Abstract of "String Dynamics and Higher Spin Anti-de Sitter Gravity" by Kewang Jin, Ph.D., Brown University, May 2011

This dissertation summarizes studies directed at a deeper understanding of the AdS/CFT correspondence, its origin and its dynamics. First, classical strings in AdS were considered and a general method for constructing string solutions was developed. The method is based on the reduction of string sigma models to integrable equations of sinh-Gordon or more generally Toda type. These equations are characterized by soliton type solutions from which the string configurations are constructed with a one-to-one correspondence between solitons of the field theory and spikes of the string. Through this correspondence the most general class of dynamical string solutions can be generated. In the case of AdS_3, these general spiky strings are characterized by an arbitrary number n of spikes and two arbitrary holomorphic functions. After fixing the conformal frame, only the soliton moduli remain, giving a specification of the string moduli. This moduli space is particle-like and it is shown that the spikes follow a closed set of equations describing the dynamics of the moduli space providing a 0-brane picture of the AdS string.

In the second part of the dissertation we pursue the direct construction of the AdS theory from the large N collective dynamics of its moduli. We accomplish this fully in the simplest sub-sector corresponding to the bi-local system of $n = 2$ spikes. It is seen that higher spin massless particles originate from the cusps of the spiky strings. The large N collective construction establishes the proposal of Klebanov and Polyakov that higher-spin AdS gravity of Vasiliev's type appears as a dual to the $O(N)$ vector model. For this an explicit mapping of the AdS_4 spacetime (and of higher-spin fields) was given from collective (bi-local) fields. This construction was deduced through the identification of isometries of $SO(2,3)$ with the conformal generators of the CFT_3 in the light-cone quantization. This mapping gives an explicit derivation of the extra spatial dimension in the AdS spacetime and reconstruction of the bulk AdS theory.

String Dynamics and Higher Spin Anti-de Sitter Gravity

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General Introduction

One of the most notable developments in Theoretical Physics in the last several decades was the realization that some of the most fundamental physical forces, such as Gravity and String Theory emerge in the large N limit of Conformal Field Theory (CFT). This scheme known as the AdS/CFT duality is characterized by the fact that the field theory is defined in d dimensional Minkowski spacetime while the dual gravitational theory is in the $d+1$ dimensional curved Anti-de Sitter (AdS) spacetime. Typical studies of the correspondence were performed in the holographic scheme where the correlators are evaluated at the boundary of AdS. One would then like to have a more fundamental understanding of the duality which does not rely on the projection of the extra dimension and where the emergence of the bulk AdS theory can be fully established. This dissertation describes research performed towards that goal.

The presentation in this dissertation is broken down into two major parts. The first part of the dissertation concerns developing methods to generate classical string solutions in AdS as well as studying their dynamics. Using a particular reduction, the nonlinear string sigma model in AdS_3 can be reduced to the sinh-Gordon equation which possesses solitonic excitations. The inverse scattering technique was employed to generate string solutions provided the sinh-Gordon soliton profiles. We found that a soliton configuration with a singularity at its location translates into a spike at the string level. This leads to a significant simplification, where the use of relevant (collective) coordinates gives the picture of N-body "partons". This investigation therefore identifies the sub-structure of the AdS string itself. The second part of the dissertation describes work on reconstructing gravity and spacetime from the established partonic sub-structure. The manner in which continuum phenomena such as gravity are reconstructed from the microscopic dynamics is argued to be associated with the phenomenon of collective motions. Studying the simplest partonic composite, consisting of a bi-local system of two particles, turned out to already produce a striking result: the appearance of one extra AdS dimension and of a sequence resonance of growing integer spins. Specifically an explicit mapping of the AdS_4 spacetime plus higher-spin fields is established from the (bi-local) collective fields of conformal field theory.

1.1 The AdS/CFT correspondence

The AdS/CFT correspondence [1, 2, 3] relates type IIB string theory on the curved background $AdS_5 \times S^5$ with $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory in four dimensions (see [4, 5, 6, 7] for a review). It is a strong-weak duality with the precise correspondence given by

$$g_{YM}^2 N = \lambda = \frac{R^4}{\alpha'^2}, \qquad \frac{1}{N} = \frac{4\pi g_s}{\lambda} \tag{1.1}$$

where g_{YM} is the Yang-Mills coupling constant, α' is the inverse string tension, λ is the effective 't Hooft coupling constant in the large N limit, g_s is the topological expansion parameter in string theory and R is the radius of the $AdS_5 \times S^5$ background.

1.1.1 Maldacena's conjecture

According to Maldacena [1], the 't Hooft limit of $\mathcal{N} = 4$ $d = 3 + 1$ SYM theory at the conformal point contains type IIB strings on $AdS_5 \times S^5$. This is shown by taking a low-energy limit of D3-branes in string theory, where the field theory on the brane decouples from the bulk. Starting with type IIB string theory with fixed string coupling g_s and consider N parallel D3-branes separated by some distance r, at low energies, we take the limit

$$\alpha' \to 0, \qquad \frac{r}{\alpha'} = \text{fixed}, \tag{1.2}$$

where the second condition is to keep the mass of the stretched strings fixed. At this decoupling limit, we bring the branes together but the Higgs expectation values corresponding to this separation remain fixed. The resulting theory on the brane is four dimensional $\mathcal{N} = 4$ U(N) SYM theory.

In more details, we consider the supergravity solution carrying D3-brane charge

$$ds^2 = f^{-1/2} dx_{||}^2 + f^{1/2}(dr^2 + r^2 d\Omega_5^2), \tag{1.3}$$

where $f = 1 + R^4/r^4$, $R^4 \equiv 4\pi g_s N \alpha'^2$ and $x_{||}$ denotes the four coordinates along the world-volume of the three-brane. In the near horizon region $r \ll R$, we can approximate $f \sim R^4/r^4$ and the geometry becomes $AdS_5 \times S^5$. Now we define the new variable $U = r/\alpha'$ and write the metric (1.3) in terms of U as

$$ds^2 = \alpha'\Big[\frac{U^2}{\sqrt{4\pi g_s N}}dx_{||}^2 + \sqrt{4\pi g_s N}\frac{dU^2}{U^2} + \sqrt{4\pi g_s N}d\Omega_5^2\Big]. \tag{1.4}$$

This metric describes the radius of $AdS_5 \times S^5$ as

$$\frac{R^2}{\alpha'} = \sqrt{4\pi g_s N}. \tag{1.5}$$

This radius is quantized because the flux of the five-form field strength on the five sphere is quantized. We can trust the supergravity solution when $g_s N \gg 1$. When N is large, we have approximately ten dimensional flat space in the neighborhood of any point.

Next we consider a near extremal black D3-brane solution in the decoupling limit (1.2) and keep the energy density on the brane world-volume theory μ fixed. We find the metric

$$ds^2 = \alpha'\Big[\frac{U^2}{\sqrt{4\pi g_s N}}[-(1-U_0^4/U^4)dt^2 + dx_i^2] + \frac{\sqrt{4\pi g_s N}dU^2}{U^2(1-U_0^4/U^4)} + \sqrt{4\pi g_s N}d\Omega_5^2\Big], \tag{1.6}$$

where $U_0^4 = \frac{2^7}{3}\pi^4 g_s^2 \mu$. Since U_0 remains finite when we take the $\alpha' \to 0$ limit, we should consider fields that propagate on the AdS background. Since the Hawking temperature is finite, there is a flux of energy from the black hole to the AdS spacetime. Since $\mathcal{N} = 4$ $d = 3+1$ U(N) SYM is a unitary theory we conclude that, for large N, it includes in its Hilbert space the states of type IIB supergravity on $AdS_5 \times S^5$.

In summary, we have started with a quantum theory and seen that it includes gravity as it is natural to think that this correspondence goes beyond the supergravity approximation. Then we are led to the conjecture that $\mathcal{N} = 4$ *U(N) Super Yang-Mills theory in 3+1 dimensions is dual to type IIB superstring theory on* $AdS_5 \times S^5$. From the physics of D-branes, we know that the SYM coupling is given by the (complex) IIB string coupling through

$$\frac{1}{g_{YM}^2} + i\frac{\theta}{8\pi^2} = \frac{1}{4\pi g_s} + i\frac{\chi}{8\pi^2}, \tag{1.7}$$

where χ is the expectation value of the Ramond-Ramond scalar.

The supersymmetry group of $AdS_5 \times S^5$ is known to be the same as the superconformal group in 3+1 dimensional spacetime, so the supersymmetries of both theories are the same. Notice the correspondence is non-perturbative in g_s and the $SL(2, Z)$ symmetry of type IIB would follow as a consequence of the $SL(2, Z)$ symmetry of SYM. It is also a strong-weak coupling correspondence in the following sense: when the effective coupling $g_s N$ becomes large we cannot trust perturbative calculations in the Yang-Mills theory but we can trust calculations in the supergravity on $AdS_5 \times S^5$.

The nontriviality of Maldacena's proposal is contained in the fact that the two theories contain very different field degrees of freedom and are defined in different spacetime dimensions. A scheme to perform comparisons between the two theories is defined by Gubser, Klebanov, Polyakov [2] and Witten [3], known as the "holographic" map according to which one is to project AdS amplitudes to the boundary of AdS spacetime and find an agreement with the CFT correlation functions. This holographic scheme was used for establishing and verifying the AdS/CFT correspondence in a number of examples, even though it gives little insight into the mechanism of the duality, the origin of bulk interactions and of the extra AdS radial dimension.

1.1.2 Correlation functions

The AdS/CFT gives a precise correspondence between conformal field theory observables and those of supergravity: correlation functions in conformal field theory are given by the dependence of the supergravity action on the asymptotic behavior at infinity. In particular, dimensions of operators in conformal field theory are given by masses of particles in supergravity. This proposal is effective and gives a practical recipe for computing large N conformal field theory correlation functions from supergravity tree diagrams.

The partition function of the AdS theory (with suitably prescribed boundary conditions for the fields) equals the generating functional of the boundary conformal field theory. Schematically, one has

$$Z_{AdS}[\phi_0] = \int_{\phi_0} \mathcal{D}\phi \exp(-I[\phi]) = Z_{CFT}[\phi_0] = \Big\langle \exp\Big(\int_{\partial\Omega} d^d x [\phi_0 \mathcal{O}] \Big) \Big\rangle. \tag{1.8}$$

The path integral on the LHS is calculated under the restriction that the field ϕ asymptotically approaches ϕ_0 on the boundary. On the other hand, the function ϕ_0 is considered as a current, which couples to the scalar density operator $\mathcal{O}$ in the boundary conformal field theory. Calculating the LHS thus allows one to explicitly obtain correlation functions of the boundary conformal field theory.

Consider the Euclidean version of the Poincaré patch of AdS_{d+1} with the metric

$$ds^2 = \frac{dx^2 + dz^2}{z^2} \tag{1.9}$$

where $x^i = (x^1, \cdots, x^d)$ and we have set the AdS radius to 1. Let $\phi(z, x)$ be a bulk scalar with mass m and consider the action [3, 8]

$$I(\phi) = \frac{1}{2}\int d^{d+1}x\sqrt{g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2], \\ = \frac{1}{2}\int d^dx dz z^{-d+1}[(\partial_z\phi)^2 + (\partial_i\phi)^2 + \frac{m^2}{z^2}\phi^2], \tag{1.10}$$

the equations of motion are

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0. \tag{1.11}$$

Its asymptotic behavior near the boundary of AdS_{d+1} is

$$\phi(z,x)|_{z\to 0} \approx z^{\Delta_-}\phi_0(x) + z^{\Delta_+}A(x) \tag{1.12}$$

where the scaling dimensions

$$\Delta_\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2+4m^2}. \tag{1.13}$$

The functions $\phi_0(x)$ and $A(x)$ are the two necessary boundary data to determine the solution of the second-order bulk equation of motion for $\phi(z,x)$. Quantizing $\phi(z,x)$ with boundary condition $A(x) = 0$ ($\phi_0(x) = 0$) would give the generating functional of the boundary operator $\mathcal{O}(x)$ with dimension Δ_+ ($\tilde{\mathcal{O}}(x)$ with dimension Δ_-). The above ambiguity does not show up in most studied cases of AdS/CFT where the operator $\tilde{\mathcal{O}}(x)$ has dimension below the unitarity bound $\Delta_- < d/2 - 1$.

Nevertheless, there exist important cases where both $\Delta_\pm$ are above the unitarity bound. Then the quantization ambiguity is present even when the asymptotic behavior of $\phi(z,x)$ is determined by one arbitrary boundary data when one requires that the bulk solution vanishes in the far interior ($z \to \infty$) of AdS. In such a case the two functions appearing in (1.12) are related by

$$A(x) = \frac{\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\nu)} \int d^d y \frac{1}{(x-y)^{2\Delta_+}} \phi_0(y), \tag{1.14}$$

with $\nu = \Delta_+ - d/2$. Then the application of AdS/CFT correspondence yields either a functional $W[\phi_0]$ of $\phi_0(x)$ or a functional $J[A]$ of $A(x)$. The first generates correlation functions of $\mathcal{O}(x)$ and the second of $\tilde{\mathcal{O}}(x)$. However, the two functionals are related by a Legendre transform

$$W[\phi_0] + 2\nu \int d^d x \phi_0(x) A(x) = J[A], \qquad \frac{\delta W[\phi_0]}{\delta \phi_0(x)} = -2\nu A(x). \tag{1.15}$$

Going back to the equations of motion (1.11), explicitly, one has

$$z^{d+1} \frac{\partial}{\partial z}\left[z^{-d+1} \frac{\partial}{\partial z} \phi(z,x) \right] + z^2 \frac{\partial^2}{\partial x^2} \phi(z,x) - m^2 \phi(z,x) = 0. \tag{1.16}$$

Witten [3] has constructed a Green's function solution which explicitly realizes the relation between the field $\phi(z,x)$ in the bulk and the boundary configuration $\phi_0(x')$. The solution to (1.16) is then related to the boundary data by

$$\phi(z,x) = \frac{\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\Delta_+ - d/2)} \int d^d x' \frac{z^{\Delta_+}}{(z^2 + |x-x'|^2)^{\Delta_+}} \phi_0(x'). \tag{1.17}$$

Plugging into the action (1.10), we find

$$I(\phi_0) = -\frac{(\Delta_+ - d/2)\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\Delta_+ - d/2)} \int d^d x d^d x' \frac{\phi_0(x)\phi_0(x')}{|x-x'|^{2\Delta_+}}. \tag{1.18}$$

This determines the two-point function of a conformal operator $\mathcal{O}$ with dimension Δ_+. Varying twice of the action (1.18) with respect to ϕ_0 we find that the two-point function of the corresponding operator is

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{(2\Delta_+ - d)\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\Delta_+ - d/2)}\frac{1}{|x-x'|^{2\Delta_+}}. \tag{1.19}$$

This evaluation can be generalized to the n-point functions.

1.2 Giant magnons

Insight into the AdS/CFT correspondence can be gained even at the semiclassical level. In particular, the semiclassical limit of string theory provides answers for the strong coupling regime of gauge theory. This has been most successfully demonstrated for the case of $\mathcal{N} = 4$ Super Yang-Mills theory where an exact Bethe ansatz solution is available [9], bridging the weak and strong couplings. The anomalous dimensions of the $\mathcal{N} = 4$ SYM operators with large R-charge can be computed using the dilatation operator [10, 11, 12], more efficiently, using a certain spin chain in the planar limit [13, 14, 15]. These spin chains have fundamental "magnon" excitations which obey a dispersion relation that is periodic in the momentum of the magnons. Using supersymmetry, the dispersion relation was found to be [16]

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}. \tag{1.20}$$

The result is exact, i.e. the one-loop superstring correction vanishes [17]. Note that the periodicity in p comes from the discreteness of the spin chain. At large 't Hooft coupling, the energy behaves as

$$E - J \sim \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right|. \tag{1.21}$$

Since this is a strong coupling result, it should be possible to reproduce it on the string theory side. Hofman and Maldacena [18] identified these magnons on the string theory side and showed how to reconcile a periodic dispersion relation with the continuum world-sheet description. The crucial idea is that the momentum is interpreted in the string theory side as a certain geometrical angle. Furthermore, the dispersion relation (1.21) is reproduced by considering a classical string in the following limit

$$J \to \infty, \quad \lambda = g^2 N = \text{fixed}, \quad p = \text{fixed}, \quad E - J = \text{fixed}. \tag{1.22}$$

This is different from the BMN limit [19] where λ was taken to infinity and $n = pJ$ was kept fixed. One nice feature of this limit is that it decouples quantum effects, which are characterized by λ, from finite J effects or finite volume effects on the string world-sheet.

1.2.1 Physical gauge

Now we consider the motion of strings in $\mathbb{R} \times S^2$, the metric is

$$ds^2 = -dt^2 + d\theta^2 + \sin^2\theta d\varphi^2 \tag{1.23}$$

where φ is the coordinate shifted by the angular momentum J. The string ground state with $E - J = 0$ corresponds to a light-like trajectory that moves along φ with $\varphi - t = \text{constant}$, that sits at $\theta = \pi/2$ and at the origin of the spatial directions of AdS_5.

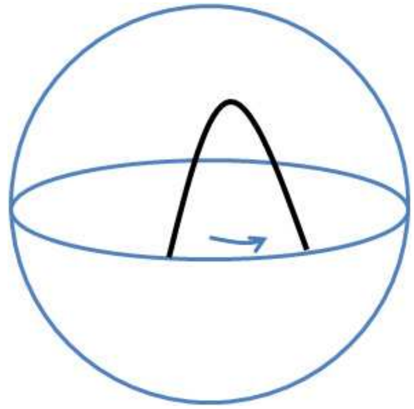

Figure 1.1: Giant magnon solution. The momentum of the state is given by the angular distance between the endpoints of the string, which are located on the equator and move at the speed of light.

Fixing the gauge by

$$t = \tau, \qquad \varphi - t = \varphi' = \sigma, \tag{1.24}$$

and we consider a configuration where θ is independent of τ. The Nambu-Goto action becomes

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \sqrt{(\dot{X}X')^2 - (\dot{X})^2(X')^2} \\ &= \frac{\sqrt{\lambda}}{2\pi}\int dt d\varphi' \sqrt{\cos^2\theta\ \theta'^2 + \sin^2\theta}. \end{aligned} \tag{1.25}$$

One can easily derive the equation of motion

$$\sin\theta\cos\theta\ \theta'' - (1+\cos^2\theta)\ \theta'^2 - \sin^2\theta = 0. \tag{1.26}$$

Integrating the equation of motion, we get

$$\sin\theta = \frac{\sin\theta_0}{\cos\varphi'}, \qquad -(\frac{\pi}{2} - \theta_0) \leq \varphi' \leq (\frac{\pi}{2} - \theta_0) \tag{1.27}$$

where $0 \leq \theta_0 \leq \pi/2$ is an integration constant.

The solution is plotted in Figure 1.1, we see that the difference in angle between the two endpoints of the string at a given time t is

$$\Delta\varphi' = \Delta\varphi = 2(\frac{\pi}{2} - \theta_0). \tag{1.28}$$

It is easy to compute the energy

$$E - J = \frac{\sqrt{\lambda}}{\pi}\cos\theta_0 = \frac{\sqrt{\lambda}}{\pi}\sin\frac{\Delta\varphi}{2}. \tag{1.29}$$

Identifying $\Delta\varphi = p$, we get

$$E - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right| \tag{1.30}$$

in perfect agreement with the large λ limit of (1.20). The sign of p is related to the orientation of the string. In summary, the string solution in physical gauge is

$$\begin{pmatrix} t \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} \tau \\ \tau + \sigma \\ \sin^{-1}[\sin\theta_0/\cos\sigma] \end{pmatrix}. \tag{1.31}$$

1.2.2 Conformal gauge

We can rewrite the solution (1.31) in a time-like conformal gauge

$$\begin{pmatrix} t \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} \tau \\ \tau + \tan^{-1}[\cot\theta_0\tanh\tilde{\sigma}] \\ \cos^{-1}[\cos\theta_0\ \mathrm{sech}\tilde{\sigma}] \end{pmatrix} \tag{1.32}$$

where $\tilde{\sigma} \equiv \frac{\sigma - \sin\theta_0\ \tau}{\cos\theta_0}$. In this case we see that the range of σ is infinite. Moreover, the energy density is a constant $\mathcal{E} = \frac{\sqrt{\lambda}}{2\pi}$. Easily, we can calculate the spacetime coordinate

$$X = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \tanh\tilde{\sigma}\sin\tau \\ \tanh\tilde{\sigma}\cos\tau \\ \mathrm{sech}\tilde{\sigma} \end{pmatrix}. \tag{1.33}$$

Using the sine-Gordon connection [20], we get

$$\alpha \equiv \cos^{-1}[(\partial_\sigma X)^2 - (\partial_\tau X)^2] = 4\tan^{-1}[\exp(\gamma(\sigma - v\tau))] \tag{1.34}$$

where $v \equiv \sin\theta_0$ and $\gamma \equiv (1-v^2)^{-1/2}$. Notice that a boost on the sine-Gordon side translates into a non-obvious classical symmetry on the $R \times S^2$ side. The solution (1.34) is exactly the one-soliton solution of the sine-Gordon equation with the energy

$$\mathcal{E}_{\text{soliton}} = \gamma. \tag{1.35}$$

Interestingly, if we define the energy of a magnon as $E - J$, we have

$$\mathcal{E}_{\text{magnon}} \equiv E - J = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\gamma}, \tag{1.36}$$

which is inversely proportional to the energy of the sine-Gordon soliton. This will have an interesting effect on the scattering phase shift of giant magnons.

1.2.3 Scattering of giant magnons

Now we consider a soliton anti-soliton pair and compute the time delay for their scattering. Since x and t coordinates are the same in the two theories, the time delay is precisely the same for the string theory magnons and for the sine-Gordon solitons.

The sine-Gordon scattering solution in the center of mass frame is

$$\alpha = 4 \tan^{-1} \left[\frac{1}{v} \frac{\sinh \gamma v t}{\cosh \gamma x} \right]. \tag{1.37}$$

The time delay is

$$\Delta T_{cm} = \frac{2}{\gamma v} \ln v. \tag{1.38}$$

The fact that the sine-Gordon scattering is dispersionless implies that the scattering of magnons is also dispersionless in the classical limit (we also expect it to be dispersionless in the quantum theory).

We now boost the configuration (1.37) to a frame where we have a soliton moving with velocity v_1 and an anti-soliton with velocity v_2, where $v_1 > v_2$. Then the time delay that particle 1 experiences as it goes through particle 2 is

$$\Delta T_{12} = \frac{2}{\gamma_1 v_1} \ln v_{cm} \tag{1.39}$$

where v_{cm} is the velocity in the center of mass frame

$$2 \ln v_{cm} = \ln \left[\frac{1 - \cos \frac{p_1 - p_2}{2}}{1 - \cos \frac{p_1 + p_2}{2}} \right]. \tag{1.40}$$

We can compute the phase shift from the formula

$$\frac{\partial \delta_{12}(\epsilon_1, \epsilon_2)}{\partial \epsilon_1} = \Delta T_{12} \tag{1.41}$$

and obtain

$$\delta_{12} = \frac{\sqrt{\lambda}}{\pi} \left\{ \left[-\cos \frac{p_1}{2} + \cos \frac{p_2}{2} \right] \ln \left[\frac{1 - \cos \frac{p_1 - p_2}{2}}{1 - \cos \frac{p_1 + p_2}{2}} \right] \right\} - p_1 \frac{\sqrt{\lambda}}{\pi} \sin \frac{p_2}{2}. \tag{1.42}$$

Note that, even though the time delay is identical to the sine-Gordon one, the phase shift is different, due to different expressions for the energy. This implies, in particular, that the phase shift is not invariant under the sine-Gordon boosts.

1.3 Gluon scattering amplitudes at strong coupling

Classical string solutions in AdS can also be used for evaluation of Yang-Mills scattering amplitudes. Alday and Maldacena [21, 22, 23] gave a prescription for computing gluon scattering amplitudes in $\mathcal{N} = 4$ SYM theory at strong coupling using AdS/CFT through Wilson loops (see [24] for a review). This prescription is equivalent to finding a classical string solution with boundary conditions determined by the gluon momenta. The value of the scattering amplitude is then related to the area of this solution

$$\mathcal{A} \sim e^{iS_{cl}} = e^{-\frac{\sqrt{\lambda}}{2\pi}(\text{Area})_{cl}} \tag{1.43}$$

where S_{cl} denotes the classical action of the classical solution of the string world-sheet equations.

On the gauge theory side, Bern, Dixon and Smirnov (BDS) [25] gave a very interesting conjecture for the all order form of the n gluon MHV scattering amplitudes. As for the four point amplitude, the conjecture takes the form

$$\mathcal{A}_4 = \mathcal{A}_4^{tree} \exp\left[(\text{IR divergent}) + \frac{f(\lambda)}{8}\left(\ln\frac{s}{t}\right)^2 + (\text{constant})\right] \tag{1.44}$$

where s, t are the Mandelstam variables, $f(\lambda)$ is the cusp anomalous dimension and the IR divergent terms are well characterized by Sudakov-like factors. At strong coupling, the anomalous dimension reads

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \cdots . \tag{1.45}$$

1.3.1 Four gluon scattering amplitude

Consider the Lorentzian AdS_5 metric in Poincaré coordinates

$$ds^2 = R^2 \left[\frac{dx_{3+1}^2 + dz^2}{z^2} \right], \tag{1.46}$$

we place a D-brane at a large value z_{IR} as the cutoff. In order to state most simply the boundary conditions for the world-sheet it is convenient to describe the solution in terms of T-dual coordinates y^μ defined in the following way. Starting with a metric that contains

$$ds^2 = \omega^2(z) dx_\mu dx^\mu + \cdots \tag{1.47}$$

where ω is the wrap factor, we define T-dual variables y^μ by

$$\partial_\alpha y^\mu = i\omega^2(z)\epsilon_{\alpha\beta}\partial_\beta x^\mu. \tag{1.48}$$

In the regime under consideration the T-dual coordinates are real and the world-sheet is Euclidean. In addition, the boundary condition for the original coordinates x^μ, which carry momentum k^μ, translates into the condition that y^μ has "winding" $\Delta y^\mu = 2\pi k^\mu$. The T-dual metric is again AdS_5 after defining $r = R^2/z$,

$$d\tilde{s}^2 = R^2 \left[\frac{dy_\mu dy^\mu + dr^2}{r^2} \right]. \tag{1.49}$$

Notice now the boundary is located at $r_{IR} = R^2/z_{IR}$. As we take the limit $z_{IR} \to \infty$, we find that the boundary of the worldsheet moves to the boundary of the T-dual metric (1.49) at $r = 0$.

For four gluons, we label the momenta as k_i, $i = 1, \cdots, 4$, where the subindex indicates the color ordering. We consider the region where particles 1 and 3 are incoming and particles 2 and 4 are outgoing. We label by k the center of mass energy or momentum of each incoming particles and we denote by φ the scattering angle in the center of mass frame; then the usual Mandelstam variables are

$$s = -(k_1 + k_2)^2 = -2k_1 \cdot k_2 = -4k^2 \sin^2 \frac{\varphi}{2}, \tag{1.50}$$

$$t = -(k_1 + k_4)^2 = -2k_1 \cdot k_4 = -4k^2 \cos^2 \frac{\varphi}{2}. \tag{1.51}$$

We will focus on the region where $s, t < 0$ which corresponds to space-like momentum transfer.

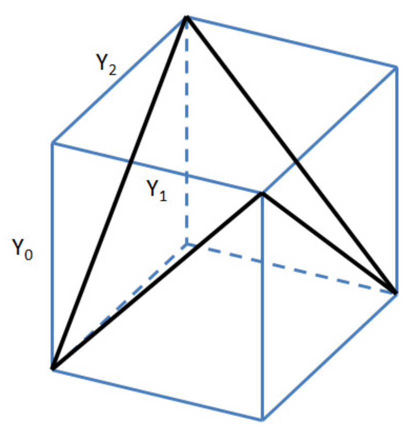

Figure 1.2: Wilson loop with four light-like boundaries. This figure lives at $r = 0$.

Next we consider a Wilson loop containing four light-like edges (see Figure 1.2). In order to write the Nambu-Goto action it is convenient to consider the Poincaré coordinates (r, y_0, y_1, y_2) while setting $y_3 = 0$, and parametrize the surface by its projection to the (y_1, y_2) plane. We consider first the case with $s = t$ where the projection of the Wilson lines is a square. The solution reads

$$y_0(y_1, y_2) = y_1 y_2, \qquad r(y_1, y_2) = \sqrt{(1-y_1^2)(1-y_2^2)}. \tag{1.52}$$

In terms of embedding coordinates, the surface is given by the equations

$$Y_3 = 0, \qquad Y_4 = 0, \qquad Y_0 Y_{-1} = Y_1 Y_2. \tag{1.53}$$

In fact, this solution is related to the single cusp solution [26] by a $SO(2,4)$ transformation. The solution for general s and t can be obtained by a simple boost, for example, in the 04 plane

$$Y_4 - vY_0 = 0, \qquad Y_{-1}\gamma(Y_0 - vY_4) = \gamma^{-1} Y_0 Y_{-1} = Y_1 Y_2, \qquad Y_3 = 0. \tag{1.54}$$

Let us now write the solutions in terms of worldsheet coordinates in conformal gauge. The induced metric on the world-sheet can be computed as

$$ds^2 = \frac{dy_1^2}{(1-y_1^2)^2} + \frac{dy_2^2}{(1-y_2^2)^2} = du_1^2 + du_2^2, \tag{1.55}$$

where $y_i = \tanh u_i$ and the metric is Euclidean. The solution with $s = t$ becomes

$$y_1 = \tanh u_1, \quad y_2 = \tanh u_2, \quad r = \frac{1}{\cosh u_1 \cosh u_2}, \quad y_0 = \tanh u_1 \tanh u_2. \tag{1.56}$$

Performing the boost (1.54) and a simple rescaling we find the solution for $s \neq t$

$$r = \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \tag{1.57}$$

$$y_0 = \frac{a\sqrt{1+b^2}\sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \tag{1.58}$$

$$y_1 = \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \tag{1.59}$$
$$y_2 = \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \tag{1.60}$$

where $b = v\gamma < 1$. The parameter a sets the overall scale of the momentum. Here a and b encode the kinematical information of the scattering as follows

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \qquad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}, \qquad \frac{s}{t} = \frac{(1+b)^2}{(1-b)^2}. \tag{1.61}$$

A common regularization scheme for computing minimal areas in AdS is to introduce a cutoff in the radial direction. The correct procedure is to impose the boundary conditions at some small $r = r_c$. It turns out, however, that in order to compute the finite piece as $r_c \to 0$ it suffices to use the original solution and cut the integral giving the area at $r = r_c$ defined by

$$\frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} = r_c. \tag{1.62}$$

The resulting integral simplifies when using the light-cone variables $u_\pm = u_1 \pm u_2$. By expanding the integrand in power series of r_c/a and integrating term by term, we arrive at the final expression for the area

$$iS = -\frac{\sqrt{\lambda}}{2\pi} A, \quad A = \frac{1}{4}\ln^2\Big(\frac{r_c^2}{-8\pi^2 s}\Big) + \frac{1}{4}\ln^2\Big(\frac{r_c^2}{-8\pi^2 t}\Big) - \frac{1}{4}\ln^2\Big(\frac{s}{t}\Big) - \frac{\pi^2}{3}, \tag{1.63}$$

which agrees perfectly with the BDS ansatz [25] including the finite constant piece.

