Initial Boundary Value Problem of the Boltzmann Equation

by

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Chanwoo Kim was born on November 1 in 1981 in Busan, South Korea. He went to Seoul National University in 2001 for undergraduate studies to major in Mathematics and minor in Physics, and graduated summa cum laude in August 2006. In September 2006, he began his doctoral work at Brown University. He have been interested in mathematical structures in physical systems and decided to research in the field of partial differential equations. During his studies ar Brown, he has been supported by University Fellowship, Several Teaching and Research Assistantship, Teaching Fellowship, and the Korean Foundation for Advanced Studies.

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Abstract of " Initial Boundary Value Problem of the Boltzmann Equation " by Chanwoo Kim, Ph.D., Brown University, May 2011

In this thesis, we study some boundary problems of the Boltzmann equation and the Boltzmann equation with the large external potential.

If the gas is contained in a bounded domain or flows past a solid bodies, the Boltzmann equation must be accompanied by boundary conditions, which describe the interaction of the gas with the solid walls. The basic boundary conditions are in-flow injection, diffuse reflection, bounce-back, and specular reflection boundary conditions.

First, we demonstrate that discontinuity of the Boltzmann solution is created at the non-convex part of the grazing boundary, then propagate only along the forward characteristics inside the domain before it hits on the boundary again with the in-flow injection, diffuse reflection and bounce-back boundary conditions. The main ingredient is the new regularizing effect of the gain term.

Second, we prove the global well-posedness of the Boltzmann equation with the specular reflection boundary condition when the domain is the cylindrical shape. The main ingredients are introducing collision-pair map to show the transversality and developing the graph comparison method to characterize grazing velocities for the non-convex domains.

If the gas particles are charged and there is background electonic potential, the Boltzmann equation is accompanied by the field term $\nabla_x \Phi \cdot \nabla_v F$ in the equation. The global well-posedness in the presence of the large external potential is an important open problem. We use Asano's work, showing the transversality in the presence of potentials, in $L^p - L^\infty$ framework to resolve this problem.

Contents

Chapter One

Introduction

In this thesis, we study some mathematical properties of the Boltzmann equation with several boundary conditions and with large external potentials.

1.1 Brief Introduction

A density of a dilute gas is governed by the Boltzmann equation

$$\partial_t F + v\cdot\nabla_x F = Q(F,F), \tag{1.1}$$

where $F(t,x,v)$ is a distribution function for the gas particles at a time $t\geq 0$, a position $x\in\Omega\subset\mathbb{R}^3$ and a velocity $v\in\mathbb{R}^3$. Throughout this paper, the collision operator takes the form

$$\begin{aligned} Q(F_1,F_2) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)F_1(u')F_2(v')d\omega du - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)F_1(u)F_2(v)d\omega du \\ &\equiv Q_+(F_1,F_2) - Q_-(F_1,F_2), \end{aligned} \tag{1.2}$$

where $u' = u+[(v-u)\cdot\omega]\omega$, $v' = v-[(v-u)\cdot\omega]\omega$ and

$$B(v-u,\omega) = |v-u|^\gamma q_0(\frac{v-u}{|v-u|}\cdot\omega),$$

with $0<\gamma\leq 1$ (hard potential) and

$$\int_{\mathbb{S}^2} q_0(\hat{u}\cdot\omega)d\omega < +\infty \quad \text{(angular cutoff)}, \tag{1.3}$$

for all $\hat{u}\in\mathbb{S}^2$.

If the gas is contained in a bounded region or flows past a solid bodies, the Boltzmann equation must be accompanied by boundary conditions, which describe the interaction of the gas molecules with the solid walls. The mathematical study of the **boundary value problem of the Boltzmann equation** : particle-boundary interaction in a bounded domain and its effect on the global dynamics is one of the fundamental problems in the Boltzmann theory.

Let the domain Ω be a smooth bounded domain. We consider three basic types of boundary conditions([27],[63]) for $F(t,x,v)$ at $(x,v)\in\partial\Omega\times\mathbb{R}^3$ with $v\cdot n(x)<0$, where $n(x)$ is an outward unit normal vector at x :

1. **In-flow injection boundary condition** : the incoming particles are prescribed ;

$$F(t,x,v) = G(t,x,v). \tag{1.4}$$

2. **Diffuse reflection boundary condition** : the incoming particles are a probability average of the outgoing particles ;

$$F(t,x,v) = c_\mu \mu(v) \int_{v'\cdot n(x)>0} F(t,x,v')\{n(x)\cdot v'\}dv', \tag{1.5}$$

with a normalized Maxwellian $\mu = e^{-\frac{|v|^2}{2}}$, a normalized constant $c_\mu > 0$ such that

$$c_\mu \int_{v'\cdot n(x)>0} \mu(v')|n(x)\cdot v'|dv' = 1. \tag{1.6}$$

3. **Bounce-back reflection boundary condition** : the particles bounce back at the reverse the velocity ;

$$F(t,x,v) = F(t,x,-v). \tag{1.7}$$

4. **Specular reflection boundary condition** : the particles bounce specularly ;

$$F(t,x,v) = F(x,v,v-2(n(x)\cdot v)n(x)) = F(x,v,R(x)v). \tag{1.8}$$

On the other hand, if the particle is charged then the dynamics of charged particles interact with the back-ground potential Φ. For this case, a density of a dilute charged gas is governed by the Boltzmann equation

$$\partial_t F + v\cdot\nabla_x F - \nabla_x\Phi(x)\cdot\nabla_v F = Q(F,F) \quad , \qquad F(0,x,v) = F_0(x,v). \tag{1.9}$$

Recently, Yan Guo develop a robust mathematical $L^p - L^\infty$ to study the time decay and continuity of Boltzmann solutions for the four basic boundary condition ([27]). In ([43]), we study the continuity and discontinuity of such Boltzmann solutions in a non-convex domains for in-flow, bounce-back and diffuse reflection boundary conditions. In ([42]), we establish the well-posedness and the exponential decay of the Boltzmann solution for the specular reflection boundary condition in 2D analytic non-convex domains. In ([44]), we use the $L^p - L^\infty$ framework to establish the stability of the local Maxwellian in the presences of large potentials.

1.2 Main Results

1.2.1 Formation and Propagation of Singularity

We assume the domain $\Omega \subset \mathbb{R}^3$ is open and bounded and connected. For simplicity, We assume that the boundary $\partial\Omega$ is smooth i.e. for each point $x_0 \in \partial\Omega$, there exists $r > 0$ and a smooth function $\Phi_{x_0} : \mathbb{R}^2 \to \mathbb{R}$ such that - upon relabeling and reorienting the coordinates axes if necessary - we have

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_3 > \Phi_{x_0}(x_1, x_2)\}. \tag{1.10}$$

The outward normal vector at $\partial\Omega$ is given by

$$n(x_1, x_2) = \frac{1}{\sqrt{1 + |\nabla_x \Phi(x_1, x_2)|^2}} (\ \partial_{x_1}\Phi_{x_0}(x_1, x_2),\ \partial_{x_2}\Phi_{x_0}(x_1, x_2),\ -1\ ).$$

Given (t, x, v), let $[X(s), V(s)] = [X(s; t, x, v), V(s; t, x, v)] = [x - (t - s)v, v]$ be a trajectory (or a characteristics) for the Boltzmann equation (3.1) :

$$\frac{dX(s)}{ds} = V(s), \qquad\qquad \frac{dV(s)}{ds} = 0, \tag{1.11}$$

with the initial condition: $[X(t; t, x, v), V(t; t, x, v)] = [x, v]$.

Definition 1. *For $(x, v) \in \bar{\Omega} \times \mathbb{R}^3$, we define the* ***backward exit time*** $t_{\mathbf{b}}(x, v) \geq 0$ *to be the last moment at which the back-time straight line $[X(s; 0, x, v), V(s; 0, x, v)]$ remains in the interior of Ω :*

$$t_{\mathbf{b}}(x, v) = \sup(\{0\} \cup \{\tau > 0 : x - sv \in \Omega \textit{ for all } \ 0 < s < \tau\}).$$

We also define the ***backward exit position*** *in* $\partial\Omega$

$$x_{\mathbf{b}}(x,v) = x - t_{\mathbf{b}}(x,v)v \in \partial\Omega,$$

and we always have $v \cdot n(x_{\mathbf{b}}(x,v)) \leq 0$.

We denote the phase boundary in the phase space $\Omega \times \mathbb{R}^3$ as $\gamma = \partial\Omega \times \mathbb{R}^3$, and split it into outgoing boundary γ_+, the incoming boundary γ_-, and the grazing boundary γ_0 :

$$\begin{aligned} \gamma_+ &= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x) \cdot v > 0\}, \\ \gamma_- &= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x) \cdot v < 0\}, \\ \gamma_0 &= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x) \cdot v = 0\}. \end{aligned}$$

We need to study the grazing boundary γ_0 more carefully.

Definition 2. *We define the* ***concave(singular) grazing boundary*** *which is a subset of the grazing boundary* γ_0 *:*

$$\gamma_0^{\mathbf{S}} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) \neq 0 \text{ and } t_{\mathbf{b}}(x,-v) \neq 0\},$$

and the ***outward inflection grazing boundary*** *in the grazing boundary* γ_0 *:*

$$\gamma_0^{I+} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) \neq 0 \ , \ t_{\mathbf{b}}(x,-v) = 0 \ , \exists \delta > 0 \ s.t. \ x + \tau v \in \bar{\Omega}^c \ \text{ for } \ \tau \in (0,\delta)\},$$

and the ***inward inflection grazing boundary*** *in the grazing boundary* γ_0 *:*

$$\gamma_0^{I-} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) = 0 \ , \ t_{\mathbf{b}}(x,-v) \neq 0 \ , \exists \delta > 0 \ s.t. \ x - \tau v \in \bar{\Omega}^c \ \text{ for } \ \tau \in (0,\delta)\},$$

and the ***convex grazing boundary*** *in the grazing boundary* γ_0 *:*

$$\gamma_0^{V} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) = 0 \text{ and } \ t_{\mathbf{b}}(x,-v) = 0\}.$$

It turns out that the concave(singular) grazing boundary $\gamma_0^{\mathbf{S}}$ is the only part at which discontinuity can be created and propagates into the interior of the phase space $\Omega \times \mathbb{R}^3$.

Definition 3. *Define the* ***discontinuity set*** *in* $[0,\infty)\times\bar{\Omega}\times\mathbb{R}^3$ *as*

$$\mathfrak{D} = \Big\{(0,\infty)\times[\ \gamma_0^{\mathbf{S}}\cup\gamma_0^{V}\cup\gamma_0^{I+}\ ]\Big\} \tag{1.12}$$

$$\cup\Big\{\ (t,x,v)\in(0,\infty)\times\{\Omega\times\mathbb{R}^3\cup\gamma_+\}:\ t\geq t_{\mathbf{b}}(x,v)\ \textit{and}\ \ (x_{\mathbf{b}}(x,v),v)\in\gamma_0^{\mathbf{S}}\ \Big\}, \tag{1.13}$$

and the ***continuity set*** *in* $[0,\infty)\times\bar{\Omega}\times\mathbb{R}^3$ *as*

$$\mathfrak{C} = \Big\{\{0\}\times\bar{\Omega}\times\mathbb{R}^3\Big\}\cup\Big\{(0,\infty)\times[\ \gamma_-\cup\gamma_0^{I-}\ ]\Big\}$$

$$\cup\Big\{\ (t,x,v)\in(0,\infty)\times\{\Omega\times\mathbb{R}^3\cup\gamma_+\}:\ t<t_{\mathbf{b}}(x,v)\quad \textit{or}\quad (x_{\mathbf{b}}(x,v),v)\in\gamma_-\cup\gamma_0^{I-}\ \Big\} \tag{1.14}$$

For bounce-back reflection boundary condition case (1.7) we need slightly different definitions : the ***bounce-back discontinuity set*** *and the* ***bounce-back continuity set*** *are*

$$\begin{aligned}
\mathfrak{D}_{bb} &= \mathfrak{D}\ \cup\ \Big\{(t,x,v)\in(0,\infty)\times\Omega\times\mathbb{R}^3: t\geq 2t_{\mathbf{b}}(x,v)+t_{\mathbf{b}}(x,-v)\ \textit{and}\ (x_{\mathbf{b}}(x,-v),-v)\in\gamma_0^{\mathbf{S}}\Big\},\\
\mathfrak{C}_{bb} &= \Big\{\{0\}\times\bar{\Omega}\times\mathbb{R}^3\Big\}\cup\Big\{(0,\infty)\times[\ \gamma_-\cup\gamma_0^{I-}\ ]\Big\}\\
&\quad\cup\Big\{(t,x,v)\in[0,\infty)\times\{\Omega\times\mathbb{R}^3\cup\gamma_+\}:\ t<t_{\mathbf{b}}(x,v)\\
&\qquad\qquad \textit{or}\quad \big[\ (x_{\mathbf{b}}(x,v),v)\in\gamma_-\cup\gamma_0^{I-}\quad \textit{and}\quad t<2t_{\mathbf{b}}(x,v)+t_{\mathbf{b}}(x,-v)\ \big]\\
&\qquad\qquad \textit{or}\quad \big[\ (x_{\mathbf{b}}(x,-v),-v)\in\gamma_-\cup\gamma_0^{I-}\quad \textit{and}\quad (x_{\mathbf{b}}(x,v),v)\in\gamma_-\cup\gamma_0^{I-}\ \big]\Big\},
\end{aligned}$$

respectively.

The discontinuity set $\mathfrak{D}$ consists of two sets : The first set of (3.90) is the grazing boundary part γ_0 of $\mathfrak{D}$. This set mainly consists of the phase boundary where the backward exit time $t_{\mathbf{b}}(x,v)$ is not continuous (Lemma 2). The second set of (3.90) is mainly the interior phase space part of $\mathfrak{D}$ i.e. $\mathfrak{D}\cap[0,\infty)\times\Omega\times\mathbb{R}^3$, which is a subset of a union of all forward trajectories in the phase space emanating from $\gamma_0^{\mathbf{S}}$. Notice that $\mathfrak{D}$ does not include the forward trajectories emanating from $\gamma_0^{V}\cup\gamma_0^{I+}$ because those forward trajectories are not in the interior phase space $[0,\infty)\times\Omega\times\mathbb{R}^3$. We also exclude the case $t<t_{\mathbf{b}}(x,v)$ from $\mathfrak{D}$. In fact, considering the pure transport equation, $t<t_{\mathbf{b}}(x,v)$ implies the transport solution at (t,x,v) equals the initial data at $(x-tv,v)$ and if the initial data is continuous, we expect the transport solution is continuous around (t,x,v). Notice that we exclude the initial plane $\{0\}\times\bar{\Omega}\times\mathbb{R}^3$ from $\mathfrak{D}$ because we assume that the Boltzmann solution at $t=0$ is continuous.

The continuity set $\mathfrak{C}$ consists of points either emanating from the initial plane or from $\gamma_- \cup \gamma_0^{I-}$, but not $\gamma_0^{\mathbf{S}}$.

Furthermore we define a set including the grazing boundary γ_0 and all forward trajectories emanating from the whole grazing boundary γ_0.

Definition 4. *The **grazing set** is defined as*

$$\mathfrak{G} \quad = \quad \{(x,v) \in \bar{\Omega} \times \mathbb{R}^3 : n(x_{\mathbf{b}}(x,v)) \cdot v = n(x - t_{\mathbf{b}}(x,v)v) \cdot v = 0\}, \tag{1.15}$$

*and the **grazing section** is*

$$\mathfrak{G}_x = \{v \in \mathbb{R}^3 : (x,v) \in \mathfrak{G}\} = \{v \in \mathbb{R}^3 : n(x_{\mathbf{b}}(x,v)) \cdot v = 0\}.$$

Obviously the grazing set $\mathfrak{G}$ includes the discontinuity set $\mathfrak{D}$. In order to study the continuity property of the Boltzmann solution we define :

Definition 5. *For a function $\phi(t,x,v)$ defined on $[0,\infty) \times \{\bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}\}$ we define the **discontinuity jump in space and velocity***

$$[\phi(t)]_{x,v} \;=\; \lim_{\delta \downarrow 0} \sup_{(x',v'),(x'',v'') \in \{\bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}\} \cap \{B((x,v);\delta) \backslash (x,v)\}} |\phi(t,x',v') - \phi(t,x'',v'')|,$$

*and the **discontinuity jump in time and space and velocity***

$$[\phi]_{t,x,v} \;=\; \lim_{\delta \downarrow 0} \sup_{\substack{t',\,t'' \in B(t;\delta) \\ (x',v'),\,(x'',v'') \in \{\bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}\} \cap \{B((x,v);\delta) \backslash (x,v)\}}} |\phi(t',x',v') - \phi(t'',x'',v'')|,$$

where $\mathfrak{G}$ is defined in Definition 4. We say a function ϕ is discontinuous in space and velocity (in time and space and velocity) at (t,x,v) if and only if $[\phi(t)]_{x,v} \neq 0$ ($[\phi]_{t,x,v} \neq 0$) and continuous in space and velocity (in time and space and velocity) at (t,x,v) if and only if $[\phi(t)]_{x,v} = 0$ ($[\phi]_{t,x,v} = 0$).

Notice that the function ϕ is only defined away from the grazing set $\mathfrak{G}$. Once the discontinuity jump of given function ϕ is zero at (t,x,v) then the function ϕ can be extended to $[0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$ near (t,x,v). Because of those definitions we can consider a function which has a removable discontinuity as a continuous function. And a non-zero discontinuity jump $[\phi]_{t,x,v} \neq 0$ means ϕ has a "real" discontinuity which is not removable.

Now we are ready to state the main theorems of Chapter 2. In order to state theorems in the unified way we use a weight function

$$w(v) = \{1 + \rho^2 |v|^2\}^{\beta},$$

such that $w^{-2}(v)\{1+|v|\}^3 \in L^1$.

Theorem 1 (Formation of Discontinuity)**.** *Let Ω be an open subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Assume Ω is non-convex, $\gamma_0^{\mathbf{S}} \neq \emptyset$. Choose any $(x_0, v_0) \in \gamma_0^{\mathbf{S}}$ with $v_0 \neq 0$. For any small $\delta > 0$,*
1. there exist $t_0 \in (0, \min\{\delta, t_{\mathbf{b}}(x_0, -v_0)\})$ and an initial datum $F_0(x,v)$ which is continuous on $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$, and an in-flow boundary datum $G(t,x,v)$ which is continuous on $[0,\infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$, satisfying

$$F_0(x,v) = G(0,x,v) \quad \textit{for} \quad (x,v) \in \gamma_- \cup \gamma_0^{\mathbf{S}}, \tag{1.16}$$

and

$$\Big|\Big| \, w \, \frac{F_0 - \mu}{\sqrt{\mu}} \Big|\Big|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} + \sup_{t\in[0,\infty)} \Big|\Big| \, w \, \frac{G(t) - \mu}{\sqrt{\mu}} \Big|\Big|_{L^\infty(\gamma_-)} < \delta, \tag{1.17}$$

such that if F on $[0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$ is Boltzmann solution of (3.1) with the in-flow boundary condition (1.4) then F is discontinuous in space and velocity at (t_0, x_0, v_0), i.e. $[F(t_0)]_{x_0,v_0} \neq 0$.
2. there exist $t_0 \in (0, \min\{\delta, t_{\mathbf{b}}(x_0, -v_0)\})$ and an initial datum $F_0(x,v)$ which is continuous on $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$, satisfying

$$F_0(x,v) = c_\mu \mu(v) \int_{v'\cdot n(x)>0} F_0(x,v')\{n(x)\cdot v'\} dv' \quad \textit{for} \quad (x,v) \in \gamma_- \cup \gamma_0^{\mathbf{S}}, \tag{1.18}$$

and

$$\Big|\Big| \, w \, \frac{F_0 - \mu}{\sqrt{\mu}} \Big|\Big|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} < \delta, \tag{1.19}$$

such that if F on $[0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$ is Boltzmann solution of (3.1) with the diffuse boundary condition (1.5) with the compatibility condition (1.18) then F is discontinuous in space and velocity at (t_0, x_0, v_0), i.e. $[F(t_0)]_{x_0,v_0} \neq 0$.
3. there exist $t_0 \in (0, \min\{\delta, t_{\mathbf{b}}(x_0, -v_0), t_{\mathbf{b}}(x_0, v_0)\})$ and an initial datum $F_0(x,v)$ which is contin-

uous on $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$*, satisfying (1.19) and*

$$F_0(x,v) = F_0(x,-v) \quad \text{for} \quad (x,v) \in \gamma_- \cup \gamma_0^{\mathbf{S}}, \tag{1.20}$$

such that if F *on* $[0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$ *is Boltzmann solution of (3.1) with the bounce-back boundary condition (1.7) then* F *is discontinuous in space and velocity at* (t_0, x_0, v_0)*, i.e.* $[F(t_0)]_{x_0,v_0} \neq 0$.

The smallness of given data (1.17), (1.19) ensures the global existence of Boltmzann solution for all boundary conditions [27]. Notice that we can observe the formation of discontinuity for any point of the concave(singular) grazing boundary $\gamma_0^{\mathbf{S}}$ of any generic non-convex domain Ω. If we assume that $F_0(x,v)$ is continuous on $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$ and $G(t,x,v)$ is continuous on $[0,\infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$, and that the compatibility conditions (1.16), (1.18) and (1.20) are valid up to $\gamma_- \cup \gamma_0^{\mathbf{S}}$, then Theorem 1 implies the continuity breaks down at the concave(singular) grazing boundary $\gamma_0^{\mathbf{S}}$ after a short time $t_0 \in (0, \min\{\delta, t_{\mathbf{b}}(x_0,-v_0)\})$ for in-flow (1.4), diffuse (1.5) boundary condition and $t_0 \in (0, \min\{\delta, t_{\mathbf{b}}(x_0,-v_0), t_{\mathbf{b}}(x_0,v_0)\})$ for bounce-back (1.7) boundary condition. For this generic cases, we said the Boltzmann solution F has a **local-in-time formation of discontinuity** at (t_0, x_0, v_0).

Once we have the formation of discontinuity at $(t_0, x_0, v_0) \in \gamma_0^{\mathbf{S}}$, we further establish that the discontinuity propagates along the forward characteristics.

Theorem 2 (Propagation of Discontinuity)**.** *Let* Ω *be an open bounded subset of* $\mathbb{R}^3$ *with a smooth boundary* $\partial\Omega$*. Let* $F(t,x,v)$ *be the Boltzmann solution of (3.1) with the initial datum* F_0 *which is continuous on* $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$*, and with one of the following boundary conditions :*

1. For in-flow boundary condition (1.4), let (1.16) and (1.17) be valid and $G(t,x,v)$ *be continuous on* $[0,\infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$*,*

2. For diffuse boundary condition (1.5), assume (1.19) and (1.18),

3. For bounce-back boundary condition (1.7), assume (1.19) and (1.20).

Then for all $t \in [t_0, t_0 + t_{\mathbf{b}}(x_0,-v_0))$ *we have*

$$[F]_{t,x_0+(t-t_0)v_0,v_0} \leq e^{-C_1(1+|v_0|)^\gamma (t-t_0)} [F(t_0)]_{x_0,v_0}, \tag{1.21}$$

where $C_1 > 0$ *only depends on* $\left|\left| w\frac{F-\mu}{\sqrt{\mu}} \right|\right|_{L^\infty([0,\infty)\times\bar{\Omega}\times\mathbb{R}^3)}$.

On the other hand, assume $[F(t_0)]_{x_0,v_0} \neq 0$*, and* $t_0 \in (0, t_{\mathbf{b}}(x_0, -v_0))$ *for in-flow and diffuse boundary conditions and* $t_0 \in (0, \min\{t_{\mathbf{b}}(x_0, -v_0), t_{\mathbf{b}}(x_0, v_0)\})$ *for bounce-back boundary condition, and a strict concavity of* $\partial\Omega$ *at* x_0 *along* v_0*, i.e.*

$$\sum_{i,j}(v_0)_i \partial_{x_i}\partial_{x_j}\Phi(x_0)(v_0)_j < -C_{x_0,v_0}. \tag{1.22}$$

Then for all $t \in [t_0, t_0 + t_{\mathbf{b}}(x_0, -v_0))$*, the Boltzmann solution* F *is discontinuous in time and space and velocity at* $(t, x_0 + (t - t_0)v_0, v_0)$*, i.e.* $[F]_{t,x_0+(t-t_0)v_0,v_0} \neq 0$ *and*

$$Ce^{-C_2(1+|v_0|)^\gamma(t-t_0)}[F(t_0)]_{x_0,v_0} \leq [F]_{t,x_0+(t-t_0)v_0,v_0}, \tag{1.23}$$

where $0 < C < 1$*, and* $C_2 = C_2(||w\frac{F-\mu}{\sqrt{\mu}}||_{L^\infty}) \in \mathbb{R}$ *which is positive for sufficiently small* $||w\frac{F-\mu}{\sqrt{\mu}}||_{L^\infty([0,\infty)\times\bar{\Omega}\times\mathbb{R}^3)}$.

The strict concavity condition (1.22) rules out some technical issue of the backward exit time $t_{\mathbf{b}}$. Our theorem characterize the propagation of discontinuity before the forward trajectory reaches the boundary. In the case that the forward trajectory reaches the boundary, i.e. $t \geq t_0 + t_{\mathbf{b}}(x_0, -v_0)$, the situation is much more complicated. Denote $x_1 = x_0 + t_{\mathbf{b}}(x_0, -v_0)v_0, t_1 = t_0 + t_{\mathbf{b}}(x_0, -v_0)$. If the trajectory his on the boundary non-tangentially, i.e. $(x_1, v_0) \in \gamma_+$, for in-flow and diffuse boundary cases, the discontinuity disappears because of the continuity of the in-flow datum and the average property of diffuse boundary operator. For bounce-back case the discontinuity is reflected and continues to propagate along the trajectory. If the trajectory hits on the boundary tangentially, i.e.$(x_1, v_0) \in \gamma_0$, there are three possibilities. Firstly if $(x_1, v_0) \in \gamma_0^{I+}$ then the situation is same as the case $(x_1, v_0) \in \gamma_+$ above. Secondly, if the trajectory is contained in the boundary for awhile, i.e. there exists $\delta > 0$ so that $x_1 + sv_0 \in \partial\Omega$ for $s \in (0, \delta)$ then it is hard to predict the propagation of discontinuity in general. Assuming certain condition on Ω for example Definition 6, we can rule such a unlikely case.

The last case is that $(x_1, v_0) \in \gamma_0^{\mathbf{S}}$. Assume we have a sequence of $\{t_n = t_{n-1} + t_{\mathbf{b}}(x_{n-1}, -v_0)\}$ and $\{x_n = x_{n-1} + t_{\mathbf{b}}(x_{n-1}, -v_0)v_0\} \in \partial\Omega$ so that $(x_n, v_0) \in \gamma_0^{\mathbf{S}}$, and a directional strict concavity (1.22) is valid for each (x_n, v_0). We can show the propagation of discontinuity also between the first and the second intersections, i.e. $[F]_{t,x_0(t-t_0)v_0,v_0} \neq 0$ for $t \in [t_1, t_2)$. For $t \geq t_2$, if we have very simple geometry, for example the first picture of Figure 2, we can show the propagation of discontinuity, i.e. $[F]_{t,x_0(t-t_0)v_0,v_0} \neq 0$ for $t \in [t_n, t_{n+1})$ even for $n = 2, 3$. But in general, for example the second domain of Figure 2, we cannot show $[F]_{t,x_0(t-t_0)v_0,v_0} \neq 0$ for $t \in [t_n, t_{n+1})$ for $n \geq 2$.

Thanks to Theorem 1 and Theorem 2, for any $(t, x, v) \in \mathfrak{D} \cap \{(0, \infty) \times \Omega \times \mathbb{R}^3\}$ we can construct continuous initial datum F_0 and continuous boundary datum G(for in-flow boundary condition) so that F has a local-in-time formation of discontinuity at $(t - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v) \in (0, \infty) \times \gamma_0^{\mathbf{S}}$ and F is discontinuous at (t, x, v).

The next result states that Theorem 1 and Theorem 2 capture all possible singularitie s(discontinuities), despite nonlinearity in the Boltzmann equation. In other words the singularity of the Boltzmann solution is propagating as the linear Boltzmann equation and no new singularities created from the nonlinearity of Boltzmann equation.

Theorem 3 (Continuity away from $\mathfrak{D}$)**.** *Let Ω be an open bounded subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Let $F(t, x, v)$ be a Boltzmann solution of (3.1) with the initial datum F_0 which is continuous on $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_+ \cup \gamma_0^{I-}\}$ and with one of*

1. In-flow boundary condition (1.4). Assume (1.17) is valid and the compatibility condition

$$F_0(x, v) = G(0, x, v) \quad \text{for} \quad (x, v) \in \gamma_- \cup \gamma_0^{I-}, \tag{1.24}$$

and $G(t, x, v)$ is continuous on $[0, \infty) \times \{\gamma_- \cup \gamma_0^{I-}\}$.

2. Diffuse boundary condition (1.5). Assume (1.19) is valid and the compatibility condition

$$F_0(x, v) = c_\mu \mu(v) \int_{v' \cdot n(x) > 0} F_0(x, v')\{n(x) \cdot v'\} dv' \quad \text{for} \quad (x, v) \in \gamma_- \cup \gamma_0^{I-}. \tag{1.25}$$

3. Bounce-back boundary condition (1.7). Assume (1.19) is valid and the compatibility condition

$$F_0(x, v) = F_0(x, -v) \quad \text{for} \quad (x, v) \in \gamma_- \cup \gamma_0^{I-}. \tag{1.26}$$

Then $F(t, x, v)$ is a continuous function on $\mathfrak{C}$ for 1,2 and a continuous function on $\mathfrak{C}_{bb}$ for 3. If the domain Ω does not include a line segment (Definition 6) then the continuity set $\mathfrak{C}$ and $\mathfrak{C}_{bb}$ are the complementary of $\mathfrak{D}$ and $\mathfrak{D}_{bb}$ respectively. Therefore $F(t, x, v)$ is continuous on $(\mathfrak{D})^c$ for 1,2 and continuous on $(\mathfrak{D}_{bb})^c$ for 3.

Definition 6. *Assume $\Omega \in \mathbb{R}^3$ be open and the boundary $\partial\Omega$ be smooth. We say the boundary $\partial\Omega$* ***does not include a line segment*** *if and only if for each $x_0 \in \partial\Omega$ and for all $(u_1, u_2) \in \mathbb{S}^1$ there is no $\delta > 0$ such that*

$$\Phi_{x_0}(\tau u_1, \tau u_2)$$

is a linear function for $\tau \in (-\delta, \delta)$ where Φ_{x_0} from (1.10).

The last theorem is a qualitative property of the gain term in (3.2). This theorem is crucial to prove Theorem 2 and Theorem 3.

Theorem 4 (Continuity of Q_+)**.** *Assume that $F(t, x, v)$ is a function defined on $(t, x, v) \in [0, T] \times \bar{\Omega} \times \mathbb{R}^3$ and is continuous away from the grazing set in (4), i.e.*

$$F \in C^0([0, T] \times (\Omega \times \mathbb{R}^3) \backslash \mathfrak{G}),$$

and

$$||\bar{w}^{-1} F||_{L^\infty([0,T] \times \bar{\Omega} \times \mathbf{R}^3)} < +\infty,$$

where $\bar{w} = \frac{e^{-\frac{|v|^2}{4}}}{(1+\rho^2|v|^2)^\beta}$ with $\rho \in \mathbb{R}$ and $\beta > 0$.

Then the gain term $Q_+(F, F)(t, x, v)$ is continuous, i.e.

$$Q_+(F, F) \in C^0([0, T] \times \Omega \times \mathbb{R}^3),$$

and

$$\sup_{[0,T] \times \bar{\Omega} \times \mathbb{R}^3} |\nu^{-1} \bar{w}^{-1} Q_+(F, F)(t, x, v)| < \infty. \tag{1.27}$$

Notice that the function F in Theorem 4 need not be a solution of the Boltzmann equation.

1.3 Specular Reflection Boundary Condition in 2D Domains

In Chapter 3, we assume the domain Ω is cylinderical, i.e. let $\underline{\Omega} \subset \mathbb{R}^2$ be connected and bounded and

$$\Omega \;\equiv\; \underline{\Omega} \times \mathbb{T}^1.$$

Moreover the boundary $\partial\underline{\Omega}$ is locally an image of unit-speed curve, i.e. for each $\underline{x}_0 \in \partial\underline{\Omega}$, there exists $r, \delta_1, \delta_2 > 0$ and a curve $\alpha : (\delta_1, \delta_2) \to \mathbb{R}^2$ such that $\dot{\alpha}(\tau) \equiv 1$ with $x_0 = \alpha(0)$ and

$$\underline{\Omega} \cap B(x_0, r) = \{\alpha(\tau) \in \mathbb{R}^2 : \tau \in (\delta_1, \delta_2)\}. \tag{1.28}$$

Moreover we always assume that $\underline{\Omega}$ is on the left side of α. For plane curve $\alpha(\tau) = (\alpha^1(\tau), \alpha^2(\tau)) \in \mathbb{R}^2$ we define the **signed curvature** of α at τ

$$\kappa(\tau) = \frac{\dot{\alpha}^1(\tau)\ddot{\alpha}^2(\tau) - \ddot{\alpha}^1(\tau)\dot{\alpha}^2(\tau)}{[\dot{\alpha}^1(\tau)^2 + \dot{\alpha}^2(\tau)^2]^{3/2}}. \tag{1.29}$$

We define two categories of domains in $\mathbb{R}^2$:

Definition 7. *If* $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1 \subset \mathbb{R}^3$ *where the periodic interval* $\mathbb{T}^1 \equiv [0,1]/\sim$ *with* $0 \sim 1$*, and* $\underline{\Omega}$ *is an open connected bounded subset of* $\mathbb{R}^2$ *and the boundary* $\partial\underline{\Omega}$ *is an image of unit-speed curve* α *with*

$$\kappa(\tau) > 0 \quad \textit{for all} \quad \tau, \tag{1.30}$$

then we say the domain Ω *is the* ***smooth strictly convex cylinderical domain****.*

Definition 8. *If* $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1 \subset \mathbb{R}^3$ *and* $\underline{\Omega}$ *is an open connected bounded subset of* $\mathbb{R}^2$ *such that there exists the family of open bounded simply connected subsets* $\underline{\Omega}_i \subset \mathbb{R}^2$ *for* $i = 0, 1, 2, ..., M < \infty$ *such that*

$$\underline{\Omega} = \underline{\Omega}_0 \backslash \{\underline{\Omega}_1 \cup \underline{\Omega}_2 \cup ... \cup \underline{\Omega}_M\}, \tag{1.31}$$

and satisfying

- $\underline{\Omega}_0 \supset cl(\underline{\Omega}_i)$ *for all* $i = 1, 2, ..., M$ *and* $cl(\underline{\Omega}_i) \cap cl(\underline{\Omega}_j) = \emptyset$ *for all* $i \neq j$ $(\in \{1, 2, ..., M\})$,
- *there exists, for each* Ω_i*, a closed unit-speed analytic curve* $\alpha_i : [a_i, b_i] \to \mathbb{R}^2$ *such that* $\partial\Omega_i$ *is an image of* α_i *and* $\underline{\Omega_i \text{ is always on the left side of } \alpha_i}_{\mathbf{L}}$.

Then we say the domain $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1$ *is the* ***analytic non-convex cylinderical domain****.*

We explain the meaning of underlined $\mathbf{L}$ in the second condition of Definition 11. It means when you walk on the α_i with $\dot{\alpha}_i$ direction $\Omega = \Omega_0 \backslash \{\Omega_1 \cup \Omega_2 \cup ... \cup \Omega_M\}$ is always the left side of you. Precisely **L** means the outer normal direction at $\alpha_i(\tau_0) \in \partial\Omega$ is always

$$n(\alpha_i(\tau_0)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{\alpha}_i(\tau_0). \tag{1.32}$$

In terms of the standard perturbation f such that $F = \mu + \sqrt{\mu} f$, the Boltzmann equation can be rewritten as

$$\{\partial_t + v \cdot \nabla + L\} f = \Gamma(f, f), \qquad f(0, x, v) = f_0(x, v),$$

where the standard linear Boltzmann operator see [27] is given by

$$Lf \equiv \nu f - Kf = -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)\} = \nu f - \int \mathbf{k}(v, v') f(v') dv' \tag{1.33}$$

with the collision frequency $\nu(v) \equiv \int |v-u|^\gamma \mu(u) q_0(\theta) du d\theta \backsim \{1 + |v|\}^\gamma$ for $0 \le \gamma \le 1$; and

$$\Gamma(f_1, f_2) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_2) \equiv \Gamma_+(f_1, f_2) - \Gamma_-(f_1, f_2). \tag{1.34}$$

In terms of f, the specular reflection is

$$f(t, x, v)|_{\gamma_-} = f(x, v, v - 2(n(x) \cdot v) n(x)) = f(x, v, R(x) v) \tag{1.35}$$

For specular reflection conditions, it is well-known that both mass and energy are conserved. Without loss of generality, we may always assume that the mass-energy conservation laws hold for $t \ge 0$, in terms of the perturbation f:

$$\int_{\Omega \times \mathbf{R}^3} f(t, x, v) \sqrt{\mu} dx dv = 0, \tag{1.36}$$

$$\int_{\Omega \times \mathbf{R}^3} |v|^2 f(t, x, v) \sqrt{\mu} dx dv = 0. \tag{1.37}$$

Moreover, if the domain Ω has *any* axis of rotation symmetry (3.12), then we further assume the corresponding conservation of angular momentum is valid for all $t \ge 0$:

$$\int_{\Omega \times \mathbf{R}^3} \{(x - x_0) \times \varpi\} \cdot v f(t, x, v) \sqrt{\mu} dx dv = 0. \tag{1.38}$$

Theorem 5. *Assume $w^{-2}\{1+|v|\}^3 \in L^1$ in (3.20). Assume that the domain is either Smooth Convex Domain (Definition 10) or Analytic Non-Convex Domain (Definition 11) and the mass (3.17) and energy (3.18) are conserved for f_0. In the case of Ω has any rotational symmetry (3.12), we require that the corresponding angular momentum (3.19) is conserved for f_0. Then there exists $\delta > 0$ such that if $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$ and $||wf_0||_\infty \leq \delta$, there exists a unique solution $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$ to the specular boundary value problem (3.16) for the Boltzmann equation (3.1) such that*

$$\sup_{0\leq t\leq\infty} e^{\lambda t}||wf(t)||_\infty \leq C||wf_0||_\infty.$$

1.4 Stability of Maxwellian with a Large external Potential

In the presence of a potential Φ, a density of a dilute charged gas is governed by the Boltzmann equation

$$\partial_t F + v\cdot\nabla_x F - \nabla_x\Phi(x)\cdot\nabla_v F = Q(F,F) \quad , \qquad F(0,x,v) = F_0(x,v), \tag{1.39}$$

where $F(t,x,v)$ is a distribution function for the gas particles at a time $t \geq 0$, a position $x \in \mathbb{T}^d$ and a velocity $v \in \mathbb{R}^d$ for $d \geq 3$. Here the external potential $\Phi(x)$ is a given function only depends on the spatial variable x in a periodic box $\mathbb{T}^d$.

Throughout Chapter 4, we study the stability of a local Maxwellian for given potential Φ :

$$\mu_E(x,v) \;=\; \exp\left\{-\frac{|v|^2}{2} - \Phi(x)\right\} \;=\; \mu(v)e^{-\Phi(x)} \;, \tag{1.40}$$

where $\mu(v) = \exp\left\{-\frac{|v|^2}{2}\right\}$ is the standard global Maxwellian in the no potential case, $\Phi\equiv 0$ ([26]). Define a perturbation distribution $f = f(t,x,v)$ by

$$F(t,x,v) = \mu_E(x,v) + \sqrt{\mu_E(x,v)}f(t,x,v). \tag{1.41}$$

Then the equation for the perturbation f is

$$\partial_t f + v\cdot\nabla_x f - \nabla\Phi\cdot\nabla_v f + e^{-\Phi(x)}Lf = e^{-\frac{1}{2}\Phi(x)}\Gamma(f,f) \;\;, \quad f(0,x,v) = f_0(x,v) = \frac{F_0-\mu_E}{\sqrt{\mu_E}} \;, \tag{1.42}$$

where the standard operators of the linearized Boltzmann theory ([25]) are

$$Lf \equiv \nu f - Kf = -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\} = \nu f - \int \mathbf{k}(v,v')f(v')dv',$$

with the collision frequency $\nu(v) \equiv \int |v-u|^\gamma \mu(u) q_0 du d\omega \sim \{1+|v|\}^\gamma$ for $0 < \gamma \le 1$; and

$$\Gamma(f_1, f_2) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}f_1, \sqrt{\mu}f_2) \equiv \Gamma_+(f_1,f_2) - \Gamma_-(f_1,f_2).$$

Let F be a solution of the Boltzmann equation (1.39) with an external potential Φ. As the no potential case ($\Phi \equiv 0$), we have the (excess) conservations of mass and energy :

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} \{F(t,x,v) - \mu_E\} dvdx = \iint_{\mathbb{T}^d\times\mathbb{R}^d} \{F_0(x,v) - \mu_E\} dvdx \equiv M_0, \quad (1.43)$$

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{R}^d} (\frac{|v|^2}{2} + \Phi(x))\{F(t,x,v) - \mu_E\} dvdx &= \iint_{\mathbb{T}^d\times\mathbb{R}^d} (\frac{|v|^2}{2} + \Phi(x))\{F_0(x,v) - \mu_E\} dvdx \\ &\equiv E_0, \quad (1.44)\end{aligned}$$

as well as the excess entropy inequality :

$$\mathcal{H}(F(t)) - \mathcal{H}(\mu_E) \le \mathcal{H}(F_0) - \mathcal{H}(\mu_E), \quad (1.45)$$

where $\mathcal{H}(g) \equiv \iint g \ \ln g \ dvdx$.

However, in the presence of an external potential Φ, the momentum conservation law is delicate. In general, the momentum is not conserved : multiplying the Boltzmann equation (1.39) by v_i and integrating over $\mathbb{T}^d \times \mathbb{R}^d$, we have

$$\frac{d}{dt}\iint v_i F(t) dxdv - \iint v_i \nabla_x \Phi(x) \cdot \nabla_v F(t) dxdv = 0.$$

Applying the integration by part to the second term, we have

$$\frac{d}{dt}\iint v_i F(t) dxdv + \iint \partial_i \Phi(x) F(t) dxdv = 0.$$

If the potential Φ does not depend on x_i, then the second integration of the above equation vanishes. Therefore we have the conservation of momentum for v_i. Otherwise, in general, we do not have such a conservation law of momentum.

More precisely, define a map

$$\Lambda : \mathbb{T}^d \ \rightarrow \ \{\text{Linear Subspaces of } \mathbb{R}^d\} \ \ , \quad \Lambda(x) = \text{span}\{\nabla\Phi(x)\} \equiv \{v \in \mathbb{R}^d : v = \tau\nabla\Phi(x) \ \ , \tau \in \mathbb{R}\}.$$

Define

$$\Lambda(\mathbb{T}^d) \ \equiv \ \bigcup_{x\in\mathbb{T}^d} \Lambda(x) \ , \tag{1.46}$$

which is a linear subspace of $\mathbb{R}^d$. Further we can decompose $\mathbb{R}^d = \Lambda(\mathbb{T}^d) \bigoplus \Lambda(\mathbb{T}^d)^\perp$. More precisely we define a **degenerate subspace** of $\nabla\Phi$ by

$$\Lambda(\mathbb{T}^d)^\perp \ \equiv \ \bigcap_{x\in\mathbb{T}^d} \Lambda(x)^\perp, \tag{1.47}$$

which is a linear subspace of $\mathbb{R}^d$. Let $n = dim \ \Lambda(\mathbb{T}^d)^\perp$. Notice that generically the degenerate subspace of $\nabla\Phi$ is a zero space $\{0\}$ and $n = dim \ \Lambda(\mathbb{T}^d)^\perp = 0$. Upon relabeling and reorienting the coordinates axes, we may assume that $\Lambda(\mathbb{T}^d)^\perp$ is spanned by $\{e_1, ..., e_n\}$, i.e.

$$\Lambda(\mathbb{T}^d)^\perp = \text{span}\{e_1, ..., e_n\} \ \ , \qquad n = dim \ \Lambda(\mathbb{T}^d)^\perp. \tag{1.48}$$

If Φ is differentiable then $\partial_{x_1}\Phi = \cdots = \partial_{x_n}\Phi \equiv 0$ and $\Phi = \Phi(x_{n+1}, \cdots, x_d)$. Further we assume $\Phi \in C^3(\mathbb{T}^d)$ and satisfies the periodic boundary condition in $\mathbb{T}^d$ and $1 \le \Phi(x) < |\Phi|_\infty$. Then we have the (excess) conservation of momentum for degenerate $\{v_1, ..., v_n\}$:

$$\begin{aligned}
&\iint_{\mathbb{T}^d\times\mathbb{R}^d} \{F(t,x,v) - \mu\}(v_1, \ldots v_n)^T \ dvdx \\
&= \iint_{\mathbb{T}^d\times\mathbb{R}^d} \{F_0(x,v) - \mu\}(v_1, \ldots v_n)^T \ dvdx \ = \ \mathbf{J}_0 \in \mathbb{R}^n.
\end{aligned} \tag{1.49}$$

Notice that generically $\Lambda(\mathbb{T}^d)^\perp$,the degenerate subspace of $\nabla\Phi$, is a zero space $\{0\}$ and $n = dim \ \Lambda(\mathbb{T}^d)^\perp = 0$ so that we do not have such a momentum conservation law as (1.49).

It is important to point out that the momentum conservation law (1.49) is necessary in order to get decay (1.52) in Theorem 1. In particular the condition (1.49) is used in (4.48) in order to show the crucial positivity of L in (4.23). Without the condition (1.49), we have the stability result (1.55) in Theorem 2 but not a decay.

We introduce the weight function for $\beta > d/2$,

$$w(x,v) = \left\{ \frac{|v|^2}{2} + \Phi(x) \right\}^{\beta/2}. \tag{1.50}$$

Theorem 6. *Assume that an external potential Φ is a periodic C^3-function on $\mathbb{T}^d$ and $\Phi = \Phi(x_{n+1}, \cdots, x_d)$ for some $n \leq d$. Assume the conservations of mass (1.43), energy (1.44) and momentum for degenerate $\{v_1, \cdots, v_n\}$ (1.49) are valid for $F_0 = \mu_E + \sqrt{\mu_E} f_0$ with*

$$(\ M_0,\ E_0,\ \mathbf{J}_0\ ) = (\ 0\ ,\ 0\ ,\ \mathbf{0}\ ) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n. \tag{1.51}$$

Then there exists $\delta > 0$ such that if $F_0(x,v) = \mu_E + \sqrt{\mu_E} f_0(x,v)$ and $||wf_0||_\infty \leq \delta$, there exists a unique solution $F(t,x,v) = \mu_E + \sqrt{\mu_E} f(t,x,v) \geq 0$ for the Boltzmann equation (1.39) such that

$$\sup_{0 \leq t \leq \infty} e^{\lambda t} ||wf(t)||_\infty \leq C ||wf_0||_\infty. \tag{1.52}$$

Without the conservation of momentum for degenerate $\{v_1, \cdots, v_n\}$ (1.49), we are not able to prove the $L^\infty-$decay (1.52). The reason is that such a momentum conservation law is crucial to show the positivity of L in Proposition 4. However, we have the $L^\infty-$stability using the natural excess entropy inequality (1.45).

Theorem 7. *Assume that an external potential Φ is a periodic C^3-function on $\mathbb{T}^d$. Assume the excess conservation of mass (1.43), energy (1.44) and the excess entropy inequality (1.45) are valid for $F_0 = \mu_E + \sqrt{\mu_E} f_0$ with*

$$|M_0|,\ |E_0|,\ |\mathcal{H}(F_0) - \mathcal{H}(\mu_E)|\ < \infty. \tag{1.53}$$

Then there exists $\delta > 0$ such that if $F_0(x,v) = \mu_E + \sqrt{\mu_E} f_0(x,v)$ and

$$||wf_0||_\infty + \sqrt{\mathcal{H}(F_0) - \mathcal{H}(\mu_E) + |M_0| + |E_0|}\ \leq\ \delta\ , \tag{1.54}$$

then there exists a unique solution $F(t,x,v) = \mu_E + \sqrt{\mu_E} f(t,x,v) \geq 0$ for the Boltzmann equation (1.39) such that

$$\sup_{0 \leq t \leq \infty} ||wf(t)||_\infty\ \leq\ C \left\{\ ||wf_0||_\infty + \sqrt{\mathcal{H}(F_0) - \mathcal{H}(\mu_E) + |M_0| + |E_0|}\ \right\}. \tag{1.55}$$

Notice that we do not need any smallness assumption for the external potential Φ in both theorems.

Chapter Two

Formation and Propagation of Singularity in Non-Convex Domains

Abstract The formation and propagation of singularities for Boltzmann equation in bounded domains has been an important question in numerical studies as well as in theoretical studies. Consider the nonlinear Boltzmann solution near Maxwellians under in-flow, diffuse, or bounce-back boundary conditions. We demonstrate that discontinuity is created at the non-convex part of the grazing boundary, then propagates only along the forward characteristics inside the domain before it hits on the boundary again.

2.1 Comments

2.1.1 Previous Works and Significance of This Work

There are many references for the mathematical study of different aspects of the boundary value problem of the Boltzmann equation, for example [35][27][47] and the references therein. In [27], an unified $L^2 - L^\infty$ theory in the near Maxwellian regime is developed to establish the existence, uniqueness and exponential decay toward a Maxwellian, for all four basic types of the boundary conditions and rather general domains.

The qualitative study of the particle-boundary interaction in a bounded domain and its effects on the global dynamics is a fundamental problem in the Boltzmann theory and the Vlasov theory. One of challenging questions is the regularity theory of kinetic equations in bounded domain. This problem is very hard because even for simplest kinetic equations with the differential operator $v\cdot\nabla_x$, the phase boundary $\partial\Omega\times\mathbb{R}^3$ is always characteristic but not uniformly characteristic at the grazing set $\gamma_0 = \{(x,v) : x \in \partial\Omega,\ \text{and}\ v\cdot n(x) = 0\}$. In a convex domain Hölder estimates of boundary problem of Vlasov equation for specular reflection boundary condition is solved recently [38][39] but Sobolev-type estimate is still widely open. In a convex domain a continuity of the Boltzmann solution away from γ_0 is established in [27] for all four basic boundary conditions. The convexity of boundary is crucial for certain regularities of both Vlasov equation and Boltzmann equation. In a convex domains, backward trajectories starting at interior points of the phase space cannot reach points of the grazing boundary γ_0, due to Velocity Lemma([30][38]), where possible singularities may exist.

In general, on the other hand, in a non-convex domain, backward trajectories starting at interior points of the phase space can reach the grazing boundary. Therefore we expect singularities will be created at some part of grazing boundary γ_0 and propagate inside of the phase space. This question has been attracting a lot of attentions from early '90s, see references in pp.91–92 in Sone's book [49]. For Boltzmann equation, most of works are numerical studies [49][50][54] and few mathematical studies.

Once we enlarge our survey to propagation of singularities which already exist on initial data or boundary data, there are some mathematical works [2][9][10][16][17] as well as numerical works [3][49]. In [2], for linear BGK model, a propagation of discontinuity ,which exists already in the boundary data, is studied mathematically and also numerically. In [9], for the full Boltzmann equation in the near vacuum regime, a propagation of Sobolev $H^{1/25}$ singularity, which exists already in the initial data, is studied and same effect has been recently shown in the near Maxwellian regime [10][17].

In Vlasov theory, we refer to [7][34][64] for the boundary value problem. Singular solutions were studied in [30] extensively. In [30], the non-convexity condition of boundary is replaced by the inward electric field which has a similar effect with non-convexity of the boundary. In convex domains, Hölder estimates of Vlasov solution with specular reflection boundary is solved recently [38][39], but Sovolev-type estimate is still widely open.

Our results give a rather complete characterization of formation and propagation of singularity for the full Boltzmann equation near Maxwellian in general domain under in-flow, diffuse, bounce-back boundary conditions. There is no restriction of the time interval. More precisely we show that for any non-convex point x of the boundary and velocity tangent to $\partial\Omega$ at x, there exists an initial datum (and in-flow datum, for in-flow boundary condition case) such that the Boltzmann solution has a jump discontinuity at (x, v). Once the discontinuity occurs at the grazing boundary, this discontinuity propagates inside along the forward trajectory until it hits the boundary again. And except those points, the grazing boundary and forward trajectories emanating from the grazing boundary, we can show that the Boltzmann solution is continuous.(Continuity away from $\mathfrak{D}$)

2.1.2 Main Ingredients of the Proofs

1. The Equality induced by Non-Convex Domain

We consider near Maxwellian regime and linearized Boltzmann equation (2.9). The formation of discontinuity is a consequence of following estimate. Assume $(x,v) \in \gamma_0$ as below picture so that for sufficiently small $t > 0$ the backward trajectory $x - tv$ is in an interior of the phase space. For simplicity we impose the trivial in-flow boundary condition $G(t,x,v) \equiv \mu(v)$ which corresponds $g(t,x,v) \equiv 0$ (2.12). Consider points (x'_n, v'_n) in γ_- and (x''_n, v''_n) missing the non-convex part near (x,v) and both sequences converge (x,v) as $n \to \infty$.

Now suppose the solution f of the linearized Boltzmann equation be continuous around (x,v). Then the Boltzmann solution f at (x'_n, v'_n)

$$f(t, x'_n, v'_n) = g(t, x'_n, v'_n) = 0,$$

and at (x''_n, v''_n),

$$f(t, x''_n, v''_n) = e^{-\nu(v''_n)t} f_0(x''_n - tv''_n, v''_n) + \int_0^t e^{-\nu(v''_n)(t-s)} \{Kf + \Gamma(f,f)\}(s, x''_n - (t-s)v''_n, v''_n) ds$$

converges each other as $n \to \infty$. Then we have the following equality

$$f_0(x - tv, v) = -\int_0^t e^{\nu(v)s} \{Kf + \Gamma(f,f)\}(s, x - (t-s)v, v) ds. \tag{2.1}$$

Thanks to [27], the pointwise estimate of f, with some standard estimates of K, Γ (Lemma 25) the right hand side of above equality has magnitude $O(t)||f_0||_\infty(1 + ||f_0||_\infty)$. If you choose $f_0(x - tv, v) = ||f_0||_\infty$ then the above equality (2.1) cannot be true for sufficiently small t unless the trivial case $f_0 \equiv 0 (F \equiv \mu)$. Therefore the Boltzmann solution f cannot be continuous at (x,v). For diffuse (1.5), bounce-back (1.7) boundary conditions we also obtain the equality induced by non-convex domain similar as (2.1).

This argument bases on the idea that free transport effect is dominant to collision effect if time $t > 0$ and the perturbation $\frac{F-\mu}{\sqrt{\mu}}$ is small.

2. Continuity of the Gain Term Q_+

The smoothing effect of the gain term Q_+ is one of the fundamental features of the Boltzmann theory. There are lots of results about the smoothing effect in Sobolev regularity, for example

$$||Q_+(\phi,\psi)||_{H^{\frac{N-1}{2}}} \leq C||\phi||_{L^1}||\psi||_{L^2},$$

with some assumption on various collision kernels [45][66][67]. To study the propagation of singularity and regularity, in the case of angular cutoff kernel (3.3), it is standard to use Duhamel formulas and combine the Velocity average lemma and the regularity of Q_+ [9]. For detail, see Villani's survey note [63] especially pp. 77–79.

In order to study the propagation of discontinuity and continuity we need a totally different smooth effect of Q_+. For the discontinuity induced by the non-convex domain, we need following : Recall the grazing set $\mathfrak{G}$ in Definition 4. A test function $\phi(t,x,v)$ is continuous on $[0,T]\times(\Omega\times\mathbb{R}^3)\backslash\mathfrak{G}$ and bounded on $[0,T]\times\Omega\times\mathbb{R}^3$. Then

$$Q_+(\phi,\phi)(t,x,v) \ \in \ C^0([0,T]\times\Omega\times\mathbb{R}^3). \tag{2.2}$$

Recall that the grazing set $\mathfrak{G}=\{(x,v)\in\bar{\Omega}\times\mathbb{R} : v\in\mathfrak{G}_x\}$. The grazing section $\mathfrak{G}_x=\{\tau u\in\mathbb{R}^3 : t\geq 0,\ u\in\mathfrak{G}_x\cap\mathbb{S}^2\}$ is a union of straight lines in velocity space $\mathbb{R}^3$ and two dimensional Lebesque measure of $\mathfrak{G}_x\cap\mathbb{S}^2$ is zero (Hongjie Dong's Lemma, Lemma 17 of [27]). Moreover, using continuous behavior of $\mathfrak{G}_x$ in x, one can invent a very effective covering of $\mathfrak{G}_x$ (Guo's covering, Lemma 18 of [27]). Because of those geometric and size restriction on $\mathfrak{G}$, even the gain term Q_+ is an integration operator in v alone, we can prove the smoothing effect of Q_+ in $C^0([0,T]\times\Omega\times\mathbb{R}^3)$ for t,x and v, see Theorem 8. Notice that those smoothing effect on $C^0_{t,x,v}$ has been believed to be true for long time without a mathematical proof in numerical communities [1], p1587 of [2], p502 of [50].

The main idea to prove the smoothing effect in $C^0_{t,x,v}$ is to use the Carleman's representation for $Q_+(\phi,\phi)(t,x,v)$ which has been a very effective tool [31][66][67].

$$\int_{\mathbb{R}^3}\phi(t,x,v')\frac{1}{|v-v'|^2}\int_{E_{vv'}}\phi(t,x,v_1')B(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|})dv_1'dv', \tag{2.3}$$

with the hyperplane $E_{vv'}=\{v_1'\in\mathbb{R}^3 : (v_1'-v)\cdot(v'-v)=0\}$. We will show the smallness of

$$|Q_+(\phi,\phi)(\bar{t},\bar{x},\bar{v})-Q_+(\phi,\phi)(t,x,v)|,$$

for $|(t,x,v)-(\bar{t},\bar{x},\bar{v})|<\delta$. Assume we have sufficient decay of ϕ for large v. Replace the integrable kernel $\frac{1}{|v-v'|^2}$ by smooth compactly supported function and cut off the singular part of $B(2v-v'-v_1', \frac{v'-v_1'}{|v'-v_1'|})$ to control the above difference as

$$O(\delta)||\phi||_\infty^2 \ + \ C\int_{|v'|<N}|\phi(t,x,v')-\phi(\bar{t},\bar{x},v'')|\int_{E_{\bar{v}v''}\cap\{|v_1''|<N\}}|\phi(\bar{t},\bar{x},v_1'')|dv_1''dv'$$
$$+\ C\int_{|v'|<N}|\phi(t,x,v')|\left\{\int_{E_{vv'}\cap\{|v_1'|<N\}}\phi(t,x,v_1')dv_1'-\int_{E_{\bar{v}v''}\cap\{|v_1''|<N\}}\phi(\bar{t},\bar{x},v_1'')dv_1''\right\}dv',$$

where $v''(v')$ is chosen to be $v'-(v-\bar{v})$, (2.21) for convenience.

One can easily control the integration at the first line. Because for the first term, integrating over v', we can cut off a small neighborhood of $\mathfrak{G}_x$ from $|v'|<N$. Away from that neighborhood, using the continuity of ϕ away from $\mathfrak{G}_x$ we can control the integrand pointwisely.

In order to control the second line integration we have to control the difference in big braces. To do that we choose a special change of variables for v_1'' ,(2.22). Under this change of variables the second line is bounded by

$$C\int_{|v'|<N}|\phi(t,x,v')|\int_{E_{vv'}\cap\{|v_1'|<N\}}|\ \phi(t,x,v_1')-\phi(\bar{t},\bar{x},v_1'')|\ dv_1'dv'.$$

The second integration term above is a function of $t,x,\bar{t},\bar{x}$ and v. Unfortunately one cannot expect a pointwise control(smallness) of the second integration for all v : even the grazing section $\mathfrak{G}_x$, where $\phi(t,x,v_1')$ might have discontinuity, is small i.e. 2 dimensional Lebesque measure of $\mathfrak{G}_x\cap\mathbb{S}_x$ is zero, the measure on the plane $E_{vv'}$ could be large(even infinite). However, in Section 3.3, we can show that those bad situation happen for very rare v' in $\{v'\in\mathbb{R}^3 : |v'|<N\}$ and use the integration over v' to control the above integration.

3. New Proof of Continuity of Boltzmann solution with Diffuse Boundary Condition

In Section 5.2 we prove a continuity away from $\mathfrak{D}$ of Boltzmann solution with diffuse boundary condition using simple iteration scheme (3.221) with iteration diffuse boundary condition (2.120). This iteration scheme has several advantages. First it preserves a continuity away from $\mathfrak{D}$ as m increasing, that is, if h^m is continuous away from $\mathfrak{D}$ then h^{m+1} is also continuous away from $\mathfrak{D}$. Second, the sequence $\{h^m\}$ has uniform L^∞ bound and moreover it is Cauchy in L^∞ for in-flow boundary condition $h^m|_{\gamma_-} = wg$. To show the second fact for the iteration diffuse boundary condition we use Guo's idea [27] : a norm of the diffuse boundary operator is less than 1 effectively, if we trace back several bounces. This approach gives simpler proof for the continuity of Boltzmann equation with diffuse boundary condition with convex domain (see Lemma 26 of [27]).

2.2 Preliminary

In this section we study continuity properties of the backward exit time $t_{\mathbf{b}}(x,v)$ and, a measure theoretic property and geometric covering of the grazing set $\mathfrak{G}$, and estimates of Boltzmann operators and the Carleman's representation.

We use Lemma 1 of [27], basic properties of the backward exist time $t_{\mathbf{b}}(x,v)$:

Lemma 1. *[27] Let Ω be an open bounded subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Let (t,x,v) be connected with $(t-t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v)$ backward in time through a trajectory of (3.13).*
1. The backward exit time $t_{\mathbf{b}}(x,v)$ is lower semicontinuous.
2 If

$$v\cdot n(x_{\mathbf{b}}(x,v)) < 0, \tag{2.4}$$

then $(t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v))$ are smooth functions of (x,v) so that

$$\begin{aligned} \nabla_x t_{\mathbf{b}} &= \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \quad \nabla_v t_{\mathbf{b}} = \frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \\ \nabla_x x_{\mathbf{b}} &= I + \nabla_x t_{\mathbf{b}} \otimes v, \quad \nabla_v x_{\mathbf{b}} = t_{\mathbf{b}} I + \nabla_v t_{\mathbf{b}} \otimes v. \end{aligned}$$

For a convex domain, if a point (x,v) is in the interior of the phase space, i.e. $(x,v)\in\Omega\times\mathbb{R}^3$ then the condition (3.169) is always satisfied and hence $t_{\mathbf{b}}(x,v)$ is smooth due to Lemma 24. However for a non-convex domain, there is a point (x,v) in $\Omega\times\mathbb{R}^3$ but $(x_{\mathbf{b}}(x,v,),v)\in\gamma_0$, i.e. $v\cdot n(x_{\mathbf{b}}(x,v)) = 0$. We further investigate a continuity property of $t_{\mathbf{b}}$ for that case. Indeed, discontinuity behavior of $t_{\mathbf{b}}(x,v)$ for $(x_{\mathbf{b}}(x,v),v)\gamma_0^{\mathbf{S}}$ is a main ingredient of the formation of discontinuity.

Lemma 2. *Let $\Omega\in\mathbb{R}^3$ be an open set with a smooth boundary $\partial\Omega$. Assume $(x_0,v_0)\in\Omega\times\mathbb{R}^3$ with*

$v_0 \neq 0$ and $t_{\mathbf{b}}(x_0, v_0) < \infty$. Consider $(x_0, v_0) \in \mathfrak{G}$, i.e. $(x_{\mathbf{b}}(x_0, v_0), v_0) \in \gamma_0$.

$$\textit{If } (x_{\mathbf{b}}(x_0, v_0), v_0) \in \gamma_0^{I-} \quad \textit{then } t_{\mathbf{b}}(x, v) \quad \textit{is continuous around } (x_0, v_0).$$

$$\textit{If } (x_{\mathbf{b}}(x_0, v_0), v_0) \in \gamma_0^{\mathbf{S}} \quad \textit{then } t_{\mathbf{b}}(x, v) \quad \textit{is not continuous around } (x_0, v_0).$$

Recall γ_0^{I-} and $\gamma_0^{\mathbf{S}}$ in Definition 12.

Proof. Throughout this proof, without loss of generality we assume that $\partial\Omega$ is a graph of Φ locally and $\Phi(0,0) = 0$ and $(\partial_{x_1}\Phi, \partial_{x_2}\Phi)(0,0) = (0,0)$. Moreover assume $x_0 = (|x_0|, 0, 0), v_0 = (|v_0|, 0, 0)$ and $t_{\mathbf{b}}(x_0, v_0) = \frac{|x_0|}{|v_0|}$ so that $x_{\mathbf{b}}(x_0, v_0) = (0,0,0) = (0, 0, \Phi(0,0))$.

First, let $(x_{\mathbf{b}}(x_0, v_0), v_0) \in \gamma_0^{I-}$. By the definition of γ_0^{I-}, we have $\Phi(-\tau, 0) > 0$ and $\Phi(\tau, 0) < 0$ for $0 < \tau << 1$. Using the continuity of Φ, choose sufficiently small $\varepsilon > 0$, $\delta > 0$ such that $\Phi(-\delta, y) > \frac{\varepsilon}{2}$ and $\Phi(\delta, y) < -\frac{\varepsilon}{2}$ for $0 < |y| < \delta$. Fix $x = (x_1, x_2, x_3) \sim x_0$ and $v = (v_1, v_2, v_3) \sim v_0$. We define

$$\Psi(x, v, t) = x_3 - tv_3 - \Phi(x_1 - tv_1, x_2 - tv_2).$$

For $t' \equiv \frac{x_1 - \delta}{v_1}$, $\Psi(x, v, t_0') = -\Phi(\delta, x_2 - v_2\frac{x_1-\delta}{v_1}) + x_3 - v_3\frac{x_1-\delta}{v_1} > \frac{\varepsilon}{4}$ for $(x_1, x_2, x_3) \sim (|x_0|, 0, 0)$, $(v_1, v_2, v_3) \sim (|v_0|, 0, 0)$. For $t'' = \frac{x_1+\delta}{v_1}$, $\Psi(x, v, t'') = -\Phi(-\delta, x_2 - \frac{x_1+\delta}{v_1}v_2) + x_3 - \frac{x_1+\delta}{v_1}v_3 < -\frac{\varepsilon}{4}$ for $(x_1, x_2, x_3) \sim (|x_0|, 0, 0)$, $(v_1, v_2, v_3) \sim (|v_0|, 0, 0)$. Using the continuity of Φ and Ψ, there exists $t_* \in (\frac{x_1}{v_1} - \frac{\delta}{v_1}, \frac{x_1}{v_1} + \frac{\delta}{v_1})$ so that $\Psi(x, v, t_*) = 0$, i.e. $t_{\mathbf{b}}(x, v) = t_*$. If $x \sim x_0$ and $v \sim v_0$ then $\frac{x_1}{v_1} - \frac{\delta}{v_1} \sim \frac{|x_0|}{|v_0|} - \frac{\delta}{|v_0|} = t_{\mathbf{b}}(x_0, v_0) - \frac{\delta}{|v_0|}$ and $\frac{x_1}{v_1} + \frac{\delta}{v_1} \sim \frac{|x_0|}{|v_0|} + \frac{\delta}{|v_0|} \sim t_{\mathbf{b}}(x_0, v_0) + \frac{\delta}{|v_0|}$ so that $t_* \in (t_{\mathbf{b}}(x_0, v_0) - \frac{\delta}{|v_0|}, t_{\mathbf{b}}(x_0, v_0) + \frac{\delta}{|v_0|})$.

Next, let $(x_{\mathbf{b}}(x, v), v) \in \gamma_0^{\mathbf{S}}$. By the definition of the concave grazing boundary $\gamma_0^{\mathbf{S}}$, we have $\Phi(-\tau, 0) > 0$ and $\Phi(\tau, 0) < 0$ for $0 < \tau << 1$. Choose a sequence $x_n = (|x_0|, 0, \frac{1}{n})$. There exists $\varepsilon > 0$ such that $t_{\mathbf{b}}(x_n, v_0) > t_{\mathbf{b}}(x_0, v_0) + \varepsilon$ for sufficiently large n. This implies that $(x_n, v_0) \to (x_0, v_0)$ but $t_{\mathbf{b}}(x_n, v_0) \nrightarrow t_{\mathbf{b}}(x_0, v_0)$ as $n \to \infty$. $\square$

In the next two lemmas, we consider the grazing set $\mathfrak{G}$ (Definition 4) including the discontinuity set $\mathfrak{D}$. The next lemma, Lemma 17 of [27] due to Hongjie Dong, is important to control a size of $\mathfrak{G}$. We denote m_2 as a standard 2-dimensional Lebesque measure and m_3 as a standard 3-dimensional measure. Recall that the grazing section $\mathfrak{G}_x$ in Definition 4.

Lemma 3. *[27] If $\partial\Omega$ is C^1 then the grazing section $\mathfrak{G}_x$ restricted to $\mathbb{S}^2$ has zero 2 dimensional Lebesque measure i.e.*

$$m_2(\mathfrak{G}_x \cap \mathbb{S}^2) = 0,$$

for all $x \in \bar{\Omega}$.

With condition $m_2(\mathfrak{G}_x \cap \mathbb{S}^2) = 0$, we can construct the Guo's covering which is little bit stronger than the original one in Lemma 18 in [27].

Lemma 4 (Guo's covering)**.** *[27] Assume $m_2(\mathfrak{G}_x \cap \mathbb{S}^2) = 0$ is valid for all $x \in \bar{\Omega}$. Let $B_N = \{v \in \mathbb{R}^3 : |v| \leq N\}$. Then for any $\varepsilon > 0$ and $N_* > 0$ there exist $\delta_{\varepsilon,N,N_*} > 0$, and $l_{\varepsilon,N,N_*,\Omega}$ balls $B(x_1; r_1), B(x_2; r_2)..., B(x_l; r_l) \subset \bar{\Omega}$, as well as open sets $O_{x_1}, O_{x_2}, ...O_{x_l}$ of B_N which are radial symmetric, i.e.*

$$O_{x_i} = \{t\hat{v} \in \mathbb{R}^3 : t \geq 0,\ \hat{v} \in O_{x_i} \cap \mathbb{S}^2\},$$

with $m_3(O_{x_i}) < \frac{\varepsilon}{N_*}$ and $m_2(O_{x_i} \cap \mathbb{S}^2) \leq \frac{\varepsilon}{N^2 N_*}$ for all $1 \leq i \leq l_{\varepsilon,N,N_*,\Omega}$, such that for any $x \in \bar{\Omega}$, there exists x_i so that $x \in B(x_i; r_i)$ and for $v \notin O_{x_i}$,

$$|v \cdot n(x_{\mathbf{b}}(x, v))| > \delta_{\varepsilon,N,N_*} > 0,$$

or equivalently

$$O_{x_i} \supset \bigcup_{x \in B(x_i; r_i)} \{v \in B_N : |v \cdot n(x_{\mathbf{b}}(x, v))| \leq \delta_{\varepsilon,N,N_*}\} \supset \bigcup_{x \in B(x_i; r_i)} \mathfrak{G}_x \cap B_N.$$

Combining Lemma 3 and Lemma 4, we have following lemma, which is useful to prove Theorem 8. Namely, a function which is continuous away from the grazing set $\mathfrak{G}$ is uniformly continuous except arbitrary small open set containing $\mathfrak{G}$.

Lemma 5. *Assume ϕ is continuous on $[0, T] \times (\Omega \times \{v \in \mathbb{R}^3 : \frac{1}{M} \leq |v| \leq N\}) \backslash \mathfrak{G}$. For fixed $x \in \Omega$ and $\varepsilon > 0$ and $N_* > 0$, there exist*

$$\delta = \delta(\phi, \Omega, \varepsilon, N_*, x, \frac{1}{M}, N) > 0, \tag{2.5}$$

and an open set $U_x \subset \{v \in \mathbb{R}^3 : \frac{1}{M} \leq |v| \leq N\}$ which is radial symmetric, i.e. $U_x = \{t\hat{v} \in \mathbb{R}^3 : t \geq$

$0\ ,\ \hat{v} \in U_x \cap \mathbb{S}^2\}$ *with* $m_3(U_x) < \frac{\varepsilon}{N_*}$ *and* $m_2(U_x \cap \mathbb{S}^2) < \frac{\varepsilon}{N_* N^2}$ *such that*

$$|\phi(t,x,v) - \phi(\bar{t},\bar{x},\bar{v})| < \frac{\varepsilon}{N_*},$$

for $v \in \{v \in \mathbb{R}^3 : \frac{1}{M} \leq |v| \leq N\} \backslash U_x$ *and* $|(t,x,v) - (\bar{t},\bar{x},\bar{v})| < \delta$.

Proof. Let $x \sim \bar{x}$. Due to Guo's covering [27], Lemma 4, we can choose $B(x_i; r_i)$ including x and $\bar{x}$, as well as $O_{x_i} \subset \mathbb{R}^3$ so that

$$O_{x_i} \supset \bigcup_{y \in B(x_i; r_i)} \mathfrak{G}_y \cap B_N \supset \bigcup_{y \in B(x;\delta)} \mathfrak{G}_y \cap B_N,$$

with $m_3(O_{x_i}) < \frac{\varepsilon}{N_*}$. Notice that $m_3(\bar{O}_{x_i}) = m_3(O_{x_i})$. We can choose an open set U_{x_i} so that $m_3(U_{x_i}) \leq 2m_3(O_{x_i})$ and $\bar{O}_{x_i} \subset U_{x_i}$. Since both of $\bar{O}_{x_i}$ and $B_N \backslash U_{x_i}$ are compact subsets of B_N, we have a positive distance between two sets, i.e.

$$0 < \mathfrak{d} = \inf\{|\zeta - \xi| : \zeta \in \bar{O}_{x_i} \text{ and } \xi \in B_N \backslash U_{x_i}\}.$$

Assume $\delta < \mathfrak{d}/2$. Fix $x \in \bar{\Omega}$ and $v \in \{v \in \mathbb{R}^3 : \frac{1}{M} \leq |v| \leq N\} \backslash U_x$. Then $|(\bar{x},\bar{v}) - (x,v)| < \delta$ implies that $\bar{v} \in \{v \in \mathbb{R}^3 : \frac{1}{M} \leq |v| \leq N\} \backslash O_{x_i}$. For such $x, v, \bar{x}$ and $\bar{v}$ consider the function ϕ as it's restriction on a compact set $[0,T] \times \bar{B}(x;\delta) \times B_N \backslash O_{x_i}$. Therefore $\phi_{[0,T] \times \bar{B}(x;\delta) \times B_N \backslash O_{x_i}}$ is uniformly continuous. Hence $|\phi(t,x,v) - \phi(\bar{t},\bar{x},\bar{v})|$ can be controlled small uniformly if $\delta > 0$ is sufficiently small. $\square$

We will use the Carleman's representation [31][66], which is a powerful tool to study the gain term Q_+, in the proof of Theorem 8 crucially. Let $Q_+(\phi,\psi)$ be defined by (3.2) and let $\psi = \psi(v)$ and $\phi = \phi(v)$, $v \in \mathbb{R}^3$ make $Q_+(\psi,\phi) < \infty$ almost everywhere. Then the **Carleman's representation** is

$$Q_+(\psi,\phi)(v) = 2\int_{\mathbb{R}^3} \psi(v') \frac{1}{|v-v'|^2} \int_{E_{vv'}} \phi(v_1') B(2v - v' - v_1', \frac{v' - v_1'}{|v' - v_1'|}) dv_1' dv', \tag{2.6}$$

where $E_{vv'}$ is a hyperplane containing $v \in \mathbb{R}^3$ and perpendicular to $\frac{v'-v}{|v'-v|} \in \mathbb{S}^2$, i.e.

$$E_{vv'} = \{v_1' \in \mathbb{R}^3 : (v_1' - v) \cdot (v' - v) = 0\}. \tag{2.7}$$

In the proof of Theorem 8 we need to control the integration over $E_{vv'}$ in (2.6) frequently :

Lemma 6. *For a rapidly decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, we have*

$$\int_{E_{vv'}} \phi(|v_1{}'|) B\big(2v - v' - v_1{}', \frac{v' - v_1{}'}{|v' - v_1{}'|}\big) dv_1{}' \leq C_\phi (1 + |v - v'|^\gamma), \tag{2.8}$$

where C_ϕ only depends on ϕ.

Proof. For fixed v' and v, let us denote $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, with $\tilde{\mathbf{e}}_3 = \frac{v'-v}{|v'-v|}$, be the orthonormal basis of $\mathbb{R}^3$ such that any $v_1' \in E_{vv'}$ can be written as $v_1' = v + \eta_1 \tilde{\mathbf{e}}_1 + \eta_2 \tilde{\mathbf{e}}_2$. Since $v' - v \perp E_{vv'}$ from (2.7), there is η_3 such that $v' - v = \eta_3 \tilde{\mathbf{e}}_3$ where $|\eta_3| = |v - v'|$. Then we can write $2v - v' - v_1' = v - v' + v - v_1' = -\eta_1 \tilde{\mathbf{e}}_1 - \eta_2 \tilde{\mathbf{e}}_2 - \eta_3 \tilde{\mathbf{e}}_3$ and $|2v - v' - v_1'|^2 = \eta_1^2 + \eta_2^2 + |v' - v|^2$. Moreover $v' - v_1' = -\eta_1 \tilde{\mathbf{e}}_1 - \eta_2 \tilde{\mathbf{e}}_2 + \eta_3 \tilde{\mathbf{e}}_3$. We can write the left hand side of (2.8) as

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\eta_1^2 + \eta_2^2 + |v|^2) \left| \begin{pmatrix} -\eta_1 \\ -\eta_2 \\ -\eta_3 \end{pmatrix} \right|^\gamma \times \frac{1}{\eta_1^2 + \eta_2^2 + |v - v'|^2} \begin{pmatrix} -\eta_1 \\ -\eta_2 \\ -\eta_3 \end{pmatrix} \cdot \begin{pmatrix} -\eta_1 \\ -\eta_2 \\ \eta_3 \end{pmatrix} d\eta_1 d\eta_2 \\
\leq & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_2^2 + |v' - v|^2)^{\frac{\gamma}{2} - 1} (\eta_1^2 + \eta_2^2 - |v' - v|^2) d\eta_1 d\eta_2 \\
\leq & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\eta_1^2 + \eta_2^2) \big(\eta_1^2 + \eta_2^2 + |v' - v|^2\big)^{\frac{\gamma}{2}} d\eta_1 d\eta_2 \\
\leq & \; C_\phi (1 + |v' - v|^\gamma).
\end{aligned}$$

□

In terms of the standard perturbation f such that $F = \mu + \sqrt{\mu} f$, the Boltzmann equation can be rewritten as

$$\{\partial_t + v \cdot \nabla + L\} f = \Gamma(f, f), \qquad f(0, x, v) = f_0(x, v), \tag{2.9}$$

where the standard linear Boltzmann operator, see [31], is given by

$$Lf \equiv \nu f - Kf,$$

with the collision frequency $\nu(v) \equiv \int |v-u|^{\gamma}\mu(u)q_0(\frac{v-u}{|v-u|}\cdot\omega)d\omega du$ for $0<\gamma\leq 1$ and

$$\begin{aligned}
& \frac{1}{C_\nu}(1+|v|)^{\gamma} \leq \nu(v) \leq C_\nu(1+|v|)^{\gamma}, \\
Kf &\equiv \int_{\mathbb{R}^3}\mathbf{k}(v,v')f(v')dv' \equiv \frac{1}{\sqrt{\mu}}Q_+(\mu,\sqrt{\mu}f)+\frac{1}{\sqrt{\mu}}Q_+(\sqrt{\mu}f,\mu)-\frac{1}{\sqrt{\mu}}Q_-(\sqrt{\mu}f,\mu), \\
\Gamma(f,f) &\equiv \frac{1}{\sqrt{\mu}}Q_+(\sqrt{\mu}f,\sqrt{\mu}f)-\frac{1}{\sqrt{\mu}}Q_-(\sqrt{\mu}f,\sqrt{\mu}f) \equiv \Gamma_+(f,f)-\Gamma_-(f,f).
\end{aligned}$$

In order to write theorems in the unified way [27] for all boundary condition cases, we use the weight function

$$w(v)=\{1+\rho^2|v|^2\}^{\beta}, \tag{2.10}$$

and define

$$h \equiv w(v)\times\frac{F-\mu}{\sqrt{\mu}}.$$

In terms of h, the Boltzmann equation (3.1) is equivalent to

$$\{\partial_t+v\cdot\nabla_x+\nu-K_w\}h = w\Gamma(\frac{h}{w},\frac{h}{w}), \tag{2.11}$$

where $K_w h \equiv wK(\frac{h}{w})$ with boundary conditions :

1. In-flow boundary condition : $\quad h(t,x,v)=w(v)g(t,x,v) \quad$ for $\ (x,v)\in\gamma_-$. (2.12)
2. Diffuse boundary condition : with a normalized constant c_μ in (1.6),
$$h(t,x,v)=w(v)\sqrt{\mu(v)}\int_{v'\cdot n(x)>0}h(t,x,v')\frac{1}{w(v')\sqrt{\mu(v')}}c_\mu\mu(v')\{n_x\cdot v'\}dv' \quad \text{for } (x,v)\in\gamma_- \tag{2.13}$$
3. Bounce-back boundary condition : $\quad h(t,x,v)=h(t,x,-v) \quad$ for $\ (x,v)\in\gamma_-$. (2.14)

We recall two estimates of operators K and Γ from [27] : The Grad estimate for hard potentials :

$$|\mathbf{k}(v,v')| \leq C_{\mathbf{k}}\{|v-v'|+|v-v'|^{-1}\}e^{-\frac{1}{8}|v-v'|^2-\frac{1}{8}\frac{||v|^2-|v'|^2|^2}{|v-v'|^2}}.$$

Recall w in (3.20). Let $0\leq\theta<\frac{1}{4}$. Then there exists $0\leq\varepsilon(\theta)<1$ and $C_\theta>0$ such that for $0\leq\varepsilon<\varepsilon(\theta)$,

$$\int\{|v-v'|+|v-v'|^{-1}\}e^{-\frac{1-\varepsilon}{8}|v-v'|^2-\frac{1-\varepsilon}{8}\frac{||v|^2-|v'|^2|^2}{|v-v'|^2}}\frac{w(v)e^{\theta|v|^2}}{w(v')e^{\theta|v'|^2}}dv' \leq \frac{C_{\mathbf{k}}}{1+|v|}. \tag{2.15}$$

For the nonlinear collision operator

$$|w\Gamma(g_1, g_2)(v)| \leq C_\Gamma(1+|v|)^\gamma ||wg_1||_\infty ||wg_2||_\infty. \tag{2.16}$$

Also we recall a standard estimate

$$\int_{\mathbb{R}^3} \phi(v')|v-v'|^\gamma dv' \sim (1+|v|)^\gamma, \tag{2.17}$$

for $\phi \in L^1(\mathbb{R}^3)$.

