Isometries on CAT(0) spaces, iteration of mapping classes and Weil-Petersson geometry

by

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Contents

Abstract

Isometries on CAT(0) spaces, iteration of mapping classes and Weil-Petersson geometry

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Let $S = S_g$ be a surface of genus $g > 1$, and Teich(S) be the Teichmüller space endowed with the Weil-Petersson metric and Mod(S) be the mapping class group of S. This dissertation mainly consists of three parts.

The first part is to study the translation lengths of parabolic isometries on complete proper visible CAT(0) spaces and their applications. We show that the translation length of parabolic isometry is always zero in the visible case. The first application is that the mapping class groups Mod(S_g) of S_g ($g \geq 3$) which properly discontinuously act on a complete proper visible CAT(0) space by isometries have zero translation length (every element in Mod(S_g) has zero translation length). We also apply the zero property to giving a criterion for closed two-dimensional manifolds with bounded geometry. The third application is to give a negative answer to P. Eberlein's conjecture which says that *a complete open manifold M with sectional curvature $-1 \leq K_M \leq 0$ and finite volume is visible if the universal covering space $\tilde{M}$ of M contains no imbedded flat half planes*.

Secondly, we show that, fix $X, Y \in$ Teich(S), for any $\phi \in$ Mod(S), there exists a positive integer k depending on ϕ such that the sequence of the directions of geodesics connecting X and $\phi^{kn} \circ Y$ is convergent in the visual sphere of X. In particular the "limit" of the sequence of geodeiscs joining X and $\phi^{kn} \circ Y$ exists in some sense, a geometric description for the limit is provided in this dissertation.

The third part is to show that the Riemannian sectional curvature operator of Teich(S) is non-positive definite. As an application we show that any twist harmonic map with respect to Mod(S) from rank-one hyperbolic spaces $H_{Q,m} = Sp(m,1)/Sp(m) \cdot Sp(1)$ or $H_{O,2} = F_4^{-20}/SO(9)$ into Teich(S) must be a constant map.

CHAPTER 1

Introduction

This dissertation has three completely independent parts. The first part is to study parabolic isometries on proper visible spaces. We give a geometric obstruction to visible manifolds with bounded geometry, which is sufficient to construct counterexamples to Eberlein's conjecture which says that weak rank one nonpositive curved manifolds with bounded geometry are visible. The second part is to study iteration of mapping classes on Teichmüller space endowed with the Weil-Petersson metric. We give a description of the limit of geodeiscs in the Teichmüller space. The third part is to study the sectional curvature operator of Weil-Petersson metric. We show that the operator is non-positive definite. As an application we will provide a rigid theorem for harmonic maps into Teichmüller space.

The first part of this dissertation is to study parabolic isometries on proper visible CAT(0) spaces and their applications. CAT(0) spaces are generalizations of Riemannian manifolds with nonpositive sectional curvature to geodesic spaces. The classification of isometries of CAT(0) spaces is similar to the classification of isometries for the hyperbolic half plane. Let M be a CAT(0) space and γ be an isometry on M. γ is called *parabolic* if the translation length $|\gamma| := \inf_{x\in M} dist(\gamma\circ x, x)$ cannot be obtained in M; otherwise it is called *semi-simple*. Let $M(\infty)$ be the ideal boundary of M defined as the asymptotic classes of rays in M (see [**10**]). M is called *visible* if for any two different points x, y in $M(\infty)$ there exists a geodesic line $c : \mathbb{R} \to M$ such that $c(-\infty) = x$ and $c(+\infty) = y$. A metric space is called *proper* if it is locally compact. Now we can state the following theorem which is one of the main results in the first part.

Theorem. *Let M be a complete proper visible CAT(0) space. Then any parabolic isometry has zero translation length, i.e., for any parabolic isometry γ of M we have $|\gamma| = 0$.*

Phan conjectured in [**48**] that *if M is a tame, finite volume, negatively curved manifold, then M is not visible if its fundamental group $\pi_1(M)$ contains a parabolic isometry of $\tilde{M}$ with positive translation length.* This result gives an affirmative answer to this conjecture.

Gromov and Eberlein constructed a two-dimensional complete open surface with Gauss curvature $K \leq 0$ and finite volume such that the fundamental group consists of isometries on the universal covering with positive translation lengths (see [**21**]). The first application of the theorem above is the following criterion for a two-dimensional manifold with bounded geometry to be closed.

Theorem. *Let M be a complete two-dimensional surface with the Gauss curvature $-1 \leq K(M) \leq 0$ and $Vol(M) < +\infty$, and let $\pi_1(M,p)$ be the fundamental group of M with a basepoint p. Then M is closed if and only if for any non-trivial deck transformation $\phi \in \pi_1(M,p)$ the translation length $|\phi| > 0$.*

In [**13**] Brock and Farb asked *whether the moduli space $\mathbb{M}_g$ of a closed surface S_g $(g \geq 2)$ (up to finite covering) admits a complete, finite volume Riemannian metric with non-positive sectional curvature.* As a second application of the zero property of parabolic isometry, the following result can partially answer Brock-Farb's question.

Theorem. *The moduli space $\mathbb{M}_g$ of a closed surface S_g $(g \geq 3)$ (up to finite covering) admits no complete, finite volume Riemannian metric whose sectional curvature is nonpositive and universal covering is visible.*

If the manifold is Gromov-hyperbolic, this result was proved in [**13**].

In [**23**] Farb conjectures that *the moduli space $\mathbb{M}_g$ of a closed surface S_g $(g \geq 2)$ (up to finite covering) admits no complete, finite volume Riemannian metric with sectional curvature* $-1 \leq K(\mathbb{M}_g) \leq 0$, which is a weaker version of Brock-Farb's question. Eberlein conjectures in [**21**] that *a complete open manifold M with sectional curvature $-1 \leq K_M \leq 0$ and finite volume is visible if the universal covering space $\tilde{M}$ of M contains no imbedded flat half planes*. From the theorem above we know that if Eberlein's conjecture is correct, then it could partially solve Farb's conjecture. If we assume that there exists a metric on $\mathbb{M}_g$ ($g \geq 3$) (up to finite covering) such that it has finite volume and sectional curvature $-1 \leq K(\mathbb{M}_g) \leq 0$ and the universal covering $\mathbb{T}_S$ contains no imbedded flat half planes (Weaker Farb conjecture), the "candidate metric" on $\mathbb{M}_g$ ($g \geq 3$) would be a counterexample of Eberlein's conjecture. The following result gives a negative answer to Eberlein's conjecture.

THEOREM. *The fundamental groups of manifolds M constructed in* [**1**] *and* [**26**] *with finite volume and sectional curvature $-1 \leq K_M < 0$ contain parabolic isometries of $\tilde{M}$ with positive translation length. In particular, M is not visible.*

In [**48**] Phan also independently proved Fujiwara's example is not visible by finding two points x, y on the visual boundary of the universal covering space of Fujiwara's example such that there does not exist a geodesic line joining x and y in the universal covering space.

Let $S = S_g$ be a closed surface of genus $g > 1$, $T(S)$ be the Teichmüller space of S (without metric) and Teich(S) be the Teichmüller space of S_g endowed with the Weil-Petersson metric. Let $X, Y \in T(S)$ and $\Gamma(X, Y)$ be the quasi-Fuchsian *Bers simultaneous uniformization* of $(X, Y) \in T(S) \times T(\overline{S})$ (see [**11**]). Then $\Gamma(X, Y)$ determines $Q(X, Y) = H^3/\Gamma(X, Y)$ as its quotient hyperbolic 3-manifold. In [**11**] it was shown that for any $\phi \in \text{Mod}(S_g)$, there is an $s \geq 1$ depending only on ϕ and bounded in terms of S so that the sequence $\{Q(\phi^{si} \circ X, Y)\}_{i \geq 1}$ converges algebraically and geometrically. Let $X, Y \in \text{Teich}(S)$ and $g(X, Y)$ denote the geodesic joining X and Y. The direction of the unique geodesic segment joining X and Y in the visual sphere of Teich(S) at X plays a role as $Q(X, Y)$. The following result is analogous to Brock's theorem.

Theorem. *Let* $\phi \in \mathrm{Mod}(S)$ *be a mapping class. Then there is an* $s \geq 1$ *only depending on* ϕ *so that the sequence of the directions of the geodesics* $\{g(X, \phi^{si} \circ Y)\}_{i \geq 1}$ *is convergent in the visual sphere of* X.

Since the limit of the directions in the theorem above exists, the geodesic starting at X with the direction of the limit is uniquely determined. A natural question is

Question 0.1. *How to describe the limit geodesic?*

We define the translation length $|\phi|$ of mapping class $\phi \in \mathrm{Mod}(S)$ by $|\phi| = \inf_{X \in \mathrm{Teich}(S)} dist(X, \phi \circ X)$. The classification of $\mathrm{Mod}(S)$ is given in [**18, 65, 70**] by using the Weil-Petersson metric, which says that every mapping class in $\mathrm{Mod}(S)$ (up to some power) is one of the following four cases: identity, multi Dehn-twists, reducible with positive translation length, or pseudo-Anosov. We study question 0.1 case by case. For identity case, the answer is trivial.

Multi Dehn twists: Before stating the result we provide some necessary background. In [**65**] Wolpert gave a compactness theorem for a sequence of geodesics in $\overline{\mathrm{Teich}(S)}$ with uniform bounded lengths. Later, in [**71**] Yamada constructed the so-called *Teichmüller-Coxeter development* $D(\overline{\mathrm{Teich}(S)}, \iota)$ (also see chaper 4) by introducing an infinite Coxeter reflection group and gluing infinite copies of $\overline{\mathrm{Teich}(S)}$ through the strata. The limit geodesic in Wolpert's compactness theorem can be well described in $D(\overline{\mathrm{Teich}(S)}, \iota)$.

Let σ be a simplex whose vertices σ^0 are mutually disjoint simple closed curves. The stratum T_σ consists of all hyperbolic surfaces with nodes along the curves in σ^0 (see [**65**]). A stratum is a convex subset in $\overline{\mathrm{Teich}(S)}$. If ϕ is a multi Dehn-twist, the limit of the geodesics $\{g(X, \phi^i \circ Y)\}_{i \geq 1}$ can be characterized by the following:

Theorem. *Let* σ *be an* m*-simplex,* $\sigma^0 = \{\alpha_1, \cdots, \alpha_{m+1}\}$*, and* τ_i *be the Dehn-twist about the curve* α_i *for* $i = 1, 2, \cdots, m+1$. *Let* $\phi = \prod_{1 \leq i \leq m+1} \tau_i \in \mathrm{Mod}(S)$, $X, Y \in \mathrm{Teich}(S)$*, and* g_n *be the unit speed geodesics* $g(X, \phi^n \circ Y)$*. Then, there exists a positive number* L*; an associated partition* $0 = t_0 < t_1 < \cdots < t_k = L$*; simplices*

$\sigma_0, \cdots, \sigma_k$; and a piecewise geodesic

$$g : [0, L] \to \overline{\mathrm{Teich}(S)}$$

with the following properties.

(1). $\sigma_i^0 \subset \sigma^0$, $\sigma_i^0 \cap \sigma_j^0$ is empty for $i \neq j$,

(2). $\sigma^0 = \bigcup_{i=1}^k \sigma_i^0$,

(3). $g(t_i) \in T_{\sigma_i}$, $i = 1, \cdots, k-1$, $g(0) = X, g(t_k) = Y$,

(4). $g_n([0, t_1])$ converges in $\overline{\mathrm{Teich}(S)}$ to the restriction $g([0, t_1])$, and for each $i = 1, \cdots, k-1$,

$$\lim_{n\to+\infty} dist(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ g_n(t), g(t)) = 0, \quad for \quad t \in [t_i, t_{i+1}],$$

where $\tau_{i,n} = \prod_{\alpha\in\sigma_i} \tau_\alpha^{-n}$, for $i = 1, \cdots, k-1$.

(5). The piecewise geodesic g is the unique minimal length path in $\overline{\mathrm{Teich}(S)}$ joining $g(0)$ to $g(L)$ and intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}}$ in order.

The first point on $g([0, L])$ meeting with the strata $g(t_1)$ is the point where the geodesic joining $(1, X)$ and $(\prod_{\alpha\in\sigma^0} \omega_\alpha, Y)$ in the Teichmüller-Coxeter development $D(\overline{\mathrm{Teich}(S)}, \iota)$ first meets with the strata, where we identify $(1, \overline{\mathrm{Teich}(S)})$ with $\overline{\mathrm{Teich}(S)}$.

If $m = 0$, this result was studied in [**12, 49**].

Reducible with positive translation length: The following theorem says that, provided ϕ is reducible and $|\phi| > 0$, the geometric limit geodesic of the sequence $g(X, \phi^{kn} \circ Y)$, as n goes to infinity, goes to an explicit stratum whose vertices consist of the boundary closed curves in proper surfaces on which ϕ^k is pseudo-Anosov for some positive integer k.

THEOREM. *Let $\phi \in \mathrm{Mod}(S)$ be reducible with $|\phi| > 0$ and k be a positive integer such that $\phi^k = \prod_{\alpha\in\sigma^0} \tau_\alpha \times \prod_j \phi_j$, where σ is a simplex, τ_α is Dehn-twist about α, and $\phi_j = \phi^k|_{PS_j}$ is pseudo-Anosov on PS_j where PS_j is a proper subsurface of S. Then for any $X, Y \in \mathrm{Teich}(S)$, there exists a geodesic ray $c : [0, +\infty) \to \mathrm{Teich}(S)$ such that*

(1). the sequence $\{g(X, \phi^{kn} \circ Y)\}$ converges to $c : [0, +\infty) \to \mathrm{Teich}(S)$.

(2). For any simple closed curve $\alpha \in \partial(\cup_j PS_j)$*, we have*

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = 0.$$

(3). There exists a positive number ϵ_0 *such that for any non-peripheral essential simple closed curve* β *in* S *but not in* $\partial(\cup_j PS_j)$*,*

$$\ell_\beta(c(t)) \geq \epsilon_0$$

for all $t \geq 0$*.*

Pseudo-Anosov: Let $\phi \in \text{Mod}(S)$ be pseudo-Anosov. It is shown in [**18, 65, 70**] that ϕ has a unique axis on which ϕ acts by a translation. The following theorem says that, as n goes to infinity, the geometric limit geodesic of the sequence $g(X, \phi^n \circ Y)$ always lies in the thick part of $\text{Teich}(S)$.

Theorem. *Let* $\phi \in \text{Mod}(S)$ *be pseudo-Anosov and* χ *be the axis for* ϕ *in* $\text{Teich}(S)$*. Then for any* $X, Y \in \text{Teich}(S)$ *there exists a geodesic ray* $c : [0, +\infty) \to \text{Teich}(S)$ *such that*

(1). the sequence $\{g(X, \phi^n \circ Y)\}$ *converges to* $c : [0, +\infty) \to \text{Teich}(S)$*.*

(2). $c([0, +\infty)$ *is strongly asymptotic to* χ*.*

(3). For any simple closed curve β *in* S*,*

$$\lim_{t \to +\infty} \ell_\beta(c(t)) = +\infty.$$

The last part is to study the negatively curved aspect of the curvature operator. The curvature of $\text{Teich}(S)$ has been studied over the past several decades. One celebrated result is that $\text{Teich}(S)$ has negative sectional curvature (see [**57, 63**]). People use Wolpert's curvature formula developed in [**63**] to show $\text{Teich}(S)$ has fruitful curvature properties (see [**32, 41, 52, 56**]).

Let $X \in \text{Teich}(S)$ and $\wedge^2 T_X(\text{Teich}(S))$ be the exterior wedge of the tangent space of $\text{Teich}(S)$ at X, and let Q be the sectional curvature operator of $\text{Teich}(S)$. Our first result is the following.

Theorem. Teich(S) *has non-positive sectional curvature operator. Moreover,* $Q(A, A) = 0$ *if and only if there exists an element* B *in* $\wedge^2 T_X(\text{Teich}(S))$ *such that* $A = B - \boldsymbol{J} \circ B$ *where* $\boldsymbol{J}$ *is the almost complex structure on* Teich(S),

where $\mathbf{J} \circ B$ is defined in chapter 5.

A harmonic map between two spaces is a critical point of the energy functional. When the domain is the Quaternionic hyperbolic space or the Cayley plane, various rigidity theorems were established in [**17, 29, 39, 47**] during the 1990s. As an application, the following rigidity theorem is established.

Theorem. *Let* Γ *be a lattice in a semisimple Lie group* G *which is either* $Sp(m, 1)$ *or* F_4^{-20}, *and* Mod(S) *be the mapping class group of* Teich(S). *Then, any twist harmonic map* f *from* G/Γ *into* Teich(S) *with respect to a homomorphism* $\rho : \Gamma \to \text{Mod}(S)$ *must be a constant.*

In particular, $\rho(\Gamma) \subset \text{Mod}(S)$ will fix the point $f(G/\Gamma) \in \text{Teich}(S)$. From a standard argument in CAT(0) geometry we know that $\rho(\Gamma) \subset \text{Mod}(S)$ must be a finite group. Hence, if we assume that there exists a twist harmonic map f with respect to any homomorphism from Γ to Mod(S), then the image of Γ in Mod(S) would be finite, which is showed by S.K.Yeung in [**72**]. The existence of a twist harmonic map requires the target space to be complete (see [**29**]). $\overline{\text{Teich}(S)}$, the completion of Teich(S), is a singular CAT(0) space which is not locally compact. It is reasonable to state the following conjecture.

Conjecture. *Let* f *be a twist harmonic map from* G/Γ *into* $\overline{\text{Teich}(S)}$ *with respect to a homomorphism* $\rho : \Gamma \to \text{Mod}(S)$ *and the image* $f(G/\Gamma)$ *contains some point in the interior* Teich(S) *of* $\overline{\text{Teich}(S)}$. *Then,* $f(G/\Gamma) \subset \text{Teich}(S)$.

Plan of the paper: Chapter 2 provides necessary backgrounds: surfaces, Teichmüller space, CAT(0) geometry, Weil-Petersson geometry, and classification of mapping classes in Teich(S).

Chapter 3 studies the parabolic isometry on complete proper CAT(0) spaces. We apply the zero property to studying Brock-Farb's question, and the relation between Farb's conjecture and Eberlein's conjecture. Counterexamples to Eberlein's conjecture are provided in this chapter.

In Chapter 4 we study the iteration mapping classes problem in the visual sphere. We also give a description of the limit geodesic through different mapping classes.

Chapter 5 applies Wolpert's curvature formula to proving the sectional curvature operator of the Weil-Petersson metric is non-positive definite, what's more, the zero-level subsets of the curvature operator are tested. As an application, a rigid harmonic map result is provided in this chapter.

CHAPTER 2

Preliminaries

1. Surfaces

Let S be a closed oriented topological surface. The 2-dimensional unit sphere $\mathbb{S}^2$ and the torus $\mathbb{S}^1 \times \mathbb{S}^1$ are the two simplest examples. All oriented surfaces are homeomorphic to the connected sum of g tori $\mathbb{S}^1 \times \mathbb{S}^1$ $(g \geq 0)$. The case $g = 0$ refers to the 2-sphere $\mathbb{S}^2$. The number g is called the *genus* of the surface. The topology of S is completely determined by g. The Euler characteristic $\chi(S)$ of S is determined by the genus via $\chi(S) = 2 - 2g$. Throughout this paper we will always assume that $g > 1$.

A *Riemann surface* X is a connected 1-dimensional complex manifold. It is a 2-dimensional real manifold locally homeomorphic to $\mathbb{C}$ with biholomorphic transition functions. The uniformization theorem[**34**] characterizes the universal covering surface $\tilde{X}$ of a Riemann surface X up to biholomorphic isomorphism as either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disk Δ. Moreover, $\tilde{X}$ is isomorphic to $\hat{\mathbb{C}}$ if and only if $X = \hat{\mathbb{C}}$, $\tilde{X}$ is isomorphic to $\mathbb{C}$ if and only if $\chi(X) = 0$, and $\tilde{X}$ is isomorphic to Δ if and only if $\chi(X) < 0$.

A *marked conformal structure* (f, X) on S is a Riemann surface X together with an orientation preserving homeomorphism $f : S \to X$. Since $\chi(S) = 2 - 2g < 0$, the universal covering surface $\tilde{X}$ is isomorphic to Δ for any marked conformal stucture (f, X) on S.

The *Poincaré metric* on Δ

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2}$$

gives Δ a complete Riemannian metric of constant curvature -1. The automorphisms $Aut(\Delta)$ of Δ preserve the Poincaré metric. The Riemann surface X can also be the quotient space Δ/Γ where Γ is a discrete subgroup of $Aut(\Delta)$, isomorphic to the fundamental group $\pi_1(X)$ of X, called a *Fuchsian group*.

Since the universal covering X is isomorphic to Δ, the Poincaré metric on Δ naturally descends to a complete Riemannian metric on X of constant curvature -1. From the Gauss-Bonnet theorem the total area of X is determined by its Euler characteristic via:

$$area(X) = 2\pi|\chi(X)|.$$

A simple closed curve on S is called *essential* if it is not freely homotopic to a point on S. Each nontrivial element in $\pi_1(S)$ is uniquely represented by an essential simple closed curve up to homotopy. Since S is closed, any marked conformal structure of (f, X) determines a discrete faithful representation $f_* : \pi_1(S) \to PSL_2(\mathbb{R}) = Isom^+(\Delta)$ up to conjugacy, such that f_* sends nontrivial elements of $\pi_1(S)$ to hyperbolic elements of $Isom^+(\Delta)$. Let $\alpha \in \pi_1(S)$. Since $f_*(\alpha)$ is hyperbolic, there exists a unique axis on which $f_*(\alpha)$ acts by isometry. After projecting this axis onto X we get a closed geodesic, the unique one representing α. On the other hand, any simple closed geodesic in X is always essential.

DEFINITION 1.1. *Let α be a simple closed geodesic in X. We define the* collar *of α to be the set*

$$N(\alpha) = \{x; \quad dist_X(x, \alpha) < \omega(\alpha)\}$$

where $\omega(\alpha) = \frac{1}{2}\ln\frac{\cosh(\ell_X(\alpha))/2+1}{\cosh(\ell_X(\alpha))/2-1}$. Actually $\omega(\alpha)$ can guarantee that $N(\alpha)$ is isometric to a hyperbolic annulus with a certain modulus depengding on $\omega(\alpha)$. One can refer to [**7, 34**] *to see more details.*

We recall a version of the Collar Lemma (see Hubbard, [**34**]) which gives a description of the portion of a hyperbolic surface with small injectivity radius.

COLLAR LEMMA. *(1) Let α and β be two disjoint essential closed geodesics. Then $N(\alpha)$ and $N(\beta)$ are disjoint.*

(2) Let α_1 and α_2 be two different essential closed geodesics with non empty intersection. Then $\ell_X(\alpha_i) \geq 2\omega(\alpha_j)$ for $i \neq j \in \{1, 2\}$.

2. Teichmüller space

The Teichmüller space can be constructed in several different ways. In this paper we introduce two equivalent constructions by using marked conformal structures and hyperbolic metrics. One can refer to [**34, 37**] for other constuctions.

Let (f, X) and (g, Y) be two marked conformal structures on S. We call (f, X) is *equivalent* to (g, Y) if and only if there is a conformal isomorphism $\phi : X \to Y$ such that $\phi \circ f$ is homotopic to g from S to Y.

DEFINITION 2.1. ***(Teichmüller space, marking)*** *Let S be a closed surface. The Teichmüller space $\mathbb{T}'_S$ modeled on S is the set of equivalence classes of pairs (f, X), where X is a Riemann surface and $f : S \to X$ is an orientation preserving homeomorphism.*

We can also construct the Teichmüller space through hyperbolic metrics. Since the genus $g > 1$, there exists a complete Riemannian metric on S whose Gauss curvature is -1. Let $\mathbb{M}_{-1}(S)$ be the set of complete Riemannian metrics on S such that the curvature is -1, which is obviously not empty. $Diff_0(S)$ is the set of self-orientation preserving diffeomorphisms of S which are isotopic to the identity. Two elements ds^2 and ds_1^2 in $\mathbb{M}_{-1}(S)$ are defined to be *equivalent* if there exists an element $f \in Diff_0(S)$ such that $f : (S, ds^2) \to (S, ds_1^2)$ is conformal.

DEFINITION 2.2. ***(Teichmüller space, metric)*** *Let S be a closed surface. The Teichmüller space $\mathbb{T}_S$ modeled on S is the set of equivalence classes of (S, ds^2), where ds^2 is complete Riemannian metric on S whose Gauss curvature is -1.*

By solving the so-called Beltrami equation we have the following property.

PROPOSITION 2.3. *(see [**37**]) For fixed surface S, $\mathbb{T}_S$ is equivalent to $\mathbb{T}'_S$, i.e., they are homeomorphic.*

From the definition above we can always assume that each element in the Teichmüller space is a Riemann surface which represents an equivalence class. In this dissertation we view the Teichm̈uller space through the metric way.

Let $X \in \mathbb{T}_S$. $\mathbb{T}_S$ has a natural complex structure, and its holomorphic cotangent space $T^*_X\mathbb{T}_S$ at X is identified with the *quadratic differentials* $Q(X) = \varphi(z)dz^2$ on X. The Fenchel-Nielsen coordinate (see [**37**]) gives a homeomorphism between $\mathbb{T}_S$ and $\mathbb{R}^{6g-6}$, which can also be proved through the Fricke coordinates (see [**37**]), harmornic maps (see [**61**]) and other ways.

Let α be an essential simple closed curve in S. For each $X = (S, ds^2) \in \mathbb{T}_S$ there exists a unique closed geodesic $[\alpha]$ representing α. We denote the length of this closed geodesic by $\ell_\alpha(X)$. This length function depends on both α and X, and it is natural to ask whether it depends smoothly on X. The following result gives an affirmative answer.

Proposition 2.4. *(see [**37**]) Let α be an essential simple closed curve in S. Then $\ell_\alpha(\cdot) : \mathbb{T}_S \to \mathbb{R}$ is analytic.*

Under a certain metric on $\mathbb{T}_S$, ℓ_α has a special property, which will be seen later.

3. CAT(0) geometry

A CAT(0) space is a geodesic metric space in which each geodesic triangle is no fatter than a triangle in the Euclidean plane with the same edge lengths.

Definition 3.1. *let M be a geodesic metric space. For any $a, b, c \in M$, three geodesics $[a, b], [b, c], [c, a]$ form a geodesic triangle Δ. Let $\overline{\Delta}(\overline{a}, \overline{b}, \overline{c}) \subset \mathbb{R}^2$ be a triangle in the Euclidean plane with the same edge lengths as Δ. Let p, q be points on $[a, b]$ and $[a, c]$ respectively, and let $\overline{p}, \overline{q}$ be points on $[\overline{a}, \overline{b}]$ and $[\overline{a}, \overline{c}]$, respectively, such that $dist_M(a, p) = dist_{\mathbb{R}^2}(\overline{a}, \overline{p}), dist_M(a, q) = dist_{\mathbb{R}^2}(\overline{a}, \overline{q})$. We call M a **CAT(0) space** if for all Δ the inequality $dist_M(p, q) \leq dist_{\mathbb{R}^2}(\overline{p}, \overline{q})$ holds.*

Complete simply connected Riemannian manifolds of non-positive curvature are CAT(0) spaces. Singular CAT(0) spaces contain trees endowed with the path metric.

One can refer to [**10**] for more examples. Similar as the angle between different smooth curves in $\mathbb{R}^2$, we can define the angle in a CAT(0) space. Let M be a complete CAT(0) space and $dist$ be the metric on it. Let $c_1 : [0, a] \to M$ and $c_2 : [0, a'] \to M$ be two geodesics of arc-length parameters issuing from the same point $p = c_1(0) = c_2(0)$. Then the *angle* $\angle_p(c_1, c_2)$ between c_1 and c_2 at p is defined as

$$\angle_p(c_1, c_2) = \lim_{t \to 0} 2 \arcsin \frac{dist(c_1(t), c_2(t))}{2t}.$$

The definition above is the same as the definition of the angle in $\mathbb{R}^2$. Throughout this paper we assume that all the geodesics use arc-length parameters.

The geodesic rays play an important role in the study of CAT(0) space. Two geodesic rays $c_1, c_2 : [0, +\infty) \to M$ are said to be *asymptotic* if there exists a constant C such that $dist(c_1(t), c_2(t)) \leq C$ for all $t \geq 0$. We start with the following property.

PROPOSITION 3.2. *(see* [**10**]*, page 261) Let M be a complete CAT(0) space and $c : [0, +\infty) \to M$ be a geodesic ray, then*

(1). for every $y \in M$ there exists a unique geodesic ray c' which issues from y and is asymptotic to $c([0, +\infty))$.

(2). The sequence of geodesics joining y and $c(n)$ converges to c'.

DEFINITION 3.3. *The ideal boundary $M(\infty)$ of M is the set of equivalence classes of geodesic rays where two geodesic rays are equivalent if and only if they are asymptotic.*

If M is a complete simply connected Riemannian manifold of non-positive sectional curvature, $M(\infty)$ is homeomorphic to the unit sphere of dimension $dim(M)-1$. In the singular case, $M(\infty)$ can be very complicated. From proposition 3.2, for each point $p \in M$ and $x \in M(\infty)$ there exists a unique geodesic ray c which represents x and starts from p. We write $c(+\infty) = x$. M is called *visible* if for any two different points x, y in $M(\infty)$ there exists a geodesic line $c : \mathbb{R} \to M$ such that $c(-\infty) = x$ and $c(+\infty) = y$. The upper half plane $\mathbb{H}^2$ is a typical example of a visible CAT(0) space, and the 2-plane $\mathbb{R}^2$ is not visible.

In $\mathbb{R}^2$ we can define the angle between two geodesic rays. We can also give a metric structure on $M(\infty)$ as follows. Given two different points x, y in $M(\infty)$ and $p \in M$, let $\angle_p(x, y)$ denote the angle at p between the unique geodesics rays which issue from p and lie in the class x and y respectively. The *angular metric* is defined by

$$\angle(x, y) := \sup_{p \in M} \angle_p(x, y).$$

The *Tits metric* Td on $M(\infty)$ is the interior metric associated to the angular metric as follows: for any two points $x, y \in M(\infty)$,

$$Td(x, y) = \inf_c \ell(c)$$

where the infimum is taken over all possible curves in $M(\infty)$ joining x and y, and $\ell(c)$ is the length of c under the angular metric.

From the definition we know that for any $x, y \in M(\infty)$,

$$Td(x, y) \geq \angle(x, y).$$

If M is visible, for any $x \neq y \in M(\infty)$, $\angle(x, y) = \pi$ and $Td(x, y) = +\infty$.

The following property gives us a way to compute the angular metric.

