Cluster integrable systems and statistical mechanics

by

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Contents

CHAPTER 1

Introduction

The subject of this thesis lies at the interface of integrable systems, cluster algebras and statistical mechanics. At the heart of it are the dimer cluster integrable systems, a beautiful class of algebraic integrable systems introduced by Goncharov and Kenyon [GK12].

The dimer model was developed by the statistical physicists Kasteleyn [Kas67], Temperley and Fischer [TF61] in the 1960's. The dimer model on the square grid is dual to domino tilings, that is, tilings with 2×1 and 1×2 rectangles. In [EKLP], Elkies, Kuperberg, Larsen and Propp introduced the domino-shuffling algorithm, a discrete time random dynamical system on domino tilings to enumerate domino tilings of certain regions of the square grid called Aztec diamonds. Underlying this algorithm is a certain three dimensional recurrence called the octahedron recurrence, whose solutions exhibit the Laurent phenomenon [Spey04]. By analogy, Propp introduced the cube recurrence [Propp01] and conjectured that it also has the Laurent property. The cube and octahedron recurrences were part of the motivation behind Fomin and Zelevinsky's introduction of cluster algebras [FZ01].

Fock and Goncharov [FG03b] defined cluster varieties, which are dual geometric objects to cluster algebras. Cluster varieties come in pairs, called the $\mathcal{X}$ and $\mathcal{A}$ cluster varieties. The $\mathcal{X}$ cluster variety comes with a canonical Poisson structure. An integrable system is a Hamiltonian system with a maximal set of functions, called Hamiltonians, that mutually commute with respect to the Poisson structure. Goncharov and Kenyon [GK12] showed that the $\mathcal{X}$ cluster variety associated to the dimer model on a torus, with its canonical Poisson structure, is an integrable system, where the Hamiltonians are given by certain dimer partition functions.

In chapter 3, we study generalizations of the domino-shuffling algorithm on the square lattice to other biperiodic planar bipartite graphs. They form a group, called the *cluster modular group*, and is the group of automorphisms of the dimer integrable system. This group was studied by Fock and Marshakov [FM16], who gave an explicit conjecture for its isomorphism type. The main result is a proof of this conjecture. This chapter has an appendix by Giovanni Inchiostro.

Resistor networks are graphs with a non-zero complex number associated to each edge, called its *conductance*. They provide the setting for discrete potential theory and are intimately related to random walks, spanning trees and discrete geometry. If the graph is embedded on the torus, we say that it is *biperiodic.* Resistor networks have a cluster-variety-like structure, with the cluster mutation replaced by the Y-Δ move (equivalently the cube recurrence). The fundamental operator in the study of networks is the discrete Laplacian. Associated to the discrete Laplacian on a biperiodic network is its *spectral data*: a curve C and a collection of points in C, that is, a divisor. In chapter 4, we show that this divisor is always a point in the Prym variety of C, and that the spectral data gives a birational isomorphism from the space of networks to a family of Prym varieties. These results for networks parallel Fock's results for the dimer model [F15].

The limit shape phenomenon is a "law of large numbers" for random surfaces: the random surface looks macroscopically like the "average surface". The first result of this kind was the celebrated arctic circle theorem for domino tilings of the aztec diamond, proved using the domino shuffling algorithm. The limit shape has macroscopic regions with different qualitative behavior, called phases and the *arctic curve* is the boundary separating the phases. The work of Kenyon, Okounkov, Sheffield [KOS06] and others has shown that periodic lattices with non-trivial Newton polygons lead to rich arctic curves with many frozen and gaseous phases. Groves are another statistical mechanical model, associated to resistor networks, that exhibits an arctic circle theorem, proved by Petersen and Speyer [PS06] using the cube recurrence. In chapter 5, we extend their proof to compute arctic curves for groves with non-trivial Newton polygons, using points on the associated resistor network cluster variety that are periodic under an element of the cluster modular group. We provide a description of asymptotic edge probabilities, combining the singular integral analysis of Baryshnikov and Pemantle [BP11] with some geometric ideas of Kenyon and Okounkov [KO07].

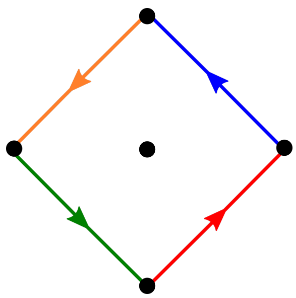
CHAPTER 2

Background

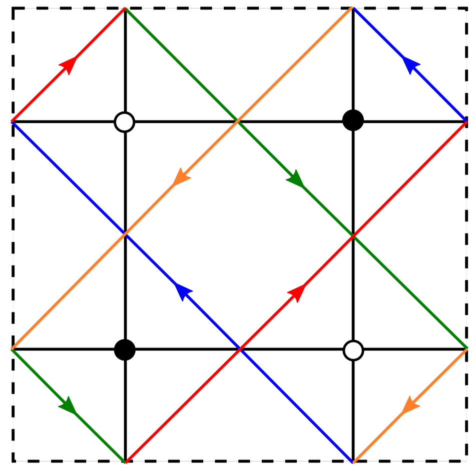
2.1 The dimer model

2.1.1 Some basic notation

Let $\mathbb{T}$ be a torus, and let $T := H_1(\mathbb{T}, \mathbb{Z})^* \otimes \mathbb{C}^*$ be the algebraic torus with group of characters $H_1(\mathbb{T}, \mathbb{Z})$. Given an convex integral polygon $N \subset H_1(\mathbb{T}, \mathbb{R})$, that is, a convex polygon whose vertices are in $H_1(\mathbb{T}, \mathbb{Z})$, we denote by V_N and E_N the vertices and edges of N respectively. A vector in $H_1(\mathbb{T}, \mathbb{R})$ whose end-points are in $H_1(\mathbb{T}, \mathbb{Z})$ is called *integral*. We say that an integral vector is *primitive* if it contains no integral points other than its end-points. Let $|E_N|$ be the number of edges of N, and for an edge $E_\rho \in E_N$, let $|E_\rho|$ be the *integral length* of E_ρ, that is the integer ℓ such that $E_\rho = \ell e_\rho$ with e_ρ a primitive parallel vector. Let u_ρ be the primitive integral vector



(a) A convex integral polygon N.



(b) A minimal bipartite graph Γ with Newton polygon N. The zig-zag paths of Γ are colored according to the corresponding edges of the Newton polygon N. Here and elsewhere, we draw zig-zag paths as loops in the medial graph of Γ.

Figure 2.1: A bipartite torus graph.

normal to E_ρ, oriented so that it points to the interior of N.

2.1.2 Bipartite torus graphs

A *bipartite graph* is a graph whose vertices are colored black or white, such that each edge is incident to one black and one white vertex. A *bipartite torus graph* is a bipartite graph Γ embedded in $\mathbb{T}$ such that the faces of Γ, that is the connected components of $\mathbb{T} \setminus \Gamma$, are contractible. We denote by $B(\Gamma)$ and $W(\Gamma)$ the black and white vertices of Γ respectively.

A *dimer cover* (or *perfect matching*) M of Γ is a subset of $E(\Gamma)$ such that each vertex of Γ is incident to exactly one edge of M, that is the edges in M match each black vertex with a white vertex. Orienting edges in a dimer cover from the black vertex to the white vertex, we obtain from each dimer cover a 1-chain in Γ which we denote also by M. If we fix a dimer cover M_0 of Γ, called the *reference dimer cover*, for each dimer cover M we obtain a homology class:

$$M \mapsto [M - M_0] \in H_1(\mathbb{T}, \mathbb{Z}).$$

The *Newton polygon* $N(\Gamma)$ of Γ is defined to be the convex hull of the homology classes of all dimer covers of Γ:

$$N(\Gamma) := \mathrm{Conv}\{[M - M_0] : M \text{ is a dimer cover of } \Gamma\} \subset H_1(\mathbb{T}, \mathbb{R}).$$

Changing the reference dimer cover M_0 results in a translation of $N(\Gamma)$.

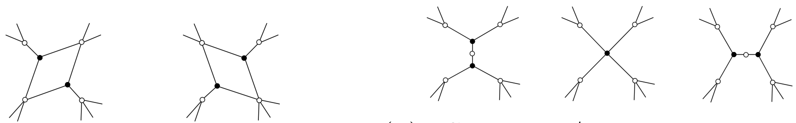
2.1.3 Boltzmann probability measures

An *edge-weight* on Γ is a function $wt : E(\Gamma) \to \mathbb{R}_{>0}$. Given an edge-weight wt, the weight of a dimer cover is given by $\prod_{e \in M} wt(e)$. The *Boltzmann probability measure* $\mathbb{P}_{wt}$ on the set of dimer covers is defined to be

$$\mathbb{P}_{wt}(M) := \frac{wt(M)}{Z},$$

where

$$Z := \sum_{\text{dimer covers } M} wt(M),$$



(a) Spider move.

(b) Shrinking/expanding 2-valent white vertices.

Figure 2.2: Elementary transformations of bipartite torus graphs.

is a normalization constant called the *partition function.* Two edge-weights wt_1, wt_2 on Γ are said to be *gauge equivalent* if there exists $b_v \in \mathbb{R}_{>0}$, for all $v \in V(\Gamma)$ such that

$$wt_1(e) = b_v wt_2(e) b_{v'},$$

where v, v' are the edges incident to e. Since M is a dimer cover, there is exactly one edge of M incident to each vertex of Γ, and therefore $\mathbb{P}_{wt_1} = \mathbb{P}_{wt_2}$.

2.1.4 Zig-zag paths and minimality

A *zig-zag path* in Γ is an oriented path in Γ that turns maximally left at white vertices and maximally right at black vertices (with respect to the orientation on the $\mathbb{T}$ induced by the one on the fundamental rectangle). The *medial graph* of Γ is the 4-valent graph that has a vertex for each edge of Γ and edge when two edges of Γ are incident to a vertex and consecutive in cyclic order. Associated to a zig-zag path is a *strand* which is an oriented path in the medial graph that passes consecutively through the edges of the zig-zag path. The components of the complement of the strands in $\mathbb{T}$ are colored black, white or gray according to whether they contain a black vertex, white vertex or face of Γ.

Γ is *minimal* if in the lift $\widetilde{\Gamma}$ of Γ to the plane, zig-zag paths have no self-intersections and there are no parallel bigons, that is, pairs of zig-zag paths oriented the same way intersecting at two points. For a minimal bipartite graph, the Newton polygon has a description in terms of zig-zag paths. Each zig-zag path defines a homology class in $H_1(\mathbb{T}, \mathbb{Z})$. The primitive integral vectors forming the boundary of the Newton polygon are in bijection with the homology classes of zig-zag paths.

There are two local rearrangements of bipartite torus graphs called *elementary transformations*:

1. Spider moves (Figure 2.2a);

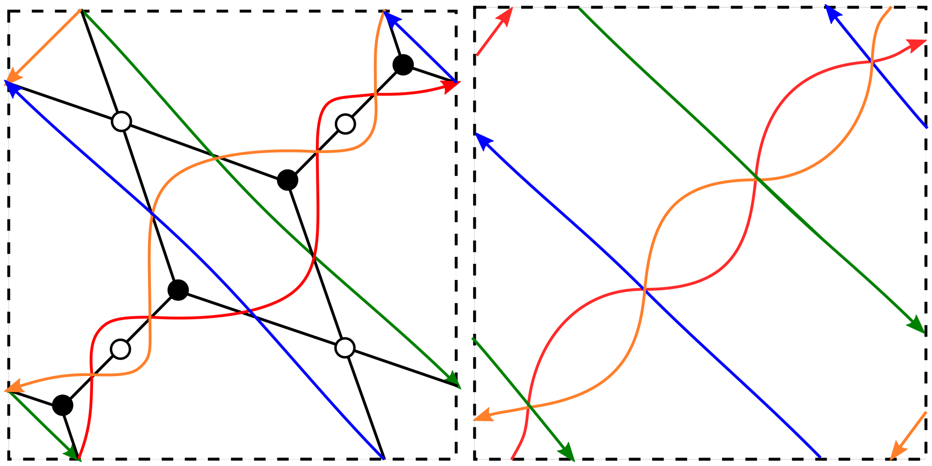
2. Shrinking/expanding 2-valent white vertices (Figure 2.2b).

Theorem 2.1.1 (Goncharov and Kenyon, 2012 [GK12, Theorem 2.5])**.** *Given a convex integral polygon N, there is a family of minimal bipartite torus graphs, each of whose Newton polygon is N. Any two minimal bipartite surface graphs with Newton polygon N are related by a sequence of elementary transformations.*

2.1.5 Triple point diagrams

A *triple point diagram* in a disk D^2 is a collection of oriented arcs called strands, defined up to isotopy, such that:

1. Three strands meet at each intersection point.



(a) Expanding the 4-valent black vertices.

(b) Minimal triple point diagram in $\mathbb{T}$.

Figure 2.3: Construction of the minimal triple point diagram in $\mathbb{T}$ from Γ.

2. The end points of each strand are distinct boundary points.

3. The orientations on the strands induce consistent orientations on the complementary regions.

Each strand starts and ends in ∂D^2, so if there are n strands, there are $2n$ points in ∂D^2, whose orientations alternate "in" and "out" as we move along ∂D^2. A triple point diagram is called *minimal* if the number of triple intersections is minimal if strands have no self intersections and parallel bigons.

There is a local move called a 2-2 move on triple point diagrams (see Figure 2.4).

Theorem 2.1.2 (Thurston, 2004 [Thur04], Postnikov, 2006 [Post06]). *1. If there are $2n$ points on the boundary of the disk, all $n!$ matchings of "in" and "out" points are achieved by triple point diagrams.*

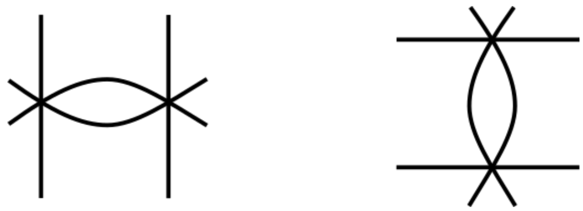


Figure 2.4: The 2-2 move.

2. *Any two minimal triple point diagrams are related by 2-2 moves.*

2.1.6 Triple point diagrams in $\mathbb{T}$

A triple point diagram in $\mathbb{T}$ is a collection of oriented curves called strands in $\mathbb{T}$, determined up to isotopy, such that:

1. Three strands meet at each intersection point.

2. Each strand has non-trivial homology in $\mathbb{T}$.

3. The orientations on the strands induce consistent orientations on the complementary regions.

A triple point diagram in $\mathbb{T}$ is *minimal* if the lift of any strand to the plane has no self intersections and the lifts of any two strands to the plane has no parallel bigons.

We recall the eqiuvalence between minimal triple point digarams in $\mathbb{T}$ and minimal bipartite torus graphs from [GK12]:

1. To convert a minimal bipartite graph to a triple point diagram, first perform a sequence of moves inverse to shrinking a 2-valent white vertex to get a bipartite

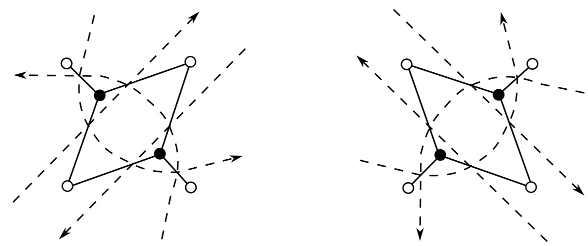


Figure 2.5: Equivalence of elementary transformations and $2-2$ moves.

graph in which all black vertices are 3-valent. Draw all zig-zag strands so that the black complementary regions are now triangles. Shrink all these black triangle regions into points to get a triple point diagram.

2. To go from triple point diagrams to bipartite graphs, resolve each triple point into a counterclockwise triangle. Put a black vertex in each complimentary region that is oriented counterclockwise and a white vertex in each complimentary region that is oriented clockwise. Edges between black and white vertices are given by the vertices of the resolved triple point diagram. Faces of the bipartite graph will be the regions where the orientations alternate.

Under this correspondence, the elementary transformations on bipartite graphs correspond to 2-2 moves as shown in Figure 2.5.

2.2 Cluster structure

We recall the construction of the dimer cluster variety from [GK12].

2.2.1 Seeds

A *seed* $\boldsymbol{s}$ is a triple $(\Lambda, (\cdot,\cdot), \{e_i\})$, where

- Λ is a lattice;
- $(\cdot,\cdot)$ is a skew symmetric integral bilinear form on Γ;
- $\{e_i\}$ is a collection of non-zero vectors in Λ.

By thickening the edges of the bipartite torus graph Γ, we can view it as a ribbon graph. The data of a ribbon graph is equivalent to the data of a cyclic order of edges around each vertex of the graph. We construct a new ribbon graph $\widehat{\Gamma}$ by reversing the cyclic order at all white vertices. The oriented surface graph $\widehat{S}_\Gamma$ obtained from $\widehat{\Gamma}$ by gluing in discs for faces is called the *conjugated surface*.

Since $\widehat{\Gamma}$ is the same as Γ as a topological space, we have a canonical isomorphism $H_1(\widehat{\Gamma}, \mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z})$. Let us denote this lattice by Λ_Γ. The embedding $\widehat{\Gamma} \hookrightarrow \widehat{S}_\Gamma$ induces a homomophism of homology groups $H_1(\widehat{\Gamma}, \mathbb{Z}) \to H_1(\widehat{S}_\Gamma, \mathbb{Z})$. The pullback of the intersection pairing on $\widehat{S}_\Gamma$ gives a skew symmetric integral bilinear form $(\cdot,\cdot)_{\widehat{\Gamma}}$ on Λ_Γ. For a face F of Γ, let $\gamma_F \in \Lambda_\Gamma$ denote the cycle ∂F, the oriented boundary of F. The seed $\boldsymbol{s}_\Gamma$ associated to Γ is $(\Lambda_\Gamma, (\cdot,\cdot)_{\widehat{\Gamma}}, \{\gamma_F\})$.

2.2.2 Seed tori

To a seed $\boldsymbol{s}$, we can associate a complex algebraic torus $\mathcal{X}_{\boldsymbol{s}} := \mathrm{Hom}(\Lambda, \mathbb{C}^*)$ called the *seed torus*. The coordinates X_i on $\mathcal{X}_{\boldsymbol{s}}$ corresponding to e_i are called *cluster variables*.

For the seed $\boldsymbol{s}_\Gamma$ associated to Γ, the seed torus is the space of edge-weights modulo gauge equivalence. While the positive-real-valued edge-weights are important for geometric and probabilistic applications, for the construction of the dimer integrable system we consider more general edge-weights that take values in $\mathbb{C}^*$. Such an edge-weight is the same thing an element of $Z^1(\Gamma, \mathbb{C}^*)$, and two edge-weights are gauge equivalent iff they differ by a coboundary. Therefore the space of gauge equivalence classes of edge-weights is identified with $H_1(\Gamma, \mathbb{C}^*)$, which we denote by $\mathcal{L}_\Gamma$. Alternately we may view a gauge equivalence class of edge-weights as a $\mathbb{C}^*(= \mathrm{GL}_1(\mathbb{C}))$-local system on Γ. The positive-real-valued points $\mathcal{L}_\Gamma(\mathbb{R}_{>0})$ of $\mathcal{L}_\Gamma$ parameterize the Boltzmann probability measures.

The *monodromy* $m(L)$ of an edge-weight wt around an oriented loop L is the product of edge-weights of edges in the loop. These monodromies generate the algebra of regular functions $H^0(\mathcal{O}_{\mathcal{L}_\Gamma})$. The intersection form $(\cdot,\cdot)_{\widehat{\Gamma}}$ gives rise to a Poisson structure on $H^0(\mathcal{O}_{\mathcal{L}_\Gamma})$:

$$\{m(L_1), m(L_2)\} := (L_1, L_2)_{\widehat{\Gamma}} m(L_1) m(L_2).$$

Let γ_x, γ_y be loops in Γ generating $H_1(\mathbb{T}, \mathbb{Z})$. For a face F of Γ, let $\partial F \in H_1(\Gamma, \mathbb{Z})$ denote the boundary of F, oriented counterclockwise. Let us denote $X_F := m(\partial F)$ and $z := m(\gamma_x), w := \gamma_y$. Then

$$H^0(\mathcal{O}_{\mathcal{L}_\Gamma}) = \mathbb{C}[z^{\pm 1}, w^{\pm 1}, X_F^{\pm 1} : F \text{ is a face of } \Gamma] / \left\langle \prod_{\text{faces } F} X_F = 1 \right\rangle,$$

These coordinates provide an isomorphism $\mathcal{L}_\Gamma \cong (\mathbb{C}^*)^{\text{number of faces of } \Gamma + 1}$.

2.2.3 Mutations

Given a seed $\boldsymbol{s}$, a mutation of $\boldsymbol{s}$ in the direction e_k is a new seed $\mu_{e_k}(\boldsymbol{s})$ given by a new collection of vectors $\{e_i'\}$:

$$e_i' := \begin{cases} e_i + (e_i, e_k)_+ e_k & \text{if } i \neq k; \\ -e_k & \text{if } i = k. \end{cases}$$

A mutation μ_{e_k} induces a Poisson birational map between seed tori $\mu_{e_k} : \mathcal{X}_{\boldsymbol{s}} \dashrightarrow \mathcal{X}_{\mu_{e_k}(\boldsymbol{s})}$ defined on cluster variables by

$$\mu_{e_k}^* : X_i \mapsto X_i(1 + X_k)^{-(e_i, e_k)}.$$

Mutations in the dimer model arise from elementary transformations of bipartite torus graphs. Shrinking/expanding 2-valent vertices $\Gamma \to \Gamma'$ gives a canonical identification between the seeds $\boldsymbol{s}_\Gamma$ and $\boldsymbol{s}_{\Gamma'}$, whereas for the spider move we have:

Lemma 2.2.1 (Goncharov and Kenyon, [GK12] Lemma 4.5)**.** *A spider move* $\Gamma \to \Gamma'$ *at a face* F *gives a mutation of seeds* $\mu_{\gamma_F} : \boldsymbol{s}_\Gamma \to \boldsymbol{s}_{\Gamma'}$.

2.2.4 The cluster $\mathcal{X}$ variety

The Poisson scheme obtained by gluing the seed tori using the birational maps induced by mutations is called the *cluster* $\mathcal{X}$ *variety.*

Given a convex integral polygon $N \subset H_1(\mathbb{T}, \mathbb{R})$, we glue together the seed tori $\mathcal{L}_\Gamma$ for all graphs Γ associated to N in Theorem 2.1.1 to get a Poisson scheme $\mathcal{X}_N$. This will be the phase space of the dimer integrable system. Mutations involve subtraction free rational functions and therefore, $\mathcal{X}_N$ inherits a well-defined notion of positive-real-valued points from the $\mathcal{L}_\Gamma$, which we denote by $\mathcal{X}(\mathbb{R}_{>0})$. We emphasize that we only glue together cluster charts related by spider moves, that is mutations at square faces rather than all possible mutations.

2.2.5 The cluster modular group

A *seed cluster transformation* is a compostion of seed isomorphisms and mutations. Mutations induce birational maps while seed isomorphisms σ induce isomorphisms between the seed tori:

$$\sigma^*(X_{\sigma(i)}) = X_i.$$

Composing the birational maps induced by mutations and seed isomorphisms, a seed cluster transformation gives a birational map between seed tori, called a *cluster transformation.* A seed cluster transformations $\boldsymbol{s} \to \boldsymbol{s}$ is *trivial* if the induced cluster transformation is the identity. The groupoid $\mathcal{G}_{\boldsymbol{s}}$ whose objects are seeds that are related to $\boldsymbol{s}$ by a seed cluster transformation and morphisms are seed cluster transformations modulo trivial seed cluster transformations is called the *cluster modular groupoid.* The fundamental group $G_{\boldsymbol{s}}$ of $\mathcal{G}_{\boldsymbol{s}}$ based at $\boldsymbol{s}$ is called the *cluster modular group.* The cluster transformations associated to elements of $G_{\boldsymbol{s}}$ give birational automorphisms of $\mathcal{X}$.

Seed isomorphisms of $\boldsymbol{s}_\Gamma$ correspond to graph automorphisms of Γ. By Theorem 2.1.1, all seeds are related by elementary transformations. Therefore the cluster

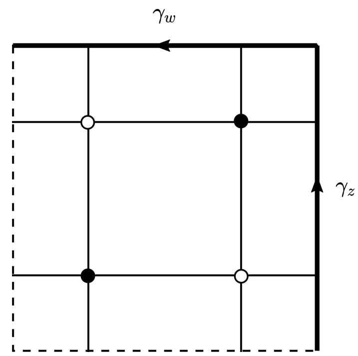


Figure 2.6: The fundamental rectangle R

modular group of $\mathcal{X}_N$ is canonically associated to N, that is, it does not depend on the choice of base seed. We denote the cluster modular group of $\mathcal{X}_N$ by G_N.

We emphasize that like in the construction of $\mathcal{X}_N$, we consider only the subset of mutations consisting of elementary transformations when defining the cluster modular group.

2.3 Algebraic complete integrability

Let R be a fundamental rectangle of $\mathbb{T}$, so that $\mathbb{T}$ is obtained by gluing together opposite sides of R. Let γ_z, γ_w the sides of R generating $H_1(\mathbb{T}, \mathbb{Z})$, oriented as shown in Figure 2.6. Let $\langle \cdot, \cdot \rangle_{\mathbb{T}}$ be the intersection pairing on $\mathbb{T}$. Let z, w denote the Poincare duals of γ_z, γ_w giving an isomorphism $T \cong (\mathbb{C}^*)^2$. For each edge e of Γ, we associate a character $z^{\langle e, \gamma_z \rangle_{\mathbb{T}}} w^{\langle e, \gamma_w \rangle_{\mathbb{T}}}$.

A *Kasteleyn sign* is an edge-weight $\kappa : E(\Gamma) \to \mathbb{C}^*$, such that the the monodromy of κ around a loop L is -1 (respectively 1) if the number of edges in L is 0 mod 4 (respectively 2 mod 4). Given an edge-weight wt and a Kasteleyn sign κ, the Kasteleyn operator is the $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$-algebra homomorphism defined as follows:

(2.1)

$$K(z,w) : \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^{B(\Gamma)} \to \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^{W(\Gamma)}$$

(2.2)

$$K(z,w)_{v_w,v_b} = \sum_{e \in E(\Gamma) \text{ incident to } v_b, v_w} wt(e)\kappa(e) z^{\langle e, \gamma_z \rangle_{\mathbb{T}}} w^{\langle e, \gamma_w \rangle_{\mathbb{T}}}, \text{ for } v_b \in B(\Gamma), v_w \in W(\Gamma).$$

Theorem 2.3.1 (Kasteleyn 1967, [Kas67])**.**

$$det\ K(z,w) = z^{i_0} w^{j_0} \sum_{M \, dimer\ cover\ of\ \Gamma} sign(M) wt(M) z^{i_M} w^{j_M},$$

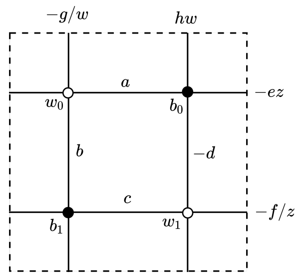
where $sign(M) \in \{\pm 1\}$ is a sign that depends on the homology class $[M - M_0]$ and κ. Here $(i_M, j_M) \in \mathbb{Z}^2$ is the homology class of $[M - M_0]$ in the basis of $H_1(\mathbb{T}, \mathbb{Z})$ given by $(-\gamma_w, \gamma_z)$ and $z^{i_0} w^{j_0}$ is an unimportant monomial.

$P(z,w) :=$ det $K(z,w)/wt(M_0)$ is called the *characteristic polynomial*, and its vanishing locus $\{P(z,w) = 0\} \subset (\mathbb{C}^*)^2$ is called the *spectral curve*. From Theorem 2.3.1, we see that N_Γ is the Newton polygon of $P(z,w)$, justifying the name.

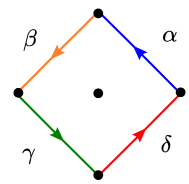
We rewrite the sum in (2.1) as:

$$P(z,w) = z^{i_0} w^{j_0} \sum_{(i,j) \in N_\Gamma \cap \mathbb{Z}^2} H_{i,j} z^i w^j,$$

where $H_{i,j} := \sum_{M:[M-M_0]=(i,j)} \mathrm{sign}(M) wt(M)/wt(M_0)$ enumerates dimer covers with homology class (i,j). The $H_{i,j}$ associated to the interior points of N_Γ are the Hamiltonians of the dimer integrable system.



(a) Edge-weights and Kasteleyn signs.



(b) Newton polygon and zig-zag paths.

Figure 2.7: A fundamental domain for the square lattice.

The Poisson center of $\mathcal{O}_{\mathcal{L}_\Gamma}$ is generated by monodromies around zig-zag paths of Γ. For a zig-zag path α, we denote by C_α the monodromy around it. These are the *Casimirs*. The symplectic leaves are the common level sets of all the Casimirs. The dimension of the generic symplectic leaf is $2g$, where g is the number of interior points of N_Γ.

Theorem 2.3.2 (Goncharov and Kenyon, [GK12] Theorem 3.7)**.** *The Hamiltonians $H_{i,j}$ for interior points $(i,j) \in N_\Gamma \cap \mathbb{Z}^2$ are independent and commute with respect to the Poisson bracket on $\mathcal{L}_\Gamma$. There are g of them, which is half the dimension of the generic symplectic leaf.*

Let us work out an explicit example. Consider the square lattice with the edge-weights and Kasteleyn signs shown in Figure 2.7a. Let us take the reference dimer M_0 to be the one with weight ac. We have:

$$K(z,w) = \begin{array}{cc} b_0 & b_1 \\ \end{array}$$

$$K(z,w) = \begin{array}{cc|c} & \begin{array}{cc} \quad b_0 \quad & \quad b_1 \quad \end{array} & \\ & \begin{pmatrix} a - ez & b - g/w \\ -d + hw & c - f/z \end{pmatrix} & \begin{array}{c} w_0 \\ w_1 \end{array} \end{array},$$

$$P(z,w) = 1 + \frac{bd}{ac} + \frac{ef}{ac} + \frac{gh}{ac} - \frac{dg}{ac}\frac{1}{w} - \frac{bh}{ac}w - \frac{f}{c}\frac{1}{z} - \frac{e}{a}z. \tag{2.3}$$

The Casimirs are given by:

$$C_\alpha = \frac{bh}{ce}, \quad C_\beta = \frac{af}{bh}, \quad C_\gamma = \frac{dg}{af}, \quad C_\delta = \frac{ce}{dg},$$

and they satisfy the relation $\prod_\rho C_\rho = 1$ in $H^1(\Gamma, \mathbb{C}^*)$, where the product is over all zig-zag paths. The Newton polygon has a single interior point, and the Hamiltonian corresponding to it is $H = 1 + \frac{bd}{ac} + \frac{ef}{ac} + \frac{gh}{ac}$.

2.4 The spectral transform

2.4.1 Toric surfaces

A *toric surface* is a normal algebraic variety of dimension 2 containing the algebraic torus $(\mathbb{C}^*)^2$ as a dense open subset, such that the action of $(\mathbb{C}^*)^2$ on itself extends. A convex integral polygon N gives rise to a projective toric surface X_N, along with an ample divisor D_N, such that the linear system $|D_N|$ is identified with curves defined by vanishing of Laurent polynomials with Newton polygon N. A generic curve $C \in |D_N|$ has genus g equal to the number of interior lattice points in N. The complement of the algebraic torus in X_N is a union of $\mathbb{P}^1$s, called *lines at infinity*, parameterized by the edges of N, and intersecting according to the combinatorics of N. We denote the line at infinity corresponding to $E_\rho \in E_N$ by D_ρ. For $C \in |D_N|$, we have $|C \cap D_\rho| = |E_\rho|$, where the points in $C \cap D_\rho$ are counted with multiplicity.

2.4.2 The spectral transform

We follow [GK12, Section 7]. A parameterization of a divisor $C = \sum_{i=1}^{k} n_i c_i$ by a set S is a function $\nu : S \to \{c_1, ..., c_k\}$ such that $|\nu^{-1}(c_i)| = n_i$. A *spectral data* is a triple (C, S, ν) where $C \in |D_N|$ is a genus g curve, S is a degree g effective divisor on $C \cap (\mathbb{C}^*)^2$ and $\nu = \{\nu_\rho\}$ are parameterizations of the divisors $D_\rho \cap C$. Let $\mathcal{S}_N$ be the moduli space parameterizing the spectral data related to N.

Fix a minimal bipartite graph Γ with Newton polygon N, and a white vertex w_0 of Γ. There is a rational map, called the *spectral transform*, defined by Kenyon and Okounkov [KO03],

$$\begin{aligned} \kappa_{\Gamma,w_0} : \mathcal{X}_N(\mathbb{C}) \supset \mathcal{L}_\Gamma(\mathbb{C}) &\dashrightarrow \mathcal{S}_N \\ wt &\mapsto (C, S, \nu), \end{aligned}$$

defined as follows:

1. C is the spectral curve.

2. S is a degree g effective divisor on C defined as follows: The cokernel of $K(z, w)$ is the pushforward of a line bundle on C for generic $wt \in \mathcal{L}_\Gamma(\mathbb{C})$. The restriction to C of the image of δ_{w_0} gives a section of this line bundle. S is the divisor of this section.

3. ν is the parameterization of the points at infinity by zig-zag paths given as follows: Consider the restriction $K|_\alpha$ of the Kasteleyn operator to the zig-zag path α. The point at infinity associated to α is the point where $K|_\alpha$ is singular.

Let us compute the spectral transform for the example in Figure 2.7. The spectral curve C is given by the vanishing of $P(z,w)$ in (2.3). Here $g = 1$, so the divisor $S = (p,q)$ is a single point on C. We can compute it explicitly by looking for the simultaneous vanishing of the w_0-column of the adjugate matrix of K, that is, it is given by:

$$c - \frac{f}{z} = -d + hw = 0,$$

so $(p,q) = (\frac{f}{c}, \frac{d}{h})$. Since there is only one zig-zag path in each homology direction in this example, the parameterization ν is irrelevant.

The main properties of the spectral transform are:

1. The spectral transform is a birational change of coordinates for $\mathcal{X}_N$ (Theorem 2.4.1).

2. The birational automorphisms of $\mathcal{X}_N$ induced by seed cluster transformations are linearized in these coordinates (Theorem 2.5.2).

We have:

Theorem 2.4.1 (Fock, 2015 [F15], George, Goncharov and Kenyon [GGK])**.** *The spectral transform is a birational isomorphism.*

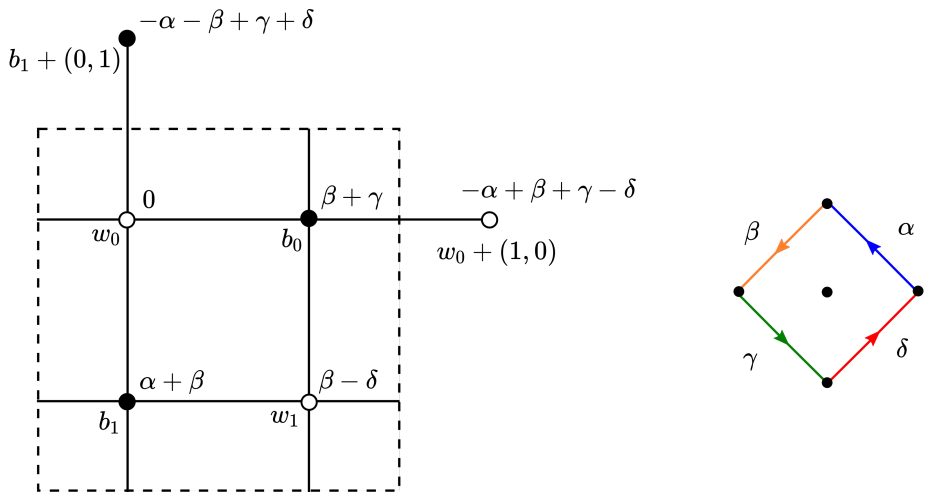


Figure 2.8: The discrete Abel map computed with $d(w_0) = 0$ for the square lattice. The zig-zag paths are labeled by their corresponding sides of the Newton polygon on the right.