2.3 Continuity of the Collision Operators

In this section we mainly prove the following Theorem :

Theorem 8 (Continuity of Q_+)**.** *Assume $F(t,x,v)$ is continuous on $[0,T]\times(\Omega\times\mathbb{R}^3)\backslash\mathfrak{G}$ and*

$$||\bar{w}^{-1}F||_{L^\infty([0,T]\times\bar{\Omega}\times\mathbf{R}^3)} < +\infty,$$

where $\bar{w} = \frac{e^{-\frac{|v|^2}{4}}}{(1+\rho^2|v|^2)^\beta}$ with $\rho \in \mathbb{R}$ and $\beta > 0$. Then $Q_+(F,F)(t,x,v)$ is continuous in $[0,T]\times\Omega\times\mathbb{R}^3$ and

$$\sup_{[0,T]\times\bar{\Omega}\times\mathbb{R}^3} |\nu^{-1}\bar{w}^{-1}Q_+(F,F)(t,x,v)| < \infty. \tag{2.18}$$

Theorem 8, a smooth effect in $C^0_{t,x,v}$, is the crucial ingredient to prove Theorem 2 and Theorem 3. This smooth effect of the gain term ensures that there is no singularity created by the nonlinearity of Botlzmann equation.

Proof of (2.18). It is easy to show the boundedness (2.18) from

$$\begin{aligned}\nu^{-1}\bar{w}^{-1}Q_+(F,F)(t,x,v) &\leq \frac{1}{\nu(v)\bar{w}(v)}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)\bar{w}(u')\bar{w}(v')d\omega du \times ||\bar{w}^{-1}F||_\infty^2 \\ &\leq \nu(v)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)\frac{e^{-\frac{|u|^2}{4}}}{(1+\rho^2|u|^2)^\beta}d\omega du \times ||\bar{w}^{-1}F||_\infty^2 \\ &\leq C\ \nu(v)^{-1}\nu(v)||\bar{w}^{-1}F||_\infty^2 \leq C||\bar{w}^{-1}F||^2_{L^\infty([0,T]\times(\bar{\Omega}\times\mathbf{R}^3)\setminus\gamma_0)}\end{aligned}$$

where we used (2.17) and $|u'|^2+|v'|^2=|u|^2+|v|^2$. □

Next we will show the continuity part of Theorem 8. The goal of following three subsections is to show

$$\begin{gathered}\text{For fixed } \varepsilon>0 \text{ and } (t,x,v)\in[0,T]\times\Omega\times\mathbb{R}^3, \text{ there is } \delta>0 \text{ such that} \\ |Q_+(\bar{w}h,\bar{w}h)(\bar{t},\bar{x},\bar{v})-Q_+(\bar{w}h,\bar{w}h)(t,x,v)|<\varepsilon \text{ for } |(\bar{t},\bar{x},\bar{v})-(t,x,v)|<\delta. \qquad (2.19)\end{gathered}$$

2.3.1 Decomposition and Change of Variables

In this subsection, we use the Carleman's representation to split $Q_+(\bar{w}h,\bar{w}h)(\bar{t},\bar{x},\bar{v})-Q_+(\bar{w}h,\bar{w}h)(t,x,v)$ in a natural way (2.20), and introduce two change of variables (2.21) and (2.22).

It is convenient to define

$$h\equiv\bar{w}^{-1}F$$

where $||h||_\infty\equiv||h||_{L^\infty([0,T]\times(\bar{\Omega}\times\mathbb{R}^3))}=||\bar{w}^{-1}F||_{L^\infty([0,T]\times(\bar{\Omega}\times\mathbf{R}^3))}$. Choose $(\bar{t},\bar{x},\bar{v})\sim(t,x,v)$. Using the Carleman's Representation (Lemma 6) we have

$Q_+(\bar{w}h,\bar{w}h)(\bar{t},\bar{x},\bar{v})-Q_+(\bar{w}h,\bar{w}h)(t,x,v)$

$$
\begin{aligned}
&= 2\int_{\mathbb{R}^3} \underbrace{\bar{w}(v'')h(\bar{t},\bar{x},v'')\frac{1}{|\bar{v}-v''|^2}}_{\mathcal{A}} \int_{E_{\bar{v}v''}} \underbrace{\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'')B\big(2\bar{v}-v''-v_1'', \frac{v''-v_1''}{|v''-v_1''|}\big)}_{\mathcal{B}} \, dv_1'' dv'' \\
&\quad -2\int_{\mathbb{R}^3} \underbrace{\bar{w}(v')h(t,x,v')\frac{1}{|v-v'|^2}}_{\mathcal{A}'} \int_{E_{vv'}} \underbrace{\bar{w}(v_1')h(t,x,v_1')B\big(2v-v'-v_1', \frac{v'-v_1'}{|v'-v_1'|}\big)}_{\mathcal{B}'} \, dv_1' dv' \\
&= 2\int_{\mathbb{R}^3} \{\mathcal{A}-\mathcal{A}'\} \int_{E_{\bar{v}v''}} \mathcal{B} \, dv_1'' dv'' \; + \; 2\int_{\mathbb{R}^3} \mathcal{A}' \int_{E_{\bar{v}v''}} \{\mathcal{B}-\mathcal{B}'\} \, dv_1'' dv' \qquad (2.20)
\end{aligned}
$$

In order to control the first term of (2.20), we need to compare arguments of $\mathcal{A}$ and $\mathcal{A}'$. For that purpose, we introduce the following change of variables :

Lemma 7. *For fixed v and $\bar{v}$ in $\mathbb{R}^3$ we define*

$$
v'' \equiv v''(v'; v, \bar{v}) = v' - (v - \bar{v}). \qquad (2.21)
$$

Then two planes $E_{\bar{v}v''}$ and $E_{vv'}$ have same normal direction. The distance between to planes is $|(\bar{v}-v)\cdot \frac{v'-v}{|v'-v|}|$.

Proof. Assume (2.21). Clearly $\frac{\partial v''(v')}{\partial v'} = I$ where I is 3×3 identity matrix. The normal direction of $E_{\bar{v}v''}$ is $\frac{v''-\bar{v}}{|v''-\bar{v}|} = \frac{v'-v}{|v'-v|}$ which is also the normal direction of $E_{vv'}$. To measure a distance between two planes $E_{vv'}$ and $E_{\bar{v}v''}$, we consider the line passing v and directing $\frac{v'-v}{|v'-v|}$, which is $v(s) = \frac{v'-v}{|v'-v|}s + v$. The solution of $v(s_*) \in E_{\bar{v}v''}$ is a solution of $0 = (v''-\bar{v})\cdot(v(s)-\bar{v}) = (v'-v)\cdot(v(s)-\bar{v}) = |v'-v|s + (v'-v)\cdot(v-\bar{v})$. Easily we have $s_* = \frac{(v'-v)\cdot(\bar{v}-v)}{|v'-v|}$. Since $v(s)$ is unit-speed line we know that $|v(s_*)-v(0)|$ is the distance between $E_{\bar{v}v''}$ and $E_{vv'}$. ☐

An important property of (2.21) is that two planes $E_{\bar{v}v''}$ and $E_{vv'}$ have the same normal directions. In order to control the second term of (2.20), we need to compare arguments of $\mathcal{B}$ and $\mathcal{B}'$ especially $v_1' \in E_{vv'}$ and $v_1'' \in E_{\bar{v}v''}$. For that purpose, we introduce the following change of variables :

Lemma 8. *For fixed v, v' and $\bar{v}$ in $\mathbb{R}^3$, we define a unit Jacobian change of variables*

$$
v_1'' \equiv v_1''(v_1'; v, v', \bar{v}) = v_1' + \frac{v'-v}{|v'-v|}\{(\bar{v}-v)\cdot\frac{v'-v}{|v'-v|}\}. \qquad (2.22)
$$

In this change of variables $v_1'' \in E_{\bar{v}v''}$ if and only if $v_1' \in E_{vv'}$.

Proof. Assume (2.21) and (2.22). Clearly $\frac{\partial v_1''(v_1')}{\partial v_1'} = I$. We can check following equality :

$$
\begin{aligned}
(v_1'' - \bar{v}) \cdot (v'' - \bar{v}) &= (v_1{}' - \bar{v} + \frac{v' - v}{|v' - v|}\{(\bar{v} - v) \cdot \frac{v' - v}{|v' - v|}\}) \cdot (v' - v) \\
&= (v_1{}' - \bar{v}) \cdot (v' - v) + |v' - v|\{(\bar{v} - v) \cdot \frac{v' - v}{|v' - v|}\} \\
&= (v_1' - v) \cdot (v' - v) + (v - \bar{v}) \cdot (v' - v) + (\bar{v} - v) \cdot (v' - v) \\
&= (v_1' - v) \cdot (v' - v).
\end{aligned}
$$

By definition, $v_1'' \in E_{\bar{v}v''}$ is equivalent to $(v_1' - v) \cdot \frac{v'-v}{|v'-v|} = 0$. Then, from the above equality, we conclude $(v_1'' - \bar{v}) \cdot \frac{v''-\bar{v}}{|v''-\bar{v}|} = 0$ which is equivalent to $v_1' \in E_{vv'}$. □

Under the first change of variables (2.21), we can rewrite the first term of (2.20) as

$$
2\int_{\mathbb{R}^3} \underbrace{\frac{1}{|v - v'|^2}\Big\{\bar{w}(v'')h(\bar{t}, \bar{x}, v'') - \bar{w}(v')h(t, x, v')\Big\}}_{(\mathcal{C})} \int_{E_{\bar{v}v''}} \underbrace{\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'')B\big(2\bar{v} - v'' - v_1'', \frac{v'' - v_1''}{|v'' - v_1''|}\big)}_{(\mathcal{D})} \, dv_1'' dv' \tag{2.23}
$$

Under the second change of variables (2.22), we can rewrite the second term of (2.20) as

$$
\begin{aligned}
&2\int_{\mathbb{R}^3} \underbrace{\bar{w}(v')h(t, x, v')\frac{1}{|v - v'|^2}}_{(\mathcal{E})} \\
&\times \int_{E_{vv'}} \underbrace{\Big\{\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'')B\big(2\bar{v} - v'' - v_1'', \frac{v'' - v_1''}{|v'' - v_1''|}\big) - \bar{w}(v_1')h(t, x, v_1')B\big(2v - v' - v_1', \frac{v' - v_1'}{|v' - v_1'|}\big)\Big\}}_{(\mathcal{F})} dv_1' dv' \qquad (2.24)
\end{aligned}
$$

We will estimate (2.23) and (2.24) separately in following two sections.

2.3.2 Estimate of (2.23)

We divide into several cases :

Case 1 : $|v| \geq N$. From Lemma 6, for $N > 0$ we can estimate

$$
\begin{aligned}
Q_+(\bar{w}h, \bar{w}h)(t, x, v)\mathbf{1}_{|v|>N} &\leq C||h||_\infty^2 \mathbf{1}_{|v|>N} \int_{\mathbf{R}^3} \bar{w}(v') \left(\frac{1}{|v - v'|^2} + \frac{1}{|v - v'|^{2-\gamma}}\right) dv' \\
&\leq C||h||_\infty^2 \left(\frac{1}{(1 + |v|)^2} + \frac{1}{(1 + |v|)^{2-\gamma}}\right) \mathbf{1}_{|v|>N} \leq \frac{C}{N}||h||_\infty^2
\end{aligned}
$$

Hence we have

$$(2.23)\mathbf{1}_{|v|\geq N} \leq \frac{C}{N}||h||_\infty^2. \tag{2.25}$$

Case 2 : $|v| \leq N$ and $|v'| \geq 2N$, or $|v| \leq N$ and $|v'| \leq \frac{1}{M}$. Also assume $0 < \delta << 1$.

$$\begin{aligned}
& 2\times \mathbf{1}_{|v|\leq N}\int_{\{|v'|\geq 2N \text{ or } |v'|\leq \frac{1}{M}\}} (\mathcal{C}) \int_{E_{\bar{v}v''}} (\mathcal{D})\, dv_1'' dv' \\
\leq\ & C\mathbf{1}_{|v|\leq N}\int_{|v'|\geq 2N} \left\{ \frac{1}{|v-v'|^2} + \frac{1}{|v-v'|^{2-\gamma}} \right\} e^{-\frac{|v'|^2}{8}} dv' e^{\frac{\delta^2}{4}} ||h||_\infty^2 \qquad (2.26) \\
& \underbrace{+C\int_{|v'|\leq \frac{1}{M}} \left\{ \frac{1}{|v'|^2} + \frac{1}{|v'|^{2-\gamma}} \right\} e^{-\frac{|v'|^2}{8}} dv'\, e^{\frac{\delta^2}{4}}}_{o(\frac{1}{M})} ||h||_\infty^2 \\
\leq\ & C\left(\frac{1}{N^2} + \frac{1}{N^{2-\gamma}} \right) ||h||_\infty^2 + o(\frac{1}{M})||h||_\infty^2 \qquad (2.27)
\end{aligned}$$

where we have used $\bar{w}(v') \leq e^{-\frac{|v'|^2}{4}}$ and $\bar{w}(v'') \leq e^{-\frac{|v'|^2}{8}} e^{\frac{\delta^2}{4}}$ and Lemma 6.

Case 3 : $|v| \leq N$ and $\frac{1}{M} \leq |v'| \leq 2N$.

$$\begin{aligned}
& 2\times \mathbf{1}_{|v|\leq N}\int_{\frac{1}{M}\leq |v'|\leq 2N} (\mathcal{C}) \int_{E_{\bar{v}v''}} (\mathcal{D})\, dv_1'' dv' \\
\leq\ & C||h||_\infty \int_{\frac{1}{M}\leq |v'|\leq 2N} \mathbf{1}_{|v|\leq N} \left(\frac{1}{|v-v'|^2} + \frac{1}{|v-v'|^{2-\gamma}} \right) \\
& \times |\bar{w}(v'')h(\bar{t},\bar{x},v'') - \bar{w}(v')h(t,x,v')|dv' \qquad (2.28)
\end{aligned}$$

Since $\left(\frac{1}{|v-v'|^2} + \frac{1}{|v-v'|^{2-\gamma}} \right)$ is integrable we can choose a smooth function $\mathbf{z}(v,v')$ with compact support such that

$$\sup_{|v|\leq N}\int_{|v'|\leq 2N} \left| \left(\frac{1}{|v-v'|^2} + \frac{1}{|v-v'|^{2-\gamma}} \right) - \mathbf{z}(v,v') \right| dv' \leq \frac{1}{N}. \tag{2.29}$$

Therefore we can bound (4.1) by two parts

$$\begin{aligned}
& C||h||_{L^\infty}^2 \int_{|v'|\leq 2N} \mathbf{1}_{|v|\leq N} \left| \left(\frac{1}{|v-v'|^2} + \frac{1}{|v-v'|^{2-\gamma}} \right) - \mathbf{z}(v,v') \right| e^{-\frac{|v'|^2}{8}} e^{\frac{\delta^2}{4}} dv' \qquad (2.30) \\
+\ & C \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{z}(v,v')| \times ||h||_{L^\infty} \\
& \times \int_{\frac{1}{M}\leq |v'|\leq 2N} \mathbf{1}_{|v|\leq N} |\bar{w}(v''(v'))h(\bar{t},\bar{x},v''(v')) - \bar{w}(v')h(t,x,v')|dv'. \qquad (2.31)
\end{aligned}$$

From (2.29), it is easy to control the first term

$$|(2.30)| \leq \frac{C}{N}||h||_\infty^2. \tag{2.32}$$

Now we are going to estimate the second term (2.31). Applying Lemma 5 to $\bar{w}(v')h(t,x,v')$, we can choose $\delta = \delta(\bar{w}h, \Omega, \varepsilon, N_*, x, \frac{1}{M}, 2N) > 0$ and an open set $U_x \subset \{\frac{1}{M} \leq |v| \leq 2N\}$ with $|U_x| < \frac{\varepsilon}{N_*}$ such that

$$|\bar{w}(v''(v'))h(\bar{t},\bar{x},v''(v')) - \bar{w}(v')h(t,x,v')| < \frac{\varepsilon}{N_*}$$

for $v' \in \{v \in \mathbb{R}^3 : \frac{1}{M} \leq |v| \leq N\} \backslash U_x$ and $|(\bar{t},\bar{x},\bar{v}) - (t,x,v)| < \delta$. Therefore we can split the second part (2.31) as integration over U_x and U_x^c and control as

$$C \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{z}(v,v')| \times ||h||_\infty^2 \times m_3(U_x)$$

$$+C||h||_\infty \int_{\{\frac{1}{M}\leq|v'|\leq 2N\}\cap U_x^c} |\bar{w}(v''(v'))h(\bar{t},\bar{x},v''(v')) - \bar{w}(v')h(t,x,v')|dv' \tag{2.33}$$

$$\leq C \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{z}(v,v')| \times ||h||_\infty^2 \frac{\varepsilon}{N_*} \ + \ C||h||_\infty N^3 \frac{\varepsilon}{N_*}. \tag{2.34}$$

In summary, combinig (2.25), (2.27), (2.32) and (2.34), we have established

$$(2.23) \leq C||h||_\infty^2 \left\{ \frac{1}{N} + o(\frac{1}{M}) + \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{z}(v,v')| \frac{\varepsilon}{N_*} \right\} + C||h||_\infty N^3 \frac{\varepsilon}{N_*}.$$

Choosing sufficiently large $N, M > 0$ and $N_* > 0$ then

$$(2.23) \leq \frac{\varepsilon}{2}. \tag{2.35}$$

2.3.3 Estimate of (2.24)

The estimate of (2.24) is much more delicate. The reason is that we cannot expect $\int_{E_{vv'}} ...dv_1'$ in (2.24) is small for all $v' \in \mathbb{R}^3$. We know that $h(t,x,v_1')$ may not be continuous on $v_1' \in \mathfrak{G}_x$. Even $\mathfrak{G}_x$ is radial symmetric and has a small measure by Lemma 3, a bad situation, the intersection of $\mathfrak{G}_x$ and $E_{vv'}$ could have large (even infinite) 2-dimensional Lebesque measure, can happen. However we can show that such bad situations only happen for very rare v''s in $\mathbb{R}^3$. Using the integration over $v' \in \mathbb{R}^3$, we are able to control (2.24) small.

Recall $(\mathcal{E})$ and $(\mathcal{F})$ in (2.24). We divide into several cases :

Case 1 : $|v| \geq N$. Follow exactly same proof of **Case 1** of the previous subsection, we conclude

$$(2.24)\mathbf{1}_{|v|\geq N} \leq \frac{C}{N}||h||_\infty^2. \tag{2.36}$$

Case 2 : $|v| \leq N$ and $|v'| \geq 2N$. We go back to original formula, the second term of (2.20), and use Lemma 6 to estimate

$$\begin{aligned}
&2\int_{|v'|\geq 2N}(\mathcal{E})\int_{E_{vv'}}(\mathcal{F})dv_1'dv'\mathbf{1}_{|v|\leq N} \\
&\leq 4||h||_\infty^2\int_{|v'|\geq 2N}\bar{w}(v')\frac{1}{|v-v'|^2}(1+|v-v'|)^\gamma dv'\mathbf{1}_{|v|\leq N} \\
&\quad \leq 4||h||_\infty^2\left(\frac{1}{N^2}+\frac{1}{N^{2-\gamma}}\right).
\end{aligned} \tag{2.37}$$

Case 3 : $|v| \leq N$, $|v'| \leq 2N$, and $|v_1'| \leq \frac{1}{N}$ or $|v_1'| \geq N$. In the case of $|v_1'| \leq \frac{1}{N}$, we have

$$\begin{aligned}
&2\times\mathbf{1}_{|v|\leq N}\int_{|v'|\geq 2N}(\mathcal{E})\int_{\{|v_1'|\leq\frac{1}{N}\}\cap E_{vv'}}(\mathcal{F})\;dv_1'dv' \\
&\leq 2||h||_\infty^2\int_{\mathbb{R}^3}\frac{\bar{w}(v')}{|v-v'|^2}dv' \\
&\times\int_{\{|v_1'|\leq\frac{1}{N}\}\cap E_{vv'}}\left\{e^{-\frac{|v_1'|^2}{8}}e^{\frac{\delta^2}{4}}(4N+\frac{1}{N}+\delta)^\gamma+e^{-\frac{|v_1'|^2}{4}}(4N+\frac{1}{N})^\gamma\right\}dv_1' \\
&\quad \leq C\frac{||h||_\infty^2}{N^{2-\gamma}}.
\end{aligned} \tag{2.38}$$

In the case of $|v_1'| \geq N$ we have

$$\begin{aligned}
&2\times\mathbf{1}_{|v|\leq N}\int_{|v'|\geq 2N}(\mathcal{E})\int_{\{|v_1'|\leq\frac{1}{N}\}\cap E_{vv'}}(\mathcal{F})\;dv_1'dv' \\
&\leq 2||h||_\infty^2\int_{\mathbb{R}^3}\frac{\bar{w}(v')}{|v-v'|^2}dv'\int_{\{|v_1'|\geq N\}\cap E_{vv'}}\left\{e^{-\frac{|v_1'|^2}{8}}e^{\frac{\delta^2}{4}}(4N+\frac{1}{N}+\delta)^\gamma+e^{-\frac{|v_1'|^2}{4}}(4N+\frac{1}{N})^\gamma\right\}dv_1' \\
&\leq C||h||_\infty^2e^{-\frac{N^2}{16}}\int_{\mathbb{R}^3}e^{-\frac{|v_1'|^2}{16}}dv'\times N^\gamma e^{-\frac{N^2}{16}}\leq C||h||_\infty^2e^{-\frac{N^2}{16}}.
\end{aligned} \tag{2.39}$$

Case 4 : $|v| \leq N, |v'| \leq 2N$, and $\frac{1}{N} \leq |v_1'| \leq N$. In order to remove the unboundedness of $\frac{1}{|v-v'|^2}$

in (2.24), we choose a positive smooth function $\mathbf{Z}(v,v')$ with compact support such that

$$\sup_{|v|\le N}\int_{|v'|\le 2N}\left|\frac{1}{|v-v'|^2}-\mathbf{Z}(v,v')\right|dv' < \frac{1}{N^{10}}. \tag{2.40}$$

Splitting $2\times \mathbf{1}_{|v|\le N}\int_{|v'|\le 2N}(\mathcal{E})\int_{\frac{1}{N}\le|v_1'|\le N}(\mathcal{F})\ dv_1'dv'$ into two parts

$$\begin{aligned}
&2\times \mathbf{1}_{|v|\le N}\int_{|v'|\le 2N}\bar{w}(v')|h(t,x,v')|\left|\frac{1}{|v-v'|^2}-\mathbf{Z}(v,v')\right|\int_{E_{vv'}\cap\{\frac{1}{N}\le|v_1'|\le N\}}(\mathcal{F})\ dv_1dv' \\
&\quad\le C||h||_\infty^2\frac{1}{N^{10}}N^{\gamma+2}, &(2.41)\\
&C\int_{|v'|\le 2N}||h||_\infty\sup_{|v|\le N,|v'|\le 2N}|\mathbf{Z}(v,v')|\int_{E_{vv'}\cap\{\frac{1}{N}\le|v_1'|\le N\}}(\mathcal{F})\ dv_1'dv', &(2.42)
\end{aligned}$$

where we used (3.253) for the first line. From now we will focus to estimate (2.42).

Case 5 : $|v|\le N, |v'|\le 2N, \frac{1}{N}\le|v_1'|\le N$ and $|2v-v'-v_1'|<\frac{1}{M}$ or $|v'-v_1'|<\frac{1}{M}$. This region included the part where the collision kernel $B(\cdot,\cdot)$ has a singular behavior.

$$\begin{aligned}
&C\int_{|v'|\le 2N}||h||_\infty\sup_{|v|\le N,|v'|\le 2N}|\mathbf{Z}(v,v')| \\
&\times\int_{E_{vv'}\cap\{\frac{1}{N}\le|v_1'|\le N\}}(\mathcal{F})\ \mathbf{1}_{\{|(2v-v')-v_1'|<\frac{1}{M}\ \text{or}\ |v'-v_1'|<\frac{1}{M}\}}(v',v_1')\ dv_1'dv' \\
\le\ & C\sup_{|v|\le N,|v'|\le 2N}|\mathbf{Z}(v,v')|\times||h||_\infty^2 &(2.43)\\
&\quad\times\int_{|v'|\le 2N}dv'e^{-\frac{|v'|^2}{4}}\int_{E_{vv'}}dv_1'\left\{\mathbf{1}_{\{|(2v-v')-v_1'|<\frac{1}{M}\}}(v_1')+\mathbf{1}_{\{|v'-v_1'|<\frac{1}{M}\}}(v_1')\right\}\times N^\gamma \\
\le\ & C\sup_{|v|\le N,|v'|\le 2N}|\mathbf{Z}(v,v')|\times||h||_\infty^2\frac{N^\gamma}{M^2}. &(2.44)
\end{aligned}$$

Case 6 : $|v|\le N, |v'|\le 2N, \frac{1}{N}\le|v_1'|\le N$ and $|2v-v'-v_1'|>\frac{1}{M}$ and $|v'-v_1'|>\frac{1}{M}$ and $0<\delta<\frac{1}{10M}$. We estimate

$$\begin{aligned}
&2\times\mathbf{1}_{|v|\le N}\int_{|v'|\le 2N}dv'\bar{w}(v')h(t,x,v')\mathbf{Z}(v,v') \\
&\times\int_{E_{vv'}\cap\{\frac{1}{N}\le|v_1'|\le N\}}\{\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'')B(2\bar{v}-v''-v_1'',\frac{v''-v_1''}{|v''-v_1''|}) \\
&-\bar{w}(v_1')h(t,x,v_1')B(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|})\}\mathbf{1}_{\{|2v-v'-v_1'|>\frac{1}{M}\}}\mathbf{1}_{\{|v'-v_1'|>\frac{1}{M}\}}dv_1' \qquad (2.45)
\end{aligned}$$

We need this step because of the singular behavior of

$$B(u_1,u_2)=|u_1|^\gamma q_0(\frac{u_1}{|u_1|}\cdot\frac{u_2}{|u_2|})=|u_1|^\gamma(q_0\circ\mathfrak{F})(u_1,u_2),$$

where $\mathfrak{F} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ with $\mathfrak{F}(u_1, u_2) = \frac{u_1}{|u_1|} \cdot \frac{u_2}{|u_2|}$. The function $\mathfrak{F}(u_1, u_2)$ is not continuous at $(u_1, u_2) = (0, 0)$ and continuous away from $(0, 0)$ i.e. the restriction of $\mathfrak{F}$ on a compact set,

$$\mathfrak{F}_{M,N} : \{\frac{1}{2M} \leq |u_1| \leq 6N\} \times \{\frac{1}{2M} \leq |u_2| \leq 4N\} \to \mathbb{R}$$

is uniformly continuous. From $|2v - v' - v_1'| > \frac{1}{M}$ and $|v - \bar{v}| < \delta < \frac{1}{10M}$ we have lower bound of

$$|2\bar{v} - v'' - v_1''| \geq \left| |2v - v' - v_1'| - |\bar{v} - v - \frac{v' - v}{|v' - v|}\{(\bar{v} - v) \cdot \frac{v' - v}{|v' - v|}\}| \right| \geq \frac{1}{2M}.$$

Similarly from $|v' - v_1'| > \frac{1}{M}$ and $|v - \bar{v}| < \delta < \frac{1}{10M}$ we have a lower bound of

$$|v'' - v_1''| \geq \left| |v' - v_1'| - |\bar{v} - v - \frac{v' - v}{|v' - v|}\{(\bar{v} - v) \cdot \frac{v' - v}{|v' - v|}\}| \right| \geq \frac{1}{2M}.$$

Therefore for any $\varepsilon > 0$, we can choose $\delta > 0$ so that

$$\begin{aligned}
& \left| B(2\bar{v} - v'' - v_1'', \frac{v'' - v_1''}{|v'' - v_1''|}) - B(2v - v' - v_1', \frac{v' - v_1'}{|v' - v_1'|}) \right| \\
= & \left| |2\bar{v} - v'' - v_1''|^\gamma (q_0 \circ \mathfrak{F})(2\bar{v} - v'' - v_1'', v'' - v_1'') - |2v - v' - v_1'|^\gamma (q_0 \circ \mathfrak{F})(2v - v' - v_1', v' - v_1') \right| < \frac{\varepsilon}{N_*},
\end{aligned} \tag{2.46}$$

for $|2v - v' - v_1'| > \frac{1}{M}$ and $|v' - v_1'| > \frac{1}{M}$ and $0 < \delta < \frac{1}{10M}$.

Now we split (2.45) by two parts

$$\begin{aligned}
& 2 \times \mathbf{1}_{|v| \leq N} \int_{|v'| \leq 2N} dv' .. \int_{E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\}} \bar{w}(v_1'') h(\bar{t}, \bar{x}, v_1'') \qquad (2.47) \\
& \left\{ B(2\bar{v} - v'' - v_1'', \frac{v'' - v_1''}{|v'' - v_1''|}) - B(2v - v' - v_1', \frac{v' - v_1'}{|v' - v_1'|}) \right\} \times \mathbf{1}_{\{|2v - v' - v_1'| > \frac{1}{M}\}} \mathbf{1}_{\{|v' - v_1'| > \frac{1}{M}\}} dv_1' \\
+ \quad & 2 \times \mathbf{1}_{|v| \leq N} \int_{|v'| \leq 2N} dv' .. \\
& \times \int_{E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\}} \left\{ \bar{w}(v_1'') h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1') h(t, x, v_1') \right\} B(2v - v' - v_1', \frac{v' - v_1'}{|v' - v_1'|}).
\end{aligned} \tag{2.48}$$

Using (2.46), the continuity of $B(\cdot, \cdot)$ away from $(0, 0)$, the first term is bounded by

$$C \sup_{v,v'} |\mathbf{Z}(v, v')| \times ||h||_\infty^2 \frac{\varepsilon}{N_*}. \tag{2.49}$$

In the remainder of this section we will focus on (2.48) :

Estimate of (2.48)

$$(2.48) \leq CN^2 ||h||_\infty \sup_{v,v'} |\mathbf{Z}(v,v')| \int_{|v'|\leq 2N} \bar{w}(v') \int_{E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} |\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'') - \bar{w}(v_1')h(t,x,v_1')| dv_1' dv', \tag{2.50}$$

where we used $\sup_{|v|\leq N, |v'|\leq 2N, |v_1'|\leq N} B(2v - v' - v_1', \frac{v'-v_1'}{|v'-v_1'|}) < \infty$. Recall our choice of v'' and v_1'' from (2.21) and (2.22) to have

$$|v_1'' - v_1'| \leq \Big| \frac{v'-v}{|v'-v|} \{(\bar{v}-v)\cdot \frac{v'-v}{|v'-v|}\} \Big| \leq |\bar{v}-v| < \delta$$

We will use the followin strategy : separate $\int_{E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} ...dv_1'$ into two parts

$$\int_{U_x\cap E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} ...dv_1' + \int_{U_x^c\cap E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} ...dv_1'.$$

The first part is the integration over U_x, a neighborhood of $\mathfrak{G}_x$ that contains possible discontinuity of h. Moreover we expect the measure of the neighborhood U_x is small so we can control the first term. For the second term, we will use the continuity of the integrand $\bar{w}h$. However if $v=0$ then $\mathfrak{G}_x$ could be a large measure set in $E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}$. For example if $\mathfrak{G}_x\cap\mathbb{S}^2 = \{u\in\mathbb{S}^2 : u_3 = 0\}$ then $\mathfrak{G}_x$ is $xy-$plane and $E_{0\mathbf{e}_3}$ is also $xy-$plane. Therefore we have to divide two cases $v\neq 0$ and $v=0$ and study separately.

Case of $\mathbf{v\neq 0}$

In the case of $v\neq 0$, assume $\varrho < |v|^2/2$ for sufficiently small $\varrho>0$. We will divide the velocity space $\mathbb{R}^3$ into

$$\mathfrak{B} = \left\{ v'\in\mathbb{R}^3 : |v| - \frac{\varrho}{|v|} \leq v'\cdot\frac{v}{|v|} \leq |v| + \frac{\varrho}{|v|} \right\} \text{ and } \mathfrak{B}^c = \left\{ v'\in\mathbb{R}^3 : \left| v'\cdot\frac{v}{|v|} - |v| \right| > \frac{\varrho}{|v|} \right\}.$$

The important property of $\mathfrak{B}$ is that if $v \in \mathfrak{B}^c$ then $E_{vv'}$ does not contain zero. We can split the integration part of (2.50) into

$$\int_{v' \in B_{2N} \cap \mathfrak{B}} \bar{w}(v') \int_{E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\}} |\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1')h(t, x, v_1')| dv_1' dv' \tag{2.51}$$

$$+ \int_{v' \in B_{2N} \setminus \mathfrak{B}} \bar{w}(v') \int_{E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\}} |\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1')h(t, x, v_1')| dv_1' dv'. \tag{2.52}$$

Notice that $\mathfrak{B} \cap B_{2N}$ has a small measure :

$$m_3(\mathfrak{B} \cap B_{2N}) \le 2\pi(2N)^2 \times 2\frac{\varrho}{|v|} \le 2\pi(2N)^2 \times 2\frac{\varrho}{\sqrt{2\varrho}} \le 2\sqrt{2}\pi(2N)^2\sqrt{\varrho}.$$

Therefore we have

$$|(2.51)| \le CN^4 ||h||_{L^\infty} \sqrt{\varrho}. \tag{2.53}$$

Now we are going to estimate (2.52). Here we use a property of $\mathfrak{B}^c$: for $v' \in \mathfrak{B}^c$ we have

$$\mathrm{dist}(0, E_{vv'}) = \left| v \cdot \frac{v' - v}{|v' - v|} \right| = \frac{|v' \cdot v - |v|^2|}{|v' - v|} > \frac{\varrho}{|v' - v|} > \frac{\varrho}{2N + |v|} \ge \frac{\varrho}{3N},$$

where we also have used $|v'| \le 2N$ and $|v| \le N$. From Lemma 5 we use U_x, an open radial symmetric subset of $\{\frac{1}{N} \le |v_1'| \le N\}$ with a small measure and $\bar{w}h$ is uniformly continuous on U_x^c, to split (2.52) into

$$\int_{v' \in B_{2N} \setminus \mathfrak{B}} \bar{w}(v') \int_{E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\} \cap U_x} |\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1')h(t, x, v_1')| dv_1' dv' \tag{2.54}$$

$$+ \int_{v' \in B_{2N} \setminus \mathfrak{B}} \bar{w}(v') \int_{E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\} \cap U_x^c} |\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1')h(t, x, v_1')| dv_1' dv'. \tag{2.55}$$

For the last line, we use Lemma 5 to know estimate $|\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1')h(t, x, v_1')| < \frac{\varepsilon}{N_*}$, for $v_1' \in E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\} \setminus U_x$ and $|v_1'' - v_1'| \le |v - \bar{v}| < \delta$. Therefore

$$|(2.55)| \le CN^2 \frac{\varepsilon}{N_*} ||h||_\infty. \tag{2.56}$$

In order to show that (2.54) is small, we introduce following projection :

Lemma 9. *Assume* $0 < \varrho < \frac{|v|^2}{2}$. *Let* $E_{vv'} = \{v_1' \in \mathbb{R}^3 : (v_1 - v) \cdot (v' - v) = 0\}$. *We define a* *projection*

$$\begin{array}{rcl} \mathbb{P} \ : \ \mathbb{S}^2 & \to & E_{vv'} \\ u \in \mathbb{S}^2 & \mapsto & \left\{ \dfrac{v \cdot (v'-v)}{u \cdot (v'-v)} \right\} u \ \in \ E_{vv'}. \end{array}$$

For $v' \in \{v' \in \mathbb{R}^3 : |v'| \leq 2N\} \backslash \mathfrak{B}$, *define the restricted projection*

$$\mathbb{P}' \equiv \mathbb{P}|_{\mathbb{P}^{-1}(E_{vv'} \cap \{1/N \leq |v'_1| \leq N\})} \ : \ \mathbb{P}^{-1}(E_{vv'} \cap \{1/N \leq |v'_1| \leq N\}) \to E_{vv'} \cap \{1/N \leq |v'_1| \leq N\}$$

Then for $v' \in B_{2N} \backslash \mathfrak{B}$ *the Jacobian of* $\mathbb{P}'$ *is bounded :*

$$Jac(\mathbb{P}') = \left| \frac{\partial \mathbb{P}'}{\partial u} \right| = \left(v \cdot \frac{v'-v}{|v'-v|} \right)^2 |\sec^2 \theta \tan \theta| \leq \frac{3N^4}{\varrho},$$

where θ *is defined by* $\cos\theta = u \cdot \frac{v'-v}{|v'-v|}$.

Proof. Without loss of generality, we may assume $\frac{v'-v}{|v'-v|} = (0,0,1)^T$. Using spherical coordinate,

$$\mathbb{P}'(u) = \frac{v \cdot (v'-v)}{u \cdot (v'-v)} u = \frac{v \cdot \frac{v'-v}{|v'-v|}}{u \cdot \frac{v'-v}{|v'-v|}} u = \frac{v \cdot \frac{v'-v}{|v'-v|}}{\cos\theta} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = v \cdot \frac{v'-v}{|v'-v|} \begin{pmatrix} \tan\theta\cos\phi \\ \tan\theta\sin\phi \\ 1 \end{pmatrix},$$

and a Jacobian matrix of $\mathbb{P}'$,

$$\frac{\partial \mathbb{P}'}{\partial(\theta,\phi)} = v \cdot \frac{v'-v}{|v'-v|} \begin{pmatrix} \sec^2\theta\cos\phi & -\tan\theta\sin\phi \\ \sec^2\theta\sin\phi & \tan\theta\cos\phi \end{pmatrix}.$$

Therefore a Jacobian of $\mathbb{P}'$ is

$$Jac(\mathbb{P}') = \left| \frac{\partial \mathbb{P}'}{\partial(\theta,\phi)} \right| = \left(v \cdot \frac{v'-v}{|v'-v|} \right)^2 \sec^2\theta |\tan\theta| \leq \mathrm{dist}(0, E_{vv'})^2 |\sec\theta|^3.$$

Notice that

$$|\sec\theta| = \frac{1}{|\cos\theta|} = \frac{1}{\left| u \cdot \frac{v'-v}{|v'-v|} \right|} = \left| \left\{ \frac{v \cdot (v'-v)}{u \cdot (v'-v)} \right\} u \right| \frac{1}{\left| v \cdot \frac{v'-v}{|v'-v|} \right|} = \frac{|\mathbb{P}'(u)|}{\mathrm{dist}(0, E_{vv'})}.$$

Because $\mathbb{P}'(u) \in \{\frac{1}{N} \leq |v_1'| \leq N\}$ and $\text{dist}(0, E_{vv'}) \geq \frac{\varrho}{3N}$ we have

$$Jac(\mathbb{P}') \leq \frac{|\mathbb{P}'(u)|^3}{|\text{dist}(0, E_{vv'})|} \leq \frac{3N^4}{\varrho}.$$

$\square$

Assume we choose $m_2(U_x \cap \mathbb{S}^2) \leq \frac{\varrho\varepsilon}{N_* N^2}$. By definition we know that $\mathbb{P}'(U_x \cap \mathbb{S}^2) = E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \cap U_x$ and the 2-dimension Lebesque measure of $E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \cap U_x$ is bounded by

$$m_2(E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \cap U_x) = m_2(\mathbb{P}'(U_x \cap \mathbb{S}^2)) \leq Jac(\mathbb{P}') \times |U_x \cap \mathbb{S}^2| \leq \frac{3N^4}{\varrho} \times \frac{\varepsilon}{N_* N^2} = \frac{3N^2}{\varrho N_*}\varepsilon.$$

Therefore we have an upper bound of (2.54) :

$$|(2.54)| \leq CN^2 \varepsilon ||h||_\infty, \tag{2.57}$$

where $C = \int_{\mathbb{R}^3} \bar{w}(v')dv'$. In case of $v \neq 0$, from (2.53), (2.56) and (2.57), we have

$$(2.48) \leq CN^2 ||h||_\infty \sup_{v,v'} |\mathbf{Z}(v,v')| \times (2.50) \leq CN^4 ||h||_\infty^2 \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{Z}(v,v')| \{N^2 \sqrt{\varrho} + (1 + \frac{3N^2}{\varrho})\frac{\varepsilon}{N_*}\}. \tag{2.58}$$

Case of $\mathbf{v = 0}$

In this case, we do not have a upper bound of the Jacobian of $\mathbb{P}'$. Instead we will use the structure of $\mathfrak{G}_x$ of Lemma 4 crucially. In case of $v = 0$, we split (2.50)

$$\int_{|v'|\leq 2N} \bar{w}(v') \int_{E_{0v'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\}} \underbrace{|\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1') - \bar{w}(v_1')h(t, x, v_1')|}_{\clubsuit} dv_1' dv'$$

$$= \int_{|v'|\leq 2N} \bar{w}(v') \underbrace{\int_{E_{0v'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\}} |\clubsuit| \times \mathbf{1}_{U_x}(v_1') dv_1'}_{\spadesuit} dv' \tag{2.59}$$

$$+ \int_{|v'|\leq 2N} \bar{w}(v') \int_{E_{0v'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\}} |\clubsuit| \times \mathbf{1}_{E_{0v'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \setminus U_x}(v_1') dv_1' dv' \tag{2.60}$$

For v', we use a spherical polar coordinates (r', θ', ϕ') so that

$$v' = (r' \sin\theta' \cos\phi', r' \sin\theta' \sin\phi', r' \cos\theta'). \tag{2.61}$$

By definition, $E_{0v'}$ is a plane containing the origin and normal to v'. We know that $E_{0v'}$ is generated by two unit vectors

$$E_{0v'} = \left\langle \begin{pmatrix} \cos\theta' \cos\phi' \\ \cos\theta' \sin\phi' \\ -\sin\theta' \end{pmatrix}, \begin{pmatrix} -\sin\phi' \\ \cos\phi' \\ 0 \end{pmatrix} \right\rangle.$$

We will use a polar coordinate (r_1', θ_1') for $v_1' \in E_{0v'}$, i.e.

$$v_1' = \begin{pmatrix} (v_1')_1 \\ (v_1')_2 \\ (v_1')_3 \end{pmatrix} (r_1', \theta_1'; \theta', \phi') \equiv r_1' \begin{pmatrix} \cos\theta' \cos\phi' & -\sin\phi' & \sin\theta' \cos\phi' \\ \cos\theta' \sin\phi' & \cos\phi' & \sin\theta' \sin\phi' \\ -\sin\theta' & 0 & \cos\theta' \end{pmatrix} \begin{pmatrix} \cos\theta_1' \\ \sin\theta_1' \\ 0 \end{pmatrix}. \tag{2.62}$$

Direct computation gives $\det\left(\frac{\partial(v_1')}{\partial(r_1', \theta_1', \theta')}\right) =$

$$(r_1')^2 \cos\theta_1' \det \begin{pmatrix} \cos\theta' \cos\phi' \cos\theta_1' - \sin\phi' \sin\theta\theta_1' & -\cos\theta' \cos\phi' \sin\theta_1' - \sin\phi' \cos\theta_1' & \sin\theta' \cos\phi' \\ \cos\theta' \sin\phi' \cos\theta_1' + \cos\phi' \sin\theta_1' & -\cos\theta' \sin\phi' \sin\theta_1' + \cos\phi' \cos\theta_1' & \sin\theta' \sin\phi' \\ \sin\theta' \cos\theta_1' & \sin\theta' \sin\theta_1' & \cos\theta' \end{pmatrix}$$
$$= (r_1')^2 \cos\theta_1'.$$

Therefore we have following identity

$$\int_{\mathbb{R}^3} \cdots dv_1' = \int_0^\infty \int_0^{2\pi} \int_0^\pi \cdots (r_1')^2 \cos\theta_1' d\theta' d\theta_1' dr_1'. \tag{2.63}$$

Recall the standard 3 dimensional polar coordinates and 2 dimensional polar coordinates :

$$\int_{|v'| \le 2N} \cdots dv' = \int_0^{2N} \int_0^{2\pi} \int_0^\pi \cdots (r')^2 \sin\theta' d\theta' d\phi' dr',$$
$$\int_{E_{0v'} \cap \{\frac{1}{N} \le |v_1'| \le N\}} \cdots dv_1' = \int_{\frac{1}{N}}^N \int_0^{2\pi} \cdots r_1' d\theta_1' dr_1',$$

and use above identities to control (2.59) by

$$\int_0^{2N} dr'(r')^2\bar{w}(r')\int_0^{2\pi} d\phi' \underbrace{\int_0^{\pi} d\theta'\sin\theta' \int_{\frac{1}{N}}^{N} dr_1' r_1' e^{-\frac{(r_1')^2}{8}} \int_0^{2\pi} d\theta_1' \mathbf{1}_{U_x}(v_1'(r_1',\theta_1';\theta',\phi'))}_{\bigotimes} ||h||_\infty. \tag{2.64}$$

We focus on the inner integration $\bigotimes$ and divide it into

$$\int_0^{\pi} d\theta'\sin\theta' \int_{\frac{1}{N}}^{N} dr_1' r_1' e^{-\frac{(r_1')^2}{8}} \int_0^{2\pi} d\theta_1' \mathbf{1}_{U_x}(v_1')\mathbf{1}_{\theta_1'\in(\frac{\pi}{2}-\varrho,\frac{\pi}{2}+\rho)\cup(\frac{3\pi}{2}-\varrho,\frac{3\pi}{2}+\varrho)} \tag{2.65}$$

$$+\int_0^{\pi} d\theta'\sin\theta' \int_{\frac{1}{N}}^{N} dr_1' r_1' e^{-\frac{(r_1')^2}{8}} \int_0^{2\pi} d\theta_1' \mathbf{1}_{U_x}(v_1')\mathbf{1}_{\theta_1'\in[0,\frac{\pi}{2}-\varrho]\cup[\frac{\pi}{2}+\varrho,\frac{3\pi}{2}-\varrho]\cup[\frac{3\pi}{2}+\varrho,2\pi]}. \tag{2.66}$$

Easily (2.65) $\leq 2\varrho(e^{-\frac{1}{8N^2}} - e^{-\frac{N^2}{8}}) \leq 4\varrho$. For (2.66), we use $1 \leq \frac{\cos\theta_1'}{\varrho}$ and $1 \leq Nr_1'$ on $\theta_1' \in [0,\frac{\pi}{2}-\varrho]\cup[\frac{\pi}{2}+\varrho,\frac{3\pi}{2}-\varrho]\cup[\frac{3\pi}{2}+\varrho,2\pi]$ and $r_1' \in [\frac{1}{N}, N]$ to have

$$(2.66) \leq \varrho^{-1}N\int_0^{\pi} d\theta' \int_{\frac{1}{N}}^{N} dr_1'(r_1')^2 \int_0^{2\pi} d\theta_1' \cos\theta_1' \mathbf{1}_{U_x}(v_1'(r_1',\theta_1';\theta',\phi')) = \varrho^{-1}N \times m_3(U_x \cap \{\frac{1}{N} \leq |v_1'| \leq N\}),$$

where we used (2.63). To sum we have

$$(2.59) \leq (3.218) \leq C||h||_\infty \left\{4\varrho + \varrho^{-1}N \times \frac{\varepsilon}{N_*}\right\}. \tag{2.67}$$

On the other hand for (2.60) we can use Lemma 5 to have

$$(2.60) \leq C\frac{\varepsilon}{N_*}. \tag{2.68}$$

From (2.67) and (2.68) we have

$$\begin{aligned}(2.48) &\leq CN^2||h||_\infty \sup_{v,v'}|\mathbf{Z}(v,v')| \times (2.50) = CN^2||h||_\infty \sup_{v,v'}|\mathbf{Z}(v,v')| \times \{(2.59)+(2.60)\} \\ &\leq CN^2||h||_\infty \sup_{v,v'}|\mathbf{Z}(v,v')| \left\{\frac{\varepsilon}{N_*} + ||h||_\infty\left\{4\varrho + \varrho^{-1}N \times \frac{\varepsilon}{N_*}\right\}\right\}. \end{aligned} \tag{2.69}$$

To summarize, from (2.36), (2.37), (2.38), (2.39), (2.41), (2.44), (2.49), (2.58) and (2.69), we have established

$$
\begin{aligned}
(2.24) \leq C||h||_{\infty}^2\{\frac{1}{N} + e^{-\frac{N^2}{16}}\} + C||h||_{\infty}^2 \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{Z}(v,v')| \frac{N^{\gamma}}{M^2} \qquad (2.70) \\
+C||h||_{\infty}^2 \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{Z}(v,v')|(N^6\sqrt{\varrho} + 4N^2\varrho) \\
+\frac{\varepsilon}{N_*} C||h||_{\infty} \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{Z}(v,v')| \left\{ N^2 + ||h||_{\infty}\Big(1 + \frac{3N^6}{\varrho} + N^4 + N^3\varrho\Big)\right\} \quad (2.71)
\end{aligned}
$$

We choose $N, M, N_* > 0$ sufficiently large and $\varrho > 0$ sufficiently small we can control $(2.24) < \frac{\varepsilon}{2}$. Combining with the result of previous subsection (2.35), we conclude (2.19) and and prove Theorem 8.

2.3.4 Continuity of Collision Operators Kf and $\Gamma(f, f)$

The following is a direct consequence of Theorem 8.

Corollary 9. *Assume $f(t, x, v)$ is continuous on $[0, T] \times (\bar{\Omega} \times \mathbb{R}^3)\backslash\mathfrak{G}$ and*

$$
w(v)f(t,x,v) = (1+\rho^2|v|^2)^{\beta} f(t,x,v) \in L^{\infty}([0,T]\times(\bar{\Omega}\times\mathbb{R}^3)).
$$

Then $Kf(t, x, v)$ and $\Gamma_+(f, f)(t, x, v)$ are continuous in $[0, T] \times \Omega \times \mathbb{R}^3$ and

$$
\sup_{[0,T]\times\bar{\Omega}\times\mathbb{R}^3} |\nu^{-1}(v)w(v)K(f)| < \infty, \qquad \sup_{[0,T]\times\bar{\Omega}\times\mathbb{R}^3} |\nu^{-1}(v)w(v)\Gamma_+(f,f)| < \infty.
$$

Proof. The above boundedness is a direct consequence of (3.174) and (2.16). Thanks to Theorem 8, we already established the continuity of Γ_+. Therefore we only need to show the continuity of

$$
\frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \mu) = e^{-\frac{|v|^2}{4}} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega) f(t,x,u) e^{-\frac{|u|^2}{4}} d\omega du.
$$

Choose $(\bar{t},\bar{x},\bar{v}) \sim (t,x,v)$ so that $|(\bar{t},\bar{x},\bar{v})-(t,x,v)| < \delta$. We will estimate

$$\begin{aligned}
& \frac{1}{\sqrt{\mu}}Q_-(\sqrt{\mu}f,\mu)(\bar{t},\bar{x},\bar{v}) - \frac{1}{\sqrt{\mu}}Q_-(\sqrt{\mu}f,\mu)(t,x,v) \\
= & \frac{1}{\sqrt{\mu}}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} e^{-\frac{|u|^2}{4}}\{B(v-u,\omega)f(t,x,v) - \underline{B(\bar{v}-u,\omega)f(\bar{t},\bar{x},u)}\}d\omega du \\
= & \frac{1}{\sqrt{\mu}}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)e^{-\frac{|u|^2}{4}}f(t,x,v)d\omega du \qquad (2.72) \\
& -\frac{1}{\sqrt{\mu}}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u',\omega)e^{-\frac{|u'-(v-\bar{v})|^2}{4}}f(\bar{t},\bar{x},u'-(v-\bar{v}))d\omega du' \\
\leq & \frac{1}{\sqrt{\mu}}\int_{u\in\mathbb{R}^3}\int_{\mathbb{S}^2} |B(v-u,\omega)||e^{-\frac{|u|^2}{4}} - e^{-\frac{|u-(v-\bar{v})|^2}{4}}|w^{-1}(u-(v-\bar{v}))||wf||_\infty d\omega du \qquad (2.73) \\
& +\frac{1}{\sqrt{\mu}}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |B(v-u,\omega)|e^{-\frac{|u|^2}{4}}|f(t,x,u)-f(\bar{t},\bar{x},u-(v-\bar{v}))|d\omega du, \qquad (2.74)
\end{aligned}$$

where we used a change of variables $u' = u + (v-\bar{v})$ for the underlined term. Using the Taylor's expansion we control

$$e^{-\frac{|u-(v-\bar{v})|^2}{4}} = e^{-\frac{|u|^2}{4}} + \frac{1}{2}|u_*|e^{-\frac{|u_*|^2}{4}}|v-\bar{v}| \leq \frac{1}{2}(|u|+\delta)e^{\frac{\delta^2}{4}}e^{-\frac{|u|^2}{4}} \times |v-\bar{v}|,$$

where $u_* = s_*\{u-(v-\bar{v})\} + (1-s_*)u$ for some $s_* \in (0,1)$ and $|v-\bar{v}| < \delta$. Therefore we control

$$\begin{aligned}
|(2.73)| \leq & e^{\frac{|v|^2}{4}}\int_{\mathbb{R}^3}|v-u|^\gamma\frac{1}{2}(|u|+\delta)e^{\frac{\delta^2}{4}}e^{-\frac{|u|^2}{4}}du \times \sup_{v,u}\int_{\mathbb{S}^2} q_0\Big(\frac{v-u}{|v-u|}\cdot\omega\Big)d\omega|v-\bar{v}| \times ||wf||_\infty \\
\leq & C(1+|v|)^\gamma e^{\frac{|v|^2}{4}}||wf||_\infty, \qquad (2.75)
\end{aligned}$$

where we have used the the angular cutoff assumption (3.3). Now we estimate (2.74) with following steps :

Case 1 : $|u| \geq N$. Since $e^{-\frac{|u|^2}{4}} \leq e^{-\frac{N^2}{8}}e^{-\frac{|u|^2}{8}}$, we estimate

$$\int_{|u|\geq N}\int_{\mathbb{S}^2}(2.74)\ d\omega du \ \leq Ce^{-\frac{N^2}{8}}\int_{\mathbb{R}^3}e^{-\frac{|u|^2}{8}}|u-v|^\gamma du \times ||wf||_\infty \leq Ce^{-\frac{N^2}{8}}\nu(v)||wf||_\infty. \qquad (2.76)$$

Case 2 : $|u| \leq N$. A function f is continuous on $[0,T]\times(\bar{\Omega}\times B(0;N))\backslash\mathfrak{G}$. By the Lemma 5, we can choose $U_x \subset B(0;N)$ with $|U_x| < \frac{\varepsilon}{N}$ with $|f(t,x,u) - f(\bar{t},\bar{x},u-(v-\bar{v}))| < \frac{\varepsilon}{N}$ for $|(t,x,u)-(\bar{t},\bar{x},u-(v-\bar{v}))| \leq \delta$ with $u \in B(0;N)\backslash U_x$. Therefore $\int_{|u|\leq N}\int_{\mathbb{S}^2}(2.74)\ d\omega du \ \leq$

$$\int_{u\in B(0;N)\cap U_x}\int_{\mathbb{S}^2}(2.74)\ d\omega du + \int_{u\in B(0;N)\backslash U_x}\int_{\mathbb{S}^2}(2.74)\ d\omega du \leq C\frac{\varepsilon}{N}\nu(v)||wf||_\infty. \qquad (2.77)$$

From (4.61), (4.62) and (4.63), we summarize

$$\frac{1}{\sqrt{\mu}}|Q_-(\sqrt{\mu}f,\mu)(\bar{t},\bar{x},\bar{v}) - Q_-(\sqrt{\mu}f,\mu)(t,x,v)| \leq (o(\delta) + e^{-\frac{N^2}{8}} + \frac{\varepsilon}{N})\frac{\nu(v)}{\sqrt{\mu}}||wf||_\infty,$$

which is less than ε for sufficiently large N and sufficiently small δ. □

2.4 In-Flow Boundary Condition

In this section we consider the Boltzmann solution with the in-flow boundary condition. First we will show the formation of discontinuity using L^∞ bound for the Boltzmann solution. Then we use the continuity of collision operators, Theorem 8, to show a continuity of solution on the continuity set $\mathfrak{C}$ and the propagation of discontinuity on the discontinuity set $\mathfrak{D}$.

2.4.1 Formation of Discontinuity

We will prove 1 of Theorem 1. Without loss of generality we may assume $x = (0,0,0)$ and $v = (1,0,0)$ and $(x,v) \in \gamma_0^{\mathbf{S}}$. Locally the boundary is a graph i.e. $\Omega \cap B(\mathbf{0};\delta) = \{(x_1,x_2,x_3) \in B(\mathbf{0};\delta) : x_3 > \Phi(x_1,x_2)\}$. The condition $(x,v) \in \gamma_0^{\mathbf{S}}$ implies $t_{\mathbf{b}}(x,v) \neq 0$ and $t_{\mathbf{b}}(x,-v) \neq 0$ which means $\Phi(\xi,0) < 0$ for $\xi \in (-\delta,\delta)$.

For simplicity we put the boundary data $g \equiv 0$. From Theorem 1 of [27] we have global existence of (2.9) with zero in-flow boundary condition : For sufficiently small $||h_0||_\infty < \delta$ we have a solution of (2.9) with zero in-flow boundary condition on $[0,T] \times \bar{\Omega} \times \mathbb{R}^3$ with

$$\sup_{t\in[0,\infty)} ||h(t)||_\infty \leq C' e^{-\lambda t}||h_0||_\infty$$

for some $\lambda > 0$. In the proof we do not use decay estimate but just boundedness

$$\sup_{t\in[0,\infty)} ||h(t)||_\infty \leq C'||h_0||_\infty \tag{2.78}$$

Recall the constants $C_{\mathbf{k}}$ and C_Γ from (3.174) and (2.16). Choose $t_* \in (0, \frac{\delta}{2})$ sufficiently small so that

$$\frac{1}{2} \leq \left(e^{-\nu(1)t_*} - t_* C_{\mathbf{k}} C' - (1 - e^{-\nu(1)t_*}) C_\Gamma (C')^2\right). \tag{2.79}$$

where $\nu(1) = \nu(v)$ for any $v \in \mathbb{R}^3$ with $|v| = 1$. This choice is possible because the right hand side of (2.79) is a continuous function of t_* and it has a value 1 when $t_* = 0$. Assume a condition for our initial data h_0 : there is sufficiently small $\delta' = \delta'(\Omega, t_*) > 0$ such that $B((-t_*, 0, 0); \delta') \subset \Omega$ and

$$h_0(x, v) \equiv ||h_0||_\infty > 0 \quad \text{for} \quad (x, v) \in B((-t_*, 0, 0); \delta') \times B((1, 0, 0); \delta'). \tag{2.80}$$

We claim the Boltzmann solution h with such an initial datum h_0 and zero in-flow boundary condition is not continuous at $(t_*, (0, 0, 0), (1, 0, 0))$. We will use a contradiction argument : suppose

$$[h(t_*)]_{x,v} = 0 \tag{2.81}$$

Choose sequences of points $(x'_n, v'_n) = ((0, 0, \frac{1}{n}), (1, 0, 0))$ and $(x_n, v_n) = ((\frac{1}{n}, 0, \Phi(\frac{1}{n}, 0)), (1, 0, \frac{1}{n}))$. Because of our choice, for sufficiently large $n \in \mathbb{N}$ we have

$$(x'_n - t_* v'_n, v'_n) = ((-t_*, 0, \frac{1}{n}), (1, 0, 0)) \in B((-t_*, 0, 0); \delta') \times B((1, 0, 0); \delta')$$

we have

$$\begin{aligned}
h(t_*, x'_n, v'_n) &= h_0(x'_n - t_* v'_n, v'_n) e^{-\nu(v'_n)t_*} + \int_0^{t_*} e^{-\nu(v'_n)(t_* - \tau)} \left\{ K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w}) \right\} (\tau, x'_n - v(t_* - \tau), v'_n) d\tau \\
&= ||h_0||_\infty e^{-\nu(v'_n)t_*} + \int_0^{t_*} e^{-\nu(v'_n)(t_* - \tau)} \left\{ K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w}) \right\} (\tau, x - v'_n(t_* - \tau), v'_n) d\tau, \\
h(t_*, x_n, v_n) &= w(v_n) g(t_*, x_n, v_n) = 0.
\end{aligned}$$

From (2.81) we know that $h(t_*, x_n, v_n) \to 0$ as $n \to 0$.
Using (2.78) we can estimate $\liminf_{n\to\infty} |h(t_*, x'_n, v'_n) - h(t_*, x_n, v_n)|$

$$\begin{aligned}
&\geq \liminf_{n\to\infty} \left| ||h_0||_\infty e^{-\nu(v'_n)t_*} - \int_0^{t_*} C_{\mathbf{k}} C' ||f_0||_\infty d\tau + \int_0^{t_*} \nu(v'_n) e^{-\nu(v'_n)(t-\tau)} C_\Gamma (C')^2 ||f_0||_\infty^2 d\tau \right| \\
&\geq ||h_0||_\infty e^{-\nu(1)t_*} - t_* C_{\mathbf{k}} C' ||f_0||_\infty - (1 - e^{-\nu(1)t_*}) C_\Gamma (C')^2 ||h_0||_\infty^2 \\
&= ||h_0||_\infty \left(e^{-\nu(1)t_*} - t_* C_{\mathbf{k}} C' - (1 - e^{-\nu(1)t_*}) C_\Gamma (C')^2\right) \geq \frac{||h_0||_\infty}{2} \neq 0
\end{aligned}$$

which is contradiction to (2.81).

2.4.2 Continuity away from $\mathfrak{D}$

We aim to prove Part 1 of Theorem 3 in this section. First we recall Lemma 19 of [27] the representation for solution operator $G(t,0)$ for the homogeneous transport equation with in-flow boundary condition.

Lemma 10. *[27] Let $h_0(x,v) \in L^\infty$ and $wg \in L^\infty$. Let $\{G(t,0)h_0\}$ be the solution to the transport equation*

$$\{\partial_t + v \cdot \nabla_x\}G(t,0)h_0 = 0, \qquad G(0,0)h_0 = h_0, \quad \{G(t,0)h_0\}_{\gamma_-} = wg.$$

For $(x,v) \notin \gamma_0 \cap \gamma_-$,

$$\{G(t,0)h_0\}(t,x,v) \quad = \quad \mathbf{1}_{t-t_{\mathbf{b}}\leq 0}h_0(x-tv,v) + \mathbf{1}_{t-t_{\mathbf{b}}>0}\{wg\}(t-t_{\mathbf{b}}, x-t_{\mathbf{b}}v, v).$$

Next we prove a generalized version of Lemma 20 in [27].

Lemma 11 (Continuity away from $\mathfrak{D}$: Transport Equation)**.** *Let Ω be an open subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$ and an initial data $h_0(x,v)$ be continuous in $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_+ \cup \gamma_0^I\}$ and a boundary data g be continuous in $[0,T]\times\{\gamma_-\cup\gamma_0^I\}$. Also assume $q(t,x,v)$ and $\phi(t,x,v)$ be continuous in the interior of $[0,T] \times \Omega \times \mathbb{R}^3$ and $\sup_{[0,T]\times\Omega\times\mathbb{R}^3} |q(t,x,v)| < \infty$ and $\sup_{[0,T]\times\Omega} |\phi(\cdot,\cdot,v)| < \infty$ for all $v \in \mathbb{R}^3$. Let $h(t,x,v)$ be the solution of*

$$\{\partial_t + v \cdot \nabla_x + \phi\}h = q \ , \quad h(0,x,v) = h_0 \ , \quad h|_{\gamma_-} = wg.$$

Assume the compatibility condition on $\gamma_- \cup \gamma_0^{I-}$

$$h_0(x,v) = w(v)g(0,x,v). \tag{2.82}$$

Then the Boltzmann solution $h(t,x,v)$ is continuous on $\mathfrak{C}$. Further, if the boundary $\partial\Omega$ does not include a line segment (6) then $h(t,x,v)$ is continuous on a complementary set of the discontinuity set i.e. $[0,T] \times \{\bar{\Omega} \times \mathbb{R}^3\}\backslash\mathfrak{D}$

Proof. Fix $(t,x,v)\in\mathfrak{C}$. Notice that

$$\left\{\frac{d}{ds}\{h(s,X(s),V(s))e^{-\int_s^t\phi(\tau,X(\tau),V(\tau))d\tau}\}-q(s,X(s),V(s))e^{-\int_s^t\phi(\tau,X(\tau),V(\tau))d\tau}\right\}\mathbf{1}_{[\max\{0,t-t_{\mathbf{b}}(x,v)\},t]}(s)=0 \tag{2.83}$$

along the characteristics $X(s;t,x,v)=x-v(t-s), V(s;t,x,v)=v$ until the characteristics hits the boundary. Choose $(\bar t,\bar x,\bar v)\sim(t,x,v)$ and use a change of variables $\bar s=s-(\bar t-t)$ with $\bar s\in[t-\bar t,t]$ to have

$$\begin{aligned}&\Big\{\frac{d}{d\bar s}\{h(\bar s+(\bar t-t),\bar X(\bar s),\bar V(\bar s))e^{-\int_{\bar s}^t\phi(\tau+(\bar t-t),\bar X(\tau),\bar V(\tau))d\tau}\}\\&\qquad-q(\bar s+(\bar t-t),\bar X(\bar s),\bar V(\bar s))e^{-\int_{\bar s}^t\phi(\tau+(\bar t-t),\bar X(\tau),\bar V(\tau))d\tau}\Big\}\mathbf{1}_{[-(\bar t-t)+\max\{0,\bar t-t_{\mathbf{b}}(\bar x,\bar v)\},t]}(s)=0\end{aligned} \tag{2.84}$$

where $\bar X(\bar s)=X(\bar s+(\bar t-t);\bar t,\bar x,\bar v)$ and $\bar V(\bar s)=V(\bar s+(\bar t-t);\bar t,\bar x,\bar v)$.

By the definition $\mathfrak{C}$ we can separate two cases : $t<t_{\mathbf{b}}(x,v)$, $(x_{\mathbf{b}}(x,v),v)\in\gamma_-\cup\gamma_0^I$.

<u>Case of $t-t_{\mathbf{b}}(x,v)<0$</u> From the assumption $t-t_{\mathbf{b}}(x,v)<0$ we know that (2.83) holds for $0\le s\le t$. Now we choose $(\bar t,\bar x,\bar v)$ near (t,x,v) so that $\bar t-t_{\mathbf{b}}(\bar x,\bar v)<0$ and $\bar X(\bar s)=X(\bar s+(\bar t-t);\bar t,\bar x,\bar v)$ is in the interior of Ω for all $\bar s\in[t-\bar t,t]$. Taking the integration over $[\min\{0,t-\bar t\},t]$ of (2.83) − (2.84) to have

$$\begin{aligned}&h(t,x,v)-h(\bar t,\bar x,\bar v)\\&=h_0(X(0),V(0))e^{-\int_0^t\phi(\tau,X(\tau),V(\tau))d\tau}-h_0(\bar X(t-\bar t),\bar V(t-\bar t))e^{-\int_{t-\bar t}^t\phi(\tau+(\bar t-t),\bar X(\tau),\bar V(\tau))d\tau}\\&\qquad+\int_{\min\{0,t-\bar t\}}^t\Big\{\mathbf{1}_{[\max\{0,t-t_{\mathbf{b}}(x,v)\},t]}(s)q(s,X(s),V(s))e^{-\int_s^t\phi(\tau,X(\tau),V(\tau))d\tau}\\&\qquad-\mathbf{1}_{[t-\bar t+\max\{0,\bar t-t_{\mathbf{b}}(\bar x,\bar v)\},t]}(s)q(s+(\bar t-t),\bar X(s),\bar V(s))e^{-\int_s^t\phi(\tau+(\bar t-t),\bar X(\tau),\bar V(\tau))d\tau}\Big\}ds\end{aligned}$$

Since h_0 and ϕ is continuous it is easy to see that the first line goes to zero when $(\bar t,\bar x,\bar v)\to(t,x,v)$. For the remainder we separate cases : $t-\bar t>0$ and $t-\bar t\le 0$. If $t-\bar t>0$ the remainder is bounded by

$$\int_{t-\bar t}^t|q(s)e^{\int_s^t\phi(\tau)\tau}-q(s+(\tau t-t))e^{-\int_s^t\phi(\tau+(\bar t-t)}|+|t-\bar t|\sup_{0\le s\le t}||q(s)||_\infty e^{t\sup_{0\le s\le t}||\phi(s)||_\infty} \tag{2.85}$$

where the first term is small using continuity of q and ϕ and the second term is small as $(\bar t,\bar x,\bar v)\to$

(t, x, v). The case $t - \bar{t} \leq 0$ is similar.

Case of $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$ We only have to consider cases of $t > t_{\mathbf{b}}(x, v)$ and $t = t_{\mathbf{b}}(x, v)$. By definition $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^I$. From Lemma 2 we know that $t_{\mathbf{b}}(x, v)$ is a continuous function on $(x, v) \notin \mathfrak{S}$. In the case of $t > t_{\mathbf{b}}(x, v)$ for $(\bar{t}, \bar{x}, \bar{v}) \sim (t, x, v)$ we have $\bar{t} > t_{\mathbf{b}}(\bar{x}, \bar{v})$. Taking the integration over $[\min\{0, t - \bar{t}\}, t]$ of (2.83) – (2.84) to have

$$\begin{aligned} h(t,x,v) - h(\bar{t},\bar{x},\bar{v}) \quad = \quad & wg(t - t_{\mathbf{b}}(x,v), X(t_{\mathbf{b}}(x,v)), V(t_{\mathbf{b}}(x,v)))e^{-\int_{t-t_{\mathbf{b}}(x,v)}^t \phi(\tau,X(\tau),V(\tau))d\tau} \\ & -wg(\bar{t} - t_{\mathbf{b}}(\bar{x},\bar{v}), X(t_{\mathbf{b}}(\bar{x},\bar{v}), V(t_{\mathbf{b}}(\bar{x},\bar{v}))))e^{-\int_{\bar{t}-t_{\mathbf{b}}(\bar{x},\bar{v})}^t \phi(\tau+(\bar{t}-t),\bar{X}(\tau),\bar{V}(\tau))d\tau} \\ & + \int_{t-t_{\mathbf{b}}(x,v)}^t q(s, X(s), V(s))e^{-\int_s^t \phi(\tau,X(\tau),V(\tau))d\tau} ds \\ & - \int_{t-t_{\mathbf{b}}(\bar{x},\bar{v})}^t q(s + (\bar{t} - t), \bar{X}(s), \bar{V}(s))e^{-\int_s^t \phi(\tau+(\bar{t}-t),\bar{X}(\tau),\bar{V}(\tau))d\tau} ds \end{aligned}$$

where using the continuity of $t_{\mathbf{b}}$ and q and ϕ it is easy to show that $|h(t, x, v) - h(\bar{t}, \bar{x}, \bar{v})| \to 0$ as $(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)$. In the case of $t = t_{\mathbf{b}}(x, v)$ we can choose $(\bar{t}, \bar{x}, \bar{v}) \sim (t, x, v)$ so that $t_{\mathbf{b}}(\bar{x}, \bar{v}) \in (t - \epsilon, t + \epsilon)$. Taking the integration over $[\min\{0, t - \bar{t}, t\}]$ of (2.83) – (2.84) to have

$$\begin{aligned} |h(t,x,v) - h(\bar{t},\bar{x},\bar{v})| \leq \; & wg(t - t_{\mathbf{b}}(x,v), X(t_{\mathbf{b}}(x,v)), V(t_{\mathbf{b}}(x,v)))e^{-\int_{t-t_{\mathbf{b}}(x,v)}^t \phi(\tau,X(\tau),V(\tau))d\tau} \\ & -\mathbf{1}_{\bar{t}>t_{\mathbf{b}}(\bar{x},\bar{v})} wg(\bar{t} - t_{\mathbf{b}}(\bar{x},\bar{v}), X(t_{\mathbf{b}}(\bar{x},\bar{v}), V(t_{\mathbf{b}}(\bar{x},\bar{v}))))e^{-\int_{\bar{t}-t_{\mathbf{b}}(\bar{x},\bar{v})}^t \phi(\tau+(\bar{t}-t),\bar{X}(\tau),\bar{V}(\tau))d\tau} \\ & -\mathbf{1}_{\bar{t}\leq t_{\mathbf{b}}(\bar{x},\bar{v})} h_0(\bar{X}(t - \bar{t}), \bar{V}(t - \bar{t}))e^{-\int_{t-\bar{t}}^t \phi(\tau+(\bar{t}-t),\bar{X}(\tau),\bar{V}(\tau))d\tau} \\ + \int_{t-t_{\mathbf{b}}(x,v)+\varepsilon}^t & \Big| q(s, X(s), V(s))e^{-\int_s^t \phi(\tau,X(\tau),V(\tau))d\tau} \\ -q(s + (\bar{t} - t), & \bar{X}(s), \bar{V}(s))e^{-\int_s^t \phi(\tau+(\bar{t}-t),\bar{X}(\tau),\bar{V}(\tau))d\tau} \Big| ds \\ & +2\varepsilon \sup_{0\leq s\leq t} ||q(s)||_\infty e^{t \sup_{0\leq s\leq t} ||\phi(s)||_\infty} \end{aligned}$$

where the first three terms can be small using the compatibility condition and continuity of h_0 in $\{\bar{\Omega} \times \mathbb{R}^3\} \backslash \{\gamma_- \cup \gamma_+ \cup \gamma_0^I\}$ and g on $\{[0, T] \times \{\partial\Omega \times \mathbb{R}^3\} \backslash \{\gamma_- \cup \gamma_0^I\}\}$ and continuity of ϕ. For the fourth term we will use the continuity of q and ϕ.

If the boundary $\partial\Omega$ does not include a line segment (6) we have $\mathfrak{C} = [0, T] \times \{\bar{\Omega} \times \mathbb{R}^3\} \backslash \{\mathfrak{D} \cup \gamma_0^V\}$. $\square$

Proof of Part 1 of Theorem 3

We will use the following iteration scheme

$$\{\partial_t + v\cdot\nabla_x + \nu\}h^{m+1} = K_w h^m + w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right) - w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right) \tag{2.86}$$

with $h^{m+1}|_{t=0} = h_0$ and $h^{m+1}(t,x,v) = wg(t,x,v)$ with $(t,x,v)\in\gamma_-\cup\gamma_0^{I-}$.

Step 1 : We claim

$$h^i \text{ is a continuous function in } \mathfrak{C}_T \tag{2.87}$$

for all $i\in\mathbb{N}$ and for any $T>0$ where

$$\mathfrak{C}_T \equiv \mathfrak{C}\cap\{[0,T]\times\bar{\Omega}\times\mathbb{R}^3\} = \{(t,x,v)\in[0,T]\times\bar{\Omega}\times\mathbb{R}^3 : t<t_{\mathbf{b}}(x,v) \text{ or } (x_{\mathbf{b}}(x,v),v)\in\gamma_-\cup\gamma_0^{I-}\}. \tag{2.88}$$

We will use mathematical induction to show (2.87). We choose $h^0=0$ then (2.87) is satisfied for $i=0$. Assume (2.87) for all $i=0,1,2,...,m$. Rewrite $w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right) = \nu\left(\sqrt{\mu}\frac{h^m}{w}\right)h^{m+1}$ then the equation of h^{m+1} is

$$\{\partial_t + v\cdot\nabla_x + \nu(v) + \nu\left(\sqrt{\mu}\frac{h^m}{w}\right)\}h^{m+1} = K_w h^m + w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right). \tag{2.89}$$

From Theorem 8 and Corollary 9 we know that $\nu\left(\sqrt{\mu}\frac{h^m}{w}\right)$ and $w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right)$ is continuous in $[0,T]\times\Omega\times\mathbb{R}^3$. Apply Lemma 11 and $\phi(t,x,v)$ corresponds to $\nu(v)+\nu(\sqrt{\nu}\frac{h^m}{w})$ and $q(t,x,v)$ corresponds to the right hand side of (2.89). Then we check (2.87) for $i=m+1$.

Step 2 : We claim that there exist $C>0$ and $\delta>0$ such that if $C\{||h_0||_\infty + \sup_{0\le s<\infty}||wg(s)||_\infty\}<\delta$ and $C||h_0||_\infty<\delta$ then there exists $T=T(C,\delta)>0$ so that

$$\sup_{0\le s\le T}||h^m(s)||_\infty \le C||h_0||_\infty \tag{2.90}$$

for all $m\in\mathbb{N}$ and $\{h^m\}_{m=0}^\infty$ is Cauchy in $L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}^3)$.

