

The Incompressible Navier-Stokes Equations: Two Regularity Criteria

by

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CHAPTER ONE

Introduction

In this thesis we discuss the incompressible Navier-Stokes equations in d spatial dimension with unit viscosity and zero external force:

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad (1.1)$$

for $x \in \Omega$ and $t > 0$ with the initial condition

$$u(0, x) = a(x) \quad \text{in } \Omega. \quad (1.2)$$

Here u is the velocity and p is the pressure. We consider several kinds of domains: the whole space $\Omega := \mathbb{R}^d$, the half space $\Omega := \mathbb{R}_+^d$, the unit ball $\Omega := B$, and the half unit ball $\Omega := B^+$.

For both cases with flat boundaries, namely $\Omega = \mathbb{R}_+^d$ and B^+ , we assume that u satisfies the zero Dirichlet boundary condition:

$$u = 0 \quad \text{on } \{x_d = 0\} \cap \partial\Omega. \quad (1.3)$$

For the three dimensional Navier-Stokes equations, the global existence of strong solutions has long been an outstanding fundamental problem in fluid dynamics and is still widely open. The local solvability is well known under the assumption that the initial data a is sufficiently regular (see [25, 17, 51, 28]). The local solution is indeed unique and smooth in both spatial and time variables.

This dissertation is concerned with two important types of weak solutions. The first type of weak solutions we are interested in is the so-called *suitable weak solutions*, which originates from the study of partial regularity by Scheffer in a series of papers [40, 41, 42]. In three space dimensions, he established various partial regular-

ity results for weak solutions satisfying the so-called local energy inequality. Later in a celebrated paper [4], Caffarelli, Kohn, and Nirenberg first introduced the notation of *suitable weak solutions*. They called a pair (u, p) a suitable weak solution if u has finite energy norm, p belongs to the Lebesgue space $L_{5/4}$, and (u, p) is a pair of weak solution to the Navier-Stokes equations and satisfies a local energy inequality. They proved that for any suitable weak solution (u, p) , there exists an open subset in which the velocity field u is Hölder continuous, and the complement of it has zero 1D Hausdorff measure. In [34], with zero external force and assuming $p \in L_{3/2}$, Lin gave a more direct and concise proof for Caffarelli, Kohn, and Nirenberg's result. A detailed treatment was later given by Ladyzhenskaya and Seregin in [31]. Thereafter, the partial regularity result for the 3D time-dependent Navier-Stokes equations was extended up to the flat boundary by Seregin [44] and to the C^2 boundary by Seregin, Shilkin, and Solonnikov [47]. The key step in the proofs of partial regularity results is to establish certain ε -regularity criteria. That is, intuitively speaking, if some scale invariant quantities are small then the solution is locally regular. Such results played a crucial part in the proof of [12]. For the higher dimensional boundary partial regularity cases, Dong and Gu [9] studied 4D time-dependent and 6D stationary incompressible Navier-Stokes equations. They proved that in both cases, the singular points sets have zero 2D Hausdorff measure up to the boundary. For the 4D time-dependent case, they obtained two boundary ε -regularity criteria [9, Theorems 1.1 and 1.2]. In Sections 3.3 and 3.4, we will extend [9, Theorem 1.2] to higher dimensions assuming the boundedness of certain norms of u and p .

In [52] Vasseur proved the following interior ε -regularity criterion, which provided an alternative proof of the well-known partial regularity result for the 3D incompressible Navier-Stokes equations proved by Caffarelli, Kohn, and Nirenberg [4]. His proof is based on the De Giorgi iteration argument originally for elliptic

equations in divergence form.

Theorem 1.0.1 (Vasseur [52]). *For any $\tilde{q} > 1$, there exists an $\varepsilon_0 = \varepsilon_0(\tilde{q}) > 0$ such that if (u, p) is a pair of suitable weak solution to (1.1)-(1.3) in Q and satisfies*

$$\sup_{t \in [-1, 0]} \int_B |u(t, x)|^2 dx + \int_Q |\nabla u|^2 dx dt + \int_{-1}^0 \|p\|_{L^{\tilde{q}}_1(B)}^{\tilde{q}} dt \leq \varepsilon_0,$$

then u is regular in $\overline{Q(1/2)}$.

Choi and Vasseur extended this result up to $\tilde{q} = 1$ in [6, Proposition 2.1] with an additional condition on the maximal function of ∇u . In [19], Phuc and Guevara further refined this result to the case with simply $\tilde{q} = 1$. Their proof exploits fractional Sobolev spaces of negative order and an inductive argument in [4] and [54]. In Chapter 2, we extend this result to the half space and use that to prove another ε -regularity criterion not involving the gradient of velocity.

The second type of weak solutions is called *Leray-Hopf* weak solutions. See Section 3.2.1 for the notation and definition. In the innovative works of Leray and Hopf, it is proved that for any divergence-free vector field $a \in L_2$, there exists at least one Leray-Hopf weak solution of the Cauchy problem (1.1)-(1.2) on $(0, \infty) \times \mathbb{R}^d$. Nevertheless, the uniqueness and regularity of Leray-Hopf weak solutions remain open. There have been extensive literatures on conditional results under various criteria, among which the most well-known condition is the so-called Ladyzhenskaya-Prodi-Serrin condition: for some $T > 0$,

$$u \in L^t_r L^x_q(\mathbb{R}^{d+1}_T), \tag{1.4}$$

where the pair (r, q) satisfies

$$2/r + d/q \leq 1, \quad q \in (d, \infty]. \quad (1.5)$$

Prodi [39] and Serrin [48] independently proved the uniqueness of Leray-Hopf weak solutions under the conditions (1.4) and (1.5). Later the smoothness was obtained by Ladyzhenskaya [30]. For further results, we hereby refer the reader to [16, 49, 50, 5] and references therein. We are intrigued by the borderline case $(r, q) = (\infty, d)$ which is not included in (1.5). This subtle case is of much interest since we cannot obtain a proof from usual methods using the local smallness of certain norms of u , which are invariant under the natural scaling

$$u(t, x) \rightarrow \lambda u(\lambda^2 t, \lambda x), \quad p(t, x) \rightarrow \lambda^2 p(\lambda^2 t, \lambda x). \quad (1.6)$$

For $d = 3$, this case was studied by Escauriaza, Seregin, and Šverák in a remarkable paper [12]. The main result of [12] is the following theorem.

Theorem 1.0.2 (Escauriaza, Seregin, and Šverák). *Let $d = 3$. Suppose that u is a Leray-Hopf weak solution of the Cauchy problem (1.1)-(1.3) in $(0, T) \times \mathbb{R}^3$ and u satisfies the condition (1.4) with $(r, q) = (\infty, 3)$. Then $u \in L_5((0, T) \times \mathbb{R}^3)$, and hence it is smooth and unique in $(0, T) \times \mathbb{R}^3$.*

In Chapter 3, we will show results on how to extend this theorem to high dimensions and domains with boundary.

The rest of the dissertation is organized as follows: In Chapter 2, we first extend an ε -regularity criterion by Vasseur [52] to the boundary case and furthermore prove a regularity criterion without involving the gradient of u . In Chapter 3, we first establish the ε -regularity criteria in both interior and boundary case. Then we prove

the regularity criterion for Leray-Hopf weak solutions in critical Lebesgue spaces. We will introduce more motivations and details of the ε -regularity criteria along with our main results in the introduction sections of the following chapters.

Throughout the dissertation, various constants are denoted by N in general, which may vary from line to line. The expression $N = N(\cdots)$ means that the given constant N depends only on the contents in the parentheses.

CHAPTER TWO

Regularity criteria for the Navier-Stokes equations in three space dimensions

2.1 Introduction and main results

In this chapter we discuss the 3-dimensional incompressible Navier-Stokes equations with unit viscosity and zero external force:

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0,$$

where $u = (u^1(t, x), u^2(t, x), u^3(t, x))$ is the velocity field and $p = p(t, x)$ is the pressure. We consider local problems: $(t, x) \in Q$ or Q^+ , where Q and Q^+ denote the unit cylinder and unit half-cylinder respectively. For the half cylinder case, we assume that u satisfies the zero Dirichlet boundary condition:

$$u = 0 \quad \text{on } \{x_3 = 0\} \cap \partial Q^+.$$

We are concerned with boundary ε -regularity criteria for suitable weak solutions for the 3D Navier-Stokes equations. We first prove a boundary version of Theorem 1.0.1. Namely,

Theorem 2.1.1. *For any $\tilde{q} > 1$, there exists a universal constant $\varepsilon_0 = \varepsilon_0(\tilde{q}) > 0$ such that if (u, p) is a pair of suitable weak solution to (1.1)-(1.3) in Q^+ with $p \in L_{\tilde{q}}^t L_1^x(Q^+)$ and satisfies*

$$\sup_{t \in [-1, 0]} \int_{B^+} |u(t, x)|^2 dx + \int_{Q^+} |\nabla u|^2 dx dt + \int_{-1}^0 \|p\|_{L_1^x(B^+)}^{\tilde{q}} dt \leq \varepsilon_0,$$

then u is regular in $\overline{Q^+(1/2)}$.

The condition $\tilde{q} > 1$ is required when we apply the coercive estimate for the linear Stokes system to estimate the pressure term. At the time of this writing, it is

not clear to us whether it is possible to take $\tilde{q} = 1$ as in the interior case.

Theorem 2.1.1 can be used to give a new proof of the boundary partial regularity result by Seregin [44]. Another consequence of the theorem is the following boundary regularity criterion, which does not involve ∇u .

Theorem 2.1.2. *For any $q > 5/2$ and $\tilde{q} > 1$, there exists a universal constant $\varepsilon_0 = \varepsilon_0(q, \tilde{q})$ such that if (u, p) is a pair of suitable weak solution to (1.1)-(1.3) in Q^+ with $p \in L_{\tilde{q}}^t L_1^x(Q^+)$ and satisfies*

$$\|u\|_{L_q(Q^+)} + \|p\|_{L_{\tilde{q}}^t L_1^x(Q^+)} < \varepsilon_0,$$

then u is regular in $\overline{Q^+(1/2)}$.

The above theorem is a special case of Theorem 2.4.1, which will be proved in Section 2.4. The corresponding interior result when $q > 20/7$ was proved recently in [19] by viewing the “head pressure” $|u|^2/2 + p$ as a signed distribution, which belongs to certain fractional Sobolev spaces of negative order. This answered a question raised by Kukavica in [29] about whether one can lower the exponent 3 in the original ε -regularity criterion in [4]. See also more recent [22] for an extension to the case when $q > 5/2$ with an application to the estimate of box dimensions of singular sets for the Navier–Stokes equations. We refer the reader to [21, 27] and references therein for various interior and boundary ε -regularity criteria for the Navier–Stokes equations.

The proofs of both Theorems 2.1.1 and 2.1.2 rely on iteration arguments. Compared to the argument in [52], our proof of Theorem 2.1.1 is much shorter and, in the conceptual level, closer to the original argument in [4]. Instead of the fractional Sobolev spaces used in [19], which does not seem to work for the boundary case,

we consider scale invariant quantities in the usual mixed-norm Lebesgue spaces, and we apply a decomposition of the pressure due to Seregin [44]. We adopt some ideas in [11, 9] on showing uniform decay rates of scale invariant quantities by induction. More precisely, we use different induction step lengths when iterating between different scale invariant quantities associated with the energy norm and the pressure respectively. In the last step, we use parabolic regularity to further improve the estimate of mean oscillation of u and conclude the Hölder continuity according to Campanato's characterization of Hölder continuous functions. By a minor modification on the proof of Theorem 2.1.1 to transform to the interior case, we also get a different proof of Theorem 1.0.1 with refined condition $\tilde{q} = 1$ obtained in [19]. The proof of Theorem 2.1.2 uses a delicate interpolation argument. We treat all the terms on the right-hand side of the generalized energy inequality in a consistent way such that they are all interpolated by the energy norms and the mixed-norm which is assumed to be small in the condition. By fitting the exponents of those energy norms slightly less than 2, we spare some space that we can borrow to use Young's inequality and proceed with an iteration to obtain the desired results.

The remaining part of the chapter is organized as follows. In the next section, we introduce some notation and the definition of suitable weak solutions to the Navier-Stokes equations. The proof of Theorem 2.1.1 is given in Section 2.3. In Sections 2.4 and 2.5, we show the proof of regularity criteria not involving ∇u for the boundary case and the interior case, respectively.

2.2 Preliminaries

2.2.1 Notation

Let $T > 0$, Ω be a domain in $\mathbb{R}^3$, $\Gamma \in \partial\Omega$, and $\Omega_T := (0, T) \times \Omega$ with the parabolic boundary

$$\partial_p \Omega_T = [0, T) \times \partial\Omega \cup \{t = 0\} \times \Omega.$$

In particular, we denote $\mathbb{R}_T^{d+1} = (0, T) \times \mathbb{R}^d$. We denote $\dot{C}_0^\infty(\Omega, \Gamma)$ to be the space of divergence-free infinitely differentiable vector fields which vanishes near Γ . Let $\dot{J}(\Omega, \Gamma)$ and $\dot{J}_2^1(\Omega, \Gamma)$ be the closures of $\dot{C}_0^\infty(\Omega, \Gamma)$ in the spaces $L_2(\Omega)$ and $W_2^1(\Omega)$, respectively.

We shall use the following notation for balls, half balls, parabolic cylinders, and half parabolic cylinders:

$$B(\hat{x}, \rho) = \{x \in \mathbb{R}^3 \mid |x - \hat{x}| < \rho\}, \quad B(\rho) = B(0, \rho), \quad B = B(1);$$

$$B^+(\hat{x}, \rho) = \{x \in B(\hat{x}, \rho) \mid x = (x_1, x_2, x_3), x_3 > \hat{x}_3\},$$

$$B^+(\rho) = B^+(0, \rho), \quad B^+ = B^+(1);$$

$$Q(\hat{z}, \rho) = (\hat{t} - \rho^2, \hat{t}) \times B(\hat{x}, \rho), \quad Q(\rho) = Q(0, \rho), \quad Q = Q(1);$$

$$Q^+(\hat{z}, \rho) = (\hat{t} - \rho^2, \hat{t}) \times B^+(\hat{x}, \rho), \quad Q^+(\rho) = Q^+(0, \rho), \quad Q^+ = Q^+(1).$$

where $\hat{z} = (\hat{t}, \hat{x})$.

We denote

$$A(\rho) = \operatorname{ess\,sup}_{-\rho^2 \leq t \leq 0} \frac{1}{\rho} \int_{B^+(\rho)} |u|^2 dx, \quad E(\rho) = \frac{1}{\rho} \int_{Q^+(\rho)} |\nabla u|^2 dz,$$

$$C_{q,r}(\rho) = \rho^{1-2/q-3/r} \|u\|_{L_q^t L_r^x(Q^+(\rho))}, \quad D_{q,r}(\rho) = \rho^{2-2/q-3/r} \|p - [p]_\rho(t)\|_{L_q^t L_r^x(Q^+(\rho))},$$

where $q, r \in [1, \infty]$ and $[p]_\rho(t)$ is the average of p with respect to x in $B^+(\rho)$. Note that all of them are scale invariant with respect to the natural scaling for (1.1):

$$u(t, x) \rightarrow \lambda u(\lambda^2 t, \lambda x), \quad p(t, x) \rightarrow \lambda^2 p(\lambda^2 t, \lambda x).$$

2.2.2 Suitable weak solutions

The definition of suitable weak solutions was introduced in [4]. We say a pair (u, p) is a suitable weak solution of the Navier-Stokes equations on the set Ω_T vanishing on $(0, T) \times \Gamma$ if

i) $u \in L_\infty(0, T; \dot{J}(\Omega, \Gamma)) \cap L_2(0, T; \dot{J}_2^1(\Omega, \Gamma))$ and $p \in L_{\tilde{q}}^t L_1^x(\Omega_T)$ for some $\tilde{q} \geq 1$;

ii) u and p satisfy equation (1.1) in the sense of distribution;

iii) For any $t \in (0, T)$ and nonnegative function $\psi \in C^\infty(\overline{\Omega_T})$ vanishing in a neighborhood of the boundary $\{t = 0\} \times \Omega$ and $(0, T) \times (\partial\Omega \setminus \Gamma)$, the integrals in the following local energy inequality are summable and the inequality holds true:

$$\begin{aligned} & \text{ess sup}_{0 \leq s \leq t} \int_{\Omega} |u(s, x)|^2 \psi(s, x) dx + 2 \int_{\Omega_t} |\nabla u|^2 \psi dx ds \\ & \leq \int_{\Omega_t} \{|u|^2(\psi_t + \Delta\psi) + (|u|^2 + 2p)u \cdot \nabla\psi\} dx ds. \end{aligned} \tag{2.1}$$

We will specify the constant $\tilde{q}$ later so that the integrals on the right-hand side of (2.1) are summable.

2.3 Proof of Theorem 2.1.1

This section is devoted to the proof of Theorem 2.1.1. We use the abbreviation

$$D(\rho) = D_{\tilde{q}, \tilde{r}}(\rho) = \frac{1}{\rho} \|p - [p]_\rho(t)\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))},$$

where $\tilde{q} \in (1, 2)$, $\tilde{r} \in (3/2, 3)$, $2/\tilde{q} + 3/\tilde{r} = 3$. We first prove a few lemmas which will be used below.

Lemma 2.3.1. *For any $\rho > 0$ and any pair of exponents (q, r) such that*

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}, \quad 2 \leq q \leq \infty, \quad 2 \leq r \leq 6,$$

we have

$$\rho^{-1/2} \|u\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} \leq N (A(\rho) + E(\rho))^{\frac{1}{2}},$$

where $N > 0$ is a universal constant.

Proof. Use the standard interpolation by the Sobolev embedding inequality and Hölder's inequality. $\square$

Lemma 2.3.2. *Let (u, p) be a pair of suitable weak solution of (1.1). For constants $\gamma \in (0, 1/2]$, $\rho > 0$, we have*

$$A(\gamma\rho) + E(\gamma\rho) \leq N \left[\gamma^2 A(\rho) + \gamma^{-2} \left((A(\rho) + E(\rho))^{\frac{3}{2}} + (A(\rho) + E(\rho))^{\frac{1}{2}} D(\rho) \right) \right],$$

where $N > 0$ is a universal constant.

Proof. The proof is more or less standard. We give the details for the sake of completeness. By scaling, we may assume $\rho = 1$. Define the backward heat kernel

as

$$\Gamma(t, x) = \frac{1}{(4\pi(\gamma^2 - t))^{3/2}} e^{-\frac{|x|^2}{4(\gamma^2 - t)}}.$$

In the energy inequality (2.1), we choose $\psi = \Gamma\phi$, where $\phi \in C_0^\infty((-1, 1) \times B(1))$ is a suitable smooth cut-off function satisfying

$$0 \leq \phi \leq 1 \quad \text{in } \mathbb{R} \times \mathbb{R}^3, \quad \phi \equiv 1 \quad \text{in } Q(1/2),$$

$$|\nabla\phi| \leq N, \quad |\nabla^2\phi| \leq N, \quad |\partial_t\phi| \leq N \quad \text{in } \mathbb{R} \times \mathbb{R}^3.$$

By using the equation

$$\Delta\Gamma + \Gamma_t = 0,$$

we have

$$\begin{aligned} & \text{ess sup}_{-1 \leq s \leq 0} \int_{B^+} |u(s, x)|^2 \Gamma(s, x) \phi(s, x) dx + 2 \int_{Q^+} |\nabla u|^2 \Gamma \phi dz \\ & \leq \int_{Q^+} \{ |u|^2 (\Gamma \phi_t + \Gamma \Delta \phi + 2 \nabla \phi \nabla \Gamma) + (|u|^2 + 2|p - [p]_1|) u \cdot (\Gamma \nabla \phi + \phi \nabla \Gamma) \} dz. \end{aligned} \quad (2.2)$$

The test function has the following properties:

- (i) For some constant $c > 0$, on $Q^+(\gamma)$ it holds that

$$\Gamma\phi = \Gamma \geq c\gamma^{-3}.$$

- (ii) For any $z \in Q^+(1)$, we have

$$|\Gamma(z)\phi(z)| \leq N\gamma^{-3}, \quad |\nabla\Gamma(z)\phi(z)| + |\Gamma(z)\nabla\phi(z)| \leq N\gamma^{-4}.$$

(iii) For any $z \in Q^+(1)$, we have

$$|\Gamma(z)\phi_t(z)| + |\Gamma(z)\Delta\phi(z)| + |\nabla\Gamma(z)\nabla\phi(z)| \leq N.$$

Therefore, (2.2) and Lemma 2.3.1 yield

$$\begin{aligned} A(\gamma) + E(\gamma) &= \gamma^{-1} \text{ess sup}_{-1 \leq t \leq 0} \int_{B^+(\gamma)} |u(t, x)|^2 dx + \gamma^{-1} \int_{Q^+(\gamma)} |\nabla u|^2 dz \\ &\leq N\gamma^2 \int_{Q^+} |u|^2 dz + N\gamma^{-2} \int_{Q^+} (|u|^2 + 2|p - [p]_1|)|u| dz \\ &\leq N\gamma^2 A(1) + N\gamma^{-2} (A(1) + E(1))^{3/2} + N(A(1) + E(1))^{1/2} D(1). \end{aligned}$$

The lemma is proved. $\square$

Lemma 2.3.3. *Let (u, p) be a pair of suitable weak solution of (1.1)-(1.3). For $\gamma \in (0, 1/2]$, $\rho > 0$, and $\varepsilon_1 \in (0, 3/\tilde{r} - 1)$, we have*

$$D(\gamma\rho) \leq N \left[\gamma^{1+\varepsilon_1} (D_{\tilde{q},1}(\rho) + (A(\rho) + E(\rho))^{\frac{1}{2}}) + \gamma^{-1} (A(\rho) + E(\rho)) \right], \quad (2.3)$$

and

$$D(\gamma\rho) \leq N \left[\gamma^{1+\varepsilon_1} (D(\rho) + (A(\rho) + E(\rho))^{\frac{1}{2}}) + \gamma^{-1} (A(\rho) + E(\rho)) \right], \quad (2.4)$$

where N is a constant independent of u , p , γ , and ρ , but may depend on $\tilde{q}$, $\tilde{r}$, and ε_1 .

Proof. By the scale-invariant property, we may also assume $\rho = 1$. We fix a domain $\tilde{B} \subset \mathbb{R}^3$ with smooth boundary so that

$$B^+(1/2) \subset \tilde{B} \subset B^+,$$

and denote $\tilde{Q} = (-1, 0) \times \tilde{B}$. Define r^* by $1/r^* = 1/\tilde{r} + 1/3 > 1$. Using Hölder's inequality, we get

$$\|u \cdot \nabla u\|_{L_{\tilde{q}}^t L_{r^*}^x(Q^+)} \leq \|\nabla u\|_{L_2^t L_2^x(Q^+)} \|u\|_{L_{q_1}^t L_{r_1}^x(Q^+)},$$

where

$$\frac{1}{\tilde{q}} = \frac{1}{2} + \frac{1}{q_1}, \quad \frac{1}{r^*} = \frac{1}{2} + \frac{1}{r_1}.$$

Because of the conditions on $(\tilde{q}, \tilde{r})$, we have $q_1 \in (2, \infty)$ and $2/q_1 + 3/r_1 = 3/2$. Thus by Lemma 2.3.1, we get

$$\|u \cdot \nabla u\|_{L_{\tilde{q}}^t L_{r^*}^x(Q^+)} \leq N(A(1) + E(1)). \quad (2.5)$$

By the solvability of the linear Stokes system with the zero Dirichlet boundary condition [35, Theorem 1.1] (see Lemma 2.4.4 below), there is a unique solution

$$v \in W_{\tilde{q}, r^*}^{1,2}(\tilde{Q}) \quad \text{and} \quad p_1 \in W_{\tilde{q}, r^*}^{0,1}(\tilde{Q})$$

to the following initial boundary value problem:

$$\begin{cases} \partial_t v - \Delta v + \nabla p_1 = -u \cdot \nabla u & \text{in } \tilde{Q}, \\ \nabla \cdot v = 0 & \text{in } \tilde{Q}, \\ [p_1]_{\tilde{B}}(t) = 0 & \text{for a.e. } t \in [-1, 0], \\ v = 0 & \text{on } \partial_p \tilde{Q}. \end{cases}$$

Moreover, we have

$$\| |v| + |\nabla v| + |p_1| + |\nabla p_1| \|_{L_{\tilde{q}}^t L_{r^*}^x(\tilde{Q})} \leq N \|u \cdot \nabla u\|_{L_{\tilde{q}}^t L_{r^*}^x(\tilde{Q})} \leq N(A(1) + E(1)), \quad (2.6)$$

where in the last inequality we used (2.5). By the Sobolev-Poincaré inequality, we have

$$\|p_1\|_{L_q^t L_{\tilde{r}}^x(Q^+)} \leq \|\nabla p_1\|_{L_q^t L_{\tilde{r}^*}^x(Q^+)} \leq N(A(1) + E(1)). \quad (2.7)$$

We set $w = u - v$ and $p_2 = p - p_1$. Then w and p_2 satisfy

$$\begin{cases} \partial_t w - \Delta w + \nabla p_2 = 0 & \text{in } \tilde{Q}, \\ \nabla \cdot w = 0 & \text{in } \tilde{Q}, \\ w = 0 & \text{on } [-1, 0) \times (\partial \tilde{B} \cap \{x_3 = 0\}). \end{cases}$$

By the Sobolev-Poincaré inequality, Hölder's inequality, and an improved integrability result for the Stokes system [46, Theorem 1.2] (see also Lemma 2.4.5 below), we have

$$\begin{aligned} \frac{1}{\gamma} \|p_2 - [p_2]_\gamma\|_{L_q^t L_{\tilde{r}}^x(Q^+(\gamma))} &\leq N \|\nabla p_2\|_{L_q^t L_{\tilde{r}}^x(Q^+(\gamma))} \\ &\leq N \gamma^{3(\frac{1}{\tilde{r}} - \frac{1}{r'})} \|\nabla p_2\|_{L_q^t L_{\tilde{r}'}^x(Q^+(\gamma))} \leq N \gamma^{3(\frac{1}{\tilde{r}} - \frac{1}{r'})} \|\nabla p_2\|_{L_q^t L_{\tilde{r}'}^x(Q^+(1/2))} \\ &\leq N \gamma^{3(\frac{1}{\tilde{r}} - \frac{1}{r'})} \left(\|p_2 - [p_2]_1\|_{L_q^t L_1^x(Q^+)} + \|w\|_{L_q^t L_1^x(Q^+)} \right) \\ &\leq N \gamma^{3(\frac{1}{\tilde{r}} - \frac{1}{r'})} \left(D_{\tilde{q},1}(1) + \|p_1\|_{L_q^t L_1^x(Q^+)} + \|v\|_{L_q^t L_1^x(Q^+)} + \|u\|_{L_q^t L_1^x(Q^+)} \right) \\ &\leq N \gamma^{3(\frac{1}{\tilde{r}} - \frac{1}{r'})} \left(D_{\tilde{q},1}(1) + A(1) + E(1) + A^{\frac{1}{2}}(1) \right), \end{aligned} \quad (2.8)$$

where $r' > \tilde{r}$ is any large number and we also used (2.6) in the last inequality. From (2.7) and (2.8), we obtain

$$D(\gamma) \leq N \left[\gamma^{3(\frac{1}{\tilde{r}} - \frac{1}{r'})} \left(D_{\tilde{q},1}(1) + (A(1) + E(1))^{\frac{1}{2}} \right) + \gamma^{-1}(A(1) + E(1)) \right].$$

Because $\tilde{r} < 3$ and r' can be arbitrarily large, ε_1 may range in $(0, 3/\tilde{r} - 1)$. Finally, (2.4) follows from Hölder's inequality. The lemma is proved. $\square$

The following is the key lemma for the proof of Theorem 2.1.1.

