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We define a category of *smooth 1-motives with torsion* over a locally noetherian base scheme and prove its Cartier duality. More precisely, we prove that the category of smooth 1-motives with torsion is equivalent to the category trivializations of particular $\mathbb{G}_m$-biextensions, and this implies the Cartier duality for smooth 1-motives with torsion. We also show that this category has realization functors when the base scheme is a spectrum of a field. Cartier duality theorem was already proved in the case of 1-motives over a field by Deligne or Ramachandran. We will extend this result to any locally noetherian base scheme and moreover to 1-motives with torsion. The category of smooth 1-motives with torsion is not an abelian cateogory, but there are many realization functors as the category of 1-motives.

1-Motives with Torsion and Cartier Duality

by

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Vita

Donghoon Park was born in Seoul, Korea in 1976. He received a B.Sc. from Seoul National University in 1999 and a M.Sc. from Seoul National University in 2001. After his military service at the Korean Naval Academy, he began to study at Brown University in 2004.

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Contents

Introduction

Since the category of all mixed motives is complicated, we can think of its subcategory. Deligne gave in [6] the category of 1-motives over a field and this is the category of universal H^1 of all algebraic curves. He defined realization functors of a 1-motive and showed that these realizations are naturally isomorphic to corresponding first cohomology groups of complex algebraic curves. He also constructed the dual 1-motive, combining many Cartier dual group schemes. According to this construction, it is not hard to show the Cartier duality theorem of 1-motives. For the complete proof, see [17]. Since the category of 1-motives is additive but not abelian, Barbieri-Viale, Rosenschon and Saito extended it by using finite abelian groups to get an abelian category. This extended category is an abelian category and its object, which is called a 1-motive with torsion has many realizations as Deligne's 1-motive does (see [2] for details). To define its dual, Barbieri-Viale and Kahn in [1] used more additive categories and showed Cartier duality in the derived category of 1-motives with torsion.

Recently, Lichtenbaum constructed the Weil-étale cohomology complex of a number field in [14] and proved the equality between the special value of a Dedekind zeta function and the Euler characteristic of this cohomology complex. He also suggested a similar cohomology complex for a 1-motive, assuming that there exists a dual 1-motive with torsion. A typical example of this object is an embedding of rational points of an abelian variety into this abelian variety itself. To define the Carier dual of it, we need an appropriate additive category containing the categoty of smooth 1-motives. This additive category will be introduced in the second chapter and see Definition 2.1.1 and 2.1.3 for objects and morphisms of it.

Deligne's proof of the Cartier duality for 1-motives is based on the fact that the category of 1-motives is equivalent to the category of (X, X', A, ϕ), where X and X' are finitely generated free abelian groups, where A is an abelian variety, and where ϕ is a biextension morphism from $X \times X'$ to the Poncaré biextension of A. This equivalence holds in more general cases, for instance, the category of smooth 1-motives over a locally noetherian base scheme. To define the torsion part of a smooth 1-motives with torsion, we need to take the extension of X by some finite group scheme over S and this new category is another example of the above equivalence. This is the statement of Proposition 2.2.6 and the Cartier duality for smooth 1-motives with torsion follows from it. As Deligne explained in the case of smooth 1-motives, we can define realization functors of smooth 1-motives with torsion and prove relations among these functors. See the last chapter for more details.

We can simply see the structure of this article as the following. The first chapter is the Cartier duality theorem for commutative group schemes which we will use. The complete proof of Cartier duality is explained in [16] for finite group schemes and abelian schemes. For twisted constant groups and groups of multiplicative type, see [7]. In the next chapter, we define additive categories of two different 1-motives with torsion and finally show that they are equivalent. Cartier duality is an easy consequence of this equivalence. In the last chapter, we compare smooth 1-motives with torsion over a field and 1-motives with torsion or cotorsion given by Barbieri-Viale in [2] and by Bertapelle in [3]. We also construct Betti, étale, l-adic, and de

Rham realization functors of a smooth 1-motives with torsion over a field and prove the functorial relations among them.

It is tempting to generalize these results further. For instance, Deligne's definition of étale realizations also makes sense in the case of a smooth 1-motive with torsion over a general base scheme. Next, according to Grothendieck's discussion about the canonical prolongation of a Poincaré biextension, we can think of a smooth integral model of a 1-motive. But we have some difficulty carrying out these works. In the first example, étale realizations are only group schemes or abelian sheaves but not abelian groups in general. So we need to find a geometric meaning of these and this is not quite simple. Defining a smooth integral model is much more complicated, since we need to prove Cartier duality of Néron models of abelian varieties and we also have to consider a smooth integral model of a torus which satisfies Cartier duality. Therefore we will not think of such problems in this article, but they still seem to be interesting questions.

Chapter 1

Cartier duality of commutative group schemes

Let S be a locally noetherian scheme. In this chapter we review how to define the dual of a commutative group scheme which is an ingredient of a smooth 1-motive with torsion over S. We also see the proof of Cartier duality theorem and the notion of a biextension that we will use in the next chapter.

1.1 $\mathcal{H}om\,(-,\mathbb{G}_m)$ and Cartier duality

Let S be a locally noetherian scheme. The first part of this section is Cartier duality of finite group schemes over S and one can read [16] I.2 for the perfect proof of each statement. The next one is Cartier duality of locally constant group schemes for the étale topology and we will follow [7] VIII and IX. In the last part, we will extend these duality results. A typical example of this general case is the group scheme extension of a locally constant group scheme by a finite group scheme, and we need to apply it to define smooth 1-motives with torsion and their duals.

Definition 1.1.1. (Finite group schemes : [16] I.2.)

N is a finite group scheme over S, if it is a commutative group scheme over S, such that the structural morphism $N \to S$ is finite, *i.e.*, one can find an open covering $\{S_\alpha\}$ of S such that $N \times_S S_\alpha = \mathrm{Spec}\,(E_\alpha)$ where E_α is finitely generated, as a module, over $\Gamma(S_\alpha, \mathcal{O}_S)$, the ring of local sections of its structure sheaf.

Let R be a commutative ring with unity. For an R-module E, define $E^D := \mathrm{Hom}_R(E, R)$ as the set of R-module homomorphisms, then E^D is an R-module again. For another R-module F and $h : E \to F$ (in other words, $h \in \mathrm{Hom}_R(E, F)$), let $h^D := \mathrm{Hom}_R(h, R) : F^D \to E^D$ be a map given by $h^D(f')(e) = f'(h(e))$ for $e \in E$ and $f' \in F^D$. If E is a finitely generated and projective R-module, then E^D is also finitely generated and projective over R. For a finitely generated and projective E and for any R-module F, define $\Phi_R(E, F) : E \otimes_R F \to \mathrm{Hom}_R(E^D, F)$ by $\big((\Phi_R(E, F))(e \otimes_R f)\big)(g) = g(e).f$, where $e \in E$, $f \in F$, and $g \in E^D$. One can show that $\Phi_R(E, F)$ is a morphism of tri-functors (in R, E, and F). If E is finitely generated and projective R-module, then $\Phi_R(E, F)$ is an isomorphism between $E \otimes_R F$ and $\mathrm{Hom}_R(E^D, F)$ for any F. See Lemma 2.2 in [16] I.2 for the proof.

If we define $\kappa_E : E \to E^{DD}$ by $(\kappa_E(a))(f) = f(a)$ for $a \in E$ and $f \in E^D$, then κ_E is functorial in E. When E is finitely generated and projective R-module, κ_E is an isomorphism between two projective modules E and E^{DD} (Corollary 2.3 in [16] I.2). We can compose the following maps:

$$\begin{tikzcd}
E^D \otimes_R E^D \arrow[r, "\Phi_R(E^D, E^D)"] \arrow[d, dashed, "\iota_E"'] & \mathrm{Hom}_R(E^{DD}, E^D) \arrow[r, "\kappa_E^*"] & \mathrm{Hom}_R(E, E^D) \arrow[d, equal] \\
(E \otimes_R E)^D \arrow[r, equal] & \mathrm{Hom}_R(E \otimes_R E, R) & \mathrm{Hom}_R(E, \mathrm{Hom}_R(E, R)) \arrow[l]
\end{tikzcd}$$

Since all these maps are isomorphisms and since they are functorial in E, We have a funtorial isomorphism $\iota_E : E^D \otimes_R E^D \to (E \otimes_R E)^D$. Define $\zeta : R^D \to R$ by $\zeta(a') = a'(1_R)$ for $a' \in R^D$ and this map is an isomorphism of R-modules.

When S is an affine scheme $\mathrm{Spec}\,(R)$, a finite group scheme N over S is an affine scheme $\mathrm{Spec}\,(E)$ where E is a finite algebra over R. Since N is a group scheme,

E is an R-bialgebra, *i.e.*, it has 5 homomorphisms (with appropriate compatibility conditions) $m_E : E \otimes_R E \to E$, $s_E : E \to E \otimes_R E$, $i_E : E \to E$, $\varepsilon_E : R \to E$, and $\epsilon_E : E \to R$. Now taking their dual homomorphisms $(-)^D$ and using ι_E, we have 5 more homomorphisms for E^D.

$$\begin{aligned}
m_{E^D} &: \ E^D \otimes_R E^D \xrightarrow{\iota_E} (E \otimes_R E)^D \xrightarrow{(s_E)^D} E^D \\
s_{E^D} &: \ E^D \xrightarrow{(m_E)^D} (E \otimes_R E)^D \xrightarrow{(\iota_E)^{-1}} E^D \otimes_R E^D \\
i_{E^D} &: \ E^D \xrightarrow{(i_E)^D} E^D \\
\varepsilon_{E^D} &: \ R \xrightarrow{\zeta^{-1}} R^D \xrightarrow{(\epsilon_E)^D} E^D \\
\epsilon_{E^D} &: \ E^D \xrightarrow{(\varepsilon_E)^D} R^D \xrightarrow{\zeta} R
\end{aligned}$$

These new homomorphisms give a bialgebra structure to E^D and $\mathrm{Spec}\,(E^D)$ is again a finite group scheme over S. It is called the Cartier dual of N and is denoted by N^D.

Proposition 1.1.1. *(Cartier duality of bialgebras)*

Let E and E' be finitely generated and projective commutative R-bialgebras and let $f : E \to E'$ be an R-bialgebra homomorphism. Then $E^D = \mathrm{Hom}_R(E, R)$ with $(m_{E^D}, s_{E^D}, i_{E^D}, \varepsilon_{E^D}, \epsilon_{E^D})$ is an R-bialgebra and $(-)^D$ is a contravariant functor.

A canonical map $\kappa_{E,E'} : \mathrm{Hom}_R(E, E') \to \mathrm{Hom}_R(E'^D, E^D)$ given by $\kappa_{E,E'}(f) := f^D$ is a functorial isomorphism in E and E', and $(-)^{DD}$ is isomorphic with the identity functor.

Proof. See proposition 2.6 of [16] I.2. □

From the above proposition, the composite of the following maps

$$\mathrm{Hom}_R(E^D, E^D) \xrightarrow{\kappa_{E,E}^{-1}} \mathrm{Hom}_R(E, E) \xrightarrow{(\kappa_E)_*} \mathrm{Hom}_R(E, E^{DD}) \tag{1.1.1}$$

is a bijection and under this map, κ_E is identified with id_{E^D}. This identification characterizes the duality homomorphism $\kappa_E : E \to E^{DD}$.

If F is flat over R of finite presentation, then F is a projective R-module (for its proof, see corollary 2.8 of [16] I.2). Therefore what we need to consider is the category of finite group schemes which is flat over S.

Definition 1.1.2. (The category of finite group schemes)

Let $\mathcal{N}_S$ be the category of finite commutative group schemes which is flat over S. When S is an affine scheme Spec (R), $\mathcal{N}_R(=\mathcal{N}_S^0)$ is the category of commutative, finitely generated as a module, flat R-bialgebras.

Now we can state Cartier duality of finite flat group schemes $\mathcal{N}_S$.

Proposition 1.1.2. *(Cartier duality of finite group schemes)*

There is a contravariant functor $(-)^{D_S} : \mathcal{N}_S \to \mathcal{N}_S$ and there is a natural isomorphism $\eta_S : \mathrm{id}_{\mathcal{N}_S} \to (-)^{D_S D_S}$. The functor $(-)^{D_S}$ commutes with base change i.e., if $T \to S$ is a morphism of (locally noetherian) schemes, and $N \in \mathcal{N}_S$, the natural homomorphism $(N \times_S T)^{D_T} \to N^{D_S} \times_S T$ is an isomorphism.

Proof. See proposition 2.9 of [16] I.2. □

Let $\mathcal{C}_S$ be the category of schemes over S and let $\widehat{\mathcal{C}_S} := \mathrm{Funct}(\mathcal{C}_S, (\mathrm{Ens}))$ be the category of contravariant functors from $\mathcal{C}_S$ to sets. For $P \in \widehat{\mathcal{C}_S}$ and $P' \in \widehat{\mathcal{C}_S}$, $\mathrm{Mor}_S(P, P')$ is defined by a set of morphisms (more precisely, natural transformations) from P to P'. A functor from $\mathcal{C}_S$ to abelian groups is called an abelian S-group. For two abelian S-groups G and G', a homomorphism of abelian S-groups $h : G \to G'$ is defined by a natural transformation such that $h(S') : G(S') \to G'(S')$ is a group homomorphism for every $S' \in \mathcal{C}_S$. A commutative group scheme over S is regarded as an abelian S-group according to Yoneda's lemma. We write $\mathrm{Hom}_S(G, G')$ to indicate a set of S-group homomorphisms (and this is denoted by $\mathrm{Hom}_{S\mathrm{gr}}(G, G')$ in [7]). Define an abelian S-group $D_S(G \to G')$ by the functor $S' \longmapsto \mathrm{Hom}_{S'}(G|_{S'}, G'|_{S'})$ for $S' \in \mathcal{C}_S$, where $G|_{S'}$ is the restriction of G to the category of S'-schemes $\mathcal{C}_{S'}$. In [7], $\underline{\mathrm{Hom}}_{\mathrm{gr}}(G, G')$ means $D_S(G \to G')$.

For a commutative ring R with unity, let R^* be its multiplicative group of units. The S-group $\mathbb{G}_{m,S}$ given by $(\mathbb{G}_{m,S})(S') = \Gamma(S', \mathcal{O}_{S'})^*$, $S' \in \mathcal{C}_S$ is representable by $\mathrm{Spec}\,\mathbb{Z}[t, t^{-1}] \times_{\mathrm{Spec}\,\mathbb{Z}} S$ which is also denoted by $\mathbb{G}_{m,S}$ and is called the **multiplicative linear group** over S. If $S' \in \mathcal{C}_S$, then $\mathbb{G}_{m,S'} = \mathbb{G}_{m,S} \times_S S'$ and $\mathbb{G}_{m,S}$ is a commutative affine group scheme over S. Let M be an abelian group and define M_S by a **constant group scheme** $\coprod_{m \in M} S_m$ over S associated with M. When $G' = \mathbb{G}_{m,S}$, we will use $D(G)$ instead of $D_S(G \to \mathbb{G}_{m,S})$. We can generalize M_S as the following:

Definition 1.1.3. (Twisted constant groups over S)

A group scheme over S is called a locally constant group over S when every point s of S admits a Zariski open neighborhood U_s such that $X \times_S U_s$ is a constant group scheme over U_s. A group scheme X over S is called a twisted constant group over S if it is locally isomorphic, for the fppf topology, to a constant group scheme. A twisted constant group over S is called quasi-isotrivial if it is locally isomorphic, for the étale topology, to a constant group scheme.

Recall that Grothendieck used the fpqc topology instead of the fppf topology in Definition 5.1 (twisted constant groups) of [7] X. He also used the big étale topology to define the quasi-isotriviality, and mentioned this definition is equivalent to: there is a surjective étale morphism $S' \to S$ so that $X \times_S S'$ is a constant group scheme over S'.

Let G and G' be S-groups. $\mathrm{BiHom}_S(G \times G', \mathbb{G}_{m,S})$ is defined by the set of bi-multiplicative natural transformations $G \times G' \to \mathbb{G}_{m,S}$ and it is denoted by $\mathrm{Hom}_{\mathrm{bil}}(G \times$

$G', \mathbb{G}_{m,S})$ in [7]. One can show that

$$\mathrm{Hom}_S(G, D(G')) \cong \mathrm{BiHom}_S(G \times G', \mathbb{G}_{m,S}) \tag{1.1.2}$$

and that this isomorphism is functorial in G and G'. For the proof, see [7] VIII.1. From this isomorphism (1.1.2), we also get a functorial bijection and replacing G' by $D(G)$, one has

$$\mathrm{Hom}_S(G, D(D(G))) \cong \mathrm{Hom}_S(D(G), D(G)). \tag{1.1.3}$$

The last isomorphism (1.1.3) gives us a canonical S-group morphism $\Upsilon_G : G \to D(D(G))$ corresponding to $\mathrm{id}_{D(G)}$. G is called **reflexive** (w.r.t. $\mathbb{G}_{m,S}$) if Υ_G is an S-group isomorphism.

Definition 1.1.4. (Groups of multiplicative type over S)

A group scheme over S is called a diagonalizable group over S when it represents the functor $D(M_S) = D_S(M_S \to \mathbb{G}_{m,S})$ for some abelian group M. A group scheme T over S is called a group of multiplicative type over S if it is locally isomorphic, for the fppf topology, to a diagonalizable group, *i.e.*, there is a surjective faithfully flat morphism $S' \to S$ of finite presentation such that $G \times_S S'$ represents $D_{S'}(M_{S'} \to \mathbb{G}_{m,S'})$ for some abelian group M. A group of multiplicative type over S is called quasi-isotrivial if it is locally isomorphic, for the étale topology, to a diagonalizable group.

To show that the above group schemes (X and T) are reflexive, we need to prove that $D(X)$ and $D(T)$ are representable. The following result guarantees this property.

Proposition 1.1.3. *(X and T are reflexive)*

(a) If X is a twisted constant group over S, then $D(X)$ is represented by a group of multiplicative type over S. X is quasi-isotrivial if and only if $D(X)$ is quasi-isotrivial.

(b) If a group T of multiplicative type over S is quasi-isotrivial, then $D(T)$ is represented by a quasi-isotrivial twisted constant group over S.

(c) The functors $X \mapsto D(X)$ and $T \mapsto D(T)$ are anti-equivalences, and quasi-inverses to each other between the category of quasi-isotrivial twisted constant groups over S and the category of quasi-isotrivial groups of multiplicative type over S.

Proof. See Proposition 5.3 in [7] X for (a). For (b) and (c), see Corollary 5.7 in the same chapter. The main point of this proof is Lemma 5.4 : Let $S' \to S$ be a faithfully flat morphism which is locally of finite presentation and let X' be a scheme over S' with the descent datum relative to $S' \to S$. Suppose that X is separated, locally of finite presentation and locally quasi-finite over S'. Then the given descent datum is effective, *i.e.*, there is a scheme X over S, and an S'-isomorphism $X \times_S S' \to X'$ compatible to the given descent datum. □

For quasi-isotrivial X and T, we call $D(X)$ the **Cartier dual** of X and $D(T)$ the Cartier dual of T. Since such X and T are reflexive, we showed Cartier duality of them. To define a 1-motive over S, it is enough to consider a group of multiplicative type that is of finite type over S. A twisted constant group over S is called **finitely generated** if it is locally (for the fppf topology) isomorphic to the constant group scheme which is given by a finitely generated abelian group.

Proposition 1.1.4. *(Groups of multiplicative type over S of finite type)*

Every group T of multiplicative type which is finite type over S is quasi-isotrivial. The functors $D(-)$ in Proposition 1.1.3 induce anti-equivalences (and quasi-inverses to each other) between the category of groups of multiplicative type and of finite type over S and the category of finitely generated twisted constant groups over S. Every finitely generated twisted constant group over S becomes quasi-isotrivial.

Proof. See Corollary 4.5 and Corollary 5.9 in [7] X. □

A **torus** over S is defined by a quasi-isotrivial group of multiplicative type over S which is locally isomorphic to a $\mathbb{G}_m^r$, where r is a finite number and is called the dimension of this torus (and note that a torus is not necessarily quasi-isotrivial in Definition 1.3 of [7] IX). Proposition 1.1.4 shows that the Cartier dual of a torus T of $\dim_S(T) = r$ is a quasi-isotrivial twisted constant group over S that is locally isomorphic to $\mathbb{Z}^r$.

The Cartier dual N^D of a finite group scheme N was not defined by $D(N)$, but we can prove that $N^D = D(N)$ and this result is called **Cartier-Shatz formula**. The notation N^{D_S} means Cartier dual of N as in Proposition 1.1.2.

Proposition 1.1.5. *(Cartier-Shatz formula)*

Let N be a flat finite group scheme over S. There exists an isomorphism

$$\Psi(S,N) : \mathrm{Hom}_S(N, \mathbb{G}_{m,S}) \to \mathrm{Mor}(S, N^{D_S}) \tag{1.1.4}$$

which is functorial in S and N. In other words, for a morphism $f : S' \to S''$ of schemes over S and for a homomorphism $g : N \to N'$ of such group schemes, we have two commutative diagrams:

$$\begin{array}{ccc}
\mathrm{Hom}_{S'}(N \times_S S', \mathbb{G}_{m,S'}) & \xrightarrow{\Psi(S',N\times_S S')} & \mathrm{Mor}_{S'}(S', (N \times_S S')^{D_{S'}}) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{Hom}_{S''}(N \times_S S'', \mathbb{G}_{m,S''}) & \xrightarrow{\Psi(S'',N\times_S S'')} & \mathrm{Mor}_{S''}(S'', (N \times_S S'')^{D_{S''}}),
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{S'}(N \times_S S', \mathbb{G}_{m,S'}) & \xrightarrow{\Psi(S',N\times_S S')} & \mathrm{Mor}_{S'}(S', (N \times_S S')^{D_{S'}}) \\
\uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle (g^{D_{S'}})_*} \\
\mathrm{Hom}_{S'}(N' \times_S S', \mathbb{G}_{m,S'}) & \xrightarrow{\Psi(S',N'\times_S S')} & \mathrm{Mor}_{S'}(S', (N' \times_S S')^{D_{S'}}).
\end{array}$$

Proof. See theorem 16.1 of [16] III. □

Therefore we have a group scheme isomorphism $D(N) \to N^{D_S}$. Using this identification, we can see that $\Upsilon_N = \eta_S(N)$ where η_S is the duality natural isomorphism in Proposition 1.1.2. Thus we will use the notation η instead of η_S and Υ. We summarize our discussion as the following. Let S be a locally noetherian scheme and let G be a commutative group scheme over S. If G is finite and flat over S or finitely generated quasi-isotrivial twisted constant for the étale topology, then its Cartier dual is defined by $G^D := D(G, \mathbb{G}_{m,S})$ and the Cartier duality homomorphism $\eta(G) : G \to G^{DD}$ is an isomorphism. A representable contravariant functor from $\mathcal{C}_S$ to sets becomes a fppf (also étale and Zariski) sheaf. See Proposition 8.1 in [4] for the proof. Let $\mathcal{H}om_S(G, \mathbb{G}_{m,S})$ be the fppf-sheaf associated to the functor (presheaf) $U \mapsto \mathrm{Hom}_U(G \times_S U, \mathbb{G}_{m,U})$. When G is a finite group scheme or a quasi-isotrivial twisted constant group scheme, $D(G)$ is representable and $D(G) = \mathcal{H}om_S(G, \mathbb{G}_{m,S})$. Therefore, for these group schemes, $\mathcal{H}om_S(-, \mathbb{G}_{m,S})$ gives their Cartier dual group schemes.

For any commutative group schemes, we can also define $(-)^D$ and $\eta(-)$ as the above. But $\eta(-)$ is not always an isomorphism and to show that it is an isomorphism, we need to use an extension group scheme of two group schemes satisfying Cartier duality. Let C_1, C_2 and C_3 be commutative group schemes over S. We say that the sequence

$$0 \to C_1 \xrightarrow{f} C_2 \xrightarrow{p} C_3 \to 0 \tag{1.1.5}$$

is **exact**, f and p being homomorphisms, if $C_1 = \mathrm{Ker}(p)$ and if

$$C_1 \times_S C_2 \underset{p_2}{\overset{m_{C_2} \circ (f, \mathrm{id}_{C_2})}{\rightrightarrows}} C_2 \xrightarrow{\quad p \quad} C_3 \tag{1.1.6}$$

is exact in the category of schemes over S where $m_{C_2} : C_2 \times_S C_2 \to C_2$ is the multiplication of C_2 and $p_2 : C_1 \times_S C_2 \to C_2$ is a projection. To find the definition of exactness of (1.1.6), see p.16 of [15].

Definition 1.1.5. (Extensions of group schemes)

For commutative group schemes C_1 and C_3 over S, a commutative group scheme C_2 over S is called an extension of C_3 by C_1 if we can find an exact sequence (1.1.5).

Let C_2 be an extension of C_3 by C_1 as in the above definition. Assume that C_1 and C_3 satisfy Cartier duality, i.e. $\eta(C_1)$ and $\eta(C_3)$ are isomorphisms. Since $(-)^D$ is a functor and since $\eta(-)$ is a natural transformation, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_1 \arrow[r, "f"] \arrow[d, "\eta(C_1)"] & C_2 \arrow[r, "p"] \arrow[d, "\eta(C_2)"] & C_3 \arrow[r] \arrow[d, "\eta(C_3)"] & 0 \\
0 \arrow[r] & C_1^{DD} \arrow[r, "f^{DD}"] & C_2^{DD} \arrow[r, "p^{DD}"] & C_3^{DD} \arrow[r] & 0
\end{tikzcd} \tag{1.1.7}$$

but to show that $\eta(C_2)$ is an isomorphism we have to prove that the bottom sequence is exact.

