Elliptic Boundary Value Problems on Irregular Domains

by

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Vita

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In this thesis, we study the elliptic boundary value problems on irregular domains.

We first obtain the $W^{2,p}$ and C^2 regularity theories for second-order, non-divergence form elliptic equations with oblique derivative boundary conditions. In both cases, the boundary smoothness requirements are relaxed by one derivative. In particular, for the $W^{2,p}$ theory, the boundary is allowed to be Lipschitz with small constant, instead of the $C^{1,1}$ condition in the classical theory. For the C^2 theory, the boundary requirement is relaxed from $C^{2,Dini}$ to $C^{1,Dini}$.

The second topic is the optimal regularity of second-order, divergence form elliptic equations with mixed Dirichlet-conormal boundary conditions. The aim is to find the minimum assumptions on the boundary and the interfacial boundary between the two boundary conditions, such that the "optimal regularity" is achieved. For this, we first develop the approximation construction to deal with locally flat domains with locally flat interfacial boundaries. Such local flatness is commonly called "Reifenberg flat" in the literature. Based on the construction, we obtain the optimal $W^{1,4-\varepsilon}$ regularity of solutions with homogeneous boundary conditions. In a subsequent work, we further generalize our method to deal with interfacial boundaries which are locally close to Lipschitz graphs in m variables with respect to the Hausdorff distance, where $m = 0, \cdots, d-2$. In this direction, we obtain the optimal $W^{1,2(m+2)/(m+1)-\varepsilon}$ regularity. Furthermore, when the domain is also Lipschitz, we obtain the unique solvability of the Laplace equation with $L^q + W^{1,q}$ boundary data, where $q \in [1, (m+2)/(m+1))$.

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Chapter One

Introduction

For linear elliptic or parabolic equations with smooth coefficients on smooth domains, the regularity theory is well established, for example the Schauder theory and the Calderon-Zygmund theory. As it was illustrated in classical examples, such regularity can be false without proper smoothness assumptions. Here a core question is: to what extend can we weaken these smoothness requirements.

In this thesis, we focus on second-order elliptic equations of divergence form

$$D_i(a_{ij}D_ju + b_iu) + \hat{b}_iD_iu + cu = f + D_if_i$$

and non-divergence form

$$a_{ij}D_{ij}u + b_iD_iu + cu = f.$$

Throughout this thesis, we adopt Einstein's summation convention. For the elliptic operators, we can ask: what is the minimum smoothness assumptions on the leading coefficients a_{ij} such that certain interior regularity is achieved. For example, the Hölder continuity, $W^{1,p}/W^{2,p}$ regularity, and $C^{1,\alpha}/C^{2,\alpha}$ regularity together with its borderline case C^1/C^2 . Meanwhile, the problem becomes more complicated as the boundary conditions come into play. Typically we can impose Dirichlet, conormal, and oblique boundaries conditions for the equations above. These are

$$u = g, \quad (a_{ij}D_{ij}u + b_iu)n_i = f_in_i, \quad \beta_iD_iu + \beta_0u = g.$$

For boundary value problems, we can ask similar questions as before: what is the minimum assumptions on the boundary such that the previous interior regularity can be obtained up to the boundary. The second question is more involved: certain boundary conditions will put extra restrictions on the best possible regularity level that we can expect. Typical situations include the appearance of corners and the meeting of two different type of boundary conditions, etc. Such restrictions do not only bring us the extra technical difficulty, but also require us to formulate the problem more carefully. For example, we can assume extra geometric or compatibility conditions such that the restrictions can be removed. Or, we can ask for the "optimal" regularity without these extra conditions.

In this thesis, we explore these questions for two particular types of boundary value problems: the oblique derivative problem and the mixed Dirichlet-conormal problem.

1.1 Problem setup and main results

In this section, we introduce the basic setup of the two boundary value problems. Then we will present our main results. For oblique derivative problems, the $W^{2,p}$ and C^2 results can be found in Chapters 2 and 3, which are based on [28] and [29], respectively. For the mixed problems, we refer to [11] and [30], of which the contents compromise Chapters 4 and 5 in this thesis.

1.1.1 Oblique derivative problem

On a Lipschitz domain Ω, we consider the following problem:

$$\begin{cases} Lu := a_{ij}D_{ij}u + a_iD_iu + a_0u = f & \text{in } \Omega, \\ Bu := b_0u + b_iD_iu = g & \text{on } \partial\Omega. \end{cases}$$

Here L is assumed to be uniform elliptic with bounded coefficients. “Oblique” means that the vector field $\boldsymbol{\beta}$ is uniformly non-tangential at the boundary, which can be quantified in our setting as

$$\beta_i n_i \leq -\delta|\boldsymbol{\beta}| < 0\,,\ a.e. \quad \text{for some } \delta \in (0,1],$$

where $\boldsymbol{n} = (n_i)_{i=1}^d$ is the unit outer normal. Simple computation shows that in order to preserve the smoothness requirements of the coefficients, the flattening procedure requires $\partial\Omega$ to be $C^{1,1}, C^{2,Dini}$, and $C^{2,\alpha}$ for the $W^{2,p},, C^2$, and $C^{2,\alpha}$ regularity, respectively. However, Lieberman in [52] proved that

$$\partial\Omega \in C^{1,\alpha} \implies u \in C^{2,\alpha}, \forall \alpha \in (0,1).$$

Here the boundary requirement is relaxed by one derivative. We aim to prove similar results for the $W^{2,p}$ and C^2 theories. Let us mention some progress before us. In [53], the $W^{1,p}$ regularity was proved, assuming $\partial\Omega \in C^{1,\alpha}$ with $\alpha > 1-1/p$. In [27], the $C^2(\overline{\Omega})$ regularity was proved under the assumption $C^{1,Dini^2}$, where $Dini^2$ means the Dini integral of the modulus of continuity is still a Dini function. Both results relaxed the boundary smoothness requirement, compared to the smooth setting $\partial\Omega \in C^{1,1}$ and $C^{2,Dini}$. However, the relaxation has not reached one derivative yet. We improve these by allowing $\partial\Omega$ to be exactly one derivative less regular.

Theorem 1.1.1 (Chapter 2, Theorem 2.2.3 and 2.2.4)**.** *For any $p \in (1,\infty)$, there exist sufficiently small θ_0, ε_0, such that the following holds. If $\partial\Omega$ is $\delta^2\theta_0$-Lipschitz, a_{ij} is θ_0-BMO, $\boldsymbol{\beta} \in C^\alpha, \alpha > 1-1/p$, and proper sign condition holds, then we have the $W^{2,p}$ well-poseness: for any $f \in L^p(\Omega)$ and $g \in W^{1-1/p,p}(\partial\Omega)$, there exists a unique $W^{2,p}$ solution u satisfying*

$$\|u\|_{W^{2,p}(\Omega)} \lesssim \|f\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\partial\Omega)}.$$

Here ε-Lipschitz means Lipschitz domain with the Lipschitz constant smaller than ε, and θ-BMO means the coefficients locally have BMO semi-norm smaller than θ. For the C^2-theory, we have

Theorem 1.1.2 (Chapter 3, Theorem 3.2.4). *Suppose that $\partial\Omega$, $\boldsymbol{\beta}$, and $g \in C^{1,Dini}$, a_{ij}, b_i, c and f L^1-mean Dini, then any $W^{2,p}(\Omega)$ solution with $p > 1$ is in $C^2(\overline{\Omega})$.*

The corresponding results also hold for concave fully nonlinear elliptic equations. See Theorem 2.6.2 and Theorem 3.2.7.

Besides the relaxation of the boundary requirement, here for the C^2 theory we assume the coefficients to be merely of L^1-mean Dini, which is weaker than the usual borderline case Schauder's theory assumption: $C^{0,Dini}$. Notice that we have the following inclusion:

$$C^{0,Dini} \subsetneq L^1 - meanDini \subsetneq \text{uniform continuous.}$$

Such mean Dini condition was first obtained in [24] for the interior estimate.

1.1.2 Mixed Dirichlet-conormal problem

Consider a domain Ω with its boundary divided into two non-intersecting, non-empty, and connected boundary portions $\partial\Omega = \mathcal{N} \cup \mathcal{D}$. Let $\Gamma = \overline{\mathcal{N}} \cap \overline{\mathcal{D}}$ be their interfacial boundary. We consider the following mixed boundary value problem

$$\begin{cases} Lu := D_i(a_{ij}D_ju + b_iu) + \hat{b}_iD_iu + cu = f + D_if_i & \text{in } \Omega, \\ Bu := (a_{ijD_ju+b_iu})n_i = f_in_i + g_{\mathcal{N}} & \text{on } \mathcal{N}, \\ u = g_{\mathcal{D}} & \text{on } \mathcal{D}. \end{cases} \tag{1.1}$$

Our goal is to find the minimum assumptions on Ω and Γ for the following regularity problems.

Problem 1(H): for (1.1) with general variable coefficients, but $g_{\mathcal{N}} = g_{\mathcal{D}} = 0$ and $f, f_i \in L^{4-\varepsilon}$, we want the solution $u \in W^{1,4-\varepsilon}$.

Problem 2(E): for (1.1) with $L = \Delta$, $B = \partial/\partial \boldsymbol{n}$, $f = f_i = 0$, $g_{\mathcal{N}} \in L^{2-\varepsilon}(\partial\Omega)$, and $g_{\mathcal{D}} \in W^{1,2-\varepsilon}(\partial\Omega)$, we want $\boldsymbol{N}(Du) \in L^{2-\varepsilon}(\partial\Omega)$.

Both regularity levels above are optimal, due to the classical example $u = \mathrm{Im}(x_1 + ix_2)^{1/2}$. In 2(E), $\boldsymbol{N}$ is the non-tangential maximal operator.

Definition 1.1.3. For $X \in \partial\Omega$ and $f \in L^1_{loc}(\Omega)$, we define the non-tangential region with opening β, the non-tangential maximal function as:

$$\Gamma_\beta(X) := \{Y \in \Omega : |X - Y| \le (1-\beta)\,\mathrm{dist}(Y, \partial\Omega)\}, \quad \boldsymbol{N}(f)(X) := \sup_{Y \in \Gamma_\beta(X)} |f(Y)|.$$

Since we bound the non-tangential function, the boundary condition in Problem 2(E) should be understood in the sense of the non-tangential limit. It is worth mentioning that, the boundary data here lies in $L^{2-\varepsilon} + W^{1,2-\varepsilon}$, which is typically of higher regularity than the trace spaces for the usual $W^{1,p}$ weak solution, which is $W^{-1/p,p} + W^{1-1/p,p}$.

Now we state our first main result regarding problem 1(H). Roughly speaking, in order to reach the optimal $W^{1,4-\varepsilon}$ regularity, we need $\partial\Omega$ and Γ to be locally flat (small Reifenberg flat).

Theorem 1.1.4 (Chapter 4, Theorem 4.2.4 and 4.2.5)**.** *For any $p \in (4/3, 4)$, there exist $\gamma_0, \theta_0 > 0$, such that, if Ω, Γ are (R_0, γ_0)-flat, $a_{ij} \in \theta_0$-BMO, and proper sign condition holds, then for any $f, f_i \in L^p(\Omega)$, there exists a unique $W^{1,p}(\Omega)$-weak solution to (1.1) with $g_{\mathcal{N}} = g_{\mathcal{D}} = 0$, satisfying*

$$\|u\|_{W^{1,p}(\Omega)} \lesssim \|f_i\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}$$

Here the (R_0, Γ_0)-flatness is commonly called Reifenberg flat or locally flat in the literature, of which the formal definition is long. Roughly speaking, it means that at all small scale ($R < R_0$), the boundary can be trapped between two parallel disks with distance $2\gamma_0 R$. In other words, boundaries of such domains are locally close to hyperplanes, not in the sense of regularity, but in the sense of Hausdorff distance. Such domains can be very rough. Typically it can contain a fractal structure, and typical examples include the "nearly flat" Koch snowflake.

Let us now briefly introduce the history and make some comparison. For problem 1(H) on rough domains, there are no comparable results on the mixed problem. Our Theorem 1.1.4 is a generalization of the $W^{1,p}$ regularity of purely Dirichlet or conormal problem in in [20] and [21] on Reifenberg flat domains with "partial VMO" coefficients—coefficients that are measurable in the "almost normal" direction and VMO in the others. Noting that neither the Reifenberg flat domains not the partial VMO structures allow change of variables, hence non-flattening techniques are needed. Intuitively, on small Reifenberg flat domains, we can "freeze" the boundary to be a hyperplane at every small scale, by designing a proper approximation problem on half space. Compared to [20] and [21], the main difficulty here is the appearance of the "transition region" near Γ where the two boundary conditions meet. For this, we design a cut-off function such that its gradient is supported on a "L-shaped" domain and then apply a reflection technique to "freeze" both $\partial\Omega$ and Γ simultaneously.

The next main result is for problem 2(E). Roughly speaking, if $\partial\Omega$ is Lipschitz and locally flat, and Γ is locally flat, then the Laplace equation is solvable for any $L^{2-\varepsilon} + W^{1,2-\varepsilon}$ boundary data.

Theorem 1.1.5 (Chapter 5, Theorem 5.2.11)**.** *Suppose that Ω is a bounded Lipschitz domain. For any $q \in (1,2)$, we can find some small constant $\gamma_0 > 0$, such that if Ω, Γ are (R_0, γ_0)-flat, then for any $g_{\mathcal{N}} \in L^q(\mathcal{N})$, $g_{\mathcal{D}} \in W^{1,q}(\mathcal{D})$, there exists a unique solution u satisfying*

$$\|\boldsymbol{N}(Du)_{L^q(\mathcal{N})}\| \lesssim \|g_{\mathcal{N}}\|_{L^q(\mathcal{N})} + \|g_{\mathcal{D}}\|_{W^{1,q}(\mathcal{D})}.$$

Following the study of Neumann problems on Lipschitz domains in [36], the corresponding estimate for the mixed problem was raised as an open problem in [37], due to the failure of the usual L^2-method. Initiated in [3], one approach is to assume an additional geometric assumption on the domain, such that the optimal regularity of $\boldsymbol{N}(Du)$ is above L^2 so that the previous method still can be applied. Such geometric assumption clearly excludes smooth domains (due to the previous counter example). There is another approach starting from the L^1 solvability with boundary data in Hardy spaces. In this direction, the best result so far is in [73], where the unique L^q solvability with q close to 1 was obtained. They assumed Ω to be any Lipschitz domain and Γ to be very rough: merely Ahlfors regular of Hausdorff dimension close to $d-2$, combined with the so-called "corkscrew conditions" on $\mathcal{D}$. Notice that such assumptions are implied by our small Reifenberg flat condition: we assume stronger conditions such that the optimal range $q < 2$ is achieved. Meanwhile, due to simple variation of the previous example, certain flatness is required for the optimal regularity. In this sense, our assumptions are sharp.

Actually in [30], we prove a more general result: the separation Γ is allowed to be a perturbation (still measured by the Hausdorff distance) of a Lipschitz graph in m variables, $m \in [0, d-2]$. The optimal range of q varies with m: when $m = d-2$,we generalize the result in [4] by allowing irregular $\partial\Omega$ and Γ. For this result, see Theorem 5.2.11 in Chapter 5.

1.2 A brief overview of ideas

In this section, we present some important ideas and methods used in this thesis.

1.2.1 Distances and decomposition

On smooth enough (at least C^2) domains, we have a C^2-distance function near the boundary. Such function is useful, for example, in constructing barrier functions and in flattening the boundary, etc. While on rough domains, the usual distance function is not regular enough. In this thesis, two different tools are used to describe a point near the boundary.

For oblique derivative problems, we use the regularized distance in the same spirit with [49]. Such distance functions can be constructed for Lipschitz graph domains, by properly mollifying the original distance function. They are smooth inside the domain, have higher derivatives with proper growth rates, and preserve important properties of the original representation functions (e.g., the smallness of gradient in Section 2.3 and certain modulus of continuity in Section 3.3.1). For one construction of such a regularized distance function, see Appendix.

For mixed problems, we are dealing with domains of which the boundaries are not local graphs—Reifenberg flat domains. For this, we take a different approach: the Whitney decomposition, i.e. locally decompose the domain into cubes. Each of these cubes has radius proportional to its distance to the boundary. Such decomposition is used in two ways. First, in Section 4.7 we use such decomposition to extend $W^{1,p}$ functions beyond certain part of the boundary. The second is about dealing with the transition region for the mixed problem near the interfacial boundary in Section 5.5, where a weighted estimate is obtained.

1.2.2 Study of the mean oscillation

One important object in studying elliptic regularity is the mean oscillation:

$$\phi(x,r) := \fint_{B_r(x)} |f - (\fint_{B_r(x)} f)|.$$

In this thesis, we work on the mean oscillation in the following two ways.

An innovated iteration. According to the Campanato's characterization of Hölder continuous functions,a function is Hölder continuous if and only if the mean oscillation decays like r^{α}. This is done by the celebrated Campanato's iteration. In Chapter 3, we prove the C^2-regularity by applying an innovated

Campanato's type iteration. We adapt the method to deal with mean Dini functions instead of Hölder continuous ones. As we have seen before, the mean Dini assumption is strictly weaker than the uniform Dini, and further strictly weaker than the Hölder continuity. Also, here an L^q version of mean oscillation is studied with $q < 1$. The idea of such iteration was first introduced in [16].

Sharp function. In [41], Krylov introduced a new method for the L^p estimate, which is based on the so-called "sharp function" estimate, the Fefferman- Stein theorem, and the celebrated Hardy-Littlewood theorem. The sharp function is simply the function $\sup_r \phi(x, r)$, and the Fefferman-Stein theorem proves the L^p-equivalence between any L^p function and its sharp function. Such method has many advantages. For example, it gives a unified proof of the L^p estimates for both parabolic and elliptic equations with VMO-in-x coefficients. Besides, this method is very suitable for the L^p estimate for fully nonlinear equations: one could simply "interpolate" the Krylov-Evans $C^{2,\alpha}$ estimate and the $W^{2,\varepsilon}$ estimate in the spirit of [56]. One such estimate is obtained in Section 2.6.

1.2.3 Level set argument and "crawling of the ink spot"

When dealing with the mixed problems in Chapters 4 and 5, one major difficulty is that the solution which we perturb from is not regular enough. More precisely, the perturbation argument requires us to "interpolate" between the $W^{1,2}$ and $W^{1,4-\varepsilon}$ spaces. Our strategy is to derive the decay rate for the measure of the level set of the Hardy-Littlewood maximal function $|\{\mathcal{M}(Du) > \lambda\}|$.

The key step in deriving the decay rate is to transform it into a local, scaling invariant estimate. This is done by using a stopping time argument which was introduced by Krylov and Safonov in [44], where they called it "crawling of the ink spot". See Section 4.5.2 for a detailed explanation and history. See also the papers [8, 74].

Chapter Two

$W^{2,p}$ estimate for oblique derivative problem in Lipschitz domains

2.1 Introduction

In this chapter, we consider W_p^2 estimate for oblique derivative problem. For the problem

$$\begin{cases} Lu := a_{ij}D_{ij}u + a_iD_iu + a_0u = f & \text{in } \Omega, \\ Bu := b_0u + b_iD_iu = g & \text{on } \partial\Omega, \end{cases}$$

we aim to prove

$$\|u\|_{W_p^2(\Omega)} \le N(\|f\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)} + \|u\|_{L_p(\Omega)}).$$

Here L is uniformly elliptic, and oblique derivative means for some $\delta \in (0,1]$ $b \cdot n \ge \delta|b|$ almost everywhere. Here n is the outer normal direction, which is defined almost everywhere on $\partial\Omega$.

As the W_p^2 estimate for elliptic equations with uniformly continuous coefficients in smooth (say $C^{1,1}$) domains has been well studied for a long time (see e.g., [34, Theorem 9.13]), people are more interested in the case of discontinuous coefficients or rough domains.

Concerning discontinuous coefficients, the case when a_{ij} belongs to the class of vanishing mean oscillation (VMO) is of particular interest. We say a function $f \in$ VMO if

$$\omega(\rho) := \sup_{x, 0<r<\rho} \fint_{B_r(x)} |f(y) - (f)_{B_r(x)}|\, dy \to 0 \text{ as } \rho \to 0,$$

where $(f)_{B_r(x)} := \frac{1}{|B_r|}\int_{B_r(x)} f(y)\, dy$ is the average of f in $B_r(x)$.

The W_p^2 estimate for equations with VMO coefficients was first established by Chiarenza, Frasca, and Longo in [9]. The method is mainly based on the representation formula: the Calderón-Zygmund theorem together with a commutator estimate. For the oblique derivative problem, we refer the reader to [13, 58]. Later in [41], based on the so-called "sharp function" estimate and the Fefferman-Stein theorem, Krylov gave a unified proof of the W_p^2 estimate for parabolic/elliptic equations with VMO-in-x coefficients. Furthermore, using this method, it is possible to relax the regularity assumptions on a_{ij}. Here we mention the following "partially VMO" condition for elliptic equations, which is extremely useful in discussing boundary value problems. Such "partially VMO" can be written as:

$$\omega'(\rho) := \sup_{x, 0<r<\rho} \fint_{x^d-r}^{x^d+r} \fint_{B_r'(x')} |f(y', y^d) - (f)_{B_r'(x')}(y^d)| \, dy' dy^d \to 0, \text{ as } \rho \to 0,$$

where $x = (x', x^d)$ and $B_r'(x')$ is $(d-1)$-dimension ball. Such theory was developed in [38] by Kim and Krylov for $p > 2$, and in [17] by the first author here for all $p \in (1, \infty)$. It is worth noting that in [17], a more general regularity assumption called "hierarchically partially VMO" was discussed.

Such results allow us to consider Dirichlet or Neumann problems with VMO coefficients in the half space $\mathbb{R}^d_+$: by simple extension and reflection we get equations in $\mathbb{R}^d$ with "partially-VMO" coefficients. Based on this, in [38] a W_p^2 estimate for the oblique derivative problem in $\mathbb{R}^d_+$ is also discussed via a perturbation argument. It turns out that the perturbation argument requires $b \in C^\alpha, \alpha > 1 - 1/p$. For details and history of discussing equations with "partially VMO" coefficients, one may refer to [18, 42].

In this chapter, we will focus on general bounded domain Ω and its regularity assumptions regarding W_p^2 estimate. First noting that, by flattening the boundary, the aforementioned W_p^2 estimate in the half space will simply lead to corresponding results in $C^{1,1}$ domains. This is because a $C^{1,1}$ change of variables will preserve all the regularity assumptions on the elliptic operator L and the boundary condition. Also, W^2_p norms under these two coordinates are comparable.

For the oblique derivative problem, however, the smoothness assumption for Ω can be relaxed. The idea is to consider an extension problem in curved domain, which will reduce the boundary condition to be homogeneous. This will compensate the lack of regularity in our change of variables when flattening the boundary. In this way, Lieberman reduced the assumption to $\Omega \in C^{1,\alpha}$, $\alpha > 1 - 1/p$ in [53, 55]. Here, with the help of Hardy's inequality, we employ a new idea of extension. Together with a perturbation argument, we get the W^2_p estimate in any small Lipschitz domain, i.e., domain with local representation function having sufficiently small Lipschitz constant.

We would like to mention that, there is also a "Schauder type" $C^{2,\alpha}$ estimate for oblique derivative problems in $C^{1,\alpha}$ domains. Such result was obtained by Lieberman in [52]. One could notice that our result is in the same spirit: the regularity assumption on $\partial\Omega$ is one derivative less than the corresponding Dirichlet problem. Later in [66], Safonov came up with an alternative proof for this problem. His proof also includes an extension problem, which actually motivates us of this result. It is worth noting that our perturbation argument in proving Theorem 2.2.3 can replace Theorem 2.1 in [66], which is to find a $C^{2,\alpha}$ diffeomorphism mapping $b \cdot D$ to $\frac{\partial}{\partial y^d}$. This can also be used to simplify the proof in [66].

The structure of the chapter is as follows. In Section 2.2 we first introduce the basic setups and notation, and then present our main result of the W^2_p estimate in Theorem 2.2.3. We state the corresponding result regarding the existence and uniqueness of W^2_p solutions in Theorem 2.2.4. Next in Section 2.3 we introduce the cylindrical neighborhood and a special choice of orthonormal systems which we are going to work with for the boundary estimate. Then as a preparation, we introduce the regularized distance which is a useful tool for rough boundaries. With all these, the proof of our main result Theorem 2.2.3 is given in Section 2.4. The solvability result Theorem 2.2.4 is proved in Section 2.5. One advantage of our proof is that it also works for nonlinear equations with proper convexity conditions. We will prove one such result for Bellman equations in Section 2.6.

2.2 Notation and main result

Let $d > 1$ be a positive integer. In this chapter, we denote a point x in $\mathbb{R}^d$ by $x = (x^1, x^2, \dots, x^d) = (x', x^d)$. Also, we denote

$$\mathbb{R}^d_+ = \{x^d > 0\}, \quad B^+_R = B_R \cap \mathbb{R}^d_+, \quad \text{and } B'_R = \{x' \in \mathbb{R}^{d-1} : |x'| < R\}.$$

Besides the usual partial derivative symbol $\frac{\partial}{\partial x^i}$, we use the following notation:

$$D_i u = u_{x^i}, \quad D_{ij} u = u_{x^i x^j}.$$

We write W^k_p for Sobolev spaces, i.e., functions themselves and all derivatives up to order k lie in L_p, and $\mathring{W}^1_p(\Omega) = \overline{C^\infty_c(\Omega)}$ under the W^1_p norm. For fractional Sobolev spaces $W^s_p, s \in (0,1)$, the interpolation definition is used. Notice that $W^{1-1/p}_p(\partial\Omega)$ is exactly the trace space of $W^1_p(\Omega)$ for C^1 (or small Lipschitz) domain. For simplicity, in this chapter sometimes we write

$$\|\cdot\|_p = \|\cdot\|_{L_p}, \quad \|\cdot\|_{p,\Omega} = \|\cdot\|_{L_p(\Omega)}, \quad \|\cdot\|_{1,p} = \|\cdot\|_{W^1_p}, \quad \text{etc.}$$

The summation convention, for instance,

$$b \cdot D = b_i D_i := \sum_{i=1}^{d} b_i D_i,$$

is adopted throughout this chapter.

For a Lipschitz continuous function f, denote

$$[f]_1 := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

for its Lipschitz constant. We write $C^{k,1}$ for the class of k-th order continuously differentiable functions with all k-th order derivatives being Lipschitz continuous.

In this chapter, we will use the following notation for the average:

$$(f)_{B_r(x)} = \fint_{B_r(x)} f(y)\, dy := \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy.$$

When proving inequalities, we will use N for the absolute constant (to be more specific, independent of the local radius parameter r or R in this chapter). In the middle steps, we will omit the dependence of N on Ω, a_{ij}, b_i, etc. Also, N may vary from line to line.

The main purpose of this chapter is to derive the W^2_p estimate in small Lipschitz domains. Let us first give the formal definition of an ε_0-Lipschitz domain.

Definition 2.2.1 (ε_0-Lipschitz domain)**.** A bounded domain $\Omega \subset \mathbb{R}^d$ is said to be ε_0-Lipschitz if for any $x_0 \in \partial\Omega$, there exist $R_0 > 0$ independent of x_0, an orthonormal coordinate system $x = (x', x^d)$ centered at x_0, and a Lipschitz function ψ_0 such that

$$\Omega \cap B_{R_0}(x_0) = \{x \in B_{R_0}(0) : x^d < \psi_0(x')\},$$

$$|\psi_0(x'_1) - \psi_0(x'_2)| < \varepsilon_0 |x'_1 - x'_2|. \tag{2.1}$$

Since Lipschitz function is almost everywhere differentiable, (2.1) is equivalent to

$$|D\psi_0(x')| < \varepsilon_0 \quad \text{a.e.} \tag{2.2}$$

Notice that this definition is given in a natural choice of coordinate system, where the x^d-axis is chosen to be close to the normal direction of $\partial\Omega$ at x_0. In the next section, we will give a coordinate system adapted to the oblique derivative boundary condition, which is more convenient to work with.

The next part gives basic conditions which will be assumed throughout this chapter. For a second-order elliptic equation

$$Lu := a_{ij} D_{ij} u + a_i D_i u + a_0 u = f,$$

we assume

$$a_{ij} \in L_\infty, \quad a_{ij} = a_{ji}, \quad \nu|\xi|^2 \le a_{ij}\xi_i\xi_j \le \nu^{-1}|\xi|^2, \tag{2.3}$$

$$a_i, a_0 \in L_\infty(\Omega), \tag{2.4}$$

where $\nu \in (0, 1]$ is a constant.

The following small BMO assumption will be assumed for a_{ij}, where θ is a constant to be specified later.

Assumption 2.2.2 (r_0, θ)**.** For a constant $\theta > 0$, there exists some $r_0 > 0$, such that

$$\sup_{x, 0<r<r_0} \fint_{B_r(x)\cap\Omega} |a_{ij}(y) - (a_{ij})_{B_r(x)\cap\Omega}| \, dy < \theta.$$

In this paper, we consider the following oblique derivative boundary condition:

$$Bu := b_0 u + b_i D_i u = g \quad \text{on } \partial\Omega. \tag{2.5}$$

As usual, this is understood in the sense of trace. For the coefficients b_0 and b_i, we assume

$$b_0, b_i \in C^\alpha(\partial\Omega), \quad \text{where } \alpha > 1 - \frac{1}{p}, \tag{2.6}$$

$$b \cdot n \geq \delta |b| \quad \text{a.e.} \quad \text{for some } \delta \in (0,1]. \tag{2.7}$$

Here (2.7) represents the obliqueness of the vector field $b = (b_i)_{i=1}^d$ and n is the unit outer normal direction. If $\partial\Omega$ locally is represented by a Lipschitz function $y^d = \psi(y')$, we can write n in terms of $D\psi$:

$$n = \frac{(D\psi, -1)}{\sqrt{|D\psi|^2 + 1}} \quad \text{a.e.} \tag{2.8}$$

Now we state our main result.

Theorem 2.2.3 (W_p^2 estimate in small Lipschitz domain)**.** *In a Lipschitz domain Ω (bounded or unbounded), suppose $u \in W_p^2(\Omega)$ solves*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ Bu = g & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Assume assumptions (2.3), (2.4), (2.6), and (2.7) hold, $f \in L_p(\Omega)$, and $g \in W_p^{1-1/p}(\partial\Omega)$. There exist $\theta_0 = \theta_0(d, p, \nu) > 0$ and $\varepsilon_0 = \varepsilon_0(d, p, \nu, \|a_i, a_0\|_\infty) > 0$ small enough, such that if Assumption 2.2.2$(r_0, \delta\theta_0)$ is satisfied, and the domain Ω is $\delta^2\varepsilon_0$-Lipschitz, then we have,

$$\|u\|_{W_p^2(\Omega)} \leq N\Big(\|u\|_{L_p(\Omega)} + \|f\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)}\Big). \tag{2.10}$$

Here $N = N(d, p, \nu, \|a_i, a_0\|_\infty, \|b_i, b_0\|_\alpha, r_0, R_0, \delta)$ is a constant. In particular, the result holds when $\Omega \in C^1$ and $a_{ij} \in VMO$.

As an application we have the corresponding solvability result. For this, we also need the following conditions:

$$a_0 \leq 0, \quad b_0 \geq 0 \quad \text{a.e.} \tag{2.11}$$

$$a_0 \not\equiv 0 \text{ or } b_0 \not\equiv 0, \tag{2.12}$$

$$\alpha > \max\{1 - 1/d, 1 - 1/p\}, \tag{2.13}$$

where α is the Hölder exponent of b.

Theorem 2.2.4. *Besides assumptions in Theorem 2.2.3 regarding Ω, L, and B, we also assume conditions (2.11)-(2.13) to hold and Ω to be bounded. Then there exist $\theta_0 = \theta_0(d, p, \nu) > 0$ and $\varepsilon_0 = \varepsilon_0(d, p, \nu, \|a_i, a_0\|_\infty) > 0$ small enough, such that, if Assumption 2.2.2 $(r_0, \delta\theta_0)$ is satisfied, and the domain Ω is $\delta^2\varepsilon_0$-Lipschitz, we have that for any $f \in L_p(\Omega)$, $g \in W_p^{1-1/p}(\Omega)$, there exists a unique solution $u \in W_p^2(\Omega)$ of (2.9) with*

$$\|u\|_{W_p^2(\Omega)} \leq N(\|f\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)}). \tag{2.14}$$

Here N is a constant independent of u.

Remark 2.2.5. Our perturbation argument still works if the regularity assumption

(2.6) is replaced by the following: for $1 \le i \le d$,

$$b_i \in \begin{cases} W^1_p(\Omega), \text{ if } p > d, \\ W^1_q(\Omega) \text{ for some } q > d, \text{ if } p = d, \\ W^1_d(\Omega), \text{ if } p < d, \end{cases} \tag{2.15}$$

and

$$b_0 \in \begin{cases} W^1_p(\Omega), \text{ if } p > d/2, \\ W^1_q(\Omega) \text{ for some } q > d/2, \text{ if } p = d/2, \\ W^1_{d/2}(\Omega), \text{ if } p < d/2. \end{cases} \tag{2.16}$$

Here $d \ge 2$ is the space dimension.

The following example in $\mathbb{R}^2$ shows that the regularity assumption (2.15)-(2.16) is sharp. Consider $\Omega = \{(x, y) : x > |y|^{1+\varepsilon}\}$, $u = (x|y|^\beta + y)\eta_R$, where $\beta \in (0, 1)$, $\varepsilon > 0$ are some constants to be determined later, η is some smooth cutoff function supported in a ball B_R, and equals to 1 in $B_{R/2}$. Direct calculation shows that if we choose

$$\max\{1/2 - (2 + \varepsilon)/p, 1 - \varepsilon + (2 + \varepsilon)/p\} < \beta \le 1 - (2 + \varepsilon)/p,$$

and

$$b = (-1, |y|^\beta),$$

then,

$$u \in L_p(\Omega), \quad \Delta u \in L_p(\Omega), \quad Bu = b \cdot Du \in W^1_p(\Omega), \quad D_{12}u \notin L_p(\Omega).$$

Hence (2.10) cannot be true in this case. Notice that (2.15) is violated since we only have $b \in W^1_q$ for $q < 2/(1 - \beta) < p$.

Remark 2.2.6. To see the importance of the small Lipschitz condition, we give the following example which is also in $\mathbb{R}^2$. We use the polar coordinates (r, θ). Let $\theta_0 \in (\pi/2, \pi)$ be a fixed angle. Consider the wedge domain

$$\Omega^{\theta_0} = \{(r, \theta) : -\theta_0 < \theta < \theta_0\} \quad \text{and} \quad \Gamma^{\theta_0} = \{\theta = -\theta_0, \theta_0\}.$$

Define $u = \operatorname{Im}\{e^{i(\alpha_0-1)\theta_0} z^{\alpha_0}\}\eta_R$, where $z = re^{i\theta}$, α_0 is some constant to be determined later, and η_R is the cutoff function in Remark 2.2.5. Noting that the opposite of the x-direction is oblique on Γ^{θ_0}, direct computation shows that on $\Gamma^{\theta_0} \cap B_{R/2}$,

$$\frac{\partial u}{\partial x} = \alpha_0 r^{\alpha_0 - 1} \sin\Big((\alpha_0 - 1)(\theta_0 + \theta)\Big).$$

Hence, if we choose $b = (-1, 0)$ and $\alpha_0 = \pi/(2\theta_0) + 1 \in (3/2, 2)$, we have $Bu = 0$ on $\Gamma^{\theta_0} \cap B_{R/2}$. Now $u \in L_p$ for all p, $\Delta u = 0$, but for $p \geq \frac{2}{2-\alpha_0}$,

$$r^{\alpha_0 - 2} \lesssim |D^2 u| \notin L_p.$$

2.3 Cylindrical neighborhood and regularized distance

In this paper, local properties near the boundary will be intensively studied. Rather than the coordinate system coming with Definition 2.2.1, it is more convenient to use the following coordinates $y = (y', y^d)$ which depend on the boundary condition (2.5). Also, it is more convenient to work with the following "cylindrical" neighborhood rather than the "half ball" neighborhood in Definition 2.2.1.

Consider an ε_0-Lipschitz domain Ω, $x_0 \in \partial\Omega$ where $D\psi_0$ exists, and a vector field b which satisfies (2.7) at x_0. We first take a rotation of the coordinates in Definition 2.2.1 to make y^d-axis lie in $b(x_0)$ direction. By (2.7), $b(x_0)$ is non-tangential at the point x_0. Taking (2.2) into account, locally $\partial\Omega$ is still a graph :$y^d = \psi(y')$. Here ψ can be obtained from ψ_0 by the implicit function theorem. The small Lipschitz condition (2.2) now can be written in terms of (y', y^d) and ψ as the smallness of the oscillation of $D\psi$. Notice that due to the rotation we will introduce a constant factor $1/\delta^2$ in front. To be specific, in Theorem 2.2.3 we assume Ω to be $\delta^2\varepsilon_0$-Lipschitz, i.e.

$$|D\psi_0(x')| < \delta^2\varepsilon_0 \quad \text{a.e. } x' \in B'_{R_0}.$$

Straightforward computation gives us, if $\varepsilon_0 < 1/8$, in the new coordinates

$$|D\psi(y') - D\psi(z')| < 3\varepsilon_0 \quad \text{a.e. } y', z' \in B'_{\delta R_0}. \tag{2.17}$$

Now due to the expression of n in terms of $D\psi$ (2.8), the obliqueness condition (2.7) at the point $x_0 = (0, \psi(0))$ can be written as:

$$|D\psi(0)| \le \sqrt{1/\delta^2 - 1}.$$

Choosing $\varepsilon_0 < 1/3$, we have

$$|D\psi| < 2/\delta \quad \text{a.e. in } B'_{\delta R_0}. \tag{2.18}$$

We further shift the y'-coordinate plane so that $x_0 = (0, \frac{3}{\delta}R)$. Here $R < \delta R_0$ is a radius parameter to be chosen small later. Due to (2.18), we have

$$\forall y' \in B'_R(0), \quad \frac{1}{\delta}R < \psi(y') < \frac{5}{\delta}R. \tag{2.19}$$

The following is the "cylindrical" neighborhoods in which the local properties will be studied:

$$\Omega_R(x_0) := \{(y', y^d) : y' \in B'_R(0), 0 < y^d < \psi(y')\},$$

$$\Gamma_R(x_0) := \{(y', y^d) : y' \in B'_R(0), y^d = \psi(y')\}.$$

$$Q_R(x_0) := \{(y', y^d) : y' \in B'_R(0), 0 < y^d < \frac{6}{\delta}R\},$$

The center x_0 will be omitted when there is no ambiguity.

The second part of this section is a useful tool for rough boundaries (say, worse than C^2). This is the regularized distance introduced by Lieberman in [49]. For our problem, we modify Theorem 2.1 in [49] to adapt to small Lipschitz domains.

Theorem 2.3.1 (Local regularized distance for small Lipschitz domain)**.** *Let Ω be a bounded Lipschitz domain with local representation ψ satisfying* (2.17) *and* (2.18) *on $\Omega_{4R}(x_0)$, $x_0 \in \partial\Omega$. Then there exists $\rho_0 \in C^\infty(Q_R \setminus \Gamma_R) \cap C^{0,1}(Q_R)$, such that:*

$$\text{there exists a constant } M > 0,\ |D\rho_0| < M,\ M^{-1} < \frac{\rho_0}{d_y} < M \text{ in } Q_R \setminus \Gamma_R, \tag{2.20}$$

$$\text{where } d_y = \begin{cases} dist(y, \partial\Omega) & y \in \Omega_R \\ -dist(y, \partial\Omega) & y \in Q_R \cap (\overline{\Omega_R})^c \end{cases},$$

$$|D\rho_0(y) - D\rho_0(z)| \leq 12\varepsilon_0 \quad \text{in } Q_R, \tag{2.21}$$

$$|D^2\rho_0(x)| \leq N\frac{\varepsilon_0}{|\rho_0(x)|} \quad \text{in } Q_R \setminus \Gamma_R. \tag{2.22}$$

Here $N > 0$ is an absolute constant, and $M = M(\delta)$.

Proof. We follow the steps in [49] for proving Theorems 1.1, 1.3, and 2.1. A sketch of the proof can be found in the appendix. ∎

In the above theorem, (2.20) means ρ_0 is a local distance function. "Regularized" refers to the fact that this distance is C^∞ in the interior. The expression (2.21) is the small Lipschitz condition for ρ_0. The ε_0 in (2.22) is important in our proof.

Regularized distance can work as a suitable function to flatten the boundary, since it is smooth in the interior with a suitable growth rate of higher order derivatives near the boundary. Besides, one could also use it for mollification:

$$\widetilde{g}(x) =: \int_{\Omega_R} g(x - \frac{\rho_0(x)}{M_1} y)\phi(y)\, dy.$$

The advantage is: besides $\|\widetilde{g}\|_{W_p^1} \lesssim \|g\|_{W_p^1}$, we have also nice control of $D^2\widetilde{g}$. Details will be given in Lemma 2.4.3 and in the proof of Theorem 2.4.5.

2.4 Proof of Theorem 2.2.3

Now we are going to prove Theorem 2.2.3. Our proof is divided into three steps. First we deal with the following model problem with a simple boundary condition.

Lemma 2.4.1. *Consider a cylindrical neighborhood together with its top boundary Ω_R, Γ_R, and the representation function ψ as in Theorem 2.3.1. Here we take $R < \frac{\delta R_0}{4}$. Assume $u \in W_p^2(\Omega_R)$ solves*

$$\begin{cases} Lu = f & \text{in } \Omega_R, \\ \frac{\partial u}{\partial y^d} = 0 & \text{on } \Gamma_R. \end{cases}$$

There exist $\theta_0 = \theta_0(d, p, \nu) > 0$ and $\varepsilon_0 = \varepsilon_0(d, p, \nu, \|a_i, a_0\|_\infty) > 0$ small enough, such that if Assumption 2.2.2 holds with $(r_0, \delta\theta_0)$, and (2.17) holds with $3\varepsilon_0$, then for any $r < R$, we have

$$\|D^2u\|_{p,\Omega_{r/2}} \le N(r^{-2}\|u\|_{p,\Omega_r} + \|f\|_{p,\Omega_r}).$$

Here $N = N(d, p, \nu, \|a_i, a_0\|_\infty, r_0, \delta)$ is a constant.

Proof. In Ω_R, we flatten the boundary using the regularized distance in Theorem 2.3.1. In other words, we take the change of variables $z = \Phi(y)$: $z' = y'$, $z^d = \rho_0(y)$. This maps curved boundary Γ_R to a flat portion of $\{z^d = 0\}$.

Write $\widetilde{u}(z) = u(y(z))$. In the z variables, the equation can be written as

$$\begin{cases} \widetilde{a_{ij}}D^z_{ij}\widetilde{u} + \widetilde{a_i}D^z_i\widetilde{u} + \widetilde{a_0}\widetilde{u} = \widetilde{f} & \text{in } \Phi(\Omega_R) \subset \mathbb{R}^d_+, \\ \frac{\partial\widetilde{u}}{\partial z^d} = 0 & \text{on } \Phi(\Gamma_R) \subset \{z^d = 0\}, \end{cases}$$

where

$$\widetilde{f} = f - a_{ij}D_{ij}\rho_0\frac{\partial\widetilde{u}}{\partial z^d},$$

and $\widetilde{a_i}(z) = a_k(y)\frac{\partial z^i}{\partial y^k}, \widetilde{a_0}(z) = a_0(y) \in L_\infty$. Noting $D\rho_0$ has small L_∞-oscillation (2.21), we can choose ε_0 and θ_0 small enough, to make $\widetilde{a_{ij}} = a_{kl}\frac{\partial z^i}{\partial y^k}\frac{\partial z^j}{\partial y^l}$ have as small BMO semi-norm as we want.

Now we apply the W^2_p estimate for second-order elliptic equations with small BMO coefficient in half space and zero Neumann boundary condition. For this, first one could find in [17] such result in $\mathbb{R}^d$. To deal with the Neumann boundary condition, we just take even extension for u and f, and correspondingly for $\widetilde{a_{ij}}$ as in [38]. Noting that the extended equation has small partially BMO coefficients, this gives us the global W^2_p estimate for small BMO coefficient in half space with zero Neumann data. One last thing to mention is that here the small BMO assumption is given in terms of the z variables, when translating back to the y variables we will have a δ factor in front due to the stretching in the change of variables that map balls to ellipses.

Localizing and using a dilation argument, we have for any $t/2 \le s < t \le R$:

$$\|D_z^2\widetilde{u}\|_{p,s} \le N\Big((t-s)^{-2}\|\widetilde{u}\|_{p,t} + \|\widetilde{f}\|_{p,t}\Big), \tag{2.23}$$

where $N = N(d, p, \nu, \|a_i, a_0\|_\infty, r_0)$. Here we used the abbreviation $\|\cdot\|_{p,s} := \|\cdot\|_{L_p(\Phi(\Omega_s))}$.

We are left to estimate $\|\widetilde{f}\|_{L_p}$. For this, we use the property of regularized distance (2.22):

$$|D^2\rho_0(x)| \le N\frac{\varepsilon_0}{|\rho_0(x)|}.$$

Combining this and Hardy's inequality, we obtain

$$\Big\|D_{ij}\rho_0\frac{\partial\widetilde{u}}{\partial z^d}\Big\|_p \le N\varepsilon_0\Big\|\frac{1}{\rho_0}\frac{\partial\widetilde{u}}{\partial z^d}\Big\|_p \le N\varepsilon_0\Big\|\frac{1}{z^d}\frac{\partial\widetilde{u}}{\partial z^d}\Big\|_p \le N\varepsilon_0\|D_z^2\widetilde{u}\|_p. \tag{2.24}$$

Here we used $z^d \lesssim \mathrm{dist}(y, \Gamma_R) \lesssim \rho_0$. Substituting into (2.23), we get

$$\|D_z^2\widetilde{u}\|_{p,s} \le N\Big((t-s)^{-2}\|\widetilde{u}\|_{p,t} + \|f\|_{p,t}\Big) + N\varepsilon_0\|a_{ij}\|_\infty\|D_z^2\widetilde{u}\|_{p,t}. \tag{2.25}$$

Choosing ε_0 small enough, such that $N\varepsilon_0\|a_{ij}\|_\infty < \frac{1}{5}$, we can use iteration argument to absorb $\|D_z^2\widetilde{u}\|_p$. Indeed, consider a sequence of balls $\{B_{r_k} : r_k = r - 2^{-k-1}r, k = 0, 1, \ldots\}$. Using (2.25) with $s = r_k, t = r_{k+1}$ and summing in k, we have

$$\sum_{k=0}^{\infty} 5^{-k}\|D_z^2\widetilde{u}\|_{p,r_k} \le \sum_{k=0}^{\infty}\Big(N\cdot 4^{k+2}\cdot 5^{-k}r^{-2}\|\widetilde{u}\|_{p,r} + 5^{-k}\|f\|_{p,r} + 5^{-k-1}\|D_z^2\widetilde{u}\|_{p,r_{k+1}}\Big).$$

This gives us:

$$\|D_z^2\widetilde{u}\|_{p,\Phi(\Omega_{r/2})} \le N(r^{-2}\|\widetilde{u}\|_{p,\Phi(\Omega_r)} + \|f\|_{p,\Phi(\Omega_r)}).$$

Now we get the desired estimate, but in the z variables. If we change back to our original y variables, we get similar singular terms, i.e., the term with $D^2\rho_0$:

$$D^y_{ij} = \frac{\partial z^k}{\partial y^i}\frac{\partial z^l}{\partial y^j}D^z_{kl} + \frac{\partial^2 z^k}{\partial y^i \partial y^j}D^z_k = \frac{\partial z^k}{\partial y^i}\frac{\partial z^l}{\partial y^j}D^z_{kl} + \frac{\partial^2 \rho_0}{\partial y^i \partial y^j}D^z_d.$$

As in (2.24), we can prove

$$\|D^y_{ij}u\|_{L_p(\Omega_{r/2})} \leq N\|\widetilde{u}\|_{W^2_p(\Phi(\Omega_{r/2}))}.$$

Notice that again the boundary condition $\frac{\partial \widetilde{u}}{\partial z^d} = 0$ is used when applying Hardy's inequality. ∎

The next part deals with an extension theorem. Our construction uses similar idea to [66]. Before we start, let us first formally introduce the mollification which we have mentioned in the previous section.

Consider $g \in W^{1-1/p}_p(\partial\Omega)$. First we extend g to the interior in the usual way,

$$E : g \in W^{1-1/p}_p(\partial\Omega) \mapsto E(g) \in W^1_p(\Omega) \tag{2.26}$$

with $\|E(g)\|_{W^1_p(\Omega)} \lesssim \|g\|_{W^{1-1/p}_p(\partial\Omega)}$. For simplicity, we will not distinguish $E(g)$ from g in the following.

Now we can give the definition of our mollification.

Definition 2.4.2 (Mollification using regularized distance)**.** Suppose Ω is a bounded domain with small Lipschitz property (2.17). If the regularized distance

ρ_0 is defined on Ω_{2R}, we can define in Ω_R:

$$\widetilde{g}(y) := \int g\Big(y - \frac{\rho_0(y)}{M_1} w\Big)\phi(w)\, dw.$$

Here,

$$\phi \in C_c^\infty(B_1) \text{ with } \phi > 0, \int \phi = 1,$$

$$M_1 := \max\Big\{\frac{3}{\delta}M, 2\|D\rho_0\|_{L_\infty}\Big\}.$$

Recall in (2.20), M is a constant such that $\rho_0(y) \leq M d_y$, where d_y is the distance to the boundary.

Clearly, $\widetilde{g} \in C^\infty(\Omega_R)$. We also have the following,

Lemma 2.4.3. *Let $g \in W_p^1(\Omega)$. Consider Ω, Ω_R with $R < \frac{\delta R_0}{8}$ and $\widetilde{g}$ defined as above. Then we have,*

$$\|\widetilde{g}\|_{L_p(\Omega_R)} \leq N\|g\|_{L_p(\Omega_{2R})}, \tag{2.27}$$

$$\|D\widetilde{g}\|_{L_p(\Omega_R)} \leq N\|Dg\|_{L_p(\Omega_{2R})}. \tag{2.28}$$

Here $N = N(p)$ is a constant.