Lemma 2.3.4. *There exists a constant $\varepsilon_2 = \varepsilon_2(\tilde{q}) > 0$ satisfying the following property. If*

$$A(1) + E(1) + D_{\tilde{q},1}(1) \leq \varepsilon_2,$$

then there exist sufficiently small $\delta > 0$ and $\rho_0 > 0$ such that

$$A(\rho) + E(\rho) \leq \rho^{2-\delta}, \quad D(\rho) \leq \rho^{1+\delta}, \quad (2.9)$$

for any $\rho \in (0, \rho_0]$.

Proof. Let $\rho_k = \rho_0^{(1+\beta)^k}$ where $\rho_0 \in (0, 1)$ and $\beta \in (0, \delta)$ are small constants to be specified later. To prove (2.9) for any $\rho \in (0, 1/2]$ (with a slightly smaller δ), it suffices to show that, for every k we have

$$A(\rho_k) + E(\rho_k) \leq \rho_k^{2-\delta}, \quad D(\rho_k) \leq \rho_k^{1+\delta}. \quad (2.10)$$

We will assume the initial step for induction later by specify some conditions on ρ_0 and ε_2 . Now suppose (2.10) is true for 0 to $k \geq l$ where l is an integer to be specified later. For $E(\rho_{k+1})$, we set $\gamma = \rho_k^\beta$ and $\rho = \rho_k$ in Lemma 2.3.2 to obtain

$$\begin{aligned} A(\rho_{k+1}) + E(\rho_{k+1}) &\leq N \left[\rho_k^{2\beta+2-\delta} + \rho_k^{-2\beta+3(2-\delta)/2} + \rho_k^{-2\beta+1+\delta+(2-\delta)/2} \right] \\ &\leq N \left[\rho_k^{2\beta+2-\delta} + \rho_k^{-2\beta+3-3\delta/2} + \rho_k^{-2\beta+2+\delta/2} \right]. \end{aligned} \quad (2.11)$$

For $\delta \ll 1$, when $\beta < \frac{3\delta}{2(4-\delta)}$, it holds that

$$\min \{2\beta + 2 - \delta, -2\beta + 3 - 3\delta/2, -2\beta + 2 + \delta/2\} > (2 - \delta)(1 + \beta). \quad (2.12)$$

For $D(\rho_{k+1})$, we let $\tilde{\beta} = (1 + \beta)^{l+1} - 1$. We set $\gamma = \rho_{k-l}^{\tilde{\beta}}$ and $\rho = \rho_{k-l}$ in (2.3) to

have

$$D(\rho_{k+1}) \leq N \left[\rho_{k-l}^{(1+\varepsilon_1)\tilde{\beta}+1+\delta} + \rho_{k-l}^{(1+\varepsilon_1)\tilde{\beta}+1-\delta/2} + \rho_{k-l}^{-\tilde{\beta}+2-\delta} \right]. \quad (2.13)$$

To satisfy

$$\min \left\{ (1+\varepsilon_1)\tilde{\beta}+1+\delta, (1+\varepsilon_1)\tilde{\beta}+1-\delta/2, -\tilde{\beta}+2-\delta \right\} > (1+\delta)(1+\tilde{\beta}), \quad (2.14)$$

we require $(\varepsilon_1 - \delta)\tilde{\beta} > 3\delta/2$. Indeed we can choose $\delta = \varepsilon_1^2$, $\beta = \varepsilon_1^2/4$, and take a sufficiently large integer l of order $1/\varepsilon_1$ so that $\tilde{\beta} \sim 3\varepsilon_1$. A direct calculation shows that both (2.12) and (2.14) are satisfied. From (2.11) and (2.13) we can find some $\xi > 0$ such that

$$A(\rho_{k+1}) + E(\rho_{k+1}) \leq N\rho_{k+1}^{2-\delta+\xi}, \quad D(\rho_{k+1}) \leq N\rho_{k+1}^{1+\delta+\xi},$$

where N is a constant independent of k and ξ . We choose ρ_0 small enough such that

$$N\rho_{k+1}^\xi < N\rho_0^\xi < 1$$

to get (2.10) for $k \geq l$.

Finally, by using (2.3) in lemma 2.3.3 with $\gamma = 1/2$ and $\rho = 1$, we have

$$A(1/2) + E(1/2) + D(1/2) \leq N\varepsilon_2^{1/2}.$$

Then by choosing ε_2 sufficiently small, we can make (2.10) to be true for $k = 0, 1, 2, \dots, l$. Therefore, by induction we conclude (2.10) is true for any integer $k \geq 0$. $\square$

Next we present a useful lemma in handling the iterations.

Lemma 2.3.5. *Suppose $f(\rho_0) \leq C_0$. If there exist $\alpha > \beta > 0$ and $C_1, C_2 > 0$ such that for any $0 < r < \rho \leq \rho_0$, it holds that*

$$f(r) \leq C_1 (r/\rho)^\alpha f(\rho) + C_2 \rho^\beta,$$

then there exist constants $C_3, C_4 > 0$ depending on $C_0, C_1, C_2, \alpha, \beta$, such that

$$f(r) \leq C_3 (r/\rho_0)^\beta f(\rho_0) + C_4 r^\beta$$

for $0 < r \leq \rho_0$.

Proof. See, for instance, [15, Chapter III, Lemma 2.1]. □

Now we are ready to give the proof of our first main result in the chapter.

Proof of Theorem 2.1.1. By Lemma 2.3.4 we have the following estimates for $\rho \in (0, \rho_0]$:

$$\|p - [p]\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} \leq \rho^{2+\delta}, \quad \| |u|^2 \|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} \leq \rho^{3-\delta}. \quad (2.15)$$

Let w be the unique weak solution to the heat equation

$$\partial_t w_i - \Delta w_i = -\partial_j(u_i u_j) - \partial_i(p - [p]_\rho) \quad \text{in } Q^+(\rho)$$

with the zero boundary condition. By the classical L_p estimate for the heat equation, we have

$$\|\nabla w\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} \leq N \| |u|^2 \|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} + N \|p - [p]_\rho\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))},$$

which together with (2.15) yields

$$\|\nabla w\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} \leq N\rho^{2+\delta}. \quad (2.16)$$

By the Poincaré inequality with zero boundary condition, we get from (2.16) that

$$\|w\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} \leq N\rho^{3+\delta}. \quad (2.17)$$

Denote $v = u - w$, which satisfies the homogeneous heat equation

$$\partial_t v - \Delta v = 0 \quad \text{in } Q^+(\rho)$$

with the boundary condition $v = u$ on $\partial_p Q^+(\rho)$. Let $0 < \gamma < \rho$. By the Poincaré inequality with zero mean value and using the fact that any Hölder norm of a caloric function in a smaller half cylinder is controlled by any L_p norm of it in a larger half cylinder. We have

$$\begin{aligned} \|v - (v)_\gamma\|_{L_q^t L_{\tilde{r}}^x(Q^+(\gamma))} &\leq N\gamma^4 \|v\|_{C^{1/2,1}(Q^+(\gamma))} \\ &\leq N(\gamma/\rho)^4 \|v - (u)_\rho\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))}, \end{aligned} \quad (2.18)$$

where $(u)_\rho$ is the average of u in $Q^+(\rho)$.

Using (2.18), (2.17), and the triangle inequality, we have

$$\begin{aligned} \|u - (u)_\gamma\|_{L_q^t L_{\tilde{r}}^x(Q^+(\gamma))} &\leq \|v - (v)_\gamma\|_{L_q^t L_{\tilde{r}}^x(Q^+(\gamma))} + 2\|w\|_{L_q^t L_{\tilde{r}}^x(Q^+(\gamma))} \\ &\leq N(\gamma/\rho)^4 \|v - (u)_\rho\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} + N\rho^{3+\delta} \\ &\leq N(\gamma/\rho)^4 \left(\|u - (u)_\rho\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} + \|w\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} \right) + N\rho^{3+\delta} \\ &\leq N(\gamma/\rho)^4 \|u - (u)_\rho\|_{L_q^t L_{\tilde{r}}^x(Q^+(\rho))} + N\rho^{3+\delta}. \end{aligned}$$

Applying Lemma 2.3.5 we obtain

$$\|u - (u)_\gamma\|_{L_q^t L_r^x(Q^+(\gamma))} \leq N\gamma^{3+\delta}$$

for any $\gamma \in (0, \rho_0)$. By Hölder's inequality, we have

$$\|u - (u)_\gamma\|_{L_1(Q^+(\gamma))} \leq N\gamma^2 \|u - (u)_\gamma\|_{L_q^t L_r^x(Q^+(\gamma))} \leq N\gamma^{5+\delta}.$$

Similar estimates can be derived for interior points using same techniques. We conclude that u is Hölder continuous in $Q^+(\rho_0)$ by Campanato's characterization of Hölder continuity. The theorem then follows by a covering argument. $\square$

2.4 A boundary regularity criterion without involving ∇u

In this section, we shall prove the following theorem, which is more general than Theorem 2.1.2.

Theorem 2.4.1. *For each triple of exponents $(q, r, \tilde{q})$ satisfying*

$$\frac{2}{q} + \frac{3}{r} < 2, \quad 2 < q < \infty, \quad \frac{3}{2} < r < \infty, \quad (2.19)$$

and

$$\frac{1}{\tilde{q}} < F(q, r) := 1 - \frac{1}{q} \cdot \frac{(1/r - 1/2)_+}{1/r - 1/6}, \quad (2.20)$$

there exists a universal constant $\varepsilon_0 = \varepsilon_0(q, r, \tilde{q})$ such that if (u, p) is a pair of suitable

weak solution to (1.1)-(1.3) in Q^+ with $p \in L_{\tilde{q}}^t L_1^x(Q^+)$ and satisfies

$$C_{q,r}(1) + D_{\tilde{q},1}(1) < \varepsilon_0, \quad (2.21)$$

then u is regular in $\overline{Q^+(1/2)}$.

Remark 2.4.2. The restriction (2.20) arises in the estimates for the pressure term below by using the coercive estimate for the linear Stokes system. It is not clear to us if this restriction can be dropped. A straightforward calculation shows that under the constraints

$$\frac{2}{q} + \frac{3}{r} \leq 2, \quad 2 < q < \infty, \quad \frac{3}{2} < r < \infty,$$

the function $F(q, r)$ attains its minimum $1 - (\sqrt{3} - \sqrt{2})^2/4$ when $1/r = 1/\sqrt{6} + 1/6$ and $1/q = 3/4 - \sqrt{6}/4$. Therefore, if

$$\tilde{q} \geq \left(1 - \frac{(\sqrt{3} - \sqrt{2})^2}{4}\right)^{-1} \approx 1.02591,$$

then (2.20) is trivial for any (q, r) satisfying (2.19). Moreover, it is easily seen that

$$\max\{1 - 1/q, 1/2 + 1/q\} < F(q, r).$$

Therefore, by decreasing $\tilde{q}$ if necessary, we may assume that

$$\max\{1 - 1/q, 1/2 + 1/q, 7/8\} < 1/\tilde{q} < F(q, r). \quad (2.22)$$

Remark 2.4.3. By a slight modification of the proof below, we have the following result. Under the conditions of Theorem 2.4.1, if instead of (2.21) we assume

$$C_{q,r}(1) + D_{\tilde{q},1}(1) \leq C_0$$

for some constant $C_0 > 0$, then

$$A(1/2) + E(1/2) \leq N(q, r, \tilde{q}, C_0) < \infty.$$

We first set up some notation and state a few lemmas that will be useful in the proof of Theorem 2.4.1. Let $\rho_k = 1 - 2^{-k}$. Denote $Q_k^+ = Q^+(\rho_k)$ and $B_k^+ = B^+(\rho_k)$ for integer $k \geq 1$. Again we denote $A_k = A(\rho_k)$ and $E_k = E(\rho_k)$. For each integer $k \geq 0$, We fix a domain $\tilde{B}_k \subset \mathbb{R}^3$ with smooth boundary so that

$$B_{k+1}^+ \subset \tilde{B}_k \subset B_{k+2}^+$$

such that the C^2 norm of $\partial \tilde{B}_k$ is bounded by $N2^k$. We also denote $\tilde{Q}_k = (-\rho_{k+1}^2, 0) \times \tilde{B}_k$.

Lemma 2.4.4. *Let $m, n \in (1, \infty)$ be two fixed numbers. Assume that $g \in L_{n,m}(\tilde{Q}_k)$. Then there exists a unique function pair (v, p) , which satisfies the following equations:*

$$\begin{cases} \partial_t v - \Delta v + \nabla p = g & \text{in } \tilde{Q}_k, \\ \nabla \cdot v = 0 & \text{in } \tilde{Q}_k, \\ [p]_{\tilde{B}_k}(t) = 0 & \text{for a.e. } t \in [-\rho_{k+1}^2, 0], \\ v = 0 & \text{on } \partial_p \tilde{Q}_k. \end{cases}$$

Moreover, v and p satisfy the following estimate:

$$\|v\|_{W_{n,m}^{1,2}(\tilde{Q}_k)} + \|p\|_{W_{n,m}^{0,1}(\tilde{Q}_k)} \leq C2^{bk} \|g\|_{L_{n,m}(\tilde{Q}_k)},$$

where the constants C and b only depend on m and n .

We refer the reader to [35, Theorem 1.1] for the proof of Lemma 2.4.4. The factor 2^{bk} can be obtained by keeping track of the constants in the localization argument in [35, Sect. 3].

Lemma 2.4.5. *Let $n, s \in (1, \infty)$ be constants and $g \in L_n^t L_s^x(Q_{k+1}^+)$. Assume that the functions $v \in W_{n,1}^{0,1}(Q_{k+1}^+)$ and $p \in L_n^t L_1^x(Q_{k+1}^+)$ satisfy the equations:*

$$\begin{cases} \partial_t v - \Delta v + \nabla p = g & \text{in } Q_{k+1}^+, \\ \nabla \cdot v = 0 & \text{in } Q_{k+1}^+, \end{cases}$$

and the boundary condition

$$v = 0 \quad \text{on } \{y \mid y = (y', 0), |y'| < \rho_{k+1}\} \times [-\rho_{k+1}^2, 0).$$

Then we have $v \in W_{n,s}^{1,2}(Q_k^+)$, $p \in W_{n,s}^{0,1}(Q_k^+)$, and

$$\|v\|_{W_{n,s}^{1,2}(Q_k^+)} + \|p\|_{W_{n,s}^{0,1}(Q_k^+)} \leq C 2^{bk} \left(\|g\|_{L_n^t L_s^x(Q_{k+1}^+)} + \|v\|_{L_n^t L_1^x(Q_{k+1}^+)} + \|p\|_{L_n^t L_1^x(Q_{k+1}^+)} \right), \quad (2.23)$$

where the constants C and b only depend on n and s .

Proof. We use a mollification argument. Denote $x' = (x_1, x_2)$ and $\hat{Q}_k = Q^+((\rho_k + \rho_{k+1})/2)$. By the Sobolev embedding theorem, we have $v \in L_n^t L_m^x(Q_{k+1}^+)$ for some $m \in (1, \min(n, s))$. Let v^ε , p^ε , and g^ε be the standard mollifications with respect to (t, x') , which satisfy the same equations as v , p , and g . By the properties of mollifications, it is clear that for sufficiently small ε ,

$$D_{x'} v^\varepsilon, D_{x'}^2 v^\varepsilon, \partial_t v^\varepsilon \in L_n^t L_m^x(\hat{Q}_k), \quad D_{x'} p^\varepsilon \in L_n^{t,x'} L_1^{x_3}(\hat{Q}_k), \quad j = 1, 2. \quad (2.24)$$

Then from the equations for v_1^ε and v_2^ε , we get $D_{x_3 x_3} v_j^\varepsilon \in L_n^{t,x'} L_1^{x_3}(\hat{Q}_k)$ for $j = 1, 2$.

By applying the Sobolev embedding theorem in the x_3 direction, we get $D_{x_3}v_j^\varepsilon \in L_n^t L_m^x(\hat{Q}_k)$, which together with (2.24) implies that $Dv_j^\varepsilon \in L_n^t L_m^x(\hat{Q}_k)$ for $j = 1, 2$. Since $D_{x'}v^\varepsilon$ satisfies the same equation, we see that $D_{x'}Dv_j^\varepsilon \in L_n^t L_m^x(\hat{Q}_k)$ for $j = 1, 2$. Now owing to $\nabla \cdot v^\varepsilon = 0$, we have

$$D_{x_3}v_3^\varepsilon, D_{x_3x_3}v_3^\varepsilon \in L_n^t L_m^x(\hat{Q}_k),$$

which together with the equation for v_3^ε further implies $D_{x_3}p^\varepsilon \in L_n^t L_m^x(\hat{Q}_k)$. Using the Sobolev embedding theorem, we then get $p^\varepsilon \in L_n^t L_m^x(\hat{Q}_k)$. By [46, Theorem 1.2] (see also [43]), we have $v^\varepsilon \in W_{n,s}^{1,2}(\hat{Q}_k)$, $p^\varepsilon \in W_{n,s}^{0,1}(\hat{Q}_k)$, and

$$\|v^\varepsilon\|_{W_{n,s}^{1,2}(\hat{Q}_k)} + \|p^\varepsilon\|_{W_{n,s}^{0,1}(\hat{Q}_k)} \leq C2^{bk}(\|g^\varepsilon\|_{L_n^t L_s^x(\hat{Q}_k)} + \|v^\varepsilon\|_{W_{n,m}^{1,0}(\hat{Q}_k)} + \|p^\varepsilon\|_{L_n^t L_m^x(\hat{Q}_k)}),$$

where C is independent of ε and k . Again the factor 2^{bk} can be obtained by keeping track of the constants in the proofs in [43]. Taking the limit as $\varepsilon \rightarrow 0$, we get

$$\|v\|_{W_{n,s}^{1,2}(\hat{Q}_k)} + \|p\|_{W_{n,s}^{0,1}(\hat{Q}_k)} \leq C2^{bk}(\|g\|_{L_n^t L_s^x(\hat{Q}_k)} + \|v\|_{W_{n,m}^{1,0}(\hat{Q}_k)} + \|p\|_{L_n^t L_m^x(\hat{Q}_k)}).$$

By interpolation inequalities, for any $\delta \in (0, 1)$ we have

$$\begin{aligned} \|v\|_{W_{n,s}^{1,2}(\hat{Q}_k)} + \|p\|_{W_{n,s}^{0,1}(\hat{Q}_k)} &\leq \delta(\|v\|_{W_{n,s}^{1,2}(\hat{Q}_k)} + \|p\|_{W_{n,s}^{0,1}(\hat{Q}_k)}) \\ &\quad + C2^{bk}\|g\|_{L_n^t L_s^x(\hat{Q}_k)} + C_\delta 2^{bk}(\|v\|_{L_n^x L_1^t(\hat{Q}_k)} + \|p\|_{L_n^t L_1^x(\hat{Q}_k)}). \end{aligned}$$

Finally, (2.23) follows by using a standard iteration argument. $\square$

We now give the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1. By replacing p with $p - [p]_1(t)$ without loss of generality,

we may assume that $[p]_1(t) = 0$ for $t \in (-1, 0)$. Let $\tau > 0$ be such that

$$\frac{2}{q} + \frac{3}{r} := 2 - \tau.$$

It is easily seen that we can find $\hat{r} \in (3/2, r)$ such that

$$\frac{2}{q} + \frac{3}{\hat{r}} := 2 - \tau_1 < 2, \quad \frac{1}{q} + \frac{2}{\hat{r}} > 1, \quad \frac{1}{q} + \frac{1}{\hat{r}} > \frac{2}{3}. \quad (2.25)$$

In the sequel, we will choose τ_1 sufficiently small by reducing $\hat{r}$.

Let $\rho_k = 1 - 2^{-k}$, $B_k = B(\rho_k)$, and $Q_k = Q(\rho_k)$ for integer $k \geq 1$. For each k , we choose a cut-off function ψ_k satisfying

$$\text{supp } \psi_k \subset Q_{k+1}, \quad \psi_k \equiv 1 \text{ on } Q_k,$$

$$|\partial_t \psi_k| \leq N2^{2k}, \quad |D\psi_k| \leq N2^k, \quad |D^2\psi_k| \leq N2^{2k}.$$

By (2.1), we have

$$\begin{aligned} & \text{ess sup}_{-\rho_k^2 \leq s \leq 0} \int_{B_k^+} |u(s, x)|^2 dx + 2 \int_{Q_k^+} |\nabla u|^2 dx ds \\ & \leq N \int_{Q_{k+1}^+} \{2^{2k}|u|^2 + 2^k(|u|^2 + 2|p|)|u|\} dx ds, \end{aligned}$$

where N is independent of k . For simplicity, we denote $A_k = A(\rho_k)$ and $E_k = E(\rho_k)$.

Thus we can rewrite the above inequality as

$$A_k + E_k \leq N2^{2k} \int_{Q_{k+1}^+} \{|u|^2 + |u|^3 + |pu|\} dz. \quad (2.26)$$

We have the following interpolation using Hölder's inequality

$$\int_{Q_{k+1}^+} |u|^3 dz \leq \|u\|_{L_q^t L_{\hat{r}}^x(Q_{k+1}^+)}^{\frac{1}{1-2\tau_1}} \|u\|_{L_{q_1}^t L_{r_1}^x(Q_{k+1}^+)}^{\frac{2-6\tau_1}{1-2\tau_1}} \leq \|u\|_{L_q^t L_r^x(Q_{k+1}^+)}^{\frac{1}{1-2\tau_1}} \|u\|_{L_{q_1}^t L_{r_1}^x(Q_{k+1}^+)}^{\frac{2-6\tau_1}{1-2\tau_1}},$$

where

$$q_1 \in [2, \infty], \quad r_1 \in [2, 6] \quad (2.27)$$

need to satisfy $2/q_1 + 3/r_1 = 3/2$. A simple calculation shows that

$$q_1 = \frac{(2 - 6\tau_1)q}{(1 - 2\tau_1)q - 1}, \quad r_1 = \frac{(2 - 6\tau_1)\hat{r}}{(1 - 2\tau_1)\hat{r} - 1}.$$

Indeed (2.27) holds because of (2.25). We then apply Lemma 2.3.1 to get

$$\int_{Q_{k+1}^+} |u|^3 dz \leq \|u\|_{L_q^t L_r^x(Q_{k+1}^+)}^{\frac{1}{1-2\tau_1}} (A_{k+1} + E_{k+1})^{\frac{1-3\tau_1}{1-2\tau_1}}. \quad (2.28)$$

Again by Hölder's inequality, we have

$$\int_{Q_{k+1}^+} |u|^2 dz \leq N \left(\int_{Q_{k+1}^+} |u|^3 dz \right)^{2/3} \leq N \|u\|_{L_q^t L_r^x(Q_{k+1}^+)}^{\frac{2}{3(1-2\tau_1)}} (A_{k+1} + E_{k+1})^{\frac{2(1-3\tau_1)}{3(1-2\tau_1)}}. \quad (2.29)$$

To deal with the last term in (2.26), we make the following decomposition. For some suitable (q', s^*) which we will specify later, there exists a pair of unique solution

$$v_k \in W_{q', s^*}^{1,2}(\tilde{Q}_k) \quad \text{and} \quad p_k \in W_{q', s^*}^{0,1}(\tilde{Q}_k)$$

to the following initial boundary value problem:

$$\begin{cases} \partial_t v_k - \Delta v_k + \nabla p_k = -u \cdot \nabla u & \text{in } \tilde{Q}_k, \\ \nabla \cdot v_k = 0 & \text{in } \tilde{Q}_k, \\ [p_k]_{\tilde{B}_k}(t) = 0 & \text{for a.e. } t \in [-\rho_{k+1}^2, 0], \\ v_k = 0 & \text{on } \partial_p \tilde{Q}_k. \end{cases} \quad (2.30)$$

We set $w_k = u - v_k$ and $h_k = p - p_k$. Then w_k and h_k satisfy

$$\begin{cases} \partial_t w_k - \Delta w_k + \nabla h_k = 0 & \text{in } \tilde{Q}_k, \\ \nabla \cdot w_k = 0 & \text{in } \tilde{Q}_k, \\ w_k = 0 & \text{on } [-\rho_{k+1}^2, 0) \times (\partial \tilde{B}_k \cap \{x_3 = 0\}). \end{cases}$$

We choose $\hat{r}_2 \in (3/2, r)$ satisfying

$$\frac{2}{q} + \frac{3}{\hat{r}_2} := 2 - \tau_2 < 2.$$

In the sequel, we will choose τ_2 sufficiently small by reducing $\hat{r}_2$.