Let me call a sheaf of abelian groups just **abelian sheaf**, briefly. It is known that for any abelian sheaf $\mathcal{I}$ over S, $\mathcal{H}om_S(-,\mathcal{I})$ is a left exact contravariant functor and we can define its derived functors $\mathcal{E}xt^i_S(-,\mathcal{I})$. If we have a short exact sequence of abelian sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, then there is a long exact sequence

$$\begin{aligned} 0 \ &\to \ \mathcal{H}om_S(\mathcal{H},\mathcal{I}) \to \mathcal{H}om_S(\mathcal{G},\mathcal{I}) \to \mathcal{H}om_S(\mathcal{F},\mathcal{I}) \\ &\to \ \mathcal{E}xt^1_S(\mathcal{H},\mathcal{I}) \to \mathcal{E}xt^1_S(\mathcal{G},\mathcal{I}) \to \mathcal{E}xt^1_S(\mathcal{F},\mathcal{I}) \to \cdots . \end{aligned} \tag{1.1.8}$$

Since C_1, C_2, C_3 and $\mathbb{G}_{m,S}$ become abelian sheaves over S, if $\mathcal{E}xt^1_S(C_1,\mathbb{G}_{m,S}) = \mathcal{E}xt^1_S(C_3,\mathbb{G}_{m,S}) = 0$, then the second row of (1.1.8) shows $\mathcal{E}xt^1_S(C_2,\mathbb{G}_{m,S}) = 0$ and the first line of (1.1.8) becomes a short exact sequence of abelian sheaves

$$0 \to C_3^D \to C_2^D \to C_1^D \to 0. \tag{1.1.9}$$

But we can not say that C_2^D is representable and we need more assumptions about C_1 and C_3.

Proposition 1.1.6. *(Representability of the extension of group schemes)*

(a) Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of sheaves of abelian groups on S_{fppf}. If $\mathcal{H}$ is represented by a scheme of finite type over S and if $\mathcal{F}$ is represented by an affine scheme of finite type over S, then $\mathcal{G}$ is also represented by a commutative group scheme over S.

(b) Let C_1 and C_3 be commutative group schemes of finite type over S. $\mathrm{Ext}^1_{S\mathrm{sch}}(C_3, C_1)$ is defined by the set of isomorphism classes of exact sequences of commutative group schemes over S

$$0 \to C_1 \to C_2 \xrightarrow{p} C_3 \to 0 \tag{1.1.10}$$

where p is faithfully flat and of finite presentation.

If C_1 is flat and affine over S then the sequence of sheaves corresponding to (1.1.10) is an element in $\mathrm{Ext}^1_{S_{\mathrm{fppf}}}(C_3, C_1)$ of fppf abelian sheaves and this map $\mathrm{Ext}^1_{S\mathrm{sch}}(C_3, C_1) \to \mathrm{Ext}^1_{S_{\mathrm{fppf}}}(C_3, C_1)$ is an isomorphism of groups.

Proof. See Proposition (17.4) and Corollary (17.5) in [16] III. □

If we apply $(-)^D$ to (1.1.9), we get an exact sequence

$$0 \to C_1^{DD} \to C_2^{DD} \to C_3^{DD} \to \mathcal{E}xt^1_S(C_1^D, \mathbb{G}_{m,S}) \tag{1.1.11}$$

and when $\mathcal{E}xt^1_S(C_1^D, \mathbb{G}_{m,S}) = 0$ again, we have another short exact sequence

$$0 \to C_1^{DD} \to C_2^{DD} \to C_3^{DD} \to 0.$$

This is the second sequence in 1.1.7 whose exactness we want to prove. We need to use the following result.

Proposition 1.1.7. *(Vanishing of $\mathcal{E}xt^1_{S_{\mathrm{fppf}}}(C, \mathbb{G}_{m,S}) = 0$)*

C is a commutative group scheme and it is

(a) a finite flat group scheme,

(b) a finitely generated twisted constant group scheme, or

(c) a group of multiplicative type and of finite type.

Then $\mathcal{E}xt^1_S(C, \mathbb{G}_{m,S}) = 0$.

Proof. For (a) and (b), see Theorem 1 of [18]. For (c), see Proposition 3.3.1 of [12] VIII. $\square$

When C_1 and C_3 are finite group schemes or quasi-isotrivial twisted constant group schemes, we can apply Proposition 1.1.7 to C_i^D and C_i^{DD}. Hence we conclude that $\mathcal{E}xt^1(C_2^D, \mathbb{G}_{m,S}) = 0$ and that $\eta(C_2)$ is an isomorphism. The following two categories will be used to define smooth 1-motives with torsion.

Definition 1.1.6. (Group schemes with Cartier duality)

Let $\mathcal{GC}_S$ be the category of commutative group schemes G over S such that G^{D_S} is an affine group scheme of finite type over S, $\eta_S(G) : G \to G^{D_S D_S}$ is an isomorphism and $\mathcal{E}xt^1_S(G, \mathbb{G}_{m,S}) = \mathcal{E}xt^1_S(G^{D_S}, \mathbb{G}_{m,S}) = 0$.

$\mathcal{GC}_S^D$ is defined by the category of group schemes given by the Cartier dual G^{D_S} of $G \in \mathcal{GC}_S$. These categories $\mathcal{GC}_S$ and $\mathcal{GC}_S^{D_S}$ have the Cartier duality isomorphisms η_S.

$\mathcal{GC}_S$ is nonempty, because any finite flat commutative group scheme satisfies the above conditions. Since $\mathcal{E}xt^1_S(\mathbb{Z}_S, \mathbb{G}_{m,S}) = \mathcal{E}xt^1_S(\mathbb{G}_{m,S}, \mathbb{G}_{m,S}) = 0$, $\mathbb{Z}_S \in \mathcal{GC}_S$ is also true. Likewise, any finitely generated twisted constant group over S is also contained in $\mathcal{GC}_S$. Moreover, a group scheme extension of them is still an object of $\mathcal{GC}_S$. This shows that $\mathcal{GC}_S$ is bigger than the category of finite flat commutative group schemes, and the category of finitely generated twisted constant groups.

1.2 $\mathcal{E}xt^1(-, \mathbb{G}_m)$ and Cartier duality

Let S be a locally noetherian scheme and let Q be a fixed scheme over S. Only in this section, T means any scheme over S, not necessarily a group of multiplicative type. The isomorphism classes of invertible sheaves on Q form an abelian group. It is called the **absolute Picard group** of X and denoted by $\mathrm{P}(Q)$. This group is isomorphic to $H^1_{\mathrm{Zar}}(Q, \mathcal{O}_Q^*)$. We define a contravariant functor $\mathrm{P}_{Q/S}$ from the category of schemes over S to the category of abelian groups by $\mathrm{P}_{Q/S}(T) := \mathrm{P}(Q \times_S T)$. For any set of schemes $\{T_i\}$ over S, we can show that $\mathrm{P}_{Q/S}(\coprod T_i) = \prod \mathrm{P}_{Q/S}(T_i)$ and we can regard this functor as a presheaf of abelian groups on the category of schemes over S. This presheaf $\mathrm{P}_{Q/S}$ is called the **absolute Picard functor** of Q over S.

Definition 1.2.1. (Relative Picard functors)

The fppf-sheaf associated to the functor $\mathrm{P}_{Q/S}$ is called the relative Picard functor of Q over S and it is denoted by $\mathrm{Pic}_{Q/S}$. For any scheme T over S, the abelian group $\mathrm{Pic}_{Q/S}(T)$ is called the relative Picard group of $Q \times_S T$ over T.

An **abelian scheme** over S is a smooth proper group scheme $f_A : A \to S$ whose geometric fibers are connected. It is a commutative group scheme of finite presentation over S (Remark 1.2 in [8] I). Since an abelian scheme is a group scheme, it has the unit section e_A of its structure morphism f_A.

Proposition 1.2.1. *(Cohomological flatness)*

Let $f : Q \to S$ be proper and flat. For a point $s \in S$, $k(s)$ means the residue class field of s. If $H^0(Q \times_S \mathrm{Spec}\,(k(s)), \mathcal{O}_{Q \times_S \mathrm{Spec}\,(k(s))}) = k(s)$ for all $s \in S$, then $f_*(\mathcal{O}_Q) = \mathcal{O}_S$ holds universally, i.e., $(f_{S'})_*(\mathcal{O}_{Q \times_S S'}) = \mathcal{O}_{S'}$ for every scheme S' over S. In this case, f is called cohomologically flat.

Proof. See Proposition (7.8.6) and Corollary (7.8.8) of [10]. □

Since $f_A : A \to S$ is proper and flat and since every geometric fibre of f_A is an abelian variety (therefore a projective variety), $f_*(\mathcal{O}_A) = \mathcal{O}_S$ holds universally. For a quasi-compact and quasi-separated $f : Q \to S$, if $f_*(\mathcal{O}_Q) = \mathcal{O}_S$ holds universally and if there is a section of f, then $\mathrm{Pic}_{Q/S}(T) = \mathrm{P}(Q \times_S T)/\mathrm{P}(T)$ for any scheme T over S (and see Proposition 4 of [4] 8.1 for the proof). Since an abelian scheme A is of finite presentation and proper over S, it is quasi-compact and separated. Thus we can conclude that $\mathrm{Pic}_{A/S}(T) = \mathrm{P}(A \times_S T)/\mathrm{P}(T)$.

Assume that $f : Q \to S$ has a section e. Let T be a scheme over S, and $\mathcal{L}$ a sheaf on $X \times_S T$. An e-**rigidification** of $\mathcal{L}$ is the choice of an isomorphism $u : \mathcal{O}_T \to (e \times \mathrm{id}_T)^*\mathcal{L}$, assuming one exists. In this case $(\mathcal{L}, u)$ is called a **rigidified invertible sheaf**. The set $(\mathrm{P}, e)(Q \times_S T)$ of isomorphism classes of rigidified invertible sheaves has a group structure $((\mathcal{L}_1, u_1)(\mathcal{L}_2, u_2) = (\mathcal{L}_1 \otimes_{\mathcal{O}_{Q \times_S T}} \mathcal{L}_2, u_1 \otimes_{\mathcal{O}_T} u_2))$ and there is an isomorphism $\rho : (\mathrm{P}, e)(Q \times_S T) \to \mathrm{P}(Q \times_S T)/\mathrm{P}(T)$ induced by the map $(\mathcal{L}, u) \mapsto \mathcal{L}$. (See Lemma 2.9 of [13] or p.204 of [4] for the proof).

Definition 1.2.2. (Rigidified Picard Functors)

Let $f : Q \to S$ has a section e and assume that $f_*(\mathcal{O}_Q) = \mathcal{O}_S$ holds universally. $(\mathrm{P}_{Q/S}, e)(T)$ is defined by the isomorphism classes of rigidified invertible sheaves on $Q \times_S T$, where T is a scheme over S. $(\mathrm{P}_{Q/S}, e)$ becomes a functor and isomorphic to the relative Picard functor of Q over S. This functor $(\mathrm{P}_{Q/S}, e)$ is called the rigidified Picard functor of Q over S.

From now, assume that A is a projective abelian scheme over S. The following theorem shows that $\mathrm{Pic}_{A/S} = (\mathrm{P}_{A/S}, e)$ is representable.

Proposition 1.2.2. *(Grothendieck's representability Theorem)*

Let $f : Q \to S$ be projective and of finitely presentation. Assume that f is flat, and that the geometric fibres of f are reduced and irreducible. Then $\mathrm{Pic}_{Q/S}$ is representable *by a separated scheme which is locally of finite presentation over S.*

Proof. See Theorem 3.1 in [11] n°232 or Theorem 1 in of [4] 8.2. $\square$

Remark 1.2.1. (Abelian algebraic spaces)

An abelian scheme is not necessarily projective and to represent it by a geometric object, we need to use an algebraic space which are more general than a scheme. This case was considered in Theorem 1.9 of [8] I, and this theorem implies that for any abelian scheme A over S, $\mathrm{Pic}_{A/S}$ is representable by a scheme and we can remove the projectiveness assumption of A to define its Picard scheme.

Therefore the relative Picard functor of A over S is represented by a scheme over S and this representing scheme is unique up to isomorphism. It is called the **Picard scheme** and denoted by $\mathbf{Pic}(A/S)$. Since $\mathrm{Pic}_{A/S}$ is a sheaf of abelian groups, $\mathbf{Pic}(A/S)$ is a commutative group scheme over S. The rigidified Picard functor of A is also represented by the same group scheme $\mathbf{Pic}(A/S)$.

Proposition 1.2.3. *(Universal rigidified invertible sheaves)*

Let $f : Q \to S$ be flat of finite presentation and let e be a section of f. Assume that $f_*\mathcal{O}_Q = \mathcal{O}_S$ holds universally. If $\mathrm{Pic}(Q/S)$ *is representable by a scheme* $\mathbf{Pic}(Q/S)$*, there exists a rigidified invertible sheaf* $(\mathcal{P}_{Q\times_S\mathbf{Pic}(Q/S)}, \epsilon_{\mathcal{P}_{Q\times_S\mathbf{Pic}(Q/S)}})$ *on* $\mathbf{Pic}(Q/S)$ *which has the following property: For any scheme T over S, and for any rigidified invertible sheaf $(\mathcal{L}, u : \mathcal{O}_T \to (e\times_S \mathrm{id}_T)^*\mathcal{L})$ on $Q\times_S T$, there exists a unique morphism $g : T \to \mathbf{Pic}(Q/S)$ such that $(\mathcal{L}, u)$ is isomorphic to the pull-back of this sheaf $(\mathcal{P}_{Q\times_S\mathbf{Pic}(Q/S)}, \epsilon_{\mathcal{P}_{Q\times_S\mathbf{Pic}(Q/S)}})$ under the morphism $\mathrm{id}_Q \times_S g : Q\times_S T \to Q\times_S \mathbf{Pic}(Q/S)$.*

Proof. See Proposition 4 in [4] 8.2. $\square$

Such a rigidified invertible sheaf $(\mathcal{P}_{Q\times_S \mathbf{Pic}(Q/S)}, \epsilon_{\mathcal{P}_{Q\times_S \mathbf{Pic}(Q/S)}})$ is unique up to the isomorphism class of rigidified invertible sheaves. $\mathcal{P}_{Q\times_S \mathbf{Pic}(Q/S)}$ is called the **universal invertible sheaf** for $(\mathrm{P}_{Q/S}, e)$ and $\epsilon_{\mathcal{P}_{Q\times_S \mathbf{Pic}(Q/S)}}$ is called the **universal rigidification**. A projective (or proper) abelian scheme A over S is a special case of the above proposition because we know the existence of the Picard scheme $f : \mathbf{Pic}(A/S) \to S$ of A and A has the unit section $e_A : S \to A$.

Suppose that $S = \operatorname{Spec} k$ where k is a field. The group scheme $\mathbf{Pic}(A/S)$ over $S = \operatorname{Spec} k$ has the identity (connected) component which is denoted by $\mathbf{Pic}^0(A/S)$. We define $\mathbf{Pic}^\tau(A/S)$ by

$$\mathbf{Pic}^\tau(A/S) := \bigcup_{n>0} \phi_n^{-1}(\mathbf{Pic}^0(A/S))$$

where $\phi_n : \mathbf{Pic}(A/S) \to \mathbf{Pic}(A/S)$ is the multiplication by n. Since ϕ_n is continuous, $\mathbf{Pic}^\tau(A/S)$ is an open subscheme of $\mathbf{Pic}(A/S)$. We can see that $\mathbf{Pic}^0(A/S)$ and $\mathbf{Pic}^\tau(A/S)$ are group subschemes of $\mathbf{Pic}(A/S)$.

For a general base scheme S, we define

$$\mathbf{Pic}^0(A/S) := \bigcup_{s\in S} (\mathbf{Pic}(A/S) \times_S \operatorname{Spec}(k(s)))^0$$

and

$$\mathbf{Pic}^\tau(A/S) := \bigcup_{s\in S} (\mathbf{Pic}(A/S) \times_S \operatorname{Spec}(k(s)))^\tau$$

as subsets of $\mathbf{Pic}_{A/S}$, where $k(s)$ is the residue class field of s. If these two sets are open in $\mathbf{Pic}^0_{A/S}$, then we can regard them as open subschemes. When A is a projective abelian scheme, they are equal and open group subschemes of $\mathbf{Pic}(A/S)$.

Proposition 1.2.4. *(Dual abelian schemes)*

Let A be a projective abelian scheme over S.

(a) $\mathbf{Pic}^\tau(A/S)$ *is a projective abelian scheme over S and is equal to* $\mathbf{Pic}^0(A/S)$*. In this case,* $\mathbf{Pic}^\tau(A/S)$ *is called the dual abelian scheme of A and denoted by A^t.*

(b) The restriction of $\mathcal{P}_{A\times_S\mathbf{Pic}(A/S)}$ to $A \times_S A^t$ gives rise to a unique (canonical) group scheme isomorphism $\eta_A : A \to A^{tt}$, where $A^{tt} := (A^t)^t$.

Proof. For (a), see Theorem 5 in [4] 8.4 or Property 5.3 in [16] I. For (b), see Theorem (20.2) in [16] III or Remark 5.24 in [13]. The key point of (b) is that we can find a rigidified invertible sheaf on $A^t \times_S A$ (depending on $\mathcal{P}_{A\times_S\mathbf{Pic}(A/S)}$) by using the short exact sequences

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & \mathrm{P}(A^t) \arrow[d] & & \\ 0 \arrow[r] & \mathrm{P}(A) \arrow[r] & \mathrm{P}(A \times_S A^t) \arrow[r] \arrow[d] & \mathrm{Pic}_{A/S}(A^t) \arrow[r] & 0. \\ & & \mathrm{Pic}_{A^t/S}(A) \arrow[d] & & \\ & & 0 & & \end{tikzcd} \tag{1.2.1}$$

□

Denote the unit section of A by $e_A : S \to A$ and the unit section of A^t by $e_{A^t} : S \to A^t$. We restrict the universal invertible sheaf $\mathcal{P}_{A\times_S\mathbf{Pic}(A/S)}$ to the subscheme $A\times_S A^t$. This sheaf is denoted by $\mathcal{P}_{A\times_S A^t}$ and is called the **Poincaré invertible sheaf** of $A \times_S A^t$. The inclusion $\iota_{A^t} : A^t \hookrightarrow \mathbf{Pic}(A/S)$ and the rigidification $\epsilon_{\mathcal{P}_{A\times_S\mathbf{Pic}(A/S)}}$ of $\mathcal{P}_A$ give the rigidification $\epsilon_1 : \mathcal{O}_{A^t} \to (e_A \times_S \mathrm{id}_{A^t})^*\mathcal{P}_{A\times_S A^t}$ of $\mathcal{P}_{A\times_S A^t} = (e_A \times_S \iota_{A^t})^*\mathcal{P}_{A\times_S\mathbf{Pic}(A/S)}$ along the section $e_A \times_S \mathrm{id}_{A^t} : S \times_S A^t \to A \times_S A^t$. Using $\mathbb{G}_{m,A\times_S A^t}$-torsor structure of $\mathcal{P}_{A\times_S A^t}$, we will find another rigidification ϵ_2.

Let G be a flat commutative group scheme over Q of finite presentation and let X be a sheaf of sets on Q_{fppf}. An **action** of G (or G-action) on X is a morphism (of sheaves of sets on Q_{fppf}) $G\times_Q X \to X$ that induces an action of the group $\mathrm{Mor}_Q(T, G)$

on the set $\mathrm{Mor}_Q(T,X)$ for any scheme T over Q. For example, the multiplication morphism $G\times_Q G\to G$ defines an action of G on its representing sheaf. A sheaf X on Q_{fppf} with an action of G is called a **torsor** under G or G-torsor if there is a covering $\{U_i\to Q\}$ for the fppf topology on Q such that, for each i, X_{U_i} with its G_{U_i}- action is isomorphic to G_{U_i} with the G_{U_i}-action induced by the multiplication morphism of G where $G_{U_i}=G\times_Q U_i$ and X_{U_i} is the sheaf on U_i which is the restriction of X.

When the base scheme Q is locally noetherian, for any affine group scheme G over Q, a G-torsor is representable (and see Theorem 4.3(a) in [15] III for its proof). Hence every $\mathbb{G}_{m,Q}$-torsor X is representable because $\mathbb{G}_{m,Q}$ is affine over Q. It is known that there is a canonical bijection between the set of (isomorphism classes of) invertible sheaves over Q and the set of (isomorphism classes of) $\mathbb{G}_{m,Q}$-torsors. This bijection is defined by $\mathcal{L}\mapsto\mathcal{I}so(\mathcal{O}_Q,\mathcal{L})$ where $\mathcal{I}so(\mathcal{O}_Q,\mathcal{L})$ is the sheafification of the presheaf $U\mapsto$ the isomorphisms of invertible sheaves $\mathcal{O}_Q|_U$ and $\mathcal{L}|_U$. Therefore, for an invertible sheaf $\mathcal{L}$ on Q, we will use the same notation $\mathcal{L}$ for its corresponding $\mathbb{G}_{m,Q}$-torsor $\mathcal{I}so(\mathcal{O}_Q,\mathcal{L})$.

For an abelian scheme A over S and for a locally noetherian scheme T over S, let E be a commutative group scheme of finite presentation over T which is given by the extension $0\to\mathbb{G}_{m,T}\xrightarrow{i}E\xrightarrow{p}A\times_S T\to 0$, where p is faithfully flat. Such an E is briefly called an **extension of A_T by $\mathbb{G}_{m,T}$**. We can show that E is a $\mathbb{G}_{m,A\times_S T}$-torsor as the following (or see the proof of Proposition (17.4) in [16] III). Since E is of finite type over T, p is of finite type. This means that E is faithfully flat of finite presentation over $A\times_S T$ because $A\times_S T$ is locally noetherian. A scheme U over $A\times_S T$ has an S-scheme structure $u_S: U\to(A\times_S T\to)S$, A-scheme structure $u_A: U\to(A\times_S T\to)A$ and T-scheme structure $u_T: U\to(A\times_S T\to)T$. Thus $u_A\times_S u_T$ is the $(A\times_S T)$-structure morphism of U and $u_{\mathbb{G}_{m,A\times_S T}}\in$

$\mathrm{Mor}_{A\times_S T}(U,\mathbb{G}_{m,A\times_S T})$ is given by $u_{\mathbb{G}_{m,A\times_S T}}\times_S u_A\times_S u_T$ where $u_{\mathbb{G}_{m,S}}$ is the composite of $u_{\mathbb{G}_{m,A\times_S T}}$ and $A\times_S T\to S$. For a U-valued point $u\in \mathrm{Mor}_{A\times_S T}(U,E)\subset \mathrm{Mor}_S(U,E)$ of E, the action $\mathrm{Act}(U)$ of $\mathrm{Mor}_{A\times_S T}(U,\mathbb{G}_{m,A\times_S T})$ on $\mathrm{Mor}_{A\times_S T}(U,E)$ is defined by $(\mathrm{Act}(U))(u_{\mathbb{G}_{m,A\times_S T}},u):=m_E(i\circ u_{\mathbb{G}_{m,S}},u)$, where m_E is the multiplication morphism of E. One can check that $\mathrm{Act}(U)$ is functorial in U and that the map $(\mathrm{Act}(U))(-,u):\mathrm{Mor}_{A\times_S T}(U,\mathbb{G}_{m,A\times_S T})\to \mathrm{Mor}_{A\times_S T}(U,E)$ is bijective for any fixed $u\in \mathrm{Mor}_{A\times_S T}(U,E)$ (because of $\mathrm{Ker}(i:\mathbb{G}_{m,T}\to E)=0$). Therefore we defined a $\mathbb{G}_{m,A\times_S T}$-action $\mathrm{Act}:\mathbb{G}_{m,A\times_S T}\times_{A\times_S T}E\to E$ on E (over $A\times_S T$) and showed that $(\mathrm{Act}(U),\mathrm{id}_{\mathrm{Mor}_{A\times_S T}(U,E)}):\mathrm{Mor}_{A\times_S T}(U,\mathbb{G}_{m,A\times_S T}\times_{A\times_S T}E)\to \mathrm{Mor}_{A\times_S T}(U,E\times_{A\times_S T}E)$ is bijective. We can conclude that $\mathrm{Act}\times_{A\times_S T}\mathrm{id}_E:\mathbb{G}_{m,A\times_S T}\times_{A\times_S T}E\to E\times_{A\times_S T}E$ is an isomorphism of $(A\times_S T)$-schemes and that E is a $\mathbb{G}_{m,A\times_S T}$-torsor. (See Proposition 4.1 in [15] III : If E is faithfully flat and locally of finite type over $A\times_S T$ and if $\mathrm{Act}\times_{A\times_S T}\mathrm{id}_E$ is an isomorphism, then E is a $\mathbb{G}_{m,A\times_S T}$-torsor.) Let $\mathcal{L}_E$ be the corresponding invertible sheaf on $A\times_S T$.

Since $(e_A\times_S \mathrm{id}_T)^*\mathcal{L}_E$ is the same as the pullback ($\mathbb{G}_{m,T}$-torsor) $e_{A_T}^*E=(e_A\times_S \mathrm{id}_T)^*E$ of the E and since $e_{A_T}^*E$ is a trivial $\mathbb{G}_{m,T}$-torsor, $(e_A\times_S\mathrm{id}_T)^*\mathcal{L}_E$ is a trivial invertible sheaf. The injective T-group scheme homomorphism $i:\mathbb{G}_{m,T}\to E$ gives rise to the isomorphism $i^*:\mathbb{G}_{m,T}\to e_{A_T}^*E$ of $\mathbb{G}_{m,T}$-torsors. The isomorphism $\mathcal{O}_T\to(e_A\times_S\mathrm{id}_T)^*\mathcal{L}_E$ corresponding to i^* is a rigidification of $\mathcal{L}_E$ along $e_A\times_S\mathrm{id}_T$ and is denoted by u_E. Therefore for any extension $E\in \mathrm{Ext}^1_{T_{\mathrm{fppf}}}(A_T,\mathbb{G}_{m,T})$, we get a unique rigidified invertible sheaf $(\mathcal{L}_E,u_E)$ on $A_T=A\times_S T$ and a unique T-valued point $g_E\in \mathrm{Mor}_S(T,\mathbf{Pic}(A/S))$ so that $(\mathcal{L}_E,u_E)=(\mathrm{id}_A\times_S g_E)^*\mathcal{P}_{A\times_S\mathbf{Pic}(A/S)}$. This construction defines a map $\varepsilon_{A/S}(T):\mathrm{Ext}^1_{T_{\mathrm{fppf}}}(A_T,\mathbb{G}_{m,T})\to \mathrm{Mor}_S(T,\mathbf{Pic}(A/S))$ and $\varepsilon_{A/S}(T)$ is functorial in A and T over S. See Proposition (17.6) and (18.4) in [16] III for more details. The important property of $\varepsilon_{A/S}$ is that it is injective and its image of $\mathrm{Ext}^1_{T_{\mathrm{fppf}}}(A_T,\mathbb{G}_{m,T})$ is equal to $\mathrm{Mor}_S(T,A^t)$. This property is called generalized

Weil-Barsotti formula and here is the precise statement.

