the relation between (Q_j,h_j) and (f_j,g_j), we find by (3.5.1) that, for all $j\geq 1$,

$$
\begin{aligned}
\partial_q f_j &= \partial_q\mathcal{G}_1(h_j) \\
&= (1+(\partial_q h_j)^2)\partial_p^2\theta_j + (\partial_q(1+(\partial_q h_j)^2))\partial_p^2 h_j + (\partial_p h_j)^2\partial_q^2\theta_j + (\partial_q^2 h_j)\partial_q(\partial_p h_j)^2 \\
&\quad - 2(\partial_q h_j)(\partial_p h_j)\partial_p\partial_q\theta_j - 2(\partial_q^2 h_j)(\partial_p h_j)(\partial_p\partial_q h_j) - 2(\partial_q h_j)(\partial_p\partial_q h_j)^2 \\
&\quad - g(\partial_p h_j)^3\rho_p\theta_j - g(h_j - d(h_j))\partial_q(\partial_p h_j)^3 + \partial_q(\partial_p h_j)^3\beta(-p), \qquad \text{in } R.
\end{aligned}
$$

Grouping terms above we may rewrite this in the form

$$
\begin{aligned}
&(1+(\partial_q^2 h_j)^2)\partial_p^2\theta_j + (\partial_p h_j)^2\partial_q^2\theta_j - 2(\partial_q h_j)(\partial_p h_j)\partial_q\partial_p\theta_j = \\
&\qquad \partial_q f_j + F(\partial_q h_j, \partial_p\partial_q h_j, \partial_q^2 h_j, \partial_p h_j, \partial_p^2 h_j, d(h)), \qquad \text{in } R,
\end{aligned} \tag{3.5.23}
$$

where F is the cubic polynomial dictated by the previous expression. Notice that the coefficients of the higher order terms above satisfy:

$$
4(1+h_q^2)h_p^2 - 4h_q^2h_p^2 = 4h_p^2 > \delta^2.
$$

Hence, the right-hand side can be viewed as a uniformly elliptic operator acting on θ.

We may likewise differentiate the equation on T to discover that, for each $j \geq 1$,

$$\begin{aligned}\partial_q g_j &= \partial_q \mathcal{G}_2(Q_j, h_j)\\ &= 2(\partial_q h_j)\partial_q\theta_j + 2(\partial_p h_j)(2\sigma\kappa[h_j] + 2g\rho h_j - Q_j)\partial_p\theta_j\\ &\qquad + 2g\rho(\partial_p h_j)^2 h_q + 2\sigma(\partial_p h_j)^2(\partial_q\kappa[h_j]), \qquad \text{on } T.\end{aligned}$$

Equivalently,

(3.5.24)

$$2\sigma(\partial_p h_j)^2\partial_q\left(\frac{\partial_q\theta_j}{(1+h_q^2)^{3/2}}\right) + 2(\partial_q h_j)\partial_q\theta_j + 2(\partial_p h_j)(2\sigma\kappa[h_j] + 2g\rho h_j - Q_j)\partial_p\theta_j =$$
$$\partial_q g_j + G(\partial_q h_j, \partial_p h_j), \qquad \text{on } T,$$

where G is the quadratic polynomial determined by the previous equation. Observe that the boundary operator present on the left-hand side above consists of a one-dimensional elliptic operator added to a uniformly oblique operator, since

$$|(2\sigma\kappa[h] + 2g\rho h - Q)h_p| \geq \delta^2 \qquad \text{on } T$$

for $(Q, h) \in \overline{\mathcal{O}_\delta}$. Finally, we note that

$$\theta_j = 0 \qquad \text{on } B. \tag{3.5.25}$$

Since $\{(f_j, g_j)\}$ is convergent in $C^{1+\alpha}_{\mathrm{per}}(\overline{R}) \times C^{1+\alpha}_{\mathrm{per}}(T)$ we have that the sequences $\{\partial_q f_j\}$, $\{\partial_q g_j\}$ are Cauchy in $C^{\alpha}_{\mathrm{per}}(\overline{R})$ and $C^{\alpha}_{\mathrm{per}}(T)$ respectively. Now, by assumption, $\{h_j\}$ is uniformly bounded in $C^{3+\alpha}_{\mathrm{per}}(\overline{R})$. From this it follows that

$$\|F(\partial_q h_j, \partial_p\partial_q h_j, \partial_q^2 h_j, \partial_p h_j, \partial_p^2 h_j, d(h_j))\|_{C^{1+\alpha}_{\mathrm{per}}(\overline{R})},\ \|G(\partial_q h_j, \partial_p h_j)\|_{C^{2+\alpha}_{\mathrm{per}}(T)} < C,$$

for some $C > 0$ independent of j. Each of these, viewed as a sequence in j, is therefore Cauchy. Then, by the compactness of the embeddings, the right-hand sides of equations (3.5.23) and (3.5.24) are strongly pre-compact in $C^{\alpha}_{\mathrm{per}}(\overline{R})$ and $C^{\alpha}_{\mathrm{per}}(T)$, respectively. Possibly passing to a subsequence, we may take both to be convergent in these spaces.

We now consider differences $\theta_j - \theta_k$, for $j, k \geq 1$. By (3.5.23) we have

$$\begin{aligned}F_{jk} &= (1 + (\partial_q^2 h_j)^2)\partial_p^2(\theta_j - \theta_k) + (\partial_p h_j)^2\partial_q^2(\theta_j - \theta_k)\\ &\qquad - 2(\partial_q h_j)(\partial_p h_j)\partial_q\partial_p(\theta_j - \theta_k), \qquad \text{in } R,\end{aligned}$$

where by our arguments in the previous paragraph we know $F_{jk} \to 0$ in $C^\alpha(\overline{R})$. Similarly, from (3.5.25) have that $\theta_j - \theta_k$ vanishes on the bottom and on the top (3.5.24) tells us

$$G_{jk} = 2\sigma(\partial_p h_j)^2 \partial_q \left(\frac{\partial_q (\theta_j - \theta_k)}{(1 + h_q^2)^{3/2}} \right) + 2(\partial_q h_j)\partial_q(\theta_j - \theta_k)$$
$$+ 2(\partial_p h_j)(2\sigma\kappa[h_j] - Q_j)\partial_p(\theta_j - \theta_k), \qquad \text{on } T.$$

Here $G_{jk} \to 0$ in $C^\alpha(T)$, again by the considerations of the preceding paragraph. We now apply the mixed-boundary condition Schauder estimates of Lemma 3.5.1 to the differences $\theta_j - \theta_k$ to deduce that there exists some constant $C > 0$ with

$$C\|\theta_j - \theta_k\|_{C^{2+\alpha}(\overline{R})} \leq \|F_{jk}\|_{C^\alpha(\overline{R})} + \|G_{jk}\|_{C^\alpha(T)} + \|\theta_j - \theta_k\|_{C^\alpha(\overline{R})}.$$

Thus, $\theta_j - \theta_k \to 0$ in $C^{2+\alpha}$ as $j, k \to \infty$. We have shown, therefore, that all third derivatives of $\{h_j\}$ are Cauchy in $C^\alpha_{\text{per}}(\overline{R})$, except possibly for $\{\partial_p^3 h_j\}$. To demonstrate the same holds for $\{\partial_p^3 h_j\}$, we use the PDE to express $\partial_p^2 h_j$ in terms of the derivatives of order less than or equal to two:

$$\partial_p^2 h_j = (1 + (\partial_q h_j)^2)^{-1} \Big(f_j - \partial_q^2 h_j (\partial_p h)^2 + 2(\partial_q h)(\partial_p h_j)(\partial_p \partial_q h_j)$$
$$- g(h_j - d(h_j))\rho_p(\partial_p h_j)^3 + (\partial_p h_j)^3 \beta(-p) \Big), \qquad \text{in } R.$$

But we have seen that the right-hand side is Cauchy in $C^{1+\alpha}(\overline{R})$, hence $\{\partial_p^3 h_j\}$ is also Cauchy in $C^\alpha_{\text{per}}(\overline{R})$. We conclude that the original sequence, $\{h_j\}$, is Cauchy in $C^{3+\alpha}$. $\square$

Lemma 3.5.3. (Fredholm Map) *For each $(Q, h) \in \mathcal{O}_\delta$, the linearized operator $\mathcal{G}_h(Q, h)$ is a Fredholm map of index 0 from X to Y.*

Proof. Let $\psi \in C^{3+\alpha}_{\text{per}}(\mathbb{R})$ be given and fix $\tau,\ Q \in \mathbb{R}$, $h \in X$. Put

$$\phi^{(i)} := \partial_q \left(\mathcal{G}^{(\tau)}_{ih}(Q, h)[\psi] \right) - \left(\partial_q \mathcal{G}^{(\tau)}_{ih}(Q, h) \right)[\psi], \qquad \text{for } i = 1, 2.$$

Here, by $(\partial_q \mathcal{G}^{(\tau)}_{ih}(Q, h))[\psi]$ we mean differentiating the coefficients of $\mathcal{G}^{(\tau)}_{ih}$ in q, then applying the resulting operator to ψ. Then $\partial_q \psi$ satisfies:

$$\begin{cases} \mathcal{G}^{(\tau)}_{1h}(h)\partial_q \psi = \phi^{(1)} & \text{on } R, \\ \\ \mathcal{G}^{(\tau)}_{2h}(Q, h)\partial_q \psi = \phi^{(2)} & \text{on } T, \\ \\ \partial_q \psi = 0 & \text{on } B, \end{cases}$$

which is a uniformly elliptic PDE with a Venttsel boundary condition. Thus there exists a constant $C > 0$, independent of ψ, such that

$$\begin{aligned} C\|\partial_q\psi\|_{C^{2+\alpha}(\overline{R})} &\le \|\partial_q\psi\|_{C^\alpha(\overline{R})} + \|\phi^{(1)}\|_{C^\alpha(\overline{R})} + \|\phi^{(2)}\|_{C^\alpha(T)} \\ &\le \|\psi\|_{C^{2+\alpha}(\overline{R})} + \|\partial_q(\mathcal{G}_{1h}^{(\tau)}(h)\psi)\|_{C^\alpha(\overline{R})} + \|\partial_q(\mathcal{G}_{2h}^{(\tau)}(Q,h)\psi)\|_{C^\alpha(T)}. \end{aligned}$$

On the other hand, we may express $\partial_p^2\psi$ via the partial differential equation to arrive at an estimate for $\partial_p^3\psi$ of the same type. Combining these estimates we find that, for some $C > 0$ and all $\psi \in X$,

$$C\|\psi\|_{C^{3+\alpha}(\overline{R})} \le \|\psi\|_{C^{2+\alpha}(\overline{R})} + \|\partial_q\phi^{(1)}\|_{C^\alpha(\overline{R})} + \|\partial_q\phi^{(2)}\|_{C^\alpha(T)}. \tag{3.5.26}$$

If we now apply Lemma 3.5.1 to ψ directly, we can estimate the $C^{2+\alpha}$–norm of ψ. Then (3.5.26) becomes

$$C\|\psi\|_X \le \|\psi\|_{Y_1} + \|\mathcal{G}_{1h}^{(\tau)}(h)\psi\|_{Y_1} + \|\mathcal{G}_{2h}^{(\tau)}(Q,h)\psi\|_{Y_2}. \tag{3.5.27}$$

This is the key estimate, from which we can conclude the range of $\mathcal{G}_{ih}^{(\tau)}$ is closed and the null space finite dimensional. The task at hand, therefore, is to prove it with $\mathcal{G}$ in place of $\mathcal{G}^{(\tau)}$. As in section 2.4, this is made possible by the following trivial but convenient fact: for any ψ as above we have

$$|d(\psi)| \le \fint_T |\psi| dq \le \|\psi\|_{C^\alpha(\overline{R})}.$$

We are therefore able to estimate terms involving d easily in spaces of Hölder continuous functions. In particular,

$$\begin{aligned} \|\mathcal{G}_{1h}(h)\psi - \mathcal{G}_{1h}^{(\tau)}(h)\psi\|_{C^\alpha(\overline{R})} &= \|3g(d(h)-\tau)h_p^3\rho_p\psi_p + g\rho_p h_p^3 d(\psi)\|_{C^\alpha(\overline{R})} \\ &\le C\|\psi\|_{C^{1+\alpha}(\overline{R})}, \end{aligned} \tag{3.5.28}$$

where the constant C above depends only on $\rho_p, \|h\|_X$ and our choice of σ. Likewise, an identical argument gives the estimates

$$\|\partial_i\mathcal{G}_{1h}(h)\psi - \partial_i\mathcal{G}_{1h}^{(\tau)}(h)\psi\|_{C^\alpha(\overline{R})} \le C\|\psi\|_{C^{2+\alpha}(\overline{R})}, \qquad i = p,\ q$$

where C depends again on $\rho_p, \|h\|_X$ and the choice of τ. Of course we do not need to make such arguments to estimate the $\mathcal{G}_{2h}^{(\tau)}(h)$ term, as it identical to $\mathcal{G}_{2h}(h)$.

Combining these observations with (3.5.27), we find that, for some $C > 0$ and all $\psi \in X$,

$$C\|\psi\|_X \leq \|\psi\|_{C^{2+\alpha}(\overline{R})} + \|\mathcal{G}_{1h}(h)\psi\|_{Y_1} + \|\mathcal{G}_{2h}(Q,h)\psi\|_{Y_2}. \tag{3.5.29}$$

Applying the arguments of LEMMA 2.4.2, we are able to conclude that $\mathcal{G}_h(Q,h)$ is Fredholm. The Fredholm index has a discrete range, hence by the connectedness of $\mathcal{O}_\delta$ it must be constant on this set. But $\mathcal{G}_h(Q_*,H_*)$ was shown to have a one-dimensional null space and range of codimension one. Since $(Q_*,H_*) \in \mathcal{O}_\delta$ by construction, the Fredholm index is uniformly 0 along the continuum $\mathcal{C}'_\delta$. □

LEMMA 3.5.4. (Spectral Properties) (i) $\forall \delta, \epsilon > 0$, $\exists c_1, c_2 > 0$ *such that for all* $(Q,h) \in \mathcal{O}_\delta$ *with* $|Q|+\|h\|_X \leq M$, *for all* $\psi \in X$ *and for all* $\mu \in \mathbb{C}$ *with* $|\mu| \geq c_2$ *and* $|\arg\, \mu| < \pi/2-\epsilon$, *we have*

$$c_1\|\psi\|_X \leq |\mu|^{\alpha/2}\|(A-\mu)\psi\|_{Y_1} + |\mu|^{\alpha/2}\|(B-\mu)\psi\|_{Y_2}$$

where $A = A(Q,h) = \mathcal{G}_{1h}(h)$ *and* $B = B(Q,h) = \mathcal{G}_{2h}(Q,h)$.

(ii) *Let* $\Sigma = \Sigma(Q,h)$ *denote the spectrum of* (A,B). *Then* Σ *consists entirely of the eigenvalues of finite multiplicity with no finite accumulation points. Furthermore, there is a neighborhood* $\mathcal{N}$ *of* $[0,+\infty)$ *in the complex plane such that* $\Sigma(\lambda,w) \cap \mathcal{N}$ *is a finite set.*

PROOF. To prove (i) we follow the argument of Agmon (cf. [**1**]). The Venttsel boundary conditions, however, necessitate some modification of the standard argument. All that we must do here is trivially modify the technical tricks of that argument. Part (ii) follows immediately from part (i) in light of the previous lemmas, so we omit the details.

Fix $\tau \in \mathbb{R}$ with $|\tau| < M$, and let $\psi \in X$, $\mu \in \mathbb{C}$ be given. Put $\theta := \arg\mu$, and suppose for some $\epsilon > 0$, $|\theta| < \pi/2+\epsilon$. Consider the operators

$$A^{(\tau)} := \mathcal{G}_{1h}^{(\tau)}(h),\ D_1^{(\tau)} := A^{(\tau)} + e^{i\theta}\partial_t^2, \qquad \text{on } \mathbb{R}\times R,$$

and

$$D_2^{(\tau)} = D_2 := B + e^{i\theta}\partial_t^2, \qquad \text{on } \mathbb{R}\times T.$$

Then, for any τ, $D_1^{(\tau)}$ is elliptic with constant of ellipticity independent of τ, while $D_2^{(\tau)}$ is of the form of LEMMA 3.5.1. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a cutoff function supported compactly in the interval $I := (-1,1)$. Put

$$e(t) := e^{i|\mu|^{1/2}t}\zeta(t), \qquad \phi(t,q,p) := e(t)\psi(q,p).$$

We apply the Schauder estimates of LEMMA 3.5.1 in $\mathbb{R}^3$ with the boundary operator $D_2^{(\tau)}$ to $\phi(t,q,p)$ to deduce

$$C\|\phi\|_{C^{2+\alpha}(I\times R)} \leq \|\phi\|_{C^{\alpha}(I\times R)} + \|D_1^{(\tau)}\phi\|_{C^{\alpha}(I\times R)} + \|D_2^{(\tau)}\phi\|_{C^{\alpha}(I\times T)} \tag{3.5.30}$$

for some constant $C > 0$ independent of ψ. In fact, since τ occurs as the coefficient of a first-order term in $\mathcal{G}_{1h}^{(\tau)}(Q,h)$, we have by LEMMA 3.5.1 that C can be chosen independently of τ (recall, $|\tau| < M$.)

A quick calculation readily confirms the existence of $C,\ C' > 0$, depending only on ζ, with

$$C\mu^{\alpha/2} \leq \|e\|_{C^{\alpha}(\mathbb{R})} \leq C'\mu^{\alpha/2}, \qquad C\mu^{(1+\alpha)/2} \leq \|e\|_{C^{1+\alpha}(\mathbb{R})} \leq C'\mu^{(1+\alpha)/2}. \tag{3.5.31}$$

Using this, we can unpack (3.5.30) to derive the following set of estimates:

$$\begin{aligned}
\|D_1^{(\tau)}\phi\|_{C^{\alpha}(I\times R)} &= \left\| e(t)\mathcal{G}_{1h}^{(\tau)}(h)[\psi] - \mu e(t)\psi + \left(i|\mu|^{1/2}\zeta' + \zeta''\right) e^{i\theta} e^{i|\mu|^{1/2}t}\psi \right\|_{C^{\alpha}(I\times R)} \\
&\leq C'|\mu|^{\alpha/2}\left(\|\mathcal{G}_{1h}(h) - \mu)\psi\|_{C^{\alpha}(R)} + \|(\mathcal{G}_{1h}(h) - \mathcal{G}_{1h}^{(\tau)}(h))\psi\|_{C^{\alpha}(R)}\right) \\
&\qquad + \|\left(i|\mu|^{1/2}\zeta' + \zeta''\right) e^{i\theta} e^{i|\mu|^{1/2}t}\psi\|_{C^{\alpha}(I\times R)} \\
&\leq C'|\mu|^{\alpha/2}\left(\|(\mathcal{G}_{1h}(h) - \mu)\,\psi\|_{C^{\alpha}(R)} + \|\psi\|_{C^{\alpha}(R)}\right) && (3.5.32) \\
&\qquad + C'\left(|\mu|^{(1+\alpha)/2} + |\mu|^{\alpha/2}\right)\|\psi\|_{C^{\alpha}(R)} && (3.5.33)
\end{aligned}$$

for $|\mu|$ sufficiently large. In transitioning from (3.5.32) to (3.5.33), we have made use of (3.5.28) taking $\tau = d(h)$ (which is valid since $d(h) < \|h\|_X < M$ and (3.5.30) holds for any τ.) A similar analysis of the boundary terms yields,

$$\begin{aligned}
\|D_2^{(\tau)}\phi\|_{C^{\alpha}(I\times T)} &= \left\| e(t)\mathcal{G}_{2h}(Q,h)[\psi] - \mu e(t)\psi + \left(|\mu|^{1/2}\zeta' + \zeta''\right) e^{i\theta} e^{i|\mu|^{1/2}t}\psi \right\|_{C^{\alpha}(I\times T)} \\
&\leq C'|\mu|^{\alpha/2}\|\left(\mathcal{G}_{2h}(Q,h) - \mu\right)\psi\|_{C^{\alpha}(T)} + C'\left(|\mu|^{(1+\alpha)/2} + |\mu|^{\alpha/2}\right)\|\psi\|_{C^{\alpha}(T)}, && (3.5.34)
\end{aligned}$$

for $|\mu|$ is sufficiently large.

