Theory and Applications of Harish-Chandra Integrals

by

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Vita

Colin Smith McSwiggen completed S.B. degrees in mathematics and physics at the Massachusetts Institute of Technology in 2011. During the 2009–2010 academic year he completed Part II of the Mathematics Tripos at Cambridge University. After his undergraduate studies, he attended the Royal College of Art and Imperial College London, graduating in 2013 with M.A. and M.Sc. degrees in design. During the following two years he worked in various roles in the technology, financial, and non-profit sectors. He entered the Ph.D. program at Brown University's Division of Applied Mathematics in the fall of 2015 and spent the 2018–2019 academic year as a Chateaubriand Fellow at Sorbonne University.

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The French educational philosopher Joseph Jacotot argued that to learn a language, it sufficed to study any one paragraph in exacting detail. By thoroughly investigating a single short text, he believed, one could master vocabulary, grammar, and eventually an entire literature — a principle he summarized in the slogan "Everything is in everything" (Jacotot, 1823). While few teachers today would recommend Jacotot's method as a way to achieve fluency in a language, this dissertation is some kind of evidence that his approach can work for learning mathematics. What follows grew out of a prolonged meditation on a few pages from Harish-Chandra's 1957 paper "Differential operators on a semisimple Lie algebra." I still have not mastered that text, but I am grateful to have had the opportunity to study it at length.

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Chapter One

Introduction

1.1 An integral over a group

This thesis is about *Harish-Chandra integrals*. These are functions of the form

$$\mathcal{H}(x,y) := \int_G e^{\langle \mathrm{Ad}_g x, y\rangle} dg, \tag{1.1.1}$$

where G is a compact Lie group, x and y lie in a Cartan subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$ or its complexification $\mathfrak{g}_{\mathbb{C}}$, dg is normalized Haar measure, and $\langle\cdot,\cdot\rangle$ is an Ad-invariant inner product. We introduce the general theory of Harish-Chandra integrals and develop several new applications to representation theory, random matrix theory, and algebraic combinatorics. Occasionally we will also study similar integrals where x and y take values in some other space carrying a G-action; in all cases we refer to these functions as *orbital integrals*, since they can be written as an integral over a G-orbit.

1.1.1 Why study Harish-Chandra integrals?

You are looking at a 200-page treatise about integrating an exponential function over a compact Lie group. Why read such a thing, let alone write it?

One answer is that these deceptively simple integrals offer an expansive vista over numerous problems of central importance in contemporary mathematics. Harish-Chandra first studied them in the 1950's with the goal of developing a theory of Fourier analysis on semisimple Lie algebras, but in the intervening decades they have far outgrown their original applications. They are now ubiquitous special functions that appear in various guises throughout representation theory, geometric analysis, random matrix theory, and physics. For this reason, they also provide unexpected links between seemingly disparate subjects: since we can study these same functions from many different perspectives, they can help us to translate techniques and ideas between different areas of mathematics, leading to applications in fields as diverse as enumerative geometry, stochastic analysis on Lie algebras, and high-dimensional statistics.

We explain a number of these connections below in Section 1.2. On a personal note, however, I hope this thesis will also convince the reader that Harish-Chandra integrals are intrinsically fascinating objects in their own right. There's way more going on in (1.1.1) than meets the eye.

1.1.2 An exact formula

A remarkable fact about the integral (1.1.1) is that it admits an exact expression as an exponential polynomial. The following formula, which holds when G is connected and semisimple, is due to Harish-Chandra (1957):

$$\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y)\int_G e^{\langle \mathrm{Ad}_g x, y\rangle} dg = \frac{[\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!]}{|W|}\sum_{w\in W}\epsilon(w)e^{\langle w(x), y\rangle}. \tag{1.1.2}$$

Here x and y lie in a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, W is the Weyl group, $\epsilon(w)$ is the sign of $w \in W$, $\Delta_{\mathfrak{g}}(x) := \prod_{\alpha\in\Phi^+}\langle \alpha, x\rangle$ is the product of the positive roots Φ^+ (which we identify with elements of $\mathfrak{g}$ via the inner product), and $[\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!]$ is a constant computed below in (2.1.3). There are many ways to interpret this formula; in Chapter 2, we illustrate several of these interpretations with five different proofs of (1.1.2).

The importance of such integrals for mathematical physics was first noted by

Itzykson and Zuber (1980), who independently discovered the formula for the case of an integral over the unitary group. The unitary integral is now known as the Harish-Chandra–Itzykson–Zuber (HCIZ) integral and has become an important and widely studied identity in quantum field theory, random matrix theory, and algebraic combinatorics. It is usually written

$$\int_{\mathrm{U}(N)} e^{\mathrm{tr}(AUBU^\dagger)} dU = \left(\prod_{p=1}^{N-1} p!\right) \frac{\det\left[e^{a_i b_j}\right]_{i,j=1}^{N}}{\Delta(A)\Delta(B)}, \tag{1.1.3}$$

where $\mathrm{U}(N)$ is the group of N-by-N unitary matrices, A and B are fixed N-by-N diagonal matrices with eigenvalues $a_1 < \ldots < a_N$ and $b_1 < \ldots < b_N$ respectively, and

$$\Delta(A) := \prod_{i<j}(a_j - a_i)$$

is the Vandermonde determinant.

Despite the existence of exact formulae such as (1.1.2) and (1.1.3), our understanding of these integrals is far from complete. In fact, for many questions of interest, the exact formulae are no help. For example, an important research program in random matrix theory has been to characterize the behavior of the HCIZ integral in various regimes as $N \to \infty$; see e.g. (Matytsin, 1994, Guionnet and Zeitouni, 2002, Guionnet, 2004a, Collins et al., 2009). However, as N grows large the determinant in the formula (1.1.3) produces factorially many terms of opposite signs, making it a poor starting point for this type of asymptotic analysis. As of this writing, the large-N behavior of more general integrals of the form (1.1.1) has still not been rigorously studied. There remain many other interesting unanswered questions about Harish-Chandra integrals, with a wide scope for new ideas and techniques.

1.2 Harish-Chandra integrals throughout mathematics and physics

Integrals of the form (1.1.1) arise all over contemporary mathematics: in harmonic analysis, they are a type of generalized Bessel function (Anker, 2015); in representation theory, they are intimately related to group characters (Kirillov, 2004); and in symplectic geometry, they are Laplace transforms of Duistermaat–Heckman measures of coadjoint orbits (Duistermaat and Heckman, 1982).

The HCIZ integral (1.1.3) is particularly significant for many reasons. It initially drew the interest of physicists because it appears in expressions for the partition functions of multi-matrix models in quantum field theory and string theory (Itzykson and Zuber, 1980, Di Francesco et al., 1995). In random matrix theory, it arises in the joint spectral densities of a number of matrix ensembles, including off-center Wigner matrices and Wishart matrices (Guionnet, 2004b, ch. 3). Since Wishart matrices model sample covariance matrices, such integrals are used by statisticians in noise models for covariance estimators for large data sets and are also used in signal processing to estimate the channel capacity of multiple-antenna transmission systems; see e.g. (Nordenvaad and Svensson, 2012, Ghaderipoor et al., 2012). In integrable systems the HCIZ integral is related to τ-functions for the 2D Toda lattice hierarchy (Zinn-Justin, 2002), while in combinatorics and enumerative geometry it is a generating function for the monotone double Hurwitz numbers (Goulden et al., 2014).

In this section, we elaborate on some of these connections to various areas of mathematics and physics.

1.2.1 Harmonic analysis

Harish-Chandra (1957) originally studied the function $\mathcal{H}(x,y)$ for the purpose of developing a theory of Fourier analysis on semisimple Lie algebras. In that context, the integral (1.1.1) plays the role of an *elementary spherical function*, which can be thought of as a multivariable generalization of a Bessel function. In particular, it appears in the integration kernel of an analogue of the radial Fourier transform on $\mathbb{R}^n$. If $f : \mathfrak{g} \to \mathbb{C}$ is an Ad-invariant Schwartz function, then it is determined by its restriction $\bar{f}$ to some Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, and we have the *radial Fourier transform*

$$\mathscr{R}[\bar{f}](\xi) := \int_{\mathfrak{t}} \bar{f}(x)\mathcal{H}(-i\xi, x)\Delta_{\mathfrak{g}}(x)^2 dx, \qquad \xi \in \mathfrak{t}, \tag{1.2.1}$$

with the inversion formula

$$\bar{f}(x) = c^{-2} \int_{\mathfrak{t}} \mathscr{R}[\bar{f}](\xi)\mathcal{H}(i\xi, x)\Delta_{\mathfrak{g}}(\xi)^2 d\xi, \tag{1.2.2}$$

where

$$c := \int_{\mathfrak{t}} e^{-\frac{|x|^2}{2}} \Delta_{\mathfrak{g}}(x)^2 dx. \tag{1.2.3}$$

In fact the radial Fourier transform (1.2.1) is a special case of a much more general type of integral transform, the *Dunkl transform*, whose kernel incorporates the generalized Bessel functions $\mathcal{B}_{k,\lambda}$ discussed below in Section 1.2.5. The Dunkl transform includes as special cases the radial Fourier transforms on all Riemannian symmetric spaces of Euclidean type. For further details, see the notes by Anker (2015).

Another way to think about elementary spherical functions on $\mathfrak{g}$ is as joint eigenfunctions of all W-invariant constant-coefficient differential operators on $\mathfrak{t}$. Lemma 2.3.3 below, which is an intermediate step in Harish-Chandra's proof of the formula (1.1.2), implies that the integral $\mathcal{H}(x,y)$ is such a joint eigenfunction. This is one possible starting point for relating Harish-Chandra integrals to integrable systems, as we discuss below in Section 1.2.5.

For an introduction to the classical Harish-Chandra theory in harmonic analysis and its applications to representation theory, see the book by Varadarajan (1977).

1.2.2 Representation theory

The characters of finite-dimensional irreducible representations of G can be expressed in terms of Harish-Chandra integrals. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ be half the sum of the positive roots, and define the function $\widehat{\Delta}_{\mathfrak{g}}$ on the maximal torus $T := \exp(\mathfrak{t})$ by

$$\widehat{\Delta}_{\mathfrak{g}}(e^x) := \prod_{\alpha\in\Phi^+}(e^{i\langle\alpha,x\rangle/2} - e^{-i\langle\alpha,x\rangle/2}), \qquad x \in \mathfrak{t}.$$

Using the inner product to identify $\mathfrak{t} \cong \mathfrak{t}^*$, we can identify weights of $\mathfrak{g}$ with elements of $\mathfrak{t}$. For a dominant integral weight λ, the Kirillov character formula for compact connected Lie groups (Kirillov, 1968, 2004) expresses the irreducible character χ_λ as:

$$\chi_\lambda(e^x) = \frac{\Delta_{\mathfrak{g}}(ix)}{\widehat{\Delta}_{\mathfrak{g}}(e^x)}\frac{\Delta_{\mathfrak{g}}(\lambda+\rho)}{\Delta_{\mathfrak{g}}(\rho)}\mathcal{H}(\lambda+\rho, ix), \tag{1.2.4}$$

for $x \in \mathfrak{t}$ with $\Delta_{\mathfrak{g}}(x) \neq 0$. We will make frequent use of the Kirillov character formula in later chapters. We discuss it further in Section 2.6.1.

1.2.3 Random matrix theory

The HCIZ integral and other analogous orbital integrals are ubiquitous in random matrix theory, because they appear in the joint spectral densities for ensembles of multiple coupled random matrices. Similarly, they arise when writing the heat kernel on a space of matrices in terms of the eigenvalues.

For example, consider the following *two-matrix model*. Let A and B be two random N-by-N Hermitian matrices with joint density

$$p(A,B) := \frac{1}{\mathcal{Z}_N} \exp\big[-\operatorname{tr} V(A) - \operatorname{tr} V(B) + \beta \operatorname{tr}(AB)\big], \tag{1.2.5}$$

where $V(x) := x^2/2 + \kappa x^4/4$, and β, κ are constants. The normalization is provided by the *partition function*

$$\mathcal{Z}_N := \iint_{\mathrm{Her}(N)^2} p(A,B)\, dA\, dB,$$

where $\mathrm{Her}(N)$ is the space of N-by-N Hermitian matrices equipped with the norm $|A|^2 = \operatorname{tr}(A^2)$. This type of interacting matrix model was originally studied by Itzykson and Zuber (1980) as a prototype for non-perturbative approaches to gauge theories, following the program of 't Hooft (1974).

Note that the measure $p(A,B)\, dA\, dB$ depends only on the eigenvalues $(a_1, \ldots, a_N)$ of A and $(b_1, \ldots, b_N)$ of B. We can integrate out the angular degrees of freedom in (1.2.5) to obtain the density $\tilde{p}(a_1, \ldots, a_N, b_1, \ldots, b_N)$ of this measure with respect to Lebesgue measure on $\mathbb{R}^N \times \mathbb{R}^N$. The resulting expression gives the joint spectral density in terms of the HCIZ integral:

$$\begin{aligned} &\tilde{p}(a_1, \ldots, a_N, b_1, \ldots, b_N) \\ &\qquad = \frac{1}{\mathcal{Z}_N} \frac{(2\pi)^{N(N-1)}}{\left(\prod_{j=1}^N j!\right)^2} \Delta(A)^2 \Delta(B)^2\, e^{-\operatorname{tr} V(A) - \operatorname{tr} V(B)} \int_{\mathrm{U}(N)} e^{\beta \operatorname{tr}(AUBU^\dagger)} dU, \end{aligned} \tag{1.2.6}$$

where here we may take $A = \operatorname{diag}(a_1, \ldots, a_N)$, $B = \operatorname{diag}(b_1, \ldots, b_N)$. Similarly, we can write the partition function as an integral over the eigenvalues:

$$\mathcal{Z}_N = \frac{(2\pi)^{N(N-1)}}{\left(\prod_{j=1}^N j!\right)^2} \int_{\mathbb{R}^{2N}} \Delta(A)^2 \Delta(B)^2 \, e^{-\operatorname{tr} V(A) - \operatorname{tr} V(B)} \int_{\mathrm{U}(N)} e^{\beta \operatorname{tr}(AUBU^\dagger)} dU \prod_{j=1}^N da_j \, db_j. \tag{1.2.7}$$

There are many other random matrix models whose spectral densities involve the HCIZ integral or its variants. Among these models, the *Wishart matrices* are important for applications to statistics, as they provide a model for sample covariance estimators. See Section 1.2.8 below, as well as (Pastur and Shcherbina, 2011, ch. 7) and (Guionnet, 2004b, sect. 3.2), for more information on Wishart matrices and statistical applications.

In Chapter 5, we study sums of two random matrices with prescribed eigenvalues, and we derive expressions for the joint spectral measures in terms of orbital integrals.

1.2.4 Symplectic geometry

The integral $\mathcal{H}(x, y)$ can also be interpreted in symplectic geometry as the Laplace transform of the Duistermaat–Heckman measure for the action of a maximal torus on a coadjoint orbit. For definitions and details of these constructions, see Section 2.5. Here we use the inner product to identify $\mathfrak{g} \cong \mathfrak{g}^*$, so that the adjoint orbit $\mathcal{O}_x$ of $x \in \mathfrak{t}$ is identified with the coadjoint orbit of the linear functional $\langle x, \cdot \rangle$, equipped with the Kostant–Kirillov–Souriau symplectic structure. Then the orthogonal projection $\phi : \mathfrak{g} \to \mathfrak{t}$ is a moment map for the action of the maximal torus $\exp(\mathfrak{t})$ on $\mathcal{O}_x$, and we have

$$\mathcal{H}(x, y) = \frac{1}{\operatorname{Vol}_\mu(\mathcal{O}_x)} \int_{\mathcal{O}_x} e^{\langle \phi(\beta), y \rangle} d\mu(\beta), \tag{1.2.8}$$

where μ is the Liouville measure on the coadjoint orbit and $\mathrm{Vol}_\mu(\mathcal{O}_x)$ is its Liouville volume.

1.2.5 Integrable systems

For $\alpha \in \Phi^+$, let s_α denote the reflection through the hyperplane

$$\{\ x \in \mathfrak{t} \mid \langle \alpha, x \rangle = 0\ \}.$$

A *multiplicity parameter* k is a complex-valued function on the roots of $\mathfrak{g}$ that is constant on Weyl orbits, i.e. $k_\alpha = k_{w\alpha}$, $w \in W$. Given such a multiplicity parameter, the *Dunkl Laplacian* L_k is a differential-difference operator acting on twice-differentiable functions $f : \mathfrak{t} \to \mathbb{C}$ by

$$L_k f(x) = \sum_{j=1}^{r} \partial_j^2 f(x) + \sum_{\alpha \in \Phi^+} \frac{2k_\alpha}{\langle \alpha, x \rangle} \partial_\alpha f(x) - \sum_{\alpha \in \Phi^+} \frac{k_\alpha |\alpha|^2}{\langle \alpha, x \rangle^2} \big[f(x) - f(s_\alpha x) \big], \quad (1.2.9)$$

where $\partial_\alpha f(x) := \frac{d}{dt} f(x + t\alpha)\big|_{t=0}$, and ∂_j, $j = 1, \ldots, r$ are derivatives along the coordinate directions with respect to some orthonormal basis of $\mathfrak{t}$.

We can consider the operator $-L_k$ as a Schrödinger operator for a quantum mechanical system of r interacting particles, the *quantum Calogero–Moser system* associated to the root system of $\mathfrak{g}$. In this interpretation, the values k_α of the multiplicity parameter determine the coupling constants for the interactions between the particles. This system is completely integrable, in the sense that the algebra of W-invariant differential operators commuting with L_k is isomorphic to a polynomial algebra of rank r. For generic multiplicity parameters k and any $\lambda \in \mathfrak{t}$, we can define the *generalized Bessel function* $\mathcal{B}_{k,\lambda}$, which is a distinguished W-invariant eigenfunction of L_k with eigenvalue $|\lambda|^2$, normalized so that $\mathcal{B}_{k,\lambda}(0) = 1$. The generalized Bessel functions play the role of energy eigenfunctions for the quantum Calogero–Moser system.

Harish-Chandra integrals are generalized Bessel functions for the multiplicity parameter with all $k_\alpha = 1$. In this case, L_k is the radial part of the Laplacian on $\mathfrak{g}$ (defined below in Section 2.2.3), and $\mathcal{B}_{k,\lambda}(x) = \mathcal{H}(\lambda, x)$. For details on the above constructions, we refer the reader to the review articles (Etingof, 2007, Anker, 2015, Opdam, 2001).

There are other known connections between Harish-Chandra integrals and integrable systems. Notably, Zinn-Justin (2002) related the large-N asymptotics of the HCIZ integral to the dispersionless 2D Toda lattice hierarchy. In Section 2.4.5 we discuss a different relationship between Harish-Chandra integrals and Calogero–Moser systems, distinct from the considerations described above, which arises in a semiclassical regime via a WKB-type ansatz.

1.2.6 Combinatorics and enumerative geometry

The HCIZ integral is a combinatorial generating function for the *genus g monotone double Hurwitz numbers* $\vec{H}_g(\alpha, \beta)$ (Goulden et al., 2014), which we now define. These numbers are labeled by a nonnegative integer g and two partitions α, β of another nonnegative integer d. They count certain walks on the Cayley graph of the symmetric group S_d as generated by transpositions.

A walk on this Cayley graph is identified with a sequence $\big((s_1\ t_1)\,, \ldots, (s_m\ t_m)\big)$ of transpositions $(s_i\ t_i)$, where $s_i, t_i \in \{1, \ldots, d\}$ and $s_i < t_i$. We say that the walk is *monotone* if the numbers t_i form a weakly increasing sequence. The number $\vec{H}_g(\alpha, \beta)$ is defined as the number of monotone walks on the Cayley graph of S_d that meet the following criteria:

1. The walk consists of $2g - 2 + \ell(\alpha) + \ell(\beta)$ steps, where $\ell(\cdot)$ indicates the length of a partition.

2. The walk begins in the conjugacy class labeled by the partition α and ends in the conjugacy class labeled by the partition β.

3. The walk's endpoints and steps together generate a transitive subgroup of S_d.

These numbers also have a geometric interpretation. If we remove the requirement that the walks be monotone, we instead get the ordinary double Hurwitz numbers $H_g(\alpha, \beta)$. By a classical result of Hurwitz (1891), $H_g(\alpha, \beta)/d!$ gives a weighted count of certain genus g branched covers of the Riemann sphere, with branching data determined by the partitions α and β. Accordingly, we can interpret $\vec{H}_g(\alpha, \beta)/d!$ as counting a subset of these branched covers satisfying the combinatorial constraint imposed by monotonicity of the corresponding walk on the Cayley graph.

Goulden et al. (2014) proved the following expansion, leading to an interpretation of the HCIZ integral as a generating function for the numbers $\vec{H}_g(\alpha, \beta)$:

$$\frac{1}{N^2} \log \int_{\mathrm{U}(N)} e^{zN \operatorname{tr}(AUBU^\dagger)} dU = \\ \sum_{d=1}^{N} \frac{z^d}{d!} \sum_{g \geq 0} \left(\frac{1}{N^2} \right)^g \sum_{\alpha, \beta \vdash d} \left(\frac{-1}{N} \right)^{\ell(\alpha)+\ell(\beta)} \prod_{i=1}^{\ell(\alpha)} \operatorname{tr}(A^{\alpha_i}) \prod_{i=1}^{\ell(\beta)} \operatorname{tr}(B^{\beta_i}) \vec{H}_g(\alpha, \beta) \\ + O(z^{N+1}), \quad (1.2.10)$$

where z is a complex parameter. Subsequently, Novak (2015) used the expansion

(1.2.10) to give a new proof of a theorem on the asymptotic distribution of vertical tiles in a uniform random lozenge tiling. It is an interesting open question whether there are analogous Hurwitz-theoretic expansions for other Harish-Chandra integrals.

In Chapters 5 and 6, we develop some further combinatorial applications of Harish-Chandra integrals, using them to derive expressions for the volumes of Berenstein–Zelevinsky polytopes and for the tensor product multiplicities of semisimple Lie algebras.

1.2.7 Theoretical physics

Orbital integrals such as the HCIZ integral were originally studied by physicists in the context of so-called *multi-matrix models* (Itzykson and Zuber, 1980), which are random matrix ensembles that can be thought of as gauge field theories in a simplified setting where spacetime consists of finitely many points. Each random matrix in the ensemble then represents the value of the gauge field at one point in spacetime. As we saw from the example of the two-matrix model in Section 1.2.3, orbital integrals appear in the partition functions of such models due to the coupling between the matrices.

In the *large-N limit* (as the size of the matrices goes to infinity), 't Hooft (1974) observed that gauge theories can be described in terms of Feynman diagrams that are represented by planar graphs. Following this idea, Brézin et al. (1978) used a model of a single random matrix to study the combinatorics of planar maps,[1] initiating a stream of research relating matrix integrals to map enumeration; see (Zvonkin, 1997) for a review. Itzykson and Zuber (1980) considered the case of two random matrices,

[1] A *map* is a graph embedded in a surface of minimal genus.

and it is in this context that they derived the HCIZ formula (1.1.3). The work of Matytsin (1994) on the large-N limit of the HCIZ integral was also motivated by gauge theory, specifically by a certain model of the strong nuclear force introduced by Kazakov and Migdal (1993).

A map can be regarded as a discretization of the underlying surface. This point of view provides a link between random matrix theory and discrete random geometry, leading to many applications in two-dimensional quantum gravity; see the review by Di Francesco et al. (1995). In particular, multi-matrix models are related to the combinatorics of *colored* maps, and they correspond to theories of a random function on a random surface ("statistical mechanics coupled to 2D gravity," in physics parlance). From this perspective, the two-matrix model (1.2.5) describes the Ising model on a certain family of random graphs. By studying the large-N limit of this model, Kazakov (1986) and Boulatov and Kazakov (1987) were able to derive a number of exact results for the Ising model on random *planar* graphs.

Aside from gauge theories and quantum gravity, the HCIZ integral has further applications in other areas of physics where random matrices play a role, perhaps most notably in quantum chaos and disordered systems (Kamenev and Mézard, 1999, Fyodorov and Khoruzhenko, 1999, Bauer et al., 2020).

1.2.8 Statistics and signal processing

Wishart matrices are a model of a statistical estimator for the covariance matrix of a random vector. Their introduction by Wishart (1928) is widely regarded as the historical genesis of random matrix theory. Multiple definitions exist in the literature, but here we consider matrices of the form

$$Y_{n,m} = X_{n,m} X_{n,m}^{\dagger}, \tag{1.2.11}$$

where $X_{n,m}$ is an n-by-m matrix whose columns are i.i.d. n-variate centered real or complex Gaussian vectors. The matrix $Y_{n,m}$ then has the distribution of a sample covariance matrix constructed from the columns of $X_{n,m}$.

The joint spectral density of a Wishart matrix can be expressed in terms of the HCIZ integral in the complex case, or in terms of an analogous integral over the orthogonal group in the real case; see (Guionnet, 2004b, sect. 3.2) for details. This has led to applications of the HCIZ formula in techniques for covariance estimation; see e.g. (Nordenvaad and Svensson, 2012, Nordenvaad, 2015, Bun et al., 2016).

The HCIZ formula has also been widely applied in signal processing for analyzing the performance of multiple-input multiple-output (MIMO) antenna arrays (Ghaderipoor et al., 2012, Alfano et al., 2014, Durisi et al., 2016, Zheng et al., 2019). In fact, many information-theoretic problems in the study of multiple-antenna transmission systems turn out to be closely related to covariance estimation: the channel capacity of a MIMO system is typically studied via the moment generating function for the mutual information between the transmitter and the receiver, which in turn is derived from the joint spectral density for a complex Wishart matrix.

1.3 Organization of the thesis

The remainder of this thesis is comprised of two chapters on the general theory of Harish-Chandra integrals, followed by three chapters on select applications in representation theory, random matrix theory, and algebraic combinatorics.

- Chapter 2 collects five different proofs of the formula (1.1.2), each illustrating a distinct perspective on the Harish-Chandra integral: Harish-Chandra's original proof via invariant differential operators, a proof by studying the heat equation on the Lie algebra $\mathfrak{g}$, a proof using a localization technique in symplectic geometry, and two representation-theoretic proofs. The text is largely based on excerpts from the papers (McSwiggen, 2018) and (McSwiggen, 2019b).

- Chapter 3, adapted from part of the paper (McSwiggen, 2018), presents detailed derivations of the specific realizations of the integral formula (1.1.2) for all compact classical groups, and discusses how (1.1.2) can be used to compute integrals over arbitrary compact Lie groups that may be neither semisimple nor connected.

- In Chapter 4, we prove a majorization inequality for Harish-Chandra integrals and use this to prove a generalization of an inequality for Schur polynomials due to Sra (2016). We also state and partially prove a conjecture on a further generalization to Heckman–Opdam polynomials. This chapter includes joint work with Jonathan Novak.

- Chapter 5 studies a probabilistic generalization of Horn's problem in linear algebra: given two orbits $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ of a linear representation of a compact Lie group, let $A \in \mathcal{O}_\alpha$, $B \in \mathcal{O}_\beta$ be independent and invariantly distributed random elements of the two orbits. The problem is to describe the probability distribution of the orbit of the sum $A + B$. We mainly consider two cases: coadjoint orbits, as well as the orbits of self-adjoint real, complex and quaternionic matrices under the conjugation actions of $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{USp}(n)$ respectively. The probability density can be expressed in terms of a function called the volume function, which can in turn be expressed in terms of Harish-Chandra integrals in the case of coadjoint orbits, or in terms of analogous orbital integrals for the various spaces of self-adjoint matrices. We relate the volume function to the symplectic or Riemannian geometry of the orbits, depending on the case, and we study its support and non-analyticities. In the coadjoint case, we also relate the volume function to representation-theoretic tensor product multiplicities for the algebra $\mathfrak{g}$ and show that it computes the volume of a family of convex polytopes introduced by Berenstein and Zelevinsky. Finally, we illustrate the above considerations with a detailed study of the volume function for the coadjoint orbits of SO(5). This chapter is adapted from the paper (Coquereaux et al., 2019a); the text and all results therein are joint work with Robert Coquereaux and Jean-Bernard Zuber.

- In Chapter 6, we develop several methods for computing the tensor product multiplicities of a semisimple Lie algebra from the volume function introduced in Chapter 5. Effectively, this leads to formulae for the tensor product multiplicities in terms of Harish-Chandra integrals. Our techniques build on box spline convolution and deconvolution formulae due to Dahmen and Micchelli (1985a) and De Concini et al. (2013). This chapter is adapted from the paper (McSwiggen, 2019a).

This thesis is not meant to be a comprehensive reference on Harish-Chandra integrals, and several important topics are omitted. Notably, we do not include a detailed discussion of large-N asymptotics, for which we refer the reader to (Matytsin, 1994, Guionnet and Zeitouni, 2002, Guionnet, 2004b, Collins et al., 2009). We also do not discuss correlation functions (Morozov, 1992, Shatashvili, 1993, Prats Ferrer et al., 2007) or integrals over non-compact groups (Rossman, 1978, Prato and Wu, 1994, Fyodorov and Strahov, 2002).

1.4 Notation

Our main notational conventions are as follows.

Throughout the text, G represents a compact real Lie group of rank r. When we make further assumptions on G, such as connectedness or semisimplicity, these will always be made explicit. We write $\mathfrak{g}$ for the Lie algebra of G, $\mathfrak{t} \subset \mathfrak{g}$ for a Cartan subalgebra, and $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ for their respective complexifications. For $x \in \mathfrak{g}$, e^x or $\exp(x) \in G$ will usually indicate the image of x under the Lie exponential map, which may differ from the matrix exponential in situations where x is represented as a matrix. We will often identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via an Ad-invariant inner product $\langle \cdot, \cdot \rangle$, by identifying $x \in \mathfrak{g}$ with the linear functional $\langle x, \cdot \rangle$. The Harish-Chandra integral will be written

$$\mathcal{H}(x, y) := \int_G e^{\langle \mathrm{Ad}_g x, y \rangle} dg, \qquad x, y \in \mathfrak{t}_{\mathbb{C}}.$$

We write Φ for the roots of $\mathfrak{g}$ and Φ^+ for the positive roots. Since we work more often with the real Lie algebra $\mathfrak{g}$ than with the complexificiation $\mathfrak{g}_{\mathbb{C}}$, it is convenient to regard the roots as real-valued linear functionals on $\mathfrak{t}$, so that they differ by a factor of i from the "complex roots" that are commonly used in the study of complex semisimple Lie algebras. Specifically, for our purposes, the roots are linear functionals $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ satisfying $[h, x] = i\alpha(h)x$ for all $h \in \mathfrak{t}_{\mathbb{C}}$ and some nonzero $x \in \mathfrak{g}_{\mathbb{C}}$. Since each $\alpha \in \Phi$ is real valued on $\mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}}$, we may regard it as an element of $\mathfrak{t}^*$, and then identify it with an element of $\mathfrak{t}$ via the inner product. The weight, coweight, root, and coroot lattices in $\mathfrak{t}^* \cong \mathfrak{t}$ will be written P, $P^\vee$, Q, and $Q^\vee$, respectively.

The discriminant of $\mathfrak{g}$ is the function $\Delta_{\mathfrak{g}}(x) := \prod_{\alpha \in \Phi^+} \langle \alpha, x \rangle$, $x \in \mathfrak{t}$. It is the infinitesimal version of the Weyl denominator $\widehat{\Delta}_{\mathfrak{g}}(e^x) := \prod_{\alpha\in\Phi^+}\left(e^{i\langle\alpha,x\rangle/2} - e^{-i\langle\alpha,x\rangle/2}\right)$. We regard $\widehat{\Delta}_{\mathfrak{g}}$ either as a function on the maximal torus $T := \exp(\mathfrak{t})$ or as a periodic function on $\mathfrak{t}$. We write W for the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$, which we frequently identify with some subgroup of G such that W acts on $\mathfrak{t}$ by the adjoint representation. The dominant Weyl chamber is the closed set

$$\mathcal{C}_+ := \{\ x \in \mathfrak{t} \mid \langle\alpha, x\rangle \geq 0,\ \forall\alpha\in\Phi^+\ \}.$$

The symbols λ, μ, ν will usually represent dominant integral weights of $\mathfrak{g}$. Then V_λ is the irreducible representation of $\mathfrak{g}$ (or G) with highest weight λ, and χ_λ is its character. We usually regard χ_λ as a function on T and write $\chi_\lambda(e^x)$, $x \in \mathfrak{t}$.

If A is a matrix with complex or quaternionic entries, we write $A^\dagger$ for its conjugate transpose and $\bar{A}$ for its untransposed conjugate.

For any manifold or vector space X, $C^k(X)$ denotes the space of functions on X with k continuous derivatives, and $C_c^\infty(X)$ denotes the space of smooth functions with compact support. For functions of two arguments, the notation $C_k^j(X \times Y)$ indicates the space of functions that are j times continuously differentiable in the first argument and k times continuously differentiable in the second argument.

A box ($\square$) indicates the end of a proof, while a triangle ($\triangle$) indicates the end of a remark, example or definition.

At some points in the text, we will adjust the above conventions for the sake of clarity. To prevent confusion and to keep the chapters relatively self-contained, key pieces of notation will often be reintroduced the first time they appear in a new chapter.

Chapter Two

Proofs of Harish-Chandra's formula

This chapter collects five proofs of the integral formula (1.1.2): Harish-Chandra's original proof, based on a simultaneous diagonalizability argument for invariant differential operators (Section 2.3); a proof by relating heat flow on a semisimple Lie algebra to heat flow on a Cartan subalgebra (Section 2.4); a proof via the Duistermaat–Heckman theorem in symplectic geometry (Section 2.5); and two representation-theoretic proofs, one using the Kirillov character formula (Section 2.6.1) and another using a character expansion for the heat kernel on G (Section 2.6.2). These five very different proofs illustrate the diverse interpretations of Harish-Chandra integrals in many areas of mathematics.

In Sections 2.1 and 2.2, we introduce notation, formally state the integral formula as a theorem, and review some facts about invariant differential operators that we will need for the proofs. We also situate the integral formula in the context of the paper (Harish-Chandra, 1957) in which it was originally published.

With the exception of Section 2.6.2, the text of this chapter is adapted from parts of the paper (McSwiggen, 2019b) and the expository article (McSwiggen, 2018). Of the five arguments presented here, only the heat equation proof is the original work of the author. To the author's knowledge, the proof via the Kirillov character formula had also not appeared prior to (McSwiggen, 2019b), however the argument would likely be obvious to an expert, so we make no claim to originality.

2.1 Preliminaries and statement of the theorem

Before explaining Harish-Chandra's formula, we first have to fix a substantial amount of notation. Let G be a compact, connected, semisimple, real Lie group of rank r

with Lie algebra $\mathfrak{g}$ and normalized Haar measure dg. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$, $\mathfrak{t}_{\mathbb{C}} := \mathfrak{t} \otimes \mathbb{C}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$.

Let W be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then W acts on $\mathfrak{t}$ as a group of linear transformations generated by reflections, and for each $w \in W$ we denote by $\epsilon(w)$ the sign of w, that is $\epsilon(w) = (-1)^{|w|}$ where $|w|$ is the number of reflections required to generate w.

Let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be an Ad-invariant inner product on $\mathfrak{g}$. The semisimplicity of $\mathfrak{g}$ and the compactness of G imply that there is a unique such inner product up to scalar multiples, so that

$$\langle x, y \rangle = -c \operatorname{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$$

for some $c > 0$. For the Lie algebras of the compact classical groups in their defining representations, $\langle x, y \rangle$ is a positive multiple of the *Hilbert–Schmidt inner product* $\operatorname{tr}(x^\dagger y)$, also known as the trace form. We extend $\langle \cdot, \cdot \rangle$ linearly to a complex-valued form on $\mathfrak{g}_{\mathbb{C}}$.

The inner product $\langle \cdot, \cdot \rangle$ induces an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ by $x \mapsto \langle x, \cdot \rangle$. Let $n = \dim \mathfrak{g}$. Given a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$, we can define coordinate functions in the usual way by setting $x_i = \langle e_i, x \rangle$, so that we may write $x = (x_1, \ldots, x_n)$. We can then identify $\mathfrak{g}$ with the real subspace of all $x \in \mathfrak{g}_{\mathbb{C}}$ that have strictly real coordinates.

For any vector space V over $\mathbb{C}$, the *symmetric algebra* $\operatorname{Sym}(V)$ is the commutative algebra of polynomials with indeterminates in V, defined by

$$\operatorname{Sym}(V) = T(V)/J$$

where $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ is the tensor algebra over V, and J is the ideal of $T(V)$ generated by all elements of the form $v \otimes w - w \otimes v$. Taking the quotient by J has the effect of forcing products in the tensor algebra to become commutative, so that

$$\mathrm{Sym}(V) \cong \mathbb{C}[v_1, \ldots, v_m]$$

where $\{v_1, \ldots, v_m\}$ is any basis of V.

The elements of the algebra $\mathrm{Sym}(\mathfrak{g})$ are *formal* polynomials over $\mathfrak{g}$. These are distinct from the actual polynomial *functions* on $\mathfrak{g}$, which are elements of $\mathrm{Sym}(\mathfrak{g}^*)$, but we identify these two algebras via the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ given by the inner product. We identify polynomial functions on $\mathfrak{g}_{\mathbb{C}}$ or $\mathfrak{t}_{\mathbb{C}}$ with their restrictions to $\mathfrak{g}$ or $\mathfrak{t}$ respectively.

Given $p(x) = \sum_\beta c_\beta x^\beta \in \mathrm{Sym}(\mathfrak{g})$ where β ranges over multi-indices, denote by $p(\partial)$ the differential operator

$$p(\partial) = \sum_\beta c_\beta \frac{\partial^{|\beta|}}{\partial x^\beta}.$$

We can extend $\langle \cdot, \cdot \rangle$ to a scalar product $[\![\cdot, \cdot]\!]$ on $\mathrm{Sym}(\mathfrak{g})$ by defining

$$[\![p, q]\!] = p(\partial) q(x) \big|_{x=0}. \tag{2.1.1}$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$, then an orthonormal basis for $\mathrm{Sym}(\mathfrak{g})$ with respect to $[\![\cdot, \cdot]\!]$ is given by monomials of the form $(\prod_i e_i^{\beta_i})/\sqrt{\beta!}$, where β is a multi-index and the multi-index factorial has the usual meaning $\beta! := \beta_1! \ldots \beta_n!$. This fact is a corollary of Lemma 2.1.2 below, which shows how to compute $[\![p, q]\!]$. Lemma 2.1.2 also shows that $[\![\cdot, \cdot]\!]$ is symmetric and nondegenerate, and that $[\![x, y]\!] = \langle x, y \rangle$ for $x, y \in \mathfrak{g}$.

Finally, let Φ^+ be a choice of positive roots of $\mathfrak{g}$. The *discriminant* of $\mathfrak{g}$ is the homogeneous polynomial $\Delta_{\mathfrak{g}} : \mathfrak{t} \to \mathbb{C}$ given by taking the product of the positive roots:

$$\Delta_{\mathfrak{g}}(x) := \prod_{\alpha \in \Phi^+} \langle \alpha, x \rangle. \tag{2.1.2}$$

The discriminant plays an important role in geometric analysis on $\mathfrak{g}$. One of its essential properties is that it is *skew* with respect to the action of W: for $w \in W$, $\Delta_{\mathfrak{g}}(w(x)) = \epsilon(w)\Delta_{\mathfrak{g}}(x)$. This follows from the fact that if α is a simple root, the reflection through the plane $\{\alpha = 0\}$ sends $\alpha \mapsto -\alpha$ and permutes the other positive roots.

With the above definitions, Harish-Chandra's formula states:

Theorem 2.1.1. *For all $x, y \in \mathfrak{t}_{\mathbb{C}}$,*

$$\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y) \int_G e^{\langle \mathrm{Ad}_g x, y\rangle} dg = \frac{[\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!]}{|W|} \sum_{w \in W} \epsilon(w) e^{\langle w(x), y\rangle}. \tag{1.1.2}$$

On the left-hand side of this equation, we have an integral over the Lie group G. On the right-hand side, we have a finite sum over the Weyl group W. Since G is compact, the action of W on $\mathfrak{t}_{\mathbb{C}}$ is represented by a finite subgroup of G acting by the adjoint representation, so that (1.1.2) has the interpretation that the integral on the left is equal to a normalized, alternating sum of the integrand's values at finitely many points. In other words, the integral is *localized* at the points of G that represent elements of W.

The following lemma shows how to evaluate $[\![p, q]\!]$ for two polynomials p, q.

Lemma 2.1.2. *Let*

$$p(x) = \sum_\beta p_\beta x^\beta \quad \textit{and} \quad q(x) = \sum_\beta q_\beta x^\beta$$

be two polynomial functions on $\mathfrak{g}$, where $(x_1, \ldots, x_n)$ are coordinates in an orthogonal basis and β runs over multi-indices. Then

$$p(\partial)q(x)\big|_{x=0} = \sum_\beta p_\beta q_\beta \beta!$$

Note that since only finitely many p_β and q_β are nonzero, the sum is finite.

Proof. Expanding out terms, we have

$$p(\partial)q(x) = \sum_{\alpha,\beta} p_\alpha q_\beta \frac{\partial^{|\alpha|} x^\beta}{\partial x^\alpha}.$$

If $|\alpha| \geq |\beta|$ and $\alpha \neq \beta$, then $\partial^{|\alpha|} x^\beta / \partial x^\alpha = 0$. If $|\alpha| < |\beta|$, then $\partial^{|\alpha|} x^\beta / \partial x^\alpha$ has positive degree and is killed by evaluating at $x = 0$, so that we are left only with terms where $\alpha = \beta$, and the sum becomes

$$p(\partial)q(x)\big|_{x=0} = \sum_\beta p_\beta q_\beta \frac{\partial^{|\beta|} x^\beta}{\partial x^\beta} = \sum_\beta p_\beta q_\beta \beta!$$

as desired. □

This lemma allows us to compute $[\![\Delta_\mathfrak{g}, \Delta_\mathfrak{g}]\!]$ from the coefficients of $\Delta_\mathfrak{g}$. Writing

$$\Delta_\mathfrak{g}(x) = \sum_{|\beta| = |\Phi^+|} \pi_\beta x^\beta$$

for some constants π_β, where $|\Phi^+|$ is the number of positive roots, we have

$$[\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!] = \left(\sum_{|\beta|=|\Phi^+|} \pi_\beta \frac{\partial^{|\beta|}}{\partial x^\beta} \right) \left(\sum_{|\beta|=|\Phi^+|} \pi_\beta x^\beta \right) = \sum_{|\beta|=|\Phi^+|} \pi_\beta^2 \beta! \tag{2.1.3}$$

2.2 Background and context of the theorem

To motivate the formula (1.1.2), it's helpful to understand what Harish-Chandra was trying to do more broadly in the paper (Harish-Chandra, 1957). Although he seems to have viewed this theorem as an auxiliary result, it nicely illustrates the central theme of the paper, which addresses the relationship between the algebras of differential operators on $\mathfrak{g}$ and on $\mathfrak{t}$. In particular, for functions $f \in C^\infty(\mathfrak{g})$ that are invariant under the adjoint action of G on $\mathfrak{g}$, Harish-Chandra's explicit goal is to express $\overline{p(\partial) f}$ in terms of $\bar{p}(\partial)$ and $\bar{f}$, where the bar indicates restriction to $\mathfrak{t}$. What he finds is that

$$\Delta_{\mathfrak{g}} \, \overline{p(\partial) f} = \bar{p}(\partial)(\Delta_{\mathfrak{g}} \bar{f}) \tag{2.2.1}$$

where $\Delta_{\mathfrak{g}}$ is the discriminant defined in (2.1.2). This result anticipates the modern theory of radial parts of differential operators, discussed below in Section 2.2.3, and appears in hindsight as a foretaste of ideas in spherical harmonic analysis that led to the development of Dunkl theory (Anker, 2015).

Many of the results in (Harish-Chandra, 1957) can be viewed as building on the Chevalley restriction theorem (Theorem 2.2.1 below), which says that the G-invariant polynomials on $\mathfrak{g}$ are isomorphic to the W-invariant polynomials on $\mathfrak{t}$, and that an isomorphism is given in the $\mathfrak{g}$-to-$\mathfrak{t}$ direction simply by restriction of functions. In effect, Harish-Chandra expands the scope of this theorem, showing that the isomorphism for invariant functions extends to a homomorphism from a particular algebra of G-invariant differential operators on $\mathfrak{g}$ to the algebra of all W-invariant differential operators on $\mathfrak{t}$. He then elaborates a number of consequences including the relation (2.2.1) and the integral formula (1.1.2).

In the rest of this section we will make these ideas concrete, starting with a formal statement of the Chevalley restriction theorem. The group G acts on functions $f : \mathfrak{g} \to \mathbb{C}$ by sending $f(x) \mapsto f(\mathrm{Ad}_{g^{-1}}x)$ for each $g \in G$. Likewise, the Weyl group W acts on functions $v : \mathfrak{t} \to \mathbb{C}$ by $v(x) \mapsto v(w^{-1}x)$ for $w \in W$. Let $I(\mathfrak{g}) \subset \mathrm{Sym}(\mathfrak{g})$ consist of polynomials on $\mathfrak{g}$ that are invariant under the adjoint action of G, and let $I(\mathfrak{t}) \subset \mathrm{Sym}(\mathfrak{t})$ consist of polynomials on $\mathfrak{t}$ that are invariant under the action of W. Then we have:

Theorem 2.2.1 (Chevalley restriction theorem)**.** *For all $p \in I(\mathfrak{g})$, the restriction $\bar{p}$ belongs to $I(\mathfrak{t})$. Moreover, $p \mapsto \bar{p}$ is an isomorphism of $I(\mathfrak{g})$ onto $I(\mathfrak{t})$.*

Example 2.2.2. In the case $G = \mathrm{U}(N)$, we have $\mathfrak{g} = \mathfrak{u}(N)$, the space of N-by-N anti-Hermitian matrices. Multiplying by i, we can identify $\mathfrak{u}(N)$ with the space of N-by-N *Hermitian* matrices. In this setting, the Chevalley restriction theorem tells us that if p is a polynomial in the entries of a Hermitian matrix M, then we have $p(M) = p(UMU^\dagger)$ for all $U \in \mathrm{U}(N)$ if and only if p can be written as a symmetric polynomial in the eigenvalues of M.

In order to explain how Harish-Chandra extends Theorem 2.2.1 to differential operators, we must introduce some more definitions and notation.

2.2.1 Generalities on differential operators

First we must clarify what exactly we mean by a "differential operator." For a finite-dimensional vector space V over $\mathbb{C}$ or $\mathbb{R}$, let $\partial\mathrm{Sym}(V) := \{p(\partial) \mid p \in \mathrm{Sym}(V)\}$, the algebra of constant-coefficient differential operators on V. The algebra $\mathcal{D}(V)$ generated by $\mathrm{Sym}(V) \cup \partial\mathrm{Sym}(V)$ is the algebra of *polynomial differential operators* on V, and these are the operators that we will primarily study. Here $p \in \mathrm{Sym}(V)$ acts by multiplication, $f \mapsto pf$. By applying the product rule, it is always possible to write any element of $\mathcal{D}(V)$ in the form $\sum_i p_i \cdot q_i(\partial)$, where p_i and q_i are polynomials.[1]

More generally, if $U \subseteq V$ is a nonempty open set, then we can define a *differential operator on* U to be any operator acting on $C^\infty(U)$ that has the form $\sum_{i=1}^n a_i \cdot q_i(\partial)$ where $a_i \in C^\infty(U)$ and $q_i \in \mathrm{Sym}(V)$. For example, in this case the coefficients a_i could blow up at the boundary of U or could be rational functions with no poles in the interior of U. When we use the term "differential operator" with no further qualification, we will mean an operator of this form. Such operators form an algebra $\mathfrak{D}(U)$, and there is a natural inclusion of $\mathcal{D}(V)$ as a subalgebra of $\mathfrak{D}(U)$ given by $p_i \cdot q_i(\partial) \mapsto p_i|_U \cdot q_i(\partial)$.

Let $W \subset V$ be a subspace and let $(w_1, \ldots, w_k)$ be linear coordinates on W. Then we can identify $\mathcal{D}(W)$ with the subalgebra of $\mathcal{D}(V)$ generated by 1, $\{w_i\}_{i=1}^k$, and $\{w_i(\partial)\}_{i=1}^k$. Thus each $D \in \mathcal{D}(W)$ can be thought of as a differential operator either on W or on V. We will ignore this distinction with impunity, since for any $f \in C^\infty(V)$ we have $(Df)|_W = D(f|_W)$. Observe also that if V is a vector space over $\mathbb{R}$, then there is a natural correspondence between differential operators on V

[1] Here the dot "$\cdot$" indicates multiplication but also emphasizes that p_i acts as a multiplication operator *following* differentiation by $q_i(\partial)$, as opposed to differentiation by $(p_iq_i)(\partial) \in \partial\mathrm{Sym}(V)$. We will sometimes also use the notation $p(\partial) \circ q$ to indicate the composition of operators, i.e. $(p(\partial) \circ q)f = p(\partial)(qf)$.

and differential operators on the complexification $V \otimes \mathbb{C}$, obtained by identifying

$$\frac{\partial}{\partial x_j} \quad \longleftrightarrow \quad \frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j} \tag{2.2.2}$$

where x_j, y_j are real coordinates on V and $z_j = x_j + iy_j$ is a complex coordinate on $V \otimes \mathbb{C}$.

2.2.2 Group actions on differential operators

We now return to the case where the underlying vector space is one of the Lie algebras $\mathfrak{g}$ or $\mathfrak{t}$. There is a natural way in which G acts on $\mathcal{D}(\mathfrak{g})$ and W acts on $\mathcal{D}(\mathfrak{t})$, extending the respective actions on $\mathrm{Sym}(\mathfrak{g})$ and $\mathrm{Sym}(\mathfrak{t})$. We define a G-action on $\partial\mathrm{Sym}(\mathfrak{g})$ by stipulating that differential operators transform in the opposite way to functions: under the map $x \mapsto \mathrm{Ad}_{g^{-1}}x$, we have

$$\frac{\partial}{\partial x} \;\mapsto\; \frac{\partial}{\partial(\mathrm{Ad}_{g^{-1}}x)} = (\mathrm{Ad}_g x)(\partial),$$

so that $p(\partial) \mapsto (p \circ \mathrm{Ad}_g)(\partial)$, where the circle "$\circ$" indicates composition of functions rather than operators. This guarantees that the actions on $\mathrm{Sym}(\mathfrak{g})$ and on $\partial\mathrm{Sym}(\mathfrak{g})$ are compatible. Since every element of $\mathcal{D}(\mathfrak{g})$ can be written as $\sum_i p_i \cdot q_i(\partial)$, the action of G on $\mathcal{D}(\mathfrak{g})$ is fully determined by the actions on $\mathrm{Sym}(\mathfrak{g})$ and $\partial\mathrm{Sym}(\mathfrak{g})$: an element $g \in G$ sends

$$p \cdot q(\partial) \mapsto (p \circ \mathrm{Ad}_{g^{-1}}) \cdot (q \circ \mathrm{Ad}_g)(\partial)\,,$$

and this action extends to all of $\mathcal{D}(\mathfrak{g})$ by linearity. The action of W on $\mathcal{D}(\mathfrak{t})$ is defined analogously. Note that the algebra homomorphism $\mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g})$ thus induced by any $g \in G$ is inverted by g^{-1}, so that for each $g \in G$ or $w \in W$ we obtain an automorphism of $\mathcal{D}(\mathfrak{g})$ or $\mathcal{D}(\mathfrak{t})$ respectively.

In what follows, we will refer to several algebras of *invariant* differential operators. Namely, let $\mathcal{I}'$ denote those elements of $\mathcal{D}(\mathfrak{g})$ that are invariant under the action of G, and let $\mathcal{I}(\mathfrak{g})$ be the subalgebra of $\mathcal{I}'$ generated by $I(\mathfrak{g}) \cup \partial I(\mathfrak{g})$, where $\partial I(\mathfrak{g}) := \{p(\partial) \mid p \in I(\mathfrak{g})\}$. Let $\mathcal{I}(\mathfrak{t})$ denote those elements of $\mathcal{D}(\mathfrak{t})$ that are invariant under the action of W. Note that by the correspondence (2.2.2), we can also consider these as algebras of invariant differential operators on the complexifications $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$.

We conclude this subsection by showing the following important property of the discriminant $\Delta_{\mathfrak{g}}$ (Harish-Chandra, 1957, Corollary to Lemma 10).

Proposition 2.2.3. *$D\Delta_{\mathfrak{g}} = 0$ for all $D \in \mathcal{I}(\mathfrak{t})$ that annihilate the constants.*

Proof. Since D is W-invariant and $\Delta_{\mathfrak{g}}$ is skew, $D\Delta_{\mathfrak{g}}$ must be skew. For each reflection $w_\alpha \in W$ we thus have $w_\alpha(D\Delta_{\mathfrak{g}}) = -D\Delta_{\mathfrak{g}}$, which implies that $D\Delta_{\mathfrak{g}}$ vanishes on the hyperplane $\{\alpha = 0\}$, so that α divides $D\Delta_{\mathfrak{g}}$. But this implies $\Delta_{\mathfrak{g}}$ divides $D\Delta_{\mathfrak{g}}$ since the positive roots are relatively prime, and $D\Delta_{\mathfrak{g}}$ is strictly lower degree than $\Delta_{\mathfrak{g}}$, so we must have $D\Delta_{\mathfrak{g}} = 0$. $\square$

We say that $\Delta_{\mathfrak{g}}$ is *W-harmonic*. In fact, the space of all W-harmonic polynomials on $\mathfrak{t}$ is exactly the linear span of the partial derivatives of $\Delta_{\mathfrak{g}}$ (Helgason, 1984, ch. 3, Theorem 3.6). In particular, since the Laplacian on $\mathfrak{t}$ is W-invariant, $\Delta_{\mathfrak{g}}$ is harmonic in the traditional sense.

2.2.3 Radial part of a differential operator

There is one final idea that we need to introduce before discussing Harish-Chandra's homomorphism of invariant differential operators. This is the notion of the *radial part* of a differential operator, which was fully developed by Helgason in the 1960s and 1970s[2] but which already plays an important role in (Harish-Chandra, 1957). We say that a submanifold $M \subset \mathfrak{g}$ is *transverse* to the adjoint orbits in $\mathfrak{g}$ if for each $p \in M$ we have a decomposition of tangent spaces

$$T_p\mathfrak{g} = T_p\mathcal{O}_p \oplus T_pM, \tag{2.2.3}$$

where $\mathcal{O}_p = \{\mathrm{Ad}_g p \mid g \in G\}$ is the adjoint orbit of p. We state without proof the following theorem, which is a special case of (Helgason, 1984, ch. 2, Theorem 3.6).

Theorem 2.2.4. *Let $M \subset \mathfrak{g}$ be a submanifold of $\mathfrak{g}$ that is transverse to the adjoint orbits. Let D be a differential operator on $\mathfrak{g}$. Then there exists a unique differential operator $\gamma(D)$ on M such that, for each function $f \in C^\infty(\mathfrak{g})$ that is locally* Ad-*invariant in the sense that $f(\mathrm{Ad}_g x) = f(x)$ for all g in some neighborhood of id_G,*

$$\overline{(Df)} = \gamma(D)\bar{f},$$

the bar indicating restriction to M.