The planar gluon scattering amplitudes at strong coupling has a dual conformal symmetry which is enough for determining the four and five gluon scattering amplitudes. The BDS ansatz fails for more gluons ($n \geq 6$) amplitudes. For more recent developments of calculating gluon scattering amplitudes using classical strings, please refer to [27]. These calculations employ the same inverse scattering technique that we will develop in the next chapter.

Chapter 2

String Dynamics in AdS

To generate classical string solutions in AdS spacetime, we use a Pohlmeyer type reduction. In this framework we find a correspondence between spikes of strings in AdS_3 and soliton profiles of the sinh-Gordon equation. This connection turns out to be most fruitful for the construction of dynamical multi-spike solutions. The inverse scattering technique was employed to generate string solutions provided the sinh-Gordon soliton profiles. We constructed the most general set of string solutions with arbitrary number of spikes using the n-soliton solution of the sinh-Gordon equation. These general spiky strings are characterized by two arbitrary holomorphic functions and a discrete set of moduli representing the soliton singularities. After fixing the conformal frame, only the soliton moduli remain, giving a specification of the string moduli.

2.1 Introduction

For states of Super Yang-Mills theory given by single trace operators with insertions of (covariant) derivatives the relevant string configurations were identified by Gubser, Klebanov and Polyakov (GKP) [28] as rotating folded strings.[1] Kruczenski gave the extension to regular n-spike configurations in [31]. These configurations in some sense represent the ground state configurations of the theory and it is of definite relevance to construct (and understand the dynamics) of more general non-static solutions [32, 33]. Ultimately one has the goal of formulating a complete dynamical picture of the moduli of strings moving in $AdS \times S$ spacetimes.

We approach the classical problem of constructing string solutions in AdS spacetime through a conformal gauge and the Pohlmeyer reduction [34] of the associated classical nonlinear sigma model. This technique was applied previously to the construction of solutions in de Sitter spacetime [35] and extended to soliton and spiky Minkowski worldsheet solutions in a series of papers [36, 37]. For the case of minimal surfaces with Euclidean worldsheet extension is applicable [38, 39]. The Pohlmeyer reduction reduces the nonlinear sigma model equations to a coupled system consisting of integrable Toda type equations and a conformal pair obeying the Cauchy-Riemann conditions.

In the Minkowski case, the integrability of the (Toda type) theories provides (singular) soliton type solutions which were identified with spikes in [36, 37] (see also [40]). Integrability of string theory on $AdS_5 \times S^5$ allows the use of algebraic methods to construct solutions of the nonlinear equations of motion. We use the inverse scattering method to construct string solutions corresponding to sinh-Gordon

[1] A semiclassical treatment of quantum fluctuations around this solution has been performed by Frolov and Tseytlin [29]. See [30] for a review of spinning strings.

solitons, antisolitons, breathers and soliton scattering solutions. We also show that the spikes of the long GKP string can be mapped to sinh-Gordon solitons at the boundary of AdS. The sigma model solutions can be constructed in terms of wavefunctions of the Pohlmeyer reduced model [41].

The advantage of this method is that it allows us to construct a string solution starting from any sinh-Gordon solution. All one has to do is to solve a linear system with coefficients depending on the chosen sinh-Gordon solution. Notice that in the dressing method [42] one is also solving a linear system, but the difference is that in the dressing method the coefficients of the system depend on the chosen vacuum solution of the string equations, whereas in this method the coefficients depend only on the sinh-Gordon or reduced system solution. This is advantageous because any sinh-Gordon solution is generally simpler than the corresponding sigma model solution.

2.1.1 AdS string as a σ-model

We will concentrate in what follows on string dynamics in purely AdS spacetime. In general, string equations in AdS_{d+1} spacetime (in conformal gauge) are described by the non-compact nonlinear sigma model on $SO(2,d)$. Defining the AdS_{d+1} space as $Y^2 = -Y_{-1}^2 - Y_0^2 + Y_1^2 + \cdots + Y_d^2 = -1$, the action reads

$$A = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Big(\partial Y \cdot \partial Y + \Lambda(\sigma,\tau)(Y \cdot Y + 1) \Big), \tag{2.1}$$

where τ, σ are the Minkowski worldsheet coordinates, the equations of motion are

$$\partial \bar{\partial} Y - (\partial Y \cdot \bar{\partial} Y) Y = 0, \tag{2.2}$$

with $z = (\sigma - \tau)/2$, $\bar{z} = (\sigma + \tau)/2$ and $\partial = \partial_\sigma - \partial_\tau$, $\bar{\partial} = \partial_\sigma + \partial_\tau$. In addition to guarantee the conformal gauge we have to impose the Virasoro conditions

$$\partial Y \cdot \partial Y = \bar{\partial} Y \cdot \bar{\partial} Y = 0. \tag{2.3}$$

It was demonstrated a number of years ago (by Pohlmeyer [34]) that nonlinear sigma models subject to Virasoro type constraints can be reduced to integrable field equations of sine-Gordon or more generally Toda type. This reduction is accomplished by concentrating on $SO(2,d)$ invariant sub-dynamics of the sigma model. The steps of the reduction were well described in [35, 36] and consist in the following. One starts by identifying first an appropriate set of basis vectors for the string coordinates

$$e_i = (Y, \partial Y, \bar{\partial} Y, B_4, \cdots, B_{d+2}), \qquad i = 1, 2, \cdots, d+2, \tag{2.4}$$

where B_i form an orthonormal set $B_i \cdot B_j = \delta_{ij}$, $B_i \cdot Y = B_i \cdot \partial Y = B_i \cdot \bar{\partial} Y = 0$. Defining the scalar field α and two sets of auxiliary fields

$$\alpha(z, \bar{z}) \equiv \ln[\partial Y \cdot \bar{\partial} Y], \tag{2.5}$$

$$u_i \equiv B_i \cdot \bar{\partial}^2 Y, \tag{2.6}$$

$$v_i \equiv B_i \cdot \partial^2 Y, \tag{2.7}$$

one can derive the equations of motion

$$\partial \bar{\partial} \alpha - e^{\alpha} - e^{-\alpha} \sum_{i=4}^{d+1} u_i v_i = 0, \tag{2.8}$$

$$\partial u_i = \sum_{j \neq i} (B_j \cdot \partial B_i) u_j, \tag{2.9}$$

$$\bar{\partial} v_i = \sum_{j \neq i} (B_j \cdot \bar{\partial} B_i) v_j. \tag{2.10}$$

In the case of AdS_3, there is only one vector B_4 so that $\partial u = 0$, $\bar{\partial} v = 0$. Therefore, the equation of motion for the scalar field α is simplified to be

$$\partial\bar{\partial}\alpha - e^{\alpha} - e^{-\alpha}u(\bar{z})v(z) = 0. \tag{2.11}$$

This is the generalized sinh-Gordon equation with two (anti)-holomorphic functions $u(\bar{z})$ and $v(z)$. In order to get the standard sinh-Gordon equation, we first shift the field

$$\alpha(z,\bar{z}) = \hat{\alpha}(z,\bar{z}) + \ln\sqrt{-u(\bar{z})v(z)}, \tag{2.12}$$

and then do a (conformal) change of variables

$$d\bar{z}' = \sqrt{2u(\bar{z})}d\bar{z}, \qquad dz' = \sqrt{-2v(z)}dz, \tag{2.13}$$

such that the equation of motion for $\hat{\alpha}$ satisfies

$$\partial'\bar{\partial}'\hat{\alpha}(z',\bar{z}') - \sinh\hat{\alpha}(z',\bar{z}') = 0. \tag{2.14}$$

2.1.2 Spikes vs Solitons

The classical motion describing a rigid rotation of a folded closed string is given by the ansatz $t = c\,\tau$, $\theta = c\,\omega\,\tau$ and $\rho = \rho(\sigma)$. The Virasoro constraints give

$$\rho'^2 = c^2(\cosh^2\rho - \omega^2\sinh^2\rho) \tag{2.15}$$

where the scaling constant c is adjusted to define the period of σ. We can set $c = 1$ and denote the position of the fold (spike) as σ_0. To demonstrate the stated correspondence with solitons we expand the solution (2.15) near the spike with $\omega = 1+2\eta$, where $\eta \ll 1$, one finds

$$\rho'^2 \sim e^{2\rho}(e^{-2\rho} - \eta). \tag{2.16}$$

Denoting $u = e^{-\rho}$, we have $u'^2 \sim u^2 - \eta$. Consider the boundary condition $u_0 = e^{-\rho_0}$ at $\sigma = \sigma_0$, one finds

$$\rho(\sigma) = -\ln\bigl(\sqrt{\eta}\cosh(\sigma - \sigma_0)\bigr), \tag{2.17}$$

so that the sinh-Gordon field becomes

$$\alpha \equiv \ln(\partial Y \cdot \bar{\partial} Y) = \ln(2\rho'^2) = \ln(2\tanh^2\sigma). \tag{2.18}$$

This is exactly the one-soliton solution to the generalized sinh-Gordon equation (2.11) with $uv = -4$. Therefore, we conclude that the finite GKP solution is a two-soliton configuration of sinh-Gordon system in which the solitons are located near the boundary of AdS. We will next describe its construction starting from the solutions of the sinh-Gordon system.

2.2 Constructing string solutions from sinh-Gordon solutions

One can next work out the equations obeyed by the elements of the basis, the derivatives of the vectors (2.4) can be expressed in terms of the basis itself

$$\bar{\partial} e_i = A_{ij}(z, \bar{z}) e_j, \qquad \partial e_i = B_{ij}(z, \bar{z}) e_j, \tag{2.19}$$

where the matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \bar{\partial}\alpha & 0 & u \\ e^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & -ue^{-\alpha} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ e^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & \partial\alpha & v \\ 0 & -ve^{-\alpha} & 0 & 0 \end{pmatrix}. \tag{2.20}$$

One finds therefore a linear system of differential equations for the vectors. The integrability condition $\partial A - \bar{\partial} B + [A, B] = 0$ is then seen to generate the equations of motion corresponding to a generalized sinh-Gordon theory. The vector equations on the other hand define the motion (and coordinates) of the string itself, they have to be solved, which leads to a scattering problem of Dirac type. These equations exhibit $SO(2, 2)$ symmetry, and can be further simplified by redefining an orthonormal basis as

$$e_1 = B, \quad e_2 = \frac{\bar{\partial} Y + \partial Y}{\sqrt{2} e^{\alpha/2}}, \quad e_3 = \frac{\bar{\partial} Y - \partial Y}{\sqrt{2} i e^{\alpha/2}}, \quad e_4 = iY. \tag{2.21}$$

One now exploits the fact that $SO(2, 2) = SO(2, 1) \times SO(2, 1)$, expanding the A, B matrices in terms of two commuting sets of $SO(2, 1)$ generators

$$A = w^i_{1,(+)} J_i + w^i_{1,(-)} K_i, \qquad B = w^i_{2,(+)} J_i + w^i_{2,(-)} K_i, \tag{2.22}$$

with $i = 1, 2, 3$ and the coefficient vectors are found to be

$$\vec{w}_{1,(\pm)} = \Big(\frac{i}{2}\bar{\partial}\alpha, \frac{-i}{\sqrt{2}}(ue^{-\alpha/2} \mp e^{\alpha/2}), \frac{-i}{\sqrt{2}}(ue^{-\alpha/2} \pm e^{\alpha/2})\Big), \tag{2.23}$$

$$\vec{w}_{2,(\pm)} = \Big(\frac{-i}{2}\partial\alpha, \frac{i}{\sqrt{2}}(ve^{-\alpha/2} \pm e^{\alpha/2}), \frac{-i}{\sqrt{2}}(ve^{-\alpha/2} \mp e^{\alpha/2})\Big). \tag{2.24}$$

Remembering $SO(2, 1) = SU(1, 1)$, we can rewrite this problem in terms of the spinor representation of the $SU(1, 1)$ group. Defining two spinors ϕ and ψ satisfying the differential equations

$$\bar{\partial}\phi = w^i_{1,(+)}\sigma_i\phi = A_1\phi, \qquad \partial\phi = w^i_{2,(+)}\sigma_i\phi = A_2\phi, \tag{2.25}$$

$$\bar{\partial}\psi = w^i_{1,(-)}\sigma_i\psi = B_1\psi, \qquad \partial\psi = w^i_{2,(-)}\sigma_i\psi = B_2\psi, \tag{2.26}$$

where σ_i are the anti-Hermitian generators of the $SU(1,1)$ group. The spinors ϕ and ψ are normalized $\phi^\dagger\phi = \phi_1^*\phi_1 - \phi_2^*\phi_2 = 1$, $\psi^\dagger\psi = \psi_1^*\psi_1 - \psi_2^*\psi_2 = 1$. The matrices A_1, A_2, B_1, B_2 become

$$A_1 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ue^{-\alpha/2} + e^{\alpha/2}) & \frac{i}{4}\bar{\partial}\alpha - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2} - e^{\alpha/2}) \\ -\frac{i}{4}\bar{\partial}\alpha - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2} - e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ue^{-\alpha/2} + e^{\alpha/2}) \end{pmatrix}, \tag{2.27}$$

$$A_2 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ve^{-\alpha/2} - e^{\alpha/2}) & -\frac{i}{4}\partial\alpha + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2} + e^{\alpha/2}) \\ \frac{i}{4}\partial\alpha + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2} + e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ve^{-\alpha/2} - e^{\alpha/2}) \end{pmatrix}, \tag{2.28}$$

$$B_1 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ue^{-\alpha/2} - e^{\alpha/2}) & \frac{i}{4}\bar{\partial}\alpha - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2} + e^{\alpha/2}) \\ -\frac{i}{4}\bar{\partial}\alpha - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2} + e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ue^{-\alpha/2} - e^{\alpha/2}) \end{pmatrix}, \tag{2.29}$$

$$B_2 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ve^{-\alpha/2} + e^{\alpha/2}) & -\frac{i}{4}\partial\alpha + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2} - e^{\alpha/2}) \\ \frac{i}{4}\partial\alpha + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2} - e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ve^{-\alpha/2} + e^{\alpha/2}) \end{pmatrix}. \tag{2.30}$$

In other words, given a solution $\alpha(z,\bar{z})$, $u(\bar{z})$ and $v(z)$ of the sinh-Gordon equation, we can find ϕ and ψ such that they solve the above linear system. Then the string solution is constructed through

$$Z_1 \equiv Y_{-1} + iY_0 = \phi_1^*\psi_1 - \phi_2^*\psi_2, \tag{2.31}$$

$$Z_2 \equiv Y_1 + iY_2 = \phi_2^*\psi_1^* - \phi_1^*\psi_2^*. \tag{2.32}$$

2.3 Special string solutions

Next let us see some explicit examples. Starting with soliton solutions of the sinh-Gordon, we construct open string solutions which touch the boundary of AdS and the spikes are located in the bulk of AdS. A one-to-one correspondence between solitons of sinh-Gordon and spikes of AdS string is established which turns out to be useful for the construction of general n-spike string solutions.

2.3.1 Vacuum

The first example is the sinh-Gordon vacuum $u = 2$, $v = -2$, $\alpha_0 = \ln 2$, the results of solving the linear system (2.25, 2.26) are

$$\phi_1 = e^{-i\tau}, \quad \phi_2 = 0, \quad \psi_1 = \cosh\sigma, \quad \psi_2 = -\sinh\sigma. \tag{2.33}$$

Then the Minkowski worldsheet solution is given by (see Figure 2.1)

$$Z_1 = e^{i\tau}\cosh\sigma, \tag{2.34}$$

$$Z_2 = e^{i\tau}\sinh\sigma. \tag{2.35}$$

This is the infinite string limit of the spinning string [28].

The Euclidean worldsheet solution is obtained by making a wick rotation $\tau \to -i\tau$. Then Y_0 and Y_2 become imaginary, thus effectively exchanging places. Therefore the

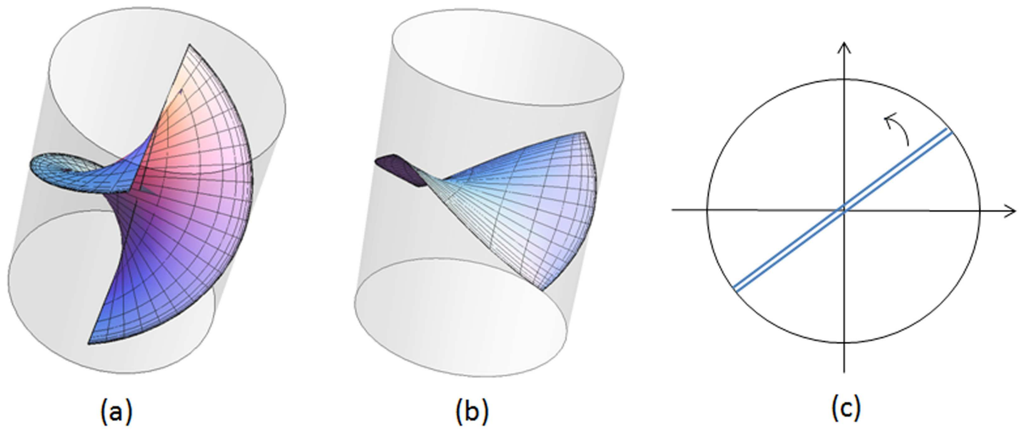

Figure 2.1: The vacuum solution in (a) Minkowskian and (b) Euclidean worldsheet plotted in AdS_3 coordinates. (c) Top view of Minkowskian vacuum solution. The boundary of the worldsheet touches the boundary of AdS space.

Euclidean vacuum solution reads

$$\vec{Y}_E = \begin{pmatrix} \cosh\sigma\cosh\tau \\ \sinh\sigma\sinh\tau \\ \sinh\sigma\cosh\tau \\ \cosh\sigma\sinh\tau \end{pmatrix} . \qquad (2.36)$$

This is the solution found in [26] and used by the authors of [21] to calculate the scattering amplitude for four gluons.

Introducing the global coordinates

$$Y = \begin{pmatrix} \cosh\rho\cos t \\ \cosh\rho\sin t \\ \sinh\rho\cos\theta \\ \sinh\rho\sin\theta \end{pmatrix} , \qquad (2.37)$$

the metric of AdS_3 can be written as

$$ds^2 = -\cosh^2\rho\; dt^2 + d\rho^2 + \sinh^2\rho\; d\theta^2. \tag{2.38}$$

If we denote $\phi^i = (t, \rho, \theta)$, the Lagrangian density becomes

$$\mathcal{L} = \frac{\sqrt{\lambda}}{2\pi}\frac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi, \tag{2.39}$$

so that the energy and angular momentum (densities) can be easily derived as

$$\mathcal{P}_t^\tau = \frac{\partial\mathcal{L}}{\partial\dot{t}} = \frac{\sqrt{\lambda}}{2\pi}\cosh^2\rho\,\dot{t}, \tag{2.40}$$

$$\mathcal{P}_\theta^\tau = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\sqrt{\lambda}}{2\pi}\sinh^2\rho\,\dot{\theta}, \tag{2.41}$$

$$E = \int d\sigma\; \mathcal{P}_t^\tau = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma\cosh^2\rho\,\dot{t}, \tag{2.42}$$

$$S = \int d\sigma\; \mathcal{P}_\theta^\tau = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma\sinh^2\rho\,\dot{\theta}. \tag{2.43}$$

For the vacuum solution, we set $t = \tau$, $\theta = \tau$, $\rho = \sigma$, and the angular momenta are

$$\mathcal{P}_t^\tau = \frac{\sqrt{\lambda}}{2\pi}\cosh^2\sigma, \qquad \mathcal{P}_\theta^\tau = \frac{\sqrt{\lambda}}{2\pi}\sinh^2\sigma. \tag{2.44}$$

The energy and angular momentum are divergent, introducing a ultraviolet cutoff $\Lambda \gg 0$, we have

$$E = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma\cosh^2\sigma = \frac{\sqrt{\lambda}}{\pi}\big(\frac{1}{4}\sinh(2\sigma) + \frac{1}{2}\sigma\big)\big|_{-\Lambda}^{\Lambda} \approx \frac{\sqrt{\lambda}}{4\pi}e^{2\Lambda}, \tag{2.45}$$

$$S = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma\sinh^2\sigma = \frac{\sqrt{\lambda}}{\pi}\big(\frac{1}{4}\sinh(2\sigma) - \frac{1}{2}\sigma\big)\big|_{-\Lambda}^{\Lambda} \approx \frac{\sqrt{\lambda}}{4\pi}e^{2\Lambda}. \tag{2.46}$$

The dispersion relation becomes

$$E - S = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma = \frac{\sqrt{\lambda}}{\pi}2\Lambda \sim \frac{\sqrt{\lambda}}{\pi}\ln\frac{4\pi}{\sqrt{\lambda}}S, \tag{2.47}$$

which agrees with the GKP solution [28].

The n-static spike solution can also be obtained by exploiting a free parameter during the process of solving the Dirac wave-equations. The more general wave-functions are

$$\phi_1 = \cosh\rho_0\; e^{-i\tau}, \quad \phi_2 = \sinh\rho_0\; e^{i\tau}, \quad \psi_1 = \cosh\sigma, \quad \psi_2 = -\sinh\sigma. \tag{2.48}$$

The string solution becomes

$$Y = \begin{pmatrix} \cosh\sigma\cos\tau\cosh\rho_0 + \sinh\sigma\cos\tau\sinh\rho_0 \\ \cosh\sigma\sin\tau\cosh\rho_0 - \sinh\sigma\sin\tau\sinh\rho_0 \\ \sinh\sigma\cos\tau\cosh\rho_0 + \cosh\sigma\cos\tau\sinh\rho_0 \\ \sinh\sigma\sin\tau\cosh\rho_0 - \cosh\sigma\sin\tau\sinh\rho_0 \end{pmatrix}. \tag{2.49}$$

Here ρ_0 has the physical meaning of the number of spikes n through the formula

$$\sinh\rho_0 = \cot\frac{\pi}{n}, \tag{2.50}$$

where $n = 2$ corresponds to the GKP case. The dispersion relation becomes

$$E - S \sim n\frac{\sqrt{\lambda}}{2\pi}\ln S, \tag{2.51}$$

which agrees with the n-spike solution by Kruczenski [31].

2.3.2 One-spike solutions

The solutions obtained in the previous subsection are static in the rotating frame. In this subsection, we will describe dynamical string solutions corresponding to one-soliton solution of the sinh-Gordon equation.

Starting with the one-soliton solution

$$\alpha_s = \ln 2 + \ln(\tanh^2 \sigma). \tag{2.52}$$

The matrices entering into the linear system (2.25, 2.26) are given by

$$A_1 = \begin{pmatrix} -i \coth 2\sigma & (i-1)\text{csch}\, 2\sigma \\ -(i+1)\text{csch}\, 2\sigma & i \coth 2\sigma \end{pmatrix}, \tag{2.53}$$

$$A_2 = \begin{pmatrix} i \coth 2\sigma & -(i+1)\text{csch}\, 2\sigma \\ (i-1)\text{csch}\, 2\sigma & -i \coth 2\sigma \end{pmatrix}, \tag{2.54}$$

$$B_1 = \begin{pmatrix} -i\text{csch}\, 2\sigma & i\text{csch}\, 2\sigma - \coth 2\sigma \\ -i\text{csch}\, 2\sigma - \coth 2\sigma & i\text{csch} 2\sigma \end{pmatrix}, \tag{2.55}$$

$$B_2 = \begin{pmatrix} i\text{csch}\, 2\sigma & -i\text{csch}\, 2\sigma - \coth 2\sigma \\ i\text{csch}\, 2\sigma - \coth 2\sigma & -i\text{csch}\, 2\sigma \end{pmatrix}. \tag{2.56}$$

The spinors that solve the linear system are

$$\phi_1 = e^{-i\tau} \cosh(\frac{1}{2} \ln \tanh \sigma), \tag{2.57}$$

$$\phi_2 = -e^{-i\tau} \sinh(\frac{1}{2} \ln \tanh \sigma), \tag{2.58}$$

$$\psi_1 = (\tau + i) \cosh(\frac{1}{2} \ln \sinh 2\sigma) - \tau \sinh(\frac{1}{2} \ln \sinh 2\sigma), \tag{2.59}$$

$$\psi_2 = -(\tau + i) \sinh(\frac{1}{2} \ln \sinh 2\sigma) + \tau \cosh(\frac{1}{2} \ln \sinh 2\sigma). \tag{2.60}$$

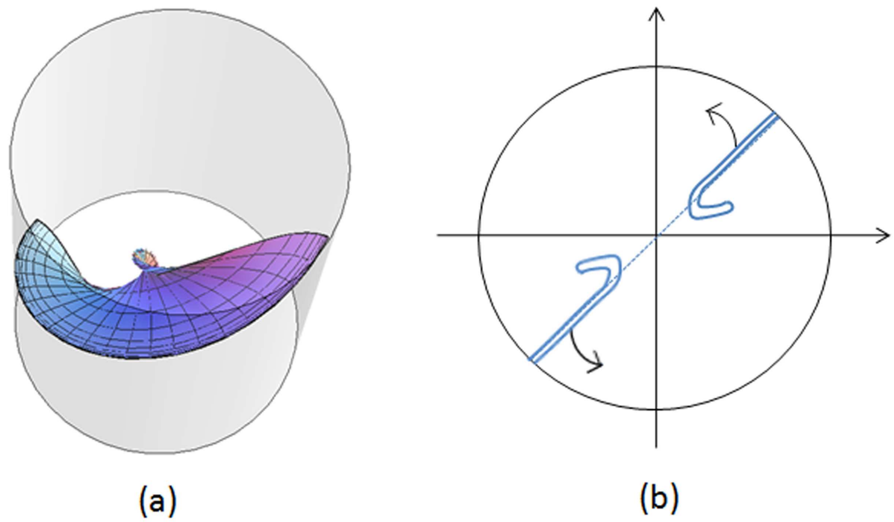

Figure 2.2: The one-soliton solution in (a) Minkowskian worldsheet plotted in AdS_3 coordinates. (b) Top view of the Minkowskian one-soliton solution. Please note the curvature of the string changes with the evolution of time.

Then we use (2.31, 2.32) to find the corresponding string solution (see Figure 2.2)

$$Z_1^s = \frac{e^{i\tau}}{2\sqrt{2}\cosh\sigma}\big(2\tau + i(\cosh 2\sigma + 2)\big), \tag{2.61}$$

$$Z_2^s = \frac{e^{i\tau}}{2\sqrt{2}\cosh\sigma}\big(-2\tau - i\cosh 2\sigma\big). \tag{2.62}$$

One can easily compute the energy and angular momentum

$$E = \int_{-\Lambda}^{\Lambda} d\sigma \frac{\sqrt{\lambda}(1 + 8\tau^2 + 4\cosh 2\sigma + \cosh 4\sigma)}{16\pi\cosh^2\sigma} \approx \frac{\sqrt{\lambda}}{\pi}(\frac{1}{8}e^{2\Lambda} + \tau^2), \tag{2.63}$$

$$S = \int_{-\Lambda}^{\Lambda} d\sigma \frac{\sqrt{\lambda}(1 + 8\tau^2 - 4\cosh 2\sigma + \cosh 4\sigma)}{16\pi\cosh^2\sigma} \approx \frac{\sqrt{\lambda}}{\pi}(\frac{1}{8}e^{2\Lambda} + \tau^2). \tag{2.64}$$

If we neglect the τ dependence since the exponential term is much larger than the square term, we have

$$E - S = \int_{-\Lambda}^{\Lambda} \frac{\sqrt{\lambda}}{2\pi}\cosh 2\sigma \,\mathrm{sech}^2\sigma d\sigma \sim \frac{\sqrt{\lambda}}{\pi}\ln\frac{8\pi}{\sqrt{\lambda}}S. \tag{2.65}$$

The energy is not conserved because there is momentum flow at the asymptotic ends of the string and the string itself is not closed.

Similarly, the one-antisoliton string solution corresponding to $\alpha_{\bar{s}}$ is given by

$$Z_1^{\bar{s}} = \frac{e^{i\tau}}{2\sqrt{2}\sinh\sigma}\big(2\tau - i\cosh 2\sigma\big), \tag{2.66}$$

$$Z_2^{\bar{s}} = \frac{e^{i\tau}}{2\sqrt{2}\sinh\sigma}\big(-2\tau + i(\cosh 2\sigma - 2)\big). \tag{2.67}$$

Notice this solution is singular at the point $\sigma = 0$.

2.3.3 Two-spike solutions

The two-soliton solutions of the sinh-Gordon equation can be obtained via the Bäcklund transformation: Let φ be a solution of the sinh-Gordon model (here $\varphi = (\alpha - \ln 2)/2$), its image under a Bäcklund transformation with spectral parameter μ is the field $\hat{\varphi} = B_\mu \cdot \varphi$ implicitly defined by the following differential equations

$$\partial(\hat{\varphi} + \varphi) = 2\mu\ \sinh(\hat{\varphi} - \varphi) \tag{2.68}$$

$$\bar{\partial}(\hat{\varphi} - \varphi) = 2\mu^{-1}\sinh(\hat{\varphi} + \varphi) \tag{2.69}$$

If φ solves the sinh-Gordon equation, so does the transformed field $\hat{\varphi}$. A remarkable property of the Bäcklund transformation is that the four solutions φ_0, φ_1, φ_2, φ_3 are linked by a purely algebraic relation

$$\tanh\Big(\frac{\varphi_3 - \varphi_0}{2}\Big) = \Big(\frac{\mu_1 + \mu_2}{\mu_1 - \mu_2}\Big)\tanh\Big(\frac{\varphi_2 - \varphi_1}{2}\Big). \tag{2.70}$$

This is called the tangent rule.

Starting with following two one-soliton solutions

$$\alpha_1 = \ln(2\tanh^2 \frac{\sigma - v_1\tau}{\sqrt{1-v_1^2}}), \qquad \alpha_2 = \ln(2\coth^2 \frac{\sigma - v_2\tau}{\sqrt{1-v_2^2}}) \tag{2.71}$$

and let $v_1 = v, v_2 = -v$, we get the two-soliton solution

$$\alpha_{ss,\bar{s}\bar{s}} = \ln 2 \pm \ln\Big[\frac{v\cosh X - \cosh T}{v\cosh X + \cosh T}\Big]^2 \tag{2.72}$$

where $X \equiv 2\gamma\sigma$, $T \equiv 2v\gamma\tau$. For α_{ss}, the spinors are

$$\phi_1 = e^{i\tau}\frac{\gamma^{-1}\cosh T + iv\sinh T}{\sqrt{\cosh^2 T - v^2\cosh^2 X}} \tag{2.73}$$

$$\phi_2 = e^{i\tau}\frac{v\sinh X}{\sqrt{\cosh^2 T - v^2\cosh^2 X}} \tag{2.74}$$

$$\psi_1 = \frac{(\gamma^{-1}\cosh X + i\sinh T)\cosh\sigma - \sinh X\sinh\sigma}{\sqrt{\cosh^2 T - v^2\cosh^2 X}} \tag{2.75}$$

$$\psi_2 = \frac{(-\gamma^{-1}\cosh X + i\sinh T)\sinh\sigma + \sinh X\cosh\sigma}{\sqrt{\cosh^2 T - v^2\cosh^2 X}} \tag{2.76}$$

The two-soliton string solution is

$$Z_1^{ss} = e^{-i\tau}\frac{v\mathrm{ch}T\mathrm{ch}\sigma + \mathrm{ch}X\mathrm{ch}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{sh}\sigma + i\gamma^{-1}\mathrm{sh}T\mathrm{ch}\sigma}{\mathrm{ch}T + v\mathrm{ch}X}, \tag{2.77}$$

$$Z_2^{ss} = e^{-i\tau}\frac{v\mathrm{ch}T\mathrm{sh}\sigma + \mathrm{ch}X\mathrm{sh}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{ch}\sigma + i\gamma^{-1}\mathrm{sh}T\mathrm{sh}\sigma}{\mathrm{ch}T + v\mathrm{ch}X}. \tag{2.78}$$

Figure 2.3 shows the shape of the two-soliton string at two different global time instants. In Figure 2.3(a), the string is folded along the x axis, whereas in Figure 2.3(b), we find the usual bulk spikes.