Proof. See Appendix. Note that because of (2.20), our choice of M_1 guarantees $y - \frac{\rho_0(y)}{M_1} w \in \Omega_{2R}$ for any $y \in \Omega_R$ and $w \in B_1$. ■

For the extension problem, we need the following inequality which is dual to Hardy's inequality.

Lemma 2.4.4. *For any $h \in L_p(0,1)$ where $p \in [1,\infty)$, we have*

$$\Big\| \int_0^x \frac{1}{1-t} h(t)\,dt \Big\|_{L_p(0,1)} \le N(p)\|h\|_{L_p(0,1)}.$$

Again the constant N only depends on p.

Proof. We prove by a duality argument. For any $\|\eta\|_{L_{p'}(0,1)} = 1$ where p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\begin{aligned} \int_0^1 \eta(x) \int_0^x \frac{1}{1-t} h(t)\,dt\,dx &= \int_0^1 h(t) \frac{1}{1-t}\Big(\int_t^1 \eta(x)\,dx \Big)\,dt && (2.29)\\ &\le \|h\|_{L_p(0,1)} \Big\| \frac{1}{s} \int_0^s \eta(1-y)\,dy \Big\|_{L_{p'}(0,1)} && (2.30)\\ &\le N(p)\|h\|_{L_p(0,1)}\|\eta\|_{L_{p'}(0,1)}. && (2.31)\end{aligned}$$

Here we used Fubini's theorem in (2.29), Hölder's inequality in (2.30), and Hardy's inequality in (2.31) noting that $p' > 1$. ∎

Now we can state our extension theorem.

Theorem 2.4.5. *Let Ω, Ω_R defined as before, $R < \frac{\delta R_0}{8}$, $g \in W_p^{1-1/p}(\partial\Omega)$. Then we can find $v \in W_p^2(\Omega_R)$, such that*

$$\begin{cases} \frac{\partial v}{\partial y^d} = g \quad \text{on } \Gamma_R, \\ \|v\|_{W_p^2(\Omega_R)} \le N(\|g\|_{W_p^{1-1/p}(\Gamma_{3R})} + R^{-1+1/p}\|g\|_{L_p(\Gamma_{3R})}), \end{cases}$$

where $N = N(\delta, p)$ is a constant.

Proof. Let $\widetilde{g}$ defined as in Definition 2.4.2. We define our extension as

$$v = \int_0^{y^d} \widetilde{g}(y', t)\, dt \quad \text{for } y' \in B_R',\ 0 < y^d < \psi(y').$$

Since $\widetilde{g}\big|_{\Gamma_R} = g$, we have $\frac{\partial v}{\partial y^d} = g$ on Γ_R. We first estimate $\|v\|_{L_p}$:

$$\|v\|_{L_p(\Omega_R)} \leq \left\| \frac{\psi}{y^d} \int_0^{y^d} \widetilde{g} \right\|_{L_p(\Omega_R)} \lesssim \frac{R}{\delta} \|g\|_{L_p(\Omega_{2R})}. \tag{2.32}$$

Here to get the last inequality, we used (2.19), Hardy's inequality, and (2.27). Similarly,

$$\begin{aligned} \|Dv\|_{L_p(\Omega_R)} &\leq \|\widetilde{g}\|_{L_p(\Omega_R)} + \left\| \int_0^{y^d} D_{y'} \widetilde{g} \right\|_{L_p(\Omega_R)} \\ &\lesssim \|g\|_{L_p(\Omega_{2R})} + \frac{R}{\delta} \|D_{y'} g\|_{L_p(\Omega_{2R})}. \end{aligned}$$

Now we estimate $D^2 v$. Noting that $\frac{\partial^2}{\partial y^i \partial y^d} v = \frac{\partial}{\partial y^i} \widetilde{g}$, we have

$$\|DD_d v\|_{L_p(\Omega_R)} \leq \|D\widetilde{g}\|_{L_p(\Omega_R)} \lesssim \|Dg\|_{L_p(\Omega_{2R})}.$$

We are left to estimate $\frac{\partial^2}{\partial y^i \partial y^j} v$ with $i, j < d$. By the chain rule,

$$\begin{aligned} &\frac{\partial^2}{\partial y^i \partial y^j} v = \int_0^{y^d} \frac{\partial^2}{\partial y^i \partial y^j} \widetilde{g}(y', t)\, dt \\ &= \int_0^{y^d} \frac{\partial^2}{\partial y^i \partial y^j} \int g\Big((y', t) - \frac{\rho_0(y', t)}{M_1} w\Big) \phi(w)\, dw\, dt \\ &= \int_0^{y^d} \frac{\partial}{\partial y^i} \int \Big[(D_j g)\Big((y', t) - \frac{\rho_0}{M_1} w\Big) + (-\frac{w^k}{M_1} D_j \rho_0)(D_k g)\Big((y', t) - \frac{\rho_0}{M_1} w\Big)\Big] \phi(w)\, dw\, dt. \end{aligned}$$

Now we make a change of variables $w \mapsto z$:

$$z = (y', t) - \frac{\rho_0(y', t)}{M_1} w.$$

In the following, for simplicity we omit the dependence of ρ_0 on (y', t) and Dg on z since there will be no ambiguity. Then we have

$$\begin{aligned}
\frac{\partial^2}{\partial y^i \partial y^j} v &= \int_0^{y^d} \frac{\partial}{\partial y^i} \int \Big(D_j g(z) - \frac{\big((y',t)-z\big)_k}{\rho_0} D_j \rho_0 D_k g(z) \Big) \phi\Big(\frac{(y',t)-z}{\rho_0} M_1\Big)\Big(\frac{M_1}{\rho_0}\Big)^d \, dz\, dt \\
&= \int_0^{y^d} \int \Big(\frac{-\delta_{ik}}{\rho_0} D_j \rho_0 D_k g + \frac{w_k}{M_1} \frac{D_i \rho_0 D_j \rho_0}{\rho_0} D_k g - \frac{w_k}{M_1} D_{ij} \rho_0 D_k g \Big) \phi(w)\, dw\, dt \\
&\quad + \int_0^{y^d} \int \Big(D_j g - \frac{w_k}{M_1} D_j \rho_0 D_k g \Big)\Big(D_i \phi \frac{M_1}{\rho_0} - D_k \phi \cdot w_k \frac{D_i \rho_0}{\rho_0} \Big)\, dw\, dt \\
&\quad + \int_0^{y^d} \int \Big(D_j g - \frac{w_k}{M_1} D_j \rho_0 D_k g \Big) \phi(w) \Big(- d \frac{D_i \rho_0}{\rho_0} \Big)\, dw\, dt.
\end{aligned}$$

From Theorem 2.3.1 we know that $D\rho_0$ is bounded, and $\frac{\partial^2 \rho_0}{\partial y^i \partial y^j} \lesssim \frac{1}{\rho_0}$ (no smallness is needed here). Also noting that ϕ has compact support in B_1, we have,

$$\Big| \int_0^{y^d} \frac{\partial^2}{\partial y^i \partial y^j} \widetilde{g}(y', t)\, dt \Big| \le N \Big(\int_0^{y^d} \frac{1}{\rho_0} \int |Dg((y', t) - \frac{\rho_0(y', t)}{M_1} w)| (|\phi| + |D\phi|)\, dw\, dt \Big).$$

Now, using properties of the regularized distance ρ_0, noting that $\rho_0(y', t), d_{(y',t)}$, and $\psi(y') - t$ all characterize the distance to the boundary, we have:

$$\frac{1}{\rho_0(y', t)} \lesssim \frac{1}{\psi(y') - t}.$$

Then,

$$\begin{aligned}
&\Big\| \int_0^{y^d} \frac{\partial^2}{\partial y^i \partial y^j} \widetilde{g}(y', t)\, dt \Big\|_{L_p(\Omega_R)}^p \\
&\le N \int_{B'_R} \int_0^{\psi(y')} \Big| \int_0^{y^d} \frac{1}{\psi(y') - t} \int_{B_1} \Big| Dg\Big((y', t) - \frac{\rho_0(y', t)}{M_1} w\Big) \Big| (|\phi| + |D\phi|)\, dw\, dt \Big|^p dy^d\, dy' \\
&\le N \int_{B'_R} \Big\| \int_{B_1} \Big| Dg\Big((y', \cdot) - \frac{\rho_0(y', \cdot)}{M_1} w\Big) \Big| (|\phi| + |D\phi|)\, dw \Big\|_{L_p((0, \psi(y')))}^p dy' \qquad (2.33)
\end{aligned}$$

$$\leq N \int_{B'_{2R}} \|Dg(y',\cdot)\|^p_{L_p((0,\psi(y')))}\,dy' \tag{2.34}$$
$$= N\|Dg\|_{L_p(\Omega_{2R})}.$$

The inequality (2.33) follows from Lemma 2.4.4 and a dilation argument with the help of (2.19). We used the Minkowski inequality to prove (2.34), which is similar to the proof of (2.27) in Appendix.

Finally, to get the estimate with only local boundary norms as in our lemma, we use a localization argument. Consider $\eta \in C_c^\infty(Q_{3R})$ with $\eta = 1$ in Q_{2R}, $D\eta \lesssim 1/R$. We have

$$\|E(\eta g)\|_{W^1_p(\Omega_{2R})} \leq N\|\eta g\|_{W_p^{1-1/p}(\partial\Omega)} \leq N(\|g\|_{W_p^{1-1/p}(\Gamma_{3R})} + R^{-1+1/p}\|g\|_{L_p(\Gamma_{3R})}).$$

Here E is the extension operator defined in (2.26). Replacing g by $E(\eta g)$ in the proof above, we reach the desired inequality. The theorem is proved. ■

Now, we have all the required ingredients for proving Theorem 2.2.3.

Proof of Theorem 2.2.3. By interpolation, we only need to prove

$$\|D^2u\|_{L_p(\Omega)} \leq N(\|f\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)} + \|u\|_{W^1_p(\Omega)}).$$

We first give a boundary estimate in $\Omega_{\delta R_0/8}(x_0), x_0 \in \partial\Omega$. We apply Theorem 2.4.5 with R replaced by $r < \frac{\delta R_0}{8}$, and g replaced by

$$h := g - \sum_{i=1}^{d}(b_i - b_i(x_0))D_iu - b_0u.$$

We find $v \in W^2_p(\Omega_r)$ such that $\frac{\partial v}{\partial y^d} = h$ on Γ_r with the following:

$$\begin{aligned}
\|v\|_{W^2_p(\Omega_r)} \leq & N(\|h\|_{W^{1-1/p}_p(\Gamma_{3r})} + r^{-1+1/p}\|h\|_{L_p(\Gamma_{3r})}) \\
\leq & N\Big((1 + r^{-1+1/p})\|g\|_{W^{1-1/p}_p(\Gamma_{3r})} + (1 + r^{-1+1/p})\|b_0 u\|_{W^{1-1/p}_p(\Gamma_{3r})} \\
& + \|(b_i - b_i(x_0))D_i u\|_{W^{1-1/p}_p(\Gamma_{3r})} + r^{-1+1/p}\|(b_i - b_i(x_0))D_i u\|_{L_p(\Gamma_{3r})}\Big) \\
\leq & N(r)\Big(\|g\|_{W^{1-1/p}_p(\Gamma_{3r})} + \|b_0\|_{C^\alpha(\Gamma_{3r})}\|u\|_{W^{1-1/p}_p(\Gamma_{3r})}\Big) + N\|b_i\|_{C^\alpha(\partial\Omega)}\|Du\|_{L_p(\Gamma_{3r})} \qquad (2.35) \\
& + \|b_i - b_i(x_0)\|_{L_\infty(\Gamma_{3r})}\|Du\|_{W^{1-1/p}_p(\Gamma_{3r})} + N(r)\|b_i\|_{L_\infty(\Omega)}\|Du\|_{L_p(\Gamma_{3r})}\Big) \\
\leq & N(r)\left(\|g\|_{W^{1-1/p}_p(\Gamma_{3r})} + \|u\|_{W^1_p(\Omega_{4r})}\right) + Nr^\alpha\|b_i\|_{C^\alpha}\|u\|_{W^2_p(\Omega_{4r})}. \qquad (2.36)
\end{aligned}$$

Here we only write down the dependence $N = N(r)$ explicitly, and omit the dependence on $d, p, \nu, \|b_i\|_{C^\alpha}$, etc. In (2.35) we used the inequality

$$\|fg\|_{W^{1-1/p}_p} \lesssim \|f\|_{C^\alpha}\|g\|_{L_p} + \|f\|_{L_\infty}\|g\|_{W^{1-1/p}_p},$$

provided that $\alpha > 1 - 1/p$. From (2.32) we also have the following estimate for lower order terms:

$$\|v\|_{L_p(\Omega_r)} \lesssim \frac{r}{\delta}\|h\|_{L_p(\Omega_{2r})} \lesssim r\|g\|_{L_p(\Omega_{2r})} + r^{1+\alpha}\|Du\|_{L_p(\Omega_{2r})} + r\|u\|_{L_p(\Omega_{2r})}. \qquad (2.37)$$

Now, $u - v$ solves

$$\begin{cases} L(u - v) = f - Lv & \text{in } \Omega_r \\ \frac{\partial(u-v)}{\partial y^d} = 0 & \text{on } \Gamma_r. \end{cases}$$

We apply Lemma 2.4.1 to obtain

$$\|D^2(u - v)\|_{p,\Omega_{r/2}} \leq N(r^{-2}\|u - v\|_{p,\Omega_r} + \|f - Lv\|_{p,\Omega_r}).$$

Then,

$$\begin{aligned}
\|D^2u\|_{p,\Omega_{r/2}} \leq & \|D^2v\|_{p,\Omega_{r/2}} + N(r^{-2}\|u\|_{p,\Omega_r} + r^{-2}\|v\|_{p,\Omega_r} + \|f\|_{p,\Omega_r} + \|Lv\|_{p,\Omega_r}) \\
\leq & N(r)(\|u\|_{p,\Omega_r} + \|f\|_{p,\Omega_r} + \|v\|_{p,\Omega_r}) + N\|v\|_{W_p^2(\Omega_r)} \\
\leq & N(r)\left(\|u\|_{W_p^1(\Omega_{4r})} + \|f\|_{L_p(\Omega_r)} + \|g\|_{W_p^{1-1/p}(\Gamma_{3r})}\right) \\
& + Nr^{\alpha}\|u\|_{W_p^2(\Omega_{4r})}.
\end{aligned} \tag{2.38}$$

Here we applied (2.36) and (2.37) to get (2.38). This gives us a local boundary estimate. Combining this and the interior W_p^2 estimate in [9, 17], we get the global estimate (2.10) using a standard partition of unity argument. For this, we cover $\bar{\Omega}$ with one interior portion and finitely many boundary "half balls" $\Omega_{r/2}$. Finally, we get

$$\|u\|_{W_p^2(\Omega)} \leq N_1(\|u\|_{L_p(\Omega)} + \|f\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)}) + N_2N_3r^{\alpha}\|u\|_{W_p^2(\Omega)}. \tag{2.39}$$

Here N_1, N_2 are both constants depending on $d, p, \nu, \|a_i, a_0\|_\infty, \|b_i, b_0\|_{C^\alpha}, r_0, \delta$, and N_1 also depends on r. The constant N_3 depends on the ratio "$\frac{\text{covering times by } \Omega_{3r}}{\text{covering times by } \Omega_{r/2}}$" which can be bounded regardless of r. Now we are left to choose r small enough such that $r < \min\{(\frac{1}{2MN_2})^{1/\alpha}, \frac{\delta R_0}{8}\}$ to absorb the last term in (2.39) into the left-hand side. ■

2.5 Application: solvability

In this section, we give the proof of Theorem 2.2.4. For this we first remove $\|u\|_{L_p}$ from the right-hand side of (2.10) for the operator $L - \lambda$ with λ large enough. We use a classical argument which can be found in [34] and [55]. As a result, in

Corollary 2.5.2 we get the W^2_p well-posedness for large λ.

Lemma 2.5.1. *Under the assumptions of Theorem 2.2.3, we can find λ_0 depending on $d, p, \nu, \|a_i, a_0\|_\infty, \|b_i, b_0\|_\alpha, r_0, R_0, \delta$ large enough, such that for any $\lambda \geq \lambda_0$ and $u \in W^2_p(\Omega)$ solving*

$$\begin{cases} (L-\lambda)u = f & \text{in } \Omega, \\ Bu = g & \text{on } \partial\Omega, \end{cases} \tag{2.40}$$

we have

$$\|u\|_{W^2_p(\Omega)} + \lambda\|u\|_{L_p(\Omega)} \leq N\Big(\|f\|_{L_p(\Omega)} + \|g\|_{W^{1-1/p}_p(\partial\Omega)} + \sqrt{\lambda}\|g\|_{L_p(\Omega)}\Big). \tag{2.41}$$

Here $N = N(d, p, \nu, \|a_i, a_0\|_\infty, \|b_i, b_0\|_\alpha, r_0, R_0, \delta)$ is a constant.

Proof. First, notice that when proving Theorem 2.2.3, actually we have proved a slightly stronger result:

Suppose Ω is a bounded domain with a "small Lipschitz" portion $T \subset \partial\Omega$, and $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \Omega \cup T$. Then

$$\|u\|_{W^2_p(\Omega')} \leq N\Big(\|u\|_{L_p(\Omega)} + \|Lu\|_{L_p(\Omega)} + \|Bu\|_{W^{1-1/p}_p(T)}\Big). \tag{2.42}$$

Introduce a new space variable x^{n+1}, and let $v := u(x)\cos(\sqrt{\lambda}x^{n+1})$. On $\Sigma := \Omega \times (-1, 1), T := \partial\Omega \times (-1, 1)$, v satisfies

$$\begin{cases} \mathcal{L}v := (L + D_{n+1,n+1})v = \cos(\sqrt{\lambda}x^{n+1})f(x) & \text{in } \Sigma, \\ \mathcal{B}v = \cos(\sqrt{\lambda}x^{n+1})Bu = \cos(\sqrt{\lambda}x^{n+1})g(x) & \text{on } T. \end{cases}$$

Applying (2.42) with Σ, T, and $\Sigma' := \Omega \times [-1/2, 1/2]$, we get

$$\|v\|_{W^2_p(\Sigma')} \leq N\Big(\|v\|_{L_p(\Sigma)} + \|\mathcal{L}v\|_{L_p(\Sigma)} + \|\mathcal{B}v\|_{W^{1-1/p}_p(T)}\Big)$$

$$\leq N\Big(\|u\|_{L_p(\Omega)} + \|(L-\lambda)u\|_{L_p(\Omega)} + \|Bu\|_{W_p^{1-1/p}(\partial\Omega)} + \sqrt{\lambda}\|Bu\|_{L_p(\Omega)}\Big).$$

Notice that $D_{n+1,n+1}v = -\lambda v$, and we can find some $C > 0$ independent of λ such that

$$C < \|\cos(\sqrt{\lambda}\cdot)\|_{L_p(-1/2,1/2)} < \|\cos(\sqrt{\lambda}\cdot)\|_{L_p(-1,1)} < 2^{1/p}, \quad \forall\lambda.$$

Then,

$$\|v\|_{W_p^2(\Sigma')} \geq C(\|u\|_{W_p^2(\Omega)} + \lambda\|u\|_{L_p(\Omega)}).$$

Substituting back and choosing λ large enough such that $C\lambda > N$, we get (2.41). ∎

Corollary 2.5.2. *Under the assumptions of Theorem 2.2.3 with ε_0 being further smaller and $\lambda > \lambda_0$ as in Lemma 2.5.1, there exists a unique W_p^2 solution to* (2.40)*.*

Proof. Uniqueness is clear from the coercive estimate (2.41). We will focus on the existence.

First, noting that if $\partial\Omega$ is smooth (say, $C^{1,1}$), the a priori estimate (2.41) immediately gives us the solvability: one can first solve

$$\begin{cases} (\Delta - \lambda)u = f & \text{in } \Omega, \\ u + \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega, \end{cases}$$

where n is the outer normal direction, then use the method of continuity. Such argument and results can be found in [59].

For Ω with the small Lipschitz property, we approximate from the interior by $\{\Omega_k \in C^{1,1}\}_k \uparrow \Omega$. Moreover, we can require that all the Ω_k are $C\delta^2\varepsilon_0$-Lipschitz, where C is a universal constant. Due to this and the continuity of b, we may further require that $b \cdot n_k \geq |b|\delta/2$ for all k, where n_k is the unit outer normal direction of

Ω_k. Now solve in $W_p^2(\Omega_k)$ for

$$\begin{cases} (L-\lambda)u_k = f & \text{in } \Omega_k, \\ Bu_k = g & \text{on } \partial\Omega_k. \end{cases}$$

Here to make sense of the boundary condition, we need to extend g which is only given on the boundary to $W_p^1(\Omega)$ as the operator E defined in Section 2.4. Notice that since the constant N in (2.41) depends on the regularity of Ω only through its Lipschitz bound and the radius in the small Lipschitz property, then $\{\|u_k\|_{W_p^2(\Omega_k)}\}_k$ are uniform bounded. We can use the following argument to get a subsequence $u_{k_i} \to u$ weakly in $W_p^2(\Omega)$.

For Ω_1, noting that $\{\|u_k\|_{W_p^2(\Omega_1)}\}_k$ are uniformly bounded, we can find a subsequence $\{u_{k_j^1}\}_j$ which converges weakly in $W_p^2(\Omega_1)$. Similarly, we can find a further subsequence $\{u_{k_j^2}\}_j$ converges weakly in $W_p^2(\Omega_2)$. Repeating this process, we find $\{u_{k_j^i}\}_{i,j}$. Now a diagonal argument will give us the required converging weakly in $W_p^2(\Omega)$ subsequence. Denote this limit function by u.

Clearly $(L-\lambda)u = f$ in Ω, we are left to check the boundary condition $Bu = g$. This is equivalent to say $Bu - g \in \mathring{W}_p^1(\Omega)$.

Since $Bu_k - g \in \mathring{W}_p^1(\Omega_k)$, we can take zero extension to $\mathring{W}_p^1(\Omega)$. Notice that $\mathring{W}_p^1(\Omega)$ is closed under the weak-W_p^1 topology, we infer that the limit u has to satisfy $Bu - g \in \mathring{W}_p^1(\Omega)$. ∎

Now we are in the position of proving Theorem 2.2.4.

Proof of Theorem 2.2.4. We aim to prove a uniform a priori estimate:

$$\|u\|_{W_p^2(\Omega)} \le N\Big(\|(L-\lambda)u\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)}\Big), \tag{2.43}$$

where $\lambda \in [0, \lambda_0]$, λ_0 is the constant given in Lemma 2.5.1, and the constant N is chosen to be independent of λ. Once we have this, the uniqueness and (2.14) can be obtained by letting $\lambda = 0$. For the existence, one only need to use the method of continuity and the large λ existence result in Corollary 2.5.2.

Now we are left to prove (2.43). Actually this can be further reduced to (2.14). First (2.14) gives us (2.43) with $N = N(\lambda)$. Then we only need to find an upper bound of $N(\lambda)$, $\lambda \in [0, \lambda_0]$. This upper bound can be found using a compactness argument: for ε sufficiently small,

$$\begin{aligned}\|u\|_{W_p^2(\Omega)} &\le N(\lambda)\Big(\|(L-\lambda)u\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)}\Big)\\ &\le N(\lambda)\Big(\|(L-\lambda\pm\varepsilon)u\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)}\Big) + N(\lambda)\varepsilon\|u\|_{L_p(\Omega)}\end{aligned}$$

implies

$$\|u\|_{W_p^2(\Omega)} \le \frac{N(\lambda)}{1-N(\lambda)\varepsilon}\Big(\|(L-\lambda\pm\varepsilon)u\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)}\Big).$$

Then for every $\lambda \ge 0$, we can find a neighborhood $(\lambda-\varepsilon, \lambda+\varepsilon)$ on which (2.43) holds with a uniform constant N. Since $[0, \lambda_0]$ is compact, a finite upper bound of N is attained.

Now we only need to prove (2.14) under additional conditions (2.11)-(2.13).

We first prove the following uniqueness result:

$$\begin{cases} u \in W_p^2(\Omega), & \\ Lu = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{has only zero solution.} \tag{2.44}$$

When $p > d$, this uniqueness result is of Aleksandrov-Bakelman-Pucci type. Such result for the oblique derivative problem can be found in [51, Corollary 2.5]: under the assumptions of Theorem 2.2.4, u must be a constant. Then we obtain $u \equiv 0$ from (2.12). Note in this step, the boundedness of Ω is needed.

Now, for general $p \in (1, \infty)$, we need to use large λ well-posedness from Lemma 2.5.1 to improve regularity, i.e., to show that $u \in W^2_{d+\varepsilon}$. From Sobolev embedding and $u \in W_p^2$, we get $u \in L_q, p < q < \frac{dp}{d-2p}$. Now rewrite the equation as $(L-\lambda)u = -\lambda u$, and take λ large enough. The W_q^2-existence and the W_p^2-uniqueness will tell us $u \in W_q^2$. Repeating if needed, we finally get $u \in W^2_{d+\varepsilon}$, where ε satisfies $1 - \frac{1}{d+\varepsilon} < \alpha$ and α is the Hölder exponent of the boundary data as in (2.13). Then we can apply the result in [51] to get $u = 0$.

Passing from (2.44) to (2.14) is a standard contradiction argument. Suppose (2.14) were not true. With help of (2.10), for all $k = 1, 2, \ldots$, there exist u_k such that

$$\|u_k\|_{L_p(\Omega)} > k(\|Lu_k\|_{L_p(\Omega)} + \|Bu_k\|_{W_p^{1-1/p}(\partial\Omega)}). \tag{2.45}$$

Without loss of generality, we take $\|u_k\|_{L_p(\Omega)} = 1$. By (2.10) and (2.45),

$$\begin{aligned} \|u_k\|_{W_p^2(\Omega)} &\le N(\|Lu_k\|_{L_p(\Omega)} + \|Bu_k\|_{W_p^{1-1/p}(\partial\Omega)} + \|u_k\|_{L_p(\Omega)}) \\ &< N\Big(\frac{1}{k} + 1\Big)\|u_k\|_{L_p} \end{aligned}$$

$$\leq 2N.$$

Now, since u_k is uniformly bounded in W_p^2, passing to a subsequence we have $u_{k_i} \to u$ weakly in $W_p^2(\Omega)$, and $u_{k_i} \to u$ strongly in $L_p(\Omega)$. Using (2.45), we obtain $\|Lu_{k_i}\|_{L_p} \to 0$, and $\|Bu_{k_i}\|_{W_p^{1-1/p}} \to 0$. We can deduce that $Lu = 0, Bu = 0$, and hence $u = 0$ by the uniqueness. We have reached a contradiction, since $u_{k_i} \to u$ strongly in $L_p(\Omega)$ implies $\|u\|_{L_p(\Omega)} = 1$. ■

2.6 Fully nonlinear equations

Similar to the Schauder estimate in [66], our method also works for fully nonlinear equations with proper convexity conditions. In this section, we show this for Bellman equations which can be written as follows:

$$\sup_{\omega}\{L^{\omega}u + f(\omega, x)\} := \sup_{\omega}\{a_{ij}(\omega, x)D_{ij}u + a_i(\omega, x)D_iu + a_0(\omega, x)u + f(\omega, x)\} = 0.$$

Compared to the linear case, we have the following assumptions which are uniform in ω: $a_{ij}(\omega, x)$ are measurable in x, symmetric, and satisfy

$$\nu|\xi|^2 \leq a_{ij}(\omega, x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2, \quad \forall x, \omega. \tag{2.46}$$

$$\exists K > 0, \text{ such that } \|a_i(\omega, \cdot), a_0(\omega, \cdot)\|_{\infty} < K, \quad \forall \omega. \tag{2.47}$$

In contrast to Assumption 2.2.2, we state the following uniformly small BMO condition.

Assumption 2.6.1 (r_0, θ)**.** For a constant $\theta > 0$, there exists an $r_0 > 0$ such that

$$\sup_{x\in\Omega, 0<r<r_0} \fint_{B_r(x)\cap\Omega} \sup_{\omega} |a(\omega, y) - (a)_{B_r(x)\cap\Omega}(\omega)|\, dy \le \theta,$$

where θ is a positive constant to be specified later.

Under these settings, we have the following result which is analogous to Theorem 2.2.3.

Theorem 2.6.2. *Assume that Ω is a Lipschitz domain and $p > d$. Let $u \in W^2_p(\Omega)$ be a solution to the Bellman equation:*

$$\begin{cases} \sup_\omega \{L^\omega u + f(\omega, x)\} = 0 & \text{in } \Omega, \\ Bu = b_0 u + b_i D_i u = g & \text{on } \partial\Omega. \end{cases} \tag{2.48}$$

Assume that (2.6), (2.7), (2.46) and (2.47) hold. Also, $\bar{f}(x) := \sup_\omega |f(\omega, x)| \in L_p(\Omega)$, $g \in W^{1-1/p}_p(\partial\Omega)$. Then there exist $\theta_0 = \theta_0(d, p, \nu) > 0$ and $\varepsilon_0 = \varepsilon_0(d, p, \nu, K) > 0$ small enough, such that if the domain Ω is $\delta^2\varepsilon_0$-Lipschitz and Assumption 2.6.1 $(r_0, \delta\theta_0)$ holds, then we have,

$$\|u\|_{W^2_p(\Omega)} \le N\Big(\|u\|_{L_p(\Omega)} + \|\bar{f}\|_{L_p(\Omega)} + \|g\|_{W^{1-1/p}_p(\partial\Omega)}\Big). \tag{2.49}$$

Here $N = N(d, p, \nu, K, \|b_0, b_0\|_\alpha, r_0, R_0, \delta)$ is a constant.

For the proof, we follow the scheme in Section 2.4 which is given for the linear case there. Recall for the linear case we prove the theorem in 3 steps: proving under the homogeneous boundary condition $\frac{\partial u}{\partial y^d} = 0$; constructing an extension; the perturbation argument. The latter two steps still work since they only deal with $\partial\Omega$ and the boundary operator B, and have nothing to do with the elliptic operator. Hence, we only need to give the proof of the first step.

Recall that in Section 2.4, we use the regularized distance to flatten the boundary, then apply Hardy's inequality and the half space result. This argument still works except that we need to prove the corresponding W_p^2 estimate for the Bellman equation in half space with the Neumann boundary condition. In the following lemma, we adopt the assumptions in Theorem 2.6.2, but in all places we replace Ω by $\mathbb{R}^d_+$.

Lemma 2.6.3. *Assume that $u \in W_p^2(\mathbb{R}^d_+)$ solves*

$$\begin{cases} \sup_\omega \{L^\omega u + f(\omega, x)\} = 0 & \text{in } \mathbb{R}^d_+, \\ \frac{\partial u}{\partial x^d} = 0 & \text{on } \partial\mathbb{R}^d_+. \end{cases}$$

There exists a constant $\theta_0 = \theta_0(d, p, \nu) > 0$, such that if Assumption 2.6.1 is satisfied with (r_0, θ_0), then we have

$$\|D^2 u\|_{L_p(\mathbb{R}^d_+)} \leq N(\|\bar{f}\|_{L_p(\mathbb{R}^d_+)} + \|u\|_{L_p(\mathbb{R}^d_+)}). \tag{2.50}$$

Again $\bar{f}(x) := \sup_\omega |f(\omega, x)|$. Here the constant N depends on d, p, ν, K, and r_0.

Before we start the proof, we would like to mention that there are similar results in [43, 26]. In [43], an interior W_p^2 estimate for Bellman equations with small BMO coefficients was established. The corresponding boundary estimate under the Dirichlet boundary condition was proved in [26]. Our proof here follows similar steps in these two papers.

Proof. Using localization techniques, we may assume u has compact support in $B^+_{r_0}(z)$. Here r_0 is the radius in Assumption 2.6.1 and $z \in \mathbb{R}^d_+$. For such u, we aim to prove

$$\fint_{B_r^+(x_0)}\fint_{B_r^+(x_0)} |D^2u(x) - D^2u(y)|^\gamma \, dx\, dy \le N\kappa^d \Big(\fint_{B_{\kappa r}^+(x_0)} (\bar{f} + |Du| + |u|)^d \, dx\Big)^{\gamma/d}$$
$$+ N\kappa^d \theta^{(1-1/\beta)\gamma/d}\Big(\fint_{B_{\kappa r}^+(x_0)} |D^2u|^{\beta d}\, dx\Big)^{\gamma/(\beta d)} + N\kappa^{-\gamma\bar{\alpha}}\Big(\fint_{B_{\kappa r}^+(x_0)} |D^2u|^d \, dx\Big)^{\gamma/d}. \tag{2.51}$$

Here $r \in (0,\infty), \beta \in (1,\infty), \kappa \ge 16$ are parameters which can be chosen arbitrarily, and $\bar{\alpha} = \bar{\alpha}(d,\nu) \in (0,1), N = N(d,\nu), \gamma = \gamma(d,\nu) \in (0,1]$ are all determined constants. The center x_0 can be any fixed point in $\mathbb{R}^d_+$.

When $B_{\kappa r}(x_0) \subset \mathbb{R}^d_+$, this is the interior estimate in [43]. Hence we only need to prove for x_0 close to the boundary. For simplicity, here we only prove for $x_0 \in \partial\mathbb{R}^d_+$, and write $B^+_{\kappa r}$ for $B^+_{\kappa r}(x_0)$. For general x_0 with $B_{\kappa r}(x_0) \not\subset \mathbb{R}^d_+$, the proof is similar.

We shall use the frozen coefficient argument. Define

$$\bar{a}^\omega_{ij} := \begin{cases} (a^\omega_{ij})_{B^+_{\kappa r}} & \text{if } \kappa r \le r_0, \\ (a^\omega_{ij})_{B^+_{r_0}(z)} & \text{if } \kappa r > r_0. \end{cases}$$

Then we decompose $u = v + w$, where v solves the boundary value problem, i.e.,

$$\begin{cases} \sup\{\bar{a}_{ij}(\omega) D_{ij} v\} = 0 & \text{in } B^+_{\kappa r}, \\ \frac{\partial v}{\partial x^d} = 0 & \text{on } \Gamma := \partial\mathbb{R}^d_+ \cap B^+_{\kappa r}, \\ v = \hat{u} := u - \sum_{i=2}^d x^i (D_i u)_{B^+_{\kappa r}} - (u)_{B^+_{\kappa r}} & \text{on } \partial B^+_{\kappa r} \setminus \Gamma. \end{cases} \tag{2.52}$$

Such $\hat{u}$ has properties: $D^2\hat{u} = D^2 u$, and by even extension and Sobolev and Poincaré inequalities,

$$\sup_{B^+_{\kappa r}} |\hat{u}| \le \kappa r \|D^2 u\|_{L_d(B^+_{\kappa r})}.$$

See [26, Lemma 2.1]. For the equation (2.52), we can find a Krylov-Evans type existence result for mixed boundary conditions in [65, Theorem 8.1]: there exists $v \in C^{2,\bar{\alpha}}_{loc}(B^+_{\kappa r} \cup \Gamma) \cap C^0(\overline{B^+_{\kappa r}})$ solving (2.52), where $\bar{\alpha} = \bar{\alpha}(d, \nu)$ is some constant between 0 and 1. The strong maximum principle tells us:

$$\sup_{B^+_{\kappa r}} |v| \le \sup_{\partial B^+_{\kappa r} \setminus \Gamma} |\hat{u}|.$$

From the equation of v, the Krylov-Evans estimate in [65, Theorem 8.1] and a dilation argument, we have: for all $x, y \in B^+_r(x_0)$,

$$\begin{aligned}
|D^2 v(x) - D^2 v(y)| &\le |x-y|^{\bar{\alpha}} [D^2 v]_{C^{\bar{\alpha}}(B^+_r(x_0))} \\
&\le N|x-y|^{\bar{\alpha}} (\kappa r - r)^{-2-\bar{\alpha}} \sup_{B^+_{\kappa r}} |v| \\
&\le N|x-y|^{\bar{\alpha}} (\kappa r)^{-2-\bar{\alpha}} \sup_{\partial B^+_{\kappa r} \setminus \Gamma} |\hat{u}| \\
&\le N|x-y|^{\bar{\alpha}} (\kappa r)^{-1-\bar{\alpha}} \|D^2 u\|_{L_d(B^+_{\kappa r})}.
\end{aligned}$$

Integrating x, y in B^+_r, we obtain

$$\fint_{B^+_r(x_0)} \fint_{B^+_r(x_0)} |D^2 v(x) - D^2 v(y)|^\gamma \, dx \, dy \le N \kappa^{-\gamma \bar{\alpha}} \Big(\fint_{B^+_{\kappa r}} |D^2 u|^d \Big)^{\gamma/d}, \quad \forall \gamma \in (0,1]. \tag{2.53}$$

Now let us consider $D^2 w$. Since $D^2 \hat{u} = D^2 u$, it is equivalent to discuss $\hat{w} := \hat{u} - v$ instead of w $(= u - v)$. From the equation of w, one can simply show that $\hat{w}$ satisfies

$$\sup\{\bar{a}_{ij}(\omega) D_{ij}\hat{w} + f(\omega, x) + a_i(\omega, x) D_i u + a_0(\omega, x) u + (a_{ij}(\omega, x) - \bar{a}_{ij}(\omega)) D_{ij} u\} \ge 0,$$
$$\inf\{\bar{a}_{ij}(\omega) D_{ij}\hat{w} + f(\omega, x) + a_i(\omega, x) D_i u + a_0(\omega, x) u + (a_{ij}(\omega, x) - \bar{a}_{ij}(\omega)) D_{ij} u\} \le 0.$$

One important observation here is that, we can rewrite the equation for $\hat{w}$ as follows,

$$\begin{cases} \hat{L}\hat{w} := \hat{a}_{ij}(x)D_{ij}\hat{w}(x) = \hat{f}(x) & \text{in } B^+_{\kappa r}, \\ \frac{\partial \hat{w}}{\partial x^d} = 0 & \text{on } \Gamma, \\ \hat{w} = 0 & \text{on } \partial B^+_{\kappa r} \setminus \Gamma. \end{cases} \tag{2.54}$$

Here $\hat{a}_{ij}$ are measurable and symmetric with $\nu|\xi|^2 \le \hat{a}_{ij}\xi_i\xi_j \le \nu^{-1}|\xi|^2$, and $\hat{f}$ is also measurable, with

$$|\hat{f}| \le \bar{f} + K(|Du| + |u|) + \sup_{\omega} |a_{ij}(\omega, x) - \bar{a}_{ij}(\omega)||D^2u|.$$

From [56], we have the following estimate for uniform elliptic operators with only measurable coefficients:

$$\|D^2h\|_{L_\gamma(B_1)} \le N\|L_\nu h\|_{L_d(B_1)}. \tag{2.55}$$

Here $h \in W^2_d(B_1), h = 0$ on ∂B_1, L_ν is any bounded and uniform elliptic operator (i.e. (2.3) holds with constant ν) of which the coefficients are only measurable, and $\gamma = \gamma(d, \nu) \in (0, 1]$. For our equation (2.54) which is in half ball with the Neumann boundary condition, to apply (2.55), we take even extension for $\hat{w}, \hat{f}$, and correspondingly for $\hat{a}_{ij}$. We derive that for some $\gamma = \gamma(d, \nu) \in (0, 1]$,

$$\begin{aligned} &\Big(\fint_{B^+_r(x_0)} |D^2\hat{w}(x)|^\gamma \, dx\Big)^{1/\gamma} \le \kappa^{d/\gamma}\Big(\fint_{B^+_{\kappa r}} |D^2\hat{w}(x)|^\gamma \, dx\Big)^{1/\gamma} \\ &\le N\kappa^{d/\gamma}\Big(\fint_{B^+_{\kappa r}} |\hat{L}\hat{w}(x)|^d \, dx\Big)^{1/d} \\ &\le N\kappa^{d/\gamma}\Big(\fint_{B^+_{\kappa r}} \Big(\bar{f} + K(|Du| + |u|) + \sup_{\omega} |a_{ij}(\omega, x) - \bar{a}_{ij}(\omega)||D^2u|\Big)^d\Big)^{1/d} \\ &\le N\kappa^{d/\gamma}\Big(\fint_{B^+_{\kappa r}} (\bar{f} + |Du| + |u|)\Big)^{1/d} + N\kappa^{d/\gamma}\theta^{(1-1/\beta)/d}\Big(\fint_{B^+_{\kappa r}} |D^2u|^{\beta d}\Big)^{1/(\beta d)}. \end{aligned}$$

Here $\beta > 1$ is any constant satisfying $\beta d < p$. In the last step, we use the fact that u

has compact support in $B^+_{r_0}(z)$, Hölder's inequality, and Assumption 2.6.1:

$$\begin{aligned}
&\|\sup_\omega |a_{ij}(\omega,x)-\bar a_{ij}(\omega)|D^2u\|_{L_d(B^+_{\kappa r})}\\
&\qquad\le \|\sup_\omega |a_{ij}(\omega,x)-\bar a_{ij}(\omega)|1_{B^+_{r_0}(z)}\|_{L_{\beta d/(\beta-1)}(B^+_{\kappa r})}\|D^2u\|_{L_{\beta d}(B^+_{\kappa r})}\\
&\qquad\le N(r_0,d)\theta^{(1-1/\beta)/d}\|D^2u\|_{L_{\beta d}(B^+_{\kappa r})}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\fint_{B^+_r(x_0)}\fint_{B^+_r(x_0)}|D^2\hat w(x)-D^2\hat w(y)|^\gamma\,dx\,dy\\
&\quad\le N\kappa^d\Big(\fint_{B^+_{\kappa r}(x_0)}(\bar f+|Du|+|u|)^d\,dx\Big)^{\gamma/d}+N\kappa^d\theta^{(1-1/\beta)\gamma/d}\Big(\fint_{B^+_{\kappa r}}|D^2u|^{\beta d}\,dx\Big)^{\gamma/(\beta d)}.
\end{aligned}\tag{2.56}$$

Combining (2.53) and (2.56), and noticing that $D^2\hat w=D^2w$, we obtain (2.51).

Once we have (2.51), take supreme in r, we have for any $\kappa\ge 16$,

$$\begin{aligned}
(D^2u)^\#_\gamma \le& N\kappa^{d/\gamma}\mathbb{M}^{1/d}(\bar f^d)+N\kappa^{d/\gamma}\mathbb{M}^{1/d}(|u|^d)+N\kappa^{d/\gamma}\mathbb{M}^{1/d}(|Du|^d)\\
&+N\kappa^{d/\gamma}\theta^{(1-1/\beta)/d}\mathbb{M}^{1/(\beta d)}(|D^2u|^{\beta d})+N\kappa^{-\bar\alpha}\mathbb{M}^{1/d}(|D^2u|^d).
\end{aligned}\tag{2.57}$$

Here $\mathbb{M}$ is the classical centered maximal function

$$\mathbb{M}f(x_0):=\sup_{r>0}\fint_{B^+_r(x_0)}|f(x)|\,dx$$

and $(D^2u)^\#_\gamma$ is the following sharp function

$$(D^2u)^\#_\gamma(x_0):=\sup_{r>0}\Big(\fint_{B^+_r(x_0)}\fint_{B^+_r(x_0)}|D^2u(x)-D^2u(y)|^\gamma\,dx\,dy\Big)^{1/\gamma}.$$

Now, taking the L_p norm for both sides of (2.57), applying the Hardy-Littlewood theorem and a Fefferman-Stein type sharp function theorem which can be found in the appendix of [43], we get for $p > \beta d$,

$$\|D^2u\|_p \le N\kappa^{d/\gamma}(\|\bar{f}\|_p + \|Du\|_p + \|u\|_p) + N(\kappa^{d/\gamma}\theta^{(1-1/\beta)/d} + \kappa^{-\bar{\alpha}})\|D^2u\|_p,$$

where $\|\cdot\|_p$ is the abbreviation for $\|\cdot\|_{L_p(\mathbb{R}^d_+)}$. We are left to choose κ sufficient large and then θ sufficient small to absorb $\|D^2u\|_p$ into the left-hand side, and then use the interpolation inequality to obtain (2.50).

Noting that in the above computation, we require u to have compact support in $B^+_{r_0}(z)$. We may reduce our original problem to this, by using partition of unity ξ_i for the covering $\{B^+_{r_0}(z_i)\}_i$ of half space. In (2.51), we substitute u by $u\xi_i$ and $\bar{f}$ by $\bar{f}\xi_i + \sup_\omega\{|[\xi_i, L^\omega]u|\}$, and then sum over all i. Here $[\cdot,\cdot]$ is the usual notation for commutators. Noting that all the extra terms are of lower order, this will only add $\|Du\|_p$ and $\|u\|_p$ to the right-hand side of our estimate. To get (2.50) we use the interpolation inequality $\|Du\|_p \le \varepsilon\|D^2u\|_p + N(\varepsilon)\|u\|_p$. ■

In a subsequent work, we will consider the W^2_p estimate and solvability for more general fully nonlinear operators with the oblique derivative boundary condition.

Chapter Three

Classical solutions of oblique derivative problem in nonsmooth domains with mean Dini coefficients

3.1 Introduction

In this chapter, we consider strong solutions to the oblique derivative problem in a bounded domain $\Omega \subset \mathbb{R}^d, d \geq 2$:

$$\begin{cases} Lu := -a_{ij}D_{ij}u + b_iD_iu + cu = f & \text{in } \Omega, \\ Bu := \beta_0 u + \beta_i D_i u = g & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

For simplicity, the notation

$$\boldsymbol{A} := (a_{ij})_{i,j=1}^d, \quad \boldsymbol{b} := (b_i)_{i=1}^d, \quad \boldsymbol{\beta} := (\beta_i)_{i=1}^d$$

are used for matrices and vectors. We also denote $\mathbb{S}^{d\times d}$ to be the set of $d \times d$ real-valued constant symmetric matrices. The following conditions regarding the elliptic operator and boundary condition will be assumed throughout this chapter. All the coefficients in the elliptic operators are assumed to be bounded and measurable, and the leading coefficients are assumed to be symmetric and uniformly elliptic with elliptic constant $\lambda > 0$:

$$\lambda|\zeta|^2 \leq a_{ij}\zeta_i\zeta_j \leq \lambda^{-1}|\zeta|^2, \ \forall \zeta \in \mathbb{R}^d, \quad |\boldsymbol{b}|, |c| \leq K, \quad a_{ij} = a_{ji},$$

where $K > 0$ is a constant. We also assume that the boundary operator is oblique, i.e., for some $\delta \in (0,1)$,

$$\beta_i n_i \leq -\delta|\boldsymbol{\beta}| < 0 \qquad \text{on } \partial\Omega, \tag{3.2}$$

where $\boldsymbol{n} = (n_i)_{i=1}^d$ is the unit outward normal direction.

Continuing [28], we discuss the minimum regularity assumptions on a_{ij} and $\partial\Omega$ regarding the regularity of strong solutions to the oblique derivative problems.

It was proved in [13, 58] that (3.1) is uniquely solvable in W^2_p for any $p \in (1, \infty)$, given that a_{ij} has vanishing mean oscillation (VMO), $\partial\Omega \in C^{1,1}$, $\boldsymbol{\beta} \in C^{0,1}$, and proper sign conditions hold. See also [14] for a result of quasilinear equations. Later in [28], the current authors proved the following results with relaxed assumptions on $\partial\Omega$ and $\boldsymbol{\beta}$. A Lipschitz domain Ω is called ε-Lipschitz, if near each boundary point the local representation function has Lipschitz constant no greater than ε. A measurable and locally integrable function A is called θ-BMO, if for some $r_0 > 0$,

$$\sup_{r<r_0, x} \fint_{B_r(x)} \left| A(y) - \Big(\fint_{B_r(x)} A \Big) \right| dy < \theta.$$

Theorem 3.1.1 ([28] Theorem 2.4)**.** *Consider* (3.1) *in a bounded Lipschitz domain* Ω *with* $\boldsymbol{\beta} \in C^\alpha$ *for some* $\alpha \in (1 - 1/d, 1)$ *and the usual sign condition* (3.3)*. Then for any* $p \in (1, 1/(1-\alpha))$*, we can find constants* θ_0 *and* ε_0 *sufficiently small, such that if* Ω *is* $\delta^2\varepsilon_0$*-Lipschitz and* a_{ij} *is* $\delta\theta_0$*-BMO, then for any* $f \in L_p(\Omega)$ *and* $g \in W_p^{1-1/p}(\partial\Omega)$*, there exists a unique solution* $u \in W^2_p(\Omega)$ *satisfying*

$$\|u\|_{W^2_p(\Omega)} \lesssim \|f\|_{L_p(\Omega)} + \|g\|_{W_p^{1-1/p}(\partial\Omega)}.$$

In particular, this result shows that if $a_{ij} \in VMO$, $\partial\Omega \in C^1$, $\boldsymbol{\beta} \in C^{0,1}$, $f \in L_\infty$, and $g \in C^1$, then any $W^2_{1+\varepsilon}$-strong solution to (3.1) is in W^2_p for any $p < \infty$.

In this chapter, we give minimal regularity assumptions on these objects such that any strong solution is $C^2(\overline{\Omega})$.