First we will show a boundedness (2.90) for all $m\in\mathbb{N}$. We use mathematical induction on m. Assume $\sup_{0\le s\le T}||h^m(s)||_\infty\le C||h_0||_\infty$ where $T>0$ will be determined later. Integrating along the trajectory we have

$$\begin{aligned}
h^{m+1}(t,x,v) &= \mathbf{1}_{t<t_{\mathbf{b}}(x,v)}e^{-\nu(v)t}h_0(x-tv,v) + \mathbf{1}_{t\ge t_{\mathbf{b}}(x,v)}e^{-\nu(v)t_{\mathbf{b}}(x,v)}w(v)g(t-t_{\mathbf{b}}(x,v),x_{\mathbf{b}}(x,v),v)\\
&+\int_{\max\{t-t_{\mathbf{b}}(x,v),0\}}^t e^{-\nu(v)(t-s)}\{K_w h^m + w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right) - w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right)\}(s,x-(t-s)v,v)ds\\
&\le ||h_0||_\infty + \sup_{0\le s\le t}||wg(s)||_\infty + tC_{\mathbf{k}}\sup_{0\le s\le t}||h^m(s)||_\infty + C_\Gamma\sup_{0\le s\le t}||h^m(s)||_\infty\sup_{0\le s\le t}\left(||h^m(s)||_\infty + ||h^{m+1}(s)||_\infty\right)
\end{aligned}$$

and

$$\begin{aligned}\sup_{0\le s\le t}||h^{m+1}(s)||_\infty &\le \frac{1+tC_{\mathbf{k}}C+C_\Gamma C\{||h_0||_\infty+\sup_s||wg(s)||_\infty\}}{1-C_\Gamma C\{||h_0||_\infty+\sup_s||wg(s)||_\infty\}}\left\{||h_0||_\infty+\sup_{0\le s\le t}||wg(s)||_\infty\right\}\\ &\le C\left\{||h_0||_\infty+\sup_{0\le s\le t}||wg(s)||_\infty\right\}.\end{aligned}$$

where we can choose $C>4$ then $\{||h_0||_\infty+\sup_{0\le s\le t}||wg(s)||_\infty\}\le\frac{1}{2C_\Gamma C}$ and then $T=\frac{C-3}{2C_{\mathbf{k}}C}$. Newt we will show the sequence is Cauchy in $L^\infty([0,T]\times\bar\Omega\times\mathbb{R}^3)$. For simplicity we define

$$q^m=K_wh^m+w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right)-w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right)\tag{2.91}$$

The equation of $h^{m+1}-h^m$ is

$$\begin{aligned}&\{\partial_t+v\cdot\nabla_x+\nu\}(h^{m+1}-h^m)=\tilde q^m\\ &(h^{m+1}-h^m)|_{t=0}=0\quad,\quad(h^{m+1}-h^m)|_{\gamma_-}=0\end{aligned}\tag{2.92}$$

where

$$\tilde q^m=K_w(h^m-h^{m-1})+w\Gamma_+\left(\frac{h^m}{w},\frac{h^m-h^{m-1}}{w}\right)-w\Gamma_+\left(\frac{h^{m-1}-h^m}{w},\frac{h^{m-1}}{w}\right)-w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}-h^m}{w}\right)+w\Gamma_-\left(\frac{h^{m-1}-h^m}{w},\frac{h^m}{w}\right).\tag{2.93}$$

From Lemma 3.15 and (3.20) we have a bound of $\sup_{0\le s\le t}||\tilde q^m(s)||_\infty$

$$\begin{aligned}&C_{\mathbf{k}}\sup_{0\le s\le t}||\{h^m-h^{m-1}\}(s)||_\infty+C_\Gamma\nu(v)\{\sup_{0\le s\le t}||\{h^m-h^{m-1}\}(s)||_\infty\\ &+\sup_{0\le s\le t}||\{h^{m+1}-h^m\}(s)||_\infty\}\times C||h_0||_\infty\end{aligned}\tag{2.94}$$

Integrating (2.92) along the trajectory, we have

$$\begin{aligned}||\{h^{m+1}-h^m\}(t)||_\infty&\le\int_0^te^{-\nu(v)(t-s)}||\tilde q^m(s,x-(t-s)v,v)||_\infty ds\le C_{\mathbf{k}}t\sup_{0\le s\le t}||\{h^m-h^{m-1}\}(s)||_\infty\\ &+CC_\Gamma\left(||h_0||_\infty+\sup_{0\le s\le t}||wg(s)||_\infty\right)\left\{\sup_{0\le s\le t}||\{h^m-h^{m-1}\}(s)||_\infty+\sup_{0\le s\le t}||\{h^{m+1}-h^m\}(s)||_\infty\right\}\end{aligned}$$

If we choose $CC_\Gamma||h_0||_\infty\le\frac14$ and $C_{\mathbf{k}}T\le\frac18$ then

$$\sup_{0\le s\le T}||\{h^{m+1}-h^m\}(s)||_\infty\le\frac12\sup_{0\le s\le T}||\{h^m-h^{m-1}\}(s)||_\infty.$$

Then we have

$$\begin{aligned}
&\sup_{0\le s\le T} \|\{h^m - h^{m-1}\}(s)\|_\infty \le \sup_{0\le s\le T} \|\{h^m - h^{m-1}\}(s)\|_\infty + \cdots + \sup_{0\le s\le T} \|\{h^{n+1} - h^n\}(s)\|_\infty \\
&\le \{\frac{1}{2^{m-n-1}} + \cdots + \frac{1}{2^0}\} \sup_{0\le s\le T} \|\{h^{n+1} - h^n\}(s)\|_\infty \le \frac{2}{2^n} \sup_{0\le s\le T} \|\{h^1 - h^0\}(s)\|_\infty \\
&\le \frac{4}{2^n} C\{\|h_0\|_\infty + \sup_{0\le s\le T} \|wg(s)\|_\infty\}
\end{aligned}$$

which means that the sequence $\{h^m\}$ is Cauchy in $L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}^3)$.

Step 3 : From previous steps we obtain that h with $\lim_{n\to\infty} h^n$ is continuous function on $\mathfrak{C}_T$. Now we claim that h is continuous in $\mathfrak{C}$. Notice that T only depends on $\|h_0\|_\infty$ and $\sup_{0\le s\le T}\|wg(s)\|_\infty$. Using unform bound of $\sup_{0\le s<\infty}\|h(s)\|_\infty$(Theorem 1 of [27]) we can obtain the continuity for h for all time by repeating $[0,T],[T,2T],....$ If the boundary $\partial\Omega$ does not include a line segment (6) then every step is valid with $[0,\infty)\times\{\bar{\Omega}\times\mathbb{R}^3\}\backslash\{\mathfrak{D}\cup\gamma_0^V\}$ instead of $\mathfrak{C}$ and $[0,T]\times\{\bar{\Omega}\times\mathbb{R}^3\}\backslash\{\mathfrak{D}\cup\gamma_0^V\}$ instead of $\mathfrak{C}_T$.

2.4.3 Propagation of Discontinuity

Proof of 1 of Theorem 2

Proof of (1.21)

In order to show the upper bound of discontinuity jump (1.21) we will show

$$[h(t)]_{x_0+(t-t_0)v_0,v_0} \le [h]_{t_0,x_0,v_0} e^{-(\frac{1}{C_\nu}+\frac{C'}{C_w}\|h_0\|_\infty)(1+|v_0|)^\gamma(t-t_0)} \tag{2.95}$$

when $(x_0,v_0)\in\gamma_0^{\mathbf{S}}$ and $t\in(0,t_0+t_{\mathbf{b}}(x_0,-v_0))$. Choose two points $(x',v'),(x'',v'')\in\{\bar{\Omega}\times\mathbb{R}^3\backslash\mathfrak{G}\}\cap B((x,v);\delta)\backslash(x,v)$ and compare the representation

$$\begin{aligned}
& |h(t,x',v') - h(t,x'',v'')| && (2.96)\\
\le\ & \Big|\mathbf{1}_{t-t_0\ge t_{\mathbf{b}}(x',v')} h(t-t_{\mathbf{b}}(x',v'), x_{\mathbf{b}}(x',v'), v') e^{-\nu(v')t_{\mathbf{b}}(x',v') - \int_{t-t_{\mathbf{b}}(x',v')}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau,x'-(t-\tau)v',v')d\tau} && (2.97)\\
& +\mathbf{1}_{t-t_0< t_{\mathbf{b}}(x',v')} h(t_0, x'-(t-t_0)v', v') e^{-\nu(v')(t-t_0) - \int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau,x'-(t-\tau)v',v')d\tau} \\
& -\mathbf{1}_{t-t_0\ge t_{\mathbf{b}}(x'',v'')} h(t-t_{\mathbf{b}}(x'',v''), x_{\mathbf{b}}(x'',v''), v'') e^{-\nu(v'')t_{\mathbf{b}}(x'',v'') - \int_{t-t_{\mathbf{b}}(x'',v'')}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau,x'-(t-\tau)v',v')d\tau} && (2.98)\\
& -\mathbf{1}_{t-t_0< t_{\mathbf{b}}(x'',v'')} h(t_0, x''-(t-t_0)v'', v'') e^{-\nu(v'')(t-t_0) - \int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau,x''-(t-\tau)v'',v'')d\tau}\Big| \\
+\ & \Big|\int_{\max\{0,t-t_0-t_{\mathbf{b}}(x',v')\}}^t \{K_w h + w\Gamma_+(\frac{h}{w},\frac{h}{w})\}(s, x'-(t-s)v', v') e^{-\nu(v')(t-s) - \int_s^t \nu(\sqrt{\mu}\frac{h}{w})(\tau,x'-(t-\tau)v',v')d\tau} ds \\
& - \int_{\max\{0,t-t_0-t_{\mathbf{b}}(x'',v'')\}}^t \{K_w h + w\Gamma_+(\frac{h}{w},\frac{h}{w})\}(s, x''-(t-s)v'', v'') e^{-\nu(v'')(t-s) - \int_s^t \nu(\sqrt{\mu}\frac{h}{w})(\tau,x''-(t-\tau)v'',v'')d\tau} ds\Big|.
\end{aligned}$$

It is easy to see that if $t-t_0 \geq t_{\mathbf{b}}(x',v')$ then as $\delta \to 0$ we have

$$t - t_{\mathbf{b}}(x',v') \to t_0 \ , \ x_{\mathbf{b}}(x',v') \to x_0 \ ,$$

and if $t-t_0 < t_{\mathbf{b}}(x',v')$ then as $\delta \to 0$ we have

$$x' - (t-t_0)v' \to x_0.$$

Therefore the first $|....|$ converges to $[h]_{t_0,x_0,v_0} \times e^{-\nu(v_0)(t-t_0) - \int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0-(t_0-\tau)v_0, v_0)d\tau}$. For the second $|....|$ using the continuity of $K_w h, \Gamma(\frac{h}{w},\frac{h}{w}), \nu(\sqrt{\mu}\frac{h}{w})$ we conclude that $|....| \to 0$. Therefore we

$$\begin{aligned}
[h(t)]_{x_0+(t-t_0)v_0,v_0} &\leq [h]_{t_0,x_0,v_0} e^{-\nu(v_0)(t-t_0) - \int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0-(t_0-\tau)v_0, v_0)d\tau} \\
&\leq [h]_{t_0,x_0,v_0} \times e^{-(\frac{1}{C_\nu} - C_w C' ||h_0||_\infty)(1+|v_0|)^\gamma (t-t_0)}
\end{aligned}$$

where we used

$$\nu_w(v) \equiv \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega) e^{-\frac{|u|^2}{4}} w^{-1}(u) d\omega du \tag{2.99}$$

$$\text{with} \quad \frac{1}{C_w}(1+|v|)^\gamma \leq \nu_w(v) \leq C_w(1+|v|)^\gamma. \tag{2.100}$$

Remark that Proof of (1.21) is valid for in-flow, diffuse and bounce-back cases.

Proof of (1.23) for a Local in time Formation of Discontinuity

Now suppose the Boltzmann solution F has a local in time formation of discontinuity at $(t_0, x_0, v_0) \in [0,\infty) \times \gamma_0^{\mathbf{S}}$. Recall that the Boltzmann solution F is continuous initially because of the continuity of the initial datum F and the in-flow datum G with the compatibility condition (1.16) but F is discontinuous after a short time $t_* \in (0, t_{\mathbf{b}}(x_0, -v_0))$. Further assume that the boundary $\partial\Omega$ is strictly concave at x_0 along v_0 direction (1.22).

Step 1 We can choose sequences $(t_n', x_n', v_n'), (t_n'', x_n'', v_n'') \in [0,\infty) \times \bar{\Omega} \times \mathbb{R}^3 \cap B((t_0,x_0,v_0); \frac{1}{n}) \backslash (t_0, x_0, v_0)$ such that $\lim_{n\to\infty} |h(t_n', x_n', v_n') - h(t_n'', x_n'', v_n'')| \geq \frac{1}{2}[h(t_0)]_{x_0,v_0} \neq 0$:

From $[h(t_0)]_{x_0,v_0} \neq 0$ we may assume

$$\sup_{(x_0',v_0'),(x_0'',v_0'') \in B((x_0,v_0);\frac{1}{n}) \backslash (x_0,v_0)} |h(t_0, x_0', v_0') - h(t_0, x_0'', v_0'')| \geq \frac{3}{4}[h(t_0)]_{x_0,v_0} \neq 0 \tag{2.101}$$

for all $n \in \mathbb{N}$. And for each $n \in \mathbb{N}$ we can choose a desired sequences.

Step 2 For given $\varepsilon > 0$ up to subsequence we may assume that

$$(x_{\mathbf{b}}(x'_n, v'_n), v'_n) \in B((x_0, v_0); \varepsilon) \backslash \mathfrak{G} \ , \ (x_{\mathbf{b}}(x''_n, v''_n), v''_n) \notin B((x_0, v_0); \varepsilon) \cup \mathfrak{G} \quad \text{for all } n \in \mathbb{N} \tag{2.102}$$

We remark that a continuity $G(t,x,v) = w(v)g(t,x,v)$ on $[0,\infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$ i.e.

$$\begin{aligned} [\ G|_{[0,\infty)\times\gamma_-}]_{t_0,x_0,v_0} &= w(v_0)[\ g|_{[0,\infty)\times\gamma_-}]_{t_0,x_0,v_0} = 0 \\ &\text{for all } \ (t_0,x_0,v_0) \in [0,\infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\} \end{aligned} \tag{2.103}$$

is crucially used in this step. In order to show the final goal (2.102) of this step, we need to prove following statement :

Assume $(x_0, v_0) \in \gamma_0^{\mathbf{S}}$ and $t_{\mathbf{b}}(x_0, v_0) > t_0$. Then for sufficiently small $\varepsilon > 0$ there exists $N > 0$ such that if $(x,v) \in B((x_0,v_0); \dfrac{1}{n})$ for $n > \mathbb{N}$ and $x_{\mathbf{b}}(x,v) \notin B((x_0,v_0);\varepsilon)$ then we have $t_{\mathbf{b}}(x,v) > t_0$. (2.104)

We will prove (2.104) later and show (2.102) using (2.104). It suffices to show that there are only finite $n \in \mathbb{N}$ such that

$$(x_{\mathbf{b}}(x'_n, v'_n), v'_n) \in B((x_0, v_0); \frac{1}{n}) \backslash \mathfrak{G} \ , \ (x_{\mathbf{b}}(x''_n, v''_n), v''_n) \in B((x_0, v_0); \frac{1}{n}) \backslash \mathfrak{G} \tag{2.105}$$

$$\text{or} \ \ (x_{\mathbf{b}}(x'_n, v'_n), v'_n) \notin B((x_0, v_0); \frac{1}{n}) \cup \mathfrak{G} \ , \ (x_{\mathbf{b}}(x''_n, v''_n), v''_n) \notin B((x_0, v_0); \frac{1}{n}) \cup \mathfrak{G} \tag{2.106}$$

Suppose there are infinitely many $n' \in \mathbb{N}$ satisfying (2.105). If $\varepsilon > 0$ is sufficiently small then (2.105) implies that $t_0 > t_{\mathbf{b}}(x'_{n'}, v'_{n'})$ and $t_0 > t_{\mathbf{b}}(x''_{n'}, v''_{n'})$. The Boltzmann solution h at $(t_0, x'_{n'}, v'_{n'})$ is

$$h(t_0, x'_{n'}, v'_{n'}) = \tag{2.107}$$

$$\underline{h(t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'}), x_{\mathbf{b}}(x'_{n'}, v'_{n'}), v'_{n'})} e^{-\nu(v'_{n'})(t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'})) - \int_{t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'})}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x'_{n'} - (t_0-\tau)v'_{n'}, v'_{n'}) d\tau} \tag{2.108}$$

$$+ \int_{t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'})}^{t_0} \{K_w h + \Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, x'_{n'} - (t_0 - s)v'_{n'}, v'_{n'}) e^{-\nu(v'_{n'})(t_0 - s) - \int_s^{t_0} \nu(F)(\tau, x'_{n'} - (t_0-\tau)v'_{n'}, v'_{n'}) d\tau} ds$$

and a similar representation for $h(t_0, x''_{n'}, v''_{n'})$. Compare representations of $h(t_0, x'_{n'}, v'_{n'})$ and $h(t_0, x''_{n'}, v''_{n'})$ to conclude

$$\begin{aligned}
& \lim_{n'\to\infty} |h(t_0, x'_{n'}, v'_{n'}) - h(t_0, x''_{n'}, v''_{n'})| \\
= & \lim_{n'\to\infty} |h(t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'}), x_{\mathbf{b}}(x'_{n'}, v'_{n'}), v'_{n'}) - h(t_0 - t_{\mathbf{b}}(x''_{n'}, v''_{n'}), x_{\mathbf{b}}(x''_{n'}, v''_{n'}), v''_{n'})| \\
& \times e^{-\nu(v_0)(t_0 - t_{\mathbf{b}}(x_0,v_0)) - \int_{t_0 - t_{\mathbf{b}}(x_0,v_0)}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0 - (t_0-\tau)v_0, v_0)d\tau} \\
\leq & [h|_{[0,\infty)\times\gamma_-}]_{t_0 - t_{\mathbf{b}}(x_0,v_0), x_{\mathbf{b}}(x_0,v_0), v_0} \times e^{-\nu(v_0)(t_0 - t_{\mathbf{b}}(x_0,v_0)) - \int_{t_0 - t_{\mathbf{b}}(x_0,v_0)}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0 - (t_0-\tau)v_0, v_0)d\tau}
\end{aligned}$$

where we used the continuity of $\nu(\sqrt{\mu}\frac{h}{w})$ and $\Gamma_+(\frac{h}{w}, \frac{h}{w})$. Further using the in-flow boundary condition $h|_{\gamma_-} = wg$, we have

$$\begin{aligned}
& \lim_{n'\to\infty} |h(t_0, x'_{n'}, v'_{n'}) - h(t_0, x''_{n'}, v''_{n'})| \\
& \leq [\, g|_{[0,\infty)\times\gamma_-}]_{t_0, x_0, v_0} w(v_0) e^{-\nu(v_0)(t_0 - t_{\mathbf{b}}(x_0,v_0)) - \int_{t_0 - t_{\mathbf{b}}(x_0,v_0)}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0 - (t_0-\tau)v_0, v_0)d\tau} = 0
\end{aligned}$$

where we used (2.103) the continuity of g on $[0,\infty) \times \{\gamma_- \cup \gamma_0\}$ at the last line. This is contradict because we choose the sequences $(x'_{n'}, v'_{n'}), (x''_{n'}, v''_{n'})$ satisfying $\lim_{n\to\infty} |h(t_0, x'_{n'}, v'_{n'}) - h(t_0, x''_{n'}, v''_{n'})| \geq \frac{1}{2}[h(t_0)]_{x_0,v_0} \neq 0$ in **Step 1**.

Now suppose there are infinitely many $n'' \in \mathbb{N}$ satisfying (2.106). Because of (2.104) we have $t_0 < t_{\mathbf{b}}(x'_{n''}, v'_{n''})$ and $t_0 < t_{\mathbf{b}}(x''_{n''}, v''_{n''})$. The Boltzmann solution h at $(t_0, x'_{n''}, v'_{n''})$ is

$$\begin{aligned}
h(t_0, x'_{n''}, v'_{n''}) &= h_0(x'_{n''} - t_0 v'_{n''}, v'_{n''}), v'_{n''}) e^{-\nu(v'_{n''})t_0 - \int_0^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x'_{n''} - (t_0-\tau)v'_{n''}, v'_{n''})d\tau} \\
+ \int_0^{t_0} \{K_w h + \Gamma_+(\frac{h}{w}, \frac{h}{w})\}&(s, x'_{n''} - (t_0 - s)v'_{n''}, v'_{n''}) e^{-\nu(v'_{n''})(t_0-s) - \int_s^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x'_{n''} - (t_0-\tau)v'_{n''}, v'_{n''})d\tau} ds
\end{aligned}$$

and same representation for $h(t_0, x''_{n''}, v''_{n''})$. Using the continuity of h_0 we see that

$$\begin{aligned}
& \lim_{n\to\infty} |h(t_0, x'_{n''}, v'_{n''}) - h(t_0, x''_{n''}, v''_{n''})| \\
& = \lim_{n\to\infty} |h_0(x'_{n''} - t_0 v'_{n''}, v'_{n''}) - h_0(x''_{n''} - t_0 v''_{n''}, v''_{n''})| w(v_0) \\
& \times e^{-\nu(v_0)(t_0 - t_{\mathbf{b}}(x_0,v_0)) - \int_{t_0 - t_{\mathbf{b}}(x_0,v_0)}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0 - (t_0-\tau)v_0, v_0)d\tau} \\
& = 0
\end{aligned}$$

which is also contradiction.

Now we prove (2.104). We can choose $\varepsilon > 0$ sufficiently small so that $\partial\Omega \cap B(x_0; \varepsilon) = \{(x_1, x_2, \Phi(x_1, x_2)) \in B(x_0; \varepsilon)\}$. From $t_{\mathbf{b}}(x_0, v_0) > t_0$ we have $\overline{x_0, x_0 - t_0 v_0} \cap \partial\Omega = \{x_0\}$ and can choose $\varrho > 0$ so large that $\bigcup_{s\in[0,t_0]} B(x_0 - sv_0; \varrho) \cap \partial\Omega \subset B(x_0;\varepsilon)$. Choose $N \in \mathbb{N}$ sufficiently large so that $\overline{x, x-t_0v} \subset \bigcup_{s\in[0,t_0]} B(x_0 - sv_0;\varrho)$ for all $(x,v) \in B((x_0,v_0);\frac{1}{n})$. If $x_{\mathbf{b}}(x,v) \notin B((x_0,v_0);\varepsilon)$ then $\overline{x, x-t_0v} \cap \partial\Omega = \emptyset$ and this implies $t_{\mathbf{b}}(x,v) > t_0$.

Step 3 Choose $t > 0$ so that $t - t_0 \in [0, t_{\mathbf{b}}(x_0,-v_0))$ and denote $x = x_0 + (t-t_0)v_0,\ v = v_0$. Then there exists $N \in \mathbb{N}$ so that $t - t_0 < t_{\mathbf{b}}(x_n', -v_n')$ for all $n > N$:

Using (2.104) we only have to prove $x_{\mathbf{b}}(x_n', -v_n') \notin B((x_0,-v_0);\varepsilon)$. From (2.102) we know that $x_{\mathbf{b}}(x_n', v_n') \in B(x_0;\varepsilon)$. We assume that $\Omega \cap B(x_0;\varepsilon) = \{x \in B(x_0;\varepsilon) : x_3 > \Phi(x_1,x_2)\}$ and $n(x_0) = (0,0,-1)$ and $v_0 = |v_0|(1,0,0)$. Let's define

$$\Psi(s) = \Phi((x_n')_1 + s(v_n')_1, (x_n')_2 + s(v_n')_2) - ((x_n')_3 + s(v_n')_2). \tag{2.109}$$

Since $x_n' \in \Omega$ we have $\Psi(0) < 0$ and $\Psi(t_{\mathbf{b}}(x_n', -v_n')) = 0 = \Psi(-t_{\mathbf{b}}(x_n', v_n'))$. Because of the strict concavity along v_0 direction at x_0 (1.22) for sufficiently large n so that $(x_n', v_n') \sim (x_0, v_0)$ we have

$$\Psi''(s) = ((v_n')_1, (v_n')_2) \begin{pmatrix} \partial_{x_1}^2\Phi & \partial_{x_1}\partial_{x_2}\Phi \\ \partial_{x_2}\partial_{x_1}\Phi & \partial_{x_2}^2\Phi \end{pmatrix} \begin{pmatrix} (v_n')_1 \\ (v_n')_2 \end{pmatrix} < -\frac{1}{2}C_{x_0,v_0} \tag{2.110}$$

where the Hessian of Φ is evaluated at $((x_n')_1 + s(v_n')_1, (x_n')_2 + s(v_n')_2)$. Since $\{x_n' + sv_n' : s \in (-t_{\mathbf{b}}(x_n', v_n'), t_{\mathbf{b}}(x_n', -v_n'))\} \subset \Omega$ we have $\Psi(s) < 0$ for $s \in (-t_{\mathbf{b}}(x_n', v_n'), t_{\mathbf{b}}(x_n', -v_n'))$. Therefore $\Phi'(-t_{\mathbf{b}}(x_n', v_n')) \leq 0$ and $\Phi'(t_{\mathbf{b}}(x_n', -v_n')) \geq 0$. This is contradiction because

$$0 \leq \Phi'(t_{\mathbf{b}}(x_n', -v_n')) = \Phi'(-t_{\mathbf{b}}(x_n', v_n')) + \int_{-t_{\mathbf{b}}(x_n', v_n')}^{t_{\mathbf{b}}(x_n', -v_n')} \Phi''(s)ds \leq 0 - \frac{1}{2}C_{x_0,v_0}\{t_{\mathbf{b}}(x_n', -v_n') + t_{\mathbf{b}}(x_n', v_n')\} < 0.$$

The consequence of this step is that for $n > N$ we have a representation of h at (t,x,v)

$$\begin{aligned} h(t, x_n' + (t-t_0)v_n', v_n') &= h(t_0, x_n', v_n') e^{-\nu(v_n')(t-t_0) - \int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_n' + (\tau - t_0)v_n', v_n')d\tau} \\ + \int_{t_0}^t \{K_w + w\Gamma_+(\frac{h}{w}, \frac{h}{w})\}&(s, x_n + (s-t_0)v_n', v_n') e^{-\nu(v_n')(t-s) - \int_s^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_n' + (\tau - t_0)v_n', v_n')d\tau} ds \end{aligned} \tag{2.111}$$

Step 4 For given $\varepsilon > 0$ there exists $\delta > 0$ so that if $|(y,u) - (x_0,v_0)| < \delta$ and $|(x,v) - (x_0,v_0)| < \delta$

and $t_0 < t_{\mathbf{b}}(y,u)$ and $t_0 < t_{\mathbf{b}}(x,v)$ then

$$|h(t_0,y,u) - h(t_0,x,v)| < \varepsilon \ : \tag{2.112}$$

We have $h(t_0,y,u) = h_0(y-t_0u,u)e^{-\nu(u)t_0 - \int_0^{t_0}\nu(\sqrt{\mu}\frac{h}{w})(\tau,y-(t_0-\tau)u,u)d\tau}$

$$+\int_0^{t_0}\{K_wh + \Gamma_+(\frac{h}{w},\frac{h}{w})\}(s,y-(t_0-s)u,u)e^{-\nu(u)(t_0-s)-\int_s^{t_0}\nu(\sqrt{\mu}\frac{h}{w})(\tau,y-(t_0-\tau)u,u)d\tau}ds$$

and similarly $h(t_0,x,v) = h_0(x-t_0v,v)e^{-\nu(v)t_0 - \int_0^{t_0}\nu(\sqrt{\mu}\frac{h}{w})(\tau,x-(t_0-\tau)v,v)d\tau}$

$$+\int_0^{t_0}\{K_wh + \Gamma_+(\frac{h}{w},\frac{h}{w})\}(s,x-(t_0-s)v,v)e^{-\nu(v)(t_0-s)-\int_s^{t_0}\nu(\sqrt{\mu}\frac{h}{w})(\tau,x-(t_0-\tau)v,v)d\tau}ds$$

Let's compare the arguments of two representations :

$$\begin{aligned}
|(y-t_0u,u)-(x-t_0v,v)| &< 2(1+t_0)\delta \quad \text{for} \quad h_0 \\
|(\tau,y-(t_0-\tau)u,u)-(\tau,x-(t_0-\tau)v,v)| &< 2(1+t_0)\delta \quad \text{for} \quad \nu(\sqrt{\mu}\frac{h}{w}) \\
|(s,y-(t_0-s)u,u)-(s,x-(t_0-s)v,v)| &< 2(1+t_0)\delta \quad \text{for} \quad K_wh+\Gamma_+(\frac{h}{w},\frac{h}{w}).
\end{aligned}$$

Using the continuity of $h_0,\ \nu(\sqrt{\mu}\frac{h}{w}),\ K_wh$ and $\Gamma_+(\frac{h}{w},\frac{h}{w})$ we can choose desired $\varepsilon > 0$ to conclude (2.112).

Step 5 Choose $t > 0$ so that $t - t_0 \in [0, t_{\mathbf{b}}(x_0,-v_0))$ and denote $x = x_0 + (t-t_0)v_0,\ v = v_0$. Let $\varepsilon \le \frac{1}{10}[h(t_0)]_{x_0,v_0}$ and $\delta > 0$ in the **Step 4**. Then we can choose $u_n'' \in \Omega$ so that $|u_n'' - v_n''| < \delta$ and $t_0 < t_{\mathbf{b}}(x_n'',u_n'')$ and $t-t_0 < t_{\mathbf{b}}(x_n'',-u_n'')$:

If there are infinitely many u_n'' so that $t_0 < t_{\mathbf{b}}(x_n'',u_n'')$ and $t-t_0 < t_{\mathbf{b}}(x_n'',-u_n'')$ then up to subsequence we can define $u_n'' = v_n''$. Therefore we may assume $t - t_0 \ge t_{\mathbf{b}}(x_n'',-v_n'')$ for all $n \in \mathbb{N}$. We assume that $\Omega \cap B(x_0;\varepsilon) = \{x \in B(x_0;\varepsilon) : x_3 > \Phi(x_1,x_2)\}$ and $n(x_0) = (0,0,-1)$ and $v_0 = |v_0|(1,0,0)$. Now we illustrate how to choose such a u_n''. Denote $x_n'' = x = (x_1,x_2,x_3)$ and $v_n'' = (v_1,v_2,v_3)$. First we will choose (u_1,u_2,u_3) and $s > 0$ so that

$$n(x_{\mathbf{b}}(x,-u))\cdot u = 0 \tag{2.113}$$

and $x_{\mathbf{b}}(x,-u) = (x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}}, \Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}}))$. The condition (2.113) implies that

$$\frac{u_3}{\sqrt{u_1^2+u_2^2}} = \frac{d}{ds}\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}}) = \frac{\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}) - x_3}{s}. \tag{2.114}$$

In order to use the implicit function theorem we define

$$\begin{aligned}\Psi(x_1,x_2,x_3;u_1,u_2;s) = \Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}}) - x_3 \\ -s\{\frac{u_1}{\sqrt{u_1^2+u_2^2}}\partial_{x_1}\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}}) \\ +\frac{u_2}{\sqrt{u_1^2+u_2^2}}\partial_{x_2}\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}})\}\end{aligned}$$

and compute $\partial_s\Psi = -s(\frac{u_1}{\sqrt{u_1^2+u_2^2}}, \frac{u_2}{\sqrt{u_1^2+u_2^2}})\begin{pmatrix}\partial_{x_1}^2\Phi & \partial_{x_1}\partial_{x_2}\Phi \\ \partial_{x_1}\partial_{x_2}\Phi & \partial_{x_2}^2\Phi\end{pmatrix}\begin{pmatrix}\frac{u_1}{\sqrt{u_1^2+u_2^2}} \\ \frac{u_2}{\sqrt{u_1^2+u_2^2}}\end{pmatrix} < -\frac{1}{2}C_{x_0,v_0}$ for $x \sim x_0$ and $v \sim v_0$ using (1.22) and the Hessian is evaluated at $(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}})$. Hence $s = s(x_1,x_2,x_3;w_1,w_2)$ is a smooth function near $x \sim x_0$ and $(u_1,u_2) \sim (v_1,v_2)$. In order to study the behavior of s we use the Taylor's expansion : from $\Psi(x_1,x_2,x_3;u_1,u_2;s) = 0$ we have

$$\begin{aligned}\Phi(x_1,x_2) - x_3 \quad = \quad & \frac{1}{u_1^2+u_2^2}\Big\{(u_1,u_2)\underbrace{\begin{pmatrix}\partial_{x_1}^2\Phi & \partial_{x_1}\partial_{x_2}\Phi \\ \partial_{x_1}\partial_{x_2}\Phi & \partial_{x_2}^2\Phi\end{pmatrix}}_{(*)}\begin{pmatrix}u_1 \\ u_2\end{pmatrix} - \\ & \frac{1}{2}(u_1,u_2)\underbrace{\begin{pmatrix}\partial_{x_1}^2\Phi & \partial_{x_1}\partial_{x_2}\Phi \\ \partial_{x_1}\partial_{x_2}\Phi & \partial_{x_2}^2\Phi\end{pmatrix}}_{(**)}\begin{pmatrix}u_1 \\ u_2\end{pmatrix}\Big\}s^2\end{aligned}$$

where the Hessian $(*)$ is evaluated at $(x_1 + s_*\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s_*\frac{u_2}{\sqrt{u_1^2+u_2^2}})$ and the Hessian $(**)$ is evaluated at $(x_1 + s_{**}\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s_{**}\frac{u_2}{\sqrt{u_1^2+u_2^2}})$ with $s_*, s_{**} \in (0,s)$. For $x \sim x_0$ and $(u_1,u_2) \sim (v_1,v_2)$ we know that the right hand side of the above equation converges to

$$-\frac{1}{2(v_1^2+v_2^2)}(v_1,v_2)\begin{pmatrix}\partial_{x_1}^2\Phi((x_0)_1,(x_0)_2) & \partial_{x_1}\partial_{x_2}\Phi((x_0)_1,(x_0)_2) \\ \partial_{x_1}\partial_{x_2}\Phi((x_0)_1,(x_0)_2) & \partial_{x_2}^2\Phi((x_0)_1,(x_0)_2)\end{pmatrix}\begin{pmatrix}v_1 \\ v_2\end{pmatrix} \neq 0$$

hence we have a control of s i.e

$$\frac{1}{C}|\Phi(x_1,x_2)-x_3|^{\frac{1}{2}} \leq s \leq C|\Phi(x_1,x_2)-x_3|^{\frac{1}{2}}. \tag{2.115}$$

From (2.114) $u_3 = \sqrt{u_1^2+u_2^2}\frac{d}{ds}\Phi(x_1+s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2+s\frac{u_2}{\sqrt{u_1^2+u_2^2}})$ is equivalent to

$$\sqrt{u_1^2+u_2^2}\begin{pmatrix} \frac{u_1}{\sqrt{u_1^2+u_2^2}} \\ \frac{u_2}{\sqrt{u_1^2+u_2^2}} \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_1}\Phi(x_1,x_2)+\frac{u_1}{\sqrt{u_1^2+u_2^2}}\partial_{x_1}^2\Phi(x_1',x_2')s+\frac{u_2}{\sqrt{u_1^2+u_2^2}}\partial_{x_1}\partial_{x_2}\Phi(x_1',x_2')s \\ \partial_{x_2}\Phi(x_1,x_2)+\frac{u_1}{\sqrt{u_1^2+u_2^2}}\partial_{x_1}\partial_{x_2}\Phi(x_1',x_2')s+\frac{u_2}{\sqrt{u_1^2+u_2^2}}\partial_{x_2}^2\Phi(x_1',x_2')s \end{pmatrix} \tag{2.116}$$

where $x_1' = x_1+s'\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2' = x_2+s'\frac{u_2}{\sqrt{u_1^2+u_2^2}}$ for some $0 < s' < s \leq C|\Phi(x_1,x_2)-x_3|^{\frac{1}{2}}$. Using the smoothness of Φ we can bound (2.116) upper and lower by

$$C|(u_1,u_2)|\left(|(x_1,x_2)|+|\Phi(x_1,x_2)-x_3|^{\frac{1}{2}}\right) \tag{2.117}$$

To sum for fixed x and direction $\frac{1}{\sqrt{u_1^2+u_2^2}}(u_1,u_2)$ we can choose u_3 such that $n(x_{\mathbf{b}}(x,-(u_1,u_2,u_3)))\cdot(u_1,u_2,u_3)=0$ and u_3 is controlled by (2.117). Finally we choose $(u_1,u_2)=\sqrt{\frac{u_1^2+u_2^2}{v_1^2+v_2^2}}(v_1,v_2)$ and find the corresponding u_3 so that $|v|=|(u_1,u_2,u_3)|$. Define $u_n''=-v+2(v\cdot\frac{(u_1,u_2,u_3)}{|(u_1,u_2,u_3)|})(u_1,u_2,u_3)$. Then we have desired u_n'' for sufficiently large $n\in\mathbb{N}$.

Step 6 To sum for $(t,x_n''+(t-t_0)u_n'',u_n'')$ we have $t-t_0 < t_{\mathbf{b}}(x_n'',-u_n'')$ and $t_0 < t_{\mathbf{b}}(x_n'',u_n'')$ and $|h(t_0,x_n'',u_n'')-h(t_0,x_n'',v_n'')| < \frac{1}{10}[h(t_0)]_{x_0,v_0}$. Hence the representation of the Boltzmann solution h at $(t,x_n''+(t-t_0)u_n'',u_n'')$ is given by

$$\begin{aligned} h(t,x_n''+(t-t_0)v_n'',u_n'') &= h(t_0,x_n'',u_n'')e^{-\nu(u_n'')(t-t_0)-\int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau,x_n''+(\tau-t_0)u_n'',u_n'')d\tau} \\ +\int_{t_0}^t\{K_w h+w\Gamma(\frac{h}{w},\frac{h}{w})\}&(s,x_n''+(s-t_0)u_n'',u_n'')e^{-\nu(u_n'')(t-s)-\int_s^t \nu(\sqrt{\mu}\frac{h}{w})(\tau,x_n''+(\tau-t_0)u_n'',u_n'')d\tau}ds \end{aligned}$$

Using (2.111) we have

$$
\begin{aligned}
&\lim_{n\to\infty} |h(t, x'_n + (t-t_0)v'_n, v'_n) - h(t, x''_n + (t-t_0)u''_n, u''_n)| \\
&= \lim_{n\to\infty} |h(t_0, x'_n, v'_n) - h(t_0, x''_n, u''_n)| e^{-\nu(v_0)(t-t_0) - \int_{t_0}^{t} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0 + (\tau - t_0)v_0, v_0) d\tau} \\
&\geq \Big\{ \lim_{n\to\infty} |h(t_0, x'_n, v'_n) - h(t_0, x''_n, v''_n)| - \lim_{n\to\infty} |h(t_0, x''_n, v''_n) - h(t_0, x''_n, u''_n)| \Big\} \\
&\qquad \times e^{-\nu(v_0)(t-t_0) - \int_{t_0}^{t} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0 + (\tau - t_0)v_0, v_0) d\tau} \\
&\geq \frac{1}{4} [h(t_0)]_{x_0, v_0} e^{-\nu(v_0)(t-t_0) - \int_{t_0}^{t} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0 + (\tau - t_0)v_0, v_0) d\tau}
\end{aligned}
$$

which implies that

$$
[h(t)]_{x,v} \geq \frac{1}{4} [h(t_0)]_{x_0, v_0} \times e^{-(C_\mu + C' C_w ||h_0||_\infty)(1+|v|)^\gamma (t - t_0)} \neq 0. \tag{2.118}
$$

Remark *Through Step 1 to Step 6, we only used the in-flow boundary datum g explicitely in Step 2. All the other steps are valid for diffuse and bounce-back boundary condition cases. In Step 2, we only used (2.103) the continuity of $G = wg$ on $[0, \infty) \times \{\gamma_- \cup \gamma_0\}$. Therefore once we have a continuity of the Boltmzann solution F restricted on*

2.5 Diffuse Reflection Boundary Condition

In spite of the averaging effect of diffuse reflection operator we can observe same phenomena of in-flow boundary case.

2.5.1 Formation of Discontinuity

The idea of proof is similar to in-flow case but we also use $|v|$ not only t_* as a parameter. We will prove Part 2 of Theorem 1. Without loss of generality we may assume $x = (0,0,0)$ and $v = (|v|, 0, 0)$ and $(x, v) \in \gamma_0^{\mathbf{S}}$. Locally the boundary is a graph i.e. $\Omega \cap B(\mathbf{0}; \delta) = \{(x_1, x_2, x_3) \in B(\mathbf{0}; \delta) : x_3 > \Phi(x_1, x_2)\}$ and $\Phi(\xi, 0) < 0$ for $\xi \in (-\delta, \delta)$.

Assume that $||h_0||_\infty < \delta$ so that the global-in-time solution $h(t, x, v)$ has uniform bound

(2.78)[27]. Choose $t_* \in (0, \frac{\delta}{2})$ sufficiently small and $|v| > 0$ sufficiently large so that

$$\frac{1}{2} \leq \left(e^{-\nu(|v|)t_*} - t_* C_{\mathbf{k}} C' - (1 - e^{-\nu(|v|)t_*}) C_\Gamma (C')^2 - C' \frac{1}{\tilde{w}(v)} \int_{\mathcal{V}} \tilde{w}(v') d\sigma(v') \right) \tag{2.119}$$

where $\nu(|v|) = \nu(v)$ and $\mathcal{V} = \mathcal{V}(x)$ for any $x \in \partial\Omega$. First we choose $|v|$ large enough to have

$$\frac{1}{\tilde{w}(v)} = \frac{(1 + \rho^2 |v|^2)^\beta}{e^{\frac{|v|^2}{4}}} \leq \frac{1}{10C'}.$$

Then we choose sufficiently small $t_* > 0$ as

$$0 < t_* = \min \left\{ \frac{\delta}{|v|}, \frac{1}{10\nu(|v|)}, \frac{1}{10C_{\mathbf{k}}C'}, \frac{1}{\nu(|v|)} \log \frac{10C_\Gamma(C')^2}{10C_\Gamma(C')^2 - 1} \right\}$$

Because of our choice $t_*|v| \leq \delta$. Denote $\tau_* = t_*|v|$. Assume a condition for our initial data h_0 : there is sufficiently small $\delta' = \delta'(\Omega, \tau_*) > 0$ such that $B((-\tau_*, 0, 0), \delta') \subset \Omega$ and (2.140).

We claim that the Boltzmann solution h with such initial data h_0 is not continuous at $(\tau_*, (0,0,0), (|v|, 0, 0))$. We will use a contradiction argument : Assume that (2.81) is valid. Choose sequences of points $(x_n', v_n') = ((0, 0, \frac{1}{n}), (|v|, 0, 0))$ and $(x_n, v_n) = ((\frac{1}{n}, 0, \Phi(\frac{1}{n}, 0)), \frac{1}{\sqrt{1+\frac{1}{n^2}}}(|v|, 0, \frac{|v|}{n}))$. Because of our choice, for sufficiently large $n \in \mathbb{N}$, we have

$$(x_n' - t_* v_n', v_n') = ((-t_*|v|, 0, \frac{1}{n}), (|v|, 0, 0)) \in B((-\tau_*, 0, 0); \delta') \times B((|v|, 0, 0); \delta').$$

Therefore,

$$\begin{aligned}
h(t_*, x_n', v_n') &= h_0(x_n' - t_* v_n', v_n') e^{-\nu(v_n')t_*} + \int_0^{t_*} e^{-\nu(v_n')(t_* - \tau)} \left\{ K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w}) \right\} (\tau, x_n' - v(t_* - \tau), v_n') d\tau \\
&= ||h_0||_\infty e^{-\nu(|v|)t_*} + \int_0^{t_*} e^{-\nu(|v|)(t_* - \tau)} \left\{ K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w}) \right\} (\tau, x - v_n'(t_* - \tau), v_n') d\tau, \\
h(t_*, x_n, v_n) &= \frac{1}{\tilde{w}(|v|)} \int_{\mathcal{V}(x_n)} h(t_*, x_n, v') \tilde{w}(v') d\sigma(v').
\end{aligned}$$

Using (2.78) and $||h_0||_\infty \leq 1$, we can estimate $|h(t_*, x_n', v_n') - h(t_*, x_n, v_n)|$

$$\begin{aligned}
\geq \;& \Big|\, ||h_0||_\infty e^{-\nu(|v|)t_*} - \int_0^{t_*} C_{\mathbf{k}} C' ||h_0||_\infty + \nu(v_n') e^{-\nu(v_n')(t-\tau)} C_\Gamma (C')^2 ||h_0||_\infty^2 d\tau \\
& \quad - C' ||h_0||_\infty \frac{1}{\tilde{w}(|v|)} \int_{\mathcal{V}} \tilde{w}(v') d\sigma v' \Big| \\
\geq \;& ||h_0||_\infty e^{-\nu(|v|)t_*} - t_* C_{\mathbf{k}} C' ||h_0||_\infty - (1 - e^{-\nu(|v|)t_*}) C_\Gamma (C')^2 ||h_0||_\infty^2 - C' ||h_0||_\infty \frac{1}{\tilde{w}(|v|)} \int_{\mathcal{V}} \tilde{w}(v') d\sigma v' \\
= \;& ||h_0||_\infty \left(e^{-\nu(|v|)t_*} - t_* C_{\mathbf{k}} C' - (1 - e^{-\nu(|v|)t_*}) C_\Gamma (C')^2 - C' \frac{1}{\tilde{w}(|v|)} \int_{\mathcal{V}} \tilde{w}(v') d\sigma v' \right) \geq \frac{||h_0||_\infty}{2} \neq 0
\end{aligned}$$

which is contradiction to (2.81).

2.5.2 Continuity away from $\mathfrak{D}$

Instead of using the argument of [27] to show continuity in the case of diffuse reflection boundary condition we will use the sequence (3.221) with the boundary condition (2.120) and Lemma 11. This argument also gives a new proof of the continuity of Boltzmann solution in a strictly convex domain with simpler way than [27].

Proof of 2 of Theorem 3

We will use the sequence (3.221) with $h^{m+1}|_{t=0} = h_0$ with following boundary condition

$$h^{m+1}(t, x, v) = \frac{1}{\tilde{w}(v)} \int_{\mathcal{V}(x)} h^m(t, x, v') \tilde{w}(v') d\sigma \tag{2.120}$$

with $(t, x, v) \in \gamma_-$.

Step 1 : We claim that

$$\frac{1}{\tilde{w}(v)} \int_{\mathcal{V}(x)} f(t, x, v') \tilde{w}(v') d\sigma(v') \tag{2.121}$$

is continuous function on $[0, T] \times \gamma$ even if $f \in L^\infty([0, T] \times \bar{\Omega} \times \mathbb{R}^3)$ is only continuous on $[0, T] \times \bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}$. We will show as $(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)$,

$$\frac{1}{\tilde{w}(v)} \int_{\mathcal{V}(x)} h^m(t, x, v') \tilde{w}(v') d\sigma(v') \to \frac{1}{\tilde{w}(\bar{v})} \int_{\mathcal{V}(\bar{x})} h^m(\bar{t}, \bar{x}, v') \tilde{w}(v') d\sigma(v'). \tag{2.122}$$

Using the fact $|\mathcal{V}(x)\backslash\mathcal{V}(\bar{x})|, |\mathcal{V}(x)\backslash\mathcal{V}(\bar{x})| \to 0$ as $\bar{x} \to x$ and the exponentially decay weight function of $\tilde{w}d\sigma$ it suffices to show that

$$\int_{\mathcal{V}(x)\cap\mathcal{V}(\bar{x})\cap\{|v'|\leq M\}} \{\tilde{w}(v)^{-1}h^m(t,x,v')\tilde{w}(v') - \tilde{w}(\bar{v})^{-1}h^m(\bar{t},\bar{x},v')\tilde{w}(v')\}d\sigma(v') \tag{2.123}$$

for sufficiently large $M > 0$. Using Lemma 5 we can choose open set $U_x \subset \{v' \in \mathbb{R}^3 : |v'| \leq M\}$ so that $|U_x|$ is small and h^m is uniformly continuous on $\{|v'| \leq M\}\backslash U_x$. Therefore we can make $\int_{\mathcal{V}(x)\cap\mathcal{V}(\bar{x})\cap\{|v'|\leq M\}\cap U_x}$ small using the smallness of U_x and make $\int_{\mathcal{V}(x)\cap\mathcal{V}(\bar{x})\cap\{|v'|\leq M\}\backslash U_x}$ small using the uniformly continuity of h^m on $\{|v'| \leq M\}\backslash U_x$. Hence (2.122) is valid.

Step 2 : We claim

$$h^i \text{ is a continuous function in } \mathfrak{C}_T \tag{2.124}$$

for all $i \in \mathbb{N}$. By induction choose $h^0 = 0$ and (2.124) is satisfied for $i = 0$. Assume (2.124) for all $i = 0, 1, 2, ..., m$. Let $w\Gamma_-\left(\frac{h^m}{w}, \frac{h^{m+1}}{w}\right) = \nu\left(\frac{h^m}{w}\right)h^{m+1}$. Then the equation of h^{m+1} is

$$\{\partial_t + v\cdot\nabla_x + \nu(v) + \nu\left(\frac{h^m}{w}\right)\}h^{m+1} = K_w h^m + w\Gamma_+\left(\frac{h^m}{w}, \frac{h^m}{w}\right).$$

From Theorem 8 and Corollary 9 we know that $\nu\left(\frac{h^m}{w}\right)$ and $w\Gamma_+\left(\frac{h^m}{w}, \frac{h^m}{w}\right)$ are both continuous in $[0,T]\times\Omega\times\mathbb{R}^3$. Because of Step 1 we know that $\frac{1}{\tilde{w}(v)}\int_{\mathcal{V}(x)} h^m(t,x,v')\tilde{w}(v')d\sigma(v')$ is also continuous function on $[0,T]\times\gamma$. So we can apply Lemma 11 to conclude (2.124) is valid for $i = m+1$.

Step 3 : We claim h^m is a Cauchy sequence in $\mathfrak{C}_T$ for some small $T > 0$. First we will compute some constants explicitly. From (1.6) the normalized constant c_μ is $\left[\int_{n(x)\cdot v'>0} e^{-\frac{|v'|^2}{2}}\{n(x)\cdot v'\}dv'\right]^{-1}$. Choose $n(x) = (1,0,0)$ and then we can compute the right hand side of above term :

$$\int_0^\infty dv_1\ v_1 e^{-\frac{v_1^2}{2}} \int_{-\infty}^\infty dv_2\ e^{-\frac{v_2^2}{2}} \int_{-\infty}^\infty dv_3\ e^{-\frac{v_3^2}{2}} = \int_0^\infty \frac{d}{dv_1}\left(-e^{-\frac{v_1^2}{2}}\right)dv_1 \times (\sqrt{2\pi})^2 = 2\pi\left[-e^{-\frac{v_1^2}{2}}\right]_0^\infty = 2\pi.$$

Therefore we have $c_\mu = \frac{1}{2\pi}$. Next we want to estimate

$$\frac{1}{\tilde{w}(v)}\underbrace{\int_{v'\cdot n(x)>0}\tilde{w}(v')d\sigma(v')}_{\spadesuit} \leq \tilde{C}_\beta\rho^{2\beta-4} \tag{2.125}$$

where $\tilde{w}(v)^{-1} = (1+\rho^2|v|^2)^{\beta} e^{-\frac{|v|^2}{4}}$. We will follow the computation of Lemma 25 in [27]. For $\frac{1}{\tilde{w}(v)}$, in the case of $\beta\rho^2 > \frac{1}{4}$ we can see that $\tilde{w}(v)^{-1}$ has a maximum value at $|v| = \sqrt{\frac{4\beta\rho^2-1}{\rho^2}}$ which is

$$(1+\rho^2|v|^2)^{\beta} e^{-\frac{|v|^2}{4}}\Big|_{|v|=\sqrt{\frac{4\beta\rho^2-1}{\rho^2}}} = 4^{\beta}\beta^{\beta}e^{-\beta}e^{\frac{1}{4\rho^2}}\rho^{2\beta} \tag{2.126}$$

and

$$\begin{aligned}
\spadesuit &\leq \int_{v'\cdot n(x)>0} \tilde{w}(v')d\sigma(v') = \frac{1}{2\pi}\int_{v_1'>0} (1+\rho^2|v'|^2)^{-\beta} e^{\frac{|v'|^2}{4}} e^{-\frac{|v'|^2}{2}} v_1' dv' \\
&= \frac{1}{2\pi}\int_{u_1>0} (1+|u|^2)^{-\beta} e^{\frac{-2|u|^2}{4\rho}} \rho^{-4} u_1 du \leq \rho^{-4} \times \frac{1}{2\pi}\int_{u_1>0} \frac{1}{(1+|u|^2)^{\beta-\frac{1}{2}}} du \\
&= C_{\beta}\rho^{-4},
\end{aligned}$$

where $\beta \geq 2$ and combining with (2.126) we conclude (2.125).

First we will show a boundedness (2.90).

Lemma 12. *Let h^m be a solution of (3.221) with $h^{m+1}_{t=0} = h_0$ and the boundary condition (2.120). Then there exist $T_*, C, \delta > 0$ such that if $||h_0||_\infty < \delta$ then*

$$\sup_{0\leq s\leq T_*} ||h^m(s)||_\infty < C||h_0||_\infty \quad \textit{for all} \quad m \in \mathbb{N}.$$

Proof. We will use mathematical induction. Choose $h^0 = h_0$ and assume $||h_0||_\infty < \delta$ and

$$\sup_{0\leq s\leq T_*} ||h^i(s)||_\infty \leq C||h_0||_\infty \tag{2.127}$$

for $i = 0, 1, 2, ..., m$, where $\delta, C, T_* > 0$ will be determined later. From Lemma 24 of [27] the representation of h^{m+1} which is a solution of (3.221) with the boundary condition (2.120) is given by

$$h^{m+1}(t,x,v) = \mathbf{1}_{t_1\leq 0}(t,x,v)\Big\{ \underbrace{h_0(x-tv,v)e^{-\nu(v)t}}_{\text{[initial data]}} + \underbrace{\int_0^t e^{-\nu(v)(t-s)} q^m(s, x-(t-s)v, v) ds}_{\mathbf{I}} \Big\} \tag{2.128}$$

$$+\mathbf{1}_{0<t_1}(t,x,v)\Big\{ \underbrace{\int_{t_1}^t e^{-\nu(v)(t-s)} q^m(s, x-(t-s)v, v) ds}_{\mathbf{II}} + \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}(v)} \int_{\prod_{j=1}^k \mathcal{V}_j} H \Big\} \tag{2.129}$$

where q^m was defined (2.91) and

$$H = \sum_{l=1}^{k} \underbrace{\mathbf{1}_{t_{l+1}\le 0<t_l} h_0(x_l - t_l v_l, v_l)}_{\text{[initial data]}} d\Sigma_l(0) \tag{2.130}$$

$$+ \underbrace{\sum_{l=1}^{k} \int_0^{t_l} \mathbf{1}_{t_{l+1}\le 0<t_l} q^{m-l}(s, x_l - (t_l - s)v_l, v_l) d\Sigma_l(s) ds}_{\mathbf{III}} \tag{2.131}$$

$$+ \underbrace{\sum_{l=1}^{k} \int_{t_{l+1}}^{t_l} \mathbf{1}_{0<t_{l+1}} q^{m-l}(s, x_l - (t_l - s)v_l, v_l) d\Sigma_l(s) ds}_{\mathbf{IV}} \tag{2.132}$$

$$+ \underbrace{\mathbf{1}_{0<t_{k+1}} h^{m-k+1}(t_{k+1}, x_{k+1}, v_k) d\Sigma_k(t_{k+1})}_{\text{[many bounces]}}. \tag{2.133}$$

Here $d\Sigma_k(t_{k+1})$ is evaluated at $s = t_{k+1}$ of

$$d\Sigma_l(s) = \{\Pi_{j=l+1}^{k} d\sigma_j\}\{e^{-\nu(v_l)(t_l - s)} \tilde{w}(v_l) d\sigma_l\} \Pi_{j=1}^{l-1}\{e^{-\nu(v_j)(t_j - t_{j+1})} d\sigma_j\}.$$

First we can estimate [initial data] in (2.128) and (2.131)

$$\begin{aligned}
&\int_{\Pi_{j=1}^{k} \mathcal{V}_j} \left\{ \mathbf{1}_{t_1\le 0} |h_0(x - tv, v)| + \frac{1}{\tilde{w}(v)} \sum_{l=1}^{k} \mathbf{1}_{t_{l+1}\le 0<t_l} |h_0(x_l - t_l v_l, v_l)| \tilde{w}(v_l) \right\} d\sigma_1 ... d\sigma_k \\
\le\ & \max\left\{1, \frac{1}{\tilde{w}(v)} \max_{1\le l\le k} \int_{\prod_{j=1}^{k} \mathcal{V}_j} \tilde{w}(v_l) d\sigma_1 ... d\sigma_k \right\} ||h_0||_\infty \\
\le\ & \left\{1 + \tilde{C}_\beta \rho^{2\beta-4}\right\} ||h_0||_\infty
\end{aligned}$$

where we used (2.125).

Next we estimate [many bounces] term in (2.133) which is crucial estimate in this proof. We use Lemma 38 in [27] to bound a contribution of [many bounces] term in (2.133) in the last term of (2.129) by

$$\begin{aligned}
&\frac{1}{\tilde{w}(v)} \int_{\prod_{j=1}^{k} \mathcal{V}_j} \mathbf{1}_{\{t_{k+1}(t,x,v,v_1,v_2,...,v_k)>0\}} \tilde{w}(v_k) d\sigma_k d\sigma_{k-1} ... d\sigma_1 \times \sup_{0\le s\le t} ||h^{m-k+1}(s)||_\infty \\
\le\ & \frac{1}{\tilde{w}(v)} \int_{\mathcal{V}_k} \tilde{w}(v_k) d\sigma_k \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k(t,x,v,v_1,...,v_{k-1})>0\}} d\sigma_{k-1} ... d\sigma_1 \times \sup_{0\le s\le t} ||h^{m-k+1}(s)||_\infty \\
\le\ & \tilde{C}_\beta \rho^{2\beta-4} \left\{\frac{1}{2}\right\}^{C_2 \rho^{5/4}} \sup_{0\le s\le t} ||h^{m-k+1}(s)||_\infty \le \tilde{C}_\beta \rho^{2\beta-4} \left\{\frac{1}{2}\right\}^{C_2 \rho^{5/4}} C ||h_0||_\infty
\end{aligned}$$

where we used (2.125) at the last step. The remainders **I**, **II**, **III** and **IV** are contributions of

$q^m, ..., q^{m-k}$. We introduce a notation

$$\begin{aligned}\mathcal{H}_i &\equiv tC_{\mathbf{k}} \sup_{0\le s\le t} ||h^i(s)||_\infty \\ &+C_\Gamma \sup_{0\le s\le t} ||h^i(s)||_\infty \left(\sup_{0\le s\le t} ||h^i(s)||_\infty + \sup_{0\le s\le t} ||h^{i+1}(s)||_\infty \right) \\ &\le C||h_0||_\infty (C_{\mathbf{k}} T_* + 2CC_\Gamma ||h_0||_\infty) \end{aligned} \tag{2.134}$$

where the above inequality holds for $0 \le t \le T_*$ and $i = 0, 1, 2, ..., m-1$ and

$$\mathcal{H}_m \le (T_* C_{\mathbf{k}} + C_\Gamma C ||h_0||_\infty) C ||h_0||_\infty + C_\Gamma C ||h_0||_\infty \sup_{0\le s\le T_*} ||h^{m+1}(s)||_\infty \tag{2.135}$$

where we used the induction hypothesis (2.127) for (2.134) and (2.135). Easily we have

$$\begin{aligned}\mathbf{I}, \mathbf{II} &\le \mathcal{H}_m \\ \mathbf{III}, \mathbf{IV} &\le \sum_{l=1}^k \frac{1}{\tilde{w}(v)} \int_{\mathcal{V}_1} d\sigma_1 .. \int_{\mathcal{V}_{l-1}} d\sigma_{l-1} \int_{\mathcal{V}_{l+1}} d\sigma_{l+1} .. \int_{\mathcal{V}_k} d\sigma_k \int_{\mathcal{V}_l} \int_0^{t_l} \mathcal{H}_{m-l} e^{-\nu(v_l)(t_l-s)} \tilde{w}(v_l) ds d\sigma_l \\ &\le \sum_{l=1}^k \mathcal{H}_{m-l} \frac{1}{\tilde{w}(v)} \int_{\mathcal{V}_l} \tilde{w}(v_l) d\sigma_l \le \tilde{C}_\beta \rho^{2\beta-4} \sum_{l=1}^k \mathcal{H}_{m-l}.\end{aligned}$$

To summarize, we can estimate all terms of representation of $h^{m+1}(t,x,v)$ in (2.128) to obtain

$$\begin{aligned}&|h^{m+1}(t,x,v)| \\ &\le ||h_0||_\infty \Big\{ C\Big[2T_* C_{\mathbf{k}} + 2C_\Gamma C ||h_0||_\infty + k\tilde{C}_\beta \rho^{2\beta-4} (C_{\mathbf{k}} T_* + 2CC_\Gamma ||h_0||_\infty) + \tilde{C}_\beta \rho^{2\beta-4} \left\{ \frac{1}{2} \right\}^{C_2 \rho^{5/4}} \Big] \\ &\qquad +1 + \tilde{C}_\beta \rho^{2\beta-4} \Big\} + C_\Gamma C ||h_0||_\infty \sup_{0\le s\le T_*} ||h^{m+1}(s)||_\infty.\end{aligned}$$

Choose $k = \rho^{5/4}$. Choose $\rho > 0$ sufficiently large so that $\tilde{C}_\beta \rho^{2\beta-4} \left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}} \le \frac{1}{30}$ and then choose $T_* > 0$ sufficiently small so that $T_* \times C_\Gamma (1 + \tilde{C}_\beta \rho^{5/4} \rho^{2\beta-4}) \le \frac{1}{30}$ and then choose $C > 0$ sufficiently large $C > 10(1 + \tilde{C}_\beta \rho^{2\beta-4})$ and choose $\delta = \min\left\{ \frac{1}{20C_\Gamma C} \ , \ \frac{1}{30C_\Gamma} \left(C\tilde{C}_\beta \rho^{5/4} \rho^{2\beta-4} \right)^{-1} \right\}$. Finally assume

$||h_0||_\infty \leq \delta$. Then we have

$$
\begin{aligned}
&\sup_{0\leq s\leq T_*} ||h^{m+1}(s)||_\infty \\
\leq\ & \frac{1}{1-C_\Gamma C||h_0||_\infty}||h_0||_\infty\Big\{1+\tilde{C}_\beta\rho^{2\beta-4}+C\Big[\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}+tC_\Gamma(1+\tilde{C}_\beta\rho^{5/4}\rho^{2\beta-4}) \\
&\qquad +C_\Gamma C||h_0||_\infty+2C_\Gamma\tilde{C}_\beta\rho^{5/4}\rho^{2\beta-4}C||h_0||_\infty\Big]\Big\} \\
\leq\ & \frac{20}{19}||h_0||_\infty\left\{\frac{C}{10}+C\Big[\frac{1}{30}+\frac{1}{30}+\frac{1}{20}+\frac{1}{15}\Big]\right\}\leq C||h_0||_\infty.
\end{aligned}
$$

□

Next we will show that h^m is a Cauchy sequence in L^∞.

Lemma 13. *Let h^m be a solution of (3.221) with $h^{m+1}|_{t=0}=h_0$ and the boundary condition (2.120). Then there exist $T_*, C, \delta > 0$ so that if $||h_0||_\infty < \delta$ then h^m is Cauchy in $L^\infty([0,T_*]\times\bar{\Omega}\times\mathbb{R}^3)$.*

Proof. The equation of $h^{m+1}-h^m$ is

$$
\begin{aligned}
\{\partial_t+v\cdot\nabla_x+\nu\}(h^{m+1}-h^m) &= \tilde{q}^m \\
\text{with}\quad \{h^{m+1}-h^m\}|_{t=0}=0 \quad , \quad \{h^{m+1}-h^m\}|_{\gamma_-} &= \frac{1}{\tilde{w}(v)}\int_{(x)}\{h^m(t,x,v')-h^{m-1}(t,x,v')\}\tilde{w}(v')d\sigma(v')
\end{aligned}
$$

where $\tilde{q}^m$ is defined at (2.93). From Lemma 24 of [27] we have the representation

$$
\begin{aligned}
\{h^{m+1}-h^m\}(t,x,v) &= \mathbf{1}_{t_1\leq 0}(t,x,v)\underbrace{\int_0^t e^{-\nu(v)(t-s)}\tilde{q}^m(s,x-(t-s)v,v)ds}_{\tilde{\mathbf{I}}} \\
&\quad +\mathbf{1}_{0<t_1}(t,x,v)\Big\{\underbrace{\int_{t_1}^t e^{-\nu(v)(t-s)}\tilde{q}^m(s,x-(t-s)v,v)ds}_{\tilde{\mathbf{II}}}+\frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}(v)}\int_{\prod_{j=1}^k\mathcal{V}_j}\tilde{H}\Big\}
\end{aligned}
\tag{2.136}
$$

where

$$
\begin{aligned}
\tilde{H} &= \underbrace{\sum_{l=1}^{k}\int_0^{t_l}\mathbf{1}_{t_{l+1}\le 0<t_l}\tilde{q}^{m-l}(s,x_l-(t_l-s)v_l,v_l)d\Sigma_l(s)ds}_{\tilde{\mathbf{III}}} \\
&\quad+\underbrace{\sum_{l=1}^{k}\int_{t_{l+1}}^{t_l}\mathbf{1}_{0<t_{l+1}}\tilde{q}^{m-l}(s,x_l-(t_l-s)v_l,v_l)d\Sigma_l(s)ds}_{\tilde{\mathbf{IV}}} \\
&\quad+\underbrace{\mathbf{1}_{0<t_{k+1}}\{h^{m-k+1}-h^{m-k}\}(t_{k+1},x_{k+1},v_k)d\Sigma_k(t_{k+1})}_{[[\text{many bounces}]]}.
\end{aligned}
$$

First using Lemma 24 of [27], we estimate [[many bounces]] term for sufficiently large $k>0$ by

$$
\begin{aligned}
&\frac{1}{\tilde{w}(v)}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}(t,x,v,v_1,v_2,\ldots,v_k)>0\}}\tilde{w}(v_k)d\sigma_k d\sigma_{k-1}\ldots d\sigma_1\times\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty \\
\le\;&\frac{1}{\tilde{w}(v)}\int_{\mathcal{V}_k}\tilde{w}(v_k)d\sigma_k\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k(t,x,v,v_1,\ldots,v_{k-1})>0\}}d\sigma_{k-1}\ldots d\sigma_1\times\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty \\
\le\;&\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty.
\end{aligned}
$$

Easily we have $\tilde{\mathbf{I}},\ \tilde{\mathbf{II}}\ \le\ \delta\mathcal{H}_m,\quad \tilde{\mathbf{III}},\ \tilde{\mathbf{IV}}\ \le\ \tilde{C}_\beta\rho^{2\beta-4}\delta\mathcal{H}_{m-l}$ where

$$
\begin{aligned}
\delta\mathcal{H}_i &\equiv tC_{\mathbf{k}}\sup_{0\le s\le t}||\{h^i-h^{i-1}\}(s)||_\infty+C||h_0||_\infty C_\Gamma\big(\sup_{0\le s\le t}||\{h^i-h^{i-1}\}(s)||_\infty+\sup_{0\le s\le t}||\{h^{i+1}-h^i\}(s)||_\infty\big) \\
&\le \frac{\tau}{4}\Big\{\sup_{0\le s\le t}||\{h^i-h^{i-1}\}(s)||_\infty \\
&\quad+\sup_{0\le s\le t}||\{h^{i+1}-h^i\}(s)||_\infty\Big\}
\end{aligned}
$$

with $\tau=4\max\{tC_{\mathbf{k}},C||h_0||_\infty C_\Gamma\}$.

To summarize, we can estimate all terms of representation of $h^{m+1}(t,x,v)-h^m(t,x,v)$ in (2.136) for any $m>k$ to obtain

$$
\begin{aligned}
&\sup_{0\le s\le t}||\{h^{m+1}-h^m\}(s)||_\infty \\
\le\;&\frac{1}{1-2\tau}\Big\{\frac{\tau}{2}\tilde{C}_\beta\rho^{2\beta-4}\sum_{l=1}^{k}\Big(\sup_{0\le s\le t}||\{h^{m-l}-h^{m-l-1}\}(s)||_\infty+\sup_{0\le s\le t}||\{h^{m-l+1}-h^{m-l}\}(s)||_\infty\Big) \\
&+\frac{\tau}{2}\sup_{0\le s\le t}||\{h^m-h^{m-1}\}(s)||_\infty+\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty\Big\}
\end{aligned}
$$

which is our starting point. Fix a small number $\tilde{\tau}>0$ chosen later. Choose $\rho>0$ sufficiently large so that $2\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}<\frac{\tilde{\tau}}{4}$ and then choose $\tau>0$ so small that $\frac{\tau/2}{1-2\tau}\tilde{C}_\beta\rho^{2\beta-4}<\frac{\tilde{\tau}}{4}$ and

$\frac{\tau/2}{1-2\tau} < \frac{\tilde{\tau}}{4}$. Then we have

$$\sup_{0\le s\le t} ||\{h^{m+1}-h^m\}(s)||_\infty \le \tilde{\tau}\left\{\sup_{0\le s\le t} ||\{h^m - h^{m-1}\}(s)||_\infty + ... + \sup_{0\le s\le t} ||\{h^{m-k+1} - h^{m-k}\}(s)||_\infty\right\} \tag{2.137}$$

Using (2.137) for $m, j \in \mathbb{N}$ so that $m-(i+1)k > 0$ and $j = 0, 1, ..., m-1$ it is easy to show

$$\sup_{0\le s\le t} ||\{h^{m-ik+1+j} - h^{m-ik+j}\}(s)||_\infty \le$$
$$\tilde{\tau}(1+\tilde{\tau})^j\left\{\sup_{0\le s\le t} ||\{h^{m-ik} - h^{m-ik-1}\}(s)||_\infty + ... + \sup_{0\le s\le t} ||\{h^{m-(i+1)k+1} - h^{m-(i+1)k}\}(s)||_\infty\right\}$$

We apply the above inequality term by term in (2.137) to have

$$\begin{aligned}
&\sup_{0\le s\le t} ||\{h^{m+1} - h^m\}(s)||_\infty \\
&\le \tilde{\tau}\{(1+\tilde{\tau})^k - 1\}\{\sup_{0\le s\le t} ||\{h^{m-k} - h^{m-k-1}\}(s)||_\infty + ... + \sup_{0\le s\le t} ||\{h^{m-2k+1} - h^{m-2k}\}(s)||_\infty\} \\
&\le \tilde{\tau}\{(1+\tilde{\tau})^k - 1\}^i\{\sup_{0\le s\le t} ||\{h^{m-ik} - h^{m-ik-1}\}(s)||_\infty + ... + \sup_{0\le s\le t} ||\{h^{m-(i+1)k+1} - h^{m-(i+1)k}\}(s)||_\infty\}.
\end{aligned}$$

Now we estimate

$$\begin{aligned}
&\sup_{0\le s\le t} ||\{h^m - h^n\}(s)||_\infty \le \sum_{l=0}^{m-n-1} \sup_{0\le s\le t} ||\{h^{m-l} - h^{m-l-1}\}(s)||_\infty \\
\le\ & \sum_{l=0}^{m-n-1} \tilde{\tau}\{(1+\tilde{\tau})^k - 1\}^i\{\sup_{0\le s\le t} ||h^{m-ik-l-1} - h^{m-ik-l-2}||_\infty + ... \\
&+ \sup_{0\le s\le t} ||h^{m-(i+1)k-l} - h^{m-(i+1)k-l-1}||_\infty\} \\
\le\ & \sum_{l=0}^{m-n-1} \tilde{\tau}\{(1+\tilde{\tau})^k - 1\}^{[\frac{m-l-1}{k}]-1}\{\sup_{0\le s\le t} ||h^{2k} - h^{2k-1}||_\infty + ... + \sup_{0\le s\le t} ||h^1 - h^0||_\infty\} \\
\le\ & \tilde{\tau}\{(1+\tilde{\tau})^k - 1\}^{[\frac{n}{k}]-1} \sum_{l=0}^{m-n-1} \{(1+\tilde{\tau})^k - 1\}^{[\frac{m-l-1}{k}]-[\frac{n}{k}]}\{\sup_{0\le s\le t} ||h^{2k} - h^{2k-1}||_\infty + ... + \sup_{0\le s\le t} ||h^1 - h^0||_\infty\} \\
\le\ & \tilde{\tau}\{(1+\tilde{\tau})^k - 1\}^{[\frac{n}{k}]-1} \frac{1}{2-(1+\tilde{\tau})^k}\{\sup_{0\le s\le t} ||h^{2k} - h^{2k-1}||_\infty + ... + \sup_{0\le s\le t} ||h^1 - h^0||_\infty\}
\end{aligned}$$

where we choose $i = \left[\frac{m-l-1}{k}\right] - 1$ so that $m-(i+1)k-l-1 \in [0, k)$. If $\tilde{\tau} > 0$ is chosen sufficiently small so that $(1+\tilde{\tau})^k - 1 \le \frac{1}{2}$ then $\{(1+\tilde{\tau})^k - 1\}^{[\frac{n}{k}]-1} \to 0$ as $n \to \infty$ which implies that

$$\sup_{0\le s\le t} ||\{h^m - h^n\}(s)||_\infty \to 0 \tag{2.138}$$

as $m, n \to \infty$. Thus h^m is Cauchy in L^∞. □

Step 4 : We claim that h is continuous in $\mathfrak{C}$. Notice that T only depends on $||h_0||_\infty$ and $\sup_{0\leq s\leq T}||wg(s)||_\infty$ (Theorem 1 of [27]). Using unform bound of $\sup_{0\leq s<\infty}||h(s)||_\infty$ we can obtain the continuity for h for all time by repeating $[0,T],[T,2T],....$

If the boundary $\partial\Omega$ does not include a line segment (6) then every step is valid with $[0,\infty)\times\{\bar{\Omega}\times\mathbb{R}^3\}\backslash\{\mathfrak{D}\cup\gamma_0^V\}$ instead of $\mathfrak{C}$ and $[0,T]\times\{\bar{\Omega}\times\mathbb{R}^3\}\backslash\{\mathfrak{D}\cup\gamma_0^V\}$ instead of $\mathfrak{C}_T$

2.5.3 Propagation of Discontinuity

Proof of 2 of Theorem 2

Proof of (1.21) : The proof is exactly same as in-flow case in Section 4.3.

Proof of (1.23) for a Local-in-time Formation of Discontinuity The proof is exactly same as the proof of in-flow case in Section 4.3 except **Step 2**. As we mentioned in Remark of **Step 2**, we need to show a continuity of a boundary datum on $\gamma_-\cup\gamma_0^{\mathbf{S}}$. In diffuse reflection boundary condition case, we need

$$\begin{aligned}0=[\,h|_{[0,\infty)\times\gamma_-}]_{t,y,v}&=\lim_{\delta\downarrow 0}\sup_{\substack{t',\,t''\,\in\,B(t;\delta)\\(y',v'),(y'',v'')\,\in\,\gamma_-\,\cap\,B((y,v);\delta)\backslash(y,v)}}|h(t',y',v')-h(t'',y'',v'')|\\&=\lim_{\delta\downarrow 0}\sup_{\substack{t',\,t''\,\in\,B(t;\delta)\\(y',v'),(y'',v'')\,\in\,\gamma_-\,\cap\,B((y,v);\delta)\backslash(y,v)}}\Big|\frac{1}{\tilde{w}(v')}\int_{\mathcal{V}(y')}h(t',y',\mathfrak{v})\tilde{w}(\mathfrak{v})d\sigma(\mathfrak{v})-\\&\qquad\frac{1}{\tilde{w}(v'')}\int_{\mathcal{V}(y'')}h(t'',y'',\mathfrak{v})\tilde{w}(\mathfrak{v})d\sigma(\mathfrak{v})\Big|\end{aligned}$$

for $(y,v)\in\gamma_-\cup\gamma_0^{\mathbf{S}}$. This is already proven in section 5.2 Continuity away from $\mathfrak{D}$.

2.6 Bounce-Back Boundary Condition

2.6.1 Formation of Discontinuity

We will prove part 3 of Theorem 1. Without loss of generality we may assume $x=(0,0,0)$ and $v=(1,0,0)$ and $(x,v)\in\gamma_0^{\mathbf{S}}$. Locally the boundary is a graph i.e. $\Omega\cap B(\mathbf{0};\delta)=\{(x_1,x_2,x_3)\in$

$B(\mathbf{0};\delta) : x_3 > \Phi(x_1,x_2)\}$. The condition $(x,v) \in \gamma_0^{\mathbf{S}}$ implies $t_{\mathbf{b}}(x,v) \neq 0$ and $t_{\mathbf{b}}(x,-v) \neq 0$ which means $\Phi(\xi,0) < 0$ for $\xi \in (-\delta,\delta)$.

Assume $||h_0||_\infty < \delta$ so that the global-in-time solution $h(t,x,v)$ has uniform bound (2.78).

Recall the constants $C_{\mathbf{k}}$ and C_Γ from (3.174) and (2.16). Choose $t_* \in (0,\frac{\delta}{2})$ sufficiently small so that

$$\frac{1}{2} \leq \left(e^{-\nu(1)t_*} - t_* C_{\mathbf{k}} C' - (1-e^{-\nu(1)t_*}) C_\Gamma (C')^2\right). \tag{2.139}$$

Assume a condition for our initial data h_0 : there is sufficiently small $\delta' = \delta'(\Omega, t_*) > 0$ such that $B((-t_*,0,0),\delta'),\ B((t_*,0,0),\delta') \subset \Omega$ and

$$\begin{aligned} h_0(x,v) &\equiv ||h_0||_\infty > 0 \quad \text{for} \ \ (x,v) \in B((-t_*,0,0);\delta') \times B((1,0,0);\delta'), \\ h_0(x,v) &\equiv -||h_0||_\infty > 0 \quad \text{for} \ \ (x,v) \in B((t_*,0,0);\delta') \times B((-1,0,0);\delta'). \end{aligned}$$

We will use a contradiction argument : assume (2.81). Choose sequences of points $(x'_n, v'_n) = ((0,0,\frac{1}{n}),(1,0,0))$ and $(x_n,v_n) = ((\frac{1}{n},0,\Phi(\frac{1}{n},0)), \frac{1}{\sqrt{1+\frac{1}{n^2}}}(1,0,\frac{1}{n}))$. Because of our choice, for sufficiently large $n \in \mathbb{N}$ we have

$$\begin{aligned} (x'_n - t_* v'_n, v'_n) &= ((-t_*,0,\frac{1}{n}),(1,0,0)) \in B((-t_*,0,0);\delta') \times B((1,0,0);\delta') \\ (x_n - t_*(-v_n), -v_n) &= ((\frac{1}{n} + \frac{t_*}{\sqrt{1+1/n^2}}, 0, \Phi(\frac{1}{n},0) + \frac{t_*}{n\sqrt{1+1/n^2}}), \frac{1}{\sqrt{1+1/n^2}}(-1,0,-\frac{1}{n})) \\ &\in B((t_*,0,0);\delta') \times B((-1,0,0);\delta') \end{aligned}$$

Using (2.78) we have

$$\begin{aligned} h(t_*, x'_n, v'_n) &\geq ||h_0||_\infty e^{-\nu(1)t_*} - t_* C_{\mathbf{k}} C' ||h_0||_\infty - (1-e^{-\nu(1)t_*}) C_\Gamma (C')^2 ||h_0||_\infty^2, \\ h(t_*, x_n, v_n) &\leq -||h_0||_\infty e^{-\nu(1)t_*} + t_* C_{\mathbf{k}} C' ||h_0||_\infty + (1-e^{-\nu(1)t_*}) C_\Gamma (C')^2 ||h_0||_\infty^2 \end{aligned}$$

where the second inequality is came from $h(t_*,x_n,v_n) = h(t_*,x_n,-v_n) =$

$$h_0(x_n - t_*(-v_n), -v_n) e^{-\nu(-v_n)t_*} + \int_0^{t_*} e^{-\nu(-v_n)(t_*-\tau)} \{K_w h + w\Gamma\left(\frac{h}{w},\frac{h}{w}\right)\}(\tau, x_n - (-v_n)(t_*-\tau), -v_n) d\tau$$

Therefore using (2.139) $h(t_*, x'_n, v'_n) - h(t_*, x_n, v_n)$

$$\geq \ 2||h_0||_\infty \left(e^{-\nu(1)t_*} - t_* C_{\mathbf{k}} C' - (1-e^{-\nu(1)t_*}) C_\Gamma (C')^2\right) \geq ||h_0||_\infty \neq 0,$$

which is contradiction to (2.81).