The two-antisoliton string solution can be constructed in the same way and it only differs from the two-soliton solution by three signs: the second and third terms

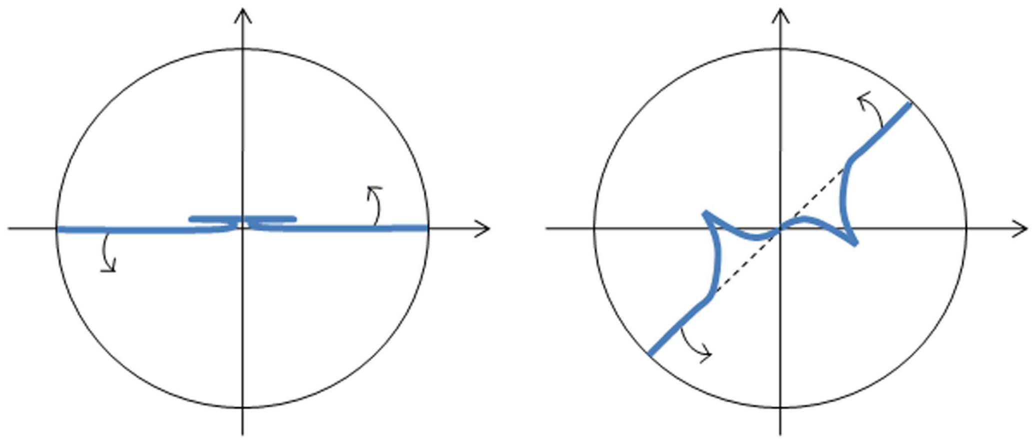

Figure 2.3: The Minkowskian two-soliton solution with $v = \frac{1}{\sqrt{5}}$ at different global time (a) $t = 0$, (b) $t = \pi/4$.

in the numerator and the second term in the denominator which makes the solution singular.

Secondly, choose the other two one-soliton solutions

$$\alpha_1 = \ln(2\tanh^2 \frac{\sigma - v_1\tau}{\sqrt{1-v_1^2}}), \qquad \alpha_2 = \ln(2\tanh^2 \frac{\sigma - v_2\tau}{\sqrt{1-v_2^2}}) \tag{2.79}$$

and let $v_1 = v, v_2 = -v$, we get the soliton-antisoliton solution

$$\alpha_{s\bar{s},\bar{s}s} = \ln 2 \pm \ln\Big[\frac{v\sinh X - \sinh T}{v\sinh X + \sinh T}\Big]^2. \tag{2.80}$$

For the soliton-antisoliton solution, the string solution is

$$Z_1^{s\bar{s}} = e^{-i\tau}\frac{v\mathrm{sh}T\mathrm{ch}\sigma \pm \mathrm{sh}X\mathrm{ch}\sigma \mp \sqrt{1-v^2}\mathrm{ch}X\mathrm{sh}\sigma + i\sqrt{1-v^2}\mathrm{ch}T\mathrm{ch}\sigma}{\mathrm{sh}T \pm v\mathrm{sh}X}, \tag{2.81}$$

$$Z_2^{s\bar{s}} = e^{-i\tau}\frac{v\mathrm{sh}T\mathrm{sh}\sigma \pm \mathrm{sh}X\mathrm{sh}\sigma \mp \sqrt{1-v^2}\mathrm{ch}X\mathrm{ch}\sigma + i\sqrt{1-v^2}\mathrm{ch}T\mathrm{sh}\sigma}{\mathrm{sh}T \pm v\mathrm{sh}X}. \tag{2.82}$$

Similarly, the above solutions are singular.

In (2.80), take v to be imaginary: $v = iw$, we get the breather solution to the sinh-Gordon equation

$$\alpha_{B,\pm} = \ln 2 \pm \ln\Big[\frac{w\sinh X_B - \sin T_B}{w\sinh X_B + \sin T_B}\Big]^2 \tag{2.83}$$

where $X_B \equiv \frac{2\sigma}{\sqrt{1+w^2}}$, $T_B \equiv \frac{2w\tau}{\sqrt{1+w^2}}$. The string solutions corresponding to the breathers are

$$\begin{aligned} Z_1^B &= \frac{e^{-i\tau}}{\sin T_B \pm w\mathrm{sh}X_B}\big\{-w\sin T_B\mathrm{sh}\sigma \pm \mathrm{sh}X_B\mathrm{sh}\sigma \\ &\quad \mp\sqrt{1+w^2}\mathrm{ch}X_B\mathrm{ch}\sigma + i\sqrt{1+w^2}\cos T_B\mathrm{sh}\sigma\big\}, \qquad (2.84) \\ Z_2^B &= \frac{e^{-i\tau}}{\sin T_B \pm w\mathrm{sh}X_B}\big\{-w\sin T_B\mathrm{ch}\sigma \pm \mathrm{sh}X_B\mathrm{ch}\sigma \\ &\quad \mp\sqrt{1+w^2}\mathrm{ch}X_B\mathrm{sh}\sigma + i\sqrt{1+w^2}\cos T_B\mathrm{ch}\sigma\big\}. \qquad (2.85) \end{aligned}$$

All these solutions are singular.

2.4 General n-spike construction

We now turn to the general construction of the AdS_3 case. We had the fact that the classical strings in AdS_3 can be reduced to a generalized sinh-Gordon model coupled to two arbitrary functions $u(\bar{z})$ and $v(z)$ which together represent a free scalar field. These functions are central to the string theory interpretation of the sinh-Gordon equation, they represent the freedom of performing general conformal transformations which are the symmetry of the conformal gauge string. In the system of coupled equations describing the Lax pair they can be transformed by a combination of conformal and gauge transformations.

After the conformal change of variables (2.13) and omitting the primes of the new fields, we can write down the two Lax pairs as

$$\bar{\partial}\phi(z,\bar{z}) = A_1\phi(z,\bar{z}), \qquad \bar{\partial}\psi(z,\bar{z}) = B_1\psi(z,\bar{z}), \tag{2.86}$$

$$\partial\phi(z,\bar{z}) = A_2\phi(z,\bar{z}), \qquad \partial\psi(z,\bar{z}) = B_2\psi(z,\bar{z}), \tag{2.87}$$

where the matrices are given by

$$A_1 = \frac{1}{4}\begin{pmatrix} -i\lambda c_1^+ & i\bar{\partial}\hat{\alpha} + \frac{i}{2}\frac{u'(\bar{z})}{u(\bar{z})} - \lambda c_1^- \\ -i\bar{\partial}\hat{\alpha} - \frac{i}{2}\frac{u'(\bar{z})}{u(\bar{z})} - \lambda c_1^- & i\lambda c_1^+ \end{pmatrix}, \tag{2.88}$$

$$A_2 = \frac{1}{4}\begin{pmatrix} i\frac{1}{\lambda}c_2^+ & -i\partial\hat{\alpha} - \frac{i}{2}\frac{v'(z)}{v(z)} - \frac{1}{\lambda}c_2^- \\ i\partial\hat{\alpha} + \frac{i}{2}\frac{v'(z)}{v(z)} - \frac{1}{\lambda}c_2^- & -i\frac{1}{\lambda}c_2^+ \end{pmatrix}, \tag{2.89}$$

$$B_1 = \frac{1}{4}\begin{pmatrix} -i\lambda c_1^- & i\bar{\partial}\hat{\alpha} + \frac{i}{2}\frac{u'(\bar{z})}{u(\bar{z})} - \lambda c_1^+ \\ -i\bar{\partial}\hat{\alpha} - \frac{i}{2}\frac{u'(\bar{z})}{u(\bar{z})} - \lambda c_1^+ & i\lambda c_1^- \end{pmatrix}, \tag{2.90}$$

$$B_2 = \frac{1}{4}\begin{pmatrix} i\frac{1}{\lambda}c_2^- & -i\partial\hat{\alpha} - \frac{i}{2}\frac{v'(z)}{v(z)} - \frac{1}{\lambda}c_2^+ \\ i\partial\hat{\alpha} + \frac{i}{2}\frac{v'(z)}{v(z)} - \frac{1}{\lambda}c_2^+ & -i\frac{1}{\lambda}c_2^- \end{pmatrix}, \tag{2.91}$$

with definitions for simpler expressions

$$c_1^+ \equiv \sqrt[4]{\frac{u}{-v}}e^{-\frac{1}{2}\hat{\alpha}} + \sqrt[4]{\frac{-v}{u}}e^{\frac{1}{2}\hat{\alpha}}, \qquad c_1^- \equiv \sqrt[4]{\frac{u}{-v}}e^{-\frac{1}{2}\hat{\alpha}} - \sqrt[4]{\frac{-v}{u}}e^{\frac{1}{2}\hat{\alpha}}, \tag{2.92}$$

$$c_2^+ \equiv \sqrt[4]{\frac{-v}{u}}e^{-\frac{1}{2}\hat{\alpha}} + \sqrt[4]{\frac{u}{-v}}e^{\frac{1}{2}\hat{\alpha}}, \qquad c_2^- \equiv \sqrt[4]{\frac{-v}{u}}e^{-\frac{1}{2}\hat{\alpha}} - \sqrt[4]{\frac{u}{-v}}e^{\frac{1}{2}\hat{\alpha}}. \tag{2.93}$$

Here we introduced the spectral parameter by rescaling $z \to \lambda z$, $\bar{z} \to \frac{1}{\lambda}\bar{z}$, which is the standard way of introducing the spectral parameter in the Lax formulation.

Our next task is to establish a relationship between the Lax pairs (2.88-2.91) found in the sigma model and the standard Lax pair of the sinh-Gordon theory

$$U = \begin{pmatrix} -i\zeta & \frac{1}{2}\bar{\partial}\hat{u} \\ \frac{1}{2}\bar{\partial}\hat{u} & i\zeta \end{pmatrix}, \qquad V = \frac{i}{4\zeta}\begin{pmatrix} \cosh\hat{u} & -\sinh\hat{u} \\ \sinh\hat{u} & -\cosh\hat{u} \end{pmatrix}, \tag{2.94}$$

which satisfies the Dirac equations

$$\bar{\partial}\varphi(z,\bar{z}) = U\varphi(z,\bar{z}), \qquad \partial\varphi(z,\bar{z}) = V\varphi(z,\bar{z}). \tag{2.95}$$

Here ζ is the spectral parameter and the sinh-Gordon field $\hat{u}$ satisfies the equation

$$\partial\bar{\partial}\hat{u}(z,\bar{z}) - \sinh\hat{u}(z,\bar{z}) = 0. \tag{2.96}$$

Defining the gauge transformation as

$$A_1 = g_A^{-1}(U - \bar{\partial})g_A, \qquad A_2 = g_A^{-1}(V - \partial)g_A, \tag{2.97}$$

for the A matrices, the transformation matrix is found to be

$$g_A = \frac{\sqrt{i}}{2}\begin{pmatrix} i\left(\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}} - \sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\right) & \left(\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}} + \sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\right) \\ i\left(\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}} + \sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\right) & \left(\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}} - \sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\right) \end{pmatrix}, \tag{2.98}$$

with the identification $\hat{u} = -\hat{\alpha}, \lambda = -2\zeta$. Similarly, for the B matrices, we find

$$g_B = -\frac{i}{2}\begin{pmatrix} i\left(\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}} + i\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\right) & \left(\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}} - i\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\right) \\ i\left(\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}} - i\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\right) & \left(\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}} + i\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\right) \end{pmatrix}, \tag{2.99}$$

with the identification $\hat{u} = -\hat{\alpha}, \lambda = -2i\zeta$.

The general solution to the sinh-Gordon equation (2.96) with n solitons can be obtained using the inverse scattering method [43] (see appendix A for details)

$$\hat{u}(z,\bar{z}) = \sinh^{-1}\Big[\frac{4\zeta}{i}\frac{\partial(\varphi_1\varphi_2)}{(\varphi_1)^2-(\varphi_2)^2}\Big], \tag{2.100}$$

where the components of spinor φ are

$$\varphi_1(\zeta,z,\bar{z}) = -\Big(\sum_{j,l=1}^{n}\frac{\lambda_j}{\zeta+\zeta_j}(1-A)^{-1}_{jl}\lambda_l\Big)e^{i\zeta\bar{z}-iz/4\zeta}, \tag{2.101}$$

$$\varphi_2(\zeta,z,\bar{z}) = \Big(1+\sum_{j,l,k=1}^{n}\frac{\lambda_j}{\zeta+\zeta_j}\frac{\lambda_j\lambda_l}{\zeta_j+\zeta_l}(1-A)^{-1}_{lk}\lambda_k\Big)e^{i\zeta\bar{z}-iz/4\zeta}, \tag{2.102}$$

with the definitions

$$A_{ij} = \sum_l a_{il}a_{lj}, \qquad a_{il} = \frac{\lambda_i\lambda_l}{\zeta_i+\zeta_l}, \qquad \lambda_k = \sqrt{c_k(0)}e^{i\zeta_k\bar{z}-iz/4\zeta_k}. \tag{2.103}$$

Here $c_k(0)$ and ζ_k are two sets of constants related to the initial positions and momenta of the n solitons.

Starting with the simplest case with $n=1$, we have the spinor φ

$$\varphi_1 = \frac{c_1(0)(2\zeta_1)^2e^{2i\zeta_1\bar{z}+iz/2\zeta_1}}{(\zeta+\zeta_1)(c_1^2(0)e^{4i\zeta_1\bar{z}}-4\zeta_1^2e^{iz/\zeta_1})}e^{i\zeta\bar{z}-iz/4\zeta}, \tag{2.104}$$

$$\varphi_2 = \left[1-\frac{c_1^2(0)(2\zeta_1)e^{4i\zeta_1\bar{z}}}{(\zeta+\zeta_1)(c_1^2(0)e^{4i\zeta_1\bar{z}}-4\zeta_1^2e^{iz/\zeta_1})}\right]e^{i\zeta\bar{z}-iz/4\zeta}. \tag{2.105}$$

Plugging into (2.100), we get the sinh-Gordon field

$$\hat{u}(z,\bar{z}) = -\sinh^{-1}\left[\frac{8c_1(0)\zeta_1(c_1^2(0)e^{4i\zeta_1\bar{z}}+4\zeta_1^2e^{iz/\zeta_1})e^{2i\zeta_1\bar{z}+iz/2\zeta_1}}{(c_1^2(0)e^{4i\zeta_1\bar{z}}-4\zeta_1^2e^{iz/\zeta_1})^2}\right], \tag{2.106}$$

where $c_1(0)$ and ζ_1 are purely imaginary in order to make the field real.

Next we proceed to write down the first spinor ϕ

$$\phi = g_A^{-1}\varphi, \qquad \hat{u} = -\hat{\alpha}, \qquad \zeta = -\lambda/2, \tag{2.107}$$

and find

$$\begin{aligned}
\phi_1 &= \frac{(1+i)}{2\sqrt{2}} e^{-\frac{1}{2}(i\lambda\bar{z} - iz/\lambda)} \Big\{ \sqrt[8]{\frac{-v}{u}} e^{\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2\zeta_1+\lambda)e^{2i\zeta_1\bar{z}} - 2\zeta_1(2\zeta_1-\lambda)e^{iz/2\zeta_1})}{(2\zeta_1-\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} + 2\zeta_1 e^{iz/2\zeta_1})} \\
&\quad + \sqrt[8]{\frac{u}{-v}} e^{-\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2\zeta_1+\lambda)e^{2i\zeta_1\bar{z}} + 2\zeta_1(2\zeta_1-\lambda)e^{iz/2\zeta_1})}{(2\zeta_1-\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} - 2\zeta_1 e^{iz/2\zeta_1})} \Big\}, \qquad (2.108) \\
\phi_2 &= \frac{(1-i)}{2\sqrt{2}} e^{-\frac{1}{2}(i\lambda\bar{z} - iz/\lambda)} \Big\{ -\sqrt[8]{\frac{-v}{u}} e^{\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2\zeta_1+\lambda)e^{2i\zeta_1\bar{z}} - 2\zeta_1(2\zeta_1-\lambda)e^{iz/2\zeta_1})}{(2\zeta_1-\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} + 2\zeta_1 e^{iz/2\zeta_1})} \\
&\quad + \sqrt[8]{\frac{u}{-v}} e^{-\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2\zeta_1+\lambda)e^{2i\zeta_1\bar{z}} + 2\zeta_1(2\zeta_1-\lambda)e^{iz/2\zeta_1})}{(2\zeta_1-\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} - 2\zeta_1 e^{iz/2\zeta_1})} \Big\}. \qquad (2.109)
\end{aligned}$$

For real λ, the components of the spinor ϕ are normalized to be $\phi_1^*\phi_1 - \phi_2^*\phi_2 = 1$. Similarly, for the second spinor ψ, we find

$$\begin{aligned}
\psi_1 &= \frac{1}{2\sqrt{2}} \left[e^{-\frac{1}{2}(\lambda\bar{z} + z/\lambda)} a_1 + e^{\frac{1}{2}(\lambda\bar{z} + z/\lambda)} b_1 \right], \qquad (2.110) \\
\psi_2 &= \frac{1}{2\sqrt{2}} \left[e^{-\frac{1}{2}(\lambda\bar{z} + z/\lambda)} a_2 - e^{\frac{1}{2}(\lambda\bar{z} + z/\lambda)} b_2 \right], \qquad (2.111)
\end{aligned}$$

where

$$\begin{aligned}
a_1 &\equiv \Big\{ \sqrt[8]{\frac{-v}{u}} e^{\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2i\zeta_1+\lambda)e^{2i\zeta_1\bar{z}} - 2\zeta_1(2i\zeta_1-\lambda)e^{iz/2\zeta_1})}{(2i\zeta_1-\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} + 2\zeta_1 e^{iz/2\zeta_1})} \\
&\quad + i\sqrt[8]{\frac{u}{-v}} e^{-\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2i\zeta_1+\lambda)e^{2i\zeta_1\bar{z}} + 2\zeta_1(2i\zeta_1-\lambda)e^{iz/2\zeta_1})}{(2i\zeta_1-\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} - 2\zeta_1 e^{iz/2\zeta_1})} \Big\}, \qquad (2.112) \\
b_1 &\equiv \Big\{ i\sqrt[8]{\frac{-v}{u}} e^{\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2i\zeta_1-\lambda)e^{2i\zeta_1\bar{z}} - 2\zeta_1(2i\zeta_1+\lambda)e^{iz/2\zeta_1})}{(2i\zeta_1+\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} + 2\zeta_1 e^{iz/2\zeta_1})} \\
&\quad + \sqrt[8]{\frac{u}{-v}} e^{-\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2i\zeta_1-\lambda)e^{2i\zeta_1\bar{z}} + 2\zeta_1(2i\zeta_1+\lambda)e^{iz/2\zeta_1})}{(2i\zeta_1+\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} - 2\zeta_1 e^{iz/2\zeta_1})} \Big\}, \qquad (2.113) \\
a_2 &\equiv \Big\{ i\sqrt[8]{\frac{-v}{u}} e^{\frac{1}{4}\hat{\alpha}} \frac{(c_1(0)(2i\zeta_1+\lambda)e^{2i\zeta_1\bar{z}} - 2\zeta_1(2i\zeta_1-\lambda)e^{iz/2\zeta_1})}{(2i\zeta_1-\lambda)(c_1(0)e^{2i\zeta_1\bar{z}} + 2\zeta_1 e^{iz/2\zeta_1})}
\end{aligned}$$

$$+\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}}\frac{(c_1(0)(2i\zeta_1+\lambda)e^{2i\zeta_1\bar{z}}+2\zeta_1(2i\zeta_1-\lambda)e^{iz/2\zeta_1})}{(2i\zeta_1-\lambda)(c_1(0)e^{2i\zeta_1\bar{z}}-2\zeta_1 e^{iz/2\zeta_1})}\Big\}, \quad (2.114)$$

$$b_2 \equiv \Big\{\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\frac{(c_1(0)(2i\zeta_1-\lambda)e^{2i\zeta_1\bar{z}}-2\zeta_1(2i\zeta_1+\lambda)e^{iz/2\zeta_1})}{(2i\zeta_1+\lambda)(c_1(0)e^{2i\zeta_1\bar{z}}+2\zeta_1 e^{iz/2\zeta_1})}$$
$$+i\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}}\frac{(c_1(0)(2i\zeta_1-\lambda)e^{2i\zeta_1\bar{z}}+2\zeta_1(2i\zeta_1+\lambda)e^{iz/2\zeta_1})}{(2i\zeta_1+\lambda)(c_1(0)e^{2i\zeta_1\bar{z}}-2\zeta_1 e^{iz/2\zeta_1})}\Big\}. \quad (2.115)$$

For real λ, the components of the spinor ψ are normalized to be $\psi_1^*\psi_1 - \psi_2^*\psi_2 = 1$. Recalling the change of variables (2.13), the one-spike string solution is then found to be

$$Z_1 = \frac{e^{\frac{1+i}{2}(i\lambda\bar{z}'-z'/\lambda)}}{2(c_1(0)e^{2i\zeta_1\bar{z}'}-2\zeta_1 e^{iz'/2\zeta_1})}\Big\{2\zeta_1 e^{iz'/2\zeta_1}(1+e^{\lambda\bar{z}'+z'/\lambda})$$
$$+c_1(0)e^{2i\zeta_1\bar{z}'}\frac{(2\zeta_1-\lambda)((2i\zeta_1+\lambda)^2+e^{\lambda\bar{z}'+z'/\lambda}(2i\zeta_1-\lambda)^2)}{(2\zeta_1+\lambda)(4\zeta_1^2+\lambda^2)}\Big\}, \quad (2.116)$$

$$Z_2 = \frac{ie^{\frac{1+i}{2}(i\lambda\bar{z}'-z'/\lambda)}}{2(c_1(0)e^{2i\zeta_1\bar{z}'}-2\zeta_1 e^{iz'/2\zeta_1})}\Big\{2\zeta_1 e^{iz'/2\zeta_1}(1-e^{\lambda\bar{z}'+z'/\lambda})$$
$$+c_1(0)e^{2i\zeta_1\bar{z}'}\frac{(2\zeta_1-\lambda)((2i\zeta_1+\lambda)^2-e^{\lambda\bar{z}'+z'/\lambda}(2i\zeta_1-\lambda)^2)}{(2\zeta_1+\lambda)(4\zeta_1^2+\lambda^2)}\Big\}, \quad (2.117)$$

It is interesting to note that $u(\bar{z})$ and $v(z)$ only come into $\bar{z}'$ and z', respectively. This is the residual conformal symmetry which can be further used to fix the time-like conformal gauge.

Now we consider the general case with arbitrary number n solitons. The first spinor ϕ is solved to be

$$\phi_1 = -\frac{(1+i)}{2\sqrt{2}}e^{-\frac{1}{2}(i\lambda\bar{z}-iz/\lambda)}\Big\{\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}(\tilde{\varphi}_2-\tilde{\varphi}_1)^1_- + \sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}}(\tilde{\varphi}_2+\tilde{\varphi}_1)^1_-\Big\},$$
$$\phi_2 = +\frac{(1-i)}{2\sqrt{2}}e^{-\frac{1}{2}(i\lambda\bar{z}-iz/\lambda)}\Big\{\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}(\tilde{\varphi}_2-\tilde{\varphi}_1)^1_- - \sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}}(\tilde{\varphi}_2+\tilde{\varphi}_1)^1_-\Big\},$$

where

$$(\tilde{\varphi}_2\pm\tilde{\varphi}_1)^1_\pm = 1\pm\sum_{j,l}\frac{\lambda_j}{\pm\frac{\lambda}{2}+\zeta_j}(1-A)^{-1}_{jl}\lambda_l+\sum_{j,l,k}\frac{\lambda_j}{\pm\frac{\lambda}{2}+\zeta_j}\frac{\lambda_j\lambda_l}{\zeta_j+\zeta_l}(1-A)^{-1}_{lk}\lambda_k. \quad (2.118)$$

The subscript $\pm$ corresponds to the $\pm$ before the spectral parameter λ. The second spinor ψ is solved to be

$$\begin{aligned}\psi_1 &= -\frac{1}{2\sqrt{2}}\Big\{\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\Big[e^{-\frac{1}{2}(\lambda\bar{z}+z/\lambda)}(\tilde{\varphi}_2-\tilde{\varphi}_1)_+^2+ie^{\frac{1}{2}(\lambda\bar{z}+z/\lambda)}(\tilde{\varphi}_2-\tilde{\varphi}_1)_-^2\Big]\\ &\quad +i\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}}\Big[e^{-\frac{1}{2}(\lambda\bar{z}+z/\lambda)}(\tilde{\varphi}_2+\tilde{\varphi}_1)_+^2-ie^{\frac{1}{2}(\lambda\bar{z}+z/\lambda)}(\tilde{\varphi}_2+\tilde{\varphi}_1)_-^2\Big], \quad (2.119)\\ \psi_2 &= -\frac{i}{2\sqrt{2}}\Big\{\sqrt[8]{\frac{-v}{u}}e^{\frac{1}{4}\hat{\alpha}}\Big[e^{-\frac{1}{2}(\lambda\bar{z}+z/\lambda)}(\tilde{\varphi}_2-\tilde{\varphi}_1)_+^2+ie^{\frac{1}{2}(\lambda\bar{z}+z/\lambda)}(\tilde{\varphi}_2-\tilde{\varphi}_1)_-^2\Big]\\ &\quad -i\sqrt[8]{\frac{u}{-v}}e^{-\frac{1}{4}\hat{\alpha}}\Big[e^{-\frac{1}{2}(\lambda\bar{z}+z/\lambda)}(\tilde{\varphi}_2+\tilde{\varphi}_1)_+^2-ie^{\frac{1}{2}(\lambda\bar{z}+z/\lambda)}(\tilde{\varphi}_2+\tilde{\varphi}_1)_-^2\Big], \quad (2.120)\end{aligned}$$

where

$$(\tilde{\varphi}_2\pm\tilde{\varphi}_1)_\pm^2=1\pm\sum_{j,l}\frac{\lambda_j}{\pm\frac{i\lambda}{2}+\zeta_j}(1-A)_{jl}^{-1}\lambda_l+\sum_{j,l,k}\frac{\lambda_j}{\pm\frac{i\lambda}{2}+\zeta_j}\frac{\lambda_j\lambda_l}{\zeta_j+\zeta_l}(1-A)_{lk}^{-1}\lambda_k. \quad (2.121)$$

Similar to (2.118), the subscript $\pm$ corresponds to the $\pm$ before the spectral parameter λ. The n-spike string solution is given by

$$\begin{aligned}Z_1 &= \frac{1-i}{4}e^{\frac{1}{2}(i\lambda\bar{z}'-iz'/\lambda)}\Big\{i(\tilde{\varphi}_2-\tilde{\varphi}_1)_+^1\Big[e^{-\frac{1}{2}(\lambda\bar{z}'+z'/\lambda)}(\tilde{\varphi}_2+\tilde{\varphi}_1)_+^2-ie^{\frac{1}{2}(\lambda\bar{z}'+z'/\lambda)}(\tilde{\varphi}_2+\tilde{\varphi}_1)_-^2\Big]\\ &\quad +(\tilde{\varphi}_2+\tilde{\varphi}_1)_+^1\Big[e^{-\frac{1}{2}(\lambda\bar{z}'+z'/\lambda)}(\tilde{\varphi}_2-\tilde{\varphi}_1)_+^2+ie^{\frac{1}{2}(\lambda\bar{z}'+z'/\lambda)}(\tilde{\varphi}_2-\tilde{\varphi}_1)_-^2\Big]\Big\}, \quad (2.122)\\ Z_2 &= \frac{1+i}{4}e^{\frac{1}{2}(i\lambda\bar{z}'-iz'/\lambda)}\Big\{i(\tilde{\varphi}_2-\tilde{\varphi}_1)_+^1\Big[e^{-\frac{1}{2}(\lambda\bar{z}'+z'/\lambda)}(\tilde{\varphi}_2+\tilde{\varphi}_1)_+^2+ie^{\frac{1}{2}(\lambda\bar{z}'+z'/\lambda)}(\tilde{\varphi}_2+\tilde{\varphi}_1)_-^2\Big]\\ &\quad +(\tilde{\varphi}_2+\tilde{\varphi}_1)_+^1\Big[e^{-\frac{1}{2}(\lambda\bar{z}'+z'/\lambda)}(\tilde{\varphi}_2-\tilde{\varphi}_1)_+^2-ie^{\frac{1}{2}(\lambda\bar{z}'+z'/\lambda)}(\tilde{\varphi}_2-\tilde{\varphi}_1)_-^2\Big]\Big\}. \quad (2.123)\end{aligned}$$

Let us summarize the properties of the general solution we just constructed. This general string configuration is characterized by two arbitrary functions $u(\bar{z})$, $v(z)$ and a discrete set of moduli representing the soliton singularities (coordinates). After fixing the conformal frame only the soliton moduli remain giving a specification of the dynamical string moduli. These represent general motions of the spikes and their locations.

2.5 Summary

We have introduced a picture where the soliton solutions are seen to be in a one-to-one correspondence with the spike solutions of the AdS string. This picture leads to some simple classical solutions for strings moving in AdS spacetime. We have studied in depth the so-called spiky string configurations and their properties. The identification of string spikes with soliton configurations explains the usefulness of the inverse scattering technique in constructing string configurations. We present the construction of general (spiky) string solutions associated with the most general n-soliton configurations on an infinite line. Our general solution is given in terms of two arbitrary functions representing the conformal frame and a discrete set of (collective) coordinates representing the solitons.

Chapter 3

Intermezzo

The investigation of classical string dynamics in AdS spacetime revealed the following structure. General classical motions can be characterized by the number of spikes, this number does not change under perturbations and remains conserved. As such it has the characteristics of a topological quantum number for solitons, where the number of solitons remains conserved in classical and quantum dynamics. The existence of this conservation number leads to the existence of superselection sectors with fixed number of spikes. This leads to great simplification of the nonlinear string dynamics in AdS spacetime. One can concentrate on a particular sector with fixed (and conserved) number of spikes, where the dynamical variables characterizing the motion are the locations of the spikes. This moduli space is particle-like and in the sector with n-spikes the moduli space becomes that of n-particles characterizing the spike locations. It is these moduli (or collective coordinates) that follow a closed set of equations describing the dynamics of the moduli space. This dynamical system provides a 0-brane picture of the AdS string and it is then expected that the full string can be reconstructed from the collective dynamics of its moduli.