The differential operator $\gamma(D)$ is called the *radial part of D with transversal manifold M*. The sets

$$\mathfrak{t}' := \{x \in \mathfrak{t} \mid \Delta_{\mathfrak{g}}(x) \neq 0\}, \qquad \mathfrak{t}'_{\mathbb{C}} := \{x \in \mathfrak{t}_{\mathbb{C}} \mid \Delta_{\mathfrak{g}}(x) \neq 0\}$$

[2] See Helgason's book (Helgason, 1984, ch. 2, sect. 3) for a detailed reference on radial parts and other geometric operations on differential operators in a general setting.

are called the *regular elements* of $\mathfrak{t}$ and $\mathfrak{t}_\mathbb{C}$ respectively. As a submanifold of $\mathfrak{t}$, $\mathfrak{t}'$ is both dense in $\mathfrak{t}$ and transverse to the adjoint orbits. To see that transversality holds, consider the root space decomposition of $\mathfrak{g}_\mathbb{C}$,

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha,$$

where α runs over the roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{t}_\mathbb{C}$. Under the usual identification $T_x\mathfrak{g} \cong \mathfrak{g}$, at $x \in \mathfrak{t}'$ we have

$$T_x\mathfrak{t}' \cong \mathfrak{t}, \quad T_x\mathcal{O}_x \cong [\mathfrak{g}, x] = \mathfrak{g} \cap \bigoplus_\alpha \mathfrak{g}_\alpha,$$

which gives the transversality property. Thus for each $D \in \mathfrak{D}(\mathfrak{g})$ we have a well-defined operator $\gamma(D) \in \mathfrak{D}(\mathfrak{t}')$, the radial part of D with transversal manifold $\mathfrak{t}'$. Moreover, we have the following fact:

Lemma 2.2.5. *The map $D \mapsto \gamma(D)$ is a homomorphism from $\mathcal{I}'$ to $\mathfrak{D}(\mathfrak{t}')$.*

Proof. We follow the proof of (Harish-Chandra, 1957, Lemma 7). We first note that γ is clearly linear from its definition. Let $x \in \mathfrak{t}'$ and let $T \subset G$ be the maximal torus in G with Lie algebra $\mathfrak{t}$. Let $g \mapsto gT$ denote the quotient map $G \to G/T$. Let $U \subset \mathfrak{t}'$, $V \subset G$ be open connected neighborhoods of x and id_G respectively, and let VT be the image of V under the quotient map. Define the function $\phi : VT \times U \to \mathfrak{g}$ by $\phi(gT, x) = \mathrm{Ad}_g x$. Then ϕ is a submersion, and since $\dim(VT \times U) = \dim \mathfrak{g}$, $N := \phi(VT \times U)$ is an open submanifold of $\mathfrak{g}$. If V and U are taken to be sufficiently small, then ϕ is bijective and defines an analytic isomorphism of $VT \times U$ onto N. For $\psi \in C^\infty(U)$, define $f_\psi \in C^\infty(N)$ by $f_\psi(\phi(gT, x)) = \psi(x)$. Then f_ψ is locally Ad-invariant. Let $D_1, D_2 \in \mathcal{I}'$. We have

$$\overline{D_1 D_2 f_\psi} = \gamma(D_1 D_2)\bar{f}_\psi = \gamma(D_1 D_2)\psi.$$

On the other hand, since D_2 is Ad-invariant, $D_2 f_\psi$ must be locally Ad-invariant, so

$$\overline{D_1 D_2 f_\psi} = \gamma(D_1)\overline{(D_2 f_\psi)},$$

and $\overline{(D_2 f_\psi)} = \gamma(D_2)\bar{f}_\psi = \gamma(D_2)\psi$, so that

$$\gamma(D_1 D_2)\psi = \gamma(D_1)\gamma(D_2)\psi.$$

Since $\psi \in C^\infty(U)$ was arbitrary, γ is a homomorphism. □

2.2.4 The δ homomorphism

We can now state the first major theorem proved in (Harish-Chandra, 1957), which Harish-Chandra calls "the central result of this paper" and which lies in the background of most of its other results. The theorem gives a meaningful sense to the idea of "restriction to $\mathfrak{t}$" for invariant differential operators on $\mathfrak{g}$, rather than merely invariant polynomials. Further, it relates the restrictions of these operators to their radial parts. Indeed, Harish-Chandra finds that Chevalley's isomorphism $I(\mathfrak{g}) \to I(\mathfrak{t})$ extends uniquely to a homomorphism $\mathcal{I}(\mathfrak{g}) \to \mathcal{I}(\mathfrak{t})$:

Theorem 2.2.6. *There exists a unique homomorphism $\delta : \mathcal{I}(\mathfrak{g}) \to \mathcal{I}(\mathfrak{t})$ such that $\delta(p) = \bar{p}$ and $\delta(p(\partial)) = \bar{p}(\partial)$ for all $p \in I(\mathfrak{g})$. Moreover, on $\mathfrak{t}'$, we have*

$$\gamma(D) = \Delta_{\mathfrak{g}}^{-1}\delta(D) \circ \Delta_{\mathfrak{g}} \tag{2.2.4}$$

for all $D \in \mathcal{I}(\mathfrak{g})$, where the circle indicates composition of operators.

The bulk of Section 3 of (Harish-Chandra, 1957) is devoted to a concrete construction of the homomorphism δ and to showing the relation (2.2.4) between δ and the radial part map. The details of Harish-Chandra's construction are beyond the scope of this discussion, since we don't actually need the full power of the δ homomorphism to prove the integral formula. Instead, it will suffice to understand the relationship between $\gamma(p(\partial))$ and $\bar{p}(\partial)$ for $p \in I(\mathfrak{g})$, as described in the following theorem (Harish-Chandra, 1957, Lemma 8).

Theorem 2.2.7. *For $p \in I(\mathfrak{g})$, $\gamma(p(\partial)) = \Delta_{\mathfrak{g}}^{-1}\bar{p}(\partial) \circ \Delta_{\mathfrak{g}}$.*

We will sketch the proof, following the argument of (Helgason, 1984, ch. 2, Theorem 5.33). Let $\omega(x) := \langle x, x\rangle$, the *quadratic Casimir polynomial* on $\mathfrak{g}$. Then the Laplacian on $\mathfrak{g}$ is $\omega(\partial)$. One first shows by a direct calculation[3] using the root space decomposition that $\gamma(\omega(\partial)) = \Delta_{\mathfrak{g}}^{-1}\bar{\omega}(\partial)\circ\Delta_{\mathfrak{g}}$. The main idea of the proof is then to extend this result from ω to all $p \in I(\mathfrak{g})$ by using a trick of taking commutators with the Laplacian, which we now show in detail.

For differential operators D_1 and D_2, we write $\{D_1, D_2\} := D_1 \circ D_2 - D_2 \circ D_1$ for their commutator. Consider the derivations

$$\mu : D \mapsto \frac{1}{2}\{\omega(\partial), D\}$$

on $\mathfrak{D}(\mathfrak{g})$ and

$$\bar{\mu} : d \mapsto \frac{1}{2}\{\gamma(\omega(\partial)), d\}$$

on $\mathfrak{D}(\mathfrak{t}')$. Then μ has the following property (Helgason, 1984, ch. 2, Lemma 5.34).

Lemma 2.2.8. *If p is a homogeneous polynomial on $\mathfrak{g}$ of degree m, then*

$$\mu^m(p) = m!\, p(\partial).$$

[3] See (Helgason, 1984, ch. 3, Proposition 3.14) for details.

Proof. The proof proceeds by induction on m. The base case $m = 1$ follows from direct calculation. Let $p = q_1 \dots q_m$, where each q_i is an arbitrary linear function, and make the inductive hypothesis that

$$\mu^{m-1}(q_1 \dots q_{m-1}) = (m-1)!\,(q_1 \dots q_{m-1})(\partial).$$

Observing that $\mu^2(q_i) = \mu(q_i(\partial)) = 0$, by the Leibniz rule for derivations we have

$$\mu^m(q_1 \dots q_{m-1} q_m) = \mu^m(q_1 \dots q_{m-1}) \circ q_m + m\mu^{m-1}(q_1 \dots q_{m-1}) \circ \mu(q_m).$$

Applying the inductive hypothesis on the right-hand side, we find that the first term vanishes and the second term is exactly $m!\,p(\partial)$. $\square$

We will also need the following commutator identity, which holds in any associative algebra.

Proposition 2.2.9. *Let A be an associative algebra. For $a \in A$, define the derivation $d_a : A \to A$ by $d_a(b) = \frac{1}{2}(ab - ba)$. If $c \in A$ is invertible and commutes with b, then*

$$d^k_{c^{-1}ac}(b) = c^{-1} d^k_a(b) c.$$

Proof. This follows from the observation that

$$d^k_a(b) = 2^{-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j a^{k-j} b a^j,$$

which is shown by an easy induction on k. $\square$

Now we can complete the proof of Theorem 2.2.7.

Proof of Theorem 2.2.7. It suffices to assume that p is homogeneous of degree m, because γ is linear. Since γ is a homomorphism on $\mathcal{I}(\mathfrak{g})$, for $D \in \mathcal{I}(\mathfrak{g})$ we have

$$\gamma(\mu(D)) = \bar{\mu}(\gamma(D)),$$

and thus by Lemma 2.2.8 we have

$$m!\,\gamma(p(\partial)) = \gamma(\mu^m(p)) = \bar{\mu}^m(\gamma(p)) = \bar{\mu}^m(\bar{p}).$$

Finally we apply Proposition 2.2.9 with $A = \mathfrak{D}(\mathfrak{t}')$, $a = \bar{\omega}(\partial)$, $b = \bar{p}$, and $c = \Delta_{\mathfrak{g}}$, observing that $c^{-1}ac = \Delta_{\mathfrak{g}}^{-1}\bar{\omega}(\partial) \circ \Delta_{\mathfrak{g}} = \gamma(\omega(\partial))$. This gives

$$m!\,\gamma(p(\partial)) = \bar{\mu}^m(\bar{p}) = (d_{c^{-1}ac})^m(\bar{p}) = \Delta_{\mathfrak{g}}^{-1} d_a^m(\bar{p}) \circ \Delta_{\mathfrak{g}} = m!\,\Delta_{\mathfrak{g}}^{-1}\bar{p}(\partial) \circ \Delta_{\mathfrak{g}},$$

which completes the proof. $\square$

2.3 Harish-Chandra's original proof

We turn now to the first proof of the integral formula (1.1.2). While I have reorganized the presentation and simplified some steps with particular help from the invaluable reference by Helgason (1984), the arguments in this section are quite close to Harish-Chandra's originals, with some additional explanation and a few modifications to notation and terminology to bring them more in line with contemporary usage.

To recap definitions and assumptions, here G is a compact, connected, semisimple real Lie group of rank r with Lie algebra $\mathfrak{g}$ and normalized Haar measure dg, $\mathfrak{t} \subset \mathfrak{g}$

is a Cartan subalgebra, W is the corresponding Weyl group, $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ are the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$ respectively, and $\langle\cdot,\cdot\rangle$ is an Ad-invariant inner product on $\mathfrak{g}$.

The proof outline goes as follows. Before proving Theorem 2.1.1 itself, we need a few preliminary lemmas. The most important of these identifies a space of analytic functions on which all operators in $\partial I(\mathfrak{t})$ are simultaneously diagonalizable. After these lemmas, the core argument of the proof proceeds in three steps:

1. Define a function $\phi_f(x) := \Delta_{\mathfrak{g}}(x) \int_G e^{\langle \mathrm{Ad}_g x, y\rangle} dg$ on $\mathfrak{t}_{\mathbb{C}}$. With the assumptions that $x \in \mathfrak{t}$ and $\Delta_{\mathfrak{g}}(y) \neq 0$, show that ϕ_f is a joint eigenfunction for all operators $q(\partial) \in \partial I(\mathfrak{t})$, and use the simultaneous diagonalization lemma to write down an explicit formula for ϕ_f containing a number of unknown constants indexed by the Weyl group W.

2. Use the same explicit formula to write down expressions for $\phi_f(w(x))$, $w \in W$, and average these over W to eliminate all but a single unknown constant.

3. Determine the value of the remaining constant by computing $\Delta_{\mathfrak{g}}(\partial)\phi_f(x)|_{x=0}$ in two different ways, and setting the resulting expressions equal to each other. This gives the integral formula for $x \in \mathfrak{t}$ and $\Delta_{\mathfrak{g}}(y) \neq 0$, and we then use the analyticity of ϕ_f to extend the result to all $x, y \in \mathfrak{t}_{\mathbb{C}}$.

The first lemma that we need relates the algebraic structures of $\mathrm{Sym}(\mathfrak{t})$ and $I(\mathfrak{t})$.

Lemma 2.3.1. *There are homogeneous elements* $v_1, \ldots, v_{|W|} \in \mathrm{Sym}(\mathfrak{t})$ *such that every* $v \in \mathrm{Sym}(\mathfrak{t})$ *can be written uniquely in the form* $v = \sum_{i=1}^{|W|} u_i v_i$ *where each* $u_i \in I(\mathfrak{t})$.

In other words, $\mathrm{Sym}(\mathfrak{t})$ is a free module of rank $|W|$ over $I(\mathfrak{t})$.

Proof. Let J_+ be the ideal in $I(\mathfrak{t})$ generated by elements of positive degree. Since $I(\mathfrak{t})$ is a subalgebra of $\mathrm{Sym}(\mathfrak{t})$, we can consider the ideal $J_+\mathrm{Sym}(\mathfrak{t})$ generated by J_+ in the larger algebra $\mathrm{Sym}(\mathfrak{t})$. A theorem by Chevalley (1955) shows that the quotient $\mathrm{Sym}(\mathfrak{t})/J_+\mathrm{Sym}(\mathfrak{t})$ is a complex vector space of dimension $|W|$, so we can choose $v_1, \dots, v_{|W|} \in \mathrm{Sym}(\mathfrak{t})$ such that

$$\mathrm{Sym}(\mathfrak{t}) = \mathrm{span}\{v_1, \dots, v_{|W|}\} + J_+\mathrm{Sym}(\mathfrak{t}).$$

Then an easy induction shows that in fact for any $m \geq 1$,

$$\mathrm{Sym}(\mathfrak{t}) = \mathrm{span}\{v_1, \dots, v_{|W|}\} \cdot I(\mathfrak{t}) + J_+^m\mathrm{Sym}(\mathfrak{t}).$$

If $p \in I(\mathfrak{t})$ and $(p)_d$ is its degree d homogeneous component, then $(p)_d \in I(\mathfrak{t})$ as well, so that we may take the elements v_i to be homogeneous. Moreover, since $\mathrm{Sym}(\mathfrak{t})$ contains no zero-divisors we can consider its field of fractions $\mathrm{Frac}(\mathrm{Sym}(\mathfrak{t}))$, and this is a field extension of degree $|W|$ over $\mathrm{Frac}(I(\mathfrak{t}))$.

Now let $v \in \mathrm{Sym}(\mathfrak{t})$ be homogeneous of degree d. We will show that v can be written uniquely in the form stated in the lemma. Let $d_i = \deg v_i$ and choose $m > d$ and $u'_1, \dots, u'_{|W|} \in I(\mathfrak{t})$ such that $v - \sum_{i=1}^{|W|} u'_i v_i \in J_+^m\mathrm{Sym}(\mathfrak{t})$. Let $u_i = (u'_i)_{d-d_i}$. Then, since

$$\left(v - \sum_{i=1}^{|W|} u'_i v_i\right)_d = 0$$

by construction and $v = (v)_d$ is homogeneous, we must have $v - \sum_{i=1}^{|W|} u_i v_i = 0$. Thus we can conclude that

$$\mathrm{Sym}(\mathfrak{t}) = \mathrm{span}\{v_1, \dots, v_{|W|}\} \cdot I(\mathfrak{t}),$$

whereby it follows that $\mathrm{Frac}(\mathrm{Sym}(\mathfrak{t}))$ is spanned over $\mathrm{Frac}(I(\mathfrak{t}))$ by the elements v_i.

But since Frac(Sym($\mathfrak{t}$)) is a degree $|W|$ extension of Frac($I(\mathfrak{t})$), the elements v_i must therefore be linearly independent, showing that the decomposition $v = \sum_{i=1}^{|W|} u_i v_i$ is unique. $\square$

The purely algebraic statement of Lemma 2.3.1 will be our main tool in proving the next lemma, which is the aforementioned simultaneous diagonalization result for $\partial I(\mathfrak{t})$. We pose an infinite system of eigenvalue problems for *each point* in $\mathfrak{t}'$, identify a family of analytic functions that solve all of them simultaneously, and then show that this family of solutions is exhaustive up to the assumption of analyticity. This result has an interesting physical interpretation in the language of quantum integrable systems, where operators in $\partial I(\mathfrak{t})$ can be viewed as quantum Hamiltonians; see (Etingof, 2007, chs. 4–5) for details on this topic.

Lemma 2.3.2. *Let U be a nonempty connected open subset of $\mathfrak{t}_{\mathbb{C}}$. Let $x_0 \in \mathfrak{t}_{\mathbb{C}}$ such that $\Delta_{\mathfrak{g}}(x_0) \neq 0$. Suppose ϕ is an analytic function on U satisfying the system of differential equations*

$$q(\partial)\phi = q(x_0)\phi, \quad \forall\, q \in I(\mathfrak{t}). \tag{2.3.1}$$

Then there exist constants $c_w \in \mathbb{C}$, $w \in W$, such that for all $x \in U$,

$$\phi(x) = \sum_{w \in W} c_w e^{\langle x, w(x_0)\rangle}.$$

Moreover, for any such ϕ, the constants c_w are uniquely determined.

In other words, the functions $e^{\langle x, w(x_0)\rangle}$ for $w \in W$ form a basis of the complex vector space of analytic solutions to the linear system (2.3.1).

Proof. First we note that $\Delta_{\mathfrak{g}}(x_0) \neq 0$ implies that the points $w(x_0)$, $w \in W$ are all distinct, so that the analytic functions

$$\phi_w(x) = e^{\langle x, w(x_0) \rangle}, \quad w \in W$$

are linearly independent on U.[4]

If we identify a point $y \in \mathfrak{t}_{\mathbb{C}}$ with the linear functional $\langle y, \cdot \rangle$ on $\mathfrak{t}$, then $y(\partial)\phi_w(x) = \langle y, w(x_0) \rangle \phi_w(x)$. Writing any $q \in \mathrm{Sym}(\mathfrak{t})$ in terms of such linear functionals, we find that $q(\partial)\phi_w = q(w(x_0))\phi_w$ for $q \in \mathrm{Sym}(\mathfrak{t})$, and in particular for $q \in I(\mathfrak{t})$. In other words, each ϕ_w solves (2.3.1).

Let E be the vector space over $\mathbb{C}$ consisting of all analytic solutions to (2.3.1). We know already that $\dim E \geq |W|$ since the ϕ_w are linearly independent. To show that the ϕ_w form a basis for E, it is therefore sufficient to show that assuming $\dim E > |W|$ leads to a contradiction.

Choose a point $x_1 \in U$ and let $v_1, \ldots, v_{|W|}$ be as in Lemma 2.3.1. If $\dim E > |W|$, we can choose $\psi \neq 0$ in E satisfying the $|W|$ linear conditions

$$v_i(\partial)\psi(x)\big|_{x=x_1} = 0, \quad 1 \leq i \leq |W|.$$

But this is impossible: by Lemma 2.3.1, for any $v \in \mathrm{Sym}(\mathfrak{t})$ we can write $v = \sum_{i=1}^{|W|} u_i v_i$ with $u_i \in I(\mathfrak{t})$, and since ψ solves (2.3.1), we have

$$v(\partial)\psi(x)\big|_{x=x_1} = \sum_{i=1}^{|W|} u_i(x_0) v_i(\partial)\psi(x)\big|_{x=x_1} = 0.$$

In other words all derivatives of ψ vanish at x_1, and since ψ is analytic it must

[4] These are well-known facts, but see e.g. (Harish-Chandra, 1956, Lemma 4), (Harish-Chandra, 1951, Lemma 41) for detailed proofs.

therefore be identically zero, which contradicts our assumption. □

Now, for any $f \in C^\infty(\mathfrak{g})$, define a function $\phi_f \in C^\infty(\mathfrak{t})$ by

$$\phi_f(x) := \Delta_{\mathfrak{g}}(x) \int_G f(\mathrm{Ad}_g x)\, dg. \tag{2.3.2}$$

We make the following observation about how ϕ_f transforms under the action of invariant differential operators.

Lemma 2.3.3. *For all $p \in I(\mathfrak{g})$,*

$$\phi_{p(\partial)f} = \bar{p}(\partial)\phi_f.$$

Proof. Let $F(x) := \int_G f(\mathrm{Ad}_g x) dg$. Then by Theorem 2.2.7, for $x \in \mathfrak{t}'$ we have $(p(\partial)F)(x) = (\Delta_{\mathfrak{g}}^{-1}\bar{p}(\partial)(\Delta_{\mathfrak{g}}\bar{F}))(x)$, so that

$$\phi_{p(\partial)f}(x) = \Delta_{\mathfrak{g}}(x)p(\partial)F(x) = \bar{p}(\partial)\phi_f(x).$$

Since $\phi_{p(\partial)f}$ and $\bar{p}(\partial)\phi_f$ are both continuous and $\mathfrak{t}'$ is dense in $\mathfrak{t}$, this equality must in fact hold for all $x \in \mathfrak{t}$. □

We now can give Harish-Chandra's original proof of the integral formula (1.1.2), following the outline of steps at the beginning of this section.

Proof (Theorem 2.1.1).

Step 1: Identify an appropriate joint eigenfunction of $\partial I(\mathfrak{t})$.

Choose $y \in \mathfrak{t}'_{\mathbb{C}}$ and define $f : \mathfrak{g} \to \mathbb{C}$ by $f(x) = e^{\langle x,y \rangle}$, so that for $x \in \mathfrak{t}$ we have

$$\phi_f(x) = \Delta_{\mathfrak{g}}(x) \int_G e^{\langle \mathrm{Ad}_g x, y\rangle} dg. \tag{2.3.3}$$

For $y_0 \in \mathfrak{g}$ we have $\partial_{y_0} f = \langle y_0, y\rangle f$, so that for $q \in \mathrm{Sym}(\mathfrak{g})$ we have $q(\partial)f = q(y)f$. But by Lemma 2.3.3, for $p \in I(\mathfrak{g})$ we have $\phi_{p(\partial)f} = \bar{p}(\partial)\phi_f$, so that

$$\bar{p}(\partial)\phi_f = \phi_{p(\partial)f} = \phi_{p(y)f} = p(y)\phi_f.$$

Since this holds for all $p \in I(\mathfrak{g})$, we may apply the isomorphism of the Chevalley restriction theorem to conclude that

$$q(\partial)\phi_f = q(y)\phi_f, \quad \forall\, q \in I(\mathfrak{t}).$$

In other words the analytic function ϕ_f satisfies the system of differential equations (2.3.1). Therefore, by Lemma 2.3.2, there is a unique choice of constants c_w, $w \in W$ such that we can write

$$\phi_f(x) = \sum_{w \in W} c_w e^{\langle w(x), y\rangle}, \qquad x \in \mathfrak{t}.$$

Step 2: Average over W to eliminate all but one unknown constant.

By the skewness of $\Delta_{\mathfrak{g}}$ and the Ad-invariance of the integral in (2.3.3), we have

$$\phi_f(w(x)) = \epsilon(w)\phi_f(x).$$

Multiplying this identity on both sides by $\epsilon(w)$ and taking the average over W gives

$$\phi_f(x) = |W|^{-1} \sum_{w \in W} \epsilon(w)\phi_f(w(x)) = |W|^{-1} c \sum_{w \in W} \epsilon(w) e^{\langle w(x), y\rangle}, \tag{2.3.4}$$

where $c = \sum_{w \in W} \epsilon(w) c_w$.

Step 3: Determine the remaining constant.

We now determine c by computing $\Delta_{\mathfrak{g}}(\partial)\phi_f(x)|_{x=0}$ in two different ways. Define, for each $w \in W$,

$$\psi_w(x) := e^{\langle w(x), y\rangle},$$

so that (2.3.4) becomes

$$\phi_f = |W|^{-1} c \sum_{w \in W} \epsilon(w)\psi_w.$$

Then for $q \in \mathrm{Sym}(\mathfrak{t})$, $q(\partial)\psi_w = q(w^{-1}(y))\psi_w$. In particular,

$$\Delta_{\mathfrak{g}}(\partial)\psi_w = \Delta_{\mathfrak{g}}(w^{-1}(y))\psi_w = \epsilon(w^{-1})\Delta_{\mathfrak{g}}(y)\psi_w = \epsilon(w)\Delta_{\mathfrak{g}}(y)\psi_w,$$

and therefore

$$\begin{aligned}\Delta_{\mathfrak{g}}(\partial)\phi_f(x)\big|_{x=0} &= \Delta_{\mathfrak{g}}(\partial)\left[|W|^{-1} c \sum_{w \in W} \epsilon(w)\psi_w(x)\right]_{x=0} \\ &= |W|^{-1} c \sum_{w \in W} \epsilon(w)^2 \Delta_{\mathfrak{g}}(y)\psi_w(0) = |W|^{-1} c\, \Delta_{\mathfrak{g}}(y)|W| = c\,\Delta_{\mathfrak{g}}(y). \end{aligned} \tag{2.3.5}$$

Now that we have one expression for the value of $\Delta_{\mathfrak{g}}(\partial)\phi_f(x)|_{x=0}$, we'll calculate it again in a different way and equate the two answers to each other. We apply the product rule to compute

$$\begin{aligned}\Delta_{\mathfrak{g}}(\partial)\phi_f(x)\big|_{x=0} &= \Delta_{\mathfrak{g}}(\partial)\left(\Delta_{\mathfrak{g}}(x)\int_G f(\mathrm{Ad}_g x)dg\right)\Big|_{x=0} \\ &= [\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!] \int_G f(0)dg + \Delta_{\mathfrak{g}}(0) \cdot \Delta_{\mathfrak{g}}(\partial) \int_G f(\mathrm{Ad}_g x)dg\Big|_{x=0} \\ &= [\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!] f(0) + 0 = [\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!], \end{aligned} \tag{2.3.6}$$

where we have used the facts that the Haar measure dg is taken to be normalized and that $\Delta_{\mathfrak{g}}(0) = 0$.

Equating (2.3.5) and (2.3.6), we see that $c = [\![\Delta_\mathfrak{g}, \Delta_\mathfrak{g}]\!]/\Delta_\mathfrak{g}(y)$. Plugging this result into (2.3.4) and multiplying both sides by $\Delta_\mathfrak{g}(y)$ gives the desired formula,

$$\Delta_\mathfrak{g}(x)\Delta_\mathfrak{g}(y)\int_G e^{\langle \mathrm{Ad}_g x, y\rangle} dg = \frac{[\![\Delta_\mathfrak{g}, \Delta_\mathfrak{g}]\!]}{|W|}\sum_{w\in W}\epsilon(w)e^{\langle w(x), y\rangle},$$

which we have established *with the assumptions* that $x \in \mathfrak{t}$ and $y \in \mathfrak{t}'_\mathbb{C}$.

To extend the result to all $x, y \in \mathfrak{t}_\mathbb{C}$, we observe that the left- and right-hand sides are both holomorphic functions on $\mathfrak{t}_\mathbb{C} \times \mathfrak{t}_\mathbb{C}$, and that these functions agree on $\mathfrak{t} \times \mathfrak{t}'_\mathbb{C}$, so that they must agree on all of $\mathfrak{t}_\mathbb{C} \times \mathfrak{t}_\mathbb{C}$. This completes the proof. $\square$

2.4 Heat equation proof

In this section, we prove Harish-Chandra's formula by relating the heat flow on $\mathfrak{g}$ to the heat flow on $\mathfrak{t}$. This proof was published in (McSwiggen, 2019b) and generalizes the methods employed by Itzykson and Zuber (1980) in their first proof of the HCIZ formula.

The heat equation proof provides insight into the asymptotics of the Harish-Chandra integral (1.1.2) as the rank of G increases. As explained in Section 2.4.5 below, the relationship between the Harish-Chandra integral and the heat equation on $\mathfrak{t}$ leads to a heuristic calculation suggesting that the large-rank asymptotics of (1.1.2) can be described in terms of a certain hydrodynamic scaling limit of a classical Calogero–Moser system associated to the root system of $\mathfrak{g}$. If rigorously proven, this would generalize known results for the $\mathrm{U}(N)$ case, where leading-order asymptotics for the HCIZ integral (1.1.3) as $N \to \infty$ were formally computed by Matytsin (1994) and rigorously justified by Guionnet and Zeitouni (2002) and Guionnet (2004a), yielding an expression for the leading-order contribution to the large-N limit in terms of a particular solution to the complex Burgers' equation. It has been shown that this phenomenon can be understood in terms of a relationship between the HCIZ integral and the Calogero–Moser system associated to the A_{N-1} root system of $\mathrm{U}(N)$ (Menon, 2017). In fact, Harish-Chandra integrals arise naturally in the study of the quantum Calogero–Moser system, where Harish-Chandra's δ homomorphism has been shown to provide a quantum Hamiltonian reduction of the algebra of differential operators on a reductive Lie algebra (Levasseur and Stafford, 1995, Etingof, 2007), though it is still unclear whether this fact is directly related to the observations in Section 2.4.5.

While it suffices for the purpose of proving the integral formula to relate heat flow on a Lie algebra to heat flow on a Cartan subalgebra, one could obtain richer information by studying the relationship between Brownian motion on the full algebra and Brownian motion confined to a Weyl chamber. This latter process is a generalization of Dyson Brownian motion and has been studied by Grabiner (1999). The arguments regarding asymptotics for the $\mathrm{U}(N)$ integral due to Guionnet and Zeitouni (2002) and Guionnet (2004a) rely on a large-deviations principle for Dyson Brownian motion, suggesting that a similar investigation of processes on general Weyl chambers could prove useful in studying the large-rank asymptotics of integrals over other groups.

Sections 2.4.1 through 2.4.4 below comprise the heat equation proof of (1.1.2).

2.4.1 Heat equations on $\mathfrak{g}$ and $\mathfrak{t}$

Let ω be the quadratic Casimir polynomial on $\mathfrak{g}$ defined by $\omega(x) := |x|^2 = \langle x, x\rangle$. The Laplacian on $\mathfrak{g}$ is the differential operator $\omega(\partial)$. A function $\phi : \mathfrak{g} \times (0, \infty) \to \mathbb{C}$

satisfies the heat equation on $\mathfrak{g}$ if

$$\left(\partial_t - \frac{1}{2}\omega(\partial_x)\right)\phi(x,t) = 0, \qquad x \in \mathfrak{g},\ t \in (0,\infty). \tag{2.4.1}$$

Similarly, $\psi : \mathfrak{t} \times (0,\infty) \to \mathbb{C}$ satisfies the heat equation on $\mathfrak{t}$ if

$$\left(\partial_t - \frac{1}{2}\bar{\omega}(\partial_x)\right)\psi(x,t) = 0, \qquad x \in \mathfrak{t},\ t \in (0,\infty), \tag{2.4.2}$$

where the bar indicates restriction to $\mathfrak{t}$.

We want to establish a relationship between the solutions of (2.4.1) and of (2.4.2). In general, if ϕ satisfies (2.4.1), it is not the case that $\bar{\phi}$ satisfies (2.4.2). However, we will show:

Lemma 2.4.1. *If $\phi \in C_1^2(\mathfrak{g} \times (0,\infty))$ solves (2.4.1) and is invariant under the adjoint action of G in the sense that*

$$\phi(\mathrm{Ad}_g x, t) = \phi(x,t), \quad \forall\, g \in G,$$

then $\Delta_{\mathfrak{g}}(x)\bar{\phi}(x,t)$ solves (2.4.2).

The proof of Lemma 2.4.1 makes use of the following expression for the radial part of the Laplacian, which is a special case of Theorem 2.2.7. Recall that the regular elements of $\mathfrak{t}$ are the subset $\mathfrak{t}' := \{\ x \in \mathfrak{t} \mid \Delta_{\mathfrak{g}}(x) \neq 0\ \}$.

Lemma 2.4.2. *The radial part of the Laplacian $\omega(\partial)$ with transversal manifold $\mathfrak{t}'$ is given by*

$$\gamma(\omega(\partial)) = \Delta_{\mathfrak{g}}^{-1}\bar{\omega}(\partial) \circ \Delta_{\mathfrak{g}}.$$

Proof of Lemma 2.4.1. Restricting (2.4.1) to $\mathfrak{t}'$ and applying Lemma 2.4.2 gives the *radial heat equation*

$$\left(\partial_t - \frac{1}{2}\Delta_{\mathfrak{g}}^{-1}(x)\bar{\omega}(\partial_x) \circ \Delta_{\mathfrak{g}}(x)\right)\bar{\phi}(x,t) = 0, \qquad x \in \mathfrak{t}', \tag{2.4.3}$$

where the bar indicates restriction to $\mathfrak{t}'$. Multiplying both sides by $\Delta_{\mathfrak{g}}$ we have

$$\left(\partial_t - \frac{1}{2}\bar{\omega}(\partial_x)\right)\Delta_{\mathfrak{g}}(x)\bar{\phi}(x,t) = 0, \qquad x \in \mathfrak{t}'. \tag{2.4.4}$$

Thus $\Delta_{\mathfrak{g}}(x)\bar{\phi}(x,t)$ solves the heat equation on $\mathfrak{t}'$ and therefore on all of $\mathfrak{t}$ by continuity since $\mathfrak{t}'$ is dense. $\square$

2.4.2 The G-averaged heat kernel

The *heat kernel* on $\mathfrak{g}$ is the function $K : \mathfrak{g}^2 \times (0,\infty) \to \mathbb{C}$ given by

$$K(x,y;t) := \left(\frac{1}{2\pi t}\right)^{\dim \mathfrak{g}/2} e^{-\frac{1}{2t}|x-y|^2}. \tag{2.4.5}$$

It solves the heat equation (2.4.1) where the spatial derivatives act in either the x or the y variables, with the boundary condition

$$\lim_{t\to 0} K(x,y;t) = \delta(x-y). \tag{2.4.6}$$

The limit is understood in the distributional sense of

$$\lim_{t\to 0}\int_{\mathfrak{g}} K(x,y;t)\varphi(y)dy = \varphi(x)$$

for $\varphi \in C_c^\infty(\mathfrak{g})$, where dy is the Lebesgue measure induced by the inner product. The choice of this integration measure is significant, as it guarantees that for all x

and t we have

$$\int_{\mathfrak{g}} K(x,y;t)\,dy = 1.$$

Following Itzykson and Zuber (1980), we define the G*-averaged heat kernel* as

$$\begin{aligned}\tilde{K}(x,y;t) &:= \int_G K(\mathrm{Ad}_g x, y; t)\,dg \\ &= \left(\frac{1}{2\pi t}\right)^{\dim \mathfrak{g}/2} e^{-\frac{1}{2t}(|x|^2+|y|^2)} I(x,y;t),\end{aligned} \tag{2.4.7}$$

where

$$I(x,y;t) := \int_G e^{\frac{1}{t}\langle \mathrm{Ad}_g x, y\rangle} dg. \tag{2.4.8}$$

Observe that $I(x,y;1) = \mathcal{H}(x,y)$ for $x, y \in \mathfrak{t}$, so that the Harish-Chandra integral appears naturally in this context.

The G-averaged heat kernel $\tilde{K}$ is constant on adjoint orbits of both x and y, and by linearity it satisfies the heat equation (2.4.1) on $\mathfrak{g}$ as well, so that by Lemma 2.4.1, $\Delta_{\mathfrak{g}}(y)\tilde{K}(x,y;t)$ satisfies the heat equation (2.4.2) on $\mathfrak{t}$ with the spatial derivatives acting in the y variables. Therefore the function

$$V(x,y;t) := (2\pi)^{(\dim \mathfrak{g} - r)/2} \Delta_{\mathfrak{g}}(x) \Delta_{\mathfrak{g}}(y) \tilde{K}(x,y;t) \tag{2.4.9}$$

also satisfies (2.4.2) and is skew with respect to the action of W on either of x or y individually. In the next step we identify the boundary conditions that V satisfies as t approaches 0. This will allow us to write an exact expression for V using the fundamental solution to the heat equation on $\mathfrak{t}$, yielding (1.1.2).

2.4.3 Boundary conditions for V

We now further assume that both $x, y \in \mathfrak{t}'$. In order to compute the distributional limit of V as t approaches 0, we use the steepest-descent method to determine the asymptotics of $I(x, y; t)$ to leading order in t. We will show:

Lemma 2.4.3. *If $x, y \in \mathfrak{t}'$, then*

$$\lim_{t \to 0} V(x, y; t) = C \sum_{w \in W} \epsilon(w)\delta(w(x) - y) \tag{2.4.10}$$

for some constant $C \in \mathbb{R}$, where the distributional sense of the limit is given by integration against test functions in $C_c^\infty(\mathfrak{t})$ with respect to the Lebesgue measure induced by the restriction of the inner product to $\mathfrak{t}$.

Proof. We first rewrite

$$V(x, y; t) = \frac{t^{-\dim \mathfrak{g}/2}}{(2\pi)^{r/2}} \Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y) e^{-\frac{1}{2t}(|x|^2+|y|^2)} I(x, y; t). \tag{2.4.11}$$

Next we rewrite $I(x, y; t)$ as follows. Let T be the maximal torus in G with Lie algebra $\mathfrak{t}$. Let dh be the normalized Haar measure on T and let $d(gT)$ be the unique left-invariant probability measure on G/T. Then by a standard Fubini-type theorem for Lie groups (Helgason, 1984, ch. 1, Theorem 1.9) we have:

$$\begin{aligned} I(x, y; t) = \int_G e^{\frac{1}{t}\langle \mathrm{Ad}_g x, y\rangle} dg = \int_{G/T} \int_T e^{\frac{1}{t}\langle \mathrm{Ad}_{gh} x, y\rangle} dh \; d(gT) \\ = \int_{G/T} e^{\frac{1}{t}\langle \mathrm{Ad}_g x, y\rangle} d(gT). \end{aligned} \tag{2.4.12}$$

We will apply the steepest-descent method to the last integral in (2.4.12). To do so, we need the following lemmas computing the critical points of the function

$gT \mapsto \langle \mathrm{Ad}_g x, y\rangle$ along with its Hessian matrix at each critical point.

Lemma 2.4.4. *The critical points of the function $gT \mapsto \langle \mathrm{Ad}_g x, y\rangle$ on G/T are the points gT such that $\mathrm{Ad}_g x = w(x)$ for some $w \in W$.*

Proof. We first note that the map $gT \mapsto \mathrm{Ad}_g x$ is a diffeomorphism of G/T onto the adjoint orbit $\mathcal{O}_x \subset \mathfrak{g}$. Thus we may equivalently find the critical points of the function $x_0 \mapsto \langle x_0, y\rangle$ for $x_0 \in \mathcal{O}_x$. The tangent space at a point $x_0 \in \mathcal{O}_x$ is $T_{x_0}\mathcal{O}_x \cong [x_0, \mathfrak{g}]$, and the partial derivative of the linear functional $\langle y, \cdot\rangle$ in the direction of a tangent vector y_0 is equal to $\langle y, y_0\rangle$. Thus the condition for $x_0 \in \mathcal{O}_x$ to be a critical point is

$$\langle y, y_0\rangle = 0, \quad \forall\, y_0 \in [x_0, \mathfrak{g}],$$

or equivalently

$$\langle [x_0, y_0], y\rangle = 0, \quad \forall\, y_0 \in \mathfrak{g}.$$

Using the antisymmetry of the bracket and the invariance of the inner product, this is equivalent to

$$\langle y_0, [x_0, y]\rangle = 0, \quad \forall\, y_0 \in \mathfrak{g}.$$

By the nondegeneracy of the inner product, this will hold if and only if $[x_0, y] = 0$, which implies $x_0 \in \mathfrak{t}$. Writing $x_0 = \mathrm{Ad}_g x$, we have $x_0 \in \mathfrak{t}$ exactly when $\mathrm{Ad}_g x = w(x)$ for some $w \in W$. $\square$

Lemma 2.4.5. *Let H be the Hessian matrix of the function $gT \mapsto \langle \mathrm{Ad}_g x, y\rangle$ at a critical point $g_0 T$, in exponential coordinates on G/T given by $e^{\xi}T \mapsto \xi + \mathfrak{t} \in \mathfrak{g}/\mathfrak{t}$. If $\mathrm{Ad}_{g_0} = w \in W$ as an operator on $\mathfrak{t}$, then*

$$\sqrt{\det(-H)} = \epsilon(w)\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y). \tag{2.4.13}$$

In order to give the correct global phase in the steepest-descent approximation, the branch of the square root in (2.4.13) is chosen by writing $\sqrt{\det(-H)} = \prod_{j=1}^{n}\sqrt{-\mu_j}$ where μ_j are the eigenvalues of H, and taking $|\arg\sqrt{-\mu_j}| < \pi/4$.

Proof. Since

$$\mathrm{Ad}_{\exp(\xi)} = \sum_{j=0}^{\infty}\frac{1}{j!}\mathrm{ad}_{\xi}^{j},$$

expanding to second order in ξ around the critical point g_0T gives

$$\langle \mathrm{Ad}_{\exp(\xi)}w(x), y\rangle = \langle w(x), y\rangle + \frac{1}{2}\langle \mathrm{ad}_{\xi}^{2}w(x), y\rangle + O(|\xi|^3). \tag{2.4.14}$$

To obtain the Hessian we must compute explicitly the second-order term in (2.4.14). For a root α, let $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\mathbb{C}}$ denote the corresponding root space. For each $\alpha \in \Phi^+$, choose $x_\alpha \in \mathfrak{g}_\alpha$, $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ normalized so that $\langle x_\alpha, x_{-\alpha}\rangle = 1$. Assuming without loss of generality that ξ lies in the orthogonal complement of $\mathfrak{t}$, we may write

$$\xi = \sum_{\alpha\in\Phi^+} c_\alpha x_\alpha + c_{-\alpha}x_{-\alpha} \tag{2.4.15}$$

for some coefficients c_α, $c_{-\alpha} \in \mathbb{C}$, and we find by a straightforward calculation[5] that

$$\frac{1}{2}\langle \mathrm{ad}_{\xi}^{2}w(x), y\rangle = -\sum_{\alpha\in\Phi^+}\alpha(w(x))\alpha(y)c_\alpha c_{-\alpha}.$$

Thus H is a block-diagonal matrix composed of 2-by-2 blocks of the form

$$\begin{bmatrix} 0 & -\alpha(w(x))\alpha(y) \\ -\alpha(w(x))\alpha(y) & 0 \end{bmatrix}$$

[5] Recall from Section 1.4 that we take the roots to be real valued on $\mathfrak{t}$, so that $[h, x_\alpha] = i\alpha(h)x_\alpha$ for $h \in \mathfrak{t}$.

for each $\alpha \in \Phi^+$. With the appropriate choice of branch for the square root, we find

$$\sqrt{\det(-H)} = \Delta_{\mathfrak{g}}(w(x))\Delta_{\mathfrak{g}}(y) = \epsilon(w)\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y)$$

as desired. $\square$

Returning now to the proof of Lemma 2.4.3 and applying the steepest-descent approximation to (2.4.12) together with Lemmas 2.4.4 and 2.4.5, we obtain

$$I(x, y; t) = C\frac{(2\pi t)^{(\dim\mathfrak{g}-r)/2}}{\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y)} \sum_{w\in W} \epsilon(w)e^{\frac{1}{t}\langle w(x),y\rangle}(1 + O(t)) \tag{2.4.16}$$

where the constant C arises from the normalization of the measure $d(gT)$. Substituting this result into (2.4.11), we find

$$V(x, y; t) = C\left(\frac{1}{2\pi t}\right)^{r/2} \sum_{w\in W} \epsilon(w)e^{-\frac{1}{2t}|w(x)-y|^2}(1 + O(t)) \tag{2.4.17}$$

as $t \to 0$, which gives the desired limit (2.4.10). $\square$

Because V solves the heat equation on $\mathfrak{t}$, taking the convolution of the boundary data (2.4.10) with the fundamental solution gives

$$V(x, y; t) = C\left(\frac{1}{2\pi t}\right)^{r/2} \sum_{w\in W} \epsilon(w)e^{-\frac{1}{2t}|w(x)-y|^2}. \tag{2.4.18}$$

Thus the higher-order terms on the right-hand side of (2.4.17) actually vanish. In physical terms, the expression (2.4.18) is analogous to a Slater determinant, with W-skewness playing the role of the antisymmetry property of fermions.

2.4.4 Normalization

It only remains to rearrange terms and determine the constant C. Evaluating V at $t = 1$, we have:

$$\begin{aligned} V(x, y; 1) &= (2\pi)^{(\dim \mathfrak{g} - r)/2} \Delta_{\mathfrak{g}}(x) \Delta_{\mathfrak{g}}(y) \tilde{K}(x, y; 1) \\ &= (2\pi)^{(\dim \mathfrak{g} - r)/2} \Delta_{\mathfrak{g}}(x) \Delta_{\mathfrak{g}}(y) \left(\frac{1}{2\pi} \right)^{\dim \mathfrak{g}/2} e^{-\frac{1}{2}(|x|^2 + |y|^2)} I(x, y; 1) \\ &= C \left(\frac{1}{2\pi} \right)^{r/2} \sum_{w \in W} \epsilon(w) e^{-\frac{1}{2}|w(x) - y|^2} \\ &= C \left(\frac{1}{2\pi} \right)^{r/2} e^{-\frac{1}{2}(|x|^2 + |y|^2)} \sum_{w \in W} \epsilon(w) e^{\langle w(x), y \rangle}. \end{aligned}$$

After cancelations, this becomes

$$\Delta_{\mathfrak{g}}(x) \Delta_{\mathfrak{g}}(y) I(x, y; 1) = C \sum_{w \in W} \epsilon(w) e^{\langle w(x), y \rangle}. \tag{2.4.19}$$

Up to this point we have assumed that $x, y \in \mathfrak{t}'$, but we can immediately remove this assumption: since both sides of (2.4.19) are analytic in x and y, this identity holds for all $x, y \in \mathfrak{t}_{\mathbb{C}}$.

Finally, we determine C. Applying $\Delta_{\mathfrak{g}}(\partial_x)$ to both sides of (2.4.19) and evaluating at $x = 0$, we obtain

$$\Delta_{\mathfrak{g}}(y) [\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!] = C |W| \Delta_{\mathfrak{g}}(y), \quad y \in \mathfrak{t}_{\mathbb{C}},$$

with $[\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!]$ defined by (2.1.1), so that $C = [\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!] / |W|$. This completes the proof of Theorem 2.1.1.

2.4.5 Relationship to Calogero–Moser systems

To illustrate the relationship between Harish-Chandra integrals and Calogero–Moser systems, we observe that by Lemma 2.4.1 the function

$$W(x,y;t) := \frac{1}{r^2}\log \tilde{K}(\sqrt{r}x, \sqrt{r}y;t) \tag{2.4.20}$$

satisfies

$$2\frac{\partial W}{\partial t} = r|\nabla W|^2 + \frac{2}{r}\nabla(\log \Delta_{\mathfrak{g}})\cdot\nabla W - \frac{1}{r}\bar{\omega}(\partial)W - \frac{1}{r^3}\Delta_{\mathfrak{g}}^{-1}\bar{\omega}(\partial)\Delta_{\mathfrak{g}}, \tag{2.4.21}$$

where spatial derivatives act in the y variables and $\Delta_{\mathfrak{g}} = \Delta_{\mathfrak{g}}(y)$. Since $\Delta_{\mathfrak{g}}$ is harmonic by Proposition 2.2.3, the last term on the right-hand side of (2.4.21) vanishes. In physical terms, the substitution (2.4.20) resembles a WKB ansatz with r^{-2} playing the role of $\hbar$, so the large-rank limit in this setting bears some similarities to a semi-classical approximation. Some caution is required in applying this analogy however, since r is not a numerical parameter but rather the dimension of the underlying space.

To compute the large-N asymptotics for the integral over $\mathrm{U}(N)$ (where $r = N$), Matytsin (1994) drops the term corresponding to $r^{-1}\bar{\omega}(\partial)W$ in (2.4.21), arguing heuristically that it should be subdominant as $N \to \infty$. If we neglect this term and make the substitution $W = S - r^{-2}\log\Delta_{\mathfrak{g}}$, then we arrive at

$$2\frac{\partial S}{\partial t} = r|\nabla S|^2 - r^{-3}|\nabla(\log\Delta_{\mathfrak{g}})|^2. \tag{2.4.22}$$

In fact (2.4.22) is (a scaling in r of) the Hamilton–Jacobi equation for a rational Calogero–Moser system associated to the root system of the algebra $\mathfrak{g}$, with r particles. From this observation we expect that the large-rank asymptotics of Harish-Chandra integrals can be understood in terms of hydrodynamic scaling limits of Calogero–Moser systems. This is known to be true for integrals over $\mathrm{U}(N)$, as in (Matytsin, 1994, Guionnet and Zeitouni, 2002) the leading-order asymptotics of the HCIZ integral are derived in terms of a particular solution to the complex Burgers' equation. As explained in (Menon, 2017), the complex Burgers' equation also arises as a hydrodynamic limit of the Calogero–Moser system associated to the A_{N-1} root system.

2.5 Symplectic geometry proof

It is also possible to prove Theorem 2.1.1 using a localization technique in symplectic geometry. While the existence of such a proof was observed already by Duistermaat and Heckman (1982), here we present the details of the argument with the goal of making this derivation accessible to non-specialists.

The symplectic geometry approach illustrates a completely different perspective on the Harish-Chandra integral. Rather than writing the integral in terms of the heat kernel on $\mathfrak{g}$, we instead view it as an oscillatory integral over a coadjoint orbit in $\mathfrak{g}^*$. The proof begins with the steepest-descent approximation (2.4.16) for $I(x, y; t)$ obtained in the proof of Lemma 2.4.3, though here we work with $I(x, y; -it^{-1})$ rather than $I(x, y; t)$, so that (2.4.16) becomes a stationary-phase approximation. Then, instead of observing that V solves the heat equation on $\mathfrak{t}$, we use the Duistermaat–Heckman theorem to deduce that the approximation is exact, after which it remains only to compute the normalization constant following the argument of Section 2.4.4 above.

We first state the Duistermaat–Heckman theorem, which was proved by Duistermaat and Heckman (1982) and later shown to be an instance of a more general principle of *equivariant localization* (Atiyah and Bott, 1984). Let (M, ω) be a compact symplectic manifold of dimension $2n$, and suppose that the r-dimensional torus T acts smoothly on M. For each $y \in \mathfrak{t}$ we define a vector field X_y on M by

$$X_y(x)f = \frac{d}{dt}\bigg|_{t=0} f(e^{ty} \cdot x), \quad x \in M,\ f \in C^\infty(M). \tag{2.5.1}$$

We assume that there is a *moment map* for the action of T on M, that is a smooth function $\phi : M \to \mathfrak{t}^*$ such that[6]

$$\omega(X_y, \,\cdot\,) = -\langle d\phi(\cdot), y\rangle, \qquad y \in \mathfrak{t}. \tag{2.5.2}$$

The *Liouville measure* μ on M is given by the volume form $\omega^{\wedge n}/n!$. Then we have the following:

Theorem 2.5.1 (Duistermaat–Heckman)**.** *The integral*

$$\int_M e^{it\langle \phi(x), y\rangle} d\mu(x) \tag{2.5.3}$$

is exactly equal to its leading-order approximation by the method of stationary phase as $t \to \infty$.[7]

To prove the exactness of (2.4.16) from Theorem 2.5.1, we re-interpret $I(x, y; t)$ in the language of symplectic geometry. Here we temporarily stop identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ in order to emphasize the fact that in symplectic geometry it is natural to study

[6]Some authors define the moment map with the opposite sign in (2.5.2).

[7]The Duistermaat–Heckman theorem is sometimes stated differently, as follows: every regular value of ϕ has a neighborhood in which $\phi_*\mu$ is equal to Lebesgue measure times a polynomial of degree at most $n - r$. Theorem 2.5.1 follows from this statement by observing that the integral (2.5.3), considered as a function of t, is the Fourier transform (with the probabilist's sign convention) of the measure $\langle \phi, y\rangle_*\mu$, i.e. the pushforward of μ by the map $\langle \phi(\cdot), y\rangle : \mathfrak{t}^* \to \mathbb{R}$.

*co*adjoint rather than adjoint orbits, but we still use $\langle \cdot, \cdot \rangle$ to indicate the duality pairing.

For $x \in \mathfrak{g}$, let $x^* \in \mathfrak{g}^*$ be its dual under the inner product, that is $x^*(y) = \langle x, y \rangle$ for $x, y \in \mathfrak{g}$. Let $\beta \in \mathfrak{t}^*$. We define the *coadjoint orbit*

$$\mathcal{O}^*_\beta := \{\mathrm{Ad}^*_g \beta \mid g \in G\} \subset \mathfrak{g}^*,$$

where Ad^* is the coadjoint representation of G on $\mathfrak{g}^*$, defined by

$$\mathrm{Ad}^*_g x^* := (\mathrm{Ad}_{g^{-1}} x)^*.$$

At a point $x^* \in \mathcal{O}^*_\beta$, under the usual identification $T_{x^*}\mathfrak{g}^* \cong \mathfrak{g}^*$, we have

$$T_{x^*}\mathcal{O}^*_\beta \cong \{[x, y]^* \mid y \in \mathfrak{g}^*\}.$$

The *Kostant–Kirillov–Souriau form* is the G-invariant 2-form ω on $\mathcal{O}^*_\beta$ defined by

$$\omega_{x^*}([x, y]^*, [x, z]^*) = \langle x, [y, z] \rangle. \tag{2.5.4}$$

This form can be shown by direct computation to be nondegenerate and closed, and it therefore makes $\mathcal{O}^*_\beta$ into a symplectic manifold (Kirillov, 2004, ch. 1), on which the maximal torus $T \subset G$ acts smoothly via the coadjoint representation.

Let $\phi : \mathcal{O}^*_\beta \to \mathfrak{t}^*$ be the orthogonal projection onto $\mathfrak{t}^*$. We will show that ϕ is a moment map for the action of T on $\mathcal{O}^*_\beta$. Observe that we have

$$\langle \phi(x^*), y \rangle = \langle x, y \rangle, \qquad y \in \mathfrak{t},\ x^* \in \mathcal{O}^*_\beta. \tag{2.5.5}$$

Moreover, $d\phi([x,z]^*)$ is also just the orthogonal projection of $[x,z]^*$ onto $\mathfrak{t}^*$, so that

$$-\langle d\phi([x,z]^*), y\rangle = -\langle [x,z], y\rangle = \langle x, [y,z]\rangle = \omega_{x^*}([x,y]^*, [x,z]^*).$$

A direct computation from (2.5.1) gives $X_y(x^*) = [x,y]^*$, so that ϕ satisfies (2.5.2) and therefore is a moment map as desired.

Next we relate the Liouville measure on $\mathcal{O}^*_\beta$ to Haar measure on G. Let $2n = \dim \mathcal{O}^*_\beta$. The Liouville measure $\omega^{\wedge n}/n!$ is G-invariant due to the invariance of ω. Pulling the Liouville measure back along the map $\mathrm{Ad}^*\beta : G \to \mathcal{O}^*_\beta$, we obtain a finite invariant measure on G, which by the uniqueness of Haar measure must equal a constant times dg. Thus for a Borel set $E \subset G$ we have

$$\int_E dg = \frac{1}{\mathrm{Vol}_\mu(\mathcal{O}^*_\beta)} \int_{\{\mathrm{Ad}^*_g\beta \ \mid \ g\in E\}} \frac{\omega^{\wedge n}}{n!}. \tag{2.5.6}$$

Putting together (2.5.5) and (2.5.6) and writing $d\mu := \omega^{\wedge n}/n!$, we can express $I(x,y;t)$ as an integral over $\mathcal{O}^*_{x^*}$:

$$I(x,y;t) = \int_G e^{\frac{1}{t}\langle \mathrm{Ad}_g x, y\rangle} dg = \frac{1}{\mathrm{Vol}_\mu(\mathcal{O}^*_{x^*})} \int_{\mathcal{O}^*_{x^*}} e^{\frac{1}{t}\langle \phi(\beta), y\rangle} d\mu(\beta).$$

This function is analytic in t for $t \in (0,\infty)$, and so by analytic continuation we can equivalently consider

$$I(x,y;-it^{-1}) = \frac{1}{\mathrm{Vol}_\mu(\mathcal{O}^*_{x^*})} \int_{\mathcal{O}^*_{x^*}} e^{it\langle \phi(\beta), y\rangle} d\mu(\beta).$$

We have now written I in the form (2.5.3), so that by Theorem 2.5.1 and (2.4.16) we have

$$I(x, y; -it^{-1}) = C\left(\frac{2\pi}{it}\right)^{(\dim\mathfrak{g}-r)/2} (\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y))^{-1}\sum_{w\in W}\epsilon(w)e^{it\langle w(x),y\rangle}$$

for some constant C. The proof concludes by evaluating at $t = -i$ and computing C as in Section 2.4.4.

2.6 Representation-theoretic proofs

Finally, we give two different proofs using ideas from representation theory: one proof via the Kirillov character formula, and a second via the character expansion for the heat kernel on the group G.

2.6.1 Via the Kirillov character formula

The Harish-Chandra integral also has an interpretation in terms of the irreducible characters of G. Here we discuss how Theorem 2.1.1 is essentially equivalent to the Kirillov character formula in the case of a compact, connected, semisimple group.