Proposition 3.4. *(see* [**10**], *page 281) Let M be a complete CAT(0) space with a basepoint p. Let $x, y \in M(\infty)$ and c, c' be two geodesic rays with $c(0) = c'(0) = p$, $c(+\infty) = x$ and $c'(+\infty) = y$. Then,*

$$2\sin\left(\frac{\angle(x, y)}{2}\right) = \lim_{t \to +\infty} \frac{dist(c(t), c'(t))}{t}.$$

Let γ be a self map of M. γ is called an *isometry* of M if and only if $\gamma : M \to M$ satisfies $dist(\gamma \circ x, \gamma \circ y) = dist(x, y)$ for any $x, y \in M$. Just as in the classification of isometry on $\mathbb{H}^2$, we classify an isometry γ of M as elliptic, hyperbolic, or parabolic. γ is called *elliptic* if it has at least one fixed point in M, and *hyperbolic* if there exists a geodesic line $c : (-\infty, +\infty) \to M$ such that γ acts on $c(\mathbb{R})$ by a non-trivial translation. If γ is neither elliptic nor hyperbolic, then we call it *parabolic*. We define the *translation length* of γ by

$$|\gamma| := \inf_{p\in M} dist(\gamma \circ p, p).$$

From the definition above we know that γ is parabolic if and only if $|\gamma|$ cannot be achieved in M (see [**5, 10**]). Otherwise γ is also called *semi-simple*. Thus semi-simple isometry is either elliptic or hyperbolic.

The following property gives us a new view point for the translation length. For elliptic and hyperbolic cases, they are obvious.

LEMMA 3.5. *Let M be a complete CAT(0) space and γ be a parabolic isometry of M. Then,*
(1). $|\gamma^2| = 2 \cdot |\gamma|$.
(2). $|\gamma| = \lim_{n\to+\infty} \frac{dist(\gamma^n\circ p,p)}{n}$ *where $p \in M$ is arbitrary.*

PROOF OF (1). For any $p \in M$, from the triangle inequality, we have

$$dist(p, \gamma^2 \circ p) \leq dist(p, \gamma \circ p) + dist(\gamma \circ p, \gamma^2 \circ p) = 2dist(p, \gamma \circ p).$$

Since p is arbitrary, we have $|\gamma^2| \leq 2|\gamma|$.

On the other hand, let p_0 be the midpoint of the geodesic connecting p and $\gamma \circ p$. Then using the definition of CAT(0) space,

$$dist(p_0, \gamma \circ p_0) \leq \frac{1}{2} dist(p, \gamma^2 \circ p).$$

By the definition of translation length we have $dist(p, \gamma^2 \circ p) \geq 2|\gamma|$. Since p is arbitrary, $|\gamma|^2 \geq 2|\gamma|$.

Hence, $|\gamma|^2 = 2|\gamma|$.

`Proof of (2)`: Denote $dist(p, \gamma^n \circ p)$ by a_n. From the triangle inequality, for any two positive integers n, m,

$$a_{n+m} \leq a_n + a_m.$$

Choose a positive integer q. For each n there exist k and r such that $n = qk + r$ where $0 \leq r < q$. Then we have

$$\frac{a_n}{n} \leq \frac{ka_q}{n} + \frac{a_r}{n}.$$

Taking the superior limit on the left hand side,

$$\limsup_{n\to+\infty} \frac{a_n}{n} \leq \frac{a_q}{q}.$$

Since q is arbitrary, taking the inferior limit on the right hand side,

$$\limsup_{n\to+\infty} \frac{a_n}{n} \leq \liminf_{q\to+\infty} \frac{a_q}{q}.$$

Hence $\lim_{n\to+\infty} \frac{a_n}{n}$ exists. It is sufficient to find a subsequence of $\{a_n\}_{n\geq 1}$ such that the limit of the subsequence is $|\gamma|$.

Let $f(p)$ be $\lim_{n\to+\infty} \frac{dist(\gamma^n \circ p, p)}{n}$. By the triangle inequality, for any p_1, p_2 in M,

$$|f(p_1) - f(p_2)| = \lim_{n\to+\infty} \frac{|dist(\gamma^n \circ p_1, p_1) - dist(\gamma^n \circ p_2, p_2)|}{n} \leq \lim_{n\to+\infty} \frac{2dist(p_1, p_2)}{n} = 0.$$

So $f(p)$ does not depend on p. By the triangle inequality,

$$dist(\gamma^n \circ p, p) \leq dist(\gamma^n \circ p, \gamma^{n-1} \circ p) + dist(\gamma^{n-1} \circ p, p) \leq \cdots \leq n \times dist(\gamma \circ p, p).$$

Hence,

$$f(p) \leq \lim_{n\to+\infty} \frac{n \times dist(\gamma \circ p, p)}{n} = dist(\gamma \circ p, p).$$

Since p is arbitrary, $f(p) \leq |\gamma|$.

Choose $k_n = 2^n$. Considering the geodesic triangle connecting $p, \gamma^{2^{n-1}} \circ p, \gamma^{2^n} \circ p$, let p_{n-1} be the midpoint of the geodesic connecting p and $\gamma^{2^{n-1}} \circ p$. Then using the definition of CAT(0) space, $dist(p_{n-1}, \gamma^{2^{n-1}} \circ p_{n-1}) \leq \frac{dist(p, \gamma^{2^n} \circ p)}{2}$. We rewrite it as

$$\frac{dist(p_{n-1}, \gamma^{2^{n-1}} \circ p_{n-1})}{2^{n-1}} \leq \frac{dist(p, \gamma^{2^n} \circ p)}{2^n}.$$

By induction on n, there exists $p_0 \in M$ such that

$$dist(p_0, \gamma \circ p_0) \leq \frac{dist(p, \gamma^{2^n} \circ p)}{2^n}.$$

Since $dist(p_0, \gamma \circ p_0) \geq |\gamma|$,

$$|\gamma| \leq \frac{dist(p, \gamma^{2^n} \circ p)}{2^n}.$$

Taking the limit, we get

$$|\gamma| \leq f(p).$$

Hence $|\gamma| = \lim_{n \to +\infty} \frac{dist(\gamma^n \circ p, p)}{n}$. □

We close this section by introducing a rigid theorem for the half plane in CAT(0) space. A metric space M is called *proper* if M is locally compact. We say a geodesic $c : \mathbb{R} \to M$ *bounds a flat half-plane* if there exists a geodesical embedding $F : \mathbb{R}^{\geq 0} \times \mathbb{R} \to M$ such that $F(\{0\} \times \mathbb{R}) = c(\mathbb{R})$. The following proposition gives a criterion for a geodesic line to bound a flat half-plane.

PROPOSITION 3.6. *(see* [**10**]*, page 290) Let M be a complete proper CAT(0) space. If $c : \mathbb{R} \to X$ is a geodesic line, then $Td(c(+\infty, -\infty)) \geq \pi$, with equality if and only if $c(\mathbb{R})$ bounds a flat half-plane.*

4. Weil-Petersson metric

People have been using different metrics on $\mathbb{T}_S$ to study Teichmüller theory for a long time [**37, 34**]. One of the most important metrics is the Weil-Petersson metric (see [**37**]).

DEFINITION 4.1. *Let $X \in \mathbb{T}_S$ and $\varphi dz^2, \psi dz^2$ be two elements in the cotangent space. The Weil-Petersson metric is the Hermitian metric on $\mathbb{T}_S$ arising from the the Petersson scalar product*

$$< \varphi, \psi >= \int_S \frac{\varphi \cdot \overline{\psi}}{\sigma^2} dz d\overline{z}$$

via duality.

We will concern ourselves primarily with its Riemannian part g_{WP}. Throughout this paper we denote the Teichmüller space endowed with the Weil-Petersson metric by Teich(S). The Weil-Petersson metric is a Kähler metric. g_{WP} has negative curvature. The path metric induced from the Weil-Petersson metric is incomplete, but geodesically convex. In chapter 5 we will study the sectional curvature operator of g_{WP}. Under this metric, the length function of a simple closed curve has the following property, which is very important in Chapter 4.

THEOREM 4.2 (Wolpert, see [**64**]). *Given a simple closed curve $\alpha \subset S$ and a Weil-Petersson geodesic $g(t)$ in* Teich(S). *Then the length $\ell_\alpha(g(t))$ is a strictly convex function of t.*

The non-completeness of the Weil-Petersson metric corresponds to finite-length geodesics in Teich(S) along which the length function for some simple closed curve converges to zero. Since Teich(S) is geodesically convex and has negative curvature, the completion $\overline{\mathrm{Teich}(S)}$ of Teich(S), also called the *augmented Teichmüller space*, is a complete CAT(0) space (see [**18, 65, 70**]). In [**44**] the *augmented Teichmüller space* is described concretely by adding strata consisting of stratum $\mathbb{T}_\sigma$ defined by the vanishing of lengths

$$\ell_\alpha = 0$$

for each $\alpha \in \sigma^0$, where σ^0 is a collection of finite mutually disjoint simple closed curves.

DEFINITION 4.3. *A k-simplex σ is a simplex whose vertices σ^0 is a set of $k+1$ distinct free homotopy classes of non-trivial mutually disjoint simple closed curves of S. We say two simplices σ and η are disjoint if σ^0 and η^0 are mutually disjoint simple closed curves.*

The topology for the stratum $\mathbb{T}_\sigma$ can be described by the so-called extended Fenchel Nielsen coordinates: Give a pants decomposition P with $\sigma^0 \subset P$, the usual coordinates map Teich(S) to $\prod_{\alpha\in P} \mathbb{R} \times \mathbb{R}^+$, where the first coordinate of each pair measures twist and the second measures the length of the corresponding simple closed curve in P. We extend the second part to 0 and take the quotient by identifying $(t, 0)$

and $(t', 0)$ in each $\mathbb{R} \times \mathbb{R}^{\geq 0}$ factor. The topology near every point in a stratum $\mathbb{T}_\sigma$ is given by these extended coordinates.

The stratum $\mathbb{T}_\sigma$ is naturally products of lower dimensional Teichmüller spaces corresponding to the nodal surfaces in $\mathbb{T}_\sigma$ (see [**44**]). Choosing suitable σ, $\mathbb{T}_\sigma$ contains geodesical embedding Euclidean spaces. Since $\overline{\text{Teich}(S)}$ is not locally compact, the aspects of geodesics can be very complicated. Let $X, Y \in \overline{\text{Teich}(S)}$. We denote the geodesic connecting X and Y by $g(X, Y)$. The following theorem is called the non-refraction property for the Weil-Petersson geodesic.

THEOREM 4.4 ([**18, 65, 70**]). *Let $X, Y \in \overline{\text{Teich}(S)}$ and σ_1 and σ_2 be the maximal collection of simple closed curves so that $X \in \mathbb{T}_{\sigma_1}$ and $Y \in \mathbb{T}_{\sigma_2}$. If $\eta = \sigma_1 \cap \sigma_2$, then*

$$int(g) \subset \mathbb{T}_\eta,$$

where $int(g)$ is the interior of $g(X, Y)$.

We remark that in the special case that both X and Y lie in $\text{Teich}(S)$, the theorem above is simply a restatement of Wolpert's geodesical convexity theorem (see [**64**]). If we allow the length function of simple closed curve to be $+\infty$, theorem 4.2 can be extended to the boundary case.

THEOREM 4.5. *Given a Weil-Petersson geodesic $g(t)$ in $\overline{\text{Teich}(S)}$ and σ a simplex such that $int(g) \subset \mathbb{T}_\sigma$. Then for any simple closed curve α in S, the length $\ell_\alpha(g(t))$ is a convex function of t.*

PROOF. If α intersects with at least one simple closed curve in σ^0, from the Collar Lemma and theorem 4.4 we know that for all t, $\ell_\alpha(g(t))$ is always infinite, which satisfies the convexity property.

If α is disjoint with σ^0, α is contained in one component of $S - \cap_{\beta \in \sigma^0} \beta$. Since T_σ is the product of lower dimensional Teichmüller spaces, the projection of $g(t)$ to each component is either a geodesic or a point. Hence, it follows from theorem 4.2 that $\ell_\alpha(g(t))$ is either strictly convex or constant. Hence, $\ell_\alpha(g(t))$ is convex.

If α is one curve in σ^0, it follows from the fact that $\ell_\alpha(g(t)) \equiv 0$. □

5. Mapping class group

Recall that $Diff_0(S)$ is the set of orientation preserving self diffeomorphisms of S which are isotopic to the identity. We define $Diff_+(S)$ to be the set of orientation preserving diffeomorphisms of S.

DEFINITION 5.1. *The mapping class group* Mod(S) *of* S *is defined as*

$$\text{Mod}(S) := Diff_+(S)/Diff_0(S).$$

Mod(S) is a discrete group which acts properly discontinuously on $\mathbb{T}_S$ and acts on Teich(S) by isometry. The whole isometry group of $\overline{\text{Teich}(S)}$ is exactly Mod(S) (see [**46**]). Mod(S) contains a torsion-free subgroup of finite index. The classification of Mod(S) can be given through different methods. In this section we provide the classification through the Weil-Petersson metric.

A mapping class is *irreducible* provided that no power fixes the free homotopy class of a simple closed curve. An irreducible mapping class is pseudo-Anosov. A mapping class is precisely one of: periodic, irreducible or reducible (see [**24**]). For a reducible mapping class h, an invariant is σ_h, the maximal simplex fixed by some power of h.

An *essential subsurface* is a submanifold $R \subset S$ whose boundary is homotopically essential. We have the following theorem due to Thurston (see [**24**], Exp. 9).

THEOREM 5.2. *(Thurston) A mapping class* $\phi \in \text{Mod}(S)$ *determines a simplex* σ_ϕ *such that*

(1). each component R *of* $S_\phi := S - \bigcup_{\alpha \in \sigma_\phi^0} \alpha$ *is an essential subsurface.*

(2). $\phi(\sigma_\phi^0) = \sigma_\phi^0$.

(3). There exists an integer k *such that for each component* R *of* $S_\phi := S - \bigcup_{\alpha \in \sigma_\phi^0} \alpha$, $\phi^k|_R$ *is either identity or pseudo-Anosov.*

Let $\phi \in \text{Mod}(S)$. We define the Weil-Petersson translation length by

$$L_{WP}(\phi) = \inf_{X \in \text{Teich}(S)} d_{WP}(X, \phi \circ X)$$

where d_{WP} is the path metric on $\mathrm{Teich}(S)$ induced by the Weil-Petersson metric. From [**18, 65, 70**], the classification of $\mathrm{Mod}(S)$ is given by the following theorem.

THEOREM 5.3. *(see [**65**]) Let γ be an element in $\mathrm{Mod}(S)$. Then γ is semi-simple if and only if $L_{WP}(\gamma)$ attains its minima at a point in $\mathrm{Teich}(S)$. In this case either γ fixes at least one point in $\mathrm{Teich}(S)$ or there exists a unique Weil-Petersson geodesic line $r(t) \subset \mathrm{Teich}(S)$ such that for all t, $\gamma \circ r(t) = r(t + L_{WP}(\gamma))$. For the latter case, $L_{WP}(\gamma) > 0$.*

γ is reducible if and only if $L_{WP}(\gamma)$ cannot attain its minima in $\mathrm{Teich}(S)$. In this case either γ fixes a point in $\overline{\mathrm{Teich}(S)}$ or there exists an integer k depending on γ such that γ^k acts on the null stratum $\mathbb{T}_{\sigma_\gamma}$ which is a product of low-dimensional Teichmüller spaces $\Pi\, T' \times \Pi\, T''$ by a product of : irreducible elements γ' on T' with axis $r_{h'}$ and the identity on each T''. For the latter case, in particular there exists a bi-infinite Weil-Petersson geodesic $r(t) \subset \overline{\mathrm{Teich}(S)}$ such that for all t, $\gamma^k \circ r(t) = r(t + kL_{WP}(\gamma))$.

For the irreducible case one can also refer to [**18, 70**].

One consequence of theorem 5.3 is the following proposition; one can also refer to [**9, 30**].

PROPOSITION 5.4. *Let ϕ be a reducible element in $\mathrm{Mod}(S)$ with $L_{WP}(\phi) = 0$. Then there exists an integer k and a simplex σ such that $\phi^k = \Pi_{\alpha_i \in \sigma^0} \tau_{\alpha_i}$ where τ_{α_i} is the Dehn-twist about α_i.*

PROOF. Since ϕ is reducible, there exists an integer k, a simplex σ, and a collection of mutually disjoint proper essential subsurfaces $\{PS_j\}$ such that

$$\phi^k = \Pi_{\alpha_i \in \sigma^0} \tau_{\alpha_i} \cdot \Pi_j \phi_j$$

where ϕ_j is pseudo-Anosov on PS_j (see [**24**]).

Since $L_{WP}(\phi) = 0$, we do not have the pseudo-Anosov part: otherwise, from theorem 5.3 we have $L_{WP}(\phi) > 0$ which is a contradiction. Hence,

$$\phi^k = \Pi_{\alpha_i \in \sigma^0} \tau_{\alpha_i}.$$

□

We close this section with a short discussion on moduli spaces.

Definition 5.5. *The moduli space* $\mathbb{M}_S$ *of* S *is defined as*

$$\mathbb{M}_S := \mathbb{T}_S/\mathrm{Mod}(S).$$

The moduli space is an orbifold. Since $\mathrm{Mod}(S)$ contains a torsion-free subgroup of finite index, there exists a finite covering of $\mathbb{M}_S$ in the orbifold sense such that it is a manifold (see [**25**]).

For a given $\epsilon > 0$, we define the *ϵ-thick part* $\mathbb{T}_{S_{\geq\epsilon}}$ of $\mathbb{T}_S$ as the following

$$\mathbb{T}_{S_{\geq\epsilon}} = \{X \in \mathbb{T}_S;\ \ \ell_\alpha(X) \geq \epsilon,\ \ \text{for all simple closed curves}\ \ \alpha\}.$$

It is easy to see that $\mathrm{Mod}(S)$ acts invariantly on $\mathbb{T}_{S_{\geq\epsilon}}$. The quotient space $\mathbb{T}_{S_{\geq\epsilon}}/\mathrm{Mod}(S)$ is called the *ϵ-thick part* of the moduli space, denoted by $\mathbb{M}_{S_{\geq\epsilon}}$.

The following compactness property is due to Mumford (see [**25**]).

Theorem 5.6. *(Mumford) For any* $\epsilon > 0$, $\mathbb{M}_{S_{\geq\epsilon}}$ *is compact.*

$\mathbb{M}_S$ itself has fruitful properties. For example $\mathbb{M}_S$ has only one end. Any simple closed curved in $\mathbb{M}_S$ can be homotopic to the outside of any compact subset in the orbifold sense. Topologically, $\mathbb{M}_S$ is simply connected. One can refer to [**25**] for details.

CHAPTER 3

Translation lengths of parabolic isometries of CAT(0) spaces and their applications

1. Introduction

CAT(0) spaces are generalizations of Riemannian manifolds with nonpositive sectional curvature to geodesic spaces. An isometry of the hyperbolic half plane $\mathbb{H}^2$ is elliptic, parabolic or hyperbolic. This can also be applied to CAT(0) spaces. Like the isometry of $\mathbb{H}^2$, an elliptic isometry of a CAT(0) space has fixed points, a parabolic isometry cannot attain its translation length, and a hyperbolic isometry of a CAT(0) space has an axis on which the isometry acts as a translation on $\mathbb{R}$ (see [**5, 10**]). If a group acts properly and cocompactly on a proper CAT(0) space by isometries, then the group consists of hyperbolic isometries. For example, the fundamental group of a hyperbolic surface S_g with genus $g \geq 2$ consists of hyperbolic isometries if we view the fundamental group as an isometry group of the hyperbolic half plane. However, if one considers an isometry group action which is not cocompact, one may have to deal with parabolic isometries. For example, the fundamental group of a hyperbolic surface $S_{g,n}$ with genus $g \geq 1$ and punctures $n \geq 1$ contains parabolic isometries if we view this group as an isometry group of the half plane.

In this chapter, we focus on parabolic isometries on CAT(0) spaces. We study the translation lengths of parabolic isometries and generalize what Buyalo did for parabolic isometries on Gromov hyperbolic spaces in [**8**]. Then we study how the mapping class groups of surfaces S_g ($g \geq 3$) act properly discontinuously on complete proper visible CAT(0) spaces. We study the zero axiom, which generalizes the curvature condition on manifolds that the sectional curvature is bounded above by a negative number, and give a description for manifolds with bounded geometry and zero axiom which generalizes what Eberlein and Schroeder did in [**21, 53**] for manifolds whose sectional curvatures are pinched by two negative numbers. At the end we also give counterexamples to Eberlein's conjecture which says that there does not exist a gap between visible manifolds and manifolds with certain geometric restrictions.

Let M be a complete CAT(0) space. An isometry γ of M is either elliptic, hyperbolic, or parabolic. Let $M(\infty)$ be the ideal boundary of M, defined as the asymptotic classes of rays in M. Recall M is visible if for any two different points x, y in $M(\infty)$ there exists at least one geodesic line $c : \mathbb{R} \to M$ such that $c(-\infty) = x$ and $c(+\infty) = y$. A metric space is called *proper* if it is locally compact. Now we can state the following theorem which is one of the main results in this chapter.

THEOREM 1.1. *Let M be a complete proper visible CAT(0) space. Then any parabolic isometry has zero translation length, i.e., for any parabolic isometry γ of M we have $|\gamma| = 0$.*

Bishop and O'Neill in [**6**] proved that any parabolic isometry of a complete simply connected manifold M with sectional curvature $K_M \leq -1$ has zero translation length. Later in [**31**] Heintze and Hof used the geometry on horospheres to prove it again. Since a manifold satisfying the sectional curvature condition $K_M \leq -1$ is visible, theorem 1.1 gives a new proof here. In 1999, Buyalo [**8**] proved that any parabolic isometry of a complete Gromov-hyperbolic CAT(0) space M has zero translation length. Theorem 1.1 generalizes this result in some sense, since a Gromov-hyperbolic CAT(0) space is visible (see [**10**]). In [**8**] M does not need to be proper.

A manifold M is called *visible* if M has non-positive sectional curvature and the universal covering space $\tilde{M}$ of M is visible. As a consequence of theorem 1.1, the classification of isometries of complete simply connected visible manifolds is given as follows.

THEOREM 1.2. *Let M be a complete simply connected visible manifold, and γ be an isometry of M. Then,*

(1). γ is elliptic if and only if γ has at least one fixed point in M.

(2). γ is hyperbolic if and only if $|\gamma| > 0$.

(3). γ is parabolic if and only if $|\gamma| = 0$ and γ does not have fixed point in M.

Mapping class groups. Let S_g be a closed surface with genus g and $\mathrm{Mod}(S_g)$ be the mapping class group of S_g, i.e. the group of isotopy classes of orientation-preserving self-homeomorphisms of S_g. Bridson in [**9**] proved that any Dehn-twist in $\mathrm{Mod}(S_g)$ has zero translation length if $\mathrm{Mod}(S_g)$ acts by isometries on a complete CAT(0) space and $g \geq 3$. Our second result in this chapter is

THEOREM 1.3. *Let M be a complete proper visible CAT(0) space and S_g be a closed surface with genus $g \geq 3$. If $\mathrm{Mod}(S_g)$ acts properly discontinuously on M by isometries, then, for any element $\sigma \in \mathrm{Mod}(S_g)$, we have $|\sigma| = 0$.*

Since every complete visible manifold M with finite volume has a nontrivial closed geodesic (see theorem 2.13 in [**4**]), the isometry which represents a nontrivial closed geodesic is a hyperbolic isometry of the universal covering space of M. In particular, it has positive translation length. Since there exists a finite covering of the moduli space $\mathbb{M}_{S_g}$ which is manifold. Applying theorem 1.3, we get the following theorem, which partially answers Brock-Farb's question which asks *whether the moduli space $\mathbb{M}_{S_g}$ of a closed surface S_g ($g \geq 2$) (up to finite covering) admits a complete, finite volume Riemannian metric with nonpositive sectional curvature.*

THEOREM 1.4. *Let S_g be a closed surface of genus $g \geq 3$. Then the moduli space $\mathbb{M}_{S_g}$ of S_g (up to finite covering) admits no complete, finite volume Riemannian metric whose sectional curvature is nonpositive and universal covering is visible.*

Ivanov in [**35**] proved that the mapping class group $\mathrm{Mod}(S_g)$ ($g \geq 3$) (up to finite index) cannot be isomorphic to the fundamental group of any complete manifold with pinched negative sectional curvature and finite volume. Later Brock and Farb in [**13**] proved that the mapping class group $\mathrm{Mod}(S_g)$ ($g \geq 3$) (up to finite index) cannot be isomorphic to the fundamental group of any complete Gromov-hyperbolic manifold of finite volume.

Two-dimensional surfaces. A hyperbolic surface is a two-dimensional Riemannian manifold with constant negative sectional curvature. A well-known result is that a hyperbolic surface M with finite volume is closed if and only if the fundamental group of M consists of hyperbolic isometries. For a general surface, it is interesting to know when the fundamental group of the surface determines the compactness of the surface.

THEOREM 1.5. *Let M be a complete two-dimensional surface with the Gauss curvature $-1 \leq K(M) \leq 0$ and $Vol(M) < +\infty$, and let $\pi_1(M,p)$ be the fundamental group of M with a basepoint p. Then M is closed if and only if for any non-trivial deck transformation $\phi \in \pi_1(M,p)$ the translation length $|\phi| > 0$.*

Two examples are given later to show that the lower bound for curvature and the finite volume are necessary for theorem 1.5.

Manifolds without visibility. In the first paragraph of page 438 of [**21**] Eberlein conjectures that *a complete open manifold M with sectional curvature $-1 \leq K_M \leq 0$ and finite volume is visible if the universal covering space $\tilde{M}$ of M contains no imbedded flat half planes*. In [**23**] Farb conjectures that *the moduli space $\mathbb{M}_{S_g}$ of closed surface S_g $(g \geq 2)$ (up to finite covering) admits no complete, finite volume Riemannian metric with sectional curvature $-1 \leq K(\mathbb{M}_{S_g}) \leq 0$, which is the weaker version of Brock-Farb's question (see* [**13**]*)*. From theorem 1.4 we know that if Eberlein's conjecture is correct, then it could partially solve Farb's conjecture. If we assume that there exists a metric on $\mathbb{M}_{S_g}$ $(g \geq 3)$ (up to a finite covering) such that it has finite volume and sectional curvature $-1 \leq K(\mathbb{M}_{S_g}) \leq 0$ and the universal covering Teich(S_g) contains no imbedded flat half planes (Weak Farb conjecture), then the "candidate metric" on $\mathbb{M}_{S_g}$ $(g \geq 3)$ would be a counterexample for Eberlein's conjecture.

The following theorem gives more evidence to Farb's conjecture.

THEOREM 1.6. *If $g \geq 3$, then $\mathrm{Mod}(S_g)$ cannot act properly discontinuously on any complete simply connected Riemannian manifold M satisfying the zero axiom and the sectional curvature $-1 \leq K_M \leq 0$,*

where the zero axiom is given in section 5.

In [**1**], the authors constructed so-called graph manifolds M with sectional curvature $-1 \leq K_M < 0$ and finite volume. Since the sectional curvature $K_M < 0$, the universal covering $\tilde{M}$ does not contain imbedded flat half planes. In the second paragraph of page 34 of [**1**] it says that the "Adjacent components" of the boundary $M^n(+\infty)$ have Tits distance equal to $\frac{\pi}{2}$. Hence $\tilde{M}$ is not visible since the Tits distance between any two points in the boundary of a visible CAT(0) space is infinity (see [**10**]). In particular Abresch and Schroeder's examples are counterexamples for Eberlein's conjecture. The authors did not provide the proof for the statement above on the Tits distance. In this paper the follow theorem gives a detailed answer to Eberlein's conjecture by using theorem 1.1.

THEOREM 1.7. *The fundamental groups of manifolds M constructed in* [**1**] *and* [**26**] *with finite volume and sectional curvature $-1 \leq K_M < 0$ contain parabolic isometries of $\tilde{M}$ with positive translation length. In particular, M is not visible.*

2. Parabolic isometries, half planes

Let M be a CAT(0) space and γ be an isometry of M. Every isometry of M can be extended to a self-homeomorphism of $\overline{M} := M \bigcup M(\infty)$ with the cone topology. Let $Fix(\gamma)$ denote the fixed points of γ in $\overline{M}$, i.e., $Fix(\gamma) := \{x \in \overline{M}; \gamma \circ x = x\}$. We define $Diam(A) := \sup_{x\in A} \sup_{y\in A} d(x, y)$ and $Rad(A) := \inf_{x\in A} \sup_{y\in A} d(x, y)$ for a metric space A with metric d, which are called the *diameter* and *radius* of A respectively. For $p \in M$, we denote by $\sum_p M$ the space of directions at p. In [**27**] Fujiwara, Nagano and Shioya proved that

THEOREM 2.1. *Let M be a proper CAT(0) space such that $\sum_p M$ is compact for every $p \in M$, and let γ be a parabolic isometry of M. Then there exists $\eta \in Fix(\gamma)$*

such that $Td(\eta, \xi) \leq \frac{\pi}{2}$ *for any* $\xi \in Fix(\gamma)$*. In particular* $Diam(Fix(\gamma)) \leq \pi$ *and* $Rad(Fix(\gamma)) \leq \frac{\pi}{2}$ *in the sense of the Tits metric on* $M(\infty)$*.*

Since a complete proper metric space has compact direction at any point, the following holds.

THEOREM 2.2. *Let* M *be a complete proper CAT(0) space, and let* γ *be a parabolic isometry of* M*. Then there exists* $x_0 \in Fix(\gamma)$ *such that* $Td(x_0, x) \leq \frac{\pi}{2}$ *for any* $x \in Fix(\gamma)$*. In particular* $Diam(Fix(\gamma)) \leq \pi$ *and* $Rad(Fix(\gamma)) \leq \frac{\pi}{2}$ *in the sense of the Tits metric on* $M(\infty)$*.*

REMARK 2.1. In the case that M is a complete manifold with nonpositive curvature, this was proved by Ballman, Gromov and Schroeder in appendix 3 of [**5**].

Now look at the following two examples.

EXAMPLE 1. Let $\mathbb{H}^2$ be the upper half plane and define $\gamma : \mathbb{H}^2 \to \mathbb{H}^2$ to be $\gamma \circ ((x, y)) = (x + 1, y)$. It is easy to see that γ is parabolic and $Fix(\gamma)$ consists of one point. Hence $Diam(Fix(\gamma)) = 0$.

EXAMPLE 2. Consider $\mathbb{R} \times \mathbb{H}^2$ endowed with the product metric where $\mathbb{H}^2$ is the upper half plane as above and define $\gamma : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{R} \times \mathbb{H}^2$ to be $\gamma \circ ((z, (x, y))) = (z, (x + 1, y))$. It is easy to see that γ is parabolic and $Diam(Fix(\gamma)) = \pi$.

In example 2 it is not hard to see that there exist two different points x, y in $Fix(\gamma) \subset (\mathbb{R} \times \mathbb{H}^2)(\infty)$ such that there exists a geodesic line $c : \mathbb{R} \to \mathbb{R} \times \mathbb{H}^2$ with $c(+\infty) = x$ and $c(-\infty) = y$ and this geodesic $c(\mathbb{R})$ bounds a flat half-plane. The following property tells us this is intrinsic in CAT(0) spaces.

