Closure and complete integrability in Burgers turbulence

by

Ravi Srinivasan

B. S., Worcester Polytechnic Institute, 2004

Sc. M., Brown University, 2005

A Dissertation submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Division of Applied Mathematics at Brown University

Providence, Rhode Island

May 2009

This dissertation by Ravi Srinivasan is accepted in its present form
by the Division of Applied Mathematics as satisfying the
dissertation requirement for the degree of Doctor of Philosophy.

Date ____________ ______________________________

Prof. Govind Menon, Director

Recommended to the Graduate Council

Date ____________ ______________________________

Prof. Govind Menon, Reader

Date ____________ ______________________________

Prof. Constantine Dafermos, Reader

Date ____________ ______________________________

Prof. Boris Rozovsky, Reader

Approved by the Graduate Council

Date ____________ ______________________________

Sheila Bonde, Dean of the Graduate School

Curriculum Vitæ

Ravi Srinivasan was born in Framingham, Massachusetts, USA on December 10, 1982. He received a Bachelor of Science degree (with high distinction) in Mathematical Sciences from Worcester Polytechnic Institute in 2004. In September of 2004 he began his doctoral studies in Applied Mathematics at Brown University, where he received a Masters of Science degree in May 2005. While at Brown he has worked under the supervision of Professor Govind Menon. From June to August 2005 he was a Geophysical Fluid Dynamics Fellow at the Woods Hole Oceanographic Institution.

In September 2009 he will join the Department of Mathematics at the University of Texas at Austin as a postdoctoral scholar.

Acknowledgements

I would first like to thank my thesis advisor, Govind Menon, for his patience and conscientiousness as a mentor. He has not only introduced me to a vast array of interesting mathematics, but has taught me how to become a better scientist in the process. I am indebted to him for his time and effort. I also give special thanks to Constantine Dafermos and Boris Rozovsky for their support during my years at Brown, and for serving as my thesis readers.

During the summer of 2005 I had the pleasure of participating in the Geophysical Fluid Dynamics program at Woods Hole, which would not have happened without the help of Rick Salmon of the Scripps Institute of Oceanography at UCSD. I am thankful to him, and to Eric Vanden-Eijnden of the Courant Institute for his mentorship that summer.

I also wish to acknowledge my undergraduate mentors at WPI, whose guidance and encouragement during my early years in mathematics motivated me to continue my studies in graduate school. In particular I thank Christopher Larsen, Suzanne Weekes, Bogdan Vernescu, and Joseph Fehribach.

I would like to express my heartfelt appreciation to my friends and family, all of whom I cannot possibly acknowledge in such a short space. I am grateful to Ruchi for her care and for many shared laughs during my time at Brown. Finally, I especially thank Vani, Prema, and Baba for their love and selfless support throughout all my endeavors. This work is dedicated to them.

Contents

List of Figures

CHAPTER 1

Introduction

The development of a statistical theory of fluid turbulence has been a long sought-after goal of modern physics. As a subject fraught with difficulties, simplified models have played a important role in isolating essential features or motivating the discovery of new mathematical tools. Often, they have proven useful for a broader class of problems in statistical physics. Our work focuses on one such model—Burgers turbulence—the study of Burgers equation

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = \nu \partial_{xx} u \tag{1.1}$$

with random initial data or forcing.

Burgers proposed his model in the 1940s to study the critical interaction of dissipative and nonlinear inertial terms in hydrodynamic turbulence [**12**]. Although (1.1) was already known to have a closed form solution by the Hopf-Cole transformation, the statistical problem associated with it was, and remains, highly nontrivial. Of particular importance is the vanishing viscosity limit ($\nu \to 0$) in which the dynamics reduce to a system of interacting shocks. Within this regime and using completely uncorrelated (white noise) data to mimic a fully turbulent initial state, Burgers made fundamental contributions to the determination of the solution statistics. As we discuss below, his work established the first steps of an effort that eventually led to the complete solution for the statistics by Groeneboom [**31**] and Frachebourg and Martin [**24**].

Burgers turbulence has limited applicability in understanding fluid turbulence because it lacks many of its essential features, particularly vortex motion and stretching. It has, however, proven useful as a phenomenological model for cluster growth and has seen a resurgence of physical and mathematical interest over the past two decades (see [**5**] for a broad review). For example, it has been studied in the context of interface deposition and domain coarsening in materials science [**36, 37**], the formation of large-scale cosmological structures in the universe [**33, 50**], among others. It also has well-established connections to the probabilistic literature, most notably with the standard additive coalescent [**1, 11**] and universality in nonparametric statistics [**17, 29, 30, 31**].

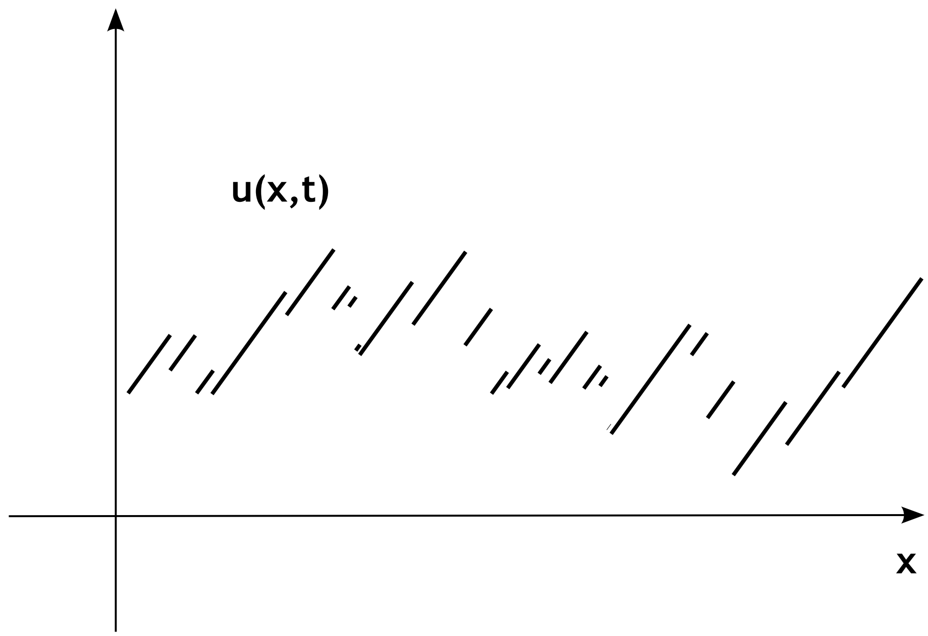

Figure 1.1. Typical sawtooth profile of a solution $u(x,t)$ to the inviscid Burgers equation. The ramp-like structures have slope t^{-1} and are interspersed with shocks that undergo ballistic aggregation.

Burgers turbulence and ballistic aggregation. For an excellent overview of Burgers equation and its associated statistical problem we direct the reader to Bertoin's article in [**4**]. We start with the inviscid Burgers equation

$$\begin{aligned} \partial_t u + \partial_x \left(\tfrac{1}{2}u^2\right) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \quad . \tag{1.2}$$

The velocity profile $u_0(x)$ describes the initial velocity of an infinitesimal particle started at x. Generically, the nonlinearity causes smooth initial data to develop shocks so (1.2) must be interpreted in a weak sense in order to yield a global solution.

The physically relevant weak solution can be obtained by considering the explicit solution $u^\nu(x,t)$ to (1.1) and taking the limit $\nu \to 0$. This yields the unique entropy solution

$$u(x,t) = \frac{x - a(x,t)}{t}, \tag{1.3}$$

given in terms of the inverse Lagrangian function

$$a(x,t) = \arg^+ \min_{s\in\mathbb{R}} \left\{ \int_0^s \left(u_0(r) - \frac{x-r}{t} \right) dr \right\} \tag{1.4}$$

where $\arg^+ \min$is the largest argument that minimizes the bracketed expression. $a(x,t)$ is nondecreasing in x and describes the initial position of the infinitesimal particle arriving at x at time t. It can be recovered by a simple visual trick: $a(x,t)$ is the last argument at which the parabola $-(x-s)^2/2t$ first touches the graph of $U_0(s) = \int_0^s u_0(r)dr$ from below. Its inverse $x(a,t) = \inf\{s \in \mathbb{R} : a(s,t) > a\}$ is called the Lagrangian function and identifies the position at time t of the infinitesimal particle started at a. Using this and (1.3) we have that

$$\partial_t x(a,t) = u(x(a,t),t).$$

That is, $u(x,t)$ is the velocity field induced by the flow of infinitesimal particles distributed uniformly at $t = 0$. The velocity field evolves according to the free movement and ballistic aggregation of such particles, where collisions are completely inelastic and conserve mass and momentum. Precisely, define the mass measure $\rho(dx,t)$ of the flow by

$$\rho((x,y],t) = a(y^+,t) - a(x^+,t).$$

Then defining u to be

$$u(x,t) = \frac{u(x^-,t) + u(x^+,t)}{2}$$

at shocks, the mass measure is a weak solution of the set of conservation laws

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0 \end{aligned}.$$

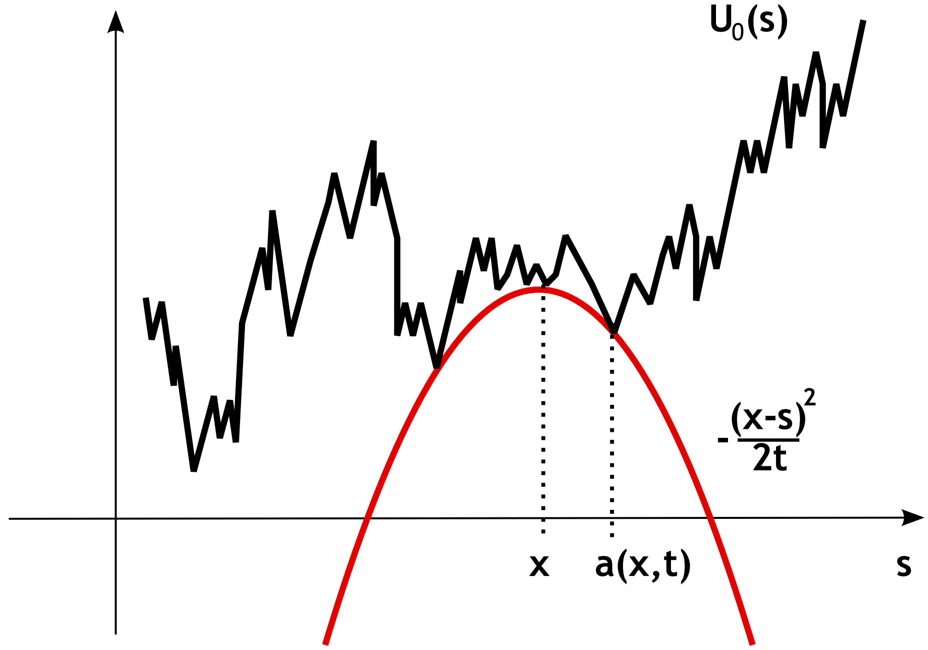

FIGURE 1.2. The parabola $-(x-s)^2/2t$ is placed below the graph of $U_0(s) = \int_0^s u_0(r)dr$ and moved upwards until it first hits the graph. The last argument where the graphs touch is $a(x,t)$.

Now suppose that $u_0(x)$ is a random process indexed by x. A prototypical case is that of u_0 a one-sided Brownian motion, which received a great deal of attention during the 1990s. In conjunction with the numerical investigations of She, Aurell, and Frisch [**46**], Sinai [**47**] proved that the solution u is a Devil's staircase with shock set dense in the real line for any positive time. Soon thereafter, Carraro and Duchon [**14, 15**] demonstrated a closure property for statistical solutions to Lévy process initial data (see Chapters 2 and 3 for definitions) which, in particular, includes the Brownian case. They determined the striking fact that the Laplace exponent $\Phi(q,t)$ of $(u(x,t)-u(0,t))_{x\geq 0}$ *also* satisfies Burgers equation, in the variables $q>0$ and t:

$$\begin{aligned} &\partial_t \Phi + \Phi \partial_q \Phi = 0 \\ &\Phi(q,0) = \Phi_0(q). \end{aligned} \tag{1.5}$$

Bertoin [**8**] subsequently obtained the same result for entropy solutions with Lévy process data having only downward jumps through an elegant derivation in terms of the first passage process of a functional of u_0.

Another well-studied class of initial data is white noise. We say u_0 is *noise* if its corresponding potential $U_0(x) = \int_0^x u_0(r)dr$ is a process with independent increments. Note here that the inverse Lagrangian (1.4) is well-defined since it only depends on u_0 through the potential U_0. The case of white noise corresponds to U_0 a two-sided Brownian motion. The numerical results of She et. al. [**46**] predicted that in this case the shock set is discrete, which was established soon thereafter by Avellaneda and E [**3**]. Analogous qualitative results regarding shock structure were obtained for stable noise initial data in [**6, 9, 28**].

The Burgers model naturally appears as the continuum limit of an wide array of sticky particle systems with random initial positions and velocities. Such systems are prevalent in the statistical physics literature. For example, consider the following model proposed by Frachebourg, Martin, and Piasecki [**25**]. As an initial configuration, place identical particles x_j with mass m on the lattice $\{j\kappa\}_{j\in\mathbb{Z}}$ with lattice constant $\kappa > 0$. Let $\rho = m/\kappa$ and prescribe the initial momenta p_j of the particles to be uncorrelated and distributed according to the Maxwell-Boltzmann distribution with inverse temperature $\beta > 0$:

$$P(p_j \in dp)/dp = \left(\frac{\beta}{2\pi m}\right)^{1/2} \exp\left(-\frac{\beta p^2}{2m}\right).$$

Suppose that particles interact via ballistic aggregation and conserve mass and momentum as the system deterministically evolves. Then in the continuum limit $m \to 0$, $\kappa \to 0$ (keeping ρ constant), the n-point correlation functions of the discrete system converge to the n-point functions of Burgers equation with white noise initial data. For a rigorous statement we refer to [**25**]. It was in this setting that the complete solution for the white noise case of Burgers turbulence was obtained by Frachebourg and Martin [**24**], who actually rederived a result originally due to Groeneboom [**31**] which we discuss below. Interestingly, an exact closure for the 2-point function of the discrete model was also demonstrated in [**25**], and in [**43**] for a related ballistic annihilation model.

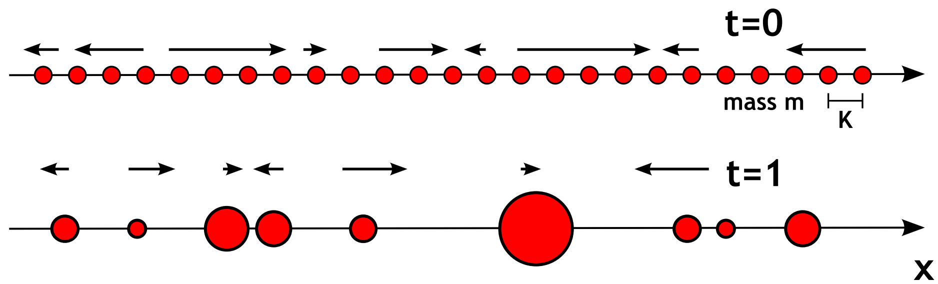

FIGURE 1.3. A discrete physical model that limits to Burgers turbulence with white noise initial data as initial particle masses $m \to 0$ with m/κ fixed. Particles do not have any spatial extent, and the size of each particle corresponds its mass for illustrative purposes.

Connections to nonparametric statistics. Quite surprisingly, Burgers turbulence also arises in an entirely unphysical context: universality in nonparametric statistics with *cube root asymptotics* [**17, 29, 30, 31**]. Specifically, it plays a central role in estimation problems that require some type of monotonicity. We illustrate this with two example: nonparametric maximum likelihood estimation (NPMLE) of a monotone density, and estimation of the mode of a unimodal density. Several other applications—in interval censoring, hazard estimation, deconvolution, estimation of least median of squares, among others—are discussed in [**32**].

In the first problem, the relevant quantity to find is maximum likelihood estimator (MLE) $(\hat{g}_n(x))_{x\geq 0}$ of an unknown, strictly decreasing density $(g(x))_{x\geq 0}$ given independent samples $X_1, \ldots, X_n$ drawn under g. The MLE can be thought of as the "most likely" approximation of g that can be constructed from the data. In the 1950s Grenander [**29**] proved that $\hat{g}_n$ is the left-derivative of the concave hull of the empirical distribution function $\hat{G}_n(x) = \frac{1}{n}\sum_i \mathbf{1}_{x\geq X_i}$. Little was known about the properties of the $\hat{g}_n$ for several decades until Rao [**44**] and later Groeneboom [**30**] demonstrated its L^1-consistency $\|\hat{g}_n - g\|_1 \to 0$ (an infinite-dimensional analogue of the law of large numbers). To obtain the corresponding fluctuation result (the analogue of the central limit theorem) it is necessary to understand the global structure of $\hat{g}_n$, which is a complicated non-Markovian process. Fortunately, an "inverse"

$$Z_n(x) = \arg^+ \max_{s \geq 0} \left\{ \hat{g}_n(s) - \frac{x}{s} \right\}$$

of $\hat{g}_n$ is well-behaved: it is an inhomogeneous Markov jump process which converges (under normalization) to the Markov process

$$Z(x) = \arg^+ \max_{s \in \mathbb{R}} \left\{ W(s) - (x-s)^2 \right\} \tag{1.6}$$

with W a two-sided Brownian motion. By (1.4) and the symmetry of Brownian motion the process $Z(x)$ has the same law as $a(x, 1/2)$ with u_0 a white noise and is directly connected to Burgers turbulence. $Z(0)$ is known as *Chernoff's distribution* [**17**]. For fixed $x \in \mathbb{R}$ such that $g'(x) \neq 0$ it was shown in [**30, 44**] that the pointwise fluctuations of $\hat{g}_n$ satisfy

$$n^{1/3} \left\{ \frac{1}{2} g(x) g'(x) \right\}^{-1/3} (\hat{g}_n(x) - g(x)) \overset{\text{law}}{\to} 2Z(0).$$

A subsequent calculation shows that fluctuations of the L^1-distance $\|\hat{g}_n - g\|_1$ can be expressed in terms of $Z(0)$ as well.

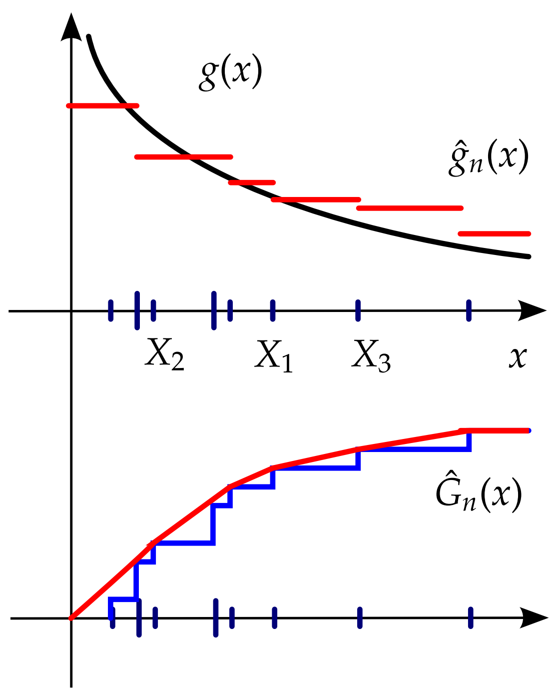

FIGURE 1.4. Construction of the Grenander MLE $\hat{g}_n$ from the data $X_1, \ldots, X_n$.

The process Z also appears in the problem of kernel density estimation of the mode of a unimodal density [**17**]. Let $h(x)$ be a continuously differentiable, unimodal density with distribution $H(x)$ and let φ be a positive kernel with unit mass. Define the mode

$$x = \arg\max_{s\in\mathbb{R}} (\varphi * h)(s)$$

of the convoluted density. Given samples $X_1, \ldots, X_n$ drawn independently from H, we construct the kernel density estimator $\hat{x}_n$ of x as follows. Let $\hat{H}_n(x) = \frac{1}{n}\sum \mathbf{1}_{X_i \geq x}$ be the empirical distribution and $\hat{h}_n(x) = \frac{d}{dx}\hat{H}_n(x) = \frac{1}{n}\sum \delta(X_i - x)$, where the last expression holds in a distributional sense. Then the kernel density estimator of the mode is

$$\hat{x}_n = \arg\max_{s\in\mathbb{R}} (\varphi * \hat{h}_n)(s).$$

We assume for convenience that the x and $\hat{x}_n$ are unique (this is obviously untrue but plays no significant role). The asymptotic behavior of $\hat{x}_n$ varies greatly depending on the regularity of the kernel φ. If φ is continuous then the fluctuations of $\hat{x}_n - x$ are $O(n^{-1/2})$ as in the central limit theorem. However, if we use the step-edge kernel $\varphi_a(x) = \frac{1}{2a}\mathbf{1}_{|x|<a}$ then the fluctuations only satisfy an $O(n^{-1/3})$ bound:

$$n^{1/3}\{h(x+a)\}^{-1/3}(\hat{x}_n - x) \overset{\text{law}}{\to} 2Z(0)$$

where $Z(0)$ is again Chernoff's distribution.

An exact solution. Several of the problems discussed above have culminated with a complete characterization of the process (1.6) by Groeneboom [**31**], for which he was awarded the 1985 Rollo Davidson prize. His work completed the effort initiated by Burgers to compute the statistics of the entropy solution to (1.2) with white noise initial data. Through a remarkable analysis of Brownian motion killed at parabolic boundaries, Groeneboom explicitly computed the 1- and 2-point statistics of $Z(x)$ in terms of Painlevé transcendents. The result is summarized in terms of the generator $\mathcal{A}$ of $u(x, 1/2) = 2(x - Z(x))$ for $\eta \in C_c^1(\mathbb{R})$:

$$\mathcal{A}\eta(u) = 2\eta'(u) + \int_{-\infty}^{u} 2^{1/3} K(2^{-1/3}(u-v)) \frac{J(2^{-1/3}v)}{J(2^{-1/3}u)} dv \qquad (1.7)$$

with the Laplace transforms of K, J given in terms of the Airy function $\mathrm{Ai}(q)$:

$$\bar{K}(q) = -2\frac{d^2}{dq^2}\log(\mathrm{Ai}(q)), \qquad \bar{J}(q) = \frac{1}{\mathrm{Ai}(q)}. \tag{1.8}$$

Outline of our results. It is clear that Burgers turbulence is a fundamental model with a number of useful applications. We now consider a generalization of (1.1) given by the 1-D scalar conservation law

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1.9}$$

with f a smooth, strictly convex flux. The Burgers case corresponds to $f(u) = u^2/2$. As we discuss in Chapter 2, the entropy solution to (1.9) is

$$u(x,t) = (f')^{-1}\left(\frac{x - a(x,t)}{t}\right) \tag{1.10}$$

with inverse Lagrangian function

$$a(x,t) = \arg^{+}\min_{s\in\mathbb{R}}\left\{\int_0^s \left(u_0(r) - (f')^{-1}\left(\frac{x-r}{t}\right)\right) dr\right\}.$$

Building upon results in the literature, our main contributions are as follows:

- At the level of the 2-point correlation function, 1-D scalar conservation laws with random initial data are *closed*: if u_0 is a Markov process in x with only downward jumps then the entropy solution (1.10) is also Markov (and has only downward jumps). The ensemble of spectrally negative Markov processes is thus conserved under the dynamics (1.9).
- The 2-point closure also holds true if the initial potential $U_0(x) = U(x,0)$ is a process with independent increments and only downward jumps. The importance of independent increments for the potential has been recognized by several authors [**3, 12, 24**]. This result broadens the closure property to spectrally negative noise initial data, for which u_0 a white noise is a special case.
- The evolution of the statistics of u under the 2-point closure can be explicitly determined, and quite remarkably yields a *completely integrable* system. This was shown by Chabanol and Duchon [**16**] for statistical solutions to Burgers equation, and we demonstrate that it is in fact true for entropy solutions to 1-D scalar

conservation laws. Specifically, suppose u is a spectrally negative Feller process in x with infinitesimal generator $\mathcal{A}(t)$:

$$\mathcal{A}(t)\eta(u) = b(u,t)\eta'(u) + \int_{(-\infty,u)} n(u,dv,t)(\eta(v) - \eta(u)), \qquad \eta \in C_c^1(\mathbb{R}).$$

Then with the associated operator $\mathcal{B}(t) = \mathcal{B}(t;\mathcal{A},f)$ given by

$$\begin{aligned}\mathcal{B}(t)\eta(u) =& b(u,t)f'(u)\eta'(u) \\ &+ \int_{(-\infty,u)} n(u,dv,t)\left(\frac{f(v)-f(u)}{v-u}\right)(\eta(v)-\eta(u)),\end{aligned}$$

the evolution equation is

$$\partial_t \mathcal{A} = [\mathcal{B},\mathcal{A}]. \tag{1.11}$$

The Lax pair implies that there is a rich structure underlying the model. Our work sets the foundation for an analysis of (1.11) through the classical theory of inverse scattering.