Proposition 1.2.5. *(Generalized Weil-Barsotti formula)*

Let A be a projective abelian scheme over S. There exist a presheaf homomorphism $\varepsilon = \varepsilon_{A/S} : \mathrm{Ext}^1_{-}(A_{-}, \mathbb{G}_{m,-}) \to \mathrm{Mor}_S(-, \mathbf{Pic}(A/S))$ and a presheaf (of ableian groups) isomorphism $\beta = \beta_{A/S} : \mathrm{Mor}_S(-, A^t) \to \mathrm{Ext}^1_{-}(A_{-}, \mathbb{G}_{m,-})$ such that the following diagram

$$\begin{tikzcd} \mathrm{Mor}_S(T, A^t) \arrow[r, "\beta(T)"] \arrow[d, hook, "\iota_{A^t}(T)"'] & \mathrm{Ext}^1_T(A_T, \mathbb{G}_{m,T}) \arrow[dl, "\varepsilon(T)"] \\ \mathrm{Mor}_S(T, \mathbf{Pic}(A/S)) & \end{tikzcd} \tag{1.2.2}$$

is commutative, where ι_{A^t} is the embedding of A^t into $\mathbf{Pic}(A/S)$ and T is a locally noetherian scheme over S. A_T means a base extension $A \times_S T$ by T.

Proof. See Proposition (17.6) in [16] III for the construction of ε and see Theorem (18.1) in [16] III for the construction of β. $\square$

In fact, $\mathrm{Ext}^1_{T_{\mathrm{fppf}}}(A_T, \mathbb{G}_{m,T})$ is an abelian group and we will see this group structure in the next chapter. Since the presheaf $\mathrm{Ext}^1_{-}(A_{-}, \mathbb{G}_{m,-})$ is represented by a commutative group scheme A^t, it is a fppf sheaf $\mathcal{E}xt^{\,1}_S(A, \mathbb{G}_{m,S})$. Note that the sheafification of $\mathrm{Ext}^i_{-}(A_{-}, \mathbb{G}_{m,-})$ is equal to $\mathcal{E}xt^{\,i}_S(A, \mathbb{G}_{m,S})$ (Remark 1.24 in [15] III).

Proposition 1.2.5 has many useful consequences. The Poincaré (rigidified) invertible sheaf $(\mathcal{P}_{A\times_S A^t}, \epsilon_1)$ gives the extension $E_{(\mathcal{P}_{A\times_S A^t}, \epsilon_1)} = E_{\epsilon_1}$ of A_{A^t} by $\mathbb{G}_{m,A^t}$ corresponding to the morphism $\mathrm{id}_{A^t} \in \mathrm{Mor}_S(A^t, A^t)$. For any locally noetherian T over S and for every extension E of A_T by $\mathbb{G}_{m,T}$, there is a unique morphism $g_E \in \mathrm{Mor}_S(T, A^t)$ such that $E = (\mathrm{id}_A \times_S g_E)^* E_{\epsilon_1} = E_{\epsilon_1} \times_{A^t} T$. This means that E_{ϵ_1} is the universal extension of A by $\mathbb{G}_m$.

When $T = S$ and $g = e_{A^t}$, the pullback extension $E_{\epsilon_1} \times_{A^t} S$ is trivial because this extension is corresponding to the unit section $e_{A^t} \in \mathrm{Mor}_S(T, A^t) \overset{\beta(S)}{=} \mathrm{Ext}^1_S(A, \mathbb{G}_{m,S})$

and as an element of $\mathrm{Ext}^1_S(A, \mathbb{G}_{m,S})$, e_{A^t} (and its corresponding extension) is trivial. Therefore the corresponding invertible sheaf $(\mathrm{id}_A \times_S e_{A^t})^* \mathcal{P}_{A\times_S A^t}$ is trivial, *i.e.*, isomorphic to $\mathcal{O}_A$ and there is a rigidification of $\mathcal{P}_{A\times_S A^t}$ along the section $\mathrm{id}_A \times_S e_{A^t} : A \to A \times_S A^t$. In the proof of Proposition 1.2.4, we see that this rigidification is unique if it exists. More explicitly, the diagram (1.2.1) shows that we have a unique (canonical) section $\mathrm{Pic}_{A^t/S}(A) \hookrightarrow \mathrm{P}(A^t \times_S A)$ defined by $(\mathcal{L}, u : \mathcal{O}_A \to (\mathrm{id}_A \times_S e_{A^t})^* \mathcal{L}) \mapsto \mathcal{L}$. This new rigidification of $\mathcal{P}_{A\times_S A^t}$ is denoted by ϵ_2, and the canonical isomorphism $\eta_A \in \mathrm{Mor}_S(A, A^{tt})$ is the A-valued point of $\mathbf{Pic}(A^t/S)$ (in fact, of A^{tt}) such that $(\mathcal{P}_{A\times_S A^t}, \epsilon_2) = \eta_A^*(\mathcal{P}_{\mathbf{Pic}(A^t/S)\times_S A^t}, \epsilon_{\mathcal{P}_{\mathbf{Pic}(A^t/S)\times_S A^t}})$.

Therefore the Poincaré invertible sheaf has two different rigidifications ϵ_1 and ϵ_2. Such an invertible sheaf is called a **birigidfied invertible sheaf**. This property plays an important role in making a biextension of $A \times_S A^t$ by $\mathbb{G}_{m,S}$ that will be defined. We can also make the Poincaré invertible sheaf of $A^t \times_S A$ (or $A^t \times_S A^{tt}$, using the biduality theorem 1.2.4 of abelian schemes), and it is easy to check that $\mathcal{P}_{A\times_S A^t} \cong \mathcal{P}_{A^t\times_S A}$ via the interchanging map $A \times_S A^t \to A^t \times_S A$. Since $(\mathcal{P}_{A\times_S A^t}, \epsilon_2)$ is coming from $(\mathcal{P}_{A^{tt}\times_S A^t}, \epsilon_{\mathrm{id}_{A^{tt}} \times_S e_{A^t}})$, it gives rise to the extension $E_{(\mathcal{P}_{A\times_S A^t}, \epsilon_2)} = E_{\epsilon_2}$ of $A^t_A = A^t \times_S A$ by $\mathbb{G}_{m,A}$ with the following universal property. For any locally noetherian T over S and for every extension E' of A^t_T by $\mathbb{G}_{m,T}$, there is a unique morphism $g_{E'} \in \mathrm{Mor}_S(T, A)$ such that $E' = (\mathrm{id}_{A^t} \times_S g_{E'})^* E_{\epsilon_2} = E_{\epsilon_2} \times_A T$.

Now we will consider another description of the Poincaré birigidified invertible sheaf $(\mathcal{P}_{A\times_S A^t}, \epsilon_1, \epsilon_2)$ over S. This description is called a biextension whose general definition requires some property of topos (for instance, the category of fppf-sheaves on S in our case). Since we need a trivial biextension or the Poincaré biextension of A and A^t by $\mathbb{G}_{m,S}$, we can use a simpler definition.

For a locally noetherian commutative group schemes P and Q over S, let B be a $\mathbb{G}_{m,P\times_S Q}$-torsor. Since $\mathbb{G}_{m,P\times_S Q}$ is affine over $P \times_S Q$, we see that if this torsor is trivial or if P and Q are of finite type over S, then B is representable. Thus we assume that B is a scheme over $P \times_S Q$. B can be regarded as a $(\mathbb{G}_{m,P}) \times_P Q_P$-torsor and can be regarded as a $(\mathbb{G}_{m,Q}) \times_Q P_Q$-torsor.

Assume that B is an extension of Q_P by $\mathbb{G}_{m,P}$ (*i.e.*, it is a commutative group scheme of finite presentation over P which is given by the extension $0 \to \mathbb{G}_{m,P} \xrightarrow{i} B \xrightarrow{p} Q_P \to 0$, where p is faithfully flat) which is compatible with the structure of $(\mathbb{G}_{m,P}) \times_P (Q_P)$-torsor and assume that B is an extension of P_Q by $\mathbb{G}_{m,Q}$ compatible with the structure of $(\mathbb{G}_{m,Q}) \times_Q P_Q$-torsor. For example, the $\mathbb{G}_{m,A\times_S A^t}$-torsor $B_{\mathcal{P}_{A\times_S A^t}}$ (corresponding to the invertible sheaf $\mathcal{P}_{A\times_S A^t}$) is an extension (group scheme over A) of A^t_A by $\mathbb{G}_{m,A}$ and is an extension (over A^t) of A_{A^t} by $\mathbb{G}_{m,A^t}$. The first extension structure *i.e.*, the (partial) multiplication $\underset{2}{+}$ of B (of Q_P by $\mathbb{G}_{m,P}$) gives rise to the following torsor isomorphism $\phi_{(p,p'),q}$ for all $p, p \in \mathrm{Mor}_S(T, P)$ and for every $q \in \mathrm{Mor}_S(T, Q)$, where T is a scheme locally of finite type over S, and where

$$\phi_{(p,p'),q} : B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',q} \xrightarrow{\sim} B_{pp',q} \tag{1.2.3}$$

is an isomorphism of $\mathbb{G}_{m,T}$-torsors satisfying associativity and commutativity. Here, $B_{p,q} (= B \times_{P\times_S Q} T)$ is the $\mathbb{G}_{m,T}$-torsor which is the pull back of (a $\mathbb{G}_{m,P\times_S Q}$-torsor) B by $(p, q) : T \to P \times_S Q$ and $B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',q}$ is a **contracted product** of $\mathbb{G}_{m,T}$-torsors which is defined by the sheafification (for the fppf topology) of the presheaf $U \mapsto (\mathrm{Mor}_T(U, B_{p,q}) \times \mathrm{Mor}_T(U, B_{p',q}))/\sim$ (for any U over T) where $(b_1, b_1') \sim (b_2, b_2')$ if and only if there is $g \in \mathrm{Mor}_T(U, \mathbb{G}_{m,T})$ so that $b_1 = g(b_2)$ and $b_2' = g(b_1')$. ϕ in (1.2.3) is functorial for the base change $T' \to T$ and the associativity of ϕ means that the following diagram

$$\begin{tikzcd}[column sep=small]
 & B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p'',q} \arrow[dl, "\phi_{(p,p'),q}\wedge \mathrm{id}"', "\sim"] \arrow[dr, "\mathrm{id}\wedge\phi_{(p',p''),q}", "\sim"'] & \\
B_{pp',q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p'',q} \arrow[dr, "\sim", "\phi_{(pp',p''),q}"'] & & B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p'p'',q} \arrow[dl, "\sim"', "\phi_{(p,p'p''),q}"] \\
 & B_{pp'p'',q} &
\end{tikzcd} \tag{1.2.4}$$

is commutative, where $p'' \in \mathrm{Mor}_S(T, P)$. The commutativity of ϕ is the commutativity of the following diagram

$$\begin{tikzcd}
B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',q} \arrow[r, "\phi_{(p,p'),q}", "\sim"'] \arrow[d, "\mathrm{sym}"', "\wr"] & B_{pp',q} \arrow[d, "\wr"', "\mathrm{id}"] \\
B_{p',q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q} \arrow[r, "\phi_{(p',p),q}", "\sim"'] & B_{p'p,q}
\end{tikzcd} \tag{1.2.5}$$

where sym is an isomorphism switching the position of components.

Remark 1.2.2. (The relation between B and the isomorphism (1.2.3))

Conversely, if B satisfies the above commutative diagrams (1.2.3), (1.2.4) and (1.2.5) for $p, p', p'' \in \mathrm{Mor}_{S_{\text{fppf}}}(\mathcal{T}, P)$ and for $q \in \mathrm{Mor}_{S_{\text{fppf}}}(\mathcal{T}, Q)$, where $\mathcal{T}$ is any fppf sheaf of sets, B is an extension of Q_P by $\mathbb{G}_{m,P}$. For the definition of a sheaf torsor, see Definition 1.4.1 in [9] III. For the definition of a contracted product, see Definition 1.3.1 in [9] III.

The other partial multiplication $\underset{1}{+}$ of B (as an extension of P_Q by $\mathbb{G}_{m,Q}$) gives rise to an isomorphism of $\mathbb{G}_{m,T}$-torsors

$$\psi_{p,(q,q')} : B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q'} \xrightarrow{\sim} B_{p,qq'} \tag{1.2.6}$$

for all T, $p \in \mathrm{Mor}_S(T, P)$ and $q, q' \in \mathrm{Mor}_S(T, Q)$, where ψ is functorial for the base change $T' \to T$. The associativity and the commutativity of ψ mean that the following diagrams are commutative for any q, q', q''.

$$\begin{tikzcd}[column sep=tiny]
& B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q'} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q''} \arrow[dl, "\psi_{p,(q,q')}\wedge\mathrm{id}"', "\sim"] \arrow[dr, "\mathrm{id}\wedge\psi_{p,(q',q'')}", "\sim"'] & \\
B_{p,qq'} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q''} \arrow[dr, "\sim", "\psi_{p,(q,q'q'')}"'] & & B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q'q''} \arrow[dl, "\sim"', "\psi_{p,(q,q'q'')}"] \\
& B_{p,qq'q''} &
\end{tikzcd} \tag{1.2.7}$$

$$\begin{tikzcd}
B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q'} \arrow[r, "\psi_{p,(q,q')}", "\sim"'] \arrow[d, "\mathrm{sym}"', "\wr"] & B_{p,qq'} \arrow[d, "\wr"', "\mathrm{id}"] \\
B_{p,q'} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q} \arrow[r, "\psi_{p,(q',q)}", "\sim"'] & B_{p,q'q}
\end{tikzcd} \tag{1.2.8}$$

Definition 1.2.3. (Biextensions of commutative group schemes by $\mathbb{G}_{m,S}$)

Let P and Q be locally noetherian commutative group schemes over S. Assume that B is a scheme and is a $\mathbb{G}_{m,P\times_S Q}$-torsor with the structure of an extension of $Q_P = Q \times_S P$ by $\mathbb{G}_{m,P}$ gives rise to the isomorphisms (1.2.3) (functorial in S, with the commutativity of (1.2.4) and (1.2.5)) and with the structure of an extension of $P_Q = P \times_S Q$ by $\mathbb{G}_{m,Q}$ gives rise to the isomorphisms (1.2.6) (functorial for T, with the commutativity of (1.2.7) and (1.2.8)).

These extension structures are called compatible if the following diagram

$$\begin{tikzcd}
 & B \times_Q B \arrow[ddr, "\underset{1}{+}"] & \\
(B \times_P B) \times_{Q \times_S Q} (B \times_P B) \arrow[ur, "(\underset{2}{+}{,}\underset{2}{+})"] \arrow[dd, "\cong"'] & & \\
 & & B \\
(B \times_Q B) \times_{P \times_S P} (B \times_Q B) \arrow[dr, "(\underset{1}{+}{,}\underset{1}{+})"'] & & \\
 & B \times_P B \arrow[uur, "\underset{2}{+}"'] &
\end{tikzcd} \tag{1.2.9}$$

is commutative. This means that for every T (locally of finite type) over S and for all $p, p' \in \mathrm{Mor}_S(T, P)$ and $q, q' \in \mathrm{Mor}_S(T, Q)$, the following diagram

$$\begin{tikzcd}
 & B_{p,qq'} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',qq'} \arrow[ddr, "\phi_{(p,p'),qq'}"] & \\
B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q'} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',q'} \arrow[ur, "\psi_{p,(q,q')} \wedge \psi_{p',(q,q')}"] \arrow[dd, "\mathrm{id} \wedge \mathrm{sym} \wedge \mathrm{id}"'] & & \\
 & & B_{pp',qq'} \\
B_{p,q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p,q'} \overset{\mathbb{G}_{m,T}}{\wedge} B_{p',q'} \arrow[dr, "\phi_{(p,p'),q} \wedge \phi_{(p,p'),q'}"'] & & \\
 & B_{pp',q} \overset{\mathbb{G}_{m,T}}{\wedge} B_{pp',q'} \arrow[uur, "\psi_{pp',(q,q')}"'] &
\end{tikzcd} \tag{1.2.10}$$

is commutative. If the above two extensions are compatible, B is called a biextension of P and Q by $\mathbb{G}_{m,S}$. (See Definition 2.1 in [12] VII for the biextension of abelian sheaves.)

We will use two examples of $\mathbb{G}_{m,S}$-biextensions. The first one is $\mathbb{G}_{m,P\times_S Q}$ as a trivial torsor and this biextension is called the **trivial biextension** of P and Q by

$\mathbb{G}_{m,S}$. The next one is called the **Poincaré biextension** of A and A^t by $\mathbb{G}_{m,S}$ and this is defined by the $\mathbb{G}_{A\times_S A^t}$-torsor $B_{\mathcal{P}_{A\times_S A^t}}$ corresponding to the Poincaré invertible sheaf $\mathcal{P}_{A\times_S A^t}$ with extension structures (and partial multiplications $\underset{1}{+}$ and $\underset{2}{+}$) coming from two rigidifications ϵ_1 and ϵ_2. We see that $B_{\mathcal{P}_{A\times_S A^t}}$ is an extension E_{ϵ_1} of A_{A^t} by $\mathbb{G}_{m,A^t}$ because this is the very statement of the generalized Weil-Barsotti formula in the case when $\mathrm{id} : A^t \to A^t$. Since A is canonically isomorphic to $(A^t)^t$ due to Proposition 1.2.4, we can apply the generalized Weil-Barsotti formula to $\eta_A : A \to A^{tt}$. Therefore B has the structure of an extension E_{ϵ_2} of A^t_A by $\mathbb{G}_{m,A}$. To show that E_{ϵ_1} and E_{ϵ_2} are compatible as in (1.2.10), see Corollary 2.9.4 and Example 2.9.5 in [12] VII.

We will use the Poincaré biextension of A and A^t instead of their birigidified Poincaré invertible sheaf. Thus $\mathcal{P}_{A\times_S A^t}$ (or simply $\mathcal{P}$) means the biextension $B_{\mathcal{P}_{A\times_S A^t}}$ from now. The following proposition is the consequence of the generalized Weil-Barsotti formula and is called the universal property of the Poincaré biextension.

Proposition 1.2.6. *(Universal property of $\mathcal{P}$)*

Let T be a locally noetherian scheme over S.

(a) For a group scheme G over T which is an extension of the abelian scheme $A\underset{S}{\times}T$ (over T) by $\mathbb{G}_{m,T}$, there is a unique morphism $g \in \mathrm{Mor}_S(T, A^t)$ such that $\mathcal{G} \cong (\mathrm{id}_A \underset{S}{\times} g)^*\mathcal{P}$ where $\mathcal{G}$ is the sheaf represented by G.

(b) For a group scheme G' over T which is an extension of the abelian scheme $T\underset{S}{\times}A^t$ (over T) by $\mathbb{G}_{m,T}$ then there is a unique morphism $g' \in \mathrm{Mor}_S(T, A)$ such that $\mathcal{G}' \cong (g' \underset{S}{\times} \mathrm{id}_{A^t})^*\mathcal{P}$ where $\mathcal{G}'$ is the sheaf represented by G'.

Due to this universal property 1.2.6, for a homomorphism $f : A_1 \to A_2$ of abelian schemes over S, there is a biextension B_f of A_1 and A_2^t given by

$$B_f = (\mathrm{id}_{A_1} \underset{S}{\times} f^t)^* \mathcal{P}_{A_1 \times_S A_2^t} = (f \underset{S}{\times} \mathrm{id}_{A_2^t})^* \mathcal{P}_{A_2 \times_S A_2^t} \tag{1.2.11}$$

where $f^t : A_2^t \to A_1^t$ is a dual homomorphism of f.

Chapter 2

Smooth 1-motives with torsion and Cartier duality

In this chapter we define two different 1-motives with torsion. One is defined by a 2-term complex of commutative group schemes, which is more convenient to consider realization functors, and the other is defined by a biextension morphism, which is called a smooth symmetric 1-motive. In the category of smooth symmetric 1-motives, we can simply define a dual smooth symmetric 1-motive and the Cartier duality theorem follows from the definition in this case. Showing that categories of two different 1-motives with torsion are equivalent, we will define a dual smooth 1-motive with torsion and prove the Cartier duality theorem in this case.

2.1 Smooth 1-motives with torsion and smooth symmetric 1-motives

Let S be a locally noetherian base scheme and we will consider only schemes locally of finite type over S. In [6] (10.1.10), Deligne defined a smooth 1-motive over S by $u : X \to G$ a group scheme homomorphism over S, where X is a finitely generated twisted constant group over S (which is quasi-isotrivial due to Proposition 1.1.4) and where G is an extension of an abelian scheme by a torus over S.

Definition 2.1.1. (Smooth 1-motives with torsion)

A 5-tuple $M = (X, T, A, G, u)$ of commutative group schemes and a group scheme homomorphism over S is called a smooth 1-motive with torsion over S if

(a) X is an object of $\mathcal{GC}_S$ in Definition 1.1.6,

(b) G is an extension (group scheme) of an abelian scheme A over S by $T \in \mathcal{GC}_S^{D_S}$ in Definition 1.1.6,

(c) $u : X \to G$ is a homomorphism of group schemes over S.

Deligne essentially considered 1-motives over a field. But he also mentioned 1-motives over any base scheme and called them smooth 1-motives. Because we will generalize smooth 1-motives once again, we need to use this longer name rather than 1-motives with torsion. Some people (for example, [1] or [2]) already extend Deligne's 1-motive over a field which also has torsion and this will be introduced and compared with a smooth 1-motive with torsion in the next chapter.

For a smooth 1-motive with torsion $M = (X, T, A, G, u)$, there is an increasing filtration which is called the **fake** weight filtration on M.

Definition 2.1.2. (Fake weight filtration of smooth 1-motives with torsion)

Let $M = (X, T, A, G, u)$ be a smooth 1-motive with torsion over S. The fake weight filtration W_* on M is given by

$$\begin{aligned}
\mathrm{W}_i(M) &= M \quad \text{for each } i \geq 0,\\
\mathrm{W}_{-1}(M) &= (0, T, A, G, 0),\\
\mathrm{W}_{-2}(M) &= (0, T, 0, T, 0),\\
\mathrm{W}_i(M) &= 0 \quad \text{for each } i \leq -3.
\end{aligned}$$

A smooth 1-motive with torsion M is denoted by $[X \xrightarrow{u} G]$ or $[X \to G]$ if its fake weight filtration is specified, *i.e.*, we know what T is. We may think of a morphism of smooth 1-motives with torsion as a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & G_1 \arrow[d] \\ X_2 \arrow[r] & G_2 \end{tikzcd}$$

but because of their fake weight filtrations, this morphism has more conditions.

Definition 2.1.3. (Morphisms of smooth 1-motives with torsion)

Let $M_i = (X_i, T_i, A_i, G_i, u_i)$ be smooth 1-motives with torsion over S for $i = 1, 2$. The set of groups scheme homomorphisms (f_X, f_T, f_A, f_G) is called a morphism of smooth 1-motives with torsion if

(a) $f_X : X_1 \to X_2$, $f_T : T_1 \to T_2$, $f_A : A_1 \to A_2$, and $f_G : G_1 \to G_2$

(b) The following two diagrams

$$\begin{tikzcd} 0 \arrow[r] & T_1 \arrow[r] \arrow[d, "f_T"] & G_1 \arrow[r] \arrow[d, "f_G"] & A_1 \arrow[r] \arrow[d, "f_A"] & 0 \\ 0 \arrow[r] & T_2 \arrow[r] & G_2 \arrow[r, "u_2"] & A_2 \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} X_1 \arrow[r, "u_1"] \arrow[d, "f_X"] & G_1 \arrow[d, "f_G"] \\ X_2 \arrow[r, "u_2"] & G_2 \end{tikzcd}$$

are commutative.

We denote a morphism of smooth 1-motives with torsion by $f : M_1 \to M_2$.

Let $\mathcal{M}_{\mathrm{sm}}$ be the category of smooth 1-motives with torsion over S with the above morphisms. $\mathcal{M}_{\mathrm{sm}}$ is an additive category, but even for $S = \mathrm{Spec}\,(\mathbb{C})$, this category is not an abelian category because of an isogeny $f_A : A_1 \to A_2$. More explicitly, if we consider a morphism $f = (0,0,f_A,f_A) : (0,0,A_1,A_1,0) \to (0,0,A_2,A_2,0)$, we can not find its kernel as a smooth 1-motive with torsion. The only possible candidate of its kernel is $(0,0,\mathrm{Ker}(f_A),\mathrm{Ker}(f_A),0)$ but the $\mathrm{Ker}(f_A)$ is not an abelian variety in general.

Remark 2.1.1. (Smooth 1-motives and smooth 1-motives with torsion)

Let $\mathcal{M}_{\mathrm{Del}}$ be the category of smooth 1-motives (without torsion) over S. $\mathcal{M}_{\mathrm{Del}}$ is a subcategory of $\mathcal{M}_{\mathrm{sm}}$ and any object of $\mathcal{M}_{\mathrm{Del}}$ has a unique (and canonical) weight filtration because every group scheme homomorphism from a torus to an abelian scheme is trivial. (For the proof, see 1.3.8 of [12] VII : Recall that $\mathrm{Mor}_S(A,T) = \mathrm{Hom}_{\mathcal{O}_S(\mathrm{alg})}(p_*(\mathcal{O}_T), p_*(\mathcal{O}_A))$ and that $p_*(\mathcal{O}_A) \cong \mathcal{O}_S$ for an abelian scheme $p : A \to S$.) Moreover, due to this property, we have that $\mathcal{M}_{\mathrm{Del}}$ is a full subcategory of $\mathcal{M}_{\mathrm{sm}}$.