PROPOSITION 2.3. *Let* M *be a complete proper CAT(0) space and* γ *be a parabolic isometry of* M*. If there exists a geodesic* $c : \mathbb{R} \to M$ *such that* $\{c(+\infty), c(-\infty)\} \subset Fix(\gamma)$*, then the geodesic* $c(\mathbb{R})$ *bounds a flat half-plane.*

PROOF. It is sufficient to show that $Td(c(+\infty), c(-\infty)) = \pi$ from proposition 3.6 in chapter 2.

Firstly from proposition 3.6 in chapter 2, we have $Td(c(+\infty), c(-\infty)) \geq \pi$.

On the other hand, since γ is parabolic and $\{c(+\infty), c(-\infty)\} \subset Fix(\gamma)$, by theorem 2.2,

$$Td(c(+\infty), c(-\infty)) \leq \pi.$$

Hence $Td(c(+\infty), c(-\infty)) = \pi$. □

REMARK 2.2. Under the assumptions of proposition 2.3, if M is visible, then the fixed point of any parabolic isometry of M is a single point (see [**27**]).

REMARK 2.3. In [**8**] Buyalo has shown that if M is a complete, not necessarily proper, Gromov-hyperbolic CAT(0) space (this is stronger than a visible CAT(0) space), then the fixed points of any parabolic isometry of M are but a single point. Actually if one carefully checks Buyalo's argument, one can show that if M is a complete visible CAT(0) space, then the fixed points of any parabolic isometry of M are either a single point or empty.

The dynamics of parabolic isometries of CAT(0) spaces are not easy to study (see the examples in [**27**]). The following theorem gives a nice description of the dynamics of a parabolic isometry with positive translation length, which is a special case of a result of Karlsson and Margulis in [**40**].

THEOREM 2.4. *[KM99] Let M be a complete CAT(0) space with a base point $p \in M$ and γ be a parabolic isometry with $|\gamma| > 0$. Then there exists a unique $x_0 \in Fix(\gamma)$ and a geodesic ray $c : \mathbb{R}^{\geq 0} \to M$ such that $c(0) = p$, $c(+\infty) = x_0$ and*

$$\lim_{n \to +\infty} \frac{dist(\gamma^n \circ p, c(|\gamma| \cdot n))}{n} = 0.$$

The following is a direct corollary.

COROLLARY 2.5. *Let M be a complete CAT(0) space with a basepoint $p \in M$ and γ be a parabolic isometry with $|\gamma| > 0$. Then $\{\gamma^n \circ p\}$ converges to a point $x \in Fix(\gamma) \subset M(\infty)$.*

Now we are ready to estimate the fixed points of a parabolic isometry of CAT(0) space.

Proposition 2.6. *Let M be a complete CAT(0) space with a basepoint $p \in M$ and γ be a parabolic isometry with $|\gamma| > 0$. Then there exist two different points $\{x, y\} \subset Fix(\gamma)$ such that $Td(x, y) \geq \pi$.*

Proof. Since $|\gamma| > 0$, by theorem 2.4 there exists a geodesic ray $c : \mathbb{R}^{\geq 0} \to M$ such that $c(0) = p$ and

$$\lim_{n\to+\infty} \frac{dist(\gamma^n \circ p, c(|\gamma| \cdot n))}{n} = 0. \tag{1}$$

Similarly, since $|\gamma^{-1}| = |\gamma| > 0$, by theorem 2.4 there exists a geodesic ray $c' : \mathbb{R}^{\geq 0} \to M$ such that $c'(0) = p$ and

$$\lim_{n\to+\infty} \frac{dist(\gamma^{-n} \circ p, c'(|\gamma| \cdot n))}{n} = 0. \tag{2}$$

By the triangle inequality,

$$\begin{aligned} \frac{dist(c(|\gamma| \cdot n), c'(|\gamma| \cdot n))}{n} &\leq \frac{dist(\gamma^n \circ p, c(|\gamma| \cdot n))}{n} \\ &+ \frac{dist(\gamma^{-n} \circ p, \gamma^n \circ p)}{n} + \frac{dist(\gamma^{-n} \circ p, c'(|\gamma| \cdot n))}{n}. \end{aligned}$$

From (1),(2) and $dist(\gamma^{-n} \circ p, \gamma^n \circ p) = dist(\gamma^{2n} \circ p, p)$, after taking the limit,

$$\lim_{n\to+\infty} \frac{dist(c(|\gamma| \cdot n), c'(|\gamma| \cdot n))}{n} \leq \lim_{n\to+\infty} \frac{dist(\gamma^{2n} \circ p, p)}{n}. \tag{3}$$

On the other hand, by the triangle inequality,

$$\begin{aligned} \frac{dist(c(|\gamma| \cdot n), c'(|\gamma| \cdot n))}{n} &\geq \frac{dist(\gamma^{-n} \circ p, \gamma^n \circ p)}{n} \\ &- \frac{dist(\gamma^n \circ p, c(|\gamma| \cdot n))}{n} - \frac{dist(\gamma^{-n} \circ p, c'(|\gamma| \cdot n))}{n}. \end{aligned}$$

From (1) and (2), after taking a limit,

$$\lim_{n\to+\infty} \frac{dist(c(|\gamma| \cdot n), c'(|\gamma| \cdot n))}{n} \geq \lim_{n\to+\infty} \frac{dist(\gamma^{2n} \circ p, p)}{n}. \tag{4}$$

Hence,

$$\lim_{n\to+\infty} \frac{dist(c(|\gamma| \cdot n), c'(|\gamma| \cdot n))}{n} = \lim_{n\to+\infty} \frac{dist(\gamma^{2n} \circ p, p)}{n}. \tag{5}$$

From lemma 3.5 in chapter 2,

$$\lim_{n\to+\infty}\frac{dist(c(|\gamma|\cdot n),c'(|\gamma|\cdot n))}{n}=|\gamma^2|=2|\gamma|. \tag{6}$$

By proposition 3.4 in chapter 2,

$$2\sin\Big(\frac{\angle(c(+\infty),c'(+\infty))}{2}\Big)=\lim_{n\to\infty}\frac{dist(c(|\gamma|\cdot n),c'(|\gamma|\cdot n))}{|\gamma|\cdot n}=2.$$

Hence,

$$\angle(c(+\infty),c'(+\infty))=\pi.$$

Since $Td(c(+\infty),c'(+\infty))\geq\angle(c(+\infty),c'(+\infty))$, $Td(c(+\infty),c'(+\infty))\geq\pi$. □

PROPOSITION 2.7. *Let M be a complete proper CAT(0) space and γ be a parabolic isometry with $|\gamma|>0$. Then there exist two different points $\{x,y\}\subset Fix(\gamma)$ such that $Td(x,y)=\pi$. In particular the diameter of $Fix(\gamma)$ $Diam(Fix(\gamma))=\pi$.*

PROOF. Since $|\gamma|>0$, by proposition 2.6 there exists $\{x,y\}\subset Fix(\gamma)$ such that $Td(x,y)\geq\pi$. On the other hand, since $\{x,y\}\subset Fix(\gamma)$, by theorem 2.2, we have $Td(x,y)\leq\pi$. Hence $Td(x,y)=\pi$. □

Now we are ready to prove theorem 1.1 and theorem 1.2.

PROOF OF THEOREM 1.1. Suppose not. We assume that $|\gamma|>0$. From proposition 2.7 there exist two different points $\{x,y\}\subset Fix(\gamma)$ such that $Td(x,y)=\pi$. Since M is a visible CAT(0) space, there exists a geodesic line $c:\mathbb{R}\to M$ such that $c(+\infty)=x, c(-\infty)=y$. By proposition 2.3 we get that $c(\mathbb{R})$ bounds a flat half-plane, which contradicts the fact that M is visible. □

REMARK 2.4. We say a manifold M is *tame* if M is homemorphic to the interior of a compact manifold $\overline{M}$ with boundary. Recently Phan conjectured in [**48**] that *if M is a tame, finite volume, negatively curved manifold, then M is not visible if the fundamental group $\pi_1(M)$ of M contains a parabolic isometry of $\tilde{M}$ with positive translation length.* Theorem 1.1 confirms this conjecture.

Since a Gromov-hyperbolic CAT(0)-space is visible, by theorem 1.1 we immediately obtain

THEOREM 2.8. *(see [**8**]) Let M be a complete proper Gromov-hyperbolic CAT(0) space. Then any parabolic isometry of M has zero translation length.*

REMARK 2.5. In the case that M is a complete simply connected manifold with curvature $K_M \leq -1$, it was showed in [**6, 31**] that the translation length function of any parabolic isometry γ along any geodesic ray whose end belongs to the fixed points of γ goes to zero.

PROOF OF THEOREM 1.2. `Proof of (1)`: By definition.

`Proof of (2)`: If γ is hyperbolic, by definition we know that $|\gamma| > 0$.

If $|\gamma| > 0$, assuming that γ is not hyperbolic, then γ should be parabolic. By theorem 1.1 we have $|\gamma| = 0$ which contradicts our assumption.

`Proof of (3)`: If γ is parabolic, it is obvious that γ does not have fixed points. $|\gamma| = 0$ follows from theorem 1.1.

If γ does not have fixed points and $|\gamma| = 0$, the conclusion that γ is parabolic follows from part (2). □

3. Mapping class group action

Let S_g be a hyperbolic surface with genus g. $\mathrm{Mod}(S_g)$ acts on the augmented Teichmüller space $\overline{\mathrm{Teich}(S)}$ by isometries. The Dehn-twists here behave as elliptic isometries whose fixed points are products of lower-dimensional Teichmüller spaces. The following theorem of Bridson (see [**9**]) says that when $\mathrm{Mod}(S_g)$ acts on a complete CAT(0) space, then the zero translation length of each Dehn twist is intrinsic, except for several cases.

THEOREM 3.1. *(see [**9**]) Whenever $\mathrm{Mod}(S_g)$ $(g \geq 3)$ acts by isometries on a complete CAT(0) space M, then each Dehn twist $\tau \in \mathrm{Mod}(S_g)$ has $|\tau| = 0$.*

A group G acting on a metric space X is said to act *properly discontinuously* if for each compact subset $K \subset X$, the set $K \cap gK$ is nonempty for only finitely many g in G. The following corollary is a direct result of theorem 3.1.

COROLLARY 3.2. *Whenever* $\mathrm{Mod}(S_g)$ $(g \geq 3)$ *acts properly discontinuously on a complete CAT(0) space* M *by isometries, each Dehn twist* $\tau \in \mathrm{Mod}(S_g)$ *acts as a parabolic isometry with* $|\tau| = 0$.

PROOF. If not, by theorem 3.1 τ is elliptic, so τ has a fixed point $x_0 \in M$ which contradicts the assumption that the action is properly discontinuous, since any Dehn twist has infinite order. □

Dehn twists on non-separate simple closed curves. In this subsection, the ideas of the statements come from Brock and Farb's paper [**13**].

LEMMA 3.3. *Let* $\mathrm{Mod}(S_g)$ *act on a complete CAT(0) space* M *by isometries. Suppose that there exists a non-separate simple closed curve* α *such that the Dehn twist on* α *has a unique fixed point in* $M(\infty)$. *Then the Dehn twist on any other non-separate simple closed curve* β *also has only one fixed point in* $M(\infty)$.

PROOF. Since both α and β are non-separate simple closed curves, there exists $\phi \in \mathrm{Mod}(S_g)$ such that $\phi(\alpha) = \beta$ (see [**25**]). Let $\tau_\bullet$ denote the Dehn twist on $\bullet$. Since

$$\phi \cdot \tau_\alpha \cdot \phi^{-1} = \tau_{\phi(\alpha)},$$

we have $\phi \cdot \tau_\alpha \cdot \phi^{-1} = \tau_\beta$. Let $x \in M(\infty)$ be the fixed point of τ_α. For any $y \in Fix(\tau_\beta)$, we have $\tau_\alpha(\phi^{-1}(y)) = \phi^{-1}(y)$. Hence $\phi^{-1}(y) = x$, that is $Fix(\tau_\beta) = \{\phi(x)\}$. □

LEMMA 3.4. *Let* $\mathrm{Mod}(S_g)$ *act on a complete CAT(0) space* M *by isometries. Suppose that* α *and* β *are disjoint non-separate simple closed curves and the Dehn twist on* α *has a unique fixed point* x *in* $M(\infty)$. *Then* $Fix(\tau_\beta) = \{x\}$.

PROOF. Since α and β are disjoint, $\tau_\alpha \cdot \tau_\beta = \tau_\beta \cdot \tau_\alpha$. Hence $\tau_\alpha(\tau_\beta(x)) = \tau_\beta(x)$. Since $Fix(\tau_\alpha) = \{x\}$, $\tau_\beta(x) = x$. By lemma 3.3 we have $Fix(\tau_\beta) = \{x\}$. □

Let us recall the Lickorish-Humphries generators for $\mathrm{Mod}(S_g)$. Let $\{\{\alpha_i\}_{i=1}^{2g+1} \bigcup \beta\}$ be $2g+2$ non-separate simple closed curves such that

$$i(\alpha_i, \alpha_j) = \begin{cases} 1, & |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$i(\alpha_i, \beta) = \begin{cases} 1, & i = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Where $i(\bullet, \bullet)$ is the geometric intersection of two curves. Then $\mathrm{Mod}(S_g)$ is generated by the Dehn twists on these curves (see [**25**]).

LEMMA 3.5. *Let* $\mathrm{Mod}(S_g)$ *act on a complete CAT(0) space* M *by isometries. Suppose that there exists a non-separate simple closed curve* α *such that the Dehn twist about* α *has a unique fixed point* x *in* $M(\infty)$*. Then* $\mathrm{Mod}(S_g)$ *fixes* x*.*

PROOF. Choose the Lickorish-Humphries generators $\{\{\alpha_i\}_{i=1}^{2g+1} \bigcup \beta\}$ for $\mathrm{Mod}(S_g)$. Since α_i are non-separate simple closed curves and $Fix(\tau_\alpha) = \{x\}$, by lemma 3.3 τ_{α_1} fixes only one point $y \in M(\infty)$. If $i > 2, Fix(\tau_{\alpha_i}) = \{y\}$ follows from lemma 3.4 and the fact that $i(\alpha_1, \alpha_i) = 0$. And $Fix(\tau_\beta) = \{y\}$ also follows from lemma 3.4 and $i(\alpha_1, \beta) = 0$. Since $i(\alpha_2, \alpha_4) = 0$, by lemma 3.4 and $Fix(\tau_{\alpha_4}) = \{y\}$ we have $Fix(\tau_{\alpha_2}) = \{y\}$. Hence, y is a common fixed point of the generator for $\mathrm{Mod}(S_g)$. Furthermore, $\mathrm{Mod}(S_g)$ fixes y. Since $Fix(\tau_\alpha) = \{x\}$, we have $y = x$. Hence, $\mathrm{Mod}(S_g)$ fixes x. □

REMARK 3.1. Using the same argument, the lemma also holds for $x \in M$. In this case M can be any metric space.

LEMMA 3.6. *Let* $\mathrm{Mod}(S_g)$ $(g \geq 3)$ *act properly discontinuously on a complete proper visible CAT(0) space* M *by isometries. Then* $\mathrm{Mod}(S_g)$ *fixes some point* $x \in M(\infty)$*.*

PROOF. Since the action is properly discontinuous, by corollary 3.2, the Dehn twist τ_α about a non-separate simple closed curve α is parabolic. Since M is proper, $Fix(\tau_\alpha)$ is not empty. The result follows from remark 2.2 and lemma 3.5. □

PROPOSITION 3.7. *Let* $\mathrm{Mod}(S_g)$ $(g \geq 3)$ *act properly discontinuously on a complete proper visible CAT(0) space* M *by isometries. Then any infinite ordered element* $\phi \in \mathrm{Mod}(S_g)$ *acts as parabolic isometry.*

PROOF. Assuming that there exists an element $\phi \in \mathrm{Mod}(S_g)$ with infinite order which acts on M as a hyperbolic isometry, there exists $x_0 \in M$ such that $d(\phi \circ x_0, x_0) = |\phi| > 0$. The geodesic line $\gamma : \mathbb{R} \to M$ extended by the geodesic segment $\overline{x_0\phi(x_0)}$ is the axis for ϕ, that is $\phi \circ \gamma(t) = \gamma(|\phi| + t)$. Since M is a visible CAT(0) space, it is not hard to see that $Fix(\phi) = \{\gamma(+\infty), \gamma(-\infty)\}$. By lemma 3.6 we can assume $\gamma(+\infty)$ is fixed by $\mathrm{Mod}(S_g)$. Let $\sigma \in \mathrm{Mod}(S_g)$. Since σ fixes $\gamma(+\infty)$ there exists a number $C > 0$ such that $dist(\sigma \circ \gamma(n \cdot |\phi|), \gamma(n \cdot |\phi|)) \leq C$ for any $n > 0$. Hence $dist((\phi^{-n} \cdot \sigma \cdot \phi^n) \circ \gamma(0), \gamma(0)) \leq C$. Since the action is properly discontinuous, there exists a subsequence $\{n_i\}$ of $\{n\}_{n \geq 1}$ such that $\phi^{-n_i} \cdot \sigma \cdot \phi^{n_i} \equiv \phi^{-n_1} \cdot \sigma \cdot \phi^{n_1}$. Hence $\phi^{n_1 - n_i} \cdot \sigma = \sigma \cdot \phi^{n_1 - n_i}$, which means that the centralizers of any two different elements have nontrivial intersection because ϕ (up to a power) belongs to the centralizer of every element. Since σ is arbitrary, we can choose two pseudo-Anosov elements σ_1, σ_2 such that $< \sigma_1, \sigma_2 >$ is a free group of rank 2 (see [**36**]). It is easy to see that the centralizers of $< \sigma_1 >$ and $< \sigma_2 >$ only have trivial intersection, which is a contradiction. □

Now we are ready to prove theorem 1.3.

PROOF OF THEOREM 1.3. Case I): suppose that σ is elliptic. Then σ fixes a point $p \in M$. Hence $|\sigma| = dist(\sigma \circ p, p) = 0$.

Case II): suppose that σ has infinite order, by proposition 3.7 we know that σ is parabolic. $|\sigma| = 0$ follows from theorem 1.1. □

REMARK 3.2. Masur-Minsky [**45**] has shown that $\mathrm{Mod}(\sum_{g,n})$ $(g \geq 1)$ acts by isometries on the complex of curves, which is a Gromov-hyperbolic space, and the pseudo-Anosov element acts as a hyperbolic element with positive translation length. We should remark that the complex of curves is not a proper space and the action is also not properly discontinuous.

4. Two-dimensional surfaces

Let M be a complete manifold with nonpositive sectional curvature and $\tilde{M}$ be the universal covering space of M. $\pi_1(M,p)$ is the fundamental group of M with basepoint $p \in M$. Each element in $\pi_1(M,p)$ is a deck transformation of $\tilde{M}$ which is an isometry of $\tilde{M}$. Theorem 1.1 tells us that $\pi_1(M,p)$ does not contain parabolic isometries with positive translation length if $\tilde{M}$ is visible. It is interesting to know when the fundamental group $\pi_1(M,p)$ of M contains a parabolic isometry with positive translation length.

At first let's look at the following example.

Example 3. Let M be the two-dimensional Riemannian manifold $(\mathbb{R} \times \mathbb{R}, ds^2)$ where $ds^2 := (e^{-y}+1)^2 dx^2 + dy^2$. Let $\phi : M \to M$ be defined by $(x,y) \mapsto (x+1,y)$. It is not hard to see that $|\phi| = \inf_{x \in M} dist(x, \phi \circ x) = 1 > 0$ and $Vol(M/ < \phi >) = \infty$.

The following theorem gives a link between the existence of a parabolic isometry with positive translation length and infinite volume.

Theorem 4.1. *Let M be a complete two-dimensional Riemannian manifold with nonpositive Gauss curvature. If the fundamental group $\pi_1(M,p)$ of M with a basepoint p contains a parabolic isometry ϕ with translation length $|\phi| > 0$, then the volume $Vol(M) = \infty$.*

Proof. Since $\pi_1(M,p)$ contains a parabolic isometry, M is non-compact. Suppose that $Vol(M) < \infty$. Then M is not flat since there does not exist a non-compact flat surface of finite area. By proposition 2.5 of [**20**] (or corollary 3.2 of [**21**]) the universal covering space $\tilde{M}$ of M is visible. Since ϕ is parabolic, by theorem 1.1 we know that $|\phi| = 0$ which is a contradiction. □

Remark 4.1. Theorem 4.1 is not true in higher dimensions. We look at the following example. Let S^1 be the unit circle and $S_{g,1}$ be the Riemannian surface of genus $g \geq 1$ and one punctured point. Consider $M = S^1 \times S_{g,1}$, endowed with the product metric. It is easy to see that M has sectional curvature $-1 \leq K(M) \leq 0$ and finite volume. It is not hard to find a parabolic isometry ϕ in the fundamental group of M such that $|\phi| > 0$.

For a finite-type complete Riemannian surface M, it is well known that M is non-compact if and only if M has cusps. It is not hard to see that there exists a unique ray in each cusp, and this ray is fixed by some parabolic isometry which has zero translation length. Theorem 1.5 gives a criterion for a closed two-dimensional manifold.

Proof of theorem 1.5. From compactness argument it is easy to see that if M is closed, then for any non-trivial deck transformation $\phi \in \pi_1(M,p)$, $|\phi| > 0$.

Assume that M is open. Then there exists a geodesic ray $r : [0,+\infty) \to M$. Since $Vol(M) < \infty$ and $K(M) \leq 0$, the injectivity radius along $r([0,+\infty))$ goes to 0. We lift the ray $r([0,+\infty))$ to a geodesic ray in the universal covering space $\tilde{M}$ of M $\tilde{r} : [0,+\infty) \to \tilde{M}$. It is easy to see that for any $\epsilon > 0$, there exists $R_0 > 0$ such that for any $t > R_0$, there exists non-trivial $\phi_t \in \pi_1(M,p)$ so that $dist(\tilde{r}(t), \phi_t \circ \tilde{r}(t)) < \epsilon$.

Claim: If ϵ is small enough, ϕ_t is parabolic for any $t > R_0$.

Assume that the claim is correct. Since the deck transformation ϕ_t is parabolic and has positive translation length, by theorem 4.1, we would have $Vol(M) = +\infty$ which is a contradiction. Hence M is compact.

Proof of Claim: Since $Vol(M) < +\infty$, using the same argument as in the proof of theorem 4.1, by proposition 2.5 of [**20**] (or Corollary 3.2 of [**21**]) the universal covering space $\tilde{M}$ of M is a visible space. Let $\epsilon > 0$ be small enough. By lemma 3.1c of [**21**] $Fix(\phi_{t_1}) = Fix(\phi_{t_2})$ for any $t_i > R_0$ $(i = 1, 2)$. Let $x \in Fix(\phi_{t_1}) \subset \tilde{M}(+\infty)$ and $\Gamma_x := \{\phi : \phi \in \pi_1(M,p), \phi(x) = x\}$.

If there exists t_0 with $t_0 > R_0$ such that ϕ_{t_0} is hyperbolic. We denote ϕ_{t_0} by ϕ. Since ϕ is hyperbolic, there exists a geodesic line $\gamma : \mathbb{R} \to \tilde{M}$ which is the axis for ϕ, that is $\phi \circ \gamma(t) = \gamma(|\phi| + t)$. Since $\tilde{M}$ is a visible CAT(0) space, it is not hard to see that $Fix(\phi) = \{\gamma(+\infty), \gamma(-\infty)\}$. Without loss of generality we set $x = \gamma(+\infty)$. For $\sigma \in \Gamma_x$, there exists a number $C > 0$ such that $dist(\sigma \circ \gamma(n \cdot |\phi|), \gamma(n \cdot |\phi|)) \leq C$ for any $n > 0$, that is $dist((\phi^{-n} \cdot \sigma \cdot \phi^n) \circ \gamma(0), \gamma(0)) \leq C$. Since the fundamental group acts properly discontinuously on the universal covering, there exists a subsequence $\{n_i\}$ such that $\phi^{-n_i} \cdot \sigma \cdot \phi^{n_i} \equiv \phi^{-n_1} \cdot \sigma \cdot \phi^{n_1}$. Hence $\phi^{n_1 - n_i} \cdot \sigma = \sigma \cdot \phi^{n_1 - n_i}$. Furthermore σ acts invariantly on $\gamma((-\infty, +\infty))$ (a reselection of axis of ϕ is possibly required), so σ is also hyperbolic and has the same axis as ϕ. Moreover we know the group generated by σ and γ is cyclic. Since $\sigma \in \Gamma_x$ is arbitrary, Γ_x is cyclic. Let α be the generator of Γ_x and translate the geodesic γ of $\tilde{M}$ by an amount $\delta > 0$, then $|\sigma| \geq \delta > 0$ for any $\sigma \in \Gamma_x$. On the other hand, since the injectivity radius along $r([0, +\infty))$ goes to 0, for any $n > 0$, there exists $\phi_n \in \Gamma_x$ such that $|\phi_n| < \frac{1}{n}$, which is a contradiction. □

Remark 4.2. The following two examples tell us that the lower bound for curvature and finite volume are necessary for theorem 1.5.

Example 4. Example 1 of page 457 in [**21**] is a two-dimensional complete noncompact surface M such that the Gauss curvature $K(M) \leq 0$, the volume $Vol(M) < +\infty$, and the fundamental group of M with basepoint p consists of hyperbolic isometries.

Example 5. Consider the upper half plane $\mathbb{H}^2$ endowed with a metric $ds^2 := (dx^2 + dy^2) + \frac{dx^2 + dy^2}{y^2}$. Let $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ defined by $(x, y) \mapsto (x + 1, y)$. Setting M to be the quotient space $\mathbb{H}^2 / < \phi >$, it is easy to check that M is complete, the sectional curvature of M satisfies $-1 \leq K_M < 0$, and $\pi_1(M)$ consists of isometries with positive translation length. But M is not closed.

Remark 4.3. I am grateful to Tam Nguyen Phan for pointing out that Gromov's example in [**28**] tells that theorem 1.5 may be incorrect in higher dimensions.

5. Negatively curved manifolds without visibility

In [**22**], Eberlein and O'Neill first introduced the so-called zero axiom. Recall that M satisfies the *zero axiom* if for any two rays $r : [0, +\infty) \to M$ and $\sigma : [0, +\infty) \to M$

with $r(+\infty) = \sigma(+\infty)$ in $M(\infty)$ we have $\lim_{t\to+\infty} dist(r(t), \sigma(\mathbb{R}^{\geq 0})) = 0$. A typical example of a space satisfying the zero axiom is a complete simply connected Riemannian manifold whose sectional curvature is bounded above by a negative number.

PROPOSITION 5.1. *Let M be a complete CAT(0) space satisfying the zero axiom, γ an infinite ordered isometry of M, and $Fix(\gamma)$ be the subset in $M(\infty)$ fixed by γ (i.e., for any $x \in Fix(\gamma), \gamma(x) = x$). Then for any geodesic ray $r : [0, +\infty] \to M$ with $r(+\infty) \in Fix(\gamma)$ we have $\lim_{t\to+\infty} dist(\gamma \circ r(t), r(t)) = |\gamma|$.*

PROOF. Let $\{p_i\}_{i\geq 1}$ be a sequence in M such that $\lim_{i\to+\infty} dist(\gamma \circ p_i, p_i) = |\gamma|$ and $r_i : [0, +\infty) \to M$ be a sequence of rays in M with $r_i(0) = p_i$ and $r_i(+\infty) = r(+\infty)$. Since M satisfies the zero axiom, for each i there exists $t_i, s_i > 0$ such that $dist(r_i(s_i), r(t_i)) < \frac{1}{i}$. By the triangle inequality,

$$dist(\gamma \circ r(t_i), r(t_i)) \leq dist(\gamma \circ r_i(s_i), r_i(s_i)) + 2 \times dist(r_i(s_i), r(t_i)). \tag{7}$$

Since $r_i(+\infty) = r(+\infty) \in Fix(\gamma)$ and the distance function between two rays in M is convex (see [**10**]), $dist(\gamma \circ r_i(t), r_i(t))$ is a decreasing function. In particular we have

$$dist(\gamma \circ r_i(s_i), r_i(s_i)) \leq dist(\gamma \circ r_i(0), r_i(0)) = dist(\gamma \circ p_i, p_i). \tag{8}$$

Combining (7) and (8),

$$dist(\gamma \circ r(t_i), r(t_i)) \leq dist(\gamma \circ p_i, p_i) + \frac{2}{i}. \tag{9}$$

Taking the limit,

$$\lim_{i\to+\infty} dist(\gamma \circ r(t_i), r(t_i)) \leq |\gamma|. \tag{10}$$

From the definition we also know that $\lim_{i\to+\infty} dist(\gamma \circ r(t_i), r(t_i)) \geq |\gamma|$. Hence $\lim_{i\to+\infty} dist(\gamma \circ r(t_i), r(t_i)) = |\gamma|$. Since $r(+\infty) \in Fix(\gamma)$, $dist(\gamma \circ r(t), r(t))$ is decreasing, so $\lim_{t\to+\infty} dist(\gamma \circ r(t), r(t)) = \lim_{i\to+\infty} dist(\gamma \circ r(t_i), r(t_i)) = |\gamma|$. □

Next now we give an affirmative answer to Farb's conjecture if the manifold satisfies the zero axiom.

Proof of theorem 1.6. Let σ be a non-separate simple closed curve. Since $g \geq 3$, we can find two intersecting simple closed curves $\sigma_1, \sigma_2 \subset (S_g - \sigma)$ such that the group generated by the two Dehn-twists τ_{σ_1} and τ_{σ_2} is free (see [**36**]).

Let τ_σ be the Dehn-twist on σ in $\mathrm{Mod}(S_g)$. We define the centralizer $N(\tau_\sigma)$ of τ_σ in the following way

$$N(\tau_\sigma) := \{\alpha \in \mathrm{Mod}(S_g) : \alpha \circ \tau_\sigma = \tau_\sigma \circ \alpha\}.$$

It is not hard to see that $< \tau_{\sigma_1}, \tau_{\sigma_2} > \subset N(\tau_\sigma)$.

If $\mathrm{Mod}(S_g)$ does act properly discontinuously on a complete simply connected Riemannian manifold M satisfying the zero axiom and the sectional curvature $-1 \leq K_M \leq 0$, then by corollary 3.2 (Bridson's theorem), the Dehn-twist τ_σ would act as a parabolic isometry on the M. By lemma 7.3 in page 87 of [**5**] we know that there exists some $x \in M(\infty)$ such that $N(\tau_\sigma)$ fixes x, that is for any $\alpha \in N(\tau_\sigma)$, $\alpha(x) = x$. Since $< \tau_{\sigma_1}, \tau_{\sigma_2} > \subset N(\tau_\sigma)$, $< \tau_{\sigma_1}, \tau_{\sigma_2} >$ fixes x.