On the other hand,

$$
\begin{aligned}
[\phi]_{2,\alpha;R\times I} &= [\partial_t^2 e(t)\psi]_{0,\alpha;R\times I} + [\partial_t e(t)\partial_q\psi]_{0,\alpha;R\times I} + [\partial_t e(t)\partial_p\psi]_{0,\alpha;R\times I} \\
&\qquad + [e(t)\partial_q^2\psi]_{0,\alpha;R\times I} + [e(t)\partial_q\partial_p\psi]_{0,\alpha;R\times I} + [e(t)\partial_p^2\psi]_{0,\alpha;R\times I} \\
&\geq C''|\mu|[\psi]_{0,\alpha;R} + [\psi]_{2,\alpha;R},
\end{aligned}
$$

$$
\|\phi\|_{C^2(I\times R)} \geq C''|\mu|\|\psi\|_{C^0(\overline{R})} + \|\psi\|_{C^2(\overline{R})}, \tag{3.5.35}
$$

for $|\mu|$ sufficiently large, and for some $C'' > 0$ independent of μ and ψ. Here, $[\cdot]_{k,\alpha;I\times R}$ denotes the usual Hölder semi-norm. Combining (3.5.30), (3.5.33), (3.5.34), and (3.5.35), we find derive the following: there exists $c_1, c_2 > 0$ such that,

$$
c_1\|\psi\|_{C^{2+\alpha}(\overline{R})} \leq |\mu|^{\alpha/2}\|\,(A-\mu)\psi\|_{C^\alpha(\overline{R})} + |\mu|^{\alpha/2}\|\,(B-\mu)\psi\|_{C^\alpha(\overline{R})}\,,
$$

for all $\mu \in \mathbb{C}$ with $|\mu| > c_2$. If we now repeat the same argument with ψ_q and ψ_p in place of ψ, we arrive at the inequality in (i). ☐

In order to complete the proof of Theorem 3.5.1 we will make use of a generalization of Leray–Schauder degree due to Kielöfer (cf. [**37, 38**]). With that in mind, we recall the following definitions.

Definition 3.5.1. (See Definition II.5.2 of [**38**].) Let W, Z be real Banach spaces with W continuously embedded in Z and let $\mathcal{W}$ be an open, bounded subset of W. We say that a map $G \in C^2(\overline{\mathcal{W}}; Y)$ is *admissible* provided that it satisfies the following:

(i) G is a proper map.

(ii) For all $w \in \mathcal{W}$, the linearized operator $G_w(w)$ is Fredholm of index 0.

(iii) There exists $\epsilon \in (0, \pi/2)$, $\alpha \in (0,1)$, and constants $c_1, c_2 > 0$ such that, for all $\psi \in X$, $w \in \mathcal{W}$, and $\mu \in \mathbb{C}$ with $|\mu| > c_2$, $|\arg\,\mu| < \pi/2 - \epsilon$, we have

$$
c_1\|\psi\|_W \leq |\mu|^{\alpha/2}\|(G_w(w)-\mu)\psi\|_Z,
$$

where we have complexified W and Z.

(iv) For each $w \in \mathcal{W}$, the spectrum Σ of $G_w(w)$ consists entirely of eigenvalues of finite multiplicity with no finite accumulation points. Furthermore, there is a neighborhood $\mathcal{N}$ of $[0,+\infty)$ in the complex plane such that $\Sigma(\lambda, w) \cap \mathcal{N}$ is a finite set.

The significance of admissible maps is that they permit us to define a degree. Suppose that G is such a map and let $z \in Z \setminus G(\partial\mathcal{W})$ be a regular value of G. We define the *(Kielöfer) degree* of G at z with respect to $\mathcal{W}$ to be the quantity

$$\deg\,(G, \mathcal{W}, z) := \sum_{w \in \mathcal{G}^{-1}(\{z\})} (-1)^{\nu(w)},$$

where $\nu(w)$ is the number of positive real eigenvalues counted by multiplicity of $G_w(w)$. By part (iii) of Definition 3.5.1, we have that $\nu(w)$ is finite. Likewise, part (i) ensures $G^{-1}(\{z\}) \cap \mathcal{W}$ is a finite set. The degree is therefore well-defined at proper values. In the usual way, this definition is extended to critical values via the Sard-Smale theorem (this is valid by since G is Fredholm.)

Our interest in degree theory stems from the fact that the degree is invariant under any homotopy that respects the boundaries of $\mathcal{W}$ in the following sense. Let $\mathcal{U} \subset [0,1] \times \mathcal{W}$ be open. Define $\mathcal{U}_t := \{w \in \mathcal{W} : (t, w) \in \mathcal{U}\}$ and likewise $\partial\mathcal{U}_t := \{w \in \mathcal{W} : (t, w) \in \partial\mathcal{U}\}$.

Lemma 3.5.5. (Homotopy Invariance) *The degree is invariant under admissible homotopies. That is, suppose $\mathcal{U} \subset [0,1] \times \mathcal{W}$ is open. If $\mathcal{H} \in C^2(\overline{\mathcal{U}}; Y)$ is proper and, for each $t \in [0,1]$, $\mathcal{H}(t, \cdot)$ is admissible, then we say $\mathcal{H}$ is an admissible homotopy and have*

$$\deg\,(\mathcal{H}(0,\cdot), \mathcal{U}_0, y) = \deg\,(\mathcal{H}(1,\cdot), \mathcal{U}_1, y)$$

provided $y \notin \mathcal{H}(t, \partial\mathcal{U}_t)$ for all $t \in [0,1]$.

The reader is directed to [**38**] for a thorough treatment of degree theory and a proof of this result.

We are now prepared to prove the main theorem of this section.

Proof of Theorem 3.5.1. From part (iii) of Definition 3.5.1, we see that in order to apply the Kielhöfer degree theory, we must first make a technical change. For $h \in C^{3+\alpha}_{\mathrm{per}}(\overline{R})$, put

$$\widetilde{h} := (h, h|_T) \in C^{3+\alpha}_{\mathrm{per}}(\overline{R}) \times C^{3+\alpha}_{\mathrm{per}}(T),$$

and

$$\widetilde{X} := \left\{(h, h|_T) : h \in C^{3+\alpha}_{\mathrm{per}}(\overline{R})\right\} \subset C^{3+\alpha}_{\mathrm{per}}(\overline{R}) \times C^{3+\alpha}_{\mathrm{per}}(T).$$

If we let π be the bijective projection $\pi : X \to \widetilde{X}$, $h \mapsto_\pi (h, h|_T)$, then we can likewise define

$$\widetilde{\mathcal{G}}(Q, \widetilde{h}) := \mathcal{G}(Q, \cdot) \circ \pi,$$

and

$$\widetilde{\mathcal{T}} := \pi(\mathcal{T}), \qquad \widetilde{\mathcal{O}}_\delta := \pi(\mathcal{O}_\delta), \qquad \widetilde{S}_\delta := \pi(S_\delta), \qquad \widetilde{\mathcal{C}}'_\delta = \pi(\mathcal{C}'_\delta).$$

Let $\mathcal{W}$ be an open bounded subset of X and fix $Q \in \mathbb{R}$. By virtue of LEMMA 3.5.2–LEMMA 3.5.4, we conclude that the map $\widetilde{\mathcal{G}}(Q, \cdot) : \overline{\pi(\mathcal{W})} \to Y$ is admissible in the sense of DEFINITION 3.5.1. This is a trivial consequence of the fact that π is a smooth bijection from X onto $\widetilde{X}$.

From our previous analysis it follows that $\widetilde{\mathcal{G}}(Q_1, \cdot) \mapsto \widetilde{\mathcal{G}}(Q_2, \cdot)$ is an admissible homotopy. Thus, in light of LEMMA 3.5.5, the degree will remain constant as we move along the continuum $\widetilde{\mathcal{C}}'_\delta$. Assuming that all three alternatives of THEOREM 3.5.1 fail, we use this feature to generate a contradiction. At this stage, we have reestablished all the relevant properties of $\widetilde{\mathcal{G}}$ that were true in the constant density case. Repeating (verbatim) the proof in [**15**] — with $\widetilde{\mathcal{G}}$ in place of $\mathcal{G}$ — we obtain the theorem. $\square$

REMARK. The motivation for choosing the Kielhöfer degree (rather than that of Healey and Simpson used, for example, in [**15**]) lies in LEMMA 3.5.4: due to the presence of higher-order derivatives, the boundary operator and interior operator are equivalent, in the sense of the Agmon-type spectral estimates of LEMMA 3.5.4 (i). This observation was first made by Wahlén in his analysis of the pure capillary wave case, where he also introduced the clever trick of considering $\widetilde{h}$, $\widetilde{X}$, etc. (cf. [**73**]).

3.6. Analytic global bifurcation

In the previous section we obtained global bifurcation results for the case when the eigenvalue at (H_*, Q_*) is simple. The basis for that method of argument, which harkens back to Rabinowitz [**60**], is the topological theory of degree. In the present section, we present an alternative theory of bifurcation that instead relies on the (real) analytic properties of the nonlinear operator $\mathcal{G}$.

This approach is originally due to Dancer (cf. [**20, 22**], or [**11**] for a survey) and has recently been applied with great success to the study of Stokes waves (cf. [**8, 9, 10**]). The major insight of Dancer's theory is that the germ of the zero-set of an analytic operator has an extremely special structure. Within the context of global continuation this means, for example, that if $\mathcal{C}$ is an analytic curve that corresponds to the zero-set of an analytic function, then there is a unique, maximal, analytic extension of $\mathcal{C}$. Lifting this fact to infinite dimensions (and assuming some additional compactness properties) provides a very powerful set of tools for global bifurcation theory.

With respect to the degree theoretic methods of the previous section, Dancer's analytic continuation method offers several advantages. The first of these is that it provides a more natural way to treat the double bifurcation case. Note that, as the multiplicity of the eigenvalue will be even, a Rabinowitz-type argument requires some additional work; in particular, special attention must be paid to the scenario where the continuations of the local branches intersect. On the other hand, since maximal analytic extensions are unique, when working within the analytic framework, we are untroubled by such intersections. Secondly, since we are using more than simply topological information, we obtain considerably better regularity for the global bifurcation branch. Indeed, we will prove that the solution continuum is, in fact, path-connected and admits a locally injective, analytic reparameterization (in the sense of Definition 3.1.1.)

Let us begin by presenting the central result of the analytic theory of global bifurcation. This was originally proved by Dancer (cf. [**20, 22**]), but we shall use a slight rephrasing of that result due to Buffoni and Toland (cf. [**11**]). For the purposes of this exposition, we shall temporarily set aside our previous notation. Let real Banach spaces W, Z, open set $U \subset \mathbb{R} \times W$ and a real analytic $\mathcal{F}\colon U \to Z$ be given. (Looking forward, we should think of U as taking the place of $\mathcal{O}_\delta$, W that of X, Z that of Y and $\mathcal{F}$ that of $\mathcal{G}$.) If the following structural properties hold

(G1) $\mathcal{T} := \{(Q,0)\colon Q \in \mathbb{R}\} \subset U$ and $\mathcal{F}(Q,0) = 0, \forall Q \in \mathbb{R}$,

(G2) $\mathcal{F}_w(Q,w)$ is a Fredholm operator of index zero for each $(Q,w) \in U$ such that $\mathcal{F}(Q,w) = 0$,

(G3) $\exists Q_* \in \mathbb{R}$ such that (i) $\mathcal{N}(\mathcal{F}_w(Q_*,0)) = \{s\phi_*\colon s \in \mathbb{R}\}$, and (ii) $\mathcal{F}_{Qw}(Q_*,0)\phi_* \notin \mathcal{R}(\mathcal{F}_w(Q_*,0))$, then, by the analytic equivalent of THEOREM 3.3.2, there exists an analytic local bifurcation curve at $(Q_*, 0)$. That is, there exists an $\epsilon > 0$ and an analytic $(\mathfrak{Q}, \mathfrak{w})\colon (-\epsilon, \epsilon) \to \mathbb{R} \times W$ such that $\mathcal{F}(\mathfrak{Q}(s), \mathfrak{w}(s)) = 0$, for all $s \in (-\epsilon, \epsilon)$, $\mathfrak{Q}(0) = Q_*$ and $\mathfrak{w}'(s) = \phi_*$. Let us denote

$$\mathcal{C}^+_{\mathrm{loc}} := \{(\mathfrak{Q}(s), \mathfrak{w}(s))\colon s \in (0, \epsilon)\}, \qquad \mathcal{S} := \{(Q, w) \in U\colon \mathcal{F}(Q, w) = 0\}. \tag{3.6.1}$$

By taking ϵ sufficiently small, we may assume that $\mathfrak{w}' \neq 0$ and $\mathcal{C}^+_{\mathrm{loc}} \subset \mathcal{S} \setminus \mathcal{T}$.

THEOREM 3.6.1. (Analytic Global Simple Bifurcation) *Let W, Z, U and $\mathcal{F}$ be given as above and assume that* (G1), (G2) *and* (G3) *hold. We may then take $(\mathfrak{Q}, \mathfrak{w})$ as before and let $\mathcal{C}^+_{\mathrm{loc}}$ and $\mathcal{S}$ be defined by* (3.6.1). *Suppose further that*

(G4) $$\mathfrak{Q}' \not\equiv 0 \text{ on } (-\epsilon, \epsilon),$$

(G5) *all bounded closed subsets of $\mathcal{S}$ are compact.*

Then there exists a unique, maximal, continuous curve $\mathcal{K}$ that extends $\mathcal{C}^+_{\mathrm{loc}}$ in the following sense.

(a) $\mathcal{K} = \{(\mathfrak{Q}(s), \mathfrak{w}(s))\colon s \in [0, \infty)\} \subset U$, *where* $(\mathfrak{Q}, \mathfrak{w})\colon [0, \infty) \to \mathbb{R} \times W$ *is continuous.*

(b) $\mathcal{C}^+_{\mathrm{loc}} \subset \mathcal{K} \subset \mathcal{S}$.

(c) The set $\{s \geq 0\colon \mathcal{N}(\mathcal{F}_w(\mathfrak{Q}(s), \mathfrak{w}(s))) \neq \{0\}\}$ *has no accumulation points.*

(d) At each point, $\mathcal{K}$ has a local analytic reparameterization.

(e) One of the following occurs.

(i) $\|(\mathfrak{Q}(s), \mathfrak{w}(s)\| \to \infty$ *as* $s \to \infty$*; or*

(ii) $(\mathfrak{Q}(s), \mathfrak{w}(s))$ *approaches the boundary of U as $s \to \infty$; or*

(iii) $\mathcal{K}$ is a closed loop, i.e. $\exists T > 0$ *such that* $(\mathfrak{Q}(s+T), \mathfrak{w}(s+T)) = (\mathfrak{Q}(s), \mathfrak{w}(s))$, *for all* $s \in [0, \infty)$. *This implies that* $\mathcal{K} = \{(\mathfrak{Q}(s), \mathfrak{w}(s))\colon s \in [0, T]\}$.

(f) If, for some $0 < s_1 < s_2$, *we have*

$$(\mathfrak{Q}(s_1), \mathfrak{w}(s_1)) = (\mathfrak{Q}(s_2), \mathfrak{w}(s_2)), \text{ and } \mathcal{N}(\mathcal{F}_w(\mathfrak{Q}(s_1), \mathfrak{w}(s_2))) = \{0\},$$

then (e)(iii) *occurs and* $s_2 - s_1$ *is an integer multiple of* T.

In order to apply this theory to our problem we need to make several benign alterations to hypotheses (G1)–(G5). First, we wish to replace $\mathcal{T}$ occurring in (G1) with that from the previous sections, namely the 1-parameter family of laminar flows. This is justified since the particular form of the trivial solutions plays no role in the proof of Theorem 3.6.1. Indeed, once the local curve $\mathcal{C}^+_{\mathrm{loc}}$ is found, no more mention need be made of $\mathcal{T}$.

Secondly, we wish to consider the case where double bifurcation occurs at the local level. Again, this is possible because the purpose of (G3) is merely to guarantee the existence of $\mathcal{C}^+_{\mathrm{loc}}$ and has no bearing on the global analysis. If, as in section 3.4, there is a *set* of local bifurcation curves $\{\mathcal{C}_{\mathrm{loc},i}\}_{i=1}^N$, then there exists a corresponding set of local parametrizations $\{(\mathfrak{Q}_i, \mathfrak{w}_i)\}_{i=1}^N$ and so we may define $\{\mathcal{C}^+_{\mathrm{loc},i}\}_{i=1}^N$ in the obvious way. Applying the argument of Theorem 3.6.1 to each of these curves yields the following corollary.

Corollary 3.6.1. (Analytic Global Double Bifurcation) *Let* W, Z, U *and* $\mathcal{F}$ *be given as before. Suppose that the following structural properties are satisfied:*

(G1') *there exists a family of trivial solutions* $\mathcal{T} \subset U \cap \mathcal{S}$,

(G2') $\mathcal{F}_w(Q, w)$ *is a Fredholm operator of index zero for each* $(Q, w) \in U \cap \mathcal{S}$,

(G3')
$$\exists (Q_*, w_*) \in \mathcal{T}, \epsilon > 0, \text{ and analytic functions } \{(\mathfrak{Q}_i, \mathfrak{w}_i)\colon (-\epsilon, \epsilon) \to \mathbb{R} \times W\}_{i=1}^N$$
such that, for all $s \in (-\epsilon, \epsilon)$, $i = 1, \ldots, N$,
$$(\mathfrak{Q}_i(0), \mathfrak{w}_i(0)) = (Q_*, w_*) \text{ and } \mathcal{F}(\mathfrak{Q}_i(s), \mathfrak{w}_i(s)) = 0,$$

(G4')
$$\mathfrak{Q}_i' \not\equiv 0 \text{ on } (-\epsilon, \epsilon), \qquad i = 1, \ldots, N,$$

(G5') *all bounded closed subset of* $\mathcal{S}$ *are compact.*

For each $i = 1, \ldots, N$, *define* $\mathcal{C}^+_{\mathrm{loc},i} := \{(\mathfrak{Q}_i(s), \mathfrak{w}_i(s))\colon s \in (0, \epsilon)\}$. *Then there exists a set of continuous curves* $\{\mathcal{K}_i\}_{i=1}^N$ *with* $\mathcal{K}_i$ *the unique maximal analytic extension of* $\mathcal{C}^+_{\mathrm{loc},i}$ *in the sense of* Theorem 3.6.1, *for* $i = 1, \ldots, N$.

This result follows directly from the proof of Theorem 3.6.1, keeping in mind our observation of the last several paragraphs. For brevity, we therefore do not provide a proof. Note, however, that this is not a new result: the multiple bifurcation case was considered by

Dancer in [**20**]. COROLLARY 3.6.1 is merely a translation of Dancer's result to the language of Buffoni and Toland.

With the background results in place, we now return to the notational conventions of section 3.5 and prove our main result. Fix $\delta > 0$ and define

$$\mathcal{O}_\delta := \left\{(Q,h) \in \mathbb{R} \times X : \; h_p > \delta \text{ in } \overline{R}, \; Q - \sigma\kappa[h] - 2g\rho h > \delta\right\},$$

$$S := \{(Q,h) \in \mathbb{R} \times X : \mathcal{G}(Q,h) = 0\}, \qquad S_\delta := \text{ closure in } \mathbb{R} \times X \text{ of } (S \cap \mathcal{O}_\delta \setminus \mathcal{T}).$$

By the local analysis of the previous sections, we may take the non-laminar solution set near (H^*, Q^*) to be comprised of a set of C^1-curves, $\{\mathcal{C}'_{\mathrm{loc},i}\}_{i=1}^N$, for some $N \geq 2$. We now prove that these curves are, in fact, analytic.

LEMMA 3.6.1. (Analyticity) (a) *The maps* $\lambda \mapsto H(\cdot;\lambda) \in C^{3+\alpha}([p_0,0])$ *and* $\lambda \mapsto Q(\lambda) \in \mathbb{R}$ *are analytic on the interval* $(-2B_{\min} + \epsilon_0, \infty)$. *Furthermore, if* I *is any subinterval of* $(-2B_{\min} + \epsilon_0, \infty)$ *that contains* λ_*, *then the operators* $\mathcal{F} : I \times X \to Y$ *and* $\mathcal{G} : \mathbb{R} \times X \to Y$ *(defined by* (3.3.20)–(3.3.21) *and* (3.5.1)–(3.5.2), *respectively) are analytic.*

(b) *There exists* $\epsilon > 0$ *and a set of analytic functions* $\{(\mathfrak{Q}_i(s), \mathfrak{h}_i(s)) : (-\epsilon,\epsilon) \mapsto \mathbb{R} \times X\}_{i=1}^N$ *such that* $\mathcal{C}'_{\mathrm{loc},i} = \{(\mathfrak{Q}_i(s), \mathfrak{h}_i(s)) : s \in (-\epsilon,\epsilon)\}$, *for* $i = 1, \ldots, N$.