To define a Cartier dual functor on $\mathcal{M}_{\mathrm{Del}}$, Deligne introduced another description of 1-motives over an algebraically closed field in [6] (10.1.12). This description works in our case.

Definition 2.1.4. (Pullback and morphisms of biextensions)

Let B be a biextension of P and Q by $\mathbb{G}_{m,S}$ as in Definition 1.2.3 and let P' and Q' be commutative group schemes over S. For group scheme homomorphisms $c : P' \to P$ and $d : Q' \to Q$, the pullback $\mathbb{G}_m$-torsor $(c,d)^*B = B \times_{P \times_S Q} (P' \times_S Q')$ is a biextension of P' and Q' by $\mathbb{G}_{m,S}$. This biextension is called the pullback of B via c and d.

Let B' be a biextension of P' and Q' by $\mathbb{G}_{m,S}$. The set of morphisms (u,v,w,f) over S is called a biextension morphism if

(a) $u : P' \to P$ and $v : Q' \to Q$ are group scheme homomorphisms.

(b) w is a group scheme homomorphism from $\mathbb{G}_{m,S}$ to $\mathbb{G}_{m,S}$.

(c) $f : B' \to B$ is a morphism that gives commutative diagrams :

$$\begin{tikzcd} \mathbb{G}_{m,Q'} \arrow[r] \arrow[d, "(w,v)"] & B' \arrow[r] \arrow[d, "f"] & P'_{Q'} \arrow[d, "(u,v)"] \\ \mathbb{G}_{m,Q} \arrow[r] & B \arrow[r] & P_Q \end{tikzcd} \qquad \begin{tikzcd} \mathbb{G}_{m,P'} \arrow[r] \arrow[d, "(w,u)"] & B' \arrow[r] \arrow[d, "f"] & Q'_{P'} \arrow[d, "(v,u)"] \\ \mathbb{G}_{m,P} \arrow[r] & B \arrow[r] & Q_P \end{tikzcd} \tag{2.1.1}$$

(d) $f \times_Q (q'_{B'}) : B' \to B \times_Q Q'$ is a group scheme homomorphism over Q' and $f \times_S (p'_{B'}) : B' \to B \times_P P'$ is a group scheme homomorphism over P', where $p'_{B'} : B' \to P'$ and $q'_{B'} : B' \to Q'$ are structure morphisms.

$P' \times_S Q'$ is called a biextension of P' and Q' by the trivial group scheme S. The set of morphisms (u, v, w, f) over S is called a biextension morphism from $P' \times_S Q'$ to B if

(a) $u : P' \to P$ and $v : Q' \to Q$ are group scheme homomorphisms.

(b) $w = e_{\mathbb{G}_{m,S}} : S \to \mathbb{G}_{m,S}$ is the unit section of $\mathbb{G}_{m,S}$.

(c) $f : P' \times_S Q' \to B$ is a morphism that gives commutative diagrams :

$$\begin{tikzcd} Q' \arrow[r, "e_{P'} \underset{S}{\times} \mathrm{id}_{Q'}"] \arrow[d, "(w,v)"] & P'_{Q'} \arrow[r, "="] \arrow[d, "f"] & P'_{Q'} \arrow[d, "(u,v)"] \\ \mathbb{G}_{m,Q} \arrow[r] & B \arrow[r] & P_Q \end{tikzcd} \qquad \begin{tikzcd} P' \arrow[r, "e_{Q'} \underset{S}{\times} \mathrm{id}_{P'}"] \arrow[d, "(w,u)"] & Q'_{P'} \arrow[r, "="] \arrow[d, "f"] & Q'_{P'} \arrow[d, "(v,u)"] \\ \mathbb{G}_{m,P} \arrow[r] & B \arrow[r] & Q_P \end{tikzcd} \tag{2.1.2}$$

where $e_{P'}$ is the unit section $S \to P'$ of P'.

(d) $f \times_Q (q'_{B'}) : P'_{Q'} \to B \times_Q Q'$ is a group scheme homomorphism over Q' and $f \times_P (p'_{B'}) : Q'_{P'} \to B \times_P P'$ is a group scheme homomorphism over P'.

When (u, v, w, f) is a biextension morphism from $P' \times_S Q'$ to B, there is a unique biextension morphism $(\mathrm{id}_{P'}, \mathrm{id}_{Q'}, e_{\mathbb{G}_{m,S}}, (c, d)^* f)$ from $P' \times_S Q'$ to $(c, d)^* B$. For a scheme $s' : S' \to S$, $(f'(S'))(p, q, g)$ is defined by $(p, q, g).(((c, d)^* f)(S'))(p, q))$ where

$p \in \mathrm{Mor}_S(S', P')$, $q \in \mathrm{Mor}_S(S', Q')$, $g \in \mathrm{Mor}_S(S', \mathbb{G}_{m,S})$, and $(p, q, \cdot).(\cdot)$ means the $\mathrm{Mor}_{P' \times_S Q'}(S', \mathbb{G}_{m, P' \times_S Q'})$-action on $\mathrm{Mor}_{P' \times_S Q'}(S', (c, d)^* B)$. One can see that $f'(S')$ is functorial in S' and this shows that $f' : P' \times_S Q' \times_S \mathbb{G}_{m,S} \to (c, d)^* B$ is a morphism of presheaves. Since the category of schemes is a full subcategory of the category of presheaves of sets, f' is a morphisms of schemes over S. We can also prove that $(\mathrm{id}_{P'}, \mathrm{id}_{Q'}, \mathrm{id}_{\mathbb{G}_{m,S}}, f')$ is a biextension morphism and this means that (u, v, w, f) gives rise to a biextension isomorphism $P' \times_S Q' \times_S \mathbb{G}_{m,S} \to (c, d)^* B$. Such an isomorphism is called a **trivialization** of $(c, d)^* B$.

Conversely, for a given trivialization of $(c, d)^* B$, using the canonical biextension morphism $(c, d)^* B \to B$, we have a unique biextension morphism $P' \times_S Q' \to B$. Thus to give a trivialization of the pullback biextension $(c, d)^* B$, it is enough to define a biextension morphism $P' \times_S Q' \to B$. We will use this fact to prove the equivalence of two different definitions of smooth 1-motives with torsion.

Definition 2.1.5. (Smooth symmetric 1-motives)

The 7-tuple $(X, X', A, A', v, v', \psi)$ of group schemes and morphisms is called a smooth symmetric 1-motive if

(a) X and X' are objects in $\mathcal{GC}_S$;

(b) A and $A' = A^t$ are abelian schemes over S dual to each other ;

(c) $v : X \to A$ and $v' : X' \to A'$ are group scheme homomorphisms ;

(d) ψ is a trivialization of the pullback $(v, v')^* \mathcal{P}_{A \times_S A'}$ via (v, v') of the Poincaré biextension $\mathcal{P}_{A \times_S A'}$ of $A \times_S A'$ by $\mathbb{G}_{m,S}$. Due to the above fact, ψ can be regarded as a biextension morphism $X \times_S X' \to \mathcal{P}_{A \times_S A'}$.

To make the collection of smooth symmetric 1-motives a category, we need to define a morphism of smooth symmetric 1-motives.

Definition 2.1.6. (Morphisms of smooth symmetric 1-motives)

Let $(X_1, X_1', A_1, A_1', v_1, v_1', \psi_1)$ and $(X_2, X_2', A_2, A_2', v_2, v_2', \psi_2)$ be smooth symmetric avatars. The 4-tuple $f = (f_{X_1}, f_{A_1}, f_{X_2'}, f_{A_2'})$ is called a morphism of smooth symmetric 1-motives if

(a) $f_{X_1} : X_1 \to X_2$, $f_{A_1} : A_1 \to A_2$, $f_{X_2'} : X_2' \to X_1'$, and $f_{A_2'} : A_2' \to A_1'$ are morphisms of group schemes.

(b) The following diagrams are commutative.

$$\begin{tikzcd} X_1 \arrow[r, "v_1"] \arrow[d, "f_{X_1}"] & A_1 \arrow[d, "f_{A_1}"] \\ X_2 \arrow[r, "v_2"] & A_2 \end{tikzcd} \quad , \quad \begin{tikzcd} X_2' \arrow[r, "v_2'"] \arrow[d, "f_{X_2'}"] & A_2' \arrow[d, "f_{A_2'}"] \\ X_1' \arrow[r, "v_1'"] & A_1' \end{tikzcd}$$

(c) f_{A_1} and $f_{A_2'}$ are dual to each other and in the following diagram via the identification $(\mathrm{id}_{A_1} \times_S f_{A_2'})^* \mathcal{P}_{A_1 \times_S A_1'} \cong (f_{A_1} \times_S \mathrm{id}_{A_2'})^* \mathcal{P}_{A_2 \times_S A_2'}$,

$$\begin{tikzcd} X_1 \times_S X_1' \arrow[r, "\psi_1"] & \mathcal{P}_{A_1 \times_S A_1'} \\ X_1 \times_S X_2' \arrow[u, "\mathrm{id}_{X_1} \times_S f_{X_2'}"] \arrow[r, dashed] \arrow[d, "f_{X_1} \times_S \mathrm{id}_{X_2'}"'] & (\mathrm{id}_{A_1} \times_S f_{A_2'})^* \mathcal{P}_{A_1 \times_S A_1'} = (f_{A_1} \times_S \mathrm{id}_{A_2'})^* \mathcal{P}_{A_2 \times_S A_2'} \arrow[u] \arrow[d] \\ X_2 \times_S X_2' \arrow[r, "\psi_2"] & \mathcal{P}_{A_2 \times_S A_2'} \end{tikzcd}$$

we have that $(\mathrm{id}_{A_1} \times_S f_{A_2'})^*(\psi_1 \circ (\mathrm{id}_{X_1} \times_S \ f_{X_2'})) = (f_{A_1} \times_S \mathrm{id}_{A_2'})^*(\psi_2 \circ (f_{X_1} \times_S \mathrm{id}_{X_2'}))$.

Remark 2.1.2. (Explanation for the condition (c))

Since we have the following commutative diagram

$$\begin{tikzcd} X_1 \times_S X'_1 \arrow[r, "(v_1,v'_1)"] & A_1 \times_S A'_1 \\ X_1 \times_S X'_2 \arrow[u, "\mathrm{id}_{X_1} \times_S f_{X'_2}"] \arrow[d, "f_{X_1} \times_S \mathrm{id}_{X'_2}"'] \arrow[r, "(v_1,v'_2)"] \arrow[ur, "h_1"] \arrow[dr, "h_2"'] & A_1 \times_S A'_2 \arrow[u, "\mathrm{id}_{A_1} \times_S f_{A'_2}"'] \arrow[d, "f_{A_1} \times_S \mathrm{id}_{A'_2}"] \\ X_2 \times_S X'_2 \arrow[r, "(v_2,v'_2)"'] & A_2 \times_S A'_2, \end{tikzcd} \tag{2.1.3}$$

we can think of the pull back of $\mathcal{P}_{A_1 \times_S A'_1}$ to $X_1 \times_S X'_2$ via h_1 and the pull back of $\mathcal{P}_{A_2 \times_S A'_2}$ to $X_1 \times_S X'_2$ via h_2. Due to the canonical equality

$$(\mathrm{id}_{A_1} \times_S f_{A'_2})^* \mathcal{P}_{A_1 \times_S A'_1} = (f_{A_1} \times_S \mathrm{id}_{A'_2})^* \mathcal{P}_{A_2 \times_S A'_2}$$

(by the universal property (1.2.11)), these two biextensions are the same. Because (2.1.3) is commutative, for h_i, there is a unique biextension morphism $\widetilde{h_i} : X_1 \times_S X'_2 \to \mathcal{P}_{A_i \times_S A'_i}$ that is a lifting of h_i. The pullback of h_i is a biextension morphism from $X_1 \times_S X'_2$ to the pullback biextension on $A_1 \times_S A'_2$ of $\mathcal{P}_{A_i \times_S A'_i}$. (c) says that this pullback morphism of $\widetilde{h_1}$ are equal to the pullback of $\widetilde{h_2}$.

2.2 Cartier duality theorem of smooth 1-motives with torsion

Let $\mathcal{M}_{\mathrm{ssym}}$ be the category of smooth symmetric 1-motives. One can check that $\mathcal{M}_{\mathrm{ssym}}$ is an additive category. We define a contravariant additive functor on $\mathcal{M}_{\mathrm{ssym}}$.

Definition 2.2.1. (Dual smooth symmetric 1-motives)

Let $N = (X, X', A, A', v, v', \psi)$ be a smooth symmetric 1-motive. Its Cartier dual N^* is defined by $(X', X, A', A, v', v, \overline{\psi})$ where $\overline{\psi} : X' \times_S X \to (v', v)^* \mathcal{P}_{A' \times_S A}$ is define by ψ (switching $X \rightleftharpoons X'$ and $A \rightleftharpoons A'$).

Cartier dual 1-motives define a functor $\mathcal{M}_{\text{ssym}} \to \mathcal{M}_{\text{ssym}}$ and this functor is contravariant and additive. For any smooth symmetric 1-motive over S, the dual of its dual is the original one itself by definition and therefore we immediately get duality (or biduality, more precisely) for this functor. Thus if we can show the equivalence of $\mathcal{M}_{\text{sm}}$ and $\mathcal{M}_{\text{ssym}}$ then we can also define a duality functor on $\mathcal{M}$. We will do it, as in [17] Chapter I and its proof is based on Proposition 1.1.6 in [17].

Proposition 2.2.1. *(The symmetric avatar of a smooth 1-motive)*

There is a nontrivial additive functor $\mathfrak{B}$ *from* $\mathcal{M}_{\text{sm}}$ *to* $\mathcal{M}_{\text{ssym}}$.

Proof. Let $M = (X, T, A, G, u)$ be a smooth 1-motive with torsion over S. For this M, we will find a smooth symmetric 1-motive. $\mathfrak{B}(M) = (X, X', A, A', v, v', \psi)$ in the following way.

We define X' to be the group scheme representing $\mathcal{H}om\,(T, \mathbb{G}_{m,S})$ with respect to fppf-topology. It is an object in the category $\mathcal{GC}_S$. We define A' by the dual abelian scheme of A *i.e.*, the group scheme which represents the sheaf $\mathcal{E}xt^{\,1}(A, \mathbb{G}_{m,S})$ with respect to fppf-topology.

The map $v : X \to A$ is defined by $v := p \circ u$ where $u : X \to G$ and $p : G \to A$ are given group scheme homomorphisms. To construct $v' : X' \to A'$, apply $\mathcal{RH}om\,(\,\cdot\,, \mathbb{G}_{m,S})$ to $0 \to T \xrightarrow{i} G \xrightarrow{p} A \to 0$ and take ∂^0 of the long exact sequence $\cdots \to \mathcal{E}xt^{\,i}(A, \mathbb{G}_{m,S}) \to \mathcal{E}xt^{\,i}(G, \mathbb{G}_{m,S}) \to \mathcal{E}xt^{\,i}(T, \mathbb{G}_{m,S}) \xrightarrow{\partial^i} \mathcal{E}xt^{\,i+1}(A, \mathbb{G}_{m,S}) \to \cdots$. Define $v' := \partial^0$ and this is a group scheme homomorphism because of Yoneda's lemma.

More explicitly, when Q is a scheme over S, the exact sequence $0 \to T \to G \to A \to 0$ gives an exact sequence $0 \to T \times_S Q \to G \times_S Q \to A \times_S Q \to 0$, by base change. A Q-valued point $q' \in \mathrm{Mor}_S(Q, X') = Hom_Q(T \times_S Q, \mathbb{G}_{m,Q})$ gives an extension of

$A \times_S Q$ by $\mathbb{G}_{m,Q}$:

$$\begin{tikzcd} 0 \arrow[r] & T \times_S Q \arrow[r] \arrow[d, "q'"] & G \times_S Q \arrow[r] \arrow[d, "q'_*"] & A \times_S Q \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\ 0 \arrow[r] & \mathbb{G}_{m,Q} \arrow[r] & G_{q'} \arrow[r] & A \times_S Q \arrow[r] & 0 \end{tikzcd} \tag{2.2.1}$$

where $G_{q'}$ is the push-out of $G \times_S Q$ and $\mathbb{G}_{m,Q}$ by $T \times_S Q$. This extension $G_{q'}$ is an element of $\mathrm{Ext}^1_Q(A \times_S Q, \mathbb{G}_{m,Q}) = \mathrm{Mor}_S(Q, A')$ and we have a map $v'(Q) : \mathrm{Mor}_S(Q, X') \to \mathrm{Mor}_S(Q, A')$ defined by $(v'(Q))(q') := G_{q'}$. We can see that $v'(Q)$ is functorial in Q and this means that $v' : X' \to A'$ is a homomorphism of presheaves. Therefore v' becomes a homomorphism of group schemes over S (Yoneda's lemma). Note that $v \times v' : X \times_S X' \to A \times_S A'$ is also a homomorphism of group schemes.

Now we need to check that $u : X \to G$ determines a trivialization ψ, which is a biextension morphism $X \times_S X' \to \mathcal{P}_{A \times_S A'}$ as we checked. Since the category $\mathrm{Sch}_{/S}$ of S-schemes locally of finite type is a full subcategory of the category Funct_S of presheves of sets on S, it is enough to define the map of sets $\psi(Q) : \mathrm{Mor}_S(Q, X) \times \mathrm{Mor}_S(Q, X') \to \mathrm{Mor}_S(Q, \mathcal{P}_{A \times_S A'})$ for all $Q \in \mathrm{Sch}_{/S}$ and need to show that ψ is functorial in Q and gives the commutative diagram (2.1.2).

$\mathcal{P}_{A \times_S A'}$ is a scheme over A and is also a scheme over A'. It is a commutative group scheme over A which is an extension :

$$0 \to \mathbb{G}_{m,A} \to \mathcal{P}_{A \times_S A'} \to A'_A \to 0 \tag{2.2.2}$$

and it is also a commutative group scheme over A' which is an extension :

$$0 \to \mathbb{G}_{m,A'} \to \mathcal{P}_{A \times_S A'} \to A_{A'} \to 0. \tag{2.2.3}$$

Here is the universal property 1.2.6 of $\mathcal{P}_{A \times_S A'}$:

(a) For a scheme U locally of finite type over S and for a group scheme extension

$$0 \to \mathbb{G}_{m,U} \to E \to A_U \to 0, \tag{2.2.4}$$

there exists a unique S-scheme morphism $a' : U \to A'$ such that $(a')^*(2.2.3)$ is isomorphic to the group scheme extension (2.2.4) and

(b) For a scheme U locally of finite type over S and for a group scheme extension

$$0 \to \mathbb{G}_{m,U} \to E' \to A'_U \to 0, \tag{2.2.5}$$

there exists a unique S-scheme morphism $a : U \to A$ such that $(a')^*(2.2.2)$ is isomorphic to the group scheme extension (2.2.5).

(In fact, a^* and $(a')^*$ are the same as fibred products $U \times_A$ and $U \times_{A'}$.)

Let Q be an S-scheme. For $q \in \mathrm{Mor}_S(Q, X)$ and for $q' \in \mathrm{Mor}_S(Q, X')$, denote $v \circ q \in \mathrm{Mor}_S(Q, A)$ by $\overline{q}$ and $v' \circ q' \in \mathrm{Mor}_S(Q, A')$ by $\overline{q'}$. The group scheme homomorphism $u : X \to G$ gives a group homomorphism $u(Q) : \mathrm{Mor}_S(Q, X) \to \mathrm{Mor}_S(Q, G)$. To define $(\psi(Q))(q, q')$ by using $(u(Q))(q)$, we need another description of $\mathrm{Mor}_S(Q, G)$. Let L' be a two term complex $v' : X' \to A'$ and consider a short exact sequence of two-term complexes $0 \to A' \to L' \to X'[1] \to 0$. In other words, it is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & X' \arrow[r, "\mathrm{id}_{X'}"] \arrow[d, "v'"] & X' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\mathrm{id}_{A'}"] & A' \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd} \tag{2.2.6}$$

The $\mathcal{E}xt^i$ long exact sequence (2.2.6) gives rise to the following short exact sequence of abelian sheaves :

$$0 \to T \to \mathcal{E}xt^1(L', \mathbb{G}_{m,S}) \to A \to 0. \tag{2.2.7}$$

More explicitly, we need to use the exact sequence $\mathcal{H}om(A, \mathbb{G}_{m,S}) \to \mathcal{H}om(X', \mathbb{G}_{m,S}) \to \mathcal{E}xt^1(L', \mathbb{G}_{m,S}) \to \mathcal{E}xt^1(A', \mathbb{G}_{m,S}) \to \mathcal{E}xt^1(X', \mathbb{G}_{m,S})$. Now $\mathcal{E}xt^1(X', \mathbb{G}_{m,S}) = 0$ because of Definition 1.1.6. For the proof of $\mathcal{H}om(A, \mathbb{G}_{m,S}) = 0$, see Remark 2.1.1 or 1.3.8 of [12] VIII. Therefore we can conclude that (2.2.7) is exact.

Lemma 2.2.2. *(The relation between G and $\mathcal{E}xt^1(L', \mathbb{G}_{m,S})$)*

The middle sheaf $\mathcal{E}xt^1(L', \mathbb{G}_{m,S})$ is represented by G.

Proof. Since T is an affine scheme, the representability theorem 1.1.6 shows that $\mathcal{E}xt^1(L', \mathbb{G}_{m,S})$ is a group scheme. For schemes V and Q over S, let me introduce the notation V_Q which means the base extension $V \times_S Q$ of V by Q. The fibred product of $\overline{q'} : Q \to A'$ and the exact sequence (2.2.3) is

$$0 \to \mathbb{G}_{m,Q} \to (\mathcal{P}_{A \times_S A'})_{/Q} \to A_{/Q} \to 0$$

where $_{/Q}$ means the base extension $\times_{A'} Q$ of an A'-scheme. $A_{/Q}$ is isomorphic to $(A \times_S A') \times_{A'} Q = A \times_S (A' \times_{A'} Q) = A \times_S Q = A_Q$. Since the structure morphism $(\mathcal{P}_{A \times_S A'})_{/Q} \to Q$ depends on the point $q' \in \mathrm{Mor}_S(Q, X')$, we can simply denote it by $\mathcal{P}_{q'}$. From the universal property (a) of $\mathcal{P}_{A \times_S A'}$, there is a canonical isomorphism

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_{m,Q} \arrow[r] \arrow[d, "\mathrm{id}_{\mathbb{G}_{m,Q}}"] & \mathcal{P}_{q'} \arrow[r] \arrow[d, "\cong"] & A_Q \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & \mathbb{G}_{m,Q} \arrow[r] & \mathrm{WB}(\overline{q'}) \arrow[r] & A_Q \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{WB}(\overline{q'})$ is the extension determined by $\overline{q'} \in \mathrm{Mor}_S(Q, A') = \mathrm{Ext}^1_Q(A_Q, \mathbb{G}_{m,Q})$. $X' = \mathcal{H}om\,(T, \mathbb{G}_m)$ and $v' : X' \to A'$, or equivalently the connecting homomorphism ∂^0, is determined by the following diagram

$$\begin{tikzcd}
0 \arrow[r] & T_Q \arrow[r] \arrow[d, "q'"] & G_Q \arrow[r] \arrow[d, "(q')_*"] & A_Q \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & \mathbb{G}_{m,Q} \arrow[r] & G_{q'} \arrow[r] & A_Q \arrow[r] & 0
\end{tikzcd} \tag{2.2.8}$$

where $G_{q'}$ is the push-out of G_Q by $q' \in \mathrm{Hom}_Q(T_Q, \mathbb{G}_{m,Q})$. In other words, $v'(Q) : \mathrm{Mor}_S(Q, X') \to \mathrm{Mor}_S(Q, A')$ sends $q' \in \mathrm{Hom}_Q(T_Q, \mathbb{G}_{m,Q})$ to $G_{q'} = (q')_*(G \times_S Q) \in \mathrm{Ext}^1_Q(A_Q, \mathbb{G}_{m,Q})$. This extension $G_{q'}$ is the same as $\mathrm{WB}(\overline{q'})$ from the fact that $(v'(Q))(q') = (v' \circ q') = \overline{q'}$.

If we can define a homomorphism $\eta : G \to \mathcal{E}xt^1(L', \mathbb{G}_{m,S})$ of abelian sheaves (or corresponding group schemes) that makes the following diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "i"] \arrow[d, "\mathrm{id}_T"] & G \arrow[r, "p"] \arrow[d, "\eta"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\ 0 \arrow[r] & T \arrow[r] & \mathcal{E}xt^1(L', \mathbb{G}_{m,S}) \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \tag{2.2.9}$$

commutative then the five lemma shows that $G = \mathcal{E}xt^1(L', \mathbb{G}_{m,S})$. We will define η by a natural transformation of abelian sheaves (*i.e.*, functors) from G to $\mathcal{E}xt^1(L', \mathbb{G}_{m,S})$.

Let $j : Q \to S$ be the structure morphism of Q over S. Since $(\ \cdot\ )_Q$ is the restriction functor of abelian sheaves, using the spectral sequence $H^p(Q, \mathcal{E}xt^q(L'_Q, \mathbb{G}_{m,Q})) \Rightarrow \mathrm{Ext}_Q^{p+q}(L'_Q, \mathbb{G}_{m,Q})$, we have $\mathcal{E}xt^1(L', \mathbb{G}_{m,S})(Q) = \mathrm{Ext}^1_Q(L'_Q, \mathbb{G}_{m,Q})$. More precisely, since j^* is an exact functor ($R^1 j^* = 0$) and since $\mathcal{H}om\,(j^*L', j^*\mathbb{G}_{m,S}) = j^*\mathcal{H}om\,(L', \mathbb{G}_{m,S})$, their first derived functors $\mathcal{E}xt^1_Q(j^*L', j^*\mathbb{G}_{m,S})(= \mathcal{E}xt^1_Q(L'_Q, \mathbb{G}_{m,Q}))$ and $j^*\mathcal{E}xt^1_S(L', \mathbb{G}_{m,S})$ are the same.