2.5 Linearization on the Jacobian

The discrete Abel map

Let $\widetilde{\Gamma}$ be the lift of Γ to the plane. Let $\mathrm{Div}_\infty(C)$ denote the divisors at infinity of C, i.e. $\mathbb{Z}$-linear combinations of the points in $\cup_\rho D_\rho \cap C$. We define the discrete Abel map d following Fock [F15] by the following rules:

$$d : \text{Vertices of } \widetilde{\Gamma} \to \mathrm{Div}_\infty(C)$$

$$d(w_0) = 0,$$

$$d(v_2) - d(v_1) = \sum_{\text{Zig-zag paths } \alpha} \langle \alpha, \gamma \rangle_{\mathbb{T}} \nu(\alpha),$$

for any path γ from v_1 to v_2. Locally d is described by the following rule, which is usually how we compute it: If bw is an edge, with zig-zag paths α, β containing bw, then

$$d(w) = d(b) - \nu(\alpha) - \nu(\beta).$$

We have an embedding

$$H_1(\mathbb{T}, \mathbb{Z}) \hookrightarrow Div_\infty(C)$$
$$m \mapsto \text{div } \chi^m|_C = \sum_{\text{Zig-zag paths } \alpha} \langle \alpha, m \rangle_{\mathbb{T}} \nu(\alpha),$$

where χ^m is the character of $T = (\mathbb{C}^*)^2$ associated to $m \in H_1(\mathbb{T}, \mathbb{Z})$. In coordinates, if $m = (i, j)$, then $\chi^m = z^i w^j$. d is $H_1(\mathbb{T}, \mathbb{Z})$-equivariant:

$$d(v + m) = d(v) + m,$$

so $d(v)$ is a well defined divisor class in C for each vertex v of Γ.

Example 2.5.1.

In Figure 2.8, we compute the discrete Abel map for the square lattice. The embedding $H_1(\mathbb{T}, \mathbb{Z}) \hookrightarrow Div_\infty(C)$ is given by

$$(1, 0) \mapsto -\alpha + \beta + \gamma - \delta;$$
$$(0, 1) \mapsto -\alpha - \beta + \gamma + \delta.$$

Let $t : \boldsymbol{s}_\Gamma \to \boldsymbol{s}_\Gamma$ be a seed cluster transformation. If ν is a parameterization of points at infinity of $\bar{C}$ and zig-zag paths of Γ, then there is an induced parameterization ν_t after doing t; the monodromy around a zig-zag path is preserved by elementary transformations. Similarly, t also induces a new discrete Abel map d_t; for

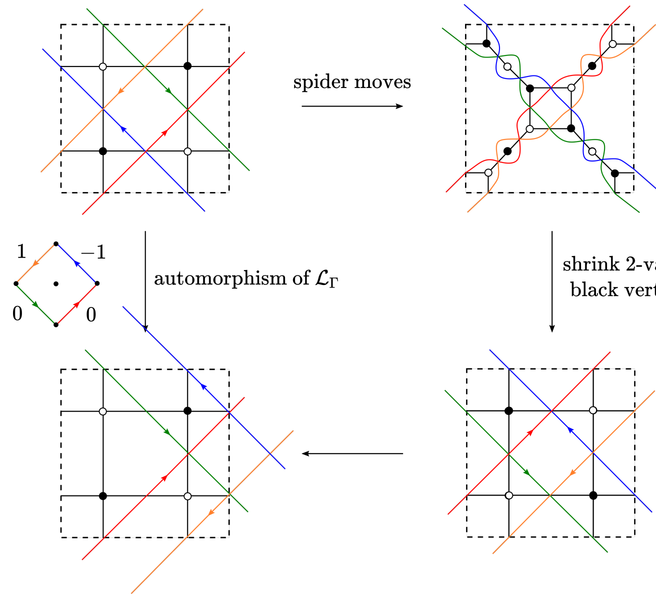


Figure 2.9: The relative positions of zig-zag paths for a seed cluster transformation.

any side E_ρ of N, the zig-zag paths in $\widetilde{\Gamma}$ divide the plane into an infinite collection of strips. The discrete Abel map of a vertex tells which of these strips contain the vertex. Since zig-zag paths corresponding to the same edge E_ρ do not cross each other during elementary transformtions, there is a bijection of strips before and after t, which gives us d_t.

Theorem 2.5.2 (Fock, 2015 [F15, Proposition 1])**.** *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{X}_N \arrow[r, dashed, "\kappa_{\Gamma,w_0}"] \arrow[d, dashed, "\mu_t"] & \mathcal{S}_N \arrow[d, dashed] \\ \mathcal{X}_N \arrow[r, dashed, "\kappa_{\Gamma,w_0}"] & \mathcal{S}_N \end{tikzcd},$$

where the map on the left is $(C, S, \nu) \mapsto (C, S_t, \nu_t)$, where S_t is the (generically) unique degree g effective divisor satisfying

$$S_t = S + d(w_0) - d_t(w_0), \text{ in } Pic^g(C).$$

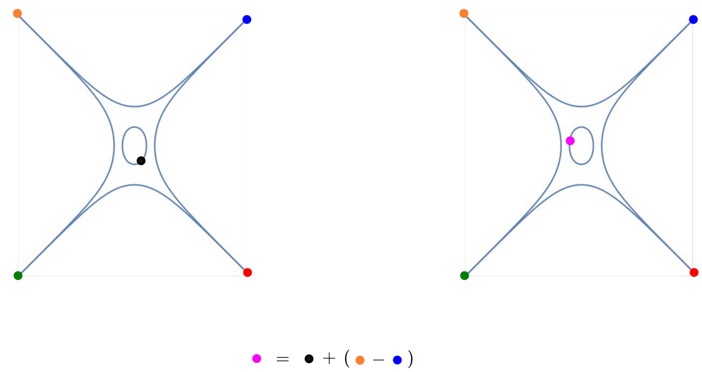


Figure 2.10: The black point on the left is the divisor on the amoeba of the spectral curve. The points at infinity of the curve are in bijection with zig-zag paths and colored according to Figure 2.9. The seed cluster transformation in Figure 2.9 maps the black point to the pink point. Fock [F15] shows that this map is the translation shown above in the Jacobian variety of the spectral curve.

Therefore, cluster modular transformations are translations on a cover of the Jacobian variety of C*.*

Example 2.5.3.

Consider t to be the cluster modular transformation of Figure 2.9 for the square lattice. Recall that we computed the discrete Abel map in Example 2.5.1. From the relative positions of zig-zag paths in Figure 2.9, we can see that the induced discrete Abel map at w_0 is given by

$$d_t(w_0) = \alpha - \beta.$$

Therefore Theorem 2.5.2 tells us that $S_t = S - \alpha + \beta$, as illustrated in Figure 2.10.

CHAPTER 3

The cluster modular group of the dimer model

Domino-shuffling is a technique introduced in [EKLP] to enumerate and generate domino tilings of the Aztec diamond graph, and was used to give the first proof of the arctic circle theorem [JPS]. Domino tilings are dual to the dimer model on the square grid. There are generalizations of domino-shuffling for other biperiodic bipartite graphs and the *cluster modular group* is a group whose elements correspond to these shufflings. This group was studied by Fock and Marshakov [FM16], who gave an explicit conjecture for its isomorphism type. The goal of this chapter is to study generalized shufflings, and in particular, to compute the cluster modular group for any biperiodic bipartite graph. The natural framework for this is the *dimer integrable system* $\mathcal{X}_N$ associated to a convex polygon N.

Generalized shufflings are dynamical systems on the space of edge-weights. Let Γ be a bipartite graph on a torus $\mathbb{T}$ and let $\mathcal{L}_\Gamma$ be the space of edge-weights on Γ (modulo gauge transformations, see Section 2.2.2 for the precise definition). There are two types of local rearrangements of bipartite graphs called *elementary transformations* (see Figure 5.2). Each elementary transformation comes with a transformation of edge weights, characterized by the property that it preserves the dimer partition function (see for example [GK12, Theorem 4.7]). Given a sequence of elementary transformations such that the initial and final graphs are both Γ (called a *seed cluster transformation*), composing the induced transformations of edge weights gives an automorphism of $\mathcal{L}_\Gamma$. The seed cluster transformation is *trivial* if the induced map on edge weights is the identity. The *cluster modular group* is the group of seed cluster transformations modulo the trival ones (cf. section 2.2.5).

A *zig-zag path* in Γ is a path that turns maximally left at white vertices and maximally right at black vertices (cf. Section 2.1.4). Associated to any biperiodic bipartite graph is a convex integral polygon N called its *Newton polygon*, whose primitive edges correspond to homology classes of zig-zag paths. The cluster modular group only depends on the polygon N. Our main result is the following conjecture of Fock and Marshakov [FM16]:

Theorem 3.0.1 (cf. Theorem 3.3.4)**.** *If the Newton polygon N has an interior lattice point, the cluster modular group G_N is isomorphic to the group*

$$\mathbb{Z}_0^{\text{Edges of } N}/H_1(\mathbb{T},\mathbb{Z}),$$

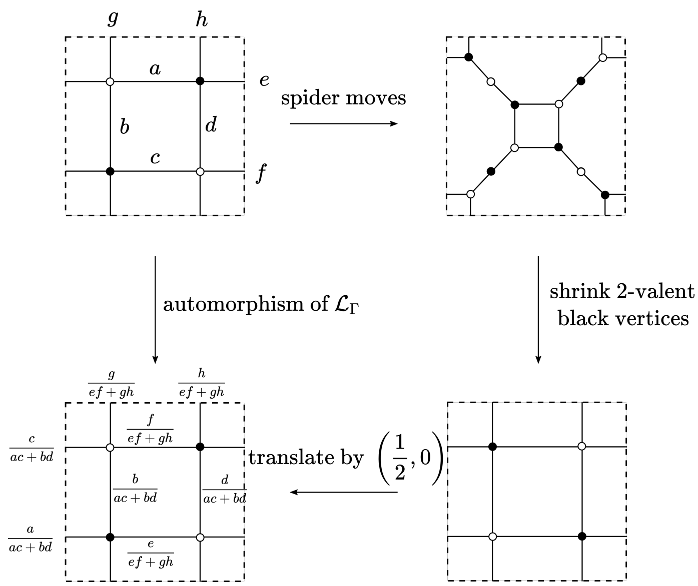


Figure 3.1: A dynamical system associated to the dimer model.

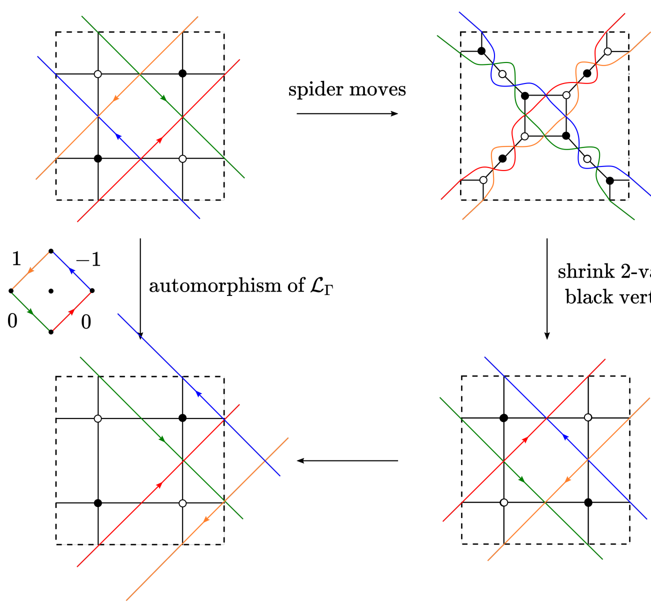


Figure 3.2: The relative positions of zig-zag paths for the seed cluster transformation from Figure 3.1. The arrow on the left shows the Newton polygon and ϕ associated to this seed cluster transformation.

defined in the paragraph below. When there is no interior lattice point, the cluster modular group is a smaller finite group.

Corollary 3.0.2. *When N contains an interior lattice point, the rank of G_N is $|E_N| - 3$, where $|E_N|$ is the number of edges of the polygon N. When N has no interior lattice points, the rank is 0.*

Elementary transformations have a description in terms of zig-zag paths (see Figure 2.5). Let $\widetilde{\Gamma}$ be the planar biperiodic graph whose quotient is Γ. If we superpose $\widetilde{\Gamma}$ over itself after a seed cluster transformation, each zig-zag path is superposed over one that is a translate of it. Following Fock and Marshakov [FM16] we can associate an integer function ϕ from the edges of the Newton polygon N as follows: For any edge E of N, the inverse image in the universal cover of the torus of all zig-zag paths corresponding to E is an infinite number of parallel zig-zag paths in $\widetilde{\Gamma}$, let us label them by $(\alpha_E^i)_{i\in\mathbb{Z}}$, ordered from left to right. Consider the zig-zag path α_E^0: after the seed cluster transformation, if we superpose $\widetilde{\Gamma}$ over itself, α_E^0 is superposed over a parallel zig-zag path, say α_E^j. Then we define $\phi(E) := -j$, the distance that any zig-zag path corresponding to E in $\widetilde{\Gamma}$ is translated by the seed cluster transformation. This function satisfies

$$\sum_{\text{Edges } E \text{ of } N} \phi(E) = 0.$$

Let us denote by $\mathbb{Z}_0^{\text{Edges of } N}$ the group of integral functions on the edges of N with sum zero over the edges. ϕ is not a well defined function since we can translate $\widetilde{\Gamma}$ by the homology $H_1(\mathbb{T},\mathbb{Z})$ of the torus before we superpose. Therefore we define ϕ to be an element of $\mathbb{Z}_0^{\text{Edges of } N}/H_1(\mathbb{T},\mathbb{Z})$, where the embedding is given by the distance zig-zag paths in $\widetilde{\Gamma}$ are translated by elements of $H_1(\mathbb{T}, \mathbb{Z})$:

$$\begin{aligned} H_1(\mathbb{T}, \mathbb{Z}) &\hookrightarrow \mathbb{Z}_0^{\text{Edges of } N} \\ m &\mapsto (E \mapsto \langle E, m \rangle)\,, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the intersection pairing in $H_1(\mathbb{T}, \mathbb{Z})$. Figure 3.2 shows the relative positions of a zig-zag path of each homology class before and after the seed cluster transformation from Figure 3.1.

The point of Theorem 3.0.1 is to provide a framework to extend results about the dimer model on the square lattice derived using domino-shuffling to general biperiodic dimer models. A drawback of working in this level of generality is that the theorem does not provide an explicit way to produce a cluster modular transformation associated to a given element of $\mathbb{Z}_0^{\text{Edges of } N}/H_1(\mathbb{T}, \mathbb{Z})$. This is to be expected; the family of minimal bipartite graphs associated to a convex integral polygon is also not explicit; both constructions rely on an existence result for triple point diagrams due to Thurston [Thur04] (cf. Section 2.1.5).

The proof of Theorem 3.0.1 has two parts. In Section 3.2, we show surjectivity, i.e. every element of $\mathbb{Z}_0^{\text{Edges of } N}/H_1(\mathbb{T}, \mathbb{Z})$ arises from a seed cluster transformation. This part is purely combinatorial. Translations by $H_1(\mathbb{T}, \mathbb{Z})$ are clearly trivial seed cluster transformations. The second part is to show that these are the only trivial seed cluster transformations. The induced transformation of edge-weights of a seed cluster transformation is a complicated rational function, and it seems difficult to

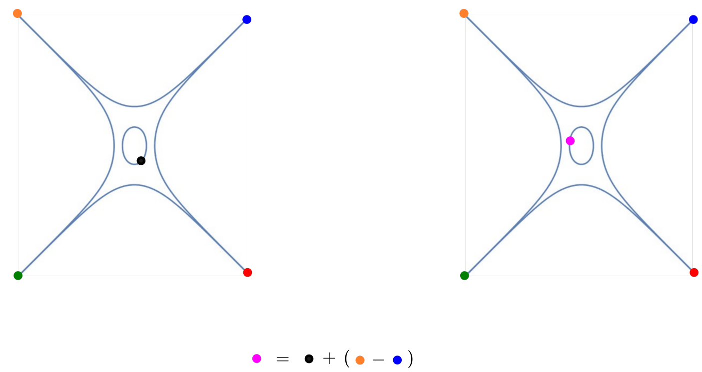


Figure 3.3: The black point on the left is the divisor on the amoeba of the spectral curve. The points at infinity of the curve are in bijection with zig-zag paths and colored according to Figure 3.2. The seed cluster transformation in Figure 3.2 maps the black point to the pink point. Fock [F15] shows that this map is the translation shown above in the Jacobian variety of the spectral curve. This translation is determined by the function ϕ shown in Figure 3.2.

check when this function is the identity. However, there is a birational change of coordinates under which this transformation becomes linear.

Kenyon and Okounkov [KO03] defined the *spectral transform* (cf. Section 2.4) of $wt \in \mathcal{L}_\Gamma$ to be a triple (C, S, ν), where $C \subset (\mathbb{C}^*)^2$ is a curve called the *spectral curve* and S is a divisor, that is a formal linear combination of points in C. C is the vanishing locus of a Laurent polynomial $P(z, w)$ which is a signed homology-class-weighted version of the partition function for dimer covers. The spectral transform is a birational isomorphism [F15, GGK], so we can view (C, S, ν) as coordinates on $\mathcal{X}_N$. In these coordinates, the seed cluster transformation acts by a translation of the divisor S in (a cover of) the Jacobian variety of C, as shown by Fock [F15] (see Section 2.5 and Figure 3.3).

This reduces the question of which seed cluster transformations are trivial to the following problem:

Theorem 3.0.3 (cf. Theorem A.0.11)**.** *Assume N has an interior lattice point. If L is a non-trivial line bundle on the projective toric surface X_N associated to N, then for a generic spectral curve C, we have $L|_C \ncong \mathcal{O}_C$.*

This is proved by Giovanni Inchiostro in the appendix.

In the last paragraph of [FM16, Section 7.3], Fock and Marshakov provide an alternate description of the group G_N; it is the group of divisor classes on the toric surface X_N that restrict to degree 0 divisors on C. However this is only true as stated for polygons whose sides are all primitive, that is, no side contains a lattice point other than the end points. Recently Treumann, Williams and Zaslow [TWZ18] gave a different version of linearization of cluster modular transformations under the spectral transform, replacing the toric variety X_N by a toric stack. Associated to N is a stacky fan $\mathbf{\Sigma}$ (see Section 3.3.1) which can be used to construct a stacky toric surface $\mathscr{X}_N$ compactifying $(\mathbb{C}^*)^2$. Let D_ρ be the toric divisor associated to the edge E_ρ of N. Let $\mathcal{L}_\rho$ denote the line bundle $\mathcal{O}(\frac{1}{|E_\rho|}D_\rho)$, where $|E_\rho|$ is the number of primitive vectors in E_ρ. The birational automorphism of $\mathcal{X}_N$ induced by a seed cluster transformation corresponds to the tensor product action on the pushforward of $\mathcal{O}_C(S)$ to $\mathscr{X}_N$ by the line bundle

$$\mathcal{L} := \bigotimes_\rho \mathcal{L}_\rho^{\otimes \phi(E_\rho)} \in \mathrm{Pic}^0(\mathscr{X}_N),$$

where $\mathrm{Pic}^0(\mathscr{X}_N)$ denotes the subgroup of the Picard group satisfying $\sum_\rho \phi(E_\rho) = 0$.

The group $\mathrm{Pic}^0(\mathscr{X}_N)$ is explicitly computed in [BH09] where they show that $\phi \mapsto \bigotimes \mathcal{L}_\rho^{\otimes\phi(E_\rho)}$ is an isomorphism $\mathbb{Z}_0^{\text{Edges of } P}/H_1(\mathbb{T},\mathbb{Z}) \cong \mathrm{Pic}^0(\mathscr{X}_N)$.

Theorem 3.0.4. *When the Newton polygon N has an interior lattice point, G_N is isomorphic to $Pic^0(\mathscr{X}_N)$.*

In Section 3.4, we study the torsion subgroup of G_N. We show that:

Theorem 3.0.5. *In the family of minimal bipartite graphs associated to N, there is a subfamily that has maximal possible translation symmetry related by elementary transformations that respect this symmetry.*

This gives the following characterization of the torison subgroup of G_N:

Lemma 3.0.6. *The torsion subgroup of the cluster modular group is the group of automorphisms induced by translations of any maximally translation invariant bipartite graph.*

Triangles

For triangular N, [IU15] Proposition 11.3 tells us that there is a unique bipartite graph with Newton polygon N and its lift to the plane is the honeycomb lattice. Since this graph does not admit any elementary transformations, the only cluster modular transformations are translation symmetries. We also see this from the explicit cluster modular group. By Corollary 3.0.2, the cluster modular group has zero rank and therefore by Lemma 3.0.6, it consists of translation symmetries.

Quadrilaterals

Corollary 3.0.2 tells us that the cluster modular group has rank one. The dimer models that have quadrilateral Newton polygons coincide with those that arise from the Speyer's "crosses and wrenches" construction [Spey04]. The octahedron recurrence studied there is the (essentially unique) non-torsion cluster modular transformation (on the $\mathcal{A}$ cluster variety). Other incarnations of this cluster modular transformation are Hirota's bilinear difference equation [Miwa82], the domino shuffling algorithm [EKLP, Propp03], the shuffling studied in [BF18] for the suspended pinch point graph and the pentagram map [FM16, Section 8.5].

The octahedron recurrence can be used to compute arctic curves [PS06, DFS14]. We observed in [G18] that part of the data needed for this technique of computing arctic curves is a cluster modular transformation along with edge weights that are periodic under the induced birational map. We hope that understanding the cluster modular group will help generalize this method beyond the quadrilateral Newton polygon case. Since higher degree polygons have cluster modular groups with rank greater than one by Corollary 3.0.2, we expect a family of arctic curves, one for each non-torsion cluster modular transformation.

Higher degree polygons

Cluster modular transformations for the dP_2 quiver, which has a pentagon Newton polygon, were explicitly studied in [GLVY16]. The dP_3 quiver with a hexagonal Newton polygon has been studied in [LMNT14, LM17, LM19]. The cube recurrence studied in [CS04, PS06] arises as the restriction to $\mathcal{R}_P$ of a cluster modular transformation on the dP_3 graph [GK12, Section 6.3].

3.1 The conjecture of Fock and Marshakov

We follow [FM16, Section 7.3]. Let $\mathbb{Z}_0^{E_N}$ be the group of integer valued functions f on E_N such that $\sum_\rho f(E_\rho) = 0$. Let $\langle *, * \rangle_{\mathbb{T}} : H_1(\mathbb{T}, \mathbb{Z}) \times H_1(\mathbb{T}, \mathbb{Z}) \to \mathbb{Z}$ be the intersection pairing on $\mathbb{T}$. We have an embedding

$$\begin{aligned} j : H_1(\mathbb{T}, \mathbb{Z}) &\hookrightarrow \mathbb{Z}_0^{E_N} \\ * &\mapsto \sum_\rho \langle |E_\rho| u_\rho, * \rangle_{\mathbb{T}} E_\rho \end{aligned}$$

Let Γ be a bipartite torus graph and let T be its triple point diagram. A seed cluster transformation of Γ is given by a sequence of triple point diagrams $T = T_1 \to T_2 \to \cdots \to T_{n-1} \to T_n \cong T$, where T_{i+1} is obtained from T_i by performing a 2-2 move. Let $\{\alpha_i\}$ be the strands of T. The sequence can be realized by a one parameter family of curves $\alpha^i(t), t \in [0,1]$ with $\alpha^i(0) = \alpha^i$ such that the intersections remain triple at all but $n-1$ parameter values where we have a quadruple intersection in the course of a 2-2 move. Using the isomorphism of triple point diagrams $T_1 \cong T_n$, we glue the end points of the parameter interval $[0,1]$ to get an S^1. Each strand α in T_1 traces out a 2-chain $S^i := \alpha^i(S^1)$ in $\mathbb{T} \times S^1$.

Let E_ρ be an edge of N and let $\{\alpha_\rho^i\}$ be the strands of T corresponding to E_ρ, and let S_i^ρ be the 2-chains they trace out. The seed cluster transformation induces a bijection of $\{\alpha_\rho^i\}$, and therefore $\partial(\sum_i S_\rho^i) = 0$. Moreover, $\sum_\rho \sum_i S_\rho^i$ is a 2-boundary; it is the boundary of the 3-chain in $\mathbb{T}\times S^1$ traced out by the regions of T corresponding to white vertices of Γ. Therefore in $H_2(\mathbb{T}\times S^1,\mathbb{Z})$, we have

$$\sum_\rho \sum_i [S^i_\rho] = 0.$$

Let $\{\gamma_x,\gamma_y\}$ be a basis for $H_1(\mathbb{T},\mathbb{Z})$ and γ_z for $H_1(S^1,\mathbb{Z})$. By the Künneth formula, $H_2(\mathbb{T}\times S^1,\mathbb{Z}) \cong \Lambda^2_{\mathbb{Z}}[\gamma_x,\gamma_y,\gamma_z]$. If each zig-zag path associated to E_ρ is translated by $a^i_\rho\gamma_x + b^i_\rho\gamma_y$, we have

$$\begin{aligned}
[S^i_\rho] &= (X_i\gamma_x + Y_i\gamma_y)\wedge(a^i_\rho\gamma_x + b^i_\rho\gamma_y + \gamma_z)\\
&= (b^i_\rho X_i - a^i_\rho Y_i)\gamma_x\wedge\gamma_y + X_i\gamma_x\wedge\gamma_z + Y_i\gamma_y\wedge\gamma_z.
\end{aligned}$$

Define a function on E_N by $g(E_\rho) = \sum_i b^i_\rho X_i - a^i_\rho Y_i$, where we take the sum over all zig-zag paths α_i associated to E_ρ. In other words, each zig-zag path is translated in the plane to a parallel zig-zag path by the seed cluster transformation. $g(E_\rho)$ is the number of steps in the direction of the normal to E_ρ pointing into the Newton polygon that any zig-zag path associated to E_ρ is translated. Since $\sum_\rho\sum_i [S^i_\rho] = 0$, we have $\sum_{E_\rho\in E_N} g(E_\rho) = 0$, that is $g \in \mathbb{Z}_0^{E_N}$.

The above construction gives us a group homomorphism

$$\psi_N : \{\text{Seed cluster transformations } \boldsymbol{s}_\Gamma \to \boldsymbol{s}_\Gamma\} \to \mathbb{Z}_0^{E_N} \to \mathbb{Z}_0^{E_N}/jH_1(\mathbb{T},\mathbb{Z}).$$

3.2 Surjectivity of ψ_N

In this section we show that ψ_N is surjective, that is, any element of $\mathbb{Z}_0^{E_N}$ can be realized by a seed cluster transformation.

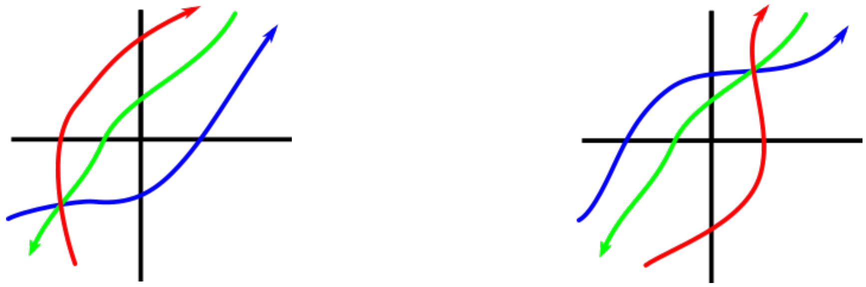
We recall the construction of a minimal bipartite graph corresponding to a polygon N from [GK12]. Start with a torus $\mathbb{T}$ constructed by gluing opposite sides of a rectangle R. We label the sides of R by $\partial R_N, \partial R_W, \partial R_S, \partial R_E$ respectively. For each edge E_ρ of N, draw loops $\{\alpha_\rho^i\}_{i=1}^{|E_\rho|}$, each with homology class $(X_\rho^i, Y_\rho^i) := E_\rho/|E_\rho|$. Isotope the loops so that the intersections of the loops with each side of R alternate in orientation. Now we use Theorem 2.1.2 in R to isotope the loops to obtain a minimal triple crossing diagram in R and follow the construction outlined in Section 2.1.6 to get the corresponding minimal bipartite graph.

We require the following lemma:

Lemma 3.2.1 (Goncharov and Kenyon, 2012 [GK12])**.** *The relative order along the boundary of R of strands associated to the same edge of N is fixed. The relative order of two incoming or outgoing strands associated to different edges of N can be interchanged by $2-2$ moves and isotopy.*

Theorem 3.2.2. *ψ_N is surjective.*

Proof. It suffices to show that the for adjacent edges $E_1, E_2 \in E_N$ of the Newton polygon with vertices V_1, V_2 and V_3 , so that $E_1 = V_1V_2$ and $E_2 = V_2V_3$, we can



(a) Initial configuration. (b) Configuration after isotopy.

Figure 3.4: Local configuration of strands near the NE corner of R in case 1. The blue strand is s_{i_1},the red strand is t_{i_m} and u is green.

construct a seed cluster transformation that ψ_N maps to the function $(1, -1, 0, .., 0) \in H_P$, since these functions generate the group $\mathbb{Z}_0^{E_N}$. Let (X_ρ, Y_ρ) be the coordinates of E_ρ. Changing the fundamental rectangle R corresponds to an action of $SL(2, \mathbb{Z})$ on the homology classes of zig-zag loops. $SL(2, \mathbb{Z})$ is generated by

$$s := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad t := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The action of s is rotation by $\pi/2$ and t is a shear. Acting by t or $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ we may assume that E_1 is neither horizontal nor vertical. Then rotating using s, we can make $X_1, Y_1 > 0$. Now using t or $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, we can assume that E_2 is neither horizontal nor vertical.

The idea is to create a simple configuration of strands near a corner of R by isotoping strands and $2-2$ moves and then pushing this configuration past ∂R.

The collection of loops $\{\alpha_\rho^i\}_{i=1}^{|E_\rho|}$ form in R a collection of strands of a minimal triple point diagram in R with end points in ∂R. Let $(s_{i_1}, ..., s_{i_n})$ and $(t_{j_1}, ..., t_{j_m})$ be the strands in R of the loops $\{\alpha_1^k\}_{k=1}^{|E_1|}$ and $\{\alpha_2^k\}_{k=1}^{|E_2|}$ respectively, in the natural cyclic order coming from their embedding in R. We want to show that we can use 2-2 moves and isotopy to change the cyclic orders of these strands to $(s_{i_2}, ..., s_{i_n}, s_{i_1})$ and $(t_{i_m}, t_{i_1}, ..., t_{i_{m-1}})$ respectively, while leaving the cyclic orders of all other strands fixed and such that if you forget the identity of the strands, the triple crossing diagram is identical to the initial one.

In the construction of the minimal bipartite graph above, we have a choice when we isotope the loops making the intersections with ∂R alternate in orientation. Therefore we may assume that intersection points of s_{i_1} with ∂R are the highest point in ∂R_W and the leftmost point in ∂R_N.

We now have four cases to consider:

1. $X_2, Y_2 > 0$. Since the Newton polygon is a closed polygon, we must have an edge E_3 with coordinates (X_3, Y_3) such that $Y_3 < 0$. By the action of $t \in SL(2, \mathbb{Z})$, we can also make $X_3 < 0$ while preserving our assumptions on E_1 and E_2. By repeatedly applying Lemma 3.2.1, we make the intersections of t_{i_m} with ∂R the lowest "out" point in ∂R_E and the rightmost "in" point in ∂R_N.

   Since ∂N is a closed path, the total homology of all loops $\sum_{\rho,i} \alpha_\rho^i$ is zero. Therefore the intersection number of the loops with any side of R is zero, that is,

(a) Initial configuration. (b) Configuration after isotopy.

Figure 3.5: Local configuration of strands near the NW corner of R in case 2. s_{i_1} is blue and t_{i_1} is red.

we have an equal number of “in" and “out" points in any side of R, alternating in orientation as we move along the side. By our assumption on s_{i_1}, the intersection point of s_{i_1} with ∂R_S is the leftmost point in ∂R_S and its orientation is “in". Therefore, the rightmost point in ∂R_S is an “out" point, which means there is an “out" point to the right of t_{i_m} in ∂R_S. For the same reason, there is an “in" point below t_{i_m} in R_E. Repeatedly using Lemma 3.2.1, we can make a strand u corresponding to E_3 pass through both these points. Using Theorem 2.1.2, we can make u and t_{i_m} run parallel to the boundary. Again using Theorem 2.1.2, we can make the three strands s_{i_1}, t_{i_m}, u meet just adjacent to the NE corner of R to obtain the local picture shown in Figure 3.4a near the NE corner. We isotope the triple point across the corner to obtain the configuration in Figure 3.4b. This achieves the change of cyclic orders. We can now use Lemma 3.2.1 and Theorem 2.1.2 in R to get back to the original triple point diagram up to this change of cyclic order.

2. $X_2, Y_2 < 0$.

   Repeatedly using lemma 3.2.1 we can make the strand t_{i_1} the leftmost “in" strand in ∂R_N and the topmost “out" strand in ∂R_W and use Theorem 2.1.2 to make s_{i_1}, t_{i_1} run parallel to the boundary to obtain the local picture shown in Figure 3.5a near the NW corner. We can then isotope to the configuration in Figure 3.5b.

3. $X_2 < 0, Y_2 > 0$.

   We can use $t \in SL(2, \mathbb{Z})$ to make $X_2 > 0$, reducing to case 1.

4. $X_2 > 0, Y_2 < 0$.

   This case is ruled out by convexity of N.

$\square$

3.3 Trivial seed cluster transformations

By Theorem 3.2.2, ψ_N is surjective. To complete the proof of Theorem 3.0.1, we need to find the kernel of ψ_N. From Theorem 2.5.2, a seed cluster transformation t is trivial if and only if $\nu_t = \nu$ and $d(w_0) - d_t(w_0) = 0$ in $\mathrm{Pic}^0(C)$ for a generic curve $C \in |D_N|$. We note that d_t and ν_t depend only on the positions of zig-zag paths before and after doing t, and since these depend only on $\psi_N(t)$, we get the important corollary:

Corollary 3.3.1. *The birational automorphism μ_t of $\mathcal{X}_N$ induced by t factors through $\psi_N(t)$:*

$$\begin{tikzcd}
\{\textit{Seed cluster transformations } \boldsymbol{s}_\Gamma \to \boldsymbol{s}_\Gamma\} \arrow[r, "\psi_N"] \arrow[dr, "t \mapsto \mu_t"'] & \mathbb{Z}_0^{E_N}/jH_1(\mathbb{T},\mathbb{Z}) \arrow[d] \\
 & Bir(\mathcal{X}_N)
\end{tikzcd},$$

where $Bir(\mathcal{X}_N)$ is the group of birational automorphisms of $\mathcal{X}_N$.

We also observe that :

Corollary 3.3.2. *Let t be a seed cluster transformation t such that $\psi_N(t)$ is a non-zero torsion element. Then μ_t is non-trivial.*

Proof. $\nu_t \neq \nu$. $\square$

We will use Theorem 2.5.2 to identify trivial seed cluster transformations. As a reality check, we see that translation by $m \in H_1(\mathbb{T},\mathbb{Z})$ is trivial: it induces $\nu_t = \nu$ and $d_t(w_0) \equiv d(w_0) - \text{div } \chi^m \equiv d(w_0)$. Therefore by Corollary 3.3.1, if $\psi_N(t) = 0$, then t is a trivial seed cluster transformation, so we have:

Lemma 3.3.3. *ker $\psi_N \subseteq \{$Trivial seed cluster transformations$\}$.*

The main theorem of this section is:

Theorem 3.3.4. *If $g = 0$, the cluster modular group is*

$$G_N \cong \mathbb{Z}_0^{E_N}/\{f \in \mathbb{Z}_0^{E_N} : f(E_\rho) \textit{ is divisible by } |E_\rho| \textit{ for all } \rho\}.$$

If $g \neq 0$, then

$$G_N \cong \mathbb{Z}_0^{E_N}/jH_1(\mathbb{T},\mathbb{Z}).$$

Proof. When $g = 0$, $S = \varnothing$, so μ_t is determined by the action of t on ν. Therefore t is trivial if an only if $\nu_t = \nu$, which happens if and only if $\psi_N(t)(E_\rho)$ is divisible by $|E_\rho|$ for all ρ.