- The statistical evolution can be given by an equivalent kinetic equation for the shock number distribution $n(u,dv,t)$ of the generator $\mathcal{A}(t)$. This equation is of Boltzmann type and reduces to a classical model if the initial velocity u_0 has independent increments: Smoluchowski's coagulation equation with additive kernel.
- The evolution (1.11) (or equivalently, the kinetic equation) admits the explicit solution (1.7) of Groeneboom

$$\mathcal{A}(t)\eta(u) = t^{-1}\eta'(u) + \int_{-\infty}^{u} t^{-1/3}K(t^{1/3}(u-v))\frac{J(t^{1/3}v)}{J(t^{1/3}u)}dv,$$

where the Laplace transforms of K, J are given as before by (1.8).

Our results provide a foundation for a large class of clustering models and demonstrate that they share a common integrable structure. In the setting of nonparametric statistics, they give a natural "differential" approach to classifying $Z(x) = a(x,1/2)$ by exploiting the self-similarity of Brownian motion and using the evolution equation (1.11). This may provide a way to move beyond the Gaussian case and classify limiting behavior for estimation problems in which stable processes and other fluxes f become relevant. Finally, in a broad sense, our exposition highlights a rich interplay between probability theory (Markov processes, random fields, first passages) and classical applied mathematics (complete integrability, inverse scattering, Riemann-Hilbert problems), one that offers a variety of exciting avenues for future research.

We aim to give a reasonably self-contained discussion of the results presented above. The remainder of the thesis is as follows. In Chapter 2 we give an overview of some preliminary topics essential to our work: entropy solutions to 1-D scalar conservation laws and the theory of Markov and Lévy processes. In Chapter 3 we discuss and prove the main results: the 2-point closure of the Burgers turbulence model and its corresponding evolution. The evolution equation is derived by two independent methods, one using the chain rule for BV functions and the other using the Hopf functional approach. In Chapter 4 we derive the kinetic interpretation of the model and show that it describes known solutions, and in particular admits the exact solution of Groeneboom. In Chapter 5 we present some tangentially related results on rates of convergence under dynamic scaling to self-similar solutions of Smoluchowski's coagulation equation. We conclude with a short discussion.

CHAPTER 2

Scalar conservation laws and statistical ensembles

1. 1-D scalar conservation laws

For completeness, we provide a short survey of entropy solutions to 1-D scalar conservation laws. We refer the reader to Dafermos [**18**] and Evans [**21**] for details and proofs of the stated results.

To begin, consider the 1-D scalar conservation law for $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$:

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{2.1}$$

where the flux f is smooth and strictly convex. In general, solutions to (2.1) are not smooth so we must consider the weak form of the equation

$$\int_0^\infty \int_{\mathbb{R}} \left[\partial_t \varphi(x,t) u(x,t) + \partial_x \varphi(x,t) \frac{1}{2} u^2(x,t) \right] dx dt + \int_{\mathbb{R}} \varphi(x,0) u_0(x) dx = 0. \tag{2.2}$$

In particular, $u \in L^1_{loc}(\mathbb{R} \times [0,\infty))$ is a *weak solution* to (2.1) if (2.2) holds for all $\varphi \in C^1_c(\mathbb{R} \times [0,\infty))$. It can be checked that without any additional constraints there exist infinitely many weak solutions. To remedy this, we use the notion of an *entropy condition*:

$$u(x+h,t) - u(x,t) \leq \frac{C}{t} h \tag{2.3}$$

for some fixed $C > 0$ and all $x \in \mathbb{R}$ and $t, h > 0$. In particular this constrains u to have only downward jumps. We then say that $u(x,t)$ is an *entropy solution* to (2.1) if it is a weak solution and satisfies (2.3). The entropy solution is unique and its form is given by the Hopf-Lax formula, which we discuss below.

1.1. Burgers equation and the vanishing viscosity method. We first consider Burgers equation ($f(u) = u^2/2$), for which the entropy solution can be recovered by the *vanishing viscosity method*. The viscous Burgers equation is

$$\begin{aligned} \partial_t u^\nu + \partial_x \left(\tfrac{1}{2} u^{\nu 2}\right) &= \nu \partial_{xx} u^\nu \\ u^\nu(x,0) &= u_0(x) \end{aligned} . \tag{2.4}$$

This equation can be explicitly solved for by a clever change of variables. With velocity potential

$$U^\nu(x,t) = \int_0^x u^\nu(r,t)dr$$

we can write (2.4) as a Hamilton-Jacobi equation with Hamiltonian $H(p) = p^2/2$:

$$\partial_t U^\nu + \frac{1}{2}(\partial_x U^\nu)^2 = \nu \partial_{xx} U^\nu. \tag{2.5}$$

The Hopf-Cole transformation

$$\phi^\nu(x,t) = \exp\left(-\frac{1}{2\nu} U^\nu(x,t)\right),$$

transforms this into the heat equation $\partial_t \phi^\nu = \nu \partial_{xx} \phi^\nu$ with initial data $\phi_0^\nu(x) = \exp\left(-\frac{1}{2\nu} U_0^\nu(x)\right)$. It has solution

$$\phi^\nu(x,t) = \frac{1}{\sqrt{4\pi t \nu}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2\nu} I(r;x,t)\right) dr,$$

where the *Hopf-Cole functional* I is defined by

$$I(s;x,t) = \int_0^s u_0(r)dr + \frac{(x-r)^2}{2t}.$$

Reverting back to u we have that

$$u^\nu(x,t) = \frac{x}{t} - \frac{1}{t} \frac{\int_{\mathbb{R}} r \exp\left(-\frac{1}{2\nu} I(r;x,t)\right) dr}{\int_{\mathbb{R}} \exp\left(-\frac{1}{2\nu} I(r;x,t)\right) dr}.$$

Taking $\nu \to 0$ we that find the probability measure

$$\frac{\exp\left(-\frac{1}{2\nu} I(r;x,t)\right) dr}{\int_{\mathbb{R}} \exp\left(-\frac{1}{2\nu} I(r;x,t)\right) dr}$$

concentrates on minimizers of the Hopf-Cole functional. This yields the solution

$$u(x,t) = \frac{x - a(x,t)}{t}$$

in terms of inverse Lagrangian function

$$a(x,t) = \arg^+ \min_{s \in \mathbb{R}} I(s;x,t)$$

where $\arg^+ \min$denotes the last argument where the expression is minimized. In order for $a(x,t)$ to be well-defined we require that the initial data satisfy $\liminf_{|s|\to\infty} u_0(s)/s \geq 0$. Note that the inverse Lagrangian function is nondecreasing and continuous everywhere except at a countable number of points. This ensures that $u \in BV_{loc}(\mathbb{R}\times[0,\infty))$ and $u(x^-,t) > u(x^+,t)$ at all such shock points. For a proof that $u(x,t)$ is indeed the entropy solution, see [**21**].

1.2. Entropy solutions and the Hopf-Lax functional. Now consider general fluxes f. As before, define the *potential*

$$U(x,t) = \int_0^x u(r,t)dr \tag{2.6}$$

with $U(x,0) = U_0(x)$. The conservation law (2.1) can be integrated to give a Hamilton-Jacobi equation with Hamiltonian $H(p) = f(p)$:

$$\begin{aligned} \partial_t U + f(\partial_x U) &= 0 \\ U(x,0) &= U_0(x) \end{aligned} \tag{2.7}$$

The solution to (2.7) is given by the *Hopf-Lax* formula

$$U(x,t) = \min_{s\in\mathbb{R}} \left\{ U_0(s) + tf^*\left(\frac{x-s}{t}\right)\right\} \tag{2.8}$$

where f^* is the Legendre transform (or convex dual) of f:

$$f^*(p) = \sup_{q\in\mathbb{R}}\{pq - f(q)\}.$$

Precisely, (2.8) is almost everywhere (a.e.) differentiable in $\mathbb{R}\times(0,\infty)$ and solves (2.7) a.e., as well as in the sense of distributions. Formally, the x-derivative of (2.8) is a prime candidate for the entropy solution to the conservation law. This is indeed the case, and is explicitly given as follows. Define the *Hopf-Lax functional*

$$I(s;x,t) = U_0(s) + tf^*\left(\frac{x-s}{t}\right) = \int_0^s \left(u_0(r) - (f')^{-1}\left(\frac{x-r}{t}\right)\right)dr \tag{2.9}$$

and the *inverse Lagrangian function*

$$a(x,t) = \sup\left\{ s\in\mathbb{R} : I(s;x,t) = \inf_{r\in\mathbb{R}} I(r;x,t)\right\}. \tag{2.10}$$

If I has no upward jumps it achieves its infimum and we write the inverse Lagrangian as

$$a(x,t) = \arg^+ \min_{s\in\mathbb{R}} I(s;x,t).$$

In particular, note that (2.9) and (2.10) are well-defined if we prescribe an initial potential U_0 instead of an initial velocity u_0. The entropy solution to (2.1) is then

$$u(x,t) = (f^{-1})' \left(\frac{x - a(x,t)}{t} \right). \tag{2.11}$$

As in the Burgers case, $a(x,t)$ is nondecreasing and continuous everywhere except on a countable set. Therefore, $u \in BV_{loc}(\mathbb{R} \times [0,\infty))$. In order for it to be finite, we require the initial data u_0 to satisfy the growth condition

$$\liminf_{|s|\to\infty} \frac{u_0(s)}{(f')^{-1}(s)} \geq 0 \tag{2.12}$$

or in terms of the potential,

$$\liminf_{|s|\to\infty} \frac{U_0(s)}{f^*(s)} \geq 0. \tag{2.13}$$

2. Markov processes

Much of our discussion depends heavily on properties unique to Markov processes, which we summarize below. Our main references are the monographs of Applebaum [**2**], Bertoin [**7**], Feller [**22**], and Revuz and Yor [**45**].

We begin with the canonical framework. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_x, X_x, \theta_x, P)$ be a stochastic process. Here, Ω is the space of càdlàg paths (those that are right-continuous and have left limits) on $(-\infty,\infty)$ endowed with the Skorohod topology and equipped with shift operators $\theta_h\omega(x) = \omega(x+h)$, and P is the law of the coordinate process $X_x(\omega) = \omega(x)$ on Ω. The natural filtration $\mathcal{F}_x = \sigma(X_y, y \leq x)$ of X is extended to be right-continuous and complete. We will denote the expected value under the measure P by E.

Two processes X and Y that generate the same measure on Ω are equal in law and denoted by $X \stackrel{\mathcal{L}}{=} Y$. If X and Y are *separable*, this is equivalent to saying that they have the same finite-dimensional distributions. We say that X is a *stationary process* if its law is invariant under translations—that is, if $\theta_h X \stackrel{\mathcal{L}}{=} X$ for all $h \in \mathbb{R}$.

2.1. Markov semigroup and the infinitesimal generator. To start, let $\mathcal{F}_\infty = \sigma\left(\bigcup_{x\in\mathbb{R}} \mathcal{F}_x\right)$. X is a *Markov process* if for any $\mathcal{F}_\infty$-measurable function Y and for all $x \in \mathbb{R}$,

$$E[Y \circ \theta_x | \mathcal{F}_x] = E[Y|X_x].$$

More simply, the Markov property states that X is conditionally independent of its past given its present state:

$$E[X_{x+h}|\mathcal{F}_x] = E[X_{x+h}|X_x].$$

This property can be generalized. We define ξ to be a *stopping time* with respect to the filtration $\mathcal{F}$ if for every $x \in \mathbb{R}$, the event $\{\xi \leq x\}$ is $\mathcal{F}_x$-measurable. The first passage time of X at a fixed set is a stopping time, while its last passage time is not. We say that X is *strong Markov* if the Markov property holds for any stopping time ξ on the set $\{\xi \neq \infty\}$.

If X is Markov, it is characterized by its 1-point probability

$$p_x(du) = P(X_x \in du) \tag{2.14}$$

and transition kernels

$$q_{x,x+h}(u, dv) = P(X_{x+h} \in dv | X_x = u), \qquad h \geq 0. \tag{2.15}$$

To see this, first denote by $C_0(\mathbb{R})$ the space of continuous functions that decay at infinity. With $x_1 < \cdots < x_n$ and $\{\eta_i\}_{1\leq i\leq n} \subset C_0(\mathbb{R})$, the *n-point correlation function* $E\left[X_{x_1} X_{x_2} \cdots X_{x_n}\right]$ can be written in terms of the 1-point probability and transition kernels by using the Markov property:

$$E\left[\prod_{i=1}^{n} \eta_i\left(X_{x_i}\right)\right] = \int_{\mathbb{R}} p_{x_1}(du_1)\eta_1(u_1) \int_{\mathbb{R}} q_{x_1,x_2}(u_1, du_2)\eta_2(u_2) \cdots \int_{\mathbb{R}} q_{x_{n-1},x_n}(u_{n-1}, du_n)\eta_n(u_n).$$

If X is stationary, then $p_x = p$ and $q_{x,x+h} = q_h$ are x-independent. In this case, $p(du)$ is an *invariant measure* for the process. The notion of stationarity is weakened by requiring only the transition kernels $q_{x,x+h} = q_h$ (but not p_x) to be independent of x, in which case we call X a *homogeneous* Markov process.

The same information can be expressed in the following manner. With $\eta \in C_0(\mathbb{R})$, define the 1- and 2-point operators

$$\mathcal{P}_x\eta = E[\eta(X_x)] = \int_{\mathbb{R}} p_x(du)\eta(u) \tag{2.16}$$

and

$$\mathcal{Q}_{x,x+h}\eta(u) = E[\eta(X_{x+h})|X_x = u] = \int_{\mathbb{R}} q_{x,x+h}(u, dv)\eta(w), \qquad h \geq 0. \tag{2.17}$$

As shown above, knowledge of the 1- and 2-point operators is equivalent to that of the 1-point probability and transition kernels. The $(\mathcal{Q}_{x,x+h})_{x\in\mathbb{R},h\geq 0}$ form an inhomogeneous L^∞-semigroup in the sense that $\mathcal{Q}_{x,x} = I$, $\mathcal{Q}_{x,y}\mathcal{Q}_{y,z} = \mathcal{Q}_{x,z}$ for every $x \leq y \leq z$, and $0 \leq \mathcal{Q}_{x,y}\eta \leq 1$ whenever $0 \leq \eta \leq 1$. Here, I is the identity operator.

To define the notion of an infinitesimal generator of the semigroup, we first require some regularity on the process—namely, the Feller property. We say that $(\mathcal{Q}_{x,x+h})_{x\in\mathbb{R},h\geq 0}$ has the *Feller property* if for every $\eta \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$,

- $\mathcal{Q}_{x,x+h}\eta \in C_0(\mathbb{R})$ for all $h \geq 0$ and
- $\lim_{h\downarrow 0} \mathcal{Q}_{x,x+h}\eta = \eta$ uniformly (that is, in the C_0-norm).

The process X characterized by this semigroup is called a *Feller process*, and can be assumed to be càdlàg without any loss of generality. Define the set

$$\begin{aligned}\mathcal{D}(\mathcal{A}) = \{\eta \in C_0(\mathbb{R}) :&\text{for each } x \in \mathbb{R}, \text{ there exists } \psi \in C_0(\mathbb{R}) \\ &\text{s.t. } \lim_{h\downarrow 0} h^{-1}(\mathcal{Q}_{x,x+h}\eta - \eta) = \psi \text{ uniformly}\}.\end{aligned}$$

It can be shown that $\mathcal{D}(\mathcal{A})$ is a dense subset of $C_0(\mathbb{R})$ [**7, 45**]. We define the *infinitesimal generator* $\mathcal{A}_x : \mathcal{D}(\mathcal{A}) \to C_0(\mathbb{R})$ of the Markov process X by the uniform limit

$$\mathcal{A}_x\eta = \lim_{h\downarrow 0} h^{-1}(\mathcal{Q}_{x,x+h}\eta - \eta).$$

$\mathcal{D}(\mathcal{A})$ is referred to as the *domain* of the generator $\mathcal{A}_x$. Formally, $\mathcal{A}_x = \partial_h\mathcal{Q}_{x,x+h}|_{h=0}$. Now define the *adjoint operator* $\mathcal{A}^\dagger$ by duality with $\mathcal{A}$ —that is, for all $\eta \in \mathcal{D}(\mathcal{A})$ and

μ a finite Radon measure

$$\int_{\mathbb{R}} \mathcal{A}^\dagger \mu(du)\eta(u) = \int_{\mathbb{R}} \mu(du)\mathcal{A}\eta(u).$$

The forward Kolmogorov (or Fokker-Planck) equation then holds in the weak sense given above:

$$\partial_x p_x(du) = \mathcal{A}_x^\dagger p_x(du) \tag{2.18}$$

If $p_x(du)$ has a density then the last display holds strongly. In the case where X is homogeneous, the semigroup $(\mathcal{Q}_{x,x+h})_{x\in\mathbb{R},h\geq 0} = (\mathcal{Q}_h)_{h\geq 0}$ is independent of x and yields a homogeneous generator $\mathcal{A}$. If, in addition, X is stationary and has an invariant measure with density $p(u)$, (2.18) reduces to the *adjoint equation*

$$\mathcal{A}^\dagger p(u) = 0. \tag{2.19}$$

Owing to the fact that the generator of a Feller semigroup satisfies the positive maximum principle (see [**2**], and also [**35, 45**]), we can use a result of Courrège to give its explicit form. Courrège's representation of the infinitesimal generator of X is

$$\begin{aligned}\mathcal{A}_x\eta(u) = d_x(u)\eta'(u) + \frac{1}{2}\sigma_x^2(u)\eta''(u)& \\ + \int_{\mathbb{R}-\{u\}} \Pi_x(u,dv)(\eta(v) - \eta(u) + \mathbf{1}_{|u-v|<1}(v-u)\eta'(u)).&\end{aligned} \tag{2.20}$$

The drift, diffusion, and jump measure terms (d_x, σ_x^2, Π_x) comprise the *characteristics* of the generator $\mathcal{A}_x$. The *Lévy kernel* Π_x satisfies for every x

$$\int_{\mathbb{R}-\{u\}} \Pi_x(u,dv)(1\wedge |u-v|^2) < \infty.$$

Π_x represents the expected number (perhaps infinite) of jumps from u to v in the infinitesimal interval $[x, x+dx)$, conditional on $X_x = u$ and divided by dx. If the measure $\Pi_x(u,dv)$ is supported on the half-interval $(-\infty, u)$ or $(u, +\infty)$, we say that X is *spectrally negative* or *spectrally positive*, respectively. Additionally, if X has sample paths of bounded variation then

$$\sigma_x \equiv 0 \qquad \text{and} \qquad \int_{\mathbb{R}-\{u\}} \Pi_x(u,dv)(1\wedge |u-v|) < \infty. \tag{2.21}$$

In this case, the generator can then be written as

$$\mathcal{A}_x\eta(u) = b_x(u)\eta'(u) + \int_{\mathbb{R}-\{u\}} n_x(u,dv)(\eta(v) - \eta(u)) \tag{2.22}$$

with the new drift and jump terms (b_x, n_x). When working with Feller processes with BV paths, we shall refer to these as the *characteristics* of the process without any ambiguity.

To see how the generator changes under certain deterministic transformations of the underlying process consider the following:

LEMMA 2.1. *Let $\bar{X}$ be a Feller process with bounded variation with generator $\bar{\mathcal{A}}_x$ and characteristics $(\bar{b}_x, \bar{n}_x)$. With $G \in C^1$ invertible, $g \in C^1$, and $\gamma > 0$ define the transformed process*

$$X_x \stackrel{\mathcal{L}}{=} G(\bar{X}_{\gamma x}) + g(x).$$

Then X is a Feller process and its generator $\mathcal{A}_x$ has characteristics (b_x, n_x) that satisfy

$$b_x(u) = \gamma G'(\bar{u})\bar{b}_{\gamma x}(\bar{u}) + g'(x), \qquad n_x(u,dv) = \gamma \bar{n}_{\gamma x}(\bar{u}, d\bar{v})$$

with transformed variable $\bar{u} = G^{-1}(u - g(x))$.

PROOF. Since G is invertible it is easy to verify the Feller property. Now with $h > 0$, let $\bar{\mathcal{Q}}_{x,x+h}$ and $\mathcal{Q}_{x,x+h}$ be the Markov semigroup of $\bar{X}$ and X, respectively. Denoting

$$\bar{\eta}_h(u) = \eta(G(u) + g(x+h))$$

we have that

$$\mathcal{Q}_{x,x+h}\eta(u) = E[\eta(X_{x+h})|X_x = u] = E[\bar{\eta}_h(\bar{X}_{\gamma x+\gamma h})|\bar{X}_{\gamma x} = \bar{u}] = \bar{\mathcal{Q}}_{\gamma x,\gamma x+\gamma h}\bar{\eta}_h(\bar{u}).$$

Therefore,

$$\begin{aligned}\mathcal{A}_x\eta(u) &= \partial_h \mathcal{Q}_{x,x+h}\eta|_{h=0}(u) \\ &= \partial_h(\bar{\mathcal{Q}}_{\gamma x,\gamma x+\gamma h}\bar{\eta}_h)|_{h=0}(\bar{u}) = \gamma\bar{\mathcal{A}}_{\gamma x}\bar{\eta}_0(\bar{u}) + \partial_h\bar{\eta}_h|_{h=0}(\bar{u}).\end{aligned}$$

Writing out the last expression explicitly we find

$$\mathcal{A}_x\eta(u) = \gamma\left\{G'(\bar{u})\bar{b}_{\gamma x}(\bar{u})\eta'(u) + \int_{\mathbb{R}} \bar{n}_{\gamma x}(\bar{u}, d\bar{v})(\eta(v) - \eta(u))\right\} + g'(x)\eta'(u).$$

Comparing this to (2.22) gives the desired result.

$\square$

2.2. Lévy processes. Lévy processes play a special role in the theory of Markov processes, simplifying their study while preserving many of their important features. In particular, they form the subclass of homogeneous Markov process with independent increments. We summarize their most basic properties, referring the reader to [**7**] for further details.

We call $(X_x)_{x\geq 0}$ a *Lévy process* started at 0 if $X_0 = 0$ P-a.s. and for every $x, h \geq 0$:

- $X_{x+h} - X_x$ is independent of $\mathcal{F}_x$ and
- $X_{x+h} - X_x \stackrel{\mathcal{L}}{=} X_h$.

If the second condition (which guarantees homogeneity of the process) is dropped, we refer to X as an *additive process*. Additive processes are exactly those that have independent increments. For Lévy processes, the transition kernels (2.15) with dv a Borel neighborhood of v simplify to $q_{x,x+h}(u, dv) = q_h(u - dv)$. Thus the transitions only depend on the initial state u through the difference $s = u - v$, illustrating the state-space invariance of Lévy processes. The law of X is characterized by its *characteristic exponent* $\Psi(\xi) : \mathbb{R} \to \mathbb{C}$ which satisfies

$$E[e^{i\xi X_x}] = \exp(-x\Psi(\xi)).$$

Owing to the fact that the distribution of X_1 is *infinitely divisible*, the characteristic exponent takes the Lévy-Khintchine form

$$\Psi(\xi) = id\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}-\{0\}} \Pi(ds)(1 - e^{i\xi s} + i\xi s\mathbf{1}_{|s|<1}), \qquad \xi \in \mathbb{R} \tag{2.23}$$

where $\int \Pi(ds)(1 \wedge |s|^2) < \infty$. This decomposes the process into three parts: a Brownian motion with variance σ^2 and drift $-d$, a compound Poisson process with jump measure $\mathbf{1}_{|s|\geq 1}\Pi(ds)$, and a pure jump martingale accounting for small jumps.

The triple (d, σ^2, Π) constitutes the *characteristics* of X. The characteristic exponent is simply the *symbol* of the generator $\mathcal{A}$ of the Lévy process

$$\begin{aligned}\mathcal{A}\eta(u) = d(u)\eta'(u) + \frac{1}{2}\sigma^2(u)\eta''(u) \\ + \int_{\mathbb{R}-\{u\}} \Pi(ds)(\eta(u+s) - \eta(u) - \mathbf{1}_{|s|<1} s\eta'(u)).\end{aligned} \tag{2.24}$$

That is, in terms of the Fourier transform $\mathcal{F}\eta(\xi) = \int_{-\infty}^{\infty} e^{i\xi u}\eta(u)du$, Ψ satisfies

$$\mathcal{F}(\mathcal{A}\eta)(\xi) = -\Psi(-\xi)\mathcal{F}\eta(\xi). \tag{2.25}$$

As before, X has sample paths of bounded variation if and only if

$$\sigma \equiv 0 \qquad \text{and} \qquad \int_{\mathbb{R}} \Pi(ds)(1 \wedge |s|) < \infty.$$

In this case,

$$\Psi(\xi) = ib\xi + \int_{\mathbb{R}-\{0\}} n(ds)(1 - e^{i\xi s})$$

$$\mathcal{A}\eta(u) = b\eta'(u) + \int_{\mathbb{R}-\{0\}} n(ds)(\eta(u+s) - \eta(u)) \tag{2.26}$$

with characteristics b and $n(ds)$.

