# A Framework for Calderón-Zygmund Singular Integral Operators on Spaces of Homogeneous Type

by

Theresa C Anderson

B.S., University of Wisconsin; Madison, WI, 2010

M.A., Brown University; Providence, RI, 2012

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics at Brown University

PROVIDENCE, RHODE ISLAND

May 2015

This dissertation by Theresa C Anderson is accepted in its present form
by Mathematics as satisfying the
dissertation requirement for the degree of Doctor of Philosophy.

Date__________________ ____________________________________

Jill Pipher, Ph.D., Advisor

Recommended to the Graduate Council

Date__________________ ____________________________________

David Cruz-Uribe, SFO, Ph.D., Reader

Date__________________ ____________________________________

Sergei Treil, Ph.D., Reader

Approved by the Graduate Council

Date__________________ ____________________________________

Peter Weber, Dean of the Graduate School

## Vitae

Theresa C. Anderson was born in New York and grew up in Wisconsin, attending the University of Wisconsin-Madison and getting Bachelor's degrees in Mathematics, Chemistry, and Spanish Literature, with honors. She went to Brown University for her masters and PhD, both in Mathematics. She earned an NSF graduate student fellowship to support her studies at Brown. Though originally set to pursue something else, Theresa was inspired by the incredible department at Wisconsin to pursue mathematics, where she will return to as an NSF postdoctoral research fellow in August of 2015.

A selected list of some of Theresa's experience and honors include:

2015-2018 NSF Mathematical Sciences Postdoctoral Research Fellow

2011-2015 NSF Graduate Fellow in Mathematics

2014 Summer School with Christoph Thiele, Lillian Pierce, and Po-Lam Yung

2013 Organizer, Special Session in Harmonic Analysis, Geometric Measure Theory and Partial Differential Equations at the Joint Math Meetings

2011 Joint Mathematics Meetings poster session prize winner

2010 UW-Madison University Bookstore award for best Undergraduate Thesis

2009-2010 UW-Madison College of Letters and Science Scholarships: David H. Dura Scholarship, Summer Senior Honors Thesis Grant, Besozzi Scholarship

2009 American Chemical Society Excellence in Physical Chemistry Award

2009 Outstanding Junior Award - Wisconsin Alumni Association

2008, 2009 UW-Madison Mathematics Department Cady Scholarship

2006 U.S. Department of Education Presidential Scholar

2006 Coca Cola Scholar

Below is a list of publications and submitted articles for the author:

Anderson, Theresa C., Hytönen, Tuomas, and Tapiola, Olli. Weak A-infinity weights and weak reverse Hölder property in a Space of Homogeneous Type. Submitted,

Anderson, Theresa C. and Damián, Wendolín. Calderón–Zygmund operators and commutators in Spaces of Homogeneous Type: weighted inequalities. Submitted.

Anderson, Theresa C. A new sufficient two-weighted bump assumption for $L^p$ boundedness of Calderon-Zygmund operators. To appear in *Proceedings of the AMS*.

Anderson, Theresa C., Cruz-Uribe, David, SFO and Moen, Kabe. Logarithmic bump conditions for Calderón-Zygmund Operators on spaces of homogeneous type. *Publicacions Mathematiques* 59(1), 2015.

Anderson, Theresa C. and Vagharshakyan, Armen. A simple proof of the sharp weighted estimate for Calderon-Zygmund operators on homogeneous spaces. *Journal of Geometric Analysis*. July 2014, Volume 24, Issue 3, pp 1276-1297.

Anderson, Theresa C., Mari-Beffa, Gloria. A completely integrable flow of star-shaped curves on the light cone in Lorenzian $R^4$. *J. Phys. A: Math. Theor.* 44 (2011) 445203. *Featured in IOP select http://Select.iop.org.

Anderson, Theresa C. Continued fractions and the Gauss map: interplay and invariance. Goettigen Summer school 2011 final project accepted by and funded by NSF.

Anderson, Theresa C., Rolen, Larry, Stoehr, Ruth E., Benford's Law for Coefficients of Modular Forms and Partition Functions. *Proceedings of the American Mathematical Society.* 139 (2011) 1533-1541.

Senior Honors Thesis: Differential Geometry of the Light Cone and Conformal Sphere - UW-Madison

## Acknowledgements

There have been so many amazing people in my life that it is hard to begin, as I am afraid that I will never end. The making of this thesis was a multi-year process of discovery and a testament to the great mathematics and mathematicians that I have met along the way. I begin professionally, to thank and acknowledge the National Science Foundation, as the work present in this thesis was done under support of an NSF Graduate student fellowship. For Chapter 3, thanks go to Jill Pipher and Anna Kairema for helpful conversations. For Chapter 4, thanks are extended to Carlos Pérez for his motivating discussions in Sevilla, Spain, as well as Tuomas Hytönen for suggestions during my visit to Helsinki. Finally, for Chapter 5, I thank Jill Pipher, David Cruz-Uribe, Cristina Pereyra, Tuomas Hytönen, Carlos Pérez and Francesco Di Plinio.

There have been many mathematical mentors on my journey. Thank you to David Cruz-Uribe for being such an enthusiastic collaborator, Carlos Pérez for many helpful discussions y por mis varias oportunidades de venir a España, mi segunda pais. Gracias a Wendy Damián por tu amistad que ha crecido por los años – fue un placer de investigar contigo. To Kabe Moen, who started as the collaborator that I never met, thank you for the opportunity to visit Alabama. I benefited from many helpful visits through the years, including to Cristina Pereyra in Albuquerque and Michael Lacey in Atlanta, two great food towns and two fantastic mathematicians that invigorated my career. Thank you to Chirstoph Thiele for many great conversations, mathematical and otherwise, in Bonn. I benefited greatly from the Thiele summer school. Also, Brett Wick has been a mathematical guide, almost from the beginning of my tenure at Brown. I would also like to thank Lesley Ward, for inspiring discussions on far numbers and the BMO class, which appear in the preliminaries. To my colleagues Amalia, Yumeng, Francesco, and Jingguo for sharing your mathematics in our informal seminar through the years. Amalia, I enjoyed our coffee conversations so much.

It goes without mention to those in the Brown community and beyond that I received great advice and knowledge from my advisor, Jill Pipher, but I also want to thank her for the great freedom to explore concepts that she fostered. Jill has always been open to exploration of many different mathematical ideas, many of which are present in my thesis, as well as providing funding for mathematical visits to hash out these ideas. Thank you to Lillian Pierce for encouraging someone you did not even know, in a completely new area that I am looking forward to continuing past my tenure at Brown.

I have had a fantastic support system during my stay in Providence. Thank you to friends Jonah Leshin, Dahlia Nadkarni, Vivian Olsiewski Healey, Wade Hindes, Amalia Culiuc, Edmund Harris, Michael Jaycox, Amanda Turek, and Wei An, who have provided great friendship in Providence that has continued, even after some of you are no longer physically present.

While at the University of Wisconsin, I made many lasting connections and feel honored to have had so many great mathematicians play such a great role, both personally and professionally, in my life. To say that you have inspired me would be an extreme understatement. A thesis is not a proper place to thank you for properly for the role that you played in my development, but I will mention some of your names: the vivacious Jordan Ellenberg, Ken Ono, under whose direction I published my first paper, Nigel Boston who made me look forward to topology, Bart Kastermans, for accepting a crazy young woman very late into your class, Alex Nagel, for continued advice, and Alex Ionescu, for challenging me in a class so difficult that we still talk about it (when I finished your class, I felt like I could do anything). Special mention to Gloria Marí-Beffa, my senior thesis (and life) advisor, for her incredible mentor-ship (in putting up with me). She saw me from a confused Pharmacy major to a confident young mathematician. And to my long time mentor Andreas Seeger. It has been quite a journey and I am honored that you have played such a major part in it.

To my family and friends: Jon, Katie, Kathy, Aunt Cathy, Mom and Dad Morris, Samantha, Nam hab Txiv, Lee, Nhia, Kou, Meng, Elizabeth, Theresa, Ger, Grandpa, Mom and Dad who have aided me on this journey. Kathy, I treasure your advice and friendship through the (seeming) ages. Mom and Dad, you have always encouraged me to be truly anything that I want to be, and I treasure the upbringing that I had where there were no gender roles, where investigation was encouraged, and confidence was inherent. These were natural to me, and together with your unconditional love, have been guiding forces in my life. To Uncle Ron, whose questions continue to drive me as a mathematician and a human. I hope you are proud of the woman that I have become.

Finally, to the love of my life, Souriya. Saying anything about you here would be to say too little.

# CONTENTS

CHAPTER **One**

# Introduction

Integral to the field of harmonic analysis are maximal functions and singular integrals. Maximal functions allow us to study behavior of functions through behavior of averages, such as the standard Hardy-Littlewood maximal function: $Mf(x) := \sup_{Q \ni x} \fint_Q |f| dx$. Singular integrals are prevalent in many applications, such as Fourier analysis, complex function theory and PDE. The integrals are called singular due to the failure of their kernels to be integrable. A classic example to keep in mind is the *Hilbert transform*, which we can understand in the principal value sense as: $Hf(x) := p.v. \frac{1}{\pi} \int_{\varepsilon < y < R} \frac{f(y)}{x-y} dy$. The Hilbert transform arises in a plethora of contexts ranging from convergence of Fourier series and potential theory to cosmology. Its discrete form is prevalent in signal processing and it was the motivating example for the Calderón and Zygmund school of analysis.

An important problem in the field was to bound maximal functions and singular integral operators on Banach and Hilbert spaces. For example, knowledge of $L^p$ bounds for the Hilbert transform tells us that partial summation operators also converge in $L^p$. Thus, a single operator controls convergence of an entire family of operators. A naturally arising generalization of the Hilbert transform is a Calderón-Zygmund Singular integral operator (CZO). Calderón-Zygmund operators have kernels which satisfy certain decay and smoothness conditions. These will be discussed in more detail later, but for now we will say that these decay and smoothness conditions mimic the decay of the Hilbert transform kernel and the decay of its derivative. Due to the delicate cancellation present in their kernels, bounding CZOs in different contexts in often difficult. Proving $L^p$ bounds for different types of CZOs was the extensive work of the Calderón-Zygmund and David-Journé programs. From here, boundedness in more general contexts than Euclidean space with Lebesgue measure were considered. For instance a question that was asked was, if $\mu$ is a measure such that $d\mu = w(x)dx$ do these $L^p$ bounds still hold? This was a natural question to ask since for the Riesz transforms to be bounded on Lebesgue space (the n-dimensional analogues of the Hilbert transform and another example of CZOs), it is required that the new measure be absolutely continuous with respect to Lebesgue measure [28]. If such a $w$ is nonnegative a.e. and locally integrable, it is called a weight. Weighted norm inequalities arise naturally in many contexts such as boundary value problems for Laplace's equations. Weights are also a key part in the theory of extrapolation. Extrapolation due to Rubio de Francia allows us to obtain $L^p$ bounds from weighted $L^2$ ones.

Muckenhoupt introduced in [60] the $A_p$ classes to answer the necessary and sufficient conditions for the boundedness of the maximal function on weighted $L^p$ spaces, that is for $1 < p < \infty$,

$$\|Mf\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)} \tag{1.1}$$

if and only if $w \in A_p$,

$$[w]_{A_p} = \sup_Q \fint_Q w \left( \fint_Q w^{1-p'} \right)^{p-1} < \infty, \tag{1.2}$$

Similar theorems do hold for some singular integrals – in fact the "if" direction holds for any CZO (see [12], [28], section 4.6 in [77]).

In this thesis, I will discuss a framework for obtaining bounds (even sharp bounds) for CZOs, started by Lerner, that I extended to a more general setting. This treatment of CZOs is the main result of this thesis and led to a wide range of applications such as sharp estimates in weighted norm inequalities (Chapters 3, 4) and a generalization of classical extrapolation (Chapter 5).

Andrei Lerner published a series of papers that culminated in a breakthrough in 2012 on how to approach CZOs. Singular integrals come in many types, such as convolution and non-convolution. Boundedness of convolution-type operators in $L^2$ is usually determined by the Fourier multiplier, but to get to $L^p$ bounds from $L^2$ is difficult – this was the purpose of the Calderón-Zygmund program. For non-convolution type, even $L^2$ is difficult – this was the purpose of the David-Journé program, via their famous $T(1)$ theorem. Lerner was able to understand all CZOs through a special collection of positive operators. Using elegant tools originally from statistics such as percentile rankings, he showed that in fact any CZO could be bounded above in norm by a suprenum of dyadic positive operators called *sparse operators* [48]. This was surprising as the kernels of CZOs are not positive, but yet by summing simple averaging operators over a distinguished collection of dyadic cubes, Lerner was able to bound the norm of a CZO in any Banach function space.

Though at first it seemed like the techniques used by Lerner were specific to $\mathbb{R}^n$, I soon discovered that they were much more general. Together with Armen Vagharshakyan, we were able to extend the tools of Lerner to *spaces of homogeneous type* (SHT) [58]. These extend Euclidean spaces, and include $\mathcal{C}^\infty$ compact Riemannian manifolds and graphs of Lipschitz functions (see [10], [14] for more examples). Our theorem is the following:

**Theorem 1** (Lerner ($\mathbb{R}^n$), TCA and Vagharshakyan (SHT))**.** *Let $T$ be a CZO. Then we have*

$$\|Tf\|_X \leqslant c \sup_{D,S} \|T^S|f|\|_X \tag{1.3}$$

*where the $T^S f = \sum_{Q\in S} \left(\fint f\right) \chi_Q$ are sparse operators, $S$ is a* sparse family *and $X$ is a Banach function space, such as $L^2(w)$.*

The supremum is taken over the finite number of dyadic grids in an SHT as well as over all sparse families, a collection of dyadic cubes such that for $Q, Q' \in S$,

$\mu\left(\bigcup_{Q' \subsetneq Q} Q'\right) \leqslant \frac{\mu(Q)}{2}$. These collections generalize the Calderón-Zygmund decomposition.

To prove 1, we introduced a more general decomposition that works in SHT which behaves like a median on a continuous set of data. This is defined for $0 < \lambda < 1$ on a finite measure space $Q$ with measure $\mu$ as $\omega_\lambda(F) = \sup\{a > 0 : \mu(f < a) \leqslant \mu(Q)\lambda\}$ for $0 < \lambda < 1$. Another feature of our proof is the use of a new modern refinement by Hytönen and Kairemaof a dyadic system on SHT by Christ [35], [10].

This treatment of CZOs is intimately connected to weighted norm inequalities. In the last decades, harmonic analysts have paid much attention to the area of weighted inequalities for singular integrals. Many have sought a deeper understanding of how the constants present in such weighted inequalities depend on $[w]_{A_p}$. Knowing sharp bounds has applications to elliptic PDE and quasiconformal mappings (see [20] and [26]). The search for these generated much research over the past 20 years. The first main result in this area was due to Buckley, who proved that

$$\|M\|_{L^p(w)} \lesssim [w]_{A_p}^{1/p}$$

where $M$ is the Hardy-Littlewoord maximal function [8]. Buckley's bound is sharp, meaning that if one lowers the exponent on $[w]_{A_p}$, then the bound does not hold.

Since then, much work was put into solving the so-called $A_2$ conjecture, which stated that the constant in the corresponding weighted norm inequality for Calderón–Zygmund singular integrals depended linearly on the $A_2$ constant, and that this bound was sharp. From there, one could extrapolate to get the $A_p$ dependence [23]. This open question was recently solved by Hytönen in [33]. See also [39] for a survey about the history of the conjecture. Hytönen used as encompassing and technical argument using Haar functions and relied on probability, random dyadic grids, and nonhomogeneous analytic techniques.

However, using Lerner's decomposition, the $A_2$ theorem follows in only a few lines ([51]).

The methodology developed in SHT, led to my work on several other problems in weighted norm inequalities. Together with Wendolín Damián, I also obtained a variety of other weighted bounds for CZOs [3], using the new technology of sparse operators to extend results to SHT as well as to greatly simplify the proofs even in $\mathbb{R}^n$. This led to other work in weighted norm inequalities such as resolving the boundedness of a certain maximal function in [1] (based on work of Pérez in [66]).

Another direction where technology involving CZOs was used is weighted norm inequalities where the weights are different. Precise conditions on the weights $w$ and $v$ that ensure the following $L^p$ bound for all CZOs

$$\int_{\mathbb{R}^n} (Tf)^p w dx \leqslant C \int_{\mathbb{R}^n} f^p v dx \tag{1.4}$$

are not known. This difficult problem has recently been attacked on an operator by operator basis, see for example [45] [44]. Classical extrapolation due to Rubio de Francia allows one to get $L^p$ bounds from weighted $L^2$ ones. These techniques are frequent in modern harmonic analysis (see the thorough reference [18]). Extrapolation techniques involving $A_p$ weights have been used often in one-weighted theory to obtain $L^p$ bounds. In the two-weighted theory there is no single natural substitute for the $A_p$ condition. However, by relying on a different kind of weight called the *reverse Hölder weights*, I introduced a new two-weighted extrapolation technique to obtain $L^p$ bounds, such as the one above, via a weighted $L^1$ bound of a CZO by a maximal function. The novelty in this extrapolation technique is that instead of working with either of the given weights $w$ or $v$, we insert a new weight $\rho$ from a different weight class [1].

The organization of this thesis is as follows: in Chapter 2 we gives basic background information about spaces of homogeneous type, dyadic systems, Calderón-Zygmund operators, weights. and norms. Chapter 3 proves the the main theorem 1 as well as the $A_2$ theorem on Spaces of Homogeneous type, and explains the deep connections between the two. Chapter 4 shows how techniques used in Chpater 3 can be used in proving a variety of other sharp weighted bounds. Finally, Chapter 5 presents new results on the two-weighted boundedness of Calderón-Zygmund operators and $L^p$ boundedness of the Orlicz maximal function, richly involved with the two-weighted theory.

*Remark* 1. Chapter 3 was partially taken from the published manuscript [5] that was jointly written by both authors. Chapter 4 was partially taken from the submitted manuscript [3] that was jointly written by both authors. Chapter 5 was partially taken from the accepted manuscript [1] that was written by the author.

CHAPTER **Two**

# Background

### 2.0.1 Spaces of homogeneous type

A space of homogeneous type is a general extension of $\mathbb{R}^n$. Specifically,

*Definition* 1. Following [10] we define a space of homogeneous type as a triple $(X, \rho, \mu)$ where $X$ is a set, $\rho$ is a quasimetric, that is:

1. $\rho(x, y) = 0$ if and only if $x = y$,

2. $\rho(x, y) = \rho(y, x)$ fo all $x, y \in X$,

3. $\rho(x, z) \leqslant C_0(\rho(x, y) + \rho(y, z))$ for all $x, y, z \in X$

for some absolute constant $C_0 > 0$ (property 3 is called the quasitriangle inequality), and the positive measure $\mu$ is *doubling*:

$$0 < \mu(B(x_0, 2r)) \leqslant C_d \mu(B(x_0, r)) < +\infty$$

for some absolute constant $C_d$.

We say that a constant is absolute if it only depends on the space $(X, \rho, \mu)$.

We will use the abbreviation SHT to refer to both the singular and plural form of space of homogeneous type. For a discussion of these spaces, see the original paper of Coifman and Weiss [15] and Christ [10].