When $g \neq 0$, if t is a seed cluster transformation such that $\psi_N(t) \neq 0$, then either:

1. $\psi_N(t)$ is a non-zero torsion element:

   Corollary 3.3.2 says that it is non-trivial.

2. $\psi_N(t)$ is not a torsion element:

   Consider the seed cluster transformation t^n, where $n = k \prod_\rho |E_\rho|, k \in \mathbb{Z}$. Then from Theorem 2.5.2 applied to t^n, we have:

   Lemma 3.3.5. *The action of t^n on spectral data is given by:*

   $$(C, S, \nu) \mapsto (C, S', \nu),$$

   where S' is the generically unique degree g effective divisor satisfying $S' \equiv S + D|_C$, where $D = n \sum_\rho \frac{\psi_N(t)}{|E_\rho|} D_\rho$.

   For sufficiently large k, $\mathcal{O}_{X_N}(D)$ is a line bundle, which we call L [CLS11, Proposition 4.2.7]. Since $\psi_N(t)$ is not a torsion element, the divisor $D = n \sum_\rho \frac{\psi_N(t)}{|E_\rho|} D_\rho$ is not a torsion element of the divisor class group of X_N either; indeed if lD is a principal divisor for some $l \in \mathbb{Z}$, then lD is the divisor of a character χ^m for some $m \in H_1(\mathbb{T}, \mathbb{Z})$. However this means that $\psi_N(t^{ln}) \in jH_1(\mathbb{T}, \mathbb{Z})$, contradicting the assumption that $\psi_N(t)$ is not a torsion element.

Therefore $L \ncong \mathcal{O}_{X_N}$, hence by Theorem A.0.11, we get $L|_C \ncong \mathcal{O}_C$ for a generic spectral curve C. Therefore by Lemma 3.3.5, t^n is a not a trivial seed cluster transformation, hence t is also not a trivial seed cluster transformation.

Therefore $\ker \psi_N = \{\text{Trivial seed cluster transformations } \boldsymbol{s}_\Gamma \to \boldsymbol{s}_\Gamma\}$. By Theorem 3.2.2, ψ_N is surjective, so by the first isomorphism theorem, the cluster modular group

$$G_N \cong \mathbb{Z}_0^{E_N}/jH_1(\mathbb{T}, \mathbb{Z}).$$

□

3.3.1 Picard group of the toric stack

Associated to N is a stacky fan $\boldsymbol{\Sigma} = (\Sigma, \hat{\Sigma}, \beta)$ where:

1. Σ is the normal fan of N in $H_1(\mathbb{T}, \mathbb{Z})^* \otimes \mathbb{R}$;
2. $\hat{\Sigma}$ is a fan in an auxiliary lattice $\mathbb{Z}^{|E_N|}$, formed by the walls of the positive orthant;
3. Let $\{e_\rho\}$ be the standard basis of Z^{E_N}. $\beta : \mathbb{Z}^{|E_N|} \to H_1(\mathbb{T}, \mathbb{Z})^*$ is the homomorphism defined by $\beta(e_\rho) = |E_\rho| u_\rho$. Note that β gives a combinatorial correspondence between cones of $\hat{\Sigma}$ and Σ.

Just as the normal fan Σ of N can be used to construct toric surface X_N, the stacky fan $\boldsymbol{\Sigma}$ gives rise to a stacky toric surface $\mathscr{X}_N$. $\mathscr{X}_N$ has coarse moduli space the toric variety X_N; let us denote by $\pi : \mathscr{X}_N \to X_N$ the projection. Let $\mathcal{O}_{\mathscr{X}_N}\left(\frac{1}{|E_\rho|}D_\rho\right)$ be the unique line bundle on $\mathscr{X}_N$ satisfying

$$\mathcal{O}_{\mathscr{X}_N}\left(\frac{1}{|E_\rho|}D_\rho\right)^{\otimes|E_\rho|} \cong \pi^*\mathcal{O}_{X_N}(D_\rho).$$

Just as the class group of X_N is generated by the toric divisors, the Picard group of $\mathscr{X}_N$ is generated by the line bundles $\mathcal{O}_{\mathscr{X}_N}\left(\frac{1}{|E_\rho|}D_\rho\right)$.

Theorem 3.3.6 (Borisov and Hua, 2009 [BH09, Proposition 3.3])**.** *The following is an isomorphism of groups:*

$$\begin{aligned}\mathbb{Z}^{E_N}/M &\to Pic\ (\mathscr{X}_N)\\ f &\mapsto \mathcal{O}_{\mathscr{X}_N}\left(\sum_\rho \frac{f(E_\rho)}{|E_\rho|}D_\rho\right)\end{aligned}$$

Corollary 3.3.7. *This isomorphism identifies $\mathbb{Z}_0^{E_N}/jH_1(\mathbb{T},\mathbb{Z})$ with the subgroup of $Pic\ (\mathscr{X}_N)$ of line bundles $\mathcal{O}_{\mathscr{X}_N}(D)$ where $D = \sum_\rho b_\rho D_\rho$, satisfying*

$$\sum_\rho |E_\rho| b_\rho = 0.$$

There is a version of Theorem 2.5.2 which illuminates this correspondence.

Theorem 3.3.8 (Treumann, Williams and Zaslow, 2018 [TWZ18])**.** *Let t be a seed cluster transformation. Let $\mathscr{C} = C \times_{X_N} \mathscr{X}_N$ and let $i : \mathscr{C} \hookrightarrow \mathscr{X}_N$ be the embedding. We have:*

$$\mathcal{O}_{\mathscr{C}}(S_t) \cong \mathcal{O}_{\mathscr{C}}(S) \otimes i^*\mathcal{O}_{\mathscr{X}_N}\left(\sum_\rho \frac{\psi_N(E_\rho)}{|E_\rho|}D_\rho\right).$$

3.4 Torsion subgroup of the cluster modular group

In this section, we study the torsion subgroup of the cluster modular group and show that it corresponds to translations in a certain sub-family of bipartite graphs in the family associated to N. Since torsion elements have degree 0, we have $\mathbb{Z}_0^{E_N}/jH_1(\mathbb{T},\mathbb{Z})^{\text{tor}} \cong \mathbb{Z}^{E_N}/jH_1(\mathbb{T},\mathbb{Z})^{\text{tor}}$. Let us denote $\mathbb{Z}^{E_N}/jH_1(\mathbb{T},\mathbb{Z})$ by A.

3.4.1

The idea here is to use the embedding $H_1(\mathbb{T},\mathbb{Z}) \xrightarrow{j} \mathbb{Z}^{E_N}$ to identify each torsion element of A with a fractional homology class.

Let $\hat{L}$ be the kernel

$$0 \to \hat{L} \to \mathbb{Z}^{E_N} \xrightarrow{\pi} A/A^{\text{tor}} \to 0.$$

Since rank $A = |E_N|-2$, by additivity of rank in exact sequences, we have rank $\hat{L} = 2$. By flatness of $\mathbb{Q}$, tensoring $H_1(\mathbb{T},\mathbb{Z}) \xrightarrow{j} \mathbb{Z}^{E_N}$ with $\mathbb{Q}$ gives an injection

$$H_1(\mathbb{T},\mathbb{Q}) \cong H_1(\mathbb{T},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \xhookrightarrow{j\otimes_{\mathbb{Z}}1} \mathbb{Z}^{E_N} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{E_N}.$$

Lemma 3.4.1. *$\hat{L}$ is contained in $j \otimes_{\mathbb{Z}} 1(H_1(\mathbb{T},\mathbb{Q}))$.*

Proof. Let $x \in \hat{L}$. Since $\pi(x) \in A^{\text{tor}}$, there is an $n \in \mathbb{Z}$ such that $nx \in jH_1(\mathbb{T},\mathbb{Z})$; let $m \in H_1(\mathbb{T},\mathbb{Z})$ such that $j(m) = nx$. Then $j \otimes_{\mathbb{Z}} 1\left(m \otimes_{\mathbb{Z}} \frac{1}{n}\right) = x$. $\square$

Therefore $L := (j \otimes_{\mathbb{Z}} 1)^{-1}(\hat{L})$ is a rank 2 lattice in $H_1(\mathbb{T}, \mathbb{Q})$ that contains $H_1(\mathbb{T}, \mathbb{Z})$.

Example 3.4.2.

Let us label the edges of the Newton polygon in Figure 2.1a by E_1, E_2, E_3, E_4 in cclw order starting from the blue edge. The homomorphism $j : M \to \mathbb{Z}^{E_N}$ is represented by the matrix

$$B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The Smith decomposition is $\widetilde{B} = UBV$, where

$$\widetilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} A &= \mathbb{Z}_2 \oplus \mathbb{Z}^2; \\ \hat{L} &= \mathbb{Z}U^{-1}(1,0,0,0)^T + \mathbb{Z}U^{-1}(0,1,0,0)^T \\ &= \mathbb{Z}(-1,1,1,-1)^T + \mathbb{Z}(0,-1,0,1)^T; \\ L &= \mathbb{Z}(1,0)^T + \mathbb{Z}\left(-\frac{1}{2}, \frac{1}{2}\right)^T. \end{aligned}$$

3.4.2 Maximally translation invariant bipartite graphs

We construct a sub-family of minimal bipartite graphs with Newton polygon N that is invariant under translations by L.

Theorem 3.4.3. *There exists a family of minimal bipartite graphs associated to N that is invariant under translations by L.*

Proof. Let $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ be generators of L. For each edge E_ρ of N, we obtain a vector

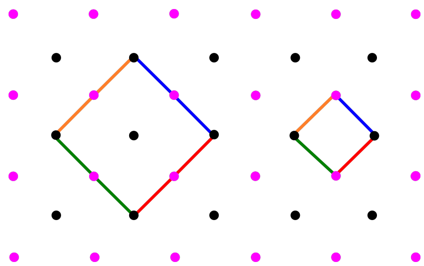
$$w_\rho = \langle j \otimes_{\mathbb{Z}} 1(\widetilde{\gamma}_2), e_\rho \rangle \widetilde{\gamma}_1 - \langle j \otimes_{\mathbb{Z}} 1(\widetilde{\gamma}_1), e_\rho \rangle \widetilde{\gamma}_2 \in L,$$

where $\langle *, * \rangle$ is the canonical pairing $\mathbb{Z}^{E_N} \times \mathbb{Z}^{E_N} \to \mathbb{Z}$, and $\{e_\rho\}$ is the standard basis of $\mathbb{Z}^{E_N}$. Since the image of $j \otimes_{\mathbb{Z}} 1$ is in $\mathbb{Z}_0^{E_N}$, we can put together the vectors $\{w_\rho\}_\rho$ in cyclic order to obtain a convex integral polygon $\widetilde{N} \subset L \otimes \mathbb{R}$. Using Theorem 2.1.1 with the polygon $\widetilde{N}$, we obtain a family of minimal bipartite graphs in the torus $L \otimes \mathbb{R}/L$. $H_1(\mathbb{T}, \mathbb{Z}) \subset L$ induces a covering map $\mathbb{T} \to L \otimes \mathbb{R}/L$. The lift of a minimal bipartite graph with Newton polygon $\widetilde{N}$ in $L \otimes \mathbb{R}/L$ is a minimal bipartite graph in $\mathbb{T}$ with Newton polygon N that is invariant under translations by L. $\square$

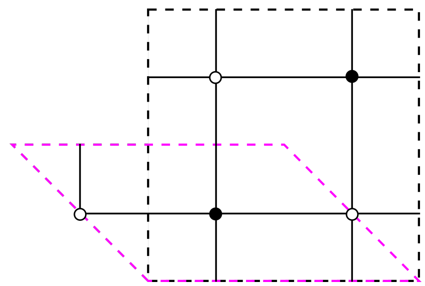
Corollary 3.4.4. *If $u \in \mathbb{Z}_0^{E_N}/jH_1(\mathbb{T}, \mathbb{Z})^{tor}$, and Γ is a graph that is invariant under L-translations, then there is a seed cluster transformation t based at Γ induced by an L-translation such that $\psi_N(t) = u$.*

Example 3.4.5.

Let us apply this construction to the polygon N from Example 3.4.2. Let us take



(a) The polygons N and $\widetilde{N}$. The lattice L consists of all the points and the sublattice M consists of the black points.



(b) The fundamental parallelograms of L and M.

Figure 3.6: A maximally translation invariant bipartite torus graph.

generators $\widetilde{\gamma}_1 = (1,0)^T, \widetilde{\gamma}_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)^T$ of L. We have

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{\gamma}_1 \\ \widetilde{\gamma}_2 \end{pmatrix}.$$

The polygon $\widetilde{N}$ is shown in Figure 3.6a and a maximally translation invariant bipartite graph is shown in Figure 3.6b. In this case, the square lattice is maximally translation invariant.

CHAPTER 4

Spectra of biperiodic planar networks

4.1 Introduction

A planar resistor network is a pair $(\tilde{G}, \tilde{c})$ where $\tilde{G}$ is a planar graph and $\tilde{c}$ is a conductance function that assigns a non-zero complex number to each edge of $\tilde{G}$, defined up to multiplication by a global constant. It is said to be biperiodic if translations by $\mathbb{Z}^2$ act on $(\tilde{G}, \tilde{c})$ by isomorphisms. This is equivalent to the data of the quotient $(G, c) := (\tilde{G}, \tilde{c})/\mathbb{Z}^2$, where G is a graph on a torus. Hereafter we assume that our networks are on a torus.

The fundamental operator in the study of networks is the discrete Laplacian. It has a certain spectrum, defined below, and the main goal of this paper is to show that this is a birational isomorphism with a certain moduli space of curves and divisors and therefore provides a way to classifiy networks. While in typical geometric or probabilistic applications the conductances are always positive real numbers, the algebraic nature of the problem leads us to consider general non-zero complex conductances.

There is a natural equivalence relation on networks, defined by certain local rearrangements of the graph and its conductances, which preserves the spectrum. To define this equivalence relation, let us start by defining a zig-zag path. A zig-zag path on G is a path that alternately turns maximally left or right. A resistor network G is minimal [CdV94, CIM98] if the lifts of any two zig-zag paths to $\tilde{G}$ do not intersect more than once and any lift of a zig-zag path has no self intersections. Minimality is a mild assumption on networks since any network may be reduced to a minimal one by certain elementary moves without affecting its electrical properties. The Newton polygon of a minimal resistor network is the unique integral polygon whose primitive edges are given by the homology classes of zig-zag paths in cyclic order. Since zig-zag paths come in pairs related by flipping the orientation, the Newton polygon of a network is always centrally symmetric.

There is a local rearrangement of resistor networks called a Y-Δ move that preserves all electrical properties outside the region where the rearrangement takes place (see Section 4.2.1 and Figure 5.2). We say that two minimal networks (G_1, c_1) and (G_2, c_2) are topologically equivalent if there is a sequence of $Y - \Delta$ moves that takes the underlying graph G_1 to the graph G_2. Topological equivalence classes of networks are parameterized by centrally symmetric Newton polygons [GK12]. In particular, any two minimal resistor networks with the same Newton polygon are related by a sequence of $Y - \Delta$ transformations.

Two networks (G_1, c_1) and (G_2, c_2) are electrically equivalent if there is a sequence of $Y - \Delta$ moves that takes the network (G_1, c_1) to the network (G_2, c_2). Goncharov and Kenyon [GK12] constructed the resistor network cluster variety $\mathcal{R}_N$ that parameterizes electrical equivalence classes of resistor networks that lie in the same topological equivalence class associated to the polygon N as follows: A centrally symmetric integral polygon N determines a finite collection of minimal resistor networks whose Newton polygon is N, related by $Y - \Delta$ transformations. To each minimal resistor network G is associated a complex torus $(\mathbb{C}^*)^{\text{number of edges of } G-1}$, which parameterizes conductance functions on G. The $Y - \Delta$ move $G_1 \to G_2$ induces a birational map between the complex tori associated to G_1 and G_2. $\mathcal{R}_N$ is obtained by gluing the complex tori using these birational maps.

Goncharov and Kenyon further showed that $\mathcal{R}_N$ can be identified with an isotropic subvariety of an algebraic completely integrable Hamiltonian system $\mathcal{X}_N$ associated to the dimer model. Let S_N be the moduli space of triples (C, S, ν) where C is the vanishing locus of a Laurent polynomial $P(z, w)$ with Newton polygon N, S is a degree g effective divisor on C (where g = number of interior lattice points in N) and ν is a parameterization of the points at infinity of C. Goncharov and Kenyon constructed the spectral map $\mathcal{X}_N \to \mathcal{S}_N$ and showed that it is a birational isomorphism. Fock [F15] constructed an explicit inverse map in terms of theta functions on the Jacobian of C. In this construction, the elementary transformation in dimer model (the spider move) is described by Fay's trisecant identity.

Associated to the Laplacian on a biperiodic planar network is its spectrum $\mathcal{R}_N \to \mathcal{S}_N$, where $\mathcal{S}_N$ is defined as in the previous paragraph, but with the divisor S now of degree $g =$ number of interior lattice points in N-1. Let $\mathcal{S}'_N$ be the subspace where $P(z,w)$ satisfies

1. $P(1,1) = 0$ and the point $(1,1)$ is a node;
2. $\sigma : (z,w) \mapsto (\frac{1}{z}, \frac{1}{w})$ is an involution on C,

and the divisor S satisfies

$$S + \sigma(S) - q_1 - q_2 = K_{\widehat{C}} \text{ in } \mathrm{Pic}^{2g-2}(\widehat{C}), \tag{4.1}$$

where $\widehat{C}$ is the normalization of C, g is the geometric genus, q_1, q_2 are the points in the fiber of the node at $(1,1)$ and $K_{\widehat{C}}$ is the canonical divisor class on $\widehat{C}$. Our main result is the following complete classification of biperiodic planar resistor networks in terms of their spectral data:

Theorem 4.1.1. *The spectral transform is a birational isomorphism $\mathcal{R}_N \to \mathcal{S}'_N$.*

The main new contribution of this paper lies in showing that the spectral divisor satisfies (4.1). Along the way, we provide an explicit description of oriented cycle rooted spanning forests of G (OCRSFs) whose homology classes are boundary lattice

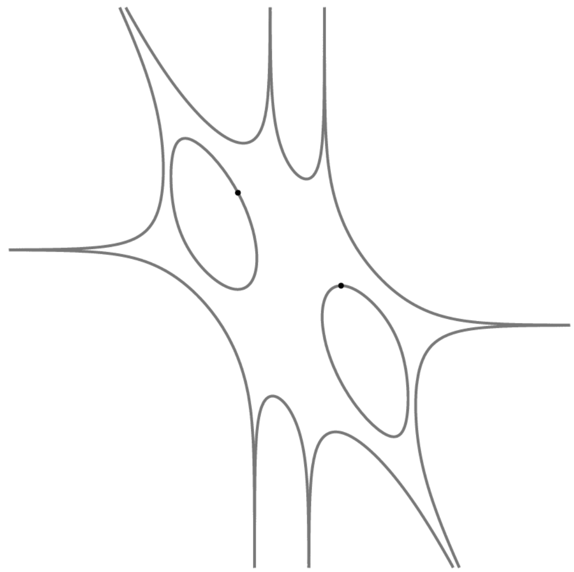


Figure 4.1: The divisor S on the amoeba of the spectral curve.

points of N (Theorem 4.3.1), analogous to results for dimers in [Bro12, GK12]. In particular, we see that every OCRSF corresponding to a boundary lattice point is a union of cycles (Corollary 4.3.2).

We construct an explicit inverse spectral map (see (4.11)). Formally this construction is the same as Fock's construction for the dimer model. We replace the Jacobian with the Prym variety and the $Y-\Delta$ transformation is described by Fay's quadrisecant identity [Fay89]. Further we show that the inverse map is compatible with $Y-\Delta$ transformations (Theorem 4.6.1).

The $Y-\Delta$ move involves subtraction free rational expressions, and therefore the set of positive-real-valued points of the cluster variety is well defined, which we denote by $\mathcal{R}_N(\mathbb{R}_{\geq 0})$. This subspace is important for probabilistic applications. For a positive real valued point, the spectrum (C, S, ν) has the following additional properties (see [K17]):

1. C is a simple Harnack curve as in [Mikh00]. Compact ovals (connected components) of C are in bijection with interior lattice points of N.

2. The oval corresponding to the origin is degenerated to a real node.

3. S has a point in each of the other compact ovals.

The spectral curves of genus zero correspond to the isoradial networks studied in [K02]. In this case, the inverse spectral map recovers Kenyon's results expressing the conductances in terms of tangents, and the quadrisecant identity reduces to the triple tangent identity. For a different generalization of isoradial networks to the case of the massive Laplacian on isoradial graphs, see [BdeTR17].

Consider the map $C(\mathbb{C}) \to \mathcal{A}(C), (z, w) \mapsto (\log|z|, \log|w|) \subset \mathbb{R}^2$ from the $\mathbb{C}-$valued points of C to its amoeba $\mathcal{A}(C)$. For a simple Harnack curve, this map is a homeomorphism from the compact ovals to the boundaries of the holes of the amoeba, and therefore provides a way to depict the divisor S (see Figure 4.1 for an example, where the network is a 2×1 fundamental domain of the triangular lattice).

A sequence of $Y - \Delta$ moves that takes a graph G to itself gives rise to a birational automorphism (called a cluster modular transformation) of $\mathcal{R}_N$, where N is the Newton polygon of G. A cluster modular transformation provides a discrete integrable system on $\mathcal{R}_N$. For example, if we consider the honeycomb lattice, and do the $Y-\Delta$ move at the downward triangles, we obtain the cube recurrence studied by Carroll and Speyer ([CS04], see also [GK12] Section 6.3). We show that cluster modular transformations are linearized on the Prym variety of C (Theorem 4.7.1). In the case of positive real conductances, we may view this as moving each point along the boundary of the corresponding hole in the amoeba.

4.2 Resistor networks

4.2.1 The resistor network cluster variety

A *resistor network* is a pair (G, c) where G is a surface graph on $\mathbb{T}$ and $c : E(G) \to \mathbb{C}^*$ is a function defined modulo global multiplication by a non-zero scalar. Associated to each surface graph G is the space $\mathcal{R}_G$ of conductance functions on it. $\mathcal{R}_G$ is non-canonically isomorphic to the complex torus $(\mathbb{C}^*)^{|E(G)|-1}$.

A $Y-\Delta$ transformation [Kenn1899] $G_1 \to G_2$ is given by replacing a Y in the graph G_1 with a triangle as shown in Figure 5.2. Any two minimal resistor networks (the definition of minimality was given in the introduction) with Newton polygon N are related by $Y-\Delta$ moves. A $Y-\Delta$ move $G_1 \to G_2$ induces a birational map $\mathcal{R}_{G_1} \to \mathcal{R}_{G_2}$, given in the notation of Figure 5.2 by

$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c}.$$

Gluing the $\mathcal{R}_G$ for all G with Newton polgyon N using these birational maps, we obtain a Poisson scheme $\mathcal{R}_N$, called the *resistor network cluster variety.*

4.2.2 The line bundle Laplacian

A *surface graph* Γ on a torus $\mathbb{T}$ is a graph embedded on $\mathbb{T}$ such that each face is contractible. A *line bundle with connection* (V, ϕ) on Γ is the data of a complex line $V_v \cong \mathbb{C}$ at each vertex v of Γ along with isomorphisms called *parallel transport* $\phi_{vv'} : V_v \to V_{v'}$ for each edge $\langle v, v' \rangle$ such that $\phi_{v'v} = \phi_{vv'}^{-1}$. Two line bundles with connection (V, ϕ) and (V', ϕ') are *gauge equivalent* if there exists isomorphisms $\psi_v : V_v \to V'_v$ such that for all edges, the following diagram commutes.

$$\begin{tikzcd} V_v \arrow[r, "\phi_{vv'}"] \arrow[d, "\psi_v"] & V_{v'} \arrow[d, "\psi_{v'}"] \\ V'_v \arrow[r, "\phi'_{vv'}"] & V'_{v'} \end{tikzcd}$$

If L is an oriented loop in Γ, the *monodromy* $m(L)$ of (V, ϕ) around L is the composition of the parallel transports around L. A line bundle with connection is *flat* if the monodromy around the boundary of any face of Γ is trivial.

The moduli space of line bundles with connection on Γ modulo isomorphisms is denoted $\mathcal{L}_\Gamma$. Let $\mathcal{L}_\Gamma^{\text{flat}}$ be the subspace of flat connections. The monodromies around loops in Γ give rise to isomorphisms such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{L}_\Gamma^{\text{flat}} \arrow[r, hook] \arrow[d, "\cong"] & \mathcal{L}_\Gamma \arrow[d, "\cong"] \\ H^1(\mathbb{T}, \mathbb{C}^*) \arrow[r, hook] & H^1(\Gamma, \mathbb{C}^*) \end{tikzcd}$$

Let (G, c) be a resistor network and let $i \in \mathcal{L}_G$ be a line bundle with connection on G. The *line bundle Laplacian* is the linear operator $\Delta = \Delta(c, i) : \mathbb{C}^{V(G)} \to \mathbb{C}^{V(G)}$ defined by

$$\Delta(f)(v) := \sum_{e:v' \to v} c(e)(f(v) - i_e f(v')),$$

where the sum is over all edges of G oriented towards v. An *oriented cycle rooted spanning forest* (OCRSF) γ of G is a collection of edges of G such that each connected component of γ has the same number of vertices and edges (so that each connected component has a unique cycle), along with a choice of orientation for each cycle in γ. Since two distinct cycles in γ cannot intersect, if η is a cycle in γ, every cycle has homology class $\pm[\eta]$. The weight of an $OCRSF$ γ is defined to be $wt(\gamma) = \prod_{e \in \gamma} c(e)$. The following result generalizes Kirchhoff's matrix tree theorem to the line bundle Laplacian:

Theorem 4.2.1 (Kenyon, 2010 [K10])**.**

$$det\ \Delta = \sum_{OCRSFs\ \gamma} wt(\gamma) \prod_{Cycles\ \eta \in \gamma} (1 - m(\eta)),$$

where $m(\eta)$ is the monodromy of i along the cycle η.

An OCRSF $\gamma^\vee$ on $G^\vee$ is *dual* to an OCRSF γ on G if no edge of $\gamma^\vee$ crosses an edge of γ. It is easy to see that $\gamma^\vee$ has the same number of cycles as γ and each cycle has homology class $\pm[\eta]$, where η is any cycle in γ. An OCRSF γ has 2^k duals where k is the number of cycles in γ, one for each choice of orientation of the dual cycles. Given a pair $(\gamma, \gamma^\vee)$ of dual OCRSFs, define its weight to be $wt(\gamma, \gamma^\vee) := wt(\gamma)$. To each pair we associate a homology class,

$$[(\gamma, \gamma')] := \frac{1}{2} \sum_{\text{Cycles } \eta \text{ in } \gamma\cup\gamma^\vee} [\eta] \in H_1(\mathbb{T}, \mathbb{Z}).$$

The *Newton polygon* of the resistor network is

$$N = \text{Conv}\,\{[(\gamma, \gamma^\vee)] \in H_1(\mathbb{T}, \mathbb{Z}) : (\gamma, \gamma^\vee) \text{ is a pair of dual OCRSFs}\}.$$

$(\gamma, \gamma') \mapsto [(\gamma, \gamma')]$ associates to each pair of dual OCRSFs an integer lattice point in the Newton polygon. N is always centrally symmetric and therefore we can center it at the origin.

Let R be a fundamental rectangle for $\mathbb{T}$, so that $\mathbb{T}$ is obtained by gluing together opposite sides of R. We label the curves in $\mathbb{T}$ forming the sides of R by γ_z, γ_w, oriented as shown in Figure 4.2a. For $z, w \in \mathbb{C}^*$, we define a flat line bundle with connection on G that is compatible with the choice of fundamental rectangle R as follows:

$$i(e) = z^{\langle e,\gamma_z\rangle_{\mathbb{T}}} w^{\langle e,\gamma_w\rangle_{\mathbb{T}}} \text{ for all } e \in E(G),$$

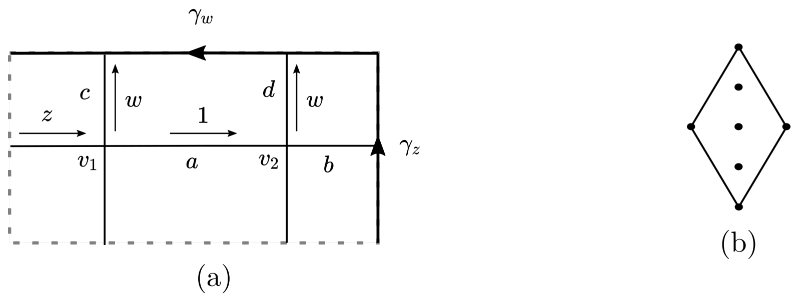


Figure 4.2: A fundamental rectangle for a resistor network on $\mathbb{T}$ and its Newton polgyon. The line bundle with connection is indicated by the arrows and the conductances are labeled by a, b, c, d.

where $\langle , \rangle_{\mathbb{T}}$ is the intersection pairing on $\mathbb{T}$.

We choose the basis $(-\gamma_w, \gamma_z)$ of $H_1(\mathbb{T}, \mathbb{Z})$ to get an isomorphism $H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$. We can rephrase Theorem 4.2.1 as

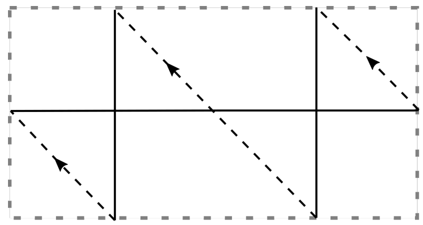
$$\det \Delta(z,w) = \sum_{\text{OCRSFs } \gamma} wt(\gamma) \prod_{\text{Cycles } \eta \in \gamma} (1 - z^{i_\eta} w^{j_\eta}), \tag{4.2}$$

where $(i_\eta, j_\eta) \in \mathbb{Z}^2$ is the homology class of η. $P(z,w) := \det \Delta(z,w)$ is called the *characteristic polynomial*. The Newton polygon of the characteristic polynomial is

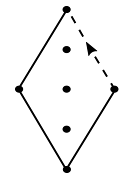
$$\text{Conv}\{(i,j) \in \mathbb{Z}^2 : \text{Coefficient of } z^i w^j \text{ is non-zero in } P(z,w)\},$$

and it coincides with the Newton polygon of the resistor network.

Let us compute the Laplacian and the characteristic polynomial for the network



(a)



(b)

Figure 4.3: An oriented zig-zag path on the network in Figure 4.2 drawn as a dashed loop in its medial graph and the corresponding primitive integral vector on the boundary of N.

in Figure 4.2. The Laplacian is given by the matrix

$$\Delta(z,w) = \begin{pmatrix} a+b+c(2-w-1/w) & -a-bz \\ -a-b/z & a+b+d(2-w-1/w). \end{pmatrix} \tag{4.3}$$

Taking the determinant, we get

$$P(z,w) = cd\left((1-w)^2 + \left(1-\frac{1}{w}\right)^2\right) + ab\left((1-z) + \left(1-\frac{1}{z}\right)\right)$$
$$+ (ac+bc+ad+bd)\left((1-w) + \left(1-\frac{1}{w}\right)\right),$$

enumerating the 12 OCRSFs of this network.

4.2.3 Zig-zag paths and minimality for resistor networks

An (oriented) *zig-zag path* on a resistor network G is a path that alternately turns maximally right or left at each vertex (see Figure 4.3a). Zig-zag paths on G come in pairs with opposite orientations. We denote the set of zig-zag paths on G by Z_G, and when G is unambiguous, by just Z. We say that G is *minimal* if the lift of any zig-zag path to the universal cover of $\mathbb{T}$ does not intersect itself and if the lifts of two different zig-zag paths intersect at most once.

If G is a minimal resistor network, associated to each $\alpha \in Z_G$ is its homology class $[\alpha] \in H_1(\mathbb{T}, \mathbb{Z})$. These homology classes are in (non-canonical) bijection with integral primitive vectors on the boundary of the Newton polygon N in counterclockwise order (see Figure 4.3b).

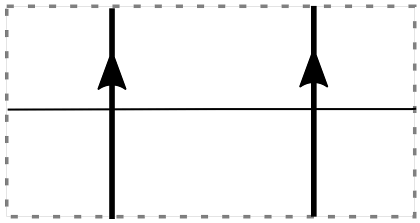
4.3 External OCRSFs

4.3.1

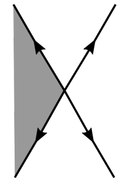
We say that a pair of dual OCRSFs F is *external* if $[F]$ is a boundary lattice point of N. It is *extremal* if $[F]$ is a vertex of N. We note that if $F = [(\gamma, \gamma^\vee)]$ is external, then the orientations of γ and $\gamma^\vee$ are uniquely determined by $[F]$ and $[\gamma] = [\gamma^\vee] = [F]$. Therefore we can define external and extremal $OCRSFs$ on G instead of pairs of dual $OCRSFs$.

4.3.2

For a vertex $v \in G$, we define the *local zig-zag fan* Σ_v at v to be the complete fan of strongly convex rational polyhedral cones in $H_1(\mathbb{T}, \mathbb{R})$ whose rays are generated by the homology classes of zig-zag paths through v that turn maximally right at v.



(a) $\gamma_{(0,2)}$.



(b) The zig-zag fan Σ along with the 2-dimensional cone σ (shaded) corresponding to the vertex $(0, 2)$ of the Newton polygon in Figure 4.2b.

Figure 4.4: An extremal OCRSF corresponding to the newtork in Figure 4.2.

The fan Σ whose rays are generated by the homology classes of all zig-zag paths on G is called the *global zig-zag fan* of G. We have the natural map of fans $i_v : \Sigma \to \Sigma_v$ for each $v \in G$. If σ is a 2-dimensional cone in Σ, $i_v(\sigma)$ is contained in a unique two dimensional cone in Σ_v, which we shall denote by σ_v. σ_v determines a unique edge e adjacent to v that is oriented away from v: e is the edge that contains the two zig-zag paths corresponding to the rays of σ_v. Let γ_{σ_v} be the 1-chain that is 1 on e, -1 on $-e$ and 0 on all other edges. We define

$$\gamma_\sigma := \sum_{v \in V(G)} \gamma_{\sigma_v}.$$

To a zig-zag path $\alpha \in Z_G$ we associate a 1-chain ω_α that is 1 on edges e in α that are oriented in the same direction as α and 0 on edges not in α. If γ is external, $[\gamma]$ lies on an edge E of N, which corresponds to a family of zig-zag paths $\{\alpha_k\}$. Let $E = \langle V_1, V_2 \rangle$, where V_1, V_2 are vertices of N such that V_2 is the vertex after V_1 when the boundary of N is traversed counterclockwise.

The following theorem explicitly describes all external OCRSFs and is key to several later results.

Theorem 4.3.1. *$\gamma_V := \gamma_\sigma$ is the unique extremal OCRSF on G such that $[\gamma_V]$ is the vertex V of N that corresponds to σ.*

Let A be a subset of the family of zig-zag paths $\{\alpha_k\}$ corresponding to E. The external OCRSFs on E are of the form

$$\gamma_A := \gamma_{V_1} + \sum_{\alpha_k \in A} \omega_{\alpha_k}.$$

In particular, $\gamma_{V_2} = \gamma_{V_1} + \sum_k \omega_{\alpha_k}$, and the number of OCRSFs corresponding to a boundary lattice point of N is a binomial coefficient.

We also need the following result later.