Due to an example given in [31, Theorem 4], $D^2 u$ might not be bounded or even BMO if we merely assume the continuity of a_{ij}. Hence certain conditions on its modulus of continuity are needed. For divergence form equations it is well known that any weak solution is C^1 if a_{ij} satisfies the so-called α-increasing Dini condition, i.e., the modulus of continuity of a_{ij} is bounded by a Dini function (see definition below) ϕ satisfying that $\phi(r)/r^\alpha$ is non-increasing for some $\alpha \in (0,1]$. This is the borderline case of the classical Campanato-type results, see [50, Section 5]. In this direction, later Li in [47] obtained the interior C^1-regularity, if

$$\varphi_A(r) := \sup_x \left(\fint_{B_r(x)} |a_{ij}(y) - a_{ij}(x)|^2 \, dy \right)^{1/2}$$

is a Dini function. Recently, in [24] the first named author and Kim generalized this result by assuming that

$$\omega_A(r) := \sup_x \fint_{B_r(x)} \left| a_{ij}(y) - \Big(\fint_{B_r(x)} a_{ij} \Big) \right| dy$$

is a Dini function, noting $\omega_A(r) \le 2\varphi_A(r)$. In the same paper, under the same type of regularity assumptions, interior C^2 estimate was also proved for equations in non-divergence form. The corresponding boundary estimate for Dirichlet problem in $C^{2,\mathrm{Dini}}$ domain can be found in [19].

Back to the oblique derivative problem, besides the same assumptions on a_{ij} as the interior case, certain regularity assumptions on $\partial\Omega$ and the boundary operator are also needed in order to obtain a global result. Direct computation shows that similar to Dirichlet problem, if $\partial\Omega \in C^{2,\mathrm{Dini}}$, we can reduce the problem to the half space case by simply flattening the boundary. However, in the same spirit of [52, 50] and [28], we expect $\partial\Omega$ to be one derivative less regular, i.e., we expect the result still holds when $\partial\Omega \in C^{1,\mathrm{Dini}}$. In this regard, a global C^2 estimate was proved in [27] given that $\partial\Omega, \beta_0, \boldsymbol{\beta} \in C^{1,\mathrm{Dini}^2}$, which means, e.g., the Dini integrals of their moduli of continuity $I\rho_\bullet$ (see definition below) are still Dini functions. The proof was based on an extension idea introduced in [50], which was also used in [28]. In this chapter, using the W^2_p estimate in [28], the regularized distance, and a delicate decomposition of solutions, we relax the regularity assumption to $\partial\Omega, \beta_0, \boldsymbol{\beta} \in C^{1,\mathrm{Dini}}$, which seems to be optimal for the global C^2 estimate. To the best of our knowledge, such result is new even when the coefficients are smooth.

We also extend our results to fully nonlinear equations with linear oblique boundary conditions, including the so-called Bellman equations (see (3.4)). Bellman equations with oblique derivative boundary condition arise naturally in the study of optimal stochastic control in domain Ω with reflecting boundary conditions. In [57], Lions and Trudinger first studied the $C^{1,1}(\overline{\Omega}) \cap C^{2,\alpha}_{loc}(\Omega)$ solvability of Bellman equations, assuming that all the coefficients, $\partial\Omega$, and the boundary operators are sufficiently smooth. Later in [50], Safonov proved the unique $C^{2,\alpha}(\overline{\Omega})$ solvability under the relaxed conditions

$$[a^{\omega}_{ij}]_{C^\alpha} + [b^{\omega}_i]_{C^\alpha} + [c^{\omega}]_{C^\alpha} \le K, \ \forall\omega, \quad \partial\Omega, \beta_0, \boldsymbol{\beta} \in C^{1,\alpha}.$$

See also [65]. Recently, there are also study of similar problems using the viscosity solution approach. For its framework and the solvability, we refer the reader to [35]. In this direction, Milakis and Silvestre in [60] studied the fully nonlinear, uniformly elliptic equation $F(D^2u) = 0$ with Neumann boundary condition on half balls. They showed the $C^\alpha, C^{1,\alpha}$ regularity of viscosity solutions, and the $C^{2,\alpha}$ regularity when F is convex. Later in [46], Li and Zhang proved a similar result for $F(D^2u) = 0$ with oblique derivative boundary condition in domain Ω. In particular, when $\partial\Omega \in C^{1,\alpha}$ and F is convex, they showed that any viscosity solution is in $C^{2,\alpha}$. The key step there is to prove a boundary Harnack inequality based on an Aleksandrov-Bakel'man-Pucci type estimate, and then design approximating problems. Here, due to the usage of the Campanato-type iteration, we are able to deal with the Dini case. Moreover, our operator F are more general depending on lower-order terms and the variable x. For fully nonlinear elliptic equations with

Dini dependence, we also refer the reader to Kovats [39] for interior C^2 estimates and Lieberman [54] for boundary weighted C^1 and C^2 estimates.

The remaining part of the chapter is organized as follows. The main results of our chapter are stated in the next section. In 3.3, we introduce some notation, recall the definitions of the regularized distance and L_p-mean oscillations, and state some auxiliary results. In Section 3.3.1, we give the mean oscillation estimates for D^2u when the lower-order terms are not presented. Section 3.5 is devoted to the proof of Theorem 3.2.4. Finally, in Section 3.6 we consider concave fully nonlinear equations and give the proof of Theorem 3.2.7.

3.2 Main results

In this section, we formally give the assumptions related to Dini conditions, and then state our main results. A function $\theta : (0, 1] \to [0, \infty)$ is called a Dini function if

$$\int_0^1 \frac{\theta(r)}{r}\, dr < \infty.$$

We write its Dini integral as

$$I\theta(r) := \int_0^r \frac{\theta(t)}{t}\, dt.$$

Both of the following notation are used for the average

$$(f)_\Omega = \fint_\Omega f.$$

We also denote

$$\Omega_r(x) := \Omega \cap B_r(x),$$

where the center x will be omitted when it is the origin.

Definition 3.2.1. For a function f defined on Ω, we consider two types of its oscillations:

$$\omega_f(r) := \sup_{x \in \overline{\Omega}} \fint_{\Omega_r(x)} |f - (f)_{\Omega_r(x)}|, \quad \rho_f(r) := \sup_{|x-y| \le r, x, y \in \Omega} |f(x) - f(y)|.$$

We say that f is of L_1-mean Dini oscillation (uniform Dini) if ω_f (ρ_f, respectively) is a Dini function.

Clearly, any uniform Dini function is L_1-mean Dini. On the other hand, according to a result by Spanne in [70], on a domain Ω with exterior measure condition, any L_1-mean Dini function f is uniformly continuous with modulus of continuity given by $I\omega_f$. Simple calculation shows that if both f and g are L_1-mean Dini functions defined on a bounded domain Ω, then fg is L_1-mean Dini.

Remark 3.2.2. It is worth mentioning that, the mean oscillation ω_f defined above is "almost increasing" given certain "doubling condition" on $|\Omega_r(x)|$. See Lemma 3.3.4. The following "increasing modification" is also commonly used in literatures:

$$\widehat{\omega}_f(r) := \sup_{t \le r} \omega_f(t).$$

Clearly the Dini condition on $\widehat{\omega}_f$ implies that f is L_1-mean Dini in our sense. The following example shows that the contrary is not always true:

$$f = 1 + \sum_{k=1}^{\infty} k^{-2} g_k(x), \quad g_k = g(200(x - a_k)/a_k), \text{ where } g(x) = (1 - |x|)^+ \text{ and } a_k = e^{-k^2}.$$

Then ω_f is a Dini function, while $\widehat{\omega}_f$ is not. Indeed, a simple calculation reveals that

$$\omega_k(r) := \omega_{g_k}(r) \sim \frac{r}{a_k} \quad \text{when } \ r \in (0, a_k], \quad \omega_k(r) \sim \frac{a_k}{r} \quad \text{when } \ r \in (a_k, 1].$$

Thus,

$$\int_0^1 \omega_k(r)\frac{dr}{r} \sim 1.$$

By the triangle inequality,

$$\omega_f(r) \le \sum_{k=1}^{\infty} k^{-2}\omega_k(r),$$

which implies that ω_f is a Dini function. On the other hand, because g_k has disjoint support,

$$\omega_f(a_k/100) \sim k^{-2}\omega_k(a_k/100) \sim k^{-2}.$$

Therefore,

$$\int_0^1 \widehat{\omega}_f(r)\frac{dr}{r} \gtrsim \sum_{k=1}^{\infty}(k+1)^{-2}\log(a_k/a_{k+1}) \gtrsim \sum_{k=1}^{\infty}(k+1)^{-1} = \infty.$$

A function ψ is said to be $C^{1,\text{Dini}}$ if it is continuous with $D\psi$ being uniform Dini. Below we give the formal definition of $C^{1,\text{Dini}}$ domains.

Definition 3.2.3. A bounded domain $\Omega \subset \mathbb{R}^d$ is said to have $C^{1,\text{Dini}}$ boundary if there exists some $C^{1,\text{Dini}}$ function $\psi_0 : \mathbb{R}^d \to \mathbb{R}$, such that

$$\Omega = \{\psi_0 > 0\}, \quad |D\psi_0| \ge 1 \text{ on } \partial\Omega.$$

Now we state our main result of this chapter.

Theorem 3.2.4. *Consider the problem (3.1) in a $C^{1,Dini}$ domain Ω. Assume that $\beta_0, \boldsymbol{\beta} \in C^{1,Dini}$, and $\boldsymbol{A}, \boldsymbol{b}, c$ are of L_1-mean Dini oscillation. Suppose for some $\varepsilon > 0$, $u \in W^2_{1+\varepsilon}(\Omega)$*

is a strong solution to (3.1) *with* $g \in C^{1,Dini}$ *and* f *being of* L_1*-mean Dini oscillation. Then* $u \in C^2(\overline{\Omega})$.

Noting the solvability in Theorem 3.1.1, we immediately obtain the following.

Corollary 3.2.5. *Besides the assumptions of Theorem 3.2.4, if we further assume*

$$c \geq 0, \quad \beta_0 \leq 0, \quad |c|^2 + |\beta_0|^2 \not\equiv 0, \tag{3.3}$$

then there exists a unique $C^2(\overline{\Omega})$ *solution to* (3.1).

Our approach is also applicable to concave fully nonlinear elliptic equations

$$F[u] := F(D^2u, Du, u, y) = 0$$

with linear oblique boundary condition $Bu = g$.

Assumption 3.2.6. The function $F(M, p, u, y)$ defined on $\mathbb{S}^{d\times d} \times \mathbb{R}^d \times \mathbb{R} \times \Omega$ satisfies

(1) $F(\cdot, p, u, y)$ is concave.

(2) There exists a constant $\lambda > 0$ such that

$$\lambda\|M\| \leq F(M + N, p, u, y) - F(N, p, u, y) \leq \lambda^{-1}\|M\|$$

for any $N \in \mathbb{S}^{d\times d}$, $p \in \mathbb{R}^d$, $u \in \mathbb{R}$, $y \in \Omega$, and $M \in \mathbb{S}^{d\times d}, M \geq 0$.

(3) $|F(0, 0, 0, y)| + [F(M, \cdot, \cdot, y)]_1 \leq K$ for any $M \in \mathbb{S}^{d\times d}$ and $y \in \Omega$, where $[\,\cdot\,]_1$ represents the Lipschitz semi-norm.

(4) *(L_d-mean Dini)* There exist $R_0 > 0$ and a Dini function ω_F, such that for any $x \in \overline{\Omega}, r \in (0, R_0]$, we can find some function $F_0 : \mathbb{S}^{d\times d} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfying

(1)-(3), such that

$$\left(\fint_{\Omega_r(x)} |F(M,p,u,y) - F_0(M,p,u)|^d \, dy\right)^{1/d} \leq (|M| + |p| + |u| + 1)\omega_F(r).$$

It is worth noting that this class includes the Bellman equations

$$\sup_{\omega\in\mathcal{A}}\{a_{ij}^{\omega}D_{ij}u + b_i^{\omega}D_i u + c^{\omega}u - f^{\omega}\} = 0, \tag{3.4}$$

where all the coefficients as well as f^{ω} are uniformly bounded satisfying the L_d-mean Dini conditions of the following type:

$$\left(\fint_{\Omega_r(x)} \sup_{\omega} |a_{ij}^{\omega}(y) - (a_{ij}^{\omega})_{\Omega_r(x)}|^d \, dy\right)^{1/d} \lesssim \omega(r).$$

We also impose the sign conditions

$$F(M,p,\cdot,y) \text{ is non-increasing and } \beta_0 \leq -\gamma < 0, \tag{3.5}$$

where $\gamma > 0$ is a constant. Now we present the regularity result for fully nonlinear equations.

Theorem 3.2.7. *In a bounded domain Ω, consider the problem $F[u] = 0$ with the oblique derivative boundary condition $Bu = g$ on $\partial\Omega$. Under the conditions $\partial\Omega$, β_0, $\boldsymbol{\beta} \in C^{1,Dini}$, Assumption 3.2.6, and (3.5), for any $C^{1,Dini}$ boundary data g, there exists a unique $C^2(\overline{\Omega})$ solution.*

3.3 Preliminary

3.3.1 Regularized distance and coordinate system

In Definition 3.2.3, a representation function ψ_0 is given for the $C^{1,\mathrm{Dini}}$ domain Ω. Here for studying problems with rough boundaries, we mollify ψ_0 properly to obtain a more suitable representation. This is the regularized distance introduced by Lieberman in [49]. Here we refer to a modified version in [27, Lemma 5.1]: there exists some function $\psi \in C^{1,\mathrm{Dini}}(\mathbb{R}^d) \cap C^\infty(\Omega)$, satisfying

$$\Omega = \{\psi > 0\}, \quad N^{-1}\psi(x) \le \mathrm{dist}(x, \partial\Omega) \le N\psi(x), \ \forall x \in \Omega,$$

$$|D\psi| \le 1 \quad \text{in } \overline{\Omega}, \quad \rho_{D\psi}(t) \le N\rho_{D\psi_0}(ct), \tag{3.6}$$

and for any multi-index l with $|l| = m \ge 2$,

$$|D^l\psi(x)| \le C\psi(x)^{1-m}\rho_{D\psi_0}(\psi(x)), \quad \forall x \in \Omega. \tag{3.7}$$

Here $C = C(d, m, \|D\psi_0\|_{L_\infty(\Omega)})$, and $N, c \in [2, \infty)$ are positive constants depending on $(d, \|D\psi_0\|_{L_\infty(\Omega)})$.

Next, we introduce the coordinate system adapted to our oblique derivative problem. For any $x_0 \in \partial\Omega$, we choose an orthonormal coordinate system $y = (y_1, \ldots, y_d)$ centered at x_0 such that the y_d-axis is in the $\boldsymbol{\beta}(x_0)$ direction. Now, noting that $D\psi$ is a inward normal on the boundary, due to the obliqueness (3.2), continuity of $D\psi$, and compactness, we can choose some $R_0 \in (0, 1)$ independent of x_0, such that

$$\frac{\beta(x_0)}{|\beta(x_0)|} D\psi = \frac{\partial\psi}{\partial y_d} \ge \frac{\delta}{2}|D\psi| \quad \text{in } \Omega_{R_0}(x_0). \tag{3.8}$$

Direct computation shows that there exists some constant $c(\delta, d) > 0$, such that for any $r \in (0, R_0]$,

$$c(\delta, d) < |\Omega_r(x_0)|/|B_r(x_0)| < 1. \tag{3.9}$$

3.3.2 Estimates on the half space

We will use the notation

$$B_r^+ := B_r \cap \{x : x_d > 0\}, \quad \Sigma_r := \{x : |x| < r, x_d = 0\}, \quad \sigma_r := \partial B_r \cap \{x : x_d > 0\}$$

throughout this chapter. The first result is a weak-type $(1, 1)$ estimate given in [27, Lemma 2.13].

Lemma 3.3.1. *Let $\overline{A} = (\overline{a_{ij}})$ be a constant symmetric matrix with elliptic constant λ. Assuming that $f \in L_2$, $u \in W_2^2(B_1^+)$ satisfy*

$$-\overline{a_{ij}} D_{ij} u = f \text{ in } B_1^+, \quad D_d u = 0 \text{ on } \Sigma_1, \quad u = 0 \text{ on } \sigma_1,$$

then for any $t > 0$, we have

$$|\{|D^2 u| > t\} \cap B_1^+| \le \frac{C}{t} \fint_{B_1^+} |f|, \tag{3.10}$$

where $C = C(d, \lambda) > 0$ is a constant.

As a corollary, we have the following strong-type $(1, p)$ estimate.

Corollary 3.3.2. *Under the assumptions of Lemma 3.3.1, for any $p \in (0, 1)$ we have*

$$\Big(\fint_{B_1^+} |D^2 u|^p \Big)^{1/p} \le C \fint_{B_1^+} |f|,$$

where $C = C(d, \lambda, p) > 0$ *is a constant.*

Proof. By using (3.10), for any $T \in (0, \infty)$,

$$\begin{aligned}\int_{B_1^+} |D^2u|^p &= \int_0^\infty pt^{p-1} |\{|D^2u| > t\} \cap B_1^+| \, dt \\ &\le \int_0^T pt^{p-1}|B_1^+| \, dt + C\int_T^\infty pt^{p-2}\Big(\fint_{B_1^+} |f|\Big) \, dt \\ &\le T^p|B_1^+| + C\frac{p}{1-p}T^{p-1}\fint_{B_1^+}|f|.\end{aligned}$$

Minimizing in T, we obtain the desired estimate. ■

We also need the following C^3 estimate.

Lemma 3.3.3. *Let* $p \in (0, 1)$. *Assume that* $u \in W_2^2(B_1^+)$ *is a strong solution to*

$$-\overline{a_{ij}} D_{ij}u = \overline{f} \quad \text{in } B_1^+, \quad D_d u = 0 \quad \text{on } \Sigma_1,$$

where $\overline{a_{ij}}$ *is a constant matrix as before and* $\overline{f}$ *is a constant. Then we have* $u \in C^3(\overline{B^+_{1/2}})$ *satisfying*

$$\|D^3u\|_{L_\infty(B^+_{1/2})} \le C(|D^2u - \boldsymbol{q}|^p)^{1/p}_{B_1^+}, \tag{3.11}$$

where $\boldsymbol{q} \in \mathbb{S}^{d\times d}$ *can be any constant symmetric matrix and* $C = C(d, p, \lambda) > 0$.

Proof. First, note that for $x' = (x_1, \ldots, x_{d-1})$, formally $U := D^2_{x'}u - \boldsymbol{q}$ is a solution to

$$-\overline{a_{ij}} D_{ij}U = 0 \quad \text{in } B_1^+, \quad D_d U = 0 \quad \text{on } \Sigma_1.$$

By using the argument of finite-difference quotients, as a corollary of the Schauder estimate for elliptic equations with the Neumann boundary condition, we have the Lipschitz estimate

$$\|DD_{x'}^2 u\|_{L_\infty(B_r^+)} \le \frac{C}{(R-r)^{1+d/2}}\|D_{x'}^2 u - \boldsymbol{q}\|_{L_2(B_R^+)},$$

for any $0 < r < R < 1$. From this, we first differentiate the equation in the x' direction to obtain the corresponding estimate for $D_d^2 D_{x'} u$. Then we differentiate in the x_d direction for $D_d^3 u$. Combining these, we have

$$\|D^3 u\|_{L_\infty(B_r^+)} \le \frac{C}{(R-r)^{1+d/2}}\|D^2 u - \boldsymbol{q}\|_{L_2(B_R^+)}.$$

From this we can obtain (3.11) using interpolation and an iteration argument which can be found, for instance, in [33, Proposition 8.19]. ∎

3.3.3 L_p-mean oscillation and more on Dini functions

In this chapter, the following L_p-mean oscillation of $D^2 u$ will be intensively studied. For any $x \in \overline{\Omega}$ and $p \in (0,1)$,

$$\phi(x,r) := \inf_{\boldsymbol{q}\in\mathbb{S}^{d\times d}} \Big(\fint_{\Omega_r(x)} |D^2 u - \boldsymbol{q}|^p \Big)^{1/p}. \tag{3.12}$$

We note a few properties of such L_p-mean oscillation:

(a) If $D^2 u \in L_{p,loc}$, for each x, r we can find at least one minimizer. In this chapter, we write $\boldsymbol{q}_{x,r}$ for such a minimizer.

(b) If D^2u is continuous at x, then $\boldsymbol{q}_{x,r} \to D^2u(x)$ as $r \to \infty$. Indeed,

$$\begin{aligned} |D^2u(x) - \boldsymbol{q}_{x,r}|^p &\le \fint_{\Omega_r(x)} |D^2u(z) - \boldsymbol{q}_{x,r}|^p \, dz + \fint_{\Omega_r(x)} |D^2u(z) - D^2u(x)|^p \, dz \\ &\le 2 \fint_{\Omega_r(x)} |D^2u(z) - D^2u(x)|^p \, dz \to 0 \text{ as } r \to 0. \end{aligned}$$

Here in the last inequality, we used the fact that $\boldsymbol{q}_{x,r}$ is a minimizer.

Later in the proof we will see, many steps in the classical Campanato's iteration for L_2-mean oscillation still work if we replace $(D^2u)_{\Omega_r(x)}$ with $\boldsymbol{q}_{r,x}$.

The following property of Dini functions is useful in our iteration argument. This iteration was introduced in [16], and was also used in [24, 19, 27] for studying equations with L_1-mean Dini coefficients. A nonnegative function ω is called comparable if there exists some constant $\kappa \in (0,1)$, such that

$$\omega(s)/\omega(t) \le C(\kappa), \quad \forall s, t \in (0,1] \text{ and } s/t \in [\kappa, \kappa^{-1}]. \tag{3.13}$$

Lemma 3.3.4. *Assume that ω is a Dini function satisfying the almost increasing condition*

$$\omega(r) \ge C\omega(s), \quad \forall \; r/2 < s < r < \infty \tag{3.14}$$

for some constant $C > 0$. Then

$$\widetilde{\omega}(r) := \sum_{i=0}^{\infty} (1/2)^i \Big(\omega(\kappa^{-i} r) 1_{\kappa^{-i} r < 1} + \omega(1) 1_{\kappa^{-i} r \ge 1} \Big) \tag{3.15}$$

is also a Dini function satisfying (3.13). Furthermore, up to a constant depending only on κ, we have

$$I\widetilde{\omega}(r) \simeq \sum_{j=0}^{\infty} \widetilde{\omega}(\kappa^j r).$$

In particular, $\widetilde{\omega}(r) \to 0$ *as* $r \to 0$.

The proof is by direct computation which can be found in [16, Lemma 1] and [24, Lemma 2.7].

Clearly, ρ_f given in Definition 3.2.1 is a non-decreasing function. One can simply check that ω_f satisfies (3.14) provided that the doubling property is satisfied, i.e., for any $x \in \overline{\Omega}$ and $r > 0$, $|\Omega_{2r}(x)| \le C_0|\Omega_r(x)|$ for some constant $C_0 > 0$.

3.4 L_p-Mean oscillation estimate

In this section, we focus on the equation without lower-order terms

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in } \Omega, \\ \beta_i D_i u = g & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

For any $x \in \overline{\Omega}$ and $p \in (0,1)$, recall the L_p-mean oscillation of D^2u in (3.12). For simplicity, we also denote

$$\begin{aligned} \omega^{(1)}(r) &:= r\|D\beta\|_{L_\infty(\Omega)} + \rho_{D\boldsymbol{\beta}}(r) + \rho_{D\psi_0}(r) + \omega_{\boldsymbol{A}}(r), \\ \omega^{(2)}(r) &:= r\|Dg\|_{L_\infty(\Omega)} + \rho_{Dg}(r) + \omega_f(r). \end{aligned}$$

Due to our assumptions, they are both Dini functions. The next proposition plays a key role in proving Theorem 3.2.4.

Proposition 3.4.1. *Assume that* $u \in C^2(\overline{\Omega})$ *solves* (3.16) *in a bounded domain* Ω. *If* $\boldsymbol{A}$ *and* f *are* L_1*-mean Dini, and* $\partial\Omega, \boldsymbol{\beta}, g \in C^{1,Dini}$, *then for any* $x \in \overline{\Omega}, p \in (0,1), \kappa \in (0,1)$, *and $r \le R_0$, we have*

$$\phi(x,\kappa r) \le C\kappa\phi(x,r)$$
$$+ C(\kappa^{-d/p}+\kappa)\Big((\|D^2u\|_{L_\infty(\Omega_r(x))} + \|Du\|_{L_\infty(\Omega_r(x))} + \|Dg\|_{L_\infty(\Omega)})\omega^{(1)}(r) + \omega^{(2)}(r)\Big), \quad (3.17)$$

where $C = C(d,p,\lambda,\delta,\|\beta\|_{C^1(\Omega)},\|D\psi_0\|_{L_\infty(\Omega)},\rho_{D\psi_0})$.

The rest of this section will be devoted to its proof.

3.4.1 Homogeneous case

We first consider the equation with constant coefficients and homogeneous boundary condition. Let $(\overline{a_{ij}})$ be a constant symmetric matrix with elliptic constant λ. For $x \in \partial\Omega$, we choose the coordinate system $y = (y_1,\ldots,y_d)$ centered at x as before. Recall that in Ω_{R_0}, we have

$$\frac{\partial\psi}{\partial y_d} \ge \delta|D\psi|/2,$$

i.e., the y_d-direction is oblique. For $r \le R_0$, denote $\Gamma_r := \partial\Omega \cap B_r$.

Lemma 3.4.2. *Assume that $v \in W^2_2(\Omega_r)$ and $f \in L_2(\Omega_r)$ satisfy*

$$\begin{cases} -\overline{a_{ij}}D_{ij}v = f & \text{in } \Omega_r, \\ \frac{\partial v}{\partial y_d} = 0 & \text{on } \Gamma_r. \end{cases} \quad (3.18)$$

Then for each $\kappa \in (0,1)$, we can find some constant matrix $\boldsymbol{V}_{\kappa r} \in \mathbb{S}^{d\times d}$ such that

$$\Big(\fint_{\Omega_{\kappa r}} |D^2v - \boldsymbol{V}_{\kappa r}|^p\Big)^{1/p} \le C\kappa\Big(\fint_{\Omega_r}|D^2v - \boldsymbol{q}|^p\Big)^{1/p}$$
$$+ C(\kappa^{-d/p}+\kappa)\Big((|f-(f)_{\Omega_r}|)_{\Omega_r} + \rho_{D\psi_0}(r)(|D^2v|^2)^{1/2}_{\Omega_r}\Big) \quad (3.19)$$

holds for any $\boldsymbol{q} \in \mathbb{S}^{d\times d}$, *where* $C = C(d, p, \lambda, \delta, \|D\psi_0\|_{L_\infty(\Omega)})$ *is a constant.*

Proof. Clearly we only need to prove (3.19) for small κ. Here we consider $\kappa < \delta/(4M)$ where $M := \max\{c\delta, 2c/\inf_{\partial\Omega}|D\psi|\}$ with c given as in (3.6).

In Ω_r, we flatten the boundary by taking the change of variables

$$y \in \Omega_r \cup \Gamma_r \to z \in \overline{\mathbb{R}^d_+}, \quad z_i(y) = y_i, \ i < d, \quad z_d(y) = \psi(y). \tag{3.20}$$

In the z-variables, (3.18) becomes

$$\begin{cases} -\overline{a_{ij}}\frac{\partial z_k}{\partial y_i}\frac{\partial z_l}{\partial y_j}\frac{\partial^2}{\partial z_k \partial z_l}v = f + \overline{a_{ij}}\frac{\partial^2 \psi}{\partial y_i \partial y_j}\frac{\partial v}{\partial z_d} & \text{in } z(\Omega_r) \subset \mathbb{R}^d_+, \\ \frac{\partial v}{\partial z_d} = 0 & \text{on } z(\Gamma_r) \subset \{z_d = 0\}. \end{cases}$$

In the sequel, we denote

$$\widetilde{\boldsymbol{A}} = (\widetilde{a_{kl}})_{k,l} := \Big(\overline{a_{ij}}\frac{\partial z_k}{\partial y_i}\frac{\partial z_l}{\partial y_j}\Big)_{k,l}.$$

Due to (3.6) and our choice of M, $\widetilde{\boldsymbol{A}}$ is uniform Dini with

$$\rho_{\widetilde{A}}\Big(\frac{\delta}{M}r\Big) \le C(d, \lambda, \|D\psi\|_{L_\infty(\Omega)})\rho_{D\psi_0}(r). \tag{3.21}$$

Since $\frac{\partial \psi}{\partial y_d} \ge \frac{\delta}{2}|D\psi|$ in Ω_r, for any two points $z^{(1)}, z^{(2)} \in z(\Omega_r)$ we have

$$\begin{aligned} |z^{(1)} - z^{(2)}|^2 &= \sum_{i<d}|y_i^{(1)} - y_i^{(2)}|^2 + |\psi(y^{(1)}) - \psi(y^{(2)})|^2 \\ &\ge \sum_{i<d}|y_i^{(1)} - y_i^{(2)}|^2 + \Big(\frac{\delta}{2}|D\psi|\Big)^2|y_d^{(1)} - y_d^{(2)}|^2 \\ &\ge \Big(\frac{\delta}{2}|D\psi|\Big)^2|y^{(1)} - y^{(2)}|^2 \ge \Big(\frac{\delta c}{M}\Big)^2|y^{(1)} - y^{(2)}|^2, \end{aligned}$$

and similarly

$$|z^{(1)} - z^{(2)}|^2 \leq 2|y^{(1)} - y^{(2)}|^2.$$

From these and $\kappa < \delta/(4M)$, we know

$$z(\Omega_{\kappa r}) \subset B^+_{2\kappa r} \subset B^+_{(\delta/2M)r} \subset B^+_{(\delta/M)r} \subset z(\Omega_{r/c}) \subset z(\Omega_r). \tag{3.22}$$

In $B^+_{(\delta/M)r}$, we decompose $v = \theta + \xi$, where $\theta \in W^2_2(B^+_{(\delta/M)r})$ is a strong solution to

$$\begin{cases} -(\widetilde{a_{kl}})_{B^+_{(\delta/M)r}} \frac{\partial^2}{\partial z_k \partial z_l}\theta & = (\widetilde{a_{kl}} - (\widetilde{a_{kl}})_{B^+_{(\delta/M)r}}) \frac{\partial^2}{\partial z_k \partial z_l} v + \overline{a_{ij}} \frac{\partial^2 \psi}{\partial y_i \partial y_j} \frac{\partial v}{\partial z_d} + f - (f)_{\Omega_r} \text{ in } B^+_{(\delta/M)r}, \\ \frac{\partial \theta}{\partial z_d} & = 0 \text{ on } \Sigma_{(\delta/M)r}, \\ \theta & = 0 \text{ on } \sigma_{(\delta/M)r}. \end{cases} \tag{3.23}$$

Recall that the notation $(f)_{\Omega_r}$ stands for the average. Due to (3.7), $\frac{\partial v}{\partial z_d} = 0$ on $z_d = 0$, and Hardy's inequality, we have

$$\left|\overline{a_{ij}} \frac{\partial^2 \psi}{\partial y_i \partial y_j} \frac{\partial v}{\partial z_d}\right| \leq C \rho_{D\psi_0}(r) \frac{1}{z_d} \left|\frac{\partial v}{\partial z_d}\right| \in L_2(B^+_{(\delta/M)r}),$$

with

$$\left\|\overline{a_{ij}} \frac{\partial^2 \psi}{\partial y_i \partial y_j} \frac{\partial v}{\partial z_d}\right\|_{L_2(B^+_{(\delta/M)r})} \leq C \rho_{D\psi_0}(r) \|D^2_z v\|_{L_2(B^+_{(\delta/M)r})}. \tag{3.24}$$

Such solution $\theta \in W^2_2(B^+_{(\delta/M)r})$ exists. Indeed, we first reduce (3.23) to a Dirichlet problem in $B_{(\delta/M)r}$ by taking the even extension in z_d for θ and the source term, and the following extension for the leading coefficients

$$\hat{a}_{kl} = \begin{cases} (\widetilde{a_{kl}})_{B^+_{(\delta/M)r}} \; z_d \geq 0, \\ \varepsilon_k \varepsilon_l (\widetilde{a_{kl}})_{B^+_{(\delta/M)r}} \; z_d < 0, \end{cases} \qquad \text{where } \varepsilon_k = \begin{cases} +1 \text{ when } k \neq d, \\ -1 \text{ when } k = d. \end{cases}$$

Note that the extended problem has measurable coefficients only depending on z_d, which are also continuous (actually equal to constants) near $z_d = \pm 1$. Then the W^2_2 solvability follows from [25, Theorem 2.8]. See also the example on [25, pp. 6483].

Using Corollary 3.3.2, (3.24), and Hölder's inequality, we obtain

$$\begin{aligned}(|D^2\theta|^p)^{1/p}_{B^+_{(\delta/M)r}} &\lesssim \rho_{\widetilde{A}}\Big(\frac{\delta}{M}r\Big)(|D^2_z v|^2)^{1/2}_{B^+_{(\delta/M)r}} + \rho_{D\psi_0}(r)(|D^2_z v|^2)^{1/2}_{B^+_{(\delta/M)r}} + (|f-(f)_{\Omega_r}|)_{B^+_{(\delta/M)r}} \\ &\lesssim \rho_{D\psi_0}(r)(|D^2_z v|^2)^{1/2}_{B^+_{(\delta/M)r}} + (|f-(f)_{\Omega_r}|)_{B^+_{(\delta/M)r}},\end{aligned} \tag{3.25}$$

where in the last inequality, we used (3.21). Here the implicit constant depends on d, p, λ, and $\|D\psi_0\|_{L_\infty(\Omega)}$.

Now $\xi := v - \theta \in W^2_2(B^+_{(\delta/M)r})$ satisfies

$$\begin{cases} -(\widetilde{a_{kl}})_{B^+_{(\delta/M)r}} \frac{\partial^2}{\partial z_k \partial z_l}\xi = (f)_{\Omega_r} & \text{in } B^+_{(\delta/M)r}, \\ \frac{\partial \xi}{\partial z_d} = 0 & \text{on } \Sigma_{(\delta/M)r}. \end{cases}$$

Noting (3.22), a rescaled version of Lemma 3.3.3 leads to

$$\begin{aligned}\Big(\fint_{B^+_{2\kappa r}} |D^2\xi - (D^2\xi)_{B^+_{2\kappa r}}|^p\Big)^{1/p} &\le C(d)\kappa r\|D^3\xi\|_{L_\infty(B^+_{2\kappa r})} \\ \le C(d)\kappa r\|D^3\xi\|_{L_\infty(B^+_{(\delta/2M)r})} &\le C(d,p,\lambda)\kappa(|D^2\xi - \widetilde{\boldsymbol{q}}|^p)^{1/p}_{B^+_{(\delta/M)r}}\end{aligned}$$

with $\widetilde{\boldsymbol{q}} \in \mathbb{S}^{d\times d}$ to be chosen later.

Combining this and (3.24), we obtain

$$\begin{aligned}&\Big(\fint_{B^+_{2\kappa r}} |D^2_z v - (D^2\xi)_{B^+_{2\kappa r}}|^p\Big)^{1/p} \\ &\le 2^{1/p-1}\Big(\Big(\fint_{B^+_{2\kappa r}} |D^2\xi - (D^2\xi)_{B^+_{2\kappa r}}|^p\Big)^{1/p} + \Big(\fint_{B^+_{2\kappa r}} |D^2_z\theta|^p\Big)^{1/p}\Big)\end{aligned}$$

$$
\begin{aligned}
&\lesssim \kappa(|D^2\xi - \widetilde{\boldsymbol{q}}|^p)^{1/p}_{B^+_{(\delta/M)r}} + \Big(\fint_{B^+_{2\kappa r}} |D_z^2\theta|^p \Big)^{1/p} \\
&\lesssim \kappa(|D_z^2 v - \widetilde{\boldsymbol{q}}|^p)^{1/p}_{B^+_{(\delta/M)r}} + (\kappa^{-d/p} + \kappa)\Big(\fint_{B^+_{(\delta/M)r}} |D_z^2\theta|^p \Big)^{1/p} \\
&\lesssim \kappa(|D_z^2 v - \widetilde{\boldsymbol{q}}|^p)^{1/p}_{B^+_{(\delta/M)r}} + (\kappa^{-d/p} + \kappa)\Big(\rho_{D\psi_0}(r)(|D_z^2 v|^2)^{1/2}_{B^+_{(\delta/M)r}} + (|f - (f)_{\Omega_r}|)_{B^+_{(\delta/M)r}} \Big) \qquad (3.26)
\end{aligned}
$$

with the constant depending on $(d, p, \lambda, \|D\psi_0\|_{L_\infty(\Omega)})$. Now we translate back to the y-coordinates. Combining

$$
dz = \left|\frac{\partial \psi}{\partial y_d}\right| dy, \quad \frac{\delta}{2}|D\psi| \le \left|\frac{\partial \psi}{\partial y_d}\right| \le 1 \ \forall y \in \Omega_r,
$$
$$
\frac{\partial^2 v}{\partial y_i \partial y_j} = \frac{\partial^2 v}{\partial z_k \partial z_l}\frac{\partial z_k}{\partial y_i}\frac{\partial z_l}{\partial y_j} + \frac{\partial v}{\partial z_d}\frac{\partial^2 \psi}{\partial y_i \partial y_j}
$$

together with (3.22), Hardy's inequality, and Hölder's inequality, we can continue the computation from (3.26):

$$
\begin{aligned}
&\Big(\fint_{\Omega_{\kappa r}} |(\frac{\partial y}{\partial z})^T D_y^2 v \frac{\partial y}{\partial z} - (D^2\xi)_{B^+_{2\kappa r}}|^p \, dy \Big)^{1/p} \\
&\lesssim \Big(\fint_{B^+_{2\kappa r}} |D_z^2 v - (D^2\xi)_{B^+_{2\kappa r}}|^p \, dz \Big)^{1/p} + \rho_{D\psi_0}(r)(|D_z^2 v|^2)^{1/2}_{B^+_{2\kappa r}} \\
&\lesssim \kappa(|D_z^2 v - \widetilde{\boldsymbol{q}}|^p)^{1/p}_{B^+_{(\delta/M)r}} + (\kappa^{-d/p} + \kappa)\Big(\rho_{D\psi_0}(r)(|D_z^2 v|^2)^{1/2}_{B^+_{(\delta/M)r}} + (|f - (f)_{\Omega_r}|)_{B^+_{(\delta/M)r}} \Big) \\
&\lesssim \kappa\Big(\fint_{\Omega_{r/c}} |(\frac{\partial y}{\partial z})^T D_y^2 v \frac{\partial y}{\partial z} - \widetilde{\boldsymbol{q}}|^p \, dy \Big)^{1/p} + (\kappa^{-d/p} + \kappa)\Big(\rho_{D\psi_0}(r)(|D_y^2 v|^2)^{1/2}_{\Omega_r} + (|f - (f)_{\Omega_r}|)_{\Omega_r} \Big).
\end{aligned} \qquad (3.27)
$$

Here besides $(d, p, \lambda, \|D\psi_0\|_{L_\infty(\Omega)})$, the constant also depends on δ. This is almost (3.19), except that we also need to deal with $\partial z/\partial y$ coming from the change of variables. By (3.6) and the (generalized) triangular inequality,

$$
\begin{aligned}
&\Big(\fint_{\Omega_{r/c}} |(\frac{\partial y}{\partial z})^T D_y^2 v \frac{\partial y}{\partial z} - \widetilde{\boldsymbol{q}}|^p \, dy \Big)^{1/p} \\
&\le 3^{1/p-1}\Big(2\rho_{D\psi}(r/c) \|\frac{\partial y}{\partial z}\|_{L_\infty(\Omega_{r/c})} (|D_y^2 v|^2)^{1/2}_{\Omega_{r/c}} \\
&\quad + \|\frac{\partial y}{\partial z}\|^2_{L_\infty(\Omega_{r/c})} \Big(\fint_{\Omega_{r/c}} \Big| D_y^2 v - \Big(\frac{\partial z}{\partial y}(0)\Big)^T \widetilde{\boldsymbol{q}} \frac{\partial z}{\partial y}(0) \Big|^p \, dy \Big)^{1/p} \Big) \\
&\le C(d, p, \delta, \|D\psi_0\|_{L_\infty(\Omega)}) \Big(\rho_{D\psi_0}(r) (|D_y^2 v|^2)^{1/2}_{\Omega_r} + \Big(\fint_{\Omega_r} \big| D_y^2 v - \boldsymbol{q} \big|^p \, dy \Big)^{1/p} \Big),
\end{aligned}
$$

if we take

$$
\widetilde{\boldsymbol{q}} = \Big(\frac{\partial y}{\partial z}(0)\Big)^T \boldsymbol{q} \frac{\partial y}{\partial z}(0).
$$

If we also take (3.8) into consideration, similar computation leads to

$$
\begin{aligned}
&\Big(\fint_{\Omega_{\kappa r}} |\Big(\frac{\partial y}{\partial z}\Big)^T D_y^2 v \frac{\partial y}{\partial z} - (D^2 \xi)_{B^+_{2\kappa r}}|^p \, dy \Big)^{1/p} \\
&\ge C(d, p, \delta, \|D\psi\|_{L_\infty(\Omega)}) \Big(\Big(\fint_{\Omega_{\kappa r}} \big| D_y^2 v - \boldsymbol{V}_{\kappa r} \big|^p \, dy \Big)^{1/p} - \kappa^{-d/2} \rho_{D\psi_0}(r) (|D_y^2 v|^2)^{1/2}_{\Omega_r} \Big),
\end{aligned}
\tag{3.28}
$$

where

$$
\boldsymbol{V}_{\kappa r} = \Big(\frac{\partial z}{\partial y}(0)\Big)^T (D^2 \xi)_{B^+_{2\kappa r}} \frac{\partial z}{\partial y}(0).
$$

Now combining (3.27)-(3.28), we immediately obtain (3.19). ∎

3.4.2 Mean oscillation estimate for $D^2 u$

Proof of Proposition 3.4.1. First, for x satisfying $B_r(x) \subset \Omega$, in which case only the interior estimates are concerned, the decay of L_p-mean oscillation can be found in

[24, pp. 427]. Actually we have

$$\phi(x,\kappa r) \le N\kappa\phi(x,r) + N(\kappa^{-d/p}+\kappa)(\|D^2u\|_{L_\infty(\Omega_r(x))} + \omega_f(r)).$$

By a standard argument, it suffices to consider the case when $x \in \partial\Omega$. Choose the coordinate system y centered at x as in Section 3.3.1. We now reduce the original problem to the homogeneous case (3.18). For this, we introduce two auxiliary functions. From Theorem 3.1.1, we can find $w \in W^2_d(\Omega)$ solving

$$\begin{cases} -\Delta w + w = 0 & \text{in}\ \Omega, \\ \eta\boldsymbol{\beta}(0)\cdot Dw + (1-\eta)\boldsymbol{\beta}\cdot Dw = -\overline{\boldsymbol{\beta}}\cdot Du - \eta(y\cdot D\boldsymbol{\beta}(0))(Du - Du(0)) + \overline{g} & \text{on}\ \partial\Omega. \end{cases} \tag{3.29}$$

where

$$\overline{\boldsymbol{\beta}} := \eta(\boldsymbol{\beta} - \boldsymbol{\beta}(0) - (y\cdot D)\boldsymbol{\beta}(0)), \quad \overline{g} := \eta(g - g(0) - (y\cdot D)g(0)).$$

In the above, $\eta \in C^\infty_c(B_r)$ is taken to be a usual cut-off function satisfying $\eta = 1$ on $B_{r/2}$ and $|D\eta| \lesssim 1/r$. Due to our previous choice of R_0, we know the boundary condition is uniform oblique because

$$\eta\boldsymbol{\beta}(0)\cdot D\psi + (1-\eta)\boldsymbol{\beta}\cdot D\psi \ge \eta\frac{\delta}{M}|\boldsymbol{\beta}(0)| + (1-\eta)\delta|\boldsymbol{\beta}| \ge \delta\min\{|\boldsymbol{\beta}(0)|/M, |\boldsymbol{\beta}|\}.$$

Applying the Aleksandrov-Bakelman-Pucci estimate and the W^2_d-a priori estimate in [28, Theorem 2.3], we obtain

$$\|w\|_{W^2_d(\Omega)} \le C\Big(\|\overline{\boldsymbol{\beta}}\cdot Du\|_{W^1_d(\Omega)} + \|\eta(y\cdot D\boldsymbol{\beta}(0))(Du - Du(0))\|_{W^1_d(\Omega)} + \|\overline{g}\|_{W^1_d(\Omega)}\Big), \tag{3.30}$$

where C is a constant depending on $(d, \delta, \|\beta\|_{C^1}, \|D\psi_0\|_{L_\infty(\Omega)}, \rho_{D\psi_0})$. From this, we obtain

$$
\begin{aligned}
&(|D^2 w|^d)^{1/d}_{\Omega_{r/2}} \\
&\le C\Big((r\|D\beta\|_{L_\infty} + \rho_{D\beta}(r))(\|D^2 u\|_{L_\infty(\Omega_r)} + \|Du\|_{L_\infty(\Omega_r)}) + r\|Dg\|_{L_\infty} + \rho_{Dg}(r)\Big).
\end{aligned} \tag{3.31}
$$

Next, we consider the parabola

$$
h := -\Big(y \cdot D\boldsymbol{\beta}(0)\Big)Du(0)y_d + g(0)y_d + y \cdot Dg(0)y_d + \frac{y_d^2(D_d\boldsymbol{\beta}(0)) \cdot Du(0) - y_d^2 D_d g(0)}{2}, \tag{3.32}
$$

which satisfies

$$
\frac{\partial h}{\partial y_d} = -\Big(y \cdot D\boldsymbol{\beta}(0)\Big)Du(0) + g(0) + y \cdot Dg(0).
$$

From our construction, $v := u - w - h$ satisfies

$$
\begin{cases}
-(a_{ij})_{\Omega_{r/2}} D_{ij} v = \widetilde{f} := -((a_{ij})_{\Omega_{r/2}} - a_{ij})D_{ij}u + f + (a_{ij})_{\Omega_{r/2}}(D_{ij}h + D_{ij}w) \quad \text{in } \Omega_{r/2}, \\
\frac{\partial v}{\partial y_d} = 0 \quad \text{on } \Gamma_{r/2}.
\end{cases}
$$

Clearly $\widetilde{f} \in L_2(\Omega_{r/2})$. By the triangular inequality and Hölder's inequality,

$$
(|\widetilde{f} - (\widetilde{f})_{\Omega_{r/2}}|)_{\Omega_{r/2}} \lesssim \omega_A(r/2)\|D^2 u\|_{L_\infty(\Omega_{r/2})} + \omega_f(r/2) + (|D^2 w|^d)^{1/d}_{\Omega_{r/2}}. \tag{3.33}
$$

Now we apply Lemma 3.4.2 with κ, r, and f replaced by 2κ, $r/2$, and $\widetilde{f}$ to obtain that there exists some $\boldsymbol{V}_{\kappa r} \in \mathbb{S}^{d\times d}$ such that for any $\boldsymbol{q} \in \mathbb{S}^{d\times d}$,

$$
\begin{aligned}
\Big(\fint_{\Omega_{\kappa r}} |D^2 v - \boldsymbol{V}_{\kappa r}|^p\Big)^{1/p} \le & C\Big(\kappa\Big(\fint_{\Omega_{r/2}} |D^2 v - \boldsymbol{q}|^p\Big)^{1/p} \\
& + (\kappa^{-d/p} + \kappa)((|\widetilde{f} - (\widetilde{f})_{\Omega_{r/2}}|)_{\Omega_{r/2}} + \rho_{D\psi_0}(r/2)(|D^2 v|^2)^{1/2}_{\Omega_{r/2}})\Big).
\end{aligned}
$$

Noting $u = v + w + h$, we can further estimate the mean oscillation of D^2u by

$$
\begin{aligned}
&\Big(\fint_{\Omega_{\kappa r}} |D^2u - (\boldsymbol{V}_{\kappa r} + D^2h)|^p\Big)^{1/p} \\
&\leq C\kappa\Big(\fint_{\Omega_{r/2}} |D^2u - (\boldsymbol{q} + D^2h)|^p\Big)^{1/p} + C(\kappa^{-d/p} + \kappa)\Big((|\widetilde{f} - (\widetilde{f})_{\Omega_{r/2}}|)_{\Omega_{r/2}} + (|D^2w|^d)^{1/d}_{\Omega_{r/2}} \\
&\quad + \rho_{D\psi_0}(r/2)(\|D^2u\|_{L_\infty(\Omega_{r/2})} + \|D\boldsymbol{\beta}\|_{L_\infty(\Omega)}\|Du\|_{L_\infty(\Omega_{r/2})} + \|Dg\|_{L_\infty(\Omega)})\Big)
\end{aligned} \tag{3.34}
$$

by applying the (generalized) triangular inequality and Hölder's inequality. Here, we also used the fact that $D^2h \in \mathbb{S}^{d\times d}$ is a constant matrix and the inequality

$$
|D^2h| \leq C\|D\boldsymbol{\beta}\|_{L_\infty(\Omega)}\|Du\|_{L_\infty(\Omega_{r/2})} + \|Dg\|_{L_\infty(\Omega)}.
$$

Now substituting the w and $\widetilde{f}$ terms in (3.34) by the corresponding estimates (3.31) and (3.33), taking infimum in $\boldsymbol{q}$, we obtain (3.17). The proposition is proved. ■

3.5 Proof of Theorem 3.2.4

With all these preparations, we are ready to give the proof of our main results. To begin with, we make some reductions. Rewrite (3.1) as

$$
\begin{cases}
-a_{ij}D_{ij}u + u = f - b_iD_iu - (c-1)u & \text{in } \Omega, \\
\beta_iD_iu = g - \beta_0u & \text{on } \partial\Omega.
\end{cases}
$$

From [70], $\int_0^1 \omega_f(r)/r\,dr < \infty$ implies that f is uniformly continuous, hence, is bounded. Also, by the Sobolev embedding, we have

$$b_i D_i u + (c-1)u,\ \beta_0 u \in L_{(1+\varepsilon)^*}(\Omega),$$

where

$$(1+\varepsilon)^* = \begin{cases} (1+\varepsilon)d/(d-1-\varepsilon) & \text{when}\, d > 1+\varepsilon, \\ d+1 & \text{when}\, d \le 1+\varepsilon. \end{cases}$$

By Theorem 3.1.1, we have $u \in W^2_{(1+\varepsilon)^*}$. Repeating if needed, in finite steps we obtain that

$$u \in W^2_{d+1} \subset C^{1,1/(d+1)}.$$

Since the coefficients $\boldsymbol{b}$ and c have L_1-mean Dini oscillations, and $\beta_0 \in C^{1,\mathrm{Dini}}$, we can deduce that $b_i D_i u$ and cu are of L_1-mean Dini, and $\beta_0 u \in C^{1,\mathrm{Dini}}$. Now, by moving all the lower-order terms to the right-hand side we only need to consider the equation (3.16). Also, due to the approximation given at the end of this section, we only need to prove an a priori estimate. In other words, we will estimate the modulus of continuity of $D^2 u$ assuming that $u \in C^2(\overline{\Omega})$. Under all these reductions, Proposition 3.4.1 applies.