2.6.2 Continuity away from $\mathfrak{D}_{bb}$

We recall some basic facts to study the bounce-back boundary condition from [27].

Definition 9. *[27] (**Bounce-Back Cycles**) Let $(t,x,v) \notin \gamma_0 \cup \gamma_-$. Let $(t_0,x_0,v_0) = (t,x,v)$ and inductively define for $k \geq 1$:*

$$(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(x_k,v_k), x_{\mathbf{b}}(x_k,v_k), -v_k).$$

We define the back-time cycles as:

$$X_{\mathbf{cl}}(s;t,x,v) = \sum_k \mathbf{1}_{[t_{k+1},t_k)}(s)\{x_k + (s-t_k)v_k\}, \quad V_{\mathbf{cl}}(s;t,x,v) = \sum_k \mathbf{1}_{[t_{k+1},t_k)}(s)v_k. \tag{2.140}$$

Clearly, we have $v_{k+1} \equiv (-1)^{k+1}v$, for $k \geq 1$,

$$x_k = \frac{1-(-1)^k}{2}x_1 + \frac{1+(-1)^k}{2}x_2, \tag{2.141}$$

where $x_1 = x - t_{\mathbf{b}}(x,v)v$ and $x_2 = x - [2t_{\mathbf{b}}(x,v) + t_{\mathbf{b}}(x,-v)](-v)$ and let $d = t_1 - t_2$, then $t_k - t_{k+1} = d \geq t_{\mathbf{b}}(t,x,v) > 0$ for $k \geq 1$, and

$$\begin{aligned}
t_1(t,x,v) &= t - t_{\mathbf{b}}(x,v) \ , \\
t_2(t,x,v) &= t_1 - t_{\mathbf{b}}(x_1,v_1) = t_1 - (t_{\mathbf{b}}(x,v) + t_{\mathbf{b}}(x_1,v_1)) = t_1 - (2t_{\mathbf{b}}(x,v) + t_{\mathbf{b}}(x,-v)) \ , \\
&\vdots \\
t_{k+1}(t,x,v) &= t_1 - k(2t_{\mathbf{b}}(x,v) + t_{\mathbf{b}}(x,-v)).
\end{aligned} \tag{2.142}$$

Lemma 14. *[27] Let $h_0 \in L^\infty(\Omega \times \mathbf{R}^3)$ and $\phi(t,x,v)$ with $\sup_{[0,T]\times\Omega} |\phi(\cdot,\cdot,v)| < \infty$. There exists a unique solution $G(t)h_0$ of*

$$\{\partial_t + v \cdot \nabla_x + \phi\}\{G(t)h_0\} = 0, \qquad \{G(0)h_0\} = h_0,$$

with the bounce-back reflection $\{G(t)h_0\}(t,x,v) = \{G(t)h_0\}(t,x,-v)$ for $x \in \partial\Omega$. For almost any

$(x,v) \in \bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$,

$$\{G(t)h_0\}(t,x,v) = \sum_k \mathbf{1}_{[t_{k+1},t_k)}(0) h_0\left(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)\right) e^{-\int_0^t \phi(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau)) d\tau} \tag{2.143}$$

where $X_{\mathbf{cl}}(\tau) = X_{\mathbf{cl}}(\tau; t,x,v)$ *and* $V_{\mathbf{cl}}(\tau) = V_{\mathbf{cl}}(\tau; t,x,v)$ *in (2.140).*

Next we prove a generalized version of Lemma 16 in [27].

Lemma 15 (Continuity away from $\mathfrak{D}_{bb}$: Transport Equation)**.** *Let* Ω *be an open subset of* $\mathbb{R}^3$ *with a smooth boundary* $\partial\Omega$ *and an initial datum* $h_0(x,v)$ *be continuous in* $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_+ \cup \gamma_0^I\}$*. Also assume* $q(t,x,v)$ *and* $\phi(t,x,v)$ *be continuous in the interior of* $[0,T] \times \Omega \times \mathbb{R}^3$ *and* $\sup_{[0,T]\times\Omega\times\mathbb{R}^3} |q(t,x,v)| < \infty$ *and* $\sup_{[0,T]\times\Omega} |\phi(\cdot,\cdot,v)| < \infty$ *for all* $v \in \mathbb{R}^3$*. Let* $h(t,x,v)$ *be the solution of*

$$\{\partial_t + v\cdot\nabla_x + \phi\}h = q\ , \quad h(0,x,v) = h_0\ , \quad h|_{\gamma_-}(t,x,v) = h(t,x,-v).$$

Assume the compatibility condition on $\gamma_- \cup \gamma_0^{I-}$

$$h_0(x,v) = h_0(x,-v).$$

Then the Boltzmann solution $h(t,x,v)$ *is continuous on* $\mathfrak{C}_{bb}$*. Further, if the boundary* $\partial\Omega$ *does not include a line segment (6) then* $h(t,x,v)$ *is continuous on a complementary set of the discontinuity set i.e.* $[0,T] \times \{\bar{\Omega} \times \mathbb{R}^3\} \backslash \mathfrak{D}_{bb}$*.*

Proof. The proof is similar to the proof of Lemma 16 of [27]. Take any point $(t,x,v) \in [0,T] \times \bar{\Omega} \times \mathbb{R}^3$ and recall its back-time cycle and (2.143). Assume $t_{m+1} \le 0 < t_m$. Using (2.143), $h(t,x,v)$ takes the form

$$\begin{aligned}
&h_0(x_m - t_m v_m, v_m) e^{-\sum_{k=0}^{m-1} \int_{t_{k+1}}^{t_k} \phi(\tau, x_k - (t_k-\tau)v_k, v_k) d\tau - \int_0^{t_m} \phi(\tau, x_m - (t_m-\tau)v_m, v_m) d\tau} \\
&\quad + \sum_{k=0}^{m-1} \int_{t_{k+1}}^{t_k} q(s, x_k - (t_k - s)v_k, v_k) e^{-\sum_{i=0}^{k-1} \int_{t_{i+1}}^{t_i} \phi(\tau, x_i - (t_i - \tau)v_i, v_i) d\tau - \int_s^{t_k} \phi(\tau, x_k - (t_k-\tau)v_k, v_k) d\tau} \\
&\quad + \int_0^{t_m} q(s, x_m - (t_m - s)v_m, v_m) e^{-\sum_{i=0}^{m-1} \int_{t_{i+1}}^{t_i} \phi(\tau, x_i - (t_i-\tau)v_i, v_i) d\tau - \int_s^{t_m} \phi(\tau, x_m - (t_m - \tau)v_m, v_m) d\tau}
\end{aligned} \tag{2.144}$$

Take any point $(t,x,v) \in \mathfrak{C}_{bb}$. By the definition of $\mathfrak{C}_{bb}$ we assume that $(x,v) \in \Omega \times \mathbb{R}^3$ or

$(x,v) \in \gamma_- \cup \gamma_0^{I-}$ and we can separate three cases : $t - t_{\mathbf{b}}(x,v) < 0$, $(x_{\mathbf{b}}(x,v),v) \in \gamma_- \cup \gamma_0^{I-}$ with $t < 2t_{\mathbf{b}}(x,v) + t_{\mathbf{b}}(x,-v)$, and $(x_{\mathbf{b}}(x,-v),-v) \in \gamma_- \cup \gamma_0^{I-}$ with $(x_{\mathbf{b}}(x,v),v) \in \gamma_- \cup \gamma_0^{I-}$.

Case of $t < t_{\mathbf{b}}(x,v)$ Simply we have $h(t,x,v) = h_0(x - tv, v)e^{-\int_0^t \phi(\tau, x-(t-\tau)v, v)d\tau} + \int_0^t q(s, x - (t-s)v, v)e^{\int_s^t \phi(\tau,x-(t-\tau)v,v)d\tau} ds$ and use the continuity of $q(t,x,v)$ and $\phi(t,x,v)$ to conclude the continuity of $h(t,x,v)$.

Case of $(x_{\mathbf{b}}(x,v),v) \in \gamma_- \cup \gamma_0^{I-}$ with $t < 2t_{\mathbf{b}}(x,v) + t_{\mathbf{b}}(x,-v)$ A representation of $h(t,x,v)$ takes the form

$$\begin{aligned} h_0(x_1 - t_1 v_1, v_1)&e^{-\int_{t_1}^t \phi(\tau,x-(t-\tau)v,v)d\tau - \int_0^{t_1} \phi(\tau,x_1-(t_1-\tau)v_1,v_1)d\tau} \\ &+ \int_{t_1}^t q(s, x-(t-s)v, v)e^{-\int_s^t \phi(\tau,x-(t-\tau)v,v)d\tau} ds \\ + \int_0^{t_1} &q(s, x_1 - (t_1 - s)v_1, v_1)e^{-\int_{t_1}^t \phi(\tau,x-(t-\tau)v,v)d\tau - \int_s^{t_1} \phi(\tau,x_1-(t_1-\tau)v_1,v_1)d\tau} ds. \end{aligned}$$

Thanks to Lemma 24 and Lemma 2, the condition $(x_{\mathbf{b}}(x,v),v) \in \gamma_- \cup \gamma_0^{I-}$ implies continuity of $x_1(x,v) = x - x_{\mathbf{b}}(x,v)$, $t_1(t,x,v) = t - t_{\mathbf{b}}(x,v)$. Therefore we can show the continuity of $h(t,x,v)$.

Case of $(x_{\mathbf{b}}(x,-v),-v) \in \gamma_- \cup \gamma_0^{I-}$ with $(x_{\mathbf{b}}(x,v),v) \in \gamma_- \cup \gamma_0^{I-}$ We have (2.144) for $h(t,x,v)$. Thanks to (2.141) and (2.142) and Lemma 24 and Lemma 2, the conditions $(x_{\mathbf{b}}(x,-v),-v) \in \gamma_- \cup \gamma_0^{I-}$ and $(x_{\mathbf{b}}(x,v),v) \in \gamma_- \cup \gamma_0^{I-}$ imply continuity of $x_k(x,v), v_k(x,v), t_k(t,x,v)$. Therefore we can show the continuity of $h(t,x,v)$. □

Proof of Part 1 of Theorem 3

Following the in-flow and diffuse cases, we use the iteration scheme (3.221) which is equivalent to (2.89) with bounce-back boundary condition $h^{m+1}|_{\gamma_-}(t,x,v) = h^{m+1}(t,x,-v)$ and an initial condition $h^{m+1}|_{t=0} = h_0$.

Step 1 : We claim that h^i is a continuous function in $\mathfrak{C}_{bb,T}$ for all $i \in \mathbb{N}$ and for any $T > 0$ where $\mathfrak{C}_{bb,T} = \mathfrak{C}_T \cap \{[0,T] \times \bar{\Omega} \times \mathbb{R}^3\}$. Choose $h^0 \equiv 0$ and use mathematical induction. Assume h^i is continuous $\mathfrak{C}_{bb,T}$ for $i = 0,1,2,...,m$. Apply Lemma 15 to conclude that h^{m+1} is continuous in $\mathfrak{C}_{bb,T}$.

Step 2 : We claim that there exist $C > 0$ and $\delta > 0$ such that if $C||h_0||_\infty < \delta$ then there exists $T = T(C,\delta) > 0$ so that $\sup_{0\le s\le T} ||h^m(s)||_\infty \le C||h_0||_\infty$ and $\{h^m\}_{m=0}^\infty$ is Cauchy in $L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}^3)$.

Fisrt we will show the boundedness using mathematical induction. Assume $\sup_{0\le s\le T}||h^m(s)||_\infty \le C||h_0||_\infty$ where $T>0$ will be chosen later. Applying Lemma 14, ϕ and q correspond with ν and the right hand side of (3.221) respectively to have a representation of $h^{m+1}(t,x,v)$

$$h_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))e^{-\nu(v)t}+\int_0^t e^{-\nu(v)(t-s)}\{K_w h^m+w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right)-w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right)\}(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))ds$$

where $\left[X_{\mathbf{cl}(s)},V_{\mathbf{cl}}(s)\right]=\left[X_{\mathbf{cl}(s;t,x,v)},V_{\mathbf{cl}}(s;t,x,v)\right]$ is in (2.140). The above term is bounded by

$$||h_0||_\infty+tC_{\mathbf{k}}\sup_{0\le s\le t}||h^m(s)||_\infty+C_\Gamma\sup_{0\le s\le t}||h^m(s)||_\infty\sup_{0\le s\le t}(||h^m(s)||_\infty+||h^{m+1}(s)||_\infty)$$

where the constants are coming from basic estimates, Lemma 25. Choose $C>4$ and $\delta<\frac{1}{2C_\Gamma}$ and $T=\frac{C-3}{2C_{\mathbf{k}}C}$. Then we have $\sup_{0\le s\le T}||h^{m+1}(s)||_\infty\le C||h_0||_\infty$.

Next we will show $\{h^m\}_{m=0}^\infty$ is Cauchy in $L^\infty([0,T]\times\bar\Omega\times\mathbb{R}^3)$. Recall $\tilde q^m(t,x,v)$ from (2.93). The equation of $h^{m+1}-h^m$ is (2.92) with a zero initial condition $(h^{m+1}-h^m)|_{t=0}=0$ and a bounce-back boundary condition $(h^{m+1}-h^m)|_{\gamma_-}(t,x,v)=(h^{m+1}-h^m)(t,x,-v)$. Applying Lemma 14 to (2.92) we have

$$(h^{m+1}-h^m)(t,x,v)=\int_0^t e^{-\nu(v)(t-s)}\tilde q^m(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))ds.$$

Then we have exactly same estimates of in-flow case to conclude $\{h^m\}$ is Cauchy.

Step 3 : Same argument as in-flow case but substitute $\mathfrak{C}_{bb,T}$, $\mathfrak{C}_{bb}$, $\mathfrak{D}_{bb,T}$, $\mathfrak{D}_{bb}$ for $\mathfrak{C}_T$, $\mathfrak{C}$, $\mathfrak{D}_T$, $\mathfrak{D}$ respectively.

2.6.3 Propagation of Discontinuity

Proof of 2 of Theorem 2

Proof of (1.21) : The proof is exactly same as in-flow case in Section 4.3.

Proof of (1.23) for a Local-in-time Formation of Discontinuity Recall that we have $[h(t_0)]_{x_0,v_0}\neq 0$ for $(x_0,v_0)\in\gamma_0^{\mathbf{S}}$ and $t_0\in(0,\min\{t_{\mathbf{b}}(x_0,-v_0),t_{\mathbf{b}}(x_0,v_0)\})$. As diffuse boundary condition case, the proof is exactly same as the proof of in-flow case in Section 4.3 except **Step 2**. We need to show a continuity of a boundary datum on $\gamma_-\cup\gamma_0^{\mathbf{S}}$. In bounce-back reflection boundary condition case, we need to show

$$0 = [\ h|_{[0,\infty)\times\gamma_-}]_{t_0,x_0,v_0} = \lim_{\delta\downarrow 0} \sup_{\substack{t',t''\in B(t;\delta)\\ (y',v'),(y'',v'')\in\gamma_-\cap B((x_0,v_0);\delta)\backslash(x_0,v_0)}} |h(t',y',v') - h(t'',y'',v'')|$$

Because (y',v') is in the incoming boundary γ_-, using the bounce-back boundary condition, we have $h(t',y',v') = h(t',y',-v')$. Further due to the condition $0 < t_0 < t_{\mathbf{b}}(x_0,-v_0)$ we have $0 < t' < t_{\mathbf{b}}(y',-v')$ and

$$\begin{aligned} h(t',y',v') &= h(t',y',-v') = h_0(y'+t'v',v')e^{-\nu(v')t' - \int_0^{t'} \nu(\sqrt{\mu}\frac{h}{w})(\tau,y'+(t'-\tau)v',v')d\tau} \\ &+ \int_0^{t'} \{K_w h + w\Gamma_+(\frac{h}{w},\frac{h}{w})\}(s,y'+(t'-s)v',v')e^{-\nu(v')(t'-s) - \int_0^{t'} \nu(\sqrt{\mu}\frac{h}{w})(\tau,y'+(t'-\tau)v',v')d\tau} ds \end{aligned}$$

and similar representation for $h(t',y',v')$. Using the continuity of $\nu(\sqrt{\mu}\frac{h}{w})$, $K_w h$ and $w\Gamma_+(\frac{h}{w},\frac{h}{w})$ we have

$$\begin{aligned} [\ h|_{[0,\infty)\times\gamma_-}]_{t_0,x_0,v_0} &= \lim_{\delta\downarrow 0} \sup_{\substack{t',t''\in B(t;\delta)\\ (y',v'),(y'',v'')\in\gamma_-\cap B((x_0,v_0);\delta)\backslash(x_0,v_0)}} |h_0(y'+t'v',v') - h_0(y''+t''v'',v'')| \\ &\quad \times e^{-\nu(v_0)t_0 - \int_0^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau,x_0+(t_0-\tau)v_0,v_0)d\tau} = 0 \end{aligned}$$

where we used the continuity of the initial datum h_0 in the last equality.

Chapter Three

Specular Reflection Boundary Condition for 2D Domains

Abstract We consider the initial-boundary value problem with specular reflection boundary condition in 2 dimensional smooth convex domain and analytic non-convex domain for the Boltzmann equation. In this paper the global existence and exponential decay in the L^∞ norm for cut-off hard potentials near an absolute Maxwellian are established.

3.1 Introduction

The two dimensional Boltzmann equation is

$$\partial_t F(t,x,v) + \underline{v} \cdot \nabla_x F(t,x,v) = Q(F,F)(t,x,v), \tag{3.1}$$

for $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2,\ v = (\underline{v}; v_3) = (v_1, v_2; v_3) \in \mathbb{R}^3$. The Boltzmann solution $F(t,x,v)$ is a distribution function for the gas particles at a time $t \geq 0$, a position $x \in \Omega \subset \mathbb{R}^2$ and a velocity $v \in \mathbb{R}^3$. Throughout this paper, the collision operator takes the form

$$\begin{aligned} Q(F_1, F_2) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega) F_1(u')F_2(v') d\omega du - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega) F_1(u)F_2(v) d\omega du \\ &\equiv Q_+(F_1,F_2) - Q_-(F_1,F_2), \end{aligned} \tag{3.2}$$

where $u' = u + [(v-u)\cdot\omega]\omega,\ v' = v - [(v-u)\cdot\omega]\omega$ and

$$B(v-u,\omega) = |v-u|^\gamma q_0\left(\frac{v-u}{|v-u|}\cdot\omega\right),$$

with $0 < \gamma \leq 1$ (hard potential) and

$$\int_{\mathbb{S}^2} q_0(\hat{u}\cdot\omega) d\omega < +\infty, \quad \text{(angular cutoff)} \tag{3.3}$$

for all $\hat{u} \in \mathbb{S}^2$.

If the gas is contained in a bounded region or flows past a solid bodies, the Boltzmann equation must be accompanied by boundary conditions, which describe the interaction of the gas molecules with the solid walls. Let the domain Ω be a smooth bounded domain. We consider the specular reflection, the most ideal boundary condition :

$$F(t,x,v)|_{\gamma_-} = F(t,x,R_x v) \tag{3.4}$$

where $R_x v = v - 2(n(x)\cdot v)n(x)$ with $n(x)$ standing the outward unit normal vector at $x \in \partial\Omega$.

The most ideal boundary condition, a wall is very clean and the gas molecule collide a wall elastically, may be the specular reflection boundary condition. In spite of the natural feature of a specular reflection boundary condition, the mathematical studies of kinetic equations with specular boundary condition in curved domain are quite rare. The reason is on the hardness to control the trajectories in the general domain. The Velocity Lemma which plays a very important role in the study of continuity for cycles in the specular case was first developed in [30][**?**], in the study of Hölder regularity of a Vlasov-Poisson (Maxwell) system with flat geometry. The machinery was generalized in [38] for the study of Hölder regularity of a Vlasov-Poisson system in a ball. Finally in [39] a regular solution of a Vlasov-Poisson system a general convex domain with specular boundary condition is constructed. Recently in [27] unified $L^2 - L^\infty$ theory in the near Maxwellian regime is developed to establish the existence, uniqueness and exponential decay toward a normalized Maxwellian $\mu = e^{-\frac{|v|^2}{2}}$, for basic types of the boundary conditions and rather general domains. However, for specular reflection boundary condition, the theorem is only established when the domain $\Omega \subset \mathbb{R}^3$ is analytic i.e. for each $x_0 \in \partial\Omega$, there exists $r > 0$ and a analytic function $\Phi_{x_0} : \mathbb{R}^2 \to \mathbb{R}$ such that - upon relabeling and reorienting the coordinates axes if necessary - we have

$$\Omega \cap B(x_0,r) = \{x \in B(x_0,r) : x_3 > \Phi_{x_0}(x_1,x_2)\},$$

and strictly convex

$$\partial_i\partial_j\Phi_{x_0}(x_1,x_2)\zeta_1\zeta_2 \geq C_\Omega|\zeta|^2,$$

for all $x \in \partial\Omega$ and all $\zeta \in \mathbb{R}^2$. The interesting open problem is removing either analyticity or convexity conditions on Ω.

The purpose of this article is to establish the well-posedness and exponential decay to Maxwellian for the nonlinear Boltzmann equation (3.1) with specular reflection boundary condition (3.4) in 2 dimensional smooth convex domain and 2 dimensional analytic general(non-convex) domain.

3.1.1 Domain and Characteristics

Throughout this paper, domains Ω are connected and bounded in $\mathbb{R}^2$. Moreover the boundary $\partial\Omega$ is locally an image of unit-speed curve i.e. for each $x_0 \in \partial\Omega$, there exists $r, \delta_1, \delta_2 > 0$ and a curve $\alpha : (\delta_1, \delta_2) \to \mathbb{R}^2$ such that $\dot{\alpha}(\tau) \equiv 1$ with $x_0 = \alpha(0)$ and

$$\Omega \cap B(x_0, r) = \{\alpha(\tau) \in \mathbb{R}^2 : \tau \in (\delta_1, \delta_2)\}. \tag{3.5}$$

Moreover we always assume that Ω is on the left side of α. For plane curve $\alpha(\tau) = (\alpha^1(\tau), \alpha^2(\tau)) \in \mathbb{R}^2$ we define the **signed curvature** of α at τ

$$\kappa(\tau) = \frac{\dot{\alpha}^1(\tau)\ddot{\alpha}^2(\tau) - \ddot{\alpha}^1(\tau)\dot{\alpha}^2(\tau)}{[\dot{\alpha}^1(\tau)^2 + \dot{\alpha}^2(\tau)^2]^{3/2}}. \tag{3.6}$$

We define two categories of domains in $\mathbb{R}^2$:

Definition 10. *If* $\Omega \subset \mathbb{R}^2$ *is an open connected bounded subset of* $\mathbb{R}^2$ *and the boundary* $\partial\Omega$ *is an image of unit-speed curve* α *with*

$$\kappa(\tau) > 0 \quad \textit{for all} \quad \tau, \tag{3.7}$$

then we say the domain Ω *is the* ***smooth convex domain****.*

Definition 11. *If* $\Omega \subset \mathbb{R}^3$ *is an open connected bounded such that*

$$\Omega \;\equiv\; \underline{\Omega} \times \mathbb{T}^1 \;, \tag{3.8}$$

where the periodic interval $\mathbb{T}^1 \equiv [0,1]/\sim$ *with* $0 \sim 1$*, and* $\underline{\Omega} \subset \mathbb{R}^2$ *such that there exists the family of open bounded simply connected subsets* $\underline{\Omega}_i \subset \mathbb{R}^2$ *for* $i = 0, 1, 2, ..., M < \infty$ *such that*

$$\underline{\Omega} = \underline{\Omega}_0 \backslash \{\underline{\Omega}_1 \cup \underline{\Omega}_2 \cup ... \cup \underline{\Omega}_M\}, \tag{3.9}$$

and satisfying

- $\underline{\Omega}_0 \supset cl(\underline{\Omega}_i)$ *for all* $i = 1, 2, ..., M$ *and* $cl(\underline{\Omega}_i) \cap cl(\underline{\Omega}_j) = \emptyset$ *for all* $i \neq j$ $(\in \{1, 2, ..., M\})$,
- *there exists, for each* Ω_i*, a closed unit-speed analytic curve* $\alpha_i : [a_i, b_i] \to \mathbb{R}^2$ *such that* $\partial\Omega_i$ *is an image of* α_i *and* $\underline{\Omega_i \textit{ is always on the left side of } \alpha}_{i_{\mathbf{L}}}$.

- *there exists the analytic function* $\xi : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\underline{\Omega} \ \equiv \ \{ \ \underline{x} = (x_1, x_2) \ : \ \xi(\underline{x}) < 0 \ \}. \tag{3.10}$$

Then we say the domain $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1$ *is the* ***analytic non-convex cylinderical domain****.*

We explain the meaning of underlined $\mathbf{L}$ in the second condition of Definition 11. It means when you walk on the α_i with $\dot{\alpha}_i$ direction $\Omega = \Omega_0 \backslash \{\Omega_1 \cup \Omega_2 \cup ... \cup \Omega_M\}$ is always the left side of you. Precisely $\mathbf{L}$ means the outer normal direction at $\alpha_i(\tau_0) \in \partial\Omega$ is always

$$n(\alpha_i(\tau_0)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{\alpha}_i(\tau_0). \tag{3.11}$$

We say that Ω has a rotational symmetry, if there are vectors x_0 and ϖ, such that for all $x \in \partial\Omega$

$$\{(x - x_0) \times \varpi\} \cdot n(x) \equiv 0. \tag{3.12}$$

We denote the phase boundary of the phase space $\Omega \times \mathbf{R}^3$ as $\gamma = \partial\Omega \times \mathbf{R}^3$, and split it into outgoing boundary γ_+, the incoming boundary γ_-, and the singular boundary γ_0 for grazing velocities:

$$\begin{aligned} \gamma_+ &= \{(x, v) = (x, v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x) \cdot v > 0\}, \\ \gamma_- &= \{(x, v) = (x, v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x) \cdot v < 0\}, \\ \gamma_0 &= \{(x, v) = (x, v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x) \cdot v = 0\}. \end{aligned}$$

We need to study the grazing boundary γ_0 more carefully.

Definition 12. *[?] We define the* ***concave(singular) grazing boundary*** *which is a subset of the grazing boundary* γ_0 *:*

$$\gamma_0^{\mathbf{S}} = \{(x, v) \in \gamma_0 : t_{\mathbf{b}}(x, v) \neq 0 \ \text{and} \ t_{\mathbf{b}}(x, -v) \neq 0\},$$

and the ***outward inflection grazing boundary*** *in the grazing boundary* γ_0 *:*

$$\gamma_0^{I+} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) \neq 0 \ \textit{and}\ t_{\mathbf{b}}(x,-v) = 0 \ \textit{and there is}\ \ \delta > 0 \ \textit{such that}\ \ x + \tau v \in \bar{\Omega}^c \ \ \textit{for}\ \ \tau \in (0,\delta)\},$$

and the ***inward inflection grazing boundary*** *in the grazing boundary* γ_0 *:*

$$\gamma_0^{I-} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) = 0 \ \textit{and}\ t_{\mathbf{b}}(x,-v) \neq 0 \ \textit{and there is}\ \ \delta > 0 \ \textit{such that}\ \ x - \tau v \in \bar{\Omega}^c \ \ \textit{for}\ \ \tau \in (0,\delta)\},$$

and the ***convex grazing boundary*** *in the grazing boundary* γ_0 *:*

$$\gamma_0^{V} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) = 0 \ \textit{and}\ \ t_{\mathbf{b}}(x,-v) = 0\}.$$

Given $(t,x,v) \in [0,\infty) \times \Omega \times \mathbb{R}^3$, let $[X(s),V(s)] = [X(s;t,x,v),V(s;t,x,v)] = [x+(s-t)\ \underline{v}\ ,v]$ be the trajectory (or the characteristics) for the 2D Boltzmann equation (3.1) :

$$\frac{dX(s)}{ds} = \underline{V}\ (s), \qquad\qquad \frac{dV(s)}{ds} = 0, \tag{3.13}$$

with the initial condition: $[X(t;t,x,v),V(t;t,x,v)] = [x,v]$ where $V = (V_1,V_2;V_3) = (\underline{V};V_3)$.

Definition 13. *For* $(x,v) \in \bar{\Omega} \times \mathbb{R}^3$*, we define the* ***backward exit time****,* $t_{\mathbf{b}}(x,v) \geq 0$ *to be the the last moment at which the back-time straight line* $X(s;0,x,v)$ *remains in the interior of* Ω *:*

$$t_{\mathbf{b}}(x,v) = \{0\} \cup \{\tau > 0, x - s\underline{v} \in \Omega \ \textit{for all}\ \ 0 \leq s \leq \tau\}.$$

We also define the ***backward exit position*** *in* $\partial\Omega$

$$x_{\mathbf{b}}(x,v) = x - t_{\mathbf{b}}(x,v)\underline{v}\ ,$$

and we always have $\underline{v} \cdot n(x_{\mathbf{b}}) \leq 0$.

3.1.2 Boundary Condition and Conservation Laws

In terms of the standard perturbation f such that $F = \mu + \sqrt{\mu} f$, the Boltzmann equation can be rewritten as

$$\{\partial_t + v \cdot \nabla + L\} f = \Gamma(f, f), \qquad f(0, x, v) = f_0(x, v),$$

where the standard linear Boltzmann operator see [**?**] is given by

$$Lf \equiv \nu f - Kf = -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)\} = \nu f - \int \mathbf{k}(v, v') f(v') dv' \tag{3.14}$$

with the collision frequency $\nu(v) \equiv \int |v-u|^\gamma \mu(u) q_0(\theta) du d\theta \backsim \{1 + |v|\}^\gamma$ for $0 \le \gamma \le 1$; and

$$\Gamma(f_1, f_2) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_2) \equiv \Gamma_+(f_1, f_2) - \Gamma_-(f_1, f_2). \tag{3.15}$$

In terms of f, the specular reflection is

$$f(t, x, v)|_{\gamma_-} = f(x, v, v - 2(n(x) \cdot v) n(x)) = f(x, v, R(x) v) \tag{3.16}$$

For specular reflection conditions, it is well-known that both mass and energy are conserved. Without loss of generality, we may always assume that the mass-energy conservation laws hold for $t \ge 0$, in terms of the perturbation f:

$$\int_{\Omega \times \mathbf{R}^3} f(t, x, v) \sqrt{\mu} dx dv = 0, \tag{3.17}$$

$$\int_{\Omega \times \mathbf{R}^3} |v|^2 f(t, x, v) \sqrt{\mu} dx dv = 0. \tag{3.18}$$

Moreover, if the domain Ω has *any* axis of rotation symmetry (3.12), then we further assume the corresponding conservation of angular momentum is valid for all $t \ge 0$:

$$\int_{\Omega \times \mathbf{R}^3} \{(x - x_0) \times \varpi\} \cdot v f(t, x, v) \sqrt{\mu} dx dv = 0. \tag{3.19}$$

3.1.3 Main Results

We introduce the weight function for $\rho > 0$ and $\beta \in \mathbf{R}^1$.

$$w(v) = (1 + \rho^2 |v|^2)^\beta \tag{3.20}$$

Theorem 10. *Assume $w^{-2}\{1+|v|\}^3 \in L^1$ in (3.20). Assume that the domain is either Smooth Convex Domain (Definition 10) or Analytic Non-Convex Domain (Definition 11) and the mass (3.17) and energy (3.18) are conserved for f_0. In the case of Ω has any rotational symmetry (3.12), we require that the corresponding angular momentum (3.19) is conserved for f_0. Then there exists $\delta > 0$ such that if $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$ and $||wf_0||_\infty \leq \delta$, there exists a unique solution $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$ to the specular boundary value problem (3.16) for the Boltzmann equation (3.1) such that*

$$\sup_{0 \leq t \leq \infty} e^{\lambda t} ||wf(t)||_\infty \leq C ||wf_0||_\infty.$$

In the case of a smooth convex domain, if $f_0(x,v)$ is continuous except on γ_0 and $f_0(x,v) = f_0(x, R_x v)$ on $\partial\Omega$ then $f(t,x,v)$ is continuous in $[0,\infty) \times \{\bar{\Omega} \times \mathbb{R}^3 \setminus \gamma_0\}$.
In the case of an analytic non-convex domain, if $f_0(x,v)$ is continuous except on γ_0 and $f_0(x,v) = f_0(x, R_x v)$ on $\partial\Omega$ then $f(t,x,v)$ is continuous in

$$[0,\infty) \times \{\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0\} \setminus \{(t,x,v) \in [0,\infty) \times \bar{\Omega} \times \mathbb{R}^3 : \exists\, s \in [0,t] \ s.t.\ (X(s;t,x,v), \underline{V}(s;t,x,v)) \in \gamma_0^{I-}\}.$$

3.1.4 $L^2 - L^\infty$ Interpolation

We follow $L^2 - L^\infty$ scheme in [27] to establish the well-posedness and exponential decay to the Maxwellian. By [27] it suffices to show L^∞ exponential decay of weighted linear Boltzmann equation

$$\{\partial_t + \underline{v} \cdot \nabla_x + \nu - K_w\} h = 0, \quad h(0,x,\underline{v}) = h_0(x,\underline{v}) \equiv wf_0 \tag{3.21}$$

with specular reflection boundary condition $h(t,x,\underline{v})_{\gamma_-} = h(t,x,R_x\underline{v})$ where

$$K_w h = wK(\frac{h}{w}).$$

The key estimate to obtain desired decay for (3.21) is

$$\int_0^t ds_1 \int_{1/N\le|\underline{v}'|\le N} d\underline{v}' \int_0^{s_1} ds \int_{1/N\le|v''|\le N} dv'' \; |f(s, X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}'), v'')| \times \mathbf{1}_{A_\varepsilon^c}(x, \underline{v}) \tag{3.22}$$

$$\lesssim \underbrace{\left(\varepsilon + \frac{1}{N}\right) \sup_{0\le s\le t} ||f(s)||_\infty}_{\mathbf{I}} + \underbrace{\int_0^t ||f(s)||_{L^2}\, ds}_{\mathbf{II}}, \tag{3.23}$$

where $A_\varepsilon \subset \bar{\Omega} \times \mathbb{R}^2$ to be determined. Once we have the above estimate, we use L^2 decay of the linear Boltzmann solution f to have L^∞ decay for (3.21).

In the case of periodic box, the estimate (3.23) is easy to show ([32]) because we have explicit formula of

$$X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}') = x - (t - \underline{s_1})\underline{v} - (s_1 - s)\underline{v}',$$

and we can verify

$$\det\left\{\frac{dX_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}')}{dv'}\right\} \neq 0, \tag{3.24}$$

for $s_1 - s \neq 0$. In order to get rid of the case $s_1 - s = 0$ we define

$$B_\varepsilon = \{(s_1, \underline{v}', s) \in [0, t] \times \{1/N \le |\underline{v}'| \le N\} \times [0, s_1] \;:\; s \in [s_1 - \varepsilon, s_1]\;\}. \tag{3.25}$$

Then we split the integrand of (3.22) as

$$\mathbf{1}_{B_\varepsilon}(s_1, \underline{v}', s) \times \text{Integrand} \;+\; (1 - \mathbf{1}_{B_\varepsilon}(s_1, \underline{v}', s)) \times \text{Integrand}$$

The first integration is bounded by $\mathbf{I}$ since

$$m_4(B_\varepsilon) \le \varepsilon \;\times (t \times N^2).$$

For the second integration, we use (4.3), precisely

$$\det\left\{\frac{dX_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}')}{dv'}\right\} = (s_1 - s)^2 \ge \varepsilon^2 \tag{3.26}$$

for $(s_1, \underline{v}', s) \in [0, t] \times \{1/N \le |\underline{v}'| \le N\} \times [0, s_1] \setminus B_\varepsilon$ so that we can apply change of variables

$y = X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}')$ to have

$$\int_0^t ds_1 \int_{\mathbb{R}^2} dy \int_0^{s_1} ds \int_{\mathbb{R}^3} dv'' \ |w(v'')f(s,y,v'')| \ \frac{1}{\det\left(\frac{dy}{d\underline{v}'}\right)} \leq \frac{t|\mathbb{T}^2|}{\varepsilon^2} ||w||_{L^2} \int_0^t ||f(s)||_{L^2} ds.$$

If the Boltzmann equation is accompanied by the specular reflection boundary condition (3.4), then showing the essential estimate (3.23) is much more hard. Let k be the bouncing number of the backward trajectory $X_{\mathbf{cl}}(s_1; t, x, v)$ and k' be the bouncing number of the backward trajectory $X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v')$. Then we can write the integration (3.22) by

$$\sum_k \int_{t_{k+1}}^{t_k} ds_1 \int_{1/N \leq |\underline{v}'| \leq N} d\underline{v}' \sum_{k'} \int_{t'_{k'+1}}^{t'_{k'}} ds \\ \times int_{1/N \leq |v''| \leq N} dv'' \ |h(s, X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), \underline{v}'), v'')| \mathbf{1}_{B_\varepsilon^c}(s_1, \underline{v}', s) \times \mathbf{1}_{A_\varepsilon^c}(x, v).$$

Because of the boundary with specular reflection boundary condition we have two main difficulties :

- The number of bounce k and k' could grow arbitrary large. We need to construct two sets $A_\varepsilon \subset \bar{\Omega} \times \mathbb{R}^2$ and $B_\varepsilon \subset [0, t] \times \{1/N \leq \underline{v}' \leq N\} \times [0, s_1]$ so that we have upper bound of bouncing number k of the billiard trajectory $X_{\mathbf{cl}}(s_1; t, x, \underline{v}\ )$ and the bouncing number k' of billiard trajectory $X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}\ ), \underline{v}')$ for $(s_1, \underline{v}', s) \in B_\varepsilon^c$ and $(x, \underline{v}\ ) \in A_\varepsilon^c$. Moreover we require that the integration over B_ε is bounded by **I** in (3.23).

- Verify the condition (4.3) on $(s_1, \underline{v}', s) \in B_\varepsilon^c$ and $(x, \underline{v}) \in A_\varepsilon^c$ so that we can apply the change of variables to bound the integration over B_ε^c by **II** in (3.23).

The main difficulty is that there is no induction argument to control the number of bounces and compute (4.3). To obtain uniform bound of the number of collisions in Billiard theory is an important question ([12],[41]). And the condition (4.3) is much stronger than Thom's transversality theory ([33]) which assures that (4.3) is valid for almost every $y = X_{\mathbf{cl}}(s_1; t, x, v)$. My main achievement os to introduce new viewpoint of billiard trajectory which leads to proof of (4.3).

3.1.5 Construct A_ε and B_ε

In section 7, We construct two sets $A_\varepsilon \subset \bar{\Omega}\times\mathbb{R}^2$ and $B_\varepsilon \subset [0,t]\times\{\underline{v}\in\mathbb{R}^2 : 1/N \leq |\underline{v}'| \leq N\}\times[0,s_1]$ such that we have

(A) An upper bound of bouncing number k of the billiard trajectory $X_{\mathbf{cl}}(s_1;t,x,\underline{v}\ )$ and the bouncing number k' of billiard trajectory $X_{\mathbf{cl}}(s;s_1,X_{\mathbf{cl}}(s_1;t,x,\underline{v}\ ),\underline{v}')$ for $(x,\underline{v}\ )\in A_\varepsilon^c$ and $(s_1,\underline{v}',s)\in B_\varepsilon^c$,

(B) $X_{\mathbf{cl}}(s;s_1.X_{\mathbf{cl}}(s_1;t,x,\underline{v}),\underline{v}')$ is differentiable with respect to $\underline{v}'$ and $X_{\mathbf{cl}}(s_1;t,x,\underline{v})$ is continuous for $(x,\underline{v}\ )\in A_\varepsilon^c$ and $(s_1,\underline{v}',s)\in B_\varepsilon^c$,

(C) The integration (3.22) over B_ε is bounded by **I** in (3.23) for all $(x,v)\in A_\varepsilon^c$. Moreover we require that the 2 dimensional Lebesque measure $m_2(A_\varepsilon(x)) = m_2(\ \{\ \underline{v}\ \in\mathbb{R}^2 : (x,\ \underline{v}\ )\in A_\varepsilon\}\ )\leq\varepsilon$ for all $x\in\bar{\Omega}$.

The main difficulty of (A) is that there is no induction argument to control the number of bounces especially in the case of non-convex domain. Notice that uniform bound of the number of collisions in Billiard theory is an important question ([12]). The main difficulty of (B) is that the backward trajectory $X_{\mathbf{cl}}$ is not differentiable when $X_{\mathbf{cl}}$ hits tangentially the non-convex part of boundary.

In the case of non-convex domain, the situation is delicate. In fact, backward trajectory is not defined after it hits the generic inflection point tangentially. Moreover if the trajectory hits near inflection points almost tangentially then it is impossible to have an upper bound of the bouncing number k or k'. Therefore the sets $A_\varepsilon, B_\varepsilon$ contain neighborhood of points possibly hitting inflection points tangentially.

For the second purpose (B), we invent the **Graph method** to identify the direction that the backward trajectory grazes the non-convex part of boundary i.e., for fixed $x\in\bar{\Omega}$ we classify the grazing directions

$$\mathfrak{G}(x) = \{u\in\mathbb{S}^1 : \exists\ s\in[0,t]\ \text{ such that } X_{\mathbf{cl}}(s;t,x,u)\in\partial\Omega \text{ and } V_{\mathbf{cl}}(s;t,x,u)\cdot n(X_{\mathbf{cl}}(s;t,x,u)) = 0\}.$$

In order to illustrate the method, we consider a simple case : for fixed $x \in \Omega$ assume there exists a direction $u_0 \in \mathbb{S}^1$ such that the backward trajectory hits the boundary non-tangentially at first but grazes next time, i.e.

$$u_0 \cdot n(x_1(x,u_0)) \neq 0 \quad \text{but} \quad v_1(x,u_0) \cdot n(x_2(x,u_0)) = 0,$$

where $x_1(x,u_0) = x - t_{\mathbf{b}}(x,u_0)u_0, v_1(x,u_0) = R_{x_1(x,u_0)}v_1(x,u_0)$ and $x_2(x,u_0) = x_1(x,u_0) - t_{\mathbf{b}}(x_1(x,u_0), v_1(x,u_0))$. Let $x_2(x,u_0)$ be a point on a non-convex part of the boundary which is parameterized by a unit-speed curve $\alpha : (-1,1) \to \mathbb{R}^2$ and $x_2(x,u_0) = \alpha(0)$ and $v_2(x,u_0) = \dot{\alpha}(0)$.

We want to classify directions $u \in \mathbb{S}^1$ such that $v_1(x,u) \cdot n(x_2(x,u)) = 0$ where $(x_2(x,u), v_1(x,u)) = (\alpha(\tau), \dot{\alpha}(\tau))$. For that purpose, we trace back from $(\alpha(\tau), -\dot{\alpha}(\tau))$ for $\tau \sim 0$ and consider the image of the billiard map $\mathcal{T}$ in (3.29)

$$(\mathcal{X}(\tau), \mathcal{V}(\tau)) \equiv \{(\mathcal{X}(\alpha(\tau), -\dot{\alpha}(\tau)), \mathcal{V}(\alpha(\tau), -\dot{\alpha}(\tau))\} = \{(\, x_{\mathbf{b}}(\alpha(\tau), -\dot{\alpha}(\tau)) \,,\, R_{x_{\mathbf{b}}(\alpha(\tau), -\dot{\alpha}(\tau))}(-\dot{\alpha}(\tau)) \,)\} \subset \partial\Omega \times \mathbb{S}^1.$$

Since the billiard map $\mathcal{T}$ is a diffeomorphism we can show $\dot{\mathcal{X}}(\tau) \neq 0$ or $\dot{\mathcal{V}}(\tau) \neq 0$. Without loss of generality we assume $\dot{\mathcal{V}}(\tau) \neq 0$ and use the inverse function theorem to have $\tau = \tau(\mathcal{V})$ which is an analytic function for $\tau \sim 0$. Then we can consider the above set as a graph of an analytic function $\mathcal{X} \circ \tau$

$$\{\ (\mathcal{X} \circ \tau(u), u) \in \partial\Omega \times \mathbb{S}^1\ )\ :\ u \sim R_{x_{\mathbf{b}}(\alpha(0), -\dot{\alpha}(0))}(-\dot{\alpha}(0))\}. \tag{3.27}$$

On the other hand, the backward trajectory at (x,u) for $u \sim u_0$ constructs

$$\{\ (x_{\mathbf{b}}(x,u), u) \in \partial\Omega \times \mathbb{S}^1\ )\ :\ u \sim u_0\}. \tag{3.28}$$

which is a graph of analytic function $x_{\mathbf{b}}(x,u)$. The crucial observation is that the backward trajectory starting from (x,u) will hit the boundary tangentially at the second bounce, i.e.

$$v_1(x,u) \cdot n(x_2(x,u)) = 0, \quad u \sim u_0,$$

if and only if two graphs in (3.27) and (3.28) are intersecting at u, i.e.

$$\mathcal{X} \circ \tau(u) = x_{\mathbf{b}}(x,u).$$

Therefore we can interpret the grazing direction $\mathfrak{G}(x)$ by the intersections of two analytic function. Use the fact that analytic function has discrete zeroes unless it is identically zero, we conclude that the grazing direction $\mathfrak{G}(x)$ is a measure zero set unless $\mathfrak{G}(x)$ forms some open interval, the continuum grazing case. Fortunately using the analyticity of $\partial\Omega$ such a pathological case is only for finite points x in $\bar{\Omega}$. Hence choose B_ε cleverly we can further utilize the integration s_1 in (3.22) to ignore these continuum grazing case.

The condition (C) is a restriction to construct sets A_ε and B_ε in the Step 1. Since we characterize bad situations that we want to avoid in (A) and (B) in very sharp way, the last condition (C) is satisfies.

3.1.6 Preliminary

In order to verify the crucial condition (4.3), instead of the standard billiard map

$$\mathcal{T} \ : \ \partial\Omega \times \mathbb{R}^2 \to \partial\Omega \times \mathbb{R}^2 \quad , \quad (x, \underline{v}\ ) \mapsto (\mathcal{T}_1(x, \underline{v}\ ), \mathcal{T}_2(x, \underline{v}\ )) \equiv (x_{\mathbf{b}}(x, \underline{v}\ ), R_{x_{\mathbf{b}}(x,\underline{v}\ )}\underline{v}\ ) \tag{3.29}$$

by $R_{x_{\mathbf{b}}(x,\underline{v}\ )}\underline{v}$ defined (3.4), we introduce a new **collision-pair map**

$$\begin{aligned} \mathfrak{T} : \partial\Omega \times \partial\Omega \quad &\rightarrow \quad \partial\Omega \times \partial\Omega \\ (x, y) \quad &\mapsto \quad (y, T(x, y)), \end{aligned}$$

where $T(x, y) = x_{\mathbf{b}}(y, R_y \left(\frac{x-y}{|x-y|}\right))$. The collision-pair is well-defined and is a diffeomorphism if and only if

$$(x, y) \pitchfork \partial\Omega \ \text{ and } \ \mathfrak{T}(x, y) \pitchfork \partial\Omega$$

where $(x, y) \pitchfork \partial\Omega$ means that $\overline{xy} \cap \partial\Omega = \{x, y\}$, a line segment $\overline{xy}$ does not have intersection with the boundary $\partial\Omega$ except its end points and the straight line passing x, y intersects the boundary non-tangentially. We can define inductively

$$\mathfrak{T}_k \ \equiv \ \underbrace{\mathfrak{T} \circ \cdots\cdots \circ \mathfrak{T}}_{k \quad times} \quad , \quad (T_{k-1}(x, y), T_k(x, y)) \equiv \mathfrak{T}_k(x, y)$$

whenever $(x,y) \pitchfork \partial\Omega$ and $\mathfrak{T}_i \pitchfork \partial\Omega$ for $i = 0,1,...,k$. Denote

$$\Delta\mathfrak{T}_i(x,y) = \Delta(T_{i-1}(x,y), T_i(x,y)) = |T_{i-1}(x,y) - T_i(x,y)|$$

and $y_0 = X_{\mathbf{cl}}(s_1; t, x, v)$ and $(\bar{x}, \bar{y}) = (x_{\mathbf{b}}(y_0, -v'), x_{\mathbf{b}}(y_0, v'))$. Then we can write $X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v')$ in (4.3) by

$$(1 - \phi(y_0; \bar{x}, \bar{y}))\ T_k(\bar{x}, \bar{y}) + \phi(y_0; \bar{x}, \bar{y})\ T_{k+1}(\bar{x}, \bar{y}) - (s_1 - s)\ |v'|\ \frac{T_k(\bar{x}, \bar{y}) - T_{k+1}(\bar{x}, \bar{y})}{\Delta\mathfrak{T}_{k+1}(\bar{x}, \bar{y})} \tag{3.30}$$

for some given smooth function $\phi(y_0; \bar{x}, \bar{y})$.

In order to compute explicitly we introduce the coordinate system. According to the chart of $\partial\Omega$ we introduce two kinds of coordinate chart of $\partial\Omega \times \partial\Omega$ near $(x,y) \in (\partial\Omega)^2$ with $(x,y) \pitchfork \partial\Omega$: the first one is the **radial coordinate** $\mathbf{R}_{y^0}$ for fixed point $y^0 \in \overline{xy}$, where $\mathbf{R}_{y^0}(\bar{x}, \bar{y})$ is a pair of angles between $\overline{\bar{x}y^0}$, $\overline{\bar{y}y^0}$ and x-axis. The second one is the **parallel coordinate** $\mathbf{P}_{x,y}$, where $[\mathbf{P}_{x,y}(\bar{x}, \bar{y})]_1$ is the parameter τ where $\overline{\bar{x}\bar{y}}$ intersects with the line $x + \tau \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{y-x}{|y-x|}$ passing x and perpendicular to $\overline{xy}$. Similarly $[\mathbf{P}_{x,y}(\bar{x}, \bar{y})]_2$ is the parameter τ where $\overline{\bar{x}\bar{y}}$ intersects with the line $y + \tau \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{y-x}{|y-x|}$ passing $x + 1 \times \frac{y-x}{|y-x|}$ and perpendicular to $\overline{xy}$. When $(x,y) \pitchfork \partial\Omega$ and $\mathfrak{T}_i(x,y) \pitchfork \partial\Omega$ for all $i = 0,1,2,...,k$ then $(\bar{x}, \bar{y}) \pitchfork \partial\Omega$ and $\mathfrak{T}_i(\bar{x}, \bar{y}) \pitchfork \partial\Omega$ for all $i = 0,1,2,...,k$ for $(\bar{x}, \bar{y}) \sim (x,y)$. Hence using those coordinate charts, we have explicit formula for $s \in (t_{k+1}, t_k)$

$$\det\left(\frac{\partial X_{\mathbf{cl}}(s; s_1, y_0, v')}{\partial v'}\right) = \frac{\mathcal{A}(\theta)}{\Delta\mathfrak{T}_{k+1}(\theta, \theta+\pi)} \frac{s_1 - s}{|v'|} + \frac{\mathcal{B}(\theta)}{(\Delta\mathfrak{T}_{k+1}(\theta, \theta+\pi))^2}(s_1 - s)^2$$

where θ is an angle between $\overline{y_0\bar{x}}$ and x-axis. From the diffeomorphic property of $\mathfrak{T}_k$ we can conclude $\mathcal{A}(\theta) \neq 0$ or $\mathcal{B}(\theta) \neq 0$ for any θ. Hence we can verify the condition (4.3) except very small measured set $\theta \in [0, 2\pi],\ s_1 \in [0,t],\ s \in [0, s_1],\ |v'| \in [1/N, N]$ with $\mathcal{B}(\theta)(s_1 - s)|v'| + \mathcal{A}(\theta)\Delta\mathfrak{T}_{k+1}(\theta, \theta+\pi) = 0$.

Hence we can verify the crucial condition (4.3) whenever $X_{\mathbf{cl}}(s_1; t, x, \underline{v})$ and $X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}')$ are well-defined and

$$n(X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}')) \cdot V_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}') \neq 0 \tag{3.31}$$

when $X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, \underline{v}), \underline{v}') \in \partial\Omega$.

3.2 Collision-Pair Map and Transversality

In this subsection we introduce the collision-pair map and compare with the standard billiard map.

Definition 14. *Let $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ so that the unit normal direction is well-defined for all $x \in \partial\Omega$. The (backward) billiard map $\mathcal{T}$ is defined*

$$\begin{aligned} \mathcal{T} : \partial\Omega \times \mathbb{R}^n &\rightarrow \partial\Omega \times \mathbb{R}^n, \\ (x,v) &\mapsto (\mathcal{X}(x,v), \mathcal{V}(x,v)) \equiv (x_{\mathbf{b}}(x,v), R_{x_{\mathbf{b}}(x,v)}v) \end{aligned} \tag{3.32}$$

where $R_{x_{\mathbf{b}}(x,v)}v = v - 2(n(x_{\mathbf{b}}(x,v)) \cdot v)n(x_{\mathbf{b}}(x,v))$.

Here we compute the jacobian matrix of the standard billiard map $\mathcal{T}(x,v)$ on $(x,v) \in \{(x,v) \in \partial\Omega \times \mathbb{R}^n : n(x) \cdot v > 0\}$. Since we only consider 2D domain Ω we do our computation in that case. Assume that the boundary $\partial\Omega$ is a unit-speed curve α. Let γ be an angle between $\dot{\alpha}(\tau)$ and x-axis, θ be an angle between α and v, γ_1 be an angle between $\dot{\alpha}(\tau_1)$ and x-axis. By the definition of γ and γ_1 we have

$$\gamma = \tan^{-1}\left(\frac{\dot{\alpha}_2(\tau)}{\dot{\alpha}_1(\tau)}\right) \;, \quad \gamma_1 = \tan^{-1}\left(\frac{\dot{\alpha}_2(\tau_1)}{\dot{\alpha}_1(\tau_1)}\right).$$

Notice that

$$v = \begin{pmatrix} \cos(\theta+\gamma) \\ \sin(\theta+\gamma) \end{pmatrix} \;, \quad n(\alpha(\tau_1)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\gamma_1 \\ \sin\gamma_1 \end{pmatrix} = \begin{pmatrix} \sin\gamma_1 \\ -\cos\gamma_1 \end{pmatrix}. \tag{3.33}$$

From $R_{\alpha(\tau_1)}v = v - 2(n(\alpha(\tau_1)) \cdot v)n(\alpha(\tau_1))$ we have

$$\begin{aligned} \begin{pmatrix} \cos(\theta_1+\gamma_1) \\ \sin(\theta_1+\gamma_1) \end{pmatrix} &= \begin{pmatrix} \cos(\theta+\gamma) \\ \sin(\theta+\gamma) \end{pmatrix} - 2\left[\begin{pmatrix} \sin\gamma_1 \\ -\cos\gamma_1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta+\gamma) \\ \sin(\theta+\gamma) \end{pmatrix}\right] \begin{pmatrix} \sin\gamma_1 \\ -\cos\gamma_1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\gamma_1-\theta-\gamma) \\ \sin(2\gamma_1-\theta-\gamma) \end{pmatrix}, \end{aligned}$$

where we used the fact that $\sin(\gamma_1-\theta-\gamma) = \sin\gamma_1\cos(-\theta-\gamma) + \cos\gamma_1\sin(-\theta-\gamma)$. In order to compute the Jacobian matrix $\frac{\partial(\tau_1,\theta_1)}{\partial(\tau,\theta)}$ we use

$$\begin{pmatrix} \alpha_1(\tau_1) \\ \alpha_2(\tau_1) \end{pmatrix} = \begin{pmatrix} \alpha_1(\tau) \\ \alpha_2(\tau) \end{pmatrix} - |\alpha(\tau)-\alpha(\tau_1)| \begin{pmatrix} \cos(\gamma+\theta) \\ \sin(\gamma+\theta) \end{pmatrix}. \tag{3.34}$$

Take τ-derivative for the first line above

$$\dot\alpha_1(\tau_1)\frac{\partial\tau_1}{\partial\tau} = \dot\alpha_1(\tau) + |\alpha(\tau)-\dot\alpha(\tau_1)|\sin(\gamma+\theta)\kappa(\tau) \tag{3.35}$$

$$-\frac{\cos(\gamma+\theta)}{|\alpha(\tau)-\alpha(\tau_1)|}\{[\alpha_1(\tau)-\alpha_1(\tau_1)][\dot\alpha_1(\tau)-\dot\alpha_1(\tau_1)\frac{\partial\tau_1}{\partial\tau}] + [\alpha_2(\tau)-\alpha_2(\tau_1)][\dot\alpha_2(\tau)-\dot\alpha_2(\tau_1)\frac{\partial\tau_1}{\partial\tau}]\} \tag{3.36}$$

where we used

$$\frac{\partial\gamma}{\partial\tau} = \kappa(\tau) \tag{3.37}$$

which is from differentiation $\tan\gamma = \dot\alpha_2(\tau)\dot\alpha_1(\tau)^{-1}$ to have $\frac{\partial\gamma}{\partial\tau} = \cos^2\gamma\{\dot\alpha_1(\tau)\ddot\alpha_2(\tau) - \dot\alpha_2(\tau)\ddot\alpha_1(\tau)\} = \kappa(\tau)$, since α is a unit-speed curve. Therefore we have

$$\begin{aligned}\frac{\partial\tau_1}{\partial\tau} &= \frac{1}{\dot\alpha_1(\tau_1) - \cos(\gamma+\theta)\frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|}\cdot\dot\alpha(\tau_1)} \\ &\times\left(\dot\alpha_1(\tau) - \cos(\gamma+\theta)\frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|}\cdot\dot\alpha(\tau) + |\alpha(\tau)-\alpha(\tau_1)|\sin(\gamma+\theta)\kappa(\tau)\right).\end{aligned}$$

Notice that $\cos\gamma_1 = \dot\alpha_1(\tau_1)$ and $\cos(\theta+\gamma-\gamma_1) = \frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|}$ and the dinominator of the above term equals

$$\begin{aligned} & \cos\gamma_1 - \cos(\gamma+\theta)\cos(\theta+\gamma-\gamma_1) & (3.38)\\ = \; & \cos\gamma_1 - \frac{1}{2}\cos(2\theta+2\gamma-\gamma_1) - \frac{1}{2}\cos\gamma_1 = -\frac{1}{2}\{\cos(2\theta+2\gamma-\gamma_1) - \cos\gamma_1\} & (3.39)\\ = \; & \sin(\theta+\gamma)\sin(-\gamma_1+\theta+\gamma). & (3.40)\end{aligned}$$

Similarly we have

$$\dot\alpha_1(\tau) - \cos(\gamma+\theta)\frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|} = \sin\theta\sin(\gamma+\theta) \tag{3.41}$$

and hence

$$\frac{\partial\tau_1}{\partial\tau} = \frac{1}{\sin(\gamma-\gamma_1+\theta)}\{\sin\theta + |\alpha(\tau)-\alpha(\tau_1)|\kappa(\tau)\}. \tag{3.42}$$

The θ-derivative of τ_1 can be computed via differentiating the equation for $\alpha_1(\tau_1)$,

$$\frac{\partial \tau_1}{\partial \theta} = \frac{1}{\dot{\alpha}_1(\tau_1) - \cos(\gamma+\theta)\frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|}\cdot\dot{\alpha}(\tau_1)}\Big(|\alpha(\tau)-\alpha(\tau_1)|\sin(\gamma+\theta)\Big) = \frac{|\alpha(\tau)-\alpha(\tau_1)|}{\sin(-\gamma_1+\theta+\gamma)}. \tag{3.43}$$

From the equality $\theta_1 = \gamma_1 - \gamma - \theta$ we have

$$\frac{\partial \theta_1}{\partial \tau} = \frac{\partial \gamma_1}{\partial \tau} - \frac{\partial \gamma}{\partial \tau} = -\kappa(\tau) + \frac{\kappa(\tau_1)}{\sin(\gamma+\theta-\gamma_1)}\Big\{\sin\theta + |\alpha(\tau)-\alpha(\tau_1)|\kappa(\tau)\Big\}. \tag{3.44}$$

Similarly we have

$$\frac{\partial \theta_1}{\partial \theta} = \frac{\partial \gamma_1}{\partial \theta} - \frac{\partial \gamma}{\partial \theta} = \frac{\partial \tau_1}{\partial \theta}\frac{\partial \gamma_1}{\partial \tau_1} - 1 = \kappa(\tau_1)\frac{|\alpha(\tau)-\alpha(\tau_1)|}{\sin(-\gamma_1+\theta+\gamma)} - 1 \tag{3.45}$$

To sum we have a Jacobian matrix $\frac{\partial(\tau_1,\theta_1)}{\partial(\tau,\theta)} = \begin{pmatrix} \frac{\partial \tau_1}{\partial \tau} & \frac{\partial \tau_1}{\partial \theta} \\ \frac{\partial \theta_1}{\partial \tau} & \frac{\partial \theta_1}{\partial \theta} \end{pmatrix}$

$$\frac{1}{\sin(\gamma+\theta-\gamma_1)} \times$$

$$\begin{pmatrix} \sin\theta + |\alpha(\tau)-\alpha(\tau_1)|\kappa(\tau) & |\alpha(\tau)-\alpha(\tau_1)| \\ \kappa(\tau_1)\{\sin\theta + |\alpha(\tau)-\alpha(\tau_1)|\kappa(\tau)\} - \kappa(\tau)\sin(\gamma+\theta-\gamma_1) & \kappa(\tau_1)|\alpha(\tau)-\alpha(\tau_1)| - \sin(-\gamma_1+\theta+\gamma) \end{pmatrix}$$

and

$$\det\left(\frac{\partial(\tau_1,\theta_1)}{\partial(\tau,\theta)}\right) = \frac{-\sin\theta}{\sin(\gamma+\theta-\gamma_1)}. \tag{3.46}$$

Further we will compute $\frac{\partial(\tau_2,\theta_2)}{\partial(\tau,\theta)}$. Use the equality

$$\alpha(\tau_2) = \alpha(\tau_1) - |\alpha(\tau_1)-\alpha(\tau_2)|\begin{pmatrix} \cos(\theta_1+\gamma_1) \\ \sin(\theta_1+\gamma_1) \end{pmatrix}. \tag{3.47}$$

Take τ-derivative to the first row of the above equality to have

$$\begin{aligned} \dot{\alpha}_1(\tau_2)\frac{\partial \tau_2}{\partial \tau} = \dot{\alpha}_1(\tau_1)\frac{\partial \tau_1}{\partial \tau} - \frac{\alpha(\tau_1)-\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|}\cdot\left(\dot{\alpha}(\tau_1)\frac{\partial \tau_1}{\partial \tau} - \dot{\alpha}(\tau_2)\frac{\partial \tau_2}{\partial \tau}\right)\cos(\theta_1+\gamma_1) \\ +|\alpha(\tau_1)-\alpha(\tau_2)|\sin(\theta_1+\gamma_1)\left(\frac{\partial \theta_1}{\partial \tau} + \frac{\partial \gamma_1}{\partial \tau}\right) \end{aligned}$$

and

$$\frac{\partial \tau_2}{\partial \tau} = \frac{1}{\dot{\alpha}_1(\tau_2) - \frac{\alpha(\tau_1)-\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|} \cdot \dot{\alpha}(\tau_2)\cos(\theta_1+\gamma_1)} \Big\{ \left(\dot{\alpha}_1(\tau_1) - \frac{\alpha(\tau_1)-\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|} \cdot \dot{\alpha}(\tau_1)\cos(\theta_1+\gamma_1) \right) \frac{\partial \tau_1}{\partial \tau} + |\alpha(\tau_1)-\alpha(\tau_2)| \sin(\theta_1+\gamma_1) \left(\frac{\partial \theta_1}{\partial \tau} + \frac{\partial \gamma_1}{\partial \tau} \right) \Big\}$$

We can compute

$$\dot{\alpha}_1(\tau_2) - \frac{\alpha(\tau_1)-\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|} \cdot \dot{\alpha}(\tau_2)\cos(\theta_1+\gamma_1) = \sin(\theta_1+\gamma_1)\sin(-\gamma_2+\theta_1+\gamma_1) \tag{3.48}$$

$$\dot{\alpha}_1(\tau_1) - \frac{\alpha(\tau_1)-\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|} \cdot \dot{\alpha}(\tau_1)\cos(\theta_1+\gamma_1) = \sin\theta_1 \sin(\gamma_1+\theta_1) \tag{3.49}$$

Therefore we have

$$\frac{\partial \tau_2}{\partial \tau} = \frac{1}{\sin(-\gamma_2+\theta_1+\gamma_1)} \left\{ \sin\theta_1 \frac{\partial \tau_1}{\partial \tau} + |\alpha(\tau_1)-\alpha(\tau_2)| \left(\frac{\partial \theta_1}{\partial \tau} + \kappa(\tau_1)\frac{\partial \tau_1}{\partial \tau} \right) \right\} \tag{3.50}$$

Similarly we have

$$\frac{\partial \tau_2}{\partial \theta} = \frac{1}{\sin(-\gamma_2+\theta_1+\gamma_1)} \left\{ \sin\theta_1 \frac{\partial \tau_1}{\partial \theta} + |\alpha(\tau_1)-\alpha(\tau_2)| \left(\frac{\partial \theta_1}{\partial \theta} + \kappa(\tau_1)\frac{\partial \tau_1}{\partial \theta} \right) \right\} \tag{3.51}$$

From the specular reflection law we have

$$\theta_2 = 2\pi - (\theta_1 + \gamma_1 - \gamma_2) \tag{3.52}$$

and

$$\frac{\partial \theta_2}{\partial \tau} = -\frac{\partial \theta_1}{\partial \tau} - \kappa(\tau_1)\frac{\partial \tau_1}{\partial \tau} - \kappa(\tau_2)\frac{\partial \tau_2}{\partial \tau} \tag{3.53}$$

$$\frac{\partial \theta_2}{\partial \theta} = -\frac{\partial \theta_1}{\partial \theta} - \kappa(\tau_1)\frac{\partial \tau_1}{\partial \theta} - \kappa(\tau_2)\frac{\partial \tau_2}{\partial \theta} \tag{3.54}$$

We first define the admissible pair : For an open subset Ω of $\mathbb{R}^n$ we say $(x, y) \in \partial\Omega \times \partial\Omega$ with $x \neq y$ a **admissible pair** if and only if

$$x_{\mathbf{b}}(x, \frac{x-y}{|x-y|}) = y. \tag{3.55}$$

These coordinates are defined locally near a transversal pair:

Definition 15. *Let Ω be a smooth bounded open subset in $\mathbb{R}^n$ i.e. there is a smooth function $\xi : \mathbb{R}^n \to \mathbb{R}$ where $\Omega = \{x \in \mathbb{R}^n : \xi(x) < 0\}$. We say $(x,y) \in \partial\Omega \times \partial\Omega$ and $x \neq y$ is a* ***transversal pair*** *if (x,y) is an admissible pair and the line segment $\overline{xy}$ meets $\partial\Omega$ only at x and y transversally i.e.*

$$\frac{x-y}{|x-y|} \cdot \nabla\xi(x) \neq 0 \quad and \quad \frac{x-y}{|x-y|} \cdot \nabla\xi(y) \neq 0. \tag{3.56}$$

Lemma 16. *Assume the same conditions in Definition 15. Suppose (x,y) be a transversal pair. Then there exists $\varepsilon > 0$ such that (x',y') is a transversal pair if $|(x,y)-(x',y')| < \varepsilon$ and $(x',y') \in \partial\Omega \times \partial\Omega$.*

Proof. From (2) of Lemma 24 it is obvious. □

Let (x,y) be a transversal pair. Imagine we have a billiard at x whose velocity is $\frac{x-y}{|x-y|}$. Since Ω is the Smooth Convex Domain $x_{\mathbf{b}}(x, \frac{x-y}{|x-y|}) = y$. After the trajectory hits y the trajectory continued with the velocity decided by the specular reflection law $R_y\left(\frac{x-y}{|x-y|}\right)$. We will denote $z = x_{\mathbf{b}}(y, R_y\left(\frac{x-y}{|x-y|}\right))$. Then x, y, z satisfy

$$\frac{y-z}{|y-z|} \quad = \quad \frac{x-y}{|x-y|} - 2\Big(\frac{x-y}{|x-y|} \cdot n(y)\Big)n(y) = R_y\Big(\frac{x-y}{|x-y|}\Big) \tag{3.57}$$

where (3.57) implies the specular reflection law. Equivalently we have a formula of z :

$$z = y - |y-z| R_y\Big(\frac{x-y}{|x-y|}\Big). \tag{3.58}$$

Notice that (3.58) implies z to be a smooth function of (x,y) if $|y-z|$ is a smooth function. We will emphasize the importance of $|y-z|$ by using the following notation : We define **a distance function** $\Delta : \partial\Omega \times \partial\Omega \to [0,\infty]$

$$\Delta(x,y) = |x-y| \tag{3.59}$$

where $(x,y) \in \partial\Omega \times \partial\Omega$ and consider x, y as elements of $\mathbb{R}^n$ $(\supset \Omega)$.

Using the distance function and (3.57) we can express z as a function of (x,y):

$$z = y - \Delta(z,y) R_y\Big(\frac{x-y}{|x-y|}\Big) = y - |y-z| \times \left\{ \frac{x-y}{|x-y|} - 2\Big(\frac{x-y}{|x-y|}\cdot n(y)\Big) n(y) \right\}. \tag{3.60}$$

Once again z is a smooth function of (x,y) if $\Delta(z,y)$ is a smooth function of (x,y). Comparing with (13) or (**??**) we easily know following relations :

$$z = x_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|})\Big) = y - t_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big)) R_y\Big(\frac{x-y}{|x-y|}\Big) \tag{3.61}$$

and $\Delta(z,y) = t_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big))$.

Definition 16. *Let Ω be a smooth domain in $\mathbb{R}^n$. For given a transversal pair (x,y) we define a **collision-point map**, $T : \partial\Omega \times \partial\Omega \to \partial\Omega$ as*

$$T(x,y) = x_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big)) = y - t_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big)) R_y\Big(\frac{x-y}{|x-y|}\Big). \tag{3.62}$$

*Further we can define a **collision-pair map**, $\mathfrak{T}$ as*

$$\mathfrak{T}(x,y) = (y, T(x,y)). \tag{3.63}$$

By Lemma 16 we T and $\mathfrak{T}$ is defined locally near (x,y). For the $\mathbf{k^{th}}$ **transversal pair** *(x,y) we can construct a family of maps inductively*

$$\mathfrak{T}_1 = id \tag{3.64}$$

$$\mathfrak{T}_2 = \mathfrak{T} \tag{3.65}$$

$$\vdots \tag{3.66}$$

$$\mathfrak{T}_n = \mathfrak{T} \circ \mathfrak{T}_{n-1} = \mathfrak{T} \circ \mathfrak{T}^{n-2} = \mathfrak{T}^{n-1} \quad \textit{for all } n \leq k-1. \tag{3.67}$$

Then $\mathbf{k^{th}}$ **collision – pair map**, *$\mathfrak{T}_k$ is defined locally near (x,v) by*

$$\mathfrak{T}_k(x,y) = \mathfrak{T}^{k+1}(x,y). \tag{3.68}$$

Let projections π_1 and π_2 $: \partial\Omega \times \partial\Omega \to \partial\Omega$ i.e. $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ where $(x,y) \in \partial\Omega \times \partial\Omega$.

We define $\mathbf{k^{th}}$ **collision – point map**, T_k *near* (x, v) *by*

$$T_k(x, y) = \pi_2 \circ \mathfrak{T}_k(x, y). \tag{3.69}$$

By definition we have

$$\mathfrak{T}_{k+1}(x, y) = \mathfrak{T}(\pi_1(\mathfrak{T}_k(x, y)), T_k(x, y)) = (T_k(x, y), T(T_{k-1}(x, y), T_k(x, y))). \tag{3.70}$$

Clearly

$$\pi_1 \circ \mathfrak{T}_{k+1}(x, y) = \pi_2 \circ \mathfrak{T}_k(x, y) = T_k(x, y). \tag{3.71}$$

Further we define $\mathbf{k^{th}}$ **collision – distance map**, D_k *by*

$$D_k(x, y) = \sum_{i=1}^{k} \Delta\mathfrak{T}_i(x, y) = |T_0(x, y) - T_1(x, y)| + |T_1(x, y) - T_2(x, y)| + ... + |T_{k-1}(x, y) - T_k(x, y)|. \tag{3.72}$$

Let Ω *be a smooth bounded open subset in* $\mathbb{R}^n$ *i.e. there is a smooth function* $\xi : \mathbb{R}^n \to \mathbb{R}$ *where* $\Omega = \{x \in \mathbb{R}^n : \xi(x) < 0\}$. *We say* $(x, y) \in \partial\Omega \times \partial\Omega$ *and* $x \neq y$ *is a* $\mathbf{k^{th}}$ **transversal pair** *if* (x, y) *is an admissible pair and*

$$\mathfrak{T}_i(x, y) \ \text{ is a transversal pair for all } \ i = 0, 1, ..., k-1. \tag{3.73}$$

The k^{th} collision-distance map $D_k(x, y)$ is a total backward traveling distance of particle which is located at x and has $\frac{x-y}{|x-y|}$ velocity initially until that particle hits the boundary k times.

Now we want to study differentiable properties of k^{th} collision-point map near k^{th} transversal pair. We assume $\partial\Omega$ is a smooth manifold and has a differential structure induced by the coordinate chart. Therefore $\partial\Omega \times \partial\Omega$ is also a smooth manifold. It is easy to see that the collision space $(\partial\Omega)^2 \backslash D$ is a smooth manifold with boundary $D = \{(x, x) \in \partial\Omega \times \partial\Omega\}$.

Lemma 17. *Let* Ω *be an open smooth subset of* $\mathbb{R}^n$. *For any* k^{th} *transversal pair* (x, y), $\mathfrak{T}_i$ *is a local diffeomorphism near* (x, y) *for all* $i = 0, 1, 2..., k$.

Proof. It suffices to show that $\mathfrak{T}$ is a local diffeomorphism near (x, y). Since (x, y) is k^{th} transversal pair, by Lemma 16 T_k and $\mathfrak{T}_k$ are well-defined near (x,y). From Lemma 24 we know that $t_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big))R_y\Big(\frac{x-y}{|x-y|}\Big)$ is a smooth function near (x,y). Clearly T is a smooth function near (x,y) from the formula (3.62).

Now we will claim that $\mathfrak{T}$ is an invertible map and $\mathfrak{T}^{-1}(x,y) = (T(y,x),x)$ so that

$$\mathfrak{T}(\mathfrak{T}^{-1}(x,y)) = \mathfrak{T}(T(y,x),x) = (x, T(T(y,x),x)) = (x,y) \tag{3.74}$$

$$\mathfrak{T}^{-1}(\mathfrak{T}(x,y)) = \mathfrak{T}^{-1}(y, T(x,y)) = (T(T(x,y),y),y) = (x,y). \tag{3.75}$$

It suffices to show that $T(T(x,y),y) = x$. Recall that $T(x,y) = x_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big))$. By definitions we have

$$T(T(x,y),y) = T(x_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big)), y) = x_{\mathbf{b}}(y, R_y\Big(\frac{x_{\mathbf{b}}-y}{|x_{\mathbf{b}}-y|}\Big)) \tag{3.76}$$

where $x_{\mathbf{b}} = x_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big))$. Let's denote $t_{\mathbf{b}} = t_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big))$. From definition we have

$$\frac{x_{\mathbf{b}}-y}{|x_{\mathbf{b}}-y|} = -t_{\mathbf{b}}(y, R_y\Big(\frac{x-y}{|x-y|}\Big))R_y\Big(\frac{x-y}{|x-y|}\Big). \tag{3.77}$$

From the positivity of $t_{\mathbf{b}}$ and $|R_y\Big(\frac{x-y}{|x-y|}\Big)| = 1$ we have

$$\frac{x_{\mathbf{b}}-y}{|x_{\mathbf{b}}-y|} = -R_y\Big(\frac{x-y}{|x-y|}\Big) \tag{3.78}$$

Now compute $R_y\Big(\frac{x_{\mathbf{b}}-y}{|x_{\mathbf{b}}-y|}\Big) =$

$$\begin{aligned}
&= \frac{x_{\mathbf{b}}-y}{|x_{\mathbf{b}}-y|} - 2\Big(\frac{x_{\mathbf{b}}-y}{|x_{\mathbf{b}}-y|}\cdot n(y)\Big)n(y) = -R_y\Big(\frac{x-y}{|x-y|}\Big) + 2\Big(R_y\Big(\frac{x-y}{|x-y|}\Big)\cdot n(y)\Big)n(y) \\
&= -\frac{x-y}{|x-y|} + 2(\frac{x-y}{|x-y|}\cdot n(y))n(y) - 2(\frac{x-y}{|x-y|}\cdot n(y))n(y) = -\frac{x-y}{|x-y|}
\end{aligned}$$

From (3.76) we have

$$T(T(x,y),y) = x_{\mathbf{b}}(y, -\frac{x-y}{|x-y|}) = x. \tag{3.79}$$

Therefore $\mathfrak{T}$ is invertible and smooth.