To define $\eta(Q) : \mathrm{Mor}_S(Q, G) \to \mathrm{Ext}^1_Q(L'_Q, \mathbb{G}_{m,Q})$, it is enough to determine $\eta(Q)(g) : X'_Q \to E'$ for every $g \in \mathrm{Mor}_S(Q, G)$, where E' is the extension given by $\overline{g} := p \circ g \in \mathrm{Mor}_S(Q, A)$:

$$0 \to \mathbb{G}_{m,Q} \to E' \to A'_Q \to 0. \tag{2.2.10}$$

We define $\eta(Q)(g)$ by a homomorphism of Q-abelian sheaves. Let U be a scheme over Q and $x' \in \mathrm{Mor}_Q(U, X'_Q)$. For the exact sequence of U-sheaves

$$0 \to T_U \to G_U \to A_U \to 0, \tag{2.2.11}$$

take the push-out of (2.2.11) by $x' \in \mathrm{Hom}_U(T_U, \mathbb{G}_{m,U})$. The following diagram

$$\begin{tikzcd} 0 \arrow[r] & T_U \arrow[r] \arrow[d, "x'"] & G_U \arrow[r] \arrow[d, "x'_*"] & A_U \arrow[r] \arrow[d, "\mathrm{id}_{A_U}"] & 0 \\ 0 \arrow[r] & \mathbb{G}_{m,U} \arrow[r] & G_{x'} \arrow[r] & A_U \arrow[r] & 0 \end{tikzcd} \tag{2.2.12}$$

is commutative and its bottom line is the exact sequence of U-group schemes

$$0 \to \mathbb{G}_{m,U} \to G_{x'} \to A_U \to 0 \tag{2.2.13}$$

where $G_{x'}$ is a group scheme over U and depends on the choice of $x' \in \mathrm{Mor}_Q(U, X'_Q)$. From the universal property of the Poincaré biextension $\mathcal{P}_{A\times_S A'}$, (2.2.13) is the pull back of the exact sequence of A'-group schemes

$$0 \to \mathbb{G}_{m,A'} \to \mathcal{P}_{A\times_S A'} \to A_{A'} \to 0 \tag{2.2.14}$$

by a unique $u' \in \mathrm{Mor}_S(U, A')$. Therefore, $G_{x'} = (\mathcal{P}_{A\times_S A'}) \times_{A'} U$ and we have a map of sets $\mathrm{Mor}_U(U, G_{x'}) \to \mathrm{Mor}_{A'}(U, \mathcal{P}_{A\times_S A'})$. Let $w : U \to Q$ be the Q-structure morphism. We can see that $g \in \mathrm{Mor}_S(Q, G) = \mathrm{Mor}_Q(Q, G_Q)$ and $g \circ w \in \mathrm{Mor}_Q(U, G_Q) = \mathrm{Mor}_U(U, G_U)$. As an element of $\mathrm{Mor}_U(U, G_U)$, $g \circ w$ becomes $(g \circ w) \times_S \mathrm{id}_U$.

Let $\pi : \mathcal{P}_{A\times_S A'} \to A \times_S A'$ be the projection of this biextension and let $k_{g,x'} : G_{g,x'} := G_{x'} \times_A U \xrightarrow{\cong} (\mathcal{P}_{A\times_S A'})_{p\circ g, v'\circ x'}$ be the canonical isomorphism of $\mathbb{G}_m$-torsors. Since $(\mathcal{P}_{A\times_S A'})_{p\circ g, v'\circ x'}$ is the fibred product $(\mathcal{P}_{A\times_S A'}) \times_{A\times_S A'} U$, there is a unique morphism $l_{p\circ g, x'} : (\mathcal{P}_{A\times_S A'})_{p\circ g, x'} \to \mathcal{P}_{A\times_S A'}$. Define $\langle g, U\rangle(x') := l_{p\circ g,x'} \circ k_{p\circ g,x'} \circ x'_* \circ g \circ w \in \mathrm{Mor}_A(U, \mathcal{P}_{A\times_S A'})$ and the A-structure morphism of U is given by $p \circ g \circ w$. Thus $\langle g, U\rangle$ is a map from $\mathrm{Mor}_Q(U, X'_Q)$ to $\mathrm{Mor}_A(U, \mathcal{P}_{A\times_S A'}) = \mathrm{Mor}_Q(U, \mathcal{P}_{A\times_S A'} \times_A Q)$. Recall that $\mathcal{P}_{A\times_S A'} \times_A Q \in \mathrm{Ext}^1_Q(A'_Q, \mathbb{G}_{m,Q})$ is given by $p \circ g \in \mathrm{Mor}_S(Q, A) = \mathrm{Mor}_Q(Q, A_Q)$ and $p \circ g$ becomes $p_Q \circ g$ as an element of $\mathrm{Mor}_Q(Q, A_Q)$.

Define $(\eta(Q)(g))(U)(x')$ by $\langle g, U\rangle(x')$, then$(\eta(Q)(g))(U)$ is a map from $\mathrm{Mor}_Q(U, X'_Q)$ to $\mathrm{Mor}_Q(U, E')$. Since $\langle g, U\rangle$ is functorial for U, $(\eta(Q)(g))(U)$ is also functorial for U over Q. Therefore $(\eta(Q)(g))$ is a morphism of schemes from X'_Q to $E' = (\mathcal{P}_{A\times_S A'}) \times_A Q$ and it is a lifting of $v'_Q : X'_Q \to A'_Q$.

We need to show that $(\eta(Q)(g)) : X'_Q \to (\mathcal{P}_{A\times_S A'}) \times_A Q$ is a Q-group scheme homomorphism and thus $(\eta(Q)(g)) \in \mathrm{Ext}^1_Q(L'_Q, \mathbb{G}_{m,Q})$. In other words, $g \mapsto (\eta(Q)(g))$

defines a map from $\mathrm{Mor}_S(Q,G)$ to $\mathrm{Ext}^1_Q(L_Q,\mathbb{G}_{m,Q})$. For $x'_1,x'_2 \in \mathrm{Mor}_Q(U,X'_Q)$, their push-out of G_U in (2.2.11) are $G_{x'_1}$ and $G_{x'_2}$ in (2.2.12). Since g is fixed, it is enough to consider G_{g,x'_1} and G_{g,x'_2} instead of $G_{x'_1}$ and $G_{x'_2}$. G_{g,x'_i} is canonically isomorphic to $(\mathcal{P}_{A\times_S A'})_{p\circ g,v'\circ x'_i}$ due to the universal property. Using the torsor isomorphism $(\mathcal{P}_{A\times_S A'})_{p\circ g,v'\circ x'_1}\wedge(\mathcal{P}_{A\times_S A'})_{p\circ g,v'\circ x'_2}\to(\mathcal{P}_{A\times_S A'})_{p\circ g,v'\circ(x'_1+x'_2)}$ coming from the partial addition $\underset{2}{+}:\mathcal{P}_{A\times_S A'}\times_A\mathcal{P}_{A\times_S A'}\to\mathcal{P}_{A\times_S A'}$, we can see that $(\eta(Q)(g))(x'_1+x'_2)=(\eta(Q)(g))(x'_1)\underset{2}{+}(\eta(Q)(g))(x'_2)\in\mathrm{Mor}_Q(U,\mathcal{P}_{A\times_S A'}\times_A Q)$. Since $\mathcal{P}_{A\times_S A'}\times_A Q$ (depending on g) becomes the second term of the extension of L'_Q by $\mathbb{G}_{m,Q}$, we get a map from $\mathrm{Mor}_S(Q,G)$ to $\mathrm{Ext}^1_Q(L_Q,\mathbb{G}_{m,Q})$.

Now we need to prove that the map $\eta(Q):\mathrm{Mor}_S(Q,G)\to\mathrm{Ext}^1_Q(L'_Q,\mathbb{G}_{m,Q})$ is a group homomorphism which makes the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Mor}_S(Q,T) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Mor}_S(Q,G) \arrow[r] \arrow[d, "\eta(Q)"] & \mathrm{Mor}_S(Q,A) \arrow[d, "\mathrm{id}"] \\ 0 \arrow[r] & \mathrm{Mor}_S(Q,T) \arrow[r] & \mathrm{Ext}^1_Q(L'_Q,\mathbb{G}_{m,Q}) \arrow[r] & \mathrm{Mor}_S(Q,A) \end{tikzcd} \tag{2.2.15}$$

commutative. For $g_1,g_2\in\mathrm{Mor}_S(Q,G)$, the corresponding elements $\eta(Q)(g_1),\eta(Q)(g_2)\in\mathrm{Ext}^1_Q(L'_Q,\mathbb{G}_{m,Q})$ are given by $X\to E'_{g_i}$ where E'_{g_i} is an extension of A'_Q by $\mathbb{G}_{m,Q}$ as in (2.2.10), and E'_{g_i} is canonically isomorphic to $(\mathcal{P}_{A\times_S A'})\times_A Q$ where the structure morphism of Q is $p\circ g_i:Q\to(G\to)A$. As in Remark 2.2.1 (or due to (2.2.15), the addition of $\mathrm{Ext}^1_Q(L'_Q,\mathbb{G}_{m,Q})$ is explained by that of $\mathrm{Mor}_S(Q,A)=\mathrm{Ext}^1_Q(A'_Q,\mathbb{G}_{m,Q})$ (coming from $\mathcal{P}_{A\times_S A'}$) and that of $\mathrm{Mor}_S(Q,T)=\mathrm{Hom}_Q(X'_Q,\mathbb{G}_{m,Q})$. Therefore, to show that $\eta(Q)$ is a homomorphism, it is enough to check that $(\eta(Q)(g_1+g_2))(x')=(\eta(Q)(g))(x'_1)\underset{1}{+}(\eta(Q)(g))(x')$ for every $x'\in\mathrm{Mor}_Q(U,X'_Q)$. This is true because x'_* is a homomorphism, and because $l_{p\circ g,x'}$ and $k_{p\circ g,x'}$ are additive for g. Since $\eta(Q)$ is functorial for Q, we have the S-group scheme homomorphism $\eta:G\to\mathcal{E}xt^1(L',\mathbb{G}_{m,S})$ which makes the diagram (2.2.9) commutative (and this follows from the definition of $\eta(Q)(g)(x')$). $\square$

To finish our proof, we need to look at the group scheme structure of the set $\mathcal{E}xt^1(L', \mathbb{G}_{m,S})(Q)$.

Remark 2.2.1. (The group structure of $\mathcal{E}xt^1(L', \mathbb{G}_{m,S})(Q)$)

We checked that $\mathcal{E}xt^1_Q(L'_Q, \mathbb{G}_{m,Q}) = \mathcal{E}xt^1_S(L', \mathbb{G}_{m,S})|_Q$ in the proof of Lemma 2.2.2 and that means $\mathcal{E}xt^1_S(L', \mathbb{G}_{m,S})(Q) = \mathcal{E}xt^1_Q(L'_Q, \mathbb{G}_{m,Q})(Q)$. The local-global spectral sequence $H^i(Q, \mathcal{E}xt^j_Q(L'_Q, \mathbb{G}_{m,Q})) \to \mathrm{Ext}^{i+j}_Q(L'_Q, \mathbb{G}_{m,Q})$ gives rise to a 5-term exact sequence $0 \to H^1(Q, \mathcal{H}om_Q(L'_Q, \mathbb{G}_{m,Q})) \to \mathrm{Ext}^1_Q(L'_Q, \mathbb{G}_{m,Q})) \to \mathcal{E}xt^1_Q(L_Q, \mathbb{G}_{m,Q}))(Q) \to H^2(Q, \mathcal{H}om_Q(L'_Q, \mathbb{G}_{m,S}))$. Since $\mathcal{H}om_Q(L'_Q, \mathbb{G}_{m,Q}) = 0$ (because of $\mathcal{H}om_S(A', \mathbb{G}_{m,S}) = 0$), we conclude that $\mathcal{E}xt^1_S(L', \mathbb{G}_{m,S}))(Q) = \mathrm{Ext}^1_Q(L'_Q, \mathbb{G}_{m,Q}))$ is the set of isomorphism classes of pairs $(E', \widetilde{v'_Q})$ where E' is an extension of A'_Q by $\mathbb{G}_{m,Q}$ and $\widetilde{v'_Q}$ is a lifting of v'_Q to E'. This pair can be visualized as :

$$\begin{tikzcd}
 & & X'_Q \arrow[r, "\mathrm{id}_{X'_Q}"] \arrow[d, "\widetilde{v'_Q}"] & X'_Q \arrow[d, "v'_Q"] & \\
0 \arrow[r] & \mathbb{G}_{m,Q} \arrow[r] & E' \arrow[r] & A'_Q \arrow[r] & 0.
\end{tikzcd} \tag{2.2.16}$$

We define a morphism of pairs (E_1, w_1) and (E_2, w_2) using a commutative diagram of pairs ($E_1 \cong E_2$ and w_1 and w_2 makes a commutative diagram)

$$\begin{tikzcd}
X'_Q \arrow[r, "w_1"] \arrow[d, "\mathrm{id}"'] & E_1 \arrow[d, "\wr"] \\
X'_Q \arrow[r, "w_2"] & E_2.
\end{tikzcd}$$

and we can prove that this morphism is always an isomorphism. The set of Q-valued points $\mathcal{E}xt^1(L', \mathbb{G}_{m,S})(Q) = \mathrm{Ext}^1(L'_Q, \mathbb{G}_{m,Q})$ has a group structure as follows. Let E be the **Baer sum** (or the contracted product) of E_1 and E_2 over A'_Q. It is also an extension of A'_Q by $\mathbb{G}_{m,Q}$ and is obtained as the following.

If we take the fibred product $E_1 \times_{A'_Q} E_2$ then it is a commutative group scheme over Q which is an extension of A'_Q by $\mathbb{G}_{m,Q} \times_Q \mathbb{G}_{m,Q}$. Let E be the push-out of

$E_1 \times_{A'_Q} E_2$ via (a homomorphism) $m : \mathbb{G}_{m,Q} \times_Q \mathbb{G}_{m,Q} \to \mathbb{G}_{m,Q}$. E is a (commutative) group scheme extension of A'_Q by $\mathbb{G}_{m,Q}$. We can check that such an E is also the contracted product of E_1 and E_2 as a $\mathbb{G}_{m,A'_Q}$-torsor.

The morphism $w_1 \times_{A'_Q} w_2 \in \mathrm{Mor}_{A'_Q}(X'_Q, E_1 \times_{A'_Q} E_2)$ is a homomorphism of group schemes over Q. Taking the composite of $w_1 \times_{A'_Q} w_2$ with the morphism $E_1 \times_{A'_Q} E_2 \to E$, we have a homomorphism $w : X'_Q \to E$ of group schemes over A'_Q. w is a lifting of $v'(Q)$ and (group valued) functor computation shows that $\mathrm{Ext}^1(L'_Q, \mathbb{G}_{m,Q})$ is a group. So the sum of (E_1, w_1) and (E_2, w_2) is defined by this pair (E, w). $\square$

For any $q \in \mathrm{Mor}(Q, X)$, $g := u(Q)(q) \in \mathrm{Mor}_S(Q, G) = \mathrm{Ext}^1(L'_Q, \mathbb{G}_{m,Q})$ gives a pair $(E'_q, \widetilde{v'_q}) \in \mathcal{E}xt^1(L', \mathbb{G}_{m,S})(Q)$. From the exact sequence (2.2.7), the image of $(u(Q))(q)$ in $\mathrm{Mor}_S(Q, A)$ is $\overline{q}$ and $E'_q = \overline{q}^*(\mathcal{P}_{A\times_S A'})$ (the universal property (b) of $\mathcal{P}_{A\times_S A'}$). The set $\mathrm{Mor}_Q(Q, E'_q)$ becomes

$$\begin{aligned}\mathrm{Mor}_Q(Q, E'_q) = \mathrm{Mor}_Q(Q, \overline{q}^*(\mathcal{P}_{A\times_S A'})) &= \mathrm{Mor}_Q(Q, \mathcal{P}_{A\times_S A'} \times_A Q) \\ &\subset \mathrm{Mor}_S(Q, \mathcal{P}_{A\times_S A'})\end{aligned}$$

where $\mathrm{Mor}_Q(Q, q^*(\mathcal{P}_{A\times_S A'}))$ is $(\pi(Q))^{-1}(\{\overline{q}\} \times \mathrm{Mor}_S(Q, A')) \subset \mathrm{Mor}_S(Q, \mathcal{P}_{A\times_S A'})$.

Since $\widetilde{v'_q}(Q) : \mathrm{Mor}_Q(Q, X'_Q) \to \mathrm{Mor}_Q(Q, E'_q)$ is a lifting of a group homomorphism $v'_Q(Q) : \mathrm{Mor}_Q(Q, X'_Q) \to \mathrm{Mor}_Q(Q, A'_Q)$ and since $v'(Q) = v'_Q(Q)$ as a group homomorphism, one can conclude that $(\widetilde{v'_q}(Q))(q') \in \pi(Q)^{-1}(\overline{q}, v' \circ q')$. If we denote $(\widetilde{v'_q}(Q))(q')$ by $(\psi(Q))(q, q')$, $\psi(Q)$ is a map from $X(Q) \times X'(Q)$ to $\mathcal{P}_{A\times_S A'}(Q)$ which is a lifting of $(v \times_S v')(Q)$ and is additive for q' (*i.e.*, $(\psi(Q))(q, q'_1 + q'_2) = (\psi(Q))(q, q'_1) \underset{2}{+} (\psi(Q))(q, q'_2)$).

We can also prove the additivity of $\psi(Q)(q, q')$ for q, using the Lemma 2.2.2. More precisely, a Q-valued point $g \in \mathrm{Mor}_S(Q, G) = \mathcal{E}xt^1(L', \mathbb{G}_{m,S})(Q)$ is the same as $(\widetilde{v'_Q})_g : X'_Q \to E'_g$ where E'_g is an element of $\mathrm{Mor}_Q(Q, A_Q) = \mathrm{Mor}_S(Q, A)$ and where $(\widetilde{v'_Q})_g$ is the lifting of $v'_Q : X'_Q \to A'_Q$. Let $\overline{g}$ be $p \circ g \in \mathrm{Mor}_S(Q, A)$ and note that E'_g is isomorphic to $\overline{g}^*\mathcal{P}_{A\times_S A'}$. For any $g_1, g_2 \in \mathrm{Mor}_S(Q,G)$, we showed that $(\widetilde{v'_Q})_{g_1+g_2} = (\widetilde{v'_Q})_{g_1} \underset{1}{+} (\widetilde{v'_Q})_{g_2}$ in Lemma 2.2.2. More precisely, $(\widetilde{v'_Q})_{g_1+g_2}(q') = (\widetilde{v'_Q})_{g_1}(Q)(q') \underset{1}{+} (\widetilde{v'_Q})_{g_2}(Q)(q')$ for every $q' \in \mathrm{Mor}_S(Q,X')$. Since $\psi(Q)(q_1+q_2,q') = (\widetilde{v'_Q})_{u(Q)(g_1+g_2)}(q')$, we see that $\psi(Q)(q_1+q_2,q') = \psi(Q)(q_1,q') \underset{1}{+} \psi(Q)(q_2,q')$ because $u : X \to G$ is a group scheme homomorphism.

The definition of ψ show that ψ is a natural transformation for Q (over S), *i.e.*, it is a morphism of S-schemes. Therefore we get a smooth symmetric 1-motive $\mathfrak{B}(M) = (X, X', A, A', v, v', \psi)$ with torsion. It is clear that $\mathfrak{B}([0 \to A]) = (0,0,A,A',0,0,0)$ is not trivial, and showing the next lemma, we can finish the proof of our proposition.

□

Lemma 2.2.3. *(Functoriality of $\mathfrak{B}$)*

$M \mapsto \mathfrak{B}(M)$ is an additive functor $\mathcal{M}_{\mathrm{sm}} \to \mathcal{M}_{\mathrm{ssym}}$.

Proof. Let $f : M_1 \to M_2$ be a morphism of smooth 1-motives with torsion. $f_A : A_1 \to A_2$ gives two homomorphisms $f_{A_1} = f_A$ and $f_{A'_2} = (f_A)^t$. Similarly, $f_{X_1} = f_X$ and $f_{X'_2}$ is given by $(f_T)^D$. All of them are group scheme homomorphisms and the following diagrams :

$$\begin{array}{ccc} X_1 & \xrightarrow{v_1} & A_1 \\ \downarrow{\scriptstyle f_{X_1}} & & \downarrow{\scriptstyle f_{A_1}} \\ X_2 & \xrightarrow{v_2} & A_2 \end{array} \quad , \qquad \begin{array}{ccc} X'_2 & \xrightarrow{v'_2} & A'_2 \\ \downarrow{\scriptstyle f_{X'_2}} & & \downarrow{\scriptstyle f_{A'_2}} \\ X'_1 & \xrightarrow{v'_1} & A'_1 \end{array}$$

are commutative. (For the first one, by the definition of f and for the other one, because of the funtoriality of $\mathcal{E}xt^i$.) $f_{A_1} := f_A$ and $f_{A'_2} := (f_A)^t$ are dual to each other.

From the property of a morphism of smooth 1-motives with torsion

$$\begin{array}{ccc} X_1 & \xrightarrow{u_1} & G_1 \\ \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_G} \\ X_2 & \xrightarrow{u_2} & G_2 \end{array} \tag{2.2.17}$$

in **Definition** (2.1.3), we have two (but the same) smooth 1-motives with torsion $u_2 \circ f_X : X_1 \to G_2$ and $f_G \circ u_1 : X_1 \to G_2$. From the first smooth 1-motive with torsion, we have a biextension morphism $X_1 \times_S X_2' \to \mathcal{P}_{A_2 \times_S A_2'}$ that is a lifting of $X_1 \times_S X_2' \to A_2 \times_S A_2'$ by taking the composite $\psi_2 \circ (f_{X_1} \times_S \ \mathrm{id}_{X_2'})$ and this lifting determines a unique biextension morphism $\overline{\psi_2} : X_1 \times_S X_2' \to (f_{A_1} \times_S \mathrm{id}_{A_2'})^* \mathcal{P}_{A_2 \times A_2'}$.

In $f_G \circ u_1 : X_1 \to G_2$, $f_G \in \mathrm{Hom}(G_1, G_2)$ gives a commutative diagram

$$\begin{tikzcd} X_2' \arrow[r, "v_2'"] \arrow[d, "f_{X_2'}"] & A_2' \arrow[d, "f_{A_2'}"] \\ X_1' \arrow[r, "v_1'"] & A_1' \end{tikzcd} \tag{2.2.18}$$

and by combining this diagram and $\psi_1 \circ (\mathrm{id}_{X_1} \times_S \ f_{X_2'})$ we get another lifting $\overline{\psi_1}$ of $(v_1, v_2') : X_1 \times_S X_2' \to A_1 \times A_2'$:

$$\begin{tikzcd} X_1 \times_S X_1' \arrow[r, "\psi_1"] & \mathcal{P}_{A_1 \times_S A_1'} \\ X_1 \times_S X_2' \arrow[u, "\mathrm{id}_{X_1} \times_S f_{X_2'}"] \arrow[r, "\overline{\psi_1}"] & (\mathrm{id}_{A_1} \times_S f_{A_2'})^* \mathcal{P}_{A_1 \times_S A_1'} . \arrow[u] \end{tikzcd} \tag{2.2.19}$$

We need to show that $\overline{\psi_1}$ gives a unique biextension morphism $X_1 \times_S X_2' \to \mathcal{P}_{A_2 \times_S A_2'}$. Since we have to check $(\mathrm{id}_{A_1} \times_S f_{A_2'})^* \mathcal{P}_{A_1 \times_S A_1'} = (f_{A_1} \times_S \mathrm{id}_{A_2'})^* \mathcal{P}_{A_2 \times_S A_2'}$, we will use the following lemma.

Lemma 2.2.4. *(Homological interpretation of biextensions)*

Let A, B, and C be fppf *abelian sheaves over S and let* $\mathrm{Biext}^1(A, B; C)$ *be the set of isomorphism classes of biextensions of A and B by C.*

$\mathrm{Biext}^1(A, B; C)$ *is a group which is isomorphic to* $\mathrm{Ext}^1(A \overset{\mathbb{L}}{\otimes} B, C)$. *This isomorphism is functorial for A, B, and C.*

Proof. See [12] VII.3.6.5 and 3.7.6. $\square$

Next, the following isomorphisms (which are functorial for B and A)

$$\mathrm{Hom}_S(A \otimes B, C) = \mathrm{Hom}_S(A, \mathcal{H}om\,(B, C))$$

$$\mathrm{Hom}_S(B \otimes A, C) = \mathrm{Hom}_S(B, \mathcal{H}om\,(A, C))$$

show that $\mathrm{RHom}_S(A \overset{\mathbb{L}}{\otimes} B, C) = \mathrm{RHom}_S(A, \mathcal{RH}om\,(B, C))$ (functorial for B), and $\mathrm{RHom}_S(A \overset{\mathbb{L}}{\otimes} B, C) = \mathrm{RHom}_S(B, \mathcal{RH}om\,(A, C))$ (functorial for A). Spectral sequences of these derived functors give five term exact sequences

$$\begin{aligned} 0 \to \mathrm{Ext}^1(A, \mathcal{H}om\,(B, C)) \to \mathrm{Biext}^1(A, B; C) \to \mathrm{Hom}(A, \mathcal{E}xt^{\,1}(B, C)) \\ \to \mathrm{Ext}^2(A, \mathcal{H}om\,(B, C)) \to \mathrm{Ext}^2(A, \mathcal{RH}om\,(B, C)) \end{aligned}$$

and

$$\begin{aligned} 0 \to \mathrm{Ext}^1(B, \mathcal{H}om\,(A, C)) \to \mathrm{Biext}^1(A, B; C) \to \mathrm{Hom}(B, \mathcal{E}xt^{\,1}(A, C)) \\ \to \mathrm{Ext}^2(B, \mathcal{H}om\,(A, C)) \to \mathrm{Ext}^2(B, \mathcal{RH}om\,(A, C)). \end{aligned}$$

When A is an abelian scheme, $B = A'$ is its dual, and $C = \mathbb{G}_{m,S}$, we have $\mathrm{Hom}_S(A, \mathcal{E}xt^{\,1}(A', \mathbb{G}_{m,S})) = \mathrm{Biext}^1(A, A'; \mathbb{G}_{m,S}) = \mathrm{Hom}_S(A', \mathcal{E}xt^{\,1}(A, \mathbb{G}_{m,S}))$ and (the isomorphism class of) $\mathcal{P}_{A \times_S A'}$ becomes $\mathrm{id}_A \in \mathrm{Hom}_S(A, \mathcal{E}xt^{\,1}(A', \mathbb{G}_{m,S}))$ and $\mathrm{id}_{A'} \in \mathrm{Hom}_S(A', \mathcal{E}xt^{\,1}(A, \mathbb{G}_{m,S}))$ because of its universal property.