Let $\lambda \in \mathfrak{t}^*$ be the highest weight of an irreducible representation of G with character χ_λ, and let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ be half the sum of the positive roots, also called the *Weyl vector* of $\mathfrak{g}$. Let μ be the Liouville measure associated to the Kostant–Kirillov–Souriau form on the coadjoint orbit $\mathcal{O}^*_{\lambda+\rho}$, and set

$$\widehat{\Delta}_{\mathfrak{g}}(e^z) := \prod_{\alpha\in\Phi^+}(e^{i\langle\alpha,z\rangle/2} - e^{-i\langle\alpha,z\rangle/2}), \qquad z \in \mathfrak{t}.$$

The *Kirillov character formula* for compact groups (Kirillov, 2004, ch. 5, Theorem 9) says:

Theorem 2.6.1. *For $\xi \in \mathfrak{t}'$,*

$$\chi_\lambda(e^\xi) = \frac{\Delta_{\mathfrak{g}}(i\xi)}{\widehat{\Delta}_{\mathfrak{g}}(e^\xi)} \int_{\mathcal{O}^*_{\lambda+\rho}} e^{i\langle\beta,\xi\rangle} d\mu(\beta). \tag{2.6.1}$$

Although here we assume that G satisfies the assumptions of Theorem 2.1.1, versions of this formula hold in a variety of situations even for non-compact groups; see the book by Kirillov (2004) for a detailed discussion.

On the other hand, the *Weyl character formula* (see e.g. (Bump, 2004, Theorem 25.4)) states:

Theorem 2.6.2. *For $\xi \in \mathfrak{t}'$,*

$$\chi_\lambda(e^\xi) = \frac{1}{\widehat{\Delta}_{\mathfrak{g}}(e^\xi)} \sum_{w\in W} \epsilon(w) e^{i\langle w(\lambda+\rho),\xi\rangle}. \tag{2.6.2}$$

For this reason, the function $\widehat{\Delta}_{\mathfrak{g}}$ is often called the *Weyl denominator*. Note that either of (2.6.1) or (2.6.2) completely determines χ_λ. Since the character is analytic it is determined on the maximal torus $\exp(\mathfrak{t})$ by its values on $\exp(\mathfrak{t}')$, and since it is a class function it is determined on all of G by its restriction to a maximal torus.

One standard proof of Theorem 2.6.1 due to Kirillov (1968) works by applying Theorem 2.6.2 to the right-hand side of (1.1.2). However, the Kirillov formula can also be proven by other methods, for example by applying the equivariant index theorem to the Dirac operator twisted by the Kostant–Souriau line bundle on the coadjoint orbit (Berline et al., 2004). Nothing stops us, therefore, from using

(2.6.1) to prove (1.1.2) instead, as there is no circularity involved. For the sake of completeness, we record this proof below.

Equating the right-hand sides of (2.6.1) and (2.6.2), writing $i\xi = y$ and $\lambda+\rho = x^*$, and using the relation (2.5.6) between Liouville measure on $\mathcal{O}^*_{\lambda+\rho}$ and Haar measure on G, we obtain:

$$\begin{aligned}\Delta_{\mathfrak{g}}(i\xi)\int_{\mathcal{O}^*_{\lambda+\rho}} e^{i\langle\beta,\xi\rangle}d\mu(\beta) &= \mathrm{Vol}_\mu(\mathcal{O}^*_{x^*})\Delta_{\mathfrak{g}}(y)\int_G e^{\langle \mathrm{Ad}_g x, y\rangle}dg \\ &= \sum_{w\in W}\epsilon(w)e^{\langle w(x),y\rangle}.\end{aligned}$$

Applying $\Delta_{\mathfrak{g}}(\partial_y)$ to the second and third expressions above and evaluating at $y=0$, we find

$$\mathrm{Vol}_\mu(\mathcal{O}^*_{x^*}) = \frac{|W|}{[\![\Delta_{\mathfrak{g}},\Delta_{\mathfrak{g}}]\!]}\Delta_{\mathfrak{g}}(x), \tag{2.6.3}$$

which recovers the integral formula (1.1.2) in the case that $y\in\mathfrak{t}'$ and $x^* = \lambda+\rho$, where λ is the highest weight of an irreducible representation of G. Since the right-hand side of (1.1.2) is W-invariant in x, the result also holds for x such that $w(x)^* = \lambda+\rho$, $w\in W$, with λ a highest weight. Such x form a lattice spanning $\mathfrak{t}$, so by scaling y and shifting the scaling onto x we obtain the result for all $y\in\mathfrak{t}'$ and x in a dense subset of $\mathfrak{t}$. Analytic continuation then gives the result for all $x, y\in\mathfrak{t}_{\mathbb{C}}$, completing the proof of Theorem 2.1.1.

Remark 2.6.3. Comparing (2.6.3) with the Kirillov character formula (2.6.1) for the trivial representation of G, we find

$$\Delta_{\mathfrak{g}}(\rho) = \frac{[\![\Delta_{\mathfrak{g}},\Delta_{\mathfrak{g}}]\!]}{|W|}, \tag{2.6.4}$$

so that (2.6.3) becomes

$$\mathrm{Vol}_\mu(\mathcal{O}^*_{x^*}) = \frac{\Delta_{\mathfrak{g}}(x)}{\Delta_{\mathfrak{g}}(\rho)}. \tag{2.6.5}$$

△

2.6.2 Via the character expansion for the heat kernel on G

This final proof, due to Altschuler and Itzykson (1991), employs both representation theory *and* heat kernel asymptotics. Here we study the heat kernel $K_G(g_1, g_2; t)$ on the group rather than on the algebra. Instead of deriving the integral formula using the fundamental solution for the heat equation on $\mathfrak{t}$, we use the fact that K_G has a known character expansion. We can then recover the Harish-Chandra formula by integrating this character expansion over the group and taking an appropriate limit.

It is instructive to compare this strategy to the heat equation proof in Section 2.4 above. There we related the Harish-Chandra integral $\mathcal{H}(x, y)$ to the G-averaged heat kernel $\tilde{K}$, defined in (2.4.7) by integrating the heat kernel on $\mathfrak{g}$ over an adjoint orbit. We then related $\tilde{K}$ to a particular solution of the heat equation on $\mathfrak{t}$, which we could write down explicitly, yielding an exact formula for $\mathcal{H}(x, y)$. In effect the proof in this subsection also works by obtaining an exact expression for $\tilde{K}$, but via a different method. Since $\mathfrak{g} = T_{\mathrm{id}_G}G$, the heat kernel on $\mathfrak{g}$ can be recovered from the local behavior of K_G near the identity. Thus it is sufficient to obtain an exact expression for the integral of K_G over G, which we can accomplish using ideas from character theory.

Now we give the proof. The heat kernel on G can be written (Fegan, 1978):

$$K_G(g_1, g_2; t) = \sum_{\lambda \in P^+} (\dim V_\lambda) \chi_\lambda(g_1 g_2^{-1}) e^{-t(|\lambda+\rho|^2 - |\rho|^2)/2}, \tag{2.6.6}$$

where P^+ is the set of dominant integral weights of G. We will first integrate K_G over G and derive an asymptotic expression for the resulting function in the limit of small g_1, g_2 and t. Then we will relate this expression to the Harish-Chandra integral. Integrating (2.6.6) over G, from the orthonormality of matrix coefficients of irreducible unitary representations, we obtain:

$$F(g_1, g_2; t) := \int_G K_G(g_1, g g_2 g^{-1}; t)\, dg = \sum_{\lambda \in P^+} \chi_\lambda(g_1) \chi_\lambda(g_2^{-1}) e^{-t(|\lambda+\rho|^2 - |\rho|^2)/2}. \tag{2.6.7}$$

The function F is clearly conjugation-invariant in both g_1 and g_2, so without loss of generality we may take $g_1 = e^x$, $g_2 = e^y$ with $x, y \in \mathfrak{t}$. Using the Weyl character formula (2.6.2) for χ_λ, followed by the Poisson summation formula, we have:

$$\begin{aligned} F(g_1, g_2; t) &= \frac{e^{t|\rho|^2/2}}{\widehat{\Delta}_{\mathfrak{g}}(e^x) \widehat{\Delta}_{\mathfrak{g}}(e^{-y})} \sum_{w \in W} \epsilon(w) \sum_{\lambda \in P} e^{i\langle \lambda, x - w(y) \rangle} e^{-t|\lambda|^2/2} \\ &= \frac{e^{t|\rho|^2/2} \nu^{1/2} (2\pi/t)^{r/2}}{\widehat{\Delta}_{\mathfrak{g}}(e^x) \widehat{\Delta}_{\mathfrak{g}}(e^{-y})} \sum_{w \in W} \epsilon(w) \sum_{\beta \in 2\pi Q^\vee} e^{-\frac{1}{2t}|x - w(y) + \beta|^2}, \end{aligned} \tag{2.6.8}$$

where P is the weight lattice, $Q^\vee$ is the coroot lattice, and ν is the index of $Q^\vee$ in P. The formula (2.6.8) was likely first derived by Frenkel (1984, sect. 4.3). Now we take a limit:

$$\lim_{\varepsilon \to 0} \varepsilon^{\dim G} F(e^{\varepsilon x}, e^{\varepsilon y}, \varepsilon^2 t) = \frac{\nu^{1/2} (2\pi/t)^{r/2}}{\Delta_{\mathfrak{g}}(x) \Delta_{\mathfrak{g}}(y)} \sum_{w \in W} \epsilon(w) e^{-\frac{1}{2t}|x - w(y)|^2}, \tag{2.6.9}$$

since $\dim G = 2|\Phi^+| + r$, and only the $\beta = 0$ term of (2.6.8) contributes to leading order.

It remains to relate $F(g_1, g_2; t)$ to the integral $I(x, y; t)$. We achieve this by relating the heat kernel K_G on G to the heat kernel K on $\mathfrak{g}$, as defined above in (2.4.5). First observe that for $\varepsilon > 0$, $K_G(e^x, e^y; t) = \lim_{\eta\to 0} F(e^{x-y}, e^{\varepsilon\eta\rho}; t)$. We thus have:

$$\lim_{\varepsilon\to 0} \varepsilon^{\dim G} K_G(e^{\varepsilon x}, e^{\varepsilon y}; \varepsilon^2 t) = \lim_{\varepsilon\to 0}\lim_{\eta\to 0} \varepsilon^{\dim G} F(e^{\varepsilon(x-y)}, e^{\varepsilon\eta\rho}; \varepsilon^2 t). \tag{2.6.10}$$

Now fix $t > 0$ and $x \neq y \in \mathfrak{t}$, and consider the function

$$\begin{aligned} f(\varepsilon, \eta) &:= \varepsilon^{\dim G} F(e^{\varepsilon(x-y)}, e^{\varepsilon\eta\rho}; \varepsilon^2 t) \\ &= \varepsilon^{\dim G} \frac{e^{\varepsilon^2 t|\rho|^2/2}\, \nu^{1/2}\, (2\pi/\varepsilon^2 t)^{r/2}}{\widehat{\Delta}_{\mathfrak{g}}(e^{\varepsilon(x-y)})\widehat{\Delta}_{\mathfrak{g}}(e^{-\varepsilon\eta\rho})} \sum_{w\in W} \epsilon(w) \sum_{\beta\in 2\pi Q^\vee} e^{-\frac{1}{2\varepsilon^2 t}|\varepsilon(x-y)-\varepsilon\eta w(\rho)+\beta|^2}. \end{aligned}$$

The denominator $\widehat{\Delta}_{\mathfrak{g}}(e^{\varepsilon(x-y)})\widehat{\Delta}_{\mathfrak{g}}(e^{-\varepsilon\eta\rho})$ vanishes to order $2|\Phi^+|$ as $\varepsilon \to 0$ and to order $|\Phi^+|$ as $\eta \to 0$, but these ostensible singularities are canceled respectively by powers of ε in the numerator and by the vanishing of the sum over the Weyl group, so that in fact f is continuous in the (ε, η) plane. Therefore we may exchange the order of limits in (2.6.10) to obtain:

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon^{\dim G} K_G(e^{\varepsilon x}, e^{\varepsilon y}; \varepsilon^2 t) &= \frac{\nu^{1/2}(2\pi)^{r/2}}{t^{\dim G/2}\Delta_{\mathfrak{g}}(\rho)} e^{-\frac{1}{2t}|x-y|^2} \\ &= \frac{\nu^{1/2}(2\pi)^{(\dim\mathfrak{g}+r)/2}}{\Delta_{\mathfrak{g}}(\rho)} K(x, y; t). \end{aligned} \tag{2.6.11}$$

Next we write:

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon^{\dim G} F(e^{\varepsilon x}, e^{\varepsilon y}, \varepsilon^2 t) &= \lim_{\varepsilon\to 0} \varepsilon^{\dim G} \int_G K_G(e^{\varepsilon x}, g e^{\varepsilon y} g^{-1}, \varepsilon^2 t)\, dg \\ &= \int_G \lim_{\varepsilon\to 0} \varepsilon^{\dim G} K_G(e^{\varepsilon x}, g e^{\varepsilon y} g^{-1}, \varepsilon^2 t)\, dg, \end{aligned}$$

where we have used bounded convergence to pull the limit under the integral. Then

(2.6.11) gives:

$$\lim_{\varepsilon\to 0}\varepsilon^{\dim G}F(e^{\varepsilon x},e^{\varepsilon y};\varepsilon^2 t)=\frac{\nu^{1/2}(2\pi)^{(\dim\mathfrak{g}+r)/2}}{\Delta_{\mathfrak{g}}(\rho)}\int_G K(\mathrm{Ad}_g x,y;t)\,dg$$
$$=\frac{\nu^{1/2}(2\pi)^{r/2}}{t^{\dim\mathfrak{g}/2}\Delta_{\mathfrak{g}}(\rho)}e^{-\frac{1}{2t}(|x|^2+|y|^2)}I(x,y;t).$$

Comparing to (2.6.9), evaluating at $t=1$, and using the observation of Remark 2.6.3 that $\Delta_{\mathfrak{g}}(\rho)=[\![\Delta_{\mathfrak{g}},\Delta_{\mathfrak{g}}]\!]/|W|$, we arrive at the Harish-Chandra formula (1.1.2) for $x\neq y\in\mathfrak{t}$. Finally, by analytic continuation we can take $x,y\in\mathfrak{t}_{\mathbb{C}}$ and remove the assumption $x\neq y$, completing the proof.

Chapter Three

Integral formulae for specific groups

This chapter provides detailed derivations of the specific forms of the Harish-Chandra integral formula (1.1.2) for all compact classical groups. We also discuss how Harish-Chandra's original result, which assumes that the group in question is connected and semisimple, extends naturally to a formula for integrals over arbitrary compact Lie groups. While some of the specific integral formulae derived in this chapter have appeared previously, for example in (Prats Ferrer et al., 2007, Forrester et al., 2019), here we show the complete calculations in a pedagogical style, making an effort not to omit details. We first study integrals over the groups $\mathrm{U}(N)$ and $\mathrm{SU}(N)$, showing how the HCIZ formula (1.1.3) can be derived from (1.1.2). Next we study integrals over $\mathrm{SO}(2N)$ and $\mathrm{O}(2N)$, $\mathrm{SO}(2N+1)$ and $\mathrm{O}(2N+1)$, $\mathrm{USp}(N)$, and finally arbitrary compact groups. The text of this chapter is adapted from part of the expository article (McSwiggen, 2018).

3.1 The HCIZ integral: the cases $G = \mathrm{U}(N)$ and $G = \mathrm{SU}(N)$

We start by showing how we can recover the HCIZ formula (1.1.3) from (1.1.2). The computation illustrates how to specialize (1.1.2) to the case of a particular group G, and our calculations of formulae for other classical groups in later sections will follow the same basic procedure.

The reader may have already remarked that Theorem 2.1.1 assumes that G is semisimple, while the group $\mathrm{U}(N)$ is not. This turns out not to matter: Theorem 2.1.1 actually allows us to compute analogous integrals over arbitrary compact G, which may be neither semisimple nor connected, by reducing to an integral over a connected semisimple group. We discuss this in detail in Section 3.5 below, but for now we simply note that the formula (1.1.2) still works as written for $\mathrm{U}(N)$.

The derivation of (1.1.3) from (1.1.2) goes as follows. Let $G = \mathrm{U}(N)$ and $\langle\cdot,\cdot\rangle$ be the Hilbert–Schmidt inner product. Then $\mathfrak{g}$ consists of N-by-N skew-Hermitian matrices, and we take $\mathfrak{t}$ to be the Cartan subalgebra of N-by-N diagonal matrices with purely imaginary entries.

First we write down the discriminant $\Delta_{\mathfrak{g}}$. The root system of $\mathrm{U}(N)$ is A_{N-1}, the same as that of $\mathrm{SU}(N)$. The matrices $e_j := iE_{jj}$, $1 \le j \le N$ are an orthonormal basis of $\mathfrak{t}$, where E_{jj} is the matrix with a single 1 in the j^{th} position on the diagonal and zeros everywhere else, and we can choose as positive roots the $N(N-1)/2$ covectors $e_j^* - e_k^*$ with $j < k$, where e_j^* indicates the linear functional $\langle e_j, \cdot\rangle \in \mathfrak{t}^*$. Writing $A = \sum_j a_j e_j \in \mathfrak{t}_{\mathbb{C}}$, we have $\Delta_{\mathfrak{g}}(A) = \prod_{i<j}(a_i - a_j) = \Delta(A)$, the Vandermonde.[1]

Next we consider the Weyl group W, which for $\mathrm{U}(N)$ is the symmetric group S_N, with $|W| = N!$. The Weyl group acts on $\mathfrak{t}$ via conjugation by the group $\mathrm{Perm}(N)$ of N-by-N permutation matrices. For $w \in W = S_N$, $\epsilon(w)$ is equal to the sign of the permutation, which is the determinant of the corresponding permutation matrix. Taking $x = A^\dagger$ and $y = B$, (1.1.2) thus becomes

$$\Delta(A)\Delta(B)\int_{\mathrm{U}(N)} e^{\mathrm{tr}(AUBU^\dagger)}dU = \frac{[\![\Delta,\Delta]\!]}{N!}\sum_{P\in\mathrm{Perm}(N)}(\det P)e^{\mathrm{tr}(APBP^\dagger)}. \tag{3.1.1}$$

We observe that the sum on the right-hand side is exactly the Leibniz formula for the determinant of the matrix $\left[e^{a_i b_j}\right]_{i,j=1}^N$, and assuming that $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$, we can divide through on both sides by $\Delta(A)\Delta(B)$.

[1]Note that in (1.1.3) we took $a_j = A_{jj}$, the j^{th} diagonal entry of A, whereas the definition we have just given differs by a factor of i: here $a_j = iA_{jj}$. This choice is convenient because it allows us to identify $\mathfrak{t}$ with the elements of $\mathfrak{t}_{\mathbb{C}}$ that have real, rather than imaginary, coordinates. One may check that the formula (1.1.3) does not depend on which of these two choices we make, since the factors of i cancel. In either case we define $\Delta(A) := \prod_{j<k}(a_j - a_k)$.

It now only remains to compute $\Delta(\partial)\Delta(x)|_{x=0}$. From the well-known formula $\Delta(x) = \det\left[x_i^{N-j}\right]_{i,j=1}^N$, it follows that the polynomial

$$\Delta(x) =: \sum_{|\beta|=\frac{N(N-1)}{2}} \pi_\beta x^\beta$$

consists of $N!$ nonzero monomial terms, each with $\beta = \sigma(0, ..., N-1)$ for some $\sigma \in S_N$, and

$$\pi_\beta = \operatorname{sgn}(\sigma) = \pm 1. \tag{3.1.2}$$

Plugging this into (2.1.3), we have

$$[\![\Delta, \Delta]\!] = \sum_{\sigma \in S_N} \left(\operatorname{sgn}(\sigma)^2 \prod_{p=0}^{N-1} \sigma(p)!\right) = N! \prod_{p=1}^{N-1} p! = \prod_{p=1}^{N} p! \tag{3.1.3}$$

which gives the correct normalization for the HCIZ formula (1.1.3), and we are done.

For the case $G = \mathrm{SU}(N)$, the formula turns out to be exactly the same:

$$\int_{\mathrm{SU}(N)} e^{\operatorname{tr}(AUBU^\dagger)} dU = \left(\prod_{p=1}^{N-1} p!\right) \frac{\det\left[e^{a_i b_j}\right]_{i,j=1}^N}{\Delta(A)\Delta(B)}. \tag{3.1.4}$$

Since $\mathrm{SU}(N)$ is semisimple unlike $\mathrm{U}(N)$, (3.1.4) is indeed a direct special case of (1.1.2) if A and B are taken to be *traceless* diagonal matrices.[2]

The similarity of the formulae for $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ is due to the fact that these two groups have the same root system. The root system of $\mathfrak{sl}(N, \mathbb{C})$ is A_{N-1}. The group $\mathrm{U}(N)$ can be decomposed as a semidirect product $\mathrm{U}(N) \cong \mathrm{SU}(N) \rtimes \mathrm{U}(1)$, and its Lie algebra is $\mathfrak{u}(N) \cong \mathfrak{su}(N) \oplus \mathfrak{u}(1)$ with complexification $\mathfrak{u}(N) \otimes \mathbb{C} = \mathfrak{gl}(N, \mathbb{C}) \cong \mathfrak{sl}(N, \mathbb{C}) \oplus \mathbb{C}$. The factor $\mathbb{C}$ lies in the center of $\mathfrak{gl}(N, \mathbb{C})$, so that it contributes no

[2]Here we identify the complexified Lie algebra $\mathfrak{su}(N) \otimes \mathbb{C} = \mathfrak{sl}(N, \mathbb{C})$ with the space of of N-by-N traceless matrices, among which the diagonal matrices form the Cartan subalgebra $\mathfrak{t}$. However, it is easily checked that (3.1.4) holds even if A and B are not traceless.

roots but adds one dimension to the Cartan subalgebra, taking us from traceless diagonal matrices in the special unitary case to arbitrary diagonal matrices in the unitary case. Thus for $\mathrm{SU}(N)$, just as in the unitary case, we find that $\Delta_{\mathfrak{g}} = \Delta$ and $W = S_N$. The calculation then proceeds exactly as above for $\mathrm{U}(N)$.

3.2 The cases $G = \mathrm{SO}(2N)$ and $\mathrm{O}(2N)$, and integrals over covering groups

In this section, we show how Theorem 2.1.1 gives an HCIZ-like formula for the special orthogonal groups $\mathrm{SO}(2N)$, and then use this formula to obtain analogous identities for $\mathrm{Spin}(2N)$ and $\mathrm{O}(2N)$. The Lie algebra of $\mathrm{SO}(2N)$ is $\mathfrak{so}(2N)$, the algebra of $2N$-by-$2N$ skew-symmetric real matrices. Its complexification is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2N, \mathbb{C})$, the algebra of $2N$-by-$2N$ skew-symmetric complex matrices. The Cartan subalgebras $\mathfrak{t}$ and $\mathfrak{t}_{\mathbb{C}}$ consist respectively of skew-symmetric real and complex matrices that are block diagonal with 2-by-2 blocks. An orthonormal basis of $\mathfrak{t}$ with respect to the Hilbert–Schmidt inner product $\langle \cdot, \cdot \rangle$ is given by

$$e_j := \frac{1}{\sqrt{2}}(E_{2j-1\ 2j} - E_{2j\ 2j-1}), \quad j = 1, \ldots, N, \tag{3.2.1}$$

where E_{jk} is the $2N$-by-$2N$ matrix with a 1 at the intersection of row j and column k and zeros everywhere else.

The root system of $\mathrm{SO}(2N)$ is D_N; as positive roots we may take the $N^2 - N$ covectors $e_j^* \pm e_k^*$ for $j < k$. The Weyl group W acts on the Cartan subalgebra by permuting the eigenvalues and changing an even number of their signs. That is, $W \cong H_{N-1} \rtimes S_N$, where H_{N-1} is the normal subgroup of $(\mathbb{Z}/2\mathbb{Z})^N$ consisting of those elements with an even number of nonzero entries. This means that each $w \in W$ can be written uniquely in the form $w = \eta\sigma$ with $\eta \in H_{N-1}$ and $\sigma \in S_N$, and $\epsilon(w) = \mathrm{sgn}(\sigma)$. Each $w \in W$ is represented in SO$(2N)$ by a matrix of the form $H_\eta P_\sigma$, where P_σ represents $\sigma \in S_N$ as a block permutation matrix with 2-by-2 blocks, and H_η represents $\eta \in H_{N-1}$ as a block-diagonal matrix with blocks equal to

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad Q_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the number of Q_2 blocks is even. The order of the Weyl group is

$$|W| = |H_{N-1}| \cdot |S_N| = 2^{N-1} N!$$

Now take $A, B \in \mathfrak{t}_{\mathbb{C}}$ and write $A = \sum_{j=1}^N a_j e_j$, $B = \sum_{j=1}^N b_j e_j$. Let $\Delta(A) = \prod_{j<k}(a_j - a_k)$ be the Vandermonde determinant, and define

$$\Xi(A) := \prod_{j<k}(a_j + a_k). \tag{3.2.2}$$

Then $\Delta_{\mathfrak{g}}(A) = \Xi(A)\Delta(A)$. Plugging all of this into (1.1.2), we have:

$$\begin{aligned} \Xi(A)\Delta(A)\Xi(B)\Delta(B) \int_{\mathrm{SO}(2N)} & e^{\mathrm{tr}(AOBO^T)} dO \\ &= \frac{[\![\Xi\Delta, \Xi\Delta]\!]}{2^{N-1}N!} \sum_{\eta \in H_{N-1}} \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) e^{\mathrm{tr}(AH_\eta P_\sigma B P_\sigma^T H_\eta^T)}. \end{aligned} \tag{3.2.3}$$

There are two steps remaining: to simplify the sum over the Weyl group, and to determine the leading constant that gives the normalization. We first turn to simplifying the sum. Here too there is a determinantal structure under the surface, albeit a more complicated one than in the unitary case. Since the trace of a product of matrices is invariant under cyclic permutations of the product, we have

$$e^{\operatorname{tr}(AH_\eta P_\sigma BP_\sigma^T H_\eta^T)} = e^{\operatorname{tr}(P_\sigma^T H_\eta^T AH_\eta P_\sigma B)} = e^{2\sum_{j=1}^N \eta^{-1}(j)a_{\sigma^{-1}(j)}b_j},$$

where we consider η as a function $\eta : \{1, \ldots, N\} \to \{\pm 1\}$. We can write $\operatorname{sgn}(\eta) = \prod_{j=1}^N \eta(j)$. Then H_{N-1} consists of $\eta \in (\mathbb{Z}/2\mathbb{Z})^N$ with $\operatorname{sgn}(\eta) = 1$.

Consider the map $A \mapsto H_\eta A H_\eta^T$. The nonzero entries of H_η are contained in N 2-by-2 blocks $L_1, \ldots, L_N$ along the diagonal, with

$$L_j = \begin{cases} I_2, & \eta(j) = 1 \\ Q_2, & \eta(j) = -1. \end{cases}$$

Under conjugation by H_η, each block L_j sends $a_j \mapsto \eta(j)a_j$, so we have

$$\sum_{\eta \in H_{N-1}} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) e^{\operatorname{tr}(AH_\eta P_\sigma BP_\sigma^T H_\eta^T)} = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \sum_{\eta \in H_{N-1}} e^{2\sum_{j=1}^N \eta(j)a_{\sigma(j)}b_j}. \tag{3.2.4}$$

We can rearrange terms to make this expression more concise, by noting that

$$\sum_{\eta \in H_{N-1}} e^{2\sum_{j=1}^N \eta(j)a_{\sigma(j)}b_j} = \frac{1}{2} \sum_{\eta \in (\mathbb{Z}/2\mathbb{Z})^N} \left[e^{2\sum_{j=1}^N \eta(j)a_{\sigma(j)}b_j} + \operatorname{sgn}(\eta) e^{2\sum_{j=1}^N \eta(j)a_{\sigma(j)}b_j} \right],$$

so that the sum over H_{N-1} can be written as an average of two sums over $(\mathbb{Z}/2\mathbb{Z})^N$, one signed and one unsigned. Each of these sums can be factored into a product of hyperbolic sines or cosines:

$$\sum_{\eta \in (\mathbb{Z}/2\mathbb{Z})^N} e^{2\sum_{j=1}^N \eta(j)a_{\sigma(j)}b_j} = \prod_{j=1}^N (e^{2a_{\sigma(j)}b_j} + e^{2a_{\sigma(j)}b_j}) = 2^N \prod_{j=1}^N \cosh(2a_{\sigma(j)}b_j), \tag{3.2.5}$$

$$\sum_{\eta\in(\mathbb{Z}/2\mathbb{Z})^N} \mathrm{sgn}(\eta) e^{2\sum_{j=1}^N \eta(j) a_{\sigma(j)} b_j} = \prod_{j=1}^N (e^{2a_{\sigma(j)}b_j} - e^{2a_{\sigma(j)}b_j}) = 2^N \prod_{j=1}^N \sinh(2a_{\sigma(j)}b_j). \tag{3.2.6}$$

These identities finally give:

$$\begin{aligned}
&\sum_{\eta\in H_{N-1}} \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) e^{\mathrm{tr}(AH_\eta P_\sigma B P_\sigma^T H_\eta^T)} \\
&\qquad = \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \frac{1}{2} \left[2^N \prod_{j=1}^N \cosh(2a_{\sigma(j)}b_j) + 2^N \prod_{j=1}^N \sinh(2a_{\sigma(j)}b_j) \right] \\
&\qquad\qquad = 2^{N-1} \Big(\det \big[\cosh(2a_j b_k)\big]_{j,k=1}^N + \det \big[\sinh(2a_j b_k)\big]_{j,k=1}^N \Big).
\end{aligned}$$

Next we turn to evaluating $[\![\Xi\Delta, \Xi\Delta]\!]$. We will find a way of rewriting the polynomial $\Xi\Delta$ that also allows us to simplify the expression $\Xi(A)\Delta(A)\Xi(B)\Delta(B)$ that appears in (3.2.3). To that end, we introduce a last piece of notation. Given $A = \sum a_j e_j \in \mathfrak{t}_\mathbb{C}$, define

$$A^{(2)} := \sum a_j^2 e_j \in \mathfrak{t}_\mathbb{C}.$$

Note that $A^{(2)}$ is in general not equal to A^2. Similarly, given a polynomial $p(x) = \sum_\beta c_\beta x^\beta$, denote by $p(\partial^2)$ the differential operator

$$p(\partial^2) := \sum_\beta c_\beta \frac{\partial^{2|\beta|}}{\partial x^{2\beta}}.$$

Now assume $a_i \neq \pm a_j$, $b_i \neq \pm b_j$ for $i \neq j$, so that $\Delta_\mathfrak{g}(A), \Delta_\mathfrak{g}(B) \neq 0$. We find that we can write

$$\Xi(A) = \prod_{i<j} (a_i + a_j) = \prod_{i<j} \frac{a_i^2 - a_j^2}{a_i - a_j} = \frac{\Delta(A^{(2)})}{\Delta(A)}.$$

That is, the polynomial Ξ is in fact a ratio of two Vandermondes, and so we have

$$\Xi(A)\Delta(A)\Xi(B)\Delta(B) = \Delta(A^{(2)})\Delta(B^{(2)}).$$

Moreover,

$$\Xi(\partial)\Delta(\partial)(\Xi\Delta)(x)\big|_{x=0} = \Delta(\partial^2)\Delta(x^{(2)})\big|_{x=0}$$

and this is an expression that we already know how to evaluate using Lemma 2.1.2 and our calculation (3.1.3) for the unitary case. With π_β as in (3.1.2) and $\beta_0 := (0, \ldots, N-1)$, we can write

$$\Delta(x^{(2)}) = \sum_{\sigma \in S_N} \pi_\beta x^{2\sigma(\beta_0)}. \tag{3.2.7}$$

Then Lemma 2.1.2 gives

$$\Delta(\partial^2)\Delta(x^{(2)})\big|_{x=0} = N! \prod_{p=1}^{N-1} (2p)!$$

Plugging all of the above results into (3.2.3) we arrive at the desired formula, which is an analogue of the HCIZ integral for $\mathrm{SO}(2N)$:

$$\int_{\mathrm{SO}(2N)} e^{\mathrm{tr}(AOBO^T)} dO = \left(\prod_{p=1}^{N-1} (2p)! \right) \frac{\det\left[\cosh(2a_j b_k)\right]_{j,k=1}^N + \det\left[\sinh(2a_j b_k)\right]_{j,k=1}^N}{\Delta(A^{(2)})\Delta(B^{(2)})}. \tag{3.2.8}$$

We now move on to the calculations for $\mathrm{Spin}(2N)$ and $\mathrm{O}(2N)$. The group $\mathrm{Spin}(2N)$ is a double cover of $\mathrm{SO}(2N)$, so that it has the same Lie algebra $\mathfrak{so}(2N)$. Since the right-hand side of (1.1.2) depends only on the algebra, we can immediately conclude that the formula for $\mathrm{SO}(2N)$ applies exactly as written:

$$\int_{\mathrm{Spin}(2N)} e^{\langle \mathrm{Ad}_g A, B \rangle} dg = \int_{\mathrm{SO}(2N)} e^{\mathrm{tr}(AOBO^T)} dO. \tag{3.2.9}$$

In general, if $\tilde{G}$ is a connected compact covering group of G, then the integral formula for G will hold as written for $\tilde{G}$ as well. This equality is a consequence of the following proposition, which we will use again below when we consider integrals over arbitrary compact groups in Theorem 3.5.4. Note that if $\tilde{G}$ is a covering group of G then their Lie algebras are isomorphic. The statement below follows from the observation that, moreover, $\tilde{G}$ and G have the same adjoint orbits.

Proposition 3.2.1. *Let G be a compact, connected Lie group, not necessarily semisimple, with Lie algebra $\mathfrak{g}$. Let $\tilde{G}$ be a compact, connected covering group of G, and let $f : \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$. Then for $x \in \mathfrak{g}_{\mathbb{C}}$,*

$$\int_{\tilde{G}} f(\mathrm{Ad}_{\tilde{g}} x)\ d\tilde{g} = \int_G f(\mathrm{Ad}_g x)\ dg, \tag{3.2.10}$$

where $d\tilde{g}$ and dg are the normalized Haar measures.

Proof. Let $\pi : \tilde{G} \to G$ be the covering homomorphism. The exponential maps $\exp_{\tilde{G}}$ and $\exp_G$ are surjective since both groups are compact and connected, so we can write $\tilde{g} \in \tilde{G}$ as $\tilde{g} = \exp_{\tilde{G}}(y)$ for some $y \in \mathfrak{g}$. We then have $\pi(\tilde{g}) = \exp_G(y)$, so that

$$\mathrm{Ad}_{\tilde{g}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}_y^k = \mathrm{Ad}_{\pi(\tilde{g})}.$$

Thus $x \in \mathfrak{g}_{\mathbb{C}}$ has the same orbit $\mathcal{O}_x$ under both adjoint actions. The pushforwards of $d\tilde{g}$ and dg to $\mathcal{O}_x$ are normalized measures that are invariant under either adjoint action, so they must be equal, and writing both sides of (3.2.10) as an integral over $\mathcal{O}_x$ gives the desired equality. $\square$

In the case $G = \mathrm{O}(2N)$, there is an additional subtlety: even though $\mathrm{O}(2N)$ is compact and has the same Lie algebra as $\mathrm{SO}(2N)$, it is not connected, so that we cannot use (1.1.2) directly. Instead we must use a trick that allows us to reduce an integral over a non-connected group to an integral over its identity component, which we will revisit in the proof of Theorem 3.5.4 below. We have

$$\mathrm{O}(2N) = \mathrm{SO}(2N) \sqcup \mathrm{O}^-(2N),$$

where $\mathrm{O}^-(2N)$ is the orientation-reversing component consisting of $2N$-by-$2N$ orthogonal matrices with determinant -1. Notice that every matrix in $\mathrm{O}^-(2N)$ can be written as $\tilde{I}O$, where $O \in \mathrm{SO}(2N)$ and $\tilde{I}$ is the matrix with $\tilde{I}_{11} = -1$, $\tilde{I}_{ii} = 1$ for $i > 1$, and $\tilde{I}_{ij} = 0$ for $i \neq j$. Thus

$$\begin{aligned}\int_{\mathrm{O}(2N)} & e^{\mathrm{tr}(AOBO^T)} dO \\ &= \int_{\mathrm{SO}(2N)} e^{\mathrm{tr}(AOBO^T)} dO + \int_{O^-(2N)} e^{\mathrm{tr}(AOBO^T)} dO \\ &= \int_{\mathrm{SO}(2N)} \left[e^{\mathrm{tr}(AOBO^T)} + e^{\mathrm{tr}(A\tilde{I}OBO^T\tilde{I}^T)} \right] dO.\end{aligned}$$

Here we take the Haar measure dO to be normalized over all of $\mathrm{O}(2N)$, so that in this case $\int_{\mathrm{SO}(2N)} dO = 1/2$ and the integration measure on $\mathrm{SO}(2N)$ differs from that used in (3.2.8) by a factor of 2.

Next, notice that by the invariance of the trace under cyclic permutations and the fact that $\tilde{I}^T = \tilde{I}$, we have

$$e^{\mathrm{tr}(A\tilde{I}OBO^T\tilde{I}^T)} = e^{\mathrm{tr}(\tilde{I}A\tilde{I}OBO^T)}.$$

Conjugation of A by $\tilde{I}$ has the effect of sending $a_1 \mapsto -a_1$. This has the consequence that for the orientation-reversing component, the sum over the Weyl group in (3.2.4) becomes a sum over $-\eta$ rather than η. Therefore in order to integrate over both components of $\mathrm{O}(2N)$, we should simply take η in (3.2.4) to run over all of $(\mathbb{Z}/2\mathbb{Z})^N$ rather than merely over H_{N-1}, and then divide by 2 to account for the different normalization. Another application of (3.2.5) then gives the desired integral formula for $\mathrm{O}(2N)$:

$$\int_{\mathrm{O}(2N)} e^{\mathrm{tr}(AOBO^T)} dO = \left(\prod_{p=1}^{N-1}(2p)!\right) \frac{\det\left[\cosh(2a_j b_k)\right]_{j,k=1}^N}{\Delta(A^{(2)})\Delta(B^{(2)})}. \tag{3.2.11}$$

3.3 The cases $G = \mathrm{SO}(2N+1)$ and $\mathrm{O}(2N+1)$

The orthogonal groups $\mathrm{SO}(2N+1)$ and $\mathrm{O}(2N+1)$ need to be treated separately from $\mathrm{SO}(2N)$ and $\mathrm{O}(2N)$, as their root system B_N differs from the system D_N of $\mathrm{SO}(2N)$.

We start with $G = \mathrm{SO}(2N+1)$. The Lie algebra of $\mathrm{SO}(2N+1)$ is $\mathfrak{g} = \mathfrak{so}(2N+1)$, the algebra of $(2N+1)$-by-$(2N+1)$ skew-symmetric real matrices, with complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2N+1, \mathbb{C})$, the algebra of $(2N+1)$-by-$(2N+1)$ skew-symmetric complex matrices. We take the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ to be the skew-symmetric block-diagonal complex matrices whose diagonal consists of N 2-by-2 blocks followed by a single zero in the bottom right corner; then $\mathfrak{t}$ consists of those matrices in $\mathfrak{t}_{\mathbb{C}}$ that have strictly real entries. An orthonormal basis of $\mathfrak{t}$ with respect to the Hilbert–Schmidt inner product is given by

$$e_j := \frac{1}{\sqrt{2}}(E_{2j-1\ 2j} - E_{2j\ 2j-1}), \quad j = 1, \ldots, N,$$

which is nearly identical to (3.2.1), except that now E_{jk} is the $(2N+1)$-by-$(2N+1)$ matrix with a 1 at the intersection of row j and column k and zeros everywhere else.

The difference from the case $G = \mathrm{SO}(2N)$ arises because B_N contains additional roots of a different length: we must consider both the $N^2 - N$ *long* positive roots $e_j^* \pm e_k^*$ for $j < k$, corresponding to the roots of D_N, and also the N *short* positive roots e_j^*, $1 \leq j \leq N$.

Since $\mathrm{SO}(2N+1)$ has more roots than $\mathrm{SO}(2N)$, its Weyl group is bigger as well. We have

$$W \cong (\mathbb{Z}/2\mathbb{Z})^N \rtimes S_N,$$

so that $|W| = |(\mathbb{Z}/2\mathbb{Z})^N| \cdot |S_N| = 2^N N!$. Like before, this means that each $w \in W$ can be written uniquely in the form $w = \eta\sigma$, with $\sigma \in S_N$ and $\eta \in (\mathbb{Z}/2\mathbb{Z})^N$. Since η can now have a negative sign, we have $\epsilon(w) = \mathrm{sgn}(\eta)\mathrm{sgn}(\sigma)$. Each element of W is then represented in $\mathrm{SO}(2N+1)$ by a matrix of the form $H_\eta P_\sigma$, where P_σ represents $\sigma \in S_N$ as a block permutation matrix with 2-by-2 blocks followed by a final 1 in the bottom right corner, and H_η represents $\eta \in (\mathbb{Z}/2\mathbb{Z})^N$ as a block-diagonal matrix with N 2-by-2 blocks equal to

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad Q_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

followed by a final ± 1 in the bottom right corner chosen so that $\det H_\eta = 1$. Because we can choose this final entry to ensure a unit determinant, unlike in the case of $\mathrm{SO}(2N)$ we do not need to require the number of Q_2 blocks to be even.

Let $A = \sum_{j=1}^N a_j e_j$, $B = \sum_{j=1}^N b_j e_j \in \mathfrak{t}_{\mathbb{C}}$. Let $\Delta(A) = \prod_{j<k}(a_j - a_k)$ be the Vandermonde, and define Ξ as in (3.2.2). Additionally, define $\Phi(A) := \prod_{j=1}^N a_j$.

Then $\Delta_{\mathfrak{g}} = \Xi\Delta\Phi$. Plugging into (1.1.2), we have:

$$\begin{aligned}\Xi(A)\Delta(A)\Phi(A)\Xi(B)\Delta(B)\Phi(B)&\int_{\mathrm{SO}(2N+1)} e^{\mathrm{tr}(AOBO^T)}dO \\ &= \frac{[\![\Xi\Delta\Phi,\Xi\Delta\Phi]\!]}{2^N N!}\sum_{\eta\in(\mathbb{Z}/2\mathbb{Z})^N}\sum_{\sigma\in S_N}\mathrm{sgn}(\eta)\mathrm{sgn}(\sigma)e^{\mathrm{tr}(AH_\eta P_\sigma BP_\sigma^T H_\eta^T)}.\end{aligned} \tag{3.3.1}$$

Proceeding as before, we next turn to simplifying the sum over the Weyl group. Again we can write

$$e^{\mathrm{tr}(AH_\eta P_\sigma BP_\sigma^T H_\eta^T)} = e^{\mathrm{tr}(P_\sigma^T H_\eta A H_\eta^T P_\sigma B)},$$

and the map $A \mapsto H_\eta A H_\eta^T$ acts on $\mathfrak{t}_{\mathbb{C}}$ in a similar way to the previous case. The nonzero entries of H_η are contained in N diagonal blocks $L_1,\dots,L_N$, each of which is equal to one of the 2-by-2 matrices I_2 or Q_2, plus the final ±1 on the diagonal, which we ignore since the final row and column of any matrix in $\mathfrak{t}_{\mathbb{C}}$ are all zeros. Just like before, under conjugation by H_η, the block L_j sends $a_j \mapsto \eta(j)a_j$. We apply (3.2.6) to conclude that

$$\begin{aligned}\sum_{\eta\in(\mathbb{Z}/2\mathbb{Z})^N}\sum_{\sigma\in S_N}\mathrm{sgn}(\eta)\mathrm{sgn}(\sigma)e^{\mathrm{tr}(AH_\eta P_\sigma BP_\sigma^T H_\eta^T)} &= \sum_{\sigma\in S_N}\mathrm{sgn}(\eta)2^N\prod_{j=1}^N \sinh(2a_{\sigma(j)}b_j) \\ &= 2^N \det\left[\sinh(2a_j b_k)\right]_{j,k=1}^N.\end{aligned}$$

It now remains to compute the constant $[\![\Xi\Delta\Phi,\Xi\Delta\Phi]\!]$. With reference to (3.2.7), we can write

$$\Xi(x)\Delta(x)\Phi(x) = \Delta(x^{(2)})\Phi(x) = \sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)x^{2\sigma(\beta_0)+\vec{1}}$$

where $\vec{1}$ is the N-component multi-index $(1, 1, \ldots, 1)$. Lemma 2.1.2 then gives

$$\Xi(\partial)\Delta(\partial)\Phi(\partial)\big[\Xi(x)\Delta(x)\Phi(x)\big]\Big|_{x=0} = \sum_{\sigma\in S_N}\left(\operatorname{sgn}(\sigma)^2\prod_{p=0}^{N-1}(2\sigma(p)+1)!\right)$$

$$= N!\prod_{p=1}^{N-1}(2p+1)!$$

We substitute this into (3.3.1) under the assumptions that $a_i \neq \pm a_j$, $b_i \neq \pm b_j$ for $i \neq j$ and that all a_i, b_i are nonzero, so that we can divide by $\Delta_{\mathfrak{g}}(A)\Delta_{\mathfrak{g}}(B)$. This yields the integral formula for $\mathrm{SO}(2N+1)$:

$$\int_{\mathrm{SO}(2N+1)} e^{\operatorname{tr}(AOBO^T)}dO \quad = \quad \left(\prod_{p=1}^{N-1}(2p+1)!\right)\frac{\det\left[\sinh(2a_jb_k)\right]_{j,k=1}^N}{\Delta(A^{(2)})\Delta(B^{(2)})\prod_{i=1}^N a_ib_i}. \quad (3.3.2)$$

Proposition 3.2.1 yields a formula for $\mathrm{Spin}(2N+1)$ that is analogous to (3.2.9).

For the case $G = \mathrm{O}(2N+1)$, we must compute the integral over the orientation-reversing component. Applying the same logic as in the case $G = \mathrm{O}(2N)$, we find

$$\int_{\mathrm{O}(2N+1)} e^{\operatorname{tr}(AOBO^T)}dO = \int_{\mathrm{SO}(2N+1)} e^{\operatorname{tr}(AOBO^T)}dO + \int_{\mathrm{O}^-(2N+1)} e^{\operatorname{tr}(AOBO^T)}dO$$

$$= \int_{\mathrm{SO}(2N+1)}\left[e^{\operatorname{tr}(AOBO^T)} + e^{\operatorname{tr}(\tilde{I}A\tilde{I}OBO^T)}\right]dO,$$

where the measure dO is now taken to be normalized over the entire group $\mathrm{O}(2N+1)$. The same argument used in the calculation for $\mathrm{O}(2N)$ shows that integrating over the orientation-reversing component amounts to sending $\eta \mapsto -\eta$ in the sum over the Weyl group on the right-hand side of (1.1.2). But since in this case we are summing over all of $(\mathbb{Z}/2\mathbb{Z})^N$, this does not actually change the sum, so that the integrals over both connected components of $\mathrm{O}(2N+1)$ are equal. Since dO is normalized over the entire group, each of these two integrals is equal to half the integral in (3.3.2), and the final formula for $\mathrm{O}(2N+1)$ is identical to the formula for $\mathrm{SO}(2N+1)$:

$$\int_{\mathrm{O}(2N+1)} e^{\mathrm{tr}(AOBO^T)} dO \quad = \quad \left(\prod_{p=1}^{N-1}(2p+1)!\right) \frac{\det\left[\sinh(2a_j b_k)\right]_{j,k=1}^N}{\Delta(A^{(2)})\Delta(B^{(2)})\prod_{i=1}^N a_i b_i}. \qquad (3.3.3)$$

3.4 The case $G = \mathrm{USp}(N)$

The unitary symplectic, or compact symplectic, group $\mathrm{USp}(N)$ is the compact real form of $\mathrm{Sp}(2N,\mathbb{C})$, the symplectic group over the complex numbers. The group $\mathrm{Sp}(2N,\mathbb{C})$ consists of $2N$-by-$2N$ complex matrices S satisfying

$$SJS^T = J,$$

where

$$J = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

is the standard symplectic matrix. (Here I_N indicates the N-by-N identity matrix.) The name "unitary symplectic" comes from the fact that

$$\mathrm{USp}(N) = \mathrm{U}(2N) \cap \mathrm{Sp}(2N,\mathbb{C}) = \left\{ \begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix} \in \mathrm{U}(2N) \right\}. \qquad (3.4.1)$$

In fact, $\mathrm{USp}(N)$ is isomorphic to the quaternionic unitary (or "hyperunitary") group $\mathrm{U}(N,\mathbb{H})$ of linear operators on $\mathbb{H}^N$ that preserve the standard Hermitian form

$$(v,w) := \sum_{i=1}^N \bar{v}_i w_i, \qquad v,w \in \mathbb{H}^N.$$

Thus we could equivalently consider $\mathrm{USp}(N)$ as a subgroup of $\mathrm{GL}(N,\mathbb{H})$, though here we regard it as a subgroup of $\mathrm{GL}(2N,\mathbb{C})$.

The complexified Lie algebra of $\mathrm{USp}(N)$ is

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(2N,\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \in \mathfrak{gl}(2n,\mathbb{C}) \;\middle|\; B^T = B,\ C^T = C \right\}.$$

From (3.4.1) we then have $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}} \cap \mathfrak{u}(2N)$, so that $\mathfrak{g}$ consists of the matrices in $\mathfrak{sp}(2N,\mathbb{C})$ that are also anti-Hermitian. We take $\mathfrak{t}$ to be the Cartan subalgebra of diagonal matrices in $\mathfrak{g}$. It is spanned by the orthonormal basis

$$e_j := \frac{i}{\sqrt{2}}(E_{jj} - E_{j+N\ j+N}), \quad 1 \le j \le N,$$

so that $\mathrm{USp}(N)$ has rank N. The root system in this case is C_N, which is similar to the root system B_N, but with the short roots multiplied by 2 (so that for C_N these are the long roots). That is, in the basis $\{e_j^*\}_{j=1}^N$ of $\mathfrak{t}^*$, the N long positive roots are given by $2e_j^*$ for $1 \le j \le N$ and the $N^2 - N$ short positive roots are given by $e_j^* \pm e_k^*$ for $j < k$. Thus for the C_N root system,

$$\Delta_{\mathfrak{g}}(x) = 2^N \Xi(x)\Delta(x)\Phi(x), \tag{3.4.2}$$

which differs from the B_N case only by a factor of 2^N. Our previous calculation for the B_N system therefore immediately gives

$$\Delta_{\mathfrak{g}}(\partial)\Delta_{\mathfrak{g}}(x)\big|_{x=0} = 2^{2N} N! \prod_{p=1}^{N-1} (2p+1)!$$

The Weyl group for C_N is isomorphic to the group for B_N:

$$W \cong (\mathbb{Z}/2\mathbb{Z})^N \rtimes S_N,$$

with $|W| = 2^N N!$, and it acts on $\mathfrak{t}$ in the same fashion. Thus if we let $A = \sum_{j=1}^N a_j e_j$, $B = \sum_{j=1}^N b_j e_j$ and take $x = A^\dagger$, $y = B$ in (1.1.2), then the sum over the Weyl group on the right-hand side is equal to

$$\sum_{\eta \in (\mathbb{Z}/2\mathbb{Z})^N} \sum_{\sigma \in S_N} \operatorname{sgn}(\eta)\operatorname{sgn}(\sigma) e^{\operatorname{tr}(AH_\eta P_\sigma B P_\sigma^T H_\eta^T)} = 2^N \det\left[\sinh(2a_j b_k)\right]_{j,k=1}^N$$

just as in the case $G = \mathrm{SO}(2N+1)$. Putting this all together in (1.1.2) and assuming as before that $a_i \neq \pm a_j$, $b_i \neq \pm b_j$ for $i \neq j$ and that all a_i, b_i are nonzero, we conclude:

$$\int_{\mathrm{USp}(N)} e^{\operatorname{tr}(ASBS^\dagger)} dS = \left(\prod_{p=1}^{N-1} (2p+1)!\right) \frac{\det\left[\sinh(2a_j b_k)\right]_{j,k=1}^N}{\Delta(A^{(2)})\Delta(B^{(2)}) \prod_{i=1}^N a_i b_i}. \tag{3.4.3}$$

Thus for the unitary symplectic integral, we end up with exactly the same expression as for the odd special orthogonal groups: the sum over the Weyl group in each case gives the same result, and although $\Delta_\mathfrak{g}$ differs from the B_N case by a factor of 2^{2N}, these factors cancel from either side of (3.4.3), so that the right-hand side matches that of (3.3.2).

3.5 Integrals over arbitrary compact groups

In this section, we show how Theorem 2.1.1 can be used to compute integrals over arbitrary compact Lie groups that may be neither semisimple nor connected, such as $\mathrm{U}(N)$ and $\mathrm{O}(N)$. As a special case, this calculation justifies the derivation of the HCIZ integral (1.1.3) from (1.1.2). The proof makes use of the classification of compact Lie groups, as stated in the following theorem.[3]

Theorem 3.5.1. *Every compact, connected real Lie group is of the form* $(K \times H)/Z$*, where* K *is connected, compact, and semisimple,* $H \cong (S^1)^d$ *is a torus, and* Z *is a finite subgroup of the center of* $K \times H$ *satisfying* $Z \cap H = \{\mathrm{id}_{K\times H}\}$*.*

The idea is that we will express an integral over an arbitrary compact group G as a sum of integrals over its identity component G_1, which we decompose as $G_1 \cong (K \times H)/Z$ following Theorem 3.5.1. We then rewrite each of these integrals over G_1 as an integral over the semisimple group K and evaluate using (1.1.2). Note that G, G_1, $K \times H$ and K all have the same root system Φ, so that the discriminant $\Delta_{\mathfrak{g}}$ is the same for all of these groups. We write W_Φ for the Weyl group generated by reflections through the root hyperplanes. Since G, G_1 and $K \times H$ have isomorphic Lie algebras, we will write $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ for the complexified Lie algebra and Cartan subalgebra of all three. Define $\mathfrak{k} := \mathrm{Lie}(K) \otimes \mathbb{C}$ and $\mathfrak{h} := \mathrm{Lie}(H) \otimes \mathbb{C}$, so that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{h}$, and let $\mathfrak{t}_K$ be the Cartan subalgebra of $\mathfrak{k}$ obtained by taking the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{t}_{\mathbb{C}}$.

Remark 3.5.2. There is no standardized definition for the Weyl group of a disconnected Lie group, since the definition for connected groups can be generalized in multiple inequivalent ways. When G is compact and connected, one usually defines $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of the maximal torus T in G. It is then a nontrivial theorem that W is isomorphic to the group W_Φ generated by reflections through the hyperplanes $\{x \in \mathfrak{t} \mid \langle \alpha, x\rangle = 0\}$ for α a simple root. However, when G is disconnected, $N_G(T)/T$ is *not* isomorphic to W_Φ but rather contains it as a proper subgroup. To avoid confusion on this point, we do not define the "Weyl group of G" in the disconnected case, choosing instead to work only with the Weyl

[3] This theorem is a slightly weakened version of (Procesi, 2007, ch. 10, sect. 7.2, Theorem 4).

group W_Φ associated to the root system Φ (or equivalently, to the Lie algebra $\mathfrak{g}$). See Remark 3.5.5 below for further discussion of the group $N_G(T)/T$ when G is disconnected. $\triangle$

First we show that the torus factor H does not change the right-hand side of the formula (1.1.2).

Lemma 3.5.3. *For all $x, y \in \mathfrak{t}_\mathbb{C}$,*

$$\Delta_\mathfrak{g}(x)\Delta_\mathfrak{g}(y)\int_{K\times H} e^{\langle \mathrm{Ad}_{(k,h)}x, y\rangle} dk\, dh = \frac{[\![\Delta_\mathfrak{g}, \Delta_\mathfrak{g}]\!]}{|W_\Phi|}\sum_{w\in W_\Phi} \epsilon(w) e^{\langle w(x), y\rangle},$$

where dk and dh are the normalized Haar measures on K and H respectively.