Estimates for p_k : For $\alpha_1 \in (0, 1)$ which we will specify later, by Hölder's inequality,

$$\int_{Q_{k+1}^+} |up_k| dz \leq \|u\|_{L_{q_2}^t L_{r_2}^x(Q_{k+1}^+)}^{1-\alpha_1} \|u\|_{L_q^t L_{\hat{r}_2}^x(Q_{k+1}^+)}^{\alpha_1} \|p_k\|_{L_{q'}^t L_{s'}^x(Q_{k+1}^+)}, \quad (2.31)$$

where the exponents need to satisfy

$$\frac{2}{q_2} + \frac{3}{r_2} = \frac{3}{2}, \quad \frac{1}{q'} + \frac{1-\alpha_1}{q_2} + \frac{\alpha_1}{q} = 1, \quad \frac{1}{s'} + \frac{1-\alpha_1}{r_2} + \frac{\alpha_1}{\hat{r}_2} = 1.$$

The system above implies that

$$\frac{2}{q'} + \frac{3}{s'} = 5 - \left((1 - \alpha_1) \cdot \frac{3}{2} + \alpha_1(2 - \tau_2) \right). \quad (2.32)$$

To make use of Lemma 2.3.1, we need $2 \leq q_2 \leq \infty$, which is equivalent to

$$\frac{1}{q'} + \frac{\alpha_1}{q} \leq 1 \leq \frac{1}{q'} + \frac{\alpha_1}{q} + \frac{1 - \alpha_1}{2}. \quad (2.33)$$

We are going to check this condition later.

Next we estimate the nonlinear term $u \cdot \nabla u$. Define another exponent s^* by

$$\frac{1}{s^*} := \frac{1}{s'} + \frac{1}{3}.$$

Using Hölder's inequality, we get

$$\|u \cdot \nabla u\|_{L_{q'}^t L_{s^*}^x(\tilde{Q}_k)} \leq \|\nabla u\|_{L_2^t L_2^x(\tilde{Q}_k)} \|u\|_{L_q^t L_{\hat{r}_2}^x(\tilde{Q}_k)}^{1-\alpha_2} \|u\|_{L_{q_3}^t L_{r_3}^x(\tilde{Q}_k)}^{\alpha_2}, \quad (2.34)$$

where

$$\begin{aligned} \frac{1}{q'} &= \frac{1}{2} + \frac{1 - \alpha_2}{q} + \frac{\alpha_2}{q_3}, \quad \frac{1}{s^*} = \frac{1}{2} + \frac{1 - \alpha_2}{\hat{r}_2} + \frac{\alpha_2}{r_3}, \\ q_3 &\in [2, \infty], \quad r_3 \in [2, 6], \quad \frac{2}{q_3} + \frac{3}{r_3} = \frac{3}{2}. \end{aligned} \quad (2.35)$$

In particular, we take $q_3 = 2$ and $r_3 = 6$ so that

$$\alpha_2 = \frac{1/q' - 1/2 - 1/q}{1/2 - 1/q}. \quad (2.36)$$

By (2.35) we have

$$\frac{2}{q'} + \frac{3}{s'} = (1 - \alpha_2)(2 - \tau_2) + (1 + \alpha_2) \cdot \frac{3}{2}. \quad (2.37)$$

From (2.32), (2.36), and (2.37), we get

$$\alpha_1 = \alpha_2 + \frac{2\tau_2}{1 - 2\tau_2} = \frac{1/q' - 1/2 - 1/q}{1/2 - 1/q} + \frac{2\tau_2}{1 - 2\tau_2}.$$

Since we have solved for α_1 , we can now go back to verify (2.33). A simple calculation gives that it indeed holds when $q' > \frac{q^2}{q^2 - q + 2}$ and τ_2 is sufficiently small. We note that there is an implicit restriction $q > 2$ contained in the conditions above. We can observe it by adding up the first inequality in (2.33) and the first equality in (2.35) and using the fact $\alpha_1 > \alpha_2$.

To make use of Lemma 2.4.4, we need to check that $s^* > 1$, which is equivalent to

$$\frac{1 - \alpha_2}{\hat{r}_2} + \frac{\alpha_2}{6} < \frac{1}{2}.$$

In the special case when $q' = \frac{q^2}{q^2 - q + 2}$, we have $\alpha_2 = 1 - 2/q \in (0, 1)$ and the above inequality becomes

$$\frac{2}{\hat{r}_2 q} + \left(1 - \frac{2}{q}\right) \frac{1}{6} < \frac{1}{2},$$

which clearly holds true because

$$2\sqrt{\frac{6}{\hat{r}_2 q}} \leq \frac{2}{q} + \frac{3}{\hat{r}_2} < 2$$

and thus $\hat{r}_2 q > 6$. By continuity, when q' is sufficiently close to $\frac{q^2}{q^2 - q + 2}$, we still have $s^* > 1$ and $\alpha_2 \in (0, 1)$.

Now by Lemma 2.4.4, we have the existence of the unique solution pair (v_k, p_k)

to (2.30) and

$$\begin{aligned} & \| |p_k| + |\nabla p_k| \|_{L_{q'}^t L_{s^*}^x(\tilde{Q}_k)} \leq N 2^{bk} \|u \cdot \nabla u\|_{L_{q'}^t L_{s^*}^x(\tilde{Q}_k)} \\ & \leq N 2^{bk} \|\nabla u\|_{L_2^t L_2^x(\tilde{Q}_k)} \|u\|_{L_q^t L_{\tilde{r}_2}^x(\tilde{Q}_k)}^{1-\alpha_2} \|u\|_{L_{q_3}^t L_{r_3}^x(\tilde{Q}_k)}^{\alpha_2}, \end{aligned}$$

where in the last inequality we used (2.34). Here and in the sequel, b is a positive constant which is independent of k and may vary from line to line. Together with the Sobolev-Poincaré inequality and Hölder's inequality, we obtain

$$\|p_k\|_{L_{q'}^t L_{s^*}^x(\tilde{Q}_k)} \leq N 2^{bk} \|\nabla u\|_{L_2^t L_2^x(\tilde{Q}_k)} \|u\|_{L_q^t L_{\tilde{r}_2}^x(\tilde{Q}_k)}^{1-\alpha_2} \|u\|_{L_{q_3}^t L_{r_3}^x(\tilde{Q}_k)}^{\alpha_2}.$$

Combining with (2.31) we have

$$\begin{aligned} \int_{Q_{k+1}^+} |up_k| dz & \leq N 2^{bk} \|u\|_{L_{q_2}^t L_{r_2}^x(Q_{k+2}^+)}^{1-\alpha_1} \|u\|_{L_q^t L_{\tilde{r}_2}^x(Q_{k+2}^+)}^{1+\alpha_1-\alpha_2} \cdot \|\nabla u\|_{L_2^t L_2^x(Q_{k+2}^+)} \|u\|_{L_{q_3}^t L_{r_3}^x(Q_{k+2}^+)}^{\alpha_2} \\ & \leq N 2^{bk} \epsilon_0^{\frac{1}{1-2\tau_2}} (A_{k+2} + E_{k+2})^{\frac{1-3\tau_2}{1-2\tau_2}}. \end{aligned} \quad (2.38)$$

Estimates for h_k : For some $\alpha_3 \in (0, 1]$ which we will specify later, analogous to (2.31) by Hölder's inequality,

$$\int_{Q_{k+1}^+} |uh_k| dz \leq \|u\|_{L_{q_4}^t L_{r_4}^x(Q_{k+1}^+)}^{1-\alpha_3} \|u\|_{L_q^t L_{\tilde{r}}^x(Q_{k+1}^+)}^{\alpha_3} \|h_k\|_{L_{\tilde{q}}^t L_{\tilde{s}}^x(Q_{k+1}^+)}, \quad (2.39)$$

where the exponents satisfy

$$\frac{2}{q_4} + \frac{3}{r_4} = \frac{3}{2}, \quad \frac{1}{\tilde{q}} + \frac{1-\alpha_3}{q_4} + \frac{\alpha_3}{q} = 1, \quad \frac{1}{\tilde{s}} + \frac{1-\alpha_3}{r_4} + \frac{\alpha_3}{r} = 1.$$

To make use of Lemma 2.3.1, we require $2 \leq q_4 \leq \infty$, which is equivalent to

$$\frac{1}{\tilde{q}} + \frac{\alpha_3}{q} \leq 1 \leq \frac{1}{\tilde{q}} + \frac{\alpha_3}{q} + \frac{1 - \alpha_3}{2}.$$

We simply choose $\alpha_3 = q(1 - 1/\tilde{q}) \in (0, 1)$ and use (2.22) to verify this condition. By Lemma 2.4.5 and the triangle inequality, we have $h_k \in W_{\tilde{q}, \tilde{s}}^{0,1}(Q_{k+2}^+)$ and the following estimate:

$$\begin{aligned} \|h_k\|_{L_{\tilde{q}}^t L_{\tilde{s}}^x(Q_{k+1}^+)} &\leq N2^{bk} \| |w_k| + |h_k| \|_{L_{\tilde{q}}^t L_1^x(Q_{k+2}^+)} \\ &\leq N2^{bk} \| |u| + |p| \|_{L_{\tilde{q}}^t L_1^x(Q_{k+2}^+)} + N2^{bk} \| |v_k| + |p_k| \|_{L_{\tilde{q}}^t L_1^x(Q_{k+2}^+)}. \end{aligned} \quad (2.40)$$

By Hölder's inequality, (2.22), and (2.21), we have

$$\| |u| + |p| \|_{L_{\tilde{q}}^t L_1^x(Q_{k+2}^+)} \leq \varepsilon_0. \quad (2.41)$$

For any $\tilde{s}^* > 1$, by Lemma 2.4.4, we have

$$\| |v_k| + |p_k| \|_{L_{\tilde{q}}^t L_1^x(\tilde{Q}_{k+1})} \leq \| |v_k| + |p_k| \|_{L_{\tilde{q}}^t L_{\tilde{s}^*}^x(\tilde{Q}_{k+1})} \leq N2^{bk} \| u \cdot \nabla u \|_{L_{\tilde{q}}^t L_{\tilde{s}^*}^x(\tilde{Q}_{k+1})}. \quad (2.42)$$

Next analogous to (2.34) by Hölder's inequality,

$$\| u \cdot \nabla u \|_{L_{\tilde{q}}^t L_{\tilde{s}^*}^x(\tilde{Q}_{k+1})} \leq \| \nabla u \|_{L_2^t L_2^x(\tilde{Q}_{k+1})} \| u \|_{L_{\tilde{q}}^t L_r^x(\tilde{Q}_{k+1})}^{1-\alpha_4} \| u \|_{L_{q_5}^t L_{r_5}^x(\tilde{Q}_{k+1})}^{\alpha_4}, \quad (2.43)$$

where the exponents satisfy

$$2 \leq q_5 \leq \infty, \quad \frac{2}{q_5} + \frac{3}{r_5} = \frac{3}{2}, \quad \frac{1}{2} + \frac{1 - \alpha_4}{q} + \frac{\alpha_4}{q_5} \leq \frac{1}{\tilde{q}}, \quad \frac{1}{2} + \frac{1 - \alpha_4}{r} + \frac{\alpha_4}{r_5} \leq \frac{1}{\tilde{s}^*}.$$

To justify the use of Lemma 2.4.4 in (2.42), we need to check that $\tilde{s}^* > 1$, which is

equivalent to

$$\frac{1 - \alpha_4}{r} + \frac{\alpha_4}{r_5} < \frac{1}{2}.$$

For later purpose, we also want $\alpha_3 = q(1 - 1/\tilde{q}) > \alpha_4$. To satisfy all of the three conditions above, we discuss two cases:

i) If $r > 2$, we simply choose $\alpha_4 = 0$. Then such $\tilde{s}^* > 1$ exists, and in view of (2.22), all the conditions are satisfied.

ii) If $r \in (3/2, 2]$, then $q > 4$. We set $q_5 = 2, r_5 = 6$. To ensure $\tilde{s}^* > 1$ and $\alpha_3 > \alpha_4$, we take

$$\frac{1/r - 1/2}{1/r - 1/6} < \alpha_4 < \min\{1/2, q(1 - 1/\tilde{q})\},$$

which is possible because of (2.20) and $r > 3/2$. Moreover, we have

$$\frac{1}{2} + \frac{1 - \alpha_4}{q} + \frac{\alpha_4}{q_5} \leq \frac{1}{2} + \frac{1 - \alpha_4}{4} + \frac{\alpha_4}{2} \leq \frac{7}{8} < \frac{1}{\tilde{q}},$$

where we used (2.22) in the last inequality.

Now plugging (2.41), (2.42), and (2.43) into (2.40), we obtain

$$\|h_k\|_{L_q^t L_s^x(Q_{k+1}^+)} \leq N 2^{bk} \left(\varepsilon_0 + \|\nabla u\|_{L_2^t L_2^x(Q_{k+3}^+)} \|u\|_{L_q^t L_r^x(Q_{k+3}^+)}^{1-\alpha_4} \|u\|_{L_{q_5}^t L_{r_5}^x(Q_{k+3}^+)}^{\alpha_4} \right).$$

Together with (2.39) we have

$$\begin{aligned} \int_{Q_{k+1}^+} |uh_k| dz &\leq N 2^{bk} \|u\|_{L_{q_4}^t L_{r_4}^x(Q_{k+3}^+)}^{1-\alpha_3} \|u\|_{L_q^t L_r^x(Q_{k+3}^+)}^{\alpha_3} \\ &\quad \cdot \left(\varepsilon_0 + \|\nabla u\|_{L_2^t L_2^x(Q_{k+3}^+)} \|u\|_{L_q^t L_r^x(Q_{k+3}^+)}^{1-\alpha_4} \|u\|_{L_{q_5}^t L_{r_5}^x(Q_{k+3}^+)}^{\alpha_4} \right). \end{aligned} \quad (2.44)$$

Note we have consistently chosen (q_n, r_n) such that

$$\frac{2}{q_n} + \frac{3}{r_n} = \frac{3}{2}, \quad 2 \leq q_n \leq \infty, \quad 2 \leq r_n \leq 6, \quad \text{for } n = 1, 2, 3, 4, 5.$$

Thus by Lemma 2.3.1, we know

$$\|u\|_{L_{q_n}^t L_{r_n}^x(Q_k^+)} \leq (A_k + E_k)^{1/2} \quad \text{for } n = 1, 2, 3, 4, 5.$$

Substituting (2.21), (2.28), (2.29), (2.38), and (2.44) into (2.26), we obtain

$$\begin{aligned} A_k + E_k &\leq N 2^{bk} \left(\sum_{j=1}^2 \varepsilon_0^{\frac{1}{1-2\tau_j}} (A_{k+3} + E_{k+3})^{\frac{1-3\tau_j}{1-2\tau_j}} + \varepsilon_0^{\frac{2}{3(1-2\tau_1)}} (A_{k+3} + E_{k+3})^{\frac{2(1-3\tau_1)}{3(1-2\tau_1)}} \right. \\ &\quad \left. + \varepsilon_0^{1+\alpha_3} (A_{k+3} + E_{k+3})^{\frac{1-\alpha_3}{2}} + \varepsilon_0^{1+\alpha_3-\alpha_4} (A_{k+3} + E_{k+3})^{\frac{2-\alpha_3+\alpha_4}{2}} \right). \end{aligned}$$

By Young's inequality, for any $\delta > 0$ we have

$$A_k + E_k \leq \delta^3 (A_{k+3} + E_{k+3}) + \tilde{N} 2^{bk} \varepsilon_0^\beta, \quad (2.45)$$

where $\tilde{N} = \tilde{N}(\delta, q, r, b) > 0$ and $\beta = \beta(q, r) \in (0, 1)$. We multiply both sides of (2.45) by δ^k and sum over integer k from 1 to infinity. By setting $\delta = 3^{-b}$, we make sure the second term on the right-hand side is summable and get

$$\begin{aligned} \sum_{k=1}^{\infty} \delta^k (A_k + E_k) &\leq \sum_{k=1}^{\infty} \delta^{k+3} (A_{k+3} + E_{k+3}) + \tilde{N} \varepsilon_0^\beta \sum_{k=1}^{\infty} (2/3)^{bk} \\ &\leq \sum_{k=3}^{\infty} \delta^k (A_k + E_k) + \tilde{N} \varepsilon_0^\beta. \end{aligned}$$

Therefore, we have

$$A_1 + E_1 \leq \tilde{N} \varepsilon_0^\beta,$$

where $\tilde{N} = \tilde{N}(\delta, q, r, b) > 0$ and $\beta = \beta(q, r) \in (0, 1)$. Together with $D_{\tilde{q},1} < \varepsilon_0$, we can use Theorem 2.1.1 to conclude that there exists a universal $\varepsilon_0 = \varepsilon_0(q, r)$ sufficiently small such that u is regular in $\overline{Q^+(1/2)}$. $\square$

2.5 An interior regularity criterion without involving ∇u

In this section, we show how to adapt our proof to the interior case where $\tilde{q}$ is allowed to be 1. We note that Theorem 2.5.1 was also obtained recently in [22] by using a different proof.

For $\rho > 0$, we define the quantities $A(\rho)$, $E(\rho)$, $C_{q,r}(\rho)$, and $D_{q,r}(\rho)$ with $B(\rho)$ and $Q(\rho)$ in place of $B^+(\rho)$ and $Q^+(\rho)$ as following:

$$A(\rho) = \operatorname{ess\,sup}_{-\rho^2 \leq t \leq 0} \frac{1}{\rho} \int_{B(\rho)} |u|^2 dx, \quad E(\rho) = \frac{1}{\rho} \int_{Q(\rho)} |\nabla u|^2 dz,$$

$$C_{q,r}(\rho) = \rho^{1-2/q-3/r} \|u\|_{L_q^t L_r^x(Q(\rho))}, \quad D_{q,r}(\rho) = \rho^{2-2/q-3/r} \|p - [p]_\rho(t)\|_{L_q^t L_r^x(Q(\rho))}.$$

Theorem 2.5.1. *For each pair of exponents (q, r) satisfying*

$$\frac{2}{q} + \frac{3}{r} < 2, \quad 1 < q < \infty, \quad \frac{3}{2} < r < \infty,$$

there exists a universal constant $\varepsilon_0 = \varepsilon_0(q, r)$ such that if (u, p) is a pair of suitable weak solution to (1.1) in Q with $p \in L_1(Q)$ and satisfies

$$C_{q,r}(1) + D_{1,1}(1) < \varepsilon_0, \tag{2.46}$$

then u is regular in $\overline{Q(1/2)}$.

Lemma 2.5.2. *For any $\rho > 0$ and a pair of exponents (q, r) such that*

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}, \quad 2 \leq q \leq \infty, \quad 2 \leq r \leq 6,$$

we have

$$\rho^{-1/2} \|u\|_{L_q^t L_r^x(Q(\rho))} \leq N (A(\rho) + E(\rho))^{\frac{1}{2}}.$$

Proof. Use the standard interpolation by the Sobolev embedding inequality and Hölder's inequality. $\square$

Proof of Theorem 2.5.1. As before we may assume that $[p]_1(t) = 0$ for $t \in (-1, 0)$.

Also we can find $\hat{r} \in (3/2, r)$ such that

$$\frac{2}{q} + \frac{3}{\hat{r}} := 2 - \tau_1 < 2, \quad \frac{1}{q} + \frac{2}{\hat{r}} > 1, \quad \frac{1}{q} + \frac{1}{\hat{r}} > \frac{2}{3}. \quad (2.47)$$

As before, we choose τ_1 sufficiently small by reducing $\hat{r}$. Following the beginning part of proof of Theorem 2.4.1, we have

$$A_k + E_k \leq N 2^{2k} \int_{Q_{k+1}} \{ |u|^2 + |u|^3 + |pu| \} dz. \quad (2.48)$$

The estimates for the first two terms on the right remain the same. For the third term, we decompose it in the following way. For each k , let $\eta_k(x)$ be a smooth cut-off function supported in B_{k+2} , $0 \leq \eta_k \leq 1$ and $\eta_k \equiv 1$ on $B(\frac{\rho_{k+1} + \rho_{k+2}}{2})$. In the sense of distribution, for a.e. $t \in (-1, 0)$, it holds that

$$\Delta p = D_{ij}(u_i u_j) \quad \text{in } B_1.$$

We consider the decomposition $p = p_k + h_k$, where p_k is the Newtonian potential of $D_{ij}(u_i u_j \eta_k)$. Then h_k is harmonic in $B(\frac{\rho_{k+1} + \rho_{k+2}}{2})$.

Estimates for p_k : Let $q' = \frac{2q}{q+1} \in (1, \infty)$, and $s' \in (1, \infty)$ and $\alpha_1 \in (0, 1)$ be constants which we will specify later. By Hölder's inequality,

$$\int_{Q_{k+1}} |up_k| dz \leq \|u\|_{L_{q_2}^t L_{r_2}^x(Q_{k+1})}^{1-\alpha_1} \|u\|_{L_q^t L_{\hat{r}}^x(Q_{k+1})}^{\alpha_1} \|p_k\|_{L_{q'}^t L_{s'}^x(Q_{k+1})}, \quad (2.49)$$

where the exponents satisfy

$$\frac{2}{q_2} + \frac{3}{r_2} = \frac{3}{2}, \quad \frac{1}{q'} + \frac{1-\alpha_1}{q_2} + \frac{\alpha_1}{q} = 1, \quad \frac{1}{s'} + \frac{1-\alpha_1}{r_2} + \frac{\alpha_1}{\hat{r}} = 1. \quad (2.50)$$

To make use of Lemma 2.5.2, we require $2 \leq q_2 \leq \infty$, which is equivalent to

$$\frac{1}{q'} + \frac{\alpha_1}{q} \leq 1 \leq \frac{1}{q'} + \frac{\alpha_1}{q} + \frac{1-\alpha_1}{2}. \quad (2.51)$$

We will come back to check this condition later. By (2.50) we also have

$$\frac{2}{q'} + \frac{3}{s'} = 5 - \left((1-\alpha_1) \cdot \frac{3}{2} + \alpha_1(2-\tau_1) \right). \quad (2.52)$$

Using the Calderón-Zygmund estimate, we have

$$\|p_k\|_{L_{q'}^t L_{s'}^x(Q_{k+1})} \leq \|p_k\|_{L_q^t L_{s'}^x(Q_{k+2})} \leq N \|u\|_{L_{2q'}^t L_{2s'}^x(Q_{k+2})}^2. \quad (2.53)$$

By Hölder's inequality, we have

$$\|u\|_{L_{2q'}^t L_{2s'}^x(Q_{k+2})}^2 \leq \|u\|_{L_q^t L_{\hat{r}}^x(Q_{k+2})} \|u\|_{L_{q_3}^t L_{r_3}^x(Q_{k+2})}, \quad (2.54)$$

where

$$q_3 = \frac{2q}{q-1} \in (2, \infty), \quad r_3 = \frac{6q}{q+2} \in (2, 6), \quad \frac{2}{q_3} + \frac{3}{r_3} = \frac{3}{2},$$

and

$$\frac{1}{s'} = \frac{1}{\hat{r}} + \frac{1}{r_3} = \frac{1}{\hat{r}} + \frac{1}{6} + \frac{1}{3q}.$$

Plugging this into (2.52) and using (2.47), we get

$$\alpha_1 = \frac{2\tau_1}{1 - 2\tau_1}.$$

Since we have solved for α_1 , we can now go back to verify (2.51), which is equivalent to

$$\frac{1}{2} + \frac{1}{2q} + \frac{\alpha_1}{q} \leq 1 \leq 1 + \frac{1}{2q} + \frac{\alpha_1}{q} - \frac{\alpha_1}{2}.$$

This indeed is satisfied when τ_1 is sufficiently small. Thus by Lemma 2.5.2, (2.49), (2.53), and (2.54), we have

$$\int_{Q_{k+1}} |up_k| dz \leq N(A_{k+2} + E_{k+2})^{\frac{1-3\tau_1}{1-2\tau_1}} \|u\|_{L_q^t L_r^x(Q_{k+2})}^{\frac{1}{1-2\tau_1}}. \quad (2.55)$$

Estimates for h_k : By Hölder's inequality, we have

$$\int_{Q_{k+1}} |uh_k| dz \leq \|u\|_{L_\infty^t L_2^x(Q_{k+1})} \|h_k\|_{L_1^t L_2^x(Q_{k+1})}. \quad (2.56)$$

Recall that h_k is harmonic in $B(\frac{\rho_{k+1} + \rho_{k+2}}{2})$. By the fact that any Sobolev norm of a harmonic function in a smaller ball can be estimated by any of its L_p norm in a greater ball, we know

$$\|h_k\|_{L_2^x(B_{k+1})} \leq N2^{bk} \|h_k\|_{L_1^x(B_{k+2})},$$

where $b > 0$ is a constant. Integrating in $t \in (-\rho_{k+1}^2, 0)$ we have

$$\|h_k\|_{L_1^t L_2^x(Q_{k+1})} \leq N2^{bk} \|h_k\|_{L_1^t L_1^x(Q_{k+2})} \leq N2^{bk} (\|p_k\|_{L_1^t L_1^x(Q_{k+2})} + \|p\|_{L_1^t L_1^x(Q_{k+2})}), \quad (2.57)$$

where the second term is small by condition (2.46). By the Calderón-Zygmund estimate and Hölder's inequality, for any $\tilde{r} > 1$ we have

$$\|p_k\|_{L_1^t L_1^x(Q_{k+2})} \leq N \|p_k\|_{L_1^t L_{\tilde{r}}^x(Q_{k+2})} \leq N \|u\|_{L_1^t L_{2\tilde{r}}^x(Q_{k+2})}^2. \quad (2.58)$$

For $q > 1$, we claim the following interpolation holds for some $\alpha, q_4, r_4 > 0$:

$$\|u\|_{L_1^t L_{2\tilde{r}}^x(Q_{k+2})}^2 \leq N \|u\|_{L_q^t L_r^x(Q_{k+2})}^{2-\alpha} \|u\|_{L_{q_4}^t L_{r_4}^x(Q_{k+2})}^\alpha, \quad (2.59)$$

where

$$q_4 \in [2, \infty], \quad r_4 \in [2, 6], \quad \frac{2}{q_4} + \frac{3}{r_4} = \frac{3}{2},$$

and they need to satisfy

$$\frac{2-\alpha}{q} + \frac{\alpha}{q_4} \leq 1, \quad \frac{2-\alpha}{r} + \frac{\alpha}{r_4} \leq 1.$$

Indeed, we can choose (α, q_4, r_4) in the following way: when $1 < q \leq 2$, we set $q_4 = \infty$, $r_4 = 2$, and $\alpha = 2 - q$; when $q > 2$, we set $q_4 = 2$, $r_4 = 6$, and $\alpha = 6/7$. Note that in both cases we have $\alpha < 1$.