PROOF. The analyticity of $\mathcal{G}$ is clear, since

$$\partial_Q^j \mathcal{G}, \; \partial_h^k \mathcal{G} \equiv 0, \qquad \forall j \geq 2, \; k \geq 5.$$

Notice also that, since

$$Q(\lambda) = \lambda + 2g\rho(0)H(0;\lambda), \qquad \mathcal{F}(\lambda, w) = \mathcal{G}(Q(\lambda), w + H(\cdot;\lambda)),$$

it suffices to prove that $\lambda \mapsto H(\cdot;\lambda)$ is analytic in order to obtain part (a).

We shall do this by extending $\lambda \mapsto H(\cdot;\lambda)$ to a function defined on the complex half-plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > -2B_{\min} + \epsilon_0\}$, and then showing that the extended function is complex analytic.

Mimicking the proof of LEMMA 3.3.1, we can prove that, for each $\lambda \in \mathbb{H}$ and $p \in [p_0, 0]$, there exists a unique $H = H(p;\lambda) \in \mathbb{C}$ satisfying the following implicit relationship

$$H(p;\lambda) = \int_{p_0}^{p} \frac{ds}{\sqrt{\lambda + G(s;\lambda)}}, \qquad p \in [p_0, 0], \; \lambda \in \mathbb{H}, \tag{3.6.2}$$

where we are taking the positive branch of the square root and

$$G(p;\lambda) = 2B(p) + 2g\int_{p_0}^{p} \rho_p(s)\left(H(s;\lambda) - H(0;\lambda)\right) ds.$$

If we denote $\lambda = \lambda_1 + i\lambda_2$ and fix $p \in [p_0, 0]$, then

$$\begin{aligned}\frac{d}{d\overline{\lambda}}H(p;\lambda) &= \frac{1}{2}\left(\frac{d}{d\lambda_1} + i\frac{d}{d\lambda_2}\right)H(p;\lambda)\\ &= -\int_{p_0}^{p}\left(g\int_{p_0}^{r}\rho_p(s)\frac{d}{d\overline{\lambda}}\left(H(s;\lambda) - H(0;\lambda)\right)ds\right)\left(\lambda + G(r;\lambda)\right)^{-3/2}\,dr.\end{aligned}$$

The last line above was found by differentiating (3.6.2) (with respect to λ_1 and λ_2), and making use of the definition of G.

It follows that, if we denote $Y(p;\lambda) := H(p;\lambda) - H(0;\lambda)$, then

$$\frac{d}{d\overline{\lambda}}Y(p;\lambda) = \int_{p}^{0}\left(g\int_{p_0}^{r}\rho_p(s)\frac{d}{d\overline{\lambda}}Y(s)ds\right)\left(\lambda + G(r;\lambda)\right)^{-3/2}\,dr. \tag{3.6.3}$$

Recalling the definition of ϵ_0 in (3.1.16), we see that, since $\lambda \in \mathbb{H}$,

$$|\lambda + G(p;\lambda)|^{-3/2} \leq |\lambda_1 + 2B_{\min}|^{-3/2} \leq (2g|p_0|^2\|\rho_p\|_\infty)^{-1}.$$

Inserting this last expression into (3.6.3), we find

$$\left\|\frac{d}{d\overline{\lambda}}Y(\cdot;\lambda)\right\|_\infty \leq \frac{1}{2}\left\|\frac{d}{d\overline{\lambda}}Y(\cdot;\lambda)\right\|_\infty,$$

and thus $dY/d\overline{\lambda} \equiv 0$. As $H(p;\lambda) = Y(p;\lambda) - Y(p_0;\lambda)$, this implies further that $dH/d\overline{\lambda} \equiv 0$, hence $\lambda \mapsto H(\cdot;\lambda)$ is complex analytic.

Now, (b) follows directly from the fact that $\mathcal{F}$ is analytic. Indeed, simply substituting the Crandall-Rabinowitz and Lyapunov-Schmidt theory of the previous sections with their analytic counterparts, we obtain the analytic parameterizations $\{(\mathfrak{Q}_i, \mathfrak{h}_i)\}_{i=1}^N$ as claimed. □

Following the notation of Theorem 3.6.1, let us denote

$$\mathcal{C}'_{\delta,i} := \{(\mathfrak{Q}_i(s), \mathfrak{h}_i(s)) : s \in (0,\epsilon)\}, \qquad i = 1, \ldots, N.$$

Note that, to make our notation more concise, we have dropped the + superscript. Our main result for this section is the following.

Theorem 3.6.2. (Analytic Global Bifurcation) *Fix $\delta > 0$. There exists a maximal, global, continuous extension $\mathcal{K}'_{\delta,i} \subset S_\delta$ of $\mathcal{C}'_{\delta,i}$ (in the sense that* Theorem 3.6.1(a)–(f) *holds for $\mathcal{K}'_{\delta,i}$), for $i = 1, \ldots, N$. Furthermore, for each i, one of the following alternatives must hold.*

(i) $\|(\mathfrak{Q}_i(s), \mathfrak{h}_i(s))\|_{\mathbb{R}\times X} \to \infty$ *as* $s \to \infty$*; or*

(ii) the closure of $\mathcal{K}'_{\delta,i}$ *contains a point* $(Q, h) \in \partial\mathcal{O}_\delta$*; or*

(iii) $\mathcal{K}'_{\delta,i}$ *is a closed loop.*

Remark. In several respects, the alternatives above are considerably stronger than their analogues in Theorem 3.5.1. If, recalling the notation of section 3.5, we define $\mathcal{C}'_\delta$ to be the connected component of S_δ containing $\bigcup_i \mathcal{C}'_{\mathrm{loc},i}$, then clearly $\bigcup_i \mathcal{K}'_{i,\delta} \subset \mathcal{C}_\delta$. In other words, the conclusions of Theorem 3.6.2 applies only to a (very special) subset of $\mathcal{C}'_\delta$, whereas the alternatives of Theorem 3.5.1 apply to the *entirety* of $\mathcal{C}'_\delta$. In that sense, the theorem above is stronger, since it identifies — in fact, it constructs — the precise path along which the alternative occurs. Moreover, the fact that $\bigcup_i \mathcal{K}'_{\delta,i}$ is path-connected and locally analytic is noteworthy in and of itself. This cannot be obtained through purely degree theoretic methods, since it requires more than topological information about the solution set.

Still, it is not possible to completely reconcile the alternatives of Theorem 3.5.1 and Theorem 3.6.2. For instance, it is entirely permissible for $\|(\mathfrak{Q}_i(s), \mathfrak{h}_i(s))\| \to \infty$ as $s \to \infty$, but, for some $s > 0$, we have $(\mathfrak{Q}_i(s), \mathfrak{h}_i(s)) \in \mathcal{T}$. Moreover, it may be that $\mathcal{C}'_{\delta,i}$ is a closed loop for all $i = 1, \ldots, N$, yet $\bigcup_i \mathcal{K}'_{\delta,i} \cap \mathcal{T} = (Q_*, H_*)$. This is possible, again, because $\bigcup_i \mathcal{K}'_{i,\delta}$ is only a subset of $\mathcal{C}'_\delta$.

Finally, let us note that a similar result can be proved for the simple bifurcation case — we need only use Theorem 3.6.1 directly, rather than Corollary 3.6.1.

Theorem 3.6.2. In light of Corollary 3.6.1 and Lemma 3.6.1, the structure of our argument is fairly straightforward: we need only confirm that $\mathcal{G}$ satisfies structural properties (G1')–(G5').

With that in mind, let X, Y, $\mathcal{O}_\delta$ and $\mathcal{G}$ stand in for W, Z, U and $\mathcal{F}$ of Corollary 3.6.1, respectively. Then the family of laminar flow, $\mathcal{T}$ restricted to $\mathcal{O}_\delta$ satisfies (G1'). Condition (G2') was verified in Lemma 3.5.3, while (G3') and (G4') follow from the analysis of section 3.3 and section 3.4 (in particular, note that since the mixed solution curves obey the scaling $|\xi| \leq C|\lambda - \lambda_*|$, for some $C > 0$, we cannot have $\mathfrak{Q}' \equiv 0$.) Finally, (G5') is a consequence of Lemma 3.5.2, since $S_\delta = \mathcal{G}^{-1}(\{0\}) \cap \mathcal{O}_\delta$. We are therefore justified in taking $\mathcal{K}'_{\delta,i}$ to be the global extension of $\mathcal{C}'_{\mathrm{loc},i}$, for $i = 1, \ldots, N$. Alternatives (i)–(iii) follow from Theorem 3.6.1(e) applied to $\mathcal{K}'_{\delta,i}$. ☐

Ideally, one would like to eliminate all but the first and second alternatives of THEOREM 3.6.2, as this would imply that either the curve is unbounded, or the problem degenerates (i.e. loses ellipticity) along $\bigcup_{\delta>0} \mathcal{K}_i'$. As we shall see in section 3.8, put into the Eulerian framework both of these possibilities lead to us to conclude that there is a curve of solutions to the original problem (3.1.1)-(3.1.6) along which either the fluid tends to stagnation at a point, or the horizontal velocity (in the moving frame) approaches $-\infty$ somewhere within the fluid.

Typically one excludes the undesirable alternative (iii) by means of a nodal property argument. That is, using maximum principles one shows that the sign of certain quantities are invariant along the curve, and then argue that this is violated if $\mathcal{K}_i'$ is a closed loop (or, if we are considering simple bifurcation, if $\mathcal{C}_\delta'$ returns to $\mathcal{T}$) (see, for example, [**21**]). Taking $\sigma = 0$, an argument of this kind was successfully carried out in the constant density case (cf. [**15**]), and, to a lesser degree, in the stratified case (see section 2.5). Unfortunately, the introduction of surface tension— in particular, the higher derivatives on the boundary that accompanies it— seems to destroy the maximum principle structure of the problem. The rather conspicuous dearth of literature on unbounded curves of capillary and capillary-gravity waves is a direct consequence of this fact.

3.7. Uniform regularity

In this section we consider the first alternatives of THEOREM 3.5.1 and THEOREM 3.6.2. As in [**15**], our goal is the establishment of uniform bounds in the Hölder norm along the continuum $\mathcal{C}_\delta'$, for the simple eigenvalue case, and along the curves $\{\mathcal{K}_{\delta,i}'\}_{i=1}^4$ for the higher multiplicity case. In that paper Constantin and Strauss were able to control the $C^{3+\alpha}$-norm along the continuum in terms $\sup_{h\in\mathcal{C}_\delta'} \sup_{\overline{R}} h_p$ (assuming that Q is likewise bounded along the continuum). However, the addition of the curvature term $\kappa[h]$ to the nonlinear boundary condition complicates matters: we must now allow for the possibility that the curvature may blowup (in L^∞) along the continuum.

THEOREM 3.7.1. (Uniform Regularity) *Let $\delta > 0$ be given and assume that $\beta \in C^{3+\alpha}((0,|p_0|))$. If*

$$\sup_{(Q,h)\in\mathcal{C}_\delta'} \sup_{\overline{R}} h_p < \infty, \qquad \sup_{(Q,h)\in\mathcal{C}_\delta'} Q < \infty \qquad \textit{and} \qquad \inf_{(Q,h)\in\mathcal{C}_\delta'} \inf_{T} \kappa[h] > -\infty,$$

then $\sup_{(Q,h)\in\mathcal{C}'_\delta} \|h\|_{C^{3+\alpha}(\overline{R})}$ *is finite.*

In the proof of Theorem 3.7.1, the most technically complicated step is obtaining bounds on the second derivatives of h along $\mathcal{C}'_\delta$. The L^∞ estimates will be a consequence of p_0 being constant along the continuum, while the higher-order terms can be expressed (via the height equation) in terms of the lower-order terms in the usual way. To treat the second-order terms, we shall employ a suite of *a priori* estimates for nonlinear elliptic equations with Venttsel boundary conditions due to Luo and Trudinger (cf. [**48**]). We shall not, however, need these results in their full generality, so instead we streamline them into a single statement. Of course, in order to apply this to our problem, we will again need to use the method of freezing the operator $\mathcal{G}$.

First note that due to the periodicity in q, the domain R can be considered as embedded on a torus, allowing us to ignore the seeming lack of smoothness at the corner points $(0,0)$, $(0,p_0)$, $(2\pi,0)$, and $(2\pi,p_0)$. We consider a differential operator $F = F(x,z,\xi,r) \in C^2(R\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{S},\mathbb{R})$ and boundary operator $G = G(x,z,\xi,r) \in C^2(R\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{S},\mathbb{R})$. Here $\mathbb{S}$ denotes the space of 2×2 real symmetric matrices, D denotes the gradient operator and D^2 the Hessian. Both of these are taken as acting on the space of smooth real-valued functions on $\overline{R}$ which are 2π-periodic in the first variable. As usual, we say that F is elliptic at $(z,\xi,r)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{S}$ provided that the matrix $F_r := [\partial F/\partial r_{ij}]_{1\le i,j\le 2}$ is positive definite at that point. Moreover, if Λ_1 and Λ_2 denote the minimum and maximum eigenvalue of F_r, respectively, then F is said to be uniformly elliptic provided that the ratio Λ_2/Λ_1 is bounded.

We now specialize this framework to the problem at hand. Let $a_{ij} = a_{ij}(z,\xi)$, $b = b(x,z,\xi) \in C^2(R\times\mathbb{R}\times\mathbb{R}^2,\mathbb{R})$ and $\mathfrak{a}_{ij}\in\mathbb{S}$, $\mathfrak{b} = \mathfrak{b}(z,\xi) \in C^2(\mathbb{R}\times\mathbb{R}^2,\mathbb{R})$ be given. We consider the case where (F,G) are of the form

$$F(z,\xi,r) = a_{ij}(z,\xi)r_{ij} + b(x,z,\xi), \qquad G(z,\xi,r) = \mathfrak{a}_{ij}r_{ij} + \mathfrak{b}(z,\xi).$$

Here a_{ij} is assumed to be such that F is uniformly elliptic, in the sense of the previous paragraph. We shall take G to be of Venttsel-type, which means the following: (i) (Uniform obliqueness) There exists some $\chi > 0$ such that, at each point $(q,0)$, the inward normal derivative $\mathfrak{b}_\xi\cdot(0,1) \ge \chi > 0$, for all $(z,\xi)\in\mathbb{R}\times\mathbb{R}^2$; (ii) (Ellipticity) $\mathfrak{a}_{ij}\zeta_i\zeta_j > 0$, for all $\zeta = (\zeta_1,\zeta_2)\in\mathbb{R}^2$ such that ζ is tangent to the lower boundary of R.

With F and G given as above we wish to consider the quasilinear elliptic (Venttsel) boundary value problem

$$
\begin{cases} F(h, Dh, D^2h) = 0, & \text{in } R \\ G(h, Dh, D^2h) = 0 & \text{on } p = 0 \\ h = 0 & \text{on } p = p_0. \end{cases} \tag{3.7.1}
$$

THEOREM 3.7.2. (Luo-Trudinger) *Let $h \in C^5(\overline{R})$ be a solution to* (3.7.1) *that is 2π-periodic in the first variable and for some constant $K > 0$ satisfies $|h| + |Dh| \leq K$ in $\overline{R}$. Suppose further that for some positive constant M the functions $F = F(z, \xi, r)$ and $G = G(z, \xi)$ satisfy the following structural conditions:*

$$\Lambda_2 \leq M\Lambda_1, \tag{3.7.2}$$

$$|F|,\ |F_\xi|,\ |F_z| \leq M\Lambda_1(|r| + 1), \tag{3.7.3}$$

$$F_{rr} \leq 0, \tag{3.7.4}$$

$$|G|, |G_z|, |G_\xi|, |G_{zz}|, |G_{z\xi}|, |G_{\xi\xi}| \leq M\chi, \tag{3.7.5}$$

$$|\partial_z^2 a_{ij}|, (1 + |\xi|)|\partial_z \partial_\xi a_{ij}|, (1 + |\xi|^2)|\partial_\xi^2 a_{ij}| \leq M\Lambda_1 \tag{3.7.6}$$

$$|b_x|, |b_z|, |b_\xi|(1 + |\xi|) \leq M\Lambda_1(1 + |\xi|^2) \tag{3.7.7}$$

$$|b_{xx}|, |b_{zx}|, |b_{zz}|, (1 + |\xi|)|b_{z\xi}|, (1 + |\xi|)|b_{x\xi}|, (1 + |\xi|^2)|b_{\xi\xi}| \leq M\Lambda_1(1 + |\xi|^2) \tag{3.7.8}$$

$$|\mathfrak{b}_{zz}|, (1 + |\xi|)|\mathfrak{b}_{z\xi}|, (1 + |\xi|)|\mathfrak{b}_{\xi\xi}| \leq M\chi(1 + |\xi|) \tag{3.7.9}$$

$$|D^3 a_{ij}|, |D^3 b| \leq M\Lambda_1 \tag{3.7.10}$$

$$|D^3 \mathfrak{b}_| \leq M\chi \tag{3.7.11}$$

for all $(z, \xi, r) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}$ such that $|z| + |\xi| \leq K$. Then there are positive constants $\mu = \mu(M)$ and $C = C(K, M)$ such that $h \in C^{2+\mu}_{per}(\overline{R})$ and

$$\|h\|_{C^{2+\mu}_{per}(\overline{R})} \leq C.$$

This is a (weaker) restatement of the results presented in [**48**]. Also, for consistency with presentation of similar theorems earlier in [**15**] and chapter 2, we have slightly altered notation. We now give a brief sketch of the proof.

Proof of Theorem 3.7.2. It was already shown in section 2.6 that (3.7.2)-(3.7.5) together amount to the "natural structural conditions" (L-T F1)–(L-T F5) presented in [**48**]. We will now show that, unsurprisingly, these further imply structure conditions (Luo-T A), (Luo-T A1), (Luo-T A2), (Luo-T B), (Luo-T B1) and (Luo-T B2) of [**48**].

First consider the conditions on F. That (L-T F1) is equivalent to (Luo-T A) is obvious. Since $F(\cdot,\cdot,\cdot,r=0)=b(\cdot,\cdot,\cdot)$, (LT F2) immediately leads to (Luo-T A1). Similarly, if we note that $D_rF=a_{ij}$, it can be readily verified that condition (L-T F4) implies the first line of inequalities in (Luo-T A2). To get the second line, we appeal to (L-T F3), noting once more that F evaluated at $r=0$ is b.

Next we turn to G. Clearly, (Luo-T B) follows from uniform obliqueness. Since $G(\cdot,\cdot,\cdot,r=0)=\beta(\cdot,\cdot,\cdot)$, evaluating (L-T G2)-(L-T G3) at $r=0$ yields the growth conditions on β in (Luo-T B1)-(Luo-T B2).

Combining this with the new structural conditions (3.7.6)-(3.7.11), the hypothesis of our theorem encompasses all the conditions of [**48**]. To arrive at the statement above, we simply apply in succession each of the results in §2–§4 of that paper. □

Our tools fully assembled, we are ready to prove Theorem 3.7.1.

Proof of Theorem 3.7.1. We begin by proving the $L^\infty(\overline{R})$ norm of h is uniformly bounded along the continuum. To do so we note that by (3.2.9)

$$0 \le h(q,p) = y + d \le \eta(0) + d,$$

and thus the definition of p_0 in (3.1.10) yields

$$\eta(0)+d \le \frac{|p_0|}{\inf_{\overline{D_\eta}}\left(\sqrt{\rho}(c-u)\right)} = |p_0| \sup_{\overline{R}} h_p$$

Thus assuming the hypothesis of the theorem, that is that h_p is uniformly bounded in L^∞ along the continuum, then the above inequalities together give the boundedness of the L^∞-norm of h.

To bound h_q we proceed as in Theorem 2.6.1 to show that h_q cannot attain an interior minimum or maximum. Indeed, since the interior equation here is identical to the one considered in that chapter, a verbatim application of their argument suffices. As $h_q \equiv 0$ on the bottom, we need only concern ourselves with the top. The nonlinear boundary condition then gives

$$\inf_T h_p^2(Q - 2\sigma\kappa[h] - 2g\rho h) \le 1 + h_q^2 \le \sup_T h_p^2(Q - 2\sigma\kappa[h] - 2g\rho h).$$

The quantity on the left-hand side is bounded uniformly from below by δ^3 on $\mathcal{C}'_\delta$. Thus $\sup_{\overline{R}} h_q = \sup_T h_q$ can be controlled by $\sup_{\mathcal{C}'_\delta} \sup_{\overline{R}} h_p$ and $\sup_{\mathcal{C}'_\delta} \sup_T(-\kappa[h])$.