Proof. For $x \in \mathfrak{t}_\mathbb{C}$, write $x = x^K + x^H$, with $x^K \in \mathfrak{t}_K$, $x^H \in \mathfrak{h}$. All of the roots vanish on $\mathfrak{h}$, so that $\Delta_\mathfrak{g}(x) = \Delta_\mathfrak{g}(x^K)$. Since $\{\mathrm{id}_K\}\times H$ lies in the center of $K\times H$, we have $\mathrm{Ad}_{(k,h)}x = x^H + \mathrm{Ad}_k x^K$. Then since $\mathfrak{h}$ is orthogonal to $\mathfrak{k}$, we find

$$\langle \mathrm{Ad}_g x, y\rangle = \langle \mathrm{Ad}_g(x^H + x^K), y^H + y^K\rangle = \langle x^H, y^H\rangle + \langle \mathrm{Ad}_g x^K, y^K\rangle.$$

Accordingly, we have:

$$\begin{aligned}
&\Delta_\mathfrak{g}(x)\Delta_\mathfrak{g}(y)\int_{K\times H} e^{\langle \mathrm{Ad}_{(k,h)}x, y\rangle} dk\, dh \\
&\qquad = \Delta_\mathfrak{g}(x^K)\Delta_\mathfrak{g}(y^K)\, e^{\langle x^H, y^H\rangle}\int_K\int_H e^{\langle \mathrm{Ad}_k x^K, y^K\rangle} dh\, dk \\
&\qquad = \Delta_\mathfrak{g}(x^K)\Delta_\mathfrak{g}(y^K)\, e^{\langle x^H, y^H\rangle}\int_K e^{\langle \mathrm{Ad}_k x^K, y^K\rangle} dk \\
&\qquad = \frac{[\![\Delta_\mathfrak{g}, \Delta_\mathfrak{g}]\!]}{|W_\Phi|}\sum_{w\in W_\Phi}\epsilon(w) e^{\langle x^H, y^H\rangle} e^{\langle w(x^K), y^K\rangle} = \frac{[\![\Delta_\mathfrak{g}, \Delta_\mathfrak{g}]\!]}{|W_\Phi|}\sum_{w\in W_\Phi}\epsilon(w) e^{\langle w(x), y\rangle}
\end{aligned}$$

as desired. $\square$

Because $K \times H$ is a compact, connected, $|Z|$-fold covering group of $(K \times H)/Z$, by Proposition 3.2.1 we have

$$\int_{K \times H} e^{\langle \mathrm{Ad}_{(k,h)} x, y\rangle} dk\, dh = \int_{(K\times H)/Z} e^{\langle \mathrm{Ad}_g x, y\rangle}\, dg_{(K\times H)/Z}, \tag{3.5.1}$$

where $dg_{(K\times H)/Z}$ is normalized Haar measure on $(K \times H)/Z$. We now have all the tools to extend Theorem 2.1.1 to arbitrary compact G.

Theorem 3.5.4. *Let G be a compact real Lie group with root system Φ, normalized Haar measure dg, and m connected components $G_1, \ldots, G_m$. For $j = 1, \ldots, m$, let $g_j \in G_j$. Then for all $x, y \in \mathfrak{t}_{\mathbb{C}}$,*

$$\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y) \int_G e^{\langle \mathrm{Ad}_g x, y\rangle} dg = \frac{1}{m} \frac{[\![\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{g}}]\!]}{|W_\Phi|} \sum_{j=1}^{m} \sum_{w \in W_\Phi} \epsilon(w) e^{\langle w(\mathrm{Ad}_{g_j} x),\, y\rangle}. \tag{3.5.2}$$

Proof. Take G_1 to be the identity component of G. First we note that

$$\begin{aligned}
\int_G e^{\langle \mathrm{Ad}_g x, y\rangle} dg &= \sum_{j=1}^{m} \int_{G_j} e^{\langle \mathrm{Ad}_g x, y\rangle} dg \\
&= \sum_{j=1}^{m} \int_{G_1} e^{\langle \mathrm{Ad}_{g g_j} x, y\rangle} dg = \frac{1}{m} \sum_{j=1}^{m} \int_{G_1} e^{\langle \mathrm{Ad}_g (\mathrm{Ad}_{g_j} x), y\rangle} dg_1,
\end{aligned} \tag{3.5.3}$$

where dg_1 is the Haar measure normalized so that the identity component G_1, rather than the entire group G, has unit volume. By Theorem 3.5.1 we can write $G_1 = (K \times H)/Z$ as above. Applying (3.5.1) and then Lemma 3.5.3 to the final expression in (3.5.3) completes the proof of the theorem. □

Remark 3.5.5. Suppose we were to define the Weyl group of a compact Lie group G as $W := N_G(T)/T$, irrespective of whether or not G is connected. As in Theorem 3.5.4, let $G_1, \ldots, G_m$ be the connected components of G. By a simple argument using Cartan's theorem on the conjugacy of maximal tori, one can show that $N_G(T)$ intersects each component G_j, and that $|W| = m|W_\Phi|$. Since T acts trivially on $\mathfrak{t}$, the adjoint action of $N_G(T)$ on $\mathfrak{t}$ descends to an action of W. If we choose each g_j in Theorem 3.5.4 to lie in $G_j \cap N_G(T)$, then each element of W acts on $\mathfrak{t}$ as $w \circ \mathrm{Ad}_{g_j}$ for some unique g_j and $w \in W_\Phi$. If we then extend the sign function ϵ from W_Φ to W by defining $\epsilon(w \circ \mathrm{Ad}_{g_j}) = \epsilon(w)$, the double sum in Theorem 3.5.4 can be rewritten as a single sum over W, so that the right-hand side of (3.5.2) looks identical to the right-hand side of (1.1.2). Thus there is a certain sense in which (1.1.2) holds "as written" for any compact Lie group, though for disconnected groups the formulation (3.5.2) is arguably more transparent. $\triangle$

Chapter Four

Majorization inequalities

4.1 Introduction

In this chapter we prove a majorization inequality for Harish-Chandra integrals and show that this implies a similar inequality for a family of rational functions associated to the irreducible characters of a compact, connected Lie group. This result generalizes a property of Schur polynomials conjectured by Cuttler et al. (2011) and proven by Sra (2016). We then conjecture a further generalization of this inequality for Heckman–Opdam polynomials, and we prove one of the two directions of implication that comprise this conjecture. Theorem 4.2.3, the main result of this chapter, forms part of forthcoming joint work with Jonathan Novak.

If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of an integer r, the *monomial symmetric polynomial* associated to λ is defined as

$$m_\lambda(x_1, \ldots, x_n) := \frac{|S_n \cdot \lambda|}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_i^{\lambda_{\sigma(i)}}, \tag{4.1.1}$$

where $|S_n \cdot \lambda|$ is the cardinality of the S_n orbit of λ. Here we consider λ as a vector in $\mathbb{R}^n$, and S_n acts by permuting the coordinates. For $x, y \in \mathbb{R}^n$, not necessarily partitions, we say that x *majorizes* y if y lies in the convex hull of the S_n orbit of x.[1] If λ, μ are two partitions of r, then λ majorizes μ if and only if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \text{ for all } i = 1, \ldots, n.$$

Muirhead's inequality (Muirhead, 1902/03) then states that

$$\frac{m_\lambda(x_1, \ldots, x_n)}{m_\lambda(1, \ldots, 1)} \geq \frac{m_\mu(x_1, \ldots, x_n)}{m_\mu(1, \ldots, 1)} \tag{4.1.2}$$

[1] An equivalent definition is that $y = Ax$ for some bistochastic matrix A.

for all $x_1, \dots, x_n \geq 0$ if and only if λ majorizes μ. This statement generalizes the well-known inequality of arithmetic and geometric means. It is one of a large family of inequalities for symmetric means that were studied systematically by Cuttler et al. (2011), who conjectured the following analogous result for Schur polynomials, which was later proven by Sra (2016). Let

$$s_\lambda(x_1, \dots, x_n) := \frac{\det[x_i^{\lambda_j+n-j}]_{i,j=1}^n}{\prod_{i<j}(x_i - x_j)} \tag{4.1.3}$$

be the Schur polynomial associated to the partition λ.

Theorem 4.1.1 (Sra's inequality)**.**

$$\frac{s_\lambda(x_1, \dots, x_n)}{s_\lambda(1, \dots, 1)} \geq \frac{s_\mu(x_1, \dots, x_n)}{s_\mu(1, \dots, 1)} \tag{4.1.4}$$

for all $x_1, \dots, x_n \geq 0$ if and only if λ majorizes μ.

Sra's proof of Theorem 4.1.1 relies on expressing s_λ in terms of the HCIZ integral (1.1.3). Here we show that by considering a more general notion of majorization, one can extend Theorem 4.1.1 to a larger class of rational functions related to irreducible characters of compact Lie groups. The proof uses the Kirillov character formula, which expresses the characters in terms of Harish-Chandra integrals.

4.2 A majorization theorem

Let G be a connected, compact Lie group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus, $\mathfrak{t} = \mathrm{Lie}(T) \subset \mathfrak{g}$ the corresponding Cartan subalgebra, and W the Weyl group.

Definition 4.2.1. Given $x, y \in \mathfrak{t}$, we say that x *W-majorizes* y, written $x \succeq y$, if y lies in the convex hull of the Weyl orbit of x. △

When $G = \mathrm{U}(n)$ so that $W \cong S_n$ and $\mathfrak{t} \cong \mathbb{R}^n$, W-majorization coincides with the usual notion of majorization for vectors. For generalities on W-majorization and its relation to majorization in the traditional sense, we refer the reader to (Giovagnoli and Wynn, 1985, Marshall et al., 2011).

Definition 4.2.2. A function $f : \mathfrak{t} \to \mathbb{R}$ is said to be *W-convex*[2] if $f(x) \geq f(y)$ whenever $x \succeq y$. △

The central result of this chapter is the following W-convexity statement for the Harish-Chandra integral

$$\mathcal{H}(x, y) := \int_G e^{\langle \mathrm{Ad}_g y, x \rangle} \, dg, \qquad x, y \in \mathfrak{t},$$

where dg is normalized Haar measure and $\langle \cdot, \cdot \rangle$ is an invariant inner product identifying $\mathfrak{g} \cong \mathfrak{g}^*$.

Theorem 4.2.3. *Fix $y, z \in \mathfrak{t}$. The following are equivalent:*

1. $y \succeq z$.
2. $\mathcal{H}(x, y) \geq \mathcal{H}(x, z)$ *for all* $x \in \mathfrak{t}$.

Proof. We first show (2) implies (1), by proving the contrapositive. Let $\Delta_{\mathfrak{g}}(x) := \prod_{\alpha \in \Phi^+} \langle \alpha, x \rangle$ where the product runs over the positive roots Φ^+, and let $x \in \mathfrak{t}$ with

[2] The term "W-increasing" might be more appropriate for this notion, however "W-convexity" is used in the literature by analogy with Schur-convexity, which coincides with the special case $W = S_n$ acting on $\mathbb{R}^n$ by permuting the coordinates.

$\Delta_{\mathfrak{g}}(x) \neq 0$. We assume for now that $\Delta_{\mathfrak{g}}(y), \Delta_{\mathfrak{g}}(z) \neq 0$ as well; later we will remove this assumption. Using the Harish-Chandra formula (1.1.2) and the identity (2.6.4) for the normalizing constant, we can write

$$\mathcal{H}(tx, z) - \mathcal{H}(tx, y) = \frac{\Delta_{\mathfrak{g}}(\rho)}{\Delta_{\mathfrak{g}}(tx)} \sum_{w \in W} \epsilon(w) \left(\frac{e^{t\langle w(x), z\rangle}}{\Delta_{\mathfrak{g}}(z)} - \frac{e^{t\langle w(x), y\rangle}}{\Delta_{\mathfrak{g}}(y)} \right), \tag{4.2.1}$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ and $\epsilon(w)$ is the sign of $w \in W$. Since $\mathcal{H}$ is W-invariant in both arguments, we can assume without loss of generality that x, y, z are all dominant. Then as $t \to \infty$ we have:

$$\mathcal{H}(tx, z) - \mathcal{H}(tx, y) = \frac{\Delta_{\mathfrak{g}}(\rho)}{\Delta_{\mathfrak{g}}(tx)} \left(\frac{e^{t\langle x, z\rangle}}{\Delta_{\mathfrak{g}}(z)} - \frac{e^{t\langle x, y\rangle}}{\Delta_{\mathfrak{g}}(y)} \right) + (\text{lower-order terms}). \tag{4.2.2}$$

Now suppose $y \not\succeq z$. Then z lies outside the convex hull of the W-orbit of y, so by the hyperplane separation theorem there is some $x_0 \in \mathfrak{t}$ and $C > 0$ such that $\langle x_0, z\rangle > C$ while $\langle x_0, w(y)\rangle < C$ for all $w \in W$. By making a small perturbation to x_0 if necessary, we can ensure that $\Delta_{\mathfrak{g}}(x_0) \neq 0$. Take x in (4.2.2) to be the dominant representative of the Weyl orbit of x_0. Then we still have $\langle x, z\rangle > C$, $\langle x, y\rangle < C$, and from (4.2.2) we find:

$$\lim_{t\to\infty} \Big[\mathcal{H}(tx, z) - \mathcal{H}(tx, y)\Big] \geq \lim_{t\to\infty} \frac{\Delta_{\mathfrak{g}}(\rho)}{\Delta_{\mathfrak{g}}(tx)} \left(\frac{e^{t\langle x, z\rangle}}{\Delta_{\mathfrak{g}}(z)} - \frac{e^{tC}}{\Delta_{\mathfrak{g}}(y)} \right) = \infty,$$

which implies $\mathcal{H}(tx, z) > \mathcal{H}(tx, y)$ for some $t > 0$, so that (2) cannot hold.

Now we remove the assumption that $\Delta_{\mathfrak{g}}(y), \Delta_{\mathfrak{g}}(z) \neq 0$. In this case the expression (4.2.1) may be singular, so we instead take a limit:

$$\mathcal{H}(tx, z) - \mathcal{H}(tx, y) = \lim_{\eta\to 0} \frac{\Delta_{\mathfrak{g}}(\rho)}{\Delta_{\mathfrak{g}}(tx)} \sum_{w \in W} \epsilon(w) \left(\frac{e^{t\langle w(x), z+\eta\rho\rangle}}{\Delta_{\mathfrak{g}}(z+\eta\rho)} - \frac{e^{t\langle w(x), y+\eta\rho\rangle}}{\Delta_{\mathfrak{g}}(y+\eta\rho)} \right). \tag{4.2.3}$$

To evaluate this limit, we apply l'Hôpital's rule as many times as needed, treating the y and z terms separately. After j applications to the y terms and k applications to the z terms for some $j, k \geq 0$, in place of (4.2.2) we find:

$$\mathcal{H}(tx, z) - \mathcal{H}(tx, y) = \frac{\Delta_{\mathfrak{g}}(\rho)}{\Delta_{\mathfrak{g}}(tx)} \left(\frac{t^k \langle \rho, x\rangle^k e^{t\langle x,z\rangle}}{\partial_\rho^k \Delta_{\mathfrak{g}}(z)} - \frac{t^j \langle \rho, x\rangle^j e^{t\langle x,y\rangle}}{\partial_\rho^j \Delta_{\mathfrak{g}}(y)} \right) + \text{(lower-order terms)}, \quad (4.2.4)$$

where $\partial_\rho^k \Delta_{\mathfrak{g}}(z) := \frac{d^k}{d\eta^k} \Delta_{\mathfrak{g}}(z + \eta\rho)\big|_{\eta=0}$. The remainder of the argument then goes through as before, and we conclude that (2) implies (1).

To show that (1) implies (2), it suffices to show that for all $x \in \mathfrak{t}$, the function $F_x(y) := \mathcal{H}(x, y)$ is W-convex. The function F_x is clearly W-invariant, and by (Giovagnoli and Wynn, 1985, Theorem 1), a W-invariant, convex function is W-convex. It therefore remains only to show that F_x is convex, and for this it is sufficient to show midpoint convexity. For $u, v \in \mathfrak{t}$ we have

$$\begin{aligned} F_x\left(\frac{u+v}{2}\right) &= \int_G e^{\langle \mathrm{Ad}_g(u+v)/2, x\rangle}\, dg = \int_G \sqrt{e^{\langle \mathrm{Ad}_g u, x\rangle} e^{\langle \mathrm{Ad}_g v, x\rangle}}\, dg \\ &\leq \int_G \frac{1}{2}\left(e^{\langle \mathrm{Ad}_g u, x\rangle} + e^{\langle \mathrm{Ad}_g v, x\rangle}\right) dg = \frac{1}{2} F_x(u) + \frac{1}{2} F_x(v), \end{aligned}$$

where in the final line we have applied the inequality of arithmetic and geometric means. This proves the theorem. □

4.3 Generalizing Sra's inequality

We now show how Theorem 4.2.3 implies a generalization of Sra's inequality.

Let $X(T) \subset \mathfrak{t}$ be the character lattice of T and write $X^+(T)$ for the cone of dominant elements of $X(T)$. Let χ_λ be the character of the irreducible G-representation V_λ with highest weight $\lambda \in X^+(T)$:

$$\chi_\lambda(e^y) = \sum_{\mu \in X(T)} \mathrm{mult}_\lambda(\mu)\, e^{i\langle \mu, y\rangle}, \qquad y \in \mathfrak{t},$$

where $\mathrm{mult}_\lambda(\mu)$ is the multiplicity of the weight μ in V_λ.

When $G = \mathrm{U}(n)$, dominant integral weights are indexed by *signatures*, which are vectors $\lambda \in \mathbb{R}^n$ with non-increasing integer entries. The function s_λ defined by the formula (4.1.3) then satisfies

$$s_\lambda(e^{iy_1}, \ldots, e^{iy_n}) = \chi_\lambda(e^y), \qquad y = (y_1, \ldots, y_n) \in \mathfrak{t} \cong \mathbb{R}^n. \tag{4.3.1}$$

If λ has only nonnegative entries then it is a partition of some integer, and s_λ is the corresponding Schur polynomial as previously defined. For general signatures λ with negative entries, s_λ is a rational function, which we call the *Schur rational function* associated to the signature λ.

We generalize this construction to arbitrary G by associating to χ_λ a W-invariant rational function p_λ as follows.

Definition 4.3.1. Let $(\omega_1, \ldots, \omega_r)$ be a lattice basis of $X(T)$, and let $(b_1, \ldots, b_r)$ be the dual basis of $\mathfrak{t}$ defined by $\langle \omega_i, b_j\rangle = \delta_{ij}$. Expand $\mu \in X(T)$ as $\mu_1\omega_1 + \ldots + \mu_r\omega_r$, and expand $x \in \mathfrak{t}$ as $x_1 b_1 + \ldots + x_r b_r$. Then we define

$$p_\lambda(x_1, \ldots, x_r) := \sum_{\mu \in X(T)} \mathrm{mult}_\lambda(\mu) \prod_{j=1}^r x_j^{\mu_j} \ \in\ \mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}], \tag{4.3.2}$$

so that for $y \in \mathfrak{t}$ we have

$$p_\lambda(e^{iy_1}, \ldots, e^{iy_r}) = \sum_{\mu \in X(T)} \mathrm{mult}_\lambda(\mu) \prod_{j=1}^{r} e^{iy_j \mu_j} = \chi_\lambda(e^y), \tag{4.3.3}$$

corresponding to the relation (4.3.1) for s_λ. We also define the normalized functions

$$\overline{p}_\lambda(x_1, \ldots, x_r) := \frac{p_\lambda(x_1, \ldots, x_r)}{p_\lambda(1, \ldots, 1)}, \tag{4.3.4}$$

so that

$$\overline{p}_\lambda(e^{iy_1}, \ldots, e^{iy_r}) = \frac{\chi_\lambda(e^y)}{\dim V_\lambda}. \tag{4.3.5}$$

△

Remark 4.3.2. Take $G = \mathrm{U}(n)$, $(\omega_1, \ldots, \omega_n)$ the standard basis of $\mathbb{R}^n$, and $\lambda \in \mathbb{R}^n$ a signature. Then $p_\lambda(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n)$ is the Schur rational function associated to λ. △

The following corollary to Theorem 4.2.3 generalizes Sra's inequality for Schur polynomials to the functions p_λ.

Corollary 4.3.3. *Let $\lambda, \mu \in X^+(T)$. The following are equivalent:*

1. $\lambda \succeq \mu$,

2. $\overline{p}_\lambda(x) \geq \overline{p}_\mu(x)$ *for all* $x \in \mathbb{R}^r$ *with* $x_1, \ldots, x_r > 0$.

The proof makes use of the following translation invariance property of W-majorization.

Lemma 4.3.4. *Let $x, y, z \in \mathfrak{t}$ all lie in the same Weyl chamber. Then $x \succeq y$ if and only if $x + z \succeq y + z$.*

Proof. Let $W \cdot x$ denote the Weyl orbit of $x \in \mathfrak{t}$, and for a subset $S \subset \mathfrak{t}$ let $\mathrm{Conv}(S)$ denote its convex hull. It is immediate from Definition 4.2.1 that $x \succeq y$ if and only if $\mathrm{Conv}(W \cdot y) \subseteq \mathrm{Conv}(W \cdot x)$. We will show that this holds if and only if $\mathrm{Conv}(W \cdot (y+z)) \subseteq \mathrm{Conv}(W \cdot (x+z))$.

Let $w_1, w_2 \in W$ and consider the following inequalities for $v \in \mathfrak{t}$:

$$\langle v, w_1(x) + w_2(z) \rangle \geq \langle v, w_1'(x) + w_2'(z) \rangle, \quad \forall\, w_1', w_2' \in W. \tag{4.3.6}$$

These inequalities are consistent if and only if $w_1(x) + w_2(z)$ is an extreme point of the Minkowski sum $(W \cdot x) + (W \cdot z)$. For any $v \in \mathfrak{t}$, (4.3.6) holds exactly when $w_1(x)$ and $w_2(z)$ both lie in the same Weyl chamber as v. Therefore, since x lies in the same chamber as z, a $v \in \mathfrak{t}$ satisfying (4.3.6) exists if and only if $w_1(x) + w_2(z) = w(x+z)$ for some $w \in W$. In other words, the extreme points of $(W \cdot x) + (W \cdot z)$ are exactly the elements of the Weyl orbit $W \cdot (x+z)$, so that these two sets have the same convex hull. Since the operation of taking convex hulls distributes across Minkowski sums, we then find:

$$\mathrm{Conv}(W \cdot (x+z)) = \mathrm{Conv}((W \cdot x) + (W \cdot z)) = \mathrm{Conv}(W \cdot x) + \mathrm{Conv}(W \cdot z).$$

Similarly, we find $\mathrm{Conv}(W \cdot (y+z)) = \mathrm{Conv}(W \cdot y) + \mathrm{Conv}(W \cdot z)$. This implies that $\mathrm{Conv}(W \cdot (y+z)) \subseteq \mathrm{Conv}(W \cdot (x+z))$ if and only if $\mathrm{Conv}(W \cdot y) \subseteq \mathrm{Conv}(W \cdot x)$, which is the desired result. $\square$

Proof of Corollary 4.3.3. Recall the Kirillov character formula (2.6.1) for compact Lie groups (Kirillov, 1968, 2004). Using the expression (2.6.5) for the Liouville volume of a coadjoint orbit, we can write the character formula in terms of a Harish-

Chandra integral:

$$\chi_\lambda(e^y) = \frac{\Delta_{\mathfrak{g}}(iy)}{\widehat{\Delta}_{\mathfrak{g}}(e^y)} \frac{\Delta_{\mathfrak{g}}(\lambda+\rho)}{\Delta_{\mathfrak{g}}(\rho)} \mathcal{H}(\lambda+\rho, iy), \qquad y \in \mathfrak{t}, \tag{4.3.7}$$

where

$$\widehat{\Delta}_{\mathfrak{g}}(e^y) := \prod_{\alpha\in\Phi^+} (e^{i\langle\alpha,y\rangle/2} - e^{-i\langle\alpha,y\rangle/2}).$$

Together with (4.3.5) and the Weyl dimension formula,

$$\dim V_\lambda = \frac{\Delta_{\mathfrak{g}}(\lambda+\rho)}{\Delta_{\mathfrak{g}}(\rho)}, \tag{4.3.8}$$

this gives:

$$\overline{p}_\lambda(e^{y_1}, \dots, e^{y_r}) = \frac{\Delta_{\mathfrak{g}}(y)}{\prod_{\alpha\in\Phi^+}(e^{\langle\alpha,y\rangle/2} - e^{-\langle\alpha,y\rangle/2})} \mathcal{H}(\lambda+\rho, y), \tag{4.3.9}$$

for $y = (y_1, \dots, y_r) \in \mathfrak{t} \cong \mathbb{R}^r$. For $x_1, \dots, x_r > 0$, we can take $y \in \mathfrak{t}$ such that $(e^{y_1}, \dots, e^{y_r}) = (x_1, \dots, x_r)$. Since λ, μ, ρ are all dominant, Lemma 4.3.4 implies that $\lambda + \rho \succeq \mu + \rho$ if and only if $\lambda \succeq \mu$, and the desired result is then immediate from (4.3.9) and Theorem 4.2.3. □

Remark 4.3.5. Although Sra's inequality holds for $x_1, \dots, x_r \geq 0$, Corollary 4.3.3 assumes the strict inequality $x_1, \dots, x_r > 0$, since in general the functions $\overline{p}_\lambda$ may have singularities along the hyperplanes where some coordinate vanishes. Schur polynomials are continuous however, so in the case that $G = U(n)$ and λ, μ are partitions, we can recover Sra's inequality from Corollary 4.3.3 by taking a limit, even if some $x_j = 0$. △

4.4 The case of Heckman–Opdam polynomials

The Heckman–Opdam polynomials $P_{k,\lambda}$ generalize various important families of symmetric and W-invariant functions, including monomial W-invariant polynomials, Jack polynomials, and irreducible characters of complex semisimple Lie algebras. They were introduced by Heckman and Opdam (1987), where they were called multivariable Jacobi polynomials. In this section, we state a majorization conjecture for Heckman–Opdam polynomials that would generalize Muirhead's inequality, Sra's inequality for Schur polynomials, and Corollary 4.3.3 above, and we prove one of the two directions of implication that comprise the conjecture. As we discuss below, the techniques of Theorem 4.2.3 cannot be straightforwardly applied to prove the other direction, since integral formulae analogous to (4.3.7) for general Heckman–Opdam polynomials are not known.

To define $P_{k,\lambda}$, we must fix some preliminary data. Here we largely follow the conventions of Opdam (1995) and Sahi (2000), and we assume that G is semisimple and simply connected, so that $X^+(T) = P^+$, the cone of dominant elements in the weight lattice P of $\mathfrak{g}$. The function $P_{k,\lambda}$ depends on a dominant weight $\lambda \in P^+$, as well as on a *multiplicity parameter*, which is a function $k : \Phi \to \mathbb{C}$, where Φ are the roots of $\mathfrak{g}$, such that $k_{w\cdot\alpha} = k_\alpha$ for $w \in W$. Although one may study Heckman–Opdam polynomials for complex values of k, here we assume that all k_α are nonnegative real numbers.

The name "Heckman–Opdam polynomial" may be confusing, since if we consider $P_{k,\lambda}$ as a function on $\mathfrak{t}$, then it is actually an *exponential* polynomial. However, we can also regard it as a polynomial in $\mathbb{R}[P]$, the group algebra of the weight lattice, by observing that $\mathbb{R}[P]$ is generated by the functions $e^{\langle\lambda,x\rangle}$, $\lambda \in P$, with pointwise addition and multiplication.

We now define $P_{k,\lambda}$ in terms of solutions to certain differential-difference equations. For $y \in \mathfrak{t}$, let s_y be the reflection through the hyperplane $\{ x \in \mathfrak{t} \mid \langle y, x\rangle = 0 \}$, and define

$$\rho^{(k)} := \frac{1}{2} \sum_{\alpha\in\Phi^+} k_\alpha \alpha.$$

Definition 4.4.1. For $y \in \mathfrak{t}$, the *Cherednik operator* $D_{k,y}$ is the differential-difference operator

$$D_{k,y} := \partial_y + \sum_{\alpha\in\Phi^+} \langle y, \alpha\rangle k_\alpha \frac{1}{1-e^{-\alpha}}(1-s_\alpha) - \langle y, \rho^{(k)}\rangle. \tag{4.4.1}$$

△

The Cherednik operators were originally defined and studied in (Cherednik, 1991, 1994). For the properties of these operators invoked below, we refer the reader to these papers as well as to (Opdam, 1995, sect. 2).

For $\lambda \in P$, write

$$\widetilde{\lambda} := \lambda + \frac{1}{2} \sum_{\alpha\in\Phi^+} k_\alpha \varepsilon_{\langle\alpha,\lambda\rangle}\alpha, \tag{4.4.2}$$

where

$$\varepsilon_t := \begin{cases} 1, & t > 0, \\ -1, & t \le 0. \end{cases}$$

Consider the infinite system of differential-difference equations

$$D_{k,y}\phi = \langle y, \widetilde{\lambda}\rangle\phi, \qquad y \in \mathfrak{t}. \tag{4.4.3}$$

The system (4.4.3) has a unique solution $\phi = E_{k,y} \in \mathbb{R}[P]$ such that the coefficient of $e^{\langle\lambda,x\rangle}$ in $E_{k,y}$ is 1.

Definition 4.4.2. The *Heckman–Opdam polynomial* $P_{k,\lambda}$ is defined by

$$P_{k,\lambda} := \frac{|W \cdot \lambda|}{|W|} \sum_{w \in W} E_{k,w\lambda}. \tag{4.4.4}$$

△

As λ ranges over P^+ with k fixed, the $P_{k,\lambda}$ form an $\mathbb{R}$-basis of the W-invariant elements $\mathbb{R}[P]^W$. For example, when all k_α are 0,

$$P_{0,\lambda}(x) = \frac{|W \cdot \lambda|}{|W|} \sum_{w \in W} e^{\langle w\lambda, x \rangle} \tag{4.4.5}$$

is the monomial W-invariant (exponential) polynomial associated to the weight λ. When $k_\alpha = 1$ for all $\alpha \in \Phi$, $P_{k,\lambda}(ix) = \chi_\lambda(e^x)$ is the character of the irreducible G-representation with highest weight λ.

We conjecture the following analogue of Theorem 4.2.3 for Heckman–Opdam polynomials.

Conjecture 4.4.3. *Let $\lambda, \mu \in P^+$. The following are equivalent:*

1. *$\lambda \succeq \mu$,*
2. *$P_{k,\lambda}(x) \geq P_{k,\mu}(x)$ for all $x \in \mathfrak{t}$.*

Here we show that (2) implies (1) using Schapira's sharp asymptotics for the Heckman–Opdam hypergeometric function (Schapira, 2008).

Proposition 4.4.4. *Let $\lambda, \mu \in P^+$. If $P_{k,\lambda}(x) \geq P_{k,\mu}(x)$ for all $x \in \mathfrak{t}$, then $\lambda \succeq \mu$.*

Proof. As in the proof of Theorem 4.2.3, we show the contrapositive. Suppose $\lambda \not\succeq \mu$, and again use hyperplane separation to obtain a $y \in \mathfrak{t}$ and $C > 0$ such that $\langle \mu, y\rangle > C$ and $\langle w(\lambda), y\rangle < C$ for all $w \in W$. Clearly both of these inequalities still hold if we replace y with the dominant representative of its Weyl orbit, so we may take y to be dominant. By (Heckman and Slichtkrul, 1994, eqn. 4.4.10),

$$F_{k,\lambda+\rho^{(k)}}(x) = c(\lambda + \rho^{(k)}, k)P_{k,\lambda}(x), \qquad x \in \mathfrak{t}, \tag{4.4.6}$$

where $F_{k,\lambda+\rho^{(k)}}(x)$ is the generalized hypergeometric function defined by Heckman and Opdam (1987), and $c(\lambda + \rho^{(k)}, k)$ is a nonnegative constant. Moreover, by (Schapira, 2008, Theorem 3.1 and Remark 3.1), we have the sharp estimate

$$F_{k,\lambda+\rho^{(k)}}(ty) \asymp \prod_{\substack{\alpha\in\Phi^+ \\ \langle\alpha,\lambda+\rho^{(k)}\rangle=0}} (1 + t\langle\alpha, y\rangle)e^{t\langle\lambda,y\rangle} \tag{4.4.7}$$

as $t \to +\infty$. Comparing to (4.4.6), we find that

$$\lim_{t\to\infty} \left[c(\mu + \rho^{(k)}, k)P_{k,\mu}(ty) - c(\lambda + \rho^{(k)}, k)P_{k,\lambda}(ty)\right] = \infty,$$

which implies that $P_{k,\lambda}(x) < P_{k,\mu}(x)$ for some $x \in \mathfrak{t}$, completing the proof. $\square$

Remark 4.4.5. We were able to deduce Corollary 4.3.3 from Theorem 4.2.3 because of the Kirillov character formula (4.3.7), which expresses the irreducible characters of G in terms of Harish-Chandra integrals. However, integral representations for general Heckman–Opdam polynomials are not known, so we cannot directly apply the same technique to prove the other direction of Conjecture 4.4.3. Although the Heckman–Opdam hypergeometric function (4.4.6) admits known integral expressions in certain cases (see e.g. (Sun, 2016, Rösler and Voit, 2016)), these are more complicated than the Kirillov character formula and have so far resisted a similar analysis. $\triangle$

Chapter Five

Horn's problem, orbital integrals, and the volume function

This self-contained chapter discusses various probabilistic generalizations of *Horn's problem*, a classical problem in linear algebra regarding the eigenvalues of the sum of two matrices. We relate Horn's problem to random matrix theory, representation theory, and symplectic and convex geometry. Harish-Chandra integrals play a central role, as they arise in the definition of a special function called the *volume function*, which will be our main object of study. This chapter is adapted from the paper (Coquereaux et al., 2019a), which is joint work of the author together with Robert Coquereaux and Jean-Bernard Zuber.

5.1 Introduction

Horn's problem is the following question. Given n-by-n Hermitian matrices A and B with known eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \ldots \geq \beta_n$, what can be said about the eigenvalues $\gamma_1 \geq \ldots \geq \gamma_n$ of their sum $C = A + B$? After decades of work by many mathematicians, the answer to this question is now well known (Horn, 1962, Klyachko, 1998, Knutson and Tao, 2001).

There is an extension of Horn's problem that is both more general and more quantitative. Let V be a representation of a compact Lie group G. To each G-orbit $\mathcal{O} \subset V$ we associate the *orbital measure at* $\mathcal{O}$, which is the unique G-invariant probability measure on V that is concentrated on $\mathcal{O}$. The *orbit space* is the topological quotient V/G, in which each point corresponds to a G-orbit. If $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ are two such orbits and we choose $A \in \mathcal{O}_\alpha$ and $B \in \mathcal{O}_\beta$ independently at random from their respective orbital measures, the sum $C = A + B$ will lie in a random orbit $\mathcal{O}_\gamma$. For each pair $(\mathcal{O}_\alpha, \mathcal{O}_\beta)$, we thus obtain a probability measure on the orbit space, called the *Horn probability measure*. Concretely, $C \in V$ is distributed according to the convolution of the orbital measures at $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, and the Horn probability measure is the pushforward of this convolution by the quotient map $V \to V/G$. The extended problem is then to give an explicit description of the Horn probability measure, whereas the original Horn's problem is to describe only the support of this measure in the specific case where V is the coadjoint representation of $\mathrm{U}(n)$.

In this chapter, we study the extended Horn's problem in two families of cases that are of special interest:

1. **Coadjoint representations:** G is an arbitrary compact, connected, semisimple Lie group acting by the coadjoint representation on the dual of its Lie algebra $\mathfrak{g}$.

2. **Spaces of self-adjoint matrices:** G is one of the classical groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ or $\mathrm{USp}(n)$ acting by conjugation on, respectively, real symmetric, complex Hermitian or quaternionic self-dual matrices. Following convention, in these cases we study the distribution of the *sorted eigenvalues* of $A+B$ rather than its orbit.[1]

We will refer to these respectively as the "coadjoint case" and the "self-adjoint case."

The main object of our study will be a function $\mathcal{J}$, called the *volume function*, which can be computed from the density of the Horn probability measure (and vice versa). This function encodes various kinds of geometric information about the orbits, and in the coadjoint case it additionally encodes combinatorial information related to tensor product multiplicities of irreducible representations of $\mathfrak{g}$. We discuss

[1] In most cases this distinction is immaterial because the spectrum of $A+B$ uniquely determines its orbit. The only exceptions are the even special orthogonal groups $G = SO(2n)$, in which case $\operatorname{diag}(x_1, \ldots, x_n)$ and $\operatorname{diag}(x_{w(1)}, \ldots, x_{w(n)})$ may lie in different orbits when $w \in S_n$ is an odd permutation.

the singular and vanishing loci of $\mathcal{J}$, its relationship to the Riemannian geometry of the orbits as submanifolds of V, and (in the coadjoint case) its interpretation as both a symplectic volume and the volume of a convex polytope, as well as identities that relate $\mathcal{J}$ to the tensor product multiplicities of $\mathfrak{g}$. Finally, we carry out a detailed case study of the coadjoint case for $\mathfrak{g} = \mathfrak{so}(5)$.

This chapter discusses several constructions related to the extended Horn's problem and to tensor product multiplicities, including a number of previously known results that we recall for the sake of completeness. However, to the author's knowledge, Propositions 5.2.1 through 5.4.3, equation (5.2.5), and the conjectured expression (5.6.2) either were new results of the paper (Coquereaux et al., 2019a) or extend in various ways several results previously obtained by Zuber (2018) and by Coquereaux and Zuber (2018, 2019). Proposition 5.4.2 is a particular instance of a more general phenomenon whereby the volumes of certain symplectic orbifolds equal the volumes of polytopes whose integer points count representation multiplicities (Heckman, 1982, Guillemin et al., 1996). As far as the author has been able to determine however, the proof in this chapter was a novel contribution of (Coquereaux et al., 2019a), and the precise statement had not appeared previously.

In the following two subsections we define $\mathcal{J}$ for the coadjoint and self-adjoint cases. To avoid overloading notation we give separate definitions in the two cases, but the concepts are analogous.

5.1.1 $\mathcal{J}$ in the coadjoint case

We first develop some preliminaries related to Horn's problem. Let $\langle \cdot, \cdot \rangle$ be the G-invariant inner product given by -1 times the Killing form. We identify $\mathfrak{g} \cong \mathfrak{g}^*$

using the inner product and we then identify the orbit space with the dominant Weyl chamber $\mathcal{C}_+$ of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, so that the quotient map $\mathfrak{g}^* \to \mathfrak{g}^*/G$ sends each orbit to its unique representative in $\mathcal{C}_+$. We identify functions on $\mathcal{C}_+$ with their unique G-invariant extension to $\mathfrak{g}$. The Jacobian of the quotient map $\mathfrak{g}^* \to \mathfrak{g}^*/G$ is equal to $\kappa_{\mathfrak{g}} \Delta_{\mathfrak{g}}(x)^2$, where $\Delta_{\mathfrak{g}}(x) := \prod_{\boldsymbol{\alpha}>0} \langle \boldsymbol{\alpha}, x \rangle$ is the product of the positive roots of $\mathfrak{g}$ and $\kappa_{\mathfrak{g}}$ is a numerical coefficient. For the classical Lie algebras, $\kappa_{\mathfrak{g}}$ may be determined by computing in two different ways a Gaussian integral over $\mathfrak{g}$ and making use of the Macdonald–Opdam integral (Macdonald, 1982, Opdam, 1989), giving

$$\kappa_{\mathfrak{g}} = \frac{(2\pi)^{N_r}}{\Delta_{\mathfrak{g}}(\rho)} = \frac{(2\pi)^{N_r}}{\prod_{i=1}^r \ell_i!} \times K \tag{5.1.1}$$

in terms of the Weyl vector $\rho = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0} \boldsymbol{\alpha}$, the rank r, the number N_r of positive roots, and the Coxeter exponents ℓ_i of $\mathfrak{g}$. The coefficient $K = \prod_{\alpha>0} \frac{\langle \boldsymbol{\theta}, \boldsymbol{\theta} \rangle}{\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle}$, where $\boldsymbol{\theta}$ is any long root, equals 1 for simply laced algebras, while for the non-simply laced cases it takes the values $K = 2^r$ for B_r $(r > 1)$, $K = 2^{r(r-1)}$ for C_r $(r > 1)$, $K = 2^{12}$ for F_4 and $K = 3^3$ for G_2 (Coquereaux, 2012).

An element $x \in \mathfrak{t}$ is said to be *regular* if $\Delta_{\mathfrak{g}}(x) \neq 0$. This term should not be confused with the more general notion of a *regular value* of a differentiable map, which we will also use frequently. For x regular, the Jacobian of the quotient map is equal to the Riemannian volume of the orbit $\mathcal{O}_x$ with respect to the metric induced by the inner product. (Note that this Riemannian volume differs from the symplectic volume discussed below.)

The Horn probability measure is supported on a convex polytope $\mathscr{H}_{\alpha\beta} \subset \mathcal{C}_+$, called the *Horn polytope*, and is absolutely continuous with respect to the induced Lebesgue measure on $\mathscr{H}_{\alpha\beta}$. We assume in what follows that α and β are regular, in which case $\dim \mathscr{H}_{\alpha\beta} = r$ and the measure has a global density on $\mathcal{C}_+$.

Our discussion of the Horn probability measure will make ubiquitous use of the *orbital integral* (also called the Harish-Chandra orbital function), defined for $\alpha, x \in \mathcal{C}_+$ as

$$\mathcal{H}(\alpha, ix) = \int_G e^{i\langle \mathrm{Ad}_g \alpha, x\rangle}\, dg, \tag{5.1.2}$$

where dg is normalized Haar measure. Considered as a G-invariant function of $x \in \mathfrak{g}$, $\mathcal{H}(\alpha, ix)$ is the Fourier transform of the orbital measure at $\mathcal{O}_\alpha$, so that the characteristic function of the convolution of orbital measures at $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ is the product $\mathcal{H}(\alpha, ix)\mathcal{H}(\beta, ix)$.

The probability density function (PDF) of the Horn probability measure can be written in terms of orbital integrals by taking the inverse Fourier transform of this characteristic function, rewriting it as a function of $\gamma \in \mathcal{C}_+$, and multiplying by the Jacobian $\kappa_\mathfrak{g}\Delta_\mathfrak{g}(\gamma)^2$ to account for the quotient map. The resulting expression for the PDF is

$$\begin{aligned} p(\gamma|\alpha,\beta) &= \frac{\kappa_\mathfrak{g}^2\, \Delta_\mathfrak{g}(\gamma)^2}{(2\pi)^{\dim \mathfrak{g}}|W|} \int_\mathfrak{t} \Delta_\mathfrak{g}(x)^2 \mathcal{H}(\alpha, ix)\mathcal{H}(\beta, ix)\big(\mathcal{H}(\gamma, ix)\big)^*\, dx \\ &= \frac{1}{(2\pi)^r|W|}\left(\frac{\Delta_\mathfrak{g}(\gamma)}{\Delta_\mathfrak{g}(\rho)}\right)^2 \int_\mathfrak{t} \Delta_\mathfrak{g}(x)^2 \mathcal{H}(\alpha, ix)\mathcal{H}(\beta, ix)\big(\mathcal{H}(\gamma, ix)\big)^*\, dx, \end{aligned} \tag{5.1.3}$$

where dx is the Lebesgue measure on $\mathfrak{t}$ associated to the inner product $\langle\cdot,\cdot\rangle$ and $|W|$ is the order of the Weyl group. Similar formulae for the convolution of orbital measures have appeared in (Dooley et al., 1993).

The volume function $\mathcal{J}(\alpha,\beta;\gamma)$ is then defined as

$$\begin{aligned}\mathcal{J}(\alpha,\beta;\gamma) &:= \frac{\Delta_{\mathfrak{g}}(\alpha)\Delta_{\mathfrak{g}}(\beta)}{\Delta_{\mathfrak{g}}(\gamma)\Delta_{\mathfrak{g}}(\rho)}\, p(\gamma|\alpha,\beta) \\ &= \frac{\Delta_{\mathfrak{g}}(\alpha)\Delta_{\mathfrak{g}}(\beta)\Delta_{\mathfrak{g}}(\gamma)}{(2\pi)^r\,|W|\,\Delta_{\mathfrak{g}}(\rho)^3}\int_{\mathfrak{t}} \Delta_{\mathfrak{g}}(x)^2 \mathcal{H}(\alpha,ix)\mathcal{H}(\beta,ix)\big(\mathcal{H}(\gamma,ix)\big)^*\,dx.\end{aligned} \tag{5.1.4}$$

For fixed α and β it is a piecewise polynomial function of γ. The last line of (5.1.4) can be used to define $\mathcal{J}$ without assuming that α or β is regular, though one finds that $\mathcal{J}$ vanishes for non-regular arguments. Note that although we take $\gamma \in \mathcal{C}_+$ above, the expression (5.1.4) extends naturally to a function on $\mathfrak{t}$ that is skew under the action of W.

The volume function is the central concern of this chapter, and it admits several interpretations. To begin with, $\mathcal{J}$ computes the symplectic (Liouville) volume of a family of symplectic orbifolds parametrized by the triple (α,β,γ). Here we take α,β regular and γ on the interior of a polynomial domain of $\mathcal{J}$. Coadjoint orbits admit a canonical G-invariant symplectic form, the Kostant–Kirillov–Souriau form (see Section 2.5 above), for which the inclusion into $\mathfrak{g}^*$ is a moment map. For $x \in \mathfrak{t}^*$ regular, by (2.6.3) and (2.6.4), the Liouville volume of the orbit $\mathcal{O}_x$ is then equal to $\Delta_{\mathfrak{g}}(x)/\Delta_{\mathfrak{g}}(\rho)$. The product of orbits $\mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_{-\gamma}$ carries a diagonal G-action with moment map $\phi : (A,B,-C) \mapsto A+B-C$. Let μ be the Liouville volume measure on $\mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_{-\gamma}$. The pushforward $\phi_*\mu$ is the Borel measure on $\mathfrak{g}^*$ defined by $\phi_*\mu(U) = \mu(\phi^{-1}(U))$, $U \subset \mathfrak{g}^*$. This measure is equal to the convolution of the three orbital measures times the volumes of the orbits, so that it has a density

(Radon–Nikodym derivative) given by

$$\begin{aligned}h_{\alpha\beta}^{\gamma}(z) &= \frac{1}{(2\pi)^{\dim\mathfrak{g}}}\frac{\Delta_{\mathfrak{g}}(\alpha)\Delta_{\mathfrak{g}}(\beta)\Delta_{\mathfrak{g}}(\gamma)}{\Delta_{\mathfrak{g}}(\rho)^3}\int_{\mathfrak{g}}\mathcal{H}(\alpha,ix)\mathcal{H}(\beta,ix)\mathcal{H}(-\gamma,ix)\,e^{-i\langle x,z\rangle}dx \\ &= \frac{\kappa_{\mathfrak{g}}}{(2\pi)^{\dim\mathfrak{g}}|W|}\frac{\Delta_{\mathfrak{g}}(\alpha)\Delta_{\mathfrak{g}}(\beta)\Delta_{\mathfrak{g}}(\gamma)}{\Delta_{\mathfrak{g}}(\rho)^3}\int_{\mathfrak{t}}\Delta_{\mathfrak{g}}(x)^2\mathcal{H}(\alpha,ix)\mathcal{H}(\beta,ix)\big[\mathcal{H}(\gamma,ix)\mathcal{H}(z,ix)\big]^*\,dx\end{aligned} \tag{5.1.5}$$

with respect to Lebesgue measure dz on $\mathfrak{g}^*$. Thus we find that

$$\mathcal{J}(\alpha,\beta;\gamma) = (2\pi)^{N_r}\Delta_{\mathfrak{g}}(\rho)\,h_{\alpha\beta}^{\gamma}(0). \tag{5.1.6}$$

By results of Duistermaat and Heckman (1982), $h_{\alpha\beta}^{\gamma}(0)$ equals the Liouville volume of $\phi^{-1}(0)/G$, the Hamiltonian reduction of $\mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_{-\gamma}$ at level 0, which is a symplectic orbifold.[2] For a more detailed discussion of these symplectic quotients in the context of the classical Horn's problem, we refer the reader to (Knutson, 2000).

The volume function is also related to tensor product multiplicities (generalized Littlewood–Richardson coefficients) in representation theory. Let V_λ, V_μ, V_ν be irreducible representations of $\mathfrak{g}$ with highest weights λ, μ, ν, which we assume to be a *compatible triple*, meaning that $\lambda + \mu - \nu$ belongs to the root lattice. Then, in the language of geometric quantization, $\mathcal{J}(\lambda,\mu;\nu)$ is a "semiclassical approximation" for the tensor product multiplicity $C_{\lambda\mu}^{\nu} := \dim \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_\mu \to V_\nu)$. Indeed, a connection with representation theory is already apparent in (5.1.4). Let a prime denote the Weyl shift of a weight: $\lambda' = \lambda + \rho$. By the Weyl dimension formula,

[2] In some cases the quotient $\phi^{-1}(0)/G$ is smooth, so that it is a genuine symplectic manifold, but in general it may have singular points.

$\dim V_\lambda = \Delta_{\mathfrak{g}}(\lambda')/\Delta_{\mathfrak{g}}(\rho)$, so that we have

$$\mathcal{J}(\lambda', \mu'; \nu') = \frac{\dim V_\lambda \dim V_\mu \dim V_\nu}{(2\pi)^r |W|} \int_{\mathfrak{t}} \Delta_{\mathfrak{g}}(x)^2 \mathcal{H}(\lambda', ix) \mathcal{H}(\mu', ix) \big(\mathcal{H}(\nu', ix)\big)^* \, dx. \tag{5.1.7}$$

We explore the representation-theoretic significance of $\mathcal{J}$ more thoroughly below in Section 5.3, where we derive two explicit relations expressing $\mathcal{J}$ in terms of the multiplicities $C_{\lambda\mu}^{\nu}$. In Section 5.4 we provide yet another perspective on the relationship between the volume function and tensor product multiplicities, by showing that $\mathcal{J}(\lambda, \mu; \nu)$ is proportional to the Euclidean volume of a polytope whose number of integer points is equal to $C_{\lambda\mu}^{\nu}$.

5.1.2 $\mathcal{J}$ in the self-adjoint case

The volume function in the self-adjoint case does not offer as rich an array of geometric interpretations as in the coadjoint case, largely due to the fact that, in general, the orbits do not carry a natural symplectic structure. The exception is when $G = \mathrm{SU}(n)$, in which case the coadjoint representation is equivalent to the action by conjugation on traceless Hermitian matrices, so that for α, β traceless the coadjoint and self-adjoint cases coincide. However, for real symmetric or quaternionic self-dual matrices we are not dealing with a coadjoint representation, and the interpretations of $\mathcal{J}$ as the volume of a symplectic orbifold or a polytope do not apply. Nevertheless, $\mathcal{J}$ still encodes substantial geometric information.

We first observe that a translation $A \mapsto A + aI$, $B \mapsto B + bI$ $(a, b \in \mathbb{R})$ merely translates the distribution of each γ_i by $a + b$. Therefore, up to a translation of its support, the Horn probability measure depends only on the trace-free parts of α and β. Accordingly, in what follows, we assume without loss of generality that A and B

are traceless.

Let $G = \mathrm{SO}(n),\ \mathrm{SU}(n)$, or $\mathrm{USp}(n)$. We label these three cases by a parameter[3] θ that respectively equals 1/2, 1 or 2. Let $\mathcal{M}_{\theta,n}$ be respectively the space of n-by-n real symmetric, complex Hermitian, or quaternionic self-dual matrices. Then G acts on $\mathcal{M}_{\theta,n}$ by conjugation. Let $\mathcal{M}^0_{\theta,n}$ be the subspace of traceless matrices in $\mathcal{M}_{\theta,n}$. We define a G-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{M}_{\theta,n}$ using the trace form $\langle A, B\rangle = \mathrm{tr}\,(AB)$. The space of spectra of matrices in $\mathcal{M}^0_{\theta,n}$ is naturally identified with the space of real diagonal matrices $\mathrm{diag}\,(x_1,\ldots,x_n)$ such that $x_1 \geq \ldots \geq x_n$ and $\sum x_i = 0$, which we also denote $\mathcal{C}_+$. We identify the space of all real traceless diagonal matrices with $\mathbb{R}^{n-1}$, and we identify functions on $\mathcal{C}_+$ with their symmetric (in the x_i variables) extensions to $\mathbb{R}^{n-1}$.

The Jacobian of the diagonalization map is equal to $\kappa_\theta|\Delta(x)|^{2\theta}$, where $\Delta(x) := \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant, and the constant κ_θ may again be determined by computing in two different ways a Gaussian integral over $\mathcal{M}_{\theta,n}$ and making use of the Mehta–Dyson integral (Mehta and Dyson, 1963)

$$\int_{\mathbb{R}^n} |\Delta(x)|^{2\theta} e^{-\frac{1}{2}\sum_i x_i^2}\, dx = (2\pi)^{n/2} \prod_{j=1}^{n} \frac{\Gamma(1+j\theta)}{\Gamma(1+\theta)}\,, \tag{5.1.8}$$

whence

$$\kappa_\theta = \frac{(2\pi)^{\frac{1}{2}n(n-1)\theta}\, n!}{\prod_{j=1}^{n} \frac{\Gamma(1+j\theta)}{\Gamma(1+\theta)}}\,. \tag{5.1.9}$$

If $\Delta(x) \neq 0$ we say that x is *regular*; this means that x is a diagonal (traceless) matrix with distinct eigenvalues. For x regular and $G \neq \mathrm{SO}(n)$ for n even, the Jacobian of the diagonalization map is equal to the Riemannian volume of the orbit

[3] In the language of β-ensembles appearing in random matrix theory, our θ is equal to $\beta/2$. We opted for this notation, which is more common in symmetric function theory, to avoid overloading the symbol β.

$\mathcal{O}_x$ with respect to the metric induced by the inner product. (When $G = \mathrm{SO}(n)$, n even, for regular elements there are two distinct orbits with the same spectrum, so that the Jacobian is equal to twice this volume.)

As before, the Horn probability measure is supported on a convex polytope $\mathscr{H}_{\alpha\beta}$ in $\mathcal{C}_+$, also called the *Horn polytope*, and is absolutely continuous with respect to the induced Lebesgue measure on $\mathscr{H}_{\alpha\beta}$. We assume for the remainder of this chapter that α and β are regular, in which case $\dim \mathscr{H}_{\alpha\beta} = n-1$, so that the Horn probability measure has a density with respect to Lebesgue measure on $\mathcal{C}_+$.

For $\alpha, x \in \mathcal{C}_+$, the orbital integral is again defined by

$$\mathcal{H}(\alpha, ix) = \int_G e^{i\langle g\alpha g^{-1}, x\rangle}\, dg. \tag{5.1.10}$$

Following an analogous procedure to the derivation of (5.1.3), we take the inverse Fourier transform of the characteristic function of the convolution of orbital measures, then multiply by the Jacobian of the diagonalization map to obtain the PDF of the Horn probability measure:

$$p(\gamma|\alpha,\beta) = \frac{\kappa_\theta^2\ \Delta(\gamma)^{2\theta}}{(2\pi)^{\dim \mathcal{M}^0_{\theta,n}} n!} \int_{\mathbb{R}^{n-1}} |\Delta(x)|^{2\theta} \mathcal{H}(\alpha, ix)\mathcal{H}(\beta, ix)\big(\mathcal{H}(\gamma, ix)\big)^*\, dx, \qquad \gamma \in \mathcal{C}_+, \tag{5.1.11}$$

where dx is the Lebesgue measure induced by identifying $\mathbb{R}^{n-1}$ with the hyperplane $\sum x_i = 0$ in $\mathbb{R}^n$.

We expect to recover (5.1.3) from (5.1.11) when $\theta = 1$. In fact for $\mathrm{SU}(n)$ we have $\kappa_{\mathfrak{su}(n)} = \kappa_1$, $\Delta_{\mathfrak{su}(n)} = \Delta$, $\dim \mathcal{M}^0_{\theta,n} = n^2 - 1 = \dim \mathfrak{su}(n)$, and $|W| = n!$, so indeed this is the case.