Let $r : [0, +\infty) \to M$ be a geodesic ray in M with $r(+\infty) = x$. Since $g \geq 3$, the translation length of any Dehn-twist is zero. Since M satisfies the zero axiom, by proposition 5.1 we have $\lim_{t\to+\infty} dist(\tau_{\sigma_1} \circ r(t), r(t)) = \lim_{t\to+\infty} dist(\tau_{\sigma_2} \circ r(t), r(t)) = 0$. Hence, for any $\epsilon > 0$ we can find t_0 such that

$$dist(\tau_{\sigma_1} \circ r(t_0), r(t_0)) < \epsilon, \qquad dist(\tau_{\sigma_2} \circ r(t_0), r(t_0)) < \epsilon.$$

Choose ϵ so that ϵ is smaller than the Margulis constant for M. After applying the Margulis Lemma (see [**5**]) at the point $r(t_0)$, we have that the group $< \tau_{\sigma_1}, \tau_{\sigma_2} >$ is a finitely generated subgroup of an almost nilpotent group which is still almost nilpotent, which contradicts the fact that $< \tau_{\sigma_1}, \tau_{\sigma_2} >$ is free. $\square$

Remark 5.1. In [**13, 35**] it was proved that the mapping class group $\mathrm{Mod}(S_{g,n})$ cannot act properly discontinuously on any complete simply connected Riemannian manifold with pinched negative sectional curvature when $3g-3+2n \geq 2$. Since a complete simply connected Riemannian manifold whose sectional curvature is bounded above by a negative number satisfies the zero axiom, theorem 1.6 generalizes these results except in several cases.

Proof of theorem 1.7. We first prove the theorem for examples in [**26**] when the dimension of M is 3. For general dimension, it can be reduced to three.

Let V be a 3-dimensional closed hyperbolic manifold and S be a simple closed geodesic in V with length $a > 0$. Let $\sigma > 0$ be small enough. Then a $\sigma-$neighborhood $N_\sigma(S)$ of S is $S \times S^1 \times (0, \sigma)$. We introduce polar coordinates (ω, θ, r) on $N_\sigma(S)$. The hyperbolic metric of V on a $\sigma-$neighborhood $N_\sigma(S)$ of V is given by

$$g_V = \cosh^2(r)d\omega^2 + \sinh^2(r)d\theta^2 + dr^2 \quad (0 \leq \theta \leq 2\pi, \ \ 0 \leq r \leq \sigma).$$

Let $M = V - S$ and g be the metric on M constructed in [**26**]:

$$g = \cosh^2(r)d\omega^2 + \sinh^2(r)d\theta^2 + f^2(r)dr^2 \quad (0 \leq \theta \leq 2\pi, \ \ 0 \leq r \leq \sigma)$$

where $f(r)$ converges to $+\infty$ as $r \to 0$, and satisfies certain properties (see [**26**]). It is showed in [**26**] that (M, g) has finite volume and sectional curvature $-1 \leq K_M < 0$.

From the definition of g we know that for any fixed positive number $c_0 \in (0, 2\pi)$, the surface $\theta = c_0$ in M is totally geodesical. The metric g restricted to $\theta = c_0$ is

$$g_{\theta=c_0} = \cosh^2(r)d\omega^2 + f^2(r)dr^2 \quad (0 \leq r \leq \sigma).$$

We denote $M|_{\theta=c_0}$ by $S\times(0, \sigma)$. The universal covering space of $S\times(0, \sigma)$ is $\mathbb{R}\times(0, \sigma)$. Let ϕ be the generator of the fundamental group of $S \times (0, \sigma)$. Since the length of S is a, it is not hard to see that, for all $(\omega, r) \in \mathbb{R} \times (0, \sigma)$, we have

$$\phi \circ (\omega, r) = (\omega + a, r) \ \ (0 \leq r \leq \sigma).$$

We claim that ϕ **is a parabolic isometry with positive translation length.**

Proof of Claim: Firstly we consider the curve $c(t) : [0, 1] \to \mathbb{R} \times (0, \sigma)$ defined by $c(t) = (\omega + t \cdot a, r)$, so we have

$$\begin{aligned} |\phi| \leq \ell(c([0,1])) &= \int_0^1 \sqrt{\cosh^2(c_2(t)) \cdot c_1'(t)^2} \\ &= \cosh(r) \cdot \int_0^1 |c_1'(t)| = a \cdot cosh(r). \end{aligned}$$

Since r is arbitrary, letting $r \to 0$ we get $|\phi| \leq a$.

Secondly, let $c(t) = (c_1(t), c_2(t)) : [0,1] \to \mathbb{R} \times (0, \sigma)$ be any smooth curve joining (ω, r) and $(\omega + a, r)$, so that in particular $c_1(0) = \omega$ and $c_1(1) = \omega + a$. The length of $c([0,1])$ is

$$\begin{aligned} \ell(c([0,1])) &= \int_0^1 \sqrt{\cosh^2(c_2(t)) \cdot c_1'(t)^2 + f^2(c_2(t)) \cdot c_2'(t)^2} \\ &\geq \int_0^1 |\cosh(c_2(t)) \cdot c_1'(t)| > \int_0^1 |c_1'(t)| \\ &\geq (c_1(1) - c_1(0)) = a > 0. \end{aligned}$$

Since $c(t)$ is arbitrary, $|\phi| \geq a > 0$. Hence $|\phi| = a > 0$. $|\phi|$ can not be attained in $\mathbb{R} \times (0, \sigma)$ since $\ell(c([0,1])) > a$ for any curve joining (ω, r) and $(\omega + a, r)$, so ϕ is parabolic.

Hence ϕ restricted to $\mathbb{R} \times (0, \sigma)$ is a parabolic isometry with positive translation length. Since $\theta = c_0$ is totally geodesical in M, $\mathbb{R} \times (0, \sigma)$ is totally geodesical in the universal covering of M. So ϕ is also a parabolic isometry with positive translation length in the universal covering of M.

From theorem 1.1 we know that M is not visible.

The proof for examples in [**1**] is similar as above; we leave it as an exercise to reader. □

REMARK 5.2. Recently, in [**48**] Phan independently proved that Fujiwara's example M is not visible by finding two points x, y on the visual boundary of $\tilde{M}$ such that there does not exist any geodesic line in $\tilde{M}$ joining x and y.

CHAPTER 4

Iteration of mapping classes and limits of geodesics

1. Introduction

Let $S = S_g$ be a closed surface of genus $g > 1$. Its *Teichmüller space* $\mathbb{T}_S$ carries various natural metrics, and for every metric people would like to draw analogies with a complete hyperbolic space of the same dimension.

$\mathbb{T}_S$ endowed with the Weil-Petersson metric is a Riemannian manifold, which is denoted by Teich(S). Wolpert and Tromba showed that Teich(S) has negative sectional curvature (see [**57, 63**]). Scott Wolpert proved that Teich(S) is non-complete (see [**16, 62**]), but geodesically convex (see [**64**]). The completion $\overline{\text{Teich}(S)}$ of Teich(S), also called the augmented Teichmüller space, is naturally a CAT(0) space. So we can study the geometry of $\overline{\text{Teich}(S)}$ through CAT(0) techniques.

A complete manifold with nonpositive sectional curvature M is compactified by the space of infinite geodesic rays starting from a given point $x \in M$, which is called the *visual sphere* at x, and basically it is the collection of directions of M at x. Likewise, for each point $X \in$ Teich(S), Teich(S) has a Weil-Petersson visual sphere, although some geodesics go to the boundary of $\overline{\text{Teich}(S)}$ in finite time. Since Teich(S) is a Riemannian manifold, the collection of directions of Teich(S) at X is a $(6g-7)-$dimensional sphere. We denote the visual sphere of Teich(S) at X by $V_X(S)$.

The mapping class group Mod(S) of S , which is the group of orientation preserving self-homeomorphisms of S up to isotopy, acts on Teich(S) by isometries. Like in the structure of isometry of a CAT(0) space, we can define the *translation length* of an element $\phi \in$ Mod(S) $|\phi|$ by $\inf_{X\in \text{Teich}(S)} dist(X, \phi \circ X)$. We say ϕ is *hyperbolic* if $|\phi|$ is attained in Teich(S); otherwise, it is called *parabolic*. In [**18, 65, 70**] it is showed that an element $\phi \in \mathrm{Mod}(S)$ is hyperbolic if and only if ϕ is finite ordered or pseudo-Anosov. Moreover, every pseudo-Anosov mapping class has positive translation length.

Let $X, Y \in T(S)$ and $\Gamma(X, Y)$ be the quasi-Fuchsian *Bers simultaneous uniformization* of $(X, Y) \in T(S) \times T(\overline{S})$. Then $\Gamma(X, Y)$ determines $Q(X, Y) = H^3/\Gamma(X, Y)$ as a quotient hyperbolic 3-manifold. In [**11**] Brock shows

THEOREM 1.1 (Brock). *Let $\phi \in \mathrm{Mod}(S)$ be a mapping class. Then there is an $s \geq 1$, depending only on ϕ and bounded in terms of S, so that the sequence $\{Q(\phi^{si}(X), Y)\}_{i \geq 1}$ converges algebraically and geometrically.*

In [**64**] it was showed that Teich(S) is geodesically convex, i.e., for any two points $X, Y \in \mathrm{Teich}(S)$, there exists a geodesic connecting X and Y, moreover the geodesic is unique because the sectional curvature of Teich(S) is negative. We denote the geodesic joining X and Y by $g(X, Y)$. Our first result is analogous to Brock's theorem.

THEOREM 1.2. *Let $\phi \in \mathrm{Mod}(S)$ be a mapping class. Then there is an $s \geq 1$, only depending on ϕ, so that the sequence of the directions of the geodesics $\{g(X, \phi^{si}(Y))\}_{i \geq 1}$ is convergent in the visual sphere of X.*

Given a collection of mutually disjoint simple closed curves, we connect them pairwise by a segment of length 1. The resulting object is called a *simplex*. Let σ be a simplex, and denote its vertices by σ^0. Recall that *the stratum* T_σ consists of all hyperbolic surfaces with nodes along the curves in σ (see [**44, 65**]). A stratum is a convex subset in $\overline{\mathrm{Teich}(S)}$. If ϕ is a Dehn-twist on a simple closed curve α, it was showed in [**12, 49**] that the limit of the sequence $\{g(X, \phi^i \circ Y)\}_{i \geq 1}$ goes to the stratum T_α. It would be interesting to know whether this property can be generalized to multi-twists, which is our second goal in this paper.

In [**65**] Wolpert gave a compactness theorem for a sequence of geodesics in $\overline{\mathrm{Teich}(S)}$ with uniform bounded lengths (see proposition 23 in [**65**]). Later Yamada in [**71**] constructed the so-called *Teichmüller-Coxeter development* $D(\overline{\mathrm{Teich}(S)}, \iota)$ through introducing an infinite Coxeter reflection group and gluing infinite copies of $\overline{\text{Teich}(S)}$ through the strata. $D(\overline{\text{Teich}(S)}, \iota)$ is a complete CAT(0) space (see [**71**]). The limit geodesic in Wolpert's compactness theorem can be well described in $D(\overline{\text{Teich}(S)}, \iota)$. The definition of $D(\overline{\text{Teich}(S)}, \iota)$ will be given in section 4 of this chapter. If ϕ is a multi Dehn-twist, the limit of the geodesics $\{g(X, \phi^i(Y))\}_{i\geq 1}$ can be characterized by the following:

THEOREM 1.3. *Let σ be an m-simplex and $\sigma^0 = \{\alpha_1, \cdots, \alpha_{m+1}\}$ and τ_i be the Dehn-twist about the curve α_i for $i = 1, 2, \cdots, m+1$. Let $\phi = \prod_{1\leq i\leq m+1} \tau_i \in \text{Mod}(S)$, $X, Y \in \text{Teich}(S)$, and g_n be the unit speed geodesics $g(X, \phi^n \circ Y)$. Then, there exists a positive number L, an associated partition $0 = t_0 < t_1 < \cdots < t_k = L$, simplices $\sigma_0, \cdots, \sigma_k$, and a piewise geodesic*

$$g : [0, L] \to \overline{\text{Teich}(S)}$$

with the following properties.

(1). $\sigma_i^0 \subset \sigma^0$, $\sigma_i^0 \cap \sigma_j^0$ is empty for $i \neq j$,

(2). $\sigma^0 = \bigcup_{i=1}^k \sigma_i^0$,

(3). $g(t_i) \in T_{\sigma_i}$, $i = 1, \cdots, k-1$, $g(0) = X, g(t_k) = Y$,

(4). $g_n[0, t_1]$ converges in $\overline{\text{Teich}(S)}$ to the restriction $g([0, t_1])$, and for each $i = 1, \cdots, k-1$,

$$\lim_{n\to+\infty} dist(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ g_n(t), g(t)) = 0, \quad \text{for } t \in [t_i, t_{i+1}],$$

where $\tau_{i,n} = \prod_{\alpha\in\sigma_i} \tau_\alpha^{-n}$, for $i = 1, \cdots, k-1$.

(5). The piecewise geodesic g is the unique minimal length path in $\overline{\text{Teich}(S)}$ joining $g(0)$ to $g(L)$ and intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}}$ in order.

The first point on $g([0, L])$ meeting with strata $g(t_1)$ is the point where the geodesic joining $(1, X)$ and $(\prod_{\alpha\in\sigma^0} \omega_\alpha, Y)$ in the Teichmüller-Coxeter development $D(\overline{\text{Teich}(S)}, \iota)$ firstly meets with the wall. Here we identify $(1, \overline{\text{Teich}(S)})$ with $\overline{\text{Teich}(S)}$.

As part of the analysis we also have the following limit result.

Theorem 1.4. *Let σ be a k-simplex and $\sigma^0 = \{\gamma_1, \cdots, \gamma_{k+1}\}$. τ_i is the Dehn-twist about the curve γ_i for $i = 1, 2, \cdots, k+1$. Let $\phi = \prod_{1 \leq i \leq k+1} \tau_i \in \mathrm{Mod}(S)$ and $g_n = g(X, \phi^n \circ Y)$. Then for any $X, Y \in \mathrm{Teich}(S)$,*

$$\lim_{n \to +\infty} \ell(g(X, \phi^n \circ Y)) \quad exists,$$

where $\ell(g(X, \phi^n \circ Y))$ is the length of the geodesic $g(X, \phi^n \circ Y)$.

Before providing further context we first recall the *Thurston-Nielsen classification* of mapping classes [**24**]. A mapping class is called *reducible* if some power fixes a collection of mutually disjoint simple closed curves in S. Reducible classes are analyzed in terms of mapping classes of proper subsurfaces. For a reducible mapping class $\phi \in \mathrm{Mod}(S)$ there exists a maximal finite collection of mutually disjoint simple closed curves $\{\alpha_i\}$ and mutually disjoint proper subsurfaces $\{PS_k\} \subset S$ such that ϕ^k is the product of Dehn-twists on $\{\alpha_i\}$ and pseudo-Anosov elements on proper subsurfaces $\{PS_k\} \subset S$ for some integer $s \geq 1$. If ϕ itself is a Dehn-twist about a simple closed curve, then there does not exist a pseudo-Anosov part on any proper subsurface. A mapping class is precisely one of: finite-ordered, reducible or pseudo-Anosov (see [**24**]).

We denote the length of a geodesic segment c by $\ell(c)$. We say a geodesic ray $c : [0, \infty) \to \mathrm{Teich}(S)$ is the *geometric limit* of geodesics $c_i : [0, \ell(c_i)) \to \mathrm{Teich}(S)$ if for any $t > 0$, $\lim_{i \to +\infty} dist(c(t), c_i(t)) = 0$, where all geodesics have unit speed. We say a geodesic ray $c : [0, +\infty) \to \mathrm{Teich}(S)$ is *strongly asymptotic* to a subset $A \subset \overline{\mathrm{Teich}(S)}$ if the distance between $c(t)$ and A satisfies

$$\lim_{t \to +\infty} dist(c(t), A) = 0.$$

If ϕ is pseudo-Anosov, the following theorem tells that the length of every simple closed curve goes to infinity along the geometric limit of $g(X, \phi^n \circ Y)$.

Theorem 1.5. *Let ϕ be a pseudo-Anosov mapping class and $X, Y \in \mathrm{Teich}(S)$. Then the geodesics $g(X, \phi^n \circ Y)$ converges to a geodesic ray $c : [0, +\infty) \to \mathrm{Teich}(S)$*

which is strongly asymptotic to the axis of ϕ in $\mathrm{Teich}(S)$*. Moreover, for any simple closed curve α in S,*

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty.$$

If ϕ is reducible with $|\phi| > 0$, the following theorem tells that the geometric limit geodesic of $g(X, \phi^n \circ Y)$ goes to an explicit stratum whose vertices consist of the boundary closed curves in proper surfaces on which ϕ is pseudo-Anosov.

THEOREM 1.6. *Let $\phi \in \mathrm{Mod}(S)$ be reducible with $|\phi| > 0$ and k be a positive integer such that $\phi^k = \prod_{\alpha \in \sigma^0} \tau_\alpha \times \prod_j \phi_j$, where σ is a simplex, τ_α is a Dehn-twist about α, and $\phi_j = \phi^k|_{PS_j}$ is pseudo-Anosov on PS_j, where PS_j is a proper subsurface of S. Then for any $X, Y \in \mathrm{Teich}(S)$, there exists a geodesic ray $c : [0, +\infty) \to \mathrm{Teich}(S)$ such that*

(1). the geodesics $g(X, \phi^n \circ Y)$ converge to $c : [0, +\infty) \to \mathrm{Teich}(S)$.

(2). For any simple closed curve $\alpha \in \partial(\cup_j PS_j)$, we have

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = 0.$$

(3). There exists a positive number ϵ_0 such that for any non-peripheral essential simple closed curve β in S but not in $\partial(\cup_j PS_j)$,

$$\ell_\beta(c(t)) \geq \epsilon_0$$

for all $t \geq 0$.

2. Weil-Petersson geodesics and the Alexandrov tangent cone

Given $X \in \overline{\mathrm{Teich}(S)}$, define

$$g_X : \overline{\mathrm{Teich}(S)} \to WP_X(S)$$

from $\overline{\mathrm{Teich}(S)}$ into the space $WP_X(S)$ of Weil-Petersson unit speed geodesics starting from X. Since $\overline{\mathrm{Teich}(S)}$ is a CAT(0) space, for all $Y \in \overline{\mathrm{Teich}(S)}$ there is a unique Weil-Petersson geodesic in $WP_X(S)$ joining X to Y. The mapping $g_X(Y)$ is given by the unique geodesic connecting X and Y. It is not hard to see that g_X gives a homeomorphism from $\overline{\text{Teich}(S)}$ to $WP_X(S)$.

Since $\overline{\text{Teich}(S)}$ is a CAT(0) space, the notion of tangent cone is available. As in [**10**], let $c_1(t)$ and $c_2(t)$ be two geodesics both of which start from X and have unit speed. The Alexandrov angle $\angle(c_1, c_2)$ between $c_1(t)$ and $c_2(t)$ at X is defined with values in $[0, \pi]$ as follows:

$$\cos\angle(c_1, c_2) = \lim_{t \to 0^+} \frac{2t^2 - dist^2(c_1(t), c_2(t))}{2t^2}.$$

We introduce the equivalence relation on $WP_X(S)$ as $c_1(t) \sim c_2(t)$ provided that $\angle(c_1, c_2) = 0$.

DEFINITION 2.1. *The Alexandrov tangent cone AC_X at X is the quotient space $WP_X(S)/\sim$.*

If $X \in \text{Teich}(S)$, AC_X coincides with the tangent space of Teich(S) at X, which is $\mathbb{R}^{6g-6}$. Let σ be a simplex and $X \in T_\sigma$. Consider a Fricke-Klein basis $\{\text{grad}\ell_\beta\}_{\beta\in\lambda}$ for T_σ around X. For a ray $r(t)$ with initial point X, we define a map $\Lambda : r(t) \to \mathbb{R}_{\geq 0}^{|\sigma|} \times T_X T_\sigma$ by

$$\Lambda(r(t)) = (\sqrt{2\pi} \cdot \frac{d\ell_\alpha^{\frac{1}{2}}(r(0^+))}{dt}, \sqrt{2\pi} \cdot \frac{d\ell_\beta(r(0^+))}{dt}).$$

The Alexandrov tangent cone at X is characterized by the following theorem (see [**67**]).

THEOREM 2.2. *(see [**67**]) For $X \in \overline{\text{Teich}(S)}$, the mapping $\Lambda : AC_X \to \mathbb{R}_{\geq 0}^{|\sigma|} \times T_X T_\sigma$ is an isometry of cones with restrictions of inner products. A geodesic $c(t)$ with $c(0) = X$ and $\frac{d\ell_\alpha^{1/2}(c(0^+))}{dt} = 0$, $\alpha \in \sigma^0$, is contained in the stratum T_α.*

The following strong reflection property is proved in the paragraph after theorem 6.9 in chapter 6 of [**69**] (or paragraph after example 4.19 in [**67**]). We will apply it later.

Proposition 2.3. *Given a simplex σ, let X, Y be two points in $\overline{\text{Teich}(S)}$. Z is a point in T_σ such that the piecewise geodesic $g(X,Z) \cup g(Z,Y)$ is the minimal length path in $\overline{\text{Teich}(S)}$ joining X and Y and intersecting the stratum T_σ. Then we have*

(1). the initial tangents of $g(Z,X)$ and $g(Z,Y)$ are equal in the components $\mathbb{R}^{|\sigma|}_{\geq 0}$.

(2). The sum of the initial tangents of $g(Z,X)$ and $g(Z,Y)$ has vanishing projection into the subcone $T_X T_\sigma$.

3. Iterated multi-twists

Let α be a homotopically essential simple closed curve on S. Let T_α denote the α-stratum of $\overline{\text{Teich}(S)}$. In the extended Fenchel-Nielsen coordinates we have

$$T_\alpha = \{X \in \overline{\text{Teich}(S)} : \quad \ell_\alpha(X) = 0, \ell_\beta(X) \neq 0 \ \text{ for all } \ \beta \in P - \alpha\},$$

where P is a pants decomposition containing α.

Let $\tau_\alpha \in \text{Mod}(S)$ be the Dehn-twist about α. The following lemma is proved in [**12**]

Lemma 3.1. *[Brock] Let X and Y lie in* $\text{Teich}(S)$*, and let α be an essential simple closed curve on S. Then there exists $X_n \in g(X, \tau_\alpha^n \circ Y)$ such that*

$$\lim_{n\to+\infty} dist(X_n, T_\alpha) = 0.$$

A perturbation argument gives the following property.

Lemma 3.2. *Let X lie in* $\text{Teich}(S)$ *and Y lie in* $\overline{\text{Teich}(S)}$*, and let α be an essential simple closed curve on S. Then there exists $X_n \in g(X, \tau_\alpha^n \circ Y)$ such that*

$$\lim_{n\to+\infty} dist(X_n, T_\alpha) = 0.$$

Proof. From lemma 3.1 it is sufficient to prove the result when Y is in a stratum. For any $\epsilon > 0$ let $Y^\epsilon \in \text{Teich}(S)$ with $dist(Y^\epsilon, Y) = \epsilon$. By lemma 3.1 there exists $X_n^\epsilon \in g(X, \tau_\alpha^n \circ Y^\epsilon)$ such that $\lim_{n\to+\infty} dist(X_n^\epsilon, T_\alpha) = 0$. Since $\overline{\text{Teich}(S)}$ is a $CAT(0)$ space, by proposition 2.2 of Chapter II.2 in [**10**], we know that there exists $X_n \in$

$g(X, \tau_\alpha^n \circ Y)$ such that $dist(X_n, X_n^\epsilon) \leq \max\{dist(\tau_\alpha^n \circ Y^\epsilon, \tau_\alpha^n \circ Y), dist(X, X)\} = dist(\tau_\alpha^n \circ Y^\epsilon, \tau_\alpha^n \circ Y) = \epsilon$. Hence,

$$dist(X_n, T_\alpha) \leq dist(X_n^\epsilon, T_\alpha) + dist(X_n^\epsilon, X_n) \leq dist(X_n^\epsilon, T_\alpha) + \epsilon$$

Taking the superior limit,

$$\limsup_{n \to +\infty} dist(X_n, T_\alpha) \leq \epsilon.$$

Since ϵ is arbitrary,

$$\lim_{n \to +\infty} dist(X_n, T_\alpha) = 0.$$

□

REMARK 3.1. If we check the argument for the proof of lemma 3.1 in [**12**], the conclusion also holds for the sequence of geodesics $g(X, \tau_\alpha^{-n} \circ Y)$. Furthermore, it is not hard to see that lemma 3.2 also holds for the sequence of geodesics $g(X, \tau_\alpha^{-n} \circ Y)$.

If we switch the position of X and Y, we have

LEMMA 3.3. *Let Y lie in* $\mathrm{Teich}(S)$ *and X lie in* $\overline{\mathrm{Teich}(S)}$ *with $\ell_\alpha(X) < +\infty$ where α is an essential simple closed curve on S. Then there exists $X_n \in g(X, \tau_\alpha^n \circ Y)$ such that*

$$\lim_{n \to +\infty} dist(X_n, T_\alpha) = 0.$$

PROOF. By remark 3.1 there exists $X_n' \in g(\tau_\alpha^{-n} \circ X, Y)$ such that

$$\lim_{n \to +\infty} dist(X_n', T_\alpha) = 0.$$

Set $X_n = \tau_\alpha^n \circ X_n'$. Since $\mathrm{Mod}(S)$ acts on $\overline{\mathrm{Teich}(S)}$ by isometry, $\tau_\alpha^{-n} \circ (g(X, \tau_\alpha^n \circ Y)) = g(\tau_\alpha^{-n} \circ X, Y)$. Hence, $X_n \in g(X, \tau_\alpha^n \circ Y)$.

$$dist(X_n, T_\alpha) = dist(\tau_\alpha^n \circ X_n', T_\alpha) = dist(X_n', T_\alpha).$$

Therefore,

$$\lim_{n \to +\infty} dist(X_n, T_\alpha) = \lim_{n \to +\infty} dist(X_n', T_\alpha) = 0.$$

□

Let $\eta \subset \sigma$ be a sub-simplex of the simplex σ. The $\eta-$stratum T_η of $\overline{\text{Teich}(S)}$ consists of all hyperbolic surfaces with nodes along the vertices of η, and is parameterized in extended Fenchel-Nielsen coordinates with respect to a pants decomposition P of S with $\sigma^0 \subset P$. Set

$$\overline{PT(S)}_\sigma = \bigcup_{\eta \subset \sigma} T_\eta.$$

By definition we know that for any simplex σ, $\text{Teich}(S) \subset \overline{PT(S)}_\sigma$ since $T_\phi = \text{Teich}(S)$ where ϕ is the empty set.

We will apply Wolpert's theorem on limits of finite length geodesics (see [**65**], proposition 23).

Theorem 3.4. *[Wolpert] Consider a sequence of unit-speed geodesics $\{g_n\}$ with initial points converging to p_0, lengths converging to L with $L > 0$ and parameter intervals converging to $[0, L]$. There exists an associated partition $0 = t_0 < t_1 < \cdots < t_k = L$ of the interval, simplices $\sigma_0, \cdots, \sigma_k$, simplices $\nu_i = \sigma_i \bigcap \sigma_{i-1}$, and a piecewise geodesic*

$$g : [0, L] \to \overline{\text{Teich}(S)}$$

with the following properties.

(1). $g(t_{i-1}, t_i) \subset T_{\nu_i}$, $i = 1, \cdots, k$.

(2). $g(t_i) \in T_{\sigma_i}$, $i = 1, \cdots, k$.

(3). There are elements $\tau_{i,n} \in Tw(\sigma_i - \nu_i \bigcup \nu_{i+1})$, for $i = 1, \cdots, k-1$, so that after passing to a subsequence, $g_n[0, t_1]$ converges in $\overline{\text{Teich}(S)}$ to the restriction $g([0, t_1])$ and for each $i = 1, \cdots, k-1$ and $t \in [t_i, t_{i+1}]$,

$$\lim_{n \to +\infty} dist(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ (g_n(t)), g(t)) = 0.$$

(4). The elements $\tau_{i,n}$ are either trivial or unbounded.

The piecewise geodesic g is the unique minimal length path in $\overline{\text{Teich}(S)}$ joining $g(0)$ to $g(L)$ and intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_k}$ in order.

Proposition 3.5. *Let σ be a k-simplex, $\sigma^0 = \{\alpha_1, \cdots, \alpha_{k+1}\}$, and τ_i be the Dehn-twist about the curve α_i for $i = 1, 2, \cdots, k+1$. Let $\phi = \prod_{1 \le i \le k+1} \tau_i \in \mathrm{Mod}(S)$. Then, for any $X, Y \in \mathrm{Teich}(S)$,*

(1). there exists $\epsilon = \epsilon(X, Y, \phi) > 0$, which depends on X, Y and ϕ, such that for any simple closed curve $\beta \notin \sigma^0$ we have

$$\inf_{Z \in g(X, \phi^n \circ Y)} \ell_\beta(Z) > \epsilon.$$

for all $n \ge 1$.

(2). Any limit of geodesics $g(X, \phi^n \circ Y)$ in the sense of theorem 3.4 goes to the stratum T_{σ_1} where $\sigma_1 \subset \sigma$.

(3). There exists X_n^i on the geodesics $g(X, \phi^n \circ Y)$ such that

$$\lim_{n \to +\infty} dist(X_n^i, T_{\alpha_i}) = 0$$

for all $1 \le i \le k+1$.

Proof. *Proof of (1):* Let β be a simple closed curve in S satisfying $\beta \notin \sigma^0$. If β is disjoint with σ^0, then $\ell_\beta(\phi^n \circ Y) = \ell_\beta(Y)$. We can always find another simple closed curve β' such that $\beta' \notin \sigma^0$ and β' intersects with β. By theorem 4.2 in chapter 2 we have

$$\ell_\beta(Z) \le max\{\ell_\beta(X), \ell_\beta(Y)\}, \quad \ell_{\beta'}(Z) \le max\{\ell_{\beta'}(X), \ell_{\beta'}(Y)\}$$

for all $Z \in g(X, \phi^n \circ Y)$. By the Collar Lemma, there exists a positive number ϵ_1 depending on X, Y, ϕ such that

$$\ell_\beta(Z) \ge \epsilon_1.$$

If β intersects at least one of σ^0. Since ϕ fixes σ^0 pointwise, for all $\alpha \in \sigma^0$, $\ell_\alpha(\phi^n \circ Y) = \ell_\alpha(Y)$, by theorem 4.2 in chapter 2 we have

$$\ell_\alpha(Z) \le max\{\ell_\alpha(X), \ell_\alpha(Y)\}$$

for all $Z \in g(X, \phi^n \circ Y)$. By the Collar Lemma, there exists a positive number ϵ_2 depending on X, Y, ϕ such that

$$\ell_\beta(Z) \geq \epsilon_2.$$

We choose $\epsilon = min\{\epsilon_1, \epsilon_2\}$.