Corollary 4.3.2. *Every external OCRSF is a disjoint union of cycles.*

Let us compute the extremal OCRSF of the network in Figure 4.2 corresponding to the vertex $(0, 2)$ of its Newton polygon (see Figure 4.4). The global zig-zag fan Σ has rays generated by $(-1, 2), (-1, -2), (1, -2), (1, 2)$ and coincides with the local zig-zag fans $\Sigma_{v_1}, \Sigma_{v_2}$. Let us consider v_1. σ is the 2-dimensional cone with rays generated by $(-1, 2)$ and $(-1, -2)$. Since $i_{v_1} : \Sigma \to \Sigma_{v_1}$ is the identity map, $\sigma_{v_1} = \sigma$. Therefore $\gamma_{\sigma_{v_1}}$ is the 1-chain that is 1 on the edge with conductance c, oriented upwards. Similarly $\gamma_{\sigma_{v_2}}$ is the edge with conductance d oriented upwards. $\gamma_{(0,2)}$ is the OCRSF given by the union of these two oriented edges (Figure 4.4a). As we expect from Corollary 4.3.2, it is a union of (two) cycles.

4.3.3

While its possible to prove Theorem 4.3.1 directly, it is easier to use Temperley's bijection to relate it to corresponding statements about the dimer model. The results of this section are not used anywhere else in the paper, and therefore may be skipped on a first reading. Let Γ be a *bipartite* surface graph on $\mathbb{T}$, that is the vertices of Γ are colored black or white, and each edge of Γ is incident to a vertex of each color.

A *dimer cover* (or *perfect matching*) of Γ is a collection of edges of Γ such that every vertex is adjacent to a unique edge in the collection. A dimer cover M on Γ gives a 1-chain ω_M on Γ. If M_0 is another dimer cover, $\omega_M - \omega_{M_0}$ is a 1-cycle and therefore determines a homology class in $H_1(\Gamma, \mathbb{Z})$. Under the projection $H_1(\Gamma, \mathbb{Z}) \to H_1(\mathbb{T}, \mathbb{Z})$, we obtain a homology class $[M] \in H_1(\mathbb{T}, \mathbb{Z})$. The *Newton polygon* of Γ is

$$N := \text{Conv}\,\{[M] \in H_1(\mathbb{T}, \mathbb{Z}) : M \text{ is a dimer cover}\}.$$

N depends on the choice of reference dimer cover M_0. Changing the reference matching corresponds to translating the polygon N. $M \mapsto [M]$ gives a well defined map from the set of dimer covers to the integer lattice points in N.

Zig-zag paths on bipartite graphs and minimality

A *zig-zag path* on a bipartite torus graph Γ is a path that turns maximally right at black vertices and maximally left at white vertices. Let us denote by Z_Γ the set of all zig-zag paths in Γ. We say that Γ is *minimal* if in the universal cover $\tilde{\Gamma}$, zig-zag paths have no self intersections and no pairs of zig-zag paths oriented in the same direction meet twice.

Suppose Γ is a minimal bipartite graph on a torus. Each path $\alpha \in Z_\Gamma$ gives us a homology class $[\alpha] \in H_1(\mathbb{T}, \mathbb{Z})$ which is an integral pimitive vector on a side of the Newton polygon N. The zig-zag paths taken in cyclic order correspond to cyclically ordered primitive integral vectors in the boundary of the Newton polygon. Therefore an edge of N corresponds to a family of zig-zag paths, each with homology class equal to the primitive integral edge vector of the edge.

Temperley's bijection on the torus

Associated to G is a bipartite graph Γ_G obtained by superposing G and its dual graph $G^\vee$. The vertices and faces of G become the black vertices of Γ_G and the edges of G become the white vertices of Γ_G. Applying Euler's formula on $\mathbb{T}$ to G we see that Γ_G has equal number of white and black vertices.

Let G be a resistor network and let Γ_G be the associated bipartite graph.

Lemma 4.3.3 (Goncharov and Kenyon, 2012 [GK12])**.** *The Newton polygon N of the resistor network G coincides with the Newton polygon of the dimer model on Γ_G. Moreover, there is a canonical homology-class-preserving bijection between zig-zag paths on G and zig-zag paths on Γ_G.*

Given a pair of dual OCRSFs $F = (\gamma, \gamma^\vee)$ on G, we can construct a dimer cover M_F on Γ_G using the rule: The oriented edge $e = \langle u, v\rangle$ is in F if and only if the edge $\langle u, e\rangle$ is in M_F.

Theorem 4.3.4 (Temperley's bijection on torus; Kenyon, Propp and Wilson, 2000 [KPW00])**.** *Let (G, c) be a resistor network on a torus. $F \mapsto M_F$ is a bijection from pairs of dual OCRSFs on G to dimer covers on Γ_G such that $[F] = [M_F]$ in N.*

External dimer covers

In this section, we collect some results about dimer covers from [Bro12, GK12]. Let Γ be a minimal bipartite graph on a torus. We say that a dimer cover M is *extremal* if $[M]$ is a vertex of the Newton polygon. If b is any black vertex in Γ, we define the *local zig-zag fan* Σ_b at b to be the complete fan of strongly convex rational polyhedral cones in $H_1(\mathbb{T}, \mathbb{Z})$ whose rays are generated by homology classes of those zig-zag paths in Γ that contain b.

The *global zig-zag fan* of Γ is the fan whose rays are generated by the homology classes of all zig-zag paths on Γ. The identity map in $H_1(\mathbb{T}, \mathbb{Z})$ defines a map of fans $i_b : \Sigma \to \Sigma_b$. If σ is any two dimensional cone in Σ, $i_b(\sigma)$ is contained in a unique two dimensional cone in Σ_b which we call σ_b. σ_b corresponds to a unique edge $\langle w, b\rangle$ incident to b, given by the intersection of the two zig-zag paths through b whose rays in Σ_b form the boundary of σ_b. Define the 1-chain $\omega(\sigma_b)$ to be 1 on the edge $\langle w, b\rangle$ and 0 on all other edges. Define

$$\omega(\sigma) = \sum_{b \in V(\Gamma) \text{ black}} \omega(\sigma_b).$$

Two dimensional cones in Σ are in bijection with vertices of the Newton polygon: If σ is a two dimensional cone in Σ, let E_1 and E_2 be the edges of N whose associated rays form the boundary of σ in Σ. Then E_1 and E_2 occur in cyclic order and therefore there is a vertex V between them in N.

Lemma 4.3.5 (Broomhead, Goncharov-Kenyon, 2012 [Bro12, GK12])**.** $\omega_V := \omega(\sigma)$ *is the unique extremal dimer cover associated to the vertex V of N that corresponds to σ.*

We say that a dimer cover M is *external* if $[M]$ is a boundary lattice point of N. To a zig-zag path α we associate a 1-form ω_α that is 1 on edges e in α that are oriented the same way as α and 0 on edges not in α. If M is external, $[M]$ lies on an edge E of N, which corresponds to a family of zig-zag paths $\{\alpha_k\}$. Let $E = \langle V_1, V_2 \rangle$, where V_1, V_2 are vertices of N such that V_2 is the vertex after V_1 when the boundary of N is traversed counterclockwise.

Lemma 4.3.6 (Broomhead, Goncharov-Kenyon, 2012 [Bro12, GK12])**.** *Let A be a subset of the family of zig-zag paths $\{\alpha_k\}$ corresponding to E. The external dimer covers on E are of the form*

$$\omega_A := \omega_{V_1} + \sum_{\alpha_k \in A} \omega_{\alpha_k}.$$

In particular, $\omega_{V_2} = \omega_{V_1} + \sum_k \omega_{\alpha_k}$, and the number of dimer covers corresponding to a boundary lattice point of N is a binomial coefficient.

Proof of Theorem 4.3.1. We use Temperley's bijection (Theorem 4.3.4), Lemmas 4.3.5 and 4.3.6, and the canonical bijection between zig-zag paths on G and Γ_G. $\square$

Proof of Corollary 4.3.2. Suppose γ_σ is an external OCRSF and let v be a vertex of G. By construction, there is a single outgoing edge from v. We show that there is also a single incoming edge. Consider the fan $-\Sigma_v$ whose rays are generated by homology classes of zig-zag paths that turn maximally left at v and let $i'_v : \Sigma \to -\Sigma_v$ be the natural map. $i'_v(\sigma)$ is contained in a unique two dimensional cone σ'_v which corresponds to a unique edge e oriented towards v. Define the 1-chain γ'_{σ_v} to be 1 on e and 0 on all other edges and define the 1-chain

$$\gamma'_\sigma := \sum_{v \in V(G)} \gamma'_{\sigma_v}.$$

Let $e = \langle u, v \rangle$ be an edge in G and let α_1 and α_2 be the two zig-zag paths through e that turn maximally left at v. Then α_1 and α_2 turn maximally right at u and therefore we have $\sigma'_v = \sigma_u$ which implies $\gamma'_{\sigma_v} = \gamma_{\sigma_u}$. Summing over all vertices, we get $\gamma'_\sigma = \gamma_\sigma$. It is clear from the definition of γ'_σ that every vertex has a unique incoming edge. It follows that γ_σ is a union of cycles.

By Theorem 4.3.1, every external OCRSF is obtained from an extremal OCRSF γ_V by adding cycles corresponding to some zig-zag paths and therefore is also a union of cycles.

$\square$

4.4 Spectral data

4.4.1

Following the definition of the dimer spectral data in [GK12], we define spectral data for resistor networks. A convex integral polygon N determines a toric surface

$\mathcal{N}$ along with an ample divisor D_N on it. The global sections of D_N can be canonically identified with Laurent polynomials with Newton polygon N. Let $|\mathcal{D}_\mathcal{N}|$ be the linear system of curves on $\mathcal{N}$ given by the vanishing loci of global sections of D_N. Let $g =$ number of interior lattice points in $N - 1$. The genus of a generic curve from $|D_N|$ is $g + 1$.

The complement of $(\mathbb{C}^*)^2$ in $\mathcal{N}$ is a union of $\mathbb{P}^1$s, called *lines at infinity*, parameterized by the edges of N and intersecting according to the combinatorics of N. For an edge E of N, let D_E denote the corresponding line at infinity. A generic curve $C \in |D_N|$ meets D_E in $|E|$ points, called the *points at infinity*, where $|E|$ is the number of primitive integral vectors in E. Therefore the number of points at infinity on D_E agrees with the number of zig-zag paths associated to E, but there is no canonical bijection between these sets. A *parameterization* $\nu = \{\nu_E\}$ of the points at infinity of C by zig-zag paths is a collection of bijections ν_E between zig-zag paths in G associated to E and points at infinity in $C \cap D_E$.

4.4.2

Let $\mathcal{S}_N$ be the moduli space of triples (C, S, ν) such that C is a curve in $|\mathcal{D}_\mathcal{N}|$, S is a degree g effective divisor on C and ν is a parameterization of the points at infinity of C by zig-zag paths. Let G be a minimal resistor network associated to N and let v be a vertex of G. The *resistor network spectral transform* is the rational map

$$\rho_{G,v} : \mathcal{R}_N \to \mathcal{S}_N,$$

described on the affine chart $\mathcal{R}_G$ as follows:

1. C is the compactification of C_0 obtained by taking the closure of the spectral curve $C_0 := \{(z, w) \in (\mathbb{C}^*)^2 : \det \Delta(z, w) = 0\}$ in $\mathcal{N}$.

2. Let $i : C_0 \hookrightarrow (\mathbb{C}^*)^2$ denote the inclusion. The Laplacian sits in the following exact sequence of sheaves on $(\mathbb{C}^*)^2$:

$$\bigoplus_{v \in V} \mathcal{O}_{(\mathbb{C}^*)^2} \xrightarrow{\Delta} \bigoplus_{v \in V} \mathcal{O}_{(\mathbb{C}^*)^2} \to \mathcal{L} \to 0. \tag{4.4}$$

   Lemma 4.4.1. *For a generic conductance, $i^*\mathcal{L}$ is a line bundle on C_0.*

   Proof. $i^*\mathcal{L}$ has one dimensional fibers over the non-singular points of C_0 [CT79, Theorem 2.2]. The fiber of $i^*\mathcal{L}$ at $(1, 1)$ is the vector space of discrete harmonic functions on G. This space is one dimensional because by the maximum principle, the only harmonic functions are the constant functions. Since C_0 is integral for a generic conductance and $i^*\mathcal{L}$ is a coherent sheaf of constant fiber dimension one, it is locally free of rank one. $\square$

   The image of the section δ_v gives a section of $i^*\mathcal{L}$. S is the divisor of zeroes of this section. Let Q be the adjugate matrix of Δ. S is given by the simultaneous vanishing of the v_0-column of Q. In fact, since corank Δ is one, it suffices to consider the simultaneous vanishing of any two entries of the v_0-column of Q.

3. ν is the parameterization of the points at infinity of C by zig-zag paths on G such that the coordinate of the point at infinity associated to a zig-zag path is determined by the product of conductances around that zig-zag path.

4.4.3

Let $W \subset |D_N|$ be the linear system of curves defined by sections $P(z,w)$ of D_N satisfying the following conditions:

1. $P(1,1)=0$, and the point $(1,1)$ is a node.

2. $\sigma : (z,w) \mapsto (\frac{1}{z}, \frac{1}{w})$ is an involution on $\{P(z,w)=0\}$.

Let $\mathcal{S}'_N$ be the moduli space of triples (C,S,ν) such that C is a curve in W, S is a degree g effective divisor on $C \setminus (1,1)$ satisfying

$$S + \sigma(S) - q_1 - q_2 = K_{\widehat{C}} \text{ in } \mathrm{Pic}^{2g-2}(\hat{C}), \tag{4.5}$$

where $\widehat{C}$ is the normalization of C, $K_{\widehat{C}}$ is the canonical divisor class of $\widehat{C}$, and ν is a parameterization of the points at infinity. The presence of the node $(1,1)$ means that a generic curve in W has geometric genus g, one less than that of a generic curve in $|D_N|$.

We determine the image of the spectral transform:

Theorem 4.4.2. *We have $\rho_{G,v}(\mathcal{R}_N) \subseteq \mathcal{S}'_N$.*

Proof. Consider the following commuting diagram:

$$\begin{tikzcd} \widehat{C_0} \arrow[r, "\phi"] \arrow[d, hook] & C_0 \arrow[r, hook, "i"] \arrow[d, hook] & (\mathbb{C}^*)^2 \arrow[d, hook] \\ \widehat{C} \arrow[r, "\pi"] & C \arrow[r, hook] & \mathcal{N} \end{tikzcd},$$

where ϕ and π are the normalization maps. We pull back (4.4) using $\phi^* i^*$ and use right-exactness of pullback to get the following exact sequence on $\widehat{C_0}$:

$$\bigoplus_{v\in V} \mathcal{O}_{\widehat{C_0}} \xrightarrow{\phi^* i^* \Delta} \bigoplus_{v\in V} \mathcal{O}_{\widehat{C_0}} \to \phi^* i^* \mathcal{L} \to 0. \tag{4.6}$$

Theorem 4.4.3 (Kenyon, 2017 [K17])**.** *For the space $\mathcal{R}_N(\mathbb{R}_{>0})$ of positive-real-valued points of $\mathcal{R}_N$, we have $(C_0, S, \nu) \in \mathcal{S}'_N$. Moreover C_0 is a simple Harnack curve.*

1. $P(z,w) = P(\frac{1}{z}, \frac{1}{w})$ follows immediately from $\Delta(z,w) = \Delta(\frac{1}{z}, \frac{1}{w})^T$.

2. $P(1,1) = 0$ follows from the observation that Δ has non-zero kernel at $(1,1)$, indeed these are the constant functions, which are discrete harmonic.

3. Differentiating the expression (4.2) for $P(z,w)$, we see that

$$\frac{\partial P(1,1)}{\partial z} = \frac{\partial P(1,1)}{\partial w} = 0,$$

   hence $(1,1)$ is a singular point. For all positive real points, Theorem 4.4.3 tells us that $(1,1)$ is a node. Since nodes are characterized by non-vanishing of the Hessian, an open condition, $(1,1)$ is a node for all points in a Zariski open subset of $\mathcal{R}_N$.

4. deg $S = g$ is proved in Corollary 4.4.10.

5. $S + \sigma(S) - q_1 - q_2 = K_{\widehat{C}}$ is Corollary 4.4.9.

□

The condition (4.5) may be interpreted as saying that there is a meromorphic 1-form on $\widehat{C}$ that has zeroes at the $2g$ points $S+\sigma(S)$ and poles at q_1, q_2. We can write down this 1-form explicitly.

Lemma 4.4.4. *Let $Q(z,w)$ be the minor of $\Delta(z,w)$ with the row and column corresponding to v_0 removed. The meromorphic 1-form*

$$\omega = \phi^* \left(\frac{Q(z,w)dz}{zw\frac{\partial P(z,w)}{\partial w}} \right),$$

satisfies

$$div_{\widehat{C}}\omega = S + \sigma(S) - q_1 - q_2.$$

Proof. For smooth $(z,w) \in C$, we have corank $\Delta(z,w) = 1$. Therefore we can write adj $\Delta(z,w) = U(z,w)V(z,w)^T$ for some $U(z,w) \in \text{Ker } \Delta(z,w), V(z,w) \in$ Coker $\Delta(z,w)$. By definition, S is the set of points in C_0 where the component $V(z,w)\cdot\delta_{v_0}$ of $V(z,w)$ vanishes. We have Ker $\Delta(z,w) \cong$ Coker $\Delta(z,w)^T =$ Coker $\Delta(\frac{1}{z}, \frac{1}{w})$, so $\sigma(S)$ are the points where the component $U(z,w)\cdot\delta_{v_0}$ vanishes. Since $Q(z,w) = (U(z,w)\cdot\delta_{v_0})(V(z,w)\cdot\delta_{v_0})$, we have

$$\text{div}_{C_0} Q(z,w) = S + \sigma(S),$$

Since C has a node at $(1,1)$, $\frac{\partial P(z,w)}{\partial w}$ has a simple zero at $(1,1)$ and so ω has simple poles at q_1, q_2. Therefore, the divisor of ω on the complement of the points at infinity is $S + \sigma(S) - q_1 - q_2$, which has degree $2g-2$. It remains to identify the zeros and poles of ω at the points at infinity.

The order of vanishing of the 1-form

$$\omega_{ij} := \frac{z^{i-1}w^{j-1}dz}{\frac{\partial P(z,w)}{\partial w}}$$

at the point at infinity corresponding to the primitive integral edge E is given by the twice the signed area of the triangle formed by E and the point (i,j) minus one (where area is positive for points (i,j) inside N). $Q(z,w)$ is the partition function of CRSFs on the graph G' obtained from G by deleting the vertex v_0. By Corollary 4.3.2, the Newton polygon of $Q(z,w)$ is strictly contained in N. Therefore the order of vanishing of ω must be non-negative at all points at infinity, that is ω has no poles at these points. The divisor of ω on the complement of the points at infinity has degree $2g-2$ (Corollary 4.4.10), which is the degree of $K_{\widehat{C}}$. Therefore ω must have an equal number of zeroes and poles at the points at infinity and therefore ω has no zeroes at infinity either. □

4.4.4 Discrete Abel and Abel-Prym maps

We denote the zig-zag path oriented opposite to α by α'. Define $d' : V(\tilde{G}) \cup F(\tilde{G}) \to \mathbb{Z}^Z$ as follows:

Set $d'(v) = 0$ for some vertex v. For any vertex or face u, let $\tilde{\gamma}$ be a path from v to u in $\tilde{G}$ and let γ be its image under the projection $\tilde{G} \to G$. Let

$$d'(u) = d'(v) + \sum_{\alpha \in Z} ([\alpha], [\gamma])_{\mathbb{T}} \alpha,$$

where $(\cdot, \cdot)_{\mathbb{T}}$ is the intersection pairing on $H_1(\mathbb{T}, \mathbb{Z})$.

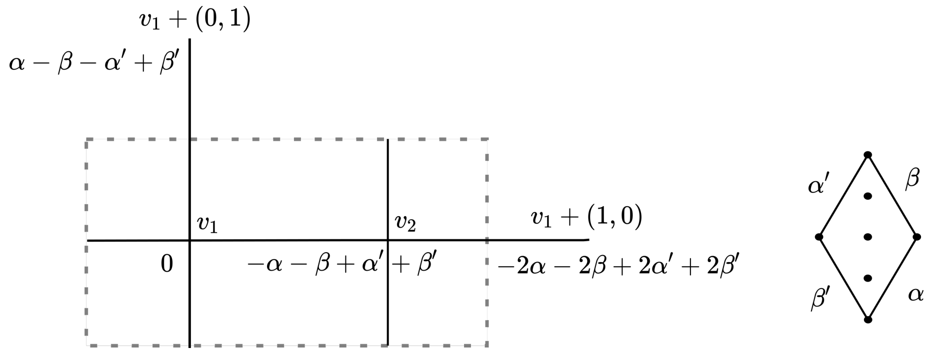


Figure 4.5: The discrete Abel map for the newtork in Figure 4.2. The labels for zig-zag paths are shown next to their corresponding edges in boundary of the Newton polygon.

Define the inclusion

$$\begin{aligned} H_1(\mathbb{T},\mathbb{Z}) &\hookrightarrow \mathbb{Z}^Z \\ h &\mapsto \sum_{\alpha\in Z}([\alpha],h)_{\mathbb{T}}\alpha. \end{aligned}$$

If $h = (i,j)$, from toric geometry we have

$$\sum_{\alpha\in Z}([\alpha],h)_{\mathbb{T}}\alpha = \operatorname{div}_{\widehat{C}} z^i w^j.$$

Abusing notation, we will denote the homology class h and its image in $\mathbb{Z}^Z$ by the same letter h. Observe that d' is equivariant with respect to the $H_1(\mathbb{T},\mathbb{Z})$ action, that is,

$$d'(h\cdot u) = h\cdot d'(u),$$

for all $u \in V(\tilde{G}) \cup F(\tilde{G})$.

We define the discrete Abel map [F15] $d : V(G) \cup F(\tilde{G}) \to \mathbb{Z}^Z$ as follows:
We identify the fundamental domain of G with the lift to $\tilde{G}$ that contains v_0 and let $d(v)$ to be $d'(v)$. We observe that for all edges $e : u \to v$ of G with pairs of oriented zig-zag paths $\alpha, \alpha', \beta, \beta'$ through e, we have

$$d(v) - d(u) = -\alpha - \beta + \alpha' + \beta' - \mathrm{div}_{\widehat{C}} z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}}.$$

We compute the discrete Abel map for the network in Figure 4.2 in Figure 4.5. We also compute $H_1(\mathbb{T}, \mathbb{Z}) \hookrightarrow \mathbb{Z}^Z$:

$$(1,0) \quad \mapsto \quad -2\alpha - 2\beta + 2\alpha' + 2\beta' \tag{4.7}$$

$$(0,1) \quad \mapsto \quad \alpha - \beta - \alpha' + \beta'. \tag{4.8}$$

For later purposes, we also define here the *discrete Abel-Prym map*

$$d_P : V(\tilde{G}) \cup F(\tilde{G}) \to \mathrm{Pr}(\widehat{C}, \sigma)$$
$$d_P = \frac{1}{2} I_P \circ d.$$

Recall the definition of the line bundle (rather its sheaf of sections) $\mathcal{O}_{\widehat{C}}(D)$ associated to a divisor D on $\widehat{C}$:

$$\mathcal{O}_{\widehat{C}}(D)(U) := \{t \in K(\widehat{C})^* : \mathrm{div}|_U t + D|_U \geq 0\} \cup \{0\}, \text{ for all } U \subset \widehat{C} \text{ open}.$$

Each rational function $t \in \mathcal{O}_{\widehat{C}}(D)(\widehat{C})$ corresponds to a regular section $\tilde{t}$ of $\mathcal{O}_{\widehat{C}}(D)$ with divisor div $t + D$.

The usefulness of the discrete Abel map stems from the following lemma:

Lemma 4.4.5. *The following is an extension of (4.6) to a morphism of vector bundles on $\widehat{C}$:*

(4.9)
$$0 \to \widetilde{\mathcal{M}} \to \bigoplus_{v \in V} \mathcal{O}_{\widehat{C}} \left(d(v) - \sum_{\alpha \in Z : v \in \alpha} \alpha - d(v_0) \right) \xrightarrow{\widetilde{\Delta}} \bigoplus_{v \in V} \mathcal{O}_{\widehat{C}}(d(v) - d(v_0)) \to \widetilde{\mathcal{L}} \to 0,$$

where $\tilde{\Delta}_{vu}$ is the section of

$$\mathcal{H}om \left(\mathcal{O}_{\widehat{C}} \left(d(u) - \sum_{\alpha \in Z : v \in \alpha} \alpha - d(v_0) \right), \mathcal{O}_{\widehat{C}}(d(v) - d(v_0)) \right)$$
$$\cong \mathcal{O}_{\widehat{C}} \left(d(v) - d(u) + \sum_{\alpha \in Z : v \in \alpha} \alpha \right),$$

corresponding to the rational function Δ_{vu}.

Proof. We need to show that for each $v, w \in V$, the component $\widetilde{\Delta}_{vu}$ is a regular section of $\mathcal{O}_{\widehat{C}}\left(d(v) - d(u) + \sum_{\alpha \in Z : v \in \alpha} \alpha\right)$ i.e. that div $\Delta_{vu} + d(v) - d(u) + \sum_{\alpha \in Z : v \in \alpha} \alpha \geq 0$. By definition, we have

$$\Delta_{vu}(z, w) = \begin{cases} \sum_{e : v' \to v : v' \neq v} c(e) + \sum_{e : v \to v} c(e)(1 - z^{\langle e, \gamma_z \rangle} w^{\langle e, \gamma_w \rangle}), \text{ if } v = u; \\ -\sum_{e : u \to v} c(e) z^{\langle e, \gamma_z \rangle} w^{\langle e, \gamma_w \rangle}, \text{otherwise.} \end{cases}$$

When $v \neq u$, recall that for each edge $e : u \to v$ we have from the definition of the discrete Abel map that

$$d(v) - d(u) = -\beta - \delta + \beta' + \delta' - \text{div } z^{\langle e, \gamma_z \rangle} w^{\langle e, \gamma_w \rangle},$$

where $\beta, \delta, \beta', \delta'$ are the oriented zig-zag paths through e, with β, β' and δ, δ' the oppositely oriented pairs. From this we get

$$\operatorname{div}\, z^{\langle e,\gamma_z\rangle} w^{\langle e,\gamma_w\rangle} + d(v) - d(u) + \sum_{\alpha\in Z: v\in\alpha} \alpha = \sum_{\alpha\in Z: v\in\alpha, \alpha\neq\beta,\delta,\beta',\delta'} \alpha \geq 0,$$

so each $z^{\langle e,\gamma_z\rangle} w^{\langle e,\gamma_w\rangle}$ is a regular section of $\mathcal{O}_{\widehat{C}}\left(d(v) - d(u) + \sum_{\alpha\in Z: v\in\alpha} \alpha\right)$. Since Δ_{vu} is a linear combination of these, the same holds for it as well.

When $v = u$, Δ_{vu} is a sum of constant terms in z, w and terms involving $z^{\langle e,\gamma_z\rangle} w^{\langle e,\gamma_w\rangle}$ as in the case $u \neq v$. $d(v) - d(u) + \sum_{\alpha\in Z: v\in\alpha} \alpha = \sum_{\alpha\in Z: v\in\alpha} \alpha \geq 0$ implies that the constant terms are also regular sections of $\mathcal{O}_{\widehat{C}}\left(d(v) - d(u) + \sum_{\alpha\in Z: v\in\alpha} \alpha\right)$.

$\square$

The extension $\widetilde{\Delta}$ has the nice property:

Lemma 4.4.6. *We have det* $\widetilde{\Delta} \in H^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(D_N|_{\widehat{C}}))$, *where*

$$\det \widetilde{\Delta} : \bigwedge_{v\in V} \mathcal{O}_{\widehat{C}}\left(d(v) - \sum_{\alpha\in Z: v\in\alpha} \alpha - d(v_0)\right) \xrightarrow{\widetilde{\Delta}} \bigwedge_{v\in V} \mathcal{O}_{\widehat{C}}(d(v) - d(v_0))$$

is the determinant line bundle.

Proof. Recall that $D_N = \sum_{E \text{ edge of } N} a_E D_E$, where $a_E \in Z$ is the distance from the origin to E along the primitive normal vector to E. Therefore

$$\begin{aligned} D_N|_{\widehat{C}} &= \sum_{E \text{ edge of } N} a_E D_E \cap \widehat{C} \\ &= \sum_{E \text{ edge of } N} \sum_{\alpha\in Z: [\alpha] || E} a_E \alpha \end{aligned}$$

.

The determinant line bundle is isomorphic to

$$\mathcal{H}om\left(\bigwedge_{v\in V}\mathcal{O}_{\widehat{C}}\left(d(v)-\sum_{\alpha\in Z:v\in\alpha}\alpha-d(v_0)\right),\bigwedge_{v\in V}\mathcal{O}_{\widehat{C}}(d(v)-d(v_0))\right)$$
$$\cong\mathcal{O}_{\widehat{C}}\left(\sum_{v\in V}(d(v)-d(v_0))-\sum_{v\in V}\left(d(v)-\sum_{\alpha\in Z:v\in\alpha}\alpha-d(v_0)\right)\right),$$

so we need to show that

$$\sum_{v\in V}(d(v)-d(v_0))-\sum_{v\in V}\left(d(v)-\sum_{\alpha\in Z:v\in\alpha}\alpha-d(v_0)\right)=D_N|_{\widehat{C}}.$$

Let ρ be a zig-zag path, let (i_1,i_2) be a vertex of N incident to the edge of N corresponding to ρ and let F be the corresponding extremal OCRSF. From our description of extremal OCRSFs (Theorem 4.3.1), we know that for each vertex $u\in V$, there is a unique outgoing edge e_u and that if ρ contains u, then $e_u\in\rho$. We pair vertices of G using e_v to rewrite the sum as

$$\sum_{e:u\to v\in F}d(v)-d(u)+\sum_{\alpha\in Z:u\in\alpha}\alpha.$$

Now we observe that if $e\in\rho$, then ρ appears twice in the summand with opposite signs and if $e\notin\rho$, then ρ does not appear in the summand, modulo contributions from the edges of F intersecting γ_z,γ_w. This latter contribution is given by

$$\begin{aligned}-\sum_{e\in F}\text{div }z^{\langle e,\gamma_z\rangle}w^{\langle e,\gamma_w\rangle}&=-z^{\langle i_1,\gamma_z\rangle}w^{i_2,\gamma_w\rangle}\\&=a_E.\end{aligned}$$

$\square$

Let $Q,\widetilde{Q}$ denote the adjugate matrices of $\Delta,\widetilde{\Delta}$ respectively. We obtain:

Corollary 4.4.7. $div_{\widehat{C}}Q_{uv}\geq D_N-d(v)+d(u)-\sum_{\alpha\in Z:u\in\alpha}\alpha.$

Proof. Q is the rational function corresponding to the section $\widetilde{Q}$ of

$$\mathcal{O}_{\widehat{C}}\left(D_N|_{\widehat{C}} - d(v) + d(u) - \sum_{\alpha \in Z : u \in \alpha} \alpha\right).$$

$\square$

Going back to our example in Figure 4.2, we check that the Laplacian that we computed in (4.3) extends to a morphism of vector bundles:

$$\mathcal{O}(-\alpha - \beta - \alpha' - \beta') \oplus \mathcal{O}(-2\alpha - 2\beta) \to \mathcal{O} \oplus \mathcal{O}(-\alpha - \beta + \alpha' + \beta'). \tag{4.10}$$

We have $\Delta(z, w)_{v_1 v_2} = -a - bz$, which we wish to show corresponds to a regular section of $\mathcal{O}(2\alpha + 2\beta)$. We check:

$$\begin{aligned} \text{div } a + 2\alpha + 2\beta &= 0 + 2\alpha + 2\beta \geq 0, \\ \text{div } bz + 2\alpha + 2\beta &= (-2\alpha - 2\beta + 2\alpha' + 2\beta') + 2\alpha + 2\beta \geq 0, \end{aligned}$$

where we have used div $z = -2\alpha - 2\beta + 2\alpha' + 2\beta'$ from (4.7). The other entries of $\Delta(z, w)$ may be checked in the same way.

In this example $D_N = 2\alpha + 2\beta + 2\alpha' + 2\beta'$. On the other hand, from (4.10) we see that the determinant line bundle is $\mathcal{O}(2\alpha + 2\beta + 2\alpha' + 2\beta')$, verifying the conclusion of Lemma 4.4.6.

Corollary 4.4.7 provides an inequality for the divisor of Q_{uv} but we need a more careful analysis of the behaviour of Q_{uv} at infinity to determine its divisor exactly. This is the goal of this section. We denote by $\tilde{s}_u, \tilde{t}_v$ the sections of $\widetilde{\mathcal{M}}^\vee \otimes \mathcal{O}_{\widehat{C}}(d(u) - \sum_{\alpha \in Z: v \in \alpha} \alpha - d(v_0))$, $\mathcal{O}_{\widehat{C}}(d(v) - d(v_0)) \otimes \widetilde{\mathcal{L}}$ respectively, given by the u-entry of the kernel map and the v-entry of the cokernel map.

Lemma 4.4.8. *We have:*

$$div_{\widehat{C}} \tilde{s}_u = \sigma(S_u) + \sum_{\alpha \in Z: u \notin \alpha} \alpha;$$

$$div_{\widehat{C}} \tilde{t}_v = S_v,$$

where S_v is the effective divisor given by the vanishing of the v-column of Q.

Proof. Let α be an oriented zig-zag path. Let x be a local parameter in a neighborhood U of α disjoint from the other points at infinity with a simple zero at α. We trivialize the line bundles in (4.9) as follows:

$$\mathcal{O}(-k\alpha)(U) \xrightarrow{\cong} \mathcal{O}(U)$$
$$f \mapsto x^{-k} f$$

Let $z = ax^m + O(x^{m+1})$ and $w = bx^n + O(x^{n+1})$ be the expansions in the local coordinate x. Let us order the vertices so that the vertices on the zig-zag path appear first. Then the Laplacian matrix at α has the following block form:

$$\widetilde{\Delta} = \begin{pmatrix} \Delta_1 & B \\ xA & \Delta_2 \end{pmatrix} + O(x),$$

where Δ_1 is the restriction of the Laplacian to the zig-zag path α and Δ_2 is the restriction to the rest of the graph, and where z and w are replaced with a and b respectively. Since we are at α, Δ_1 is singular. For smooth $\widehat{C}$, dim Ker $\Delta_1 = 1$ and Δ_2 is invertible.

Let $v \in$ Ker Δ_1^*. Then we have

$$\text{Ker } \widetilde{\Delta}^* = (v, -(\Delta_2^*)^{-1} B^* v) + O(x).$$

Since generically none of the entries in Ker $\widetilde{\Delta}^*$ is 0, and since these entries are the cofactors of Δ, we see that $\tilde{t}_v$ has no poles or zeros at α. Since α was arbitrary, $\tilde{t}_v$ has no zeroes or poles at infinity.

Now let $v \in$ Ker Δ_1. We have

$$\text{Ker } \widetilde{\Delta} = (v, -x\Delta_2^{-1} A v) + O(x),$$

from which we see that $\tilde{s}_u$ has a simple zero at α for $u \notin \alpha$ and no other zeroes or poles at α for $u \in \alpha$. $\square$

Corollary 4.4.9. *We have* $div_{\widehat{C}} Q_{uv} = S_v + \sigma(S_u) - D_N|_{\widehat{C}} + d(v) - d(u) + \sum_{\alpha \in Z} \alpha$*, and therefore*

$$S + \sigma(S) - q_1 - q_2 = K_{\widehat{C}} \textit{ in } Pic^{2g-2}(\widehat{C}).$$

Proof. Since $\widetilde{Q}_{uv} = \tilde{s}_u \tilde{t}_v$, we have

$$\mathrm{div}_{\widehat{C}} \widetilde{Q}_{uv} = S_v + \sigma(S_u) + \sum_{\alpha \in Z : u \notin \alpha} .$$

From Corollary 4.4.7, we know that Q is the rational function corresponding to the section $\widetilde{Q}$ of $\mathcal{O}_{\widehat{C}}\left(D_N|_{\widehat{C}} - d(v) + d(u) - \sum_{\alpha \in Z : u \in \alpha} \alpha\right)$, from which we obtain the first statement.