First we will derive the decay of L_p-mean oscillation $\phi(x,r)$ from (3.17). From now on, we fix some $p \in (0,1)$ and then choose $\kappa \in (0,1)$ small enough such that $C\kappa < 1/2$ in (3.17) to get, for any $r \le R_0$,

$$\phi(x,\kappa r) \le \frac{1}{2}\phi(x,r) + C(\kappa)\Big((\|D^2 u\|_{L_\infty(\Omega)} + \|Du\|_{L_\infty(\Omega)} + \|Dg\|_{L_\infty(\Omega)})\omega^{(1)}(r) + \omega^{(2)}(r)\Big).$$

Applying this j times, we have

$$
\begin{aligned}
&\phi(x,\kappa^j r) \le (1/2)^j \phi(x,r) \\
&\quad + C\Big((\|D^2u\|_{L_\infty(\Omega)} + \|Du\|_{L_\infty(\Omega)} + \|Dg\|_{L_\infty(\Omega)})\widetilde{\omega}^{(1)}(\kappa^{j-1}r) + \widetilde{\omega}^{(2)}(\kappa^{j-1}r)\Big),
\end{aligned}
\tag{3.35}
$$

where $\widetilde{\omega}^1$ and $\widetilde{\omega}^{(2)}$ are Dini functions derived from $\omega^{(1)}$ and $\omega^{(2)}$ as in (3.15).

Second, we estimate $\|D^2u\|_{L_\infty(\Omega)}$. For any point $x \in \overline{\Omega}$, we take $\boldsymbol{q}_{x,r}$ as a minimizer in $\phi(x,r)$, which exists as explained in Section 3.3.3. By the triangular inequality,

$$
|\boldsymbol{q}_{x,\kappa^{j+1}r} - \boldsymbol{q}_{x,\kappa^j r}|^p \le |\boldsymbol{q}_{x,\kappa^{j+1}r} - D^2u(z)|^p + |D^2u(z) - \boldsymbol{q}_{x,\kappa^j r}|^p
$$

holds for any integer $j \ge 0$. Taking the average for $z \in \Omega_{\kappa^{j+1}r}(x)$, we obtain

$$
|\boldsymbol{q}_{x,\kappa^{j+1}r} - \boldsymbol{q}_{x,\kappa^j r}| \le 2^{1/p-1}\Big(\phi(x,\kappa^{j+1}r) + C(d)\kappa^{-d/p}\phi(x,\kappa^j r)\Big).
$$

Now taking summation in j, and using the property

$$
\boldsymbol{q}_{x,\kappa^j r} \to D^2u(x) \quad \text{as } j \to \infty
$$

as noted in Section 3.3.3, we have

$$
\begin{aligned}
&|D^2u(x) - \boldsymbol{q}_{x,r}| \le \sum_{j=0}^{\infty} |\boldsymbol{q}_{x,\kappa^{j+1}r} - \boldsymbol{q}_{x,\kappa^j r}| \le C(d,p,\kappa) \sum_{j=0}^{\infty} \phi(x,\kappa^j r) \\
&\lesssim C_1(d,p,\kappa)\phi(x,r) + C_2\Big((\|D^2u\|_{L_\infty(\Omega)} + \|Du\|_{L_\infty(\Omega)} + \|Dg\|_{L_\infty(\Omega)})I\widetilde{\omega}^{(1)}(r) + I\widetilde{\omega}^{(2)}(r)\Big),
\end{aligned}
$$

where $C_2 = C_2(d,p,\lambda,\delta,\|\beta\|_{C^1},\|D\psi_0\|_{L_\infty(\Omega)},\rho_{D\psi_0},\kappa)$. The last inequality follows from (3.35) and Lemma 3.3.4, noting that both $\omega^{(1)}$ and $\omega^{(2)}$ satisfy (3.14). Using the interpolation inequality, we can further deduce that

$$
\begin{aligned}
&|D^2u(x) - \boldsymbol{q}_{x,r}| \\
&\le C_1\phi(x,r) + C_2\Big((\|D^2u\|_{L_\infty(\Omega)} + \|Du\|_{L_1(\Omega)} + \|Dg\|_{L_\infty(\Omega)})I\widetilde{\omega}^{(1)}(r) + I\widetilde{\omega}^{(2)}(r)\Big).
\end{aligned} \tag{3.36}
$$

By the definition of $\phi(x,r)$, Hölder's inequality, and (3.9), we get

$$
\phi(x,r) \le \Big(\fint_{\Omega_r(x)} |D^2u|^p\Big)^{1/p} \le \fint_{\Omega_r(x)} |D^2u| \lesssim r^{-d}\|D^2u\|_{L_1(\Omega)}. \tag{3.37}
$$

Similarly,

$$
|\boldsymbol{q}_{x,r}| \le 2^{1/p-1}\Big(\phi(x,r) + \Big(\fint_{\Omega_r(x)} |D^2u|^p\Big)^{1/p}\Big) \lesssim r^{-d}\|D^2u\|_{L_1(\Omega)}, \tag{3.38}
$$

where the first inequality follows by taking the average for $z \in \Omega_r(x)$ on both sides of

$$
|\boldsymbol{q}_{x,r}|^p \le |\boldsymbol{q}_{x,r} - D^2u(z)|^p + |D^2u(z)|^p.
$$

Using the triangular inequality, (3.37), and (3.38), we can derive from (3.36) that

$$
\begin{aligned}
|D^2u(x)| &\le Cr^{-d}\|D^2u\|_{L_1(\Omega)} \\
&\quad + C_2\Big((\|D^2u\|_{L_\infty(\Omega)} + \|Du\|_{L_1(\Omega)} + \|Dg\|_{L_\infty(\Omega)})I\widetilde{\omega}^{(1)}(r) + I\widetilde{\omega}^{(2)}(r)\Big).
\end{aligned}
$$

Because Ω is bounded and $D^2u \in C(\overline{\Omega})$, we can find some point $\bar{x} \in \overline{\Omega}$ such that

$$
|D^2u(\bar{x})| = \|D^2u\|_{L_\infty(\Omega)}.
$$

Since $\widetilde{\omega}^{(1)}$ is a Dini function, we can choose r small enough (denoted by $\bar{r}$) such that

$$
C_2 I\widetilde{\omega}^{(1)}(\bar{r}) \le 1/2
$$

to absorb $\|D^2u\|_{L_\infty(\Omega)}$ term. Finally we reach

$$\|D^2u\|_{L_\infty(\Omega)} \le C(\|u\|_{W^2_1(\Omega)} + \|Dg\|_{L_\infty(\Omega)} + I\widetilde{\omega}^{(2)}(\bar{r})). \tag{3.39}$$

Next for any $x, y \in \Omega$, $x \neq y$, we estimate $D^2u(x) - D^2u(y)$. Let $r = |x - y|$. Due to (3.39), we only need to focus on the case $r < R_0/2$. As before, we take the minimizers $\boldsymbol{q}_{x,2r}$ and $\boldsymbol{q}_{y,r}$ for $\phi(x, 2r)$ and $\phi(y, r)$. By the triangular inequality

$$\begin{aligned}
&|D^2u(x) - D^2u(y)|^p \\
&\le |D^2u(x) - \boldsymbol{q}_{x,2r}|^p + |D^2u(y) - \boldsymbol{q}_{y,r}|^p + |\boldsymbol{q}_{x,2r} - D^2u(z)|^p + |\boldsymbol{q}_{y,r} - D^2u(z)|^p.
\end{aligned}$$

Taking the average for $z \in \Omega_r(y)$, noting that $\Omega_r(y) \subset \Omega_{2r}(x)$, we have

$$\begin{aligned}
&|D^2u(x) - D^2u(y)|^p \\
&\le |D^2u(x) - \boldsymbol{q}_{x,2r}|^p + |D^2u(y) - \boldsymbol{q}_{y,r}|^p + C(d, \delta/M)\phi^p(x, 2r) + \phi^p(y, r).
\end{aligned}$$

Now we apply (3.36), (3.39), and the generalized triangular inequality to obtain

$$\begin{aligned}
&|D^2u(x) - D^2u(y)| \\
&\lesssim \phi(x, 2r) + \phi(y, r) + \Big((\|u\|_{W^2_1(\Omega)} + \|Dg\|_{L_\infty(\Omega)} + I\widetilde{\omega}^{(2)}(\bar{r}))I\widetilde{\omega}^{(1)}(r) + I\widetilde{\omega}^{(2)}(r)\Big).
\end{aligned} \tag{3.40}$$

Let j be the integer such that

$$2r \in (\kappa^{j+1}R_0, \kappa^j R_0].$$

From (3.35), interpolation inequalities, (3.39), and (3.9), we obtain

$$\phi(x, 2r)$$

$$
\begin{aligned}
&\le (\frac{1}{2})^j \phi(x, 2\kappa^{-j} r) + C(\|u\|_{W_1^2(\Omega)} + \|Dg\|_{L_\infty(\Omega)} + I\widetilde{\omega}^{(2)}(\bar{r}))\widetilde{\omega}^{(1)}(2\kappa^{-1} r) + C\widetilde{\omega}^{(2)}(2\kappa^{-1} r)\\
&\lesssim (\frac{r}{R_0})^{\alpha_0} \phi(x, 2\kappa^{-j} r) + (\|u\|_{W_1^2(\Omega)} + \|Dg\|_{L_\infty(\Omega)} + I\widetilde{\omega}^{(2)}(\bar{r}))\widetilde{\omega}^{(1)}(2\kappa^{-1} r) + \widetilde{\omega}^{(2)}(2\kappa^{-1} r)\\
&\lesssim (\frac{r}{R_0})^{\alpha_0} R_0^{-d} \|D^2 u\|_{L_1(\Omega)} + (\|u\|_{W_1^2(\Omega)} + \|Dg\|_{L_\infty(\Omega)} + I\widetilde{\omega}^{(2)}(\bar{r}))\widetilde{\omega}^{(1)}(2\kappa^{-1} r)\\
&\qquad + \widetilde{\omega}^{(2)}(2\kappa^{-1} r),
\end{aligned}
$$

where $\alpha_0 = \log(2)/\log(1/\kappa) > 0$. Similarly, we can obtain the decay rate of $\phi(y, r)$. Substituting these into (3.40) and using Lemma 3.3.4 to bound $\widetilde{\omega}$ by $I\widetilde{\omega}$, we obtain that for any $|x - y| < R_0/2$,

$$
\begin{aligned}
&|D^2 u(x) - D^2 u(y)| \lesssim \|D^2 u\|_{L_1(\Omega)} |x - y|^{\alpha_0}\\
&\quad + (\|u\|_{W_1^2(\Omega)} + \|Dg\|_{L_\infty(\Omega)} + I\widetilde{\omega}^{(2)}(\bar{r})) I\widetilde{\omega}^{(1)}(|x - y|) + I\widetilde{\omega}^{(2)}(|x - y|).
\end{aligned}
\tag{3.41}
$$

Clearly, the right-hand-side goes to zero as r goes to zero, which gives us the desired estimate for the modulus of continuity of $D^2 u$.

Now it remains to remove the assumption $u \in C^2(\overline{\Omega})$. For $u \in W_{d+1}^2(\Omega)$, we consider the mollified problem:

$$
\begin{cases}
-a_{ij}^{(n)} D_{ij} u_n + u_n = f^{(n)} + u & \text{in } \Omega^{(n)},\\
\beta^{(n)} \cdot D u_n = g^{(n)} & \text{on } \partial\Omega^{(n)},
\end{cases}
$$

where for some fixed $\alpha \in (0, 1)$,

$$
\Omega^{(n)} \nearrow \Omega, \quad \Omega^{(n)}, \boldsymbol{\beta}^{(n)}, g^{(n)} \in C^{1,\alpha}, \quad a_{ij}^{(n)}, f^{(n)} \in C^\alpha
$$

with corresponding moduli of continuity (either in the L_1 or L_∞ sense), which are uniform with respect to n. According to [50], we can find a unique solution $u_n \in C^{2,\alpha}(\Omega^{(n)})$ for each n. Due to the same reason for (3.30), $\{u_n\}$ are uniformly bounded in W^2_d, hence in W^2_1. Now, using (3.39), (3.41), the Arzela-Ascoli theorem, and a diagonal argument, we can obtain a subsequence which converges in $C^2(\overline{\Omega^{(k)}})$ for every k. Clearly the limit $u_\infty \in C^2(\overline{\Omega})$ and satisfies the equation. To see that u_∞ satisfies the boundary condition, we extend

$$G^{(n)} := \boldsymbol{\beta}^{(n)} \cdot Du_n - g^{(n)} \in \mathring{W}^1_d(\Omega^{(n)})$$

to be zero outside $\Omega^{(n)}$, so that $G^{(n)} \in \mathring{W}^1_d(\Omega)$. Since the $W^1_d(\Omega)$ norm of $G^{(n)}$ is uniformly bounded, by passing to a further subsequence and noting that $\mathring{W}^1_d(\Omega)$ is weakly closed in $W^1_d(\Omega)$, we obtain $\boldsymbol{\beta} \cdot Du_\infty - g = 0$. Due to the uniqueness part of Theorem 3.1.1, $u_\infty = u$. This finishes the proof of Theorem 3.2.4.

3.6 Fully nonlinear equations

Our method can also be applied to derive the $C^2(\overline{\Omega})$ regularity for fully nonlinear equations. In this section, we prove Theorem 3.2.7. As preparation, we first introduce two lemmas, which can be viewed as the nonlinear version of Lemma 3.3.3 and Corollary 3.3.2, as well as an interpolation inequality. The first lemma deals with the function $F : \mathbb{S}^{d\times d} \to \mathbb{R}$, satisfying

$$F \text{ is concave}, \quad \lambda\|M\| \le F(M+N) - F(N) \le \lambda^{-1}\|M\|, \ \forall M, N \in \mathbb{S}^{d\times d}, \text{ and } M \ge 0.$$

Lemma 3.6.1. *For any continuous function φ and constant C, there exists a unique solution u in $C^2(B_1^+ \cup \Sigma_1) \cap C(\overline{B_1^+})$ to*

$$F(D^2u) = C \ in\, B_1^+, \quad \frac{\partial u}{\partial x^d} = 0 \ on\, \Sigma_1, \quad u = \varphi \ on\, \sigma_1. \tag{3.42}$$

Furthermore, there exists some constant $\overline{\alpha} = \overline{\alpha}(d, \lambda) \in (0, 1)$, *such that* $u \in C^{2,\overline{\alpha}}_{loc}(B_1^+ \cup \Sigma_1)$, *and*

$$[D^2 u]_{C^{\overline{\alpha}}(B^+_{1/2})} \le N(d, \lambda, p)(|D^2 u - \boldsymbol{q}|^p)^{1/p}_{B_1^+} \tag{3.43}$$

holds for any $p \in (0, 1)$ *and any constant matrix* $\boldsymbol{q} \in \mathbb{S}^{d\times d}$.

Proof. The unique solvability of (3.42) and the following boundary estimate of the Evans-Krylov type for Neumann problem are classical:

$$[D^2 u]_{C^{\overline{\alpha}}(B^+_{1/2})} \le N(d, \lambda)\|u\|_{L_2(B_1^+)}. \tag{3.44}$$

See, for example, [65, Theorem 8.1]. Now we prove

$$[D^2 u]_{C^{\overline{\alpha}}(B^+_{1/2})} \le N(d, \lambda)(|D^2 u - \boldsymbol{q}|^2)^{1/2}_{B_1^+}, \quad \forall \boldsymbol{q} \in \mathbb{S}^{d\times d}. \tag{3.45}$$

For this, consider

$$v(x) := u(x) - \langle x, \overline{\boldsymbol{q}} x\rangle/2 - l(x'), \quad \overline{\boldsymbol{q}} := \begin{bmatrix} \boldsymbol{q}' & 0 \\ 0 & q_{dd} \end{bmatrix}.$$

where $\boldsymbol{q}' \in \mathbb{S}^{(d-1)\times(d-1)}$ is the first $(d-1)\times(d-1)$ submatrix of $\boldsymbol{q}$, $x' = (x_1, \ldots, x_{d-1})$, and $l(x')$ is the affine function chosen suitably to make $(v)_{B^+_{3/4}} = (D_{x'} v)_{B^+_{3/4}} = 0$. Then $D^2 v = D^2 u - \overline{\boldsymbol{q}}$, and v satisfies

$$G(D^2 v) := F(D^2 v + \overline{\boldsymbol{q}}) = C \text{ in } B_1^+, \quad \frac{\partial v}{\partial x^d} = 0 \text{ on } \Sigma_1.$$

From a rescaled version of (3.44) for v, the Sobolev-Poincaré inequality, and the boundary Poincaré inequality, we obtain

$$[D^2 v]_{C^{\bar\alpha}(B^+_{1/2})} \lesssim (|v|^2)^{1/2}_{B^+_{3/4}}$$
$$\lesssim (|D_{x'}v|^2)^{1/2}_{B^+_{3/4}} + (|D_d v|^2)^{1/2}_{B^+_{3/4}} \lesssim (|DD_{x'}v|^2)^{1/2}_{B^+_{3/4}} + (|D_d^2 v|^2)^{1/2}_{B^+_{3/4}}. \tag{3.46}$$

We can remove the $D_{x'}D_d v$ term from the right-hand side. Indeed, by applying the boundary W^1_2 estimate of the Dirichlet problem

$$-\Delta(D_d v) = \operatorname{div}(-(0,\ldots,\Delta v)) \text{ in } B^+_1, \quad D_d v = 0 \text{ on } \Sigma_1,$$

we have

$$(|DD_d v|^2)^{1/2}_{B^+_{3/4}} \lesssim (|\Delta v|^2)^{1/2}_{B^+_1} + (|D_d v|^2)^{1/2}_{B^+_1} \lesssim (|\Delta v|^2)^{1/2}_{B^+_1} + (|D_d^2 v|^2)^{1/2}_{B^+_1},$$

where in the last inequality, again we use the boundary Poincaré inequality. Substituting this into (3.46), we obtain (3.45):

$$[D^2 u]_{C^{\bar\alpha}(B^+_{1/2})} = [D^2 v]_{C^{\bar\alpha}(B^+_{1/2})} \lesssim (|D^2_{x'}v|^2)^{1/2}_{B^+_1} + (|D^2_d v|^2)^{1/2}_{B^+_1}$$
$$= (|D^2_{x'}u - \bar{\boldsymbol{q}}|^2)^{1/2}_{B^+_1} + (|D^2_d u - q_{dd}|^2)^{1/2}_{B^+_1} \le (|D^2 u - \boldsymbol{q}|^2)^{1/2}_{B^+_1}.$$

From (3.45), we obtain (3.43) using standard scaling and iteration argument as mentioned in the proof of Lemma 3.3.3. Notice that the corresponding interior version of (3.45) can be obtained by a similar technique by applying the interior Evans-Krylov estimate and the Sobolev-Poincaré inequality to the function

$$v(x) := u(x) - \langle x, qx\rangle/2 - l(x),$$

where $l(x)$ is the affine function such that $(v)_{B_{3/4}} = (Dv)_{B_{3/4}} = 0$. The lemma is proved. ■

The second lemma is a boundary W^2_ε-estimate in the spirit of [56]. See, for example, [28, pp. 19].

Lemma 3.6.2. *Suppose that $u \in W^2_d(B_1^+)$ and $f \in L_d(B_1^+)$ satisfy*

$$a_{ij}D_{ij}u = f \text{ in } B_1^+, \quad \frac{\partial u}{\partial x^d} = 0 \text{ on } \Sigma_1, \quad u = 0 \text{ on } \sigma_1, \tag{3.47}$$

where a_{ij} is symmetric, bounded measurable, and uniformly elliptic with constant λ. Then there exists some $\varepsilon = \varepsilon(d, \lambda) \in (0, 1)$, such that

$$\|D^2u\|_{L_\varepsilon(B_1^+)} \le N(d, \lambda)\|f\|_{L_d(B_1^+)}.$$

Recall the notation $\rho_{D^2u}(r) := \sup_{|x-y|\le r, x,y\in\Omega} |D^2u(x) - D^2u(y)|$. Proceeding as in [40, Lemma 3.1.4, Theorem 3.2.1], we have the following interpolation inequality.

Lemma 3.6.3. *Let Ω be a domain in $\mathbb{R}^d$ satisfying the interior cone condition with opening $\theta(\Omega) > 0$ and height $h(\Omega) > 0$. Then for any $u \in C^2(\overline{\Omega})$ and $\tau \in (0, \frac{\theta(\Omega)}{\theta(\Omega)+\pi}h(\Omega))$,*

$$\|D^2u\|_{L_\infty(\Omega)} \le C(d, \theta(\Omega))(\rho_{D^2u}(\tau) + \tau^{-2}\|u\|_{L_\infty(\Omega)}).$$

Now we turn to the proof of Theorem 3.2.7. This is similar to that of Theorem 3.2.4, which we will sketch here.

Proof of Theorem 3.2.7. The proof is spitted into several steps. We first derive the a priori estimates for ρ_{D^2u} corresponding to (3.41), assuming $u \in C^2(\overline{\Omega})$. Then we use the interpolation and the Aleksandrov-Bakel'man-Pucci (ABP) maximum principle in [55, Theorem 6.1] to obtain the estimate for $\|D^2u\|_{L_\infty(\Omega)}$ and remove all the u terms on the right-hand side of the estimates. Lastly, we construct a

$C^{2,\alpha}$-approximating sequence. Using the uniform estimates, we can show that the limit exists, solves the problem, and is in $C^2(\overline{\Omega})$.

Step 1: The a priori estimate. The key step is to derive the L_ε-mean oscillation estimate corresponding to (3.17): for any $x \in \overline{\Omega}, \kappa \in (0,1)$, and $r \in (0, R_0]$,

$$\begin{aligned} &\phi(x, \kappa r) \\ &\le N\kappa^{\overline{\alpha}}\phi(x,r) + N(\kappa)\Big((\|D^2u\|_{L_\infty(\Omega)} + \|Du\|_{L_\infty(\Omega)} + \|u\|_{L_\infty(\Omega)})\omega^{(3)}(r) + \omega^{(4)}(r)\Big). \end{aligned} \tag{3.48}$$

Here ϕ is defined in (3.12) with p replaced by ε which is given in Lemma 3.6.2, and $\overline{\alpha}$ is introduced in Lemma 3.6.1. The Dini functions $\omega^{(3)}$ and $\omega^{(4)}$ are defined as follows

$$\omega^{(3)}(r) := r(\|D\beta\|_{L_\infty} + 1) + \rho_{D\beta}(r) + \rho_{D\psi_0}(r) + \omega_F(r),$$
$$\omega^{(4)}(r) := r\|Dg\|_{L_\infty} + \rho_{Dg}(r) + \rho_{D\psi_0}(r)\|Dg\|_{L_\infty} + \omega_F(r).$$

Clearly, it suffices to prove (3.48) for two cases: $x \in \partial\Omega$ or $B_r(x) \subset \Omega$. We only focus on the first one, since the same argument below dealing with the Neumann problem in half balls will still work for the interior case. As before, we take the coordinates y centered at x.

For $r \le R_0$, as in Section 3.4.2 we find $w \in W^2_d(\Omega)$ solving (3.29) satisfying the estimate (3.31), as well as the parabola h defined in (3.32). Now $v := u - w - h$ satisfies

$$\begin{cases} G_0(D^2v(y)) = f(y) - F_0(D^2h, (Du)_{\Omega_r}, (u)_{\Omega_r}) & \text{in } \Omega_r, \\ \partial v/\partial y_d = 0 & \text{on } \Gamma_r, \end{cases} \tag{3.49}$$

where F_0 is the function chosen in Assumption 3.2.6 and

$$G_0(M) := F_0(M + D^2h, (Du)_{\Omega_r}, (u)_{\Omega_r}) - F_0(D^2h, (Du)_{\Omega_r}, (u)_{\Omega_r}),$$
$$f(y) := F_0(D^2u(y) - D^2w(y), (Du)_{\Omega_r}, (u)_{\Omega_r}).$$

As in the linear case, we first prove the mean oscillation estimate for D^2v,

$$\begin{aligned}(\fint_{\Omega_{\kappa r}} |D^2v - \boldsymbol{V}_{\kappa r}|^{\varepsilon})^{1/\varepsilon} &\lesssim \kappa^{\overline{\alpha}}(\fint_{\Omega_r} |D^2v - \boldsymbol{q}|^{\varepsilon})^{1/\varepsilon} \\ &+ C(\kappa)\left(\omega^{(3)}(r)(\|D^2u\|_{L_\infty} + \|Du\|_{L_\infty} + \|u\|_{L_\infty}) + \omega^{(4)}(r)\right),\end{aligned} \tag{3.50}$$

which can be compared to (3.19). To prove this, we flatten the boundary for the problem (3.49) using the change of variables (3.20). For $z \in B^+_{(\delta/M)r}$, the equation becomes

$$\widetilde{G_0}(D_z^2v) := G_0((\frac{\partial z}{\partial y}(0))^TD_z^2v\frac{\partial z}{\partial y}(0)) = \widetilde{f} - F_0(D^2h, (Du)_{\Omega_r}, (u)_{\Omega_r}), \tag{3.51}$$

where

$$\widetilde{f} := G_0((\frac{\partial z}{\partial y}(0))^TD_z^2v\frac{\partial z}{\partial y}(0)) - G_0((\frac{\partial z}{\partial y})^TD_z^2v\frac{\partial z}{\partial y} + \frac{\partial v}{\partial z_d}D^2\psi) + f.$$

We decompose $v = \theta + \xi$, where $\xi \in C^2(B^+_{(\delta/M)r} \cup \Sigma_{(\delta/M)r}) \cap C(\overline{B^+_{(\delta/M)r}})$ solves

$$\begin{cases}\widetilde{G_0}(D_z^2\xi) = -F_0(D^2h, (Du)_{\Omega_r}, (u)_{\Omega_r}) & \text{in } B^+_{(\delta/M)r}, \\ \partial\xi/\partial z_d = 0 & \text{on } \Sigma_{(\delta/M)r}, \\ \xi = v & \text{on } \sigma_{(\delta/M)r}.\end{cases} \tag{3.52}$$

Such ξ exists and satisfies the boundary $C^{2,\overline{\alpha}}$ estimate according to Lemma 3.6.1. Recalling (3.22), we can further deduce from a rescaled version of (3.43) that for any $\widetilde{\boldsymbol{q}} \in \mathbb{S}^{d\times d}$,

$$\Big(\fint_{B^+_{2\kappa r}} |D^2\xi - (D^2\xi)_{B^+_{2\kappa r}}|^\varepsilon\Big)^{1/\varepsilon} \lesssim (2\kappa r)^{\bar\alpha}[D^2\xi]_{C^{\bar\alpha}(B^+_{2\kappa r})} \lesssim \kappa^{\bar\alpha}(|D^2\xi - \widetilde{\boldsymbol{q}}|^p)^{1/p}_{B^+_{(\delta/M)r}}. \tag{3.53}$$

Taking the difference between (3.51) and the first line of (3.52), noting that $\widetilde{G}_0(0) = G_0(0) = 0$, we see that θ satisfies (3.47) with λ, f, and B_1^+ replaced by λ/d, $\widetilde{f}$, and $B^+_{\delta/M}$. By Lemma 3.6.2,

$$(|D^2\theta|^\varepsilon)^{1/\varepsilon}_{B^+_{(\delta/M)r}} \lesssim (|\widetilde{f}|^d)^{1/d}_{B^+_{(\delta/M)r}}. \tag{3.54}$$

Using $F[u] = 0$, Assumption 3.2.6, Hardy's inequality, and (3.31), we can estimate $\widetilde{f}$ as follows

$$\begin{aligned}
(|\widetilde{f}|^d)^{1/d}_{B^+_{(\delta/M)r}} \lesssim &(|G_0((\frac{\partial z}{\partial y}(0))^T D_z^2 v\frac{\partial z}{\partial y}(0)) - G_0((\frac{\partial z}{\partial y})^T D_z^2 v\frac{\partial z}{\partial y} + \frac{\partial v}{\partial z_d}D^2\psi)|^d)^{1/d}_{B^+_{(\delta/M)r}} \\
&+ (|F_0(D^2u - D^2w, (Du)_{\Omega_r}, (u)_{\Omega_r}) - F_0(D^2u, (Du)_{\Omega_r}, (u)_{\Omega_r})|^d)^{1/d}_{\Omega_{r/2}} \\
&+ (|F_0(D^2u, (Du)_{\Omega_r}, (u)_{\Omega_r}) - F(D^2u, (Du)_{\Omega_r}, (u)_{\Omega_r}, y)|^d)^{1/d}_{\Omega_{r/2}} \\
&+ (|F(D^2u, (Du)_{\Omega_r}, (u)_{\Omega_r}, y) - F(D^2u, Du, u, y)|^d)^{1/d}_{\Omega_{r/2}} \\
\lesssim &\rho_{D\psi_0}(r)(|D^2v|^d)^{1/d}_{\Omega_{r/2}} + (|D^2w|^d)^{1/d}_{\Omega_{r/2}} \\
&+ \omega_F(r)(\|D^2u\|_{L_\infty} + \|Du\|_{L_\infty} + \|u\|_{L_\infty} + 1) + r(\|D^2u\|_{L_\infty} + \|Du\|_{L_\infty}) \\
\lesssim &\omega^{(3)}(r)(\|D^2u\|_{L_\infty} + \|Du\|_{L_\infty} + \|u\|_{L_\infty}) + \omega^{(4)}(r).
\end{aligned}$$

Combining (3.53) and (3.54), and following the proof of Lemma 3.4.2, we obtain (3.50). Then, the same steps as in the proof of Proposition 3.4.1 leads to (3.48). The iteration argument in the proof of Theorem 3.2.4 gives, for any $r \in (0, R_0/2)$,

$$\rho_{D^2u}(r) \le C\Big(\|D^2u\|_{L_\infty(\Omega)}r^{\alpha_0} + (\|D^2u\|_{L_\infty(\Omega)} + \|u\|_{L_\infty(\Omega)})I\widetilde{\omega}^{(3)}(r) + I\widetilde{\omega}^{(4)}(r)\Big), \tag{3.55}$$

where as before, $\alpha_0 = \log(2)/\log(1/\kappa) > 0$.

Step 2: Interpolation and maximum principle. In this step, we aim to bound $\|D^2u\|_{L_\infty(\Omega)}$ and remove all the u terms from the right-hand side of (3.55). Noting (3.8), we can choose the parameters

$$\theta(\Omega) = 2\arcsin(\delta/2), \quad h(\Omega) = \sqrt{3}R_0/2$$

for the interior cone at each point. Using Lemma 3.6.3 and (3.55), we obtain, for $0 < \tau < \sqrt{3}R_0\frac{\arcsin(\delta/2)}{2\arcsin(\delta/2)+\pi}$,

$$\begin{aligned}\|D^2u\|_{L_\infty(\Omega)} &\le C(d,\delta)(\rho_{D^2u}(\tau) + \tau^{-2}\|u\|_{L_\infty(\Omega)}) \\ &\le C\Big((\tau^{\alpha_0} + I\widetilde{\omega}^{(3)}(\tau))\|D^2u\|_{L_\infty(\Omega)} + (\tau^{-2} + I\widetilde{\omega}^{(3)}(\tau))\|u\|_{L_\infty(\Omega)} + I\widetilde{\omega}^{(4)}(\tau)\Big).\end{aligned}$$

Now, choose $\tau > 0$ sufficiently small to absorb the first term on the right-hand side. Then, noting the sign condition (3.5), we can use the ABP estimate in [55, Theorem 6.1] to bound $\|u\|_{L_\infty(\Omega)}$. This leads to

$$\|D^2u\|_{L_\infty(\Omega)} \lesssim \|F(0,0,0,\cdot)\|_{L_d(\Omega)} + \|g\|_{L_\infty(\Omega)} + I\widetilde{\omega}^{(4)}(1). \tag{3.56}$$

Substituting back into (3.55) and using the ABP estimate again, we conclude

$$\rho_{D^2u}(r) \lesssim (\|F(0,0,0,\cdot)\|_{L_d(\Omega)} + \|g\|_{L_\infty}(\Omega) + I\widetilde{\omega}^{(4)}(1))(r^{\alpha_0} + I\widetilde{\omega}^{(3)}(r)) + I\widetilde{\omega}^{(4)}(r). \tag{3.57}$$

Step 3: Approximation. We fix some constant $\alpha \in (0,\overline{\alpha})$, and take the mollification

$$\Omega^{(n)} \nearrow \Omega, \quad \Omega^{(n)}, \beta_0^{(n)}, \boldsymbol{\beta}^{(n)}, g^{(n)} \in C^{1,\alpha}, \quad [F^{(n)}(M,p,u,\cdot)]_\alpha \le K_n(|M| + |p| + |u| + 1),$$

with corresponding moduli of continuity (in the sense of L_∞ or Assumption 3.2.6), which are uniform with respect to n. According to [65, Theorem 3.3], for each n, there exists a unique solution $u_n \in C^{2,\alpha}(\Omega^{(n)})$ to

$$F^{(n)}(D^2u, Du, u, y) = 0 \text{ in } \Omega^{(n)}, \quad B^{(n)}u = g^{(n)} \text{ on } \partial\Omega^{(n)}.$$

Notice that (3.56) and (3.57) give us the $C^2(\overline{\Omega^{(k)}})$ pre-compactness of the family $\{u^{(n)}\}_{n\geq k}$. Similar compactness argument as in the linear case gives us the unique $C^2(\overline{\Omega})$ solution to the original problem. ■

Chapter Four

Optimal regularity of a Dirichlet-Neumann problem in Reifenberg flat domain

4.1 Introduction

In this chapter, we discuss the mixed boundary value problem for second-order elliptic operators:

$$\begin{cases} Lu = f + D_i f_i & \text{in } \Omega, \\ Bu = f_i n_i & \text{on } \mathcal{N}, \\ u = 0 & \text{on } \mathcal{D}, \end{cases} \tag{4.1}$$

where Ω is a domain (not necessarily bounded) in $\mathbb{R}^d$, $d \geq 2$ with the boundary divided into two non-intersecting portions $\mathcal{D}$ and $\mathcal{N}$. The differential operator L is in divergence form acting on real valued functions u as follows:

$$Lu = D_i(a_{ij}(x)D_j u + b_i(x)u) + \hat{b}_i(x)D_i u + c(x)u.$$

Here, all the coefficients are assumed to be bounded measurable, and the leading coefficients a_{ij} are symmetric and uniformly elliptic. We denote by $Bu = (a_{ij}D_j u + b_i u)n_i$ the conormal derivative of u on $\mathcal{N}$ associated with the operator L. Dirichlet and conormal boundary conditions are prescribed on the portions $\mathcal{D}$ and $\mathcal{N}$ respectively, which are separated by their relative boundary $\Gamma \subset \partial\Omega$. Both the equation and the boundary conditions are understood in the weak sense. For precise definition, see Definition 4.2.1.

As is well known, solutions to purely Dirichlet/conormal boundary value problems are smooth when coefficients, data, and boundaries of domains are smooth. However, for mixed boundary value problems, such a regularity result does not hold near the interface Γ, and the regularity of solutions depends also on that of Γ and the way two boundary conditions meet (e.g., the meeting angle and certain compatibility conditions). For instance, the best possible regularity of derivatives of solutions to (4.1) is

$$Du \in L_p \quad \text{for } p < 4$$

when the two boundary portions meet tangentially (the angle between $\mathcal{D}$ and $\mathcal{N}$ is π); see Example 4.2.6 for a classical counterexample. In this chapter, we investigate minimal regularity assumptions of a_{ij}, $\partial\Omega$, and Γ, which guarantee the above optimal regularity as well as the solvability of the mixed problem (4.1).

Regularity theory for mixed problems has been studied for a long time. For the case when the two boundary portions $\mathcal{D}$ and $\mathcal{N}$ meet tangentially, we refer the reader to Shamir [68] and Savaré [67]. In [68], the author proved $W^{1,4-\varepsilon}$ regularity for non-divergence form elliptic equations with smooth coefficients in half space. He also obtained $W^{s,p}$ regularity on a smooth bounded domain with the indices $p > 4$ and $s < 1/2 + 2/p$. At one end, the optimal $C^{1/2-\varepsilon}$-Hölder regularity can be obtained by passing $p \nearrow \infty$, which improved a general Hölder regularity result of De Giorgi's type by Stamppachia in [71]. It is also worth mentioning that in [67], the author proved optimal regularity in Besov space $B^{3/2}_{2,\infty}$ for the divergence form elliptic equations with Lipschitz coefficients on a $C^{1,1}$ domain.

For the case when $\mathcal{D}$ and $\mathcal{N}$ do not meet tangentially, we refer the reader to I. Mitrea-M. Mitrea [61], where the authors studied the mixed problem (4.1) with $Lu = \Delta u$. They proved the $W^{1,p}$ solvability with

$$\frac{3}{2+\varepsilon} < p < \frac{3}{1-\varepsilon} \quad \text{for some } \varepsilon = \varepsilon(\Omega, \mathcal{D}, \mathcal{N}) \in (0,1) \tag{4.2}$$

on the so-called creased domains in $\mathbb{R}^d$, $d \geq 3$, which means that $\mathcal{D}$ and $\mathcal{N}$ are separated by a Lipschitz interface and the angle between $\mathcal{D}$ and $\mathcal{N}$ is less than π. This class of domains was introduced by Brown in [3] to answer a question raised by Kenig in [37] regarding the non-tangential maximal function estimate

$$\|(\nabla u)^*\|_{L_2(\partial\Omega)} \le C\Big(\|\nabla_{\tan} u\|_{L_2(\mathcal{D})} + \|\partial u/\partial n\|_{L_2(\mathcal{N})}\Big)$$

of harmonic functions. As mentioned in [37], the above regularity result can be false when Ω is smooth so that $\mathcal{D}$ and $\mathcal{N}$ meet tangentially, whereas it holds for purely Dirichlet/Neumann problem. For further work in this direction, see [5, 62] and the references therein.

In this chapter, we work on the so-called "Reifenberg flat" domain, which is, roughly speaking, at every small scale the boundary is close to certain hyperplane. A Reifenberg flat domain is much more general than a Lipschitz domain with small Lipschitz constant: locally it is not given by a graph, and typically it contains fractal structures. The Reifenberg flat domain was introduced by Reifenberg in [63] when he worked on the Plateau problem. Since then, there has been a lot of work on Reifenberg flat domains regarding minimal surfaces, harmonic measures, regularity of free boundaries, and divergence form elliptic/parabolic equations. An important fact for studying divergence form equation in such domains is that any small Reifenberg flat domain is a $W^{1,p}$-extension domain for every $p \in [1,\infty]$. Hence we have all the Sobolev inequalities up to the first order. For this result and the history of studying Reifenberg flat domains, one may refer to [45].

Notice that although on Reifenberg flat domain, neither the outer normal nor the trace operator of $W^{1,p}$ is defined, the weak formulation in Definition 4.2.1 still makes sense due to the fact that no boundary integral term appears when Ω is smooth enough so that the outer normal and the trace operator are well defined.

We prove the solvability in Sobolev spaces $W^{1,p}$ and the L_p-estimates with p

being in the optimal range $4/3 < p < 4$ for the mixed problem (4.1) with BMO coefficients on Reifenberg flat domains. The two boundary portions $\mathcal{D}$ and $\mathcal{N}$ are assumed to meet *almost* tangentially, which means $\mathcal{D}$ and $\mathcal{N}$ are separated by some Reifenberg flat set of co-dimension 2 on the boundary. We note that our result holds for both bounded and unbounded domains. For the bounded domain case, we can further relax the assumptions on the source term. As mentioned before, since Lipschitz domains with small Lipschitz constant are Reifenberg flat, our results can be applied also on creased domains. Therefore, we see that in the restriction (4.2), the best possible range of ε is $0 < \varepsilon < 1/4$ for creased domains with small Lipschitz constant.

This chapter is a continuation of [20, 21], in which elliptic systems on Reifenberg flat domains with rough coefficients and purely Dirichlet/conormal boundary conditions were studied. See also the series [6, 7] regarding second-order equations on bounded domains. Our proof is mainly based on a perturbation argument suggested in [8] by Caffarelli and Peral, by studying the level sets of maximal functions. The key step in our proof is to carefully design an approximation function near Γ, which combines the cut-off and reflection techniques in [20, 21]. Compared to the purely Dirichlet or purely conormal problems, the approximation function in our problem is less regular, which is only $W^{1,4-\varepsilon}$, not Lipschitz. This situation is similar to [23], where dedicated decay rates of the level sets are required.

The chapter is organized as follows. In Section 2, we introduce the basic notation, definitions, and assumptions. Our main results are given in Theorem 4.2.4 for both bounded and unbounded domains and in Theorem 4.2.5 for bounded domains. In Section 3, we prove two useful tools for our problem: the local Sobolev-Poincaré inequality and the reverse Hölder inequality. Then in Section 4, we study a model problem, which is the $W^{1,4-\varepsilon}$ regularity of harmonic functions on the upper half space with mixed boundary conditions. With all these preparation, the proof of the main theorem including the approximation via cut-off and reflection, and the level set argument is presented in Section 5. In Section 6, we relax the regularity assumptions on the source term for the bounded domain case, mainly by solving a divergence form equation.

4.2 Notation and main Results

Let d be the space dimension. We write a typical point $x \in \mathbb{R}^d$ as $x = (x', x'')$, where

$$x' = (x_1, x_2) \in \mathbb{R}^2, \quad x'' = (x_3, \cdots, x_d) \in \mathbb{R}^{d-2}.$$

In the same spirit, for a domain $\Omega \subset \mathbb{R}^d$ and $p, q \geq 1$, we define the anisotropic space $L_{p,x'}L_{q,x''}(\Omega)$ as the set of all measurable functions u on Ω having a finite norm

$$\|u\|_{L_{p,x'}L_{q,x''}(\Omega)} = \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^{d-2}} |u|^q \mathbb{I}_\Omega \, dx'' \right)^{p/q} dx' \right)^{1/p},$$

where $\mathbb{I}$ is the usual indicator function. We abbreviate $L_{p,x'}L_{p,x''}(\Omega) = L_p(\Omega)$. We will also use the notation

$$\mathbb{R}^d_+ = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_1 > 0\}, \quad B_R^+ = \mathbb{R}^d_+ \cap B_R(0), \tag{4.3}$$

$$B_R' = \{x' \in \mathbb{R}^2 : |x'| < R\}, \quad (B_R')^+ = \mathbb{R}^2_+ \cap B_R',$$

and $\Omega_R(x) = \Omega \cap B_R(x)$ for all $x \in \mathbb{R}^d$ and $R > 0$.

Now we formulate our mixed boundary value problem. We consider domain

$\Omega \subset \mathbb{R}^d$ with boundary divided into two non-intersecting portions, and Γ being the boundary of $\mathcal{D}$ relative to $\partial\Omega$:

$$\partial\Omega = \mathcal{D} \cup \mathcal{N}, \quad \mathcal{D} \cap \mathcal{N} = \emptyset, \quad \Gamma = \partial_{\partial\Omega}\mathcal{D}.$$

We need the following notation for Sobolev spaces with boundary conditions prescribed on the whole or part of the boundary. For $1 \leq p \leq \infty$, we denote by $W^{1,p}(\Omega)$ the usual Sobolev space and by $W_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, where $C_0^\infty(\Omega)$ is the set of all smooth, compactly supported functions in Ω. Similarly, we let $W_{\mathcal{D}}^{1,p}(\Omega)$ be the completion of $C_{\mathcal{D}}^\infty(\Omega)$ in $W^{1,p}(\Omega)$, where $C_{\mathcal{D}}^\infty(\Omega)$ is the set of all smooth functions on $\overline{\Omega}$ which vanish in a neighborhood of $\mathcal{D}$.

Let L be a second-order elliptic operator in divergence form

$$Lu = D_i(a_{ij}(x)D_j u + b_i(x)u) + \hat{b}_i(x)D_i u + c(x)u,$$

where the coefficients $\boldsymbol{A} = (a_{ij})_{i,j=1}^d$, $\boldsymbol{b} = (b_1, \ldots, b_d)$, $\hat{\boldsymbol{b}} = (\hat{b}_1, \ldots, \hat{b}_d)$, and c are bounded measurable functions defined on $\overline{\Omega}$: for some positive constants Λ and K, we have

$$|\boldsymbol{A}| \leq \Lambda^{-1}, \quad |\boldsymbol{b}| + |\hat{\boldsymbol{b}}| + |c| \leq K.$$

Note that the summation convention is adopted throughout this chapter. The leading coefficients $\boldsymbol{A} = (a_{ij})$ are also assumed to be symmetric, satisfy the uniformly ellipticity condition:

$$\sum_{i,j=1}^d a_{ij}(x)\xi_j\xi_i \geq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \overline{\Omega}.$$

We denote by

$$Bu = (\boldsymbol{A}Du + \boldsymbol{b}u) \cdot n = (a_{ij}D_j u + b_i u)\, n_i$$

the conormal derivative operator on the boundary of Ω associated with the operator L, where $n = (n_1, \ldots, n_d)$ is the outward unit normal to $\partial\Omega$. We will see that in the weak formulation, this boundary condition is still well defined even when the outer unit normal is not defined point-wise. Now we give the formal definition of weak solutions. Let $p \in (1, \infty)$.

Definition 4.2.1 (Weak Solution)**.** For $f, f_i \in L_p(\Omega)$, $i \in \{1, \ldots, d\}$, we say that $u \in W^{1,p}_{\mathcal{D}}(\Omega)$ is a weak solution to the mixed boundary value problem

$$\begin{cases} Lu = f + D_i f_i & \text{in } \Omega, \\ Bu = f_i n_i & \text{on } \mathcal{N}, \\ u = 0 & \text{on } \mathcal{D}, \end{cases} \tag{4.4}$$

if

$$\int_\Omega (-a_{ij} D_j u - b_i u) D_i \phi + (\hat{b}_i D_i u + cu)\phi \, dx = \int_\Omega f\phi \, dx - \int_\Omega f_i D_i \phi \, dx$$

holds for any $\phi \in W^{1,p/(p-1)}_{\mathcal{D}}(\Omega)$.

In this chapter, we will work on the so-called Reifenberg flat domains, which is defined below in (i). In (ii), we assume that locally the two types of boundary conditions are almost separated: the relative boundary Γ is also Reifenberg flat.

Assumption 4.2.2 (γ)**.** There exists a positive constant R_1 such that the following hold.

(i) For any $x_0 \in \partial\Omega$ and $R \in (0, R_1]$, there is a coordinate system depending on x_0 and R such that in this new coordinate system (called the coordinate system associated with (x_0, R)), we have

$$\{y : x_{01} + \gamma R < y_1\} \cap B_R(x_0) \subset \Omega_R(x_0) \subset \{y : x_{01} - \gamma R < y_1\} \cap B_R(x_0).$$

(ii) Let Γ be the boundary (relative to $\partial\Omega$) of $\mathcal{D}$. If $x_0 \in \Gamma$ and $R \in (0, R_1]$, we can further require that the coordinate system defined in (i) satisfy

$$\Gamma \cap B_R(x_0) \subset \{y : |y' - x'| < \gamma R\} \cap B_R(x_0),$$

$$\Big(\partial\Omega \cap B_R(x_0) \cap \{y : y_2 > x_{02} + \gamma R\}\Big) \subset \mathcal{D},$$

$$\Big(\partial\Omega \cap B_R(x_0) \cap \{y : y_2 < x_{02} - \gamma R\}\Big) \subset \mathcal{N}.$$

In this chapter, we always assume that $\mathcal{D}, \mathcal{N} \neq \emptyset$, since otherwise the boundary condition becomes purely conormal or Dirichlet. Corresponding results have been included in [20, 21]. See also [6, 7].

We consider the equations with small "BMO" leading coefficients with a small parameter $\theta \in (0, 1)$ to be specified later.

Assumption 4.2.3 (θ)**.** There exists $R_2 \in (0, 1]$ such that for any $x \in \overline{\Omega}$ and $r \in (0, R_2]$, we have

$$\fint_{\Omega_r(x)} |a_{ij}(y) - (a_{ij})_{\Omega_r(x)}| \, dy < \theta.$$

In the following, we denote $R_0 := \min\{R_1, R_2\}$.

Now we can present our main result. First, in Ω (bounded or unbounded) we consider the existence and uniqueness of $W^{1,p}_{\mathcal{D}}$ weak solution to the following equation:

$$\begin{cases} Lu - \lambda u = f + D_i f_i & \text{in } \Omega, \\ Bu = f_i n_i & \text{on } \mathcal{N}, \\ u = 0 & \text{on } \mathcal{D}. \end{cases} \tag{4.5}$$

Compared to (4.4), here we introduce the $-\lambda u$ term to create the required decay at infinity for the unbounded domain case. For simplicity, we will use the following notation with $\lambda > 0$:

$$U := |Du| + \sqrt{\lambda}|u|, \quad F := \sum_{i=1}^{d} |f_i| + \frac{1}{\sqrt{\lambda}}|f|.$$

Theorem 4.2.4. *For any $p \in (4/3, 4)$, we can find positive constants*

$$(\gamma_0, \theta_0) = (\gamma_0, \theta_0)(d, p, \Lambda), \quad \lambda_0 = \lambda_0(d, p, \Lambda, R_0, K),$$

such that the following holds. If Assumptions 4.2.2 (γ_0) and 4.2.3 (θ_0) are satisfied, and $\lambda > \lambda_0$, then for any $(f_i)_{i=1}^d \in (L_p(\Omega))^d$, $f \in L_p(\Omega)$ there exists a unique weak solution $u \in W_{\mathcal{D}}^{1,p}(\Omega)$ to (4.5) satisfying

$$\|U\|_{L_p(\Omega)} \le N\|F\|_{L_p(\Omega)}, \tag{4.6}$$

where $N = N(d, p, \Lambda)$ is a constant.