By analogy with (5.1.4), we define the volume function $\mathcal{J}(\alpha,\beta;\gamma)$ by

$$\begin{aligned}\mathcal{J}(\alpha,\beta;\gamma) &:= \frac{(2\pi)^{\theta n(n-1)}}{\kappa_\theta^2\,\Delta_{\mathfrak{g}}(\rho)^3}\left(\frac{\Delta(\alpha)\Delta(\beta)}{\Delta(\gamma)}\right)^\theta p(\gamma|\alpha,\beta) &(5.1.12)\\ &= \frac{\big(\Delta(\alpha)\Delta(\beta)\Delta(\gamma)\big)^\theta}{(2\pi)^{n-1}n!\,\Delta_{\mathfrak{g}}(\rho)^3}\int_{\mathbb{R}^{n-1}}|\Delta(x)|^{2\theta}\mathcal{H}(\alpha,ix)\mathcal{H}(\beta,ix)\big(\mathcal{H}(\gamma,ix)\big)^*\,dx. &(5.1.13)\end{aligned}$$

Here $\Delta_{\mathfrak{g}}(\rho)$ indicates the same quantity as in the coadjoint case for the group G, so that this expression for $\mathcal{J}$ recovers (5.1.4) for $\theta = 1$. It is clear that $\mathcal{J}$ depends in both cases on the choice of G, but for the sake of brevity we choose *not* to append this information to the notation $\mathcal{J}$ and will instead specify the case and group under discussion whenever necessary.

Note that $\mathcal{J}$ is defined in (5.1.12) only for traceless α,β,γ, but since in (5.1.13) $\mathbb{R}^{n-1}$ is understood as the space of traceless diagonal matrices x, $\mathcal{H}(\alpha,ix)$ is invariant under translation of all α_i by the same constant a:

$$\mathcal{H}(\alpha+aI,ix) = e^{ia\sum x_i}\mathcal{H}(\alpha,ix) = \mathcal{H}(\alpha,ix)\,, \tag{5.1.14}$$

so that one may extend $\mathcal{J}$ to arbitrary α,β,γ, even relaxing the conservation of traces.

Organization of the chapter

- In Section 5.2 we consider the self-adjoint case. We first show in Section 5.2.1 that $\mathcal{J}$ is real analytic away from a particular collection of hyperplanes, and that the equations defining these hyperplanes are the same in all three cases of real symmetric, complex Hermitian, and quaternionic self-dual matrices.

Next, in Section 5.2.2 we relate $\mathcal{J}$ to the Riemannian geometry of the orbits, considered as submanifolds of V, and we explain how this interpretation helps to understand the nature of the divergences that arise in $\mathcal{J}$ in the case of real symmetric matrices (Coquereaux and Zuber, 2019).

- For the remainder of the chapter after Section 5.2, we restrict our attention to the coadjoint case. In Section 5.3 we discuss the relationship between $\mathcal{J}$ and tensor product multiplicities, and we derive two different identities that express $\mathcal{J}$ in terms of tensor product multiplicities when the arguments are particular triples of highest weights.

- In Section 5.4.1, we relate $\mathcal{J}$ to the Euclidean volume of the *BZ polytope*, which provides a polyhedral model for tensor product multiplicities. This point of view explains geometrically the relationship between $\mathcal{J}$ and tensor product multiplicities, and also provides insight into the nature of the non-analyticities of $\mathcal{J}$. It also leads to a proof that $\mathcal{J}$ does not vanish in the interior of the Horn polytope.

- Section 5.5 is a detailed case study of the above considerations for B_2, i.e. the case $\mathfrak{g} = \mathfrak{so}(5)$.

5.2 The self-adjoint case

In this section we take $G = \mathrm{SO}(n)$, $\mathrm{SU}(n)$ or $\mathrm{USp}(n)$, and we respectively fix $\theta = 1/2$, 1, or 2 and let $\mathcal{M}^0_{\theta,n}$ be the set of traceless n-by-n real symmetric, complex Hermitian or quaternionic self-dual matrices. We study the action of G on $\mathcal{M}^0_{\theta,n}$ by conjugation.

In Section 5.2.1 we present an argument showing that non-analyticities of $\mathcal{J}$ lie along the same hyperplanes in all three cases. In Section 5.2.2 we write $\mathcal{J}$ in terms of quantities related to the Riemannian geometry of the orbits, which can help to understand the origin of the divergences that appear in both $\mathcal{J}$ and the Horn PDF in the real symmetric case.

5.2.1 Singular loci and nature of the non-analyticities

What follows is essentially an argument due to Vergne (2018), based on a technique originally used to identify the singular loci of Duistermaat–Heckman densities in symplectic geometry. It is well known (Duistermaat and Heckman, 1982) that for a Hamiltonian G-action on a symplectic manifold, the associated Duistermaat–Heckman measure on $\mathbb{R}^r$, $r = \mathrm{rank}(G)$, has a piecewise polynomial density with non-analyticities along certain hyperplanes. In the coadjoint case the Horn probability measure is equal to a polynomial times the Duistermaat–Heckman measure for the diagonal G-action on $\mathcal{O}_\alpha \times \mathcal{O}_\beta$, so these symplectic methods can be used to identify the singular locus of the density. In this section we show that an analogous technique works even in cases where the orbits do not carry a symplectic structure.

For $A, B \in \mathcal{M}^0_{\theta,n}$ we can write $A = g_1 \alpha g_1^{-1}$, $B = g_2 \beta g_2^{-1}$ for some $g_1, g_2 \in G$, where α and β are diagonalizations of A and B. Given α and β ordered and regular, and g_1, g_2 drawn independently at random from the Haar probability measure on G, we are interested in the distribution of $\gamma = \mathrm{diag}\,(\gamma_i)$, where $\gamma_1 \geq \ldots \geq \gamma_n$ are the eigenvalues of $C = A + B$.

Proposition 5.2.1. *The distribution of γ has a piecewise real analytic density. Non-analyticities occur only when γ lies on a hyperplane defined by an equation of the* *form*

$$\sum_{j\in K}\gamma_j = \sum_{j\in I}\alpha_j + \sum_{j\in J}\beta_j\,, \tag{5.2.1}$$

where the subsets I, J, K have the same cardinality: $|I| = |J| = |K|$.

Proof. Following Vergne (2018), we consider the map $\Phi: \;\; G\times G \to \mathcal{M}^0_{\theta,n}$ that sends $(g_1, g_2) \mapsto C = A + B$. The pushforward by Φ of the product of the Haar measures is a G-invariant measure on the image, which has a density $\rho_{A,B}(C)$. The map Φ is proper and real analytic, so that for C_0 any regular value of Φ, a result of Shiga (1964, p. 133) guarantees the existence of a neighborhood $E \ni C_0$ such that $\Phi^{-1}(E) \cong \Phi^{-1}(C_0)\times E$ as real analytic manifolds. Let (z, C) be local analytic coordinates on $\Phi^{-1}(C_0)\times E$. In these coordinates the Haar measure has a real analytic density $h(z, C)$, and for $C \in E$, $\rho_{A,B}$ can be written as

$$\rho_{A,B}(C) = \int_{\Phi^{-1}(C_0)} h(z, C)\; dz,$$

which is clearly a real analytic function on E. It follows that $\rho_{A,B}$ is analytic in a neighborhood of every regular value of Φ. As before, let $\mathcal{C}_+ \subset \mathcal{M}^0_{\theta,n}$ denote the cone of traceless diagonal matrices $\mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ with $\gamma_1 \geq \ldots \geq \gamma_n$. The restriction of $\rho_{A,B}$ to $\mathcal{C}_+$ equals the volume function $\mathcal{J}$, up to a normalizing factor depending on α and β. All non-analyticities of $\mathcal{J}$ must therefore occur at non-regular values of Φ, i.e. γ such that the differential $d\Phi$ fails to be surjective at some point in the preimage $\Phi^{-1}(\gamma)$. The claim will now follow by identifying all non-regular values of Φ.

At the point (g_1, g_2), the differential is

$$d_{(g_1,g_2)}\Phi(X, Y) = [X, A] + [Y, B], \quad X, Y \in \mathfrak{g},$$

where we have identified the tangent space $T_{(g_1,g_2)}(G \times G)$ with $\mathfrak{g} \oplus \mathfrak{g}$. At a point where $d\Phi$ is not surjective this operator has a nontrivial kernel, corresponding to a nonzero solution $Z \in \mathcal{M}^0_{\theta,n}$ of

$$\forall X, Y \in \mathfrak{g}, \quad \mathrm{tr}\,(Z \cdot ([X, A] + [Y, B])) = 0\,. \tag{5.2.2}$$

Thus we must determine for which (g_1, g_2) such a Z exists.

(1) Using the invariance $\rho_{A,B} = \rho_{gAg^{-1},\,gBg^{-1}}$, we can reduce to the case $g_1 = I$ (at the price of redefining X, Y, Z). Then taking $Y = 0$, the condition (5.2.2) reduces to

$$\forall X \in \mathfrak{g}, \quad \mathrm{tr}\,(X \cdot [Z, \mathrm{diag}\,(\alpha_i)]) = 0,$$

so that we must have $[Z, \mathrm{diag}\,(\alpha_i)] = 0$. Since the eigenvalues α_i are assumed distinct, this implies that Z is diagonal.

(2) Having taken $g_1 = I$, we rewrite $g_2 = g$. For $X = 0$, Y arbitrary, the condition (5.2.2) reads $[Z, B] = 0$.

If Z is regular, this implies that B is diagonal, and since β is regular, this implies that g acts as a permutation: $B = \mathrm{diag}\,(\beta_{w_i})$ for some $w \in S_n$. Thus the ordered eigenvalues γ_i of C are:

$$\gamma_i = \alpha_{w'_i} + \beta_{w_i} \qquad w, w' \in S_n, \quad i = 1, \cdots, n\,, \tag{5.2.3}$$

which is a particular case of (5.2.1).

On the other hand, if Z is not regular, i.e. Z has repeated eigenvalues, with eigenvalue z_ℓ of multiplicity r_ℓ for $\ell = 1, \cdots, s$, then one may only assert that B is block diagonal, with s blocks of size $r_1, r_2, \cdots r_s$. For each block of size 1, one is back to the situation described in (5.2.3). For each block B_ℓ of size $r_\ell > 1$, the partial trace over the corresponding block in $\mathrm{diag}\,(\alpha) + B$ is a sum of r_ℓ eigenvalues γ_j, which we write $\sum^{(\ell)} \gamma_j = \sum^{(\ell)} \alpha_j + \mathrm{tr}\, B_\ell$. But since the trace of the block B_ℓ is just the sum of the β_j pertaining to that block, $\mathrm{tr}\, B_\ell = \sum^{(\ell)} \beta_j$, we have $\sum^{(\ell)} \gamma_j = \sum^{(\ell)} \alpha_j + \sum^{(\ell)} \beta_j$, which we can rewrite in the form (5.2.1). Such a linear relation on the coordinates γ_j defines a hyperplane in $\mathbb{R}^{n-1}$, since we have assumed that A, B and C were traceless. □

Remark 5.2.2.

a) The possible singularities identified in (5.2.1) include the hyperplanes that contain the facets of the Horn polytope (other than the walls of the Weyl chamber), where the Horn PDF and volume function vanish in a non-C^∞ way. The reader will recognize in (5.2.1) the form of Horn's (in)equalities.

b) The singular hyperplanes in γ-space depend only on α and β and not on θ. This is in agreement with the argument of Fulton (2000) that Horn's inequalities are the same for all three cases considered in this section. This also justifies the empirical observation made previously that the singular locus, for given α and β, is the same in all three cases (Zuber, 2018, Coquereaux and Zuber, 2019).

c) Equation (5.2.1) is only a *necessary* condition for a non-analyticity. It doesn't tell us on which hyperplanes a non-analyticity does in fact occur. Also, it doesn't tell us that all the points of that hyperplane are singular. An example is provided by the case $n = 3$ where some singularities occur along half-lines in the (γ_1, γ_2)–plane (Zuber, 2018, Coquereaux and Zuber, 2019).

d) The argument above doesn't tell us anything about the *nature* of the singularity. Indeed, much stronger singularities appear in the real symmetric case (where $\mathcal{J}$ can actually diverge) than in the complex Hermitian or quaternionic self-dual cases (Coquereaux and Zuber, 2019). For $\theta = 1$ or 2, and for all of the coadjoint cases, we can use explicit formulae for the orbital integrals to write $\mathcal{J}$ as a sum of Fourier transforms. A power-counting argument (differentiating under the integral sign and using the Riemann–Lebesgue lemma) then yields a lower bound on the number of continuous derivatives of $\mathcal{J}$. In the complex Hermitian case with $n \geq 3$, one expects the volume function to be *at least* of differentiability class C^{n-3} (Zuber, 2018). For example, it is continuous but non-differentiable for SU(3), and at least once continuously differentiable for SU(4). For SU(2), $\mathcal{J}$ is the indicator function of $\mathscr{H}_{\alpha,\beta}$ and is therefore discontinuous at the boundary. For a geometric interpretation of these singularities, see Section 5.4.4 below. $\triangle$

5.2.2 Riemannian interpretation of $\mathcal{J}$, and singularities in the real symmetric case

In this section we interpret $\mathcal{J}$ in terms of the Riemannian geometry of the orbits. The Riemannian interpretation can help to understand the origin of the divergences that can appear in $\mathcal{J}$ when $\theta = 1/2$. It was observed by Zuber (2018) and Coquereaux and Zuber (2019) that for SO(2) and SO(3) acting on real symmetric matrices, $\mathcal{J}(\alpha, \beta; \gamma)$ actually tends to infinity as γ approaches certain singular hyperplanes. It is unknown whether $\mathcal{J}$ diverges for $\theta = 1/2$ and $n > 3$. We follow the notation of Section 5.1.2, and we assume as before that α and β are regular and traceless.

The inner product $\langle A, B\rangle = \mathrm{tr}(AB)$ gives a G-invariant Riemannian metric on $\mathcal{M}_{\theta,n}$, so that we obtain G-invariant induced metrics on the orbits $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$. The associated Riemannian volume measures μ_α and μ_β are also G-invariant, so they must respectively equal $\mu_\alpha(\mathcal{O}_\alpha)$ and $\mu_\beta(\mathcal{O}_\beta)$ times the unique invariant probability measure on each orbit. If α and β are both regular as we assume, then we have $\mu_\alpha(\mathcal{O}_\alpha) = \kappa'_\theta \Delta(\alpha)^{2\theta}$, $\mu_\beta(\mathcal{O}_\beta) = \kappa'_\theta \Delta(\beta)^{2\theta}$, where Δ is the Vandermonde determinant, and the constant κ'_θ equals κ_θ except when $\theta = 1/2$ and n is even, in which case $\kappa'_\theta = \frac{1}{2}\kappa_\theta$.

Let $\tilde{\Psi} : \mathcal{O}_\alpha \times \mathcal{O}_\beta \to \mathcal{C}_+$ be the map that sends (A, B) to the diagonalization of $A + B$ with non-increasing entries down the diagonal. Define the measure ν on $\mathcal{O}_\alpha \times \mathcal{O}_\beta$ as the product measure of the normalized G-invariant measures on each orbit. If we endow $\mathcal{O}_\alpha \times \mathcal{O}_\beta$ with the product metric of the induced Riemannian metrics, so that its Riemannian volume measure is the product $\mu_\alpha \otimes \mu_\beta$, then we find $\nu = \kappa'^{-2}_\theta (\Delta(\alpha)\Delta(\beta))^{-2\theta}(\mu_\alpha \otimes \mu_\beta)$. The Horn probability measure on $\mathcal{C}_+$ is the pushforward $\tilde{\Psi}_*\nu = \kappa'^{-2}_\theta (\Delta(\alpha)\Delta(\beta))^{-2\theta}\tilde{\Psi}_*(\mu_\alpha \otimes \mu_\beta)$.

We can rewrite the measure in a simpler form by eliminating one of the orbits from the domain of $\tilde{\Psi}$. Recalling that $\tilde{\Psi}$ is invariant under the diagonal G-action on $\mathcal{O}_\alpha \times \mathcal{O}_\beta$, it suffices to consider the case $A = \alpha$ and the "reduced" map $\Psi : \mathcal{O}_\beta \to \mathcal{C}_+$ that sends $B \in \mathcal{O}_\beta$ to the diagonalization of $\alpha + B$ with non-increasing entries down the diagonal. The Horn probability measure is then equal to $\kappa'^{-1}_\theta \Delta(\beta)^{-2\theta}\Psi_*\mu_\beta$.

If γ_0 is a regular value of Ψ, then for a sufficiently small coordinate neighborhood $E \ni \gamma_0$ all fibers of Ψ over E are diffeomorphic, and $\Psi^{-1}(E)$ is diffeomorphic to $\Psi^{-1}(\gamma_0) \times E$. Let $z_1, \ldots, z_m$ be local coordinates on the fiber $\Psi^{-1}(\gamma_0)$, where $m = \dim \mathcal{M}_{\theta,n} - n + 1$. Then (z, γ) are local coordinates on $\Psi^{-1}(E)$, so that for γ

sufficiently close to γ_0 we can write the Horn PDF as the fiber integral

$$p(\gamma|\alpha,\beta) = \kappa_\theta'^{-1}\Delta(\beta)^{-2\theta}\int_{\Psi^{-1}(\gamma_0)}\sqrt{g_\beta(z,\gamma)}\,dz, \tag{5.2.4}$$

where g_β is the determinant of the induced metric on $\mathcal{O}_\beta$ in our chosen coordinates. Accordingly, we have

$$\mathcal{J}(\alpha,\beta;\gamma) = \frac{(2\pi)^{\theta n(n-1)}}{\kappa_\theta'\kappa_\theta^2\,\Delta_{\mathfrak{g}}(\rho)^3}\left(\frac{\Delta(\alpha)}{\Delta(\beta)\Delta(\gamma)}\right)^\theta\int_{\Psi^{-1}(\gamma_0)}\sqrt{g_\beta(z,\gamma)}\,dz \tag{5.2.5}$$

for γ sufficiently close to any regular value γ_0 of Ψ.

Equation (5.2.5) gives the desired Riemannian interpretation of $\mathcal{J}$ and provides some geometric insight into the origin of the volume function's singularities. In particular, this point of view helps to explain why $\mathcal{J}$ can actually diverge in the real symmetric case. The integral appearing in (5.2.4) and (5.2.5) looks almost like the induced volume of the (compact) fiber $\Psi^{-1}(y)$, so it may be surprising at first that for $\theta = 1/2$, $\mathcal{J}$ can tend to infinity on the interior of $\mathscr{H}_{\alpha\beta}$. However, this integral is not the volume of the fiber. If $g_\beta^\top$ and $g_\beta^\perp$ are the determinants, respectively, of the restriction of the induced metric on $\mathcal{O}_\beta$ to the tangent bundle and normal bundle of $\Psi^{-1}(\gamma)$, then we have

$$\mathrm{Vol}(\Psi^{-1}(\gamma)) = \int_{\Psi^{-1}(\gamma_0)}\sqrt{g_\beta^\top(z,\gamma)}\,dz,$$

whereas

$$\int_{\Psi^{-1}(\gamma_0)}\sqrt{g_\beta(z,\gamma)}\,dz = \int_{\Psi^{-1}(\gamma_0)}\sqrt{g_\beta^\top(z,\gamma)\,g_\beta^\perp(z,\gamma)}\,dz.$$

The factor $g_\beta^\perp$ can blow up as γ approaches a non-regular value of Ψ. This merely reflects a singularity of the idiosyncratic choice of coordinates (z,γ), but it will cause $\mathcal{J}$ to diverge if $g_\beta^\top$ doesn't diminish sufficiently to compensate.

To illustrate this idea, we consider the simple example of SO(2) acting on 2-by-2 real symmetric matrices, relaxing temporarily our assumption that α and β are traceless. In this case, the orbits of regular elements are circles embedded in $\mathcal{M}_{\frac{1}{2},2} \cong \mathbb{R}^3$. If we parametrize SO(2) as rotation matrices

$$R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \quad 0 \leq \phi < 2\pi,$$

then the orbit of $\beta = \operatorname{diag}(\beta_1, \beta_2)$ is

$$R(\phi)\beta R(\phi)^T = \begin{bmatrix} \beta_1\cos^2\phi + \beta_2\sin^2\phi & (\beta_1-\beta_2)\cos\phi\sin\phi \\ (\beta_1-\beta_2)\cos\phi\sin\phi & \beta_1\sin^2\phi + \beta_2\cos^2\phi \end{bmatrix}, \quad 0 \leq \phi < 2\pi.$$

We have $\mathcal{C}_+ = \{\operatorname{diag}(\gamma_1, \gamma_2) \mid \gamma_1 \geq \gamma_2 \text{ and } \gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2\}$, so that $\mathcal{C}_+ \cong [0, \infty)$ is parametrized by the single coordinate $\gamma_{12} := \gamma_1 - \gamma_2$. The map Ψ sends $\phi \in [0, 2\pi)$ to $\gamma_1 - \gamma_2$ where γ_1, γ_2 are the eigenvalues of $\alpha + R(\phi)\beta R(\phi)^T$, and we have

$$\Psi(\phi) = \sqrt{\alpha_{12}^2 + \beta_{12}^2 + 2\alpha_{12}\beta_{12}\cos(2\phi)},$$

where $\alpha_{12} := \alpha_1 - \alpha_2$, $\beta_{12} := \beta_1 - \beta_2$. The image of Ψ is the interval

$$\big[\ |\alpha_{12} - \beta_{12}|,\ \alpha_{12} + \beta_{12}\ \big].$$

We assume that $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$, as otherwise this image is just a single point. An explicit computation yields:

$$\mathcal{J}(\alpha, \beta; \gamma) = \begin{cases} \frac{2}{\pi^2}\sqrt{\frac{\alpha_{12}\beta_{12}\gamma_{12}}{((\alpha_{12}+\beta_{12})^2 - \gamma_{12}^2)(\gamma_{12}^2 - (\alpha_{12}-\beta_{12})^2)}}, & \gamma_{12} \in \big[|\alpha_{12} - \beta_{12}|, \alpha_{12} + \beta_{12}\big], \\ 0, & \text{otherwise.} \end{cases}$$

Figure 5.1 shows the plot of $\mathcal{J}$ as a function of γ_{12} for $\alpha_{12} = 1$, $\beta_{12} = 2$.

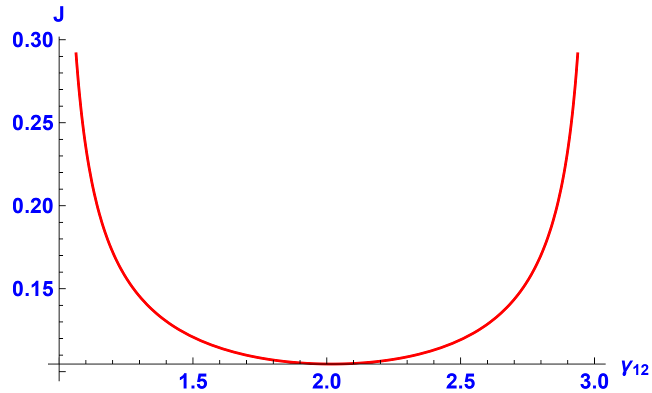


Figure 5.1: The $\mathcal{J}$ function for $\alpha_{12} = 1$, $\beta_{12} = 2$.

$\mathcal{J}$ diverges at the endpoints $|\alpha_{12} - \beta_{12}| = \Psi(0)$ and $\alpha_{12} + \beta_{12} = \Psi(\pi)$, which are the non-regular values of Ψ. The fiber of Ψ over a regular value consists of two points, one in each of the open sets of $\mathcal{O}_\beta$ parametrized by $0 < \phi < \pi$ and $\pi < \phi < 2\pi$. In this case we may think of Ψ as giving a coordinate chart on either of these two open sets; the density of the Riemannian volume form in these coordinates is just $\sqrt{g_\beta^\perp}$, which in this case is a function only of γ_{12}, for α_{12}, β_{12} fixed. Since the fibers are 0-dimensional, the induced Riemannian volume on each fiber is just the counting measure, so that $\mathrm{Vol}(\Psi^{-1}(\gamma_{12})) = 2$ when γ_{12} is a regular value. Inverting the prefactor before the integral in (5.2.5) and dividing by 2 to account for the volume of the fiber, we find that on either submanifold $\phi \in (0, \pi)$ or $\phi \in (\pi, 2\pi)$, the density of the Riemannian volume on $\mathcal{O}_\beta$ can be expressed in terms of the local coordinate γ_{12} as

$$\sqrt{g_\beta^\perp(\gamma_{12})} = \frac{\pi^3}{\sqrt{2}} \sqrt{\frac{\beta_{12}\gamma_{12}}{\alpha_{12}}} \mathcal{J}(\alpha_{12}, \beta_{12}; \gamma_{12}).$$

The divergences of $\mathcal{J}$ at the endpoints indicate that this coordinate becomes singular as it approaches a non-regular value of Ψ.

5.3 Coadjoint case: $\mathcal{J}$-LR relations

For the remainder of this chapter, we restrict our attention to the coadjoint case. Let G be a compact, connected, semisimple Lie group, $\mathfrak{g}$ its Lie algebra. Here and in Section 5.4 below, we will always make the further assumption that $\mathfrak{g}$ contains no simple summands isomorphic to $\mathfrak{su}(2)$. This assumption can be removed, but requires some additional care due to the discontinuity of $\mathcal{J}$ at the boundary of the Horn polytope in the $\mathfrak{su}(2)$ case; see (Coquereaux and Zuber, 2018, sect. 4.1.1).

For x in a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, define[4]

$$\Delta_{\mathfrak{g}}(x) := \prod_{\boldsymbol{\alpha}>0} \langle \boldsymbol{\alpha}, x \rangle, \qquad \widehat{\Delta}_{\mathfrak{g}}(e^x) := \prod_{\boldsymbol{\alpha}>0} \left(e^{\frac{i}{2}\langle \boldsymbol{\alpha}, x \rangle} - e^{-\frac{i}{2}\langle \boldsymbol{\alpha}, x \rangle} \right), \tag{5.3.1}$$

where we regard $\widehat{\Delta}_{\mathfrak{g}}$ as a function on the maximal torus $T := \exp(\mathfrak{t})$. Using (2.6.5), the Kirillov formula (2.6.1) for the character χ_λ of an irreducible representation V_λ with highest weight λ can be written

$$\frac{\chi_\lambda(e^x)}{\dim V_\lambda} = \frac{\Delta_{\mathfrak{g}}(ix)}{\widehat{\Delta}_{\mathfrak{g}}(e^x)} \mathcal{H}(\lambda + \rho, ix), \tag{5.3.2}$$

with $\mathcal{H}$ the Harish-Chandra integral

$$\mathcal{H}(\lambda, ix) = \int_G e^{i\langle \lambda, \mathrm{Ad}_g x \rangle} \, dg. \tag{5.3.3}$$

Let Q be the root lattice of $\mathfrak{g}$, P the weight lattice, and $P^\vee$ the coweight lattice. If (λ, μ, ν) is a *compatible* triple of highest weights, i.e. $\lambda + \mu - \nu \in Q$, and if we denote by primes λ', μ', ν' the shift of λ, μ, ν by the Weyl vector ρ, i.e. $\lambda' = \lambda + \rho$, etc., we

[4] In the following, we use boldface $\boldsymbol{\alpha}$ to denote roots, not to confuse them with eigenvalues α_i.

have:

$$\begin{aligned}\mathcal{J}(\lambda',\mu';\nu') &= \frac{\dim V_\lambda \dim V_\mu \dim V_\nu}{(2\pi)^r|W|}\int_{\mathfrak{t}} |\Delta_{\mathfrak{g}}(x)|^2\, \mathcal{H}(\lambda',ix)\mathcal{H}(\mu',ix)(\mathcal{H}(\nu',ix))^*\,dx \\ &= \frac{1}{(2\pi)^r|W|}\int_{\mathfrak{t}} |\widehat{\Delta}_{\mathfrak{g}}(e^x)|^2\, \frac{\widehat{\Delta}_{\mathfrak{g}}(e^x)}{\Delta_{\mathfrak{g}}(ix)}\,\chi_\lambda(e^x)\chi_\mu(e^x)(\chi_\nu(e^x))^*\,dx \\ &= \int_{\mathfrak{t}/2\pi P^\vee}\sum_{\delta\in 2\pi P^\vee}\frac{\widehat{\Delta}_{\mathfrak{g}}(e^{x+\delta})}{\Delta_{\mathfrak{g}}(i(x+\delta))}\,\chi_\lambda(e^{x+\delta})\chi_\mu(e^{x+\delta})(\chi_\nu(e^{x+\delta}))^*\,\frac{|\widehat{\Delta}_{\mathfrak{g}}(e^{x+\delta})|^2}{(2\pi)^r|W|}\,dx.\end{aligned} \tag{5.3.4}$$

For $\delta\in 2\pi P^\vee$, a direct computation yields the identity $\chi_\lambda(e^{x+\delta}) = e^{i\langle\lambda,\delta\rangle}\chi_\lambda(e^x)$, and the compatibility of the triple (λ,μ,ν) implies that $\exp i\langle(\lambda+\mu-\nu),\delta\rangle = 1$. Moreover, $\mathfrak{t}/2\pi P^\vee \cong T/Z$, where Z is the center of G. Using these facts, we can rewrite (5.3.4) as an integral over the maximal torus T:

$$\begin{aligned}\mathcal{J}(\lambda',\mu';\nu') &= \int_{\mathfrak{t}/2\pi P^\vee}\left(\sum_{\delta\in 2\pi P^\vee} e^{i\langle\rho,\delta\rangle}\frac{\widehat{\Delta}_{\mathfrak{g}}(e^x)}{\Delta_{\mathfrak{g}}(i(x+\delta))}\right)\chi_\lambda(e^x)\chi_\mu(e^x)(\chi_\nu(e^x))^*\,\frac{|\widehat{\Delta}_{\mathfrak{g}}(e^x)|^2}{(2\pi)^r|W|}\,dx \\ &= \int_T\left(\sum_{\delta\in 2\pi P^\vee} e^{i\langle\rho,\delta\rangle}\frac{\widehat{\Delta}_{\mathfrak{g}}(e^x)}{\Delta_{\mathfrak{g}}(i(x+\delta))}\right)\chi_\lambda(e^x)\chi_\mu(e^x)(\chi_\nu(e^x))^*\,dt,\end{aligned} \tag{5.3.5}$$

where

$$dt = \frac{|\widehat{\Delta}_{\mathfrak{g}}(e^x)|^2}{(2\pi)^r|W||Z|}\,dx.$$

By a similar calculation, we find that for (λ,μ,ν) compatible,

$$\mathcal{J}(\lambda,\mu;\nu) = \int_T\left(\sum_{\delta\in 2\pi P^\vee}\frac{\widehat{\Delta}_{\mathfrak{g}}(e^x)}{\Delta_{\mathfrak{g}}(i(x+\delta))}\right)\chi_{\lambda-\rho}(e^x)\,\chi_{\mu-\rho}(e^x)\,(\chi_{\nu-\rho}(e^x))^*\,dt. \tag{5.3.6}$$

Now, as observed by Coquereaux and Zuber (2018) in the case of $G=\mathrm{SU}(n)$ and proved in full generality by Etingof and Rains (2018), the two sums over the coweight lattice $P^\vee$ that appear in (5.3.5, 5.3.6) can each be expressed as a finite sum of characters over a set K, respectively $\hat{K}$, of highest weights:

$$R := \sum_{\delta\in 2\pi P^\vee} e^{i\langle\rho,\delta\rangle}\frac{\widehat{\Delta}_{\mathfrak{g}}(e^x)}{\Delta_{\mathfrak{g}}(i(x+\delta))} = \sum_{\kappa\in K} r_\kappa \chi_\kappa(e^x), \tag{5.3.7}$$

$$\hat{R} := \sum_{\delta\in 2\pi P^\vee} \frac{\widehat{\Delta}_{\mathfrak{g}}(e^x)}{\Delta_{\mathfrak{g}}(i(x+\delta))} = \sum_{\kappa\in \hat{K}} \hat{r}_\kappa \chi_\kappa(e^x)\,, \tag{5.3.8}$$

where r_κ and $\hat{r}_\kappa$ are coefficients to be determined, see below. According to Etingof and Rains (2018), in order for the character χ_κ to occur in the sum R defined in (5.3.7), κ should belong to the interior of the convex hull of the Weyl group orbit of ρ. More precisely, K is the set of dominant weights occurring in the irreducible representation of highest weight $\rho-\xi$, where

$$\xi = \begin{cases} \sum_{i=1}^r \boldsymbol{\alpha}_i \quad \text{(sum of simple roots)} & \text{if } \rho\in Q \\ \sum_{i=1}^r k_i\boldsymbol{\alpha}_i & \text{if } \rho\notin Q, \text{ with } \; k_i = \begin{cases} 1 & \text{if } \langle\rho,\omega_i^\vee\rangle\in\mathbb{Z} \\ \frac{1}{2} & \text{if } \langle\rho,\omega_i^\vee\rangle\notin\mathbb{Z} \end{cases} \end{cases} \tag{5.3.9}$$

where the ω_i, $i = 1,\cdots,r$, are the fundamental weights. Observe that $\rho-\xi$, and therefore all weights $\kappa\in K$, must lie in the root lattice. Obviously $\hat{R}$ and R differ only if $\rho\notin Q$, in which case $\hat{K}$ consists of the dominant weights occurring in the irreducible representation of highest weight $\rho-\hat{\xi}$, where $\hat{\xi}$ is obtained by swapping the two lines above:

$$\hat{\xi} = \sum_{i=1}^r \hat{k}_i\boldsymbol{\alpha}_i \qquad \text{with} \quad \hat{k}_i = \begin{cases} \frac{1}{2} & \text{if } \langle\rho,\omega_i^\vee\rangle\in\mathbb{Z} \\ 1 & \text{if } \langle\rho,\omega_i^\vee\rangle\notin\mathbb{Z}. \end{cases} \tag{5.3.10}$$

Note that the trivial weight 0 always occurs in K, but never occurs in $\hat{K}$ when $\rho\notin Q$.

Example 5.3.1. For SU(3) we have $\rho-\xi = 0$, and $R = \hat{R}$, given by the right-hand side of (5.3.7), equals 1. For SU(4) we have $\rho - \xi = \omega_1 + \omega_3$. One finds $K = \{(0,0,0),\ (1,0,1)\}$ in the fundamental weight basis, with $r_0 = 9/24$ and $r_{\rho-\xi} = 1/24$; see (Coquereaux and Zuber, 2018, eqn. 62a). One finds also $\rho - \hat{\xi} = \omega_2$, $\hat{K} = \{\omega_2\}$, and $\hat{r}_{\omega_2} = 1/6$; see (Coquereaux and Zuber, 2018, eqn. 62b).

More generally, for the A_r series, one has $\xi = \sum_{i\,\text{odd}} \boldsymbol{\alpha}_i/2 + \sum_{i\,\text{even}} \boldsymbol{\alpha}_i$ if r is odd, and $\xi = \sum_i \boldsymbol{\alpha}_i$ if r is even. Moreover $\hat{\xi} = \xi$ if r is even, and $\hat{\xi} = \sum_i \boldsymbol{\alpha}_i$ if r is odd. Explicit results for R (i.e. for χ_κ and r_κ) in the cases of SU(5) and SU(6) are also given in (Coquereaux and Zuber, 2018, sects. 4.2.2, 4.2.3, 4.2.4), along with results for $\hat{R}$ in the case of SU(6).

For SO(5), $\rho = \frac{1}{2}(3\boldsymbol{\alpha}_1 + 4\boldsymbol{\alpha}_2)$, $\xi = \frac{1}{2}\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$, $\rho - \xi = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \omega_1$, $K = \{(0,0),(1,0)\}$, and $R = \frac{1}{8}(3\chi_0 + \chi_{\omega_1})$. One also finds $\hat{\xi} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$, $\hat{K} = \{(0,1)\}$, and $\hat{r}_{(0,1)} = 1/4$.

For SO(7), we find $K = \{(0,0,0),(1,0,0),(0,1,0),(2,0,0),(0,0,2),(1,1,0),(1,0,2)\}$ and the corresponding list of coefficients r_κ: $\frac{1}{92160}\{7230, 3995, 1651, 85, 479, 29, 1\}$. One also finds $\hat{\xi} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3$, $\hat{K} = \{(0,0,1),(1,0,1),(0,1,1)\}$, and $\hat{r}_\kappa = \frac{1}{2880}\{190, 26, 1\}$. $\triangle$

Introducing (generalized) Littlewood–Richardson (LR) coefficients

$$C_{\lambda\,\mu}^{\nu} := \dim \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_\mu \ \to V_\nu),$$

$$C_{\lambda\,\mu\,\kappa}^{\nu} := \dim \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_\mu \otimes V_\kappa \to V_\nu),$$

and putting (5.3.7, 5.3.8) into (5.3.5, 5.3.6), we finally obtain:

Proposition 5.3.2. *Let $(\lambda, \mu; \nu)$ be a compatible triple of highest weights, i.e. such that $\lambda + \mu - \nu \in Q$. Let $(\lambda', \mu'; \nu')$ be the corresponding triple shifted by the Weyl* *vector ρ: $\lambda' = \lambda + \rho$, etc. Then we have the two $\mathcal{J}$-**LR relations**:*

$$\mathcal{J}(\lambda', \mu'; \nu') = \sum_{\kappa\in K,\,\tau} r_\kappa C_{\lambda\,\mu}^{\tau} C_{\tau\,\kappa}^{\nu} = \sum_{\kappa\in K} r_\kappa C_{\lambda\,\mu\,\kappa}^{\nu} \tag{5.3.11}$$

$$\mathcal{J}(\lambda, \mu; \nu) = \sum_{\kappa\in \hat{K},\,\tau} \hat{r}_\kappa C_{(\lambda-\rho)\,(\mu-\rho)}^{\tau} C_{\tau\,\kappa}^{\nu-\rho} = \sum_{\kappa\in \hat{K}} \hat{r}_\kappa C_{(\lambda-\rho)\,(\mu-\rho)\,\kappa}^{\nu-\rho}\,. \tag{5.3.12}$$

Remark 5.3.3.

a) The above derivation generalizes and simplifies substantially the discussion given in (Coquereaux and Zuber, 2018) for the case of $G = \mathrm{SU}(n)$.

b) The coefficients r_κ, $\hat{r}_\kappa$ may be determined either by a direct calculation of the sums in (5.3.7, 5.3.8) as it was done in (Coquereaux and Zuber, 2018), or by a geometric argument (Etingof and Rains, 2018), or by noticing that at the following special points, (5.3.11, 5.3.12) reduce to:[5]

$$r_\kappa = \mathcal{J}(\rho, \rho, \kappa + \rho), \qquad \kappa \in K, \tag{5.3.13}$$

$$\hat{r}_\kappa = \mathcal{J}(\rho, \rho, \kappa + \rho), \qquad \kappa \in \hat{K}. \tag{5.3.14}$$

c) Taking the limit $x \to 0$ in (5.3.7, 5.3.8), we see that[6]

$$\sum_{\kappa\in K} r_\kappa \dim V_\kappa = 1 \qquad \text{and} \qquad \sum_{\kappa\in \hat{K}} \hat{r}_\kappa \dim V_\kappa = 1\,. \tag{5.3.15}$$

d) For ν deep enough in the dominant Weyl chamber, so that all the weights $\nu - \rho - k$ are dominant when k runs over the set $\mathrm{wt}(\kappa)$ of weights of each V_κ,

[5] The right-hand side of (5.3.13) or (5.3.14), interpreted as a volume as we shall see in the next section, can be read, for instance, from the (stretched) LR polynomial defined by the triples that appear as arguments of $\mathcal{J}$, or, for low rank, computed from explicit expressions such as those in (Zuber, 2018) or below in Section 5.5.1.

[6] One may use this relation and the dimensions $\dim V_\kappa$, $\kappa \in K$, to check the coefficients r_κ. In the above examples of B_2 and B_3, the dimensions $\dim V_\kappa$ for the representations with $\kappa \in K$ are respectively $\{1, 5\}$ and $\{1, 7, 21, 27, 35, 105, 189\}$.

the right-hand sides of (5.3.11, 5.3.12) may be written explicitly as

$$\mathcal{J}(\lambda,\mu;\nu) = \sum_{\kappa\in\hat{K}} \hat{r}_\kappa \sum_{k\in\mathrm{wt}(\kappa)} \mathrm{mult}_\kappa(k)\, C_{(\lambda-\rho)\,(\mu-\rho)}^{\nu-\rho-k} \tag{5.3.16}$$

and a similar formula for $\mathcal{J}(\lambda',\mu';\nu')$. See examples for A_r in (Coquereaux and Zuber, 2018) and for B_2 in Section 5.5.5.2 below.

e) Multiplying (5.3.11) by $\dim V_\nu = \Delta_{\mathfrak{g}}(\nu')/\Delta_{\mathfrak{g}}(\rho)$, summing over ν, and using (5.3.15) along with the identity $\sum_\nu C_{\lambda\mu}^\nu \dim V_\nu = \dim V_\lambda \dim V_\mu$, one finds:

$$\sum_\nu \mathcal{J}(\lambda',\mu';\nu')\frac{\Delta_{\mathfrak{g}}(\nu')\Delta_{\mathfrak{g}}(\rho)}{\Delta_{\mathfrak{g}}(\lambda')\Delta_{\mathfrak{g}}(\mu')} = \sum_\nu p(\nu'|\lambda',\mu') = 1, \tag{5.3.17}$$

so that the PDF (5.1.3) also satisfies a discrete normalization condition. $\triangle$

5.4 Coadjoint case: $\mathcal{J}$ as the volume of a polytope

In this section we show that $\mathcal{J}(\alpha,\beta;\gamma)$ is equal to the (relative) volume of a certain convex polytope, the *Berenstein–Zelevinsky (BZ) polytope* $H_{\alpha\beta}^\gamma$, and we explore some consequences of this fact. The primary importance of the BZ polytope is that the tensor product multiplicity $C_{\lambda\mu}^\nu$ is equal to the number of integer points in $H_{\lambda\mu}^\nu$. This fact provides another perspective on the link between $\mathcal{J}$ and tensor product multiplicities. In Section 5.4.1 we recall the definition of the BZ polytope, and we show in Section 5.4.2 that $\mathcal{J}$ computes its volume. In Section 5.4.3 we use this geometric interpretation to show that $\mathcal{J}$ cannot vanish on the interior of the Horn polytope, and in Section 5.4.4 we discuss how the non-analyticities of $\mathcal{J}$ arise from changes in the geometry of $H_{\alpha\beta}^\gamma$ as γ varies.

5.4.1 The BZ polytope

Following Berenstein and Zelevinsky (1988, 1992, 2001), one may determine the tensor product multiplicity $C^{\nu}_{\lambda\mu}$ for a compact or complex semisimple Lie algebra $\mathfrak{g}$ by counting the number of integer points of a certain convex polytope, the *BZ polytope* (in the A_r case it is closely related to the *hive polytope*[7] of Knutson and Tao (1999)), which we denote $H^{\nu}_{\lambda\mu}$. We will show below that the volume function $\mathcal{J}(\lambda,\mu;\nu)$ is proportional to the Euclidean volume of this polytope. Intuitively, for a compatible triple of highest weights (λ,μ,ν), if the polytope $H^{\nu}_{\lambda\mu}$ is very large then we expect that the number of its integer points should give a very good approximation of its volume. In practice there are some additional subtleties because we want to count the integer points in a space of higher dimension than $H^{\nu}_{\lambda\mu}$, however at a heuristic level this intuition illustrates geometrically why $\mathcal{J}(\lambda,\mu;\nu)$ can be considered as a semiclassical approximation of $C^{\nu}_{\lambda\mu}$.

The BZ construction hinges on the result that $C^{\nu}_{\lambda\mu}$ equals the number of ways of decomposing the weight $\sigma=\lambda+\mu-\nu$ as a positive integer combination of positive roots, such that the decomposition also satisfies some additional combinatorial constraints. To construct the polytope $H^{\nu}_{\lambda\mu}$, one therefore starts by introducing real parameters $u_1,\ldots,u_{N_r}$, where $r=\mathrm{rank}(\mathfrak{g})$ and $N_r=|\Phi^+|$ denotes the number of *positive* roots $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_{N_r}$ of $\mathfrak{g}$. A decomposition of σ as a positive linear combination of positive roots corresponds to a point in the polytope of $\mathfrak{g}$-*partitions with weight* σ,

$$\mathrm{Part}_{\mathfrak{g}}(\sigma) := \left\{(u_1,\ldots,u_{N_r})\in\mathbb{R}^{N_r} \;\middle|\; \sum_{a=1}^{N_r} u_a\boldsymbol{\alpha}_a=\sigma\right\}\cap\mathbb{R}^{N_r}_+, \tag{5.4.1}$$

where $\mathbb{R}^{N_r}_+$ is the positive orthant in $\mathbb{R}^{N_r}$. Since σ lies in an r-dimensional space,

[7] Actually the BZ polytope is the image of the hive polytope under an injective lattice-preserving linear map (Pak and Vallejo, 2005), so that one can identify them for the purpose of counting arguments; however, the Euclidean volumes of the two polytopes differ by an r-dependent constant.

we have $\dim \mathrm{Part}_{\mathfrak{g}}(\sigma) = N_r - r =: d_r$. Positive *integer* decompositions of σ correspond to integer points of $\mathrm{Part}_{\mathfrak{g}}(\sigma)$. Additional linear constraints must still be imposed on these integer decompositions, so that the BZ polytope $H_{\lambda\mu}^{\nu}$ is finally obtained by intersecting $\mathrm{Part}_{\mathfrak{g}}(\sigma)$ with some number of half-spaces. Generically we have $\dim H_{\lambda\mu}^{\nu} = d_r$, but for non-generic triples we may have $\dim H_{\lambda\mu}^{\nu} =: d \leq d_r$.

One may alternatively introduce the quantity $f_r := N_r + 2r$, which in the case of A_r is the number of "independent fundamental intertwiners" (see (Coquereaux and Zuber, 2014, sect. 4)), and then impose $3r$ conditions on the three weights λ, μ, ν recovering $d_r = f_r - 3r = N_r - r$. Finally, the d_r independent parameters are again subject to certain linear inequalities, thus defining the convex polytope $H_{\lambda\mu}^{\nu}$.

Note that we have defined $H_{\lambda\mu}^{\nu}$ as the solution set of a system of equations and inequalities that depend linearly on λ, μ and ν. We can therefore talk about the BZ polytope $H_{\alpha\beta}^{\gamma}$ associated to *any* triple of points (α, β, γ) in the dominant Weyl chamber, which may not be compatible integral weights or even rational points.

In the case that (λ, μ, ν) is indeed a compatible triple, $H_{\lambda\mu}^{\nu}$ is not in general integral, but it is always rational. Upon scaling by a positive integer s, $P_{\lambda\mu}^{\nu}(s) := C_{s\lambda\, s\mu}^{s\nu}$ is a *quasi-polynomial* of the variable s, and is the Ehrhart quasi-polynomial[8] of the polytope $H_{\lambda\mu}^{\nu}$. It is sometimes called the *stretching quasi-polynomial* or *LR quasi-polynomial* of the triple (λ, μ, ν).

Remark 5.4.1.

a) The definition of $P_{\lambda\mu}^{\nu}(s)$ makes sense whether or not (λ, μ, ν) is a compatible triple. In many instances in the literature, like the construction of the BZ

[8] See (Stanley, 1997) for background on Ehrhart quasi-polynomials.

polytope and the discussion of saturation or of the properties of the stretching polynomial, compatibility of the triple is generally assumed. Since in the present setting we do not limit our consideration to compatible triples,[9] we will make clear the hypothesis of compatibility whenever it is necessary.

b) When discussing such topics (stretching polynomials, saturation property, etc.) in terms of the representation theory of the group G rather than the algebra $\mathfrak{g}$, one should be specific about the group under consideration since the conclusions will usually differ if one compares two Lie groups with the same Lie algebra but different fundamental groups. For example, in this chapter we consider the coadjoint representation of the orthogonal group $\mathrm{SO}(n)$, however the tensor product multiplicities that arise in this setting for algebras of types B or D actually correspond to those of the simply connected group $\mathrm{Spin}(n)$. $\triangle$

$\mathfrak{g}$	N_r	f_r	d_r	δ_r
A_r	$\frac{1}{2}r(r+1)$	$\frac{1}{2}r(r+5)$	$\frac{1}{2}r(r-1)$	$(r+1)^{r-1}$
B_r	r^2	$r(r+2)$	$r(r-1)$	$(2r-1)^r$
C_r	r^2	$r(r+2)$	$r(r-1)$	$2^{r-2}(r+1)^r$
D_r	$r(r-1)$	$r(1+r)$	$r(r-2)$	$2^{r-2}(r-1)^r$
E_6	36	48	30	$2^{12}3^5$
E_7	63	77	56	$2^6 3^{14}$
E_8	120	136	112	$2^8 3^8 5^8$
F_4	24	32	20	$2^2 3^8$
G_2	6	10	4	$2^4 3$

Table 5.1: The numbers N_r, f_r, d_r, δ_r for the various simple algebras. The quantity δ_r, discussed in the appendix to this chapter, is a conjectured value of the squared covolume of the affine lattice Λ defined in Section 5.4.2.

[9] For instance, as noted in the previous section, for some algebras if a triple is compatible then the corresponding Weyl shifted triple is not. This occurs for example in the case of B_2.

5.4.2 $\mathcal{J}$ and the Euclidean volume of the BZ polytope

In this section we show that $\mathcal{J}(\alpha, \beta; \gamma)$ is proportional to the Euclidean d_r-volume of the BZ polytope $H_{\alpha\beta}^{\gamma}$ for an arbitrary compact or complex semisimple Lie algebra $\mathfrak{g}$ with no $\mathfrak{su}(2)$ summands. Specifically, we show that $\mathcal{J}$ equals the *relative* d_r*-volume* of the BZ polytope, defined as the Euclidean d_r-volume divided by the covolume (volume of a fundamental domain) of the lattice $\Lambda := \text{aff}(H_{\alpha\beta}^{\gamma}) \cap \mathbb{Z}^{N_r}$. Here $\text{aff}(H_{\alpha\beta}^{\gamma})$ indicates the affine span of $H_{\alpha\beta}^{\gamma}$, i.e. the minimal affine subspace of $\mathbb{R}^{N^r}$ containing $H_{\alpha\beta}^{\gamma}$, and $\mathbb{Z}^{N_r}$ is the integer lattice of the parameters u_a appearing in (5.4.1). When $\dim H_{\alpha\beta}^{\gamma} = d_r$ we have $\text{aff}(H_{\alpha\beta}^{\gamma}) = \text{aff}(\text{Part}_{\mathfrak{g}}(\sigma))$, and the covolume of Λ is a constant $c_{\mathfrak{g}}$ that depends only on the root system of the algebra $\mathfrak{g}$, so that we have

$$\text{Vol}_{\text{rel}}(H_{\alpha\beta}^{\gamma}) = \frac{1}{c_{\mathfrak{g}}} \text{Vol}(H_{\alpha\beta}^{\gamma}), \tag{5.4.2}$$

where Vol_{rel} is the relative d_r-volume and Vol is the Euclidean d_r-volume. We discuss the covolumes $c_{\mathfrak{g}}$ in more detail in the appendix at the end of this chapter, where we explain the conjectured values for $\delta_r := c_{\mathfrak{g}}^2$ appearing above in Table 5.1. Below when we refer to the "volume" of the BZ polytope, it will be understood that we mean the relative volume.[10]

In the A_r case (i.e. $\text{SU}(r+1)$ acting on traceless Hermitian matrices), the relationship between the volume function $\mathcal{J}(\alpha, \beta; \gamma)$ and the Euclidean volume of $H_{\alpha\beta}^{\gamma}$ follows from the well-known fact that the stretching quasi-polynomials (for compatible triples) are genuine polynomials (Derksen and Weyman, 2002, Rassart, 2004). In fact for A_r, if one considers the hive polytope rather than the BZ polytope, it turns out that $\mathcal{J}$ exactly computes the Euclidean volume, without a covolume fac-

[10] Note that the relative volume is not the same thing as the *normalized volume* considered in (Coquereaux and Zuber, 2018).

tor. Since the A_r case has been treated elsewhere, we merely refer the reader to the paper by Coquereaux and Zuber (2018) for a more detailed discussion, as well as to Section 6.4.3 below, where we give a new proof of the polynomiality property for stretched Littlewood–Richardson coefficients. In the remainder of this section we treat the general case, namely:

Proposition 5.4.2. *Let $\mathfrak{g}$ be any compact semisimple Lie algebra without $\mathfrak{su}(2)$ summands. Then for any $\alpha, \beta, \gamma \in \mathcal{C}_+$,*

$$\mathcal{J}(\alpha, \beta; \gamma) = \mathrm{Vol}_{\mathrm{rel}}(H_{\alpha\beta}^{\gamma}). \tag{5.4.3}$$

Proof. We begin by assuming that we are working with a compatible triple (λ, μ, ν) with $C_{\lambda\mu}^{\nu} \neq 0$, though later we will remove this assumption. The stretching quasi-polynomial of $H_{\lambda\mu}^{\nu}$ can be written[11]

$$P_{\lambda\mu}^{\nu}(s) = \sum_{k=1}^{d} a_k(s) s^k, \tag{5.4.4}$$

where $d = \dim H_{\lambda\mu}^{\nu} \leq d_r$ and each a_k is a rational-valued periodic function on $\mathbb{N}$.

Step 1:

Using the assumption that $C_{\lambda\mu}^{\nu} \neq 0$, it follows from results of McMullen (1978/79) that the leading coefficient of the quasi-polynomial $P_{\lambda\mu}^{\nu}$ is constant. At the end of this section we will sketch an intuitive argument for why this must be so, which does not require knowledge of the theory in (McMullen, 1978/79). The fact that

[11]For example, by inspection of the BZ inequalities for B_2 (see (5.5.5) below), it is easy to see that in this case $H_{\lambda\mu}^{\nu}$ may have corners at integer or half-integer points, hence $P_{\lambda\mu}^{\nu}(s)$ is a quasi-polynomial of period 2. In fact it is well known that in the B_r, C_r and D_r cases, the period of $P_{\lambda\mu}^{\nu}(s)$ is at most 2 (De Loera and McAllister, 2006).

a_d is constant implies that it equals the d-volume of $H_{\lambda\mu}^{\nu}$, by the following simple observation.

Since $H_{\lambda\mu}^{\nu}$ is a rational polytope, we can choose $m \geq 1$ such that the m-fold dilation $mH_{\lambda\mu}^{\nu}$ is an integral polytope, whose Ehrhart polynomial is equal to

$$P_{\lambda\mu}^{\nu}(ms) = a_d(ms)^d + \sum_{k=1}^{d-1} a_k(m)(ms)^k.$$

By the standard result that the relative volume of an integral polytope equals the leading coefficient of its Ehrhart polynomial, we then have $\mathrm{Vol}_{\mathrm{rel}}(mH_{\lambda\mu}^{\nu}) = a_d m^d$, and thus

$$\mathrm{Vol}_{\mathrm{rel}}(H_{\lambda\mu}^{\nu}) = a_d. \tag{5.4.5}$$

Step 2:

We may now use (5.4.5) to identify the function $\mathcal{J}$ with the d_r-volume of the BZ polytope. Upon dilation of λ, μ, ν by a factor s, relation (5.3.11) gives

$$\mathcal{J}(s\lambda + \rho, s\mu + \rho; s\nu + \rho) = \sum_{\substack{\kappa \in K \\ \tau}} r_\kappa C_{s\lambda\, s\mu}^{\tau} C_{\tau\,\kappa}^{s\nu}\,. \tag{5.4.6}$$

Since $\mathfrak{g}$ has no $\mathfrak{su}(2)$ summands, we can determine from the Riemann–Lebesgue lemma (see Remark 5.2.2(d)) that $\mathcal{J}$ is a continuous function of its three arguments. For $s \gg 1$, we use the continuity and homogeneity of $\mathcal{J}$ to approximate the left-hand side of (5.4.6) by $\mathcal{J}(s\lambda, s\mu; s\nu) = s^{d_r}\mathcal{J}(\lambda, \mu; \nu)$. On the right-hand side, we observe that for s large, any weight $\tau = s\nu - k$, where k runs over the set $\mathrm{wt}(\kappa)$ of weights of V_κ, is dominant and contributes $\mathrm{mult}_\kappa(k)$ to $C^{s\nu}_{\tau\,\kappa}$, so that

$$\sum_{\substack{\kappa\in K\\ \tau}} r_\kappa C^{\tau}_{s\lambda\, s\mu} C^{s\nu}_{\tau\,\kappa} \overset{s\gg 1}{=} \sum_{\substack{\kappa\in K\\ k\in \mathrm{wt}(\kappa)}} r_\kappa \,\mathrm{mult}_\kappa(k)\, C^{s\nu-k}_{s\lambda\, s\mu} \approx \sum_{\kappa\in K} r_\kappa \dim V_\kappa\, C^{s\nu}_{s\lambda\, s\mu}, \tag{5.4.7}$$

where we have approximated $C^{s\nu-k}_{s\lambda\, s\mu} \approx C^{s\nu}_{s\lambda\, s\mu}$. This approximation is justified by the geometric observation that, since the equations and inequalities defining the BZ polytope depend linearly on (λ,μ,ν), the difference between the number of integral points in $H^{s\nu}_{s\lambda\, s\mu}$ and in $H^{s\nu-k}_{s\lambda\, s\mu}$ must be lower order than s^{d_r} for s large. Finally, using relation (5.3.15), we obtain

$$\sum_{\kappa\in K} r_\kappa C^{s\nu}_{s\lambda\, s\mu\,\kappa} \approx C^{s\nu}_{s\lambda\, s\mu} = P^{\nu}_{\lambda\mu}(s) = a_{d_r} s^{d_r} + \text{ (lower-order terms)},$$

whence the identification

$$\mathcal{J}(\lambda,\mu;\nu) = a_{d_r} = \mathrm{Vol}_{\mathrm{rel}}(H^{\nu}_{\lambda\mu})\,. \tag{5.4.8}$$

Note that $\mathcal{J}(\lambda,\mu;\nu)$ may vanish for a compatible triple, but in this case by the above argument we have $a_{d_r} = 0$, so that these are exactly the cases when $d < d_r$ and the d_r-volume of $H^{\nu}_{\lambda\mu}$ vanishes.