The canonical divisor of the toric variety $\mathcal{N}$ is given by $-\sum_E D_E$. Therefore the adjunction formula for nodal curves [ACGH85, Appendix A.8] gives $K_{\widehat{C}} = (\sum_E D_E + D_N)|_{\widehat{C}} - q_1 - q_2$, from which we get the second statement. □

Corollary 4.4.10. *deg* $S_v = g$ *for all* $v \in V(G)$.

Proof. We take degrees on both sides of $S_v + \sigma(S_v) - q_1 - q_2 = K_{\widehat{C}}$, and use deg $K_{\widehat{C}} = 2g - 2$. □

Corollary 4.4.11. *The cokernel line bundle* $\widetilde{\mathcal{L}} \cong \mathcal{O}_{\widehat{C}}(S_{v_0})$. *Moreover*

$$S_v + d(v) - d(u) = S_u \text{ in } Pic^g(\widehat{C}),$$

for all $u, v \in V(G)$.

Proof. $\tilde{t}_{v_0}$ which is the image of the regular section δ_{v_0} under the cokernel map is a regular section of $\widetilde{L}$, so $\widetilde{L} \cong \mathcal{O}_{\widehat{C}}(S_{v_0})$.
Similarly, the image of the rational section δ_v for $v \in V(G)$ is a rational section of $\widetilde{L}$, and has divisor $S_v + d(v) - d(v_0)$. □

Now that we have identified the cokernel line bundle, we can find the cokernel map. Each component of the cokernel map is given by a rational section of $\mathcal{O}(S)$ with prescribed order of vanishing at infinity. These can be written down explicitly in terms of Prym theta functions and the prime form (see Appendix for notation).

Let $e = \frac{1}{2}\pi_1(I(S) - I(q_1) - I(q_2) - \pi^*\Delta_C) + d_P(v_0)$ (see Lemma B.1.2). Define for each vertex $v \in G$,

$$\psi_v(x) := \frac{\eta(x + d_P(v) - e)}{\eta(d_P(v) - e)} E_{d(v)-d(v_0)}(x),$$

where d_P is the discrete Abel-Prym map (see Section 4.4.4).

Lemma 4.4.12. *The cokernel map is given by $\delta_v \mapsto \psi_v$.*

Proof. If D is a generic degree g effective divisor, the Jacobi inversion theorem (Theorem B.0.2) implies that $H^0(\widehat{C}, \mathcal{O}(D))$ is 1-dimensional. The cokernel map in (4.9) is given by a collection of global sections of $\mathcal{H}om(\mathcal{O}(d(v) - d(v_0)), \mathcal{O}(S)) \cong \mathcal{O}(S + d(v_0) - d(v)) \cong \mathcal{O}(S_v)$, and therefore uniquely determined up to scaling each component once we specify the image of δ_v for all v. The scaling is fixed by the requirement that the cokernel at q_1 and q_2 be $(1, 1, ..., 1)$. $\square$

4.5 Inverse spectral map

In this section, we construct the inverse of the spectral map. We begin by describing the normalization map π in terms of the prime form.

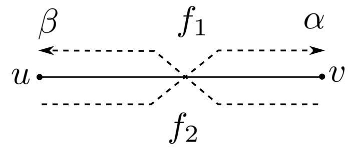


Figure 4.6: Vertices, faces and zig-zag paths in the definition of the conductance function.

Lemma 4.5.1. *The following diagram commutes:*

$$\begin{tikzcd}
\widehat{C} \arrow[dr, dashed] \arrow[drr, dashed, bend left, "x\mapsto(E_{(1,0)}(x),E_{(0,1)}(x))"] \arrow[ddr, bend right, "\pi"'] & & \\
& C_0 \arrow[r, hook] \arrow[d, hook] & (\mathbb{C}^*)^2 \arrow[d, hook] \\
& C \arrow[r, hook] & \mathcal{N}
\end{tikzcd}$$

Proof. The functions z and w on $(\mathbb{C}^*)^2$ restrict to rational functions on C, which pull back to rational functions π^*z and π^*w on $\widehat{C}$. We have

$$\mathrm{div}_{\widehat{C}}\pi^* z = \mathrm{div}_{\widehat{C}} E_{(1,0)}(x),$$

so they agree up to multiplication by a constant. Since $E_{(1,0)}(q_1) = \pi^* z(q_1) = 1$, the constant is 1, and therefore we have $\pi^* z = E_{(1,0)}(x)$. By the same argument applied to w, we get $\pi^* w = E_{(0,1)}(x)$. $\square$

Let uv be an edge in $\tilde{G}$, f_1 and f_2 be the faces adjacent to uv and let α, β be the zig-zag paths as shown in Figure 4.6. Define the conductance function

$$c_{u,v} := \frac{\eta(d_P(u)-e)\eta(d_P(v)-e)}{\eta(d_P(f_1)-e)\eta(d_P(f_2)-e)} \frac{E(\alpha,\beta)}{E(\alpha,\beta')}. \tag{4.11}$$

We note the similarity of (4.11) with [GK12, (58)].

Lemma 4.5.2. *$c_{u,v}$ has the following properties:*

1. *$c_{u,v} = c_{v,u}$;*

2. *$c_{u,v}$ is compatible with taking the dual graph, that is, $c_{f_1,f_2} = 1/c_{u,v}$.*

3. *$c_{u,v}$ is $H_1(\mathbb{T},\mathbb{Z})$-periodic and therefore descends to a conductance function c on G.*

Proof. 1. Follows from the symmetry $E(\alpha,\beta) = E(\alpha',\beta')$.

2. Clear.

3. Let $h \in H_1(\mathbb{T},\mathbb{Z})$. We have

$$I_P(d_P(u+h) - d_P(u)) = \frac{1}{2}\pi_1 I(h) = 0,$$

since $h = (i,j) = \mathrm{div}_{\widehat{C}} z^i w^j$.

□

Theorem 4.5.3. *The rational map $\rho_{G,v_0} : (C,S,\nu) \mapsto V(c)$ is the inverse of κ_{G,v_0}. Therefore $\mathcal{R}_N$ is birational to $\mathcal{S}'_N$.*

Proof. 1. $\kappa_{G,v_0} \circ \rho_{G,v_0} = \mathrm{id}$:

Let u be a vertex in Γ and let $v_1, ..., v_n$ be the vertices adjacent to u in G. Let $\alpha_1, ..., \alpha_n$ be the zig-zag paths as shown in Figure 4.7. Note that

$$i_{v,u}^{-1}\psi_v(x) = \psi_u(x).$$

Using Theorem B.1.3 with $z = q_1, t = d_P(u) - e, x_k = \alpha_k$, we get

$$\sum_{v_k \sim u} c_{u,v_k} = \frac{\eta\left(d_P(u) - e - \sum_{i=1}^{k}\alpha_k\right)\eta(d_P(u)-e)^2}{\prod_{k=1}^{n}\eta(d_P(u)-e-\alpha_k)} \prod_{k=1}^{n}\frac{E(\alpha_k,\alpha_{k+1})}{E(\alpha_k,\alpha'_{k+1})}. \tag{4.12}$$

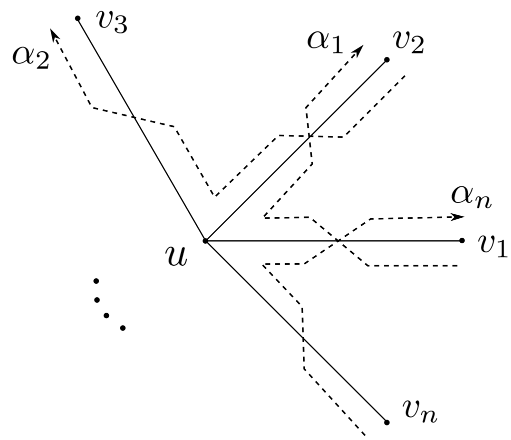


Figure 4.7: Local configuration near a vertex u.

Using Theorem B.1.3 with $z = x, t = d_P(u) - e, x_k = \alpha_k$ and (4.12), we get

$$\sum_{v_k \sim u} c_{u,v_k}(\psi_u(x) - i^{-1}_{v_k,u}\psi_{v_k}(x)) = 0,$$

so the following is sequence is exact:

$$0 \to \text{Ker } \phi^* i^* \Delta^T \xrightarrow{1 \mapsto (\psi_v)_v} \bigoplus_{v \in V} \mathcal{O}_{\widehat{C}}(-d(v) + d(v_0)) \xrightarrow{\phi^* i^* \Delta^T} \bigoplus_{v \in V} \mathcal{O}_{\widehat{C}}(-d(v) + \sum_{\alpha \in Z : v \in \alpha} \alpha + d(v_0)).$$

Since this is the transpose of (4.9), the cokernel map in (4.9) is $\delta_v \mapsto \psi_v$ and we recover $S = \text{div}_{\widehat{C_0}} \psi_{v_0}$ as the divisor.

2. $\rho_{G,v_0} \circ \kappa_{G,v_0} = \text{id}$:

   Suppose c' is a conductance function such that $\kappa_{G,v_0}(c') = (C, S, \nu)$. By Lemma 4.4.12, the cokernel map is determined by S and is given by $\delta_v \mapsto \psi_v$. Taking

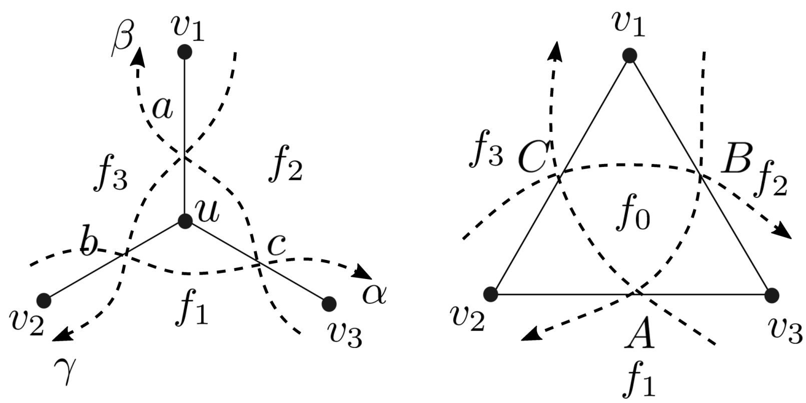


Figure 4.8: Y-Delta transformation.

transpose, the equation of $\phi^* i^* \Delta^T$ becomes

$$\sum_{v_k \sim u} c'_{u,v_k}(\psi_u(x) - i^{-1}_{v_k,u}\psi_{v_k}(x)) = 0.$$

Since the coefficients of the quadrisecant identity are uniquely determined up to a constant, comparing with Theorem B.1.3 with $z = x, t = d_P(u) - e, x_k = \alpha_k$, we see that c' agrees with c up to a multiplicative constant.

□

4.6 Compatibility with $Y - \Delta$ transformations

A $Y - \Delta$ transformation is induced by sliding a zig-zag path through the crossing of two other zig-zag paths as shown in Figure 5.2. Therefore discrete Abel and discrete Abel-Prym maps d, d_P on G_1 induce discrete Abel and discrete Abel-Prym maps on G_2, which we will also denote by d, d_P respectively.

Theorem 4.6.1. *Let $G_1 \to G_2$ be a $Y - \Delta$ transformation and let v_1 and v_2 be* *vertices of G_1 and G_2 respectively. The following diagram commutes:*

$$\begin{tikzcd}
 & \mathcal{R}_N \arrow[dl, "\kappa_{G_1,v_1}"'] \arrow[dr, "\kappa_{G_2,v_2}"] & \\
\mathcal{S}'_N \arrow[rr, "s"] & & \mathcal{S}'_N
\end{tikzcd}$$

The birational map s is defined as $(C, S_1, \nu_1) \mapsto (C, S_2, \nu_2)$, where

1. *There is a natural bijection between zig-zag paths on G_2 and G_1 induced by $Y - \Delta$ transformation. ν_2 is obtained by composing this bijection with ν_1.*

2. *S_2 is the generically unique degree g effective divisor satisfying $S_2 \equiv S_1 + d(v_1) - d(v_2)$.*

Proof. The $Y - \Delta$ transformation preserves the spectral curve. The local picture is shown in Figure 5.2. Let $e = \frac{1}{2}\pi_1(I(S_1) - I(q_1) - I(q_2) - \pi^*\Delta_C) + d_P(v_1)$. We show that $\kappa_{G_1,v_1}^{-1} = \kappa_{G_2,v_2}^{-1} \circ s$. We have

$$\begin{aligned}
a = \kappa_{G_1,v_1}^{-1}(C, S_1, \nu_1)_{uv_1} &= \frac{\eta(d_P(u) - e)\eta(d_P(v_1) - e)}{\eta(d_P(f_2) - e)\eta(d_P(f_3) - e)} \frac{E(\beta, \gamma)}{E(\beta, \gamma')};\\
b = \kappa_{G_1,v_1}^{-1}(C, S_1, \nu_1)_{uv_2} &= \frac{\eta(d_P(u) - e)\eta(d_P(v_2) - e)}{\eta(d_P(f_1) - e)\eta(d_P(f_3) - e)} \frac{E(\gamma, \alpha)}{E(\gamma, \alpha')};\\
c = \kappa_{G_1,v_1}^{-1}(C, S_1, \nu_1)_{uv_3} &= \frac{\eta(d_P(u) - e)\eta(d_P(v_3) - e)}{\eta(d_P(f_1) - e)\eta(d_P(f_2) - e)} \frac{E(\alpha, \beta)}{E(\alpha, \beta')}.
\end{aligned}$$

Note that by the definition of s,

$$\begin{aligned}
&\frac{1}{2}\pi_1(I(S_2) - I(q_1) - I(q_2) - \pi^*\Delta_C) + d_P(v_2)\\
&= \frac{1}{2}\pi_1(I(S_1 + d(v_1) - d(v_2)) - I(q_1) - I(q_2) - \pi^*\Delta_C) + d_P(v_2)\\
&= e.
\end{aligned}$$

Therefore

$$A = \kappa_{G_2,v_2}^{-1} \circ s(C, S_2, \nu_2)_{v_2 v_3} = \frac{\eta(d_P(v_2) - e)\eta(d_P(v_3) - e)}{\eta(d_P(f_0) - e)\eta(d_P(f_1) - e)} \frac{E(\gamma, \alpha')}{E(\gamma, \alpha)}.$$

Equation (4.12) becomes

$$a + b + c = \frac{\eta(d_P(u) - e)^2 \eta(d_P(f_0) - e)}{\eta(d_P(f_1) - e)\eta(d_P(f_2) - e)\eta(d_P(f_3) - e)} \frac{E(\alpha, \beta)E(\beta, \gamma)E(\gamma, \alpha)}{E(\alpha, \beta')E(\beta, \gamma')E(\gamma, \alpha')}.$$

Plugging in these expressions, we see that $\frac{bc}{a+b+c} = A$, which is the transition map between the G_1 and G_2 affine charts. $\square$

4.7 Discrete integrable systems from $Y - \Delta$ moves

Let T be a sequence of $Y - \Delta$ moves on a graph G such that the resulting graph $T \cdot G$ is isomorphic to G as graphs. Let $\phi_T : G \to T \cdot G$ be the isomorphism. The composition

$$\mathcal{R}_N \supset \mathcal{R}_G \to \mathcal{R}_{T \cdot G} \xrightarrow{\simeq} \mathcal{R}_G \subset \mathcal{R}_N$$

defines a birational automorphism of $\mathcal{R}_N$, which we denote by μ_T. It is a cluster modular transformation as defined in [FG03b]. Using Theorem 4.6.1, we construct the follwing commuting diagram:

$$\begin{tikzcd}
\mathcal{R}_N \supset \mathcal{R}_G \arrow[r] \arrow[d, "\kappa_{G,v}"] \arrow[rr, bend left, "\mu_T"] & \mathcal{R}_{T \cdot G} \arrow[r, "\simeq"] \arrow[d, "\kappa_{T \cdot G, \phi_T^{-1}(v)}"] & \mathcal{R}_G \subset \mathcal{R}_N \arrow[d, "\kappa_{G,v}"] \\
\mathcal{S}'_N \arrow[r, "s"] & \mathcal{S}'_N \arrow[r, "t"] & \mathcal{S}'_N
\end{tikzcd},$$

where s is the map in Theorem 4.6.1 and t is the natural map induced by the graph isomorphism ϕ_T, that is $(C,S,\nu) \mapsto (C,S,\nu')$, where ν' is obtained from ν by composing with ϕ_T. We have shown:

Theorem 4.7.1. *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{R}_N \arrow[r, "\mu_T"] \arrow[d, "\kappa_{G,v}"] & \mathcal{R}_N \arrow[d, "\kappa_{G,v}"] \\ \mathcal{S}'_N \arrow[r, "s_T"] & \mathcal{S}'_N \end{tikzcd},$$

where the birational map s_T is defined as $(C,S,\nu) \mapsto (C,S_T,\nu_T)$ where S_T is the (generically) unique degree g effective divisor satisfying $S_T \equiv S + d(v) - d(\phi_T^{-1}(v))$ and $\nu_T = \nu \circ \phi_T^{-1}$.

For a fixed C, the fiber of the projection $(C,S,\nu) \mapsto C$ over C is a cover of the space of degree g effective divisors on C satisfying (4.5), which is birational to a cover of $\mathrm{Prym}(\widehat{C},\sigma)$. Therefore, along with Lemma B.1.2, Theorem 4.7.1 tells us that the discrete integrable system arising from T is linearized on a finite cover of $\mathrm{Prym}(\widehat{C},\sigma)$.

4.8 Further questions

We end by listing some directions that we believe deserve further study.

1. Liouville integrability: Goncharov and Kenyon [GK12] proved that the dimer cluster variety is an algebraic integrable system, with its natural Poisson structure. We expect the same to be true for the resistor network cluster variety. Find a Poisson structure compatible with the Y-Δ transformation that makes

the resistor network cluster variety an algebraic integrable system and with respect to which the fibration by Prym varieties given by the spectral transform is Lagrangian. More generally, the Y-Δ move belongs to the framework of Lam and Pylyavskyy's Laurent phenomenon algebras [LP16], for which we can ask the same question.

2. Massive Laplacian: Boutillier, de Tilière and Raschel [BdeTR17] proved analogous results for the massive Laplacian in the isoradial case, that is in the case where the spectral curve has genus one. We expect that there is a common generalization of their results and this paper to the massive Laplacian where the spectral curve has arbitrary genus. We expect that the massive Y-Δ move might be described by a generalization of the Beauville-Debarre quadrisecant identity [BD87].

3. Relation to the dimer spectral transform: Let G be a minimal resistor network, Γ_G be the associated bipartite graph. Recall the dimer spectral data $\kappa_{\Gamma_G,v}$: $\mathcal{X}_N \to \mathcal{S}_N$ as defined in [GK12, Proposition 7.2]. By [GK12, Theorem 1.4] or [F15], $\kappa_{\Gamma_G,v}$ is a birational isomorphism. We conjecture that the map t that makes the diagram below commute is $(C, S, \nu) \mapsto (C, S + (1, 1), \nu)$.

$$\begin{tikzcd} \mathcal{R}_N \arrow[r, "\kappa_{G,v}"] \arrow[d, hook] & \mathcal{S}'_N \arrow[d, "t"] \\ \mathcal{X}_N \arrow[r, "\kappa_{\Gamma_G,v}"] & \mathcal{S}_N \end{tikzcd}$$

4. Connections to representation theory: Fock and Marshokov [FM16] showed that the dimer integrable sytems coincide with integrable systems on the Poisson-Lie groups $\widehat{\mathrm{PGL}}$. Is there an analogous construction for resistor networks?

CHAPTER 5

Arctic curves for groves from periodic cluster modular transformations

5.1 Introduction

A function $f : \mathbb{Z}^3 \to \mathbb{C}$ satisfies the *cube recurrence* (also known as the *Miwa equation* or the *discrete BKP equation* [Miwa82]) if for all $(i, j, k) \in \mathbb{Z}^3$

$$f_{i,j,k} f_{i-1,j-1,k-1} = f_{i-1,j,k} f_{i,j-1,k-1} + f_{i,j-1,k} f_{i-1,j,k-1} + f_{i,j,k-1} f_{i-1,j-1,k}.$$

We denote by $\mathcal{F}$ the set of functions satisfying the cube recurrence. Define the *lower cone* of $(i, j, k) \in \mathbb{Z}^3$ to be $C(i, j, k) := \{(i', j', k') \in \mathbb{Z}^3_{\leq 0} : i' \leq i, j' \leq j, k' \leq k\}$. Let $\mathcal{L}$ be a subset of $\mathbb{Z}^3_{\leq 0}$ such that $\mathbb{Z}^3_{\leq 0} \setminus \mathcal{L}$ is finite and if $(i, j, k) \in \mathcal{L}$ then we have $C(i, j, k) \subseteq \mathcal{L}$. Let $\mathcal{U} := \mathbb{Z}^3_{\leq 0} \setminus \mathcal{L}$. A *set of initial conditions* is defined to be

$\mathcal{I} := \{(i,j,k) \in \mathcal{L} : (i+1, j+1, k+1) \notin \mathcal{L}\}$. Let $\mathfrak{I}$ denote the set of all sets of initial conditions. The set of initial conditions corresponding to $\mathcal{L} = \{(i,j,k) \in \mathbb{Z}_{\leq 0} : i+j+k \leq 1-n\}$ will be denoted by $I(n)$ and is called the standard initial conditions of order n.

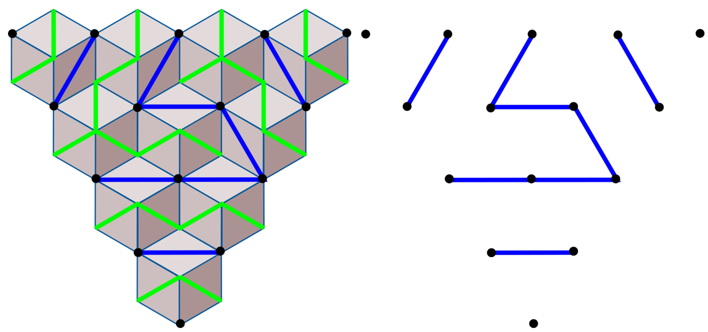
If we assign formal variables $f_{i,j,k} := x_{i,j,k}$ for (i,j,k) in a set of initial conditions and solve for $f_{i,j,k}$ where $(i,j,k) \in \mathcal{U}$, we obtain rational functions in $x_{i,j,k}$. In [FZ01], Fomin and Zelevinsky showed using cluster algebra techniques that these rational functions are Laurent polynomials in $x_{i,j,k}$ with coefficients in $\mathbb{Z}$.

In [CS04], Carroll and Speyer studied a more general version of the cube recurrence, which they call the edge-variable version. Define *edge variables* $a_{i,j}, b_{i,k}, c_{i,j}$ for each $i,j,k \in \mathbb{Z}_{\leq 0}$. A function $g : \mathbb{Z}^3_{\leq 0} \to \mathbb{R}_{>0}$ satisfies the *edge-variable version* of the cube recurrence if

$$g_{i,j,k}g_{i-1,j-1,k-1} = b_{i,k}c_{i,j}g_{i-1,j,k}g_{i,j-1,k-1} + a_{i,k}c_{i,j}g_{i,j-1,k}g_{i-1,j,k-1} + a_{j,k}b_{i,k}g_{i,j,k-1}g_{i-1,j-1,k},$$

for $(i,j,k) \in \mathbb{Z}^3_{\leq 0}$. Carroll and Speyer constructed combinatorial objects called *groves* (See Figure 5.1 (a) for an example), which they showed are in bijection with the monomials in the Laurent polynomial generated by the edge-variable version of the cube recurrence. This was used to give a combinatorial proof of the Laurent property.

Groves on the standard initial conditions $I(n)$ are in bijection with spanning forests of a portion of the triangular lattice where each component of the forest connects boundary vertices in a prescribed manner (see Figure 5.1 (b)). Petersen and Speyer [PS06] proved an arctic circle theorem for groves: For large n, a uniformly



(a) A grove on $I(3)$. (b) The corresponding simplified grove.

Figure 5.1

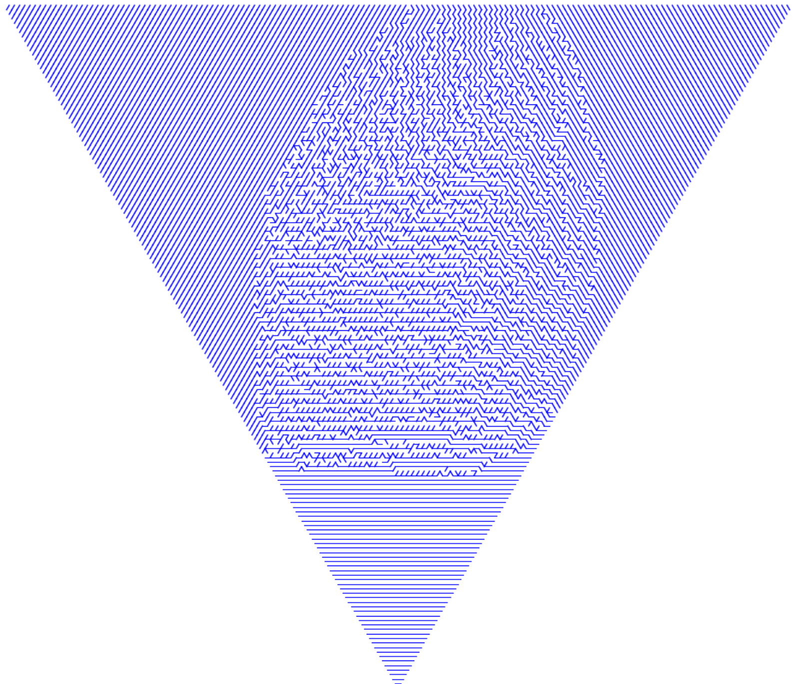
random simplified grove on $I(n)$, rescaled by a factor of n so that it is now supported on the unit triangle, appears deterministic outside the circle inscribed in the triangle.

In the present paper, we extend the arctic circle theorem to a large class of probability measures on groves. There are two natural probability measures one can consider on groves:

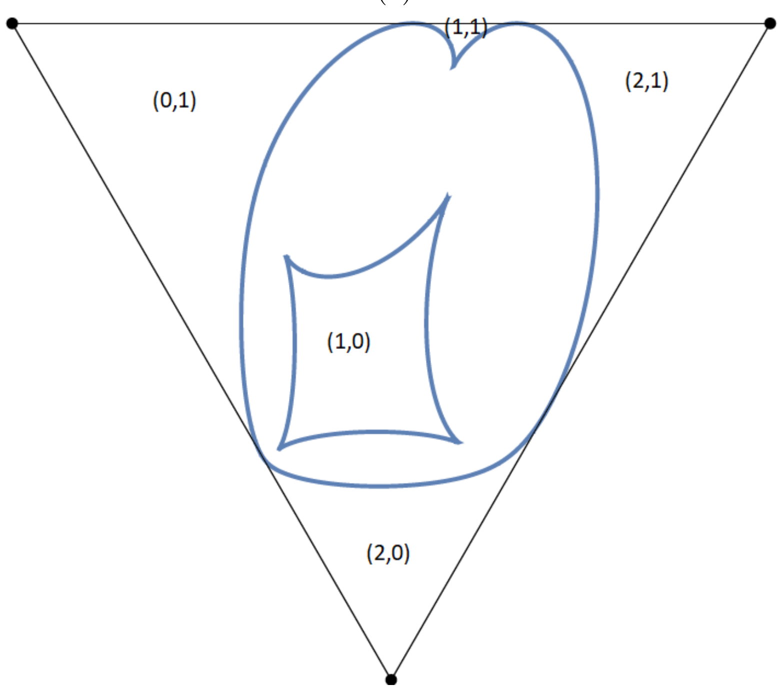
- Given a positive real-valued function f satisfying the cube recurrence, we can put a probability measure on groves where each grove gets a probability proportional to the value of the monomial associated to it in the bijection of Carroll and Speyer. We denote this probability measure by $\mathbb{P}^f_{\mathcal{I}}$.

- We can define a conductance function C, a positive real-valued function on the edges of the triangular lattice, and consider the Boltzmann distribution, assigning to a grove G the probability,

$$\mathbb{P}^C_{\mathcal{I}}(G) \propto \prod_{\text{Edges } e \in G} C(e).$$



(a)



(b) The arctic curve, along with macroscopic regions labeled according to the points of the Newton polygon that correspond to the EGM describing local statistics in the region.

Figure 5.2

This is the natural measure to put on spanning forests from the point of view of statistical mechanics and generalizes the spanning tree measure.

There is a way to associate a conductance function C^f to a function $f : \mathbb{Z}^3 \to \mathbb{R}$ due to Fomin and Zelevinsky ([FZ01], see also [GK12]), such that the cube recurrence for f becomes the resistor network Y-Δ transformation (due to Kennelly [Kenn1899]) for C^f. We show that under this change of variables, the probability measures $\mathbb{P}^f_{\mathcal{I}}$ and $\mathbb{P}^C_{\mathcal{I}}$ coincide (see Theorem 5.2.4) and that the map $f \mapsto C^f$ is surjective. This lets us define our class of probability measures on groves in terms of conductance functions, but still allows us to exploit the algebraic structure of the f variables and the cube recurrence to compute the edge probability generating functions as in [PS06].

The class of probability measures we consider comes from periodic conductance functions on the triangular lattice. This however leads to an infinite system of linear equations for the edge-probability generating functions. We further impose the condition that the conductance function is periodic under a cluster modular transformation (defined in Section 5.3.5) to obtain a finite linear system (Theorem 5.4.1).

Starting from any conductance function that is periodic in both these senses, we derive asymptotic edge probabilities using the machinery developed by Baryshnikov, Pemantle and Wilson [PW02, PW04, BP11, PW13]. We obtain generating functions that have isolated singularities with degree greater than 2 and therefore fall outside the class of quadratic singularities considered in [BP11], but for specific examples, we see that their techniques still work. This in particular leads to explicit computations of arctic curves (see for example Figure 5.2.2).

By analogy with the dimer model [KOS06, KO07], a generic conductance function on a $\mathbb{Z}^2$-periodic resistor network is expected to give rise to a limit shape where there are macroscopic regions corresponding to each lattice point in the Newton polygon of the resistor network (see sections 5.3.2 and 5.3.6). Figure 5.2 suggests that although the class of conductances functions we consider lies in a closed subvariety of $\mathbb{Z}^2$-periodic conductances, it is still sufficiently general to exhibit all the possible macroscopic phases.

5.2 Groves and the cube recurrence

5.2.1 Groves

We recall some some terminology and basic properties of groves from [CS04]. A *rhombus* is any set in one of the following three forms for $(i,j,k) \in \mathbb{Z}^3_{\leq 0}$:

$$r_a(i,j,k) := \{(i,j,k),(i,j-1,k),(i,j,k-1),(i,j-1,k-1)\}$$

$$r_b(i,j,k) := \{(i-1,j,k),(i,j,k),(i,j,k-1),(i-1,j,k-1)\}$$

$$r_c(i,j,k) := \{(i,j,k),(i-1,j,k),(i,j-1,k),(i-1,j-1,k)\}.$$

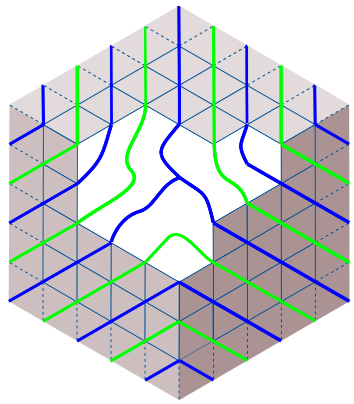


Figure 5.3: The connectivity of a grove.

We call the edges $E_a(i,j,k) := \{(i,j-1,k),(i,j,k-1)\}$ and $e_a(i,j,k) := \{(i,j,k),(i,j-1,k-1)\}$ the *long diagonal* and the *short diagonal* of the rhombus $r_a(i,j,k)$, and define analogously the edges E_b, e_b, E_c and e_c, where the pattern of -1 shifts is easily evinced from the equations above. We denote the set of all diagonals of rhombi by $\mathcal{D}$.

Let $\Gamma_{\mathcal{I}}$ be the graph with vertex-set $\mathcal{I}$ and edge-set constituted by the long and short diagonals appearing in each rhombus in $\mathcal{I}$. Then an $\mathcal{I}$-*grove* is a subgraph G of $\Gamma_{\mathcal{I}}$ with the following properties:

- The vertex-set of G is all of $\mathcal{I}$.
- For each rhombus in $\mathcal{I}$, exactly one of the two diagonals occurs in G.
- There exists an integer N such that, if all the vertices of a rhombus satisfies $i+j+k < -N$, the short diagonal occurs.
- For N large enough, every component of G contains exactly one of the following sets of vertices, and each such set is contained in a component of G (Figure

5.3),

- $\{(0,p,q),(p,0,q)\}$, $\{(p,q,0),(0,q,p)\}$, and $\{(q,0,p),(q,p,0)\}$ for all p,q with $0 > p > q$ and $p+q \in \{-N-1,-N-2\}$;
- $\{(0,p,p),(p,0,p),(p,p,0)\}$ for $2p \in \{-N-1,-N-2\}$;
- $\{(0,0,q)\}$, $\{(0,q,0)\}$, and $\{(q,0,0)\}$ for $q \leq -N-1$.

It is shown in [CS04] that groves on standard initial conditions $I(n)$ are completely determined by their long-diagonal edges. Therefore, we can represent groves as a spanning forest of a finite portion of the triangular lattice (see Figure 5.1), which is called a *simplified grove*.

Suppose $\mathcal{I} \in \mathfrak{I}$ is a set of initial conditions. The edge-variable version of the cube recurrence gives $g_{0,0,0}$ as a rational function in the variables $\{a_{j,k}, b_{i,k}, c_{i,j}, g_{i,j,k}\}_{(i,j,k)\in\mathcal{I}}$. The following is the main result of [CS04].

Theorem 5.2.1 (Carroll and Speyer, 2004 [CS04])**.**

$$g_{0,0,0} = \sum_{G\in\mathcal{G}(\mathcal{I})} M(G),$$

where

$$M(G) = \left(\prod_{e_a(i,j,k)\in E(G)} a_{j,k}\right)\left(\prod_{e_b(i,j,k)\in E(G)} b_{i,k}\right)\left(\prod_{e_c(i,j,k)\in E(G)} c_{i,j}\right) m_g(G)$$

and

$$m_g(G) = \prod_{(i,j,k)\in\mathcal{I}} g_{i,j,k}^{deg(i,j,k)-2},$$

where $deg(i,j,k)$ is the degree of the vertex (i,j,k) in the (unsimplified) grove G.

Now suppose $f : \mathbb{Z}_{\leq 0} \to \mathbb{R}_{>0}$ satisfies the cube recurrence. Since $f_{i,j,k}$ are positive real numbers, by Theorem 5.2.1,

$$\mathbb{P}^f_{\mathcal{I}}(G) := \frac{m_f(G)}{f_{0,0,0}}$$

defines a probability measure on $\mathcal{G}(\mathcal{I})$. Therefore any function f that satisfies the cube recurrence induces a family of probability measures $\{\mathbb{P}^f_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$.