For the second-order terms we need to make use THEOREM 3.7.2, and so we shall once again freeze the operator $\mathcal{G}$ in the sense of section 3.5. Fix $K > 0$, $\tau \in \mathbb{R}$. Let $\zeta = \zeta(\xi)$ be a smooth cutoff function with $0 \le \zeta \le 1$, $\zeta \equiv 1$ on $B_K(0) \cap \{\xi : \xi_2 > \delta\}$ and $\operatorname{supp} \zeta \subset B_{2K}(0) \cap \{\xi : \xi_2 > \delta/2\}$. For each $x = (x_1, x_2) \in \mathbb{R}^2$, $z \in \mathbb{R}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, put

$$\begin{aligned} a_{ij}(\xi) &:= \begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 \\ -\xi_1\xi_2 & 1 + \xi_1^2 \end{pmatrix} \zeta(\xi) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (1 - \zeta(\xi)), \\ b(x, z, \xi) &:= \xi_1^3 \left(\beta(-x_2) - g(z - \tau)\rho_p(x_2)\right) \zeta(\xi), \end{aligned}$$

and

$$\mathfrak{a}_{ij} := \begin{pmatrix} 2\sigma & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathfrak{b}(z, \xi) := \left(1 + \xi_1^2\right)^{\frac{3}{2}} \left(Q - 2g\rho(0)z - \frac{1 + \xi_1^2}{\xi_2^2}\right) \zeta(\xi) + (1 - \zeta(\xi))\xi_2.$$

Suppose that for some τ, (h, Q) is a solution to (3.7.1) with F and G as in THEOREM 3.7.2, that is,

$$\begin{aligned} F(x, z, \xi, r) &= \left(\xi_2^2 r_{11} + (1 + \xi_1^2) r_{22} - 2\xi_1\xi_2 r_{12} + \xi_1^3 \left(\beta(-x_2) - g(z - \tau)\rho_p(x_2)\right)\right) \zeta(\xi) \\ &\quad + r_{11}(1 - \zeta(\xi)) \\ G(z, \xi, r) &= 2\sigma r_{11} + \left((1 + \xi_1^2)^{3/2} \left(Q - 2g\rho(0)z - \frac{1 + \xi_1^2}{\xi_2^2}\right)\right) \zeta(\xi) + \xi_2(1 - \zeta(\xi)). \end{aligned}$$

It follows that each $(h,\ Q) \in \mathcal{C}'_\delta$ with $|h| + |Dh| < K$ satisfies the Venttsel boundary value problem (3.7.1) for the above choice of F and G. The next task is verifying the structural conditions. It is easy to see that F is uniformly elliptic. To show uniform obliqueness note that, at any point $x \in \partial R$, the inward normal is given by:

$$\mathfrak{b}_\xi \cdot (0, 1) = \frac{2\left(1 + \xi_1^2\right)^{\frac{5}{2}}}{\xi_2^3} \zeta(\xi) - (\partial_{\xi_2}\zeta(\xi))\xi_2 + 1 - \zeta(\xi).$$

Inside the region where $\zeta \equiv 1$, this is clearly uniformly positive. Also, since we may take ζ to be monotonic decreasing in the ξ_2-direction, the last three terms guarantee that the innward normal derivative is uniformly bounded away from zero in the entirety of $\mathbb{R}^2$.

The final structural conditions are easy. Since a_{ij}, b, $\mathfrak{a}_{ij}$ and $\mathfrak{b}$ are all suitably smooth, by compactness we can choose a sufficiently large M so that (3.7.2)-(3.7.11) hold for $\xi \in \overline{B_K(0)} \cap \{\xi : \xi_2 \geq \delta\}$. (Note that here we make use of the assumption $\gamma \in C^3$ to get (3.7.10).)

The more serious step is to guarantee that h has enough regularity to begin with. That is, we know $h \in C^{3+\alpha}(\overline{R})$, however THEOREM 3.7.2 requires it to be of class C^5. In order to conclude that $\mathcal{C}'_\delta$ lies in $C^5(\overline{R})$ we shall apply standard linear elliptic theory. Setting aside the question of β for the moment, the fact that F is quasilinear tells us that, if $h \in C^{k+\alpha}(\overline{R})$ with $k > 2$, then the coefficients of the problem are at worst $C^{k-1+\alpha}(\overline{R})$. Thus, if we assume $\beta \in C^{3+\alpha}((0, |p_0|))$, we have $a_{ij},\ b \in C^{2+\alpha}$, which implies by the estimates of LEMMA 3.5.1 for the linear Venttsel problem that $h \in C^{4+\alpha}(\overline{R})$. This means, in turn, that the coefficients are actually $C^{3+\alpha}$. Turning the crank once more, we conclude $h \in C^{5+\alpha}(\overline{R})$. THEOREM 3.7.2 is therefore applicable, and thus the second order derivatives are bounded.

Thus (F, G) satisfy the structural requirements of THEOREM 3.7.2. We conclude that there exists a constants $C = C(K, M, \tau)$ and $\mu = \mu(K, M, \tau)$ so that, for any solution (h, Q) of (3.7.1) with $|h| + |Dh| < K$,

$$\|h\|_{C^{2+\mu}(\overline{R})} \leq C.$$

Note that every $h \in \mathcal{C}'_\delta$ is such a solution for $\tau = d(h)$. In order to complete the estimates of the second-order terms, therefore, requires removing the dependency on τ from C. Toward that end we recall that $\tau = d(h) \leq \sup_R |h|$. Thus we may select K independently of τ. Moreover, since in the definition of F, τ does not occur in any of the second-order terms a_{ij}, it does not affect Λ_1 and Λ_2. Indeed, the characteristic polynomial satisfied by the eigenvalues is

$$\Lambda_i^2 - (1 + \xi_1^2 + \xi_2^2)\Lambda_i - (1 + \xi_1^2)\xi_2^2 - 4\xi_1^2\xi_2^2 = 0, \qquad i = 1, 2$$

which does not depend on τ. This is an obvious consequence of the fact that the effects of density variation in the height equation are consigned to the lower-order terms. The term in F that does contain τ is b, and it can be easily estimated in terms of K. Thus C and μ can be chosen independently of τ, which gives the second-derivative bounds.

Finally we establish third derivative bounds. Denoting $\theta := h_q$, we differentiate the height equation in q to find,

$$
(3.7.12) \qquad \begin{cases}
(1+h_q^2)\theta_{pp} - 2h_p h_q \theta_{pq} + h_p^2 \theta_{qq} \\
\qquad = C(h, h_p, h_q, h_{pp}, h_{pq}, h_{qq}), & \text{in } R, \\[2ex]
2\sigma h_p^2 \left(\theta_q (1+h_q^2)^{-3/2}\right)_q + 2h_q\theta_q + (2g\rho h + 2\sigma\kappa[h] - Q)h_p\theta_p \\
\qquad = D(h_q, h_p) & \text{on } T, \\[2ex]
\theta = 0 & \text{on } B.
\end{cases}
$$

Thus θ solves a second-order elliptic PDE with $C^{1+\mu}_{\mathrm{per}}(\overline{R})$ coefficients and right-hand sides $C \in C^{\mu}_{\mathrm{per}}(\overline{R})$, and $D \in C^{\mu}_{\mathrm{per}}(T)$. Since we have already proved h is bounded uniformly in $C^{2+\mu}_{\mathrm{per}}(\overline{R})$, it follows that the right-hand side of (3.7.12) is bounded uniformly in $C^{\mu}_{\mathrm{per}}(\overline{R})$ along the continuum. As we have seen in section 3.5, so long as h_p is bounded uniformly away from 0 along the continuum, problem (3.7.12) above will be uniformly elliptic with a nondegenerate Venttsel-type boundary condition. We may therefore apply the linear theory for such problems (e.g. LEMMA 3.5.1) to obtain uniform *a priori* estimates of θ in $C^{2+\mu}_{\mathrm{per}}(\overline{R})$ over all of $\mathcal{C}'_\delta$.

It remains only to bound h_{ppp} in $C^{\mu}_{\mathrm{per}}(\overline{R})$. By means of the height equation, we may express h_{pp} in terms of the lower order derivatives of h:

$$
h_{pp} = -(1+h_q^2)^{-1}\left(h_p^3\gamma(-p) + h_{qq}h_p^2 - 2h_q h_p h_{pq}\right)
$$

But the right-hand side above is in $C^{1+\mu}_{\mathrm{per}}(\overline{R})$ by the arguments of the previous paragraphs. Thus h_{ppp} in $C^{\mu}_{\mathrm{per}}(\overline{R})$ as desired.

To transition back to the original Hölder exponent, α, we merely note that if $h \in C^{3+\mu}_{\mathrm{per}}(\overline{R})$, then it is certainly in $C^{2+\alpha}_{\mathrm{per}}(\overline{R})$. The arguments we have used to derive the third derivative bounds in no way relied on the particular value of μ, so running through them with α instead we find $h \in C^{3+\alpha}_{\mathrm{per}}(\overline{R})$. $\square$

3.8. Proof of the main results

With the uniform regularity results we established in the previous section, we are now prepared to begin the task of proving the main theorems. THEOREM 3.5.1 gave us three possibilities for the continuation of the local bifurcation curve(s). In order to prove THEOREM 3.1.1, we need merely to make a careful study of these alternatives in light of THEOREM 3.7.1.

PROOF OF THEOREM 3.1.1. Fix $\delta > 0$. If $\mathcal{C}_\delta'$ is unbounded in $\mathbb{R} \times X$, then at least one of the following must occur:

(i) there exists a sequence $(Q_n, h_n) \in \mathcal{C}_\delta'$ with $\lim_{n\to\infty} Q_n = \infty$; or

(ii) there exists a sequence $(Q_n, h_n) \in \mathcal{C}_\delta'$ with $\lim_{n\to\infty} \min_T \kappa[h_n] = -\infty$; or

(iii) there exists a sequence $(Q_n, h_n) \in \mathcal{C}_\delta'$ with $\lim_{n\to\infty} \max_{\overline{R}} \partial_p h_n = \infty$,

where the second and third possibilities are a consequence of THEOREM 3.7.1. If $\mathcal{C}_\delta'$ contains a point of $\partial\mathcal{O}_\delta$, then either

(iv) there exists a $(Q, h) \in \mathcal{C}_\delta'$ with $\partial_p h = \delta$ somewhere in $\overline{R}$, or

(v) there exists a $(Q, h) \in \mathcal{C}_\delta'$ with $2g\rho h + 2\sigma\kappa[h] = Q - \delta$ somewhere on T.

Thus, THEOREM 3.5.1 tell us that one of the five alternatives above occurs, or else the bifurcation curve intersects $\mathcal{T}$ at a point other than (Q^*, H^*).

Consider the first alternative (i). By the definition of p_0 in (3.1.10), we can estimate

$$\inf_{y\in[-d,\eta(x)]} \sqrt{\rho(x,y)}\Big(c - u(x,y)\Big) \leq \frac{p_0}{\eta(x)+d} \leq \sup_{y\in[-d,\eta(x)]} \sqrt{\rho(x,y)}\Big(c - u(x,y)\Big),$$

and thus, appealing to (3.2.5), we deduce

$$\begin{aligned} Q &= \rho(0,\eta(0))\Big(c - u(0,\eta(0))\Big)^2 + 2g\rho(0,\eta(0))(\eta(0)+d) \\ &\leq \sup_{\overline{D_\eta}} \rho(c-u)^2 + \frac{2g\rho(0,\eta(0))p_0}{\inf_{\overline{D_\eta}} \sqrt{\rho}(c-u)}. \end{aligned} \tag{3.8.1}$$

If for some $\delta > 0$ the first alternative holds, we conclude from (3.8.1) that for the corresponding sequence $(u_n, v_n, \rho_n, \eta_n)$ of solutions to (3.1.7)-(3.1.8), either $\sup_{\overline{D_{\eta_n}}} \sqrt{\rho_n}(c - u_n) \to \infty$, or $\inf_{\overline{D_{\eta_n}}} \sqrt{\rho_n}(c - u_n) \to 0$.

Similarly, if the third alternative (iii) holds, then, simply by rearranging the change of variables equations in (3.2.11), we have

$$\partial_p h_n = \frac{1}{\sqrt{\rho_n}(c - u_n)},$$

whence $\max_{\overline{R}} \partial_p h_n \to \infty$ if and only if $\inf_{\overline{D_{\eta_n}}} \sqrt{\rho_n}(c - u_n) \to 0$.

Now consider the second alternative (ii) wherein, along some sequence, we have that a point of infinite (negative) curvature is developing. Then, by the dynamic boundary condition (3.1.4), we see that

$$P_n - P_{\mathrm{atm}} = -\sigma \eta_n'' \left(1 + (\eta_n')^2\right)^{-3/2},$$

and so $\max_{\eta_n}(P_n - P_{\mathrm{atm}}) \to \infty$. In other words, as the free surface develops a corner point, the pressure at that points (relative to the atmospheric pressure) goes to positive infinity. This, we anticipate, leads to a blow-up in the energy on the free surface. Indeed, from the definition of Q in the Eulerian framework, we obtain

$$\begin{aligned} Q_n &= 2\left(E_n|_{\eta_n} - P_{\mathrm{atm}} + g\rho_n|_{\eta_n} d_n\right) \\ &= 2\left(P_n - P_{\mathrm{atm}}\right) + \left[\rho_n\left((u_n - c)^2 + v_n^2\right) + 2g\rho_n\left(\eta_n + d_n\right)\right]\Big|_{\eta_n} \\ &> 2\left(P_n - P_{\mathrm{atm}}\right). \end{aligned}$$

Thus alternative (ii) reduces to alternative (i).

Next, suppose that for some decreasing sequence $\delta_n \to 0$ the fourth (iv) alternative holds. Again, appealing to (3.2.11), we see immediately that the corresponding sequence in the Eulerian framework, $(u_n, v_n, \rho_n, \eta_n)$, must satisfy $\sup_{\overline{D_{\eta_n}}} \sqrt{\rho_n}(c - u_n) \to \infty$ as $n \to \infty$.

Finally, if we have a sequence $\delta_n \to 0$ for which the fifth alternative (v) holds, then there exists a sequence $\{(Q_n, h_n)\}$ with $(Q_n, h_n) \in \mathcal{C}'_{\delta_n}$ for each $n \geq 1$ and $\sup_T(Q_n - 2g\rho h_n 2\sigma\kappa[h_n]) \to 0$. If we evaluate the boundary condition on T for this sequence, we find

$$\max_T \left(\frac{1}{(\partial_p h_n)^2}\right) \leq \max_T \left(\frac{1 + (\partial_q h_n)^2}{(\partial_p h_n)^2}\right) = \max_T \left(Q_n - 2g\rho_n h_n - 2\sigma\kappa[h_n]\right).$$

We infer, therefore, that $\max_{\overline{R}} \partial_p h_n \to \infty$, and hence the fifth alternative implies the third.

By construction, the family of continua $\mathcal{C}'_\delta$ indexed by $\delta > 0$ is increasing as δ decreases. We may therefore define $\mathcal{C}' := \bigcup_{\delta>0} \mathcal{C}'_\delta$ to be the maximal continuum. Then by the considerations of the §2, and in particular Lemma 3.2.1, there exists a connected set $\mathcal{C}$ of solutions to (3.1.1)-(3.1.6) corresponding to $\mathcal{C}'$. The arguments of the preceding paragraphs, moreover, imply that along some sequence in $\mathcal{C}$, either (i) $\sup_{\overline{D_{\eta_n}}} u_n \to c$, or (ii) $\inf_{\overline{D_{\eta_n}}} u_n \to -\infty$, or (iii) $\mathcal{C}$ contains more than one laminar flow solution. This proves part (a).

Part (b) follows from part (a), taking into account the remarks following THEOREM 3.6.2. That is, for each $\delta > 0$, we have a path-connected set $\mathcal{K}'_\delta \subset \mathcal{C}'_\delta$ where, along some path one of alternative (i)–(v) occurs or $\mathcal{K}'_\delta$ is a closed loop. Arguing as before, it is clear that, if $\mathcal{K}' := \bigcup_{\delta>0} \mathcal{K}'_\delta \subset \mathcal{C}'$, then there is a corresponding path-connected set $\mathcal{K} \subset \mathcal{C}$ of solutions to (3.1.1)-(3.1.6). It follows that either $\mathcal{K}$ is a closed loop, or along some path in $\mathcal{K}$ we have either (i) $\sup_{\overline{D_\eta}} u \to c$, or (ii) $\inf_{\overline{D_\eta}} u \to -\infty$.

Finally, (c) is an automatic consequence of the definition of the space X. The proof is complete. □

PROOF OF THEOREM 3.1.3. Fix $\delta > 0$ and $i \in \{1, 2, 3, 4\}$. In view of THEOREM 3.6.2 and THEOREM 3.7.1, the following list of alternatives is exhaustive.

(i) $\mathfrak{Q}_i(s) \to \infty$, as $s \to \infty$; or

(ii) $\min_T \kappa[\mathfrak{h}_i(s)] \to -\infty$, as $s \to \infty$; or

(iii) $\max_{\overline{R}} \partial_p \mathfrak{h}_i(s) \to \infty$, as $s \to \infty$; or

(iv) $\min_{\overline{R}} \partial_p \mathfrak{h}_i(s) \to \delta$, as $s \to \infty$; or

(v) $\min_T (\mathfrak{Q}_i(s) - 2g\rho\mathfrak{h}_i(s) - 2\sigma\kappa[\mathfrak{h}_i(s)]) \to \delta$, as $s \to \infty$; or

(vi) $\mathcal{K}_{\delta,i}$ is a closed loop.

Here $(\mathfrak{Q}_i(\cdot), \mathfrak{h}_i(\cdot))$ is a global parameterization of $\mathcal{K}_{\delta,i}$ in the sense of THEOREM 3.6.1 (by abuse of notation we have suppressed the dependency of the parameterization on δ).

In light of LEMMA 3.2.1, to $\mathcal{K}'_{\delta,i}$ there is associated a path-connected set $\mathcal{K}_{\delta,i}$ of solutions to the problem in its Eulerian formulation (3.1.1)-(3.1.8). By the same arguments as in the proof of THEOREM 3.1.1, we deduce that the first three alternatives (i)–(iii) imply that one of the following occurs:

$$\sup_{\overline{D_{\eta_n}}} u_n \to c \qquad \text{or} \qquad \inf_{\overline{D_{\eta_n}}} u_n \to -\infty,$$

for some sequence $\{(u_n, v_n, \rho_n, \eta_n, Q_n)\} \subset \mathcal{K}_{\delta,i}$. The same holds true, again by the considerations of THEOREM 3.1.1, if either alternative (iv) or alternative (v) hold for a decreasing sequence δ_n with $\delta_n \to 0$. If we take $\mathcal{K}_i := \bigcup_{\delta>0} \mathcal{K}_{\delta,i}$ and $\mathcal{K} := \bigcup_i \mathcal{K}_i$, then the theorem is proved. □

REMARK. Naively, one might hope that since each $\mathcal{K}'_{\delta,i}$ has an analytic parameterization, the same might be true of the associated curves $\mathcal{K}_{\delta,i}$. Unfortunately, this does not seem to be true in general. The difficulty lies in LEMMA 3.2.1, where the key step in transitioning from the height equation formulation back to the Eulerian is to reconstruct ψ from (h,Q). In [**15, 76**], this is accomplished as follows. Let $(\mathfrak{Q},\mathfrak{h})$ be an analytic parameterization of $\mathcal{K}'_{\delta,i}$ and, fixing $s\in\mathbb{R}^+$, let $h=\mathfrak{h}(s)$. In view of (3.2.11), for each $x_0\in\mathbb{R}$, we define $\psi(x_0,\cdot;s)$ to be the unique solution of the first-order ODE:

$$\begin{cases} \psi_y(x_0,y;s) = -h_p^{-1}\left(x_0,-\psi(x_0,y;s)\right) \\ \psi(x_0,h(x_0,0)-d(h);s)=0. \end{cases} \tag{3.8.2}$$

Following the arguments of [**15, 76**], we can show that this ψ indeed corresponds to the pseudo-stream function for a solution of (3.1.1)–(3.1.8), thus obtaining Lemma 3.2.1 for fixed s. Varying s gives a curve of pseudo-stream functions and the path-connected set $\mathcal{K}_{\delta,i}$. But, since $\mathfrak{h}(s)$ need not be analytic in p, we *cannot* conclude from (3.8.2) that $\psi(\cdot,\cdot;s)$ is analytic in s.