Step 3:

We now use a simple approximation technique to remove the assumption of compatibility and show that $\mathcal{J}(\alpha,\beta;\gamma) = \mathrm{Vol}_{\mathrm{rel}}(H^{\gamma}_{\alpha\beta})$ for arbitrary points α,β,γ in the dominant chamber. If $\gamma \notin \mathscr{H}_{\alpha\beta}$ then $H^{\gamma}_{\alpha\beta}$ is the empty set (see the proof of Proposition 5.4.3 below). Accordingly we may assume $\gamma \in \mathscr{H}_{\alpha\beta}$. For any $\varepsilon > 0$

we can find a compatible triple (λ, μ, ν) with $C^{\nu}_{\lambda\mu} \neq 0$ and $N \in \mathbb{N}$ such that $|\alpha - \lambda/N| + |\beta - \mu/N| + |\gamma - \nu/N| < \varepsilon$. We have:

$$\mathcal{J}(\lambda/N, \mu/N; \nu/N) = N^{-d_r}\mathcal{J}(\lambda, \mu; \nu) = N^{-d_r}\mathrm{Vol}_{\mathrm{rel}}(H^{\nu}_{\lambda\mu}) = \mathrm{Vol}_{\mathrm{rel}}(H^{\nu/N}_{\lambda/N\ \mu/N}),$$

and $|\mathrm{Vol}_{\mathrm{rel}}(H^{\nu/N}_{\lambda/N\ \mu/N}) - \mathrm{Vol}_{\mathrm{rel}}(H^{\gamma}_{\alpha\beta})| = O(\varepsilon)$ for ε small. Since $\mathcal{J}$ is continuous, letting $\varepsilon \to 0$ we obtain $\mathcal{J}(\alpha, \beta; \gamma) = \mathrm{Vol}_{\mathrm{rel}}(H^{\gamma}_{\alpha\beta})$. $\square$

We end this subsection by sketching a brief argument that the leading coefficient a_d of $P^{\nu}_{\lambda\mu}$ must be constant when (λ, μ, ν) is a compatible triple with $C^{\nu}_{\lambda\mu} \neq 0$. For the sake of simplicity we will assume that $P^{\nu}_{\lambda\mu}$ has period 2, so that it has the form

$$P^{\nu}_{\lambda\mu}(s) = \sum_{k=1}^{d}(a^{+}_{k} + (-1)^{s}a^{-}_{k})s^{k} \tag{5.4.9}$$

where a^{+}_{k}, a^{-}_{k} are rational coefficients. However, the discussion easily extends to an arbitrary period.

Since $C^{\nu}_{\lambda\mu} \neq 0$, $\mathrm{aff}(H^{\nu}_{\lambda\mu})$ obviously contains at least one integer point. Moreover $\mathrm{aff}(H^{\nu}_{\lambda\mu})$ is a d-dimensional rational affine subspace, so the fact that it contains one integer point implies that it contains a rank d affine sublattice $\Lambda \subset \mathbb{Z}^{N_r}$. Choose a linear transformation $\psi : \mathbb{R}^{N_r} \to \mathbb{R}^{d}$ that maps Λ bijectively to $\mathbb{Z}^{d}$. For all $s = 1, 2, \ldots$, the number of integer points in $sH^{\nu}_{\lambda\mu}$ is equal to the number of integer points in $s\psi(H^{\nu}_{\lambda\mu})$, so that these two polytopes have the same Ehrhart quasi-polynomial $P^{\nu}_{\lambda\mu}(s)$. Thus we have reduced the problem to studying dilations of a *full* polytope (that is, a d-dimensional polytope in $\mathbb{R}^{d}$ rather than the higher-dimensional space $\mathbb{R}^{N_r}$).

Now suppose for the sake of contradiction that the leading coefficient of $P^{\nu}_{\lambda\mu}(s)$

were not a constant, i.e. $a_d^- \neq 0$ in (5.4.9), and compute the difference

$$P^{\nu}_{\lambda\mu}(s+1) - P^{\nu}_{\lambda\mu}(s) = 2a_d^- s^d + \text{ (lower-order terms)}.$$

This equals the difference between the numbers of integer points in the two polytopes $(s+1)\psi(H^{\nu}_{\lambda\mu})$ and $s\psi(H^{\nu}_{\lambda\mu})$. It is easy to see that this number is bounded by a multiple of the $(d-1)$-dimensional surface area of the larger polytope and is therefore $O(s^{d-1})$, in contradiction with the previous expression. This proves that the leading coefficient of $P^{\nu}_{\lambda\mu}(s)$ must be a constant.

5.4.3 Nonvanishing of $\mathcal{J}$ on the interior of the Horn polytope

Proposition 5.4.2 leads to a proof of the following proposition, which generalizes a result that was shown for the A_r case by Coquereaux and Zuber (2018).

Proposition 5.4.3. *For all $\alpha, \beta \in \mathcal{C}_+$ and γ in the interior of $\mathscr{H}_{\alpha\beta}$ (in the topology of $\mathfrak{t}$), $\mathcal{J}(\alpha, \beta; \gamma) > 0$.*

Proof. We may assume that $\dim \mathscr{H}_{\alpha\beta} = r$; otherwise its interior is empty. It follows from Proposition 5.4.2 that $\mathcal{J}(\alpha, \beta; \gamma) > 0$ exactly when $\dim H^{\gamma}_{\alpha\beta} = d_r$. There is at least one point $\gamma_\star$ in the interior such that $\dim H^{\gamma_\star}_{\alpha\beta} = d_r$, since $\mathcal{J}$ is locally a polynomial of degree $d_r > 0$ and cannot vanish everywhere. It thus suffices to show that $d(\gamma) := \dim H^{\gamma}_{\alpha\beta}$ is constant on the interior of $\mathscr{H}_{\alpha\beta}$.

The BZ polytope $H^{\gamma}_{\alpha\beta}$ is the simultaneous solution set of the equation

$$\sum_{a=1}^{N_r} u_a \boldsymbol{\alpha}_a = \alpha + \beta - \gamma \tag{5.4.10}$$

and a collection of m linear inequalities that can all be written in the form

$$v_j(u_1, \ldots, u_{N_r}) \geq h_j(\alpha, \beta, \gamma), \quad j = 1, \ldots, m, \tag{5.4.11}$$

where each v_j is a linear functional on $\mathbb{R}^{N_r}$ and h_j is a linear functional on $\mathbb{R}^{3r}$. Moving all terms involving γ to the left-hand side, we can rewrite (5.4.10) and (5.4.11) as

$$\sum_{a=1}^{N_r} u_a \boldsymbol{\alpha}_a + \gamma = \alpha + \beta, \tag{5.4.12}$$

$$v'_j(u_1, \ldots, u_{N_r}, \gamma) \geq h'_j(\alpha, \beta), \quad j = 1, \ldots, m, \tag{5.4.13}$$

where each v'_j is a linear functional on $\mathbb{R}^{N_r+r}$ and h'_j is a linear functional on $\mathbb{R}^{2r}$. Considering γ as an independent variable, the simultaneous solution set of (5.4.12) and (5.4.13) is a polytope in $\mathbb{R}^{N_r+r}$, which we denote $U_{\alpha\beta}$. The projection $\mathbb{R}^{N_r+r} \to \mathbb{R}^r$ onto the γ coordinates maps the relative interior of $U_{\alpha\beta}$ onto the interior of $\mathscr{H}_{\alpha\beta}$. Let P be this map on the interiors. Each fiber $P^{-1}(\gamma)$ is in turn the relative interior of $H^\gamma_{\alpha\beta}$, so that $\dim P^{-1}(\gamma) = \dim H^\gamma_{\alpha\beta}$. Moreover the map P is a global submersion, so that its fibers all have the same dimension (see e.g. (Lee, 2003, ch. 7)). This completes the proof. □

5.4.4 Geometric origin and nature of the non-analyticities of $\mathcal{J}$

We now explain how the non-analyticities of $\mathcal{J}$ can be understood in terms of the geometry of $H^\gamma_{\alpha\beta}$. The facets of $H^\gamma_{\alpha\beta}$ are cut out by some number of hyperplanes in $\mathbb{R}^{N_r}$ corresponding to the BZ inequalities. For fixed α and β, these hyperplanes undergo linear translations when γ varies in the dominant chamber. As γ varies, an inequality may become redundant, meaning that the corresponding hyperplane no longer intersects the polytope, so that the polytope has one fewer facet; or alternatively, a previously redundant inequality may become relevant, meaning that the corresponding hyperplane intersects the polytope, forming a new facet. Non-analyticities of the volume $\mathcal{J}$ as a function of γ occur at such points, where one of the hyperplanes defined by the BZ inequalities hits the polytope $H_{\alpha\beta}^{\gamma}$, or conversely does not intersect it anymore.

Since we know that $\mathcal{J}$ is a piecewise polynomial function of γ, these non-analyticities take the form of a change of polynomial determination (i.e., the local polynomial form of $\mathcal{J}$). For a point γ at a distance ε from a non-analyticity hyperplane (see Proposition 5.2.1), the change of determination is of the form $\Delta\mathcal{J} = O(\varepsilon^m)$, which is the volume of the piece of the polytope chopped off by the incident hyperplane. In the neighborhood of that non-analyticity hyperplane, the function is thus of differentiability class C^{m-1}. Obviously the integer m is bounded from above by d_r, but also from below due to known lower bounds on the number of continuous derivatives of $\mathcal{J}$ (see Remark 5.2.2(d)). One finds $r-1 \leq m \leq d_r = \frac{1}{2}r(r-1)$ for the A_r cases, and $2(r-1) \leq m \leq d_r = r(r-1)$ for B_r. For example, in the case of SU(4), an explicit (unpublished) calculation with $\alpha = \beta = (3,2,1,0)$ has revealed non-analyticities of class C^1 and C^2. See Figure 5.2 for illustration, where the rightmost case is in fact prohibited by the above bound on m. One may convince oneself that this bound on m guarantees that any non-analyticity of the volume of the polytope must involve the appearance or disappearance of a facet, and not merely a rearrangement of lower-dimensional faces.

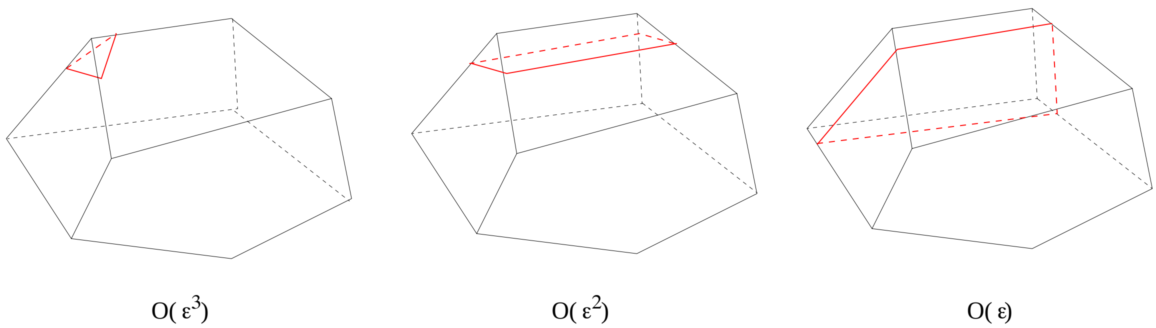


Figure 5.2: Artist's view of the intersection of a $(d-1)$-hyperplane with the polytope $H^{\gamma}_{\alpha\beta}$, here in $d=3$ dimensions, chopping off a volume of order $O(\varepsilon^3)$, $O(\varepsilon^2)$ or $O(\varepsilon)$. We argue in the text that the rightmost case cannot occur.

5.5 The case of B_2

In this section, we illustrate the previous considerations in the case of the B_2 root system, i.e. $\mathfrak{g} = \mathfrak{so}(5)$.

5.5.1 The function $\mathcal{J}$ for $\mathrm{SO}(5)$

Consider two *skew*–symmetric 5×5 real matrices, in the block-diagonal form

$$A = \mathrm{diag}\left(\begin{bmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{bmatrix}_{i=1,2}, 0 \right)$$

and likewise for B. For $\mathrm{SO}(5)$ orbits of these matrices, the function $\mathcal{J}$ has been written in (Zuber, 2018) in the form

$$\begin{aligned} &\mathcal{J}(\alpha, \beta; \gamma) = \\ &\frac{1}{64\pi^2} \int \frac{1}{st(t^2-s^2)} \Big(\big[\sin s(\alpha_1+\alpha_2) \sin t(\alpha_1-\alpha_2) - \sin s(\alpha_1-\alpha_2) \sin t(\alpha_1+\alpha_2) \big] \\ &\qquad\qquad \times \big[\text{same with } \beta\big] \big[\text{same with } \gamma\big] \Big) ds\, dt, \end{aligned} \tag{5.5.1}$$

which may then be written explicitly as a degree 2 piecewise polynomial. (Here the PDF, normalized on the Horn polytope $\mathscr{H}_{\alpha\beta}$, is given by $\frac{3}{2}\frac{|\Delta_O(\gamma)|}{|\Delta_O(\alpha)|\,|\Delta_O(\beta)|}\mathcal{J}$.) Recall that the B_2 Weyl group $W = \mathrm{Dih}(4) = S_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ acts on the 2-vector (α_1, α_2) by a change of sign of either component, or by swapping them. Denote the action of $(w, w', w'') \in W^3$ on the vector $\alpha + \beta - \gamma$ by $\sigma := w(\alpha) + w'(\beta) - w''(\gamma)$, so that $\sigma_i = w(\alpha)_i + w'(\beta)_i - w''(\gamma)_i$, $i = 1, 2$. Let $\epsilon(w)$ denote the sign of a Weyl group element. Then

$$\mathcal{J} = \frac{1}{2^8} \sum_{w,w',w''\in W} \epsilon(w)\epsilon(w')\epsilon(w'')\Big(4\sigma_1|\sigma_1| - 4\sigma_2|\sigma_2| - 2(\sigma_1-\sigma_2)|\sigma_1-\sigma_2|\Big)\,\mathrm{sign}\,(\sigma_1+\sigma_2)\,. \tag{5.5.2}$$

(Note that one may eliminate one of the three summations over W, fixing $w'' = \mathrm{id}$ and multiplying the result by a factor $|W| = 8$, which simplifies the actual computation.)

It follows from this expression that $\mathcal{J}$ is of differentiability class C^1. Recall that one may always assume that $\alpha_1 > \alpha_2 > 0$, $\beta_1 > \beta_2 > 0$ and $\gamma_1 \geq \gamma_2 \geq 0$. The support of $\mathcal{J}$ is determined by generalized Horn inequalities of B_2 type. To write them down, we note that if A is a skew–symmetric real matrix, then iA is a complex Hermitian matrix. Thus the inequalities of B_2 type follow from the classical Horn's inequalities (Horn, 1962, Fulton, 2000) of A_4 type (i.e. for 5×5 complex Hermitian matrices), applied to matrices of the form $\mathrm{diag}(\alpha_1, \alpha_2, 0, -\alpha_2, -\alpha_1)$, and likewise for β and γ. One finds:

$$\begin{aligned}
\max(|\alpha_1-\beta_1|, |\alpha_2-\beta_2|) \leq \quad & \gamma_1 && \leq \alpha_1+\beta_1 \\
\max(0, \alpha_2-\beta_1, -\alpha_1+\beta_2) \leq \quad & \gamma_2 && \leq \min(\alpha_1+\beta_2, \alpha_2+\beta_1) \\
|\alpha_1-\beta_1| + |\alpha_2-\beta_2| \leq \quad & \gamma_1+\gamma_2 && \leq \alpha_1+\alpha_2+\beta_1+\beta_2 \\
\max(0, \alpha_1-\alpha_2-\beta_1-\beta_2, \beta_1-\beta_2-\alpha_1-\alpha_2) \leq \quad & \gamma_1-\gamma_2 && \leq \alpha_1+\beta_1-|\alpha_2-\beta_2|\,.
\end{aligned} \tag{5.5.3}$$

These inequalities, supplemented by $\gamma_1 \geq \gamma_2 \geq 0$, define the ($B_2$–generalized) Horn polygon; see Figure 5.3 for examples.

The singular lines of $\mathcal{J}$ may be determined by the same kind of argument as in Section 5.2.1. Alternatively, following the same reasoning as in the computation of (5.5.3) above, they can be computed as a special case of the singular lines of A_4 type. Thus the *possible* lines of non-C^2 differentiability are those among

$$\begin{aligned}
\gamma_1 = \gamma_{1s} \quad &\in \quad \{\ \alpha_1+\beta_2\,,\quad \alpha_2+\beta_1\,,\quad \alpha_2+\beta_2\,,\quad |\alpha_1-\beta_2|\,,\quad |\alpha_2-\beta_1|\ \}\,,\\
\gamma_2 = \gamma_{2s} \quad &\in \quad \{\ \alpha_2+\beta_2\,,\quad |\alpha_1-\beta_2|\,,\quad |\alpha_2-\beta_1|\,,\quad |\alpha_2-\beta_2|\,,\quad |\alpha_1-\beta_1|\ \},\\
\gamma_1+\gamma_2 = \gamma_{1+2\,s} \quad &\in \quad \{\ \alpha_1+\alpha_2+\beta_1-\beta_2\,,\ |\alpha_1+\alpha_2-\beta_1+\beta_2|\,,\ \alpha_1-\alpha_2+\beta_1+\beta_2\,,\\
&\qquad\qquad |-\alpha_1+\alpha_2+\beta_1+\beta_2|\,,\ \alpha_1-\alpha_2+\beta_1-\beta_2\ \},\\
\gamma_1-\gamma_2 = \gamma_{1-2\,s} \quad &\in \quad \{\ |-\alpha_1+\alpha_2+\beta_1+\beta_2|\,,\ |\alpha_1+\alpha_2-\beta_1+\beta_2|\,,\ \alpha_1-\alpha_2+\beta_1-\beta_2\,,\\
&\qquad\qquad |\alpha_1-\alpha_2-\beta_1+\beta_2|\,,\ |\alpha_1+\alpha_2-\beta_1-\beta_2|\ \}
\end{aligned} \tag{5.5.4}$$

that intersect the Horn polygon.

One may find an explicit piecewise polynomial representation of the volume function $\mathcal{J}$. At the price of possibly swapping α and β, one may always assume that $|\beta_1-\alpha_2| \geq |\alpha_1-\beta_2|$. The lines (5.5.4) have 4-fold intersections at four vertices, denoted I, J, K, L, with coordinates

$$I = (\alpha_1+\beta_2,\, |\alpha_2-\beta_1|); \quad J = (\alpha_2+\beta_1,\, |\alpha_1-\beta_2|);$$

$$K = (|\beta_1-\alpha_2|,\, |\alpha_1-\beta_2|); \quad L = (\alpha_2+\beta_2,\, |\alpha_1-\beta_1|),$$

some of which may be outside the Horn polygon.

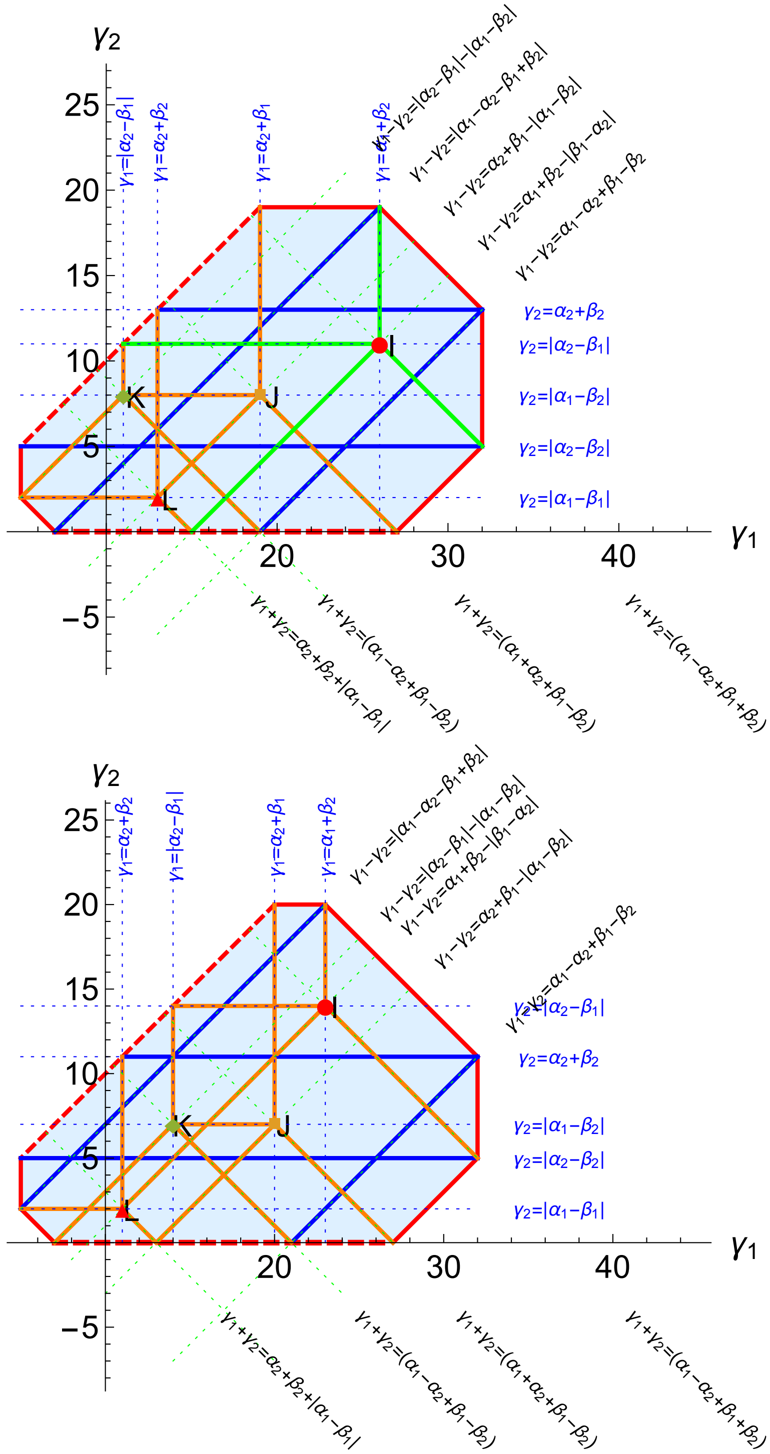


Figure 5.3: Two examples of the Horn polytope and of the singular lines for $\alpha = (17, 4)$, $\beta = (15, 9)$ and for $\alpha = (15, 3)$, $\beta = (17, 8)$. The function $\mathcal{J}$ has a quadratic change of determination across the solid lines, and a linear change across the two dashed edges of the Horn polytope.

Then two cases arise, depending on whether $\alpha_1 \geq \beta_1$ or $\beta_1 \geq \alpha_1$. In the latter case, and only then, I and L belong to the same diagonal $\gamma_1 - \gamma_2 =$ constant. A fairly generic example of each case is depicted in Figure 5.3. (For other values of α or β, some of the lines might merge or disappear from the figure.)

The heavy solid lines shown in the figure are the loci of singularities, where the polynomial determination changes. Note that these lines join at "four-prong vertices," either inside the polygon at some of the points I, J, K or L (Figure 5.4, left), or at its boundary (Figure 5.4, right).

Rather than giving a detailed polynomial expression in each sector arising from this decomposition, it is simpler to give an empirically observed set of rules that determine the *change* of polynomial determination. These rules are as follows.

Starting from the exterior of the polygon, where $\mathcal{J}$ vanishes, enter the polygon through any of the solid red lines. Crossing any of the solid lines along the direction of an arrow shown in Figure 5.4 increases $\mathcal{J}$ by $\frac{1}{2}\Delta^2$, where $\Delta = (\gamma_i - \gamma_{is})$ for a vertical or horizontal line of equation $\gamma_i = \gamma_{is}$, and $\Delta = \frac{1}{\sqrt{2}}(\gamma_1 \pm \gamma_2 - \gamma_{1\pm 2\,s})$ for an oblique line of equation $\gamma_1 \pm \gamma_2 = \gamma_{1\pm 2\,s}$. When two lines merge, the differences $\frac{1}{2}\Delta^2$ add up.

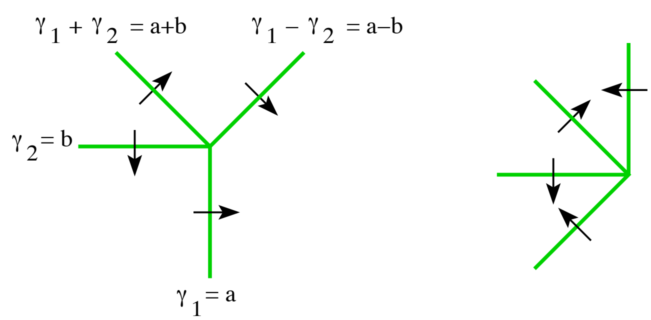


Figure 5.4: Prescriptions for changes of $\mathcal{J}$ across the lines emanating from a four-prong vertex. This holds for any rotated configuration of these two types.

Now the reader will verify that this prescription is consistent: following a closed loop around a four-prong vertex (see Figure 5.4), we return to our initial expression for $\mathcal{J}$, thanks to the trivial identity

$$x^2 + y^2 = \Big(\frac{x+y}{\sqrt{2}}\Big)^2 + \Big(\frac{x-y}{\sqrt{2}}\Big)^2,$$

where $x = \gamma_1 - \gamma_{1s}$ and $y = \gamma_2 - \gamma_{2s}$. Thus the rules give a well-defined function. By considering a path that crosses the polygon, we find in particular that $\mathcal{J}$ returns to 0 outside the polygon.

Also note that along the edges $\gamma_1 - \gamma_2 = 0$ or $\gamma_2 = 0$ of the polygon, which are *not* dictated by Horn's inequalities but rather reflect our choice to work in the dominant Weyl chamber, $\mathcal{J}$ may vanish only linearly. This is why those lines have been depicted by dashed lines in Figure 5.3, and is also why no prescription is given for the crossing of those dashed lines.

At this stage, this piecewise polynomial construction of $\mathcal{J}$ is just a conjecture that has been checked on many examples. In principle, establishing it should follow from a careful examination of equation (5.5.2) and of its possible changes of determination across the singular lines given in (5.5.4). Obtaining this construction of $\mathcal{J}$ from the complicated expression (5.5.2) has so far resisted our attempts.

5.5.2 Geometry of the B_2 root space. Possible vanishing of $C_{\lambda\mu}^{\nu}$ and saturation property

Recall that if e_i, $i = 1, 2$, are the standard basis of $\mathbb{R}^2$, the two simple roots of B_2 may be written as $\boldsymbol{\alpha}_1 = e_1 - e_2$, $\boldsymbol{\alpha}_2 = e_2$, while the fundamental weights are $\omega_1 = e_1 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$, $\omega_2 = \frac{1}{2}(e_1 + e_2)$. (See Figure 5.5.) In what follows, weights will usually be specified by their coordinates in the (ω_1, ω_2) basis ("Dynkin labels"). Occasionally we will use the simple root basis ("Kac indices") or the e_i basis.

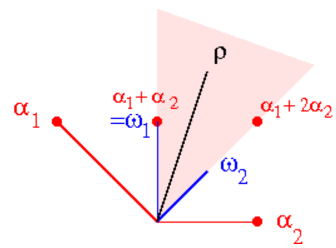


Figure 5.5: Roots and weights of B_2: the positive roots in red, the two fundamental weights in blue, and the Weyl vector in black. The shaded octant is the dominant Weyl chamber.

In the case of B_2, the compatibility condition $\sigma := \lambda + \mu - \nu \in Q$ amounts to $\sigma_2 = 0 \bmod 2$. As an example, for $\lambda = \mu = \nu = \omega_2$ (the spinorial representation), $\sigma = \omega_2 = \boldsymbol{\alpha}_1/2 + \boldsymbol{\alpha}_2 \notin Q$, and $C_{\lambda\,\mu}^{\nu} = 0$ as this is a non-compatible triple. By way of contrast, if $\lambda = \mu = \nu = \omega_1$ (the vectorial representation), the triple is compatible since $\sigma = \omega_1 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \in Q$, but nevertheless we still have $C_{\lambda\,\mu}^{\nu} = 0$. Thus in both cases $C_{\lambda\,\mu}^{\nu} = 0$, while $C_{2\lambda\,2\mu}^{2\nu} = 1$.

A semisimple algebra $\mathfrak{g}$ is said to satisfy the *saturation property* if $C_{N\lambda\,N\mu}^{N\nu} \neq 0$ implies $C_{\lambda\,\mu}^{\nu} \neq 0$ whenever $N \in \mathbb{N}$ and (λ, μ, ν) is a compatible triple. The saturation property is proved for A_n but fails for B_n, in particular for B_2, as the example just given shows.

5.5.3 Berenstein–Zelevinsky parameters

For the convenience of the reader, we reproduce here the result of Theorem 2.4 of (Berenstein and Zelevinsky, 2001), specialized to the case of B_2. Introduce four real parameters $t_0^{(0)}$, $t_{-1}^{(1)}$, $t_0^{(1)}$, $t_1^{(1)}$ such that $\sigma = \lambda + \mu - \nu$ has the following decomposition in terms of the positive roots:

$$\sigma = (t_{-1}^{(1)} - t_0^{(1)} + 2t_1^{(1)})\boldsymbol{\alpha}_1 + t_0^{(0)}\boldsymbol{\alpha}_2 + (t_0^{(1)} - 2t_1^{(1)})(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2) + t_1^{(1)}(\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2)\,.$$

Note that these parameters are linear combinations of the u_a associated to the (over-complete) family of positive roots in (5.4.1). The parameters $t_i^{(j)}$ are subject to the conditions:

$$\begin{aligned}
2t_{-1}^{(1)} \geq t_0^{(1)} \geq 2t_1^{(1)} \geq 0 \quad &\text{and} \quad t_0^{(0)} \geq 0\,,\\
\sigma = \lambda + \mu - \nu \quad &= \quad (t_1^{(1)} + t_{-1}^{(1)})\boldsymbol{\alpha}_1 + (t_0^{(0)} + t_0^{(1)})\boldsymbol{\alpha}_2\,,\\
\lambda_1 \geq \max(t_1^{(1)}, t_0^{(1)} - t_{-1}^{(1)}, t_{-1}^{(1)} - t_0^{(0)}) \quad &\text{and} \quad \lambda_2 \geq t_0^{(0)}\,,\\
\mu_1 \geq \max(t_{-1}^{(1)} + 2t_1^{(1)} - t_0^{(1)}, t_1^{(1)}) \quad &\text{and} \quad \mu_2 \geq \max(t_0^{(0)} + 2(t_0^{(1)} - t_{-1}^{(1)} - t_1^{(1)}), t_0^{(1)} - 2t_1^{(1)})\,.
\end{aligned} \tag{5.5.5}$$

Then $C_{\lambda\mu}^{\nu}$ is equal to the number of integer solutions of (5.5.5).

5.5.4 Stretching. Parameter polytope

Under stretching, $P_{\lambda\mu}^{\nu}(s) := C_{s\lambda\, s\mu}^{s\nu}$ is in general not a polynomial in s but rather a quasi-polynomial. For example, for $\lambda = (5,6),\ \mu = (3,4),\ \nu = (5,6)$, we find

$$P_{\lambda\mu}^{\nu}(s) = 6s^2 + \frac{7}{2}s + \frac{1}{4}(3 + (-1)^s)\,. \tag{5.5.6}$$

The 2-dimensional BZ polytope defined by (5.5.5) is therefore in general *not* an integral polytope.

For these same values $\lambda = (5,6),\ \mu = (3,4),\ \nu = (5,6)$, after eliminating[12] the

[12] Note however that when counting integer points to determine the tensor product multiplicity, we must make sure to count only points such that these eliminated parameters are also integers.

parameters $t_{-1}^{(1)}$ and $t_0^{(1)}$ through the 2 equalities $\lambda+\mu-\nu=\cdots$, we find the polytope (here a polygon) in the $(t_0^{(0)}, t_1^{(1)})$ plane defined by the inequalities

$$\Big\{ \quad 0 \le t_0^{(0)} < 6 \quad \text{and} \quad 0 \le t_1^{(1)} \le 3 \quad \text{and} \quad t_0^{(0)} + 3 \ge 2t_1^{(1)} \\ \text{and} \quad 3 \le t_0^{(0)} + 2t_1^{(1)} \le 7 \quad \text{and} \quad t_0^{(0)} + t_1^{(1)} \le 5 \quad \Big\},$$

as depicted in Figure 5.6(a). It has $C_{\lambda\mu}^{\nu} = 10$ integer points, but its corners are not integral, and its Ehrhart quasi-polynomial is given in (5.5.6). Note that its (relative) volume, here an area, is $V = 6$, in accordance with the computation of $\mathcal{J}$.

In contrast, for $\lambda = (5,6),\ \mu = (3,4),\ \nu = (6,4)$, we find $C_{\lambda\,\mu}^{\nu} = 10$, $V = 11/2$, and

$$P_{\lambda\,\mu}^{\nu}(s) = \frac{11}{2}s^2 + \frac{7}{2}s + 1\,. \tag{5.5.7}$$

The BZ polygon, defined by

$$\Big\{ \quad 2 \le t_0^{(0)} \le 6 \quad \text{and} \quad 0 \le t_1^{(1)} \le 3 \quad \text{and} \quad t_0^{(0)} + 2 \ge 2t_1^{(1)} \\ \text{and} \quad 4 \le t_0^{(0)} + 2t_1^{(1)} \le 8 \quad \text{and} \quad t_0^{(0)} + t_1^{(1)} \le 6 \quad \Big\},$$

is shown in Figure 5.6(b). Note that the polygon is again not integral, although its Ehrhart quasi-polynomial is the genuine polynomial (5.5.7).

There are also cases where the polygon is integral. For example, still with $\lambda = (5,6),\ \mu = (3,4)$, and now $\nu = (2,10)$, we get $C_{\lambda\,\mu}^{\nu} = 8$ and

$$P_{\lambda\,\mu}^{\nu}(s) = \frac{7}{2}s^2 + \frac{7}{2}s + 1\,, \tag{5.5.8}$$

see Figure 5.6(c).

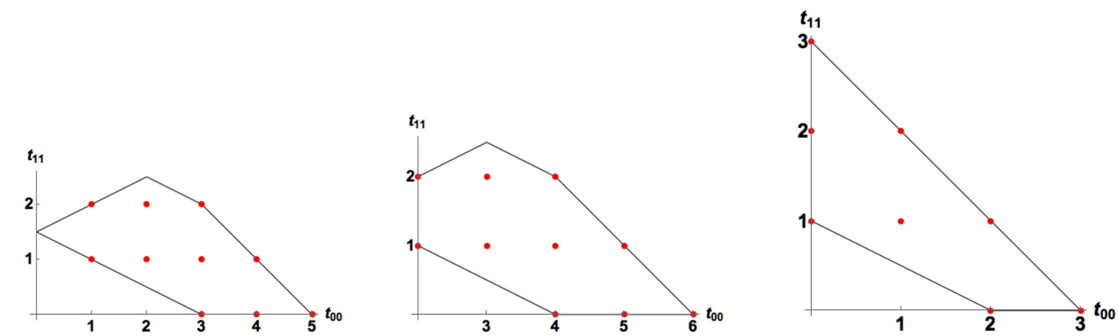
By the principle of *Ehrhart–Macdonald reciprocity*, the number of integer points on the *interior* of a d-dimensional rational polytope with Ehrhart quasi-polynomial P is equal to $(-1)^d P(-1)$; see (Stanley, 1997). In the three previous examples, we indeed find that $P^{\nu}_{\lambda\mu}(-1)$ gives the number of interior integer points: respectively, 3, 3 and 1.

In order to make a few simple comments of a geometrical nature, we assume in the rest of this subsection that the sub-leading coefficient of $P(s)$ is constant. We have found no counter-example of this property for B_2 BZ polygons, but we do not offer a proof. In other words we assume that (nondegenerate) B_2 stretching polynomials read $P(s) = Vs^2 + L/2s + (p + (-1)^s(1-p))$ where V is the (relative) area of the parameter polygon, L is the (relative) length of its boundary and p is some scalar. For a genuine (i.e. not merely quasi-) polynomial, one has $p = 1$. The polytope being closed and convex, we know that $P(0) = 1$. Let $C := C^{\nu}_{\lambda\mu} = P(1)$ be the tensor product multiplicity, i.e. the total number of integer points of the polytope. Then $i := P(-1)$ is the number of interior points (by Ehrhart–Macdonald reciprocity), and $b := C - i$ is the number of integer points belonging to the boundary. Evaluation of P at $+1$ and -1 gives $C + i = 2(V + (2p-1))$ and $C - i = L$; together these two equations imply

$$2C - 2V - L = 2(2p-1). \tag{5.5.9}$$

Moreover the two relations $C = b + i$ and $C = L + i$ imply $L = b$. All these relations can be checked in the three examples above (see Figure 5.6).

Notice that the polytope is integral only if $p = 1$, i.e. if $2C - 2V - L = 2$; this is what happens in the third example above, where $C = 8$. In general, p may be read off from (5.5.9). For instance in the first example where $V = 6$, $L = 7$ and $C = 10$, we find $2C - 2V - L = 1$ and $p = 3/4$, in agreement with the expression of the stretching polynomial. When the polytope is integral, Pick's theorem, written $V = i + b/2 - 1$, applies, and this is of course equivalent to (5.5.9) with $p = 1$.



(a) (b) (c)

Figure 5.6: The BZ polytope in the $(t_0^{(0)}, t_1^{(1)})$ plane for $\lambda = (5,6)$, $\mu = (3,4)$ and (a) $\nu = (5,6)$; (b) $\nu = (6,4)$; (c) $\nu = (2,10)$.

Finally there are cases where the polygon degenerates, either to a point (whenever $C_{\lambda\mu}^{\nu} = 1$), or to a segment. In the A_r case, $C_{\lambda\mu}^{\nu} = 1$ implies $C_{s\lambda\, s\mu}^{s\nu} = 1$ for all s, as proved by Knutson et al. (2004). Whether this holds true for other cases like B_r seems to be still an open question. The polytope may also degenerate to a segment, as occurs for example when we keep $\lambda = (5,6)$, $\mu = (3,4)$, but take $\nu = (0,10)$. Then we find $C_{\lambda\mu}^{\nu} = 3$; the segment has length 2, and upon dilation $P_{\lambda\mu}^{\nu}(s) = 2s + 1$.

5.5.5 Volumes and multiplicities in practice

5.5.5.1 Determination of multiplicities in the B_2 case

In order to determine the generalized Littlewood-Richardson coefficients in the B_2 case, one can use the Racah–Speiser algorithm (Racah, 1964, Speiser, 1964) which works for any semisimple Lie algebra (even for affine Lie algebras), or equivalently Klimyk's formula (Klimyk, 1968), which is implemented in several computer algebra packages such as LiE (van Leeuwen et al., 1992). Another possibility, since the

Kostant partition function is known for B_2 (Tarski, 1964, Capparelli, 2003), is to use the Kostant–Steinberg formula (Steinberg, 1961); see (6.5.19) below. A third possibility is to count the integer points in Berenstein–Zelevinsky polytopes, as explained in Sections 5.4.1 and 5.5.3. We implemented these three methods in Mathematica (Wolfram Research, Inc., 2012), which was also used for most formal manipulations and figures in this chapter.

5.5.5.2 The many ways to compute the volume of a BZ polytope

Let (λ, μ, ν) be a given compatible triple of highest weights of $\mathfrak{g}$. We are interested in the (relative) volume V of the associated BZ polytope.

There are at least four ways to compute this:

1. One can use the volume function $\mathcal{J}_r(\lambda, \mu, \nu)$ when it is explicitly known, as is the case for A_r, with $r = 1, 2, 3, 4$ (Zuber, 2018, Coquereaux and Zuber, 2018), and for B_2 via the expression (5.5.2).

2. One can use the $\mathcal{J}$-LR relation (5.3.12), which expresses V in terms of a finite number of tensor product multiplicities as well as the constants $\hat{r}_\kappa$ (defined in Section 5.3) associated to $\mathfrak{g}$.

3. One can determine the stretching quasi-polynomial defined by the triple, by calculating stretched multiplicities $C_{s\lambda\, s\mu}^{s\nu}$ up to some s of the order of $\varpi \times d_r$, where ϖ is the period of the quasi-polynomial (not more than 2 for classical Lie algebras) and d_r is the degree (see Table 5.1 in Section 5.4). Then V is the coefficient of the leading-order term.

4. When the BZ polytope is explicitly defined in terms of appropriate parameters

such as those described previously for B_2, one can compute its volume by integrating the constant function 1 over the polytope, possibly after eliminating redundant parameters. When using the u_a parameters of (5.4.1), this method will compute the Euclidean volume, so to obtain the relative volume one must divide by the covolume $c_{\mathfrak{g}} = \delta_r^{1/2}$, see Table 5.1.

For illustration, let us consider the following triple for B_2: $\lambda = (4,7)$, $\mu = (5,3)$, $\nu = (2,4)$, in the basis of fundamental weights. Equivalently, in the e_i basis, $\lambda = (15/2, 7/2)$, $\mu = (13/2, 3/2)$, $\nu = (4,2)$. This triple has multiplicity 5. In what follows, coordinates of weights are written in the fundamental weight basis.

1. Direct evaluation of (5.5.2) with the above arguments gives $V = 7/4$.

2. The set $\hat{K}$ contains only one element, $\kappa = (0,1)$, with $r_\kappa = 1/4$. The nonzero contribution to (5.3.12) comes from the following weights (with multiplicities) that enter the decomposition of the tensor product of irreducible representations with highest weights $(3,6) = (4,7) - (1,1)$ and $(4,2) = (5,3) - (1,1)$:

$$(0,4) \times 1, \quad (1,2) \times 1, \quad (1,4) \times 3, \quad (2,2) \times 2,$$

so that one obtains $V = (1+1+3+2)/4 = 7/4$. More generally, for ν deep enough (see (5.3.16)), one finds

$$\mathcal{J}(\lambda, \mu; \nu) = \frac{1}{4} \sum_k C^{\nu - k}_{(\lambda-\rho)(\mu-\rho)},$$

where k runs over the set $\{(2,0),\ (1,2),\ (1,0),\ (0,2)\}$.

3. For scaling factors $s = 0, 2, 4$, the multiplicities are respectively $1, 13, 39$. For scaling factors $s = 1, 3, 5$, the multiplicities are respectively $5, 24, 57$. The

quasi-polynomial reads:

$$\begin{cases} 7s^2/4 + 5s/2 + 1, & s = 0 \bmod 2, \\ 7s^2/4 + 5s/2 + 3/4, & s = 1 \bmod 2. \end{cases}$$

As expected, the leading coefficient of both polynomials is $V = 7/4$.

4. In the $(t_0^{(0)}, t_1^{(1)})$ plane, this BZ polytope is defined by the inequalities:

$$\begin{aligned} &(t_1^{(1)} = 2 \quad \text{and} \quad t_0^{(0)} = 6), \quad \text{or} \\ &(2 < t_1^{(1)} \leq 7/2 \quad \text{and} \quad 10 - 2t_1^{(1)} \leq t_0^{(0)} \leq 8 - t_1^{(1)}), \quad \text{or} \\ &(7/2 < t_1^{(1)} \leq 4 \quad \text{and} \quad 3 \leq t_0^{(0)} \leq 8 - t_1^{(1)}). \end{aligned}$$

It is displayed below (Figure 5.7); note that it is not an integral polygon. Its area ($V = 7/4$) can be readily calculated.

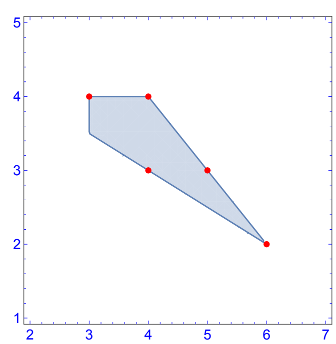


Figure 5.7: The BZ polytope in the $(t_0^{(0)}, t_1^{(1)})$ plane for $\lambda = (4, 7)$, $\mu = (5, 3)$, $\nu = (2, 4)$.

5.5.5.3 Determination of the coefficients r_κ and $\hat{r}_\kappa$

As explained in Section 5.3, the formulae (5.3.13, 5.3.14) can be used to determine the constants r_κ and $\hat{r}_\kappa$. Take for example $\kappa = (0, 0)$ in B_2; the triple $(\rho, \rho, \kappa + \rho)$

appearing in (5.3.13) is never compatible. One computes the multiplicity of the scaled triples $(s(1,1), s(1,1), s(1,1))$, which is 0 when s is odd (in particular for $s = 1$), and equal to $(1, 4, 10, \ldots)$ when $s = 0, 2, 4, \ldots$; the quasi-polynomial is 0 when s is odd, and $3s^2/8 + 3s/4 + 1$ when s is even. The leading coefficient obtained for even values of s gives $r_{(0,0)} = 3/8$, as desired.

Likewise, for B_3, the coefficient r_κ associated with $\kappa = (0, 0, 0)$ can be determined from the scaled triples $(s(1,1,1), s(1,1,1), s(1,1,1))$. Here the period of the stretching quasi-polynomial is 4, a result which is not in contradiction with known theorems, since the triple is not compatible. The LR quasi-polynomial vanishes for odd s, whereas one gets

$$\frac{241}{3072}s^6 + \frac{241}{512}s^5 + \frac{4165}{3072}s^4 + \frac{19}{8}s^3 + \frac{523}{192}s^2 + 2s + 1$$

when $s = 0 \bmod 4$, and

$$\frac{241}{3072}s^6 + \frac{241}{512}s^5 + \frac{4165}{3072}s^4 + \frac{281}{128}s^3 + \frac{839}{384}s^2 + \frac{19}{16}s + \frac{35}{64}$$

when $s = 2 \bmod 4$. Therefore $r_{(0,0,0)} = 241/3072 = 7230/92160$ as given in Section 5.3. Notice that here one has to use scaling factors up to $s = 6 \times 4 = 24$ and $s = 6 \times 4 + 2 = 26$ to obtain the two nontrivial polynomial determinations of this quasi-polynomial.

5.6 Appendix: Covolumes

Let us sketch how the covolume $c_{\mathfrak{g}}$ may be determined for the affine lattice Λ of integer points of $\mathrm{aff}(H^\nu_{\lambda\mu})$, in the case that $\dim H^\nu_{\lambda\mu} = d_r$ so that $\mathrm{aff}(H^\nu_{\lambda\mu}) = \mathrm{aff}(\mathrm{Part}_{\mathfrak{g}}(\sigma))$, see (5.4.1). This affine span has codimension r in the Euclidean space $\mathbb{R}^{N_r}$ generated as the formal span of the positive roots, which are taken to form a canonical basis, i.e. $\langle \boldsymbol{\alpha}_a, \boldsymbol{\alpha}_b \rangle = \delta_{ab}$, $a, b = 1, \cdots, N_r$. Since the covolume of Λ is independent of the value of σ, we may take $\sigma = 0$. Then Λ is the lattice of integer combinations of the positive roots $\boldsymbol{\alpha}_a$ satisfying the condition

$$\sum_{a=1}^{N_r} u_a \boldsymbol{\alpha}_a = 0 \in \mathfrak{t}^*. \tag{5.6.1}$$

Suppose that $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r$ are the simple roots, and denote by A the $(N_r - r) \times r$ matrix that expresses the non-simple roots in terms of the simple ones:

$$\boldsymbol{\alpha}_a = \sum_{i=1}^{r} A_{ai}\, \boldsymbol{\alpha}_i, \quad a = r+1, \ldots, N_r.$$

We can eliminate the parameters u_i for $i = 1, \ldots, r$ in the condition (5.6.1), using the relation $u_i = -\sum_{a=r+1}^{N_r} u_a A_{ai}$. We then find that Λ consists of integer combinations of the form $\sum_{a=r+1}^{N_r} u_a(-\sum_i A_{ai}\, \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_a)$, so that the $N_r - r$ vectors $w_a := (-\sum A_{ai} \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_a)$ form a lattice basis. *Under the assumption* that the parallelepiped spanned by these vectors is a fundamental domain of Λ, the covolume of Λ is equal to the volume of this parallelepiped, which we can compute as follows. The Gram matrix of the vectors w_a in the Euclidean space $\mathbb{R}^{N_r}$ reads:

$$G_{ab} := \langle w_a, w_b \rangle = \delta_{ab} + \sum_{i=1}^{r} A_{ai} A_{bi}\,.$$

The covolume is then the square root of the determinant δ_r of G, $c_{\mathfrak{g}} = \delta_r^{1/2}$. We have computed these determinants for the classical algebras A_r, B_r, C_r and D_r up to $r = 8$, and the values of δ_r listed in Table 5.1 are extrapolations of these numbers. The values for the exceptional algebras are also included in Table 5.1. A last observation is that for all of the simple algebras, a single formula encompasses the expressions found in the table:

$$\delta_r = \frac{(h^\vee)^r}{\det C} \prod_{i=1}^{r} \frac{\langle \boldsymbol{\theta}, \boldsymbol{\theta} \rangle}{\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i \rangle}\,, \tag{5.6.2}$$

where C is the Cartan matrix, $h^\vee$ is the dual Coxeter number, $\boldsymbol{\theta}$ is any long root, and the product runs over the simple roots. We thus make the following conjecture.

Conjecture 5.6.1. *For all simple Lie algebras, the formula (5.6.2) gives the correct value of the squared covolume of the lattice* Λ.

Chapter Six

Computing multiplicities from the volume function

6.1 Introduction

In the previous chapter, we introduced the tensor product multiplicities $C_{\lambda\mu}^{\nu}$ of a compact semisimple Lie algebra $\mathfrak{g}$, or equivalently of the connected, simply connected group G with Lie algebra $\mathfrak{g}$. These are the multiplicities arising in the decomposition

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} C_{\lambda\mu}^{\nu} V_\nu,$$

where V_λ, V_μ, V_ν are finite-dimensional irreducible representations of $\mathfrak{g}$, and λ, μ, ν are the corresponding highest weights. Since characters combine multiplicatively under the tensor product and additively under the direct sum, the tensor product multiplicities are also the structure constants of the algebra generated by the irreducible characters of G:

$$\chi_\lambda \chi_\mu = \sum_{\nu} C_{\lambda\mu}^{\nu} \chi_\nu.$$

Computing these multiplicities is an important combinatorial problem in representation theory. There are a number of known algorithms for calculating $C_{\lambda\mu}^{\nu}$, although in general the problem is #P-complete (Narayanan, 2006). The most widely studied case is $\mathfrak{g} = \mathfrak{su}(n)$, where the multiplicities are usually called Littlewood–Richardson coefficients as they are described by the famous Littlewood–Richardson rule (Littlewood and Richardson, 1934).

In this chapter we study the relationship between the multiplicities $C_{\lambda\mu}^{\nu}$ and the volume function $\mathcal{J}$ for the coadjoint representation of G. The definition of $\mathcal{J}$, introduced in Chapter 5, is recalled in Definition 6.2.1 below. We develop multiple methods for computing $C_{\lambda\mu}^{\nu}$ from $\mathcal{J}$, and we demonstrate how each of these two objects can be used to study the other, leading to new results as well as new proofs of known theorems.

The original motivation for this study was a question posed by Coquereaux and Zuber in (Coquereaux and Zuber, 2018, sect. 2.3), where they showed that certain values of the volume function for $\mathfrak{su}(n)$ can be expressed algebraically in terms of Littlewood–Richardson coefficients, leading ultimately to the $\mathcal{J}$-LR relations (5.3.11, 5.3.12). Coquereaux and Zuber asked whether these relations might be inverted, yielding an expression for $C_{\lambda\mu}^{\nu}$ in terms of $\mathcal{J}$. We will answer this question in the affirmative, although the formulae that we obtain will be more or less explicit depending on $\mathfrak{g}$ and on the highest weights (λ, μ, ν).

Recall from Proposition 5.4.2 that $\mathcal{J}(\lambda, \mu; \nu)$ equals the volume of the BZ polytope $H_{\lambda\mu}^{\nu}$, whose number of integer points is equal to $C_{\lambda\mu}^{\nu}$. Recovering $C_{\lambda\mu}^{\nu}$ from $\mathcal{J}$ therefore amounts to computing the number of integer points in $H_{\lambda\mu}^{\nu}$ given the volumes of the whole family of polytopes $\{H_{\alpha\beta}^{\gamma}\}_{\alpha,\beta,\gamma\in\mathfrak{t}}$. In the terminology of geometric quantization, $\mathcal{J}(\lambda, \mu; \nu)$ is a semiclassical approximation of $C_{\lambda\mu}^{\nu}$, so that an expression for $C_{\lambda\mu}^{\nu}$ in terms of $\mathcal{J}$ can be interpreted as exactly recovering a "quantum" object from its classical limit. General methods for counting lattice points in polytopes based on this type of volume data have been developed by Brion and Vergne (1997) and by Szenes and Vergne (2003), but here we take a different approach.

Our methods are based on the idea of box spline deconvolution. It follows from a formula of De Concini et al. (2013) that for certain fixed values of α and β, $\mathcal{J}(\alpha, \beta; \gamma)$ can be represented as a convolution of two measures on $\mathfrak{t}$: a finitely supported measure that encodes the multiplicities $C_{\lambda\mu}^{\nu}$, and a continuous measure called a box spline (see Definition 6.2.3). Therefore one way to think about computing $C_{\lambda\mu}^{\nu}$ from $\mathcal{J}$ is as a deconvolution problem. To this end, it is necessary to study the box spline itself in some detail. The problem of inverting the convolution with the box spline was studied in depth by Dahmen and Micchelli (1985a,b) and more recently by De Concini et al. (2013), Duflo and Vergne (2011), and Vergne (2013).

These later authors introduced the idea of using box spline deconvolution to study representation-theoretic multiplicity problems, and here we build on their work. We will review a number of their results below.

The main new technique in this chapter, introduced in Section 6.5, is to simplify the problem by restricting $\mathcal{J}$ to a lattice. By moving to this discrete setting we lose no relevant information, but the deconvolution problem becomes more tractable, leading to new formulae for the multiplicities. We also obtain a reformulation of the $\mathcal{J}$-LR relations in terms of a finite difference operator $\mathcal{D}$ that we call the box spline Laplacian. This finally allows us, in Theorem 6.5.22, to derive the following concise expression for Littlewood–Richardson coefficients of $\mathfrak{su}(n)$, which holds when the triple (λ, μ, ν) lies sufficiently far from a certain hyperplane arrangement in $\mathfrak{t}^3$:

$$C_{\lambda\mu}^{\nu} = \sum_{k=0}^{\lfloor d/2 \rfloor} \Big(-\frac{1}{2}\mathcal{D}\Big)^k \mathcal{J}(\lambda', \mu'; \nu'), \tag{6.1.1}$$

where $d = \frac{1}{2}(n-1)(n-2)$ and the primes indicate the shift by the Weyl vector. The precise sense of (6.1.1), including notational details and the required assumptions on (λ, μ, ν), is explained below in Sections 6.5.3 and 6.5.4.