If X is monotone increasing it is called a *subordinator*. In particular, it is of bounded variation. In this case, it is easier to work with the *Laplace exponent* $\Phi(q) : [0, \infty) \to [0, \infty)$ of X, which satisfies

$$E[e^{-qX_x}] = \exp(-x\Phi(q)).$$

In terms of characteristics,

$$\Phi(q) = \Psi(-iq) = bq + \int_{(0,\infty)} n(ds)(1 - e^{-qs}) \tag{2.27}$$

and with Laplace transform $\mathcal{L}\eta(q) = \int_0^{\infty} e^{-qu}\eta(u)du$, we have

$$\mathcal{L}(\mathcal{A}\eta)(q) = \Phi(q)\mathcal{L}\eta(q). \tag{2.28}$$

To finish, we define X to be a *stable process* with index $\alpha \in (0, 2]$ if for every $\kappa > 0$ and $\xi \in \mathbb{R}$ its characteristic exponent has the scaling property $\Psi(\kappa\xi) = \kappa^{\alpha}\Psi(\xi)$. Then, X is self-similar:

$$(\kappa^{-1/\alpha}X_{\kappa x})_{x\geq 0} \stackrel{\mathcal{L}}{=} (X_x)_{x\geq 0}.$$

Stable processes play a central role in the generalized invariance principle as limiting distributions in Skorohod space.

2.3. Splitting times. We now provide a brief survey of the decomposition of Markov processes at splitting times. Significant effort has been devoted to classifying those random times L for which the Markov property holds, the simplest one being the set of stopping times. Remarkably, the Markov property remains intact for a much broader class of random times. In particular it holds at the time of last passage at a fixed set. We refer the interested reader to [**41**] for further discussion and an excellent survey of the probabilistic literature.

Two results will be particularly useful for our purposes, which we give without proof. They hold in much greater generality than we present here. The first theorem is due to Millar [**42**] and concerns splitting at the ultimate minimum of a functional of the process:

THEOREM 2.2 (Millar, 1978). *Let X_s be a càdlàg strong Markov process and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous. Let L be the last time the process $g(X_{s-}, X_s)$ hits its ultimate minimum. Then $\{X_s\}_{s \geq L}$ is conditionally independent of $\{X_s\}_{s<L}$ given X_L and the auxiliary variable $m = \inf_s g(X_{s-}, X_s) = g(X_{L-}, X_L)$.*

The second result is given in Getoor [**26**] and includes Millar's theorem as a special case:

THEOREM 2.3 (Getoor, 1979). *Let $M \subset \mathbb{R}$ be a fixed set and let $L = \sup\{x \in \mathbb{R} : x \in M\}$ be the end of M. Then given a càdlàg strong Markov process X_x, the post-L process $\{X_x\}_{x \geq L}$ is independent of $\{X_x\}_{x<L}$ given X_L.*

We remark that in both theorems, the transition semigroup of the pre- and post-L processes can be explicitly determined from that of X. It is an interesting and subtle fact that these transition semigroups are typically different from each other.

CHAPTER 3

Integrability in statistical closures of Burgers turbulence

1. Closure for the 2-point correlation function

In this section we demonstrate that 1-D scalar conservation laws with random initial data are closed at the level of the 2-point function. More precisely, we show that the entropy solution to

$$\begin{aligned} &\partial_t u + \partial_x f(u) = 0 \\ &u(x,0) = u_0(x) \end{aligned} \tag{3.1}$$

with f a smooth, strictly convex flux and $u_0(x)$ a spectrally negative Feller process in x is also of this type. Note that x plays here the role of "time" as it is traditionally used in theory of stochastic processes. The time parameter t essentially acts as a constant in the following discussion.

Recall the canonical framework. Let $u_0 = (\Omega, \mathcal{F}, \mathcal{F}_s, u_0(s), \mu_0)$ be a spectrally negative Markov process on the space Ω of càdlàg paths on $(-\infty, \infty)$ endowed with the Skorohod topology. Here, μ_0 is the law of the coordinate process $u_0(\omega; s) = \omega(s)$ on Ω and $\mathcal{F}(s) = \sigma(u_0(r), r \leq s)$ is the natural filtration of u_0 extended to be right-continuous and complete.

Recall the Hopf-Lax functional and inverse Lagrangian function

$$I(s;x,t) = \int_0^s \left(u_0(r) - (f')^{-1}\left(\frac{x-r}{t}\right)\right) dr \tag{3.2}$$

$$a(x,t) = \arg^+ \min_{s\in\mathbb{R}} I(s;x,t). \tag{3.3}$$

To ensure that (3.3) is well-defined and finite, we assume that the mild condition (2.12) holds a.s. for u_0. This condition is automatically satisfied if u_0 is a stationary random process. Since $u(x,t)$ is an isomorphism of $a(x,t)$ for $t > 0$, to obtain the closure property we only need to show that $a(x,t)$ is a (strong) Markov process. We prove the following:

THEOREM 3.1. *Let u_0 be a spectrally negative Markov process that satisfies (2.12) a.s. Under the law μ_0 and for any fixed $t > 0$, the inverse Lagrangian process $a(x,t)$ is Markov. Thus $u(x,t)$ is a spectrally negative Markov process.*

PROOF. Fix $x \in \mathbb{R}$. Without loss of generality it suffices to take $t = 1$. The proof consists of three steps, the first two of which are entirely deterministic. The first uses that the inverse Lagrangian function is increasing to show that for any s, $(a(x-h) - a(x))_{h>0}$ and $(a(x+h) - a(x))_{h\geq 0}$ depend only on $(u_0(s))_{s<a(x)}$ and $(u_0(s))_{s\geq a(x)}$, respectively. The second step demonstrates that the presence of only downward jumps in u_0 uniquely identifies the value of $u_0(a(x))$. Finally, the last step demonstrates that the Markov property of u_0 holds at $a(x)$: $(u_0(s))_{s<a(x)}$ and $(u_0(s))_{s\geq a(x)}$ are statistically independent given $a(x)$. Together these give the desired result.

1. *Dependence structure of* $(a(x))_{x\in\mathbb{R}}$: In the following we will make frequent use of (3.2), which is well-defined since the inverse Lagrangian (3.3) is finite almost surely. Since $a(x)$ is increasing, for $h > 0$

$$\begin{aligned} a(x+h) &= \arg^+ \min_{s\in\mathbb{R}} I(s; x+h) \\ &= \arg^+ \min_{s\geq a(x)} \{I(a(x); x+h) + (I(s; x+h) - I(a(x); x+h))\} \\ &= a(x) + \arg^+ \min_{s\geq 0} \left\{ \int_{a(x)}^{a(x)+s} \big(u_0(r) - (f')^{-1}(x-r)\big)\, dr \right\}. \end{aligned} \tag{3.4}$$

$(a(x+h) - a(x))_{h>0}$ therefore depends on u_0 only through $(u_0(s))_{s>a(x)}$. The same argument gives that $(a(x-h) - a(x))_{h>0}$ depends only on $(u_0(s))_{s<a(x)}$.

2. *Spectral negativity determines* $u_0(a(x))$: The restriction to only downward jumps in the initial velocity u_0 severely constrains its behavior at minimizers of the Hopf-Lax functional (3.2). Specifically, it implies that $\partial_s I(a(x); x) = 0$, or equivalently,

$$u_0(a(x)) = (f')^{-1}(x - a(x)). \tag{3.5}$$

Alternatively, it can be said that $a(x)$ is required to be a Lagrangian regular point—that is, it is *not* a Lagrangian shock point. This only occurs for u_0 spectrally negative.

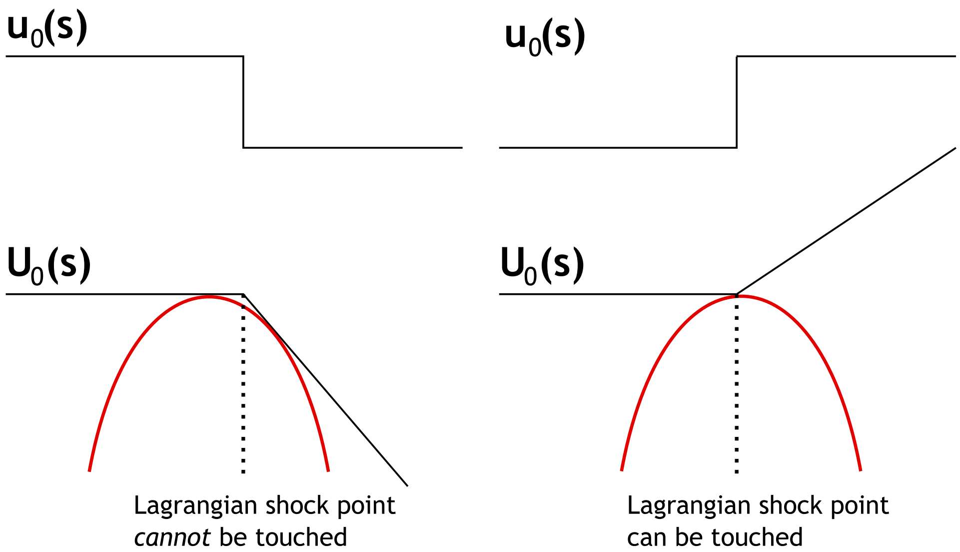

FIGURE 3.1. Spectral negativity of u_0 from the perspective of the potential U_0. This condition is equivalent to requiring that $a(x)$ be a Lagrangian regular point.

3. *Markov property of* u_0 *at* $a(x)$: That $a(x)$ constitutes a Markov time for u_0 is not obvious. It is certainly not a stopping time since the event $\{a(x) \leq s\}$ is not $\mathcal{F}_s$-measurable (this is seen by noting that $a(x)$ is the last time (3.2) is minimized and hence depends on $(u_0(r))_{r>s}$ as well). $a(x)$ is, however, a splitting time for a certain functional of u_0. To see this, first define

$$m(s;x) = \min_{r \leq s} I(r;x), \qquad D(s;x) = (I-m)(s;x).$$

Dropping the explicit dependence on x from the notation, consider the bivariate process (u_0, D). Since $D(s)$ is entirely dependent on $(u_0(r))_{r\leq s}$, $(u_0, D)(s)$ is $\mathcal{F}_s$-measurable. For any $\mathcal{F}$-stopping time ξ

$$m(\xi+s) = m(\xi) \wedge \left(\min_{\xi < r \leq \xi+s} I(r) \right) = I(\xi) + \left\{ -D(\xi) \wedge \min_{0<r\leq s} (I(\xi+r) - I(\xi)) \right\}$$

and

$$D(\xi+s) = (I(\xi+s) - I(\xi)) - \left\{ -D(\xi) \wedge \min_{0<r\leq s} (I(\xi+r) - I(\xi)) \right\}.$$

For $s > 0$ the increment $I(\xi + s) - I(\xi)$ depends on $\mathcal{F}_\xi$ only through $u_0(\xi)$ by the strong Markov property of u_0. Therefore, (u_0, D) is also a càdlàg, strong Markov process.

We now show that $a(x)$ is a splitting time for (u_0, D). By definition (3.3) and step 2,

$$a(x) = \sup\left\{s \in \mathbb{R} : (u_0, D)(s) = \left((f')^{-1}(x-s), 0\right)\right\}.$$

Thus, $a(x)$ is the last time (u_0, D) hits the fixed set $\{((f')^{-1}(x-s), 0) : s \in \mathbb{R}\}$ and we can use Theorem 2.3 to split u_0 at $a(x)$:

$$(u_0, D)_{s<a(x)} \text{ and } (u_0, D)_{s>a(x)},$$

and consequently,

$$(u_0(s))_{s<a(x)} \text{ and } (u_0(s))_{s>a(x)}$$

are conditionally independent given $(f')^{-1}(x - a(x))$ —that is, given $a(x)$.

Putting together the first and third steps of the proof we find that $a(x)$ is Markov, noting that the law of the increments of the process varies with x. Therefore, $u(x) = (f')^{-1}(x - a(x))$ is a spectrally negative Markov process. $\square$

An additional class of initial data that we would like to consider is noise (the generalized derivative of a process with independent increments). Noise initial data naturally arises in the context of renormalized u_0, for which short-range correlations of the velocity profile vanish. As we shall see, the closure result also extends to this case. Here we prescribe the law of the data at the level of the potential

$$U_0(x) = \int_0^x u_0(r)dr.$$

Let

$$U_0(x) = \begin{cases} X_x & x \geq 0 \\ -\tilde{X}_{-x^-} & x < 0 \end{cases}, \tag{3.6}$$

where $X, \tilde{X}$ are independent copies of a one-sided spectrally negative additive process starting at 0. If the process has stationary increments this reduces to the Lévy case.

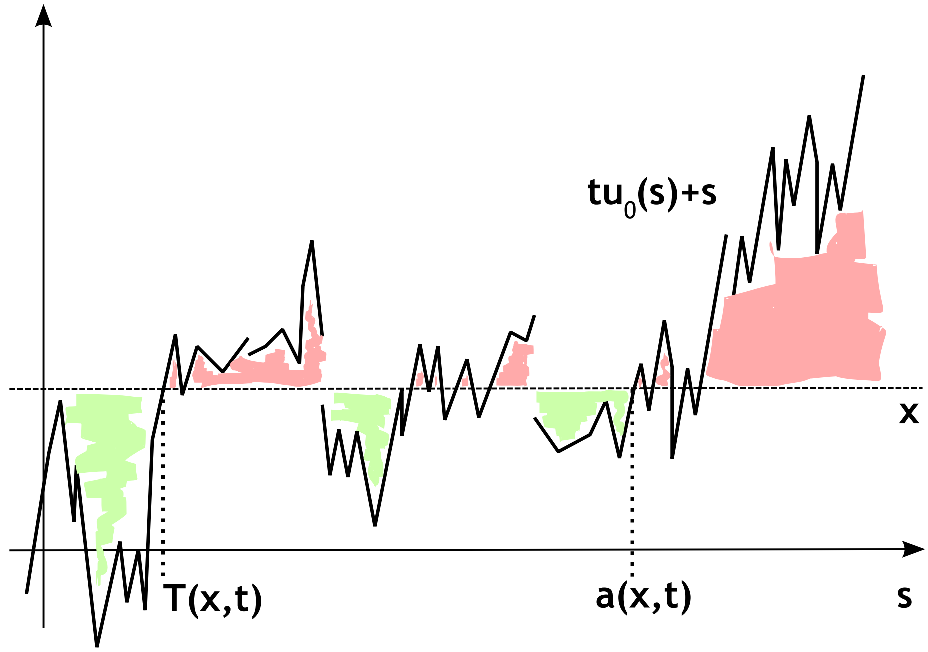

FIGURE 3.2. Splitting of u_0 at the last hitting time $a(x)$ of (u_0, D) in the case of Burgers flux. In this case, $a(x)$ is the last time the integral $\int_0^s (u_0(r) + r - x)dr$ is minimized. This cannot occur at a downward jump of u_0, so we necessarily have that $u_0(a(x)) = x - a(x)$. Also shown is the first passage time $T(x)$ of u_0 at x.

In addition, assume U_0 satisfies the growth condition (2.13) almost surely. Then we have the following:

THEOREM 3.2. *Suppose $U_0 = (\Omega, \mathcal{G}, \mathcal{G}_s, U_0(s), M_0)$ is a two-sided spectrally negative process with independent increments and satisfies (2.13) a.s. Then under M_0 and for all $t > 0$, $u(x,t)$ is a spectrally negative Markov process.*

PROOF. Again we fix $x \in \mathbb{R}$, and take $t = 1$ without loss of generality.

1. *Dependence structure of* $(a(x))_{x\in\mathbb{R}}$: As before, for $h>0$

$$
\begin{aligned}
a(x+h) &= \arg^+\min_{s\in\mathbb{R}} I(s;x+h) \\
&= a(x) + \arg^+\min_{s\geq 0}\{I(a(x)+s;x+h) - I(a(x);x+h)\} \\
&= a(x) + \arg^+\min_{s\geq 0}\{U_0(a(x)+s) - U_0(a(x)) \qquad (3.7)\\
&\qquad\qquad + f^*\left(x+h-(a(x)+s)\right) - f^*\left(x+h-a(x)\right)\}
\end{aligned}
$$

Given $a(x)$, the increment $a(x+h)-a(x)$ therefore depends on U_0 only through the increment $U_0(a(x)+s)-U_0(a(x))$. The same argument gives that $(a(x-h)-a(x))_{h\geq 0}$ depends only on $(U_0(s))_{s<a(x)}$.

2. *Spectral negativity of* U_0: If U_0 is spectrally negative, $U_0(s)\leq U_0(s^-)$. In particular

$$U_0(a(x)^-)\wedge U_0(a(x)) = U_0(a(x)).$$

Without spectral negativity, $U_0(a(x)^-)\wedge U_0(a(x))$ cannot be reduced to any simpler form.

3. *Markov property of* U_0 *at* $a(x)$: We use Theorem 2.2. Consider the functional

$$
\begin{aligned}
g(s) &= \big(U_0(s^-) + f^*(x-s)\big) \wedge (U_0(s) + f^*(x-s)) \\
&= (U_0(s^-)\wedge U_0(s)) + f^*(x-s).
\end{aligned}
$$

Then the inverse Lagrangian satisfies

$$a(x) = \sup\left\{s\in\mathbb{R} : g(s) = \min_{r\in\mathbb{R}} g(r)\right\}.$$

Since U_0 is Markov, Theorem 2.2 implies that $(U_0(s))_{s\geq a(x)}$ is independent of $(U_0(s))_{s<a(x)}$ given $(U_0(a(x)), g(a(x)))$ —that is, given

$$\big(U_0(a(x)), \{U_0(a(x)^-)\wedge U_0(a(x))\} + f^*(x-a(x))\big). \qquad (3.8)$$

Now we use the spectral negativity of U_0. Since $U_0(a(x)^-)\wedge U_0(a(x)) = U_0(a(x))$, we have that the information in (3.8) is equivalent to that of $(U_0(a(x)), a(x))$. Note that if we did not have spectral negativity then we would have to be given $g(a(x))$.

Putting together steps 1 and 3, we have that the increment $a(x+h) - a(x)$ is independent of $(a(x-h)-a(x))_{h\geq 0}$ given $(U_0(a(x)), a(x))$. Since $a(x+h)-a(x)$ only depends on U_0 through the increment $U_0(a(x)+s) - U_0(a(x))$, it is independent of $U_0(a(x))$ as well. Therefore, $a(x)$ is a Markov process and $u(x)$ is spectrally negative Markov. Notice that if U_0 was not spectrally negative or did not have independent increments, then $a(x+h)-a(x)$ would depend on $g(a(x))$ or $U_0(a(x))$ in addition to $a(x)$, hence destroying the Markov property. $\square$

Remark 3.1. For symmetric fluxes $f(u) = f(-u)$, the condition of spectral negativity of U_0 is somewhat artificial since it is entirely dependent on the continuity properties of the potential. If we switch from càdlàg paths to those that are left-continuous with right limits, then the closure argument holds true for U_0 spectrally positive. In essence, given a fixed coordinate framework we will inevitably be restricted to a condition of spectral monotonicity for the potential. Deterministically, this arises due to the invariance of the conservation law under the change of variables $x \to -x$, $u \to -u$. Probabilistically, it is needed for the statistics of U_0 to split without requiring an auxiliary variable aside from $a(x)$.

We would ideally like to extend the closure theorems to include the Feller property in order to be able to define an infinitesimal generator for the solution process for any fixed $t > 0$. Essentially, this corresponds to the propagation in time of a certain spatial regularity of the initial data u_0. In the case of noise initial data, an analogous result must be shown. At the present time we do not have a proof of this, although we strongly believe it to be true. For now we shall make the assumption that it holds:

Assumption 3.1. In addition to the conditions of Theorem 3.1, suppose u_0 satisfies the Feller property. Then $a(x,t)$ and $u(x,t)$ are Feller processes as well.

To conclude the section, we make several remarks contrasting the given results to those available in the literature.

Remark 3.2. Theorem 3.1 is similar to previous results by Bertoin [**8**] (with Burgers flux and u_0 a one-sided Lévy process) and Winkel [**49**] (in the case where the

Hopf-Lax functional is modified and u_0 is a one-sided, spectrally negative, Markov process). Our result reduces to that of [**8**] if $f(u) = u^2/2$ (Burgers flux) and u_0 is a one-sided Lévy process. To see this, use (3.5) in (3.4):

$$\begin{aligned} a(x+h) - a(x) &= \arg^+ \min_{s \geq 0} \left\{ \int_{a(x)}^{a(x)+s} (u_0(r) - u_0(a(x)) + r - a(x))\, dr \right\} \\ &= \arg^+ \min_{s \geq 0} \left\{ \int_0^s (u_0(a(x)+r) - u_0(a(x)) + r)\, dr \right\}. \end{aligned}$$

Since u_0 has independent increments, the integrand is independent of $(u_0(s))_{s<a(x)}$ and has the same law as $u_0(r) - u_0(0) + r$. So, the law of $a(x+h) - a(x)$ is independent of $a(x-h) - a(x))_{h \geq 0}$ and does not vary with x. Note that it is necessary to use the Burgers flux to obtain this result, and to show the increments of $a(x)$ are identical in law to those of the first hitting process $T(x) = \inf\{s \geq 0 : u_0(s) + s \geq x\}$.

Remark 3.3. The two-point closure for Burgers equation with white noise initial data was implicitly derived by Burgers himself [**12**]. An explicit proof was given in [**3**]. The exact computation of the statistics was first given in Groeneboom [**31**], followed by [**24**]. Theorem 3.2 extends the closure proof to general fluxes and a large class of noise initial data using the same methods.

2. Evolution equation for the 2-point statistics

The closure property of 1-D scalar conservation laws with spectrally negative, Feller initial data allows us to develop an evolution (or master) equation describing how the statistics of the entropy solution $u(x,t)$ evolve in time. Since Feller processes are completely characterized by their infinitesimal generator, the evolution is given in terms of the generator of the process. We now derive this result by two independent methods—BV calculus and the Hopf functional approach—which are detailed below. The main result is as follows.

Recall from Section 2 that the Markov process $u(x,t)$ is characterized by its 1- and 2-point (inhomogeneous) operators $(\mathcal{P}(t))_{x\in\mathbb{R}}$ and $(\mathcal{Q}_{x,x+h}(t))_{x\in\mathbb{R},h\geq 0}$. If u_0 is a stationary process it is easy to see that u is as well, and these reduce to $\mathcal{P}(t)$ and

$(\mathcal{Q}_h(t))_{h\geq 0}$ with $\mathcal{Q}_h = \mathcal{Q}_{0,h}$. We will consider this case throughout the remainder of the section.

Since $u(x,t)$ is Feller, for $t > 0$ there exists a generator $\mathcal{A}(t)$ of the semigroup $(\mathcal{Q}_h(t))_{h\geq 0}$. By the representation (2.20), for all test functions $\eta \in C_c^1(\mathbb{R})$

$$\mathcal{A}(t)\eta(u) = b(u,t)\eta'(u) + \int_{(-\infty,u)} n(u,dv,t)(\eta(v) - \eta(u)). \tag{3.9}$$

$b(u,t)$ is the drift and $n(u,dv,t)$ the jump measure of the process. Since sample paths are solutions to (3.1), they are of bounded variation and the diffusion and compensation terms of $\mathcal{A}(t)$ have vanished. In addition, nontrivial solutions have non-negative drift and the jump measure satisfies (2.21):

$$b(u,t) \geq 0 \qquad \text{and} \qquad \int_{(-\infty,u)} n(u,dv,t)(1 \wedge |u-v|) < \infty. \tag{3.10}$$

The integrability condition restricts the number of vanishingly small jumps of the process. Note that $n(u,dw,t)$ is supported on $(-\infty,u)$ due to the entropy condition of the solution.

