Vertex Models, Ising Models and Fisher Graphs

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Department of Mathematics at Brown University

PROVIDENCE, RHODE ISLAND

May 2011

This dissertation by Zhongyang Li is accepted in its present form
by Department of Mathematics as satisfying the
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Acknowledgements

I would like to thank my advisor, Richard Kenyon, for his invaluable mentoring over the past four years. He suggested these problems, and in regular meetings, provided ideas and guidance that ultimately led to their completion.

Thanks also to Kavita Ramanan, and David Wilson for serving on the defense and reading committees.

I would also like to thank David Wilson, Béatrice de Tilière and Sunil Chhita for comments on the early drafts of this work.

Abstract of "Vertex Models, Ising Models and Fisher Graphs " by Zhongyang Li, Ph.D., Brown University, May 2011

In Chapter 1, we study planar "vertex" models, which are probability measures on edge subsets of a planar graph, satisfying certain constraints at each vertex, examples including the dimer model, and the 1-2 model, which we will define. We express the local statistics of a large class of vertex models on a finite hexagonal lattice as a linear combination of the local statistics of dimers on the corresponding Fisher graph, with the help of a generalized holographic algorithm. Using an $n \times n$ torus to approximate the periodic infinite graph, we give an explicit integral formula for the free energy and local statistics for configurations of the vertex model on an infinite bi-periodic graph. As an example, we simulate the 1-2 model by the technique of Glauber dynamics.

In Chapter 2, we study the spectral curves of dimer models on periodic Fisher graphs, defined by the zero locus of the determinant of a modified weighted adjacency matrix. We prove that either they are disjoint from the unit torus ($\mathbb{T}^2 = \{(z, w) : |z| = 1, |w| = 1\}$) or they intersect $\mathbb{T}^2$ at a single real point. As an application, we prove that the single edge probability of dimer models on periodic Fisher graphs is unique under any translation invariant Gibbs measure.

A periodic Ising model is one endowed with interactions that are invariant under translations of members of a full-rank sublattice $\mathfrak{L}$ of $\mathbb{Z}^2$. In Chapter 3, we give an exact, quantitative description of the critical temperature, defined by the supremum of the temperatures at which the spontaneous magnetization of a periodic, Ising ferromagnets is nonzero, as the solution of a certain algebraic equation, namely, the condition that the spectral curve of the corresponding dimer model on the Fisher graph has a real node on the unit torus. A simple proof for the exponential decay of spin-spin correlations above the critical temperature for the symmetric, periodic Ising ferromagnet, as well as the exponential decay of the edge-edge correlations for all non-critical edge weights of the dimer model on periodic Fisher graphs, is obtained by our technique.

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Chapter One

Local Statistics of Realizable Vertex Models

1.1 Introduction

A **vertex model** is a graph $G = (V, E)$ where we associate to each vertex $v \in V$ a **signature** r_v. A **local configuration** at a vertex v is a subset of incident edges of v. A **configuration** of the graph G is an edge subset of G. The signature r_v at a vertex v is a function which assigns a nonnegative real number (**weight**) to each local configuration at v. The **partition function** of the vertex model is the weighted sum of configurations $X \in \{0, 1\}^E$, where the weight of a configuration is the product of weights of local configurations, obtained by restricting the configuration at each vertex. Dimers, loop models, and random tiling models are some special examples of vertex models.

Direct computations of the partition function of a general vertex model usually require exponential time. On the other hand, using the Fisher-Kasteleyn-Temperley method [13, 14], we can efficiently count the number of perfect matchings (dimer configurations) of a finite planar graph. The idea of **generalized holographic reduction** is to reduce a vertex model on a planar graph to a dimer model on another planar graph, essentially by a linear base change, see section 3. For an original version of the holographic reduction (Valiant's Algorithm), see [29].

However, not all satisfying assignment problems can be reduced to a perfect matching problem ("realized") using the holographic algorithm. We study the realizability problem of the generalized algorithm for vertex models on the hexagonal lattice and prove that the signature of realizable models form a submanifold of positive codimension of the manifold of all signatures, see Theorem 3.6. An example of realizable models is the **1-2 model**, which is a signature on the honeycomb lattice, only one or two edges allowed to be present in each local configuration, see Figure

13, 14. The realizability problem of Valiant's algorithm is studied by Cai [4], and the realizability problem of **uniform 1-2 model** (not-all-equal relation), a special 1-2 model which assigns all the configurations weight 1, under Valiant's algorithm is studied by Schwartz and Bruck [28].

Realizable vertex models may be reduced to dimer models in more than one way, that is, using different bases. However, all the dimer models corresponding to the same vertex model are shown to be **gauge equivalent**, i.e. obtained from one another by a trivial reweighting.

One of the simplest vertex configuration models is a graph with the same signature at all vertices. Using the singular value decomposition, we prove that such models on a hexagonal lattice are realizable if and only if they are realizable under orthogonal base change. Moreover, the orthogonal realizability condition takes a very nice form; see section 3.2.

We compute the local statistics of realizable vertex models on a hexagonal lattice with the help of the generalized holographic reduction, i.e. for the natural probability measure, we compute the probabilities of given configurations at finitely many fixed vertices, which are proved to be computable by sums of finitely many Pfaffians, see Theorem 5.1 and Theorem 5.2.

The weak limit of probability measures of the vertex model on finite graphs are of considerable interest. Using an $n \times n$ torus to approximate the infinite periodic graph, we give an explicit integral formula for the probability of a specific local configuration at a fixed vertex, see section 6. These results follow from a study of the zeros of the **characteristic polynomial**, or the **spectral curve**, on the unit torus $\mathbb{T}^2$. For a more general result about the intersection of the spectral curve with

$\mathbb{T}^2$, see [24]. For example, using our method, we compute the probability that a $\{001\}$ dimer occurs and for uniform 1-2 model, and the probability that a $\{011\}$ configuration occurs at a vertex for critical 1-2 model, see Examples 7.2 and 7.3.

The main result of this paper can be stated in the following theorems

Theorem 1.1. *For a periodic, realizable, positive-weight vertex model on a hexagonal lattice G with period 1×1, assume the corresponding Fisher graph has positive edge weights, then the free energy of G is*

$$F := \lim_{n\to\infty} \frac{1}{n^2} \log S(G_n) = \frac{1}{8\pi^2} \iint_{|z|=1,|w|=1} \log P(z,w) \frac{dz}{iz}\frac{dw}{iw}$$

where G_n is the quotient graph $G/(n\mathbb{Z}\times n\mathbb{Z})$, $P(z,w)$ is the characteristic polynomial.

Theorem 1.2. *Assume the periodic vertex model on hexagonal lattice is realizable to the dimer model on a Fisher graph with positive, periodic edge weights, and assume the entries of the corresponding base change matrices are nonzero. Let λ_n be the probability measure defined for configurations on toroidal hexagonal lattice G_n. Then for a configuration c at a vertices v*

$$\lim_{n\to\infty} \lambda_n(c,v) = \sum_{d_j} [\prod_{i=1}^{p} w_{d_j}] |\mathrm{Pf}(K_\infty^{-1})_{V(d_j)}|$$

where

$$K_\infty^{-1}((u,x_1,y_1),(v,x_2,y_2)) = \quad \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} z^{x_1-x_2} w^{y_1-y_2} \frac{\mathrm{Cof}(K(z,w))_{u,v}}{P(z,w)} \frac{dz}{iz}\frac{dw}{iw}$$

d_j are local dimer configurations on the gadget of the Fisher graph corresponding to v. $V(d_j)$ is the set of vertices involved in the configuration d_j.

1.2 Background

1.2.1 Vertex Models

Let $\{0,1\}^k$ denote the set of all binary sequences of length k. A **vertex model** is a graph $G=(V,E)$ where we associate to each vertex $v \in V$ a function

$$r_v : \{0,1\}^{deg(v)} \to \mathbb{R}^+$$

r_v is called the **signature** of the vertex model at vertex v. We give a linear ordering on the edges adjacent to v, and we fix such an ordering around each vertex once and for all. This way the binary sequences of length $\deg(v)$ are in one-to-one correspondence with the local configurations at v. Each edge corresponds to a digit; if that edge is included in the configuration, the corresponding digit is 1, otherwise the corresponding digit is 0. Hence we can also consider r_v as a column vector indexed by local configurations at v:

$$r_v = \begin{pmatrix} r_v(0...00) \\ r_v(0...01) \\ r_v(0...10) \\ ... \\ r_v(1...11) \end{pmatrix}$$

Example 1.3 (signature of the vertex model at a vertex)**.** *Assume we have a degree-2* *vertex with signature*

$$r_v = \begin{pmatrix} r_v(00) \\ r_v(01) \\ r_v(10) \\ r_v(11) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

that means we give weights to the four different local configurations as in Figure 1:

Figure 1.1: Relation at a Vertex

Assume G is a finite graph. We define a probability measure with sample space the set of all configurations, $\Omega = \{0, 1\}^{|E|}$. The probability of a specific configuration $\mathcal{R}$ is

$$\lambda(\mathcal{R}) = \frac{1}{S} \prod_{v \in V} r_v(\mathcal{R}). \tag{1.1}$$

The product is over all vertices. $r_v(\mathcal{R})$ is the weight of the local configuration obtained by restricting $\mathcal{R}$ to the vertex v, and S is a normalizing constant called the **partition function** for vertex models, defined to be

$$S = \sum_{\mathcal{R} \in \Omega} \prod_{v \in V} r_v(\mathcal{R}).$$

The sum is over all possible configurations of G.

Now we consider a vertex model on a $\mathbb{Z}^2$-periodic planar graph G. By this we mean that G is embedded in the plane so that translations in $\mathbb{Z}^2$ act by signature-preserving isomorphisms of G. Examples of such graphs are the square and Fisher lattices, as shown in Figure 7. Let G_n be the quotient of G by the action of $n\mathbb{Z}^2$. It is a finite graph embedded into a torus. Let S_n be the partition function of the vertex model on G_n. The **free energy** of the infinite periodic vertex model G is defined to be

$$F := \lim_{n\to\infty} \frac{1}{n^2} \log S_n$$

1.2.2 Matchgates, Matchgrids

A **matchgate** Γ is a planar local graph including a set X of external vertices, i.e. vertices located along the boundary of the local graph. The external vertices are ordered anti-clockwise on the boundary. Γ is called an odd(even) matchgate if it has an odd(even) number of vertices in total(counting both internal and external vertices). For example, the matchgate in Example 2.2 is an even matchgate.

The **signature of the matchgate** is a vector indexed by subsets of external vertices, $\{0,1\}^{|X|}$. For a subset $X_0 \subset X$, the entry of the signature at X_0 is the partition function of dimer configurations on a subgraph of the matchgate. The subgraph is obtained from the matchgate by removing all the external vertices in X_0.

Example 1.4 (signature of a matchgate)**.** *Assume we have a matchgate with external vertices 1, 2, 3, and edge weights as illustrated in the following figure:*

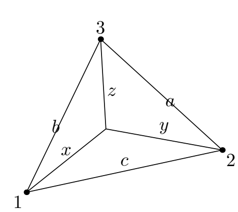

Figure 1.2: Matchgate

then the signature of the matchgate is

$$\mathfrak{m}\begin{pmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{pmatrix} = \begin{pmatrix} ax+by+cz \\ 0 \\ 0 \\ x \\ 0 \\ y \\ z \\ 0 \end{pmatrix}$$

A **matchgrid** M is a weighted planar graph consisting of a collection of matchgates and connecting edges. Each connecting edge has weight 1 and joins an external vertex of a matchgate with an external vertex of another matchgate, so that every external vertex is incident to exactly one connecting edge.

1.3 Generalized Holographic Reduction

In this section we introduce a generalized holographic algorithm to compute the partition function of the vertex model on a finite planar graph in terms of the partition function for perfect matchings on a matchgrid. The idea is using a matchgate to replace each vertex, and perform a change of basis, such that after the base change process, the signature of a vertex becomes the signature of the corresponding matchgate. We describe the algorithm in detail as follows.

For a finite graph G, we associate to each oriented edge e a 2-dimensional vector space V_e. To the edge with the reversed orientation, the associated vector space is the dual space, i.e. $V_{-e} = V_e^*$. Give a set basis $\{f_e^0, f_e^1\}$ for each V_e, satisfying

$$f_{-e}^j(f_e^i) = \delta_{ij}.$$

Let v be a vertex with incident edges $e_{l_1}, ..., e_{l_k}$, oriented away from v. The signature of the vertex model at a vertex v, r_v, can be considered as an element in $W_v = V_{e_{l_1}} \otimes ... \otimes V_{e_{l_k}}$. Hence r_v has representations under bases $F = \{f_e^0, f_e^1\}_{e\in E}$ and $B = \{b_e^0, b_e^1\}_{e\in E}$ as follows

$$r_v = \sum_{c_{l_1},...,c_{l_k}} r_v(c_{l_1} \cdots c_{l_k}) b_{e_{l_1}}^{c_{l_1}} \otimes \cdots \otimes b_{e_{l_k}}^{c_{l_k}} \tag{1.2}$$

$$= \sum_{c_{l_1},...,c_{l_k}} r_{v,f}(c_{l_1} \cdots c_{l_k}) f_{e_{l_1}}^{c_{l_1}} \otimes \cdots \otimes f_{e_{l_k}}^{c_{l_k}} \tag{1.3}$$

where B are the set of standard bases for each V_e

$$b_e^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad b_e^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$c_{l_i} \in \{0, 1\}$. $c_{l_1} \cdots c_{l_k}$ are binary sequences of length k. From the definition of the signature of vertex models, $r_v(c_{l_1} \cdots c_{l_k})$ is the weight of the configuration $c_{l_1} \cdots c_{l_k}$ at vertex v.

We construct a matchgrid M as follows. We replace each vertex v by a matchgate $\mathcal{D}_v$, such that the number of external vertices of $\mathcal{D}_v$ is the same as the degree of v, and the edges of the vertex model graph G become connecting edges joining different matchgates in the matchgrid M. Examples of such replacements are illustrated in the following Figure.

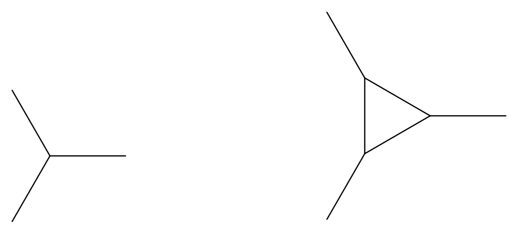

Figure 1.3: degree-3 vertex and matchgate

Figure 1.4: degree-4 vertex and matchgate

If the signature m_v of the matchgate $\mathcal{D}_v$ satisfies

$$m_v = \sum_{c_{l_1} \cdots c_{l_k}} r_{v,f}(c_{l_1} \cdots c_{l_k}) b_{e_{l_1}}^{c_{l_1}} \otimes \cdots \otimes b_{e_{l_k}}^{c_{l_k}}, \tag{1.4}$$

that is, the representation of m_v under bases B is the same as the representation of r_v under bases F, then we have the following theorem:

Theorem 1.5. *Under the above base change process, the partition function of the vertex model of G is equal to the partition function of the dimer model of M.*

Proof. There is a natural mapping Φ from $\otimes_{v\in V} W_v$ to $\mathbb{C}$ induced by $\otimes_{e\in E}\phi_e$, where ϕ_e is the natural pairing from $V_e \otimes V_e^*$ to $\mathbb{C}$. Note that in $\otimes_{v\in V} W_v$, each V_e and V_{-e} appear exactly once. Since the representation of m_v under bases B is the same as the representation of r_v under bases F, we have

$$\Phi(\otimes_{v\in V} m_v) = \Phi(\otimes_{v\in V} r_v). \tag{1.5}$$

(1.5) follows from the fact that each $\phi_e : V_e \otimes V_e^* \to \mathbb{C}$ is independent of bases as long as we choose the dual basis for the dual vector space. However, the left side of (1.5) is exactly the partition function of the dimer model of M, while the right side of (1.5) is exactly the partition function of the vertex model of G. $\square$

Define the **base change matrix at edge** e, $T_e = \begin{pmatrix} f_e^0 & f_e^1 \end{pmatrix}$. The **base change matrix at vertex** v, T_v, acting on W_v by multiplication, is defined to be

$$T_v = \otimes_{\{e|e \text{ is incident to } v, \text{ and oriented away from } v\}} T_e.$$

In order for a vertex model problem to be reduced to dimer model problem, one sufficient condition is that at each vertex, the signature of the vertex model r_v under the bases F is the same as the signature of the matchgate m_v under the standard bases. Namely,

$$T_v m_v = r_v. \tag{1.6}$$

(1.6) follows directly from (1.3) and (1.4).

Example 1.6. *Consider the graph in Figure 5 with standard dimer signature*

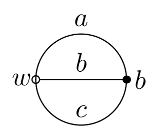

Figure 1.5: Graph of Example 1.6

$$r_w = r_b \;=\; \begin{pmatrix} r_{000} & r_{001} & r_{010} & r_{011} & r_{100} & r_{101} & r_{110} & r_{111} \end{pmatrix}^t$$

$$= \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^t$$

Define the base change matrix on edges a, b, c to be

$$T_a = T_b = T_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

By definition, $T_w = T_a \otimes T_b \otimes T_c$. Note that $T_w = (T_w^t)^{-1} = T_b$. After the base change, we have the standard loop signature

$$\tilde{r}_w = \tilde{r}_b = T_w \cdot r_w = T_b \cdot r_b$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}^t$$

For instance, after the base change, the dimer configuration 001 *with only c-edge* *occupied becomes*

$$T_w \cdot (b_0 \otimes b_0 \otimes b_1) \\ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

which is the configuration 110*, the loop configuration with* a*-edge and* b*-edge occupied.*

However, since the number of vertices in a matchgate is either even or odd, at a vertex of degree d, the signature of a matchgate must be a 2^{d-1} dimensional subspace of $\mathbb{C}^{2^d}$, with those 2^{d-1} entries being 0. These are the entries correspond to the partition function of dimer configurations on a subgraph of the matchgate with an odd number of vertices, see example 2.1. This is the **parity constraint**. As a result, by dimension count we can see that it is not possible to use holographic algorithm to reduce all vertex models into dimer models. To characterize the special class of vertex models applicable to holographic reduction, we introduce the following definition.

Definition 1.7. *A network of relations on a finite graph is **realizable**, if there exists a system of bases reducing the model to the set of perfect matchings of a matchgrid.*

Definition 1.8. *A network of relations is **bipartite realizable**, if it is realizable and the corresponding matchgrid is a bipartite graph.*

Remark. The above generalizes Valiant's algorithm [29] in the sense that our basis can be different from edge to edge. As a consequence, our approach results in an enlargement of the dimension of realizable submanifold, which will be shown in the next section.

1.3.1 Realizability

We are interested in periodic vertex models on the honeycomb lattice with period $n \times n$. The quotient graph can be embedded on a torus $\mathbb{T}^2 = S^1 \times S^1$. We classify all the edges into a-type, b-type and c-type according to their direction, and assume b-type and c-type edges have the same direction as the two basic homology cycles $(1, 0)$ and $(0, 1)$ of torus, respectively, see Figure 6.

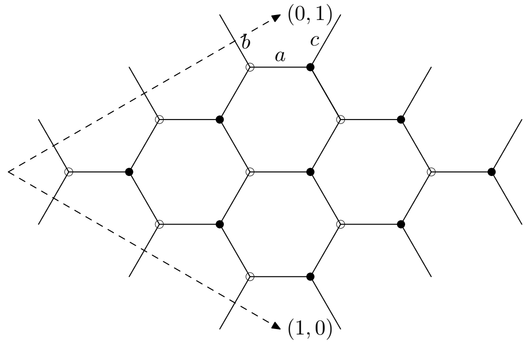

Figure 1.6: Periodic Honeycomb Lattice

Assume the vertex model is realizable, then each corresponding matchgate may have either an even or an odd number of vertices. By enlarging the fundamental domain, we can always assume that there are an even number of matchgates with an even number of vertices. Then by permuting rows of matrices on a finite number of edges, we can always have all the matchgates having an odd number of vertices. For example, assume we have a pair of adjacent matchgates with base change matrix on the connecting edge e, $T_e = \begin{pmatrix} f_0 & f_1 \end{pmatrix}$, as illustrated in Figure 7. Then if we

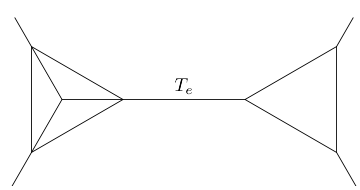

Figure 1.7: Permutation of Basis Vectors 1

assume $\tilde{T}_e = \begin{pmatrix} f_1 & f_0 \end{pmatrix}$ obtained by permutating two basis vectors of T_e, we actually interchange the roles of 0 and 1 at the corresponding digit of the binary sequences as indices of signatures of the matchgate. Namely, for given vertex signature r_v, assume $T_v m_v = r_v$, and $\tilde{T}_v \tilde{m}_v = r_v$, where $\tilde{T}_v = \tilde{T}_a \otimes T_b \otimes T_c$. Then for any binary sequence $c_1 c_2 c_3$, the entry $m_{v\{c_1 c_2 c_3\}}$ is the same as $\tilde{m}_{v\{(1-c_1)c_2 c_3\}}$, according to equation (1.6). If originally we have an even matchgate at v, by parity constraint, the $001, 010, 100, 111$ entries of m_v are zero. After the permutation of basis vectors, $\tilde{m}_v$ will have $101, 110, 000, 011$ entries to be zero. Hence $\tilde{m}_v$ has to be an odd matchgate, as illustrate in Figure 8. By finitely many times of such permutations, we can move

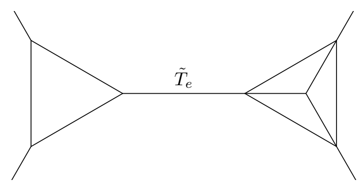

Figure 1.8: Permutation of Basis Vectors 2

two even matchgates to adjacent positions. Then we permutate the basis on the connecting edge, we can decrease the number of even matchgates by 2.

Since we are considering a vertex model with periodic signatures, one period of which can be embedded into a torus; if for one period, we have an odd number of vertices whose signatures are realizable to an odd matchgate, then after enlarging the fundamental domain by 2, we can always have an even number of even matchgates in the bigger torus. Hence if we repeat the process described above, in the end, all matchgates will be odd.

Assumption 1.9. *From now on, we make the following assumption:*

- *All entries of base change matrices are nonzero.*
- *All entries of the matchgate signature are nonzero.*

Since the honeycomb lattice is a bipartite graph, we can color all the vertices in black and white such that black vertices are adjacent only to white vertices and vice versa. Assume vertex signatures at black and white vertices are as follows:

$$r_w^{ij} = \begin{pmatrix} x_{000}^{ij} & x_{001}^{ij} & x_{010}^{ij} & x_{011}^{ij} & x_{100}^{ij} & x_{101}^{ij} & x_{110}^{ij} & x_{111}^{ij} \end{pmatrix}^t \tag{1.7}$$

$$r_b^{ij} = \begin{pmatrix} y_{000}^{ij} & y_{001}^{ij} & y_{010}^{ij} & y_{011}^{ij} & y_{100}^{ij} & y_{101}^{ij} & y_{110}^{ij} & y_{111}^{ij} \end{pmatrix}^t \tag{1.8}$$

where $(i,j), 1 \le i \le n, 1 \le j \le n$ is the row and column index of white and black vertices. Assume $000 = 1$, then entries of signatures are indexed from 1 to 8. Associate to an a-edge, b-edge and c-edge incident to a white or black vertices, a basis

$$T_a^{(i,j,k)} = \begin{pmatrix} n_{0a}^{(i,j,k)} & p_{0a}^{(i,j,k)} \\ n_{1a}^{(i,j,k)} & p_{1a}^{(i,j,k)} \end{pmatrix} = \begin{pmatrix} \mathbf{n}_a^{(i,j,k)} & \mathbf{p}_a^{(i,j,k)} \end{pmatrix}$$

$$T_b^{(i,j,k)} = \begin{pmatrix} n_{0b}^{(i,j,k)} & p_{0b}^{(i,j,k)} \\ n_{1b}^{(i,j,k)} & p_{1b}^{(i,j,k)} \end{pmatrix} = \begin{pmatrix} \mathbf{n}_b^{(i,j,k)} & \mathbf{p}_b^{(i,j,k)} \end{pmatrix}$$

$$T_c^{(i,j,k)} = \begin{pmatrix} n_{0c}^{(i,j,k)} & p_{0c}^{(i,j,k)} \\ n_{1c}^{(i,j,k)} & p_{1c}^{(i,j,k)} \end{pmatrix} = \begin{pmatrix} \mathbf{n}_c^{(i,j,k)} & \mathbf{p}_c^{(i,j,k)} \end{pmatrix}$$

where $k = 1$ for white vertices and $k = 0$ for black vertices.

By the realizability equation (1.6), and the fact that all matchgates are odd, the 1st, 4th, 6th, 7th entries of the matchgate signatures m_v are 0, so we have a system of 4 algebraic equations at each vertex v. For example, at each black vertex, we have

$$m_{i,j,0} = T_a^{i,j,0} \otimes T_b^{i,j,0} \otimes T_c^{i,j,0} \cdot r_b^{ij} \tag{1.9}$$

and the fact that the 1st entry of $m_{i,j,0}$ is 0 gives the following equation

$$n_{0a}n_{0b}n_{0c}y_1 + n_{0a}n_{0b}p_{0c}y_2 + n_{0a}p_{0b}n_{0c}y_3 + n_{0a}p_{0b}p_{0c}y_4$$
$$+p_{0a}n_{0b}n_{0c}y_5 + p_{0a}n_{0b}p_{0c}y_6 + p_{0a}p_{0b}n_{0c}y_7 + p_{0a}p_{0b}p_{0c}y_8 = 0$$

That the 4th, 6th, 7th entries of $m_{i,j,0}$ are 0 give three other similar equations. Similarly at each white vertex, we have

$$m_{i,j,1} = [(T_a^{i,j,1} \otimes T_b^{i,j,1} \otimes T_c^{i,j,1})^t]^{-1} \cdot r_w^{ij} \tag{1.10}$$

the same process gives a system of 4 equations at the white vertex.

Let $\alpha_{lm}^{ij} = \frac{n_{lm}^{(i,j,1)}}{p_{lm}^{(i,j,1)}}$,$\beta_{lm}^{ij} = \frac{n_{lm}^{(i,j,0)}}{p_{lm}^{(i,j,0)}}$, $l \in \{0,1\}$,$m \in \{a,b,c\}$. Then the equations we get are linear with respect to α_{0a}^{ij}, α_{1a}^{ij}, β_{0a}^{ij}, β_{1a}^{ij}, which can be solved explicitly.

$$\alpha_{0a}^{ij} = \frac{r^{ij} \cdot u^{ij}}{r^{ij} \cdot v^{ij}} = \frac{q^{ij} \cdot u^{ij}}{q^{ij} \cdot v^{ij}} \tag{1.11}$$

$$\alpha_{1a}^{ij} = \frac{s^{ij} \cdot u^{ij}}{s^{ij} \cdot v^{ij}} = \frac{p^{ij} \cdot u^{ij}}{p^{ij} \cdot v^{ij}} \tag{1.12}$$

$$\beta_{0a}^{ij} = -\frac{\xi^{ij} \cdot t^{ij}}{\xi^{ij} \cdot w^{ij}} = -\frac{\kappa^{ij} \cdot t^{ij}}{\kappa^{ij} \cdot w^{ij}} \tag{1.13}$$

$$\beta_{1a}^{ij} = -\frac{\sigma^{ij} \cdot t^{ij}}{\sigma^{ij} \cdot w^{ij}} = -\frac{\lambda^{ij} \cdot t^{ij}}{\lambda^{ij} \cdot w^{ij}} \tag{1.14}$$

where

$$\begin{cases} p^{ij} = \begin{pmatrix} \alpha_{0b}^{ij}\alpha_{0c}^{ij} & \alpha_{0b}^{ij} & \alpha_{0c}^{ij} & 1 \end{pmatrix} & q^{ij} = \begin{pmatrix} \alpha_{0b}^{ij}\alpha_{1c}^{ij} & \alpha_{0b}^{ij} & \alpha_{1c}^{ij} & 1 \end{pmatrix} \\ r^{ij} = \begin{pmatrix} \alpha_{1b}^{ij}\alpha_{0c}^{ij} & \alpha_{1b}^{ij} & \alpha_{0c}^{ij} & 1 \end{pmatrix} & s^{ij} = \begin{pmatrix} \alpha_{1b}^{ij}\alpha_{1c}^{ij} & \alpha_{1b}^{ij} & \alpha_{1c}^{ij} & 1 \end{pmatrix} \\ \xi^{ij} = \begin{pmatrix} \beta_{0b}^{ij}\beta_{0c}^{ij} & \beta_{0b}^{ij} & \beta_{0c}^{ij} & 1 \end{pmatrix} & \sigma^{ij} = \begin{pmatrix} \beta_{0b}^{ij}\beta_{1c}^{ij} & \beta_{0b}^{ij} & \beta_{1c}^{ij} & 1 \end{pmatrix} \\ \lambda^{ij} = \begin{pmatrix} \beta_{1b}^{ij}\beta_{0c}^{ij} & \beta_{1b}^{ij} & \beta_{0c}^{ij} & 1 \end{pmatrix} & \kappa^{ij} = \begin{pmatrix} \beta_{1b}^{ij}\beta_{1c}^{ij} & \beta_{1b}^{ij} & \beta_{1c}^{ij} & 1 \end{pmatrix} \\ u^{ij} = \begin{pmatrix} x_4^{ij} & -x_3^{ij} & -x_2^{ij} & x_1^{ij} \end{pmatrix} & v^{ij} = \begin{pmatrix} x_8^{ij} & -x_7^{ij} & -x_6^{ij} & x_5^{ij} \end{pmatrix} \\ w^{ij} = \begin{pmatrix} y_1^{ij} & y_2^{ij} & y_3^{ij} & y_4^{ij} \end{pmatrix} & t^{ij} = \begin{pmatrix} y_5^{ij} & y_6^{ij} & y_7^{ij} & y_8^{ij} \end{pmatrix} \end{cases} \tag{1.15}$$

Since $\alpha_{0m} \neq \alpha_{1m}$ and $\beta_{0m} \neq \beta_{1m}$ as a consequence of the invertibility of the base change matrices, after clearing denominators and some reducing, for any (i, j), equations (1.11)-(1.14) are equivalent to

$$2y_2y_8 - 2y_6y_4 + (\beta_{1c} + \beta_{0c})(y_1y_8 + y_2y_7 - y_5y_4 - y_6y_3) + 2(y_1y_7 - y_5y_3)\beta_{1c}\beta_{0c} = 0 \tag{1.16}$$

$$-2x_1x_7 + 2x_5x_3 + (\alpha_{1c} + \alpha_{0c})(x_2x_7 + x_1x_8 - x_4x_5 - x_6x_3) + 2(x_6x_4 - x_2x_8)\alpha_{1c}\alpha_{0c} = 0 \tag{1.17}$$

$$2y_8y_3 - 2y_4y_7 + (\beta_{1b} + \beta_{0b})(y_1y_8 + y_3y_6 - y_4y_5 - y_7y_2) + 2\beta_{1b}\beta_{0b}(y_6y_1 - y_2y_5) = 0 \tag{1.18}$$

$$2x_5x_2 - 2x_1x_6 + (x_1x_8 - x_2x_7 - x_5x_4 + x_3x_6)(\alpha_{1b} + \alpha_{0b}) + 2(x_7x_4 - x_3x_8)\alpha_{0b}\alpha_{1b} = 0 \tag{1.19}$$

If we solve $\alpha_{0b}, \alpha_{1b}, \beta_{0b}, \beta_{1b}$ explicitly, a similar process yields

$$2x_3x_2 - 2x_1x_4 + (x_5x_4 + x_1x_8 - x_2x_7 - x_3x_6)(\alpha_{0a} + \alpha_{1a}) + 2(x_6x_7 - x_5x_8)\alpha_{0a}\alpha_{1a} = 0 \tag{1.20}$$

$$2y_6y_7 - 2y_5y_8 + (y_3y_6 + y_2y_7 - y_1y_8 - y_4y_5)(\beta_{0a} + \beta_{1a}) + 2(y_2y_3 - y_1y_4)\beta_{0a}\beta_{1a} = 0 \tag{1.21}$$

We get two equations per edge, one involving the a-variables, the other involving the b-variables. But a-variables and b-variables are actually the same thing for each single edge. From equation (1.16),(1.17), we can solve basis entries $\alpha_{lc}^{i,j} = \beta_{lc}^{i,j-1}$; from equation (1.18),(1.19), we can solve basis entries $\alpha_{lb}^{i,j} = \beta_{1b}^{i-1,j}$. Finally from (1.11)-(1.14), we can solve α_{la} and β_{la}, the only constraint left will be $\alpha_{la}^{i,j} = \beta_{la}^{i,j}$, which is two polynomial equations with respect to relation signature at each a-edge. Together with Theorem 2 in appendix, we have the following theorem

Theorem 1.10. *Under assumption 3.5, the realizable signatures on the $n \times n$ periodic honeycomb lattice form a $14n^2$ dimensional submanifold of the $16n^2$ dimensional manifold of all positive signatures; the bipartite realizable signatures on the $n \times n$ periodic honeycomb lattice form a $12n^2$ dimensional submanifold of the $16n^2$ dimensional manifold of all positive signatures.*

For the exact realizability condition, see the appendix.

Under assumption 3.5, the weight $\{111\}$ is non-vanishing at each matchgate. Since the probability measure will not change if all entries of the signature at one vertex are multiplied by a constant, we can assume at each matchgate, the weight of $\{111\}$ is 1. Therefore, we have

Theorem 1.11. *A realizable vertex model on a finite hexagonal lattice G can always be transformed to dimers on M, a Fisher graph as shown in Figure 9, with the partition function of the vertex model of G equal to the partition function of the dimer model of M, up to multiplication of a constant.*

It is possible to construct a matchgrid with different weights to produce the same vertex model. Since the holographic reduction is an invertible process, by which we mean that because the base change matrices are nonsingular, we can always achieve the matchgate signature from the vertex signature and vice versa, there is an equivalence relation on dimer models producing the same vertex model.

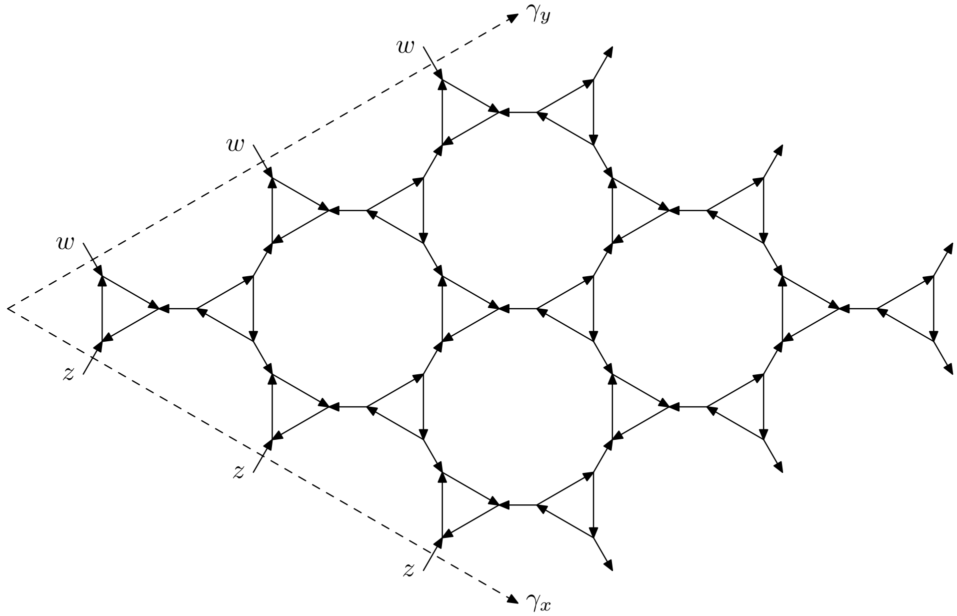

Figure 1.9: Matchgrid with 3× 3 Fundamental Domain

Definition 1.12. *A vertex model is holographically equivalent to a dimer model on a matchgrid, if it can be reduced to the dimer model on the matchgrid using the holographic algorithm, in such a way that partition function of the vertex model corresponds to perfect matchings of the matchgrid. Two matchgrids are holographically equivalent, if they are holographically equivalent to the same vertex model.*

Proposition 1.13. *Holographically equivalent matchgrids give rise to gauge equivalent dimer models. Therefore, they have the same probability measure.*

Proof. See the appendix. □

1.4 Orthogonal Realizability

Definition 1.14. *A vertex model is orthogonally realizable if it is realizable by an orthonormal base change matrix on each edge.*

Consider a single vertex. To the incident edges of the vertex, associate matrices U_1, U_2, U_3. Without loss of generality, we can assume $U_1, U_2, U_3 \in SO(2)$. In fact, if $\det U_i = -1$, we multiply the first row of U_i by -1. The new signature of the matchgate will be multiplied by -1. This change does not violate the parity constraint.

Assume $U_i = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}$. Assume $U = \otimes_{i=1}^{3} U_i$, then $U \in SO(8)$, each term of which is product of three of $\cos\alpha_i, \sin\alpha_i$. Moreover, the eigenvalues are $e^{(\pm\alpha_1\pm\alpha_2\pm\alpha_3)\sqrt{-1}}$. Using trigonometric identities, each entry is a linear combination of $\cos(\pm\alpha_1 \pm \alpha_2 \pm \alpha_3)$ and $\sin(\pm\alpha_1 \pm \alpha_2 \pm \alpha_3)$. If we further define

$$g = \begin{pmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & g_8 \end{pmatrix}'$$

$$h = \begin{pmatrix} 0 & h_2 & h_3 & 0 & h_5 & 0 & 0 & h_8 \end{pmatrix}'$$

$$\gamma = \alpha_1 + \alpha_2 - \alpha_3;\ \psi = \alpha_1 + \alpha_3 - \alpha_2;\ \varphi = \alpha_2 + \alpha_3 - \alpha_1$$

$$j_1 = g_4 + g_6 + g_7 - g_1;\ j_2 = g_1 + g_6 + g_7 - g_4;$$

$$j_3 = g_1 + g_4 + g_7 - g_6;\ j_4 = g_1 + g_4 + g_6 - g_7;$$

$$j_5 = g_3 + g_5 + g_8 - g_2;\ j_6 = g_2 + g_5 + g_8 - g_3;$$

$$j_7 = g_2 + g_3 + g_8 - g_5;\ j_8 = g_2 + g_3 + g_5 - g_8$$

$$P = j_2\cos(\varphi) + j_7\sin(\varphi);\ \ Q = j_3\cos(\psi) + j_6\sin(\psi);\ \ R = j_4\cos(\gamma) + j_5\sin(\gamma)$$
$$K = j_1\cos(\varphi+\psi+\gamma) - j_8\sin(\varphi+\psi+\gamma)$$
$$H = j_7\cos(\varphi) - j_2\sin(\varphi);\ \ L = j_6\cos(\psi) - j_3\sin(\psi);\ \ M = j_5\cos(\gamma) - j_4\sin(\gamma)$$
$$N = j_8\cos(\varphi+\psi+\gamma) + j_1\sin(\varphi+\psi+\gamma)$$

then if $Ug = h$, we have

$$0 = P = Q = R = K \tag{1.22}$$
$$4h_2 = H + L - M + N \tag{1.23}$$
$$4h_3 = H - L + M + N \tag{1.24}$$
$$4h_5 = N + M + L - H \tag{1.25}$$
$$4h_8 = H + L + M - N \tag{1.26}$$

By (1.22), we have $\tan\varphi = -\frac{j_2}{j_7}$,$\tan\psi = -\frac{j_3}{j_6}$,$\tan\gamma = -\frac{j_4}{j_5}$,$\tan(\varphi+\psi+\gamma) = \frac{j_1}{j_8}$. If we define

$$\begin{aligned} t(u,v) &= \frac{u+v}{1-uv} \\ tt(a,b,c,d) &= \frac{a+b+c+d-abc-abd-acd-bcd}{1-ab-ac-ad-bc-bd-cd+abcd} \end{aligned}$$

Then the following theorem holds:

Theorem 1.15. *A vertex model on a periodic honeycomb lattice with period $n \times n$ is orthogonally realizable if and only if its signatures satisfy the following system of* *equations*

$$
\begin{aligned}
&tt(-\frac{z_7^{ijk}}{z_2^{ijk}}, -\frac{z_6^{ijk}}{z_3^{ijk}}, -\frac{z_4^{ijk}}{z_5^{ijk}}, -\frac{z_1^{ijk}}{z_8^{ijk}}) = 0 \qquad \forall i,j,k \\
&\left.\begin{aligned}
t(-\tfrac{z_6^{ij0}}{z_3^{ij0}}, -\tfrac{z_4^{ij0}}{z_5^{ij0}}) &= t(-\tfrac{z_6^{ij1}}{z_3^{ij1}}, -\tfrac{z_4^{ij1}}{z_5^{ij1}}) \\
t(-\tfrac{z_3^{ij-1,0}}{z_6^{ij-1,0}}, -\tfrac{z_2^{ij-1,0}}{z_7^{ij-1,0}}) &= t(-\tfrac{z_3^{ij1}}{z_6^{ij1}}, -\tfrac{z_2^{ij,1}}{z_7^{ij1}}) \\
t(-\tfrac{z_7^{i-1,j0}}{z_2^{i-1,j0}}, -\tfrac{z_4^{i-1,j0}}{z_5^{i-1,j0}}) &= t(-\tfrac{z_7^{ij1}}{z_2^{ij1}}, -\tfrac{z_4^{ij1}}{z_5^{ij1}})
\end{aligned}\right\} \to \forall i,j
\end{aligned}
$$

where $z_1^{ij1} = x_4 + x_6 + x_7 - x_1, z_2^{ij1} = x_3 + x_5 + x_8 - x_2, z_3^{ij1} = x_2 + x_5 + x_8 - x_3, z_4^{ij1} = x_1 + x_6 + x_7 - x_4, z_5^{ij1} = x_2 + x_3 + x_8 - x_5, z_6^{ij1} = x_1 + x_4 + x_7 - x_6, z_7^{ij1} = x_1 + x_4 + x_6 - x_7, z_8^{ij1} = x_2 + x_3 + x_5 - x_8$, *and the same relation for* z_l^{ij0} *and* $y_1, ..., y_8$. $x's$ *and* $y's$ *are defined by* (1.7) *and* (1.8).

It is trivial to verify these equations in any given situation.

Definition 1.16. *A vertex model on a periodic honeycomb lattice with period* $n \times n$ *is positively orthogonally realizable if it is orthogonal realizable and for each vertex* (i, j, k)*, there exists angles* $\varphi^{ijk}, \psi^{ijk}, \gamma^{ijk}$*, such that*

$$
\begin{aligned}
\sin\varphi &= \frac{z_4^{ijk}}{\sqrt{(z_4^{ijk})^2 + (z_5^{ijk})^2}}; \\
\sin\psi &= \frac{z_6^{ijk}}{\sqrt{(z_3^{ijk})^2 + (z_6^{ijk})^2}}; \\
\sin\gamma &= \frac{z_7^{ijk}}{\sqrt{(z_7^{ijk})^2 + (z_2^{ijk})^2}} \\
\sin(-\gamma - \varphi - \psi) &= \frac{z_1^{ijk}}{\sqrt{(z_1^{ijk})^2 + (z_8^{ijk})^2}}
\end{aligned}
$$

Proposition 1.17. *If a vertex model on a bi-periodic hexagonal lattice is orthogo-*

nally realizable, then the corresponding dimer configuration has positive edge weights.

Proof. Under the assumption that the vertex model have nonnegative signature, we have

$$z_1 \le z_4 + z_6 + z_7;\; z_4 \le z_1 + z_6 + z_7;\; z_6 \le z_1 + z_4 + z_7;\; z_7 \le z_4 + z_6 + z_7$$
$$z_2 \le z_3 + z_5 + z_8;\; z_3 \le z_2 + z_5 + z_8;\; z_5 \le z_2 + z_3 + z_8;\; z_8 \le z_2 + z_3 + z_5$$

at all vertices. Since at least three of z_1, z_4, z_6, z_7 are nonnegative, similarly for z_2, z_3, z_5, z_8, if we take absolute value for all z_i, the above inequalities also hold. therefore

$$\sqrt{z_4^2 + z_5^2} + \sqrt{z_6^2 + z_3^2} + \sqrt{z_2^2 + z_7^2} - \sqrt{z_1^2 + z_8^2} \ge 0$$
$$\sqrt{z_4^2 + z_5^2} + \sqrt{z_6^2 + z_3^2} + \sqrt{z_1^2 + z_8^2} - \sqrt{z_2^2 + z_7^2} \ge 0$$
$$\sqrt{z_4^2 + z_5^2} + \sqrt{z_1^2 + z_8^2} + \sqrt{z_2^2 + z_7^2} - \sqrt{z_3^2 + z_6^2} \ge 0$$
$$\sqrt{z_1^2 + z_8^2} + \sqrt{z_6^2 + z_3^2} + \sqrt{z_2^2 + z_7^2} - \sqrt{z_4^2 + z_5^2} \ge 0$$

Under the assumption that both the vertex model are positively orthogonal realizable, the left side of the above inequalities are exactly edge weights of corresponding matchgates. □

Theorem 1.18. *If all matchgates have the same signature, for a generic choice of signature, a vertex model on a hexagonal lattice is realizable if and only if it is orthogonal realizable.*

Proof. Obviously orthogonal realizability implies realizability, we only need to show that realizability implies orthogonal realizability.