The box spline convolution and deconvolution identities of De Concini et al. (2013) are general statements in index theory, and they presumably could be used to extend the techniques of this chapter to the more general problem of decomposing a representation of G into irreducible representations of an arbitrary closed connected subgroup. We do not pursue this line of reasoning in full, but we illustrate it at the end of the chapter by showing how the same techniques can be used to compute weight multiplicities of an irreducible representation of G, leading to an analogue of (6.1.1) for Kostka numbers.

The text of this chapter is adapted from the paper (McSwiggen, 2019a).

Organization of the chapter

In Section 6.2 we define the volume function and the box spline associated to $\mathfrak{g}$, and we review some of their properties. We then recall a formula of De Concini et al. (2013) that represents the volume function as a convolution of the box spline and a finitely supported measure that encodes the tensor product multiplicities. In Section 6.3 we review some generalities on box spline deconvolution due to Dahmen and Micchelli (1985a), De Concini et al. (2013), and Duflo and Vergne (2011). We discuss in particular the case $\mathfrak{g} = \mathfrak{su}(n)$, which is special among the cases that we consider because it admits a particularly straightforward expression for the deconvolution operator, leading to an integrodifferential formula for the Littlewood–Richardson coefficients.

Most of the statements in Sections 6.2 and 6.3 are fairly direct consequences either of the deconvolution theorems of Dahmen and Micchelli (1985a), De Concini et al. (2013), and Duflo and Vergne (2011), or of results on the volume function shown in (Coquereaux et al., 2019a, Coquereaux and Zuber, 2018). Accordingly, these two sections should be regarded primarily as a review of known results, with the goal of working out in detail an important special case of the more general considerations in (De Concini et al., 2013, Duflo and Vergne, 2011). We have, however, recorded several explicit formulae and derivations that do not seem to have appeared previously in the literature.

In Section 6.4, we use these representations of $\mathcal{J}$ and $C_{\lambda\mu}^{\nu}$ to give new proofs of three known results: the differentiability class of $\mathcal{J}$ (Corollary 6.4.1), the semiclassical asymptotics expressing $\mathcal{J}$ as a scaling limit of tensor product multiplicities (Corollary 6.4.5), and a theorem of Rassart (2004) and Derksen and Weyman (2002) stating that for fixed (λ, μ, ν), the Littlewood–Richardson coefficient $C_{N\lambda\,N\mu}^{N\nu}$ of $\mathfrak{su}(n)$ is a polynomial in the variable $N \in \mathbb{N}$ (Theorem 6.4.6). We include these proofs mainly for illustrative purposes: the first two demonstrate the wide-ranging consequences of the De Concini–Procesi–Vergne convolution formula, while the third illustrates geometric ideas that we will revisit in the proof of Theorem 6.5.22.

Section 6.5 contains our main new results. Here we replace the volume function and the box spline with discrete approximations and study convolutions on the root lattice or the weight lattice rather than on the entire Cartan subalgebra. This simplifies the deconvolution problem; in fact, for any given (λ, μ, ν), only finitely many values of $\mathcal{J}$ are needed to compute the tensor product multiplicity. We give two methods for calculating $C_{\lambda\mu}^{\nu}$ in this discrete setting: an algorithm (Theorem 6.5.3) and an integral formula (Theorem 6.5.5). Since $\mathcal{J}$ can be expressed in terms of the Harish-Chandra integral (1.1.2), for $\mathfrak{g} = \mathfrak{su}(n)$ these results give the Littlewood–Richardson coefficients in terms of the HCIZ integral (1.1.3).

Next we derive some identities for the discretized box spline on the root lattice, and we introduce a finite difference operator called the box spline Laplacian that provides a convenient representation of discrete convolution with the box spline (Proposition 6.5.14). As a consequence, we obtain an alternate formulation of the $\mathcal{J}$-LR relations. Finally, we draw on results from all of the previous sections to prove a formula for Littlewood–Richardson coefficients as a linear combination of values of $\mathcal{J}$ (Theorem 6.5.22), which holds for "typical" dominant weights (λ, μ, ν) of $\mathfrak{su}(n)$.

Section 6 sketches how the techniques in this chapter are equally applicable to the simpler problem of computing weight multiplicities.

6.2 The volume function and the box spline

Let G be a compact, semisimple, connected, simply connected Lie group of rank r with Lie algebra $\mathfrak{g}$, and $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra with Weyl group W. Fix a G-invariant inner product $\langle \cdot, \cdot \rangle$ identifying $\mathfrak{g} \cong \mathfrak{g}^*$, as well as a choice $\Phi^+ \subset \mathfrak{t}$ of positive roots.

We recall the definition of the volume function $\mathcal{J}$ associated to $\mathfrak{g}$. It is defined in terms of Harish-Chandra orbital integrals,

$$\mathcal{H}(x, y) := \int_G e^{\langle \mathrm{Ad}_g y, x \rangle} dg, \qquad x, y \in \mathfrak{t}_{\mathbb{C}},$$

where dg is the normalized Haar measure. Using the expression (2.6.4) for the normalizing constant, we can write the Harish-Chandra formula (1.1.2) as:

$$\Delta_{\mathfrak{g}}(x)\Delta_{\mathfrak{g}}(y)\mathcal{H}(x, y) = \Delta_{\mathfrak{g}}(\rho) \sum_{w \in W} \epsilon(w) e^{\langle w(y), x \rangle}, \qquad x, y \in \mathfrak{t}_{\mathbb{C}}, \tag{6.2.1}$$

where $\Delta_{\mathfrak{g}}(x) := \prod_{\alpha \in \Phi^+} \langle \alpha, x \rangle$ is the discriminant of $\mathfrak{g}$, $\epsilon(w)$ is the sign of $w \in W$, and $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ is the Weyl vector.

Definition 6.2.1. For $\alpha, \beta, \gamma \in \mathfrak{t}$ we define the *volume function* as

$$\mathcal{J}(\alpha, \beta; \gamma) := \frac{\Delta_{\mathfrak{g}}(\alpha)\Delta_{\mathfrak{g}}(\beta)\Delta_{\mathfrak{g}}(\gamma)}{(2\pi)^r \, |W| \, \Delta_{\mathfrak{g}}(\rho)^3} \int_{\mathfrak{t}} \Delta_{\mathfrak{g}}(x)^2 \mathcal{H}(ix, \alpha)\mathcal{H}(ix, \beta)\mathcal{H}(ix, -\gamma) \, dx. \tag{6.2.2}$$

It is a homogeneous piecewise polynomial function of the triple $(\alpha, \beta, \gamma) \in \mathfrak{t}^3$, of degree $|\Phi^+| - r$. We usually take α, β fixed and regard $\mathcal{J}(\alpha, \beta; \gamma)$ as a W-skew function of $\gamma \in \mathfrak{t}$, in which case it is supported on a union of $|W|$ convex polytopes in $\mathfrak{t}$. △

As a first illustration of the relationship between $\mathcal{J}$ and the tensor product multiplicities $C_{\lambda\mu}^{\nu} := \dim \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_\mu \to V_\nu)$, we recall the $\mathcal{J}$-LR relation (5.3.11), which expresses certain values of $\mathcal{J}$ in terms of $C_{\lambda\mu}^{\nu}$.

Let $Q \subset \mathfrak{t}$ be the root lattice and $\mathcal{C}_+ \subset \mathfrak{t}$ the dominant Weyl chamber. We will say that a triple (λ, μ, ν) of dominant weights of $\mathfrak{g}$ is *compatible* if $\lambda + \mu - \nu \in Q$, as this is a well-known necessary condition for $C_{\lambda\mu}^{\nu} \neq 0$. Let a prime denote the shift of a weight by the Weyl vector: $\lambda' = \lambda + \rho$. Let $K \subset Q$ be the set of dominant elements of the root lattice that lie on the interior of the convex hull of the Weyl orbit of ρ, and for $\kappa \in K$ define $r_\kappa := \mathcal{J}(\rho, \rho; \kappa')$. Then for (λ, μ, ν) compatible, we have (Coquereaux et al., 2019a, eqn. 28):

$$\mathcal{J}(\lambda', \mu'; \nu') = \sum_{\kappa \in K} \sum_{\substack{\tau \in \lambda+\mu+Q \\ \cap\ \mathcal{C}_+}} r_\kappa C_{\lambda\mu}^{\tau} C_{\tau\kappa}^{\nu} = \sum_{\kappa \in K} r_\kappa C_{\lambda\mu\kappa}^{\nu}, \tag{6.2.3}$$

where $C_{\lambda\mu\kappa}^{\nu} := \dim \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_\mu \otimes V_\kappa \to V_\nu)$ is the multiplicity of V_ν in the triple tensor product. The index τ runs over all dominant elements of the translated root lattice $\lambda + \mu + Q$. The second $\mathcal{J}$-LR relation (5.3.12) gives a similar formula for $\mathcal{J}(\lambda, \mu; \nu)$ with unshifted arguments.

Remark 6.2.2. The $\mathcal{J}$-LR relations offer a starting point for deducing properties of $\mathcal{J}$ from those of $C_{\lambda\mu}^{\nu}$ and vice versa. For example, Coquereaux and Zuber (2011) showed that the total multiplicity of a tensor product of two irreducible representations is unchanged by conjugating one of the highest weights:

$$\sum_{\nu \in \lambda + \overline{\mu} + Q} C_{\lambda\overline{\mu}}^{\nu} = \sum_{\nu \in \lambda + \mu + Q} C_{\lambda\mu}^{\nu}, \tag{6.2.4}$$

where $\overline{\mu} := -w_0(\mu)$ for w_0 the longest element of W. Summing over ν in the $\mathcal{J}$-LR relations (5.3.11, 5.3.12) and using (6.2.4), we obtain analogous properties for $\mathcal{J}$:

$$\sum_{\substack{\nu\in\lambda+\overline{\mu}+Q\\ \cap\ \mathcal{C}_+}} \mathcal{J}(\lambda', \overline{\mu}'; \nu') = \sum_{\substack{\nu\in\lambda+\mu+Q\\ \cap\ \mathcal{C}_+}} \mathcal{J}(\lambda', \mu'; \nu'), \tag{6.2.5}$$

$$\sum_{\substack{\nu\in\lambda+\overline{\mu}+Q\\ \cap\ \mathcal{C}_+}} \mathcal{J}(\lambda, \overline{\mu}; \nu) = \sum_{\substack{\nu\in\lambda+\mu+Q\\ \cap\ \mathcal{C}_+}} \mathcal{J}(\lambda, \mu; \nu). \tag{6.2.6}$$

△

The work in this chapter grew out of a desire to answer a question posed by Coquereaux and Zuber (2018): is there an "inverse" formula to (6.2.3) that expresses $C_{\lambda\mu}^{\nu}$ in terms of $\mathcal{J}$? We will find that in fact we can compute $C_{\lambda\mu}^{\nu}$ from $\mathcal{J}$ in multiple ways. All of them involve a type of piecewise polynomial measure called a box spline.

Definition 6.2.3. For $X \subset \mathfrak{t}$ a finite collection of vectors, we define a measure $B_c[X]$ on $\mathfrak{t}$ by

$$\int_{\mathfrak{t}} f\ dB_c[X] = \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} f\Big(\sum_{\alpha\in X} t_\alpha \alpha\Big) \prod_{\alpha\in X} dt_\alpha, \qquad f \in C^0(\mathfrak{t}). \tag{6.2.7}$$

This measure $B_c[X]$ is the *centered box spline* associated to the set X. △

We will mainly study $B_c[\Phi^+]$, which is a probability measure supported on the convex hull of the Weyl orbit of ρ. It has a density b with respect to the Lebesgue measure dx on $\mathfrak{t}$ induced by the inner product $\langle\cdot,\cdot\rangle$, so that

$$\int_{\mathfrak{t}} f\, dB_c[\Phi^+] = \int_{\mathfrak{t}} f(x)\, b(x)\, dx, \qquad f \in C^0(\mathfrak{t}).$$

The density b is a piecewise polynomial function of degree $|\Phi^+| - r$. From the definition (6.2.7) we find the following symmetries:

$$b(-x) = b(x), \qquad x \in \mathfrak{t}, \tag{6.2.8}$$

$$b(w(x)) = b(x), \qquad x \in \mathfrak{t},\ w \in W. \tag{6.2.9}$$

For a general introduction to box splines, see the seminal paper by de Boor and Höllig (1982/83) or the review by Prautzsch et al. (2002).

We take the probabilist's sign convention for the Fourier transform of a Borel measure μ on $\mathfrak{t}$,

$$\mathscr{F}[\mu](x) := \int_{\mathfrak{t}} e^{i\langle \xi, x\rangle} d\mu(\xi), \qquad x \in \mathfrak{t}.$$

A direct calculation then reveals that

$$\mathscr{F}\left[B_c[\Phi^+]\right](x) = \prod_{\alpha\in\Phi^+} \frac{e^{i\langle\alpha,x\rangle/2} - e^{-i\langle\alpha,x\rangle/2}}{i\langle\alpha, x\rangle} = j_{\mathfrak{g}}^{1/2}(x), \tag{6.2.10}$$

where $j_{\mathfrak{g}}$ is the function sometimes called the Jacobian of the exponential map, well known to representation theorists due to its appearance in the Kirillov character formula, recalled in (6.2.12) below.

The box spline $B_c[\Phi^+]$ controls the relationship between the volume function and the tensor product multiplicities. This is a consequence of a very general formula in index theory due to De Concini et al. (2013, Proposition 5.14) (see also (Duflo and Vergne, 2011, sect. 3)), which relates the box spline to a quantity called the infinitesimal index of a transversally elliptic symbol. As a special case, their formula implies the following expression for $\mathcal{J}$ as the convolution of $B_c[\Phi^+]$ and a finitely supported W-skew measure that encodes the coefficients $C_{\lambda\mu}^{\nu}$.

Let δ_x denote the measure assigning unit mass to the point $x \in \mathfrak{t}$. Where it improves ease of reading, we will sometimes write $f(x) * \mu$ rather than $(f * \mu)(x)$ for the convolution of a function f and a measure μ, that is, $f(x) * \mu = \int_{\mathfrak{t}} f(x-y)\, d\mu(y)$.

Proposition 6.2.4 (De Concini–Procesi–Vergne)**.** *Let $\lambda, \mu \in \mathfrak{t}$ be dominant weights of $\mathfrak{g}$. Then*

$$\mathcal{J}(\lambda', \mu'; \gamma) = b(\gamma) \, * \, \Bigg(\sum_{\substack{\nu \in (\lambda+\mu)+Q \\ \cap \, \mathcal{C}_+}} C_{\lambda\mu}^{\nu} \sum_{w \in W} \epsilon(w) \, \delta_{w(\nu')} \Bigg). \tag{6.2.11}$$

Recall that Q denotes the root lattice and $\mathcal{C}_+$ the dominant Weyl chamber, so that the sum over ν in (6.2.11) runs over dominant weights satisfying the compatibility criterion $\lambda + \mu - \nu \in Q$.

Although one could deduce Proposition 6.2.4 from (De Concini et al., 2013, Proposition 5.14) by invoking index-theoretic constructions, it is more instructive to give a detailed proof specialized to the context that we consider here. The argument below follows a method sketched by Paradan and Vergne (2018, Lemma 5.2).

Remark 6.2.5. If $\mathfrak{g}$ contains no simple summands isomorphic to $\mathfrak{su}(2)$, then $\mathcal{J}$ and b are both continuous functions of $\gamma \in \mathfrak{t}$. In these cases, (6.2.11) and all equations relating $\mathcal{J}$ and b below hold pointwise on $\mathfrak{t}$. If $\mathfrak{g}$ does contain one or more $\mathfrak{su}(2)$ summands, then both $\mathcal{J}$ and b have jump discontinuities on the boundaries of their respective supports, and all expressions for $\mathcal{J}$ in terms of b should be understood to hold almost everywhere with respect to Lebesgue measure. Away from these boundary discontinuities we identify $\mathcal{J}$ and b with their locally continuous versions, so that we may talk about their pointwise values at all other points of $\mathfrak{t}$. $\triangle$

Proof of Proposition 6.2.4. Recall the Kirillov character formula (2.6.1) for compact connected Lie groups (Kirillov, 1968, 2004), which we now write as:

$$j_{\mathfrak{g}}^{1/2}(x)\chi_\lambda(e^x) = \int_{\mathcal{O}_{\lambda'}} e^{i\langle \xi, x\rangle} d\beta_{\mathcal{O}_{\lambda'}}(\xi), \quad x \in \mathfrak{t}. \tag{6.2.12}$$

Here χ_λ is the irreducible character of G with highest weight λ, $\mathcal{O}_{\lambda'}$ is the coadjoint orbit of λ', and $d\beta$ is the Liouville measure of the Kostant–Kirillov–Souriau symplectic form on the coadjoint orbit.

Consider now the direct product $G \times G$. This group is also compact and connected, and its irreducible representations take the form $V_\lambda \otimes V_\mu$, where V_λ and V_μ are irreducible representations of G. Denote the character of such a representation by $\tilde{\chi}_{\lambda\otimes\mu}$. The diagonal subgroup $G_\Delta \subset G \times G$ acts on $V_\lambda \otimes V_\mu$ by the usual representation of G on the tensor product, so identifying $G \cong G_\Delta$ we have $\tilde{\chi}_{\lambda\otimes\mu}|_{G_\Delta} = \chi_\lambda \chi_\mu$. To compute the decomposition of $V_\lambda \otimes V_\mu$ into irreducible representations V_ν, it thus suffices to decompose $\tilde{\chi}_{\lambda\otimes\mu}|_{G_\Delta}$ into irreducible characters χ_ν.

We start by using the Kirillov character formula (6.2.12) to obtain an expression for $\tilde{\chi}_{\lambda\otimes\mu}$. The coadjoint orbit of $G\times G$ corresponding to the irreducible representation $V_\lambda \otimes V_\mu$ is $\mathcal{O}_{\lambda'} \times \mathcal{O}_{\mu'}$. The positive roots of $G \times G$ are $\Phi^+ \sqcup \Phi^+$, i.e. we count each positive root of G twice, once for each factor, so that $j_{\mathfrak{g}\oplus\mathfrak{g}}^{1/2}(x, y) = j_{\mathfrak{g}}^{1/2}(x) j_{\mathfrak{g}}^{1/2}(y)$. Putting all this into (6.2.12), we get:

$$j_{\mathfrak{g}}^{1/2}(x)\, j_{\mathfrak{g}}^{1/2}(y)\, \tilde{\chi}_{\lambda\otimes\mu}(e^{(x,y)}) \;=\; \int_{\mathcal{O}_{\lambda'}\times\mathcal{O}_{\mu'}} e^{i(\langle \xi, x\rangle + \langle \eta, y\rangle)}\, d\beta_{\mathcal{O}_{\lambda'}\times\mathcal{O}_{\mu'}}(\xi, \eta),$$

$$(x, y) \in \mathfrak{t} \oplus \mathfrak{t}.$$

Restricting to the diagonal $x = y$, this becomes:

$$\begin{aligned} j_{\mathfrak{g}}(x)\tilde{\chi}_{\lambda\otimes\mu}(e^{(x,x)}) = j_{\mathfrak{g}}(x)\chi_\lambda(e^x)\chi_\mu(e^x) = j_{\mathfrak{g}}(x)\sum_\nu C^\nu_{\lambda\mu}\chi_\nu(e^x) \\ = \int_{\mathcal{O}_{\lambda'}\times\mathcal{O}_{\mu'}} e^{i\langle\xi+\eta,x\rangle}\, d\beta_{\mathcal{O}_{\lambda'}\times\mathcal{O}_{\mu'}}(\xi,\eta), \qquad x \in \mathfrak{t}. \quad (6.2.13) \end{aligned}$$

Using the fact (2.6.3) that the symplectic volume of $\mathcal{O}_{\nu'}$ is equal to $\Delta_{\mathfrak{g}}(\nu')/\Delta_{\mathfrak{g}}(\rho)$, we can reduce the integral over $\mathcal{O}_{\lambda'} \times \mathcal{O}_{\mu'}$ to two integrals over G with respect to the normalized Haar measure dg, so that (6.2.13) becomes:

$$j_{\mathfrak{g}}(x)\sum_\nu C^\nu_{\lambda\mu}\chi_\nu(e^x) = \frac{\Delta_{\mathfrak{g}}(\lambda')\Delta_{\mathfrak{g}}(\mu')}{\Delta_{\mathfrak{g}}(\rho)^2}\mathcal{H}(ix,\lambda')\mathcal{H}(ix,\mu'). \quad (6.2.14)$$

Applying Kirillov's formula (6.2.12) again to each χ_ν and then using the Harish-Chandra integral formula (6.2.1), we find that the left-hand side of (6.2.14) can be rewritten as:

$$\begin{aligned} j_{\mathfrak{g}}(x)\sum_\nu C^\nu_{\lambda\mu}\chi_\nu(e^x) &= j_{\mathfrak{g}}^{1/2}(x)\sum_\nu C^\nu_{\lambda\mu}\int_{\mathcal{O}_{\nu'}} e^{i\langle\xi,x\rangle}\, d\beta_{\mathcal{O}_{\nu'}}(\xi) \\ &= j_{\mathfrak{g}}^{1/2}(x)\sum_\nu C^\nu_{\lambda\mu}\frac{\Delta_{\mathfrak{g}}(\nu')}{\Delta_{\mathfrak{g}}(\rho)}\mathcal{H}(ix,\nu') \\ &= \frac{j_{\mathfrak{g}}^{1/2}(x)}{\Delta_{\mathfrak{g}}(ix)}\sum_\nu C^\nu_{\lambda\mu}\sum_{w\in W}\epsilon(w)e^{i\langle w(\nu'),x\rangle}. \quad (6.2.15) \end{aligned}$$

Equating this last expression to the right-hand side of (6.2.14) and multiplying through by $\Delta_{\mathfrak{g}}(ix)$, we finally obtain:

$$j_{\mathfrak{g}}^{1/2}(x)\sum_\nu C^\nu_{\lambda\mu}\sum_{w\in W}\epsilon(w)e^{i\langle w(\nu'),x\rangle} = \frac{\Delta_{\mathfrak{g}}(\lambda')\Delta_{\mathfrak{g}}(\mu')}{\Delta_{\mathfrak{g}}(\rho)^2}\Delta_{\mathfrak{g}}(ix)\mathcal{H}(ix,\lambda')\mathcal{H}(ix,\mu'). \quad (6.2.16)$$

Now we take the inverse Fourier transform, over $\mathfrak{t}$, of each side of (6.2.16). On the left-hand side, using (6.2.10) we find:

$$\mathscr{F}^{-1}\left[j_{\mathfrak{g}}^{1/2}(x)\sum_{\nu} C_{\lambda\mu}^{\nu}\sum_{w\in W}\epsilon(w)e^{i\langle w(\nu'),x\rangle}\right](\gamma)$$
$$= b(\gamma) * \left(\sum_{\nu} C_{\lambda\mu}^{\nu}\sum_{w\in W}\epsilon(w)\delta_{w(\nu')}\right). \quad (6.2.17)$$

On the right-hand side, we have:

$$\begin{aligned}
&\mathscr{F}^{-1}\left[\frac{\Delta_{\mathfrak{g}}(\lambda')\Delta_{\mathfrak{g}}(\mu')}{\Delta_{\mathfrak{g}}(\rho)^2}\Delta_{\mathfrak{g}}(ix)\mathcal{H}(ix,\lambda')\mathcal{H}(ix,\mu')\right](\gamma)\\
&= \frac{1}{(2\pi)^r}\frac{\Delta_{\mathfrak{g}}(\lambda')\Delta_{\mathfrak{g}}(\mu')}{\Delta_{\mathfrak{g}}(\rho)^2}\int_{\mathfrak{t}}\Delta_{\mathfrak{g}}(ix)\mathcal{H}(ix,\lambda')\mathcal{H}(ix,\mu')e^{-i\langle x,\gamma\rangle}dx\\
&= \frac{\Delta_{\mathfrak{g}}(\lambda')\Delta_{\mathfrak{g}}(\mu')}{(2\pi)^r\,|W|\,\Delta_{\mathfrak{g}}(\rho)^2}\int_{\mathfrak{t}}\Delta_{\mathfrak{g}}(ix)\mathcal{H}(ix,\lambda')\mathcal{H}(ix,\mu')\left(\sum_{w\in W}\epsilon(w)e^{\langle w(ix),-\gamma\rangle}\right)dx\\
&= \frac{\Delta_{\mathfrak{g}}(\lambda')\Delta_{\mathfrak{g}}(\mu')\Delta_{\mathfrak{g}}(\gamma)}{(2\pi)^r\,|W|\,\Delta_{\mathfrak{g}}(\rho)^3}\int_{\mathfrak{t}}\Delta_{\mathfrak{g}}(x)^2\mathcal{H}(ix,\lambda')\mathcal{H}(ix,\mu')\mathcal{H}(ix,-\gamma)\,dx, \qquad (6.2.18)
\end{aligned}$$

where in the last line we have again applied the Harish-Chandra formula (6.2.1) and have used the fact that $\Delta_{\mathfrak{g}}$ is homogeneous of degree $|\Phi^+| = (\dim\mathfrak{g} - r)/2$ to cancel factors of -1 and i.

Comparing (6.2.18) to the definition (6.2.2) of $\mathcal{J}$, we see that this last expression is equal to $\mathcal{J}(\lambda',\mu';\gamma)$, completing the proof. □

To conclude this section, we quickly derive a more compact expression for $\mathcal{J}$. For $\mathfrak{g} = \mathfrak{su}(n)$, the identity below reduces to (Zuber, 2018, eqn. 18).

Proposition 6.2.6. *Define the function $\psi_{\alpha\beta} : \mathfrak{t} \to \mathbb{C}$ by*

$$\psi_{\alpha\beta}(x) := (-i)^{|\Phi^+|}\sum_{w,w'\in W}\frac{\epsilon(ww')}{\Delta_{\mathfrak{g}}(x)}e^{i\langle x,w(\alpha)+w'(\beta)\rangle}. \qquad (6.2.19)$$

Then $\mathcal{J}(\alpha,\beta;\gamma) = \mathscr{F}^{-1}[\psi_{\alpha\beta}](\gamma)$.

Remark 6.2.7. Some care is required in interpreting both the definition of $\psi_{\alpha\beta}$ and the inverse Fourier transform in Proposition 6.2.6. For x such that $\Delta_{\mathfrak{g}}(x) = 0$, we understand the expression (6.2.19) as the limit $\psi_{\alpha\beta}(x) := \lim_{t\to 0}\psi_{\alpha\beta}(x+t\rho)$; the double sum over the Weyl group then introduces cancelations such that $\psi_{\alpha\beta}$ vanishes. Moreover when $\mathfrak{g}$ contains a simple summand isomorphic to $\mathfrak{su}(2)$, the usual integral representation of the inverse Fourier transform of $\psi_{\alpha\beta}$ is not absolutely convergent and must instead be interpreted as a Cauchy principal value. △

Proof. We may assume that $\Delta_{\mathfrak{g}}(\alpha)$ and $\Delta_{\mathfrak{g}}(\beta)$ are both nonzero, since otherwise $\mathcal{J}$ and $\psi_{\alpha\beta}$ both vanish and there is nothing to prove. Applying Harish-Chandra's formula (6.2.1) to the definition (6.2.2) and canceling factors of $\Delta_{\mathfrak{g}}$, we can rewrite

$$\begin{aligned}
&\mathcal{J}(\alpha,\beta;\gamma) = \\
&\frac{(-i)^{|\Phi^+|}}{(2\pi)^r\,|W|}\int_{\mathfrak{t}}\frac{1}{\Delta_{\mathfrak{g}}(x)}\left(\sum_{w\in W}\epsilon(w)e^{i\langle w(x),\alpha\rangle}\right)\left(\sum_{w'\in W}\epsilon(w')e^{i\langle w'(x),\beta\rangle}\right)\left(\sum_{w''\in W}\epsilon(w'')e^{-i\langle w''(x),\gamma\rangle}\right)dx \\
&\qquad = \frac{(-i)^{|\Phi^+|}}{(2\pi)^r\,|W|}\int_{\mathfrak{t}}\sum_{w,w',w''\in W}\frac{\epsilon(ww'w'')}{\Delta_{\mathfrak{g}}(x)}e^{i\langle x, w(\alpha)+w'(\beta)-w''(\gamma)\rangle}dx,
\end{aligned}$$

where the integral is interpreted according to the discussion in Remark 6.2.7.

Using the W-invariance of the inner product and the W-skewness of $\Delta_{\mathfrak{g}}$, we can reindex the triple sum as a double sum, giving

$$\begin{aligned}
\mathcal{J}(\alpha,\beta;\gamma) &= \frac{(-i)^{|\Phi^+|}}{(2\pi)^r}\int_{\mathfrak{t}}\sum_{w,w'\in W}\frac{\epsilon(ww')}{\Delta_{\mathfrak{g}}(x)}e^{i\langle x, w(\alpha)+w'(\beta)-\gamma\rangle}dx \\
&= \mathscr{F}^{-1}[\psi_{\alpha\beta}](\gamma) \qquad (6.2.20)
\end{aligned}$$

as desired. □

6.3 Unimodularity and the $\hat{A}(\Phi^+)$ operator

As we will see below in Section 6.5, the coefficients $C_{\lambda\mu}^{\nu}$ can always be recovered from $\mathcal{J}$. In other words, the convolution in Proposition 6.2.4 is invertible. However, when $\mathfrak{g} = \mathfrak{su}(n)$, the inverse admits a particularly convenient form that is more explicit than in most other cases. This is due to the fact that the positive roots of $\mathfrak{su}(n)$ are *unimodular*,[1] meaning that any collection of positive roots spanning $\mathfrak{t}$ also generates the root lattice Q. Direct calculations with the other classical and exceptional root systems reveal that the series $\mathfrak{su}(n)$ are the only compact simple Lie algebras with this property.

Unimodularity of Φ^+ implies – in a delicate sense that we will shortly make precise – that convolution with $B_c[\Phi^+]$ can be inverted by a differential operator. By applying this operator to $\mathcal{J}$ and taking a limit, we can compute the coefficients $C_{\lambda\mu}^{\nu}$. This result is an application of a deconvolution formula originally proved by Dahmen and Micchelli (1985a) and dramatically expanded by Vergne and collaborators in (Brion and Vergne, 1997, Szenes and Vergne, 2003, Duflo and Vergne, 2011, De Concini et al., 2013, Vergne, 2013). In fact similar deconvolution results hold even when the positive roots are not unimodular, however in these situations the formulae in terms of differential operators do not allow $C_{\lambda\mu}^{\nu}$ to be recovered from $\mathcal{J}$ alone but rather require knowledge of a larger family of piecewise polynomial functions corresponding to Duistermaat–Heckman measures for various submanifolds of $\mathcal{O}_{\lambda'} \times \mathcal{O}_{\mu'}$. Accordingly we do not treat such cases here and instead refer the reader to (Duflo and Vergne, 2011, Vergne, 2013) for the relevant results. In Section 6.5 we will develop a different approach to deconvolution for both the unimodular and non-unimodular cases.

[1] This use of the term "unimodular" is not to be confused with the different notion of a unimodular (i.e. self-dual) lattice. Indeed the root lattice of $\mathfrak{su}(n)$ is not unimodular.

To start, we review some generalities on convolution with the box spline. Let $\mathbb{C}^Q$ be the space of complex-valued functions on Q and let $L^1_{\mathrm{loc}}(\mathfrak{t})$ be the space of locally integrable functions on $\mathfrak{t}$. Consider the operator $\mathcal{T} : \mathbb{C}^Q \to L^1_{\mathrm{loc}}(\mathfrak{t})$ defined by

$$\mathcal{T}m(\gamma) := b(\gamma) * \sum_{\nu \in Q} m(\nu)\, \delta_\nu, \qquad m : Q \to \mathbb{C}. \tag{6.3.1}$$

If $\mathcal{T}m \neq 0$, then $\mathcal{T}m$ is a piecewise polynomial function of degree $d := |\Phi^+| - r$.

Proposition 6.2.4 expresses $\mathcal{J}(\lambda', \mu'; \gamma)$ in the form (6.3.1), up to an inconsequential translation of Q by $\lambda + \mu$. Therefore, if we hope to recover $C^\nu_{\lambda\mu}$ from $\mathcal{J}$, the first question we should ask is whether $\mathcal{T}$ is injective. The answer depends on whether or not Φ^+ is unimodular. In particular, from (Dahmen and Micchelli, 1985b, Theorem 3.2) and (Dahmen and Micchelli, 1985a, Theorem 4.1) we have:

Theorem 6.3.1 (Dahmen–Micchelli)**.** *The operator $\mathcal{T}$ is injective on $\mathbb{C}^Q$ if and only if Φ^+ is unimodular. Moreover, if Φ^+ is not unimodular, then the kernel of $\mathcal{T}$ intersects nontrivially with $\ell^\infty(Q)$.*

Fortunately, to compute $C^\nu_{\lambda\mu}$ from $\mathcal{J}$, we only need $\mathcal{T}$ to be injective on finitely supported functions, and indeed this always holds (see Section 6.5.1). However, Theorem 6.3.1 makes the unimodular case much simpler to handle. It allows many nice results for $\mathfrak{g} = \mathfrak{su}(n)$ that do not hold in other cases, including the deconvolution formula that we now develop.

For any $\mathfrak{g}$, we can write down an infinite-order differential operator $\hat{A}(\Phi^+)$, which inverts $\mathcal{T}$ on a specific class of functions that we define below. The operator $\hat{A}(\Phi^+)$ is defined by

$$\hat{A}(\Phi^+) := \prod_{\alpha \in \Phi^+} \frac{\partial_\alpha}{e^{\frac{1}{2}\partial_\alpha} - e^{-\frac{1}{2}\partial_\alpha}}, \tag{6.3.2}$$

where $\partial_\alpha f(x) := \frac{d}{dt}\big|_{t=0} f(x+t\alpha)$ for $f \in C^1(\mathfrak{t})$. We interpret this operator as a series expansion, recognizing that we can write

$$\hat{A}(\Phi^+) = \prod_{\alpha\in\Phi^+} \hat{a}_\alpha(\partial),$$

where

$$\hat{a}_\alpha(x) := \frac{\langle\alpha, x\rangle}{2}\operatorname{csch}\left(\frac{\langle\alpha, x\rangle}{2}\right)$$

and csch is the hyperbolic cosecant. We have the Taylor series

$$\operatorname{csch}(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)B_{2n}}{(2n)!} z^{2n-1}$$

where

$$B_{2n} = \sum_{k=0}^{2n}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{j^{2n}}{k+1}, \qquad n = 1, 2, \ldots$$

is the $(2n)^{\text{th}}$ Bernoulli number, with $B_0 = 1$. This gives

$$\hat{A}(\Phi^+) = \prod_{\alpha\in\Phi^+}\left(\sum_{n=0}^{\infty}\frac{(2^{1-2n}-1)B_{2n}}{(2n)!}\partial_\alpha^{2n}\right) = \sum_{\vec{n}}\left(\prod_{\alpha\in\Phi^+}\frac{(2^{1-2n_\alpha}-1)B_{2n_\alpha}}{(2n_\alpha)!}\right)\partial^{2\vec{n}} \tag{6.3.3}$$

where in this last expression $\vec{n} = (n_\alpha)$ runs over multi-indices with $|\Phi^+|$ components, and $\partial^{2\vec{n}} := \prod_{\alpha\in\Phi^+}\partial_\alpha^{2n_\alpha}$.

The significance of $\hat{A}(\Phi^+)$ is that it allows us to invert $\mathcal{T}$ on suitable polynomials. Concretely, let $D(\Phi^+)$ denote the space of polynomials in the image of $\mathcal{T}$:

$$D(\Phi^+) := \{\ p \in \Pi(\mathfrak{t}) \mid p = \mathcal{T}m \text{ for some } m \in \mathbb{C}^Q\ \}, \tag{6.3.4}$$

where $\Pi(\mathfrak{t})$ denotes the space of polynomial functions on $\mathfrak{t}$. Dahmen and Micchelli (1985b) characterized $D(\Phi^+)$ as the solution space of a certain system of partial differential equations and computed its dimension.[2] In (Dahmen and Micchelli, 1985a), they proved the following (see also (De Concini et al., 2013, Theorem 2.23)):

Theorem 6.3.2 (Dahmen–Micchelli). *The map $p \mapsto \mathcal{T}(p\big|_Q)$ is a linear isomorphism of $D(\Phi^+)$, with inverse $\hat{A}(\Phi^+)$.*

The above holds for any $\mathfrak{g}$, but in the unimodular case where $\mathcal{T}$ is injective not merely on $D(\Phi^+)$ but on all of $\mathbb{C}^Q$, we can use $\hat{A}(\Phi^+)$ to construct a global inverse of $\mathcal{T}$, giving a general deconvolution formula (see (Dahmen and Micchelli, 1985a) and also (Duflo and Vergne, 2011, Theorem 2.1)):

Theorem 6.3.3 (Dahmen–Micchelli, Duflo–Vergne). *Suppose that Φ^+ is unimodular and let $m : Q \to \mathbb{C}$. Choose a vector η in the positive cone generated by Φ^+ such that η does not lie in any hyperplane spanned by elements of Φ^+. Then*

$$m(\nu) = \lim_{t \to 0^+} \hat{A}(\Phi^+)\,\mathcal{T} m(\nu + t\eta), \qquad \nu \in Q. \tag{6.3.5}$$

Remark 6.3.4. The limit in (6.3.5) ensures that we only ever evaluate $\hat{A}(\Phi^+)\,\mathcal{T}m$ on the interior of a polynomial domain. This allows us to truncate the series expansion (6.3.3) so that $\hat{A}(\Phi^+)$ acts locally as a finite-order operator. △

Observing that the translation $Q \to \lambda + \mu + Q$ in (6.2.11) is inconsequential for the deconvolution formula, we obtain the following corollary to Proposition 6.2.4 and Theorem 6.3.3, which is the sought-after expression for the Littlewood–Richardson coefficients.

Corollary 6.3.5. *Let $\mathfrak{g} = \mathfrak{su}(n)$ and choose η as in Theorem 6.3.3. For a compatible triple (λ, μ, ν) of dominant weights, the Littlewood–Richardson coefficient is expressed*

[2] In fact the results of Dahmen and Micchelli (1985a,b) are much more general than what we state here, as they concern arbitrary box splines not necessarily associated to root systems.

in terms of the volume function as

$$C_{\lambda\mu}^{\nu} = \lim_{t\to 0^+} \hat{A}(\Phi^+)\, \mathcal{J}(\lambda', \mu'; \nu' + t\eta), \tag{6.3.6}$$

where the operator $\hat{A}(\Phi^+)$ acts in the third argument of $\mathcal{J}$.

Note that $\mathcal{J}$ is a piecewise polynomial of degree d, so that following Remark 6.3.4 above, we can replace $\hat{A}(\Phi^+)$ in (6.3.6) with its series expansion to this same order. Thus we have expressed $C_{\lambda\mu}^{\nu}$ as a finite sum of derivatives of $\mathcal{J}$. Putting together (6.3.3), (6.3.6) and Proposition 6.2.6, we can write Corollary 6.3.5 in a more explicit form:

Proposition 6.3.6. *For $\mathfrak{g} = \mathfrak{su}(n)$, η as in Theorem 6.3.3, (λ, μ, ν) a compatible triple of dominant weights, and $\psi_{\lambda'\mu'}$ as in (6.2.19),*

$$C_{\lambda\mu}^{\nu} = \lim_{t\to 0^+} \sum_{|\vec{n}|\leq\lfloor d/2\rfloor} \left(\prod_{\alpha\in\Phi^+} \frac{(2^{1-2n_\alpha} - 1)B_{2n_\alpha}}{(2n_\alpha)!} \right) \partial^{2\vec{n}} \mathscr{F}^{-1}[\psi_{\lambda'\mu'}](\nu' + t\eta). \tag{6.3.7}$$

Example 6.3.7. For $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, $d = 0$ and 1 respectively, so only the degree 0 term of $\hat{A}(\Phi^+)$ acts nontrivially, and we recover the well-known result (see (Coquereaux and Zuber, 2018)):

$$C_{\lambda\mu}^{\nu} = \mathcal{J}(\lambda', \mu'; \nu'). \tag{6.3.8}$$

The first nontrivial case is $\mathfrak{su}(4)$, where we have $d = 3$, so that we must expand $\hat{A}(\Phi^+)$ to second order:

$$C_{\lambda\mu}^{\nu} = \lim_{t\to 0^+} \left(1 - \frac{1}{24} \sum_{\alpha\in\Phi^+} \partial_\alpha^2 \right) \mathcal{J}(\lambda', \mu'; \nu' + t\eta). \tag{6.3.9}$$

△

6.4 First applications

In this section we use Propositions 6.2.4 and 6.3.6 to give new proofs of three known results: the number of continuous derivatives of $\mathcal{J}$, the semiclassical limit by which $\mathcal{J}(\lambda, \mu; \nu)$ approximates $C^{\nu}_{\lambda,\mu}$ for λ, μ, ν large, and a polynomiality property of the Littlewood–Richardson coefficients. These proofs illustrate the wide range of phenomena that can be understood as consequences of the box spline convolution identity (6.2.11). The considerations of Sections 6.4.1 and 6.4.2 depend only on Proposition 6.2.4 and apply to arbitrary compact semisimple $\mathfrak{g}$, while in Section 6.4.3 we take $\mathfrak{g} = \mathfrak{su}(n)$.

6.4.1 Regularity of $\mathcal{J}$

In (Coquereaux and Zuber, 2018, Coquereaux et al., 2019a), the number of continuous derivatives of $\mathcal{J}$ was determined via Fourier-analytic arguments, essentially by studying the decay of the function $\psi_{\alpha\beta}$ defined in (6.2.19). The following corollary to Proposition 6.2.4 provides an alternative proof of the degree of differentiability of $\mathcal{J}$. Recall that the density $b(\gamma)$ of $B_c[\Phi^+]$ is a piecewise polynomial function of degree $d = |\Phi^+| - r$, supported on the convex hull of the Weyl orbit of ρ.

Corollary 6.4.1. *Fix arbitrary $\alpha, \beta \in \mathfrak{t}$. As a function of γ, $\mathcal{J}(\alpha, \beta; \gamma)$ has at least as many continuous derivatives as $b(\gamma)$.*

Proof. By the W-skewness of $\mathcal{J}$ in all three arguments, it suffices to consider the case that α, β, γ all lie in the dominant Weyl chamber. Proposition 6.2.4 implies the statement in the case that $\alpha = \lambda'$, $\beta = \mu'$ for λ, μ a pair of dominant weights.

The statement for arbitrary α, β then follows by an approximation argument. First note that if $\mathfrak{g}$ has a simple summand isomorphic to $\mathfrak{su}(2)$, then b is discontinuous and there is nothing to prove. Thus we may assume $\mathfrak{g}$ contains no $\mathfrak{su}(2)$ summands, so that b is continuous. For any $\varepsilon > 0$ we can then choose $s_\varepsilon > 0$ and dominant weights $\lambda_\varepsilon, \mu_\varepsilon$ such that

$$|\mathcal{J}(\alpha, \beta, \gamma) - \mathcal{J}(\lambda'_\varepsilon/s_\varepsilon, \mu'_\varepsilon/s_\varepsilon, \gamma)| < \varepsilon.$$

By homogeneity, $\mathcal{J}(\lambda'_\varepsilon/s_\varepsilon, \mu'_\varepsilon/s_\varepsilon, \gamma) = s_\varepsilon^{-d}\mathcal{J}(\lambda'_\varepsilon, \mu'_\varepsilon, s_\epsilon\gamma)$, so that $\mathcal{J}(\lambda'_\varepsilon/s_\varepsilon, \mu'_\varepsilon/s_\varepsilon, \gamma)$ also has at least the same regularity in γ as b does. Let D be any constant-coefficient differential operator such that Db is continuous. If we apply D to $\mathcal{J}$ in the third argument, then $D\mathcal{J}(\lambda'/s, \mu'/s, \gamma)$ is a continuous function of γ whenever λ, μ are dominant weights and $s > 0$. For this to be the case, clearly $\mathcal{J}(\lambda'/s, \mu'/s, \gamma)$ must also be continuous in γ, and from the symmetry of the definition (6.2.2) of $\mathcal{J}$, both $\mathcal{J}$ and $D\mathcal{J}$ must be continuous in their first two arguments as well. Since points of the form λ'/s are dense in the dominant chamber, we find that $D\mathcal{J}$ defines a continuous piecewise polynomial function on $\mathfrak{t}^3$. In particular, $D\mathcal{J}(\lambda'_\varepsilon/s_\varepsilon, \mu'_\varepsilon/s_\varepsilon, \gamma)$ converges uniformly in γ as $\varepsilon \to 0$, so that its limit is a continuous function of γ and equals the derivative $D\mathcal{J}(\alpha, \beta, \gamma)$. □

In general the degree of differentiability of b depends on $\mathfrak{g}$, but it may be determined based on the following standard fact about box splines, which is proved for example in (Prautzsch et al., 2002, sect. 1.5):

Theorem 6.4.2. *If all subsets of Φ^+ obtained by deleting $k + 1$ roots span $\mathfrak{t}$, then $b \in C^k(\mathfrak{t})$.*

Using Theorem 6.4.2, we can determine the degree of differentiability of $\mathcal{J}$ for all of the classical series of compact Lie algebras.

Example 6.4.3. When $\mathfrak{g} = \mathfrak{su}(n)$, the subset $\{e_i - e_j \mid 1 \leq i < j \leq n-1\} \subset \Phi^+$, obtained by removing the $n-1$ roots of the form $e_i - e_n$, does not span $\mathfrak{t}$. On the other hand, a simple induction argument reveals that all subsets of Φ^+ obtained by removing $n-2$ roots do span $\mathfrak{t}$. For $\mathfrak{su}(n)$ we thus find that $\mathcal{J} \in C^{n-3}(\mathfrak{t})$ in agreement with the results obtained by Fourier analysis in (Coquereaux and Zuber, 2018, Coquereaux et al., 2019a), while we expect in general that $\mathcal{J} \notin C^{n-2}(\mathfrak{t})$.

When $\mathfrak{g} = \mathfrak{so}(2n)$, the subset $\{e_i \pm e_j \mid 1 \leq i < j \leq n-1\} \subset \Phi^+$, obtained by removing the $2n-2$ roots of the form $e_i \pm e_n$, does not span $\mathfrak{t}$, whereas all subsets of Φ^+ obtained by removing $2n-3$ roots do span $\mathfrak{t}$. We thus find that $\mathcal{J} \in C^{2n-4}(\mathfrak{t})$, while we expect in general that $\mathcal{J} \notin C^{2n-3}(\mathfrak{t})$.

For $\mathfrak{g} = \mathfrak{so}(2n+1)$, the subset $\{e_i,\ e_i \pm e_j \mid 1 \leq i < j \leq n-1\} \subset \Phi^+$, obtained by removing $2n-1$ roots (the short root e_n and all roots of the form $e_i \pm e_n$), does not span $\mathfrak{t}$, whereas all subsets of Φ^+ obtained by removing $2n-2$ roots do span $\mathfrak{t}$. We thus find that $\mathcal{J} \in C^{2n-3}(\mathfrak{t})$, while in general $\mathcal{J} \notin C^{2n-2}(\mathfrak{t})$. Again this result agrees with remarks in (Coquereaux et al., 2019a, sect. 4.4) based on Fourier analysis.

Repeating the argument above but replacing the short roots e_i of B_n with the long roots $2e_i$ of C_n, we find the same result for $\mathfrak{sp}(2n)$ as for $\mathfrak{so}(2n+1)$. $\triangle$

6.4.2 Stretched multiplicities and the semiclassical limit

As discussed in Chapter 5, there are various ways in which $\mathcal{J}$ can be understood to provide a "semiclassical approximation" of the multiplicities $C_{\lambda\mu}^{\nu}$. In the framework of geometric quantization developed by Guillemin and Sternberg (1982), one can realize $\mathcal{J}(\lambda, \mu; \nu)$ as the symplectic volume of the classical phase space of a certain Hamiltonian system, and $C_{\lambda\mu}^{\nu}$ as the dimension of the state space of a corresponding quantum-mechanical system. The semiclassical approximation can also be understood in more combinatorial terms: Berenstein and Zelevinsky (2001) constructed polytopes $H_{\lambda\mu}^{\nu}$ such that $C_{\lambda\mu}^{\nu}$ is equal to the number of integer points in $H_{\lambda\mu}^{\nu}$, while in (Coquereaux et al., 2019a) it was shown that $\mathcal{J}(\lambda, \mu; \nu)$ equals the volume of $H_{\lambda\mu}^{\nu}$, leading to an asymptotic equality between $\mathcal{J}(\lambda, \mu; \nu)$ and $C_{\lambda\mu}^{\nu}$ as λ, μ, ν grow large. Here we show how the box spline convolution identity for $\mathcal{J}$ provides yet one more perspective on the semiclassical approximation.

Given a triple of highest weights (λ, μ, ν) with $\lambda + \mu - \nu \in Q$, we can study the *stretched multiplicities* $C_{N\lambda\, N\mu}^{N\nu}$ for positive integers N. Much is known about the stretched multiplicities: for example, since the inequalities defining the polytope $H_{\lambda\mu}^{\nu}$ are linear in (λ, μ, ν), $C_{N\lambda\, N\mu}^{N\nu}$ is equal to the number of integer points in the dilated polytope $NH_{\lambda\mu}^{\nu}$. By standard results in the geometry of polytopes, it then follows that the stretched multiplicities for fixed (λ, μ, ν) are given by a quasi-polynomial function of N, and that

$$\mathcal{J}(\lambda, \mu; \nu) = \lim_{N \to \infty} \frac{1}{N^d} C_{N\lambda\, N\mu}^{N\nu}. \tag{6.4.1}$$

This is what is usually meant by the statement that $\mathcal{J}$ is a "semiclassical limit" of tensor product multiplicities. The main observation in this subsection is that the asymptotic relationship (6.4.1) can also be understood in terms of the following exact relationship between $\mathcal{J}$ and $C_{N\lambda\, N\mu}^{N\nu}$.

Proposition 6.4.4. *For λ, μ dominant weights of $\mathfrak{g}$,*

$$\mathcal{J}(\lambda+\rho/N, \mu+\rho/N; \gamma) = \frac{1}{N^d} b(N\gamma) * \left(\sum_{\substack{\nu \in (\lambda+\mu)+N^{-1}Q \\ \cap\ \mathcal{C}_+}} C^{N\nu}_{N\lambda\, N\mu} \sum_{w \in W} \epsilon(w)\, \delta_{w(\nu+\rho/N)} \right), \tag{6.4.2}$$

where the sum over ν runs over those elements of the scaled and translated root lattice $(\lambda+\mu)+N^{-1}Q$ that lie in the dominant Weyl chamber $\mathcal{C}_+$.

Proof. Proposition 6.2.4 gives

$$\begin{aligned} \mathcal{J}(N\lambda+\rho, N\mu+\rho; \gamma) &= b(\gamma) * \left(\sum_{\substack{\nu \in (N\lambda+N\mu)+Q \\ \cap\ \mathcal{C}_+}} C^{\nu}_{N\lambda\, N\mu} \sum_{w \in W} \epsilon(w) \delta_{w(\nu+\rho)} \right) \\ &= b(\gamma) * \left(\sum_{\substack{\nu \in (\lambda+\mu)+N^{-1}Q \\ \cap\ \mathcal{C}_+}} C^{N\nu}_{N\lambda\, N\mu} \sum_{w \in W} \epsilon(w) \delta_{w(N\nu+\rho)} \right). \end{aligned}$$

Scaling γ by N on both sides, we obtain

$$\mathcal{J}(N\lambda+\rho, N\mu+\rho; N\gamma) = b(N\gamma) * \left(\sum_{\substack{\nu \in (\lambda+\mu)+N^{-1}Q \\ \cap\ \mathcal{C}_+}} C^{N\nu}_{N\lambda\, N\mu} \sum_{w \in W} \epsilon(w) \delta_{w(\nu+\rho/N)} \right).$$

The homogeneity of $\mathcal{J}$ then gives (6.4.2). $\square$

The following immediate corollary is a distributional version of the semiclassical limit (6.4.1). Let $d\gamma$ denote Lebesgue measure on $\mathfrak{t}$ and let $\implies$ denote weak convergence of finite signed measures.

Corollary 6.4.5. *For λ, μ dominant weights of $\mathfrak{g}$,*

$$\frac{1}{N^{|\Phi^+|}} \sum_{\substack{\nu \in (\lambda+\mu)+N^{-1}Q \\ \cap\ \mathcal{C}_+}} C_{N\lambda\, N\mu}^{N\nu} \sum_{w \in W} \epsilon(w)\, \delta_{w(\nu)} \implies \mathcal{J}(\lambda, \mu; \gamma)\, d\gamma. \tag{6.4.3}$$

Proof. Letting $N \to \infty$ in (6.4.2) and recalling that $d = |\Phi^+| - r$, we obtain

$$\begin{aligned} \mathcal{J}(\lambda, \mu; \gamma) &= \lim_{N \to \infty} \frac{1}{N^d} b(N\gamma) * \left(\sum_{\substack{\nu \in (\lambda+\mu)+N^{-1}Q \\ \cap\ \mathcal{C}_+}} C_{N\lambda\, N\mu}^{N\nu} \sum_{w \in W} \epsilon(w)\, \delta_{w(\nu+\rho/N)} \right) \\ &= \lim_{N \to \infty} N^r b(N\gamma) * \left(\frac{1}{N^{|\Phi^+|}} \sum_{\substack{\nu \in (\lambda+\mu)+N^{-1}Q \\ \cap\ \mathcal{C}_+}} C_{N\lambda\, N\mu}^{N\nu} \sum_{w \in W} \epsilon(w)\, \delta_{w(\nu)} \right) \end{aligned}$$

almost everywhere $d\gamma$.

Since $||b||_{L^1} = 1$, the sequence of functions $N^r b(N\gamma)$ is an approximation to the identity as $N \to \infty$, which implies the desired result. □

6.4.3 The Littlewood–Richardson polynomial

In this subsection we take $\mathfrak{g} = \mathfrak{su}(n)$. For a compatible triple (λ, μ, ν) of dominant weights of $\mathfrak{su}(n)$, the function $P_{\lambda\mu}^{\nu}(N) = C_{N\lambda\, N\mu}^{N\nu}$ is in fact a polynomial in N, called the *Littlewood–Richardson polynomial*, rather than merely a quasi-polynomial. This polynomiality property of the stretched Littlewood–Richardson coefficients was proven by Rassart (2004) using vector partition functions and separately by Derksen and Weyman (2002) using semi-invariants of quivers. Both Derksen and Weyman (2002) and Rassart (2004) mention a third, unpublished proof by Knutson using ideas from symplectic geometry. The symplectic geometry proof follows from the "quantization commutes with reduction" theorem in geometric quantization (Knutson, 2019); see (Guillemin and Sternberg, 1982, Meinrenken and Sjaamar, 1999, Vergne, 2002) for background on this theorem. These proofs confirmed part of a conjecture by King et al. (2004), who later studied the degrees and factorization properties of the Littlewood–Richardson polynomials (King et al., 2006). Here we will use box spline deconvolution to give yet another proof of the polynomiality of $P^{\nu}_{\lambda\mu}$.

We essentially follow Rassart's recipe with different ingredients. In particular, like Rassart, we take as given the quasi-polynomiality of $P^{\nu}_{\lambda\mu}(N)$, which follows already from the construction of Berenstein–Zelevinsky polyopes. In a sense therefore we only give a novel approach to the final step required to show polynomiality, since we do not offer a new proof of quasi-polynomiality. Nonetheless, this proof illustrates a number of ideas that we will use again later, in particular the relationship between the domains of polynomiality of $C^{\nu}_{\lambda\mu}$ and those of $\mathcal{J}$, which will be important in the proof of Theorem 6.5.22.

Before giving the proof, we first recall some definitions and briefly sketch Rassart's argument. For our purposes, a *hyperplane arrangement* $\mathcal{HA}$ in a Euclidean space V is a finite union of (affine) hyperplanes. The *chamber complex* associated to $\mathcal{HA}$ is a partition of V, consisting of all connected components of $V \setminus \mathcal{HA}$ together with the common refinement of the hyperplanes. Each set in this partition is called a *chamber*. If $\mathcal{HA}$ is *centered*, meaning that all of the hyperplanes pass through the origin, then the chamber complex partitions V into convex polyhedral cones.