To find an evolution equation for the generator we need only determine the evolution of the 1- and 2-point correlation functions with $\eta,\psi \in C_c^1(\mathbb{R})$and $\alpha > 0$:

$$\mathcal{P}(t)\eta = E[\eta(u(x,t))], \qquad \mathcal{P}(t)\eta\mathcal{Q}_\alpha(t)\psi = E[\eta(u(x,t))\psi(u(x+\alpha,t))]. \tag{3.11}$$

Define the operator $\mathcal{B}(t)$ associated to $\mathcal{A}(t)$ and flux f for $\eta \in C_c^1(\mathbb{R})$ by

$$[\mathcal{B}(t)\eta](u) = b(u,t)f'(u)\eta'(u) + \int_{(-\infty,u)} n(u,dv,t)\frac{f(v)-f(u)}{v-u}(\eta(v) - \eta(u)). \tag{3.12}$$

Then the evolution satisfies the following:

Theorem 3.3. *Let u_0 be a spectrally negative, stationary Feller process for which (2.12) holds a.s. For all $\eta,\psi \in C^1(\mathbb{R})$ and $\alpha > 0$ the 1- and 2-point functions of $(u(\cdot,t))_{t>0}$ satisfy*

$$\partial_t \mathcal{P}\eta(u) = -\mathcal{P}\mathcal{B}\eta(u) \tag{3.13}$$

$$\partial_t(\mathcal{P}\eta\mathcal{Q}_\alpha\psi) = -\mathcal{P}(\mathcal{B}\eta\mathcal{Q}_\alpha\psi - \eta\mathcal{B}\mathcal{Q}_\alpha\psi + \eta\mathcal{Q}_\alpha\mathcal{B}\psi). \tag{3.14}$$

Therefore, one has that

$$\partial_t\mathcal{P} = -\mathcal{P}\mathcal{B} \tag{3.15}$$

$$\partial_t \mathcal{Q}_\alpha = [\mathcal{B}, \mathcal{Q}_\alpha] \tag{3.16}$$

$$\partial_t \mathcal{A} = [\mathcal{B}, \mathcal{A}]. \tag{3.17}$$

The Lax pair in (3.17) implies that Burgers turbulence under the 2-point closure naturally leads to a completely integrable system. It is therefore a Hamiltonian system with an infinite number of conserved quantities. This integrable structure is quite remarkable and indicates that classical techniques such as inverse scattering analysis can be used to derive solutions for the evolution equation. As we shall see in Chapter 4, hints of this structure again appear when considering an explicitly known exact solution to (3.17). We will not pursue this direction any further in the present work.

Remark 3.4. Understanding the $t \downarrow 0$ limit for the statistics of $u(x,t)$ poses an interesting and difficult challenge. Here we would like to classify the form of all generators $\mathcal{A}(0)$ that can arise in the singular limit, particularly those corresponding to u_0 of unbounded variation (such as an Ornstein-Uhlenbeck process) or u_0 a noise. In the noise case, this problem would also provide valuable insight on the corresponding Hamilton-Jacobi equation (for the potential) with random initial data. The only result we know of in this direction is the determination of *eternal solutions* to Burgers equation with Lévy process initial data, as discussed in Bertoin [**10**]. A relevant result for the white noise case is given in [**27**].

We now give two derivations of Theorem 3.3, the first using a chain rule for functions of bounded variation originally given by Vol'pert [**48**] and the second using a functional approach pioneered by Hopf [**34**].

2.1. BV calculus derivation. Recall the canonical framework presented in Sections 2 and 1. Let $\Omega' = \Omega \cap BV_{loc}(\mathbb{R})$ be the space of càdlàg paths of locally bounded variation. Starting with random initial data u_0 with law μ_0, consider the evolution of the probability measures $(\mu_t)_{t>0}$ on $\Omega' \subset \Omega$ induced by the entropy solution $(u(\cdot, t; u_0))_{t>0}$. Here, u_0 is assumed to satisfy the conditions of Theorem (3.1) so that

u is a spectrally negative Feller process. We will derive the equations (3.13-3.14) for the 1- and 2-point functions in the following manner. In the case of the 1-point function, we seek to evaluate the quantity $\partial_t \mathcal{P}\eta(u) = \partial_t E[\eta(u)]$. Naively, if the process u were a classical solution to (3.1) we would bring the time derivative inside the expected value and use the conservation law to obtain

$$\partial_t E[\eta(u)] = -E[\eta'(u)\partial_x f(u)] = -E[f'(u)\partial_x \eta(u)].$$

Approximating the spatial derivative in the last expression by finite differences, we could then express this using the Markov semigroup of u:

$$\begin{aligned} -E[f'(u)\partial_x \eta(u)] &= \lim_{h\downarrow 0} h^{-1} E[f'(u(x))\{\eta(u(x)) - \eta(u(x-h))\}] \\ &= \lim_{h\downarrow 0} h^{-1} \mathcal{P}\{-\eta \mathcal{Q}_h f' + \mathcal{Q}_h(\eta f')\} \end{aligned}$$

This limit can be expressed in terms of the generator $\mathcal{A}$ of the process. A similar formal analysis can also be given for the 2-point function. It should be obvious that these calculations do not hold since they completely disregard the most important part of the structure of u, namely, the shocks. As we shall see, however, since u is of bounded variation we can use an argument of the same flavor to obtain the proper evolution equations. See [**20**] for a related approach in the derivation of kinetic hierarchies for forced Burgers turbulence.

To begin, fix $t > 0$. We will make frequent use of the chain rule for BV functions introduced in [**48**]. To be explicit, let $h > 0$ and define the functions

$$u_{-h}(x,t) = u(x-h,t), \qquad u_-(x,t) = \lim_{h\downarrow 0} u_{-h}(x,t)$$

where the pointwise limit is taken. Note that this is well-defined and that we need not define an analogous quantity for the right limit since u is càdlàg. Then for $\eta \in C^1(\mathbb{R})$, the x-component $[\eta'](u(x,t))$ of $[\nabla_{(x,t)}\eta](u(x,t))$ and the averaged quantity $\bar{\eta}$ equal

$$[\eta'](u) = \int_0^1 \eta'(u_- + \beta(u - u_-))d\beta = \begin{cases} \frac{\eta(u)-\eta(u_-)}{u-u_-} & \text{if } u \neq u_- \\ \eta'(u) & \text{if } u = u_- \end{cases}$$

$$\bar{\eta}(u) = \frac{1}{2}(\eta(u) + \eta(u_-)).$$

We will make use of natural approximations of these quantities, defined by

$$[\eta']_h(u) = \int_0^1 \eta'(u_{-h} + \beta(u - u_{-h}))d\beta = \begin{cases} \frac{\eta(u)-\eta(u_{-h})}{u-u_{-h}} & \text{if } u \neq u_{-h} \\ \eta'(u) & \text{if } u = u_{-h} \end{cases}$$

$$\bar{\eta}_h(u) = \frac{1}{2}(\eta(u) + \eta(u_{-h})).$$

With $\psi \in C^1(\mathbb{R})$, the following chain rules hold in the sense of equality of measures on Borel sets:

$$\partial_x \eta = [\eta']\partial_x u \tag{3.18}$$

$$\partial_x(\eta\psi) = \bar{\eta}\partial_x\psi + \partial_x\eta\bar{\psi} = \bar{\eta}([\psi']\partial_x u) + ([\eta']\partial_x u)\bar{\psi}. \tag{3.19}$$

The proof of Theorem 3.3 is now given, wherein the closure theorem and the previous lemma are used to compute the evolution of $\mathcal{P}$, $\mathcal{Q}_\alpha$, and the generator $\mathcal{A}$.

Proof. (of Theorem 3.3)

1. *Derivation for the 1-point function*: As before, for $\eta \in C_c^1(\mathbb{R})$

$$\mathcal{P}(t)\eta = E[\eta(u(x,t))] = \int_{\Omega'} \eta(u(x,t;u_0))\mu_0(du_0)$$

where du_0 is a Borel set in Ω' and we have made the dependence of u on u_0 explicit. The above integral makes sense since for fixed x and t, $u(x,t;u_0) : \Omega' \to \mathbb{R}$. Given this interpretation we could ask for an evolution equation for $E[\eta(u(x,t))]$ by testing its time derivative against $\varphi \in C_c^1(\mathbb{R} \times [0,\infty))$:

$$\begin{aligned} &\int_{\mathbb{R}\times[0,\infty)} \varphi(x,t)\partial_t E[\eta(u(x,t))]dxdt \\ &\qquad = \int_{\mathbb{R}\times[0,\infty)} \varphi(x,t)\partial_t \left(\int_{\Omega'} \eta(u(x,t;u_0))\mu_0(du_0)\right) dxdt \\ &\qquad = \int_{\Omega'} \left(\int_{\mathbb{R}\times[0,\infty)} \varphi(x,t)\partial_t\eta(u(x,t;u_0))dxdt\right) \mu_0(du_0). \end{aligned} \tag{3.20}$$

Here we have used integration by parts and Fubini's theorem for the second equality. Since $u \in BV_{loc}(\mathbb{R}\times[0,\infty))$, we can use the BV chain rule (3.18) and Radon-Nikodym theorem with (3.1) to get

$$\partial_t\eta = [\eta']\partial_t u = -[\eta']\partial_x f = -[\eta'][f']\partial_x u = -[f']\partial_x\eta$$

in the sense of equality of measures on Borel sets in $\mathbb{R} \times [0, \infty)$. The last expression in (3.20) is then

$$-\int_{\Omega'} \left(\int_{\mathbb{R}\times[0,\infty)} \varphi(x,t)[f'(u(x,t;u_0))]\partial_x \eta(u(x,t;u_0))dxdt \right) \mu_0(du_0). \tag{3.21}$$

In order to write this in terms of finite differences of the process u, we make the following assumption which remains unproven at the present time. It is a statement about the product of two weakly converging sequences and is therefore nontrivial:

Assumption 3.2. With $\varphi \in C_c^1(\mathbb{R} \times [0, \infty))$, $\eta \in C_c^1(\mathbb{R})$, f strictly convex and $u \in BV_{loc}(\mathbb{R} \times [0, \infty))$,

$$\begin{aligned} &\int_{\Omega'} \left(\int_{\mathbb{R}\times[0,\infty)} \varphi(x,t)[f'(u(x,t;u_0))]\partial_x \eta(u(x,t;u_0))dxdt \right) \mu_0(du_0) \\ &= \int_{\Omega'} \left(\int_{\mathbb{R}\times[0,\infty)} \varphi \lim_{h\downarrow 0} \frac{1}{h}\{[f']_h(\eta(u) - \eta(u_{-h}))\}dxdt \right) \mu_0(du_0) \end{aligned}$$

Now using Assumption 3.2 in (3.21) with another application of the Fubini and dominated convergence theorems gives that

$$\begin{aligned} &-\int_{\Omega'} \left(\int_{\mathbb{R}\times[0,\infty)} \varphi \lim_{h\downarrow 0} \frac{1}{h}\{[f']_h(\eta(u) - \eta(u_{-h}))\}dxdt \right) \mu_0(du_0) \\ &= -\int_{\mathbb{R}\times[0,\infty)} \varphi \left(\lim_{h\downarrow 0} \frac{1}{h} \int_{\Omega'} \{[f']_h(\eta(u) - \eta(u_{-h}))\}\mu_0(du_0) \right) dxdt \\ &= -\int_{\mathbb{R}\times[0,\infty)} \varphi \lim_{h\downarrow 0} \frac{1}{h} E[[f']_h(\eta(u) - \eta(u_{-h}))]dxdt. \end{aligned}$$

Therefore we have

$$\partial_t E[\eta(u(x,t))] = -\lim_{h\downarrow 0} \frac{1}{h} E[[f']_h(\eta(u) - \eta(u_{-h})] \tag{3.22}$$

in the sense of measures (or strongly, if the above expression is smooth enough). We show now that

$$\lim_{h\downarrow 0} \frac{1}{h} E[[f']_h(\eta(u) - \eta(u_{-h}))] = \mathcal{PB}\eta \tag{3.23}$$

which then yields the weak form of (3.15).

It is enough to prove (3.23) for a monomial $f(u) = u^m$. Since $\mathcal{B}$ depends linearly on f, (3.23) also holds for polynomials and by approximation for C^1 fluxes. Thus, assume $f(u) = u^m$ and expand the jump term

$$[f']_h = \frac{f(u) - f(u_{-h})}{u - u_{-h}} = \sum_{j=0}^{m-1} u_{-h}^{m-1-j} u^j$$

The limit in (3.23) is a sum of terms of the form

$$(\eta(u) - \eta(u_{-h}))u_{-h}^{m-1-j}u^j = u_{-h}^{m-1-j}\left(\eta(u)u^j - \eta(u_{-h})u_{-h}^j - \eta(u_{-h})\left(u^j - u_{-h}^j\right)\right). \tag{3.24}$$

Using the Markov semigroup, each of these terms have the corresponding limit

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}E\left[(\eta(u)\right. & \left.-\eta(u_{-h}))u_{-h}^{m-1-j}u^j\right] \\ &= \lim_{h\to 0}\frac{1}{h}\mathcal{P}\left(u_{-h}^{m-1-j}\left(\mathcal{Q}_h(u^j\eta) - u^j\eta - \eta\left(\mathcal{Q}_h u^j - u^j\right)\right)\right) \\ &= \mathcal{P}\left(u^{m-1-j}\left(\mathcal{A}(u^j\eta) - \eta\mathcal{A}(u^j)\right)\right). \end{aligned} \tag{3.25}$$

The term $\mathcal{A}(u^j\eta) - \eta\mathcal{A}(u^j)$ can be computed using (3.9). The drift term is

$$u^{m-1-j}\left(b(u^j\eta)' - \eta b(u^j)'\right) = bu^{m-1}\eta'.$$

Similarly, the jump term simplifies to

$$u^{m-1-j}\int_{\mathbb{R}} n(u, dw)w^j(\eta(w) - \eta(u)).$$

Summing over all the terms in the expansion of (3.23) and using the definition (3.12) for $\mathcal{B}$, we obtain (3.13), which is the weak form of (3.15):

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}E[[f']_h(\eta(u) - \eta(u_{-h}))] & \\ &= \sum_{j=0}^{m-1}\mathcal{P}\left(u^{m-1-j}\left(\mathcal{A}(u^j\eta) - \eta\mathcal{A}(u^j)\right)\right) \\ &= \mathcal{P}\left(\sum_{j=0}^{m-1} bu^{m-1}\eta' + \int_{\mathbb{R}} n(u, dw)u^{m-1-j}w^j(\eta(w) - \eta(u))\right) \\ &= \mathcal{P}\left(bmu^{m-1}\eta' + \int_{\mathbb{R}} n(u, dw)\left(\frac{w^m - u^m}{w - u}\right)(\eta(w) - \eta(u))\right) \\ &= \mathcal{P}\mathcal{B}\eta. \end{aligned} \tag{3.26}$$

2. *Derivation for the 2-point function*: Let $\alpha > 0$ and consider the correlation of u at (x,t) and $(x+\alpha,t)$. As before, take η and ψ to be arbitrary test functions in $C_c^1(\mathbb{R})$. The 2-point function is given by

$$\mathcal{P}(t)(\eta\mathcal{Q}_\alpha(t)\psi) = E\left[\eta(u(x,t))\psi(u(x+\alpha,t))\right].$$

As before we combine the chain rule (3.18) and the conservation law to obtain

$$\partial_t\eta(u) = -[f']\partial_x\eta(u), \qquad \partial_t\psi(u) = -[f']\partial_x\psi(u).$$

The product rule (3.19) then yields

$$\begin{aligned}\partial_t(\eta(u(x,t))\psi(u(x+\alpha,t))) &= (\partial_t\eta)(x,t)\bar{\psi}(x+\alpha,t) + \bar{\eta}(x,t)(\partial_t\psi)(x+\alpha,t)\\ &= -([f']\partial_x\eta)(x,t)\bar{\psi}(x+\alpha,t) - \bar{\eta}([f']\partial_x\psi)(x+\alpha,t).\end{aligned} \tag{3.27}$$

We can compute the expected value of these expressions exactly as we did for the 1-point function. Consider the first term on the r.h.s. of the (3.27). Again using Assumption 3.2

$$\begin{aligned}&E[([f']\partial_x\eta)(x,t)\bar{\psi}(x+\alpha,t)]\\ &\qquad = \lim_{h\to 0}\frac{1}{h}E\left[[f']_h(u)(\eta(u)-\eta(u_{-h}))\left(\frac{\psi(\theta_\alpha u)+\psi(\theta_\alpha u_{-h})}{2}\right)\right],\end{aligned} \tag{3.28}$$

where $\theta_h u = u(x,t;\theta_h u_0) = u(x+h,t;u_0)$ is the shift operator on path space. The mean value plays no role in the limit as it simply converges to $\psi(\theta_\alpha u)$. We therefore replace it in the expression above and use the Markov semigroup and (3.26) to find that (3.28) is

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{h}E\left[[f']_h(u)(\eta(u)-\eta(u_{-h}))\mathcal{Q}_\alpha\psi(u)\right]\\ &= \lim_{h\to 0}\frac{1}{h}E\left[[f']_h(u)(\{(\eta\mathcal{Q}_\alpha\psi)(u)-(\eta\mathcal{Q}_\alpha\psi)(u_{-h})\}\right.\\ &\qquad\qquad \left.-\eta(u_{-h})\{(\mathcal{Q}_\alpha\psi)(u)-(\mathcal{Q}_\alpha\psi)(u_{-h})\})\right]\\ &= \mathcal{P}(\mathcal{B}(\eta\mathcal{Q}_\alpha\psi)-\eta\mathcal{B}\mathcal{Q}_\alpha\psi).\end{aligned} \tag{3.29}$$

We now compute the second term on the right hand side of (3.27):

$$E[\bar{\eta}(x,t)([f']\partial_x\psi)(x+\alpha,t)] \\ = \lim_{h\to 0}\frac{1}{h}E\left[\left(\frac{\eta(u)+\eta(u_{-h})}{2}\right)[f']_h(\theta_\alpha u)(\psi(\theta_\alpha u)-\psi(\theta_\alpha u_{-h}))\right]. \tag{3.30}$$

Again we can do away with the mean value and use a derivation nearly identical to (3.26) to obtain

$$\lim_{h\to 0}\frac{1}{h}E\left[\eta(u)\mathcal{Q}_\alpha([f']_h(u)(\psi(u)-\psi(u_{-h})))\right] = \mathcal{P}(\eta\mathcal{Q}_\alpha\mathcal{B}\psi). \tag{3.31}$$

Combining (3.29) and (3.31), we obtain (3.14):

$$\partial_t(\mathcal{P}(\eta\mathcal{Q}_\alpha\psi)) = -\mathcal{P}(\mathcal{B}(\eta\mathcal{Q}_\alpha\psi) - \eta\mathcal{B}\mathcal{Q}_\alpha\psi + \eta\mathcal{Q}_\alpha\mathcal{B}\psi).$$

3. *Derivation for the generator*: The left hand side of the evolution equation for the 2-point function (3.14) is simply

$$(\partial_t\mathcal{P})(\eta\mathcal{Q}_\alpha\psi) + \mathcal{P}(\eta(\partial_t\mathcal{Q}_\alpha)\psi),$$

and by (3.13),

$$(\partial_t\mathcal{P})(\eta\mathcal{Q}_\alpha\psi) = -\mathcal{P}\mathcal{B}(\eta\mathcal{Q}_\alpha\psi).$$

Thus, by substituting into (3.14) yields

$$\mathcal{P}(\eta(\partial_t\mathcal{Q}_\alpha)\psi) = \mathcal{P}(\eta(\mathcal{B}\mathcal{Q}_\alpha - \mathcal{Q}_\alpha\mathcal{B})\psi).$$

This equation holds for all η and ψ, so it can be written as

$$\partial_t\mathcal{Q}_\alpha = (\mathcal{B}\mathcal{Q}_\alpha - \mathcal{Q}_\alpha\mathcal{B}) = [\mathcal{B},\mathcal{Q}_\alpha],$$

Now take the limit $\alpha\to 0$ to get the main result:

$$\partial_t\mathcal{A} = [\mathcal{B},\mathcal{A}].$$

□

Lastly, we rederive the equation for the generator without the unnecessary assumption of stationarity of the underlying process. The equations will depend on x in addition to t. It is easy to show that a nearly identical derivation holds for the 1- and 2-point (inhomogeneous) operators $\mathcal{P}_x$, $\mathcal{Q}_{x,x+\alpha}$ using the (now inhomogeneous) generator $\mathcal{A}_x$ and associated generator

$$\mathcal{B}_x(t)\eta(u) = b_x(u,t)f'(u)\eta'(u) + \int_{(-\infty,u)} n_x(u,dv,t)\frac{f(v)-f(u)}{v-u}(\eta(v)-\eta(u)).$$

Therefore, we give the following general result without proof:

THEOREM 3.4. *Let u_0 be a spectrally negative Feller process. For all $\eta, \psi \in C_c^1(\mathbb{R})$ and $\alpha > 0$ the 1- and 2-point functions satisfy*

$$(\partial_t P_x)\eta = -P_x\mathcal{B}_x\eta \tag{3.32}$$

$$\partial_t(P_x\eta Q_{x,x+\alpha}\psi) = -P_x\{\mathcal{B}_x(\eta Q_{x,x+\alpha}\psi) - \eta\mathcal{B}_x Q_{x,x+\alpha}\psi + \eta Q_{x,x+\alpha}\mathcal{B}_{x+\alpha}\psi\} \tag{3.33}$$

so that the following relations hold:

$$\partial_t P_x = -P_x\mathcal{B}_x \tag{3.34}$$

$$\partial_t Q_{x,x+\alpha} = \mathcal{B}_x Q_{x,x+\alpha} - Q_{x,x+\alpha}\mathcal{B}_{x+\alpha} \tag{3.35}$$

$$\partial_t\mathcal{A}_x + \partial_x\mathcal{B}_x = [\mathcal{B}_x, \mathcal{A}_x]. \tag{3.36}$$

REMARK 3.5. To find the equation for $\mathcal{A}_x$, the only additional step is to use that

$$Q_{x,x+\alpha}\partial_\alpha\mathcal{B}_{x+\alpha}\,|_{\alpha=0} = Q_{x,x+\alpha}\partial_x\mathcal{B}_{x+\alpha}\big|_{\alpha=0} = \partial_x\mathcal{B}_x.$$

REMARK 3.6. The transport equation (3.36) is easily seen to be true for the trivial flux $f(u) = cu$. In this case, $\mathcal{B}_x \equiv c\mathcal{A}_x$ and (3.36) is

$$\partial_t\mathcal{A}_x + c\partial_x\mathcal{A}_x = 0$$

with solution $\mathcal{A}_x(t) = \mathcal{A}_{x-ct}(0)$. This simply corresponds to translation of the statistics with speed c, corresponding to sample paths of the solution being translated without modification.

2.2. Hopf functional derivation. An alternative, formal derivation of Theorem 3.3 can be given via the Hopf functional approach originally proposed by Hopf [**34**] and discussed in [**23**]. This method was used in [**16**] in the context of so-called statistical solutions to Burgers equation. It should be emphasized that the notion of a statistical solution is considerably weaker than that of the entropy solution, as we shall see below. Despite this, combined with the rigorous closure of Section (1) this method leads to the correct evolution equation.

Again we start the canonical framework as presented in the previous subsection. As before, we drop notational dependence of x or t whenever possible. Recall that the initial data u_0 has law μ_0 on Ω. Let $u(t; u_0) : [0, \infty) \to \Omega$ be a weak solution of (3.1) with initial data u_0, which induces a sequence of probability measures $(\mu_t)_{t>0}$ on Ω. To derive an equation for the flow of measures $(\mu_t)_{t>0}$, make the assumption that u is differentiable in t so that the conservation law can be written for all $\varphi \in C_c^\infty(\mathbb{R})$ as

$$\partial_t \langle u(t; u_0), \varphi\rangle - \langle f(u(t; u_0)), \varphi'\rangle = 0. \tag{3.37}$$

Here, $' = \partial_x$ and $\langle \cdot, \cdot\rangle$ is the standard duality pairing. This is of course incorrect since the solution has shocks, and (3.37) should technically be replaced by a continuous dynamical system with a suitable weak topology.