Some examples of SHT include:

1. $\mathbb{R}^n$ with Lebesque measure
2. $\mathcal{C}^\infty$ compact Riemannian manifolds
3. Graphs of Lipschitz functions F where $\mu(F(E)) = |E|$
4. Certain Cantor sets using Hausdorff measure (fractal connection)
5. $\mathbb{Z}$ with counting measure
6. Homogeneous space from algebra/Riemann surface theory (A group G acting continuously and transitively on a topological space X), is a particular case

These spaces are important since SHT are some of the most general extensions of $\mathbb{R}^n$ where all of the following hold:

1. Vitali covering lemma: a way to decompose a set into simpler geoemtric objects
2. Whitney covering lemma - a different decompostion of sets into simpler objects
3. Lebesgue dominated convergence
4. Calderón-Zygmund decomposition - a way of bounding functions on certain spaces, to be discussed later
5. $T(1)$ theorem - a famous theorem of David and Journé on testing simple classes of functions to obtain bounds of operators

6. Lebesgue Differentiation Theorem - describes the almost everywhere convergence of averages

The Lebesgue Differentiation theorem has been proved with additional related assumptions, but as far as we are aware was proved for the first time in [2]. We include this proof here, along with two remarks and a corollary, mentioning that the proof of the lemma consists of assembling pieces already present in the literature and can easily be derived from Toledano [79]. We use the notation $⨍_Q = \frac{1}{\mu(Q)} \int_Q$

**Lemma 1.** *Given an SHT $(X, \rho, \mu)$, the Lebesgue differentiation theorem holds: for $\mu$-almost every $x \in X$,*

$$\lim_{r \to 0} \frac{1}{\mu(B_\rho(x, r))} \int_{B_\rho(x,r)} |f(y) - f(x)|\, d\mu(y) = 0. \tag{2.1}$$

*Proof.* Macías and Segovia, building on their earlier work in [58], showed in [57] that given any SHT $(X, \rho, \mu)$, there exists an equivalent quasidistance $\delta$ (i.e., there exist constants $c_1$, $c_2$ depending on $X$ such that for all $x$, $y \in X$, $c_1\rho(x, y) \leqslant \delta(x, y) \leqslant c_2\rho(x, y)$), such that given any ball $B_\delta$ with respect to $\delta$, then $(B_\delta, \delta, \mu)$ is again a space of homogeneous type, and the constants are independent of the ball $B_\delta$.

Toledano [79] proved that since $\mu(B_\delta) < \infty$, the measure $\mu$ when restricted to $B_\delta$ is regular. The Lebesgue differentiation theorem holds for regular measures: this follows from the standard argument (cf. Rudin [75, Chapter 7]) using the fact that the maximal operator is weak $(1, 1)$ on $L^1(B_\delta, \mu)$ (Christ [10]) and that smooth functions of compact support are dense in $L^1(B_\delta, \mu)$ ([75, Chapter 3]). Therefore, we have that for $\mu$-almost every $x \in B_\delta$,

$$\lim_{r \to 0} ⨍_{B_\delta(x,r)} |f(y) - f(x)|\, d\mu(y) = 0.$$

Since $\rho$ and $\delta$ are equivalent and $\mu$ is doubling, it follows immediately that (2.1) holds in $B_\rho$. Since $X$ can be covered by a countable collection of $\rho$-balls, it holds for $\mu$-almost every $x \in X$. $\square$

*Remark* 2. As a corollary to this proof we also have that $C_c^\infty(X)$ is dense in $L^1(X, \mu)$. This fact, together with Lemma 1, can be used to simplify the hypotheses for results in a number of papers: see, for example, [6, 9].

**Corollary 1.** *Given an SHT* $(X, \rho, \mu)$ *and a dyadic grid* $D$ *that satisfies the hypotheses of Theorem 10, then for* $\mu$*-almost every* $x \in X$*, if* $\{Q_k\}$ *is the sequence of dyadic cubes in* $D$ *such that* $\cap_k Q_k = \{x\}$*, then*

$$\lim_{k\to\infty} \fint_{Q_k} |f(y) - f(x)|\, d\mu(y) = 0. \tag{2.2}$$

*Proof.* First note that since $\rho$ is a quasi-distance and $\mu$ is doubling, if $x \in B_\rho(x_0, r)$, then there exists a constant $K$ such that $B_\rho(x_0, r) \subset B_\rho(x, 2Kr)$ and $\mu(B_\rho(x, 2Kr)) \approx \mu(B_\rho(x_0, r))$. Hence,

$$\fint_{B_\rho(x_0,r)} |f(y) - f(x)|\, d\mu(y) \leqslant C \fint_{B_\rho(x,2Kr)} |f(y) - f(x)|\, d\mu(y).$$

Therefore, if $B_k$ is a sequence of balls such that $\bigcap_k B_k = \{x\}$, then it follows from Lemma 1 that

$$\lim_{k\to\infty} \fint_{B_k} |f(y) - f(x)| d\mu(y) = 0. \tag{2.3}$$

Now for any $k$, by Theorem 10 there exists a ball $B_k$ such that $x \in Q_k \subset B_k$ and $\mu(B_k) \leqslant C\mu(Q_k)$. Then (2.2) follows at once from (2.3). □

*Remark* 3. Corollary 1 was stated in [5] without proof and with a reference to [79]. However, as we noted, this result was only implicit there. This result is used in chapter 3.

### 2.0.2 Dyadic systems

On the real line, dyadic cubes are intervals $[n2^k, (n+1)2^k)$ for $n, k \in \mathbb{Z}$ These have key structural properties, together making up a dyadic system. Michael Christ developed a construction of a dyadic system in SHT that preserved many key properties found in the dyadic intervals ([10]). A further sophisticated refinement was completed by Hytönen and Kairema [35].

**Theorem 2** (Hytönen and Kairema)**.** *There exists a family of sets* $D = \cup_{k\in\mathbb{Z}} D_k$*, called a dyadic decomposition of* $X$*, constants* $0 < C, \epsilon < \infty$*, and a corresponding family of points* $\{x_c(Q)\}_{Q\in D}$ *such that:*

1. $X = \bigcup_{Q \in D_k} Q$, *for all* $k \in \mathbb{Z}$.

2. *If* $Q_1 \cap Q_2 \neq \emptyset$, *then* $Q_1 \subseteq Q_2$ *or* $Q_2 \subseteq Q_1$.

3. *For every* $Q \in D_k$ *there exists at least one child cube* $Q_c \in D_{k-1}$ *such that* $Q_c \subseteq Q$.

4. *For every* $Q \in D_k$ *there exists exactly one parent cube* $Q_p \in D_{k+1}$ *such that* $Q \subseteq Q_p$.

5. *If* $Q_2$ *is a child of* $Q_1$ *then* $\mu(Q_2) \geqslant \epsilon\mu(Q_1)$.

6. $B(x_c(Q), \delta^k) \subset Q \subset B(x_c(Q), C\delta^k)$.

The motivation for using the Hytönen-Kairema cubes over the Christ cubes in this thesis was mainly due to property 4, as it was critical in a stopping time argument used in Chapter 3 that every cube have precisely one parent. In the Christ construction, only cubes on certain levels $k$ are stated to have exactly one parent (note the indexing for the Chrsit cubes is different from the one above).

We also need to use the fact that there are a finite number of *distinct* dyadic systems in SHT [35]. This is a generalization of the "1/3" trick on the real line that says that any interval can be contained in a dyadic interval or a shifted dyadic interval (by $2^{-k} \cdot \frac{1}{3}$ for level $k$) of approximately the same size. We will call a set of dyadic systems *distinct* if it satisfies Mei's criterion that every cube is contained in a cube from one of the dyadic systems of approximately the same size (described more precisely below). Therefore the dyadic and shifted dyadic system described above on the real line are distinct, and in general there are $n+1$ distinct dyadic systems in $\mathbb{R}^n$, and this number is sharp [16]. In a SHT we will use the analogous result of Hyrönen and Kairema [35] to be described further in Chapter 3 .

A way to generate a distinct dyadic system in $\mathbb{R}^n$ is to use a $\delta$-modulated dyadic grid $D_\delta$, where $\delta$ is *far from the dyadic rationals* [54]. We define $D_\delta$ to be

$$D_\delta = \bigcup_{m \in \mathbb{Z}} D_\delta^m$$

where for $m \geqslant 0$

$$D_\delta^n = \{I + \delta : I \in D_\delta^m\}$$

and for $m < 0$, $m$ even

$$D_\delta^m = \left\{ \left[ \frac{k}{2^m} + \delta + \sum_{j=(m/2)+1}^{0} 2^{-2j}, \frac{k+1}{2^n} + \delta + \sum_{j=(m/2)+1}^{0} 2^{-2j} \right] : k \in \mathbb{Z} \right\}$$

and along with the nesting property of dyadic systems completely determines $D_\delta^m$ for $m$ odd.

We say that a number is *q-far* if it is distant enough from all rationals of the form $\frac{i}{q^k}$ for $i, k \in \mathbb{Z}$. More precisely,

*Definition* 2. We say that a real number $\delta$ is q-far if there exists a $C(\delta)$ such that for all $n \in \mathbb{Z}$ and for all $k \in \mathbb{Z}$,

$$|\delta - k/q^n| \geqslant C/q^n \tag{2.4}$$

$C$ is a positive constant that may depend on $\delta$ only. We now generate some q-far numbers.

**Lemma 2.** *Let $q$ be prime, and $p$ be a prime such that $q \neq p$. We have that $1/p$ is q-far.*

*Proof.* We must show 2.4, which is equivalent to showing that

$$\left| \frac{q^n - kp}{q^n} \right| \geqslant Cp/q^n$$

or that

$$q^n \geqslant p(C + k) \text{ or } q^n \leqslant p(k - C).$$

Now let $C = 1/p$. Then for all $n, k$, we need $q^n \geqslant 1 + kp$ or $q^n \leqslant pk - 1$. Since $q^n \in \mathbb{Z}$, the only time our requirement would not be satisfied is when $q^n = kp$. So for $1/p$ to be q-far, we need

$$q^n \not\equiv 0 \pmod{p}$$

for all $n$. However, we have that $gcd(p, q) = 1$, so $gcd(q^n, p) = 1$, which implies that $q^n \not\equiv 0$

(mod $p$) for all $n$.

$\square$

When $q = 2$, we also call 2-far "far from the dyadic rationals". Note that $1/3$ is 2-far.

*Remark* 4. The choice of $C$ above is the best possible. This is due to an argument using Fermat's Little Theorem (FLT). We need to show that if $C > 1/p$ then the criterion 2.4 fails for $p$ and $q$ (where we still have $\gcd(p, q) = 1$). So if $C = 1/p + \varepsilon$ for any $\varepsilon > 0$, then we would need to show that

$$q^n \geqslant pk + 1 + p\varepsilon \text{ or } q^n \leqslant pk - 1 - p\varepsilon.$$

Thus we would need to prevent $q^n = pk + 1$ and $q^n = pk - 1$ for all $n$ and $k$. But by FLT,

$$q^{p-1} \equiv 1 \pmod{p},$$

so

$$q^{p-1} = pk + 1,$$

a contradiction. Thus $C = 1/p$ is the best possible. This $C$ of course corresponds to $d(\delta)$.

*Remark* 5. In a similar way, we can also prove that $1/m$ is q-far where $q$ does not divide $m$.

**Lemma 3.** *We have that* $\frac{h}{q^j} + \frac{1}{p}\frac{m}{q^j}$ *is q-far for* $\gcd(p, qm) = 1$.

*Proof.* We must show that

$$\left|\frac{h}{q^j} + \frac{1}{p}\frac{m}{q^j} - \frac{k}{q^n}\right| \geqslant C/q^n$$

or in other words, that

$$\left|\frac{hq^{n-j}p + mq^{n-j} - kp}{q^n}\right| \geqslant Cp/q^n.$$

When we work out the inequalities, we must show that there are no integers in the range $(\frac{q^{n-j}(m+hp)}{p} - C, \frac{q^{n-j}(m+hp)}{p} + C)$. Letting $C = 1/p$, we can easily see that there are no integers in the range $(\frac{q^{n-j}(m+hp)-1}{p}, \frac{q^{n-j}(m+hp)+1}{p})$ since $\gcd(p, q^{n-j}(m+hp)) = 1$ so $(q^{n-j}(m+hp))^{p-1} \equiv 1 \pmod{p}$. $\square$

*Remark* 6. An easy observation is that any repeating decimal expansion in base $q$ is q-far. This is because any $q$-rational has a finite decimal expansion, so upon subtraction of any finite decimal, a repeating decimal will always be bounded away from 0.

In [54] is the fact that on $\mathbb{R}$, if $\delta$ is far from the dyadic rationals, then $D_\delta$ satisfies a generalization of the "1/3 trick;" this is called Mei's criterion since it was originally done for the torus by T. Mei (note also that Mei uses a slightly different $D_\delta$ [59]). Mei's criterion says that any cube in is comparable to a cube in $D$ or $D_\delta$. More precisely, stated here for our $D_\delta$ on the real line:

**Proposition 1.** *Let $\delta \in (0,1)$ be far from the dyadic rationals. Then there exists a constant $K(\delta)$ such that for each interval $Q \in \mathbb{R}$, there exists an interval $I \in \mathbb{R}$ such that*

$$i) Q \subset I$$

$$ii) |I| \leqslant K(\delta)|Q|$$

$$iii) I \in D \text{ or } I \in D_\delta$$

*This constant $K(\delta)$ can be taken to be $2/\mathcal{C}(\delta)$*

*Remark* 7. Note that in the statement in [54], $K(\delta)$ is stated differently, but can be easily seen to be equivalent to $2/\delta$. Their proof follows Mei's proof for the torus [54].

*Remark* 8. Note that one can easily extend this result to $\mathbb{R}^n$ by repeating the argument for each of the $n$ sides of the cube $Q$ and show that each side is contained in some interval $I \in D$ or $I \in D_\delta$. There are $2^n$ choices of n-tuples of $D$ and $D_\delta$'s, so with $2^n$ grids, Mei's criterion holds in $\mathbb{R}^n$. This number was achieved by Hytönen and Pérez [36].

We use a variant of Mei's criterion in a key point in Chapter 3.

Given a dyadic grid $D$, we define a dyadic maximal function

$$M^D f(x) = \sup_{Q \ni x, Q \in D} \fint_Q |f| d\mu = \sup_{q \ni x, Q \in D} \|f\|_Q.$$

We will also need a *Calderón-Zygmund decomposition.* This is a way of decomposing a function into level sets. We use a decomposition used by the author, David Cruz-Uribe, and Kabe Moen in [2]. The following description of this decomposition was taken from this article.

To begin to define a Calderón-Zygmund deomposition, we need the dyadic maximal function to construct level sets.

**Theorem 3.** *Given an SHT $(X, \rho, \mu)$ such that $\mu(X) = \infty$, a dyadic grid $D$, suppose that $f$ is a measurable function such that $\|f\|_Q := \fint_Q |f| d\mu \to 0$ as $\mu(Q) \to \infty$. Then the following are true:*

1. *For each $\lambda > 0$, there exists a collection $\{Q_j\} \subset D$ that is pairwise disjoint, maximal with respect to inclusion, and such that*

$$\Omega_\lambda = \{x \in X : M^D f(x) > \lambda\} = \bigcup_j Q_j.$$

*Moreover, there exists a constant $C(X)$ such that for every $j$,*

$$\lambda < \|f\|_{Q_j} \leqslant C(X)\,\lambda.$$

2. *Given $a > 2/\epsilon$, where $\epsilon$ is as in Theorem 2, for each $k \in \mathbb{Z}$ let $\{Q_j^k\}_j$ be the collection of maximal dyadic cubes in* (1) *with*

$$\Omega_k = \{x \in X : M^D f(x) > a^k\} = \bigcup_j Q_j^k.$$

*Then the set of cubes $S = \{Q_j^k\}$ is sparse (see definition 7), and $E(Q_j^k) = Q_j^k \backslash \Omega_{k+1}$.*

*If $\mu(X) < \infty$, then* (1) *holds provided that $\lambda > \fint_X |f(x)|\, d\mu(x)$, and* (2) *holds for all $k$ such that $a^k > \fint_X |f(x)|\, d\mu(x)$.*

The proof of Theorem 3 is essentially identical to that in the Euclidean in case: see, for example, [18, Appendix A.1]. The constant $C(X)$ in (1) depends on the doubling constant of

$\mu$. When $\mu(X) < \infty$ some minor modifications to the proof are necessary; these correspond to what is often referred to as a "local" Calderón-Zygmund decomposition. To make them it suffices to note that in this case $X$ is bounded (see [29]). Therefore, by Theorem 2, for all dyadic cubes $Q$ sufficiently large, $X = Q$, and so the argument for (1) still holds if we take $\lambda > \fint_X |f(x)|\, d\mu(x) = \fint_Q |f(x)|\, d\mu(x)$.

**Theorem 4.** *Given an SHT $(X, \rho, \mu)$ such that $\mu(X) = \infty$, and a dyadic grid $D$, suppose $f$ is a function such that $\|f\|_Q \to 0$ as $\mu(Q) \to \infty$. Then for any $\lambda > 0$ there exists a pairwise disjoint family $\{Q_j\} \subset D$ and functions $b$ and $g$ such that:*

1. *$f = b + g$;*
2. *$g = f\chi_{\{x:M^D f(x) \leqslant \lambda\}} + \sum_j f_{Q_j}$;*
3. *for $\mu$-a.e. $x \in X$, $|g(x)| \leqslant C(X)\lambda$;*
4. *$b = \sum_j b_j$, where $b_j = (f - f_{Q_j})\chi_{Q_j}$;*
5. *the support of $b_j$ is a subset of $Q_j$ and $\fint_{Q_j} b_j(x)\, d\mu(x) = 0$.*

*If $\mu(X) < \infty$, then this decomposition still exists if we take $\lambda > \fint_X |f(x)|\, d\mu(x)$.*

Theorem 4 is proved exactly as in the Euclidean case, taking $\{Q_j\}$ to be the cubes from (1) in Theorem 3. The proof that $g \in L^\infty$ requires the Lebesgue differentiation theorem, Corollary 1.

### 2.0.3 Weights and operators

The basis of much of our analysis is on $L^p$ spaces, the space of functions on $(X, \mu)$ with bounded $L^p$ norm:

$$\|f\|_{L^p(X)} := \left(\int_X |f|^p d\mu\right)^{1/p} < \infty.$$

We typically take $1 < p < \infty$, (for $p < 1$, this quantity is no longer a norm, but we will not be using this case).

We will also need the $L^\infty$ norm, which measures the supremum of a function outside a set of measure 0:

$$\|f\|_{L^\infty(X)} = ess\sup\{|f(x)|\}$$

Then $L^\infty(X) = \{f : \|f\|_{L^\infty(X)} < \infty\}$.

We typically use the shorthand $\|f\|_{L^p}$ or $\|f\|_p$ and similarly for $L^\infty$.

We also use the space BMO, which is the space a bounded mean osciallation. It contains $L^\infty$ and useful in many applications since many operators map $L^\infty$ to BMO.

*Definition* 3. Let $\|f\|_{BMO} = \sup_Q \fint_Q |f - \fint_Q |f||d\mu$. We say $f \in BMO$ if $\|f\|_{BMO} < \infty$.

We define a Calderón-Zygmund Kernel in both $\mathbb{R}^n$ and SHT. First in $\mathbb{R}^n$:

*Definition* 4. We'll say that $K : \mathbb{R}^n \times \mathbb{R}^n \backslash \{x = y\} \to R$ is a *Calderón-Zygmund kernel* if there exist $\varepsilon > 0$ and $C < \infty$ such that for all $x_0 \neq y \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ it satisfies the decay condition:

$$|K(x_0, y)| \leqslant \frac{C}{|x_0 - y|^n} \tag{2.5}$$

and the smoothness condition for $2|x_0 - x| \leqslant |x_0 - y|$:

$$|K(x, y) - K(x_0, y)| \leqslant \frac{C|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon}}, \tag{2.6}$$

$$|K(y, x) - K(y, x_0)| \leqslant \frac{C|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon}}.$$

And in SHT:

*Definition* 5. We'll say that $K : X \times X \backslash \{x = y\} \to R$ is a *Calderón-Zygmund kernel* if there exist $\eta > 0$ and $C < \infty$ such that for all $x_0 \neq y \in X$ and $x \in X$ it satisfies the decay condition:

$$|K(x_0, y)| \leqslant \frac{C}{\mu(B(x_0, \rho(x_0, y)))} \tag{2.7}$$

and the smoothness condition for $\rho(x_0, x) \leqslant \eta\rho(x_0, y)$:

$$|K(x,y) - K(x_0,y)| \leqslant \left(\frac{\rho(x,x_0)}{\rho(x_0,y)}\right)^{\eta} \frac{1}{\mu(B(x_0,\rho(x_0,y)))}, \tag{2.8}$$

$$|K(y,x) - K(y,x_0)| \leqslant \left(\frac{\rho(x,x_0)}{\rho(x_0,y)}\right)^{\eta} \frac{C}{\mu(B(x_0,\rho(x_0,y)))}.$$

Now for both $\mathbb{R}^n$ and SHT we have

*Definition* 6. Let $T$ be a singular integral operator associated to Calderón-Zygmund kernel $K$. If in addition $T$ is $L^2$ bounded, we say that $T$ is a *Calderón-Zygmund operator*.