Assume $r = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{pmatrix}^t$ is a realizable signature for the vertex model. By lemma 4.1, we can assume the corresponding bases on all edges have real entries. Consider the singular value decomposition for the base change matrix on each edge. Since $T_{-e} = (T_e^{-1})^t$, multiplying the matrices T by a scalar does not change anything, we can assume that of of the entries of the diagonal matrix is 1. Namely,

$$T_i = U_i \begin{pmatrix} 1 & 0 \\ 0 & \lambda_i \end{pmatrix} V_i \quad i = 1, 2, 3$$

where $U_i, V_i \in O(2)$, and λ_i is a nonnegative real number. Assume

$$v = (\otimes_{i=1}^3 V_i) r = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{pmatrix}^t$$

Then we can consider $(\otimes_{i=1}^3 \begin{pmatrix} 1 & 0 \\ 0 & \lambda_i \end{pmatrix})v$ and $(\otimes_{i=1}^3 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix})v$ to be signatures of black and white vertices that are orthogonal realizable. Then by the conditions of orthogonal realizability, we have

$$\frac{v_1 + \lambda_1\lambda_3 v_6 + \lambda_1\lambda_2 v_7 - \lambda_2\lambda_3 v_4}{\lambda_3 v_2 + \lambda_2 v_3 + \lambda_1\lambda_2\lambda_3 v_8 - \lambda_1 v_5} = \frac{v_1 + \frac{1}{\lambda_1\lambda_3} v_6 + \frac{1}{\lambda_1\lambda_2} v_7 - \frac{1}{\lambda_2\lambda_3} v_4}{\frac{1}{\lambda_3} v_2 + \frac{1}{\lambda_2} v_3 + \frac{1}{\lambda_1\lambda_2\lambda_3} v_8 - \frac{1}{\lambda_1} v_5} \tag{1.27}$$

$$\frac{v_1 + \lambda_1\lambda_3 v_6 - \lambda_1\lambda_2 v_7 + \lambda_2\lambda_3 v_4}{-\lambda_3 v_2 + \lambda_2 v_3 + \lambda_1\lambda_2\lambda_3 v_8 + \lambda_1 v_5} = \frac{v_1 + \frac{1}{\lambda_1\lambda_3} v_6 - \frac{1}{\lambda_1\lambda_2} v_7 + \frac{1}{\lambda_2\lambda_3} v_4}{-\frac{1}{\lambda_3} v_2 + \frac{1}{\lambda_2} v_3 + \frac{1}{\lambda_1\lambda_2\lambda_3} v_8 + \frac{1}{\lambda_1} v_5} \tag{1.28}$$

$$\frac{v_1 - \lambda_1\lambda_3 v_6 + \lambda_1\lambda_2 v_7 + \lambda_2\lambda_3 v_4}{\lambda_3 v_2 - \lambda_2 v_3 + \lambda_1\lambda_2\lambda_3 v_8 + \lambda_1 v_5} = \frac{v_1 - \frac{1}{\lambda_1\lambda_3} v_6 + \frac{1}{\lambda_1\lambda_2} v_7 + \frac{1}{\lambda_2\lambda_3} v_4}{\frac{1}{\lambda_3} v_2 - \frac{1}{\lambda_2} v_3 + \frac{1}{\lambda_1\lambda_2\lambda_3} v_8 + \frac{1}{\lambda_1} v_5} \tag{1.29}$$

$$\frac{-v_1 + \lambda_1\lambda_3 v_6 + \lambda_1\lambda_2 v_7 + \lambda_2\lambda_3 v_4}{\lambda_3 v_2 + \lambda_2 v_3 - \lambda_1\lambda_2\lambda_3 v_8 + \lambda_1 v_5} = \frac{-v_1 + \frac{1}{\lambda_1\lambda_3} v_6 + \frac{1}{\lambda_1\lambda_2} v_7 + \frac{1}{\lambda_2\lambda_3} v_4}{\frac{1}{\lambda_3} v_2 + \frac{1}{\lambda_2} v_3 - \frac{1}{\lambda_1\lambda_2\lambda_3} v_8 + \frac{1}{\lambda_1} v_5} \tag{1.30}$$

If we transform fractions into polynomials, and consider (1.27)+(1.28)−(1.29)−(1.30), (1.27)+(1.29)−(1.28)−(1.30), and (1.27)+(1.30)−(1.28)−(1.29), then

$$(\lambda_1 + 1)(\lambda_1 - 1)(v_4 v_8 + v_1 v_5 + v_3 v_7 + v_2 v_6) = 0$$

$$(\lambda_2 + 1)(\lambda_2 - 1)(v_3 v_4 + v_5 v_6 + v_1 v_2 + v_7 v_8) = 0$$

$$(\lambda_3 + 1)(\lambda_3 - 1)(v_4 v_2 + v_7 v_5 + v_3 v_1 + v_8 v_6) = 0$$

Then for a generic choice of signature, we must have $\lambda_1 = \lambda_2 = \lambda_3 = 1$. $\square$

1.5 Characteristic Polynomial

Assume all entries of the vertex signatures are strictly positive. In this section we prove some interesting properties of the characteristic polynomial.

Lemma 1.19. *For realizable vertex model with positive signature and period* 1×1, *there exists a realization of base change over* $\mathbf{GL}_2(\mathbb{R})$, *(i. e. one can take base change matrices to be real), with the property that, at each edge* e, $\alpha_{0e}\alpha_{1e} < 0$ *(*$\frac{n_{0e}n_{1e}}{p_{0e}p_{1e}} < 0$*).*

Proof. For 1×1 fundamental domain, $a_{lk} = b_{lk}$, $\forall\ l, k$. By (1.11)-(1.14), we have

$$\alpha_{0a} = -\frac{p \cdot t}{p \cdot w} = -\frac{s \cdot t}{s \cdot w} = \frac{r \cdot u}{r \cdot v} = \frac{q \cdot u}{q \cdot v}$$
$$\alpha_{1a} = -\frac{q \cdot t}{q \cdot w} = -\frac{r \cdot t}{r \cdot w} = \frac{s \cdot u}{s \cdot v} = \frac{p \cdot u}{p \cdot v},$$

From which we derive

$$(p \cdot t)(r \cdot v) + (p \cdot w)(r \cdot u) - (r \cdot t)(p \cdot v) - (r \cdot w)(p \cdot u) = 0 \tag{1.31}$$

$$(s \cdot t)(q \cdot v) + (s \cdot w)(q \cdot u) - (q \cdot t)(s \cdot v) - (q \cdot w)(s \cdot u) = 0 \tag{1.32}$$

$$(p \cdot t)(q \cdot v) + (p \cdot w)(q \cdot u) - (q \cdot t)(p \cdot v) - (q \cdot w)(p \cdot u) = 0 \tag{1.33}$$

$$(s \cdot t)(r \cdot v) + (s \cdot w)(r \cdot u) - (r \cdot t)(s \cdot v) - (r \cdot w)(s \cdot u) = 0. \tag{1.34}$$

Since each base change matrix is invertible, we have $\alpha_{0i} \neq \alpha_{1i}$. Plugging (1.15) into (1.31)-(1.34), and factor out $(\alpha_{0b} - \alpha_{1b})$, or $(\alpha_{0c} - \alpha_{1c})$, we obtain that α_{0c}, α_{1c} are two roots of quadratic polynomial

$$(x_6y_5 + y_7x_8 + y_1x_2 + y_3x_4)z^2 + (-y_7x_7 - y_3x_3 + y_2x_2 + y_4x_4 + x_6y_6 - x_5y_5 + y_8x_8 - y_1x_1)z$$
$$-y_6x_5 - x_3y_4 - y_2x_1 - x_7y_8 = 0,$$

and α_{0b}, α_{1b} are two roots of quadratic polynomial

$$(x_3y_1 + y_6x_8 + y_2x_4 + y_5x_7)z^2 + (y_8x_8 - x_6y_6 - x_5y_5 + y_7x_7 - y_2x_2 + y_3x_3 + y_4x_4 - y_1x_1)z$$
$$-y_8x_6 - y_7x_5 - x_1y_3 - y_4x_2 = 0$$

Under the assumption that x_i,y_j are positive, these polynomials have real roots, since the products of two roots are always negative. Then the lemma follows. $\square$

By definition, the characteristic polynomial $P(z, w)$ is the determinant of an 6×6 matrix $K(z, w)$ whose rows and columns are indexed by the 6 vertices in a 1×1 fundamental domain, see Figure 8:

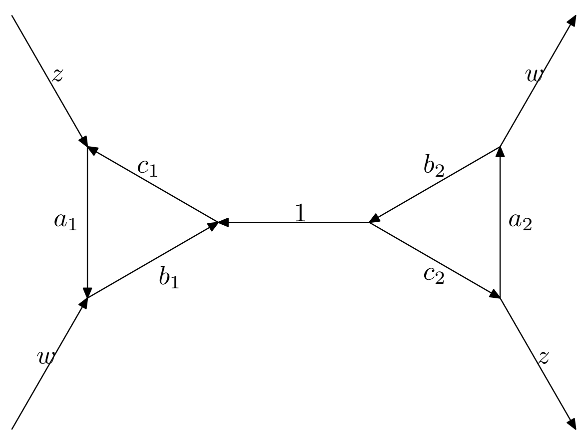

Figure 1.10: Weighted 1×1 Fundamental Domain

$$P(z,w) = \det \begin{pmatrix} 0 & c_1 & -b_1 & -1 & 0 & 0 \\ -c_1 & 0 & a_1 & 0 & -\frac{1}{z} & 0 \\ b_1 & -a_1 & 0 & 0 & 0 & -\frac{1}{w} \\ 1 & 0 & 0 & 0 & c_2 & -b_2 \\ 0 & z & 0 & -c_2 & 0 & a_2 \\ 0 & 0 & w & b_2 & -a_2 & 0 \end{pmatrix}$$
$$= (z+\frac{1}{z})(ab-c) + (w+\frac{1}{w})(ac-b) + (\frac{z}{w}+\frac{w}{z})(bc-a) + a^2 + b^2 + 1 + c^2$$

where

$$a = a_1a_2 \qquad b = b_1b_2 \qquad c = c_1c_2 \tag{1.35}$$

and we consider the non-degenerate case, which means $ac - b \neq 0$, $bc - a \neq 0$, $ab - c \neq 0$. We have the following lemma:

Lemma 1.20. *For realizable vertex model with period* 1×1, *let* a, b, c, d *be the product of dimer weights (1.35) after holographic reduction, under assumption 3.4, we have the following inequalities:*

$$(a+b-c-d)(a+c-b-d)(a+d-b-c) > 0$$
$$a+b+c+1 > 0.$$

Here $d = d_1d_2$. d_1 *and* d_2 *are original edge weights in the* $\{111\}$ *position of the white and the black matchgates obtained from equation (6). When we only care about the underlying probability measure, we can multiply all the edge weights by a constant and assume* $d = 1$.

Proof. For generic choice of vertex signature, we can assume quotients of basis entries $\alpha_{ij} = \frac{n_{ij}}{p_{ij}}$ are finite. Apply the realizability equation (1.6) to a 1×1 quotient graph, we have 8 equations for each black vertex, and 8 equations for each white vertex. At each black vertex, 4 equations has 0 on the right side, and 4 equations with a_1, b_1, c_1, d_1 on the right side. We divide the 8 equations into 4 groups, each of which has 1 equation with 0 on the right, 1 equation with a non-vanishing edge weight on the right. We take the difference of the two equations in each group, and we get 4 new equations with a non-vanishing edge weight on the right. We perform the same procedure for each white vertex. Then we multiply and add those equations correspondingly, we obtain

$$a + d - b - c = (-\alpha_{0c} + \alpha_{1c})(y_3x_4 + x_8y_7 + x_6y_5 + y_1x_2)$$

$$a + c - b - d = (\alpha_{0b} - \alpha_{1b})(y_1x_3 + y_2x_4 + y_6x_8 + y_5x_7)$$

$$a + b - c - d = (\alpha_{0a} - \alpha_{1a})(y_1x_5 + y_4x_8 + y_2x_6 + y_3x_7)$$

Moreover, since holographic reduction leaves the partition function invariant,

$$a + b + c + d = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 + x_6y_6 + x_7y_7 + x_8y_8$$

where the left side is the partition function of dimers of the 1×1 quotient graph, and the right side is the partition function of the vertex model, see Figure 10. According to lemma 5.1, α_{0k} and α_{1k} are two real numbers with opposite sign. Among the three terms $\alpha_{0a}, \alpha_{0b}, \alpha_{0c}$, two of them have the same sign, that can be made positive by properly changing the signs of T. Without loss of generality, we can assume α_{0b} and α_{0c} are both positive. At the black vertices, the $\{000\}$ entry of matchgate signature corresponds to partition functions of perfect matchings of the generator with all output vertices kept. Since there are odd number of vertices, no perfect matching exists, therefore

$$\alpha_{0a} = -\frac{\alpha_{0b}\alpha_{0c}y_5 + \alpha_{0b}y_6 + \alpha_{0c}y_7 + y_8}{\alpha_{0b}\alpha_{0c}y_1 + \alpha_{0b}y_2 + \alpha_{0c}y_3 + y_4} < 0,$$

hence we have

$$-(\alpha_{0a} - \alpha_{1a})(\alpha_{0b} - \alpha_{1b})(\alpha_{0c} - \alpha_{1c}) > 0$$

The lemma follows from the assumption that entries of relation signatures are positive. □

The expression of the characteristic polynomial shows that $P(z, w)$ is a smooth function on $\mathbb{T}^2 = \{(z, w)||z| = 1, |w| = 1\}$. We are interested in the intersection of spectral curve $P(z, w) = 0$ with unit torus $\mathbb{T}^2$, because it has implications on the convergence rate of correlations. Theorem 4.4 describes the behavior of $P(z, w)$ on $\mathbb{T}^2$. Before stating the theorem, we mention an elementary lemma:

Lemma 1.21. *Let $f(\phi) = A\sin\phi + B\cos\phi + C$, where A, B, C are real. If $C \geq 0$ and $A^2 + B^2 - C^2 \leq 0$, then $f(\phi) \geq 0$ for any $\phi \in \mathbb{R}$.*

Theorem 1.22. *Under assumption 3.5, either the spectral curve after holographic reduction is disjoint from $\mathbb{T}^2$, or intersects $\mathbb{T}^2$ at a single real node, that is, for some $(z_0, w_0) = (\pm 1, \pm 1)$,*

$$P(z, w) = \alpha(z - z_0)^2 + \beta(z - z_0)(w - w_0) + \gamma(w - w_0)^2 + ...,$$

where $\beta^2 - 4\alpha\gamma \leq 0$.

Proof. Let

$$
\begin{aligned}
Q(\theta,\phi) &= P(e^{i\theta}, w^{i\phi}) \\
&= [2(ac-b)+2(bc-a)\cos\theta]\cos\phi + 2(bc-a)\sin\theta\sin\phi \\
&+ 2(ab-c)\cos\theta + a^2+b^2+c^2+1.
\end{aligned}
$$

Consider $Q(\theta,\phi)$ as a trigonometric polynomial with respect to ϕ,

$$Q(\theta,\phi) = A\sin\phi + B\cos\phi + C$$

where

$$
\begin{aligned}
&A = 2(ac-b)+2(bc-a)\cos\theta \\
&B = 2(bc-a)\sin\theta \\
&C = a^2+b^2+c^2+1+2(ab-c)\cos\theta \geq 0.
\end{aligned}
$$

Define

$$
\begin{aligned}
g(\cos\theta) &= C^2 - A^2 - B^2 \\
&= 4(ab-c)^2\cos^2\theta + 4(ab+c)(a^2+b^2-c^2-1)\cos\theta \\
&- 4(bc-a)^2 - 4(ac-b)^2 + (a^2+b^2+c^2+1)^2,
\end{aligned}
$$

then g is a quadratic polynomial with respect to $\cos\theta$ with discriminant

$$\Delta = 64abc(a+b-c-1)(a+b+c+1)(a+1-b-c)(a+c-b-1).$$

The minimal value of $g(t)$ is attained at

$$t_0 = -\frac{(ab+c)(a^2+b^2-c^2-1)}{2(ab-c)^2},$$

where

$$g(t_0) = -\frac{\Delta}{16(ab-c)^2}$$

If $abc > 0$,

$$g(\cos\theta) = [2(ab+c)\cos\theta + a^2+b^2-c^2-1]^2 + 16abc\sin^2\theta \geq 0$$

therefore $Q(\theta,\phi) \geq 0$, and $Q(\theta,\phi) = 0$ only if $\sin\theta = \sin\phi = 0$.

If $abc < 0$, lemma 4.2 implies that $\Delta < 0$, then $g(\cos\theta) > 0$, therefore $Q(\theta,\phi) > 0$.

If $abc = 0$, we have $\left|\frac{ab+c}{ab-c}\right| = 1$. Moreover,

$$(a^2+b^2-c^2-1)^2 - 4(ab-c)^2 = (a+1+b+c)(a-1-b+c)(a+1-b-c)(a-1+b-c) > 0,$$

which implies

$$|t_0| = \left|\frac{ab+c}{ab-c}\right| \left|\frac{a^2+b^2-c^2-1}{2(ab-c)}\right| > 1$$

then $g(\cos\theta) > 0$, and $Q(\theta,\phi) > 0$.

Therefore the only possible intersection of the spectral curve with $\mathbb{T}^2$ is real point. It is trivial to check that $\frac{\partial P}{\partial z}\big|_{(\pm1,\pm1)} = 0$, $\frac{\partial P}{\partial w}\big|_{(\pm1,\pm1)} = 0$, and $Hessian[P(z,w)]$ is positive definite wherever a real intersection of $P(z,w) = 0$ and $\mathbb{T}^2$ exists, then the theorem follows. $\square$

1.6 Local Statistics

1.6.1 Configuration at Single Vertex

We are interested in the probability of a specified configuration at a single vertex. Consider a realizable vertex model on a planar finite hexagonal lattice, whose partition function is S. Assume both the vertex with specified signature and its neighbors lie in the interior of the graph. We work out an explicit example here; other cases are very similar. If only the configuration $\{011\}$ is allowed at a black vertex v, we get a new vertex model, assume partition function is S_{011}. Then the probability that the local configuration $\{011\}$ appear at v is

$$Pr(\{011\}, v) = \frac{S_{011}}{S}.$$

Hence the local statistics problem reduces to the problem of how to compute S_{011} efficiently. We will show that S_{011} can also be computed by the dimer technique with the help of the holographic algorithm.

Let us denote the adjacent vertices of v by v_a, v_b, v_c, according to the direction of the connecting edges, as illustrated in Figure 11.

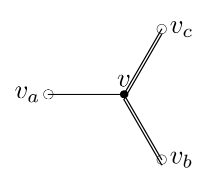

Figure 1.11: Configuration 011 at a Black Vertex

Let $r_v,\ r_a,\ r_b,\ r_c$ denote the signatures of relations at vertices $v,\ v_a,\ v_b,\ v_c$, then we have

$$\begin{pmatrix} r_{v\{000\}} \\ r_{v\{001\}} \\ r_{v\{010\}} \\ r_{v\{011\}} \\ r_{v\{100\}} \\ r_{v\{101\}} \\ r_{v\{110\}} \\ r_{v\{111\}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} r_{a\{000\}} \\ r_{a\{001\}} \\ r_{a\{010\}} \\ r_{a\{011\}} \\ r_{a\{100\}} \\ r_{a\{101\}} \\ r_{a\{110\}} \\ r_{a\{111\}} \end{pmatrix} = \begin{pmatrix} x_1^a \\ x_2^a \\ x_3^a \\ x_4^a \\ \star \\ \star \\ \star \\ \star \end{pmatrix}$$

$$\begin{pmatrix} r_{b\{000\}} \\ r_{b\{001\}} \\ r_{b\{010\}} \\ r_{b\{011\}} \\ r_{b\{100\}} \\ r_{b\{101\}} \\ r_{b\{110\}} \\ r_{b\{111\}} \end{pmatrix} = \begin{pmatrix} \star \\ \star \\ x_3^b \\ x_4^b \\ \star \\ \star \\ x_7^b \\ x_8^b \end{pmatrix} \qquad \begin{pmatrix} r_{c\{000\}} \\ r_{c\{001\}} \\ r_{c\{010\}} \\ r_{c\{011\}} \\ r_{c\{100\}} \\ r_{c\{101\}} \\ r_{c\{110\}} \\ r_{c\{111\}} \end{pmatrix} = \begin{pmatrix} \star \\ x_2^c \\ \star \\ x_4^c \\ \star \\ x_6^c \\ \star \\ x_8^c \end{pmatrix}$$

Here $\star$ means the entry at the position can be arbitrary. This is because we only allow the configuration $\{011\}$ at v, therefore the only configurations which actually affect the partition function of the vertex model will be those who do not occupy the a-edge of v_a, and occupy both the b-edge for v_b and the c-edge for v_c. We will split each of r_a, r_b, r_c into 2 parts, namely, $r_l = r_l(0) + r_l(1),\ l = a, b, c$, below. By definition, the partition function of the vertex model with signature r, r_a, r_b, r_c is a sum of 8 terms, each of which is the partition function of a vertex model with signature $r_v, r_a(i), r_b(j), r_c(k)$ $i, j, k \in \{0, 1\}$. That is,

$$S_{011} = S_{\{r_v, r_a, r_b, r_c\}} = \sum_{i,j,k \in \{0,1\}} S_{\{r_v, r_a(i), r_b(j), r_c(k)\}} \tag{1.36}$$

For each $S_{\{r_v, r_a(i), r_b(j), r_c(k)\}}$, we give new bases on edges adjacent to v, such that $S_{\{r_v, r_a(i), r_b(j), r_c(k)\}}$ become the partition function of certain local dimer configurations. Namely

$$S_{\{r_v, r_a(i), r_b(j), r_c(k)\}} \overset{base\ change}{=} Z_{\{m_v(i,j,k), m_a(i), m_b(j), m_c(k)\}} \tag{1.37}$$

where $Z_{\{m_v(i,j,k), m_a(i), m_b(j), m_c(k)\}}$ is the partition function of the dimer model on a matchgrid with signature $m_v(i, j, k), m_a(i), m_b(j), m_c(k)$. Assume the original matchgates u_a, u_b, u_c have weights

$$m_l = \begin{pmatrix} 0 & a_2^l & b_2^l & 0 & c_2^l & 0 & 0 & d_2^l \end{pmatrix}^t, \quad l = a, b, c$$

see Figure 12. We require

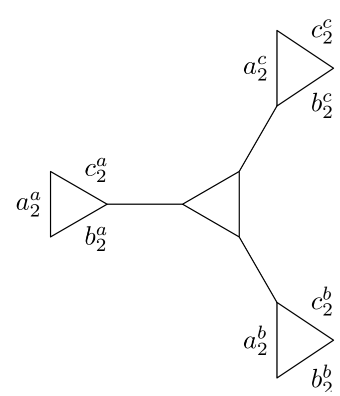

Figure 1.12: Weighted Matchgate

$$m_a(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & c_2^a & 0 & 0 & d_2^a \end{pmatrix}^t \tag{1.38}$$

$$m_a(0) = \begin{pmatrix} 0 & a_2^a & b_2^a & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t \tag{1.39}$$

$$m_b(1) = \begin{pmatrix} 0 & 0 & b_2^b & 0 & 0 & 0 & 0 & d_2^b \end{pmatrix}^t \tag{1.40}$$

$$m_b(0) = \begin{pmatrix} 0 & a_2^b & 0 & 0 & c_2^b & 0 & 0 & 0 \end{pmatrix}^t \tag{1.41}$$

$$m_c(1) = \begin{pmatrix} 0 & a_2^c & 0 & 0 & 0 & 0 & 0 & d_2^c \end{pmatrix}^t \tag{1.42}$$

$$m_c(0) = \begin{pmatrix} 0 & 0 & b_2^c & 0 & c_2^c & 0 & 0 & 0 \end{pmatrix}^t. \tag{1.43}$$

Notice that for any $l \in \{a, b, c\}$, $m_l(0)$ corresponds to configurations with an unoccupied l-edge; and $m_l(1)$ corresponds to configurations with an occupied l-edge. We are trying to determine the new base change matrices on incident edges of v, for each $r_v, r_a(i), r_b(j), r_c(k)$, such that (1.36),(1.37), (1.38)-(1.43) are satisfied simultaneously.

Assume on the adjacent edges of $v_a,\ v_b,\ v_c$ we have original base change matrix

$$T_k^j = \begin{pmatrix} n_{0k}^j & n_{1k}^j \\ p_{0k}^j & p_{1k}^j \end{pmatrix} \qquad k = 1, 2, 3;\ j = a, b, c$$

On an a-type edge adjacent to v, we consider 2 possible base change matrices

$$S_1^0 = \begin{pmatrix} m_{01}^0 & 0 \\ q_{01}^0 & q_{11}^0 \end{pmatrix} \qquad S_1^1 = \begin{pmatrix} 0 & m_{11}^1 \\ q_{01}^1 & q_{11}^1 \end{pmatrix}$$

On a b-type edge adjacent to v, we consider 2 possible base change matrices

$$S_2^0 = \begin{pmatrix} m_{02}^0 & m_{12}^0 \\ q_{02}^0 & 0 \end{pmatrix} \qquad S_2^1 = \begin{pmatrix} m_{02}^1 & m_{12}^1 \\ 0 & q_{12}^1 \end{pmatrix}$$

On a c-type edge adjacent to v, we consider 2 possible base change matrices

$$S_3^0 = \begin{pmatrix} m_{03}^0 & m_{13}^0 \\ q_{03}^0 & 0 \end{pmatrix} \qquad S_3^1 = \begin{pmatrix} m_{03}^1 & m_{13}^1 \\ 0 & q_{13}^1 \end{pmatrix}$$

For signatures $r_v, r_a(i), r_b(j), r_c(k)$, we choose basis S_1^i, S_2^j, S_3^k on edges a, b, c. By the realizability condition at vertex v, we have

$$m_v(i,j,k) = (S_1^i \otimes S_2^j \otimes S_3^k)^t \cdot r_v = e_{\{ijk\}} w_{ijk},$$

where

$$w_{ijk} = m_{i1}^i q_{j2}^j q_{k3}^k y_4,$$

and $e_{\{ijk\}}$ is the 8 dimensional vector with entry 1 at the position labeled by the binary sequence $\{ijk\}$, and 0 elsewhere. Obviously the choice of new base change matrices, namely, the position of zeros in the new base change matrices, results in the single configuration $\{ijk\}$ at the matchgate corresponding to v. Now the problem is to determine the entries of the base change matrices satisfying the equations.

According to realizability conditions at vertices v_a, v_b, v_c,

$$r_a = \Sigma_{i=1}^2 r_a(i) = \Sigma_{i=1}^2 S_1^i \otimes T_2^a \otimes T_3^a \cdot m_a(i) \tag{1.44}$$

$$r_b = \Sigma_{j=1}^2 r_a(j) = \Sigma_{i=1}^2 T_1^b \otimes S_2^j \otimes T_3^b \cdot m_b(j) \tag{1.45}$$

$$r_c = \Sigma_{k=1}^2 r_c(k) = \Sigma_{k=1}^2 T_1^c \otimes T_2^c \otimes S_3^k \cdot m_c(k). \tag{1.46}$$

For the original basis, we have the realizability condition as follows:

$$r_l = T_1^l \otimes T_2^l \otimes T_3^l \cdot m_l. \tag{1.47}$$

Substituting r_a, r_b, r_c by (1.47) in (1.46) we have the following system of linear equations with respect to entries of S

$$T_2^a \otimes T_3^a \begin{pmatrix} m_{11}^1 c_2^a \\ m_{01}^2 a_2^a \\ m_{01}^2 b_2^a \\ m_{11}^1 d_2^a \end{pmatrix} = \begin{pmatrix} x_1^a \\ x_2^a \\ x_3^a \\ x_4^a \end{pmatrix} = T_2^a \otimes T_3^a \begin{pmatrix} n_{11}^a c_2^a \\ n_{01}^a a_2^a \\ n_{01}^a b_2^a \\ n_{11}^a d_2^a \end{pmatrix}$$

$$T_1^b \otimes T_3^b \begin{pmatrix} q_{12}^1 b_2^b \\ q_{02}^2 a_2^b \\ q_{02}^2 c_2^b \\ q_{12}^1 d_2^b \end{pmatrix} = \begin{pmatrix} x_3^b \\ x_4^b \\ x_7^b \\ x_8^b \end{pmatrix} = T_1^b \otimes T_3^b \begin{pmatrix} p_{12}^b b_2^b \\ p_{02}^b a_2^b \\ p_{02}^b c_2^b \\ p_{12}^b d_2^b \end{pmatrix}$$

$$T_1^c \otimes T_2^c \begin{pmatrix} q_{13}^1 a_2^c \\ q_{03}^2 b_2^c \\ q_{03}^2 c_2^b \\ q_{13}^1 d_2^b \end{pmatrix} = \begin{pmatrix} x_2^c \\ x_4^c \\ x_6^c \\ x_8^c \end{pmatrix} = T_1^c \otimes T_2^c \begin{pmatrix} p_{13}^c a_2^c \\ p_{03}^c b_2^c \\ p_{03}^c c_2^b \\ p_{13}^c d_2^b \end{pmatrix}$$

Under the assumption that original base change matrices are invertible, we have

$$m_{11}^1 = n_{11}^a, \qquad m_{01}^2 = n_{01}^a$$
$$q_{12}^1 = p_{12}^b, \qquad q_{02}^2 = p_{02}^b$$
$$q_{13}^1 = p_{13}^c, \qquad q_{03}^2 = p_{03}^c$$

Therefore, we only need to choose the nonzero entries of the new base change matrices to be the same as the original ones.

For the other seven configurations at vertex v, we can use the same method to achieve a similar result. By splitting each one of r_a, r_b, r_c into 2 parts, we express the partition function of the satisfying assignments as the sum of 8 terms, each one of which is the partition function of the relations with signature $r_v,\ r_a(i),\ r_b(j), r_c(k)$, $i, j, k \in \{0, 1\}$. We apply the holographic reduction to each part separately, and we derive that the partition function of the relations with signature $r_v,\ r_a(i),\ r_b(j), r_c(k)$, $i, j, k \in \{0, 1\}$ is equal to the partition function of the dimer configurations on the corresponding matchgrids with signature $m_v(i, j, k), m_a(i), m_b(j), m_c(k)$, which corresponds to the local configuration $\{ijk\}$. This may not be a dimer configuration; to make it a dimer configuration, let us divide those 8 terms into two groups: one consists of all $\{i, j, k\}$ satisfying $i + j + k \equiv 0 \pmod 2$, namely $\{000\}, \{011\}, \{101\}, \{110\}$; the other consists of all $\{i, j, k\}$ satisfying $i + j + k \equiv 1 \pmod 2$. The sum of partition functions in the first group is equal to the partition function of dimer configurations in a matchgrid with weights illustrated in the left graph of Figure 13, where $t = \frac{w_{000}}{w_{011}+w_{101}+w_{110}}$; the sum of partition functions corresponding to $\{001\}, \{010\}, \{100\}, \{111\}$ is equal to the partition function of dimer configurations in a matchgrid with weights as illustrated in the right graph of Figure 13, up to a multiplication constant w_{111}.

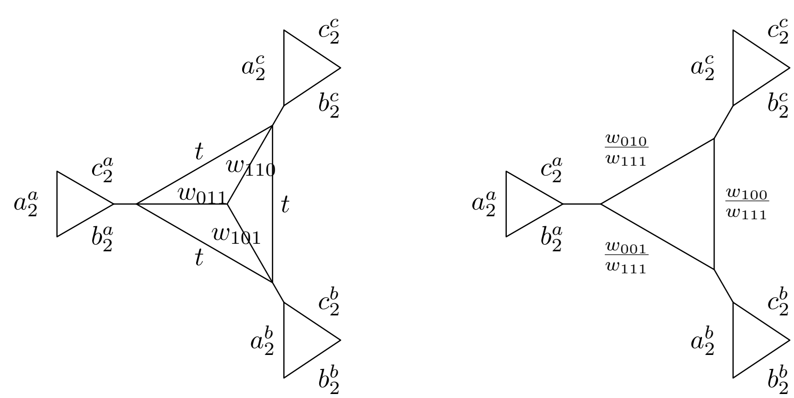

Figure 1.13: Even and Odd Matchgates

However, in the left graph we change the parity of the total number of vertices, as a result, there is no dimer. Therefore we have the following theorem

Theorem 1.23. *Assume we have a vertex model $\mathcal{H}_0$ on a finite hexagonal lattice, which is holographic equivalent to the dimer model on a Fisher graph $\mathcal{F}_0$. If at a fixed interior vertex v of $\mathcal{H}_0$, only one local configuration is allowed, then the partition function of the new vertex model $\mathcal{H}_1$ can be computed by the partition function of the dimer model on a Fisher graph $\mathcal{F}_1$, multiplied by a non-zero constant. $\mathcal{F}_0$ and $\mathcal{F}_1$ have the same edge weights except for the matchgate corresponding to v.*

1.6.2 Configuration at Finitely Many Vertices

We are interested in the probability that one specified configuration is allowed at each of the given finitely many vertices for a realizable vertex model on a hexagonal lattice. For simplicity, assume all of them are black. Let V_0^b be the set of all black vertices with specified configuration. To compute the probability, we first give a criterion to construct the new base change matrices on incident edges of V_0^b according to whether the edge is present in the dimer configuration and in the vertex model configuration. Assume $\begin{pmatrix} n_0 & n_1 \\ p_0 & p_1 \end{pmatrix}$ is the original base change matrix on the edge,

Base Change Matrices	Presence in Vertex Configuration	Presence in Dimer
$\begin{pmatrix} 0 & n_1 \\ p_0 & p_1 \end{pmatrix}$	No	Yes
$\begin{pmatrix} n_0 & 0 \\ p_0 & p_1 \end{pmatrix}$	No	No
$\begin{pmatrix} n_0 & n_1 \\ 0 & p_1 \end{pmatrix}$	Yes	Yes
$\begin{pmatrix} n_0 & n_1 \\ p_0 & 0 \end{pmatrix}$	Yes	No

whether the edges incident to vertices in V_0^b are present in the configuration of the vertex model is known, given all the specified configurations at V_0^b. As before, we are going to split the signature of each adjacent white vertex into several parts, and we apply the holographic reduction to each parts separately. Our expectation is that after the reduction process, each part will be equivalent to the partition function of a single local dimer configuration. The new base change matrix on each edge is chosen according to whether we want the edge to be present in the dimer configuration or not after the reduction. As before, all the non-vanishing entries of the new bases change matrices are equal to the original ones; we will see how it works below.

Consider an arbitrary white vertex $w \in \Gamma(V_0^b)$, the neighbors of V_0^b. Assume r_w is the vertex signature at w, and m_w is the original dimer signature at the corresponding matchgate. Assume

$$m_w = \begin{pmatrix} 0 & a_w & b_w & 0 & c_w & 0 & 0 & d_w \end{pmatrix}^t$$

Let $D(w)$ denote the number of adjacent vertices of w with specified configuration. We classify $\Gamma(V_0^b)$ according to D.

If $D(w) = 1$, we split $r_w = r_w(1) + r_w(2)$. Without loss of generality, assume the left-digit edge(a-type edge, horizontal edge) connects w with $b \in V_0^b$, and the edge wb is present in the configuration of the vertex model. Assume

$$T_1 = \begin{pmatrix} n_{01} & n_{11} \\ p_{01} & p_{11} \end{pmatrix} \qquad T_2 = \begin{pmatrix} n_{02} & n_{12} \\ p_{02} & p_{12} \end{pmatrix} \qquad T_3 = \begin{pmatrix} n_{03} & n_{13} \\ p_{03} & p_{13} \end{pmatrix}$$

are original base change matrices on the edges adjacent to w. According to the table, the presence of wb in the vertex model configuration implies two possible choices of the new basis on the horizontal edge, namely,

$$S_1^1 = \begin{pmatrix} n_{01} & n_{11} \\ p_{01} & 0 \end{pmatrix} \qquad S_1^2 = \begin{pmatrix} n_{01} & n_{11} \\ 0 & p_{11} \end{pmatrix}$$

while on the two other incident edges of w we have the original bases T_2, T_3. Assume

$$m_w(1) = \begin{pmatrix} 0 & a_w & b_w & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$m_w(2) = \begin{pmatrix} 0 & 0 & 0 & 0 & c_w & 0 & 0 & d_w \end{pmatrix}$$

Notice that the nonzero entries $m_w(1)$ have indices $\{001\}$ and $\{010\}$, which corresponds to configurations without the a-edge present; the non-vanishing entries of $m_w(2)$ have indices $\{100\}$ and $\{111\}$, which corresponds to configurations with the a-edge present. Choose

$$r_w(1) = S_1^1 \otimes T_2 \otimes T_3 \cdot m_w(1)$$
$$r_w(2) = S_1^2 \otimes T_2 \otimes T_3 \cdot m_w(2)$$

Then we can check

$$r_w(1) + r_w(2) = \otimes_{i=1}^{3} T_i \cdot m_w = r_w$$

If $D(w) = 2$, we split $r_w = \sum_{i=1}^{4} r_w(i)$. Without loss of generality, assume the middle-digit(b-type edge) and right-digit(c-type edge) edges connects w to $b_2, b_3 \in V_0^b$, wb_2 is present while wb_3 is not present in the configuration of the vertex model. According to our criteria listed in the table, on wb_2 we have two different base change matrices

$$S_2^1 = \begin{pmatrix} n_{02} & n_{12} \\ p_{02} & 0 \end{pmatrix} \qquad S_2^2 = \begin{pmatrix} n_{02} & n_{12} \\ 0 & p_{12} \end{pmatrix}$$

On wb_3 we have two different base change matrices

$$S_3^1 = \begin{pmatrix} n_{03} & 0 \\ p_{03} & p_{13} \end{pmatrix} \qquad S_3^2 = \begin{pmatrix} 0 & n_{13} \\ p_{03} & p_{13} \end{pmatrix}$$

while on the left-digit edge, we keep original basis T_1, the original basis on the middle and right digit edge are T_2, T_3, as in the $D(w) = 1$ case. Assume

$$m_w(1) = \begin{pmatrix} 0 & a_w & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$m_w(2) = \begin{pmatrix} 0 & 0 & b_w & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$m_w(3) = \begin{pmatrix} 0 & 0 & 0 & 0 & c_w & 0 & 0 & 0 \end{pmatrix}$$
$$m_w(4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_w \end{pmatrix}$$

Obviously, $m_w(4)$ corresponds to both edge being present, $m_w(3)$ corresponds to neither being present, $m_w(2)$ corresponds to b-edge being present and c-edge not, and $m_w(1)$ corresponds to b-edge not present and c-edge present. Again according to the table, choose

$$r_w(1) = T_1 \otimes S_2^1 \otimes S_3^2 \cdot m_w(1)$$
$$r_w(2) = T_1 \otimes S_2^2 \otimes S_3^1 \cdot m_w(2)$$
$$r_w(3) = T_1 \otimes S_2^1 \otimes S_3^1 \cdot m_w(3)$$
$$r_w(4) = T_1 \otimes S_2^2 \otimes S_3^2 \cdot m_w(4)$$

we can check

$$\sum_{i=1}^{4} r_w(i) = \otimes_1^3 T_i \cdot m_w = r_w$$

If $D(w) = 3$, we split $r_w = \sum_{i=1}^4 r_w(i)$, and assume $m_w(i), i = 1, ..., 4$ as in $D(w) = 2$. Without loss of generality, assume S_1^1, S_1^2 as in $D(w) = 1$, $S_2^1, S_2^2, S_3^1, S_3^2$ as in $D(w) = 2$; that is, we have vertex-model signature $\{110\}$ at w. Choose

$$r_w(1) = S_1^1 \otimes S_2^1 \otimes S_3^2 \cdot m_w(1)$$
$$r_w(2) = S_1^1 \otimes S_2^2 \otimes S_3^1 \cdot m_w(2)$$
$$r_w(3) = S_1^2 \otimes S_2^1 \otimes S_3^1 \cdot m_w(3)$$
$$r_w(4) = S_1^2 \otimes S_2^2 \otimes S_3^2 \cdot m_w(4)$$

again we can check $r_w = \sum_{i=1}^4 r_w(i)$ as in $D(w) = 2$.

For all the other local configurations, the same technique works, and we will have a similar result. Assume $V_0^b = \{v_1, ..., v_p\}, \Gamma(V_0^b) = \{w_1, ..., w_k\}$, then

$$\begin{aligned} S(\tilde{r}_{v_1}, ..., \tilde{r}_{v_p}) = & \quad [\otimes_{j=1}^{k} r_{w_j} \otimes \otimes_{w \notin \Gamma(V_0^b)} r_w] \cdot [\otimes_{q=1}^{p} \tilde{r}_{v_q} \otimes \otimes_{b \notin V_0^b} r_b] \\ = & \sum_{i_1,...,i_k} [\otimes_{j=1}^{k} r_{w_j}(i_j) \otimes \otimes_{w \notin \Gamma(V_0^b)} r_w] \cdot [\otimes_{q=1}^{p} \tilde{r}_{v_q} \otimes \otimes_{b \notin V_0^b} r_b] \end{aligned}$$

where $\tilde{r}_{v_q}$ is the specified configuration at the vertex v_q. The first equality follows from the definition of the partition function of satisfying assignments. When we compute the tensor product of relation signatures of all black (white) vertices, we get a vector of dimension $2^{|E(G)|}$, where $|E(G)|$ is the number of edges in the planar finite graph G. This vector can be indexed by binary sequences of length $|E(G)|$. Each binary sequence corresponds to a configuration, and the entry there is the product of weights of the local configurations obtained from the configuration restricted to each black (white) vertices. Obviously the inner product of the vector at black vertices and the vector at white vertices are exactly the partition function of the satisfying assignments. The second equality follows from the multi-linearity of the tensor product. For $1 \leq j \leq k$, if $D(w_j) = 1$, $i_j \in \{1, 2\}$; if $D(w_j) = 2, 3$, $i_j \in \{1, 2, 3, 4\}$. For each summand, we choose a basis on incident edges of vertices in V_0^b according to whether the edge is present in the dimer configuration and relation configuration, and keep the original bases on all the other edges. as described above. This is realizable because for each $b \in V_0^b$, whether its incident edges are occupied by dimers are completely determined in each part of the sum. In other words, each term on the right side of the second equality corresponds to a configuration on all the edges incident to vertices in V_0^b, which is a local dimer configuration at each odd matchgate corresponding to white vertices w's with $D(w) = 3$, and can be extended to local dimer configurations at each odd matchgate corresponding to white vertices w's with $D(w) = 1$ and $D(w) = 2$. To make them be dimer configurations at each black matchgate, we divide those configurations into groups according to the following criterion: two configurations are in the same group if and only if the parity of the number of occupied incident edges at each black vertex in V_0^b is the same. If zero or two incident edges are occupied, we construct an even matchgate with modified weights at the black vertex; if one or three incident edges are occupied, we construct an odd matchgate with modified weights. Notice that the modified weights depend only on the local configuration on the incident edges of the vertex. Since $|V_0^b| = p$, we have 2^p different constructions in total, but only 2^{p-1} of them admit dimer cover, each of which has even number of even matchgates. Therefore we have

Theorem 1.24. *Assume we have a realizable vertex model $\mathcal{H}_0$ on a finite hexagonal lattice, which is holographic equivalent to the dimer model on a Fisher graph $\mathcal{F}_0$. Let V_0^b be a subset of black vertices. If for each $v \in V_0^b$, we only allow one local configuration, this way we obtain a new vertex model $\mathcal{H}_1$. The partition function of $\mathcal{H}_1$ is equal to the sum of partition functions of 2^{p-1} dimer models on matchgrids $\mathcal{F}_1, ..., \mathcal{F}_{2^{p-1}}$, each of which has an even number of even matchgates. $\mathcal{F}_i (1 \leq i \leq 2^{p-1})$ are the same $\mathcal{F}_0$, except for the matchgates corresponding to vertices in V_0^b.*

1.6.3 Ising Model and Vertex Model

Consider the Ising model on a finite Kagome lattice, embedded into a torus, as illustrated in the following figure.

The associated honeycomb lattice is illustrated in the Figure by the dashed line. Each vertex of the Kagome lattice corresponds to an edge of the honeycomb lattice; hence each Ising spin configuration on the Kagome lattice corresponds to an edge subset of the honeycomb lattice. If the spin is "+", then the corresponding edge is

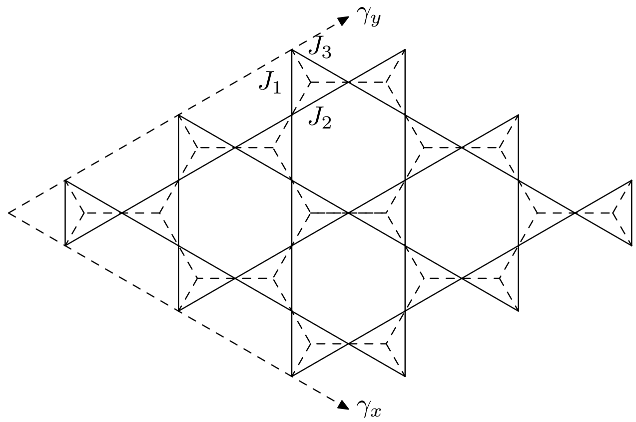

Figure 1.14: Kagome Lattice and Honeycomb Lattice

included in the subset; otherwise the edge is not included. Assume the bonds of the Kagome lattice have interactions J_1, J_2, J_3, as illustrated in Figure 14. We define a vertex model on the honeycomb lattice with signature at all vertices as follows.

$$\begin{pmatrix} r_{v,000} \\ r_{v,001} \\ r_{v,010} \\ r_{v,011} \\ r_{v,100} \\ r_{v,101} \\ r_{v,110} \\ r_{v,111} \end{pmatrix} = \begin{pmatrix} e^{2(J_1+J_2+J_3)} \\ e^{2J_3} \\ e^{2J_2} \\ e^{2J_1} \\ e^{2J_1} \\ e^{2J_2} \\ e^{2J_3} \\ e^{2(J_1+J_2+J_3)} \end{pmatrix}$$

This way the probability measure of the Ising model on the Kagome lattice is equivalent to the probability measure of the vertex model on the honeycomb lattice. It is trivial to check that this vertex model is orthogonally realizable.

1.7 Asymptotic Behavior

In this section, we prove the main theorems concerning the asymptotic behavior of realizable vertex models on the periodic hexagonal lattice, as stated in the introduction. Consider an infinite periodic graph G, with period 1×1, see Figure 9. Our technique to deal with such a graph is to consider a graph G_n with n^2 1×1 fundamental domains, embed G_n into a torus, and consider the limit when $n \to \infty$, see Page 5. Our first theorem is about the free energy of the infinite periodic hexagonal lattice.