When Ω is bounded, we have better results: instead of taking large λ, we can assume the usual sign condition $L1 \le 0$, which is understood in the weak sense:

$$\int_\Omega (-b_i D_i \phi + c\phi)\, dx \le 0$$

for any $\phi \in W_{\mathcal{D}}^{1,p/(p-1)}(\Omega)$ satisfying $\phi \ge 0$. Also, the integrability of the non-divergence form source term f can be generalized to L_{p_*}, where

$$p_* = \begin{cases} pd/(p+d) & \text{when } p > d/(d-1), \\ 1 + \varepsilon & \text{when } p \le d/(d-1) \end{cases} \tag{4.7}$$

for any $\varepsilon > 0$.

Theorem 4.2.5. *Let Ω be a bounded domain in $\mathbb{R}^d$. For any $p \in (4/3, 4)$, we can find positive constants γ_0, θ_0 depending on (d, p, Λ), such that the following holds. If Assumptions 4.2.2 (γ_0) and 4.2.3 (θ_0) are satisfied, and $L1 \le 0$ in the weak sense, then for any $(f_i)_{i=1}^d \in (L_p(\Omega))^d$, $f \in L_{p_*}(\Omega)$ there exists a unique weak solution $u \in W_{\mathcal{D}}^{1,p}(\Omega)$ to (4.4) satisfying*

$$\|u\|_{W^{1,p}(\Omega)} \le N\left(\sum_{i=1}^{d} \|f_i\|_{L_p(\Omega)} + \|f\|_{L_{p_*}(\Omega)}\right), \tag{4.8}$$

where N is a constant independent of u, f_i and f.

In the above theorems, we always assume that

$$p \in (4/3, 4), \quad \boldsymbol{A} = (a_{ij})_{i,j=1}^d \text{ is symmetric.} \tag{4.9}$$

Indeed, by the Lax-Milgram Lemma and the reverse Hölder's inequality, when p is close to 2, the symmetry of $\boldsymbol{A}$ is not needed. Otherwise, by the following two examples, we see that the restrictions in (4.9) are optimal for the solvability of mixed boundary value problems. Precisely, based on a duality argument, Example 4.2.6 shows the restriction $p \in (4/3, 4)$ is optimal, and Example 4.2.7 shows the symmetry of $\boldsymbol{A}$ is required for the solvability in $W^{1,p}(\Omega)$ when p is away from 2. Here, for the reader's convenience, we temporarily set

$$\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\},$$

which is different from that in (4.3). Note that the examples below are applicable to higher dimensional cases by a trivial extension.

Example 4.2.6. In $\mathbb{R}^2_+$, let $u(x_1, x_2) = \operatorname{Im}(x_1 + ix_2)^{1/2}$. One can simply check that

$$\Delta u = 0 \text{ in } \mathbb{R}^2_+, \quad u = 0 \text{ on } \partial\mathbb{R}^2_+ \cap \{x_1 > 0\}, \quad \frac{\partial u}{\partial x_2} = 0 \text{ on } \partial\mathbb{R}^2_+ \cap \{x_1 < 0\}.$$

Since Du is of order $r^{-1/2}$, one could also check that near the origin $Du \in L_p$ for any $p \in [1, 4)$, but $Du \notin L_4$.

Example 4.2.7. In $\mathbb{R}^2_+$, let $u(x_1, x_2) = \operatorname{Im}(x_1 + ix_2)^s$ with $s \in (0, 1/2)$. We have

$$D_i(a_{ij}D_ju) = 0 \text{ on } \mathbb{R}^2_+, \quad u = 0 \text{ on } \partial\mathbb{R}^2_+ \cap \{x_1 > 0\}, \quad a_{ij}D_jun_i = 0 \text{ on } \partial\mathbb{R}^2_+ \cap \{x_1 < 0\},$$

where

$$(a_{ij})^2_{i,j=1} = \begin{bmatrix} 1 & \cot(\pi s) \\ -\cot(\pi s) & 1 \end{bmatrix}.$$

Since Du is of order r^{s-1}, near the origin we only have $Du \in L_p$ only if $p < \frac{2}{1-s}$. Note that $\frac{2}{1-s} < 4$ and $\frac{2}{1-s} \searrow 2$ as $s \searrow 0$.

4.3 Local Poincaré inequality and reverse Hölder's inequality

In this section, we introduce two useful tools for our problem. The first one is the local Sobolev-Poincaré inequality. Notice that a Reifenberg flat domain intersecting with a ball might no longer be Reifenberg flat. We cannot simply localize to obtain the required local version, although Sobolev inequalities of $W^{1,p}$ hold for the Reifenberg flat domain since it is an extension domain.

Theorem 4.3.1 (Local Sobolev-Poincaré inequality)**.** *Let $\gamma \in [0, 1/48]$ and $\Omega \subset \mathbb{R}^d$ be a Reifenberg flat domain satisfying Assumption 4.2.2 (γ) (i). Let $x_0 \in \partial\Omega$ and $R \in (0, R_1/4]$. Then, for any $p \in (1, d)$ and $u \in W^{1,p}(\Omega_{2R}(x_0))$, we have*

$$\|u - (u)_{\Omega_R(x_0)}\|_{L_{dp/(d-p)}(\Omega_R(x_0))} \le N\|Du\|_{L_p(\Omega_{2R}(x_0))},$$

where $N = N(d, p)$.

Proof. See [10, Theorem 3.5]. ■

Corollary 4.3.2. *Let* $\gamma \in [0, 1/48]$ *and* $\Omega \subset \mathbb{R}^d$ *be a Reifenberg flat domain satisfying Assumption 4.2.2* (γ) (i)*. Let* $x_0 \in \overline{\Omega}$, $R \in (0, R_1/4]$, *and* $\mathcal{D} \subset \partial\Omega$ *with* $\mathcal{D} \cap B_R(x_0) \neq \emptyset$*. If there exist* $z_0 \in \mathcal{D} \cap B_R(x_0)$ *and* $\alpha \in (0, 1)$ *such that*

$$B_{\alpha R}(z_0) \subset B_R(x_0), \quad \Big(\partial\Omega \cap B_{\alpha R}(z_0)\Big) \subset \Big(\mathcal{D} \cap B_R(x_0)\Big), \tag{4.10}$$

then the following hold.

(a) *For any* $p \in (1, d)$ *and* $u \in W^{1,p}_{\mathcal{D}}(\Omega)$*, we have*

$$\|u\|_{L_{dp/(d-p)}(\Omega_R(x_0))} \leq N \|Du\|_{L_p(\Omega_{2R}(x_0))},$$

where $N = N(d, p, \alpha)$.

(b) *For any* $u \in W^{1,2}_{\mathcal{D}}(\Omega)$*, we have*

$$\|u\|_{L_2(\Omega_R(x_0))} \leq NR \|Du\|_{L_2(\Omega_{2R}(x_0))},$$

where $N = N(d, \alpha)$.

Proof. The assertion (b) is a simple consequence of the assertion (a). Indeed, by taking $p \in \left(\frac{2d}{d+2}, 2\right)$, and using Hölder's inequality and the assertion (a), we have

$$\begin{aligned}\|u\|_{L_2(\Omega_R(x_0))} &\leq NR^{d/2-d/p+1} \|u\|_{L_{dp/(d-p)}(\Omega_R(x_0))} \\ &\leq NR^{d/2-d/p+1} \|Du\|_{L_p(\Omega_{2R}(x_0))} \leq NR \|Du\|_{L_2(\Omega_{2R}(x_0))}.\end{aligned}$$

To prove the assertion (a), we extend u by zero on $B_{\alpha R}(z_0) \setminus \Omega$ so that $u \in W^{1,p}(B_{\alpha R}(z_0))$. Since $|B_{\alpha R}(z_0) \setminus \Omega| \geq N(d)(\alpha R)^d$, by the boundary Poincaré inequality, we have

$$\|u\|_{L_{dp/(d-p)}(\Omega_{\alpha R}(z_0))} \leq N(d,q)\|Du\|_{L_p(\Omega_{\alpha R}(z_0))}. \tag{4.11}$$

Notice from the triangle inequality and Hölder's inequality that

$$\begin{aligned}
&\|u\|_{L_{dp/(d-p)}(\Omega_R(x_0))} \\
&\leq \|u - (u)_{\Omega_R(x_0)}\|_{L_{dp/(d-p)}(\Omega_R(x_0))} + \|(u)_{\Omega_R(x_0)} - (u)_{\Omega_{\alpha R}(z_0)}\|_{L_{dp/(d-p)}(\Omega_R(x_0))} \\
&\quad + \|(u)_{\Omega_{\alpha R}(z_0)}\|_{L_{dp/(d-p)}(\Omega_R(x_0))} \\
&\leq N\alpha^{1-d/p}\Big(\|u - (u)_{\Omega_R(x_0)}\|_{L_{dp/(d-p)}(\Omega_R(x_0))} + \|u\|_{L_{dp/(d-p)}(\Omega_{\alpha R}(z_0))}\Big).
\end{aligned}$$

This combined with Theorem 4.3.1 and (4.11) gives the desired estimate. ■

In the rest of the section, we shall prove the reverse Hölder's inequality for the following mixed boundary value problem without lower order terms

$$\begin{cases} D_i(a_{ij}D_ju) - \lambda u = f + D_if_i & \text{in } \Omega, \\ a_{ij}D_jun_i = f_in_i & \text{on } \mathcal{N}, \\ u = 0 & \text{on } \mathcal{D}. \end{cases} \tag{4.12}$$

Here, we do not impose any regularity assumption (including the symmetry condition) on the coefficients a_{ij}. Recall the notation that for $\lambda > 0$,

$$U := |Du| + \sqrt{\lambda}|u|, \quad F := \sum_{i=1}^d |f_i| + \frac{1}{\sqrt{\lambda}}|f|.$$

Lemma 4.3.3. *Let $\gamma \in (0, 1/48]$, $\frac{2d}{d+2} < p < 2$, and $\Omega \subset \mathbb{R}^d$ be a Reifenberg flat domain satisfying Assumption 4.2.2 (γ). Suppose that $u \in W^{1,2}_{\mathcal{D}}(\Omega)$ satisfies (4.12) with*

$f_i, f \in L_2(\Omega)$. Let $x_0 \in \overline{\Omega}$ and $R \in (0, R_1]$, satisfying either

$$B_{R/16}(x_0) \subset \Omega \quad \text{or} \quad x_0 \in \partial\Omega.$$

Then, when $\lambda > 0$, we have

$$\int_{\Omega_{R/32}(x_0)} U^2\,dx \leq NR^{d(1-2/p)}\bigg(\int_{\Omega_R(x_0)} U^p\,dx\bigg)^{2/p} + N\int_{\Omega_R(x_0)} F^2\,dx.$$

When $\lambda = 0$ and $f \equiv 0$, we have

$$\int_{\Omega_{R/32}(x_0)} |Du|^2\,dx \leq NR^{d(1-2/p)}\bigg(\int_{\Omega_R(x_0)} |Du|^p\,dx\bigg)^{2/p} + N\int_{\Omega_R(x_0)} |f_i|^2\,dx.$$

In the above, the constant N depends only on d, p, and Λ.

Proof. Here we only prove for the case $\lambda > 0$. When $\lambda = 0$, the proof still works if we replace U by $|Du|$ and F by $\sum_i |f_i|$. Also, we prove only the case $x_0 \in \partial\Omega$ because the proof for the interior case is similar to the one in case (ii) for purely conormal boundary conditions. Without loss of generality, we assume that $x_0 = 0$. Let us fix $R \in (0, R_1]$. We consider the following two cases:

$$B_{R/16} \cap \Gamma \neq \emptyset, \quad B_{R/16} \cap \Gamma = \emptyset.$$

i) $B_{R/16} \cap \Gamma \neq \emptyset$. We take $y_0 \in \Gamma$ such that $\operatorname{dist}(0, \Gamma) = |y_0|$, and observe that

$$B_{R/16} \subset B_{R/8}(y_0) \subset B_{R/2}(y_0) \subset B_R. \tag{4.13}$$

Since $u \in W^{1,2}_{\mathcal{D}}(\Omega)$, as a test function to (4.12), we can use $\eta^2 u \in W^{1,2}_{\mathcal{D}}(\Omega)$, where

η is a smooth function on $\mathbb{R}^d$ satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } B_{R/8}(y_0), \quad \operatorname{supp} \eta \subset B_{R/4}(y_0), \quad |\nabla \eta| \le NR^{-1}.$$

Now, using Hölder's inequality and Young's inequality, we have

$$\int_{\Omega_{R/4}(y_0)} \eta^2 U^2 \, dx \le \frac{N}{R^2} \int_{\Omega_{R/4}(y_0)} |u|^2 \, dx + N \int_{\Omega_{R/4}(y_0)} F^2 \, dx, \tag{4.14}$$

where $N = N(d, \Lambda)$. We fix a coordinate system associated with $(y_0, R/4, \Gamma)$ satisfying the properties in Assumption 4.2.2 (γ) (ii). Since we have

$$\left(\partial\Omega \cap B_{R/4}(y_0) \cap \{y : y_2 > \gamma R/4\}\right) \subset \mathcal{D},$$

there exists $z_0 \in \mathcal{D}$ satisfying

$$B_{R/16}(z_0) \subset B_{R/4}(y_0), \quad \left(\partial\Omega \cap B_{R/16}(z_0)\right) \subset \left(\mathcal{D} \cap B_{R/4}(y_0)\right).$$

Note that because $\frac{2d}{d+2} < p < 2$, we have $\frac{dp}{d-p} > 2$. Then by Hölder's inequality and Corollary 4.3.2 (a), we see that

$$\begin{aligned} \frac{1}{R^2} \int_{\Omega_{R/4}(y_0)} |u|^2 \, dx &\le NR^{d(1-2/p)} \left(\int_{\Omega_{R/4}(y_0)} |u|^{dp/(d-p)} \, dx \right)^{2(d-p)/dp} \\ &\le NR^{d(1-2/p)} \left(\int_{\Omega_{R/2}(y_0)} |Du|^p \, dx \right)^{2/p}, \end{aligned} \tag{4.15}$$

where $N = N(d, p)$. Combining this inequality and (4.14), and using (4.13), we obtain the desired estimate.

ii) $B_{R/16} \cap \Gamma = \emptyset$. Then $\partial\Omega \cap B_{R/16}$ is contained in either $\mathcal{D}$ or $\mathcal{N}$. When it is in $\mathcal{D}$, the proof for the previous case still works if we simply choose any $y_0 \in \partial\Omega \cap B_{R/16}$. When it is contained in $\mathcal{N}$, as a test function to (4.4), we can use $\zeta^2(u-c) \in W^{1,2}_{\mathcal{D}}(\Omega)$, where $c = (u)_{\Omega_{R/16}}$ and ζ is a smooth function on $\mathbb{R}^d$ satisfying

$$0 \le \zeta \le 1, \quad \zeta \equiv 1 \text{ on } B_{R/32}, \quad \operatorname{supp} \zeta \subset B_{R/16}, \quad |\nabla \zeta| \le NR^{-1}.$$

By testing (4.12) with $\zeta^2(u-c)$, we have

$$\int_{\Omega_{R/16}} (\zeta U)^2 \, dx \le \frac{N}{R^2} \int_{\Omega_{R/16}} \left| u - (u)_{\Omega_{R/16}} \right|^2 dx + \frac{N}{R^d} \left(\int_{\Omega_{R/16}} \sqrt{\lambda} |u| \, dx \right)^2 + N \int_{\Omega_{R/16}} F^2 \, dx,$$

where $N = N(d, \Lambda)$. Similar to (4.15), we get from Theorem 4.3.1 that

$$\frac{1}{R^2} \int_{\Omega_{R/16}} \left| u - (u)_{\Omega_{R/16}} \right|^2 dx \le NR^{d(1-2/p)} \left(\int_{\Omega_{R/8}} |Du|^p \, dx \right)^{2/p},$$

where $N = N(d, p)$. By Hölder's inequality, we also have

$$\frac{1}{R^d} \left(\int_{\Omega_{R/16}} \sqrt{\lambda} |u| \, dx \right)^2 \le NR^{d(1-2/p)} \left(\int_{\Omega_{R/16}} \left(\sqrt{\lambda} |u| \right)^p dx \right)^{2/p}.$$

Combining these together, we obtain the desired estimate.

The lemma is proved. ■

Based on Lemma 4.3.3 and Gehring's lemma, we get the following reverse Hölder's inequality.

Lemma 4.3.4 (Reverse Hölder's inequality)**.** *Let $\gamma \in (0, 1/48]$, $p > 2$, and $\Omega \subset \mathbb{R}^d$ be a Reifenberg flat domain satisfying Assumption 4.2.2 (γ). Suppose that $u \in W^{1,2}_{\mathcal{D}}(\Omega)$ satisfies (4.12) with $f_i, f \in L_p(\Omega) \cap L_2(\Omega)$. Then there exist constants $p_0 \in (2, p)$ and $N > 0$, depending only on d, p, and Λ, such that for any $x_0 \in \mathbb{R}^d$ and $R \in (0, R_1]$, the* *following hold. When $\lambda > 0$, we have*

$$\left(\overline{U}^{p_0}\right)^{1/p_0}_{B_{R/2}(x_0)} \le N\left(\overline{U}^2\right)^{1/2}_{B_R(x_0)} + N\left(\overline{F}^{p_0}\right)^{1/p_0}_{B_R(x_0)}.$$

When $\lambda = 0$ and $f \equiv 0$, we have

$$\left(|D\overline{u}|^{p_0}\right)^{1/p_0}_{B_{R/2}(x_0)} \le N\left(|D\overline{u}|^2\right)^{1/2}_{B_R(x_0)} + N\left(|\overline{f_i}|^{p_0}\right)^{1/p_0}_{B_R(x_0)},$$

where $\overline{U}$, $\overline{F}$, $D\overline{u}$, and $\overline{f_i}$ are the extensions of U, F, Du, and f_i to $\mathbb{R}^d$ so that they are zero on $\mathbb{R}^d \setminus \Omega$.

Proof. Again, we only prove for the case $\lambda > 0$. Let us fix a constant $p_1 \in \left(\frac{2d}{d+2}, 2\right)$, and set

$$\Phi = \overline{U}^{p_1}, \quad \Psi = \overline{F}^{p_1}.$$

Then by Lemma 4.3.3, we have

$$\int_{B_{R/112}(x_0)} \Phi^{2/p_1}\,dx \le NR^{d(1-2/p_1)}\left(\int_{B_R(x_0)} \Phi\,dx\right)^{2/p_1} + N\int_{B_R(x_0)} \Psi^{2/p_1}\,dx \tag{4.16}$$

for any $x_0 \in \mathbb{R}^d$ and $R \in (0, R_1]$, where $N = N(d, \Lambda, p) = N(d, \Lambda)$. Indeed, if $B_{R/56}(x_0) \subset \Omega$, then (4.16) follows from Lemma 4.3.3. In the case when $B_{R/56}(x_0) \cap \partial\Omega \neq \emptyset$, there exists $y_0 \in \partial\Omega$ such that $|x_0 - y_0| = \text{dist}(x_0, \partial\Omega)$ and

$$B_{R/112}(x_0) \subset B_{3R/112}(y_0) \subset B_{6R/7}(y_0) \subset B_R(x_0).$$

Using this together with Lemma 4.3.3, we get (4.16). If $B_{R/56}(x_0) \subset \mathbb{R}^d \setminus \Omega$, by the definition of $\overline{U}$, (4.16) holds.

By (4.16) and a covering argument, we have

$$\fint_{B_{R/2}(x_0)} \Phi^{2/p_1}\,dx \le N\Big(\fint_{B_R(x_0)} \Phi\,dx\Big)^{2/p_1} + N\fint_{B_R(x_0)} \Psi^{2/p_1}\,dx$$

for any $x_0 \in \mathbb{R}^d$ and $R \in (0, R_1]$, where $N = N(d, \Lambda)$. Therefore, by Gehring's lemma (see, for instance, [32, Ch. V]), we get the desired estimate. The lemma is proved. ■

4.4 Harmonic functions in half space with mixed boundary condition

In this section, we prove a regularity result for harmonic functions with mixed Dirichlet-Neumann boundary conditions on half space. We denote

$$B_R = B_R(0), \quad \Gamma_R^+ := B_R \cap \{x_1 = 0, x_2 > 0\}, \quad \Gamma_R^- := B_R \cap \{x_1 = 0, x_2 < 0\}.$$

Theorem 4.4.1. *Suppose $u \in W^{1,2}_{\Gamma_R^+}(B_R^+)$ is a weak solution to*

$$\begin{cases} \Delta u - \lambda u = 0 & \text{in } B_R^+, \\ \frac{\partial u}{\partial x_1} = 0 & \text{on } \Gamma_R^-, \\ u = 0 & \text{on } \Gamma_R^+, \end{cases}$$

where $\lambda > 0$. Then for any $p \in [2, 4)$, we have $u \in W^{1,p}(B_{R/4}^+)$ with

$$(U^p)_{B_{R/4}^+}^{1/p} \le N(d, p)(U^2)_{B_R^+}^{1/2}.$$

In the case when $\lambda = 0$, the same estimate holds with $|Du|$ in place of U.

Remark 4.4.2. In Theorem 4.4.1, the boundary condition is only prescribed on the flat part of the boundary. Hence the meaning of "weak solution" is slightly different. In the theorem and throughout the chapter, a $W^{1,2}_{\Gamma_R^+}(B_R^+)$ weak solution means: for any $\phi \in W^{1,2}(B_R^+)$ satisfying $\phi = 0$ on $\partial B_R^+ \setminus \Gamma_R^-$,

$$\int_{B_R^+} (\nabla u \cdot \nabla \phi + \lambda u \phi)\, dx = 0.$$

It is clear that, as a test function, one can use ηu, where $\eta \in C_c^\infty(B_R)$.

For the proof of Theorem 4.4.1, we will use the following two dimensional regularity result.

Lemma 4.4.3. *In the half ball $B_R^+ \subset \mathbb{R}^2$, consider $u \in W^{1,2}_{\Gamma_R^+}(B_R^+)$ which solves*

$$\begin{cases} \Delta u = f & \text{in } B_R^+, \\ \frac{\partial u}{\partial x_1} = 0 & \text{on } \Gamma_R^-, \\ u = 0 & \text{on } \Gamma_R^+, \end{cases}$$

where $f \in L_2(B_R^+)$. Then for any $p \in [2,4)$, we have $u \in W^{1,p}(B_{R/2}^+)$ with

$$(|Du|^p)^{1/p}_{B_{R/2}^+} \le N(p)\Big((|Du|^2)^{1/2}_{B_R^+} + R(|f|^2)^{1/2}_{B_R^+}\Big). \tag{4.17}$$

From the proof below, it is clear that in Lemma 4.4.3, $R/2$ can be replaced with any $r \in (0,R)$. In this case, the constant N also depends on r and R.

Proof of Lemma 4.4.3. By a scaling argument, we may assume $R = 1$. We consider the following change of variable: $(y_1, y_2) \in B_1 \cap \{y_1 > 0, y_2 > 0\} \mapsto (x_1, x_2) \in B_1^+$:

$$x_1 = 2y_1y_2, \quad x_2 = y_2^2 - y_1^2,$$

or in complex variables:

$$x_2 + ix_1 = (y_2 + iy_1)^2.$$

Write $\widetilde{u}(y_1, y_2) = u(x_1, x_2)$ and $\widetilde{f}(y_1, y_2) = f(x_1, x_2)$. Then we can rewrite the equation as

$$\begin{cases} \Delta_y \widetilde{u} = 4|y|^2 \widetilde{f} & \text{in } B_1^{++}, \\ \frac{\partial \widetilde{u}}{\partial y_2} = 0 & \text{on } B_1 \cap \{y_1 > 0, y_2 = 0\}, \\ \widetilde{u} = 0 & \text{on } B_1 \cap \{y_1 = 0, y_2 > 0\}, \end{cases}$$

where $B_1^{++} := B_1 \cap \{y_1 > 0, y_2 > 0\}$. Next, we take an even extension of $\widetilde{u}$ and $\widetilde{f}$ with respect to y_2-variable. Still denote the extended functions on B_1^+ by $\widetilde{u}$ and $\widetilde{f}$. Then the following equation is satisfied:

$$\begin{cases} \Delta_y \widetilde{u} = 4|y|^2 \widetilde{f} & \text{in } B_1^{+}, \\ \widetilde{u} = 0 & \text{on } B_1 \cap \{y_1 = 0\}. \end{cases}$$

Note that

$$|D_x u| \le \frac{N}{|y|} |D_y \widetilde{u}|, \quad dx = 4|y|^2 \, dy.$$

By the local W_2^2 estimate for elliptic equations and the Sobolev embedding theorem, we obtain

$$\begin{aligned} \|D_y \widetilde{u}\|_{L_q(B^+_{\sqrt{2}/2})} \le \|\widetilde{u}\|_{W_2^2(B^+_{\sqrt{2}/2})} &\le N\Big(\|D_y \widetilde{u}\|_{L_2(B_1^+)} + \|4|y|^2 \widetilde{f}\|_{L_2(B_1^+)}\Big) \\ &\le N\Big(\|D_y \widetilde{u}\|_{L_2(B_1^{++})} + \|4|y|^2 \widetilde{f}\|_{L_2(B_1^{++})}\Big), \end{aligned}$$

where $N = N(p) > 0$ and $q = q(p)$ is a constant with

$$q > \frac{2p}{4-p} \geq p. \tag{4.18}$$

Here we used the fact that $\widetilde{u}$ and $\widetilde{f}$ are both even functions in y_2. Translating back to x-variables, we obtain

$$\begin{aligned} \||x|^{\frac{q-2}{2q}} D_x u\|_{L_q(B^+_{1/2})} &\leq N\Big(\|D_x u\|_{L_2(B^+_1)} + \||x|^{1/2} f\|_{L_2(B^+_1)}\Big) \\ &\leq N\Big(\|D_x u\|_{L_2(B^+_1)} + \|f\|_{L_2(B^+_1)}\Big). \end{aligned} \tag{4.19}$$

By Hölder's inequality and (4.18), we get from (4.19) that

$$\begin{aligned} \|D_x u\|_{L_p(B^+_{1/2})} &\leq \||x|^{\frac{q-2}{2q}} D_x u\|_{L_q(B^+_{1/2})} \||x|^{-\frac{q-2}{2q}}\|_{L_{qp/(q-p)}(B^+_{1/2})} \\ &\leq N\||x|^{\frac{q-2}{2q}} D_x u\|_{L_q(B^+_{1/2})}. \end{aligned}$$

Combining these together, we obtain

$$\|D_x u\|_{L_p(B^+_{1/2})} \leq N\Big(\|D_x u\|_{L_2(B^+_1)} + \|f\|_{L_2(B^+_1)}\Big),$$

which is exactly (4.17). The lemma is proved. ∎

We are now ready to present the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. We first prove the theorem for $\lambda = 0$. By a scaling argument and Lemma 4.4.3, we may assume $R = 1$ and $d \geq 3$. Noting that we can differentiate both the equation and the boundary condition in x''-direction, the following

Caccioppoli type inequality holds:

$$\|D_x(D_{x''}^k u)\|_{L_2(B_s^+)} \le \frac{N(d,k)}{|t-s|^k}\|Du\|_{L_2(B_t^+)}$$

for $0 < s < t \le 1$ and $k \in \{0, 1, 2, \ldots\}$. Thus by anisotropic Sobolev embedding, we can increase the integrability in x''-variables so that

$$D_{x'}u \in L_{2,x'}L_{p,x''}(B_r^+), \quad D_{x''}u,\ D_{x''}^2 u \in L_p(B_r^+) \quad \forall r < 1,$$

with the estimate

$$\|D_{x'}u\|_{L_{2,x'}L_{p,x''}(B_r^+)} + \|D_{x''}u\|_{L_p(B_r^+)} + \|D_{x''}^2 u\|_{L_p(B_r^+)} \le N(d,p,r)\|Du\|_{L_2(B_1^+)}. \tag{4.20}$$

It remains to estimate $D_{x'}u$. From (4.20), for almost every $|x''| < 1/2$, we have

$$D_{x''}^2 u(\cdot, x'') \in L_2\Big((B'_{2/3})^+\Big).$$

Now we rewrite the equation as a 2-dimension problem in x'-variables:

$$\begin{cases} \Delta_{x'}u(\cdot,x'') = -\Delta_{x''}u(\cdot,x'') & \text{in } (B'_{2/3})^+, \\ \frac{\partial u}{\partial x_1} = 0 & \text{on } B'_{2/3} \cap \{x_1 = 0, x_2 < 0\}, \\ u = 0 & \text{on } B'_{2/3} \cap \{x_1 = 0, x_2 > 0\}. \end{cases}$$

We apply a properly rescaled version of Lemma 4.4.3 to see that for almost every $|x''| < 1/2$,

$$\|D_{x'}u(\cdot,x'')\|_{L_p((B'_{1/2})^+)} \le N(p)\Big(\|D_{x'}u(\cdot,x'')\|_{L_2((B'_{2/3})^+)} + \|\Delta_{x''}u(\cdot,x'')\|_{L_2((B'_{2/3})^+)}\Big).$$

Taking L_p norm in $\{x'' \in \mathbb{R}^{d-2} : |x''| < 1/2\}$ for both sides, and using the Minkowskii inequality and (4.20) with $r = \sqrt{3}/2$, we obtain $D_{x'}u \in L_p(B^+_{1/2})$ and

$$\|D_{x'}u\|_{L_p(B^+_{1/2})} \le N(d,p)\|Du\|_{L_2(B^+_1)}.$$

This gives the desired estimate for $\lambda = 0$.

For a general $\lambda > 0$, we use an idea by S. Agmon. We define

$$v(x,\tau) = u(x)\cos(\sqrt{\lambda}\tau + \pi/4),$$

and observe that v satisfies

$$\begin{cases} \Delta_{(x,\tau)}v = 0 & \text{in } \hat{B}^+_1, \\ \frac{\partial v}{\partial x_1} = 0 & \text{on } \hat{\Gamma}^-_1, \\ v = 0 & \text{on } \hat{\Gamma}^+_1. \end{cases} \tag{4.21}$$

where

$$\hat{B}_1 = \{(x,\tau) \in \mathbb{R}^{d+1} : |(x,\tau)| < 1\}, \quad \hat{B}^+_1 = \hat{B}_1 \cap \{x_1 > 0\},$$

$$\hat{\Gamma}^+_1 = \hat{B}_1 \cap \{x_1 = 0, x_2 > 0\}, \quad \hat{\Gamma}^-_1 = \hat{B}_1 \cap \{x_1 = 0, x_2 < 0\}.$$

By applying the result for $\lambda = 0$ to (4.21), we have

$$(|D_{(x,\tau)}v|^p)^{1/p}_{\hat{B}^+_{1/2}} \le N(|D_{(x,\tau)}v|^2)^{1/2}_{\hat{B}^+_1}, \tag{4.22}$$

where $N = N(d,p)$. Note that the function Φ given by

$$\Phi(\lambda) = \int_0^{1/4} \left|\cos(\sqrt{\lambda}\tau + \pi/4)\right|^p d\tau$$

has a positive lower bound depending only on p. Thus by using (4.22) and the fact that

$$\left|Du(x)\cos(\sqrt{\lambda}\tau+\pi/4)\right| \le |D_{(x,\tau)}v(x,\tau)| \le U(x),$$

we have

$$\begin{aligned}\int_{B^+_{1/4}} |Du|^p\,dx &\le N\int_0^{1/4}\int_{B^+_{1/4}} |Du|^p\left|\cos(\sqrt{\lambda}\tau+\pi/4)\right|^p dx\,d\tau\\ &\le N\int_{\hat{B}^+_{1/2}} |D_{(x,\tau)}v|^p\,dx\,d\tau \le N\left(\int_{B^+_1} U^2\,dx\right)^{p/2},\end{aligned}$$

where $N=N(d,p)$. Similarly, from the fact that

$$\left|\sqrt{\lambda}u(x)\sin(\sqrt{\lambda}\tau+\pi/4)\right| \le |D_{(x,\tau)}v(x,\tau)| \le U(x),$$

we obtain

$$\int_{B^+_{1/4}} \left|\sqrt{\lambda}u\right|^p dx \le N\left(\int_{B^+_1} U^2\,dx\right)^{p/2}.$$

Combining these together we get the desired estimate. The theorem is proved. ■

4.5 Regularity of $W^{1,2}_{\mathcal{D}}$ weak solutions

The crucial step in proving unique $W^{1,p}$ solvability is the following improved regularity result. As in Theorem 4.2.4, we consider a domain $\Omega\subset\mathbb{R}^d$ which can be bounded or unbounded, together with nonempty boundary portions $\mathcal{D},\mathcal{N}$. Again, recall the notation that for $\lambda>0$,

$$U := |Du| + \sqrt{\lambda}|u|, \quad F := \sum_{i=1}^d |f_i| + \frac{1}{\sqrt{\lambda}}|f|.$$

Proposition 4.5.1 (Regularity of $W^{1,2}_{\mathcal{D}}$ weak solutions)**.** *For any $p\in(2,4)$, we can find* *positive constants γ_0, θ_0 depending on (d, p, Λ), such that if Assumptions 4.2.2 (γ_0) and 4.2.3 (θ_0) are satisfied, the following holds. For any $W^{1,2}_{\mathcal{D}}(\Omega)$ weak solution u to (4.12) with $\lambda > 0$ and $f_i, f \in L_p(\Omega) \cap L_2(\Omega)$, we have $u \in W^{1,p}_{\mathcal{D}}(\Omega)$ and*

$$\|U\|_{L_p(\Omega)} \leq N(R_0^{d(1/p-1/2)}\|U\|_{L_2(\Omega)} + \|F\|_{L_p(\Omega)}). \tag{4.23}$$

Furthermore, if we also have $f \equiv 0$, then we can take $\lambda = 0$, and the following estimate holds:

$$\|Du\|_{L_p(\Omega)} \leq N(R_0^{d(1/p-1/2)}\|Du\|_{L_2(\Omega)} + \|f_i\|_{L_p(\Omega)}). \tag{4.24}$$

In the above, the constant N only depends on d, p, and Λ.

Based on Proposition 4.5.1, we obtain the following a priori estimate for the equations with lower order terms and large λ, which will be useful for the unique solvability in Theorem 4.2.4.

Corollary 4.5.2. *Let $p \in (2, 4)$ and γ_0, θ_0 be the constants from Proposition 4.5.1. Under Assumptions 4.2.2 (γ_0) and 4.2.3 (θ_0), there exists a positive constant λ_1 depending on (d, p, Λ, R_0, K) such that if u is a $W^{1,p}_{\mathcal{D}}$ weak solution to the equation (4.5) with $f_i, f \in L_p(\Omega)$ and $\lambda > \lambda_1$, then we have*

$$\|U\|_{L_p(\Omega)} \leq N\|F\|_{L_p(\Omega)}, \tag{4.25}$$

where $N = N(d, p, \Lambda)$.

The rest of this section is devoted to the proofs of Proposition 4.5.1 and Corollary 4.5.2.

4.5.1 Decomposition of Du

We will use an interpolation argument to prove Proposition 4.5.1. The key step is the following decomposition (approximation).

Proposition 4.5.3. *Suppose that* $u \in W^{1,2}_{\mathcal{D}}(\Omega)$ *satisfies* (4.12) *with* $\lambda > 0$ *and* $f_i, f \in L_p(\Omega) \cap L_2(\Omega)$, *where* $p > 2$. *Then under Assumptions* 4.2.2 (γ) *and* 4.2.3 (θ) *with* $\gamma < 1/(32\sqrt{d+3})$ *and* $\theta \in (0,1)$, *for any* $x_0 \in \overline{\Omega}$ *and* $R < R_0$, *there exist positive functions* $W, V \in L_2(\Omega_{R/32}(x_0))$ *such that*

$$U \le W + V \quad \text{in } \Omega_{R/32}(x_0).$$

Moreover, we have for any $q < 4$,

$$(W^2)^{1/2}_{\Omega_{R/32}(x_0)} \le N\Big((\theta^{\frac{1}{2\mu'}} + \gamma^{\frac{1}{2\mu'}})(U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{\frac{1}{2\mu}}_{\Omega_R(x_0)}\Big), \tag{4.26}$$

$$(V^q)^{1/q}_{\Omega_{R/32}(x_0)} \le N\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{\frac{1}{2\mu}}_{\Omega_R(x_0)}\Big). \tag{4.27}$$

Here μ *is a constant satisfying* $2\mu = p_0$, *where* $p_0 = p_0(d,p,\Lambda) > 2$ *comes from Lemma* 4.3.4, *and* μ' *satisfies* $1/\mu + 1/\mu' = 1$. *The constant* N *only depends on* d, p, q, *and* Λ.

The rest of Section 4.5.1 will be devoted to the proof of this proposition.

Proof. According to the relative position of x_0 to $\mathcal{D}, \mathcal{N}$, we will discuss the following 3 cases.

Case 1: $\operatorname{dist}(x_0, \partial\Omega) \ge R/32$.

In this case, our decomposition is only concerned with the interior of Ω. We can do the usual "freezing coefficient" approximation. The existence of such W, V

can be found in [20, Lemma 8.3 (i)] or [21, Lemma 5.1 (i)].

In the next two cases, we also need to approximate the Reifenberg flat boundary by hyperplane and deal with corresponding boundary conditions.

Case 2: $\mathrm{dist}(x_0, \partial\Omega) < R/32$, $\mathrm{dist}(x_0, \Gamma) \geq R/24$.

In this case, either we have $B_{R/24}(x_0) \cap \mathcal{N} = \emptyset$ or $B_{R/24}(x_0) \cap \mathcal{D} = \emptyset$. Correspondingly, we only deal with purely Dirichlet or purely conormal boundary condition. The functions W and V are constructed in $\Omega_{R/24}(x_0)$. Then for the estimates, we need to shrink the radius to $R/32$. Such construction and estimates can be found in [20, Lemma 8.3 (ii)] and [21, Lemma 5.1 (ii)]. Briefly, we approximate the neighborhood of $\Omega_{R/24}(x_0)$ by a half ball thanks to the small Reifenberg flat assumption. Then we apply a cutoff technique for the Dirichlet case, or a reflection technique for the conormal case. All these two techniques will be introduced in Case 3 below.

In both Cases 1 and 2, actually we can take $q = \infty$ in (4.27).

Case 3: $\mathrm{dist}(x_0, \partial\Omega) < R/32$, $\mathrm{dist}(x_0, \Gamma) < R/24$.

In this case, we deal with the "mixed" boundary condition. Take $y_0 \in \Gamma$ with $\mathrm{dist}(y_0, x_0) < R/24$. Consider the coordinate system associated with $(y_0, R/4)$ as in Assumption 4.2.2 (γ). For simplicity, we shift the origin in $x' = (x_1, x_2)$−hyperplane, such that

$$\begin{aligned} &\partial\Omega \cap B_{R/4}(y_0) \subset \{-\gamma R/2 < x_1 < 0\}, \\ &\Gamma \cap B_{R/4}(y_0) \subset \{-\gamma R/2 < x_2 < 0\}. \end{aligned} \tag{4.28}$$

In the following, we will omit the center when it is y_0. For example,

$$\Omega_{R/4} := \Omega \cap B_{R/4}(y_0), \quad \Omega^+_{R/4} := \Omega_{R/4} \cap \mathbb{R}^d_+, \quad \Omega^-_{R/4} := \Omega_{R/4} \cap \mathbb{R}^d_-,$$

where $\mathbb{R}^d_- = \{x \in \mathbb{R}^d : x_1 < 0\}$. Note that this is slightly different from the usual convention that we omit the center when it is the coordinate origin. The following inclusion relation will be useful:

$$\Omega_{R/32}(x_0) \subset \Omega_{R/8} \subset \Omega_{R/2} \subset \Omega_R(x_0). \tag{4.29}$$

Now we start to construct the decomposition. First, we introduce a cut-off function $\chi \in C^\infty(\mathbb{R}^d)$ with $D\chi$ supported in a "L-shaped" domain, satisfying

$$\begin{cases} \chi = 0, \quad \text{on } \{x_2 > -\gamma R\} \cap \{x_1 < \gamma R\}, \\ \chi = 1, \quad \text{on } \{x_2 < -2\gamma R\} \cup \{x_1 > 2\gamma R\}, \\ 0 \le \chi \le 1, |D\chi| \le \frac{2}{\gamma R}. \end{cases}$$

The following two lemmas should be read as parts of the proof of Proposition 4.5.3. The first one is an important estimate of a typical term in our proof. Both the inequality itself and the decomposition technique in the proof will be used later.

Lemma 4.5.4. *We have*

$$(|D\chi u|^2)^{1/2}_{\Omega_{R/4}} \le N\gamma^{1/(2\mu')}\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}\Big),$$

where $N = N(d, p, \Lambda)$.

Proof. From the construction of χ, we have

$$\|D\chi u\|_{L_2(\Omega_{R/4})} \leq \frac{2}{\gamma R}\|\mathbb{I}_{\mathrm{supp}\{D\chi\}}u\|_{L_2(\Omega_{R/4})}.$$

Now we decompose the set $\mathrm{supp}\{D\chi\} \cap \Omega_{R/4}$ to obtain the required smallness. Consider the following grid points on $\partial\mathbb{R}^d_+$:

$$\mathcal{D}_{grid} := \{z \in \mathbb{R}^d : z = (0, k\gamma R) \text{ for } k = (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}, k_2 \geq -1\} \cap \Omega_{R/4}.$$

Clearly $\bigcup_{z\in\mathcal{D}_{grid}} \Omega_{\sqrt{d+3}\gamma R}(z)$ covers $\mathrm{supp}\{D\chi\} \cap \Omega_{R/4}$, and because $\gamma < 1/(32\sqrt{d+3})$

$$\bigcup_{z\in\mathcal{D}_{grid}} \Omega_{\sqrt{d+3}\gamma R}(z) \subset \Omega_{R/3},$$

with each point covered by at most $N(d)$ of such neighborhoods. Due to (4.28), we know that in each $\Omega_{\sqrt{d+3}\gamma R}(z)$, (4.10) is satisfied with

$$(z_0, \alpha) = (z + (c, (-1 + \sqrt{d+3})\gamma R/2, 0, \cdots, 0), 1/4),$$

where $c \in (-\gamma R/2, 0)$ is chosen carefully to guarantee $z_0 \in \partial\Omega$. Hence we can apply the Poincaré inequality stated in Corollary 4.3.2 and Hölder's inequality to obtain

$$\begin{aligned}
\|\mathbb{I}_{\mathrm{supp}\{D\chi\}}u\|^2_{L_2(\Omega_{R/4})} &\leq N \sum_{z\in\mathcal{D}_{grid}} \|u\|^2_{L_2(\Omega_{\sqrt{d+3}\gamma R}(z))} \\
&\leq N(\gamma R)^2 \sum_{z\in\mathcal{D}_{grid}} \|Du\|^2_{L_2(\Omega_{2\sqrt{d+3}\gamma R}(z))} \\
&\leq N(\gamma R)^2 \|\mathbb{I}_{|x_1|<2\sqrt{d+3}\gamma R} Du\|^2_{L_2(\Omega_{R/3})} &(4.30)\\
&\leq N(\gamma R)^2 \cdot (\gamma R^d)^{1/\mu'} \|Du\|^2_{L_{2\mu}(\Omega_{R/3})}. &(4.31)
\end{aligned}$$

To obtain (4.31), we used Hölder's inequality. Now we rewrite this using the notation of average and use a properly rescaled version of Lemma 4.3.4 as well as (4.29) to obtain

$$
\begin{aligned}
(|D\chi u|^2)^{1/2}_{\Omega_{R/4}} &\le N\gamma^{1/(2\mu')}(|Du|^{2\mu})^{1/(2\mu)}_{\Omega_{R/3}} \\
&\le N\gamma^{1/(2\mu')}\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}\Big).
\end{aligned}
$$

The lemma is proved. ∎

The second lemma shows how we "freeze" the boundary to be a hyperplane using a cut-off technique together with a reflection.

Lemma 4.5.5. *The function $\chi u \in W^{1,2}(\Omega^+_{R/4})$ satisfies the following equation in the weak sense*

$$
\begin{cases}
D_i(a_{ij}D_j(\chi u)) - \lambda\chi u = D_i g_i^{(1)} + D_i g_i^{(2)} + g_i^{(3)} D_i\chi + g_i^{(4)} D_i\widetilde{\chi} + g^{(5)} & \text{in } \Omega^+_{R/4}, \\
a_{ij}D_j(\chi u)n_i = g_i^{(1)}n_i + g_i^{(2)}n_i & \text{on } \Gamma^-, \\
\chi u = 0 & \text{on } \Gamma^+,
\end{cases} \tag{4.32}
$$

where

$$
\begin{aligned}
&g_i^{(1)} = a_{ij}uD_j\chi + f_i\chi, \quad g_i^{(2)} = (-\varepsilon_i\varepsilon_j\widetilde{a_{ij}}\widetilde{\chi}D_j\widetilde{u} + \varepsilon_i\widetilde{\chi}\widetilde{f_i})\mathbb{I}_{(-x_1,x_2,x'')\in\Omega^-_{R/4}}, \\
&g_i^{(3)} = a_{ij}D_ju - f_i, \quad g_i^{(4)} = (\varepsilon_i\varepsilon_j\widetilde{a_{ij}}D_j\widetilde{u} - \varepsilon_i\widetilde{f_i})\mathbb{I}_{(-x_1,x_2,x'')\in\Omega^-_{R/4}}, \\
&g^{(5)} = \chi f + \widetilde{\chi}\widetilde{f}\mathbb{I}_{(-x_1,x_2,x'')\in\Omega^-_{R/4}} + \lambda\widetilde{\chi}\widetilde{u}\mathbb{I}_{(-x_1,x_2,x'')\in\Omega^-_{R/4}}, \\
&\Gamma^+ = \partial\Omega^+_{R/4} \cap \{x_1 = 0, x_2 > 0\}, \quad \Gamma^- = \partial\Omega^+_{R/4} \cap \{x_1 = 0, x_2 < 0\}.
\end{aligned}
$$

Here we denote $\widetilde{f}(x_1,x_2,x'') := f(-x_1,x_2,x'')$, and similarly for $\widetilde{a_{ij}}$, $\widetilde{\chi}$, and $\widetilde{u}$. We also use *the following notation*

$$\varepsilon_i := \begin{cases} -1 & \textit{if } \ i = 1, \\ 1 & \textit{if } \ i \neq 1. \end{cases}$$

Proof. Take any test function $\psi \in W^{1,2}_{\partial\Omega^+_{R/4}\setminus\Gamma^-}(\Omega^+_{R/4})$. We extend ψ to $\mathbb{R}^d_+$ by setting $\psi \equiv 0$ on $\mathbb{R}^d_+ \setminus \Omega^+_{R/4}$, and then, we again extend evenly to $\mathbb{R}^d$. Denote this extended function by $\mathcal{E}\psi$, and note that $\chi\mathcal{E}\psi \in W^{1,2}_{\mathcal{D}}(\Omega)$. Testing (4.12) with $\chi\mathcal{E}\psi$ and rearranging terms will give us (4.32). ∎

We continue the proof of Proposition 4.5.3. Solve the following equation

$$\begin{cases} D_i(\overline{a_{ij}}D_j\hat{w}) - \lambda\hat{w} = D_i((\overline{a_{ij}} - a_{ij})D_j(\chi u)) + D_i g_i^{(1)} + D_i g_i^{(2)} \\ \qquad\qquad + g_i^{(3)} D_i\chi + g_i^{(4)} D_i\widetilde{\chi} + g^{(5)} & \text{in } \Omega^+_{R/4}, \\ \overline{a_{ij}}D_j\hat{w}\cdot n_i = (\overline{a_{ij}} - a_{ij})D_j(\chi u)n_i + g_i^{(1)}n_i + g_i^{(2)}n_i & \text{on } \Gamma^-, \\ \hat{w} = 0 & \text{on } \partial\Omega^+_{R/4}\setminus\Gamma^-, \end{cases} \tag{4.33}$$

for $\hat{w} \in W^{1,2}_{\partial\Omega^+_{R/4}\setminus\Gamma^-}(\Omega^+_{R/4})$, where $\overline{a_{ij}} = (a_{ij})_{\Omega_{R/4}}$ are constants. Due to the Lax-Milgram lemma, such $\hat{w}$ exists. For simplicity, we denote

$$\hat{W} := |D\hat{w}| + \sqrt{\lambda}|\hat{w}|.$$

Testing (4.33) by $\hat{w}$, and using the ellipticity and Hölder's inequality, we have

$$\|\hat{W}\|^2_{L_2(\Omega^+_{R/4})} \le \|(\overline{a_{ij}} - a_{ij})D_j(\chi u)\|_{L_2(\Omega^+_{R/4})}\|D\hat{w}\|_{L_2(\Omega^+_{R/4})} \tag{4.34}$$

$$+ \left\|g_i^{(1)}\right\|_{L_2(\Omega^+_{R/4})}\|D\hat{w}\|_{L_2(\Omega^+_{R/4})} + \left\|g_i^{(2)}\right\|_{L_2(\Omega^+_{R/4})}\|D\hat{w}\|_{L_2(\Omega^+_{r/4})} \tag{4.35}$$

$$+ \left\|\mathbb{I}_{\operatorname{supp}\{D\chi\}} g_i^{(3)}\right\|_{L_2(\Omega^+_{R/4})}\|D_i\chi\hat{w}\|_{L_2(\Omega^+_{R/4})} + \left\|g_i^{(4)}\right\|_{L_2(\Omega^+_{R/4})}\left\|D_i\widetilde{\chi}\hat{w}\mathbb{I}_{(-x_1,x_2,x'')\in\Omega^-_{R/4}}\right\|_{L_2(\Omega^+_{R/4})} \tag{4.36}$$

$$+ \left\| \lambda^{-1/2} g^{(5)} \right\|_{L_2(\Omega^+_{R/4})} \|\lambda^{1/2} \hat{w}\|_{L_2(\Omega^+_{R/4})}. \tag{4.37}$$

For the term in (4.34), we use Assumption 4.2.3, Hölder's inequality, Lemma 4.3.4, and Lemma 4.5.4. Noting that $|\Omega^+_{R/4}|$, $|\Omega_{R/4}|$ and $|\Omega_R(x_0)|$ are all comparable to R^d, we have:

$$\begin{aligned}
\Big(|(\overline{a_{ij}} - a_{ij}) D_j(\chi u)|^2\Big)^{1/2}_{\Omega^+_{R/4}} &\le N\Big(|\overline{a_{ij}} - a_{ij}|^{2\mu'}\Big)^{1/(2\mu')}_{\Omega_{R/4}} \Big(|Du|^{2\mu}\Big)^{1/(2\mu)}_{\Omega_{R/4}} + N\Big(|D\chi u|^2\Big)^{1/2}_{\Omega^+_{R/4}} \\
&\le N\Big(|\overline{a_{ij}} - a_{ij}|\Big)^{1/(2\mu')}_{\Omega_{R/4}} \Big(|Du|^{2\mu}\Big)^{1/(2\mu)}_{\Omega_{R/4}} + N\Big(|D\chi u|^2\Big)^{1/2}_{\Omega^+_{R/4}} \\
&\le N\Big(\theta^{1/(2\mu')} + \gamma^{1/(2\mu')}\Big)\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}\Big),
\end{aligned} \tag{4.38}$$

where $N = N(d, p, \Lambda)$ is a constant.