By associated we mean that we have the inner product

$$\langle Tf, g\rangle = \int_X \int_X K(x,y) f(x) g(y) d\mu(x) d\mu(y)$$

where $f$ and $g$ have disjoint supports.

Now we list some basic background results for operators on SHT.

**Theorem 5.** *[58] We can define an equivalent quasimetric $\rho'$, induced by the topology of the balls of $X$. If all balls are open, then $\rho$ is equivalent to $\rho'$.*

**Theorem 6.** *[10] The Hardy-Littlewood maximal functional $M(f) = \sup_B \fint_B |f(x)|$ for balls $B(x,r)$ is weak (1,1), strong (2,2), and bounded on $L^2(w)$.*

**Theorem 7.** *[13] Let $T$ be a Calderon-Zygmund operator on an SHT. Then $T$ is weak $L_1$.*

We will be invoking sparse operators to bound our Calderón-Zygmund operators using the formula originally due to Lerner [51]. Before we discuss this, we need to define a sparse family on a dyadic grid $D = \cup_k D_k$.

*Definition* 7. A sparse family $S = \cup_k S_k$, $S_k \in D_k$ on $D$ is a collection of dyadic cubes such that for $Q', Q \in S$,

$$\mu(\bigcup_{Q' \subsetneq Q} Q') \leqslant \frac{\mu(Q)}{2}.$$

*Definition* 8. Given a sparse family $S$, we define a sparse operator as follows

$$T^S(f) = \sum_{Q \in S} (\fint_Q f) \cdot \chi_Q.$$

A way to construct a disjoint family from a sparse one, where the measure of each cube in $S$ is approximately the same as a disjoint partner, is the following:

*Definition* 9. Let

$$E(Q) = Q \backslash \bigcup_{Q' \subsetneq Q, Q' \in S} Q'.$$

It is easy to see using the definitions that $E(Q)$ is disjoint and that

$$\mu(E(Q)) \leqslant \mu(Q) \leqslant 2\mu(E(Q)).$$

And finally, we introduce the definitions of a $BMO$ function and the iterated commutators of Calderón–Zygmund operators with functions in $BMO$ in spaces of homogeneous type.

*Definition* 10. For a locally integrable function $b : X \longrightarrow \mathbb{R}$ we define

$$||b||_{BMO} = \sup_Q \frac{1}{\mu(Q)} \int_Q |b(y) - b_Q| d\mu(y) < \infty,$$

where the supremum is taken over all dyadic cubes in $X$, and

$$b_Q = \frac{1}{\mu(Q)} \int_Q b(y) d\mu(y).$$

*Definition* 11. Given a Calderón–Zygmund operator $T$ with kernel $K$ and a function $b$ in $BMO$, we define the k-th order commutator with $b$, for an integer $k \geqslant 0$, as follows

$$T_b^k(f)(x) = \int_X (b(x) - b(y))^k K(x,y) f(y) d\mu(y).$$

In the particular case when $k = 1$, $T_b^1$ is the classic commutator and we will denote it by $T_b$.

Here we define weights:

A weight $w$ is a nonnegative locally integrable function on $(X, \mu)$ that takes values in $(0, \infty)$ almost everywhere. For any $1 < p < \infty$ we define the $A_p$ constant of the weight $w$ on the space of homogeneous type $X$ as follows

$$[w]_{A_p} := \sup_Q \left( \frac{1}{\mu(Q)} \int_Q w(x) d\mu \right) \left( \frac{1}{\mu(Q)} \int_Q w(x)^{1-p'} d\mu \right)^{p-1} . \tag{2.9}$$

where $Q$ is a dyadic cube. We could also define this with respect to balls as

$$\sup_{r>0} \fint_{B(x,r)} w \left( \fint_{B(x,r)} w^{1-p'} \right)^{p-1} = [w]_{A_2,B} < \infty.$$

Note that if $w \in A_p$ (with respect to balls), then $w \in A_p$ with respect to cubes (see remark 12).

### 2.0.4 Orlicz classes

The Orlicz classes and Young functions arise in Chapter 5. In [67], Carlos Pérez defined a property of Young functions termed "the $B_p$ condition", which is important to boundedness of different types of maximal functions, important in weighted theory and discussed in Chapter 5.

*Definition* 12. A Young function is a a convex, increasing, continuous function $A : [0, \infty) \to [0, \infty)$ such that $A(0) = 0$ (see [64]).

We sometimes normalize $A(1) = 1$.

*Definition* 13. Let $A$ be a Young function. The complementary Young function $\bar{A}$ is defined as:

$$\bar{A}(s) = \sup_t \{st - A(t)\},$$

(see [18]).

Now, we can use a Young function to define the Orlicz norm.

*Definition* 14. Given a Young function $A$ and a set $E$ such that $\mu(E) > 0$, define the Orlicz space norm of a function $v \in L^1_{loc}$,

$$\|v\|_{A,E} = inf\{\lambda > 0 : \fint_E A\left(\frac{|v(x)|}{\lambda}\right) d\mu \leqslant 1\}.$$

*Definition* 15. The corresponding Orlicz maximal function is

$$M_A(v(x)) = \sup_{B \ni x} \|v\|_{A,B},$$

where $B$ ranges over balls.

Note that sometimes we define this maximal function with respect to cubes or to dyadic cubes.

*Remark* 9. Note that if $A = t^p$, this norm becomes the normalized $L^p$ norm, that is: $\|f\|_{p,Q} = \left(\fint_Q f^p\right)^{1/p}$.

All Young functions satisfy *Young's inequality*, that is

$$ab \leqslant A(a) + \bar{A}(b).$$

The reason why is

$$ab = A(a) + ab - A(a) \leqslant A(a) + \sup_a\{ab - A(a)\} = A(a) + \bar{A}(b).$$

This property underlies a geometrical argument using convexity: if we graph the function $A'(t)$ and integrate, we see that

$$ab \leqslant \int_0^a A'(t)dt + \int_0^b \bar{A}'(s)ds = A(a) + \bar{A}(b)$$

since $ab$, the area of the rectangle, is less than or equal to the sum of integrals by convexity.

We also have the *generalized Hölder's inequality*:

$$\int_X fg \leqslant \|f\|_A \|g\|_{\bar{A}}.$$

This can even be stated in the context of a Banach function space (to be discussed later in Chapter 5).

*Definition* 16. A Young function $A(t)$ is doubling if

$$A(2t) \leqslant CA(t)$$

for all $t$ and for a fixed constant $C$.

Some examples of Young functions include power functions $t^s$, log bumps

$$\frac{t^p}{\ln^{1+\delta}(1+t)}$$

and loglog bumps

$$\frac{t^p}{\ln(1+t)(\ln\ln(1+t))^{1+\delta}}$$

for $\delta > 0$.

For notation, we use the standard $H \lesssim I$ to mean $H \leqslant CI$, where $C$ is a constant. We typically use the letters $C$ and $K$ to denote constants, even if they change from line to line.

CHAPTER **Three**

# A framework for representing Calderón-Zygmund operators on spaces of homogeneous type and connections to the $A_2$ theorem

Here we show that Lerner's method of local mean oscillation gives a simple proof of the $A_2$ conjecture for spaces of homogeneous type: that is, the linear dependence on the $A_2$ characteristic for weighted $L^2$ bounds of CZOs. Weighted norm estimates for singular integral operators in Euclidean space and, more generally, spaces of homogeneous type are classical results. More recently, a variety of methods have been developed which yield the sharp (linear) dependence of norm bounds for such operators with respect to the Muckenhoupt $A_2$ weight class. In [33], Hytönen gave the first complete proof of these linear bounds, the so-called $A_2$ conjecture, and in [62], the $A_2$ conjecture was proven for geometric doubling spaces by adapting the technology of random dyadic grids. Subsequently, an extremely elegant and different approach due to Lerner yielded another proof of the conjecture in the Euclidean setting [51].

We show here that Lerner's simple approach to the $A_2$ conjecture also works in the setting of spaces of homogeneous type. The main tool that is necessary to carry out this approach is the additional dyadic structure developed in [35], specifically a version of T. Mei's family of dyadic spaces whereby any ball can be covered by some "dyadic cube" of roughly the same size. Our objective then is to give a simple proof of this theorem:

**Theorem 8.** *Let $T$ be a Calderon-Zygmund operator and $X$ a SHT. Then for any $w \in A_2$,*

$$\|T\|_{L^2(w)} \leqslant C(T, X)[w]_{A_2}.$$

The simplicity of our arguments comes from a version of Lerner's local mean oscillation decomposition. This concept appeared in the work of [42] and [78] and has gained popularity through its development in several papers of Lerner [49],[51],[50]. The technique requires very few assumptions on the measure space.

The strategy of the proof is to adapt Lerner's elegant maximal decomposition of a Calderson-Zygmund operator, together with the additional dyadic structure provided by Mei's lemma, which was shown in [35]. The proof we give is completely self contained. This chapter is organized in the following way: relying on few *a priori* assumptions, we begin section 1 with a review of the median and its generalization. Spaces of homogeneous type appear next, leading to our maximal decomposition. We discretize the operator in section 3. Finally, section 4 is devoted to proving 8 by showing a linear $A_2$ bound for an operator from our maximal decomposition. The structure of the proof is as in Lerner [51], but steps 3 and 4 of Lemma 29 and Lemma 32 are new. We finally remark that all absolute constants in this chapter depend only on the space $X$ and the operator $T$.

Earlier important contributions to this subject can be found in: [81], [7], [8], [17], [20], [33], [38], [40], [41], [47], [62], [71], [72], [68], and [73].

## 3.1 Median

We collect some standard properties of the median and its generalizations. Note that few assumptions are required on the measure. Let $(Q, A, |\cdot|)$ be a measure space where $|Q| < \infty$. Let $\mu$ be the corresponding normalized measure so that $\mu(Q) = 1$. Note that for this section the measure is not yet assumed to be doubling. For a measurable function:

$$f : Q \to \mathbb{R}$$

denote by $F(a) = \mu(f < a)$ its distribution function.

*Definition* 17. The median of $f$, $m(f)$, is defined as

$$m(f) = \sup\{a \in \mathbb{R} \colon F(a) \leqslant 1/2\}.$$

Now we'll point out several properties of the median that we'll be using.

**Lemma 4.** *We have*

$$\{a \colon F(a) \leqslant 1/2\} = (-\infty; m(f)],$$

*in particular,*

$$F(m(f)) \leqslant 1/2.$$

*Proof.* The fact that the set $\{a \colon F(a) \leqslant 1/2\}$ includes its supremum follows from the left continuity of the distribution function $F$. □

**Lemma 5.** *We have*

$$\mu(f \leqslant m(f)) \geqslant 1/2.$$

*Proof.* Observe that $\mu(f \leqslant m(f)) = \lim_{\epsilon \to 0+} F(m(f) + \epsilon)$ and $F(m(f) + \epsilon) > 1/2$ by 17, thus the claim follows. □

**Lemma 6.** *If $f \leqslant g$, then $m(f) \leqslant m(g)$.*

**Lemma 7.** $|m(f)| \leqslant m(|f|)$.

*Proof.* By (6) we have $m(f) \leqslant m(|f|)$. On the other hand by (5),

$$\mu(f \leqslant m(f)) \geqslant 1/2$$

so that $\mu(f > m(f)) \leqslant 1/2$. Equivalently $\mu(-f < -m(f)) \leqslant 1/2$, hence by the definition of the median of $-f$,

$$-m(f) \leqslant m(-f).$$

This coupled with (6) gives:

$$-m(f) \leqslant m(-f) \leqslant m(|f|)$$

□

**Lemma 8.** *For any $c \in \mathbb{R}$,*

$$m(f - c) = m(f) - c$$

**Lemma 9.** *We have*

$$m(|f - m(f)|) \leqslant 2 \inf_{c \in R} m(|f - c|)$$

*Proof.* We have

$$m(|f - m(f)|) = m(|f - c - m(f - c)|) \leqslant m(|f - c| + |m(f - c)|)$$

$$= m(|f - c|) + |m(f - c)| \leqslant 2m(|f - c|)$$

where we applied 8, 6, 8 again and finally 7. □

It is important to note that the triangle inequality fails to hold for the median, as illustrated by the following example.

**Example 1.** *Let $Q = [0, 1]$ with $|\cdot|$ as Lebesgue measure. Let $f(x) = 1\chi_{[0,3/8]} + \frac{1}{4}\chi_{[3/8,1]}$ and $g(x) = \frac{1}{4}\chi_{[0,5/8]} + 1\chi_{[5/8,1]}$. Then $m(f + g) = 5/4$, but $m(f) = m(g) = 1/4$.*

However, we have the following:

**Lemma 10.**

$$m(|f|) \leqslant \|f-g\|_\infty + m(|g|)$$

*Proof.*

$$m(|f|) \leqslant m(|f-g|+|g|) \leqslant m(\|f-g\|_\infty + |g|) = \|f-g\|_\infty + m(|g|)$$

using 6 twice, followed by 8. Note that we could replace $|f-g|$ with $\|f-g\|_\infty$ inside the median since the median does not change upon removing a set of measure 0. $\square$

Lerner discusses a generalization of the median value, $\omega_\lambda$.

*Definition* 18. For $0 < \lambda < 1$ and define

$$\omega_\lambda(f) = \sup(a : F(a) \leqslant 1-\lambda).$$

*Remark* 10. The median is a special case: $\omega_{1/2}(f) = m(f)$.

If unclear from the context, we'll write $\omega_\lambda(f, Q)$ instead of $\omega_\lambda(f)$ to emphasize the domain of $f$.

The analogues of 4, 5, 6, 10 and 8 for $\omega_\lambda$ hold, for example the analogue of 10 is:

**Lemma 11.** *For any* $0 < \lambda < 1$

$$\omega_\lambda(|f|) \leqslant \|f-g\|_\infty + \omega_\lambda(|g|)$$

The analogue of 9 holds only for a range of $\lambda$'s.

**Lemma 12.** *For all* $0 < \lambda \leqslant 1/2$ *we have:*

$$\omega_\lambda(|f-m(f)|) \leqslant 2 \inf_{c\in\mathbb{R}} \omega_\lambda(|f-c|)$$

*Proof.* Arguing just like in 9 we get

$$\omega_\lambda(|f-m(f)|)=\omega_\lambda(|f-c-m(f-c)|)\leqslant\omega_\lambda(|f-c|+|m(f-c)|)\leqslant\omega_\lambda(|f-c|)+m(|f-c|),$$

but $\omega_\lambda(|f-c|)+m(|f-c|)\leqslant 2\omega_\lambda(|f-c|)$ since $m(|f|)\leqslant\omega_\lambda(|f|)$ only for $0\leqslant\lambda\leqslant 1/2$. □

We additionally point out the following lemma that will relate $\omega_\lambda$ to the maximal operator:

**Lemma 13.** *For $0<\lambda<1$ we have*

$$\lambda\omega_\lambda(|f|,Q)\leqslant\frac{1}{|Q|}\int_Q|f|$$

*Proof.* Let

$$a=\frac{1}{\lambda|Q|}\int_Q|f|.$$

We only need to consider the case $a>0$.

By Chebychev's inequality, we get

$$\mu\left(|f(x)|\geqslant a\right)=\frac{|\{x\in Q:|f(x)|\geqslant a\}|}{|Q|}\leqslant\frac{1}{a}\cdot\frac{1}{|Q|}\int_Q|f|=\lambda.$$

Consider the two possible cases

$$\mu(|f(x)|\geqslant a)<\lambda \text{ and } \mu(|f(x)|\geqslant a)=\lambda.$$

In the first case, $\mu(|f(x)|<a)>1-\lambda$, hence $\omega_\lambda(|f|,Q)\leqslant a$.

In the second case, when the Chebychev inequality is an equality, the function $|f|$ can take only two values: 0 and $a$ (up to a set of measure zero), so then $|\{x:|f|=a\}|=\lambda|Q|$. This implies $\omega_\lambda(|f|,Q)=a$ and

$$\lambda\omega_\lambda(|f|,Q)=\frac{1}{|Q|}\int_Q|f|.$$

$\square$

The following lemma relates $\omega_\lambda$ to the weak $L^{1,\infty}$ norm:

**Lemma 14.** *For any* $0 < \lambda < 1$ *we have*

$$\omega_\lambda(|f|) \leqslant \frac{1}{\lambda|Q|}||f||_{1,\infty}$$

*Proof.* Using the definition of $\omega_\lambda$, taking the complement of the set involved, we have,

$$\lambda \leqslant \mu(|f| \geqslant \omega_\lambda(|f|)) = \frac{|\{x \in Q : |f| \geqslant \omega_\lambda(|f|)\}|}{|Q|} \leqslant \frac{||f||_{1,\infty}}{\omega_\lambda(|f|)|Q|}$$

$\square$

The final lemma of this section relates $\omega_\lambda$ to the values of function via the Lebesgue differentiation theorem.

**Lemma 15.** *Let* $0 < \lambda < 1$ *and* $x_0 \in Q_n$ *where* $Q_n$ *is a sequence of sets such that*

$$\fint_{Q_n} |f - f(x_0)| \to 0.$$

*Then*

$$\omega_\lambda(f, Q_n) \to f(x_0).$$

*Proof.* By 13 we have

$$|\omega_\lambda(f, Q_n) - f(x_0)| = \omega_\lambda(|f - f(x_0)|, Q_n) \leqslant \frac{1}{\lambda}\fint_{Q_n} |f - f(x_0)| \to 0$$

$\square$

Finally we note that we can define an equivalent BMO norm using the median values as our sequence of constants instead of averages. To be precise, first recall another BMO definition:

**Lemma 16.** *Let* $\|f\|_* = \sup_Q \fint_Q |f - \fint_Q f|$ *be the usual BMO norm. Then*

$$\frac{1}{2}\|f\|_* \leqslant \sup_Q \inf_{a\in\mathbb{R}} \fint_Q |f-a| \leqslant \|f\|_*,$$

*so* $\sup_Q \inf_{a\in\mathbb{R}} \fint_Q |f-a|$ *defines an equivalent norm.*

The proof can be found in [24].

Now we define the space $BMO_m$ and show that it is equivalent to BMO. Let $m_Q(f)$ be the median with respect to the cube $Q$, that is $m(f) = \sup\{a \in \mathbb{R} \colon F(a) \leqslant |Q|/2\}$.

*Definition* 19. Let $\|f\|_m = \sup_Q \fint_Q |f - m_Q(f)|$. We say $f \in BMO_m$ if $\|f\|_m < \infty$.