5.2.2 Conductance variables and the Y-Δ transformation

A function $C : \mathcal{D} \to \mathbb{C}$ satisfying $C(E_q(i,j,k)) = 1/C(e_q(i,j,k))$ for all $q \in \{a,b,c\}, (i,j,k) \in \mathbb{Z}_{\leq 0}$ is called a *conductance function*. We simplify notation by writing $C_q(i,j,k) = C(E_q(i,j,k))$ and $c_q(i,j,k) = C(e_q(i,j,k))$. A positive real-valued conductance function C determines a family of Boltzmann probability measures on groves $\{\mathbb{P}^C_{\mathcal{I}}\}_{\mathcal{I}\in\mathfrak{I}}$:

$$\mathbb{P}^C_{\mathcal{I}}(G) := \frac{w(G)}{Z},$$

for $G \in \mathcal{G}(\mathcal{I})$, where $w(G) = \prod_{E_q(i,j,k)\in G} C_q(i,j,k)$ is the product of the conductances of the long edges appearing in G and $Z_{\mathcal{I}}$ is the partition function,

$$Z_{\mathcal{I}} = \sum_{G\in\mathcal{G}(\mathcal{I})} \prod_{E_q(i,j,k)\in G} C_q(i,j,k).$$

Let us denote

$$\Delta(i,j,k) := \frac{1}{C_b(i,j,k)C_c(i,j,k) + C_a(i,j,k)C_c(i,j,k) + C_a(i,j,k)C_b(i,j,k)}.$$

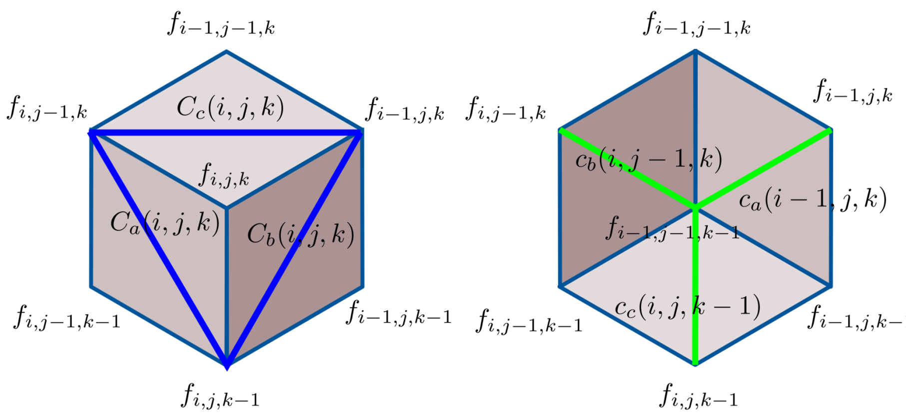


Figure 5.4: The Y-Δ transformation.

A conductance function C is *Y-Δ consistent* if for all q and (i,j,k), we have

$$C_a(i,j,k)c_a(i-1,j,k) = \frac{1}{\Delta(i,j,k)};$$
$$C_b(i,j,k)c_b(i,j-1,k) = \frac{1}{\Delta(i,j,k)};$$
$$C_c(i,j,k)c_c(i,j,k-1) = \frac{1}{\Delta(i,j,k)}. \tag{5.1}$$

We will denote by $\mathcal{C}$ the set of Y-Δ consistent conductance functions.

Let $f \in \mathcal{F}$ and let us define a conductance function C^f (see Figure 5.4),

$$C_a^f(i,j,k) = \frac{f_{i,j-1,k}f_{i,j,k-1}}{f_{i,j,k}f_{i,j-1,k-1}},$$
$$C_b^f(i,j,k) = \frac{f_{i-1,j,k}f_{i,j,k-1}}{f_{i,j,k}f_{i-1,j,k-1}},$$
$$C_c^f(i,j,k) = \frac{f_{i-1,j,k}f_{i,j-1,k}}{f_{i,j,k}f_{i-1,j-1,k}}. \tag{5.2}$$

It was observed in [FZ01, GK12] that the Y-Δ equations (5.1) for C^f reduce to the cube recurrence for f, and therefore C^f is Y-Δ consistent. Therefore we obtain a function

$$\begin{aligned} p : \mathcal{F} &\to \mathcal{C} \\ f &\mapsto C^f. \end{aligned}$$

Lemma 5.2.2. *$p : \mathcal{F} \to \mathcal{C}$ is surjective.*

Proof. Given $C \in \mathcal{C}$, we can construct a function f such that $p(f) = C$ as follows: We define for all $i, j, k \in \mathbb{Z}_{\leq 0}$,

$$f(i, 0, 0) = f(0, j, 0) = f(0, 0, k) = 1.$$

The equations (5.2) now uniquely define f on $(i, j, 0), (0, j, k), (i, j, 0)$ for $i, j, k \in \mathbb{Z}_{\leq 0}$. We define f everywhere else using the cube recurrence. □

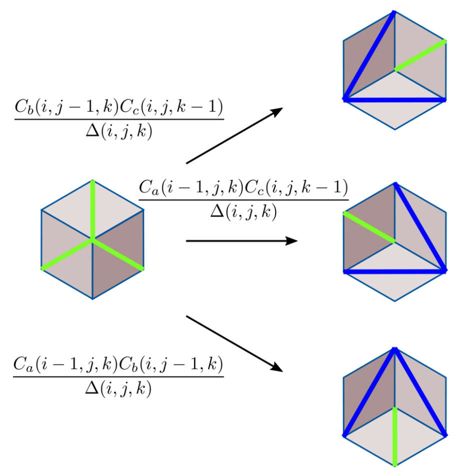
5.2.3 Grove Shuffling

Let C be a Y-Δ consistent conductance function. Note that

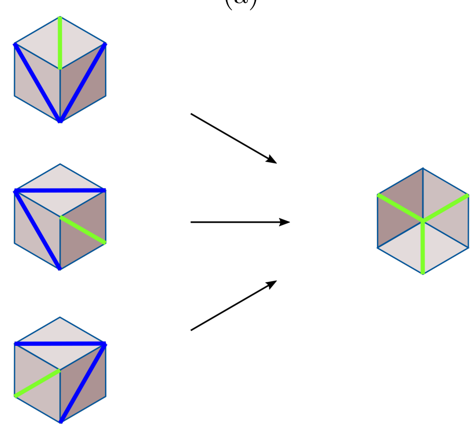
$$\begin{aligned} \Delta(i, j, k) = (&C_b(i, j-1, k) C_c(i, j, k-1) + C_a(i-1, j, k) C_c(i, j, k-1) \\ &+ C_a(i-1, j, k) C_b(i, j-1, k)), \end{aligned}$$

as a consequence of (5.1).

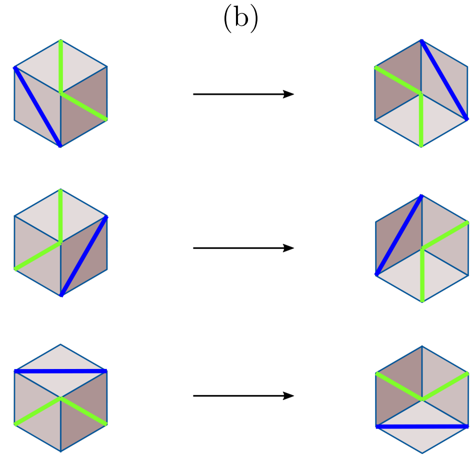
Grove shuffling is a local move that generates groves with measure $\mathbb{P}^C_{\mathcal{I}}(G)$ and couples the probability measures for different initial conditions in a convenient way (see Figure 5.5). Grove shuffling takes a cube, removes it and replaces a configuration



(a)



(b)



(c)

Figure 5.5: Grove shuffling

on the left in Figure 5.5 with a corresponding configuration on the right. The only random part is (a) where the configuration on the left is replaced with one of the configurations on the right with probabilities indicated on the arrows.

We can generate a random grove on initial conditions $\mathcal{I}$ as follows. Start with the unique grove on $\{(i,j,k) \in \mathbb{Z}^3_{\leq 0} : \max\{i,j,k\} = 0\}$. Use grove shuffling to remove cubes until you end up with initial conditions $\mathcal{I}$. The following lemma shows that this can always be done.

Lemma 5.2.3 (Carroll and Speyer, 2004 [CS04])**.** *Suppose* $(0,0,0) \in \mathcal{I}$*. Then there exist* $i,j,k \leq 0$ *such that* $(i-1,j,k),(i,j-1,k-1),(i,j-1,k),(i-1,j,k-1),(i,j,k-1),(i-1,j-1,k) \in \mathcal{I}$ *(and so* $(i,j,k) \in \mathcal{U}$*)*

We define a generalization of the edge-variable version of the cube recurrence:

$$\begin{aligned} g_{i,j,k}g_{i-1,j-1,k-1} = \frac{1}{\Delta(i,j,k)}&(C_b(i,j-1,k)C_c(i,j,k-1)g_{i-1,j,k}g_{i,j-1,k-1} \\ &+ C_a(i-1,j,k)C_c(i,j,k-1)g_{i,j-1,k}g_{i-1,j,k-1} \\ &+ C_a(i-1,j,k)C_b(i,j-1,k)g_{i,j,k-1}g_{i-1,j-1,k}), \end{aligned}$$

for $(i,j,k) \in \mathbb{Z}^3_{\leq 0}$. The key input in the computation of the limit shape is the following theorem that generalizes Theorem 5.2.1.

Theorem 5.2.4. *Suppose* C *is a Y-*Δ *consistent conductance function and let* f *be the solution to the cube recurrence such that* $p(f) = C$ *from lemma 5.2.2. The following are true:*

- *The generalized cube recurrence satisfies for all $\mathcal{I} \in \mathfrak{I}$,*

$$g_{0,0,0} = \sum_{G \in \mathcal{G}(\mathcal{I})} \mathbb{P}^C_{\mathcal{I}}(G) m_g(G).$$

- *Grove shuffling generates groves with probability measure $\mathbb{P}^C_{\mathcal{I}}$, regardless of the order in which the cubes are shuffled.*

- *The probability measures $\mathbb{P}^C_{\mathcal{I}}$ and $\mathbb{P}^f_{\mathcal{I}}$ are the same.*

-

$$Z_{\mathcal{I}} = \prod \Delta(i,j,k),$$

 where the product is over all (i,j,k) such that the cube at (i,j,k) is removed to reach $\mathcal{I}$.

Proof. The proof is by induction on $|\mathcal{U}|$. If $\mathcal{U} = \emptyset$ then it is clear. Suppose $\mathcal{U}$ is not empty. Choose (i,j,k) as in lemma 5.2.3. We obtain the initial conditions $\mathcal{I}$ by shuffling the cube with vertex (i,j,k) in $\mathcal{I}'$. We first show that

$$Z_{\mathcal{I}} = Z_{\mathcal{I}'} \Delta(i,j,k).$$

Consider any $\mathcal{I}$ grove G. Since $(i-1,j-1,k-1)$ belongs to three rhombi, it has degree 3, 2 or 1.

Suppose $(i-1,j-1,k-1)$ has degree 1. Then G belongs to a triple of $\mathcal{I}$ groves, say $\{G_1, G_2, G_3\}$ in the order shown in Figure 5.5 (a), all of which are obtained from a single $\mathcal{I}'$ grove G' by shuffling. We have

$$w(G_1) = C_b(i, j-1, k)C_c(i, j, k-1)w(G'),$$
$$w(G_2) = C_a(i-1, j, k)C_c(i, j, k-1)w(G'),$$
$$w(G_3) = C_a(i-1, j, k)C_b(i, j-1, k)w(G').$$

Therefore

$$w(G_1) + w(G_2) + w(G_3) = w(G')\Delta(i, j, k).$$

Suppose $(i-1, j-1, k-1)$ has degree 3. Then there are three $\mathcal{I}'$ groves, say G_1,G_2 and G_3 (in the order in Figure 5.5 (b)) that upon shuffling the cube at (i, j, k) yields G. We have

$$\begin{aligned} w(G) &= \frac{w(G_1)}{C_b(i, j, k)C_c(i, j, k)}, \\ &= \frac{w(G_2)}{C_a(i, j, k)C_c(i, j, k)}, \\ &= \frac{w(G_3)}{C_a(i, j, k)C_b(i, j, k)}. \end{aligned}$$

Therefore

$$\begin{aligned} w(G) &= \frac{w(G_1) + w(G_2) + w(G_3)}{C_b(i, j, k)C_c(i, j, k) + C_a(i, j, k)C_c(i, j, k) + C_a(i, j, k)C_b(i, j, k)} \\ &= (w(G_1) + w(G_2) + w(G_3))\Delta(i, j, k). \end{aligned}$$

Suppose $(i-1, j-1, k-1)$ has degree 2. Up to simple symmetry considerations, it is sufficient to concentrate only on the first situation in figure 5.5(C). We have

$$\begin{aligned}
w(G) &= w(G')\frac{C_a(i,j,k-1)}{C_a(i,j,k)} \\
&= \frac{w(G')}{C_b(i,j,k)C_c(i,j,k)+C_a(i,j,k)C_c(i,j,k)+C_a(i,j,k)C_b(i,j,k)} \\
&= w(G')\Delta(i,j,k).
\end{aligned}$$

Since each of them is multiplied by the same factor, we have shown that

$$Z_{\mathcal{I}} = Z_{\mathcal{I}'}\Delta(i,j,k).$$

Now we will check that $\mathbb{P}^C = \mathbb{P}^f$. Suppose $(i-1,j-1,k-1)$ has degree 1 and let $\{G_1, G_2, G_3\}$ be the triple of groves obtained from a single $\mathcal{I}'$ grove G' shown in Figure 5.5 (a).

$$\begin{aligned}
\mathbb{P}^f_{\mathcal{I}}(G) &= \frac{m_f(G_1)}{f_{0,0,0}} \\
&= \frac{m_f(G')}{f_{0,0,0}}\frac{f_{i-1,j,k}f_{i,j-1,k-1}}{f_{i-1,j-1,k-1}f_{i,j,k}} \\
&= \mathbb{P}^f_{\mathcal{I}'}(G')\frac{f_{i-1,j,k}f_{i,j-1,k-1}}{f_{i-1,j-1,k-1}f_{i,j,k}} \\
&= \mathbb{P}^C_{\mathcal{I}'}C(G')\frac{f_{i-1,j,k}f_{i,j-1,k-1}}{f_{i-1,j-1,k-1}f_{i,j,k}} \\
&= \frac{w(G')}{Z_{\mathcal{I}'}}\frac{C_b(i,j-1,k)C_c(i,j,k-1)}{\Delta(i,j,k)} \\
&= \frac{w(G)}{Z_{\mathcal{I}}} \\
&= \mathbb{P}^C_{\mathcal{I}}(G),
\end{aligned}$$

where we have used

$$\frac{C_b(i,j-1,k)C_c(i,j,k-1)}{\Delta(i,j,k)} = \frac{f_{i-1,j,k}f_{i,j-1,k-1}}{f_{i-1,j-1,k-1}f_{i,j,k}},$$

which may be checked by direct substitution.

Since each shuffle is independent, the probability of obtaining G_1 is

$$\mathbb{P}^C_{\mathcal{I}'}(G').\frac{C_b(i,j-1,k)C_c(i,j,k-1)}{\Delta(i,j,k)} = \mathbb{P}^C_{\mathcal{I}}(G).$$

The last thing to check is that $g_{0,0,0}$ has the stated form. Let $g'_{0,0,0}$ be the expression obtained by solving the generalized cube recurrence on $\mathcal{I}'$. By induction hypothesis,

$$g'_{0,0,0} = \sum_{G' \in \mathcal{G}(\mathcal{I}')} \mathbb{P}^C_{\mathcal{I}'}(G')m_g(G'),$$

and $g_{0,0,0}$ is obtained from $g'_{0,0,0}$ be substituting the generalized cube recurrence for $g_{i,j,k}$. We know that

$$\begin{aligned} m_g(G_1) &= m_g(G')\frac{g_{i-1,j,k}g_{i,j-1,k-1}}{g_{i-1,j-1,k-1}g_{i,j,k}} \\ \mathbb{P}^C_{\mathcal{I}}(G_1) &= \mathbb{P}^C_{\mathcal{I}'}(G')\frac{C_b(i,j-1,k)C_c(i,j,k-1)}{\Delta(i,j,k)}, \end{aligned}$$

and similarly for G_2 and G_3. Therefore we see that $\mathbb{P}^C_{\mathcal{I}}(G_1)m_g(G_1)+\mathbb{P}^C_{\mathcal{I}}(G_2)m_g(G_2)+\mathbb{P}^C_{\mathcal{I}}(G_3)m_g(G_3)$ is obtained from $\mathbb{P}^C_{\mathcal{I}'}(G')m_g(G')$ by substituting the generalized cube recurrence for $g_{i,j,k}$.

The argument when the degree of $(i-1,j-1,k-1)$ is 2 and 3 is similar. □

Let us denote by $\mathbb{E}_{\mathcal{I}}$ the expectation with respect to the measure $\mathbb{P}_{\mathcal{I}}$. Recall that the exponent of the variable g_{i_0,j_0,k_0} in $g_{0,0,0}$ is $\deg(i_0,j_0,k_0)-2$ (Theorem 5.2.1). We immediately obtain:

Corollary 5.2.5. *Let* $(i_0, j_0, k_0) \in \mathcal{I}$.

$$\mathbb{E}_{\mathcal{I}}\left[deg(i_0, j_0, k_0) - 2\right] = \left.\frac{\partial g_{0,0,0}}{\partial g_{i_0,j_0,k_0}}\right|_{g|_{\mathcal{I}}=1}.$$

5.2.4 Creation rates

Let $(i, j, k) \in \mathbb{Z}_{\leq 0}$. Let $\mathcal{I}$ be a set of initial conditions such that $r_a(i, j, k) \subset \mathcal{I}$ and let G have distribution $\mathbb{P}_{\mathcal{I}}$. Define the long-edge probabilities

$$p(i, j, k) = \mathbb{P}_{\mathcal{I}}(E_a(i, j, k) \in G).$$

These are well defined since if $\mathcal{I}'$ is another set of initial conditions, then we can use grove shuffling to move between $\mathcal{I}$ and $\mathcal{I}'$ leaving the rhombus $r_a(i, j, k)$ intact. Similarly define

$$q(i, j, k) = \mathbb{P}_{\mathcal{I}}(E_b(i, j, k) \in G);$$

$$r(i, j, k) = \mathbb{P}_{\mathcal{I}}(E_c(i, j, k) \in G),$$

and the *creation rates*

$$E(i, j, k) = 1 - p(i, j, k) - q(i, j, k) - r(i, j, k).$$

It was shown in [PS06] that

$$E(i_0, j_0, k_0) = \mathbb{E}_{\mathcal{I}}\left[\deg(i_0, j_0, k_0) - 2\right],$$

and therefore by Corollary 5.2.5,

$$E(i_0, j_0, k_0) = \left. \frac{\partial g_{0,0,0}}{\partial g_{i_0,j_0,k_0}} \right|_{g|_{\mathcal{I}}=1}. \tag{5.3}$$

The following result lets us obtain the generating function for $p(i, j, k)$ from that of $E(i, j, k)$. Let us introduce for convenience the notation:

$$U(i,j,k) = \frac{C_b(i,j-1,k)C_c(i,j,k-1)}{\Delta(i,j,k)},$$
$$V(i,j,k) = \frac{C_b(i,j-1,k)C_c(i,j,k-1)}{\Delta(i,j,k)},$$
$$W(i,j,k) = \frac{C_b(i,j-1,k)C_c(i,j,k-1)}{\Delta(i,j,k)}.$$

Lemma 5.2.6 (Petersen and Speyer, 2005 [PS06], Theorem 2)**.** *The edge probabilities are given recursively by*

$$p(i,j,k) = p(i+1,j,k) + (V(i+1,j,k) + W(i+1,j,k))E(i+1,j,k);$$

$$q(i,j,k) = q(i,j+1,k) + (U(i,j+1,k) + W(i,j+1,k))E(i,j+1,k);$$

$$r(i,j,k) = r(i,j,k+1) + (U(i,j,k+1) + V(i,j,k+1))E(i,j,k+1).$$

5.2.5 Creation-rate generating functions

Given a Y-Δ consistent conductance function C, for all $\mu = (i_0, j_0, k_0) \in \mathbb{Z}^3_{\leq 0}$, we define a conductance function C^μ by:

$$C^\mu_q(i,j,k) = C_q(i+i_0, j+j_0, k+k_0).$$

Let g^μ denote the corresponding solution to the generalized cube recurrence and $E^\mu, p^\mu, q^\mu, r^\mu$ the corresponding creation rates and edge probabilities. Let $F^\mu(x,y,z) = \sum_{i,j,k\geq 0} E^\mu(-i,-j,-k)x^i y^j z^k$ be the generating functions for the creation rates.

Lemma 5.2.7. *Let (i_1,j_1,k_1) and (i_2,j_2,k_2) be such that (i_1,j_1,k_1) is in the lower cone $C(i_2,j_2,k_2)$. Then*

$$\left.\frac{\partial g^\mu_{i_2,j_2,k_2}}{\partial g^\mu_{i_1,j_1,k_1}}\right|_{g^\mu|_{\mathcal{I}}=1} = E^{\mu+(i_2,j_2,k_2)}(i_1-i_2, j_1-j_2, k_1-k_2).$$

Proof. Translate so that (i_2,j_2,k_2) goes to $(0,0,0)$. $\square$

Theorem 5.2.8. *$F^\mu(x,y,z)$ satisfy the following infinite system of linear equations over $\mathbb{C}(x,y,z)$:*

$$\begin{aligned}
&F^\mu + xyzF^{\mu+(-1,-1,-1)} - U^\mu(0,0,0)(xF^{\mu+(-1,0,0)} + yzF^{\mu+(0,-1,-1)})\\
&- V^\mu(0,0,0)(yF^{\mu+(0,-1,0)} + xzF^{\mu+(-1,0,-1)})\\
&- W^\mu(0,0,0)(zF^{\mu+(0,0,-1)} + xyF^{\mu+(-1,-1,0)})) = 1,
\end{aligned}$$

for all $\mu \in \mathbb{Z}^3_{\leq 0}$.

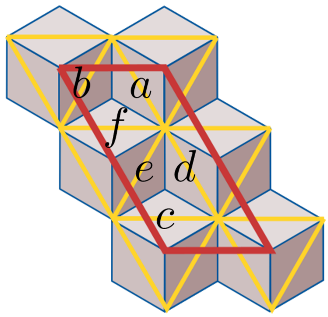
Proof. Let $(i,j,k) \in \mathbb{Z}^3_{\leq 0}$ and $(i_0,j_0,k_0) \in C(i,j,k)$. Differentiating the generalized cube recurrence with respect to $g^\mu(i_0,j_0,k_0)$, setting $g^\mu|_{\mathcal{I}} = 1$ and using lemma 5.2.7, we obtain

$$
\begin{aligned}
&E^{\mu+(i,j,k)}(i_0-i, j_0-j, k_0-k) + E^{\mu+(i-1,j-1,k-1)}(i_0-i+1, j_0-j+1, k_0-k+1) \\
&= U^\mu(i,j,k)(E^{\mu+(i-1,j,k)}(i_0-i+1, j_0-j, k_0-k) \\
&\quad + E^{\mu+(i,j-1,k-1)}(i_0-i, j_0-j+1, k_0-k+1)) \\
&+ V^\mu(i,j,k)(E^{\mu+(i,j-1,k)}(i_0-i, j_0-j+1, k_0-k) \\
&\quad + E^{\mu+(i-1,j,k-1)}(i_0-i+1, j_0-j, k_0-k+1)) \\
&+ W^\mu(i,j,k)(E^{\mu+(i,j,k-1)}(i_0-i, j_0-j, k_0-k+1) \\
&\quad + E^{\mu+(i-1,j-1,k)}(i_0-i+1, j_0-j+1, k_0-k))).
\end{aligned}
$$

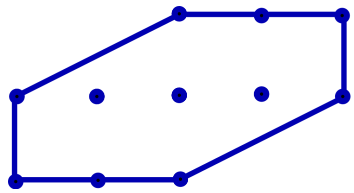
Letting $i_0 - i = r, j_0 - j = s, k_0 - k = t$ and relabeling $\mu + (i,j,k)$ as μ, we have for all $r, s, t < 0, \mu \in \mathbb{Z}^3_{\leq 0}$:

$$
\begin{aligned}
&E^\mu(r,s,t) + E^{\mu+(-1,-1,-1)}(r+1, s+1, t+1) \\
&= U^\mu(0,0,0)(E^{\mu+(-1,0,0)}(r+1,s,t) + E^{\mu+(0,-1,-1)}(r, s+1, t+1)) \\
&+ V^\mu(0,0,0)(E^{\mu+(0,-1,0)}(r,s+1,t) + E^{\mu+(-1,0,-1)}(r+1, s, t+1)) \\
&+ W^\mu(0,0,0)(E^{\mu+(0,0,-1)}(r,s,t+1) + E^{\mu+(-1,-1,0)}(r+1, s+1, t)).
\end{aligned} \tag{5.4}
$$

Near the boundary, for $(r,s,t) \in \partial\mathbb{Z}^3_{\leq 0}$, equation (5.4) holds if we set $E^{\mu'}(r', s', t') = 0$ for $\mu' \in \mathbb{Z}^3_{\leq 0}$ and $(r', s', t') \notin \mathbb{Z}^3_{\leq 0}$. Upon multiplying by $x^r y^s z^t$, summing up over all $(r,s,t) \in \mathbb{Z}^3_{\leq 0}$, we obtain the linear equations for F^μ. $\square$



(a) The fundamental domain with a conductance function.



(b) Newton polygon.

Figure 5.6: The graph $T_{1,2}$.

5.3 The resistor network model on a torus

5.3.1 Quotients of the triangular lattice

Consider the triangular lattice T embedded in the plane $x+y+z=-1$ in $\mathbb{R}^3$ with vertices at $\{(i,j,k)\in\mathbb{Z}^3 : i+j+k=0\}$. We have a $\mathbb{Z}^2$-action defined by translations

$$\tau_{(1,0)}\cdot(i,j,k)=(i-1,j+1,k),$$
$$\tau_{(0,1)}\cdot(i,j,k)=(i,j+1,k-1).$$

Let $T_{m,n} := T/(m\mathbb{Z}\times n\mathbb{Z})$ be the quotient. It is a finite graph on a torus $\mathbb{T}$ with mn vertices and forms an $m\times n$-cover of $T_{1,1}$. The parallelogram with vertices at $(0,0,0),(-m,m,0),(0,n,-n),(-m,m+n,-n)$ gives a fundamental domain for the torus. Figure 5.6 (a) shows the fundamental domain for $T_{1,2}$.

5.3.2 The vector bundle Laplacian

The notion of the vector bundle Laplacian was introduced and studied in [K10]. We report here the facts that we need in this paper. Let Γ be a finite graph on a torus embedded such that every face is a topological disk. Let c be a *conductance function* on Γ, i.e. a positive real-valued function on the edges of Γ defined modulo global scaling. A pair (Γ, c) is called a *resistor network*. A *line bundle with connection* (V, i) on Γ is the data of a complex line V_v at each vertex v of Γ along with an isomorphism, called *parallel transport* $i_{vv'} : V_v \to V_{v'}$ for each edge $\langle v, v'\rangle$ such that $i_{v'v} = i_{vv'}^{-1}$. Two line bundles with connection (V, i) and (V', i') are isomorphic if there exists a collection of isomorphisms $\psi_v : V_v \to V_v'$ such that for all edges vv', the following diagram commutes.

$$\begin{array}{ccc} V_v & \xrightarrow{i_{v,v'}} & V_{v'} \\ \downarrow{\scriptstyle \psi_v} & & \downarrow{\scriptstyle \psi_{v'}} \\ V_v' & \xrightarrow{i'_{v,v'}} & V_{v'}' \end{array}$$

A connection is *flat* if the monodromies around the faces of Γ, that is, the products of the $i_{vv'}$'s in cyclic order around the face, are trivial. The Laplacian is a linear operator $\Delta : \bigoplus_v V_v \to \bigoplus_v V_v$ defined by

$$\Delta(f)(v) := \sum_{v' \sim v} c(v, v')(f(v) - i_{v'v} f(v')).$$

If the monodromies of all (contractible) faces are trivial, the monodromies z, w in the two homology directions of the torus are univocally defined. Suppose we have a flat connection. Then $P(z, w) := \det\Delta(z, w)$ is a Laurent polynomial and is called the *characteristic polynomial*. The compactification of the curve $\{(z, w) \in (\mathbb{C}^*)^2 :$

$P(z,w)=0\}$ is called the *spectral curve.* The convex integral polygon

$$N = \mathrm{Conv}\{(i,j) \in \mathbb{Z}^2 : z^i w^j \text{ has non-zero coefficient in } P(z,w)\}$$

is called the *Newton polygon.* It is always centrally symmetric.

A zig-zag path on Γ is an unoriented path that alternately turns maximally left or right at each vertex. Each zig-zag path gives rise to a pair of homology classes $\pm[\alpha] \in H_1(\mathbb{T},\mathbb{Z})$, where $[\alpha]$ is the homology of the path α equipped with an orientation. There is a unique centrally symmetric integral polygon $N(G) \subset H_1(\mathbb{T},\mathbb{Z}) \cong \mathbb{Z}^2$ centered at the origin such that the sides of $N(G)$ are given by the vectors $\pm[\alpha]$.

Lemma 5.3.1 (Goncharov and Kenyon, 2012 [GK12])**.** *$N(G)$ coincides with the Newton polygon.*

For the graphs $T_{m,n}$, we choose the connection as follows: For every oriented edge vv', we will have $i_{vv'} = z^\alpha w^\beta$, with $\alpha,\beta \in \{0,\pm1\}$. If an edge crosses the side $(0,0,0),(-m,m,0)$ of the fundamental parallelogram, we multiply by a factor of w. If an edge crosses the side $(-m,m,0),(-m,m+n,-n)$, we multiply by z. Consistently with the rule $i_{vv'} = i_{v'v}^{-1}$, if an edge crosses the sides of the parallelogram parallel to the ones above for the case z and w, we multiply by a factor of z^{-1} or w^{-1} respectively. The Laplacian may then be represented by a matrix with entries in $\mathbb{C}[z^{\pm1},w^{\pm1}]$. The Newton polygon of $T_{m,n}$ is a hexagon with vertices at

$$(\pm n,0),(0,\pm m),(n,m),(-n,-m).$$

For $T_{1,2}$ with conductance function as shown in Figure 5.6 (a), the Laplacian is

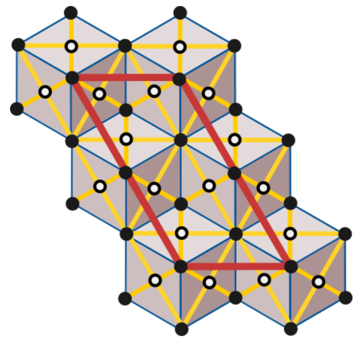


Figure 5.7: Generalized Temperley's bijection for $T_{1,2}$.

$$\Delta(z,w) = \begin{pmatrix} a+b+d+e+f(2-z-\frac{1}{z}) & -aw-bzw-d-\frac{e}{z} \\ -\frac{a}{w}-\frac{b}{zw}-d-ez & a+b+d+e+c(2-z-\frac{1}{z}) \end{pmatrix},$$

and the Newton polygon is the hexagon in Figure 5.6 (b).

5.3.3 Templerley's bijection

Given a resistor network Γ embedded on a torus $\mathbb{T}$, the generalized Temperley's trick [KPW00] gives a bipartite graph G_Γ on $\mathbb{T}$ as follows:

Superimpose Γ and its dual graph, declare the vertices of Γ and its dual black and put a white vertex at intersections of the edges of Γ and its dual. For $T_{1,2}$, the resulting bipartite graph is shown in Figure 5.7.

5.3.4 Cluster Poisson variety associated to the resistor network model

We recall the resistor network cluster Poisson variety as defined by Goncharov and Kenyon in [GK12]. The moduli space of line bundles with connection on G_Γ modulo isomorphisms is denoted $\mathcal{L}_{G_\Gamma}$. Let $\widehat{G_\Gamma}$ be the conjugate surface graph obtained by reversing the cyclic order of edges at each white vertex. The *Poisson structure on* $\mathcal{O}(\mathcal{L}_{G_\Gamma})$ is defined to be the canonical Poisson structure on $\mathcal{O}(\mathcal{L}_{\widehat{G_\Gamma}})$ coming from the intersection pairing on $\widehat{G_\Gamma}$ under the natural isomorphism

$$\mathcal{L}_{G_\Gamma} \cong \mathcal{L}_{\widehat{G_\Gamma}}.$$

The monodromies W_F around the faces of Γ along with the monodromies around generators of $H_1(\mathbb{T}^2, \mathbb{Z})$ form a coordinate system on $\mathcal{L}_\Gamma$, subject to the single relation $\prod_F W_F = 1$.

A conductance function c on Γ determines $V(c) \in \mathcal{L}_{G_\Gamma}$ as follows:
The fiber over each vertex is identified with $\mathbb{C}$. The connection is defined to be the identity map if the edge comes from the incidence of a face and edge of Γ. If the edge of G_Γ goes from a vertex of Γ to the mid point of an edge E in Γ, then the connection is defined to be $* \mapsto * \times c_E$. The moduli space of line bundles with connections arising from conductance functions forms a subvariety $\mathcal{R}_\Gamma \subset \mathcal{L}_{G_\Gamma}$.

A graph Γ is *minimal* if, in the universal cover, the lifts of any two zig-zag paths

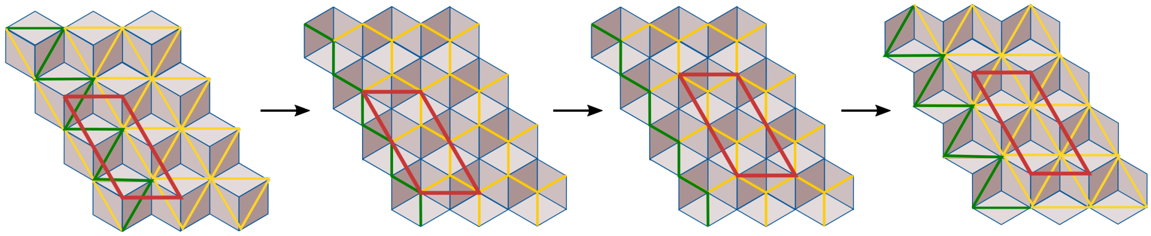


Figure 5.8: The cluster modular transformation T on $T_{1,2}$. We have colored in green a zig-zag path with homology $(0,1)$ to show that it goes around the torus. Zig-zag paths with other homology classes have no net displacement. The first step is a Y-Δ move at each downward triangle, the second step is a translation of the entire graph by the vector $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and the third step is taking the dual graph.

intersect at most once and the lift of any zig-zag path has no self intersections.

Theorem 5.3.2 (Goncharov and Kenyon, 2012 [GK12])**.** *Any two minimal graphs with the same Newton polgygon are related by Y-Δ moves up to taking the dual graph.*

A Y-Δ transformation $\Gamma \to \Gamma'$ induces a birational isomorphism

$$\mu_{Y-\Delta} : \mathcal{R}_\Gamma \dashrightarrow \mathcal{R}_{\Gamma'}.$$

Gluing the $\mathcal{R}_\Gamma$ with the Newton polygon N using these birational maps gives the *cluster Poisson variety of the resistor network model* $\mathcal{R}_N$.

5.3.5 Cluster modular transformations

A birational automorphism of $\mathcal{R}_N$ induced by a sequence of Y-Δ moves taking Γ to itself up to taking the dual graph is called a *cluster modular transformation.* The group of cluster modular transformations is called the *cluster modular group* (see [GK12], Section 6.2).

Cluster modular transformations are easier to describe in terms of zig-zag paths. A Y-Δ move is induced by moving a zig-zag path across the crossing of two other zig-zag paths. For each centrally symmetric pair of edges E, E' of N, we have zig-zag paths $(\alpha_i)_{i=1}^k$ in cyclic order in the fundamental domain of the torus with homology class given by the vector of the edge E up to orientation. By minimality, these paths do not intersect. Isotope them cyclically around the torus in the direction specified by the outward normal (out from N) to the edge vector of E, so that $(\alpha_1, ..., \alpha_k) \to (\alpha_2, ..., \alpha_k, \alpha_1)$ or $(\alpha_1, ..., \alpha_k) \to (\alpha_k, \alpha_1, \alpha_2, ..., \alpha_{k-1})$, leaving the other strands unchanged. This induces a sequence of Y-Δ moves corresponding to moving α_i through simple crossings of two zig-zag paths, which transforms Γ back to itself. The composition of the birational maps induced by these Y-Δ moves gives a cluster modular transformation T_E. Note that $T_{E'} = T_E^{-1}$.