$\square$

Proof. We will show that the jacobian of $\mathfrak{T}$ is non-zero to prove $\mathfrak{T}$ is a local diffeomorphism. Assume that

$$\mathfrak{T}\begin{pmatrix}\alpha(\tau)\\ \alpha(\tau_1)\end{pmatrix}=\begin{pmatrix}\alpha(\tau_1)\\ \alpha(\tau_2)\end{pmatrix} \tag{3.80}$$

and the jacobian matrix of $\mathfrak{T}$ is

$$\begin{pmatrix}\frac{\partial\tau_1}{\partial\tau} & \frac{\partial\tau_1}{\partial\tau_1}\\ \frac{\partial\tau_2}{\partial\tau} & \frac{\partial\tau_2}{\partial\tau_1}\end{pmatrix}=\begin{pmatrix}0 & 1\\ \frac{\partial\tau_2}{\partial\tau} & \frac{\partial\tau_2}{\partial\tau_1}\end{pmatrix} \tag{3.81}$$

where $\frac{\partial\tau_2}{\partial\tau}$ satisfies

$$\frac{\partial\tau_2}{\partial\tau}\times\left\{-\frac{\dot\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|}+\left[\frac{\dot\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|}\cdot\frac{\alpha(\tau_1)-\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|}\right]\frac{\alpha(\tau_1)-\alpha(\tau_2)}{|\alpha(\tau_1)-\alpha(\tau_2)|}\right\} \tag{3.82}$$

$$=\frac{\dot\alpha(\tau)}{|\alpha(\tau)-\alpha(\tau_1)|}-\left[\frac{\dot\alpha(\tau)}{|\alpha(\tau)-\alpha(\tau_1)|}\cdot\frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|}\right]\frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|} \tag{3.83}$$

$$-2\left\{\left(\frac{\dot\alpha(\tau)}{|\alpha(\tau)-\alpha(\tau_1)|}-\left[\frac{\dot\alpha(\tau)}{|\alpha(\tau)-\alpha(\tau_1)|}\cdot\frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|}\right]\frac{\alpha(\tau)-\alpha(\tau_1)}{|\alpha(\tau)-\alpha(\tau_1)|}\right)\cdot\begin{pmatrix}-\dot\alpha_2(\tau_1)\\ \dot\alpha_1(\tau_1)\end{pmatrix}\right\}\begin{pmatrix}-\dot\alpha_2(\tau_1)\\ \dot\alpha_1(\tau_1)\end{pmatrix} \tag{3.84}$$

Directly we can compute that

$$\left|\frac{\partial\tau_2}{\partial\tau}\right|\frac{1}{|\alpha(\tau_1)-\alpha(\tau_2)|}|\sin\theta_2|=\frac{1}{|\alpha(\tau)-\alpha(\tau_2)|} \tag{3.85}$$

and

$$\left|\frac{\partial\tau_2}{\partial\tau}\right|=\frac{1}{|\sin\theta_2|} \tag{3.86}$$

which is a jacobian of $\mathfrak{T}$. Similarly for kth collision-pair map we have

$$D\mathfrak{T}^k\begin{pmatrix}x\\ y\end{pmatrix}=D\mathfrak{T}(\mathfrak{T}^{k-1}\begin{pmatrix}x\\ y\end{pmatrix})\cdot D\mathfrak{T}(\mathfrak{T}^{k-2}\begin{pmatrix}x\\ y\end{pmatrix})\cdots D\mathfrak{T}\begin{pmatrix}x\\ y\end{pmatrix} \tag{3.87}$$

□

Lemma 18. *Let Ω be an open smooth subset of $\mathbb{R}^n$. For any k^{th} transversal pair (x,y), D_i is a smooth function near (x,y) for all $i=0,1,...,k$.*

Proof. It suffice to represent $D_k(x,y)$ as a function of T of $\mathfrak{T}$. First we will show the following for

$k \geq 2$

$$\mathfrak{T}^k(x,y) = (T(\mathfrak{T}^{k-2}(x,y)), T(\mathfrak{T}^{k-1}(x,y))). \tag{3.88}$$

We see that for $k = 2$ (3.88) is hold because

$$\mathfrak{T}^2(x,y) = \mathfrak{T}(y, T(x,y)) = (T(x,y), T(y, T(x,y))). \tag{3.89}$$

Assume (3.88) is valid for $l = 2, 3, ..., k-1$.

$$\begin{aligned} \mathfrak{T}^k(x,y) &= \mathfrak{T}(\mathfrak{T}^{k-1}(x,y)) = \mathfrak{T}(T(\mathfrak{T}^{k-3}(x,y)), T(\mathfrak{T}^{k-2}(x,y))) \\ &= (T(\mathfrak{T}^{k-2}(x,y)), T(T(\mathfrak{T}^{k-3}(x,y)), T(\mathfrak{T}^{k-2}(x,y)))) = (T(\mathfrak{T}^{k-2}(x,y)), T(\mathfrak{T}^{k-1}(x,y))) \end{aligned}$$

Therefore k^{th} collision-distance map

$$D_k(x,y) = |x-y| + |y - T(x,y)| + \sum_{i=2}^{k-1} |T(\mathfrak{T}^{k-2}(x,y)) - T(\mathfrak{T}^{k-1}(x,y))|. \tag{3.90}$$

By Velocity Lemma we know that

$$\text{If } \ x \neq y \ \text{ then } \ T(\mathfrak{T}^{k-2}(x,y)) - T(\mathfrak{T}^{k-1}(x,y)) \neq 0 \tag{3.91}$$

and each term in (3.90) is smooth. Therefore we get the lemma. □

From now let the domain Ω be a subset of $\mathbb{R}^2$ and investigate the representation of $\mathfrak{T}$ with respect to two special coordinates. These coordinates are defined locally near a transversal pair :

Lemma 19. *Let Ω be a smooth bounded open subset in $\mathbb{R}^2$. Let $(x,y) \in \partial\Omega \times \partial\Omega$ be a transversal pair. Choose $x^0 \in \overline{xy} \cap \Omega$. Then there exist $\varepsilon_1 = \varepsilon_1(\Omega, x, y, x^0) > 0$, $\varepsilon_2 = \varepsilon_2(\Omega, x, y, x^0) > 0$ and a* **radial coordinate map**

$$\mathbf{R}_{x^0} : \{\partial\Omega \cap B(x;\varepsilon_1)\} \times \{\partial\Omega \cap B(y;\varepsilon_2)\} \longrightarrow [0, 2\pi) \times [0, 2\pi) \tag{3.92}$$

which is an injection for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$.

Proof. First define a family of mappings for given a transversal pair $(x,y) \in \partial\Omega \times \partial\Omega$. By Lemma 16 there exists $\varepsilon > 0$ such that $(x', y') \in \partial\Omega \times \partial\Omega$ is also a transversal pair if $(x', y') \in \{B(x;\varepsilon) \cap$

$\partial\Omega\} \times \{B(x;\varepsilon) \cap \partial\Omega\}$. Further we can choose sufficiently small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so that

$$\nabla\xi(x') \cdot \frac{x' - x^0}{|x' - x^0|} \neq 0 \quad , \quad \nabla\xi(y') \cdot \frac{y' - x^0}{|y' - x^0|} \neq 0 \tag{3.93}$$

for all $x' \in B(x;\varepsilon_1) \cap \partial\Omega$ and $y' \in B(y;\varepsilon_2) \cap \partial\Omega$. Then we define a mapping

$$\mathbf{R}_{x^0} : \{B(x;\varepsilon_1(x,y)) \cap \partial\Omega\} \times \{B(y;\varepsilon_2(x,y)) \cap \partial\Omega\} \to [0,2\pi) \times [0,2\pi) \tag{3.94}$$

$$\mathbf{R}_{x^0}(x',y') = \left(\tan^{-1}\left(\frac{x_2' - x_2^0}{x_1' - x_1^0}\right), \tan^{-1}\left(\frac{y_2' - x_2^0}{y_1' - x_1^0}\right)\right). \tag{3.95}$$

Fix another transversal pair $(\bar{x},\bar{y})$ near (x,y) so that

$$B(x;\varepsilon(x,y)) \cap B(\bar{x};\varepsilon(\bar{x},\bar{y})) \cap \partial\Omega \neq \emptyset \tag{3.96}$$

$$\text{and} \quad B(y;\varepsilon(x,y)) \cap B(\bar{y};\varepsilon(\bar{x},\bar{y})) \cap \partial\Omega \neq \emptyset. \tag{3.97}$$

We want to show that $\mathbf{R}_{x^0} \circ \mathbf{R}_{\bar{x}^0}^{-1}$ and $\mathbf{R}_{\bar{x}^0} \circ \mathbf{R}_{x^0}^{-1}$ is differentiable. Choose $(X,Y) \in \{B(x;\varepsilon(x,y)) \cap B(\bar{x};\varepsilon(\bar{x},\bar{y})) \cap \partial\Omega\} \times \{B(y;\varepsilon(x,y)) \cap B(\bar{y};\varepsilon(\bar{x},\bar{y})) \cap \partial\Omega\}$. Denote

$$\mathbf{R}_{x^0}(X,Y) = (\theta,\varphi) \quad , \quad \mathbf{R}_{\bar{x}^0}(X,Y) = (\bar{\theta},\bar{\varphi}). \tag{3.98}$$

Without loss of generality we may assume $X = (X_1,X_2) = (X_1,\Phi(X_1))$ where Φ is a graph of $\partial\Omega$. By definition we have

$$\tan\theta = \frac{\Phi(X_1) - x_2^0}{X_1 - x_1^0} \quad , \quad \tan\bar{\theta} = \frac{\Phi(X_1) - \bar{x}_2^0}{X_1 - \bar{x}_1^0}. \tag{3.99}$$

From the first equation we have $\Psi(X_1,\theta) = 0$ where

$$\Psi(X_1,\theta) = \left[X_1 - x_1^0\right]\tan\theta - \Phi(X_1) + x_2^0. \tag{3.100}$$

We compute

$$\partial_{X_1}\Psi(X_1,\theta) = \tan\theta - \Phi'(X_1) \neq 0 \tag{3.101}$$

since (X,Y) is a transversal pair. By the implicit function theorem we have a differentiable function $X_1 = X_1(\theta)$ which satisfies the first equation of (3.99). Then we have

$$\bar{\theta} = \bar{\theta}(\theta) = \tan^{-1}\left(\frac{\Phi(X_1(\theta)) - \bar{x}_2^0}{X_1(\theta) - \bar{x}_1^0}\right) \tag{3.102}$$

which is a differentiable function. Similarly we have

$$\bar{\varphi} = \bar{\varphi}(\varphi) = \tan^{-1}\left(\frac{\bar{\Phi}(Y_1(\varphi)) - \bar{x}_1^0}{Y_1(\varphi) - \bar{x}_1^0}\right) \tag{3.103}$$

where $\bar{\Phi}$ is a local graph of $\partial\Omega$. Therefore $(\mathbf{R}_{\bar{x}^0} \circ \mathbf{R}_{x^0}^{-1})(\theta, \varphi) = (\bar{\theta}(\theta), \bar{\varphi}(\varphi))$ is differentiable. Similarly we can show $(\mathbf{R}_{x^0} \circ \mathbf{R}_{\bar{x}^0}^{-1})$ is differentiable. Therefore the family

$$(\{\partial\Omega \times B(x;\varepsilon_1(x,y))\} \times \{\partial\Omega \times B(y;\varepsilon_2(x,y))\}\ ,\ \mathbf{R}_{x,y}) \tag{3.104}$$

is a coordinate system near any transversal pair (x, y). □

We need another map near a transversal pair.

Lemma 20. *Let Ω be a smooth bounded open subset in $\mathbb{R}^2$. Let $(x, y) \in \partial\Omega \times \partial\Omega$ be a transversal pair. Then there exist $\varepsilon_1 = \varepsilon_1(\Omega, x, y) > 0, \varepsilon_2 = \varepsilon_2(\Omega, x, y) > 0$ and a* **parallel coordinate map**

$$\mathbf{P}_{x,y} : \{\partial\Omega \cap B(x;\varepsilon_1)\} \times \{\partial\Omega \cap B(y;\varepsilon_2)\} \longrightarrow \mathbb{R} \times \{0\} \times \mathbb{R} \times \{1\} \tag{3.105}$$

which is an injection for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$.

Proof. We define a family of mappings for given a transversal pair $(x, y) \in \partial\Omega \times \partial\Omega$. By Lemma 16 there exists $\varepsilon > 0$ such that $(x', y') \in \partial\Omega \times \partial\Omega$ is also a transversal pair if $(x', y') \in \{B(x;\varepsilon) \cap \partial\Omega\} \times \{B(x;\varepsilon) \cap \partial\Omega\}$. Therefore we know that

$$\nabla\xi(x') \cdot \frac{x' - x^0}{|x' - x^0|} \neq 0 \quad , \quad \nabla\xi(y') \cdot \frac{y' - x^0}{|y' - x^0|} \neq 0 \tag{3.106}$$

for all $x' \in B(x;\varepsilon_1) \cap \partial\Omega$ and $y' \in B(y;\varepsilon_2) \cap \partial\Omega$ with $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Now we will define the mapping $\mathbf{P}_{x,y}$ near the transversal pair on $\{\partial\Omega \cap B(x;\varepsilon_1)\} \times \{\partial\Omega \cap B(y;\varepsilon_2)\}$. First rotate $\overline{xy}$ to parallel to $y-$axis and translate x to an origin which is

$$Ax' = A\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{|x-y|}\begin{pmatrix} -x_2 + y_2 & x_1 - y_1 \\ -x_1 + y_1 & -x_2 + y_2 \end{pmatrix}\begin{pmatrix} x_1' - x_1 \\ x_2' - x_2 \end{pmatrix}. \tag{3.107}$$

Denote the matrix

$$\mathbb{A}_{x,y} = \begin{pmatrix} -x_2 + y_2 & x_1 - y_1 \\ -x_1 + y_1 & -x_2 + y_2 \end{pmatrix}. \tag{3.108}$$

Then consider a equation of line passing Ax' and Ay'

$$\mathfrak{L}(X, Y) = Y - [Ax']_2 - \frac{[A(y' - x')]_2}{[A(y' - x')]_1}(X - [Ax']_1) = 0 \tag{3.109}$$

and find out the value X to make $Y = 0, 1$. These are

$$P_1(x', y') = [Ax']_1 - [Ax']_2 \frac{[A(y' - x')]_1}{[A(y' - x')]_2} \tag{3.110}$$

$$P_2(x', y') = [Ax']_1 + (1 - [Ax']_2) \frac{[A(y' - x')]_1}{[A(y' - x')]_2}. \tag{3.111}$$

so that $\mathfrak{L}(P_1(x', y'), 0) = 0 = \mathfrak{L}(P_2(x', y'), 1)$. Define

$$\mathbf{P}_{x,y} : \{\partial\Omega \cap B(x; \varepsilon_1)\} \times \{\partial\Omega \cap B(y; \varepsilon_2)\} \to \mathbb{R} \times \{0\} \times \mathbb{R} \times \{1\} \tag{3.112}$$

$$\mathbf{P}_{x,y}(x', y') = (P_1(x', y'), 0, P_2(x', y'), 1). \tag{3.113}$$

Fix another transversal pair $(\bar{x}, \bar{y})$ near (x, y) so that

$$B(x; \varepsilon_1(x, y)) \cap B(\bar{x}; \varepsilon_1(\bar{x}, \bar{y})) \cap \partial\Omega \neq \emptyset \tag{3.114}$$

$$\text{and} \quad B(y; \varepsilon_2(x, y)) \cap B(\bar{y}; \varepsilon_2(\bar{x}, \bar{y})) \cap \partial\Omega \neq \emptyset. \tag{3.115}$$

We want to show that $\mathbf{P}_{x,y} \circ \mathbf{P}^{-1}_{\bar{x},\bar{y}}$ and $\mathbf{P}_{\bar{x},\bar{y}} \circ \mathbf{P}^{-1}_{x,y}$ is differentiable. Choose $(x', y') \in \{B(x; \varepsilon_1(x, y)) \cap B(\bar{x}; \varepsilon_1(\bar{x}, \bar{y})) \cap \partial\Omega\} \times \{B(y; \varepsilon_2(x, y)) \cap B(\bar{y}; \varepsilon_2(\bar{x}, \bar{y})) \cap \partial\Omega\}$. In order to define the inverse function $\mathbf{P}^{-1}_{x,y}$ temporally we define notations

$$\tilde{P}_1 = \begin{pmatrix} P_1(x', y') \\ 0 \end{pmatrix}, \ \tilde{P}_2 = \begin{pmatrix} P_2(x', y') \\ 1 \end{pmatrix}, \ \tilde{A}_1 = \begin{pmatrix} [A_{x,y}x']_1 \\ [A_{x,y}x']_2 \end{pmatrix}, \ \tilde{A}_2 = \begin{pmatrix} [A_{x,y}y']_1 \\ [A_{x,y}y']_2 \end{pmatrix}$$

where we used $A_{x,y}$ emphasizing (x, y) dependency of A. We can write (3.110) and (3.111) in the matrix form

$$\begin{pmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{pmatrix} = \frac{1}{[A_{x,y}y']_2 - [A_{x,y}x']_2} \begin{pmatrix} [A_{x,y}y']_2 & -[A_{x,y}x']_2 \\ -1 + [A_{x,y}y']_2 & 1 - [A_{x,y}x']_2 \end{pmatrix} \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} \tag{3.116}$$

Clearly we have

$$\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} = \begin{pmatrix} 1 - [A_{x,y}x']_2 & [A_{x,y}x']_2 \\ 1 - [A_{x,y}y']_2 & [A_{x,y}y']_2 \end{pmatrix} \begin{pmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{pmatrix}. \tag{3.117}$$

i.e.

$$\begin{pmatrix} [A_{x,y}x']_1 \\ [A_{x,y}x']_2 \end{pmatrix} = (1 - [A_{x,y}x']_2) \begin{pmatrix} P_1(x', y') \\ 0 \end{pmatrix} + [A_{x,y}x']_2 \begin{pmatrix} P_2(x', y') \\ 1 \end{pmatrix} \tag{3.118}$$

$$\begin{pmatrix} [A_{x,y}y']_1 \\ [A_{x,y}y']_2 \end{pmatrix} = (1 - [A_{x,y}y']_2) \begin{pmatrix} P_1(x', y') \\ 0 \end{pmatrix} + [A_{x,y}y']_2 \begin{pmatrix} P_2(x', y') \\ 1 \end{pmatrix}. \tag{3.119}$$

From definition of $A_{x,y}$ and above equations we have

$$0 = \Psi_1(x'_1, P_1, P_2) \equiv [A_{x,y}x']_1(x'_1) - (1 - [A_{x,y}x']_2(x'_1))P_1 - [Ax']_2(x'_1)P_2 \tag{3.120}$$

$$0 = \Psi_2(y'_1, P_1, P_2) \equiv [A_{x,y}y']_1(y'_1) - (1 - [A_{x,y}y']_2(y'_1))P_1 - [A_{x,y}y']_2(y'_1)P_2. \tag{3.121}$$

Recall that

$$[Ax']_1 = \frac{1}{|x-y|}\{(-x_2 + y_2)(x'_1 - x_1) + (x_1 - y_1)(\Phi_1(x'_1) - x_2)\} \tag{3.122}$$

$$[Ax']_2 = \frac{1}{|x-y|}\{(-x_1 + y_1)(x'_1 - x_1) + (-x_2 + y_2)(\Phi_1(x'_1) - x_2)\} \tag{3.123}$$

and similar formula for $[Ay']_1$ and $[Ay']_2$. In order to use the implicit function theorem we compute

$$\frac{\partial \Psi_1}{\partial x'_1} = [A_{x,y}x']'(x'_1) + P_1[A_{x,y}x']'(x'_1) - P_2[A_{x,y}x']'_2(x'_1)$$

$$= \frac{1}{|x-y|}\left\{ \underbrace{\begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \Phi'_1(x'_1) \end{pmatrix}}_{\mathbf{I}} + (-P_1 + P_2) \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \Phi'_1(x'_1) \end{pmatrix} \right\}.$$

We know that $\mathbf{I} \neq 0$ because (x, y) and (x', y') are transversal pairs. Since we can make $(-P_1 + P_2)$

small choosing small $\varepsilon > 0$ we show that $\frac{\partial \Psi_1}{\partial x_1'} \neq 0$. By the implicit function theorem we conclude that x_1' is a smooth function of (P_1, P_2). Similarly we conclude that y_1' is a smooth function of (P_1, P_2). To sum we have a smooth inverse function

$$\mathbf{P}_{x,y}^{-1} : (-\varepsilon, \varepsilon) \times \{0\} \times (-\varepsilon, \varepsilon) \times \{1\} \to \{\partial\Omega \cap B(x; \varepsilon_1)\} \times \{\partial\Omega \cap B(y; \varepsilon_2)\} \tag{3.124}$$

$$\mathbf{P}_{x,y}^{-1}(P_1, 0, P_2, 1) = \left(\begin{pmatrix} x_1' \\ \Phi_1(x_1') \end{pmatrix}, \begin{pmatrix} y_1' \\ \Phi_2(y_1') \end{pmatrix} \right). \tag{3.125}$$

We can explicitly compute

$$\mathbf{P}_{\bar{x},\bar{y}}(\mathbf{P}_{x,y}^{-1}(P_1, 0, P_2, 1)) = \mathbf{P}_{\bar{x},\bar{y}}(\begin{pmatrix} x_1'(P_1, P_2) \\ \Phi_1(x_1'(P_1, P_2)) \end{pmatrix}, \begin{pmatrix} y_1'(P_1, P_2) \\ \Phi_2(y_1'(P_1, P_2)) \end{pmatrix}) \tag{3.126}$$

$$= \left(\left([A_{\bar{x},\bar{y}} x'(P_1, P_2)]_1 - [A_{\bar{x},\bar{y}} x'(P_1, P_2)]_2 \frac{[A_{\bar{x},\bar{y}}(y'(P_1, P_2) - x'(P_1, P_2))]_1}{[A_{\bar{x},\bar{y}}(y'(P_1, P_2) - x'(P_1, P_2))]_2} \right) \right. , 0 \tag{3.127}$$

$$\left. , [A_{\bar{x},\bar{y}} x'(P_1, P_2)]_1 + (1 - [A x'(P_1, P_2)]_2) \frac{[A_{\bar{x},\bar{y}}(y'(P_1, P_2) - x'(P_1, P_2))]_1}{[A_{\bar{x},\bar{y}}(y'(P_1, P_2) - x'(P_1, P_2))]_2} , 1 \right). \tag{3.128}$$

Clearly we know that $\mathbf{P}_{\bar{x},\bar{y}} \circ \mathbf{P}_{x,y}^{-1}$ is smooth. □

Transversality

In the section we will prove so called the transversality (**??**) locally. Precisely let $y \in \Omega$ and $u \in \mathbb{R}^2 \backslash \{0\}$ such that $(x_{\mathbf{b}}(y, -u), x_{\mathbf{b}}(y, u))$ is a k^{th} transversal pair and $t_{k+1}(s_1, y, u) < s < t_k(s_1, y, u)$. For any $\varepsilon > 0$ there is $\delta(N, \varepsilon, T_0) > 0$ and an open ball $B(s_1, y, u; r_{s_1,y,u}) \subset [0, T_0] \times \mathbb{R}^2 \times \mathbb{R}^2$ and an open set $O_{s_1,y,u} \subset \mathbb{R} \times \mathbb{R}^2$ with $|O_{s_1,y,u}| < \varepsilon$ such that

$$J(\bar{s}; \bar{s}_1, \bar{y}, \bar{u}) = \det \left\{ \frac{\partial X_{\mathbf{cl}}(\bar{s}; \bar{s}_1, \bar{y}, \bar{u})}{\partial \bar{u}} \right\} \geq \delta > 0 \tag{3.129}$$

for all $(\bar{s}_1, \bar{y}, \bar{u}) \in B(s_1, y, u; r_{s_1,y,u})$ and $(\bar{s}, \bar{u})$ in

$$O_{s_1,y,u}^c \cap [0, T_0] \times \{|\bar{u}| \leq 2N\}. \tag{3.130}$$

In the case of $s \in \mathbb{R}$, $x \in \Omega$ and $v' \in \mathbb{R}^2$ such that $X_{\mathbf{cl}}(s; 0, x, v') \in \Omega$ and $(x_{\mathbf{b}}(x, v'), x_{\mathbf{b}}(x, -v')) \in \partial\Omega \times \partial\Omega$ is an transversal pair we will prove the transversality (**??**) holds . The transversality (**??**)

is very crucial step in $L^2 - L^\infty$ theory of Boltzmann equation near Maxwellian regime.

The main idea is to write $X_{\mathbf{cl}}(s; s_1, y_0, u)$ in terms of k^{th} collision-pair map and k^{th} collision-point map and k^{th} collision-distance map. Let $y_0 \in \Omega$ and $u \in \mathbb{R}^2 \backslash \{0\}$ such that $(x_{\mathbf{b}}(y_0, -u), x_{\mathbf{b}}(y_0, u))$ is k^{th} transversal pair and $t_{k+1}(s_1, y_0, u) < s < t_k(s_1, y_0, u)$. For sufficiently small $\varepsilon > 0$, $(x_{\mathbf{b}}(\bar{y}_0, -\bar{u}), x_{\mathbf{b}}(\bar{y}_0, \bar{u}))$ is also k^{th} transversal pair for $|(\bar{y}_0, \bar{u}) - (y_0, u)| < \varepsilon$. Define $\bar{x} = x_{\mathbf{b}}(\bar{y}_0, -\bar{u})$ and $\bar{y} = x_{\mathbf{b}}(\bar{y}_0, \bar{u})$. Then we have

$$\begin{aligned} \bar{t}_0 &= \bar{s}_1 \\ \bar{t}_1 &= \bar{s}_1 - [D_1(\bar{x}, \bar{y}) - |\bar{x} - \bar{y}_0|] \frac{1}{\bar{\xi}} \\ \bar{t}_2 &= \bar{s}_1 - [D_2(\bar{x}, \bar{y}) - |\bar{x} - \bar{y}_0|] \frac{1}{\bar{\xi}} \\ &\vdots \\ \bar{t}_k &= \bar{s}_1 - [D_k(\bar{x}, \bar{y}) - |\bar{x} - \bar{y}_0|] \frac{1}{\bar{\xi}}. \end{aligned}$$

Now we want to represent $v_k(\bar{y}_0, \bar{u})$ as a function of points. From $x_{k+1}(\bar{y}_0, \bar{u}) = x_k(\bar{y}_0, \bar{u}) - t_{\mathbf{b}}(x_k(\bar{y}_0, \bar{u}), v_k(\bar{y}_0, \bar{u})) v_k(\bar{y}_0, \bar{u})$ we get

$$\begin{aligned} v_k(\bar{y}_0, \bar{u}) = \frac{x_k(\bar{y}_0, \bar{u}) - x_{k+1}(\bar{y}_0, \bar{u})}{t_{\mathbf{b}}(x_k(\bar{y}_0, \bar{u}), v_k(\bar{y}_0, \bar{u}))} &= \frac{x_k(\bar{y}_0, \bar{u}) - x_{k+1}(\bar{y}_0, \bar{u})}{|x_k(\bar{y}_0, \bar{u}) - x_{k+1}(\bar{y}_0, \bar{u})|} \bar{\xi} && (3.131) \\ &= \frac{T_k(\bar{x}, \bar{y}) - T_{k+1}(\bar{x}, \bar{y})}{|\Delta \mathfrak{T}_{k+1}(\bar{x}, \bar{y})|} \bar{\xi}. && (3.132) \end{aligned}$$

where $\bar{\xi} = |\bar{u}| = |v_i(\bar{y}_0, \bar{u})|$ for all $i = 0, 1, 2, ..., k$. Now we can rewrite

$$\begin{aligned} X_{\mathbf{cl}}(\bar{s}; \bar{s}_1, \bar{y}_0, \bar{u}) &= x_k(\bar{y}_0, \bar{u}) - [t_k(\bar{s}_1, \bar{y}_0, \bar{u}) - \bar{s}] v_k(\bar{y}_0, \bar{u}) \\ &= T_k(\bar{x}, \bar{y}) + \left[\bar{s} - \bar{s}_1 + \frac{D_k(\bar{x}, \bar{y}) - |\bar{x} - \bar{y}_0|}{\bar{\xi}} \right] \frac{T_k(\bar{x}, \bar{y}) - T_{k+1}(\bar{x}, \bar{y})}{\Delta \mathfrak{T}_{k+1}(\bar{x}, \bar{y})} \bar{\xi} \\ &= \left(1 - \frac{|\bar{x} - \bar{y}_0| - D_k(\bar{x}, \bar{y})}{\Delta \mathfrak{T}_{k+1}(\bar{x}, \bar{y})} \right) T_k(\bar{x}, \bar{y}) + \frac{|\bar{x} - \bar{y}_0| - D_k(\bar{x}, \bar{y})}{\Delta \mathfrak{T}_{k+1}(\bar{x}, \bar{y})} T_{k+1}(\bar{x}, \bar{y}) \\ &\quad - (\bar{s}_1 - \bar{s}) \, \bar{\xi} \, \frac{T_k(\bar{x}, \bar{y}) - T_{k+1}(\bar{x}, \bar{y})}{\Delta \mathfrak{T}_{k+1}(\bar{x}, \bar{y})}. && (3.133) \end{aligned}$$

For fixed $(\bar{s}, \bar{s}_1, \bar{y}_0)$ we know that $X_{\mathbf{cl}}$ is a function of $\bar{x}, \bar{y}$ and $\bar{\xi}$ as (3.133). Since (x, y) is a transversal pair we can use the radial coordinate. Denote

$$\mathbf{R}_{\bar{y}_0}(\bar{x}, \bar{y}) = (\bar{\theta}, \bar{\varphi}). \tag{3.134}$$

Use the notation $p(\bar{x},\bar{y};\bar{y}_0) = \frac{|\bar{x}-\bar{y}_0|-D_k(\bar{x},\bar{y})}{\Delta\mathfrak{T}_{k+1}(\bar{x},\bar{y})}$ and

$$p(\bar{\theta},\bar{\varphi}) = p(\bar{\theta},\bar{\varphi};\bar{y}_0) = (p\circ\mathbf{R}_{\bar{y}_0}^{-1})(\bar{\theta},\bar{\varphi}). \tag{3.135}$$

Similarly we denote

$$T_k(\bar{\theta},\bar{\varphi}) = (T_k\circ\mathbf{R}_{\bar{y}_0}^{-1})(\bar{\theta},\bar{\varphi}) \quad , \quad T_{k+1}(\bar{\theta},\bar{\varphi}) = (T_{k+1}\circ\mathbf{R}_{\bar{y}_0}^{-1})(\bar{\theta},\bar{\varphi}). \tag{3.136}$$

and

$$X_{\mathbf{cl}}(\bar{\theta},\bar{\varphi},\bar{\xi}) = X_{\mathbf{cl}}(\bar{s},\bar{s}_1,\bar{y}_0;\bar{\theta},\bar{\varphi},\bar{\xi}) = X_{\mathbf{cl}}(\bar{s},\bar{s}_1,\bar{y}_0;\mathbf{R}_{\bar{y}_0}^{-1}(\bar{\theta},\bar{\varphi}),\bar{\varphi}) \tag{3.137}$$

$$= (1-p(\bar{\theta},\bar{\varphi}))T_k(\bar{\theta},\bar{\varphi}) + p(\bar{\theta},\bar{\varphi})T_{k+1}(\bar{\theta},\bar{\varphi}) - \frac{(\bar{s}_1-s)\bar{\xi}}{\Delta\mathfrak{T}_{k+1}(\bar{\theta},\bar{\varphi})}(T_k(\bar{\theta},\bar{\varphi}) - T_{k+1}(\bar{\theta},\bar{\varphi})). \tag{3.138}$$

In order to use the parallel coordinate we take a rigid motion

$$\mathbf{A}_{k,k+1} = \mathbf{A}_{T_k(x,y),T_{k+1}(x,y)} = \mathbf{A}_{x_k(y_0,u),x_{k+1}(y_0,u)} \tag{3.139}$$

as (3.107) to have

$$\mathbf{A}_{k,k+1}X_{\mathbf{cl}}(\bar{\theta},\bar{\varphi},\bar{\xi}) = \frac{1}{|T_k(x,y)-T_{k+1}(x,y)|}\Bigg\{(1-p(\bar{\theta},\bar{\varphi}))\mathbb{A}_{k,k+1}\begin{pmatrix}[T_k(\bar{\theta},\bar{\varphi})]_1-[T_k(x,y)]_1\\ [T_k(\bar{\theta},\bar{\varphi})]_2-[T_k(x,y)]_2\end{pmatrix}$$

$$+p(\bar{\theta},\bar{\varphi})\mathbb{A}_{k,k+1}\begin{pmatrix}[T_{k+1}(\bar{\theta},\bar{\varphi})]_1-[T_k(x,y)]_1\\ [T_{k+1}(\bar{\theta},\bar{\varphi})]_2-[T_k(x,y)]_2\end{pmatrix} - \frac{(\bar{s}_1-\bar{s})\bar{\xi}}{\Delta\mathfrak{T}_{k+1}(\bar{\theta},\bar{\varphi})}\mathbb{A}_{k,k+1}\begin{pmatrix}[T_k(\bar{\theta},\bar{\varphi})]_1-[T_{k+1}(\bar{\theta},\bar{\varphi})]_1\\ [T_k(\bar{\theta},\bar{\varphi})]_2-[T_{k+1}(\bar{\theta},\bar{\varphi})]_2\end{pmatrix}\Bigg\}$$

where $\mathbb{A}_{k,k+1} = \mathbb{A}_{T_k(x,y),T_{k+1}(x,y)}$. By definition of a parallel coordinate map we have

$$\mathbf{A}_{k,k+1}X_{\mathbf{cl}}(\bar{\theta},\bar{\varphi},\bar{\xi}) = (1-p(\bar{\theta},\bar{\varphi}))\begin{pmatrix}P_1(\bar{\theta},\bar{\varphi})\\ 0\end{pmatrix} + p(\bar{\theta},\bar{\varphi})\begin{pmatrix}P_2(\bar{\theta},\bar{\varphi})\\ 1\end{pmatrix} \tag{3.140}$$

$$-\frac{(\bar{s}_1-\bar{s})\bar{\xi}}{\Delta\mathfrak{T}_{k+1}(\bar{\theta},\bar{\varphi})}\begin{pmatrix}P_1(\bar{\theta},\bar{\varphi})-P_2(\bar{\theta},\bar{\varphi})\\ -1\end{pmatrix}. \tag{3.141}$$

where

$$\mathbf{A}_{k,k+1}\circ T_k\circ\mathbf{R}_{\bar{y}_0}^{-1} = \begin{pmatrix}P_1\\ 0\end{pmatrix} \quad , \quad \mathbf{A}_{k,k+1}\circ T_{k+1}\circ\mathbf{R}_{\bar{y}_0}^{-1} = \begin{pmatrix}P_2\\ 1\end{pmatrix}. \tag{3.142}$$

Since $\bar{x} = x_{\mathbf{b}}(\bar{y}_0, -\bar{u})$ and $\bar{y} = x_{\mathbf{b}}(\bar{y}_0, \bar{u})$ we have $\bar{\varphi} = \bar{\theta} + \pi$. Denote $\mathbf{A}X_{\mathbf{cl}}(\bar{\theta}, \bar{\xi}) = \mathbf{A}_{k,k+1}X_{\mathbf{cl}}(\bar{\theta}, \bar{\theta} + \pi, \bar{\xi})$. By direct computation we have

$$
\begin{aligned}
\frac{\partial [\mathbf{A}X_{\mathbf{cl}}]_1}{\partial \bar{\theta}} = &\underbrace{\left(-p'(\bar{\theta}) + \frac{(\bar{s}_1 - \bar{s})\ \bar{\xi}\ \partial_{\bar{\theta}}(\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi))}{(\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi))^2}\right)}_{\mathbf{I}} \{P_1(\bar{\theta}, \bar{\theta}+\pi) - P_2(\bar{\theta}, \bar{\theta}+\pi)\} \\
&+ (1 - p(\bar{\theta}), p(\bar{\theta})) \begin{pmatrix} \partial_1 P_1(\bar{\theta}, \bar{\theta}+\pi) & \partial_2 P_1(\bar{\theta}, \bar{\theta}+\pi) \\ \partial_1 P_2(\bar{\theta}, \bar{\theta}+\pi) & \partial_2 P_2(\bar{\theta}, \bar{\theta}+\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&- \frac{(\bar{s}_1 - \bar{s})\bar{\xi}}{\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi)} (1, -1) \begin{pmatrix} \partial_1 P_1(\bar{\theta}, \bar{\theta}+\pi) & \partial_2 P_1(\bar{\theta}, \bar{\theta}+\pi) \\ \partial_1 P_2(\bar{\theta}, \bar{\theta}+\pi) & \partial_2 P_2(\bar{\theta}, \bar{\theta}+\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned}
$$

$$
\frac{\partial [\mathbf{A}X_{\mathbf{cl}}]_1}{\partial \bar{\xi}} = -\frac{\bar{s}_1 - \bar{s}}{\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi)} (P_1(\bar{\theta}, \bar{\theta}+\pi) - P_2(\bar{\theta}, \bar{\theta}+\pi)),
$$

$$
\frac{\partial [\mathbf{A}X_{\mathbf{cl}}]_2}{\partial \bar{\theta}} = \underbrace{p'(\bar{\theta}) - \frac{(\bar{s}_1 - \bar{s})\ \bar{\xi}\ \partial_{\bar{\theta}}(\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi))}{(\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi))^2}}_{\mathbf{II}},
$$

$$
\frac{\partial [\mathbf{A}X_{\mathbf{cl}}]_2}{\partial \bar{\xi}} = \frac{\bar{s}_1 - \bar{s}}{\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi)}.
$$

Clearly every geometric information is in $\partial_{\bar{\theta}}(\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi))$ and $\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi)$ which is extremely hard to compute or control. However because of the fact $\mathbf{I} = -\mathbf{II}$ the jacobian enjoys crucial cancelation and does not include the geometric information $\mathbf{I}$ or $\mathbf{II}$. Precisely we can compute

$$
\begin{aligned}
\det\left(\frac{\partial \mathbf{A}X_{\mathbf{cl}}}{\partial(\bar{\theta}, \bar{\xi})}\right) = &(1 - p(\bar{\theta}), p(\bar{\theta})) \begin{pmatrix} \partial_1 P_1(\bar{\theta}, \bar{\theta}+\pi) & \partial_2 P_1(\bar{\theta}, \bar{\theta}+\pi) \\ \partial_1 P_2(\bar{\theta}, \bar{\theta}+\pi) & \partial_2 P_2(\bar{\theta}, \bar{\theta}+\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{\bar{s}_1 - \bar{s}}{(\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi))^{1/2}} \\
&+ (-1, 1) \begin{pmatrix} \partial_1 P_1(\bar{\theta}, \bar{\theta}+\pi) & \partial_2 P_1(\bar{\theta}, \bar{\theta}+\pi) \\ \partial_1 P_2(\bar{\theta}, \bar{\theta}+\pi) & \partial_2 P_2(\bar{\theta}, \bar{\theta}+\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{(\bar{s}_1 - \bar{s})^2}{\Delta\mathfrak{T}_{k+1}(\bar{\theta}, \bar{\theta}+\pi)} \bar{\xi}.
\end{aligned}
$$

Returning to our purpose we want to calculate $\det\left(\frac{\partial X_{\mathbf{cl}}(\bar{s}; \bar{s}_1, \bar{y}, \bar{u})}{\partial \bar{u}}\right)$ which equals

$$
\frac{1}{\det\left(\mathbb{A}_{T_k(x,y), T_{k+1}(x,y)}\right)} \cdot \det\left(\frac{\partial(\bar{\theta}, \bar{\xi})}{\partial \bar{u}}\right) \cdot \det\left(\frac{\partial \mathbf{A}X_{\mathbf{cl}}}{\partial(\bar{\theta}, \bar{\xi})}\right)
$$

From direct computation we know that $\det\left(\mathbb{A}_{T_k(x,y), T_{k+1}(x,y)}\right) = 1$ and $\det\left(\frac{\partial(\bar{\theta}, \bar{\xi})}{\partial \bar{u}}\right) = 1/\bar{\xi}$.

Therefore we have $\det\left(\frac{\partial X_{\mathbf{cl}}(\bar{s};\bar{s}_1,\bar{y},\bar{u})}{\partial \bar{u}}\right) =$

$$\underbrace{(1-p(\bar\theta),p(\bar\theta))\begin{pmatrix}\partial_1 P_1(\bar\theta,\bar\theta+\pi) & \partial_2 P_1(\bar\theta,\bar\theta+\pi)\\ \partial_1 P_2(\bar\theta,\bar\theta+\pi) & \partial_2 P_2(\bar\theta,\bar\theta+\pi)\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}}_{\mathcal{A}(\bar\theta)}\frac{\bar{s}_1-\bar{s}}{\Delta\mathfrak{T}_{k+1}(\bar\theta,\bar\theta+\pi)}\frac{1}{\bar\xi} \tag{3.143}$$

$$\underbrace{+(-1,1)\begin{pmatrix}\partial_1 P_1(\bar\theta,\bar\theta+\pi) & \partial_2 P_1(\bar\theta,\bar\theta+\pi)\\ \partial_1 P_2(\bar\theta,\bar\theta+\pi) & \partial_2 P_2(\bar\theta,\bar\theta+\pi)\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}}_{\mathcal{B}(\bar\theta)}\frac{(\bar{s}_1-\bar{s})^2}{(\Delta\mathfrak{T}_{k+1}(\bar\theta,\bar\theta+\pi))^2} \tag{3.144}$$

$$= \frac{\mathcal{A}(\bar\theta)}{\Delta\mathfrak{T}_{k+1}(\bar\theta,\bar\theta+\pi)}\frac{\bar{s}_1-\bar{s}}{\bar\xi} + \frac{\mathcal{B}(\bar\theta)}{(\Delta\mathfrak{T}_{k+1}(\bar\theta,\bar\theta+\pi))^2}(\bar{s}_1-\bar{s})^2. \tag{3.145}$$

The essential observation is following :

Lemma 21. *Let Ω be an open smooth subset of $\mathbb{R}^2$. Suppose (x,y) be k^{th} transversal pair and $x^0 \in \overline{xy}\cap\Omega$. For sufficiently small $\varepsilon > 0$ if $|\bar{x}-x|+|\bar{y}-y|+|\bar{y}_0-x^0| < \varepsilon$ then*

$$\nabla\xi(\bar{x})\cdot\frac{\bar{x}-\bar{y}_0}{|\bar{x}-\bar{y}_0|} \neq 0 \quad , \quad \nabla\xi(\bar{y})\cdot\frac{\bar{y}-\bar{y}_0}{|\bar{y}-\bar{y}_0|} \neq 0 \tag{3.146}$$

and denote $\mathbf{R}_{\bar{y}_0}^{-1}(\bar{x},\bar{y}) = (\bar\theta,\bar\varphi)$ and

$$(\mathbf{A}_{T_k(x,y),T_{k+1}(x,y)} \circ T_k \circ \mathbf{R}_{\bar{y}_0}^{-1})(\bar\theta,\bar\varphi) = \begin{pmatrix}P_1(\bar\theta,\bar\varphi)\\0\end{pmatrix}, \tag{3.147}$$

$$(\mathbf{A}_{T_k(x,y),T_{k+1}(x,y)} \circ T_{k+1} \circ \mathbf{R}_{\bar{y}_0}^{-1})(\bar\theta,\bar\varphi) = \begin{pmatrix}P_2(\bar\theta,\bar\varphi)\\1\end{pmatrix}. \tag{3.148}$$

Then

$$\begin{pmatrix}\partial_1 P_1(\bar\theta,\bar\varphi) & \partial_2 P_1(\bar\theta,\bar\varphi)\\ \partial_1 P_2(\bar\theta,\bar\varphi) & \partial_2 P_2(\bar\theta,\bar\varphi)\end{pmatrix} \tag{3.149}$$

is invertible.

Proof. From Lemma 17 we know that $\mathfrak{T}_k$ is a local diffeomorphism. Since the radial coordinate map and the parallel coordinate map are coordinate maps by Lemma 19 and Lemma 20 we have (3.149). $\square$

Because of (3.149) we know that

$$\begin{pmatrix} \partial_1 P_1 & \partial_2 P_1 \\ \partial_1 P_2 & \partial_2 P_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.150}$$

Since

$$(1 - p(\bar\theta), p(\bar\theta)) \quad , \quad (-1, 1) \tag{3.151}$$

cannot be parallel, we know that $\mathcal{A}(\bar\theta) \neq 0$ or $\mathcal{B}(\bar\theta) \neq 0$ for any $\bar\theta$.

If $\mathcal{B}(\bar\theta) = 0$ for some $\bar\theta$ then $\mathcal{A}(\bar\theta) \neq 0$. Therefore $\det\left(\frac{\partial X_{\mathbf{cl}}(\bar s;\bar s_1,\bar\theta,\bar\xi)}{\partial \bar u}\right) = 0$ only for $\bar s_1 = \bar s$. If $\mathcal{A}(\theta) = 0$ for some θ then $\mathcal{B}(\theta) \neq 0$ and $\det\left(\frac{\partial X_{\mathbf{cl}}(\bar s;\bar s_1,\bar\theta,\bar\xi)}{\partial \bar u}\right) = 0$ only for $\bar s_1 = \bar s$. In the last case of $\mathcal{A}(\bar\theta) \neq 0$ and $\mathcal{B}(\bar\theta) \neq 0$ then $\det\left(\frac{\partial X_{\mathbf{cl}}(\bar s;\bar s_1,\bar\theta,\bar\xi)}{\partial \bar u}\right) = 0$ for $\bar s, \bar s_1, \bar\theta$ and $\bar\xi$ such that

$$(\bar s_1 - \bar s)\bar\xi = -\frac{\mathcal{A}(\bar\theta)}{\mathcal{B}(\bar\theta)}\ \Delta\mathfrak{T}_{k+1}(\bar\theta, \bar\theta + \pi). \tag{3.152}$$

Example 11. *We will construct one example of the totally grazing point. Assume that*

$$\{(-2 + \cos\theta, \sin\theta) \in \mathbb{R}^2 \ : \ \theta \in [0, 2\pi]\} \subset \partial\Omega.$$

Fix a point $(\frac{3}{2}, \frac{\sqrt{3}}{2}) \in \mathbb{R}^2$. *We will construct a function* $f : (-\delta, \delta) \to \mathbb{R}$ *so that*

$$\{(x, f(x)) \in \mathbb{R}^2 \ : \ x \in (-\delta, \delta)\} \in \partial\Omega$$

and there exist $\delta_1 > 0$ *so that*

$$\begin{aligned} x_2((\frac{3}{2}, \frac{\sqrt{3}}{2}), (\cos\theta, \sin\theta)) &\in \{(-2 + \cos\theta, \sin\theta) \in \mathbb{R}^2 \ : \ \theta \in [0, 2\pi]\} \\ v_1((\frac{3}{2}, \frac{\sqrt{3}}{2}), (\cos\theta, \sin\theta)) \cdot n(x_2((\frac{3}{2}, \frac{\sqrt{3}}{2}), (\cos\theta, \sin\theta))) &= 0 \end{aligned}$$

for all $\theta \in (\frac{\pi}{3} - \delta_1, \frac{\pi}{3} + \delta_1)$.

The unknown function f has to satisfy two equations : the first equation due to the specular reflection condition

$$
\begin{aligned}
&\frac{1}{[(x-\frac{3}{2})^2+(f(x)-\frac{\sqrt{3}}{2})^2]^{1/2}}\begin{pmatrix} x-\frac{3}{2} \\ f(x)-\frac{\sqrt{3}}{2}\end{pmatrix} \\
&= \frac{1}{[(-2+\cos\theta-x)^2+(\sin\theta-f(x))^2]^{1/2}}\begin{pmatrix} -2+\cos\theta-x \\ \sin\theta-f(x)\end{pmatrix} \\
&\quad -\frac{2}{[1+(f'(x))^2]^{1/2}}\begin{pmatrix} f'(x) \\ -1\end{pmatrix} \\
&\times\left[\frac{1}{[1+(f'(x))^2]^{1/2}}\begin{pmatrix} f'(x) \\ -1\end{pmatrix}\cdot\frac{1}{[(-2+\cos\theta-x)^2+(\sin\theta-f(x))^2]^{1/2}}\begin{pmatrix} -2+\cos\theta-x \\ \sin\theta-f(x)\end{pmatrix}\right]
\end{aligned} \tag{3.153}
$$

and the second equation is due to the grazing condition

$$
\begin{pmatrix} \cos\theta \\ \sin\theta\end{pmatrix}\cdot\begin{pmatrix} -2+\cos\theta-x \\ \sin\theta-f(x)\end{pmatrix}=0.
$$

From the second equation we have a formula for $f(x)=\sin^{-1}\theta\{1-2\cos\theta-x\cos\theta\}$. Notice that from this equation if we have a formula of $x=x(\theta)$ then we have a formula of $f(x)$. In order to study $x=x(\theta)$ we take θ-derivative to have :

$$
\frac{dx}{d\theta}=\frac{1}{f'(x)+\frac{\cos\theta}{\sin\theta}}\left\{2+x-\frac{\cos\theta}{\sin^2\theta}(1-2\cos\theta-x\cos\theta)\right\}
$$

where we derive the formula of f from the other equation. Multiplying the second raw of (3.153) by

$$
[(-2+\cos\theta-x)^2+(\sin\theta-f(x))^2]^{1/2}[(x-\frac{3}{2})^2+(f(x)-\frac{\sqrt{3}}{2})^2]^{1/2}
$$

to have

$$
\begin{aligned}
\mathfrak{A}(f'(x))^2-2[(x-\frac{3}{2})^2+(f(x)-\frac{\sqrt{3}}{2})^2]^{1/2}(-2+\cos\theta-x)f'(x) \\
+2[(x-\frac{3}{2})^2+(f(x)-\frac{\sqrt{3}}{2})^2]^{1/2}(\sin\theta-f(x))+\mathfrak{A}=0
\end{aligned}
$$

where

$$\mathfrak{A} = (\frac{1-2\cos\theta - x\cos\theta}{\sin\theta} - \frac{\sqrt{3}}{2})[(-2+\cos\theta - x)^2 + (\sin\theta - \frac{1-2\cos\theta - x\cos\theta}{\sin\theta})^2]^{1/2}$$
$$-(\sin\theta - \frac{1-2\cos\theta - x\cos\theta}{\sin\theta})[(x-\frac{3}{2})^2 + (\frac{1-2\cos\theta - x\cos\theta}{\sin\theta} - \frac{\sqrt{3}}{2})^2]^{1/2}.$$

Notice that $\mathfrak{A}|_{x=0,\theta=\frac{\pi}{3}} = -3$. We can solve (3.154) using the formula of the solutions of the quadratic equation. When we apply the formula we have to choose a 'proper' sign. Since we require that $f'(0)=0$ the formula for $f'(x)$ is

$$\begin{aligned} f'(x) = \mathfrak{B} \quad \equiv \quad & \mathfrak{A}^{-1}\Big\{[(x-\frac{3}{2})^2 + (f(x) - \frac{\sqrt{3}}{2})^2]^{1/2}(-2+\cos\theta - x) \\ & + \Big[[(x-\frac{3}{2})^2 + (\frac{1-2\cos\theta - x\cos\theta}{\sin\theta} - \frac{\sqrt{3}}{2})^2](-2+\cos\theta - x)^2 \\ & -\mathfrak{A}\{\mathfrak{A} + 2\big[(x-\frac{3}{2})^2 + (\frac{1-2\cos\theta - x\cos\theta}{\sin\theta} - \frac{\sqrt{3}}{2})^2\big]^{1/2}(\sin\theta - f(x))\}\Big]^{\frac{1}{2}}\Big\}. \end{aligned}$$

Notice that $\mathfrak{B}|_{x=0,\theta=\frac{\pi}{3}} = 0$. Therefore the equation of $x = x(\theta)$ is

$$\frac{dx}{d\theta} = F(x,\theta),$$

where

$$F(x,\theta) = \frac{1}{\mathfrak{B} + \frac{\cos\theta}{\sin\theta}}\Big\{2 + x - \frac{\cos\theta}{\sin^2\theta}(1-2\cos\theta - x\cos\theta)\Big\}.$$

In order to apply the Picard existence theory(Section 1.10 of [11]) we need to check the fact

$$F(x,\theta)\ , \quad \frac{\partial F}{\partial x}(x,\theta)$$

are continuous near $(x,\theta) = (0,\frac{\pi}{3})$. Clearly $F(x,\theta)$ is continuous and

$$F(0,\frac{\pi}{3}) = 2\sqrt{3} \neq 0$$

Directly we compute

$$
\begin{aligned}
\frac{\partial \mathfrak{A}}{\partial x} &= \Big[(-2+\cos\theta-x)^2+(\sin\theta-\frac{1-2\cos\theta-x\cos\theta}{\sin\theta})^2\Big]^{-1/2} \\
&\quad \times\Big\{-\frac{\cos\theta}{\sin\theta}\Big[(-2+\cos\theta-x)^2+(\sin\theta-\frac{1-2\cos\theta-x\cos\theta}{\sin\theta})^2\Big] \\
&\qquad\qquad +\Big(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)\frac{2+x-\cos\theta}{\sin^2\theta}\Big\} \\
&+ \Big[(x-\frac{3}{2})^2+\Big(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)^2\Big]^{-1/2} \\
&\qquad \times\Big\{-\frac{\cos\theta}{\sin\theta}\Big[(x-\frac{3}{2})^2+\Big(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)^2\Big] \\
&\quad -\Big(\sin\theta-\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}\Big)\frac{2+x-\cos\theta+\frac{\sqrt{3}}{2}\cos\theta\sin\theta-\frac{7}{2}\sin^2\theta}{\sin^2\theta}\Big\}
\end{aligned}
$$

and

$$
\frac{\partial \mathfrak{A}}{\partial x}\Big|_{x=0,\theta=\frac{\pi}{3}} = -\frac{5}{2}. \tag{3.154}
$$

Further directly we have

$$
\frac{\partial F}{\partial x} = \frac{1}{\sin^2\theta\,\big(\mathfrak{B}+\frac{\cos\theta}{\sin\theta}\big)} - \frac{2+x-\frac{\cos\theta}{\sin^2\theta}(1-2\cos\theta-x\cos\theta)}{\big(\mathfrak{B}+\frac{\cos\theta}{\sin\theta}\big)^2}\frac{\partial \mathfrak{B}}{\partial x} \tag{3.155}
$$

where

$$
\begin{aligned}
\frac{\partial \mathfrak{B}}{\partial x}\times\{-\mathfrak{A}\} &= \frac{\partial \mathfrak{A}}{\partial x}\times\mathfrak{B}+(-2+\cos\theta-x)\frac{(x-\frac{3}{2})+(f(x)-\frac{\sqrt{3}}{2})f'(x)}{[(x-\frac{3}{2})^2+(f(x)-\frac{\sqrt{3}}{2})^2]^{1/2}}-[(x-\frac{3}{2})^2+(f(x)-\frac{\sqrt{3}}{2})^2]^{1/2} \\
&+ \frac{1}{2}\Big[[(x-\frac{3}{2})^2+(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2})^2](-2+\cos\theta-x)^2 \\
&\qquad -\mathfrak{A}\big\{\mathfrak{A}+2\big[(x-\frac{3}{2})^2+(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2})^2\big]^{1/2}(\sin\theta-f(x))\big\}\Big]^{-1/2} \\
&\quad\times\Bigg(\Big[2x-3-\frac{2\cos\theta}{\sin\theta}\Big(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)\Big](-2+\cos\theta-x)^2 \\
&\qquad -2(-2+\cos\theta-x)\Big[\Big(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)^2+\Big(x-\frac{3}{2}\Big)^2\Big] \\
&\qquad -\frac{\partial \mathfrak{A}}{\partial x}\Big\{\mathfrak{A}+2(\sin\theta-f(x))\Big[(x-\frac{3}{2})^2+\Big(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)^2\Big]^{1/2}\Big\} \\
&\qquad -\mathfrak{A}\Big\{\frac{\partial \mathfrak{A}}{\partial x}+2\frac{(\sin\theta-f(x))\Big[x-\frac{3}{2}-\frac{\cos\theta}{\sin\theta}\Big(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)\Big]}{\Big[\big(x-\frac{3}{2}\big)^2+\Big(\frac{1-2\cos\theta-x\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)^2\Big]^{1/2}} \\
&\qquad -2\mathfrak{B}\Big[\big(x-\frac{3}{2}\big)^2+\Big(\frac{1-2\cos\theta-2\cos\theta}{\sin\theta}-\frac{\sqrt{3}}{2}\Big)^2\Big]^{1/2}\Big\}\Bigg).
\end{aligned}
$$

Using the fact $\mathfrak{A}|_{x=0,\theta=\frac{\pi}{3}} = -3$ we know that $\frac{\partial \mathfrak{B}}{\partial x}$ is continuous function near $(x,\theta) = (0,\frac{\pi}{3})$. Moreover from the formula of $F(x,\theta)$ it is clear that $F(x,\theta)$ is a real analytic function. From the Picard theory we can solve $x = x(\theta)$ for $\theta \in (\frac{\pi}{3} - \delta, \frac{\pi}{3} + \delta)$. Moreover from the fact $F(0,\frac{\pi}{3}) \neq 0$ we can use the inverse function theorem to solve $\theta = \theta(x)$ locally. Further from the formula of f we have the desired analytic function

$$f(x) = \frac{1}{\sin\theta(x)} \{1 - 2\cos\theta(x) - x\cos\theta(x)\} \tag{3.156}$$

Lemma 22. *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with*

$$\frac{\partial^k \Phi}{\partial y^k}(x_0, y_0) \neq 0 \tag{3.157}$$

for some $k \in \mathbb{N}$. Then there exists $\varepsilon > 0$ and continuous functions $y_i : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ such that $y_i(x_0) = y_0$ for $i = 1, 2, ..., k-1$ and

$$\{(x,y) : x \in (x_0 - \varepsilon, x_0 + \varepsilon) \text{ and } \Phi(x,y) = 0\} = \bigcup_{i=1}^{k-1} \{(x,y) : x \in (x_0 - \varepsilon, x_0 + \varepsilon)\ ,\ y = y_i(x)\} \tag{3.158}$$

there are at most $k-1$ continuous functions $x_i = x_i(y)$ for $i = 1, 2, ..., k-1$ such that

Proof. Without loss of generality we may assume $\frac{\partial^i \Phi}{\partial x^i}(x_0, y_0) = 0$ for $i = 0, 1, 2, ..., k-1$. Write $x = x_0 + \delta x$ and $y = y_0 + \delta y$ and use the Taylor expansion to get

$$0 = \Phi(x_0 + \delta x, y_0 + \delta y) = \underline{\Phi(x_0, y_0)} + \underline{\frac{\partial \Phi}{\partial x}\Big|_{(x_0,y_0)}} \delta x + \frac{\partial \Phi}{\partial y}\Big|_{(x_0,y_0)} \delta y \tag{3.159}$$

$$+ \underline{\frac{1}{2!}\frac{\partial^2 \Phi}{\partial x^2}\Big|_{(x_0,y_0)}} \delta x^2 + \frac{1}{2!}\frac{\partial^2 \Phi}{\partial y^2}\Big|_{(x_0,y_0)} \delta y^2 + \frac{\partial^2 \Phi}{\partial x \partial y}\Big|_{(x_0,y_0)} \delta x \delta y + \cdots \tag{3.160}$$

$$+ \underline{\frac{1}{(k-1)!}\frac{\partial^{k-1} \Phi}{\partial x^{k-1}}\Big|_{(x_0,y_0)}} (\delta x)^{k-1} + \cdots + \frac{1}{(k-1)!}\frac{\partial^{k-1} \Phi}{\partial y^{k-1}}\Big|_{(x_0,y_0)} (\delta y)^{k-1} \tag{3.161}$$

$$+ \frac{1}{k!}\frac{\partial^k \Phi}{\partial x^k}\Big|_{(x_0,y_0)} (\delta x)^k + \cdots + \frac{1}{k!}\frac{\partial^k \Phi}{\partial y^k}\Big|_{(x_0,y_0)} (\delta y)^k + O(|\delta x|^{k+1} + |\delta y|^{k+1}). \tag{3.162}$$

By the assumption all underlined terms are zero. Therefore we have $0 = \Phi(x_0 + \delta x, y_0 + \delta y) =$

$$\frac{1}{k!}\frac{\partial^k \Phi}{\partial x^k}\Big|_{(x_0,y_0)} (\delta x)^k + \underline{\sum_{i=1}^{k}\sum_{j=0}^{i-1} C_{k,i,j} \frac{\partial^i \Phi}{\partial x^j \partial y^{i-j}}\Big|_{(x_0,y_0)} (\delta x)^j (\delta y)^{i-j}} + O(|\delta x|^{k+1}) + O(|\delta y|^{k+1}) \tag{3.163}$$

where underlined term can be written as

$$\left(\frac{1}{k!}\left|\frac{\partial^k\Phi}{\partial x^k}\Big|_{(x_0,y_0)}\right|\right)^{j/k}(\delta x)^j \times C_{k,i,j}\frac{\left|\frac{\partial^i\Phi}{\partial x^j\partial y^{i-j}}\Big|_{(x_0,y_0)}\right|}{\left(\frac{1}{k!}\left|\frac{\partial^k\Phi}{\partial x^k}\Big|_{(x_0,y_0)}\right|\right)^{j/k}}(\delta y)^{i-j} \tag{3.164}$$

Using the Young's inequality with ε we have $\frac{1}{k!}\frac{\partial^k\Phi}{\partial x^k}\Big|_{(x_0,y_0)}(\delta x)^k + O(|\delta x|^{k+1}) \leq$

$$O(|\delta y|^{k+1}) + \sum_{i=1}^{k}\sum_{j=0}^{i-1}\frac{\varepsilon}{k^2}\frac{1}{k!}\left|\frac{\partial^k\Phi}{\partial x^k}\Big|_{(x_0,y_0)}\right|(\delta x)^k + \sum_{i=1}^{k}\sum_{j=0}^{i-1}C(\varepsilon)\left\{C_{k,i,j}\frac{\left|\frac{\partial^i\Phi}{\partial x^j\partial y^{i-j}}\Big|_{(x_0,y_0)}\right|}{\frac{1}{k!}\left|\frac{\partial^k\Phi}{\partial x^k}\Big|_{(x_0,y_0)}\right|^{j/k}}\right\}^{\frac{k}{k-j}}(\delta y)^{\frac{k(i-j)}{k-j}} \tag{3.165}$$

For sufficiently small $|\delta x| \leq O(\left|\frac{\partial^k\Phi}{\partial x^k}\Big|_{(x_0,y_0)}\right|)$ we have

$$|\delta x| \leq \frac{1}{\sqrt[k]{\frac{2-\varepsilon}{k!}\left|\frac{\partial^k\Phi}{\partial x^k}\Big|_{(x_0,y_0)}\right|}} \times \left\{|\delta y|^{k+1} + \sum_{i=1}^{k}\sum_{j=0}^{i-1}\tilde{C}_{k,i,j}\ \delta y^{\frac{k(i-j)}{k-j}}\right\}^{1/k} \tag{3.166}$$

where $\tilde{C}_{k,i,j} = C(\varepsilon)\left\{C_{k,i,j}\frac{\left|\frac{\partial^i\Phi}{\partial x^j\partial y^{i-j}}\Big|_{(x_0,y_0)}\right|}{\frac{1}{k!}\left|\frac{\partial^k\Phi}{\partial x^k}\Big|_{(x_0,y_0)}\right|^{j/k}}\right\}^{\frac{k}{k-j}}$. Therefore $\delta x \to 0$ and $\delta y \to 0$. □

Lemma 23. *Let $\Omega = \Omega' \times [0,1]/\sim$ be the Smooth Convex Domain defined in (1.22). Define the functional along the trajectories $\frac{d\bar{X}(s)}{ds} = \bar{V}(s)$, $\frac{dV(s)}{ds} = 0$ in (**??**) as:*

$$\alpha(s) \equiv \xi^2(\bar{X}(s)) + [\bar{V}(s)\cdot\nabla_{\bar{x}}\xi(\bar{X}(s))]^2 - 2\{\bar{V}(s)\cdot\nabla_{\bar{x}}^2\xi(\bar{X}(s))\cdot\bar{V}(s)\}\xi(\bar{X}(s)). \tag{3.167}$$

Let $\bar{X}(s) \in \Omega'$ for $t_1 \leq s \leq t_2$. Then there exists constant $C_\xi > 0$ such that

$$\begin{aligned} e^{C_\xi(|\bar{V}(t_1)|+1)t_1}\alpha(t_1) &\leq e^{C_\xi(|\bar{V}(t_1)|+1)t_2}\alpha(t_2); \\ e^{-C_\xi(|\bar{V}(t_1)|+1)t_1}\alpha(t_1) &\geq e^{-C_\xi(|\bar{V}(t_1)|+1)t_2}\alpha(t_2). \end{aligned} \tag{3.168}$$

Proof. Under the convexity assumption (1.22), we notice that $\alpha(s) \geq 0$ for $t_1 \leq s \leq t_2$. Since $\frac{dV(s)}{ds} \equiv 0$ by (3.13), we compute the derivative of $\alpha(s)$ in (3.167) as

$$\begin{aligned} \frac{d\alpha(s)}{ds} &= 2\xi(\bar{X}(s))[\nabla_{\bar{x}}\xi(\bar{X}(s))\cdot\bar{V}(s)] + 2[\bar{V}(s)\nabla_{\bar{x}}^2\xi(\bar{X}(s))\bar{V}(s))][\bar{V}(s)\cdot\nabla_{\bar{x}}\xi(\bar{X}(s))] \\ &\quad -2\{\bar{V}(s)\cdot\nabla_{\bar{x}}^2\xi(\bar{X}(s))\cdot\bar{V}(s)\}[\nabla_{\bar{x}}\xi(\bar{X}(s))\cdot\bar{V}(s)] \\ &\quad -2[\bar{V}(s)\cdot\nabla_{\bar{x}}^3\xi(\bar{X}(s))\bar{V}(s)\cdot\bar{V}(s)]\xi(\bar{X}(s)). \end{aligned}$$

Note that the second and the third terms on the right hand side cancel exactly. By the convexity (1.22), there exists $C_\xi > 0$ so that we can further bound the last term by $\alpha(s)$ as:

$$\begin{aligned} & |2[\bar{V}(s)\cdot\nabla_{\bar{x}}^3\xi(\bar{X}(s))\bar{V}(s)\cdot\bar{V}(s)]\xi(\bar{X}(s))| \\ \leq\ & C_\xi|\bar{V}(t_1)|\times|\{-2\bar{V}(s)\cdot\nabla_{\bar{x}}^2\xi(\bar{X}(s))\cdot\bar{V}(s)\}\xi(\bar{X}(s))| \\ \leq\ & C_\xi|\bar{V}(t_1)|\alpha(s). \end{aligned}$$

We therefore have from the definition (3.167):

$$\begin{aligned} |\frac{d\alpha(s)}{ds}| &\leq \xi^2(\bar{X}(s))+[\nabla_{\bar{x}}\xi(\bar{X}(s)\cdot\bar{V}(s)]^2+C_\xi|\bar{V}(t_1)|\alpha(s) \\ &\leq \{C_\xi|\bar{V}(t_1)|+1\}\alpha(s). \end{aligned}$$

Our lemma thus follows from the standard Gronwall inequality. □

Lemma 24. *Let (t,x,v) be connected with $(t-t_{\mathbf{b}},x_{\mathbf{b}},v)$ backward in time through a trajectory of (3.13).*

(1) The backward exit time $t_{\mathbf{b}}(x,v)$ is lower semicontinuous.

(2) If

$$v\cdot n(x_{\mathbf{b}}) = v\cdot\frac{\nabla\xi(x_{\mathbf{b}})}{|\nabla\xi(x_{\mathbf{b}})|}<0, \tag{3.169}$$

then $(t_{\mathbf{b}}(x,v),x_{\mathbf{b}}(x,v))$ are smooth functions of (x,v) so that

$$\begin{aligned} \nabla_x t_{\mathbf{b}} &= \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \quad \nabla_v t_{\mathbf{b}} = \frac{t_{\mathbf{b}}n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \\ \nabla_x x_{\mathbf{b}} &= I+\nabla_x t_{\mathbf{b}}\otimes v, \quad \nabla_v x_{\mathbf{b}} = t_{\mathbf{b}}I+\nabla_v t_{\mathbf{b}}\otimes v. \end{aligned} \tag{3.170}$$

Furthermore, if ξ is real analytic, then $(t_{\mathbf{b}}(x,v),x_{\mathbf{b}}(x,v))$ are also real analytic.

(3)

(4) Let $x_i\in\partial\Omega$, for $i=1,2$, and let (t_1,x_1,v) and (t_2,x_2,v) be connected with the trajectory $\frac{dX(s)}{ds}=V(s)$, $\frac{dV(s)}{ds}=0$ which lies inside $\bar{\Omega}$. Then there exists a constant $C_\xi>0$ such that

$$|t_1-t_2|\geq\frac{|n(x_1)\cdot v|}{C_\xi|v|^2}. \tag{3.171}$$

(5) Define the boundary mapping

$$\Phi_{\mathbf{b}} : (t, x, v) \to (t - t_{\mathbf{b}}, x_{\mathbf{b}}(x, v), v) \in \mathbf{R} \times \{\gamma_0 \cup \gamma_-\}. \tag{3.172}$$

Then $\Phi_{\mathbf{b}}$ *and* $\Phi_{\mathbf{b}}^{-1}$ *maps zero measure sets to zero-measure sets between either* $\{t\} \times \Omega \times \mathbf{R}^3$ *and* $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\}$ *or* $\mathbf{R} \times \{\gamma_0 \cup \gamma_+\} \to \mathbf{R} \times \{\gamma_0 \cup \gamma_-\}$.

Lemma 25. *Recall (3.14) and the Grad estimate [Gr2] for hard potentials:*

$$|\mathbf{k}(v, v')| \leq C\{|v - v'| + |v - v'|^{-1}\} e^{-\frac{1}{8}|v-v'|^2 - \frac{1}{8}\frac{||v|^2 - |v'|^2|^2}{|v-v'|^2}}. \tag{3.173}$$

Recall w *in (3.20). Let* $0 \leq \theta < \frac{1}{4}$. *Then there exists* $0 \leq \varepsilon(\theta) < 1$ *and* $C_\theta > 0$ *such that for* $0 \leq \varepsilon < \varepsilon(\theta)$,

$$\int \{|v - v'| + |v - v'|^{-1}\} e^{-\frac{1-\varepsilon}{8}|v-v'|^2 - \frac{1-\varepsilon}{8}\frac{||v|^2 - |v'|^2|^2}{|v-v'|^2}} \frac{w(v) e^{\theta|v|^2}}{w(v') e^{\theta|v'|^2}} dv' \leq \frac{C}{1 + |v|}. \tag{3.174}$$

Lemma 26. *Recall (3.20) and (3.15). We have*

$$|w\Gamma[g_1, g_2](v)| \leq C(|v| + 1)^\gamma ||wg_1||_\infty ||wg_2||_\infty.$$

The following theorem is L^2 decay theory of the linear Boltzmann equation:

Theorem 12. *Let* $f(t, x, v) \in L^2$ *be the (unique) solution to the linear Boltzmann equation (**??**) with trace* $f_\gamma \in L^2_{loc}(\mathbb{R}_+; L^2(\gamma))$. *If* f *satisfy the specular reflection condition (3.16) and the mass and energy conservation laws (3.17) and (3.18). If* Ω *has any axis of rotational symmetry (3.12) we further require that the corresponding conservation of the angular momentum (3.19). Then there exists* $\lambda > 0$ *and* $C > 0$ *such that* $\sup_{0 \leq t \leq \infty}\{e^{2\lambda t}||f(t)||^2\} \leq 2||f_0||^2$.

The proof of Theorem 14 is same or simpler than Section 3 of [27].

Definition 17. *Let* Ω *be convex (1.22). Fix any point* $(t, x, v) \notin \gamma_0 \cap \gamma_-$, *and define* $(t_0, x_0, v_0) = (t, x, v)$, *and for* $k \geq 1$

$$(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(t_k, x_k, v_k), x_{\mathbf{b}}(x_k, v_k), R(x_{k+1})v_k), \tag{3.175}$$

where $R(x_{k+1})v_k = v_k - 2(v_k \cdot n(x_{k+1}))n(x_{k+1})$. *And we define the specular back-time cycle*

$$X_{\mathbf{cl}}(s) \equiv \sum_{k=1}^{\infty} \mathbf{1}_{[t_k,t_{k+1})}(s)\left\{x_k + v_k(s-t_k)\right\}, \quad V_{\mathbf{cl}}(s) \equiv \sum_{k=1}^{\infty} \mathbf{1}_{[t_k,t_{k+1})}(s)v_k. \tag{3.176}$$

Lemma 27. *Let* Ω *be convex (1.22). Let* $h_0 \in L^\infty$ *and* $G(t)h_0$ *solves*

$$\{\partial_t + v\cdot\nabla_x + \nu\}\{G(t)h_0\} = 0,\ \{G(0)h_0\} = h_0 \tag{3.177}$$

with the specular boundary condition $h(t,x,v) = h(t,x,R(x)v)$ *for* $x \in \partial\Omega$. *Then for almost all* $(x,v) \notin \gamma_0$,

$$\begin{aligned} &\{G(t)h_0\}(t,x,v) = e^{-\nu(v)t}h_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) \\ = \ &\sum_k \mathbf{1}_{[t_{k+1},t_k)}(0)e^{-\nu(v)t}h_0(x_k - t_kv_k, v_k). \end{aligned} \tag{3.178}$$

And $e^{\nu_0 t}||G(t)h_0||_\infty \le ||h_0||_\infty$.

Lemma 28. *Let* ξ *be convex as in (1.22). Let* h_0 *be continuous in* $\bar\Omega \times \mathbf{R}^3 \setminus \gamma_0$ *and* $q(t,x,v)$ *be continuous in the interior of* $[0,\infty)\times\Omega\times\mathbf{R}^3$ *and* $\sup_{[0,\infty)\times\Omega\times\mathbf{R}^3}|\frac{q(t,x,v)}{\nu(v)}| < \infty$. *Assume that on* γ_-, $h_0(x,v) = h_0(x,R(x)v)$. *Then the solution* $h(t,x,v)$ *to*

$$\{\partial_t + v\cdot\nabla_x + \nu\}h = q,\ h(0,x,v) = h_0 \tag{3.179}$$

with $h(t,x,v) = h(t,x,R(x)v)$, $x \in \partial\Omega$ *is continuous on* $[0,\infty)\times\{\bar\Omega\times\mathbf{R}^3\setminus\gamma_0\}$.

3.3 Smooth Convex Cylinderical Domain

3.3.1 Bad Sets and Properties

near-Boundary Set

We define *the near-boundary set*, the $\varepsilon-$neighborhood of points in the phase space whose position x is near to the boundary and velocity v is close to the tangential direction of the boundary. For convenience, we introduce the following notation.

Notation : For $\underline{x} \in \mathbb{R}^2, r > 0$, the open cylinder $\mathcal{C}(\underline{x}; r)$ is defined by $\underline{B}(\underline{x}; r) \times \mathbb{T}^1$ where $\underline{B}(\underline{x}; r) = \{\underline{x}' \in \mathbb{R}^2 : |\underline{x}' - \underline{x}| < r\}$.

Lemma 29 (Near-Boundary Set)**.** *Assume that the domain $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1$ with a smooth boundary $\partial\Omega = \partial\underline{\Omega} \times \mathbb{T}^1$. For $\varepsilon, N > 0$, there exist finite points $\underline{x}_1^{nB}, \cdots, \underline{x}_{l_{nB}}^{nB}$ in $\underline{\Omega}$ and open cylinders $\mathcal{C}(\underline{x}_1^{nB}; r_1^{nB}), \cdots, \mathcal{C}(\underline{x}_{l_{nB}}^{nB}; r_{l_{nB}}^{nB})$, as well as open sets $\mathcal{O}_i^{nB}, \cdots, \mathcal{O}_{l_{nB}}^{nB}$ in $\{v = (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \leq |\underline{v}| \leq N/2\ ,\ -N/2 \leq v_3 \leq N/2\}$ with $m_3(\mathcal{O}_i^{nB}) < \varepsilon$ for all $i = 1, \cdots, l_{nB}$ such that for every $x \in$ cl(Ω) there exists $i \in \{1, \cdots, l_{nB}\}$ with $x \in \mathcal{C}(\underline{x}_i^{nB}; r_i^{nB})$ and satisfies either $\mathcal{C}(x_i^{nB}; r_i^{nB}) \cap \partial\Omega = \emptyset$ or $|v' \cdot n(x')| > \varepsilon/N^4$ for all $x' \in \mathcal{C}(\underline{x}_i; r_i) \cap \partial\Omega$ and $v' \in \{v = (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \leq |\underline{v}| \leq N/2\ ,\ -N/2 \leq v_3 \leq N/2\} \backslash \mathcal{O}_i^{nB}$.*

Proof. We construct such sets in the projected 2-dimensional setting. Notice that $\partial\underline{\Omega} \in \mathbb{R}^2$ is a compact set in $\mathbb{R}^2$ and a union of the images of finite smooth curves. For $\underline{x} \in \underline{\Omega}$, we define $r_{\underline{x}} > 0$ such that $\underline{B}(\underline{x}; r_{\underline{x}}) \cap \partial\underline{\Omega} = \emptyset$. For each $\underline{x} \in \partial\underline{\Omega}$, we can define the outer normal direction $n(\underline{x})$ and define the outer normal angle $\theta_{\mathbf{n}}(\underline{x}) \in [0, 2\pi)$ specified uniquely by $n(\underline{x}) = (\cos\theta_{\mathbf{n}}(\underline{x}),\ \sin\theta_{\mathbf{n}}(\underline{x}))^T$. Using the smoothness of the boundary $\partial\underline{\Omega}$, for each $\underline{x} \in \partial\underline{\Omega}$ we can choose $r_{\underline{x}} > 0$ such that the outer normal direction in the $r_{\underline{x}}$-neighborhood of $\underline{x}$ is close to the outer normal direction of $\underline{x}$:

$$| \ \theta_{\mathbf{n}}(\underline{x}') - \theta_{\mathbf{n}}(\underline{x}) \ | < \varepsilon/4N^3 \quad \text{for all} \ \ \underline{x}' \in \underline{B}(\underline{x}; r_{\underline{x}}) \cap \partial\underline{\Omega}$$

; and $\underline{B}(\underline{x}; r_{\underline{x}}) \cap \partial\underline{\Omega}$ has the only one connected component of $\partial\underline{\Omega}$. Further, using the compactness (extracting finite sub-coverings from the the open coverings), we have finite numbers $l_1 > l > 0$ such that 2-dimensional open balls $\underline{B}(\underline{x}_1; r_1), \cdots \underline{B}(\underline{x}_l; r_l)$ are coverings of the boundary $\partial\underline{\Omega}$; and 2-dimensional open balls $\underline{B}(\underline{x}_{l+1}; r_1), \cdots \underline{B}(\underline{x}_{l_1}; r_{l_1})$ are coverings of the interior $\underline{\Omega}$ and $| \ \theta_{\mathbf{n}}(\underline{x}') - \theta_{\mathbf{n}}(\underline{x}_i) \ | < \varepsilon/4N^3 \quad$ for all $\ \underline{x}' \in \underline{B}(\underline{x}_i; r_{\underline{x}_i}) \cap \partial\underline{\Omega}$. Now for each $i = 1, \cdots, l$, we define an open set $\underline{O}_i \in \mathbb{R}^2$ that is an $\varepsilon-$neighborhood of $\theta_i \pm \pi/2$, the tangential direction of the boundary. Precisely, denoting $\theta_i = \theta_{\mathbf{n}}(\underline{x}_i)$,

$$\begin{aligned} \underline{O}_i \equiv \Big\{ v = \Big(|\underline{v}| \cos\theta, |\underline{v}| \sin\theta \Big)^T : |\underline{v}| \in [\frac{1}{N}, \frac{N}{2}] \text{ and} \\ \theta \notin \Big(\theta_i - \frac{\pi}{2} - \frac{\varepsilon}{N^3},\ \theta_i - \frac{\pi}{2} + \frac{\varepsilon}{N^3} \Big) \cup \Big(\theta_i + \frac{\pi}{2} - \frac{\varepsilon}{N^3},\ \theta_i + \frac{\pi}{2} + \frac{\varepsilon}{N^3} \Big) \Big\}. \end{aligned}$$

From the 2-dimensional setting, we prove the lemma. Define

$$\mathcal{C}(\underline{x}_i; r_i) \ \equiv \ \underline{B}(\underline{x}_i; r_i) \times \mathbb{T}^1 \ , \quad O_i \ \equiv \ \underline{O}_i \times \{v_3 \in \mathbb{R} : -N/2 < v_3 < N/2\}.$$

We can check that $m_3(O_i) \le m_2(\underline{O}_i) \times N \le N^2 \frac{\varepsilon}{N^3} \times N \le \varepsilon$ and

$$\begin{aligned} |v' \cdot n(x')| &\ge |\underline{v}' \cdot \underline{n}(\underline{x}')| \ge |\underline{v}'| \times \left|(\cos\theta', \sin\theta')^T \cdot (\cos\theta_{\mathbf{n}}(\underline{x}'), \sin\theta_{\mathbf{n}}(\underline{x}'))^T\right| \\ &\ge \frac{2}{N} \times |\cos\left(-\frac{\pi}{2} + \frac{\pi}{2} + \theta' - \theta_{\mathbf{n}}(\underline{x}')\right)| = \frac{2}{N} |\sin\left(+\frac{\pi}{2} + \theta' - \theta_{\mathbf{n}}(\underline{x}_i)\right)| \\ &\ge \frac{2}{N}\left\{|\frac{\pi}{2} + \theta' - \theta_i| - |\theta_i - \theta_{\mathbf{n}}(\underline{x}')|\right\} \ge \frac{2}{N} \times \{\frac{\varepsilon}{N^3} - \frac{\varepsilon}{4N^3}\} \ge \frac{\varepsilon}{N^4}, \end{aligned}$$

for $x' \in \mathcal{C}(\underline{x}_i; r_i)$ and $v' \in \{v = (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2\ ,\ -N/2 \le v_3 \le N/2\} \backslash \mathcal{O}_i$. $\square$

Infinite-Bounces Set

Definition 18 (Collision Space)**.** *Assume $\Omega \subset \mathbb{R}^n$ be the Smooth Convex Domain as Definition 10. Then the boundary $\partial\Omega$ is $(n-1)-$dimensional manifold without boundary and $(\partial\Omega)^2 = \partial\Omega \times \partial\Omega$ is a smooth $(2n-2)-$dimensional manifold. The diagonal set $D = \{(x,x) \in (\partial\Omega)^2\}$ is $(n-1)-$dimensional hyper-surface in $(\partial\Omega)^2$. Then we define the collision space*

$$\mathfrak{C} \equiv (\partial\Omega)^2 \backslash D \tag{3.180}$$

which is a smooth manifold with boundary.