**Lemma 17.** *Let* $f \in BMO$*. Given* $\{m_Q(f)\}$*, a sequence of medial values, we have that*

$$\sup_Q \fint_Q |f - m_Q(f)| \leqslant 3\|f\|_*.$$

*Proof.* By 8, we have

$$\fint_Q |f - m_Q(f)| = \fint_Q |f - c - m_Q(f-c)| \leqslant \fint_Q |f-c| + \fint_Q |m_Q(f-c)| \leqslant \frac{3}{|Q|}\int_Q |f-c|$$

by 7 and 13. Hence

$$\sup_Q \fint_Q |f - m_Q(f)| \leqslant 3 \sup_Q \inf_{a\in\mathbb{R}} \fint_Q |f-a|.$$

$\square$

Now the main result:

**Theorem 9.** *We have that* $BMO_m \approx BMO$*, that is* $\|f\|_* \approx \|f\|_m$*.*

*Proof.* Since clearly $\sup_Q \inf_{a\in\mathbb{R}} \fint_Q |f-a| \leqslant \sup_Q \fint_Q |f - m_Q(f)|$, using the lemma and putting everything together, we have

$$\frac{1}{2}\|f\|_* \leqslant \sup_Q \inf_{a\in\mathbb{R}} \fint_Q |f-a| \leqslant \sup_Q \fint_Q |f - m_Q(f)| \leqslant 3\|f\|_*,$$

completing the proof. □

*Remark* 11. Instead of using $\{m_Q(f)\}$ we could also use $\{\omega_\lambda(f,Q)\}_Q$ to define BMO. The proofs follow in the same way, except instead of the constant 3, we get $1 + 1/\lambda$.

## 3.2 Maximal decomposition

We prove the analogue of Theorem 2.3 in [51] for SHT. We recall the definition of a dyadic system with the notation that we will use in the proof. We now assume $(X, \rho, |\cdot|)$ is an SHT.

**Theorem 10.** *[10] and [35] There exist absolute constants $C > 0$, $\delta > 0$, $0 < \epsilon < 1$, a family of sets $D = \cup_{k\in\mathbb{Z}} D_k$ (called a dyadic decomposition of $X$) and a corresponding family of points $\{x_c(Q)\}_{Q\in D}$ that satisfy the following properties:*

$$X = \bigcup_{Q\in D_k} Q, \textit{ for all } k \in \mathbb{Z} \tag{3.1}$$

$$\textit{If } Q_1, Q_2 \in D \textit{ then either } Q_1 \cap Q_2 = \varnothing \textit{ or one is contained in the other.} \tag{3.2}$$

$$\textit{For any } Q_1 \in D_k \textit{ there exists at least one } Q_2 \in D_{k+1} \textit{ so that } Q_2 \subset Q_1 \textit{ (called a child of } Q_1) \tag{3.3}$$

$$\textit{and there exists exactly one } Q_3 \in D_{k-1} \textit{ so that } Q_1 \subset Q_3 \textit{ (called the parent of } Q_1)\,.$$

$$\textit{If } Q_2 \textit{ is a child of } Q_1 \textit{ then } |Q_2| \geqslant \epsilon |Q_1|. \tag{3.4}$$

$$B(x_c(Q), \delta^k) \subset Q \subset B(x_c(Q), C\delta^k) \tag{3.5}$$

*Definition* 20. For a $Q \in D$ we'll denote by $k(Q) \in Z$ the only integer for which $Q \in D_{k(Q)}$.

*Definition* 21. For a.e. $x \in X$ for any $k \in Z$ there exists a unique set denoted by $Q_k(x)$ for whom $x \in Q_k(x) \in D_k$.

We emphasize that though much of the dyadic terminology is the same for SHT and with Euclidean space, the usual concepts of centers and side lengths, and dilations of cubes do not exist as in Euclidean spaces.

*Definition* 22. Let $D$ be a dyadic decomposition of the space $X$ and $Q \in D_k$. For a number $\lambda > 1$ we'll denote by $\lambda Q$ the "dilated" set

$$\lambda Q = B(x_c(Q), \lambda C \delta^k),$$

where the constant $C$ is the one provided by 10. The set $\lambda Q$ will play the role of the dilation of the set $Q$ by the number $\lambda$. However, note that even in the Euclidean case, $\lambda Q$ is not necessarily a dilation with respect to the center of $Q$.

*Remark* 12. From 3.5 and the fact that the measure is doubling it follows that if $w$ is an $A_2$ weight then for any $Q \in D$,

$$⨍_Q w ⨍_Q w^{-1} \leqslant D_0[w]_{A_2}$$

where $D_0$ is an absolute constant.

Let $D$ be a dyadic decomposition of $X$. Fix $Q_0 \in D$ and a measurable function $f : Q_0 \to \mathbb{R}$. In the rest of this section we'll construct the maximal decomposition of $f$ (stated in 12). For convenience denote $f_m = |f - m(f, Q_0)|$.

*Definition* 23. Denote by $M$ those elements $Q$ of $D$ that lie in $Q_0$ and that are maximal under inclusion with respect to the following property:

$$m(f_m, Q) > \omega_{\varepsilon/4}(f_m, Q_0),$$

(where $0 < \epsilon < 1$ is the number provided by Theorem 10).

Take the set of parents of $M$, then take all the maximal elements in this set under usual inclusion of cubes. Denote this set by $\hat{M}$. Note that the elements of $\hat{M}$ are disjoint and for $Q \in \hat{M}$:

$$m(f_m, Q) \leqslant \omega_{\varepsilon/4}(f_m, Q_0). \tag{3.6}$$

Also,

**Lemma 18.** *We have the following:*

$$\sum_{Q \in \hat{M}} |Q| \leqslant \frac{|Q_0|}{2} \tag{3.7}$$

*Proof.* For $Q \in M$ we have that $m(f_m, Q) > \omega_{\varepsilon/4}(f_m, Q_0)$, so that

$$\frac{|Q|}{2} \leqslant |\{x \in Q \colon f_m(x) \geqslant m(f_m, Q)\}| \leqslant |\{x \in Q \colon f_m(x) > \omega_{\varepsilon/4}(f_m, Q_0)\}|.$$

where the first inequality is by 5. The elements of $M$ are disjoint, so summing over these elements we get

$$\frac{1}{2} \sum_{Q \in M} |Q| \leqslant |\{x \in Q_0 \colon f_m(x) > \omega_{\varepsilon/4}(f_m, Q_0)\}| \leqslant \varepsilon \frac{|Q_0|}{4}$$

where the second inequality is due to the analogue of 5 for $\omega_\lambda$. By 3.4,

$$\frac{\varepsilon}{2} \sum_{Q \in \hat{M}} |Q| \leqslant \frac{1}{2} \sum_{Q \in M} |Q| \leqslant \frac{\varepsilon |Q_0|}{4}.$$

□

*Remark* 13. Note that 3.7 implies that $Q_0 \notin \hat{M}$.

*Remark* 14. For any $x \in Q_0$ we have the following decomposition

$$f(x) - m(f, Q_0) = (f(x) - m(f, Q_0))\chi_{Q_0 \setminus \cup_{Q \in \hat{M}} Q}(x) +$$

$$+ \sum_{Q \in \hat{M}} (f(x) - m(f, Q))\, \chi_Q(x) + \sum_{Q \in \hat{M}} (m(f, Q) - m(f, Q_0))\chi_Q.$$

We'll be estimating parts of that decomposition. We first need the following results: see Chapter 2.

**Theorem 11.** *We call $x$ a* Lebesgue point *if $⨍_{B(x,r)} |f(x)| \to f(x)$ as $|B| \to 0$ (note that $|\cdot|$ is the homogeneous measure). If $f \in L^1(X)$ then almost every point is Lebesgue.*

**Lemma 19.** *Let $x \in Q_0$ be a Lebesgue point of $f$, then*

$$\frac{1}{|Q_k(x)|} \int_{Q_k(x)} |f - f(x)| \to 0, \qquad as\ k \to +\infty,$$

*where $Q_k(x)$ is defined in 21.*

*Proof.* To prove this, we use a few key facts. If $k$ is sufficiently large then $Q_k(x) \subset Q_0$. Since

$$B(x_c(Q_k), \delta^k) \subseteq Q_k(x) \subseteq B(x_c(Q_k), C\delta^k),$$

by applying the doubling property, we can arrive at the result. $\square$

The following lemma estimates the first term in 14.

**Lemma 20.** *For a.e. $x \in Q_0 \backslash \cup_{Q \in \hat{M}} Q$*

$$|f(x) - m(f, Q_0)| \leqslant \omega_{\varepsilon/4}(f_m, Q_0)$$

*Proof.* For an $x \in Q_0 \backslash \cup_{Q \in \hat{M}} Q$, we have that $f_m(x) > \omega_{\varepsilon/4}(f_m, Q_0)$. Since $x$ is a Lebesgue point for $f_m$, then by 19,

$$m(f_m, Q_k(x)) > \omega_{\varepsilon/4}(f_m, Q_0) \text{ for a sufficiently large } k,$$

implying that $Q_k(x)$ is a subset of an element of $M$, thus a subset of an element of $\hat{M}$. In particular, $x \in \cup_{Q \in \hat{M}} Q$, a contradiction. $\square$

*Remark* 15. Also, using 3.6 we get that for any $Q \in \hat{M}$:

$$|m(f, Q) - m(f, Q_0)| = |m(f - m(f, Q_0), Q)| \leqslant m(f_m, Q) \leqslant \omega_{\varepsilon/4}(f_m, Q_0).$$

*Remark* 16. Using the estimates 15, 20 in 14 we get that for a.e. $x \in Q_0$:

$$|f(x) - m(f, Q_0)| \leqslant$$

$$M^{\#}_{\varepsilon/4,Q_0}(f)(x)\chi_{Q_0\setminus\cup_{Q\in\hat{M}}Q}(x)+\omega_{\varepsilon/4}(|f-m(f,Q_0)|,Q_0)\chi_{Q_0}(x)+\sum_{Q\in\hat{M}}|f-m(f,Q)|\chi_Q(x),$$

where

$$M^{\#}_{\lambda,Q_0}(f)(x)=\sup\{\omega_\lambda(|f-m(f,Q)|,Q)\colon\ x\in Q\subset Q_0, Q\in D\},$$

following Lerner's notation.

**Theorem 12.** *Applying the estimate of 16 inductively (to the last term in the sum) and using 3.7 we get that for a.e. $x\in Q_0$:*

$$|f(x)-m(f,Q_0)|\leqslant M^{\#}_{\varepsilon/4,Q_0}(f)(x)+\sum_{Q\in D(Q_0)}\omega_{\varepsilon/4}(|f-m(f,Q)|,Q)\chi_Q(x),$$

*where the sum is spread over a family of sets $D(Q_0)$ which is* sparse *in $Q_0$ with respect to the dyadic decomposition $D$. Namely we have*

$$D(Q_0)=\bigcup_{n=0,1,\dots}S_n$$

*(where $n$ represents the step in the induction), where:*

*1. Each element of $D(Q_0)$ belongs to $D$ and is a subset of $Q_0$.*

*2. The elements of each family $S_n$ are disjoint.*

*2. $S_0=\{Q_0\}$.*

*3. If $n>0$ then each $Q\in S_n$ is a subset of an element of $S_{n-1}$*

*5. For any $Q'\in S_n$ we have that:*

$$\left|\bigcup_{Q'\in S_{n+1},Q'\subsetneq Q}Q'\right|\leqslant\frac{|Q|}{2}. \tag{3.8}$$

*Condition 5 is the sparsity criterion. Note how this relates to 3.7.*

## 3.3 Discretization of operator

In this section, we follow Lerner's strategy in [51] in order to reduce estimates of a Calderon-Zygmund operator to the case of a discrete operator.

**Lemma 21.** *Let $T$ be a Calderon-Zygmund operator (see 5), $Q \in D$ be an element of a dyadic decomposition $D$ of the space $X$ (see 10) and $0 < \epsilon < 1$ be a constant provided by 10. Note that we can assume that $\eta < 1$. We then have the following:*

$$\omega_{\varepsilon/4}(|Tf - m(Tf, Q)|, Q) \leqslant D_1 \sum_{i=1}^{\infty} \frac{1}{2^{i\eta}} \fint_{2^i Q} |f(y)|dy,$$

*where $2^i Q$ is defined in 22 and $D_1$ is an absolute constant.*

*Proof.* For compactness denote $x_c = x_c(Q)$ (see 21) and $k = k(Q)$ (see 20). By 22, define $Q^* = \frac{1}{\eta}Q = B(x_c, \frac{C}{\eta}\delta^k)$where $\eta$ is the constant provided by the smoothness condition for $T$ (see 5). For a function $f$ denote

$$f_1 = f \cdot \chi_{Q^*} \text{ and } f_2 = f \cdot \chi_{X \backslash Q^*}$$

By 12 and 11 we have

$$\omega_{\varepsilon/4}(|Tf - m(Tf, Q)|, Q) \leqslant 2\omega_{\varepsilon/4}(|Tf - (Tf_2)(x_c)|, Q) \tag{3.9}$$

$$\leqslant 2\omega_{\varepsilon/4}(|Tf_1 + Tf_2 - (Tf_2)(x_c)|, Q) \leqslant 2\omega_{\varepsilon/4}(|Tf_1|, Q) + 2\|(Tf_2)(x) - (Tf_2)(x_c)\|_{L_\infty(Q)}$$

For the first term in the right side of 3.9 note that the operator $T$ is of weak type $(1, 1)$, thus by 14.:

$$\omega_{\varepsilon/4}(|Tf_1|, Q) \leqslant \frac{4||Tf_1||_{1,\infty}}{\epsilon|Q|} \leqslant \frac{4||T||_{1,\infty}}{\epsilon} \cdot \frac{||f_1||_1}{|Q|} \leqslant \frac{4||T||_{1,\infty}}{\epsilon} \cdot \frac{|Q^*|}{|Q|} \fint_{Q^*} |f|$$

Note that by 3.5,

$$\frac{|Q^*|}{|Q|} \leqslant \frac{|B(x_c, C\delta^k/\eta)|}{|B(x_c, \delta^k)|},$$

which is bounded by an absolute constant due to the doubling property.

In second term in the right side of 3.9, for $x \in Q$ we have:

$$\rho(x, x_c) < C\delta^k$$

for $y \in X \backslash Q^*$ we have:

$$\frac{C}{\eta}\delta^k \leqslant \rho(x_c, y), \tag{3.10}$$

and due to 3.10 there exists unique $i > 1$ so that:

$$2^{i-1}\delta^k \leqslant \rho(x_c, y) \leqslant 2^i\delta^k.$$

We can then apply the kernel estimate

$$\begin{aligned}
|K(x,y) - K(x_c,y)| \leqslant & \left(\frac{\rho(x,x_c)}{\rho(x_c,y)}\right)^\eta \frac{1}{|B(x_c, \rho(x_c,y))|} \\
\leqslant & \frac{(2C)^\eta}{2^{i\eta}} \cdot \frac{1}{|B(x_c, 2^{i-1}\delta^k)|}\chi_{B(x_c,2^i\delta^k)}(y) \\
\leqslant & \frac{(2C)^\eta}{2^{i\eta}} \cdot \frac{|B(x_c, 2^iC\delta^k)|}{|B(x_c, 2^{i-1}\delta^k)|} \cdot \frac{1}{|2^iQ|}\chi_{2^iQ}(y),
\end{aligned}$$

where the second factor is bounded by an absolute constant due to the doubling property. Thus

$$\begin{aligned}
|(Tf_2)(x) - (Tf_2)(x_c)| \leqslant & \int_{X\backslash Q^*} |f_2(y)||K(x,y) - K(x_c,y)|dy \\
\leqslant & D_1 \sum_{i=1}^{\infty} \frac{1}{2^{i\eta}|2^iQ|} \int_{X\backslash Q^*} |f_2(y)|\chi_{2^iQ}(y)dy \\
= & D_1 \sum_{i=1}^{\infty} \frac{1}{2^{i\eta}} ⨍_{2^iQ} |f(y)|dy,
\end{aligned}$$

for some absolute constant $D_1$. □

Let $T$ be a Calderon-Zygmund operator (see 5). Let $D$ be a dyadic decomposition of the space $X$ (see 10). Fix an arbitrary $Q_0 \in D$. By (12) for a.e. $x \in Q_0$ we have:

$$|(Tf)(x) - m(Tf, Q_0)| \leqslant M^{\#}_{\varepsilon/4, Q_0}(Tf)(x) + \sum_{Q \in D(Q_0)} \omega_{\varepsilon/4}(|Tf - m(Tf,Q)|, Q)\chi_Q(x).$$

By (21), following Lerner's approach,

$$|(Tf)(x)| \leqslant |m(Tf,Q_0)| + D_2 \cdot M(f)(x) + D_1 \sum_{i=1}^{\infty} \frac{1}{2^{i\eta}} \sum_{Q \in D(Q_0)} \fint_{2^i Q} |f| \chi_Q(x),$$

where $D_2$ is a certain constant depending on the operator $T$ (on the parameter $\tau$ to be precise). Let $w$ be an $A_2$ weight. Then

$$||Tf||_{L^2(w,Q_0)} \leqslant ||m(Tf,Q_0)||_{L^2(w,Q_0)} + D_2||M(f)||_{L^2(w,Q_0)} + D_1 \sum_{i=1}^{\infty} \frac{1}{2^{i\eta}} \left\| \sum_{Q \in D(Q_0)} \fint_{2^i Q} |f| \chi_Q \right\|_{L^2(w)}$$

Note that the sharp bound for the maximal function is due to the result in [35] (stated for metric spaces, but holds for SHT as well), a generalization of Buckley's result (see 1). We'll now estimate the last term. By 14:

$$|m(Tf,Q_0)| \leqslant \frac{||Tf||_{L_{1,\infty}(Q_0)}}{|Q_0|} \leqslant ||T||_{1,\infty} \frac{||f||_{L_1(Q_0)}}{|Q_0|}$$

Applying Holder's inequality and recalling that $[w]_{A_2} \geqslant 1$, we get:

$$||m(Tf,Q_0)||_{L^2(w,Q_0)} \leqslant \frac{||T||_{1,\infty}}{|Q_0|} \left( \int_{Q_0} w \right)^{1/2} ||f||_{L_1(Q_0)} \leqslant \tag{3.11}$$

$$||T||_{1,\infty}||f||_{L^2(w,Q_0)} \left( \int_{Q_0} w^{-1} \right)^{1/2} \left( \int_{Q_0} w \right)^{1/2} \frac{1}{|Q_0|} \leqslant ||T||_{1,\infty}||f||_{L^2(w,Q_0)}[w]_{A_2}^{1/2}$$
$$\leqslant ||T||_{1,\infty}||f||_{L^2(w,Q_0)}[w]_{A_2}.$$

Thus, in order to prove 8 (i.e. the linear $A_2$ bound for $T$) if suffices to show a linear $A_2$ bound for the discrete operator:

$$A_i(f) = \sum_{Q \in D(Q_0)} \fint_{2^i Q} |f| \chi_Q, \quad i = 1, 2, \dots \tag{3.12}$$

with a norm estimate depending linearly on the parameter $i$ ($i$ is called the complexity of the discrete operator $A_i$). This will be proved in the next section.

## 3.4 Linear bound with respect to complexity of discrete operators

By Theorem 4.1 in [35] there exist constants $J \in N$ and $D > 0$ depending only the space $X$ and there exists a corresponding collection of dyadic decompositions $\{D^j : j = 1, 2 \ldots, J\}$ of the space $X$ such that for every ball $B(x, r)$ there exists some $1 \leqslant j \leqslant J$ and $Q^* \in D^j$ with $B(x, r) \subset Q^*$ and $diam(Q^*) \leqslant Dr$.

Hence in order to prove a linear $A_2$ bound for the operator 3.12 it suffices to prove a linear $A_2$ bound, depending linearly on the complexity $i$, for the operators of the following form:

$$B_{j,i}(f) = \sum_{Q \in D'(Q_0)} \frac{1}{|Q^*|} \int_{Q^*} f \cdot \chi_Q \quad j = 1, 2, \ldots, J \tag{3.13}$$

The sum in the definition of $B_{i,j}$ is spread over $D'(Q_0)$ which is a certain subset of $D(Q_0)$ which has the following property: for any $Q \in D'(Q_0)$ there exists a set in $D^j$ denoted by $Q^* \in D^j$ so that

$$2^i Q = B(x_c(Q), 2^i C \delta^{k(Q)}) \subset Q^* \tag{3.14}$$

and

$$diam(Q^*) \leqslant D \cdot 2^i C \delta^{k(Q)}, \tag{3.15}$$

where $k(Q)$ was defined in 20. Thus $D'(Q_0) = \cup S'_n$ where some levels may be empty.