For the terms in (4.35), we first estimate $g_i^{(1)}$. This is simply due to Lemma 4.5.4 and Hölder's inequality:

$$\begin{aligned}
\Big(\Big|g_i^{(1)}\Big|^2\Big)^{1/2}_{\Omega^+_{R/4}} &\le N(|uD\chi|^2)^{1/2}_{\Omega_{R/4}} + N(|f_i|^2)^{1/2}_{\Omega_{R/4}} \\
&\le N\gamma^{1/(2\mu')}(U^2)^{1/2}_{\Omega_R(x_0)} + N(F^{2\mu})^{\frac{1}{2\mu}}_{\Omega_R(x_0)}.
\end{aligned} \tag{4.39}$$

Now we estimate $g_i^{(2)}$ as follows:

$$\begin{aligned}
\Big(\Big|g_i^{(2)}\Big|^2\Big)^{1/2}_{\Omega^+_{R/4}} &\le \Big(\Big|\widetilde{a_{ij}}\widetilde{\chi} D_j \widetilde{u} \mathbb{I}_{(-x_1, x_2, x'') \in \Omega^-_{R/4}}\Big|^2\Big)^{1/2}_{\Omega^+_{R/4}} + \Big(\Big|\widetilde{\chi}\widetilde{f_i} \mathbb{I}_{(-x_1, x_2, x'') \in \Omega^-_{R/4}}\Big|^2\Big)^{1/2}_{\Omega^+_{R/4}} \\
&\le N\Big(\Big|\mathbb{I}_{\Omega^-_{R/4}} Du\Big|^2\Big)^{1/2}_{\Omega_{R/4}} + N(|f_i|^2)^{1/2}_{\Omega_{R/4}} \\
&\le N\gamma^{1/(2\mu')}(|Du|^{2\mu})^{1/(2\mu)}_{\Omega_{R/4}} + N(|f_i|^{2\mu})^{1/(2\mu)}_{\Omega_{R/4}} \\
&\le N\gamma^{1/(2\mu')}\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}\Big) + N(|f_i|^{2\mu})^{1/(2\mu)}_{\Omega_{R/4}},
\end{aligned} \tag{4.40}$$

where in the last line, we used Lemma 4.3.4.

For the terms in (4.36), we first use the same decomposition technique together with Poincaré's inequality as in the proof of Lemma 4.5.4 (until the step (4.30)) to obtain:

$$\|D_i\chi\hat{w}\|_{L_2(\Omega^+_{R/4})} \le N\|D\hat{w}\|_{L_2(\Omega^+_{R/4})}.$$

To avoid the problem of increased integrating domain, here we need to modify the decomposition to be

$$\bigcup_{z\in\mathcal{D}_{grid}} \big(\Omega_{\sqrt{d+3}\gamma R}(z)\cap\Omega^+_{R/4}\big),$$

and the same proof still applies. Again, with the help of Hölder's inequality and Lemma 4.3.4, we can estimate $g_i^{(3)}$ as follows:

$$\begin{aligned}\Big\|\mathbb{I}_{\mathrm{supp}\{D\chi\}}g_i^{(3)}\Big\|_{L_2(\Omega^+_{R/4})} &\le N\Big\|\mathbb{I}_{\mathrm{supp}\{D\chi\}}Du\Big\|_{L_2(\Omega^+_{R/4})} + N\Big\|\mathbb{I}_{\mathrm{supp}\{D\chi\}}f_i\Big\|_{L_2(\Omega^+_{R/4})}\\ &\le N\gamma^{1/(2\mu')}(\|U\|_{L_2(\Omega_R(x_0))} + R^{d/(2\mu')}\|F\|_{L_{2\mu}(\Omega_R(x_0))}).\end{aligned}$$

Hence,

$$\begin{aligned}&\Big\|\mathbb{I}_{\mathrm{supp}\{D\chi\}}g_i^{(3)}\Big\|_{L_2(\Omega^+_{R/4})}\|D_i\chi\hat{w}\|_{L_2(\Omega^+_{R/4})}\\ &\qquad\le N\gamma^{1/(2\mu')}(\|U\|_{L_2(\Omega_R(x_0))} + R^{d/(2\mu')}\|F\|_{L_{2\mu}(\Omega_R(x_0))})\|D\hat{w}\|_{L_2(\Omega^+_{R/4})}.\end{aligned} \tag{4.41}$$

Using similar techniques as in (4.40), we can deduce that

$$\begin{aligned}&\Big\|g_i^{(4)}\Big\|_{L_2(\Omega^+_{R/4})}\Big\|D_i\widetilde{\chi}\hat{w}\mathbb{I}_{(-x_1,x_2,x'')\in\Omega^-_{R/4}}\Big\|_{L_2(\Omega^+_{R/4})}\\ &\qquad\le N\gamma^{1/(2\mu')}(\|U\|_{L_2(\Omega_R(x_0))} + R^{d/(2\mu')}\|F\|_{L_{2\mu}(\Omega_R(x_0))})\|D\hat{w}\|_{L_2(\Omega^+_{R/4})}.\end{aligned} \tag{4.42}$$

We are left to estimate the one last term in (4.37):

$$\begin{aligned}&\Big\|\lambda^{-1/2}g^{(5)}\Big\|_{L_2(\Omega^+_{R/4})}\\ &\le \Big\|\lambda^{-1/2}\chi f\Big\|_{L_2(\Omega^+_{R/4})} + \Big\|\lambda^{-1/2}\widetilde{f}\widetilde{\chi}\mathbb{I}_{(-x_1,x_2,x'')\in\Omega^-_{R/4}}\Big\|_{L_2(\Omega^+_{R/4})} + \Big\|\lambda^{1/2}\widetilde{\chi}\widetilde{u}\mathbb{I}_{(-x_1,x_2,x'')\in\Omega^-_{R/4}}\Big\|_{L_2(\Omega^+_{R/4})}\end{aligned}$$

$$\leq 2\|F\|_{L_2(\Omega_{R/4})} + N\gamma^{1/(2\mu')}(\|U\|_{L_2(\Omega_R(x_0))} + \|F\|_{L_2(\Omega_R(x_0))}), \tag{4.43}$$

where for the last term, we applied similar techniques as we did to estimate $g_i^{(2)}$.

Substituting (4.38)-(4.43) back, we obtain

$$(\hat{W}^2)^{1/2}_{\Omega^+_{R/4}} \leq N\Big((\theta^{\frac{1}{2\mu'}} + \gamma^{\frac{1}{2\mu'}})(U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{\frac{1}{2\mu}}_{\Omega_R(x_0)}\Big). \tag{4.44}$$

Now we define

$$W := \begin{cases} \hat{W} + |D((1-\chi)u)| + \sqrt{\lambda}|(1-\chi)u| & \text{in } \Omega_{R/32}(x_0) \cap \mathbb{R}^d_+, \\ |Du| + \sqrt{\lambda}|u| & \text{in } \Omega_{R/32}(x_0) \cap \mathbb{R}^d_-. \end{cases}$$

Using (4.44), Hölder's inequality, Lemma 4.3.4, and Lemma 4.5.4, we can obtain (4.26).

To construct V, we set

$$v := \chi u - \hat{w}.$$

Clearly $v \in W^{1,2}_{\Gamma^+}(\Omega^+_{R/4})$. Simple computation using Lemma 4.5.5 and (4.33) shows that v satisfies

$$\begin{cases} D_i(\overline{a_{ij}} D_j v) - \lambda v = 0 & \text{in } \Omega^+_{R/4}, \\ \overline{a_{ij}} D_j v \cdot n_i = 0 & \text{on } \Gamma^-, \\ v = 0 & \text{on } \Gamma^+. \end{cases}$$

Now we define

$$V := \begin{cases} |Dv| + \sqrt{\lambda}|v| & \text{in } \Omega^+_{R/4}, \\ 0 & \text{in } \Omega^-_{R/4}. \end{cases}$$

Then we have

$$U \le W + V \quad \text{in } \Omega_{R/32}(x_0)$$

from the fact that

$$u = v + \hat{w} + (1-\chi)u \quad \text{in } \Omega_{R/32}(x_0) \cap \mathbb{R}^d_+, \quad W = U \quad \text{in } \Omega_{R/32}(x_0) \cap \mathbb{R}^d_-.$$

Using (4.29), we can apply a properly rescaled version of (4.4.1) to obtain that for any $q \in [2, 4)$, $V \in L_q(\Omega_{R/32}(x_0))$ satisfying

$$\begin{aligned}
(V^q)^{1/q}_{\Omega_{R/32}(x_0)} \le &(V^q)^{1/q}_{\Omega^+_{R/8}} \le N(V^2)^{1/2}_{\Omega^+_{R/4}} \\
\le &N((|D(\chi u)|^2)^{1/2}_{\Omega^+_{R/4}} + \sqrt{\lambda}(|\chi u|^2)^{1/2}_{\Omega^+_{R/4}} + |\hat{W}^2|^{1/2}_{\Omega^+_{R/4}}) \\
\le &N(U^2)^{1/2}_{\Omega_R(x_0)} + N\Big((\theta^{1/(2\mu')} + \gamma^{1/(2\mu')})(U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}\Big) \\
\le &N(U^2)^{1/2}_{\Omega_R(x_0)} + N(F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}.
\end{aligned} \tag{4.45}$$

Here we used the estimates for $uD\chi$ and $\hat{W}$ in previous steps. Clearly, from (4.45) we obtain (4.27). This finishes the proof of Proposition 4.5.3. ∎

4.5.2 Level set argument

In previous steps, we treat the perturbation problem by decomposing U into two parts, with L_2 and L_q estimates respectively. Now we interpolate using a level set argument to obtain the required L_p estimate for Proposition 4.5.1. Such argument was suggested by Caffarelli in [8] for a "kernel free" approach to $W^{1,p}$ estimate of divergence form second-order elliptic equations. Note that our estimate is not an a priori estimate, i.e., we do not need to assume $Du \in L_p$ in advance.

Define

$$\mathcal{A}(s) := \{x \in \Omega : \mathcal{M}_\Omega(U^2)^{1/2} > s\},$$
$$\mathcal{B}(s) := \{x \in \Omega : (\gamma^{1/(2\mu')} + \theta^{1/(2\mu')})^{-1}\mathcal{M}_\Omega(F^{2\mu})^{1/(2\mu)} + \mathcal{M}_\Omega(U^2)^{1/2} > s\},$$

where $\mu, \mu' \in (1, \infty)$ are the constants from Proposition 4.5.3. Here we denote $\mathcal{M}_\Omega$ to be the Hardy-Littlewood maximal operator restricted on Ω, i.e., for $f \in L_{1,\text{loc}}(\Omega)$ and $x \in \Omega$:

$$\mathcal{M}_\Omega(f)(x) := \sup_{r>0} \fint_{B_r(x)} |f|\mathbb{I}_\Omega.$$

By the Hardy-Littlewood theorem, for any $f \in L_q(\Omega)$ with $q \in [1, \infty)$, we have

$$|\{x \in \Omega : \mathcal{M}_\Omega(f)(x) > s\}| \le N\frac{\|f\|^q_{L_q(\Omega)}}{s^q}, \tag{4.46}$$

where $N = N(d, q)$.

Proposition 4.5.3 leads to the following lemma.

Lemma 4.5.6. *Under the same hypothesis of Proposition 4.5.3, for any $q \in [2, 4)$, there exists a constant N depending on (d, p, q, Λ), such that for all $\kappa > 2^{d/2}$ and $s > 0$, the following holds: if for some $R < R_0, x_0 \in \overline{\Omega}$,*

$$|\Omega_{R/128}(x_0) \cap \mathcal{A}(\kappa s)| \ge N\Big(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big)|\Omega_{R/128}(x_0)|, \tag{4.47}$$

then $\Omega_{R/128}(x_0) \subset \mathcal{B}(s)$.

Proof. Without loss of generality, we assume $s = 1$. We also extend U and F to be zero outside Ω. We will prove the contrapositive of the above statement.

Suppose there exists a point z_0, with

$$z_0 \in \Omega_{R/128}(x_0), \quad z_0 \notin \mathcal{B}(1),$$

then by the definition of $\mathcal{B}$, we have

$$(\gamma^{1/(2\mu')} + \theta^{1/(2\mu')})^{-1}\mathcal{M}_\Omega(F^{2\mu})^{1/(2\mu)}(z_0) + \mathcal{M}_\Omega(U^2)^{1/2}(z_0) \le 1.$$

In particular, for any $r > 0$, we have

$$(\gamma^{1/(2\mu')} + \theta^{1/(2\mu')})^{-1}(F^{2\mu})^{1/(2\mu)}_{B_r(z_0)} + (U^2)^{1/2}_{B_r(z_0)} \le 1.$$

Using Proposition 4.5.3 with z_0 in place of x_0, we can find W, V defined on $\Omega_{R/32}(z_0)$, such that for any $q \in [2, 4)$,

$$\begin{aligned} &U \le V + W \quad \text{in } \Omega_{R/32}(z_0), \\ &(W^2)^{1/2}_{\Omega_{R/32}(z_0)} \le N(\gamma^{1/(2\mu')} + \theta^{1/(2\mu')}), \quad (V^q)^{1/q}_{\Omega_{R/32}(z_0)} \le N. \end{aligned} \tag{4.48}$$

Notice that we have the following inclusion

$$\Omega_{R/128}(x_0) \subset \Omega_{R/64}(z_0) \subset \Omega_{R/32}(z_0). \tag{4.49}$$

Now for any $y_0 \in \Omega_{R/128}(x_0) \cap \mathcal{A}(\kappa)$, by the definition of $\mathcal{A}$, we can find some $r > 0$ such that

$$\left(\fint_{B_r(y_0)} U^2\,dx\right)^{1/2} > \kappa.$$

We claim that $r < R/64$. Otherwise noting $y_0 \in \Omega_{R/64}(z_0)$, we have $\Omega_r(y_0) \subset \Omega_{2r}(z_0)$. Hence we can deduce that

$$\left(\fint_{B_r(y_0)} U^2\,dx\right)^{1/2} \le 2^{d/2}\left(\fint_{B_{2r}(z_0)} U^2\,dx\right)^{1/2} \le 2^{d/2}\mathcal{M}_\Omega(U^2)^{1/2}(z_0) \le 2^{d/2} < \kappa,$$

which is a contradiction.

Now, since $r < R/64$, the decomposition $U \le W + V$ is defined in $\Omega_r(y_0) \subset \Omega_{R/32}(z_0)$. Extending W and V to be zero outside Ω, we have

$$\begin{aligned}\left(\fint_{B_r(y_0)} U^2\,dx\right)^{1/2} &\le \left(\fint_{B_r(y_0)} W^2\,dx\right)^{1/2} + \left(\fint_{B_r(y_0)} V^2\,dx\right)^{1/2}\\ &\le \mathcal{M}_\Omega(W^2\mathbb{I}_{\Omega_{R/32}(z_0)})^{1/2}(y_0) + \mathcal{M}_\Omega(V^2\mathbb{I}_{\Omega_{R/32}(z_0)})^{1/2}(y_0).\end{aligned}$$

Then by (4.46), (4.48) and (4.49), we obtain

$$\begin{aligned}|\Omega_{R/128}(x_0) \cap \mathcal{A}(\kappa)| \le& |\Omega_{R/32}(z_0) \cap \mathcal{A}(\kappa)|\\ \le& \left|\{\mathcal{M}_\Omega(W^2\mathbb{I}_{\Omega_{R/32}(z_0)})^{1/2} > \kappa/2\}\right| + \left|\{\mathcal{M}_\Omega(V^2\mathbb{I}_{\Omega_{R/32}(z_0)})^{1/2} > \kappa/2\}\right|\\ \le& N\frac{\|W\|^2_{L_2(\Omega_{R/32}(z_0))}}{(\kappa/2)^2} + N\frac{\|V\|^q_{L_q(\Omega_{R/32}(z_0))}}{(\kappa/2)^q}\\ \le& N|\Omega_{R/32}(z_0)|\Big(\kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'}) + \kappa^{-q}\Big)\\ \le& N\Big(\kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'}) + \kappa^{-q}\Big)|\Omega_{R/128}(x_0)|.\end{aligned}$$

Here $N = N(d, p, q, \Lambda)$ is exactly what we aim to find. ■

Using a lemma in measure theory called "crawling of the ink spot" which was first introduced by Krylov and Safonov in [44, 64], we obtain the following decay estimate from Lemma 4.5.6.

Corollary 4.5.7 (Decay of $\mathcal{A}(s)$)**.** *Under the same hypothesis of Proposition 4.5.3, for any $q \in [2, 4)$, there exists a constant N depending on (d, p, q, Λ), such that for any $\kappa > \max\{2^{d/2}, \kappa_0\}$ and*

$$s > s_0(d, p, q, \Lambda, \kappa, R_0, \|U\|_{L_2(\Omega)}) := \left(\frac{\|U\|^2_{L_2(\Omega)}}{N\kappa^2(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'}))|B_{R_0/128}|}\right)^{1/2}, \quad (4.50)$$

we have

$$|\mathcal{A}(\kappa s)| \leq N\Big(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big)|\mathcal{B}(s)|,$$

where κ_0 is the constant satisfying

$$N\Big(\kappa_0^{-q} + \kappa_0^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big) < 1/3. \tag{4.51}$$

Here we only sketch the proof. The key idea is to use a stopping time argument (or the Calderón-Zygmund decomposition as in [8]). Different from Krylov and Safonov's original version, we cover Ω by balls instead of dyadic cubes. For any $x_0 \in \mathcal{A}(\kappa s)$, by (4.46), (4.50), and (4.51), we see that (4.47) does not hold with R_0 in place of R. We shrink the "ball" $\Omega_{R/128}(x_0)$ from $R = R_0$ until the first time (4.47) holds. Due to (4.50), (4.51) and the Lebesgue differentiation theorem, such R exists and $R \in (0, R_0)$. We are left to use the Vitali covering lemma to pick a "almost disjoint" cover.

4.5.3 Proof of Proposition 4.5.1 and Corollary 4.5.2

Now we are ready to give the proof of Proposition 4.5.1.

Proof of Proposition 4.5.1. Let us fix $p \in (2, 4)$, and let γ, θ, and κ be positive constants to be chosen later, such that

$$\gamma < 1/(32\sqrt{d+3}), \quad \theta < 1, \quad \kappa > \max\{2^{d/2}, \kappa_0\},$$

where $\kappa_0 = \kappa_0(d, p, \Lambda)$ is a constant satisfying (4.51) with $q = (p+4)/2$. It suffices to prove

$$\lim_{S\to\infty}\int_0^S p|\mathcal{A}(s)|s^{p-1}\,ds \le N\Big(R_0^{d(1-p/2)}\|U\|^p_{L_2(\Omega)} + \|F\|^p_{L_p(\Omega)}\Big) \tag{4.52}$$

under Assumptions 4.2.2 (γ) and 4.2.3 (θ). The left-hand side becomes

$$\lim_{S\to\infty}\int_0^{S/\kappa} p\kappa^p|\mathcal{A}(\kappa s)|s^{p-1}\,ds.$$

We bound the integrand by using Corollary 4.5.7 with $q=(p+4)/2$ when $s>s_0$. For $s\le s_0$, we apply Chebyshev's inequality. Then we have

$$\begin{aligned}
&\int_0^{S/\kappa}|\mathcal{A}(\kappa s)|\kappa^p s^{p-1}\,ds\\
&\le N\int_0^{s_0}\frac{\|U\|^2_{L_2(\Omega)}}{(\kappa s)^2}\kappa^p s^{p-1}\,ds + N\Big(\kappa^{-q}+\kappa^{-2}(\gamma^{\frac{1}{\mu'}}+\theta^{\frac{1}{\mu'}})\Big)\int_0^{S/\kappa}|\mathcal{B}(s)|\kappa^p s^{p-1}\,ds\\
&\le N_0R_0^{d(1-p/2)}\|U\|^p_{L_2(\Omega)} + N_1\Big(\kappa^{-q}+\kappa^{-2}(\gamma^{\frac{1}{\mu'}}+\theta^{\frac{1}{\mu'}})\Big)\kappa^p\int_0^{S/\kappa}|\mathcal{A}(s/2)|s^{p-1}\,ds + N\|F\|^p_{L_p(\Omega)},
\end{aligned}$$

where $N_1=N_1(d,p,\Lambda)$ and N_0 depends also on κ. Here in the last line, we used the following relationship:

$$\mathcal{B}(s)\subset\mathcal{A}(s/2)\cup\{(\gamma^{1/(2\mu')}+\theta^{1/(2\mu')})^{-1}\mathcal{M}_\Omega(F^{2\mu})^{1/(2\mu)}>s/2\}$$

and the Hardy-Littlewood inequality, noting that $2\mu<p$. Now we choose κ sufficient large such that $N_1\kappa^{p-q}<2^{-p-2}$, and then θ and γ sufficient small such that $N_1(\kappa^{p-2}(\gamma^{1/\mu'}+\theta^{1/\mu'}))<2^{-p-2}$. Then we have

$$\int_0^S p|\mathcal{A}(s)|s^{p-1}\,ds\le NR_0^{d(1-p/2)}\|U\|^p_{L_2(\Omega)}+N\|F\|^p_{L_p(\Omega)}+\frac{p}{2}\int_0^{S/(2\kappa)}|\mathcal{A}(s)|s^{p-1}\,ds,$$

where $N=N(d,p,\Lambda)$. This yields (4.52). Hence, we have $u\in W^{1,p}(\Omega)$ satisfying (4.23). Note that all the previous proof including the reverse Hölder inequality, the estimate for harmonic functions, the decomposition lemma, and the level set argument, also work when $\lambda = 0$ if we substitute U by $|Du|$ and F by $\sum_i |f_i|$. Thus we can also obtain (4.24) when $\lambda = 0$ and $f = 0$. ■

To end this section, we give the proof of Corollary 4.5.2.

Proof of Corollary 4.5.2. Consider the usual smooth cut-off function $\zeta \in C_c^\infty(B_\varepsilon)$ with $\zeta \in [0,1], |D\zeta| \le N/\varepsilon$. From (4.5), we can obtain the equation for ζu:

$$\begin{cases} D_i(a_{ij}D_j(u\zeta)) - \lambda(u\zeta) = D_i(f_i\zeta + h_i) + (f\zeta + h) & \text{in } \Omega, \\ a_{ij}D_j(u\zeta)n_i = (f_i\zeta + h_i)n_i & \text{on } \mathcal{N}, \\ u\zeta = 0 & \text{on } \mathcal{D}, \end{cases}$$

where h_i and h are given as follows:

$$\begin{aligned} h_i &:= a_{ij}uD_j\zeta - b_iu\zeta, \\ h &:= a_{ij}D_juD_i\zeta + b_iuD_i\zeta - \hat{b}_iD_iu\zeta - cu\zeta - f_iD_i\zeta. \end{aligned}$$

Hence by (4.23), for any $\lambda > 0$, we have

$$\begin{aligned} &\|D(u\zeta)\|_{L_p(\Omega)} + \sqrt{\lambda}\|u\zeta\|_{L_p(\Omega)} \\ &\le N_0R_0^{d(1/p-1/2)}\Big(\|D(u\zeta)\|_{L_2(\Omega)} + \sqrt{\lambda}\|u\zeta\|_{L_2(\Omega)}\Big) \\ &\quad + N_0\|f_i\zeta\|_{L_p(\Omega)} + \frac{N_0}{\sqrt{\lambda}}\Big(\|f_iD\zeta\|_{L_p(\Omega)} + \|f\zeta\|_{L_p(\Omega)}\Big) \\ &\quad + N_1\Big(\|uD\zeta\|_{L_p(\Omega)} + \|u\zeta\|_{L_p(\Omega)}\Big) \\ &\quad + \frac{N_1}{\sqrt{\lambda}}\Big(\|Du\cdot D\zeta\|_{L_p(\Omega)} + \|uD\zeta\|_{L_p(\Omega)} + \|Du\zeta\|_{L_p(\Omega)} + \|u\zeta\|_{L_p(\Omega)}\Big), \end{aligned} \tag{4.53}$$

where $N_0 = N_0(d, p, \Lambda)$ and N_1 depends also on K. Using Hölder's inequality, we obtain

$$\|D(u\zeta)\|_{L_2(\Omega)} + \sqrt{\lambda}\|u\zeta\|_{L_2(\Omega)} \le \varepsilon^{d/2-d/p}\Big(\|D(u\zeta)\|_{L_p(\Omega)} + \sqrt{\lambda}\|u\zeta\|_{L_p(\Omega)}\Big).$$

Thus by taking $\varepsilon = \varepsilon(d, p, \Lambda, R_0) > 0$ sufficiently small such that $N_0(\varepsilon/R_0)^{d/2-d/p} < 1/2$, we can absorb the first two terms on the right-hand side of (4.53) to the left-hand side. Then by using the standard partition of unity technique and choosing λ large enough, we conclude (4.25). The corollary is proved. ∎

4.6 Solvability and general p

With the regularity result in hand, we are now going to prove Theorem 4.2.4 concerning the solvability. Note that in this section, we deal with more general cases $p \in (4/3, 4)$. We first state the following L_2 well-posedness result, which is a direct consequence of the Lax-Milgram lemma.

Lemma 4.6.1. *Let Ω be a domain with $\partial\Omega = \mathcal{D} \cup \mathcal{N}$. Consider the equation (4.5) with $b_i, \hat{b}_i, c \in L_\infty(\Omega)$. Then for any*

$$f, f_i \in L_2(\Omega), \quad \lambda > \lambda_2 := 4\Big(\|b_i\|^2_{L_\infty(\Omega)}/\Lambda + \|\hat{b}_i\|^2_{L_\infty(\Omega)}/\Lambda + \|c\|_{L_\infty(\Omega)}\Big),$$

there exists a unique $W^{1,2}_{\mathcal{D}}(\Omega)$ weak solution u to (4.5), satisfying

$$\|U\|_{L_2(\Omega)} \le N\|F\|_{L_2(\Omega)},$$

where $N = N(\Lambda)$ is a constant.

Proof of Theorem 4.2.4. We prove by three cases under Assumptions 4.2.2 (γ_0) and

4.2.3 (θ_0), where γ_0, θ_0 are the constants from Proposition 4.5.1. Assume that $\lambda > \lambda_0$, where λ_0 is a constant to be chosen below, which satisfies

$$\lambda_0 \geq \max\{\lambda_1, \lambda_2\}.$$

Here, λ_1 and λ_2 are the constants from Corollary 4.5.2 and Lemma 4.6.1, respectively.

Case 1: $p = 2$. This is Lemma 4.6.1.

Case 2: $p \in (2, 4)$. Due to the method of continuity and the a priori estimate (see Corollary 4.5.2), it suffices to prove the theorem when all the lower order coefficients are zero, i.e., $b_i \equiv \hat{b}_i \equiv c \equiv 0$.

Now we approximate f, f_i by $f^{(n)}, f_i^{(n)}$ strongly in $L_p(\Omega)$, where $f^{(n)}, f_i^{(n)} \in L_2(\Omega) \cap L_p(\Omega)$. Then by Lemma 4.6.1, there exist $W^{1,2}_{\mathcal{D}}(\Omega)$ weak solutions $u^{(n)}$ to (4.5) (without lower order terms) with $f^{(n)}, f_i^{(n)}$ in place of f, f_i. Moreover, it follows from Proposition 4.5.1 and Corollary 4.5.2 that $\{u^{(n)}\}$ is a Cauchy sequence in $W^{1,p}_{\mathcal{D}}(\Omega)$. Denote its limit by $u \in W^{1,p}_{\mathcal{D}}(\Omega)$. Clearly u is a weak solution, and (4.6) is satisfied.

Case 3: $p \in (4/3, 2)$. We first use a duality argument to prove the a priori estimate, i.e. assuming $u \in W^{1,p}_{\mathcal{D}}(\Omega)$ is a solution to (4.5) with $f, f_i \in L_p(\Omega)$, we are to prove the estimate (4.6). For simplicity, we consider the following equivalent norm for the space $L_p(\Omega) \times (L_p(\Omega))^d$ with $\lambda > 0$:

$$\|(f, (f_i)_{i=1}^d)\|_{p,\lambda} := \lambda^{-1/2}\|f\|_{L_p(\Omega)} + \sum_{i=1}^d \|f_i\|_{L_p(\Omega)},$$

and its dual space $L_{p'}(\Omega) \times (L_{p'}(\Omega))^d$:

$$\|(f, (f_i)_{i=1}^d)\|_{p',1/\lambda} := \lambda^{1/2}\|f\|_{L_{p'}(\Omega)} + \sum_{i=1}^{d} \|f_i\|_{L_{p'}(\Omega)},$$

where $p' \in (2, 4)$ satisfying $1/p + 1/p' = 1$.

By duality, to prove (4.6), it suffices to prove

$$\sup_{\substack{\varphi_i, \varphi \in C_c^\infty(\Omega) \\ \|(\varphi, (\varphi_i)_{i=1}^d)\|_{p',1/\lambda} = 1}} \left| \int_\Omega (D_i u \varphi_i + \lambda u \varphi)\, dx \right| \le N \left(\|f_i\|_{L_p(\Omega)} + \lambda^{-1/2}\|f\|_{L_p(\Omega)} \right).$$

For this, we solve for $v \in W^{1,p'}_{\mathcal{D}}(\Omega)$ to the following adjoint problem:

$$\begin{cases} D_i(a_{ji} D_j v - \hat{b}_i v) - b_i D_i v + cv - \lambda v = \lambda \varphi - D_i \varphi_i & \text{in } \Omega, \\ a_{ji} D_j v n_i - \hat{b}_i v n_i = -\varphi_i \cdot n_i & \text{on } \mathcal{N}, \\ v = 0 & \text{on } \mathcal{D}. \end{cases} \tag{4.54}$$

Noting that u is a test function for (4.54), and v is a test function for (4.5), we have

$$\begin{aligned} \left| \int_\Omega (D_i u \varphi_i + \lambda u \varphi)\, dx \right| &= \left| \int_\Omega (-a_{ji} D_j v D_i u + \hat{b}_i v D_i u - b_i D_i v u + cvu - \lambda v u)\, dx \right| \\ &= \left| \int_\Omega (-f_i D_i v + f v)\, dx \right| \\ &\le \Big(\|f_i\|_{L_p(\Omega)} + \lambda^{-1/2}\|f\|_{L_p(\Omega)} \Big) \Big(\|Dv\|_{L_{p'}(\Omega)} + \lambda^{1/2}\|v\|_{L_{p'}(\Omega)} \Big) \\ &= N \Big(\|f_i\|_{L_p(\Omega)} + \lambda^{-1/2}\|f\|_{L_p(\Omega)} \Big). \end{aligned}$$

Here, we use the following $W^{1,p'}$ estimate for the equation (4.54):

$$\|D_i v\|_{L_{p'}(\Omega)} + \sqrt{\lambda}\|v\|_{L_{p'}(\Omega)} \le N \Big(\|\varphi_i\|_{L_{p'}(\Omega)} + \lambda^{-1/2}\|\lambda \varphi\|_{L_{p'}(\Omega)} \Big) = N.$$

This gives us the $W^{1,p}$-a priori estimate, i.e., (4.6) when $p \in (4/3, 4)$.

To see the solvability, we approximate f, f_i by

$$f^{(n)}, f_i^{(n)} \in C_c^\infty (\subset L_p), \quad f^{(n)} \to f,\ f_i^{(n)} \to f_i \quad \text{in } L_p.$$

Let $u^{(n)}$ be the unique $W^{1,2}_{\mathcal{D}}$ weak solution associated with $f^{(n)}$ and $f_i^{(n)}$. Due to the $W^{1,p}$-a priori estimate that we just obtained, and the same argument as in case 2, it is enough to show $u^{(n)} \in W^{1,p}_{\mathcal{D}}(\Omega)$. Due to Hölder's inequality, this can be further reduced to showing the following:

$$\sum_k \|u\|_{W^{1,2}(\Omega_{k+1}\setminus\Omega_k)} \cdot k^{(d-1)(1/p-1/2)} < \infty. \tag{4.55}$$

Here, we denoted $\Omega_k := \Omega_k(0)$. For this, we use a classical "hole-filling" technique. Take $\eta \in C_c^\infty(B_k^c)$, $\eta = 1$ in B_{k+1}^c, $|D\eta| \le 2$. Testing the equation by $u\eta^2$ and rearranging terms, we obtain that there exits some $\lambda_0 = \lambda_0(d, p, \Lambda, R_0, \|b_i\|_\infty, \|\hat{b}_i\|_\infty, \|c\|_\infty)$, such that for $\lambda > \lambda_0$,

$$\int_{\Omega_{k+1}^c} (|Du|^2 + \lambda|u|^2)\,dx \le N \int_{\Omega_{k+1}\setminus\Omega_k} |u|^2\,dx.$$

Clearly, this leads to

$$\|Du\|^2_{L_2(\Omega_{k+1}^c)} + \lambda\|u\|^2_{L_2(\Omega_{k+1}^c)} \le \frac{N}{N+\lambda}(\|Du\|^2_{L_2(\Omega_k^c)} + \sqrt{\lambda}\|u\|^2_{L_2(\Omega_k^c)}).$$

Hence, $\|u\|_{W^{1,2}(\Omega_{k+1}\setminus\Omega_k)}$ decays exponentially and in particular, (4.55) holds. This finishes our proof. ∎

4.7 Bounded domain case

In this section, we deal with the bounded domain case, i.e., Theorem 4.2.5. First, we reduce the problem to the case $f = 0$ by solving a divergence equation. This reduction has also been used in [10]. Note that in the following, we use a key fact that a Reifenberg flat domain is also a so-called John domain, which can be found in [22, Remark 3.3].

Let us first recall the definition of John domains.

Definition 4.7.1 (John domain, [1])**.** A bounded set $\Omega \subset \mathbb{R}^d$ is a John domain, if there exist $x_0 \in \Omega$ and $\lambda > 0$ such that for every $x \in \Omega$ there exists a continuous rectifiable curve $\gamma : [0, 1] \mapsto \Omega$, such that $\gamma(0) = x, \gamma(1) = x_0$, and

$$\mathrm{dist}(\gamma(t), \Omega^c) \geq \lambda \cdot |\gamma[0, t]| \tag{4.56}$$

for all $t \in [0, 1]$, where $|\gamma[0, t]|$ represents the arc length.

Lemma 4.7.2. *Assume Ω is a bounded Reifenberg flat domain with $\partial\Omega = \mathcal{D} \cup \mathcal{N}, \mathcal{D}, \mathcal{N} \neq \emptyset$, satisfying Assumption 4.2.2 (γ), $\gamma < 1/2$. Let $p > 1$ and p_* be given as in (4.7). Then for every $f \in L_{p_*}(\Omega)$, there exists $\phi = (\phi_1, \cdots, \phi_d) \in (W^{1,p_*}_{\mathcal{N}}(\Omega))^d$ ($\subset (L_p(\Omega))^d$, by the Sobolev embedding), such that*

$$D_i\phi_i = f \text{ in } \Omega, \quad \|\phi_i\|_{L_p(\Omega)} \leq N\|f\|_{L_{p_*}}(\Omega), \tag{4.57}$$

where $N = N(d, p, \mathrm{diam}(\Omega), R_1)$ is a constant.

Proof. Noting that $\mathcal{D}, \mathcal{N} \neq \emptyset$, we can choose a point $x_0 \in \Gamma$. Taking the coordinate system in $B_{R_1}(x_0)$ from Assumption 4.2.2, we extend Ω beyond $\mathcal{D}$ as follows.

We first take the Whitney decomposition of the open set

$$\widetilde{\Omega}_{R_1} := \Omega_{R_1}(x_0) \cap \{x_{02} + 23/32R_1 < y_2 < x_{02} + 25/32R_1\}$$

as in [72, Chapter IV], i.e., $\widetilde{\Omega}_{R_1} = \cup_k Q_k$, where the disjoint cubes Q_k satisfy

$$\operatorname{diam}(Q_k) \leq \operatorname{dist}(\overline{Q_k}, (\widetilde{\Omega}_{R_1})^c) \leq 4 \operatorname{diam}(Q_k).$$

Denote the center of Q_k to be x_k. We extend Q_k to $\hat{Q}_k$ in the way that

$$\hat{Q}_k - x_k = 8(Q_k - x_k).$$

Let $\hat{\Omega} = \Omega \cup (\cup_k \hat{Q}_k)$. It is easy to see that

$$\mathcal{N} \subset \partial\hat{\Omega}, \quad C_1 R_1^d \leq |\hat{\Omega} \setminus \Omega| \leq C_2 R_1^d,$$

where C_1, C_2 are constants only depending on the space dimension d. Next, we check that $\hat{\Omega}$ is still a John domain, i.e., for any $\hat{x} \in \hat{\Omega}$ we construct the path connecting $\hat{x}$ and x_0, which satisfies the conditions in Definition 4.7.1.

Case 1: $\hat{x} \in \Omega$. Noting that any Reifenberg flat domain is also a John domain, we take the same path as in Definition 4.7.1. Noting that for any $x \in \Omega$, $\operatorname{dist}(x, \hat{\Omega}^c) \geq \operatorname{dist}(x, \Omega^c)$, (4.56) is satisfied with the same λ.

Case 2: $\hat{x} \in \hat{\Omega} \setminus \Omega$. We assume that $\hat{x}$ lies in the extended cube $\hat{Q}_k$ with center x_k and $\operatorname{diam}(\hat{Q}_k) = 8r_k$. Let x_0 be the point defined in Definition 4.7.1. If $x_0 \in Q_k$, we can take the straight line path. In this case, (4.56) is satisfied with the constant

$7/(9\sqrt{d})$. Now, if $x_0 \notin Q_k$, we first consider the straight line path

$$\gamma_1 : [0, 1/2] \mapsto \hat{Q}_k, \quad \gamma_1(0) = \hat{x}, \quad \gamma_1(1/2) = x_k.$$

Since $\hat{x} \notin Q_k$, we have

$$\frac{1}{2} r_k \leq |\gamma_1[0, 1/2]| \leq 4\sqrt{d} r_k.$$

Noting that $x_k \in \Omega$, we consider the re-parametrized path coming from Definition 4.7.1:

$$\gamma_2 : [0, 1/2] \mapsto \Omega, \quad \gamma_2(0) = x_k, \quad \gamma_2(1/2) = x_0.$$

Take $\gamma = \gamma_1 \circ \gamma_2$ be the path connecting γ_1 and γ_2. Now, when $t \in [0, 1/2]$, again (4.56) is satisfied with the constant $1/\sqrt{d}$. When $t \in [1/2, 1]$, we consider the following two cases: $\gamma(t) \in Q_k$ or $\gamma(t) \notin Q_k$.

If $\gamma(t) \in Q_k$, we have

$$\begin{aligned}
\frac{|\gamma[0, t]|}{\operatorname{dist}(\gamma(t), \hat{\Omega}^c)} &\leq \frac{|\gamma[0, 1/2]|}{\operatorname{dist}(\gamma(t), \hat{\Omega}^c)} + \frac{|\gamma[1/2, t]|}{\operatorname{dist}(\gamma(t), \Omega^c)} \\
&\leq \frac{4\sqrt{d} r_k}{7 r_k/2} + \lambda^{-1} = \frac{8\sqrt{d}}{7} + \lambda^{-1}.
\end{aligned}$$

When $\gamma(t) \notin Q_k$, we have $|\gamma[1/2, t]| \geq r_k/2$. Hence,

$$\begin{aligned}
\frac{|\gamma[0, t]|}{\operatorname{dist}(\gamma(t), \hat{\Omega}^c)} &= \frac{|\gamma[0, t]|}{|\gamma[1/2, t]|} \cdot \frac{|\gamma[1/2, t]|}{\operatorname{dist}(\gamma(t), \hat{\Omega}^c)} \\
&= (1 + \frac{|\gamma[0, 1/2]|}{|\gamma[1/2, t]|}) \cdot \frac{|\gamma[1/2, t]|}{\operatorname{dist}(\gamma(t), \hat{\Omega}^c)} \\
&\leq (1 + \frac{4\sqrt{d} r_k}{r_k/2}) \cdot \frac{|\gamma[1/2, t]|}{\operatorname{dist}(\gamma(t), \hat{\Omega}^c)} \\
&\leq (1 + 8\sqrt{d})\lambda^{-1}.
\end{aligned}$$

With all above, we have proved that $\hat{\Omega}$ is still a John domain. Now we extend f to

$\hat{\Omega}$ as

$$\begin{cases} \hat{f} := f & \text{in } \Omega, \\ \hat{f} := -\frac{1}{|\hat{\Omega}\setminus\Omega|}\int_\Omega f & \text{in } \hat{\Omega}\setminus\Omega. \end{cases}$$

Then we have

$$\int_{\hat{\Omega}} \hat{f} = 0, \quad \|\hat{f}\|_{L_{p_*}(\hat{\Omega})} \le N(R_1, |\Omega|)\|f\|_{L_{p_*}(\Omega)}.$$

Since $\hat{\Omega}$ is a John domain, we apply the result in [1, Theorem 4.1] to find $\phi = (\phi_1, \cdots, \phi_d) \in (W_0^{1,p_*}(\hat{\Omega}))^d$ satisfying

$$D_i\phi_i = \hat{f} \quad \text{in } \hat{\Omega}, \quad \|\phi_i\|_{W^{1,p_*}(\hat{\Omega})} \le N(\text{diam}(\hat{\Omega}), d, p)\|\hat{f}\|_{L_{p_*}(\hat{\Omega})}.$$

Now by Sobolev inequalities and our construction of $\hat{\Omega}, \hat{f}$, we obtain that $\phi \in (W_{\mathcal{N}}^{1,p_*}(\Omega))^d$, and

$$\|\phi_i\|_{L_p(\Omega)} \le N(\text{diam}(\Omega), R_1, d, p)\|\hat{f}\|_{L_{p_*}(\hat{\Omega})}.$$

The lemma is proved ■

Now we are ready to give the proof of Theorem 4.2.5.

Proof of Theorem 4.2.5. Using Lemma 4.7.2, for every $f \in L_{p_*}(\Omega)$, we can find

$$(\phi_i)_{i=1}^d \in (W_{\mathcal{N}}^{1,p_*}(\Omega))^d \subset (L_p(\Omega))^d$$

satisfying (4.57). Now we consider the following problem:

$$\begin{cases} Lu = D_i(f_i + \phi_i) & \text{in } \Omega, \\ Bu = (f_i + \phi_i)\, n_i & \text{on } \mathcal{N}, \\ u = 0 & \text{on } \mathcal{D}. \end{cases} \tag{4.58}$$

Since $\phi = 0$ on $\mathcal{N}$, one can easily check that any solution to (4.58) is also a solution to (4.4). Hence, without loss of generality, we may assume $f = 0$.

We aim to use the Fredholm alternative. For this, we first introduce some operators. From Theorem 4.2.4, for fixed large enough λ, we can find a unique weak solution $u \in W^{1,p}_{\mathcal{D}}(\Omega)$ to (4.5) satisfying (4.6), and hence (4.8).

We write $R(\lambda, L)$ as this solution operator, i.e.,

$$R(\lambda, L) : (L_p(\Omega))^d \times L_p(\Omega) \mapsto W^{1,p}_{\mathcal{D}}(\Omega), \quad R(\lambda, L)(f_i, f) = u.$$

In particular, for any L_p function f, we write

$$R_\lambda(f) := R(\lambda, L)(0, f).$$

From (4.8), R_λ is a bounded linear operator from $L_p(\Omega)$ to $W^{1,p}_{\mathcal{D}}(\Omega)$. Denote I as the compact embedding from $W^{1,p}_{\mathcal{D}}(\Omega)$ to $L_p(\Omega)$. Now, we write

$$T : W^{1,p}_{\mathcal{D}}(\Omega) \to W^{1,p}_{\mathcal{D}}(\Omega), \quad T(u) = R_\lambda \circ I(u).$$

From our construction, T is a compact operator. Noting that we have assumed that $f = 0$, applying the operator $R(\lambda, L)$ to both sides of

$$(L-\lambda)u + \lambda u = D_i f_i,$$

we can rewrite (4.4) as

$$(Id + \lambda T)u = R(\lambda, L)(f_i, 0). \tag{4.59}$$

By the Fredholm alternative, (4.59) has a unique $W^{1,p}_{\mathcal{D}}(\Omega)$ solution satisfying

$$||u||_{W^{1,p}(\Omega)} \leq N||f_i||_{L_p(\Omega)},$$

if the following homogeneous equation only has zero solution

$$\begin{cases} Lv = 0 & \text{in } \Omega, \\ Bv = 0 & \text{on } \mathcal{N}, \\ v = 0 & \text{on } \mathcal{D}. \end{cases} \tag{4.60}$$

When $v \in W^{1,2}(\Omega)$, this is true due to the weak maximum principle, noting that the proof in [34, Section 8.1] actually shows that $\sup_{\overline{\Omega}}|v|$ has to be achieved at the Dirichlet boundary. Hence the uniqueness of (4.60) is proved for the case $p \geq 2$.

When $p < 2$, we can use Theorem 4.2.4 and a bootstrap argument to improve the regularity. Suppose $v \in W^{1,p}(\Omega)$ is a solution to (4.60). Take λ large enough, noting that v is also a $W^{1,p}$ solution to

$$\begin{cases} (L-\lambda)v = -\lambda v & \text{in } \Omega, \\ Bv = 0 & \text{on } \mathcal{N}, \\ v = 0 & \text{on } \mathcal{D}. \end{cases}$$

By the Sobolev embedding, $-\lambda v \in L_{pd/(d-p)}(\Omega)$. Take $p^* = \min\{pd/(d-p), 2\}$. By the uniqueness of W^{1,p^*} solutions in Theorem 4.2.4, we obtain $v \in W^{1,p^*}$. Repeating this process if needed, in finite steps, we can reach $v \in W^{1,2}$. Hence we can use the weak maximum principle to deduce $v = 0$ as before. This finishes the proof of the uniqueness. Hence, by the Fredholm alternative, Theorem 4.2.5 is proved. ■

Chapter Five

The Dirichlet-conormal problem with homogeneous and inhomogeneous boundary conditions

5.1 Introduction

In this paper, we continue our discussion in [11] on the mixed Dirichlet-conormal boundary value problems. On a domain $\Omega \subset \mathbb{R}^d$, we consider the following second-order symmetric divergence form elliptic equation, with two types of boundary conditions prescribed on two different parts of the boundary:

$$\begin{cases} Lu = f + D_i f_i & \text{in } \Omega, \\ Bu = f_i n_i + g_{\mathcal{N}} & \text{on } \mathcal{N}, \\ u = g_{\mathcal{D}} & \text{on } \mathcal{D}. \end{cases} \tag{5.1}$$

Here, the boundary is decomposed into two non-empty and non-intersection portions $\mathcal{N}$ and $\mathcal{D}$, separated by their interfacial boundary Γ:

$$\partial\Omega = \mathcal{N} \cup \mathcal{D}, \quad \Gamma := \overline{\mathcal{N}} \cap \overline{\mathcal{D}}.$$

The elliptic operator L and the associated conormal derivative operator B are defined as

$$\begin{aligned} &Lu := D_i(a_{ij}(x)D_j u + b_i(x)u) + \hat{b}_i(x)D_i u + c(x)u, \\ &Bu := (a_{ij}D_j u + b_i u)n_i, \end{aligned} \tag{5.2}$$

where $\boldsymbol{n} = (n_i)_{i=1}^d$ is the outer normal direction and $a_{ij} = a_{ji}$. We always assume for some constants $\Lambda \in (0, 1]$ and $K > 0$,

$$\Lambda|\xi|^2 \le \sum_{i,j=1}^d a_{ij}(x)\xi_j\xi_i \le \Lambda^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d \text{ and } x \in \overline{\Omega}, \quad |b_i| + |\hat{b}_i| + |c| \le K.$$

Unlike the purely Dirichlet or conormal boundary value problem, solutions to (5.1) can be non-smooth near Γ even if the domain, coefficients, and boundary data are all smooth. For example, the function $u = \operatorname{Im}\sqrt{z}$ (with $z = x + iy$) is harmonic in the upper half-plane with zero Dirichlet data on the positive real axis and zero Neumann data on the negative real axis, but u is only in $C^{1/2}$ and $W^{1,4-\varepsilon}$. Our aim is to find minimum smoothness assumptions on Ω and the separation $\Gamma = \overline{\mathcal{N}} \cap \overline{\mathcal{D}}$, such that certain "optimal regularity" is achieved.