Now we plug (2.57), (2.58), and (2.59) into (2.56) to obtain

$$\begin{aligned} \int_{Q_{k+1}} |uh_k| dz &\leq N2^{bk} \left((A_{k+2} + E_{k+2})^{\frac{1}{2}} \|p\|_{L_1^t L_1^x(Q_{k+2})} \right. \\ &\quad \left. + (A_{k+2} + E_{k+2})^{\frac{1+\alpha}{2}} \|u\|_{L_q^t L_r^x(Q_{k+2})}^{2-\alpha} \right). \end{aligned} \quad (2.60)$$

By (2.48), (2.28), (2.29), (2.55), (2.60), and condition (2.46) we conclude that

$$\begin{aligned} A_k + E_k &\leq N2^{bk} \left(\varepsilon_0^{\frac{1}{1-2\tau_1}} (A_{k+2} + E_{k+2})^{\frac{1-3\tau_1}{1-2\tau_1}} + \varepsilon_0^{\frac{2}{3(1-2\tau_1)}} (A_{k+2} + E_{k+2})^{\frac{2(1-3\tau_1)}{3(1-2\tau_1)}} \right. \\ &\quad \left. + \varepsilon_0 (A_{k+2} + E_{k+2})^{\frac{1}{2}} + \varepsilon_0^{2-\alpha} (A_{k+2} + E_{k+2})^{\frac{1+\alpha}{2}} \right). \end{aligned}$$

Now similar to the proof in Section 2.4, we obtain

$$A_1 + E_1 \leq \tilde{N} \varepsilon_0^\beta.$$

Together with $D_{1,1} < \varepsilon_0$, we can apply [19, Theorem 1.5] to conclude that there exists a universal $\varepsilon_0 = \varepsilon_0(q, r)$ sufficiently small such that u is regular in $\overline{Q(1/2)}$. $\square$

CHAPTER THREE

Regularity criteria for the Navier-Stokes equations in high space dimensions

3.1 Introduction and main results

Before present our main results in this chapter, we would like to revisit Theorem 1.0.2 to see the motivation and difficulties we have when moving to higher space dimensions and domains with boundary. The proofs in [12] are highly nontrivial and rely on the ε -regularity criteria in the light of [4], [34], and [31]. These regularity criteria may break down when the dimension increases, which inspires us to search for a way to modify and generalize the argument. Another main ingredient of the proof is a backward uniqueness theorem of heat equations with bounded lower order coefficients in the half space (see [13]). We will also use this part of argument in the proof of our theorems. Under an additional assumption on the pressure, there are some extensions of Theorem 1.0.2 to the half space case and the bounded domain case; we refer the reader to [45] and [37] for some results in this direction. See also [26, 14, 55, 3, 1, 38, 2] and the references therein for other related results. It is worth mentioning that the ε -regularity criteria are crucially used in all these references.

As for the extension to the higher dimensional Navier-Stokes equations, in [8], Dong and Du used Schoen's trick to establish another regularity criterion similar to Theorem 1.0.2 and extended this result to $\mathbb{R}^d$ for $d \geq 3$. Unfortunately, there is a gap in the proof of [8, Lemma 3.2]. We borrow the idea from [8], that is, to find a priori L_∞ bound only depends on the $L_\infty^t L_d^x$ norm and the dimension. Instead of using Schoen's trick, we prove two ε -regularity criteria adapted from [9, Theorem 1.2] to provide a unified approach to obtain results in the spirit of Theorem 1.0.2 for both whole space $\mathbb{R}^d$ and half space $\mathbb{R}_+^d$ for $d \geq 4$.

We now state the main results in this chapter. The notation in Theorems 3.1.1, 3.1.2, and 3.1.3 is introduced in Section 3.2. A remark is that though we state and

prove the following theorems for $d \geq 4$, with a minor modification of the exponents in the scale invariant quantities we defined in Section 3.2.3, we can give an alternative proof of the case when $d = 3$ which has been proved before in [12, 45].

Theorem 3.1.1. *Let $d \geq 4$ be an integer. Suppose (u, p) is a pair of Leray-Hopf weak solution to the Cauchy problem in $(0, T) \times \mathbb{R}$ and u satisfies the following condition for some $K > 0$,*

$$\|u\|_{L_\infty^t L_d^x((0, T) \times \mathbb{R}^d)} \leq K.$$

Then $u \in L_{d+2}((0, T) \times \mathbb{R}^d)$, hence the solution is smooth and unique in $(0, T) \times \mathbb{R}^d$. Moreover, if $T = \infty$, we have

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{L_\infty^x} = 0.$$

Theorem 3.1.2. *Let $d \geq 4$ be an integer. Let $\mathbb{R}_+^d := \{x \in \mathbb{R}^d : x_d > 0\}$ and $T \in (0, \infty]$. Suppose (u, p) is a pair of Leray-Hopf weak solution to the Cauchy problem in $(0, T) \times \mathbb{R}_+^d$ and u satisfies the following condition for some $K > 0$,*

$$\|u\|_{L_\infty^t L_d^x((0, T) \times \mathbb{R}_+^d)} \leq K. \tag{3.1}$$

Let u satisfy the vanishing boundary condition

$$u(t, x) = 0 \quad \text{on } x_d = 0, \quad 0 \leq t \leq T.$$

We consider two types of local conditions on the pressure term p :

i) If we assume

$$p \in L_{2-1/d, \text{loc}}([0, T] \times \overline{\mathbb{R}_+^d}), \tag{3.2}$$

then u is regular up to the boundary in $(0, T) \times \mathbb{R}_+^d$.

ii) If we assume a stronger uniform local condition on p : for any $z_0 \in (0, T) \times \mathbb{R}_+^d$,

$$\|p\|_{L_{2-1/d}(Q(z_0, 1) \cap (0, T) \times \mathbb{R}_+^d)} \leq K, \quad (3.3)$$

then $u \in L_{d+2}((0, T) \times \mathbb{R}_+^d)$, hence the solution is smooth and unique. In this case, if $T = \infty$, we have

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{L_\infty^x(\mathbb{R}_+^d)} = 0. \quad (3.4)$$

Theorem 3.1.3. *Let $d \geq 4$ be an integer. Suppose (u, p) is a pair of Leray-Hopf weak solution to the Navier-Stokes problem (1.1) in $(-1, 0) \times \mathbb{R}_+^d$. Let $Q^+ := \{z = (t, x) : x \in \mathbb{R}^d, |x| < 1, x_d > 0, -1 < t < 0\}$. Assume (u, p) satisfies the conditions:*

$$u \in L_\infty^t L_d^x(Q^+), \quad p \in L_{2-1/d}(Q^+),$$

and the boundary condition

$$u(t, x) = 0 \quad \text{on } x_d = 0, \quad |x| \leq 1, \quad -1 \leq t \leq 0.$$

Then u is Hölder continuous in the closure of the set

$$Q^+(1/4) := \{z = (t, x) : x \in \mathbb{R}^d, |x| < 1/4, x_d > 0, -(1/4)^2 < t < 0\}.$$

Theorem 3.1.1 provides an alternative approach to prove the main result in [8]. As for Theorem 3.1.2, we consider two different types of local integrability conditions on the pressure, which we believe that are necessary for the half-space and also for local problems in Theorem 3.1.3. In fact, stronger assumptions such as assuming the pressure belongs to certain Sobolev spaces have been made in previous papers [45, 37, 44, 47]. To justify the generalized energy inequality in the definition of

suitable weak solutions, it also requires the local integrability conditions. We note that by using the argument in the proofs of [4, Sec. 2C], [7, Lemma 2.9], or [20, Lemma 4.2], it is possible to slightly relax such integrability conditions. However, in order not to overburden the proof we do not pursue to find an optimal condition for the pressure here.

Next we give a brief description of our argument for Theorems 3.1.1 and 3.1.2. By adding conditions (3.1)-(3.3), we extend [9, Theorem 1.2] to an ε -regularity criterion which reads that if certain scale invariant quantities are small then the solution is locally Hölder continuous. As in [12], we start with proof by contradiction and blow up the solution (u, p) near a singular point at the first blow-up time. We can show the scale invariant quantities are uniformly bounded along a blow-up sequence (u_k, p_k) , hence this implies there exists a pair of limiting suitable weak solution (u_∞, p_∞) to the Navier-Stokes equations. Furthermore, outside of a large cylinder, we can show the scale invariant quantities are indeed uniformly small for all (u_k, p_k) 's. Thus we can use the ε -regularity criterion to get local Hölder continuity and uniform local L_∞ bound for u_k 's. Together with L_p -convergence, we can show the local boundedness of u_∞ as well as $u_\infty(0, \cdot) = 0$ by reversing the blow-up procedure. Then by applying the backward uniqueness theorem proved in [13] to the vorticity equation, we can see that $\text{curl } u_\infty = 0$ in the outside region for all time, which further implies that $u_\infty \equiv 0$ by using the spatial analyticity of strong solutions and the weak-strong uniqueness of the Navier-Stokes equations. The rest part of the proof follows the approach in [8]. Utilizing the ε -regularity criteria proved in Sections 3.3 and 3.4, we first show there exists a u_{k_0} that is regular around the origin, hence this contradicts with the assumption that u blows up near a singular point. Next we bound the sup norm of u to conclude $u \in L_{d+2}((0, T) \times \Omega)$. For $T = \infty$, a key observation is that u is in $L_4((0, \infty) \times \mathbb{R}^d)$, which implies the smallness of its L_4 norm in $(T, \infty) \times \mathbb{R}^d$ for large

T and furthermore the smallness of the scale invariant quantities on any cylinder beyond time T . Again we can apply the ε -criteria to get a uniform L_∞ bound on the scaled solutions beyond time T . We finally prove the decay with respect to the time by scaling back to the original u .

The remaining part of the chapter is organized as follows. We introduce notation and terminologies in Section 3.2. Sections 3.3 and 3.4 are devoted to ε -regularity criteria in the whole space and half space, respectively. We use a three-step approach to obtain the ε -regularity criteria for both the whole space and the half space. In the first step, we give some estimates of the scale invariant quantities, which are by now standard and essentially follow the arguments in [34, 7]. In the second step, we establish a decay estimate of certain scale invariant quantities by using an iteration argument based on the estimates we proved in the first step. In the third step, we apply parabolic regularity to get an estimate of $L_{2-\frac{1}{d}}$ -mean oscillations of u , which yields the Hölder continuity of u according to Campanato's characterization of Hölder continuous functions. The main difference between the two cases lies in the treatment of the pressure term. In the interior case, the pressure can be decomposed into a sum of a harmonic function and a term controlled by u using the Calderón-Zygmund estimate. In the boundary case, we need the additional assumption (3.3) on the pressure to use classical L_p estimates for linear Stokes system to get a more subtle control of the pressure. In Section 3.5, we present the proof of Theorem 3.1.2 via the blow-up procedure mentioned previously. Theorem 3.1.1 follows a similar argument of 3.1.2 so we omit the details. Theorem 3.1.3 is another application of the ε -regularity criteria we proved in Sections 3.3 and 3.4. We briefly describe the proof of Theorem 3.1.3 in Section 3.6.

3.2 Preliminaries

3.2.1 Leray-Hopf weak solutions.

By a Leray-Hopf weak solution of (1.1)-(1.2) in Ω_T , we mean a vector field u such that:

i) $u \in L_\infty(0, T; \dot{J}(\Omega)) \cap L_2(0, T; \dot{J}_2^1(\Omega));$

ii) the function $t \rightarrow \int_\Omega u(t, x) \cdot w(x) dx$ is continuous on $[0, T]$ for any $w \in L_2(\Omega);$

iii) the equation (1.1) holds weakly in the sense that for any $w \in \dot{C}_0^\infty(\Omega_T),$

$$\int_{\Omega_T} (-u \cdot \partial_t w - u \otimes u : \nabla w + \nabla u : \nabla w) dx dt = 0;$$

iv) The energy inequality:

$$\frac{1}{2} \int_\Omega |u(t, x)|^2 dx + \int_{\Omega_t} |\nabla u(s, x)|^2 dx ds \leq \frac{1}{2} \int_\Omega |a(x)|^2 dx$$

holds for any $t \in [0, T]$, and we have

$$\|u(t, \cdot) - a(\cdot)\|_{L_2^x} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

When $\Omega = \mathbb{R}^d$ or $\mathbb{R}_+^d$, for any $a \in \dot{J}(\Omega)$, there exists at least one Leray-Hopf weak solution to the Cauchy problem (1.1)-(1.2) on $(0, \infty) \times \Omega$. See [32] and [23].

3.2.2 Suitable weak solutions

We revisit the definition of suitable weak solutions. Here we make a slightly different condition on the pressure term compared with that in Section 2.2.2. We say a pair (u, p) is a suitable weak solution of the Navier-Stokes equations on the set Ω_T if

$$i) \ u \in L_\infty(0, T; \dot{J}(\Omega, \Gamma)) \cap L_2(0, T; \dot{J}_2^1(\Omega, \Gamma)) \text{ and } p \in L_{2-1/d}(\Omega_T);$$

$$ii) \ u \text{ and } p \text{ satisfy equation (1.1) in the sense of distribution;}$$

iii) For any $t \in (0, T)$ and nonnegative function $\psi \in C^\infty(\overline{\Omega_T})$ vanishing in a neighborhood of the boundary $\{t = 0\} \times \Omega$ and $(0, T) \times (\partial\Omega \setminus \Gamma)$, the integrals in the following local energy inequality are summable and the inequality holds true:

$$\begin{aligned} \text{ess sup}_{0 \leq s \leq t} \int_{\Omega} |u(s, x)|^2 \psi(s, x) dx + 2 \int_{\Omega_t} |\nabla u|^2 \psi dx ds \\ \leq \int_{\Omega_t} \{|u|^2(\psi_t + \Delta\psi) + (|u|^2 + 2p)u \cdot \nabla\psi\} dx ds. \end{aligned} \quad (3.5)$$

3.2.3 Scale invariant quantities

In this chapter, we write a point in $[0, T] \times \mathbb{R}^d$ as $z = (t, x) = (t, x_1, x_2, \dots, x_d) = (t, x', x_d)$, where $x' = (x_1, x_2, \dots, x_{d-1})$. We shall use the following notation for balls, half balls, spheres, half spheres, parabolic cylinders, half parabolic cylinders, and parabolic boundaries:

$$B(\hat{x}, r) = \{x \in \mathbb{R}^d \mid |x - \hat{x}| < r\}, \quad B_r = B(r) = B(0, r), \quad B = B(1);$$

$$B^+(\hat{x}, r) = \{x \in B(\hat{x}, r) \mid x = (x', x_d), x_d > \hat{x}_d\},$$

$$B_r^+ = B^+(r) = B^+(0, r), \quad B^+ = B^+(1);$$

$$\begin{aligned}
S(\hat{x}, r) &= \{x \in \mathbb{R}^d \mid |x - \hat{x}| = r\}, \quad S_r = S(r) = S(0, r), \quad S = S(1); \\
S^+(\hat{x}, r) &= \{x \in S(\hat{x}, r) \mid x = (x', x_d), x_d > \hat{x}_d\}, \\
S_r^+ &= S^+(r) = S^+(0, r), \quad S^+ = S^+(1); \\
Q(\hat{z}, r) &= (\hat{t} - r^2, \hat{t}) \times B(\hat{x}, r), \quad Q_r = Q(r) = Q(0, r), \quad Q = Q(1); \\
Q^+(\hat{z}, r) &= (\hat{t} - r^2, \hat{t}) \times B^+(\hat{x}, r), \quad Q_r^+ = Q^+(r) = Q^+(0, r), \quad Q^+ = Q^+(1); \\
\partial_p Q(\hat{z}, r) &= [\hat{t} - r^2, \hat{t}) \times S(\hat{x}, r) \cup \{t = \hat{t} - r^2\} \times B(\hat{x}, r),
\end{aligned}$$

where $\hat{z} = (\hat{t}, \hat{x})$ and $\hat{x}_d$ is the d -th coordinate of $\hat{x}$. Moreover,

$$\Omega(\hat{x}, r) = B(\hat{x}, r) \cap \Omega, \quad \omega(\hat{z}, r) = Q(\hat{z}, r) \cap \Omega_T.$$

We denote mean values of summable functions as follows:

$$[u]_{\hat{x}, r}(t) = \frac{1}{|\Omega(\hat{x}, r)|} \int_{\Omega(\hat{x}, r)} u(t, x) dx, \quad (u)_{\hat{z}, r} = \frac{1}{|\omega(\hat{z}, r)|} \int_{\omega(\hat{z}, r)} u(z) dz,$$

where $|A|$ as usual denotes the Lebesgue measure of the set A .

Now we introduce the following important quantities:

i) When $\Omega = \mathbb{R}^d$,

$$\begin{aligned}
A(r, z_0) &= \text{ess sup}_{t_0 - r^2 \leq t \leq t_0} \frac{1}{r^{d-2}} \int_{B(x_0, r)} |u|^2 dx, \\
E(r, z_0) &= \frac{1}{r^{d-2}} \int_{Q(z_0, r)} |\nabla u|^2 dz, \\
C(r, z_0) &= \frac{1}{r^{d+2/d-2}} \int_{Q(z_0, r)} |u|^{2(2d-1)/d} dz, \\
D(r, z_0) &= \frac{1}{r^{d+2/d-2}} \int_{Q(z_0, r)} |p - [p]_{x_0, r}(t)|^{(2d-1)/d} dz.
\end{aligned}$$

ii) When $\Omega = \mathbb{R}_+^d$ or Q^+ ,

$$\begin{aligned} A^+(r, z_0) &= \operatorname{ess\,sup}_{t_0-r^2 \leq t \leq t_0} \frac{1}{r^{d-2}} \int_{\Omega(x_0, r)} |u|^2 dx, \\ E^+(r, z_0) &= \frac{1}{r^{d-2}} \int_{\omega(z_0, r)} |\nabla u|^2 dz, \\ C^+(r, z_0) &= \frac{1}{r^{d+2/d-2}} \int_{\omega(z_0, r)} |u|^{2(2d-1)/d} dz, \\ D^+(r, z_0) &= \frac{1}{r^{d+2/d-2}} \int_{\omega(z_0, r)} |p - [p]_{x_0, r}(t)|^{(2d-1)/d} dz. \end{aligned}$$

We notice that these quantities are all invariant under the natural scaling (1.6).

In the later part of the chapter, we use notation $A^{(+)}$ to represent either A or A^+ depending on $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}_+^d$ when there is no confusion and similarly for $E^{(+)}, C^{(+)}, D^{(+)}, Q^{(+)}$, etc. We omit z_0 , the argument for center, from the above expressions and write $A(r)$, $E(r)$, $C(r)$, and $D(r)$ when there is no ambiguity.

3.2.4 Strong solutions and spatial analyticity.

We recall the following local solvability of (1.1)-(1.2) (see, for instance, [25, 17, 51, 28, 56]), and spatial analyticity of strong solutions (see, for instance, [18, 10, 24, 36]).

Proposition 3.2.1. *For any divergence-free initial data $a \in L_p(\Omega)$ with $p \geq d$, where $\Omega = \mathbb{R}^d$ or $\mathbb{R}_+^d$, the Cauchy problem (1.1)-(1.2) has a unique strong solution $u \in C([0, \delta); L_p(\Omega))$ for some $\delta > 0$. Moreover, u is infinitely differentiable and spatial analytic for $t \in (0, \delta)$.*

In the following two sections, we will show the Hölder continuity of u given the scale invariant quantities defined previously are sufficiently small. The main

difference between the interior estimate and the boundary estimate results from the different estimates of quantities D and D^+ .

3.3 Interior Hölder continuity estimate

In this section, we consider $\Omega = \mathbb{R}^d$. We take a three-step approach to prove the following ε -regularity criterion in the whole space.

Theorem 3.3.1. *Let (u, p) be a pair of suitable weak solution of (1.1)-(1.2) in $(0, T) \times \mathbb{R}^d$ satisfying (3.1). There exists a universal constant ε_0 satisfying the following property. Assume that for a point $z_0 \in \mathbb{R}_T^{d+1}$ we have*

$$A(\rho_0, z_0) + E(\rho_0, z_0) + D(\rho_0, z_0) \leq \varepsilon_0$$

for some small $\rho_0 > 0$. Then u is Hölder continuous near z_0 .

3.3.1 Step 1

We present several inequalities of the scale invariant quantities. We will make use of the following interpolation inequality from [8, Lemma 2.1] substantially.

Lemma 3.3.2. *For any function $u \in W_2^1(B_r)$, $r > 0$, and $q \in [2, 2d/(d-2)]$, we have*

$$\begin{aligned} \int_{B_r} |u|^q dx &\leq N(q, d) \left(\int_{B_r} |\nabla u|^2 dx \right)^{\frac{d}{2}(\frac{q}{2}-1)} \left(\int_{B_r} |u|^2 dx \right)^{\frac{q}{2}-\frac{d}{2}(\frac{q}{2}-1)} \\ &\quad + N(q, d) r^{-d(\frac{q}{2}-1)} \left(\int_{B_r} |u|^2 dx \right)^{\frac{q}{2}}. \end{aligned} \tag{3.6}$$

Proof. Without loss of generality, we assume $r = 1$. For $q \in [2, 2d/(d-2)]$, we use Hölder's inequality inside the unit ball B ,

$$\|u\|_{L_q(B)} \leq \|u\|_{L_2(B)}^{1-\frac{d}{q}(\frac{q}{2}-1)} \|u\|_{L_{\frac{2d}{d-2}}(B)}^{\frac{d}{q}(\frac{q}{2}-1)},$$

which together with Sobolev embedding theorem gives (3.6). The lemma is proved. $\square$

The next lemma is an application of the interpolation inequality proved in Lemma 3.3.2.

Lemma 3.3.3. *For $\alpha \in [0, 1]$ and $p \in [2 + \frac{4\alpha}{d}, d + (4-d)\alpha]$, suppose $r > 0$, $Q(z_0, r) \subset \mathbb{R}_T^{d+1}$, and u satisfies the condition (3.1). Then we have*

$$\frac{1}{r^{d+2-p}} \int_{Q(z_0, r)} |u|^p dz \leq N (A(r, z_0) + E(r, z_0))^{\frac{d-p+2\alpha}{d-2}}.$$

In particular, taking $\alpha = 1$ and $p = 4 - 2/d$ we have

$$C(r, z_0) \leq N (A(r, z_0) + E(r, z_0))^{\frac{d-2+2/d}{d-2}}.$$

Proof. Without loss of generality, we assume $r = 1$. Because $p \in [2 + \frac{4\alpha}{d}, d + (4-d)\alpha]$, we know

$$\frac{pd - 2d - 4\alpha}{d(d-2)} \geq 0 \quad \text{and} \quad q := \frac{2d(d-p+2\alpha)}{d^2 - dp + 4\alpha} \in \left[2, \frac{2d}{d-2}\right].$$

By using Hölder's inequality and (3.6) with this q , we have

$$\int_B |u|^p dx \leq \left(\int_B |u|^{\frac{2d(d-p+2\alpha)}{d^2 - dp + 4\alpha}} dx \right)^{\frac{d^2 - dp + 4\alpha}{d(d-2)}} \left(\int_B |u|^d dx \right)^{\frac{pd - 2d - 4\alpha}{d(d-2)}}$$

$$\leq N \left(\int_B |\nabla u|^2 dx \right)^\alpha \left(\int_B |u|^2 dx \right)^{\frac{(1-\alpha)d+4\alpha-p}{d-2}} + N \left(\int_B |u|^2 dx \right)^{\frac{d-p+2\alpha}{d-2}},$$

where we used (3.1) in the last inequality. Integrating in time yields the desired result. $\square$

Lemma 3.3.4. *Let (u, p) be a pair of suitable weak solution of (1.1). For $\rho > 0$ and $Q(z_0, \rho) \subset \mathbb{R}_T^{d+1}$, we have*

$$A(\rho/2) + E(\rho/2) \leq N \left(C(\rho)^{\frac{d}{2d-1}} + C(\rho)^{\frac{3d}{2(2d-1)}} + C(\rho)^{\frac{d}{2(2d-1)}} D(\rho)^{\frac{d}{2d-1}} \right),$$

where N is independent of z_0 and ρ .