We are interested in the cluster modular transformation $T := T_{\langle(-n,0),(-n,-m)\rangle}$ on the graph $T_{m,n}$. In the case of $T_{1,2}$, this cluster modular transformation is illustrated in Figure 5.8. In the coordinates of Figure 5.6, it is given by:

$$T(a,b,c,d,e,f) = \left(a\Delta_{abc}, b\Delta_{abc}, f\Delta_{def}, d\Delta_{def}, e\Delta_{def}, c\Delta_{abc}\right),$$

where

$$\Delta_{abc} = \frac{1}{ab+bc+ac}, \quad \Delta_{def} = \frac{1}{de+ef+df}.$$

The cluster modular transformation T is said to be N-*periodic* if $T^N = \mathrm{id}$. Since conductances are defined modulo scaling, this means that T^N leaves the conductance function invariant modulo scaling and in particular, preserves the probability measure.

5.3.6 Ergodic Gibbs measures

Let $\tilde{\Gamma}$ be the lift of Γ to the universal cover of the torus. An *essential spanning forest* (ESF) on $\tilde{\Gamma}$ is a spanning forest in which every component is infinite. Kenyon proved the following classification for ergodic Gibbs measures (EGMs) on ESFs on $\tilde{\Gamma}$, extending to groves models the results in [KOS06] for the dimer model.

Theorem 5.3.3 (Kenyon 2017, [K17])**.** *For each centrally symmetric pair* $((s,t),(-s,-t)) \in N(P)$*, there exists a unique EGM on ESFs of* $\tilde{\Gamma}$ *with components having average density* (s,t) *in the two coordinate directions.*

An ergodic Gibbs measure is in the solid phase if some edge correlation is deterministic. It is in the liquid phase if the edge correlations decay quadratically with distance and gaseous if the decay is exponential. The solid phases are in bijection with boundary lattice points of the Newton polygon, and the gaseous phases are in bijection with the interior lattice points, unless the corresponding compact oval in $P(z,w)$ degenerates to a real node, in which case it is in the liquid phase (see [K17]). This always happens for the central point, which corresponds to the UST measure. For what concerns our running example, $T_{1,2}$, the Newton polygon is in Figure 5.6 (b). Therefore, there are four EGMs in the solid phase and one EGM in the gaseous phase. By analogy with dimer limit shapes (see [KOS06, KO07]), we expect to see macroscopic regions where the local statistics are described by each of the solid and gaseous EGMs in a generic limit shape.

5.4 Edge-probability generating functions

Starting with a conductance function C^t on the $T_{m,n}$ that is N-periodic under the cluster modular transformation T, we construct a conductance function on $\mathbb{Z}^3$ which is Y-Δ consistent as follows:
Fix a scale factor for C^t and define $C|_{\{i+j+k=-1\}} = C^t$. Extend to all of $\mathbb{Z}^3$ using the Y-Δ transformation.

From $T^N C^t = C^t$ up to scaling, we have for all k,

$$U^{\mu+k(-N,0,0)} = U^{\mu},$$

$$V^{\mu+k(-N,0,0)} = V^{\mu},$$

$$W^{\mu+k(-N,0,0)} = W^{\mu},$$

which implies that

$$F^{\mu+k(-N,0,0)} = F^{\mu}.$$

Moreover, since C^t comes from $T_{m,n}$, we also obtain for all $k \in \mathbb{Z}$,

$$C^{\mu+k(-m,m,0)} = C^{\mu},$$

$$C^{\mu+k(0,n,-n)} = C^{\mu},$$

from which we get,

$$F^{\mu+k(-m,m,0)} = F^{\mu},$$

$$F^{\mu+k(0,n,-n)} = F^{\mu}.$$

Let us introduce an equivalence relation $\sim$ on $\mathbb{Z}^3$: For all k,

$$\mu \sim \mu + k(-N,0,0),$$

$$\mu \sim \mu + k(-m,m,0),$$

$$\mu \sim \mu + k(0,n,-n).$$

$\mathcal{M} := \mathbb{Z}^3/\sim$ parameterizes the distinct F^{μ}. The infinite linear system of equations in Theorem 5.2.8 reduces to a finite linear system and so we obtain a matrix $A = (A_{[\mu],[\nu]})$ for $[\mu],[\nu] \in \mathcal{M}$, such that the linear system may be written as

$$A(F^{[\mu]})_{[\mu]\in\mathcal{M}} = \mathbf{1},$$

where $\mathbf{1}$ is the constant vector of 1s.

Let

$$G_p^{\mu}(x,y,z) = \sum_{i,j,k\geq 0} p^{\mu}(-i,-j,-k)x^i y^j z^k,$$

$$G_q^{\mu}(x,y,z) = \sum_{i,j,k\geq 0} q^{\mu}(-i,-j,-k)x^i y^j z^k,$$

$$G_r^{\mu}(x,y,z) = \sum_{i,j,k\geq 0} r^{\mu}(-i,-j,-k)x^i y^j z^k,$$

be the generating functions for edge probabilities.

Theorem 5.4.1. *The edge probability generating functions satisfy the following linear system of equations:*

$$A\left(G_p^{[\mu]}\right)_{[\mu]\in\mathcal{M}} = \frac{x}{1-x}(Q^{[\mu]}(0,0,0) + R^{[\mu]}(0,0,0))_{[\mu]\in\mathcal{M}},$$
$$A\left(G_p^{[\mu]}\right)_{[\mu]\in\mathcal{M}} = \frac{y}{1-y}(P^{[\mu]}(0,0,0) + R^{[\mu]}(0,0,0))_{[\mu]\in\mathcal{M}},$$
$$A\left(G_p^{[\mu]}\right)_{[\mu]\in\mathcal{M}} = \frac{z}{1-z}(P^{[\mu]}(0,0,0) + Q^{[\mu]}(0,0,0))_{[\mu]\in\mathcal{M}}. \tag{5.5}$$

Proof. We will derive the first equation, the other two may be derived in the same way. Let $\alpha^{[\mu]}(i,j,k) = p^{[\mu]}(i-1,j,k) - p^{[\mu]}(i,j,k)$. By Lemma 5.2.6, $\alpha^{[\mu]}(i,j,k) = (V^{[\mu]}(i,j,k) + W^{[\mu]}(i,j,k))E^{[\mu]}(i,j,k)$. We have

$$\begin{aligned}
&\alpha^{[\mu-v]}(r+v,s+v,t+v)\\
&= (V^{[\mu-v]}(r+v,s+v,t+v) + W^{[\mu-v]}(r+v,s+v,t+v))E^{[\mu-v]}(r+v,s+v,t+v)\\
&= (V^{[\mu]}(r,s,t) + W^{[\mu]}(r,s,t))E^{[\mu-v]}(r+v,s+v,t+v).
\end{aligned}$$

In particular, we observe that the factor $(V^{[\mu]}(r,s,t) + W^{[\mu]}(r,s,t))$ does not depend on v. Therefore from equation (5.4), we obtain

$$\begin{aligned}
&\alpha^{[\mu]}(r,s,t) + \alpha^{[\mu+(-1,-1,-1)]}(r+1,s+1,t+1)\\
&= U^{\mu}(0,0,0)(\alpha^{[\mu+(-1,0,0)]}(r+1,s,t) + \alpha^{[\mu+(0,-1,-1)]}(r,s+1,t+1))\\
&+ V^{\mu}(0,0,0)(\alpha^{[\mu+(0,-1,0)]}(r,s+1,t) + \alpha^{[\mu+(-1,0,-1)]}(r+1,s,t+1))\\
&+ W^{\mu}(0,0,0)(\alpha^{[\mu+(0,0,-1)]}(r,s,t+1) + \alpha^{[\mu+(-1,-1,0)]}(r+1,s+1,t)).
\end{aligned}$$

Therefore the generating functions $H^{[\mu]}(x,y,z) = \sum_{i,j,k\geq 0} \alpha^{[\mu]}(-i,-j,-k)x^i y^j z^k$

satisfy the linear system of equations,

$$A(H^{[\mu]})_{[\mu]\in\mathcal{M}} = (Q^{[\mu]}(0,0,0) + R^{[\mu]}(0,0,0))_{[\mu]\in\mathcal{M}}.$$

From $\alpha^{[\mu]}(i,j,k) = p^{[\mu]}(i-1,j,k) - p^{[\mu]}(i,j,k)$, we have

$$G_p^{[\mu]}(x,y,z) = \frac{x}{1-x} H^{[\mu]}(x,y,z) + \sum_{(0,j,k)\in\mathbb{Z}^3_{\geq 0}} p^{[\mu]}(0,-j,-k) y^j z^k.$$

Observe that for all $j,k \geq 0$, $p^{[\mu]}(0,-j,-k) = 0$ and therefore we get

$$\sum_{(0,j,k)\in\mathbb{Z}^3_{\geq 0}} p^{[\mu]}(0,-j,-k) y^j z^k = 0.$$

□

5.5 Arctic curves

Following the theory of asymptotics of multivariate generating functions developed in [PW02, PW04, BP11, PW13], we compute the asymptotic edge probabilities in the grove model.

Solving the linear system (5.5), we obtain

$$G_p^{(0,0,0)}(x,y,z) = \frac{x}{1-x} \frac{P_p(x,y,z)}{Q(x,y,z)},$$

where P_p and Q are polynomials and $Q = \det(A)$. Note that the matrix A is always singular at $x = 1, y = 1, z = 1$ because the sum of the columns of A vanishes. We denote by $\tilde{P}$ and $\tilde{Q}$ the homogeneous parts of these polynomials at the singular point $(1,1,1)$.

We are interested in the behavior of the coefficients $p^{[(0,0,0)]}(-i,-j,-k)$ of $G^{[(0,0,0)]}(x,y,z)$

for (i,j,k) large i.e. we are interested in computing the limit,

$$p(\hat{\boldsymbol{r}}) = \lim_{\substack{i,j,k\to\infty \\ \frac{(-i,-j,-k)}{\sqrt{i^2+j^2+k^2}}\to\hat{\boldsymbol{r}}}} p^{[(0,0,0)]}(-i,-j,-k),$$

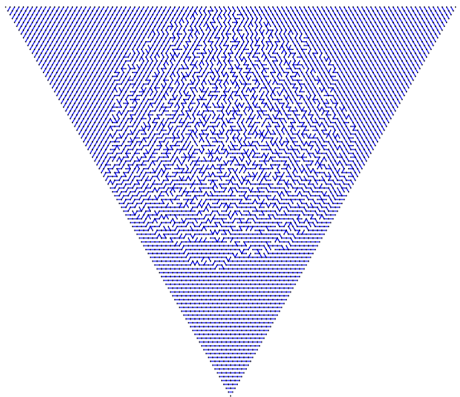
for $\hat{\boldsymbol{r}} \in \mathbb{R}^3_{\leq 0}$ such that $|\hat{\boldsymbol{r}}| = 1$.

For a homogeneous polynomial $f(x,y,z)$ in three variables, let $Z(f)$ be the plane curve $\{P \in \mathbb{P}^2_{\mathbb{C}} : f(P) = 0\}$ and let $C(f) \subset \mathbb{C}^3$ be the affine cone over $Z(f)$. The dual cone to $C(f)$ is denoted $C^\vee(f)$ and is equal to $C(f^\vee)$ where by $f^\vee$ we mean the projective dual of f, which may be computed by setting $z = -ux - vy$ in $f(x,y,z)$ and eliminating x and y from the system of equations,

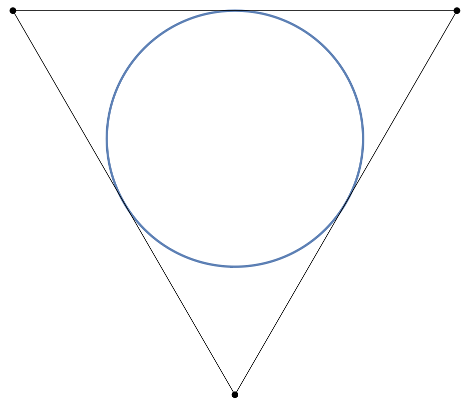
$$f = 0, \qquad \frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0.$$

.

The computation of asymptotic edge probabilities leads to explicit expressions for arctic curves. We consider simplified groves on standard initial conditions of order n, so that they are supported on an equilateral triangle in the plane $i+j+k = -n$ with vertices at $(-n,0,0), (0,-n,0)$ and $(0,0,-n)$. We rescale so that the vertices are now at $(-1,0,0), (0,-1,0)$ and $(0,0,-1)$, obtaining an equilateral triangle ∇ in the plane $i+j+k = -1$. For n large, we observe macroscopic regions in the triangle with different qualitative behavior (see Figures 5.2, 5.9 and 5.11). The arctic curve is the boundary separating the macroscopic regions in different phases.



(a) Simulation of a uniformly random grove on $\mathcal{I}(100)$.



(b) The arctic circle.

Figure 5.9: Uniform groves on $T_{1,1}$.

5.5.1 $T_{1,1}$

On $T_{1,1}$, $N = 1$ is forced. Let us take the conductance function on $T_{1,1}$ to be the constant function 1. This gives rise to the uniform probability measure on groves. See Figure 5.9 for a simulation of a random (simplified) grove on standard initial conditions of order 100. Equation (5.5) gives

$$G_p^{[(0,0,0)]}(x,y,z) = \frac{2x}{3(1-x)} \frac{1}{1 + xyz - \frac{1}{3}(x+y+z+yz+xz+xy)}.$$

Here $P_p(x,y,z) = \frac{2}{3}$ and $Q(x,y,z) = 1 + xyz - \frac{1}{3}(x+y+z+yz+xz+xy)$. The homogeneous parts at the singular point $(1,1,1)$ are

$$\begin{aligned} \tilde{P}_p(x,y,z) &= \frac{2}{3}, \\ \tilde{Q}(x,y,z) &= \frac{2}{3}(yz+xz+xy). \end{aligned} \tag{5.6}$$

The dual curve is $\tilde{Q}^\vee(u,v,w) = vw+uw+uv-\frac{1}{2}(u^2+v^2+w^2)$. Let K be the region bounded by the cone $C(\tilde{Q}^\vee)$.

Theorem 5.5.1 (The (weak) arctic circle theorem, Petersen and Speyer, 2005 [PS06])**.** *$p(-i,-j,-k) \to 0$ exponentially fast outside convex-hull$(K \cup \{(u,v,w) \in \mathbb{R}^3 : v = w = 0\})$.*

Let us denote by $P(\hat{\boldsymbol{r}})$ the point in ∇ obtained by intersecting the line in the direction $\hat{\boldsymbol{r}}$ with the plane $u+v+w=-1$. $\hat{\boldsymbol{r}} \mapsto P(\hat{\boldsymbol{r}})$ is clearly a bijection. Let $C^\vee$ be the curve inscribed in ∇ obtained by the intersection of $C(\tilde{Q}^\vee)$ with $u+v+w=-1$. Observe that for a point $P(\hat{\boldsymbol{r}})$ outside the region bounded by $C^\vee$, there are two (real) tangents through $P(\hat{\boldsymbol{r}})$ to $C^\vee$ while from a point inside $C^\vee$, there are no (real) tangents to $C^\vee$. What is happening is that as we approach the boundary of $C^\vee$ from the outside, the two real tangents merge into a pair of complex conjugate tangents. Under projective duality, this pair of complex conjugate tangents gives us two complex conjugate points t_1, t_2 on $Z(\tilde{Q})$, where we assume t_1 has positive imaginary part.

Theorem 5.5.2 (Baryshnikov and Pemantle, 2011 [BP11]))**.** *For $(i,j,k) \in \mathbb{Z}^3$ large* *such that for*

$$\hat{\boldsymbol{r}} = \frac{(-i,-j,-k)}{\sqrt{i^2+j^2+k^2}},$$

$P(\hat{\boldsymbol{r}})$ *is in the interior of* $C^\vee$*, we have*

$$p(-i,-j,-k) = \frac{1}{2\pi i}\int_{\delta(\hat{\boldsymbol{r}})} \omega + O\left(\frac{1}{\sqrt{i^2+j^2+k^2}}\right),$$

where in the affine coordinates $X = \frac{x}{z}, Y = \frac{y}{z}$*,* ω *is the meromorphic 1-form*

$$\omega = \frac{\tilde{P}_p(X,Y,1)dX}{X\frac{\partial \tilde{Q}(X,Y,1)}{\partial Y}}. \tag{5.7}$$

The chain of integration $\delta(\hat{\boldsymbol{r}})$ *is a simple path from* t_1 *to* t_2 *passing through the arc between* $[0:1:0]$ *and* $[0:0:1]$ *containing* $[1:0:0]$ *in the real part of* $Z(\tilde{Q})$*. In particular, we have*

$$p(\hat{\boldsymbol{r}}) = \frac{1}{2\pi i}\int_{\delta(\hat{\boldsymbol{r}})} \omega.$$

Note that the only dependence on $\hat{\boldsymbol{r}}$ is through the chain of integration. Note also that this shows that $C^\vee$ is the strict boundary for exponential decay of two of the asymptotic edge probabilities. In particular, this shows that the arctic curve is $C^\vee$.

In our case, plugging in (5.6), we obtain the 1-form

$$\omega = \frac{dX}{X(X+1)},$$

which has poles at $[0:1:0]$ and $[0:0:1]$ with residues -1 and 1 respectively. We are led to the following description of the arctic curve: As $P(\hat{\boldsymbol{r}})$ approaches the curve

$C^\vee$, the two complex tangents from $P(\hat{\boldsymbol{r}})$ to $C^\vee$ merge into a real double tangent. Under projective duality, on $Z(\tilde{Q})$, the two points t_1 and t_2 merge into a point on the real part of $Z(\tilde{Q})$ and therefore $\delta(\hat{\boldsymbol{r}})$ becomes a closed loop. Using the residue theorem, the asymptotic edge probabilities in the frozen region $P(\hat{\boldsymbol{r}})$ approaches may be read from the residue divisor of ω.

If we take a non-constant conductance function on $T_{1,1}$, it was shown in [PS06] that the arctic curve is an ellipse inscribed in the triangle ∇.

5.5.2 $T_{1,2}$ with $N = 1$

In this section, we work out the computation of the arctic curve for a specific T-invariant conductance function on $T_{1,2}$, although the approach works for all such conductance functions. Consider as an example the following T-invariant conductance function on $T_{1,2}$ given in the notation of Figure 5.6 (A) by:

$$a = \frac{1}{2}; \qquad b = \frac{1}{8}; \qquad c = \frac{3}{2}; \qquad d = \frac{1}{8}; \qquad e = \frac{1}{2}; \qquad f = \frac{3}{2}.$$

The linear system from (5.5) is:

(5.8)
$$\begin{pmatrix} -\frac{3x}{16} - \frac{\mathrm{xy}}{16} - \frac{3y}{4} + 1 & xyz - \frac{3xz}{4} - \frac{3yz}{16} - \frac{z}{16} \\ xyz - \frac{3xz}{16} - \frac{3yz}{4} - \frac{z}{16} & -\frac{xy}{16} - \frac{3x}{4} - \frac{3y}{16} + 1 \end{pmatrix} \begin{pmatrix} G^{(0,0,0)}(x,y,z) \\ G^{(0,0,-1)}(x,y,z) \end{pmatrix} = \frac{x}{1-x} \begin{pmatrix} \frac{13}{16} \\ \frac{1}{4} \end{pmatrix}.$$

We compute

$$\tilde{P}_p(x,y,z) = \frac{185x}{256} + \frac{13y}{32},$$
$$\tilde{Q}(x,y,z) = \frac{1}{256}(255x^2y + 255xy^2 + 104x^2z + 370xyz + 104y^2z).$$

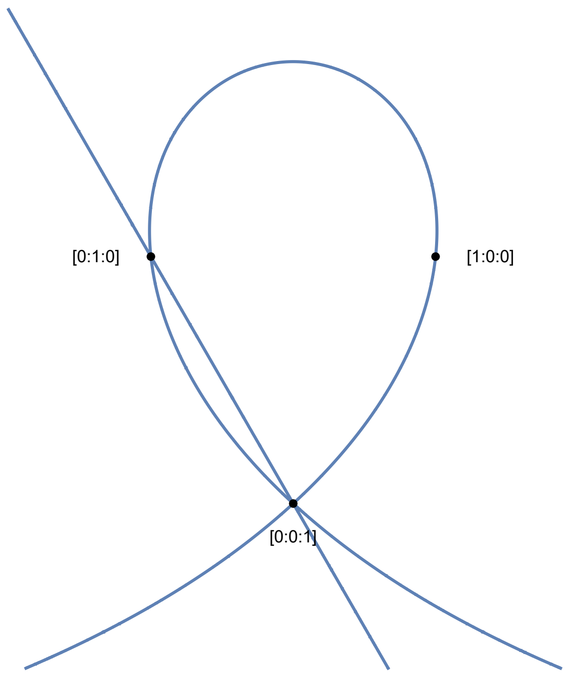
The dual curve is

$$\begin{aligned}\tilde{Q}^\vee(u,v,w) = 6619392u^4 - 47099520u^3v + 97021584u^2v^2 - 47099520uv^3 +\\ 6619392v^4 - 38301120u^3w - 3164400u^2vw - 3164400uv^2w -\\ 38301120v^3w + 73033700u^2w^2 + 6779600uvw^2 + 73033700v^2w^2\\ -57655500uw^3 - 57655500vw^3 + 27635625w^4.\end{aligned}$$

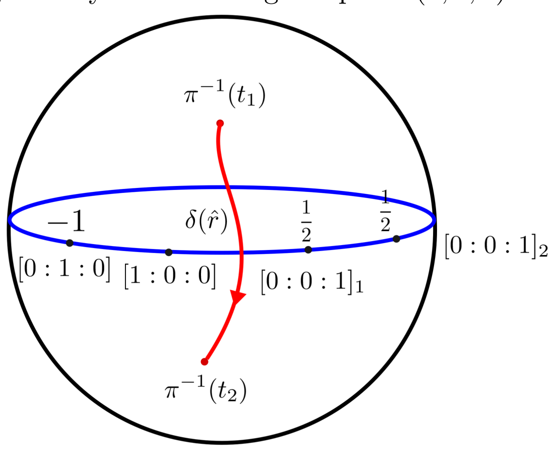
$Z(\tilde{Q})$ is singular with a node at $[0:0:1]$ (See Figure 5.10 (A)). This is outside the class of quadratic singularities studied in [BP11], but as observed in Section 7 of that paper, the techniques used still go through with minor modifications. Theorem 5.5.1 still holds, so we still have exponential decay outside the dual curve (see [BP11], Proposition 2.23).

We need the following notions from [KO07]: A degree d real algebraic curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is *winding* if:

- it intersects every line $L \subset \mathbb{P}^2_{\mathbb{R}}$ in at least $d-2$ points counting multiplicity, and
- there exists a point $p_0 \in \mathbb{P}^2_{\mathbb{R}} \setminus C$ called the center, such that every line through p_0 intersects C in d points.

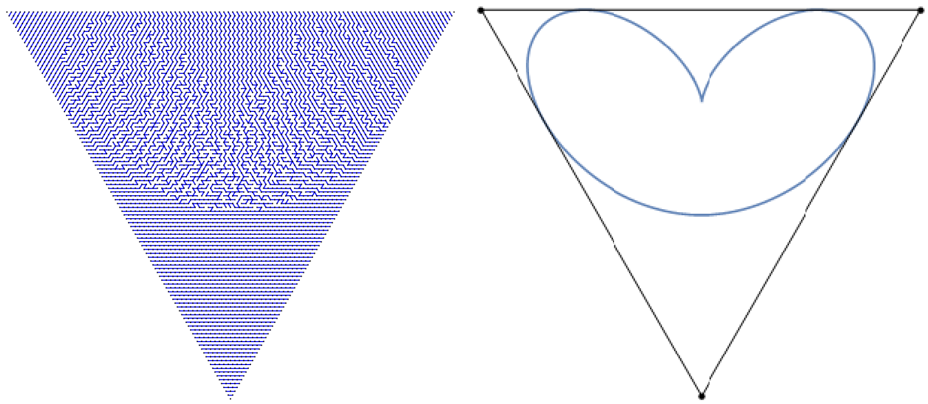


(a) $C(x\tilde{Q}) \cap \{(x, y, z) \in \mathbb{R}^3 : x+y+z = -1\}$ illustrating the geometry near the singular point $(1, 1, 1)$.



(b) The residue divisor of ω on $X \cong \mathbb{P}^1_{\mathbb{C}}$. The blue curve is the real part of X and is isomorphic to $\mathbb{P}^1_{\mathbb{R}}$

Figure 5.10: $\tilde{Q}(x, y, z)$ and its normalization X.



(a) Simulation of a random grove on $\mathcal{I}(100)$.

(b) The arctic curve $C^\vee$.

Figure 5.11: $T_{1,2}$ with $N = 1$.

The dual of a winding curve C is called a *cloud curve*. $C^\vee$ separates $\mathbb{P}^2_{\mathbb{R}}$ into two regions, formed by the lines that intersect C in d and $d-2$ points, which we call the exterior and interior respectively. A cloud curve $C^\vee$ has a unique pair of complex conjugate tangents through any point in its interior which under projective duality gives a pair of complex conjugate points on C.

Theorem 5.5.3. *The curve $Z(\tilde{Q})$ is winding. Let $\pi : X \to Z(\tilde{Q})$ be the normalization of $Z(\tilde{Q})$, where we denote by $[0:0:1]_1$ and $[0:0:1]_2$ the two points in X in the fiber above the node $[0:0:1]$ of $Z(\tilde{Q})$, such that in cyclic order, we have $[0:1:0], [1:0:0], [0:0:1]_1, [0:0:1]_2$ in the real part of X.*

Let $\hat{\mathbf{r}}$ be as in Theorem 5.5.2. Let t_1, t_2 be the pair of points in $Z(\tilde{Q})$ corresponding, under projective duality, to the unique pair of complex conjugate tangents. The conclusions of Theorem 5.5.2 hold with the following modifications:

- *The 1-form ω (defined in (5.7)) is replaced by its pullback to X.*
- *The chain $\delta(\hat{\boldsymbol{r}})$ is also pulled back to X so that it is now a simple path from $\pi^{-1}(t_1)$ to $\pi^{-1}(t_2)$ passing through the arc between $[0:1:0]$ and $[0:0:1]_1$. In particular, the asymptotic edge probability is given by*

$$p(\hat{\boldsymbol{r}}) = \frac{1}{2\pi i}\int_{\pi^{-1}\delta(\hat{\boldsymbol{r}})} \pi^*\omega.$$

Proof. The curve $Z(\tilde{Q})$ is a winding curve, where we may take the center to be $[1:1:1]$ (This is also easily seen from the dual picture: $C^\vee$ is a cardioid (Figure 5.11 (b)) and there is a unique real tangent to $C^\vee$ from a point in its interior, whereas there are three real tangents from its exterior). Therefore, for any point $P(\hat{\boldsymbol{r}})$ in the interior of $C^\vee$ we have a pair of complex conjugate points t_1, t_2 on $Z(\tilde{Q})$. This is exactly the hypothesis needed in the proof of Lemma 6.15 in [BP11] to determine the boundary of $\delta(\hat{\boldsymbol{r}})$. □

Since

$$\eta = \frac{\tilde{P}_p(X,Y,1)dX}{\frac{\partial\tilde{Q}(X,Y,1)}{\partial Y}},$$

is a holomorphic 1-form, and $\omega = \frac{1}{X}\eta$, the poles of ω are supported on the intersection of $Z(\tilde{Q})$ with the line $Z(x)$, which is a finite number of points. By computing Puiseux expansions at these points, we see that $\pi^*\omega$ has the residue divisor shown in Figure 5.10 (b). We can explain the new frozen region as follows: As $P(\hat{\boldsymbol{r}})$ approaches that region, $\pi^{-1}(t_1)$ and $\pi^{-1}(t_2)$ merge into a point on the real part of X

on the arc between $[0:0:1]_1$ and $[0:0:1]_2$, thereby enclosing a pole with residue $\frac{1}{2}$.

Note that we don't see a macroscopic region in the gaseous phase. The reason is that our choice of conductance is not generic and on the T-invariant subvariety, the compact oval in the spectral curve $P(z,w)$ corresponding to the gaseous phase degenerates to a real node. Therefore we need to consider conductances that have a higher T-periodicity to see generic limit shapes.

5.5.3 $T_{1,2}$ with $N = 2$.

By a simple computation, we can see that there are no $T-2-$periodic solutions that are not T-invariant for $T_{1,2}$, and therefore this case is subsumed by the previous one.

5.5.4 $T_{1,2}$ with $N = 3$

The following analysis works for any choice of a T-3-periodic conductance function, but for clarity and ease of computation, we only work out a specific example here. Consider the T-3-periodic conductance function on $T_{1,2}$ given in the notation of Figure 5.6 (a) by:

$$a = \frac{1}{2}; \qquad b = \frac{1}{3}; \qquad c = 1; \qquad d = \frac{10}{3}; \qquad e = \frac{1}{4}; \qquad f = \frac{2}{43}.$$

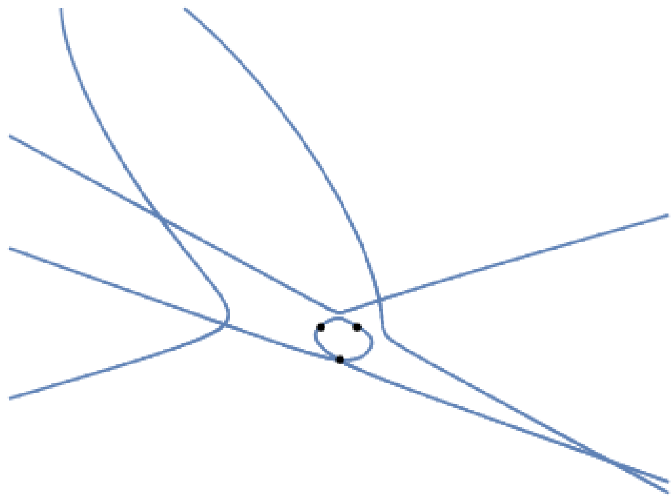
In the linear system from (5.5), we have:

$$A = \begin{pmatrix} 1 & xyz & -\frac{x}{2}-\frac{y}{3} & -\frac{z}{6} & -\frac{xy}{6} & -\frac{xz}{3}-\frac{yz}{2} \\ xyz & 1 & -\frac{5z}{6} & -\frac{20x}{129}-\frac{y}{86} & -\frac{xz}{86}-\frac{20yz}{129} & -\frac{5xy}{6} \\ -\frac{43xy}{53} & -\frac{4xz}{53}-\frac{6yz}{53} & 1 & xyz & -\frac{6x}{53}-\frac{4y}{53} & -\frac{43z}{53} \\ -\frac{3xz}{53}-\frac{40yz}{53} & -\frac{10xy}{53} & xyz & 1 & -\frac{10z}{53} & -\frac{40x}{53}-\frac{3y}{53} \\ -\frac{6x}{53}-\frac{4y}{53} & -\frac{43z}{53} & -\frac{43xy}{53} & -\frac{4xz}{53}-\frac{6yz}{53} & 1 & xyz \\ -\frac{10z}{53} & -\frac{40x}{53}-\frac{3y}{53} & -\frac{3xz}{53}-\frac{40yz}{53} & -\frac{10xy}{53} & xyz & 1 \end{pmatrix},$$

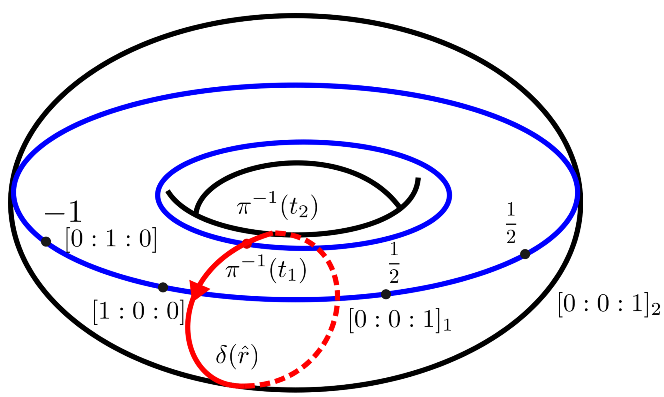
$$\left(G_p^{[\mu]}\right)_{[\mu]\in\mathcal{M}} = \begin{pmatrix} G^{(0,0,0)} \\ G^{(-2,0,-1)} \\ G^{(-1,0,0)} \\ G^{(0,0,-1)} \\ G^{(-2,0,0)} \\ G^{(-1,0,-1)} \end{pmatrix} \text{ and } (Q^{[\mu]}(0,0,0)+R^{[\mu]}(0,0,0))_{[\mu]\in\mathcal{M}} = \begin{pmatrix} \frac{1}{2} \\ \frac{109}{129} \\ \frac{47}{53} \\ \frac{13}{53} \\ \frac{47}{53} \\ \frac{13}{53} \end{pmatrix}.$$

We obtain

$$\begin{aligned} \tilde{P}_p(x,y,z) = (&-8376157535x^3 - 27465850948x^2y - 37792606090x^2z - 32422312230xy^2 \\ &- 81250160702xyz - 41078137290xz^2 - 12081677400y^3 - 37378399260y^2z \\ &- 26396541912yz^2)/2035744098; \end{aligned}$$



(a) A plot of $C(\tilde{Q}) \cap \{(x, y, z) \in \mathbb{R}^3 : x + y + z = -1\}$. The three dots are the points $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$, as in Figure 5.10 (A).



(b) The residue divisor of ω on X and the chain of integration $\delta(\hat{\boldsymbol{r}})$ when $P(\hat{\boldsymbol{r}})$ is on the irreducible component bounding the gaseous region. The blue curves are the two irreducible components of $\tilde{Q}(x, y, z)$ viewed as a real algebraic curve in $\mathbb{P}^2_{\mathbb{R}}$.

Figure 5.12: $\tilde{Q}(x, y, z)$ and its normalization X.

$$\begin{aligned}\tilde{Q}(x,y,z) = (&-2195435870x^4y - 4213162175x^4z - 8636813573x^3y^2 - 26901515220x^3yz \\ &- 18270472400x^3z^2 - 8949558855x^2y^3 - 44782155243x^2y^2z - 62350371390x^2yz^2 \\ &- 19642088100x^2z^3 - 2785734900xy^4 - 25376048920xy^3z - 53016222846xy^2z^2 \\ &- 27385424860xyz^3 - 4027225800y^4z - 12459466420y^3z^2 \\ &- 8798847304y^2z^3)/678581366.\end{aligned}$$

As a real algebraic curve, we observe that $\tilde{Q}(x,y,z)$ is winding with center $(1,1,1)$ and has two irreducible components (see Figure 5.12). Let us denote by V_1 the component that contains the axes and by V_2 the other one. Under duality, we obtain two dual real components $V_1^\vee$ and $V_2^\vee$, where $V_2^\vee$ is in the interior of $V_1^\vee$ (see Figure 5.2 (B)). The region bounded by $V_2^\vee$ is a gaseous phase. The local statistics in this region are expected to be described by the ergodic Gibbs measure of slope $(1,0)$.

Let K be the cone over the region in the interior of $V_1^\vee$. Then it follows from [BP11] (see also [PW13], Theorem 11.3.8) that $p(-i,-j,-k)$ decays exponentially quickly outside convex-hull$(K \cup \{(u,v,w) \in \mathbb{R}^3 : v = w = 0\})$.

$Z(\tilde{Q})$ has genus 1 and therefore its normalization is topologically a torus. The 1-form $\pi^*\omega$ in Theorem 5.5.3 has the residue divisor shown in Figure 5.12 (B). We observe that as $P(\hat{\boldsymbol{r}})$ approaches $V_2^\vee \cap \{(u,v,w) \in \mathbb{R}^3 : u+v+w = -1\}$, the points $\pi^{-1}(t_1)$ and $\pi^{-1}(t_2)$ merge to a point on the inverse image of V_1 in X and therefore $\delta(\hat{\boldsymbol{r}})$ becomes a loop with non-trivial homology on the torus (see Figure 5.12 (B)).

5.6 Further questions

We are able to compute several interesting examples of arctic curves but there are several questions that remain.