3.1 Sinh-Gordon soliton dynamics

The sinh-Gordon soliton dynamics can be summarized by an associated n-body problem [44]. One way to deduce the dynamics of sinh-Gordon solitons is to follow the poles of the Hamiltonian density [45]. Here we denote the sinh-Gordon field as ϕ and use the variables t, x, the general solution to the sinh-Gordon equation can be written in the form

$$\phi = \ln(f/g)^2. \tag{3.1}$$

Plugging the above ansatz into the Hamiltonian density, we find

$$\mathcal{H} = \frac{2}{f^2 g^2}\Big[(fg_x - gf_x)^2 + (fg_t - gf_t)^2 + \frac{1}{4}(f^2 - g^2)^2\Big]. \tag{3.2}$$

One has the poles of the Hamiltonian density located at

$$fg = 0. \tag{3.3}$$

Firstly, let us consider the one-soliton solution to the sinh-Gordon equation

$$\phi_{s,\bar{s}} = \pm \ln\Big[\tanh\frac{(x - x_0) - vt}{2\sqrt{1 - v^2}}\Big]^2, \tag{3.4}$$

where x_0 is the initial position of the soliton and v is the velocity of the soliton. The motion of poles are easily determined by (3.3) and we get $x(t) = x_0 + vt$, which represents a free motion of the pole. The rest mass of the soliton diverges at $x = 0$. However, in this whole analysis of dynamics, the rest mass turns out to be an overall multiplier and we can set $m = 1$.

Secondly, for the two-soliton solution in the center-of-mass frame

$$\phi_{ss} = \ln\Big[\frac{v\cosh(\gamma x) - \cosh(\gamma v t)}{v\cosh(\gamma x) + \cosh(\gamma v t)}\Big]^2, \tag{3.5}$$

where $\gamma = (1-v^2)^{-1/2}$ and v is the relative velocity of the two solitons, the poles are located at

$$fg = v^2\cosh^2(\gamma x) - \cosh^2(\gamma v t) = 0, \tag{3.6}$$

so that the trajectories are

$$x(t) = \pm\frac{1}{\gamma}\cosh^{-1}\Big[\frac{1}{v}\cosh(\gamma v t)\Big]. \tag{3.7}$$

These trajectories are the same as the sine-Gordon solitons [45]. Similarly, the time delay of two-soliton scattering can be easily worked out as

$$\Delta t = \lim_{L\to\infty}\left(\frac{1}{\gamma v}\cosh^{-1}[v\cosh(\gamma x)]|_{-L}^{L} - \frac{2L}{v}\right) = \frac{2}{\gamma v}\ln v. \tag{3.8}$$

A classical relativistic particle is most conveniently described not with its customary momentum p and position x, but rather with is rapidity θ and the canonically conjugate generalized position q, which are defined by

$$p \equiv \sinh\theta, \qquad x \equiv q/\cosh\theta. \tag{3.9}$$

We see that the trajectories and time delays are the same with those of the sine-Gordon theory. The dynamics therefore can be summarized by a n-body Hamiltonian of Ruijsenaars and Schneider [46] form

$$H = \sum_{j=1}^{N}\cosh\theta_j\prod_{k\neq j} f(q_j - q_k). \tag{3.10}$$

In the case of two particles, one defines the center-of-mass variables

$$s = q_1 + q_2, \qquad \varphi = \tfrac{1}{2}(\theta_1 + \theta_2), \tag{3.11}$$

$$q = q_1 - q_2, \qquad \theta = \tfrac{1}{2}(\theta_1 - \theta_2), \tag{3.12}$$

the soliton dynamics is governed by the reduced Hamiltonian

$$H_r = \cosh\theta \; W(q). \tag{3.13}$$

The potential for soliton-soliton scattering is given by

$$W(q) = \left|\coth\left(\frac{q}{2}\right)\right|. \tag{3.14}$$

A quick way to see the above potential gives the time delay (3.8) is to note first that (3.13) implies

$$\dot{q}^2 + W^2(q) = H_r^2. \tag{3.15}$$

Now H_r is a constant of motion, so that we may plug in the potential (3.14) and get

$$\dot{q}^2 + 1/\sinh^2(q/2) = E, \tag{3.16}$$

where $E = H_r^2 - 1 = \sinh^2\theta$. Then the time delay is calculated as

$$\Delta t = \lim_{L\to\infty}\left(\int_{-L}^{L}\frac{dq}{\sqrt{\sinh^2\theta - 1/\sinh^2[q/2]}} - \frac{2L}{\tanh\theta}\right) = \frac{2}{\sinh\theta}\ln[\tanh\theta], \tag{3.17}$$

which agrees with (3.8) by noticing $v = \tanh\theta$.

The generators

$$H = P^0 = 2\cosh\varphi\cosh\theta\ W(q), \tag{3.18}$$

$$P = P^1 = 2\sinh\varphi\cosh\theta\ W(q), \tag{3.19}$$

$$J = -(q_1 + q_2) = -s, \tag{3.20}$$

closes the 2D Poincaré algebra. A manifestly relativistic description of this system can be given as follows. Consider the coordinates $x_\mu^{(i)}$, $i = 1, 2$ and the constraints

$$p_1^2 - M^2[(x_1 - x_2)^2] = 0, \tag{3.21}$$

$$p_2^2 - M^2[(x_1 - x_2)^2] = 0. \tag{3.22}$$

Here $M(x^2)$ is a function of relative distance to be specified later. In the center-of-mass frame, defining the variables $P = \frac{1}{2}(p_1 + p_2)$, $p = \frac{1}{2}(p_1 - p_2)$, $X = x_1 + x_2$, $x = x_1 - x_2$, the constraints become

$$P^2 + p^2 - M^2(x^2) = 0, \tag{3.23}$$

$$P \cdot p = 0. \tag{3.24}$$

The commutation of (3.23) and (3.24) leads to an extra constraint

$$P \cdot x = 0 \tag{3.25}$$

This description involves two times: $x_0^{(1)}$ and $x_0^{(2)}$. The second class constraints (3.24) and (3.25) can be used for elimination of $x_0^{(1)} - x_0^{(2)}$ and the corresponding momentum and allow a single time $x_0 = \frac{1}{2}(x_0^{(1)} + x_0^{(2)})$ description. Specifically,

$$x_\mu = P_\mu \bar{u} + b_\mu u, \tag{3.26}$$

$$p_\mu = \frac{P_\mu}{P^2}\bar{\pi} + b_\mu \pi, \tag{3.27}$$

with $b \cdot P = 0$ reduces the constraints (3.24, 3.25) to $\bar{u} = \bar{\pi} = 0$. What remains is the center-of-mass variables X_μ, P_μ and the relative $\{\pi, u\} = 1$ degrees of freedom plus the equation

$$P_\mu^2 + \pi^2 - M^2(u^2) = 0. \tag{3.28}$$

To deduce the function $M(u^2)$ just take the Ruijsenaars-Schneider Hamiltonian and find

$$P_0^2 - P_1^2 = \cosh^2\theta + \frac{\cosh^2\theta}{q^2}. \tag{3.29}$$

Identifying $\pi = \sinh\theta$, $u = \frac{q}{\cosh\theta}$, this has the form of (3.28) with

$$M^2(u^2) = 1 + \frac{1}{u^2}. \tag{3.30}$$

This shows how the two-body sine-Gordon system is described in manifestly relativistic terms with constraints (3.23, 3.24, 3.25).

3.2 String dynamics in flat spacetime

It is useful to summarize the construction of a much simpler case of flat three dimensional string representing the $R \to \infty$ limit of the AdS spacetime. In this case the reduced theory is given by the Liouville equation [47, 48] whose general solutions are explicitly given.

The conformal gauge equations of motion for strings in flat spacetime are

$$\partial_+\partial_- X = 0, \tag{3.31}$$

as well as the Virasoro constraints

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0, \tag{3.32}$$

where $X^\mu = (X^0, X^1, X^2)$ with the flat metric $\eta_{\mu\nu} = \{-1, 1, 1\}$. Here we follow the notation $\sigma^\pm = \tau \pm \sigma$ so that $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/2$, where τ, σ are the Minkowski worldsheet coordinates of the string. Defining the scalar field

$$\alpha(\sigma^+, \sigma^-) \equiv -\ln[\partial_+ X \cdot \partial_- X], \tag{3.33}$$

we find the equation of motion for α to be

$$\partial_+\partial_-\alpha(\sigma^+, \sigma^-) - u(\sigma^+)v(\sigma^-)e^\alpha = 0, \tag{3.34}$$

where $u(\sigma^+)$ and $v(\sigma^-)$ are two arbitrary functions. The general solution to the Liouville equation (3.34) reads

$$\alpha = \ln\Big[\frac{2}{u(\sigma^+)v(\sigma^-)} \frac{f'(\sigma^+)g'(\sigma^-)}{[f(\sigma^+) + g(\sigma^-)]^2}\Big]. \tag{3.35}$$

In order to understand the most general form of the string solution we note that in the conformal gauge one still has a residual symmetry. Both the equations of motion (3.31) and the Virasoro constraints (3.32) are invariant with respect to the conformal transformations $\sigma^+ \to f(\sigma^+)$, $\sigma^- \to g(\sigma^-)$. We can, without loss of generality, specify the conformal frame by the following conditions

$$\dot{X}_{,+}^2 = u^2(\sigma^+), \qquad \dot{X}_{,-}^2 = v^2(\sigma^-). \tag{3.36}$$

The general solution to the equations of motion (3.31) satisfying the Virasoro constraints (3.32) is now constructed following [47] as

$$X^\mu(\sigma^+, \sigma^-) = \psi_+^\mu(\sigma^+) + \psi_-^\mu(\sigma^-), \tag{3.37}$$

where $\psi_+^{'\mu}$ and $\psi_-^{'\mu}$ being the isotropic vectors

$$(\psi_\pm')^2 = 0. \tag{3.38}$$

The prime implies the differentiation with respect to the function argument. Substituting (3.37) into (3.36), we obtain one more condition on $\psi_\pm^\mu$

$$(\psi_+^{''})^2 = u^2(\sigma^+), \qquad (\psi_-^{''})^2 = v^2(\sigma^-). \tag{3.39}$$

The conditions (3.38) and (3.39) can easily be satisfied by expanding the vectors $\psi_\pm^{'\mu}$ in a special basis. In the case of three dimensions, we choose the basis

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.40}$$

The expansion for $\psi_\pm'(\sigma^\pm)$ in this basis can be written as

$$\psi_+'(\sigma^+) = +\frac{u(\sigma^+)}{f'(\sigma^+)}\Big[e_1 + \frac{1}{2}f^2(\sigma^+)e_2 + f(\sigma^+)e_3\Big], \tag{3.41}$$

$$\psi_-'(\sigma^-) = -\frac{v(\sigma^-)}{g'(\sigma^-)}\Big[e_1 + \frac{1}{2}g^2(\sigma^-)e_2 - g(\sigma^-)e_3\Big]. \tag{3.42}$$

The spikes are located at $(X')^2 = -2e^{-\alpha} = 0$, which generates the condition

$$f(\sigma^+) + g(\sigma^-) = 0. \tag{3.43}$$

As an example, take the spiky string solution in [31], we have

$$\psi_+(\sigma^+) = \begin{pmatrix} \lambda(n-1)\sigma^+ \\ \lambda\cos((n-1)\sigma^+) \\ \lambda\sin((n-1)\sigma^+) \end{pmatrix}, \quad \psi_-(\sigma^-) = \begin{pmatrix} \lambda(n-1)\sigma^- \\ \lambda(n-1)\cos(\sigma^-) \\ \lambda(n-1)\sin(\sigma^-) \end{pmatrix}, \tag{3.44}$$

so that

$$u = \lambda(n-1)^2, \qquad v = \lambda(n-1). \tag{3.45}$$

Using the representation (3.41) and (3.42), we find

$$f(\sigma^+) = \sqrt{2}\cot\frac{(n-1)\sigma^+}{2}, \qquad g(\sigma^-) = -\sqrt{2}\cot\frac{\sigma^-}{2}. \tag{3.46}$$

The condition (3.43) generates the locations of spikes

$$(n-1)\sigma^+ - \sigma^- = 2\pi m, \tag{3.47}$$

which of course agrees with [31].

As we have understood that spikes in the string configuration are associated with locations of solitons (singularities) in the scalar field solution. For the Liouville theory a detailed study of the dynamics of singularities was given in [48]. They are determined by the equation (3.43) giving a description of the world lines of dynamical particles. This interpretation is suggested by the time-like nature of the singularity lines and the fact that each line is characterized by the initial data σ_j^0 and v_j. Explicitly, if $\sigma_j(\tau)$ is the equation of the j-th singularity line, then we have $\sigma_j(0) = \sigma_j^0$,

$\dot{\sigma}_j(0) = v_j$.

To summarize the discussion of [48], one starts with arbitrary functions $f(\sigma^+)$ with N_A singularities and $g(\sigma^-)$ with N_B singularities

$$f(\sigma^+) = \sum_{j=1}^{N_A} \frac{c_j}{y_j - \sigma^+}, \qquad g(\sigma^-) = \sum_{j=1}^{N_B} \frac{d_j}{z_j - \sigma^-}, \tag{3.48}$$

where c_j, d_j, y_j, z_j are constants. The number of singularities determined by (3.43) is $N = N_A + N_B - 1$, which we denote by $\sigma_i^{\pm}(\tau)$, where τ is the variable parameterizing the lines, and for convenience we assume that

$$\sigma_i^{\pm}(0) = \pm\sigma_i^0, \qquad \dot{\sigma}_i^{\pm}(0) = 1 \pm v_i, \qquad i = 1, 2, \cdots, N. \tag{3.49}$$

Thus, we obtain the system

$$\sum_{j=1}^{N_A} \frac{c_j}{y_j - \sigma_i^+} + \sum_{j=1}^{N_B} \frac{d_j}{z_j - \sigma_i^-} = 0, \qquad i = 1, 2, \cdots, N. \tag{3.50}$$

The constants c_j, d_j, y_j, z_j in this system must be determined from the initial data of the Liouville field. Here, this can be seen directly. We differentiate (3.50) with respect to τ,

$$\dot{\sigma}_i^+ \sum_{j=1}^{N_A} \frac{c_j}{(y_j - \sigma_i^+)^2} + \dot{\sigma}_i^- \sum_{j=1}^{N_B} \frac{d_j}{(z_j - \sigma_i^-)^2} = 0, \qquad i = 1, 2, \cdots, N. \tag{3.51}$$

Setting now $\tau = 0$ in (3.50) and (3.51) and using (3.49), we obtain a system of $2N = 2(N_A + N_B) - 2$ equations for determining the $2(N_A + N_B)$ constants c_j, d_j, y_j, z_j. The remaining two-parameter arbitrariness exactly coincides with the arbitrariness of the restricted Bianchi transformation.

To obtain the dynamical equations of motion whose solutions are to be these singularities lines, we can differentiate further (3.51) with respect to τ. These really are the equations of motion, since the constants c_j, d_j, y_j, z_j can be expressed in accordance with (3.50) and (3.51) in terms of $\sigma_i^+, \dot{\sigma}_i^+, \sigma_i^-, \dot{\sigma}_i^-$ not only for $\tau = 0$ but also at any time τ, in particular the time at which we consider the system equations of motion.

Turning now to the string solution, it was given generally by (3.41, 3.42) with two arbitrary functions $f(\sigma^+)$ and $g(\sigma^-)$ and the functions $u(\sigma^+)$ and $v(\sigma^-)$ representing the conformal frame. After fixing these, one can in principle integrate (3.41, 3.42) to determine the string solution. A particular interesting class of these functions are those with singularities described above. These singularities in field theory will translate to spikes in string theory. To exemplify this connection, we will describe the simplest cases with one and two singularities, i.e., one and two spikes.

In the case of one singularity, set $N = N_A = N_B = 1$ in (3.48), using the initial data (3.49), we can solve for the constants

$$c_1 = -d_1 \frac{1 + v_1}{1 - v_1}, \qquad y_1 = \frac{2\sigma_1^0}{1 - v_1} + z_1 \frac{1 + v_1}{1 - v_1}. \tag{3.52}$$

so that the trajectory of the singularity is

$$\sigma_1(\tau) = \sigma_1^0 + v_1 \tau. \tag{3.53}$$

By integrating (3.41) and (3.42), we get the string solution

$$
\begin{aligned}
X^0 &= \frac{u}{\sqrt{2}d_1\tilde{v}_1}\Big(\frac{1}{3}(\tilde{\sigma}^+)^3 + \frac{1}{2}d_1^2\tilde{v}_1^2\tilde{\sigma}^+\Big) + \frac{v}{\sqrt{2}d_1}\Big(\frac{1}{3}(\tilde{\sigma}^-)^3 + \frac{1}{2}d_1^2\tilde{\sigma}^-\Big), && (3.54)\\
X^1 &= \frac{u}{\sqrt{2}d_1\tilde{v}_1}\Big(\frac{1}{3}(\tilde{\sigma}^+)^3 - \frac{1}{2}d_1^2\tilde{v}_1^2\tilde{\sigma}^+\Big) + \frac{v}{\sqrt{2}d_1}\Big(\frac{1}{3}(\tilde{\sigma}^-)^3 - \frac{1}{2}d_1^2\tilde{\sigma}^-\Big), && (3.55)\\
X^2 &= \frac{u}{2}(\tilde{\sigma}^+)^2 + \frac{v}{2}(\tilde{\sigma}^-)^2, && (3.56)
\end{aligned}
$$

where, for simplicity, u and v are chosen to be constants and the redefinitions

$$
\tilde{\sigma}^+ \equiv \sigma^+ - \frac{2\sigma_1^0}{1-v_1} - z_1\frac{1+v_1}{1-v_1}, \qquad \tilde{\sigma}^- \equiv \sigma^- - z_1, \qquad \tilde{v}_1 \equiv \frac{1+v_1}{1-v_1}. \tag{3.57}
$$

We can generalize the above case to the 'periodic' case with the identity

$$
\cot\frac{x}{2} = \sum_{n=-\infty}^{\infty}\frac{2}{x+2\pi n}, \tag{3.58}
$$

and find

$$
f(\sigma^+) = \frac{d_1\tilde{v}_1}{2}\cot\frac{\tilde{\sigma}^+}{2}, \qquad g(\sigma^-) = -\frac{d_1}{2}\cot\frac{\tilde{\sigma}^-}{2}. \tag{3.59}
$$

After the integration, the string solution is found to be

$$
\begin{aligned}
X^0 &= -\frac{\sqrt{2}u}{d_1\tilde{v}_1}\Big((\tilde{\sigma}^+ - \sin\tilde{\sigma}^+) + \frac{d_1^2\tilde{v}_1^2}{8}(\tilde{\sigma}^+ + \sin\tilde{\sigma}^+)\Big)\\
&\quad -\frac{\sqrt{2}v}{d_1}\Big((\tilde{\sigma}^- - \sin\tilde{\sigma}^-) + \frac{d_1^2}{8}(\tilde{\sigma}^- + \sin\tilde{\sigma}^-)\Big), && (3.60)\\
X^1 &= -\frac{\sqrt{2}u}{d_1\tilde{v}_1}\Big((\tilde{\sigma}^+ - \sin\tilde{\sigma}^+) - \frac{d_1^2\tilde{v}_1^2}{8}(\tilde{\sigma}^+ + \sin\tilde{\sigma}^+)\Big)\\
&\quad -\frac{\sqrt{2}v}{d_1}\Big((\tilde{\sigma}^- - \sin\tilde{\sigma}^-) - \frac{d_1^2}{8}(\tilde{\sigma}^- + \sin\tilde{\sigma}^-)\Big), && (3.61)\\
X^2 &= u\cos\tilde{\sigma}^+ + v\cos\tilde{\sigma}^-. && (3.62)
\end{aligned}
$$

It is interesting to notice the special case where

$$
v_1 = 0, \qquad \tilde{v}_1 = 1, \qquad d_1 = 2\sqrt{2}, \tag{3.63}
$$

the string solution (3.60-3.62) reduces to the spiky strings in [31] with two spikes ($n = 2$) if we identify

$$\tilde{\sigma}^{+} = \frac{\pi}{2} - \sigma_{+}, \qquad \tilde{\sigma}^{-} = \frac{\pi}{2} - \sigma_{-}, \qquad u = \lambda, \qquad v = \lambda. \tag{3.64}$$

3.3 AdS space dynamics

Here we focus on two simple examples where the energies can be explicitly evaluated. First, for one soliton, we choose the parameters to be

$$u(\bar{z}) = 2, \quad v(z) = -2, \quad c_1(0) = -2\zeta_1 = -i\tilde{v}_1, \tag{3.65}$$

where $\tilde{v}_1 \equiv \sqrt{(1 - v_1)/(1 + v_1)}$ and v_1 is the velocity of soliton on the worldsheet.[1] This will correspond to the one-soliton solution of the sinh-Gordon equation

$$\alpha = \ln\Big[2\tanh^2\big[\frac{\sigma - v_1\tau}{\sqrt{1 - v_1^2}}\big]\Big]. \tag{3.66}$$

In terms of the worldsheet coordinates τ and σ, the string solution is given by

$$\begin{aligned} Z_1 &= \frac{e^{i\tau}}{e^{(\sigma-\tau)/\tilde{v}_1} + e^{-(\sigma+\tau)\tilde{v}_1}}\Big\{e^{(\sigma-\tau)/\tilde{v}_1}\cosh\sigma \\ &\quad + e^{-(\sigma+\tau)\tilde{v}_1}\frac{(1 - i\tilde{v}_1)^2((1 + \tilde{v}_1^2)\cosh\sigma + 2\tilde{v}_1\sinh\sigma)}{1 - \tilde{v}_1^4}\Big\}, \end{aligned} \tag{3.67}$$

$$\begin{aligned} Z_2 &= \frac{-ie^{i\tau}}{e^{(\sigma-\tau)/\tilde{v}_1} + e^{-(\sigma+\tau)\tilde{v}_1}}\Big\{e^{(\sigma-\tau)/\tilde{v}_1}\sinh\sigma \\ &\quad + e^{-(\sigma+\tau)\tilde{v}_1}\frac{(1 - i\tilde{v}_1)^2((1 + \tilde{v}_1^2)\sinh\sigma + 2\tilde{v}_1\cosh\sigma)}{1 - \tilde{v}_1^4}\Big\}. \end{aligned} \tag{3.68}$$

[1] Notice, in general, the worldsheet time τ is different from the global time t.

The momenta densities are calculated as

$$\mathcal{P}_t^\tau = \frac{\sqrt{\lambda}}{2\pi}\mathrm{Im}(\dot{Z}_1^* Z_1), \qquad \mathcal{P}_t^\sigma = \frac{\sqrt{\lambda}}{2\pi}\mathrm{Im}(Z_1'^* Z_1), \tag{3.69}$$

$$\mathcal{P}_\theta^\tau = \frac{\sqrt{\lambda}}{2\pi}\mathrm{Im}(\dot{Z}_2^* Z_2), \qquad \mathcal{P}_\theta^\sigma = \frac{\sqrt{\lambda}}{2\pi}\mathrm{Im}(Z_2'^* Z_2), \tag{3.70}$$

where λ is the coupling constant. One can explicitly check $\mathcal{P}_t^\sigma$ and $\mathcal{P}_\theta^\sigma$ vanish asymptotically at $\sigma = \pm\infty$ as long as the string solution is regular. That is, in the singular case when $v_1 = 0$, we found nonvanishing momentum flow with τ dependence at the boundary of the string [36].

By the current conservation $\partial_\tau \mathcal{P}_{t,\theta}^\tau - \partial_\sigma \mathcal{P}_{t,\theta}^\sigma = 0$, we can calculate the energy and angular momentum at any convenient τ. For instance, the energy and momentum densities at $\tau = 0$ are simplified to be

$$\mathcal{P}_t^\tau = \frac{\sqrt{\lambda}}{2\pi(e^{\sigma/\tilde{v}_1} + e^{-\sigma\tilde{v}_1})^2}\Big\{e^{2\sigma/\tilde{v}_1}\cosh^2\sigma + e^{-2\sigma\tilde{v}_1}\Big(\frac{\cosh\sigma + \epsilon_1^{-1}\sinh\sigma}{v_1}\Big)^2\Big\}, \tag{3.71}$$

$$\mathcal{P}_\theta^\tau = \frac{\sqrt{\lambda}}{2\pi(e^{\sigma/\tilde{v}_1} + e^{-\sigma\tilde{v}_1})^2}\Big\{e^{2\sigma/\tilde{v}_1}\sinh^2\sigma + e^{-2\sigma\tilde{v}_1}\Big(\frac{\sinh\sigma + \epsilon_1^{-1}\cosh\sigma}{v_1}\Big)^2\Big\}, \tag{3.72}$$

where $\epsilon_1 \equiv (1 - v_1^2)^{-1/2}$ is the energy of the soliton. Introduce a cutoff Λ for the σ integration, up to the subleading term, the energy and angular momentum are

$$E = \int_{-\Lambda}^{\Lambda} \mathcal{P}_t^\tau d\sigma \sim \frac{\sqrt{\lambda}}{2\pi}\Big[\frac{1}{4(1+\epsilon_1^{-1})}e^{2\Lambda} + \Lambda\Big], \tag{3.73}$$

$$S = \int_{-\Lambda}^{\Lambda} \mathcal{P}_\theta^\tau d\sigma \sim \frac{\sqrt{\lambda}}{2\pi}\Big[\frac{1}{4(1+\epsilon_1^{-1})}e^{2\Lambda} - \Lambda\Big]. \tag{3.74}$$

Therefore, the difference between E and S can be calculated as

$$E - S = \frac{\sqrt{\lambda}}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{\cosh[2\epsilon_1\sigma]}{1+\cosh[2\epsilon_1\sigma]}d\sigma \sim \frac{\sqrt{\lambda}}{2\pi}\Big[\ln\frac{8\pi S}{\sqrt{\lambda}} + \ln(1+\epsilon_1^{-1}) - \epsilon_1^{-1}\Big]. \tag{3.75}$$

Measuring the energy from the infinite GKP string, the dispersion relation can be written as

$$E - S = E_0 + \frac{\sqrt{\lambda}}{2\pi}\left[\frac{1}{2}\ln\frac{1+\epsilon_1^{-1}}{1-\epsilon_1^{-1}} - \epsilon_1^{-1}\right] \tag{3.76}$$

with the excitation energy

$$E^1_{\text{spike}}(\epsilon_1) \equiv E - S - E_0 = \frac{\sqrt{\lambda}}{2\pi}\left[\frac{1}{2}\ln\frac{1+\epsilon_1^{-1}}{1-\epsilon_1^{-1}} - \epsilon_1^{-1}\right]. \tag{3.77}$$

The inverse power in the soliton energy shows a similarity with the case of giant magnons on $R \times S^2$ [18] where the excitation energy of the string (and the giant magnon) is equal to the inverse of the sine-Gordon soliton energy.[2] Here in the AdS case we find extra logarithmic terms pointing to a more complicated dynamical system governing the spike dynamics in AdS as compared with $R \times S^2$.

Our next explicit example is for two solitons, where the parameters are chosen to be

$$c_1(0) = 2\zeta_1\frac{\zeta_1+\zeta_2}{\zeta_1-\zeta_2}, \quad c_2(0) = 2\zeta_2\frac{\zeta_1+\zeta_2}{\zeta_1-\zeta_2}, \quad \zeta_1 = -\frac{i}{2}\tilde{v}_1, \quad \zeta_2 = \frac{i}{2}\tilde{v}_2^{-1}, \tag{3.78}$$

where $\tilde{v}_{1,2} \equiv \sqrt{(1-v_{1,2})/(1+v_{1,2})}$ and $v_{1,2}$ are the magnitudes of the velocities of the two solitons moving towards each other. The exact expressions for the momenta densities are lengthy to write them down, but the leading terms at $\tau = 0$ are

$$\mathcal{P}^\tau_t \sim \frac{\sqrt{\lambda}}{2\pi}\left[\frac{1}{4}\left(\frac{1-\tilde{v}_1}{1+\tilde{v}_1}\right)^2 e^{2\sigma} + \frac{1}{4}\left(\frac{1-\tilde{v}_2}{1+\tilde{v}_2}\right)^2 e^{-2\sigma} + \frac{1}{2}\right], \tag{3.79}$$

$$\mathcal{P}^\tau_\theta \sim \frac{\sqrt{\lambda}}{2\pi}\left[\frac{1}{4}\left(\frac{1-\tilde{v}_1}{1+\tilde{v}_1}\right)^2 e^{2\sigma} + \frac{1}{4}\left(\frac{1-\tilde{v}_2}{1+\tilde{v}_2}\right)^2 e^{-2\sigma} - \frac{1}{2}\right]. \tag{3.80}$$

In the special case where $v_1 = v_2 = v$, the dispersion relation is

$$E - S = \frac{\sqrt{\lambda}}{2\pi}\left[\ln\frac{8\pi S}{\sqrt{\lambda}} + \ln\frac{1+\epsilon^{-1}}{1-\epsilon^{-1}} - 2\epsilon^{-1} + \cdots\right]. \tag{3.81}$$

[2] There one uses a time-like gauge $t = \tau$.

For different velocities, using the symmetric ρ regularization, we obtain the result

$$E - S = \frac{\sqrt{\lambda}}{2\pi}\Big[\ln\frac{8\pi S}{\sqrt{\lambda}} + \frac{1}{2}\ln\frac{1+\epsilon_1^{-1}}{1-\epsilon_1^{-1}} + \frac{1}{2}\ln\frac{1+\epsilon_2^{-1}}{1-\epsilon_2^{-1}} - \epsilon_1^{-1} - \epsilon_2^{-1}\Big]. \tag{3.82}$$

Following the definition of spike energy (3.77), the excitation energy of the two-spike solution is given by the sum of two individual spike energies

$$E^2_{\text{spike}}(\epsilon_1, \epsilon_2) = E^1_{\text{spike}}(\epsilon_1) + E^1_{\text{spike}}(\epsilon_2). \tag{3.83}$$

In the center of mass frame $v_1 = v_2 = v$, the energy is $E^2_{\text{spike}}(\epsilon) = 2E^1_{\text{spike}}(\epsilon)$.

In general the sinh-Gordon singularities behave as particles and follow interacting particle trajectories. Through our explicit transformations this dynamics translates into the spike dynamics of the AdS_3 string. Concretely, given the trajectories of N solitons $x_i(t), i = 1, 2, \cdots, N$, we can in principle by direct substitution (2.122, 2.123) with $\sigma_i(\tau)$ construct the trajectories of N spikes by

$$Z_1^i(\tau) = Z_1(\tau, \sigma_i(\tau)), \qquad Z_2^i(\tau) = Z_2(\tau, \sigma_i(\tau)), \tag{3.84}$$

where τ acts like the proper time. We therefore have a mapping where on the left hand side the index i labels the string spikes while on the right side it denotes the solitons/singularities. This construction is straightforward in principle, with the map provided by the known wavefunctions of the scattering problem.

Some general features of the dynamics that emerge from the construction can be deduced. First of all the N-body field theory dynamics being integrable it is automatic that the corresponding string theory system defined by our inverse scattering map is also integrable. The soliton N-body interactions have the characteristic that they are of Calogero type (as compared to the Toda, nearest neighbor interactions). It is not obvious, and remains to be established which of these two possible integrable schemes are associated with the spike dynamics of closed AdS strings. Here we also have the analogy and lessons from the recent study of N-body description of magnons on $R \times S^2$. In the magnon case the map between the soliton dynamical system and the "string" system was established in [49]. It involves the multi-Hamiltonian and multi-Poisson structures of the integrable N-body Ruijsenaars-Schneider system [50]. Our present construction implies that such a correspondence is also expected to hold in the AdS case.