3.9. Examples

In this section, we consider the implications of our results in a a few of the most important special cases for the given functions ρ and β.

3.9.1. Pure capillary waves.

Examining (3.1.7)–(3.1.8), it is apparent that the contribution of surface tension comes in the form of a term involving second-order spatial derivatives, whereas the gravitational constant appears as the coefficient of first-order derivative terms. When considering flows in which the wave length is small with respect to σ, therefore, we expect the motion to be driven largely by the effects of capillarity. In such circumstances, it is often appropriate to discount the presence of gravity entirely, both in the interior of the fluid and on the free surface. A common example is that of a pebble being thrown into a quiescent body of water; the ripples that result are typically small enough in width as to be accurately modeled as pure capillary waves.

Strictly speaking, of course, pure capillary waves lie outside of the regime we are considering in this chapter, as we have taken g to be a strictly positive constant. Nonetheless, our arguments can be easily altered to include this scenario. Indeed, defining the pseudostream function and proceeding as before, one arrives at a height equation formulation analogous to (3.2.13):

$$
\begin{cases}
(1+h_q^2)h_{pp}+h_{qq}h_p^2-2h_qh_ph_{pq}=-h_p^3\beta(-p) & p_0<p<0,\\
1+h_q^2+h_p^2(2\sigma\kappa[h]-Q)=0 & p=0,\\
h=0 & p=p_0.
\end{cases}
\tag{3.9.1}
$$

As one would expect — and, indeed, (3.9.1) confirms — ignoring gravitational effects completely negates the significance of stratification. That is, for every solution of (3.9.1) and any choice of streamline density function, there corresponds a solution to (3.1.1)–(3.1.8) (cf. [**82**]).

The system of equations (3.9.1) was first considered by Wahlén in [**71**], where he used it to give a systematic study of small amplitude solutions. In his thesis, Wahlén expanded upon these results, proving a global bifurcation theorem in the same vein as THEOREM 3.5.1 (cf. [**73**]). As an easy byproduct of our work in sections 3.6, 3.7 and 3.8, we are now able to significantly extend upon this result.

Let us first begin by presenting the key results for the local theory developed in [**71**]. In keeping with our previous notation, we have written these in terms of the Bernoulli function β, but it is worth noting that, since $\rho_p \equiv 0$ for pure capillary waves, β reduces to the familiar vorticity function γ of [**15, 16, 71, 70, 72**].

The first step is to establish the existence of a curve of laminar flow solutions, i.e. those where $h_q \equiv 0$. In light of the boundary condition on the top, this task is somewhat simpler than for capillary-gravity waves. In fact, there is no need to introduce an additional parametrizing variable.

LEMMA 3.9.1. (Wahlén, [**71**]) *For each Q with $0 \le -2B_{min} < Q$, the corresponding laminar flow solution $H = H(p;Q)$ of (3.2.13) is given by*

$$H(p;Q) := \int_{p_0}^{p} \frac{ds}{\sqrt{Q+2B(s)}}, \qquad p_0 \le p \le 0.$$

The existence of small amplitude solutions is proved by a bifurcation argument as was carried out in section 3.3, with the end result being the following.

THEOREM 3.9.1. (Wahlén, [**71**]) *Let the wave speed $c>0$, the relative mass flux $p_0<0$, coefficient of surface tension σ and the Bernoulli function $\beta \in C^{1+\alpha}([0,|p_0|])$ and $\alpha \in (0,1)$*

be given. Then, if

$$\sup_{p_1 \in M} \left(\frac{\sigma p_1^2 - \int_{p_1}^0 (p-p_1)^2 (2B(p) - 2B_{min})^{1/2} dp}{\int_{p_1}^0 (2B(p) - 2B_{min})^{3/2} dp} \right)^{1/2} > 1, \tag{3.9.2}$$

where

$$M := \left\{ p_1 \in [p_0, 0] : \sigma p_1^2 > \int_{p_1}^0 (p-p_1)^2 (2B(p) - 2B_{min}) dp \right\},$$

there exists a C^1-curve $\mathcal{D}_{loc} \subset \mathscr{S}$ of small amplitude traveling wave solutions (u, v, η, Q) of (3.1.1)-(3.1.6) *with period 2π, speed c and relative mass flux p_0 , satisfying $u < c$ throughout the fluid. The curve $\mathcal{D}_{loc}$, moreover, exhibits the structural properties of* Theorem 3.3.1 (i)–(iii).

Note that the statement above differs from the theorem presented in [**71**] in two ways. First, we have specialized to the case where $L = 2\pi$. This is done simply as a matter of taste and to stress similarities with [**15**] and chapter 2. More importantly, we have increased the required regularity of the solutions $(u, v, \eta) \in \mathcal{D}_{\text{loc}}$ and the vorticity function by one (that this is valid is straightforward to show, by the remarks following the statement of (3.1.1).) The motivation for working with more regular solutions is merely to allow us to later employ Theorem 3.7.1.

Also, let us draw attention to the fact that, for pure capillary waves, the bifurcation from $\mathcal{T}$ is always simple. This is to be expected as sending g to zero in (3.3.16), we find $\sigma_c = 0$ so that $\Sigma_1 = \mathbb{R}^+$ and $\Sigma_2 \cup \Sigma_3 = \emptyset$. Interestingly, then, double bifurcation is truly a capillary-gravity wave phenomenon: it is absent in both pure capillary wave and gravity wave regimes.

Now, observe that the arguments of section 3.5 does not in any way rely on assuming g to be strictly positive. In fact, by taking $g = 0$ and replacing the local theory of section 3.3 with that above, we recreate exactly the global bifurcation theorem for pure capillary waves proved by Wahlén in [**73**]. Let $\delta > 0$ be given and define $\mathcal{D}_\delta'$, $\mathcal{O}_\delta$ and S_δ as in section 3.5, but with $\mathcal{D}_{\text{loc}}$ in place of $\mathcal{C}_{\text{loc}}$.

Theorem 3.9.2. (Wahlén, [**73**]) *Let $\delta > 0$ be given and let $\mathcal{D}_\delta$ be as described above. One of the following alternatives must hold:*

(i) $\mathcal{D}_\delta'$ is unbounded in $\mathbb{R} \times X$.

(ii) $\mathcal{D}_\delta'$ intersects $\mathcal{T}$ at more than one point.

(iii) $\mathcal{D}'_\delta$ *contains a point* $(Q, h) \in \partial\mathcal{O}_\delta$.

Fortunately, our arguments in the remaining sections are similarly indifferent to the strict positivity of g. For instance, taking $g = 0$ and running through the proofs of section 3.7 verbatim yields a theorem identical to THEOREM 3.7.1 but for pure capillary waves. One way of interpreting this robustness is to note that, from a technical standpoint, the difficulty introduced by stratification and capillarity are neatly confined to the interior and boundary, respectively. They must therefore be treated more-or-less independently, and hence we have already done the heavy lifting necessary to obtain the corresponding results for the pure capillary case.

Combining these results (and likewise generalizing the proof of THEOREM 3.1.1) yields the following.

THEOREM 3.9.3. *Fix a wave speed* $c > 0$, *wavelength* $L > 0$, *relative mass flux* $p_0 < 0$ *and coefficient of surface tension* $\sigma > 0$. *Fix any* $\alpha \in (0, 1)$, *and let the function* $\beta \in C^{3+\alpha}([0, |p_0|])$ *be given such that condition* (3.9.2). *Let* $\rho \in C^{1+\alpha}([p_0, 0]; \mathbb{R}^+)$ *be any choice of streamline density function.*

There exists a connected set $\mathcal{D} \subset \mathscr{S}$ *of solutions* (u, v, ρ, η, Q) *of the traveling, stratified, pure capillary wave problem. The set* $\mathcal{D}$, *moreover, exhibits all of the structural properties described in* THEOREM 3.1.1 (a)–(c).

This theorem represents a substantial strengthening of THEOREM 3.9.2, elucidating as it does the meaning of alternatives (i) and (iii) in the Eulerian framework, as well as proving the existence of a path-connected subset.

3.9.2. Irrotational homogeneous capillary-gravity waves. This simplest case to consider with $\sigma > 0$ is when $\beta \equiv 0$ and $\rho \equiv \rho_0$, a positive constant. Since $\rho_p \equiv 0$, the laminar problem simplifies greatly and we are able to solve it explicitly. In particular, for $\lambda \geq 0$, $p_0 < p < 0$, we have

$$H(p; \lambda) = \frac{p - p_0}{\sqrt{\lambda}}, \qquad Q(\lambda) = \lambda + \frac{2g\rho_0|p_0|}{\sqrt{\lambda}}.$$

Thus $\lambda_0 = (g\rho_0|p_0|)^{2/3}$ and $a = \sqrt{\lambda}$. Furthermore, since we are working in the constant density regime, λ_c must coincide with λ_0, whence

$$\sigma_c = \frac{1}{3}(g\rho_0)^2|p_0|^3\lambda_0^{-5/2} = \frac{1}{3}(g\rho_0)^{1/3}|p_0|^{4/3}.$$

Arguing as we have in the previous sections, we see that there exists a local curve of non-laminar solutions of period $2\pi/n$ bifurcating from $\mathcal{T}$ at $(H(\cdot;\lambda_*),\lambda_*)$, for some $n \in \mathbb{N}^\times$, $\lambda_* > 0$, provided there exists $M = M(p)$ satisfying

$$\begin{cases} a^3 M'' = n^2 a M, & p_0 < p < 0, \\ M(p_0) = 0, \\ \lambda_*^{3/2} M'(0) = (g\rho_0 + n^2\sigma) M(0). \end{cases}$$

From the first equation above we infer that any such M must take the form

$$M(p) = \sinh\left(\frac{n(p-p_0)}{\sqrt{\lambda}}\right), \qquad p_0 < p < 0.$$

Imposing the boundary conditions on the top, it follows that bifurcation will occur for (n,λ_*) satisfying the following relation

$$\lambda_* = \frac{n^2\sigma + g\rho_0}{n} \tanh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right). \tag{3.9.3}$$

Note that, for fixed n, there is always a unique choice of λ_* for which (3.9.3) will hold. This corresponds to the fact that, for constant density and zero vorticity, the Capillary-Gravity Local Bifurcation Condition is satisfied automatically. However, it may very well be the case that (n_1,λ_*) and (n_2,λ_*) each satisfy (3.9.3) for some $\lambda_* > 0$ and distinct $n_1, n_2 \in \mathbb{N}^\times$.

For definiteness, take $n_1 = 1$ and let λ_* be chosen so that $(1,\lambda_*)$ satisfies relation (3.9.3). If there is no $n_2 > 1$ such that (3.9.3) holds for (n_2,λ_*), then $\sigma \in \Sigma_1$ and we are in the simple bifurcation setting described by Theorem 3.3.1. In view of Lemma 3.3.6 we know that simple bifurcation will occur whenever $\sigma \geq \sigma_c$. This fact can be seen directly from (3.9.3) since, for any such σ, we must have that λ is strictly increasing as a function of n.

On the other hand, there is a set Σ_2 of σ lying to the left of σ_c for which there *will* be an $n_2 > 1$ such that (n_2,λ_*) satisfies relation (3.9.3). In this case, double bifurcation occurs at $(H_*, Q(\lambda_*))$ and we are in the setting of Theorem 3.1.2. As we might expect, the set Σ_2 is highly non-generic. Indeed, the following result shows that it has measure zero.

Proposition 3.9.1. *For homogeneous, irrotational, capillary-gravity waves, Σ_2 is countable. In fact, it consists entirely of isolated points.*

Proof. Let $\sigma_* \in \Sigma_2$ be given. Since (L-B) is always satisfied for irrotational waves, for each $\sigma \in \mathbb{R}^+$, there exists a unique $\lambda_* > \epsilon_0$ for which

$$-(a^3\phi_1')' + g\rho'\phi_1 = \mu a \phi_1, \text{ for } p_0 < p < 0, \qquad \phi(p_0) = 0, \ \lambda_*^{3/2}\phi_1'(0) = (-\mu\sigma + g\rho(0))\phi_1(0).$$

has a (nontrivial) solution ϕ_1, where $\mu = \mu_1 = -1$. By continuous dependence on parameters, we may view λ_* as a smooth function of σ.

As σ_* is a point of double bifurcation, there exists precisely one other nonpositive integer, $\mu_2 = -n_2^2$, for which the above eigenvalue problem has nontrivial solution with $\mu = \mu_2$. Clearly $n_2 \neq 1$, since this would entail simple bifurcation. Let us first consider the case where $n_2 \geq 2$.

Again appealing to continuous dependence, we may consider the negative, real eigenvalues μ_1 and μ_2 to be smooth functions of (λ, σ) defined in a sufficiently small neighborhood $\mathcal{U}_1 \times \mathcal{U}_2$ of (λ_*, σ_*). Unravelling definitions, we see that $\sigma \in \mathcal{U}_2$ is an element of Σ_2 if and only if $\sqrt{-\mu_2(\lambda_*(\sigma), \sigma)}$ is a positive integer. It follows that to show Σ_2 is countable it suffices to confirm

$$\left.\frac{\partial(\mu_2)}{\partial\sigma}\right|_{(\lambda_*, \sigma_*)} \neq 0.$$

We now prove this claim. Evaluating (3.9.3) for $n = 1$, we find

$$\lambda_* = (\sigma + g\rho_0) \tanh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right). \tag{3.9.4}$$

Differentiating the relation in σ and rearranging terms, we are able to compute

$$\begin{aligned} \frac{d\lambda_*}{d\sigma} &= \tanh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\left(1 + (\sigma + g\rho_0)\frac{|p_0|}{2\lambda_*^{3/2}}\operatorname{sech}^2\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\right)^{-1} \\ &= \lambda_*\left(\sigma + g\rho_0 + (\sigma + g\rho_0)^2\frac{|p_0|}{2\lambda_*^{3/2}}\operatorname{sech}^2\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\right)^{-1}, \end{aligned} \tag{3.9.5}$$

which is strictly positive. (In fact, we already knew this in the general case by Lemma 3.3.7.)

Let us denote $n(\sigma) := \sqrt{-\mu_2(\lambda_*(\sigma), \sigma)}$, for each $\sigma \in \mathcal{U}_2$ (in particular, observe that $n(\sigma_*) = n_2$.) From (3.9.3), therefore, we have

$$n\lambda_* = (n^2\sigma + g\rho_0) \tanh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right). \tag{3.9.6}$$

Differentiating (3.9.6) by σ, we obtain

$$\begin{aligned} \frac{dn}{d\sigma}\lambda_* + n\frac{d\lambda_*}{d\sigma} &= \left(2n\frac{dn}{d\sigma}\sigma + n^2\right)\tanh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right) \\ &\quad + (n^2\sigma + g\rho_0)\operatorname{sech}^2\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)\left(\frac{|p_0|}{\sqrt{\lambda_*}}\frac{dn}{d\sigma} - \frac{n|p_0|}{2\lambda_*^{3/2}}\frac{d\lambda_*}{d\sigma}\right). \end{aligned}$$

Upon combining like terms, this simplifies to

$$C_1 \frac{dn}{d\sigma} = C_2,$$

where

$$\begin{aligned} C_1 &:= \lambda_* - 2n\sigma \tanh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right) - (n^2\sigma + g\rho_0)\frac{|p_0|}{\sqrt{\lambda_*}}\operatorname{sech}^2\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right), \\ C_2 &:= n^2 \tanh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right) - n\frac{d\lambda_*}{d\sigma} - (n^2\sigma + g\rho_0)\frac{n|p_0|}{2\lambda_*^{3/2}}\operatorname{sech}^2\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)\frac{d\lambda_*}{d\sigma}. \end{aligned} \tag{3.9.7}$$

All that remains is to prove $C_2 \neq 0$.

Using (3.9.5), we may write

$$\begin{aligned} C_2 &= n\left[n\tanh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right) - \frac{d\lambda_*}{d\sigma}\left(1 + (n^2\sigma + g\rho_0)\frac{|p_0|}{2\lambda_*^{3/2}}\operatorname{sech}^2\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)\right)\right] \\ &= n\tanh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\left[\frac{n\tanh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)}{\tanh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)} - \frac{1 + (n^2\sigma + g\rho_0)\frac{|p_0|}{2\lambda_*^{3/2}}\operatorname{sech}^2\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)}{1 + (\sigma + g\rho_0)\frac{|p_0|}{2\lambda_*^{3/2}}\operatorname{sech}^2\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)}\right] \\ &=: n\tanh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\left[C_3 - C_4\right]. \end{aligned} \tag{3.9.8}$$

Returning to (3.9.4) and (3.9.6), we see

$$(n^2\sigma + g\rho_0)\operatorname{sech}^2\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right) = n\lambda_* \operatorname{sech}\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)\operatorname{csch}\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right),$$

$$(\sigma + g\rho_0)\operatorname{sech}^2\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right) = \lambda_* \operatorname{sech}\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\operatorname{csch}\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right).$$

Thus, C_4 can be estimated

$$\begin{aligned} C_4 &= \frac{\sinh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\cosh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)}{\sinh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\cosh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right) + \frac{|p_0|}{2\lambda_*^{3/2}}} + n\frac{\sinh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\cosh\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)}{\sinh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)\cosh\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)} \\ &\leq 1 + C_3\sinh^2\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\operatorname{csch}^2\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right). \end{aligned} \tag{3.9.9}$$

Observe, however, that the map

$$n \mapsto \sinh^2\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right)\operatorname{csch}^2\left(\frac{n|p_0|}{\sqrt{\lambda_*}}\right)$$

has a global upper bound of $1/4$ for $n \geq 2$ and any values of $|p_0|, \lambda_* > 0$. Also, since $\tanh\left(n|p_0|/\sqrt{\lambda_*}\right) > \tanh\left(|p_0|/\sqrt{\lambda_*}\right)$, we have that $C_3 > n \geq 2$. From (3.9.9) it then follows that $C_4 < 3C_3/4$, whence (3.9.8) implies $C_2 > 0$. Altogether, then, we have proved that $dn/d\sigma < 0$. It follows that the points $\sigma \in \Sigma_2$ where $n_2 \geq 2$ are isolated. $\square$

Let us investigate further the case when $\sigma \in \Sigma_2$. For simplicity, let us suppose $n_2 > 2$ and fix $\lambda = \lambda_*$. Note that, by evaluating the right-hand side of (3.9.3) for $n = 1$ and $n = 2$, we find that nondeneracy condition (3.4.1) is equivalent to

$$\frac{4\sigma + g\rho_0}{2\sigma + 2g\rho_0} \tanh\left(\frac{2|p_0|}{\sqrt{\lambda_*}}\right) \operatorname{cotanh}\left(\frac{|p_0|}{\sqrt{\lambda_*}}\right) \neq 1.$$

Now, recalling our earlier notation, we compute, for $i = 1, 2$,

$$\begin{aligned} \frac{1}{\pi}\Psi_{ii} &= -\frac{n_i^2|p_0|}{2\sqrt{\lambda}} - g\rho_0\lambda^{-1}\sinh^2\left(\frac{n_i|p_0|}{\sqrt{\lambda}}\right) - \sigma n_i^2\lambda^{-1} \\ &= -\frac{n_i^2|p_0|}{2\sqrt{\lambda}} - \frac{n_i}{2}\sinh\left(\frac{2n_i|p_0|}{\sqrt{\lambda}}\right), \end{aligned} \tag{3.9.10}$$

$$\begin{aligned} \frac{1}{\pi}\Theta_{iiii} &= \frac{n_i^2\sqrt{\lambda}}{4}\left(\frac{7|p_0|}{4} - \frac{\sqrt{\lambda}}{4n_i}\sinh\left(\frac{2n_i|p_0|}{\sqrt{\lambda}}\right) - \frac{5\sqrt{\lambda}}{16n_i}\sinh\left(\frac{4n_i|p_0|}{\sqrt{\lambda}}\right)\right) \\ &\quad + \frac{3n_i^2}{4\sqrt{\lambda}}\sinh\left(\frac{n_i|p_0|}{\sqrt{\lambda}}\right)\cosh^3\left(\frac{n_i|p_0|}{\sqrt{\lambda}}\right), \end{aligned} \tag{3.9.11}$$

$$\begin{aligned} \frac{1}{\pi}\Theta_{iijj} &= \frac{1}{2}\sqrt{\lambda}\left(\frac{1}{2}n_i^2n_j^2|p_0| - \frac{1}{4}n_in_j^2\sqrt{\lambda}\sinh\left(\frac{2n_i|p_0|}{\sqrt{\lambda}}\right)\right) \\ &\quad + n_in_j^3\sqrt{\lambda}\left(\frac{\sinh\left(\frac{2(n_i-n_j)|p_0|}{\sqrt{\lambda}}\right)}{8(n_j-n_i)} + \frac{\sinh\left(\frac{2(n_i+n_j)|p_0|}{\sqrt{\lambda}}\right)}{8(n_i+n_j)}\right) \\ &\quad + \frac{n_in_j^2}{2\sqrt{\lambda}}\sinh\left(\frac{n_i|p_0|}{\sqrt{\lambda}}\right)\cosh\left(\frac{n_i|p_0|}{\sqrt{\lambda}}\right)\cosh^2\left(\frac{n_j|p_0|}{\sqrt{\lambda}}\right). \end{aligned} \tag{3.9.12}$$

Observe that the quantity in (3.9.10) is strictly negative, as expected. The signs and relative sizes of the Θ_{iiii} and Θ_{iijj} are more difficult to ascertain in general, but inserting (3.9.10)–(3.9.12) into (3.4.2) yields an explicitly verifiable condition under which Theorem 3.1.2 holds. We omit the details.