- The projective duals of curves arising as limit shapes in the grove model and in the dimer model are expected to be winding. In dimers, in cases where $\tilde{Q}(x, y, z)$ is rational, this is proved in [KO07]. In [K17], groves were shown to satisfy a variational principle that is algebraically identical to the one in [CKP01] for dimers, and therefore the same holds. Can we prove that the polynomials $\tilde{Q}(x, y, z)$ are winding for all genus from the generating function?

- We have seen that Lemma 6.15 from [BP11] can be extended to tackle our examples and that the necessary assumption was that $\tilde{Q}(x, y, z)$ is winding. This motivates the following problem: Extend the machinery of [BP11] to describe the asymptotics of generating functions with higher degree isolated singularities where the local geometry is described by a winding curve.

- Periods of the 1-form ω encode asymptotic probabilities of the different solid and gaseous phases. Since we know what these measures are, these asymptotic probabilities are easy to compute from and depend only on the Newton polgyon. Can we prove a description of the residue divisor of ω for general $T_{m,n}$ and N in terms of the Newton polygon?

- Can we generalize the results of this paper to groves on other $\mathbb{Z}^2$-periodic networks?

- What can we say about the subvariety of T-N-periodic points of $\mathcal{R}_N$?

APPENDIX A

Toric surfaces ruled by lines, by Giovanni Inchiostro

The goal of this appendix is to relate the Picard group of a projective toric surface X, with an equivariant embedding $X \to \mathbb{P}^n$, with the one of a generic hyperplane section. This will be achieved in Theorem A.0.11. Since one can study such a projective toric surface by looking at its associated polygon, we will use the combinatorics of the polygons to prove Theorem A.0.11.

In this appendix, all polygons will be convex, integral and in $\mathbb{R}^2$.

Definition A.0.1. *We define a* building block *polygon to be a polygon* $\Delta \subseteq \mathbb{R}^2$ *with a single interior lattice point, and with at most five lattice points.*

A building block polygon has either three or four edges (see Figure A.1).

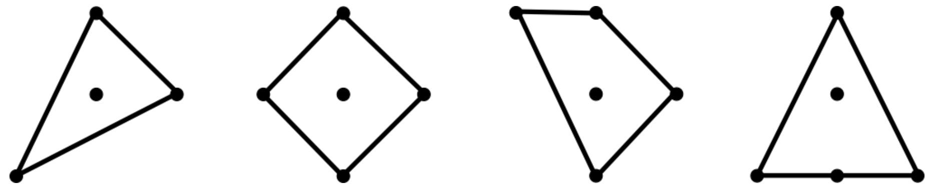


Figure A.1: The four building block polygons modulo lattice equivalence.

Our first goal is to show that given any polygon P with an interior lattice point, one can find a building block polygon Δ with $\Delta \subseteq P$ (Proposition A.0.4). We will find Δ as a polygon with the least number of lattice points among all polygons which are both contained in P and have at least one interior lattice point.

We begin with a few preparatory lemmas.

Lemma A.0.2. *Consider a polygon P with an lattice interior point. Then there is a polygon $Q \subseteq P$ which has an interior lattice point and at most four edges.*

Proof. Let x be an interior lattice point of P. Pick a polygon $Q \subseteq P$ which is minimal among all polygons contained in P containing x as an interior lattice point, with the partial order being the inclusion, and we aim at showing that Q has either three or four edges. If not, let $a_1, ..., a_n$ be the edges of Q with $n > 4$, labeled in clockwise order. Consider the segments joining a_1 with a_3, and a_3 with a_5. They divide Q into three smaller polygons, each with fewer lattice points, and these segments intersect only at a_3, since by assumption a_5 is distinct from a_3 and a_1. Therefore x is an interior point of one of these three sub-polytopes, contradicting the minimality of Q. □

Lemma A.0.3. *Consider a polygon P with an interior lattice point. Then there is a polygon $Q \subseteq P$ which has a exactly one interior lattice point.*

Proof. Consider Q a subpolygon of P, which has at least two interior lattice points, x and y. Up to shrinking Q, we can assume from Lemma A.0.2 that either Q is a triangle or it has four edges.

If Q is a triangle, consider the line through x and y. It must meet an edge ℓ of Q and let a and b be the vertices of ℓ. Up to swapping x and y, we can assume that the distance between x and ℓ is less than the distance between y and ℓ. Then the triangle with vertices y, a and b, with an interior point (namely x) has fewer interior lattice points than Q (y is not an interior point).

If Q has four edges, consider the two diagonals of Q. They have a single intersection point, so there must be a diagonal ℓ which does not contain both x and y. Then ℓ divides Q into two smaller polygons, and one of them must have an interior point.

Therefore, if we consider a polygon which is minimal for the inclusion and has an interior point, it must have a exactly one interior point. □

Proposition A.0.4. *Given any polygon P with an interior lattice point, one can find a building block polygon Δ such that $\Delta \subseteq P$.*

Proof. Consider a polygon $Q \subseteq P$. From Lemma A.0.3 and Lemma A.0.2, we can assume, up to shrinking Q, that Q has at most four edges, and a single interior point x. If Q has four edges and five points, we are done, otherwise there is an edge ℓ with a point $y \in \ell$ which is not a vertex. Let a, b be the two vertices of Q not contained in ℓ. Then the segments $\overline{ya}$ and $\overline{yb}$ intersect only at y, and divide Q into three smaller polygon. One of them must contain x in its interior. Therefore if Q is minimal and has four edges, it must be a building block polygon.

If instead Q is a triangle, assume it has more than five lattice points. Then there are two lattice points p, q which are on the boundary of Q, but are not vertices. If they belong to the same edge ℓ, let a be the vertex of Q not contained in ℓ. Then the segments $\ell_1 := \overline{ap}$ and $\ell_2 := \overline{aq}$ meet only at a, and divide Q into smaller polygons. Then there must be one among ℓ_1 and ℓ_2 which does not contain x, and which is the side of a smaller polygon contained in Q and with an interior point. Similarly if p and q do not belong to the same edge, let a be the vertex not contained in the edge containing p. Then the segments $\ell_1 := \overline{ap}$ and $\ell_2 := \overline{aq}$ intersect only at a and divide Q into smaller polygons. One of them must have an interior point. $\square$

Lemma A.0.5. *Consider X the projective toric surface corresponding to the polygon P, and let $Q \subseteq P$ a subpolygon of P, with corresponding projective toric surface Y. The two polygons give projective embeddings $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$. There is a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$ which gives an equivariant rational map $X \dashrightarrow Y$, which is an isomorphism on $(\mathbb{C}^*)^2$.*

Proof. Consider the characters $\chi_0, ..., \chi_n$ corresponding to the points in P, and let $\chi_0, ..., \chi_m$ be those corresponding to the points in Q, with $m < n$. Then one can consider the map $\Phi : (\mathbb{C}^*)^2 \to \mathbb{P}^n$ sending $p \mapsto [\chi_0(p), ..., \chi_n(p)]$. The variety X is the closure of the image of Φ, and Y is the closure of the map $\Phi : (\mathbb{C}^*)^2 \to \mathbb{P}^m$ sending $p \mapsto [\chi_0(p), ..., \chi_m(p)]$. Then the projection from the last $n - m$ coordinates gives the desired rational map. $\square$

Theorem A.0.6. *Consider a projective toric surface corresponding to a building block polygon P, with the points of P corresponding to the characters $\chi_0, ..., \chi_n$. Then there is no line $\ell \subseteq \mathbb{P}^n$ passing through the identity of $T := (\mathbb{C}^*)^2 \subseteq X$.*

Proof. We can write any line in $\mathbb{P}^n$ as the intersection of $n-1$ *linearly independent* hyperplanes $H_1, ..., H_{n-1}$. When we restrict these to the torus T, they can be written as $(H_i)|_T = \sum_j a_{i,j} x^i y^j$ where (x, y) are the coordinates on $(\mathbb{C}^*)^2$. Now, if f is the equation of $\ell|_T$, then we can factor $(H_i)|_T = f g_i$ for $g_i \in \mathbb{C}[x^{\pm}, y^{\pm}]$. We can then consider the Newton polygon P_f associated to f: if we write $f := \sum c_{i,j} x^{a_i} y^{b_j}$, P_f is the convex hull of the points (i, j) such that $c_{i,j} \neq 0$. Similarly, if we denote with P_i the one associated to g_i, from [Ost76, Theorem VI] we have that $P_f + P_i$ has vertices corresponding to the points $\chi_{i,j}$. In particular,

$$P_f + P_i \text{ is a subpolygon of } P.$$

Now, since the hyperplanes H_i are linearly independent, the set $\bigcup_i P_i$ has at least $n-1$ elements, say $x_1, ..., x_{n-1}$. Moreover, since f vanishes on the identity, f has at least two non-zero coefficients, so P_f contains a segment s. Therefore the segments $x_i + s$ belong to P. Checking the four building block polygons, one can see that this is not possible. □

Corollary A.0.7. *Any projective toric surface X whose polygon P has an interior point is not ruled by lines.*

Proof. From Proposition A.0.4, there is a building block polygon $Q \subseteq P$ corresponding to a toric surface Y. From Lemma A.0.5, there is a rational map $\pi : X \dashrightarrow Y$, which sends lines to lines. So if X is ruled by lines, there is a line ℓ passing through the identity of the torus in X. Then $\pi(\ell)$ would be a line through the identity, contradicting Theorem A.0.6. □

We are finally ready to prove the main theorem of this appendix:

Proposition A.0.8. *Consider a projective toric surface X, equivariantly embedded into $\mathbb{P}^n$, with a non-trivial line bundle L. Assume that the polygon of the embedding $X \hookrightarrow \mathbb{P}^n$ has an interior point. Then there is an hyperplane section $C \subseteq X$ such that $L|_C$ is not trivial, and C is irreducible and smooth.*

Proof. We have:

Lemma A.0.9 ([Lop91, Lemma II.2.4])**.** *An irreducible non-degenerate surface $S \subset \mathbb{P}^n, n \geq 3$, has an $(n-1)$-dimensional family of reducible hyperplane sections if and only if S is either ruled by lines, or is the Veronese surface, or its general projection in $\mathbb{P}^4$, or its general projection in $\mathbb{P}^3$ (the Steiner surface).*

The Veronese surface is toric and corresponds to the Newton polygon with vertices

$$\mathrm{Conv}\{(0,0),(2,0),(0,2)\},$$

and so has no interior points. Its general projections have hyperplane sections of zero genus. Since the generic hyperplane sections of X have genus 1, it is not the Veronese or its projections. Therefore by Lemma A.0.9 and Corollary A.0.7, we can find a generic pencil of hyperplane sections with all the members irreducible, and with generic member which is smooth. Such a pencil Π gives a rational map $X \dashrightarrow \mathbb{P}^1$, we can blow-up the toric surface $\pi : Y \to X$ to resolve the indeterminacy locus, and have a morphism $f : Y \to \mathbb{P}^1$ with fibers the members of the pencil Π.

Such a morphism is flat since it is dominant with target a smooth curve, proper since the source is proper and the target separated, and generically smooth since the generic member of Π is smooth. We want to show that $f_*(\mathcal{O}_Y) = \mathcal{O}_{\mathbb{P}^1}$. For that, first observe that $f_*(\mathcal{O}_Y)$ is a torsion free sheaf, since Y is integral. Therefore, since the local rings of $\mathcal{O}_{\mathbb{P}^1}$ are DVRs and torsion free modules over a DVR are free, the sheaf $f_*(\mathcal{O}_Y)$ is locally free: it is a vector bundle. To check its rank, observe that there is a fiber Y_p of f at a point p which is smooth and connected (the smooth member of Π). Therefore $h^0(\mathcal{O}_{Y_p}) = 1$, and from [Vak17, 28.1.1] there is an open subset $U \subseteq \mathbb{P}^1$ such that for $x \in U$ we have $h^0(\mathcal{O}_{Y_x}) = 1$. Then from [Vak17, 28.1.5] this is the rank of $f_*(\mathcal{O}_Y)$ at x. In particular, the latter is a line bundle. But from the definition of push forward, $H^0(\mathbb{P}^1, f_*(\mathcal{O}_Y) = H^0(Y, \mathcal{O}_Y) \cong \mathbb{C}$: we have that $f_*(\mathcal{O}_Y)$ is a line bundle on $\mathbb{P}^1$ with a single global section. From the description of the line bundles on $\mathbb{P}^1$ we have the desired isomorphism $f_*(\mathcal{O}_Y) \cong \mathcal{O}_{\mathbb{P}^1}$.

Now, assume that for every member C of Π we have $L|_C \cong \mathcal{O}_C$. The members of Π are the fibers of f, thus for every fiber F of f we have $\pi^*(L)|_F \cong \mathcal{O}_F$. Then from [Vak17, Proposition 28.1.11], there is a line bundle G on $\mathbb{P}^1$ such that $\pi^*(L) \cong f^*(G)$. Now we can proceed as in the second paragraph of [Sta16].

In particular, there is a fiber F of f such that $\pi^*(L)|_F$ is not trivial. Then from Lemma A.0.10, there is an open subset $U \subseteq \mathbb{P}^1$ where for every $p \in U$ we have $\pi^*(L)|_{Y_p}$ is not trivial, and Y_p is smooth (since being smooth is an open condition). □

The following Lemma is well known, we provide a proof for completeness.

Lemma A.0.10. *Consider a flat proper morphism $X \to B$ with integral fibers, and let L be a line bundle on X. Then the set $\{b \in B$ such that $L|_{X_b} \cong \mathcal{O}_{X_b}\}$ is closed.*

Proof. From the upper-semicontinuity theorems [Vak17, 28.1.1] the set $b \in B$ where $h^0(L|_{X_b}) > 0$ and $h^0(L^{-1}|_{X_b}) > 0$ is closed. It suffices to prove that if one has a line bundle G on an integral proper (over $\mathbb{C}$) scheme Y, then $h^0(G) > 0$ and $h^0(G^{-1}) > 0$ imply $G \cong \mathcal{O}_Y$. This is the first paragraph of the proof of the Seesaw theorem [Mum74]. □

Theorem A.0.11. *With the same assumptions of Proposition A.0.8, assume that the embedding $X \to \mathbb{P}^n$ is non-degenerate. Then if C is the generic hyperplane section, we have $L|_C \neq \mathcal{O}_C$.*

Proof. If we denote with $(\mathbb{P}^n)^\vee$ the projective dual projective space of $\mathbb{P}^n$, we can construct the generic hyperplane section $\mathcal{H} \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ as

$$\{(x, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee : x \in H\}.$$

We have the closed embedding $X \hookrightarrow \mathbb{P}^n$ which in turn gives the closed embedding $X \times (\mathbb{P}^n)^\vee \hookrightarrow \mathbb{P}^n \times (\mathbb{P}^n)^\vee$. We can construct the fibred product $\mathcal{C} := \mathcal{H} \times_{\mathbb{P}^n \times (\mathbb{P}^n)^\vee} X \times (\mathbb{P}^n)^\vee$. Observe that $\mathcal{C} \to \mathcal{H}$ is a closed embedding as well, since being a closed embedding is stable under base change. Moreover, $\mathcal{H} \to (\mathbb{P}^n)^\vee$ is proper. So the composition $\pi : \mathcal{C} \to (\mathbb{P}^n)^\vee$ is proper as well.

We understand the space $\mathcal{C}$ via its morphism $\pi : \mathcal{C} \to (\mathbb{P}^n)^\vee$: a fiber of π over the point of $(\mathbb{P}^n)^\vee$ corresponding to the hyperplane H is the intersection $H \cap X$, i.e. it is a hyperplane section in X. To check that the morphism π is flat, it suffices to check that all the fibers have the same Hilbert polynomial [Vak17, 24.7.A, (d)]. But for every hyperplane section H, we have an embedding $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}$, which gives the following exact sequence where $C := X \cap H$, since X is non-degenerate:

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$

Then by definition of Hilbert polynomial, we see that the Hilbert polynomial of C does not depend on H. Therefore all the fibers of π have the same Hilbert polynomial, so π is flat.

We can take the pull-back of L to $X \times (\mathbb{P}^n)^\vee$ and to $\mathcal{C}$ to get a line bundle G on $\mathcal{C}$ which along each fiber $C = H \cap X$ of π restricts to $L|_C$. From Proposition A.0.8, there is a smooth fiber F of π such that $G|_F \ncong \mathcal{O}_F$. We can replace $(\mathbb{P}^n)^\vee$ with the locus $U \subseteq (\mathbb{P}^n)^\vee$ where π is smooth (which is open, and contains the fiber F). Then Lemma A.0.10 applies, giving the desired result. $\square$

APPENDIX B

Curves and their Jacobians

Let C be a smooth proper connected complex algebraic curve of genus g. Let $(A_i, B_i)_{i=1}^g$ be a canonical basis for $H_1(C, \mathbb{Z})$, so that

$$A_i \cdot A_j = 0, B_i \cdot B_j = 0, A_i \cdot B_j = \delta_{ij},$$

where $\cdot$ is the intersection pairing on C. Let $(\omega_i)_{i=1}^g$ be the dual basis of $H^0(C, K_C)$:

$$\int_{a_i} \omega_j = \delta_{ij}.$$

We have the period map

$$\begin{aligned} H_1(C, \mathbb{Z}) &\hookrightarrow \mathbb{C}^g \\ \sigma &\mapsto \left(\int_\sigma \omega_i \right)_{i=1}^g , \end{aligned}$$

identifying $H_1(C,\mathbb{Z})$ with a lattice in $\mathbb{C}^g$, called the *period lattice*. The *Jacobian variety* of C is defined as

$$J(C) := \mathbb{C}^g / H_1(C,\mathbb{Z}).$$

If p_0 is a basepoint, we define the *Abel map*:

$$\begin{aligned} I : C &\to J(C) \\ p &\mapsto \left(\int_{p_0}^{p} \omega_i\right)_{i=1}^{g} \text{ modulo } H_1(C,\mathbb{Z}). \end{aligned}$$

The d-fold symmetric powers of C is defined as $C^{(d)} := C^d/S_d$, the quotient of C^d by the natural action of the symmetric group or equivalently, the set of degree d effective divisors on C. The Abel map naturally extends to:

$$\begin{aligned} I : C^{(d)} &\to J(C) \\ \sum_{i=1}^{d} p_i &\mapsto \sum_{i=1}^{d} (I(p_i)). \end{aligned}$$

Theorem B.0.1 (Abel's theorem)**.** *Two effective divisors D and D' of degree d on a smooth curve C are linearly equivalent if and only if $I(D) = I(D')$. Equivalently, the fibers of the Abel map are complete linear systems:*

$$I^{-1}(I(D)) = |D|.$$

Theorem B.0.2 (Jacobi inversion)**.** *Let C be a smooth curve of genus g. Then $I : C^{(g)} \to J(C)$ is surjective and birational. Therefore for a generic degree g effective divisor D, the complete linear system $I^{-1}(I(D)) = |D|$ is one dimensional.*

B.1 Prym varieties

For background on the material collected here, see [Fay73, Fay89, Mum1, Mum2, Taim97]. Let $\pi : \widehat{C} \to C$ be a ramified double covering of genus $\widehat{g}$ of a smooth curve of genus g with branch points q_1, q_2. By the Riemann-Hurwitz theorem, $\widehat{g} = 2g$. Let $\sigma : \widehat{C} \to \widehat{C}$ be the involution permuting the branches of the covering with fixed points at q_1, q_2 and let $x' = \sigma(x)$ denote the conjugate point of $x \in \widehat{C}$. We can choose a canonical homology basis for $H_1(\widehat{C}, \mathbb{Z})$

$$A_1, B_1, A_2, B_2, ..., A_{2g}, B_{2g},$$

such that $(\pi_*(A_i), \pi_*(B_i))_{i=1}^g$ is a basis for $H_1(C, \mathbb{Z})$ and such that

$$\sigma(A_k) + A_k = \sigma(B_k) + B_k = 0, \quad 1 \leq k \leq g.$$

If the dual basis of holomorphic differentials on $\widehat{C}$ is

$$u_1, ..., u_{2g},$$

then for $1 \leq k \leq g$ we have

$$\sigma^* u_k + u_{g+k} = 0.$$

A holomorphic differential ω on $\widehat{C}$ is called a *Prym differential* if $\sigma^*(\omega) + \omega = 0$. For $1 \leq k \leq g$,

$$\omega_k = \sigma^* u_k + u_k$$

is a basis for Prym differentials on $\widehat{C}$. Let Π be the matrix of periods of the Prym differentials around the b-cycles of $\widehat{C}$:

$$\Pi_{jk} = \int_{B_k} u_j.$$

The *Prym variety* $\mathrm{Pr}(\widehat{C}, \sigma)$ is defined to be

$$\frac{\mathbb{C}^g}{\mathbb{Z}^g + \Pi\mathbb{Z}^g}.$$

Let $J(\widehat{C}), J$ be the Jacobian of $\widehat{C}$ and let $I : \widehat{C} \to J(\widehat{C})$ be the Abel map with base-point $q_0 \in \widehat{C}$. The involution σ induces an involution $\sigma_* : J(\widehat{C}) \to J(\widehat{C})$: Given $\zeta \in J(\widehat{C})$, let $D \in \widehat{C}^{(2g)}$ (which exists by Jacobi inversion) such that $I(D) = \zeta$ and let $\sigma_*(\zeta) = I(\sigma(D))$. In coordinates, σ_* is given by

$$(z_1, ..., z_{2g}) \mapsto (-z_{g+1}, ..., -z_{2g}, -z_1, ..., -z_g).$$

The Prym variety is embedded by $\phi : \mathrm{Pr}(\widehat{C}, \sigma) \hookrightarrow J(\widehat{C})$:

$$(z_1, ..., z_g) \mapsto (z_1, ..., z_g, z_1, ..., z_g).$$

. We also have the projection $\pi_1 : J(\widehat{C}) \to \mathrm{Pr}(\widehat{C}, \sigma)$ given by

$$\pi_1(z_1, ..., z_{2g}) = (z_1 + z_{g+1}, ..., z_g + z_{2g}).$$

Define the Abel-Prym map with base-point q_1:

$$\begin{aligned} I_P &: \widehat{C} \to \mathrm{Pr}(\widehat{C}, \sigma) \\ &x \mapsto \left(\int_{q_1}^{x} \omega_1, ..., \int_{q_1}^{x} \omega_g\right) \text{ modulo } \mathbb{Z}^g + \Pi\mathbb{Z}^g, \text{ for } x \in \widehat{C}. \end{aligned}$$

Note that $I_P = \pi_1 \circ I$. Let $\eta(z)$ be the theta function on $\mathrm{Pr}(\widehat{C}, \sigma)$. Note that for $e \in \mathrm{Pr}(\widehat{C}, \sigma)$, we have

$$e = \frac{1}{2}\pi_1(\phi(e)).$$

Theorem B.1.1. *If $e \in Pr(\widehat{C}, \sigma)$, then either $\eta(I_P(x) - e) \equiv 0$ for all $x \in \widehat{C}$ or $div_{\widehat{C}}\eta(I_P(x) - e) = D$ is a degree $\widehat{g}$ effective divisor satisfying*

$$\phi(e) = I(D) - I(q_1) - I(q_2) - \pi^*\Delta_C \quad \textit{in } J(\widehat{C}),$$

where $\Delta_C \in J(C)$ is the vector of Riemann constants on C, and

$$D + \sigma(D) - q_1 - q_2 = K_{\widehat{C}},$$

where $K_{\widehat{C}}$ is the canonical class of $\widehat{C}$. Moreover, such a D is uniquely determined by these conditions.

Lemma B.1.2. *If $D \in \widehat{C}^{(\widehat{g})}$ such that*

$$D + \sigma(D) - q_1 - q_2 = K_{\widehat{C}},$$

then

$$I(D) - q_1 - q_2 - \pi^*\Delta_C \in \phi(Pr(\widehat{C}, \sigma)).$$

Proof. By definition $\phi(\mathrm{Pr}(\widehat{C},\sigma)) = \{\zeta \in J(\widehat{C}) : \sigma_*\zeta + \zeta = 0\}$. We have

$$\begin{aligned}
&I(D) - q_1 - q_2 - \pi^*\Delta_C + \sigma_*(I(D) - q_1 - q_2 - \pi^*\Delta_C)\\
&= I(D + \sigma(D)) - 2q_1 - 2q_2 - 2\pi^*\Delta_C\\
&= I(K_{\widehat{C}} - q_1 - q_2) - I(\pi^*K_C)\\
&= 0,
\end{aligned}$$

where we have used $\sigma_*\pi^*\Delta_C = \pi^*\Delta_C$, $I(K_C) = 2\Delta_C$ and $K_{\widehat{C}} = \pi^*K_C - q_1 - q_2$. $\square$

Let $E(x,y)$ denote the prime form on $\widehat{C}$. $E(x,y)$ has the symmetry $E(x,y) = E(x',y')$ for all $x,y \in \widehat{C}$. For a divisor $D = \sum_i a_i - \sum_j b_j$ on $\widehat{C}$, we define

$$E_D(x) := \frac{\prod_i E(x,a_i)}{\prod_j E(x,b_j)}.$$

It is a section of the line bundle $\mathcal{O}_{\widehat{C}}(D)$ with divisor D.

Theorem B.1.3 (Fay's quadrisecant identity [Fay89])**.** *Let $t \in Pr(\widehat{C},\sigma)$, $z \in \widehat{C}$ and suppose $x_k \in \widehat{C}$ for $k \in \mathbb{Z}/n\mathbb{Z}$.*

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\eta(t + I_P(z) - I_P(x_k) - I_P(x_{k+1}))}{\eta(t - I_P(x_k))\eta(t - I_P(x_{k+1}))} \frac{E(x_k,x_{k+1})}{E(x_k,x'_{k+1})} \frac{E(z,x'_k)E(z,x'_{k+1})}{E(z,x_k)E(z,x_{k+1})}\\
&\qquad = \frac{\eta\left(t - \sum_{i=1}^{k} I_P(x_k)\right)\eta(t + I_P(z))}{\prod_{k=1}^{n}\eta(t - I_P(x_k))} \prod_{k=1}^{n} \frac{E(x_k,x_{k+1})}{E(x_k,x'_{k+1})}.
\end{aligned}$$

Bibliography

[ACGH85] Arbarello E., Cornalba M., Griffiths P., Harris J. *Geometry of algebraic curves.*, Vol. I, volume 267 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985.

[BP11] Baryshnikov Y., Pemantle R. *Asymptotics of multivariate sequences, part III: quadratic points.* Adv. Math. 228 (2011), 3127-3206.

[BD87] Beauville A., Debarre O. *Sur le problème de Schottky pour les variétés de Prym.* Annali della Scuola Normale Super. di Pisa. Ser. IV. 14 (1987), 613–623. MR 89m:14014.

[BF18] Borodin A., Ferrari P. *Random tilings and Markov chains for interlacing particles.* Markov Processes Relat. Fields 24 (2018), no. 3.

[BH09] Borisov L., Hua Z. *On the conjecture of King for smooth toric Deligne-Mumford stacks.* Adv. Math., 221 (1) (2009), pp. 277-301.

[BdeTR17] Boutillier C., de Tilière B., Raschel K. *The Z-invariant massive Laplacian on isoradial graphs*, Invent. math. (2017) 208: 109.

[Bro12] Broomhead N. *Dimer models and Calabi-Yau algebras*, Mem. Amer. Math. Soc. 215 (2012), no 1011.

[CKP01] Cohn H., Kenyon R., Propp J. *A variational principle for domino tilings.* J. Amer. Math. Soc. 14 (2001), no. 2, 297-346.

[CdV94] Colin de Verdière Y. *Réseaux électriques planaires. I.* Comment. Math. Helv. 69 (1994), no. 3, 351-374.

[CIM98] Curtis E. B., Ingerman D., Morrow J. A. *Circular planar graphs and resistor networks.* Linear Algebra Appl. 283 (1998), no. 1-3, 115?150.

[CS04] Carroll G., Speyer D. *The cube recurrence.* Electr. J. Comb., 2004, volume 11.

[CT79] Cook R. J., Thomas A. D. *Line bundles and homogeneous matrices.* Quart. J. Math. Oxford (2), 30 (1979), 423–429.

[CLS11] Cox D. A., Little J. B., Schenck H. K. *Toric varieties.* (Graduate Studies in Mathematics). American Mathematical Society, Providence (2011).

[DFS14] Di Francesco P., Soto-Garrido R. *Arctic curves of the octahedron equation.* Journal of Physics A: Mathematical and Theoretical, Volume 47, Number 28 (2014).

[EKLP] Elkies N., Kuperberg G., Larsen M., Propp J. *Alternating sign matrices and domino tilings.* J. Algebraic Combin. 1 (1992), no. 2, 111–132; J. Algebraic Combin. 1 (1992), no. 3, 219-234.

[Fay73] Fay J. *Theta functions on Riemann surfaces.* Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin, 1973.

[Fay89] Fay J. *Schottky relations on* $\frac{1}{2}(C-C)$. Proc. Symp. in Pure Math, 49 Part 1, 485-502.

[F15] Fock V. *Inverse spectral problem for GK integrable system.* arXiv:1503.00289.

[FG03b] Fock V., Goncharov A.B. *Cluster ensembles, quantization and the dilogarithm.* Ann. Sci. L'Ecole Norm. Sup. (2009). arXiv: math.AG/0311245.

[FM16] Fock V., Marshakov A. *Loop Groups, Clusters, Dimers and Integrable Systems.* Geometry and Quantization of Moduli Spaces. Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser, Cham (2016).

[FZ01] Fomin S., Zelevinsky A. *Cluster algebras. I.* J. Amer. Math. Soc. 15 (2002), no. 2, 497–529. arXiv:math/0104151.

[GLVY16] Gao, Y., Li, Z., Vuong, T. D., Yang, L. *Toric Mutations in the dP_2 Quiver and Subgraphs of the dP_2 Brane Tiling.* arXiv:1611.05320.

[G18] George T. *Grove arctic curves from periodic cluster modular transformations.* arXiv:1711.00790, 2018.

[G19] George T. *Spectra of biperiodic planar networks.* arXiv:1901.06353, 2019.

[GGK] George T., Goncharov A. B., Kenyon R. *Dimers and the Beauville integrable system.* In preparation.

[GK12] Goncharov A. B., Kenyon R. *Dimers and cluster integrable systems.* Ann. ENS (2012).

[IU15] Ishii A., Ueda K. *Dimer models and the special McKay correspondence.* Geom. Topol. Volume 19, Number 6 (2015), 3405-3466.

[JPS] Jockusch W., Propp J. and Shor P. *Random domino tilings and the arctic circle theorem.* arXiv:math/9801068.

[Kas67] Kasteleyn P. *Graph theory and crystal physics.* 1967 Graph Theory and Theoretical Physics pp. 43–110 Academic Press, London.

[Kenn1899] Kennelly A.E. *Equivalence of triangles and stars in conducting networks*, Electrical World and Engineer, 34 (1899), 413-414.

[K02] Kenyon R. *The Laplacian and Dirac operators on critical planar graphs.* Invent. Math. 150 (2002), no. 2, 409?439.

[K09] Kenyon R. *Lectures on dimers.* Statistical mechanics, 191–230, IAS/Park City Math. Ser., 16, Amer. Math. Soc., Providence, RI, 2009.

[K10] Kenyon R. *Spanning forests and the vector bundle laplacian*, Ann. Probab. 39 (2011), no. 5, 1983?2017. arxiv:1001.4028.

[K12] Kenyon R. *The Laplacian on planar graphs and graphs on surfaces.* (2012), arXiv:1203.1256.

[K17] Kenyon R. *Determinantal spanning forests on planar graphs* , arXiv:1702.03802.

[KO03] Kenyon R., Okounkov A. *Planar dimers and Harnack curves.* Duke Math. J. 131 (2006), no. 3, 499–524. arXiv:math/0311062.

[KOS06] Kenyon R., Okounkov A., Sheffield S. *Dimers and amoebae.* Ann. Math. 163 , no. 3, 1019–1056.

[KO07] Kenyon R., Okounkov A. *Limit shapes and the complex Burgers equation.* Acta Math (2007), 199: 263.

[KPW00] Kenyon, R., Propp, J., Wilson, D.B. *Trees and matchings.* Electron. J. Combin. 7 (2000), Research Paper 25, 34 pp. (electronic).

[LM17] Lai T., Musiker G. *Beyond Aztec Castles: Toric Cascades in the dP_3 Quiver.* Comm. in Math. Phys. (2017), Volume 356, Issue 3, 823–881.

[LM19] Lai T., Musiker G. *Dungeons and Dragons: Combinatorics for the dP_3 Quiver.* arXiv:1805.09280.

[LP16] Lam T., Pylyavskyy P. *Laurent phenomenon algebras.* Cambridge Journal of Mathematics, 4(1):121–162, 2016.

[LMNT14] Leoni M., Musiker G., Neel S., Turner P. *Aztec Castles and the dP_3 Quiver.* J. Phys. A: Math. Theor. 47 (2014), 474011.

[Lop91] Lopez A. F. *Noether-Lefschetz theory and the Picard group of projective surfaces.* American Mathematical Soc., volume 438, 1991.

[Mikh00] Mikhalkin G. *Real algebraic curves, the moment map and amoebas*, Ann. of Math. (2) 151 (2000), no. 1, 309-326. arXiv:math/0010018.

[Miwa82] Miwa T. *On Hirota's difference equations.* Proc. Japan Acad. Ser A, (Math Sci), Volume 58, Number 1 (1982), 9-12.

[Mum74] Mumford, D. *Abelian varieties.* Volume 108, 1974, Oxford university press.

[Mum1] Mumford D. *Tata lectures on theta I.* Birkhäuser (2007).

[Mum2] Mumford D. *Tata lectures on theta II.* Birkhäuser (2007).

[Ost76] Ostrowski, A. M. *On multiplication and factorization of polynomials (I). Lexicographic orderings and extreme aggregates of terms* Aequationes Math., 14, 1976, pages 1–31.

[PS06] Petersen T., Speyer D. *An arctic circle theorem for Groves.* Journal of Combinatorial Theory, Series A. 111. 137-164. 10.1016/j.jcta.2004.11.013.

[PW02] Pemantle R., Wilson M. *Asymptotics of multivariate sequences, part I: Smooth points of the singular variety.* J. Combin. Theory Ser. A (2002), 97 (1), 129-161.

[PW04] Pemantle R., Wilson M. *Asymptotics of multivariate sequences, part II: Multiple points of the singular variety.* Combin. Probab. Comput. (2004) 13, pages 735–761.

[PW13] Pemantle R., Wilson M. *Analytic Combinatorics in Several Variables.* Cambridge University Press, Cambridge, UK (2013).

[Post06] Postnikov, A. *Total positivity, Grassmannians and networks.* arXiv:math/0609764.

[Propp01] Propp J. *The Many Faces of Alternating-Sign Matrices.* Discrete Mathematics and Theoretical Computer Science Proceedings AA (DM-CCG) (2001), 43-58.

[Propp03] Propp, J. *Generalized domino-shuffling.* Theoretical Computer Science, Volume 303, Issues 2–3, 2003, Pages 267-301, ISSN 0304-3975.

[Spey04] Speyer D. *Perfect matchings and the octahedron recurrence.* J. Algebraic Combin. 25 (2007), no. 3, 309–348. arXiv:math/0402452.

[stacks-project] The Stacks project authors, *The Stacks project.* `https://stacks.math.columbia.edu`, 2019.

[Sta16] Starr, J. *Restriction of the Picard group of a surface to a curve*, MathOverflow, `https://mathoverflow.net/q/233170(version:2016-03-09)`.

[Taim97] Taimanov I. A. *Secants of Abelian varieties, theta functions, and soliton equations*, Uspekhi Mat. Nauk, 52:1(313) (1997), 149–224; Russian Math. Surveys, 52:1 (1997), 147–218.

[TF61] Temperley W., Fisher M. *Dimer problem in statistical mechanics—an exact result.* Philos. Mag. (8) 6 (1961) 1061–1063.

[Thur04] Thurston D. *From dominos to hexagons.* arXiv:math/0405482.

[TWZ18] Treumann D., Williams H., Zaslow E. *Kasteleyn operators from mirror symmetry.* arXiv:1810.05985.

[Vak17] Vakil, R. *The rising sea: Foundations of algebraic geometry.* Preprint, 2017.