Define the Hopf characteristic functional $\hat{\mu}_t$ of the law μ_t

$$\hat{\mu}_t(\varphi) = \int_\Omega e^{i\langle u,\varphi\rangle} \mu_t(du) \tag{3.38}$$

which evolves according to

$$\begin{aligned} \partial_t \hat{\mu}_t(\varphi) = \partial_t \int_\Omega e^{i\langle u,\varphi\rangle} \mu_t(du) &= \partial_t \int_\Omega e^{i\langle u(t;u_0),\varphi\rangle} \mu_0(du_0) \\ &= \int_\Omega i \left\langle f(u(t; u_0), \varphi'\right\rangle e^{i\langle u(t;u_0),\varphi\rangle} \mu_0(du_0) \\ &= \int_\Omega i \left\langle f(u), \varphi'\right\rangle e^{i\langle u,\varphi\rangle} \mu_t(du). \end{aligned} \tag{3.39}$$

Any set of probability measures $(\mu_t)_{t>0}$ on Ω that satisfies (3.39) for all $\varphi \in C_c^\infty(\mathbb{R})$ is defined to be a *statistical solution* of the scalar conservation law—in particular, the flow of measures generated by the entropy solution is a statistical solution. Assuming that all moments of μ_t are finite, the exponential can be expanded in series (denoting $d\mathbf{x}_n = \Pi_{j=1}^n dx_j$) as

$$e^{i\langle u,\varphi\rangle} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^n} \prod_{j=1}^{n} u(x_j)\varphi(x_j) d\mathbf{x}_n.$$

Let

$$E_{\mu_t}\left[\prod_{i=1}^{n} \eta_i(u(x_i))\right] = \int_{\Omega} \prod_{i=1}^{n} \eta_i(u(x_i))\mu_t(du).$$

Substituting the expansion for the exponential into (3.39) yields the infinite hierarchy

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^n} \partial_t E_{\mu_t}\left[\prod_{j=1}^{n} u(x_j)\varphi(x_j)\right] d\mathbf{x}_n \\
&= \sum_{n=0}^{\infty} \frac{i^{n+1}}{n!} \int_{\mathbb{R}^{n+1}} E_{\mu_t}\left[f(u)\varphi'(x)\prod_{j=1}^{n} u(x_j)\varphi(x_j)\right] dx d\mathbf{x}_n. \qquad (3.40)
\end{aligned}$$

We will need the following calculus trick:

Lemma 3.5. *If $g(x_1,\ldots,x_n) = g(x_{\sigma(1)},\ldots x_{\sigma(n)})$ for all permutations σ of $\{1,\ldots,n\}$ then*

$$\int_{\mathbb{R}^n} g(x_1,\ldots,x_n)d\mathbf{x}_n = n! \int_{x_1<x_2<\cdots<x_n} g(x_1,\ldots,x_n)d\mathbf{x}_n. \qquad (3.41)$$

Proof. Proceed by induction. For $n = 2$ one has

$$\begin{aligned}
\int_{\mathbb{R}^2} g(x_1,x_2)dx_1dx_2 &= \int_{x_1<x_2} g(x_1,x_2)dx_1dx_2 + \int_{x_2<x_1} g(x_1,x_2)dx_2dx_1 \\
&= 2\int_{x_1<x_2} g(x_1,x_2)dx_1dx_2
\end{aligned}$$

by exchanging x_1 and x_2 in the second integral and using $g(x_2,x_1) = g(x_1,x_2)$. Assume the proposition holds for the case $n-1$ and consider the case n:

$$\begin{aligned}
\int_{\mathbb{R}^n} g(x_1,\ldots,x_n)d\mathbf{x}_n &= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} g(x_1,\ldots,x_{n-1},x_n)d\mathbf{x}_{n-1}dx_n \\
&= (n-1)!\int_{\mathbb{R}}\int_{x_1<x_2<\cdots<x_{n-1}} g(x_1,\ldots,x_{n-1},x_n)d\mathbf{x}_{n-1}dx_n \\
&= (n-1)!\left[\sum_{j=1}^{n}\int_{x_1<\cdots<x_{j-1}<x_n<x_j<\cdots<x_{n-1}} g(x_1,\ldots,x_{n-1},x_n)d\mathbf{x}_n\right] \\
&= n!\int_{x_1<x_2<\cdots<x_n} g(x_1,\ldots,x_n)d\mathbf{x}_n
\end{aligned}$$

where the last line follows from exchanging indices in each term of the sum (with $x_0 = -\infty$). $\square$

Resuming the derivation, use

$$g(x_1, \ldots, x_{n+1}) = \frac{1}{n} \sum_{k=1}^{n+1} E_{\mu_t} \left[f(u(x_k))\varphi'(x_k) \prod_{j \neq k} u(x_j)\varphi(x_j) \right],$$

in (3.41) and substitute into (3.40) to obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} i^n \int_{x_1 < \cdots < x_n} \partial_t E_{\mu_t} \left[\prod_{j=1}^{n} u(x_j)\varphi(x_j) \right] d\mathbf{x}_n \\ &= \sum_{n=0}^{\infty} i^{n+1} \int_{x_1 < \cdots < x_{n+1}} \sum_{k=1}^{n+1} E_{\mu_t} \left[f(u(x_k))\varphi'(x_k) \prod_{j \neq k} u(x_j)\varphi(x_j) \right] d\mathbf{x}_n. \end{aligned} \tag{3.42}$$

Re-indexing the l.h.s. of (3.42) and subtracting the r.h.s. gives that

$$\begin{aligned} \sum_{n=1}^{\infty} i^n \int_{x_1 < \cdots < x_n} &\left\{ \partial_t E_{\mu_t} \left[\prod_{j=1}^{n} u(x_j) \right] \prod_{j=1}^{n} \varphi(x_j) \right. \\ &\left. - \sum_{k=1}^{n} E_{\mu_t} \left[f(u(x_k)) \prod_{j \neq k} u(x_j) \right] \varphi'(x_k) \prod_{j \neq k} \varphi(x_j) \right\} d\mathbf{x}_n = 0. \end{aligned} \tag{3.43}$$

Finally, the statistical hierarchy (3.43) can be considerably simplified upon making the closure assumption that $(\mu_t)_{t>0}$ is the law of a stationary Feller process with 1- and 2-point operators $\mathcal{P}$ and $(\mathcal{Q}_\alpha)_{\alpha>0}$. Recall that in terms of the 1-point and transition measures these are

$$\mathcal{P}\eta = \int_{\mathbb{R}} p(du)\eta(u), \qquad [\mathcal{Q}_h\eta](u) = \int_{\mathbb{R}} q_h(u, dv)\eta(u).$$

With $h_i = x_i - x_{i-1}$ and $\mathcal{Q}_i = \mathcal{Q}_{h_i}$ for $i = 2, \ldots, n$,

$$\begin{aligned} &E_{\mu_t} \left[\prod_{i=1}^{n} \eta_i(u(x_i)) \right] \\ &\quad = \int_{\mathbb{R}} p(du_1, t)\eta_1(u_1) \int_{\mathbb{R}} q_{h_2}(u_1, du_2, t)\eta_2(u_2) \cdots \int_{\mathbb{R}} q_{h_n}(u_{n-1}, du_n, t)\eta_n(u_n) \\ &\quad = \mathcal{P}\eta_1\mathcal{Q}_2\eta_2 \cdots \mathcal{Q}_n\eta_n. \end{aligned}$$

Assume the transition measures q_h are differentiable in h. Transferring the derivative on the test function $\varphi'(x_k)$ onto $q_{h_k}(u_{k-1}, du_k)$ in (3.42) and using integration by parts, the statistical hierarchy is

$$\sum_{n=1}^{\infty} i^n \int_{x_1<\cdots<x_n} \{\partial_t E_{\mu_t}\left[\prod_{j=1}^{n} u(x_j)\right] + \sum_{k=1}^{n} \partial_{x_k} E_{\mu_t}\left[f(u(x_k))\prod_{j\neq k} u(x_j)\right]\Bigg\} \prod_{j=1}^{n} \varphi(x_j) d\mathbf{x}_n = 0.$$

Note that the boundary terms in the previous equation vanish due to cancellation of terms and the fact that $\varphi \in C_c^{\infty}(\mathbb{R})$ has compact support. The density of tensor products of the form $\phi(x_1, \ldots, x_n) = \prod_{j=1}^{n} \varphi(x_j)$ in the space of test functions on $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 < \cdots < x_n\}$ implies that for $n \in \mathbb{N}$ and $x_1 < x_2 < \cdots < x_n$, the following infinite set of equations hold:

$$\partial_t E_{\mu_t}\left[\prod_{j=1}^{n} u(x_j)\right] = -\sum_{k=1}^{n} \partial_{x_k} E_{\mu_t}\left[f(u(x_k))\prod_{j\neq k} u(x_j)\right]. \tag{3.44}$$

In terms of the operators $\mathcal{Q}_i' = \partial_{h_i}\mathcal{Q}_i$ (denoting $h_1 = x_1$, $\mathcal{Q}_1 = \mathcal{P}$, $\mathcal{Q}_{n+1} = I$) this is

$$\partial_t E_{\mu_t}\left[\prod_{j=1}^{n} u(x_j)\right] = \sum_{k=1}^{n}\{\mathcal{P}u\mathcal{Q}_2 u\cdots\mathcal{Q}_k f(u)\mathcal{Q}_{k+1}' u\cdots\mathcal{Q}_n u - \mathcal{P}u\mathcal{Q}_2 u\cdots\mathcal{Q}_k' f(u)\mathcal{Q}_{k+1} u\cdots\mathcal{Q}_n u\},$$

or equivalently, in terms of the generator $\mathcal{A}$ (using $\mathcal{Q}_i' = \mathcal{A}\mathcal{Q}_i$)

$$\partial_t E_{\mu_t}\left[\prod_{j=1}^{n} u(x_j)\right] = \sum_{k=1}^{n} \mathcal{P}u\mathcal{Q}_2 u\cdots\mathcal{Q}_k\left[f(u), \mathcal{A}\right]\mathcal{Q}_{k+1} u\cdots\mathcal{Q}_n u.$$

One obtains the evolution equation for the 1-point function by taking $h_i \to 0$ for all i (i.e., $\mathcal{Q}_i = I$ for all i), or the equation for the 2-point function by taking $h_{l+1} = \alpha$ for $l \in \{1, \ldots, n-1\}$ and $h_i \to 0$ for all $i \neq l+1$ (i.e., $\mathcal{Q}_{l+1} = \mathcal{Q}_\alpha$ and $\mathcal{Q}_i = I$ for all $i \neq l+1$):

$$\partial_t E_{\mu_t}\left[(u(x_1))^n\right] = \sum_{k=1}^{n} \mathcal{P}u^{k-1}\left[f(u), \mathcal{A}\right] u^{n-k} \tag{3.45}$$

$$\partial_t E_{\mu_t}\left[(u(x_1))^l(u(x_1+\alpha))^{n-l}\right] = \sum_{k=1}^{l} \mathcal{P}u^{k-1}\left[f(u),\mathcal{A}\right]u^{l-k}\mathcal{Q}_\alpha u^{n-l} + \sum_{k=l+1}^{n} \mathcal{P}u^l\mathcal{Q}_\alpha u^{k-(l+1)}\left[f(u),\mathcal{A}\right]u^{n-k} \tag{3.46}$$

Using the expression (3.9) for the generator $\mathcal{A}$, (3.45) and (3.46) simplify:

$$\begin{aligned}\partial_t\mathcal{P}u^n &= -\sum_{k=1}^{n}\mathcal{P}u^{k-1}\left\{b(u)f'(u)u^{n-k} + \int_{\mathbb{R}} n(u,dv)(f(v)-f(u))v^{n-k}\right\}\\ &= -\mathcal{P}\left\{b(u)f'(u)(u^n)' + \int_{\mathbb{R}} n(u,dv)\frac{f(v)-f(u)}{v-u}(v^n-u^n)\right\}\end{aligned}$$

$$\begin{aligned}&\partial_t(\mathcal{P}u^l\mathcal{Q}u^{n-l})\\ &= -\sum_{k=1}^{l}\mathcal{P}u^{k-1}\left\{b(u)f'(u)u^{l-k}\mathcal{Q}_\alpha u^{n-l} + \int_{\mathbb{R}} n(u,dv)(f(v)-f(u))v^{l-k}\mathcal{Q}_\alpha v^{n-l}\right\}\\ &- \sum_{k=l+1}^{n}\mathcal{P}u^l\mathcal{Q}_\alpha u^{k-(l+1)}\left\{b(u)f'(u)u^{n-k} + \int_{\mathbb{R}} n(u,dv)(f(v)-f(u))v^{n-k}\right\}\\ &= -\mathcal{P}\left\{b(u)f'(u)(u^l)'\mathcal{Q}_\alpha u^{n-l} + \int_{\mathbb{R}} n(u,dv)\frac{f(v)-f(u)}{v-u}(v^l-u^l)\mathcal{Q}_\alpha v^{n-l}\right\}\\ &- \mathcal{P}u^l\mathcal{Q}_\alpha\left\{b(u)f'(u)(u^{n-l})' + \int_{\mathbb{R}} n(u,dv)\frac{f(v)-f(u)}{v-u}(v^{n-l}-u^{n-l}).\right.\end{aligned}$$

In terms of the operator $\mathcal{B}$ given in (3.12) the above expressions read

$$\partial_t\mathcal{P}u^n = -\mathcal{P}\mathcal{B}u^n, \qquad \partial_t(\mathcal{P}u^l\mathcal{Q}_\alpha u^{n-l}) = -\mathcal{P}(\mathcal{B}(u^l\mathcal{Q}_\alpha u^{n-l}) - u^l\mathcal{B}\mathcal{Q}_\alpha u^{n-l} + u^l\mathcal{Q}_\alpha\mathcal{B}u^{n-l}).$$

Approximating $\eta,\psi \in C_c^\infty(\mathbb{R})$ by polynomials and using that $\mathcal{B}$ is a linear operator, this yields the evolution equation for the 1- and 2-point operators as before:

$$\partial_t\mathcal{P}\eta = -\mathcal{P}\mathcal{B}\eta, \qquad \partial_t(\mathcal{P}\eta\mathcal{Q}_\alpha\psi) = -\mathcal{P}(\mathcal{B}(\eta\mathcal{Q}_\alpha\psi) - \eta\mathcal{B}\mathcal{Q}_\alpha\psi + \eta\mathcal{Q}_\alpha\mathcal{B}\psi).$$

Therefore we have the evolution $\partial_t\mathcal{A} = [\mathcal{B},\mathcal{A}]$ exactly as in the proof of Theorem 3.3.

CHAPTER 4

Kinetic equation and self-similar solutions

1. Derivation of kinetic equation

In this section we demonstrate that the evolution equation $\partial_t \mathcal{A} = [\mathcal{B}, \mathcal{A}]$ has a natural kinetic description in terms of the drift and jump components of the generator. To be precise, recall that in the stationary case the generator of the solution process $u(x,t)$ is characterized by its drift $b(u,t)$ and jump measure $n(u, dv, t)$. The restriction to stationary processes is, as before, for convenience and can be dropped. Using the notation

$$[f']_{a,b} = \frac{f(a) - f(b)}{a - b}$$

we have the following:

Remark 4.1. The evolution (3.17) of the generator (3.9) is given in terms of its characteristics by

$$\partial_t b(u,t) = -b^2(u,t) f''(u) \tag{4.1}$$

$$\partial_t n(u, dv, t) = D_f(b,n) + Q_f(n,n). \tag{4.2}$$

The free-streaming and collision operators in (4.2) are

$$\begin{aligned} D_f(b,n) =& \partial_u \left\{ n(u,dv) b(u) \left(f'(u) - [f']_{u,v} \right) \right\} + \partial_v \left\{ n(u,dv) b(v) \left(f'(v) - [f']_{v,u} \right) \right\} \\ & - n(u,dv) \left\{ b(u) f''(u) + b'(u) \left(f'(u) - [f']_{u,v} \right) \right\} \end{aligned} \tag{4.3}$$

$$\begin{aligned} Q_f(n,n) =& \int_{(v,u)} n(u,dw) n(w,dv) \left([f']_{u,w} - [f']_{w,v} \right) \\ & - n(u,dv) \int_{(-\infty,v)} n(v,dw) \left([f']_{u,v} - [f']_{v,w} \right) \\ & - n(u,dv) \int_{(-\infty,u)} n(u,dw) \left([f']_{u,w} - [f']_{u,v} \right). \end{aligned} \tag{4.4}$$

The above result yields a partial differential equation for the family of measures $(n(u, dv, t))_{u\in\mathbb{R}}$, which are coupled through the collision term (4.4). Since (4.1-4.2) are equivalent to (3.17), given initial data $(b_0(u), n_0(u, dv))$ that satisfy (3.10) we *automatically* have existence of weak (measure-valued) solutions to these equations. This is seen as follows. The characteristics (b_0, n_0) uniquely define (up to a μ_0-measure zero set on Ω) a Feller process $u_0(\omega; x)$. Each sample path admits a unique entropy solution $u(\omega; x, t)$, and the closure theorem implies that the process $u(x, t)$ is also Feller. Finally, the evolution equation ensures that $u(x, t)$ has characteristics $(b(t), n(t))$, which must therefore satisfy (4.1-4.2).

We now derive the kinetic equation. Let $\eta \in C_c^1(\mathbb{R})$. The l.h.s. of (3.17) is

$$(\partial_t \mathcal{A})\, \eta(u) = \partial_t b(u,t)\eta'(u) + \int_{\mathbb{R}} \partial_t n(u, dv, t)(\eta(v) - \eta(u)). \tag{4.5}$$

To evaluate the r.h.s. of (3.17) we first decompose the operators into their differential and integral parts. That is, let

$$\mathcal{A}(t) = \Delta_{\mathcal{A}} + \Sigma_{\mathcal{A}}, \qquad \mathcal{B}(t) = \Delta_{\mathcal{B}} + \Sigma_{\mathcal{B}}$$

where

$$\Delta_{\mathcal{A}}\eta(u) = b(u,t)\partial_u \eta(u), \qquad \Sigma_{\mathcal{A}}\eta(u) = \int_{\mathbb{R}} n(u, dv, t)(\eta(v) - \eta(u))$$
$$\Delta_{\mathcal{B}}\eta(u) = b(u,t)f'(u)\partial_u \eta(u), \qquad \Sigma_{\mathcal{B}}\eta(u) = \int_{\mathbb{R}} n(u, dv, t)[f']_{u,v}(\eta(v) - \eta(u)).$$

With this decomposition the bracket splits into

$$[\mathcal{B}, \mathcal{A}] = [\Delta_{\mathcal{B}}, \Delta_{\mathcal{A}}] + [\Delta_{\mathcal{B}}, \Sigma_{\mathcal{A}}] + [\Sigma_{\mathcal{B}}, \Delta_{\mathcal{A}}] + [\Sigma_{\mathcal{B}}, \Sigma_{\mathcal{A}}].$$

We show that the first term yields the r.h.s. of (4.1), the second and third terms give (4.3), and the last term yields (4.4). Evaluate each term separately:

$$[\Delta_{\mathcal{B}}, \Delta_{\mathcal{A}}]\eta(u) = -b^2(u,t)f''(u)\eta'(u) \tag{4.6}$$

$$\begin{aligned}[\Delta_{\mathcal{B}}, \Sigma_{\mathcal{A}}]\eta(u) = &\int_{\mathbb{R}} \partial_u n(u, dv) b(u) f'(u)(\eta(v) - \eta(u)) \\ &- \int_{\mathbb{R}} n(u, dv) b(v) f'(v) \eta'(v) \end{aligned} \tag{4.7}$$

$$
\begin{aligned}
[\Sigma_{\mathcal{B}}, \Delta_{\mathcal{A}}]\eta(u) = &- \int_{\mathbb{R}} \partial_u n(u,dv) b(u) [f']_{u,v} (\eta(v) - \eta(u)) \\
&+ \int_{\mathbb{R}} n(u,dv) b(v) [f']_{u,v} \eta'(v) \\
&- \int_{\mathbb{R}} n(u,dv) b(u) (\partial_u [f']_{u,v}) (\eta(v) - \eta(u))
\end{aligned} \tag{4.8}
$$

We combine (4.7) and (4.8) and integrate by parts to get

$$
\begin{aligned}
&([\Delta_{\mathcal{B}}, \Sigma_{\mathcal{A}}] + [\Sigma_{\mathcal{B}}, \Delta_{\mathcal{A}}])\, \eta(u) \\
&\quad = \int_{\mathbb{R}} \partial_u \left\{ n(u,dv) b(u) \left(f'(u) - [f']_{u,v} \right) \right\} (\eta(v) - \eta(u)) \\
&\quad + \int_{\mathbb{R}} \partial_v \left\{ n(u,dv) b(v) \left(f'(v) - [f']_{u,v} \right) \right\} (\eta(v) - \eta(u)) \qquad (4.9) \\
&\quad - \int_{\mathbb{R}} n(u,dv) \left\{ b(u) f''(u) + b'(u) \left(f'(u) - [f']_{u,v} \right) \right\} (\eta(v) - \eta(u)).
\end{aligned}
$$

Finally, the remaining term is

$$
\begin{aligned}
[\Sigma_{\mathcal{B}}, \Sigma_{\mathcal{A}}]\eta(u) = &\int_{\mathbb{R}} n(u,dv) \int_{\mathbb{R}} n(v,dw) \left([f']_{u,v} - [f']_{v,w} \right) (\eta(w) - \eta(v)) \\
&- \int_{\mathbb{R}} n(u,dv) \int_{\mathbb{R}} n(u,dw) \left([f']_{u,v} - [f']_{u,w} \right) (\eta(w) - \eta(u)).
\end{aligned}
$$

To put this into an appropriate form, split the first term of the r.h.s. into two parts by writing $\eta(w) - \eta(v)$ as $(\eta(w) - \eta(u)) - (\eta(v) - \eta(u))$. For the two terms that now contain $\eta(w) - \eta(u)$, switch the labels w and v and exchange the order of integration. This yields:

$$
\begin{aligned}
[\Sigma_{\mathcal{B}}, \Sigma_{\mathcal{A}}]\eta(u) = &\int_{\mathbb{R}} \int_{\mathbb{R}} n(u,dw) n(w,dv) \left([f']_{u,w} - [f']_{w,v} \right) (\eta(v) - \eta(u)) \\
&- \int_{\mathbb{R}} n(u,dv) \int_{\mathbb{R}} n(v,dw) \left([f']_{u,v} - [f']_{v,w} \right) (\eta(v) - \eta(u)) \qquad (4.10) \\
&- \int_{\mathbb{R}} n(u,dv) \int_{\mathbb{R}} n(u,dw) \left([f']_{u,w} - [f']_{u,v} \right) (\eta(v) - \eta(u)).
\end{aligned}
$$

Combining (4.5), (4.6), (4.9), and (4.10) we have that

$$
\begin{aligned}
&\partial_t b(u,t) \eta'(u) + \int_{\mathbb{R}} \partial_t n(u,dv,t) (\eta(v) - \eta(u)) \\
&\qquad = -b^2(u,t) f''(u) \eta'(u) + \int_{\mathbb{R}} (D_f(b,n) + Q_f(n,n)) (\eta(v) - \eta(u)).
\end{aligned} \tag{4.11}
$$

As this holds true for all test functions η, the terms corresponding to the drift and jump measures decouple and we arrive at (4.1) and (4.2).

The drift term is easily solved for using (4.1):

$$b(u,t) = (b(u,t_0)^{-1} + (t-t_0)f''(u))^{-1}. \tag{4.12}$$

In particular, for the special case of noise initial data ($b(u,t_0) = +\infty$) the drift takes the form $b(u,t) = (t-t_0)^{-1}f''(u)^{-1}$.

REMARK 4.2. Fix $t_0 = 0$ and consider Burgers equation with noise initial data. The drift satisfies $b(u,t) = t^{-1}$ and the free streaming and collision terms are

$$D(b,n) = t^{-1}\left(\frac{u-v}{2}\right)(\partial_u n - \partial_v n) \tag{4.13}$$

$$\begin{aligned} Q(n,n) = {} & \left(\frac{u-v}{2}\right)\int_{\mathbb{R}} n(u,dw)n(w,dv) \\ & - n(u,dv)\int_{\mathbb{R}} n(v,dw)\left(\frac{u-w}{2}\right) - n(u,dv)\int_{\mathbb{R}} n(u,dw)\left(\frac{w-v}{2}\right). \end{aligned} \tag{4.14}$$

Alternatively, we can write an equation for the jump measure $\bar{n}_x$ of the inverse Lagrangian function as follows. By Lemma 2.1, the generator of $a(x,t)$ has characteristics

$$\bar{b}_x(u) = -tb(t^{-1}(x-u)) + 1 = 0$$

$$\bar{n}_x(u,dv,t) = n(t^{-1}(x-u), t^{-1}(x-dv), t).$$

Then using (4.13) and (4.14) in (4.2), $\bar{n}_x$ solves its own kinetic equation:

$$\partial_t \bar{n}_x(u,dv,t) = \bar{D}(\bar{b}_x,\bar{n}_x) + \bar{Q}(\bar{n}_x,\bar{n}_x)$$

where

$$\bar{D}(\bar{b}_x,\bar{n}_x) = t^{-2}\left(\frac{u+v}{2} - x\right)(\partial_u\bar{n}_x + \partial_v\bar{n}_x), \qquad \bar{Q}(\bar{n}_x,\bar{n}_x) = t^{-2}Q(\bar{n}_x,\bar{n}_x).$$

2. Self-similar solutions for Burgers equation

2.1. Lévy process initial data. Recall from Chapter 1 that the Laplace exponent $\Phi(q,t)$ of $(u(x,t)-u(0,t))_{x\geq 0}$ satisfies Burgers equation in $q>0$ and t:

$$\begin{aligned} &\partial_t \Phi + \Phi \partial_q \Phi = 0 \\ &\Phi(q,0) = \Phi_0(q). \end{aligned} \tag{4.15}$$

These results are a prototype of the results of Chapter 3 and have strongly motivated their development. Since Lévy processes are a subclass of Markov processes, we can derive (4.15) from (3.17) as follows. A similar analysis is found in [**16**].