Proof. By a scaling argument, we may assume $\rho = 1$. In the energy inequality (3.5), we put $t = t_0$ and choose a suitable smooth cut-off function ψ such that

$$\psi \equiv 0 \text{ in } \mathbb{R}_{t_0}^{d+1} \setminus Q(z_0, 1), \quad 0 \leq \psi \leq 1 \text{ in } \mathbb{R}_{t_0}^{d+1},$$

$$\psi \equiv 1 \text{ in } Q(z_0, 1/2), \quad |\nabla \psi| + |\partial_t \psi| + |\nabla^2 \psi| < N \text{ in } \mathbb{R}_{t_0}^{d+1}.$$

By using (3.5), we get

$$A(1/2) + 2E(1/2) \leq N \int_{Q(z_0, 1)} |u|^2 dz + N \int_{Q(z_0, 1)} (|u|^2 + 2|p - [p]_{x_0, 1}|) |u| dz.$$

Due to Hölder's inequality, we can obtain

$$\int_{Q(z_0, 1)} |u|^2 dz \leq N(C(1))^{\frac{d}{2d-1}}, \quad \int_{Q(z_0, 1)} |u|^3 dz \leq N(C(1))^{\frac{3d}{2(2d-1)}},$$

and

$$\begin{aligned} \int_{Q(z_0,1)} |p||u| dz &\leq \left(\int_{Q(z_0,1)} |u|^{\frac{2d-1}{d-1}} dz \right)^{\frac{d-1}{2d-1}} \left(\int_{Q(z_0,1)} |p - [p]_{x_0,1}|^{\frac{2d-1}{d}} dz \right)^{\frac{d}{2d-1}} \\ &\leq C(1)^{\frac{d}{2(2d-1)}} D(1)^{\frac{d}{2d-1}}. \end{aligned}$$

The conclusion of Lemma 3.3.4 follows immediately. $\square$

Lemma 3.3.5. *Let (u, p) be a pair of suitable weak solution of (1.1) when $d \geq 4$. For constants $\gamma \in (0, 1/2]$, $\rho > 0$, and $Q(z_0, \rho) \subset \mathbb{R}_T^{d+1}$, we have*

$$\begin{aligned} &A(\gamma\rho) + E(\gamma\rho) \\ &\leq N \left[\gamma^2 A(\rho) + \gamma^{-d+1} \left((A(\rho) + E(\rho))^{\frac{d-1}{d-2}} + (A(\rho) + E(\rho))^{\frac{1}{2}} D(\rho)^{\frac{d}{2d-1}} \right) \right]. \end{aligned} \quad (3.7)$$

Proof. Assume $\rho = 1$. Define the backward heat kernel as

$$\Gamma(t, x) = \frac{1}{(4\pi(\gamma^2 + t_0 - t))^{d/2}} e^{-\frac{|x-x_0|^2}{4(\gamma^2+t_0-t)}},$$

In the energy inequality (3.5), we choose $\psi = \Gamma\phi$, where $\phi \in C_0^\infty((t_0 - 1, t_0 + 1) \times B(x_0, 1))$ is a suitable smooth cut-off function satisfying

$$0 \leq \phi \leq 1 \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \quad \phi \equiv 1 \quad \text{in } Q(z_0, 1/2),$$

$$|\nabla\phi| \leq N, \quad |\nabla^2\phi| \leq N, \quad |\partial_t\phi| \leq N \quad \text{in } \mathbb{R} \times \mathbb{R}^d.$$

By using the equation

$$\Delta\Gamma + \Gamma_t = 0,$$

we have

$$\begin{aligned}
& \text{ess sup}_{t_0-1 \leq t \leq t_0} \int_{B(x_0,1)} |u(t,x)|^2 \Gamma(t,x) \phi(t,x) dx + 2 \int_{Q(z_0,1)} |\nabla u|^2 \Gamma \phi dz \\
& \leq \int_{Q(z_0,1)} \{ |u|^2 (\Gamma \phi_t + \Gamma \Delta \phi + 2 \nabla \phi \nabla \Gamma) + (|u|^2 + 2|p - [p]_{x_0,1}|) u \cdot (\Gamma \nabla \phi + \phi \nabla \Gamma) \} dz.
\end{aligned} \tag{3.8}$$

The test function has the following properties:

(i) For some constant $c > 0$, on $Q(z_0, \gamma)$ it holds that

$$\Gamma \phi = \Gamma \geq c \gamma^{-d}.$$

(ii) For any $z \in Q(z_0, 1)$, we have

$$|\Gamma(z) \phi(z)| \leq N \gamma^{-d}, \quad |\nabla \Gamma(z) \phi(z)| + |\Gamma(z) \nabla \phi(z)| \leq N \gamma^{-d-1}.$$

(iii) For any $z \in Q(z_0, 1)$, we have

$$|\Gamma(z) \phi_t(z)| + |\Gamma(z) \Delta \phi(z)| + |\nabla \Gamma(z) \nabla \phi(z)| \leq N.$$

Therefore (3.8) yields

$$\begin{aligned}
A(\gamma) + E(\gamma) &= \gamma^{-d+2} \text{ess sup}_{t_0-\gamma^2 \leq t \leq t_0} \int_{B(x_0,\gamma)} |u(t,x)|^2 dx + \gamma^{-d+2} \int_{Q(z_0,\gamma)} |\nabla u|^2 dz \\
&\leq N \gamma^2 \int_{Q(z_0,1)} |u|^2 dz + N \gamma^{-d+1} \int_{Q(z_0,1)} (|u|^2 + 2|p - [p]_{x_0,1}|) |u| dz.
\end{aligned}$$

Recall that $d \geq 4$. Applying Lemma 3.3.3 with $\alpha = 1$ and $p = 3$ we have

$$\int_{Q(z_0,1)} |u|^3 dz \leq N (A(1, z_0) + E(1, z_0))^{\frac{d-1}{d-2}},$$

and again applying Lemma 3.3.3 with $\alpha = \frac{d}{4(d-1)}$ and $p = \frac{2d-1}{d-1}$, we have

$$\begin{aligned} & \int_{Q(z_0,1)} |p - [p]_{x_0,1}| |u| dz \\ & \leq \left(\int_{Q(z_0,1)} |u|^{\frac{2d-1}{d-1}} dz \right)^{\frac{d-1}{2d-1}} \left(\int_{Q(z_0,1)} |p - [p]_{x_0,1}|^{\frac{2d-1}{d}} dz \right)^{\frac{d}{2d-1}} \\ & \leq N (A(1, z_0) + E(1, z_0))^{\frac{1}{2}} D(1, z_0)^{\frac{d}{2d-1}}. \end{aligned}$$

The lemma is proved. $\square$

Lemma 3.3.6. *Let (u, p) be a pair of Leray-Hopf weak solution of (1.1). For constants $\gamma \in (0, 1/2]$, $\rho > 0$, and $Q(z_0, \rho) \subset \mathbb{R}_T^{d+1}$, we have*

$$D(\gamma\rho) \leq N(d)[\gamma^{-d-2/d+2}C(\rho) + \gamma^{4-3/d}D(\rho)]. \quad (3.9)$$

Proof. Let $r = \gamma\rho$ and $\eta(x)$ be a smooth cut-off function supported in $B(1)$, $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\bar{B}(2/3)$. In the sense of distribution, for a.e. $t \in (t_0 - \rho^2, t_0)$, one has

$$\Delta p = D_{ij}(u_i u_j).$$

We consider the decomposition

$$p = p_{x_0, \rho} + h_{x_0, \rho},$$

where $p_{x_0, \rho}$ is the Newtonian potential of

$$D_{ij}(u_i u_j \eta((x - x_0)/\rho)).$$

Then $h_{x_0, \rho}$ is harmonic in $B(x_0, 2\rho/3)$.

By using the Calderón-Zygmund estimate, we have

$$\int_{Q(z_0, r)} |p_{x_0, \rho}|^{\frac{2d-1}{d}} dz \leq \int_{Q(z_0, \rho)} |p_{x_0, \rho}|^{\frac{2d-1}{d}} dz \leq N \int_{Q(z_0, \rho)} |u|^{\frac{2(2d-1)}{d}} dz. \quad (3.10)$$

From the fact that any Sobolev norm of harmonic function $h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}$ in a smaller ball can be estimated by any of its L_p norm in $B(x_0, 2\rho/3)$, one obtains

$$\begin{aligned} \int_{B(x_0, r)} |h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}|^{\frac{2d-1}{d}} dx &\leq N r^{d + \frac{2d-1}{d}} \sup_{B(x_0, r)} |\nabla h_{x_0, \rho}|^{\frac{2d-1}{d}} \\ &\leq N \left(\frac{r}{\rho}\right)^{d + \frac{2d-1}{d}} \int_{B(x_0, \rho)} |h_{x_0, \rho} - [p]_{x_0, \rho}|^{\frac{2d-1}{d}} dx. \end{aligned} \quad (3.11)$$

Integrating (3.11) in $t \in (t_0 - r^2, t_0)$, we obtain

$$\begin{aligned} &\int_{Q(z_0, r)} |h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}|^{\frac{2d-1}{d}} dz \\ &\leq N \left(\frac{r}{\rho}\right)^{d + \frac{2d-1}{d}} \int_{Q(z_0, \rho)} \left(|p - [p]_{x_0, \rho}|^{\frac{2d-1}{d}} + |p_{x_0, \rho}|^{\frac{2d-1}{d}} \right) dz \\ &\leq N \left(\frac{r}{\rho}\right)^{d + \frac{2d-1}{d}} \int_{Q(z_0, \rho)} \left(|p - [p]_{x_0, \rho}|^{\frac{2d-1}{d}} + |u|^{\frac{2(2d-1)}{d}} \right) dz, \end{aligned} \quad (3.12)$$

where we used (3.10) in the last inequality. We combine (3.12), (3.10), and use the triangle inequality to have

$$\begin{aligned} &\int_{Q(z_0, r)} |p - [p]_{x_0, r}|^{\frac{2d-1}{d}} dz \\ &\leq N \int_{Q(z_0, r)} |p - [h_{x_0, \rho}]_{x_0, r}|^{\frac{2d-1}{d}} dz \\ &\leq N \int_{Q(z_0, r)} \left(|h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}|^{\frac{2d-1}{d}} + |p_{x_0, \rho}|^{\frac{2d-1}{d}} \right) dz \\ &\leq N \left(\frac{r}{\rho}\right)^{d + \frac{2d-1}{d}} \int_{Q(z_0, \rho)} \left(|p - [p]_{x_0, \rho}|^{\frac{2d-1}{d}} + |u|^{\frac{2(2d-1)}{d}} \right) dz. \end{aligned}$$

The lemma is proved. $\square$

Corollary 3.3.7. *Let (u, p) be a pair of suitable weak solution of (1.1). For some $z_0 \in \mathbb{R}_T^{d+1}$, suppose there exist $\rho_0 > 0$ and $C_1 > 0$, such that $Q(z_0, \rho_0) \subset \mathbb{R}_T^{d+1}$ and $C(\rho, z_0) \leq C_1$ for all $\rho \in (0, \rho_0]$ and $D(\rho_0, z_0) \leq C_1$. Then we can find $C = C(C_1, d) > 0$ such that $D(\rho, z) \leq C$ for all $z \in Q(z_0, 1/2)$ and $\rho \in (0, \rho_0/2]$.*

Proof. Note that $Q(z, \rho_0/2) \subset Q(z_0, \rho_0)$ for $z \in Q(z_0, 1/2)$, hence $D(\rho_0/2, z) \leq ND(\rho_0, z_0)$. By fixing γ small enough that $N\gamma^{4-3/d} \leq 1/2$ and using (3.9), we have

$$D(\gamma\rho_0/2, z) \leq \frac{1}{2}D(\rho_0/2, z) + C_2,$$

where $C_2 = N\gamma^{-d-2/d+2}C_1$. Inductively, for any integer k we have

$$D(\gamma^k\rho_0/2, z) \leq \frac{1}{2^k}D(\rho_0/2, z) + C_2 \sum_{j=0}^{k-1} \frac{1}{2^j} \leq NC_1 + 2C_2 := C_3.$$

Now for any $\gamma^{k+1}\rho_0 \leq \rho < \gamma^k\rho_0$, we can control $D(\rho/2, z)$ by

$$D(\rho/2, z) \leq \left(\frac{\gamma^k\rho_0}{\gamma^{k+1}\rho_0} \right)^{d+2/d-2} D(\gamma^k\rho_0/2, z) \leq \gamma^{-d-2/d+2}C_3 := C.$$

The corollary is proved because C is independent of k . □

3.3.2 Step 2

We will find some decay rates for the scale invariant quantities with respect to the radius assuming the quantities are initially small.

Lemma 3.3.8. *There exists a universal constant $\varepsilon_0 > 0$ satisfying the following property. Suppose that for some $z_0 = (t_0, x_0)$ and $\rho_0 > 0$, it holds that $Q(z_0, \rho_0) \subset$*

$\mathbb{R}_T^{d+1}$ and

$$A(\rho_0, z_0) + E(\rho_0, z_0) + D(\rho_0, z_0) \leq \varepsilon_0. \quad (3.13)$$

Then fixing any $\alpha_0 \in (0, 2)$, there exists $N > 0$ such that for any $\rho \in (0, \rho_0/2)$ and $z \in Q(z_0, \rho_0/2)$, the following estimate holds uniformly

$$A(\rho, z) + E(\rho, z) + C(\rho, z)^{\frac{d-2}{d-2+2/d}} + D(\rho, z) \leq N\varepsilon_0^{\alpha_0/2} \left(\frac{\rho}{\rho_0} \right)^{\alpha_0}, \quad (3.14)$$

where N is a positive constant depending on α_0 , but independent of ε_0 , ρ_0 , ρ , and z .

Proof. For any $z \in Q(z_0, \rho_0/2)$, by (3.13) and

$$Q(z, \rho_0/2) \subset Q(z_0, \rho_0) \subset \mathbb{R}_T^{d+1},$$

we get

$$A(\rho_1, z) + E(\rho_1, z) + D(\rho_1, z) \leq N\varepsilon_0, \quad (3.15)$$

where $\rho_1 = \rho_0/2$. By Lemma 3.3.3,

$$C(\rho_1, z) \leq (N\varepsilon_0)^{\frac{d-2+2/d}{d-2}}. \quad (3.16)$$

Next we fix an auxiliary parameter $\alpha \in (\alpha_0, 2)$. By a scaling argument, we first discuss a special case when $\rho_1^\alpha = N\varepsilon_0 < 1$. In this case, we can prove the following decay rates inductively:

$$A(\rho_k) + E(\rho_k) \leq \rho_k^\alpha, \quad C(\rho_k)^{\frac{d-2}{d-2+2/d}} \leq \rho_k^\alpha, \quad D(\rho_k) \leq \rho_k^\alpha, \quad (3.17)$$

where $\rho_{k+1} = \rho_k^{1+\beta} = \rho_1^{(1+\beta)^k}$ and β is a small number to be specified. For $k = 1$, the statement follows from (3.15), (3.16), and our assumption that $\rho_1^\alpha = N\varepsilon_0$. Next by

choosing $\gamma = \rho_k^\beta$ and $\rho = \rho_k$ in (3.7) and (3.9), we have

$$A(\rho_{k+1}) + E(\rho_{k+1}) \leq N \left[\rho_k^{2\beta+\alpha} + \rho_k^{(-d+1)\beta} \left(\rho_k^{\frac{d-1}{d-2}\alpha} + \rho_k^{\left(\frac{1}{2} + \frac{d}{2d-1}\right)\alpha} \right) \right],$$

$$D(\rho_{k+1}) \leq N \left[\rho_k^{(-d-\frac{2}{d}+2)\beta + \left(1 + \frac{2}{d(d-2)}\right)\alpha} + \rho_k^{\left(4-\frac{3}{d}\right)\beta + \alpha} \right].$$

We choose β satisfying

$$\beta < \min \left\{ \frac{\alpha}{(d-2)(d+\alpha-1)}, \frac{\alpha}{2(2d-1)(d+\alpha-1)}, \frac{2\alpha}{d(\alpha+d+\frac{2}{d}-2)(2d-2)} \right\}.$$

Then all the exponents on the right-hand sides are greater than $(1+\beta)\alpha$.

Now we can find $\xi > 0$ depending on β such that

$$A(\rho_{k+1}) + E(\rho_{k+1}) \leq N \rho_{k+1}^{\alpha+\xi}, \quad D(\rho_{k+1}) \leq N \rho_{k+1}^{\alpha+\xi},$$

where N is a constant independent of k and ξ . By taking ε_0 small enough such that $N \rho_{k+1}^\xi < N \rho_1^\xi < N(N\varepsilon_0)^{\xi/2} < 1$, we obtain

$$A(\rho_{k+1}) + E(\rho_{k+1}) \leq \rho_{k+1}^\alpha, \quad D(\rho_{k+1}) \leq \rho_{k+1}^\alpha.$$

By induction, we have justified (3.17) for the case when $\rho_1^\alpha = N\varepsilon_0$.

For convenience, we additionally assume that the parameter β satisfying

$$\alpha_0 < \min \left\{ \frac{1}{1+\beta}(\alpha - (d-2)\beta), \frac{1}{(1+\beta)} \left(\alpha - \left(d + \frac{2}{d} - 2 \right) \beta \right) \right\}.$$

There always exists feasible β because $\alpha > \alpha_0$.

Now for any $\rho \in (0, \rho_0/2)$, we can find a positive integer k such that $\rho_{k+1} \leq \rho <$

ρ_k . Then

$$A(\rho) + E(\rho) \leq (\rho_k/\rho_{k+1})^{d-2} (A(\rho_k) + E(\rho_k)) \leq \rho_k^{\alpha-(d-2)\beta} = \rho_{k+1}^{\frac{1}{1+\beta}(\alpha-(d-2)\beta)} \leq \rho^{\alpha_0},$$

$$D(\rho) \leq (\rho_k/\rho_{k+1})^{d+\frac{2}{d}-2} D(\rho_k) \leq \rho_k^{\alpha-(d+\frac{2}{d}-2)\beta} = \rho_{k+1}^{\frac{1}{1+\beta}(\alpha-(d+\frac{2}{d}-2)\beta)} \leq \rho^{\alpha_0}.$$

Hence we have proved (3.14) when $\rho_1^\alpha = N\varepsilon_0$. For the general case, we use the scale invariant property of the quantities and apply the previous results with an additional scaling factor $N\varepsilon_0^{\alpha_0/\alpha} \rho_0^{-\alpha_0}$ on the right-hand side. $\square$

3.3.3 Step 3

In this final step, we first use parabolic L_p estimates to further improve the decay rate and then conclude the result by using Campanato's characterization of Hölder continuity.

By Lemma 3.3.8 we know, for any small $\delta_1 > 0$, the following estimates are true for all $\rho \in (0, \rho_0/2)$ sufficiently small and $z_1 := (t_1, x_1) \in Q(z_0, \rho_0/2)$:

$$\int_{Q(z_1, \rho)} |u|^{4-\frac{2}{d}} dz \leq N\rho^{d+\frac{2}{d-2}-\delta_1}, \quad (3.18)$$

$$\int_{Q(z_1, \rho)} |p - [p]_{x_1, \rho}|^{2-\frac{1}{d}} dz \leq N\rho^{d+\frac{2}{d}-\delta_1}. \quad (3.19)$$

Let v be the unique weak solution to the heat equation

$$\partial_t v - \Delta v = 0 \quad \text{in } Q(z_1, \rho),$$

with the boundary condition $v = u$ on $\partial_p Q(z_1, \rho)$. Let $0 < r < \rho$. By the Poincaré inequality with zero mean value and using the fact that the $C^{1/2,1}$ semi-norm of a

caloric function in a smaller cylinder is controlled by any L_p norm of it in a larger cylinder. We have

$$\begin{aligned} \int_{Q(z_1, r)} |v - (v)_{z_1, r}|^{2-\frac{1}{d}} dz &\leq r^{d+4-\frac{1}{d}} \|v - (u)_{z_1, \rho}\|_{C^{1/2, 1}(Q(z_1, r))}^{2-\frac{1}{d}} \\ &\leq N \left(\frac{r}{\rho}\right)^{d+4-\frac{1}{d}} \int_{Q(z_1, \rho)} |v - (u)_{z_1, \rho}|^{2-\frac{1}{d}} dz. \end{aligned} \quad (3.20)$$

Denote $w = u - v$. Then w satisfies the inhomogeneous heat equation

$$\partial_t w_i - \Delta w_i = -\partial_j(u_i u_j) - \partial_i(p - [p]_{x_1, \rho}) \quad \text{in } Q(z_1, \rho)$$

with the zero boundary condition. By the classical L_p estimate for the heat equation, we have

$$\|\nabla w\|_{L_{2-\frac{1}{d}}(Q(z_1, \rho))} \leq N \| |u|^2 \|_{L_{2-\frac{1}{d}}(Q(z_1, \rho))} + N \|p - [p]_{x_1, \rho}\|_{L_{2-\frac{1}{d}}(Q(z_1, \rho))},$$

which together with (3.18) and (3.19) yields

$$\int_{Q(z_1, \rho)} |\nabla w|^{2-\frac{1}{d}} dz \leq N \rho^{d+\frac{2}{d}-\delta_1}. \quad (3.21)$$

By the Poincaré inequality with zero boundary condition, we get from (3.21) that

$$\int_{Q(z_1, \rho)} |w|^{2-\frac{1}{d}} dz \leq N \rho^{d+\frac{2}{d}-\delta_1+2-\frac{1}{d}} = N \rho^{d+2+\frac{1}{d}-\delta_1}. \quad (3.22)$$

Using (3.20), (3.22), and the triangle inequality, we have

$$\begin{aligned} &\int_{Q(z_1, r)} |u - (u)_{z_1, r}|^{2-\frac{1}{d}} dz \\ &\leq \int_{Q(z_1, r)} |v - (v)_{z_1, r}|^{2-\frac{1}{d}} dz + \int_{Q(z_1, r)} |w - (w)_{z_1, r}|^{2-\frac{1}{d}} dz \end{aligned}$$

$$\begin{aligned}
&\leq N \left(\frac{r}{\rho} \right)^{d+4-\frac{1}{d}} \int_{Q(z_1, \rho)} |v - (u)_{z_1, \rho}|^{2-\frac{1}{d}} dz + N \rho^{d+2+\frac{1}{d}-\delta_1} \\
&\leq N \left(\frac{r}{\rho} \right)^{d+4-\frac{1}{d}} \left(\int_{Q(z_1, \rho)} |u - (u)_{z_1, \rho}|^{2-\frac{1}{d}} dz + \int_{Q(z_1, \rho)} |w|^{2-\frac{1}{d}} dz \right) + N \rho^{d+2+\frac{1}{d}-\delta_1} \\
&\leq N \left(\frac{r}{\rho} \right)^{d+4-\frac{1}{d}} \int_{Q(z_1, \rho)} |u - (u)_{z_1, \rho}|^{2-\frac{1}{d}} dz + N \rho^{d+2+\frac{1}{d}-\delta_1}.
\end{aligned}$$

Applying Lemma 2.3.5 and choosing $\delta_1 = \frac{1}{2d}$, we obtain

$$\int_{Q(z_1, r)} |u - (u)_{z_1, r}|^{2-\frac{1}{d}} dz \leq N r^{d+2+\frac{1}{2d}},$$

for any $r \in (0, \rho_0/4)$ and $z_1 \in Q(z_0, \rho_0/4)$. We then conclude that u is Hölder continuous near z_0 by Campanato's characterization of Hölder continuity.

3.4 Boundary Hölder continuity estimate

In this section, we consider the case when $\Omega = \mathbb{R}_+^d$. We again use a three-step approach to prove an ε -regularity criterion near boundary. The main difference from the interior estimate is the iteration dealing with the pressure term.

Theorem 3.4.1. *Let (u, p) be a suitable weak solution of (1.1)-(1.2) in $(0, T) \times \mathbb{R}_+^d$ satisfying (3.1). There exists a universal constant $\hat{\varepsilon}_0$ satisfying the following property. Assume that for a point $\hat{z} = (\hat{t}, \hat{x})$, where $\hat{x} = (x', 0)$, and for some $\rho_0 > 0$ we have $Q^+(\hat{z}, \rho_0) \subset (0, T) \times \mathbb{R}_+^d$ and*

$$A^+(\rho_0, \hat{z}) + E^+(\rho_0, \hat{z}) + D^+(\rho_0, \hat{z}) \leq \hat{\varepsilon}_0.$$

Then u is Hölder continuous near $\hat{z}$.

3.4.1 Step 1

We present several inequalities for the scale invariant quantities.

Lemma 3.4.2. *Suppose function $u \in W_2^1(B_r^+)$ with $r > 0$ vanish on the boundary $x_d = 0$. For any $q \in [2, 2d/(d-2)]$, we have*

$$\int_{B_r^+} |u|^q dx \leq N(q, d) \left(\int_{B_r^+} |\nabla u|^2 dx \right)^{d(q/4-1/2)} \left(\int_{B_r^+} |u|^2 dx \right)^{q/2-d(q/4-1/2)}.$$

Proof. Modify the proof of Lemma 3.3.2 using the Poincaré inequality with odd extension for functions vanishing on the flat boundary

$$\int_{B_r^+} |u|^2 dx \leq Nr^2 \int_{B_r^+} |\nabla u|^2 dx$$

to absorb the second term on the right-hand side. $\square$

We recall the following two important lemmas which are useful in handling the estimates for the pressure p and the proofs are essentially the same as those of Lemmas 2.4.4 and 2.4.5.

Lemma 3.4.3. *Let $\mathcal{D} \subset \mathbb{R}^d$ be a domain with smooth boundary and $T > 0$ be a constant. Let $1 < m < +\infty, 1 < n < +\infty$ be two fixed numbers. Assume that $g \in L_n^t L_m^x(\mathcal{D}_T)$. Then there exists a unique function pair (v, p) , which satisfies the following equation:*

$$\begin{cases} \partial_t v - \Delta v + \nabla p = g & \text{in } \mathcal{D}_T, \\ \nabla \cdot v = 0 & \text{in } \mathcal{D}_T, \\ [p]_{\mathcal{D}}(t) = 0 & \text{for a.e. } t \in [0, T], \\ v = 0 & \text{on } \partial_p \mathcal{D}_T. \end{cases}$$

Moreover, v and p satisfy the following estimate:

$$\|v\|_{W_{n,m}^{1,2}(\mathcal{D}_T)} + \|p\|_{W_{n,m}^{0,1}(\mathcal{D}_T)} \leq C \|g\|_{L_n^t L_m^x(\mathcal{D}_T)},$$

where the constant C only depends on m, n, T , and $\mathcal{D}$.