Proof of Theorem 1.1 Assume M_n is the corresponding matchgrid with respect to G_n, and $P_n(z, w)$ is the characteristic polynomial. Obviously M_n is also a quotient graph of a periodic infinite graph modulo a subgraph of $\mathbb{Z}^2$ generated by $(n, 0)$ and $(0, n)$. The corresponding Kasteleyn matrices here are defined given the orientation of Figure 9. For even n, the crossing orientation can be obtained from the orientation of Figure 9 by reversing all the z-edges and w-edges. By Theorem 3.6, M_n is a Fisher graph, and the partition of the vertex model can be expressed as follows according to the principle of holographic reduction:

$$S(G_n) = Z(M_n) = \frac{1}{2}|-\mathrm{Pf}K_n(1,1) + \mathrm{Pf}K_n(1,-1) + \mathrm{Pf}K_n(-1,1) + \mathrm{Pf}K_n(-1,-1)|$$

Thus

$$\max_{u,v\in\{-1,1\}} |\mathrm{Pf}P_n(u,v)| \leq Z(M_n) \leq 2 \max_{u,v\in\{-1,1\}} |\mathrm{Pf}P_n(u,v)|$$

On the other hand, according to the formula of enlarging fundamental domains,

$$\frac{1}{n^2} \log \max_{u,v\in\{-1,1\}} |\mathrm{Pf}K_n(u,v)| = \max_{u,v\in\{-1,1\}} \frac{1}{2n^2} \sum_{z^n=u} \sum_{w^n=v} \log |P(z,w)|$$

By Theorem 4.4, either $P(z,w)$ has no zeros on $\mathbb{T}^2$, or it has a single real node, in which case any sample point in $\max_{u,v\in\{-1,1\}} \frac{1}{2n^2}\sum_{z^n=u}\sum_{w^n=v}\log|P(z,w)|$ is at least $\frac{C}{n}$ from the real node, for some constant $C>0$. Therefore

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n^2}S(G_n) = & \lim_{n\to\infty}\max_{u,v\in\{-1,1\}}\frac{1}{2n^2}\sum_{z^n=u}\sum_{w^n=v}\log|P(z,w)| \\
= & \frac{1}{8\pi^2}\iint_{|z|=1,|w|=1}\log P(z,w)\frac{dz}{iz}\frac{dw}{iw}
\end{aligned}$$

□

For each G_n, a measure λ_n is defined as in (1.1). Assume at a fixed vertices v, we only allow configuration c_v. Let μ_n be the Boltzmann measure of dimer configurations on M_n. Let $\tilde{M}_n$ be the matchgrid corresponding to the vertex model which only allows c_v at v, as described in Theorem 5.1. Let m_v be the matchgate of $\tilde{M}_n$, corresponding to v, and d_j be a local dimer configuration at m_v, w_{d_j} be product of weights of matchgate edges included in d_j, and V_{d_j} be the set of external vertices of matchgates m_v occupied by dimer configuration d_j. In our graph, every matchgate has 3 external vertices. Our second theorem is about the asymptotic behavior of the measure λ_n.

Proof of Theorem 1.2 According to Theorem 5.1

$$\begin{aligned}
\lambda_n(c_1,\ldots,c_p) = & \frac{Z(\tilde{M}_n)}{Z(M_n)} = \sum_{d_j}\frac{Z(\tilde{M}_n(d_j))}{Z(M_n)} \\
= & \sum_{d_j} w_{d_j}\frac{Z(M_n\setminus V(d_j))}{Z(M_n)}
\end{aligned}$$

where the sum is over all local dimer configurations d_j. Since the number of local configurations is finite, it suffices to consider $\lim_{n\to\infty}\frac{Z(M_n\setminus V(d_j))}{Z(M_n)}$. Note that $\tilde{M}_n$ differs from M_n only on edge weights of m_v, hence $M_n \setminus V(d_j)$ and $\tilde{M}_n \setminus V(d_j)$ are the same.

Given d_j, let $W_{n,d_j} = M_n \setminus V(d_j)$ be the subgraph of M_n by removing all vertices occupied by the configuration (d_j), as well as their incident edges. Then

$$\begin{aligned}\frac{Z(W_{n,d_j})}{Z(M_n)} = \frac{1}{2Z_n}| &- \mathrm{Pf}(K_n^{11}(W_{n,d_j})) + \mathrm{Pf}(K_n^{-1,1}(W_{n,d_j})) \\ &+\mathrm{Pf}(K_n^{1,-1}(W_{n,d_j})) + \mathrm{Pf}(K_n^{-1,-1}(W_{n,d_j}))|\end{aligned}$$

First of all, let us assume $P(z,w)$ has no zeros on $\mathbb{T}^2$. According to the formula of enlarging fundamental domains, for any $(\theta,\tau)\in\{-1,1\}$, $Pf(K_n^{\theta,\tau})\neq 0$. Then

$$|\mathrm{Pf}(K_n^{\theta,\tau})_{E^c}| = |\mathrm{Pf}(K_n^{\theta,\tau})^{-1}_E||\mathrm{Pf}(K_n^{\theta,\tau})|$$

In [6, 16], it was proved that given two vertices (u,x_1,y_1) and (v,x_2,y_2)

$$(K_n^{\theta,\tau})^{-1}((u,x_1,y_1),(v,x_2,y_2)) = \frac{1}{n^2}\sum_{z^n=\theta}\sum_{w^n=\tau} z^{x_1-x_2}w^{y_1-y_2}\frac{\mathrm{Cof}(K(z,w))_{u,v}}{P(z,w)}$$

Since $P(z,w)$ has no zeros on $\mathbb{T}^2$, we have

$$\begin{aligned}\lim_{n\to\infty}(K_n^{\theta,\tau})^{-1}((u,x_1,y_1),(v,x_2,y_2)) &= \frac{1}{4\pi^2}\iint_{\mathbb{T}^2} z^{x_1-x_2}w^{y_1-y_2}\frac{\mathrm{Cof}(K(z,w))_{u,v}}{P(z,w)}\frac{dz}{iz}\frac{dw}{iw} \\ &= K_\infty^{-1}((u,x_1,y_1),(v,x_2,y_2))\end{aligned}$$

As $n\to\infty$, each entry of $(K_n^{\theta\tau})^{-1}$ is convergent, so is the Pfaffian of a finite order sub-matrix $(K_n^{\theta\tau})^{-1}_{V(d_j)}$, and we have

$$\lim_{n\to\infty}\frac{Z(M_n\setminus V(d_j))}{Z(M_n)} = |\mathrm{Pf}(K_\infty^{-1})_{V(d_j)}|$$

If $P(z,w)$ has a real node on $\mathbb{T}^2$, without loss of generality, we can assume the real node is $(1,1)$. It was proved in [3] that if $P(z,w)=0$ has a node at $(1,1)$, then for any fixed finite subset E

$$\lim_{n\to\infty}\frac{1}{2Z_n}[\mathrm{Pf}(K_n^{-1,1})_{E^c}+\mathrm{Pf}(K_n^{1,-1})_{E^c}+\mathrm{Pf}(K_n^{-1,-1})_{E^c}]=\mathrm{Pf}(K_\infty^{-1})_E$$

and

$$\lim_{n\to\infty}\frac{\mathrm{Pf}(K_n^{1,1})_{E^c}}{2Z_n}=0$$

If we take $E=V(d_{i,j},...,d_{p,j})$, our theorem follows. $\square$

1.8 Simulations of 1-2 Model

Assume we have 1-2 model with signature $r_s = \begin{pmatrix} 0 & c & b & a & a & b & c & 0 \end{pmatrix}^t$ at all vertices. In other words, at each vertex, either one or two edges are allowed to be present in a local configuration. By Section 3.2, the signature is orthogonally realizable with base change matrix $T = \begin{pmatrix} \cos\frac{3\pi}{4} & \sin\frac{3\pi}{4} \\ -\sin\frac{3\pi}{4} & \cos\frac{3\pi}{4} \end{pmatrix}$ on all edges.

We define a discrete-time, time-homogeneous Markov chain $\mathfrak{M}_t$ with state space the set of all configurations of the 1-2 model. For an $n \times n$ honeycomb lattice embedded into a torus, the state space is finite, and let us denote it by $\{i_k\}_{k=1}^{K}$. Let $\Gamma(i_k)$ be the set of configurations that can be obtained from configuration i_k by adding or deleting a single edge. Assume p, q, r are edge variables, namely $p, q, r \in \{a, b, c\}$, and $\{p, q, r\} = \{a, b, c\}$. Define $\Gamma_{p,+q}(i_k)$ be the set of configurations of 1-2 model which can be obtained from i_k by adding a single q-edge uv; before adding uv, only a p-edge is present at both u and v. Define $\Gamma_{p,-q}(i_k)$ be the set of configurations of 1-2 model which can be obtained from i_k by deleting a single q-edge uv; after deleting uv, only a p-edge is present at both u and v. Define $\Gamma_0(i_k) = \Gamma(i_k) \setminus \{\cup_{p,q \in a,b,c, p\neq q} \Gamma_{p,+q}(i_k) \cup_{p,q \in a,b,c, p\neq q} \Gamma_{p,-q}(i_k)\}$. Define entries of transition matrix for

$\mathfrak{M}_t$ as follows:

$$P(i_l|i_k) = \begin{cases} \frac{1}{3n^2} & \text{if } i_l \in \Gamma_0(i_k) \\ \frac{1}{3n^2} & \text{if } i_l \in \Gamma_{p,+q}(i_k) \text{ and } r \geq p \\ \frac{1}{3n^2}\frac{r^2}{p^2} & \text{if } i_l \in \Gamma_{p,+q}(i_k) \text{ and } r < p \\ \frac{1}{3n^2} & \text{if } i_l \in \Gamma_{p,-q}(i_k) \text{ and } r \leq p \\ \frac{1}{3n^2}\frac{p^2}{r^2} & \text{if } i_l \in \Gamma_{p,-q}(i_k) \text{ and } r > p \\ 1 - \sum_{i_j \in \Gamma(i_k)} P(i_j|i_k) & \text{if } i_l = i_k \\ 0 & \text{else} \end{cases}$$

Obviously, $\mathfrak{M}_t$ is aperiodic. For more information on Markov chain, see [18, 23]. Moreover, we have

Proposition 1.25. $\mathfrak{M}_t$ *is irreducible.*

Proof. By definition, we only need to prove that any two configurations communicate to each other. We claim that any two dimer configuration can be obtained from each other by finite steps. In fact, the symmetric difference of any two dimer configurations is a union of finitely many loops. Obviously one dimer configuration can be obtained from any other dimer configuration by first adding finitely many edges to achieve their union, then deleting alternating edges of loops. Notice that we have a 1-2 configuration at each step. Hence we only need to prove that any configuration of 1-2 model can reach a dimer by finite steps, each of which is adding or deleting one single edge.

Let us start with an arbitrary configuration of 1-2 model. There are 3 types of connected local configurations: loops with even number of edges; zigzag paths with odd number of edges; zigzag paths with even number of edges. For the first and second types, we can always achieve dimers by deleting alternating edges. Hence we can assume that all the zigzag paths with even number of edges are of length 2, and all the other connected local configurations are dimers. There are two types of length-2 paths. One has vertices black-white-black (BWB), and the other has vertices white-black-white (WBW). For each fixed configuration, the number of BWB paths is the same as the number of WBW paths; otherwise the complement graph cannot be covered by dimers. Consider an arbitrary WBW path in a fixed configuration, as illustrated in Figure 15.

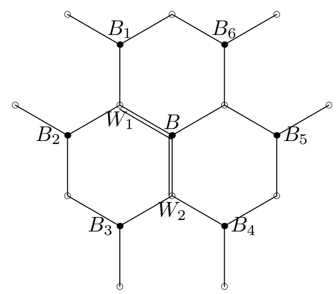

Figure 1.15: WBW Path

One can easily check that no matter what the configuration is, it communicates with a configuration satisfying one of the following two conditions: 1. the number of length-2 edges is equal to the current number of length-2 edges minus 2; 2. it can be moved to each of a WBW configurations containing B_i, for all $1 \leq i \leq 6$. Where B_i is the nearest black vertices to B, as in Figure 15. Hence if the number of length-2 paths does not decrease, one can transverse all black vertices without meeting a BWB path, because our graph is finite. However, this is impossible because a BWB path always exists as long as a WBW path exists. $\square$

For an irreducible, aperiodic Markov chain $\mathfrak{M}_t$ with transition matrix P and stationary distribution π, let x_0 be an arbitrary initial distribution. Then

$$\lim_{n\to\infty} x_0 P^n = \pi.$$

Therefore, in order to sample a configuration, we can approximately sample according to the distribution x_0P^N, with large N. To that end, first we choose a fixed dimer configuration with probability 1 as the initial distribution x_0, then we randomly change the configuration by adding or deleting a single edge according to the conditional probability specified by the transition matrix P. If neither adding nor deleting u_1u_2 ends up with a satisfying configuration, we just keep the previous configuration; else we get a new configuration by adding or deleting u_1u_2. Then we repeat the process for N steps. This way we get a sample for distribution x_0P^N, if N is sufficiently large, this is approximately a sample for distribution π, which is exactly the distribution given by 1-2 model.

Example 1.26. *(Uniform 1-2 Model) Consider the 1-2 model with $a = b = c = 1$. After the holographic reduction, the signature of each matchgate is*

$$\begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{3\sqrt{2}}{2} \end{pmatrix}'$$

It is gauge equivalent to a positive-weight dimer model on Fisher graph, whose spectral curve does not intersect $\mathbb{T}^2$. We are interested in the probability that a $\{001\}$ dimer occurs, that is, at a pair of adjacent vertices v_1, v_2 connected by an a-edge, only the configuration $\{001\}$ is allowed. By the technique described in section 5 and section 6, the partition function of the configurations in which a dimer is present at v_1v_2 is equal to the sum of two partition functions, one corresponds to both v_1 and v_2 are replaced by an odd matchgate with signatures

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} \end{pmatrix};$$

the other corresponds to both v_1 and v_2 are replaced by an even matchgate with *signatures*

$$\begin{pmatrix} \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 \end{pmatrix}$$

Both of them have the same partition function as a graph with positive weights. Moreover, if we give a base change matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *on* v_1v_2 *edge, and apply holographic reduction, we see that the two graphs with positive weights are holographic equivalent. Hence it suffices to consider the one with a pair of odd matchgates with modified weights. Then*

$$\Pr(\text{a dimer is present at } v_1v_2) = \frac{Z_{001}}{18Z}$$

where Z_{001} *is the partition function of dimer configurations with weight 1 on triangles corresponding to* v_1 *and* v_2 *and weight* $\frac{1}{3}$ *on all the other triangles, and* Z *is the partition function with weight* $\frac{1}{3}$ *on all the triangles. Moreover*

$$\begin{aligned} Z_{001} = \quad & Z^{001001} + Z^{001111} + Z^{010010} + Z^{010100} + Z^{100100} + Z^{100010} + Z^{111001} + Z^{111111} \\ = \quad & Z^{001001} + 2Z^{001111} + 2Z^{010010} + 2Z^{010100} + Z^{111111} \end{aligned}$$

where $Z^{ijk,\tilde{i}\tilde{j}\tilde{k}}$ *is the partition function of dimer configurations with fixed configuration* $ijk, \tilde{i}\tilde{j}\tilde{k}$ *on triangles corresponding to* v_1 *and* v_2*. The second equality follows from symmetry. Meanwhile*

$$\frac{1}{9}Z^{001001} + \frac{2}{3}Z^{001111} + \frac{2}{9}Z^{010010} + \frac{2}{9}Z^{010100} + Z^{111111} = Z,$$

then

$$\begin{aligned} \Pr(\text{a dimer is present at } v_1v_2) = \quad & \frac{1}{18}\left(1 + \frac{8Z^{001001}}{9Z} + \frac{4Z^{001111}}{3Z} + \frac{16Z^{010010}}{9Z} + \frac{16Z^{010100}}{9Z}\right) \\ = \quad & \frac{1}{18}\left(1 + \frac{4}{3}|\mathrm{Pf}(K_\infty^{-1})_{23}| + \frac{4}{9}|\mathrm{Pf}(K_\infty^{-1})_{2356}| + \frac{16}{3}|\mathrm{Pf}(K_\infty^{-1})_{13}|\right) \end{aligned}$$

where

$$
\begin{aligned}
(K_\infty^{-1})_{23} = &\quad \frac{3}{16\pi^2}\iint_{\mathbb{T}^2}\frac{w(zw+1+z-9w)}{2(z^2+w^2)+2(z+w)+2zw(z+w)-21zw}\frac{dz}{iz}\frac{dw}{iw} = (K_\infty^{-1})_{56}\\
(K_\infty^{-1})_{13} = &\quad -\frac{3}{16\pi^2}\iint_{\mathbb{T}^2}\frac{w(w+z+z^2-9zw)}{2(z^2+w^2)+2(z+w)+2zw(z+w)-21zw}\frac{dz}{iz}\frac{dw}{iw}\\
(K_\infty^{-1})_{25} = &\quad \frac{1}{16\pi^2}\iint_{\mathbb{T}^2}\frac{8z+8zw-82w+9w^2+9}{2(z^2+w^2)+2(z+w)+2zw(z+w)-21zw}\frac{dz}{iz}\frac{dw}{iw} = (K_\infty^{-1})_{36}\\
(K_\infty^{-1})_{26} = &\quad \frac{1}{16\pi^2}\iint_{\mathbb{T}^2}\frac{-z-zw-w+9}{2(z^2+w^2)+2(z+w)+2zw(z+w)-21zw}\frac{dz}{iz}\frac{dw}{iw} = (K_\infty^{-1})_{35}
\end{aligned}
$$

The entries of the inverse matrix can be expressed as elliptic functions. The probability that a $\{001\}$ *dimer occurs is approximately* 6%. *A sample of the uniform 1-2 model is illustrated in Figure 16.*

Example 1.27. *(Critical 1-2 Model) Consider the 1-2 model with* $a = 4, b = c = 1$. *After the holographic reduction, it has the same partition function as a positive-weight dimer model on Fisher graph, whose spectral curve has a single real node on* $\mathbb{T}^2$. *The probability of the configuration* $\{011\}$

$$
\Pr(\{011\}) = \frac{Z_{011}}{3Z} = \frac{1}{3}(|(K_\infty^{-1})_{12}| + |(K_\infty^{-1})_{23}| + |(K_\infty^{-1})_{13}|) + |\mathrm{Pf}(K_\infty^{-1})_{123456}|)
$$

Z_{011} *is the partition function on a Fisher graph with weights* $1, 1, 1$ *on one triangle, and weights* $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$ *on all the other triangles.* Z *is the partition function with weights*

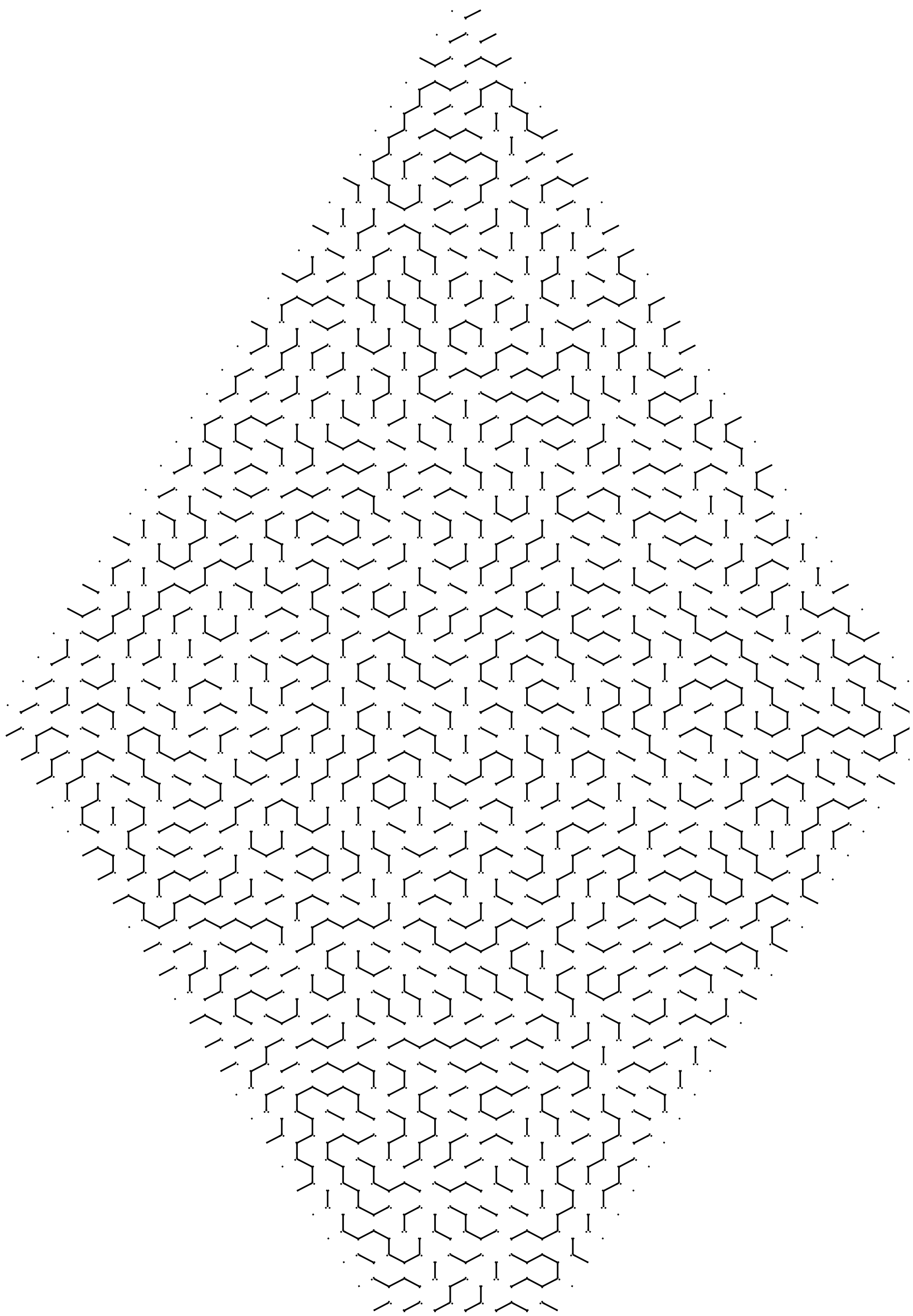

Figure 1.16: Sample of Uniform 1-2 Model

$\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$ *on all the triangles. By symmetry* $|(K_\infty^{-1})_{12}| = |(K_\infty^{-1})_{23}|$. *Then*

$$
\begin{aligned}
\Pr(\{011\}) &= \frac{1}{3} + \frac{2}{9}|(K_\infty^{-1})_{12}| + \frac{2}{9}|(K_\infty^{-1})_{23}| \\
&= \frac{1}{3} + \frac{1}{6\pi^2}\left|\iint_{\mathbb{T}^2} \frac{w(4wz+4+z-9w)}{-7(w^2+z^2)+32wz(w+z)+32(w+z)-114wz}\frac{dz}{iz}\frac{dw}{iw}\right| \\
&\quad + \frac{1}{3\pi^2}\left|\iint_{\mathbb{T}^2} \frac{z(9wz-4z-w^2-4w)}{-7(w^2+z^2)+32wz(w+z)+32(w+z)-114wz}\frac{dz}{iz}\frac{dw}{iw}\right| \\
&= \frac{1}{3} + \frac{1}{12\pi}\left|\int_{|z|=1} \frac{8}{32z-7} - \frac{18}{z(32z-7)} + \frac{16z^2-85z+29}{z(32z-7)\sqrt{4z^2-17z+4}}dz\right| \\
&\quad + \frac{1}{3\pi}\left|\int_{|w|=1} \frac{9}{(32w-7)} - \frac{4}{w(32w-7)} + \frac{22w^2-35w+8}{w(32w-7)\sqrt{4w^2-17w+4}}dw\right| \\
&= \frac{23}{48} - \frac{25}{112\pi}\arctan\frac{4}{3} - \frac{65}{336\pi}\arctan\frac{44}{117} \approx 39\%.
\end{aligned}
$$

For $\sqrt{\cdot}$ *we choose a branch with positive real part. This integral can be evaluated explicitly because the graph has critical edge weights. A sample of the critical 1-2 model is illustrated in Figure 17.*

Question: How large should N be as a function of the size of the graph?

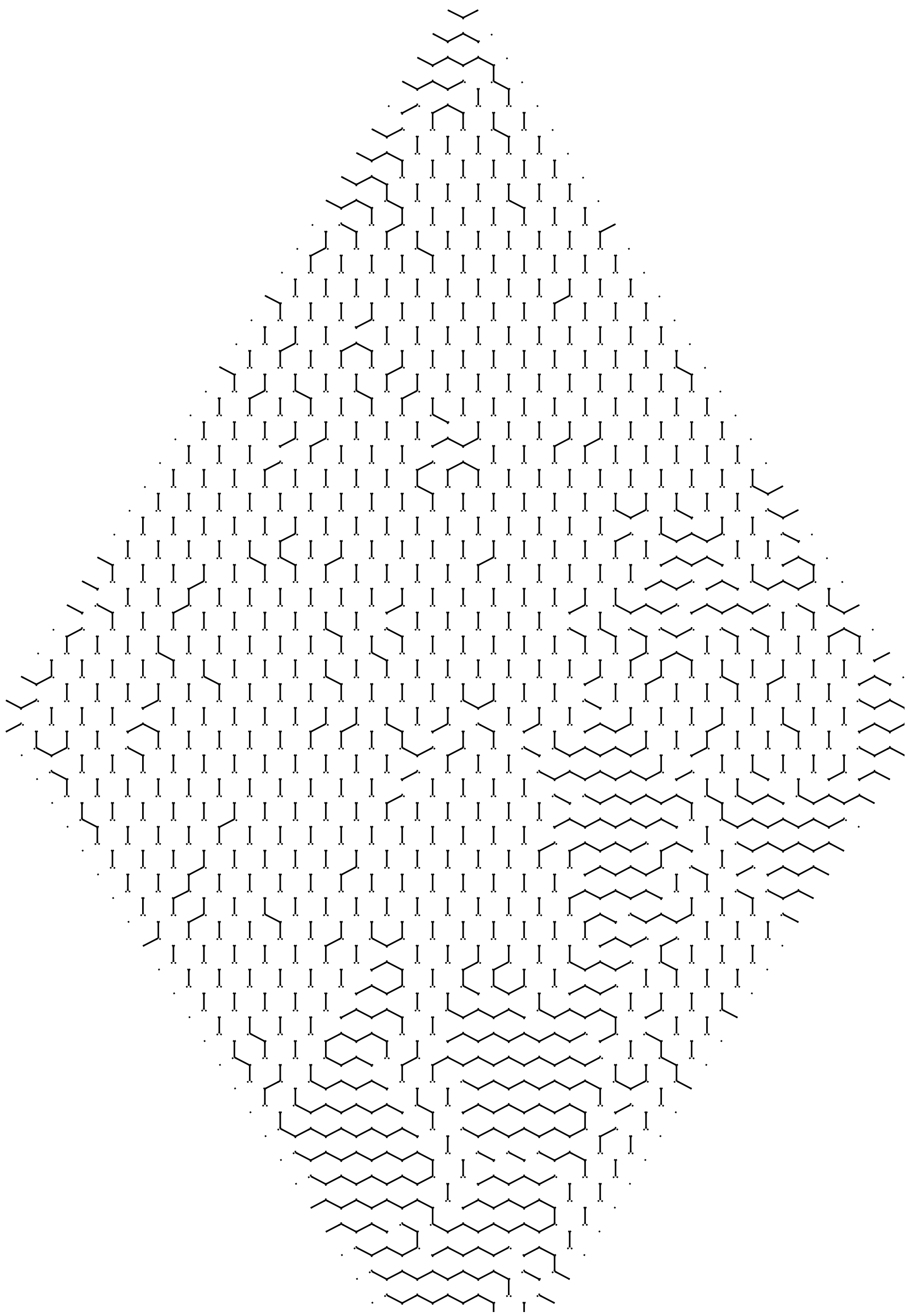

Figure 1.17: Sample of Critical 1-2 Model

1.9 Realizability Condition

In this section we give explicit realizability conditions for periodic honeycomb lattice embedded on a $n \times n$ domain. Let us consider an arbitrary vertex x with adjacent vertices y, z, w. Assume relations have following signatures:

$$r_x = \begin{pmatrix} x_1 & \dots & x_8 \end{pmatrix}^t \quad r_y = \begin{pmatrix} y_1 & \dots & y_8 \end{pmatrix}^t$$
$$r_z = \begin{pmatrix} z_1 & \dots & z_8 \end{pmatrix}^t \quad r_w = \begin{pmatrix} w_1 & \dots & w_8 \end{pmatrix}^t$$

According to the realizability equation (1), after a lengthy computation, we have

Theorem 1.28. *A periodic network of relation on $n \times n$ honeycomb lattice is relizable if and only if at each vertex x with adjacent vertices y, z, w connected by a, b, c edge, respectively, the signatures satisfy the following equation:*

$$\sum_{i,j,k \in \{0,1,2\}} X_{ijk} Y_i Z_j W_k = 0$$

where

$$Y_0 = y_1 y_4 - y_2 y_3 \qquad Y_1 = y_1 y_8 + y_4 y_5 - y_2 y_7 - y_3 y_6 \qquad Y_2 = y_5 y_8 - y_6 y_7$$
$$Z_0 = z_1 z_6 - z_2 z_5 \qquad Z_1 = z_1 z_8 + z_3 z_6 - z_4 z_5 - z_2 z_7 \qquad Z_2 = z_3 z_8 - z_4 z_7$$
$$W_0 = w_1 w_7 - w_3 w_5 \quad W_1 = w_1 w_8 + w_2 w_7 - w_5 w_4 - w_3 w_6 \quad W_2 = w_2 w_8 - w_6 w_4$$

and

$$X_{000} = x_1^2(x_3x_6 + x_2x_7 + x_4x_5 - x_1x_8) - 2x_1x_2x_3x_5$$
$$X_{001} = x_1^2(x_6x_4 - x_2x_8) + x_2^2(x_1x_7 - x_3x_5)$$
$$X_{002} = x_2^2(x_2x_7 - x_1x_8 - x_3x_6 - x_4x_5) + 2x_1x_2x_4x_6$$
$$X_{010} = x_3^2(x_1x_6 - x_2x_5) + x_1^2(x_7x_4 - x_3x_8)$$
$$X_{011} = x_1x_2(x_4x_7 - x_3x_8) + x_3x_4(x_1x_6 - x_2x_5)$$
$$X_{012} = x_2^2(x_4x_7 - x_3x_8) + x_4^2(x_1x_6 - x_2x_5)$$
$$X_{020} = x_3^2(x_3x_6 - x_2x_7 - x_4x_5 - x_1x_8) + 2x_1x_4x_3x_7$$
$$X_{021} = x_4^2(x_1x_7 - x_3x_5) + x_3^2(x_6x_4 - x_2x_8)$$
$$X_{022} = x_4^2(x_2x_7 + x_1x_8 + x_3x_6 - x_4x_5) - 2x_2x_3x_4x_8$$

$$X_{100} = x_5^2(x_1x_4 - x_2x_3) + x_1^2(x_6x_7 - x_5x_8)$$
$$X_{101} = x_1x_2(x_6x_7 - x_5x_8) + x_5x_6(x_1x_4 - x_2x_3)$$
$$X_{102} = x_6^2(x_1x_4 - x_2x_3) + x_2^2(x_6x_7 - x_5x_8)$$
$$X_{110} = x_1x_3(x_7x_6 - x_5x_8) + x_5x_7(x_1x_4 - x_2x_3)$$
$$X_{111} = x_1x_4x_6x_7 - x_2x_3x_5x_8$$
$$X_{112} = x_4x_8(x_1x_6 - x_2x_5) + x_2x_6(x_4x_7 - x_3x_8)$$
$$X_{120} = x_7^2(x_1x_4 - x_2x_3) + x_3^2(x_6x_7 - x_5x_8)$$
$$X_{121} = x_3x_7(x_4x_6 - x_2x_8) + x_4x_8(x_1x_7 - x_3x_5)$$
$$X_{122} = x_8^2(x_1x_4 - x_2x_3) + x_4^2(x_6x_7 - x_5x_8)$$

$$\begin{aligned}
X_{200} &= && x_5^2(x_4x_5 - x_1x_8 - x_3x_6 - x_2x_7) + 2x_1x_5x_6x_7 \\
X_{201} &= && x_5^2(x_6x_4 - x_2x_8) + x_6^2(x_1x_7 - x_3x_5) \\
X_{202} &= && x_6^2(x_1x_8 + x_2x_7 + x_4x_5 - x_3x_6) - 2x_5x_6x_2x_8 \\
X_{210} &= && x_5^2(x_4x_7 - x_3x_8) + x_7^2(x_1x_6 - x_2x_5) \\
X_{211} &= && x_6x_8(x_1x_7 - x_3x_5) + x_5x_7(x_4x_6 - x_2x_8) \\
X_{212} &= && x_8^2(x_1x_6 - x_2x_5) + x_6^2(x_4x_7 - x_3x_8) \\
X_{220} &= && x_7^2(x_1x_8 + x_3x_6 + x_4x_5 - x_2x_7) - 2x_3x_8x_5x_7 \\
X_{221} &= && x_8^2(x_1x_7 - x_3x_5) + x_7^2(x_6x_4 - x_2x_8) \\
X_{222} &= && x_8^2(x_1x_8 - x_3x_6 - x_2x_7 - x_4x_5) + 2x_7x_8x_6x_4
\end{aligned}$$

Theorem 1.29. *A periodic network of relation with $n \times n$ fundamental domain is bipartite realizable if and only if it is realizable and at any vertex v, the signature satisfy*

$$\begin{gathered}
v_1^2v_8^2 + v_2^2v_7^2 + v_3^2v_6^2 + v_4^2v_5^2 - 2v_1v_8v_2v_7 - 2v_1v_8v_3v_6 - 2v_1v_8v_4v_5 \\
-2v_2v_7v_3v_6 - 2v_2v_7v_4v_5 - 2v_3v_6v_4v_5 + 4v_1v_4v_6v_7 + 4v_2v_3v_5v_8 = 0
\end{gathered}$$

Proof. Without loss of generality, we assume at each matchgate, the signature $\{111\}$ is 0, and assume v is a black vertex. The $\{111\}$ entry of the signature of a black vertex is 0 gives us

$$\alpha_{1a}\alpha_{1b}\alpha_{1c}v_1 + \alpha_{1a}\alpha_{1b}v_2 + \alpha_{1a}\alpha_{1c}v_3 + \alpha_{1a}v_4 + \alpha_{1b}\alpha_{1c}v_5 + \alpha_{1b}v_6 + \alpha_{1c}v_7 + v_8 = 0 \tag{1.48}$$

The parity constraint implies that the $\{110\}$ entry is also 0, namely

$$\alpha_{1a}\alpha_{1b}\alpha_{0c}v_1 + \alpha_{1a}\alpha_{1b}v_2 + \alpha_{1a}\alpha_{0c}v_3 + \alpha_{1a}v_4 + \alpha_{1b}\alpha_{0c}v_5 + \alpha_{1b}v_6 + \alpha_{0c}v_7 + v_8 = 0 \tag{1.49}$$

From (A.1)(A.2), we can solve $\alpha_1 a$, and solution has following form

$$\alpha_{1a} = \frac{N_1}{D_1} = \frac{N_2}{D_2}$$

Then $N_1 D_2 - N_2 D_1 = 0$. Under the assumption that $\alpha_{0c} \neq \alpha_{1c}$, we have

$$(-v_2 v_5 + v_6 v_1)\alpha_{1b}^2 + (v_8 v_1 + v_6 v_3 - v_4 v_5 - v_2 v_7)\alpha_{1b} + v_8 v_3 - v_4 v_7 = 0 \tag{1.50}$$

From (12), we have

$$2(v_1 v_6 - v_2 v_5)\alpha_{0b}\alpha_{1b} + (v_1 v_8 + v_3 v_6 - v_4 v_5 - v_2 v_7)(\alpha_{0b} + \alpha_{1b}) + 2(v_3 v_8 - v_4 v_7) = 0 \tag{1.51}$$

$2\times$(A.3)$-$(A.4), under the assumption that $\alpha_{0b} \neq \alpha_{1b}$, we have

$$\alpha_{1b} = -\frac{v_1 v_8 + v_3 v_6 - v_4 v_5 - v_2 v_7}{2(v_1 v_6 - v_2 v_5)}$$

which implies that equation (A.3) has double real roots, and its discriminant is 0. That is exactly the statement of the theorem. □

Proof of Proposition 3.9 Since holographic reduction is an invertible process, two matchgrids M and $\hat{M}$ are holographically equivalent if and only if there exists a basis for each edge, such that one can be transformed to the other using holographic reduction. Obviously holographic equivalent matchgrids have the same dimer partition function. Assume weights m_b, m_w of M and $\hat{m_b}, \hat{m_w}$ of $\hat{M}$ are as follows:

$$m_w^{ij} = \begin{pmatrix} 0 & c_1^{ij} & b_1^{ij} & 0 & a_1^{ij} & 0 & 0 & d_1^{ij} \end{pmatrix}^t$$
$$m_b^{ij} = \begin{pmatrix} 0 & c_2^{ij} & b_2^{ij} & 0 & a_2^{ij} & 0 & 0 & d_2^{ij} \end{pmatrix}^t$$
$$\hat{m}_w^{ij} = \begin{pmatrix} 0 & \hat{c}_1^{ij} & \hat{b}_1^{ij} & 0 & \hat{a}_1^{ij} & 0 & 0 & \hat{d}_1^{ij} \end{pmatrix}^t$$
$$\hat{m}_b^{ij} = \begin{pmatrix} 0 & \hat{c}_2^{ij} & \hat{b}_2^{ij} & 0 & \hat{a}_2^{ij} & 0 & 0 & \hat{d}_2^{ij} \end{pmatrix}^t$$

Then by equations (1.16)-(1.21), and the uniqueness of the basis on each edge, we have

$$\frac{a_1^{ij} b_1^{ij}}{d_1^{ij} c_1^{ij}} = \frac{d_2^{i,j-1} c_2^{i,j-1}}{b_2^{i,j-1} a_2^{i,j-1}} = a_{03}^{ij} a_{13}^{ij} = b_{03}^{i,j-1} b_{13}^{i,j-1} \tag{1.52}$$
$$\frac{a_1^{ij} c_1^{ij}}{b_1^{ij} d_1^{ij}} = \frac{d_2^{i-1,j} b_2^{i-1,j}}{a_2^{i-1,j} c_5^{i-1,j}} = a_{02}^{ij} a_{12}^{ij} = b_{02}^{i-1,j} b_{12}^{i-1,j} \tag{1.53}$$
$$\frac{b_1^{ij} c_1^{ij}}{a_1^{ij} d_1^{ij}} = \frac{d_2^{ij} a_2^{ij}}{b_2^{ij} c_2^{ij}} = a_{01}^{ij} a_{11}^{ij} = b_{01}^{ij} b_{11}^{ij} \tag{1.54}$$

Plugging in (A.5)-(A.7) to (1.9)-(1.10), we have

$$\begin{cases} \hat{c}_2^{ij} = c_2^{ij} n_{01}^{i,j,0} n_{02}^{i,j,0} p_{13}^{i,j,0} \cdot C_1^{ij} \\ \hat{b}_2^{ij} = b_2^{ij} n_{01}^{i,j,0} p_{12}^{i,j,0} n_{03}^{i,j,0} \cdot C_1^{ij} \\ \hat{a}_2^{ij} = a_2^{ij} p_{11}^{i,j,0} n_{02}^{i,j,0} n_{03}^{i,j,0} \cdot C_1^{ij} \\ \hat{d}_2^{ij} = d_2^{ij} p_{11}^{i,j,0} p_{12}^{i,j,0} p_{13}^{i,j,0} \cdot C_1^{ij} \end{cases}$$

$$\begin{cases} \hat{c}_1^{ij} = \frac{c_1^{ij}}{n_{01}^{i,j,1} n_{02}^{i,j,1} p_{13}^{i,j,1}} \cdot C_2^{ij} \\ \hat{b}_1^{ij} = \frac{b_1^{ij}}{n_{01}^{i,j,1} p_{12}^{i,j,1} n_{03}^{i,j,1}} \cdot C_2^{ij} \\ \hat{a}_1^{ij} = \frac{a_1^{ij}}{p_{11}^{i,j,1} n_{02}^{i,j,1} n_{03}^{i,j,1}} \cdot C_2^{ij} \\ \hat{d}_1^{ij} = \frac{d_1^{ij}}{p_{11}^{i,j,1} p_{12}^{i,j,1} p_{13}^{i,j,1}} \cdot C_2^{ij} \end{cases}$$

To prove that probability measures of M and $\hat{M}$ are identical, we only need to prove that for any dimer configuration, the products of weights differ by the same constant factor. Each dimer configuration corresponds to a binary sequence of length N, where $N = 3n^2$ is the number of connecting edges. Choose an arbitrary edge with basis $\begin{pmatrix} n_0 & n_1 \\ p_0 & p_1 \end{pmatrix}$. If the edge is occupied, then from M to $\hat{M}$, adjacent generator weight is divided by p_1, while adjacent recognizer weight is multiplied by p_1. If the edge is unoccupied, then from M to $\hat{M}$, adjacent generator weight is divided by n_0, while the adjacent recognizer weight is multiplied by n_0. Therefore, the total effect is for any particular dimer configuration ϖ

$$\frac{\text{Partition}(\varpi \textit{ on } \hat{M})}{\text{Partition}(\varpi \textit{ on } M)} = \prod_{i,j} C_1^{ij} C_2^{ij}$$

which is a constant independent of configuration ϖ. $\square$

Chapter Two

Spectral Curve of Periodic Fisher Graphs

2.1 Introduction

The **Fisher graph** we consider in this paper is a graph with each vertex of the honeycomb lattice replaced by a triangle, see Figure 1.

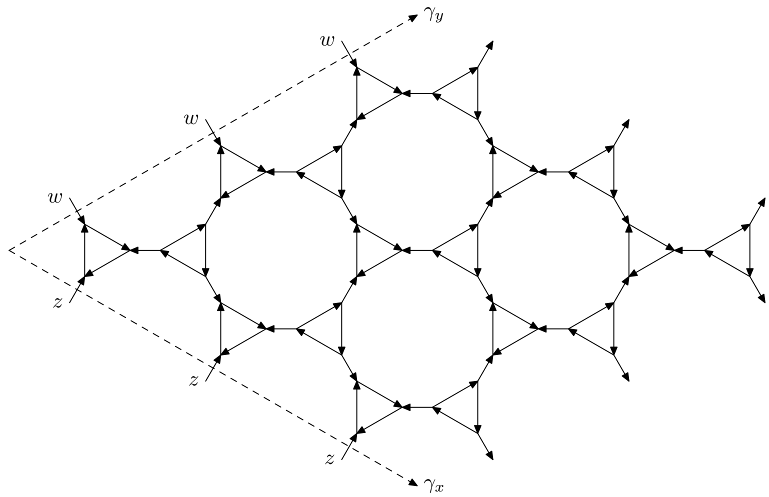

Figure 2.1: Fisher graph

We also discuss periodic graph on a strip in this paper. Assume we have a Fisher graph, periodic in the z direction, finite in the w direction, by removing all w-edges and $\frac{1}{w}$ edges. An example of such a graph is shown in Figure 2. Assume it is $m \times n$,

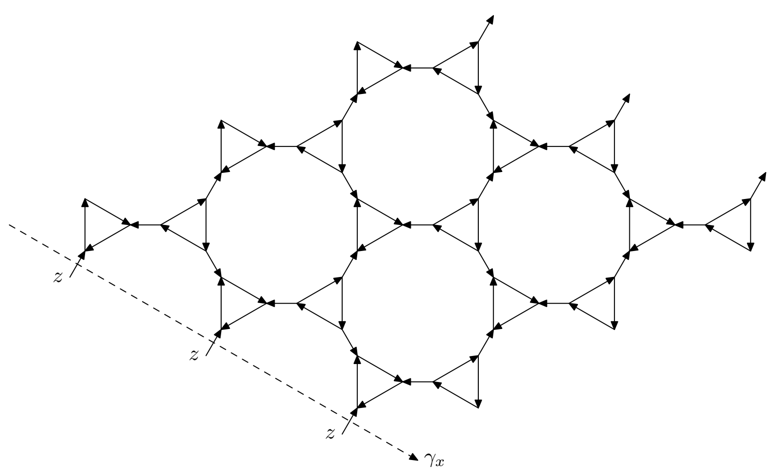

Figure 2.2: Fisher graph on a cylinder

meaning that it has width m with respect to z, but period n with respect to w. We embed this graph into a cylinder. Let $K(z)$ be the corresponding weighted adjacency matrix with an orientation shown in Figure 2. $K(z)$ is obtained from $K(z,w)$ by giving all edges crossed by γ_y weight 0. Let $P(z) = \det K(z)$.

Kenyon and Okounkov ([15]) proved that the spectral curves of any periodic bipartite graph with positive edge weights are **Harnack curves** [26, 27], whose intersection with $\mathbb{T}_{x,y} = \{(z,w) : |z| = e^x, |w| = e^y\}$ can only be: *i*. no intersection; *ii*. a pair of conjugate points, each of which is of multiplicity 1; *iii*. a single real node. Since the Fisher graph we consider in this paper is non-bipartite, no previous result is known. We prove the following theorems

Theorem 2.1. *Consider the dimer model on a banded graph, which is periodic in one direction, and finite on the other direction. Its quotient graph under the translation can be embedded into a cylinder, as illustrated in Figure 2. The intersection of* $P(z) = 0$ *and* $\mathbb{T}$ *is either empty or a single real point.*

Theorem 2.2. *Consider the dimer model on a bi-periodic Fisher Graph, given edge-orientations as illustrated in Figure 1. Assume all the edge weights are strictly positive, the only possible intersection of* $P(z,w) = 0$ *with the unit torus* $\mathbb{T}^2 = \{(z,w) : |z| = 1, |w| = 1\}$ *is a single real point. If the real zero exists on* $\mathbb{T}^2$*, for generic choice of edge weights, the zero is of multiplicity 2.*

Our result is very general. Firstly, it leads to important properties of the dimer model on cylindrical graphs, such as weak convergence of Boltzmann measures and convergence rate of correlations for an infinite, periodic graph with finite width along one direction, see Proposition 3.4. Secondly, since the dimer model on the Fisher graph is closely related to the Ising model [25] and the vertex model [**?**], our result gives a quantitative way to understand the phase transition for those important physical models [24].

2.2 Combinatorial and Analytic Properties

To compute the characteristic polynomial $P(z,w)$ of the Fisher graph, we give an orientation to the edges as illustrated in Figure 1.

Lemma 2.3. *The characteristic polynomial P(z,w) for a periodic Fisher graph with period $(m,0)$ and $(0,n)$ is a Laurent polynomial of the following form:*

$$P(z,w) = \sum_{i,j} P_{ij}(z^i w^j + \frac{1}{w^j z^i})$$

where (i,j) are integral points of the Newton polygon with vertices $(\pm m, 0), (0, \pm n), (\pm m, \mp n)$:

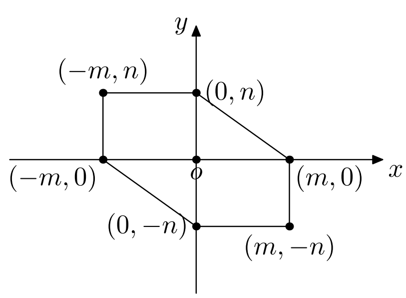

Figure 2.3: Newton polygon

Proof. Let $p = 6mn$. By definition,

$$P(z,w) = \det K(z,w) = \sum_{i_1,\ldots,i_p} (-1)^{\sigma(i_1,\ldots,i_p)} k_{1,i_1} \times \cdots \times k_{p,i_p}.$$

Here σ is the number of even cycles of the permutation $(i_1, ..., i_p)$. The sum is over all possible permutations of p elements. Each term of $P(z,w)$ corresponds to an oriented loop configuration occupying each vertex exactly twice. For the graph G_1 with m z-edges and n w-edges, $P(z,w)$ is a Laurent polynomial with leading terms

$z^m, \frac{1}{z^m}, w^n, \frac{1}{w^n}, \frac{z^m}{w^n}, \frac{w^n}{z^m}$. $z^i w^j$ corresponds to loops of homology class (i, j). $P(z, w)$ is symmetric with respect to $z^i w^j$ and $\frac{1}{w^j z^i}$. That is because for each term of $z^i w^j$, if we reverse the orientation of all loops, we get a term of $\frac{1}{w^j z^i}$ with coefficients of the same absolute value, corresponding to the product of weights of edges included in the configuration. The sign of the term is multiplied by $(-1)^p = 1$.