In the future we'll write $B$ instead of $B_{j,i}$, suppressing the dependence on the parameters $i$ and $j$. We'll also need the formal adjoint of $B$:

$$B^*(f) = \sum_{Q \in D'(Q_0)} \frac{1}{|Q^*|} \int_Q f \cdot \chi_{Q^*}$$

The rest of the paper will be devoted to the proof of a linear $A_2$ bound of the discrete operator $B$ depending linearly on the complexity $i$, proved in 34, thus completing the proof of 8. We follow the outline from Lerner, but emphasize that due to the design of SHT, our argument in steps 3 and 4 of 29 is different. In 32, we give an argument for an estimate which is implicit, but not stated in [51].

**Lemma 22.** *For a constant $\gamma > 0$ depending on the space $X$ only,*

$$||B(f)||_2 = ||B^*(f)||_2 \leqslant \gamma ||f||_2$$

*Proof.* For any $Q \in C_n'$ denote $E(Q) = Q \backslash \bigcup_{Q_1 \in C_{n+1}} Q_1$. The sets $E(Q)$ are pairwise disjoint, and by 3.8 we have: $|E(Q)| \leqslant |Q|/2$, thus for any $f, g \in L^2(X)$:

$$\int_X B(f) \cdot g \leqslant \sum_{Q \in D'(Q_0)} \fint_{Q^*} |f| \cdot \fint_Q |g| \cdot |Q| \leqslant$$

$$2 \sum_{Q \in D'(Q_0)} \fint_{Q^*} |f| \cdot \fint_Q |g||E(Q)| \leqslant 2 \sum_{Q \in D'(Q_0)} \int_{E(Q)} (Mf \cdot Mg) \leqslant 2 \int_X (Mf \cdot Mg) \leqslant \gamma ||f||_2 ||g||_2$$

which finishes the proof. $\square$

**Lemma 23.** *There exists an absolute constant $\beta$ so that we have:*

$$||B^*(f)||_{1,\infty} \leqslant \beta i ||f||_1,$$

*Proof.* **Step 1.**

The set $\Omega = \{(Mf)(x) > a\}$ is open and bounded (since $|Q| < \infty$). Denote by $\{W_i\}_{i \in I}$ the family of elements $W_i$ of the dyadic decomposition $D$ that lie in the set $\Omega$ and which are maximal with respect to the following property $diam(W_i) \leqslant dist(W_i, \Omega^c)$. The sets $W_i$ are disjoint by maximality. Additionally $\Omega = \cup_i W_i$.

Indeed, for a point $x \in \Omega$ consider the sets $Q_k(x)$ introduced in 21. We have that $diam(Q_k(x)) \to 0$ and by quasitriangle inequality $dist(Q_i(x), \Omega^c) \geqslant dist(x, \Omega^c)/C_0$ as $k \to +\infty$. Thus $Q_k(x) \subset \Omega$ and $diam(Q_k(x)) < dist(Q_k(x), \Omega^c)$ for a sufficiently large $k \in Z$.

**Step 2.**

Set

$$b_i = \left( f - \fint_{W_i} f \right) \chi_{W_i}, \quad b = \sum_{i \in I} b_i, \quad g = f - b$$

For any $a > 0$ we have:

$$|\{x\colon |(B^*f)(x)| > a\}| \leqslant |\Omega| + |\{x\colon |(B^*g)(x)| > a/2\}| + |\{x \in \Omega^c\colon |(B^*b)(x)| > a/2\}|. \qquad (3.16)$$

In order to prove 29 we estimate the terms on the right of 3.25. For the first term we use that the maximal function is weak $L_1$, thus:

$$|\Omega| \leqslant \frac{D_2}{a}||f||_1,$$

for an absolute constant $D_2$.

**Step 3.**

For the second term we first note that

$$g(x) \leqslant D_3 \cdot a \qquad (3.17)$$

for some absolute constant $D_3 > 0$. Indeed, if $x \in \Omega^c$ then $|g(x)| = |f(x)| \leqslant a$.

Now if $x \in \Omega$ then $x \in W_i$ for some $i \in I$. Denote by $W_i^p$ the parent of $W_i$ in the dyadic decomposition $D$. Two cases are possible:

**Case 1.** $diam(W_i^p) > dist(W_i^p, \Omega^c)$.

**Case 2.** $W_i^p$ contains a point in $\Omega^c$.

Consider case 1. There exist $y \in \Omega^c$ and $x_1 \in W_i^p$ so that

$$diam(W_i^p) > \rho(x_1, y).$$

Let $x_2$ be an arbitrary point in $W_i$. By quasitriangle inequality:

$$\rho(x_2, y) \leqslant C_0(\rho(x_2, x_1) + \rho(x_1, y)) < 2C_0 diam(W_i^p) \qquad (3.18)$$

Hence

$$W_i \subset B(y, 2C_0 diam(W_i^p)) =: B \qquad (3.19)$$

But $y \in \Omega^c$ so that $(Mf)(y) \leqslant a$. Hence, we can estimate:

$$|g(x)| = \left| \fint_{W_i} f \right| \leqslant \frac{1}{|W_i|} \int_B |f| \leqslant \frac{|B|}{|W_i|} \fint_B |f| \leqslant$$

$$\leqslant \frac{|B|}{|W_i|}(Mf)(y) \leqslant \frac{|B|}{|W_i|} a$$

In order to estimate the fraction $|B|/|W_i|$, take $x_2$ to be $x_c(W_i)$ in 3.27 to claim:

$$\rho(x_c(W_i), y) < 2C_0 diam(W_i^p)$$

which together with 3.28 implies that

$$\rho(x_c(W_i^p), 2C_0 diam(W_i^p)) \leqslant C_0(\rho(y, x_c) + \rho(y, 2C_0 diam(W_i^p))) \leqslant 2C_0(2C_0 diam(W_i^p)$$

so it follows that

$$B \subset B(x_c(W_i), 4C_0^2 diam(W_i^p)).$$

So that we can estimate.

$$\frac{|B|}{|W_i|} \leqslant \frac{|B(x_c(W_i), 4C_0^2 diam(W_i^p))|}{|W_i|}$$

But due to 3.5 that quantity is bounded by an absolute constant by the doubling property.

Note that in case 2 it is still true that for a certain $y \in \Omega^c$ we have $(Mf)(y) \leqslant a$ and $W_i \subset B$ so that case 2 can be treated similarly.

Now we are ready to estimate the second term of 3.25. Use 22 and 3.26 to write:

$$|\{x \colon |(B^*g)(x)| > a/2| \leqslant \frac{4}{a^2}||B^*(g)||_2^2 \leqslant \frac{4\gamma}{a^2}||g||_2^2 \leqslant \frac{4\gamma D_3}{a}||g||_1 \leqslant \frac{4\gamma D_3}{a}||f||_1$$

**Step 4.** In order to estimate the third term of 3.25 fix $x \in \Omega^c$ and consider:

$$(B^*b)(x) = \sum_{i \in l} \sum_{Q} \left( \frac{1}{|Q^*|} \int_Q b_i \right) \cdot \chi_{Q^*}(x) \tag{3.20}$$

Assume that for certain $Q$ and $W_i$ we have that:

$$\left(\int_Q b_i\right) \cdot \chi_{Q^*}(x) \neq 0$$

Then $Q \cap W_i \neq \varnothing$. But also $W_i \not\subset Q$ as $\int_{W_i} b_i = 0$ by definition. Finally both $W_i$ and $Q$ belong to the same dyadic decomposition $D$. Thus we must have that $Q \subset W_i$. Also $\chi_{Q^*}(x) \neq 0$ implies $x \in Q^*$. Using all of these and 3.14, 3.15 we can estimate:

$$diam(W_i) \leqslant dist(W_i, \Omega^c) \leqslant dist(x_c(Q), x) \leqslant diam(Q^*) \leqslant D \cdot 2^i C \delta^{k(Q)} \leqslant 2^i \cdot DC \cdot diam(Q) \quad (3.21)$$

Recall that by 20 $k(W_i)$ denotes the only integer for whom $W_i \in D_{k(W_i)}$. We have:

$$diam(W_i) \geqslant \delta^{k(W_i)} \leqslant 2^i \cdot DC diam(Q) \leqslant 2^i \cdot DC \cdot 2C_0 C \delta^{k(Q)} \quad (3.22)$$

We have that for some absolute constants $D_4, D_5 > 0$,

$$k(W_i) \leqslant k(Q) < k(W_i) + D_4 \cdot i + D_5$$

where the left hand side follows from $Q \subset W_i$ and right hand side follows from plugging 3.31 into 3.30. Thus, for our fixed $x \in \Omega^c$ we can write

$$(B^*b)(x) \leqslant \sum_{l \in I} \sum_{Q\ :\ k(W_l) \leqslant k(Q) \leqslant D_4 i + k(W_l) + D_5} \left(\frac{1}{|Q^*|}\int_Q |b_l|\right) \cdot \chi_{Q^*}(x),$$

but the sets $Q$ for which the number $k(Q)$ is the same are disjoint, thus

$$\int_{\Omega^c} |(B^*b)(x)| \leqslant \sum_{l \in I} \sum_{Q\ :\ k(W_l) \leqslant k(Q) \leqslant D_4 i + k(W_l) + D_5} \left(\int_Q |b_l|\right) \leqslant$$

$$\sum_{l \in I} (D_4 i + D_5) \int_{W_l} |b_l| \leqslant 2(D_4 k + D_5)||f||_1$$

So that we can estimate the third term in 3.25 as follows:

$$|\{x \in \Omega^c \colon |(B^*b)(x)| > a/2\}| \leqslant \frac{2}{a}||B^*b||_{L_1(\Omega^c)} \leqslant \frac{2(D_4 + D_5)i||f||_1}{a}$$

$\square$

**Lemma 24.** *For any $Q_1 \in D^j$ we have that:*

$$\omega_{\epsilon/4}(|B^*(f) - m(B^*(f))|, Q_1) \leqslant D_6 i \fint_{Q_1} |f|,$$

*where $\epsilon$ is the constant provided by 10, $B = B_{i,j}$ and $D_6$ is an absolute constant.*

*Proof.* Note that the function

$$\sum_{Q\colon Q \in D'(Q_0), Q_1 \subsetneq Q^*} \left( \frac{1}{|Q^*|} \int_Q f \right) \chi_{Q^*}(x)$$

is constant on $Q_1$ as both $Q_1$ and the sets $Q^*$ involved in the sum are in $D^j$. Thus, using 12 we can write:

$$\begin{aligned}
\omega_{\epsilon/4}(|B^*(f) - m(B^*(f))|, Q_1) \leqslant \quad & \omega_{\epsilon/4}\left( \left| B^*(f) - \sum_{Q\colon Q \in D'(Q_0), Q_1 \subsetneq Q^*} \left( \frac{1}{|Q^*|} \int_Q f \right) \chi_{Q^*}(x) \right|, Q_1 \right) \\
= \quad & \omega_{\epsilon/4}\left( \left| \sum_{Q\colon Q \in D'(Q_0), Q^* \subset Q_1} \left( \frac{1}{|Q^*|} \int_Q f \right) \chi_{Q^*}(x) \right|, Q_1 \right) \\
= \quad & \omega_{\epsilon/4}\left( |B^*(f \cdot \chi_{Q_1})|, Q_1 \right) \leqslant \omega_{\epsilon/4}\left( B^*(|f| \cdot \chi_{Q_1}), Q_1 \right)
\end{aligned}$$

Further by 14 and 29 we have

$$\omega_{\epsilon/4}\left( B^*(|f| \cdot \chi_{Q_1}), Q_1 \right) \leqslant \frac{4}{\epsilon |Q_1|} ||B^*(|f| \cdot \chi_{Q_1})||_{L_{1,\infty}(Q_1)} \leqslant \frac{4}{\epsilon |Q_1|} \beta i ||f||_{L_1(Q_1)} \leqslant \frac{4\beta}{\epsilon} i \fint_{Q_1} |f|$$

$\square$

**Lemma 25.**

$$||B^*(f)||_{L^2(w)} \leqslant D_7 \cdot i[w]_{A_2} ||f||_{L^2(w)},$$

*where $B = B_{i,j}$ and $D_7$ is an absolute constant.*

*Proof.* Let $Q_2 \in D^j$ be a set satisfying:

$$diam(Q_2) > D \cdot 2^i C \delta^{k(Q_0)},$$

where $Q_0$ is the set involved in the definition of $B$ (see 3.13). (This can always be achieved by taking $Q_2$ from a sufficiently high level of the dyadic decomposition $D^j$). Then for any $Q^*$ from 3.13 using 3.15 we have that $diam(Q^*) < diam(Q_2)$ and hence

$$Q_2 \not\subset Q^* \tag{3.23}$$

Apply the maximal decomposition 12 for the function $B^*(f)$ taking the $Q_0$ of 12 to be $Q_2$ and apply the estimates obtained in 30 to estimate:

$$|(B^*f)(x)| \leqslant |m(B^*f, Q_2)| + D_6 \cdot k \cdot (Mf)(x) + D_6 \cdot i \cdot \sum_{Q_1 \in D^j(Q_2)} \fint_{Q_1} |f| \quad \text{for a.e. } x \in Q_2$$

And thus:

$$\begin{aligned} ||B^*f||_{L^2(w,Q_2)} &\leqslant ||m(B^*f, Q_2)||_{L^2(w,Q_2)} + D_6 \cdot i \cdot ||Mf||_{L^2(w,Q_2)} \\ &+ D_6 \cdot i \cdot \left|\left| \sum_{Q_1 \in D^j(Q_2)} \fint_{Q_1} |f| \chi_{Q_1} \right|\right|_{L^2(w,Q_2)} \end{aligned} \tag{3.24}$$

The first term of 3.33 will be estimated in 32 and the second term in 33, thus finishing the proof of 31.

**Lemma 26.**

$$||m(B^*f, Q_2)||_{L^2(w,Q_2)} \leqslant \beta i ||f||_{L^2(w,Q_2)} [w]_{A_2},$$

*Proof.* Decompose:

$$B^*f = \sum_{Q \in D'(Q_0), Q^* \subset Q_2} \frac{1}{|Q^*|} \int_Q f \cdot \chi_{Q^*} + \sum_{Q \in D'(Q_0), Q_2 \subsetneq Q^*} \frac{1}{|Q^*|} \int_Q f \cdot \chi_{Q^*}$$

$$+ \sum_{Q \in D'(Q_0), Q^* \cap Q_2 = \varnothing} \frac{1}{|Q^*|} \int_Q f \cdot \chi_{Q^*}.$$

Here the third term will be identically zero on $Q_2$ and the second term is zero due to 3.32. Thus for $m(B^*f, Q_2)$ only the first term matters. Using $Q \subset Q^*$ we get:

$$m(B^*f, Q_2) = m(B^*(f \cdot \chi_{Q_2}), Q_2)$$

By 14 and 29:

$$|m(B^*(f \cdot \chi_{Q_2}), Q_2)| \leqslant \frac{||B^*(f \cdot \chi_{Q_2})||_{L_{1,\infty}(Q_2)}}{|Q_2|} \leqslant \beta i \frac{||f||_{L_1(Q_2)}}{|Q_2|}$$

so that applying Holder's inequality and 12 we get

$$||m(B^*f, Q_2)||_{L^2(w,Q_2)} \leqslant \beta i \frac{||f||_{L_1(Q_2)}}{|Q_2|} \left( \int_{Q_2} w \right)^{1/2} \leqslant \beta i ||f||_{L^2(w,Q_2)} [w]_{A_2}.$$

$\square$

**Lemma 27.**
$$\left\| \sum_{Q_1 \in D^j(Q_2)} \left( \fint_{Q_1} |f| \right) \chi_{Q_1}(x) \right\|_{L^2(w,Q_2)} \leqslant D_7 [w]_{A_2}$$

We first note that for any function $g \in L^2(w^{-1})$,

$$\left| \int_X \sum_{Q_1 \in D^j(Q_2)} \fint |f| \chi_{Q_1}(x) \cdot g(x) dx \right| \leqslant \sum_{Q_1 \in D^j(Q_2)} \fint_{Q_1} |f| \fint_{Q_1} |g| \cdot |Q_1| \leqslant$$

$$2 \sum_{Q_1 \in D^j(Q_2)} \fint_{Q_1} |f| \fint_{Q_1} |g| \cdot |E(Q_1)|$$

where for every $Q_1 \in D^j(Q_2)$ we denoted

$$E(Q_1) = Q_1 \backslash \bigcup_{Q \in D(Q_2), Q \subsetneq Q_1} Q$$

and used 3.8. Now let $w(Q) = \int_Q w$ and $M_{w^{-1}}(f) = \sup_Q \frac{1}{w(Q)} \int |fw| w^{-1}$. We further estimate:

$$\begin{aligned}
\sum_{Q_1 \in D^j(Q_2)} \fint_{Q_1} |f| \fint_{Q_1} |g||E(Q_1)| &\leqslant [w]_{A_2} \sum_{Q_1 \in D^j(Q_2)} \left( \frac{1}{w^{-1}(Q_1)} \int_{Q_1} |f| \right) \left( \frac{1}{w(Q)} \int_{Q_1} |g| \right) |E(Q_1)| \\
&\leqslant [w]_{A_2} \sum_{Q_1 \in D^j(Q_2)} \int_{E(Q_1)} M_{w^{-1}}(fw) M_w(gw^{-1}) \\
&\leqslant [w]_{A_2} \int_X M_{w^{-1}}(fw) M_w(gw^{-1}) \\
&\leqslant [w]_{A_2} ||M_{w^{-1}}(fw)||_{L^2(w^{-1})} ||M_w(gw^{-1})||_{L^2(w)} \\
&\leqslant D_7 [w]_{A_2} ||f||_{L^2(w)} ||g||_{L^2(w^{-1})}
\end{aligned}$$

Where we use the fact that $M_w$ is bounded on $L^2(w)$ indepedent of $[w]_{A_2}$. Taking the suprenum over $||g||_{L^2(w^{-1})} = 1$,

$$\left\| \sum_{Q_1 \in D^j(Q_2)} \left( \fint_{Q_1} |f| \right) \chi_{Q_1}(x) \right\|_{L^2(w, Q_2)} = \left\| \sum_{Q_1 \in D^j(Q_2)} \left( \fint_{Q_1} |f| \right) \chi_{Q_1}(x) \right\|_{L^2(w)} \leqslant D_7 [w]_{A_2}$$

□

**Lemma 28.**

$$||B(f)||_{L^2(w)} \leqslant D_7 \cdot i[w]_{A_2} ||f||_{L^2(w)},$$

*for* $B = B_{i,j}$.

*Proof.* For any $g \in L^2(w^{-1})$ using 31 we have:

$$\int_X B(f) g = \int_X f B^*(g) = \int_X f w^{1/2} B^*(g) w^{-1/2} \leqslant ||f||_{L^2(w)} ||B^* g||_{L^2(w^{-1})}$$

$$\leqslant D_7 i [w^{-1}]_{A_2} ||f||_{L^2(w)} ||g||_{L^2(w^{-1})}$$

so that the claim of the lemma and our main theorem follows. □

**Lemma 29.** *There exists an absolute constant* $\beta$ *so that we have:*

$$||B^*(f)||_{1,\infty} \leqslant \beta i ||f||_1,$$

*Proof.* **Step 1.**

The set $\Omega = \{(Mf)(x) > a\}$ is open and bounded (since $|Q| < \infty$). Denote by $\{W_l\}_{i \in I}$ the family of elements $W_l$ of the dyadic decomposition $D$ that lie in the set $\Omega$ and which are maximal with respect to the following property $diam(W_l) \leqslant dist(W_l, \Omega^c)$. The sets $W_l$ are disjoint by maximality. Additionally $\Omega = \cup_l W_l$.