3.4 Spikes and Singletons

A group theoretic description of the spikes was given by Sundell *et. al.* [51] in terms of singletons. Singletons are ultra-short unitary irreducible representations (UIR) of $so(2,d)$ whose weights form single lines in weight space. This phenomena was first discovered by Dirac [52] in the case of the scalar and spinor singletons in $d = 3$. Since $SO(2,3)$ can be realized as the group of the isometries of AdS_4, singletons play a key role in the study of AdS physics. Singletons also play an important role in higher-spin gauge theory, where the presently known full higher-spin field equations, due to Vasiliev, are based on gauging higher-spin algebras given by subalgebras of the enveloping algebra of $so(2,d)$ obtained by factoring out ideals given by singleton annihilators.

The main characteristic of singletonic particles is that they are unobservable in the bulk of AdS and, as isolated objects, they can only be observed on the boundary. They can be naturally described by a conformal particle on the zero radius limit of AdS, known as Dirac's hypercone, leading to an $sp(2)$-gauged sigma model. However, their composites are indeed observable in the bulk, since they are the ordinary massless and massive particles in AdS. In fact, a fundamental result for quantum field theory in AdS space is the compositeness theorem by Flato and Fronsdal [53] that the product of two (scalar or spinor) singleton representations decomposes into an infinite sum over all possible massless representations. This result can be interpreted by saying that any massless particle in AdS is a composite object made of two singletons.

Writing the generators of the $SO(2,3)$ group as $L_{\alpha\beta}$, an UIR of $so(2,3)$ is denoted by $D(E_0, s)$ where E_0 is the lowest energy eigenvalue of L_{05} and s is the spin eigenvalue of L_{12}. (The elements L_{05}, L_{12} generate a compact Cartan subalgebra.) Massless particles in AdS space are composite: Each state of a massless particle, with arbitrary spin, may be regarded as a state of two Dirac singletons. The two positive energy Dirac singletons are called Di and Rac

$$\text{Di} = D(1, 1/2), \qquad \text{Rac} = D(1/2, 0). \tag{3.85}$$

The spectrum of the Cartan subalgebra is given by

$$\text{Di}: \qquad E = J + \frac{1}{2}, \qquad J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ... \tag{3.86}$$

$$\text{Rac}: \qquad E = J + \frac{1}{2}, \qquad J = 0, 1, 2, ... \tag{3.87}$$

Here J is the total angular momentum defined by $L_{12}^2 + L_{23}^2 + L_{31}^2 = J(J+1)$. The dimension of each L_{05}-eigenspace is precisely $2J + 1$.

Massless particles are associated with the family $D(s+1, s)$ and $D(2, 0)$. In these representations, the spectra of the Cartan subalgebra are given by

$$D(s+1,s): \qquad E = J+1, J+2, ... \qquad s>0, \tag{3.88}$$

$$D(1,0): \qquad E = J+1, J+3, ... \tag{3.89}$$

$$D(2,0): \qquad E = J+2, J+4, ... \tag{3.90}$$

and $J = s, s+1, ...$ in every case. All these representations are unitary and irreducible. The singletons have the following wonderful properties

$$\mathrm{Di} \otimes \mathrm{Di} = \oplus_{s=1,2,...} D(s+1,s) \oplus D(2,0) \tag{3.91}$$

$$\mathrm{Di} \otimes \mathrm{Rac} = \oplus_{2s=1,3,...} D(s+1,s) \tag{3.92}$$

$$\mathrm{Rac} \otimes \mathrm{Rac} = \oplus_{s=0,1,2,...} D(s+1,s) \tag{3.93}$$

which can be naively proven by calculating the characters.

3.4.1 The Rac and composites

The Rac is the unitary, irreducible representation $D(1/2,0)$. The anti-Rac or $\overline{\mathrm{Rac}}$ is the corresponding negative-energy representation $D(-1/2,0)$. These representations are related to the "massless" representations $D(s+1,s)$ by the formulas

$$\mathrm{Rac} \otimes \mathrm{Rac} = \oplus_s D(s+1,s), \tag{3.94}$$

$$\overline{\mathrm{Rac}} \otimes \overline{\mathrm{Rac}} = \oplus_s D(-s-1,s). \tag{3.95}$$

The Rac particle is a particle on the cone $u^2 = u_0^2 + u_5^2 - \vec{u}^2 = 0$, it satisfies the equations

$$\partial_u^2 \phi = g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = 0, \qquad (u^\alpha\partial_\alpha + \frac{1}{2})\phi = 0. \tag{3.96}$$

Parametrizing the cone as

$$u_0 = U\sin t, \quad u_5 = U\cos t, \quad \vec{u} = U\hat{u}, \tag{3.97}$$

and use the ansatz $\phi(u) = U^{-1/2}\phi(t,\hat{u})$, we find the equation of motion in three dimensional subspace

$$(\partial_t^2 + \vec{L}^2 + \frac{1}{4})\phi(t,\hat{u}) = 0. \tag{3.98}$$

This equation can be obtained by variation of the invariant action

$$A = \int_{S^1\times S^2} dt d\hat{u} \frac{1}{2}\Big[\phi_t^2 - \phi_\theta^2 - \frac{1}{\sin^2\theta}\phi_\phi^2 - \frac{1}{4}\phi^2\Big]. \tag{3.99}$$

The positive energy, stationary solution is

$$\phi_{lm}(t,\hat{u}) = (2l+1)^{-1/2} e^{-it(l+1/2)} Y_{lm}(\hat{u}) \tag{3.100}$$

where $Y_{lm}(\hat{u})$ are the spherical harmonics.

Naively, the 2-Rac field $\Phi(u,v)$ is a bi-local scalar field satisfying the system of equations

$$\begin{aligned}(u\cdot\partial_u + \frac{1}{2})\Phi = (v\cdot\partial_v + \frac{1}{2})\Phi = 0,\\ \partial_u^2\Phi = \partial_v^2\Phi = 0.\end{aligned} \tag{3.101}$$

A close examination shows adding one more constraint $\partial_u \cdot \partial_v \Phi = 0$ will get rid of the mixed eigenstates of energy and angular momentum in $\mathrm{Rac} \oplus \overline{\mathrm{Rac}}$. Therefore, the 2-Rac field carrying the unitary representation $(\mathrm{Rac}\oplus\mathrm{Rac})\otimes(\overline{\mathrm{Rac}}\oplus\overline{\mathrm{Rac}})$ must satisfy the following set of equations

$$
\begin{aligned}
&(u\cdot\partial_u+\frac{1}{2})\Phi=(v\cdot\partial_v+\frac{1}{2})\Phi=0,\\
&\partial_u^2\Phi=\partial_v^2\Phi=\partial_u\cdot\partial_v\Phi=0.
\end{aligned}
\tag{3.102}
$$

3.4.2 Two singletons equals massless higher spin particles

To describe a particle in AdS with spin, one can use a two-particle system corresponding to the two-Rac's. This model possesses two gauge symmetries expressing strong conservation of the phase-space counterparts of the second- and fourth-order Casimir operators for $so(2,3)$. We have the generators

$$
J_{AB}=y_Ap^y_B-y_Bp^y_A+z_Ap^z_B-z_Bp^z_A. \tag{3.103}
$$

where y^A and z^A represent two separate objects and $A,B=0,1,2,3,5$ with the metric $\eta_{AB}=\mathrm{diag}(-,+,+,+,-)$. The second- and fourth-order Casimir operators are given by

$$
\begin{aligned}
\Omega_1 &= \frac{1}{2}J_{AB}J^{AB}=y^2p_y^2-(y\cdot p_y)^2+z^2p_z^2-(z\cdot p_z)^2\\
&\quad +2(y\cdot z)(p_y\cdot p_z)-2(y\cdot p_z)(z\cdot p_y), \qquad (3.104)\\
\Omega_2 &= \frac{1}{4}J_{AB}J^B{}_CJ^C{}_DJ^{DA}-\frac{1}{2}\Big(\frac{1}{2}J_{AB}J^{AB}\Big)^2\\
&= +y^2(p_z^2(zp_y)^2+p_y^2(zp_z)^2-2(p_yp_z)(zp_y)(zp_z))\\
&\quad +z^2(p_z^2(yp_y)^2+p_y^2(yp_z)^2-2(p_yp_z)(yp_y)(yp_z))\\
&\quad +y^2z^2((p_yp_z)^2-p_y^2p_z^2)+(yz)^2(p_y^2p_z^2-(p_yp_z)^2)\\
&\quad -(yp_z)^2(zp_y)^2-(yp_y)^2(zp_z)^2+2(yp_y)(yp_z)(zp_y)(zp_z)\\
&\quad +2(p_yp_z)(yp_y)(yz)(zp_z)+2(p_yp_z)(yp_z)(yz)(zp_y)\\
&\quad -2p_y^2(yp_z)(yz)(zp_z)-2p_z^2(yp_y)(yz)(zp_y). \qquad (3.105)
\end{aligned}
$$

They are constrained to

$$\Omega_1 + E_0^2 + s^2 = 0, \tag{3.106}$$

$$\Omega_2 + E_0^2 s^2 = 0. \tag{3.107}$$

A solution to the constraints leads to

$$y \cdot p_y = -E_0, \tag{3.108}$$

$$y \cdot p_z = 0, \tag{3.109}$$

$$p_z^2 = 0, \tag{3.110}$$

$$z \cdot p_z = s, \tag{3.111}$$

$$p_y^2 = 0, \tag{3.112}$$

$$p_y \cdot p_z = 0. \tag{3.113}$$

The massless case corresponds to $E_0 = s + 1$. These constraints will be seen to agree with Fronsdal's covariant formulation of higher-spin theory that will be explored in the next chapter.

3.5 Summary

In this middle chapter we have presented a heuristic discussion of various aspects of the spiky string configuration moduli space. We have mainly concentrated on the simplest $n = 2$ (two-body) case describing its Hamiltonian dynamics and also a possible group theoretic interpretation in terms of "singletons" of Dirac and Fronsdal. Through this insight one is led to an emerging higher-spin picture that bound singletons (or spikes) dynamically generate massless, higher spin degrees of freedom in AdS spacetime. This picture is only heuristic, it is based on the compositeness argument of Flato and Fronsdal where a direct product of two simple (Rac) representations generates an infinite sequence of states with growing spins. What is lacking in this picture is a dynamical mechanism for the formation of bound states. Such dynamical mechanism will be provided by the large N collective field theory framework introduced in the 80's for systematically representing field theories with large ($N \to \infty$) degrees of freedom. It will be seen that for a many body problem of large number of spikes, or a large number of scalar fields, a collective picture results in generation of a curved AdS spacetime (with one extra dimension) and of the interacting Higher Spin Gravity.

Chapter 4

Higher Spin Anti-de Sitter Gravity

The second part of the Dissertation describes work on reconstructing gravity and spacetime from the partonic sub-structure. We will first give a review of the higher spin gauge theory. The Lagrangian description of free higher-spin fields was found by Fronsdal. The gauge invariance of the free equation of motion can be shown by requiring the tensor fields to be symmetric and double traceless. The cubic interaction was proposed by Fradkin and Vasiliev following by writing the higher-spin fields using the spinor notation and following the MacDowell-Mansuri action of Einstein gravity. Extension to the full nonlinear level is a highly nontrival problem. However, the full set of nonlinear equations of motion was later found by Vasiliev. By taking the Vasiliev's equations to linear order, we recover Fronsdal's free equation of motion.

4.1 Fronsdal's free higher-spin gravity

The theory of free massless fields of all spins is now developed in full detail. Note that in $d = 3+1$ flat spacetime, the physical fields of spin $s \leq 2$ are part of the family of the symmetric tensors. Their well know equations of motion and gauge transformations are reproduced below in a form suitable for the generalization to the case of higher spin fields

- $s = 0$: $\phi(x)$-scalar fields, $\partial_\mu \partial^\mu \phi = \partial^2 \phi = 0$, matter field, no gauge symmetry;
- $s = 1$: $A_\mu(x)$-Maxwell field, $\partial^\mu F_{\mu\nu} = \partial^2 A_\nu - \partial_\nu \partial_\mu A^\mu = 0$, $\delta A_\mu = \partial_\mu \xi(x)$;
- $s = 2$: $g_{\mu_1\mu_2}(x)$-graviton, $R_{\mu_1\mu_2} = 0$, $\delta g_{\mu_1\mu_2} = D_{\mu_1}\xi_{\mu_2} + D_{\mu_2}\xi_{\mu_1}$, where $D_\mu = \partial_\mu + \Gamma^\rho_{\mu\nu}$ is the covariant derivative and $\Gamma_{\mu\nu,\rho} = \frac{1}{2}(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\mu\rho})$ is the Christoffel connection.

Except for the scalar fields, all other massless fields are gauge fields. So it is natural to assume that all massless higher spin fields are gauge fields. In the case of AdS space, the partial derivative ∂ is replaced by the covariant derivative $\bigtriangledown$.

The higher spin field with spin s is described using the symmetric tensor field of rank s: $h_{\nu_1 \ldots \nu_s}$. The quadratic actions $S_s^{(2)}$ for free higher-spin fields can be fixed unambiguously (up to an overall factor) by the requirement of gauge invariance under the transformations $\delta h_{\nu_1\ldots\nu_s} = \bigtriangledown_{\{\nu_1}\epsilon_{\nu_2\ldots\nu_s\}}$ with the parameters $\epsilon_{\nu_1\ldots\nu_{s-1}}$ being traceless ${\epsilon^\rho}_{\rho\nu_3\ldots\nu_{s-1}} = 0$. The final result is [54]

$$\begin{aligned} S_s^{(2)} &= \int \sqrt{-g} d^4x \Big[\frac{1}{2} g^{\mu\nu} (\bigtriangledown_\mu h)\cdot(\bigtriangledown_\nu h) - \frac{1}{2} s (\bigtriangledown \cdot h)\cdot(\bigtriangledown \cdot h) \\ &+ \frac{1}{2} s(s-1)(\bigtriangledown \cdot h')\cdot(\bigtriangledown \cdot h) - \frac{1}{4} s(s-1) g^{\mu\nu}(\bigtriangledown_\mu h')\cdot(\bigtriangledown_\nu h') \end{aligned}$$

$$-\frac{1}{8}s(s-1)(s-2)(\bigtriangledown\cdot h')\cdot(\bigtriangledown\cdot h')-\frac{\rho}{2}(s^2-2s-2)h\cdot h$$
$$+\frac{1}{4}s(s-1)(s^2-3)h'\cdot h'\Big], \tag{4.1}$$

where ρ is the curvature parameter of the AdS space, h' is the single trace of h and the double trace h'' vanishes identically. The gauge invariant free equation of motion can be deduced from the quadratic action and we get

$$\Box h-\sum_1\bigtriangledown(\bigtriangledown\cdot h)+\frac{1}{2}\sum_1\bigtriangledown\sum_1\bigtriangledown h'+(s^2-2s-2)\rho h$$
$$+\sum_2 g[\bigtriangledown\bigtriangledown h-\Box h'-\frac{1}{2}\sum_1\bigtriangledown(\bigtriangledown\cdot h')-(s^2-3)\rho h']=0, \tag{4.2}$$

where $\Box$ is the covariant d'Alembertian. The sums are over all unequal orderings of the indices; thus $\sum_1$ contains s terms and $\sum_2$ contains $\frac{1}{2}s(s-1)$ terms, since h and g are symmetric. A more digestable form of the free equation of motion can be written as [55]

$$\bigtriangledown_\rho\bigtriangledown^\rho h_{\mu_1\ldots\mu_s}-s\bigtriangledown_\rho\bigtriangledown_{\mu_1}h^\rho_{\mu_2\ldots\mu_s}+\frac{1}{2}s(s-1)\bigtriangledown_{\mu_1}\bigtriangledown_{\mu_2}h^\rho{}_{\rho\mu_3\ldots\mu_s}$$
$$+2(s-1)(s+d-3)h_{\mu_1\ldots\mu_s}=0. \tag{4.3}$$

4.1.1 Covariant gauge

In covariant formulation, the gauge conditions are

$$g^{\mu\nu}h_{\mu\nu\ldots}=0,\qquad g^{\mu\nu}\bigtriangledown_\mu h_{\nu\ldots}=0. \tag{4.4}$$

Using the embedding coordinates $y^\alpha y_\alpha = y_0^2 - y_1^2 - y_2^2 - y_3^2 + y_5^2 = 1/\rho$, the new tensor fields $k_{\alpha\cdots}$ are related to $h_{\mu\cdots}$ by

$$h_{\mu\cdots}(x) = y_\mu{}^\alpha \cdots k_{\alpha\cdots}(y(x)), \tag{4.5}$$

where $y_\mu{}^\alpha = \partial y^\alpha / \partial x^\mu$. In order to guarantee the same degrees of freedom, the tensor fields k must satisfy the transversality and homogeneity conditions

$$y^\alpha k_{\alpha\cdots} = 0, \qquad k(\lambda y) = \lambda^N k(y). \tag{4.6}$$

Furthermore, the gauge condition (4.4) can be written neatly as

$$\partial^\alpha k_{\alpha\cdots} = 0. \tag{4.7}$$

Now we can reformulate the results using h in terms of k. The double traceless condition $h'' = 0$ directly translates to $k'' = 0$. The degree of homogeneity of k is fixed by (4.6) so that

$$(\hat{N} - N)k = 0, \tag{4.8}$$

where $\hat{N} = y \cdot \partial_y = y_\alpha(\partial/\partial y_\alpha)$. This condition is consistent with the wave equation. In the case of $s = 0$, the only non-vanishing Casimir operator is

$$Q = \frac{1}{2} L^{\alpha\beta} L_{\alpha\beta}, \tag{4.9}$$

where $L_{\alpha\beta}$ is the symmetry generators of the $SO(2,3)$ group. The wave equation reads

$$(Q - \langle Q \rangle)k = 0, \tag{4.10}$$

where $\langle Q \rangle$ is the value of Q in $D(E_0, 0)$

$$\langle Q \rangle = E_0(E_0 - 3). \tag{4.11}$$

Then the wave equation (4.10) takes the explicit form

$$[(\hat{N} + E_0)(\hat{N} - E_0 + 3) - y^2 \partial^2] k = 0. \tag{4.12}$$

In the general case $s \neq 0$, it is not enough to fix the values of Q and $\hat{N}$. If E_0 is large enough, then the representation $D(E_0, s)$ is carried by the subspace of fields that satisfy

$$(Q - \langle Q \rangle) k = 0, \qquad (\hat{N} - N) k = 0, \tag{4.13}$$

$$\partial \cdot k = 0, \qquad y \cdot k = 0, \qquad k' = 0, \tag{4.14}$$

where

$$\langle Q \rangle = E_0(E_0 - 3) + s(s + 1). \tag{4.15}$$

The other Casimir operators are fixed by these equations and need not to be considered separately.

In order to describe all integer spins, we introduce another set of variables z_α ($\alpha = 0, 1, 2, 3, 5$) and let $K(y, z)$ denote the formal series

$$K(y, z) = \sum_s z^{\alpha_1} \cdots z^{\alpha_s} k_{\alpha_1 \cdots \alpha_s}(y). \tag{4.16}$$

The complete set of wave equations and subsidiary conditions for all integer spins can now be re-expressed in terms of the field K

$$y \cdot k = 0 \iff y \cdot \partial_z K = 0 \quad \text{(transversality)}, \tag{4.17}$$

$$\partial \cdot k = 0 \iff \partial_y \cdot \partial_z K = 0 \quad \text{(gauge conditions)}, \tag{4.18}$$

$$k' = 0 \iff \partial_z^2 K = 0 \quad \text{(tracelessness)}. \tag{4.19}$$

The Casimir operator $\langle Q \rangle - Q$ takes the form

$$(\langle Q \rangle - Q)K = [y^2 \partial_y^2 + 2y \cdot z \partial_y \cdot \partial_z + z^2 \partial_z^2 + (\hat{n} - \hat{N} - 2)(\hat{n} + \hat{N} + 1)]K, \tag{4.20}$$

where $\hat{n} = z \cdot \partial_z$. Therefore, the representation $\oplus_s[D(s+1,s) \oplus D(-s-1,s)]$ is realized on the space of solutions of

$$\begin{aligned} &(\langle Q \rangle - Q)K = 0, \qquad (\hat{n} + \hat{N} + 1)K = 0, \\ &\partial_y \cdot \partial_z K = 0, \qquad y \cdot \partial_z K = 0, \qquad \partial_z^2 K = 0 \end{aligned} \tag{4.21}$$

modulo the space of gauge solutions.

4.1.2 The intertwining map

We shall construct an operator $F : K \to \Phi$ that intertwines between the representation $(\mathrm{Rac} \oplus \mathrm{Rac}) \otimes (\overline{\mathrm{Rac}} \oplus \overline{\mathrm{Rac}})$, realized on bi-local fields satisfying (3.102), and the representation $\oplus_s[D(s+1,s) \oplus D(-s-1,s)]$, realized on a formal series that satisfies (4.21). Specifically, the map is given by

$$\Phi(u,v) = (FK)(y,z) \tag{4.22}$$

where $y = u + v$, $z = u - v$ and the kernel is

$$F = \sum_k (4^k k!)^{-1} (z \cdot \partial y)^{2k} / (\hat{n} + 1)(\hat{n} + 2) \cdots (\hat{n} + k). \tag{4.23}$$

It is easy to show that

$$(y \cdot \partial_y + z \cdot \partial_z + 1)K = 0 \Rightarrow (u \cdot \partial_u + v \cdot \partial_v + 1)\Phi = 0. \tag{4.24}$$

Furthermore, by using the identity $F z_\alpha y \cdot \partial_z = (u - v)_\alpha (u \cdot \partial_u - v \cdot \partial_v)F$, one finds

$$y \cdot \partial_z K = 0 \Rightarrow (u \cdot \partial_u - v \cdot \partial_v)\Phi = 0, \tag{4.25}$$

From (4.24, 4.25), we have

$$(u \cdot \partial_u + \frac{1}{2})\Phi = (v \cdot \partial_v + \frac{1}{2})\Phi = 0. \tag{4.26}$$

The equation of motion, gauge condition and the traceless condition

$$\partial_y^2 K = \partial_y \cdot \partial_z K = \partial_z^2 K = 0, \tag{4.27}$$

together translates into

$$\partial_u^2 \Phi = \partial_u \cdot \partial_v \Phi = \partial_v^2 \Phi = 0, \tag{4.28}$$

using the mapping (4.22). In summary, one can show the correspondence

$$\left\{ \begin{matrix} y \cdot \partial_z K = 0 \\ (\hat{n} + \hat{N} + 1)K = 0 \end{matrix} \right\} \Longleftrightarrow \left\{ \begin{matrix} (u \cdot \partial_u + \frac{1}{2})\Phi = 0 \\ (v \cdot \partial_v + \frac{1}{2})\Phi = 0 \end{matrix} \right\}, \tag{4.29}$$

$$\left\{ \begin{matrix} (Q - \langle Q \rangle)K = 0 \\ \partial_z^2 K = \partial_y \cdot \partial_z K = 0 \end{matrix} \right\} \Longleftrightarrow \left\{ \begin{matrix} \partial_u^2 \Phi = \partial_v^2 \Phi = 0 \\ \partial_u \cdot \partial_v \Phi = 0 \end{matrix} \right\}. \tag{4.30}$$

This finishes the proof of two singletons generate the massless higher spin fields, at the free level.

4.2 The interaction problem

We have seen that free massless fields of higher spin are gauge fields. In order to preserve gauge invariance, we need to formulate the dynamics of higher spin fields on the AdS background in a form which is convenient for finding the symmetry of the higher spins [56]. In the case of four dimensions, it is convenient to use the formalism of two-component spinors (see appendix B for the conventions). Lower or upper indices denoted by a single letter are understood to correspond to symmetrization. Instead of writing $A_{\alpha_1 \dots \alpha_s}$, we shall write $A_{\alpha(s)}$. For example, the quantity $B_{\alpha(n)}{}^{\beta(m)}$ denotes the multi-spinor

$$\frac{1}{n!m!}\{B_{\alpha_1 \dots \alpha_n}{}^{\beta_1 \dots \beta_m} + (n!-1) \text{ permutations of } \alpha + (m!-1) \text{ permutations of } \beta\}. \quad (4.31)$$

4.2.1 Linearized curvature for physical fields

In general, the Lorentz-covariant curvatures linearized on an AdS background for the systems of fields corresponding to spin s have the form

$$R^L_{\nu\mu,\alpha(n),\dot{\beta}(m)} = [D^L_\nu \omega_{\mu,\alpha(n),\dot{\beta}(m)} + \lambda(n h_{\nu\alpha\dot{\delta}} \omega_{\mu,\alpha(n-1),\dot{\beta}(m)}{}^{\dot{\delta}} + m h_{\nu\gamma\dot{\beta}} \omega_{\mu,\alpha(n)}{}^{\gamma}{}_{,\dot{\beta}(m-1)})] - [\nu \leftrightarrow \mu], \quad (4.32)$$

where λ is the inverse of the AdS radius and the spin s is related to the indices by $n+m=2(s-1)$, D^L_ν is the Lorentz-covariant derivative for the background connection

$$D^L_\nu\omega_{\mu,\alpha(n),\dot\beta(m)} = \partial_\nu\omega_{\mu,\alpha(n),\dot\beta(m)} + nw_{\nu\alpha\gamma}\omega_{\mu,\alpha(n-1)}{}^{\gamma}{}_{,\dot\beta(m)} + m\bar w_{\nu\dot\beta\dot\delta}\omega_{\mu,\alpha(n),\dot\beta(m-1)}{}^{\dot\delta}. \quad (4.33)$$

The gauge transformation corresponding to the curvature (4.32) has the form

$$\delta^L\omega_{\nu,\alpha(n),\dot\beta(m)} = D^L_\nu\xi_{\alpha(n),\beta(m)} + \lambda(nh_{\nu\alpha\dot\delta}\xi_{\mu,\alpha(n-1),\dot\beta(m)}{}^{\dot\delta} + mh_{\nu\gamma\dot\beta}\xi_{\mu,\alpha(n)}{}^{\gamma}{}_{,\dot\beta(m-1)}). \quad (4.34)$$

The action for a free massless field of spin s on the AdS background is

$$S^{(2)}_s = \frac{\gamma^s}{2}\sum_{n,m}\frac{i^{n+m+1}}{n!m!}\delta(n+m-2(s-1))\epsilon(n-m)\int d^4x\epsilon^{\nu\mu\rho\sigma}R^L_{\nu\mu,\alpha(n),\dot\beta(m)}R^L_{\rho\sigma}{}^{\alpha(n),\dot\beta(m)}, \quad (4.35)$$

where γ^s is some normalization coefficient and $\epsilon(n-m)$ picks up the sign of $n-m$. This form of the action follows closely the MacDowell-Mansuri action of Einstein gravity [85] (see appendix B for more details).

An important property of this quadratic action is that its variation with respect to $\omega(n,m)$ with $|n-m|>2$ vanishes identically due to the Bianchi identities

$$\epsilon^{\nu\mu\rho\sigma}D^L_\rho R^L_{\nu\mu,\alpha(n),\dot\beta(m)} = -\lambda\epsilon^{\nu\mu\rho\sigma}(nh_{\rho\alpha\dot\delta}R^L_{\nu\mu,\alpha(n-1),\dot\beta(m)}{}^{\dot\delta} + mh_{\rho\gamma\dot\beta}R^L_{\nu\mu,\alpha(n)}{}^{\gamma}{}_{,\dot\beta(m-1)}). \quad (4.36)$$

This means the action contains only fields $\omega(n,m)$ with $|n-m|\le 2$. In the bosonic case, these fields are $\omega(s-2,s),\omega(s,s-2),\omega(s-1,s-1)$. The corresponding equations of motion read

$$\epsilon^{\nu\mu\rho\sigma}h_{\rho\alpha\dot\delta}R^L_{\nu\mu,\alpha(s-1),\dot\beta(s-2)}{}^{\dot\delta} = 0, \quad (4.37)$$

$$\epsilon^{\nu\mu\rho\sigma}h_{\rho\gamma\dot\beta}R^L_{\nu\mu,\alpha(s-2)}{}^{\gamma}{}_{,\dot\beta(s-1)} = 0, \quad (4.38)$$

$$\epsilon^{\nu\mu\rho\sigma}[h_{\rho\alpha\dot{\delta}}R^L_{\nu\mu,\alpha(s-2),\dot{\beta}(s-1)}{}^{\dot{\delta}} - h_{\rho\gamma\dot{\beta}}R^L_{\nu\mu,\alpha(s-1)}{}^{\gamma}{}_{,\dot{\beta}(s-2)}] \;=\; 0. \tag{4.39}$$

The first two equations are equivalent to the equation

$$R^L_{\nu\mu,\alpha(s-1),\dot{\beta}(s-1)} = 0, \tag{4.40}$$

which is analogous to the equation of zero curvature in gravitation. The fields $\omega(s, s-2)$ and $\omega(s-2, s)$ are the analogs of the Lorentz connection and the field $\omega(s-1, s-1)$ is the analog of the tetrad. Similar to what occurs in gravitation for the Lorentz connection, the fields $\omega(s, s-2)$ and $\omega(s-2, s)$ are auxiliary fields up to a purely gauge piece, which corresponds to the parameters $\xi(s+1, s-3)$ and $\xi(s-3, s+1)$ and drops out of the equations of motion, can be expressed in terms of the first derivatives of the dynamical fields $\omega(s-1, s-1)$ by solving (4.37, 4.38) or equivalently (4.40). Plugging into (4.39), one obtains a second-order equation for the dynamical field $\omega(s-1, s-1)$ which describes a massless field of spin s.

When the interaction is switched on "naively" the variation of the action with respect to the "extra" fields $\omega(n, m)$ with $|n-m| > 2$ is nonzero, so the extra fields correspond to essentially nonlinear equations. It is desirable that the extra fields somehow can be expressed in terms of the physical fields already at the linearized level. This can be done by requiring that the fields $\omega(n, m)$ satisfy the following system of motions

$$R^L_{\nu\mu,\alpha(n),\dot{\beta}(m)} \;=\; 0, \qquad n > 0, m > 0, n+m = 2(s-1) \tag{4.41}$$

$$\epsilon^{\nu\mu\rho\sigma}R^L_{\nu\mu,\alpha(2s-2),\dot{\beta}(0)}h_{\rho}{}^{\alpha}{}_{\dot{\beta}} \;=\; 0, \tag{4.42}$$

$$\epsilon^{\nu\mu\rho\sigma}R^L_{\nu\mu,\alpha(0),\dot{\beta}(2s-2)}h_{\rho\alpha}{}^{\dot{\beta}} \;=\; 0. \tag{4.43}$$

It can be shown that these equations contain the equations for the physical fields and express all the extra fields in terms of the physical fields without imposing any additional constraints on the latter. At the linearized level the role of (4.42-4.43) can be replaced by the equations

$$\epsilon^{\nu\mu\rho\sigma} R^{L}_{\nu\mu,\alpha(n),\dot\beta(m)} h_{\rho}{}^{\alpha}{}_{\dot\beta} = 0, \qquad m \geq n \geq 1, \tag{4.44}$$

$$\epsilon^{\nu\mu\rho\sigma} R^{L}_{\nu\mu,\alpha(n),\dot\beta(m)} h_{\rho\alpha}{}^{\dot\beta} = 0, \qquad n \geq m \geq 1. \tag{4.45}$$

The solution leads to recurrence relations which express successively all "extra fields" in terms of the dynamical fields [56]. A consequence of this mechanism is that higher-spin interactions for the dynamical fields contain higher derivatives. The same mechanism leads to the non-analyticity of the interaction terms in the cosmological constant.