Lemma 30. *Assume $\Omega \subset \mathbb{R}^n$ be the Smooth Convex Domain. If $(x,y) \in \mathfrak{C}$ i.e. $x, y \in \partial\Omega \times \partial\Omega$ and $x \neq y$ then (x,y) is a transversal pair. Moreover for any $k \in \mathbb{N}$*

$$\mathfrak{T}_k(x,y) \in \mathfrak{C} \tag{3.181}$$

and is a transversal pair.

3.3.2 Wellposedness $L^\infty-$Decay Estimate

Definition 19. *Let $\Omega \subset \mathbb{R}^2$ be the Smooth Convex Domain. Fix any point $(t,x,v) \notin \gamma_0 \cap \gamma_-$, and define $(t_0, x_0, v_0) = (t,x,v)$, and for $k \ge 1$*

$$(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(t_k, x_k, \bar{v}_k), x_{\mathbf{b}}(x_k, \bar{v}_k), R(x_{k+1})v_k), \tag{3.182}$$

where $R(x_{k+1})v_k = v_k - 2(v_k \cdot n(x_{k+1}))n(x_{k+1})$. *And we define the specular back-time cycle*

$$X_{\mathbf{cl}}(s) \equiv \sum_{k=1}^{\infty} \mathbf{1}_{[t_k,t_{k+1})}(s)\left\{x_k - (t_k - s)\bar{v}_k\right\}, \quad V_{\mathbf{cl}}(s) \equiv \sum_{k=1}^{\infty} \mathbf{1}_{[t_k,t_{k+1})}(s)v_k. \tag{3.183}$$

Lemma 31. *Let* $\Omega \subset \mathbb{R}^2$ *be the Smooth Convex Domain. Let* $h_0 \in L^\infty$ *and* $G(t)h_0$ *solves (3.177) with specular boundary condition* $h(t,x,v) = h(t,x,R(x)v)$ *for* $x \in \partial\Omega$. *Then for almost all* $(x,v) \notin \gamma_0$,

$$\begin{aligned}\{G(t)h_0\}(t,x,v) &= e^{-\nu(v)t}h_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \\ &= \sum_k \mathbf{1}_{[t_{k+1},t_k)}(0)e^{-\nu(v)t}h_0(x_k - t_k v_k, v_k).\end{aligned} \tag{3.184}$$

And $e^{\nu_0 t}||G(t)h_0||_\infty \le ||h_0||_\infty$.

Lemma 32. *Let* $\Omega \subset \mathbb{R}^2$ *be the Smooth Convex Domain. Let* h_0 *be continuous in* $\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$ *and* $q(t,x,v)$ *be continuous in the interior of* $[0,\infty) \times \Omega \times \mathbf{R}^3$ *and* $\sup_{[0,\infty)\times\Omega\times\mathbf{R}^3} |\frac{q(t,x,v)}{\nu(v)}| < \infty$. *Assume that on* γ_-, $h_0(x,v) = h_0(x,R(x)v)$. *Then the specular solution* $h(t,x,v)$ *to (3.179) is continuous on* $[0,\infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

We now fix any point (t,x,v) so that $(x,v) \notin \gamma_0$. Let the back-time specular cycle of (t,x,v) be $[X_{\mathbf{cl}}(s_1), V_{\mathbf{cl}}(s_1)]$. By (**??**), we use twice (3.184) to derive $h(t,x,v) =$

$$e^{-\nu(v)t}h_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))$$

$$+\int_0^t e^{-\nu(v)(t-s_1)}\int \mathbf{k}_w(V_{\mathbf{cl}}(s_1),v')e^{-\nu(v')s_1}h_0\left(X'_{\mathbf{cl}}(0), V'_{\mathbf{cl}}(0)\right)dv'ds_1 \tag{3.185}$$

$$+\int_0^t\int_0^{s_1}\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{-\nu(v)(t-s_1)-\nu(v')(s_1-s)}\mathbf{k}_w(V_{\mathbf{cl}}(s_1),v')\mathbf{k}_w(V'_{\mathbf{cl}}(s),v'')h\left(s, X'_{\mathbf{cl}}(s), v''\right)dv'dv''dsds_1 \tag{3.186}$$

where the back-time specular cycle from $(s_1, X_{\mathbf{cl}}(s_1), v')$ is denoted by

$$X'_{\mathbf{cl}}(s) = X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1), v'), \qquad V'_{\mathbf{cl}}(s) = V_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1), v'). \tag{3.187}$$

More explicitly, let t_k and $t'_{k'}$ be the corresponding times for both specular cycles as in (3.182). For $t_{k+1} \le s_1 < t_k$, $t'_{k'+1} \le s < t'_{k'}$

$$X'_{\mathbf{cl}}(s) = X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1), v') \equiv x'_{k'} + (s - t'_{k'})v'_{k'} \tag{3.188}$$

where $x'_{k'} = X_{\mathbf{cl}}(t_{k'}; s_1, x_k + (s_1 - t_k)v_k, v')$, $v'_{k'} = V_{\mathbf{cl}}(t_{k'}; s_1, x_k + (s_1 - t_k)v_k, v')$. Recall α in (3.167) and define naturally

$$\alpha(x,v) \equiv \alpha(t) = \xi^2(\bar{x}) + [\bar{v}\cdot\nabla\xi(\bar{x})]^2 - [\bar{v}\nabla^2\xi(\bar{x})\bar{v}]\xi(\bar{x}). \tag{3.189}$$

We define the main set

$$A_\alpha = \{(x,v) : x \in \bar{\Omega},\ \frac{1}{N} \leq |v| \leq N,\ \text{and } \alpha(x,v) \geq \frac{1}{N}\}. \tag{3.190}$$

Lemma 33. *Assume Ω is a Smooth Convex Domain. Fix k and k'. Define for $t_{k+1} \leq s_1 \leq t_k$, $s \in \mathbb{R}$*

$$J \equiv J_{k,k'}(t,x,v,s_1,s,v') \equiv \det\left(\frac{\partial\{x'_{k'} - (t'_{k'} - s)v'_{k'}\}}{\partial v'}\right). \tag{3.191}$$

For any $\varepsilon > 0$ sufficiently small, there is $\delta(N,\varepsilon,T_0,k,k') > 0$ and an open covering $\bigcup_{i=1}^m B(t_i,x_i,v_i;r_i)$ of $[0,T_0]\times A_\alpha$ and corresponding open sets (with $x_i \in \bar{\Omega}$) O_{t_i,x_i,v_i} for $[t_{k+1}+\varepsilon, t_k+\varepsilon]\times\mathbb{R}\times\mathbb{R}^3$ with $O_{t_i,x_i,v_i} < \varepsilon$, such that

$$|J_{k,k'}(t,x,v,s_1,s,v')| \geq \delta > 0, \tag{3.192}$$

for $0 \leq t \leq T_0$, $(x,v) \in A_\alpha$ and (s_1,s,v') in

$$O^c_{t_i,x_i,v_i} \cap [t_{k+1}+\varepsilon, t_k-\varepsilon]\times[0,T_0]\times\{|v'| \leq 2N\}. \tag{3.193}$$

Proof. Fix (t,x,v) such that $x,v \in A_\alpha$ in (3.190), and fix k,k'. Since $x,v \notin \gamma_0$, by Velocity Lemma 23, we deduce that $\alpha(t_k) = \{n(x_k)\cdot v_k\}^2 \neq 0$ so that $t_k - t_{k+1} > 0$ from (3.171). We note since $|v'| \geq \frac{1}{N}$ from (3.190), $t_k - t_{k+1} \leq N\mathrm{diam}\Omega$. Since for $t_{k+1} + \frac{\varepsilon}{2} \leq s_1 \leq t_k - \frac{\varepsilon}{2}$, $x_k - (s_1 - t_k)v_k \in \Omega$, the interior of the domain with ε sufficiently small. From (3.189) $\alpha(x_k + (s_1 - t_k)v_k, v') > 0$ for all v'. This implies that along its back-time specular cycle $[X'_{\mathbf{cl}}(s), V'_{\mathbf{cl}}(s)]$, $\alpha(t'_{l'}) > 0$ and $v'_{l'}\cdot n(x'_{l'}) \neq 0$ from the Velocity Lemma 23. Clearly, by the Velocity Lemma 23 and part (2) of Lemma 24, t'_l, x'_l, v'_l are smooth functions of s_1, s, v'. Therefore the function $J_{k,k'}(t,x,v,s_1,s,v')$ is well-defined, and smooth for all $v' \in \mathbf{R}^3$, $s \in \mathbf{R}$, and $t_{k+1} + \frac{\varepsilon}{2} \leq s_1 \leq t_k - \frac{\varepsilon}{2}$. Also for

$(\bar{t},\bar{x},\bar{v},\bar{s}_1,\bar{s},\bar{v}')\sim(t,x,v,s_1,s,v')$we know that

$$\pi_1\left(X_{\mathbf{cl}}(\bar{s};\bar{s}_1,X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),\bar{v}')\right)=\pi_1\left(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v})\right)-(\bar{s}_1-\bar{s})\pi_1(\bar{v}') \tag{3.194}$$

$$=\pi_1(\bar{x})-(\bar{t}-\bar{s}_1)\pi_1(\bar{v})-(\bar{s}_1-\bar{s})\pi_1(\bar{v}')\ , \tag{3.195}$$

$$\pi_2\left(X_{\mathbf{cl}}(\bar{s};\bar{s}_1,X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v})),\bar{v}'\right)=\pi_2\left(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v})\right)-(\bar{s}_1-\bar{s})\bar{v}_3' \tag{3.196}$$

$$=\bar{x}_3-(\bar{t}-\bar{s}_1)\bar{v}_3-(\bar{s}_1-\bar{s})\bar{v}_3'. \tag{3.197}$$

Since (4.54) is independent of $\bar{v}_3'$ we have $J(\bar{t},\bar{x},\bar{v},\bar{s}_1,\bar{s},\bar{v}')=\det\left(\frac{\partial X_{\mathbf{cl}}(\bar{s};\bar{s}_1,X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),\bar{v}')}{\partial\bar{v}'}\right)=$

$$\det\begin{pmatrix}\frac{\partial(\pi_1(\bar{x})-(\bar{t}-\bar{s}_1)\pi_1(\bar{v})-(\bar{s}_1-\bar{s})\pi_1(\bar{v}'))}{\partial(\pi_1(\bar{v}'))} & \vec{0}\\ \vec{0}^T & \frac{\partial(\bar{x}_3-(\bar{t}-\bar{s}_1)\bar{v}_3-(\bar{s}_1-\bar{s})\bar{v}_3')}{\partial\bar{v}_3'}\end{pmatrix} \tag{3.198}$$

$$=-(\bar{s}_1-\bar{s})\times\underbrace{\det\left(\frac{\partial(\pi_1(\bar{x})-(\bar{t}-\bar{s}_1)\pi_1(\bar{v})-(\bar{s}_1-\bar{s})\pi_1(\bar{v}'))}{\partial(\pi_1(\bar{v}'))}\right)}_{\bigotimes} \tag{3.199}$$

where $\vec{0}=\begin{pmatrix}0\\0\end{pmatrix}$. Since $\bigotimes=$ (3.145) from Section 3.2 we obtain that $J(\bar{t},\bar{x},\bar{v},\bar{s}_1,\bar{s},\bar{v}')=$

$$-(\bar{s}_1-\bar{s})\times\left\{\frac{\mathcal{A}(\bar{\theta}(\bar{v}'))}{\Delta\mathfrak{T}_{k+1}(\bar{\theta}(\bar{v}'),\bar{\theta}(\bar{v}')+\pi)}\frac{\bar{s}_1-\bar{s}}{\bar{\xi}(\bar{v}')}+\frac{\mathcal{B}(\bar{\theta}(\bar{v}'))}{[\Delta\mathfrak{T}_{k+1}(\bar{\theta}(\bar{v}'),\bar{\theta}(\bar{v}')+\pi)]^2}(\bar{s}_1-\bar{s})^2\right\} \tag{3.200}$$

where

$$\mathcal{A}(\bar{\theta}(\bar{v}'))=(1-p(\bar{\theta}(\bar{v}')),p(\bar{\theta})(\bar{v}'))\begin{pmatrix}\partial_1P_1(\bar{\theta}(\bar{v}')) & \partial_2P_1(\bar{\theta}(\bar{v}'))\\ \partial_1P_2(\bar{\theta}(\bar{v}')) & \partial_2P_2(\bar{\theta}(\bar{v}'))\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix},$$

$$\mathcal{B}(\bar{\theta}(\bar{v}'))=(-1,1)\begin{pmatrix}\partial_1P_1(\bar{\theta}(\bar{v}')) & \partial_2P_1(\bar{\theta}(\bar{v}'))\\ \partial_1P_2(\bar{\theta}(\bar{v}')) & \partial_2P_2(\bar{\theta}(\bar{v}'))\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

with

$$p(\bar{\theta}(\bar{v}')) = p(x_{\mathbf{b}}(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),-\bar{v}'),x_{\mathbf{b}}(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),\bar{v}');X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}))$$
$$= \frac{|x_{\mathbf{b}}(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),-\bar{v}') - X_{\mathbf{cl}}(\bar{s}_1,\bar{t},\bar{x},\bar{v})| - D_k(x_{\mathbf{b}}(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),-\bar{v}'),x_{\mathbf{b}}(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),\bar{v}'))}{\Delta\mathfrak{T}_{k+1}(x_{\mathbf{b}}(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),-\bar{v}'),x_{\mathbf{b}}(X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v}),\bar{v}'))},$$
$$\begin{pmatrix} P_1(\bar{\theta}(\bar{v}'),\bar{\theta}(\bar{v}')+\pi) \\ 0 \end{pmatrix} = \mathbf{A}_{x_k(X'_{\mathbf{cl}}(s),v'),x_{k+1}(X'_{\mathbf{cl}}(s),v')}(T_k(\mathbf{R}^{-1}_{X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v})}(\bar{\theta}(\bar{v}'),\bar{\theta}(\bar{v}')+\pi))),$$
$$\begin{pmatrix} P_2(\bar{\theta}(\bar{v}'),\bar{\theta}(\bar{v}')+\pi) \\ 1 \end{pmatrix} = \mathbf{A}_{x_k(X'_{\mathbf{cl}}(s),v'),x_{k+1}(X'_{\mathbf{cl}}(s),v')}(T_{k+1}(\mathbf{R}^{-1}_{X_{\mathbf{cl}}(\bar{s}_1;\bar{t},\bar{x},\bar{v})}(\bar{\theta}(\bar{v}'),\bar{\theta}(\bar{v}')+\pi)))$$

with $X'_{\mathbf{cl}}(s) = X'_{\mathbf{cl}}(s;s_1,X_{\mathbf{cl}}(s_1;t,x,v),v')$. Clearly from the argument of Section 3.2 for fixed $(t,x,v)\in[0,T_0]\times A_\alpha$,

$$J(t,x,v,\bar{s}_1,\bar{s},\bar{v}') \neq 0 \tag{3.201}$$

for $(\bar{s}_1,\bar{s},\bar{v}')\in O^c_{t,x,v}\cap[t_{k+1}+\varepsilon,t_k-\varepsilon]\times[0,T_0]\times\{|v'|\le 2N\}$ which is a compact set. Since J is a continuous function of (s_1,s,v') on a compact set we have a lower bound :

$$J_{k,k'}(t,x,v;\bar{s}_1,\bar{s},\bar{v}') > \delta_{\ k,k',t,x,v}/2 > 0 \tag{3.202}$$

for $(\bar{s}_1,\bar{s},\bar{v}')\in O^c_{t,x,v}\cap[t_{k+1}+\varepsilon,t_k-\varepsilon]\times[0,T_0]\times\{|v'|\le 2N\}$. Since $J_{k,k'}$ is continuous with respect to (t,x,v,s_1,s,v') we can choose an open ball $B(t,x,v;r_{t,x,v})\subset[0,T_0]\times A_\alpha$ such that

$$J_{k,k'}(\bar{t},\bar{x},\bar{v};\bar{s}_1,\bar{s},\bar{v}') > \delta_{\ k,k',t,x,v} > 0 \tag{3.203}$$

for $(\bar{t},\bar{x},\bar{v})\in B(t,x,v;r_{t,x,v})$ and $(\bar{s}_1,\bar{s},\bar{v}')\in O^c_{t,x,v}\cap[t_{k+1}+\varepsilon,t_k-\varepsilon]\times[0,T_0]\times\{|v'|\le 2N\}$. Now by a finite covering for the compact set $[0,T_0]\times A_\alpha$ by such $B(t,x,v;r_{t,x,v})$, there are $[t_1,x_1,v_1],...,[t_m,x_m,v_m]$ such that $[0,T_0]\times A_\alpha\subset\cup_{i=1}^m B(t_i,x_i,v_i;r_i)$. For any point (t,x,v), there is i so $(t,x,v)\in B(t_i,x_i,v_i;r_i(\varepsilon))$ and

$$|J_{k,k'}(t,x,v,s_1,s,v')| > \min_{1\le i\le m}\frac{\delta_{i,T_0,N,\varepsilon,k,k'}}{2} > 0 \tag{3.204}$$

for $(s_1,s,v')\in O^c_{t_i,x_i,v_i}\cap[t_{k+1}+\varepsilon,t_k-\varepsilon]\times[0,T_0]\times\{|v'|\le 2N\}$. □

Theorem 13. *Assume $w^{-2}\{1+|v|\}\in L^1$. Assume that $\Omega=\Omega'\times[0,1]/\sim$ is the Smooth Convex*

*Domain, and the mass (3.17) and energy (3.18) are conserved. In the case of Ω has rotational symmetry (3.12), we also assume conservation of corresponding angular momentum (3.19). Let $h_0 \in L^\infty$. There exits a unique solution to both the (**??**) and (**??**) with boundary specular condition, and the exponential decay*

$$e^{\lambda t}||U(t)h_0||_\infty \leq C||h_0||_\infty \tag{3.205}$$

is valid.

For the well-posedness for both problems, we know from the Duhamel principle (**??**), there exists a L^∞ solution $h(t) = U(t)h_0$ to the weighted linear Boltzamnn equation (**??**). By Ukai's trace theorem, it follows that h_γ is also in L^∞. Therefore, since $w^{-2}\{1+|v|\} \in L^1$, $f = \frac{h}{w} \in L^2$ and $f_\gamma = \frac{h_\gamma}{w} \in L^2_{loc}(L^2(\gamma))$ is a solution to the original linear Boltzmann equation (**??**), which is unique by the standard energy estimate. Based on the L^2 decay estimate for f, to prove the decay estimate, it suffices to establish a finite-time estimate (3.206) as [27].

Lemma 34. *Assume that there exists $\lambda > 0$ so that the solution $f(t,x,v)$ of (**??**) satisfies $e^{\lambda t}||f(t)|| \leq C||f_0||$. Let $h_0 = wf_0 \in L^\infty$ and $h(t) = U(t)h_0 = wf(t)$ is the solution of (**??**) where $w^{-2} \in L^1$. Assume there exist $T_0 > 0$ and $C_{T_0} > 0$ such that the satisfies*

$$||U(T_0)h_0||_\infty \leq e^{-\lambda T_0}||h_0||_\infty + C_{T_0}\int_0^{T_0} ||f(s)||ds. \tag{3.206}$$

Then (3.205) is valid.

Proof. The well-posedness follows from the exact argument in the proof of Theorem **??**. Thanks to Lemma 34, we only need to establish the finite time estimate (3.206). Recall A_α in (3.190).

STEP 1: Estimate of $h(t,x,v)\mathbf{1}_{A_\alpha}(x,v)$. We first express and estimate the main part $h(t,x,v)\mathbf{1}_A$ through (3.185). The first term of (3.185) is bounded by

$$e^{-\nu(v)t}|h_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))| \leq e^{-\nu_0 t}||h_0||_\infty \tag{3.207}$$

and also the second term is bounded by

$$\int_0^t e^{-\nu_0(t-s_1)}\int_{\mathbb{R}^3}\mathbf{k}_w(V_{\mathbf{cl}}(s_1), v')dv' e^{-\nu_0 s_1}||h_0||_\infty \leq te^{-\nu_0 t}||h_0||_\infty \leq e^{-\frac{\nu_0}{2}t}||h_0||_\infty. \tag{3.208}$$

For the third main contribution in (3.185), notice that along the back-time specular cycles $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ and $[X'_{\mathbf{cl}}(s), V'_{\mathbf{cl}}(s)]$ in (3.187), $|V_{\mathbf{cl}}(s_1)| \equiv |v|$ and $|V'_{\mathbf{cl}}(s_1)| \equiv |v'|$.

CASE 1: For $|v| \geq N$. Since from (3.174) with $\varepsilon = 0$ in Lemma 25,

$$\int\int \mathbf{k}_w(v,v')\mathbf{k}_w(v',v'')dv'dv'' \leq \frac{C_K}{1+|v|} \leq \frac{C_K}{N},$$

By Lemma 25 again, the double-time integration $\int_0^t \int_0^{s_1}$ for $|v| \geq N$ is controlled by

$$\begin{aligned} \frac{C_K}{N}\int_0^t\int_0^{s_1} e^{-\nu_0(t-s)}||U(s,0)h_0||_\infty ds ds_1 \quad &\leq \qquad\qquad (3.209)\\ \frac{C_K e^{-\frac{\nu_0 t}{2}}}{N}\sup_s\{e^{\frac{\nu_0 s}{2}}||U(s,0)h_0||_\infty\}\int_0^t\int_0^{s_1} e^{-\frac{\nu_0(t-s)}{2}}dsds_1 \quad &\leq \quad \frac{C_K e^{-\frac{\nu_0 t}{2}}}{N}\sup_s\{e^{\frac{\nu_0 s}{2}}||U(s,0)h_0||_\infty\}, \end{aligned}$$

where we have split the exponent as

$$e^{-\nu_0(t-s)} = e^{-\frac{\nu_0 t}{2}}e^{-\frac{\nu_0(t-s)}{2}}e^{\frac{\nu_0 s}{2}}, \qquad (3.210)$$

and used the fact $\int_0^t\int_0^{s_1} e^{-\frac{\nu_0(t-s)}{2}}dsds_1 < +\infty$ by a direct computation.

CASE 2: For $|v| \leq N$, $|v'| \geq 2N$, or $|v'| \leq 2N$, $|v''| \geq 3N$. Notice that we have either $|v'-v| \geq N$ or $|v'-v''| \geq N$, and either one of the following are valid correspondingly:

$$|\mathbf{k}_w(v,v')| \leq e^{-\frac{\varepsilon}{8}N^2}|\mathbf{k}_w(v,v')e^{\frac{\varepsilon}{8}|v-v'|^2}|, \qquad |\mathbf{k}_w(v',v'')| \leq e^{-\frac{\varepsilon}{8}N^2}|\mathbf{k}_w(v',v'')e^{\frac{\varepsilon}{8}|v'-v''|^2}|. \quad (3.211)$$

From (3.174) in Lemma 25, both $\int|\mathbf{k}_w(v,v')e^{\frac{\varepsilon}{8}|v-v'|^2}|$ and $\int|\mathbf{k}_w(v',v'')e^{\frac{\varepsilon}{8}|v'-v''|^2}|$ are still finite. By (3.210), we use (3.211) to combine the cases of $|v'-v| \geq N$ or $|v'-v''| \geq N$ as:

$$\begin{aligned} &\int_0^t\int_0^{s_1}\left\{\int_{|v|\leq N,|v'|\geq 2N,} + \int_{|v'|\leq 2N,|v''|\geq 3N}\right\}\\ \leq\ & C_K\int_0^t\int_0^{s_1}||U(s,0)h_0||_\infty\left\{\int_{|v|\leq N,|v'|\geq 2N,}|\mathbf{k}_w(v,v')|dv' + \sup_{v'}\int_{|v'|\leq 2N,|v''|\geq 3N}|\mathbf{k}_w(v',v'')|dv''\right\}\\ \leq\ & C_{\varepsilon,K}e^{-\frac{\varepsilon}{8}N^2}\int_0^t\int_0^{s_1} e^{-\nu_0(t-s)}||U(s,0)h_0||_\infty dsds_1\\ \leq\ & C_{\varepsilon,K}e^{-\frac{\varepsilon}{8}N^2}e^{-\frac{\nu_0 t}{2}}\sup_{s\geq 0}\{e^{\frac{\nu_0}{2}s}||U(s,0)h_0||_\infty\}. \qquad (3.212) \end{aligned}$$

CASE 3: For $|v| \leq N$, $|v'| \leq 2N$, $|v''| \leq 3N$. This is the last remaining case, because if $|v'| > 2N$, it is included in Case 2, while if $|v''| > 3N$, either $|v'| \leq 2N$ or $|v'| \geq 2N$ are also included in

Case 2. We now can bound (3.186) by

$$C\int_0^t\int_0^{s_1}\int_B e^{-\nu_0(t-s)}|\mathbf{k}_w(V_{\mathbf{cl}}(s_1),v')\mathbf{k}_w(V'_{\mathbf{cl}}(s),v'')h(s,X'_{\mathbf{cl}}(s),v'')|dv'dv''dsds_1 \tag{3.213}$$

where $B=\{|v'|\le 2N,\ |v''|\le 3N\}$. By (3.173) , $\mathbf{k}_w(v,v')$ has possible integrable singularity of $\frac{1}{|v-v'|}$, we can choose $\mathbf{k}_N(v,v')$ smooth with compact support such that

$$\sup_{|p|\le 3N}\int_{|v'|\le 3N}|\mathbf{k}_N(p,v')-\mathbf{k}_w(p,v')|dv'\le\frac{1}{N}. \tag{3.214}$$

Splitting

$$\begin{aligned}\mathbf{k}_w(v,v')\mathbf{k}_w(v',v'') &= \{\mathbf{k}_w(v,v')-\mathbf{k}_N(v,v')\}\mathbf{k}_w(v',v'')\\ &\quad+\{\mathbf{k}_w(v',v'')-\mathbf{k}_N(v',v'')\}\mathbf{k}_N(v,v')+\mathbf{k}_N(v,v')\mathbf{k}_N(v',v''),\end{aligned}$$

we can use such an approximation (3.214) to bound the above s_1,s integration by

$$\frac{Ce^{-\frac{\nu_0 t}{2}}}{N}\sup_s\{e^{\frac{\nu_0}{2}s}||h(s)||_\infty\}\times\left\{\sup_{|v'|\le 2N}\int|\mathbf{k}_w(v',v'')|dv''+\sup_{|v|\le 2N}\int|\mathbf{k}_N(v,v')|dv'\}\right\} \tag{3.215}$$

$$+C\int_0^t\int_0^{s_1}\int_B e^{-\nu_0(t-s)}|\mathbf{k}_N(V_{\mathbf{cl}}(s_1),v')\mathbf{k}_N(V'_{\mathbf{cl}}(s),v'')|h(s,X'_{\mathbf{cl}}(s),v'')\mathbf{1}_{A_\alpha}(x,v)|dv'dv''dsds_1. \tag{3.216}$$

Since $\mathbf{k}_N(v,v')$ and $\mathbf{k}_N(v',v'')$ are smooth functions on a compact set B they are bounded :

$$\begin{aligned}(3.216) &\le C_N\int_0^t\int_0^{s_1}\int_{|v'|\le 2N,|v''|\le 3N}e^{-\nu_0(t-s)}\mathbf{1}_{A_\alpha}|h\left(s,X'_{\mathbf{cl}}(s),v''\right)|dv'dv''ds_1ds\\ &= C_N\int_{\substack{\alpha(X_{\mathbf{cl}}(s_1),v')<\varepsilon\\|v'|\le 2N,|v''|\le 3N}}+C_N\int_{\substack{\alpha(X_{\mathbf{cl}}(s_1),v')\ge\varepsilon\\|v'|\le 2N,|v''|\le 3N}}\end{aligned}$$

where we have further spit into $\alpha(X_{\mathbf{cl}}(s_1),v')\le\varepsilon$ and $\alpha(X_{\mathbf{cl}}(s_1),v')>\varepsilon$.

In the case $\alpha(X_{\mathbf{cl}}(s_1),v')\le\varepsilon$, $\xi^2(\bar{X}_{\mathbf{cl}}(s_1))+[\bar{v}'\cdot\nabla\xi(\bar{X}_{\mathbf{cl}}(s_1))]^2\le\varepsilon$. Hence for ε small, $X_{\mathbf{cl}}(s_1)\backsim\partial\Omega$, and $|\nabla\xi(X_{\mathbf{cl}}(s_1))|\ge\frac{1}{2}$. The first part integral is bounded by

$$\begin{aligned}&C_N\int_0^t\int_0^{s_1}e^{-\nu_0(t-s)}||h(s)||_\infty dsds_1\int_{\substack{\alpha(X_{\mathbf{cl}}(s_1),v')\le\varepsilon\\|v'|\le 2N,|v''|\le 3N,}}\\ \le\ &C_N\sup_{t\ge s}e^{-\frac{\nu_0}{2}(t-s)}||h(s)||_\infty\int_{|v'\cdot\frac{\nabla\xi(X_{\mathbf{cl}}(s_1))}{|\nabla\xi(X_{\mathbf{cl}}(s_1))|}|\le 2\sqrt{\varepsilon},|v'|\le 2N,|v''|\le 3N}\le C_N\sqrt{\varepsilon}\sup_{t\ge s}e^{-\frac{v_0}{2}(t-s)}||h(s)||_\infty.\end{aligned}$$

Finally, from (3.188), we bound the first main term $\alpha(X_{\mathbf{cl}}(s_1), v') \geq \varepsilon$ as

$$\begin{aligned}
& C_N \int_0^t e^{-\nu_0(t-s)} \int_0^{s_1} \int_{\substack{\alpha(X_{\mathbf{cl}}(s_1),v')\geq\varepsilon \\ |v'|\leq 2N, |v''|\leq 3N}} \mathbf{1}_{A_\alpha} |h\left(s, X'_{\mathbf{cl}}(s), v''\right)| dv' dv'' \\
= \; & C_N \sum_{k,k'} \int_{t_{k+1}}^{t_k} \int_{t'_{k'+1}}^{t'_{k'}} \int_{\substack{\alpha(X_{\mathbf{cl}}(s_1),v')\geq\varepsilon \\ |v'|\leq 2N, |v''|\leq 3N}} \mathbf{1}_{A_\alpha} e^{-\nu_0(t-s)} |h\left(s, x'_{k'} + (s - t'_{k'})v'_{k'}, v''\right)|,
\end{aligned}$$

where $[t'_{k'}, x'_{k'}, v'_{k'}]$ is the back-time cycle of $(s_1, x_k + (s_1 - t_k)v_k, v_k)$, for $t_{k+1} \leq s_1 \leq t_k$.

We now study $x'_{k'} + (s - t'_{k'})v'_{k'}$. By the repeatedly using Velocity Lemma 23, we deduce for $(t, x, v) \in A_\alpha$ and $\alpha(X(s_1), v') \geq \varepsilon, |v'| \leq 2N$:

$$\begin{aligned}
\alpha(t_l) &= \{v_l \cdot n_{x_l}\}^2 \geq e^{-\{C_\xi N - 1\}T_0} \alpha(s_1) \geq C_{T_0,\xi,N} > 0; \\
\alpha(t'_l) &= \{v'_l \cdot n_{x'_l}\}^2 \geq e^{-\{C_\xi N - 1\}T_0} \alpha(X_{\mathbf{cl}}(s_1), v') \geq C_{T_0,N,\xi}\varepsilon > 0.
\end{aligned}$$

Therefore, applying (3.171) in Lemma 24 yields $t_l - t_{l+1} \geq \frac{c_{T_0,\xi,N}}{N^2}$ and $t'_l - t'_{l+1} \geq \frac{c_{T_0,\xi,N}\varepsilon}{4N^2}$ so that

$$k \leq \frac{T_0 N^2}{c_{T_0,\xi,N}} = C_{T_0,\xi,N}, \quad k' \leq \frac{T_0 N^2}{c_{T_0,\xi,N}\varepsilon} = C_{T_0,\xi,N,\varepsilon}. \tag{3.217}$$

We therefore further split the s_1-integral as

$$\begin{aligned}
C_N \quad & \sum_{k \leq C_{T_0,\xi,N}} \int_{t_{k+1}(t,x,v)}^{t_k(t,x,v)} ds_1 \int_{|v'|\leq 2N} dv' \sum_{k' \leq C_{T_0,\xi,N,\varepsilon}} \int_{t'_{k'+1}(t,x,v;s_1,v')}^{t'_{k'}(t,x,v;s_1,v')} ds \int_{|v''|\leq 3N} dv'' \\
& \times \mathbf{1}_{A_\alpha}(x,v) \; e^{-\nu_0(t-s)} |h\left(s, x'_{k'} + (s - t'_{k'})v'_{k'}, v''\right)| \\
= \quad & \int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon} + \int_{t_k}^{t_k-\varepsilon} + \int_{t_{k+1}}^{t_{k+1}+\varepsilon}.
\end{aligned}$$

Since $k-$summation is finite and $\sum_{k'} \int_{t'_{k'+1}}^{t'_{k'}} = \int_0^{s_1}$, the last two terms make small contribution as

$$\varepsilon C_N C_{T_0,\xi,N} \sup_{0\leq s\leq t} e^{-\nu_0(t-s)} ||h(s)||_\infty \int_0^{T_0} \int_{|v'|\leq 2N, |v''|\leq 3N} = \varepsilon C_N C_{T_0,\xi,N} \sup_{0\leq s\leq t} e^{-\nu_0(t-s)} ||h(s)||_\infty.$$

For the main contribution $\int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon}$, By Lemma 33, on the set $(O^{k,k'}_{t_i,x_i,v_i})^c \cap [t_{k+1}+\varepsilon, t_k - \varepsilon] \times$

$[0,T_0]\times\{|v'|\leq N\}$, we can define a change of variable

$$y\equiv x'_{k'}+(s-t'_{k'})v'_{k'},$$

so that $\det(\frac{\partial y}{\partial v'})>\delta$ on the same set. By the Implicit Function Theorem, there are a finite open covering $\cup_{j=1}^{m_{k,k'}}V_j^{k,k'}$ of $(O^{k,k'}_{t_i,x_i,v_i})^c\cap\ [t_{k+1}+\varepsilon,t_k-\varepsilon]\times[0,T_0]\times\{|v'|\leq N\}$, and smooth function F_j such that $v'=F_j(t,x,v,y,s_1,s)$ in $V_j^{k,k'}$. We therefore have

$$C_N\sum_{k\leq C_{T_0,\xi,N}}\int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon}ds_1\int_{|v'|\leq 2N}dv'\sum_{k'\leq C_{T_0,\xi,N,\varepsilon}}\int_{t'_{k'+1}}^{t'_{k'}}ds\int_{|v''|\leq 3N}dv''\mathbf{1}_{A_\alpha}e^{-\nu_0(t-s)}|h(s,y,v'')|$$

$$\leq C_N\sum_{k,k'}\int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon}ds_1\int_{|v'|\leq 2N}dv'\int_{t'_{k'+1}}^{t'_{k'}}ds\int_{|v''|\leq 3N}dv''\mathbf{1}_{A_\alpha}e^{-\nu_0(t-s)}|h(s,y,v'')|\mathbf{1}_{O^{k,k'}_{t_i,x_i,v_i}}(s_1,s,v') \tag{3.218}$$

$$+C_N\sum_{k,k'}\int_{t_{k+1}+\varepsilon}^{t_k-\varepsilon}ds_1\int_{|v'|\leq 2N}dv'\int_{t'_{k'+1}}^{t'_{k'}}ds\int_{|v''|\leq 3N}dv''\mathbf{1}_{A_\alpha}e^{-\nu_0(t-s)}|h(s,y,v'')|\mathbf{1}_{V_j^{k,k'}}(s_1,s,v') \tag{3.219}$$

Since $k-$ summation and $k'-$summation are finite and $|O^{k,k'}_{t_i,x_i,v_i}|<\varepsilon$ we have

$$(3.218)\leq\varepsilon C_N C_{T_0,\xi,N}C_{T_0,\xi,N,\varepsilon}\sup_{0\leq s\leq t}e^{-\nu_0(t-s)}||h(s)||_\infty. \tag{3.220}$$

The second part (3.219) can be controlled by

$$\begin{aligned}
& C_N C_{T_0,\xi,N,\varepsilon}\sum_k\sup_{k'\leq C_{T_0,\xi,N,\varepsilon}}\int_{t_{k+1}(t,x,v)+\varepsilon}^{t_k(t,x,v)-\varepsilon}ds_1\int_0^{T_0}ds\int_{\substack{\alpha(X_{\mathbf{cl}}(s_1),v')\geq\varepsilon\\|v'|\leq 2N}}dv'\int_{|v''|\leq 3N}dv''\\
& \times\mathbf{1}_{A_\alpha}e^{-\nu_0(t-s)}|h(s,y,v'')|\mathbf{1}_{V_j^{k,k'}}(s_1,s,v')\\
\leq\ & C_N C_{T_0,\xi,N,\varepsilon}C_{T_0,\xi,N}\sup_{k,k'}\sum_j\int_{V_j^{k,k'}}\int_{|v''|\leq 3N}e^{-\nu_0(t-s)}|h(s,x'_{k'}-(t'_{k'}-s)v'_{k'},v'')|dv''dv'dsds_1\\
\leq\ & C_N C_{T_0,\xi,N,\varepsilon}C_{T_0,\xi,N}\sup_{k,k'}\sum_j\int_{V_j^{k,k'}}\int_{|v''|\leq 3N}e^{-\nu_0(t-s)}|h(s,y,v'')|\frac{1}{|\det\{\frac{\partial y}{\partial v'}\}|}dydv''dsds_1\\
\leq\ & C_N C_{T_0,\xi,N,\varepsilon}C_{T_0,\xi,N}\frac{1}{\min_{k,k'}\delta_{k,k'}}\int_0^t\int_0^{s_1}e^{-\nu_0 t}\int_{|v''|\leq 3N}e^{\nu_0 s}\left\{\int_\Omega|h(s,y,v'')|^2dy\right\}^{1/2}dv''dsds_1\\
\leq\ & C_{\varepsilon,T_0,N}\int_0^t||f(s)||_{L^2}ds
\end{aligned}$$

where $f = \frac{h}{w}$. We therefore conclude

$$||h(t,x,v)\mathbf{1}_{A_\alpha}||_\infty \leq \{1 + C_K t\}e^{-\nu_0 t}||h_0||_\infty + \{\frac{C}{N} + C_{N,T_0}\varepsilon\}\sup_s e^{-\frac{\nu_0}{2}\{t-s\}}||h(s)||_\infty + C_{\varepsilon,N,T_0}\int_0^t ||f(s)||ds. \tag{3.221}$$

STEP 2: Estimate of $h(t,x,v)$. We now further plug (3.221) back in: $h(t,x,v) =$

$$G(t,0)h_0 + \int_0^t G(t,s_1)\{K_w h(s_1)\}ds_1 \tag{3.222}$$

$$= h_0(X_{\mathbf{cl}}(0;t,x,v), V_{\mathbf{cl}}(0;t,x,v))e^{-\nu(v)t} \tag{3.223}$$

$$+ \int_0^t ds_1 \int_{\mathbb{R}^3} dv' e^{-\nu(v)(t-s_1)}\mathbf{k}_w(v', V_{\mathbf{cl}}(s_1;t,x,v))h(s_1, X_{\mathbf{cl}}(s_1;t,x,v), v') \tag{3.224}$$

to get

$$||h(t)||_\infty \leq e^{-\nu_0 t}||h_0||_\infty + \int_0^t e^{-\nu_0\{t-s_1\}}||K_w h||_\infty(s_1)ds. \tag{3.225}$$

But $\{K_w h\}(s_1, X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) = \int_{\mathbb{R}^3}\mathbf{k}_w(v', V_{\mathbf{cl}}(s;t,x,v))h(s_1, X_{\mathbf{cl}}(s;t,x,v), v')dv'$ and we split it as

$$\int_{\mathbf{R}^3} e^{-\nu(v)(t-s_1)}\mathbf{k}_w(v', V_{\mathbf{cl}}(s_1;t,x,v))h(s_1, X_{\mathbf{cl}}(s_1;t,x,v), v')\{1 - \mathbf{1}_{A_\alpha(X_{\mathbf{cl}}(s_1;t,x,v),v')}\}dv'$$
$$+ \int_{\mathbf{R}^3} e^{-\nu(v)(t-s_1)}\mathbf{k}_w(v', V_{\mathbf{cl}}(s_1;t,x,v))h(s_1, X_{\mathbf{cl}}(s_1;t,x,v), v')\mathbf{1}_{A_\alpha(X_{\mathbf{cl}}(s_1;t,x,v),v')}dv'.$$

By the definition of A_α in (3.190), the first term is bounded by

$$\left(\int_{|v'|\geq N,\ \text{or}\ |v'|\leq\frac{1}{N}} |\mathbf{k}_w(v,v')|dv' + \int_{\alpha(x,v')\leq\frac{1}{N}} |\mathbf{k}_w(v,v')|\right)||h(s_1)||_\infty.$$

By (3.174) and (3.214) if necessary, $\int_{|v'|\geq N,\ \text{or}\ |v'|\leq\frac{1}{N}} |\mathbf{k}_w(v,v')|dv' = o(1)$ as $N \to \infty$. From $\alpha(x,v') \leq \frac{1}{N}$, $\xi^2(\bar{x}) + [\bar{v}'\cdot\nabla\xi(\bar{x})]^2 \leq \frac{1}{N}$. For N large, $x \backsim \partial\Omega$ and $|\nabla\xi(x)| \geq \frac{1}{2}$ so that

$$\int_{\alpha(x,v')\leq\frac{1}{N}} |\mathbf{k}_w(v,v')|dv' \leq \int_{|\bar{v}'\cdot\frac{\nabla\xi(\bar{x})}{|\nabla\xi(\bar{x})|}|\leq\frac{2}{\sqrt{N}}} |\mathbf{k}_w(v,v')|dv' = o(1)$$

as $N \to \infty$. We apply (3.221) to the second term to bound $||\{K_w h\}(s_1)||_\infty$ as

$$\{1 + C_K s_1\}e^{-\nu_0 s_1}||h_0||_\infty + \{o(1) + \frac{C}{N} + C_{N,T_0}\varepsilon\}\sup_s e^{-\frac{\nu_0}{2}\{s_1-s\}}||h(s)||_\infty + C_{\varepsilon,N,T_0}\int_0^{s_1} ||f(s)||ds.$$

Hence, by (3.225), $||h(t)||_\infty$ is bounded by

$$
\begin{aligned}
& e^{-\nu_0 t}||h_0||_\infty + \int_0^t e^{-\nu_0 t}\{1+C_K s_1\}||h_0||_\infty + \\
& + \int_0^t e^{-\nu_0\{t-s_1\}}\{o(1)+\frac{C}{N}+C_{N,T_0}\varepsilon\}\sup_{s\le t} e^{-\frac{\nu_0}{2}\{s_1-s\}}||h(s)||_\infty + C_{\varepsilon,N,T_0}\int_0^{s_1}||f(s)||ds\}ds_1 \\
\le\ & \{1+C_K t^2\}e^{-\nu_0 t}||h_0||_\infty + C\{o(1)+\frac{1}{N}+C_{N,T_0}\varepsilon\}\sup_{s\le t}\{e^{-\frac{\nu_0}{4}\{t-s\}}||h(s)||_\infty\} + C_{\varepsilon,N,T_0}\int_0^t ||f(s)||ds.
\end{aligned}
$$

We choose T_0 large such that $2\{1+C_K T_0^2\}e^{-\frac{\nu_0}{4}T_0} = e^{-\lambda T_0}$, for some $\lambda > 0$. We then further choose N large, and then ε sufficiently small such that $C\{o(1)+\frac{1}{N}+C_{N,T_0}\varepsilon\} < \frac{1}{2}$. We there have

$$
\sup_{0\le s\le t}\{e^{\frac{\nu_0}{4}s}||h(s)||_\infty\} \le 2\{1+C_K t^2\}||h_0||_\infty + C_{T_0}\int_0^t ||f(s)||ds.
$$

Choosing $s = t = T_0$, we deduce the finite-time estimate (3.206), and our theorem follows from Lemma 34. $\square$

3.4 Analytic Non-Convex Cylinderical Domain

3.4.1 Bad Sets and Properties

Thought this section, we assume the domain Ω is *the analytic non-convex cylinderical domain*, specified in Definition 2. In the presence of the non-convex part of the boundaries, there are two main difficulties in order to show the crucial condition (CRUCIAL LEMMA) : First, there is no quantitative estimates of the number of bounces. Second, the specular back-time cycle $[X_{\mathbf{b}}(s;t,x,v), V_{\mathbf{b}}(s;t,x,v)]$ is not differentiable with respect to v, when the cycle hits the boundary tangentially. We construct *the infinite-bounces set* (Section 5.1) and *the grazing set* (Section 5.2) to characterize the points in the phase space cause such two difficulties. Then we show that two sets are small enough and the crucial condition (CRUCIAL LEMMA) is satisfied on the complementary of the union of those two sets, the infinite-bounce set and the grazing set. Finally we are able to show the basic estimate (BASIC ESTIMATE) of $L^p - L^\infty$ framework works.

We state the main result of Section 4.1. The proof of Lemma 35 will be established in Section 4.1.1 – 4.1.8.

Lemma 35. *Assume that the domain* $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1$ *is the analytic non-convex cylinderical domain. For* $\varepsilon > 0, N > 0$*, there exist a finite number* $K > 0$ *and finite open coverings* $\mathcal{U}_1, \cdots, \mathcal{U}_l$ *of* $cl(\Omega)$*, as well as open sets* $\mathcal{O}_1, \cdots, \mathcal{O}_l$ *in* $\left\{ (\underline{v}, v_3) \in \mathbb{R}^3 \ : \ 1/N \le |\underline{v}| \le N/2 \ , \quad -N/2 \le v_3 \le N/2 \right\}$ *with* $m_3(\mathcal{O}_\rangle) < \varepsilon$ *for all* $i = 1, 2, \cdots, l$ *such that, for every* $x \in cl(\Omega)$ *there exists* $i \in \{1, \cdots, l\}$ *so that* $x \in \mathcal{U}_i$*, and for* $v \in \left\{ (\underline{v}, v_3) \in \mathbb{R}^3 \ : \ 1/N \le |\underline{v}| \le N/2 \ , \quad -N/2 \le v_3 \le N/2 \right\} \setminus \mathcal{O}_i$*, the following properties hold :*

- *The specular backward trajectory* $(X_{\mathbf{b}}(s; t, x, v), \ V_{\mathbf{b}}(s; t, x, v))$ *is well-defined for all* $0 \le s \le t \le T_0$.
- *The specualr backward cycle* $(t_i(x, v), x_i(x, v), v_i(x, v))$ *is well-defined for all* $i = 0, 1, 2, \cdots, K$ *so that* $s \in [t_{i+1}(x, v), t_i(x, v))$ *and* $X_{\mathbf{b}}(s; t, x, v) = x_i(x, v) - (t_i(x, v) - s) v_i(x, v), \ V_{\mathbf{b}}(s; t, x, v) = v_i(x, v)$.
- *If* $X_{\mathbf{b}}(s; t, x, v) \in \partial\Omega$ *, then* $n(X_{\mathbf{b}}(s; t, x, v)) \cdot V_{\mathbf{b}}(s; t, x, v) \neq 0$.

Decomposition of Domain

The domain $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1$ is the analytic non-convex cylinderical domain as Definition 11. Notice that κ stands the signed curvature in Definition 3.6 which is a convenient notion to classify convex parts and concave parts of the boundary $\partial\underline{\Omega}$. Assume unit-speed curve $\alpha = (\alpha_1, \alpha_2)$ is a part of the boundary $\partial\Omega$ with $\alpha(\tau) \in \partial\Omega$. Because of *the condition* **L** in Definition 11, it is clear that $\partial\underline{\Omega}$ is *strictly convex* at $\alpha(\tau)$ if and only if $|\ddot{\alpha}(\tau)| \neq 0$ and $(\ddot{\alpha}_1(\tau), \ddot{\alpha}_2(\tau))^T = |\ddot{\alpha}(\tau)| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\dot{\alpha}_1(\tau), \dot{\alpha}_2(\tau))^T = |\ddot{\alpha}(\tau)| \left(-\dot{\alpha}_2(\tau), \dot{\alpha}_1(\tau)\right)^T$. By the definition of the signed curvature, in this case, the signed curvature has the positive sign :

$$\kappa(\tau) \ \equiv \ \dot{\alpha}_1(\tau)\ddot{\alpha}_2(\tau) - \ddot{\alpha}_1(\tau)\dot{\alpha}_2(\tau) \ = \ |\ddot{\alpha}(\tau)| \times \{\dot{\alpha}_1(\tau)\dot{\alpha}_1(\tau) + \dot{\alpha}_2(\tau)\dot{\alpha}_2(\tau)\} \ = \ |\ddot{\alpha}(\tau)| \ > \ 0.$$

Similarly, $\partial\underline{\Omega}$ is *strictly concave* at $\alpha(\tau)$ if and only if $|\ddot{\alpha}(\tau)| \neq 0$ and $(\ddot{\alpha}_1(\tau), \ddot{\alpha}_2(\tau))^T = -|\ddot{\alpha}(\tau)| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\dot{\alpha}_1(\tau), \dot{\alpha}_2(\tau))^T = -|\ddot{\alpha}(\tau)| \left(-\dot{\alpha}_2(\tau), \dot{\alpha}_1(\tau)\right)^T$. In this case, the signed curvature has the negative sign :

$$\kappa(\tau) \ \equiv \ \dot{\alpha}_1(\tau)\ddot{\alpha}_2(\tau) - \ddot{\alpha}_1(\tau)\dot{\alpha}_2(\tau) \ = - \ |\ddot{\alpha}(\tau)| \times \{\dot{\alpha}_1(\tau)\dot{\alpha}_1(\tau) + \dot{\alpha}_2(\tau)\dot{\alpha}_2(\tau)\} \ = \ - \ |\ddot{\alpha}(\tau)| \ < \ 0.$$

Recall that

$$\underline{\Omega} \;\equiv\; \underline{\Omega}^0 \setminus \{\underline{\Omega}^1 \cup \cdots \cup \underline{\Omega}^M\} \;,$$

where each $\underline{\Omega}^i$ is an image of an analytic curve $\alpha^i : [a^i, b^i] \to \mathbb{R}^2$. Consider a fixed i and $\underline{\Omega}^i$. Since the signed curvature κ is continuous, the set $\{\tau \in [a^i, b^i] : \kappa(\tau) < 0\}$ is an open subset of the interval $[a^i, b^i]$, and it is a countable union of disjoint open intervals, i.e.

$$\{\tau \in [a^i, b^i] \;:\; \kappa(\tau) < 0\} \;=\; \bigsqcup_{j=1}^{\infty} (a^{i,j}, b^{i,j}) \;.$$

It is clear that $\kappa(a^{i,j}) = 0 = \kappa(b^{i,j})$ for all i, j : Suppose not, then there exists $\epsilon > 0$ such that $(a^{i,j} - \epsilon, b^{i,j}) \in \{\tau \in [a^i, b^i] : \kappa(\tau) < 0\}$ or $(a^{i,j}, b^{i,j} + \epsilon) \in \{\tau \in [a^i, b^i] : \kappa(\tau) < 0\}$ which is a contradiction. Since the curve α^i is analytic, it's signed curvature κ is also analytic. If κ is identically zero then α is a straight line so α cannot be a boundary of a bounded set Ω. Since the analytic function κ on a compact set $[a^i, b^i]$ only allow finite zeroes, there is a finite number $N_i < \infty$ such that

$$\{\tau \in [a^i, b^i] \;:\; \kappa(\tau) < 0\} \;=\; \bigsqcup_{j=1}^{N_i} (a^{i,j}, b^{i,j}) \;,$$

which is a finite union of disjoint open intervals. The closure of $\{\tau \in [a^i, b^i] : \kappa(\tau) < 0\}$ is a union of closed intervals and there may exist two closed intervals which have same end point. For example $[a^{1,1}, b^{1,1}]$ and $[a^{1,2}, b^{1,2}]$ can have the same end point $b_{1,1} = a_{1,2}$. In this case we can rewrite $[a^{1,1}, b^{1,1}] \cup [a^{1,2}, b^{1,2}] \;\equiv\; [\tilde{a}^{1,1}, \tilde{b}^{1,1}]$ with $\tilde{a}^{1,1} = a^{1,1}$ and $\tilde{b}^{1,1} = b^{1,2}$. Then we can rewrite

$$cl(\{\tau \in [a^i, b^i] \;:\; \kappa(\tau) < 0\}) \;=\; \bigcup_{j=1}^{N_i} [a^{i,j}, b^{i,j}] \;=\; \bigsqcup_{j=1}^{M_i} [\tilde{a}^{i,j}, \tilde{b}^{i,j}] \tag{3.226}$$

and N_i ($\geq M_i$) is the number of intervals $[a^{i,j}, b^{i,j}]$ and $cl(A)$ stands the closure of the set A.

Definition 20. *Let* $\Omega \;\equiv\; \underline{\Omega} \times \mathbb{T}^1$ *be the analytic non-convex cylinderical domain. We decompose the boundary* $\partial\underline{\Omega}$ *into three parts;*

$$\begin{aligned}
\partial\underline{\Omega}_C \;&=\; \bigsqcup_{i=0}^{M} \bigsqcup_{j=1}^{M_i} \alpha^i(\tilde{a}^{i,j}, \tilde{b}^{i,j}) \;=\; \bigsqcup_{l=1}^{M_C} \partial\underline{\Omega}_{C,l} && \textit{(Concave part)},\\
\partial\underline{\Omega}_I \;&=\; \bigsqcup_{i=1}^{M} \{\alpha^i(\tilde{a}^{i,j}), \alpha^i(\tilde{b}^{i,j})\} && \textit{(Inflection part)},\\
\partial\underline{\Omega}_V \;&=\; \partial\underline{\Omega} \setminus (\partial\underline{\Omega}_C \cup \partial\underline{\Omega}_I) && \textit{(Convex part)},
\end{aligned}$$

where $\alpha^i(\tilde{a}^{i,j}, \tilde{b}^{i,j})$ *is an image of* α^i *of an open interval* $(\tilde{a}^{i,j}, \tilde{b}^{i,j})$*. The number* $M_C \;\equiv\; \sum_{i=0}^{M} M_i$

and the l-th concave part $\partial\underline{\Omega}_{C,l}$ *for* $l = 1, 2, ..., M_C$ *is renumbered sequence of* $\alpha^i(\tilde{a}^{i,j}, \tilde{b}^{i,j})$*. We can further split* $\partial\Omega_I = \partial\Omega_{I^+} \cup \partial\Omega_{I^-}$ *where* $\partial\Omega_{I^+} = \bigsqcup_{i=0}^{M} \partial\Omega_{I^+,i}$ *and* $\partial\Omega_{I^-} = \bigsqcup_{i=0}^{M} \partial\Omega_{I^-,i}$ *with*

$$\partial\Omega_{I^+,i} = \{\alpha_i(\tau) \in \partial\Omega_{I,i} : \exists\ \varepsilon > 0 \text{ such that } k_i(\tau') > 0 \text{ for } \tau' \in (\tau - \varepsilon, \tau) \text{ and } k_i(\tau') < 0 \text{ for } \tau' \in (\tau, \tau + \varepsilon)\}$$
$$\partial\Omega_{I^-,i} = \{\alpha_i(\tau) \in \partial\Omega_I^i : \exists\ \varepsilon > 0 \text{ such that } k_i(\tau') < 0 \text{ for } \tau' \in (\tau - \varepsilon, \tau) \text{ and } k_i(\tau') > 0 \text{ for } \tau' \in (\tau, \tau + \varepsilon)\}$$

We say the concave part $\partial\Omega_C$*, the convex part* $\partial\Omega_V$ *and the inflection points* $\partial\Omega_I$*. It is easy to see that* $\overline{\partial\Omega}_C = \bigcup_{i=0}^{M} \overline{\{t \in [a_i, b_i] : k(t) < 0\}} = \bigsqcup_{i=1}^{M} \bigsqcup_{j=1}^{M_i} [a'_{i,j}, b'_{i,j}]$*.*

For convenience we abuse the notation as $\partial\Omega_C = \bigsqcup_{i=0}^{M} \bigsqcup_{j=1}^{M_i} (a'_{i,j}, b'_{i,j})$ and $\partial\Omega_I = \bigsqcup_{i=1}^{M} \{a'_{i,j}, b'_{i,j}\}$ and $\partial\Omega_V = \bigsqcup_{i=0}^{M} [a_i, b_i] \backslash (\partial\Omega_C \cup \partial\Omega_I)$. We also use notations $\partial\Omega_{C,i} = \partial\Omega_C \cap \partial\Omega_i$, $\partial\Omega_{I,i} = \partial\Omega_I \cap \partial\Omega_i$ and $\partial\Omega_{V,i} = \partial\Omega_V \cap \partial\Omega_i$.

near-Boundary Set

The *near-Boundary set* for the analytic non-convex cylinderical domain is exactly same as one of the smooth convex cylinderical domain.

near-Inflection Set

In this section, we define *the near-inflection set*, the $\varepsilon-$neighborhood of points in the phase space whose specular back-time cycle hits the inflection point of boundaries tangentially. Precisely, we want to characterize

$$\left\{(x, v) \in cl(\Omega) \times \mathbb{R}^3 : 1/N \le |v| \le N, (X_{\mathbf{b}}(s; T_0, x, v), V_{\mathbf{b}}(s; T_0, x, v)) \in \mathfrak{I} \text{ for some } s \in [0, T_0]\right\},$$

and find out it's $\epsilon-$neighborhood.

Lemma 36 (near-Inflection Set)**.** *Assume that the domain* $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1$ *is the analytic non-convex cylindrical domain. For* $\varepsilon, N > 0$*, there exist finite points* $\underline{x}_1^{nI}, \cdots, \underline{x}_{l_{nI}}^{nI}$ *in* $\underline{\Omega}$ *and open cylinders* $\mathcal{C}(\underline{x}_1^{nI}; r_1^{nI}), \cdots, \mathcal{C}(\underline{x}_{l_{nI}}^{nI}; r_{l_{nI}}^{nI})$*, as well as open sets* $\mathcal{O}_1^{nI}, \cdots, \mathcal{O}_{l_{nI}}^{nI}$ *in* $\{v = (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2\ ,\ -N/2 \le v_3 \le N/2\}$ *with* $m_3(\mathcal{O}_i^{nI}) < \varepsilon$ *for all* $i = 1, \cdots, l_{nI}$ *such that for every* $x \in cl(\Omega)$ *there exists* $i \in \{1, \cdots, l_{nI}\}$ *with* $x \in \mathcal{C}(\underline{x}_i^{nI}; r_i^{nI})$ *and, for* $v \in \{v = (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le$

$N/2$, $-N/2 \le v_3 \le N/2\} \setminus \mathcal{O}_i^{nI}$, *satisfies*

$$\left(X_{\mathbf{b}}(s;T_0,x,v), \frac{V_{\mathbf{b}}(s;T_0,x,v)}{|V_{\mathbf{b}}(s;T_0,x,v)|} \right) \notin \mathfrak{I} \quad \text{for all } s \in [0,T_0] \ . \tag{3.227}$$

Now we are going to establish some preliminary facts in order to prove Lemma 36. In the next lemma, we show that for each $(x,v) \in cl(\Omega) \times \mathbb{R}^3$, the number of bounces of the specular cycle starting at (x,v) is finite for the finite time.

Lemma 37. *Assume the domain $\Omega = \underline{\Omega} \times \mathbb{T}^1$ is the analytic non-convex cylinderical domain. For any $\underline{x} \in cl(\underline{\Omega})$ and $\hat{v} \in \mathbb{S}^1$, if $(x_i(\underline{x},\hat{v}), v_i(\underline{x},\hat{v})) \notin \underline{\mathfrak{I}}$ for all $i = 0,1,2,\cdots$, then*

$$\sum_{i=0}^{\infty} \mid x_i(\underline{x},\hat{v}) - x_{i+1}(\underline{x},\hat{v}) \mid = +\infty \ . \tag{3.228}$$

Similarly, if $(x_i^{\mathbf{f}}(\underline{x},\hat{v}), v_i^{\mathbf{f}}(\underline{x},\hat{v})) \notin \underline{\mathfrak{I}}$ for all $i = 0,1,2,\cdots$, then $\sum_{i=0}^{\infty} \mid x_i^{\mathbf{f}}(\underline{x},\hat{v}) - x_{i+1}^{\mathbf{f}}(\underline{x},\hat{v}) \mid = +\infty$.

Proof. We prove the lemma via a contradiction. Notice that we always consider the specular back-time (or forward) cycle in the projected 2-dimensional setting. Suppose $(x_i(\underline{x},\hat{v}), v_i(\underline{x},\hat{v}) \notin \mathfrak{I}$ for all $i = 0,1,2,...$ and $\sum_{i=0}^{\infty} |x_i(\underline{x},\hat{v}) - x_{i+1}(\underline{x},\hat{v})| = M < \infty$. Since there are infinite number of bounces in a finite time length M, there is an accumulation point $x_\infty = \lim_{i\to\infty} x_i(\underline{x},\hat{v})$. Since the boundary $\partial\underline{\Omega}$ is a closed set in $\mathbb{R}^2$, we have $\lim_{i\to\infty} x_i(\underline{x},\hat{v}) \in \partial\underline{\Omega}$. Assume that $\lim_{i\to\infty} x_i(\underline{x},\hat{v}) = \alpha(\tau_\infty)$. For sufficiently large $i \in \mathbb{N}$, we assume $x_i(\underline{x},\hat{v}) = \alpha(\tau_i)$ and $\tau_i \in (\tau_\infty - \varepsilon, \tau_\infty]$ for small $\varepsilon > 0$. Considering the sign of the signed curvature κ, there are three cases :

1. $\kappa(\tau_\infty) < 0$: It means that $\alpha(\tau_\infty)$ is on some concave part of the boundary. This case cannot happen. Suppose $\kappa < 0$ on $(\tau_\infty - \varepsilon, \tau_\infty + \varepsilon)$. For $\tau_i, \tau_{i+1} \in (\tau_\infty - \varepsilon, \tau_\infty + \varepsilon)$, the line $\overline{\alpha(\tau_1)\alpha(\tau_2)}$ cannot be included in $cl(\Omega)$. Therefore there is no $\hat{v} \in \mathbb{S}^1$ such that $\alpha(\tau_2) = x_{\mathbf{b}}(\alpha(\tau_1),\hat{v})$. This is a contradiction.

2. $\kappa(\tau_\infty) = 0$: There are three cases.

   - $\kappa < 0$ on $(\tau_\infty - \varepsilon, \tau_\infty)$: This case cannot happen with the same reason of Case 1 $\kappa(\tau_\infty) < 0$.
   - $\kappa = 0$ on $(\tau_\infty - \varepsilon, \tau_\infty)$: This case cannot happen because of the analyticity of α. If κ is identically zero on $(\tau_\infty - \varepsilon, \tau_\infty)$ then $\kappa(\tau) \equiv 0$ for all τ and the image of α is a straight

line. But the boundary cannot contain the straight line, because $\underline{\Omega}$ is bounded.

- $\underline{\kappa > 0 \ \text{ on } \ (\tau_\infty - \varepsilon, \tau_\infty)}$

3. $\underline{\kappa(\tau_\infty) > 0}$

The remaining cases are two underlined cases. In order to treat two cases at the same time, from now, we assume that $\kappa > 0$ on $(\tau_\infty - \varepsilon, \tau_\infty)$.

Step 1 : We show that the curvature κ must vanish at τ_∞.

Suppose that the curvature κ has a lower bound $\kappa_0 > 0$ near τ_∞. Then by the Velocity Lemma in [27], the number of bounces is bounded uniformly. For detail see the page 778 of [27].

Step 2 : We use the argument in Chapter 6. The Non-Convex Case, an Example of [58]. The curvature must vanish to infinite order at τ_∞, i.e. $\frac{d^k\kappa}{dx_1^k}\Big|_{\tau_\infty} = 0$ for all $k \in \mathbb{N}$. This implied that the curve is straight line which is contradiction because the domain is bounded. If this is not the case, clearly $k(\tau)$ must decrease monotonically to zero for $\tau \sim \tau_\infty$. But for this case $l_{k+1} > l_k$ which contradicts convergence of $\sum l_k$. $\square$

Definition 21. *For $\underline{x} \in cl(\underline{\Omega})$ and $\hat{v} \in \mathbb{S}^1$, we define*

$$\sharp_{\mathbf{b}}^{\mathfrak{I}}(\underline{x}, \hat{v}) \ = \ \inf\{ i \in \mathbb{N} : (x_i(\underline{x}, \hat{v}), v_i(\underline{x}, \hat{v})) \in \underline{\mathfrak{I}} \} \ ,$$

which is the number of bounces before the specular back-time cycle hitting the inflection set $\underline{\mathfrak{I}} \subset \partial\underline{\Omega} \times \mathbb{R}^2$. We also define

$$\sharp_{\mathbf{b}}^{>L}(\underline{x}, \hat{v}) \ = \ \inf \left\{ i \in \mathbb{N} : i \leq \sharp_{\mathbf{b}}^{\mathfrak{I}}(\underline{x}, \hat{v}) \ \textit{and} \ \sum_{k=0}^{i} |x_k(\underline{x}, \hat{v}) - x_{k+1}(\underline{x}, \hat{v})| > L \right\} \ ,$$

which is the smallest number of bounces that the traveling distance of the specular back-time cycle exceeds the chosen $L > 0$. Similarly we define

$$\begin{aligned} \sharp_{\mathbf{f}}^{\mathfrak{I}}(\underline{x}, \hat{v}) \ &= \ \inf\{ \ i \in \mathbb{N} : (x_i^{\mathbf{f}}(\underline{x}, \hat{v}), v_i^{\mathbf{f}}(\underline{x}, \hat{v})) \in \underline{\mathfrak{I}} \ \} \ , \\ \sharp_{\mathbf{f}}^{>L}(\underline{x}, \hat{v}) \ &= \ \inf \left\{ i \in \mathbb{N} : i \leq \sharp_{\mathbf{f}}^{\mathfrak{I}}(\underline{x}, \hat{v}) \ \textit{and} \ \sum_{k=0}^{i} \left| x_k^{\mathbf{f}}(\underline{x}, \hat{v}) - x_{k+1}^{\mathbf{f}}(\underline{x}, \hat{v}) \right| > L \right\} \ . \end{aligned}$$

Notice that

$$\sharp_{\mathbf{b}}^{>L}(\underline{x}, \hat{v}) \ < \infty \ ,$$

because of Lemma 37. By Definition 21, for $\underline{x} \in cl(\underline{\Omega})$ and $\hat{v} \in \mathbb{S}^1$, the specular back-time cycle $(\ x_i(\underline{x}, \hat{v}),\ v_i(\underline{x}, \hat{v})\ )$ is well-defined for all $i = 0, 1, \cdots, \sharp_{\mathbf{b}}^{\mathfrak{I}}(\underline{x}, \hat{v})$. Moreover, if $\sharp_{\mathbf{b}}^{\mathfrak{I}}(\underline{x}, \hat{v})$ is strictly bigger than $\sharp_{\mathbf{b}}^{>L}(\underline{x}, \hat{v})$, then

$$\sum_{k=0}^{\sharp_{\mathbf{b}}^{>L}(\underline{x}, \hat{v})} |x_k(\underline{x}, \hat{v}) - x_{k+1}(\underline{x}, \hat{v})| \ \geq L\ .$$

The same properties hold for the specular forward cycle $(\ x_i^{\mathbf{f}}(\underline{x}, \hat{v}),\ v_i^{\mathbf{f}}(\underline{x}, \hat{v})\ )$ and $\sharp_{\mathbf{f}}^{>L}(\underline{x}, \hat{v})$.

Definition 22. *For $\underline{x} \in \underline{\Omega}$ and $\hat{v} \in \mathbb{S}^1$, we define the* ***backward L-track***

$$\mathbf{Trk_b}(\underline{x}, \hat{v}; L) = \Big\{ (\underline{y}, \hat{u}) \in cl(\underline{\Omega}) \times \mathbb{S}^1 : \underline{y} \in \overline{x_i(\underline{x}, \hat{v}) x_{i+1}(\underline{x}, \hat{v})} \quad \textit{and} \ \ \hat{u} = v_i(\underline{x}, \hat{v}) \quad \textit{for} \ \ i = 0, 1, \cdots, \sharp_{\mathbf{b}}^{>L}(\underline{x}, \hat{v}) \Big\}\ ,$$

and the ***forward L-track***

$$\mathbf{Trk_f}(\underline{x}, \hat{v}; L) = \Big\{ (\underline{y}, \hat{u}) \in cl(\underline{\Omega}) \times \mathbb{S}^1 : \underline{y} \in \overline{x_i^{\mathbf{f}}(\underline{x}, \hat{v}) x_{i+1}^{\mathbf{f}}(\underline{x}, \hat{v})} \quad \textit{and} \ \ \hat{u} = v_i^{\mathbf{f}}(\underline{x}, \hat{v}) \quad \textit{for} \ \ i = 0, 1, \cdots, \sharp_{\mathbf{f}}^{>L}(\underline{x}, \hat{v}) \Big\}\ .$$

Notice that the unit-velocity projected 2-dimensional inflection set $\hat{\underline{\mathfrak{I}}} = \{ (\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}}) : i = 1, 2, \cdots, N^{\mathfrak{I}} \}$ is a finite set. Let's define a union of the whole forward L-tracks with $L = NT_0$ from the unit-velocity projected 2-dimensional inflection set $\hat{\underline{\mathfrak{I}}}$: $\bigcup_{i=1}^{N^{\mathfrak{I}}} \mathbf{Trk_f}(\ \underline{x}_i^{\mathfrak{I}},\ \hat{v}_i^{\mathfrak{I}}\ ; NT_0\ )$. Now, we fix $(\underline{x}, \underline{v}) \in cl(\underline{\Omega}) \times \{ \underline{v} \in \mathbb{R}^2 : 1/N \leq |\underline{v}| \leq N/2 \}$. Suppose $\big(\underline{X}_{\mathbf{b}}(s; T_0, \underline{x}, \underline{v}),\ \underline{V}_{\mathbf{b}}(s; T_0, \underline{x}, \underline{v}) \big) \in \underline{\mathfrak{I}}$ for some $s \in [0, T_0]$. Therefore for $K = \sharp_{\mathbf{b}}^{>L}(\underline{x}, \underline{v}/|\underline{v}|) < \infty$, we have $(x_K(\underline{x}, \underline{v}), v_K(\underline{x}, \underline{v})) \in \mathfrak{I}$. Hence we have $(\underline{x}, \underline{v}) \in \bigcup_{i=1}^{N^{\mathfrak{I}}} \mathbf{Trk_f}(\ \underline{x}_i^{\mathfrak{I}},\ \hat{v}_i^{\mathfrak{I}}\ ; NT_0\ )$. To sum, we conclude that

$$\Big\{ (\underline{x}, \underline{v}) \in cl(\underline{\Omega}) \times \mathbb{R}^2 : 1/N \leq |\underline{v}| \leq N,\ \big(\underline{X}_{\mathbf{b}}(s; T_0, \underline{x}, \underline{v}),\ \underline{V}_{\mathbf{b}}(s; T_0, \underline{x}, \underline{v}) \big) \in \mathfrak{I} \text{ for some } s \in [0, T_0] \Big\} \subset \bigcup_{i=1}^{N^{\mathfrak{I}}} \mathbf{Trk_f}(\ \underline{x}_i^{\mathfrak{I}},\ \hat{v}_i^{\mathfrak{I}}\ ; NT_0\ ).$$

Now we give the proof of Lemma 36. The basic idea to prove Lemma 36 is to characterize the $\varepsilon-$neighborhood of the whole specular forward cycles starting from the inflection set $\mathfrak{I}$:

$$\Big\{ \Big(X_{\mathbf{f}}(s; T_0, x, v), V_{\mathbf{f}}(s; T_0, x, v) \Big) \in cl(\Omega) \times \mathbb{R}^3 : 1/N \leq |v| \leq N, (x, \frac{v}{|v|}) \in \mathfrak{I} \ \text{ for } s \in [0, T_0] \Big\}.$$

Proof of Lemma 36. We construct such sets in the projected 2-dimensional setting. Notice that $\partial\underline{\Omega} \subset \mathbb{R}^2$ is a compact set in $\mathbb{R}^2$ and a union of the image of finite smooth curves. Consider the union of all forward L-tracks starting from the inflection set $\mathfrak{I}$: $\bigcup_{i=1}^{N^{\mathfrak{I}}}$ $\mathbf{Trk_f}$ $(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}}; NT_0)$ which is

$$\bigcup_{i=1}^{N^{\mathfrak{I}}} \bigcup_{k=0}^{\sharp_{\mathbf{f}}^{>L}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})} \overline{x_k^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\ x_{k+1}^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})} \times \{v_k(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\} \tag{3.229}$$

Here, we set $L = NT_0$, the longest distance that the specular backward cycle can travel for time T_0 with velocity $|v| \leq N$. Denote $\theta_{i,k} \in [0, 2\pi)$, specified by $(\cos\theta_{i,k}, \sin\theta_{i,k})^T = v_k(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})$.

For $\underline{x} \in \bigcup_{i=1}^{N^{\mathfrak{I}}} \bigcup_{k=0}^{\sharp_{\mathbf{f}}^{>L}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})} \overline{x_k^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\ x_{k+1}^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})}$. Then there exists $i \in \{1, 2, \cdots, N^{\mathfrak{I}}\}$ and $k \in \{0, 1, \cdots, \sharp_{\mathbf{f}}^{>L}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\}$ so that $\underline{x}$ is on the line segment of $\overline{x_k^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\ x_{k+1}^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})}$. So, there are corresponding $\{\ \theta_{i,k} \in [0, 2\pi)\ :\ \underline{x} \in x_{k+1}^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\}$ and define

$$\underline{O}_{\underline{x}} \equiv \left\{ \underline{v} = \Big(|\underline{v}|\cos\theta, |\underline{v}|\sin\theta \Big)^T : |\underline{v}| \in \big[\frac{1}{N}, N\big] \text{ and } \theta \in \Big(\theta_{i,k} - \frac{\varepsilon}{40N^3 N^{\mathfrak{I}} \sharp_{\mathbf{f}}^{>L}(\mathfrak{I})},\ \theta_{i,k} + \frac{\varepsilon}{40N^3 N^{\mathfrak{I}} \sharp_{\mathbf{f}}^{>L}(\mathfrak{I})} \Big) \right\},$$

where $\sharp_{\mathbf{f}}^{>L}(\mathfrak{I}) = \max_{i=1,\cdots,N^{\mathfrak{I}}} \sharp_{\mathbf{f}}^{>L}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}}) < +\infty$. Choose

$$0 < r_{\underline{x}} < \frac{1}{10} \min_{i=1,\cdots,N^{\mathfrak{I}}} \min_{k=0,1,\cdots,\sharp_{\mathbf{f}}^{>L}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})} |x_k(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}}) - x_{k+1}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})|.$$

We can check that the 2 dimensional Lebesque measure of $\underline{O}_{\underline{x}}$ is less than $\varepsilon/10$:

$$m_2(\underline{O}_{\underline{x}}) \leq N^2 \times \frac{2\varepsilon}{40N^3 N^{\mathfrak{I}} \sharp_{\mathbf{f}}^{>L}(\mathfrak{I})} \times N^{\mathfrak{I}} \times 2\sharp_{\mathbf{f}}^{>L}(\mathfrak{I}) \leq \frac{\varepsilon}{10N}. \tag{3.230}$$

For $\underline{x} \notin \bigcup_{i=1}^{N^{\mathfrak{I}}} \bigcup_{k=0}^{\sharp_{\mathbf{f}}^{>L}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})} \overline{x_k^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\ x_{k+1}^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})}$. Choose $r_{\underline{x}} > 0$ so that $\underline{B}(\underline{x}; r_{\underline{x}}) \cap \overline{x_k^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\ x_{k+1}^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})} = \emptyset$ for all $i = 1, \cdots, N^{\mathfrak{I}}$ and $k = 0, 1, \cdots, \sharp_{\mathbf{f}}^{>L}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})$. And define $\underline{O}_{\underline{x}} = \emptyset$ in this case.

Therefore, for each $\underline{x} \in cl(\underline{\Omega})$, we have $\{\underline{B}(\underline{x}; r_{\underline{x}}) : \underline{x} \in cl(\underline{\Omega})\}$, open coverings of $cl(\underline{\Omega})$ and open sets $\underline{O}_{\underline{x}} \subset \mathbb{R}^2$. Using the compactness of $cl(\underline{\Omega})$ and $\bigcup_{i=1}^{N^{\mathfrak{I}}} \bigcup_{k=0}^{\sharp_{\mathbf{f}}^{>L}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})} \overline{x_k^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\ x_{k+1}^{\mathbf{f}}(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})}$, we extract finite coverings. Therefore we have $\underline{x}_1^{nI}, \cdots, \underline{x}_{l_{nI}}^{nI}$ in $cl(\underline{\Omega})$ such that $\underline{B}(\underline{x}_i^{nI})$ is open coverings of $cl(\underline{\Omega})$ and $\underline{O}_i^{nI} \subset \mathbb{R}^2$ with $m_2(\underline{O}_i^{nI}) \leq \frac{\varepsilon}{10}$.

From the 2-dimensional stting, we prove the lemma. Define

$$\mathcal{C}(\underline{x}_i^{nI}) \ \equiv\ \underline{B}(\underline{x}_i^{nI}; r_i^{nI}) \times \mathbb{T}^1\ , \quad O_i^{nI}\ \equiv\ \underline{O}_i^{nI} \times \{v_3 \in \mathbb{R} : -N/2 < v_3 < N/2\}.$$

We can check that $m_3(O_i^{nI}) \leq m_2(\underline{O}_i^{nI}) \times N \leq \frac{\varepsilon}{10}$ from (3.230). Now we will prove (3.232). Suppose we have $x \in \mathcal{C}(\underline{x}_i^{nI}; r_i^{nI})$ and, for $v \in \{v = (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N \ , \ -N \le v_3 \le N\} \setminus \mathcal{O}_i^{nI}$ such that $\left(\underline{X}_{\mathbf{b}}(s; T_0, \underline{x}, \underline{v}), \frac{\underline{V}_{\mathbf{b}}(s;T_0,\underline{x},\underline{v})}{|\underline{V}_{\mathbf{b}}(s;T_0,\underline{x},\underline{v})|}\right) \in \underline{\mathfrak{I}}$ for some $s \in [0, T_0]$. Let $j \in \{1, \cdots, N^{\mathfrak{I}}\}$ so that $(\underline{x}_j^{\mathfrak{I}}, \hat{v}_j^{\mathfrak{I}}) = \left(\underline{X}_{\mathbf{b}}(s; T_0, \underline{x}, \underline{v}), \frac{\underline{V}_{\mathbf{b}}(s;T_0,\underline{x},\underline{v})}{|\underline{V}_{\mathbf{b}}(s;T_0,\underline{x},\underline{v})|}\right)$. Then we can find $k \in \mathbb{N}$ and $k \le \sharp_{\mathbf{f}}^{>L}(\mathfrak{I})$ so that $(\underline{x}, \underline{v}) = (x_k^{\mathbf{f}}(\underline{x}_j^{\mathfrak{I}}, \hat{v}_j^{\mathfrak{I}}), v_k^{\mathbf{f}}(\underline{x}_j^{\mathfrak{I}}, \hat{v}_j^{\mathfrak{I}}))$ so that $\underline{v} \in \underline{O}_i^{nI}$ which is a contradiction. $\square$

Infinite-Bounces Set

Now, we define the infinite-bounces set and prove it's crucial properties. The following is direct consequence of Lemma 29 and Lemma 36.