To show that all the powers (i, j) lie in the polygon, we multiply all the b-edges by z (or $\frac{1}{z}$), and all the c-edges by w(or $\frac{1}{w}$), according to their orientation. This way the corresponding characteristic polynomial becomes $P(z^n, w^m)$. Let $(\tilde{i}, \tilde{j})$ be a power of monomial in $P(z^n, w^m)$. At each triangle, all the possible contributions of local configurations to the power of the monomial can only be $(0, 0), (0, \pm 1), (\pm 1, 0), (\pm 1, \mp 1)$. Examples are illustrated in the following Figure 4. The left graph has two doubled edges, and the contribution to the power of the monomial is $(0, 0)$, the right graph has a loop winding from the z-edge to the w-edge, and the contribution to the power of the monomial is $(1, -1)$.

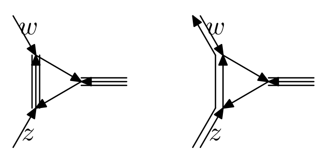

Figure 2.4: local configurations

Consider all the $2mn$ triangles. For each edge, we considered its contribution twice, since we counted it from both triangles it connected. Hence we have

$$-2mn \leq 2\tilde{i} \leq 2mn$$

$$-2mn \leq 2\tilde{j} \leq 2mn$$

$$-2mn \leq 2(\tilde{i} + \tilde{j}) \leq 2mn$$

Since $\tilde{i} = ni$, $\tilde{j} = mj$, and the Newton polygon $N(P)$ is defined to be

$$N(P) = \text{convex hull}\{(i,j) \in \mathbb{Z}^2 | z^i w^j \text{ is a monomial in P(z, w)}\},$$

the lemma follows □

Lemma 2.4. *For configurations with odd loops corresponding to a non-vanishing term in $P(z,w)$, all odd loops have non-trivial homology, and the total number of odd loops is even.*

Proof. For any loop configuration on the Fisher graph embedded on a torus, the number of odd loops is always even. That is because the total number of vertices are even, while odd loops always involve odd number of vertices. Any term in P_{ij} including odd loops can appear only when odd loops have non-trivial homology. That is because for a contractible odd loop, we can reverse the orientation of that loop to negate the sign of that term. The term with reversed orientation on the odd loop cancels with the original term, because the homology class $(0,0)$ of the configurations is not changed given the odd loop is contractible. □

2.3 Graph on a Cylinder

2.3.1 Spectral Curve

Proof of Theorem 2.1 Without loss of generality, assume the period n, or the circumference of the cylinder, is even. Assume $P(z) = 0$ has a non-real zero $z_0 \in \mathbb{T}$. Assume

$$z_0 = e^{i\alpha_0 \pi}, \qquad \alpha_0 \in (0,1) \cup (1,2)$$

We classify all the real numbers in $(0,1) \cup (1,2)$ into 3 types

1. $\alpha_0 = \frac{p}{q}$ where p, q are positive integers with no common factors, p is odd
2. α_0 is irrational
3. $\alpha_0 = \frac{p}{q}$ where p, q are positive integers with no common factors, p is even

First let us consider Case 1 and Case 2. There exists a sequence $\ell_k \in \mathbb{N}$, such that

$$\lim_{k \to \infty} z_0^{\ell_k} = -1$$

In other words, if we assume $z_0^{\ell_k} = e^{i\alpha_k \pi}$ where $\alpha_k \in [0,2)$, then

$$\lim_{k \to \infty} \alpha_k = 1$$

According to the formula of enlarging the fundamental domain,

$$P(z_0) = 0$$

implies

$$P_{\ell_k}(z_0^{\ell_k}) = 0 \qquad \forall k$$

Since the cylindrical graph is actually planar, if we reverse the orientation of all the edges crossed by γ_x, we get a clockwise-odd orientation, given that n is even. Hence

$$P_{\ell_k}(-1) = Z_{\ell_k}^2$$

where Z_{ℓ_k} is the partition function of dimer configurations of the cylinder with circumference $n\ell_k$ and height m. Therefore we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \left| \frac{P_{\ell_k}(z_0^{\ell_k}) - P_{\ell_k}(-1)}{P_{\ell_k}(-1)} \right| && (2.1) \\ &= \lim_{k\to\infty} \frac{1}{Z_{\ell_k}^2} |P_{\ell_k}(e^{i\alpha_k\pi}) - P_{\ell_k}(e^{i\pi})| && (2.2) \\ &= \lim_{k\to\infty} \frac{1}{Z_{\ell_k}^2} \left| \sum_{t=1}^{\infty} \frac{[\pi(\alpha_k - 1)]^{2t}}{(2t)!} \frac{\partial^{2t} P_{\ell_k}(e^{i\theta})}{\partial \theta^{2t}} |_{\theta=\pi} \right| && (2.3) \end{aligned}$$

Since

$$P_{\ell_k}(z) = \sum_{0\le j\le m} P_j^{(\ell_k)} \left(z_j + \frac{1}{z^j} \right)$$

where $P_j^{(\ell_k)}$ is the signed sum of loop configurations winding exactly j times around the cylinder (see Lemma 2.1), we have

$$\frac{\partial^{2t} P_{\ell_k}(e^{i\theta})}{\partial \theta^{2t}} |_{\theta=\pi} = \sum_{1\le j\le m} 2j^{2t}(-1)^t P_j^{(\ell_k)},$$

and

$$\lim_{k\to\infty}\frac{1}{Z_{\ell_k}^2}\left|\sum_{t=1}^{\infty}\frac{[\pi(\alpha_k-1)^{2t}]}{(2t)!}\frac{\partial^{2t}P_{\ell_k}(e^{i\theta})}{\partial\theta^{2t}}|_{\theta=\pi}\right|$$
$$\leq \lim_{k\to\infty}\frac{1}{Z_{\ell_k}^2}\sum_{t=1}^{\infty}\frac{[\pi m(\alpha_k-1)]^{2t}}{(2t)!}2\sum_{1\leq j\leq m}|P_j^{(\ell_k)}|$$

To estimate $|P_j^{(\ell_k)}|$, let us divide the $m\times\ell_k n$ cylinder into $\ell_k n$, $m\times 1$ layers. Connecting edges between two layers may be occupied once, unoccupied, or occupied twice (appear as doubled edges). Choose one layer L_0, we construct an equivalent class of loop configurations. Two loop configurations are equivalent if they differ from each other only in L_0, and coincide on all the other layers and boundary edges of L_0.

We claim that for any given equivalent class, there is at least one configuration including only even loops. To see that we choose an arbitrary configuration in that equivalent class including odd loops. Let $\mathcal{T}$ be the set of all triangles in L_0 belonging to some loop crossing L_0. We choose an odd loop s_1. Choose an triangle $\Delta_1\in s_1\cap L_0$. Moving along L_0 until we find a triangle Δ_2 belonging to a different non-planar odd loop s_2. This is always possible because by Lemma 2.2, any loop configuration including odd loops always has an even number of odd loops, and all the odd loops are winding once along the cylinder. Change path through Δ_1. Starting from Δ_1, we change the doubled edge configuration to alternating edges along a path in L_0, and change the path through all triangles, belonging to even loops, or s_1, between Δ_1 and Δ_2, then we change the path through Δ_2. For any even loop between Δ_2 and Δ_2, we change paths through an even number of triangles of that even loop, hence it is still an even loop. However, for s_1 and s_2, we change paths through an odd number of triangles, then both of them become even. We can continue this process until we eliminate all the odd loops, because there are always an even number of odd loops in the configuration. An example of such an path change process is illustrated in the following figures.

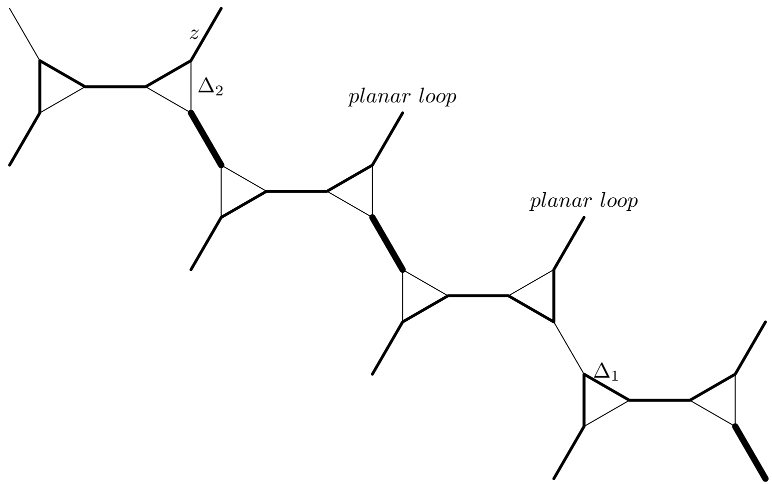

Figure 2.5: before path change

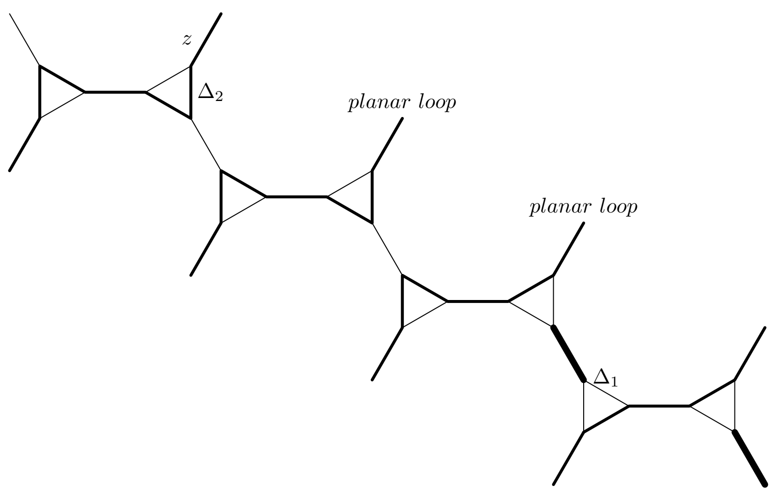

Figure 2.6: after path change

Since we have at most 2^m different configurations in each equivalent class, and each equivalent class has at least one configuration including only even loops, we have

$$\frac{\#\text{ of loop configurations including odd loops}}{\#\text{ of even loop configurations}} < 2^m$$

We have a finite number of different edge weights, each of which is positive, hence the quotient of any two weights is bounded by a constant C_1, then

$$\begin{aligned}\sum_{1\leq j\leq m} |P_j^{(\ell_k)}| \leq &\text{ Partition function of configurations including nonplanar odd loops}\\ &+\text{Partition function of configurations with only even loops}\\ &\leq (2^m C_1^{6m}+1)\text{Partition function of even loop configurations} \leq C_2^m Z_{\ell_k}^2\end{aligned}$$

As a result

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{Z_{\ell_k}^2}\left|\sum_{t=1}^{\infty}\frac{[\pi(\alpha_k-1)]^{2t}}{(2t)!}\frac{\partial^{2t}P_k(e^{i\theta})}{\partial\theta^{2t}}\right|\\ &\leq \lim_{k\to\infty}\frac{1}{Z_{\ell_k}^2}\sum_{t=1}^{\infty}\frac{[m\pi(\alpha_k-1)]^{2t}}{(2t)!}C_2^m Z_{\ell_k}^2\end{aligned}$$

Since m is a constant, and $\lim_{k\to\infty}\alpha_k = 1$, we have

$$\lim_{k\to\infty}\sum_{t=1}^{\infty}\frac{[m\pi(\alpha_k-1)]^{2t}}{(2t)!}C_2^m = 0$$

which is a contradiction to (2.3). Hence for all α_0 in Case 1 and Case 2, $P(e^{i\alpha_0\pi}) \neq 0$.

Now let us consider α_0 in Case 3. From the argument above we derive that as long as

$$|\alpha - 1| < \delta,$$

where δ is a small positive number depending on the edge weights and m, but not on n,

$$P(e^{i\alpha\pi}) \neq 0.$$

Consider $\alpha_0 = \frac{p}{q}$, where p, q have no common factors and p is even. We claim that as long as the denominator q is sufficiently large, $P(z_0) \neq 0$. To see that, if

$$\frac{1}{q} < \frac{\delta}{2},$$

then there exists an integer k, such that

$$\left| \frac{1}{\pi} \operatorname{Arg}[e^{ik\alpha_0\pi}] - 1 \right| < \delta.$$

Hence after enlarging the fundamental domain to an $m \times kn$ cylinder, we have

$$P_k(e^{ik\alpha_0\pi}) \neq 0,$$

hence

$$P(e^{i\alpha_0\pi}) \neq 0.$$

Now we consider $\frac{1}{q} \geq \frac{\delta}{2}$, only finitely many q's satisfy this condition. Let ℓ be a prime number satisfying $\ell > \left[\frac{2}{\delta}\right] + 1$, then after enlarging the fundamental domain to an $m \times \ell n$ cylinder, the corresponding δ will not change because it depends only on m. For any

$$1 \geq \frac{1}{q} \geq \frac{\delta}{2},$$

we have

$$\frac{\delta}{2} > \frac{1}{\ell q}.$$

Let $\ell\alpha_1(\mod 2) = \alpha_0$. Namely,

$$\alpha_1 = \frac{p + 2sq}{\ell q} \qquad \text{for } s = 1, 2, ..., \ell$$

in reduced form. By the previous argument $P(e^{\alpha_1 i\pi}) \neq 0$, hence $P_\ell(e^{\alpha_0\pi\sqrt{-1}}) \neq 0$. Hence when α_0 is in Case 3, $P_\ell(z_0) \neq 0$ after enlarging the fundamental domain to $m \times n\ell$, where ℓ depends only on m, and is independent of α_0. If $P(z_0) = 0$, z_0 is not real, then we enlarge the fundamental domain to $m \times n\ell$, where ℓ is a big prime number depending only on m, we derive that $P_\ell(z_0^\ell) = 0$, however this is impossible since $P_\ell(z) = 0$ can have only real root on $\mathbb{T}$. □

Proposition 2.5. *Enumerations of all terms in P(z) show that when $m \leq 2$ (m is the height of the cylinder), the only possible intersections of $P(z) = 0$ with $\mathbb{T}$ is a single real point.*

Proof. Let $P(z)$ be the characteristic polynomial of the cylindrical graph. We modify the weighted adjacency matrices in other ways and obtain other polynomials, as described below. If an edge uv is crossed by γ_x, multiply both the entries (u, v) and (v, u) by z, we get a matrix $\mathcal{H}(z)$. If we multiply both the entries (u, v) and (v, u) by $\frac{1}{z}$, we get a matrix $\mathcal{J}(z)$. Both $\mathcal{J}(z)$ and $\mathcal{H}(z)$ are anti-symmetric matrices, hence we can define

$$s(z) = \mathrm{Pf}\mathcal{H}(z)$$

$$t(z) = \mathrm{Pf}\mathcal{J}(z)$$

Obviously $s(z)$ and $t(z)$ are Laurent polynomials with respect to z, and if $z \in \mathbb{T}$, $t(z)$ is the conjugate of $s(z)$. Let us consider

$$r(z) = P(z) - s(z)t(z)$$

Each monomial of $s(z)$, $t(z)$ corresponds to a dimer configuration, and each monomial of $P(z)$ corresponds to a loop configuration. Hence terms in $r(z)$ are configurations satisfy one of the following conditions

1. The configuration includes odd loops.

2. All the loops in the configuration are even. Yet there exists a contractible loop $\mathfrak{L}$ passing z-edges twice from opposite directions. However, when decomposing into dimer configurations $\mathcal{D}_1$ and $\mathcal{D}_2$, let N_1^z and N_2^z be the number of z-edges coming from the loop $\mathfrak{L}$ in $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, we have

$$N_1^z = 2$$
$$N_2^z = 0$$

For case 1, according to Lemma 2.2, all the odd loops are non-contractible and we have an even number of odd loops in total. Moreover, for loops on the cylindrical graph, any single non-planar loop has the homology class $(1, 0)$, that is, winding around the cylinder exactly once. Hence when $m \leq 2$, if odd loops exist, the total number of odd loops is 2. Consider an arbitrary fixed configuration of Case 1. Without loss of generality, we assume that except those odd loops, all the other parts of the graph are covered by doubled edge configuration. We also assume that the circumference of the circle, n, is even. Since the cylindrical graph is actually planar, if we reverse the orientation of all edges crossed by γ_x, we get a clockwise-odd orientation for all edges, including the face formed by the circle on top of the cylinder. Under this orientation, all the even loop configurations have a positive sign.

Any loop configuration, including odd loops, can always be obtained from an even loop configuration, by changing the path of loops through triangles, as described in the proof of Theorem 1.1. Any time we change the path through a pair of triangles, we negate the sign of the monomial corresponding to the configuration. Hence the contribution of odd loops are

$$\prod_{e \in \text{odd loops}} w_e(-1)(\frac{1}{z} - z)^2 \tag{2.4}$$

Let $z = e^{\sqrt{-1}\theta}$, (2.4) can be written as

$$\prod_{e \in \text{odd loops}} w_e 2^2 \sin^2 \theta,$$

which is nonnegative.

Let $\mathfrak{L}$ be a loop as described in case 2. Let $e_1, \cdots, e_{2l}$ be all the edges involved in the loop. Since

$$N_1^z - N_2^z = 2$$

Then the contribution of $\mathfrak{L}$ in $r(z)$ is

$$\prod_{j=1}^{2l} w_{e_j}(2 - z^2 - \frac{1}{z^2})$$

which is nonnegative given $z \in \mathbb{T}$.

Therefore $r(z) > 0$, if $z \in \mathbb{T}$, and $z \notin \mathbb{R}$. When $z \in \mathbb{T}$, $s(z)$ and $t(z)$ are complex conjugates to each other, hence $s(z)t(z) \geq 0$. As a result $P(z) > 0$ if z is not real. Since $P(-1)$ is the square of the dimer partition function, the only possible zero of $P(z)$ on $\mathbb{T}$ is 1, give that the period n is even. □

Corollary 2.6. *For any non-real $z \in \mathbb{T}$, all eigenvalues of $K(z)$ are of the form $i\lambda_j, j = 1, ..., 6mn$, where i is the imaginary unit. $3mn$ of the λ_j's are positive, the* *other λ_j's are negative.*

Proof. From the definition of $K(z)$, $iK(z)$ is a Hermitian matrix. We claim that $iK(z)$ has $3mn$ positive and $3mn$ negative eigenvalues for any non-real $z \in \mathbb{T}$. In fact, for a planar graph with no z vertices, K_0 is a anti-symmetric real matrix with eigenvalues $\pm i\lambda_j$, $1 \le j \le 3mn - m$, and $\det K_0 > 0$, because $|PfK_0|$ is the partition function of dimer configurations with positive edge weights, and $\det K_0 = (PfK_0)^2$ By the previous theorem, every time we add a pair of boundary vertices connected by a z edge, we have a anti-Hermitian matrix K_{r+1} of order $6mn - 2m + 2(r+1)$ with K_r as a principal minor. By induction hypothesis iK_r has the same number of positive and negative eigenvalues, the interlacing theorem implies that iK_{r+1} has at least $3mn - m + r$ positive and $3mn - m + r$ negative eigenvalues. By previous theorem, $\det K_{r+1} > 0$ for non-real z, so the other two eigenvalues of iK_{r+1} can only be one positive and one negative. ∎

Lemma 2.7. *If $P(1) = 0$, then $\left.\frac{\partial P(z)}{\partial z}\right|_{z=1} = 0$. Let $z = e^{\sqrt{-1}\theta}$, then $\left.\frac{\partial P(e^{\sqrt{-1}\theta})}{\partial \theta}\right| = 0$. For generic choice of edge weights, $\left.\frac{\partial^2 P(z)}{\partial z^2}\right|_{z=1} \neq 0$, $\left.\frac{\partial^2 P(e^{\sqrt{-1}\theta})}{\partial \theta^2}\right| \neq 0$*

Proof. We prove the result for derivatives with respect to z, the derivatives with respect to θ are very similar. Since

$$P(z) = \sum_j P_j(z^j + \frac{1}{z^j})$$

we have

$$\begin{aligned} \left.\frac{\partial P}{\partial z}\right|_{z=1} &= \sum_j j\ P_j(z^{j-1} - \frac{1}{z^{j+1}})\Big|_{z=1} = 0 \\ \left.\frac{\partial^2 P}{\partial z^2}\right|_{z=1} &= \sum_j 2j^2 P_j \end{aligned}$$

Each monomial in $P(1)$ corresponds to a loop configuration including only even loops. Consider a configuration including $2k$ even loops, each of which winding exactly once around the cylinder. Except those non-contractible even loops, the rest of the graph is covered by doubled-edge configuration. Consider $P(1)$ as a polynomial of edge weights. The monomial corresponding to that configuration has a coefficient

$$S_{0,k} = \pm 2^{2k}$$

because each single loop can have two different orientations, corresponding to two terms in the expansion of the determinant. However, the coefficient of the monomial corresponding to the same configuration in $\left.\frac{\partial^2 P(z)}{\partial z^2}\right)_{z=1}$ is

$$S_{2,k} = (2k)^2 + 2k(2k-2)^2 + \begin{pmatrix} 2k \\ 2 \end{pmatrix} (2k-4)^2 + \cdots + \begin{pmatrix} 2k \\ 2k \end{pmatrix} (2k-4k)^2$$

Obviously, $\frac{S_{2,k}}{S_{0,k}}$ is a number depending on k. Therefor $P(1)$ and $\left.\frac{\partial^2 P(z)}{\partial z^2}\right|_{z=1}$ cannot divide each other. Let

$$\begin{aligned} W_2 &= \{(w_e)_{e\in E} : \left.\frac{\partial^2 P(z)}{\partial z^2}\right|_{z=1} = 0\} \\ W_0 &= \{(w_e)_{e\in E} : P(1) = 0\} \end{aligned}$$

The intersection of W_2 and W_0 forms a proper subvariety of W. Hence if $P(1) = 0$, and we choose the edge weights generically, $\left.\frac{\partial^2 P(z)}{\partial z^2}\right|_{z=1} \neq 0$. □

Remark. Lemma 3.3 implies that if $P(1) = 0$, for generic choice of edge weights, the intersection is of multiplicity 2.

Proposition 2.8. *Let*

$$F(r) = \frac{1}{2\pi}\int_0^{2\pi} \log P(re^{i\theta})d\theta$$

If $P(z) = 0$ does not intersect the unit torus $\mathbb{T}$, $F(r)$ is differentiable at $r = 1$; If $P(z) = 0$ has a real zero of multiplicity 2 at $z = 1$,

$$\lim_{r\to 1+} \frac{\partial F(r)}{\partial r} - \lim_{r\to 1-} \frac{\partial F(r)}{\partial r} = 2$$

Proof. Since $P(z)$ is a Laurent polynomial in z, according to the Jensen's formula

$$\log|P(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|P(re^{i\theta})|d\theta$$

where $a_1, \cdots, a_n$ are the zeros of P in the interior of the disk $\{z : |z| < r\}$. If $P(z)$ has no zeros on the circle $\{z : |z| = r\}$, then

$$\begin{aligned}
\frac{\partial F}{\partial r} &= \lim_{\Delta r\to 0} \frac{F(r+\Delta r) - F(r)}{\Delta r} \\
&= \lim_{\Delta r\to 0} \frac{1}{\Delta r}\sum_{k=1}^{n} \log\left(\frac{r+\Delta r}{r}\right) \\
&= \frac{n}{r}
\end{aligned}$$

The proposition follows from substituting $r = 1$, and the fact that the intersection of $P(z) = 0$ with $\mathbb{T}$ can only be a single real point. □

Remark. Jensen's formula implies that the curve $P(z) = 0$ is Harnack if and only if

$$r_0(\lim_{r\to r_0+} - \lim_{r\to r_0-})\frac{\partial F(r)}{\partial r} \leq 2,$$

for any $r_0 > 0$. In fact, if the height of the cylinder $m \leq 3$, we can always derive that the corresponding spectral curve is Harnack. To see that, first of all, being

Harnack is a closed condition, for generic choice of edge weights, the intersection of $P(z) = 0$ with $|z| = 1$ is at most two points(counting multiplities). Without loss of generality, assume for some $r(0 < r < 1)$, $P(z) = 0$ intersects $|z| = r$ at 3 different points z_1, z_2, z_3, then $\bar{z}_1, \bar{z}_2, \bar{z}_3$, lie also on the intersection of $P(z) = 0$ and $|z| = r$, given that $P(z)$ is a real-coeffient polynomial. Hence the intersection of $P(z) = 0$ with $|z| = r$ is at least 4 points. Moreover, by symmetry, the reciprocal of those points are also roots of $P(z) = 0$, then $P(z) = 0$ has at least 8 roots, which is a contradiction to the fact $m \leq 3$.

2.3.2 Limit Measure of Cylindrical Approximation

From the proof we know that all terms in $P_{m\times n}(-1)$ are positive, if n is even. We can always enlarging the fundamental domain in z direction without changing edge weights to get

$$P_{m\times 2n}(-1) = P_{m\times n}(i)P_{m\times n}(-i) = P^2_{m\times n}(i) = |Pf_{m\times 2n}K(-1)|^2$$

Therefore $P_{m\times n}(i)$ is the partition function of dimer configurations of the $m \times 2n$ cylinder graph. According to the formula of enlarging the fundamental domain,

$$P_{m\times 2ln}(-1) = \prod_{z^{2l}=-1} P_{m\times n}(z),$$

we have

$$\begin{aligned} \lim_{l\to\infty} \frac{1}{4l} \log P_{m\times 2ln}(-1) &= \lim_{l\to\infty} \frac{1}{4\pi} \frac{2\pi}{2l} \sum_{z^{2l}=-1} \log P_{m\times n}(z) && (2.5) \\ &= \frac{1}{4\pi} \int_{\mathbb{T}} \log P_{m\times n}(z) \frac{dz}{iz} && (2.6) \end{aligned}$$

The convergence of the Riemann sums to the integral follows from the fact that the only possible zeros of $P(z)$ on $\mathbb{T}$ is a single real node. (2.6) is defined to be the **partition function per fundamental domain**.

Any probability measure on the infinite banded graph, with depth m on one direction, and period n on the other direction, is determined by the probability of cylindrical sets. Namely, we choose a finite number of edges $e_1, e_2, \cdots, e_k$ arbitrarily, and the probabilities

$$Pr(e_1 \& e_2 \& \cdots \& e_k)$$

that $e_1, e_2, \cdots, e_k$ occur in the dimer configuration simultaneously for all finite edge sets determines the probability measure. We consider the measures on the infinite graph as weak limits of measures on cylindrical graphs. First of all, we prove a lemma about the entries of the inverse Kasteleyn matrix using the cylindrical approximation.

Lemma 2.9.

$$\lim_{l\rightarrow\infty} K^{-1}_{m\times 2ln}(-1)_{(k_v,s_v),(k_w,s_w)} = \frac{1}{2\pi} p.v. \int_{\mathbb{T}} z^{k_v-k_w} \frac{cofactor K_{m\times n}(s_v,s_w)(z)}{P_{m\times n}(z)} \frac{dz}{iz}$$

Proof. To that end, we construct a transition matrix S to make $S^{-1}K_{m\times 2ln}S$ block diagonal, with each block corresponding to a $m\times n$ quotient graph. Define

$$S = (e_0^1, ..., e_0^{6mn}, e_1^1, ..., e_1^{6mn}, ..., e_{2l-1}^1, ..., e_{2l-1}^{6mn})$$

where

$$e_k^s(j,t) = \begin{cases} e^{\frac{i\pi(2j+1)(2k+1)}{4l}} & s = t \\ 0 & s \neq t \end{cases}$$

then

$$S^{-1}K_{m\times 2ln}S = \begin{pmatrix} K_{m\times n}(e^{\frac{i\pi}{2l}}) & & & \text{\Large 0} \\ & K_{m\times n}(e^{\frac{3i\pi}{2l}}) & & \\ & & \ddots & \\ \text{\Large 0} & & & K_{m\times n}(e^{\frac{i(4l-1)\pi}{2l}}) \end{pmatrix}$$

Since $S^{-1} = \frac{1}{2l}\bar{S}^t$, we have

$$K^{-1}_{m\times 2ln}(-1)_{(k_v,s_v),(k_w,s_w)} = \frac{1}{2l}\sum_{j=0}^{2l-1} \frac{cofactor K_{m\times n}(s_v, s_w)}{P_{m\times n}(e^{\frac{(2j+1)i\pi}{2l}})} e^{\frac{i(2j+1)\pi(k_v-k_w)}{2l}}$$

where k_v is the index of the fundamental domain for vertex v and s_v is the index of vertex in the fundamental domain.

If $P_{m\times n}(z)$ has no zero on $\mathbb{T}$, we have

$$\lim_{l\longrightarrow\infty} K^{-1}_{m\times 2ln}(-1)_{(k_v,s_v),(k_w,s_w)} = \frac{1}{2\pi}\int_{\mathbb{T}} z^{k_v-k_w}\frac{cofactor K_{m\times n}(s_v, s_w)(z)}{P_{m\times n}(z)}\frac{dz}{iz} \tag{2.7}$$

If 1 is an order-2 zero of $P_{m\times n}(z)$, let

$$Q(z) = z^{k_v-k_w}\frac{cofactor K_{m\times n}(s_v, s_w)(z)}{P_{m\times n}(z)},$$

Since $\det K(1)$ is an anti-symmetric real matrix of even order, and non-invertible, the dimension of its null space is non-zero and even. Hence $AdjK(1)$ is a zero matrix, and 1 is at least a zero of order 1 for $cofactor K_{m\times n}(s_v, s_w)(z)$. Then in a neighborhood of 1,

$$Q(z) = \frac{Res_{z=1}Q(z)}{z-1} + R(z),$$

where $R(z)$ is analytic at 1.

$$\lim_{l\to\infty} K^{-1}_{m\times 2ln}(-1)_{(k_v,s_v),(k_w,s_w)} \tag{2.8}$$

$$= \lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}\left(\sum_{0\le j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{\frac{l\delta}{\pi}-\frac{1}{2}\le j\le 2l-\frac{1}{2}-\frac{l\delta}{\pi}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\le 2l-1}\right) Q(e^{\frac{(2j+1)i\pi}{2l}}) \tag{2.9}$$

For the second term,

$$\lim_{\delta\to 0+}\lim_{l\to 0}\frac{1}{2l}\sum_{\frac{l\delta}{\pi}-\frac{1}{2}\le j\le 2l-\frac{1}{2}-\frac{l\delta}{\pi}} Q(e^{\frac{(2j+1)i\pi}{2l}}) \tag{2.10}$$

$$= \lim_{\delta\to 0+}\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} Q(e^{i\theta})d\theta \tag{2.11}$$

$$= p.v.\frac{1}{2\pi}\int_{\mathbb{T}} Q(z)\frac{dz}{iz} \tag{2.12}$$

$$= \frac{1}{2}Res_{z=1}Q(z) + \sum Res_{|z|<1}\frac{Q(z)}{z} \tag{2.13}$$

For the first and the third term

$$\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}\left(\sum_{0\le j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\le 2l-1}\right) Q(e^{\frac{(2j+1)i\pi}{2l}})$$

$$= \lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}\left(\sum_{0\le j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\le 2l-1}\right)\left(\frac{Res_{z=1}Q(z)}{e^{\frac{(2j+1)i\pi}{2l}}-1} + R(e^{\frac{(2j+1)i\pi}{2l}})\right)$$

Since $R(z)$ is analytic in a neighborhood of 1,

$$\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}\left(\sum_{0\le j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\le 2l-1}\right) R(e^{\frac{(2j+1)i\pi}{2l}}) = 0 \tag{2.14}$$

Moreover,

$$\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}Res_{z=1}Q(z)\left(\sum_{0\leq j<\frac{l\delta}{\pi}-\frac{1}{2}}+\sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\leq 2l-1}\right)\frac{1}{e^{\frac{(2j+1)i\pi}{2l}}-1} \tag{2.15}$$

$$= \lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}Res_{z=1}Q(z)\sum_{0\leq j<\frac{l\delta}{\pi}-\frac{1}{2}}\left(\frac{1}{e^{\frac{(2j+1)i\pi}{2l}}-1}+\frac{1}{e^{-\frac{(2j+1)i\pi}{2l}}-1}\right) \tag{2.16}$$

$$= -\lim_{\delta\to 0+}\frac{1}{2l}Res_{z=1}Q(z)\left[\frac{l\delta}{\pi}+\frac{1}{2}\right]=0 \tag{2.17}$$

And the theorem follows from (2.7), (2.9), (2.13),(2.14), (2.17). □

Proposition 2.10. *Using a large cylinder to approximate the infinite periodic banded graph, we derive that the weak limit of probability measures of the dimer model exists. The probability of a cylindrical set under this limit measure is*

$$Pr(e_1\&e_2\&\cdots\&e_k)=\lim_{l\to\infty}\prod_{j=1}^{k}w_{e_j}\sqrt{\det K^{-1}_{m\times 2ln}\begin{pmatrix} u_1 & v_1 & \cdots & u_k & v_k \\ u_1 & v_1 & \cdots & u_k & v_k \end{pmatrix}}$$

where w_{e_j} is the weight of e_j, and u_j, v_j are the two vertices of e_j.

Proof. On a finite cylindrical graph of $m\times 2ln$

$$Pr(e_1\&e_2\&\cdots\&e_k)=\frac{Z_{e_1,\cdots,e_k}}{Z}$$

where Z is the partition function of dimer configurations on that graph, and $Z_{e_1,\cdots,e_k}$ is the partition function of dimer configurations for which $e_1,\cdots e_k$ appear simultaneously. Since

$$Z^2 = \det K_{m\times 2ln}(-1) \tag{2.18}$$

$$Z^2_{e_1,\cdots,e_k} = \prod_{j=1}^{k}w_{e_j}\,\text{cofactor}\,K_{m\times 2ln,(u_1,v_1,\cdots,u_k,v_k)}(-1) \tag{2.19}$$

(2.19) follows from the fact that if originally we have a clockwise-odd orientation, we still have a clockwise-odd orientation when removing edges and ending vertices, while keeping the orientation on the rest of the graph. The proposition follows from Jacobi's formula for the determinant of minor matrices. □

We consider two edges e_1 and e_2 with weight x_1 and x_2 on $m \times 2ln$ cylinder, and compute the covariance.

$$\begin{aligned}
& Pr_{m\times 2ln}(e_1\&e_2) - Pr_{m\times 2ln}(e_1)Pr_{m\times 2ln}(e_2) && (2.20)\\
= \; & x_1x_2\sqrt{\det K^{-1}_{m\times 2ln}\begin{pmatrix} v_1 & w_1 & v_2 & w_2 \\ v_1 & w_1 & v_2 & w_2 \end{pmatrix}(-1)} && (2.21)\\
& -x_1x_2\sqrt{\det K^{-1}_{m\times 2ln}\begin{pmatrix} v_1 & w_1 \\ v_1 & w_1 \end{pmatrix}(-1)\cdot \det K^{-1}_{m\times 2ln}\begin{pmatrix} v_2 & w_2 \\ v_2 & w_2 \end{pmatrix}(-1)} && (2.22)\\
= \; & x_1x_2(|K^{-1}_{m\times 2ln}(v_1,w_1)K^{-1}_{m\times 2ln}(v_2,w_2) + K^{-1}_{m\times 2ln}(v_1,w_2)K^{-1}_{m\times 2ln}(w_1,v_2) && (2.23)\\
& -K^{-1}_{m\times 2ln}(v_1,v_2)K^{-1}_{m\times 2ln}(w_1,w_2)| - |K^{-1}_{m\times 2ln}(v_1,w_1)K^{-1}_{m\times 2ln}(v_2,w_2)|) && (2.24)
\end{aligned}$$

In order to compute the covariance as $l \nearrow \infty$, we only need to compute the entries of $K^{-1}_{m\times 2ln}(-1)$ as $l \nearrow \infty$.

Then we have the following proposition

Proposition 2.11. *Consider the dimer model on a Fisher graph, embedded into an $m \times ln$ cylinder, as illustrated in Figure 4. Let the circumference of the cylinder go to infinity, i.e. $l \to \infty$, and keep the height of the cylinder unchanged. Consider two edges e_1 and e_2. As $|e_1 - e_2| \nearrow \infty$: if $P(z) = 0$ does not intersect $\mathbb{T}$, the edge-edge correlation decays exponentially ; if $P(z) = 0$ has a node at 1, the edge-edge correlation tends to a constant.*

Proof. The theorem follows from formula (3.23), and the estimates of the entries of inverse Kasteleyn matrix. By (2.13), the second term goes to 0 as $|e_1 - e_2| \to \infty$, the first term is 0, if no zero exists on $\mathbb{T}$, the first term is a nonzero constant if the spectral curve has a real node at 1. $\square$

Example 2.12. *(1 × 2 cylindrical graph)Assume we have a dimer model on a Fisher graph embedded into an infinite cylinder of height 1 and period 2. One period of the graph is illustrated in Fig 5. The characteristic polynomial of the model is*

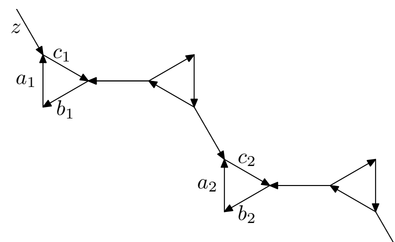

Figure 2.7: 1 × 2 cylindrical graph

$$P(z) = (b_1b_2z - a_1a_2)(\frac{b_1b_2}{z} - a_1a_2).$$

The probability that an a_1-edge occurs is

$$\begin{aligned} Pr(a_1) &= \frac{a_1a_2}{4\pi}\left(p.v.\int_{\mathbb{T}} \frac{1}{a_1a_2 - b_1b_2z}\frac{dz}{\sqrt{-1}z} + p.v.\int_{\mathbb{T}} \frac{1}{a_1a_2 - \frac{b_1b_2}{z}}\frac{dz}{\sqrt{-1}z}\right) \\ &= \begin{cases} 1 & \text{if } a_1a_2 > b_1b_2 \\ 0 & \text{if } a_1a_2 < b_1b_2 \\ \frac{1}{2} & \text{if } a_1a_2 = b_1b_2 \end{cases} \end{aligned}$$

$P(z) = 0$ has a real node at 1 if and only if $a_1a_2 = b_1b_2$. At the critical case, the covariance of an a_1 edge and a b_2 edge, as their distance goes to infinity, is

$$Pr(a_1\&b_2) - Pr(a_1)Pr(b_2) = 0 - \frac{1}{4} = -\frac{1}{4}$$

2.4 Graph on a Torus

In this section we consider a bi-periodic Fisher lattice whose quotient graph can be embedded into a torus.

Lemma 2.13. *Fix* $(\alpha, \beta) \in \mathbb{C}^2$*, consider* $P = \det K(\alpha, \beta)$ *as a polynomial of edge weights. Assume* $\mathcal{S}_0 = \{(\alpha, \beta) : (\alpha, \beta) = (\pm 1, \pm 1)\}$*. If* $(\alpha, \beta) \notin \mathcal{S}_0$*,* $P(\alpha, \beta)$ *is irreducible, as a polynomial of edge weights.*

Proof. $P = \det K(\alpha, \beta)$ is a quadratic polynomial in each edge weight $\nu(e)$. Assume we could write $P = P_1 P_2$. Each monomial in P corresponds to a configuration which occupies each vertex exactly twice. Fix a monomial ξ in P_1. We can divide all the vertices of G_1 into 3 types: Type A consists of all the vertices appearing in the monomial twice. Type B consists of all the vertices occupied by the monomial once, Type C consists of all the vertices that do not appear in the monomial at all. Then we claim that for all the monomials in P_1, they occupy A-vertices twice, B-vertices once, and do not occupy C-vertices. For all the monomials in P_2, they do not occupy A-vertices, occupy B-vertices once and C-vertices twice.

To see that, for the fixed monomial ξ in P_1, A-vertices are occupied twice. Hence for all the monomials in P_2, A-vertices are not occupied. Then for all the monomials in P_1, A-vertices are occupied twice. Similarly, we can derive that all the monomials in P_2 occupy C-vertices twice, and all the monomials in P_1 do not occupy C-vertices. If there exists another monomial λ in P_1, occupying some B-vertices twice, then by the argument above, all the monomials in P_1 should occupy those vertices twice, which is a contradiction to the fact that ξ occupies all the B-vertices exactly once. Similarly, there are no monomials in P_1 that do not occupy B-vertices. Hence all the monomials in P_1 and P_2 occupy B-vertices exactly once.

We claim that A-vertices are not connected to B-vertices or C-vertices. To see that, if there is an edge e connecting an A-vertex and a B-vertex, then any doubled edge configuration of e can not be written as a product of a monomial in P_1 and a monomial in P_2. The connectivity of G_1 implies that all the vertices are of the same type. If all of them are of Type A, then $P_2 = 1$; if all of them are of Type C, then $P_1 = 1$; if all of them are of Type B, then consider a configuration including odd loops. No matter how we divide its occupied edges, we always end up with a factor with a B-vertex occupied twice, which is a contradiction. Under the condition that $(\alpha, \beta) \notin \mathcal{S}_0$, odd loops cannot completely cancel. Hence P is an irreducible polynomial in edge weights for each fixed $(\alpha, \beta) \in \mathbb{C}^2 \setminus \mathcal{S}_0$. $\square$

Lemma 2.14. *If $\sum_{q=1}^{\infty} \frac{1}{\chi(q)}$ is convergent, then the set of ξ which satisfy*

$$\left| \frac{p}{q} - \xi \right| < \frac{1}{q\chi(q)}$$

for infinitely many q's is a set of measure 0.

Lemma 2.15. *If ξ is irrational, then there are infinitely many fractions $\frac{p}{q}$ which satisfy*

$$\left| \frac{p}{q} - \xi \right| < \frac{1}{q^2}$$

For proof of these two lemmas, see [11].

Lemma 2.16. *Let $P(z, w)$ be the characteristic polynomial for the graph embedded into an $m \times n$ torus, let*

$$I_1 = \frac{1}{4\pi} \int_0^{2\pi} \log P(e^{i\theta}, 1) d\theta \qquad (2.25)$$

$$I_2 = \frac{1}{4\pi} \int_0^{2\pi} \log P(e^{i\theta}, -1) d\theta \qquad (2.26)$$

If $I_1 > I_2$, the intersection of $P(z,1)=0$ with the unit torus $\mathbb{T}=\{z:|z|=1\}$ can only be a real point. If $I_2 > I_1$, the intersection of $P(z,-1)=0$ with the unit torus $\mathbb{T}$ can only be a real point.