Lemma 3.4.4. *Let $1 < m \leq 2, 1 < n \leq 2$, and $m \leq s < +\infty$ be constants and $g \in L_n^t L_s^x(Q^+)$. Assume that the functions $v \in W_{n,m}^{0,1}(Q^+)$ and $p \in L_n^t L_m^x(Q^+)$ satisfy the equations:*

$$\begin{cases} \partial_t v - \Delta v + \nabla p = g & \text{in } Q^+, \\ \nabla \cdot v = 0 & \text{in } Q^+, \end{cases}$$

and the boundary condition

$$v = 0 \quad \text{on } \{y \mid y = (y', 0)0, \|y'\| < 1\} \times [-1, 0).$$

Then, we have $v \in W_{n,s}^{1,2}(Q^+(1/2))$, $p \in W_{n,s}^{0,1}(Q^+(1/2))$, and

$$\|v\|_{W_{n,s}^{1,2}(Q^+(1/2))} + \|p\|_{W_{n,s}^{0,1}(Q^+(1/2))} \leq C \left(\|g\|_{L_n^t L_s^x(Q^+)} + \|v\|_{W_{n,m}^{0,1}(Q^+)} + \|p\|_{L_n^t L_m^x(Q^+)} \right),$$

where the constant C only depends on m, n , and s .

The following three lemmas are analogous to Lemma 3.3.3, 3.3.4, 3.3.5, and the proofs are similar.

Lemma 3.4.5. *For $\alpha \in [0, 1]$, $p \in [2 + 4\alpha/d, d + (4 - d)\alpha]$, suppose $r > 0$, $\hat{x} \in \partial\Omega$, $\omega(\hat{z}, r) = Q^+(\hat{z}, r)$, and u satisfies the condition (3.1). Then we have*

$$\frac{1}{r^{d+2-p}} \int_{Q^+(\hat{z}, r)} |u|^p dz \leq N \left(A^+(r, \hat{z}) + E^+(r, \hat{z}) \right)^{\frac{d-p+2\alpha}{d-2}}.$$

In particular, taking $\alpha = 1$ and $p = 4 - 2/d$ we have

$$C^+(r, \hat{z}) \leq N \left(A^+(r, \hat{z}) + E^+(r, \hat{z}) \right)^{\frac{d-2+2/d}{d-2}}.$$

Lemma 3.4.6. *Let (u, p) be a pair of suitable weak solution of (1.1). For $\hat{x} \in \partial\Omega$ and $\omega(\hat{z}, \rho) = Q^+(\hat{z}, \rho)$, we have*

$$A^+(\rho/2) + E^+(\rho/2) \leq N \left(C^+(\rho)^{\frac{d}{2d-1}} + C^+(\rho)^{\frac{3d}{2(2d-1)}} + C^+(\rho)^{\frac{d}{2(2d-1)}} D^+(\rho)^{\frac{d}{2d-1}} \right), \quad (3.23)$$

where N is independent of $\hat{z}$ and ρ .

Lemma 3.4.7. *Let (u, p) be a pair of suitable weak solution of (1.1) with $d \geq 4$. For constant $\gamma \in (0, 1/2]$, $\rho > 0$ and $\hat{x} \in \partial\Omega$, $\omega(\hat{z}, \rho) = Q^+(\hat{z}, \rho)$, we have*

$$\begin{aligned} & A^+(\gamma\rho) + E^+(\gamma\rho) \\ & \leq N \left[\gamma^2 A^+(\rho) + \gamma^{-d+1} \left((A^+(\rho) + E^+(\rho))^{\frac{d-1}{d-2}} + (A^+(\rho) + E^+(\rho))^{\frac{1}{2}} D^+(\rho)^{\frac{d}{2d-1}} \right) \right]. \end{aligned} \quad (3.24)$$

At last, we present an estimate for quantity $D^+(\rho)$, which is essentially different from Lemma 3.3.6 for the interior case.

Lemma 3.4.8. *Let (u, p) be a pair of Leray-Hopf weak solution of (1.1) and u satisfies condition (3.1). Let $\gamma \in (0, 1/4]$ and $\rho > 0$ be constants. Suppose that $\hat{x} \in \partial\Omega$ and $\omega(\hat{z}, \rho) = Q^+(\hat{z}, \rho)$. Then given any small $\delta_2 > 0$, we have*

$$\begin{aligned} D^+(\gamma\rho) & \leq N \left[\gamma^{-d-2/d+2} (A^+(\rho) + E^+(\rho))^{1+\frac{2}{d(d-2)}} \right. \\ & \quad \left. + \gamma^{4-3/d-\delta_2} \left(D^+(\rho) + A^+(\rho)^{1-\frac{1}{2d}} + E^+(\rho)^{1-\frac{1}{2d}} \right) \right], \end{aligned} \quad (3.25)$$

where N is a constant independent of γ, ρ , and $\hat{z}$, but may depend on δ_2 .

Proof. Without loss of generality, by shifting the coordinate we may assume that $\hat{z} = (0, 0)$. By the scale-invariant property, we may also assume $\rho = 1$. We fix a domain $\tilde{B} \subset \mathbb{R}^d$ with smooth boundary so that

$$B^+(1/2) \subset \tilde{B} \subset B^+,$$

and denote $\tilde{Q} = (-1, 0) \times \tilde{B}$. Using Hölder's inequality, Lemma 3.4.2 with $q = \frac{2d(d+2)}{d^2+4}$, and (3.1), we get

$$\begin{aligned} & \left(\int_{B^+} |u \cdot \nabla u|^{\frac{d(2d-1)}{d^2+2d-1}} dx \right)^{\frac{d^2+2d-1}{d^2}} \\ & \leq \left(\int_{B^+} |\nabla u|^2 dx \right)^{1-\frac{1}{2d}} \left(\int_{B^+} |u|^{\frac{2d(d+2)}{d^2+4}} dx \right)^{\frac{d^2+4}{2d^2(d-2)}} \left(\int_{B^+} |u|^d dx \right)^{\frac{2(d-3)}{d(d-2)}} \\ & \leq N \left(\int_{B^+} |\nabla u|^2 dx \right) \left(\int_{B^+} |u|^2 dx \right)^{\frac{2}{d(d-2)}}. \end{aligned} \quad (3.26)$$

Integrating in t , we have $u \cdot \nabla u \in L^t_{\frac{2d-1}{d}} L^x_{\frac{d(2d-1)}{d^2+2d-1}}(\tilde{Q})$. By Lemma 3.4.3, there is a unique solution

$$v \in W^{\frac{1,2}{\frac{2d-1}{d}, \frac{d(2d-1)}{d^2+2d-1}}}(\tilde{Q}) \quad \text{and} \quad p_1 \in W^{0,1}_{\frac{2d-1}{d}, \frac{d(2d-1)}{d^2+2d-1}}(\tilde{Q})$$

to the following initial boundary value problem:

$$\begin{cases} \partial_t v - \Delta v + \nabla p_1 = -u \cdot \nabla u & \text{in } \tilde{Q}, \\ \nabla \cdot v = 0 & \text{in } \tilde{Q}, \\ [p_1]_{\tilde{B}}(t) = 0 & \text{for a.e. } t \in [0, T], \\ v = 0 & \text{on } \partial_p \tilde{Q}. \end{cases}$$

Moreover, we have

$$\begin{aligned}
& \| |v| + |\nabla v| + |p_1| + |\nabla p_1| \|_{L^t_{\frac{2d-1}{d}} L^x_{\frac{d(2d-1)}{d^2+2d-1}}(\tilde{Q})} \leq N \|u \cdot \nabla u\|_{L^t_{\frac{2d-1}{d}} L^x_{\frac{d(2d-1)}{d^2+2d-1}}(\tilde{Q})} \\
& \leq N \left(\int_{-1}^0 \left(\int_{B^+} |\nabla u|^2 dx \right) \left(\int_{B^+} |u|^2 dx \right)^{\frac{2}{d(d-2)}} dt \right)^{d/(2d-1)}, \tag{3.27}
\end{aligned}$$

where in the last inequality we used (3.26).

We set $w = u - v$ and $p_2 = p - p_1 - [p]_{0,1/2}$. Then w and p_2 satisfy

$$\begin{cases} \partial_t w - \Delta w + \nabla p_2 = 0 & \text{in } \tilde{Q}, \\ \nabla \cdot w = 0 & \text{in } \tilde{Q}, \\ w = 0 & \text{on } [-1, 0) \times \{\partial \tilde{B} \cap \partial \Omega\}. \end{cases}$$

By Lemma 3.4.4 and the triangle inequality, fixing some $s > 0$ large enough to be specified later, we have $p_2 \in W^{0,1}_{\frac{2d-1}{d},s}(Q^+(1/4))$ and

$$\begin{aligned}
& \|\nabla p_2\|_{L^t_{\frac{2d-1}{d}} L^x_s(Q^+(1/4))} \leq N \| |w| + |\nabla w| + |p_2| \|_{L^t_{\frac{2d-1}{d}} L^x_{\frac{d(2d-1)}{d^2+2d-1}}(Q^+(1/2))} \\
& \leq N \| |u| + |\nabla u| + |p - [p]_{0,1/2}| \|_{L^t_{\frac{2d-1}{d}} L^x_{\frac{d(2d-1)}{d^2+2d-1}}(Q^+(1/2))} \\
& \quad + N \| |v| + |\nabla v| + |p_1| \|_{L^t_{\frac{2d-1}{d}} L^x_{\frac{d(2d-1)}{d^2+2d-1}}(Q^+(1/2))}.
\end{aligned}$$

Together with (3.27), we obtain

$$\begin{aligned}
& \|\nabla p_2\|_{L^t_{\frac{2d-1}{d}} L^x_s(Q^+(1/4))} \leq N \| |u| + |\nabla u| + |p - [p]_{0,1/2}| \|_{L^t_{\frac{2d-1}{d}} L^x_{\frac{d(2d-1)}{d^2+2d-1}}(Q^+(1/2))} \\
& \quad + N \left(\int_{-1}^0 \left(\int_{B^+} |\nabla u|^2 dx \right) \left(\int_{B^+} |u|^2 dx \right)^{\frac{2}{d(d-2)}} dt \right)^{d/(2d-1)}. \tag{3.28}
\end{aligned}$$

By making s large such that $\frac{2d-1}{s} < \delta_2$, we finish the proof. $\square$

Corollary 3.4.9. *Let (u, p) be a pair of suitable weak solution of (1.1) with $d \geq 4$. Suppose there exist $\hat{x} \in \partial\Omega$, $C_1 > 0$, and $\rho_0 > 0$ such that $\omega(\hat{z}, \rho) = Q^+(\hat{z}, \rho)$, $C^+(\rho, \hat{z}) \leq C_1$ for all $\rho \in (0, \rho_0]$ and $D^+(\rho_0, \hat{z}) \leq C_1$. Then we can find $C := C(C_1) > 0$ such that $D^+(\rho, z^*) \leq C$ for all $z^* \in Q(\hat{z}, \rho_0/2) \cap \{x_d = 0\}$ and $\rho \in (0, \rho_0/4)$.*

Proof. Because $Q^+(z^*, \rho_0/2) \subset Q^+(\hat{z}, \rho_0)$ for $z^* \in Q(\hat{z}, \rho_0/2) \cap \{x_d = 0\}$, we have

$$D^+(\rho_0/2, z^*) \leq ND^+(\rho_0, \hat{z}).$$

From (3.23) and (3.25) with $\delta_2 = 1$, we can get an estimation for D^+ :

$$\begin{aligned} D^+\left(\frac{\gamma\rho}{4}, z^*\right) &\leq N \left[\gamma^{-d-2/d+2} \left(A^+\left(\frac{\rho}{4}, z^*\right) + E^+\left(\frac{\rho}{4}, z^*\right) \right)^{1+\frac{2}{d(d-2)}} \right. \\ &\quad \left. + \gamma^{3-3/d} \left(D^+\left(\frac{\rho}{4}, z^*\right) + A^+\left(\frac{\rho}{4}, z^*\right)^{1-\frac{1}{2d}} + E^+\left(\frac{\rho}{4}, z^*\right)^{1-\frac{1}{2d}} \right) \right] \\ &\leq N(C_1) \left[\gamma^{-d-2/d+2} \left(1 + D^+\left(\frac{\rho}{2}, z^*\right)^{\frac{d}{2d-1}} \right)^{1+\frac{2}{d(d-2)}} \right. \\ &\quad \left. + \gamma^{3-3/d} \left(D^+\left(\frac{\rho}{4}, z^*\right) + \left(1 + D^+\left(\frac{\rho}{2}, z^*\right)^{\frac{d}{2d-1}} \right)^{1-\frac{1}{2d}} \right) \right] \\ &\leq N(C_1) \left[\gamma^{-d-2/d+2} \left(1 + D^+\left(\frac{\rho}{2}, z^*\right)^{\frac{d^2-2d+2}{2d^2-5d+2}} \right) + \gamma^{3-3/d} \left(D^+\left(\frac{\rho}{2}, z^*\right) + 1 \right) \right]. \end{aligned} \tag{3.29}$$

For any $\varepsilon > 0$ and $\delta := 1 - \frac{d^2-2d+2}{2d^2-5d+2}$, which is positive when $d \geq 4$, again by Young's inequality we get

$$D^+(\rho/2, z^*)^{1-\delta} = (\varepsilon D^+(\rho/2, z^*))^{1-\delta} \frac{1}{\varepsilon^{1-\delta}} \leq (1-\delta)\varepsilon D^+(\rho/2, z^*) + \delta \varepsilon^{-\frac{1-\delta}{\delta}}.$$

We can choose γ and ε small such that $N\gamma^{3-3/d} < 1/8$ and $N\gamma^{-d-2/d+2}(1-\delta)\varepsilon < 1/8$.

The two inequalities above implies that (3.29) can be written into such form:

$$D^+(\gamma\rho/4, z^*) \leq \frac{1}{2}D^+(\rho/2, z^*) + C.$$

The rest of the proof is a handy modification of Corollary 3.3.7. $\square$

3.4.2 Step 2

We will find some decay rates for the scale invariant quantities with respect to the radius of the cylinder assuming the quantities are initially small.

Lemma 3.4.10. *There exists a universal constant $\hat{\varepsilon}_0 > 0$ satisfying the following property. Suppose that for some $\hat{z} = (\hat{x}, \hat{t})$, where $\hat{x} = (x', 0)$, and for some $\rho_0 > 0$, it holds that $\omega(\hat{z}, \rho_0) = Q^+(\hat{z}, \rho_0)$ and*

$$A^+(\rho_0, \hat{z}) + E^+(\rho_0, \hat{z}) + D^+(\rho_0, \hat{z})^{1/\tau} \leq \hat{\varepsilon}_0, \quad (3.30)$$

where $\tau = 1 - \frac{1}{2d} + \varepsilon$, $\varepsilon \sim \mathcal{O}(\frac{1}{d^2})$. Then fixing any $\alpha_0 \in (0, 2)$, we can find $N > 0$ such that for any $\rho \in (0, \rho_0/4)$ and $z^* \in \overline{Q^+(\hat{z}, \rho_0/4)}$, the following estimate holds uniformly

$$A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*)^{\frac{d-2}{d-2+2/d}} + D^+(\rho, z^*) \leq N \hat{\varepsilon}_0^{\alpha_0^2 \tau/4} \left(\frac{\rho}{\rho_0} \right)^{\alpha_0^2 \tau/2}, \quad (3.31)$$

where N is a positive constant depending on α_0 , but independent of $\hat{\varepsilon}_0$, ρ_0 , ρ , and z^* .

Proof. We divide the proof into two parts. In the first part, we only consider z^* on the boundary $Q(\hat{z}, \rho_0/2) \cap \{x_d = 0\}$. In the second part, we use an iteration

argument to close the proof for general $z^* \in Q^+(\hat{z}, \rho_0/4)$.

i) First we assume $z^* \in Q(\hat{z}, \rho_0/2) \cap \{x_d = 0\}$. In this case we will prove a slightly stronger estimate than (3.31):

$$A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*)^{\frac{d-2}{d-2+2/d}} + D^+(\rho, z^*)^{1/\tau} \leq N\hat{\varepsilon}_0^{\alpha_0/2} \left(\frac{\rho}{\rho_0}\right)^{\alpha_0}. \quad (3.32)$$

By (3.30) and

$$Q^+(z^*, \rho_0/2) \subset Q^+(\hat{z}, \rho_0),$$

we get

$$A^+(\rho_1, z^*) + E^+(\rho_1, z^*) + D^+(\rho_1, z^*)^{1/\tau} \leq N\hat{\varepsilon}_0, \quad (3.33)$$

where $\rho_1 = \rho_0/2$. By Lemma 3.4.5,

$$C^+(\rho_1, z^*) \leq (N\hat{\varepsilon}_0)^{\frac{d-2+2/d}{d-2}}. \quad (3.34)$$

Next we fix an auxiliary parameter $\alpha \in (\alpha_0, 2)$. By a scaling argument, we first discuss a special case when $\rho_1^\alpha = N\hat{\varepsilon}_0 < 1$. In this case, we can prove the following decay rates inductively:

$$A^+(\rho_k) + E^+(\rho_k) \leq \rho_k^\alpha, \quad C^+(\rho_k)^{\frac{d-2}{d-2+2/d}} \leq \rho_k^\alpha, \quad D^+(\rho_k) \leq \rho_k^{\alpha\tau}, \quad (3.35)$$

where $\rho_{k+1} = \rho_k^{1+\beta} = \rho_1^{(1+\beta)^k}$, and β is a small number to be specified. For $k = 1$, the statement follows from (3.33) and (3.34). Next by choosing $\gamma = \rho_k^\beta$ and $\rho = \rho_k$ in (3.24), we have

$$A^+(\rho_{k+1}) + E^+(\rho_{k+1}) \leq N \left[\rho_k^{2\beta+\alpha} + \rho_k^{(-d+1)\beta} \left(\rho_k^{\frac{d-1}{d-2}\alpha} + \rho_k^{\left(\frac{1}{2} + \frac{d}{2d-1}\tau\right)\alpha} \right) \right].$$

We choose β satisfying

$$\beta < \min \left\{ \frac{\alpha}{(d-2)(d+\alpha-1)}, \frac{\left(\frac{d\tau}{2d-1} - \frac{1}{2}\right)\alpha}{d+\alpha-1} \right\} \sim \mathcal{O}\left(\frac{1}{d^2}\right).$$

Then all the exponents on the right-hand sides are greater than $(1+\beta)\alpha$.

To estimate the remaining term $D^+(\rho_{k+1})$, we apply Lemma 3.4.8 but with different step size. Let $\beta_1 = (1+\beta)^{n_0+1} - 1$, where n_0 is an integer to be specified later. Instead of plugging in the result of one last previous step, we plug in (3.25) with $\gamma = \rho_{k-n_0}^{\beta_1}$ and $\rho = \rho_{k-n_0}$, and we have

$$D^+(\rho_{k+1}) \leq N \left[\rho_{k-n_0}^{(-d-2/d+2)\beta_1 + (1+\frac{2}{d(d-2)})\alpha} + \rho_{k-n_0}^{(4-3/d-\delta_2)\beta_1} \left(\rho_{k-n_0}^{\alpha\tau} + \rho_{k-n_0}^{\alpha(1-\frac{1}{2d})} \right) \right]. \quad (3.36)$$

Our goal is to choose an appropriate β_1 such that all three exponents on right-hand side of (3.36) are greater than $(1+\beta_1)\alpha\tau$. We hence obtain an upper bound and a lower bound for β_1 :

$$\begin{aligned} \beta_1 &> \frac{\left(\tau - \left(1 - \frac{1}{2d}\right)\right)\alpha}{4 - \frac{3}{d} - \alpha\tau} = \frac{\varepsilon\alpha}{4 - \frac{3}{d} - \alpha\tau}, \\ \beta_1 &< \frac{\left(1 + \frac{2}{d(d-2)} - \tau\right)\alpha}{d + \frac{2}{d} - 2 + \alpha\tau} = \frac{\left(\frac{1}{2d} + \frac{2}{d(d-2)} - \varepsilon\right)\alpha}{d + \frac{2}{d} - 2 + \alpha\tau} \sim \mathcal{O}\left(\frac{1}{d^2}\right). \end{aligned}$$

To ensure such β_1 exists, we make $\varepsilon \sim \mathcal{O}\left(\frac{1}{d^2}\right)$ small such that the upper bound is greater than the lower bound of β_1 . As long as β is small enough, there exists an integer n_0 such that $\beta_1 = (1+\beta)^{n_0+1} - 1$ satisfies the conditions above.

Now we can find $\xi > 0$ depending on β and that

$$A^+(\rho_{k+1}) + E^+(\rho_{k+1}) \leq N\rho_{k+1}^{\alpha+\xi}, \quad D^+(\rho_{k+1}) \leq N\rho_{k+1}^{\alpha\tau+\xi},$$

where N is a constant independent of k and ξ . By taking $\hat{\varepsilon}_0$ small enough such that

$$N\rho_{k+1}^\xi < N\rho_1^\xi < N(N\hat{\varepsilon}_0)^{\xi/2} < 1,$$

we obtain

$$A^+(\rho_{k+1}) + E^+(\rho_{k+1}) \leq \rho_{k+1}^\alpha, \quad D^+(\rho_{k+1}) \leq \rho_{k+1}^{\alpha\tau}.$$

By induction, we have justified (3.35) for the case when $\rho_1^\alpha = N\hat{\varepsilon}_0$.

For convenience, we additionally assume that the parameter β satisfying

$$\alpha_0 < \min \left\{ \frac{1}{1+\beta}(\alpha - (d-2)\beta), \frac{1}{(1+\beta)\tau} \left(\alpha\tau - \left(d + \frac{2}{d} - 2 \right) \beta \right) \right\}.$$

There always exists feasible β because $\alpha > \alpha_0$.

Now for any $\rho \in (0, \rho_0/2)$, we can find a positive integer k such that $\rho_{k+1} \leq \rho < \rho_k$. Then

$$A^+(\rho) + E^+(\rho) \leq \left(\frac{\rho_k}{\rho_{k+1}} \right)^{d-2} (A^+(\rho_k) + E^+(\rho_k)) \leq \rho_k^{\alpha-(d-2)\beta} = \rho_{k+1}^{\frac{1}{1+\beta}(\alpha-(d-2)\beta)} \leq \rho_{k+1}^{\alpha_0},$$

$$D^+(\rho) \leq \left(\frac{\rho_k}{\rho_{k+1}} \right)^{d+\frac{2}{d}-2} D^+(\rho_k) \leq \rho_k^{\alpha\tau-(d+\frac{2}{d}-2)\beta} = \rho_{k+1}^{\frac{1}{1+\beta}(\alpha\tau-(d+\frac{2}{d}-2)\beta)} \leq \rho_{k+1}^{\alpha_0\tau}.$$

Hence we have proved the statement of the lemma when $\rho_1^\alpha = N\hat{\varepsilon}_0$. For general $\rho_0 > 0$, we use the scale invariant property of the quantities and yield similar results with an additional scaling factor $N\hat{\varepsilon}_0^{\alpha_0/\alpha} \rho_0^{-\alpha_0}$ on the right-hand side. Hence (3.32) is true.

ii) To deal with $z^* \in Q^+(\hat{z}, \rho_0/4)$, we need to discuss two cases as comparing x_d^* , the distance of z^* to the boundary, with ρ , the radius of the cylinder.

When $\rho \geq x_d^*$, we denote the projection of z^* on the boundary by $\hat{z}^*$. Because $\omega(z^*, \rho) \subset Q^+(\hat{z}^*, 2\rho)$, by definition we have

$$\begin{aligned} & A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*)^{\frac{d-2}{d-2+2/d}} \\ & \leq N \left(A^+(2\rho, \hat{z}^*) + E^+(2\rho, \hat{z}^*) + C^+(2\rho, \hat{z}^*)^{\frac{d-2}{d-2+2/d}} \right). \end{aligned}$$

By the triangle inequality, we have

$$\int_{\omega(z^*, \rho)} |p - [p]_{x^*, \rho}(t)|^{2-1/d} dz \leq N \int_{Q^+(\hat{z}^*, 2\rho)} |p - [p]_{\hat{z}^*, 2\rho}(t)|^{2-1/d} dz.$$

Hence $D^+(\rho, z^*) \leq ND^+(2\rho, \hat{z}^*)$. From part (i) we know

$$A^+(2\rho, \hat{z}^*) + E^+(2\rho, \hat{z}^*) + C^+(2\rho, \hat{z}^*)^{\frac{d-2}{d-2+2/d}} + D^+(2\rho, \hat{z}^*)^{1/\tau} \leq N\hat{\varepsilon}_0^{\alpha_0/2} \left(\frac{\rho}{\rho_0} \right)^{\alpha_0}.$$

The three inequalities above together imply that

$$A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*)^{\frac{d-2}{d-2+2/d}} + D^+(\rho, z^*)^{1/\tau} \leq N\hat{\varepsilon}_0^{\alpha_0/2} \left(\frac{\rho}{\rho_0} \right)^{\alpha_0}.$$

When $\rho < x_d^*$, we have $\omega(z^*, \rho) = Q(z^*, \rho) \subset Q(z^*, x_d^*) \subset Q^+(\hat{z}^*, 2x_d^*)$. By the proof above, we have

$$A^+(x_d^*, z^*) + E^+(x_d^*, z^*) + C^+(x_d^*, z^*)^{\frac{d-2}{d-2+2/d}} + D^+(x_d^*, z^*)^{1/\tau} \leq N\hat{\varepsilon}_0^{\alpha_0/2} \left(\frac{x_d^*}{\rho_0} \right)^{\alpha_0}.$$

With $\hat{\varepsilon}_0$ small such that $N\hat{\varepsilon}_0^{\alpha_0/2} \left(\frac{x_d^*}{\rho_0} \right)^{\alpha_0} < \varepsilon_0^{1/\tau}$ where ε_0 is from Lemma 3.3.8, we can apply the interior result of Lemma 3.3.8 to obtain

$$A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*)^{\frac{d-2}{d-2+2/d}} + D^+(\rho, z^*) \leq N\hat{\varepsilon}_0^{\alpha_0^2\tau/4} \left(\frac{\rho}{\rho_0} \right)^{\alpha_0^2\tau/2}.$$

The proof is complete. $\square$

3.4.3 Step 3

In this final step, we first use parabolic L_p estimates to further improve the decay rate and then conclude the result by using Campanoto's characterization of Hölder continuity. By (3.31) from the previous step, we know the following estimates are true for all $\rho > 0$ sufficiently small and $z^* = (t^*, x^*) \in Q(\hat{z}, \rho_0/4) \cap \{x_d = 0\}$:

$$\int_{Q^+(z^*, \rho)} |u|^{4-\frac{2}{d}} dz \leq N \rho^{d+\frac{2}{d-2}-\frac{1}{d}+\varepsilon}, \quad (3.37)$$

$$\int_{Q^+(z^*, \rho)} |p - [p]_{x^*, \rho}|^{2-\frac{1}{d}} dz \leq N \rho^{d+\frac{1}{d}+\varepsilon}, \quad (3.38)$$

where $\varepsilon \sim \mathcal{O}\left(\frac{1}{d^2}\right)$.