When $\partial\Omega$, Γ, and all the coefficients are sufficiently smooth, there are quite a few results in the literature concerning the optimal regularity. See, for example, the $W^{1,4-\varepsilon}$-regularity on half space and the more general result of $W^{s,p}$-regularity on smooth domains in [68]. In [67], the optimal $B^{3/2}_{2,\infty}$-regularity on $C^{1,1}$ domains was obtained. For more details and history, see [11] and the references therein.

On irregular domains, it turns out that the only requirement for reaching the optimal regularity is certain "flatness" of $\partial\Omega$ and Γ. Indeed, for the problem with homogeneous boundary condition, i.e., (5.1) with $g_{\mathcal{N}} = g_{\mathcal{D}} = 0$, in [11] we proved the $W^{1,4-\varepsilon}$ regularity for weak solutions, assuming both $\partial\Omega$ and Γ to be Reifenberg flat (see Assumptions 5.2.2 and 5.2.5(a)). The boundaries of Reifenberg flat domains are only flat in the sense that they are close to hyperplanes in the Hausdorff distance, not in the sense the regularity. Typically such domains can have fractal structures. Our result in [11] is a generalization of the $W^{1,p}$ regularity of purely Dirichlet or conormal problem with "partially VMO" coefficients in [20] and [21]. The first objective of the current paper is to generalize the result in [11] by allowing Γ to be perturbations of Lipschitz graphs, not just hyperplanes. See Theorem 5.2.8.

The second objective of the paper is to study the extension problem of the inhomogeneous boundary data $g_{\mathcal{N}}$ and $g_{\mathcal{D}}$. Stronger than the usual trace spaces, we consider almost everywhere defined boundary data: $g_{\mathcal{N}} \in L^q(\mathcal{N})$ and $g_{\mathcal{D}} \in W^{1,q}(\mathcal{D})$. We aim to derive an L^q non-tangential maximal function estimate for the Laplace equation. Such non-tangential maximal function estimate also implies the unique solvability: for any $g_{\mathcal{N}} \in L^q(\mathcal{N})$ and $g_{\mathcal{D}} \in W^{1,q}(\mathcal{D})$, we can find a unique harmonic function satisfying the boundary conditions in the sense of "non-tangential limit".

In this direction, the study of the above problem is an extension of that of the Neumann problem with L^q data or the Dirichlet problem with $W^{1,q}$ data on Lipschitz domains, for which the results were obtained in [36] for $q = 2$ and in [12] for $q < 2 + \varepsilon$. Note that the range $q < 2 + \varepsilon$ is optimal. Due to the failure of the usual L^2-method, the corresponding estimate for the mixed problem was raised as an open problem in [37]. Initiated in [3], one approach is to assume an additional geometric condition on the domain such that the optimal regularity of the non-tangential maximal function is above L^2, hence the classical L^2-method still applies. See also [61]. Such geometric assumption clearly excludes smooth domains. The other approach started from the L^1-solvability with boundary data in Hardy spaces. In this direction, the best result so far is the unique $L^{1+\varepsilon}$-solvability obtained in [73]. For this, they allowed Ω to be any Lipschitz domain and Γ to be very rough: merely Ahlfors regular of Hausdorff dimension close to $d-2$, in addition to the so-called "corkscrew conditions" on $\mathcal{D}$. Following [73], recently an explicit solvability range $q < d/(d-1)$ was obtained in [4], assuming $\partial\Omega \in C^{1,1}$ and Γ to be a Lipschitz graph.

In the current paper, we prove that for $m = 0, \ldots, d-2$, the L^q non-tangential maximal function estimate holds when $q \in (1, (m+2)/(m+1))$, under the conditions that $\partial\Omega$ is Lipschitz and Reifenberg flat, and Γ is a perturbation (measured by the Hausdorff distance) of a Lipschitz graph in m variables, $m \in [0, d-2]$. See Theorem 5.2.11. In particular, when $m = d-2$, our result generalized the one in [4] by allowing rougher $\partial\Omega$ and Γ. In another special case when $m = 0$, i.e., Γ is Reifenberg flat of co-dimension 2, the optimal range $q < 2 - \varepsilon$ is achieved.

In this paper, we only consider the non-tangential maximal function estimate for the Laplace equation. In a subsequent work, we plan to discuss the perturbation theory where elliptic operators with variable coefficients are considered, with the aim to generalize the corresponding result for the conormal problem in [15].

The rest of the paper is organized as follows. In Section 5.2, we will set up the problems and then give our main results: Theorem 5.2.8 for $W^{1,p}$-estimate of weak solutions and Theorem 5.2.11 for the L^q non-tangential maximal function estimate. In Sections 5.3 and 5.4, we give the proof of Theorem 5.2.8, where Section 5.3 is for the estimates on the half space and Section 5.4 is for the perturbation argument. Finally, we prove Theorem 5.2.11 in Section 5.5. In the proof, Corollary 5.4.2 from the previous homogeneous boundary data part plays a key role in improving the regularity.

5.2 Problem set up and main results

To begin with, let us give all the definitions of domains which we will be discussing throughout this paper.

Assumption 5.2.1 (M-Lipschitz)**.** There exists some constant $R_0 > 0$, such that, for any $x_0 \in \partial\Omega$, there exists a Lipschitz function $\psi_0 : \mathbb{R}^{d-1} \to \mathbb{R}$ such that in some coordinate system (upto rotation and translation),

$$\Omega_{R_0}(x_0) := \Omega \cap B_{R_0}(x_0) = \{x \in B_{R_0}(0) : x^1 > \psi_0(x')\} \quad \text{and} \quad |D\psi_0(x')| < M \quad \text{a.e.}$$

Assumption 5.2.2 (γ-flat)**.** There exists a constant $R_1 > 0$, such that, for any $x_0 \in \partial\Omega$ and $R \in (0, R_1]$, in some coordinate system (upto rotation and translation) depending on x_0 and R, we have

$$\{x : x^1 > x_0^1 + \gamma R\} \cap B_R \subset \Omega_R(x_0) \subset \{x : x^1 > x_0^1 - \gamma R\} \cap B_R.$$

Clearly, γ-Lipschitz implies γ-flat. However, the following example shows that Lipschitz and Reifenberg-flat do not imply small Lipschitz.

Example 5.2.3. We construct a curve by gluing infinitely many congruent copies of a "S"-shaped curve, while the k-th copy is of size 2^{-k}. The "S"-shaped curve is designed as follows. We start from a horizontal line segment with length $1/2$. Then on its right end we connect a line segment with slope $1/m$ and length $1/4$. Repeat this process until we link $m+1$ line segments together, where the jth segment has slope $(j-1)/m$ and length 2^{-j}. The last step is to extend it symmetrically beyond the right end point.

By choosing m large enough, the curve is ε-flat, while the Lipschitz constant is at least $\tan(\pi/8)$.

As shown in [45], any small Reifenberg flat domain is a $W^{1,p}$-extension domain for every $p \in [1, \infty]$. Hence we have all the Sobolev inequalities up to the first order. In the following, we state a local Poincacé inequality for functions vanishing only on part of the boundary, which can be found in [11, Corollary 3.2 (b)]. Throughout the paper, for $\mathcal{D} \subset \partial\Omega$ and $p \in [1, \infty)$, we denote the space $W^{1,p}_{\mathcal{D}}$ to be the closure of $C^\infty_c(\overline{\Omega} \setminus \mathcal{D})$ under the usual $W^{1,p}$-norm.

Lemma 5.2.4 (Poincaré inequality with mixed boundary condition)**.** *Let* $\gamma \in [0, 1/48]$ *and* $\Omega \subset \mathbb{R}^d$ *be a Reifenberg flat domain satisfying Assumption* 5.2.2 (γ)*. Let* $x_0 \in \overline{\Omega}$*,* $R \in (0, R_1/4]$*, and* $\mathcal{D} \subset \partial\Omega$ *with* $\mathcal{D} \cap B_R(x_0) \neq \emptyset$*. If there exist* $z_0 \in \mathcal{D} \cap B_R(x_0)$ *and* $\alpha \in (0, 1)$ *such that*

$$B_{\alpha R}(z_0) \subset B_R(x_0), \quad \Big(\partial\Omega \cap B_{\alpha R}(z_0)\Big) \subset \Big(\mathcal{D} \cap B_R(x_0)\Big),$$

then, for any $u \in W^{1,2}_{\mathcal{D}}(\Omega)$*, we have*

$$\|u\|_{L^2(\Omega_R(x_0))} \leq CR\|Du\|_{L^2(\Omega_{2R}(x_0))},$$

where $C = C(d, \alpha)$*.*

The assumptions for the interfacial boundary Γ are given as follows.

Assumption 5.2.5 ((γ, m)-flat separation)**.** Let Ω be a domain satisfying either Assumption 5.2.1 or 5.2.2, with $\partial\Omega$ divided into two non-intersecting portions $\mathcal{D}$ and $\mathcal{N}$. Let Γ be the boundary (relative to $\partial\Omega$) of $\mathcal{D}$. We call Γ a (γ, m)-flat separation if for any $x_0 \in \Gamma$ and $R \in (0, R_1]$, the following holds.

(a) When $m = 0$, the coordinate system given in the previous assumption satisfies

$$\Big(\partial\Omega \cap B_R(x_0) \cap \{x : x^2 > x^2_0 + \gamma R\}\Big) \subset \mathcal{D},$$

$$\Big(\partial\Omega \cap B_R(x_0) \cap \{x : x^2 < x^2_0 - \gamma R\}\Big) \subset \mathcal{N}.$$

(b) When $1 \leq m \leq d - 2$, there is a Lipschitz function $\phi : \mathbb{R}^m \to \mathbb{R}$ with Lipschitz constant M, such that the coordinate system satisfy

$$\Big(\partial\Omega \cap B_R(x_0) \cap \{x : x^2 > \phi(x^3, \ldots, x^{m+2}) + \gamma R\}\Big) \subset \mathcal{D},$$

$$\Big(\partial\Omega \cap B_R(x_0) \cap \{x : x^2 < \phi(x^3, \ldots, x^{m+2}) - \gamma R\}\Big) \subset \mathcal{N}.$$

Now we formulate our first main results. Recall the elliptic operator L and its associated conormal derivative operator B in (5.2). For any constant λ, we denote $L_\lambda := L - \lambda$. The following problem will be discussed:

$$\begin{cases} L_\lambda u = f + D_i f_i & \text{in } \Omega, \\ Bu = f_i n_i & \text{on } \mathcal{N}, \\ u = 0 & \text{on } \mathcal{D}. \end{cases} \tag{5.3}$$

We consider weak solutions to (5.3).

Definition 5.2.6 ($W^{1,p}_{\mathcal{D}}$-weak solution)**.** For $\Omega, \mathcal{N}, \mathcal{D}, \Gamma$ given as before, and $p \in [1, \infty)$, we call $u \in W^{1,p}_{\mathcal{D}}$ a weak solution to (5.3), if for any $\zeta \in C^\infty_c(\Omega \cup \mathcal{N})$ (or equivalently, $W^{1,p/(p-1)}_{\mathcal{D}}(\Omega)$),

$$\int_\Omega (-a_{ij} D_j u - b_i u) D_i \zeta + (\hat{b}_i D_i u + cu)\zeta \, dx = \int_\Omega f\zeta \, dx - \int_\Omega f_i D_i \zeta \, dx.$$

Since there is no boundary term in the integral form, the weak solution is still well defined for very rough $\partial\Omega$. In particular, we can still discuss weak solution for Reifenberg flat domains where $\boldsymbol{n}$ is not point-wise defined and there is no notion of the trace space.

For the leading coefficients, the following small BMO condition will be considered.

Assumption 5.2.7 (θ-BMO)**.** There exists $R_1 \in (0, 1]$ such that for any $x \in \overline{\Omega}$ and $r \in (0, R_1]$, we have

$$\fint_{\Omega_r(x)} |a_{ij}(y) - (a_{ij})_{\Omega_r(x)}|\, dy < \theta.$$

In the above assumption and thoughout this paper, we use the following notation for the average:

$$(f)_{\Omega_r(x)} = \fint_{\Omega_r(x)} f(y)\, dy := \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y)\, dy.$$

In one of our main results, the following sign condition will be assumed: $L_0 1 \le 0$ is satisfied in the weak sense, if

$$\int_\Omega (-b_i D_i \zeta + c\zeta) \le 0, \quad \forall \zeta \in C_c^\infty(\Omega),\ \zeta \ge 0.$$

We also denote the following index p_*:

$$p_* := \begin{cases} pd/(p+d) & \text{when } p > d/(d-1), \\ 1 + \varepsilon & \text{when } p \le d/(d-1), \end{cases}$$

where ε can be any positive constant. In this paper, for simplicity, when $\lambda > 0$ we use the following notation

$$U := |Du| + \sqrt{\lambda}|u|, \quad F := \sum_i |f_i| + \frac{1}{\sqrt{\lambda}}|f|.$$

Now we state our first main result regarding equations with homogeneous boundary conditions.

Theorem 5.2.8. *For any integer $m \in [0, d-2]$ and constant $p \in (2(m+2)/(m+3), 2(m+$*

2)/(m + 1)), we can find positive constants $(\gamma_0, \theta_0) = (\gamma_0, \theta_0)(d, p, M, \Lambda)$*, such that if Assumptions 5.2.2* (γ_0)*, 5.2.5* (γ_0, m)*, and 5.2.7* (θ_0) *are satisfied, the following hold.*

(i) There exists $\lambda_0 = \lambda_0(d, p, M, \Lambda, R_1, K)$ *such that for any* $(f_i)_{i=1}^d \in (L^p(\Omega))^d$ *and* $f \in L^p(\Omega)$*, the problem (5.3) with* $\lambda > \lambda_0$ *has a unique solution* $u \in W^{1,p}_{\mathcal{D}}(\Omega)$*. The solution satisfies the estimate*

$$\|U\|_{L^p(\Omega)} \le C\|F\|_{L^p(\Omega)},$$

where $C = C(d, p, M, \Lambda)$ *is a constant.*

(ii) On a bounded domain Ω*, if* $L_0 1 \le 0$ *in the weak sense, then for any* $(f_i)_{i=1}^d \in (L^p(\Omega))^d$ *and* $f \in L^{p_*}(\Omega)$*, (5.3) with* $\lambda = 0$ *has a unique solution* $u \in W^{1,p}_{\mathcal{D}}(\Omega)$ *satisfying*

$$\|u\|_{W^{1,p}(\Omega)} \le C\left(\sum_{i=1}^d \|f_i\|_{L^p(\Omega)} + \|f\|_{L^{p_*}(\Omega)}\right),$$

where C *is a constant independent of* u*,* f*, and* f_i*.*

The following example in [11] shows that the symmetry of $\boldsymbol{A}$ is required for the $W^{1,p}$-regularity when p is away from 2.

Example 5.2.9. In $\mathbb{R}^2_+ := \{(x, y) : y > 0\}$, let $u(x, y) = \mathrm{Im}(x + iy)^s$ with $s \in (0, 1/2)$. We have

$$D_i(a_{ij}D_j u) = 0 \text{ on } \mathbb{R}^2_+, \quad u = 0 \text{ on } \partial\mathbb{R}^2_+ \cap \{x > 0\}, \quad a_{ij}D_j u n_i = 0 \text{ on } \partial\mathbb{R}^2_+ \cap \{x < 0\},$$

where

$$(a_{ij})_{i,j=1}^2 = \begin{bmatrix} 1 & \cot(\pi s) \\ -\cot(\pi s) & 1 \end{bmatrix}.$$

Since Du is of order r^{s-1}, near the origin we have $Du \in L^p$ only for $p < \frac{2}{1-s}$. Note that $\frac{2}{1-s} < 4$ and $\frac{2}{1-s} \searrow 2$ as $s \searrow 0$.

Next we formulate the problem with inhomogeneous boundary data. To make sense of the boundary condition, we introduce the following concepts.

Definition 5.2.10. Let $\beta > 0$ be a constant. For a domain Ω and a point $x \in \partial\Omega$, we define the non-tangential approach region at x with opening β as

$$\Gamma_\beta(x) := \{y \in \Omega : |x - y| \leq (1+\beta)\,\mathrm{dist}(y, \partial\Omega)\}.$$

The corresponding non-tangential maximal function is defined as

$$\boldsymbol{N}(u)(x) := \sup_{Y \in \Gamma_\beta(x)} |u(y)|.$$

In this paper, we will omit the dependence of the opening β.

For some $q \in [1, \infty)$, we consider the following boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g_{\mathcal{N}} & \text{on } \mathcal{N}, \\ u = g_{\mathcal{D}} & \text{on } \mathcal{D}, \\ \boldsymbol{N}(Du) \in L^q(\partial\Omega). \end{cases} \tag{5.4}$$

For $g_{\mathcal{N}} \in L^q(\mathcal{N})$ and $g_{\mathcal{D}} \in W^{1,q}(\mathcal{D})$, we call u a solution to (5.4), if $u \in W^{1,2}(\partial\Omega)$ is a weak solution in the usual sense, and $\boldsymbol{N}(Du)$ is controlled. For such solution, the Dirichlet boundary condition is satisfied in the sense of non-tangential limit and the conormal boundary condition is satisfied in the sense of "L^q-weak non-tangential limit". See, for instance, [37, Theorem 1.8.1].

The second main result of the paper is the following theorem.

Theorem 5.2.11. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain satisfying Assumption 5.2.1(M). For* *any* $q \in (1, \frac{m+2}{m+1})$, *we can find sufficiently small* $\gamma_1 = \gamma_1(d, q, M) > 0$, *such that if* Ω *and* Γ *satisfy the Assumptions 5.2.2* (γ_1) *and 5.2.5* (γ_1, m), *then for any* $g_{\mathcal{N}} \in L^q(\mathcal{N}), g_{\mathcal{D}} \in W^{1,q}(\mathcal{D})$, *there exists a unique solution* u *to the problem (5.4) satisfying*

$$\|\boldsymbol{N}(Du)\|_{L^q(\partial\Omega)} \leq C\|g_{\mathcal{N}}\|_{L^q(\mathcal{N})} + C\|g_{\mathcal{D}}\|_{W^{1,q}(\mathcal{D})},$$

where $C = C(d, M, R_0, R_1, diam(\Omega), q)$ *is a constant.*

5.3 Harmonic function on half space

Let ϕ be a Lipschitz function $\mathbb{R}^m \to \mathbb{R}$ with $\phi(0) = 0$. Let

$$\mathbb{R}^d_+ := \{x^1 > 0\}, \quad B^+_R := B_R(0) \cap \{x^1 > 0\}.$$

Throughout this section, for $m = 0, \ldots, d-2$, we use the notation

$$\Gamma := \{x^1 = 0, x^2 = \phi(x^3, \ldots, x^{m+2})\}, \quad \mathcal{D} := \{x^1 = 0, x^2 > \phi\}, \quad \mathcal{N} := \{x^1 = 0, x^2 < \phi\}.$$

When $m = 0$, we just set $\phi = 0$. The main result in this section is the following reverse Hölder inequality.

Theorem 5.3.1. *Let* $u \in W^{1,2}(B^+_R)$ *be a weak solution to the following problem with* $\lambda > 0$*:*

$$\begin{cases} -\Delta u + \lambda u = 0 & \text{in } B^+_R, \\ \frac{\partial u}{\partial x^1} = 0 & \text{on } \mathcal{N} \cap B_R, \\ u = 0 & \text{on } \mathcal{D} \cap B_R, \end{cases}$$

Then for any $p < \frac{2(m+2)}{m+1}$, we have $u \in W^{1,p}(B^+_{R/2})$ satisfying

$$(U^p)^{1/p}_{B^+_{R/2}} \le C(U^2)^{1/2}_{B^+_R},$$

where $C = C(d, p, M)$.

The case $m = 0$ was proved in [11, Theorem 4.1]. In the following, we first prove a lemma which contains the result when $m = d - 2$. From this lemma, we can derive the case when m is in between. In this section, we do not distinguish the geometric objects on $\mathbb{R}^{m+2}$ or $\mathbb{R}^d$.

Lemma 5.3.2. *On $\mathbb{R}^{m+2}$, let $u \in W^{1,2}(B^+_R)$ be a weak solution to*

$$\begin{cases} -\Delta u = f & \text{in } B^+_1, \\ \frac{\partial u}{\partial x^1} = 0 & \text{on } \mathcal{N} \cap B_1, \\ u = 0 & \text{on } \mathcal{D} \cap B_1, \end{cases} \tag{5.5}$$

where $f \in L^2(B^+_1)$. Then for any $p \in \left[2, \frac{2(m+2)}{m+1}\right)$, we have $u \in W^{1,p}(B^+_{1/2})$ and

$$\|\nabla u\|_{L^p(B^+_{1/2})} \le C(m, p)(\|\nabla u\|_{L^2(B^+_1)} + \|f\|_{L^2(B^+_1)}).$$

Proof. The case when $m = 0$ is proved in [11, Lemma 4.3]. Here we only consider the case when $m \ge 1$. In the following, we construct a problem on the half space $\mathbb{R}^{m+2}_+$, which coincides with (5.5) in $B^+_{1/2}$. Hence the Besov space regularity in [4, Theorem 3.3] applies. For this, first we take zero extension of u on $\mathbb{R}^{m+2}_+ \setminus B^+_1$. Then let

$$\eta \in C^\infty_c(B_1), \quad \eta = 1 \text{ in } B_{1/2}, \quad \text{even in } x^1.$$

Now $u\eta \in W^{1,2}(\mathbb{R}^{m+2}_+)$, and solves

$$\begin{cases} -\Delta(u\eta) = f\eta - 2\nabla u \cdot \nabla\eta - u\Delta\eta & \text{in } \mathbb{R}^{m+2}_+, \\ \frac{\partial(u\eta)}{\partial x^1} = 0 & \text{on } \mathcal{N}, \\ u\eta = 0 & \text{on } \mathcal{D}. \end{cases}$$

We have

$$\begin{aligned} &\|\nabla u\|_{L^p(B^+_{1/2})} = \|\nabla(u\eta)\|_{L^p(B^+_{1/2})} \\ &\lesssim \|\nabla(u\eta)\|_{B^{1/2}_{2,\infty}(\mathbb{R}^{m+2}_+)} && (5.6) \\ &\lesssim \|f\eta - 2\nabla u \cdot \nabla\eta - u\Delta\eta\|_{L^2(\mathbb{R}^{m+2}_+)} && (5.7) \\ &\lesssim \|f\|_{L^2(B^+_1)} + \|u\|_{W^{1,2}(B^+_1)} \\ &\lesssim \|f\|_{L^2(B^+_1)} + \|\nabla u\|_{L^2(B^+_1)}. && (5.8) \end{aligned}$$

Here, (5.6) is due to the Besov-Sobolev embedding, where $p < \frac{2(m+2)}{m+1}$ is required, [4, Theorem 3.3] is applied to obtain (5.7), and the estimate (5.8) is obtained by applying the usual Poincaré inequality on Lipschitz domains with the homogeneous Dirichlet condition imposed on part of the boundary (see Lemma 5.2.4). ■

Now we turn to the proof of the main result of this section.

Proof of Theorem 5.3.1. As mentioned before, we only need to prove when $0 < m < d-2$. For simplicity, here we introduce the notation $x = (\bar{x}, \hat{x})$, where $\bar{x} := (x^1, \ldots, x^{m+2})$ and $\hat{x} := (x^{m+3}, \ldots, x^d)$, and for $p, q \in [1, \infty]$ we denote

$$\|u\|_{L^p_{\bar{x}} L^q_{\hat{x}}} = \left\| \|u\|_{L^q_{\hat{x}}} \right\|_{L^p_{\bar{x}}}.$$

The proof is similar to that of [11, Theorem 4.1]. By applying Agmon's idea as in the proof of [11, Theorem 4.1] and scaling, it suffices to prove when $\lambda = 0, R = 1$. Also, since the case when $p \le 2$ is trivial, in the following we assume $2 < p < \frac{2(m+2)}{m+1}$.

We first rewrite (5.5) as an equation in the $\bar{x}$-variables: for every $|\hat{x}| < 1/2$,

$$\begin{cases} \Delta_{\bar{x}} u(\cdot, \hat{x}) = -\Delta_{\hat{x}} u(\cdot, \hat{x}) & \text{in } B^+_{2/3}, \\ \frac{\partial u}{\partial x^1} = 0 & \text{on } \mathcal{N} \cap B_{2/3}, \\ u = 0 & \text{on } \mathcal{D} \cap B_{2/3}. \end{cases}$$

Now, a properly rescaled version of Lemma 5.3.2 leads to

$$\|D_{\bar{x}} u(\cdot, \hat{x})\|_{L^p(B^+_{1/2})} \lesssim \|D_{\bar{x}} u(\cdot, \hat{x})\|_{L^2(B^+_{2/3})} + \|\Delta_{\hat{x}} u(\cdot, \hat{x})\|_{L^2(B^+_{2/3})}, \quad \forall\, |\hat{x}| < 1/2.$$

Taking L^p norm for $\hat{x} \in B_{1/2}$, we obtain

$$\|D_{\bar{x}} u\|_{L^p(B^+_{1/2})} \lesssim \|D_{\bar{x}} u\|_{L^p_{\hat{x}} L^2_{\bar{x}}(B^+_{3/4})} + \|\Delta_{\hat{x}} u\|_{L^p_{\hat{x}} L^2_{\bar{x}}(B^+_{3/4})}. \tag{5.9}$$

Due to the translation invariance in $\hat{x}$, we can differentiate both the equation and boundary conditions in the $\hat{x}$-direction. Hence we obtain the following estimate by applying the Caccioppoli inequality k times:

$$\|DD^k_{\hat{x}} u\|_{L^2(B^+_s)} \le \frac{C(d,k)}{|t-s|^k} \|Du\|_{L^2(B^+_t)}$$

for $0 < s < t \le 1$ and $k \in \{0, 1, 2, \ldots\}$. From this and the anisotropic Sobolev embedding (see, for instance, [2, Sec. 18.12]),

$$\|D_{\bar{x}} u\|_{L^2_{\bar{x}} L^p_{\hat{x}}(B^+_r)} + \|D_{\hat{x}} u\|_{L^p(B^+_r)} + \|D^2_{\hat{x}} u\|_{L^p(B^+_r)} \le C(d,p,r) \|Du\|_{L^2(B^+_1)}, \quad \forall r \in (0,1),$$

where $p \in (2, 2(m+2)/(m+1))$. Combining this, (5.9), and the Minkowski inequality, we reach the desired estimate. ■

5.4 Proof of Theorem 5.2.8: regularity of $W^{1,2}$-weak solutions

The key step in proving Theorem 5.2.8 is the regularity result Proposition 5.4.1 stated below. The following equation with $b_i = \hat{b}_i = c = 0$ will be discussed:

$$\begin{cases} L_\lambda^{(0)} u := D_i(a_{ij} D_j u) - \lambda u = f + D_i f_i & \text{in } \Omega, \\ a_{ij} D_j u n_i = f_i n_i & \text{on } \mathcal{N}, \\ u = 0 & \text{on } \mathcal{D}. \end{cases} \tag{5.10}$$

Proposition 5.4.1. *For any $p \in (2, \frac{2(m+2)}{m+1})$, we can find positive constants γ_0 and θ_0 depending on (d, p, M, Λ), such that if Assumptions 5.2.2 (γ_0), 5.2.5 (γ_0, m), and 5.2.7 (θ_0) are satisfied, the following holds. For any $W^{1,2}_{\mathcal{D}}(\Omega)$ weak solution u to (5.10) with $\lambda > 0$ and $f_i, f \in L^p(\Omega) \cap L^2(\Omega)$, we have $u \in W^{1,p}_{\mathcal{D}}(\Omega)$ and*

$$\|U\|_{L^p(\Omega)} \le C\Big(R_1^{d(1/p-1/2)} \|U\|_{L^2(\Omega)} + \|F\|_{L^p(\Omega)}\Big). \tag{5.11}$$

Furthermore, if we also assume $f \equiv 0$, then we can take $\lambda = 0$. In this case, the following estimate holds:

$$\|Du\|_{L^p(\Omega)} \le C\Big(R_1^{d(1/p-1/2)} \|Du\|_{L^2(\Omega)} + \|f_i\|_{L^p(\Omega)}\Big).$$

In the above, the constant C depends on d, p, M, and Λ.

We give the following corollary which will be useful for the inhomogeneous boundary condition case later.

Corollary 5.4.2. *Let* $0 < r < R \leq diam(\Omega)$*. Consider a harmonic function in* Ω_R *with the corresponding mixed boundary conditions on* $\partial\Omega \cap B_R$*. Under the assumptions for* p, Ω*, and* Γ *as in Proposition 5.4.1, we have for any* $r < R$*,*

$$\|Du\|_{L^p(\Omega_r)} \leq C(R-r)^{-d/p'}\|Du\|_{L^1(\Omega_R)},$$

where p' *satisfies* $1/p + 1/p' = 1$ *and* $C = C(d, p, M, \Lambda, R_1)$.

The proof is standard, which we will only sketch here. We may certainly assume that $r \geq R/2$. By using a covering argument (with balls of radius $(R-r)/2$ centered at, say $x_1, \ldots, x_k \in \Omega_r$), we first localize the result in Proposition 5.4.1 to obtain an $L^2 - L^p$ estimate: for $j = 1, \ldots, k$,

$$(|Du|^p)^{1/p}_{\Omega_{(R-r)/4}(x_j)} \leq C\Big((|Du|^2)^{1/2}_{\Omega_{(R-r)/2}(x_j)} + (|u|^2)^{1/2}_{\Omega_{(R-r)/2}(x_j)}\Big).$$

To remove the L^2-norm of u on the right-hand side, we apply the Poincaré inequality. When $\Omega_{(R-r)/2}(x_j) \cap \mathcal{D} = \emptyset$, we replace u with $u - (u)_{\Omega_{(R-r)/2}(x_j)}$, which locally satisfies the same equation with the conormal boundary condition, and then apply the usual Poincaré inequality. Otherwise, we need to apply the Poincaré inequality in Lemma 5.2.4. Finally, to replace $\|Du\|_{L^2}$ with $\|Du\|_{L^1}$, we use Hölder's inequality and an iteration argument. Summing in j, we get the desired estimate.

The remainder of this section will be mostly devoted to the proof of Proposition 5.4.1. We first sketch the idea. By constructing a cut-off function and applying a reflection technique, at all small scales we decompose the solution into two parts (Lemma 5.4.4). One part (up to rotation) solves the problem in Theorem 5.3.1, hence can reach the optimal $W^{1,\frac{2(m+2)}{m+1}-\varepsilon}$-regularity. The other part deals with all the perturbation terms measured by a $W^{1,2}$-estimate.

Next we "interpolate" these two part to reach the regularity of u in between, by applying the level set argument introduced in [8]. The key idea is to use a measure theoretical lemma called "crawling of the ink spots" in [44]: from the decomposition and a Chebyshev-type inequality, we first deduce a local property at certain small scales R, regarding the level set of the maximal function (Lemma 5.4.5). Then the "crawling of the ink spots" lemma leads to a global decay estimate of the measure of the level sets (Corollary 5.4.6). From this decay estimate, the desired L^p-estimate can be obtained by an integral representation of the L^p norm and the Hardy-Littlewood maximal function theorem.

The proof follows similar steps as in [11, Section 5], where detailed computation can be found. Essentially the new ingredient here is the construction of a cut-off function when proving Lemma 5.4.4.

5.4.1 A reverse Hölder inequalities

Using the local Sobolev-Poincaré inequality in Lemma 5.2.4 and Gehring's Lemma, we first improve the regularity of a weak solution from $W^{1,2}$ to $W^{1,2+\varepsilon}$.

Proposition 5.4.3 (Reverse Hölder's inequality)**.** *Let $\gamma \in (0, 1/48]$, $p > 2$, and the integer $m \in [0, d-2]$. Assume on $\mathbb{R}^d$, Ω, and Γ satisfy Assumptions 5.2.2 (γ) and 5.2.5 (γ, m), and the function $u \in W^{1,2}_{\mathcal{D}}(\Omega)$ satisfies (5.10) with $f_i, f \in L^p(\Omega) \cap L^2(\Omega)$. Then there exist constants $p_0 \in (2, p)$ and $C > 0$, depending only on d, p, M, and Λ, such that* *for any* $x_0 \in \mathbb{R}^d$ *and* $R \in (0, R_1]$, *the following hold. When* $\lambda > 0$, *we have*

$$\left(\overline{U}^{p_0}\right)^{1/p_0}_{B_{R/2}(x_0)} \le C\left(\overline{U}^2\right)^{1/2}_{B_R(x_0)} + C\left(\overline{F}^{p_0}\right)^{1/p_0}_{B_R(x_0)}.$$

When $\lambda = 0$ *and* $f \equiv 0$, *we have*

$$\left(|D\overline{u}|^{p_0}\right)^{1/p_0}_{B_{R/2}(x_0)} \le C\left(|D\overline{u}|^2\right)^{1/2}_{B_R(x_0)} + C\left(|\overline{f_i}|^{p_0}\right)^{1/p_0}_{B_R(x_0)},$$

where $\overline{U}$, $\overline{F}$, $D\overline{u}$, *and* $\overline{f_i}$ *are the zero extensions of* U, F, Du, *and* f_i *to* $\mathbb{R}^d$.

The proof can be simply adapted from the one of [11, Lemma 3.4], which we omit here.

5.4.2 Decomposition of solutions

In this subsection, we prove the following lemma.

Lemma 5.4.4. *Suppose that* $u \in W^{1,2}_{\mathcal{D}}(\Omega)$ *satisfies* (5.10) *with* $\lambda > 0$ *and* $f_i, f \in L^p(\Omega) \cap L^2(\Omega)$, *where* $p > 2$. *Then under Assumptions* 5.2.2 (γ), 5.2.5 (γ, m), *and* 5.2.7 (θ) *with* $\gamma < 1/(32\sqrt{d+3})$ *and* $\theta \in (0,1)$, *for any* $x_0 \in \overline{\Omega}$ *and* $R < R_1$, *there exist nonnegative functions* $W, V \in L^2(\Omega_{R/32}(x_0))$ *such that*

$$U \le W + V \quad \text{in } \Omega_{R/32}(x_0).$$

Moreover, we have for any $q < \frac{2(m+2)}{m+1}$,

$$(W^2)^{1/2}_{\Omega_{R/32}(x_0)} \le C\Big((\theta^{\frac{1}{2\mu'}} + \gamma^{\frac{1}{2\mu'}})(U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{\frac{1}{2\mu}}_{\Omega_R(x_0)}\Big), \tag{5.12}$$

$$(V^q)^{1/q}_{\Omega_{R/32}(x_0)} \le C\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{\frac{1}{2\mu}}_{\Omega_R(x_0)}\Big). \tag{5.13}$$

Here μ is a constant satisfying $2\mu = p_0$, where $p_0 = p_0(d, p, M, \Lambda) > 2$ comes from Proposition 5.4.3, and μ' satisfies $1/\mu + 1/\mu' = 1$. The constant C only depends on d, p, q, M, and Λ.

Proof of Lemma 5.4.4. When $\operatorname{dist}(x_0, \Gamma) \geq R/16$, we either only deal with the interior case or the case with purely Dirichlet/conormal boundary condition (depending on $B_{R/16}(x_0) \cap \mathcal{N} = \emptyset$ or $B_{R/16}(x_0) \cap \mathcal{D} = \emptyset$). The interior $W^{1,p}$-estimates for equations with VMO coefficients are by now standard, while the corresponding estimates for purely Dirichlet/conormal problems on Reifenberg flat domains can be found in [20, 21]. Also, one may refer to [11, page 22]. In the following, we focus on the case when $\operatorname{dist}(x_0, \Gamma) < R/16$, where the boundary condition is "mixed".

Pick a point $y_0 \in \Gamma$ with $\operatorname{dist}(y_0, x_0) < R/16$. Now we take the coordinate system associated with y_0 and $R/4$ as in Assumptions 5.2.2 and 5.2.5, so that $y_0 = 0$. Denote

$$\begin{aligned}
&\Omega_{R/4} := \Omega \cap B_{R/4},\\
&\Omega^+_{R/4} := \Omega_{R/4} \cap \{x^1 > \gamma R/4\}, \quad \Omega^-_{R/4} := \Omega_{R/4} \cap \{x^1 < \gamma R/4\},\\
&\Gamma^+ := \{x^1 = \gamma R/4, x^2 > \phi - \gamma R/4\} \cap B_{R/4},\\
&\Gamma^- := \{x^1 = \gamma R/4, x^2 < \phi - \gamma R/4\} \cap B_{R/4}.
\end{aligned}$$

We take

$$W := |Du| + \sqrt{\lambda}|u| \quad \text{and} \quad V := 0 \quad \text{on } \Omega^-_{R/4}. \tag{5.14}$$

Due to Hölder's inequality, Assumption 5.2.2, and Proposition 5.4.3, we obtain

$$\begin{aligned}
(W^2 \mathbb{I}_{\Omega^-_{R/4}})^{1/2}_{\Omega_{R/32}(x_0)} &\leq \frac{|\Omega^-_{R/4} \cap \Omega_{R/32}(x_0)|^{1/(2\mu')}}{|\Omega_{R/32}(x_0)|^{1/(2\mu')}} |(W^{2\mu} \mathbb{I}_{\Omega^-_{R/4}})^{1/(2\mu)}_{\Omega_{R/32}(x_0)}\\
&\leq C(d, M)\gamma^{1/(2\mu')}(U^{2\mu})^{1/(2\mu)}_{\Omega_{R/32}(x_0)}\\
&\leq C(d, p, M, \Lambda)\gamma^{1/(2\mu')}\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}\Big).
\end{aligned} \tag{5.15}$$

In the following, we mainly focus on constructing W and V on $\Omega^+_{R/4}$. For this, we introduce a cut-off function $\chi \in C^\infty(\mathbb{R}^d)$:

$$\begin{cases} \chi = 0 \quad \text{on } \{x^2 > \phi - \gamma R/2\} \cap \{x^1 < \gamma R/2\}, \\ \chi = 1 \quad \text{on } \{x^2 < \phi - \gamma R\} \cup \{x^1 > 2\gamma R\}, \\ 0 \le \chi \le 1, \quad |D\chi| \le \frac{4\sqrt{1+M^2}}{\gamma R}. \end{cases}$$

We have the following two estimates:

$$(|(1-\chi)U|^2)^{1/2}_{\Omega_{R/4}} \le C_1 \gamma^{1/(2\mu')}\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}\Big), \tag{5.16}$$

$$(|uD\chi|^2)^{1/2}_{\Omega_{R/4}} \le C_2 \gamma^{1/(2\mu')}(U^2)^{1/2}_{\Omega_R(x_0)}, \tag{5.17}$$

where $C_1 = C_1(d, p, \Lambda)$ and $C_2 = C_2(d, p, M, \Lambda)$. The estimate (5.16) is a direct consequence of Hölder's inequality and Proposition 5.4.3, by noting that

$$|\Omega_{R/4} \cap \operatorname{supp}(1-\chi)| \le C(d)\gamma R^d.$$

The estimate (5.17) can be obtained as follows: we first decompose

$$\operatorname{supp}\{D\chi\} \cap \Omega_{R/4} \subset \bigcup_{z \in \mathcal{D}_{grid}} \Omega_{2\sqrt{d+3}\gamma R}(z),$$

where

$$\mathcal{D}_{grid} := \Big\{ z \in \mathbb{R}^d : z = (\gamma R/4, k\gamma R) \ \text{ for } \ k = (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}, \\ \text{where } \Omega_{\sqrt{d+3}\gamma R}(z) \cap \{x^2 > \phi - 3\gamma R/2\} \cap \Omega_{R/4} \neq \emptyset \Big\}.$$

Note that on each $\Omega_{2\sqrt{d+3}\gamma R}(z)$, the conditions of Lemma 5.2.4 are satisfied with a uniform $\alpha \geq \alpha(d, M) > 0$. Hence we can apply Lemma 5.2.4 to obtain

$$\|uD\chi\|_{L^2(\Omega_{2\sqrt{d+3}\gamma R}(z))} \lesssim \frac{1}{\gamma R}\|u\|_{L^2(\Omega_{2\sqrt{d+3}\gamma R}(z))} \lesssim \|Du\|_{L^2(\Omega_{4\sqrt{d+3}\gamma R}(z))}.$$

Using $\gamma < 1/(32\sqrt{d+3})$ and the definition of $\mathcal{D}_{grid}$, we have $\Omega_{4\sqrt{d+3}\gamma R}(z) \subset \Omega_R(x_0)$. Since each point is covered by at most $N(d)$ such balls, (5.17) is proved.

A straightforward but tedious calculation gives the following equation for χu on $\Omega^+_{R/4}$.

$$\begin{cases} D_i(a_{ij}D_j(\chi u)) - \lambda \chi u = D_i g_i^{(1)} + D_i g_i^{(2)} + g_i^{(3)} D_i \chi + g_i^{(4)} D_i \widetilde{\chi} + g^{(5)} & \text{in } \Omega^+_{R/4}, \\ a_{ij} D_j(\chi u) n_i = g_i^{(1)} n_i + g_i^{(2)} n_i & \text{on } \Gamma^-, \\ \chi u = 0 & \text{on } \Gamma^+, \end{cases} \tag{5.18}$$

where

$$\begin{aligned} g_i^{(1)} &= a_{ij} u D_j \chi + f_i \chi, \quad g_i^{(2)} = (-\varepsilon_i \varepsilon_j \widetilde{a_{ij}} \widetilde{\chi} D_j \widetilde{u} + \varepsilon_i \widetilde{\chi} \widetilde{f_i}) \mathbb{I}_{Ex \in \Omega^-_{R/4}}, \\ g_i^{(3)} &= a_{ij} D_j u - f_i, \quad g_i^{(4)} = (\varepsilon_i \varepsilon_j \widetilde{a_{ij}} D_j \widetilde{u} - \varepsilon_i \widetilde{f_i}) \mathbb{I}_{Ex \in \Omega^-_{R/4}}, \\ g^{(5)} &= \chi f + \widetilde{\chi} \widetilde{f} \mathbb{I}_{Ex \in \Omega^-_{R/4}} + \lambda \widetilde{\chi} \widetilde{u} \mathbb{I}_{Ex \in \Omega^-_{R/4}}. \end{aligned}$$

Here we denote the reflection operator and reflected function as

$$E(x^1, x^2, \ldots, x^d) := (\gamma R/2 - x^1, x^2, \ldots, x^d), \quad \widetilde{f} := f \circ E,$$

and similarly for $\widetilde{a_{ij}}$, $\widetilde{\chi}$, and $\widetilde{u}$. We also use the following notation for sign functions

$$\varepsilon_i := \begin{cases} -1 & \text{if } \; i = 1, \\ 1 & \text{if } \; i \neq 1. \end{cases}$$

The function χu satisfies (5.18) in the weak sense: the usual integral identity is satisfied if we take any test function $\psi \in W^{1,2}_{\partial\Omega^+_{R/4}\setminus\Gamma^-}(\Omega^+_{R/4})$.

Now we are in position of constructing the decomposition. By the Lax-Milgram lemma, the following equation has a unique solution $w \in W^{1,2}_{\partial\Omega^+_{R/4}\setminus\Gamma^-}(\Omega^+_{R/4})$:

$$\begin{cases} \begin{aligned} D_i(\overline{a_{ij}}D_j w) - \lambda w = D_i((\overline{a_{ij}} - a_{ij})D_j(\chi u)) + D_i g_i^{(1)} + D_i g_i^{(2)} \\ + g_i^{(3)} D_i\chi + g_i^{(4)} D_i\widetilde{\chi} + g^{(5)} \end{aligned} & \text{in } \Omega^+_{R/4}, \\ \overline{a_{ij}}D_j w \cdot n_i = (\overline{a_{ij}} - a_{ij})D_j(\chi u)n_i + g_i^{(1)}n_i + g_i^{(2)}n_i & \text{on } \Gamma^-, \\ w = 0 & \text{on } \partial\Omega^+_{R/4} \setminus \Gamma^-, \end{cases} \tag{5.19}$$

where we take $\overline{a_{ij}} := (a_{ij})_{\Omega_{R/4}}$. Taking the difference between (5.18) and (5.19), we obtain that $v := \chi u - w$ satisfies

$$\begin{cases} D_i(\overline{a_{ij}}D_j v) - \lambda v = 0 & \text{in } \Omega^+_{R/4}, \\ \overline{a_{ij}}D_j v \cdot n_i = 0 & \text{on } \Gamma^-, \\ v = 0 & \text{on } \Gamma^+. \end{cases}$$

We define

$$W := |Dw| + \sqrt{\lambda}|w| + |D((1-\chi)u)| + \sqrt{\lambda}|(1-\chi)u|, \quad V := |Dv| + \sqrt{\lambda}|v| \quad \text{on } \Omega^+_{R/4}. \tag{5.20}$$

By our construction (5.14) and (5.20), clearly we have $U \le W + V$ on $\Omega_{R/32}(x_0) \subset \Omega_{R/4}$. Now we estimate W. By (5.16) and (5.17), we have

$$\begin{aligned} &(|D((1-\chi)u)|^2)^{1/2}_{\Omega_{R/32}(x_0)} + \sqrt{\lambda}(|(1-\chi)u|^2)^{1/2}_{\Omega_{R/32}(x_0)} \\ &\lesssim (|D((1-\chi)u)|^2)^{1/2}_{\Omega_{R/4}} + \sqrt{\lambda}(|(1-\chi)u|^2)^{1/2}_{\Omega_{R/4}} \\ &\lesssim \gamma^{1/(2\mu')}\Big((U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}\Big). \end{aligned} \tag{5.21}$$

The estimate for $|Dw| + \sqrt{\lambda}|w|$ requires more work. For simplicity, here we denote

$$\widehat{W} := |Dw| + \sqrt{\lambda}|w|.$$

Testing (5.19) by w, using the ellipticity condition and Hölder's inequality, and then rearranging terms, we obtain

$$\begin{aligned}
&(\widehat{W}^2)_{\Omega^+_{R/4}} \\
&\lesssim\Big((|(\overline{a_{ij}} - a_{ij})D_j(\chi u)|^2)^{1/2}_{\Omega^+_{R/4}} + (|uD\chi|^2)^{1/2}_{\Omega^+_{R/4}} + (|\chi F|^2)^{1/2}_{\Omega_{R/4}} + (|\mathbb{I}_{Ex\in\Omega^-_{R/4}} U|^2)^{1/2}_{\Omega_{R/4}}\Big)(\widehat{W}^2)^{1/2}_{\Omega^+_{R/4}} \\
&\quad + \Big((|\mathbb{I}_{\mathrm{supp}(D\chi)} U|^2)^{1/2}_{\Omega^+_{R/4}} + (|\mathbb{I}_{\mathrm{supp}(D\chi)} F|^2)^{1/2}_{\Omega^+_{R/4}}\Big)(|wD\chi|^2)^{1/2}_{\Omega^+_{R/4}} \\
&\quad + \Big((|\mathbb{I}_{Ex\in\Omega^-_{R/4}} U|^2)^{1/2}_{\Omega^+_{R/4}} + (|\mathbb{I}_{Ex\in\Omega^-_{R/4}} F|^2)^{1/2}_{\Omega^+_{R/4}}\Big)(|wD\widetilde{\chi}|^2)^{1/2}_{\Omega^+_{R/4}}.
\end{aligned} \tag{5.22}$$

Applying the decomposition argument in the proof of (5.17) and the Poincaré inequality in Lemma 5.2.4 again, we have

$$(|wD\chi|^2)^{1/2}_{\Omega^+_{R/4}} + (|wD\widetilde{\chi}|^2)^{1/2}_{\Omega^+_{R/4}} \lesssim (|Dw|^2)^{1/2}_{\Omega^+_{R/4}}. \tag{5.23}$$

Notice that here we do not need to increase the domain of integration since we can take zero extension of w outside $\Omega^+_{R/4}$, which is still a $W^{1,2}$ function on the upper half space.