If $A = A_1$, $B = A_2$, $C = \mathbb{G}_{m,S}$, and $f : A_1 \to A_2$ a homomorphism, then $\mathrm{Hom}_S(A_1, \mathcal{E}xt^{\,1}(A_2', \mathbb{G}_{m,S})) = \mathrm{Biext}^1(A_1, A_2'; \mathbb{G}_{m,S}) = \mathrm{Hom}_S(A_2', \mathcal{E}xt^{\,1}(A_1, \mathbb{G}_{m,S}))$. The pull back $(\mathrm{id}_{A_1} \times_S f_{A_2'})^* \mathcal{P}_{A_1 \times_S A_1'}$ is corresponding to $f_{A_2'} \in \mathrm{Hom}_S(A_2', A_1')$ and $(f_{A_1} \times_S \mathrm{id}_{A_2'})^* \mathcal{P}_{A_2 \times_S A_2'}$ is is corresponding to $f_{A_1} \in \mathrm{Hom}_S(A_1, A_2)$. Since the (functorial) isomorphism $\mathrm{Hom}_S(A_1, A_2) \cong \mathrm{Hom}_S(A_2', A_1')$ sends f_{A_1} to its dual homomorphism $f_{A_2'}$, $(f_{A_1} \times_S \mathrm{id}_{A_2'})^* \mathcal{P}_{A_2 \times_S A_2'} = (\mathrm{id}_{A_1} \times_S f_{A_2'})^* \mathcal{P}_{A_1 \times_S A_1'}$. The biextension morphism $\overline{\psi_1} : X_1 \times_S X_2' \to (f_{A_1} \times_S \mathrm{id}_{A_2'})^* \mathcal{P}_{A_2 \times_S A_2'}$ gives a unique biextension morphism $\widetilde{\psi_1} : X_1 \times_S X_2' \to \mathcal{P}_{A_2 \times_S A_2'}$ and the commutativity of the diagram (2.2.17) $f_G \circ u_1 =$

$u_2 \circ f_X : X_1 \to G_2$ implies that $\widetilde{\psi_1} = \psi_2 : X_1 \times_S X_2' \to \mathcal{P}_{A_2 \times_S A_2'}$. Due to the universal property of the inverse image (or the fibred product of schemes in our case), we have $\overline{\psi_1} = \overline{\psi_2}$. In other words, the following diagram is commutative.

$$\begin{tikzcd}
X_1 \times_S X_1' \arrow[rr, "\psi_1"] & & \mathcal{P}_{A_1 \times_S A_1'} \\
X_1 \times_S X_2' \arrow[u, "\mathrm{id}_{X_1} \times_S f_{X_2'}"] \arrow[d, "f_{X_1} \times_S \mathrm{id}_{X_2'}"'] \arrow[r] & (\mathrm{id}_{A_1} \times_S f_{A_2'})^* \mathcal{P}_{A_1 \times_S A_1'} = (f_{A_1} \times_S \mathrm{id}_{A_2'})^* \mathcal{P}_{A_2 \times_S A_2'} \arrow[ur] \arrow[dr] & \\
X_2 \times_S X_2' \arrow[rr, "\psi_2"] & & \mathcal{P}_{A_2 \times_S A_2'}
\end{tikzcd}$$

(The additivity of $\mathfrak{B}$ can be seen by its definition.) $\square$

Now we need to prove that $\mathfrak{B}$ is an equivalence functor between $\mathcal{M}_{\mathrm{sm}}$ and $\mathcal{M}_{\mathrm{ssym}}$. So we need to define a functor $\mathfrak{C}$ from $\mathcal{M}_{\mathrm{ssym}}$ to $\mathcal{M}_{\mathrm{sm}}$ such that $\mathfrak{C} \circ \mathfrak{B} : \mathcal{M}_{\mathrm{sm}} \to \mathcal{M}_{\mathrm{sm}}$ is $\mathrm{id}_{\mathcal{M}}$ and $\mathfrak{B} \circ \mathfrak{C} : \mathcal{M}_{\mathrm{ssym}} \to \mathcal{M}_{\mathrm{ssym}}$ is $\mathrm{id}_{\mathcal{M}_{\mathrm{ssym}}}$ up to isomorphisms. First, let me define an additive functor from $\mathcal{M}_{\mathrm{ssym}}$ to $\mathcal{M}_{\mathrm{sm}}$.

Proposition 2.2.5. *(Construction of $\mathfrak{B}^{-1}$)*

There is a nontrivial additive functor $\mathfrak{C} : \mathcal{M}_{\mathrm{ssym}} \to \mathcal{M}_{\mathrm{sm}}$.

Proof. Let $N = (X, X', A, A', v, v', \psi)$ be a symmetric avatar. X and A in the smooth 1-motive $u : X \to G$ are given by X and A in N. Let T be the group scheme which represents the sheaf $X'^D = \mathcal{H}om_S(X', \mathbb{G}_{m,S})$. One can show that $T^D \in \mathcal{GC}_S$. We need to decide G and u.

To define G, look at $L' : X' \xrightarrow{v'} A'$. In this complex, X' and A' are regarded as the abelian sheaves (for the S_{fppf}-topology) and then the group scheme homomorphism $v' : X' \to A'$ is a morphism of abelian sheaves. Apply the functor $\mathcal{RH}om(\,\cdot\,, \mathbb{G}_{m,S})$ to the short exact sequence

$$0 \longrightarrow A' \longrightarrow L' \longrightarrow X'[1] \longrightarrow 0\,, \qquad (2.2.20)$$

we get a short exact sequence (as a part of the long exact sequence) :

$$0 \to \mathcal{H}om\,(X', \mathbb{G}_{m,S}) \to \mathcal{E}xt^{\,1}(L', \mathbb{G}_{m,S}) \to \mathcal{E}xt^{\,1}(A', \mathbb{G}_{m,S}) \to 0. \tag{2.2.21}$$

The first sheaf is represented by T that is an affine scheme. The last sheaf is represented by A (the Weil-Barsotti formula 1.2.5). Since the first sheaf is represented by an affine scheme, the representability of the middle sheaf $\mathcal{E}xt^{\,1}(L, \mathbb{G}_{m,S})$ follows from Proposition 1.1.6. G is defined by this group scheme which is an extension of A by T in the sense of the first section (and see equations (1.1.5) and (1.1.6) for more details).

Finally we need to determine the lifting $u : X \to G$ of $v : X \to A$. We will define it by a natural transformation (of group functors). Let Q be a scheme over S. Fix $q \in \mathrm{Mor}_S(Q, X)$ and $\overline{q} := v \circ q \in \mathrm{Mor}_S(Q, A)$. Since $A = (A')'$ is the dual of A', we have a Q-group scheme extension :

$$0 \to \mathbb{G}_{m,Q} \to E_{\overline{q}} \to A'_Q \to 0$$

depending on the Q-valued point $\overline{q} \in \mathrm{Mor}_S(Q, A)$ and thus $q \in \mathrm{Mor}_S(Q, X)$. Let me use the abbreviation $\mathcal{P}_q = (\mathcal{P}_{A \times_S A'}) \times_A Q$ for $\overline{q} : Q \to A$. We get $E_{\overline{q}} = \mathcal{P}_q$ because of the universal property of $\mathcal{P}_{A \times_S A'}$.

Since $\psi : X \times_S X' \to \mathcal{P}_{A \times_S A'}$ is a biextension morphism (and thus it is additive for X'), $\psi_q : X'_Q \to \mathcal{P}_q$ is a group scheme homomorphism over Q. Since $\mathcal{P}_q$ is the extension of A'_Q by $\mathbb{G}_{m,Q}$, this lifting $\widetilde{L'_q} := X'_Q \overset{\psi_q}{\to} \mathcal{P}_q$ (of $L'_Q : X'_Q \overset{v'_Q}{\to} A'_Q$) is an element of $\mathcal{E}xt^{\,1}_S(L', \mathbb{G}_{m,S})(Q)$. Since ψ is a biextension morphism (and thus it is $(\underset{1}{+})$-additive for X), $\psi_{q_1+q_2} = \psi_{q_1} \underset{1}{+} \psi_{q_2}$ for $q_1, q_2 \in \mathrm{Mor}_S(Q, X)$. Define $u(Q)(q) := \widetilde{L'_q}$ and then $u(Q) : \mathrm{Mor}_S(Q, X) \to \mathrm{Mor}_S(Q, G)$ is a homomorphism. Since $u(Q)$ is functorial for Q over S, we have a group scheme homomorphism $u : X \to G$ over S. This construction gives a smooth 1-motive (X, X'^D, A, G, u) with torsion which is denoted by $\mathfrak{C}(X, X', A, A', v, v', \psi)$.

To prove $\mathfrak{C}$ is a functor, we need to check about morphisms of smooth symmetric 1-motives. When we have a morphism $g : N_1 \to N_2$ of two smooth symmetric 1-motives where $N_i = (X_i, X'_i, A_i, A'_i, v_i, v'_i, \psi_i)$ and $g = (f_{X_1}, f_{X'_2}, f_{A_1}, f_{A'_2})$, it gives

$$\begin{tikzcd} X_1 \arrow[r, "u_1"] \arrow[d, "f_X"] & G_1 \arrow[d, "f_G"] \\ X_2 \arrow[r, "u_2"] & G_2 \end{tikzcd} \tag{2.2.22}$$

where $f_X = f_{X_1}$ and f_G is defined by the following way. Regard commutative group schemes as abelian sheaves and use the notation $L'_i = [X'_i \to A'_i]$ as in (2.2.20). In the following commutative diagram (of group schemes and morphisms)

$$\begin{tikzcd} X'_2 \arrow[r, "v'_2"] \arrow[d, "f_{X'_2}"] & A'_2 \arrow[d, "f_{A'_2}"] \\ X'_1 \arrow[r, "v'_1"] & A'_1 \end{tikzcd} \tag{2.2.23}$$

gives homomorphisms of abelian sheaves $f_{X'_2} : X'_2 \to X'_1$, $f_{A'_2} : A'_2 \to A'_1$ and a morphism of complexes $f_{L'_2} : L'_2 \to L'_1$. Therefore the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A'_2 \arrow[r] \arrow[d, "f_{A'_2}"] & L'_2 \arrow[r] \arrow[d, "f_{L'_2}"] & X'_2[1] \arrow[r] \arrow[d, "f_{X'_2}[1]"] & 0 \\ 0 \arrow[r] & A'_1 \arrow[r] & L'_1 \arrow[r] & X'_1[1] \arrow[r] & 0 \end{tikzcd} \tag{2.2.24}$$

is commutative where $C[1]$ is a left translation by degree 1 for any complex C. Applying $\mathcal{E}xt^j(\,\cdot\,, \mathbb{G}_{m,S})$ to (2.2.24), we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_1 \arrow[r] \arrow[d, "\mathcal{E}xt^0(f_{X'_2}, \mathbb{G}_{m,S})"] & \mathcal{E}xt^1(L'_1, \mathbb{G}_{m,S}) \arrow[r] \arrow[d, "\mathcal{E}xt^1(f_{L'_2}, \mathbb{G}_{m,S})"] & A_1 \arrow[r] \arrow[d, "f_{A_1}"] & 0 \\ 0 \arrow[r] & T_2 \arrow[r] & \mathcal{E}xt^1(L'_2, \mathbb{G}_{m,S}) \arrow[r] & A_2 \arrow[r] & 0 \end{tikzcd} \tag{2.2.25}$$

where T_i represents the sheaf $\mathcal{H}om\,(X'_i, \mathbb{G}_{m,S}) = \mathcal{E}xt^0(X'_i, \mathbb{G}_{m,S})$ and $A_i = \mathcal{E}xt^1(A'_i, \mathbb{G}_{m,S})$, $f_{A_1} = \mathcal{E}xt^1(f_{A'_2}, \mathbb{G}_{m,S})$ by the Weil-Barsotti formula 1.2.5. Since $\mathcal{H}om\,(A'_i, \mathbb{G}_{m,S}) = 0$ and $\mathcal{E}xt^1(X'_i, \mathbb{G}_{m,S}) = 0$ (see Proposition 1.1.7), each row of (2.2.25) is a short exact sequence. $\mathcal{E}xt^1(L_i', \mathbb{G}_{m,S})$ is represented (because of Proposition 1.1.6) by a commutative scheme G_i that we already constructed. $f_G : G_1 \to G_2$ is defined by the morphism corresponding the middle map $\mathcal{E}xt^1(f_{L_2'}, \mathbb{G}_{m,S})$ in (2.2.25).

To show that the diagram (2.2.22) commutes, we need to compare a biextension morphism induced by $\psi_1 \circ (\mathrm{id}_{X_1} \times_S \ f_{X_2'})$ and the other biextension morphism $\psi_2 \circ (f_{X_1} \times_S \ \mathrm{id}_{X_2'})$. Since g is a morphism of smooth symmetric 1-motives, they are equal and we conclude that the diagram (2.2.22) is commutative.

The functor $\mathfrak{C}$ is nontrivial ($\mathfrak{C}(0,0,A,A',0,0,0) = (0,0,A,A,0)$) and one can check that it is additive. $\square$

We need to verify that these functors $\mathfrak{B}$and $\mathfrak{C}$ is in fact an equivalence between the category of smooth 1-motives with torsion and the category of smooth symmetric 1-motives.

Proposition 2.2.6. *(Equivalence of $\mathcal{M}_{\mathrm{sm}}$ and $\mathcal{M}_{\mathrm{ssym}}$)*

The additive functor $\mathfrak{B}$ from $\mathcal{M}_{\mathrm{sm}}$ to $\mathcal{M}_{\mathrm{ssym}}$ induces an equivalence of additive categories.

Proof. By Lemma 2.2.2, we can identify a smooth 1-motive with torsion (X, T, A, G, u) with the morphism of sheaves $\mathfrak{B}_0(u) : X \to \mathcal{E}xt^1([v' : T^D \to A'], \mathbb{G}_{m,S})$, where v' is defined as in the proof of Proposition 2.2.1. In other words, the following diagram

$$\begin{tikzcd}
X \arrow[rr, equal, "\mathrm{id}_X"', "\mathfrak{B}_0(X)"] \arrow[d, "u"] & & X \arrow[d, "\mathfrak{B}_0(u) := \mathfrak{B}_0(G)\circ u"] \\
G \arrow[rr, "\cong", "\mathfrak{B}_0(G)"'] & & \mathcal{E}xt^1([v' : T^D \to A'], \mathbb{G}_{m,S})
\end{tikzcd}$$

is commutative and $\mathfrak{B}_0 : \mathcal{M}_{\mathrm{sm}} \to \mathcal{M}_{\mathrm{sm}}$ is an equivalence functor whose quasi-inverse is the inclusion $\iota_{\mathfrak{B}_0(\mathcal{M}_{\mathrm{sm}})} : \mathfrak{B}_0(\mathcal{M}_{\mathrm{sm}}) \hookrightarrow \mathcal{M}_{\mathrm{sm}}$.

For a smooth 1-motive with torsion $(X, (X')^D, A, \mathcal{E}xt^1([v' : X' \to A'], \mathbb{G}_{m,S}), u)$ in $\mathfrak{B}_0(\mathcal{M}_{\mathrm{sm}})$ and for a locally noetherian scheme Q over S, the group homomorphism

$u(Q) : \mathrm{Mor}_S(Q, X) \to (\mathcal{E}xt^1([v' : X' \to A'], \mathbb{G}_{m,S}))(Q)$ is given by $q \mapsto [\widetilde{v'_q} : X' \times_S Q \to (\mathcal{P}_{A\times_S A'}) \times_A Q]$, where the structure morphism $Q \to A$ is $v \circ q$. Now fix a point $q' \in \mathrm{Mor}_Q(Q, X' \times_S Q) = \mathrm{Mor}_S(Q, X')$. At the end of the proof of Proposition 2.2.1, $\psi(Q)(q, q')$ was defined by $(\widetilde{v'_q}(Q))(q')$ and we showed that $\psi(Q) : \mathrm{Mor}_S(Q, X \times_S X') \to \mathrm{Mor}_A(Q, (\mathcal{P}_{A\times_S A'}) \times_A Q) \hookrightarrow \mathrm{Mor}_S(Q, \mathcal{P}_{A\times_S A'})$ is a map of biextension sets which is functorial for Q. Therefore we get a biextension morphism $\mathfrak{B}_1(u) := \psi : X \times_S X' \to \mathcal{P}_{A\times_S A'}$ and we already proved that $\mathfrak{B}_1 : \mathfrak{B}_0(\mathcal{M}_{\mathrm{sm}}) \to \mathcal{M}_{\mathrm{ssym}}$ is a functor in Lemma 2.2.3.

Conversely, when $(X, X', A, A', v, v', \psi)$ is a smooth symmetric 1-motive and $q \in \mathrm{Mor}_S(Q, X)$ is a Q-valued point, the group scheme homomorphism $((\mathfrak{C}_1(\psi))(Q))(q) := \psi_q : X' \times_S Q \to (\mathcal{P}_{A\times_S A'}) \times_A Q$ gives (functorially for Q) an element in $(\mathcal{E}xt^1([v' : X' \to A'], \mathbb{G}_{m,S}))(Q)$, where the structure morphism $Q \to A$ is $v \circ q$. This means that $\mathfrak{C}_1(\psi) : X \to \mathcal{E}xt^1([v' : X' \to A'], \mathbb{G}_{m,S})$ is a group scheme homomorphism due to Lemma 2.2.2 and it is also proved that $\mathfrak{C}_1 : \mathcal{M}_{\mathrm{ssym}} \to \mathfrak{B}_0(\mathcal{M}_{\mathrm{sm}})$ is a functor in Proposition 2.2.5. We can see that $\mathfrak{B}_1$ and $\mathfrak{C}_1$ are quasi-inverses to each other and thus $\mathfrak{B} := \mathfrak{B}_1 \circ \mathfrak{B}_0$ and $\mathfrak{C} := \iota_{\mathfrak{B}_0(\mathcal{M}_{\mathrm{sm}})} \circ \mathfrak{C}_1$ are also quasi-inverses to each other. In fact, up to Cartier duality isomorphisms of commutative group schemes, $\mathfrak{B}_0$ is the same as $\mathrm{id}_{\mathcal{M}_{\mathrm{sm}}}$, and $\mathfrak{B}_1$ is the inverse functor of $\mathfrak{C}_1$. $\square$

We call $\mathfrak{B}(M)$ the **symmetric avatar** of the smooth 1-motive with torsion M and we can define a dual smooth 1-motive with torsion by using $\mathfrak{B}$ and $\mathfrak{C}$.

Definition 2.2.2. (Dual smooth 1-motives with torsion)

Let M be a smooth 1-motive with torsion over S then its Cartier dual M^* is defined by $\mathfrak{C}(\mathfrak{B}(M)^*)$.

From the definition of Cartier dual of smooth symmetric 1-motives and the equivalence of $\mathcal{M}_{\mathrm{sm}}$ and $\mathcal{M}_{\mathrm{ssym}}$, we showed the following statement.

Theorem 2.2.7. *(Cartier duality of smooth 1-motives with torsion)*

$^* : \mathcal{M}_{\mathrm{sm}} \to \mathcal{M}_{\mathrm{sm}}$ *defined by* $M \mapsto M^*$ *is a contravariant functor.* $M^{**} = (M^*)^* \cong M$ *via the composite of* $\mathfrak{B}$, $\mathfrak{C}$, *and the identity map of any* 7*-tuple in* $\mathcal{M}_{\mathrm{ssym}}$ *and thus* * *is (anti)equivalence functor.*

Proof. $^* : \mathcal{M}_{\mathrm{ssym}} \to \mathcal{M}_{\mathrm{ssym}}$ is an anti-equivalence because it is a contravariant additive functor and $N^{**} = N$. Proposition 2.2.6 means that $\mathfrak{B}$ and $\mathfrak{C}$ are contravariant additive functors and equivalences. Since M^* is defined by their composite, it is also an anti-equivalence. $\square$

Chapter 3

Smooth 1-motives with torsion over a field

In this chapter we consider the category of smooth 1-motives with torsion over a spectrum of a field. In the first section we will see one more way to define a 1-motive with torsion. See [1], [2], or [3] for details. But we also see that this category is different from the category of smooth 1-motives with torsion. Although the category of smooth 1-motives with torsion over a field is not an abelian category, there are many realization functors. In the second section we define these realizations and show the relation between them when the base field is $\mathbb{C}$.

3.1 1-motives with torsion over a field

In this section, we compare $\mathcal{M}_{\mathrm{sm}}$ with the category ${}^t\mathcal{M}^{\text{ét}}$ of (étale) 1-motives with torsion defined in [2] and developed for the last few years. To do so, all of definitions and results in this section are just repetitions of the first section of [2] and sections 2 and 4 of [3].

Recently Bertapelle revised the definition of ${}^t\mathcal{M}^{\text{ét}}$, and gave ${}^t\mathcal{M}^{\text{fl}}$ over any perfect field, allowing finite connected k-group schemes in the component of the first term of

1-motives. She also proved that ${}^t\mathcal{M}^{\mathrm{fl}}$ is an abelian category and the category $\mathcal{M}_{\mathrm{Del}}$ of Deligne's 1-motives is a full exact subcategory of ${}^t\mathcal{M}^{\mathrm{fl}}$ as in [2].

First of all, let k be a field of characteristic 0, and hence perfect.

Definition 3.1.1. (Definition 1.1.1 in [1])

A commutative group scheme L over k is called **discrete** if it is locally constant and finitely generated for the étale topology.

A **lattice** is a locally constant group scheme over k, isomorphic to a finitely generated **free abelian** group, for the étale topology. In other words, A lattice is a torsion free discrete group scheme.

Before defining 1-motives with torsion, we need to do some work.

Definition 3.1.2. (Effective étale 1-motives with torsion)

An **effective** étale 1-motive (with torsion) over k is a complex $M = [u\colon X \to G]$ of k-group schemes where X is a discrete commutative group scheme and where G is a semi-abelian variety. An **effective morphism** $M \to M'$ is a map of complexes (f, g), with $f\colon X \to X'$, $g\colon G \to G'$ morphisms of group schemes. The abelian group of effective morphisms of étale 1-motives is denoted by $\mathrm{Hom}_{\mathrm{eff,\acute{e}t}}(M, M')$ and the category of effective étale 1-motives is denoted by ${}^t\mathcal{M}_k^{\mathrm{eff,\acute{e}t}}$ and more briefly, ${}^t\mathcal{M}^{\mathrm{eff,\acute{e}t}}$.

We can show that ${}^t\mathcal{M}^{\mathrm{eff,\acute{e}t}}$ is an additive category and an effective morphism (f, g) has a cokernel by defining $\mathrm{Cok}(f, g) := [\mathrm{Cok}(f) \to \mathrm{Cok}(g)]$. Note that $\mathrm{Ker}(f)$ and $\mathrm{Cok}(f)$ are discrete commutative group schemes and $\mathrm{Cok}(g)$ is a semiabelian variety. To show the last assertion, we need to use the fact that the category of commutative group schemes of finite type over any field to be an abelian category (and for the proof, see [11], pp 212-217, corollary 7.4). When $X = X' = 0$, $f = 0$, and g is an isogeny, $\mathrm{Ker}(g)$ is no longer connected, and therefore the complex $[\mathrm{Ker}(f) \to \mathrm{Ker}(g)]$

can not be an effective étale 1-motives. To make an abelian category from ${}^t\mathcal{M}^{\mathrm{eff},\mathrm{\acute{e}t}}$, we need to think of the following morphisms.

Definition 3.1.3. (Quasi-isomorphisms of effective étale 1-motives)

An effective morphism of effective étale 1-motives $M \to M'$, here $M = [X \xrightarrow{u} G]$ and $M' = [X' \xrightarrow{u'} G']$, is a **quasi-isomorphism** of 1-motives if it yields a pull-back diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ F & \xrightarrow{u''} & F' \\ \downarrow & & \downarrow \\ X & \xrightarrow{u} & G \\ \downarrow & & \downarrow \\ X' & \xrightarrow{u'} & G' \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{3.1.1}$$

where F and F' are finite étale group schemes over k and u'' is an isomorphism.

We can make a category ${}^t\mathcal{M}^{\mathrm{\acute{e}t}}$ with the same objects as ${}^t\mathcal{M}^{\mathrm{eff},\mathrm{\acute{e}t}}$ but taking

$$\mathrm{Hom}_{{}^t\mathcal{M}^{\mathrm{\acute{e}t}}}(M, M') = \varinjlim \mathrm{Hom}_{\mathrm{eff}}(\widetilde{M}, M') \tag{3.1.2}$$

where the direct limit is taken over isogenies $\widetilde{G} \to G$, and $\widetilde{M} = [\widetilde{X} \to \widetilde{G}]$ with $\widetilde{X} = X \times_G \widetilde{G}$. To show that ${}^t\mathcal{M}^{\mathrm{\acute{e}t}}$ is really a category, we need the following result.