Proof of (2): Let $W \in T_\sigma$, we have $\phi \circ W = W$. By the triangle inequality,

$$\begin{aligned} Length(g(X, \phi^n \circ Y)) &\leq dist(X, W) + dist(W, \phi^n \circ Y) \\ &= dist(X, W) + dist(W, Y) < +\infty. \end{aligned}$$

By theorem 3.4 any limit of $g(X, \phi^n \circ Y)$ goes to some stratum T_{σ_1} for some simplex σ_1. From part 1 we know that either $\sigma_1 \subset \sigma$ or σ_1 is empty. We exclude the empty case for the following reason. If σ_1 is empty, after passing to a subsequence, the geodesics $g(X, \phi^n \circ Y)$ converge to a geodesic in Teich(S). In particular, $\{\phi^n \circ Y\}_{n \geq 1}$, passing to a subsequence, converges to a point in Teich(S), which is impossible since ϕ has infinite order and Mod(S) acts properly discontinuously on Teich(S).

Proof of (3): We prove it by induction on k.

If $k = 0$, it follows from lemma 3.1.

Assume that the conclusion holds for any $k < k_0$, we prove it for $k = k_0$. Assume the result is incorrect. Passing to a subsequence (without of generality), we can assume that there exists a positive number ϵ_0 such that

$$dist(g(X, \phi^n \circ Y), T_{\alpha_1}) \geq \epsilon_0$$

for all n.

Let $W \in T_\sigma$. We have $\phi \circ W = W$. By the triangle inequality,

$$\begin{aligned} Length(g(X, \phi^n \circ Y)) &\leq dist(X, W) + dist(W, \phi^n \circ Y) \\ &= dist(X, W) + dist(W, Y) < +\infty. \end{aligned}$$

By theorem 3.4 and part 1, any subsequence of the geodesics $g(X, \phi^n \circ Y)$ can only converge to a stratum T_{η_1} where $\eta_1^0 \subset \sigma^0$. Hence, without of generality we assume that the geodesics $g(X, \phi^n \circ Y)$ converge to a stratum T_{η_1} with $p_1 \in T_{\eta_1}$. Without loss of generality, let $\eta_1^0 = \{\alpha_2, \cdots, \alpha_{k_1}\}$ and $\eta_2^0 = \{\alpha_1, \alpha_{k_1+1}, \cdots, \alpha_{k+1}\})$ such that the distance between the piecewise geodesic $g(X, p_1) \bigcup g(p_1, \phi^n \circ Y)$ and the geodesic $g(X, \phi^n \circ Y)$ goes to 0 as n approaches to ∞. Since $\phi \in \text{Mod}(S)$ and $\text{Mod}(S)$ acts on $\text{Teich}(S)$ by isometries, $g(p_1, \phi^n \circ Y)$=$g(\omega_1^n \circ p_1, \phi^n \circ Y)$=$\omega_1^n \circ g(p_1, \omega_2^n \circ Y)$ where $\omega_1 = \prod_{\gamma \in \eta_2^0} \tau_\gamma, \omega_2 = \prod_{\gamma \in \eta_1^0} \tau_\gamma$.

For any $\epsilon > 0$ let $p_1^\epsilon \in \text{Teich}(S)$ with $dist(p_1^\epsilon, p_1) = \epsilon$. From our assumption there exists $X_n^\epsilon \in g(p_1^\epsilon, \omega_2^n \circ Y)$ such that

$$\lim_{n \to +\infty} dist(X_n^\epsilon, T_{\alpha_1}) = 0.$$

Since $\overline{\text{Teich}(S)}$ is a $CAT(0)$ space, by proposition 2.2 of Chapter II.2 in [**10**] we know that there exists $X_n \in g(p_1, \omega_2^n \circ Y)$ such that $dist(X_n, X_n^\epsilon) \leq \max\{dist(\omega_2^n \circ Y, \omega_2^n \circ Y), dist(p_1, p_1^\epsilon)\} = \epsilon$. Hence,

$$dist(X_n, T_{\alpha_1}) \leq dist(X_n^\epsilon, T_{\alpha_1}) + dist(X_n^\epsilon, X_n) \leq dist(X_n^\epsilon, T_{\alpha_1}) + \epsilon$$

Taking the superior limit,

$$\limsup_{n \to +\infty} dist(X_n, T_{\alpha_1}) \leq \epsilon.$$

Since $X_n \in g(p_1, \omega_2^n \circ Y)$, $\omega_1^n \circ X_n \in g(p_1, \phi^n \circ Y)$. When n is big enough, there exists $Y_n \in g(X, \phi^n \circ Y)$ such that $dist(Y_n, \omega_1^n \circ X_n) < \frac{\epsilon_0}{2}$ because the distance between the piecewise geodesic $g(X, p_1) \bigcup g(p_1, \phi^n \circ Y)$ and the geodesic $g(X, \phi^n \circ Y)$ approaches 0 as n goes to ∞. On the other hand,

$$dist(Y_n, T_{\alpha_1}) \leq dist(Y_n, \omega_1^n \circ X_n) + dist(\omega_1^n \circ X_n, T_{\alpha_1}) < \frac{\epsilon_0}{2} + dist(X_n, T_{\alpha_1}).$$

Taking the limit,

$$\limsup_{n \to +\infty} dist(Y_n, T_{\alpha_1}) \leq \frac{\epsilon_0}{2} + \epsilon.$$

Since ϵ is arbitrary, if we choose $\epsilon = \frac{\epsilon_0}{4}$, the inequality above contradicts our assumption that $dist(g(X, \phi^n \circ Y), T_{\alpha_1}) \geq \epsilon_0$ for all n. $\square$

We extend proposition 3.5 to the case that X is in a stratum.

PROPOSITION 3.6. *Let σ be a k-simplex, $\sigma^0 = \{\alpha_1, \cdots, \alpha_{k+1}\}$ and $\phi = \prod_{1\leq i\leq k+1} \tau_i \in$ $\mathrm{Mod}(S)$, where τ_i is the Dehn-twist about the curve α_i for $i = 1, 2, \cdots, k+1$. Given a simplex σ' disjoint with σ and two points $X \in T_{\sigma'}$ and $Y \in \mathrm{Teich}(S)$, then we have*

(1). there exists $\epsilon = \epsilon(X, Y, \phi) > 0$ which depends on X, Y and ϕ such that for any simple closed curve $\beta \notin \sigma^0 \cup \sigma'^0$ we have

$$\inf_{Z\in g(X,\phi^n\circ Y)} \ell_\beta(Z) > \epsilon.$$

for all $n \geq 1$.

(2). Any limit of geodesics $g(X, \phi^n \circ Y)$ in the sense of theorem 3.4 goes to a stratum T_{σ_1} where $\sigma_1^0 \subset \sigma^0$.

(3). There exists X_n^i on the geodesic $g(X, \phi^n \circ Y)$ such that

$$\lim_{n\to+\infty} dist(X_n^i, T_{\alpha_i}) = 0$$

for all $1 \leq i \leq k+1$.

PROOF. *Proof of (1):* Let β be a simple closed curve in S satisfying $\beta \notin \sigma^0 \cup \sigma'^0$. If β is disjoint with $\sigma^0 \cup \sigma'^0$, then $\ell_\beta(\phi^n \circ Y) = \ell_\beta(Y)$. We can always find another simple closed curve β' such that $\beta' \notin \sigma^0 \cup \sigma'^0$ and β' intersects with β. By theorem 4.2 in chapter 2 we have

$$\ell_\beta(Z) \leq max\{\ell_\beta(X), \ell_\beta(Y)\}, \quad \ell_{\beta'}(Z) \leq max\{\ell_{\beta'}(X), \ell_{\beta'}(Y)\}$$

for all $Z \in g(X, \phi^n \circ Y)$. From the Collar Lemma, there exists a positive ϵ_1 depending on X, Y, ϕ such that

$$\ell_\beta(Z) \geq \epsilon_1.$$

If β intersects at least one of $\sigma^0 \cup \sigma'^0$. Since σ and σ' are disjoint, ϕ fixes $\sigma^0 \cup \sigma'^0$ pointwise, for all $\alpha \in \sigma^0 \cup \sigma'^0$, $\ell_\alpha(\phi^n \circ Y) = \ell_\alpha(Y)$. From theorem 4.2 we have

$$\ell_\alpha(Z) \leq max\{\ell_\alpha(X), \ell_\alpha(Y)\}$$

for all $Z \in g(X, \phi^n \circ Y)$. By the Collar Lemma, there exists a positive ϵ_2 depending on X, Y, ϕ such that

$$\ell_\beta(Z) \geq \epsilon_2.$$

We choose $\epsilon = min\{\epsilon_1, \epsilon_2\}$.

Proof of (2): Assume that the conclusion is incorrect.

At first by part 1), any limit of geodesics $g(X, \phi^n \circ Y)$ in the sense of theorem 3.4 goes to a stratum T_η, where either η is empty or $\eta^0 \subset \sigma^0 \cup \sigma'^0$. We exclude the empty case for the following reason. It is similar to the proof of part (1) in the proposition above. By theorem 3.4, passing to a subsequence, the geodesics $g(X, \phi^n \circ Y)$ converge to a geodesic in Teich(S). In particular, $\{\phi^n \circ Y\}$, passing to a subsequence, converges to a point in Teich(S), which is impossible since ϕ has infinite order and Mod(S) acts properly discontinuously on Teich(S).

We denote the point on which the limit first meets the strata by Z. If η is not a subsimplex of σ. By part (1), there exists a simple closed curve $\beta \in \eta^0 \cap \sigma'^0$. Since $g(0) = X \in T_{\sigma'} \subset T_\beta$, by theorem 4.4, $g([0, t_1]) \subset T_\beta$. Hence, $g(X, Z) \subset T_\beta$, the initial tangent vector of $g(Z, X)$ vanish in the component $\mathbb{R}^\beta_{\geq 0}$. Since ϕ is reducible, it is not hard to see that $k \geq 2$, otherwise it contradicts lemma 3.3. Since the piecewise geodesic g is the unique minimal length path in $\overline{\text{Teich}(S)}$ joining $g(0)$ to $g(L)$ intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_k}$ in order, the piecewise geodesic $g(X, Z) \cup g(Z, g(t_2))$ is the minimal length path in $\overline{\text{Teich}(S)}$ joining X and $g(t_2)$ and intersecting the stratum T_η. From theorem 2.3 we know that the initial tangent vector of $g(Z, X)$ and $g(Z, g(t_2))$ are equal in the components $\mathbb{R}^{|\eta|}_{\geq 0}$, so the initial tangent vector of $g(Z, g(t_2))$ vanish in the component $\mathbb{R}^\beta_{\geq 0}$. By theorem 2.2, we have that the geodesic $g(Z, g(t_2))$ is contained in T_β. In particular $g(t_2) \in T_\beta$. By induction on t_i, we have

$$g(t_i) \in T_\beta, \quad \text{for all} \quad i = 1, 2, \cdots, k.$$

In particular $g(t_k) \in T_\beta$.

On the other hand we claim $g(t_k) \in \text{Teich}(S)$.

If the claim is correct, we get a contradiction since T_β does not have any intersection with $\text{Teich}(S)$.

Proof of the claim: From part (1) we know that $\sigma_i \subset \sigma \cup \sigma'$ for all $i = 1, \cdots, k$. So $\tau_{k-1,n} \circ \cdots \circ \tau_{1,n}$ is a product of Dehn-twists about the curves in $\sigma \cup \sigma'$. Since $g_n(t_k) = \phi^n \circ Y$, we have

$$\lim_{n \to +\infty} dist(\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ (\phi^n \circ Y), g(t_k)) = 0.$$

Since ϕ^n is a product of Dehn-twists about the curves in σ, $\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ \phi^n$ is a product of Dehn-twists about the curves in $\sigma \cup \sigma'$. If we project both $\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ (\phi^n \circ Y)$ and $g(t_k)$ onto the moduli space of S, we get

$$\lim_{n \to +\infty} dist_{\mathbb{M}_S}(\pi(\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ (\phi^n \circ Y)), \pi(g(t_k))) = 0$$

where π is the quotient map from $\text{Teich}(S)$ onto $\mathbb{M}_S$ and $dist_{\mathbb{M}_S}$ is the path-metric on the moduli space. Since $\pi(\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ (\phi^n \circ Y)) = \pi(Y)$,

$$dist_{\mathbb{M}_S}(\pi(Y), \pi(g(t_k))) = 0$$

which means that $g(t_k)$ is pre-image of Y, which is in $\text{Teich}(S)$.

Proof of (3): For any $\epsilon > 0$, let $X^\epsilon \in \text{Teich}(S)$ such that $dist(X^\epsilon, X) = \epsilon$. From proposition 3.5 we know that, for each $\alpha_i \in \sigma^0$, there exists $X_n^{i,\epsilon}$ such that

$$\lim_{n \to +\infty} dist(X_n^{i,\epsilon}, T_{\alpha_i}) = 0.$$

Since $\overline{\text{Teich}(S)}$ is a $CAT(0)$ space, from proposition 2.2 of Chapter II.2 in [**10**] we know that there exists $X_n^i \in g(X, \phi^n \circ Y)$ such that $dist(X_n^i, X_n^{i,\epsilon}) \le \max\{dist(X^\epsilon, X), dist(\phi^n \circ Y, \phi^n \circ Y)\} = \epsilon$. Hence,

$$dist(X_n^i, T_{\alpha_i}) \le dist(X_n^{i,\epsilon}, T_{\alpha_i}) + dist(X_n^{i,\epsilon}, X_n^i) \le dist(X_n^{i,\epsilon}, T_{\alpha_i}) + \epsilon$$

Taking the superior limit,

$$\limsup_{n\to+\infty} dist(X_n^i, T_{\alpha_i}) \leq \epsilon.$$

Since ϵ is arbitrary,

$$\lim_{n\to+\infty} dist(X_n^i, T_{\alpha_i}) = 0.$$

$\square$

Let $X, Y \in \text{Teich}(S)$ and τ_α be the Dehn-twist about α, where α is a simple closed curve. Lemma 4.1 in [**12**] gives a nice description about the limit of geodesics $g(X, \tau_\alpha^n \circ Y)$ in the sense of Wolpert's convergence. For completeness we restate it here. From proposition 3.5, any limit of the geodesics $g(X, \tau_\alpha^n \circ Y)$ goes to the stratum T_α. Let $g(t_1) \in T_\alpha$ in theorem 3.4. Then, up to a subsequence, we have, for all t

$$\lim_{n\to+\infty} dist(g(X, \tau_\alpha^n \circ Y)(t), g(X, g(t_1)) \cup \tau_\alpha^n \circ g(g(t_1), Y)(t)) = 0$$

where the piecewise geodesic segment $g(X, g(t_1)) \cup \tau_\alpha^n \circ g(g(t_1), Y)(t)$ also uses arc-length parameter.

The following result generalizes this observation into multi-twists.

Proposition 3.7. *Let σ be an m-simplex, $\sigma^0 = \{\alpha_1, \cdots, \alpha_{m+1}\}$, and τ_i be the Dehn-twist about the curve α_i, for $i = 1, 2, \cdots, m+1$. Let $\phi = \prod_{1\leq i\leq m+1} \tau_i \in \text{Mod}(S)$, $X, Y \in \text{Teich}(S)$, and g_n be the unit speed geodesics $g(X, \phi^n \circ Y)$. Then, after passing to a subsequence, there exists a positive number L, an associated partition $0 = t_0 < t_1 < \cdots < t_k = L$, simplices $\sigma_0, \cdots, \sigma_k$, and a piecewise geodesic*

$$g : [0, L] \to \overline{\text{Teich}(S)}$$

with the following properties.

(1). $\sigma_i^0 \subset \sigma^0$, $\sigma_i^0 \cap \sigma_j^0$ is empty for $i \neq j$.

(2). $\sigma^0 = \bigcup_{i=1}^k \sigma_i^0$.

(3). $g(t_i) \in T_{\sigma_i}$, $i = 1, \cdots, k-1$, $g(0) = X, g(t_k) = Y$.

(4). There are elements $\tau_{i,n} \in Tw(\sigma_i)$, for $i = 1, \cdots, k-1$ so that $g_n[0, t_1]$ converges in $\overline{\text{Teich}(S)}$ to the restriction $g([0, t_1])$, and for each $i = 1, \cdots, k-1$ and $t \in [t_i, t_{i+1}]$

$$\lim_{n \to +\infty} dist(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ (g_n(t)), g(t)) = 0.$$

In particular, $\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} = \phi^{-n}$ when n is big enough.

The piecewise geodesic g is the unique minimal length path in $\overline{\text{Teich}(S)}$ joining $g(0)$ to $g(L)$ and intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}}$ in order.

PROOF. Choose a point $W \in T_\sigma$. $\phi \circ W = W$, hence

$$\begin{aligned} Length(g(X, \phi^n \circ Y)) &\leq dist(X, W) + dist(W, \phi^n \circ Y) \\ &= dist(X, W) + dist(W, Y) < +\infty. \end{aligned}$$

By theorem 3.4, after passing to a subsequence, there exists $t_1 > 0$ such that

(1). $g(X, \phi^n \circ Y)([0, t_1])$ converges to a geodesic $g([0, t_1])$ in $\overline{\text{Teich}(S)}$,

(2). there exists a simplex σ_1 such that $g(t_1) \in T_{\sigma_1}$.

From part (2) of proposition 3.5 we know that σ_1 is a subsimplex of σ, i.e., $\sigma_1^0 \subset \sigma^0$. Set

$$\tau_{1,n} = \prod_{\gamma \in \sigma_1^0} \tau_\gamma^n.$$

Hence, the piecewise geodesic $g(X, g(t_1)) \cup g(g(t_1), \phi^n \circ Y)$ and the geodesic $g(X, \phi^n \circ Y)$ coincide with each other as n goes to infinity. Since $g(t_1) \in T_{\sigma_1}$,

$$g(g(t_1), \phi^n \circ Y) = \tau_{1,n} \circ g(g(t_1), \Pi_{\gamma \in (\sigma^0 - \sigma_1^0)} \tau_\gamma^n \circ Y).$$

Let σ_1' be a simplex with vertices $\sigma_1'^0 = \sigma - \sigma_1$, so σ_1 is disjoint with σ_1'. The geodesics $g(g(t_1), \Pi_{\gamma \in (\sigma^0 - \sigma_1^0)} \tau_\gamma^n \circ Y)$ satisfy the conditions of proposition 3.6. Hence, by proposition 3.6 there exists a $t_2 > 0$ and a simplex σ_2 such that

(1). $g(g(t_1), \Pi_{\gamma \in (\sigma^0 - \sigma_1^0)} \tau_\gamma^n \circ Y)$ converges to a geodesic $g([t_1, t_2])$ in $\overline{\text{Teich}(S)}$,

(2). $g(t_2) \in \sigma_2 \subset \sigma_1'$ which indicates that $\sigma_2 \subset \sigma$ and $\sigma_1 \cap \sigma_2 = \Phi$.

Set

$$\tau_{2,n} = \prod_{\gamma\in\sigma_2^0} \tau_\gamma^n.$$

The piecewise geodesic $g(g(t_1), g(t_2)) \cup g(g(t_2), \Pi_{\gamma\in(\sigma_1'^0)}\tau_\gamma^n \circ Y)$ and $g(X, \Pi_{\gamma\in(\sigma_1'^0)}\tau_\gamma^n \circ Y)$ coincide with each other as n goes to infinity. Since the piecewise geodesic $g(X, g(t_1)) \cup g(g(t_1), \phi^n \circ Y)$ and the geodesic $g(X, \phi^n \circ Y)$ coincide with each other as n goes to infinity, we have, for all $t \in [t_1, t_2]$,

$$\lim_{n\to+\infty} dist(\tau_{2,n} \circ \tau_{1,n} \circ (g_n(t)), g(t)) = 0.$$

Continuing this process, by theorem 3.4, after finitely many steps, we get a sequence of simplexes $\{\sigma_i\}_{i=1,\cdots,k}$ and a sequence of positive numbers $\{t_i\}_{i=1,\cdots,k}$ such that

(1). $\sigma_i^0 \subset \sigma^0$, $\sigma_i^0 \cap \sigma_j^0$ is empty for $i \neq j$.

(2). $\sigma^0 = \bigcup_{i=1}^k \sigma_i^0$.

(3). $g(t_i) \in T_{\sigma_i}$, $i = 1, \cdots, k$, $g(0) = X, g(t_k) = Y$.

(4). There are elements $\tau_{i,n} \in Tw(\sigma_i)$, for $i = 1, \cdots, k-1$ so that $g_n[0, t_1]$ converges in $\overline{\text{Teich}(S)}$ to the restriction $g([0, t_1])$, and for each $i = 1, \cdots, k-1$ and $t \in [t_i, t_{i+1}]$,

$$\lim_{n\to+\infty} dist(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ (g_n(t)), g(t)) = 0.$$

Since $g(t_k) = Y$,

$$\lim_{n\to+\infty} dist(\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ \phi^n \circ Y, Y) = 0.$$

This only happens when $\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ \phi^n$ is always identity after some time.

It follows from theorem 3.4 that the piecewise geodesic g is the unique minimal length path in $\overline{\text{Teich}(S)}$ joining $g(0)$ to $g(L)$ and intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}}$ in order. □

4. Teichmüller-Coxeter development and geodesic limits

In [**71**], Yamada constructed Teichmüller-Coxeter complex $D(\overline{\text{Teich}(S)}, \iota)$ by introducing an infinite Coxeter reflection group and gluing infinite copies of $\overline{\text{Teich}(S)}$

through the strata. In this section, we apply $D(\overline{\mathrm{Teich}(S)}, \iota)$ to studying geodesic limits.

To each simplex σ, we associate a formal reflection group W_σ with one reflection generator ω_α for each simple closed curve $\alpha \in \sigma^0$, with $\omega_\alpha^2 = id$ and commuting generators. For an inclusion of simplices $\sigma \subset \tau$, associate the natural injective homomorphism $\psi_{\tau\sigma} : W_\sigma \to W_\tau$ satisfying

$$\psi_{\tau\rho} = \psi_{\tau\sigma}\psi_{\sigma\rho} \quad for \quad \rho \subset \sigma \subset \tau.$$

The system of groups and monomorphisms $\{W_\sigma, \psi_{\tau\sigma}\}$ has a direct limit $\hat{W}$, called the *Coxeter group of curves*. The injectivity of the homomorphisms ensures that the homomorphisms: $i_\sigma : W_\sigma \to \hat{W}$ are injective. The *Teichmüller-Coxeter development* $D(\overline{\mathrm{Teich}(S)}, \iota)$ is the quotient of $\hat{W} \times \overline{\mathrm{Teich}(S)}$ by the equivalence relation

$$(\omega, Y) \sim (\omega', Y') \;\; provided \;\; Y = Y' \;\; and \;\; \omega^{-1}\omega' \in W_{\sigma(Y)}$$

where $\sigma(Y)$ is the simplex of null lengths for the surface Y. One can refer to [**69, 71**] for details.

The following theorem is proved in [**71**].

Theorem 4.1 (Yamada). *1): $D(\overline{\mathrm{Teich}(S)}, \iota)$ is a complete CAT(0) space. In particular, for any two points $(\omega_1, y_1), (\omega_2, y_2) \in D(\overline{\mathrm{Teich}(S)}, \iota)$ there exists a unique geodesic joining (ω_1, y_1) and (ω_2, y_2).*

2): Let σ be a simplex and $Z \in T_\sigma$. The Alexandrov tangent cone at $(1, Z)$ in $D(\overline{\mathrm{Teich}(S)}, \iota)$ is the vector space $\mathbb{R}^{|\sigma|} \times T_Z T_\sigma$.

Let σ be a simplex and Z be a point in T_σ. By the definition of $D(\overline{\mathrm{Teich}(S)}, \iota)$, $(1, Z) = (\Pi_{\alpha\in\sigma^0}\omega_\alpha, Z)$ since $\Pi_{\alpha\in\sigma^0}\omega_\alpha \in W_{\sigma(Z)}$. The following lemma gives a another viewpoint for proposition 2.3 in the Teichmüller-Coxeter development.

Lemma 4.2. *Given a simplex σ, and X, Y be two points in $\overline{\mathrm{Teich}(S)}$. Let Z be a point in T_σ such that the piecewise geodesic $g(X, Z) \cup g(Z, Y)$ is the minimal length path in $\overline{\mathrm{Teich}(S)}$ joining X and Y and intersecting the stratum T_σ. Then, the* *piecewise geodesic* $g((1,X),(1,Z)) \cup g((\omega,Z),(\omega,Y))$ *is a global geodesic segment in the Teichmüller-Coxeter development* $D(\overline{\mathrm{Teich}(S)},\iota)$*, where* $\omega = \Pi_{\alpha\in\sigma^0}\omega_\alpha$.

PROOF. Since the Alexandrov tangent cone of X is a vector space, it is sufficient to show that the direction of the geodesic $g((1,Z),(1,X))$ at $(1,Z)$ is opposite to the direction of $g((\omega,Z),(\omega,Y)$ at (ω,Z). By proposition 2.3 the directions of the geodesics $g((Z,X))$ and $g(Z,Y)$ in $\overline{\mathrm{Teich}(S)}$at Z satisfies

(a). the initial tangents of $g(Z,X)$ and $g(Z,Y)$ are equal in the components $\mathbb{R}^{|\sigma|}_{\geq 0}$.

(b). The sum of the initial tangents of $g(Z,X)$ and $g(Z,Y)$ has vanishing projection into the subcone $T_X T_\sigma$.

From the construction of the Teichmüller-Coxeter development $D(\overline{\mathrm{Teich}(S)},\iota)$ (see [**71**]) we know that the direction of the geodesic $g((\omega,Z),(\omega,Y))$ in $D(\overline{\mathrm{Teich}(S)},\iota)$ satisfies

(1). the initial tangents of $g((\omega,Z),(\omega,Y))$ and $g((1,Z),(1,Y))$ are opposite in the components $\mathbb{R}^{|\sigma|}$.

(2). The sum of the initial tangents of $g((\omega,Z),(\omega,Y))$ and $g((1,Z),(1,Y))$ has vanishing projection into the subcone $T_X T_\sigma$.

(1) and (2) says exactly that the direction of the geodesic $g((1,Z),(1,X))$ at $(1,Z)$ is opposite to the direction of $g((\omega,Z),(\omega,Y))$ at (ω,Z).

□

The following proposition gives a another viewpoint for proposition 3.7 in the Teichmüller-Coxeter development.

PROPOSITION 4.3. *Let* σ *be an* m*-simplex,* $\sigma^0 = \{\alpha_1,\cdots,\alpha_{m+1}\}$*, and* τ_i *be the Dehn-twist about the curve* α_i *for* $i = 1,2,\cdots,m+1$*. Let* $\phi = \prod_{1\leq i\leq m+1}\tau_i \in \mathrm{Mod}(S)$*,* $X, Y \in \mathrm{Teich}(S)$*, and* g_n *be the unit speed geodesics* $g(X,\phi^n \circ Y)$*. Then, for any limit of* $g(X,\phi^n\circ Y)$ *in the sense of proposition 3.7, we have there exists a positive number* L*, an associated partition* $0 = t_0 < t_1 < \cdots < t_k = L$*, simplices* $\sigma_0,\cdots,\sigma_k$*, and a piecewise geodesic*

$$g : [0,L] \to \overline{\mathrm{Teich}(S)}$$

such that the piecewise geodesic

$$\bigcup_{j=0}^{k-2} g((\prod_{i=0}^{j} \Pi_{\alpha\in\sigma_i}\omega_\alpha, g(t_j)), (\prod_{i=0}^{j+1} \Pi_{\alpha\in\sigma_i}\omega_\alpha, g(t_{j+1})))$$

is a global geodesic joining $(1, X)$ *and* $(\prod_{i=0}^{k-1} \Pi_{\alpha\in\sigma_i}\omega_\alpha, g(t_{j+1}))$ *passing through the points* $(\prod_{i=0}^{j+1} \Pi_{\alpha\in\sigma_i}\omega_\alpha, g(t_k)), j = 1, \cdots, k-2$.

PROOF. Combine proposition 3.7 and lemma 4.2. ☐

Now we are ready to show that the sequence of the directions of the geodesics $g(X, \phi^n \circ Y)$ is convergent in $V_X(S)$ if ϕ is a multi-twists.

THEOREM 4.4. *Let* σ *be an* m*-simplex,* $\sigma^0 = \{\alpha_1, \cdots, \alpha_{m+1}\}$*, and* τ_i *be the Dehn-twist about the curve* α_i *for* $i = 1, 2, \cdots, m+1$*. Let* $\phi = \prod_{1\le i\le m+1} \tau_i \in \mathrm{Mod}(S)$*,* $X, Y \in \mathrm{Teich}(S)$*, and* g_n *be the unit speed geodesics* $g(X, \phi^n \circ Y)$*. Then, there exists a positive number* t_1*, a subsimplex* σ_1 *of* σ*, and a geodesic* $g : [0, t_1] \to \overline{\mathrm{Teich}(S)}$ *such that* $g(t_1) \in T_{\sigma_1}$ *and* $g_n[0, t_1]$ *converges to* $g([0, t_1])$ *as* n *goes to infinity. In particular, the sequence of the directions of the geodesics* $g(X, \phi^n \circ Y)$ *is convergent in* $V_X(S)$*.*

PROOF. It is sufficient to show that the sequence of the directions of the geodesics $g(X, \phi^n \circ Y)$ is convergent in $V_X(S)$. Assume that the sequence of the directions of the geodesics $g(X, \phi^n \circ Y)$ is not convergent in $V_X(S)$. Since the direction at X in $\mathrm{Teich}(S)$ is S^{6g-7} which is compact, there exist two subsequences $\{g(X, \phi^{n_k^1} \circ Y)\}_{k\ge1}$ and $\{g(X, \phi^{n_k^2} \circ Y)\}_{k\ge1}$ of $\{g(X, \phi^n \circ Y)\}_{n\ge1}$ such that the limits in $V_X(S)$ are different.

We first consider the sequence of geodesics $\{g(X, \phi^{n_k^1} \circ Y)\}_{n\ge1}$. From proposition 3.7, after passing to a subsequence of $\{n_k^1\}_{k\ge1}$(we still denote it by $\{n_k^1\}_{k\ge1}$), there exists a positive number L_1, an associated partition $0 = t_0 < t_1 < \cdots < t_k = L_1$, simplices $\sigma_0, \cdots, \sigma_k$, and a piecewise geodesic

$$g^1 : [0, L] \to \overline{\mathrm{Teich}(S)}$$

with the following properties.

(1). $\sigma_i^0 \subset \sigma^0$, $\sigma_i^0 \cap \sigma_j^0$ is empty for $i \ne j$.

(2). $\sigma^0 = \bigcup_{i=1}^k \sigma_i^0$.

(3). $g^1(t_i) \in T_{\sigma_i}$, $i = 1, \cdots, k-1$, $g^1(0) = X, g^1(t_k) = Y$.