Proof. First we prove that if $I_1 > I_2$, for generic choice of edge weights, $P(z,1)\neq 0$, for any $z\in\mathbb{T}$. We prove by contradiction. Assume $P(z,1)=0$ has roots $z_0, z_1, \cdots, z_k$ on $\mathbb{T}$. For generic choice of edge weights, we can assume $\alpha_i = \frac{Arg z_i}{\pi}\in[0,2)$ are irrational. Let $P_l(z,w)$ be the characteristic polynomial after enlarging the fundamental domain to $m\times ln$ without changing edge weights, and Z_l the corresponding partition function. When l is even, we have

$$Z_l = \frac{1}{2}(-\mathrm{Pf}K_l(1,1) + \mathrm{Pf}K_l(1,-1) + \mathrm{Pf}K_l(-1,1) + \mathrm{Pf}K_l(-1,-1))$$

Explicit computations show that

$$\begin{aligned}
-\mathrm{Pf}K_l(1,1) &= A_l + B_l + C_l - D_l\\
\mathrm{Pf}K_l(1,-1) &= A_l + B_l - C_l + D_l\\
\mathrm{Pf}K_l(-1,1) &= A_l - B_l + C_l + D_l\\
\mathrm{Pf}K_l(-1,-1) &= -A_l + B_l + C_l + D_l
\end{aligned}$$

where A_l, B_l, C_l, D_l are sums of dimer configurations. Given that all the edge weights are strictly positive, each one of A_l, B_l, C_l, D_l is strictly positive. Let

$$\{S_{l,1}, S_{l,2}, S_{l,3}, S_{l,4}\} = \{-\mathrm{Pf}K_l(1,1), \mathrm{Pf}K_l(-1,1), \mathrm{Pf}K_l(1,-1), \mathrm{Pf}K_l(-1,-1)\},$$

satisfying

$$S_{l,1} \geq S_{l,2} \geq S_{l,3} \geq S_{l,4},$$

then

$$S_{l,2} \leq Z_l \leq 3S_{l,2},$$

To see that, for example when $A_l \geq B_l \geq C_l \geq D_l$, we have

$$\begin{aligned} Z_l &= A_l + B_l + C_l + D_l \\ S_{l,2} &= A_l + B_l + D_l - C_l \end{aligned}$$

$P(z, \pm 1)$ are Laurent polynomials with respect to z, therefore improper integrals I_1 and I_2, given by (2.25) and (2.26) are finite. According to the formula of enlarging the fundamental domain

$$\begin{aligned} P_l(1,1) &= \prod_j P(e^{\frac{2\pi j i}{l}}, 1) \\ P_l(1,-1) &= \prod_j P(e^{\frac{2\pi j i}{l}}, -1) \\ P_l(-1,1) &= \prod_j P(e^{\frac{\pi(2j+1)i}{l}}, 1) \\ P_l(-1,-1) &= \prod_j P(e^{\frac{\pi(2j+1)i}{l}}, -1) \end{aligned}$$

then $\frac{1}{2l}\log P_l(\pm 1, 1)$ are Riemann sums for I_1, and $\frac{1}{2l}\log P_l(\pm 1, -1)$ are Riemann sums for I_2. According to Lemmas 5.2 and 5.3, for generic choice of edge weights, we have infinitely many l's satisfying

$$\frac{1}{l^3} < \left|\frac{j_{0,s}}{l} - \alpha_0\right| < \frac{1}{l^{\frac{3}{2}}} \tag{2.27}$$

$$\frac{1}{l^3} < \left|\frac{j}{l} - \alpha_i\right| \qquad \forall j \in \mathbb{N}, 1 \leq i \leq k \tag{2.28}$$

We denote the corresponding subsequence of $\mathbb{N}$ by $\{l_s\}_{s=1}^{\infty}$. Then on that subsequence

$$\lim_{s\to\infty} \frac{1}{2l_s} \log P_{l_s}(1,1) = I_1.$$

That is, the Riemann sum $\frac{1}{2l_s}\log P_{l_s}(1,1)$ is convergent to the integral I_1. Given $I_1 > I_2$, when s is sufficiently large, we have

$$\sqrt{P_{l_s}(1,1)} \geq S_{l,2} \geq \frac{1}{3}Z_{l_s} > 0 \tag{2.29}$$

Assume $z_0^{l_s} = e^{i\phi_s}$, where $\phi_s \in [0, 2\pi)$. We claim that $\lim_{s\to\infty} z_0^{l_s} = 1$. In fact, if we multiply (2.27) by l_s, we can see that the distance between $l_s\alpha_0$ and an integer j_{0,l_s} is less than $l_s^{-\frac{1}{2}}$. According to the formula of enlarging the fundamental domain, $P(z_0, 1) = 0$ implies

$$P_{l_s}(e^{i\phi_s}, 1) = 0.$$

Hence we have

$$1 = \left|\frac{P_{l_s}(e^{i\phi_s},1) - P_{l_s}(1,1)}{P_{l_s}(1,1)}\right| \leq \sum_{t=1}^{\infty} \frac{|\phi_s|^{2t}}{(2t)!|P_{l_s}(1,1)|}\left|\frac{\partial^{2t}P_{l_s}(e^{i\theta},1)}{\partial\theta^{2t}}|_{\theta=0}\right| = \sum_{t=1}^{\infty}\frac{R_t}{(2t)!} \tag{2.30}$$

In the Taylor series expansion around $\theta = 0$, only even order derivatives appear, because $P_{l_s}(z,1)$ is symmetric with respect to z and $\frac{1}{z}$, all the odd order derivatives vanish at $\theta = 0$. Assume

$$P_{l_s}(e^{i\theta}, 1) = \sum_{j=0}^{m} 2Q_j^{(l_s)} \cos j\theta,$$

using (2.29), we have

$$R_t = \left|\frac{\sum_j 2Q_j^{l_s} j^{2t}}{P_{l_s}(1,1)}\right| |\phi_s|^{2t} \leq C\left|\frac{\sum_{j\neq 0} 2Q_j^{l_s}}{Z_{l_s}^2}\right| |\phi_s|^{2t}(m\phi_s)^{2t}$$

Each $Q_i^{(\ell_s)}$ corresponds to a signed sum of loop configurations with nonzero homology i in z-direction. Using the same technique as in the proof of Theorem 1.1, we can derive

$$\sum_{j\neq 0} |Q_j^{(l_s)}| \leq C(m) Z_{l_s}^2,$$

where $C(m)$ is a constant depending only on m. As a result,

$$R_t \leq C(m)(m\phi_s)^{2t},$$

then

$$\lim_{s\to\infty} \left| \frac{P_{l_s}(e^{i\phi_s}, 1) - P_{l_s}(1,1)}{P_{l_s}(1,1)} \right| \leq C(m) \lim_{s\to\infty} \sum_{t=1}^{\infty} \frac{(m\phi_s)^{2t}}{(2t)!}.$$

The limit is zero because $\lim_{s\to\infty} \phi_s = 0$ and m is fixed. This is a contradiction to (2.30). Hence when $I_1 > I_2$, for generic choice of edge weights, $P(z, 1)$ has no zeros on $\mathbb{T}$. Since $P(1, 1) > 0$ is the determinant of an anti-symmetric matrix, by taking limits of edge weights, we derive that when $I_1 > I_2$, $z \in \mathbb{T}$ $P(z, 1) \geq 0$ for all positive edge weights. Moreover, we can derive the following claim

Claim 2.17. *If $I_1 \geq I_2$, then $P(z, 1) \geq 0$ for all positive edge weights.*

Assume $P(z_*, 1) = 0$, and z_* is not real. Fix an edge e, and let w_e be its weight. Fix z_*, we have

$$P(z_*, 1) = D_2^e(z_*, 1)w_e^2 + D_1^e(z_*, 1)w_e + D_0^e$$

The first term corresponds to the configurations which have a doubled edge at e, the second term and the third term corresponds to a configuration for which e is part of a loop, the third term corresponds to configurations that do not occupy e. if we consider $P(z_*, 1)$ as a polynomial of edge weights.

$$\frac{\partial P(z_*, 1)}{\partial w_e} = 2D_2^e(z_*, 1)w_e + D_1^e(z_*, 1) =: P_e(z_*, 1)$$

Because z_* is not real, $P(z_*, 1)$ is irreducible, according to Lemma 2.5.1. The intersection of the zero set of P_e and the zero set of P forms a proper subvariety of the zero set of P. Under a small perturbation of edge weights along the subvariety $P(z_*,1)=0$, we can always make $P_e(z_*,1)\neq 0$, another small perturbation of w_e can result in $P(z_*,1)<0$. When the perturbation is small enough, we can keep $I_1>I_2$. But this is a contradiction to the fact that $P(z_*,1)\geq 0$ for any edge weights given $I_1>I_2$.

Similarly we can prove that if $I_1<I_2$, the only possible intersection of $P(z,-1)=0$ with $\mathbb{T}$ is a single real point. $\square$

Proof of Theorem 2.2 We need to prove that if one of (α,β) is not-real, $(\alpha,\beta)\in\mathbb{T}^2$, then $P(\alpha,\beta)\neq 0$. We prove by contradiction. Without loss of generality, assume $P(\alpha,\beta)=0$ and α is not real. Fix an edge e with weight w_e. As before

$$\begin{aligned} P(\alpha,\beta) &= D_2^e(\alpha,\beta)w_e^2+D_1^e(\alpha,\beta)w_e+D_0^e(\alpha,\beta) \\ \frac{\partial P(\alpha,\beta)}{\partial w_e} &= 2D_2^e(\alpha,\beta)w_e+D_1^e(\alpha,\beta)=:P_e(\alpha,\beta) \end{aligned}$$

Fix (α,β). When α is not real, the intersection of the zero sets of $P_e(\alpha,\beta)$ and $P(\alpha,\beta)$ forms a proper subvariety of the zero set of $P(\alpha,\beta)$. Under a small perturbation of edge weights along the variety $P(\alpha,\beta)=0$, we can make $P_e(\alpha,\beta)\neq 0$. Another small perturbation of w_e can result in $P(\alpha,\beta)<0$. Hence there exists a neighborhood $(\beta-h,\beta+h)$, such that

$$P(\alpha,\beta_*)<0 \qquad \forall\beta_*\in(\beta-h,\beta+h). \tag{2.31}$$

Assume

$$I_3 = \frac{1}{2\pi}\int_0^{2\pi}\log P(1,e^{i\theta})d\theta \qquad I_4=\frac{1}{2\pi}\int_0^{2\pi}\log P(-1,e^{i\theta})d\theta$$

Without loss of generality, assume $I_3 \geq I_4$. Claim 2.17 implies that $P(1, w) \geq 0$, for all $w \in \mathbb{T}$. Since the set $\{w \in \mathbb{T}^2 : P(1, w) = 0\}$ consists of finitely many discrete points, there exists another open interval $(\beta_1, \beta_2) \subset (\beta - h, \beta + h)$, such that

$$P(1, \tilde{\beta}) > 0 \qquad \forall \tilde{\beta} \in (\beta_1, \beta_2) \tag{2.32}$$

Hence $P(z, \tilde{\beta}) = 0$, as a polynomial in z has non-real zeros on $\mathbb{T}$ for all $\tilde{\beta} \in (\beta_1, \beta_2)$, according to (2.31), (2.32) and the connectivity of $\mathbb{T}$. Namely, if $\operatorname{Arg}\alpha \in (0, \pi)$, then the argument of the root z satisfies $0 < \operatorname{Arg} z < \operatorname{Arg}\alpha$; if $\operatorname{Arg}\alpha \in (\pi, 2\pi)$, then the argument of the root z satisfies $\operatorname{Arg}\alpha < \operatorname{Arg} z < 2\pi$, obviously the root is not real. Under a small perturbation of edge weights, we can assume $I_1^r \neq I_2^r$ for any $r \in \mathbb{N}$ after enlarging the fundamental domain to $rm \times n$, and the statement that $P(z, \tilde{\beta}) = 0$ has non-real zeros in $\mathbb{T}$ for all $\beta \in (\beta_1, \beta_2)$ is still true. Let $\{r_1^k\}$ (resp. $\{r_2^k\}$) be the subset of $\mathbb{N}$ such that $I_1^r > I_2^r$ (resp. $I_1^r < I_2^r$). Then at least one of $\{r_1^k\}$ and $\{r_2^k\}$ is infinite. Without loss of generality, assume r_1^k is infinite, then the r_1^kth root of 1 is dense in $\mathbb{T}$. Hence there exists $\hat{\beta} \in (\beta_1, \beta_2)$, and $r_0 \in \{r_1^k\}_{k=1}^{\infty}$, such that $\hat{\beta}^{r_0} = 1$. Then after enlarging the fundamental domain to $r_0 m \times n$, we have $P_{r_0}(z, 1) = 0$, for non-real z. But this is impossible given $I_1^{r_0} > I_2^{r_0}$, and Lemma 5.4.
□

Example 2.18. *Assume the Fisher graph has period* $1 \times n$, *and all edge weights are strictly positive. The intersection of* $P(z, w) = 0$ *with* $\mathbb{T}^2$ *is either empty or a single real node. That is, they intersect at one of the point* $(\pm 1, \pm 1)$ *and the intersection is of multiplicity 2.*

Proof. Without loss of generality, assume $P(z, w)$ is linear with respect to z and $\frac{1}{z}$, then all terms in $P(z, w)$ fall into two categories

i) occupy z-edge exactly once

$ii)$ does not occupy z-edge or occupy z-edge twice

$1 \times n$ fundamental domain is comprised of n 1×1 blocks, for each block, we give weights $a_{i1}, b_{i1}, c_{i1}, a_{i2}, b_{i2}, c_{i2}$ for triangle edges, assume $a_i = a_{i1}a_{i2}$, $b_i = b_{i1}b_{i2}$, $c_i = c_{i1}c_{i2}$, $1 \leq i \leq n$. See Figure 6. Obviously such an arrangement of edge weights is gauge equivalent to the arrangement giving all triangle edges weight 1 and non-triangle edges positive edge weights.

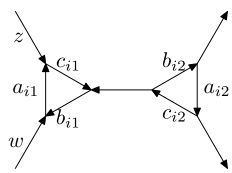

Figure 2.8: Fisher graph on a cylinder

Each configuration in case $i)$ consists of a single even loop and some double edges, the partition of all configurations in i is

$$P_1 = -\prod_{i=1}^{n}[(a_i - b_ic_i)w + (c_i - a_ib_i)]\frac{1}{z} - \prod_{i=1}^{n}[(a_i - b_ic_i)\frac{1}{w} + (c_i - a_ib_i)]z$$

Configurations in case $ii)$ depend on configurations of each 1×1 block, which are determined by configurations of boundary edges. Let $V_i^{0,0}$ denote the partition at $i-th$ block when both edges in z direction are unoccupied, $V_i^{2,0}$ denote the partition at $i-th$ block when left edge in z direction is occupied twice, while right edge is unoccupied, similarly for $V_i^{0,2}$ and $V_i^{2,2}$. Then we have

$$\begin{aligned}
V_i^{0,0} &= a_i^2 + c_i^2 + a_ic_i(w + \frac{1}{w}) \\
V_i^{0,2} &= a_{i1}c_{i1}b_{i2}(\frac{1}{w} - w) \\
V_i^{2,0} &= b_{i1}a_{i2}c_{i2}(w - \frac{1}{w}) \\
V_i^{2,2} &= b_i^2 - b_i(w + \frac{1}{w}) + 1
\end{aligned}$$

Let $k_i \in \{0,2\}$, $1 \le i \le n$, then partition of all configurations in ii is

$$P_{0,2} = \sum_{k_1,\ldots,k_n} \prod_{i=1}^{n} V_i^{k_i,k_{i+1}}$$

Assume $w = e^{\sqrt{-1}\phi}$, then $V_i^{0,0} \ge 0$, $V_i^{2,2} \ge 0$. Periodicity implies that $V_i^{0,2}$ and $V_i^{2,0}$ always appear in pairs, then all terms in $P_{0,2}$ where $\sin\phi$ appears depend only on $\sin^2\phi$, moreover, given all edge weights are positive, each term with $\sin\phi$ is nonnegative, therefore

$$\begin{aligned} P(z,w) &= P_1 + P_{0,2} \\ &= -\prod_{i=1}^{n}[(a_i - b_ic_i)w + (c_i - a_ib_i)]\frac{1}{z} - \prod_{i=1}^{n}[(a_i - b_ic_i)\frac{1}{w} + (c_i - a_ib_i)]z \\ &\quad + \prod_{i=1}^{n}[a_i^2 + c_i^2 + a_ic_i(w + \frac{1}{w})] + \prod_{i=1}^{n}[b_i^2 - b_i(w + \frac{1}{w}) + 1] + F(w) \end{aligned}$$

where $F(w) \ge 0$. Assume $G(z,w) = z[P(z,w) - F(w)]$. Fix w, $G(z,w)$ is a quadratic polynomial with respect to z. Assume

$$\begin{aligned} C_0 &= \prod_{i=1}^{n}[(a_i - b_ic_i)\frac{1}{w} + (c_i - a_ib_i)] \\ C_1 &= \prod_{i=1}^{n}[a_i^2 + c_i^2 + a_ic_i(w + \frac{1}{w})] + \prod_{i=1}^{n}[b_i^2 - b_i(w + \frac{1}{w}) + 1] \end{aligned}$$

then the root for $G(z,w) = 0$ will be

$$z_{1,2} = \frac{-C_1 \pm (C_1^2 - 4|C_0|^2)^{\frac{1}{2}}}{2C_0}$$

On the other hand,

$$C_1^2 - 4|C_0|^2 \geq 4[\prod_{i=1}^{n}(a_i^2 + c_i^2 + 2a_ic_i\cos\phi)(b_i^2 + 1 - 2b_i\cos\phi)$$
$$- \prod_{i=1}^{n}[(a_i - b_ic_i)^2 + (c_i - a_ib_i)^2 + 2(a_i - b_ic_i)(c_i - a_ib_i)\cos\phi]]$$

$$\begin{aligned} & (a_i^2 + c_i^2 + 2a_ic_i\cos\phi)(b_i^2 + 1 - 2b_i\cos\phi) - [(a_i - b_ic_i)^2 + (c_i - a_ib_i)^2 + 2(a_i - b_ic_i)(c_i - a_ib_i)\,\mathrm{co} \\ = \; & 4a_ib_ic_i\sin^2\phi \geq 0 \end{aligned}$$

Therefore, $C_1^2 - 4|C_0|^2 \geq 0$. Given $a_i, b_i, c_i > 0$, equality holds only if w is real. If $C_1^2 - 4|C_0| > 0$, then $|z_{1,2}| \neq 1$, $G(z, w)$ has no zeros on $\mathbb{T}^2$. Since $G(1,1) = \det K(1,1) > 0$, $P(z,w) - F(w) > 0$ on $\mathbb{T}^2$, so $P(z,w) > 0$ on $\mathbb{T}^2$. If $C_1^2 - 4|C_0|^2 = 0$, then w is real, $z_1 = z_2$ are real, $P(z,w)$ has real node on $\mathbb{T}^2$. $\square$

2.5 Single Edge Probability

Consider a periodic Fisher graph obtained by replacing each vertex of a honeycomb lattice by a triangle. Assume all the triangle edges have weight 1, and all the other edges have strictly positive edge weights, not equal to 1. Let $\mathcal{F}$ be the partition function per fundamental domain (free energy). That is

$$\mathcal{F} := \frac{1}{|\Lambda|} \lim_{\Lambda \to \infty} \log Z_\Lambda,$$

where Λ is a finite box, $|\Lambda|$ is the number of fundamental domains in the finite box, and Z_Λ is the partition function.

Lemma 2.19. *$\mathcal{F}$ is independent of boundary conditions as long as*

$$\lim_{\Lambda \to \infty} \frac{|\partial \Lambda|}{|\Lambda|} = 0$$

If we raise the weight of a non-triangle edge e to the power β, and let $\mathcal{F}_e(\beta)$ be the new free energy, then $\mathcal{F}_e(\beta)$ is convex in β.

Proof. Explicit computations. See the proof of Theorem 3.3. □

Lemma 2.20. *$\frac{\partial F_e(\beta)}{\partial \beta}$ is continuous.*

Proof. Since $\mathcal{F}$ is independent of boundary conditions, we can impose bi-periodic boundary condition to compute the free energy explicitly

$$\mathcal{F}_e(\beta) = \frac{1}{8\pi^2} \iint_{\mathbb{T}^2} \log P_e(z, w, \beta) \frac{dz}{iz} \frac{dw}{iw}$$

where $P_e(z, w, \beta)$ is the characteristic polynomial. It is proved in theorem 2.2 that for bi-periodic Fisher graph with strictly positive weights, either $P(z,w,\beta)=0$ does not intersect $\mathbb{T}^2$, or it intersects $\mathbb{T}^2$ at a single real node.

Fix β_0, if $P_e(z,w,\beta_0)$ does not intersect $\mathbb{T}^2$, by continuity there exists $\eta>0$, such that $P_e(z,w,\beta)$ is bounded away from 0 in $\mathbb{T}^2\times[\beta_0-\eta,\beta_0+\eta]$. Therefore

$$\frac{\log P_e(z,w,\beta_0+\Delta\beta)-\log P_e(z,w,\beta_0)}{\Delta\beta}=\frac{1}{P_e(z,w,\beta_0+\theta\Delta\beta)}\frac{\partial P}{\partial\beta}\Big|_{(z,w,\beta_0+\theta\Delta\beta)}$$

is uniformly bounded in $\mathbb{T}^2\times[\beta_0-\eta,\beta_0+\eta]$. According to the dominated convergence theorem, we have

$$\begin{aligned}\frac{\partial F_e}{\partial\beta}\Big|_{\beta_0} &= \lim_{\Delta\beta\to 0}\frac{1}{8\pi^2}\iint_{\mathbb{T}^2}\frac{\log P_e(z,w,\beta+\Delta\beta)-\log P_e(z,w,\beta_0)}{\Delta\beta}\frac{dz}{iz}\frac{dw}{iw}\\ &= \frac{1}{8\pi^2}\iint_{\mathbb{T}^2}\lim_{\Delta\beta\to 0}\frac{\log P_e(z,w,\beta+\Delta\beta)-\log P_e(z,w,\beta_0)}{\Delta\beta}\frac{dz}{iz}\frac{dw}{iw}\\ &= \frac{1}{8\pi^2}\iint_{\mathbb{T}^2}\frac{1}{P_e(z,w,\beta_0)}\,\frac{\partial P_e}{\partial\beta}\Big|_{(z,w,\beta_0)}\frac{dz}{iz}\frac{dw}{iw},\end{aligned}$$

which is continuous at β_0.

If $P_e(z,w,\beta_0)$ has a real node on $\mathbb{T}^2$, without loss of generality, assume the real node is $(1,1)$. Define

$$\begin{aligned}S_{\delta,n} &= \left\{(x,y)\in\mathbb{Z}^2\Big|\,|x|<\frac{n\delta}{\pi}-\frac{1}{2},|y|<\frac{n\delta}{\pi}-\frac{1}{2}\right\}\\ U_{\delta,n} &= \left\{\left(\frac{(2x+1)\pi}{2n},\frac{(2y+1)\pi}{2n}\right)\Big|\,(x,y)\in S_{\delta,n}\right\}\end{aligned}$$

We have

$$\begin{aligned}F_e(\beta) &= \lim_{n\to\infty}\lim_{\delta\to 0+}\frac{1}{8n^2}\left(\sum_{(x,y)\in S_{\delta,n}}+\sum_{(x,y)\in[-n,n-1]^2\setminus S_{\delta,n}}\right)\log P_e\left(e^{\frac{(2x+1)\pi i}{2n}},e^{\frac{(2y+1)\pi i}{2n}},\beta\right)\\ &= \lim_{n\to\infty}\lim_{\delta\to 0+}\mathcal{F}^1_{e,n,\delta}(\beta)+\mathcal{F}^2_{e,n,\delta}(\beta)\end{aligned}$$

$$\frac{\partial \mathcal{F}^2_{e,n,\delta}(\beta)}{\partial \beta} = \frac{1}{8n^2} \sum_{(x,y)\in[-n,n-1]^2\setminus S_{\delta,n}} \frac{1}{P_e}\frac{\partial P_e}{\partial \beta}\Big|_{(e^{\frac{(2x+1)\pi i}{2n}},e^{\frac{(2y+1)\pi i}{2n}},\beta)}$$

there exists $\eta > 0$, such that $\frac{1}{P_e}\frac{\partial P_e}{\partial \beta}$ is uniformly continuous in $\{(e^{\theta i}, e^{\phi i}, \beta) | \max\{|\theta|, |\phi|\} \geq \delta, |\beta - \beta_0| \leq \eta\}$, therefore $\frac{\partial \mathcal{F}^2_{e,n,\delta}}{\partial \beta}$ converges uniformly to

$$\frac{1}{8\pi^2}\int_{\theta,\phi\in[-\pi,\pi],\max\{|\theta|,|\phi|\geq\delta\}} \frac{1}{P_e}\frac{\partial P_e}{\partial \beta}(e^{\theta i}, e^{\phi i}, \beta) d\theta d\phi$$

in $[\beta_0 - \eta, \beta_0 + \eta]$.

$$\frac{\partial \mathcal{F}^1_{e,n,\delta}(\beta)}{\partial \beta} = \frac{1}{8n^2} \sum_{\theta,\phi\in U_{\delta,n}} \frac{1}{P_e}\frac{\partial P_e}{\partial \beta}\Big|_{(e^{i\theta},e^{i\phi},\beta)}$$

In $U_{\delta,n} \times [\beta_0 - \eta, \beta_0 + \eta]$

$$\begin{aligned} P_e(\theta,\phi) &= A_0 + A_1\theta^2 + A_2\theta\phi + A_3\phi^2 + O(|\theta|^3, |\phi|^3) \\ \frac{\partial P_e(\theta,\phi)}{\partial \beta} &= B_0 + B_1\theta^2 + B_2\theta\phi + B_3\phi^2 + O(|\theta|^3 + |\phi|^3) \end{aligned}$$

where $A_0 = P_e(0,0)$ is the determinant of an anti-symmetric matrix, hence $A_0(\beta)$ is the perfect square of a function in β. Since $(1,1)$ is a real node for $P_e(z,w,\beta_0)$, we have

$$\begin{aligned} &A_0(\beta_0) = B_0(\beta_0) = 0 \\ &A_0(\beta) > 0 \qquad B_0(\beta) \neq 0 \qquad \forall \beta \in [\beta_0 - \eta, \beta_0 + \eta] \setminus \{\beta_0\} \end{aligned}$$

Moreover,

$$P_e(\theta,\phi,\beta_0) = A_1\theta^2 + A_2\theta\phi + A_3\phi^2 + O(|\theta|^3 + |\phi|^3) > 0 \qquad \forall (\theta,\phi) \neq (0,0)$$

therefore,

$$\max\{A_1, A_3\} > 0, \qquad A_2^2 - 4A_3A_1 < 0 \qquad \forall \beta \in [\beta_0 - \eta, \beta_0 + \eta]$$

the discriminant is negative follows from the fact that $P(z, w) > 0$ for any $(z, w) \in \mathbb{T}^2 \setminus \{1, 1\}$. then

$$\frac{1}{P_e}\frac{\partial P_e}{\partial \beta}(\theta, \phi, \beta) \leq \begin{cases} \frac{(|B_1|+|B_2|+|B_3|)\phi^2}{\left(\frac{4A_3A_5 - A_4^2}{4A_1}\right)\phi^2} & \text{if } |\phi| \geq |\theta| \\ \frac{(|B_1|+|B_2|+|B_3|)\theta^2}{\left(\frac{4A_3A_5 - A_4^2}{4A_3}\right)\theta^2} & \text{if } |\phi| < |\theta| \end{cases}$$

in $U_{\delta,n} \times [\beta_0 - \eta, \beta_0 + \eta]$. We have

$$\begin{aligned} &\frac{1}{8n^2} \sum_{\theta,\phi \in U_{\delta,n}} \frac{1}{P_e}\frac{\partial P_e}{\partial \beta}\bigg|_{e^{\theta i}, e^{\phi i}, \beta} \\ &\leq \frac{C}{n^2} \sum_{\theta,\phi \in U_{n,\delta}, |\theta| > |\phi|} 1 + \sum_{(\theta,\phi) \in U_{n,\delta}, |\phi| > |\theta|} 1 \\ &\leq C\delta^2, \end{aligned}$$

where C is a constant. Hence

$$\begin{aligned} &\forall \epsilon, \delta > 0 \ \exists N, \ \forall n > N, \ \forall \beta \in [\beta_0 - \eta, \beta_0 + \eta] \\ &\left| \frac{\partial F^1_{e,n,\delta}(\beta)}{\partial \beta} + \frac{\partial F^2_{e,n,\delta}}{\partial \beta} - \frac{1}{8\pi^2} \int_{(\theta,\phi) \in [-\pi,\pi]^2, \max\{|\theta,\phi| \geq \delta\}} \frac{1}{P_e}\frac{\partial P_e}{\partial \beta}(e^{\theta\sqrt{-1}}, e^{\phi\sqrt{-1}}, \beta) d\theta d\phi \right| < \epsilon + C\delta^2, \end{aligned}$$

Then

$$\begin{aligned} &\forall \epsilon, \delta > 0 \ \exists N, \ \forall m, n > N, \ \forall \beta \in [\beta_0 - \eta, \beta_0 + \eta] \\ &\left| \frac{\partial \mathcal{F}_{e,n}(\beta)}{\partial \beta} - \frac{\partial \mathcal{F}_{e,m}(\beta)}{\partial \beta} \right| < \epsilon + C\delta^2 \end{aligned}$$

That is, $\frac{\partial \mathcal{F}_{e,n}}{\partial \beta}$ converges uniformly in $[\beta_0 - \eta, \beta_0 + \eta]$, therefore

$$\frac{\partial \mathcal{F}_e(\beta)}{\partial \beta} = \lim_{n \to \infty} \frac{\partial \mathcal{F}_{e,n}(\beta)}{\partial \beta} \qquad \forall \beta \in [\beta_0 - \eta, \beta_0 + \eta],$$

and the limit $\frac{\partial \mathcal{F}_e}{\partial \beta}$ is continuous at β_0. $\square$

Theorem 2.21. *The weak limit of single edge probability for any edge, whose weight is strictly positive and not equal to 1, is unique.*

Proof. The convexity of $\mathcal{F}_e(\beta)$ with respect to β implies that

$$\mathcal{F}_e(\beta) \geq \mathcal{F}_e(0) + \int \log w_e \mathbf{1}_e d\mu \tag{2.33}$$

where μ is an arbitrary translation invariant Gibbs measure, and $\mathbf{1}_e$ is the indicator that the edge e is present in the dimer configuration. For a proof of 2.33, see [17]. Since $\frac{\partial \mathcal{F}_e}{\partial \beta} \mid$ is continuous, and $\log w_e \neq 0$, the single edge probability under any translation invariant Gibbs measure is unique. $\square$

Chapter Three

Critical Temperature of Periodic Ising Models

3.1 Introduction

Phase transitions are among the most interesting and central topics of statistical mechanics. A system undergoes a first-order phase transition whenever, for some value of the temperature and other relevant thermodynamic parameters, two or more phases can coexist in equilibrium. The different properties of the pure phases manifest themselves also as discontinuities in certain observables as a function of the the appropriate thermodynamic variables, e.g., discontinuity of the magnetization as a function of the magnetic field in a ferromagnet. Of special interest is the temperature at which the phase transition first occurs, that is, the **critical temperature**.

In [21], Lebowitz and Martin-Löf defined a critical temperature, T_c, for Ising ferromagnets as the supreme of the temperatures such that the spontaneous magnetization is nonzero. In [19], Lebowitz identified T_c with the self-dual point for a square grid Ising model with the same interactions on all edges. In [1], Aizenman, Bursky and Fernández characterized the phase transition by the exponential decay of two point spin correlations above T_c. Their technique was based on various correlation inequalities and differential inequalities.

Another approach to solve the two-dimensional Ising model is to apply the Fisher correspondence ([7]), which is a measure-preserving bijection between the spin-spin even correlation functions of the Ising model on a graph G, and the edge probabilities of a dimer model on a decorated graph, or the **Fisher graph**. Since Fisher, dimer techniques have been a powerful tool for solving pertinent questions about the Ising model, see for example the paper of Kasteleyn [14], and the book of McCoy and Wu [25]. However, due to the complexity in computing large matrices, the two-dimensional Ising model has acquired a notorious reputation for difficulty. On the mathematical side, Kenyon has proven numerous spectacular results about the dimer model on bipartite graphs in recent years. Those results make it possible to look at the two-dimensional Ising model from a new perspective. In this paper, we follow the dimer approach to the Ising model, and explicitly identify the critical temperature, defined as in [21] and [1], with the zero of an algebraic equation, see the following theorem.

Theorem 3.1. *The critical temperature of the periodic Ising model, defined by the lowest temperature such that the spontaneous magnetization is nonzero, is finite, nonzero, and can be identified with the condition that the spectral curve of the corresponding dimer model on the Fisher graph has a real node on* $\mathbb{T}^2$.

Theorem 1.1 actually allows us to compute the critical temperature of arbitrary periodic Ising ferromagnets accurately, by solving an algebraic equation, see Example 6.14. The proof of Theorem 1.1 is of 3 steps:

Step 1: Applying Lebowitz's technique [20] and the FKG inequality, we prove that the weak limit of the spin-spin even correlation functions is independent of the boundary conditions as the size of the graph goes to infinity. For the uniqueness theorem about the Gibbs measure of dimer models on a more general class of non-bipartite graphs, see Corollary 2.14.

Step 2: Using an $n \times n$ torus to approximate the infinite bi-periodic graph, we express the two-point spin correlation functions as the determinants of certain block Toeplitz matrices. For basics of Toeplitz and Hankel matrices, see the appendix. We prove that the determinant of the symbol of such block Toeplitz matrices is identically 1. Using the FKG inequality, we derive that there is a unique $T_{c,p}$, defined to be the lowest temperature such that the limit of the two-point spin correlation function is 0. The analytic property for the limit of two-spin correlation function when the entries of the block Toeplitz matrices are analytic with respect to the reciprocal temperature follows from Widom's theorem ([30]) and a result from operator analysis [9]. Hence $T_{c,p}$ has to satisfy the condition that the spectral curve has a real node on $\mathbb{T}^2$.

Step 3: Identification of T_c and the condition that the spectral curve has a real node on $\mathbb{T}^2$(critical dimer weights). Since the two point spin correlation functions are unique under any translation invariant Gibbs measure, we derive $T_{c,p} = T_c \subset \{critical\ dimer\ weights\}$. We also prove that as β (reciprocal temperature) increases from 0 to ∞, there is a unique β_0 $(0 < \beta_0 < \infty)$, such that the spectral curve has a real node on $\mathbb{T}^2$. The uniqueness of the critical dimer weights implies the identification of T_c with it.

Here is a description of the structure of the paper. Section 2 proves the uniqueness of spin-spin even correlation functions for periodic ferromagnetic Ising models. In section 3 we discuss different correspondence between Ising models on square grid and dimer models on the Fisher graph, and prove that the spectral curve is invariant under the duality transformation. In Section 4, we complete Step 2 and Step 3 to prove the main theorem. The block Toeplitz technique and the transfer matrix technique also give a simple proof of exponential decay of the spin-spin correlations in high temperature for symmetric interactions, see Section 5. For a proof of exponential decay for general interactions, see [1, 2]. In the appendix, we discuss several basic facts of Toeplitz and Hankel matrices for the readers' reference.

3.2 Translation Invariant Gibbs Measure

3.2.1 FKG Inequality for Periodic Interactions

We consider a Ising spin system with a ferromagnetic pair interaction in a finite box Λ embedded on the whole plane, i.e, at each point p of the lattice there is a spin $\sigma_p = \pm 1$, and the conditional probability of a spin configuration in the box Λ given a configuration outside it is proportional to

$$e^{-E_\Lambda(\sigma)} = e^{\beta(\frac{1}{2}\sum_{p \neq q \in \Lambda, |p-q|=1} J_{pq}\sigma_p\sigma_q + \sum_{p\in\Lambda} h\sigma_p + \sum_{p\in\Lambda, q\in\Lambda^c} J_{pq}\sigma_p\sigma_q)} \tag{3.1}$$

where $J_{pq} > 0$ is the pair interaction. Assume J_{pq} has period (m, n), i.e, for any $i, j \in \mathbb{Z}$, $J_{pq} = J_{p+(im,jn),q+(im,jn)}$. h is the uniform external magnetic field, and β is the reciprocal temperature. A boundary condition for the box Λ, is specified by giving a probability distribution b_Λ for the configurations outside Λ. Let Γ be the set of all Ising spin configurations on Λ. Under the partial order relation "$-$"<"$+$", the partial order set Γ is a **lattice**, since any two configurations x and y in Γ have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$. Moreover, Γ is a **finite distributive lattice**, see [8] for a definition. We have the following theorem:

Theorem 3.2. *Let f be an increasing function with respect to the partial order relation on Γ, μ_1, μ_2 be two probability measures, if all the following conditions are satisfied*

- *the boundary condition $\delta_1 \geq \delta_2$;*
- *the external magnetic field $h_1 \geq h_2$;*
- *the pair interaction $J^1_{pq} \geq J^2_{pq}$;*

Then the expected values of f satisfy

$$\langle f\rangle_{\mu_1} \geq \langle f\rangle_{\mu_2}$$

Proof. Since Γ is a finite distributive lattice, according to [12], if any spin configuration $A, B \in \Gamma$ satisfy

$$\mu_1(A \vee B)\mu_2(A \wedge B) \geq \mu_1(A)\mu_2(B), \tag{3.2}$$

then

$$\langle f\rangle_{\mu_1} \geq \langle f\rangle_{\mu_2}$$

for any increasing functions f on Γ.

Let us prove condition (3.2) for $\delta_1 \geq \delta_2$. If A does not have boundary configuration δ_1 or B does not have boundary configuration δ_2, condition (3.2) is satisfied automatically since the right hand side is zero while the left hand side is nonnegative. Hence to prove condition (3.2), it suffices to prove that for any A with boundary configuration δ_1 and B with boundary configuration δ_2,

$$\mu(A \vee B)\mu(A \wedge B) \geq \mu(A)\mu(B) \tag{3.3}$$

where μ is the probability measure on Γ with free boundary condition. Condition (3.3) is true if energy for all bonds $J_e \geq 0$. Notice that J_e may be different from edge to edge. It is trivial to check condition (3.3) for each single bond with 16 different combinations of spin configurations for A,B. If the inequality is true for the contribution of each edge in the probability, the inequality is true for the probability, since the probability is the product of contributions of each edge, and the partial order relation is defined bond by bond.

The same method can be applied to check condition (3.2) spin by spin or bond by bond for the cases $h_1 \geq h_2$ and $J^1_{pq} \geq J^2_{pq}$. $\square$

Remark. Condition (2) is called the FKG inequality.

3.2.2 Free Energy

As before, let Λ be a finite box of an infinite periodic square grid. The equilibrium state of the system in Λ is the probability distribution for configurations in Λ defined by (3.1) together with boundary condition b_Λ, or equivalently, by the family of correlation functions

$$\langle \sigma_A \rangle_{h,\Lambda,b_\Lambda} = \langle \prod_{p \in A} \sigma_p \rangle_{h,\Lambda,b_\Lambda} \quad for\ any\ A \subset \Lambda$$

An equilibrium state of the infinite system is defined to be a family of correlation functions obtained as the limit of correlation functions for a sequence of finite boxes with some boundary conditions

$$\langle \sigma_A \rangle_{h,b} = \lim_{\Lambda \to \infty} \langle \sigma_A \rangle_{h,\Lambda,b_\Lambda}$$

We say a sequence of boxes $\{\Lambda_n\}_{n=1}^{\infty}$ tends to infinity if

1. each Λ_n is connected and $\Lambda_n \subset \Lambda_{n+1}$;

2. $\bigcup_n \Lambda_n = \mathbf{T}$;

3. for any $l \geq 0$, let Λ^l be the set of points in Λ with distance at most l from its boundary, and $|\Lambda|$ denote the number of points in Λ, then

$$\lim_{n\to\infty} \frac{|\Lambda_n^l|}{|\Lambda_n|} = 0 \quad \text{(Van Hove convergence)}$$

The letter b indicates some arbitrary boundary condition, and $b_\Lambda =$ "+" or $b_\Lambda =$ "−" will indicate the boundary conditions defined by putting all spins outside Λ " + " or "−". The statement $\langle \sigma_A \rangle_{h,b}$ does not depend on the boundary condition is equivalent to the statement that the equilibrium is unique.

For a finite system, the free energy is defined as follows

$$F(h, \Lambda, b_\Lambda) = \frac{1}{|\Lambda|} \log(\sum_\sigma e^{-E_\Lambda(\sigma)}),$$

where $E_\Lambda(\sigma)$ is defined in (3.1). We have the following theorem about the free energy

Theorem 3.3. *As $\Lambda \to \infty$, the limit of $F(h, \Lambda, b_\Lambda)$ exists and is independent of the boundary condition. Assume*

$$\lim_{\Lambda\to\infty} F(h, \Lambda, b_\Lambda) = F(h)$$

then $F(h)$ is convex, and analytic for $h \neq 0$

Proof. First of all, for each fixed Λ and b_Λ, $F(h, \Lambda, b_\Lambda)$ is a convex function of h. Assume

$$\begin{aligned} A_\sigma &= e^{-E_\Lambda(\sigma)} \geq 0 \\ B_\sigma &= \sum_{p\in\Lambda} \beta\sigma_p \end{aligned}$$

then

$$\frac{\partial^2 F(h,\Lambda,b_\Lambda)}{\partial h^2} = \frac{(\sum_\sigma A_\sigma)(\sum_\sigma A_\sigma B_\sigma^2) - (\sum_\sigma A_\sigma B_\sigma)^2}{|\Lambda|(\sum_\sigma A_\sigma)^2} \geq 0$$

The inequality follows from applying the Cauchy-Schwartz to $\{\sqrt{A_\sigma}\}_\sigma$ and $\{\sqrt{A_\sigma}B_\sigma\}_\sigma$. Equality holds only if all the B_σ are equal, which is impossible, hence $\frac{\partial^2 F(h,\Lambda,b_\Lambda)}{\partial h^2} > 0$, $F(h,\Lambda,b_\Lambda)$ is convex.

We claim that if $\lim_{\Lambda\to\infty} F(h,\Lambda,b_\Lambda)$ exists, it is independent of the boundary condition b_Λ. Assume $G(h,\Lambda)$ is the free energy for the free boundary conditions, namely

$$G(h,\Lambda) = -\frac{1}{|\Lambda|}\log\sum_\sigma e^{\beta(\frac{1}{2}\sum_{p\neq q\in\Lambda,|p-q|=1} J_{pq}\sigma_p\sigma_q + \sum_{p\in\Lambda} h\sigma_p)}$$

then

$$-G(h,\Lambda) - \frac{|\partial\Lambda|}{|\Lambda|}\beta\max_{p,q} J_{pq} \leq -F(h,\Lambda,b_\Lambda) \leq -G(h,\Lambda) + \frac{|\partial\Lambda|}{|\Lambda|}\beta\max_{p,q} J_{pq}$$

Under the assumption that $\Lambda\to\infty$ in the sense of Van Hove,

$$\lim_{\Lambda\to\infty}\frac{|\partial\Lambda|}{|\Lambda|} = 0$$

Since J_{pq} has finite period, we have

$$\lim_{\Lambda\to\infty} F(h,\Lambda,b_\Lambda) = \lim_{\Lambda\to\infty} G(h,\Lambda)$$

which is independent of boundary conditions.

It is convenient to introduce a new variable

$$z = e^{-2\beta h}$$

then

$$e^{-|\Lambda|G} = Q(z)e^{|\Lambda|h\beta}$$

where

$$Q(z) = \sum Q_n z^n \qquad (n = 0, 1, ..., |\Lambda|) \tag{3.4}$$

The coefficients Q_n are the contribution to the partition function of the Ising lattice in zero external field from configurations with the number of " $-$ " spins equal to n. The following lemma is proved by Lee and Yang [22]

Lemma 3.4. *Let $x_{\alpha\beta} = x_{\beta\alpha}(\alpha \neq \beta, \alpha, \beta = 1, 2, ..., n)$ be real numbers whose absolute values are less than or equal to 1. Divide the integers $1, 2, ..., n$ into 2 groups a and b so that there are γ integers in group a and $(n - \gamma)$ integers in group b. Consider the product of all $x_{\alpha\beta}$ where α belongs to group a and β belongs to group b. We shall denote by Q_γ the sum of all such products over all the $\frac{n!}{\gamma!(n-\gamma)!}$ possible ways of dividing the n integers, in other words*

$$Q_\gamma = \sum_{a \subset V, |a| = \gamma} \prod_{\alpha \in a} \prod_{\beta \in V \setminus a} x_{\alpha\beta}$$

Consider the polynomial

$$Q(z) = 1 + Q_1 z + ... + Q_n z^n$$

Then all the roots of the equation $Q = 0$ are on the unit circle.

Apply the lemma to (3.4), we obtain that the roots of $Q(z) = 0$ are on the unit circle. The distribution of roots of $Q(z) = 0$ as $\Lambda \to \infty$ may be described by a measure $d\mu(\theta)$, so that $|\Lambda| d\mu(\theta)$ is the number of roots between $e^{i\theta}$ and $e^{i(\theta+d\theta)}$. Since $Q(z)$ has real coefficients, we have

$$d\mu(\theta) = d\mu(-\theta)$$

Taking the logarithm of (3.4), and taking the limit as $\Lambda \to \infty$, we have

$$\begin{aligned} F(h) &= h\beta + \int_0^{2\pi} \log(z - e^{i\theta}) d\mu(\theta) \\ &= h\beta + \int_0^{\pi} \log(z^2 - 2z\cos\theta + 1) d\mu(\theta) \end{aligned}$$

The singularities of $F(h)$ corresponds to zeros of $z^2 - 2z + 1$. Since $z = e^{-2\beta h} > 0$, the only possible singularity of $F(h)$ happens at $h = 0, z = 1$. The convexity of $F(h)$ follows from the convexity of $F(h, \Lambda, b_\Lambda)$. $\square$

3.2.3 High Temperature

Define the average magnetization as follows

$$m(h, \Lambda, b_\Lambda) = \langle \frac{1}{|\Lambda|} \sum_{p \in \Lambda} \sigma_p \rangle_{h,\Lambda,b_\Lambda} = \frac{1}{\beta} \frac{\partial F(h, \Lambda, b_\Lambda)}{\partial h}$$

We have the following lemma about the average magnetization

Lemma 3.5. *As $\Lambda \to \infty$, the limit of $m(h, \Lambda, b_\Lambda)$ exists and is independent of* *boundary conditions. Moreover,*

$$\lim_{\Lambda\to\infty} m(h,\Lambda,b_\Lambda) = \frac{dF(h)}{dh} \qquad \forall h \neq 0$$

Proof. $\frac{1}{|\Lambda|}\sum_{p\in\Lambda}\sigma_p$ is an increasing function on Γ_Λ, and the FKG inequality says that

$$m(h,\Lambda,-) \leq m(h,\Lambda,b_\Lambda) \leq m(h,\Lambda,+) \tag{3.5}$$

For any $\Lambda' \supseteq \Lambda$, $m(h,\Lambda,+)$ can be obtained from $m(h,\Lambda',+)$ by adding an infinite positive external magnetic field on $\Lambda' \setminus \Lambda$. By the FKG inequality we have

$$m(h,\Lambda',+) \leq m(h,\Lambda,+)$$

Hence $m(h,\Lambda,+)$ is decreasing as Λ increases. Therefore $\lim_{\Lambda\to\infty} m(h,\Lambda,+)$ exists, denoted by $m(h,+)$, for all h. Moreover, since $|m(h,\Lambda,b_\Lambda)| \leq 1$, according to the Dominated Convergence Theorem, we have

$$\int_{h_0}^{h} m(\hbar,+)d\hbar = \lim_{\Lambda\to\infty}\int_{h_0}^{h} m(\hbar,\Lambda,+)d\hbar = F(h) - F(h_0)$$

where $h > h_0 > 0$ or $0 > h > h_0$. Since $F(h)$ is analytic in h when $h \neq 0$, we have

$$m(h,+) = \frac{dF(h)}{dh}$$

Similar process shows that

$$m(h,-) = \frac{dF(h)}{dh}$$

And the lemma follows from 3.5. $\square$

Since $F(h)$ is analytic in h when $h \neq 0$, we define

$$m^* = \lim_{h \to 0+} \frac{dF(h)}{dh}$$

Applying FKG inequality and follow exactly the same process by Lebowitz and Martin-Löf [21], we have the following lemma:

Lemma 3.6. *When the external magnetic field is zero there is a unique equilibrium state for the periodic, ferromagnetic infinite system if and only if $m^* = 0$.*

$\frac{1}{|\Lambda|}\sum_{p\in\Lambda}\sigma_p$ is an increasing function on Γ, according to the FKG inequality, $m(h, \Lambda, b_\Lambda) = \langle \frac{1}{|\Lambda|}\sum_{p\in\Lambda}\sigma_p\rangle_{\Lambda, b_\Lambda}$ is an increasing function in $J_{i,j}$ and h, hence if we fix $J_{i,j}$ and h, it is an increasing function in β. When $\beta = 0$, we have a uniform distribution for all configurations, therefore

$$m(h, \Lambda, b_\Lambda)|_{\beta=0} = 0$$

As a result $m(h, \Lambda, b_\Lambda) \geq 0$ for any β. As its limit m^* is nonnegative and increasing in β. The critical temperature β_c is uniquely defined by the conditions $m^*(\beta) = 0$ for $\beta < \beta_c$ and $m^*(T) > 0$ for $\beta > \beta_c$. Then we have the following theorem

Theorem 3.7. *When $\beta < \beta_c$, there exists a unique probability measure of ferromagnetic, periodic Ising spin system without external magnetic field.*

3.2.4 Uniqueness of Spin-Spin Even Correlations

This section is devoted to the proof of Theorem 2.7. The proof is divided into proving several lemmas, as specified below.