Proposition 3.1.1. *(Composition of morphisms)*

For any effective morphism $w : \widetilde{M} \to M'$ and any quasi-isomorphism $q' : \widetilde{M}' \to M'$, there exists a quasi-isomorphism $r : \widehat{M} \to \widetilde{M}$ together with a morphism $v : \widehat{M} \to \widetilde{M}'$ forming a commutative diagram. Furthermore, v is uniquely determined by the other morphisms and the commutativity. In particular, we have a well-defined *composition of morphisms of* 1*-motives*

$$\mathrm{Hom}(M, M') \times \mathrm{Hom}(M', M'') \to \mathrm{Hom}(M, M'').$$

Proof. See lemma 1.2 in [2]. $\square$

In other words, any morphisms $M \to M'$ and $M' \to M''$ are formally represented by wq^{-1} and $w'(q')^{-1}$ as in the following diagram and moreover we can find $\widehat{M}$, r and v making this diagram :

$$\begin{tikzcd}
 & & \widehat{M} \arrow[dl, dashed, "r"'] \arrow[dr, dashed, "v"] & & \\
 & \widetilde{M} \arrow[dl, "q"'] \arrow[dr, "w"] & & \widetilde{M'} \arrow[dl, "q'"'] \arrow[dr, "w'"] & \\
M & & M' & & M''
\end{tikzcd}$$

commutative. Therefore we have the composite of given morphisms wq^{-1} and $w'(q')^{-1}$ and it is represented by $(w' \circ v)(q \circ r)^{-1}$.

Recall that the cokernel of an effective morphism is stable under quasi-isomorphisms since any quasi-isomorphism (f, g) is surjective by its definition. Likewise, all of the axioms, except the existence of kernels, of an abelian category can be proved due to this stability under quasi-isomorphisms. So the only problem is how to define the kernel of a morphism in ${}^t\mathcal{M}^{\text{ét}}$ and we need to use one more result. Before stating it, let me give a definition.

Definition 3.1.4. (Strict morphisms)

An effective morphism $(f, g)\colon M \to M'$ is **strict** if g has (smooth) connected kernel, *i.e.*, the kernel of g is still semiabelian.

The following statement tells us how to define the kernel of a morphism of ${}^t\mathcal{M}^{\text{ét}}$ and, as a result, the relation between ${}^t\mathcal{M}^{\text{ét}}_{\mathbb{C}}$ and the category of mixed Hodge structures.

Proposition 3.1.2. *(Kernels in ${}^t\mathcal{M}^{\text{ét}}$ and the Betti realization)*

Let $w = (f, g) : M \to M'$ be an effective morphism of effective étale 1-motives. Then there exists an effective étale 1-motive $\widetilde{M'}$ and there is a quasi-isomorphism $q' : \widetilde{M'} \to M'$ such that w is lifted to an effective morphism $\widetilde{w} = (\widetilde{f}, \widetilde{g}) : M \to \widetilde{M'}$ such that $\mathrm{Ker}\left(\widetilde{g} : G \to \widetilde{G'}\right)$ is connected. In particular, ${}^t\mathcal{M}^{\text{ét}}$ is an abelian category.

If $k = \mathbb{C}$, then Betti realization $\mathrm{H}_{\mathbb{Z}}(M) := X \times_G \mathrm{Lie}\,(G)$ gives an equivalence of ${}^t\mathcal{M}^{\text{ét}} \to \mathrm{MHS}_1$, where MHS_1 is the category of mixed Hodge structures with $\mathbb{Z}$-coefficients of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$ such that $\mathrm{Gr}^W_{-1}(\mathrm{H} \otimes_{\mathbb{Z}} \mathbb{Q})$ is polarizable. For the definition of a polariztion, see definition 2.1.15 in [5].

Proof. See proposition 1.3 and 1.5 in [2]. □

The first part of the above proposition means that, for any effective morphism, if we fix M, M' and $w = (f, g)$ in the following diagram and moreover g is surjective (that is enough to decide what the kernel is), we can find a unique $\widetilde{M'}$ such that q and $\widetilde{w}$ are also determined uniquely, and the following

$$\begin{tikzcd} & \widetilde{M'} \arrow[d, dashed, "q'"] \\ \widehat{M} \arrow[ur, dashed, "\widetilde{w}"] \arrow[r, "w"] & M'. \end{tikzcd}$$

is commutative. For the proof of this uniqueness part, see proposition C.4.2 in [1], but the essential reason is $\widetilde{g}$ should be bijective when g is surjective. Even if $\mathrm{Ker}(g)$ is not connected, $\mathrm{Ker}(\widetilde{g})$ is connected, and $\mathrm{Ker}(\widetilde{w}) : \mathrm{Ker}(\widetilde{f}) \to \mathrm{Ker}(\widetilde{g})$ is an effective étale 1-motive. For the well-definedness of $\mathrm{Ker}(w) \in {}^t\mathcal{M}^{\text{ét}}$, the following commutative diagram

$$\begin{tikzcd}
 & \widehat{M'} \arrow[d, dashed, "\widehat{q}'"] \\
M \arrow[ur, dashed, "\widehat{w}"] \arrow[r, "\widetilde{w}q"] \arrow[d, "q"'] & \widetilde{M'} \arrow[d, "q'"] \\
M \arrow[ur, "\widetilde{w}"] \arrow[r, "w"] & M'.
\end{tikzcd}$$

explains why $\mathrm{Ker}(w) \cong \mathrm{Ker}(\widehat{w})$, and at last, $\mathrm{Ker}(w)$ is well defined and it is isomorphic to $\mathrm{Ker}(f) \cap u^{-1}(\mathrm{Ker}(g)^0) \to \mathrm{Ker}(g)^0$ in ${}^t\mathcal{M}^{\text{ét}}$, where $M = [X \xrightarrow{u} G]$ and where $\mathrm{Ker}(g)^0$ is the identity connected component of $\mathrm{Ker}(g)$.

Therefore, ${}^t\mathcal{M}^{\text{ét}}$ is an abelian category and the authors of [2] could prove the equvalence between ${}^t\mathcal{M}^{\text{ét}}$ and the category of any (*i.e.*, not necessarily torsion free) mixed Hodge structures with $\mathbb{Z}$-coefficients of type $\{(0,0),(0,-1),(-1,0),(-1,-1)\}$ by extending Betti realizations of Deligne's 1-motives to those of objects in ${}^t\mathcal{M}^{\text{ét}}$. This is the other part of the Proposition 3.1.2. To construct this abelian category ${}^t\mathcal{M}^{\text{ét}}$ we need the connectedness condition of G in M, but this brings one more difficulty when we try to extend Cartier dual functor. After giving one more definition, I will do that.

Definition 3.1.5. (Étale 1-motives with torsion)

The category ${}^t\mathcal{M}^{\text{ét}}$ is called the category of étale 1-motives (with torsion) and its object is called an étale 1-motive. The morphisms between two étale 1-motives M and M' is given as in (3.1.2).

Although ${}^t\mathcal{M}^{\text{ét}}$ is big enough to be abelian category, Cartier dual on the category $\mathcal{M}_{\text{Del}}$ of 1-motives can not be extended to an anti-equivalence on the category of ${}^t\mathcal{M}^{\text{ét}}$ itself. The reason is that we need to consider Cartier dual of étale group scheme, it can make the second term of its dual 1-motive not connected. Thus we need to define its dual category and this new category is called the category of **1-motives with**

cotorsion as in [1], 1.8.

Definition 3.1.6. (Effective multiplicative 1-motives with cotorsion)

Let ${}_t\mathcal{M}^{\mathrm{eff,mult}}$ be the category of complexes $M = [u\colon X \to G]$ where X is a torsion free, discrete commutative group scheme over k and G is a commutative group scheme which is an extension of an abelian scheme A by a commutative k-group Q that is product of a group scheme of multiplicative type of finite type over k. An effective morphism $w = (f, g) : M \to M'$ is a morphism of complexes and its kernel is defined by $\mathrm{Ker}(f) \to \mathrm{Ker}(g)$.

${}_t\mathcal{M}^{\mathrm{eff,mult}}$ is an additive category without cokernel, because $\mathrm{Cok}(f)$ might have torsion. As ${}^t\mathcal{M}^{\mathrm{eff,ét}}$, we can also define (dually) quasi-isomorphisms, $\mathrm{Hom}_{{}_t\mathcal{M}^{\mathrm{eff,mult}}}(M, M')$, and ${}_t\mathcal{M}^{\mathrm{mult}}$ as in Definition 3.1.3, (3.1.2), and Definition 3.1.5. All dual statements of Proposition 3.1.1 and 3.1.2 are also true. Let me list (in fact, repeat) all of these statements without proof.

Definition 3.1.7. (Quasi-isomorphisms of effective multiplicative 1-motives)

An effective morphism of effective multiplicative 1-motives $M \to M'$, here $M = [X \stackrel{u}{\to} G]$ and $M' = [X' \stackrel{u'}{\to} G']$, is a quasi-isomorphism of 1-motives if it yields a pull-back diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ X & \xrightarrow{u} & G' \\ \downarrow & & \downarrow \\ X' & \xrightarrow{u'} & G' \\ \downarrow & & \downarrow \\ F & \xrightarrow{u''} & F' \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{3.1.3}$$

where F and F' are group of multiplicative type which is finite over k and u'' is an isomorphism. Since $char(k) = 0$, they are of course étale schemes.

We can also make a category ${}_t\mathcal{M}^{\text{mult}}$ with the same objects as ${}_t\mathcal{M}^{\text{eff,mult}}$ but morphisms are

$$\mathrm{Hom}_{{}_t\mathcal{M}^{\text{mult}}}(M, M') = \varinjlim \mathrm{Hom}_{{}_t\mathcal{M}^{\text{eff,mult}}}(M, \widetilde{M}') \tag{3.1.4}$$

where the direct limit is taken over an injection $X' \to \widetilde{X}'$, and $\widetilde{M} = [\widetilde{X}' \to \widetilde{G}']$ with $\widetilde{G}'$ is a push out of $\widetilde{X}'$ and $\widetilde{G}'$.

Proposition 3.1.3. *(Composition of morphisms of ${}_t\mathcal{M}^{\text{mult}}$)*

For any effective morphism $w' : M' \to \widetilde{M}''$ and any quasi-isomorphism $q' : M' \to \widetilde{M}'$, there exists a quasi-isomorphism $r : \widetilde{M}'' \to \widehat{M}''$ together with a morphism $v : \widetilde{M}' \to \widehat{M}''$ forming a commutative diagram. Furthermore, v is uniquely determined by the other morphisms and the commutativity. A morphisms $M \to M'$ and $M' \to M''$ are formally represented by $(q')^{-1}w$ and $(q'')^{-1}w'$ as in the following diagram and we can find $\widehat{M}''$, q'' and v making the following diagram

$$\begin{tikzcd}
M \arrow[dr, "w"'] & & M' \arrow[dl, "q'"'] \arrow[dr, "w'"] & & M'' \arrow[dl, "q''"'] \\
& \widetilde{M}' \arrow[dr, dashed, "v"] & & \widetilde{M}'' \arrow[dl, dashed, "r"'] & \\
& & \widehat{M}'' & &
\end{tikzcd}$$

commutative. Therefore we have the composite of given morphisms $(q')^{-1}w$ and $(q'')^{-1}w'$ and it is represented by $(r \circ q'')^{-1}(v \circ w')$.

Since the kernel of an effective morphism is stable under quasi-isomorphisms and all of the axioms, except the existence of cokernels, of an abelian category can be proved due to this stability under quasi-isomorphisms, we only need to define kernels.

Definition 3.1.8. (Strict morphisms)

An effective morphism $(f, g)\colon M \to M'$ is strict if $\mathrm{Cok}(f)$ is **torsion free**.

Here is a dual statement of **Proposition** 3.1.2, but we can not say anything about the Betti realization in this case.

Proposition 3.1.4. *(Cokernels in ${}_t\mathcal{M}^{\mathrm{mult}}$)*

Let $w = (f, g) : M \to M'$ be an effective morphism of effective étale 1-motives. Then there exists an effective multiplicative 1-motive $\widetilde{M}$ and there is a quasi-isomorphism $q : M \to \widetilde{M}$ and there is an effective morphism $\widetilde{w} = (\widetilde{f}, \widetilde{g}) : \widetilde{M} \to M'$ such that $w = \widetilde{w} \circ q$ and $\mathrm{Cok}(\widetilde{f}$ is torsion free.

When an effective morphism $w = (f, g) : M \to M'$ is strict and f is injective (this is enough to decide what the cokernel is), if $w = \widetilde{w} \circ q$, then q is an isomorphism. Moreover, in this case, $\mathrm{Cok}(f) \to \mathrm{Cok}(g)$ is stable under taking any quasi-isomorphism and assigning the strict morphism.

Therefore for a morphism (represented by $w = (f, g)$) of ${}_t\mathcal{M}^{\mathrm{mult}}$, we can define cok$(w)$ in ${}_t\mathcal{M}^{\mathrm{mult}}$, and in particular, ${}^t\mathcal{M}^{\text{ét}}$ is an abelian category.

Though we can show ${}_t\mathcal{M}^{\mathrm{mult}}$ is an abelian category, it shoud be a dual category to ${}^t\mathcal{M}^{\text{ét}}$ and defining the Betti realization functor is not simple. But we finally get another abelian category and let me state its definition to construct the Cartier dual functor.

Definition 3.1.9. (Multiplicative 1-motives with torsion)

The category ${}_t\mathcal{M}^{\mathrm{mult}}$ is called the category of multiplicative 1-motives (with cotorsion) and its object is called an multiplicative 1-motive. The morphisms between two étale 1-motives M and M' is given as in (3.1.4).

Recall that ${}^t\mathcal{M}^{\text{eff,ét}}$ and ${}_t\mathcal{M}^{\text{eff,mult}}$ can be regarded as subcategories in $\mathcal{M}_{\mathrm{sm}}$ by restricting their morphisms preserving the fake weight filtration of $\mathcal{M}_{\mathrm{sm}}$ and then we can follow Definition 2.2.2. From this definition and Theorem 2.2.7, we know that a dual of effective étale 1-motive is effective multiplicative and a dual of effective multiplicative one is effective étale. Of course, morphisms can be sent to morphisms of duals (contravariantly) and hence taking such a dual gives two contravariant functors ${}^t\mathcal{M}^{\text{eff,ét}} \to_t \mathcal{M}^{\text{eff,mult}}$ and ${}_t\mathcal{M}^{\text{eff,mult}} \to {}^t\mathcal{M}^{\text{eff,ét}}$. But more interesting fact is that these functors preserve quasi-isomorphisms.

Proposition 3.1.5. *(Cartier dual and quasi-isomorphisms)*

By the above way, Cartier duality on $\mathcal{M}_{\text{Del}}$ *extends to contravariant additive functors* $(\ )^* \colon {}^t\mathcal{M}^{\text{eff,ét}} \to {}_t\mathcal{M}^{\text{eff,mult}}$*, and* $(\ )^* \colon {}_t\mathcal{M}^{\text{eff,mult}} \to {}^t\mathcal{M}^{\text{eff,ét}}$ *which send quasi-isomorphisms to quasi-isomorphisms.*

Proof. For $(\ )^* \colon {}^t\mathcal{M}^{\text{eff,ét}} \to {}_t\mathcal{M}^{\text{eff,mult}}$, see lemma 1.8.3 in [1] and the other statement can be shown dually. $\square$

So we can get two contravariant functors $(\ )^* \colon {}^t\mathcal{M}^{\text{ét}} \to {}_t\mathcal{M}^{\text{mult}}$, and $(\ )^* \colon {}_t\mathcal{M}^{\text{mult}} \to {}^t\mathcal{M}^{\text{ét}}$. The only problem is that we need to show that these are anti-equivalence functors. This is the very statement of proposition 1.8.4 in [1] and we can conclude the Cartier duality result.

When k is any perfect field, ${}^t\mathcal{M}^{\text{ét}}$ is not enough to be an abelian category, because there are connected finite flat commutative group schemes but not étale. Moreover, since they can be the kernel of some isogeny, we need to extend ${}^t\mathcal{M}^{\text{ét}}$ to a bigger category containing every finite flat commutative group schemes. Let $\mathcal{CE}$ be the category of commutative k-group schemes extension of a discrete group scheme by a finite flat commutative connected group scheme. Recall that such connected group schemes are flat and that the extension is automatically split. Over any perfect field, the proposition 2.14 in [16] I.2 tells us that a flat and finite commutative group scheme is the direct sum of reduced and nipotent group schemes.

Definition 3.1.10. (Effective flat 1-motives with torsion)

An effective flat 1-motive with torsion, or briefly, an effective 1-motive is a complex $M = [u\colon X \to G]$ of k-group schemes where X is an object in $\mathcal{CE}$ and where G is a semi-abelian variety. An effective morphism $M \to M'$ is a map of complexes (f, g), with $f\colon X \to X'$, $g\colon G \to G'$ morphisms of group schemes.

Denote by ${}^t\mathcal{M}^{\mathrm{eff,fl}}$ the category of effective flat 1-motives and now ${}^t\mathcal{M}^{\mathrm{eff,ét}}$ is its full subcategory. One can also check that the category $\mathcal{M}_{\mathrm{Del}}$ of Deligne's 1-motives is the full subcategory of ${}^t\mathcal{M}^{\mathrm{eff,ét}}$ consisting of those M with X torsion-free.

All terms and results for ${}^t\mathcal{M}^{\mathrm{eff,ét}}$ also work for ${}^t\mathcal{M}^{\mathrm{eff,fl}}$ by adding finite flat commutative group schemes. For instance, we can define what the quasi-isomorphism of ${}^t\mathcal{M}^{\mathrm{eff,fl}}$ is by taking $F \cong F'$ finite flat and so what ${}^t\mathcal{M}^{\mathrm{fl}}$ is. Of course, we can show ${}^t\mathcal{M}^{\mathrm{fl}}$ is an abelian category and for its proof, see theorem 2.1.9 in [3].

Similarly, we can also define the category ${}_t\mathcal{M}^{\mathrm{eff,fl}}$ of **effective flat 1-motive with cotorsion**, their quasi-isomorphisms, and ${}_t\mathcal{M}^{\mathrm{eff,fl}}$. Finally, we can extend Cartier dual functors to ${}^t\mathcal{M}^{\mathrm{eff,fl}} \to_{t\mathcal{M}^{\mathrm{eff,fl}}}$ and ${}_t\mathcal{M}^{\mathrm{eff,fl}} \to {}^t\mathcal{M}^{\mathrm{eff,fl}}$ and get anti-equivalences ${}^t\mathcal{M}^{\mathrm{fl}} \to {}_t\mathcal{M}^{\mathrm{fl}}$ and ${}_t\mathcal{M}^{\mathrm{fl}} \to {}^t\mathcal{M}^{\mathrm{fl}}$. For the proof of anti-equivalence part, see proposition 4.1.4 in [3]. Using derived categories of them, we can show that $\mathcal{D}({}^t\mathcal{M}^{\mathrm{fl}})$ and $\mathcal{D}({}_t\mathcal{M}^{\mathrm{eff,fl}})$ are equivalent and for its proof, see lemma 2.3.5 and theorem 4.1.5 in [3].

Let me finish this section by comparing $\mathcal{M}_{\mathrm{sm}}(k)$ with ${}^t\mathcal{M}^{\mathrm{fl}}(k)$. First of all, ${}^t\mathcal{M}^{\mathrm{fl}}(k)$ is an abelian category but $\mathcal{M}_{\mathrm{sm}}(k)$ is not. But to prove the above result for ${}^t\mathcal{M}^{\mathrm{fl}}(k)$, we need to use the fact that the category of commutative group schemes over Spec (k) is an abelain category. When the base scheme S is not a spectrum of some field, this category of commutative group schemes over S is no longer an abelian category, and also ${}^t\mathcal{M}^{\mathrm{fl}}_{/S}$ is not. Therefore, for the purpose to construct the category of 1-motives with torsion over any base scheme, we need to give up the condition that our category is abelian, and $\mathcal{M}_{\rm sm}(k)$ is still good in this point of view.

Second, both categories have Cartier dual functors and Cartier duality isomorphisms. But $\mathcal{M}_{\rm sm}(k)$ is closed under this dual functor, and ${}^t\mathcal{M}^{\rm fl}(k)$ is not. Considering a couple of derived categories and their equivalence functors, we can make a triangulated category containing ${}^t\mathcal{M}^{\rm fl}(k)$, closed under this extended dual functor.

Finally, looking at the proof of Proposition 3.1.1 and 3.1.3, to get an abelian category containing $\mathcal{M}_{\rm Del}$, we have to abandon torsion of one term in the complex $X \to G$. If we want to think about a 2-term complex $F_1 \to F_2$ of étale groups, we need to consider both terms, and this 2-term complex is a typical example which shows that $\mathcal{M}_{\rm sm}(k)$ is useful. Of course, ${}^t\mathcal{M}^{\rm fl}(k)$ is good enough to understand the 1-motives of algebraic varieties over k when k is algebraically closed. But if we consider more arithmetic problems, $\mathcal{M}_{\rm sm}(k)$ is also useful.

3.2 Realization functors

In this section, I will follow Deligne's definition in [6] as much as possible, to define realization functors, *i.e.*, Betti (or Hodge according to Deligne himself), étale (and l-adic when l is invertible), and de Rham realizations.

Let $S = \mathrm{Spec}\,(\mathbb{C})$ then any object in $\mathcal{GC}_S = \mathcal{GC}_{\mathrm{Spec}\,(\mathbb{C})}$ becomes a locally étale schemes. In other words, if $X \in \mathcal{GC}_{\mathrm{Spec}\,(\mathbb{C})}$, then its Cartier dual (in the sense of Definition 1.1.6) is always a group of multiplicative type over $\mathbb{C}$ due to $Char(\mathbb{C}) = 0$. Moreover, since $\mathbb{C}$ is algebraically closed, we can regard X as a finitely generated ableian group $\mathbb{Z}^n\oplus$(finite torsion part).

Assume that a commutative group scheme G over $\mathbb{C}$ is given by an extension of an abelian variety A by $T \in \mathcal{GC}^{D_{\mathrm{Spec}\,(\mathbb{C})}}_{\mathrm{Spec}\,(\mathbb{C})}$, *i.e.*, $0 \to T \to G \to A \to 0$ as in Definition 1.1.5, where T can be regarded as a group of multiplicative type over $\mathbb{C}$. In this case, $G(\mathbb{C})$ has a complex Lie group structure and we can define its exponential map $\exp_{G(\mathbb{C})} : \mathrm{Lie}(G(\mathbb{C})) \to G(\mathbb{C})$, where $\mathrm{Lie}(G(\mathbb{C}))$ is the Lie algebra of $G(\mathbb{C})$.

Now, let $M = [u : X \to G]$ be a smooth 1-motive with torsion over $\mathbb{C}$. From the above argument,we have a diagram of abelian groups

$$\begin{tikzcd} X(\mathbb{C}) \arrow[dr, "u(\mathbb{C})"] & & \mathrm{Lie}(G(\mathbb{C})) \arrow[dl, "\exp_{G(\mathbb{C})}"'] \\ & G(\mathbb{C}) \arrow[dl] \arrow[dr] & \\ \mathrm{Cok}(\exp_{G(\mathbb{C})}) & & \mathrm{Cok}(u(\mathbb{C})). \end{tikzcd} \tag{3.2.1}$$

One can easily see that $\exp_{G(\mathbb{C})} : \mathrm{Lie}(G(\mathbb{C})) \to G(\mathbb{C})^0$ is surjective where $G(\mathbb{C})^0$ is the identity connected component of $G(\mathbb{C})$. Since $G(\mathbb{C})$ has only finitely many connected components, $\mathrm{Cok}(\exp_{G(\mathbb{C})})$ is a finite abelian group and obviously a Lie group. But $\mathrm{Cok}(u(\mathbb{C}))$ is not necessarily a Lie group.

Definition 3.2.1. (Betti realizations)

Let $M = [u : X \to G]$ be a smooth 1-motive with torsion over $\mathbb{C}$ as above.

(a) The first Betti realization is defined by (as 10.1.10 in [6])

$$\begin{aligned} H^1_{\mathbb{Z}}(M) &:= X \times_{G(\mathbb{C})} \mathrm{Lie}(G(\mathbb{C})) \\ &:= \{(x, v) \in X \times \mathrm{Lie}(G(\mathbb{C})) | u(\mathbb{C})(x) = \exp_{G(\mathbb{C})}(v)\} \end{aligned}$$

(b) The second Betti realization is defined by

$$\begin{aligned} H^2_{\mathbb{Z}}(M) &:= \mathrm{Cok}(u(\mathbb{C})) +_{G(\mathbb{C})} \mathrm{Cok}(\exp_{G(\mathbb{C})}) \\ &:= \Big(\mathrm{Cok}(u(\mathbb{C})) \times \mathrm{Cok}(\exp_{G(\mathbb{C})})\Big)\Big/ f(G) \end{aligned}$$

where $f : G \to \mathrm{Cok}(u(\mathbb{C})) \times \mathrm{Cok}(\exp_{G(\mathbb{C})})$ is defined by

$$f(g) = \Big(g + u(X), g + \exp_{G(\mathbb{C})}(\mathrm{Lie}(G(\mathbb{C}))\Big).$$

In other words, $H^1_{\mathbb{Z}}(M)$ is the **fibred product** of the top part of (3.2.1) and $H^2_{\mathbb{Z}}(M)$ is the **fibred sum** of the bottom part of (3.2.1). We can regard $H^2_{\mathbb{Z}}(M)$ as the double quotient $u(\mathbb{C})(X(\mathbb{C})) \setminus G(\mathbb{C}) / \exp_{G(\mathbb{C})}(\mathrm{Lie}(G(\mathbb{C}))$.

Assume that there is a short exact sequence in $\mathcal{M}_{\mathrm{sm}}(\mathbb{C})$

$$0 \longrightarrow M_1 \xrightarrow{h} M_2 \xrightarrow{k} M_3 \longrightarrow 0$$

i.e., the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X_1 \arrow[r, "h_1"] \arrow[d, "u_1"] & X_2 \arrow[r, "k_1"] \arrow[d, "u_2"] & X_3 \arrow[r] \arrow[d, "u_3"] & 0 \\ 0 \arrow[r] & G_1 \arrow[r, "h_2"] & G_2 \arrow[r, "k_2"] & G_3 \arrow[r] & 0 \end{tikzcd} \tag{3.2.2}$$

where horizontal sequences are exact in the sense of (1.1.6).