(4). There are elements $\tau_{i,n} \in Tw(\sigma_i)$, for $i = 1, \cdots, k-1$, so that $g_n[0, t_1]$ converges in $\overline{\text{Teich}(S)}$ to the restriction $g^1([0, t_1])$, and for each $i = 1, \cdots, k-1$ and $t \in [t_i, t_{i+1}]$,

$$\lim_{n \to +\infty} dist(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ (g_n(t)), g^1(t)) = 0.$$

In particular, $\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} = \phi^{-n}$ when n is big enough.

The piecewise geodesic g^1 is the unique minimal length path in $\overline{\text{Teich}(S)}$ joining $g^1(0)$ to $g^1(L_1)$ and intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}}$ in order.

Since $\sigma_i^0 \cap \sigma_j^0$ is empty for $i \neq j$ and $\sigma^0 = \bigcup_{i=1}^k \sigma_i^0$,

$$\prod_{j=0}^{k-1} (\prod_{\gamma \in \sigma_i^0} \omega_\gamma) = \prod_{\alpha \in \sigma^0} \omega_\alpha.$$

From proposition 4.3 and the equation above we know that there exist positive numbers t_1, L_1 and a geodesic $l_1 : [0, L_1] \to D(\overline{\text{Teich}(S)}, \iota)$ satisfying

(1). $l_1(0) = (1, X)$, $l_1(L_1) = (\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$.

(2). The wall l_1 first crosses is T_{σ_1}, and the point in T_{σ_1} is $g^1(t_1) = l_1(t_1)$, where we identify $(1, \overline{\text{Teich}(S)})$ with $\overline{\text{Teich}(S)}$.

We make the same argument as above for geodesics $\{g(X, \phi^{n_k^2} \circ Y)\}_{k \geq 1}$, after passing to a subsequence of $\{n_k^2\}_{k \geq 1}$(we still denote it by $\{n_k^2\}_{k \geq 1}$): there exist positive numbers s_1, L_2, a subsimplex β_1 of σ, and a geodesic $l_2 : [0, L_2] \to D(\overline{\text{Teich}(S)}, \iota)$ satisfying

(1). $l_2(0) = (1, X)$, $l_2(L_2) = (\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$.

(2): The wall l_2 first crosses is T_{β_1}, and the point in T_{β_1} is $l_2(s_1)$, where we also identify $(1, \overline{\text{Teich}(S)})$ with $\overline{\text{Teich}(S)}$.

Since the limit of the directions of the geodesics $\{g(X, \phi^{n_k^1} \circ Y)\}_{k \geq 1}$ and $\{g(X, \phi^{n_k^2} \circ Y)\}_{k \geq 1}$ is different in $V_X(S)$, $l_1(t_1) \neq l_2(s_1)$. On the other hand, both l_1 and l_2 are geodesics joining $(1, X)$ and $(\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$. Hence they coincide with each other, which contradicts the fact that they cross different points on the wall for the first time.

□

REMARK 4.1. Using the same argument as in the proof of theorem 4.4, the result is still true if X belongs to a stratum T_η where η is disjoint with σ.

Now we are ready to prove theorems 1.2, 1.3 and 1.4.

PROOF OF THEOREM 1.2. From theorem 5.3 in chapter 2, we know that ϕ is in one of the four cases: semi-simple with either $|\phi| = 0$ or $|\phi| > 0$, or reducible with either $|\phi| = 0$ or $|\phi| > 0$.

Case I: ϕ is semi-simple and $\delta = L_{WP}(\phi) = 0$

From theorem 5.3 in chapter 2 we know that there exists an integer k with $\phi^k = id$. Hence, $\lim_{n\to+\infty} g(X, \phi^{kn} \circ Y) = g(X, Y)$. So the limit of the sequence of the directions of the geodesics $g(X, \phi^{kn} \circ Y)$ is the direction of $g(X, Y)$.

Case II: ϕ is semi-simple and $\delta = L_{WP}(\phi) > 0$

By theorem 5.3 in chapter 2, there exists a unique bi-infinite Weil-Petersson geodesic γ such that $\phi \circ \gamma(t) = \gamma(t + \delta)$ for all $t \in \mathbb{R}$. Consider the following triangles $\Delta(X, \phi^n \circ Y, \phi^n \circ \gamma(0))$ which are geodesic triangles in Teich(S) with vertices $\{X, \phi^n \circ Y, \phi^n \circ \gamma(0)\}$. Since Mod($S$) acts on Teich($S$) by isometries, $dist(\phi^n \circ Y, \phi^n \circ \gamma(0)) = dist(Y, \gamma(0))$. On the other hand

$$\begin{aligned} dist(X, \phi^n \circ \gamma(0)) &\geq dist(\gamma(0), \phi^n \circ \gamma(0)) - dist(X, \phi^n \circ \gamma(0)) \\ &= n \cdot \delta - dist(X, \phi^n \circ \gamma(0)). \end{aligned}$$

Hence $dist(X, \phi^n \circ \gamma(0))$ goes to $+\infty$ as n goes to $+\infty$. From the standard argument in CAT(0) geometry (see [**10**]) we know that the limit of the sequence of the directions of $g(X, \phi^n \circ \gamma(0))$ exists in the visual sphere at X. Since $dist(\phi^n(Y), \phi^n \circ \gamma(0))$ is bounded above and $\lim dist(X, \phi^n \circ \gamma(0)) = +\infty$, the limit of the sequence of the directions of $g(X, \phi^n \circ Y)$ exists, and equals the limit of the sequence of the directions of $g(X, \phi^n \circ \gamma(0))$ in $V_X(S)$.

Case III: ϕ is reducible and $\delta = L_{WP}(\phi) > 0$

By theorem 5.3 in chapter 2, there exists an integer k and a bi-infinite Weil-Petersson geodesic $\gamma(t) \subset \overline{\text{Teich}(S)}$ such that for all t $\phi^k \circ \gamma(t) = \gamma(t + k\delta)$. We consider the triangles $\Delta(X, \phi^{kn}(Y), \phi^{kn}(\gamma(0)))$, which are geodesic triangles in $\overline{\text{Teich}(S)}$ with vertices $\{X, \phi^{kn}(Y), \phi^{kn}(\gamma(0))\}$. By the same argument as in Case II the limit of the sequence of the directions of $g(X, \phi^{kn} \circ Y)$ exists, and equals the limit of the sequence of the directions of $g(X, \phi^{kn} \circ \gamma(0))$ in $V_X(S)$.

Case IV: ϕ is reducible and $\delta = L_{WP}(\phi) = 0$

By proposition 5.4 in chapter 2 there exists an integer k such that ϕ^k is the product of Dehn-Twists about disjoint closed curves, i.e., there exists a positive integer n_0 and the Dehn-Twists τ_i about simple closed curve γ_i such that $\phi^k = \prod_{1 \leq i \leq n_0} \tau_i$. From theorem 4.4 we know that the limit of the sequence of the directions of $g(X, \phi^{kn} \circ Y)$ exists as $n \to +\infty$. □

Proof of theorem 1.3. At first it follows from theorem 4.4 that there exists a positive number t_1, a subsimplex σ_1 of σ, and a geodesic $g : [0, t_1] \to \overline{\text{Teich}(S)}$ such that $g(t_1) \in T_{\sigma_1}$ and $g_n([0, t_1])$ converges to $g([0, t_1])$ as n goes to infinity. Considering the geodesics $g(g(t_1), \prod_{\alpha \in \sigma^0 - \sigma_1^0} \tau_\alpha^n \circ Y)$, it follows from remark 4.1 that there exists a positive number t_2', a subsimplex σ_2 of σ, and a geodesic $g, : [0, t_2'] \to \overline{\text{Teich}(S)}$ such that

(1). $\sigma_2 \cap \sigma_1 =$ empty.

(2). $g'(0) = g(t_1)$, $g'(t_2') \in T_{\sigma_2}$ and $g(g(t_1), \prod_{\alpha \in \sigma^0 - \sigma_1^0} \tau_\alpha^n \circ Y)([0, t_2'])$ converges to $g'([0, t_2'])$ as n goes to infinity.

Since $g(g(t_1), \phi^n \circ Y) = \prod_{\alpha \in \sigma_1^0} \tau_\alpha^n \circ g(g(t_1), \prod_{\alpha \in \sigma^0 - \sigma_1^0} \tau_\alpha^n \circ Y)$, setting $\prod_{\alpha \in \sigma_1^0} \tau_\alpha^{-n} = \tau_{1,n}$, we have

$$\lim_{n \to +\infty} dist(\tau_{1,n} \circ g(g(t_1), \phi^n \circ Y)(t + t_1), g'(t)) = 0, \quad for \;\; t \in [0, t_2'].$$

Setting $t_2 = t_1 + t_2'$ and $g(t) = g'(t + t_1)$, we get a piecewise geodesic $g(X, g(t_1)) \cup g(g(t_1), g(t_2))$ satisfying

(1). $\sigma_i^0 \subset \sigma^0$ for $i = 1, 2$, $\sigma_1^0 \cap \sigma_2^0$ is empty,

(2). $g(t_i) \in T_{\sigma_i}$, $i = 1, 2$, $g(0) = X$,

(3). $g_n([0, t_1])$ converges in $\overline{\text{Teich}(S)}$ to the restriction $g([0, t_1])$, and

$$\lim_{n \to +\infty} dist(\tau_{1,n} \circ g_n(t), g(t)) = 0, \quad for \quad t \in [t_1, t_2],$$

where $\tau_{1,n} = \prod_{\alpha \in \sigma_1} \tau_\alpha^{-n}$.

If we only consider the geodesics $g(X, \phi^n \circ Y)[0, t_2]$, it follows from theorem 3.4 that the piecewise geodesic g is the unique minimal length path in $\overline{\text{Teich}(S)}$ joining $g(0)$ to $g(t_2)$ and intersecting the closures of the stratum T_{σ_1}.

Continuing this process, it ends after finitely many steps since $|\sigma| = k < +\infty$. Parts (1), (2), (3), (4) and (5) of the conclusion follow after finitely steps.

From proposition 4.3 we know that the geodesic joining $(1, X)$ and $(\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$ in the Teichmüller-Coxeter development $D(\overline{\text{Teich}(S)}, \iota)$ passes through $(1, g(t_1))$. $\square$

Remark 4.2. Both theorem 4.1 and theorem 1.3 hold for geodesics $g(X, \phi^{k_n} \circ Y)$ where $\{k_n\}$ is a subsequence of $\{n\}_{n \geq 1}$.

Proof of theorem 1.4. Assume that the conclusion is incorrect.

There exist two subsequences of $\{n\}_{n \geq 1}$ $\{k_n^1\}_{n \geq 1}$ and $\{k_n^2\}_{n \geq 1}$ such that

$$\lim_{n \to +\infty} length(g(X, \phi^{k_n^1} \circ Y)) \neq \lim_{n \to +\infty} length(g(X, \phi^{k_n^2} \circ Y)).$$

By theorem 1.3, $\{g(X, \phi^{k_n^1} \circ Y)\}$ induces a geodesic joining $(1, X)$ and $(\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$ in the Teichmüller-Coxeter development $D(\overline{\text{Teich}(S)}, \iota)$ whose length is the same as $\lim_{n \to +\infty} length(g(X, \phi^{k_n^1} \circ Y))$. Similarly, $\{g(X, \phi^{k_n^2} \circ Y)\}$ also induces a geodesic joining $(1, X)$ and $(\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$ in the Teichmüller-Coxeter development $D(\overline{\text{Teich}(S)}, \iota)$ whose length is the same as $\lim_{n \to +\infty} length(g(X, \phi^{k_n^2} \circ Y))$. Since $D(\overline{\text{Teich}(S)}, \iota)$ is a CAT(0) space, the geodesic joining $(1, X)$ and $(\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$ is unique, in particular

$$\lim_{n \to +\infty} length(g(X, \phi^{k_n^1} \circ Y)) = \lim_{n \to +\infty} length(g(X, \phi^{k_n^2} \circ Y))$$

which is a contradiction. □

5. Geometric limits of geodesics $g(X, \phi^n \circ Y)$

Let M be a CAT(0) space. Given a geodesic ray in M, for any point $p \in M$, there exists a unique ray emanating from p which has finite Hausdorff distance with a given ray (see [**10**]). In [**14**], the authors use the Gauss-Bonnet formula and ruled surface technique to show that two equivalent rays in Teich(S) are strongly asymptotic to each other if at least one of them is recurrent, where a geodesic ray $c : [0, +\infty) \to \text{Teich}(S)$ is recurrent if there exists a positive number ϵ and a sequence $\{t_n\}$ with $t_n \to +\infty$ such that $c(t_n) \in \text{Teich}(S)_{\geq\epsilon}$, where

$$\text{Teich}(S)_{\geq\epsilon} = \{X \in \text{Teich}(S) : \ell_\alpha(X) \geq \epsilon, \quad \text{for all simple closed curve} \quad \alpha\}.$$

Proposition 5.1. *Let $c : [0, +\infty) \to \text{Teich}(S)$ be a recurrent ray and $X \in \overline{\text{Teich}(S)}$. Then the unique ray $c' : [0, +\infty) \to \overline{\text{Teich}(S)}$ emanating from X with finite Hausdorff distance from $c(t)$ is strongly asymptotic to $c(t)$.*

Proof. Let $g : [0, dist(X, c(0))] \to \overline{\text{Teich}(S)}$ be the geodesic joining X and $c(0)$ with unite speed. Since $c(0) \in \text{Teich}(S)$, by theorem 4.4 in chapter 2, $g(0, dist(X, c(0))] \subset \text{Teich}(S)$.

Assume the conclusion is incorrect.

There exists a positive number δ such that for all $t \geq 0$, we have the distance between $c(t)$ and $c'(\mathbb{R}^{\geq 0})$ is greater than δ. Choose a point $g(\frac{\delta}{2})$ and consider the geodesic ray $c'' : [0, +\infty) \to \text{Teich}(S)$ emanating from $g(\frac{\delta}{2})$ with finite Hausdorff distance from $c(t)$. Since $g(\frac{\delta}{2}) \in \text{Teich}(S)$, the distance between $c(t)$ and $c''(\mathbb{R}^{\geq 0})$ goes to zero as t goes to infinity (see theorem 4.1 in [**14**]). Since the distance function between two convex sets is convex (see [**10**]), the distance between $c'(t)$ and $c''(\mathbb{R}^{\geq 0})$ is less than $dist(c''(0), c'(0)) = \frac{\delta}{2}$ if t is big enough. Hence, when t is big enough, the distance between $c(t)$ and $c'(\mathbb{R}^{\geq 0})$ is less than $\frac{\delta}{2}$ which contradicts the assumption. □

Proposition 5.2. *Let $c : [0, +\infty) \to \text{Teich}(S)$ be a recurrent ray. Then, for any simple closed curve α in S,*

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty.$$

Proof. By theorem 4.5 in chapter 2, $\lim_{t\to+\infty} \ell_\alpha(c(t))$ is either infinite or finite. Assume the conclusion is incorrect. There exists a number $C \geq 0$ such that $\lim_{t\to+\infty} \ell_\alpha(c(t)) = C$. It is not hard to see that the second derivative of ℓ_α satisfies

$$\lim_{t \to +\infty} \ell''_\alpha(c(t)) = 0.$$

On the other hand, given an $\epsilon > 0$, there exists a number $c > 0$ such that at each t for which $c(t) \in \text{Teich}(S)_{\geq \epsilon}$ we have $\lim_{t\to+\infty} \ell''_\alpha(c(t)) \geq c \cdot \ell_\alpha(c(t)) \geq c\epsilon > 0$ (see [**66**]). Since $c(t)$ is recurrent, there exists a positive number ϵ and a sequence $\{t_n\}$ with $t_n \to +\infty$ such that $r(t_n) \in \text{Teich}(S)_{\geq\epsilon}$. Hence,

$$\lim_{t \to +\infty} \ell''_\alpha(c(t)) \neq 0,$$

which is a contradiction. □

Now we are ready to prove theorem 1.5.

Proof of theorem 1.5. Since ϕ is pseudo-Anosov, by theorem 5.3 in chapter 2 there exists a bi-infinite unit speed Weil-Petersson geodesic $r : (-\infty, +\infty) \to \text{Teich}(S)$ such that $\phi^n \circ r(0) = r(n \cdot |\phi|)$. In particular $r(t)$ is recurrent.

Since the limit of the geodesics $\{g(X, r(n \cdot |\phi|))\}$ is a geodesic ray $c : [0, +\infty) \to \text{Teich}(S)$ satisfying $c(0) = X$ and $c(t)$ is asymptotic to the geodesic ray $r[0, +\infty)$ (see [**10**]). Since $\text{Mod}(S)$ acts on $\text{Teich}(S)$ by isometries,

$$dist(\phi^n \circ Y, r(n \cdot |\phi|)) = dist(\phi^n \circ Y, \phi^n \circ r(0)) = dist(Y, r(0)) < +\infty.$$

Hence, the geodesics $\{g(X, \phi^n \circ Y)\}$ and $\{g(X, r(n \cdot |\phi|))\}$ have the same limit ray $c : [0, +\infty) \to \text{Teich}(S)$, which is asymptotic to the geodesic ray $r[0, +\infty)$ (see [**10**]). Since $r(t)$ is recurrent, by theorem 4.1 in [**14**] $c : [0, +\infty) \to \text{Teich}(S)$ is strongly asymptotic to $r[0, +\infty)$. This implies that $c([0, +\infty))$ is also recurrent. By proposition 5.2 we have, for any simple closed curve α in S, $\lim_{t\to+\infty} \ell_\alpha(c(t)) = +\infty$. □

Recall that ϕ is called reducible if there exists a collection of mutually disjoint simple closed curves II such that $\phi(II) = II$. Thurston's classification takes the following form (see [**24**]).

THEOREM 5.3 (Thurston). *Any reducible mapping class $\phi \in \mathrm{Mod}(S)$ determines a maximal collection of simple closed curves $\{\alpha_i\}$ and a maximal collection of proper subsurfaces $\{PS_j\}$ of S such that there exists a positive integer k such that*

$$\phi^k = (\prod_i \tau_{\alpha_i}) \cdot (\prod_j \phi_j)$$

where τ_i is the Dehn-twist about α_i and $\phi_j = \phi^k|_{PS_j}$ is pseudo-Anosov on PS_j.

Since the translation length of multi Dehn-twists in $\overline{\mathrm{Teich}(S)}$ is zero, the pseudo-Anosov part is not empty if $|\phi| > 0$ ($|\phi^k| = |k||\phi|$). Let σ be a simplex such that $\sigma^0 = (\cup_i \alpha_i) \cup (\cup_j \cup_{\beta \in \partial(PS_j)} \beta)$, where $\partial(PS_j)$ is the boundary of PS_j. The stratum T_σ is a product of low dimensional Teichmüller spaces $\prod T' \times \prod_j T''_j$ with ϕ^k fixing the factors, acting by a product of : the identity on T' and pseudo-Anosov elements ϕ_j on T''_j with axis c_j.

REMARK 5.1. If $c : (-\infty, +\infty) \to \mathrm{Teich}(S)$ is an axis for a pseudo-Anosov mapping class, there exists a positive number ϵ such that $c(R) \subset Teich_{\geq \epsilon}$. By theorem 1.5 in chapter 2, for any simple closed curve α we have $\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty$.

REMARK 5.2. Let $\phi \in \mathrm{Mod}(S)$ be a reducible mapping class with $|\phi| > 0$ and $c : (-\infty, +\infty) \to \overline{\mathrm{Teich}(S)}$ be an axis for ϕ^k. By theorem 5.3, it is not hard to see that the projection of $c(R)$ onto T''_j is the geodesic line which is the axis for ϕ_j on T''_j. Adding the remark above, for any non-peripheral essential simple closed curve α in PS_j we also have $\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty$.

Firstly let us consider the case that ϕ does not have a twist part. That is, there exists a positive integer k and a maximal collection of proper subsurfaces $\{PS_j\}$ of S such that $\phi^k = \prod_j \phi_j$ where $\phi_j = \phi^k|_{PS_j}$ is pseudo-Anosov on PS_j.

Given a simplex σ, the distance from a point $X \in \mathrm{Teich}(S)$ to the stratum T_σ is estimated in terms of the geodesic length sum

$$\ell = \sum_{\alpha \in \sigma^0} \ell_\alpha(X)$$

of the lengths of simple closed curves in σ^0 on X by

$$dist(X, T_\sigma) = \sqrt{2\pi\ell} + O(\ell^2) \tag{11}$$

(see([**65**], cor.21)).

Then we have the following

PROPOSITION 5.4. *Let $\phi \in \mathrm{Mod}(S)$ be reducible and k be a positive number such that $\phi^k = \prod_j \phi_j$, where $\phi_j = \phi^k|_{PS_j}$ is pseudo-Anosov on PS_j. Then for any $X, Y \in$* Teich(S)*, the geodesics $g(X, \phi^{kn} \circ Y)$ converge to a geodesic ray $c([0, +\infty))$ in* Teich(S) *with $c(0) = X$ as $n \to +\infty$.*

Moreover, (1). for any non-peripheral essential simple closed curve α in PS_j we have

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty.$$

(2). There exists a positive number ϵ_0 such that for any non-peripheral essential simple closed curve β in the complement $(S - \cup_j PS_j)$ of $\cup_j PS_j$ we have, for all $t \geq 0$,

$$\frac{1}{\epsilon_0} \geq \ell_\beta(c(t)) \geq \epsilon_0.$$

(3). There exists a positive number ϵ_1 such that for any non-peripheral essential simple closed curve γ which intersects with at least one of $\cup_j \cup_{\alpha \in \partial(PS_j)} \alpha$ we have, for all $t \geq 0$,

$$\ell_\gamma(c(t)) \geq \epsilon_1.$$

PROOF. Proof of (1): Let σ be a simplex with $\sigma^0 = \cup_j(\cup_{\beta \in \partial(PS_j)}\beta)$ and $r : (-\infty, +\infty) \to T_\sigma$ be the axis for ϕ^k with $\phi^k \circ r(t) = r(t + k|\phi|)$ for all $t \in \mathbb{R}$. Since $\mathrm{Mod}(S)$ acts on $\overline{\mathrm{Teich}(S)}$ by isometries,

$$dist(\phi^{kn} \circ Y, r(kn|\phi|)) = dist(\phi^{kn} \circ Y, \phi^{kn} \circ r(0)) = dist(Y, c(0)) < +\infty.$$

So the limit of the geodesics $g(X, \phi^{kn} \circ Y)$ is the same as the limit of the geodesics $g(X, r(kn|\phi|))$ as $n \to +\infty$, which is the unique ray emanating from X with finite Hausdorff distance to the ray $r(\mathbb{R}^{\geq 0})$ (see [**10**]). We denote this ray by $c([0, +\infty))$. By theorem 4.4 in chapter 2, $c([0, +\infty))$ is contained in Teich(S).

Assume there exists a non-peripheral essential simple closed curve α in PS_j for some j such that $\lim_{t\to+\infty} \ell_\alpha(c(t)) \neq +\infty$. Since the length of a simple closed curve along Weil-Petersson geodesic is convex, there exists a number $C_1 \geq 0$ such that $\lim_{t\to+\infty} \ell_\alpha(c(t)) = C_1 < +\infty$ and ℓ_α is decreasing along $c(\mathbb{R}^{\geq 0})$. From equation (11) there exists a positive number C_2 such that, for all $t \geq 0$, we have

$$dist(c(t), T_\alpha) \leq C_2.$$

Let $\{Z_i\}$ be a sequence of points in T_α with $dist(c(i), Z_i) \leq C_2 + 1$. So there exists a positive number C_3 such that for any positive integer i,

$$dist(r(i), Z_i) \leq C_3.$$

Hence the geodesics $g(r(0), Z_i)$ converge to $r(\mathbb{R}^{\geq 0})$ (see [**10**]). By theorem 4.5 in chapter 2 we have

$$\max_{Z \in g(r(0), Z_i)} \ell_\alpha(Z) \leq \max\{\ell_\alpha(r(0)), \ell_\alpha(Z_i)\} = \ell_\alpha(r(0)).$$

Since ℓ_α is continuous on $\overline{\text{Teich}(S)}$, taking the limit, we have for all $t \geq 0$,

$$\ell_\alpha(r(t)) \leq \ell_\alpha(r(0)).$$

On the other hand, from remark 5.2, we should have

$$\ell_\alpha(r(t)) = +\infty,$$

which is a contradiction.

Proof of (2): For any non-peripheral essential simple closed curve β in the complement $(S - \cup_j PS_j)$ of $\cup_j PS_j$, there exists another non-peripheral essential simple closed curve β' in the complement of $\cup_j PS_j$ such that β' intersects with β. Since ϕ^k acts as identity on $(S - \cup_j PS_j)$, for all $t \geq 0$,

$$\ell_\beta(r(t)) = \ell_\beta(r(0)), \quad \ell_{\beta'}(r(t)) = \ell_{\beta'}(r(0)).$$

Since $c(\mathbb{R}^{\geq 0})$ is the limit of the geodesics $g(X, r(i))$ as $i \to +\infty$, by theorem 4.5 in chapter 2 we have

$$\max_{Z \in g(X,r(i))} \ell_{\beta'}(Z) \leq \max\{\ell_{\beta'}(X), \ell_{\beta'}(r(i))\} = \max\{\ell_{\beta'}(X), \ell_{\beta'}(r(0))\}.$$

After taking the limit, the conclusion follows from the Collar Lemma.

Proof of (3): Let $\alpha_0 \in \cup_j \cup_{\alpha \in \partial(PS_j)} \alpha$ such that γ intersects with α_0. Since $\ell_{\alpha_0}(r(i)) = 0$, by theorem 4.5 in chapter 2 we have

$$\max_{Z \in g(X,r(i))} \ell_{\alpha_0}(Z) \leq \max\{\ell_{\alpha_0}(X), \ell_{\alpha_0}(r(i))\} = \ell_\alpha(X),$$

Since $c(\mathbb{R}^{\geq 0})$ is the limit of the geodesics $g(X, r(i))$ as $i \to +\infty$, after taking the limit, the conclusion follows from the Collar Lemma. □

The non-refraction property implies that the interior of any geodesic in $\overline{\text{Teich}(S)}$ is contained in one single stratum. In [**65**] the flat subspaces of $\overline{\text{Teich}(S)}$ are well studied. As an application Wolpert gives a new proof for Brock-Farb's theorem (see [**13**]) that $\overline{\text{Teich}(S)}$ is in general not Gromov-hyperbolic. Recall that a metric space is called Gromov-hyperbolic if there exists a positive number δ such that for each geodesic triangle the $\delta-$neighborhood of any two sides contains the third side(see [**10**]). We say a geodesic triangle Δ is *flat* if Δ is isometric to a triangle in two-dimensional Euclidean space $\mathbb{R}^2$.

PROPOSITION 5.5 (see [**65**], prop. 16). *Let Δ be a flat geodesic triangle in $\overline{\text{Teich}(S)}$. Then there exists a simplex σ such that the interior of Δ is contained in the stratum T_σ. Moreover, the projection of Δ to each component Teichmüller space of T_σ is a point or a geodesic segment.*

LEMMA 5.6. *Let $\phi \in \text{Mod}(S)$ be reducible and k be an integer such that $\phi^k = \prod_j \phi_j$ where $\phi_j = \phi^k|_{PS_j}$ are pseudo-Anosov on PS_j, and let σ be a simplex with*

$\sigma^0 = \cup_j(\cup_{\beta\in\partial(PS_j)}\beta)$ and $r : (-\infty, +\infty) \to T_\sigma$ be the axis for ϕ^k such that for all t $\phi^k \circ r(t) = r(t + k|\phi|)$. Then there does not exist any flat geodesic triangle Δ in $\overline{\text{Teich}(S)}$ whose three vertices X, Y, Z satisfy $X = r(0), Y = r(|k\phi|)$ and $Z \in T_\rho$, where ρ is a simplex satisfying $\rho^0 \subset \cup_j PS_j$ and $\rho \neq \sigma$.

PROOF. Assume there exists a flat geodesic triangle Δ in $\overline{\text{Teich}(S)}$ whose three vertices X, Y, Z satisfy $X = r(0), Y = r(|k\phi|)$ and $Z \in T_\rho$ where ρ is a simplex satisfying $\rho^0 \subset \cup_j PS_j$ and $\rho \neq \sigma$.

Since $g(X, Y)$ is a geodesic segment of the axis $r(\mathbb{R})$, $g(X, Y) \subset T_\sigma$. Let Z_0 be the midpoint of $g(X, Y)$. By theorem 4.4 in chapter 2 the interior of the geodesic $g(Z, Z_0)$ $int(g(Z, Z_0)) \subset T_{\sigma\cap\rho}$. It follows from proposition 5.5 that the interior of Δ is contained in $T_{\sigma\cap\rho}$. Since $\rho^0 \subset \cup_j PS_j$, either ρ^0 has nonempty intersection with the interior of $\cup_j PS_j$ or ρ^0 is a subset of the boundary of $\cup_j PS_j$.

Case I): there exists a simple closed curve $\gamma \in \rho^0$ but $\gamma \notin \cup_j\partial(PS_j)$. Since $\rho^0 \subset \cup_j PS_j$, without loss of generality, assume that $\gamma \subset PS_1$. Since $T_\sigma \subset \overline{T_{\sigma\cap\rho}}$, there exists a component T of $T_{\sigma\cap\rho}$ such that $\text{Teich}(PS_1) \subset \overline{T}$.

Firstly since $Z \in T_\rho \subset \overline{T_\gamma}$ and $\gamma \subset PS_1$, the projection Z' of Z onto T is contained in $T_\gamma \cap \overline{T}$.

Secondly, it is not hard to see that $\phi^k|_T = \prod_{j;PS_j\subset T}\phi_j$. Since $g(X, Y)$ is a geodesic segment on the axis of ϕ^k and $\phi^k|_{PS_j}$ is pseudo-Anosov, the projection of $g(X, Y)$ onto T is a geodesic segment on the axis of $\phi^k|_T$ in $\overline{T}$. Since $\phi^k|_{PS_1}$ is pseudo-Anosov, the projection of the axis of $\phi^k|_T$ onto $\text{Teich}(PS_1)$ is a geodesic segment on the axis of $\phi^k|_{PS_1}$. From theorem 5.3 in chapter 2 we know that the axis of $\phi^k|_{PS_1}$ is contained in $\text{Teich}(PS_1)$. Since $\gamma \subset PS_1$, the axis of $\phi^k|_T$ does not intersect $\overline{T_\gamma}$. From lemma 5.6 we know that the projection of Δ onto T is a geodesic. Let X', Y' and Z' be the projection of X, Y and Z onto $\overline{T}$. Hence we get a geodesic segment $g(X', Y') \cup g(Y', Z')$ in $\overline{\text{Teich}(S)}$ satisfying $Y' \in \overline{T_\gamma}$ and $X', Y' \notin \overline{T_\gamma}$. On the other hand, since the geodesic in $\overline{\text{Teich}(S)}$ does not have refraction (see theorem 4.4 in chapter 2), $g(X', Y') \cup g(Y', Z')$ is not a geodesic segment in $\overline{\text{Teich}(S)}$, which is a contradiction.