To obtain the dispersion relation for these solutions, let U_* denote the horizontal velocity corresponding to the laminar flow H_*. Then, by the definition of p_0,

$$|p_0| = \int_{-d}^0 \sqrt{\rho_0}\,(c - U_*)\,dy = \int_{-d}^0 (\partial_p H_*)^{-1}\,dy = d\sqrt{\lambda_*}.$$

Writing (3.9.3) in terms of the depth and the horizontal velocity, therefore, we obtain

$$c - U_* = \sqrt{\frac{n^2\sigma + g\rho_0}{n} \tanh{(nd)}}.$$

CHAPTER 4

Symmetry of stratified steady water waves

4.1. Introduction

One of the characteristic features of ocean waves is their propensity to display stratification — a heterogeneous distribution of density that can result from the interplay between gravity and the salinity of the water. In a typical scenario, one observes a region just below the surface where density increases quickly with depth, followed by a larger region extending to the ocean bed where it is essentially constant (cf. [**57**]). Qualitatively, flows with heterogeneous density may differ quite strikingly from their homogeneous counterparts. For instance, slowly moving stratified traveling waves have a strong tendency to be two-dimensional. That is, the motion is identical along any line perpendicular to the direction of propagation and the vertical axis [**82**]. Remarkably, this columnar behavior will persist even with a foreign body, such as an infinitely long cylinder, placed vertically within the fluid.

Unsurprisingly, therefore, stratification has been the subject of a great deal of scholarly interest, especially in the oceanography and geophysical fluid dynamics communities. In [**76, 75**], the author developed an existence theory for two-dimensional stratified steady and periodic gravity waves, with or without surface tension (presented here in chapters 2–3.) It was shown that, under certain unrestrictive assumptions, there is a global continuum of such waves that are classical. This was done via a bifurcation argument, with the 1-parameter family of laminar flows playing the role of the trivial solutions. As a consequence of this construction, it was shown that each member of the continuum has monotonic surface profile and is symmetric. That is, the wave profile possesses a single crest and a single trough per period and is monotonic between them. Moreover, both the horizontal velocity and surface are symmetric across the crest line, whereas the vertical velocity is antisymmetric. In this work we address the complementary question: Under what circumstances are stratified steady water waves with monotonic profiles necessarily symmetric? While at first blush it may seem that waves of this type are quite special, our main result implies that, in fact, they are the only possible solutions within certain size regimes.

Let us now briefly review the setup for stratified waves in chapter 2. Fix a Cartesian coordinate system so that the x-axis points in the direction of propagation, and the y-axis is vertical. We assume that the floor of the ocean is flat and occurs at $y = -d$. Let $y = \eta(x,t)$ be the free surface. We shall normalize η by choosing the axes so that the free surface is oscillating around the line $y = 0$. As usual we let $u = u(x,y,t)$ and $v = v(x,y,t)$ denote the horizontal and vertical velocities, respectively, and let $\rho = \rho(x,y,t) > 0$ be the density.

For water waves, it is appropriate to suppose that the flow is incompressible. Mathematically, this assumption manifests itself as the requirement that the vector field be divergence-free for all time,

$$u_x + v_y = 0. \tag{4.1.1}$$

Taking the fluid to be inviscid, conservation of mass implies that the density of a fluid particle remains constant as it follows the flow. This is expressed by the continuity equation

$$\rho_t + u\rho_x + v\rho_y = 0. \tag{4.1.2}$$

Conservation of momentum is described by Euler's equations

$$\begin{cases} \rho u_t + \rho u u_x + \rho v u_y &= -P_x, \\ \rho v_t + \rho u v_x + \rho v v_y &= -P_y - g\rho, \end{cases} \tag{4.1.3}$$

where $P = P(x,y,t)$ denotes the pressure and g is the gravitational constant. Here we are assuming that gravity is the only external force acting on the fluid.

On the free surface, the dynamic boundary condition requires the pressure of the fluid to match atmospheric pressure, that we shall denote P_{atm}. Thus,

$$P = P_{\mathrm{atm}}, \qquad \text{on } y = \eta(x,t). \tag{4.1.4}$$

The corresponding condition for the vector field is the so-called kinematic condition

$$v = \eta_t + u\eta_x, \qquad \text{on } y = \eta(x,t). \tag{4.1.5}$$

In essence, (4.1.5) states that fluid particles on the free surface remain there as the flow develops. Finally, we assume that the ocean bed is impermeable and thus,

$$v = 0, \qquad \text{on } y = -d. \tag{4.1.6}$$

Note that, since we are in the inviscid regime, we do not impose any conditions on u along the bed.

We are considering classical traveling periodic wave solutions (u, v, ρ, P, η) to (4.1.1)–(4.1.6). More precisely, we take this to mean that, for some fixed $c > 0$, the solution appears steady in time and periodic in the x-direction when observed in a frame that moves with constant speed c to the right. The vector field will thus take the form $u = u(x - ct, y)$, $v = v(x - ct, y)$, where each of these is L-periodic in the first coordinate. Likewise for the scalar quantities: $\rho = \rho(x - ct, y)$, $P = P(x - ct, y)$, and $\eta = \eta(x - ct)$, again with L-periodicity in the first coordinate. We therefore take moving coordinates

$$(x - ct, y) \mapsto (x, y),$$

which eliminates time dependency from the problem. In the moving frame (4.1.1)–(4.1.3) become

$$\begin{cases} u_x + v_y & = & 0 \\ (u-c)\rho_x + v\rho_y & = & 0 \\ \rho(u-c)u_x + \rho v u_y & = & -P_x \\ \rho(u-c)v_x + \rho v v_y & = & -P_y - g\rho \end{cases} \tag{4.1.7}$$

throughout the fluid domain. Meanwhile, the reformulated boundary conditions are

$$\begin{cases} v & = & (u-c)\eta_x & \text{on } y = \eta(x) \\ v & = & 0 & \text{on } y = -d \\ P & = & P_{\mathrm{atm}} & \text{on } y = \eta(x) \end{cases} \tag{4.1.8}$$

where (u, v, ρ, P) are taken to be functions of x and y, η is a function of x, and all of them are L-periodic in x.

Waves of this type will be called *symmetric* provided that there exists $x_0 \in \mathbb{R}$ such that (u, ρ, η) are symmetric over the line $x = x_0$, while v is antisymmetric. We shall call any local maximum of η a *crest*, and any local minimum a *trough*.

In the event that $u = c$ somewhere in the fluid we say that *stagnation* has occurred. This is a slight abuse of the term, since we do not know that the vertical velocity vanishes at the point as well. As will become apparent later, stagnation points may produce some degeneracy or instability in the problem. For that reason, we shall restrict our attention to the case where $u < c$ throughout the fluid.

Recall that we have chosen our axes so that η oscillates around the line $y = 0$. In other words,

$$\fint_{-L/2}^{L/2} \eta(x) dx = 0. \tag{4.1.9}$$

We shall also denote

$$\eta_{\max} := \max_{x \in \mathbb{R}} \eta(x) + d, \qquad \eta_{\min} := \min_{x \in \mathbb{R}} \eta(x) + d.$$

These are the maximum and minimum distances between the surface and the bed, respectively.

Observe that, by conservation of mass and incompressibility, ρ is transported and the vector field is divergence free. Therefore we may introduce a (relative) *pseudo-stream function* $\psi = \psi(x, y)$, defined uniquely up to a constant by:

$$\psi_x = -\sqrt{\rho} v, \qquad \psi_y = \sqrt{\rho}(u - c).$$

Here we have the addition of a ρ factor to the typical definition of the stream function for an incompressible fluid. This neatly captures the inertial effects of the heterogeneity of the flow [**82**]. The particular choice of $\sqrt{\rho}$ is merely an algebraic nicety.

It is a straightforward calculation to check that ψ is indeed a (relative) stream function in the usual sense, i.e. its gradient is orthogonal to the vector field in the moving frame at each point in the fluid domain. As usual, we shall refer to the level sets of ψ as the *streamlines* of the flow. Observe that, in assuming $u < c$, we have guaranteed that the streamlines are not closed. For definiteness we choose $\psi \equiv 0$ on the free boundary, so that $\psi \equiv p_0$ on $y = -d$, where p_0 is the (relative) *pseudo-volumetric mass flux*,

$$p_0 := \int_{-d}^{\eta(x)} \sqrt{\rho(x, y)} \, [u(x, y) - c] \, dy. \tag{4.1.10}$$

It is easy to check that p_0 is well-defined, i.e., the integral on the right-hand side above is independent of x. Examining (4.1.10), it is clear that p_0 describes the rate of fluid flowing through any vertical line extending from the bed to the free surface and with respect to the transformed vector field $(\sqrt{\rho}(u - c), \sqrt{\rho} v)$.

Since ρ is transported, it must be constant on the streamlines. Abusing notation we may therefore let $\rho \in C^2([p_0, 0]; \mathbb{R}^+)$ be given such that

$$\rho(x, y) = \rho(-\psi(x, y)) \tag{4.1.11}$$

throughout the fluid. When there is risk of confusion, we shall refer to the ρ occurring on the right-hand side above as the *streamline density function.* We shall focus our attention on the case where the density is nondecreasing as depth increases. This assumption is physically well-motived, since it is a prerequisite for hydrodynamic stability. Note that the level set $-\psi = p_0$ corresponds to the flat bed, and the set where $-\psi = 0$ corresponds to the free surface. For that reason, we shall say the flow is *stably stratified* provided that the streamline density function is nonincreasing.

By Bernoulli's theorem, the quantity

$$E := P + \frac{\rho}{2}\left((u-c)^2 + v^2\right) + g\rho y,$$

is constant along streamlines. To see this, note that by the conservation of mass (4.1.2),

$$\begin{aligned}((u-c)\partial_x + v\partial_y)\, E = (u-c)P_x + vP_y + \rho\Big(&(u-c)^2 u_x + (u-c)vv_x \\ &+ (u-c)vu_y + v^2 v_y\Big) + g\rho v.\end{aligned}$$

But, (4.1.7) implies

$$(u-c)P_x = -\rho(u-c)^2 u_x - \rho(u-c)vu_y, \qquad vP_y = -\rho(u-c)vv_x - \rho v^2 v_y - g\rho v,$$

whence

$$(u-c, v)\cdot \nabla E = 0.$$

Then, under the assumption that $u < c$ throughout the fluid, there exists a function $\beta \in C^1([0, |p_0|]; \mathbb{R})$ such that

$$\frac{dE}{d\psi}(x,y) = -\beta(\psi(x,y)). \tag{4.1.12}$$

For want of a better name we shall refer to β as the *Bernoulli function* corresponding to the flow. Physically it describes the variation of specific energy as a function of the streamlines. It is worth noting that when ρ is a constant, β reduces to the vorticity function.

We now define several quantities describing various properties of the wave. Let

$$D_\eta := \left\{(x,y) \in \mathbb{R}^2 : -d < y < \eta(x)\right\}$$

denote the fluid domain (in the moving frame) and, for each $k \in \mathbb{N}$, domain $\Omega \subset \mathbb{R}^2$, we introduce the space

$$C^k_{\mathrm{per}}(\overline{\Omega}) := \left\{ f \in C^k(\overline{\Omega}) : f(x+L, y) = f(x,y),\ \forall x, y \in \overline{\Omega} \right\}.$$

Put

$$a_0 := \min_{\overline{D_\eta}} \frac{\sqrt{\rho}(c-u)}{2} \left(1 + |\nabla\psi|^2 - \sqrt{(1+|\nabla\psi|^2)^2 - 4\left(\sqrt{\rho}\,(c-u)\right)^2}\right)$$

and

$$A_0 := \max_{\overline{D_\eta}} \sqrt{\rho}(c-u).$$

Roughly speaking, the first of these gauges how far away from stagnation the flow is (relative to the magnitude of the vector field), while the second gives a bound on u^-, the motion of the fluid counter to the direction of propagation. Note that the quantities in parentheses above are always positive. Next, we let

(4.1.13)
$$M := \max\left\{ \max_{\overline{D_\eta}} \left|\frac{v}{u-c}\right|,\ \max_{\overline{D_\eta}} \left|\left(\partial_x + \frac{v}{u-c}\partial_y\right)\frac{v}{u-c}\right|,\ \max_{\overline{D_\eta}} \left|\frac{1}{\sqrt{\rho}(u-c)}\partial_y \frac{v}{u-c}\right|\right\}.$$

M describes (in a sense that will be made clear later) the variation dy/dx following along a streamline. In particular, $M = 0$ only for the family of laminar flows, these being shear flows where the streamlines are parallel to the bed and $\eta \equiv 0$. The motivation here will become significantly more transparent in the next section, in particular, see (4.2.7) and the accompanying discussion.

Our main result is the following.

THEOREM 4.1.1. *Consider a stably stratified steady periodic wave train propagating at fixed speed c over a flat bed at $y = -d$ with relative pseudo-mass flux $p_0 < 0$. Let $\rho \in C^2([p_0, 0]; \mathbb{R}^+)$ and $\beta \in C^1([0, |p_0|])$ be the streamline density function and Bernoulli function associated with the flow, and let $(u, v) \in C^2_{per}(\overline{D_\eta}) \times C^2_{per}(\overline{D_\eta})$ be the vector field. Assume that the wave profile $y = \eta(x)$ is monotonic between crests and troughs with period L and that $u - c < 0$. Each of the following is a sufficient condition for the wave to be symmetric.*

(S1)
$$\sup \beta' + g\eta_{\max} \sup|\rho''| < \max\left\{\frac{\pi^2}{\eta_{\max}^2},\ \exp\left(-\min\{L,\ \eta_{\min}\}\right)\right\},$$

or

(S2)
$$\begin{cases} \epsilon\sqrt{\frac{\epsilon}{a_0}} + g\sup|\rho'| < \exp\left(-a_0^{-1/2}\min\{L, |p_0|\}\right), \\ \epsilon < a_0, \end{cases}$$

where

$$\epsilon := a_0^{-1}A_0^3M + 2A_0^3M^2 + \frac{3}{4}a_0^{-2}A_0^3\left(\sup|\beta| + g\eta_{\max}\sup|\rho'|\right) \\ + 2a_0^{-1}A_0^3M^3 + \frac{1}{2}a_0^{-2}A_0^3M^2 + 2A_0^2M + \frac{1}{4}a_0^{-3}A_0^3\left(\sup|\beta| + g\eta_{\max}\sup|\rho'|\right).$$

Remark. The first condition (S1) is a direct generalization of the main result of [**14**]. We mention that, in particular, if $\beta' \leq 0$ and $g\eta_{\max}\rho'' \leq 1$ then condition (S1) immediately applies. For β' or ρ'' small but positive, the Main Theorem implies that the wave is symmetric provided that the maximum elevation of the profile and/or the period are sufficiently small.

On the other hand, condition (S2) gives a criteria based on the distance from stagnation, the degree to which the flow is laminar, as well the magnitudes of the volumetric mass flux, Bernoulli function, density variation and the period. Generally, it states that solutions exhibiting small β and ρ' and which are sufficiently far from stagnant are necessarily symmetric.

Let us now discuss the history of this problem. The basic machinery we employ is the moving planes method, which was originated by Alexandroff [**2**]. Since then, this argument has been used extensively by a number of authors, notably Gidas, Nirenberg and Ni [**31**] and Serrin [**63**]. The first significant application of a moving planes argument to the study of water waves was by Craig and Sternberg [**17**], who considered the case of solitary waves in the irrotational setting.

The direct inspiration for our work is a number of recent breakthroughs in the study of symmetry for two-dimensional, constant density, rotational gravity water waves, beginning with a result of Constantin and Escher [**14**] analogous to condition (S1) of our main theorem. This was a generalization of an earlier result of Okamoto and Shōji, who showed in [**55**] that symmetry for irrotational waves must occur when the profile has a single crest and is monotonically decreasing. Soon thereafter, Hur, building on the work of Garabedian [**29**] and Toland [**67**] for irrotational waves, proved symmetry for arbitrary vorticity functions under the assumption that every streamline attains a minimum on the trough line [**33**]. While this last condition was not entirely physical, she also introduced the idea of working within the Dubreil–Jacotin formulation, which proved to be extremely convenient. As we shall see, the main advantage of Dubreil–Jacotin is that it demotes the vorticity function from a semilinear forcing term to the coefficient of a first-order term in a quasilinear equation, thereby nullifying its importance for symmetry. Refining Hur's argument, Constantin, Ehrnström and Wahlén were able to significantly improve upon the result in [**14**]. In [**13**], they showed that *every* two-dimensional rotational steady gravity wave with a monotonic profile is symmetric. In other words, symmetry follows without any assumptions on the vorticity function.

Work on the heterogeneous case has been comparatively sparse. The most complete results are due to Maia [**49**], who considered solitary internal waves confined to an infinite horizontal strip. It was assumed that the waves were quiescent at $+\infty$ and connected to a laminar flow at $-\infty$, where a piece-wise continuous density distribution was prescribed. Maia proved that, in this setting, waves of elevation are symmetric. Her argument, however, relies heavily on both the geometry of the domain and the type of boundary data at $\pm\infty$, and therefore does not extend to the free surface and periodic case which we consider.

It is worth noting that, in the stratified, deep water regime, there is a family of explicit solutions due to Dubreil–Jacotin [**26**]. This was done by building up from the well-known family of Gerstner waves, which are homogeneous, rotational solutions to the infinite depth analogue of (4.1.7)–(4.1.8) (cf. [**30, 12**]). Dubreil–Jacotin's construction hinged on the fact that for Gerstner waves the pressure is a function of the streamlines, a remarkable feature that is especially convenient in the heterogeneous density setting. We note that, like those constructed in chapter 2, Dubreil-Jacotin's explicit solutions are symmetric in the sense described above.

The structure of our argument is as follows. In §4.2.1 we begin by providing two scalar reformulations of (4.1.7)–(4.1.8); these are the natural generalizations to heterogeneous flows of the equations studied in [**13**] and [**14**]. In §4.2.2, we then derive estimates on the given quantities that ensure solutions of these equations satisfy a maximum principle. This sets the stage for the moving planes argument that we provide in §4.3.

4.2. Preliminaries

4.2.1. Two reformulations. It will be convenient to replace the system (4.1.7)-(4.1.8) with an equivalent scalar PDE for the stream function. Differentiating the expression for E and appealing to the definition of ψ, one can prove the following identity, known as Yih's equation or the Yih–Long equation [**82**],

$$\frac{dE}{d\psi} = \Delta\psi + gy\frac{d\rho}{d\psi}.$$

Rewriting the above expression in terms of the Bernoulli function β and streamline density function yields the following

$$-\beta(\psi) = \Delta\psi - gy\rho'(-\psi). \tag{4.2.1}$$

Moreover, evaluating Bernoulli's theorem on the free surface $\psi \equiv 0$, we find

$$|\nabla\psi|^2 + 2g\rho(-\psi)\,(\eta(x) + d) = Q, \qquad \text{on } y = \eta(x) \tag{4.2.2}$$

where the constant $Q := 2(E|_\eta - P_{\text{atm}} + gd)$. Note that Q gives roughly the energy density along the free surface of the fluid. Together with the fact that $\psi = -p_0$ on $y = -d$ and $\psi = 0$ on the free surface, (4.2.1)–(4.2.2) provide a complete reformulation of the problem as a semilinear elliptic equation.