Indeed, for a point $x \in \Omega$ consider the sets $Q_k(x)$ introduced in 21. We have that $diam(Q_k(x)) \to 0$ and by quasitriangle inequality $dist(Q_i(x), \Omega^c) \geqslant dist(x, \Omega^c)/C_0$ as $k \to +\infty$. Thus $Q_k(x) \subset \Omega$ and $diam(Q_k(x)) < dist(Q_k(x), \Omega^c)$ for a sufficiently large $k \in Z$.

**Step 2.**

Set

$$b_l = \left(f - \fint_{W_l} f\right) \chi_{W_l}, \quad b = \sum_{l \in I} b_l, \quad g = f - b$$

For any $a > 0$ we have:

$$|\{x \colon |(B^* f)(x)| > a\}| \leqslant |\Omega| + |\{x \colon |(B^* g)(x)| > a/2\}| + |\{x \in \Omega^c \colon |(B^* b)(x)| > a/2\}|. \tag{3.25}$$

In order to prove 29 we estimate the terms on the right of 3.25. For the first term we use that the maximal function is weak $L_1$, thus:

$$|\Omega| \leqslant \frac{D_2}{a} ||f||_1,$$

for an absolute constant $D_2$.

**Step 3.**

For the second term we first note that

$$g(x) \leqslant D_3 \cdot a \tag{3.26}$$

for some absolute constant $D_3 > 0$. Indeed, if $x \in \Omega^c$ then $|g(x)| = |f(x)| \leqslant a$.

Now if $x \in \Omega$ then $x \in W_l$ for some $l \in I$. Denote by $W_l^p$ the parent of $W_l$ in the dyadic decomposition $D$. Two cases are possible:

**Case 1.** $diam(W_l^p) > dist(W_l^p, \Omega^c)$.

**Case 2.** $W_l^p$ contains a point in $\Omega^c$.

Consider case 1. There exist $y \in \Omega^c$ and $x_1 \in W_l^p$ so that

$$diam(W_l^p) > \rho(x_1, y).$$

Let $x_2$ be an arbitrary point in $W_l$. By quasitriangle inequality:

$$\rho(x_2, y) \leqslant C_0(\rho(x_2, x_1) + \rho(x_1, y)) < 2C_0 diam(W_l^p) \tag{3.27}$$

Hence

$$W_l \subset B(y, 2C_0 diam(W_l^p)) =: B \tag{3.28}$$

But $y \in \Omega^c$ so that $(Mf)(y) \leqslant a$. Hence, we can estimate:

$$|g(x)| = \left| \fint_{W_l} f \right| \leqslant \frac{1}{|W_l|} \int_B |f| \leqslant \frac{|B|}{|W_l|} \fint_B |f| \leqslant$$

$$\leqslant \frac{|B|}{|W_l|}(Mf)(y) \leqslant \frac{|B|}{|W_l|} a.$$

Let $z \in B$. In order to estimate the fraction $|B|/|W_l|$, take $x_2$ to be $x_c(W_l)$ in 3.27 to claim:

$$\rho(x_c(W_l), y) < 2C_0 diam(W_l^p)$$

which together with 3.28 implies that

$$\rho(x_c(W_l^p), z) \leqslant C_0(\rho(y, x_c) + \rho(y, z)) \leqslant 2C_0(2C_0 diam(W_l^p)$$

so it follows that

$$B \subset B(x_c(W_l), 4C_0^2 diam(W_l^p)).$$

We can estimate.

$$\frac{|B|}{|W_l|} \leqslant \frac{|B(x_c(W_l), 4C_0^2 diam(W_l^p))|}{|W_l|}.$$

But due to 3.5 that quantity is bounded by an absolute constant by the doubling property.

Note that in case 2 it is still true that for a certain $y \in \Omega^c$ we have $(Mf)(y) \leqslant a$ and $W_l \subset B$ so that case 2 can be treated similarly.

Now we are ready to estimate the second term of 3.25. Use 22 and 3.26 to write:

$$|\{x\colon |(B^*g)(x)| > a/2| \leqslant \frac{4}{a^2}||B^*(g)||_2^2 \leqslant \frac{4\gamma}{a^2}||g||_2^2 \leqslant \frac{4\gamma D_3}{a}||g||_1 \leqslant \frac{4\gamma D_3}{a}||f||_1.$$

**Step 4.** In order to estimate the third term of 3.25 fix $x \in \Omega^c$ and consider:

$$(B^*b)(x) = \sum_{l\in l}\sum_{Q}\left(\frac{1}{|Q^*|}\int_Q b_l\right)\cdot \chi_{Q^*}(x) \tag{3.29}$$

Assume that for certain $Q$ and $W_l$ we have that:

$$\left(\int_Q b_l\right)\cdot \chi_{Q^*}(x) \neq 0.$$

Then $Q \cap W_l \neq \varnothing$. But also $W_l \not\subset Q$ as $\int_{W_l} b_l = 0$ by definition. Finally both $W_l$ and $Q$ belong to the same dyadic decomposition $D$. Thus we must have that $Q \subset W_l$. Also $\chi_{Q^*}(x) \neq 0$ implies $x \in Q^*$. Using all of these and 3.14, 3.15 we can estimate:

$$diam(W_l) \leqslant dist(W_l,\Omega^c) \leqslant dist(x_c(Q),x) \leqslant diam(Q^*) \leqslant D\cdot 2^i C\delta^{k(Q)} \leqslant 2^i\cdot DC\cdot diam(Q) \tag{3.30}$$

Recall that by 20 $k(W_l)$ denotes the unique integer for which $W_l \in D_{k(W_l)}$. We have:

$$diam(W_l) \geqslant \delta^{k(W_l)} \leqslant 2^i\cdot DCdiam(Q) \leqslant 2^i\cdot DC\cdot 2C_0C\delta^{k(Q)} \tag{3.31}$$

We have that for some absolute constants $D_4, D_5 > 0$,

$$k(Q) \leqslant k(W_l) < k(Q) + D_4\cdot i + D_5$$

where the left hand side follows from $Q \subset W_l$ and right hand side follows from plugging 3.31 into

3.30. Thus, for our fixed $x \in \Omega^c$ we can write

$$(B^*b)(x) \leqslant \sum_{l \in I} \sum_{Q\;:\;\; k(Q) \leqslant k(W_l) \leqslant D_4 i + k(Q) + D_5} \left( \frac{1}{|Q^*|} \int_Q |b_l| \right) \cdot \chi_{Q^*}(x),$$

but the sets $Q$ for which the number $k(Q)$ is the same are disjoint, thus

$$\int_{\Omega^c} |(B^*b)(x)| \leqslant \sum_{l \in I} \sum_{Q\;:\;\; k(Q) \leqslant k(W_l) \leqslant D_4 i + k(Q) + D_5} \left( \int_Q |b_l| \right) \leqslant$$

$$\sum_{l \in I} (D_4 i + D_5) \int_{W_l} |b_l| \leqslant 2(D_4 i + D_5) ||f||_1$$

so that we can estimate the third term in 3.25 as follows:

$$|\{x \in \Omega^c \colon |(B^*b)(x)| > a/2\}| \leqslant \frac{2}{a} ||B^*b||_{L_1(\Omega^c)} \leqslant \frac{2(D_4 + D_5) i ||f||_1}{a}$$

$\square$

**Lemma 30.** *For any $Q_1 \in D^j$ we have that:*

$$\omega_{\epsilon/4}(|B^*(f) - m(B^*(f))|, Q_1) \leqslant D_6 i \fint_{Q_1} |f|,$$

*where $\epsilon$ is the constant provided by 10, $B = B_{j,i}$ and $D_6$ is an absolute constant.*

*Proof.* Note that the function

$$\sum_{Q\colon\, Q \in D'(Q_0), Q_1 \subsetneq Q^*} \left( \frac{1}{|Q^*|} \int_Q f \right) \chi_{Q^*}(x)$$

is constant on $Q_1$ as both $Q_1$ and the sets $Q^*$ involved in the sum are in $D^j$. Thus, using 12 we can write:

$$\begin{aligned}\omega_{\epsilon/4}(|B^*(f)-m(B^*(f))|,Q_1) \leqslant\ & \omega_{\epsilon/4}\left(\left|B^*(f)-\sum_{Q:\ Q\in D'(Q_0),Q_1\subsetneq Q^*}\left(\frac{1}{|Q^*|}\int_Q f\right)\chi_{Q^*}(x)\right|,Q_1\right)\\ =\ & \omega_{\epsilon/4}\left(\left|\sum_{Q:\ Q\in D'(Q_0),Q^*\subset Q_1}\left(\frac{1}{|Q^*|}\int_Q f\right)\chi_{Q^*}(x)\right|,Q_1\right)\\ =\ & \omega_{\epsilon/4}\left(|B^*(f\cdot\chi_{Q_1})|,Q_1\right)\leqslant \omega_{\epsilon/4}\left(B^*(|f|\cdot\chi_{Q_1}),Q_1\right)\end{aligned}$$

Further by 14 and 29 we have

$$\omega_{\epsilon/4}\left(B^*(|f|\cdot\chi_{Q_1}),Q_1\right)\leqslant\frac{4}{\epsilon|Q_1|}||B^*(|f|\cdot\chi_{Q_1})||_{L_{1,\infty}(Q_1)}\leqslant\frac{4}{\epsilon|Q_1|}\beta i||f||_{L_1(Q_1)}\leqslant\frac{4\beta}{\epsilon}i⨍_{Q_1}|f|$$

□

**Lemma 31.**

$$||B^*(f)||_{L^2(w)}\leqslant D_7\cdot i[w]_{A_2}||f||_{L^2(w)},$$

*where $B=B_{j,i}$ and $D_7$ is an absolute constant.*

*Proof.* Let $Q_2\in D^j$ be a set satisfying:

$$diam(Q_2)>D\cdot 2^iC\delta^{i(Q_0)},$$

where $Q_0$ is the set involved in the definition of $B$ (see 3.13). (This can always be achieved by taking $Q_2$ from a sufficiently high level of the dyadic decomposition $D^j$). Then for any $Q^*$ from 3.13 using 3.15 we have that $diam(Q^*)<diam(Q_2)$ and hence

$$Q_2\notin Q^* \tag{3.32}$$

Apply the maximal decomposition 12 for the function $B^*(f)$ taking the $Q_0$ of 12 to be $Q_2$ and apply the estimates obtained in 30 to estimate:

$$|(B^*f)(x)| \leqslant |m(B^*f,Q_2)| + D_6\cdot i\cdot (Mf)(x) + D_6\cdot i\cdot \sum_{Q_1\in D^j(Q_2)} \fint_{Q_1} |f| \quad \text{for a.e. } x\in Q_2$$

And thus:

$$||B^*f||_{L^2(w,Q_2)} \leqslant ||m(B^*f,Q_2)||_{L^2(w,Q_2)} + D_6\cdot i\cdot ||Mf||_{L^2(w,Q_2)} \tag{3.33}$$
$$+D_6\cdot i\cdot \left\|\sum_{Q_1\in D^j(Q_2)} \fint_{Q_1} |f|\chi_{Q_1}\right\|_{L^2(w,Q_2)}$$

The first term of 3.33 will be estimated in 32 and the third term in 33, thus finishing the proof of 31.

**Lemma 32.**

$$||m(B^*f,Q_2)||_{L^2(w,Q_2)} \leqslant \beta i||f||_{L^2(w,Q_2)}[w]_{A_2},$$

*Proof.* Decompose:

$$B^*f = \sum_{Q\in D'(Q_0),Q^*\subset Q_2} \frac{1}{|Q^*|}\int_Q f\cdot\chi_{Q^*} + \sum_{Q\in D'(Q_0),Q_2\subsetneq Q^*} \frac{1}{|Q^*|}\int_Q f\cdot\chi_{Q^*}$$
$$+\sum_{Q\in D'(Q_0),Q^*\cap Q_2=\varnothing} \frac{1}{|Q^*|}\int_Q f\cdot\chi_{Q^*}.$$

Here the third term will be identically zero on $Q_2$ and the second term is zero due to 3.32. Thus for $m(B^*f,Q_2)$ only the first term matters. Using $Q\subset Q^*$ we get:

$$m(B^*f,Q_2) = m(B^*(f\cdot\chi_{Q_2}),Q_2)$$

By 14 and 29:

$$|m(B^*(f\cdot\chi_{Q_2}),Q_2)| \leqslant \frac{||B^*(f\cdot\chi_{Q_2})||_{L_{1,\infty}(Q_2)}}{|Q_2|} \leqslant \beta i\frac{||f||_{L_1(Q_2)}}{|Q_2|}$$

so that applying Holder's inequality and 12 we get

$$||m(B^*f,Q_2)||_{L^2(w,Q_2)} \leqslant \beta i \frac{||f||_{L_1(Q_2)}}{|Q_2|}\left(\int_{Q_2} w\right)^{1/2} \leqslant \beta i ||f||_{L^2(w,Q_2)}[w]_{A_2}.$$

□

**Lemma 33.**

$$\left\|\sum_{Q_1\in D^j(Q_2)}\left(\fint_{Q_1}|f|\right)\chi_{Q_1}(x)\right\|_{L^2(w,Q_2)} \leqslant D_7[w]_{A_2}$$

We first note that for any function $g \in L^2(w^{-1})$,

$$\left|\int_X \sum_{Q_1\in D^j(Q_2)} \fint |f|\chi_{Q_1}(x)\cdot g(x)dx\right| \leqslant \sum_{Q_1\in D^j(Q_2)} \fint_{Q_1}|f| \fint_{Q_1}|g|\cdot|Q_1| \leqslant$$

$$2\sum_{Q_1\in D^j(Q_2)} \fint_{Q_1}|f| \fint_{Q_1}|g|\cdot|E(Q_1)|$$

where for every $Q_1 \in D^j(Q_2)$ we denoted

$$E(Q_1) = Q_1\backslash \bigcup_{Q\in D(Q_2),Q\subsetneq Q_1} Q$$

and used 3.8. Now let $w(Q) = \int_Q w$ and $M_{w^{-1}}(f) = \sup_Q \frac{1}{w(Q)}\int |fw|w^{-1}$. We further estimate:

$$\begin{aligned}
\sum_{Q_1\in D^j(Q_2)} \fint_{Q_1}|f| \fint_{Q_1}|g||E(Q_1)| \leqslant\ & [w]_{A_2}\sum_{Q_1\in D^j(Q_2)}\left(\frac{1}{w^{-1}(Q_1)}\int_{Q_1}|f|\right)\left(\frac{1}{w(Q)}\int_{Q_1}|g|\right)|E(Q_1)| \\
\leqslant\ & [w]_{A_2}\sum_{Q_1\in D^j(Q_2)}\int_{E(Q_1)} M_{w^{-1}}(fw)M_w(gw^{-1}) \\
\leqslant\ & [w]_{A_2}\int_X M_{w^{-1}}(fw)M_w(gw^{-1}) \\
\leqslant\ & [w]_{A_2}||M_{w^{-1}}(fw)||_{L^2(w^{-1})}||M_w(gw^{-1})||_{L^2(w)} \\
\leqslant\ & D_7[w]_{A_2}||f||_{L^2(w)}||g||_{L^2(w^{-1})}
\end{aligned}$$

where we use the fact that $M_w$ is bounded on $L^2(w)$ independent of $[w]_{A_2}$. Taking the suprenum over $||g||_{L^2(w^{-1})} = 1$,

$$\left\| \sum_{Q_1 \in D^j(Q_2)} \left( \fint_{Q_1} |f| \right) \chi_{Q_1}(x) \right\|_{L^2(w,Q_2)} = \left\| \sum_{Q_1 \in D^j(Q_2)} \left( \fint_{Q_1} |f| \right) \chi_{Q_1}(x) \right\|_{L^2(w)} \leqslant D_7[w]_{A_2}$$

□

**Lemma 34.**

$$||B(f)||_{L^2(w)} \leqslant D_7 \cdot i[w]_{A_2} ||f||_{L^2(w)},$$

*for* $B = B_{j,i}$.

*Proof.* For any $g \in L^2(w^{-1})$ using 31 we have:

$$\int_X B(f)g = \int_X fB^*(g) = \int_X fw^{1/2}B^*(g)w^{-1/2} \leqslant ||f||_{L^2(w)}||B^*g||_{L^2(w^{-1})}$$

$$\leqslant D_7 i[w^{-1}]_{A_2}||f||_{L^2(w)}||g||_{L^2(w^{-1})}$$

so that the claim of the lemma and our main theorem follows. □

Underlying the arguments employed above (and from [5]), is the trademark decomposition of a CZO into sparse operators, done first by Lerner in $\mathbb{R}^n$, and written out is full detail in [2]:

**Theorem 13.** *Given an SHT* $(X, \rho, \mu)$ *and a Calderón-Zygmund operator* $T$*, then for any Banach function space* $Y$ *(see 5.1 for a discussion),*

$$\|T(f)\|_Y \leqslant C(X,T) \sup_{D,S} \|T^S(f)\|_Y, \tag{3.34}$$

*where the supremum is taken over every dyadic grid* $D$ *in* $(X, \rho, \mu)$ *and over every sparse family* $S$ *in* $D$*.*

Taking $Y = L^2(w)$ is the case of interest and easily yields the $A_2$ theorem by reduction to the sharp constants for sparse operators.

We include the proof that appeared in [2] for completion, based on original notes of the author derived from [5].

*Proof.* As was proved in [34, Proposition 4.3], if we fix a point $x_0 \in X$, we can construct a dyadic grid $D^*$ satisfying Theorem 10 that contains a sequence of nested dyadic cubes $\{Q_N\}$ such that $x_0$ is the center of each cube $Q_N$ and such that $\bigcup_N Q_N = X$. Therefore, by duality and Fatou's lemma, there exists $g$ in the associate space $Y'$, $\|g\|_{Y'} = 1$, such that

$$\|Tf\|_Y = \int_X |Tf(x)|g(x)\,d\mu(x) \leqslant \liminf_{N\to\infty} \int_{Q_N} |Tf(x)|g(x)\,d\mu(x).$$

Fix $N > 0$; we will prove that the final integral is bounded by $C \sup \|T^S f\|_Y$, where the supremum is taken over some collection of $S$ and $D$, but the constant is independent of these and also independent of $N$.