Case II): ρ is a proper subsimplex of σ. There exists a simple closed curve $\gamma \in (\sigma^0 - \rho^0)$ such that $\ell_\gamma(Z) \neq 0$. Let T be a component of $T_{\sigma\cap\rho} = T_\rho$ containing γ. It is easy to see that the projection Z' of Z onto $\overline{T}$ has $\ell_\gamma(Z') > 0$. Since $\phi^k|_T = \prod_{j;PS_j\subset T}\phi_j$ and $g(X,Y)$ is a geodesic segment on the axis of ϕ^k in $\overline{\mathrm{Teich}(S)}$, it is not hard to see that the projection $g(X',Y')$ of $g(X,Y)$ onto $\overline{T}$ is a geodesic segment which is contained in $\cap_{j;PS_j\subset T}\cap_{\alpha\in\partial(PS_j)}\overline{T_\alpha}$. Since $\gamma \subset T$, we have $g(X',Y') \subset T_\gamma$. By proposition 5.5 we get a geodesic segment $g(X',Y')\cup g(Y',Z')$ in $\overline{\mathrm{Teich}(S)}$ satisfying $Z' \notin \overline{T_\gamma}$ and $X',Y' \in \overline{T_\gamma}$. On the other hand, by the non-fraction property of geodesic (see theorem 4.4 in chapter 2), $g(X',Y')\cup g(Y',Z')$ is not a geodesic in $\overline{\mathrm{Teich}(S)}$, which is a contradiction. □

The following lemma is basic in CAT(0) geometry.

LEMMA 5.7. *Let M be a CAT-(0) space and $r_i : [0,1] \to M$ be two different geodesics with unit speed, i=1,2. If $dist(r_1(t), r_2(t))$ is a constant for all $t \in [0,1]$, then the convex hull of $r_1([0,1]) \cup r_2([0,1])$ is isometric to a parallelogram in two-dimensional Euclidean space $\mathbb{R}^2$.*

PROOF. Exercise for the reader. □

Before proving theorem 1.6, we show the following weaker statement.

THEOREM 5.8. *Let $\phi \in \mathrm{Mod}(S)$ be reducible and k be a positive number such that $\phi^k = \prod_j \phi_j$ where $\phi_j = \phi^k|_{PS_j}$ is pseudo-Anosov on PS_j. Then for any $X, Y \in \mathrm{Teich}(S)$, the limit geodesic ray $c : [0,+\infty) \to \mathrm{Teich}(S)$ of geodesics $g(X, \phi^{kn}\circ Y)$ in proposition 5.4 satisfies the property that for any simple closed curve $\alpha \in \partial(\cup_j PS_j)$, we have*

$$\lim_{t\to+\infty} \ell_\alpha(c(t)) = 0.$$

PROOF. Firstly we claim that there exists a simple closed curve $\alpha \in \partial(\cup_j PS_j)$ such that

$$\lim_{t\to+\infty} \ell_\alpha(c(t)) = 0.$$

Supposing not, by proposition 5.4 there exists a positive number ϵ such that $c(\mathbb{R}^{\geq 0})$ lies in $\text{Teich}(S)_{\geq\epsilon}$. In particular $c(\mathbb{R}^{\geq 0})$ is recurrent. Let $r : (-\infty, +\infty) \to \overline{\text{Teich}(S)}$ be an axis for ϕ^k. Since the Hausdorff distance between $c(\mathbb{R}^{\geq 0})$ and $r(\mathbb{R}^{\geq 0})$ is finite, by proposition 5.1 $r(\mathbb{R}^{\geq 0})$ is strongly asymptotic to $c(\mathbb{R}^{\geq 0})$. In particular $c(\mathbb{R}^{\geq 0}) \subset \text{Teich}(S)_{\geq\epsilon}$, which is a contradiction because $r(\mathbb{R}^{\geq 0})$ is contained in a stratum.

Assume there exists a simple closed curve $\beta \in \partial(\cup_j PS_j)$ such that

$$\lim_{t\to+\infty} \ell_\beta(c(t)) > 0.$$

Since ℓ_β is convex on $\text{Teich}(S)$, by part (2) of proposition 5.4, there exists a positive number C such that $\lim_{t\to+\infty} \ell_\beta(c(t)) = C$. Since $\ell_\beta(r(t)) \equiv 0$, there exists a positive number C_1 such that

$$\lim_{t\to+\infty} dist(r(t), c([0, +\infty))) = C_1 > 0. \tag{12}$$

Consider a sequence of quadrilaterals $\{\Lambda_n\}$, where Λ_n is a quadrilatera whose vertices are $\{r(kn|\phi|), r(k(n+1)|\phi|), c(k(n+1)|\phi|), c(kn|\phi|)\}$. From equation (12) and lemma 5.7 we know that $\{\Lambda_n\}$ will converge to a parallelogram which is isometric to a flat parallelogram in two-dimensional Euclidean space $\mathbb{R}^2$. The pulled-back quadrilaterals $\Lambda'_n = \phi^{-kn} \circ \Lambda_n$ have a common edge $g(r(0), r(k|\phi|))$. We consider a sequence of geodesics $\phi^{-kn} \circ g(r(kn|\phi|), c(kn|\phi|)) = g(r(0), \phi^{-kn} \circ c(kn|\phi|))$. From equation (12) we know that

$$\lim_{n\to+\infty} length(g(r(0), \phi^{-kn} \circ c(kn|\phi|))) = C_1. \tag{13}$$

We denote the geodesics $g(r(0), \phi^{-kn} \circ c(kn|\phi|))$ by g_n. Let σ be a simplex with $\sigma^0 = \partial(\cup_j PS_j)$. From Wolpert's Compactness theorem (theorem 3.4), after passing to a subsequence of $\{g_n\}$, there exists a positive number t_1, a point Z_1, a simplex σ_1, and a sequence of product Dehn-twists $\tau_n \in Tw(\sigma^0 - \sigma^0 \cap \sigma_1^0)$ such that $Z_1 \in T_{\sigma_1}$ and $\tau_n \circ g(r(0), g_n(t_1))$ converges in $\overline{\text{Teich}(S)}$ to the geodesic $g(r(0), Z_1)$. In particular $\tau_n \circ g_n(t_1) \to Z_1$ as $n \to +\infty$. Since $\{\Lambda_n\}$ converges to a flat parallelogram, so does $\{\Lambda'_n\}$. Hence the sequence of geodesic triangles with vertices $\{r(0), r(k|\phi|), \tau_n \circ g_n(t_1)\}$

converges to a geodesic triangle which is isometric to a flat triangle in $\mathbb{R}^2$ (may be a singular triangle, i.e., a geodesic segment!).

Claim: $\sigma_1^0 \subset \cup_j PS_j$ **and** $\sigma_1 \neq \sigma$**.**

If the claim is correct, the conclusion follows since we get a contradiction with lemma 5.6.

Proof of Claim: Firstly we show $\sigma_1 \neq \sigma$. If not, we get $k = 1$ in Wolpert's compactness theorem. So τ_n is trivial, $t_1 = C_1$, and $\phi^{-kn} \circ c(kn|\phi|)$ converges to $Z_1 \in T_\sigma$. On the other hand, since ϕ fixes β, $\ell_\beta(\phi^{-kn} \circ c(kn|\phi|)) = \ell_\beta(c(kn|\phi|))$, by our assumption we have $\lim_{n\to+\infty} \ell_\beta(c(kn|\phi|)) > 0$. Hence,

$$\lim_{t\to+\infty} \ell_\beta(Z_1) > 0,$$

which contradicts with $Z_1 \in T_\sigma$.

Secondly, we show that $\sigma_1^0 \subset \cup_j PS_j$. For any simple closed curve $\alpha \notin \cup_j PS_j$, either γ is a non-peripheral essential simple closed curve in the complement of $\cup_j PS_j$ or γ is a non-peripheral essential simple closed curve intersecting $\cup_j \partial(PS_j)$.

Assuming γ is non-peripheral essential in the complement of $\cup_j PS_j$, there exists another non-peripheral essential simple closed curve γ' which is also in the complement of $\cup_j PS_j$. Since ϕ fixes γ', by proposition 5.4, $\ell_{\gamma'}(\phi^{-kn} \circ c(kn|\phi|)) = \ell_{\gamma'}(c(kn|\phi|)) \leq \frac{1}{\epsilon}$. From Wolpert's convexity theorem (theorem 4.5) we have

$$\begin{aligned}\max_{Z\in g(r(0),\phi^{-kn}\circ c(kn|\phi|))} \ell_{\gamma'}(Z) &\leq \max\{\ell_{\gamma'}(r(0)), \ell_{\gamma'}(\phi^{-kn} \circ c(kn|\phi|))\} \\ &\leq \max\{\ell_{\gamma'}(r(0)), \frac{1}{\epsilon}\} < +\infty.\end{aligned}$$

Since γ intersects γ', it follows from the Collar Lemma that there exists a positive number ϵ_1 such that

$$\max_{Z\in g(r(0),\phi^{-kn}\circ c(kn|\phi|))} \ell_{\gamma'}(Z) \geq \epsilon_1$$

for all non-peripheral essential simple closed curves in the complement of $\cup_j PS_j$.

Assume that γ is a non-peripheral essential intersecting with $\cup_j \partial(PS_j)$. Let $\alpha \in \cup_j \partial(PS_j)$ such that α intersects γ. Since ϕ fixes α, $\ell_\alpha(\phi^{-kn} \circ c(kn|\phi|)) = \ell_\alpha(c(kn|\phi|))$

by Wolpert's convexity theorem (theorem 4.5) we have

$$\begin{aligned}\max_{Z\in g(r(0),\phi^{-kn}\circ c(kn|\phi|))}\ell_\alpha(Z) &\le \max\{\ell_\alpha(r(0)),\ell_\alpha(c(kn|\phi|))\}\\ &= \ell_\alpha(c(kn|\phi|)).\end{aligned}$$

Using the same argument as in the proof of part (3) in proposition 5.4 we have

$$\max_{Z\in g(r(0),\phi^{-kn}\circ c(kn|\phi|))}\ell_\alpha(Z)\le \ell_\alpha(c(kn|\phi|))\le \ell_\alpha(X)<+\infty.$$

Since γ intersects with α, it follows from the Collar Lemma that there exists a positive number ϵ_2 such that

$$\max_{Z\in g(r(0),\phi^{-kn}\circ c(kn|\phi|))}\ell_{\gamma'}(Z)\ge\epsilon_2$$

for all non-peripheral essential simple closed curves intersecting $\cup_j\partial(PS_j)$.

Hence each simple closed curve along geodesics $g(r(0),\phi^{-kn}\circ c(kn|\phi|))$ pinching to zero length is contained in $\cup_j PS_j$. In particular we have $\sigma_1^0\subset\cup_j PS_j$.

□

Now we are ready to prove theorem 1.6.

Proof of theorem 1.6. Proof of (1): Since $|\phi|>0$, by theorem 5.3 in chapter 2 there exists a unit speed geodesic line $r:(-\infty,+\infty)\to\overline{\mathrm{Teich}(S)}$ such that $\phi^n\circ r(0)=r(n|\phi|)$ for all integers n. Since $\mathrm{Mod}(S)$ acts on $\overline{\mathrm{Teich}(S)}$ by isometries, $dist(\phi^n\circ Y,\phi^n\circ r(0))=dist(Y,r(0))<+\infty$. Hence $g(X,\phi^n\circ Y)$ converges to a unique geodesic ray emanating from X which has finite Hausdorff distance to $r(\mathbb{R}^{\ge0})$ (see [**10**]). From theorem 4.4 in chapter 2 we know that $c(\mathbb{R}^{\ge0})\subset\mathrm{Teich}(S)$.

Proof of (2): Since $g(X,\phi^n\circ Y)$ is convergent, $g(X,\phi^{kn}\circ Y)$ also converges to $c:[0,+\infty)\to\mathrm{Teich}(S)$. Setting $\phi'=\prod_j\phi_j$, we have

$$dist(\phi^{kn}\circ Y,\phi'^n\circ Y)=dist(\prod_{\alpha\in\sigma^0}\tau_\alpha^n\circ Y,Y).$$

By theorem 1.4, there exists a positive number C such that

$$dist(\phi^{kn}\circ Y,\phi'^n\circ Y)\le C.$$

So the limit of the geodesics $g(X, \phi^n \circ Y)$ is the same as the limit of the geodesics $g(X, \phi'^n \circ Y)$, which is also $c : [0, +\infty) \to \text{Teich}(S)$. By theorem 5.8 we have

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = 0,$$

for any simple closed curve $\alpha \in \partial(\cup_j PS_j)$.

Proof of (3): Since $c : [0, +\infty) \to \text{Teich}(S)$ is also the limit of the geodesics $g(X, \phi'^n \circ Y)$, by proposition 5.4 there exists positive number ϵ_0 such that

$$\ell_\beta(c(t)) \geq \epsilon_0$$

for any non-peripheral essential simple closed curve $\beta \notin \partial(\cup_j PS_j)$.

□

CHAPTER 5

The Riemannian sectional curvature operator of the Weil-Petersson Metric annd its applications

1. Introduction

Let S be a closed surface of genus g, where $g > 1$, and $\mathbb{T}_S$ be the Teichmüller space of S. $\mathbb{T}_S$ carries two important metrics. One is the Teichmüller metric which is a complete Finsler metric. The other is the Weil-Petersson metric, which is an incomplete Kähler metric [**2, 62**]. We let Teich(S) denote $\mathbb{T}_S$ endowed with the Weil-Petersson metric. For Weil-Petersson geometry one can refer to Wolpert's recent nice book [**69**].

The curvature of Teich(S) has been studied over the past several decades. Ahlfors in [**2**] showed the holomorphic sectional curvatures are negative. Tromba [**57**] and Wolpert [**63**] independently showed the sectional curvatures are negative, and in [**63**] a curvature formula is established to prove Royden'conjecture which says that the holomorphic curvatures are bounded above by a negative number which only depends on the topology of the surface. Schumacher [**52**] used Wolpert's curvature formula to show that Teich(S) has strongly negative curvature in the sense of Siu [**54**] which is stronger than negative sectional curvature. Huang [**32**] showed that there is no negative upper bound for the sectional curvature. Let $X \in$ Teich(S) and Δ be the Beltrami-Laplace operator on X. Recently Liu-Sun-Yau [**41**] also used Wolpert's curvature formula and the positivity of the Green function of the operator $(\Delta-2)^{-1}$ to show that Teich(S) has dual Nakano negative curvature, which says that the complex curvature operator on the dual tangent bundle is positive in some sense. For some other related problems one can refer to [**13, 32, 33, 42, 43, 56, 65, 68**].

One of our purposes is to present some explicit formulas in different subspaces of the exterior wedge of the tangent space of Teich(S) to understand the Riemannian sectional curvature operator of Teich(S). The method in this paper is highly influenced by the methods in [**41, 52, 63**].

Let X be a point in Teich(S), and $T_X\text{Teich}(S)$ be the tangent space at this point. Let $\{\mu_i\}_{i=1}^{3g-3}$ be the harmonic Beltrami differentials which form a basis of $T_X\text{Teich}(S)$, and $\frac{\partial}{\partial t_i}$ be the vector fields corresponding to μ_i. Locally t_i is a holomorphic coordinate, and we set $t_i = x_i + \mathbf{i}y_i$. Let Q be the Riemannian curvature operator of Teich(S) which is defined in section 3.

Our first result in this chapter is the nonpositivity of Q.

Theorem 1.1. *Let Q be the curvature operator on $\wedge^2 T_X\text{Teich}(S)$. Then, Q is nonpositive definite, and $Q(A,A) = 0$ if and only if there exists an element B in $\wedge^2 T_X\text{Teich}(S)$ such that $A = B - \boldsymbol{J}\circ B$, where $\boldsymbol{J}$ is the almost complex structure on* Teich(S) *and defined in section 4.*

A direct corollary is that the sectional curvature of Teich(S) is negative [**2, 57, 63**]. Normally a metric of negative curvature may not also have nonpositive curvature operator (see [**3**]).

In the second part of this paper we will study harmonic maps from rank-one spaces into Teich(S). For harmonic maps, there are a lot of very beautiful results in the case when the target is a manifold with nonpositive curvature or nonpositive curvature operator (see [**17, 18, 39, 47, 70**]). For harmonic maps into Teich(S), one can refer to the nice survey [**19**]. In this paper we establish the following rigid result.

Theorem 1.2. *Let Γ be a lattice in a semisimple Lie group G which is either $Sp(m,1)$ or F_4^{-20}, and* Mod(S) *be the mapping class group of* Teich(S)*. Then, any twist harmonic map f from G/Γ into* Teich(S) *with respect to each homomorphism $\rho : \Gamma \to \text{Mod}(S)$ must be a constant.*

2. Preliminaries

Let $X \in \text{Teich}(S)$ and $D = -2(\Delta - 2)^{-1}$ where Δ is the Beltrami-Laplace operator on X, hence we have $D^{-1} = -\frac{1}{2}(\Delta - 2)$. The following property has been proved in many literatures. For completeness, we still state the proof here.

PROPOSITION 2.1. *Let D be the operator above. Then*
(1). D is self-adjoint.
(2). D is positive.

PROOF OF (1). Let f and g be two real-valued smooth functions on X, and $u = Df$, $v = Dg$. Then

$$\begin{aligned}\int_S Df \cdot g dA &= \int_S u \cdot (-\frac{1}{2}(\Delta - 2)v) dA = -\frac{1}{2}\int_S u \cdot (\Delta - 2) v dA \\ &= -\frac{1}{2}\int_S v \cdot (\Delta - 2) u dA = \int_S Dg \cdot f dA\end{aligned}$$

where the equality in the second row follows from the fact that Δ is self-adjoint on closed surfaces. For the case that f and g are complex-valued, we can prove it for the real and imaginary part by using the same argument.

Proof of (2): Let f be a real-valued smooth functions on X, and $u = Df$. Then

$$\begin{aligned}\int_S Df \cdot f dA &= \int_S u \cdot (-\frac{1}{2}(\Delta - 2)u) dA = -\frac{1}{2}\int_S (u \cdot (\Delta u) - 2u^2) dA \\ &= \frac{1}{2}(\int_S |\nabla u|^2 + 2u^2 dA) \geq 0,\end{aligned}$$

where the equality in the second row follows from the Stoke's Theorem. The last equality holds if and only if $u = 0$, i.e., D is positive. For the case that f is complex-valued, we also show it for the real and imaginary part. □

For the Green function of the operator $-2(\Delta - 2)^{-1}$, we have

PROPOSITION 2.2. *Let D be the operator above. Then there exists a Green function $G(w, z)$ for D satisfying:*

(1). $G(w,z)$ is positive.

(2). $G(w,z)$ is symmetric, i.e, $G(w,z)=G(z,w)$.

Proof. One can refer to [**51**] and [**63**]. □

The Riemannian tensor of the Weil-Petersson metric. The curvature tensor is given by the following. Let μ_α, μ_β be two elements in the tangent space at X, and

$$g_{\alpha\overline{\beta}} = \int_X \mu_\alpha \cdot \overline{\mu_\beta} dA$$

where dA is the area element for X.

Let us study the curvature tensor in these local coordinates. First of all, for the inverse of $(g_{i\bar{j}})$, we use the convention

$$g^{i\bar{j}} g_{k\bar{j}} = \delta_{ik}.$$

The curvature tensor is given by

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2}{\partial t^k \partial \overline{t^l}} g_{i\bar{j}} - g^{s\bar{t}} \frac{\partial}{\partial t^k} g_{i\bar{t}} \frac{\partial}{\partial \overline{t^l}} g_{s\bar{j}}.$$

Since Ahlfors showed that the first derivatives of the metric tensor vanish at the base point X in these coordinates, at X we have

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2}{\partial t^k \partial \overline{t^l}} g_{i\bar{j}}. \tag{14}$$

By the same argument in Kähler geometry we have

Proposition 2.3. *For any indices i, j, k, l, we have*

$$(1). R_{ij\overline{kl}} = R_{\overline{ij}kl} = 0,$$

$$(2). R_{i\bar{j}k\bar{l}} = -R_{i\bar{j}\bar{l}k},$$

$$(3). R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}},$$

$$(4). R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}}.$$

Proof. These follow from formula (14) and the first Bianchi identity, one can refer to [**38**]. □

Now let us state Wolpert's curvature formula, which is crucial in this chapter.

Theorem 2.4. *(see [**63**]) The curvature tensor is given by*

$$R_{i\bar{j}k\bar{l}} = \int_X D(\mu_i\mu_{\bar{j}}) \cdot (\mu_k\mu_{\bar{l}})dA + \int_X D(\mu_i\mu_{\bar{l}}) \cdot (\mu_k\mu_{\bar{j}})dA. \tag{15}$$

Definition 2.5. *Let μ_* be elements $\in T_X\mathrm{Teich}(S)$. Set*

$$(i\bar{j}, k\bar{l}) := \int_X D(\mu_i\mu_{\bar{j}}) \cdot (\mu_k\mu_{\bar{l}})dA.$$

From theorem 2.4 and the definition above,

Lemma 2.6. $R_{i\bar{j}k\bar{l}} = (i\bar{j}, k\bar{l}) + (i\bar{l}, k\bar{j})$.

3. Curvature operator on subspaces of $\wedge T_X^2\mathrm{Teich}(S)$

Before we study the curvature operator, let us set some notations. We set $t_i = x_i + \mathbf{i}y_i$. Since $(t_1, t_2, \cdots, t_{3g-3})$ is a local holomorphic coordinate in a neighborhood U of X, $(x_1, x_2, \cdots, x_{3g-3}, y_1, y_2, \cdots, y_{3g-3})$ is a real smooth coordinate in U. Furthermore, we have

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}, \quad \frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \overline{t_i}}).$$

Let $T\mathrm{Teich}(S)$ be the real tangent bundle of $\mathrm{Teich}(S)$. On the bundle $\wedge^2 T\mathrm{Teich}(S)$, the *curvature operator* Q is defined by

$$Q(V_1 \wedge V_2, V_3 \wedge V_4) = R(V_1, V_2, V_3, V_4)$$

and extended linearly, where V_i are real vectors. It is easy to see that Q is a bilinear symmetric form.

Since $(x_1, x_2, \cdots, x_{3g-3}, y_1, y_2, \cdots, y_{3g-3})$ is a real coordinate in U, for any $X \in U$, $T_X\mathrm{Teich}(S) = Span\{\frac{\partial}{\partial x_i}(X), \frac{\partial}{\partial y_j}(X)\}_{1 \leq i,j \leq 3g-3}$. Furthermore,

$$\wedge^2 T\mathrm{Teich}(S) = Span\{\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}\}.$$

Set

$$\wedge^2 T_X^1 \text{Teich}(S) = Span\{\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\},$$
$$\wedge^2 T_X^2 \text{Teich}(S) = Span\{\frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_l}\},$$
$$\wedge^2 T_X^3 \text{Teich}(S) = Span\{\frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}\}.$$

Hence,

$$\wedge^2 T_X \text{Teich}(S) = Span\{\wedge^2 T_X^1 \text{Teich}(S), \wedge^2 T_X^2 \text{Teich}(S), \wedge^2 T_X^3 \text{Teich}(S)\}.$$

3.1. The curvature operator on $\wedge^2 T_X^1 \text{Teich}(S)$. Now we start to prove theorem 3.1, which is influenced by theorem 4.1 in [**41**]. Let $\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ be an element in $\wedge^2 T_X^1 \text{Teich}(S)$, where a_{ij} are real. Set

$$F(z,w) = \sum_{i,j=1}^{3g-3} a_{ij} \mu_i(w) \cdot \overline{\mu_j(z)}.$$

THEOREM 3.1. *Let Q be the curvature operator and $D = -2(\Delta - 2)^{-1}$ where Δ is the Beltrami-Laplace operator on X. G is the Green function of D, and $\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ is an element in $\wedge^2 T_X^1 \text{Teich}(S)$ where a_{ij} are real. Then we have*

$$\begin{aligned} & Q(\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}) \\ = & \int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)}) dA(z) \\ - & 2 \cdot \int_{X \times X} G(z,w) |F(z,w))|^2 dA(w) dA(z) \\ + & 2 \cdot Re\{\int_{X \times X} G(z,w) F(z,w) F(w,z) dA(w) dA(z)\}, \end{aligned}$$

where $F(z,w) = \sum_{i,j=1}^{3g-3} a_{ij} \mu_i(w) \cdot \overline{\mu_j(z)}$.

PROOF. Since $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}$, from proposition 2.3,

$$
\begin{aligned}
& Q(\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}) \\
= & \sum_{i,j,k,l} a_{ij} a_{kl} (R_{i\bar{j}k\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{\bar{i}j\bar{k}l}) \\
= & \sum_{i,j,k,l} a_{ij} a_{kl} (R_{i\bar{j}k\bar{l}} - R_{i\bar{j}l\bar{k}} - R_{j\bar{i}k\bar{l}} + R_{j\bar{i}l\bar{k}}) \\
= & \sum_{i,j,k,l} a_{ij} a_{kl} ((i\bar{j}, k\bar{l}) + (i\bar{l}, k\bar{j}) - (i\bar{j}, l\bar{k}) - (i\bar{k}, l\bar{j}) \\
- & (j\bar{i}, k\bar{l}) - (j\bar{l}, k\bar{i}) + (j\bar{i}, l\bar{k}) + (j\bar{k}, l\bar{i})) \quad \text{(by lemma 2.6)} \\
= & \sum_{i,j,k,l} a_{ij} a_{kl} (i\bar{j} - j\bar{i}, k\bar{l} - l\bar{k}) \\
+ & \sum_{i,j,k,l} a_{ij} a_{kl} ((i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k})) \\
- & \sum_{i,j,k,l} a_{ij} a_{kl} ((i\bar{k}, l\bar{j}) + (j\bar{l}, k\bar{i})).
\end{aligned}
$$

For the first term, from definition 2.5,

$$
\begin{aligned}
& \sum_{i,j,k,l} a_{ij} a_{kl} (i\bar{j} - j\bar{i}, k\bar{l} - l\bar{k}) \\
= & \int_X D(\sum_{ij} a_{ij} \mu_i \overline{\mu_j} - \sum_{ij} a_{ij} \mu_j \overline{\mu_i})(\sum_{ij} a_{ij} \mu_i \overline{\mu_j} - \sum_{ij} a_{ij} \mu_j \overline{\mu_i}) dA(z) \\
= & \int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)}) dA(z).
\end{aligned}
$$

For the second term, since D is self adjoint, using the Green function G,

$$
\begin{aligned}
& \sum_{i,l} a_{ij} a_{kl} ((i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k})) = 2 \cdot Re\{\sum_{i,l} a_{ij} a_{kl} ((i\bar{l}, k\bar{j})\} \\
= & 2 \cdot Re\{\int_X D(\sum_i a_{ij} \mu_i \overline{\mu_l})(\sum_k a_{kl} \mu_k \overline{\mu_j}) dA(z)\} \\
= & 2 \cdot Re\{\int_X \int_X G(w,z) \sum_i a_{ij} \mu_i(w) \overline{\mu_l(w)} (\sum_k a_{kl} \mu_k(z) \overline{\mu_j}(z)) dA(z) dA(w)\}.
\end{aligned}
$$

From the definition of $F(z,w)$,

$$\begin{aligned}
&\sum_{i,j,k,l} a_{ij}a_{kl}((i\bar{l},k\bar{j})+(l\bar{i},j\bar{k})) \\
= \;& 2\cdot Re\{\int_{X\times X} G(z,w)F(z,w)F(w,z)dA(w)dA(z)\}.
\end{aligned}$$

For the last term, just as with the second term,

$$\begin{aligned}
&\sum_{i,k} a_{ij}a_{kl}((i\bar{k},l\bar{j})+(k\bar{i},j\bar{l})) = 2\cdot Re\{\sum_{i,k} a_{ij}a_{kl}((i\bar{k},l\bar{j})\} \\
= \;& 2\cdot Re\{\int_X D(\sum_i a_{ij}\mu_i\overline{\sum_k a_{kl}\mu_k})(\mu_l\overline{\mu_j})dA(z)\} \\
= \;& 2\cdot Re\{\int_X\int_X G(w,z)\sum_i a_{ij}\mu_i(w)\overline{\sum_k a_{kl}\mu_k(w)}(\mu_l(z)\overline{\mu_j}(z))dA(z)dA(w)\}
\end{aligned}$$

From the definition of $F(z,w)$,

$$\begin{aligned}
&\sum_{i,j,k,l} a_{ij}a_{kl}((i\bar{k},l\bar{j})+(k\bar{i},j\bar{l})) \\
= \;& 2\cdot Re\{\int_{X\times X} G(z,w)F(z,w)\overline{F(z,w)}dA(w)dA(z)\} \\
= \;& 2\cdot\int_{X\times X} G(z,w)|F(z,w)|^2 dA(w)dA(z).
\end{aligned}$$

Combining the three terms above, we get the theorem. □

Using the Green function's positivity and symmetry,

THEOREM 3.2. *Under the conditions of theorem 3.1, Q is negative definite on $\wedge^2 T^1_X\mathrm{Teich}(S)$.*

PROOF. By theorem 3.1 we have

$$\begin{aligned}
&Q(\sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},\sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}) \\
= \;& \int_X D(F(z,z)-\overline{F(z,z)})(F(z,z)-\overline{F(z,z)})dA(z) \\
- \;& 2\cdot(\int_{X\times X} G(z,w)|F(z,w))|^2 dA(w)dA(z) \\
- \;& Re\{\int_{X\times X} G(z,w)F(z,w)F(w,z)dA(w)dA(z)\}).
\end{aligned}$$

For the first term, since $F(z,z) - \overline{F(z,z)} = 2\mathbf{i}Im\{F(z,z)\}$, by the positivity of the operator D,

$$\begin{aligned}
& \int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)})dA(z) \\
= \quad & -4 \cdot \int_X D(Im\{F(z,z)\})(Im\{F(z,z)\})dA(z) \leq 0.
\end{aligned}$$

For the second term, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
& |\int_{X\times X} G(z,w)F(z,w)F(w,z)dA(w)dA(z)| \\
\leq \quad & \int_{X\times X} |G(z,w)F(z,w)F(w,z)|dA(w)dA(z) \\
\leq \quad & \sqrt{\int_{X\times X} |G(z,w)||F(z,w)|^2 dA(w)dA(z)} \\
\times \quad & \sqrt{\int_{X\times X} |G(z,w)||F(w,z)|^2 dA(w)dA(z)} \\
= \quad & \int_{X\times X} G(z,w)|F(z,w)|^2 dA(w)dA(z),
\end{aligned}$$

since G is positive and symmetric.

Combining these three terms we get that Q is nonpositive on $\wedge^2 T^1_X \mathrm{Teich}(S)$.