In terms of the Fourier transform $\mathcal{F}\eta(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\eta(u)du$, the generator $\mathcal{A}$ and characteristic exponent Ψ of $(u(x,t)-u(0,t))_{x\geq 0}$ (formally) satisfy:

$$\mathcal{F}(\mathcal{A}(t)\eta)(\xi) = -\Psi(-\xi,t)\mathcal{F}\eta(\xi). \tag{4.16}$$

This simply expresses the relationship between the generator and its symbol (see Section 2). Now suppose that $\eta \in C_c^1(0,\infty)$ and consider the Laplace transform for $q>0$:

$$\mathcal{L}\eta(q) = \int_0^{\infty} e^{qu}\eta(u)du.$$

Since $u(x,t)-u(0,t)$ is spectrally negative, (4.16) can be written in terms of the Laplace exponent $\Phi(q,t) = -\Psi(-iq,t)$ for $q>0$ as

$$\mathcal{L}(\mathcal{A}(t)\eta)(q) = \Phi(q,t)\mathcal{L}\eta(q). \tag{4.17}$$

Differentiating (4.17) in t and using the equation for the generator we shall obtain the desired result.

For convenience we now drop the explicit dependence on t from the notation. To begin, recall from (2.26) the generator $\mathcal{A}$ of the Lévy process $u(x,t)$ is of the form

$$\mathcal{A}\eta(u) = b\eta'(u) + \int_{\mathbb{R}_-} n(ds)(\eta(u+s)-\eta(u))$$

with drift $b(t)$ and jump measure $n(ds,t)$ independent of u. Its adjoint is easily shown to satisfy

$$\mathcal{A}^{\dagger}\eta(u) = -b\eta'(u) + \int_{\mathbb{R}_-} n(ds)(\eta(u-s)-\eta(u)).$$

Then for $q \geq 0$, (4.17) implies that the adjoint operator satisfies for $u \geq 0$

$$\mathcal{A}^{\dagger} e^{-qu} = \Phi(q) e^{-qu}. \tag{4.18}$$

More precisely, in (4.18) we are actually considering functions of the form $e^{-qu}\psi(u) \in \mathcal{D}(\mathcal{A}^{\dagger})$ where ψ is a smooth positive test function that is supported on (a, ∞) for some $a < 0$ and equals 1 for all $u \geq 0$. Since the Laplace transform (4.17) is over the region $u \geq 0$ we are only concerned with $(\mathcal{A}^{\dagger}(e^{-qu}\psi))_{u\geq 0}$, which coincides with $(\mathcal{A}^{\dagger} e^{-qu})_{u\geq 0}$. Clearly, $e^{-qu}\psi(u) \in \mathcal{D}(\mathcal{A}^{\dagger})$ since the BV condition $\int_{\mathbb{R}} n(ds)s < \infty$ gives that

$$\left| \int_{\mathbb{R}_-} n(ds) \left(e^{-q(u-s)}\psi(u-s) - e^{-qu}\psi(u) \right) \right| < \|\psi\|_{\infty} e^{-qu} \left| \int_{\mathbb{R}_-} n(ds)(e^{qs} - 1) \right| < \infty.$$

We can thus regard e^{-qu} to be in $\mathcal{D}(\mathcal{A}^{\dagger})$ if we restrict ourselves to $u \geq 0$, as we will do throughout the remainder of this section. By explicit calculation, we have

$$\begin{aligned} \mathcal{A}^{\dagger} e^{-qu} &= \left(qb + \int_{\mathbb{R}_-} n(ds)(e^{qs} - 1) \right) e^{-qu} \\ &= \Phi(q) e^{-qu} \end{aligned} \tag{4.19}$$

Since $f(u) = \frac{1}{2}u^2$, the associated generator and its adjoint satisfy

$$\mathcal{B}\eta(u) = bu\eta'(u) + \int_{\mathbb{R}_-} n(ds) \left(u + \frac{1}{2}s \right) (\eta(u+s) - \eta(u))$$

$$\mathcal{B}^{\dagger}\eta(u) = -b(u\eta)' + \int_{\mathbb{R}_-} n(ds) \left(u - \frac{1}{2}s \right) (\eta(u-s) - \eta(u))$$

Again we regard $e^{-qu} \in \mathcal{D}(\mathcal{B}^{\dagger})$ for $u \geq 0$ to get that

$$\begin{aligned} \mathcal{B}^{\dagger} e^{-qu} &= \left(b(qu - 1) + u \int_{\mathbb{R}_-} n(ds)(e^{qs} - 1) - \frac{1}{2} \int_{\mathbb{R}_-} n(ds)s(e^{qs} - 1) \right) e^{-qu} \\ &= \rho(u, q) e^{-qu}. \end{aligned} \tag{4.20}$$

Now write $\rho(u, q) = \Phi(q)u - \varphi(q)$ with

$$\varphi(q) = b + \frac{1}{2} \int_{\mathbb{R}_-} n(ds)s(e^{qs} - 1).$$

The last step is to evaluate $[\mathcal{A}^\dagger, \mathcal{B}^\dagger]e^{-qu}$. Using

$$\begin{aligned}\mathcal{A}^\dagger\left(ue^{-qu}\right) &= \left(b(qu-1) + \int_{\mathbb{R}_-} n(ds)\left((u-s)e^{qs} - u\right)\right)e^{-qu}\\ &= (\Phi(q)u - \partial_q\Phi(q))e^{-qu}\end{aligned}$$

along with (4.19) and (4.20),

$$\mathcal{A}^\dagger\mathcal{B}^\dagger e^{-qu} = \Phi(q)\{\Phi(q)u - \partial_q\Phi(q) - \varphi(q)\}e^{-qu}$$

$$\mathcal{B}^\dagger\mathcal{A}^\dagger e^{-qu} = \Phi(q)\{\Phi(q)u - \varphi(q)\}e^{-qu}.$$

This yields

$$[\mathcal{A}^\dagger, \mathcal{B}^\dagger]e^{-qu} = -\Phi(q)\partial_q\Phi(q)e^{-qu}. \tag{4.21}$$

Combining (4.17) and (4.21) with the evolution equation, for any $\eta \in C_c^\infty(0,\infty)$:

$$\begin{aligned}\partial_t\Phi(q)\mathcal{L}\eta(q) = \partial_t\mathcal{L}(\mathcal{A}\eta)(q) &= \int_0^\infty e^{-qu}(\partial_t\mathcal{A})\eta(u)du\\ &= \int_0^\infty \left([\mathcal{A}^\dagger, \mathcal{B}^\dagger]e^{-qu}\right)\eta(u)du = -\Phi(q)\partial_q\Phi(q)\mathcal{L}\eta(q).\end{aligned}$$

This implies that the Laplace exponent Φ (formally) satisfies (4.15).

Remark 4.3. In this case, it can be seen from (4.3-4.4) and the form of the generator (2.26) that (up to a change of variable and rescaling) the kinetic equation for the jump density $n(s)$ is Smoluchowski's coagulation equation with additive kernel:

$$\partial_t n(s,t) = \frac{1}{2}\int_0^s sn(s-s',t)n(s',t)ds' - \int_0^\infty (s+s')n(s,t)n(s',t)ds' \tag{4.22}$$

There is an abundance of literature on self-similar solutions to (4.22) and their domains of attraction under dynamic scaling (see, e.g. [**39, 38**]). In Chapter 5 we present a result that determines rates of convergence to such solutions. For additional discussion on connections between Burgers turbulence and Smoluchowski's equation we refer the reader to [**8, 40**].

2.2. Noise initial data and self-similar solutions. In the following we restrict ourselves to the special case of Burgers equation. We consider the choice of stable noise initial data, which plays a crucial role in determining the limiting behavior of a large class of velocity fields $u(x,t)$ under dynamic scaling in time. This case also arises when studying continuum limits of 1-D discrete aggregation models with nearest neighbor interactions [**25**].

As discussed in [**40**], there is a one-to-one correspondence between **(a)** stable Lévy processes, **(b)** statistically self-similar solutions to Burgers turbulence with independent increments, and **(c)** self-similar solutions to Smoluchowski's coagulation equation. In direct analogy with the Lévy case, we now show that there is a similar correspondence between **(a)** stable noise, **(b)** statistically self-similar solutions to Burgers turbulence with the Markov property, and **(c)** self-similar solutions to the kinetic equation (4.2), which can be regarded as a state-dependent coagulation equation. The relationship is seen as follows.

Suppose u_0 is a spectrally negative α-stable noise—that is, the potential U_0 is a two-sided spectrally negative stable process. As in [**9**], we take $\alpha \in (1/2, 2]$ to ensure $U_0(x) = o(x^2)$ a.s. as $|x| \to \infty$ so that (2.13) is automatically satisfied. Then U_0 has stationary and independent increments whose distribution scales as

$$U_0(x) \overset{\text{law}}{=} x^{1/\alpha} U_0(1).$$

Thus, for all $\kappa > 0$

$$(U_0(x))_{x\in\mathbb{R}} \overset{\mathcal{L}}{=} (\kappa U_0(\kappa^{-\alpha} x))_{x\in\mathbb{R}}.$$

Recall by Theorem 3.2 that $u(x,t)$ is Markov. Using the definition (3.3) of the inverse Lagrangian function we find that

$$u(x,t) \overset{\mathcal{L}}{=} t^{1/\beta-1} u\left(t^{-\beta}x, 1\right), \qquad \beta = \frac{2\alpha-1}{\alpha}. \tag{4.23}$$

This expresses the statistical self-similarity of the random field evolving under Burgers flux. We emphasize that (4.23) *does not* correspond to spatial self-similarity of the solution at any fixed positive time. That is, $u(x,t)$ is not self-similar in x alone since for each $t > 0$ there is a typical characteristic length scale given by the average separation between shocks.

Applying Lemma 2.1 to (4.23), the characteristics of the generator $\mathcal{A}$ of $u(x,t)$ scale as

$$b(u,t) = t^{-1} b\left(t^{1-1/\beta} u, 1\right) \tag{4.24}$$

$$n(u, dv, t) = t^{-1/\beta} n\left(t^{1-1/\beta} u, t^{1-1/\beta} dv, 1\right). \tag{4.25}$$

By (4.12) the drift is $b(u,t) = t^{-1}$. In terms of rescaled variables

$$n^*(\hat{u}, d\hat{v}) = n(\hat{u}, d\hat{v}, 1), \qquad \hat{u} = t^{1-1/\beta} u$$

we can write the weak form of the kinetic equation (4.2) with Burgers flux as

$$-\gamma n^* + \frac{1}{2}(\hat{v} - \gamma \hat{u}) \partial_{\hat{u}} n^* - \frac{1}{2}(\hat{u} - \gamma \hat{v}) \partial_{\hat{v}} n^* = \hat{Q}(n^*, n^*), \qquad \gamma = (2\alpha - 1)^{-1}. \tag{4.26}$$

Here, $\hat{Q}$ is the same as (4.14) with n and u, v, w replaced by n^* and $\hat{u}$, $\hat{v}$, $\hat{w}$, respectively:

$$\begin{aligned}
\hat{Q}(n^*, n^*) = &\left(\frac{\hat{u} - \hat{v}}{2}\right) \int_{\mathbb{R}} n^*(\hat{u}, d\hat{w}) n^*(\hat{w}, d\hat{v}) \\
&- n^*(\hat{u}, d\hat{v}) \int_{\mathbb{R}} n^*(\hat{v}, d\hat{w}) \left(\frac{\hat{u} - \hat{w}}{2}\right) - n^*(\hat{u}, d\hat{v}) \int_{\mathbb{R}} n^*(\hat{u}, d\hat{w}) \left(\frac{\hat{w} - \hat{v}}{2}\right).
\end{aligned}$$

To conclude, note that if the jump measure has a density $n^*(\hat{u}, \hat{v})$ then (4.26) holds pointwise in $n^*(\hat{u}, \hat{v})$.

2.3. White noise initial data and Groeneboom's solution. In [**31**], Groeneboom explicitly derived the solution to Burgers equation with white noise initial data through a detailed analysis of Brownian motion constrained by parabolic barriers. A similar computation with additional information on shock statistics was later given by Frachebourg and Martin [**24**].

We now demonstrate that Groeneboom's solution verifies the evolution equation (3.17). To begin, define the density

$$n^*(\hat{u}, \hat{v}) = K(\hat{u} - \hat{v}) \frac{J(\hat{v})}{J(\hat{u})}. \tag{4.27}$$

The functions K and J are defined by their Fourier-Laplace transforms for $q \in \bar{\mathbb{C}}_+$ in terms of the Airy function:

$$\bar{K}(q) = \mathcal{L}[K](q) = -2\frac{d^2}{dq^2}\log(\mathrm{Ai}(q)), \qquad \bar{J}(q) = \mathcal{L}[J](q) = \frac{1}{\mathrm{Ai}(q)}. \tag{4.28}$$

In terms of the generator of $u(x,t)$, Groeneboom's solution is

$$\mathcal{A}(t)\eta(u) = t^{-1}\eta'(u) + \int_{-\infty}^{u} t^{-1/3}K(t^{1/3}(u-v))\frac{J(t^{1/3}v)}{J(t^{1/3}u)}(\eta(v)-\eta(u))dv. \tag{4.29}$$

The case of white noise corresponds to solving (4.26) with $\alpha = 2$ (that is, $\gamma = 1/3$). Substituting (4.27) into (4.26) yields:

$$\begin{aligned}
-\frac{2}{3} + \left(\hat{v} - \frac{1}{3}\hat{u}\right)&\left[\left(\frac{K'}{K}\right)(\hat{u}-\hat{v}) - \left(\frac{J'}{J}\right)(\hat{u})\right] \\
-\left(\hat{u} - \frac{1}{3}\hat{v}\right)&\left[\left(\frac{K'}{K}\right)(\hat{u}-\hat{v}) - \left(\frac{J'}{J}\right)(\hat{v})\right] \\
&= (\hat{u}-\hat{v})\left(\frac{K*K}{K}\right)(\hat{u}-\hat{v}) \\
&\quad + \left[\left(\frac{(zK)*J}{J}\right)(\hat{u}) - (\hat{u}-\hat{v})\left(\frac{K*J}{J}\right)(\hat{u})\right] \\
&\quad - \left[\left(\frac{(zK)*J}{J}\right)(\hat{v}) - (\hat{v}-\hat{u})\left(\frac{K*J}{J}\right)(\hat{v})\right].
\end{aligned} \tag{4.30}$$

Here, z is a dummy spatial variable. In order to use (4.28) to verify the equation above, we first derive a set of fundamental identities for convolutions of K, zK, and J. We obtain these in Fourier-Laplace space by using the Airy equation

$$\mathrm{Ai}''(q) - q\mathrm{Ai}(q) = 0.$$

The identities for $q \in \bar{\mathbb{C}}_+$ with $' = \partial_q$ are

$$\begin{aligned}
\bar{J}'' &= \bar{K}\bar{J} + q\bar{J} \\
\tfrac{3}{2}\bar{K}'\bar{J} &= (\bar{K}\bar{J})' - \bar{J} \\
-\bar{K}''' &= 3(\bar{K}^2)' + 4q\bar{K} + 2\bar{K}.
\end{aligned} \tag{4.31}$$

Reverting to the spatial variable z and rearranging terms, these read as

$$\begin{aligned} (K * J)(z) &= z^2 J(z) - J'(z) \\ ((zK) * J)(z) &= \left(\tfrac{2}{3}z^3 + \tfrac{2}{3}\right) J(z) - \tfrac{2}{3} z J'(z) \\ z(K * K)(z) &= \left(\tfrac{1}{3}z^3 - \tfrac{2}{3}\right) K(z) - \tfrac{4}{3} z K'(z). \end{aligned} \tag{4.32}$$

By substituting the identities (4.32) into the r.h.s. of (4.30), we find that the equation is indeed verified. In other words, (4.29) solves (3.17).

We can also verify the adjoint equation $\mathcal{A}^\dagger(t)p(u,t) = 0$ (see (2.19)) for the 1-point probability given in [**31**]. It is otherwise known as *Chernoff's distribution*:

$$p(u,t) = J(t^{1/3}u)J(-t^{1/3}u). \tag{4.33}$$

We only need to check this for $t = 1$. The adjoint operator is

$$\mathcal{A}^\dagger(1)\eta(u) = -\eta'(u) + \int_u^\infty n(v,u)\eta(v)dv - \left(\int_{-\infty}^u n(u,v)dv\right)\eta(u).$$

Substituting for (4.33) and using (4.32) gives the desired result:

$$\begin{aligned} &\mathcal{A}^\dagger(1)p(u,1) \\ &\quad = p(u,1)\left\{\left[-\left(\frac{J'}{J}\right)(u) + \left(\frac{J'}{J}\right)(-u)\right] + \left(\frac{K*J}{J}\right)(-u) - \left(\frac{K*J}{J}\right)(u)\right\} = 0. \end{aligned}$$

Remark 4.4. Hints of the completely integrable structure appear in the terms (4.28) of Groeneboom's solution. The form of $\bar{K}(q)$ is exactly of the form given by Dyson's formula

$$-2\frac{d^2}{dq^2}\log(\mathcal{D}(q))$$

with $\mathcal{D}(q) = \mathrm{Ai}(q)$ the Fredholm determinant of an appropriate integral operator. It is quite likely that a spectral analysis of (3.17) via inverse scattering will lead to an independent derivation of (4.29).

CHAPTER 5

Berry-Esséen theorems for Smoluchowski's coagulation equation

In this chapter we take a slight departure from our previous discussion. Here, we establish near optimal rates of convergence under dynamic scaling to self-similar solutions of Smoluchowski's coagulation equation

$$\partial_t n(t,x) = \frac{1}{2}\int_0^x K(x-y,y)n(t,x-y)n(t,y)dy - \int_0^\infty K(x,y)n(t,x)n(t,y)dy$$

with kernels $K = 2$, $x + y$, and xy. Recall that the case of $K = x + y$ appears in Burgers turbulence as the kinetic equation (4.22) for the jump density of shocks with Lévy process initial data. Our method is a simple analogue of the Berry-Esséen theorem in classical probability and requires minimal assumptions on the initial data, namely that of an extra finite moment condition. We conjecture that these rates are actually achieved for monodisperse initial data.

1. Introduction

Smoluchowski's coagulation equation

$$\partial_t n(t,x) = \frac{1}{2}\int_0^x K(x-y,y)n(t,x-y)n(t,y)dy - \int_0^\infty K(x,y)n(t,x)n(t,y)dy \quad (5.1)$$

is a fundamental mean-field model for cluster growth. Here, $n(t,x)$ is the number density of clusters of mass x per unit volume at time t and $K(x,y)$ is a symmetric rate kernel. In the spatially discrete case this reads as

$$\partial_t n(t,x) = \frac{1}{2}\sum_{y=1}^{x-1} K(x-y,y)n(t,x-y)n(t,y) - \sum_{y=1}^{\infty} K(x,y)n(t,x)n(t,y)dy. \quad (5.2)$$

The continuous and discrete cases can be considered together via the weak formulation of Smoluchowski's coagulation equation, given in terms of a moment identity for the number distribution $n(t, dx)$:

$$\partial_t \int_0^\infty \phi(x)n(t,dx) = \frac{1}{2}\int_0^\infty \int_0^\infty (\phi(x+y)-\phi(y)-\phi(x))K(x,y)n(t,dy)n(t,dx) \quad (5.3)$$

where ϕ is a suitable test function. We direct the interested reader to [**1, 38, 39**] for further discussion. Note that throughout this paper we try to keep the same notation as in [**39**].

The present work is restricted to the homogeneous, 'solvable' kernels $K = 2, x+y, xy$ (corresponding to $K(\alpha x, \alpha y) = \alpha^\gamma K(x,y)$ with $\gamma = 0, 1, 2$). It has been shown in [**38**] that (5.3) admits a one-parameter family of self-similar solutions whose domains of attraction are characterized by the tails of the initial data. In particular, there is a unique self-similar solution with finite $(\gamma+1)$th moment, which has exponentially decaying tails and attracts all initial data that satisfy this finite moment condition. In [**39**], it was shown that with an additional integrability hypothesis on the Fourier transform of the initial data one has uniform convergence of densities (in the continuous case) or coefficients (in the discrete case) to a self-similar solution as t approaches the time horizon T_γ, where $T_\gamma = \infty$ for $\gamma = 0, 1$ and $T_\gamma = 1$ for $\gamma = 2$. The proof of this result is similar to that of uniform convergence of densities in the central limit theorem (see Feller [**22**], Section XV.5).

Here we establish a near-optimal rate of convergence to the exponentially decaying self-similar solution of Smoluchowski's coagulation equation with $K = 2, x+y, xy$. Continuing the analogy with the classical CLT, this result corresponds to the Berry-Esséen theorem for rates of convergence to the normal law [**22**], XVI.5. The method is simple and robust: it holds for all initial distributions, requiring only an additional finite moment condition beyond those needed for convergence to the self-similar profile. In particular, it is true when the initial distribution has a density or is a lattice measure. Our work improves upon several recent results of Canizo et. al [**13**] since it holds for a large class of initial data and for all of the solvable kernels.

The main results can be stated in a unified framework. With $t_0 = 1$ for $\gamma = 0$, $t_0 = 0$ for $\gamma = 1, 2$, and $n_0(dx) = n(t_0, dx)$, define the moments $\mu_j = \int_0^\infty x^j n_0(dx)$ of the initial data. Assume that the γth and $(\gamma+1)$th moments are finite, for which we scale x and n_0 so that $\mu_\gamma = \mu_{\gamma+1} = 1$. Finiteness of the γth moment ensures well-posedness of (5.3). Finiteness of the $(\gamma+1)$th moment guarantees that the initial data are in the domain of attraction of the self-similar solution with exponential decaying tails:

$$n(t,x) = \frac{m_\gamma(t)}{\lambda_\gamma(t)^{\gamma+1}} \hat{n}_{*,\gamma}\left(\frac{x}{\lambda_\gamma(t)}\right) \tag{5.4}$$

where the profiles $\hat{n}_{*,\gamma}$ for $\hat{x} \geq 0$ take the form

$$\hat{n}_{*,0}(\hat{x}) = e^{-\hat{x}}, \qquad \hat{x}\hat{n}_{*,1}(\hat{x}) = \hat{x}^2\hat{n}_{*,2}(\hat{x}) = \frac{1}{\sqrt{2\pi}}\hat{x}^{-1/2}e^{-\hat{x}/2} \tag{5.5}$$

and the time-dependent moments m_γ and scalings λ_γ are

$$m_0(t) = t^{-1}, \qquad m_1(t) = 1, \qquad m_2(t) = (1-t)^{-1} \tag{5.6}$$

$$\lambda_0(t) = t, \qquad \lambda_1(t) = e^{2t}, \qquad \lambda_2(t) = (1-t)^{-2}. \tag{5.7}$$

If, in addition, we assume finiteness of the $(\gamma+2)$th moment then with time parameter $\tau_\gamma(t) = \int_{t_0}^t m_\gamma(s)ds$ explicitly given by

$$\tau_0(t) = \log(t), \qquad \tau_1(t) = t, \qquad \tau_2(t) = \log(1-t)^{-1} \tag{5.8}$$

we have exponentially fast convergence to self-similar form:

Theorem 5.1. *Let n_0 be a positive measure with $\int_{(0,\infty)} x^\gamma n_0(dx) = \int_{(0,\infty)} x^{\gamma+1} n_0(dx) = 1$. Suppose that the additional finite moment assumption $\int_0^\infty x^{\gamma+2}n_0(dx) < \infty$ holds and let $n(t,dx)$ be the measure-valued solution to Smoluchowski's equation with $K = 2$, $x+y$, or xy and initial data $n_0(dx)$. With rescaled solution*

$$\hat{n}(\tau_\gamma, d\hat{x}) := \frac{\lambda_\gamma(t)^\gamma}{m_\gamma(t)} n(t, \lambda_\gamma(t)d\hat{x}) \tag{5.9}$$

define the corresponding cumulative distribution functions

$$F_\gamma(\tau_\gamma, \hat{x}) = \int_{(0,\hat{x})} \hat{y}^\gamma \hat{n}(\tau_\gamma, d\hat{y}), \qquad F_{*,\gamma}(\hat{x}) = \int_{(0,\hat{x})} \hat{y}^\gamma \hat{n}_{*,\gamma}(d\hat{y}). \tag{5.10}$$

Then for $\tau_\gamma(t) \in [0,\infty)$,

$$\sup_{\hat{x}>0} |F_\gamma(\tau_\gamma,\hat{x}) - F_{*,\gamma}(\hat{x})| \leq C(\mu_{\gamma+2})(1+\tau_\gamma)e^{-\tau_\gamma}$$

where $C(\mu_{\gamma+2})$ is a constant that depends only on $\mu_{\gamma+2}$.