By the construction of Berenstein and Zelevinsky (2001), the multiplicity $C^{\nu}_{\lambda\mu}$ equals the number of integer points in the polytope $H^{\nu}_{\lambda\mu}$. By standard results about vector partition functions (see e.g. (Sturmfels, 1995)), this implies that there is a minimal centered hyperplane arrangement $\mathcal{LR}_n \subset \mathfrak{t}^3$, called the *Littlewood–Richardson arrangement*, such that $C^{\nu}_{\lambda\mu}$ is a quasi-polynomial function on each chamber of the associated chamber complex. In particular, $C^{\nu}_{\lambda\mu}$ is quasi-polynomial on rays starting at the origin, so that $P^{\nu}_{\lambda\mu}(N)$ is a quasi-polynomial function of N.

Rassart defines the *Steinberg arrangement* $\mathcal{SA}_n$ to be the union of all affine hyperplanes of the form

$$\langle \sigma(\alpha+\rho)+\tau(\beta+\rho)-(\gamma+2\rho), \theta(\omega_j)\rangle = 0, \qquad \alpha,\beta,\gamma\in\mathfrak{t}, \tag{6.4.4}$$

for $\sigma,\tau,\theta\in W=S_n$ (the symmetric group) and ω_j a fundamental weight. By analyzing the Kostant–Steinberg formula (Steinberg, 1961) for the multiplicities $C^{\nu}_{\lambda\mu}$ (see (6.5.19) below), he concludes that $C^{\nu}_{\lambda\mu}$ is in fact polynomial in (λ,μ,ν) on each connected component of $\mathfrak{t}^3\setminus\mathcal{SA}_n$. He then shows that each cone in the chamber complex of $\mathcal{LR}_n$ contains an arbitrarily large ball lying in some component of $\mathfrak{t}^3\setminus\mathcal{SA}_n$, implying that the quasi-polynomial determining $C^{\nu}_{\lambda\mu}$ on each cone must in fact coincide with one of the polynomials determining $C^{\nu}_{\lambda\mu}$ on $\mathfrak{t}^3\setminus\mathcal{SA}_n$.

The proof below is similar in structure to Rassart's argument but uses different techniques at each step. Proposition 6.3.6 plays the role of the Kostant–Steinberg formula, while a different hyperplane arrangement, which we call the ρ-shifted Duistermaat–Heckman arrangement, plays the role of $\mathcal{SA}_n$.

Theorem 6.4.6 (Rassart, Derksen–Weyman)**.** *For a fixed compatible triple (λ,μ,ν) of dominant weights of $\mathfrak{su}(n)$, $P^{\nu}_{\lambda\mu}(N)$ is a polynomial in N with degree at most $\frac{1}{2}(n-1)(n-2)$.*

Proof. We identify $\mathfrak{su}(n)$ with the space of n-by-n traceless anti-Hermitian matrices, and $\mathfrak{t}$ with the space of traceless diagonal matrices with imaginary entries. By Proposition 5.2.1, the boundaries of the domains of polynomiality of $\mathcal{J}$ are contained in the hyperplanes

$$\sum_{i\in I}\alpha_i+\sum_{j\in J}\beta_j-\sum_{k\in K}\gamma_k=0, \qquad \alpha,\beta,\gamma\in\mathfrak{t}, \tag{6.4.5}$$

where $I,J,K\subset\{1,\dots,n\}$ with $|I|=|J|=|K|$, and $\alpha=i\,\mathrm{diag}(\alpha_1,\dots,\alpha_n)$, etc. We call the hyperplane arrangement (6.4.5) the *Duistermaat–Heckman arrangement* $\mathcal{DH}_n$.

In our chosen parametrization of $\mathfrak{t}$, we find $\rho=\frac{i}{2}\mathrm{diag}(n-2j+1)_{j=1}^n$. Applying the Weyl shift to the arguments in (6.4.5) we find that as a function of $(\alpha,\beta,\gamma)\in\mathfrak{t}^3$, the regions of polynomiality of $\mathcal{J}(\alpha',\beta';\gamma')$ are cut out by the *ρ-shifted Duistermaat–Heckman arrangement* $\mathcal{DH}_n^\rho$ consisting of the hyperplanes

$$\sum_{i\in I}(\alpha_i-i)+\sum_{j\in J}(\beta_j-j)-\sum_{k\in K}(\gamma_k-k)=-\frac{1}{2}(n+1)|I|, \tag{6.4.6}$$

where again $|I|=|J|=|K|$. The arrangement $\mathcal{DH}_n^\rho$ is not centered, but its hyperplanes do all pass through the point $-(\rho,\rho,\rho)$, so that its chamber complex consists of convex polyhedral cones with this point as their common apex.

Define $P(\alpha,\beta,\gamma):=\lim_{t\to0^+}\hat{A}(\Phi^+)\mathcal{J}(\alpha',\beta';\gamma'+t\eta)$, with η defined as in Theorem 6.3.3. By Corollary 6.3.5, for (λ,μ,ν) a compatible triple of dominant weights, we have $C_{\lambda\mu}^\nu=P(\lambda,\mu,\nu)$. For (α,β,γ) not lying on one of the hyperplanes (6.4.6), $P(\alpha,\beta,\gamma)$ can be represented as a finite linear combination of $\mathcal{J}(\alpha',\beta';\gamma')$ and its derivatives. If (α,β,γ) does lie on one of the hyperplanes (6.4.6), then taking the limit in (6.3.7) we find that $P(\alpha,\beta,\gamma)$ can be represented as a finite linear combination of ϕ and its derivatives, where $\phi(\alpha,\beta,\gamma)$ is the local polynomial expression of $\mathcal{J}(\alpha',\beta';\gamma')$ on one of the adjacent connected components of $\mathfrak{t}^3\setminus\mathcal{DH}_n^\rho$, as a function of the *unshifted* triple (α,β,γ).

On every connected component of $\mathfrak{t}^3 \setminus \mathcal{DH}_n^\rho$, either $\mathcal{J}(\alpha', \beta'; \gamma')$ vanishes or it is a polynomial of degree $|\Phi^+| - (n-1) = \frac{1}{2}(n-1)(n-2)$ in the variables (α, β, γ). Moreover, by (6.3.3) the zeroth-order term of $\hat{A}(\Phi^+)$ is 1. Thus we find that on every chamber of $\mathcal{DH}_n^\rho$, $P(\alpha, \beta, \gamma)$ is a polynomial of degree at most $\frac{1}{2}(n-1)(n-2)$, and its degree is exactly $\frac{1}{2}(n-1)(n-2)$ except on chambers where $\mathcal{J}(\alpha', \beta'; \gamma')$ vanishes or which lie on the boundary of a chamber where $\mathcal{J}(\alpha', \beta'; \gamma')$ vanishes.

We have now shown the existence of a locally polynomial expression for $C_{\lambda\mu}^{\nu}$. However, we cannot yet conclude the polynomiality of $P_{\lambda\mu}^{\nu}(N)$, because the arrangement $\mathcal{DH}_n^\rho$ is not centered, so that a ray starting at the origin may pass through multiple polynomial domains. To finish the proof, we compare $\mathcal{DH}_n^\rho$ to the centered arrangement $\mathcal{LR}_n$, following a similar geometric intuition to Rassart (2004, Theorem 4.1) but using a different argument.

Consider the union of hyperplane arrangements $\mathcal{LR}_n \cup \mathcal{DH}_n^\rho$. The chamber complex of this arrangement is the common refinement of the chamber complexes of $\mathcal{LR}_n$ and $\mathcal{DH}_n^\rho$. Let C be any cone of the chamber complex of $\mathcal{LR}_n$. Then C is partitioned into finitely many convex polyhedral chambers of $\mathcal{LR}_n \cup \mathcal{DH}_n^\rho$, each of which is contained in a single chamber of $\mathcal{DH}_n^\rho$. There must be at least one chamber R of this partition that is unbounded with $\dim R = \dim C$. This region R lies in a single polynomial domain of the locally polynomial function P defined above. Thus we know that $C_{\lambda\mu}^{\nu}$ is expressed as a quasi-polynomial $q(\lambda, \mu, \nu)$ on C and by a polynomial $P(\lambda, \mu, \nu)$ on R, so that at all lattice points (λ, μ, ν) in R, $q(\lambda, \mu, \nu)$ and $P(\lambda, \mu, \nu)$ must agree. It follows that on the entire cone C, q is in fact a polynomial and equals the polynomial that expresses P locally on R.

Finally, fix a compatible triple (λ, μ, ν) of dominant weights. The points $(N\lambda, N\mu, N\nu)$ lie on a ray starting at the origin and therefore lie in a single cone of

$\mathcal{LR}_n$, so that $P^{\nu}_{\lambda\mu}(N) = q(N\lambda, N\mu, N\nu)$ for some polynomial q as above. Thus $P^{\nu}_{\lambda\mu}$ is a polynomial in N with $\deg P^{\nu}_{\lambda\mu} \leq \deg q \leq \frac{1}{2}(n-1)(n-2)$, as desired. □

6.5 Discrete convolution and deconvolution

In this section we consider a discrete analogue of Proposition 6.2.4, which yields several new results. First we use this discrete convolution identity to develop two different methods for computing the multiplicities $C^{\nu}_{\lambda\mu}$ from finitely many values of $\mathcal{J}$, which work irrespective of unimodularity. In the process, we make some novel observations on the R-polynomial introduced in Section 5.3, which turns out to be closely related to the box spline $B_c[\Phi^+]$. Next we derive some combinatorial identities involving the box spline density, which we use to relate Proposition 6.2.4 to the $\mathcal{J}$-LR relation (6.2.3) and to express the discrete convolution with b on the root lattice in terms of a finite difference operator called the box spline Laplacian. Finally we prove that for $\mathfrak{g} = \mathfrak{su}(n)$, "typical" Littlewood–Richardson coefficients can be computed as an explicit finite linear combination of values of $\mathcal{J}$.

Our starting point is the observation that if we restrict attention to $\gamma = \nu'$ with $\nu \in \lambda + \mu + Q$, then Proposition 6.2.4 says the following:

Corollary 6.5.1.

$$\sum_{\nu \in \lambda+\mu+Q} \mathcal{J}(\lambda', \mu'; \nu')\, \delta_{\nu'} = \left(\sum_{\tau \in Q} b(\tau)\, \delta_\tau \right) * \sum_{\substack{\tau \in \lambda+\mu+Q \\ \cap\ \mathcal{C}_+}} C^{\tau}_{\lambda\mu} \sum_{w \in W} \epsilon(w)\, \delta_{w(\tau')}. \tag{6.5.1}$$

Proof. All measures appearing in (6.5.1) are supported on the weight lattice $P \subset \mathfrak{t}$.

The convolution of two arbitrary measures on P may be written

$$\left(\sum_{\tau\in P} f_\tau\,\delta_\tau\right) * \left(\sum_{\tau\in P} g_\tau\,\delta_\tau\right) = \sum_{\tau\in P}\sum_{\nu\in P} f_\nu g_{\tau-\nu}\,\delta_\tau, \tag{6.5.2}$$

where $\{f_\tau, g_\tau\}_{\tau\in P}$ are some coefficients.

From Proposition 6.2.4 we have

$$\mathcal{J}(\lambda',\mu';\nu') = \sum_{\substack{\tau\in\lambda+\mu+Q\\ \cap\ \mathcal{C}_+}} C^\tau_{\lambda\mu} \sum_{w\in W} \epsilon(w) b(\nu' - w(\tau')), \tag{6.5.3}$$

so that

$$\sum_{\nu\in\lambda+\mu+Q} \mathcal{J}(\lambda',\mu';\nu')\,\delta_{\nu'} = \sum_{\nu\in\lambda+\mu+Q}\ \sum_{\substack{\tau\in\lambda+\mu+Q\\ \cap\ \mathcal{C}_+}} C^\tau_{\lambda\mu} \sum_{w\in W} \epsilon(w) b(\nu' - w(\tau'))\,\delta_{\nu'}.$$

Comparing the right-hand side above to (6.5.2) and observing that $\nu' - w(\tau')$ runs over Q for each fixed ν, we conclude (6.5.1). $\square$

Corollary 6.5.1 has the same form as Proposition 6.2.4 but involves measures that we can think of as discrete approximations of $\mathcal{J}$ and b. In Section 6.5.2 we show that this statement is actually equivalent to the $\mathcal{J}$-LR relation (6.2.3). One consequence of Corollary 6.5.1, which we show in the next subsection, is that tensor product multiplicities can be computed from the volume function in all cases, whether or not Φ^+ is unimodular. Moreover, for a given triple (λ,μ,ν), only finitely many values of $\mathcal{J}(\lambda',\mu';\gamma)$ are required, and it is not necessary to compute derivatives of $\mathcal{J}$ as in Corollary 6.3.5.

6.5.1 Finite deconvolution on a lattice

In this subsection we give two methods for computing $C_{\lambda\mu}^{\nu}$ from $\mathcal{J}$: a constructive algorithm for computing the multiplicities algebraically, and a method based on Fourier analysis leading to an integral representation of the multiplicities. These two methods are complementary, since the integral representation is easy to write down but may be difficult to evaluate analytically, while the algebraic procedure is straightforward to execute computationally but does not lead a priori to a clean formula. We also make some new observations on the vanishing locus of the R-polynomial of $\mathfrak{g}$, whose definition we recall in Definition 6.5.4 below.

We start with the algebraic approach. The key insight is that no information is lost by considering only the discrete measure in (6.5.2) rather than the full function $\mathcal{J}$. This is immediate from the following lemma, which provides a general deconvolution algorithm for finitely supported functions on a lattice. The result below must be well known, but the precise statement that we need was difficult to find in the literature, so we prove it here.

Lemma 6.5.2. *Let Λ be a lattice, let $f : \Lambda \to \mathbb{C}$ and $g : \Lambda \to \mathbb{C}$ be finitely supported functions, and suppose that f is not uniformly zero. Let $h := f * g$ be their convolution, i.e.,*

$$h(\lambda) := \sum_{\mu\in\Lambda} f(\mu)g(\lambda-\mu), \quad \lambda \in \Lambda. \tag{6.5.4}$$

Then g can be computed from f and h using finitely many arithmetic operations.

Proof. The algorithm follows a recursive procedure. At each iteration we solve a linear programming problem that allows us to determine one additional nonzero value of g. When all nonzero values of g have been computed, the procedure terminates.

First observe that $||h||_{\ell^1} = ||f||_{\ell^1}||g||_{\ell^1}$. Therefore if h is uniformly zero then g must be uniformly zero, and we are done. Suppose therefore that h is not uniformly zero.

Let $V := \Lambda \otimes \mathbb{R}$ be the real span of Λ. Given a function $\varphi : \Lambda \to \mathbb{C}$, let $\mathrm{supp}(\varphi) \subset \Lambda$ denote its support and $\mathrm{Conv}(\varphi) \subset V$ denote the convex hull of $\mathrm{supp}(\varphi)$. Since $\mathrm{supp}(h)$ is finite and non-empty, $\mathrm{Conv}(h)$ has at least one vertex. Choose such a vertex τ; we can then find a linear functional $\ell \in V^*$ such that τ maximizes ℓ uniquely over $\mathrm{Conv}(h)$. From this unique maximization property and the fact that $\mathrm{supp}(h)$ is contained in the Minkowski sum $\mathrm{supp}(f) + \mathrm{supp}(g)$, it follows that there is a unique way of writing $\tau = \tau_1 + \tau_2$ with $\tau_1 \in \mathrm{supp}(f)$ and $\tau_2 \in \mathrm{supp}(g)$. Moreover, τ_1 and τ_2 are the unique maximizers of ℓ over $\mathrm{Conv}(f)$ and $\mathrm{Conv}(g)$ respectively, so we can compute τ_1 by linear programming and τ_2 as $\tau - \tau_1$.

By the definition (6.5.4), we have

$$h(\tau) = \sum_{\mu \in \Lambda} f(\mu) g(\tau - \mu) = f(\tau_1) g(\tau_2) + \sum_{\mu \neq \tau_1} f(\mu) g(\tau - \mu).$$

But we have just argued that τ_1 is the unique $\mu \in \mathrm{supp}(f)$ such that $\tau - \mu \in \mathrm{supp}(g)$, so the last sum above vanishes, giving $h(\tau) = f(\tau_1)g(\tau_2)$. We conclude that $g(\tau_2) = h(\tau)/f(\tau_1)$.

Having determined $g(\tau_2)$, we can remove τ_2 from $\mathrm{supp}(g)$ and repeat the procedure. That is, we apply the same algorithm to the function

$$h'(\lambda) := h(\lambda) - g(\tau_2) f(\lambda - \tau_2) = (f * g')(\lambda),$$

where $g'(\lambda) = g(\lambda)$ for $\lambda \neq \tau_2$ and $g'(\tau_2) = 0$. Since $\mathrm{supp}(g)$ was assumed finite, after $|\mathrm{supp}(g)|$ iterations we obtain the zero function, at which point we know that all values of g have been determined, and the algorithm terminates. $\square$

We can identify each $f \in \ell^1(P)$ with the complex measure $\sum_{\tau \in P} f(\tau)\, \delta_\tau$, so that convolution of measures as in (6.5.2) is equivalent to convolution of functions as in (6.5.4). Recognizing that Corollary 6.5.1 then expresses the restriction of $\mathcal{J}(\lambda', \mu'; \gamma)$ to $\lambda + \mu + \rho + Q$ as a convolution of two finitely supported functions on P, we see that we have shown the following.

Theorem 6.5.3. *For any $\mathfrak{g}$ and any dominant weights λ, μ, ν, the multiplicity $C_{\lambda\mu}^{\nu}$ can be computed constructively as an arithmetic expression in finitely many values of $\mathcal{J}(\lambda', \mu'; \gamma)$ and $b(\gamma)$.*

Although Lemma 6.5.2 does provide a constructive method for computing $C_{\lambda\mu}^{\nu}$ in terms of $\mathcal{J}$, the procedure may be quite lengthy, and the form of the expression thus obtained may depend on λ and μ in a complicated way. It will not necessarily be the case that the algorithm above leads to a compact or transparent identity relating the volume function to the multiplicities. However, for many triples of highest weights of $\mathfrak{su}(n)$, a simple expression does indeed exist for $C_{\lambda\mu}^{\nu}$ as a linear combination of values of $\mathcal{J}$. We return to this topic in Section 6.5.4 below.

It should also be noted that Theorem 6.5.3 depends crucially on the fact that $\mathcal{J}(\lambda', \mu'; \gamma)$ is compactly supported. On the other hand, even though Theorem 6.3.3 and its analogue for the non-unimodular case in (Duflo and Vergne, 2011) require more information in a sense (derivatives of $\mathcal{J}$, as well as additional piecewise polynomial functions when Φ^+ is not unimodular), they also apply in a more general setting, as they do not assume compact support for the discrete measure appearing in the convolution with b.

Next we give the Fourier-analytic approach. We will derive an inversion formula involving a quantity called the *R-polynomial* of $\mathfrak{g}$, which was introduced in Section 5.3. The R-polynomial was defined in (Coquereaux and Zuber, 2018) and further studied in (Etingof and Rains, 2018, Coquereaux et al., 2019a, Coquereaux, 2019). The following definition is equivalent to (5.3.7):

Definition 6.5.4. The *R-polynomial* of $\mathfrak{g}$ is the function

$$R(x) := \sum_{\eta \in 2\pi P^\vee} j_\mathfrak{g}^{1/2}(x+\eta), \qquad x \in \mathfrak{t}, \tag{6.5.5}$$

where $P^\vee \subset \mathfrak{t}$ is the coweight lattice and $j_\mathfrak{g}^{1/2}$ is defined by (6.2.10). △

The R-polynomial can be regarded either as a trigonometric polynomial on $\mathfrak{t}$ or as a genuine polynomial on the torus $\exp(\mathfrak{t}) \cong \mathfrak{t}/2\pi P^\vee$. As a trigonometric polynomial on $\mathfrak{t}$, its coefficients are given by the values of b at points of the root lattice:

$$R(x) = \sum_{\tau \in Q} b(\tau) \cos(\langle \tau, x \rangle), \qquad x \in \mathfrak{t}. \tag{6.5.6}$$

This follows immediately from the definition (6.5.5). Since $P^\vee$ is the dual lattice of Q and $j_\mathfrak{g}^{1/2} = \mathscr{F}[b]$, the Poisson summation formula gives

$$\sum_{\eta \in 2\pi P^\vee} j_\mathfrak{g}^{1/2}(x+\eta) = \sum_{\tau \in Q} b(\tau) e^{i\langle \tau, x \rangle}, \tag{6.5.7}$$

and (6.5.6) then follows from the symmetry $b(\tau) = b(-\tau)$. Note that the sum in (6.5.6) is actually finite, since b is compactly supported.

Another expression for the R-polynomial was proved by Etingof and Rains (2018):

$$R(x) = \sum_{\kappa \in K} r_\kappa \chi_\kappa(e^x), \tag{6.5.8}$$

where $K \subset Q$ consists of all dominant elements of the root lattice that lie on the interior of the convex hull of the Weyl orbit of ρ, and $r_\kappa := \mathcal{J}(\rho, \rho, \kappa')$.

Recall that the dual of the weight lattice P is the coroot lattice $Q^\vee$. Let $|Q^\vee|$ denote the volume of a fundamental domain of $Q^\vee$. We have the following integral representation for tensor product multiplicities:

Theorem 6.5.5. *For any dominant weights λ, μ, ν of $\mathfrak{g}$,*

$$C_{\lambda\mu}^{\nu} = \frac{1}{(2\pi)^r |Q^\vee|} \int_{\mathfrak{t}/2\pi Q^\vee} \frac{1}{R(x)} \sum_{\tau \in \lambda+\mu+Q} \mathcal{J}(\lambda', \mu'; \tau')\, e^{i\langle \tau - \nu, x\rangle}\, dx. \tag{6.5.9}$$

Proof. We think of P as the character group of a torus $T \cong \mathfrak{t}/2\pi Q^\vee$. Taking Fourier transforms on both sides of (6.5.1), we obtain an equality of functions on T:

$$\begin{aligned}\sum_{\tau \in \lambda+\mu+Q} \mathcal{J}(\lambda', \mu'; \tau')\, e^{i\langle \tau', x\rangle} &= \left(\sum_{\tau \in Q} b(\tau)\, e^{i\langle \tau, x\rangle}\right)\left(\sum_{\substack{\tau \in \lambda+\mu+Q \\ \cap\, \mathcal{C}_+}} C_{\lambda\mu}^{\tau} \sum_{w \in W} \epsilon(w)\, e^{i\langle w(\tau'), x\rangle}\right) \\ &= R(x)\left(\sum_{\substack{\tau \in \lambda+\mu+Q \\ \cap\, \mathcal{C}_+}} C_{\lambda\mu}^{\tau} \sum_{w \in W} \epsilon(w)\, e^{i\langle w(\tau'), x\rangle}\right).\end{aligned}$$

Since $R(x)$ is a nonzero trigonometric polynomial, it can vanish at most on a set of measure zero. Therefore the following holds almost everywhere with respect to Haar measure on T:

$$\sum_{\substack{\tau \in \lambda+\mu+Q \\ \cap\, \mathcal{C}_+}} C_{\lambda\mu}^{\tau} \sum_{w \in W} \epsilon(w)\, e^{i\langle w(\tau'), x\rangle} = \frac{1}{R(x)} \sum_{\tau \in \lambda+\mu+Q} \mathcal{J}(\lambda', \mu'; \tau')\, e^{i\langle \tau', x\rangle}. \tag{6.5.10}$$

The left-hand side of this expression is a finite sum of bounded functions, so it is integrable on T. Therefore we can recover the coefficient $C_{\lambda\mu}^{\nu}$ by integrating the right-hand side against $e^{-i\langle\nu',x\rangle}$ with respect to the normalized Haar measure $(2\pi)^{-r}|Q^\vee|^{-1}dx$, where dx is Lebesgue measure on a fundamental domain of $2\pi Q^\vee$ in $\mathfrak{t}$. This completes the proof. □

Even though $R(x)$ may have zeros, the integral in (6.5.9) is always absolutely convergent. Indeed, (6.5.10) implies that the integrand is nonsingular, so any zeros of $R(x)$ in the denominator must be canceled in the numerator by zeros of the sum $\sum_{\tau\in\lambda+\mu+Q}\mathcal{J}(\lambda',\mu';\tau')\,e^{i\langle\tau-\nu,x\rangle}$. We cannot necessarily pull the summation out of the integral, since without these cancelations the individual terms $R(x)^{-1}\mathcal{J}(\lambda',\mu';\tau')\,e^{i\langle\tau-\nu,x\rangle}$ may fail to be integrable. If we could pull the sum out, however, then we could rewrite (6.5.9) in the much cleaner form:

$$C_{\lambda\mu}^{\nu} = \sum_{\tau\in\lambda+\mu+Q}\mathcal{J}(\lambda',\mu';\tau')\,c(\nu-\tau), \tag{6.5.11}$$

where

$$c(\tau) := \frac{1}{(2\pi)^r|Q^\vee|}\int_{\mathfrak{t}/2\pi Q^\vee}\frac{e^{-i\langle\tau,x\rangle}}{R(x)}\,dx, \qquad \tau\in Q. \tag{6.5.12}$$

In fact the function c would then provide a universal deconvolution kernel, such that

$$\left(\sum_{\tau\in Q}c(\tau)\delta_\tau\right)*\left(\sum_{\tau\in Q}b(\tau)\delta_\tau\right)=\delta_0.$$

This would be convenient, but in general it is impossible. The integral in (6.5.12) is absolutely convergent if and only if $R(x)$ is nowhere vanishing, and the following proposition shows that this fails in the non-unimodular case.

Proposition 6.5.6. *If Φ^+ is not unimodular, then $R(x)=0$ for some $x\in\mathfrak{t}$.*

Proof. By (6.5.7), the R-polynomial is the Fourier transform of $b|_Q$. Therefore, by Wiener's Tauberian theorem (Wiener, 1932), $R(x)$ is nonvanishing if and only if the translates of $b|_Q$ are dense in $\ell^1(Q)$. Since $\ell^1(Q)^* \cong \ell^\infty(Q)$, this in turn is equivalent to the statement that there is no nonzero $a \in \ell^\infty(Q)$ such that

$$\sum_{\tau \in Q} a(\tau) b(\nu - \tau) = 0$$

for all $\nu \in Q$. We know from Theorem 6.3.1 however that if Φ^+ is not unimodular then such an $a \in \ell^\infty(Q)$ does exist, so that $R(x)$ must vanish somewhere on $\mathfrak{t}$. $\square$

Therefore in the non-unimodular case, (6.5.12) is not well defined a priori, and we cannot expect a further simplification of (6.5.9).

The case of $\mathfrak{su}(n)$ is less clear, however. For $n = 2$ or 3, $R(x) \equiv 1$, giving $c(0) = 1$ and $c(\tau) = 0$ for $\tau \neq 0$, which recovers (6.3.8). For larger n it quickly becomes difficult to study the R-polynomial analytically, due to an explosion in the number of terms. Nonetheless, numerical investigations up to $n = 7$, using formulae for $R(x)$ derived by Coquereaux (2019), suggest that the R-polynomial remains strictly positive in these cases. We venture the following conjecture.

Conjecture 6.5.7. *If Φ^+ is unimodular, then $R(x) > 0$ for all $x \in \mathfrak{t}$.*

Remark 6.5.8. The proof of Proposition 6.5.6 amounts to the observation that $R(x)$ is nonvanishing if and only if $\ker(\mathrm{Res}_Q \circ \mathcal{T}) \cap \ell^\infty(Q) = \{0\}$, where $\mathcal{T}$ is the operator defined in (6.3.1) and Res_Q is restriction to Q. In the unimodular case Theorem 6.3.1 only tells us that $\ker(\mathcal{T}) = \{0\}$, which is not immediately enough to conclude that $\ker(\mathrm{Res}_Q \circ \mathcal{T}) = \{0\}$. $\triangle$

6.5.2 Identities for the box spline and the R-polynomial

We next record some identities regarding the coefficients r_κ and the values of the box spline density b at points of the root lattice Q. One implication of these results is that Corollary 6.5.1 is actually equivalent to the $\mathcal{J}$-LR relation (6.2.3). In the following subsection we will use these identities to express the discrete convolution with b in terms of finite difference operators, which will be a key ingredient in the proof of Theorem 6.5.22 in Section 6.5.4 below.

First, it is a well-known fact that the integer translates of a box spline form a partition of unity; see (Prautzsch et al., 2002, sect. 2.1). This immediately implies:

Proposition 6.5.9.

$$R(0) = \sum_{\tau \in Q} b(\tau) = 1. \tag{6.5.13}$$

Second, we can express the values $b(\tau)$ in terms of the coefficients r_κ and vice versa. For λ, μ dominant weights of $\mathfrak{g}$, let $\mathrm{mult}_\lambda(\mu)$ represent the multiplicity of the weight μ in the irreducible representation V_λ.

Proposition 6.5.10.

$$b(\tau) = \sum_{\kappa \in K} r_\kappa \mathrm{mult}_\kappa(\tau), \qquad \tau \in Q, \tag{6.5.14}$$

$$r_\kappa = \sum_{w \in W} \epsilon(w)\, b(\kappa' - w(\rho)), \qquad \kappa \in K. \tag{6.5.15}$$

Proof. Since $K \subset Q$, we can write $\chi_\kappa(e^x) = \sum_{\tau \in Q} \mathrm{mult}_\kappa(\tau) e^{i\langle \tau, x\rangle}$. Then (6.5.14) follows by equating (6.5.6) and (6.5.8). The relation (6.5.15) is immediate from Proposition 6.2.4 and the definition $r_\kappa = \mathcal{J}(\rho, \rho, \kappa')$. $\square$

Comparing (6.5.14) and (6.5.15), we find the following further relations:

Corollary 6.5.11.

$$\begin{aligned} b(\tau) &= \sum_{w\in W} \epsilon(w) \sum_{\kappa\in K} b(\kappa' - w(\rho)) \,\mathrm{mult}_\kappa(\tau), \qquad \tau \in Q, & (6.5.16)\\ r_\kappa &= \sum_{w\in W} \epsilon(w) \sum_{\xi\in K} r_\xi \,\mathrm{mult}_\xi(\kappa' - w(\rho)), \qquad \kappa \in K. & (6.5.17)\end{aligned}$$

Example 6.5.12. For $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, we have $K = Q \cap \mathrm{supp}(b) = \{0\}$ and $b(0) = 1$.

For $\mathfrak{su}(4)$ we have $K = \{0, \alpha_+\}$, where α_+ is the single root that lies in $\mathcal{C}_+$, while $Q \cap \mathrm{supp}(b)$ consists of 0 and the 12 roots. We find $r_0 = 9/24$, $r_{\alpha_+} = 1/24$, and $\mathrm{mult}_0(0) = 1$, $\mathrm{mult}_0(\alpha_+) = 0$, $\mathrm{mult}_{\alpha_+}(\alpha_+) = 1$, $\mathrm{mult}_{\alpha_+}(0) = 3$. From Proposition 6.5.10 we thus obtain $b(0) = 1/2$ and $b(\alpha) = 1/24$ for α any root, confirming that

$$\sum_{\tau\in Q} b(\tau) = \frac{1}{2} + 12 \cdot \frac{1}{24} = 1.$$

For $\mathfrak{su}(5)$, the $\tau \in Q$ with $b(\tau) \neq 0$ are 0, the 20 roots, and the Weyl orbit consisting of the 30 points of the form $\alpha + \beta$ where α and β are any two orthogonal roots. Computing as above we find $b(0) = 1/4$, $b(\alpha) = 1/30$ for α any root, and $b(\alpha + \beta) = 1/360$ for β any root orthogonal to α, again giving $\sum_{\tau\in Q} b(\tau) = 1$.

For $\mathfrak{so}(5)$, there are two simple roots: a long root α_1 and a short root α_2. We have $K = \{0, \alpha_1 + \alpha_2\}$, with $r_0 = 3/8$ and $r_{\alpha_1+\alpha_2} = 1/8$, and $Q \cap \mathrm{supp}(b)$ consists of 0 and the 4 short roots. We find $\mathrm{mult}_0(0) = 1$, $\mathrm{mult}_0(\alpha_2) = 0$, $\mathrm{mult}_{\alpha_1+\alpha_2}(0) = \mathrm{mult}_{\alpha_1+\alpha_2}(\alpha_2) = 1$. All together, this gives $b(0) = 1/2$ and $b(\alpha) = 1/8$ for α any short root, giving $\sum_{\tau\in Q} b(\tau) = 1$ as expected. △

We now recall a couple of classical multiplicity formulae that we will use below.

Let Part : $Q \to \mathbb{N}$ be the Kostant partition function, which counts the number of distinct ways that an element of the root lattice can be decomposed as a positive integer linear combination of the positive roots. Then we have the Kostant multiplicity formula (Kostant, 1959):

$$\mathrm{mult}_\lambda(\mu) = \sum_{w \in W} \epsilon(w)\,\mathrm{Part}(w(\lambda') - \mu'). \tag{6.5.18}$$

In particular, (6.5.18) implies that $\mathrm{mult}_\lambda(\mu) = 0$ whenever μ lies outside the convex hull of the Weyl orbit of λ.

We also have the Kostant–Steinberg formula for $C^\nu_{\lambda\mu}$ (Steinberg, 1961):

$$C^\nu_{\lambda\mu} = \sum_{w,w' \in W} \epsilon(ww')\,\mathrm{Part}(w(\lambda') + w'(\mu') - \nu' - \rho). \tag{6.5.19}$$

Putting (6.5.18) and (6.5.19) together, we get:

$$C^\nu_{\lambda\mu} = \sum_{w \in W} \epsilon(w)\,\mathrm{mult}_\lambda(w(\mu') - \nu'). \tag{6.5.20}$$

To end this subsection, we observe that from (6.5.20) and Proposition 6.5.10, we obtain:

$$\sum_{w \in W} \epsilon(w)\, b(w(\tau') - \nu') = \sum_{\kappa \in K} r_\kappa C^\nu_{\tau\kappa}. \tag{6.5.21}$$

Comparing this with the discrete convolution formula (6.5.1) for $\mathcal{J}$, we recover the $\mathcal{J}$-LR relation (6.2.3), and vice versa. We have thus shown that (6.5.1) and (6.2.3) are equivalent.

6.5.3 The box spline Laplacian

In this subsection we introduce a finite difference operator called the box spline Laplacian, which allows us to give a convenient new representation of the discrete convolution with b. This leads in turn to a reformulation of the convolution identity in Corollary 6.5.1 and to a representation of $\hat{A}(\Phi^+)$ as a finite difference operator on the space $D(\Phi^+)$ defined in (6.3.4), both of which will be useful in Section 6.5.4 below.

Definition 6.5.13. For τ in the weight lattice, let Δ_τ and ∇_τ denote respectively the forwards and backwards finite difference operators in the direction of τ:

$$\begin{aligned}
\Delta_\tau f(x) &:= f(x+\tau) - f(x), \\
\nabla_\tau f(x) &:= f(x) - f(x-\tau), \qquad f : \mathfrak{t} \to \mathbb{C}.
\end{aligned}$$

Define the *box spline Laplacian* $\mathcal{D}$ by

$$\mathcal{D} := \sum_{\tau \in Q} b(\tau) \nabla_\tau \Delta_\tau. \tag{6.5.22}$$

The term $b(0)\nabla_0$, while formally included, contributes nothing. We will sometimes consider Δ_τ, ∇_τ and $\mathcal{D}$ as operators on $\mathfrak{t}^3$, in which case we will always assume that they act in the third argument, so that e.g.:

$$\nabla_\tau \mathcal{J}(\lambda', \mu'; \nu') = \mathcal{J}(\lambda', \mu'; \nu') - \mathcal{J}(\lambda', \mu'; \nu' - \tau).$$

△

The significance of $\mathcal{D}$ is that we can use it to represent the discrete convolution with b as a finite difference operator.

Proposition 6.5.14. *Let $m : Q \to \mathbb{C}$ and let*

$$m_b(\nu) := \sum_{\tau \in Q} b(\tau) m(\nu - \tau), \qquad \nu \in Q$$

be its discrete convolution with b on Q. Then

$$m_b(\nu) = \left(1 + \frac{1}{2}\mathcal{D}\right) m(\nu), \qquad \nu \in Q. \tag{6.5.23}$$

Proof. Proposition 6.5.9 gives:

$$\begin{aligned}
m_b(\nu) &= b(0)m(\nu) + \sum_{\tau \neq 0} b(\tau) m(\nu - \tau) \\
&= \Big(1 - \sum_{\tau \neq 0} b(\tau)\Big) m(\nu) + \sum_{\tau \neq 0} b(\tau) m(\nu - \tau) \\
&= m(\nu) - \sum_{\tau \in Q} b(\tau)(m(\nu) - m(\nu - \tau)) \\
&= \Big(1 - \sum_{\tau \in Q} b(\tau) \nabla_\tau\Big) m(\nu).
\end{aligned}$$

By the reflection symmetry $b(\tau) = b(-\tau)$ and the fact that $\nabla_0 = 0$, we can rewrite this last expression to get

$$m_b(\nu) = \Big(1 - \frac{1}{2} \sum_{\tau \in Q} b(\tau)(\nabla_\tau + \nabla_{-\tau})\Big) m(\nu).$$

The claim then follows from the observation that

$$(\nabla_\tau + \nabla_{-\tau}) m(\nu) = -m(\nu + \tau) + 2m(\nu) - m(\nu - \tau) = -\nabla_\tau \Delta_\tau m(\nu).$$

□

Proposition 6.5.14 leads to a useful reformulation of the discrete convolution identity in Corollary 6.5.1. For any $\lambda \in P$, not necessarily dominant, let $\lambda_+ \in \mathcal{C}_+$ denote the unique dominant element in the Weyl orbit of λ, and let $\lambda_* := (\lambda')_+ - \rho$. Define a function $\mathscr{C}$ on P^3 by

$$\mathscr{C}(\lambda, \mu, \nu) := C_{\lambda_* \mu_*}^{\nu_*}, \tag{6.5.24}$$

observing that for λ, μ, ν all dominant, $\mathscr{C}(\lambda, \mu, \nu) = C_{\lambda\mu}^{\nu}$. Combining Corollary 6.5.1 and Proposition 6.5.14, we get:

Corollary 6.5.15. *For (λ, μ, ν) a compatible triple of dominant weights of $\mathfrak{g}$,*

$$\mathcal{J}(\lambda', \mu'; \nu') = \left(1 + \frac{1}{2}\mathcal{D}\right) \mathscr{C}(\lambda, \mu, \nu). \tag{6.5.25}$$

Recall that in the expression above, $\mathcal{D}$ acts in the third argument of $\mathscr{C}$.

Example 6.5.16. For $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, $\mathcal{D} = 0$, so that (6.5.25) just reads $\mathcal{J}(\lambda', \mu'; \nu') = C_{\lambda\mu}^{\nu}$, which we already know from (6.3.8).

For $\mathfrak{su}(4)$, from the values of b computed in Example 6.5.12, we find $\mathcal{D} = \frac{1}{12} \sum_{\alpha \in \Phi^+} \nabla_\alpha \Delta_\alpha$, and (6.5.25) reads

$$\mathcal{J}(\lambda', \mu'; \nu') = \left(1 + \frac{1}{24} \sum_{\alpha \in \Phi^+} \nabla_\alpha \Delta_\alpha\right) C_{\lambda\mu}^{\nu}, \tag{6.5.26}$$

which was already observed by Coquereaux and Zuber (2018, sect. 4.2.2).

For $\mathfrak{su}(5)$, again using the values of b computed in Example 6.5.12, after some manipulation we find

$$\mathcal{D} = \frac{1}{15} \sum_{\alpha \in \Phi^+} \left(\nabla_\alpha \Delta_\alpha + \frac{1}{12} \sum_{\substack{\beta \in \Phi^+ \\ \langle \beta, \alpha \rangle = 0}} \left(\nabla_{\alpha+\beta} \Delta_{\alpha+\beta} + \nabla_{\alpha-\beta} \Delta_{\alpha-\beta} \right) \right). \tag{6.5.27}$$

For $\mathfrak{so}(5)$ we find $\mathcal{D} = \frac{1}{4}(\nabla_\alpha \Delta_\alpha + \nabla_\beta \Delta_\beta)$, where α and β are the two short positive roots, which are orthonormal vectors in $\mathfrak{t} \cong \mathbb{R}^2$. Thus $\mathcal{D}$ is just 1/4 times the usual discrete Laplacian in two dimensions. △

Remark 6.5.17. Proposition 6.5.14 might lead one to hope that the discrete convolution with b could be inverted via the Neumann series

$$\Big(1 + \frac{1}{2}\mathcal{D}\Big)^{-1} = 1 - \frac{1}{2}\mathcal{D} + \frac{1}{4}\mathcal{D}^2 - \cdots, \tag{6.5.28}$$

but in general this is not the case. For (6.5.28) to hold on any given Banach space, the series on the right-hand side must converge in the operator norm, which occurs if and only if the spectral radius of $\mathcal{D}$ is strictly less than 2. Except in the trivial cases of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ where $\mathcal{D} = 0$, one can show that this fails on many spaces of interest such as $\ell^p(Q)$, $1 \le p \le \infty$, as well as the space $c_0(Q)$ of functions on Q that decay to zero at infinity. However, (6.5.28) *does* hold on the space $D(\Phi^+)$ defined in (6.3.4): $(1 + \frac{1}{2}\mathcal{D})$ is invertible on $D(\Phi^+)$ due to Theorem 6.3.2, and since $D(\Phi^+)$ is a space of polynomials of degree $d = |\Phi^+| - r$, the following lemma shows that the Neumann series truncates after $\lfloor d/2 \rfloor$ terms. △

Lemma 6.5.18. *Let p be a polynomial on $\mathfrak{t}$. If $\deg p \ge 2$ then $\deg \mathcal{D}p \le \deg p - 2$, and if $\deg p < 2$ then $\mathcal{D}p = 0$.*

Proof. This is immediate from the facts that

$$\deg \nabla_\tau p = \deg \Delta_\tau p \le \max(\deg p - 1, 0)$$

and that both ∇_τ and Δ_τ annihilate the constants. □

Therefore, combining Proposition 6.5.14, Theorem 6.3.2, and (6.5.28), we find the following representation of $\hat{A}(\Phi^+)$ as a finite difference operator on $D(\Phi^+)$.

Proposition 6.5.19. *For $p \in D(\Phi^+)$,*

$$\hat{A}(\Phi^+)p = \sum_{k=0}^{\lfloor d/2 \rfloor} \Big(-\frac{1}{2}\mathcal{D}\Big)^k p. \tag{6.5.29}$$

6.5.4 An explicit algebraic formula for $\mathfrak{g} = \mathfrak{su}(n)$

In this subsection we take $\mathfrak{g} = \mathfrak{su}(n)$. We show that, given an assumption on the highest weights (λ, μ, ν) that holds for "typical" triples, we can write $C_{\lambda\mu}^{\nu}$ explicitly as a linear combination of values of $\mathcal{J}(\lambda', \mu'; \gamma)$ at lattice points. This result can be thought of as a partial inverse to the $\mathcal{J}$-LR relation (6.2.3).

Before stating the theorem, we need to give another definition.

Definition 6.5.20. Let $d = |\Phi^+| - r = \frac{1}{2}(n-1)(n-2)$. We will say that a compatible triple (λ, μ, ν) of dominant weights of $\mathfrak{su}(n)$ is *shielded* if the points $\nu' + \lfloor d/2 \rfloor w(\rho)$, $w \in W$ are dominant and all lie in the interior of a single polynomial domain of $\mathcal{J}(\lambda', \mu'; \gamma)$. △

Remark 6.5.21. For $n = 2$, all compatible triples are shielded. For $n > 2$ there are infinitely many non-shielded triples, but shielded triples are "typical" in the following sense. The non-analyticities of $\mathcal{J}$ are contained within a centered hyperplane arrangement in $\mathfrak{t}^3$, and any compatible triple (λ, μ, ν) such that (λ', μ', ν') lies further than a distance $\lfloor d/2 \rfloor |\rho|$ from each of these hyperplanes is shielded. (In fact this condition is much stronger and excludes many shielded triples.) In particular, as λ and μ both grow large, the ratio

$$\frac{\#\{\, \nu \mid C_{\lambda\mu}^{\nu} \neq 0,\ (\lambda, \mu, \nu) \text{ shielded} \,\}}{\#\{\, \nu \mid C_{\lambda\mu}^{\nu} \neq 0 \,\}}$$

goes to 1. △

The main result of this subsection is then:

Theorem 6.5.22. *For (λ, μ, ν) a shielded triple of dominant weights of $\mathfrak{su}(n)$,*

$$C_{\lambda\mu}^{\nu} = \sum_{k=0}^{\lfloor d/2 \rfloor} \Big(-\frac{1}{2}\mathcal{D}\Big)^{k} \mathcal{J}(\lambda', \mu'; \nu'). \tag{6.5.30}$$

Proof. The polynomial domains of $\mathcal{J}$ in $\mathfrak{t}^3$ are cones with apex at the origin. This implies that for (λ, μ, ν) shielded, there exists a polynomial q on $\mathfrak{t}^3$ and an open cone R with apex $-(\rho, \rho, \rho) \in \mathfrak{t}^3$, such that $\mathcal{J}(\alpha', \beta'; \gamma') = q(\alpha, \beta, \gamma)$ for all $(\alpha, \beta, \gamma) \in R$, and also $\mathcal{J}(\lambda', \mu'; \nu' + \tau) = q(\lambda, \mu, \nu + \tau)$ for all $\tau \in Q$ lying in the convex hull of the Weyl orbit of $\lfloor d/2 \rfloor \rho$.

Since all points $(\alpha', \beta', \gamma')$ with $(\alpha, \beta, \gamma) \in R$ lie on the interior of the same polynomial domain of $\mathcal{J}$, Proposition 6.3.6 yields a polynomial $p(\alpha, \beta, \gamma) := \hat{A}(\Phi^+) q(\alpha, \beta, \gamma)$ such that $C_{\eta\xi}^{\theta} = p(\eta, \xi, \theta)$ for all compatible triples $(\eta, \xi, \theta) \in R$. Moreover, by (6.5.25), at any such point we have

$$q(\eta, \xi, \theta) = \Big(1 + \frac{1}{2}\mathcal{D}\Big)\mathscr{C}(\eta, \xi, \theta),$$

and if (η, ξ, θ) lies farther than a distance $|\rho|$ from the boundary of R, then

$$\Big(1 + \frac{1}{2}\mathcal{D}\Big)\mathscr{C}(\eta, \xi, \theta) = \Big(1 + \frac{1}{2}\mathcal{D}\Big)p(\eta, \xi, \theta).$$

Therefore we must have the equality of polynomials $q = (1 + \frac{1}{2}\mathcal{D})p$, as this holds at all compatible triples in R that lie sufficiently far from the boundary.

By Lemma 6.5.18, it is impossible that $\mathcal{D}f = -2f$ for a nonzero polynomial

f, which means that $1 + \frac{1}{2}\mathcal{D}$ is injective on polynomials. Also by Lemma 6.5.18, $\mathcal{D}^{\lfloor \deg f/2 \rfloor + 1} f = 0$, so that the Neumann series (6.5.28) truncates, and we find

$$C_{\lambda\mu}^{\nu} = p(\lambda, \mu, \nu) = \sum_{k=0}^{\lfloor d/2 \rfloor} \Big(-\frac{1}{2}\mathcal{D}\Big)^{k} q(\lambda, \mu, \nu).$$

Each term $(-\frac{1}{2}\mathcal{D})^k q(\lambda, \mu, \nu)$ in the sum above is a linear combination of the values $q(\lambda, \mu, \nu + \tau)$, where $\tau \in Q$ lies in the convex hull of the Weyl orbit of $k\rho$. Since $q(\lambda, \mu, \nu+\tau) = \mathcal{J}(\lambda', \mu'; \nu'+\tau)$ for all such τ, this gives the desired result (6.5.30). $\square$

Remark 6.5.23. The polynomiality property of $C_{\lambda\mu}^{\nu}$ for $\mathfrak{su}(n)$, discussed in Section 6.4.3, is crucial in the proof above. For arbitrary $\mathfrak{g}$ we know that $C_{\lambda\mu}^{\nu}$ is expressed by a piecewise quasi-polynomial function, but the Neumann series (6.5.28) for $1+\frac{1}{2}\mathcal{D}$ may not converge when applied to quasi-polynomials rather than genuine polynomials. This is why we need to take $\mathfrak{g} = \mathfrak{su}(n)$ in Theorem 6.5.22.

The same concerns about convergence motivate the notion of a shielded triple. Since (6.5.28) may also fail to converge when applied to piecewise polynomials, Definition 6.5.20 is designed to ensure that we only ever need to apply $\mathcal{D}$ to a single local polynomial expression for $\mathcal{J}$. $\triangle$

Example 6.5.24. Comparing with Example 6.5.16, we find that for (λ, μ, ν) a shielded triple of $\mathfrak{su}(4)$,

$$C_{\lambda\mu}^{\nu} = \left(1 - \frac{1}{24} \sum_{\alpha \in \Phi^+} \nabla_\alpha \Delta_\alpha \right) \mathcal{J}(\lambda', \mu', \nu'). \tag{6.5.31}$$

Note the similarity to equation (6.3.9) computing $C_{\lambda\mu}^{\nu}$ from $\mathcal{J}$ using $\hat{A}(\Phi^+)$ rather than finite difference operators. In light of Proposition 6.5.19, this comes as no surprise.

Similarly, for (λ, μ, ν) a shielded triple of $\mathfrak{su}(5)$, we have:

$$C_{\lambda\mu}^{\nu} = \sum_{k=0}^{3} \left[-\frac{1}{30} \sum_{\alpha\in\Phi^+} \left(\nabla_\alpha \Delta_\alpha + \frac{1}{12} \sum_{\substack{\beta\in\Phi^+ \\ \langle\beta,\alpha\rangle=0}} \left(\nabla_{\alpha+\beta}\Delta_{\alpha+\beta} + \nabla_{\alpha-\beta}\Delta_{\alpha-\beta} \right) \right) \right]^k \mathcal{J}(\lambda', \mu', \nu'). \tag{6.5.32}$$

$\triangle$

6.6 Weight multiplicities

We conclude this chapter by showing how the ideas of the preceding sections can also be used to compute the weight multiplicities of irreducible representations of $\mathfrak{g}$, leading in particular to a formula for Kostka numbers that is analogous to Theorem 6.5.22. The theory for weight multiplicities is simpler than for tensor product multiplicities, so we only sketch the proofs, as they amount to simplified versions of arguments that we have already given above.

From the form (6.5.20) of the Kostant–Steinberg formula we find that for sufficiently large $k \in \mathbb{N}$,

$$\mathrm{mult}_\lambda(\mu) = C_{\lambda\,(k\rho)}^{\mu+k\rho}. \tag{6.6.1}$$

Thus one can think of the weight multiplicities as degenerations of the tensor product multiplicities that depend only on two parameters rather than three. All of the constructions that we have developed for tensor product multiplicities have analogues in this setting.

First we define the analogue of the volume function,

$$\mathcal{I}(\alpha;\beta) := \frac{\Delta_{\mathfrak{g}}(\alpha)}{\Delta_{\mathfrak{g}}(\rho)} \mathscr{F}^{-1}\big[\mathcal{H}(i\,\cdot\,,\alpha)\big](\beta), \qquad \alpha,\beta \in \mathfrak{t}. \tag{6.6.2}$$

As a function of $(\alpha,\beta) \in \mathfrak{t}^2$, $\mathcal{I}$ is a homogeneous piecewise polynomial of degree $d = |\Phi^+| - r$. We will usually fix α and consider $\mathcal{I}(\alpha;\beta)$ as a function of β. When $\Delta_{\mathfrak{g}}(\alpha) \neq 0$, $\mathcal{I}(\alpha;\beta)$ is the density of the Duistermaat–Heckman measure for the action of the maximal torus on the coadjoint orbit $\mathcal{O}_\alpha$. Equivalently, $(\Delta_{\mathfrak{g}}(\rho)/\Delta_{\mathfrak{g}}(\alpha))\mathcal{I}(\alpha;\beta)$ is the probability density for the orthogonal projection onto $\mathfrak{t}$ of a uniform random element of $\mathcal{O}_\alpha$. When $\mathfrak{g} = \mathfrak{su}(n)$, this is the joint probability density of the diagonal entries of a uniform random traceless Hermitian matrix with eigenvalues $(\alpha_1, \dots, \alpha_n)$; see (Coquereaux et al., 2019b, sect. 2.3).

Just like tensor product multiplicities, each weight multiplicity $\mathrm{mult}_\lambda(\mu)$ equals the number of integer points in a certain polytope (Billey et al., 2004, Bliem, 2008), and it can be shown by the method of Coquereaux et al. (2019a, Proposition 3) that the d-dimensional volume of this polytope is equal to $\mathcal{I}(\lambda;\mu)$.

Taking Fourier transforms on both sides of the Kirillov character formula (6.2.12), we obtain an analogue of Proposition 6.2.4:

$$\mathcal{I}(\lambda';\beta) = b(\beta) * \sum_{\mu\in\lambda+Q} \mathrm{mult}_\lambda(\mu)\,\delta_\mu. \tag{6.6.3}$$

This formula appeared in the literature at least as early as (Dooley et al., 1993, eqn. 5.3). Restricting to the shifted root lattice $\lambda + Q$, we obtain a discrete version,

$$\sum_{\mu\in\lambda+Q} \mathcal{I}(\lambda';\mu)\,\delta_\mu = \left(\sum_{\tau\in Q} b(\tau)\,\delta_\tau\right) * \sum_{\tau\in\lambda+Q} \mathrm{mult}_\lambda(\tau)\,\delta_\tau, \tag{6.6.4}$$

and by the same Fourier-analytic method as in Theorem 6.5.5 we find:

Theorem 6.6.1. *For any dominant weights λ, μ of $\mathfrak{g}$,*

$$\mathrm{mult}_\lambda(\mu) = \frac{1}{(2\pi)^r |Q^\vee|} \int_{\mathfrak{t}/2\pi Q^\vee} \frac{1}{R(x)} \sum_{\tau \in \lambda + Q} \mathcal{I}(\lambda'; \tau)\, e^{i\langle \tau - \mu, x\rangle}\, dx. \tag{6.6.5}$$

It also follows that $\mathrm{mult}_\lambda(\mu)$ can be computed algebraically from $\mathcal{I}$ using the algorithm of Lemma 6.5.2.

We now take $\mathfrak{g} = \mathfrak{su}(n)$. In this case the weight multiplicities are usually called Kostka numbers, and we write them as K_λ^μ rather than $\mathrm{mult}_\lambda(\mu)$. From Theorem 6.3.3 we have:

Corollary 6.6.2. *Let $\mathfrak{g} = \mathfrak{su}(n)$ and choose η as in Theorem 6.3.3. For dominant weights λ, μ with $\mu \in \lambda + Q$,*

$$K_\lambda^\mu = \lim_{t \to 0^+} \hat{A}(\Phi^+)\, \mathcal{I}(\lambda'; \mu + t\eta), \tag{6.6.6}$$

where the operator $\hat{A}(\Phi^+)$ acts in the second argument of $\mathcal{I}$.

By (Billey et al., 2004, Theorem 5.1), there is a piecewise polynomial function P of degree $d = \frac{1}{2}(n-1)(n-2)$ on $\mathfrak{t}^2$ such that $K_\lambda^\mu = P(\lambda, \mu)$. Moreover, (Billey et al., 2004, Theorem 3.2) shows that the domains of polynomiality of P are the same as those of $\mathcal{I}$. This leads to the following analogue of Definition 6.5.20:

Definition 6.6.3. We will say that a pair of weights (λ, μ) of $\mathfrak{su}(n)$ is *shielded* if λ is dominant, $\mu \in \lambda + Q$, and the points $\mu + \lfloor d/2 \rfloor w(\rho)$, $w \in W$ all lie in the interior of a single polynomial domain of $\mathcal{I}(\lambda'; \beta)$. $\triangle$

Then the same technique used to prove Theorem 6.5.22 gives:

Theorem 6.6.4. *For* (λ, μ) *a shielded pair of weights of* $\mathfrak{su}(n)$,

$$K_\lambda^\mu = \sum_{k=0}^{\lfloor d/2 \rfloor} \Big(-\frac{1}{2}\mathcal{D}\Big)^k \mathcal{I}(\lambda'; \mu). \tag{6.6.7}$$

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