Substituting (5.23) back into (5.22) and then applying Hölder's inequality, (5.17), and Proposition 5.4.3, we obtain

$$\begin{aligned}
&(|\widehat{W}|^2)^{1/2}_{\Omega^+_{R/4}} \\
&\lesssim(|(\overline{a_{ij}} - a_{ij})D_j(\chi u)|^2)^{1/2}_{\Omega^+_{R/4}} + (|uD\chi|^2)^{1/2}_{\Omega^+_{R/4}} + (|\chi F|^2)^{1/2}_{\Omega_{R/4}} + (|\mathbb{I}_{Ex\in\Omega^-_{R/4}} U|^2)^{1/2}_{\Omega_{R/4}} \\
&\quad + (|\mathbb{I}_{\mathrm{supp}(D\chi)} U|^2)^{1/2}_{\Omega^+_{R/4}} + (|\mathbb{I}_{\mathrm{supp}(D\chi)} F|^2)^{1/2}_{\Omega^+_{R/4}} + (|\mathbb{I}_{Ex\in\Omega^-_{R/4}} F|^2)^{1/2}_{\Omega^+_{R/4}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim (|(\overline{a_{ij}} - a_{ij})|^{2\mu'})^{1/(2\mu')}_{\Omega^+_{R/4}} (|Du|^{2\mu})^{1/(2\mu)}_{\Omega^+_{R/4}} + (|uD\chi|^2)^{1/2}_{\Omega_{R/4}} + (F^{2\mu})^{1/(2\mu)}_{\Omega_{R/4}} \\
&\quad + (|\mathbb{I}_{Ex\in\Omega^-_{R/4}} + \mathbb{I}_{\mathrm{supp}(D\chi)}|^{2\mu'})^{1/(2\mu')}_{\Omega_{R/4}} (U^{2\mu})^{1/(2\mu)}_{\Omega_{R/4}} \\
&\lesssim (\theta^{1/(2\mu')} + \gamma^{1/(2\mu')})(U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}.
\end{aligned}
\tag{5.24}
$$

Hence we reach (5.12) by combining (5.15), (5.21), and (5.24):

$$
\begin{aligned}
&(W^2)^{1/2}_{\Omega_{R/32}(x_0)} \\
&\lesssim (|W\mathbb{I}_{\Omega^-_{R/4}}|^2)^{1/2}_{\Omega_{R/32}(x_0)} + (|W\mathbb{I}_{\Omega^+_{R/4}}|^2)^{1/2}_{\Omega_{R/32}(x_0)} \\
&\lesssim (|W\mathbb{I}_{\Omega^-_{R/4}}|^2)^{1/2}_{\Omega_{R/32}(x_0)} + (\widehat{W}^2)^{1/2}_{\Omega^+_{R/4}} + (|D((1-\chi)u)|^2)^{1/2}_{\Omega_{R/32}(x_0)} + \sqrt{\lambda}(|(1-\chi)u|^2)^{1/2}_{\Omega_{R/32}(x_0)} \\
&\lesssim (\theta^{1/(2\mu')} + \gamma^{1/(2\mu')})(U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}.
\end{aligned}
$$

The estimate for V follows from a linearly transformed and rescaled version of Theorem 5.3.1, (5.17), and (5.24). Since $V \le \widehat{W} + |D(\chi u)| + \sqrt{\lambda}|\chi u|$, for any $q \in \left(2, \frac{2(m+2)}{m+1}\right)$,

$$
\begin{aligned}
(V^q)^{1/q}_{\Omega_{R/32}(x_0)} &\le (V^q)^{1/q}_{\Omega^+_{R/8}} \lesssim (V^2)^{1/2}_{\Omega^+_{R/4}} \\
&\lesssim (|D(\chi u)|^2)^{1/2}_{\Omega^+_{R/4}} + \sqrt{\lambda}(|\chi u|^2)^{1/2}_{\Omega^+_{R/4}} + (\widehat{W}^2)^{1/2}_{\Omega^+_{R/4}} \\
&\lesssim (U^2)^{1/2}_{\Omega_R(x_0)} + N(\theta^{1/(2\mu')} + \gamma^{1/(2\mu')})(U^2)^{1/2}_{\Omega_R(x_0)} + (F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)} \\
&\lesssim (U^2)^{1/2}_{\Omega_R(x_0)} + N(F^{2\mu})^{1/(2\mu)}_{\Omega_R(x_0)}.
\end{aligned}
$$

Hence Lemma 5.4.4 is proved. ■

5.4.3 Level set argument and proof of Proposition 5.4.1

In this subsection, we prove Proposition 5.4.1, by deriving a decay estimate of the following two level sets:

$$\mathcal{A}(s) := \{x \in \Omega : \mathcal{M}_\Omega(U^2)^{1/2} > s\},$$
$$\mathcal{B}(s) := \{x \in \Omega : (\gamma^{1/(2\mu')} + \theta^{1/(2\mu')})^{-1}\mathcal{M}_\Omega(F^{2\mu})^{1/(2\mu)} + \mathcal{M}_\Omega(U^2)^{1/2} > s\}.$$

Here $\mathcal{M}_\Omega$ is the Hardy-Littlewood maximal operator on Ω, by taking zero extension outside: for $f \in L^1_{\mathrm{loc}}(\Omega)$ and $x \in \Omega$,

$$\mathcal{M}_\Omega(f)(x) := \sup_{r>0} \fint_{B_r(x)} F\mathbb{I}_\Omega.$$

By the Hardy-Littlewood theorem, for any $f \in L^q(\Omega)$ with $q \in [1, \infty)$, we have

$$|\{x \in \Omega : \mathcal{M}_\Omega(f)(x) > s\}| \leq C\frac{\|f\|^q_{L^q(\Omega)}}{s^q}, \tag{5.25}$$

where $C = C(d, q)$.

As mentioned before, we first prove the following local property at certain small scales.

Lemma 5.4.5. *Under the conditions of Lemma 5.4.4, for any $q \in [2, 2(m+2)/(m+1))$, there exists a constant C depending on (d, p, q, M, Λ), such that for all $\kappa > 2^{d/2}$ and $s > 0$, the following holds: if for some $R < R_1, x_0 \in \overline{\Omega}$,*

$$|\Omega_{R/128}(x_0) \cap \mathcal{A}(\kappa s)| \geq C\Big(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big)|\Omega_{R/128}(x_0)|, \tag{5.26}$$

then $\Omega_{R/128}(x_0) \subset \mathcal{B}(s)$.

Proof. It suffices to prove the contrapositive of the statement with $s = 1$: suppose that there exists a point $z_0 \in \Omega_{R/128}(x_0), z_0 \notin \mathcal{B}(1)$, we aim to prove that there exists some constant $C = C(d, p, q, M, \Lambda)$, such that

$$|\Omega_{R/128}(x_0) \cap \mathcal{A}(\kappa)| < C\Big(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big)|\Omega_{R/128}(x_0)|. \tag{5.27}$$

We decompose

$$\begin{aligned}
&\Omega_{R/128}(x_0) \cap \mathcal{A}(\kappa) \\
&= \{y \in \Omega_{R/128}(x_0) : \sup_{r \in A_y^\kappa} r > R/64\} \cup \{y \in \Omega_{R/128}(x_0) : \sup_{r \in A_y^\kappa} r \le R/64\},
\end{aligned}$$

where

$$A_y^\kappa := \{r : (U^2)^{1/2}_{B_r(y)} > \kappa\}.$$

We claim $\{y \in \Omega_{R/128}(x_0) : \sup_{r \in A_y^\kappa} r > R/64\} = \emptyset$. By contradiction, if there exists one such y, take any $r \in A_y^\kappa, r > R/64$. From $B_r(y) \subset B_{2r}(z_0)$ and $z_0 \notin \mathcal{B}(1)$, we obtain

$$(U^2)^{1/2}_{B_r(y)} \le 2^{d/2}(U^2)^{1/2}_{B_{2r}(z_0)} \le 2^{d/2}\mathcal{M}_\Omega(U^2)^{1/2}(z_0) \le 2^{d/2} \le \kappa.$$

This contradicts the definition of A_y^κ. We are left to estimate $|\{y \in \Omega_{R/128}(x_0) : \sup_{r \in A_y^\kappa} r \le R/64\}|$. For any y in this set and $r \in A_y^\kappa$, by noting that $B_r(y) \subset B_{R/32}(z_0)$, we apply Lemma 5.4.4 on $\Omega_{R/32}(z_0)$ to obtain the decomposition $U \le V + W$, satisfying

$$(W^2)^{1/2}_{\Omega_{R/32}(z_0)} \le C(\gamma^{1/(2\mu')} + \theta^{1/(2\mu')}), \quad (V^q)^{1/q}_{\Omega_{R/32}(z_0)} \le C, \tag{5.28}$$

where the constant $C = C(d, p, q, M, \Lambda)$. Here, to reach (5.28) from (5.12) and (5.13), we used the fact that $z_0 \notin \mathcal{B}(1)$. Now, by $r \in A_y^\kappa$ and $U \le V + W$ on $B_{R/32}(z_0)$, we have for any point in $\{y \in \Omega_{R/128}(x_0) : \sup_{r\in A_y^\kappa} r \le R/64\}$,

$$\kappa < (U^2)^{1/2}_{B_r(y)} \le \mathcal{M}_\Omega(|W\mathbb{I}_{\Omega_{R/32}(z_0)}|^2)^{1/2}(y) + \mathcal{M}_\Omega(|V\mathbb{I}_{\Omega_{R/32}(z_0)}|^2)^{1/2}(y).$$

Hence, we get

$$\begin{aligned}
&|\{y \in \Omega_{R/128}(x_0) : \sup_{r\in A_y^\kappa} r \le R/64\}| \\
&\quad \le |\{\mathcal{M}_\Omega(|W\mathbb{I}_{\Omega_{R/32}(z_0)}|^2)^{1/2} > \kappa/2\}| + |\{\mathcal{M}_\Omega(|V\mathbb{I}_{\Omega_{R/32}(z_0)}|^2)^{1/2} > \kappa/2\}| \\
&\quad \le C\Big((\kappa/2)^{-2}\|W\|^2_{L^2(\Omega_{R/32}(z_0))} + (\kappa/2)^{-q}\|V\|^q_{L^q(\Omega_{R/32}(z_0))}\Big) && (5.29)\\
&\quad \le C|\Omega_{R/32}(z_0)|(\kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'}) + \kappa^{-q}) && (5.30)\\
&\quad \le C\Big(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big)|\Omega_{R/128}(x_0)|,
\end{aligned}$$

where we applied (5.25) to obtain (5.29), and used (5.28) to reach (5.30). The constant $C = C(d, p, q, M, \Lambda)$. This proves (5.27) and hence the lemma. ∎

The local property in Lemma 5.4.5 leads to the following global estimate by using the “crawling of the ink spots” lemma in [44]. For any $x_0 \in \mathcal{A}(\kappa s)$, we shrink the ball $\Omega_{R/128}(x_0)$ from $R = R_1$ until the first time (5.26) holds. By (5.25), (5.31), (5.32), and the Lebesgue differentiation theorem, such R exists and $R \in (0, R_1)$. Now we cover $\mathcal{A}(\kappa s)$ by such balls which are almost disjoint (using the Vitali covering lemma). On each ball $\Omega_{R/128}(x_0)$, by Lemma 5.4.5 we have $\Omega_{R/128}(x_0) \subset \mathcal{B}(s)$. Since this is an almost disjoint cover of $\mathcal{A}(\kappa s)$, the following corollary is proved.

Corollary 5.4.6. *Under the same hypothesis of Lemma 5.4.5, for any $q \in [2, 2(m+2)/(m+1))$, there exists a constant C depending on (d, p, q, M, Λ), such that for any $\kappa > \max\{2^{d/2}, \kappa_0\}$ and*

$$s > s_0(d, p, q, M, \Lambda, R_1, \kappa, \gamma, \theta, \|U\|_{L^2(\Omega)})$$

$$:= \left(\frac{3\|U\|_{L^2(\Omega)}^2}{C\kappa^2(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'}))|B_{R_1/128}|} \right)^{1/2}, \tag{5.31}$$

we have

$$|\mathcal{A}(\kappa s)| \leq C\Big(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big)|\mathcal{B}(s)|,$$

where κ_0 *is a constant satisfying*

$$C\Big(\kappa_0^{-q} + \kappa_0^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big) < 1. \tag{5.32}$$

Now we give the proof of Proposition 5.4.1.

Proof of Proposition 5.4.1. For any $W^{1,2}$-solution u, in order to prove $u \in W^{1,p}$ and the estimate (5.11), it suffices to prove

$$\lim_{S\to\infty} p \int_0^S |\mathcal{A}(s)|s^{p-1}\, ds \leq C\Big(R_1^{d(1-p/2)}\|U\|_{L^2(\Omega)}^p + \|F\|_{L^p(\Omega)}^p\Big). \tag{5.33}$$

For this, we will apply Corollary 5.4.6 with constants $q = p/2 + (m+2)/(m+1)(> p > 2\mu)$ and (κ, γ, θ) to be determined later satisfying

$$\gamma < 1/(32\sqrt{d+3}), \quad \theta \in (0,1), \quad \kappa > \max\{2^{d/2}, \kappa_0\}.$$

Changing the integral variable in (5.33) from s to κs, applying (5.25) and Corollary 5.4.6, we obtain

$$\begin{aligned}
p \int_0^S |\mathcal{A}(s)|s^{p-1}\, ds &= p\kappa^p \int_0^{S/\kappa} |\mathcal{A}(\kappa s)|s^{p-1}\, ds \\
&\leq C_0\kappa^p \int_0^{s_0} \frac{\|U\|_{L^2(\Omega)}^2}{(\kappa s)^2} s^{p-1}\, ds + C\kappa^p\Big(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big) \int_{s_0}^{S/\kappa} |\mathcal{B}(s)|s^{p-1}\, ds \\
&\leq C_1(R_1^{d(1-p/2)}\|U\|_{L^2(\Omega)}^p + \|F\|_{L^p(\Omega)}^p)
\end{aligned}$$

$$+ C\kappa^p\Big(\kappa^{-q} + \kappa^{-2}(\gamma^{1/\mu'} + \theta^{1/\mu'})\Big)\int_0^{S/(2\kappa)} |\mathcal{A}(s)| s^{p-1}\, ds,$$

where $C_0 = C_0(d,p)$, $C_1 = C_1(d,p,q,M,\Lambda,\kappa,\gamma,\theta)$, and $C = C(d,p,q,M,\Lambda)$. Here for the last inequality, we used the relation

$$\mathcal{B}(s) \subset \mathcal{A}(s/2) \cup \Big\{(\gamma^{1/(2\mu')} + \theta^{1/(2\mu')})^{-1}\mathcal{M}_\Omega(F^{2\mu})^{1/(2\mu)} > s/2\Big\}$$

and the Hardy-Littlewood theorem, then we changed the integral variable from s to $s/2$. Now (5.33) is proved if we first choose κ large enough and then choose γ and θ small enough to absorb the last term on the right-hand side involving $\mathcal{A}(s)$, and finally pass S to the infinity. ■

5.4.4 Proof of Theorem 5.2.8

We close this section by giving the proof of our first main result Theorem 5.2.8. First, from Proposition 5.4.1, we can deduce the following a priori estimate for the original equation (5.3), without the L^2 norm on the right-hand side.

Corollary 5.4.7. *Let $p \in (2, 2(m+2)/(m+1))$, γ_0, θ_0 be the constants from Proposition 5.4.1, and $u \in W^{1,p}_{\mathcal{D}}$ be a weak solution the equation (5.3) with $f_i, f \in L^p(\Omega)$. Under Assumptions 5.2.2 (γ_0), 5.2.5 (γ_0, m), and 5.2.7 (θ_0), there exists a positive constant λ_1 depending on (d,p,M,Λ,R_1,K) such that if $\lambda > \lambda_1$, the following estimate holds:*

$$\|U\|_{L^p(\Omega)} \le C\|F\|_{L^p(\Omega)},$$

where $C = C(d,p,M,\Lambda)$.

The proof of the above corollary is the same as [11, Corollary 5.2], which we will sketch below. We first localize by considering the equation for $u\zeta$, where $\zeta \in C_0^\infty(B_\varepsilon(x_0))$ with $x_0 \in \overline{\Omega}$ and ε/R_1 sufficiently small. To apply Proposition 5.4.1, notice that $u\zeta$ still satisfies a mixed Dirichlet-conormal problem in the form of (5.10), if we move all the lower order terms on both the equation and the conormal boundary condition to the right-hand side. Now we can absorb the L^2 norm in (5.11) by applying Hölder's inequality and choosing ε sufficiently small. Then we use the partition of unity and choose λ sufficiently large to remove the cut-off and absorb the L^p norms of Du and u on the right-hand side as usual.

Now we are ready to give the proof of Theorem 5.2.8.

Proof of Theorem 5.2.8. We first prove (i).

Case 1: $p = 2$. This is by applying the Lax-Milgram lemma directly. In this case, actually we only need to choose $\lambda > \lambda_2(d, K, \Lambda)$ for a sufficiently large λ_2, without imposing any regularity assumptions on the coefficients and domain.

Case 2: $p \in (2, 2(m+2)/(m+1)$. For $\lambda > \lambda_0 := \max\{\lambda_1, \lambda_2\}$, we have both $W^{1,2}$-solvability and $W^{1,p}$-a priori estimate Corollary 5.4.7. Hence we only need to prove the existence of $W^{1,p}$ solutions. We first prove the solvabilty for (5.10), noting that the only problem here is $L^p \not\subset L^2$ since our domain can be unbounded. This is solved by approximation. We approximate f and f_i by $f^{(n)}$ and $f_i^{(n)}$ strongly in $L^p(\Omega)$ with $f^{(n)}, f_i^{(n)} \in L^p \cap L^2$. Associated to these right-hand side functions, we can find solution $u^{(n)} \in W^{1,2}_{\mathcal{D}}(\Omega)$. By Proposition 5.4.1, we have $u^{(n)} \in W^{1,p}(\Omega)$. Furthermore, by Corollary 5.4.7 $\{u^{(n)}\}_n$ is a Cauchy sequence in the space $W^{1,p}(\Omega)$. Clearly the limit $u \in W^{1,p}(\Omega)$ must be a solution to (5.10).

From this, the original problem (5.3) can be solved by applying the method of continuity to the family of operators $tL_\lambda^{(0)} + (1-t)L_\lambda$ (and the corresponding mixed boundary condition).

Case 3: $p \in (2(m+2)/(m+3), 2)$. This can be obtained from Case 2 by duality. Such argument can be found in [11, Proof of Theorem 2.4].

Now we prove (ii). We need to lower the integrability condition for f and reduce the large λ condition to the usual sign condition $L_0 1 \le 0$. For the first one, we solve the follow divergence form equation for $f \in L^{p_*}(\Omega)$:

$$\operatorname{div} \varphi = f \text{ in } \Omega, \quad \text{where } \varphi = (\varphi_i)_{i=1}^d \in W_{\mathcal{N}}^{1,p_*}(\Omega).$$

Such φ_i exists due to [11, Lemma 7.2], and it satisfies

$$\|\varphi_i\|_{L^p(\Omega)} \le C(\operatorname{diam}(\Omega), R_1, d, p, M)\|f\|_{L^{p_*}(\Omega)}.$$

Then we solve the equation with right-hand side $D_i(f_i + \varphi_i)$. Finally, we apply the Fredholm alternative to obtain the unique solvability as well as the estimate, noting that on bounded domain the weak maximum principle holds. ■

5.5 Inhomogeneous boundary data

In this section, we consider the inhomogeneous boundary value problem (5.4) and prove Theorem 5.2.11. For this, we first solve the problem in $L^{1+\varepsilon}$ by applying [73, Theorem 1.1]. Then we improve the regularity of this solution with the help of the reverse Hölder inequality in Corollary 5.4.2.

5.5.1 Notation

Let Ω be a Lipschitz domain locally represented by $x^1 > \psi_0(x')$. For $x = (x^1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$, we denote the lifting up and the projection maps as

$$\Psi_0(x') := (\psi_0(x'), x'), \quad \mathbb{P}_{d-1}(x^1, x') := x'.$$

By the definition, for any (γ, m)-flat Γ, the projection $\mathbb{P}_{d-1}\Gamma$ locally is also (γ, m)-flat, of co-dimension 1.

We consider two types of neighborhoods on the boundary: surface cubes and surface balls. We say that $Q \subset \partial\Omega$ is a surface cube if $Q = \Psi_0(Q')$, where Q' is a cube in $\mathbb{R}^{d-1}$. For surface cubes, we denote $kQ := \Psi_0(kQ')$. We also consider the "cylinder" over Q:

$$T(Q) := \{y \in \Omega : \mathrm{dist}(y, Q) < r_Q, \mathbb{P}(y) \in \mathbb{P}(Q)\}, \quad \text{where } r_Q = \mathrm{diam}(Q).$$

For any point $x \in \partial\Omega$, we denote

$$\Delta_r(x) := B_r(x) \cap \partial\Omega$$

to be the surface ball and

$$\delta(x) := \mathrm{dist}(x, \Gamma).$$

In this section, we need the following truncated non-tangential maximal functions:

$$\boldsymbol{N}_r(f)(x) := \sup_{y \in \Gamma_\beta(x) \cap B_r(x)} |f(y)|, \quad \boldsymbol{N}^r(f)(x) := \sup_{y \in \Gamma_\beta(x) \cap B_r^c(x)} |f(y)|. \tag{5.34}$$

5.5.2 An $L^{1+\varepsilon}$-solvability

In this part we check that our assumptions on Γ imply those in [73, Theorem 1.1]. As a corollary, the following $L^{1+\varepsilon}$-solvability holds.

Lemma 5.5.1. *Let $m \in \{0, 1, \dots, d-2\}$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain satisfying Assumption 5.2.1(M). Then we can find $\gamma = \gamma(d, M)$, such that if Γ is (γ, m)-flat, the following hold.*

(a) For any $p > 1$, (5.4) has at most one solution.

(b) There exists some constant $q_0 = q_0(d, M) > 1$, such that for any $g_{\mathcal{N}} \in L^{q_0}(\mathcal{N})$ and $g_{\mathcal{D}} \in W^{1,q_0}(\mathcal{D})$, the problem (5.4) has a unique solution u satisfying

$$\|\boldsymbol{N}(Du)\|_{L^{q_0}(\partial\Omega)} \le C\|g_{\mathcal{N}}\|_{L^{q_0}(\mathcal{N})} + C\|g_{\mathcal{D}}\|_{W^{1,q_0}(\mathcal{D})},$$

where $C = C(d, M, R_0, R_1, diam(\Omega))$ is a constant.

For this, first we can simply check that if Γ is (γ, m)-flat, then $\mathcal{D}$ satisfies the corkscrew condition relative to $\partial\Omega$ in [73] with the parameter $2/(1-\gamma)$. Now fix any small $\varepsilon > 0$, we are going to find the constant $\gamma(\varepsilon, d)$, such that (γ, m)-flat implies Ahlfors $(d-2+\varepsilon)$-regular.

For any $x \in \Gamma$ and $r \in (0, R_1]$, due to the definition of (γ, m)-flatness at radius r, it can be easily verified that the surface ball $\Delta_r(x) \cap \Gamma$ can be covered by C/γ^{d-2} balls of radius γr, with C depending only on (d, M). Iterating k times, we get $(C/\gamma^{d-2})^k$ number of balls of radius $\gamma^k r$, the union of which covers $\Delta_r(x) \cap \Gamma$. Thus,

$$\mathcal{H}^{d-2+\varepsilon}(\Delta_r(x) \cap \Gamma) \le \sup_k \{C_0 (C/\gamma^{d-2})^k (\gamma^k r)^{d-2+\varepsilon}\} = \sup_k \{C_0 (C\gamma^{\varepsilon})^k r^{d-2+\varepsilon}\}.$$

If we take $\gamma^{\varepsilon} < 1/C$, the above quantity is less than $C_0 r^{d-2+\varepsilon}$. This means that Γ is Ahlfors $(d-2+\varepsilon)$-regular. Now we have verified the conditions for [73, Theorem 1.1], and thus Lemma 5.5.1 holds.

5.5.3 A reverse Hölder inequality on boundary

On a surface cube Q_0 lying in a coordinate chart, consider a function v satisfying

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on } \mathcal{N} \cap Q_0, \\ v = 0 & \text{on } \mathcal{D} \cap Q_0, \\ \boldsymbol{N}(Dv) \in L^{q_1}(\partial\Omega), \end{cases} \tag{5.35}$$

where $q_1 \in (1, q)$ is some constant. In this subsection, we prove the following result which plays a key role in proving Theorem 5.2.11.

Proposition 5.5.2. *Suppose that v satisfies (5.35). Then for any $q < (m+2)/(m+1)$, we can find sufficiently small $\gamma_1 > 0$, such that if Ω and Γ satisfy the Assumptions 5.2.2(γ_1) and 5.2.5(γ_1, m), then for any surface cube with $8Q \subset Q_0$, we have $\boldsymbol{N}(Dv) \in L^q(Q)$ satisfying*

$$\left(\fint_Q \boldsymbol{N}(Dv)^q \right)^{1/q} \lesssim \fint_{8Q} \boldsymbol{N}(Dv). \tag{5.36}$$

We start with the following lemma which relates the boundary norm and the interior norm, for harmonic functions with purely Dirichlet or conormal boundary data.

Lemma 5.5.3. *Suppose that v is a harmonic function in $T(2Q)$ with either $\partial v/\partial \boldsymbol{n} = 0$ or*

$v = 0$ on $2Q$, then we have $\mathbf{N}_{r_Q}(Du) \in L^2(Q)$ satisfying

$$\left(\mathbf{N}_{r_Q}(Dv)^2\right)_Q^{1/2} \leq C(|Dv|^2)_{T(2Q)}^{1/2},$$

where $C = C(d, M)$ is a constant.

See [62, Lemma 4.4, 4.8]. The proof is by localizing the global L^2-result in [36], noting that for any $x \in Q$ we have $\Gamma_\beta(x) \cap B_{r_Q}(x) \subset T(2Q)$. Using the above lemma, we can prove the following weighted estimate for harmonic functions with mixed boundary condition. Notice that $\mathbf{N}(Dv) \in L^{q_1}$ guarantees that Dv is an almost everywhere defined L^{q_1} function on $\partial\Omega$.

Lemma 5.5.4. *Suppose that v satisfies (5.35). In any surface cube Q with $2Q \subset Q_0$ and $4Q \cap \Gamma \neq \emptyset$, for any ε', we have*

$$\int_Q |Dv(y)|^2 \delta(y)^{1-\varepsilon'}\, dy \lesssim \int_{T(2Q)} |Dv(z)|^2 \delta(z)^{-\varepsilon'}\, dz.$$

In this statement, the case $+\infty \leq +\infty$ is allowed.

Proof. We first construct a non-intersecting decomposition $Q \setminus \Gamma = \bigcup_j Q_j$, where the surface balls $\{Q_j\}_j$ satisfy

$$c''\delta(y) \leq r_{Q_j} \leq c'\delta(y),\ \ \forall y \in \partial\Omega, \tag{5.37}$$

$$\sum \chi_{T(2Q_j)} \leq C(d, M, c''), \tag{5.38}$$

where $c' = 1/(4\sqrt{d})$ and $c'' = 1/(16\sqrt{d})$. This can be done by considering the usual Whitney decomposition $\bigcup_j Q_j'$ of $\mathbb{P}(4Q) \setminus \mathbb{P}(\Gamma)$ on the coordinate plane $\mathbb{R}^{d-1}$, then "lifting up" back to surface cubes by the map Ψ_0. Such decomposition can also be found in [62, Lemma 4.9].

Now since $c' < 1/4$, we have $4Q_j \cap \Gamma = \emptyset$, which means only one type of boundary condition is prescribed on $4Q_j$. Then applying Lemma 5.5.3, we obtain

$$\int_{Q_j} \boldsymbol{N}_{r_{Q_j}}(Dv)^2 \leq C r_{Q_j}^{-1} \int_{T(2Q_j)} |Dv|^2,$$

where $\boldsymbol{N}_{r_{Q_j}}$ is the truncated non-tangential maximal operator defined in (5.34). Using (5.37), the point-wise inequality $|Dv(y)| \leq \boldsymbol{N}_{r_{Q_j}}(Dv)(y)$ for each $y \in Q_j$, and $\delta(y) \approx \delta(z) \approx r_{Q_j}$ for each $y \in Q_j$ and $z \in T(2Q_j)$, we have

$$\int_{Q_j} |Dv(y)|^2 \delta(y)^{1-\varepsilon'} \leq C \int_{T(2Q_j)} |Dv(z)|^2 \delta(z)^{-\varepsilon'}.$$

The lemma is proved by simply summing over j and using (5.38). ■

Now we are in the position of proving Proposition 5.5.2.

Proof of Proposition 5.5.2. We first claim that for any surface cube Q with $4Q \subset Q_0$,

$$\left(\fint_Q |Dv|^q \right)^{1/q} \lesssim \fint_{4Q} \boldsymbol{N}(Dv). \tag{5.39}$$

Case 1: $4Q \cap \Gamma = \emptyset$. Noting that $q < (m+2)/(m+1) < 2$, we apply Hölder's inequality, Lemma 5.5.3, and a rescaled version of Corollary 5.4.2 with $p = 2$ to obtain

$$\begin{aligned}\left(\fint_Q |Dv|^q \right)^{1/q} &\lesssim \left(\fint_Q \boldsymbol{N}_{r_Q}(Dv)^q \right)^{1/q} \lesssim \left(\fint_Q \boldsymbol{N}_{r_Q}(Dv)^2 \right)^{1/2} \lesssim \left(\fint_{T(2Q)} |Dv|^2 \right)^{1/2} \\ &\lesssim \fint_{T(4Q)} |Dv| \lesssim \fint_{4Q} |\boldsymbol{N}(Dv)|.\end{aligned}$$

Here in the first and the last inequality, we used the point-wise inequality in the form

$$|Dv(y)| \le |\boldsymbol{N}_{r_Q}(Dv(\psi_0(y'), y'))| \le |\boldsymbol{N}(Dv(\psi_0(y'), y'))|, \ \forall y = (y^1, y') \in T(4Q).$$

Case 2: $4Q \cap \Gamma \neq \emptyset$. In this case, we estimate as follows, with $p \in (1, 2(m+2)/(m+1))$ and ε' to be chosen later:

$$\left(\fint_Q |Dv|^q\right)^{1/q} \le \left(\fint_Q |\nabla v|^2 \delta^{1-\varepsilon'}\right)^{1/2} \left(\int_Q \delta^{(\varepsilon'-1)/(2/q-1)}\right)^{1/q-1/2} \tag{5.40}$$

$$\lesssim \left(\fint_{T(2Q)} |\nabla v|^2 \delta^{-\varepsilon'}\right)^{1/2} \left(\fint_Q \delta^{(\varepsilon'-1)/(2/q-1)}\right)^{1/q-1/2} \tag{5.41}$$

$$\lesssim \left(\fint_{T(2Q)} |\nabla v|^p\right)^{1/p} \left(\fint_{T(2Q)} \delta^{(-\varepsilon')\frac{1}{1-2/p}}\right)^{1/2-1/p} \left(\fint_Q \delta^{(\varepsilon'-1)/(2/q-1)}\right)^{1/q-1/2} \tag{5.42}$$

$$\lesssim \fint_{T(4Q)} |\nabla v| \lesssim \fint_{4Q} \boldsymbol{N}(Dv). \tag{5.43}$$

Here, for the inequalities (5.40) and (5.42) we applied Hölder's inequality. In (5.41), we applied Lemma 5.5.4. In order to obtain (5.43), we used Corollary 5.4.2. To make sure the two integrals in (5.42) involving δ cancel, we only require that they are both finite. For this, we fix

$$p = \frac{m+2}{m+1} + q \quad \text{and} \quad \varepsilon' = 2 - 1/q - 2/p$$

so that

$$\frac{-\varepsilon'}{1-2/p} > -2, \quad \frac{\varepsilon'-1}{2/q-1} > -1.$$

Now we reach (5.39). Note that the only difference between (5.39) and (5.36) is that we only controlled Dv instead of $\boldsymbol{N}(Dv)$. But for harmonic functions actually they are equivalent. For $t \in (1/2, 2)$, since $v \in W^{1,q}(\partial T((1+t)Q)$ and $T((1+t)Q)$ is a

Lipschitz domain, we can apply the classical L^q-estimate of the Dirichlet regularity problem in [36] to obtain

$$\left(\fint_Q \boldsymbol{N}_{r_Q/2}(Dv)^q\right)^{1/q} \lesssim \left(\fint_{\partial T(1+t)Q} |D_T u|^q\right)^{1/q} \lesssim \left(\fint_{\partial T(1+t)Q} |Du|^q\right)^{1/q},$$

where D_T is the tangential derivative, and we also used the fact that

$$\Gamma_\beta(x) \cap B_{r_Q/2}(x) \subset T(1+t)Q, \ \forall x \in Q \text{ and } t > 1/2.$$

Now we can take the average for $t \in (1/2, 2)$ and apply Corollary 5.4.2 to obtain

$$\begin{aligned}\left(\fint_Q \boldsymbol{N}_{r_Q/2}(Dv)^q\right)^{1/q} &\lesssim \left(\fint_{T(2Q)} |Dv|^q\right)^{1/q} + \left(\fint_{2Q} |Dv|^q\right)^{1/q} \\ &\lesssim \left(\fint_{T(4Q)} |Dv|\right) + \left(\fint_{2Q} |Dv|^q\right)^{1/q} \lesssim \fint_{4Q} \boldsymbol{N}(Dv) + \left(\fint_{2Q} |Dv|^q\right)^{1/q}.\end{aligned} \tag{5.44}$$

The estimate for $\boldsymbol{N}^{r_Q/2}(Dv)$ is easier. For $z \in \Omega$, consider the region

$$\Lambda_\beta(z) := \{y \in \partial\Omega : z \in \Gamma_\beta(y)\}.$$

Since $\partial\Omega$ is Lipschitz, for any $z \in \Gamma_\beta(x)$ with $x \in Q, |z - x| \geq r_Q/2$, we have

$$|\Lambda_\beta(z) \cap 2Q| \geq Cr^{d-1} \geq C|Q|, \tag{5.45}$$

where $C = C(d, M)$ is a constant. Hence, for any $x \in Q$ and $z \in \Gamma_\beta(X) \cap B^c_{r_Q/2}(x)$, we have

$$|Dv(z)| \leq \fint_{\Lambda_\beta(z)\cap 2Q} \boldsymbol{N}(Dv) \lesssim \fint_{2Q} \boldsymbol{N}(Dv),$$

where in the last inequality, we used (5.45). Taking sup in $z \in \Gamma_\beta(x) \cap B^c_{r_Q/2}(x)$, we obtain

$$\boldsymbol{N}^{r_Q/2}(Dv)(x) \lesssim \fint_{2Q} \boldsymbol{N}(Dv), \ \ \forall x \in Q.$$

Now taking L^q-average for $x \in Q$, we have

$$\left(\fint_Q \boldsymbol{N}^{r_Q/2}(Dv)^q\right)^{1/q} \lesssim \fint_{2Q} \boldsymbol{N}(Dv). \tag{5.46}$$

Combining (5.44) and (5.46), then applying (5.39) with Q replaced by $2Q$, we have

$$\begin{aligned}\left(\fint_Q \boldsymbol{N}(Dv)^q\right)^{1/q} &\le \left(\fint_Q \boldsymbol{N}_{r_Q/2}(Dv)^q\right)^{1/q} + \left(\fint_Q \boldsymbol{N}^{r_Q/2}(Dv)^q\right)^{1/q}\\ &\lesssim \fint_{4Q} \boldsymbol{N}(Dv) + \left(\fint_{2Q} |Dv|^q\right)^{1/q} \lesssim \fint_{8Q} \boldsymbol{N}(Dv).\end{aligned}$$

The proposition is proved. ∎

5.5.4 Proof of Theorem 5.2.11

We first state an interpolation result that will be useful in our proof. It can be simply deduced from [69, Theorem 3.2]. See also Remark 3.3 in the same paper.

Theorem 5.5.5 ([69]). *Let Q_0 be a surface cube and $F \in L^1(Q_0)$. Let $p_1 > 1$ and $f \in L^{p_2}(Q_0)$ for some $1 < p_2 < p_1$. Suppose that for each (dyadic) surface cube $Q \subset \frac{1}{4}Q_0$, there exist two integrable functions F_Q and R_Q on Q such that $|F| \le |F_Q| + |R_Q|$ on Q, and*

$$\left(\fint_Q |R_Q|^{p_1}\right)^{1/p_1} \le C_1\left(\fint_{16Q} |F| + \sup_{Q' \supset Q} \fint_{Q'} |f|\right),$$

$$\fint_Q |F_Q| \le C_2 \sup_{Q' \supset Q} \fint_{Q'} |f|.$$

Then $F \in L^{p_2}(1/4Q_0)$ *and*

$$\left(\fint_{1/4Q_0} |F|^{p_2}\right)^{1/p_2} \leq C\left(\fint_{Q_0} |F| + (\fint_{Q_0} |f|^{p_2})^{1/p_2}\right),$$

where $C = C(d, M, p_1, p_2, C_1, C_2)$.

Essentially this is the boundary version of the argument used by us earlier in Section 5.4.

Before giving the proof of Theorem 5.2.11, we first make some reduction. Since $\mathcal{D}$ is a $W^{1,q}$-extension domain, we can extend $g_{\mathcal{D}} \in W^{1,q}(\mathcal{D})$ to $\overline{g}_{\mathcal{D}} \in W^{1,q}(\partial\Omega)$. Also, we can extend $g_{\mathcal{N}} \in L^q(\mathcal{N})$ by zero to $\overline{g}_{\mathcal{N}} \in L^q(\partial\Omega)$. According to [37, Corollary 2.12], the following L^q Dirichlet regularity problem has a unique solution for any $1 < q < 2 + \varepsilon$:

$$\begin{cases} -\Delta\overline{u} = 0 & \text{in } \Omega, \\ \overline{u} = \overline{g}_{\mathcal{D}} & \text{on } \partial\Omega, \\ \boldsymbol{N}(D\overline{u}) \in L^q(\partial\Omega), \end{cases}$$

with

$$\|\boldsymbol{N}(D\overline{u})\|_{L^q(\partial\Omega)} \lesssim \|\overline{g}_{\mathcal{D}}\|_{W^{1,q}(\partial\Omega)} \lesssim \|g_{\mathcal{D}}\|_{W^{1,q}(\mathcal{D})}.$$

Hence, without loss of generality, we can assume $g_{\mathcal{D}} = 0$ and $g_{\mathcal{N}} \in L^q(\partial\Omega)$.

Proof of Theorem 5.2.11. From Lemma 5.5.1, we can find some q_0 slightly larger than 1 and a solution to the mixed problem with $g_{\mathcal{N}} \in L^q \subset L^{q_0}$ and $g_{\mathcal{D}} = 0$, satisfying

$$\|\boldsymbol{N}(Du)\|_{L^{q_0}(\partial\Omega)} \lesssim \|g_{\mathcal{N}}\|_{L^{q_0}(\mathcal{N})}. \tag{5.47}$$

We are left to show that this solution satisfies $\boldsymbol{N}(Du) \in L^q$ for any $q \in (1, (m +$

2)/(m + 1)) and

$$\|\boldsymbol{N}(Du)\|_{L^q(\partial\Omega)} \lesssim \|g_{\mathcal{N}}\|_{L^q(\mathcal{N})},$$

if we choose the parameters γ_1 in the Assumptions 5.2.2 and 5.2.5 sufficiently small.

For any surface cubes Q_0 and Q with $16Q \subset Q_0$, we take a cutoff function $\eta \in C_c^\infty(16Q)$ with $\eta = 1$ in $8Q$, and solve the following equation

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = \eta g_{\mathcal{N}} & \text{on } \mathcal{N}, \\ w = 0 & \text{on } \mathcal{D}, \\ \boldsymbol{N}(Dw) \in L^{q_0}(\partial\Omega). \end{cases}$$

Again by Lemma 5.5.1 such w exists and satisfies the following estimate

$$\left(\fint_{8Q} \boldsymbol{N}(Dw)^{q_0}\right)^{1/q_0} \lesssim \left(\fint_{16Q} |g_{\mathcal{N}}|^{q_0}\right)^{1/q_0}.$$

Then $v := u - w$ satisfies (5.35) with Q_0 replaced by $8Q$. Hence from Proposition 5.5.2 with q replaced by some $q + \varepsilon \in (q, (m+2)/(m+1))$, we have

$$\begin{aligned} \left(\fint_{Q} \boldsymbol{N}(Dv)^{q+\varepsilon}\right)^{1/(q+\varepsilon)} &\lesssim \fint_{8Q} \boldsymbol{N}(Dv) \lesssim \fint_{8Q} \boldsymbol{N}(Du) + \fint_{8Q} \boldsymbol{N}(Dw) \\ &\lesssim \fint_{8Q} \boldsymbol{N}(Du) + \left(\fint_{8Q} \boldsymbol{N}(Dw)^{q_0}\right)^{1/q_0} \lesssim \fint_{8Q} \boldsymbol{N}(Du) + \left(\fint_{16Q} |g_{\mathcal{N}}|^{q_0}\right)^{1/q_0}. \end{aligned}$$

Now we apply Theorem 5.5.5 with

$$F = \boldsymbol{N}(Du)^{q_0}, \quad F_Q = 2^{q_0-1}\boldsymbol{N}(Dw)^{q_0}, \quad R_Q = 2^{q_0-1}\boldsymbol{N}(Dv)^{q_0},$$

$$f = |g_{\mathcal{N}}|^{q_0}, \quad p_1 = (q+\varepsilon)/q_0, \quad p_2 = q/q_0$$

to obtain $\boldsymbol{N}(Du) \in L^q(Q_0/4)$ with

$$\left(\fint_{Q_0/4} \boldsymbol{N}(Du)^q\right)^{1/q} \lesssim \left(\fint_{Q_0} \boldsymbol{N}(Du)^{q_0}\right)^{1/q_0} + \left(\fint_{Q_0} |g_{\mathcal{N}}|^q\right)^{1/q}.$$

Since the choice of the surface cube Q_0 is arbitrary, the theorem is proved by using a covering argument, (5.47), and Hölder's inequality. ■

Appendix A

Regularized distance on small Lipschitz domains

In this appendix, we construct the regularized distances for small Lipschitz domains and state some useful properties.

A.1 Construction

In this section, we sketch the proof for Theorem 2.3.1. In [49, Theorem 1.1], the construction of regularized distance in Lipschitz domain is given, along with property (2.20). As for properties (2.21) and (2.22) concerning the derivatives, we modify the proof of [49, Theorem 1.3], which requires $\partial\Omega \in C^1$.

Proof of Theorem 2.3.1. Let $g(y) = \psi(y') - y^d$. Since Ω is Lipschitz, as in [49, Section 2], if we take $M = 2(4/\delta^2 + 1)^{1/2}$, then

$$[g]_1 \leq M/2, \quad (M/2)^{-1} \leq g/d_y \leq M/2 \quad \text{in } Q_{4R}(x_0) \setminus \Gamma_{4R}.$$

Here d_y is the distance between y and the boundary.

We construct the regularized distance as the fixed point of the following:

$$\rho(y) = G(y, \rho(y)), \tag{A.1}$$

where G is the mollification of g,

$$G(y, \tau) := \int_{|w|<1} g(y - \frac{\tau}{M} w)\phi(w)\, dw.$$

Here $\phi \in C_c^\infty(B_1), \phi \geq 0, \int \phi = 1$.

Clearly $[G(\cdot,\tau)]_1 \le [g]_1$. The key fact of such G is that: $[G(y,\cdot)]_1 \le 1/2$. From the contraction mapping theorem, we get the unique fixed point in C^0, denoted by ρ_0. We first show $\rho_0 \in C^{0,1}(Q_R) \cap C^\infty(Q_R \setminus \Gamma_R)$ and (2.20).

Using $[G(y,\cdot)]_1 \le 1/2$, from

$$\begin{aligned}|\rho_0(y) - \rho_0(z)| &\le |G(y,\rho_0(y)) - G(y,\rho_0(z))| + |G(y,\rho_0(z)) - G(z,\rho_0(z)|) \\ &\le [G(y,\cdot)]_1|\rho_0(y) - \rho_0(z)| + [G(\cdot,\rho(z))]_1,\end{aligned}$$

we can obtain

$$[\rho_0]_1 \le 2[G(\cdot,t)]_1 \le 2[g]_1 \le M.$$

Hence $\rho_0 \in C^{0,1}(Q_R)$. The fact that $\rho_0 \in C^\infty(Q_R \setminus \Gamma_R)$ follows from implicit function theorem and the fact that $\phi \in C^\infty$.

For (2.20), we write

$$|\rho_0(y) - G(y,0)| = |G(y,\rho_0(y)) - G(y,0)| \le \frac{1}{2}|\rho_0(y)|.$$

Noting that $G(y,0) = g(y)$, we have $\frac{2}{3} \le \frac{\rho_0}{g(y)} \le 2$. Since g is a distance function, (2.20) is proved.

Now we turn to proving (2.21) and (2.22). We modify the proof of [49, Theorem 1.3], which works for C^1 domains, to adapt to the case of small Lipschitz domains. Denote

$$G_i(y,\tau) := \frac{\partial}{\partial y^i}G(y,\tau), \quad i = 1,\ldots,d, \quad G_{d+1}(y,\tau) := \frac{\partial}{\partial \tau}G(y,\tau).$$

Differentiating both sides of (A.1) at the fixed point ρ_0, and applying the chain rule, we get

$$|D_i\rho_0(y) - D_i\rho_0(z)| \leq |G_i(y,\rho_0(y)) - G_i(z,\rho_0(z))| + |D_i\rho_0(y)||G_{d+1}(y,\rho_0(y)) - \\ - G_{d+1}(z,\rho_0(z))| + ||G_{d+1}(z,\rho_0(z))|D_i\rho_0(y) - D_i\rho_0(z)|.$$

Again, using $[G(z,\cdot)]_1 \leq 1/2, [\rho_0]_1 \leq M$, we have

$$\begin{aligned} &|D_i\rho_0(y) - D_i\rho_0(z)| \\ &\leq 2|G_i(y,\rho_0(y)) - G_i(z,\rho_0(z))| + 2M|G_{d+1}(y,\rho_0(y)) - G_{d+1}(z,\rho_0(z))|. \end{aligned} \tag{A.2}$$

Now we use the explicit expression for g, i.e., $g = \psi(y') - y^d$, and the small Lipschitz property (2.17) given in terms of $D\psi$. Noting that for g Lipschitz continuous, we can still differentiate under the integral sign in (A.1).

$$G_i(y,\rho_0(y)) = \begin{cases} \int_{|w|<1} -\phi(w)\,dw, & i = d, \\ \int_{|w|<1} D_i\psi(y' - \frac{\rho_0(y)}{M}w')\phi(w)\,dw, & i < d, \end{cases}$$

$$G_{d+1}(y,\rho_0(y)) = \int_{|w|<1} -D_k\psi\Big(y' - \frac{\rho_0(y)}{M}w'\Big)\frac{w^k}{M}\phi(w)\,dw.$$

Substituting back to (A.2) and using (2.17): $|D\psi(y') - D\psi(z')| < 3\varepsilon_0$, we obtain

$$|D_i\rho(y) - D_i\rho(z)| \leq 2\cdot 3\varepsilon_0 + 2M\cdot\frac{3\varepsilon_0}{M} = 12\varepsilon_0.$$

The proof for (2.22) is similar. Differentiating both sides of $\rho_0(y) = G(y,\rho_0(y))$, we get

$$D_{ij}\rho_0(y) = G_{ij} + G_{i,d+1}D_j\rho_0 + G_{j,d+1}D_i\rho_0 + G_{d+1,d+1}D_i\rho_0 D_j\rho_0 + G_{d+1}D_{ij}\rho_0.$$

Here $G_{ij}(y,\tau) = \frac{\partial^2}{\partial y^i \partial y^j} G(y,\tau)$, and so on. Using $[G(z,\cdot)]_1 \le 1/2$, $[\rho_0]_1 \le M$, we have

$$|D_{ij}\rho_0(y)| \le 2(|G_{ij}| + M|G_{i,d+1}| + M|G_{j,d+1}| + M^2|G_{d+1,d+1}|).$$

Thus we only need to estimate D^2G. Here we will only compute G_{ij} with $i, j < d$, while the others are similar.

$$\begin{aligned} G_{ij}(y,\tau) &= D_i \int_{|w|<1} D_j\psi\Big(y' - \frac{\tau}{M}w'\Big)\phi(w)\,dw \\ &= D_i \int_{|w|<1} \frac{M}{\tau}\psi\Big(y' - \frac{\tau}{M}w'\Big)D_j\phi(w)\,dw \qquad \text{(A.3)} \\ &= \int_{|w|<1} \frac{M}{\tau}D_i\psi\Big(y' - \frac{\tau}{M}w'\Big)D_j\phi(w)\,dw. \qquad \text{(A.4)} \end{aligned}$$

Here in (A.3) we integrated by parts, noticing that

$$\frac{\partial\psi(y' - \frac{\tau}{M}w')}{\partial w^i} = -\frac{\tau}{M}D_i\psi\Big(y' - \frac{\tau}{M}w'\Big).$$

Since $\int_{|w|<1} D_j\phi(w)\,dw = 0$, we may rewrite (A.4) as follows:

$$\int_{|w|<1} \frac{M}{\tau}\Big(D_i\psi(y' - \frac{\tau}{M}w') - D_i\psi(y')\Big)D_j\phi(w)\,dw.$$

Now we can apply (2.17) to deduce

$$\begin{aligned} |G_{ij}(y,\rho_0(y))| &\le \int_{|w|<1} \frac{M}{|\rho_0(y)|}\Big|D_i\psi(y' - \frac{\rho_0(y)}{M}w') - D_i\psi(y')\Big||D_j\phi(w)|\,dw \\ &\le N\frac{\varepsilon_0}{|\rho_0(y)|}. \end{aligned}$$

The theorem is proved. ∎

A.2 Inequalities for regularized mollification

We give the proof of Lemma 2.4.3:

Proof. The proof of (2.27) follows from the Minkowski inequality,

$$\left\| \int f(x,\cdot)\,dx \right\|_p \le \int \|f(x,\cdot)\|_p\,dx.$$

From the definition of $\widetilde{g}$, we obtain

$$\begin{aligned} \|\widetilde{g}(\cdot)\|_{L_p(\Omega_R)} &= \left\| \int g(\cdot - \frac{\rho_0(\cdot)}{M_1} w)\phi(w)\,dw \right\|_{L_p(\Omega_R)} \\ &\le \int \|g(\cdot - \frac{\rho_0(\cdot)}{M_1} w)\|_{L_p(\Omega_R)}\phi(w)\,dw \\ &\le 2^{1/p}\|g\|_{L_p(\Omega_{2R})}\|\phi\|_{L_1}. \end{aligned} \tag{A.5}$$

Here (A.5) is due to $y - \frac{\rho_0(y)}{M_1} w \in \Omega_{2R}, \forall y \in \Omega_R, w \in B_1$, and the Jacobian of the change of variables $y \mapsto (y - \frac{\rho_0(y)}{M_1} w)$ has a positive lower bound. One could compute this Jacobian as follows,

$$\frac{\partial(y - \frac{\rho_0(y)}{M_1} w)}{\partial y} = I_d - \frac{D\rho_0}{M_1} w.$$

Thus we have $|I_d - \frac{D\rho_0(y)}{M_1} w| \ge \frac{1}{2}$ from our choice of M_1 in Definition 2.4.2.

The proof of (2.28) is similar, since

$$D\widetilde{g}(y) = \int \Big(I_d - \frac{D\rho_0}{M_1} w\Big) Dg(y - \frac{\rho_0(y)}{M_1} w)\phi(w)\,dw$$

and $D\rho_0$ is bounded. ∎

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