4.2.2 Linearized curvature for auxiliary fields

Consistent linearized curvatures describing auxiliary fields were constructed in [57], which contain systems of auxiliary fields $a_{\nu,\alpha(n),\dot\beta(m)}$ with $n \geq 0$, $m \geq 0$, $n - m = k$ where $k = 0, \pm 1, \pm 2...$ is an arbitrary fixed integer

$$A_{\nu\mu,\alpha(n),\dot\beta(m)} = D^L_\nu a_{\mu,\alpha(n),\dot\beta(m)} - i\lambda h_{\nu\gamma\dot\delta} a_{\mu,\alpha(n)}{}^{\gamma}{}_{,\dot\beta(m)}{}^{\dot\delta} + i\lambda nm h_{\nu\alpha\dot\beta} a_{\mu,\alpha(n-1),\dot\beta(m-1)} - (\nu \leftrightarrow \mu) \tag{4.46}$$

where D^L_ν is the background Lorentz covariant derivative

$$D^L_\nu a_{\mu,\alpha(n),\dot\beta(m)} = \partial_\nu a_{\mu,\alpha(n),\dot\beta(m)} + n w_{\nu\alpha\gamma} a_{\mu,\alpha(n-1)}{}^{\gamma}{}_{,\dot\beta(m)} + m \bar{w}_{\nu\dot\beta\dot\delta} a_{\mu,\alpha(n),\dot\beta(m-1)}{}^{\dot\delta}. \tag{4.47}$$

Linearized (abelian) gauge transformations read

$$\delta a_{\nu,\alpha(n),\dot\beta(m)} = D^L_\nu \eta_{\alpha(n),\dot\beta(m)} - i\lambda h_{\nu\gamma\dot\delta} \eta_{\alpha(n)}{}^{\gamma}{}_{,\dot\beta(m)}{}^{\dot\delta} + i\lambda nm h_{\nu\alpha\dot\beta} \eta_{\alpha(n-1),\dot\beta(m-1)}. \tag{4.48}$$

The curvatures (4.46) are consistent in the sense that they are invariant under the gauge transformation (4.48). In addition, they satisfy the Bianchi identities

$$\epsilon^{\mu\nu\rho\sigma} D^L_\mu A_{\rho\sigma,\alpha(n),\dot\beta(m)} = i\lambda\epsilon^{\nu\mu\rho\sigma}[h_{\mu\gamma\dot\delta} A_{\rho\sigma,\alpha(n)}{}^{\gamma}{}_{,\dot\beta(m)}{}^{\dot\delta} - nmh_{\mu\alpha\dot\beta} A_{\rho\sigma,\alpha(n-1),\dot\beta(m-1)}]. \quad (4.49)$$

The gauge invariant action takes the form

$$S_{uv} = \frac{i}{2\lambda} \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \delta(n-m-u+v)\theta(u+v-n-m) \int d^4x \epsilon^{\nu\mu\rho\sigma} A_{\nu\mu,\alpha(n),\dot\beta(m)} A_{\rho\sigma,}{}^{\alpha(n),\dot\beta(m)} + h.c. \quad (4.50)$$

One can see that the action really contains only the fields $a(u,v)$, $a(u+1,v+1)$ and their conjugates $b(v,u)$, $b(v+1,u+1)$. The equations of motion for $a(u,v)$ and $a(u+1,v+1)$ are

$$\epsilon^{\nu\mu\rho\sigma} A_{\nu\mu,\alpha(u)}{}^{\gamma}{}_{,\dot\beta(v)}{}^{\dot\delta} h_{\rho\gamma\dot\delta} = 0, \quad (4.51)$$

$$\epsilon^{\nu\mu\rho\sigma} A_{\nu\mu,\alpha(u),\dot\beta(v)} h_{\rho\alpha\dot\beta} = 0. \quad (4.52)$$

For arbitrary integer $u \geq 0$, $v \geq 0$, the action describes the systems of auxiliary fields, i.e. that the number of independent functions of three spatial coordinates is reduced to zero after imposing of full gauge invariance. In order to have a closed system of equations for the whole set of fields $a(n,m)$ with $n \geq 0$, $m \geq 0$, $n - m = k$ for any fixed k, one can impose the following constraints

$$\epsilon^{\nu\mu\rho\sigma} A_{\nu\mu,\alpha(n),\dot\beta(m)} h_{\rho\alpha\dot\beta} = 0. \quad (4.53)$$

We suppose now that the constraints (4.53) are imposed simultaneously with the dynamic equations (4.51, 4.52) for $u = k, v = 0$ or $u = 0, v = -k$. It is clear that for $k \neq 0$ all fields $a(n,m)$ with $n - m = k$ will be auxiliary. Indeed, (4.52) is contained among (4.53) and provides no new information. As for (4.51), one can show by virtue of the Bianchi identity (4.49) that the following equations must be satisfied

$$
\begin{aligned}
A_{\nu\mu,\alpha(n),\dot\beta(m)} &= 0, \qquad n>0, m>0, n-m=k\neq 0, && (4.54)\\
\epsilon^{\nu\mu\rho\sigma}A_{\nu\mu,\alpha(k),\dot\beta(0)}h_{\rho\alpha\dot\beta} &= 0, \qquad k>0, && (4.55)\\
\epsilon^{\nu\mu\rho\sigma}A_{\nu\mu,\alpha(0),\dot\beta(-k)}h_{\rho\alpha\dot\beta} &= 0, \qquad k<0, && (4.56)
\end{aligned}
$$

which reminds us (4.41-4.43). This is seen most easily by induction. At the linearized level, for some $n=n_0>0$, $m=m_0>0$, the roles of last two equations (4.55-4.56) can be played by

$$
\begin{aligned}
\epsilon^{\nu\mu\rho\sigma}h_{\rho}{}^{\alpha\dot\beta}A_{\nu\mu,\alpha(n),\dot\beta(m)} &= 0, && (4.57)\\
\epsilon^{\nu\mu\rho\sigma}h_{\rho\alpha\dot\beta}A_{\nu\mu,\alpha(n),\dot\beta(m)} &= 0, && (4.58)
\end{aligned}
$$

which is similar to (4.44-4.45).

4.2.3 The complete description

A complete description of the linearized equations of motion (4.41-4.43) should include two auxiliary fields C and $\bar C$ (which are conjugate to each other) [58]

$$
R^L_{\nu\mu,\alpha(n),\dot\beta(m)} = \delta(m)h_{\nu}{}^{\gamma}{}_{\dot\beta}h_{\mu}{}^{\gamma\dot\beta}C_{\alpha(n)\gamma(2)} + \delta(n)h_{\nu\gamma}{}^{\dot\delta}h_{\mu}{}^{\gamma\dot\delta}\bar C_{\dot\beta(m)\dot\delta(2)}. \qquad (4.59)
$$

The fields ω are assumed to obey the hermiticity conditions $(\omega_{\nu,\alpha(n),\dot\beta(m)})^\dagger = \omega_{\nu,\beta(m),\dot\alpha(n)}$. These gauge fields are assumed to be (anti-) commuting if the number of spinor indices $n+m$ is (odd) even. It follows from the consistency conditions that C and $\bar C$

obey the restrictions

$$\epsilon^{\nu\mu\rho\sigma} h_{\nu}{}^{\gamma}{}_{\dot\beta} h_{\mu}{}^{\gamma\dot\beta} D^L_\rho C_{\alpha(n)\gamma(2)} = 0, \tag{4.60}$$

$$\epsilon^{\nu\mu\rho\sigma} h_{\nu\alpha}{}^{\dot\delta} h_{\mu}{}^{\alpha\dot\delta} D^L_\rho \bar C_{\dot\beta(m)\dot\delta(2)} = 0. \tag{4.61}$$

It is not difficult to make sure the above two equations are equivalent to

$$D^L_\rho C_{\alpha(n),\dot\beta(m)} - i h_\rho{}^{\gamma\dot\delta} C_{\alpha(n)\gamma,\dot\beta(m)\dot\delta} + inm h_{\rho\alpha\dot\beta} C_{\alpha(n-1),\dot\beta(m-1)} = 0, \tag{4.62}$$

$$D^L_\rho \bar C_{\alpha(n),\dot\beta(m)} - i h_\rho{}^{\gamma\dot\delta} \bar C_{\alpha(n)\gamma,\dot\beta(m)\dot\delta} + inm h_{\rho\alpha\dot\beta} \bar C_{\alpha(n-1),\dot\beta(m-1)} = 0. \tag{4.63}$$

Any spin s is described by the fields $\omega(n,m)$ with $n+m=2(s-1)$ when $s\geq 1$, $C(n,m)$ with $n-m=2s$ and $\bar C(n,m)$ with $m-n=2s$. Physical spin-s fields are identified with the 1-forms $\omega(n,m)$ at $|n-m|\leq 1$ when $s\geq 1$ or with $C(2s,0)$ and $\bar C(0,2s)$ when $s<1$. All other fields belonging to the chains above can be expressed algebraically in terms of the physical fields and their derivatives by means of the equations (4.59,4.62,4.63).

Let us now demonstrate that free equations of motion for the auxiliary fields $a_{\nu,\alpha(n),\dot\beta(m)}$ can be dealt with in a quite similar fashion. Linearized curvatures for these fields satisfy the equation

$$A^L_{\nu\mu,\alpha(n),\dot\beta(m)} = \delta(m)\theta(n-2) h_{\nu\alpha\dot\delta} h_{\mu\alpha}{}^{\dot\delta} \mathcal{D}_{\alpha(n-2)} + \delta(n)\theta(m-2) h_{\nu\gamma\dot\beta} h_{\mu}{}^{\gamma}{}_{\dot\beta} E_{\dot\beta(m-2)}. \tag{4.64}$$

Here $\mathcal{D}(n-2,0)$ and $E(0,m-2)$ are "auxiliary Weyl 0-forms" which can be viewed as some new independent variables analogous to the higher-spin Weyl 0-forms introduced previously. Quite similar to the case of massless fields, one can make sure that (4.64) leads to the following chains of consistency conditions

$$D^L_\rho \mathcal{D}_{\alpha(n),\dot\beta(m)} + nh_{\rho\alpha\dot\delta}\mathcal{D}_{\alpha(n-1),\dot\beta(m)}{}^{\dot\delta} + mh_{\rho\gamma\dot\beta}\mathcal{D}_{\alpha(n)}{}^{\gamma}{}_{,\dot\beta(m-1)} = 0, \quad (4.65)$$

$$D^L_\rho E_{\alpha(n),\dot\beta(m)} + nh_{\rho\alpha\dot\delta}E_{\alpha(n-1),\dot\beta(m)}{}^{\dot\delta} + mh_{\rho\gamma\dot\beta}E_{\alpha(n)}{}^{\gamma}{}_{,\dot\beta(m-1)} = 0. \quad (4.66)$$

Note that these two equations contain no additional dynamic conditions and merely express all 0-forms $\mathcal{D}(n,m)$ and $E(n,m)$ in terms of $\mathcal{D}(k,0)$ and $E(0,l)$.

Next we observe the structure of the equations and find that both 1-forms (ω, a) and Wyel 0-forms $(C, \bar{C}, \mathcal{D}, E)$ belong to the adjoint representation of some Lie superalgebra shsa(1) incorporating both the massless fields and the auxiliary fields. This infinite-dimensional superalgebra gives rise to the set of gauge fields $\omega_{\nu,}{}^{AB}{}_{\alpha(n),\dot\beta(m)}$ with the indices A, B taking values 0 or 1. The curvatures of shsa(1) read

$$\begin{aligned} R_{\nu\mu,}{}^{AB}{}_{\alpha(n),\dot\beta(m)} = & \Big(\partial_\nu \omega_{\mu,}{}^{AB}{}_{\alpha(n),\dot\beta(m)} + \frac{1}{2}\sum_{p,q,s,k,l,t,C,\mathcal{D},F,G} \delta(n-p-q) \\ & \times\delta(m-k-l)\delta(|A+C+F|_2)\delta(|B+\mathcal{D}+G|_2) \\ & \times i^{s+t-1}(-1)^{F(p+s)+G(k+t)}\frac{n!m!}{p!q!s!k!l!t!} \\ & \times\omega_{\nu,}{}^{C\mathcal{D}}{}_{\alpha(p)\gamma(s),\dot\beta(k)\dot\delta(t)}\omega_{\mu,}{}^{FG}{}_{\alpha(q)}{}^{\gamma(s)}{}_{,\dot\beta(l)}{}^{\dot\delta(t)}\Big) - (\nu \leftrightarrow \mu), \end{aligned} \quad (4.67)$$

where $|n|_2 = 0$ for $n = 2k$ and $|n|_2 = 1$ for $n = 2k+1$. The fields $\omega^{AA}(n,m)$ with $A = 0,1$ are identified with the massless fields, while the fields $\omega^{AB}(n,m)$ with $A + B = 1$ are auxiliary fields. This identification is due to the fact that, after linearization, the curvature (4.67) leads to (4.59) when $A = B$ and (4.64) when $A + B = 1$. All the Weyl 0-forms introduced above will be assumed to belong to the adjoint representation of shsa(1) which is described by the quantities $C^{AB}{}_{\alpha(n),\dot\beta(m)}$.

One can now rewrite all the linearized equations for massless and auxiliary fields in the following uniform way

$$R^{L}_{\nu\mu,}{}^{AB}{}_{\alpha(n),\dot\beta(m)} + \eta_1\delta(n)\delta(|A+B|_2)C^{|A+1|_2 B}{}_{\alpha(0),\dot\beta(m)\dot\delta(2)}h_{\nu\gamma}{}^{\dot\delta}h_{\mu}{}^{\gamma\dot\delta}$$
$$+\bar\eta_1\delta(m)\delta(|A+B|_2)C^{A|A+1|_2}{}_{\alpha(n)\gamma(2),\dot\beta(0)}h_{\nu}{}^{\gamma}{}_{\dot\delta}h_{\mu}{}^{\gamma\dot\delta}$$
$$-\eta_1\delta(n)m(m-1)\delta(|A+B+1|_2)C^{BB}{}_{\alpha(0),\dot\beta(m-2)}h_{\nu\gamma\dot\beta}h_{\mu}{}^{\gamma}{}_{\dot\beta}$$
$$-\bar\eta_1\delta(m)n(n-1)\delta(|A+B+1|_2)C^{AA}{}_{\alpha(n-2),\dot\beta(0)}h_{\nu\alpha\dot\delta}h_{\mu\alpha}{}^{\dot\delta} = 0, \quad (4.68)$$
$$\mathcal{D}^{L}_{\nu}C^{AB}{}_{\alpha(n),\dot\beta(m)} = 0, \quad (4.69)$$

where $\mathcal{D}^{L}_{\nu}$ includes three terms: the covariant derivative D^{L}_{ν} plus two background terms.

4.2.4 The cubic interaction

An extension of the quadratic action (4.35), which will describe the interactions of massless higher spin fields, does exist at least in the first nontrivial order [59, 60]. The corresponding curvature and infinitesimal gauge transformations with parameter ϵ are

$$\begin{aligned}
R_{\nu\mu,\alpha(n),\dot\beta(m)} &= \partial_\nu\omega_{\mu,\alpha(n),\dot\beta(m)} - \partial_\mu\omega_{\nu,\alpha(n),\dot\beta(m)} \\
&\quad + \sum_{p,q,s,k,l,t=0}^{\infty} i^{s+t-1}\frac{n!m!}{p!q!s!k!l!t!}\delta(n-p-q)\delta(m-k-l) \\
&\quad \times\lambda^{1+\frac{1}{2}(|n-m|-|p+s-k-t|-|q+s-l-t|)} \\
&\quad \times\delta(|(p+k)(q+l)+(p+k)(s+t)+(q+l)(s+t)+1|_2) \\
&\quad \omega_{\nu,\alpha(p)\gamma(s),\dot\beta(k)\dot\delta(t)}\omega_{\mu,\alpha(q)}{}^{\gamma(s)}{}_{,\dot\beta(l)}{}^{\dot\delta(t)}, \qquad (4.70) \\
\delta\omega_{\nu,\alpha(n),\dot\beta(m)} &= \partial_\nu\epsilon_{\alpha(n),\dot\beta(m)} \\
&\quad + \sum_{p,q,s,k,l,t=0}^{\infty} i^{s+t-1}\frac{n!m!}{p!q!s!k!l!t!}\delta(n-p-q)\delta(m-k-l) \\
&\quad \times\lambda^{1+\frac{1}{2}(|n-m|-|p+s-k-t|-|q+s-l-t|)}
\end{aligned}$$

$$\times\delta(|(p+k)(q+l)+(p+k)(s+t)+(q+l)(s+t)+1|_2)$$
$$\omega_{\nu,\alpha(p)\gamma(s),\dot\beta(k)\dot\delta(t)}\epsilon_{\alpha(q)}{}^{\gamma(s)}{}_{,\dot\beta(l)}{}^{\dot\delta(t)}, \tag{4.71}$$

where we use the notation $\delta(n)=1$ for $n=0$, zero otherwise; $\theta(n)=1$ for $n\geq 0$, zero otherwise. The proposal for the action is

$$S=\frac{1}{2}\sum_{n+m>0} i^{n+m+1}\frac{1}{n!m!}\beta(n+m)\epsilon(n-m)\lambda^{-|n-m|}\int d^4x\epsilon^{\nu\mu\rho\sigma}R_{\nu\mu,\alpha(n),\dot\beta(m)}R_{\rho\sigma,}{}^{\alpha(n),\dot\beta(m)}, \tag{4.72}$$

where $\beta[2(s-1)]$ is a normalization coefficient for the free action of spin s. This action is a generalization of the MacDowell-Mansouri action for a (super) gravity with a cosmological term. The part of the action which depends on only the gravitational fields $\omega(n,m)$ with $n+m=2$ is the same as the action of the pure gravity.

As we mentioned before, an important property of the action is that in the quadratic approximation its variation over all the "extra" fields $\omega(n,m)$ with $|n-m|>2$ vanishes identically. As a result, the free higher spins are described exclusively by "dynamic" $\omega(n,m)$ with $|n-m|\leq 2$. When the interaction is turned on, however, the variation of the action in terms of the extra fields is nonzero, so that it cannot be interpreted in a reasonable way if the extra fields are assumed to be independent dynamic variables. A way out of this difficulty is to express all the extra fields from the outset in terms of dynamical fields by means of certain constraints. Constraints making this possible at a linearized level are (4.44-4.45). Remarkably, even the linearized constraints are sufficient to prove the invariance of the action (4.72) in the cubic approximation. The extra fields appear only in nonlinear combinations of the type $R^l\omega\omega$, so that it is sufficient to know their linearized expressions in the cubic approximation. In summary, the action (4.72) supplemented by constraints (4.44-4.45), gives a non-contradictory description of the dynamics of all massless fields with higher spins in the cubic approximation.

4.3 Vasiliev's full nonlinear theory

The extension of the cubic action to the full nonlinear level is highly nontrivial. However, the full nonlinear set of equations of motion was found by Vasiliev [61]. To describe on-shell higher-spin dynamics in $d = 3+1$, we introduce the following set of generating functions

$$W = dx^{\nu} W_{\nu}(x|y, \bar{y}, z, \bar{z}), \tag{4.73}$$

$$S = dz^{\alpha} S_{\alpha}(x|y, \bar{y}, z, \bar{z}) + d\bar{z}^{\dot{\alpha}} \bar{S}_{\dot{\alpha}}(x|y, \bar{y}, z, \bar{z}), \tag{4.74}$$

$$B = B(x|y, \bar{y}, z, \bar{z}), \tag{4.75}$$

where $Y = (y_{\alpha}, \bar{y}_{\dot{\alpha}})$ and $Z = (z_{\alpha}, \bar{z}_{\dot{\alpha}})$ are two independent sets of auxiliary spinor variables and x denotes the spacetime coordinates. The Vasiliev's equation for higher-spin fields are

$$dW = W * W, \tag{4.76}$$

$$dB = W * B - B * W, \tag{4.77}$$

$$dS = W * S - S * W, \tag{4.78}$$

$$S * S = -i\{dz_{\alpha} dz^{\alpha}[1 + F(B) * \kappa] + d\bar{z}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}}[1 + \bar{F}(B) * \bar{\kappa}]\}, \tag{4.79}$$

$$S * B = B * S, \tag{4.80}$$

where the operator $d = dx^{\nu}(\partial/\partial x^{\nu})$ and the star-product is defined as

$$f(Y, Z) * g(Y, Z) = \int d^4U d^4V e^{i(u_{\alpha} v^{\alpha} + \bar{u}_{\dot{\alpha}} \bar{v}^{\dot{\alpha}})} f(Y + U, Z + U) g(Y + V, Z - V), \tag{4.81}$$

where $U = (u_{\alpha}, \bar{u}_{\dot{\alpha}})$ and $V = (v_{\alpha}, \bar{v}_{\dot{\alpha}})$ are the integration variables.

The operators κ and $\bar{\kappa}$ takes the form

$$\kappa = k\exp(i\langle z, y\rangle), \qquad \bar{\kappa} = \bar{k}\exp(i\langle \bar{z}, \bar{y}\rangle). \tag{4.82}$$

By definition, the Klein operators k and $\bar{k}$ anticommute with all undotted and dotted spinors, respectively,

$$kf(z,\bar{z},y,\bar{y};dz,d\bar{z};K) = f(-z,\bar{z},-y,\bar{y};-dz,d\bar{z};K)k, \tag{4.83}$$

$$\bar{k}f(z,\bar{z},y,\bar{y};dz,d\bar{z};K) = f(z,-\bar{z},y,-\bar{y};dz,-d\bar{z};K)\bar{k}. \tag{4.84}$$

In addition, it is required that

$$k^2 = \bar{k}^2 = 1, \quad [k,\bar{k}] = 0, \quad [k, dx^\nu] = [\bar{k}, dx^\nu] = 0. \tag{4.85}$$

The equations (4.76-4.80) are explicitly invariant under the gauge transformations

$$\delta W = d\epsilon - W * \epsilon + \epsilon * W, \tag{4.86}$$

$$\delta B = \epsilon * B - B * \epsilon, \tag{4.87}$$

$$\delta S = \epsilon * S - S * \epsilon. \tag{4.88}$$

The field variables W, B, S are assumed to obey the (anti)hermiticity (reality) conditions

$$W^\dagger = -W, \qquad B^\dagger = B, \qquad S^\dagger = -S, \tag{4.89}$$

defined by the relations

$$(z_\alpha)^\dagger = -\bar{z}_{\dot{\alpha}}, \quad (dz_\alpha)^\dagger = d\bar{z}_{\dot{\alpha}}, \quad (y_\alpha)^\dagger = \bar{y}_{\dot{\alpha}}, \quad (dx_\nu)^\dagger = dx_\nu, \quad k^\dagger = \bar{k}. \tag{4.90}$$

To complete the explanation of the Vasiliev's equations, $F(B)$ and $\bar{F}(B)$ are arbitrary functions of the form

$$F(B) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n \underbrace{B * ... * B}_{n}, \qquad \bar{F}(B) = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{f}_n \underbrace{B * ... * B}_{n}, \tag{4.91}$$

where f_n are arbitrary complex coefficients. We will mainly study the *minimal* Vasiliev's theory where $F(B) = f_1 B$, $\bar{F}(B) = \bar{f}_1 B$.

4.3.1 Linear approximation

In this subsection, we will expand Vasiliev's full nonlinear theory to the linearized order and find agreement with Fronsdal's free higher-spin theory. First of all, the vacuum solution is

$$B_0 = 0, \tag{4.92}$$

$$S_0 = dz^\alpha z_\alpha + d\bar{z}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}, \tag{4.93}$$

$$W_0 = \frac{1}{4i}[{\omega_0}^{\alpha\beta}(x) y_\alpha y_\beta + {\bar{\omega}_0}^{\dot{\alpha}\dot{\beta}}(x) \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2{h_0}^{\alpha\dot{\beta}}(x) y_\alpha \bar{y}_{\dot{\beta}}], \tag{4.94}$$

where $\omega_0, \bar{\omega}_0, h_0$ describe the background AdS spacetime.

The first-order equations are

$$dW^{(1)} = W_0 * W^{(1)} + W^{(1)} * W_0, \tag{4.95}$$

$$dB^{(1)} = W_0 * B^{(1)} - B^{(1)} * W_0, \tag{4.96}$$

$$dS^{(1)} = W_0 * S^{(1)} - S^{(1)} * W_0 + W^{(1)} * S_0 - S_0 * W^{(1)}, \tag{4.97}$$

$$S_0 * S^{(1)} + S^{(1)} * S_0 = -i\{dz_\alpha dz^\alpha f_1 B^{(1)} * \kappa + d\bar{z}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}} \bar{f}_1 B^{(1)} * \bar{\kappa}\}, \tag{4.98}$$

$$S_0 * B^{(1)} = B^{(1)} * S_0. \tag{4.99}$$

The last equation (4.99) says

$$[S_0, B^{(1)}]_* = -2i\Big(dz^\alpha \frac{\partial}{\partial z^\alpha} + d\bar{z}^{\dot{\alpha}} \frac{\partial}{\partial \bar{z}^{\dot{\alpha}}}\Big) B^{(1)} = 0. \tag{4.100}$$

Therefore, the first-order B field is independent of Z as we denote it as

$$B^{(1)}(Z;Y;K|x) = C(Y;K|x), \tag{4.101}$$

which satisfy the second equation (4.96) of the first-order equations

$$dC = W_0 * C - C * W_0. \tag{4.102}$$

The field S is purely auxiliary with the first-order takes the form

$$S^{(1)} = dz^\alpha S^{(1)}_\alpha + d\bar{z}^{\dot{\alpha}} \bar{S}^{(1)}_{\dot{\alpha}} = -dz_\alpha S^{(1)\alpha} - d\bar{z}_{\dot{\alpha}} \bar{S}^{(1)\dot{\alpha}}. \tag{4.103}$$

Notice the anti-commuting of spinorial differentials, the left hand side of the fourth equation (4.98) can be written as

$$S_0 * S^{(1)} + S^{(1)} * S_0 = -2i\Big(dz_\alpha dz^\alpha \frac{\partial}{\partial z^\alpha} S^{(1)\alpha} + d\bar{z}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}} \frac{\partial}{\partial \bar{z}^{\dot{\alpha}}} \bar{S}^{(1)\dot{\alpha}}\Big), \tag{4.104}$$

where we emphasize that the four α or $\dot{\alpha}$ are the same and summed over 1,2. Therefore, comparing to the right side of the fourth equation, one gets

$$\frac{\partial}{\partial z^\alpha} S^{(1)\alpha} = f_1 C(-z, \bar{y}; K) k \exp(i\langle z, y\rangle), \tag{4.105}$$

$$\frac{\partial}{\partial \bar{z}^{\dot{\alpha}}} \bar{S}^{(1)\dot{\alpha}} = \bar{f}_1 C(y, -\bar{z}; K) \bar{k} \exp(i\langle \bar{z}, \bar{y}\rangle), \tag{4.106}$$

where we used the property of star product that κ changes $y \to -z$, $z \to -y$, similarly for $\bar{\kappa}$. Using the formula

$$\frac{\partial}{\partial z^\alpha} f^\alpha(z) = g(z) \Longrightarrow f_\alpha(z) = \int_0^1 dt t z_\alpha g(tz), \tag{4.107}$$

the first-order S field which can be evaluated in terms of C as

$$S^{(1)} = \int_0^1 dt t[dz^\alpha z_\alpha f_1 C(-tz, \bar{y}; K) k \exp(it\langle z, y\rangle) + d\bar{z}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}} \bar{f}_1 C(y, -t\bar{z}; K) \bar{k} \exp(it\langle \bar{z}, \bar{y}\rangle)]. \tag{4.108}$$

The field W can be splitted into Z-independent part ω and the Z-dependent part. From the third equation (4.97), the Z-dependent part satisfies the equation

$$[S_0, W^{(1)}]_* = -2i\Big(dz^\alpha \frac{\partial}{\partial z^\alpha} + d\bar{z}^{\dot{\alpha}} \frac{\partial}{\partial \bar{z}^{\dot{\alpha}}}\Big) W^{(1)} = -dS^{(1)} + [W_0, S^{(1)}]_*. \tag{4.109}$$

The general solution to the equation $(\partial/\partial z^\alpha)\varphi(z) = \chi_\alpha(z)$ satisfying the condition $(\partial/\partial z^\alpha)\chi^\alpha(z) = 0$ is

$$\varphi(z) = \int_0^1 dt z^\alpha \chi_\alpha(tz). \tag{4.110}$$

The integration over $dS^{(1)}$ vanishes because $z^\alpha z_\alpha = \bar{z}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}} = 0$. Therefore, the first-order W field can be written as

$$W^{(1)}(Z; Y; K) = \omega(Y; K) + \frac{i}{2} \int_0^1 dt \{z^\alpha [W_0, S^{(1)}_\alpha]_*(tz, \bar{z}; Y; K) + \bar{z}^{\dot{\alpha}} [W_0, \bar{S}^{(1)}_{\dot{\alpha}}]_*(z, t\bar{z}; Y; K)\}. \tag{4.111}$$

Now we are in the position to calculate various star products. It turns out to be convenient to split the field $C(k, \bar{k})$ into even or odd functions of $k, \bar{k}$. Consider the expansion

$$C(k, \bar{k}) = C^{00} + kC^{10} + \bar{k}C^{01} + k\bar{k}C^{11}, \tag{4.112}$$

we introduce

$$C(-k,-\bar{k}) = C^{00} - kC^{10} - \bar{k}C^{01} + k\bar{k}C^{11}. \tag{4.113}$$

Therefore, plugging

$$kC^{10} + \bar{k}C^{01} = \frac{1}{2}[C(k,\bar{k}) - C(-k,-\bar{k})], \tag{4.114}$$

$$C^{00} + k\bar{k}C^{11} = \frac{1}{2}[C(k,\bar{k}) + C(-k,-\bar{k})], \tag{4.115}$$

into (4.108, 4.111) and finally the first equation (4.95), the field $\omega(Y;K)$ satisfies the equation

$$\begin{aligned} d\omega(Y;K) &= W_0 * \omega(Y;K) + \omega(Y;K) * W_0 \\ &+\frac{i}{8}\Big(f_1 h_\alpha{}^{\dot\beta} \wedge h^{\alpha\dot\delta} k \frac{\partial}{\partial \bar{y}^{\dot\beta}} \frac{\partial}{\partial \bar{y}^{\dot\delta}} [C(0,\bar{y};k,\bar{k}) - C(0,\bar{y};-k,-\bar{k})] \\ &+\bar{f}_1 h^\alpha{}_{\dot\beta} \wedge h^{\gamma\dot\beta} \bar{k} \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y^\gamma} [C(y,0;k,\bar{k}) - C(y,0;-k,-\bar{k})] \\ &- f_1 h_\alpha{}^{\dot\beta} \wedge h^{\alpha\dot\delta} k \bar{y}_{\dot\beta} \bar{y}_{\dot\delta} [C(0,\bar{y};k,\bar{k}) + C(0,\bar{y};-k,-\bar{k})] \\ &- \bar{f}_1 h^\alpha{}_{\dot\beta} \wedge h^{\gamma\dot\beta} \bar{k} y_\alpha y_\gamma [C(y,0;k,\bar{k}) + C(y,0;-k,-\bar{k})] \Big). \end{aligned} \tag{4.116}$$

To make contact with the conventional formulation of the dynamics of massless fields one should insert the expansions of the fields ω and C (which satisfy the equations (4.116, 4.102)) in powers of the auxiliary variables

$$f(Y;K|x) = \sum_{A,B=0}^{1} \sum_{n,m=0}^{\infty} \frac{1}{2in!m!} (k)^A (\bar{k})^B y_{\alpha_1} ... y_{\alpha_n} \bar{y}_{\dot\beta_1} ... \bar{y}_{\dot\beta_m} f^{AB\alpha_1...\alpha_n,\dot\beta_1...\dot\beta_m}(x), \tag{4.117}$$

which results in (4.68) and (4.69). Here massless fields are described by the fields ω^{AA} and $C^{A\ 1-A}$ while auxiliary fields are described by $\omega^{A\ 1-A}$ and C^{AA}. This finishes the derivation of the free theory from the linear order of the Vasiliev's nonlinear equations of motion. For other relevant work, review and recent developments on higher spin theory, please refer to [62] for more details.