Lemma 38 (Infinite-Bounces Set). *Assume that the domain $\Omega \equiv \underline{\Omega} \times \mathbb{T}^1$ is the analytic non-convex cylinderical domain. For $\varepsilon, N > 0$, there exist finite points $\underline{x}_1^{IB}, \cdots, \underline{x}_{l_{IB}}^{IB}$ in $cl(\underline{\Omega})$ and open cylinders $\mathcal{C}(\underline{x}_1^{IB}; r_1^{IB}), \cdots, \mathcal{C}(\underline{x}_{l_{IB}}^{IB}; r_{l_{IB}}^{IB})$, as well as open sets $\mathcal{O}_1^{IB}, \cdots, \mathcal{O}_{l_{IB}}^{IB}$ in $\{v = (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2 \ , \ -N/2 \le v_3 \le N/2\}$ with $m_3(\mathcal{O}_i^{IB}) < \varepsilon/5$ for all $i = 1, \cdots, l_{IB}$ such that for every $x \in cl(\Omega)$, there exists $i \in \{1, \cdots, l_{IB}\}$ with $x \in \mathcal{C}(\underline{x}_i^{IB}; r_i^{IB})$ and, for $v \in \{v = (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2 \ , \ -N/2 \le v_3 \le N/2\} \backslash \mathcal{O}_i^{IB}$,*

$$|v \cdot n(x)| > \frac{\varepsilon}{N^4} \ , \tag{3.231}$$

for all $x \in \partial\Omega \cap \mathcal{C}(\underline{x}_i^{IB}; r_i^{IB})$ and

$$\left(X_{\mathbf{b}}(s; T_0, x, v), \frac{V_{\mathbf{b}}(s; T_0, x, v)}{|V_{\mathbf{b}}(s; T_0, x, v)|}\right) \notin \mathfrak{I} \quad \text{for all } s \in [0, T_0] \ . \tag{3.232}$$

We denote *the infinite-bounces set* $\mathfrak{IB}$ as

$$\begin{aligned} \mathfrak{IB} \ \equiv \ \Big\{ & (x, v) \in cl(\Omega) \times \big\{(\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2, \ -N/2 \le v_3 \le N/2\big\} \\ & : \ v \notin \mathcal{O}_i^{IB} \ \text{ for all } \ i \in \{1, 2, \cdots, l_{IB}\} \ \text{ satisfying } \ x \in \mathcal{C}(\underline{x}_i^{IB}; r_i^{IB})\Big\} \end{aligned} \tag{3.233}$$

and simply denote *the complementary set of* $\mathfrak{IB}$ as

$$\mathfrak{IB}^C \ \equiv \ cl(\Omega) \times \big\{(\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2, \ -N/2 \le v_3 \le N/2\big\} \backslash \mathfrak{IB} \ . \tag{3.234}$$

The corresponding 2-dimensional setting for above sets are

$$\begin{aligned}\underline{\mathfrak{IB}} &\equiv \Big\{ (\underline{x},\underline{v}) \in cl(\underline{\Omega}) \times \big\{\underline{v} \in \mathbb{R}^2 : 1/N \leq |\underline{v}| \leq N/2\big\} \qquad (3.235)\\ &\qquad : \ \underline{v} \notin \underline{\mathcal{O}}_i^{IB} \ \text{ for all } \ i \in \{1,2,\cdots,l_{IB}\} \ \text{ satisfying } \ \underline{x} \in \underline{B}(\underline{x}_i^{IB}; r_i^{IB})\Big\} ,\\ \mathfrak{IB}^C &\equiv cl(\underline{\Omega}) \times \big\{\underline{v} \in \mathbb{R}^2 : 1/N \leq |\underline{v}| \leq N/2\big\} \backslash \underline{\mathfrak{IB}} . \qquad (3.236)\end{aligned}$$

The most important property of the infinite-bounces set is that the bouncing number of the specular backward trajectories on the complementary set of the infinite-bounces is uniformly bounded.

Lemma 39. *Assume the assumptions of Lemma 38. Let $L \equiv NT_0$. Then*

$$\sup_{(x,v)\in\mathfrak{IB}^C} \sharp_{\mathbf{b}}^{>L}(x,v) \ < \ +\infty \ , \qquad (3.237)$$

where the supremum is taken on $\mathfrak{IB}$, the complementary set of $\mathfrak{IB}$, i.e. for all $i = 1,2,\cdots,l_{IB}$, and $x \in \mathcal{C}(\underline{x}_i^{IB}; r_i^{IB})$ and $v \in \{\ (\underline{v}, v_3) \in \mathbb{R}^3 : 1/N \leq |\underline{v}| \leq N/2\ ,\ -N/2 \leq v_3 \leq N/2\ \} \backslash \mathcal{O}_i^{IB}$.

Proof. It suffices to show $\sup \ \sharp_{\mathbf{b}}^{>L}(\underline{x},\underline{v}) \ < \ +\infty$ for $L \equiv T_0$ where the supremum is taken for each $i \in \{1,2,\cdots,l_{IB}\}$ and $\underline{x} \in cl(\underline{B}(\underline{x}_i^{IB}; r_i^{IB}))$ and $\underline{v} \in \mathbb{S}^1 \backslash \mathcal{O}_i^{IB}$. Notice that both $cl(\underline{B}(\underline{x}_i^{IB}; r_i^{IB}))$ and $\mathbb{S}^1 \backslash \mathcal{O}_i^{IB}$ are compact sets.

Step 1 : Fix $i \in \{1,2,\cdots,l_{IB}\}$. For $\underline{x} \in cl(\underline{B}(\underline{x}_i^{IB}; r_i^{IB}))$ and $\underline{v} \in \mathbb{S}^1 \backslash \mathcal{O}_i^{IB}$, there is an open subset $U_{\underline{x},\underline{v}}$ of $cl(\underline{B}(\underline{x}_i^{IB}; r_i^{IB})) \times \mathbb{S}^1 \backslash \mathcal{O}_i^{IB}$ such that

$$\sup_{\left\{cl(\underline{B}(\underline{x}_i^{IB}; r_i^{IB})) \times \mathbb{S}^1 \backslash \mathcal{O}_i^{IB}\right\} \backslash U_{\underline{x},\underline{v}}} \sharp_{\mathbf{b}}^{>L}(\underline{x},\underline{v}) \ < \ +\infty \ .$$

Fix $(\underline{x},\underline{v}) \in cl(\underline{B}(\underline{x}_i^{IB}; r_i^{IB})) \times \mathbb{S}^1 \backslash \mathcal{O}_i^{IB}$. Due to Lemma 37, we have $\sharp_{\mathbf{b}}^{>L}(\underline{x},\underline{v}) < \infty$. Denote the specular backward cycle as $(\underline{x}_k,\underline{v}_k)$ for $k = 1,2,\cdots,\sharp_{\mathbf{b}}^{>L}(\underline{x},\underline{v})$. From (3.232) of Lemma 38, either $\underline{v}_k \cdot n(\underline{x}_k) \neq 0$ or $\underline{v}_k \cdot n(\underline{x}_k) = 0$ with $(\underline{x}_k,\underline{v}_k) \notin \mathfrak{I}$. We denote the later case using the superscript $*$, i.e. the numbering of the specular backward cycle $\{1,2,\cdots,k_1-1,k_1^*,\cdots,k_2^*,k_2+1,\cdots,k_l^*,\cdots,\sharp_{\mathbf{b}}^{>L}(\underline{x},\underline{v})\}$ implies that $\underline{v}_{k_1^*} \cdot n(\underline{x}_{k_1^*}) = \underline{v}_{k_2^*} \cdot n(\underline{x}_{k_2^*}) = \underline{v}_{k_l^*} \cdot n(\underline{x}_{k_l^*}) = 0$ with $(\underline{x}_{k_1^*},\underline{v}_{k_1^*}), (\underline{x}_{k_2^*},\underline{v}_{k_2^*}), (\underline{x}_{k_l^*},\underline{v}_{k_l^*}) \notin \mathfrak{I}$; and $\underline{v}_k \cdot n(\underline{x}_k) \neq 0$ for non-superscripted k's.

Step 2 : Using the compactness of $cl(\underline{B}(\underline{x}_i^{IB}; r_i^{IB})) \times \mathbb{S}^1 \backslash \mathcal{O}_i^{IB}$, we have finite $(\underline{x}_j,\underline{v}_j)$ and $U_{\underline{x}_j,\underline{v}_j}$

with $\bigcup_j U_{\underline{x}_j,\underline{v}_j} = cl(\underline{B}(\underline{x}_i^{IB};r_i^{IB})) \times \mathbb{S}^1 \backslash \mathcal{O}_i^{IB}$. Therefore

$$\sup_j \sup_{U_{\underline{x}_j,\underline{v}_j}} \sharp_{\mathbf{b}}^{>L}(\underline{x},\underline{v}) < +\infty , \tag{3.238}$$

and we prove the lemma. $\square$

Now we establish an important property of the infinite-bounce set. First, we show there is a lower bound of the distance between $(X_{\mathbf{b}}(s;T_0,x,v), V_{\mathbf{b}}(s;T_0,x,v))$ and the inflection set $\mathfrak{I} = \{(\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}})\}_{i=1}^{i=N^{\mathfrak{I}}}$ in the following sense :

Lemma 40. *Assume the same conditions in Lemma 38. Then there exist $\delta_1, \delta_2 > 0$ such that for $(x,v) \in \mathfrak{IB}^C$,*

$$\begin{aligned}
&\textit{if} \quad X_{\mathbf{b}}(s;T_0,x,v) \in \partial\Omega \textit{ and } \Big| V_{\mathbf{b}}(s;T_0,x,v)\cdot n(X_{\mathbf{b}}(s;T_0,x,v)) \Big| < \delta_1 , \\
&\qquad \textit{then } \Big| \underline{X_{\mathbf{b}}}(s;T_0,x,v) - \underline{x}_i^{\mathfrak{I}} \Big| > \delta_2 \textit{ for all } i = 1,\cdots,N^{\mathfrak{I}}, \\
&\textit{if} \quad X_{\mathbf{b}}(s;T_0,x,v) \in \partial\Omega \textit{ and } \Big| \underline{X_{\mathbf{b}}}(s;T_0,x,v) - \underline{x}_i^{\mathfrak{I}} \Big| < \delta_1 \textit{ for some } s \in [0,T_0] \textit{ and } (\underline{x}_i^{\mathfrak{I}}, \hat{v}_i^{\mathfrak{I}}) \in \mathfrak{I}, \\
&\qquad \textit{then } \Big| V_{\mathbf{b}}(s;T_0,x,v)\cdot n(X_{\mathbf{b}}(s;T_0,x,v)) \Big| > \delta_2 .
\end{aligned}$$

Proof. Since the second statement is equivalent to the first one, we are proving the first one. The proof is via a contradiction. Suppose, for all $k \in \mathbb{N}$, there are $s_k \in [0,T_0],\ x_k \in \mathcal{C}(\underline{x}_i^{IB};r_i^{IB}),\ v_k \in \{v = (\underline{v},v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2\ ,\ -N/2 \le v_3 \le N/2\} \setminus \mathcal{O}_i^{IB}$ and $i = 1,\cdots,N^{\mathfrak{I}}$ such that $X_{\mathbf{b}}(s_k,T_0,x_k,v_k) \in \partial\Omega$ and

$$\Big| V_{\mathbf{b}}(s_k;T_0,x_k,v_k)\cdot n(X_{\mathbf{b}}(s_k;T_0,x_k,v_k)) \Big| < \frac{1}{k} \quad \text{and} \quad \Big| \underline{X_{\mathbf{b}}}(s_k;T_0,x_k,v_k) - \underline{x}_i^{\mathfrak{I}} \Big| < \frac{1}{k} .$$

Define $y_k \equiv \underline{X_{\mathbf{b}}}(s_k;T_0,x_k,v_k) \in \partial\underline{\Omega}$ and $u_k \equiv \frac{V_{\mathbf{b}}(s_k;T_0,x_k,v_k)}{|V_{\mathbf{b}}(s_k;T_0,x_k,v_k)|} \in \mathbb{S}^1$ so that y_k converges to $\underline{x}_i^{\mathfrak{I}} \in \partial\underline{\Omega}$ and $u_k \cdot \underline{n}(y_k) < N/k$. Using the compactness of $[0,T_0],\ cl(\mathcal{C}(\underline{x}_j^{IB};r_j^{IB}))$ and $\{v = (\underline{v},v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2\ ,\ -N/2 \le v_3 \le N/2\} \setminus \mathcal{O}_i^{IB}$, we conclude that the sequence (s_k,x_k,v_k) has a subsequence converging to $(s_\infty,x_\infty,v_\infty) \in [0,T_0] \times cl(\mathcal{C}(\underline{x}_j^{IB};r_j^{IB})) \times \{v = (\underline{v},v_3) \in \mathbb{R}^3 : 1/N \le |\underline{v}| \le N/2\ ,\ -N/2 \le v_3 \le N/2\} \setminus \mathcal{O}_i^{IB}$. Moreover, we have $u_\infty \cdot \underline{n}(y_\infty) = 0$. Hence u_∞ should be either $+\ u_{\mathfrak{I}}(\underline{x}_j^{\mathfrak{I}})$ or $-\ u_{\mathfrak{I}}(\underline{x}_j^{\mathfrak{I}})$.

We show that $u_\infty = +\ u_{\mathfrak{I}}(\underline{x}_j^{\mathfrak{I}})$, which is a contradiction to (3.232) in Lemma 38. Suppose $u_\infty = -\ u_{\mathfrak{I}}(\underline{x}_j^{\mathfrak{I}})$. From the continuity of $\underline{X_{\mathbf{b}}}$ away from the infinite-bounces set, we have $y_\infty =$

$\underline{X_{\mathbf{cl}}}(s_\infty;T_0,x_\infty,v_\infty)$. From Lemma 39, there is $k\in\mathbb{N}$ so that $(y_\infty,u_\infty)=\Big(x_k(x_\infty,v_\infty),\frac{v_k(x_\infty,v_\infty)}{|v_k(x_\infty,v_\infty)|}\Big)$. If $y_\infty\neq x_\infty$, then the line segment connecting $y_\infty=x_k(x_\infty,v_\infty)$ and $x_{k-1}(x_\infty,v_\infty)$ has a direction $-u_{\mathfrak{I}}(\underline{x}_j^{\mathfrak{I}})$ and should be in $cl(\Omega)$, which is impossible. On the other hand, if $y_\infty=x_\infty$, then $(x_\infty,v_\infty)\in\mathcal{C}(\underline{x}_i^{IB};r_i^{IB})\times\mathcal{O}_i^{IB}$, which is also a contradiction. $\square$

Grazing Set

In this section, we characterize the points of $\mathfrak{IB}^C$ in (3.234) whose specular backward cycle grazes the boundary (hits the boundaries tangentially) at some moment. More precisely

Definition 23. *For $T_0>0$ and $(x,v)\in cl(\Omega)\times\mathbb{R}^3$, we define **the First Grazing Time** as*

$$t_{\mathbf{g}}(T_0,x,v)\ \equiv\ \max\Big\{\ s\in[0,T_0]\ :\ X_{\mathbf{cl}}(s;T_0,x,v)\in\partial\Omega\ \ \text{and}\ \ V_{\mathbf{cl}}(s;T_0,x,v)\cdot n(X_{\mathbf{cl}}(s;T_0,x,v))=0\ \Big\}\tag{3.239}$$

*If the above set is empty, let $t_{\mathbf{g}}(T_0,x,v)=-\infty$. For the analytic non-convex cylinderical domain, **the Grazing Set on $\mathfrak{IB}^C$** is defined as*

$$\mathfrak{G}\ \equiv\ \Big\{\ (x,v)\ \in\mathfrak{IB}^C\ :\ t_{\mathbf{g}}(T_0,x,v)\ \in\ [0,T_0]\ \Big\}\ .\tag{3.240}$$

According to the decomposition of the boundaries of the domain Ω, we have corresponding decomposition for the grazing set $\underline{\mathfrak{G}}$.

Lemma 41 (Decomposition of $\underline{\mathfrak{G}}$)**.** *Assume that the domain Ω is the analytic non-convex cylinderical domain. Then*

$$\underline{\mathfrak{G}}\ =\ \underline{\mathfrak{G}}_C\ =\ \Big\{\ (\underline{x},\underline{v})\in\underline{\mathfrak{G}}\ :\ \underline{X_{\mathbf{b}}}(t_{\mathbf{g}}(T_0,\underline{x},\underline{v});T_0,\underline{x},\underline{v})\ \in\ \partial\underline{\Omega}_C\ \Big\}\ ,$$

and further

$$\underline{\mathfrak{G}}_C\ =\ \bigcup_j\underline{\mathfrak{G}}_C^j\ =\ \bigcup_j\bigcup_{l=1}^{M_C}\underline{\mathfrak{G}}_C^{j,l}\ ,\tag{3.241}$$

where $j \in \mathbb{N}$ is a finite index and the above unions are disjoint unions and

$$\underline{\mathfrak{G}}_C^j \equiv \Big\{ (\underline{x}, \underline{v}) \in \underline{\mathfrak{G}}_C \ : \ t_{\mathbf{g}}(T_0, \underline{x}, \underline{v}) = t_j(T_0, \underline{x}, \underline{v}) \Big\} , \tag{3.242}$$

$$\underline{\mathfrak{G}}_C^{j,l} \equiv \Big\{ (\underline{x}, \underline{v}) \in \underline{\mathfrak{G}}_C^j \ : \ \underline{X}_{\mathbf{cl}}(t_{\mathbf{g}}(T_0, \underline{x}, \underline{v}); T_0, \underline{x}, \underline{v}) \ \in \ \partial \underline{\Omega}_C^l \Big\} . \tag{3.243}$$

We call $\underline{\mathfrak{G}}_C^j$ *as* $\mathbf{j}^{st}-$***Grazing Set****.*

Proof. From the decomposition of the boundaries in Definition 20, we have $\underline{\mathfrak{G}} = \underline{\mathfrak{G}}_I \cup \underline{\mathfrak{G}}_V \cup \underline{\mathfrak{G}}_C$, where

$$\underline{\mathfrak{G}}_I = \{ \ (\underline{x}, \underline{v}) \in \underline{\mathfrak{G}} \ : \ \underline{X}_{\mathbf{b}}(t_{\mathbf{g}}(T_0, \underline{x}, \underline{v}); T_0, \underline{x}, \underline{v}) \ \in \ \partial \underline{\Omega}_I \ \} ,$$

$$\underline{\mathfrak{G}}_V = \{ \ (\underline{x}, \underline{v}) \in \underline{\mathfrak{G}} \ : \ \underline{X}_{\mathbf{b}}(t_{\mathbf{g}}(T_0, \underline{x}, \underline{v}); T_0, \underline{x}, \underline{v}) \ \in \ \partial \underline{\Omega}_V \ \}.$$

We claim that the above two sets are empty. Recall that the grazing set $\mathfrak{G}$ is a subset of $\mathfrak{IB}^C$. By (3.232) in Lemma 38, we conclude that $\underline{\mathfrak{G}}_I$ is an empty set. Now we claim $\underline{\mathfrak{G}}_V = \emptyset$. Choose $(\underline{x}, \underline{v}) \in \underline{\mathfrak{G}}_V$. From Lemma 39, there is $k \in \mathbb{N}$ so that $t_{\mathbf{g}}(T_0, \underline{x}, \underline{v}) = t_k(T_0, \underline{x}, \underline{v})$. Since $\underline{X}_{\mathbf{b}}(t_{\mathbf{g}}(T_0, \underline{x}, \underline{v}); T_0, \underline{x}, \underline{v}), = x_k(\underline{x}, \underline{v}) \in \partial\Omega_C$ and $v_k(\underline{x}, \underline{v}) \cdot n(x_k(\underline{x}, \underline{v})) = 0$, the line segment connecting $x_{k-1}(\underline{x}, \underline{v})$ and $x_k(\underline{x}, \underline{v})$ is not in $cl(\underline{\Omega})$, which is a contradiction.

The indices $j \in \mathbb{N}$ is finite because of the uniform bound for the number of bounces on $\mathfrak{IB}^C$, in Lemma 39. Here the index $j \in \mathbb{N}$ cannot exceed the finite number $\sup_{(x,v)\in\mathfrak{IB}^C} \ \sharp_{\mathbf{b}}^{>L}(x, v) \ < \ +\infty$ for $L = N \times T_0$ in (3.237). ☐

Remark If $(\underline{x}, \underline{v}) \in \bigcup_j \underline{\mathfrak{G}}_C^{j,l}$ then there exists $\tau \in (a^l, b^l)$ such that $\underline{X}_{\mathbf{b}}(t_{\mathbf{g}}(T_0, \underline{x}, \underline{v}); T_0, \underline{x}, \underline{v}) = \alpha^l(\tau)$. Notice that all $\mathfrak{G}_C^{j,l} \ \subset \ \mathfrak{IB}^C$. Due to Lemma 40 such τ cannot be close arbitrarily to the end points a^l, b^l which are inflection points. Lemma 40 implies that there exists $a_*^l > a^l$ and $b_*^l < b^l$ such that

$$\Big\{ \ \tau \in (a^l, b^l) \ : \ \underline{X}_{\mathbf{b}}(t_{\mathbf{g}}(T_0, \underline{x}, \underline{v}); T_0, \underline{x}, \underline{v}) \ = \ \alpha^l(\tau) \ \text{ for } \ (\underline{x}, \underline{v}) \in \bigcup_j \underline{\mathfrak{G}}_C^{j,l} \ \Big\} \ \subset \ [\ a_*^l, \ b_*^l \ ] \ . \tag{3.244}$$

Now we will state the abstract and generalized version of Lemma 18 in [27], which is an important tool for following Lemma 43, 44, 45.

Lemma 42. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact Euclidian spaces. Denote an element of each set as* $\mathbf{x} \in \mathcal{X}$

and $\mathbf{y} \in \mathcal{Y}$. *Let*

$$\Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \ , \tag{3.245}$$

be the well-defined function and continuous away from it's zero set $\mathcal{S} \equiv \{(\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y} : \Phi(\mathbf{x}, \mathbf{y}) = 0\}$. *Denote* $m_{\mathcal{Y}}$ *be the Lebesque measure on* $\mathcal{Y}$ *and* $[\mathcal{S}]_{\mathbf{x}} \equiv \{\mathbf{y} \in \mathcal{Y} : (\mathbf{x}, \mathbf{y}) \in \mathcal{S}\}$. *Suppose*

$$m_{\mathcal{Y}}([\mathcal{S}]_{\mathbf{x}}) \ = \ 0 \ , \tag{3.246}$$

for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{F}$ *where* $\mathcal{F} = \{\xi_1, \xi_2, \cdots, \xi_N\} \subset \mathcal{X}$ *has finite elements in* $\mathcal{X}$. *Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *and* $l \in \mathbb{N}$ *balls* $B(\mathbf{x}_1; r_1), B(\mathbf{x}_2; r_2), \cdots, B(\mathbf{x}_l; r_l) \subset \mathcal{X}$ *with* $\mathbf{x}_i \in \mathcal{X}$ *for* $i = 1, 2, \cdots, l$, *as well as open sets* $\mathcal{O}_{\mathbf{x}_1}, \mathcal{O}_{\mathbf{x}_2}, \cdots, \mathcal{O}_{\mathbf{x}_l} \subset \mathcal{Y}$ *with* $m_{\mathcal{Y}}(\mathcal{O}_{\mathbf{x}_i}) < \varepsilon$ *for all* $i = 1, 2, \cdots, l$, *such that for any* $\mathbf{x} \in \mathcal{X}$, *there exists* $\mathbf{x}_i$ *or* $\xi_j \in \mathcal{F}$ *so that* $\mathbf{x} \in B(\mathbf{x}_i; r_i)$ *or* $\mathbf{x} \in B(\xi_j; \varepsilon)$. *Further if* $\mathbf{x} \in B(\mathbf{x}_i; r_i)$ *then, for* $\mathbf{y} \in \mathcal{Y} \setminus \mathcal{O}_{\mathbf{x}_i}$, $|\Phi(\mathbf{x}, \mathbf{y})| \ > \ \delta \ > \ 0$. *Denote such an* $\varepsilon-$***neighborhood of*** $\mathcal{S}$ *as*

$$(\mathcal{S})_\varepsilon \ \equiv \ \bigcup_{i=1}^{l} B(\mathbf{x}_i; r_i) \times \mathcal{O}_{\mathbf{x}_i} \ \ \cup \ \ \bigcup_{j=1}^{N} B(\xi_j; \varepsilon) \times \mathcal{Y} \ . \tag{3.247}$$

$1^{\text{st}}-$Grazing Set, $\underline{\mathfrak{G}}_C^1$

It is clear from the definition of $\underline{\mathfrak{G}}_C^{1,l}$ and (3.244),

$$\begin{aligned} \underline{\mathfrak{G}}_C^{1,l} \ \ \subset \ \ & \Big\{ \big(\alpha^l(\tau) + s\dot{\alpha}^l(\tau), |\underline{v}|\dot{\alpha}^l(\tau)\big) \ : \ \tau \in [a_*^l, b_*^l] \ , \ 1/N \le |\underline{v}| \le N/2 \ , \ s \in [0, t_{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau))] \Big\} \\ & \cup \Big\{ \big(\alpha^l(\tau) + s(-\dot{\alpha}^l(\tau)), -|\underline{v}|\dot{\alpha}^l(\tau)\big) \ : \ \tau \in [a_*^l, b_*^l] \ , \ 1/N \le |\underline{v}| \le N/2 \ , \ s \in [0, t_{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau))] \Big\} \ . \end{aligned}$$

Denote $[\underline{\mathfrak{G}}_C^{1,l}]_{\underline{x}} \equiv \{\underline{v} \in \mathbb{R}^2 : (\underline{x}, \underline{v}) \in \underline{\mathfrak{G}}_C^{1,l}\}$. Since the signed curvature κ is negative but finite points, $\mathbb{S}^1 \cap [\underline{\mathfrak{G}}_C^{1,l}]_{\underline{x}}$ is at most two points set. Since

$M_C < \infty$, $\mathbb{S}^1 \cap [\underline{\mathfrak{G}}_C^1]_{\underline{x}}$ is a finite points which has $2 \times M_C$ points so that

$$m_2([\underline{\mathfrak{G}}_C^1]_{\underline{x}}) \ = \ 0 \ . \tag{3.248}$$

On the other hand,

$$\phi^1(\underline{x}, \underline{v}) \ \equiv \ \underline{v} \ \cdot \ n(x_{\mathbf{b}}(\underline{x}, \underline{v})) \tag{3.249}$$

is well-defined and smooth if $\phi^1(\underline{x},\underline{v}) \neq 0$. Since $\phi^1(\underline{x},\underline{v}) \neq 0$ for $(\underline{x},\underline{v}) \in \mathfrak{IB}^C \setminus \underline{\mathfrak{G}}^1_C$, the function ϕ^1 is smooth on $\mathfrak{IB}^C \setminus \underline{\mathfrak{G}}^1_C$.

Now we apply Lemma 42 : $cl(\underline{\Omega})$, $\{\underline{v} \in \mathbb{R}^2 \ : \ 1/N \leq |\underline{v}| \leq N\}$ correspond to $\mathcal{X}$, $\mathcal{Y}$ and $\underline{\mathfrak{G}}^1_C$, ϕ^1 correspond to $\mathcal{S}$, Φ and $\mathcal{F} = \emptyset$ in this case :

Lemma 43. *For any $\varepsilon > 0$, we have the $\varepsilon-$neighborhood of $\underline{\mathfrak{G}}^1_C$:*

$$(\underline{\mathfrak{G}}^1_C)_\varepsilon \ = \ \bigcup_{i=1}^{l_1} B(\underline{x}^{1,i}; r^{1,i}) \times \ \mathcal{O}^1_{\underline{x}^{1,i}} \ , \tag{3.250}$$

with $m_2(\mathcal{O}^1_{\underline{x}^{1,i}}) < \varepsilon$ for all $i = 1,2,\cdots l_1$ and for any $(\underline{x},\underline{v}) \in \mathfrak{IB}^C$, there exists $\underline{x}^{1,i}$ so that $\underline{x} \in B(\underline{x}^{1,i}; r^{1,i})$ and for $\underline{v} \in \{\underline{v} \in \mathbb{R}^2 : 1/N \leq |\underline{v}| \leq N/2\} \setminus \mathcal{O}^1_{\underline{x}^{1,i}}$,

$$| \ \underline{v} \ \cdot \ n(x_{\mathbf{b}}(\underline{x},\underline{v})) \ | \ > \ \delta \ . \tag{3.251}$$

$\mathbf{2^{nd}}$**−Grazing Set** on $\{\mathfrak{IB}^C \setminus (\underline{\mathfrak{G}}^1_C)_\varepsilon\}$, $\underline{\mathfrak{G}}^2_C \setminus (\underline{\mathfrak{G}}^1_C)_\varepsilon$

It is clear from the definition of $\underline{\mathfrak{G}}^{2,l}_C$ and (3.244), the set $\underline{\mathfrak{G}}^{2,l}_C \setminus (\underline{\mathfrak{G}}^1_C)_\varepsilon$ is a subset of

$$\begin{aligned}
\Bigg(\quad & \Big\{ \big(x^{\mathbf{f}}_1(\alpha^l(\tau),\dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_1(\alpha^l(\tau),\dot{\alpha}^l(\tau)), |\underline{v}| v^{\mathbf{f}}_1(\alpha^l(\tau),\dot{\alpha}^l(\tau)) \big) \\
: \ & \underline{\underline{\tau \ \in \ [\ a^l_* \ , \ b^l_* \ ]}} \ , \ 1/N \leq |\underline{v}| \leq N/2 \ , \ s \in [t^{\mathbf{f}}_1(\alpha^l(\tau),\dot{\alpha}^l(\tau)), \ t^{\mathbf{f}}_2(\alpha^l(\tau),\dot{\alpha}^l(\tau))] \Big\} \\
\cup \ & \Big\{ \big(x^{\mathbf{f}}_1(\alpha^l(\tau),-\dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_1(\alpha^l(\tau),-\dot{\alpha}^l(\tau)), |\underline{v}| v^{\mathbf{f}}_1(\alpha^l(\tau),-\dot{\alpha}^l(\tau)) \big) \\
: \ & \underline{\underline{\tau \ \in \ [\ a^l_* \ , \ b^l_* \ ]}} \ , \ 1/N \leq |\underline{v}| \leq N/2 \ , \ s \in [t^{\mathbf{f}}_1(\alpha^l(\tau),-\dot{\alpha}^l(\tau)), \ t^{\mathbf{f}}_2(\alpha^l(\tau),-\dot{\alpha}^l(\tau))] \Big\} \Bigg) \Big\backslash \ (\underline{\mathfrak{G}}^1_C)_\varepsilon.
\end{aligned} \tag{3.252}$$

In order to apply Lemma 42, we check several delicate facts :

Step 1 : $\underline{\underline{\tau \ \in \ [\ a^l_* \ , \ b^l_* \ ]}}$ can be substituded by $\tau \in [\ a^l_* \ , \ b_1 \ ) \ \cup \ (\ a_2 \ , \ b_2 \ ) \ \cup \ \cdots \ \cup \ (\ a_{m-1} \ , \ b_{m-1} \ ) \ \cup \ (\ a_m \ , \ b^l_* \ ]$.

Notice that the above set (3.252) is a subset of $\mathfrak{IB}^C \setminus (\underline{\mathfrak{G}}^1_C)_\varepsilon$. Therefore if $\dot{\alpha}^l(\tau) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau),\dot{\alpha}^l(\tau))) = 0$ or $-\dot{\alpha}^l(\tau) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau),-\dot{\alpha}^l(\tau))) = 0$, then for such $\tau \in [a^l_*, b^l_*]$,

$$\big(x^{\mathbf{f}}_1(\alpha^l(\tau),\dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_1(\alpha^l(\tau),\dot{\alpha}^l(\tau)), |\underline{v}| v^{\mathbf{f}}_1(\alpha^l(\tau),\dot{\alpha}^l(\tau)) \big)$$

cannot be in $\underline{\mathfrak{G}}_C^2 \setminus (\underline{\mathfrak{G}}_C^1)_\varepsilon$ or $\big(x_1^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)) + s v_1^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)), |\underline{v}| v_1^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau))\big)$ cannot be in $\underline{\mathfrak{G}}_C^2 \setminus (\underline{\mathfrak{G}}_C^1)_\varepsilon$. Hence, it suffices to show that

$$\{\tau \in [a_*^l, b_*^l] : \dot{\alpha}^l(\tau) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau))) \neq 0\} \ \subset \ [a_*^l, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_{m-1}, b_{m-1}) \cup (a_m, b_*^l].$$

is a union of finite connected open subsets of $[a_*^l, b_*^l]$. For $\tau \in [a_*^l, b_*^l]$, if $\dot{\alpha}^l(\tau) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau))) \neq 0$ then there exists there exists $\delta_-, \delta_+ > 0$ so that $\dot{\alpha}^l(\tau') \cdot n(x^{\mathbf{f}}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) \neq 0$ for all $\tau' \in (\tau - \delta_-, \tau + \delta_+)$. If we choose the maximum of such δ_- and δ_+, then $\dot{\alpha}^l(\tau') \cdot n(x^{\mathbf{f}}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) = 0$ for $\tau' = \tau - \delta_-$ and $\tau' = \tau - \delta_+$: otherwise we can extend more so $\delta_, \delta_+$ cannot be maximums. Therefore we conclude that $\{\tau \in [a_*^l, b_*^l] : \dot{\alpha}^l(\tau) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau))) \neq 0\}$ is in a union of countable open connected intervals.

We only have to show that such union is finite. We prove this claim by the contradiction argument. Suppose the union is not finite. Then the complementary of such union is also a union of countable connected closed intervals. We choose one point τ_i from each connected closed interval. We can extract a convergent subsequence and $\tau_i \to \tau_\infty \in [a_*^l . b_*^l]$. If $\dot{\alpha}^l(\tau_\infty) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau_\infty), \dot{\alpha}^l(\tau_\infty))) \neq 0$ then $\dot{\alpha}^l(\tau') \cdot n(x^{\mathbf{f}}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) \neq 0$ for $\tau' \sim \tau_\infty$. Therefore $\dot{\alpha}^l(\tau_\infty) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau_\infty), \dot{\alpha}^l(\tau_\infty))) = 0$. Then by Lemma 48, τ_∞ is the sticky grazing or the isolated grazing. If it is sticky grazing then there is an open neighborhood containing τ_∞ and for all τ' in that interval, $\dot{\alpha}^l(\tau_\infty) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau_\infty), \dot{\alpha}^l(\tau_\infty))) = 0$. Then τ_∞ cannot be an accumulation point of τ_i because we choose τ_i for each connected closed interval. On the other hand, if it is the isolated grazing, then there is an open neighborhood of τ_∞ such that $\dot{\alpha}^l(\tau') \cdot n(x^{\mathbf{f}}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) \neq 0$ for all τ' in the neighborhood except τ_∞. Then it couldn't be an accumulation points of τ_i chosen above.

Step 2 : Indeed, $\underline{\mathfrak{G}}_C^2 \setminus (\ \underline{\mathfrak{G}}_C^1\ )_\varepsilon$ is a subset of

$$\begin{aligned}
&\Big\{ \big(x_1^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) + s v_1^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau)), |\underline{v}| v_1^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau))\big) : \\
&\qquad s \in [t_1^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau)),\ t_2^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau))] \ , \qquad\qquad (3.253)\\
&\qquad 1/N \leq |\underline{v}| \leq N/2 \ , \ \underline{\tau \ \in \ [a_*^l, b_{1,*}] \cup [a_{2,*}, b_{2,*}] \cup \cdots \cup [a_{m-1,*}, b_{m-1,*}] \cup [a_{m,*}, b_*^l]} \Big\} \\
&\cup \Big\{ \big(x_1^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)) + s v_1^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)), |\underline{v}| v_1^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau))\big) : \\
&\qquad s \in [t_1^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)),\ t_2^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau))] \ , \\
&\qquad 1/N \leq |\underline{v}| \leq N/2 \ , \ \underline{\tau \ \in \ [a_*^l, b_{1,*}] \cup [a_{2,*}, b_{2,*}] \cup \cdots \cup [a_{m-1,*}, b_{m-1,*}] \cup [a_{m,*}, b_*^l]} \Big\},
\end{aligned}$$

and for all $\tau \ \in \ [a^l_*, b_{1,*}] \cup [a_{2,*}, b_{2,*}] \cup \cdots \cup [a_{m-1,*}, b_{m-1,*}] \cup [a_{m,*}, b^l_*]$

$$\dot\alpha^l(\tau) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau), \dot\alpha(\tau))) \neq 0 \ , \quad -\dot\alpha^l(\tau) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau), -\dot\alpha(\tau))) \neq 0 \ .$$

This result is from (3.232).

Step 3 : Construct $\mathbf{2^{nd}}-$**Sticky Grazing Set** $\underline{\mathcal{SG}}^{2,l}_C$ corresponding to $\mathcal{F} \subset \mathcal{X}$ in Lemma 42. Consider only the first set of two sets in (3.253). Choose any interval $[a_{i,*}, b_{i,*}]$ and temporally define

$$\underline{G}^{2,l,i}_C \quad \equiv \quad \Big\{ \ \big(x^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau)) + s v^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau)), |\underline{v}| v^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau))\big) \\ : \ s \in [t^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau)), \ t^{\mathbf{f}}_2(\alpha^l(\tau), \dot\alpha^l(\tau))] \ , \ \frac{1}{N} \leq |\underline{v}| \leq \frac{N}{2} \ , \ \underline{\tau \ \in \ [a_{i,*}, b_{i,*}]} \ \Big\} \ .$$

Notice that for all $\tau \in [a_{i,*}, b_{i,*}]$, we have $\dot\alpha^l(\tau) \cdot n(x^{\mathbf{f}}(\alpha^l(\tau), \dot\alpha(\tau))) \neq 0$. Fix $\underline{x} \in cl(\underline{\Omega})$. If there does not exist $\tau \in [a_{i,*}, b_{i,*}]$ and $s \in [t^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau)), \ t^{\mathbf{f}}_2(\alpha^l(\tau), \dot\alpha^l(\tau))]$ satisfying $x^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau)) + s v^{\mathbf{f}}_1(\alpha^l(\tau)$ then $[\ \underline{G}^{2,l,i}_C \ ]_{\underline{x}} = \emptyset$ and $m_2(\ [\ \underline{G}^{2,l,i}_C \ ]_{\underline{x}} \ ) = 0$. Now suppose there exist such τ and s. Due to Lemma 48, there are only two cases: **sticky grazing** : for all $\tau \in [a_{i,*}, b_{i,*}]$ there exists $s \in [t^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau)), \ t^{\mathbf{f}}_2(\alpha^l(\tau), \dot\alpha^l(\tau))]$ so that

$$\underline{x} = x^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau)) + s v^{\mathbf{f}}_1(\alpha^l(\tau) \tag{3.254}$$

or **isolated grazing** : there exists $\delta_-, \delta_+ > 0$ so that for $\tau' \in (\tau - \delta_-, \tau + \delta_+) \backslash \{\tau\}$ there is no s satisfying (3.254).

$2^{\text{nd}}-$sticky grazing set $\underline{\mathcal{SG}}^{2,l}_C$ is collection of the sticky grazing case $\underline{x} \in cl(\underline{\Omega})$.

Definition 24. *Consider (3.253) and underlined union of intervals* $[a^l_*, b_{1,*}) \cup \cdots \cup [a_{i,*}, b_{i,*}] \cup \cdots \cup [a_{m,*}, b^l_*]$. *There are finite* $i \in I^{2,l}_{\mathbf{sg}} \subset \{1, 2, \cdots, m\}$ *so that*

$$\bigcap_{\tau \in [a_{i,*}, b_{i,*}]} \overline{x^{\mathbf{f}}_1(\alpha^l(\tau), \dot\alpha^l(\tau)) \ x^{\mathbf{f}}_2(\alpha^l(\tau), \dot\alpha^l(\tau))} \ = \ \underline{x}^{2,l,i}_{\mathbf{sg}} \ \textit{is one point.} \tag{3.255}$$

The $2^{nd}-$*sticky grazing set is the collection of such points :*

$$\underline{\mathcal{SG}}^2_C = \bigcup_{l=1}^{M_C} \underline{\mathcal{SG}}^{2,l}_C \ \equiv \ \bigcup_{l=1}^{M_C} \{ \ \underline{x}^{2,l,i}_{\mathbf{sg}} \ \in \ cl(\underline{\Omega}) \ : \ i \in I^{2,l}_{\mathbf{sg}} \ \} \ .$$

Step 4 : Show $m_2\big(\ [\underline{\mathfrak{G}}^{2,l}_C \setminus (\underline{\mathfrak{G}}^1_C)_\varepsilon]_{\underline{x}}\ \big) = 0$ for all $\underline{x} \in cl(\underline{\Omega}) \setminus \underline{\mathcal{SG}}^{2,l}_C$

Consider again the set (3.253). For any $i \in \{1, 2, \cdots, m\} \setminus I^{2,l}_{\mathbf{sg}}$, there is no $\tau \in [a_{i,*}, b_{i,*}]$ satisfying

$$\bigcap_{\tau' \in (\tau-\delta_-, \tau+\delta_+)} \overline{x^{\mathbf{f}}_1(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))\ x^{\mathbf{f}}_2(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))} \neq \emptyset\ , \tag{3.256}$$

for some $\delta_-, \delta_+ > 0$. Therefore for all $\underline{x} \in cl(\underline{\Omega}) \setminus \underline{\mathcal{SG}}^{2,l}_C$, $[\underline{\mathfrak{G}}^{2,l}_C \setminus (\underline{\mathfrak{G}}^1_C)_\varepsilon]_{\underline{x}} \cap \mathbb{S}^1$ is a finite set.

Step 5 : Apply Lemma 42

Further define

$$\phi^2(\underline{x}, \underline{v})\ \equiv\ v_1(\underline{x}, \underline{v})\ \cdot\ n(\ x_2(\underline{x}, \underline{v})\ )\ . \tag{3.257}$$

Now we can apply Lemma 42 : $\underline{\mathfrak{G}}^2_C = \underline{\mathfrak{G}}^2_C \setminus (\underline{\mathfrak{G}}^1_C)_\varepsilon \cup (\underline{\mathfrak{G}}^1_C)_\varepsilon$, $\underline{\mathcal{SG}}^2_C$ and ϕ^2 correspond to $\mathcal{S}$, $\mathcal{F}$ and Φ. In step 3, we show that $\underline{\mathcal{SG}}^2_C$ is finite subset of $cl(\underline{\Omega})$. In step 4, we show that $m_2\big(\ [\underline{\mathfrak{G}}^2_C \setminus (\underline{\mathfrak{G}}^1_C)_\varepsilon]_{\underline{x}}\ \big) = 0$ for all $\underline{x} \in cl(\underline{\Omega}) \setminus \underline{\mathcal{SG}}^2_C$ and combining with (3.248) to conclude $m_2\big(\ [\ \underline{\mathfrak{G}}^{2,l}_C\ ]_{\underline{x}}\ \big) = 0$. By the definition, $\phi^2(\underline{x}, \underline{v})$ is well-defined and continuous and $\phi^2(\underline{x}, \underline{v}) \neq 0$ if $(\underline{x}, \underline{v})\ \in \mathfrak{IB}^C \setminus \underline{\mathfrak{G}}^2_C$.

Lemma 44. *For any $\varepsilon > 0$, we have the $\varepsilon-$neighborhood of $\underline{\mathfrak{G}}^2_C$:*

$$(\ \underline{\mathfrak{G}}^2_C\ )_\varepsilon\ =\ \bigcup_{i=1}^{l_2} B(\underline{x}^{2,i}; r^{2,i}) \times \mathcal{O}^2_{\underline{x}^{2,i}}\ \ \cup\ \ \bigcup_{j=1}^{m_2} B(\underline{x}^{2,j}_{\mathbf{sg}}; \varepsilon) \times \{\underline{v} \in \mathbb{R}^2 : 1/N \leq |\underline{v}| \leq N/2\}\ . \tag{3.258}$$

with $m_2(\ \mathcal{O}^2_{\underline{x}^{2,i}}\ ) < \varepsilon$ for all $i = 1, 2, \cdots, l_2$, and for any $(\underline{x}, \underline{v}) \in \mathfrak{IB}^C$, there exists either $\underline{x}^{2,i}$, $\underline{x} \in B(\underline{x}^{2,i}; r^{2,i})$ or there exists $\underline{x}^{2,j}_{\mathbf{sg}}$, $\underline{x} \in B(\underline{x}^{2,j}_{\mathbf{sg}}; \varepsilon)$. If $\underline{x} \in B(\underline{x}^{2,i}; r^{2,i})$ then for $\underline{v} \in \{\underline{v} \in \mathbb{R}^2\ :\ 1/N \leq |\underline{v}| \leq N/2\} \setminus \mathcal{O}^2_{\underline{x}^{2,i}}$,

$$|\ v_1(\underline{x}, \underline{v})\ \cdot\ n(x_2(\underline{x}, \underline{v}))\ |\ > \delta_2\ ,\quad |\ \underline{v}\ \cdot\ n(x_1(\underline{x}, \underline{v}))\ |\ > \delta_1\ .$$

$\mathbf{k}^{\mathrm{st}}-$Grazing Set, $\underline{\mathfrak{G}}^k_C$

Now we are going to construct, for $k > 2$, the $\mathbf{k}^{\mathrm{st}}-$**Grazing Set** and it's $\varepsilon-$neighborhood. We construct such sets via the mathematical induction. Assume that there exist $\underline{\mathfrak{G}}^i_C$ and their $\varepsilon-$neighborhood, $(\underline{\mathfrak{G}}^h_C)_\varepsilon$ for all $h = 1, 2, \cdots, k-1$ and satisfy the following especially for $h = k-1$:

For any $\varepsilon > 0$, wa have the $\varepsilon-$neighborhood of $\underline{\mathfrak{G}}_C^{k-1}$:

$$(\underline{\mathfrak{G}}_C^{k-1})_\varepsilon \ = \ \bigcup_{i=1}^{l_{k-1}} B(\underline{x}^{k-1,i}; r^{k-1,i}) \times \mathcal{O}^{k-1}_{\underline{x}^{k-1,i}} \ \cup \ \bigcup_{j=1}^{m_{k-1}} B(\underline{x}_{\mathbf{sg}}^{k-1,j}; \varepsilon) \times \{ \underline{v} \in \mathbb{R}^2 : 1/N \leq |\underline{v}| \leq N/2 \} \ , \tag{3.259}$$

with $m_{k-1}(\ \mathcal{O}^{k-1}_{\underline{x}^{k-1,i}} \ ) < \varepsilon$ for all $i = 1, 2, \cdots, l_{k-1}$, and for any $(\underline{x}, \underline{v}) \in \mathfrak{IB}^C$, there exists either $\underline{x}^{k-1,i}$, $\underline{x} \in B(\underline{x}^{k-1,i}; r^{k-1,i})$ or there exists $\underline{x}_{\mathbf{sg}}^{k-1,j}$, $\underline{x} \in B(\underline{x}_{\mathbf{sg}}^{k-1,j}; \varepsilon)$. If $\underline{x} \in B(\underline{x}^{k-1,i}; r^{k-1,i})$ then for $\underline{v} \in \{ \underline{v} \in \mathbb{R}^2 : 1/N \leq |\underline{v}| \leq N/2 \} \setminus \mathcal{O}^{k-1}_{\underline{x}^{k-1,j}}$, for all $h = 1, 2, \cdots, k-1$,

$$| \ v_h(\underline{x}, \underline{v}) \ \cdot \ n(x_h(\underline{x}, \underline{v})) \ | \ > \ \delta_h \ . \tag{3.260}$$

Step 1 : Express the $\mathbf{j}^{\text{th}}-$Grazing Set on $\{ \mathfrak{IB}^C \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon \}$.

It is clear from the definition of $\underline{\mathfrak{G}}_C^j$ and (3.232), the set $\underline{\mathfrak{G}}_C^{j,l} \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon$ is a subset of

$$\begin{aligned} \Bigg(\Big\{ & \big(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)), |\underline{v}| v^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \big) \\ & : \ \underline{\underline{\tau \ \in \ [\ a^l_* \ , \ b^l_* \ ]}} \ , \ 1/N \leq |\underline{v}| \leq N/2 \ , \ s \in [t^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)), \ t^{\mathbf{f}}_j(\alpha^l(\tau), \dot{\alpha}^l(\tau))] \Big\} \\ \cup \Big\{ & \big(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_{j-1}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)), |\underline{v}| v^{\mathbf{f}}_{j-1}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)) \big) \\ & : \ \underline{\underline{\tau \ \in \ [\ a^l_* \ , \ b^l_* \ ]}} \ , \ 1/N \leq |\underline{v}| \leq N/2 \ , \ s \in [t^{\mathbf{f}}_{j-1}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)), \ t^{\mathbf{f}}_j(\alpha^l(\tau), -\dot{\alpha}^l(\tau))] \Big\} \ \Bigg) \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon . \end{aligned} \tag{3.261}$$

Step 2 : $\underline{\underline{\tau \ \in \ [\ a^l_* \ , \ b^l_* \ ]}}$ can be substituded by $\tau \ \in \ [\ a^l_* \ , \ b_{1,*} \ ] \ \cup \ [\ a_{2,*} \ , \ b_{2,*} \ ] \ \cup \ \cdots \ \cup \ [\ a_{m-1,*} \ , \ b_{m-1,*} \ ] \ \cup \ [\ a_{m,*} \ , \ b^l_* \ ]$.

Notice that the above set (3.261) is a subset of $\mathfrak{IB}^C \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon$. Therefore if $v^{\mathbf{f}}_{h-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \cdot n(x^{\mathbf{f}}_h(\alpha^l(\tau), \dot{\alpha}^l(\tau))) = 0$ or $v^{\mathbf{f}}_{h-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \cdot n(x^{\mathbf{f}}_h(\alpha^l(\tau), \dot{\alpha}^l(\tau))) = 0$ for some $h \in \{1, 2, \cdots, j-1\}$, then for such $\tau \in [a^l_*, b^l_*]$, $\big(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)), |\underline{v}| v^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \big)$ cannot be in $\underline{\mathfrak{G}}_C^{j,l} \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon$ or $\big(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)), |\underline{v}| v^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \big)$ cannot be in $\underline{\mathfrak{G}}_C^{j,l} \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon$. Hence, it suffices to show that

$$\begin{aligned} \bigcap_{h=1}^{j-1} & \big\{ \tau \in [a^l_*, b^l_*] \ : \ v^{\mathbf{f}}_{h-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \cdot n(x^{\mathbf{f}}_h(\alpha^l(\tau), \dot{\alpha}^l(\tau))) \neq 0 \big\} \\ & \subset [a^l_*, b_{1,*}] \cup [a_{2,*}, b_{2,*}] \cup \cdots \cup [a_{m-1,*}, b_{m-1,*}] \cup [a_{m,*}, b^l_*] \end{aligned} \tag{3.262}$$

which is a union of finite connected closed subsets of $[a^l_*, b^l_*]$. For fixed $j \in \mathbb{N}$, we use the mathematical induction. For $h = 1$, (3.262) is true from Step 1 and Step 2 in $\mathbf{2^{nd}}$**−Grazing Set**. Assume that (3.262) holds up to $h = j-2$, i.e. $\bigcap_{h=1}^{j-2}\{\cdots\} \subset [a^l_*, b_{1,*}] \cup [a_{2,*}, b_{2,*}] \cup \cdots \cup [a_{m-1,*}, b_{m-1,*}] \cup [a_{m,*}, b^l_*]$.

Now we consider any one interval $[a_{i,*}, b_{i,*}]$ of the above union. By the induction hypothesis, $v^{\mathbf{f}}_{h-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \cdot n(x^{\mathbf{f}}_h(\alpha^l(\tau), \dot{\alpha}^l(\tau))) \neq 0$ for all $\tau \in [a_{i,*}, b_{i,*}]$ for all $h = 1, 2, \cdots, j-2$. If $v^{\mathbf{f}}_{j-2}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau))) \neq 0$ for $\tau \in [a_{i,*}, b_{i,*}]$ then there exists there exists $\delta_-, \delta_+ > 0$ so that $v^{\mathbf{f}}_{j-2}(\alpha^l(\tau'), \dot{\alpha}^l(\tau')) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) \neq 0$ for all $\tau' \in (\tau - \delta_-, \tau + \delta_+)$. If we choose the maximum of such δ_- and δ_+, then $v^{\mathbf{f}}_{j-2}(\alpha^l(\tau'), \dot{\alpha}^l(\tau')) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) = 0$ for $\tau' = \tau - \delta_-$ and $\tau' = \tau - \delta_+$: otherwise we can extend more so δ_-, δ_+ cannot be maximums. Therefore we conclude that $\{\tau \in [a^l_*, b^l_*] \ : \ v^{\mathbf{f}}_{j-2}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau))) \neq 0 \}$ is in a union of countable open connected intervals.

Next we show that such union is finite. We prove this claim by the contradiction argument. Suppose the union is not finite. Then the complementary of such union is also a union of countable connected closed intervals. We choose one point τ_i from each connected closed interval. We can extract a convergent subsequence and $\tau_i \to \tau_\infty \in [a^l_*.b^l_*]$. If $v^{\mathbf{f}}_{j-2}(\alpha^l(\tau_\infty), \dot{\alpha}^l(\tau_\infty)) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau_\infty), \dot{\alpha}^l(\tau_\infty))) \neq 0$ then $v^{\mathbf{f}}_{j-2}(\alpha^l(\tau'), \dot{\alpha}^l(\tau')) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) \neq 0$ for $\tau' \sim \tau_\infty$. Therefore $v^{\mathbf{f}}_{j-2}(\alpha^l(\tau'), \dot{\alpha}^l(\tau')) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) = 0$. Then by Lemma 48, τ_∞ is the sticky grazing or the isolated grazing. If it is sticky grazing then there is an open neighborhood containing τ_∞ and for all τ' in that interval, $v^{\mathbf{f}}_{j-2}(\alpha^l(\tau_\infty), \dot{\alpha}^l(\tau_\infty)) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau_\infty), \dot{\alpha}^l(\tau_\infty))) \neq 0$. Then τ_∞ cannot be an accumulation point of τ_i because we choose τ_i for each connected closed interval. On the other hand, if it is the isolated grazing, then there is an open neighborhood of τ_∞ such that $v^{\mathbf{f}}_{j-2}(\alpha^l(\tau'), \dot{\alpha}^l(\tau')) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau'), \dot{\alpha}^l(\tau'))) \neq 0$ for all τ' in the neighborhood except τ_∞. Then it couldn't be an accumulation points of τ_i chosen above.

Finally, from (3.232), we conclude that $\{\tau \in [a^l_*, b^l_*] \ : \ v^{\mathbf{f}}_{j-2}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \cdot n(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau), \dot{\alpha}^l(\tau))) \neq 0 \}$ is in a union of countable closed connected intervals. It is a proof of (3.262).

Step 3 : Construct $\mathbf{j^{nd}}$**−Sticky Grazing Set** $\underline{\mathcal{SG}}^{j,l}_C$ corresponding to $\mathcal{F} \subset \mathcal{X}$ in Lemma 42. Consider only the first set of two sets in (3.261). Choose any interval $[a_{i,*}, b_{i,*}]$ and temporally define

$$\underline{G}_C^{j,l,i} \equiv \Big\{ \big(x^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)), |\underline{v}| v^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)) \big) \\ : \ s \in [t^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)),\ t^{\mathbf{f}}_{j}(\alpha^l(\tau),\dot{\alpha}^l(\tau))] \ , \ \frac{1}{N} \le |\underline{v}| \le \frac{N}{2} \ , \ \underline{\tau \in [a_{i,*}, b_{i,*}]} \ \Big\} .$$

Notice that for all $\tau \in [a_{i,*}, b_{i,*}]$ and for all $h = 1, 2, \cdots, j-2$, we have $v^{\mathbf{f}}_{h-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)) \cdot n(x^{\mathbf{f}}_{h}(\alpha^l(\tau),\dot{\alpha}^l(\tau))) \neq 0$. Fix $\underline{x} \in cl(\underline{\Omega})$. If there does not exist $\tau \in [a_{i,*}, b_{i,*}]$ and

$$s \in [t^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)),\ t^{\mathbf{f}}_{j}(\alpha^l(\tau),\dot{\alpha}^l(\tau))]$$

satisfying $\underline{x} = x^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau))$ then $[\ \underline{G}_C^{j,l,i}\ ]_{\underline{x}} = \emptyset$ and $m_2(\ [\ \underline{G}_C^{j,l,i}\ ]_{\underline{x}}\ ) = 0$. Now suppose there exist such τ and s and let $\underline{x} = x^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)) + s v^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau))$. Due to Lemma 48, there are only two cases: the point $\underline{x}$ is **the sticky grazing point** : for all $\tau \in [a_{i,*}, b_{i,*}]$

$$\bigcap_{\tau \in [a_{i,*}, b_{i,*}]} \overline{x^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau))\ x^{\mathbf{f}}_{j}(\alpha^l(\tau),\dot{\alpha}^l(\tau))} \ = \ \underline{x} \quad \text{is one point and,}$$
$$[\ \underline{G}_C^{j,l,i}\ \cap\ \mathbb{S}^1\ ]_{\underline{x}} \ = \ \{ v^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)) \in \mathbb{S}^1 \ : \ \tau \in [a_{i,*}, b_{i,*}]\ \}$$

or the point $\underline{x}$ is **the isolated grazing point** : there exists $\delta_-, \delta_+ > 0$ so that for $\tau' \in (\tau - \delta_-, \tau + \delta_+) \backslash \{\tau\}$

$$\underline{x} \ \notin \ \overline{x^{\mathbf{f}}_{j-1}(\alpha^l(\tau'),\dot{\alpha}^l(\tau'))\ x^{\mathbf{f}}_{j}(\alpha^l(\tau'),\dot{\alpha}^l(\tau'))} \quad \text{for all} \ \ \tau' \in (\tau - \delta_-, \tau + \delta_+) \backslash \{\tau\} \ ,$$
$$[\ \underline{G}_C^{j,l,i}\ \cap\ \{ v^{\mathbf{f}}_{j-1}(\alpha^l(\tau'),\dot{\alpha}^l(\tau')) \in \mathbb{S}^1 \ : \ \tau' \in (\tau - \delta_-, \tau + \delta_+) \}\ ]_{\underline{x}} \ = \ v^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau)) \ .$$

$j^{\text{th}}-$sticky grazing set $\underline{\mathcal{SG}}_C^{j,l}$ is collection of the sticky grazing case $\underline{x} \in cl(\underline{\Omega})$.

Definition 25. *Consider (3.261) and underlined union of intervals* $[a^l_*, b_{1,*}) \cup \cdots \cup [a_{i,*}, b_{i,*}] \cup \cdots \cup [a_{m,*}, b^l_*]$. *There are finite* $i \in I^{j,l}_{\mathbf{sg}} \subset \{1, 2, \cdots, m\}$ *so that*

$$\bigcap_{\tau \in [a_{i,*}, b_{i,*}]} \overline{x^{\mathbf{f}}_{j-1}(\alpha^l(\tau),\dot{\alpha}^l(\tau))\ x^{\mathbf{f}}_{j}(\alpha^l(\tau),\dot{\alpha}^l(\tau))} \ = \ \underline{x}^{j,l,i}_{\mathbf{sg}} \ \textit{is one point.} \tag{3.263}$$

The $j^{th}-$*sticky grazing set is the collection of such points :*

$$\underline{\mathcal{SG}}_C^{j} = \bigcup_{l=1}^{M_C} \underline{\mathcal{SG}}_C^{j,l} \ \equiv \ \bigcup_{l=1}^{M_C} \{\ \underline{x}^{j,l,i}_{\mathbf{sg}} \ \in \ cl(\underline{\Omega}) \ : \ i \in I^{j,l}_{\mathbf{sg}}\ \} \ .$$

Step 4 : Show $m_2\big(\ [\underline{\mathfrak{G}}_C^{j,l} \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon]_{\underline{x}}\ \big) = 0$ for all $\underline{x} \in cl(\underline{\Omega}) \setminus \underline{\mathcal{SG}}_C^{j,l}$

Consider again the set (3.261). For any $i \in \{1, 2, \cdots, m\} \setminus I_{\mathbf{sg}}^{j,l}$,

$$\Big[\quad \Big\{ \big(x_{j-1}^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) + s v_{j-1}^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau)), |\underline{v}| v_{j-1}^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau)) \big)$$
$$: \underline{\tau \ \in \ [\ a_{i,*}\ ,\ b_{i,*}\ ]}\ ,\ 1/N \le |\underline{v}| \le N/2\ ,\ s \in [t_{j-1}^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau)),\ t_j^{\mathbf{f}}(\alpha^l(\tau), \dot{\alpha}^l(\tau))] \Big\}$$
$$\cup \Big\{ \big(x_{j-1}^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)) + s v_{j-1}^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)), |\underline{v}| v_{j-1}^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)) \big)$$
$$: \underline{\tau \ \in \ [\ a_{i,*}\ ,\ b_{i,*}\ ]}\ ,\ 1/N \le |\underline{v}| \le N/2\ ,\ s \in [t_{j-1}^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau)),\ t_j^{\mathbf{f}}(\alpha^l(\tau), -\dot{\alpha}^l(\tau))] \Big\} \ \Big]_{\underline{x}}$$

is finite or empty set for all $\underline{x} \in cl(\underline{\Omega})$.

Step 5 : Apply Lemma 42

Further define

$$\phi^j(\underline{x}, \underline{v}) \ \equiv \ v_{j-1}(\underline{x}, \underline{v}) \ \cdot \ n(\ x_j(\underline{x}, \underline{v})\ )\ . \tag{3.264}$$

Now we can apply Lemma 42 : $\underline{\mathfrak{G}}_C^j = \underline{\mathfrak{G}}_C^j \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon \ \cup \ (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon$, $\underline{\mathcal{SG}}_C^j$ and ϕ^j correspond to $\mathcal{S}$, $\mathcal{F}$ and Φ. In step 3, we show that $\underline{\mathcal{SG}}_C^2$ is finite subset of $cl(\underline{\Omega})$. In step 4, we show that $m_2\big(\ [\underline{\mathfrak{G}}_C^j \setminus (\underline{\mathfrak{G}}_C^{j-1})_\varepsilon]_{\underline{x}}\ \big) = 0$ for all $\underline{x} \in cl(\underline{\Omega}) \setminus \underline{\mathcal{SG}}_C^j$ and combining with (3.248) to conclude $m_2\big(\ [\ \underline{\mathfrak{G}}_C^{2,l}\ ]_{\underline{x}}\ \big) = 0$. By the definition, $\phi^j(\underline{x}, \underline{v})$ is well-defined and continuous and $\phi^j(\underline{x}, \underline{v}) \neq 0$ if $(\underline{x}, \underline{v}) \in \mathfrak{IB}^C \setminus \underline{\mathfrak{G}}_C^j$.

Lemma 45. *For any $\varepsilon > 0$, we have the $\varepsilon-$neighborhood of $\underline{\mathfrak{G}}_C^j$:*

$$(\ \underline{\mathfrak{G}}_C^j\ )_\varepsilon \ = \ \bigcup_{i=1}^{l_j} B(\underline{x}^{j,i}; r^{j,i}) \times \mathcal{O}_{\underline{x}^{j,i}}^j \ \cup \ \bigcup_{h=1}^{m_j} B(\underline{x}_{\mathbf{sg}}^{j,h}; \varepsilon) \times \{\underline{v} \in \mathbb{R}^2 : 1/N \le |\underline{v}| \le N/2\}\ . \tag{3.265}$$

with $m_2(\ \mathcal{O}_{\underline{x}^{j,i}}^j\ ) < \varepsilon$ for all $i = 1, 2, \cdots, l_j$, and for any $(\underline{x}, \underline{v}) \in \mathfrak{IB}^C$, there exists either $\underline{x}^{j,i}$, $\underline{x} \in B(\underline{x}^{j,i}; r^{j,i})$ or there exists $\underline{x}_{\mathbf{sg}}^{j,h}$, $\underline{x} \in B(\underline{x}_{\mathbf{sg}}^{j,h}; \varepsilon)$. If $\underline{x} \in B(\underline{x}^{j,h}; r^{j,h})$ then for $\underline{v} \in \{\underline{v} \in \mathbb{R}^2\ :\ 1/N \le |\underline{v}| \le N/2\} \setminus \mathcal{O}_{\underline{x}^{j,h}}^j$,

$$|\ v_{h-1}(\underline{x}, \underline{v}) \ \cdot \ n(x_h(\underline{x}, \underline{v}))\ | \ > \delta_h\ ,$$

for all $h = 1, 2, \cdots, j$.

Chapter Four

Stability with a Large Potential

Abstract The stability of the Maxwellian of the Boltzmann equation with a large amplitude external potential Φ has been an important open problem. In this paper, we resolve this problem with a large C^3-potential in a periodic box $\mathbb{T}^d$, $d \geq 3$. We use [5] in $L^p - L^\infty$ framework to establish the well-posedness and the $L^\infty-$stability of the Maxwellian $\mu_E(x,v) = \exp\left\{-\frac{|v|^2}{2} - \Phi(x)\right\}$.

4.1 Comments

There are some investigations about the dynamical problems of the Boltzmann equation with an external potential. The local well-posedness was established in [18] and [6]. Near Maxwellian regime, the global well-posedness was established in [46], [60], [20], [21], [68] and [22] with some smallness assumptions for the external potential Φ using the nonlinear energy method. In [55], using the semi-group approach, the global well-posedness was established with some smallness assumptions for the external potential Φ in a periodic box. This result was later generalized([57][56]) in the case of an unbounded external potential in $\mathbb{R}^3$ with a spherically symmetric assumption. Near vacuum regime, the global well-posedness was established in [24] with a small (self-consistent) external potential and in [19] with a large external potential Φ with some special conditions. In the case of 1-dimensional Boltzmann equation ($x \in \mathbb{R},\ v \in \mathbb{R}^3$) near Maxwellian regime, the well-posedness and stability are established in [23] with a large amplitude external potential.

In the presence of a large amplitude external potential, the key difficulty is the collapse of Sobolev estimate in higher order energy norms. The derivatives of the Boltzmann solution can grow in time unless the potential is small. In order to overcome this difficulty, we use the weighted L^∞ formulation without any derivatives([27][28]).

With the conservation of momentum for degenerate $\{v_1, \cdots, v_n\}$, we use the $L^2 - L^\infty$ framework([27]) which consists of two parts : First, establish L^2-decay for the linear Boltzmann equation(Section 3) ; Second, establish the L^∞-decay for the nonlinear Boltzmann solution using the Vidav's idea and the L^2-decay of the first part(Section 4).

For proving the linear L^2-decay (Section 3), the main difficulty is the absence of the momentum conservation laws for all velocity components $\{v_1, \cdots, v_d\}$. The key ingredient to prove the linear L^2-decay is the positivity of the linear Boltzmann operator L (Proposition 3). Following [25], we establish such a positivity of L by the contradiction argument. The consequential limiting function is non-zero and only has the hydrodynamic part which is the null space of L spanned by the basis $\{\ \sqrt{\mu},\ v\sqrt{\mu},\ |v|^2\sqrt{\mu}\ \}$. The coefficients of the limiting function satisfy the macroscopic equation (4.38) $-$ (4.42). Crucially using the conservation of momentum (1.49) for degenerate $\{v_1,\cdots,v_n\}$ and the periodicity in $\mathbb{T}^d$, we are able to show that all the coefficients are zero, which is a contradiction.

For nonlinear $L^\infty-$decay (Section 4), we use Vidav's idea. We denote $X(s;t,x,v)$ the backward trajectory at time s, starting at time $t\in[0,\infty)$, position $x\in\mathbb{T}^d$ with velocity $v\in\mathbb{R}^d$. Similarly $X(s_1;s,X(s;t,x,v),v')$ is the backward trajectory at time s_1 starting at time $s\in[0,t]$, position $X(s;t,x,v)\in\mathbb{T}^d$ with velocity $v'\in\mathbb{R}^d$. The goal is to establish the following estimate for the solution of the Boltzmann equation :

$$\int_0^t ds\int_{\frac{1}{N}\le|v'|\le N}dv'\int_0^s ds_1\int dv''\ |f(s_1,X(s_1;s,X(s;t,x,v),v'),v'')| \tag{4.1}$$

$$\lesssim\left(\varepsilon+\frac{1}{N}\right)\sup_{0\le s_1\le t}||f(s_1)||_\infty+\int_0^t||f(s_1)||_{L^p}\ ds_1\ , \tag{4.2}$$

where $p=1$ or $p=2$. Once we have the estimate (4.2) for $p=2$, using the established L^2-decay, we are able to show the $L^\infty-$decay. The basic idea to show the desired estimate (4.2) is to establish

$$\det\left\{\frac{dX(s_1;s,X(s;t,x,v),v')}{dv'}\right\}\neq 0\ , \tag{4.3}$$

for almost every $(s_1,s,v')\in(0,t)\times(0,t)\times\mathbb{R}^d$ for all $X(s;t,x,v)\in\mathbb{T}^d$. Then we apply the change of variables

$$v'\ \rightarrow\ X(s_1;s,X(s;t,x,v),v')\ ,$$

for main part of $(0,t)\times(0,t)\times\mathbb{R}^d$ to bound (4.1) by the L^2-term in (4.2) and to bound (4.1) by $L^\infty-$term in (4.2) for the small remainder part. In this paper, we use Asano's result in [5] to verify the crucial condition (4.3) for smooth external potentials. In [5], using the symplectic geometric approach, the points that fail to satisfy the condition (4.3) are characterized by the eigenvalues of some symmetric matrix. Because of this formulation, using the standard min-max principle, Lemma 46 was established in [5]. The condition (4.3) has been proved in many other cases. In [27], the condition (4.3) has been shown in the case of bounded domains $\Omega\subset\mathbb{R}^d$ with several boundary conditions without an external field($\nabla\Phi\equiv 0$). Notice that the characteristics are determined according to the boundary conditions. In the case of in-flow, bounce-back, diffuse reflection boundary conditions, the condition (4.3) was proved for bounded domains $\Omega \subset \mathbb{R}^3$ with the smooth boundary $\partial\Omega$. For the specular reflection boundary condition case, the condition (4.3) was established for analytic and strictly convex domains $\Omega \subset \mathbb{R}^3$. In [42], for the specular reflection case, the condition (4.3) was established for analytic and non-convex, 2-dimension domains $\Omega \subset \mathbb{R}^2$. Without boundary condition : $\Omega = \mathbb{R}^d$, in [29], the condition (4.3) was shown for self-consistent electric fields if the perturbation f is small. In 1-dimensional case $\Omega = \mathbb{R}$, the condition is proved for external potentials with a large amplitude ([23]).

Without the conservation of momentum for degenerate $\{v_1, \cdots, v_n\}$, we are not able to prove the linear L^2-decay. Instead, we use the natural excess entropy inequality (1.45) to obtain L^1-stability of the Boltzmann equation([28]). The basic idea is that

$$\iint \{F \ln F - \mu_E \ln \mu_E\} \sim \iint (1 + \ln \mu_E)(F - \mu_E) + \iint \frac{1}{2\mu_E}(F - \mu_E)^2 \ .$$

Notice that the first term is controlled via the excess entropy inequality (1.45) and the second term is controlled via the conservation of mass (1.43) and energy (1.44). Therefore we can control the third term and the L^1-norm of $|F - \mu_E|$. Indeed, the L^p-term in (4.2) with $p = 1$ is bounded by the mass and energy and entropy. Therefore, we obtain the $L^\infty-$boundedness(stability) of the Boltzmann solution F.

Our paper is organized as follows. In section 2, we state the Asano's result(Lemma 1) and construct an open covering of the points that fail to satisfy (4.3). In section 3, we establish the linear L^2-decay (Theorem 3). In section 4, we use Vidav's idea to bootstrap the nonlinear L^∞-decay (Theorem 1) from linear L^2-decay. In section 5, we use the entropy-energy estimate([28]) to prove the L^∞-stability (Theorem 2).

4.2 Characteristics and Transversality

In this section we study the characteristcs for the Boltzmann equation with an external field (1.39). The hamiltonian of the system is given by

$$H(x,v) = \frac{|v|^2}{2} + \Phi(x).$$

We consider the Hamilton flow determined by the hamiltonian H ,that is, the characteristic curve satisfying the differential equation:

$$\frac{dX(\tau;t,x,v)}{d\tau} = V(\tau;t,x,v) \quad , \quad \frac{dV(\tau;t,x,v)}{d\tau} = -\nabla_x \Phi(X(\tau;t,x,v)), \tag{4.4}$$

with $[X(t;t,x,v), V(t;t,x,v)] = [x,v]$. Clearly the hamiltonian is constant along the characteristics, i.e.
$H(X(\tau;t,x,v), V(\tau;t,x,v)) = H(x,v)$ for all τ. Therefore we have an equality $\frac{1}{2}|V(s)|^2 + \Phi(X(s)) = \frac{1}{2}|v|^2 + \Phi(x)$ and further we have

$$|V(s)| = \sqrt{|v|^2 + 2\Phi(x) - 2\Phi(X(s))} \leq \sqrt{|v|^2 + 2|\Phi|_\infty} \leq |v| + \sqrt{2|\Phi|_\infty} \ .$$

On the other hand we have

$$|V(s)| = \sqrt{|v|^2 + 2\Phi(x) - 2\Phi(X(s))} \geq \sqrt{|v|^2 - 2|\Phi|_\infty} \geq |v| - \sqrt{2|\Phi|_\infty} \ .$$

Hence we know that

$$\Big| \, |V(\tau;t,x,v)| - |v| \, \Big| \ \leq \ 2|\Phi|_\infty^{1/2} \ , \tag{4.5}$$

for all (τ, t, x, v).

We will use the following geometric result of [5] crucially in this paper.

Lemma 46. *([5]) Assume that* $\Phi \in C^3(\mathbb{R}^d)$. *Suppose* $\det\left(\frac{dX(s_0;T_0,x_0,v_0)}{dv}\right) = 0$ *for some* $(s_0;T_0,x_0,v_0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$. *Then there exist* $\delta > 0$ *and an open neighborhood* $U_0 \subset \mathbb{R}^d\times\mathbb{R}^d$ *of* (x_0,v_0), *and a family of Lipschitz continuous functions on* U_0, $\{\psi_j : U_0 \to \mathbb{R}\}_{j=1}^d$ *and* $\psi_j(0,0) = 0$ *so that*

$$\det\left(\frac{dX(s;T_0,x,v)}{dv}\right) = 0, \tag{4.6}$$

if and only if

$$s = s_0 + \psi_j(x, v), \tag{4.7}$$

in $(s, x, v) \in (s_0 - \delta_0, s_0 + \delta_0) \times U_0$.

Using Lemma 46 we construct the ε-neighborhood of the set of points (s, x, v) satisfying (4.6).

Lemma 47. *Assume that* $\Phi \in C^3(\mathbb{T}^d)$ *is a periodic function. Fix* $T_0 > 0$ *and* $N > 0$*. There are disjoint open interval partitions of the time interval* $[0, T_0]$ *:* $\mathfrak{D}^1_{I^1} \subset [0, T_0]$ *for* $i_1 \in \{1, 2, \cdots, M^1\}$ *and disjoint open box partitions of the periodic box* $\mathbb{T}^d$ *:* $\mathfrak{D}^2_{i^2} \subset \mathbb{T}^d$ *for multi-index* $I^2 = (i^2_1, i^2_2, \cdots, i^2_d) \in \{1, 2, \cdots, M^2\}^d$ *; and disjoint open box partitions of* $\{v \in \mathbb{R}^d : v_i \in [-4N, 4N]$ *for all* $i = 1, 2 \cdots, d\}$ *:* $\mathfrak{D}^3_{I^3}$ *for* $I^3 = (i^3_1, i^3_2, \cdots, i^3_d) \in \{1, 2, \cdots, M^3\}^d$*. For each* i^1, I^2 *and* I^3 *we have* $t_{j,i^1,I^2,I^3} \in \mathfrak{D}^1_{i^1}$ *for* $j = 1, 2, ..., d$ *so that*

$$\left\{ s \in \mathfrak{D}^1_{i_1} : \det\left(\frac{dX(s; T_0, x, v)}{dv}\right) = 0 \right\} \subset \bigcup_{j=1}^{d} \left\{ s \in \left(t_{j,i^1,I^2,I^3} - \frac{\varepsilon}{4M^1}\ ,\ t_{j,i^1,I^2,I^3} + \frac{\varepsilon}{4M^1}\right) \right\},$$

for all $(x, v) \in \mathfrak{D}^2_{I^2} \times \mathfrak{D}^3_{I^3}$ *and*

$$\det\left(\frac{dX(s; T_0, x, v)}{dv}\right) > \delta_* \quad \text{for} \quad s \notin \bigcup_{j=1}^{d} \left(t_{j,i^1,I^2,I^3} - \frac{\varepsilon}{4M^1}\ ,\ t_{j,i^1,I^2,I^3} + \frac{\varepsilon}{4M^1}\right), \tag{4.8}$$

if $(s, x, v) \in \mathfrak{D}^1_{i^1} \times \mathfrak{D}^2_{I^2} \times \mathfrak{D}^3_{I^3}$ *for all* i^1, I^2 *and* I^3.

Proof. Choose $(t_0, x_0, v_0) \in [0, T_0] \times \mathbb{T}^d \times \{v \in \mathbb{R}^d : v_i \in [-4N, 4N], \quad \text{for } i = 1, 2, ..., d\}$.