Let v be the unique weak solution to the heat equation

$$\partial_t v - \Delta v = 0 \quad \text{in } Q^+(z^*, \rho)$$

with the boundary condition $v = u$ on $\partial_p Q^+(z^*, \rho)$.

Let $0 < r < \rho$. By the Poincaré inequality with zero boundary condition and using the fact that the L_∞ norm of the gradient of a caloric function in a small half cylinder is controlled by any L_p norm of it in a larger half cylinder, we have

$$\begin{aligned} \int_{Q^+(z^*, r)} |v|^{2-\frac{1}{d}} dz &\leq N r^{d+4-\frac{1}{d}} \|\nabla v\|_{L_\infty(Q^+(z^*, r))} \\ &\leq N \left(\frac{r}{\rho}\right)^{d+4-\frac{1}{d}} \int_{Q^+(z^*, \rho)} |v|^{2-\frac{1}{d}} dz. \end{aligned} \quad (3.39)$$

Denote $w = u - v$. Then w satisfies the inhomogeneous heat equation

$$\partial_t w_i - \Delta w_i = -\partial_j(u_i u_j) - \partial_i(p - [p]_{x^*, \rho}) \quad \text{in } Q^+(z^*, \rho)$$

with the zero boundary condition. By the classical L_p estimate for the heat equation, we have

$$\|\nabla w\|_{L_{2-\frac{1}{d}}(Q^+(z^*, \rho))} \leq N \| |u|^2 \|_{L_{2-\frac{1}{d}}(Q^+(z^*, \rho))} + N \|p - [p]_{x^*, \rho}\|_{L_{2-\frac{1}{d}}(Q^+(z^*, \rho))},$$

which together with (3.37) and (3.38) yields

$$\int_{Q^+(z^*, \rho)} |\nabla w|^{2-\frac{1}{d}} dz \leq N \rho^{d+\frac{1}{d}+\varepsilon}.$$

By the Poincaré inequality with zero boundary condition, we get

$$\int_{Q^+(z^*, \rho)} |w|^{2-\frac{1}{d}} dz \leq N \rho^{d+2+\varepsilon}. \quad (3.40)$$

Using (3.39), (3.40), and the triangle inequality, we have

$$\begin{aligned} \int_{Q^+(z^*, r)} |u|^{2-\frac{1}{d}} dz &\leq N \int_{Q^+(z^*, r)} |v|^{2-\frac{1}{d}} dz + N \int_{Q^+(z^*, r)} |w|^{2-\frac{1}{d}} dz \\ &\leq N \left(\frac{r}{\rho}\right)^{d+4-\frac{1}{d}} \int_{Q^+(z^*, \rho)} |v|^{2-\frac{1}{d}} dz + N r^{d+2+\varepsilon} \\ &\leq N \left(\frac{r}{\rho}\right)^{d+4-\frac{1}{d}} \int_{Q^+(z^*, \rho)} |u|^{2-\frac{1}{d}} dz + N \left(\frac{r}{\rho}\right)^{d+4-\frac{1}{d}} \int_{Q^+(z^*, \rho)} |w|^{2-\frac{1}{d}} dz + N r^{d+2+\varepsilon} \\ &\leq N \left(\frac{r}{\rho}\right)^{d+4-\frac{1}{d}} \int_{Q^+(z_0, \rho)} |u|^{2-\frac{1}{d}} dz + N \rho^{d+2+\varepsilon}. \end{aligned}$$

Applying Lemma 2.3.5, we obtain

$$\int_{Q^+(z^*, r)} |u|^{2-\frac{1}{d}} dz \leq N r^{d+2+\varepsilon} \quad (3.41)$$

for any $r \leq \rho_0/4$ and $z^* \in Q(\hat{z}, \rho_0/4) \cap \{x_d = 0\}$.

Consider any $\tilde{z} = (\tilde{t}, \tilde{x}) \in Q^+(\hat{z}, \rho_0/8)$. Let $z^* = (\tilde{t}, \tilde{x}', 0)$ be the projection of $\tilde{z}$ on the boundary. Note that $z^* \in Q(\hat{z}, \rho_0/8) \cap \{x_d = 0\}$. We consider two cases either the radius of the parabolic ball around z^* is smaller or larger than $\tilde{x}_d$.

Case 1: $\tilde{x}_d \leq r$. In this case, we have $\omega(\tilde{z}, r) \subset Q^+(z^*, 2r)$. Thus by (3.41), we have

$$\int_{\omega(\tilde{z}, r)} |u - (u)_{\tilde{z}, r}|^{2-1/d} dz \leq N \int_{Q^+(z^*, 2r)} |u|^{2-1/d} dz \leq Nr^{d+2+\varepsilon}.$$

Case 2: $r < \tilde{x}_d$. For $r < \rho \leq \tilde{x}_d$, from the proof of Theorem 3.3.1, we have

$$\int_{Q(\tilde{z}, r)} |u - (u)_{\tilde{z}, r}|^{2-1/d} dz \leq N \left(\frac{r}{\rho} \right)^{d+4-\frac{1}{d}} \int_{Q(\tilde{z}, \rho)} |u - (u)_{\tilde{z}, \rho}|^{2-1/d} dz + N\rho^{d+2+\varepsilon}. \quad (3.42)$$

We apply Lemma 2.3.5 to (3.42) to get

$$\int_{Q(\tilde{z}, r)} |u - (u)_{\tilde{z}, r}|^{2-1/d} dz \leq N \left(\frac{r}{\tilde{x}_d} \right)^{d+2+\varepsilon} \int_{Q(\tilde{z}, \tilde{x}_d)} |u - (u)_{\tilde{z}, \tilde{x}_d}|^{2-1/d} dz + Nr^{d+2+\varepsilon}. \quad (3.43)$$

By Case 1, we have

$$\int_{Q(\tilde{z}, \tilde{x}_d)} |u - (u)_{\tilde{z}, \tilde{x}_d}|^{2-1/d} dz \leq N\tilde{x}_d^{d+2+\varepsilon}.$$

Plug this into (3.43) to get

$$\int_{Q(\tilde{z}, r)} |u - (u)_{\tilde{z}, r}|^{2-1/d} dz \leq Nr^{d+2+\varepsilon}.$$

By Campanato's characterization of Hölder continuity near a flat boundary (see,

for instance, [33, Lemma 4.11]), we can conclude that u is Hölder continuous in a neighborhood of $\hat{z}$.

3.5 Proofs of Theorems 3.1.1 and 3.1.2

In this section, we start with a construction on a sequence of suitable weak solutions which converges to a limiting solution. Let (u, p) be a pair of Leray-Hopf weak solution of the Cauchy problem (1.1)-(1.2) on $\mathbb{R}^d$ or $\mathbb{R}_+^d$. Because of the local strong solvability for smooth data and the weak-strong uniqueness (see, for instance, [53]), we know that u is regular for $t \in (0, T_0)$ for some $T_0 \in (0, T]$. Suppose T_0 is the first blowup time of u , and

$$Z_0 = (T_0, X_0) = (T_0, X_{0,1}, X_{0,2}, \dots, X_{0,d}) = (T_0, X'_0, X_{0,d})$$

is a singular point. We take a decreasing sequence $\{\lambda_k\}$ converging to 0 and rescale the pair (u, p) at time T_0 . Define

$$u_k(t, x) = \lambda_k u(T_0 + \lambda_k^2 t, X_0 + \lambda_k x), \quad p_k(t, x) = \lambda_k^2 p(T_0 + \lambda_k^2 t, X_0 + \lambda_k x),$$

for each $k = 1, 2, \dots$. We will show that each (u_k, p_k) is a suitable weak solution of (1.1)-(1.2) and u_k is smooth for $t \in (-\lambda_k^{-2} T_0, 0)$.

To prove this, the first observation is the property of uniform boundedness of the scale invariant quantities after rescaling. In this section, we use the ambiguous notation as we mentioned before in the preliminaries. By $\mathbb{R}_{(+)}^d$ we mean either $\mathbb{R}^d$ or $\mathbb{R}_+^d$ depending on which domain we are talking about: the whole space or the half space. The same goes for $A^{(+)}, E^{(+)}, C^{(+)}, D^{(+)}$, etc.

Lemma 3.5.1. *Under the assumptions in Theorem 3.1.2, there exists $N > 0$ such that for any $z_0 \in (-\infty, 0] \times \mathbb{R}_{(+)}^d$ and $0 < r \leq 1$,*

$$\limsup_{k \rightarrow \infty} C^{(+)}(r, z_0, u_k, p_k) \leq N, \quad (3.44)$$

and

$$\limsup_{k \rightarrow \infty} D^{(+)}(r, z_0, u_k, p_k) \leq N. \quad (3.45)$$

Remark 3.5.2. If we merely assume (3.2), then N may depend on the choice of Z_0 . If we assume the stronger condition (3.3), then N is independent the choice of Z_0 .

Proof. For $z_0 \in (-\infty, 0] \times \mathbb{R}_{(+)}^d$, denote

$$z_0^k = (t_0^k, x_0^k) = (T_0 + \lambda_k^2 t_0, X_0 + \lambda_k x_0).$$

For convenience, we first assume $\Omega = \mathbb{R}^d$ to prove (3.44). Since C and D are invariant, we have

$$C(r, z_0, u_k, p_k) = C(\lambda_k r, z_0^k, u, p) \quad \text{and} \quad D(r, z_0, u_k, p_k) = D(\lambda_k r, z_0^k, u, p).$$

Using Hölder's inequality we have

$$C(r, z_0, u, p) = \frac{1}{r^{d+2/d-2}} \int_{Q(z_0, r)} |u|^{\frac{2(2d-1)}{d}} dz \leq N \|u\|_{L_\infty^t L_d^x(Q(z_0, r))}^{\frac{2(2d-1)}{d}}. \quad (3.46)$$

Substituting r with $\lambda_k r$ and z_0 with z_0^k , we have

$$C(\lambda_k r, z_0^k, u, p) \leq N \|u\|_{L_\infty^t L_d^x(Q(z_0^k, \lambda_k r))}^{\frac{2(2d-1)}{d}} \leq N,$$

where in the last inequality we used (3.1). This part of proof can easily be adapted

to the case $\Omega = \mathbb{R}_+^d$.

To prove (3.45), we need to consider several cases separately:

i) $\Omega = \mathbb{R}^d$, by using the Calderón-Zygmund estimate, one has

$$\|p\|_{L_\infty^t L_{d/2}^x((0,T) \times \mathbb{R}^d)} \leq K.$$

Since $\frac{d}{2} \geq 2 - \frac{1}{d}$, following the same reasoning in (3.46) we can reach (3.45).

ii) $\Omega = \mathbb{R}_+^d$ and $X_{0,d} > 0$, that is the half-space case when Z_0 does not lie on the boundary. The domain of u_k will expand to the whole space, therefore $z_0 \in (-\infty, 0] \times \mathbb{R}^d$. When $X_{0,d} \geq 1/4$, i.e., Z_0 is away from the boundary, from (3.2) or (3.3) we know $D(1/4, Z_0, u, p) \leq N$. When k is large, $Q(z_0^k, \lambda_k r) \subset Q(Z_0, 1/4)$. By Corollary 3.3.7, we know $D(r, z_0, u_k, p_k) = D(\lambda_k r, z_0^k, u, p)$ is uniformly bounded. When $X_{0,d} < 1/4$, i.e., Z_0 is close to the boundary, recall our notation $\omega(Z_0, 1) = Q(Z_0, 1) \cap (0, T) \times \mathbb{R}_+^d$. Denote $\hat{Z}_0 = (X'_0, 0, T_0)$ to be the projection of Z_0 on the boundary and $\hat{z}_0^k$ to be projection of z_0^k . Note when k is large, $\lambda_k r \ll X_{0,d}$, hence

$$Q(z_0^k, \lambda_k r) \subset Q(Z_0, X_{0,d}) \subset Q^+(\hat{Z}_0, 2X_{0,d}) \subset Q^+(\hat{Z}_0, 1/2) \subset \omega(Z_0, 1).$$

From the proofs of Corollaries 3.3.7 and 3.4.9, we know that

$$D(r, z_0, u_k, p_k) = D(\lambda_k r, z_0^k, u, p) \leq ND(X_{0,d}, Z_0, u, p) + C, \quad (3.47)$$

$$D^+(2X_{0,d}, \hat{Z}_0, u, p) \leq ND^+(1/2, \hat{Z}_0, u, p) + C \leq ND^+(1, Z_0, u, p) + C \leq C. \quad (3.48)$$

We use (3.3) in the last inequality. Moreover, we have

$$\int_{Q(Z_0, X_{0,d})} |p - (p)_{Z_0, X_{0,d}}|^{2-1/d} dz \leq N \int_{Q^+(\hat{Z}_0, 2X_{0,d})} |p - (p)_{\hat{Z}_0, 2X_{0,d}}|^{2-1/d} dz, \quad (3.49)$$

which implies

$$D(X_{0,d}, Z_0, u, p) \leq ND^+(2X_{0,d}, \hat{Z}_0, u, p).$$

Together with (3.47) and (3.48), we again deduce that $D(r, z_0, u_k, p_k)$ is uniformly bounded.

iii) $\Omega = \mathbb{R}_+^d$ and $X_{0,d} = 0$, that is the half-space case when Z_0 lies on the boundary. The domain of u_k will expand to the half space, therefore $z_0 \in (-\infty, 0] \times \mathbb{R}_+^d$. We compare the radius of the cylinder against the distance from x_0 to the boundary. When $r \geq x_{0,d}$, we have $\omega(z_0^k, \lambda_k r) \subset Q^+(\hat{z}_0^k, 2\lambda_k r) \subset Q^+(Z_0, 1)$ when k is large. By (3.3), (3.49), and the proof of Corollary 3.4.9, we have

$$\begin{aligned} D^+(r, z_0, u_k, p_k) &= D^+(\lambda_k r, z_0^k, u, p) \leq ND^+(2\lambda_k r, \hat{z}_0^k, u, p) \\ &\leq ND^+(1, Z_0, u, p) + C \leq C. \end{aligned}$$

When $r \leq x_{0,d}$, we have

$$\omega(z_0^k, \lambda_k r) = Q(z_0^k, \lambda_k r) \subset Q(z_0^k, \lambda_k x_{0,d}) \subset Q^+(\hat{z}_0^k, 2\lambda_k x_{0,d}) \subset Q^+(Z_0, 1)$$

when k is large. From (3.3), (3.49), and the proofs of Corollaries 3.3.7, and 3.4.9, we know that

$$\begin{aligned} D(r, z_0, u_k, p_k) &= D(\lambda_k r, z_0^k, u, p) \leq ND(\lambda_k x_{0,d}, \hat{z}_0^k, u, p) + C \\ &\leq ND^+(2\lambda_k x_{0,d}, \hat{z}_0^k, u, p) + C \leq ND^+(1, Z_0, u, p) + C \leq C. \end{aligned}$$

Therefore, we have proved that $D^+(r, z_0, u_k, p_k)$ is uniformly bounded by the $L_\infty^t L_d^x$ condition in (3.1) and the local pressure condition in (3.3). $\square$

Next we want to show, up to passing to a subsequence, $\{(u_k, p_k)\}_{k=1}^\infty$ converge to a limiting solution (u_∞, p_∞) . We modify [8, Proposition 3.5] and state the results on $\mathbb{R}^d$ in next proposition. These results can be easily extended to $\mathbb{R}_+^d$. To make the statement concise, we hereby introduce the following notation: $L_{p,\text{unif}}(\Omega_T)$, which means that the L_p norm in $Q(z_0, 1) \cap \Omega_T$ for any $z_0 \in \Omega_T$ are uniformly bounded independent of the choice of z_0 .

Proposition 3.5.3. *i) There is a subsequence of (u_k, p_k) , which is still denoted by (u_k, p_k) , such that*

$$u_k \rightarrow u_\infty \text{ in } C([t_0 - 1/4^2, t_0]; L_{q_1}(B(x_0, 1/4))), \quad (3.50)$$

$$p_k \rightharpoonup p_\infty \text{ weakly in } L_{2-\frac{1}{d}}(Q(z_0, 1/4)). \quad (3.51)$$

for any $z_0 \in (-\infty, 0] \times \mathbb{R}^d$ and $q_1 \in [1, d)$.

ii) Furthermore, (u_∞, p_∞) is a suitable weak solution of (1.1) in $(-\infty, 0) \times \mathbb{R}^d$, and

$$u_\infty \in L_{q_2}^t L_d^x(((-T_1, 0) \times \mathbb{R}^d)), \quad p_\infty \in L_{2-\frac{1}{d}, \text{unif}}((-T_1, 0) \times \mathbb{R}^d).$$

for any $T_1 > 0$ and $q_2 \in [1, \infty)$.

Proof. First we fix a $z_0 \in (-\infty, 0] \times \mathbb{R}^d$. By the previous Lemma 3.5.1, p_k 's have a uniform bound of the $L_{2-\frac{1}{d}}(Q(z_0, 1))$ norm, so there is a subsequence, which is still denoted by $\{p_k\}$, such that (3.51) holds. Similarly,

$$\|u_k\|_{L_\infty^t L_d^x(Q(z_0, 1))} \leq \|u_k\|_{L_\infty^t L_d^x((t_0-1, t_0) \times \mathbb{R}^d)} \leq N,$$

where N is independent of k . By Lemmas 3.3.4 and 3.5.1, we have

$$A(1/2, z_0, u_k, p_k) + E(1/2, z_0, u_k, p_k) \leq N.$$

From (3.1) and the weak continuity of Leray-Hopf weak solutions, we can conclude that

$$\|u_k(t, \cdot)\|_{L_d(B(z_0, 1/2))} \leq N, \quad (3.52)$$

for each $t \in [-\lambda_k^{-2}T_0, 0)$. By using Lemma 3.3.2 with $q = 2d/(d-2)$ and $r = 1/2$, we have

$$\|u_k\|_{L_2^t L_{2d/(d-2)}^x(Q(z_0, 1/2))} \leq N,$$

which together with (3.52) and Hölder's inequality yields

$$\|u_k\|_{L_4(Q(z_0, 1/2))} \leq N, \quad \|u_k \cdot \nabla u_k\|_{L_{4/3}(Q(z_0, 1/2))} \leq N.$$

Following the coercive estimate for the Stokes system (see, for instance, [35]) we have

$$\partial_t u_k, D^2 u_k, \nabla p_k \in L_{4/3}(Q(z_0, 1/4))$$

with uniform norms. Therefore, we can find a subsequence still denoted by $\{u_k\}$ such that

$$u_k \rightarrow u_\infty \text{ in } C([t_0 - 1/4^2, t_0]; L_{4/3}(B(x_0, 1/4))).$$

This together with (3.52) gives (3.50) by using Hölder's inequality. To finish the proof of Part i), it suffices to use a Cauchy diagonal argument. Part ii) then follows from Part i) and the fact that p_k 's have a uniform bound of the $L_{2-\frac{1}{d}}$ norm in $Q(z_0, 1)$, which is independent of the choice of z_0 . $\square$

Corollary 3.5.4. *When $d \geq 4$, for any $\varepsilon > 0$, $r > 0$, and $T_1 \geq 1$, we can find $R \geq 1$*

such that, for any $z_0 \in (-T_1 - 1, 0] \times (\mathbb{R}_{(+)}^d \setminus B_{R+1}^{(+)})$,

$$\limsup_{k \rightarrow \infty} C^{(+)}(r, z_0, u_k, p_k) \leq \varepsilon. \quad (3.53)$$

Proof. For simplicity let us assume $\Omega = \mathbb{R}^d$, as the other case is similar. Due to Proposition 3.5.3 ii), for any $r > 0$, and $T_1 \geq 1$, we can find R large such that

$$\frac{1}{r^2} \int_{(-T_1-2, 0) \times (\mathbb{R}^d \setminus B_R)} |u_\infty|^d dz$$

is sufficiently small. Thus by Hölder's inequality when $d \geq 4$, for any $z_0 \in (-T_1 - 1, 0] \times (\mathbb{R}^d \setminus B_{R+1})$,

$$\frac{1}{r^{d+2/d-2}} \int_{Q(z_0, r)} |u_\infty|^{2(2-1/d)} dz$$

is sufficiently small. This together with Proposition 3.5.3 i) proves (3.53). $\square$

Lemma 3.5.5. *When $d \geq 4$, for any $\varepsilon_1 > 0$ and $T_1 \leq 1$, we can find $R \geq 1$ and $r_0 > 0$ such that, for any $z_0 \in (-T_1 - 1, 0] \times (\mathbb{R}_{(+)}^d \setminus B_{R+2}^{(+)})$,*

$$\begin{aligned} \limsup_{k \rightarrow \infty} (A^{(+)}(r_0, z_0, u_k, p_k) + E^{(+)}(r_0, z_0, u_k, p_k) \\ + C^{(+)}(r_0, z_0, u_k, p_k) + D^{(+)}(r_0, z_0, u_k, p_k)) \leq \varepsilon_1. \end{aligned} \quad (3.54)$$

Proof. This lemma is to improve the boundedness property of C and D we achieved in Lemma 3.5.1 to smallness. We first prove the interior case. From Lemma 3.5.1, we have

$$\limsup_{k \rightarrow \infty} D(r, z_0, u_k, p_k) \leq N,$$

for any $z_0 \in (-\infty, 0] \times \mathbb{R}^d$. Now let $r = 1$. From Corollary 3.5.4, for any $\varepsilon_0 > 0$, we can find $R > 0$ such that for any $z_0 \in (-T_1 - 1, 0] \times (\mathbb{R}^d \setminus B_{R+2})$,

$$\limsup_{k \rightarrow \infty} C(1, z_0, u_k, p_k) \leq \varepsilon_0.$$

By Lemma 3.3.6, we immediately know that there exists $\gamma > 0$ such that

$$\limsup_{k \rightarrow \infty} D(\gamma, z_0, u_k, p_k) \leq \frac{N\varepsilon_0}{\gamma^{d+2/d-2}} + N\gamma^{4-3/d}. \quad (3.55)$$

Then

$$\limsup_{k \rightarrow \infty} C(\gamma, z_0, u_k, p_k) \leq \frac{\varepsilon_0}{\gamma^{d+2/d-2}}. \quad (3.56)$$

Using Lemma 3.3.4 we can get

$$\limsup_{k \rightarrow \infty} (A(\gamma/2, z_0, u_k, p_k) + E(\gamma/2, z_0, u_k, p_k)) \leq N \left(\frac{\varepsilon_0}{\gamma^{d+2/d-2}} \right)^{1/4}. \quad (3.57)$$

We add (3.56), (3.55), and (3.57) together to obtain

$$\begin{aligned} & \limsup_{k \rightarrow \infty} (A(\gamma/2, z_0, u_k, p_k) + E(\gamma/2, z_0, u_k, p_k) \\ & + C(\gamma/2, z_0, u_k, p_k) + D(\gamma/2, z_0, u_k, p_k)) \\ & \leq N \left(\frac{\varepsilon_0}{\gamma^{d+2/d-2}} \right)^{1/4} + \frac{N\varepsilon_0}{\gamma^{d+2/d-2}} + N\gamma^{4-3/d}. \end{aligned} \quad (3.58)$$

For any $\varepsilon_1 > 0$, fix some γ such that $N\gamma^{4-3/d} < \varepsilon_1/2$. Next by choosing $R > 0$ large enough, we can make ε_0 small such that

$$N \left(\frac{\varepsilon_0}{\gamma^{d+2/d-2}} \right)^{1/4} + \frac{N\varepsilon_0}{\gamma^{d+2/d-2}} < \varepsilon_1/2.$$

Letting $r_0 = \gamma/2$, by (3.58) we have proved (3.54) for the interior case.

For the boundary case, we do a similar discussion on different shapes of $\omega(z_0^k, \lambda_k r)$ as we did in the proof of Lemma 3.5.1. We hereby omit the repeated proof. $\square$

Next we show that u_∞ is identically equal to zero. We modify the proof of [8,

Proposition 5.3] by replacing Schoen's trick with the Hölder continuity proved in Sections 3.3 and 3.4. We state and prove the following proposition for $\Omega = \mathbb{R}_+^d$. The results readily generalize to problem on $\Omega = \mathbb{R}^d$.

Proposition 3.5.6. *Under the assumptions of Theorem 3.1.2, let (u_∞, p_∞) be the suitable weak solution constructed in this section. Then*

$$u_\infty(t, \cdot) \equiv 0 \quad \forall t \in (-\infty, 0).$$

Proof. Let $\hat{\varepsilon}_0$ be the constant in Theorem 3.4.1. Note that we can assume $\hat{\varepsilon}_0$ is smaller than ε_0 in Theorem 3.3.1. Fix some $T_1 \geq 1$. Owing to Lemma 3.5.5, we can find $R \geq 1$ and $r_0 > 0$ such that for any $z_0 \in [-T_1 - 1, 0] \times (\mathbb{R}_+^d \setminus B_{R+1}^+)$,

$$\limsup_{k \rightarrow \infty} (A^+(r_0, z_0, u_k, p_k) + E^+(r_0, z_0, u_k, p_k) + D^+(r_0, z_0, u_k, p_k)) \leq \hat{\varepsilon}_0.$$

Thus Theorems 3.3.1 and 3.4.1 yields that

$$\limsup_{k \rightarrow \infty} |u_k(z_0)| \leq N(d)$$

for a.e. $z_0 \in [-T_1 - 1, 0] \times (\mathbb{R}_+^d \setminus B_{R+2}^+)$. By Proposition 3.5.3, we obtain

$$|u_\infty(z_0)| \leq N(d)$$

for a.e. $z_0 \in [-T_1 - 1, 0] \times (\mathbb{R}_+^d \setminus B_{R+2}^+)$. Upon using the regularity results for linear Stokes systems, one can estimate higher derivatives

$$|D^j u_\infty(z_0)| \leq N(d, j) \tag{3.59}$$

for any $j \geq 1$ and a.e. $z_0 \in [-T_1 - 1, 0] \times (\mathbb{R}_+^d \setminus B_{R+3}^+)$.