Chapter 5

Collective Field Reconstruction of Higher-Spin Gravity

In this chapter, we pursue the construction of higher-spin theory in AdS_4 from CFT_3 in terms of canonical collective fields. In null-plane quantization an exact map is established between the two spaces. The coordinates of the AdS_4 space-time are generated from the collective coordinates of the bi-local field. This, in the light-cone gauge, provides an exact one-to-one reconstruction of bulk AdS_4 space-time and higher-spin fields.

5.1 Introduction

The AdS/CFT correspondence is characterized by the conjectured emerging dimensions of space-time. In $\mathcal{N} = 4$ Super Yang-Mills theory the $D = 10$ of the string in $AdS_5 \times S^5$ background emerges. While the main understanding of the duality itself is provided by 't Hooft's large N expansion (which establishes $1/N$ as the string coupling constant) the origin of the extra spatial dimension is less clearly understood, one speaks of them as being holographic and have a relationship (in the case of radial AdS dimension) with renormalization group scaling parameters.

One framework for analytical understanding of the large N limit in general introduced several decades ago is based on the notion of collective fields [63]. They capture the relevant degrees of freedom and a general method for describing their effective dynamics both at the Hamiltonian and Lagrangian level was given. This approach has been successful in analytical treatment as well as in exhibiting the relevant physics in various model theories. In the $c = 1$ matrix model collective dynamics naturally led to (one) extra dimension relevant in establishing the model as a 2D non-critical string theory [64]. It has re-emerged in the sub-dynamics of the $\mathcal{N} = 4$ Yang-Mills problem in the 1/2 BPS sub-sector. Through certain matrix model truncations (of $\mathcal{N} = 4$ Yang-Mills theory) the construction of dual string theory Hamiltonian was attempted [65].

For further understanding of this mechanism it is useful to concentrate on exactly solvable theories. The simplest field theory model for which one can build the AdS/CFT correspondence is that of N-component vector theory. It was originally pointed out by Klebanov and Polyakov [66] that the conformal fixed points of the theory are naturally described in four dimensional AdS space-time. In particular, the expected dual is to be given by Vasiliev's higher spin theory. (For other relevant work, see [67].) An impressive comparison of three-point boundary correlators between the two theories was performed recently by Giombi and Yin [68].

The relevance of collective fields for higher-spin holography was discussed by Das and Jevicki [69]. There the framework of covariant bi-local collective fields was employed and it was shown that they decompose into an infinite sequence of integer spin fields in one extra dimension. In [70], we sharpened this picture concentrating on the canonical formulation with the goal of establishing the correspondence directly at the Hamiltonian level. It will be advantageous to work in null-plane quantization, since it gives a physical description of higher-spin gauge theory. In this framework, we produced an exact one-to-one map between (collective) coordinates of the large N field and the AdS_4 coordinates of the higher-spin theory. It is shown how collective fields provide a construction of bulk (rather than boundary) fields of the AdS theory. In particular it is demonstrated that all the bulk AdS space-time transformation symmetries are recovered from transformations of the bi-local collective field.

5.1.1 Collective vs conformal fields

The basis of the holographic map is in a (complete) set of primary operators of the $SO(2,d)$ group. They are built as composite operators from the basic fields of the theory and obey current conservation once the field equations are used. They are used as sources at the boundary and their correlators are then shown to be in agreement with the AdS amplitudes projected to the boundary of AdS space. The N-component vector model field theory with the Lagrangian

$$L = \int d^d x \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) + v(\phi \cdot \phi), \quad a = 1, ..., N \tag{5.1}$$

possesses two critical points: the UV fixed point at zero value of the coupling and an IR fixed point at nonzero coupling. For the UV case corresponding to the free theory where the potential $v = 0$, a full set of conformal currents is explicitly given by [68]

$$\mathcal{O}(\vec{x}, \vec{\epsilon}) = \phi^a(x-\epsilon) \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(2\epsilon^2 \overleftarrow{\partial_x} \cdot \overrightarrow{\partial_x} - 4(\epsilon \cdot \overleftarrow{\partial_x})(\epsilon \cdot \overrightarrow{\partial_x})\right)^n \phi^a(x+\epsilon) \tag{5.2}$$

where $\vec{\epsilon}$ is a null polarization vector $\vec{\epsilon}^{\,2} = 0$. These currents are conserved and in the holographic scheme of GKP-W [2, 3] their correlators are compared with the AdS boundary amplitudes.

Collective fields for large N theories are introduced in a very different manner. They are to represent a (complete) set of invariants under the $O(N)$ or $U(N)$ (gauge) symmetry group. The meaning of completeness is established in two not unrelated ways. First one has completeness in group theoretic terms, namely that any other invariant can be expressed in terms of them. Second is the requirement of closure under (quantum) equations of motion. This leads to the most important fact, namely that they provide a complete dynamical description [63] of the large N theory where $1/N$ is seen to emerge as the natural expansion parameter.

In the $O(N)$ vector model one simply has the bi-local collective field

$$\Psi(x^\mu, y^\mu) = \sum_{a=1}^{N} \phi^a(x) \cdot \phi^a(y) \tag{5.3}$$

in the covariant formalism [69, 71]. It is the case for the $O(N)$ model, and also more generally that the set of collective fields is actually over-complete. This property has significant implications on the emerging space-time, when implemented it naturally leads to space-time cutoffs and ultimately non-commutativity.

As far as the relationship between the conformal and collective fields we have the following. Clearly any conformal field is contained in the collective (bi-local) field, one has a prescription with derivatives given above. But the converse is not true, collective fields represent a more general set. This property will have important implications on the bulk vs boundary description of the theory. It has already seen in approximate manner [69] that the relative coordinate in the bi-local field into angles generating a sequence of spins and the radial part which plays the role of an extra dimension. What prevented a precise identification however was the fact that higher-spin is a gauge theory, whose dynamical form depends on the gauge chosen. Consequently for establishing a precise one-to-one map, one has to bring both theories to the same gauge. This will be accomplished in the present work in a canonical description.

The canonical formalism for collective fields is based (in equal-time quantization) on the observables

$$\Psi(t;\vec{x},\vec{y}) = \sum_a \phi^a(t,\vec{x})\cdot\phi^a(t,\vec{y}) \equiv \Psi_{xy} \tag{5.4}$$

which are local in time but bi-local in $d-1$ dimensional space. These observables (collective fields) are characterized by the fact that they represent a complete set of $O(N)$ invariant canonical variables (obtained through scalar product). To deduce the dynamics obeyed by these fields, one performs an operator change of variables [63] from $\phi^a(t,\vec{x})$ to the bi-local field $\Psi(t;\vec{x},\vec{y})$ using the chain rule

$$\frac{\delta}{\delta\phi(\vec{x})} = \frac{\delta\Psi(\vec{y},\vec{z})}{\delta\phi(\vec{x})}\frac{\delta}{\delta\Psi(\vec{y},\vec{z})}. \tag{5.5}$$

Starting from the canonical Hamiltonian

$$H = \int\Big(-\frac{1}{2}\frac{\delta}{\delta\phi^a(\vec{x})}\frac{\delta}{\delta\phi^a(\vec{x})} + \frac{1}{2}\bigtriangledown_x\phi^a\bigtriangledown_x\phi^a + v(\vec{\phi}\cdot\vec{\phi})\Big)d\vec{x}, \tag{5.6}$$

one deduces an equivalent representation in terms of collective variables

$$\begin{aligned} H &= 2\mathrm{Tr}(\Pi\Psi\Pi) + \frac{N^2}{8}\mathrm{Tr}\Psi^{-1} + \int d\vec{x}v(\Psi(\tilde{x},\tilde{y})|_{\tilde{x}=\tilde{y}}) \\ &\quad + \frac{1}{2}\int d\vec{x}[-\bigtriangledown_x^2\Psi(\tilde{x},\tilde{y})|_{\tilde{x}=\tilde{y}}] + \Delta V \end{aligned} \tag{5.7}$$

where we have the conjugate momentum denoted by

$$\Pi(\vec{x},\vec{y}) = -i\frac{\delta}{\delta\Psi(\vec{x},\vec{y})} \tag{5.8}$$

and ΔV summarizes ordering terms which are lower order in $1/N$

$$\Delta V = -\frac{N}{2}\Big(\int dx\delta(0)\Big)\mathrm{Tr}\Psi^{-1} + \frac{1}{2}\Big(\int dx\delta(0)\Big)^2\mathrm{Tr}\Psi^{-1}. \tag{5.9}$$

The product of two bi-local fields is defined by

$$AB = \int d\vec{y}A(\vec{x},\vec{y})B(\vec{y},\vec{z}) \tag{5.10}$$

and the trace of a bi-local field means

$$\mathrm{Tr}(A) = \int d\vec{x}A(\vec{x},\vec{x}). \tag{5.11}$$

For more details on this representation, including the fact that it generates correctly the large N Schwinger-Dyson equations, the reader should consult Refs. [63, 72].

5.1.2 Expansion

The main feature of the collective representation in terms of the Hamiltonian (5.7) is that it can be expanded in series of $1/N$ with an infinite number of polynomial vertices to generate systematically the $1/N$ expansion. This is seen by a simple rescaling of field variables: $\Psi \to N\Psi$, $\Pi \to \Pi/N$ whereby N factorizes in front of the action. The terms in ΔV are seen to be of lower order, consequently they provide counter-terms in the systematic $1/N$ expansion.

To generate the expansion, one first evaluates the static large N background $\psi_0(\vec{x},\vec{y})$ obtained from the time-independent equations of motion

$$\frac{\partial V}{\partial \Psi(\vec{x},\vec{y})} = 0, \tag{5.12}$$

where we have set $v = 0$ and the effective potential reads

$$V = \frac{1}{8}\mathrm{Tr}\Psi^{-1} + \frac{1}{2}\int d\vec{x}[- \bigtriangledown_x^2 \Psi(\tilde{x},\tilde{y})|_{\tilde{x}=\tilde{y}}]. \tag{5.13}$$

One performs a shift

$$\Psi = \psi_0 + \frac{1}{\sqrt{N}}\eta, \quad \Pi = \sqrt{N}\pi \tag{5.14}$$

generating an infinite sequence of vertices

$$\mathrm{Tr}\Psi^{-1} = \mathrm{Tr}\psi_0^{-1} + \sum_{n=1}^{\infty}\frac{(-1)^n}{N^{\frac{n}{2}}}\mathrm{Tr}(\psi_0^{-1}(\eta\psi_0^{-1})^n). \tag{5.15}$$

The quadratic and cubic terms in the Hamiltonian are seen to be given by

$$H^{(2)} = 2\mathrm{Tr}(\pi\psi_0\pi) + \frac{1}{8}\mathrm{Tr}(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}), \tag{5.16}$$

$$H^{(3)} = \frac{2}{\sqrt{N}}\mathrm{Tr}(\pi\eta\pi) - \frac{1}{8\sqrt{N}}\mathrm{Tr}(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}). \tag{5.17}$$

The higher order vertices are obtained directly from the expansion (5.15).

We now discuss the evaluation of the spectrum which follows from diagonalization of $H^{(2)}$. In doing this we follow closely [72]. Using a Fourier transform

$$\psi^0_{xy} = \int d\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\psi^0_k, \tag{5.18}$$

with

$$\psi_k^0 = \frac{1}{2\sqrt{\vec{k}^2}}, \tag{5.19}$$

and for the fields

$$\eta_{xy} \equiv \int d\vec{k}_1 d\vec{k}_2 e^{-i\vec{k}_1\cdot\vec{x}} e^{+i\vec{k}_2\cdot\vec{y}} \eta_{k_1k_2}, \tag{5.20}$$

$$\pi_{xy} \equiv \int d\vec{k}_1 d\vec{k}_2 e^{+i\vec{k}_1\cdot\vec{x}} e^{-i\vec{k}_2\cdot\vec{y}} \pi_{k_1k_2}, \tag{5.21}$$

the quadratic Hamiltonian now becomes

$$H^{(2)} = 2\int d\vec{k}_1 d\vec{k}_2 \psi_{k_1}^0 \pi_{k_1k_2}\pi_{k_1k_2} + \frac{1}{16}\int d\vec{k}_1 d\vec{k}_2 \eta_{k_1k_2}({\psi_{k_1}^0}^{-2}{\psi_{k_2}^0}^{-1} + {\psi_{k_2}^0}^{-2}{\psi_{k_1}^0}^{-1})\eta_{k_1k_2}. \tag{5.22}$$

Redefining

$$\pi_{k_1k_2} \to \frac{1}{2}{\psi_{k_1}^0}^{-1/2}\pi_{k_1k_2} \quad \eta_{k_1k_2} \to 2{\psi_{k_1}^0}^{+1/2}\eta_{k_1k_2} \tag{5.23}$$

one has the quadratic Hamiltonian

$$H^{(2)} = \frac{1}{2}\int d\vec{k}_1 d\vec{k}_2 \pi_{k_1k_2}\pi_{k_1k_2} + \frac{1}{8}\int d\vec{k}_1 d\vec{k}_2 \eta_{k_1k_2}({\psi_{k_1}^0}^{-1} + {\psi_{k_2}^0}^{-1})^2\eta_{k_1k_2} \tag{5.24}$$

from which one reads off the frequencies

$$\omega_{k_1k_2} = \frac{1}{2}{\psi_{k_1}^0}^{-1} + \frac{1}{2}{\psi_{k_2}^0}^{-1} = \sqrt{\vec{k}_1^2} + \sqrt{\vec{k}_2^2}. \tag{5.25}$$

To summarize, the quadratic Hamiltonian and momentum can be written in use of bi-local fields as

$$H^{(2)} = \int d\vec{x}d\vec{y}\Psi^\dagger(\vec{x},\vec{y})\Big(\sqrt{-\nabla_x^2} + \sqrt{-\nabla_y^2}\Big)\Psi(\vec{x},\vec{y}), \tag{5.26}$$

$$P^{(2)} = \int d\vec{x}d\vec{y}\Psi^\dagger(\vec{x},\vec{y})(\nabla_x + \nabla_y)\Psi(\vec{x},\vec{y}). \tag{5.27}$$

In the light-cone quantization, we have the quadratic Hamiltonian

$$P^{-(2)} = H^{(2)} + P^{(2)} = \int dx_1^- dx_2^- d\vec{x}_1 d\vec{x}_2 \Psi^\dagger \Big(-\frac{\nabla_1^2}{2p_1^+} - \frac{\nabla_2^2}{2p_2^+} \Big) \Psi. \tag{5.28}$$

Here $\Psi(x^+; x_1^-, x_2^-; \vec{x}_1, \vec{x}_2)$ is a bi-local field where $1, 2$ refer to the two space points.

5.2 Conformal transformations of the collective fields

Our goal is to demonstrate that the collective field contains all the necessary information and is in a one-to-one map with the physical fields of the higher-spin theory in AdS_4. For this comparison to be done it is advantageous to work in the light-cone gauge, where the physical degrees of freedom of a gauge theory are most transparent [73]. Our strategy is to compare directly the action of the conformal group of the $d = 3$ field theory with that of the Anti-de Sitter higher spin field. This comparison is similar to the study in D-brane case and $\mathcal{N} = 4$ Super Yang-Mills theory performed in [74]. In this direct comparison we will see that as expected we have very different set of space-time variables and a different realization of SO(2,3). The number of canonical variables however will be shown to be identical and one can search for a (canonical) transformation to establish a one-to-one relation between the two representations.

One can work out the conformal transformations in light-cone notation ($x^+ = t$)

for any dimension d. As for the linear momenta, we have

$$P^- = H = \int d\vec{x}\Big(-\frac{1}{2}(\partial_i\phi)^2\Big), \tag{5.29}$$

$$P^+ = \int d\vec{x}\big(\pi^2\big), \tag{5.30}$$

$$P^i = \int d\vec{x}\big(\pi\partial_i\phi\big), \tag{5.31}$$

where $\pi = \partial^+\phi$ is the conjugate momentum and i is the transverse index (for the specific case when $d = 3$, the index i runs over a single value). Similarly, for Lorentz transformations, the conserved charges are

$$M^{+-} = tH - \int d\vec{x}\Big(x^-\pi^2\Big), \tag{5.32}$$

$$M^{+i} = \int d\vec{x}\Big(t\pi\partial_i\phi - x^i\pi^2\Big), \tag{5.33}$$

$$M^{-i} = \int d\vec{x}\Big(x^-\pi\partial_i\phi - x^i\,\mathcal{H}\Big), \tag{5.34}$$

$$M^{ij} = \int d\vec{x}\Big(x^i\pi\partial_j\phi - x^j\pi\partial_i\phi\Big). \tag{5.35}$$

The Dilatation operator takes the form

$$D = tH + \int d\vec{x}\Big(\pi(d_\phi + x^i\partial_i)\phi + x^-\pi^2\Big), \tag{5.36}$$

where $d_\phi = \frac{d-2}{2}$ is the scaling dimension of the ϕ field. The special conformal generators are

$$K^- = \int d\vec{x}\Big(x^-\,\mathcal{D} - \frac{1}{2}(2tx^- + x^jx_j)\mathcal{H} - \frac{1}{2}d_\phi\phi^2\Big), \tag{5.37}$$

$$K^+ = tD - \int d\vec{x}\Big(\frac{1}{2}(2tx^- + x^jx_j)\pi^2\Big), \tag{5.38}$$

$$K^i = \int d\vec{x}\Big(x^i\,\mathcal{D} - \frac{1}{2}(2tx^- + x^jx_j)\pi\partial_i\phi\Big), \tag{5.39}$$

where $\mathcal{D}$ and $\mathcal{H}$ are the densities of these two operators.

The dynamical variables in the light-cone formulation are (x^-, x^i). The momentum conjugate to x^- is p^+. In the massless case, the energy can be expressed as

$$p^- = -\frac{p^i p^i}{2p^+}. \tag{5.40}$$

To define the mode expansion, we perform a Fourier transform of the fields $\phi(x^-, x^i)$ and $\pi(x^-, x^i)$ along the x^- direction. The creation and annihilation operators are defined in terms of

$$\phi(x^-, x^i) = \int_0^\infty \frac{dp^+}{\sqrt{2\pi}} \frac{1}{\sqrt{2p^+}} \Big(a(p^+, x^i) e^{ip^+ x^-} + a^\dagger(p^+, x^i) e^{-ip^+ x^-} \Big), \tag{5.41}$$

$$\pi(x^-, x^i) = -i \int_0^\infty \frac{dp^+}{\sqrt{2\pi}} \sqrt{\frac{p^+}{2}} \Big(a(p^+, x^i) e^{ip^+ x^-} - a^\dagger(p^+, x^i) e^{-ip^+ x^-} \Big). \tag{5.42}$$

The actions of linear momenta now take the form

$$P^- : \quad \delta a(p^+, x^i) = \frac{\partial_i^2}{2p^+} a(p^+, x^i), \tag{5.43}$$

$$P^+ : \quad \delta a(p^+, x^i) = p^+ a(p^+, x^i), \tag{5.44}$$

$$P^i : \quad \delta a(p^+, x^i) = i\partial_i a(p^+, x^i). \tag{5.45}$$

For the Lorentz generators, one has

$$M^{+-} : \quad \delta a(p^+, x^i) = \Big(t\frac{\partial_i^2}{2p^+} - i\sqrt{p^+} \frac{\partial}{\partial p^+} \sqrt{p^+} \Big) a(p^+, x^i), \tag{5.46}$$

$$M^{+i} : \quad \delta a(p^+, x^i) = \Big(it\partial_i - x^i p^+ \Big) a(p^+, x^i), \tag{5.47}$$

$$M^{-i} : \quad \delta a(p^+, x^i) = \Big(-\partial_i \frac{\partial}{\partial p^+} - \frac{\partial_j x^i \partial_j}{2p^+} \Big) a(p^+, x^i), \tag{5.48}$$

$$M^{ij} : \quad \delta a(p^+, x^i) = \Big(ix^i \partial_j - ix^j \partial_i \Big) a(p^+, x^i). \tag{5.49}$$

and the Dilatation operator

$$D: \quad \delta a(p^+, x^i) \;=\; \Big(t\frac{\partial_i^2}{2p^+} + i\Big[d_\phi + x^i\partial_i + \sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\Big]\Big)a(p^+, x^i). \quad (5.50)$$

Finally, for the special conformal generators

$$\begin{aligned} K^-: \quad \delta a(p^+, x^i) &= \Big\{-\frac{\partial_j x^i x^i \partial_j}{4p^+} - \sqrt{p^+}\frac{\partial}{\partial p^+}\frac{\partial}{\partial p^+}\sqrt{p^+} - x^i\partial_i\frac{\partial}{\partial p^+} \\ &\quad -d_\phi\frac{1}{\sqrt{p^+}}\frac{\partial}{\partial p^+}\sqrt{p^+}\Big\}a(p^+, x^i), \quad (5.51) \\ K^+: \quad \delta a(p^+, x^i) &= \Big\{t^2\frac{\partial_i^2}{2p^+} + it(d_\phi + x^i\partial_i) - \frac{1}{2}x^i x^i p^+\Big\}a(p^+, x^i), \quad (5.52) \\ K^i: \quad \delta a(p^+, x^i) &= \Big\{t\frac{\partial_j x^i \partial_j}{2p^+} + t\partial_i\frac{\partial}{\partial p^+} - \frac{i}{2}x^j x^j \partial_i + ix^i\Big[d_\phi + x^j\partial_j \\ &\quad + \sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\Big]\Big\}a(p^+, x^i). \quad (5.53) \end{aligned}$$

We next deduce the transformation for the collective fields. In creation-annihilation form $A(x_1^-, x_2^-, \vec{x}_1, \vec{x}_2) = a(x_1^-, \vec{x}_1)a(x_2^-, \vec{x}_2)$, we have $\delta A(1,2) = \delta a(1)a(2) + a(1)\delta a(2)$ and any conformal generator

$$G = \int dx_1^- dx_2^- d\vec{x}_1 d\vec{x}_2 A^\dagger \hat{g} A = \int dx_1^- dx_2^- d\vec{x}_1 d\vec{x}_2 A^\dagger (\hat{g}_1 + \hat{g}_2) A. \quad (5.54)$$

Denoting the conjugate momenta as $(p_1^+, p_2^+, p_1^i, p_2^i)$, we can write down the following generators

$$\begin{aligned} \hat{p}^- &= p_1^- + p_2^- = -\Big(\frac{p_1^i p_1^i}{2p_1^+} + \frac{p_2^i p_2^i}{2p_2^+}\Big), \quad (5.55) \\ \hat{p}^+ &= p_1^+ + p_2^+, \quad (5.56) \\ \hat{p}^i &= p_1^i + p_2^i, \quad (5.57) \end{aligned}$$

$$
\begin{aligned}
\hat{m}^{+-} &= t\hat{p}^- - x_1^- p_1^+ - x_2^- p_2^+, && (5.58)\\
\hat{m}^{+i} &= t\hat{p}^i - x_1^i p_1^+ - x_2^i p_2^+, && (5.59)\\
\hat{m}^{-i} &= x_1^- p_1^i + x_2^- p_2^i + x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_2^i \frac{p_2^j p_2^j}{2p_2^+}, && (5.60)\\
\hat{m}^{ij} &= x_1^i p_1^j - x_1^j p_1^i + x_2^i p_2^j - x_2^j p_2^i, && (5.61)\\
\hat{d} &= t\hat{p}^- + x_1^- p_1^+ + x_2^- p_2^+ + x_1^i p_1^i + x_2^i p_2^i + 2d_\phi, && (5.62)\\
\hat{k}^- &= x_1^i x_1^i \frac{p_1^j p_1^j}{4p_1^+} + x_2^i x_2^i \frac{p_2^j p_2^j}{4p_2^+} + x_1^- (x_1^- p_1^+ + x_1^i p_1^i + d_\phi)\\
&\quad + x_2^- (x_2^- p_2^+ + x_2^i p_2^i + d_\phi), && (5.63)\\
\hat{k}^+ &= t^2 \hat{p}^- + t(x_1^i p_1^i + x_2^i p_2^i + 2d_\phi) - \frac{1}{2} x_1^i x_1^i p_1^+ - \frac{1}{2} x_2^i x_2^i p_2^+, && (5.64)\\
\hat{k}^i &= -t\Big(x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_2^i \frac{p_2^j p_2^j}{2p_2^+} + x_1^- p_1^i + x_2^- p_2^i \Big)\\
&\quad - \frac{1}{2} x_1^j x_1^j p_1^i - \frac{1}{2} x_2^j x_2^j p_2^i + x_1^i (x_1^- p_1^+ + x_1^j p_1^j + d_\phi)\\
&\quad + x_2^i (x_2^- p_2^+ + x_2^j p_2^j + d_\phi). && (5.65)
\end{aligned}
$$

5.3 Mapping to AdS$_4$

The correspondence introduced in [66] is specific for $CFT_3 \leftrightarrow AdS_4$. We will from now on consider the case of $d = 3$ for the vector model. In the light-cone notation, there is only one transverse dimension $x^i = x$ and $x^\mu = (x^+, x^-, x)$. The AdS$_4$ spacetime coordinates in the light-cone notation ($x^+ = t$) are denoted with the Poincaré metric

$$
ds^2 = \frac{2dtdx^- + dx^2 + dz^2}{z^2}. \tag{5.66}
$$

The lowercase transverse index $i = 1$ denotes x only, while the uppercase transverse index $I = (1, 2)$ denotes (x, z). In AdS$_4$ higher-spin theory, the generators were worked out by Metsaev in [73] which we now summarize.

5.3.1 Conformal generators from higher-spin theory

The four-dimensional case has the unique property that, after fixing light-cone gauge [75], the only physical states are the $\pm s$ helicity states [76]. Let us now explain how to fix the light-cone gauge. Starting from the covariant notation

$$|\Phi\rangle = \sum_{s=1}^{\infty} \Phi^{\mu_1 \ldots \mu_s} a^{\dagger}_{\mu_1} \ldots a^{\dagger}_{\mu_s} |0\rangle. \tag{5.67}$$

where $\mu = (0, 1, z, 3)$ in the case of AdS_4, one fixes the light-cone gauge in two steps. First, we drop the oscillators $a^{\pm} = a^0 \pm a^3$ and keep only the transverse oscillators $a^I, a^{\dagger J}$ including the z component. The oscillators satisfy the commutators

$$[a^I, a^{\dagger J}] = \delta^{IJ}, \quad [a^I, a^J] = [a^{\dagger I}, a^{\dagger J}] = 0. \tag{5.68}$$

The spin matrix of the Lorentz algebra now takes the form

$$M^{IJ} = a^{\dagger I} a^J - a^{\dagger J} a^I. \tag{5.69}$$

The next step is to impose a further constraint

$$T|\Phi\rangle = 0, \qquad T = a^I a^I \tag{5.70}$$

so that only two components will survive. With the complex oscillators

$$\alpha = \frac{1}{\sqrt{2}}(a_1 + i a_2), \qquad \alpha^{\dagger} = \frac{1}{\sqrt{2}}(a_1^{\dagger} + i a_2^{\dagger}), \tag{5.71}$$

$$\bar{\alpha} = \frac{1}{\sqrt{2}}(a_1 - i a_2), \qquad \bar{\alpha}^{\dagger} = \frac{1}{\sqrt{2}}(a_1^{\dagger} - i a_2^{\dagger}), \tag{5.72}$$

we find the simple expansion for $|\Phi\rangle$

$$|\Phi\rangle = \sum_{\lambda=1}^{\infty}\Big(\Phi_{(\lambda)}(\bar{\alpha}^{\dagger})^{\lambda} + \bar{\Phi}_{(\lambda)}(\alpha^{\dagger})^{\lambda}\Big)|0\rangle. \tag{5.73}$$

This expansion obviously satisfies the constraint

$$T|\Phi\rangle = 0, \qquad T = \bar{\alpha}\alpha. \tag{5.74}$$

The spin matrix $M = \alpha^{\dagger}\bar{\alpha} - \bar{\alpha}^{\dagger}\alpha$ also reduces to (5.69). In four dimensions, the only non-vanishing spin matrix is M^{xz}. One can represent $\alpha = e^{i\theta}$, $\bar{\alpha} = e^{-i\theta}$. In a coherent basis, the operator M^{xz} becomes $\frac{\partial}{\partial\theta}$. Then we have $\Phi(x^{\mu}, z, \theta)$ or in light-cone notation $\Phi(x^{+}, x^{-}, x, z; \theta)$. The generators can be written as

$$G = \int dx^{-}dxdzd\theta\ \bar{\Phi}\hat{g}\Phi. \tag{5.75}$$

Denoting the conjugate momenta as $(p^{+}, p^{x}, p^{z}, p^{\theta})$, one has [73]

$$\hat{p}^{-} = -\frac{p^{x}p^{x} + p^{z}p^{z}}{2p^{+}}, \tag{5.76}$$

$$\hat{p}^{+} = p^{+}, \tag{5.77}$$

$$\hat{p}^{x} = p^{x}, \tag{5.78}$$

$$\hat{m}^{+-} = t\hat{p}^{-} - x^{-}p^{+}, \tag{5.79}$$

$$\hat{m}^{+x} = tp^{x} - xp^{+}, \tag{5.80}$$

$$\hat{m}^{-x} = x^{-}p^{x} - x\hat{p}^{-} + \frac{p^{\theta}p^{z}}{p^{+}}, \tag{5.81}$$

$$\hat{d} = t\hat{p}^{-} + x^{-}p^{+} + xp^{x} + zp^{z} + d_{a}, \tag{5.82}$$

$$\hat{k}^{-} = -\frac{1}{2}(x^{2} + z^{2})\hat{p}^{-} + x^{-}(x^{-}p^{+} + xp^{x} + zp^{z} + d_{a})$$

$$+\frac{1}{p^+}\big((xp^z - zp^x)p^\theta + (p^\theta)^2\big), \tag{5.83}$$

$$\hat{k}^+ = t^2\hat{p}^- + t(xp^x + zp^z + d_a) - \frac{1}{2}(x^2+z^2)p^+, \tag{5.84}$$

$$\begin{aligned}\hat{k}^x &= t(x\hat{p}^- - x^-p^x - \frac{p^\theta p^z}{p^+}) + \frac{1}{2}(x^2 - z^2)p^x \\ &\quad + x(x^-p^+ + zp^z + d_a) + zp^\theta,\end{aligned} \tag{5.85}$$

where the scaling dimension $d_a = 1$ in the case of AdS_4.

5.3.2 The map: canonical transformation

We will now show how the two pictures are related by a canonical transformation. At this point, we will give the classical transformation (it can be specified in its full quantum version also). So in what follows we do not compare terms with d_ϕ which will receive quantum corrections (due to ordering).