<u>First Case</u> : If

$$\det\left(\frac{dX(t_0; T_0, x_0, v_0)}{dv}\right) \neq 0,$$

then there exist positive numbers $\{\tau^0; \xi^0_1, \cdots, \xi^0_d; \eta^0_1, \cdots \eta^0_d\}$ such that

$$\begin{aligned} \det\left(\frac{dX(t; T_0, x, v)}{dv}\right) \neq 0, \quad \text{for all} \quad & (t; x_1, \cdots, x_d; v_1, \cdots, v_d) \in (t_0 - \tau^0, t_0 + \tau^0) \times \\ & \times \big\{((x_0)_1 - \xi^0_1, (x_0)_1 + \xi^0_1) \times \cdots \times ((x_0)_d - \xi^0_d, (x_0)_d + \xi^0_d)\big\} \\ & \times \big\{((v_0)_1 - \eta^0_1, (v_0)_1 + \eta^0_1) \times \cdots ((v_0)_d - \eta^0_d, (v_0)_d + \eta^0_d)\big\}. \end{aligned}$$

<u>Second Case</u> : If

$$\det\left(\frac{dX(t_0;T_0,x_0,v_0)}{dv}\right)=0,$$

then by Lemma 46, there exist positive numbers $\{\tau^0;\xi_1^0,\cdots,\xi_d^0;\eta_1^0,\cdots\eta_d^0\}$ and Lipschitz functions ψ_j^0 defined on

$$\big\{((x_0)_1-\xi_1^0,(x_0)_1+\xi_1^0)\times\cdots\times((x_0)_d-\xi_d^0,(x_0)_d+\xi_d^0)\big\}\times\big\{((v_0)_1-\eta_1^0,(v_0)_1+\eta_1^0)$$
$$\times\cdots((v_0)_d-\eta_d^0,(v_0)_d+\eta_d^0)\big\}$$

and $\psi_j^0(0,0)=0$ so that, for $(t;x_1,\cdots,x_d;v_1,\cdots,v_d)\in(t_0-\tau^0,t_0+\tau^0)\times\big\{((x_0)_1-\xi_1^0,(x_0)_1+\xi_1^0)\times\cdots\times((x_0)_d-\xi_d^0,(x_0)_d+\xi_d^0)\big\}\times\big\{((v_0)_1-\eta_1^0,(v_0)_1+\eta_1^0)\times\cdots\times((v_0)_d-\eta_d^0,(v_0)_d+\eta_d^0)\big\}$,

$$\det\left(\frac{dX(t;T_0,x,v)}{dv}\right)=0\ , \tag{4.9}$$

if and only if

$$t=t_0+\psi_j^0(x_1,\cdots,x_d;v_1,\cdots v_d)\ \text{ for some }\ j=1,2,\cdots,d. \tag{4.10}$$

Notice that from the first and second case, we obtain an open covering of $[0,T_0]\times\mathbb{T}^d\times\{v\in\mathbb{R}^d: v_i\in[-4N,4N]\}$. Since $[0,T_0]\times\mathbb{T}^d\times\{v\in\mathbb{R}^d: v_i\in[-4N,4N]\}$ is a compact set, we can choose finite points $(t_i;(x_i)_1,\cdots,(x_i)_d;(v_i)_1,\cdots(v_i)_d)$ and positive numbers $\{\tau^i;\xi_1^i,\cdots,\xi_d^i;\eta_1^i,\cdots,\eta_d^i\}$ for finite index i 's. Notice that

$$\begin{aligned}(t_i-\tau^i,t_i+\tau^i)\quad&\times\quad\Big\{\big(\,(x_i)_1-\xi_1^i\ ,\ (x_i)_1+\xi_1^i\,\big)\times\cdots\times\big(\,(x_i)_d-\xi_d^i\ ,\ (x_i)_d+\xi_d^i\,\big)\Big\}\\&\times\quad\Big\{\big(\,(v_i)_1-\eta_1^i\ ,\ (v_i)_1+\eta_1^i\,\big)\times\cdots\times\big(\,(v_i)_d-\eta_d^i\ ,\ (v_i)_d+\eta_d^i\,\big)\Big\},\end{aligned}$$

forms an open covering for finite i 's. Define a refined grid via relabeling as

$$\{\ 0=a_1<a_2<\cdots<a_{M_1}=T_0\ \}\ =\ \{\ t_i\pm\tau^i\ \text{ for all }\ i\ 's\ \}, \tag{4.11}$$
$$\{\ -1=\tilde b_1<\tilde b_2<\cdots<\tilde b_{\tilde M_2}=1\ \}\ =\ \{\ (x_i)_k\pm\xi_k^i\ \text{ for all }\ i\ 's,\ \ k=1,2,\cdots,d\ \}, \tag{4.12}$$
$$\{\ -4N=\tilde c_1<\tilde c_2<\cdots<\tilde c_{\tilde M_3}=4N\ \}\ =\ \{\ (v_i)_k\pm\eta_k^i\ \text{ for all }\ i\ 's,\ \ k=1,2,\cdots,d\ \}. \tag{4.13}$$

We use the index $i^1=1,2,\cdots,M^1$ and $\tilde i_k^2=1,2,\cdots,\tilde M^2$ and $\tilde i_k^3=1,2,\cdots,\tilde M^3$ for all $k=1,2,\cdots,d$, and multi-index $\tilde I^2=(\tilde i_1^2,\tilde i_2^2,\cdots,\tilde i_d^2)\in\{1,2,\cdots,\tilde M^2\}^d$ and $\tilde I^3=(\tilde i_1^3,\tilde i_2^3,\cdots,\tilde i_d^3)\in$

$\{1,2,\cdots,\tilde{M}^3\}^d$. Notice that for each $(j,i^1,\tilde{I}^2,\tilde{I}^3)$ we have $t_{j,i^1,\tilde{I}^2,\tilde{I}^3}\in(a_{i^1},a_{i^1+1})$ and a Lipschitz function $\psi_{j,i^1,\tilde{I}^2,\tilde{I}^3}$. Using the Lipschitz continuity, we choose a constant $C>0$ so that

$$| \ \psi_{j,i^1,\tilde{I}^2,\tilde{I}^3}(x,v)-\psi_{j,i^1,\tilde{I}^2,\tilde{I}^3}(\bar{x},\bar{v}) \ | \ \leq \ C \ | \ (x,v)-(\bar{x},\bar{v}) \ | \ ,$$

for all $j,i^1,\tilde{I}^2,\tilde{I}^3$ which are finite indices or finite multi-indices. From the above inequality we have

$$| \ \psi_{j,i^1,\tilde{I}^2,\tilde{I}^3}(x,v)-\psi_{j,i^1,\tilde{I}^2,\tilde{I}^3}(\bar{x},\bar{v}) \ | \ \leq \ \frac{\varepsilon}{8M^1} \ , \tag{4.14}$$

for $|(x,v)-(\bar{x},\bar{v})|\leq\frac{\varepsilon}{8M^1C}$. Therefore we further refine the grid of (4.12) and (4.13) as (if necessary we may put more points to make the grid finer)

$$\begin{aligned}\{-1=b_1<b_2<\cdots<b_{M^2}=1\} \quad &\supset \quad \{-1=\tilde{b}_1<\tilde{b}_2<\cdots<\tilde{b}_{\tilde{M}^2}=1\},\\ \{-4N=c_1<c_2<\cdots<c_{M^3}=4N\} \quad &\supset \quad \{-4N=\tilde{c}_1<\tilde{c}_2<\cdots<\tilde{c}_{\tilde{M}^3}=4N\},\end{aligned}$$

and denote multi-indices $I^2=(i^2_1,i^2_2,\cdots,i^2_d)\in\{1,2,\cdots,M^2\}^d$ and $I^3=(i^3_1,i^3_2,\cdots,i^3_d)\in\{1,2,\cdots,M^3\}^d$ and define

$$\mathfrak{D}^1_{i^1}\equiv(a_{i^1},a_{i^1+1}), \tag{4.15}$$

$$\mathfrak{D}^2_{I^2}=\mathfrak{D}^2_{i^2_1,\cdots,i^2_d}\equiv(b_{i^2_1},b_{i^2_1+1})\times\cdots\times(b_{i^2_d},b_{i^2_d+1}), \tag{4.16}$$

$$\mathfrak{D}^3_{I^3}=\mathfrak{D}^3_{i^3_1,\cdots,i^3_d}\equiv(c_{i^3_1},b_{i^3_1+1})\times\cdots\times(c_{i^3_d},c_{i^3_d+1}), \tag{4.17}$$

so that $|b_{i^2_1+1}-b_{i^2_1}|+\cdots+|b_{i^2_d+1}-b_{i^2_d}|+|c_{i^3_1+1}-c_{i^3_1}|+\cdots+|c_{i^3_d+1}-c_{i^3_d}|\leq\frac{\varepsilon}{8M^1C}$ and (4.14) is valid for all $(x,v),(\bar{x},\bar{v})\in\mathfrak{D}^2_{I^2}\times\mathfrak{D}^3_{I^3}$. From (4.10), for each j,i^1,I^2,I^3 there exist $t_{j,i^1,I^2,I^3}\in\mathfrak{D}^1_{i^1}$ so that

$$t_{j,i^1,I^2,I^3}+\psi_{j,i^1,I^2,I^3}(x,v) \ \in \ \bigcup_{j=1}^{d}(\ t_{j,i^1,I^2,I^3}-\frac{\varepsilon}{2M^1} \ ,t_{j,i^1,I^2,I^3}+\frac{\varepsilon}{2M^1} \ ) \ , \quad \text{for all} \quad (x,v)\in\mathfrak{D}^2_{I^2}\times\mathfrak{D}^3_{I^3}.$$

Therefore

$$\det\left(\frac{dX(t;T_0,x,v)}{dv}\right)\neq 0 \ ,$$

holds for $t\in\mathfrak{D}^1_{i^1}\backslash(\ t_{j,i^1,I^2,I^3}-\frac{\varepsilon}{2M^1} \ ,t_{j,i^1,I^2,I^3}+\frac{\varepsilon}{2M^1} \ )$ and for $(x,v)\in\mathfrak{D}^2_{I^2}\times\mathfrak{D}^3_{I^3}$. Notice that

$\det\left(\frac{dX(t;T_0,x,v)}{dv}\right)$ is continuous and non-zero on a compact set

$$\left\{ \overline{\mathfrak{D}^1_{i^1}} \Big\backslash \bigcup_{j=1}^{d} \left(t_{j,i^1,I^2,I^3} - \frac{\varepsilon}{4M^1} , t_{j,i^1,I^2,I^3} + \frac{\varepsilon}{4M^1} \right) \right\} \times \overline{\mathfrak{D}^2_{I^2}} \times \overline{\mathfrak{D}^3_{I^3}} . \tag{4.18}$$

Hence there exists positive number $\delta_{i^1,I^2,I^3} > 0$ so that

$$\det\left(\frac{dX(t;T_0,x,v)}{dv}\right) > \delta_{i^1,I^2,I^3} ,$$

on the set (4.18). Define $\delta_* = \min_{i^1,I^2,I^3} \delta_{i^1,I^2,I^3} > 0$ where i^1, I^2, I^3 are finite indices. This proves the Lemma. $\square$

4.3 Linear L^2 Decay

In this section, we will show the L^2-decay of the solution of the linear Boltzmann equation :

$$\partial_t f + v\cdot\nabla_x f - \nabla\Phi\cdot\nabla_v f + e^{-\Phi(x)} Lf = 0 \quad , \quad f(0,x,v) = f_0(x,v), \tag{4.19}$$

assuming the conservation of mass (1.43) and energy (1.44) and momentum for degenerate $\{v_1,\cdots,v_n\}$ (1.49) with $(M_0,E_0,\mathbf{J}_0) = (0,0,\mathbf{0}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$. Here, the number n in the degenerate $\{v_1,\cdots,v_n\}$ is a dimension of the degenerate subspace of $\nabla\Phi$ in (1.48) where, in general, the degenerate subspace of $\nabla\Phi$ is a zero space $\{0\}$ and $n=0$. Therefore, in this section, we are showing actually the linear L^2-decay only with the conservation of mass and energy for the generic external potential Φ in the periodic box $\mathbb{T}^d$.

For notational simplicity, we use $\langle\cdot,\cdot\rangle$ to denote the standard L^2-inner product in $\mathbb{T}^d\times\mathbb{R}^d$. We also define

$$\langle g_1, g_2\rangle_\nu \equiv \langle \nu(v) g_1, g_2\rangle.$$

We shall use $||\cdot||$ and $||\cdot||_\nu$ to denote their corresponding L^2 norms.

Theorem 14. *Assume that the external potential Φ is a periodic C^3-function on $\mathbb{T}^d$ and $\Phi = \Phi(x_{n+1},\cdots,x_d)$. Let $f(t,x,v)\in L^2$ be the (unique) solution to the linear Boltzmann equation (4.19). Assume that f satisfies the conservations of mass (1.43) and energy (1.44) and momentum* *for degenerate* $\{v_1, \cdots, v_n\}$ *(1.49) with* $(M_0, E_0, \mathbf{J}_0) = (0, 0, \mathbf{0}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$*. Then there exists* $\lambda > 0$ *and* $C > 0$ *such that*

$$\sup_{0 \le t \le \infty} e^{\lambda t} ||f(t)|| \ \le \ C||f(0)|| \ .$$

Proof. We will show that Proposition 15 implies Theorem 14, following the proof of Theorem 5 in [27]. Assume Proposition 15 is true. Let $0 \le N \le t \le N+1$, N being an integer. We split $[0, t] = [0, N] \cup [N, t]$. First we establish the L^2 energy estimate for any solution f to the linear Boltzmann equation (4.19) on the time interval $[N, t]$ as

$$||f(t)||^2 + 2\int_N^t \int_{\mathbb{T}^d} e^{-\Phi(x)} \int_{\mathbb{R}^d} Lf \cdot f \ dv \ dx \ ds = ||f(N)||^2. \tag{4.20}$$

From (4.19), we have the equation for $e^{\lambda t} f(t)$:

$$\{\partial_t + v \cdot \nabla_x - \nabla_x \Phi \cdot \nabla_v + e^{-\Phi(x)} L\}\{e^{\lambda t} f\} - \lambda e^{\lambda t} f = 0.$$

For the time interval $[0, N]$, we multiply the above equation by $e^{\lambda t} f$ to derive the L^2-energy estimate on the time interval $[0, N]$:

$$e^{2\lambda N}||f(N)||^2 + 2\int_0^N e^{2\lambda s} \int_{\mathbb{T}^d} e^{-\Phi(x)} \int_{\mathbb{R}^d} Lf \cdot f dv dx ds - \lambda \int_0^N e^{2\lambda s}||f(s)||^2 ds = ||f(0)||^2.$$

Divide the time interval $[0, N]$ into $\bigcup_{k=0}^{N-1}[k, k+1)$ and define $f_k(s, x, v) \equiv f(k+s, x, v)$ for $k = 0, 1, 2, ..., N-1$. Then we can rewrite the above equation as

$$e^{2\lambda N}||f(N)||^2 + \underbrace{\sum_{k=0}^{N-1} \int_0^1 e^{2\lambda\{k+s\}} \int_{\mathbb{T}^d} e^{-\Phi(x)} \int_{\mathbb{R}^d} Lf_k(s) \cdot f_k(s) \ dv dx ds}_{(\mathbf{A})} \tag{4.21}$$

$$= ||f(0)||^2 + \lambda \underbrace{\sum_{k=0}^{N-1} \int_0^1 e^{2\lambda\{k+s\}}||f_k(s)||^2 ds}_{(\mathbf{B})} . \tag{4.22}$$

Notice that $f_k(k+s, x, v)$ satisfies the linear Boltzmann equation (4.19) in the time interval $[0, 1]$.

By Proposition 15, we have a lower bound for $(\mathbf{A})$ as

$$(\mathbf{A}) \geq \sum_{k=0}^{N-1} e^{2\lambda k} e^{-|\Phi|_\infty} \int_0^1 \langle Lf_k(s), f_k(s)\rangle ds \geq \sum_{k=0}^{N-1} e^{2\lambda k} e^{-|\Phi|_\infty} M \int_0^1 ||f_k(s)||_\nu^2 ds$$
$$\geq \nu_0 M e^{-|\Phi|_\infty} \sum_{k=0}^{N-1} e^{2\lambda k} \int_0^1 ||f_k(s)||^2 ds.$$

Using the fact $e^{2\lambda(k+s)} \leq e^{2\lambda} e^{2\lambda k}$ for $s \in [0,1]$, we have $(\mathbf{B}) \leq \lambda e^{2\lambda} \sum_{k=0}^{N-1} \int_0^1 e^{2\lambda k} ||f_k(s)||^2 ds$. Thus, for sufficiently small $\lambda > 0$, we have a positive lower bound of $(\mathbf{A}) - (\mathbf{B})$ as

$$(\mathbf{A}) - (\mathbf{B}) \geq (\nu_0 M e^{-|\Phi|_\infty} - \lambda e^{2\lambda}) \sum_{k=0}^{N-1} \int_0^1 e^{2\lambda k} ||f_k(s)||^2 ds \geq 0.$$

Therefore from (4.22), we have

$$e^{2\lambda N} ||f(N)||^2 \ \leq \ ||f(0)||^2.$$

Further we can choose $\lambda > 0$ small so that $e^{2\lambda(t-N)} \leq 2$ for all $t \in [N, N+1]$. Hence, multiply (4.20) by $e^{2\lambda t}$ and combine with the above inequality to conclude

$$e^{2\lambda t} ||f(t)||^2 \ \leq \ e^{2\lambda t} ||f(N)||^2 \ \leq \ e^{2\lambda\{t-N\}} ||f_0||^2 \ \leq \ 2||f_0||^2.$$

□

Proposition 15. *Assume that the external potential Φ is a periodic C^3-function on $\mathbb{T}^d$ and $\Phi = \Phi(x_{n+1}, ..., x_d)$. Let $f(t,x,v)$ be any solution to the linear Boltzmann equation (4.19) satisfying the conservations of mass (1.43) and energy (1.44) and momentum for degenerate $\{v_1, \cdots, v_n\}$ (1.49) with $(M_0, E_0, \mathbf{J}_0) = (0, 0, \mathbf{0}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$. Then there exists $M > 0$ such that f satisfies*

$$\int_0^1 \langle Lf(s), f(s)\rangle ds \ \geq \ M \int_0^1 ||f(s)||_\nu^2 ds. \tag{4.23}$$

Proof. We prove the Proposition by the contradiction argument. If the inequality of (4.23) is not true, then a sequence of solutions $f_k(t,x,v)$ (not identically zero) to (4.19) exists so that

$$\int_0^1 \langle Lf_k(s), f_k(s)\rangle ds \leq \frac{1}{k} \int_0^1 ||f_k(s)||_\nu^2 ds.$$

Equivalently, in terms of normalization $Z_k(t,x,v) = \frac{f_k(t,x,v)}{\sqrt{\int_0^1 ||f_k(s)||_\nu^2 ds}}$, we have

$$\int_0^1 \langle LZ_k(s), Z_k(s)\rangle ds \leq \frac{1}{k}, \tag{4.24}$$

and from $\nu(v) \geq \nu_0$, we have $\nu_0 \int_0^1 ||Z_k(s)||^2 ds \leq \int_0^1 ||Z_k(s)||_\nu^2 ds = 1$. Due to the weak compactness in L^2 space, there exists $Z(t,x,v)$ with $\int_0^1 ||Z(s)||_\nu ds \leq 1$ such that

$$Z_k \rightharpoonup Z \quad \text{weakly in} \quad \int_0^1 ||\cdot||_\nu^2 ds \quad \text{and} \quad \int_0^1 ||\cdot||^2 ds. \tag{4.25}$$

On the other hand, since $f_k(t,x,v)$ solves (4.19), $Z_k(t,x,v)$ satisfies the same equation

$$\{\partial_t + v\cdot\nabla_x - \nabla\Phi(x)\cdot\nabla_v\}Z_k(t,x,v) + e^{-\Phi(x)}LZ_k(t,x,v) = 0, \tag{4.26}$$

and hence $Z_k(t,x,v)$ satisfies same conservation laws (1.43), (1.44), and (1.49) with $(M_0, E_0, \mathbf{J}_0) = (0,0,\mathbf{0}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ as $f_k(t,x,v)$ does. Using the weak convergence in (4.25), we conclude that for almost every $t \in [0,1]$, the limiting function $Z(t,x,v)$ also satisfies

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} Z(t,x,v)\sqrt{\mu_E(x,v)}dvdx \;=\; 0\;, \tag{4.27}$$

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} \left(\Phi(x) + \frac{|v|^2}{2}\right) Z(t,x,v)\sqrt{\mu_E(x,v)}dvdx \;=\; 0\;, \tag{4.28}$$

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} (v_1,\cdots,v_n)^T\; Z(t,x,v)\sqrt{\mu_E(x,v)}dvdx \;=\; 0\;. \tag{4.29}$$

Step 1 : $\mathbf{P}Z_k \rightharpoonup \mathbf{P}Z$ weakly in $\int_0^1 ||\cdot||_\nu^2 ds$ and $\int_0^1 ||\cdot||^2 ds$.

It suffices to show that

$$\int_0^1 \langle \mathbf{P}Z_k, \phi_1(s,x)\phi_2(v)\rangle_\nu ds \to \int_0^1 \langle \mathbf{P}Z, \phi_1(s,x)\phi_2(v)\rangle_\nu ds, \tag{4.30}$$

for any smooth functions $\phi_1(s,x)$ on $[0,1]\times\mathbb{T}^d$ and $\phi_2(v)$ on $\mathbb{R}^d$ with compact supports.

Temporally denote $(m_0, m_1, \cdots, m_d, m_{d+1}) \equiv (1, v_1, \cdots, v_d, |v|^2)$. Then the left-hand side of

(4.30) is written componentwisely as

$$\int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} m_i(u)\sqrt{\mu(u)}\; Z_k(s,x,u)du \right) m_i(v)\sqrt{\mu(v)}\; \nu(v)\phi_1(s,x)\phi_2(v)dvdxds$$
$$= \int_{\mathbb{R}^d} \underline{\left(\int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} m_i\sqrt{\mu}\; Z_k\phi_1 dudxds \right) m_i(v)\sqrt{\mu(v)}\nu(v)\phi_2(v)}dv. \tag{4.31}$$

Notice that the underlined integrand of (4.31) is bounded by L^1 function uniformly in k 's , i.e.

$$\left(\int_0^1 ||m_i\phi_1\sqrt{\mu}||^2_{L^2_{x,v}} ds \right)^{\frac{1}{2}} \left(\int_0^1 ||Z_k(s)||^2_{L^2_{x,v}} ds \right)^{\frac{1}{2}} m_i(v)\nu(v)\phi_2(v)\sqrt{\mu(v)} \leq Cm_i(v)\nu(v)\phi_2(v)\sqrt{\mu(v)} \in L^1(\mathbb{R}^d),$$

and the underlined integrand of (4.31) converges to $\left(\int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} m_i\sqrt{\mu}\; Z\phi_1 dudxds \right) m_i(v)\sqrt{\mu(v)}\nu(v)\phi_2(v)$ for almost every $v \in \mathbb{R}^d$. By the Lebesque convergence theorem, we conclude the convergence of (4.30).

Step 2 : $K(Z_k) \to K(Z) \quad \text{in} \quad L^2([0,1] \times \mathbb{T}^d \times \mathbb{R}^d)$.

The proof of this step is same as Step 1 in the proof of Lemma 4.1 in [25] page 1124. Denote $k(u,v)$ as the kernel of the operator K and define it's approximation $k_m(u,v) \equiv k(u,v)\mathbf{1}_{\{(u,v):|u-v|\geq \frac{1}{m}, |v|\leq m\}}$. Since $k_m \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ we can choose smooth functions with compact supports to satisfy

$$\kappa_\varepsilon(u,v) = \kappa_1(u)\kappa_2(v) \quad \text{such that} \quad ||k_m - \kappa_\varepsilon||^2 \leq \varepsilon.$$

In order to prove this step, we only need to change (4.8) in [25] by

$$[\partial_t + v\cdot\nabla_x - \nabla_x\Phi(x)\cdot\nabla_v]\{\kappa_1(v)\chi(t,x)Z_k(t,x,v)\}$$
$$= [\partial_t + v\cdot\nabla_x - \nabla_x\Phi(x)\cdot\nabla_v]\{\kappa_1(v)\chi(t,x)\}Z_k(t,x,v), \tag{4.32}$$

where $\chi(t,x)$ is a smooth cut-off function in $(0,1)\times\mathbb{R}^d$ such that $\chi(t,x) \equiv 1$ in $[\varepsilon, 1-\varepsilon]\times\mathbb{T}^d$. It is strainghforward to verify that the right hand side of (4.32) is uniformly bounded in $L^2([0,1] \times \mathbb{R}^d \times \mathbb{R}^d)$. From the velocity average lemma([8],[37]), for some $\alpha > 0$,

$$\int_{\mathbb{R}^d} \chi(t,x)\kappa_1(u)Z_k(t,x,u)du \;\in\; H^\alpha([0,1]\times\mathbb{R}^d).$$

It follows that, up to a subsequence,

$$\int_{\mathbb{R}^d} \kappa_1(u) Z_k(s,x,u)du \to \int_{\mathbb{R}^d} \kappa_1(u) Z(s,x,v)du,$$

strongly in $L^2([\varepsilon, 1-\varepsilon] \times \mathbb{T}^d)$ and this suffices to prove **Step 2**. For detail, see the proof of Lemma 4.1 in [25].

Step 3 : $Z(t,x,v) = \{a(t,x) + v \cdot \mathbf{b}(t,x) + |v|^2 c(t,x)\}\sqrt{\mu_E(x,v)} \;\neq\; 0.$

From the definitions of $L = \nu - K$ and Z_k, we have

$$1 - \int_0^1 \langle KZ_k(s), Z_k(s)\rangle ds = \int_0^1 ||Z_k(s)||_\nu ds - \int_0^1 \langle KZ_k(s), Z_k(s)\rangle ds = \int_0^1 \langle LZ_k(s), Z_k(s)\rangle ds \tag{4.33}$$

Combining the strong convergence in **Step 2** with the weak convergence of Z_k, we conclude

$$\lim_{k\to\infty} \int_0^1 \langle KZ_k(s), Z_k(s)\rangle = \int_0^1 \langle KZ(s), Z(s)\rangle.$$

Combining the above relation with (4.33) and $0 \le \int_0^1 \langle LZ_k(s), Z_k(s)\rangle ds \le \frac{1}{k}$, we conclude

$$0 = 1 - \int_0^1 \langle KZ(s), Z(s)\rangle ds.$$

Using the above equation, with the non-negativity of L and $\int_0^1 ||Z(s)||_\nu ds \le 1$, we have

$$0 \le \int_0^1 \langle LZ(s), Z(s)\rangle ds = \int_0^1 ||Z(s)||_\nu ds - \int_0^1 \langle KZ(s), Z(s)\rangle ds \le 1 - \int_0^1 \langle KZ(s), Z(s)\rangle ds = 0,$$

to conclude

$$\int_0^1 \langle LZ(s), Z(s)\rangle ds = 0 \ , \qquad \int_0^1 ||Z(s)||_\nu ds = 1. \tag{4.34}$$

Now use the standard property of L :

$$\int_{\mathbb{R}^d} LZ(s,x,v) \cdot Z(s,x,v) dv \ge C \int_{\mathbb{R}^d} \nu(v) |\{\mathbf{I} - \mathbf{P}\} Z(s,x,v)|^2 dv, \tag{4.35}$$

and combine with (4.34) to conclude

$$\{\mathbf{I} - \mathbf{P}\} Z(t,x,v) = 0,$$

for almost every $(t,x,v) \in [0,1]\times\mathbb{T}^d\times\mathbb{R}^d$. Hence $Z = \mathbf{P}Z = \{\tilde{a}(t,x)+v\cdot\tilde{\mathbf{b}}(t,x)+|v|^2\tilde{c}(t,x)\}\sqrt{\mu(v)}$ where $[\tilde{a}(t,x),\tilde{\mathbf{b}}(t,x),\tilde{c}(t,x)]$ is linear combinations of $\left[\int Z(t,x,\cdot)\sqrt{\mu}dv, \int vZ(t,x,\cdot)\sqrt{\mu}dv, \int |v|^2 Z(t,x,\cdot)\sqrt{\mu}dv\right]$. Define $[a(t,x),\mathbf{b}(t,x),c(t,x)] = [\tilde{a}(t,x),\tilde{\mathbf{b}}(t,x),\tilde{c}(t,x)]\times e^{\frac{\Phi(x)}{2}}$ then we have

$$Z(t,x,v) = \{a(t,x)+v\cdot\mathbf{b}(t,x)+|v|^2c(t,x)\}\sqrt{\mu_E}\quad . \tag{4.36}$$

From (4.34) we conclude that $Z(t,x,v)$ is not identically zero.

Step 4 : $Z\equiv 0$

This leads to a contradiction to **Step 3**. Notice that from (4.24) and (4.35), we have

$$\int_0^1 ||\{\mathbf{I}-\mathbf{P}\}Z_k(s)||_\nu^2 ds \;\le\; \frac{1}{C}\int_0^1 \langle LZ_k(s), Z_k(s)\rangle ds \;\to\; 0,$$

to conclude $\{\mathbf{I}-\mathbf{P}\}Z_k \to 0$ strongly in $\int_0^1 ||\cdot||_\nu ds$. From $LZ_k = L\{\mathbf{I}-\mathbf{P}\}Z_k$, we have $\int_0^1\langle L\{\mathbf{I}-\mathbf{P}\}Z_k,\varphi\rangle ds = \int_0^1\langle LZ_k,\varphi\rangle ds \to 0$ for all $\varphi\in C_c^\infty([0,1]\times\mathbb{T}^d\times\mathbb{R}^d)$. Hence letting $k\to\infty$ in (4.26), we have, in the sense of distribution,

$$\partial_t Z + v\cdot\nabla_x Z - \nabla\Phi(x)\cdot\nabla_v Z = 0.$$

We plug (4.36) into the above equation and expand as the products of a polynomial in v_i :

$$\begin{aligned}\{\dot{a}-\nabla_x\Phi\cdot b\}\sqrt{\mu_E} + \{\dot{b}+\nabla_x a - 2c\nabla_x\Phi\}\cdot v\sqrt{\mu_E} + \sum_i(\dot{c}+\partial_{x_i}b_i)|v_i|^2\sqrt{\mu_E}&\\ +\sum_{i\neq j}\{\partial_{x_i}b_j\}v_iv_j\sqrt{\mu_E} + \nabla_x c\cdot v|v|^2\sqrt{\mu_E} = 0.&\end{aligned}\tag{4.37}$$

Since $\sqrt{\mu}, v_i\sqrt{\mu}, v_jv_k\sqrt{\mu}$ and $v_lv_mv_n\sqrt{\mu}$ are linearly independent, we deduce that in the sense of distributions, all the coefficients on the left hand side of (4.37) should be zero. We therefore obtain the **macroscopic equations** of $a(t,x),\mathbf{b}(t,x)$ and $c(t,x)$, which was introduced in [25] :

$$\dot{a}(t,x) - \nabla_x\Phi(x)\cdot\mathbf{b}(t,x) = 0 \tag{4.38}$$

$$\dot{\mathbf{b}}(t,x) + \nabla_x a(t,x) - 2c(t,x)\nabla_x\Phi(x) = 0 \tag{4.39}$$

$$\dot{c}(t,x) + \partial_{x_i}b_i = 0 \quad \text{for all } i \tag{4.40}$$

$$\partial_{x_j}b_i + \partial_{x_i}b_j = 0 \;,\quad i\neq j \tag{4.41}$$

$$\nabla_x c(t,x) = 0 \tag{4.42}$$

We obtain the Laplace equation of $b_i(t,x)$

$$\begin{aligned}
\Delta b_i &= \sum_{j=1}^{d} \partial_j\partial_j b_i = \sum_{j\neq i} \partial_j \underline{\partial_j b_i}_{*} + \partial_i \underline{\partial_i b_i}_{**} = \sum_{j\neq i} \partial_j \underline{(-\partial_i b_j)}_{\diamond} + \partial_i \underline{(-\dot{c})}_{\diamond\diamond} \\
&= -\partial_i \left(\sum_{j\neq i} \underline{\partial_j b_j}_{**} \right) - \partial_i \dot{c} = -(d-1)\partial_i \underline{(-\dot{c})}_{\diamond\diamond} - \partial_i \dot{c} = (d-2)\underline{\partial_i \dot{c} \ = \ 0}_{***} \qquad (4.43)
\end{aligned}$$

where we used (4.41) for $* = \diamond$ and used (4.40) for $** = \diamond\diamond$ and used (4.42) for $***$. From (4.36) we have, for all $i = 1, 2, .., d$,

$$\int_{\mathbb{R}^d} v_i Z(t,x,v)dv = \sum_j \int_{\mathbb{R}^d} v_i v_j b_j(t,x) e^{-\frac{|v|^2}{4}} e^{-\frac{\Phi(x)}{2}} dv = b_i(t,x) e^{-\frac{\Phi(x)}{2}} \int_{\mathbb{R}^d} |v_i|^2 e^{-\frac{|v|^2}{4}} dv.$$

Since $Z(t,x,v)$ is periodic in x, we conclude that $\mathbf{b}(t,x)$ is also periodic in x. Using the periodicity of $\mathbf{b}$, multiply (4.43) by b_i and integrate to yield

$$0 = \int_{\mathbb{T}^d} \Delta b_i b_i dx = -\int_{\mathbb{T}^d} |\nabla b_i|^2 dx,$$

and

$$\nabla_x b_i(t,x) = 0 \ , \quad \mathbf{b}(t,x) = \mathbf{b}(t) \quad \text{for almost all} \ \ (t,x).$$

Combining the above equality with (4.40) and (4.42), we conclude that

$$c(t,x) = c_0 \quad \text{for almost all} \ \ (t,x). \tag{4.44}$$

Further integrate (4.39) on $\mathbb{T}^d$ to have

$$0 = \int_{\mathbb{T}^d} (4.39) \ dx = \dot{\mathbf{b}}(t) + \int_{\mathbb{T}^d} \nabla_x a(t,x)dx - 2c_0 \int_{\mathbb{T}^d} \nabla_x \Phi(x)dx = \dot{\mathbf{b}}(t),$$

where we used the periocity in $x \in \mathbb{T}^d$ of $a(t,x)$ and $\Phi(x)$. Therefore we conclude $\mathbf{b}(t,x) = \mathbf{b}_0$. From (4.38) and (4.39), we have equations of $a(t,x)$:

$$\nabla_x a(t,x) = 2c_0 \nabla_x \Phi(x) \ , \quad \dot{a}(t,x) = \mathbf{b}_0 \cdot \nabla_x \Phi(x).$$

From the first equation above, we have $a(t,x) = 2c_0\Phi(x) + \varphi(t)$ for some φ. Plugging this formula into the second equation above, we get

$$\dot{\varphi}(t) = \dot{a}(t,x) = \mathbf{b}_0 \cdot \nabla_x \Phi(x).$$

Since the left hand side of the above equation is a function of t only and the right hand side is a function of x only, we conclude that both of them are constant. In order to show that the both sides are zero actually, we utilize the periodicity of the external potential Φ : take the integration over $x \in \mathbb{T}^d$ to yield

$$\int_{\mathbb{T}^d} \mathbf{b}_0 \cdot \nabla_x \Phi(x) dx = \int_{\mathbb{T}^d} \nabla_x \cdot \{\Phi(x)\mathbf{b}_0\} dx = 0.$$

Therefore we conclude that, for all $t \in \mathbb{R}$ and $x \in \mathbb{T}^d$, we have $\varphi(t) \ \equiv \ \varphi_0$ and

$$\begin{aligned} a(t,x) \ &\equiv \ 2c_0\Phi(x) + \varphi_0 \ , \\ \mathbf{b}_0 \cdot \nabla_x \Phi(x) \ &\equiv \ 0 \ \text{ for all } \ x \in \mathbb{T}^d \ . \end{aligned} \tag{4.45}$$

Recall $\Lambda(\mathbb{T}^d)$, the degenerate subspace of $\nabla\Phi$ in (1.47). By the definition, we have

$$\mathbf{b}_0 \in \Lambda(\mathbb{T}^d)^{\perp}, \quad \text{and hence} \quad \mathbf{b}_0 = ((\mathbf{b}_0)_1, \cdots, (\mathbf{b}_0)_n, 0, \cdots, 0). \tag{4.46}$$

For simplicity we think $\mathbf{b}_0$ as a vector in $\mathbb{R}^n$. To sum, we plug (4.44), (4.45) and (4.46) into (4.36) to conclude

$$Z(t,x,v) = \big\{\varphi_0 + \mathbf{b}_0 \cdot (v_1, \cdots, v_n) + 2c_0\big(\Phi(x) + \frac{|v|^2}{2}\big)\big\}\sqrt{\mu_E(x,v)}. \tag{4.47}$$

Notice that we obtain the above formula for Z only using the macroscopic equations and periodicity in $x \in \mathbb{T}^d$.

In order to conclude $Z \equiv 0$, we use the conservations of mass (4.27) and energy (4.28) and momentums for degenerate $\{v_1, \cdots, v_n\}$ (4.29) crucially. From the conservation of momentum for degenerate $\{v_1, \dots, v_n\}$, we have

$$0 = \int_0^1 \iint_{\mathbb{T}^d \times \mathbb{R}^d} v_i Z(s,x,v)\sqrt{\mu_E(x,v)} dv dx ds = (\mathbf{b}_0)_i \int_0^1 \iint_{\mathbb{T}^d \times \mathbb{R}^d} (v_i)^2 \mu_E(x,v) dv dx ds, \tag{4.48}$$

for all $i = 1, \cdots, n$, so that $\mathbf{b}_0 \equiv \mathbf{0} \in \mathbb{R}^n$.

From the conservation of mass for Z in (4.27), we have

$$0 = \varphi_0 \iint_{\mathbb{T}^d \times \mathbb{R}^d} \mu_E(x,v) dv dx + 2c_0 \iint_{\mathbb{T}^d \times \mathbb{R}^d} \left(\Phi(x) + \frac{|v|^2}{2} \right) \mu_E(x,v) dv dx,$$

and from the conservation of energy for Z in (4.28), we have

$$0 = \varphi_0 \iint_{\mathbb{T}^d \times \mathbb{R}^d} \left(\Phi(x) + \frac{|v|^2}{2} \right) \mu_E(x,v) dv dx + 2c_0 \iint_{\mathbb{T}^d \times \mathbb{R}^d} \left(\Phi(x) + \frac{|v|^2}{2} \right)^2 \mu_E(x,v) dv dx.$$

Using the notation $\langle \cdot, \cdot \rangle$ for $L^2(\mathbb{T}^d \times \mathbb{R}^3)$ inner product, the above two equations are

$$\begin{pmatrix} \big\langle \sqrt{\mu_E}, \sqrt{\mu_E} \,\big\rangle & \big\langle (\Phi(x) + |v|^2/2)\sqrt{\mu_E},\ \sqrt{\mu_E} \,\big\rangle \\ \big\langle (\Phi(x) + |v|^2/2)\sqrt{\mu_E},\ \sqrt{\mu_E} \,\big\rangle & \big\langle (\Phi(x) + |v|^2/2)\sqrt{\mu_E},\ (\Phi(x) + |v|^2/2)\sqrt{\mu_E} \,\big\rangle \end{pmatrix} \begin{pmatrix} \varphi_0 \\ 2c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Once we show that the determinant of the above matrix is not zero :

$$\big\langle \sqrt{\mu_E}, \sqrt{\mu_E} \,\big\rangle \big\langle (\Phi(x) + |v|^2/2)\sqrt{\mu_E},\ (\Phi(x) + |v|^2/2)\sqrt{\mu_E} \,\big\rangle - \big\langle (\Phi(x) + |v|^2/2)\sqrt{\mu_E},\ \sqrt{\mu_E} \,\big\rangle^2 \neq 0, \tag{4.49}$$

then we conclude $\varphi_0 = 0$ and $c_0 = 0$ and hence $Z \equiv 0$. From the Cauchy-Schwartz inequality

$$\big\langle (\Phi(x) + |v|^2/2)\sqrt{\mu_E},\ \sqrt{\mu_E} \,\big\rangle^2 \lneqq \big\langle \sqrt{\mu_E}, \sqrt{\mu_E} \,\big\rangle \big\langle (\Phi(x) + |v|^2/2)\sqrt{\mu_E},\ (\Phi(x) + |v|^2/2)\sqrt{\mu_E} \,\big\rangle,$$

if $\sqrt{\mu_E}$ and $(\Phi(x) + |v|^2/2)\sqrt{\mu_E}$ are linearly independent which is obvious. Thus we conclude (4.49). $\square$

4.4 Nonlinear L^∞ Decay

In this section, we prove Theorem 1, especially the nonlinear $L^\infty-$decay in (1.52). Recall the weight function by $w(x,v) = \{\frac{|v|^2}{2} + \Phi(x)\}^{\beta/2}$ in (3.20) and define a weighted perturbation by

$$h(t,x,v) = w(x,v) \times \frac{F(t,x,v) - \mu_E(x,v)}{\sqrt{\mu_E(x,v)}}. \tag{4.50}$$

Notice that $h(t,x,v) = w(x,v) f(t,x,v)$. Then h satisfies

$$\{\partial_t + v \cdot \nabla_x - \nabla\Phi(x) \cdot \nabla_v\} h(t,x,v) + e^{-\Phi(x)} \nu h(t,x,v) - e^{-\Phi(x)} K_w h = e^{-\frac{\Phi(x)}{2}} w \Gamma(\frac{h}{w}, \frac{h}{w}), \tag{4.51}$$

where $L_w h = \nu(v)h - K_w h$ with $K_w h = wK(\frac{h}{w})$ ([27]). Notice that via Lemma 19 in [27], assuming $\sup_{0\le s\le T_0} e^{\lambda s}||h(s)||_\infty$ is small, we only have to show that there exist $\lambda > 0$ and $T_0 > 0$ and $C_{T_0} > 0$ such that

$$||h(T_0)||_\infty \le e^{-\lambda T_0}||h_0||_\infty + C_{T_0}\int_0^{T_0} ||f(s)||_{L^2} ds, \tag{4.52}$$

in order to show the nonlinear $L^\infty-$decay, i.e.

$$\sup_{0\le t\le\infty} e^{\lambda t}||h(t)||_\infty \le C||h_0||_\infty, \tag{4.53}$$

which is equivalent to (1.52). Once we establish (4.53), proving the existence and uniqueness, positivity of the Boltzmann solution F were established in [27].

For any (t, x, v), integrating along its backward trajectory $\frac{dX(s)}{ds} = V(s)$, $\frac{dV(s)}{ds} = -\nabla\Phi(X(s))$ in (4.4), we express

$$\begin{aligned} h(t,x,v) &= e^{-\int_0^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau} h(0, X(0), V(0)) && (4.54)\\ &+ \int_0^t e^{-\Phi(X(s))-\int_s^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau} K_w h(s, X(s), V(s))ds && (4.55)\\ &+ \int_0^t e^{-\frac{\Phi(X(s))}{2}-\int_s^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau} w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s, X(s), V(s))ds. && (4.56) \end{aligned}$$

Easily we can control the first line above by

$$(4.54) \le e^{-t\nu_0 e^{-|\Phi|_\infty}}||h(0)||_{L^\infty_{x,v}}. \tag{4.57}$$

Next we estimate (4.56). We can bound the loss term in (4.56) :

$$\begin{aligned} & w(X(s), V(s))\Gamma_-\left(\frac{h}{w}, \frac{h}{w}\right)(s, X(s), V(s)) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} q(\omega, |u - V(s)|)\frac{e^{-\frac{|u|^2}{4}}}{w(X(s), u)} h(s, X(s), u)h(s, X(s), V(s))d\omega du \\ &\le \left[\iint q(\omega, |u - V(s)|)e^{-\frac{|u|^2}{4}} w^{-1}(X(s), u)dud\omega\right] \times ||h(s)||^2_{L^\infty_{x,v}} \le \nu(V(s))||h(s)||^2_{L^\infty_{x,v}}, \end{aligned}$$

and for the gain term

$$
\begin{aligned}
& w(X(s),V(s))\Gamma_+\left(\frac{h}{w},\frac{h}{w}\right)(s,X(s),V(s)) \\
&= w(X(s),V(s))\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} q(\omega,|u-V(s)|)e^{-\frac{|u|^2}{4}}\frac{h(s,X(s),u')}{w(X(s),u')}\frac{h(s,X(s),v')}{w(X(s),v')}d\omega du \\
&\le \Phi(X(s))^{-\beta/2}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} q(\omega,|u-V(s)|)e^{-\frac{|u|^2}{4}}d\omega du \times ||h(s)||^2_{L^\infty_{x,v}} \le \nu(V(s))||h(s)||^2_{L^\infty_{x,v}},
\end{aligned}
$$

where $v' = v'(u,V(s))$, $u' = u'(u,V(s))$ and we used $|u|^2+|V(s)|^2 = |u'|^2+|v'|^2$ so that

$$
\begin{aligned}
& w(X(s),u')w(X(s),v') \\
&= (\Phi(X(s))+\frac{|u'|^2}{2})^{\beta/2}(\Phi(X(s))+\frac{|v'|^2}{2})^{\beta/2} \\
&\ge \left\{\Phi(X(s))^2+\Phi(X(s))\left(\frac{|u'|^2}{2}+\frac{|v'|^2}{2}\right)\right\}^{\beta/2} \\
&= \left\{\Phi(X(s))^2+\Phi(X(s))\left(\frac{|u|^2}{2}+\frac{|V(s)|^2}{2}\right)\right\}^{\beta/2} \ge \left\{\Phi(X(s))\left(\Phi(X(s))+\frac{|V(s)|^2}{2}\right)\right\}^{\beta/2} \\
&\ge \Phi(X(s))^{\beta/2}w(X(s),V(s)).
\end{aligned}
$$

Note that

$$
\begin{aligned}
e^{-\frac{1}{2}\int_s^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau} &\le e^{-\frac{1}{2}\nu_0 e^{-|\Phi|_\infty}(t-s)}, \\
2\frac{d}{ds}\left\{e^{-\frac{1}{2}\int_s^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau}\right\} &= \nu(V(s))e^{-\Phi(X(s))}e^{-\frac{1}{2}\int_s^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau}.
\end{aligned}
$$

Using above relations, we have an upper bound of the integrand of (4.56) as

$$
\begin{aligned}
& e^{-\frac{\Phi(X(s))}{2}}e^{-\frac{1}{2}\int_s^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau}e^{-\frac{1}{2}\nu_0 e^{-|\Phi|_\infty}(t-s)}\nu(V(s))||h(s)||^2_\infty \\
\le\ & e^{\frac{|\Phi|_\infty}{2}}\nu(V(s))e^{-\Phi(X(s))}e^{-\frac{1}{2}\int_s^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau}\times\{e^{-\frac{1}{4}\nu_0 e^{-|\Phi|_\infty}(t-s)}||h(s)||_\infty\}^2 \\
=\ & e^{\frac{|\Phi|_\infty}{2}}\times 2\frac{d}{ds}\left\{e^{-\frac{1}{2}\int_s^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau}\right\}\times\{e^{-\frac{1}{4}\nu_0 e^{-|\Phi|_\infty}(t-s)}||h(s)||_\infty\}^2.
\end{aligned}
$$

Therefore we have a control of the integration of (4.56) by

$$
\begin{aligned}
(4.56) &\le 2e^{\frac{|\Phi|_\infty}{2}}\{1-e^{-\frac{1}{2}\int_0^t e^{-\Phi(X(\tau))}\nu(V(\tau))d\tau}\}\times\sup_{0\le s\le t}\{e^{-\frac{1}{4}\nu_0 e^{-|\Phi|_\infty}(t-s)}||h(s)||_\infty\}^2 \\
&\le 2e^{\frac{|\Phi|_\infty}{2}}\times\sup_{0\le s\le t}\{e^{-\frac{1}{4}\nu_0 e^{-|\Phi|_\infty}(t-s)}||h(s)||_\infty\}^2. \qquad (4.58)
\end{aligned}
$$

From now, we concentrate to estimate (4.55). Let $\mathbf{k}(v,v')$ be the corresponding kernel associated with K. Notice that in the integrand of (4.55)

$$\{K_w h\}(s,X(s),V(s)) = \int_{\mathbb{R}^d} \mathbf{k}_w(V(s),v')\underline{h(s,X(s),v')}dv'. \tag{4.59}$$

Now we use the representation of the underlined $\underline{h(s,X(s),v')}$ again to evaluate (4.59). We need a following crucial inequality, Lemma 3 in [27] :

Lemma 48. *([27])*

$$|\mathbf{k}(v,v')| \le C\{|v-v'|+|v-v'|^{-1}\}\exp\left\{-\frac{1}{8}|v-v'|^2 - \frac{1}{8}\frac{||v|^2-|v'|^2|^2}{|v-v'|^2}\right\}.$$

Let $0 \le \theta < \frac{1}{4}$. Then there exists $0 \le \varepsilon(\theta) < 1$ and $C_\theta > 0$ such that for $0 \le \varepsilon < \varepsilon(\theta)$,

$$\int_{\mathbb{R}^d}\{|v-v'|+|v-v'|^{-1}\}\exp\left\{-\frac{1-\varepsilon}{8}|v-v'|^2 - \frac{1-\varepsilon}{8}\frac{||v|^2-|v'|^2|^2}{|v-v'|^2}\right\}\frac{w(x,v)e^{\theta|v|^2}}{w(x,v')e^{\theta|v'|^2}}dv' \le \frac{C}{1+|v|}.$$

Proof. We can check

$$\begin{aligned}\left|\frac{w(x,v)}{w(x,v')}\right| &= \left\{\frac{\frac{|v|^2}{2}+\Phi(x)}{\frac{|v'|^2}{2}+\Phi(x)}\right\}^{\beta/2} \le \left\{1+\frac{\frac{|v'|^2}{2}+\frac{|v-v'|^2}{2}}{\Phi(x)+\frac{|v'|^2}{2}}\right\}^{\beta/2} \le \left\{1+\frac{\frac{|v'|^2}{2}}{\Phi(x)+\frac{|v'|^2}{2}}+\frac{\frac{|v-v'|^2}{2}}{\Phi(x)+\frac{|v'|^2}{2}}\right\}^{\beta/2} \\ &\le \{1+1+C_\Phi|v-v'|^2\}^{\beta/2} \le C(1+|v-v'|^2)^{\beta/2}.\end{aligned}$$

The remainder of the proof is exactly same as the proof of Lemma 3 in [27]. ☐

In order to simplify notations, we define

$$\begin{cases} \Psi_1(t) = \int_0^t e^{-\Phi(X(\tau;t,x,v))}\nu(V(\tau;t,x,v))d\tau \\ \Psi_2(s,t) = \Phi(X(s;t,x,v)) + \int_s^t e^{-\Phi(X(\tau;t,x,v))}\nu(V(\tau;t,x,v))d\tau \ge \nu_0 e^{-|\Phi|_\infty}(t-s) \\ \Psi_3(s,t) = \frac{1}{2}\Phi(X(s;t,x,v)) + \int_s^t e^{-\Phi(X(\tau;t,x,v))}\nu(V(\tau;t,x,v))d\tau \\ \Psi_1'(s) = \int_0^s e^{-\Phi(X(\tau;s,X(s),v'))}\nu(V(\tau;s,X(s),v'))d\tau \ge \nu_0 e^{-|\Phi|_\infty}s \\ \Psi_2'(s_1,s) = \Phi(X(s_1;s,X(s),v')) + \int_{s_1}^s e^{-\Phi(X(\tau;s,X(s),v'))}\nu(V(\tau;s,X(s),v'))d\tau \ge \nu_0 e^{-|\Phi|_\infty}(s-s_1) \\ \Psi_3'(s_1,s) = \frac{1}{2}\Phi(X(s_1;s,X(s),v')) + \int_{s_1}^s e^{-\Phi(X(\tau;s,X(s),v'))}\nu(V(\tau;s,X(s),v'))d\tau \end{cases}$$

We can rewrite $\underline{h(s, X(s), v')}$ in (4.59) as

$$\begin{aligned} e^{-\Psi_1'(s)} h(0, X'(0), V'(0)) \quad &+ \quad \int_0^s e^{-\Psi_2'(s_1,s)} \int_{\mathbb{R}^d} \mathbf{k}_w(V'(s_1), v'') h(s_1, X'(s_1), v'') dv'' ds_1 \\ &+ \quad \int_0^s e^{-\Psi_3'(s_1,s)} w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s_1, X'(s_1), V'(s_1)) ds_1. \end{aligned}$$

We plug the above formula into (4.59) and (4.55) to have

$$(4.55) \tag{4.60}$$

$$= \int_0^t \int_{\mathbb{R}^d} e^{-\Psi_2(t,s)} e^{-\Psi_1'(s)} \mathbf{k}_w(V(s), v') h_0(X'(0), V'(0)) dv' ds \tag{4.61}$$

$$+ \int_0^t \int_{\mathbb{R}^d} \int_0^s \int_{\mathbb{R}^d} e^{-\Psi_2(t,s)} e^{-\Psi_2'(s_1,s)} \mathbf{k}_w(V(s), v') \mathbf{k}_w(V'(s_1), v'') h(s_1, X'(s_1), v'') dv'' ds_1 dv' ds \tag{4.62}$$

$$+ \int_0^t \int_{\mathbb{R}^d} \int_0^s e^{-\Psi_2(t,s)} e^{-\Psi_3'(s_1,s)} \mathbf{k}_w(V(s), v') w\Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s_1, X'(s_1), V'(s_1)) ds_1 dv' ds. \tag{4.63}$$

For the first line, we have

$$(4.61) \le \int_0^t e^{-t\nu_0 e^{-|\Phi|_\infty}} ||h_0||_\infty \int_{\mathbb{R}^d} \mathbf{k}_w(V(s), v') dv' ds \tag{4.64}$$

$$\le \left(t e^{-\frac{t}{2}\nu_0 e^{-|\Phi|_\infty}}\right) e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2} t} ||h_0||_\infty \le C e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2} t} ||h_0||_\infty, \tag{4.65}$$

and the third line (4.63) is bounded by

$$\int_0^t \int_{\mathbb{R}^d} \int_0^s e^{-\Phi(X(s))} \underbrace{e^{-\int_s^t e^{-\Phi(X(\tau))} \nu(V(\tau)) d\tau}}_{\mathbf{I}} \tag{4.66}$$

$$\times \underbrace{e^{-\frac{1}{2}\Phi(X'(s_1))} e^{-\int_{s_1}^s e^{-\Phi(X'(\tau))} \nu(V'(\tau)) d\tau} \nu(V'(s_1))}_{\mathbf{II}} \mathbf{k}_w(V(s), v') ||h(s_1)||_\infty^2 ds_1 dv' ds$$

$$\le 2\, e^{\frac{|\Phi|_\infty}{2}} \int_0^t e^{-\frac{\nu_0(t-s)}{2e^{|\Phi|_\infty}}} \int_{\mathbb{R}^d} \mathbf{k}_w(V(s), v') \int_0^s \frac{d}{ds_1}\left\{e^{-\frac{1}{2}\int_{s_1}^s e^{-\Phi(X'(\tau))} \nu(V'(\tau)) d\tau}\right\} ds_1 dv' ds \tag{4.67}$$

$$\times \sup_{0 \le s_1 \le t} \{e^{-\frac{\nu_0(t-s_1)}{4e^{|\Phi|_\infty}}} ||h(s_1)||_\infty\}^2$$

$$\le 2 e^{\frac{|\Phi|_\infty}{2}} \frac{2}{\nu_0} e^{|\Phi|_\infty} \times \sup_{0 \le s_1 \le t} \{e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{4}(t-s_1)} ||h(s_1)||_\infty\}^2 \tag{4.68}$$

$$\le \quad \frac{4}{\nu_0} e^{\frac{3}{2}|\Phi|_\infty} \sup_{0 \le s_1 \le t} \{e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{4}(t-s_1)} ||h(s_1)||_\infty\}^2, \tag{4.69}$$

where we used

$$
\begin{aligned}
\mathbf{I} &\leq e^{-(t-s)\nu_0 e^{-|\Phi|_\infty}}, \\
\mathbf{II} &\leq 2e^{\frac{|\Phi|_\infty}{2}} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(s-s_1)} \times \frac{1}{2} e^{-\Phi(X'(s_1))} e^{-\frac{1}{2}\int_{s_1}^{s} e^{-\Phi(X'(\tau))}\nu(V'(\tau))d\tau} \nu(V'(s_1)) \\
&= 2e^{\frac{|\Phi|_\infty}{2}} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(s-s_1)} \times \frac{d}{ds_1}\left\{ e^{-\frac{1}{2}\int_{s_1}^{s} e^{-\Phi(X'(\tau))}\nu(V'(\tau))d\tau} \right\}.
\end{aligned}
$$

Now we concentrate on the second term, (4.62).

4.4.1 Estimate of (4.62)

CASE 1 : $|v| \geq N$ with $N >> \sqrt{|\Phi|_\infty}$. Using (4.5), we have $|V(s)| \geq \frac{N}{2}$ so that

$$
\iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{k}_w(V(s), v')\mathbf{k}_w(V'(s_1), v'')dv''dv' \leq \frac{C}{1+|V(s)|} \leq \frac{C}{N}.
$$

Thus in this case, (4.62) is bounded by

$$
\begin{aligned}
&\int_0^t \int_0^s e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{k}_w(V(s), v')\mathbf{k}_w(V'(s_1), v'')dv''dv'ds_1 ds \\
&\qquad times \sup_{0\leq s_1 \leq t} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)} ||h(s_1)||_\infty \\
&\leq \left(\frac{2}{\nu_0 e^{-|\Phi|_\infty}}\right)^2 \times \frac{C}{N} \times \sup_{0\leq s_1 \leq t} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)} ||h(s_1)||_\infty && (4.70) \\
&\leq \frac{C}{N}\frac{4}{\nu_0^2} e^{2|\Phi|_\infty} \sup_{0\leq s_1 \leq t} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)} ||h(s_1)||_\infty, && (4.71)
\end{aligned}
$$

where we used the fact

$$
\begin{aligned}
&\int_0^t \int_0^s e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)} ds_1 ds \\
&= \frac{2}{\nu_0 e^{-|\Phi|_\infty}} \int_0^t ds \int_0^s \frac{d}{ds_1}\left\{ e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)} \right\} ds_1 = \frac{2}{\nu_0 e^{-|\Phi|_\infty}} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}t} \int_0^t \{ e^{\frac{\nu_0 e^{-|\Phi|_\infty}}{2}s} - 1\} ds \\
&= \frac{2}{\nu_0 e^{-|\Phi|_\infty}} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}t} \left(\frac{2}{\nu_0 e^{-|\Phi|_\infty}} e^{\frac{\nu_0 e^{-|\Phi|_\infty}}{2}t} - t - \frac{2}{\nu_0 e^{-|\Phi|_\infty}} \right) \leq \left(\frac{2}{\nu_0 e^{-|\Phi|_\infty}}\right)^2.
\end{aligned}
$$

CASE 2 : $|v| \leq N, |v'| \geq 2N$, or $|v'| \leq 2N, |v''| \geq 3N$. Observe that

$$
\begin{aligned}
|V(s) - v'| &\geq |v' - v| - |V(s) - v| \geq |v'| - |v| - |V(s) - v|, \\
|V'(s_1) - v''| &\geq |v'' - v'| - |V'(s_1) - v'| \geq |v''| - |v'| - |V'(s_1) - v'|,
\end{aligned}
$$

and $|V(s)-v|, |V'(s_1)-v'| \le 2|\Phi|_\infty^2$ from (4.5), thus we have either $|V(s)-v'| \ge \frac{N}{2}$ or $|V'(s_1)-v''| \ge \frac{N}{2}$ and either one of the followings are valid correspondingly for $\eta > 0$:

$$\begin{aligned} \mathbf{k}_w(V(s),v') &\le e^{-\frac{\eta}{8}N^2}\mathbf{k}_w(V(s),v')e^{\frac{\eta}{8}|V(s)-v'|^2}, \\ \mathbf{k}_w(V'(s_1),v'') &\le e^{-\frac{\eta}{8}N^2}\mathbf{k}_w(V'(s_1),v'')e^{\frac{\eta}{8}|V'(s_1)-v''|^2}. \end{aligned}$$

From Lemma 48, both $\int \mathbf{k}_w(V(s),v')e^{\frac{\eta}{8}|V(s)-v'|^2}dv'$ and $\int \mathbf{k}_w(V'(s_1),v'')e^{\frac{\eta}{8}|V'(s_1)-v''|^2}dv''$ are still finite for sufficiently small $\eta > 0$. Therefore (4.62) is bounded by

$$\begin{aligned} &\int_0^t ds \int_0^s ds_1 e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)} \iint dv''dv'\mathbf{k}_w(V(s),v')\mathbf{k}_w(V'(s_1),v'') \\ &\times \sup_{0\le s_1\le t} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)}||h(s_1)||_\infty \\ &\le \left(\frac{2}{\nu_0 e^{-|\Phi|_\infty}}\right)^2 e^{-\frac{\eta}{8}N^2} \sup_{0\le s_1\le t} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)}||h(s_1)||_\infty \qquad (4.72) \\ &\le e^{-\frac{\eta}{8}N^2}\frac{4}{\nu_0^2}e^{2|\Phi|_\infty} \sup_{0\le s_1\le t} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)}||h(s_1)||_\infty, \qquad (4.73) \end{aligned}$$

where we used the fact

$$\begin{aligned} &\iint \mathbf{k}_w(V(s),v')\mathbf{k}_w(V'(s_1),v'')dv''dv' \\ &\le e^{-\frac{\eta}{8}N^2}\iint \mathbf{k}_w(V(s),v')\mathbf{k}_w(V'(s_1),v'')\{e^{\frac{\eta}{8}|V(s)-v'|^2}+e^{\frac{\eta}{8}|V'(s_1)-v''|^2}\} \le Ce^{-\frac{\eta}{8}N^2}. \end{aligned}$$

CASE 3 : $|v| \le N, |v'| \le 2N, |v''| \le 3N$. This is the last remaining case because if $|v'| > 2N$, it is included in Case 2; while if $|v''| > 3N$, either $|v'| \le 2N$ or $|v'| \ge 2N$ are also included in Case 2. We can bound (4.62) by

$$\int_0^t \int_{|v'|\le 2N}\int_0^s e^{-\nu_0 e^{-|\Phi|_\infty}(t-s_1)}\int_{|v''|\le 3N} \underbrace{\mathbf{k}_w(V(s),v')\mathbf{k}_w(V'(s_1),v'')}_{\odot}|h(s_1,X'(s_1),v'')|dv''ds_1dv'ds \qquad (4.74)$$

Since $\mathbf{k}_w(v,v')$ has possible integrable singularity of $\frac{1}{|v-v'|}$, we can choose a smooth function with compact support $\mathbf{k}_N(v,v')$ such that

$$\sup_{|p|\le 3N}\int_{|v'|\le 3N}|\mathbf{k}_w(p,v')-\mathbf{k}_N(p,v')|dv' \le \frac{1}{N}.$$

Splitting $\mathbf{k}_w(V(s),v')\mathbf{k}_w(V'(s_1),v'')$ in $\bigodot$ by

$$\mathbf{k}_N(V(s),v')\mathbf{k}_N(V'(s_1),v'')\ , \tag{4.75}$$

$$\{\mathbf{k}_w(V(s),v')-\mathbf{k}_N(V(s),v')\}\mathbf{k}_w(V'(s_1),v'')+\{\mathbf{k}_w(V'(s_1),v'')-\mathbf{k}_N(V'(s_1),v'')\}\mathbf{k}_N(V(s),v') \tag{4.76}$$

We can bound (4.74), in the case of $\bigodot$ = (4.76), by

$$\frac{C}{N}\frac{4}{\nu_0^2}e^{2|\Phi|_\infty}\sup_{0\le s_1\le t}e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)}||h(s_1)||_\infty. \tag{4.77}$$

In the case of $\bigodot$ = (4.75), we can bound (4.74) by

$$C_N\int_0^t ds\int_{|v'|\le 2N}dv'\int_0^s ds_1 e^{-\nu_0 e^{-|\Phi|_\infty}(t-s_1)}\int_{|v''|\le 3N}|h(s_1,X(s_1;s,X(s;t,x,v),v'),v'')|dv''. \tag{4.78}$$

Recall that we need to show the decay for $t=T_0$ from (4.52). Since the potential is time-independent we have

$$X(s_1;s,X(s;T_0,x,v),v')=X(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v'),$$

for $0\le s_1\le s\le T_0$. From Lemma 47, we split (4.78) by

$$\begin{aligned}&C_N\sum_{i^1}^{M^1}\sum_{I^2}^{(M^2)^d}\sum_{I^3}^{(M^3)^d}\int_0^{T_0}ds\int_0^s ds_1\mathbf{1}_{\{X(s_1-s+T_0;T_0,x,v)\in\mathfrak{D}^2_{I^2}\}}(s_1,s)\ \underbrace{\mathbf{1}_{\mathfrak{D}^1_{i^1}}(s_1-s+T_0)}_{\otimes}e^{-\nu_0 e^{-|\Phi|_\infty}(T_0-s_1)}\\&\qquad\times\int_{|v'|\le 2N}dv'\mathbf{1}_{\mathfrak{D}^3_{I^3}}(v')\int_{|v''|\le 3N}|h(s_1,X(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v'),v'')|dv''\end{aligned} \tag{4.79}$$

From Lemma 47, we have

$$\begin{aligned}&\{(s_1-s+T_0,X(T_0;2T_0-s,x,v),v')\in\mathfrak{D}^1_{i^1}\times\mathfrak{D}^2_{I^2}\times\mathfrak{D}^3_{I^3}\\&\quad: det\left(\frac{\partial X}{\partial v'}\right)(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v')=0\}\\&\subset\bigcup_j^d\{(s_1-s+T_0,X(T_0;2T_0-s,x,v),v')\in\mathfrak{D}^1_{i^1}\times\mathfrak{D}^2_{I^2}\times\mathfrak{D}^3_{I^3}\\&\qquad:\ s_1-s+T_0\in(t_{j,i^1,I^2,I^3}-\frac{\varepsilon}{4M^1},t_{j,i^1,I^2,I^3}+\frac{\varepsilon}{4M^1})\}.\end{aligned}$$

For each i^1, I^2, I^3 and j, we split $\bigotimes$ as

$$\mathbf{1}_{\mathfrak{D}^1_{i^1}}(s_1-s+T_0)\mathbf{1}_{(t_{j,i^1,I^2,I^3}-\frac{\varepsilon}{4M^1},t_{j,i^1,I^2,I^3}+\frac{\varepsilon}{4M^1})}(s_1-s+T_0)\ , \tag{4.80}$$

$$\mathbf{1}_{\mathfrak{D}^1_{i^1}}(s_1-s+T_0)\{1-\mathbf{1}_{(t_{j,i^1,I^2,I^3}-\frac{\varepsilon}{4M^1},t_{j,i^1,I^2,I^3}+\frac{\varepsilon}{4M^1})}(s_1-s+T_0)\}. \tag{4.81}$$

CASE 3a : In the case of $\bigotimes = (4.80)$, the integration (4.79) is bounded by

$$\begin{aligned} C_N \sum_{i^1}^{M^1}\sum_{I^2}^{(M^2)^d}\sum_{I^3}^{(M^3)^d}\int_0^{T_0} ds \int_0^s ds_1 \mathbf{1}_{\{X(s_1-s+T_0;T_0,x,v)\in\mathfrak{D}^2_{I^2}\}}(s_1,s)\ \mathbf{1}_{\mathfrak{D}^1_{i^1}}(s_1-s+T_0)\underline{e^{-\nu_0 e^{-|\Phi|_\infty}(T_0-s_1)}}_{*} \\ \times \mathbf{1}_{(t_{j,i^1,I^2,I^3}-\frac{\varepsilon}{4M^1},t_{j,i^1,I^2,I^3}+\frac{\varepsilon}{4M^1})}(s_1-s+T_0) \\ \times\int_{|v'|\le 2N} dv'\mathbf{1}_{\mathfrak{D}^3_{I^3}}(v')\int_{|v''|\le 3N}|h(s_1,X(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v'),v'')|dv''. \end{aligned}$$

We split

$$\underline{e^{-\nu_0 e^{-|\Phi|_\infty}(t-s_1)}}_{*} = e^{-\nu_0 e^{-|\Phi|_\infty}2(t-s)}e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(s-s_1)}\times e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(t-s_1)}.$$

and rewrite the above integration as

$$\begin{aligned} C_N \sum_{i^1}^{M^1}\sum_{I^2}^{(M^2)^d}\sum_{I^3}^{(M^3)^d}\int_0^{T_0} ds \mathbf{1}_{\{X(s_1-s+T_0;T_0,x,v)\in\mathfrak{D}^2_{I^2}\}}(s_1,s)e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(T_0-s)} \\ \times\int_0^s ds_1 e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(s-s_1)}\ \mathbf{1}_{\mathfrak{D}^1_{i^1}}(s_1-s+T_0)\mathbf{1}_{(t_{j,i^1,I^2,I^3}-\frac{\varepsilon}{4M^1},t_{j,i^1,I^2,I^3}+\frac{\varepsilon}{4M^1})}(s_1-s+T_0) \\ \times\int_{|v'|\le 2N} dv'\mathbf{1}_{\mathfrak{D}^3_{I^3}}(v')\int_{|v''|\le 3N} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(T_0-s_1)}||h(s_1)||_\infty dv''. \end{aligned}$$

For fixed i^1, I^2, I^3, using the fact that $e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(s-s_1)}$ is an increasing function of $s_1\in[0,s]$, the second line of the above term is bounded by

$$\begin{aligned} \int_{s-\frac{\varepsilon}{4M^1}}^{s}\frac{2}{\nu_0 e^{-|\Phi|_\infty}}\frac{d}{ds_1}\left\{e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(s-s_1)}\right\}ds_1 &= \frac{2}{\nu_0 e^{-|\Phi|_\infty}}\{1-e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}\frac{\varepsilon}{4M^1}}\} \\ &\sim \frac{2}{\nu_0 e^{-|\Phi|_\infty}}\frac{\nu_0 e^{-|\Phi|_\infty}}{2}\frac{\varepsilon}{4M^1} = \frac{\varepsilon}{4M^1}. \end{aligned}$$

Therefore we can bound (4.79) by

$$\begin{aligned}
&C_N \int_0^{T_0} \underbrace{\sum_{I^2} \mathbf{1}_{\{X(s_1-s+T_0;T_0,x,v)\in\mathfrak{D}^2_{i^2}\}}(s_1,s)\quad e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(T_0-s)}}_{**} ds \times \sum_{i^1=1}^{M^1} \frac{\varepsilon}{4M^1} \\
&\quad \times \int_{|v'|\le 2N} dv' \underbrace{\sum_{I^3} \mathbf{1}_{\mathfrak{D}^3_{I^3}}(v')}_{***} \quad \times (3N)^3 \sup_{0\le s_1\le T_0} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(T_0-s_1)} ||h(s_1)||_\infty \\
&\le\ C_N \int_0^{T_0} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(T_0-s)} ds \times \frac{\varepsilon}{4} \int_{|v'|\le 2N} dv'(3N)^3 \sup_{0\le s_1\le T_0} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(T_0-s_1)} ||h(s_1)||_\infty \\
&\le\ C_N \frac{2}{\nu_0 e^{-|\Phi|_\infty}} \frac{\varepsilon}{4}(2N)^3(3N)^3 \sup_{0\le s_1\le T_0} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(T_0-s_1)} ||h(s_1)||_\infty \\
&\le\ \varepsilon \frac{C_N e^{|\Phi|_\infty}}{\nu_0} \sup_{0\le s_1\le T_0} e^{-\frac{\nu_0 e^{-|\Phi|_\infty}}{2}(T_0-s_1)} ||h(s_1)||_\infty,
\end{aligned} \tag{4.82}$$

where we used the fact that

$$\begin{aligned}
&\underline{**} \qquad \sum_{I^2}^{(M^2)^d} \mathbf{1}_{\{X(s_1-s+T_0;T_0,x,v)\in\mathfrak{D}^2_{I^2}\}}(s_1,s) \\
&\qquad\qquad = \mathbf{1}_{\{X(s_1-s+T_0;T_0,x,v)\in\mathbb{T}^d\}}(s_1,s) = \mathbf{1}_{\{0\le s\le T_0\}}(s)\mathbf{1}_{\{0\le s_1\le s\}}(s_1), \\
&\underline{***} \qquad \sum_{I^3}^{(M^3)^d} \mathbf{1}_{\mathfrak{D}^3_{I^3}}(v')\mathbf{1}_{|v'|\le 2N}(v') = \mathbf{1}_{|v'|\le 2N}(v').
\end{aligned}$$

CASE 3b : In the case of $\bigotimes = (4.81)$, the integration (4.79) is bounded by

$$\begin{aligned}
&C_N \sum_{i^1}^{M^1} \sum_{I^2}^{(M^2)^d} \sum_{I^3}^{(M^3)^d} \int_0^{T_0} ds \int_0^s ds_1 \mathbf{1}_{\{X(s_1-s+T_0;T_0,x,v)\in\mathfrak{D}^2_{I^2}\}}(s_1,s)\ \mathbf{1}_{\mathfrak{D}^1_{i^1}}(s_1-s+T_0) e^{-\nu_0 e^{-|\Phi|_\infty}(T_0-s_1)} \\
&\times \left\{ 1 - \mathbf{1}_{(t_{j,i^1,I^2,I^3}-\frac{\varepsilon}{4M^1},\, t_{j,i^1,I^2,I^3}+\frac{\varepsilon}{4M^1})}(s_1-s+T_0) \right\} \\
&\times \int_{|v'|\le 2N} \mathbf{1}_{\mathfrak{D}^3_{I^3}}(v') \int_{|v''|\le 3N} |h(s_1, \underline{X(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v')}, v'')| dv' dv''.
\end{aligned} \tag{4.83}$$

By Lemma 47, we can apply a change of variables :

$$v' \to y \equiv X(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v'),$$

$$Jac\left(\frac{\partial X}{\partial v'}\right)(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v') > \delta_*.$$

Therefore the last line of the above term is bounded by

$$\begin{aligned}\frac{1}{\delta_*}\int_{x\in\mathbb{T}^d}\int_{|v''|\leq 3N} w(x,v'')|f(s_1,x,v'')|dv''dx &\leq \frac{1}{\delta_*}\left(\int_{|v''|\leq 3N}\int_{\mathbb{T}^d} w(x,v'')^2 dxdv''\right)^{\frac{1}{2}}||f(s_1)||_{L^2}\\ &\leq \frac{1}{\delta_*}C(N,|\Phi|_\infty)||f(s_1)||_{L^2}.\end{aligned}$$

Therefore, in the case of $\bigotimes = (4.81)$, we have an upper bound of (4.79) as

$$C(M^1,M^2,M^3,\delta_*,N,|\Phi|_\infty,\nu_0)\int_0^{T_0}||f(s_1)||_{L^2}ds_1. \tag{4.84}$$

To summarize, let $\lambda = \frac{\nu_0 e^{-|\Phi|_\infty}}{4}$ and from (4.57), (4.58), (4.65), (4.69), (4.71), (4.73), (4.77), (4.82) and (4.84) we conclude

$$\begin{aligned}||h(T_0)||_\infty &\leq e^{-\lambda T_0}||h_0||_\infty + C(\frac{1}{N}+\varepsilon+e^{-\frac{\eta}{8}N^2})\sup_{0\leq s\leq T_0}\{e^{-\lambda(T_0-s)}||h(s)||_\infty\}\\ &\quad + C\sup_{0\leq s\leq T_0}\{e^{-\lambda(T_0-s)}||h(s)||_\infty\}^2 + C_{T_0}\int_0^{T_0}||f(s_1)||_{L^2}ds_1.\end{aligned}$$

Assume $\sup_{0\leq s\leq T_0}\{e^{\lambda s}||h(s)||_\infty\}$ is sufficiently small. Choose sufficiently large $N>0$ and small $\varepsilon>0$ and small $||h_0||_\infty$. Then we conclude (4.52).

4.5 Nonlinear L^∞ Stability

In this section, we prove Theorem 2 and establish the nonlinear L^∞ stability in (1.55). The following lemma, which has been established in [28], plays a crucial rule in the proof of the nonlinear stability (1.55) without the conservation of momentum.

Lemma 49. *([28]) Let $\mu_E(x,v)=\exp\{-\frac{|v|^2}{2}-\Phi(x)\}$. Assume F satisfies the conservation of mass (1.43), energy (1.44) and the entropy inequality (1.45). For $0<\delta<1$, we have*

$$\iint |F(t)-\mu_E|\mathbf{1}_{|F(t)-\mu_E|\geq\delta\mu_E} \leq \frac{4}{\delta}\{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|\}. \tag{4.85}$$

Proof. The proof is almost same as the argument in Page 147 of [28]. The difference is the fact that $\ln\mu_E = -\frac{|v|^2}{2}-\Phi(x)$ is only bounded by the energy $\frac{|v|^2}{2}+\Phi(x)$. We now make use of the entropy inequality (1.45). Recall from the Taylor expansion,

$$\begin{aligned}\mathcal{H}(F(t))-\mathcal{H}(\mu_E) &= \iint\{F(t)\ln F(t)-\mu_E\ln\mu_E\} = \underline{\iint(\ln\mu_E+1)\{F(t)-\mu_E\}}+\iint\frac{\{F(t)-\mu_E\}^2}{2\tilde{F}}\\ &\le \mathcal{H}(F_0)-\mathcal{H}(\mu_E)\ ,\end{aligned}$$

where $\tilde{F}$ is a number between $F(t)$ and μ_E. Notice that the underlined term is bounded by the mass and energy of $F(t)$. Hence, from the conservation of mass (1.43) and energy (1.44), we get

$$\iint\frac{\{F(t)-\mu_E\}^2}{2\tilde{F}} \le\ \mathcal{H}(F_0)-\mathcal{H}(\mu_E)\ +\ |M_0|\ +\ |E_0|.$$

The rest of the proof is exactly same as the argument of Page 147 of [28]. $\square$

In order to obtain (1.55), we do estimate a weighted perturbation h in (4.50) satisfying the linearized Boltzmann equation (4.51). The proof is exactly same as Section 4 except **CASE 3b**. Consider (4.83) in **CASE 3b**. We introduce the indicator functions $\mathbf{1}_{|F(t)-\mu_E|\le\delta\mu_E}$ and $\mathbf{1}_{|F(t)-\mu_E|\ge\delta\mu_E}$ and split the last line of (4.83) into

$$\begin{aligned}\int_{|v'|\le 2N}\int_{|v''|\le 3N}&\mathbf{1}_{\mathfrak{D}^3_{I^3}}|h(s_1,X(s_1-s+T_0),v'')|\mathbf{1}_{|F(t)-\mu_E|\le\delta\mu_E}\\ &+\mathbf{1}_{\mathfrak{D}^3_{I^3}}|h(s_1,X(s_1-s+T_0),v'')|\mathbf{1}_{|F(t)-\mu_E|\ge\delta\mu_E}.\end{aligned}$$

The first integration is bounded by

$$\delta\int_{|v'|\le 2N}\int_{|v''|\le 3N}\mathbf{1}_{\mathfrak{D}^3_{I^3}}w(X(s_1-s+T_0),v'')\sqrt{\mu_E(X(s_1-s+T_0),v'')}.$$

Using Lemma 4, the second integration is bounded by

$$\begin{aligned}\int_{|v'|\le 2N}\int_{|v''|\le 3N}&\mathbf{1}_{\mathfrak{D}^3_{I^3}}\frac{w}{\sqrt{\mu_E}}(X(s_1-s+T_0),v'')\\ \times|F(s_1,X(s_1-s+T_0),v'')&-\mu_E(X(s_1-s+T_0),v'')|\mathbf{1}_{|F(t)-\mu_E|\ge\delta\mu_E}dv''dv'.\end{aligned}\tag{4.86}$$

By Lemma 2, we apply the change of variables

$$v'\to y = X(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v'),$$

$$Jac\left(\frac{\partial X}{\partial v'}\right)(s_1-s+T_0;T_0,X(T_0;2T_0-s,x,v),v')>\delta_*.$$

to bound (4.86) by

$$\frac{C_{N,\Phi}}{\delta_*}\int_{|v'|\le 2N}\int_{y\in\mathbb{T}^d}\mathbf{1}_{\mathfrak{D}^3_{I^3}}|F(s_1,y,v'')-\mu_E(s_1,y,v'')|\mathbf{1}_{|F(t)-\mu_E|\ge\delta\mu_E}dydv'. \tag{4.87}$$

Combining these two cases, using Lemma 4, the whole integration (4.83) is bounded by

$$\begin{aligned}
& C_{N,\Phi}\int_0^{T_0}ds\int_0^s ds_1 e^{-\nu_0 e^{-|\Phi|_\infty}(T_0-s_1)}\int_{|v'|\le 2N} \\
& \times\int_{\mathbb{T}^d}\{\delta+\frac{1}{\delta_*}|F(s_1,y,v'')-\mu_E(s_1,y,v'')|\mathbf{1}_{|F(t)-\mu_E|\ge\delta\mu_E}\}dydv' \\
& \quad\le\ C_{N,\Phi}\left[\delta+\frac{1}{\delta_*\delta}\{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|\}\right] \\
& \quad\le\ C_{N,\Phi}\delta_*^{-\frac{1}{2}}\sqrt{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|}\ .
\end{aligned}$$

We also have optimized δ such that (for sufficiently small $|\mathcal{H}(F_0)-\mathcal{H}(\mu_E)|+|M_0|+|E_0|$),

$$\delta=\frac{1}{\delta_*\delta}\{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|\}.$$

To summarize, from the last part of Section 4, we conclude

$$\begin{aligned}
\sup_{0\le t\le T_0}||h(t)||_\infty\ &\le\ e^{-\lambda T_0}||h_0||_\infty+C(\frac{1}{N}+\varepsilon+e^{-\frac{\eta}{8}N^2})\sup_{0\le s\le T_0}\{e^{-\lambda(T_0-s)}||h(s)||_\infty\} \\
&+\ C\sup_{0\le s\le T_0}\{e^{-\lambda(T_0-s)}||h(s)||_\infty\}^2+C_{N,\Phi}\delta_*^{-\frac{1}{2}}\sqrt{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|}.
\end{aligned}$$

Assume $\sup_{0\le s\le T_0}||h(s)||_\infty$ and $\varepsilon>0$ sufficiently small and T_0,N,η sufficiently large to conclude

$$||h(T_0)||_\infty\le\frac{1}{2}||h_0||_\infty+C_{T_0}\sqrt{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|}. \tag{4.88}$$

From this finite time estimate, we use the argument in page 23 of [23] to establish a large time estimate. Apply (4.88) repeatedly to get

$$\begin{aligned}
||h(nT_0)||_\infty\ &\le\ \frac{1}{2}||h_0||_\infty+C_{T_0}\sqrt{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|} \\
&\le\ \frac{1}{4}||h_0||_\infty+\{1+\frac{1}{2}\}C_{T_0}\sqrt{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|} \\
&\le\ \ldots \\
&\le\ \frac{1}{2^n}||h_0||_\infty+\{1+\frac{1}{2}+\frac{1}{4}+\ldots\}C_{T_0}\sqrt{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|} \\
&\le\ \frac{1}{2^n}||h_0||_\infty+2C_{T_0}\sqrt{\mathcal{H}(F_0)-\mathcal{H}(\mu_E)+|M_0|+|E_0|}.
\end{aligned}$$

For any $t > 0$, we can find n such that $nT_0 \leq t \leq \{n+1\}T_0$ and form L^∞ estimate on $[0, T_0]$, we conclude (1.55) by

$$||h(t)||_\infty \leq C_{T_0}||h(nT_0)||_\infty \leq C\left\{\ ||h_0||_\infty + \sqrt{\mathcal{H}(F_0) - \mathcal{H}(\mu_E) + |M_0| + |E_0|}\ \right\}.$$

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