Furthermore, equality holds precisely when

$$Q(\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}) = 0,$$

that is there exists a constant complex number k such that both of the following hold:

$$\begin{cases} F(z,z) = \overline{F(z,z)}, \\ F(z,w) = k \cdot \overline{F(w,z)}. \end{cases}$$

If we let $z = w$, we get $k = 1$. Hence, the last equation is equivalent to

$$\sum_{ij} (a_{ij} - a_{ji}) \mu_i(w) \overline{\mu_j}(z) = 0.$$

Since $\{\mu_i\}_{i\geq 1}$ is a basis,

$$a_{ij} = a_{ji}.$$

This means $\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = 0$, so Q does not have any non-trivial eigenvectors corresponding to 0. Hence, Q is negative definite on $\wedge^2 T^1_X \mathrm{Teich}(S)$. □

3.2. The curvature operator on $\wedge^2 T^2_X \mathrm{Teich}(S)$. Let b_{ij} be real and $\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} \in \wedge^2 T^2_X \mathrm{Teich}(S)$. Set

$$H(z,w) = \sum_{i,j=1}^{3g-3} b_{ij} \mu_i(w) \cdot \overline{\mu_j(z)}.$$

Using a similar computation as for theorem 3.1, the formula for the curvature operator on $\wedge^2 T^2_X \mathrm{Teich}(S)$ is given as follows.

THEOREM 3.3. *Let Q be the curvature operator and D be the same operator as shown in theorem 3.1. Let $\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T^2_X \mathrm{Teich}(S)$, where b_{ij} are real. Then we have*

$$\begin{aligned} & Q(\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}) \\ = & -\int_X D(H(z,z) + \overline{H(z,z)})(H(z,z) + \overline{H(z,z)}) dA(z) \\ - & \ 2 \cdot \int_{X \times X} G(z,w) |H(z,w))|^2 dA(w) dA(z) \\ - & \ 2 \cdot Re\{\int_{X \times X} G(z,w) H(z,w) H(w,z) dA(w) dA(z), \} \end{aligned}$$

where $H(z,w) = \sum_{i,j=1}^{3g-3} b_{ij} \mu_i(w) \cdot \overline{\mu_j(z)}$.

PROOF. Since $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}$ and $\frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \overline{t_i}})$, from proposition 2.3,

$$\begin{aligned}
& Q(\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}, \sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}) \\
= & -\sum_{i,j,k,l} b_{ij}b_{kl}(R_{i\bar{j}k\bar{l}} - R_{i\bar{j}\bar{k}l} - R_{\bar{i}jk\bar{l}} + R_{\bar{i}j\bar{k}l}) \\
= & -\sum_{i,j,k,l} b_{ij}b_{kl}(R_{i\bar{j}k\bar{l}} + R_{i\bar{j}l\bar{k}} + R_{j\bar{i}k\bar{l}} + R_{j\bar{i}l\bar{k}}). \\
= & -\sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{j}, k\bar{l}) + (i\bar{l}, k\bar{j}) + (i\bar{j}, l\bar{k}) + (i\bar{k}, l\bar{j}) \quad \text{(by lemma 2.6)} \\
+ & (j\bar{i}, k\bar{l}) + (j\bar{l}, k\bar{i}) + (j\bar{i}, l\bar{k}) + (j\bar{k}, l\bar{i})) \\
= & -\sum_{i,j,k,l} b_{ij}b_{kl}(i\bar{j} + j\bar{i}, k\bar{l} + l\bar{k}) \\
- & \sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k})) \\
- & \sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{k}, l\bar{j}) + (j\bar{l}, k\bar{i})).
\end{aligned}$$

For the first term, from definition 2.5,

$$\begin{aligned}
& -\sum_{i,j,k,l} b_{ij}b_{kl}(i\bar{j} + j\bar{i}, k\bar{l} + l\bar{k}) \\
= & -\int_X D(\sum_{ij} b_{ij}\mu_i\overline{\mu_j} + \sum_{ij} b_{ij}\mu_j\overline{\mu_i})(\sum_{ij} b_{ij}\mu_i\overline{\mu_j} + \sum_{ij} b_{ij}\mu_j\overline{\mu_i})dA(z) \\
= & -\int_X D(H(z,z) + \overline{H(z,z)})(H(z,z) + \overline{H(z,z)})dA(z).
\end{aligned}$$

For the second term and the third term, just as in the proof of theorem 3.1 we have

$$\begin{aligned}
& \sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k})) \\
= & 2\cdot Re\{\int_{X\times X} G(z,w)H(z,w)H(w,z)dA(w)dA(z)\}, \\
& \sum_{i,j,k,l} b_{ij}b_{kl}((i\bar{k}, l\bar{j}) + (k\bar{i}, j\bar{l})) \\
= & 2\cdot\int_{X\times X} G(z,w)|H(z,w)|^2 dA(w)dA(z).
\end{aligned}$$

Combining these three terms we get the theorem. □

Using the same method as for theorem 3.2, we prove the following nonpositivity result.

THEOREM 3.4. *Under the conditions of theorem 3.3, then Q is nonpositive definite on $\wedge^2 T_X^2 \mathrm{Teich}(S)$, and the zero level subsets of $Q(\cdot,\cdot)$ are $\{\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j};\ b_{ij}=-b_{ji}\}$.*

PROOF. Let $\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T_X^2 \mathrm{Teich}(S)$, from theorem 3.3 we have

$$\begin{aligned}
& Q(\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j},\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}) \\
= & -\int_X D(H(z,z)+\overline{H(z,z)})(H(z,z)+\overline{H(z,z)})dA(z) \\
- & \ 2(\cdot\int_{X\times X} G(z,w)|H(z,w))|^2 dA(w)dA(z) \\
+ & \ \cdot Re\{\int_{X\times X} G(z,w)H(z,w)H(w,z)dA(w)dA(z)\}).
\end{aligned}$$

For the first term, since $H(z,z)+\overline{H(z,z)} = 2\cdot Re\{H(z,z)\}$, by the positivity of the operator D,

$$\begin{aligned}
& -\int_X D(H(z,z)+\overline{H(z,z)})(H(z,z)+\overline{H(z,z)})dA(z) \\
= & -4\int_X D(Re\{H(z,z)\})(Re\{H(z,z)\})dA(z)\leq 0.
\end{aligned}$$

For the second term, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
& |\int_{X\times X} G(z,w)H(z,w)H(w,z)dA(w)dA(z)| \\
\leq & \int_{X\times X} |G(z,w)H(z,w)H(w,z)|dA(w)dA(z) \\
\leq & \sqrt{\int_{X\times X} |G(z,w)||H(z,w)|^2 dA(w)dA(z)} \\
\times & \sqrt{\int_{X\times X} |G(z,w)||H(w,z)|^2 dA(w)dA(z)} \\
= & \int_{X\times X} G(z,w)|H(z,w)|^2 dA(w)dA(z),
\end{aligned}$$

since G is positive and symmetric.

Combining these two terms, we get Q is always nonpositive on $\wedge^2 T_X^2 \mathrm{Teich}(S)$.

As in the proof of theorem 3.2,

$$Q(\sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \sum_{ij} b_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}) = 0$$

if and only if there exists a constant complex number k such that both of the following hold:

$$\begin{cases} H(z,z) = -\overline{H(z,z)}, \\ H(z,w) = k \cdot \overline{H(w,z)}. \end{cases}$$

If we let $z = w$, we get $k = -1$. Hence, the last equation is equivalent to

$$\sum_{ij} (b_{ij} + b_{ji}) \mu_i(w) \overline{\mu_j}(z) = 0.$$

Since $\{\mu_i\}_{i \geq 1}$ is a basis,

$$b_{ij} = -b_{ji}.$$

□

3.3. The curvature operator on $\wedge^2 T_X^3 \mathrm{Teich}(S)$. Let $\mathbf{J}$ be the almost complex structure on Teich(S). Since $\{t_i\}$ is a holomorphic coordinate, we have

$$\mathbf{J} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}.$$

Since the Weil-Petersson metric is a Kähler metric, $\mathbf{J}$ is an isometry on the tangent space. In particular we have

$$\begin{aligned} R(V_1, V_2, V_3, V_4) &= R(\mathbf{J}V_1, \mathbf{J}V_2, \mathbf{J}V_3, \mathbf{J}V_4) \\ &= R(\mathbf{J}V_1, \mathbf{J}V_2, V_3, V_4) = R(V_1, V_2, \mathbf{J}V_3, \mathbf{J}V_4) \end{aligned}$$

where R is the curvature tensor and V_i are real tangent vectors in $T_X \mathrm{Teich}(S)$.

Let $C = \sum_{ij} c_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T_X^3 \mathrm{Teich}(S)$, where c_{ij} are real. Set

$$K(z,w) = \sum_{i,j=1}^{3g-3} c_{ij} \mu_i(w) \cdot \overline{\mu_j(z)}.$$

THEOREM 3.5. *Let Q be the curvature operator, and $\sum_{ij} c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T^3_X \mathrm{Teich}(S)$. Then we have*

$$
\begin{aligned}
& Q(\sum_{ij} c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}, \sum_{ij} c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}) \\
= & \int_X D(K(z,z)-\overline{K(z,z)})(K(z,z)-\overline{K(z,z)})dA(z) \\
- & 2\cdot\int_{X\times X} G(z,w)|K(z,w))|^2 dA(w)dA(z) \\
+ & 2\cdot Re\{\int_{X\times X} G(z,w)K(z,w)K(w,z)dA(w)dA(z)\}.
\end{aligned}
$$

PROOF. Since $\frac{\partial}{\partial y_i} = \mathbf{J}\frac{\partial}{\partial t_i} + \mathbf{J}\frac{\partial}{\partial \overline{t_i}}$ and $\mathbf{J}$ is an isometry, by proposition 2.3,

$$
\begin{aligned}
& Q(\sum_{ij} c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}, \sum_{ij} c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}) \\
= & \sum_{i,j,k,l} c_{ij}c_{kl}(R_{i\bar{j}k\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{\bar{i}j\bar{k}l}) \\
= & Q(\sum_{ij} c_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \sum_{ij} c_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j})
\end{aligned}
$$

By theorem 3.1,

$$
\begin{aligned}
& Q(\sum_{ij} c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}, \sum_{ij} c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}) \\
= & \int_X D(K(z,z)-\overline{K(z,z)})(K(z,z)-\overline{K(z,z)})dA(z) \\
- & 2\cdot\int_{X\times X} G(z,w)|K(z,w))|^2 dA(w)dA(z) \\
+ & 2\cdot Re\{\int_{X\times X} G(z,w)K(z,w)K(w,z)dA(w)dA(z)\}.
\end{aligned}
$$

□

The same proof as that of theorem 3.2 shows

THEOREM 3.6. *Let Q be the curvature operator as above, then Q is a negative definite operator on $\wedge^2 T^3_X \mathrm{Teich}(S)$.*

4. Curvature operator on $\wedge^2 T_X \text{Teich}(S)$

Every element in $\wedge^2 T_X \text{Teich}(S)$ can be represented by $\sum_{ij}(a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j} + c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j})$.

PROPOSITION 4.1. *Let Q be the curvature operator. Then*

$$\begin{aligned} & Q(\sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j} + c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}, \\ & \sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j} + c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}) \\ = \; & Q(\sum_{ij} d_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}, \sum_{ij} d_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} + b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}),\end{aligned}$$

where $d_{ij} = a_{ij} + c_{ij}$.

PROOF. Direct computation by using the isometric property of the almost complex structure **J**. □

We want to know if the curvature operator Q is nonpositive definite. By proposition 4.1 it is sufficient to see if Q is nonpositive definite on $Span\{\wedge^2 T_X^1 \text{Teich}(S), \wedge^2 T_X^2 \text{Teich}(S)\}$. Firstly, let us establish the following formula.

LEMMA 4.2. *Let Q be the curvature operator, $\sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ be an element in $\wedge^2 T_X^1 \text{Teich}(S)$, and $\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}$ be an element in $\wedge^2 T_X^2 \text{Teich}(S)$. Then we have*

$$\begin{aligned} & Q(\sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}) \\ = \; & \boldsymbol{i}\cdot\int_X D(F(z,z) - \overline{F(z,z)})(H(z,z) + \overline{H(z,z)})dA(z) \\ - \; & 2\cdot Im\{\int_{X\times X} G(z,w)F(z,w)\overline{H(z,w)})dA(w)dA(z)\} \\ - \; & 2\cdot Im\{\int_{X\times X} G(z,w)F(z,w)H(w,z)dA(w)dA(z)\}.\end{aligned}$$

Proof. Since $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \overline{t_i}}$ and $\frac{\partial}{\partial y_i} = \mathbf{i}(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial \overline{t_i}})$, by proposition 2.3,

$$\begin{aligned}
& Q(\sum_{ij} a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}) \\
= \ & (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-R_{i\bar{j}k\bar{l}} + R_{i\bar{j}\bar{k}l} - R_{\bar{i}jk\bar{l}} + R_{\bar{i}j\bar{k}l}) \\
= \ & (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-R_{i\bar{j}k\bar{l}} - R_{i\bar{j}l\bar{k}} + R_{j\bar{i}k\bar{l}} + R_{j\bar{i}l\bar{k}}) \\
= \ & (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-(i\bar{j}, k\bar{l}) - (i\bar{l}, k\bar{j}) - (i\bar{j}, l\bar{k}) - (i\bar{k}, l\bar{j}) \\
+ \ & (j\bar{i}, k\bar{l}) + (j\bar{l}, k\bar{i}) + (j\bar{i}, l\bar{k}) + (j\bar{k}, l\bar{i})) \quad \text{(by lemma 2.6)} \\
= \ & (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(j\bar{i} - i\bar{j}, k\bar{l} + l\bar{k}) \\
+ \ & (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-(i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k})) \\
+ \ & (-\mathbf{i})\sum_{i,j,k,l} a_{ij}b_{kl}(-(i\bar{k}, l\bar{j}) + (j\bar{l}, k\bar{i})).
\end{aligned}$$

For the first term, by definition 2.5,

$$\begin{aligned}
& \sum_{i,j,k,l} a_{ij}b_{kl}(j\bar{i} - i\bar{j}, k\bar{l} + l\bar{k}) \\
= \ & \int_X D(\sum_{ij} a_{ij}\overline{\mu_i}\mu_j - \sum_{ij} a_{ij}\mu_i\overline{\mu_j})(\sum_{kl} b_{kl}\mu_k\overline{\mu_l} + \sum_{kl} b_{kl}\mu_l\overline{\mu_k})dA(z) \\
= \ & \int_X D(\overline{F(z,z)} - F(z,z))(H(z,z) + \overline{H(z,z)})dA(z).
\end{aligned}$$

For the second term, since D is self adjoint, using the Green function G,

$$\begin{aligned}
& \sum_{i,l} a_{ij}b_{kl}(-(i\bar{l}, k\bar{j}) + (l\bar{i}, j\bar{k})) \\
= \ & 2\mathbf{i}\cdot Im\{\sum_{i,l} a_{ij}b_{kl}(-(i\bar{l}, k\bar{j})\} \\
= \ & -2\mathbf{i}\cdot Im\{\int_X D(\sum_i a_{ij}\mu_i\overline{\mu_l})(\sum_k b_{kl}\mu_k\overline{\mu_j})dA(z)\} \\
= \ & -2\mathbf{i}\cdot Im\{\int_X\int_X G(w,z)\sum_i a_{ij}\mu_i(w)\overline{\mu_l(w)}(\sum_k b_{kl}\mu_k(z)\overline{\mu_j}(z))dA(z)dA(w)\} \\
= \ & -2\mathbf{i}\cdot Im\{\int_{X\times X} G(z,w)F(z,w)H(w,z)dA(w)dA(z)\}.
\end{aligned}$$

For the last term,

$$\begin{aligned}
&\sum_{i,k} a_{ij}b_{kl}(-(i\bar{k},l\bar{j})+(k\bar{i},j\bar{l})) = -2\mathbf{i}\cdot Im\{\sum_{i,k} a_{ij}b_{kl}(i\bar{k},l\bar{j})\}\\
= \quad & -2\mathbf{i}\cdot Im\{\int_X D(\sum_i a_{ij}\mu_i\overline{\sum_k b_{kl}\mu_k})(\mu_l\overline{\mu_j})dA(z)\}\\
= \quad & -2\mathbf{i}\cdot Im\{\int_X\int_X G(w,z)\sum_i a_{ij}\mu_i(w)\overline{\sum_k b_{kl}\mu_k(w)}(\mu_l(z)\overline{\mu_j}(z))dA(z)dA(w)\}\\
= \quad & -2\mathbf{i}\cdot Im\{\int_{X\times X} G(z,w)F(z,w)\overline{H(z,w)}dA(w)dA(z)\}.
\end{aligned}$$

Combining these three terms above, we get the lemma. ☐

Now we study the curvature operator Q on $Span\{\wedge^2T^1_X\text{Teich}(S),\wedge^2T^2_X\text{Teich}(S)\}$. Setting

$$A=\sum_{ij}a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, B=\sum_{ij}b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j},$$

on $Span\{\wedge^2T^1_X\text{Teich}(S),\wedge^2T^2_X\text{Teich}(S)\}$, we have

Theorem 4.3. *Let Q be the curvature operator and D as above. Let $A=\sum_{ij}a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ and $B=\sum_{ij}b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}$. Then we have*

$$\begin{aligned}
&Q(A+B,A+B)=\\
-4\quad&\int_X DIm\{F(z,z)+\boldsymbol{i}H(z,z)\}(Im\{F(z,z)+\boldsymbol{i}H(z,z)\})dA(z)\\
-2\quad&\int_{X\times X} G(z,w)|F(z,w)+\boldsymbol{i}H(z,w))|^2dA(w)dA(z)\\
+2\quad&Re\{\int_{X\times X} G(z,w)(F(z,w)+\boldsymbol{i}H(z,w))(F(w,z)+\boldsymbol{i}H(w,z))dA(w)dA(z)\},
\end{aligned}$$

where $F(z,w)=\sum_{i,j=1}^{3g-3}a_{ij}\mu_i(w)\cdot\overline{\mu_j(z)}$ and $H(z,w)=\sum_{i,j=1}^{3g-3}b_{ij}\mu_i(w)\cdot\overline{\mu_j(z)}$.

Proof. Since $Q(A,B)=Q(B,A)$,

$$Q(A+B,A+B)=Q(A,A)+2Q(A,B)+Q(B,B).$$

By theorem 3.1, theorem 3.3 and lemma 4.2 we have

$$
\begin{aligned}
& Q(A+B, A+B) \\
= \ & (\int_X D(F(z,z) - \overline{F(z,z)})(F(z,z) - \overline{F(z,z)})dA(z) \\
- \ & \int_X D(H(z,z) + \overline{H(z,z)})(H(z,z) + \overline{H(z,z)})dA(z) \\
+ \ & 2\mathbf{i} \cdot \int_X D(F(z,z) - \overline{F(z,z)})(H(z,z) + \overline{H(z,z)})dA(z)) \\
(- \ & 2 \cdot \int_{X\times X} G(z,w)|F(z,w))|^2 dA(w)dA(z) \\
- \ & 2 \cdot \int_{X\times X} G(z,w)|H(z,w))|^2 dA(w)dA(z) \\
- \ & 4 \cdot Im\{\int_{X\times X} G(z,w)F(z,w)\overline{H(z,w)})dA(w)dA(z)\}) \\
(+ \ & 2 \cdot Re\{\int_{X\times X} G(z,w)F(z,w)F(w,z)dA(w)dA(z)\} \\
- \ & 2 \cdot Re\{\int_{X\times X} G(z,w)H(z,w)H(w,z)dA(w)dA(z)\} \\
- \ & 4 \cdot Im\{\int_{X\times X} G(z,w)F(z,w)H(w,z)dA(w)dA(z)\})
\end{aligned}
$$

The sum of the first three terms is exactly

$$-4\int_X D(Im\{F(z,z) + \mathbf{i}H(z,z)\})(Im\{F(z,z) + \mathbf{i}H(z,z)\})dA(z).$$

Just as $|a+\mathbf{i}b|^2 = |a|^2 + |b|^2 + 2\cdot Im(a\cdot\overline{b})$, where a and b are two complex numbers, the sum of the second three terms is exactly

$$-2\cdot\int_{X\times X} G(z,w)|F(z,w) + \mathbf{i}H(z,w))|^2 dA(w)dA(z).$$

For the last three terms, since

$$Im(F(z,w)\cdot H(w,z)) = -Re(F(z,w)\cdot(\mathbf{i}H(w,z))),$$

the sum is exactly

$$2\cdot Re\{\int_{X\times X} G(z,w)(F(z,w) + \mathbf{i}H(z,w))(F(w,z) + \mathbf{i}H(w,z))dA(w)dA(z)\}.$$

□

Furthermore,

THEOREM 4.4. *Under the conditions of theorem 4.3, Q is nonpositive definite on $Span\{\wedge^2 T^1_X \mathrm{Teich}(S), \wedge^2 T^2_X \mathrm{Teich}(S)\}$, and the zero level subsets of $Q(\cdot,\cdot)$ are $\{\sum_{ij} b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j};\ b_{ij}=-b_{ji}\}$ in $Span\{\wedge^2 T^1_X \mathrm{Teich}(S), \wedge^2 T^2_X \mathrm{Teich}(S)\}$.*

PROOF. Let us estimate the terms in theorem 4.3 separately. For the first term, since D is a positive operator,

$$-\int_X D(Im\{F(z,z)+\mathbf{i}H(z,z)\})(Im\{F(z,z)+\mathbf{i}H(z,z)\})dA(z)\leq 0.$$

For the third term, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&|\int_{X\times X} G(z,w)(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))dA(w)dA(z)|\\
\leq&\int_{X\times X} G(z,w)|(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))|dA(w)dA(z)\\
\leq&\sqrt{\int_{X\times X} G(z,w)|(F(z,w)+\mathbf{i}H(z,w))|^2 dA(w)dA(z)}\\
\times&\sqrt{\int_{X\times X} G(z,w)|(F(w,z)+\mathbf{i}H(w,z))|^2 dA(w)dA(z)}.
\end{aligned}$$

Since $G(z,w)=G(w,z)$,

$$\begin{aligned}
&|\int_{X\times X} G(z,w)(F(z,w)+\mathbf{i}H(z,w))(F(w,z)+\mathbf{i}H(w,z))dA(w)dA(z)|\\
=&\sqrt{\int_{X\times X} G(z,w)|(F(z,w)+\mathbf{i}H(z,w))|^2 dA(w)dA(z)}\\
\times&\sqrt{\int_{X\times X} G(w,z)|(F(w,z)+\mathbf{i}H(w,z))|^2 dA(w)dA(z)}\\
=&\int_{X\times X} G(z,w)|(F(z,w)+\mathbf{i}H(z,w))|^2 dA(w)dA(z).
\end{aligned}$$

Combining the two inequalities above and the second term in theorem 3.3, we see that Q is a nonpositive operator on $Span\{\wedge^2 T^1_X \mathrm{Teich}(S), \wedge^2 T^2_X \mathrm{Teich}(S)\}$. Furthermore, $Q(A+B,A+B)=0$ if and only if there exists a constant k such that both of the following hold:

$$\begin{cases} Im\{F(z,z)+\mathbf{i}H(z,z)\}=0,\\ F(z,w)+\mathbf{i}H(z,w)=k\cdot\overline{(F(w,z)+\mathbf{i}H(w,z))}. \end{cases}$$

If we let $z=w$, we get $k=1$. Hence, the second equation is equivalent to

$$\sum_{ij}(a_{ij}-a_{ji}+\mathbf{i}(b_{ij}+b_{ji}))\mu_i(w)\overline{\mu_j}(z)=0.$$

Since $\{\mu_i\}_{i\geq 1}$ is a basis,

$$a_{ij}=a_{ji}, b_{ij}=-b_{ji}.$$

That is, $A=0$ and $B=\sum_{ij}b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}$, where $b_{ij}=-b_{ji}$.

Conversely, if $A=0$ and $B\in\{\sum_{ij}b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j};\ b_{ij}=-b_{ji}\}$, by theorem 3.4 we have $Q(A+B,A+B)=0$. □

Before we prove the main theorem, let us define a natural action of $\mathbf{J}$ on $\wedge^2 T_X\mathrm{Teich}(S)$ by

$$\begin{cases} \mathbf{J}\circ\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}:=\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j},\\ \mathbf{J}\circ\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}:=-\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial x_j}=\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial y_i},\\ \mathbf{J}\circ\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}:=\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \end{cases}$$

and extend it linearly. It is easy to see that $\mathbf{J}\circ\mathbf{J}=id$.

Now we are ready to prove theorem 1.1.

Proof of Theorem 1.1. Combining proposition 4.1 and theorem 3.4, we get that Q is nonpositive definite.

If $A=C-\mathbf{J}\circ C$ for some C in $\wedge^2 T_X\mathrm{Teich}(S)$, from the isometry of $\mathbf{J}$ it is easy to see that $Q(A,A)=0$.

Assume that $A\in\wedge^2 T_X\mathrm{Teich}(S)$ such that $Q(A,A)=0$. Since $\wedge^2 T\mathrm{Teich}(S)=Span\{\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial y_l},\frac{\partial}{\partial y_m}\wedge\frac{\partial}{\partial y_n}\}$, there exists a_{ij},b_{ij} and c_{ij} such that

$$A=\sum_{ij}a_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}+b_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j}+c_{ij}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}.$$

Since $Q(A, A) = 0$, by proposition 4.1 and theorem 3.4 we must have

$$a_{ij} + c_{ij} = a_{ji} + c_{ji}, b_{ij} = -b_{ji}.$$

That is,

$$a_{ij} - a_{ji} = -(c_{ij} - c_{ji}), b_{ij} = -b_{ji}.$$

Set

$$C = \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}.$$

Claim: $A = C - \mathbf{J} \circ C$.

Since $\sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \sum_{i<j} (a_{ij} - a_{ji}) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, we have

$$\mathbf{J} \circ \sum_{ij} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \sum_{i<j} (a_{ij} - a_{ji}) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$$

$$= \quad -\sum_{i<j} (c_{ij} - c_{ji}) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} = -\sum c_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}.$$

Similarly,

$$\mathbf{J} \circ \sum \left(\frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}\right) = -\sum \frac{b_{ij}}{2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}.$$

The claim follows from the two equations above. □

5. Application

In this section we study the twist-harmonic maps from some domains into the Teichmüller space. Before we go to the rank-one hyperbolic space case, let us state the following lemma, which is influenced by lemma 5 in [**72**].

LEMMA 5.1. *The rank-one Hyperbolic spaces* $H_{Q,m} = Sp(m,1)/Sp(m) \cdot Sp(1)$ *and* $H_{O,2} = F_4^{-20}/SO(9)$ *cannot be totally geodesically immersed into* Teich(S).

PROOF. On quaternionic hyperbolic manifolds $H_{Q,m} = Sp(m,1)/Sp(m)$, assume that there is a totally geodesic immersion of $H_{Q,m}$ into Teich(S). We may select $p \in H_{Q,m}$. Choose a quaternionic line l_Q on $T_p H_{Q,m}$, and we may assume that l_Q is spanned over R by v, Iv, Jv and Kv. Without loss of generality, we may assume that J on $l_Q \subset T_pH_{Q,m}$ is the same as the complex structure on Teich(S). Choose an element

$$v \wedge Jv + Kv \wedge Iv \in \wedge^2 T_pH_{Q,m}.$$

Let $Q^{H_{Q,m}}$ be the curvature operator on $H_{Q,m}$.

$$\begin{aligned} & Q^{H_{Q,m}}(v \wedge Jv + Kv \wedge Iv, v \wedge Jv + Kv \wedge Iv) \\ = \ & R^{H_{Q,m}}(v, Jv, v, Jv) + R^{H_{Q,m}}(Kv, Iv, Kv, Iv) + 2 \cdot R^{H_{Q,m}}(v, Jv, Kv, Iv). \end{aligned}$$

Since I is an isometry, we have

$$\begin{aligned} R^{H_{Q,m}}(Kv, Iv, Kv, Iv) &= R^{H_{Q,m}}(IKv, IIv, IKv, IIv) \\ &= R^{H_{Q,m}}(-Jv, -v, -Jv, -v) = R^{H_{Q,m}}(v, Jv, v, Jv). \end{aligned}$$

Similarly,

$$\begin{aligned} R^{H_{Q,m}}(v, Jv, Kv, Iv) &= R^{H_{Q,m}}(v, Jv, IKv, IIv) \\ &= R^{H_{Q,m}}(v, Jv, -Jv, -v) = -R^{H_{Q,m}}(v, Jv, v, Jv). \end{aligned}$$

Combining the terms above, we have

$$Q^{H_{Q,m}}(v \wedge Jv + Kv \wedge Iv, v \wedge Jv + Kv \wedge Iv) = 0.$$

Since f is a geodesical immersion,

$$Q^{\text{Teich}(S)}(v \wedge Jv + Kv \wedge Iv, v \wedge Jv + Kv \wedge Iv) = 0.$$

On the other hand, by theorem 1.1, there exists C such that

$$v \wedge Jv + Kv \wedge Iv = C - \mathbf{J} \circ C.$$

Hence,

$$\begin{aligned} & \mathbf{J} \circ (v \wedge Jv + Kv \wedge Iv) \\ = \ & \mathbf{J} \circ (C - \mathbf{J} \circ C) = \mathbf{J} \circ C - \mathbf{J} \circ \mathbf{J} \circ C = \mathbf{J} \circ C - C \\ = \ & -(v \wedge Jv + Kv \wedge Iv). \end{aligned} \tag{16}$$

Since $\mathbf{J}$ is the same as J in $H_{Q,m}$, we also have

$$\mathbf{J} \circ (v \wedge Jv + Kv \wedge Iv) = (Jv \wedge JJv + JKv \wedge JIv) \tag{17}$$
$$= \quad Jv \wedge (-v) + Iv \wedge (-Kv) = v \wedge Jv + Kv \wedge Iv.$$

From equations (16) and (17) we get

$$v \wedge Jv + Kv \wedge Iv = 0$$

which is a contradiction since l_Q is spanned over R by v, Iv, Jv and Kv.

In the case of the Cayley hyperbolic plane $H_{O,2} = F_4^{20}/SO(9)$, the argument is the same, replacing the argument above of a quaternionic line by a Cayley line ([**15**]). □

Now we are ready to prove theorem 1.2.

Proof of theorem 1.2. Since the sectional curvature operator on Teich(S) is nonpositive, Teich(S) also has nonpositive Riemannian sectional curvature in the complexified sense as stated in [**47**]. Assume f is not constant. From theorem 2 in [**17, 47**], we know that f should be a totally geodesic immersion. On the other hand, by lemma 5.1, there does not exist any totally geodesic immersion from G/Γ into Teich(S). Hence, f must be a constant. □

Remark 5.1. In [**72**] it is showed that the image of any homomorphism ρ from Γ to Mod(S) is finite. Hence, $\rho(\Gamma)$ must have a fixed point in Teich(S) from classical theory (Teich(S) is contractible). If we assume that there exists a twist harmonic map f with respect to this homomorphism, then by theorem 1.2 we know $\rho(\Gamma) \subset$ Mod(S) will fix the point $f(G/\Gamma) \in$ Teich(S).

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