As was proved in [5, Section 5], there exist $C_1$, $C_2$, $\eta > 0$ such that for $\mu$-a.e. $x \in Q_N$,

$$|Tf(x)| \leqslant C_1 Mf(x) + C_2 \sum_{i=1}^{\infty} \frac{1}{2^{i\eta}} A_i f(x), \tag{3.35}$$

where

$$A_i f(x) = \sum_{Q \in S_N} ⨍_{2^i Q} f(y)\,d\mu(y) \cdot \chi_Q(x)$$

and $S_N$ is a sparse subset of $D^*$ that consists of dyadic sub-cubes of $Q_N$. The constants depend only on $X$ and $T$; in particular $C_1$ depends on the fact (see [10]) that $T : L^1(X,\mu) \to L^{1,\infty}(X,\mu)$. Therefore, by Hölder's inequality (with respect to $Y$ and $Y'$),

$$\int_{Q_N} |Tf(x)|g(x)\,d\mu(x) \leqslant C_1 \|Mf \cdot \chi_{Q_N}\|_Y + C_2 \sum_i 2^{-i\eta} \|A_i f \cdot \chi_{Q_N}\|_Y = I_1 + I_2.$$

To estimate $I_1$ we give a pointwise estimate for $Mf(x)$. By [35, Theorem 7.9] there exists a constant $K = K(X)$ and a collection $D^1, \ldots, D^K$ of dyadic grids such that for every $x \in X$,

$$Mf(x) \leqslant C(X) \sum_{k=1}^{K} M^k f(x), \tag{3.36}$$

where $M^k = M^{D^k}$ is the dyadic maximal operator defined with respect to $D^k$. We claim that for each $k$ there exists a sparse subset $S_k$ (depending on $f$) such that

$$M^k f(x) \leqslant C(X) T^{S_k} f(x). \tag{3.37}$$

This follows from (2) in Theorem 3. With the notation of this result, let $S_k = \{Q_j^i\} \in D^k$ be the sparse family. Then given $x \in \Omega_i \backslash \Omega_{i+1}$, there exists $Q_j^i$ such that

$$Mf(x) \leqslant a^{i+1} < a \fint_{Q_j^i} f(y)\, d\mu(y);$$

hence, for $\mu$-a.e. $x$,

$$M^k f(x) \leqslant a \sum_{i,j} \fint_{Q_j^i} f(y)\, d\mu(y) \cdot \chi_{Q_j^i}(x) = a T^{S_k} f(x).$$

If we now combine inequalities (3.36) and (3.37), we have that

$$I_1 \leqslant C(T, X) \|Mf\|_Y \leqslant C(T, X) \sum_{k=1}^{K} \|T^{S_k} f\|_Y \leqslant C(T, X) \sup_{D,S} \|T^S f\|_Y.$$

To estimate $I_2$ we will decompose each $A_i f$ and apply duality. By [35, Theorem 4.1] there exists a family of dyadic grids $D^1, \ldots, D^J$, satisfying the properties of Theorem 10 with the additional property that given any ball $B_\rho$, there exists $j$ and $Q^* \subset D^j$ such that $B_\rho \subset Q^*$ and $\mu(B_\rho) \approx \mu(Q^*)$, with constants depending only on $X$. Recall that $2^i Q$ is defined to be a ball. Therefore, if we define

$$S_N^j = \{Q \in S_N : \exists Q^* \in D^j, 2^i Q \subset Q^*\},$$

then

$$A_i f(x) \leqslant C(X) \sum_{j=1}^{J} \sum_{\substack{Q \in S_N^j \\ 2^i Q \subset Q^*}} \fint_{Q^*} f(y)\, d\mu(y) \cdot \chi_Q(x) = C(X) \sum_{j=1}^{J} B_{i,j} f(x).$$

Arguing as in [5, Section 6] (see especially Lemmas 6.5 and 6.13) we apply the same argument used to prove (3.35) for $T$ to the adjoint operators $B_{i,j}^*$. Key to this is the fact that adjoint operators are weak $(1,1)$ with a constant that is linear in $i$. This yields the following pointwise estimate:

$$\begin{aligned} B_{i,j}^* f(x) &\leqslant iC_1(X) M f(x) + iC_2(X) \sum_{Q \in S_*^j} \fint_Q f(y)\, d\mu(y) \cdot \chi_Q(x) \\ &= iC_1(X) M f(x) + iC_2(X) T_{i,j} f(x), \end{aligned}$$

where $S_*^j \subset D^j$ is sparse. Therefore, repeating the above argument for bounding the maximal operator, we have that $B_{i,j}^*$ is bounded pointwise by a finite sum of sparse operators $T^{S^l}$, $1 \leqslant l \leqslant L$ (defined with respect to a finite collection of dyadic grids $D$). We can now estimate $I_2$ by duality using the fact that the operators $T^{S^l}$ are self-adjoint: there exists a collection of $g_i \in Y'$, $\|g_i\|_{Y'} = 1$, such that

$$
\begin{aligned}
I_2 &= C(T,X)\sum_i 2^{-i\eta}\|A_i f\cdot\chi_{Q_N}\|_Y \\
&= C(T,X)\sum_i 2^{-i\eta}\int_{Q_N} A_i f(x) g_i(x)\,d\mu(x) \\
&\leqslant C(T,X)\sum_i 2^{-i\eta}\sum_{j=1}^{J}\int_{Q_N} B_{i,j} f(x) g_i(x)\,d\mu(x) \\
&\leqslant C(T,X)\sum_i 2^{-i\eta}\sum_{j=1}^{J}\int_{X} f(x) B_{i,j}^{*} g_i(x)\,d\mu(x) \\
&\leqslant C(T,X)\sum_i i2^{-i\eta}\sum_{j=1}^{J}\sum_{l=1}^{L}\int_{X} f(x) T^{S^l} g_i(x)\,d\mu(x) \\
&= C(T,X)\sum_i i2^{-i\eta}\sum_{j=1}^{J}\sum_{l=1}^{L}\int_{X} T^{S^l} f(x) g_i(x)\,d\mu(x) \\
&\leqslant C(T,X)\sum_i i2^{-i\eta}\sum_{j=1}^{J}\sum_{l=1}^{L}\|T^{S^l} f\|_Y\|g_i\|_{Y'} \\
&\leqslant C(T,X)\sup_{D,S}\|T^S f\|_Y.
\end{aligned}
$$

This completes the proof of Theorem 13. $\square$

CHAPTER **Four**

# More sharp weighted estimates and applications of the techniques of Lerner

After the solution of the $A_2$ conjecture, an improvement of this result was obtained in [36]. This new result can be better understood if we first consider the case of Buckley's estimate for the maximal function. For the case $p = 2$, the maximal estimate is given by [8]

$$\|M\|_{L^2(w)} \leqslant c_n\,[w]_{A_2}.$$

An idea is to replace a portion of the $A_2$ constant by another smaller constant defined in terms of the $A_\infty$ constant given by the functional

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q). \tag{4.1}$$

To be more precise, the improvement of Buckley's theorem is the following "$A_2 - A_\infty$" estimate

$$\|M\|_{L^2(w)} \leqslant c_n\,[w]_{A_2}^{1/2}\,[w^{-1}]_{A_\infty}^{1/2}$$

which can be found in [36] along with its $L^p$ counterpart. The $A_\infty$ constant as given by (4.1) was originally introduced by Fujii in [27] and rediscovered later by Wilson in [82]. This definition is more suitable than the more classical condition due to Hruščěv [32], which is defined by the expression

$$[w]_{A_\infty}^H = \sup_Q \left( \frac{1}{|Q|} \int_Q w(t)dt \right) exp \left( \frac{1}{|Q|} \int_Q \log w(t)^{-1} dt \right),$$

since

$$[w]_{A_\infty} \leqslant c_n \, [w]_{A_\infty}^H,$$

as it was observed in [36]. In fact it is shown in the same paper with explicit examples that $[w]_{A_\infty}$ is much smaller (actually exponentially smaller) than $[w]_{A_\infty}^H$.

Considering the case of singular integrals, the mixed sharp $A_2 - A_\infty$ result below obtained in [36] improves the $A_2$ theorem:

$$\|T\|_{L^2(w)} \leqslant c_n \, [w]_{A_2}^{1/2} \, ([w^{-1}]_{A_\infty}^{1/2} + [w]_{A_\infty}^{1/2}).$$

This is the right estimate when compared with $M$ since this reflects the property that $T$ is (essentially) self-adjoint.

In this paper we follow this idea of replacing a portion of the $A_p$ constant by the $A_\infty$ constant for the problems considered in [52] and improved in [53], within the context of spaces of homogeneous type.

To do this we will prove first the following proposition which follows essentially from [53], but in this stated form can be found in [70].

**Theorem 14.** *Let $T$ be a Calderón–Zygmund operator and let $1 < p < \infty$. Then for any weight $w$ and $r > 1$,*

$$||Tf||_{L^p(w)} \leqslant Cpp'(r')^{\frac{1}{p'}} ||f||_{L^p(M_r w)}. \tag{4.2}$$

This mixed theorem leads to sharp mixed $A_1 - A_\infty$ weighted bounds for Calderón–Zygmund operators using simple properties of $A_p$ weights:

$$||Tf||_{L^p(w)} \leqslant Cpp'[w]_{A_\infty}^{1/p'}||f||_{L^p(Mw)}, \tag{4.3}$$

and if $w \in A_1$,

$$||Tf||_{L^p(w)} \leqslant Cpp'[w]_{A_\infty}^{1/p'}[w]_{A_1}^{1/p}||f||_{L^p(w)}, \tag{4.4}$$

where $C$ is a dimensional constant that also depends on $T$.

Recent results relating to the simple proof of the $A_2$ conjecture have allowed us not only to simplify and streamline the proof of this theorem, but to also extend it to the versatile spaces of homogeneous type. This extension is due to the fact that every Calderón–Zygmund operator is bounded from above by a supremum of sparse operators in spaces of homogeneous type. Recall that a sparse operator is an averaging operator over a sparse family of cubes of the following form,

$$T^S f = \sum_{Q \in S} (\fint_Q f)\chi_Q,$$

where a sparse family $S$ is defined by the property that if $Q \in S$,

$$\mu(\bigcup_{Q' \subsetneq Q,\, Q' \in S}) \leqslant \frac{\mu(Q)}{2}.$$

Then the decomposition

$$\|Tf\|_Y \leqslant \sup_{S,D} \|T^S f\|_Y,$$

where $Y$ is a Banach function space, and $D$ a dyadic grid, was shown to hold in $\mathbb{R}^n$ by Lerner and was key in his proof of many weighted conjectures [51], as it was in the extension of these results to spaces of homogeneous type [2] and [5].

A similar approach can also be done with sharp weak bounds. We have the following sharp bound of Lerner, Ombosi and Pérez in [53],

$$\|T\|_{L^p(w)} \leqslant Cpp'[w]_{A_1},$$

which led to the following endpoint estimates in [36]. Namely,

1. If $w \in A_\infty$
$$||Tf||_{L^{1,\infty}(w)} \leqslant C \log(e + [w]_{A_\infty}) ||f||_{L^1(M_r w)}.$$

2. If $w \in A_1$
$$||Tf||_{L^{1,\infty}(w)} \leqslant C[w]_{A_1} \log(e + [w]_{A_\infty}) ||f||_{L^1(w)}.$$

Again, we can update this proof by applying recent estimates in sharp weighted theory as well as extend to SHT. We will employ a new sharp reverse Hölder inequality for spaces of homogeneous type (see Lemma 41). By a Calderón-Zygmund decomposition and an estimate of the maximal function that comes about from the use of sparse operators, we arrive at the result.

We are also able to extend the sharp $L^p$ and $A_q$ bounds from Duoandikoetxea [25] to SHT in as we show in the following result.

**Corollary 2.** *Let $T$ be an operator such that*

$$||Tf||_{L^p(w)} \leqslant CN([w]_{A_1}) ||f||_{L^p(w)}$$

*for all weights $w \in A_1$ and all $1 < p < \infty$, with $C$ independent of $w$. Then we have*

$$||Tf||_{L^p(w)} \leqslant CN([w]_{A_q}) ||f||_{L^p(w)} \tag{4.5}$$

*where $N$ is an increasing function, for all $w \in A_q$ and $1 \leqslant q < p < \infty$, with $C$ independent of $w$,*

Finally, we show sharp weighted bounds for commutators of Calderón–Zygmund operators with functions in $BMO$ and their iterates. These questions have been considered before in [11] and lately improved in [36] but our bounds are new in the context of spaces of homogeneous type.

Moreover, we adapt the necessary lemmas to our situation, such as the sharp John-Nirenberg inequality.

The organization of this chapter will be as follows. In Section 4.1 we give some background and definitions which will help us to prove our main results, listed in Section 4.2. Finally, Section 4.3 contains all the proofs as well as some remarks.

## 4.1 Preliminaries

Thankfully, many basic constructions and tools for classical harmonic analysis still exist in some form in spaces of homogeneous type, such as certain covering lemmas. We will especially use the Lebesgue Differentiation Theorem 1.

We also rely on a dyadic grid decomposition, and use the new construction of [35] described in the background section.

Recall the $A_p$ constant of the weight $w$ on the space of homogeneous type $X$

$$[w]_{A_p} := \sup_Q \left(\frac{1}{\mu(Q)}\int_Q w(x)d\mu\right)\left(\frac{1}{\mu(Q)}\int_Q w(x)^{1-p'}d\mu\right)^{p-1}. \tag{4.6}$$

It is crucial to note that we will take this constant with respect to cubes (by which we always will mean dyadic cubes), because the $A_\infty$ constant below is not comparable between cubes and balls in the same way, even in the classical Euclidean case.

We define the Fujii–Wilson $A_\infty$ constant in a space of homogeneous type as follows:

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)}\int_Q M(w\chi_Q)d\mu.$$

While this $A_\infty$ constant is comparable using dyadic cubes or balls, the constant of comparison depends on the measure $w$ (which is doubling since $w \in A_\infty$). Hence to achieve sharp bounds in SHT we cannot simply switch between these constants defined with respect to cubes or balls since we introduce a $w$-dependent factor. This is reflected in the difference between the sharp reverse Hölder inequalities involving this $A_\infty$ constant: if defined with respect to cubes we get a sharp reverse Hölder (see Lemma 41), but if defined with respect to balls we only get a sharp weak version (see [37]). This motivates the definition of a weak $A_\infty$ class in [4]; see this reference for more discussion.

## 4.2 Main results

Our goal is to prove the following results.

**Theorem 15.** *Let $T$ be a Calderón–Zygmund operator and let $1 < p < \infty$. Then for any weight $w$ and $r > 1$,*

$$||Tf||_{L^p(w)} \leqslant Cpp'(r')^{\frac{1}{p'}}||f||_{L^p(M_r w)}, \tag{4.7}$$

*where $C$ is an absolute constant that also depends on $T$.*

Applying a sharp version of a reverse Hölder inequality, from the previous theorem we obtain the following estimates as immediate corollaries.

**Corollary 3.** *Let $T$ be a Calderón–Zygmund operator and let $1 < p < \infty$. Then if $w \in A_\infty$ we obtain*

$$||Tf||_{L^p(w)} \leqslant Cpp'[w]_{A_\infty}^{1/p'}||f||_{L^p(Mw)}, \tag{4.8}$$

*and if $w \in A_1$,*

$$||Tf||_{L^p(w)} \leqslant Cpp'[w]_{A_\infty}^{1/p'}[w]_{A_1}^{1/p}||f||_{L^p(w)}, \tag{4.9}$$

*where $C$ is an absolute constant that also depends on $T$.*

As an application of (4.7) we obtain the following endpoint estimate.

**Theorem 16.** *Let $T$ be a Calderón–Zygmund operator. Then for any weight $w$ and $r > 1$,*

$$||Tf||_{L^{1,\infty}(w)} \leqslant C\log(e + r')||f||_{L^1(M_r w)}, \tag{4.10}$$

*where $C$ is an absolute constant that also depends on $T$.*

Additionally we get the following estimates as corollaries of the above result choosing $r$ as the sharp exponent in the reverse Hölder inequality for weights in the $A_\infty$ class in the setting of SHT (see Lemma 41) and taking into account that $r' \approx [w]_{A_\infty}$.

**Corollary 4.** *Let $T$ be a Calderón–Zygmund operator. Then*

1. *If $w \in A_\infty$*

$$||Tf||_{L^{1,\infty}(w)} \leqslant C\log(e + [w]_{A_\infty})||f||_{L^1(M_r w)}.$$

2. *If $w \in A_1$*

$$||Tf||_{L^{1,\infty}(w)} \leqslant C[w]_{A_1}\log(e + [w]_{A_\infty})||f||_{L^1(w)}.$$

*In both cases $C$ is an absolute constant that also depends on $T$.*

A natural extension of our earlier listed results for Calderón–Zygmund operators is a version of an extrapolation theorem from Duoandikoetxea [25], for SHT. Here we get sharp bounds involving the $A_p$ weight constant through an initial $A_{p_0}$ boundedness assumption.

**Theorem 17.** *Assume that for some family of pairs of nonnegative functions $(f, g)$, for some $p_0 \in [1, \infty)$, and for all $w \in A_{p_0}$ we have*

$$\left(\int_X g^{p_0} w\right)^{1/p_0} \leqslant CN([w]_{A_{p_0}}) \left(\int_X f^{p_0} w\right)^{1/p_0},$$

*where $N$ is an increasing function and the constant $C$ does not depend on $w$. Then for all $1 < p < \infty$ and all $w \in A_p$ we have*

$$\left(\int_X g^{p} w\right)^{1/p} \leqslant CK(w) \left(\int_X f^{p} w\right)^{1/p},$$

*where*

$$K(w) = \begin{cases} N[w]_{A_p}(2\|M\|_{L^p(w)})^{p_0-p}, & \text{if } p < p_0; \\ N[w]_{A_p}^{\frac{p_0-1}{p-1}}(2\|M\|_{L^{p'}(w^{1-p'})})^{\frac{p-p_0}{p-1}}, & \text{if } p > p_0. \end{cases}$$

*In particular, $K(w) \leqslant C_1 N(C_2[w]_{A_p}^{\max\{1, \frac{p_0-1}{p-1}\}})$, for $w \in A_p$ where $C_2$ is an absolute constant.*

As an application of the last result we obtain an estimate for $L^p(w)$ norms with $A_q$ weights for $q < p$.

**Corollary 5.** *Let $T$ be an operator such that*

$$||Tf||_{L^p(w)} \leqslant CN([w]_{A_1})||f||_{L^p(w)}$$

*for all weights $w \in A_1$ and all $1 < p < \infty$, with $C$ independent of $w$. Then we have*

$$||Tf||_{L^p(w)} \leqslant CN([w]_{A_q})||f||_{L^p(w)} \tag{4.11}$$

*for all $w \in A_q$ and $1 \leqslant q < p < \infty$, with $C$ independent of $w$. In particular,* (4.11) *holds with $N(t) = t$ if $T$ is a Calderón–Zygmund operator.*

We also prove the following bound for a Calderón–Zygmund operator that is useful to get sharp $A_2 - A_\infty$ bounds for the commutators in SHT.

**Theorem 18.** *Let $T$ be a Calderón–Zygmund operator and $w \in A_2$. Then the following sharp weighted bound in an space of homogeneous type holds:*

$$\|T\|_{L^2(w)} \leqslant C[w]_{A_2}^{1/2}([w]_{A_\infty} + [\sigma]_{A_\infty})^{1/2}.$$

And finally as a corollary of the previous result and using a precise version of the John–Nirenberg inequality proved in Section 4.3.4, we prove the following generalized sharp weighted bound for the k-th iterate commutator of a Calderón–Zygmund operator.

**Corollary 6.** *Let $T$ be a Calderón–Zygmund operator defined on a space of homogeneous type and $b \in BMO$. Then*

$$\|T_b^k(f)\|_{L^2(w)} \leqslant C[w]_{A_2}^{1/2}([w]_{A_\infty} + [\sigma]_{A_\infty})^{k+1/2}\|b\|_{BMO}\|f\|_{L^2(w)}. \tag{4.12}$$

*where $C$ is an absolute constant. In particular, for the classical commutator we get the following estimate*

$$\|T_b(f)\|_{L^2(w)} \leqslant C[w]_{A_2}^{1/2}([w]_{A_\infty} + [\sigma]_{A_\infty})^{3/2}\|b\|_{BMO}\|f\|_{L^2(w)}. \tag{4.13}$$

*Remark* 17. The optimality of the exponents in these results follow from the corresponding results in $\mathbb{R}^n$ which were obtained by building specific examples of weights for each operator. However, a new approach to derive the optimality of the exponents without building explicit examples can be found in [56].

## 4.3 Proofs

### 4.3.1 Proofs of Theorem 15 and Corollary 3

First, we will prove an inequality of Coifman–Fefferman type, bounding a CZO by a maximal function.

**Proposition 2.** *Let $T$ be a Calderón–Zygmund operator and let $1 < p < \infty$. If $w \in A_p$ then*

$$\int_X |Tf(x)| w(x) d\mu(x) \leqslant C[w]_{A_p} \int_X Mf(x) w(x) d\mu(x), \tag{4.14}$$

*where $C$ is an absolute constant that depends also on $T$.*

Before proving Proposition 2 we need to recall the following lemma that will allow us to obtain the precise constant in (4.14) and that can be found in [31, Ex. 9.2.5] as well as in [28, p. 388] in the context of two weights.