By relating (5.56-5.59) to (5.77-5.80), one can easily solve for

$$x^- = \frac{x_1^- p_1^+ + x_2^- p_2^+}{p_1^+ + p_2^+}, \tag{5.86}$$

$$p^+ = p_1^+ + p_2^+, \tag{5.87}$$

$$x = \frac{x_1 p_1^+ + x_2 p_2^+}{p_1^+ + p_2^+}, \tag{5.88}$$

$$p^x = p_1 + p_2. \tag{5.89}$$

From (5.76, 5.81, 5.82, 5.84), we get

$$z^2 = \frac{(x_1 - x_2)^2 p_1^+ p_2^+}{(p_1^+ + p_2^+)^2}, \tag{5.90}$$

$$p^z p^z = \frac{(p_1 p_2^+ - p_2 p_1^+)^2}{p_1^+ p_2^+}, \tag{5.91}$$

$$zp^z = \frac{(x_1 - x_2)(p_1 p_2^+ - p_2 p_1^+)}{(p_1^+ + p_2^+)}, \tag{5.92}$$

$$p^\theta p^z = (x_1^- - x_2^-)(p_1 p_2^+ - p_2 p_1^+) + (x_1 - x_2)\Big(\frac{p_2^+(p_1)^2}{2p_1^+} - \frac{p_1^+(p_2)^2}{2p_2^+}\Big). \tag{5.93}$$

The solution to (5.90-5.93) can be written as

$$z = \frac{(x_1 - x_2)\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}, \tag{5.94}$$

$$p^z = \sqrt{\frac{p_2^+}{p_1^+}}p_1 - \sqrt{\frac{p_1^+}{p_2^+}}p_2, \tag{5.95}$$

$$p^\theta = \sqrt{p_1^+ p_2^+}(x_1^- - x_2^-) + \frac{x_1 - x_2}{2}\Big(\sqrt{\frac{p_2^+}{p_1^+}}p_1 + \sqrt{\frac{p_1^+}{p_2^+}}p_2\Big). \tag{5.96}$$

A nontrivial check of the consistency is given by comparing (5.83, 5.85) with (5.63, 5.65). We now turn to the construction of θ. The condition that θ Poisson commutes with p^x implies θ is a function of $x_1 - x_2$ and the condition that θ Poisson commutes with p^+ implies that θ is a function of $x_1^- - x_2^-$. Requiring that θ Poisson commutes with x^-, x, z and p^z as well as θ and p^θ Poisson commute to give 1 we obtain

$$\theta = 2\arctan\sqrt{\frac{p_2^+}{p_1^+}}. \tag{5.97}$$

An important consistency check on the correctness of the map that we have constructed is that all the Poisson brackets of the derived variables (like z and p^z etc.) take the canonical form with distinct canonical sets commuting with each other. One can confirm the Poisson brackets $\{x^-, p^+\} = \{x, p^x\} = \{z, p^z\} = 1$ and others vanish.

Finally, as a consequence of the above map it follows that the wave equation in the collective picture has a map [77] to the wave equation of higher-spin gravity in four-dimensional AdS background. This follows from the generators (5.55) and

(5.76) coinciding after the canonical transformation. The canonical transformation can be understood as a point transformation in the momentum space (if we interpret θ as momentum (5.97), the other momenta are given by (5.87, 5.89, 5.95)). Consequently, the transformation between the higher-spin field and bi-local field is simple in momentum space

$$\begin{aligned}
&\Phi(x^-, x, z, \theta) = \int dp^+ dp^x dp^z e^{i(x^- p^+ + xp^x + zp^z)} \\
&\int dp_1^+ dp_2^+ dp_1 dp_2 \delta(p_1^+ + p_2^+ - p^+)\delta(p_1 + p_2 - p^x) \\
&\delta\Big(p_1 \sqrt{p_2^+/p_1^+} - p_2 \sqrt{p_1^+/p_2^+} - p^z\Big) \\
&\delta\big(2 \arctan \sqrt{p_2^+/p_1^+} - \theta\big) \tilde{\Psi}(p_1^+, p_2^+, p_1, p_2)
\end{aligned} \tag{5.98}$$

where $\tilde{\Psi}(p_1^+, p_2^+, p_1, p_2)$ is the Fourier transform of the bi-local field $\Psi(x_1^-, x_2^-, x_1, x_2)$.

5.4 Origin of the extra dimension

The main contribution here is an explicit one-to-one map between the collective field (in the case of the O(N) vector model) and the field of higher-spin gravity in 4D AdS space-time. This map is defined by the canonical transformation which establishes the relationship between the coordinates of the bi-local collective field and the coordinates of the AdS_4 space-time plus spin variables. The map is one to one, in particular the most telling formula is the one for the extra radial coordinate of AdS space-time z.

Here we have an explicit expression, in terms of the collective coordinates contained in the bi-local field. The physical picture for this extra dimension is much like the (collective) coordinates of solitons, which are contained in the field itself but are nontrivial to exhibit. Their origin is again through a canonical map from the existing field degrees of freedom. Naturally, if the boundary conditions are too restrictive then these degrees will be absent. In more recent phenomenological studies of scattering processes in QCD, a dipole picture [78] was used which can have a relation to the construction presented. It is interesting to confront this collective mechanism for the emerging dimension with other viewpoints such as holographic [79], Feynman diagrams [80] and stochastic quantization [81]. There is also the important question of locality in the bulk of AdS [82].

The collective field theory gives a bulk Hamiltonian representation for the higher-spin gravity which is bi-local in Minkowski spacetime. It specifies an infinite set of bulk interacting vertices, which can be explicitly evaluated. These can be compared with the higher spin approaches, in particular Vasiliev's and we expect to find agreement. Therefore in this construction the complete Higher Spin Gravity (in a particular gauge) is seen to emerge. The interactions are seen to be given by the $1/N$ parameter of the $O(N)$ vector theory. This contains definite implications on the question of loop and quantum corrections in Higher Spin Gravity.

Our collective field construction gives a strong operator representation of AdS bulk higher spin fields. It contains the extra radial AdS dimension z explicitly and an important check on the validity of this off-shell construction is the projection of our formula to $z = 0$. We now demonstrate that the collective field indeed correctly reduces to the conformal primary operators of the vector model field theory. In the light-cone gauge, these primary operators for a particular spin s take the form [83]

$$\mathcal{O}^s = \sum_{k=0}^{s} \frac{(-1)^k \Gamma(s+1/2)\Gamma(s+1/2)}{k!(s-k)!\Gamma(s-k+1/2)\Gamma(k+1/2)} (\partial_+)^k \varphi (\partial_+)^{s-k} \varphi. \tag{5.99}$$

On the other hand at $z = 0$, through the bi-local mapping (5.98), the collective field reduces to

$$\Phi(x^-, x, z, \theta) = \int dp_1^+ dp_2^+ e^{ix^-(p_1^+ + p_2^+)} \delta(\theta - 2\tan^{-1}\sqrt{p_2^+/p_1^+}) \Psi(p_1^+, p_2^+, x, x). \quad (5.100)$$

Expanding the delta function in Fourier series, we find the binomial expansion

$$\left(\sqrt{p_1^+} - i\sqrt{p_2^+}\right)^{2s} = \frac{(-1)^k (2s)!}{(2k)!(2s-2k)!} (p_1^+)^k (p_2^+)^{s-k}. \quad (5.101)$$

This expansion agrees with (5.99) up to an overall normalization constant by noticing the identity

$$\frac{(2s)!}{(2k)!(2s-2k)!} = \frac{s!\Gamma(s+1/2)\Gamma(1/2)}{k!(s-k)!\Gamma(s-k+1/2)\Gamma(k+1/2)}. \quad (5.102)$$

We therefore see that the collective field at $z = 0$ reduces to the conformal primary operators.

5.5 A symmetric gauge

To establish the full agreement between higher spin and collective theory, it is useful to exhibit a symmetric gauge formulation of Vasiliev's theory. Such a formulation exists and is given as the $W = 0$ gauge. Starting with the nonlinear equations of motion

$$dW = W * W, \quad (5.103)$$

$$dB = W * B - B * \tilde{W}, \quad (5.104)$$

$$dS = W * S - S * W, \quad (5.105)$$

$$S * S = dz_\alpha dz^\alpha (i + B * \kappa) + d\bar{z}^{\dot\alpha} d\bar{z}_{\dot\alpha} (i + B * \bar\kappa), \tag{5.106}$$

$$S * B - B * \tilde{S} = 0, \tag{5.107}$$

where $\kappa = e^{iz_\alpha y^\alpha}$, $\bar\kappa = e^{i\bar{z}_{\dot\alpha}\bar{y}^{\dot\alpha}}$ and $\sim$ changes a sign of all undotted spinors

$$\tilde{f}(dz, d\bar{z}; z, \bar{z}, y, \bar{y}) = f(-dz, d\bar{z}; -z, \bar{z}, -y, \bar{y}). \tag{5.108}$$

Since W is a flat connection in spacetime, at least locally we can always go to a gauge in which W is set to zero. We will denote by S' and B' the corresponding master fields in this gauge. The equations of motion then states that S' and B' are independent of the spacetime coordinate x^μ, and are functions of Y, Z only. Explicitly, we can write

$$W(x|Y,Z) = g^{-1}(x|Y,Z) * d_x g(x|Y,Z), \tag{5.109}$$

$$S(x|Y,Z) = g^{-1}(x|Y,Z) * S'(Y,Z) * g(x|Y,Z), \tag{5.110}$$

$$B(x|Y,Z) = g^{-1}(x|Y,Z) * B'(Y,Z) * \pi(g(x|Y,Z)). \tag{5.111}$$

The equations for S' and B' now take the form

$$S' * S' = dz^\alpha dz_\alpha (i + B' * \kappa) + d\bar{z}^{\dot\alpha} d\bar{z}_{\dot\alpha} (i + B' * \bar\kappa), \tag{5.112}$$

$$S' * B' = B' * \pi(S'). \tag{5.113}$$

This system has the residual gauge symmetry

$$S'(Y,Z) = g^{-1}(Y,Z) * S''(Y,Z) * g(Y,Z), \tag{5.114}$$

$$B'(Y,Z) = g^{-1}(Y,Z) * B''(Y,Z) * \pi(g(Y,Z)). \tag{5.115}$$

Omitting the primes and shift of the S field as

$$S_\alpha = z_\alpha + \hat{S}_\alpha, \qquad \bar{S}_{\dot{\alpha}} = \bar{z}_{\dot{\alpha}} + \hat{\bar{S}}_{\dot{\alpha}}, \tag{5.116}$$

we find the equations of motion in components

$$i\partial_\alpha S^\alpha - S_\alpha * S^\alpha = B * \kappa, \tag{5.117}$$

$$i\bar{\partial}_{\dot{\alpha}} \bar{S}^{\dot{\alpha}} - \bar{S}_{\dot{\alpha}} * \bar{S}^{\dot{\alpha}} = B * \bar{\kappa}, \tag{5.118}$$

$$i\partial_\alpha \bar{S}_{\dot{\beta}} - i\bar{\partial}_{\dot{\beta}} S_\alpha - [S_\alpha, \bar{S}_{\dot{\beta}}]_* = 0, \tag{5.119}$$

$$i\partial_\alpha B - S_\alpha * B - B * \pi(S_\alpha) = 0, \tag{5.120}$$

$$i\bar{\partial}_{\dot{\alpha}} B - \bar{S}_{\dot{\alpha}} * B - B * \bar{\pi}(\bar{S}_{\dot{\alpha}}) = 0, \tag{5.121}$$

where we omitted the hats and rescaled both fields S, B by a factor of 2. A useful symmetry property for the bosonic case is

$$\pi(S_\alpha) = -\bar{\pi}(S_\alpha), \qquad \pi(\bar{S}_{\dot{\alpha}}) = -\bar{\pi}(\bar{S}_{\dot{\alpha}}). \tag{5.122}$$

The last two equations of the B field is not independent. Using the solution for the B field from the first two equations, one can verify that the B equations are automatically satisfied. Therefore, one can totally get rid of the B field and find five equations for the S field

$$F_{\alpha\dot{\beta}} \equiv i\partial_\alpha \bar{S}_{\dot{\beta}} - i\bar{\partial}_{\dot{\beta}} S_\alpha - [S_\alpha, \bar{S}_{\dot{\beta}}]_* = 0, \tag{5.123}$$

$$(i\partial_\alpha S^\alpha - S_\alpha * S^\alpha) * \kappa - (i\bar{\partial}_{\dot{\alpha}} \bar{S}^{\dot{\alpha}} - \bar{S}_{\dot{\alpha}} * \bar{S}^{\dot{\alpha}}) * \bar{\kappa} = 0. \tag{5.124}$$

Next we introduce an ansatz

$$S_1 = -iM^{-1} * \partial_1 M, \qquad S_2 = -i\bar{M} * \partial_2 \bar{M}^{-1}, \tag{5.125}$$

$$\bar{S}_1 = -i\bar{M} * \bar{\partial}_1 \bar{M}^{-1}, \qquad \bar{S}_2 = -iM^{-1} * \bar{\partial}_2 M, \tag{5.126}$$

where we used the notation $M^\dagger = \bar{M}$. This ansatz solves the $F_{1\dot{2}} = F_{2\dot{1}} = 0$ equations automatically. The other two equations $F_{1\dot{1}} = F_{2\dot{2}} = 0$ give a simple form

$$\bar{\partial}_1 (J^{-1} * \partial_1 J) = 0, \tag{5.127}$$

$$\partial_2 (J^{-1} * \bar{\partial}_2 J) = 0, \tag{5.128}$$

where we have defined the gauge invariant quantity $J = M * \bar{M}$. The last equation $F_{11} * \kappa - F_{22} * \bar{\kappa} = 0$ gives

$$\partial_2 (J^{-1} * \partial_1 J) * \kappa + \bar{\partial}_1 (J^{-1} * \bar{\partial}_2 J) * \bar{\kappa} = 0, \tag{5.129}$$

where we used the symmetry property $M(Y, Z) = M(-Y, -Z)$ in the bosonic case.

We have presented a completely symmetric reformulation of Vasiliev's Higher Spin Gravity in terms of a single (bi-local) scalar field $J(Y, Z)$. This formulation consists of an equation of motion and two additional constraints. It can be shown that the additional constraints reduce the dimensionalities of the Y and Z spaces from $4 + 4$ to $3 + 3$. This then agrees with the dimensionality of the bi-local collective field constructed from CFT_3. Further studies of this nonlinear system of equations can then establish a full nonlinear map between the two formulations of the theory. This can be accomplished by a nonlinear field redefinition. Once established the field correspondence will provide an exact demonstration of the AdS/CFT correspondence in a particular case of 3D conformal field theory.

Chapter 6

Conclusion

This dissertation presents an in-depth study of the AdS/CFT correspondence. In the first part, a detailed investigation of the classical string and its dynamics in AdS spacetime was performed. General methods (for construction of classical solutions) were developed and applied in detail to the case of AdS_3. For this the inverse scattering technique was adopted and a useful correspondence with field theoretic soliton solutions was formulated. As a result a most general set of "spiky" string configurations was obtained. Based on the explicit solutions we were able to discuss the form of the moduli space and its dynamics. Locations of spikes provided the collective coordinates for describing the moduli of the general string solutions, it was established that there is a one-to-one correspondence with the locations of solitons in the corresponding reduced field theory of sinh-Gordon type. This produced a "partonic" picture of the AdS string, where the partons are localized at the locations of "spikes" and can be described in terms of an n-body dynamics. Through the inverse scattering construction it was seen that this n-body dynamics is related to the n-body dynamics of soliton coordinates of the sinh-Gordon field theory.

The second part of the dissertation describes work on constructing AdS gravity from the large N partonic system. The manner in which continuum phenomena such as gravity are reconstructed from the microscopic dynamics is argued to be associated with the phenomenon of collective motions. Studying the simplest partonic composite, consisting of a bi-local system of two particles turned out to produce a striking result: the appearance of one extra AdS dimension and of a sequence resonances of growing integer spins. Specifically an explicit mapping of the AdS_4 spacetime plus higher-spin fields is given from the (bi-local) collective fields of the free conformal theory. Higher spin massless fields were seen to be associated with the cusps of the spiky strings. The main result described in the second part is an explicit one-to-one map between bi-local collective field (of the 3D $O(N)$ vector model) and the fields of higher-spin gravity in 4D AdS spacetime. This construction is based on equating the isometries of $SO(2,3)$ with the conformal generators of the CFT_3 in the light-cone gauge. The mapping itself gives a general understanding on the (collective) origin of the extra spatial dimension in the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence. It is to be compared with the "holographic" approaches where the extra spatial dimension of the AdS spacetime is typically projected out or argued to be related to the renormalization group scale. Our construction demonstrates explicitly the origin of the extra AdS dimension and the emergence of Higher-Spin General Relativity in AdS spacetime. It has the potential for a complete demonstration of the AdS/CFT correspondence in the case of the 3D vector model.

Appendix A

Inverse scattering and the spinors

In this appendix, we summarize the inverse scattering method [43] to solve the Lax pair equations (2.95) with the matrices (2.94). Choose the boundary condition of the field as $\hat{u}, \bar{\partial}\hat{u} \to 0$ as $\bar{z} \to \pm\infty$. The spinor φ can be written as an integral form

$$\varphi(\zeta, \bar{z}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta\bar{z}} + \int_{\bar{z}}^{\infty} K(\bar{z}, s) e^{i\zeta s} ds, \tag{A.1}$$

where the kernel satisfies the GLM equation

$$\bar{K}(\bar{z}, y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(\bar{z} + y) + \int_{\bar{z}}^{\infty} K(\bar{z}, s) F(s + y) ds = 0, \tag{A.2}$$

with the function $F(x)$ defined to be

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk - i \sum_{j=1}^{N} c_j e^{i\zeta_j x}, \tag{A.3}$$

where $r(k)$ is the reflection coefficient, c_j and ζ_j are constants. In the case of sinh-Gordon with real field $\hat{u}$, the kernels are

$$K = \begin{pmatrix} K_1(\bar{z}, s) \\ K_2(\bar{z}, s) \end{pmatrix}, \qquad \bar{K} = \begin{pmatrix} K_2(\bar{z}, s) \\ K_1(\bar{z}, s) \end{pmatrix}, \tag{A.4}$$

with K_1 and K_2 real.

Now we try to solve the GLM equation (A.2) by some ansatz. Consider the soliton solutions to the sinh-Gordon equation which has $r(k) = 0$ and plug in the ansatz

$$K_i(\bar{z}, s) = \sum_{j=1}^{N} \sqrt{c_j} f_{ij}(\bar{z}) e^{i\zeta_j s}, \qquad i = 1, 2, \tag{A.5}$$

we get

$$f_{1j} = i \sum_{k=1}^{N} (1 - A)^{-1}_{jk} \lambda_k, \tag{A.6}$$

$$f_{2j} = -i \sum_{l,k=1}^{N} \frac{\lambda_j \lambda_l}{\zeta_j + \zeta_l} (1 - A)^{-1}_{lk} \lambda_k, \tag{A.7}$$

where the matrix A is defined as

$$A_{ij} = \sum_{l} a_{il} a_{lj}, \qquad a_{il} = \frac{\lambda_i \lambda_l}{\zeta_i + \zeta_l}, \qquad \lambda_k = \sqrt{c_k} e^{i\zeta_k \bar{z}}. \tag{A.8}$$

The wavefunctions are solved to be

$$\varphi_1(\zeta, \bar{z}) = -\Big(\sum_{j,l} \frac{\lambda_j}{\zeta + \zeta_j} (1 - A)^{-1}_{jl} \lambda_l \Big) e^{i\zeta\bar{z}}, \tag{A.9}$$

$$\varphi_2(\zeta, \bar{z}) = \Big(1 + \sum_{j,l,k} \frac{\lambda_j}{\zeta + \zeta_j} \frac{\lambda_j \lambda_l}{\zeta_j + \zeta_l} (1 - A)^{-1}_{lk} \lambda_k \Big) e^{i\zeta\bar{z}}. \tag{A.10}$$

Adding the z dependence, we get

$$\varphi_1(\zeta, z, \bar{z}) = -\Big(\sum_{j,l} \frac{\lambda_j}{\zeta + \zeta_j}(1-A)^{-1}_{jl}\lambda_l\Big)e^{i\zeta\bar{z}-iz/4\zeta}, \tag{A.11}$$

$$\varphi_2(\zeta, z, \bar{z}) = \Big(1 + \sum_{j,l,k} \frac{\lambda_j}{\zeta + \zeta_j}\frac{\lambda_j\lambda_l}{\zeta_j + \zeta_l}(1-A)^{-1}_{lk}\lambda_k\Big)e^{i\zeta\bar{z}-iz/4\zeta}, \tag{A.12}$$

with $c_j(z) = c_j(0)e^{-iz/2\zeta_j}$. The sinh-Gordon field $\hat{u}(z, \bar{z})$ is found to be

$$\hat{u}(z, \bar{z}) = \sinh^{-1}\Big[\frac{4\zeta}{i}\frac{\partial(\varphi_1\varphi_2)}{(\varphi_1)^2 - (\varphi_2)^2}\Big]. \tag{A.13}$$

Appendix B

Einstein gravity

In this appendix, we review various formulations of Einstein gravity.

The first order veilbein formulation:

The natural vielbein variables are the 1-forms $e^a = dx^\mu e^a_\mu$ and the Lorentz affinities $w^{ab} = dx^\mu w^{ab}_\mu = -w^{ba}$, where a, b denote Lorentz indices running from $0, ..., d-1$ and μ, ν are world indices also ranging from $0, ..., d-1$. The torsion tensor is defined as

$$T_{\mu\nu}{}^a = D_\mu e^a_\nu - D_\nu e^a_\mu \tag{B.1}$$

where covariant Lorentz derivatives are given by

$$D_\mu v^a = \partial_\mu v^a + v^b w_{\mu b}{}^a. \tag{B.2}$$

Taking two derivatives one obtains

$$[D_\mu, D_\nu]v^a = -R_{\mu\nu}{}^a{}_b v^b \tag{B.3}$$

where the Riemann tensor is defined as

$$R_{\mu\nu a}{}^b = \partial_\mu w_{\nu a}{}^b - \partial_\nu w_{\mu a}{}^b - w_\mu{}^b{}_c w_{\nu a}{}^c + w_\nu{}^b{}_c w_{\mu a}{}^c. \tag{B.4}$$

Contracting once and twice with the veilbein gives $R_{\mu a} = e_b{}^\nu R_{\mu\nu a}{}^b$ and $R = e^{a\mu} R_{\mu a}$ respectively. The first order action we start with is [84]

$$\kappa^2 S = \langle \omega_\mu^{ab} \partial_\nu (e e_{ab}{}^{\mu\nu}) + \frac{1}{2} \omega_{\mu\nu}{}^{ab} e e_{ab}{}^{\mu\nu} \rangle, \tag{B.5}$$

where $e = \det e_\mu^a$, $e_{ab}{}^{\mu\nu} = e_a^\mu e_b^\nu - e_a^\nu e_b^\mu$ and $\omega_{\mu\nu}{}^{ab} = \omega_\mu{}^{ac} \omega_\nu{}^b{}_c - \omega_\mu{}^{bc} \omega_\nu{}^a{}_c$. Variations with respect to e_a^μ lead to

$$R_\mu{}^a - \frac{1}{2} e_\mu^a R = 0 \tag{B.6}$$

and variations with respect to $\omega_\mu{}^{ab}$ yield

$$D_\nu [e e^{ab\mu\nu}] = e T^{ab,\mu} = e e^{a\nu} e^{b\rho} T_{\nu\rho}{}^\mu = 0 \tag{B.7}$$

showing that the torsion vanishes.

To make contact with the metric formulation of gravity, one must assume that the frame e_μ^a has maximal rank d so that it gives rise to the non-degenerate metric tensor $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$. From the torsion-free equation $T_a = 0$, one solves this constraint and expresses the Lorentz connection in terms of the frame field $w = w(e, \partial e)$. It can be checked that the tensor $R_{\rho\sigma,\mu\nu} = e_\mu^a e_\nu^b R_{\rho\sigma,ab}$ is then expressed solely in terms of the metric, and is the Riemann tensor.

Gravity as a gauge theory:

It is well known that gravity can be interpreted as a gauge theory corresponding to an appropriate space-time symmetry algebra g. Vierbein $h_\nu{}^a$ and Lorentz connection $\omega_\nu{}^{ab}$ can be identified with the connection 1-forms of g. For example, in the four-dimensional space-time one can choose g to be the AdS algebra $so(2,3)$, which gives rise to the gauge field $A_\nu{}^{\hat{a}\hat{b}} = -A_\nu{}^{\hat{b}\hat{a}}$ with $\hat{a}, \hat{b} = 0 \div 4$, and one can set $\omega_\nu{}^{ab} = A_\nu{}^{ab}$ and $h_\nu{}^a = \lambda^{-1} A_\nu{}^{a4}$ with $a, b = 0 \div 3$. The $so(2,3)$ Yang-Mills strengths read in these terms

$$
\begin{aligned}
R_{\nu\mu}{}^{ab} &= \partial_\nu \omega_\mu{}^{ab} + \omega_\nu{}^a{}_c \omega_\mu{}^{cb} + \lambda^2 h_\nu{}^a h_\mu{}^b - (\nu \leftrightarrow \mu), && \text{(B.8)} \\
R_{\nu\mu}{}^{a} &= \partial_\nu h_\mu{}^a + \omega_\nu{}^a{}_c h_\mu{}^c - (\nu \leftrightarrow \mu). && \text{(B.9)}
\end{aligned}
$$

From (B.9) one recognizes that $R_{\nu\mu}{}^a$ has a form of the torsion tensor in the vierbein formulation of gravity. The constraint $R_{\nu\mu}{}^a = 0$ expresses the Lorentz connection $\omega_\nu{}^{ab}$ in terms of (derivatives of) the vierbein $h_\nu{}^a$ provided that $h_\nu{}^a$ is a non-degenerate matrix. Substituting these expressions back into the Lorentz components of the field strength (B.8), one can make sure that, up to the cosmological-type terms $\lambda^2 hh$, $R_{\nu\mu}{}^{ab}$ coincides with the Riemann tensor in gravity. Then one observes that the equations $R_{\nu\mu}{}^{ab} = 0$ and $R_{\nu\mu}{}^a = 0$ describe AdS space of radius λ^{-1}. In fact, this is the way how AdS space appears as a vacuum solution of the higher-spin equations considered below.

A remarkable observation by MacDowell and Mansouri [85] is that Einstein-Hilbert action with the cosmological term can be formulated in terms of the curvatures (B.8) in the form

$$
S^{MM} = -\frac{1}{4\kappa^2\lambda^2} \int d^4x \epsilon^{\nu\mu\rho\sigma} \epsilon^{abcd} R_{\nu\mu,ab} R_{\rho\sigma,cd}. \qquad \text{(B.10)}
$$

Let us note that the terms proportional to λ^{-2} in S^{MM}, which involve higher derivatives, combine into a topological term and do not affect the equations of motion. The λ-dependent term and the term proportional to λ^2 reduce to the scalar curvature and the cosmological term, respectively.

Gravity in spinor notation:

A useful form of the Einstein gravity which can be generalized to higher-spin case is summarized in [86] using the formalism of two-component spinors. The spinor indices are $\alpha, \beta = 1,2$ and $\dot\alpha, \dot\beta = 1,2$. They are raised or lowered using the symplectic forms

$$A^\alpha = \epsilon^{\alpha\beta} A_\beta, \qquad A_\alpha = -\epsilon_{\alpha\beta} A^\beta, \qquad B^{\dot\beta} = \epsilon^{\dot\beta\dot\alpha} B_{\dot\alpha}, \qquad B_{\dot\beta} = -\epsilon_{\dot\beta\dot\alpha} B^{\dot\alpha}, \tag{B.11}$$

where the Levi-Civita symbols are

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{B.12}$$

The corresponding curvatures can be written as

$$R_{\nu\mu\alpha(2)} = \partial_\nu \omega_{\mu\alpha(2)} - \partial_\mu \omega_{\nu\alpha(2)} + 2\omega_{\nu\alpha\gamma}\omega_{\mu\alpha}{}^{\gamma} + 2\lambda^2 h_{\nu\alpha\dot\delta} h_{\mu\alpha}{}^{\dot\delta}, \tag{B.13}$$

$$\bar R_{\nu\mu\dot\beta(2)} = \partial_\nu \bar\omega_{\mu\dot\beta(2)} - \partial_\mu \bar\omega_{\nu\dot\beta(2)} + 2\bar\omega_{\nu\dot\beta\dot\delta}\bar\omega_{\mu\dot\beta}{}^{\dot\delta} + 2\lambda^2 h_{\nu\gamma\dot\beta} h_{\mu}{}^{\gamma}{}_{\dot\beta}, \tag{B.14}$$

$$R_{\nu\mu\alpha\dot\beta} = [\partial_\nu \omega_{\mu\alpha\dot\beta} + \omega_{\nu\alpha\gamma} h_{\mu}{}^{\gamma}{}_{\dot\beta} + \bar\omega_{\nu\dot\beta\dot\delta} h_{\mu\alpha}{}^{\dot\delta}] - [\nu \leftrightarrow \mu]. \tag{B.15}$$

The fields $\omega_{\nu\alpha(2)}$ and $\bar\omega_{\nu\dot\beta(2)}$ in the spinor terms describe the Lorentz connection $\omega_{\nu,ab}$ and the field $h_{\nu\alpha\dot\beta}$ describes the tetrad $h_{\nu,a}$. The parameter λ is inverse of the radius of the AdS space.

The action (B.10) can be written in the form

$$S = -\frac{i}{4\kappa^2\lambda^2}\int d^4x\epsilon^{\nu\mu\rho\sigma}[R_{\nu\mu\alpha(2)}R_{\rho\sigma}{}^{\alpha(2)} - \bar{R}_{\nu\mu\dot{\beta}(2)}\bar{R}_{\rho\sigma}{}^{\dot{\beta}(2)}], \quad \text{(B.16)}$$

where $\epsilon^{0123} = 1$. This action is invariant under the Lorentz gauge subgroup of the original AdS group. The fields $\omega_{\nu\alpha(2)}$ and $\bar{\omega}_{\nu\dot{\beta}(2)}$ can be expressed in terms of the tetrad $\omega_{\nu\alpha\dot{\beta}}$ using the corresponding equations of motion $R_{\nu\mu\alpha\dot{\beta}} = 0$. AdS background is described by a tetrad and connection satisfying the equations

$$R_{\nu\mu\alpha(2)} = 0, \qquad \bar{R}_{\nu\mu\dot{\beta}(2)} = 0, \qquad R_{\nu\mu\alpha\dot{\beta}} = 0. \quad \text{(B.17)}$$

One can solve these equations and fix the fields $\omega_{\nu\alpha\dot{\beta}}$, $\omega_{\nu\alpha(2)}$, $\omega_{\nu\dot{\beta}(2)}$ where we denote them as $h_{\nu\alpha\dot{\beta}}$, $w_{\nu\alpha(2)}$, $\bar{w}_{\nu\dot{\beta}(2)}$.[1] Treating the fields $h, w, \bar{w}$ as background fields and assuming that the deviations of the fields ω from them are small, we can expand the curvatures (B.13-B.15) in powers of these deviations and keep only the linear terms. The substitution of the corresponding linearized curvatures into the action (B.16) gives the action for a massless field of spin 2 in the AdS space.

[1] Please distinguish ω and w.

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