Theorem 3.8. *Given a bi-periodic, ferromagnetic Ising model, the spin-spin even correlation functions are unique under any translation invariant Gibbs measure. That is, fix a finite subset A of the vertices of the graph such that $|A|$ is even, for any translation invariant Gibbs measure μ, $\langle\sigma_A\rangle_\mu$ is independent of μ.*

Lemma 3.9. $\langle\rho_A\rangle_\pm = \lim_{\Lambda\to\infty}\langle\rho_A\rangle_{\Lambda,\pm}$ *exists and is translation invariant. That is* $\langle\rho_{A+g}\rangle_\pm = \langle\rho_A\rangle_\pm$*, where g is a translation vector.*

Proof. The FKG inequality applies to $f = \rho_A$, because this is an increasing function in each spin variable. This implies that $\langle\rho_A\rangle_{\Lambda',+} \leq \langle\rho_A\rangle_{\Lambda,+}$, if $\Lambda' \supseteq \Lambda$, by the FKG inequality. Hence

$$\langle\rho_A\rangle_+ = \lim_{\Lambda\to\infty}\langle\rho_A\rangle_{\Lambda,+}$$

exists and $\langle\rho_A\rangle_+ \leq \langle\rho_A\rangle_{\Lambda,+}$. The translation invariance of the $\langle\rho_A\rangle_\pm$ follows from the uniqueness of taking the limit $\Lambda\to\infty$. □

Lemma 3.10.

$$\langle\sigma_A\rangle_{\Lambda,b_\Lambda} \leq \langle\sigma_A\rangle_{\Lambda,+}$$

Proof. Consider

$$\langle\sigma_A\rangle = \frac{\sum_\sigma \sigma_A e^{\sum_{i\sim j, i,j\in\Lambda^0} J_{ij}\sigma_i\sigma_j + \sum_{k\in\partial\Lambda} J_k\sigma_k}}{Z}$$
$$\langle\sigma_A\rangle' = \frac{\sum_\sigma \sigma_A e^{\sum_{i\sim j} J_{ij}\sigma_i\sigma_j + \sum_{k\in\partial\Lambda} J'_k\sigma_k}}{Z'}$$

where

$$Z = \sum_\sigma e^{\sum_{i\sim j, i,j\in\Lambda^0} J_{ij}\sigma_i\sigma_j + \sum_{k\in\partial\Lambda} J_k\sigma_k}$$
$$Z' = \sum_\sigma e^{\sum_{i\sim j, i,j\in\Lambda^0} J_{ij}\sigma_i\sigma_j + \sum_{k\in\partial\Lambda} J'_k\sigma_k}$$

then

$$\langle\sigma_A\rangle - \langle\sigma_A\rangle' = \frac{\sum_{\sigma,\sigma'}(\sigma_A - \sigma'_A)e^{\sum_{i\sim j,i,j\in\Lambda^0} J_{ij}(\sigma_i\sigma_j+\sigma'_i\sigma'_j)+\sum_{k\in\partial\Lambda}(J_k\sigma_k+J'_k\sigma'_k)}}{ZZ'} \tag{3.6}$$

$$= \frac{\sum_t(1-t_A)\sum_\sigma \sigma_A e^{\sum_{i\sim j,i,j\in\Lambda^0} J_{ij}(1+t_it_j)\sigma_i\sigma_j+\sum_{k\in\partial\Lambda}(J_k+t_kJ'_k)\sigma_k}}{ZZ'} \tag{3.7}$$

where $t_i = \sigma_i\sigma'_i$. It follows from the ferromagnetic condition that when

$$J_k \geq |J'_k| \qquad \forall k \in \partial\Lambda$$

the right side of (3.7) is nonnegative. □

Let $A = B \triangle C = B \cup C \setminus B \cap C$ be the symmetric difference between the sets B and C, then $t_A = t_Bt_C = \pm 1$, and

$$1 - t_Bt_C = \pm(t_B - t_C) \tag{3.8}$$

Substituting (3.8) on the right of (3.7) and going back to the σ' variables, we obtain our basic inequality for $\sigma_A = \sigma_B\sigma_C$:

$$\langle\sigma_B\sigma_C\rangle - \langle\sigma_B\sigma_C\rangle' \geq |\langle\sigma_B\rangle\langle\sigma_C\rangle' - \langle\sigma_B\rangle'\langle\sigma_C\rangle| \tag{3.9}$$

Lemma 3.11.

$$\lim_{\beta'\to\beta+}\langle\rho_A\rangle_{\beta',+} = \langle\rho_A\rangle_{\beta,+} \tag{3.10}$$

$$\lim_{\beta'\to\beta-}\langle\rho_A\rangle_{\beta',-} = \langle\rho_A\rangle_{\beta,-} \tag{3.11}$$

Proof. Since $\langle \rho_A \rangle_{\beta,+} \leq \langle \rho_A \rangle_{\beta,\Lambda,+}$

$$\lim_{\beta' \to \beta+} \langle \rho_A \rangle_{\beta',+} \leq \lim_{\beta' \to \beta+} \langle \rho_A \rangle_{\beta',\Lambda,+} = \langle \rho_A \rangle_{\beta,\Lambda,+}$$

Letting $\Lambda \to \infty$, we have

$$\lim_{\beta' \to \beta+} \langle \rho_A \rangle_{\beta',+} \leq \langle \rho_A \rangle_{\beta,+}$$

But F.K.G inequality implies that

$$\langle \rho_A \rangle_{\beta,+} \leq \langle \rho_A \rangle_{\beta',+}$$

for $\beta \leq \beta'$, so

$$\langle \rho_A \rangle_{\beta,+} \leq \lim_{\beta' \to \beta+} \langle \rho_A \rangle_{\beta',+}$$

Then (3.10) is proved, and (3.11) can be proved analogously. □

Lemma 3.12. $\frac{\partial F}{\partial \beta}$ *is continuous for all the ferromagnetic interactions.*

Proof. First of all, notice that $\langle \sigma_i \sigma_j \rangle_+ = \langle \sigma_i \sigma_j \rangle_-$ by symmetry. Let $\frac{\partial F}{\partial \beta^+} \left(\frac{\partial F}{\partial \beta^-} \right)$ be the right(left) derivative of F with respect to β. The assertion will follow if we prove that

$$\frac{\partial F}{\partial \beta^{\pm}} = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \sum_{i \sim j, i,j \in \Lambda} \langle \sigma_i \sigma_j \rangle_{\Lambda,\pm} = \frac{1}{|\Lambda_1|} \sum_{i \sim j, i,j \in \Lambda_1} \langle \sigma_i \sigma_j \rangle_{\pm}, \tag{3.12}$$

where Λ_1 is the quotient graph of the infinite periodic graph with respect to translation. Now we prove (3.12). Consider first the boundary condition $+$. For any $\epsilon > 0$, let Λ_ϵ be a box containing all the edges of Λ_1, such that $\langle \sigma_i \sigma_j \rangle_{\Lambda_\epsilon,+} \leq \langle \sigma_i \sigma_j \rangle_+ + \epsilon$, for all $i \sim j$, $i, j \in \Lambda_1$. Then for any translation vector g, and Λ such that $\Lambda_\epsilon + g \subseteq \Lambda$, we have for all $i \sim j, i, j \in \Lambda_0$,

$$\begin{aligned} \langle \sigma_i \sigma_j \rangle_+ &= \langle \sigma_{i+g} \sigma_{j+g} \rangle_+ \le \langle \sigma_{i+g} \sigma_{j+g} \rangle_{\Lambda,+} \\ &\le \langle \sigma_{i+g} \sigma_{j+g} \rangle_{\Lambda_{\epsilon+g},+} = \langle \sigma_i \sigma_j \rangle_{\Lambda_\epsilon,+} \le \langle \sigma_i \sigma_j \rangle_+ + \epsilon \end{aligned} \tag{3.13}$$

Using the decomposition

$$\frac{1}{|\Lambda|} \sum_{i\sim j, i,j\in\Lambda} \langle \sigma_i \sigma_j \rangle_{\Lambda,\pm} = \frac{1}{|\Lambda|} \sum_{i,j\in\Lambda_1, i\sim j} \sum_{i+g\in\Lambda, j+g\in\Lambda} \langle \sigma_{i+g} \sigma_{j+g} \rangle_{\Lambda,\pm},$$

where the second sum is over g. $\lim_{\Lambda\to\infty} \frac{|\partial\Lambda|}{|\Lambda|} = 0$ and (3.13) implies that

$$\begin{aligned} \frac{1}{|\Lambda_1|} \sum_{i,j\in\Lambda_1, i\sim j} \langle \sigma_i \sigma_j \rangle_+ &\le \liminf \frac{1}{|\Lambda|} \sum_{i\sim j} \langle \sigma_i \sigma_j \rangle_{\Lambda,+} \\ \le \limsup \frac{1}{|\Lambda|} \sum_{i\sim j} \langle \sigma_i \sigma_j \rangle_{\Lambda,+} &\le \frac{1}{|\Lambda_1|} \sum_{i,j\in\Lambda_1, i\sim j} \langle \sigma_i \sigma_j \rangle + \epsilon \end{aligned}$$

for any ϵ, which means that

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \sum_{i\sim j, i,j\in\Lambda} \langle \sigma_i \sigma_j \rangle_{\Lambda,+} = \lim_{\Lambda\to\infty} \frac{\partial F(\Lambda,+)}{\partial\beta} = \frac{1}{|\Lambda_1|} \sum_{i,j\in\Lambda_1, i\sim j} \langle \sigma_i \sigma_j \rangle_+$$

and similarly for the boundary condition "$-$". Since F is convex with respect to β, the discontinuous points of $\frac{\partial F}{\partial\beta}$ is countably many β's, at most. For any finite box Λ of a locally finite periodic graph

$$\left| \frac{1}{|\Lambda|} \sum_{i\sim j, i,j\in\Lambda} \langle \sigma_i \sigma_j \rangle \right| \le \max_{v\in\Lambda_1} \deg(v)$$

According to the Dominated Convergence Theorem

$$\begin{aligned} \int_{\beta_0}^{\beta_1} \lim_{\Lambda\to\infty} \frac{\partial F(\Lambda,+)}{\partial\beta} d\beta &= \lim_{\Lambda\to\infty} \int_{\beta_0}^{\beta_1} \frac{\partial F(\Lambda,+)}{\partial\beta} d\beta \\ = \lim_{\Lambda\to\infty} [F(\Lambda,\beta_1,+) - F(\Lambda,\beta_0,+)] &= F(\beta_1) - F(\beta_0) \end{aligned}$$

The last equality follows from the fact that F is independent of boundary conditions. Let $\beta' \to \beta$ from the right side ($\beta' > \beta$). We have

$$\frac{\partial F}{\partial \beta^+} = \lim_{\beta' \to \beta+} \left.\frac{\partial F}{\partial \beta}\right|_{\beta=\beta'} = \lim_{\beta' \to \beta+} \frac{1}{|\Lambda_1|} \sum_{i \sim j, i,j \in \Lambda_1} \langle \sigma_i \sigma_j \rangle_+ =$$
$$\frac{1}{|\Lambda_1|} \sum_{i \sim j, i,j \in \Lambda_1} \langle \sigma_i \sigma_j \rangle_{\beta',+} = \frac{1}{|\Lambda_1|} \sum_{i \sim j, i,j \in \Lambda_1} \langle \sigma_i \sigma_j \rangle_{\beta,+}$$

So (3.12) is proved for the boundary condition +, and − is treated analogously. □

Lemma 3.13. $\langle \sigma_i \sigma_j \rangle_\mu = \langle \sigma_i \sigma_j \rangle_+$ *for any translation invariant limit probability measure* μ *and any* $i \sim j$*, if* $\frac{\partial F}{\partial \beta}$ *is continuous.*

Proof. Let $\mathbf{K}$ be an arbitrary translation invariant interaction, define

$$A_{\mathbf{K}} = \frac{1}{|\Lambda_1|} \sum_{i \sim j, i,j \in \Lambda_1} K_{ij} \sigma_i \sigma_j$$

Then for any translation invariant probability measure μ

$$F(\mathbf{J} + \mathbf{K}) \geq F(\mathbf{J}) + \int A_{\mathbf{K}} d\mu \tag{3.14}$$

That is $\int A_{\mathbf{K}} d\mu$ is a tangent vector of F at $\mathbf{J}$ along the direction $\mathbf{K}$. In other words, the expected value of $A_{\mathbf{K}}$ under any translation invariant probability measure corresponds to a tangent vector. For the proof of (3.14), see [17]. If $\left.\frac{\partial F(\mathbf{J}+t\mathbf{J})}{\partial t}\right|_{t=0}$ is continuous, the tangent vector is unique and its slope is equal to the corresponding derivative, which means the expected values of $A_{\mathbf{K}}$ under any translation invariant probability measure are the same. Hence if $\frac{\partial F}{\partial \beta}$ is continuous, we have

$$\sum_{i \sim j, i,j \in \Lambda_1} J_{ij} \langle \sigma_i \sigma_j \rangle_\mu = \sum_{i \sim j, i,j \in \Lambda_1} J_{ij} \langle \sigma_i \sigma_j \rangle_+$$

Given Lemma 2.9, under the assumption that $J_{ij} > 0$, we have $\langle \sigma_{ij} \rangle_+ = \langle \sigma_{ij} \rangle_\mu$ for all translation invariant limit measure μ. $\square$

Lemma 3.14. *Assume $J_{ij} > 0$, if $\langle \sigma_i \sigma_j \rangle = \langle \sigma_i \sigma_j \rangle' \neq 0$, for all $i \sim j$, then $\langle \sigma_E \rangle = \langle \sigma_E \rangle'$, for all sets E containing an even number of sites.*

Proof. Any translation invariant Gibbs measure can be considered as the weak limit of Boltzman measures on finite graphs with given boundary conditions, as the size of the graph goes to infinity. Any boundary conditions can be transformed to " + " boundary conditions by changing the coupling constant on the edges incident to an boundary site. If the configuration at the boundary site is "+", then the new coupling constant on incident edges is unchanged; otherwise the new coupling constant is $-J_e$. Given $J_{ij} > 0$, we have $J_{ij} \geq |J_{ij}|$, we derive that the inequality (3.9) is true. Since $\langle \sigma_B \sigma_C \rangle - \langle \sigma_B \sigma_C \rangle' \geq |\langle \sigma_B \rangle \langle \sigma_C \rangle' - \langle \sigma_B \rangle' \langle \sigma_C \rangle|$, let $A = B \triangle C$, we have

$$\langle \sigma_A \rangle - \langle \sigma_A \rangle' \geq |\langle \sigma_B \rangle \langle \sigma_A \sigma_B \rangle' - \langle \sigma_B \rangle' \langle \sigma_A \sigma_B \rangle|$$

Then $\langle \sigma_A \rangle = \langle \sigma_A \rangle'$, $\langle \sigma_B \rangle = \langle \sigma_B \rangle' \neq 0$ implies $\langle \sigma_A \sigma_B \rangle = \langle \sigma_A \sigma_B \rangle'$. The result follows by induction. $\square$

Proof of Theorem 2.7 From Lemma 2.13, it suffices to prove that $\langle \sigma_i \sigma_j \rangle \neq 0$ and is independent of boundary conditions for all $i \sim j$. From Lemma 2.12, this is true if and only if $\frac{\partial F}{\partial \beta}$ is continuous, and $\frac{\partial F}{\partial \beta}$ is always continuous, given lemma 2.11. $\square$

The uniqueness of the spin-spin even correlation functions can lead to the uniqueness of translation invariant Gibbs measures of dimer models on a large class of non-bipartite graphs. Interested readers may look at the appendix for more about this topic.

3.3 Fisher Correspondence

Correspondence 1: Measure-Preserving Correspondence

An Ising model on a square grid is associated to closed polygon configurations on the **dual** square grid as follows: if two adjacent spins are of the same sign, the dual edge separating them is not present in the closed polygon configuration, otherwise the dual edge is present in the closed polygon configuration. An example of this correspondence is illustrated in Figure 4:

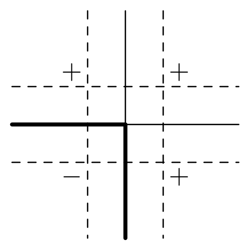

Figure 3.1: Measure-Preserving Correspondence

The dashed line are bonds of the Ising model, and the black line are edges of the dual grid. If two adjacent spins are of the same sign, the dual edge separating them is not present in the closed polygon configuration; otherwise the dual edge is present in the closed polygon configuration. This correspondence is two-to-one because negating the sign of all spins will end up with the same closed polygon configuration on the square grid. Combining with the one-to-one correspondence of closed polygon configurations on the square grid and the dimer configurations on Fisher graph, by assigning edge weights e^{-2J_e} to the **dual** edges, we get a measure-preserving correspondence between the Ising spin system and the dimer system, namely,

$$Pr(D(\sigma)) = Pr(\sigma) + Pr(-\sigma) = 2Pr(\sigma).$$

where $D(\sigma)$ is the dimer configuration corresponding to the spin configuration σ.

Correspondence 2: Correspondence Based on High-Temperature Expansion

Let $\mathcal{G}$ be a finite square grid, the high temperature expansion of the Ising model on $\mathcal{G}$ is

$$\begin{aligned} Z_{G,I} &= \sum_{\sigma} \prod_{e=uv\in E(G)} \exp(J_e\sigma_u\sigma_v) \\ &= \sum_{\sigma} \prod_{e=uv\in E(G)} (\cosh J_e + \sigma_u\sigma_v \sinh J_e) \\ &= (\prod_{e=uv\in E(G)} \cosh J_e) \sum_{\sigma} \prod_{e=uv\in E(G)} (1 + \sigma_u\sigma_v \tanh J_e) \\ &= (\prod_{e=uv\in E(G)} \cosh J_e) \sum_{C\in S} \prod_{e\in C} 2^{n^2} \tanh J_e, \end{aligned}$$

where S is the set of all closed polygon configurations of $\mathcal{G}$. This way, up to a multiplicative constant, the Ising partition function is the same as the partition function of the closed polygon configuration of the **same** square grid. There is no one-to-one correspondence between configurations in this case, hence although in this case the partition function is invariant, it is not measure-preserving. Combining with the correspondence between the closed polygon configurations on the square grid and the dimer configurations on the Fisher graph, we obtain a correspondence between Ising models on square grid and dimer models on Fisher graph, by giving the corresponding edge of the Fisher graph $\tanh J_e$.

3.3.1 Duality Transformation

Let G_n be the quotient graph of the square grid on the plane, as defined on Page 4. Let G_n^* be the dual graph of G_n. Define an Ising model on G_n with interactions $\{J_e\}_{e\in E(G_n)}$. Assume the Ising model on G_n has partition function $Z_{G_n,I}$. Then $Z_{G_n,I}$ can be written as, up to a constant multiple, following from the measure-preserving correspondence, we have

$$Z_{G_n,I} = 2\prod_{e\in E(G_n)}\exp(J_e)\sum_{C^*\in S_{00}^*}\prod_{e\in C^*}\exp(-2J_e) := 2\prod_{e\in E(G_n)}\exp(J_e)Z_{F_n,D_{00}},$$

where S_{00}^* is the set of closed polygon configurations of G_n^*, with an even number of occupied bonds crossed by both γ_x and γ_y. The sum is over all configurations in S_{00}^*. Similarly, we can define $S_{01}^*(S_{10}^*, S_{11}^*)$ to be the set of closed polygon configurations of G_n^*, with an even(odd,odd) number of occupied bonds crossed by γ_x, and an odd(even,odd) number of occupied bonds crossed by γ_y. F_n is the Fisher graph obtained from G_n^* by replacing each vertex with a gadget, as described in Figure 3. Let $Z_{F_n,D}$ be the partition function of dimer configurations on F_n, with weights e^{-2J_e} on dual edges of G_n, and weight 1 on all the other edges. Then

$$Z_{F_n,D} = Z_{F_n,D_{00}} + Z_{F_n,D_{01}} + Z_{F_n,D_{10}} + Z_{F_n,D_{11}},$$

where

$$Z_{F_n,D_{\theta,\tau}} = \sum_{C^*\in S_{\theta,\tau}^*}\prod_{e\in C^*}\exp(-2J_e).$$

For example, $Z_{F_n,D_{01}}$ is the dimer partition function on F_n with an even number of occupied edges crossed by γ_x, and an odd number of occupied edges crossed by γ_y. It also corresponds to an Ising model which has the same configuration on the two boundaries parallel to γ_y, and the opposite configurations on the two boundaries parallel to γ_x. Similar results hold for all the $Z_{F_n,D_{\theta,\tau}}$, $\theta,\tau \in \{0,1\}$.

On the other hand, if we consider the high temperature expansion of the Ising model on G_n, we have

$$Z_{G_n,I} = (\prod_{e=uv\in E(G_n)} \cosh J_e) \sum_{C\in S} \prod_{e\in C} 2^{n^2} \tanh J_e,$$

where S is the set of all closed polygon configurations of G_n. Let $\tilde{F}_n$ be a Fisher graph embedded into an $n \times n$ torus, obtained from G_n by the correspondence in Figure 3, with weights $\tanh J_e$ on edges of G_n, and weight 1 on all the other edges. In other words, the edge with weight $\tanh J_e$ of $\tilde{F}_n$ and the edge with weight e^{-2J_e} of F_n are dual edges. Then we have

$$Z_{G_n,I} = 2 \prod_{e\in E(G_n)} \exp(J_e) Z_{F_n,D_{00}} = 2^{n^2} \prod_{e\in E(G_n)} \cosh J_e Z_{\tilde{F}_n,D}$$

where $Z_{\tilde{F}_n,D}$ is the partition function of dimer configurations on $\tilde{F}_n$.

Similarly, we can expand all the $Z_{F_n,D_{\theta\tau}}$ as follows:

$$Z_{F_n,D_{\theta,\tau}} = \frac{1}{2^{n^2+1}} \prod_{e\in E_{G_n}} (1+\exp(-2J_e)) Z_{\tilde{F}_n,D}((-1)^{\tau},(-1)^{\theta}). \tag{3.15}$$

$Z_{\tilde{F}_{n,D}}(-1,1)$ is the dimer partition function of $\tilde{F}_n$ with weights of edges dual to the edges crossed by γ_x multiplied by -1. let us consider Similarly for $Z_{\tilde{F}_{n,D}}(1,-1)$ and $Z_{\tilde{F}_{n,D}}(-1,-1)$. To see why (3.15) is true, let us consider, for example $Z_{F_n,D_{10}}$, an odd number of present edges are crossed by γ_x, and an even number of present edges are crossed by γ_y. The corresponding Ising model has boundary condition such that winding once along γ_x, the spins change sign, while winding once along γ_y, the spins are invariant. This is the same as for a row of edges crossed by γ_y, we change the coupling constant from J_e to $-J_e$, and give the periodic boundary conditions. After the duality transformation, the edge weights $\tanh J_e$, if e is crossed by γ_y will be the opposite.

Without loss of generality, assume n is even. Let $K(z,w)$ be the Kasteleyn matrix of the toroidal graph, as defined on page 4. Given the orientation in Figure 1, we have

$$\begin{aligned}
\mathrm{Pf}K_{F_n}(1,1) &= Z_{F_n,D_{00}} - Z_{F_n,D_{01}} - Z_{F_n,D_{10}} - Z_{F_n,D_{11}} \\
&= \frac{1}{2^{n^2+1}} \prod_{e\in E(G_n)} (1+\exp(-2J_e))[Z_{\tilde{F}_{n,D}}(1,1) - Z_{\tilde{F}_{n,D}}(-1,1) - Z_{\tilde{F}_{n,D}}(1,-1) - Z_{\tilde{F}_{n,D}}(-1,-1)] \\
&= \frac{1}{2^{n^2+2}} \prod_{e\in E(G_n)} (1+\exp(-2J_e)) \\
&\quad \{[-\mathrm{Pf}K_{\tilde{F}_n}(1,1) + \mathrm{Pf}K_{\tilde{F}_n}(1,-1) + \mathrm{Pf}K_{\tilde{F}_n}(-1,1) + \mathrm{Pf}K_{\tilde{F}_n}(-1,-1)] \\
&\quad -[-\mathrm{Pf}K_{\tilde{F}_n}(-1,1) + \mathrm{Pf}K_{\tilde{F}_n}(-1,-1) + \mathrm{Pf}K_{\tilde{F}_n}(1,1) + \mathrm{Pf}K_{\tilde{F}_n}(1,-1)] \\
&\quad -[-\mathrm{Pf}K_{\tilde{F}_n}(1,-1) + \mathrm{Pf}K_{\tilde{F}_n}(1,1) + \mathrm{Pf}K_{\tilde{F}_n}(-1,-1) + \mathrm{Pf}K_{\tilde{F}_n}(-1,1)] \\
&\quad -[-\mathrm{Pf}K_{\tilde{F}_n}(-1,-1) + \mathrm{Pf}K_{\tilde{F}_n}(-1,1) + \mathrm{Pf}K_{\tilde{F}_n}(1,-1) + \mathrm{Pf}K_{\tilde{F}_n}(1,1)]\} \\
&= -\frac{1}{2^{n^2}} \prod_{e\in E(G_n)} (1+\exp(-2J_e))\mathrm{Pf}K_{\tilde{F}_n}(1,1) \qquad (3.16)
\end{aligned}$$

If we perform the same expansion for all the $\mathrm{Pf}K_{F_n}((-1)^\theta,(-1)^\tau)$, for $\theta,\tau\in\{0,1\}$, we get

$$\mathrm{Pf}K_{F_n}((-1)^\theta,(-1)^\tau) = \frac{1}{2^{n^2}} \prod_{e\in E(G_n)} (1+\exp(-2J_e))\mathrm{Pf}K_{\tilde{F}_n}((-1)^\theta,(-1)^\tau) \qquad \text{if } \{\theta,\tau\}\neq(0,0).$$

Proposition 3.15. *Consider a Fisher graph F_n embedded into an $n\times n$ torus. If an edge e is parallel to γ_x,γ_y, give e weight $w_e = e^{-2J_e}$. For all the other edges give weight 1. Define the duality transformation of F_n to be another Fisher graph $\tilde{F}_n$,* *embedded into an* $n \times n$ *torus. If an edge* e *is parallel to* γ_x, γ_y, *give* e *weight* $\tanh J_e$. *For all the other edges, give weight 1. Let* $P(z, w) = 0$ *denote the spectral curve. Then*

$$P_{F_n}((-1)^{\theta}, (-1)^{\tau}) = 0 \Longleftrightarrow P_{\tilde{F}_n}((-1)^{\theta}, (-1)^{\tau}) = 0$$

3.4 Critical Temperature and Spectral Curve

The purpose of this section is to identify the critical temperature of the periodic Ising model with the condition that the spectral curve as a single real node on $\mathbb{T}^2$. Using the correspondence based on the high-temperature expansion, we express the spin-spin correlation of the periodic Ising model as the determinant of a block Toeplitz matrix. The analysis of the holomorphic properties of the matrix leads to the result.

Lemma 3.16. *(Kelly and Sherman[10]) Given a Hamiltanonian*

$$\mathcal{H} = -\sum_{A \subset \Lambda} J_A \sigma_A$$

with ferromagnetic interactions, that is

$$J_A \geq 0$$

for all $A \subset \Lambda$, then

$$\frac{1}{\beta} \frac{\partial \langle \sigma_B \rangle}{\partial J_C} = \langle \sigma_B \sigma_C \rangle - \langle \sigma_B \rangle \langle \sigma_C \rangle \geq 0$$

The high temperature expansion of the partition function gives

$$Z = \prod_e 2 \cosh J_e \sum_{\mathcal{C}} \prod_{e \in \mathcal{C}} \tanh J_e$$

where the sum is over all closed polygon of the square grid. Assume $\tau_e = \tanh J_e$. We consider the spin located at $(0,0)$, σ_{00} and the spin located at $(0,N)$, σ_{0N}. Then

$$(\sigma_{00}\sigma_{0N}) = (\sigma_{00}\sigma_{01})(\sigma_{01}\sigma_{02})...(\sigma_{0,N-1}\sigma_{0N})$$

Let e_l denote the edge connecting $(0,l)$ and $(0,l+1)$. Since

$$\sigma_{0,l}\sigma_{0,l+1}(1+\tau_{e_l}\sigma_{0,l}\sigma_{0,l+1}) = \tau_{e_l}(1+\frac{1}{\tau_{e_l}}\sigma_{0l}\sigma_{0,l+1}),$$

we have

$$\langle\sigma_{00}\sigma_{0N}\rangle = Z^{-1}\prod_e 2\cosh J_e \prod_{l=0}^{N-1}\tau_{e_l}\sum_\sigma\prod_{l=0}^{N-1}(1+\frac{1}{\tau_{e_l}}\sigma_{0l}\sigma_{0,l+1}){\prod_e}'(1+\tau_e\sigma_p\sigma_q)$$

where $\prod'$ means the terms corresponding to e_l, $0 \le l \le N-1$ are omitted. This expression is of the form of a partition function for some Ising counting lattice with bonds τ_{e_l} on the straight line connecting sites $(0,0)$ and $(0,N)$, replaced by $\frac{1}{\tau_{e_l}}$. We transform the square grid G_n to the Fisher Graph, using the technique as described in Figure 3. Assume the edge weights of the Fisher graph are $\tanh J_e$ for all edges corresponding to edges of G_n, and all the other edges have weight 1.

Let $\gamma_x(\gamma_y)$ denote a path winding horizontally(vertically) once on the torus, multiply all the edge weights crossed by $\gamma_x(\gamma_y)$ with z or $\frac{1}{z}$, (w or $\frac{1}{w}$), according to the orientation of edges.

Let $K_{mn}(z,w)$ denote the corresponding weighted adjacency matrix. If both m and n are even, we have

$$\langle\sigma_{00}\sigma_{0N}\rangle_{m,n} = \prod_{l=0}^{N-1}\tau_{e_l}\frac{-\mathrm{Pf}K'_{mn}(1,1)+\mathrm{Pf}K'_{mn}(-1,1)+\mathrm{Pf}K'_{mn}(1,-1)+\mathrm{Pf}K'_{mn}(-1,-1)}{-\mathrm{Pf}K_{mn}(1,1)+\mathrm{Pf}K_{mn}(-1,1)+\mathrm{Pf}K_{mn}(1,-1)+\mathrm{Pf}K_{mn}(-1,-1)},$$

where $K'_{mn}(z,w)$ denote the corresponding weighted adjacency matrix by changing the edge weights τ_{e_l} to $\frac{1}{\tau_{e_l}}$.

Theorem 3.17. *For the Ising model on the infinite periodic square grid, we can express* $\lim_{N\to\infty}\langle\sigma_{00}\sigma_{0N}\rangle^2$ *as the determinant of a block Toeplitz matrix multiplied by* *a function of weights of all edges on the straight line connecting* σ_{00} *and* σ_{0N}.

Proof. We consider $\frac{\mathrm{Pf}K'_{mn}(u,v)}{\mathrm{Pf}K_{mn}(u,v)}$ where $u, v \in \{1, -1\}$. Assume $\delta = K'_{mn}(u,v) - K_{mn}(u,v)$, then δ is given by

$$\delta(0,l;0,l+1) = -\delta^T(0,l+1;0,l) = \begin{array}{cc} & \begin{array}{cccc} R & L & U & D \end{array} \\ \begin{array}{c} R \\ L \\ U \\ D \end{array} & \begin{pmatrix} 0 & \frac{1}{\tau_{e_l}} - \tau_{e_l} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

If $0 \leq l \leq N-1$ and zero otherwise. Thus if we define y as the $2N \times 2N$ sub-matrix of δ in the subspace where δ does not vanish identically, and if we define Q as the $2N \times 2N$ sub-matrix of K_{mn}^{-1} in the same subspace, we find

$$\frac{\mathrm{Pf}K'_{mn}(u,v)}{\mathrm{Pf}K_{mn}(u,v)} = (-1)^{k_1}\mathrm{Pf}(\delta + K_{mn}(u,v))\mathrm{Pf}(K_{mn}^{-1}(u,v)) = (-1)^{k_2}\mathrm{Pf}y\mathrm{Pf}(y^{-1} + Q(u,v))$$

where k_1, k_2 are constants depending only on m, n, N, and independent of u, v, and

y is given by

$$y = \begin{array}{cc} & \begin{array}{cccccccc} 0,0 & 0,1 & ... & 0,N-1 & 0,1 & 0,2 & ... & 0,N \\ R & R & & R & L & L & & L \end{array} \\ \begin{array}{cc} 0,0 & R \\ 0,1 & R \\ \vdots & \\ 0,N-1 & R \\ 0,1 & L \\ 0,2 & L \\ \vdots & \\ 0,N & L \end{array} & \left(\begin{array}{cccc|cccc} & & & & \frac{1}{\tau_{e_1}} - \tau_{e_1} & & & \\ & \mathbf{0} & & & & \frac{1}{\tau_{e_2}} - \tau_{e_2} & & \\ & & & & & & \ddots & \\ & & & & & & & \frac{1}{\tau_{e_N}} - \tau_{e_N} \\ \hline \tau_{e_1} - \frac{1}{\tau_{e_1}} & & & & & & & \\ & \tau_{e_2} - \frac{1}{\tau_{e_2}} & & & & \mathbf{0} & & \\ & & \cdots & & & & & \\ & & & \tau_{e_N} - \frac{1}{\tau_{e_N}} & & & & \end{array}\right) \end{array}$$

so that

$$\mathrm{Pf} y = \prod_{l=0}^{N-1} (\frac{1}{\tau_{e_l}} - \tau_{e_l})(-1)^{\frac{N(N-1)}{2}}$$

Hence we have

$$\langle \sigma_{00}\sigma_{0N} \rangle_{m,n} = \prod_{l=0}^{N-1} \tau_{e_l} \sum_{u,v \in \{-1,1\}} \frac{\mathrm{Pf} K'_{mn}(u,v)}{\mathrm{Pf} K_{mn}(u,v)} \frac{(-1)^{s(u,v)} \mathrm{Pf} K_{m,n}(u,v)}{Z}$$

where $s(u,v)$ is $+1$ or -1 depending on u, v. We choose a subsequence $m', n' \to \infty$ such that k_2 are always even, then

$$\lim_{m',n' \to \infty} \langle \sigma_{00}\sigma_{0N} \rangle^2_{m,n} = \prod_{l=0}^{N-1} (1 - \tau^2_{e_l})^2 \det T_N$$

If we use K to denote the weighted adjacency matrix of the infinite graph, we can express T_N as follows:

$$T_N = \begin{pmatrix} 0 & \cdots & K^{-1}_{0,0,R;0,N-1,R} & K^{-1}_{0,0,R;0,1,L} - \frac{\tau_{e_1}}{1-\tau^2_{e_1}} & \cdots & K^{-1}_{0,0,R;0,N,L} \\ K^{-1}_{1,1,R;0,0,R} & \cdots & K^{-1}_{0,1,R;0,N-1,R} & K^{-1}_{0,1,R;0,1,L} & \cdots & K^{-1}_{0,1,R;0,N,L} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K^{-1}_{0,N-1,R;0,0,R} & \cdots & 0 & K^{-1}_{0,N-1,R;0,1,L} & \cdots & K^{-1}_{0,N-1,R;0,N,L} - \frac{\tau_{e_N}}{1-\tau^2_{e_N}} \\ K^{-1}_{0,1,L;0,0,R} + \frac{\tau_{e_1}}{1-\tau^2_{e_1}} & \cdots & K^{-1}_{0,1,L;0,N-1,R} & 0 & \cdots & K^{-1}_{0,1,L;0,N,L} \\ K^{-1}_{0,2,L;0,0,R} & \cdots & K^{-1}_{0,2,L;0,N-1,R} & K^{-1}_{0,2,L;0,1,L} & \cdots & K^{-1}_{0,2,L;0,N,L} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K^{-1}_{0,N,L;0,0,R} & \cdots & K^{-1}_{0,N,L;0,N-1,R} + \frac{\tau_{e_N}}{1-\tau^2_{e_N}} & K^{-1}_{0,N,L;0,1,R} & \cdots & 0 \end{pmatrix}$$

It is well-known that T_N is independent of u, v, since the entries of K^{-1} can be expressed as follows

$$K^{-1}(j,k,p;j',k',q) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{i\phi_1(k-k')}e^{i\phi_2(j-j')}K_{11}^{-1}(e^{i\phi_1}, e^{i\phi_2})_{p,q}d\phi_1 d\phi_2 \quad (3.17)$$

where $K_{11}(z,w)$ is the weighted adjacency matrix of one fundamental domain embedded into a torus. And

$$\lim_{N\to\infty}\langle\sigma_{00}\sigma_{0N}\rangle^2 = \lim_{N\to\infty}\det T_N \prod_{0\le l\le N-1}(1-\tau^2_{e_l})^2 \quad (3.18)$$

Let l_0 denote the length of one period, assume N is a multiple of l_0. We rearrange the rows and columns of T_N, in such a way that the determinant of T_N will not change after this manipulation, neither will the limit as $N \to \infty$. Then we group the matrix after row and column rearrangements, denoted still by T_N, into $\frac{N}{l_0} \times \frac{N}{l_0}$ blocks, and each block is a $2l_0 \times 2l_0$ submatrix, such that the (j, j') block of T_N has a row range from $(0, (j-1)l_0, R)$ to $(0, jl_0 - 1, R)$, from $(0, (j-1)l_0 + 1.L)$ to $(0, jl_0, L)$, and a column range from $(0, (j'-1)l_0, R)$ to $(0, j'l_0 - 1, R)$, from $(0, (j'-1)l_0 + 1.L)$ to $(0, j'l_0, L)$.

If we define ψ, a matrix-valued function on the unit circle, as follows

$$\psi(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} K^{-1}(\zeta, e^{i\phi_2})_{0,0,R;0,0,R} & \cdots & K^{-1}(\zeta, e^{i\phi_2})_{0,0,R;0,l_0-1,R} & K^{-1}(\zeta, e^{i\phi_2})_{0,0,R;0,1,L} - \frac{\tau_{e_1}}{1-\tau_{e_1}^2} & \cdots & K^{-1}(\zeta, e^{i\phi_2})_{0,0,R;0,l_0,L} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K^{-1}(\zeta, e^{i\phi_2})_{0,l_0-1,R;0,0,R} & \cdots & K^{-1}(\zeta, e^{i\phi_2})_{0,l_0-1,R;0,l_0-1,R} & K^{-1}(\zeta, e^{i\phi_2})_{0,l_0-1,R;0,1,L} & \cdots & K^{-1}(\zeta, e^{i\phi_2})_{0,l_0-1,R;0,l_0,L} - \frac{\tau_{e_{l_0}}}{1-\tau_{e_{l_0}}^2} \\ K^{-1}(\zeta, e^{i\phi_2})_{0,1,L;0,0,R} + \frac{\tau_{e_1}}{1-\tau_{e_1}^2} & \cdots & K^{-1}(\zeta, e^{i\phi_2})_{0,1,L;0,l_0-1,R} & K^{-1}(\zeta, e^{i\phi_2})_{0,1,L;0,1,L} & \cdots & K^{-1}(\zeta, e^{i\phi_2})_{0,1,L;0,l_0,L} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K^{-1}(\zeta, e^{i\phi_2})_{0,l_0,L;0,0,R} & \cdots & K^{-1}(\zeta, e^{i\phi_2})_{0,l_0,L;0,l_0-1,R} + \frac{\tau_{e_{l_0}}}{1-\tau_{e_{l_0}}^2} & K^{-1}(\zeta, e^{i\phi_2})_{0,l_0,L;0,1,L} & \cdots & K^{-1}(\zeta, e^{i\phi_2})_{0,l_0,L;0,l_0,L} \end{pmatrix} d\phi_2$$

Then the (j, j') block of ψ is the $(j - j')$ Fourier coefficient of ψ, given (3.17). Let T denote the infinite matrix as $N \to \infty$. Namely,

$$(T_N)_{j,j'} = \frac{1}{2\pi}\int_{|\zeta|=1} \zeta^{j-j'}\psi(\zeta)\frac{d\zeta}{i\zeta}$$

where the integral is evaluated entry by entry. Let T denote the infinite matrix as $N \to \infty$, then T is a block Toeplitz matrix with symbol ψ. $\square$

Lemma 3.18. *([24]) Let* $\mathbb{T}^2 = \{z, w||z| = 1, |w| = 1\}$ *be the unit torus, either* $\det K(z, w) = 0$ *has no zeros on* $\mathbb{T}^2$*, or it has a single real node on* $\mathbb{T}^2$*. That is,* $\det K(z, w) = 0$ *intersect* $\mathbb{T}^2$ *at one of the points* $(\pm 1, \pm 1)$*, and both* $\frac{\partial \det K(z,w)}{\partial z}$ *and* $\frac{\partial \det K(z,w)}{\partial w}$ *vanish at the same point.*

To study the asymptotic behavior of $\langle \sigma_{00}\sigma_{0N}\rangle$ as $N \to \infty$, we first introduce the following lemmas

Lemma 3.19. *(Widom[30]) Let $T_n(\psi)$ be a block Toeplitz matrix with symbol ψ, and $n \times n$ blocks in total. If*

$$\sum_{k=-\infty}^{\infty} \|\psi_k\| + (\sum_{k=-\infty}^{\infty} |k| \|\psi_k\|^2)^{\frac{1}{2}} < \infty$$
$$\det \psi(e^{i\theta}) \neq 0, \qquad \frac{1}{2\pi}\Delta_{0\leq\theta\leq 2\pi} \arg \det \psi(e^{i\theta}) = 0,$$

here $\|\psi_k\|$ denotes the Hilbert-Schmidt norm of the matrix ψ_k, and $\frac{1}{2\pi}\Delta_{0\leq\theta\leq 2\pi} \arg \det \psi(e^{i\theta})$ is the index of the point 0 with respect to the curve $\{\det \psi(e^{i\theta}) : 0 \leq \theta \leq 2\pi\}$, namely

$$\frac{1}{2\pi}\Delta_{0\leq\theta\leq 2\pi} \arg \det \psi(e^{i\theta}) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{1}{\det \psi(\zeta)} \frac{\partial \det \psi(\zeta)}{\partial \zeta} d\zeta$$

then

$$\lim_{n\to\infty} \frac{\det T_n(\psi)}{G[\psi]^{n+1}} = E[\psi]$$

with

$$\begin{aligned} G[\psi] &= \exp\{\frac{1}{2\pi}\int_0^{2\pi} \log \det \psi(e^{i\theta}) d\theta\} \\ E[\psi] &= \det T[\psi] T[\psi^{-1}] \end{aligned}$$

where the last det *refers to the determinant defined for operators on Hilbert space differing from the identity by an operator of trace class.*

Lemma 3.20. *For ferromagnetic, periodic Ising model with nearest neighbor pair interactions on a square grid, if $\det K(z,w)$ has no zeros on $\mathbb{T}^2$, then*

$$\prod_{l=1}^{l_0} (1 - \tau_{e_l}^2)^2 \det \psi(e^{i\theta}) = 1$$

for any $\theta \in [0, 2\pi)$

Proof. Let us consider an $m \times n$ torus, where m and n are the numbers of periods along each direction. The previous technique to compute the spin-spin correlations implies that

$$\langle \sigma_{00}\sigma_{0n} \rangle = \prod_{1 \leq l \leq l_0} \tau_{e_l}^n \frac{Z'_{mn}}{Z_{mn}}$$

where Z'_{mn} is the partition function of dimer configurations on an $m \times n$ Fisher graph with wights $\tau_{e_l} (1 \leq l \leq n)$ changing to $\frac{1}{\tau_{e_l}}$. Since the graph is embedded into an $m \times n$ torus, σ_{00} and σ_{0n} are actually the same spin, therefore we have $\langle \sigma_{00}\sigma_{0n} \rangle = 1$, that is

$$\prod_{1 \leq l \leq l_0} \tau_{e_l}^n Z'_{mn} = Z_{mn}$$

Let $K'_{mn}(K_{mn})$ be the corresponding Kasteleyn matrix with respect to $Z'_{mn}(Z_{mn})$. We are going to construct a correspondence between monomials in $\tau_{e_l}^2 \det K'_{m,1}(1, w)$ and $\det K_{m,1}(-1, w)$, where w is any number on the unit circle. Each term in the expansion of the determinant corresponds to an oriented loop configuration in the $m \times 1$ torus, where each vertex is occupied by the configuration exactly twice. The correspondence is constructed by changing configurations on the first row of the graph, while keeping the configurations on all the other rows invariant. Consider the first row. For each pair of nearest components, which are parts of non-doubled edge loops crossing the first row, there exists a pair of nearest triangles, one on each component. We change their paths through the pair of nearest triangles. This way, in the new configuration, all the e_l occupied once originally will still be occupied once, with wieght $\frac{1}{\tau_{e_l}}$ changed to τ_{e_l}. Any e_l unoccupied will be a doubled edge, with weight 1 changed to $\tau_{e_l}^2$, any e_l occupied twice will be unoccupied, with weight $\frac{1}{\tau_{e_l}^2}$ changed to 1. Hence the corresponding terms in $\prod_{1 \leq l \leq l_0} \tau_{e_l}^2 \det K'_{m,1}(1, w)$ and $\det K_{m,1}(-1, w)$ have the same absolute value. They are equal if the signs are also the same. For each component, which is part of a non-doubled-edge loop crossing the first row of the torus, if we change the path through a pair of triangles (corresponding to two nearest components on both sides), the sign will change by a factor of -1. This sign change will cancel if we change the sign of the z-edge passed by the loop simultaneously. Hence we have

$$\prod_{1\leq l\leq l_0} \tau_{e_l}^2 \det K'_{m,1}(1,w) = \det K_{m,1}(-1,w)$$

An example of such a path change is illustrated in Figure 5 and Figure 6. Similarly we have

$$\prod_{1\leq l\leq l_0} \tau_{e_l}^2 \det K'_{m,1}(-1,w) = \det K_{m,1}(1,w)$$

An example of such a change of configurations along an n-cycle is illustrated in Figure 5 and Figure 6.