Proposition 3.2.1. *(Snake Lemma for Betti realizations)*

There is a connecting homomorphism $d : H^1_{\mathbb{Z}}(M_3) \to H^2_{\mathbb{Z}}(M_1)$ which makes a long exact sequence from (3.2.2)

$$\begin{array}{cccccccc} 0 & \longrightarrow & H^1_{\mathbb{Z}}(M_1) & \xrightarrow{h^1_{\mathbb{Z}}} & H^1_{\mathbb{Z}}(M_2) & \xrightarrow{k^1_{\mathbb{Z}}} & H^1_{\mathbb{Z}}(M_3) & \\ & \xrightarrow{d} & H^2_{\mathbb{Z}}(M_1) & \xrightarrow{h^2_{\mathbb{Z}}} & H^2_{\mathbb{Z}}(M_2) & \xrightarrow{k^2_{\mathbb{Z}}} & H^2_{\mathbb{Z}}(M_3) & \longrightarrow 0. \end{array}$$

Proof. For simplicity, only in this proof, let me use the following notations: X_i means $X_i(\mathbb{C})$, G_i means $G_i(\mathbb{C})$, $\mathrm{Lie}(G_i)$ means $\mathrm{Lie}(G_i(\mathbb{C}))$, u_i means $u_i(\mathbb{C})$ and $\exp_i$ means $\exp_{G_i(\mathbb{C})}$.

From the second sequence in (3.2.2) we can get a short exact sequence

$$0 \longrightarrow \mathrm{Lie}\,(G_1) \xrightarrow{h_3} \mathrm{Lie}\,(G_2) \xrightarrow{k_3} \mathrm{Lie}\,(G_3) \longrightarrow 0$$

and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}\,(G_1) \arrow[r, "h_3"] \arrow[d, "\exp_1"] & \mathrm{Lie}\,(G_2) \arrow[r, "k_3"] \arrow[d, "\exp_2"] & \mathrm{Lie}\,(G_3) \arrow[r] \arrow[d, "\exp_3"] & 0 \\ 0 \arrow[r] & G_1 \arrow[r, "h_2"] & G_2 \arrow[r, "k_2"] & G_3 \arrow[r] & 0 \end{tikzcd} \tag{3.2.3}$$

where the map $h^1_{\mathbb{Z}} : H^1_{\mathbb{Z}}(M_1) \to H^1_{\mathbb{Z}}(M_2)$ is defined by $(h_1, h_3) \mid_{H_{\mathbb{Z}}(M_1)}$ and obviously injective by the injectivity of h_1 and k_1.

$k^1_{\mathbb{Z}} : H_{\mathbb{Z}}(M_2) \longrightarrow H_{\mathbb{Z}}(M_3)$ is also defined by $(k_1, k_3) \mid_{H_{\mathbb{Z}}(M_2)}$ and because of $k_i \circ h_i = 0$ we get $k^1_{\mathbb{Z}} \circ h^1_{\mathbb{Z}} = 0$. If $(x_2, g'_2) \in \mathrm{Ker}(k^1_{\mathbb{Z}})$ then $k^1_{\mathbb{Z}}(x_2, g'_2) = (k_1(x_2), k_3(g'_2)) = (0, 0)$ tells that the existence of $(x_1, g'_1) \in X_1 \times \mathrm{Lie}\,(G_1)$ such that $h_1(x_1) = x_2$ and $h_3(g'_1) = g'_2$. By the commutativity of (3.2.2) and (3.2.3) this (x_1, g'_1) is contained in $H^1_{\mathbb{Z}}(M_1)$ and this shows the exactness of the $H^1_{\mathbb{Z}}(M_2)$ part.

Now let's take $(x_3, g'_3) \in H^1_{\mathbb{Z}}(M_3)$ to define $d : H^1_{\mathbb{Z}}(M_3) \to H^2_{\mathbb{Z}}(M_1)$. We can choose $(x_2, g'_2) \in X_2 \times \mathrm{Lie}\,(G_2)$ such that $k_1(x_2) = x_3$ and $k_3(g'_2) = g'_3$ up to the elements in $X_1 \times \mathrm{Lie}\,(G_1)$ from the exactness of (3.2.2) and (3.2.3). Because of the commutativity of (3.2.2) and (3.2.3), $k_2(u_2(x_2)) = k_2(\exp_2(g'_2))$ that is

$$g_2 = (u_2(x_2) - \exp_2(g'_2)) \in \mathrm{Ker}(k_2) \tag{3.2.4}$$

and we can find only one $g_1 \in G_1$ whose image under h_2 is g_2. This g_1 determines one element $\overline{g_1}$ in $H^2_{\mathbb{Z}}(M_1)$, the double quotient of G_1 by $u_1(X_1)$ and $\exp_1(\mathrm{Lie}\,(G_1))$. We define $d(x_3, g'_3) = \overline{g_1}$. For the well definedness, if we take another representative $(v_2, e'_2) \in X_2 \times \mathrm{Lie}\,(G_2)$ the difference

$$\begin{aligned}
g_2 - e_2 &= (u_2(x_2) - \exp_2(g'_2)) - (u_2(v_2) - \exp_2(e'_2)) \\
&= u_2(x_2 - v_2) - \exp_2(g'_2 - e'_2) \\
&= u_2(h_1(w_1)) - \exp_2(h_3(f_1)) \\
&= h_2(u_1(w_1)) - h_2(\exp_1(f_1)) \\
&= h_2(u_1(w_1) - \exp_1(f_1))
\end{aligned}$$

is in the image of the the subgroup of G_1 generated by $u_1(X_1)$ and $\exp_1(\mathrm{Lie}\,(G_1)$ where $(w_1, f_1) \in X_1 \times \mathrm{Lie}\,(G_1)$ is a unique element that goes to $(x_2 - v_2, g'_2 - e'_2)$. Therefore this map d is well defined.

Since $(x_2, g_2') \in H^1_{\mathbb{Z}}(M_2)$ means $u_2(x_2) = \exp_2(g_2') \in G_2$ we can get

$$\begin{aligned} d(k^1_{\mathbb{Z}}(x_2, g_2')) &= d(k_1(x_2), k_3(g_2')) \\ &= h_2^{-1}(u_2(x_2) - \exp_2(g_2')) \\ &= h_2^{-1}(0) \\ &= 0 \end{aligned}$$

i.e., $d \circ k^1_{\mathbb{Z}} = 0$. If $(x_3, g_3') \in \mathrm{Ker}(d)$ we can pick $(x_2, g_2') \in X_2 \times \mathrm{Lie}\,(G_2)$ such that

$$h_2^{-1}(u_2(x_2) - \exp_2(g_2')) = g_1 = u_1(x_1) - \exp_1(g_1')$$

as in (3.2.4). From this equality we can find

$$u_2(x_2) - h_2(u_1(x_1)) = \exp_2(g_2') - h_2(\exp_1(g_1'))$$

and

$$u_2(x_2 - h_1(x_1)) = \exp_2(g_2' - h_1(g_1')) \in G_2$$

where $k_1(x_2 - h_1(x_1)) = x_3$ and $k_3(g_2' - h_1(g_1')) = g_3'$. Hence $(x_3, g_3') \in \mathrm{Im}(k^1_{\mathbb{Z}})$ and this proves the exactness at $H^1_{\mathbb{Z}}(M_3)$. Since $h^2_{\mathbb{Z}}(g_1) = h_2(g_1) \in H^2_{\mathbb{Z}}(M_2)$, for any $(x_3, g_3') \in H^1_{\mathbb{Z}}(M_3)$

$$h^2_{\mathbb{Z}}(d((x_3, g_3'))) = h^2_{\mathbb{Z}}(g_1) = \overline{h_2(g_1)} = \overline{u_2(x_2) - \exp_2(g_2') = 0}$$

by definition of d and $H^2_{\mathbb{Z}}(M_2)$ and thus $h^2_{\mathbb{Z}} \circ d = 0$. Let's assume that $\overline{g_1} \in \mathrm{Ker}(h^2_{\mathbb{Z}})$. This implies $h_2(g_1) = u_2(x_2) - \exp_2(g_2')$ which is contained in the subgroup of G_2 generated by $u_2(X_2)$ and $\exp_2(\mathrm{Lie}\,(G_2))$. So if we take $(x_3, g_3') = (k_1(x_2), k_3(g_2'))$ then it's clearly in $H^1_{\mathbb{Z}}(M_3)$ and $d(x_3, g_3') = \overline{g_1}$ by the construction of the map d. Hence we have checked

$$H^1_{\mathbb{Z}}(M_3) \longrightarrow H^2_{\mathbb{Z}}(M_1) \longrightarrow H^2_{\mathbb{Z}}(M_2)$$

is exact.

Because $G_1 \xrightarrow{h} G_2 \xrightarrow{k} G_3$ is exact $k^2_{\mathbb{Z}} \circ h^2_{\mathbb{Z}} = 0$ is clear. Suppose that $\overline{g_2} \in \mathrm{Ker}(k^2_{\mathbb{Z}})$, in other words

$$k_2(g_2) = u_3(x_3) - \exp_3(g'_3)$$

for some $x_3 \in X_3$ and $g'_3 \in \mathrm{Lie}\,(G_3)$. There are x_2 and g'_2 so that $k_1(x_2) = x_3$ and $k_3(g'_2) = g'_3$ and moreover

$$\begin{aligned} & k_2(g_2 - (u_2(x_2) - \exp_2(g'_2))) \\ = \;& k_2(g_2) - k_2(u_2(x_2)) + k_2(\exp_2(g'_2)) \\ = \;& u_3(x_3) - \exp_3(g'_3) - u_3(k_1(x_2)) + \exp_3(k_3(g'_2)) \\ = \;& 0. \end{aligned}$$

Since $\overline{g_2 - (u_2(x_2) - \exp_2(g'_2))} = \overline{g_2}$ we can choose $g_1 \in G_1$ such that

$$h_2(g_1) = g_2 - (u_2(x_2) - \exp_2(g'_2))$$

i.e. $h^2_{\mathbb{Z}}(g_1) = h_2(g_1) = g_2$ and we showed the exactness at $H^2_{\mathbb{Z}}(M_2)$. Finally $k^2_{\mathbb{Z}}$ is surjective because k_2 is surjective and $k^2_{\mathbb{Z}}$ is well defined. This finishes our proof. $\square$

Let k be any field and $S = \mathrm{Spec}\,(k)$, then as in 10.1.5 of [6] we identify $M \in \mathcal{M}_{\mathrm{sm}}(S)$ to the complex of S-groups (and so sheaves) $[X \xrightarrow{u} G]$ of degree 0 and 1 and for an integer n let $[\mathbb{Z} \xrightarrow{n} \mathbb{Z}]$ be the complex of degree -1 and 0.

Definition 3.2.2. (Étale realizations)

Let K be the complex $[X \xrightarrow{u} G] \otimes [\mathbb{Z} \xrightarrow{n} \mathbb{Z}]$ of degree -1, 0, and 1. K can be identified with $M \overset{\mathbb{L}}{\otimes} \mathbb{Z}/n\mathbb{Z}$ in the derived category of sheaves over S. Define

$$\begin{aligned} H^0_{\mathbb{Z}/n\mathbb{Z}(M)} &:= h^{-1}(K) \\ H^1_{\mathbb{Z}/n\mathbb{Z}(M)} &:= h^{0}(K) \\ H^2_{\mathbb{Z}/n\mathbb{Z}(M)} &:= h^{1}(K) \end{aligned}$$

where $h^i(K)$ is the i-th cohomological object of K.

Since $h^{-1}(K)$ is just a kernel of $(n,u): X \to X \oplus G$, it is a subgroup scheme of X and obviously a group scheme. Let me recall that the category of commutative group schemes of finite type over k is an abelian category, and therefore $h^0(K)$ can be represented by a commutative group scheme, because $n: G \to \mathrm{Im}(n) = n(G)$ is an isogeny of group schemes of finite type. $h^1(K)$ is also represented by some group scheme, since $\mathrm{Cok}(n)$ is a finite group scheme over k.

For $n = md$, we can define transition morphisms

$$\begin{aligned} \phi^0_{m,n} &: H^0_{\mathbb{Z}/n\mathbb{Z}(M)} \longrightarrow H^0_{\mathbb{Z}/m\mathbb{Z}(M)} \\ \phi^1_{m,n} &: H^1_{\mathbb{Z}/n\mathbb{Z}(M)} \longrightarrow H^0_{\mathbb{Z}/m\mathbb{Z}(M)} \\ \phi^2_{m,n} &: H^2_{\mathbb{Z}/n\mathbb{Z}(M)} \longrightarrow H^2_{\mathbb{Z}/m\mathbb{Z}(M)} \end{aligned} \tag{3.2.5}$$

which are induced from

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "(m,u)"] \arrow[d, "id"] & X \oplus G \arrow[r, "u-m"] \arrow[d, "(d,id)"] & G \arrow[r] \arrow[d, "d"] & 0\,. \\ 0 \arrow[r] & X \arrow[r, "(n,u)"] & X \oplus G \arrow[r, "u-n"] & G \arrow[r] & 0 \end{tikzcd}$$

So for any prime number l we get projective systems of $H^0_{\mathbb{Z}/l^n\mathbb{Z}}(M)$, $H^1_{\mathbb{Z}/l^n\mathbb{Z}}(M)$, $H^2_{\mathbb{Z}/l^n\mathbb{Z}}(M)$. Moreover l is of course invertible over $S = \mathrm{Spec}\,(k)$, and we can have the limits of these projective systems as $\mathbb{Z}_l$-sheaves over k.

Definition 3.2.3. (l-adic realizations)

Let $M \in \mathcal{M}_{\mathrm{sm}}(k)$, then its l-adic realizations are given by

$$\begin{aligned} H^0_l(M) &:= \varprojlim H^0_{\mathbb{Z}/l^n\mathbb{Z}}(M) \\ H^1_l(M) &:= \varprojlim H^1_{\mathbb{Z}/l^n\mathbb{Z}}(M) \\ H^2_l(M) &:= \varprojlim H^2_{\mathbb{Z}/l^n\mathbb{Z}}(M), \end{aligned}$$

where these projective systems are given in (3.2.5).

When we consider $M \in \mathcal{M}_{\mathrm{sm}}(\mathbb{C})$, we use the notations in the proof of Proposition 3.2.1 and additionally, $H^0_{\mathbb{Z}/n\mathbb{Z}}(M)$, $H^1_{\mathbb{Z}/n\mathbb{Z}}(M)$, and $H^2_{\mathbb{Z}/n\mathbb{Z}}(M)$ to denote the sets of complex points of them.

Proposition 3.2.2. *(Comparison between Betti and étale)*

We have a long exact sequence

$$\begin{aligned} 0 \to H^0_{\mathbb{Z}/n\mathbb{Z}}(M) \quad &\to H^1_{\mathbb{Z}}(M) \xrightarrow{n} H^1_{\mathbb{Z}}(M) \to H^1_{\mathbb{Z}/n\mathbb{Z}}(M) \\ &\to H^2_{\mathbb{Z}}(M) \xrightarrow{n} H^2_{\mathbb{Z}}(M) \to H^2_{\mathbb{Z}/n\mathbb{Z}}(M) \quad \to 0. \end{aligned} \tag{3.2.6}$$

In particular $H^2_{\mathbb{Z}/n\mathbb{Z}}(M) \cong H^2_{\mathbb{Z}}(M)/n(H^2_{\mathbb{Z}}(M))$ and moreover if

$$n : H^2_{\mathbb{Z}}(M) \to H^2_{\mathbb{Z}}(M)$$

is injective then it's also true that $H^1_{\mathbb{Z}/n\mathbb{Z}}(M) \cong H^1_{\mathbb{Z}}(M)/n(H^1_{\mathbb{Z}}(M))$.

Proof. By definition $H^0_{\mathbb{Z}/n\mathbb{Z}}(M) = \mathrm{Ker}(u) \cap \mathrm{Ker}(n) \subset \mathrm{Ker}(u)$ and we can view this as a subset of $H^1_{\mathbb{Z}}(M) \cap \mathrm{Ker}(u)$. Thus the first map exists and injective.

Because $H^0_{\mathbb{Z}/n\mathbb{Z}(M)} \subset \mathrm{Ker}(n)$ the composite of $n : H^1_{\mathbb{Z}}(M) \to H^1_{\mathbb{Z}}(M)$ and the above map is trivial. Moreover $n : \mathrm{Lie}\,(G) \to \mathrm{Lie}\,(G)$ is always injective and this means $(x, g') \in \mathrm{Ker}(n : H^1_{\mathbb{Z}}(M) \to H^1_{\mathbb{Z}}(M))$ should be $(x, 0)$ and $x \in \mathrm{Ker}(u) \cap \mathrm{Ker}(n) = H^0_{\mathbb{Z}/n\mathbb{Z}}(M)$. This shows the exactness at the first $H^1_{\mathbb{Z}}(M)$.

Let me define $f^1 : H^1_{\mathbb{Z}}(M) \to H^1_{\mathbb{Z}/n\mathbb{Z}}(M)$ by

$$f^1(x, g') := \overline{\left(\left(x, \exp\frac{g'}{n}\right)\right)} = \left(x, \exp\left(\frac{g'}{n}\right)\right) + \{(nx, u(x)) | x \in X\},$$

then we have

$$f^1 \circ n(x, g') = f^1(nx, ng') = \overline{\left(nx, \exp\left(\frac{ng'}{n}\right)\right)} = \overline{(nx, \exp(g'))} = 0$$

from the construction of $H^1_{\mathbb{Z}}(M)$. Conversely if $(x, g) \in \mathrm{Ker}(f^1)$ there is $x' \in X$ such that $x = nx'$ and $\exp(g'/n) = u(x')$. Thus $(x', g'/n) \in H^1_{\mathbb{Z}}(M)$ and $(x, g') = n(x', g'/n)$ i.e. this part is exact.

Now if we define $\partial : H^1_{\mathbb{Z}/n\mathbb{Z}}(M) \to H^2_{\mathbb{Z}}(M)$ by

$$\partial(x,g) := \overline{g} = g + (u(X) + \exp(\mathrm{Lie}\,(G))$$

then this map is obviously well defined and

$$\partial \circ f^1(x,g') = \partial\overline{\left(x, \exp\left(\frac{g'}{n}\right)\right)} = \overline{\exp\left(\frac{g'}{n}\right)} = 0.$$

So $\mathrm{Im}(f^1) \subset \mathrm{Ker}(\partial)$. Suppose $(x,g) \in \mathrm{Ker}(\partial)$ then we get $g \in (u(X) + \exp(\mathrm{Lie}\,(G))$ i.e. there are $x' \in X$ and $g' \in \mathrm{Lie}\,(G)$ such that $g = u(x') + \exp(g')$. By definition of $H^1_{\mathbb{Z}/n\mathbb{Z}}(M)$, of course $u(x) - ng = 0$, and we have

$$u(x) = ng = nu(x') + n\exp(g') = u(nx') + \exp(ng')$$

That is $(x - nx', ng') \in H^1_{\mathbb{Z}}(M)$ and its image under f^1 is contained in the same class as (x,g). This shows $\mathrm{Ker}(\partial) \subset \mathrm{Im}(f^1)$.

For any $(x,g) \in H^1_{\mathbb{Z}/n\mathbb{Z}}(M)$ we can find $n \circ \partial(x,g) = n(\overline{g}) = \overline{ng}$ and $ng = u(x)$. Thus $n \circ \partial\overline{(x,g)} = \overline{u(x)} = 0$ and this means $n \circ \partial = 0$. If $n(\overline{g}) = 0$ there is a pair (x,g') so that $ng = u(x) + \exp(g')$. Hence

$$\overline{\left(x, g - \exp\left(\frac{g'}{n}\right)\right)} \in H^1_{\mathbb{Z}/n\mathbb{Z}}(M)$$

and

$$\overline{g} = \overline{\left(x, g - \exp\left(\frac{g'}{n}\right)\right)}$$

Thus we have $\mathrm{Ker}(\partial) = \mathrm{Im}(f^1)$.

Finally define $f^2 : H^2_{\mathbb{Z}}(M) \to H^2_{\mathbb{Z}/n\mathbb{Z}}(M)$ by

$$f^2(g) := f^2(g + (u(X) + \exp(\mathrm{Lie}\,(G))) = \overline{\overline{g}} = g + (u(X) + nG)$$

This map is well defined because the multiplication by n is an isogeny and its restriction to the identity component of G is surjective. By this definition $f^2 \circ n = 0$ and f^2 is clearly surjective. Now suppose $f^2(g) = 0$. That is $g = u(x) + ng_1$ for some $x \in X$ and $g_1 \in G$. Thus $g = ng_1 = ng_1$ and we can conclude the exactness of (3.2.6). $\square$

To make de Rham realizations as in 10.1.7 of [6], let k be a field of charateristic 0. In this case, for any multiplicative group scheme T,

$$\mathcal{E}xt^{1}(T, \mathbb{G}_a) = 0. \tag{3.2.7}$$

The short exact sequence of $\mathcal{M}_{\mathrm{sm}}(k)$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] & G \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

gives the long exact sequence of sheaves

$$\to \mathcal{E}xt^{i-1}(X, \mathbb{G}_a) \to \mathcal{E}xt^{i}(M, \mathbb{G}_a) \to \mathcal{E}xt^{i}(G, \mathbb{G}_a) \to \mathcal{E}xt^{i}(X, \mathbb{G}_a) \to$$

and because $\mathcal{H}om\,(G, \mathbb{G}_a) = \mathcal{E}xt\,(X, \mathbb{G}_a) = 0$ we have

$$0 \to \mathcal{H}om\,(X, \mathbb{G}_a) \to \mathcal{E}xt^{1}(M, \mathbb{G}_a) \to \mathcal{E}xt^{1}(G, \mathbb{G}_a) \to 0.$$

First of all, $\mathcal{H}om\,(X, \mathbb{G}_a)$ is represented by a vector group over k (and so by an affine group scheme of finite type). Since $\mathcal{E}xt^{1}(G, \mathbb{G}_a) \cong \mathcal{E}xt^{1}(A, \mathbb{G}_a)$ is represented by Lie(A), $\mathcal{E}xt^{1}(M, \mathbb{G}_a)$ is also represented by a commutative group scheme. We can also get $\mathcal{H}om\,(T, \mathbb{G}_a) = 0$ and hence according to (10.1.7) in [6] there is a universal extension $M^{\natural} = [X \to G^{\natural}]$ of M by a vector group

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}xt^{1}(M, \mathbb{G}_a)^{*} \arrow[r] & G^{\natural} \arrow[r] & G \arrow[r] & 0,
\end{tikzcd}$$

where $\mathcal{E}xt^{1}(M, \mathbb{G}_a)^{*} := \mathcal{H}om_{\,\mathcal{O}_S}(\mathcal{E}xt^{1}(M, \mathbb{G}_a), \mathcal{O}_S)$.

Definition 3.2.4. (De Rham realizations)

Let $M \in \mathcal{M}_{\mathrm{sm}}(k)$ and let $Char(k) = 0$. Its de Rham realization is given by

$$H^{1}_{dR}(M) := \mathrm{Lie}\,(G^{\natural}).$$

Its Hodge filtration is defined by

$$\mathrm{F}^0 T_{dR}(M) = \mathrm{Ker}(\mathrm{Lie}\,(G^\natural) \to \mathrm{Lie}\,(G)) \cong \mathcal{E}xt^{\,1}(M, \mathbb{G}_a)^*$$

and its weight filtration is given by

$$\begin{aligned} W_{-2} H^1_{dR}(M) &= \mathrm{Lie}\,(T) \\ W_{-1} H^1_{dR}(M) &= \mathrm{Lie}\,(E(G)) \\ W_0\ \ H^1_{dR}(M) &= H^1_{dR}(M) \end{aligned}$$

where $E(G)$ is the universal extension of G itself (instead of M).

Now we can get a generalization of (10.1.8) in [6].

Proposition 3.2.3. *(Comparison between de Rham and Betti)*

If $M = [X \to G] \in \mathcal{M}_{\mathrm{sm}}(\mathbb{C})$, *then*

$$(H^1_{dR}(M), F, W) \cong (H^1_{\mathbb{Z}}(M) \otimes \mathbb{C}, F, W).$$

Proof. In fact, this proof is the same as (10.1.8) in [6].

For $i = 0$ or 1, $\mathrm{Ext}^i((X, \mathbb{G}_a)$, $\mathrm{Ext}^i((A, \mathbb{G}_a)$, and $\mathrm{Ext}^i((T, \mathbb{G}_a)$ are the same in the algebraic and analytic categories. Taking the composite of $H^1_{\mathbb{Z}}(M) \otimes \mathbb{C} \to \mathrm{Lie}(G(\mathbb{C}))$ and $\exp_{G(\mathbb{C})}$, we get a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & (H^1_{\mathbb{Z}}(M) \otimes \mathbb{C})/H_1(G) \arrow[d] \\ X \arrow[r, "u"] & G \end{tikzcd}$$

giving an extension of $M \in \mathcal{M}^{\mathrm{an}}_{\mathrm{tor}}(\mathbb{C})$ by the vector group $\mathrm{F}^0 T_{dR}(M)$, and therefore an extension of $M \in \mathcal{M}_{\mathrm{sm}}(\mathbb{C})$ by this group. To show that this extension is the universal extension, it is enough to prove that the category of extensions of $X \to (H^1_{\mathbb{Z}}(M) \otimes \mathbb{C})/H_1(G)$ by $\mathbb{G}_a$ is trivial and unique up to automorphisms. This is also the category of extensions of $(H^1_{\mathbb{Z}}(M)\otimes\mathbb{C})/H^1_{\mathbb{Z}}(M)$ by $\mathbb{G}_a$, and one gets $\mathrm{Ext}((H^1_{\mathbb{Z}}(M)\otimes\mathbb{C})/H^1_{\mathbb{Z}}(M), \mathbb{G}_a) = 0$ for $i = 0$ or 1. $\square$

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