REMARK 5.1. The rates given above are near optimal. Although we do not show this here, we can demonstrate $e^{-\tau_\gamma}$ is a lower bound for the decay rate in the case of monodisperse initial data $n_0(dx) = \delta_1(dx)$ for each of the kernels $K = 2, x + y, xy$.

Since we are primarily concerned with an asymptotic rate of decay to self-similarity, we make no effort in optimizing the constant in front of the exponential term. We now present a smoothing argument given in [**22**], XV.5, that allows us to consider a broad class of initial data:

LEMMA 5.2. *Let F and F_* be probability distribution functions and assume $|F_*'(x)| \leq 1$. Consider the Fourier transforms of their corresponding measures,*

$$u(ik) = \int_{(-\infty,\infty)} e^{-ikx} F(dx), \qquad u_*(ik) = \int_{(-\infty,\infty)} e^{-ikx} F_*(dx).$$

Assume that $u'(0) = u_*'(0)$—that is, equality of the first moment of $F(dx)$ and $F_*(dx)$. With the mollifier

$$\psi_T(x) = \frac{1-\cos(Tx)}{\pi T x^2}$$

define

$$\Delta(x) = F(x) - F_*(x), \qquad \Delta_T(x) = \psi_T(x) * \Delta.$$

Then

$$\sup_x |\Delta(x)| \leq 2 \sup_x |\Delta_T(x)| + \frac{24}{\pi T}. \tag{5.11}$$

In terms of Fourier transforms, (5.11) is

$$\sup_x |F(x) - F_*(x)| \leq \frac{1}{\pi} \sup_x \left| \int_{-T}^{T} \frac{e^{ikx}}{s} (u(ik) - u_*(ik)) dk \right| + \frac{24}{\pi T}. \tag{5.12}$$

We note that the distribution functions (5.10) satisfy the assumptions of the lemma.

2. Rate of convergence for the constant kernel $K = 2$

In this section we restate and prove Theorem 5.1 for the constant kernel case. Explicitly, we prove:

THEOREM 5.3. *Let n_0 be a positive measure, $\int_{(0,\infty)} n_0(dx) = \int_{(0,\infty)} x n_0(dx) = 1$, and $\mu_2 = \int_{(0,\infty)} x^2 n_0(dx) < \infty$. With $\tau = \log t$ and in terms of the rescaling (5.6-5.7), let $\hat{n}(\tau, d\hat{x})$ be the rescaled solution (5.9) to Smoluchowski's equation with $K = 2$ and initial data $n_0(dx)$. Then for*

$$F(\tau, \hat{x}) = \int_{(0,\hat{x})} \hat{n}(\tau, d\hat{y}), \qquad F_*(\hat{x}) = \int_{(0,\hat{x})} \hat{n}_{*,0}(d\hat{y}).$$

and $\tau \in [0, \infty)$,

$$\sup_{\hat{x}>0} |F(\tau, \hat{x}) - F_*(\hat{x})| \le C(\mu_2)(1 + \tau)e^{-\tau} \tag{5.13}$$

where $C(\mu_2)$ is a constant that depends only on μ_2.

Proof of rate of convergence for $K = 2$. The proof of the theorem can be summarized as follows. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and $\bar{\mathbb{C}}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) \ge 0\}$. For $s \in \bar{\mathbb{C}}_+$ define the Fourier-Laplace transforms

$$u(\tau, s) = \int_{(0,\infty)} e^{-s\hat{x}} \hat{n}(\tau, d\hat{x}), \qquad u_*(s) = \int_{(0,\infty)} e^{-s\hat{x}} \hat{n}_*(d\hat{x})$$

and let $u_0(s) = u(0, s)$. By (5.12), we have that

$$\sup_{\hat{x}>0} |F(\tau, \hat{x}) - F_*(\hat{x})| \le \frac{1}{\pi} \sup_{\hat{x}>0} \left| \int_{-iT}^{iT} \frac{e^{\sigma\hat{x}}}{\sigma} (u(\tau, \sigma) - u_*(\sigma)) d\sigma \right| + \frac{24}{\pi T} \tag{5.14}$$

The uniform rate of convergence of the distribution function is thus given by convergence rate of $u(\tau, s)$ to $u_*(s)$. As shown in [**38, 39**], u satisfies a linear first-order PDE and can be solved for in terms of u_0 by the method of characteristics in $\bar{\mathbb{C}}_+$. The finiteness condition on μ_2 is a statement about the tail of $\hat{n}_0$, and therefore, about the regularity of u_0 near the origin. Using a Taylor expansion to estimate the difference $u_0(s) - u_*(s)$ near $s = 0$, we can propagate this estimate outward at the natural growth rate of the characteristics of u_0 and u_*. Combining this with Lemma 5.2 gives us the desired result.

1. Evolution of characteristics: As shown in [**38**], the Fourier-Laplace transform $u(\tau, s)$ by solves the PDE

$$\partial_\tau u + s\partial_s u = -u(1-u). \tag{5.15}$$

This equation can be derived directly from (5.3), or from (5.1) if the initial number measure has a density as in [**39**]. Fix an initial time τ_0 and an initial condition $s_0 \in \bar{\mathbb{C}}_+$. The solution to (5.15) is given in terms of the characteristics $s(\tau; \tau_0, s_0)$, which satisfy

$$s(\tau; \tau_0, s_0) = e^{\tau - \tau_0} s_0 \in \bar{\mathbb{C}}_+. \tag{5.16}$$

Note that the equation for the characteristics *is independent of* u_0 and therefore holds for any choice of u_0. Along characteristics we have that

$$\frac{du}{d\tau} = -u(1-u),$$

which can be integrated to yield the solution

$$u(\tau, s) = e^{-\tau} \frac{u_0(s_0)}{1 - u_0(s_0)(1 - e^{-\tau})}. \tag{5.17}$$

We can perform a similar analysis for the Fourier-Laplace transform of the self-similar solution, which satisfies

$$u_*(s) = \frac{1}{1+s}. \tag{5.18}$$

As noted above, with $s_0^* \in \bar{\mathbb{C}}_+$ the corresponding characteristics for initial data u_* to (5.15) are also

$$s^*(\tau; \tau_0, s_0^*) = e^{\tau - \tau_0} s_0^* \in \bar{\mathbb{C}}_+.$$

We emphasize here that for fixed $\sigma_0 \in \bar{\mathbb{C}}_+$ ($\sigma \in \bar{\mathbb{C}}_+$) the forward (backward) characteristics of $u(\tau, s)$ and $u_*(s)$ are *identical*

$$s^*(\tau; \tau_0, \sigma_0) \equiv s(\tau; \tau_0, \sigma_0), \qquad s_0^*(\tau_0; \tau, \sigma) \equiv s_0(\tau_0, \tau, \sigma). \tag{5.19}$$

Since $u_*(s)$ is a stationary solution to (5.15) (or by using (5.18) in (5.17)), we have that the self-similar solution satisfies

$$u_*(s) = e^{-\tau} \frac{u_*(s_0^*)}{1 - u_*(s_0^*)(1 - e^{-\tau})}. \tag{5.20}$$

2. Estimates near the origin: Now we utilize the additional finite moment assumption

$$\mu_2 = \int_0^\infty x^2 n_0(dx) = \int_0^\infty \hat{x}^2 \hat{n}_0(dx) < \infty$$

to obtain an approximation for u_0 near the origin. We use

$$u_0(0) = -u_0'(0) = 1, \qquad u_0''(0) = \mu_2$$
$$u_*(0) = -u_*'(0) = 1, \qquad u_*''(0) = 2$$

and a Taylor expansion about the origin to get the estimate

$$|u_0(\sigma_0) - u_*(\sigma_0)| \leq \left(1 + \frac{\mu_2}{2}\right) |\sigma_0|^2. \tag{5.21}$$

3. Propagation of estimates: We now bound the r.h.s. of (5.14) with $T = \delta e^{\tau}$, where $\delta = \delta(\mu_2) > 0$ is a length scale characterizing the region about the origin where the difference between u_0 and u_* is small. We will then propagate the estimate (5.21) from this region. Explicitly, define

$$2\delta(\mu_2) = \sqrt{1 + 2\left(1 + \frac{\mu_2}{2}\right)^{-1}} - 1 \tag{5.22}$$

so that $\left(1 + \frac{\mu_2}{2}\right)\delta = \frac{1}{2}(1+\delta)^{-1}$. Note that if $\mu_2 \to \infty$ then $\delta \to 0$. Substituting (5.17), (5.20) into the r.h.s. of (5.14) and changing variables to the backward characteristics $\sigma_0(0;\tau,s)$, where $\sigma_0 = s_0 = s_0^*$, we have:

$$\begin{aligned}
&\sup_{\hat{x}} |F(\tau,\hat{x}) - F_*(\hat{x})| \\
&\leq \frac{1}{\pi} \sup_{\hat{x}} \left| \int_{-i\delta}^{i\delta} \frac{e^{s(\tau;0,\sigma_0)\hat{x}}}{s(\tau;0,\sigma_0)} \frac{(u_0(\sigma_0) - u_*(\sigma_0))}{(1 - u_*(\sigma_0)(1-e^{-\tau}))(1 - u_0(\sigma_0)(1-e^{-\tau}))} d\sigma_0 \right| + \frac{24}{\pi\delta} e^{-\tau}
\end{aligned} \tag{5.23}$$

We bound the above integral by splitting it into the two regions $0 \leq |\sigma_0| < \delta e^{-\tau}$ and $\delta e^{-\tau} \leq |\sigma_0| \leq \delta$.

4. Bounds for $0 \leq |\sigma_0| < \delta e^{-\tau}$: In this region we use that $|u_0|, |u_*| \leq 1$ and estimate the terms in the denominator by the naive lower bound

$$|1 - z(1-e^{-\tau})| \geq 1 - |z|(1 - e^{-\tau}) \geq e^{-\tau}, \qquad |z| \leq 1.$$

Using (5.21) for the numerator and the explicit form (5.19) of $s(\tau; 0, \sigma_0)$ this yields

$$\begin{aligned}
&\left| \int_{0 \le |\sigma_0| < \delta e^{-\tau}} \frac{e^{s(\tau;0,\sigma_0)\hat{x}}}{s(\tau;0,\sigma_0)} \frac{(u_0(\sigma_0) - u_*(\sigma_0))}{(1 - u_*(\sigma_0)(1-e^{-\tau}))(1 - u_0(\sigma_0)(1-e^{-\tau}))} d\sigma_0 \right| \\
&\qquad \le 2\left(1 + \frac{\mu_2}{2}\right) e^{\tau} \int_0^{\delta e^{-\tau}} |\sigma_0| d|\sigma_0| \\
&\qquad = \left(1 + \frac{\mu_2}{2}\right) \delta^2 e^{-\tau}. \qquad (5.24)
\end{aligned}$$

5. Bounds for $\delta e^{-\tau} \le |\sigma_0| \le \delta$: In this region, we have by (5.18) and using that σ_0 is pure imaginary that

$$|1 - u_*(\sigma_0)(1 - e^{-\tau})| = \left| \frac{e^{-\tau} + \sigma_0}{1 + \sigma_0} \right| \ge \frac{|\sigma_0|}{1 + \delta}$$

and by (5.21),

$$|1 - u_0(\sigma_0)(1-e^{-\tau})| \ge |1 - u_*(\sigma_0)(1-e^{-\tau})| - \left(1 + \frac{\mu_2}{2}\right)|\sigma_0|^2 \ge \frac{1}{2} \cdot \frac{|\sigma_0|}{1+\delta}.$$

Therefore,

$$\begin{aligned}
&\left| \int_{\delta e^{-\tau} \le |\sigma_0| < \delta} \frac{e^{s(\tau;0,\sigma_0)\hat{x}}}{s(\tau;0,\sigma_0)} \frac{(u_0(\sigma_0) - u_*(\sigma_0))}{(1 - u_*(\sigma_0)(1-e^{-\tau}))(1 - u_0(\sigma_0)(1-e^{-\tau}))} d\sigma_0 \right| \\
&\qquad \le 4\left(1 + \frac{\mu_2}{2}\right)(1+\delta)^2 e^{-\tau} \int_{\delta e^{-\tau}}^{\delta} \frac{1}{|\sigma_0|} d|\sigma_0| \\
&\qquad = 4\left(1 + \frac{\mu_2}{2}\right)(1+\delta)^2 \tau e^{-\tau}. \qquad (5.25)
\end{aligned}$$

5. Rate of convergence: By combining (5.23), (5.24), and (5.25), we obtain the desired result (5.13).

3. Rate of convergence for the additive kernel $K = x + y$

We now consider the case of the additive kernel:

THEOREM 5.4. *Let n_0 be a positive measure, $\int_0^\infty x n_0(dx) = \int_0^\infty x^2 n_0(dx) = 1$, and $\mu_3 = \int_0^\infty x^3 n_0(dx) < \infty$. With $\hat{n}(t, dx)$ the rescaled solution to Smoluchowski's equation with $K = x + y$ and initial data $n_0(dx)$ let*

$$F(t, \hat{x}) = \int_0^{\hat{x}} \hat{y} \hat{n}(t, d\hat{y}), \qquad F_*(\hat{x}) = \int_0^{\hat{x}} \hat{y} \hat{n}_{*,1}(d\hat{y}).$$

Then for $t \in [0, \infty)$,

$$\sup_{\hat{x}>0} |F(t,\hat{x}) - F_*(\hat{x})| \le C(\mu_3)(1+t)e^{-t} \tag{5.26}$$

where $C(\mu_3)$ *is a constant that depends only on* μ_3.

Proof of convergence rate for $K = x+y$. The analysis for the additive kernel is identical in spirit to that of the constant kernel but is complicated by the fact that the characteristics of the Fourier-Laplace transform are no longer rays. We use a contour deformation argument to overcome this difficulty.

To start, consider for $s \in \bar{\mathbb{C}}_+$ the characteristic exponent of the rescaled number measures

$$\varphi(t,s) = \int_{(0,\infty)} (1 - e^{-s\hat{x}})\hat{n}(t, d\hat{x}), \qquad \varphi_*(s) = \int_{(0,\infty)} (1 - e^{-s\hat{x}})\hat{n}_*(d\hat{x})$$

and let $\varphi_0(s) = \varphi(0,s)$. The Fourier-Laplace transform of the mass measure then satisfies

$$u(t,s) = \partial_s \varphi(t,s) = \int_{(0,\infty)} e^{-s\hat{x}}\hat{x}\hat{n}(t,d\hat{x}), \qquad u_*(s) = \int_{(0,\infty)} e^{-s\hat{x}}\hat{x}\hat{n}_*(d\hat{x})$$

with $u_0(s) = u(0,s)$. We have by (5.12)

$$\sup_{\hat{x}>0} |F(t,\hat{x}) - F_*(\hat{x})| \le \frac{1}{\pi} \sup_{\hat{x}>0} \left| \int_{-iT}^{iT} \frac{e^{\sigma\hat{x}}}{\sigma}(u(t,\sigma) - u_*(\sigma))d\sigma \right| + \frac{24}{\pi T}. \tag{5.27}$$

Note that by the definition of φ and u we have that for $s \in \bar{\mathbb{C}}_+$

$$|\varphi(t,s)| \le |s|, \qquad |u(t,s)| \le 1. \tag{5.28}$$

1. Evolution of characteristics: The evolution of φ and u are given by

$$\partial_t \varphi + (2s - \varphi)\partial_s \varphi = \varphi \tag{5.29}$$

$$\partial_t u + (2s - \varphi)\partial_s u = -u(1-u). \tag{5.30}$$

As with the constant kernel, these equations can be derived from (5.3), or from (5.1) if the initial number measure has a density [**39**]. The solution to these PDE are in terms of the characteristics $s(t; t_0, s_0)$ for $s_0 \in \bar{\mathbb{C}}_+$, which satisfy

$$\frac{ds}{dt} = 2s - \varphi. \tag{5.31}$$

Along these characteristic curves we have that

$$\frac{d\varphi}{dt} = \varphi, \qquad \frac{du}{dt} = -u(1-u)$$

so that by integrating:

$$\varphi(t,s) = e^t \varphi_0(s_0), \qquad u(t,s) = e^{-t} \frac{u_0(s_0)}{1 - u_0(s_0)(1-e^{-t})}. \tag{5.32}$$

The explicit form for the first set of characteristics $s_0 \mapsto s$ is then

$$s(t; 0, s_0) = e^{2t}(s_0 - \varphi_0(s_0)(1 - e^{-t})), \tag{5.33}$$

which can also be expressed as

$$\frac{\varphi(t,s)}{s} = e^{-t} \frac{(\varphi_0(s_0)/s_0)}{1 - (\varphi_0(s_0)/s_0)(1-e^{-t})}. \tag{5.34}$$

In addition, (5.33) implies that

$$\frac{ds}{ds_0} = e^{2t}(1 - u_0(s_0)(1 - e^{-t})). \tag{5.35}$$

Note that the equations for these characteristics depend on φ_0, and thus on u_0. We can perform the same analysis for the Fourier-Laplace transform of the self-similar solution

$$u_*(s) = \frac{1}{\sqrt{1+2s}}, \qquad \varphi_*(s) = \sqrt{1+2s} - 1 \tag{5.36}$$

Here we are working with a second pair of characteristics $s_0^* \mapsto s^*$,

$$s^*(t; 0, s_0^*) = e^{2t}(s_0^* - \varphi_*(s_0^*)(1 - e^{-t})) \tag{5.37}$$

corresponding to the limiting self-similar profile. Note that

$$\frac{ds^*}{ds_0^*} = e^{2t}(1 - u_*(s_0^*)(1 - e^{-t})). \tag{5.38}$$

Since $u_*(s)$ is a stationary solution along these characteristics, it satisfies

$$u_*(s) = e^{-t} \frac{u_*(s_0^*)}{1 - u_*(s_0^*)(1-e^{-t})}. \tag{5.39}$$

2. Estimates near the origin: We now use the additional finite moment assumption

$$\mu_3 = \int_0^\infty x^3 n_0(dx) = \int_0^\infty \hat{x}^3 \hat{n}_0(dx) < \infty$$

to find an a good approximation for u_0 near the origin. Since

$$u_0(0) = -u_0'(0) = 1, \qquad u_0''(0) = \mu_3$$
$$u_*(0) = -u_*'(0) = 1, \qquad u_*''(0) = 3$$

we can use a Taylor expansion about the origin to get

$$|u_0(\sigma_0) - u_*(\sigma_0)| \leq \left(\frac{3}{2} + \frac{\mu_3}{2}\right) |\sigma_0|^2. \tag{5.40}$$

In addition, using that

$$\varphi_0(0) = 0, \qquad \varphi_0'(0) = 1, \qquad \varphi_0''(0) = -1, \qquad \varphi_0'''(0) = \mu_3$$
$$\varphi_*(0) = 0, \qquad \varphi_*'(0) = 1, \qquad \varphi_*''(0) = -1, \qquad \varphi_*'''(0) = 3$$

we can supplement (5.40) with the estimate

$$|\varphi_0(\sigma_0) - \varphi_*(\sigma_0)| \leq \left(\frac{1}{2} + \frac{\mu_3}{6}\right) |\sigma_0|^3. \tag{5.41}$$

This also gives an estimate for the distance between the characteristics s, s^* starting at σ_0:

$$\begin{aligned} |s(t;0,\sigma_0) - s^*(t;0,\sigma_0)| &= e^{2t}|\varphi_0(\sigma_0) - \varphi_*(\sigma_0)|(1 - e^{-t}) \\ &\leq \left(\frac{1}{2} + \frac{\mu_3}{6}\right)(1 - e^{-t})e^{2t}|\sigma_0|^3. \end{aligned} \tag{5.42}$$

Using (5.28) we have the bound

$$|s^*(t;0,\sigma_0)| = e^{2t}\left|\sigma_0 - \varphi_*(\sigma_0)(1 - e^{-t})\right| \leq e^{2t}|\sigma_0| \tag{5.43}$$

3. Contour deformation: The mappings $s_0 \mapsto s$ and $s_0^* \mapsto s^*$ no longer form rays. Consider the images $\bar{\Omega}_t$ and $\bar{\Omega}_t^*$ of $\bar{\mathbb{C}}_+$ under these mappings. Their boundaries Γ_t and Γ_t^* are images of the imaginary axis. In particular, Γ_t and Γ_t^* are no longer stationary curves in $\bar{\mathbb{C}}_-$ but instead evolve in time and can be considered graphs over the imaginary axis (see [**39**] for further discussion regarding the geometry of characteristics). To straighten the picture we use a contour deformation argument.

Fix $\delta > 0$. We evaluate the r.h.s. of (5.27) as follows. Let

$$\Sigma_0 = \{\sigma_0 \in \mathbb{C} : \sigma_0 = iy, y \in [-\delta, \delta]\},$$

so that Σ_0 is a piece of the imaginary axis of length 2δ. Define the image of Σ_0 under the mappings $s_0 \mapsto s$ and $s_0^* \mapsto s^*$:

$$S_t = s(t; 0, \Sigma_0), \qquad S_t^* = s^*(t; 0, \Sigma_0).$$

S_t and S_t^* are finite length curves in $\bar{\mathbb{C}}_-$. Finally, define

$$\Sigma_t = \{\sigma \in \mathbb{C} : \sigma = iy, y \in [-\delta e^{2t}, \delta e^{2t}]\}. \tag{5.44}$$

To evaluate the integral (5.27), first split it into its individual components

$$\int_{\Sigma_t} \frac{e^{\sigma\hat{x}}}{\sigma}(u(t,\sigma) - u_*(\sigma))d\sigma = \int_{\Sigma_t} \frac{e^{\sigma\hat{x}}}{\sigma}u(t,\sigma)d\sigma - \int_{\Sigma_t} \frac{e^{\sigma\hat{x}}}{\sigma}u_*(\sigma)d\sigma. \tag{5.45}$$

Since the integrands are holomorphic in $\bar{\Omega}_t$ and $\bar{\Omega}_t^*$, respectively, we can deform the contours and apply Cauchy's theorem:

$$\int_{\Sigma_t} \frac{e^{\sigma\hat{x}}}{\sigma}u(t,\sigma)d\sigma = \int_{S_t} \frac{e^{s\hat{x}}}{s}u(t,s)ds + \int_{\mathcal{E}_t} \frac{e^{s\hat{x}}}{s}u(t,s)ds, \tag{5.46}$$

and

$$\int_{\Sigma_t} \frac{e^{\sigma\hat{x}}}{\sigma}u_*(\sigma)d\sigma = \int_{S_t^*} \frac{e^{s^*\hat{x}}}{s^*}u_*(s^*)ds^* + \int_{\mathcal{E}_t^*} \frac{e^{s^*\hat{x}}}{s^*}u_*(s^*)ds^*. \tag{5.47}$$

Here, the error terms are integrals over the contours $\mathcal{E}_t$ and $\mathcal{E}_t^*$ that connect S_t to Σ_t and S_t^* to Σ_t, respectively (see Figure 5.1). We choose $\mathcal{E}_t$ and $\mathcal{E}_t^*$ so that their backward images

$$E_t = s(0; t, \mathcal{E}_t), \qquad E_t^* = s^*(0; t, \mathcal{E}_t^*) \tag{5.48}$$

are straight lines. Lastly we make the change of variables $s \mapsto s_0$ and $s^* \mapsto s_0^*$, respectively, and use (5.32), (5.35), (5.38), and (5.39) to get:

$$\begin{aligned} &\int_{S_t} \frac{e^{s\hat{x}}}{s}u(t,s)ds - \int_{S_t^*} \frac{e^{s^*\hat{x}}}{s^*}u_*(s^*)ds^* \\ &= e^{-t}\int_{\Sigma_0}\left(e^{s(t;0,\sigma_0)\hat{x}}\frac{e^{2t}}{s(t;0,\sigma_0)}u_0(\sigma_0) - e^{s^*(t;0,\sigma_0)\hat{x}}\frac{e^{2t}}{s^*(t;0,\sigma_0)}u_*(\sigma_0)\right)d\sigma_0. \end{aligned} \tag{5.49}$$

4. Propagation of estimates: Upon inspection of (5.33) and (5.37) we see that the natural growth rate of both sets of characteristics is e^t near the origin, and e^{2t} at an $O(1)$ distance away from the origin. We will bound the r.h.s. of (5.27) with $T = \delta e^{2\tau}$, where $\delta = \delta(\mu_2) > 0$ is a length scale characterizing the region about the

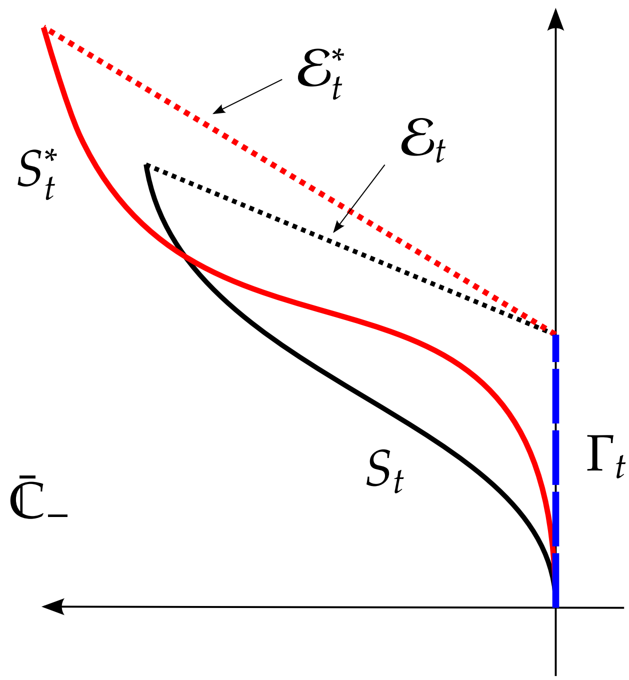

FIGURE 5.1. Contour deformation of the imaginary axis under the maps $s_0 \mapsto s$ and $s_0^* \mapsto s^*$. Here, the endpoints of the curves S_t and S_t^* are $s(t; 0, i\delta)$ and $s^*(t; 0, i\delta)$, respectively.

origin where the difference between u_0 and u_* is small. We will then propagate the estimate (5.40) from this region. To begin, let $\rho_3 = \max\{\mu_3, 3\}$ and define

$$\delta(\mu_3) = \sqrt{1 + \frac{1}{2}\left(\frac{1}{12} + \frac{\rho_3}{6}\right)^{-1}} - 1 \tag{5.50}$$

so that $\left(\frac{1}{12} + \frac{\rho_3}{6}\right)\delta = \frac{1}{2}(2+\delta)^{-1}$. Note that if $\mu_3 \to \infty$ then $\delta \to 0$.