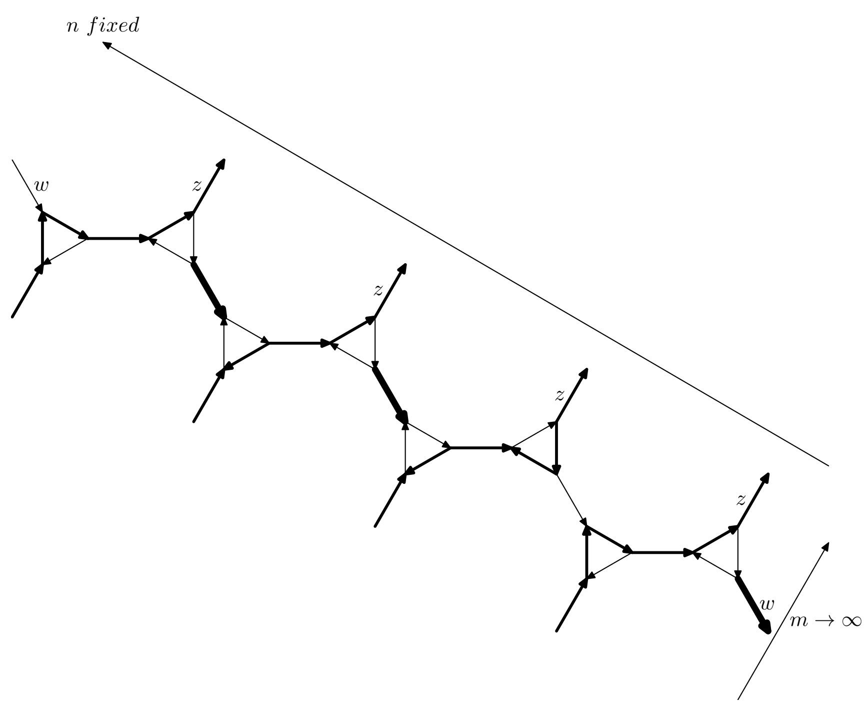

Figure 3.2: Configuration before change

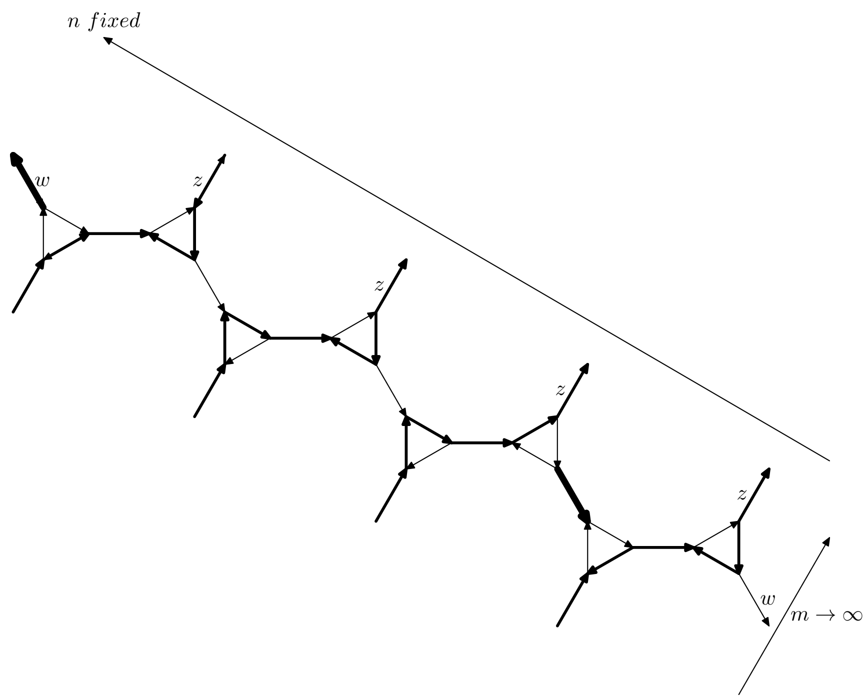

Figure 3.3: Configuration after change

Therefore, we have

$$\begin{aligned}[\prod_{l=1}^{l_0}(1-\tau_{e_l}^2)^2]\det\psi(e^{i\theta}) &= \lim_{m\to\infty}\frac{(\prod_{l=1}^{l_0}\tau_{e_l}^2)\det K'_{m,1}(1,e^{i\theta})}{\det K_{m,1}(1,e^{i\theta})}\\ &= \lim_{m\to\infty}\frac{\det K_{m1}(-1,e^{i\theta})}{\det K_{m1}(1,e^{i\theta})}\end{aligned}$$

Let $v_1,\cdots,v_{2nl_0}$ be vertices which are endpoints of those edges changing weights from τ_{e_ℓ} to $\frac{1}{\tau_{e_\ell}}$. Let us consider

$$\begin{aligned}\lim_{m\to\infty}\frac{\det K'_{m,1}(1,e^{i\theta})}{\det K_{m,1}(1,e^{i\theta})} &= \lim_{m\to\infty}\sum_{u_1,\cdots,u_{2nl_0},u_j\sim v_j}\prod_{j=1}^{2n}K'_{m1}(1,e^{i\theta})_{v_ju_j}\frac{\det K'_{m1}(1,e^{i\theta})_{E^c\{u_1,\cdots,u_{2nl_0}\}}}{\det K_{m1}(1,e^{i\theta})}\\ &= \lim_{m\to\infty}\sum_{u_1,\cdots,u_{2nl_0},u_j\sim v_j}\prod_{j=1}^{2n}K'_{m1}(1,e^{i\theta})_{v_ju_j}\det K^{-1}(1,e^{i\theta})_{E^c\{u_1,\cdots,u_{2nl_0}\}}\end{aligned}$$

where $K'_{mn}(1,e^{i\theta})_{E^c\{u_1,\cdots,u_{2nl_0}\}}$ is the submatrix of $K'_{mn}(1,e^{i\theta})$ deleting rows $v_1,\cdots,v_{2nl_0}$ and columns $u_1,\cdots,u_{2nl_0}$. This submatrix is also a submatrix of $K_{mn}(1,e^{i\theta})$, because all the edges with changed weights have been deleted. $K'_{mn}(1,e^{i\theta})_{v_j,u_j}$ are edge weights, which are independent of the first variable - weights of edges crossed by γ_x. As $m\to\infty$, the limit of entries of $K^{-1}_{mn}(1,e^{i\theta})$ is independent of the first variable too, since $\lim_{m\to\infty} K^{-1}_{mn}(z,e^{i\theta})$ can always be expressed as the same complex integral no matter where z is located on the unit circle. As a result,

$$\lim_{m\to\infty}\frac{\det K'_{m1}(1,e^{i\theta})}{\det K_{m1}(1,e^{i\theta})}=\lim_{m\to\infty}\frac{\det K'_{m1}(-1,e^{i\theta})}{\det K_{m1}(-1,e^{i\theta})}$$

Hence we have

$$\begin{aligned}\lim_{m\to\infty}\frac{\prod_{\ell=1}^{\ell_0}\tau_{e_\ell}^2\det K'_{m1}(1,e^{i\theta})}{\det K_{m1}(1,e^{i\theta})} &= \lim_{m\to\infty}\frac{\det K_{m1}(-1,e^{i\theta})}{\det K_{m1}(1,e^{i\theta})}\\ &= \lim_{m\to\infty}\frac{\det K_{m1}(1,e^{i\theta})}{\det K_{m1}(-1,e^{i\theta})}\\ &= \lim_{m\to\infty}\frac{\prod_{\ell=1}^{\ell_0}\tau_{e_\ell}^2\det K'_{m1}(-1,e^{i\theta})}{\det K_{m1}(-1,e^{i\theta})}\end{aligned}$$

Therefore

$$[\prod_{\ell=1}^{\ell_0}(1-\tau_{e_\ell}^2)^2\det\psi(e^{i\theta})]^2=1, \tag{3.19}$$

When $\theta=0$, the quantity is the quotient of two skew-symmetric matrix, hence it is positive and equal to 1. The lemma follows from the continuity of $[\prod_{\ell=1}^{\ell_0}(1-\tau_{e_\ell}^2)^2\det\psi(e^{i\theta})]$.

□

Lemma 3.21. *(Gohberg,Krein[9]) Let $\mathfrak{F}_1$ denote the set of all completely continuous operator A for which*

$$\sum_{j=1}^{\infty}s_j(A)<\infty$$

where s_j are singular values of A. Let $A(\mu)$ be an operator-function with values in $\mathfrak{F}_1$ and holomorphic in some region. Then the determinant $\det(I - A(\mu))$ *is holomorphic in the same region.*

Lemma 3.22. *If we allow complex edge weights, then the spin correlation*

$$\lim_{N\to\infty} \langle \sigma_{00}\sigma_{0N}\rangle_p^2$$

under toroidal boundary conditions is analytic with respect to edge weights except for the edge weights $\tanh J_{i,j}$ *such that the spectral curve has a real node on* $\mathbb{T}^2$

Proof. Let $H[\psi]$ denote the semi-infinite Hankel matrices with kernel ψ, namely,

$$H[\psi] = (\psi_{i+j+1}) \qquad 0 \leq i, j < \infty$$

then

$$T[\psi]T[\psi^{-1}] = I - H[\psi]H[\tilde{\psi}^{-1}]$$

where

$$\tilde{\psi}(\zeta) = \psi(\frac{1}{\zeta})$$

Since ψ is an anti-Hermitian matrix on the unit circle, its determinant will always be real, hence we have

$$\Delta_{0\leq\theta\leq 2\pi} \arg\det \psi(e^{i\theta}) = 0$$

For any edge weights which do not lie on the critical surface, the condition of Theorem

4.4, and 4.5 will satisfy(Widom[30]). Then we have

$$\lim_{N\to\infty}\langle\sigma_{00}\sigma_{0N}\rangle_p^2 = \lim_{N\to\infty}\prod_{0\le l\le N-1}(1-\tau_{e_l}^2)^2\det T_N[\psi] = \det(I - H[\psi]H[\tilde{\psi}^{-1}])$$

Since the entries of $H[\psi]H[\tilde{\psi}^{-1}]$ will be analytic everywhere except on the spectral curve, so is the spin correlation $\lim_{N\to\infty}\langle\sigma_{00}\sigma_{0N}\rangle_p^2$. □

Lemma 3.23. *The weak limit of the Boltzmann measures $\mathcal{P}_n$ on G_n, the quotient graph defined on page 4, is a translation invariant Gibbs measure $\mathcal{P}$. The probability of occurrence of a subset of edges $e_1 = u_1v_1, ..., e_m = u_mv_m$ of G in a dimer configuration of G chose with respect to Gibbs measure $\mathcal{P}$, is*

$$\mathcal{P}(e_1, ..., e_m) = \prod_{i=1}^{m} w_{u_iv_i}|\mathrm{Pf}(K^{-1})_{e_1,...,e_m}|$$

where K^{-1} is the inverse of the infinite Kasteleyn matrix, defined as in (3.17), and $(K^{-1})_{e_1,...,e_m}$ is the sub-matrix of K^{-1}, whose lines and columns are defined by the vertices of the edges $e_1, ..., e_m$.

Proof. The dimer partition function Z_n of the graph G_n, for n even, is

$$Z_n = \frac{1}{2}[-\mathrm{Pf}(K_n^{00}) + \mathrm{Pf}(K_n^{10}) + \mathrm{Pf}(K_n^{01}) + \mathrm{Pf}(K_n^{11})]$$

Let $E = \{e_1 = u_1v_1, ..., e_m = u_mv_m\}$ be a subset of edges of G_n, then the probability $\mathcal{P}_n(e_1, ..., e_m)$ of these edges occurring in a dimer configuration of G_n chosen with respect to the Boltzmann measure $\mathcal{P}_n$, for n even, is

$$\frac{\prod_{i=1}^{m} w_{u_iv_i}}{2Z_n}[-\mathrm{Pf}(K_n^{00})_{E^c} + \mathrm{Pf}(K_n^{10})_{E^c} + \mathrm{Pf}(K_n^{01})_{E^c} + \mathrm{Pf}(K_n^{11})_{E^c}]$$

where $E^c = V(G_n) \setminus \{u_1v_1, ..., u_mv_m\}$ and $(K_n^{\theta\tau})_{E^c}$ is the sub-matrix of $K_n^{\theta\tau}$ whose lines and columns are indexed by E^c. Let $P(z,w) = \det K(z,w)$. By lemma 4.3, either the spectral curve $P(z,w) = 0$ does not intersect $\mathbb{T}^2$, or it has a single real node on $\mathbb{T}^2$.

If $P(z,w) = 0$ does not intersect $\mathbb{T}^2$,

$$(K_n^{\theta\tau})^{-1}_{(x,y,v),(x',y',v')} = \frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} e^{\frac{i(2j+\theta)\pi(x-x')}{n}} e^{\frac{i(2k+\tau)\pi(y-y')}{n}} \frac{Cof[K(e^{\frac{i\pi(2j+\theta)}{n}}, e^{\frac{i\pi(2k+\tau)}{n}})]_{v,v'}}{P(e^{\frac{i\pi(2j+\theta)}{n}}, e^{\frac{i\pi(2k+\tau)}{n}})}$$

The right side is a Riemann sum in non-critical case, it converges to an integral for any $\theta, \tau \in \{-1, 1\}$, that is,

$$K^{-1}_{(x,y,v),(x',y',v')} = \lim_{n\to\infty} (K_n^{\theta,\tau})^{-1}_{(x,y,v),(x',y',v')} \tag{3.20}$$

$$= \frac{1}{4\pi^2}\iint_{\mathbb{T}^2} z^{x-x'} w^{y-y'} \frac{Cof[K(z,w)]_{v,v'}}{P(z,w)} \frac{dz}{iz}\frac{dw}{iw}. \tag{3.21}$$

We also have

$$\mathrm{Pf}(K_n^{\theta\tau})^{-1}_E = (-1)^{\kappa(n)} \frac{(\mathrm{Pf}K_n^{\theta\tau})_{E^c}}{\mathrm{Pf}K_n^{\theta\tau}}$$

where $\kappa : \mathbb{N} \to \{0,1\}$ is a function. We can choose a subsequence $\{n'\}$ such that $\kappa(n')$ is a constant κ_0. Then

$$\begin{aligned}
\mathcal{P}(e_1, ..., e_m) &= \lim_{n'\to\infty} \mathcal{P}_{n'}(e_1, ..., e_m) \\
&= \lim_{n'\to\infty} (-1)^{\kappa_0} \frac{\prod_{i=1}^m w_{u_iv_i}}{2Z_n} [-\mathrm{Pf}(K_{n'}^{00})^{-1}_E \mathrm{Pf}K_{n'}^{00} + \mathrm{Pf}(K_{n'}^{01})^{-1}_E \mathrm{Pf}K_{n'}^{01} \\
&\quad +\mathrm{Pf}(K_{n'}^{10})^{-1}_E \mathrm{Pf}K_{n'}^{10} + \mathrm{Pf}(K_{n'}^{11})^{-1}_E \mathrm{Pf}K_{n'}^{11}] \\
&= \prod_{i=1}^m w_{u_iv_i} |\mathrm{Pf}(K^{-1})_E|
\end{aligned}$$

Note that the convergence actually holds for all n, because on the complement subsequence $\mathbb{N} \setminus \{n'\}$, κ is also a constant given that κ is a two-valued function. The sign does not matter since we take the absolute value in the end.

If $P(z,w) = 0$ has a single real node on $\mathbb{T}^2$, see [3] for a proof. The weak convergence of $\mathcal{P}_n$ for $n \in \mathbb{N}$ follows from the fact that there is a unique translation invariant Gibbs measure, and the weak limit of $\mathcal{P}_n$ on any subsequence is translation invariant. $\square$

Lemma 3.24. *Let $\mathcal{P}$ be the weak limit of $\mathcal{P}_n$, Boltzmann measures defined for graph G_n with periodic boundary conditions. $P(z,w) = 0$ does not intersect $\mathbb{T}^2$, if and only if the edge-edge correlation*

$$\mathcal{P}(e_1 \& e_2) - \mathcal{P}(e_1)\mathcal{P}(e_2)$$

converges to 0 exponentially fast as $|e_1 - e_2| \to \infty$.

Proof. By Lemma 4.8,

$$\mathcal{P}(e_1 \& e_2) - \mathcal{P}(e_1)\mathcal{P}(e_2) = w_{u_1v_1}w_{u_2v_2}\left[\left|\mathrm{Pf}K^{-1}\begin{pmatrix} u_1 & v_1 & u_2 & v_2 \\ u_1 & v_1 & u_2 & v_2 \end{pmatrix}\right| - |K^{-1}_{u_1,v_1}K^{-1}_{u_2,v_2}|\right] \tag{3.22}$$

$$= w_{u_1v_1}w_{u_2v_2}[|K^{-1}_{u_1,v_1}K^{-1}_{u_2,v_2} + K^{-1}_{u_1,v_2}K^{-1}_{v_1,u_2} - K^{-1}_{u_1,u_2}K^{-1}_{v_1,v_2}| - |K^{-1}_{u_1,v_1}K^{-1}_{u_2,v_2}|] \tag{3.23}$$

If $P(z,w) = 0$ has no zeros on $\mathbb{T}^2$, (3.21) and (3.23) imply that as $|e_1 - e_2| \to \infty$, $\mathcal{P}(e_1 \& e_2)$ converges to 0 exponentially. If $P(z,w)$ has a real node on $\mathbb{T}^2$, then

$$\frac{Cof(K(z,w))_{v,v'}}{P(z,w)}$$

is not an analytic function in edge weights, hence its Fourier coefficients does not decay exponentially fast. To see that, if the Fourier coefficients decays exponentially, then the function is the limit of a locally uniform convergent power series, hence the function has to be analytic. □

Lemma 3.25. *If $P(z,w)=0$ has a single real node on $\mathbb{T}^2$, and if*

$$\det\begin{pmatrix} \frac{\partial^2 P(e^{i\theta},e^{i\phi})}{\partial\theta^2} & \frac{\partial^2 P(e^{i\theta},e^{i\phi})}{\partial\theta\partial\phi} \\ \frac{\partial^2 P(e^{i\theta},e^{i\phi})}{\partial\theta\partial\phi} & \frac{\partial^2 P(e^{i\theta},e^{i\phi})}{\partial\phi^2} \end{pmatrix}\Bigg|_{(\theta,\phi)=(0,0)} \neq 0$$

then the edge-edge correlation

$$\mathcal{P}(e_1\&e_2)-\mathcal{P}(e_1)\mathcal{P}(e_2)$$

converges to 0 quadratically as $|e_1-e_2|\rightarrow\infty$

Proof. Without loss of generality, assume the real node is $(1,1)$.

$$0=P(1,1)=(\mathrm{Pf}K(1,1))^2$$

where $\mathrm{Pf}K(1,1)$ is linear with respect to each edge weight. Fix an edge $e=uv$, and assume the weight of edge e is w_e, then

$$P(1,1)=\mathcal{C}_2w_e^2+\mathcal{C}_1w_e+\mathcal{C}_0$$

where $\mathcal{C}_2$ is the signed weighted sum of loop configurations in which e is a doubled edge. $\mathcal{C}_1$ consists of two parts: $\mathcal{C}_1=\mathcal{D}_1+\mathcal{D}_2$. $\mathcal{D}_1$ is the signed weighted sum of loop configurations in which e is part of a loop oriented from u to v; $\mathcal{D}_2$ is the signed weighted sum of loop configurations in which e is part of a loop oriented from v to u. When $z=w=1$, $\mathcal{D}_1=\mathcal{D}_2$. $\mathcal{C}_0$ is the signed weighted sum of loop configurations in which e does not appear at all. The determinant of the submatrix of $K(1,1)$ by removing the row indexed by u and the column indexed by v is exactly $\mathcal{C}_2 w_e + \mathcal{D}_1$.

Since $P(1,1)$ is a perfect square

$$(\mathcal{D}_1)^2 = \mathcal{C}_0 \mathcal{C}_2$$

As a result,

$$\begin{aligned} P(1,1) &= (\mathcal{C}_2 w_e + \mathcal{D}_1) w_e + (\mathcal{D}_1 w_e + \frac{\mathcal{D}_1^2}{\mathcal{C}_2}) \\ &= \frac{1}{\mathcal{C}_2} (\mathcal{C}_2 w_e + \mathcal{D}_1)^2 \end{aligned}$$

$P(1,1) = 0$ implies $\mathcal{C}_2 w_e + \mathcal{D}_1 = 0$, hence the determinant of a submatrix of $K(1,1)$ by removing the row indexed by u and the column indexed by v is 0, if uv is an edge. Now, let us consider $Adj K(1,1)$, i.e. the adjugate matrix of $K(1,1)$ whose (i,j) entry is $(-1)^{i+j}$ multiplying the determinant of the submatrix of $K(1,1)$ by removing the jth row and the ith column. Since $\det K(1,1) = 0$, $Adj K(1,1)$ is a matrix of rank at most 1. As explained above, the entries of $Adj K(1,1)$ corresponding to an edge is 0. This implies that $Adj K(1,1)$ has 0 entries on each row and each column. Therefore $Adj K(1,1)$ is a zero matrix, and any minor of $K(1,1)$ by removing a row and a column is 0.

In a neighborhood of $(\theta, \phi) = (0,0)$ P has the expansion

$$P(e^{i\theta}, e^{i\phi}) = a\theta^2 + b\theta\phi + c\phi^2 + \cdots$$

where $a, b, c \in \mathbb{R}$ are real numbers and $\cdots$ denotes terms of order at least 3. Then near a node, a point on $P = 0$ satisfies either $\theta = \lambda\phi + O(\phi^2)$ or $\theta = \bar{\lambda}\phi + O(\phi^2)$, where $\lambda, \bar{\lambda}$, which are necessarily non-real, are the roots of $ax^2 + bx + c = 0$, since

$P(z,w) > 0$, for any $(z,w) \in \mathbb{T}^2 \setminus \{(1,1)\}$([24]), and the assumption that the Hessian matrix at $(\theta,\phi)=(0,0)$ is invertible. This implies

$$b^2 - 4ac < 0, \qquad a = \left.\frac{\partial^2 P}{\partial \theta^2}\right|_{(\theta,\phi)=(0,0)} > 0, \qquad c = \left.\frac{\partial^2 P}{\partial \phi^2}\right|_{(\theta,\phi)=(0,0)} > 0$$

Since $P(z,w) = \det K(z,w)$, we apply the differentiation rule for the determinant and obtain

$$\frac{\partial P}{\partial \theta} = \sum_v \det K_v$$

where K_v is a matrix which are the same as K except the row indexed by v. For each entry on the row indexed by v, we take the derivative with respect to θ. The sum is over all the vertices of the quotient graph. Then we expand $\det K_v$ with respect to the row indexed by v. Obviously at most one entry on that row of K_v is nonzero. Hence $\det K_v = 0$ or $\det K_v = \pm i e^{\pm i\theta}(AdjK)_{v,w}$, where vw is an z-edge or $\frac{1}{z}$-edge. Then we take the second order derivative at $\theta = 0$, $\phi = 0$,

$$\frac{\partial \det K_v}{\partial \theta} = -e^{\pm i\theta}(AdjK(1,1))_{v,w} \pm e^{\pm i\theta} \left.\frac{d(AdjK)_{v,w}}{d\theta}\right|_{(\theta,\phi)=(0,0)}$$

The first term is 0. However, since $\left.\frac{\partial^2 P}{\partial \theta^2}\right|_{(\theta,\phi)=(0,0)} \neq 0$, there exists at least one z-edge vv' such that $\left.\frac{d(AdjK)_{v,w}}{d\theta}\right|_{(\theta,\phi)=(0,0)} \neq 0$. When (θ,ϕ) lies in a neighborhood of $(0,0)$, we have

$$\begin{aligned}
\frac{Cof[K(e^{i\theta},e^{i\phi})]_{v,v'}}{P(e^{i\theta},e^{i\phi})} &= \frac{p\theta + q\phi}{a\theta^2 + b\theta\phi + c\phi^2} + O(1) \\
&= \frac{1}{\lambda - \bar{\lambda}} \left[\frac{p\lambda - q}{a(\theta - \lambda\phi)} + \frac{q - p\bar{\lambda}}{a(\theta - \bar{\lambda}\phi)} \right] + O(1) \\
&= \frac{1}{\lambda - \bar{\lambda}} \left[\frac{p\lambda - q}{a(z-1) - a\lambda(w-1)} + \frac{q - p\bar{\lambda}}{a(z-1) - a\bar{\lambda}(w-1)} \right] + O(1)
\end{aligned}$$

where $z = e^{i\theta}, w = e^{i\phi}$. It is proved in [16] that if

$$R(z,w) = \alpha(z - z_0) + \beta(w - w_0) + O(|z - z_0|^2 + |w - w_0|^2),$$

then we have the following asymptotic formula for the Fourier coefficients of R^{-1}

$$\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\frac{w^x z^y}{R(z,w)}\frac{dz}{z}\frac{dw}{w} = \frac{-w_0^x z_0^y}{2\pi i(x\alpha z_0 - y\beta w_0)} + O(\frac{1}{x^2+y^2})$$

Therefore we have

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\frac{w^x z^y Cof[K(z,w)]_{v,v'}}{P(z,w)}\frac{dz}{z}\frac{dw}{w} \\
= &\frac{1}{\lambda - \bar{\lambda}}\left[\frac{q - p\lambda}{2\pi i(xa + ya\lambda)} + \frac{p\bar{\lambda} - q}{2\pi i(xa + ya\bar{\lambda})}\right] + O(\frac{1}{x^2+y^2})
\end{aligned}$$

Hence $K^{-1}_{v,v'}$ decays linearly as $|v - v'| \to \infty$, as a result, the edge-edge correlation decays quadratically as $|v - v'| \to \infty$. $\square$

Lemma 3.26. *Let $\mathcal{P}_I$ be a Gibbs measure of the ferromagnetic Ising model on the periodic square grid, $\mathcal{P}$ be the induced Gibbs measure of the dimer model by the Fisher correspondence in Figure 2. If under $\mathcal{P}_I$ the spin-spin correlation decays exponentially, then under $\mathcal{P}$, the edge-edge correlation decays exponentially.*

Proof. Let $\rho_i = \frac{\sigma_i+1}{2}$ be the lattice gas variable. Let A, B be two finite subsets of the vertices of the square grid. Define

$$S_A = \sum_{i\in A}\rho_i$$
$$\rho_A = \prod_{i\in A}\rho_i$$

Clearly S_A, ρ_A and $S_A - \rho_A$ are nonnegative, monotone functions of the $\{\rho_i\}$ and thus also of the $\{\sigma_i\}$. According to the $F.K.G$ inequality, we have

$$\langle \rho_B(S_A - \rho_A)\rangle \geq \langle \rho_B\rangle\langle S_A - \rho_A\rangle \geq 0 \tag{3.24}$$

and

$$\langle S_A(S_B - \rho_B)\rangle \geq \langle S_A\rangle\langle S_B - \rho_B\rangle \geq 0 \tag{3.25}$$

Combining (3.24) and (3.25), we have

$$\begin{aligned} 0 \leq \quad & \langle \rho_A\rho_B\rangle - \langle \rho_A\rangle\langle \rho_B\rangle \leq \langle S_A\rho_B\rangle - \langle S_A\rangle\langle \rho_B\rangle \leq \langle S_A S_B\rangle - \langle S_A\rangle\langle S_B\rangle \\ = \quad & \sum_{i\in A}\sum_{j\in B}\langle \rho_i\rho_j\rangle - \langle \rho_i\rangle\langle \rho_j\rangle = \frac{1}{4}\sum_{i\in A}\sum_{j\in B}\langle \sigma_i\sigma_j\rangle - \langle \sigma_i\rangle\langle \sigma_j\rangle \end{aligned}$$

Let $e_1(e_2)$ be an edge with endpoints $u_1, v_1(u_2, v_2)$. According to the Fisher correspondence illustrated in Figure 2, we have

$$\begin{aligned} \mathcal{P}(e_1\&e_2) - \mathcal{P}(e_1)\mathcal{P}(e_2) = \quad & \langle \rho_{u_1}(1-\rho_{v_1})\rho_{u_2}(1-\rho_{v_2})\rangle - \langle \rho_{u_1}(1-\rho_{v_1})\rangle\langle \rho_{u_2}(1-\rho_{v_2})\rangle \\ & +\langle \rho_{v_1}(1-\rho_{u_1})\rho_{v_2}(1-\rho_{u_2})\rangle - \langle \rho_{v_1}(1-\rho_{u_1})\rangle\langle \rho_{v_2}(1-\rho_{u_2})\rangle \\ & +\langle \rho_{u_1}(1-\rho_{v_1})\rho_{v_2}(1-\rho_{u_2})\rangle - \langle \rho_{u_1}(1-\rho_{v_1})\rangle\langle \rho_{v_2}(1-\rho_{u_2})\rangle \\ & +\langle \rho_{v_1}(1-\rho_{u_1})\rho_{u_2}(1-\rho_{v_2})\rangle - \langle \rho_{v_1}(1-\rho_{u_1})\rangle\langle \rho_{u_2}(1-\rho_{v_2})\rangle \end{aligned}$$

and

$$\begin{aligned}
& |\langle \rho_{u_1}(1-\rho_{v_1})\rho_{u_2}(1-\rho_{v_2})\rangle - \langle \rho_{u_1}(1-\rho_{v_1})\rangle\langle \rho_{u_2}(1-\rho_{v_2})\rangle| \\
= \quad & |(\langle \rho_{u_1}\rho_{u_2}\rangle - \langle \rho_{u_1}\rangle\langle \rho_{u_2}\rangle) - (\langle \rho_{u_1}\rho_{u_2}\rho_{v_2}\rangle - \langle \rho_{u_1}\rangle\langle \rho_{u_2}\rho_{v_2}\rangle) \\
& -(\langle \rho_{u_1}\rho_{v_1}\rho_{u_2}\rangle - \langle \rho_{u_1}\rho_{v_1}\rangle\langle \rho_{u_2}\rangle) + (\langle \rho_{u_1}\rho_{v_1}\rho_{u_2}\rho_{v_2}\rangle - \langle \rho_{u_1}\rho_{v_1}\rangle\langle \rho_{u_2}\rho_{v_2}\rangle)| \\
\leq \quad & 4|\langle \rho_{u_1}\rho_{u_2}\rangle - \langle \rho_{u_1}\rangle\langle \rho_{u_2}\rangle| + 2|\langle \rho_{u_1}\rho_{v_2}\rangle - \langle \rho_{u_1}\rangle\langle \rho_{v_2}\rangle| \\
& +2|\langle \rho_{u_2}\rho_{v_1}\rangle - \langle \rho_{u_2}\rangle\langle \rho_{v_1}\rangle| + |\langle \rho_{v_1}\rho_{v_2}\rangle - \langle \rho_{v_1}\rangle\langle \rho_{v_2}\rangle|
\end{aligned}$$

The other terms of $\mathcal{P}(e_1 \& e_2) - \mathcal{P}(e_1)\mathcal{P}(e_2)$ can be bounded in a similar way. Hence if the spin-spin correlation decays exponentially, the edge-edge correlation decays exponentially after the Fisher correspondence described in Figure 2. □

Lemma 3.27. *Given interactions* $\{J_k\}_{k=1}^{2n^2}$*, when* β *increases from* 0 *to* $+\infty$*,* β_c *corresponds to the smallest* β *that the spectral curve for dimer configurations with weights* $\tanh \beta J_k$ *has a real node on* $\mathbb{T}^2$

Proof. Consider the weak limit of Boltzmann measures of the Ising model under periodic boundary conditions,

$$\begin{aligned}
\langle \sigma_i\sigma_j\rangle_p &= \langle \sigma_i\sigma_j\rangle_p - \langle \sigma_i\rangle_p\langle \sigma_j\rangle_p \\
&= 4(\langle \rho_i\rho_j\rangle_p - \langle \rho_i\rangle_p\langle \rho_j\rangle_p) = 4\langle \rho_i\rho_j\rangle_p - 1
\end{aligned}$$

Since $\rho_i\rho_j$ is an increasing function with respect to each single variable σ, $\langle \sigma_i\sigma_j\rangle_p$ is an increasing function with respect to β. In particular $\lim_{n\to\infty}\langle \sigma_{00}\sigma_{0n}\rangle_p$ is increasing in β. There is a unique critical temperature $\beta_{c,p}$, for any given interactions

$\{J_{i,j}\}_{i,j}$, such that

$$\lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle_p = 0 \qquad \text{if } \beta < \beta_{c,p}$$
$$\lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle_p > 0 \qquad \text{if } \beta > \beta_{c,p}$$

Hence at the edge weights $\tanh\beta_{c,p}J$, $\lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle_p$ fails to be an analytic function of β(complex value of β are also allowed). By Lemma 4.7, $\tanh\beta_{c,p}J$ is a subset of the edge weights such that $P(z,w)=0$ has a real node on $\mathbb{T}^2$. By Theorem 1.1, the uniqueness of the translation invariant Gibbs measure implies that

$$\lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle_+ = \lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle_p$$

Therefore for $\beta < \beta_{c,p}$

$$0 \le \lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle_+ - \langle\sigma_{00}\rangle_+^2 \le \lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle = 0,$$

where the first inequality follows from Lemma 6.1. Hence for $\beta < \beta_{c,p}$, the spontaneous magnetization

$$m^* = \frac{1}{|\Lambda_1|}\sum_{i\in\Lambda_1}\langle\sigma_i\rangle_+ = 0$$

Λ_1, the quotient graph consisting of one period. Therefore $\beta_{c,p}\le\beta_c$. β_c, as defined before, is the reciprocal of the lowest temperature such that $m^*=0$.

It is proved in [1, 2] that for $\beta<\beta_c$, the spin-spin correlation decays exponentially, therefore after the Fisher correspondence, the edge-edge correlation on a graph with weights $e^{-2\beta J}$ decays exponentially. We derive that $\beta_c\le\beta_{c,p}$, because at $\beta_{c,p}$, $P(z,w)=0$ has a real node on $\mathbb{T}^2$, the edge-edge correlation does not decay exponentially, see Lemma 4.9. Combining with the previous argument, we have $\beta_{c,p}=\beta_c$. When $\beta<\beta_c$, the corresponding spectral curve with weights $e^{-2\beta J}$ does not intersect $\mathbb{T}^2$. By proposition 3.1, the spectral curve of dimer configurations with weights $\tanh \beta J$ does not intersect $\mathbb{T}^2$ either, if $\beta < \beta_c$. This completes the proof. $\square$

Lemma 3.28. *Assume the Ising model has period $n \times n$, n is even. If the weights of all edges parallel to γ_x, γ_y lie in the open interval $(0, 1)$, and all the other edges have weight 1. Given an orientation to the Fisher graph as illustrated in Figure 1, the only possible real node of the spectral curve $P(z, w) = 0$ on $\mathbb{T}^2$ is $(1, 1)$.*

Proof. It suffices to prove that under the assumption of the lemma, $\mathrm{Pf}K(1, -1) \neq 0$, $\mathrm{Pf}K(-1, 1) \neq 0$ and $\mathrm{Pf}K(-1, -1) \neq 0$. Given an orientation as in Figure 1,

$$\mathrm{Pf}K(-1, -1) = Z_{D,00} + Z_{D,01} + Z_{D,10} - Z_{D,11}, \tag{3.26}$$

where $Z_{D,01}$ is the partition function of dimer configurations with an even number of occupied edges crossed by γ_x and an odd number of occupied edges crossed by γ_y. $Z_{D,00}, Z_{D,10}, Z_{D,11}$ are defined similarly. The expression of $P(-1, -1)$ follows from the fact that if we reverse the orientation of all the edges crossed by γ_x and γ_y, the resulting orientation is an crossing orientation, as defined on Page 4. Similarly, we have

$$\begin{aligned} \mathrm{Pf}K(1, -1) &= Z_{D,00} + Z_{D,01} - Z_{D,10} + Z_{D,11} && (3.27) \\ \mathrm{Pf}K(-1, 1) &= Z_{D,00} - Z_{D,01} + Z_{D,10} + Z_{D,11} && (3.28) \end{aligned}$$

$Z_{D,01}$ also corresponds to an Ising partition function with the opposite spin configurations along the two boundaries parallel to γ_y, and the same spin configuration along the two boundaries parallel to γ_x. After the duality transformation, $Z_{D,01}$ becomes, up to a constant multiple, the partition function of dimer configurations on the dual Fisher graph $\tilde{F}$, with weights of edges crossed by γ_x multiplied by -1, see

(3.15). Since the duality transformation does not change the partition function, up to a multiplicative constant, and after the duality transformation $Z_{D,00}$, $Z_{D,01}$, $Z_{D,10}$, $Z_{D,11}$ are partition function of the dimer model on the same graph with the same edge weights except on the boundary edges, and $Z_{D,00}$ is the only one of them such that all the edge weights are positive, as a result, $Z_{D,00}$ is the biggest of $\{Z_{D,\theta\tau}\}_{\theta,\tau\in\{0,1\}}$, because all the others are partition function of dimer configurations of the same graph with negative weights. As a result of (3.26), (3.27) and (3.28), and the fact that each one of $\{Z_{D,\theta\tau}\}_{\theta,\tau\in\{0,1\}}$ is positive as the partition function of dimer model on a graph with only positive edge weights,

$$\mathrm{Pf}K((-1)^{\theta},(-1)^{\tau})>0,\ if\ (\theta,\tau)\neq(0,0)$$

□

Lemma 3.29. *Assume n is even. For any given $\{J_k\}_{k=1}^{2n^2}$, $0<J_k<\infty$, the curve*

$$\gamma(t)=(\tanh tJ_1,\cdots,\tanh tJ_{2n^2})\qquad 0\leq t\leq+\infty$$

in the edge weight space(a $2n^2$-dimensional vector space with coordinates given by edge weights) intersect $P(1,1)=0$ at a unique point $\gamma(t_0)$, $0<t_0<+\infty$

Proof. First of all, we claim that

$$\gamma(t)\bigcap\{P(1,1)=0\}\neq\emptyset.$$

Note that $P(1,1)=(\mathrm{Pf}K(1,1))^2$. Given edge weights $\gamma(0)=(0,\cdots,0)$, $\mathrm{Pf}K(1,1)=1$. Consider the edge weights $\lim_{t\to+\infty}\gamma(t)=(1,\cdots,1)$. After the duality transformation, all the edges parallel to γ_x, γ_y have weight 0. Since $Z_{D,01}$, $Z_{D,10}$, $Z_{D,11}$ corresponds to dimer configurations by negating the weights $z-(w-,z-$ and $w-)$

edges, and all such edges have weight 0, we obtain

$$Z_{D,00} = Z_{D,01} = Z_{D,10} = Z_{D,11} = 2^{n^2-1}$$

for weights $(1, \cdots, 1)$, see (3.15). Hence $\lim_{t\to+\infty} \mathrm{Pf} K(1,1) < 0$. There exists $0 < t_0 < \infty$, such that $\gamma(t_0) \in \{P(1,1) = 0\}$.

Now we prove the uniqueness of the intersection. Since $P(1,1) = 0$ is invariant under the duality transformation, i.e, any edge weights lie on the surface $P(1,1) = 0$ if and only if the images after duality transformation lie on the surface $P(1,1) = 0$, it suffices to prove that the image of $\gamma(t)$

$$\gamma^*(t) = (\exp(-2tJ_1), \cdots, \exp(-2tJ_{2n^2}))$$

intersects $P(1,1) = 0$ at a unique point. Assume $f(t)$ is the function obtained by plugging edge weights $(\exp(-2tJ_1), \cdots, \exp(-2tJ_{2n^2}))$ into $\mathrm{Pf} K(1,1)$, and $g(t) = 2\prod_{k=1}^{2n^2} \exp(tJ_k) f(t)$, it suffices to prove that there is a unique t_0 in $(0, +\infty)$ such that $g(t_0) = 0$. Given interactions $(tJ_1, \cdots, tJ_{2n^2})$ on edges of the quotient graph of G_n, let $Z^{00}_{G_n,I}$ be the partition function of the Ising model on G_n. Let $Z^{10}_{G_n,I}(Z^{01}_{G_n,I}, Z^{10}_{G_n,I}, Z^{11}_{G_n,I})$ be the partition function of an Ising model, with the interactions on edges crossed by $\gamma_x(\gamma_y$, γ_x and $\gamma_y)$ multiplied by -1. Then we have

$$\begin{aligned} g(t) &= Z^{00}_{G_n,I} - Z^{01}_{G_n,I} - Z^{10}_{G_n,I} - Z^{11}_{G_n,I} \\ g'(t) &= Z^{00}_{G_n,I} \sum_{e=uv\in E(G_n)} J^{00}_e \langle \sigma_u\sigma_v\rangle_{00} - Z^{01}_{G_n,I} \sum_{e=uv\in E(G_n)} J^{01}_e \langle \sigma_u\sigma_v\rangle_{01} \\ &\quad - Z^{10}_{G_n,I} \sum_{e=uv\in E(G_n)} J^{10}_e \langle \sigma_u\sigma_v\rangle_{10} - Z^{11}_{G_n,I} \sum_{e=uv\in E(G_n)} J^{11}_e \langle \sigma_u\sigma_v\rangle_{11} \end{aligned}$$

where $\{J^{\theta,\tau}_e\}_{e\in E(G_n)}$ are corresponding interactions for $Z^{\theta\tau}_{G_n,I}$ and $\langle \quad \rangle_{\theta\tau}$ are the expected values with respect to Boltzmann measure defined by $\{J^{\theta\tau}_e\}_{e\in E(G_n)}$. Let E_0

be the subset of $E(G_n)$ consisting of edges intersecting neither γ_x nor γ_y, and $E_x(E_y)$ be subsets of $E(G_n)$ consisting of edges crossed by $\gamma_x(\gamma_y)$.

$$\begin{aligned}
g'(t) &= \sum_{e=uv\in E_0} J_e^{00}(Z_{G_n,I}^{00}\langle\sigma_u\sigma_v\rangle_{00} - Z_{G_n,I}^{01}\langle\sigma_u\sigma_v\rangle_{01} - Z_{G_n,I}^{10}\langle\sigma_u\sigma_v\rangle_{10} - Z_{G_n,I}^{11}\langle\sigma_u\sigma_v\rangle_{11}) \\
&\quad + \sum_{e=uv\in E_x} J_e^{00}(Z_{G_n,I}^{00}\langle\sigma_u\sigma_v\rangle_{00} - Z_{G_n,I}^{01}\langle\sigma_u\sigma_v\rangle_{01} + Z_{G_n,I}^{10}\langle\sigma_u\sigma_v\rangle_{10} + Z_{G_n,I}^{11}\langle\sigma_u\sigma_v\rangle_{11}) \\
&\quad + \sum_{e=uv\in E_y} J_e^{00}(Z_{G_n,I}^{00}\langle\sigma_u\sigma_v\rangle_{00} + Z_{G_n,I}^{01}\langle\sigma_u\sigma_v\rangle_{01} - Z_{G_n,I}^{10}\langle\sigma_u\sigma_v\rangle_{10} + Z_{G_n,I}^{11}\langle\sigma_u\sigma_v\rangle_{11})
\end{aligned}$$

We claim that for $(\theta,\tau)\neq(0,0)$

$$|\langle\sigma_u\sigma_v\rangle_{\theta\tau}| < \langle\sigma_u\sigma_v\rangle_{00}. \tag{3.29}$$

First of all,

$$\langle\sigma_u\sigma_v\rangle_{\theta\tau} < \langle\sigma_u\sigma_v\rangle_{00} \tag{3.30}$$

by F.K.G inequality. The inequality (3.30) is strict because the graph is finite. Moreover,

$$\begin{aligned}
\langle\sigma_u\sigma_v\rangle_{\theta\tau} + \langle\sigma_u\sigma_v\rangle_{00} &= \frac{1}{Z_{G_n,I}^{00} Z_{G_n,I}^{\theta\tau}} \sum_{\sigma,\sigma'} [(\sigma_u\sigma_v + \sigma_u'\sigma_v')\exp(\sum_{e\in E(G_n)}(J_e^{\theta\tau}+J_e^{00}))\sigma_u\sigma_v] \\
&= \frac{1}{Z_{G_n,I}^{00} Z_{G_n,I}^{\theta\tau}} \sum_{t}(1+t_ut_v)\sum_{\sigma}[\sigma_u\sigma_v\exp(\sum_{e\in E(G_n)}(J_e^{\theta\tau}+J_e^{00}))\sigma_u\sigma_v]
\end{aligned}$$

where we have introduced the Ising variables $\sigma_i'=\pm1$ and $t_i=\sigma_i'\sigma_i$. Since $J_e^{00}+J_e^{\theta\tau}\geq 0$, the ferromagnetic condition and finiteness of G_n imply

$$\langle\sigma_u\sigma_v\rangle_{\theta\tau} + \langle\sigma_u\sigma_v\rangle_{00} > 0 \tag{3.31}$$

(3.29) follows from (3.30) and (3.31). Hence

$$g'(t) > (Z^{00}_{G_n,I} - Z^{01}_{G_n,I} - Z^{10}_{G_n,I} - Z^{11}_{G_n,I}) \sum_{e=uv\in E(G_n)} J^{00}_e \langle \sigma_u \sigma_v \rangle_{00}$$

Since $J^{00}_e > 0$, $\langle \sigma_u \sigma_v \rangle_{00} > 0$.

$$Z^{00}_{G_n,I} - Z^{01}_{G_n,I} - Z^{10}_{G_n,I} - Z^{11}_{G_n,I} = \frac{\mathrm{Pf}K(1,1)}{2\prod_{k=1}^{2n^2} \exp(tJ_k)}$$

We have if $\mathrm{Pf}K(1,1) \geq 0$, $g'(t) > 0$, $g(t)$ is strictly increasing as t increases. Since the duality transformation changes the sign of $\mathrm{Pf}K(1,1)$, we have $g(0) < 0$, see (3.16), and $\lim_{t\to\infty} g(t) > 0$. When t increases from 0 to ∞, originally $g(t)$ is negative, and there $g(t) = 0$ is possible for some t. Consider the first such t that $g(t) = 0$. From then on, $g(t)$ will always be positive since $g'(t) > 0$. As a result, there is a unique t_0, such that $g(t_0) = 0$. This proves the lemma. □

Proof of Theorem 1.1 Theorem 1.2 follows directly from Lemmas 4.11, 4.12 and 4.13.

Example 3.30. *(1 × 2 Ising model) Consider an Ising model on a square grid whose interactions have period 1 × 2, as illustrated in Figure 7. The critical reciprocal*

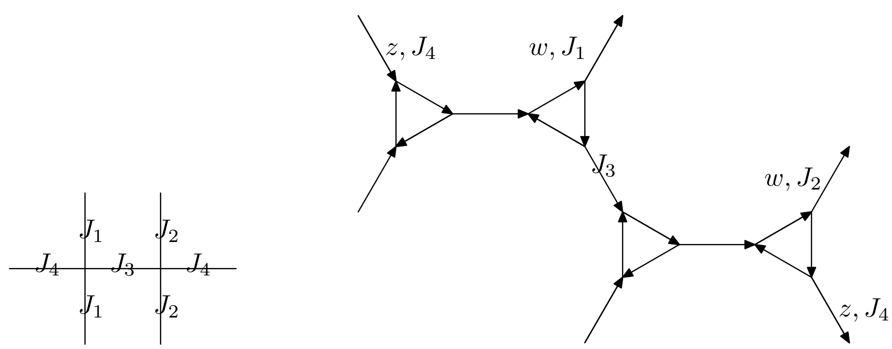

Figure 3.4: 1 × 2 Ising model and Fisher graph

temperature β_c is the solution of the following equation:

$$\begin{aligned} P(z=1,w=-1) &= \det K(1,-1) \\ &= [b_2 + c_1 b_2 c_2 - b_1 b_2 + c_1 c_2 b_1 b_2 - 1 + c_2 c_1 + b_1 + c_1 b_1 c_2]^2 = 0 \end{aligned}$$

where

$$b_1 = \tanh(\beta_c J_4) \qquad b_2 = \tanh(\beta_c J_3)$$
$$c_1 = \tanh(\beta_c J_1) \qquad c_2 = \tanh(\beta_c J_2)$$

In this case, $w = -1$, because the size of the period is odd$\times$even.