It will also prove useful to consider the semi-Lagrangian (or Dubreil–Jacotin) formulation, wherein we trade some additional nonlinearity in exchange for a fixed domain. Consider the alternate coordinate system, (q, p), where

$$q = x, \qquad p = -\psi(x, y).$$

Then, under the transformation $(x, y) \mapsto (q, p)$, the closed fluid domain is mapped to the rectangle

$$\overline{R} := \{(q, p) \in \mathbb{R}^2 : -L/2 \le q \le L/2,\ p_0 \le p \le 0\}.$$

The purpose of the minus sign is simply to flip the rectangle so that the free surface will correspond to the top of R, while the flat bed will map to the bottom. Given this, we shall denote

$$T := \{(q, p) \in R : p = 0\}, \qquad B := \{(q, p) \in R : p = p_0\}.$$

Note that, in light of (4.1.11) and (4.1.12), we have that

$$\beta = \beta(-p), \qquad \rho = \rho(p).$$

Moreover, the assumption that the streamline density function is nonincreasing becomes

$$\rho' \le 0. \tag{4.2.3}$$

Next, following the ideas of Dubreil–Jacotin, define

$$h(q,p) := y + d \tag{4.2.4}$$

which gives the height above the flat bottom on the streamline corresponding to p and at $x = q$. We calculate:

$$\psi_y = -\frac{1}{h_p}, \qquad \psi_x = \frac{h_q}{h_p}, \tag{4.2.5}$$

and

$$\partial_q = \partial_x + h_q \partial_y, \qquad \partial_p = h_p \partial_y. \tag{4.2.6}$$

Working within the Dubreil–Jacotin formulation, we can now make precise the way in which M in (4.1.13) describes the variation of dy/dx along the streamlines. Using (4.2.5) to write (4.2.6) in terms of the Eulerian quantities (u, v), we see that (4.1.13) is equivalent to

$$M = \|h_q\|_{C^1(\overline{R})}. \tag{4.2.7}$$

Of course, h_q describes the variation of the height along a fixed streamline. For a laminar flow, where the streamlines are all flat, we therefore have $h_q, h_{qq}, h_{qp} \equiv 0$. In light of (4.2.7), we see that M gives the distance of the flow from laminar. The rather complicated expression for M in (4.1.13) is simply a consequence of re-expressing (4.2.7) in Eulerian variables.

Observe that (4.2.5) also implies $h_p > 0$, because we have stipulated that $u < c$ throughout the fluid. Yih's equation (4.2.1) becomes the following

$$-h_p^3 \beta(-p) = (1 + h_q^2) h_{pp} + h_{qq} h_p^2 - 2 h_q h_p h_{pq} - g(h - d) h_p^3 \rho' \tag{4.2.8}$$

where we have used (4.2.4) to write $y = h - d$. Recall, however, that we have normalized η so that it has mean zero. Taking the mean of (4.2.4) along T, we obtain

$$d = d(h) = \fint_{-L/2}^{L/2} h(q, 0)\, dq. \tag{4.2.9}$$

That is, the average depth d must be viewed as a linear operator acting on h. Namely, it is the average value of h over T. Where there is no risk of confusion, we shall suppress this dependency and simply write d.

Next consider the boundaries of the transformed domain. On the bed we must have, by the definition of h, that

$$h \equiv 0, \qquad \text{on } B. \tag{4.2.10}$$

In the new coordinates, the definition of Q given in (4.2.2) becomes the requirement

$$1 + h_q^2 + h_p^2\,(2g\rho h - Q) = 0, \qquad \text{on } T. \tag{4.2.11}$$

Altogether, then, combining (4.2.8), (4.2.10) and (4.2.11) we have that the fully reformulated problem is the following. Find $(h,\ Q) \in C^3_{\text{per}}(\overline{R}) \times \mathbb{R}$ satisfying

$$\begin{cases} (1+h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_qh_ph_{pq} & \\ \qquad -g(h-d(h))h_p^3\rho' = -h_p^3\beta(-p) & p_0 < p < 0 \\ 1 + h_q^2 + h_p^2(2g\rho h - Q) = 0 & p = 0, \\ h = 0 & p = p_0, \end{cases} \tag{4.2.12}$$

where $h_p > 0$. Here $\rho \in C^2([p_0, 0]; \mathbb{R}^+)$ and $\beta \in C^1([0, |p_0|])$ are given functions with $\rho' \leq 0$. The equivalence of (4.2.12) to the original formulation is proved in Lemma 2.2.1 by a straightforward generalization of the argument given by Constantin and Strauss for constant density rotational waves in [**15**].

4.2.2. Maximum principle. In anticipation of our argument in the next section, we now attempt to find an elliptic problem satisfied by the difference of two solutions of the height equation. Fix $h, \tilde{h}$ solutions to (4.2.12) with the same value of Q and suppose that $d(h) = d(\tilde{h})$. Define

$$\begin{aligned} \mathcal{L} \;:=\; & (1+h_q^2)h_p^{-3}\partial_p^2 + h_p^{-1}\partial_q^2 - 2h_qh_p^{-2}\partial_p\partial_q \\ & + \left(\tilde{h}_{qq}(h_p+\tilde{h}_p) - 2\tilde{h}_q\tilde{h}_{pq} + \beta(-p)(h_p^2 + h_p\tilde{h}_p + \tilde{h}_p^2)\right) h_p^{-3}\partial_p \\ & + \left(\tilde{h}_{pp}(h_q+\tilde{h}_q) - 2h_p\tilde{h}_{pq}\right) h_p^{-3}\partial_q \\ & - g\rho'(\tilde{h} - d(\tilde{h}))(h_p^2 + h_p\tilde{h}_p + \tilde{h}_p^2)h_p^{-3}\partial_p \\ & - g\rho'. \end{aligned} \tag{4.2.13}$$

Note that the first three lines of (4.2.13) are the same as the analogous operator found in [**13**], except that we have divided through by h_p^3; the stratification manifests itself only in the last two lines. Also, since $\rho' \leq 0$, the sign of the zeroth-order term "goes the wrong way" in the sense that, even taking for granted that $\mathcal{L}$ is elliptic, the maximum principle does not generally apply. The goal of this section is to find suitable restrictions on h so that a kind of maximum principle result *will* hold. Our main tool for this is a number of results due to Berestycki, Nirenberg and Varadhan [**5**]. We begin by summarizing the most relevant of these.

DEFINITION 1. Let $\mathcal{L}$ be an elliptic second-order linear differential operator with bounded continuous coefficients. The *principal eigenvalue* $\lambda_1 = \lambda_1(\mathcal{L})$ is defined as follows

$$\lambda_1 := \sup\{\lambda \in \mathbb{R} : \exists \phi \in C^2(R) \cap C(\overline{R}) \text{ satisfying } \phi > 0 \text{ and } (\mathcal{L}+\lambda)\phi \leq 0 \text{ in } R\}.$$

The fact that λ_1 is well-defined, i.e., that the supremum above exists, is a classical result. Our interest in λ_1 stems from the following theorem.

THEOREM 4.2.1 (Beresycki–Varadhan–Nirenberg, [**5**]). *Let $\mathcal{L}$ be an elliptic second-order linear differential operator with bounded continuous coefficients. Then $\mathcal{L}$ has a (refined) maximal principle if and only if $\lambda_1(\mathcal{L}) > 0$.*

The meaning of "refined" above is made precise in [**5**], but for our purposes it can be safely said to coincide with the traditional definition of the (weak) maximum principle.

In light of Theorem 4.2.1, our objective is clear: We seek to prove lower bounds on the principal eigenvalue of the operator in (4.2.13). The most basic such estimate is given by the following result of Protter and Weinberger [**59**].

THEOREM 4.2.2 (Protter–Weinberger, [**59**]). *Let $\mathcal{L}$ be a general second order linear differential operator with bounded coefficients. Then, if $\phi \in C^2(R) \cap C^1(\overline{R})$ is a strictly positive function on R,*

$$\lambda_1(\mathcal{L}) \geq \inf_R \left(-\frac{\mathcal{L}\phi}{\phi}\right).$$

Using this with together with a construction in [**5**] we can prove a very rough but explicit bound on λ_1 in the case where there is no zeroth-order term.

Corollary 4.2.1. *Let $M := a_{ij}\partial_i\partial_j + b_i\partial_i$ be a second-order elliptic linear differential operator with bounded continuous coefficients. Let a_0 denote the lower ellipticity constant and let b be any constant satisfying $b^2 \geq \sup_R \sum b_i^2$. Let $\sigma > 0$ solve the equation*

$$a_0\sigma^2 - b\sigma - b = 1.$$

Then

$$\lambda_1(M) \geq \exp\left(-\sigma \min\{L, |p_0|\}\right). \tag{4.2.14}$$

Proof. In [**5**] it was shown that there exists a function $u_0 \in C^2(R) \cap C(\overline{R})$ satisfying $Mu_0 = -1$ in R, $u_0 > 0$ in R and $u_0|_{\partial R} = 0$. If we further examine the actual construction of u_0, we find that $0 < u_0 \leq e^{\sigma \min\{L, |p_0|\}}$, with σ as above (see §3 of [**5**], particularly p. 61). Then (4.2.14) follows directly from Theorem 4.2.2 with $\phi = u_0$. □

Finally, for reference we repackage two important estimates from [**5**] regarding the dependence of λ_1 on the magnitudes of the first- and zeroth-order terms.

Proposition 4.2.1 (Beresycki–Varadhan–Nirenberg, [**5**]). *Let $M := a_{ij}\partial_i\partial_j + b_i\partial_i$ and $M' := a_{ij}\partial_i\partial_j + b_i'\partial_i$ be a second-order elliptic differential operators with smooth coefficients. Let a_0 be the lower ellipticity constant and let b be any constant satisfying $b^2 \geq \sup_R \sum_i b_i^2$. Suppose that δ is chosen so that*

$$\sum_i \left(b_i' - b_i\right)^2 \leq \delta^2 \leq b a_0,$$

then

$$\lambda_1(M') \geq \lambda_1(M) - \sqrt{\frac{b}{a_0}}\delta. \tag{4.2.15}$$

Proof. This is essentially estimate (5.2) of [**5**] (used to prove Proposition 5.1 of that paper) with $c \equiv 0$. □

Proposition 4.2.2 (Beresycki–Varadhan–Nirenberg, [**5**]). *In its dependence on the zeroth-order term, λ_1 is Lipschitz continuous in the L^∞-norm with Lipschitz constant 1.*

That said, we are now prepared to prove the key lemma of this section.

LEMMA 4.2.1. (Properties of $\mathcal{L}$) *Let* $h,\ \tilde{h} \in C^{3+\alpha}_{\text{per}}(\overline{R})$, $Q \in \mathbb{R}$ *be given such that* $d(h) = d(\tilde{h})$ *and* (h,Q), $(\tilde{h},Q)$ *are solutions of* (4.2.12) *with* $\sup_{\overline{R}} h_p^{-1} = \sup_{\overline{R}} \tilde{h}_p^{-1}$, $\inf_{\overline{R}} h_p^{-1} = \inf_{\overline{R}} \tilde{h}_p^{-1}$ *. Then the operator* $\mathcal{L}$ *defined in* (4.2.13) *satisfies the following.*

(a) $\mathcal{L}(h-\tilde{h}) = 0$.

(b) $\mathcal{L}$ *is a uniformly elliptic differential operator with lower ellipticity constant*

$$a_0 = \inf_{\overline{R}} \left(\frac{1+h_q^2+h_p^2}{2h_p^3} - \frac{1}{2h_p^3}\sqrt{\left(1+h_q^2+h_p^2\right)^2 - 4h_p^2} \right).$$

(c) Let $\lambda_1(\mathcal{L})$ *denote the principle eigenvalue of* $\mathcal{L}$ *as in* (1)*. Define*

$$\begin{aligned} b := \sup_{\overline{R}} \Bigg\{ \Big[& h_p^{-3}\tilde{h}_{qq}(h_p+\tilde{h}_p) - 2h_p^{-3}\tilde{h}_q\tilde{h}_{pq} + \beta(-p)h_p^{-3}(h_p^2+h_p\tilde{h}_p+\tilde{h}_p^2) \\ & - h_p^{-3} g\rho'(\tilde{h}-d(\tilde{h}))(h_p^2+h_p\tilde{h}_p+\tilde{h}_p^2)\Big]^2 \\ & + \Big[h_p^{-3}\tilde{h}_{pp}(h_q+\tilde{h}_q) - 2h_p^{-2}\tilde{h}_{pq}\Big]^2 \Bigg\}^{\frac{1}{2}}. \end{aligned} \tag{4.2.16}$$

and let σ *be as in Corollary 4.2.1. Then* $\lambda_1(\mathcal{L})$ *is strictly positive provided that*

$$g \sup_{[p_0,0]} |\rho'| < \exp\left(-\sigma \min\{L,\ |p_0|\}\right) \tag{4.2.17}$$

Alternatively, we have that $\lambda_1(\mathcal{L})$ *is strictly positive provided*

$$b \le a_0 \tag{4.2.18}$$

and

$$b\sqrt{\frac{b}{a_0}} + g \sup_{[p_0,0]} |\rho'| < \exp\left(-a_0^{-1/2}\min\{L,\ |p_0|\}\right). \tag{4.2.19}$$

PROOF. A tedious but easy calculation readily confirms that $\mathcal{L}(h-\tilde{h}) = 0$, proving (a). Similarly, the second-order terms of $\mathcal{L}$ are identical to those of the differential operator in the height equation which has been proven to be uniformly elliptic section 2.4. Calculating the precise lower ellipticity constant is a simple exercise that we omit for brevity.

The interesting part is, of course, proving (c), which is done as follows. By Proposition 4.2.2 we know that λ_1 is Lipschitz in its dependence on the zeroth-order coefficients with

Lipschitz constant 1. That is, if we denote as $\mathcal{L}_0$ the differential operator found by dropping the zeroth-order term in $\mathcal{L}$, then

$$\lambda_1(\mathcal{L}) > \lambda_1(\mathcal{L}_0) - \sup|g\rho'|. \tag{4.2.20}$$

In order to show $\lambda_1(\mathcal{L})$ is positive, therefore, we must produce lower bounds for $\mathcal{L}_0$. This can be achieved by using Corollary 4.2.1 directly, proving $\lambda_1(\mathcal{L})$ is positive under condition (4.2.17).

To arrive at the more involved conditions (4.2.18)-(4.2.19) we instead begin by chopping off the first-order terms using Proposition 4.2.1. Let $\mathcal{L}_1$ denote the operator operator consisting of the second-order terms of $\mathcal{L}$. Applying Corollary 4.2.1 to $\mathcal{L}_1$ yields

$$\lambda_1(\mathcal{L}_1) \geq e^{-a_0^{-1/2}\min\{L,\ |p_0|\}}.$$

The final step is to estimate the difference between $\lambda_1(\mathcal{L}_0)$ and $\lambda_1(\mathcal{L}_1)$. Let b be as in (4.2.16) so that (4.2.18) implies $b \leq a_0$. Then, taking $\delta = b$, we have by (4.2.15):

$$\lambda_1(\mathcal{L}_0) \geq \lambda_1(\mathcal{L}_1) - b\sqrt{\frac{b}{a_0}}. \tag{4.2.21}$$

Using this information in concert with (4.2.20) we obtain

$$\begin{aligned}\lambda_1(\mathcal{L}) &> \lambda_1(\mathcal{L}_1) - \sup|g\rho'| - b\sqrt{\frac{b}{a_0}} \\ &> \exp\left(-a_0^{-1/2}\min\{L,\ |p_0|\}\right) - \sup|g\rho'| - b\sqrt{\frac{b}{a_0}}.\end{aligned}$$

Observe that the quantity on the right-hand side above is positive by (4.2.19), hence $\lambda_1(\mathcal{L}) > 0$. This completes the proof of (c) and the lemma. □

4.3. Symmetry

We are now prepared to prove the main theorem of this chapter. This shall be done by employing an adapted moving plane method (cf. [**13, 14, 31, 63**].) The necessary maximum principle properties will follow from the results of the previous section.

Proof of Theorem 4.1.1. First consider the case when (S1) holds. We shall structure our argument on those of [**14**]. Without loss of generality, suppose that the trough of the wave train occurs at $x = \pm L/2$. By assumption, we may choose $\delta \in (0, 1/2)$ satisfying

$$\beta'(s) + gy(-s)\rho''(-s) \leq \frac{\pi^2(1-2\delta)^2}{\eta_{\max}^2}, \qquad 0 \leq s \leq |p_0|. \tag{4.3.1}$$

We may therefore define a function $\alpha \in C^2([0, \eta_{\max}])$ by

$$\alpha(y) := \sin \pi \left((1-2\delta)\frac{y}{\eta_{\max}} + \delta \right), \qquad 0 \le y \le \eta_{\max}.$$

One can readily verify that α has the following two properties

$$0 < \sin(\pi\delta) \le \alpha \le 1, \qquad \frac{\alpha_{yy}}{\alpha} = -\frac{\pi^2(1-2\delta)^2}{\eta_{\max}^2} \ge -\frac{\pi^2}{\eta_{\max}^2}. \tag{4.3.2}$$

For each $\lambda \in (-L/2, 0]$ consider the truncated fluid domain

$$D_\lambda := D \cap \{(x,y) : -L/2 < x < \lambda\}.$$

Since the trough occurs at $x = -L/2$, for λ sufficiently near $-L/2$ the surface profile is monotonically increasing, hence the transformation $(x,y) \mapsto (2\lambda - x, y)$ maps D_λ into D. For such λ we define the reflected domain

$$D_\lambda^R := D_\lambda \cup \{(2\lambda - x, y) : (x,y) \in D_\lambda\} \subset D.$$

Incrementing λ, we eventually reach a point where D_λ^R is no longer contained in D. Therefore, if we put

$$\lambda_0 := \sup\left\{\lambda \in (-L/2, 0] : D_\lambda^R \not\subset D\right\},$$

We consider two cases, the first being that $\lambda_0 = 0$. If this fails to be true, then $\lambda_0 < 0$ and either λ_0 occurs at a crest, or λ_0 lies strictly to the left of a crest. Notice, however, that the symmetry of the equations allow us to assume the crest occurs in the interval $[0, L/2)$, without loss of generality. Thus, when $\lambda_0 < 0$, we need only consider the case when λ_0 occurs to the left of a crest.

Case 1: $\lambda_0 = 0$. For each $(x,y) \in D_{\lambda_0}^R =: \Omega$, put

$$w(x,y) := \frac{\psi(-x,y) - \psi(x,y)}{\alpha(y)},$$

where ψ is the pseudo-stream function. Proving symmetry of the wave is equivalent to showing that $w \equiv 0$ in Ω. Note that $\psi(-x,y) = \psi(x,y)$ on the free surface, by construction, and $\psi(-x,y) = \psi(x,y)$ on the bed and under the trough line, by periodicity. In other words, $w = 0$ on $\partial\Omega$. This suggests that, in order to prove w vanishes identically, we need to show that w satisfies a maximum principle. Naturally, this will have to come from (4.2.1).

With that in mind, we compute

$$\Delta w + 2\frac{\alpha_y}{\alpha}\partial_y w = \frac{\Delta(\alpha w)}{\alpha} - \frac{\alpha_{yy}}{\alpha}w, \qquad \text{in } \Omega.$$

Since $\alpha(y)w(x,y) = \psi(-x,y) - \psi(x,y)$, from (4.2.1) and (4.3.2) we conclude

$$\Delta w + 2\frac{\alpha_y}{\alpha}\partial_y w + \frac{\overline{\beta} - gy\overline{\rho'}}{\alpha} - \frac{\pi^2(1-2\delta)^2}{\eta_{\max}^2}w = 0, \qquad \text{in } \Omega,$$

where, for all $(x,y) \in \Omega$,

$$\overline{\beta}(x,y) := \frac{\beta(\psi(-x,y)) - \beta(\psi(x,y))}{\psi(-x,y) - \psi(x,y)}, \qquad \overline{\rho'}(x,y) := \frac{\rho'(-\psi(-x,y)) - \rho'(-\psi(x,y))}{\psi(-x,y) - \psi(x,y)}.$$

But, by the mean value theorem, for each $(x,y) \in \Omega$, there exist $s_1 = s_1(x,y) \in (0,|p_0|)$, $s_2 = s_2(x,y) \in (p_0, 0)$, such that

$$\overline{\beta}(x,y) = \beta'(s_1(x,y)), \qquad \overline{\rho'}(x,y) = -\rho''(s_2(x,y)).$$

Therefore, if we define

$$c(x,y) := \beta'(s_1(x,y)) + gy\rho''(s_2(x,y)) - \frac{\pi^2(1-2\delta)^2}{\eta_{\max}^2}, \qquad \forall (x,y) \in \Omega,$$

then

$$\Delta w + 2\frac{\alpha_y}{\alpha}\partial_y w + cw = 0, \qquad \text{in } \Omega.$$

Observe that, by condition (S1), c is non-positive. We are therefore able to apply a minimum principle and conclude either $w > 0$, or $w \equiv 0$ in Ω.