Using the basic inequality $|a_1 b_1 - a_2 b_2| \le |b_1 - b_2||a_1| + |a_1 - a_2||b_2|$, we can bound the magnitude of (5.49) by the sum $\Lambda_1 + \Lambda_2$, where

$$\Lambda_1 = e^{-t}\int_0^{\delta} \left| \frac{e^{2t}}{s} u_0(\sigma_0) - \frac{e^{2t}}{s^*} u_*(\sigma_0) \right| |e^{s^*\hat{x}}| d|\sigma_0| \tag{5.51}$$

$$\Lambda_2 = e^{-t}\int_0^{\delta} |e^{s\hat{x}} - e^{s^*\hat{x}}| \left| \frac{e^{2t}}{s^*} u_*(\sigma_0) \right| d|\sigma_0| \tag{5.52}$$

We now evaluate each of these terms. For Λ_1 the analysis is almost exactly as in the constant kernel case.

5. Bounds for Λ_1: To begin, consider Λ_1. Since $s^* \in \bar{\mathbb{C}}_-$, we have $|e^{s^*\hat{x}}| \le 1$. Using the explicit form (5.33), (5.37) of the characteristics, we rearrange terms in the integrand of (5.51) to get

$$\begin{aligned}\left|\frac{e^{2t}}{s}u_0(\sigma_0) - \frac{e^{2t}}{s^*}u_*(\sigma_0)\right| \\ = \left|\frac{1}{\sigma_0}\frac{\left(\frac{e^{-2t}s^*}{\sigma_0}u_0(\sigma_0) - \frac{e^{-2t}s}{\sigma_0}u_*(\sigma_0)\right)}{\left(1-\frac{\varphi_*(\sigma_0)}{\sigma_0}(1-e^{-t})\right)\left(1-\frac{\varphi_0(\sigma_0)}{\sigma_0}(1-e^{-t})\right)}\right|. \qquad (5.53)\end{aligned}$$

Notice that $\varphi_0(\sigma_0)/\sigma_0$ and $\varphi_*(\sigma_0)/\sigma_0$ play the same role here as $u_0(\sigma_0)$ and $u_*(\sigma_0)$ do in the constant kernel. Using that $|u_*| \leq 1$ and the estimates (5.42), (5.43), the numerator in (5.53) satisfies

$$\begin{aligned}\left|\frac{e^{-2t}s^*}{\sigma_0}u_0(\sigma_0) - \frac{e^{-2t}s}{\sigma_0}u_*(\sigma_0)\right| \\ \leq \left|\frac{e^{-2t}s^*}{\sigma_0} - \frac{e^{-2t}s}{\sigma_0}\right||u_*(\sigma_0)| + |u_0(\sigma_0) - u_*(\sigma_0)|\left|\frac{e^{-2t}s^*}{\sigma_0}\right| \\ \leq 7\left(\frac{1}{2}+\frac{\mu_3}{6}\right)|\sigma_0|^2. \qquad (5.54)\end{aligned}$$

To estimate the denominator, we split the integral in Λ_1 into the two regions $0 \leq |\sigma_0| < \delta e^{-t}$ and $\delta e^{-t} \leq |\sigma_0| \leq \delta$.

5a. Bounds for Λ_1 in $0 \leq |\sigma_0| < \delta e^{-t}$: Here we use that $|\varphi_0(\sigma_0)/\sigma_0|, |\varphi_*(\sigma_0)/\sigma_0| \leq 1$ and estimate the terms in the denominator of (5.53) by the naive lower bound

$$|1 - z(1-e^{-t})| \geq 1 - |z|(1-e^{-t}) \geq e^{-t}, \qquad |z| \leq 1.$$

Substituting this and (5.54) into (5.53) one obtains

$$\begin{aligned}e^{-t}\int_{0\leq|\sigma_0|<\delta e^{-t}}\left|\frac{e^{2t}}{s}u_0(\sigma_0) - \frac{e^{2t}}{s^*}u_*(\sigma_0)\right||e^{s^*\hat{x}}|d|\sigma_0| \\ \leq 7\left(\frac{1}{2}+\frac{\mu_3}{6}\right)e^t\int_0^{\delta e^{-t}}|\sigma_0|d|\sigma_0| \\ = \frac{7}{2}\left(\frac{1}{2}+\frac{\mu_3}{6}\right)\delta^2 e^{-t}. \qquad (5.55)\end{aligned}$$

5b. Bounds for Λ_1 in $\delta e^{-t} \le |\sigma_0| \le \delta$: In this region we use Taylor expansion, the fact that σ_0 is pure imaginary, and the value (5.50) of δ to get

$$\begin{aligned}\left|1 - \frac{\varphi_*(\sigma_0)}{\sigma_0}(1-e^{-t})\right| &\ge \left|\frac{2e^{-t}+\sigma_0}{2+\sigma_0}\right| - \left(\frac{1}{12}+\frac{3}{6}\right)|\sigma_0|^2 \\ &\ge \frac{|\sigma_0|}{2+\delta} - \left(\frac{1}{12}+\frac{3}{6}\right)|\sigma_0|^2 \ge \frac{1}{2}\cdot\frac{|\sigma_0|}{2+\delta} \qquad (5.56)\end{aligned}$$

and

$$\begin{aligned}\left|1 - \frac{\varphi_0(\sigma_0)}{\sigma_0}(1-e^{-t})\right| &\ge \left|\frac{2e^{-t}+\sigma_0}{2+\sigma_0}\right| - \left(\frac{1}{12}+\frac{\mu_3}{6}\right)|\sigma_0|^2 \\ &\ge \frac{|\sigma_0|}{2+\delta} - \left(\frac{1}{12}+\frac{\mu_3}{6}\right)|\sigma_0|^2 \ge \frac{1}{2}\cdot\frac{|\sigma_0|}{2+\delta}. \qquad (5.57)\end{aligned}$$

Therefore, using (5.56-5.57) and (5.54) in (5.53) gives

$$\begin{aligned}&e^{-t}\int_{\delta e^{-t}\le|\sigma_0|\le\delta}\left|\frac{e^{2t}}{s}u_0(\sigma_0) - \frac{e^{2t}}{s^*}u_*(\sigma_0)\right| |e^{s^*\hat{x}}| d|\sigma_0| \\ &\le 4\cdot 7\left(\frac{1}{2}+\frac{\mu_3}{6}\right)(2+\delta)^2 e^{-t}\int_{\delta e^{-t}}^{\delta}\frac{1}{|\sigma_0|}d|\sigma_0| \\ &= 4\cdot 7\left(\frac{1}{2}+\frac{\mu_3}{6}\right)(2+\delta)^2 t e^{-t}. \qquad (5.58)\end{aligned}$$

6. Bounds for Λ_2: Now we consider Λ_2. Using $|u_*| \le 1$ and (5.37) we can rewrite this as

$$\Lambda_2 \le e^{-t}\int_0^{\delta}\frac{|e^{s\hat{x}} - e^{s^*\hat{x}}|}{|\sigma_0|}\left|\frac{1}{\left(1 - \frac{\varphi_*(\sigma_0)}{\sigma_0}(1-e^{-t})\right)}\right| d|\sigma_0|. \qquad (5.59)$$

Note that

$$\frac{|e^{s\hat{x}} - e^{s^*\hat{x}}|}{|s-s^*|} \le \sup_{\hat{x}>0}|\hat{x}e^{s^*\hat{x}}| \le \frac{1}{\mathrm{Re}(s^*)} = \frac{e^{-2t}}{\mathrm{Re}(\varphi_*(\sigma_0))(1-e^{-t})}. \qquad (5.60)$$

By (5.36), $\varphi_*(\sigma_0) = \sqrt{1+2\sigma_0} - 1$. From this we obtain the lower bound

$$|\mathrm{Re}(\varphi_*(\sigma_0))| \ge \frac{1}{4}|\sigma_0^2|. \qquad (5.61)$$

Combining (5.42), (5.60), and (5.61) gives

$$\frac{|e^{s\hat{x}} - e^{s^*\hat{x}}|}{|\sigma_0|} \le 4\left(\frac{1}{2}+\frac{\mu_3}{6}\right). \qquad (5.62)$$

For the remaining term in (5.59), we split the integral into the two regions $0 \leq |\sigma_0| < \delta e^{-t}$ and $\delta e^{-t} \leq |\sigma_0| \leq \delta$.

6a. Bounds for Λ_2 in $0 \leq |\sigma_0| < \delta e^{-t}$: Using the naive lower bound for the term in the denominator as we did in Step 4 and (5.62), we get

$$e^{-t} \int_0^{\delta e^{-t}} \frac{|e^{s\hat{x}} - e^{s^*\hat{x}}|}{|\sigma_0|} \left| \frac{1}{\left(1 - \frac{\varphi_*(\sigma_0)}{\sigma_0}(1 - e^{-t})\right)} \right| d|\sigma_0| \leq 4 \left(\frac{1}{2} + \frac{\mu_3}{6}\right) \delta e^{-t}. \tag{5.63}$$

6b. Bounds for Λ_2 in $\delta e^{-t} \leq |\sigma_0| \leq \delta$: For the term in the denominator, we use the estimate (5.56). Then,

$$\begin{aligned} e^{-t} \int_{\delta e^{-t}}^{\delta} \frac{|e^{s\hat{x}} - e^{s^*\hat{x}}|}{|\sigma_0|} & \left| \frac{1}{\left(1 - \frac{\varphi_*(\sigma_0)}{\sigma_0}(1 - e^{-t})\right)} \right| d|\sigma_0| \\ & \leq 2 \cdot 4 \left(\frac{1}{2} + \frac{\mu_3}{6}\right) (2+\delta) e^{-t} \int_{\delta e^{-t}}^{\delta} \frac{1}{|\sigma_0|} d|\sigma_0| \\ & = 2 \cdot 4 \left(\frac{1}{2} + \frac{\mu_3}{6}\right) (2+\delta) t e^{-t}. \end{aligned} \tag{5.64}$$

7. Bounds for $\Lambda_1 + \Lambda_2$: Putting together (5.55), (5.58), (5.63), and (5.64), we finally find a bound for (5.51-5.52):

$$\Lambda_1 + \Lambda_2 \leq 40 \left(\frac{1}{2} + \frac{\mu_3}{6}\right) (2+\delta)^2 (1+t) e^{-t}. \tag{5.65}$$

8. Error terms: Lastly, we bound the error terms obtained during the contour deformation argument (5.46-5.47). We need only estimate

$$\left| \int_{\mathcal{E}_t} \frac{e^{s\hat{x}}}{s} u(t,s) ds \right|$$

since the corresponding integral over $\mathcal{E}_t^*$ is estimated by an identical argument. First note that the endpoints of the line segment E_t are $i\delta$ and $s(0; t, i\delta e^{2t})$, the latter of which converges to the point $\rho_0 \in \mathbb{C}_+$ that solves

$$\rho_0 - \varphi_0(\rho_0) = i\delta.$$

Then we have the estimate

$$\begin{aligned}\left|\int_{\mathcal{E}_t} \frac{e^{s\hat{x}}}{s} u(t,s)ds\right| &= e^{-t}\left|\int_{E_t} e^{s(t;0,s_0)\hat{x}} \frac{u_0(s_0)}{s_0 - \varphi_0(s_0)(1-e^{-t})} ds_0\right| \\ &\leq e^{-t}|E_t| \sup_{s_0 \in E_t} |s_0 - \varphi_0(s_0)(1-e^{-t})|^{-1} \\ &\leq C_1 e^{-t}\end{aligned}$$

for some constant C_1 that depends only on the initial data u_0. This is similar to the argument for the additive kernel in [**39**].

9. Rate of convergence: By combining (5.27) with (5.65) and the estimate for the error terms, we obtain the desired result (5.26).

4. Rate of convergence for the multiplicative kernel $K = xy$

THEOREM 5.5. *Let n_0 be a positive measure, $\int_0^\infty x^2 n_0(dx) = \int_0^\infty x^3 n_0(dx) = 1$, and $\mu_4 = \int_0^\infty x^4 n_0(dx) < \infty$. With $\tau = \log(1-t)^{-1}$ and $\hat{n}(t,dx)$ the rescaled solution to Smoluchowski's equation with $K = xy$ and initial data $n_0(dx)$ let*

$$F(\tau,\hat{x}) = \int_0^{\hat{x}} \hat{y}^2 \hat{n}(\tau, d\hat{y}), \qquad F_*(\hat{x}) = \int_0^{\hat{x}} \hat{y}^2 \hat{n}_{*,2}(d\hat{y}).$$

Then for $\tau \in [0,\infty)$,

$$\sup_{\hat{x}>0} |F(\tau,\hat{x}) - F_*(\hat{x})| \leq C(\mu_4)(1+\tau)e^{-\tau} \tag{5.66}$$

where $C(\mu_4)$ is a constant that depends only on μ_4.

Proof of convergence rate for $K = xy$. The rate for the multiplicative kernel can be recovered from that of the additive kernel by a change of variable, as discussed in [**19, 39**]. The measure-valued solutions $n_2(t,dx)$ and $n_1(t,dx)$ to (5.3) with $K = xy$ and $K = x+y$, respectively, are related by

$$xn_2(t,dx) = (1-t)^{-1} n_1(\tau(t), dx).$$

With the similarity variables

$$\hat{x}_2 = \frac{x}{\lambda_2(t)}, \qquad \hat{x}_1 = \frac{x}{\lambda_1(\tau)}$$

given by (5.7) and the rescaled number distributions defined by (5.9), we have that

$$\hat{x}_2^2 \hat{n}_2(\tau, d\hat{x}_2) = \hat{x}_1 \hat{n}_1(\tau, d\hat{x}_1).$$

Therefore,

$$F_2(\tau, \hat{x}) = \int_0^{\hat{x}} \hat{y}_2^2 \hat{n}_2(\tau, d\hat{y}_2) = \int_0^{\hat{x}} \hat{y}_1 \hat{n}_1(\tau, d\hat{y}_1) = F_1(\tau, \hat{x}).$$

Since (5.5) implies that $F_{*,2}(\hat{x}) = F_{*,1}(\hat{x})$, we combine this with the previous equation to get (5.66)

$$\sup_{\hat{x}>0} |F_2(\tau, \hat{x}) - F_{*,2}(\hat{x})| = \sup_{\hat{x}>0} |F_1(\tau, \hat{x}) - F_{*,1}(\hat{x})| \leq C(\mu_4)(1+\tau)e^{-\tau}.$$

The constant C here is the same as that obtained in the case $K = x + y$, with μ_3 replaced by μ_4. The convergence rate is nearly optimal by the near optimality of the rate for the additive kernel.

Conclusions

We have demonstrated that at the level of the 2-point correlation function, Burgers turbulence is closed and yields a completely integrable system. This holds true for all 1-D scalar conservation laws. Additionally, we have shown that the model admits Groeneboom's explicit solution and therefore has a rich family of self-similar solutions. Our work provides some of the initial steps necessary to obtain an full understanding of this model, but leaves many more questions unanswered. What is the probabilistic interpretation of the evolution equation? Can it be derived by a first-passage argument for spectrally negative Markov processes, analogous to Bertoin's analysis for Lévy processes? How do we use the integrable structure to rederive Groeneboom's solution? Are there universal distributions other than Chernoff's that can be recovered using this model? These questions, among many others, pose interesting challenges for future study.

Bibliography

[1] D.J. Aldous. Deterministic and stochastic models for coalescence (aggregation and coagulation): A review of the mean-field theory for probabilists. *Bernoulli-London*, 5(1):3–48, 1999.

[2] D. Applebaum. *Lévy processes and stochastic calculus*. Cambridge University Press, 2004.

[3] M. Avellaneda and W. E. Statistical properties of shocks in Burgers turbulence. *Communications in Mathematical Physics*, 172(1):13–38, 1995.

[4] O.E. Barndorff-Nielsen, T. Mikosch, and S.I. Resnick. *Lévy processes: theory and applications*. Birkhauser, 2001.

[5] J. Bec and K. Khanin. Burgers turbulence. *Physics Reports*, 447(1-2):1–66, 2007.

[6] J. Bertoin. Large-deviations estimates in Burgers turbulence with stable noise initial data. *Journal of Statistical Physics*, 91(3):655–667, 1998.

[7] J. Bertoin. *Lévy processes*. Cambridge university press, 1998.

[8] J. Bertoin. The inviscid Burgers equation with Brownian initial velocity. *Communications in Mathematical Physics*, 193(2):397–406, 1998.

[9] J. Bertoin. Structure of shocks in Burgers turbulence with stable noise initial data. *Communications in Mathematical Physics*, 203(3):729–741, 1999.

[10] J. Bertoin. Eternal solutions to Smoluchowski's coagulation equation with additive kernel and their probabilistic interpretations. *Annals of Applied Probability*, pages 547–564, 2002.

[11] J. Bertoin. Some aspects of additive coalescents, In: Proceedings of the International Congress of Mathematicians, Beijing 2002, Vol. III, pp. 15-23, Beijing Higher Ed. Press. 2002.

[12] J.M. Burgers. *The nonlinear diffusion equation*. Reidel, 1974.

[13] J.A. Canizo, S. Mischler, and C. Mouhot. Rate of convergence to self-similarity for Smoluchowski's coagulation equation with constant coefficients. *Arxiv preprint arXiv:0811.1169*, 2008.

[14] L. Carraro and J. Duchon. Solutions statistiques intrinsèques de l'équation de Burgers et processus de Lévy. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 319(8):855–858, 1994.

[15] L. Carraro and J. Duchon. Équation de Burgers avec conditions initiales à accroissements indépendants et homogènes. *Annales de l'institut Henri Poincaré (C) Analyse non linéaire*, 15(4):431–458, 1998.

[16] M.L. Chabanol and J. Duchon. Markovian solutions of inviscid Burgers equation. *Journal of Statistical Physics*, 114(1):525–534, 2004.

[17] H. Chernoff. Estimation of the mode. *Annals of the Institute of Statistical Mathematics*, 16(1):31–41, 1964.

[18] C.M. Dafermos. *Hyperbolic conservation laws in continuum physics*. Springer, 2005.

[19] R.L. Drake. A general mathematical survey of the coagulation equation. *Topics in Current Aerosol Research (Part 2)*, pages 201–376, 1972.

[20] W. E and E. Vanden-Eijnden. Statistical Theory for the Stochastic Burgers Equation in the Inviscid Limit. *Communications on Pure and Applied Mathematics*, 53(7):852–901, 2000.

[21] L.C. Evans. *Partial differential equations*. American Mathematical Society, 1998.

[22] W. Feller. *An Introduction to Probability Theory and Its Applications: Volume II*. John Wiley & Sons, 1971.

[23] C. Foiaş. Statistical study of Navier-Stokes equations. I, II. *Rend. Sem. Mat. Univ. Padova*, 48:219–348 (1973); ibid. 49 (1973), 9–123, 1972.

[24] L. Frachebourg and P.A. Martin. Exact statistical properties of the Burgers equation. *Journal of Fluid Mechanics*, 417:323–349, 2000.

[25] L. Frachebourg, P.A. Martin, and J. Piasecki. Ballistic aggregation: a solvable model of irreversible many particles dynamics. *Physica A: Statistical Mechanics and its Applications*, 279(1-4):69–99, 2000.

[26] R.K. Getoor. Splitting times and shift functionals. *Probability Theory and Related Fields*, 47(1):69–81, 1979.

[27] C. Giraud. Genealogy of Shocks in Burgers Turbulencewith White Noise Initial Velocity. *Communications in Mathematical Physics*, 223(1):67–86, 2001.

[28] C. Giraud. On regular points in Burgers turbulence with stable noise initial data. In *Annales de l'Institut Henri Poincaré/Probabilités et statistiques*, volume 38, pages 229–251. Elsevier, 2002.

[29] U. Grenander. On the theory of mortality measurement, part II. *Skandinavisk Aktuarietidskrift*, 39:125–153, 1956.

[30] P. Groeneboom. Estimating a monotone density, In: Le Cam, L.E., and Olshen, R.A. (eds.), Proceedings of the Berkeley conference in honor of Jerzy Neyman and Jack Kiefer, Vol. II, pp. 539-555, Wadsworth, Monterey. 1985.

[31] P. Groeneboom. Brownian motion with a parabolic drift and Airy functions. *Probability Theory and Related Fields*, 81(1):79–109, 1989.

[32] P. Groeneboom and J.A. Wellner. Computing Chernoff's distribution. *Journal of Computational and Graphical Statistics*, 10(2):388–400, 2001.

[33] S.N. Gurbatov and A.I. Saichev. Probability distribution and spectra of potential hydrodynamic turbulence. *Radiophysics and Quantum Electronics*, 27(4):303–313, 1984.

[34] E. Hopf. Statistical hydrodynamics and functional calculus. *J. Rat. Mech. Anal*, 1:87–123, 1952.

[35] N. Jacob and R.L. Schilling. Levy-type processes and pseudo-differential operators. *Levy processes: Theory and Applications, Birkhauser, Boston*, pages 139–167, 2001.

[36] M. Kardar, G. Parisi, and Y.C. Zhang. Dynamic scaling of growing interfaces. *Physical Review Letters*, 56(9):889–892, 1986.

[37] J. Krug and H. Spohn. Universality classes for deterministic surface growth. *Physical Review A*, 38(8):4271–4283, 1988.

[38] G. Menon and R.L. Pego. Approach to self-similarity in Smoluchowski's coagulation equations. *Communications on Pure and Applied Mathematics*, 57(9):1197–1232, 2004.

[39] G. Menon and R.L. Pego. Dynamical Scaling in Smoluchowski's Coagulation Equations: Uniform Convergence. *SIAM Review*, 48(4):745–768, 2006.

[40] G. Menon and R.L. Pego. Universality classes in Burgers turbulence. *Communications in Mathematical Physics*, 273(1):177–202, 2007.

[41] P.W. Millar. Random times and decomposition theorems. *Probability*, 31:91, 1977.

[42] P.W. Millar. A path decomposition for Markov processes. *The Annals of Probability*, 6(2):345–348, 1978.

[43] J. Piasecki. Ballistic annihilation in a one-dimensional fluid. *Physical Review E*, 51(6):5535–5540, 1995.

[44] B.L.S. Prakasa Rao. Estimation of a unimodal density. *Sankhya Ser. A*, 31:23–36, 1969.

[45] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Springer, 1999.

[46] Z.S. She, E. Aurell, and U. Frisch. The inviscid Burgers equation with initial data of Brownian type. *Communications in Mathematical Physics*, 148(3):623–641, 1992.

[47] Y.G. Sinai. Statistics of shocks in solutions of inviscid Burgers equation. *Communications in Mathematical Physics*, 148(3):601–621, 1992.

[48] A.I. Vol'pert. The spaces BV and quasilinear equations. *Mathematics of the USSR-Sbornik*, 2:225–267, 1967.

[49] M. Winkel. Limit clusters in the inviscid Burgers turbulence with certain random initial velocities. *Journal of Statistical Physics*, 107(3):893–917, 2002.

[50] Y.B. Zel'dovich. Gravitational instability: An approximate theory for large density perturbations. *Astron. Astrophys.*, 5:84–89, 1970.