3.5 Simple Proof of Exponential Decay of the Spin-Spin Correlation

The goal of this section is to prove the exponential decay of spin-spin correlations for symmetric interactions above the critical temperature. Namely

$$\langle \sigma_i \sigma_j \rangle \leq e^{-\alpha|i-j|}$$

where $\alpha > 0$ is a constant. The symmetric interactions are those who are invariant under the reflexion by the horizontal axis $y = 0$ and the vertical axis $x = 0$, see Figure 8. Lemma 5.1 proves the exponential decay along the lattice direction, and Lemma

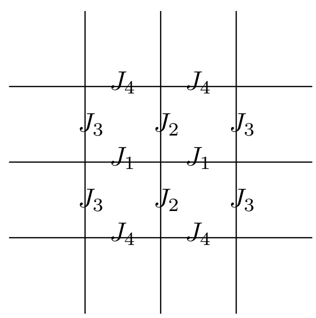

Figure 3.5: 1×2 Ising model and Fisher graph

5.2 proves that the spin-spin correlations along arbitrary directions are dominated by those along the lattice direction. Hence the result follows.

Lemma 3.31. *Under the assumption that* $\det K(z, w) \neq 0$

$$\langle \sigma_{00} \sigma_{0N} \rangle^2 - \lim_{N \to \infty} \langle \sigma_{00} \sigma_{0N} \rangle^2 \leq K e^{-\alpha N}$$

where $0 < K, \alpha < \infty$ *are positive constants.*

Proof. According to (3.18), it suffices to consider the asymptotic behavior of $\prod_{0 \leq l \leq N-1}(1-$

$\tau_{e_l}^2)^2 \det T_N[\psi]$, as N large. This is the determinant of a block Toeplitz matrix, obtained from T_N by multiplying the jth row $(1 \leq j \leq N)$ by $(1-\tau_{e_{j-1}}^2)^2$. The determinant of its symbol is 1, according to Lemma 6.5, since τ_{e_j} changes periodically. In the proof of this lemma, we denote the new Toeplitz matrix by the same notation T_N, and the new symbol by the same notation ψ, which is the original ψ multiplied by a constant for each row. Assume

$$\begin{aligned}
\tilde{\psi} &= \psi(z^{-1}) \\
T[\psi] &= \lim_{n\to\infty} T_n[\psi] \\
T[\tilde{\psi}] &= \lim_{n\to\infty} T_n[\tilde{\psi}]
\end{aligned}$$

Define $A_r \bigcap K_r$ the Banach algebra under the norm

$$\|\phi\| = \sum_{k=-\infty}^{\infty} \|\phi_k\| + \{\sum_{k=-\infty}^{\infty} |k| \|\phi_k\|^2\}^{\frac{1}{2}}$$

where ϕ is an $r \times r$ matrix-valued function. $\|\cdot\|$ on the right side is any norm for $r \times r$ matrices, and ϕ_k is the kth Fourier coefficient of ϕ. First consider the case that the Toeplitz operator $T[\tilde{\psi}]$ is invertible, which is equivalent to the matrix-valued function ψ has a factorization.

$$\psi = \psi_+ \psi_-$$

where $\psi_\pm$ are invertible in $A_r \bigcap K_r$, and ψ_+ and ψ_- have Fourier coefficients that vanish for negative, resp. positive indices. The Hankel operator $H[\psi]$ is Hilbert-Schmidt, the product of any two such is of trace class.

$$\begin{aligned}
H[\psi] &= (\psi_{i+j+1}) \qquad 0 \leq i, j \leq \infty && (3.32) \\
T_n[\psi\phi] - T_n[\psi]T_n[\phi] &= P_n H[\psi] H[\tilde{\phi}] P_n + Q_n H[\tilde{\psi}] H[\phi] Q_n && (3.33)
\end{aligned}$$

where P_n and Q_n are defined by

$$P_n(f_0, f_1, ...) = (f_0, ..., f_n, 0, ..., 0)$$
$$Q_n(f_0, f_1, ...) = (f_n, ..., f_0, 0, ..., 0)$$

If we replace ϕ by ψ_-^{-1} in 3.33, since $H[\psi_-^{-1}] = 0$, we have

$$T_n[\psi_+] - T_n[\psi]T_n[\psi_-^{-1}] = P_nH[\psi]H[\tilde{\psi}_-^{-1}]P_n$$

Multiply the above equation by $T_n[\psi_+^{-1}]$, we have

$$T_n[\psi_+]T_n[\psi_+^{-1}] - T_n[\psi]T_n[\psi_-^{-1}]T_n[\psi_+^{-1}] = P_nH[\psi]H[\tilde{\psi}_-^{-1}]P_nT_n[\psi_+^{-1}]$$

Use ψ_+, ψ_+^{-1} to substitute ψ, ϕ in 3.33, we have

$$I_n - T_n[\psi_+]T_n[\psi_+^{-1}] = P_nH[\psi_+]H[\tilde{\psi}_+^{-1}]P_n + Q_nH[\tilde{\psi}_+]H[\psi_+^{-1}]Q_n$$

Since $H[\tilde{\psi}_+^{-1}] = 0$ and $H[\tilde{\psi}_+] = 0$, we have

$$T_n[\psi_+]T_n[\psi_+^{-1}] = I_n$$

Therefore

$$\begin{aligned} T_n[\psi]T_n[\psi_-^{-1}]T_n[\psi_+^{-1}] &= I_n - P_nH[\psi]H[\tilde{\psi}_-^{-1}]P_nT_n[\psi_+^{-1}] \\ &= P_n(I - H[\psi]H[\tilde{\psi}_-^{-1}]P_nT[\psi_+^{-1}])P_n \end{aligned}$$

Since $T_n[\psi_\pm^{-1}]$ are block triangular matrices, one sees that the left side has determinant exactly

$$D_n[\psi]G[\psi_-^{-1}]^{n+1}G[\psi_+^{-1}]^{n+1} = \frac{D_n[\psi]}{G[\psi]^{n+1}}$$

Moreover, explicit computations show that

$$H[\psi]H[\tilde{\psi}_-^{-1}]T[\psi_+^{-1}] = H[\psi]H[\tilde{\psi}^{-1}]$$

Let $\|\cdot\|_1$ denote the norm of operators in $\mathfrak{F}_1$(see Lemma 4.6, and the appendix). The uniform norm $|\cdot|$ of an operator A is the number

$$|A| = \sup_{|\phi|=1} |A\phi|$$

We have

$$\begin{aligned}
& \|P_nH[\psi]H[\tilde{\psi}_-^{-1}]P_nT[\psi_+^{-1}]P_n - H[\psi]H[\tilde{\psi}^{-1}]\|_1 \\
\leq \; & \|P_nH[\psi]H[\tilde{\psi}_-^{-1}]P_nT[\psi_+^{-1}]P_n - P_nH[\psi]H[\tilde{\psi}_-^{-1}]T[\psi_+^{-1}]P_n\|_1 \\
& + \|P_nH[\psi]H[\tilde{\psi}^{-1}]P_n - H[\psi]H[\tilde{\psi}^{-1}]\| \leq Ke^{-\alpha n}
\end{aligned}$$

where $0 < K, \alpha < \infty$ are positive constants. Define

$$\begin{aligned}
A_n &= P_nH[\psi]H[\tilde{\psi}_-^{-1}]P_nT[\psi_+^{-1}]P_n \\
A &= H[\psi]H[\tilde{\psi}^{-1}]
\end{aligned}$$

then

$$\|A_n - A\|_1 \leq Ke^{-\alpha n}$$

$$\begin{aligned}
D_{A_n}(\mu) &= \det(I - \mu A_n) \\
D_A(\mu) &= \det(I - \mu A) \\
A(\mu) &= A(I - \mu A)^{-1} \\
A_n(\mu) &= A_n(I - \mu A_n)^{-1}
\end{aligned}$$

We shall call the complex number μ an F-regular point of the operator A, if the operator $I - \mu A$ has an inverse. Let Γ be a simple rectifiable contour, consisting of F-regular points of the operator A, which encloses the point $\mu = 0$ and the point $\mu = 1$. By virtue of the maximum modulus principle, if we can prove that for any $\|A_n - A\|_1 < Ke^{-\alpha n}$,

$$\max_{\mu \in \Gamma} |D_A(\mu) - D_{A_n}(\mu)| < CKe^{-\alpha n}$$

then

$$|D_A(1) - D_{A_n}(1)| < CKe^{-\alpha n}$$

Let us denote by L a simple rectifiable curve consisting of F-regular points of the operator A, which connects the point $\mu = 0$ with some point of the contour Γ. Let $\Gamma_\mu(\mu \in \Gamma \bigcup L)$ denote the shortest path along the curves Γ and L which connects the point $\mu = 0$ to the point μ. From the equality

$$(I - \mu A_n)^{-1} = (I - \mu A)^{-1}[I - \mu(A_n - A)(I - \mu A)^{-1}]^{-1}$$

It follows that when the condition

$$\|A_n - A\|_1 < \min_{\mu \in \Gamma \bigcup L} [|\mu||(I - \mu A)^{-1}|]^{-1}$$

is fulfilled, all points μ of the curves $\Gamma \bigcap L$ are F-regular points of the operator A_n and

$$\max_{\mu \in \Gamma \bigcup L} |(I - \mu A_n)^{-1}| < C \max_{\mu \in \Gamma \bigcup L} |(I - \mu A)^{-1}|$$

where C is a constant depending only on the curves Γ and L and the operator A. Since

$$A(\mu) - A_n(\mu) = (I - \mu A)^{-1}(A - A_n)(I - \mu A_n)^{-1}$$

we have

$$\|A(\mu) - A_n(\mu)\|_1 \leq |(I - \mu A)^{-1}| \|A - A_n\|_1 |(I - \mu A_n)^{-1}|$$

for any $\mu_0 \in \Gamma \bigcup L$ Then

$$\left| \int_{\Gamma_{\mu_0}} \mathrm{tr}(A(\mu) - A_n(\mu)) d\mu \right| \leq \int_{\Gamma_{\mu_0}} \|A(\mu) - A_n(\mu)\|_1 d\mu < C \|A - A_n\|_1$$

where C is another constant depending only on the curves Γ and L and the operator A, and $tr(K)$ is the sum of all eigenvalues of operator K(for details, see the appendix). Γ_{μ_0} is the shortest path along the curves Γ and L connecting the points 0 and μ_0. Then

$$|D_A(\mu_0) - D_{A_n}(\mu_0)| = \left| D_A(\mu_0) \left[1 - \frac{D_{A_n}(\mu_0)}{D_A(\mu_0)} \right] \right|$$

and

$$\begin{aligned} \left(\log \frac{D_{A_n}(\mu)}{D_A(\mu)} \right)' &= \frac{D'_{A_n}(\mu)}{D_{A_n}(\mu)} - \frac{D'_A(\mu)}{D_A(\mu)} \\ &= -\sum_{j=1}^{r(A)} \frac{\lambda_j(A)}{1 - \mu \lambda_j(A)} + \sum_{j=1}^{r(A_n)} \frac{\lambda_j(A_n)}{1 - \mu \lambda_j(A_n)} \\ &= \mathrm{tr}(A(\mu) - A_n(\mu)) \end{aligned}$$

where $r(K)$ is the dimension of operator K, namely, the dimension of the closure of the range of K. Hence

$$|D_A(\mu) - D_{A_n}(\mu)| = |D_A(\mu_0) \{ 1 - \exp[\int_{\Gamma_{\mu_0}} \mathrm{tr}(A(\mu) - A_n(\mu))] \}| < C \|A - A_n\|_1$$

where C is another constant depending only on the curves Γ and L and the operator A, then we have

$$|D_{A_n}(1) - D_A(1)| < CKe^{-\alpha n}$$

Now we consider the case $T[\tilde{\psi}]$ is not invertible. According to [30], there is a ϕ with only finitely many non-vanishing Fourier coefficients such that $T[\tilde{\psi} + \varepsilon\tilde{\phi}]$ i s invertible for all sufficiently small nonzero ε. Let $\eta_\varepsilon = \psi + \varepsilon\phi$, then the previous process shows that

$$\|P_n H[\eta_\varepsilon] H[\tilde{\eta_\varepsilon}_-^{-1}] P_n T[{\eta_\varepsilon}_+^{-1}] P_n - H[\eta_\varepsilon] H[\tilde{\eta_\varepsilon}^{-1}]\|_1 < Ke^{-\alpha n}$$

where K, α are independent of ε since ϕ has only finitely many non-vanishing Fourier coefficients. We consider ε on the boundary of a disk D

$$\max_{\varepsilon \in \partial D} |D_{A_n,\varepsilon}(1) - D_{A,\varepsilon}(1)| < CKe^{-\alpha n}$$

where

$$\begin{aligned} A_\varepsilon &= H[\eta_\varepsilon] H[\tilde{\eta}_\varepsilon^{-1}] \\ A_{n,\varepsilon} &= P_n H[\eta_\varepsilon] H[\tilde{\eta_\varepsilon}_-^{-1}] P_n T[{\eta_\varepsilon}_+^{-1}] P_n \end{aligned}$$

Then the maximal modulus theorem says that

$$|D_{A_n}((1) - D_A(1)| < CKe^{-\alpha n}$$

Since $G[\psi] = 1$, we have

$$|\det T_n[\psi] - \lim_{n\to\infty} \det T[\psi]| < Ke^{-\alpha n}$$

where $0 < K, \alpha < \infty$ are constants. □

Lemma 3.32. *Assume each fundamental domain has symmetric edge weights with respect to a center vertex, as illustrated in Figure 8. Assume (p_0, q_0) is a vertex on* *the boundary of a fundamental domain centered at* $(0,0)$*. If* i *is even,*

$$\langle\sigma_{p_0q_0}\sigma_{p_0+im,q_0+jn}\rangle_p \leq \langle\sigma_{p_0q_0}\sigma_{p_0+im,q0}\rangle_p$$

Proof. Let W be a layer of cylinder consisting t fundamental domains. In other words, if we use the number of fundamental domains as a measure for the size of W, then the circumference of W is t, and the width of W is 1. The transfer matrix Q_t is defined to be a square matrix whose rows (columns) are labeled by the upper(lower) boundary configurations of W. Namely, let X be an upper boundary configuration, and Y be an lower boundary configuration, then

$$\begin{aligned} Q_t(X,Y) &= \sum_{\{\sigma_r|r\in W\backslash\partial W\}} \exp[-\beta(\frac{1}{2}\sum_{\{a,b|a\sim b,\sigma_a,\sigma_b\in X\}} J_{ab}\sigma_a\sigma_b + \frac{1}{2}\sum_{\{c,d|c\sim d,\sigma_c,\sigma_d\in Y\}} J_{cd}\sigma_c\sigma_d \\ &\quad + \sum_{e\in E(W\backslash\partial W)} J_e\sigma_u\sigma_v)]. \end{aligned}$$

Without the external magnetic field,

$$\begin{aligned} \langle\sigma_{p_0q_0}\sigma_{p_0+im,q_0+jn}\rangle_p &= \langle\sigma_{p_0q_0}\sigma_{p_0+im,q_0+jn}\rangle_p - \langle\sigma_{p_0q_0}\rangle_p\langle\sigma_{p_0+im,q_0+jn}\rangle_p \\ &= 4(\langle\rho_{p_0q_0}\rho_{p_0+im,q_0+jn}\rangle_p - \langle\rho_{p_0q_0}\rangle_p\langle\rho_{p_0+im,q_0+jn}\rangle_p) \end{aligned}$$

Hence it suffices to prove that

$$\langle\rho_{p_0q_0}\rho_{p_0+im,q_0+jn}\rangle_p - \langle\rho_{p_0q_0}\rangle_p\langle\rho_{p_0+im,q_0+jn}\rangle_p \leq \langle\rho_{p_0q_0}\rho_{p_0+im,q_0}\rangle_p - \langle\rho_{p_0q_0}\rangle_p\langle\rho_{p_0+im,q_0}\rangle_p$$

Assume we have an $s\times t$ torus, where s and t are the number of fundamental domains along each direction, then

$$\langle\rho_{p_0q_0}\rangle_{st} = \frac{\sum_{\{\sigma|\sigma_{p_0q_0}=1\}}\exp[-\beta\sum_{\{u,v|u\sim v\}} J_{uv}\sigma_u\sigma_v]}{\sum_\sigma \exp[-\beta\sum_{\{u,v|u\sim v\}} J_{uv}\sigma_u\sigma_v]} = \frac{\mathrm{tr}(F_tQ_t^s)}{\mathrm{tr}(Q_t^s)}$$

where F_t is a diagonal matrix with

$$F_t(X,X) = \begin{cases} 1 & \text{if } \sigma_{\mathrm{p_0q_0}} = 1 \text{ in X} \\ 0 & \text{else} \end{cases}$$

Since Q_t is a symmetric matrix, all its eigenvalues are real, there exists an orthogonal matrix S_t, such that $\tilde{Q}_t = S_t^{-1}Q_tS_t$ is diagonal, and the modulus of its eigenvalues decreases along the diagonal. Let $\tilde{F}_t = S_t^{-1}F_tS_t$. Moreover, since Q_t has strict positive entries, the Perron-Frobenius Theorem says that the largest eigenvalue in modulus of Q_t, $\lambda_{1,t}$, is simple and strictly positive. Hence

$$\lim_{s\to\infty}\langle\rho_{p_0q_0}\rangle_{st} = \lim_{s\to\infty}\frac{\mathrm{tr}(\tilde{F}_t\tilde{Q}_t^s)}{\mathrm{tr}(Q_t^s)} = \tilde{F}_t(1,1)$$

Similarly,

$$\langle\rho_{p_0+im,q_0+jn}\rangle_{st} = \mathrm{tr}(Q_t^iG_tQ_t^{s-i}),$$

and G_t is a diagonal matrix with

$$G_t(X,X) = \begin{cases} 1 & \text{if} \sigma_{\mathrm{p_0+im,q_0+jn}} = 1 \text{ in X} \\ 0 & \text{else} \end{cases}$$

. Assume $\tilde{G}_t = S_t^{-1}G_tS_t$, then

$$\lim_{s\to\infty}\langle\rho_{p_0+im,q_0+jn}\rangle_{st} = \tilde{G}_t(1,1).$$

Finally we have

$$\begin{aligned}
\lim_{s\to\infty}\langle\rho_{p_0q_0}\rho_{p_0+im,q_0+jn}\rangle_{st} &= \lim_{s\to\infty}\frac{\mathrm{tr}(F_tQ_t^iG_tQ_t^{s-i})}{trQ_t^s}\\
&= \frac{\tilde{F}_t\tilde{Q}_t^i\tilde{G}_t(1,1)}{\lambda_{1,t}^i}\\
&= \frac{\sum_k \tilde{F}_t(1,k)\lambda_{k,t}^i\tilde{G}_t(k,1)}{\lambda_{1,k}^i}
\end{aligned}$$

therefore

$$\begin{aligned}
&\lim_{s\to\infty}(\langle\rho_{p_0q_0}\rho_{p_0+im,q_0+jn}\rangle_{st}-\langle\rho_{p_0q_0}\rangle_{st}\langle\rho_{p_0+im,q_0+jn}\rangle_{st})\\
=\;& \frac{\sum_{k\neq 1}\tilde{F}_t(1,k)\lambda_{k,t}^i\tilde{G}_t(1,k)}{\lambda_{1,k}^i}\\
\leq\;& \frac{[\sum_{k\neq 1}\tilde{F}_t^2(1,k)\lambda_{k,t}^i\sum_{k\neq 1}\tilde{G}_t^2(1,k)\lambda_{k,t}^i]^{\frac{1}{2}}}{\lambda_{1,t}^i}\\
=\;& \lim_{s\to\infty}\sqrt{(\langle\rho_{p_0q_0}\rho_{p_0+im,q_0}\rangle_{st}-\langle\rho_{p_0q_0}\rangle_{st}\langle\rho_{p_0+im,q_0}\rangle_{st})}\\
&\sqrt{(\langle\rho_{p_0,q_0+jn}\rho_{p_0+im,q_0+jn}\rangle_{st}-\langle\rho_{p_0+im,q_0+jn}\rangle_{st}\langle\rho_{p_0,q_0+jn}\rangle_{st})}\\
=\;& \lim_{s\to\infty}\langle\rho_{p_0q_0}\rho_{p_0+im,q0}\rangle_{st}-\langle\rho_{p_0q_0}\rangle_{st}\langle\rho_{p_0+im,q0}\rangle_{st}
\end{aligned}$$

The inequality follows from the Cauchy-Schwartz, it is an inner product since i is even, and the last equality follows from translation invariance. The result follows by letting $t\to\infty$, and the fact that the limit of the spin-spin correlation is independent of the order of $s\to\infty$ and $t\to\infty$. $\square$

Assume we have a finite graph. Let $\gamma_{x,y}$ be a path connecting $(0,0)$ and (x,y), then

$$\langle\sigma_{00}\sigma_{xy}\rangle = Z^{-1}\prod_{e\in\gamma_{x,y}}\tanh J_e\sum_{\sigma}\prod_{e\in\gamma_{x,y}}(1+\frac{1}{\tanh J_e}\sigma_u\sigma_v)\prod_{e\in\gamma_{x,y}^c}(1+\tanh J_e\sigma_u\sigma_v)$$

The denominator corresponds to the closed polygon configurations while the numerator corresponds to configurations on the graph which have an odd number of present edges at vertices $(0,0)$ and (x,y), and an even number of present edges at all the other vertices. Hence $\langle \sigma_{0,0}\sigma_{x,y}\rangle$ can be expressed as a sum of monomer-monomer correlations the Fisher graph, that is, the partition function of dimer configurations on a graph with two vertices removed, divided by the partition function of the original graph. We have the following corollary

Corollary 3.33. *Above the critical temperature, the monomer-monomer correlation on symmetric, periodic, ferromagnetic Fisher graph decays to zero exponentially fast. The decay rate is rotationally invariant.*

Appendix A

Realizability Conditions

In this section we give explicit realizability conditions for periodic honeycomb lattice embedded on a $n \times n$ domain. Let us consider an arbitrary vertex x with adjacent vertices y, z, w. Assume relations have following signatures:

$$r_x = \begin{pmatrix} x_1 & ... & x_8 \end{pmatrix}^t \quad r_y = \begin{pmatrix} y_1 & ... & y_8 \end{pmatrix}^t$$
$$r_z = \begin{pmatrix} z_1 & ... & z_8 \end{pmatrix}^t \quad r_w = \begin{pmatrix} w_1 & ... & w_8 \end{pmatrix}^t$$

According to the realizability equation (1), after a lengthy computation, we have

Theorem A.1. *A periodic network of relation on $n \times n$ honeycomb lattice is relizable if and only if at each vertex x with adjacent vertices y, z, w connected by a, b, c edge, respectively, the signatures satisfy the following equation:*

$$\sum_{i,j,k \in \{0,1,2\}} X_{ijk} Y_i Z_j W_k = 0$$

where

$$\begin{array}{ccc} Y_0 = y_1y_4 - y_2y_3 & Y_1 = y_1y_8 + y_4y_5 - y_2y_7 - y_3y_6 & Y_2 = y_5y_8 - y_6y_7 \\ Z_0 = z_1z_6 - z_2z_5 & Z_1 = z_1z_8 + z_3z_6 - z_4z_5 - z_2z_7 & Z_2 = z_3z_8 - z_4z_7 \\ W_0 = w_1w_7 - w_3w_5 & W_1 = w_1w_8 + w_2w_7 - w_5w_4 - w_3w_6 & W_2 = w_2w_8 - w_6w_4 \end{array}$$

and

$$\begin{aligned}
X_{000} &= x_1^2(x_3x_6 + x_2x_7 + x_4x_5 - x_1x_8) - 2x_1x_2x_3x_5 \\
X_{001} &= x_1^2(x_6x_4 - x_2x_8) + x_2^2(x_1x_7 - x_3x_5) \\
X_{002} &= x_2^2(x_2x_7 - x_1x_8 - x_3x_6 - x_4x_5) + 2x_1x_2x_4x_6 \\
X_{010} &= x_3^2(x_1x_6 - x_2x_5) + x_1^2(x_7x_4 - x_3x_8) \\
X_{011} &= x_1x_2(x_4x_7 - x_3x_8) + x_3x_4(x_1x_6 - x_2x_5) \\
X_{012} &= x_2^2(x_4x_7 - x_3x_8) + x_4^2(x_1x_6 - x_2x_5) \\
X_{020} &= x_3^2(x_3x_6 - x_2x_7 - x_4x_5 - x_1x_8) + 2x_1x_4x_3x_7 \\
X_{021} &= x_4^2(x_1x_7 - x_3x_5) + x_3^2(x_6x_4 - x_2x_8) \\
X_{022} &= x_4^2(x_2x_7 + x_1x_8 + x_3x_6 - x_4x_5) - 2x_2x_3x_4x_8
\end{aligned}$$

$$\begin{aligned}
X_{100} &= x_5^2(x_1x_4 - x_2x_3) + x_1^2(x_6x_7 - x_5x_8) \\
X_{101} &= x_1x_2(x_6x_7 - x_5x_8) + x_5x_6(x_1x_4 - x_2x_3) \\
X_{102} &= x_6^2(x_1x_4 - x_2x_3) + x_2^2(x_6x_7 - x_5x_8) \\
X_{110} &= x_1x_3(x_7x_6 - x_5x_8) + x_5x_7(x_1x_4 - x_2x_3) \\
X_{111} &= x_1x_4x_6x_7 - x_2x_3x_5x_8 \\
X_{112} &= x_4x_8(x_1x_6 - x_2x_5) + x_2x_6(x_4x_7 - x_3x_8) \\
X_{120} &= x_7^2(x_1x_4 - x_2x_3) + x_3^2(x_6x_7 - x_5x_8) \\
X_{121} &= x_3x_7(x_4x_6 - x_2x_8) + x_4x_8(x_1x_7 - x_3x_5) \\
X_{122} &= x_8^2(x_1x_4 - x_2x_3) + x_4^2(x_6x_7 - x_5x_8)
\end{aligned}$$

$$X_{200} = x_5^2(x_4x_5 - x_1x_8 - x_3x_6 - x_2x_7) + 2x_1x_5x_6x_7$$

$$X_{201} = x_5^2(x_6x_4 - x_2x_8) + x_6^2(x_1x_7 - x_3x_5)$$

$$X_{202} = x_6^2(x_1x_8 + x_2x_7 + x_4x_5 - x_3x_6) - 2x_5x_6x_2x_8$$

$$X_{210} = x_5^2(x_4x_7 - x_3x_8) + x_7^2(x_1x_6 - x_2x_5)$$

$$X_{211} = x_6x_8(x_1x_7 - x_3x_5) + x_5x_7(x_4x_6 - x_2x_8)$$

$$X_{212} = x_8^2(x_1x_6 - x_2x_5) + x_6^2(x_4x_7 - x_3x_8)$$

$$X_{220} = x_7^2(x_1x_8 + x_3x_6 + x_4x_5 - x_2x_7) - 2x_3x_8x_5x_7$$

$$X_{221} = x_8^2(x_1x_7 - x_3x_5) + x_7^2(x_6x_4 - x_2x_8)$$

$$X_{222} = x_8^2(x_1x_8 - x_3x_6 - x_2x_7 - x_4x_5) + 2x_7x_8x_6x_4$$

Theorem A.2. *A periodic network of relation with $n \times n$ fundamental domain is bipartite realizable if and only if it is realizable and at any vertex v, the signature satisfy*

$$v_1^2v_8^2 + v_2^2v_7^2 + v_3^2v_6^2 + v_4^2v_5^2 - 2v_1v_8v_2v_7 - 2v_1v_8v_3v_6 - 2v_1v_8v_4v_5$$
$$-2v_2v_7v_3v_6 - 2v_2v_7v_4v_5 - 2v_3v_6v_4v_5 + 4v_1v_4v_6v_7 + 4v_2v_3v_5v_8 = 0$$

Proof. Without loss of generality, we assume at each matchgate, the signature $\{111\}$ is 0, and assume v is a black vertex. The $\{111\}$ entry of the signature of a black vertex is 0 gives us

$$\alpha_{1a}\alpha_{1b}\alpha_{1c}v_1 + \alpha_{1a}\alpha_{1b}v_2 + \alpha_{1a}\alpha_{1c}v_3 + \alpha_{1a}v_4 + \alpha_{1b}\alpha_{1c}v_5 + \alpha_{1b}v_6 + \alpha_{1c}v_7 + v_8 = 0 \quad \text{(A.1)}$$

The parity constraint implies that the $\{110\}$ entry is also 0, namely

$$\alpha_{1a}\alpha_{1b}\alpha_{0c}v_1 + \alpha_{1a}\alpha_{1b}v_2 + \alpha_{1a}\alpha_{0c}v_3 + \alpha_{1a}v_4 + \alpha_{1b}\alpha_{0c}v_5 + \alpha_{1b}v_6 + \alpha_{0c}v_7 + v_8 = 0 \quad \text{(A.2)}$$

From (A.1)(A.2), we can solve $\alpha_1 a$, and solution has following form

$$\alpha_{1a} = \frac{N_1}{D_1} = \frac{N_2}{D_2}$$

Then $N_1 D_2 - N_2 D_1 = 0$. Under the assumption that $\alpha_{0c} \neq \alpha_{1c}$, we have

$$(-v_2 v_5 + v_6 v_1)\alpha_{1b}^2 + (v_8 v_1 + v_6 v_3 - v_4 v_5 - v_2 v_7)\alpha_{1b} + v_8 v_3 - v_4 v_7 = 0 \tag{A.3}$$

From (12), we have

$$2(v_1 v_6 - v_2 v_5)\alpha_{0b}\alpha_{1b} + (v_1 v_8 + v_3 v_6 - v_4 v_5 - v_2 v_7)(\alpha_{0b} + \alpha_{1b}) + 2(v_3 v_8 - v_4 v_7) = 0 \tag{A.4}$$

$2\times$(A.3)$-$(A.4), under the assumption that $\alpha_{0b} \neq \alpha_{1b}$, we have

$$\alpha_{1b} = -\frac{v_1 v_8 + v_3 v_6 - v_4 v_5 - v_2 v_7}{2(v_1 v_6 - v_2 v_5)}$$

which implies that equation (A.3) has double real roots, and its discriminant is 0. That is exactly the statement of the theorem. $\square$

Proof of Proposition 3.9 Since holographic reduction is an invertible process, two matchgrids M and $\hat{M}$ are holographically equivalent if and only if there exists a basis for each edge, such that one can be transformed to the other using holographic reduction. Obviously holographic equivalent matchgrids have the same dimer partition function. Assume weights m_b, m_w of M and $\hat{m_b}, \hat{m_w}$ of $\hat{M}$ are as follows:

$$m_w^{ij} = \begin{pmatrix} 0 & c_1^{ij} & b_1^{ij} & 0 & a_1^{ij} & 0 & 0 & d_1^{ij} \end{pmatrix}^t$$
$$m_b^{ij} = \begin{pmatrix} 0 & c_2^{ij} & b_2^{ij} & 0 & a_2^{ij} & 0 & 0 & d_2^{ij} \end{pmatrix}^t$$
$$\hat{m}_w^{ij} = \begin{pmatrix} 0 & \hat{c}_1^{ij} & \hat{b}_1^{ij} & 0 & \hat{a}_1^{ij} & 0 & 0 & \hat{d}_1^{ij} \end{pmatrix}^t$$
$$\hat{m}_b^{ij} = \begin{pmatrix} 0 & \hat{c}_2^{ij} & \hat{b}_2^{ij} & 0 & \hat{a}_2^{ij} & 0 & 0 & \hat{d}_2^{ij} \end{pmatrix}^t$$

Then by equations (1.16)-(1.21), and the uniqueness of the basis on each edge, we have

$$\frac{a_1^{ij}b_1^{ij}}{d_1^{ij}c_1^{ij}} = \frac{d_2^{i,j-1}c_2^{i,j-1}}{b_2^{i,j-1}a_2^{i,j-1}} = a_{03}^{ij}a_{13}^{ij} = b_{03}^{i,j-1}b_{13}^{i,j-1} \tag{A.5}$$
$$\frac{a_1^{ij}c_1^{ij}}{b_1^{ij}d_1^{ij}} = \frac{d_2^{i-1,j}b_2^{i-1,j}}{a_2^{i-1,j}c_5^{i-1,j}} = a_{02}^{ij}a_{12}^{ij} = b_{02}^{i-1,j}b_{12}^{i-1,j} \tag{A.6}$$
$$\frac{b_1^{ij}c_1^{ij}}{a_1^{ij}d_1^{ij}} = \frac{d_2^{ij}a_2^{ij}}{b_2^{ij}c_2^{ij}} = a_{01}^{ij}a_{11}^{ij} = b_{01}^{ij}b_{11}^{ij} \tag{A.7}$$

Plugging in (A.5)-(A.7) to (1.9)-(1.10), we have

$$\begin{cases} \hat{c}_2^{ij} = c_2^{ij} n_{01}^{i,j,0} n_{02}^{i,j,0} p_{13}^{i,j,0} \cdot C_1^{ij} \\ \hat{b}_2^{ij} = b_2^{ij} n_{01}^{i,j,0} p_{12}^{i,j,0} n_{03}^{i,j,0} \cdot C_1^{ij} \\ \hat{a}_2^{ij} = a_2^{ij} p_{11}^{i,j,0} n_{02}^{i,j,0} n_{03}^{i,j,0} \cdot C_1^{ij} \\ \hat{d}_2^{ij} = d_2^{ij} p_{11}^{i,j,0} p_{12}^{i,j,0} p_{13}^{i,j,0} \cdot C_1^{ij} \end{cases}$$

$$\begin{cases} \hat{c}_1^{ij} = \frac{c_1^{ij}}{n_{01}^{i,j,1} n_{02}^{i,j,1} p_{13}^{i,j,1}} \cdot C_2^{ij} \\ \hat{b}_1^{ij} = \frac{b_1^{ij}}{n_{01}^{i,j,1} p_{12}^{i,j,1} n_{03}^{i,j,1}} \cdot C_2^{ij} \\ \hat{a}_1^{ij} = \frac{a_1^{ij}}{p_{11}^{i,j,1} n_{02}^{i,j,1} n_{03}^{i,j,1}} \cdot C_2^{ij} \\ \hat{d}_1^{ij} = \frac{d_1^{ij}}{p_{11}^{i,j,1} p_{12}^{i,j,1} p_{13}^{i,j,1}} \cdot C_2^{ij} \end{cases}$$

To prove that probability measures of M and $\hat{M}$ are identical, we only need to prove that for any dimer configuration, the products of weights differ by the same constant factor. Each dimer configuration corresponds to a binary sequence of length N, where $N = 3n^2$ is the number of connecting edges. Choose an arbitrary edge with basis $\begin{pmatrix} n_0 & n_1 \\ p_0 & p_1 \end{pmatrix}$. If the edge is occupied, then from M to $\hat{M}$, adjacent generator weight is divided by p_1, while adjacent recognizer weight is multiplied by p_1. If the edge is unoccupied, then from M to $\hat{M}$, adjacent generator weight is divided by n_0, while the adjacent recognizer weight is multiplied by n_0. Therefore, the total effect is for any particular dimer configuration ϖ

$$\frac{\text{Partition}(\varpi \ on \ \hat{M})}{\text{Partition}(\varpi \ on \ M)} = \prod_{i,j} C_1^{ij} C_2^{ij}$$

which is a constant independent of configuration ϖ. $\square$

Appendix B

Toeplitz and Hankel Matrices

If ϕ is a matrix-valued function defined on the unit circle with Fourier coefficients ϕ_k, then $T[\phi]$, $H[\phi]$ are, respectively, the semi-infinite Toeplitz and Hankel matrices,

$$\begin{aligned} T[\phi] &= (\phi_{i-j}) \qquad 0 \leq i, j < \infty \\ H[\phi] &= (\phi_{i+j+1}) \qquad 0 \leq i, j < \infty \end{aligned}$$

ϕ is called the symbol of the Toeplitz and Hankel matrices. If ϕ is bounded, these may be thought as operators from the Hilbert space ℓ_2 to ℓ_2. In addition we writie

$$\tilde{\phi}(z) = \phi(z^{-1})$$

There is a simply identity relating Toeplitz and Hankel operators

$$T[\phi\psi] - T[\phi]T[\psi] = H[\phi]H[\tilde{\psi}^{-1}] \tag{B.1}$$

Identity (B.1) is trivial. The left side has (i,j) block

$$\sum_{k=-\infty}^{\infty} \phi_{i-k}\psi_{k-j} - \sum_{k=0}^{\infty} \phi_{i-k}\psi_{k-j} = \sum_{k=-\infty}^{-1} \phi_{i-k}\psi_{k-j} = \sum_{k=0}^{\infty} \phi_{i+k+1}\psi_{-k-j-1}$$

Two applications of the identity (B.1) give (I=identity matrix)

$$\begin{aligned} T[\phi]T[\phi^{-1}] &= I - H[\phi]H[\tilde{\phi}^{-1}] \\ T[\phi^{-1}]T[\phi] &= I - H[\phi^{-1}]H[\tilde{\phi}] \end{aligned}$$

Let X and Y be Banach spaces. A bounded linear operator $T : X \rightarrow Y$ is called completely continuous if, for any weakly convergent sequence (x_n) from X, the sequence Tx_n is norm-convergent in Y.

A compact operator L from X to Y is a linear operator such that the image under L of any bounded subset of X is a relatively compact subset of Y. Compact operators on Banach spaces are always completely continuous.

The collection $\mathfrak{F}_p (1 \leq p < \infty)$ consists of all completely continuous operators A for which

$$\sum_{j=1}^{\infty} s_j^p(A) < \infty$$

where s_j are singular values of A. The norm in $\mathfrak{F}_p$ is defined by

$$\|A\|_p = (\sum_{j=1}^{\infty} s_j^p(A))^{\frac{1}{p}}$$

When $p = 1$ the operator is called a trace-class operator. Here is an equivalent definition for the trace-class operator. A bounded linear operator A over a separable Hilbert space H is said to be in the trace class if for some orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of H, the sum

$$\|A\|_1 = \sum_k < (A^*A)^{\frac{1}{2}} e_k, e_k >$$

is finite, where A^* is the adjoint operator of A under the inner product in H, and $< \cdot, \cdot >$ denote the inner product. In this case

$$\sum_k < Ae_k, e_k >$$

is absolutely convergent and is independent of the choice of orthonormal basis. This limit is denoted by $\mathrm{tr}(A)$. When $A \in \mathfrak{F}_1$, this limit is equal to the sum of all eigenvalues of the operator A.

When $p = 2$, the operator is a Hilbert-Schmidt operator. A trace-class operator is always a Hilbert-Schmidt operator. A Hilbert-Schmidt operator is always compact.

Given a Hankel matrix with continuous symbol ϕ. The Fourier coefficients of ϕ decays exponentially fast. In this case, the Hankel operator is a trace-class operator.

Appendix C

Uniqueness of Gibbs Measure on Dimer Models of Non-bipartite Graphs

One correspondence between the Ising spin configurations and dimer configfruations is described in [5]. Consider the dual graph $\mathcal{G}^*$ of the graph $\mathcal{G}$ where the original Ising model is defined. Replace each vertex of the dual graph by a gadget.

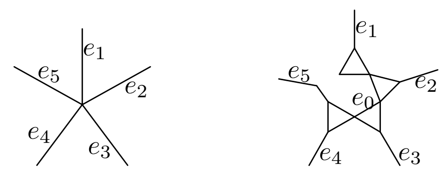

Figure C.1: Dual Ising Graph and Decorations

Assume all the edges which do not correspond to the edges of the dual graph have weight 1. For those edges corresponding to edges of the dual graph, assume an even number of such edges, surrounding each gadget, have weight less than 1, all the other dual edges have weights greater than or equal to 1. For example, in Figure 1, edges e_1, e_2, e_3, e_4, e_5 have corresponding edges in the dual graph(dual edges), while the edge e_0 does not correspond to any edge in the dual graph at all. The two-to-one correspondence between the Ising spin configurations and the dimer configurations is described as follows:

1. If two adjacent spins have the same sign, and the weight of the edge of $\mathcal{G}^*$ separating the two spins is bigger than 1, then the edge is present in the dimer configuration;

2. If two adjacent spins have the same sign, and the weight of the edge of $\mathcal{G}^*$ separating the two spins is less than 1, then the edge in is not present in the dimer configuration;

3. If two adjacent spins have the opposite sign, and the weight of the edge of $\mathcal{G}^*$ separating the two spins is less than 1, then the edge in is present in the dimer configuration;

4. If two adjacent spins have the opposite sign, and the weight of the edge of $\mathcal{G}^*$ separating the two spins is bigger than 1, then the edge in is not presnet in the dimer configuration;

Corollary C.1. *For a graph obtained from the dual graph of a ferromagnetic Ising model as described above, there is a unique translation-invariant Gibbs measure defined on dimers.*

Proof. It suffices to prove the result on any finite cylindrical set. Let ρ_e be the variable associated to an edge e, that is, if e is present in the dimer configuration, $\rho_e = 1$, otherwise $\rho_e = 0$. If e is not an dual edge, the configuration of e is uniquely determined by dual edges. For example, in Figure

$$\rho_{e_0} = \rho_{e_3}\rho_{e_4}(1-\rho_{e_5}) + \rho_{e_3}(1-\rho_{e_4})\rho_{e_5}$$

Hence it suffices to prove the result on an arbitrary cylindrical set consisting of dual edges. When e is a dual edge let σ_u, σ_v be spins on endpoints of the dual edge e^*, then

$$\rho_e = \begin{cases} \frac{1-\sigma_u\sigma_v}{2} & \text{if } w_e < 1 \\ \frac{1+\sigma_u\sigma_v}{2} & \text{if } w_e > 1 \end{cases}$$

Hence $\langle \rho_{e_1} \cdots \rho_{e_k} \rangle$ depends only on spin-spin even correlation functions. The uniqueness of spin-spin even correlation functions implies the uniqueness of the translation-invariant Gibbs measure on dimers. $\square$

Bibliography

[1] M. Aizenman, D. J. Barsky, and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *J. Statist. Phys.*, 47(3-4):343–374, 1987.

[2] Michael Aizenman. Rigorous studies of critical behavior. In *Applications of field theory to statistical mechanics (Sitges, 1984)*, volume 216 of *Lecture Notes in Phys.*, pages 125–139. Springer, Berlin, 1985.

[3] Cédric Boutillier and Béatrice de Tilière. The critical Z-invariant Ising model via dimers: the periodic case. *Probab. Theory Related Fields*, 147(3-4):379–413, 2010.

[4] Jin-Yi Cai. Holographic algorithms. In *Current developments in mathematics, 2005*, pages 111–150. Int. Press, Somerville, MA, 2007.

[5] David Cimasoni. A generalized kac-ward formular. *J. Stat. Mech.*, 2010, 2010.

[6] Henry Cohn, Richard Kenyon, and James Propp. A variational principle for domino tilings. *J. Amer. Math. Soc.*, 14(2):297–346 (electronic), 2001.

[7] Michael E. Fisher. On the dimer solution of planar ising modles. *J. Math. Phys.*, 7(10):1776–1781, 1966.

[8] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22:89–103, 1971.

[9] I. C. Gohberg and M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators*. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.

[10] Robert B. Griffiths. Rigorous results and theorems. In *Phase transition and critical phenomena*. Academic Press, New York, 1972.

[11] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, Oxford, sixth edition, 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

[12] Richard Holley. Remarks on the FKG inequalities. *Comm. Math. Phys.*, 36:227–231, 1974.

[13] P. W. Kasteleyn. The statistics of dimers on a lattice: I. the number of dimer arrangements on a quadratic lattice. *Physica*, 27(12):1209–1225, 1961.

[14] P. W. Kasteleyn. Graph theory and crystal physics. In *Graph Theory and Theoretical Physics*, pages 43–110. Academic Press, London, 1967.

[15] Richard Kenyon and Andrei Okounkov. Planar dimers and Harnack curves. *Duke Math. J.*, 131(3):499–524, 2006.

[16] Richard Kenyon, Andrei Okounkov, and Scott Sheffield. Dimers and amoebae. *Ann. of Math. (2)*, 163(3):1019–1056, 2006.

[17] Oscar E Lanford, III. Entropy and equilibrium states in classical statistical mechanics. In *Statistical mechanics and mathematical problems*, pages 1–113. Springer, 1973.

[18] Gregory F. Lawler. *Introduction to stochastic processes*. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2006.

[19] Joel L. Lebowitz. Bounds on the correlations and analyticity properties of ferromagnetic Ising spin systems. *Comm. Math. Phys.*, 28:313–321, 1972.

[20] Joel L. Lebowitz. Coexistence of phases in Ising ferromagnets. *J. Statist. Phys.*, 16(6):463–476, 1977.

[21] Joel L. Lebowitz and Anders Martin-Löf. On the uniqueness of the equilibrium state for Ising spin systems. *Comm. Math. Phys.*, 25:276–282, 1972.

[22] T. D. Lee and C. N. Yang. Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model. *Physical Rev. (2)*, 87:410–419, 1952.

[23] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[24] Zhongyang Li. Spectral curve of periodic fisher graphs.

[25] B. McCoy and T. T. Wu. *The two-dimensional Ising Model*. Harvard University Press, Oxford, first edition, 1973.

[26] Grigory Mikhalkin. Amoebas of algebraic varieties and tropical geometry. In *Different faces of geometry*, volume 3 of *Int. Math. Ser. (N. Y.)*, pages 257–300. Kluwer/Plenum, New York, 2004.

[27] Grigory Mikhalkin and Hans Rullgård. Amoebas of maximal area. *Internat. Math. Res. Notices*, (9):441–451, 2001.

[28] Moshe Schwartz and Jehoshua Bruck. Constrained codes as networks of relations. *IEEE Trans. Inform. Theory*, 54(5):2179–2195, 2008.

[29] Leslie G. Valiant. Holographic algorithms. *SIAM J. Comput.*, 37(5):1565–1594, 2008.

[30] Harold Widom. Asymptotic behavior of block Toeplitz matrices and determinants. II. *Advances in Math.*, 21(1):1–29, 1976.