Let us denote the corner point $C := (-L/2, \eta(-L/2))$. Then, by periodicity of the wave profile, we must have

$$w(C) = w_y(C) = w_{yy}(C) = w_{xx}(C) = 0.$$

Likewise, differentiating the relation $\psi = 0$ on $y = \eta(x)$, we find

$$\psi_x + \psi_y \eta' = 0, \qquad \text{on } y = \eta(x).$$

Since the trough occurs at $x = -L/2$, we have $\eta'(-L/2) = \psi_x(C) = 0$. It follows that $w_x(C) = 0$.

Finally, consider $w_{xy}(C)$. From the nonlinear boundary condition (4.2.2) we compute

$$\psi_x(\psi_{yy} + \psi_{xy}\eta') + \psi_y(\psi_{xy} + \psi_{yy}\eta') + g\rho(-\psi)\eta' = 0, \qquad \text{on } y = \eta(x).$$

Evaluating this at C, we conclude $\psi_{xy}(C)\psi_y(C) = 0$. But, since $\psi_y < 0$, this can only be the case provided ψ_{xy} vanishes at C. In light of this, and the fact that $w_x(C) = 0$, it follows that

$$w_{xy} = -2\frac{\psi_{xy}}{\alpha} - \frac{\alpha_y}{\alpha} w_x = 0, \qquad \text{at } C.$$

Thus all derivatives of w of order up to two vanish at C. This violates Serrin's edge lemma (cf. [**28**]), unless $w \equiv 0$ in Ω, as desired.

Case 2: $\lambda_0 < 0$. As we remarked earlier, we may assume that the crest occurs somewhere in the interval $[0, L/2)$. From the definition of λ_0 it is clear that $D^R_{\lambda_0}$ is internally tangent to the upper boundary of D at the point of where $D^R_{\lambda_0}$ contacts D, call it $(\xi, \eta(\xi))$.

Put $\lambda_1 := 2\lambda_0 + L/2$, $\lambda_2 := \lambda_0 + L/2$. We wish to consider the domain that results from reflecting D_{λ_1} over the line $x = \lambda_1$, and the set $D \setminus D_{\lambda_2}$ over the line $x = \lambda_2$. More precisely, let us define

$$\tilde{\eta}(x) := \begin{cases} \eta(2\lambda_0 - x) & \lambda_0 \le x \le \lambda_1, \\ \eta(2\lambda_2 - x) & \lambda_1 \le x \le \lambda_2, \end{cases}$$

and

$$\Omega := \{(x,y) \in \mathbb{R}^2 : \lambda_0 < x < \lambda_2,\ -d < y < \tilde{\eta}(x)\}.$$

Observe that ξ must necessarily lie in some interval where the free surface is decreasing. This is a simple consequence of the fact that, left of the line $x = \lambda_0$, the boundary of $D^R_{\lambda_0}$ is negatively sloped. We conclude, therefore, that $\Omega \subset D$. Let us further define, for all $(x,y) \in \Omega$,

$$w(x,y) := \begin{cases} \dfrac{\psi(x,y) - \psi(2\lambda_0 - x, y)}{\alpha(y)} & \lambda_0 \le x \le \lambda_1, \\[2ex] \dfrac{\psi(x,y) - \psi(2\lambda_2 - x, y)}{\alpha(y)} & \lambda_1 \le x \le \lambda_2. \end{cases}$$

We know that $\psi = 0$ on the free surface and $\psi \ge 0$ in the interior of the fluid. It follows that $w \ge 0$ on the upper boundary of Ω. As before, periodicity ensures that ψ vanishes on the lateral boundaries of Ω, by construction. Finally, we know that $\psi = p_0$ on $y = -d$, hence w vanishes identically on the lower boundary of Ω.

As in the previous case, a simple calculation shows that w is a solution in Ω with respect to a linear elliptic operator of the form $\Delta + \alpha^{-1}\alpha_y\partial_y + c$, where $c \le 0$ when (S1) holds. It follows that $w > 0$ in Ω, or $w \equiv 0$ in $\overline{\Omega}$.

Suppose that we can prove $w \equiv 0$ in $\overline{\Omega}$. Explicitly, this means $\psi(x,y) = \psi(2\lambda_0 - x, y)$ in Ω, for $\lambda_0 \le x \le \lambda_1$, and $\psi(x,y) = \psi(2\lambda_2 - x, y)$ in Ω for $\lambda_1 \le x \le \lambda_2$. Using the fact that $\psi_y < 0$, we can differentiate the relation $\psi(x,\eta(x)) = 0$ to conclude $\eta(x) = \tilde{\eta}(x)$, for $x \in (\lambda_0, \lambda_2)$. But this is impossible, as the free surface is assumed to be monotonic. Thus $w \equiv 0$ in $\overline{\Omega}$ implies $\lambda_0 = 0$, and so we are in the first case, for which we have already proved symmetry holds.

Seeking a contradiction, suppose instead that $w > 0$ in Ω. Let $C := (\xi, \eta(\xi))$ denote the contact point and note that it is in the interior of the upper boundary of Ω. We claim further that $w(C) = 0$. To see why this is the case, observe that by definition of ξ, we have $\eta(\xi) = \tilde{\eta}(\xi)$. Therefore, in light of the fact that $\psi = 0$ on the free surface,

$$\psi(\xi, \eta(2\lambda_0 - \xi)) = \psi(\xi, \eta(\xi)) = 0 = \psi(2\lambda_0 - \xi, \eta(2\lambda_0 - \xi)).$$

Since $w(C) = 0$, w attains a minimum at C. As $D^R_{\lambda_0}$ is internally tangent to D at C, we must have that the interior sphere condition is satisfied there. The hypotheses for Hopf's boundary lemma (cf. [**28**]) now completely verified, we infer that

$$\left.\frac{\partial w}{\partial \nu}\right|_C < 0,$$

where ν is the normal vector at C.

We will produce a contradiction by showing that, in fact, the normal derivative must vanish at C. Towards that end let us consider $\xi_0 := 2\lambda_0 - \xi$, that is, the reflection of ξ over the line $x = \lambda_0$. By definition of ξ, we must have that $\eta(\xi) = \eta(\xi_0)$, and $\eta'(\xi) = -\eta'(\xi_0)$. Differentiating the relation $\psi(x,\eta(x)) = 0$ and taking these observations into account, we arrive at the following identity

$$\left.\frac{\psi_x}{\psi_y}\right|_{(\xi_0,\eta(\xi_0))} = -\left.\frac{\psi_x}{\psi_y}\right|_C. \tag{4.3.3}$$

Here we again made use of the fact that $\psi_y = u - c < 0$. Also, the nonlinear boundary condition (4.2.2) implies

$$\Big(|\nabla\psi|^2 + g\rho(-\psi)(\eta + d)\Big)\Big|_{(\xi_0,\eta(\xi_0))} = Q = \Big(|\nabla\psi|^2 + g\rho(-\psi)(\eta + d)\Big)\Big|_C,$$

hence, since ρ is constant on the free surface,

$$|\nabla\psi(C_0)|^2 = |\nabla\psi(C)|^2, \tag{4.3.4}$$

where we let $C_0 := (\xi_0, \eta(\xi_0))$. From (4.3.3)-(4.3.4) we conclude further that

$$\psi_x(C_0) = -\psi_x(C), \qquad \psi_y(C_0) = \psi_y(C).$$

Let the normal vector at C be $\nu = (\nu_1, \nu_2)$. Then,

$$\left.\frac{\partial w}{\partial \nu}\right|_C = \nu_1 \frac{\psi_x(C) + \psi_x(C_0)}{\alpha(\eta(\xi))} + \nu_2 \frac{\psi_y(C) - \psi_y(C_0)}{\alpha(\eta(\xi))} - \nu_2 \frac{\alpha_y(\eta(\xi))}{\alpha(\eta(\xi))} w(C),$$

which is zero, by the above considerations. We conclude that $w \equiv 0$ in $\overline{\Omega}$.

Up to this point, we have seen that symmetry holds if

$$\sup \beta' + g\eta_{\max} \sup |\rho''| < \frac{\pi^2}{\eta_{\max}^2}.$$

To complete the proof for (S1), we must now show that symmetry holds when the right-hand side above is replaced by $\exp(-\min\{L, \eta_{\min}\})$. We note that the only use of (S1) in the preceding argument was to ensure that a minimum principle holds for w. If we take $\alpha \equiv 1$, then w will be a solution of a linear elliptic operator of the form $\Delta + c$. Since the ellipticity constant here is one, Corollary 4.2.1 and Proposition 4.2.2 together show that the requirement

$$\sup_{\Omega} c^+ < \exp(-\min\{L,\ \eta_{\min}\}) \tag{4.3.5}$$

implies w has a minimum principle, and hence the wave is symmetric. The second term on the right-hand side of condition (S1) is chosen precisely so that (4.3.5) holds.

The last condition, (S2), will require more effort. In particular, we shall work with the height equation formulation (4.2.12) and make use of the operator $\mathcal{L}$. In doing this we are closely following the path set forth by [**13**].

Let h be the height function for a steady periodic gravity water wave as in the statement. If h is flat there is nothing to prove, so without loss of generality we suppose that a crest occurs in the interval $[0, L/2)$ and a trough at $q = -L/2$. This is permissible, as (4.2.12) is invariant under the transformation $q \mapsto -q$.

For each $\lambda \in (-L/2, 0)$, $(q,p) \in [\lambda, 2\lambda + L/2] \times [p_0, 0]$, put

$$q^\lambda := 2\lambda - q, \qquad w(q,p;\ \lambda) := h(q,p) - h(q^\lambda, p).$$

Then, by construction, for $\lambda \in [-L/2, L/2]$, $(q,p) \in [\lambda, 2\lambda + L/2] \times [0, p_0]$,

$$w(\lambda, p; \lambda) = h(\lambda, p) - h(q^\lambda, p) = 0, \tag{4.3.6}$$

and

$$w(q,p_0;\lambda) = h(q,p_0) - h(q^\lambda, p_0) = 0. \tag{4.3.7}$$

Also we observe that

$$d\left(w(q,p;\lambda)\right) = \fint_T \Big(h(q,0) - h(q^\lambda,0)\Big)\, dq = 0, \tag{4.3.8}$$

owing to the periodicity of h.

Fix q. Then, by monotonicity of the profile, $w(q,0;\lambda) \geq 0$ for λ near $-L/2$. Put

$$\lambda_0 := \sup\left\{\lambda \in [-L/2,0] :\ w(q,0;\lambda) \geq 0,\ \text{for } q \in (\lambda, 2\lambda + L/2)\right\}.$$

There are two cases to consider.

Case 3: $\lambda_0 = 0$. Denote $\Omega := (0,L/2) \times (p_0,0)$ and, to be concise, fix $\lambda = \lambda_0$ for w. Identity (4.3.7) implies that $w \equiv 0$ on the set $\partial\Omega_b := \{(q,p_0) : 0 < q < L/2\}$. Likewise, by periodicity of h we have $w(L/2,p) = h(L/2,p) - h(-L/2,p) = 0$, for all $p_0 \leq p \leq 0$. Thus w vanishes identically on the right boundary, $\partial\Omega_r := \{(L/2,p) : p_0 < p < 0\}$. Taking λ and replacing it with $\lambda_0 = 0$ in (4.3.6), we obtain the same result on the left boundary, $\partial\Omega_l := \{(0,p) : p_0 < p < 0\}$. Finally, on the top of Ω, which we denote $\partial\Omega_t := \{(q,0) : 0 < q < L/2\}$, we have $w \geq 0$ by the definition of λ_0. Combining then, we see $w \geq 0$ on the entire boundary of Ω.

Suppose for now that w has a determined sign in some region $\Omega' \subset \Omega$ whose boundary includes the corner point $(L/2,0)$ and which is blunt in the sense that the Serrin edge lemma holds there (see, for instance, Definition E3 of [**28**]). By (4.3.6) and periodicity, $w(L/2,0;\lambda_0) = w_p(L/2,0;\lambda_0) = w_q(L/2,0;\lambda_0) = 0$. Using these facts we can differentiate (4.2.11) in q and evaluate at the corner to

$$2h_p h_{pq}\left(2g\rho h - Q\right) = 0, \qquad \text{at } (L/2,0),$$

which implies h_{pq} — and thus w_{pq} — vanishes at $(L/2,0)$.

As w is a solution for $\mathcal{L}$ in Ω' with determined sign, applying Serrin's edge lemma and the arguments of the previous paragraph we see that at least one of w_{qq} and w_{pp} must be nonvanishing at $(L/2,0)$. This leads to a contradiction, as $w(q,p;\lambda_0) = h(q,p) - h(-q,p)$, and thus

$$\left.\begin{array}{rcl} w_{qq}(L/2,0) & = & h_{qq}(L/2,0)-h_{qq}(-L/2,0) \\ w_{pp}(L/2,0) & = & h_{pp}(L/2,0)-h_{pp}(-L/2,0) \end{array}\right\} = 0.$$

We conclude that no such region Ω' can exist.

Case 4: $\lambda_0 \in (-L/2, 0)$. In this case, there also exists a first point $q_0 \in (\lambda_0, 2\lambda_0 + L/2]$ for which $w(q_0, 0; \lambda_0) = 0$, and $w(q, 0; \lambda_0) > 0$ for $\lambda_0 \leq q < q_0$ This will be attained at the point where the graphs of the functions $q \mapsto h(q, 0)$, $q \mapsto h(q^{\lambda_0}, 0)$ are tangent to each other.

First observe that, by definition of λ_0, we cannot have $w(q, 0; \lambda_0) > 0$ for $q > \lambda_0$. We may continuously extend w by letting

$$w(q,p;\lambda_0) := h(q,p) - h(q^{\lambda_0} + L, p), \qquad \text{for } (q,p) \in \left[2\lambda_0 + \frac{L}{2},\ \lambda_0 + \frac{L}{2}\right] \times [p_0, 0].$$

It is important to observe that the extended w satisfies $\mathcal{L}w = 0$ in the interior of its domain of definition. Redefine

$$\Omega := \left(\lambda_0,\ \lambda_0 + \frac{L}{2}\right) \times (p_0, 0),$$

and, as in [**13**], periodicity implies $w \in C^2(\overline{\Omega})$. Also, if $2\lambda + L/2$ lies to the left of the crest line, $w(q, 0; \lambda) > 0$ holds for all $\lambda \leq q \leq 2\lambda + L/2$. Thus $2\lambda_0 + L/2$ must be to the right of (or coinciding with) the crest line. Monotonicity of the profile then ensures that h is decreasing for $q \in (2\lambda_0 + L/2, L/2)$. This implies that on the top

$$w(q, 0; \lambda_0) \geq 0, \qquad \forall q \in \left[\lambda_0,\ \lambda_0 + \frac{L}{2}\right].$$

On the boundary, we therefore have the following sign information

$$\left\{\begin{array}{rcll} w(\lambda_0, p; \lambda_0) & = & w\left(\lambda_0 + \frac{L}{2}, p; \lambda_0\right) = 0, & \text{for } p \in [p_0, 0], \\ w(q, p_0; \lambda_0) & = & 0 \quad \text{and} \quad w(q, 0; \lambda_0) \geq 0, & \text{for } q \in \left[\lambda_0, \lambda_0 + \frac{L}{2}\right]. \end{array}\right.$$

Suppose there exists a region $\Omega'' \subset \Omega$ where $w \geq 0$ in Ω'', $(q_0, 0) \in \partial\Omega''$ and $\partial\Omega''$ is smooth at $(q_0, 0)$. Since the graphs of $q \mapsto h(q, 0)$, $q \mapsto h(q^{\lambda_0}, 0)$ are tangent at the point $(q_0, 0)$ this point, $w_q(q_0, 0; \lambda_0) = 0$. In particular, $h_q(q_0, 0) = -h_q(q^{\lambda_0}, 0)$. Using this to evaluate (4.2.11) at $(q_0, 0)$ and $(q_0^{\lambda_0}, 0)$ we find $h_p(q_0, 0) = h_p(q_0^{\lambda_0}, 0)$. Thus $\nabla w(q_0, 0; \lambda_0) = 0$. However this contradicts the Hopf lemma (which is applicable, despite the adverse sign of the zeroth-order term of $\mathcal{L}$, because of the predetermined sign of w in Ω'', cf. Remark 2.16 of [**28**]). Thus no such region Ω'' can exist.

Proving both Case 3 and Case 4, therefore, involves showing that either Ω' or Ω'' must exist, unless $w \equiv 0$ in Ω. Since we know that $\mathcal{L}w = 0$, this can be done easily provided we know $\mathcal{L}$ satisfies a maximum principle. That is, in both of these cases we have demonstrated that $w \geq 0$ on $\partial\Omega$ (where, we note, w and Ω are defined differently in both cases). Hence, if $\mathcal{L}$ satisfies a maximum principle, then by definition $w \equiv 0$ or $w > 0$ in Ω, so that we may take $\Omega' = \Omega'' = \Omega$.

Unfortunately, it will not in general be true that $\mathcal{L}$ satisfies a kind of maximum principle. However, if we let $\tilde{h}(q,p) := h(q^\lambda, p)$, then by Theorem 4.2.1 we know that the maximum principle will hold precisely when $\lambda_1(\mathcal{L}) > 0$. Lemma 4.2.1 then tells us that this will occur provided (4.2.18) and (4.2.19) hold. Making this assumption, the rest of the proof of the theorem is immediate. All that remains to show, therefore, is that (S2) implies (4.2.18)–(4.2.19).

Let b be defined as in Lemma 4.2.1 taking $\tilde{h}(q,p) := h(q^{\lambda_0}, p)$ for Case 1, or $h(q^{\lambda_0} + L, p)$ for Case 2. First note that, by definition, $a_0 \leq \inf_{\overline{R}} 1/(2h_p)$, whence

$$\sup_{\overline{R}} h_p,\ \sup_{\overline{R}} \tilde{h}_p \leq \frac{1}{2} a_0^{-1}.$$

Then, using the equation (4.2.12) satisfied by $\tilde{h}$, we can estimate

$$\begin{aligned}
\sup_{\overline{R}} |\tilde{h}_{pp}| &\leq \sup_{\overline{R}} \left| 2\tilde{h}_p \tilde{h}_q \tilde{h}_{qp} - \tilde{h}_p^2 \tilde{h}_{qq} + g(\tilde{h} - d(\tilde{h}))\rho' \tilde{h}_p^3 - \beta(-p)\tilde{h}_p^3 \right| \\
&\leq a_0^{-1} M^2 + \frac{1}{4} a_0^{-2} M + \frac{1}{8}\left(\sup|\beta| + g\eta_{\max} \sup|\rho'|\right) a_0^{-3},
\end{aligned}$$

where we take $M := \|h_q\|_{C^1(\overline{R})}$, which, in view of (4.2.7), is precisely the same as in (4.1.13). From the above inequality we find,

$$\begin{aligned}
b \leq\ & a_0^{-1} A_0^3 M + 2A_0^3 M^2 + \frac{3}{4} a_0^{-2} A_0^3 \left(\sup|\beta| + g\eta_{\max}\sup|\rho'|\right) \\
& + 2a_0^{-1} A_0^3 M^3 + \frac{1}{2} a_0^{-2} A_0^3 M^2 + 2A_0^2 M + \frac{1}{4} a_0^{-3} A_0^3 \left(\sup|\beta| + g\eta_{\max}\sup|\rho'|\right).
\end{aligned}$$

The right-hand side, of course, we recognize as the ϵ occurring in the theorem's statement. Therefore (4.2.18)–(4.2.19) follow immediately from (S2), which completes the theorem. $\square$

Remark. From the above proof we can see quite clearly why providing criteria for the symmetry of stratified waves is dramatically more difficult than for the homogeneous case: it is *precisely* the appearance of density variation that breaks the maximum principle structure. Naturally, this leads us to wonder whether or not it is reasonable to expect that highly stratified waves are necessarily symmetric. While we are not prepared to offer any conjectures on this question, it seems clear that, at the very least, making significant progress will require a new point of view.

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