We now claim $u_\infty(0, \cdot) = 0$ by adapting the argument in the proof of [12, Theorem 1.4]. For any $x_0 \in \mathbb{R}_+^d$, by using (3.50),

$$\begin{aligned}
& \int_{\Omega(x_0, 1/4)} |u_\infty(0, x)| \, dx \\
& \leq \int_{\Omega(x_0, 1/4)} |u_k(0, x) - u_\infty(0, x)| \, dx + \int_{\Omega(x_0, 1/4)} |u_k(0, x)| \, dx \\
& \leq \|u_k - u\|_{C([-1/4^2, 0]; L_1(\Omega(x_0, 1/4)))} + N(d) \|u_k(0, \cdot)\|_{L_d(\Omega(x_0, 1/4))} \\
& = \|u_k - u\|_{C([-1/4^2, 0]; L_1(\Omega(x_0, 1/4)))} + N(d) \|u(T_0, \cdot)\|_{L_d(\Omega(\lambda_k x_0/4 + X_0, \lambda_k/4))}.
\end{aligned}$$

The right-hand side of the above inequality goes to zero as $k \rightarrow \infty$, which proves the claim.

Because of (3.59), the vorticity $\omega = \text{curl } u_\infty$ satisfies the differential inequality

$$|\partial_t \omega - \Delta \omega| \leq N(|\omega| + |\nabla \omega|),$$

on $(-T_1, 0) \times (\mathbb{R}_+^d \setminus B_{R+3}^+)$. We apply the half-space backward uniqueness theorem proved in [12, 13] on the open half space $\{x_d > R + 3\}$ to reach

$$\omega(z) = 0 \quad \text{on } (-T_1, 0] \times \{x_d > R + 3\}. \quad (3.60)$$

Now we fix a $t_0 \in (-T_1, 0)$. Take an increasing sequence $\{t_k\}_{k=0}^\infty \subset (-T_1, 0)$ converging to t_0 . For each k , we consider equation (1.1) with initial data $u_\infty(t_k, \cdot)$. By Proposition 3.2.1, one can locally find a strong solution

$$v_k \in C([t_k, t_k + \delta_k]; L_d(\mathbb{R}_+^d))$$

for some small δ_k , and $v_k(t, \cdot)$ is spatial analytic for $t \in (t_k, t_k + \delta_k)$. We may assume

that $t_k + \delta_k < t_0$. By the weak-strong uniqueness, $v_k \equiv u_\infty$ for $t \in [t_k, t_k + \delta_k)$. Therefore, $\omega(t, \cdot)$ is also spatial analytic for $t \in (t_k, t_k + \delta_k)$. Because of (3.60), we get

$$\omega(z) = 0 \quad \text{on } (t_k, t_k + \delta_k) \times \mathbb{R}_+^d,$$

which implies that $u_\infty \equiv 0$ in the same region. In particular, we can take a sequence $\{s_k\}$ such that $t_k < s_k < t_k + \delta_k$. Then $\{s_k\}$ converges to t_0 and $u_\infty(s_k, \cdot) \equiv 0$. This together with the weak continuity of u_∞ yields that $u_\infty(t_0, \cdot) \equiv 0$. Since $t_0 \in (-T_1, 0)$ and $T_1 \geq 1$ are both arbitrary, we complete the proof of the theorem. $\square$

We now show the proof of Theorem 3.1.2 which also works for Theorem 3.1.1. Again this proof is a modification of [8, Section 5] by replacing Schoen's trick with the Hölder continuity proved in Sections 3.3 and 3.4.

Proof of Theorem 3.1.2. We prove the theorem in four steps.

Step 1. In this step, we assume condition (3.2) holds. First we show that u is regular for $t \in (0, T]$. Owing to Propositions 3.5.3 and 3.5.6,

$$u_k \rightarrow 0 \text{ in } C([-1/4^2, 0]; L_{2(2-1/d)}(B^+(1/4))).$$

Hence

$$\limsup_{k \rightarrow \infty} C^+(r, z_0, u_k, p_k) = 0$$

for $r \in (0, 1/4)$. Also recall that $D^+(r, z_0, u_k, p_k)$ has a uniform bound for $r \in (0, 1)$. Following the proof of Lemma 3.5.5 we have: for any $\varepsilon > 0$, there is a $r_0 > 0$ small and a positive integer k_0 such that, for any $z_0 \in (-2, 0] \times B^+(2)$,

$$A^+(r_0, z_0, u_{k_0}, p_{k_0}) + E^+(r_0, z_0, u_{k_0}, p_{k_0}) + D^+(r_0, z_0, u_{k_0}, p_{k_0}) \leq \varepsilon.$$

We choose ε sufficiently small and apply Theorem 3.4.1 to get, for some $\tilde{r} > 0$,

$$\sup_{(-\tilde{r}^2, 0) \times B^+(\tilde{r})} |u_{k_0}| < \infty,$$

which implies that

$$\sup_{\omega(Z_0, \lambda_{k_0} \tilde{r})} |u| < \infty.$$

This contradicts the assumption that $Z_0 = (T_0, X_0)$ is a blowup point. Therefore, u is regular up to the boundary for $t \in (0, T]$.

Step 2. From this step on, we assume condition (3.3) holds. We bound the sup norm of u in this step. Fix some small $\delta \in (0, T)$. Since

$$\|u\|_{L_\infty^t L_d^x((0, T) \times \mathbb{R}_+^d)} \leq N, \quad \|p\|_{L_{2-1/d, \text{unif}}((0, T) \times \mathbb{R}_+^d)} \leq N,$$

by the same reasoning as at the beginning of the proof of Proposition 3.5.6, we see that there exists a large $R \geq 1$ such that

$$\sup_{[\delta, T] \times (\mathbb{R}_+^d \setminus B^+(R))} |u| \leq N. \tag{3.61}$$

Next we estimate the sup norm of u in $[\delta, T] \times B^+(R)$. Fix a $z_0 = (t_0, x_0)$ in $[\delta, T] \times \bar{B}^+(R)$. In the construction of u_k , we replace (T_0, X_0) by (t_0, x_0) . By the same reasoning as in the first step, for some $\tilde{r} = \tilde{r}(t_0, x_0) > 0$, we have

$$\sup_{\omega(z_0, \tilde{r})} |u| \leq N(t_0, x_0).$$

Utilizing the compactness of $[\delta, T] \times \bar{B}^+(R)$, we see that

$$\sup_{[\delta, T] \times \bar{B}^+(R)} |u| \leq N.$$

This together with (3.61) yields

$$\sup_{[\delta, T] \times \mathbb{R}_+^d} |u| \leq N.$$

Step 3. Since we can choose the δ in step 2 to be arbitrarily small, we can then utilize the local strong solvability of (1.1) to find some $T_1 > \delta$ such that $u \in L_{d+2}((0, T_1) \times \mathbb{R}_+^d)$. From Step 2, for $t \in [T_1, T]$ the solution is uniformly bounded and belongs to $L_\infty^t L_d^x((T_1, T) \times \mathbb{R}_+^d)$, thus $u \in L_{d+2}((0, T) \times \mathbb{R}_+^d)$. The uniqueness follows from the Ladyzhenskaya-Prodi-Serrin criterion.

Step 4. Now it remains to prove (3.4). We use a scaling argument. Let $\lambda > 0$ be a constant to be specified later. We define

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x).$$

Then (u_λ, p_λ) is also a Leray-Hopf weak solution of (1.1) in $(0, \infty) \times \mathbb{R}_+^d$, and u_λ satisfies (3.1) with the same constant K due to the scale invariant property. Due to the weak continuity of Leray-Hopf weak solutions,

$$\|u_\lambda(t, \cdot)\|_{L_d(\mathbb{R}_+^d)} \leq N, \tag{3.62}$$

for each $t \in (0, \infty)$. By using Lemma 3.3.2 with $q = 2d/(d-2)$ and $r = \infty$, we have

$$\|u_\lambda\|_{L_2^t L_{2d/(d-2)}^x((0, \infty) \times \mathbb{R}_+^d)} \leq N. \tag{3.63}$$

Putting together (3.62) and (3.63) and using Hölder's inequality yield

$$\|u_\lambda\|_{L_4((0, \infty) \times \mathbb{R}_+^d)} \leq N.$$

Thus there exists $T > 0$ such that $\|u_\lambda\|_{L_4((T,\infty)\times\mathbb{R}_+^d)}$ is sufficiently small, which together with Hölder's inequality implies the smallness of $C^+(1, z_0, u_\lambda, p_\lambda)$ for $z_0 \in (T, \infty) \times \mathbb{R}_+^d$. Again let $\hat{\varepsilon}_0$ denote the constant in Theorem 3.4.1. Following the argument in Lemma 3.5.5 we can find a large $T = T_\lambda$ and $r_0 > 0$ such that

$$A^+(r_0, z_0, u_\lambda, p_\lambda) + E^+(r_0, z_0, u_\lambda, p_\lambda) + D^+(r_0, z_0, u_\lambda, p_\lambda) \leq \hat{\varepsilon}_0$$

for any $z_0 \in [T, \infty) \times \mathbb{R}_+^d$. Owing to Theorems 3.3.1 and 3.4.1, we conclude

$$\sup_{[T,\infty)\times\mathbb{R}_+^d} |u_\lambda(z)| \leq N,$$

where $N = N(d)$ is independent of λ . Therefore,

$$\sup_{[\lambda^2 T, \infty)\times\mathbb{R}_+^d} |u(z)| \leq N/\lambda.$$

Sending $\lambda \rightarrow \infty$ yields the desired result. $\square$

3.6 Proof of Theorem 3.1.3

In this section, we prove the local result mentioned in Theorem 3.1.3. Most part of the proof remains the same with last section. We omit some repeated details in similar proofs from last section. We again start with the blow-up procedure: Suppose (u, p) is a pair of Leray-Hopf weak solution (1.1)-(1.2) in $(-1, 0) \times \mathbb{R}_+^d$ and we study the local problem in Q^+ . Correspondingly, we modify the notation $\omega(z, r) = Q(z, r) \cap Q^+$. Suppose T_0 is the first blowup time of u in $\overline{Q_{1/4}^+}$, and $Z_0 = (T_0, X_0) = (T_0, X_{0,1}, X_{0,2}, \dots, X_{0,d}) = (T_0, X'_0, X_{0,d})$ is a singular point in $\overline{Q_{1/4}^+}$. Take a decreasing sequence $\{\lambda_k\}$ converging to 0 and rescale the pair (u, p) at time

T_0 . By defining

$$u_k(t, x) = \lambda_k u(T_0 + \lambda_k^2 t, X_0 + \lambda_k x), \quad p_k(t, x) = \lambda_k^2 p(T_0 + \lambda_k^2 t, X_0 + \lambda_k x),$$

for each $k = 1, 2, \dots$, (u_k, p_k) is a suitable weak solution of (1.1)-(1.2) for $t \in (-\lambda_k^{-2} T_0, 0)$. The first observation is the property of uniform boundedness of C^+ and D^+ after the rescaling.

Lemma 3.6.1. *Under the conditions in Theorem 3.1.3, there exists $N > 0$ such that for any $z_0 \in (-\infty, 0] \times \mathbb{R}_{(+)}^d$ and $0 < r \leq 1$,*

$$\limsup_{k \rightarrow \infty} C^+(r, z_0, u_k, p_k) \leq N,$$

and

$$\limsup_{k \rightarrow \infty} D^+(r, z_0, u_k, p_k) \leq N. \tag{3.64}$$

Proof. Following the proof of Lemma 3.5.1, we easily have

$$C^+(\lambda_k r, z_0^k, u, p) \leq N \|u\|_{L_\infty^t L_d^x(\omega(z_0^k, \lambda_k r))} \leq N.$$

To prove (3.64), we consider three cases on whether Z_0 is on the flat boundary and the shape of the $\omega(z_0^k, \lambda_k r)$:

i) Consider when $Z_0 \in Q_{1/4}^+$, i.e., $X_{0,d} > 0$. Based on the blow-up procedure, u_k 's are defined on a larger and larger domain which eventually expands to the whole space, so we can always assume u_k has definition on $Q(z_0, r)$ when k is large. Denote $\hat{Z}_0 = (X'_0, 0, T_0)$ to be the projection of Z_0 on the flat boundary and $\hat{z}_0^k$ to be the

projection of z_0^k . Note when k is large, $\lambda_k r \ll X_{0,d}$, we have

$$Q(z_0^k, \lambda_k r) \subset Q(Z_0, X_{0,d}) \subset Q^+(\hat{Z}_0, 2X_{0,d}) \subset Q^+(\hat{Z}_0, 1/2) \subset Q^+.$$

From Corollaries 3.3.7 and 3.4.9, we know that

$$\begin{aligned} D(r, z_0, u_k, p_k) &= D(\lambda_k r, z_0^k, u, p) \leq ND(X_{0,d}, Z_0, u, p) + C \\ &\leq ND^+(2X_{0,d}, \hat{Z}_0, u, p) + C \leq ND^+(1/2, \hat{Z}_0, u, p) + C \\ &\leq N\|p\|_{L_{2-1/d}(Q^+)}^{2-1/d} + C. \end{aligned}$$

ii) Consider when $Z_0 \in Q_{1/4}^+$, $X_{0,d} = 0$, and $r \geq x_{0,d}$. When k is large, we have

$$\omega(z_0^k, \lambda_k r) \subset Q^+(\hat{z}_0^k, 2\lambda_k r) \subset Q^+(Z_0, 1/2) \subset Q^+.$$

From the proof of Corollary 3.4.9, we have

$$\begin{aligned} D^+(r, z_0, u_k, p_k) &= D^+(\lambda_k r, z_0^k, u, p) \leq ND^+(2\lambda_k r, \hat{z}_0^k, u, p) \\ &\leq ND^+(1/2, Z_0, u, p) + C \leq N\|p\|_{L_{2-1/d}(Q^+)}^{2-1/d} + C. \end{aligned}$$

iii) Consider when $Z_0 \in Q_{1/4}^+$, $X_{0,d} = 0$ and $r < x_{0,d}$. When k is large, we have

$$\omega(z_0^k, \lambda_k r) = Q(z_0^k, \lambda_k r) \subset Q(\hat{z}_0^k, \lambda_k x_{0,d}) \subset Q^+(Z_0, 1/2) \subset Q^+.$$

From the proofs of Corollaries 3.3.7 and 3.4.9 we have

$$\begin{aligned} D^+(r, z_0, u_k, p_k) &= D(\lambda_k r, z_0^k, u, p) \leq ND(\lambda_k x_{0,d}, z_0^k, u, p) + C \\ &\leq ND(2\lambda_k x_{0,d}, \hat{z}_0^k, u, p) + C \leq ND^+(1/2, Z_0, u, p) + C \leq N\|p\|_{L_{2-1/d}(Q^+)}^{2-1/d} + C. \end{aligned}$$

Therefore, $D^+(r, z_0, u_k, p_k)$ is uniformly bounded by $\|p\|_{L_{2-1/d}(Q^+)}$ plus some constant.

□

After we proved the boundedness, we can again prove the results of Proposition 3.5.3, Corollary 3.5.4, Lemma 3.5.5 and Proposition 3.5.6 in the same fashion as previous. Here we show the proof of Theorem 3.1.3 in the boundary case $X_{0,d} = 0$. The interior case is similar.

Proof of Theorem 3.1.3. Owing to Propositions 3.5.3 and 3.5.6, we have

$$u_k \rightarrow 0 \text{ in } C([-1/4^2, 0]; L_{2(2-1/d)}(B^+(1/4))).$$

Hence

$$\limsup_{k \rightarrow \infty} C^+(r, z_0, u_k, p_k) = 0$$

for $r \in (0, 1/4)$. Also recall that $D^+(r, z_0, u_k, p_k)$ has a uniform bound for $r \in (0, 1/4)$.

Following the proof of Lemma 3.5.5 we have: for any $\varepsilon > 0$, there is a $r_0 > 0$ small and a positive integer k_0 such that, for any $z_0 \in (-1/4^2, 0] \times B^+(1/4)$,

$$A^+(r_0, z_0, u_{k_0}, p_{k_0}) + E^+(r_0, z_0, u_{k_0}, p_{k_0}) + D^+(r_0, z_0, u_{k_0}, p_{k_0}) \leq \varepsilon,$$

which implies

$$A^+(\lambda_{k_0} r_0, z_0^{k_0}, u, p) + E^+(\lambda_{k_0} r_0, z_0^{k_0}, u, p) + D^+(\lambda_{k_0} r_0, z_0^{k_0}, u, p) \leq \varepsilon.$$

We choose ε sufficiently small and apply Theorem 3.4.1 to deduce that u is Hölder continuous around Z_0 since $z_0^{k_0}$ is an arbitrary point around Z_0 . We reach a contradiction and finish the proof. □

Bibliography

- [1] D. ALBRITTON, *Blow-up criteria for the Navier-Stokes equations in non-endpoint critical Besov spaces*, Anal. PDE, 11 (2018), pp. 1415–1456.
- [2] D. ALBRITTON AND T. BARKER, *Global weak Besov solutions of the Navier-Stokes equations and applications*, ArXiv e-prints, (2018).
- [3] T. BARKER AND G. SEREGIN, *A necessary condition of potential blowup for the Navier-Stokes system in half-space*, Math. Ann., 369 (2017), pp. 1327–1352.
- [4] L. CAFFARELLI, R. KOHN, AND L. NIRENBERG, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math., 35 (1982), pp. 771–831.
- [5] A. CHESKIDOV AND R. SHVYDKOY, *The regularity of weak solutions of the 3D Navier-Stokes equations in $B_{\infty,\infty}^{-1}$* , Arch. Ration. Mech. Anal., 195 (2010), pp. 159–169.
- [6] K. CHOI AND A. F. VASSEUR, *Estimates on fractional higher derivatives of weak solutions for the Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), pp. 899–945.
- [7] H. DONG AND D. DU, *Partial regularity of solutions to the four-dimensional Navier-Stokes equations at the first blow-up time*, Comm. Math. Phys., 273 (2007), pp. 785–801.
- [8] ———, *The Navier-Stokes equations in the critical Lebesgue space*, Comm. Math. Phys., 292 (2009), pp. 811–827.
- [9] H. DONG AND X. GU, *Boundary partial regularity for the high dimensional Navier-Stokes equations*, J. Funct. Anal., 267 (2014), pp. 2606–2637.
- [10] H. DONG AND D. LI, *Optimal local smoothing and analyticity rate estimates for the generalized Navier-Stokes equations*, Commun. Math. Sci., 7 (2009), pp. 67–80.
- [11] H. DONG AND R. M. STRAIN, *On partial regularity of steady-state solutions to the 6D Navier-Stokes equations*, Indiana Univ. Math. J., 61 (2012), pp. 2211–2229.

- [12] L. ESCAURIAZA, G. A. SEREGIN, AND V. ŠVERÁK, *$L_{3,\infty}$ -solutions of Navier-Stokes equations and backward uniqueness*, Uspekhi Mat. Nauk, 58 (2003), pp. 3–44.
- [13] —, *On backward uniqueness for parabolic equations*, Arch. Ration. Mech. Anal., 169 (2003), pp. 147–157.
- [14] I. GALLAGHER, G. S. KOCH, AND F. PLANCHON, *Blow-up of critical Besov norms at a potential Navier-Stokes singularity*, Comm. Math. Phys., 343 (2016), pp. 39–82.
- [15] M. GIAQUINTA, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, vol. 105 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1983.
- [16] Y. GIGA, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations, 62 (1986), pp. 186–212.
- [17] Y. GIGA AND T. MIYAKAWA, *Solutions in L_r of the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal., 89 (1985), pp. 267–281.
- [18] Y. GIGA AND O. SAWADA, *On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem*, in Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday. Vol. 1, 2, Kluwer Acad. Publ., Dordrecht, 2003, pp. 549–562.
- [19] C. GUEVARA AND N. C. PHUC, *Local energy bounds and ϵ -regularity criteria for the 3D Navier-Stokes system*, Calc. Var. Partial Differential Equations, 56 (2017), pp. Art. 68, 16.
- [20] —, *Local energy bounds and ϵ -regularity criteria for the 3D Navier-Stokes system*, Calc. Var. Partial Differential Equations, 56 (2017), pp. Art. 68, 16.
- [21] S. GUSTAFSON, K. KANG, AND T. TSAI, *Interior regularity criteria for suitable weak solutions of the Navier-Stokes equations*, Comm. Math. Phys., 273 (2007), pp. 161–176.
- [22] C. HE, Y. WANG, AND D. ZHOU, *New ϵ -regularity criteria and application to the box dimension of the singular set in the 3D Navier-Stokes equations*, arXiv e-prints, (2017), p. arXiv:1709.01382.
- [23] E. HOPF, *über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr., 4 (1951), pp. 213–231.
- [24] C. KAHANE, *On the spatial analyticity of solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal., 33 (1969), pp. 386–405.
- [25] T. KATO, *Strong L^p -solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions*, Math. Z., 187 (1984), pp. 471–480.
- [26] C. E. KENIG AND G. S. KOCH, *An alternative approach to regularity for the Navier-Stokes equations in critical spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28 (2011), pp. 159–187.

- [27] J. KIM AND M. KIM, *Local regularity of the Navier-Stokes equations near the curved boundary*, J. Math. Anal. Appl., 363 (2010), pp. 161–173.
- [28] H. KOCH AND D. TATARU, *Well-posedness for the Navier-Stokes equations*, Adv. Math., 157 (2001), pp. 22–35.
- [29] I. KUKAVICA, *Regularity for the Navier-Stokes equations with a solution in a Morrey space*, Indiana Univ. Math. J., 57 (2008), pp. 2843–2860.
- [30] O. A. LADYŽENSKAJA, *Uniqueness and smoothness of generalized solutions of Navier-Stokes equations*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 5 (1967), pp. 169–185.
- [31] O. A. LADYZHENSKAYA AND G. A. SEREGIN, *On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations*, J. Math. Fluid Mech., 1 (1999), pp. 356–387.
- [32] J. LERAY, *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique*, NUMDAM, [place of publication not identified], 1933.
- [33] G. M. LIEBERMAN, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
- [34] F. LIN, *A new proof of the Caffarelli-Kohn-Nirenberg theorem*, Comm. Pure Appl. Math., 51 (1998), pp. 241–257.
- [35] P. MAREMONTI AND V. A. SOLONNIKOV, *On estimates for the solutions of the nonstationary Stokes problem in $S. L.$ Sobolev anisotropic spaces with a mixed norm*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 222 (1995), pp. 124–150, 309.
- [36] K. MASUDA, *On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation*, Proc. Japan Acad., 43 (1967), pp. 827–832.
- [37] A. S. MIKHAILOV AND T. N. SHILKIN, *$L_{3,\infty}$ -solutions to the 3D-Navier-Stokes system in the domain with a curved boundary*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 336 (2006), pp. 133–152, 276.
- [38] N. C. PHUC, *The Navier-Stokes equations in nonendpoint borderline Lorentz spaces*, J. Math. Fluid Mech., 17 (2015), pp. 741–760.
- [39] G. PRODI, *Un teorema di unicità per le equazioni di Navier-Stokes*, Ann. Mat. Pura Appl. (4), 48 (1959), pp. 173–182.
- [40] V. SCHEFFER, *Partial regularity of solutions to the Navier-Stokes equations*, Pacific J. Math., 66 (1976), pp. 535–552.
- [41] —, *Hausdorff measure and the Navier-Stokes equations*, Comm. Math. Phys., 55 (1977), pp. 97–112.
- [42] —, *The Navier-Stokes equations on a bounded domain*, Comm. Math. Phys., 73 (1980), pp. 1–42.

- [43] G. A. SEREGIN, *Some estimates near the boundary for solutions to the non-stationary linearized Navier-Stokes equations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 271 (2000), pp. 204–223, 317.
- [44] —, *Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary*, J. Math. Fluid Mech., 4 (2002), pp. 1–29.
- [45] —, *On smoothness of $L_{3,\infty}$ -solutions to the Navier-Stokes equations up to boundary*, Math. Ann., 332 (2005), pp. 219–238.
- [46] —, *A note on local boundary regularity for the Stokes system*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 370 (2009), pp. 151–159, 221–222.
- [47] G. A. SEREGIN, T. N. SHILKIN, AND V. A. SOLONNIKOV, *Boundary partial regularity for the Navier-Stokes equations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 310 (2004), pp. 158–190, 228.
- [48] J. SERRIN, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal., 9 (1962), pp. 187–195.
- [49] —, *The initial value problem for the Navier-Stokes equations*, in Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962), Univ. of Wisconsin Press, Madison, Wis., 1963, pp. 69–98.
- [50] M. STRUWE, *On partial regularity results for the Navier-Stokes equations*, Comm. Pure Appl. Math., 41 (1988), pp. 437–458.
- [51] M. E. TAYLOR, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations, 17 (1992), pp. 1407–1456.
- [52] A. F. VASSEUR, *A new proof of partial regularity of solutions to Navier-Stokes equations*, NoDEA Nonlinear Differential Equations Appl., 14 (2007), pp. 753–785.
- [53] W. VON WAHL, *The equations of Navier-Stokes and abstract parabolic equations*, Aspects of Mathematics, E8, Friedr. Vieweg & Sohn, Braunschweig, 1985.
- [54] W. WANG AND Z. ZHANG, *On the interior regularity criteria and the number of singular points to the Navier-Stokes equations*, J. Anal. Math., 123 (2014), pp. 139–170.
- [55] W. WANG AND Z. ZHANG, *Blow-up of critical norms for the 3-D Navier-Stokes equations*, Sci. China Math., 60 (2017), pp. 637–650.
- [56] F. B. WEISSLER, *The Navier-Stokes initial value problem in L^p* , Arch. Rational Mech. Anal., 74 (1980), pp. 219–230.