**Lemma 35.** *Let $\mu$ be a positive doubling measure and $1 < p < \infty$. If $w \in A_p$ then*

$$\left(\frac{\mu(A)}{\mu(Q)}\right)^p \leqslant [w]_{A_p} \frac{w(A)}{w(Q)}, \tag{4.15}$$

*where $A \subset Q$ is a $\mu$-measurable set and $Q$ is a cube.*

*Proof of Proposition 2.* We have that

$$\int_X |Tf(x)| w(x) d\mu(x) \leqslant C_{X,T} \sup_{S,D} \int_X \left| \sum_{Q \in S} \left( ⨍_Q f(x) \right) \chi_Q(x) \right| w(x) d\mu(x)$$

by the formula of Lerner proved in the homogeneous setting in [2]. Using (4.15), we obtain that

$$\begin{aligned} \int_X |Tf(x)| w(x) d\mu(x) &\leqslant C_{X,T} \sup_{S,D} \sum_{Q \in S} \left( ⨍_Q |f(x)| \right) w(Q) \\ &\leqslant C_{X,T,p} \sup_{S,D} [w]_{A_p} \sum_{Q \in S} \left( ⨍_Q f(x) \right) w(E(Q)) \\ &\leqslant C_{X,T,p} [w]_{A_p} \sup_{D,S} \sum_{Q \in S} \int_{E(Q)} Mf(x) w(x) d\mu(x), \end{aligned}$$

Finally, since the family $E(Q)$ is disjoint, we can bound the above by

$$\int_X |Tf(x)|w(x)d\mu(x) \leqslant C[w]_{A_p} \sup_{D,S} \int_X Mf(x)w(x)d\mu(x)$$
$$\leqslant C[w]_{A_p} \int_X Mf(x)w(x)d\mu(x),$$

where $C$ is an absolute constant that depends also on $T$ and $p$, proving (4.14) as wanted.

□

Now we prove the following lemma.

**Lemma 36.** *Let $w$ be any weight and let $1 \leqslant p, r < \infty$. Then there is a constant $C = C_{X,T}$ such that*

$$||Tf||_{L^p((M_r w)^{1-p})} \leqslant Cp||Mf||_{L^p((M_r w)^{1-p})}.$$

The proof of this lemma is based in a variation of the Rubio de Francia algorithm that can be found in [70].

*Proof of Lemma 36.* We want to prove

$$\left|\left|\frac{Tf}{M_r w}\right|\right|_{L^p(M_r w)} \leqslant Cp\left|\left|\frac{Mf}{M_r w}\right|\right|_{L^p(M_r w)}.$$

By duality we have

$$\left|\left|\frac{Tf(x)}{M_r w(x)}\right|\right|_{L^p(M_r w)} = \left|\int_X Tf(x)h(x)d\mu(x)\right| \leqslant \int_X |Tf(x)||h(x)|d\mu(x),$$

for some $h$ such that $||h||_{L^{p'}(M_r w)} = 1$. By a variation of Rubio de Francia's algorithm adapted to SHT (see [70, Lemma 4.4]) with $s = p'$ and $v = M_r w$ there exists an operator $R$ such that

1. $0 \leqslant h \leqslant R(h)$.
2. $||R(h)||_{L^{p'}(M_r w)} \leqslant 2C_{X,p}||h||_{L^{p'}(M_r w)}$.
3. $[R(h)(M_r w)^{1/p'}]_{A_1} \leqslant C_X p$.

Let us recall two facts: First, if two weights $w_1, w_2 \in A_1$, then $w = w_1 w_2^{1-p} \in A_p$ and $[w]_{A_p} \leqslant [w_1]_{A_1}[w_2]_{A_1}^{p-1}$. Second, by the Coifman-Rochberg theorem in spaces of homogeneous type (this follows [18, Prop. 5.32]), if $r > 1$ then $(Mf)^{1/r} \in A_1$ and $[(Mf)^{1/r}]_{A_1} \leqslant C_X r'$. Combining these facts and (3) we obtain

$$\begin{aligned}
[R(h)]_{A_3} &= [R(h)(M_r w)^{1/p'}((M_r w)^{1/2p'})^{-2}]_{A_3} \\
&\leqslant [R(h)(M_r w)^{1/p'}]_{A_1}[(M_r w)^{1/2p'}]_{A_1}^2 \\
&\leqslant C_X p((2p'r)')^2 \leqslant C_X p,
\end{aligned}$$

since $(2p'r)' < 2$.

Thus, by Proposition 4.14 and using (1) and (2), we obtain

$$\begin{aligned}
\int_X |Tf(x)|h(x)d\mu(x) &\leqslant \int_X |Tf(x)|R(h)(x)d\mu(x) \\
&\leqslant C[R(h)]_{A_3} \int_X M(f)(x)R(h)(x)d\mu(x) \\
&= C[R(h)]_{A_3} \int_X M(f)(x)R(h)(x) \\
&\times (M_r w(x))^{-1} M_r w(x) d\mu(x) \\
&\leqslant C[R(h)]_{A_3} \left|\left|\frac{M(f)}{M_r w}\right|\right|_{L^p(M_r w)} ||R(h)||_{L^{p'}(M_r w)} \\
&\leqslant C[R(h)]_{A_3} \left|\left|\frac{M(f)}{M_r w}\right|\right|_{L^p(M_r w)} ||h||_{L^{p'}(M_r w)} \\
&\leqslant Cp \left|\left|\frac{M(f)}{M_r w}\right|\right|_{L^p(M_r w)},
\end{aligned}$$

and we are done.

$\square$

And finally using Lemma 36 applied to $T^*$ we can prove Theorem 15.

*Proof of Theorem 15.* First we are going to prove the following inequality

$$||Tf||_{L^p(w)} \leqslant Cpp'(r')^{1-\frac{1}{pr}}||f||_{L^p(M_r w)}, \tag{4.16}$$

from which follows (4.7). We have

$$1 - \frac{1}{pr} = 1 - \frac{1}{p} + \frac{1}{p} - \frac{1}{pr} = \frac{1}{p'} - \frac{1}{pr'},$$

and since $t \geqslant 1$, it follows that

$$(r')^{1-\frac{1}{pr}} = (r')^{\frac{1}{p'}-\frac{1}{pr'}} \leqslant r^{\frac{1}{p'}}.$$

Next consider the dual estimate of (4.16), namely

$$||T^*f||_{L^{p'}((M_r w)^{1-p'})} \leqslant Cpp'(r')^{1-\frac{1}{pr}}||f||_{L^{p'}(w^{1-p'})},$$

where $T^*$ is the adjoint operator of $T$. Since $T$ is also a CZO, we can apply Lemma 36 to $T^*$, we get

$$\left|\left|\frac{T^*f}{M_r w}\right|\right|_{L^{p'}(M_r w)} \leqslant Cp' \left|\left|\frac{Mf}{M_r w}\right|\right|_{L^{p'}(M_r w)}.$$

Using Hölder's inequality with exponent $pr$ we have

$$\frac{1}{|Q|}\int_Q fw^{-1/p}w^{1/p}d\mu \leqslant \left(\frac{1}{|Q|}\int_Q w^r d\mu\right)^{1/pr}\left(\frac{1}{|Q|}\int_Q (fw^{-1/p})^{(pr)'}d\mu(x)\right)^{1/(pr)'},$$

and hence,

$$M(f)^{p'} \leqslant (M_r w)^{\frac{p'}{p}} M((fw^{-1/p})^{(pr)'})^{p'/(pr)'}.$$

From this and the boundedness of the unweighted maximal function in SHT that can be easily obtained from the proof in [30], changing the dimensional constant for a geometric one, we obtain

$$\begin{aligned}
\left(\int_X \frac{M(f)^{p'}}{(M_r w)^{p'-1}}d\mu\right)^{1/p'} &\leqslant \left(\int_X M((fw^{-1/p})^{(pr)'})^{p'/(pr)'}d\mu\right)^{1/p'} \\
&\leqslant C\left(\frac{p'}{p'-(pr)'}\right)^{1/(pr)'}\left(\int_X f^{p'}w^{1-p'}d\mu\right)^{1/p'} \\
&= C\left(\frac{p'}{p'-(pr)'}\right)^{1/(pr)'}\left\|\frac{f}{w}\right\|_{L^{p'}(w)} \\
&= C\left(\frac{rp-1}{r-1}\right)^{1-1/pr}\left\|\frac{f}{w}\right\|_{L^{p'}(w)} \\
&\leqslant Cp\left(\frac{r}{r-1}\right)^{1-1/pr}\left\|\frac{f}{w}\right\|_{L^{p'}(w)},
\end{aligned}$$

proving (4.16) and therefore (4.7).

$\square$

*Proof of Corollary 3.* The proofs of (4.8) and (4.9) follow immediately. In the first case, the estimate is derived by applying the sharp reverse Hölder inequality for weights in the $A_\infty$ class proved in Lemma 41 to (4.7) and using the fact that $r' \approx [w]_{A_\infty}$. Corlollary (4.9) is is a direct consequence of (4.8) since $w \in A_1$. □

### 4.3.2 Proof of Theorem 16

First we establish a lemma which follows similar ideas of [28, Ch. 4, Lemma 3.3], that we will need for the proof of Theorem 16.

**Lemma 37.** *Let $T$ be a CZO. Let $w$ be a weight and $a \in L^1(w)$ be supported in a cube $Q$ with $\int_Q a(y)d\mu(y) = 0$ and $\eta$ be the smoothness constant from the definition of a CZO kernel in SHT. Then, if we set $\tilde{Q} = LQ$ for a large $L \geqslant 1/\eta > 0$, the following inequality holds*

$$\int_{X\setminus\tilde{Q}} |T(a)(x)|w(x)d\mu(x) \leqslant C\int_X |a(x)|Mw(x)d\mu(x), \tag{4.17}$$

*with $C$ an absolute constant depending on the kernel $K$.*

*Proof.* Fix $y_0 \in X$ and assume for simplicity that $Q = B(y_0, R)$, with $R > 0$. Now making use of the cancellation property of $a$, we obtain

$$\begin{aligned}\int_{X\setminus\tilde{Q}} |T(a)(x)|w(x)d\mu(x) &= \int_{X\setminus\tilde{Q}} \left|\int_Q K(x,y)a(y)d\mu(y)\right| w(x)d\mu(x)\\ &\leqslant \int_Q\int_{X\setminus\tilde{Q}} |K(x,y)-K(x,y_0)|w(x)d\mu(x)|a(y)|d\mu(y)\\ &= \int_Q I(y)|a(y)|d\mu(y).\end{aligned}$$

We only need to prove that $I$ is bounded by $CMw(y)$ where $C = C_{X,K}$ is an absolute constant also depending on the kernel $K$. For every $y \in Q$, using the smoothness property of $K$ in the second variable (since $\rho(y, y_0) \leqslant \eta\rho(x, y_0)$), we obtain

$$
\begin{aligned}
I(y) &= \int_{X\setminus\tilde{Q}} |K(x,y) - K(x,y_0)| w(x) d\mu(x) \\
&\lesssim \int_{X\setminus\tilde{Q}} \left(\frac{\rho(y,y_0)}{\rho(x,y_0)}\right)^{\eta} \frac{1}{\mu(B(y_0,\rho(x,y_0)))} w(x) d\mu(x) \\
&\leqslant C_{X,K} \sum_{l=1}^{\infty} \int_{2^l Q\setminus 2^{l-1}Q} \left(\frac{\rho(y,y_0)}{\rho(x,y_0)}\right)^{\eta} \frac{1}{\mu(B(y_0,\rho(x,y_0)))} w(x) d\mu(x)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C_{X,K} \sum_{l=1}^{\infty} \int_{2^l Q} \frac{2^{\eta}}{2^{l\eta}} \frac{\mu(B(y_0, 2^l R))}{\mu(B(y_0, 2^{l-1}R))} \frac{1}{\mu(2^l Q)} w(x) d\mu(x) \\
&\leqslant C_{X,K} \sum_{l=1}^{\infty} \frac{1}{2^{l\eta}} \frac{1}{\mu(2^l Q)} \int_{2^l Q} w(x) d\mu(x) \\
&\leqslant C_{X,K} Mw(y).
\end{aligned}
$$

Above we have used the fact that $\rho(y, y_0) < R$ and since $\rho(x, y_0) > LR$, there exists $l > 1$, so that $2^{l-1}R < \rho(x, y_0) \leqslant 2^l R$. Thus we have shown that (4.17) holds.

□

*Proof of Theorem 16.* The proof of Theorem 16 is based on several ingredients that we will mention as we need them. We follow the proof of Theorem 1.6 in [65]. We claim that the following inequality holds: for any $1 < p, r < \infty$

$$
\begin{aligned}
||Tf||_{L^{1,\infty}(w)} &= \lambda \sup_{\lambda>0} w(\{y \in X : |Tf(y)| > \lambda\}) \\
&\leqslant Cp^p (p')^p (r')^{p-1} ||f||_{L^1(M_r w)},
\end{aligned} \tag{4.18}
$$

and this claim implies (4.10). Indeed, it suffices to fix $r > 1$ and choose $p = 1 + \frac{1}{\log r'}$ (then $p^p = C$).

We then obtain

$$||Tf||_{L^{1,\infty}(w)} \leqslant C(p')^{p-1}(1+\log r')(r')^{p-1}||f||_{L^1(w)}$$
$$\leqslant C\log{(e+r')}||f||_{L^1(M_r w)}.$$

since $p' = 1+\log r'$, $(p')^{p-1} = (1+\log(r'))^{1/\log(r')} \leqslant e$, $(r')^{p-1} = (r')^{1/\log(r')} = e$ and $1+\log r' = \log{(er')} \leqslant 2\log(e+r')$.

We now prove (4.18). By the Calderón–Zygmund decomposition of a function $f \in \mathcal{C}_0^\infty(X)$ at a level $\lambda$ (see background section), we obtain a family of non-overlapping dyadic cubes $\{Q_j\}$ satisfying

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j}|f(x)|d\mu(x) \leqslant C_X\lambda.$$

Let $\Omega = \cup_j \Omega_j$ and let us denote $\widetilde{Q}_j = 2Q_j$ and $\widetilde{\Omega} = \cup_j \widetilde{Q}_j$. We write $f = g+b$ where $g(x) = \sum_j f_{Q_j}\chi_{Q_j}(x) + f(x)\chi_{\Omega^c}$ and $b = \sum_j b_j$ with $b_j(x) = (f(x)-f_{Q_j})\chi_{Q_j}(x)$. We have

$$w(\{y\in X : |Tf(y)| > \lambda\}) \leqslant w(\widetilde{\Omega}) + w(\{y \in (\widetilde{\Omega})^c : |Tg(y)| > \lambda/2\})$$
$$+ w(\{y \in (\widetilde{\Omega})^c : |Tb(y)| > \lambda/2\}) \equiv I + II + III.$$

Now

$$I = w(\widetilde{\Omega}) \leqslant C\sum_j \frac{w(\widetilde{Q}_j)}{\mu(\widetilde{Q}_j)}\mu(Q_j) \leqslant \frac{C}{\lambda}\sum_j \frac{w(\widetilde{Q}_j)}{\mu(\widetilde{Q}_j)}\int_{Q_j}|f(x)|d\mu(x)$$
$$\leqslant \frac{C}{\lambda}\sum_j\int_{Q_j}|f(x)|Mw(x)d\mu(x) \leqslant \frac{C}{\lambda}\int_X |f(x)|Mw(x)d\mu(x)$$
$$\leqslant \frac{C}{\lambda}\int_X |f(x)|M_r w(x)d\mu(x).$$

The second term is estimated as follows using Chebyshev's inequality and (4.7). For each $p > 1$

we get

$$\begin{aligned}
II &= w(\{x \in (\widetilde{\Omega})^c : |T(g)(x)| > \lambda/2\}) \\
&\leqslant Cp^p(p')^p(r')^{p/p'} \frac{1}{\lambda^p} \int_X |g|^p M_r(w\chi_{(\widetilde{\Omega})^c}) d\mu(x) \\
&\leqslant Cp^p(p')^p(r')^{p/p'} \frac{1}{\lambda} \int_X |g| M_r(w\chi_{(\widetilde{\Omega})^c}) d\mu(x) \\
&\leqslant Cp^p(p')^p(r')^{p/p'} \frac{1}{\lambda} \left( \int_{X\setminus\Omega} |g| M_r(w\chi_{(\widetilde{\Omega})^c}) d\mu(x) + \int_{\Omega} |g| M_r(w\chi_{(\widetilde{\Omega})^c}) d\mu(x) \right) \\
&= Cp^p(p')^p(r')^{p/p'} \frac{1}{\lambda} (II_1 + II_2).
\end{aligned}$$

It is clear that

$$II_1 = \int_{X\setminus\Omega} |g| M_r(w\chi_{(\widetilde{\Omega})^c}) d\mu(x) \leqslant \int_X |f| M_r w d\mu(x).$$

Next, we estimate $II_2$ as follows

$$\begin{aligned}
II_2 &= \int_{\Omega} |g| M_r(w\chi_{(\widetilde{\Omega})^c})(x) d\mu(x) \\
&\leqslant \sum_j \int_{Q_j} |f_{Q_j}| M_r(w\chi_{(\widetilde{\Omega})^c})(x) d\mu(x) \\
&\leqslant \sum_j \int_{Q_j} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)| d\mu(y) M_r(w\chi_{(\widetilde{\Omega})^c})(x) d\mu(x) \\
&\leqslant C \sum_j \int_{Q_j} |f(y)| d\mu(y) \frac{1}{\mu(Q_j)} \inf_{Q_j} M_r(w\chi_{(\widetilde{\Omega})^c}) \mu(Q_j) \\
&\leqslant C \sum_j \int_{Q_j} |f(y)| M_r w(y) d\mu(y) \\
&\leqslant C \int_X |f(y)| M_r w(y) d\mu(y),
\end{aligned}$$

where we have used that for any $r > 1$, non-negative function $w$ with $M_r w(x) < \infty$ a.e., cube $Q_j$ and $x \in Q_j$ we have

$$M_r(u\chi_{\tilde{\Omega}^c})(x) \lesssim \inf_{y \in Q_j} M_r(u\chi_{\tilde{\Omega}^c})(y) \tag{4.19}$$

where the implicit constant depends only on the doubling constant $C_d$. This is proved in [2]. In the classical case with the Hardy–Littlewood maximal operator $M$, see for instance [28, p. 159].

Combining both estimates we obtain

$$w(\{x \in (\widetilde{\Omega})^c : |T(g)(x)| > \lambda/2\}) \leqslant \frac{C}{\lambda} p^p (p')^p (r')^{p-1} ||f||_{L^1(M_r w)}.$$

Next we estimate the term $III$, using the estimate in Lemma 4.17 and replacing $w$ by $w\chi_{X \backslash \tilde{Q}_j}$ we have

$$\begin{aligned}
III &= w(\{y \in X \backslash (\widetilde{\Omega})^c : |T(b)(y)| > \frac{\lambda}{2}\}) \\
&\leqslant \frac{C}{\lambda} \int_{X \backslash \tilde{\Omega}} |b(y)| w(y) d\mu(y) \\
&\leqslant \frac{C}{\lambda} \sum_j \int_{X \backslash \tilde{Q}_j} |b_j(y)| w(y) d\mu(y) \\
&\leqslant \frac{C}{\lambda} \sum_j \int_X |b_j(y)| M(w\chi_{X \backslash \tilde{Q}_j})(y) d\mu(y) \\
&\leqslant \frac{C}{\lambda} \sum_j \int_{Q_j} |b(y)| M(w\chi_{X \backslash \tilde{Q}_j})(y) d\mu(y) \\
&\leqslant \frac{C}{\lambda} \sum_j \left( \int_{Q_j} |f(y)| Mw(y) d\mu(y) + \int_{Q_j} |g(y)| M(w\chi_{X \backslash \tilde{Q}_j})(y) d\mu(y) \right) \\
&= \frac{C}{\lambda} (III_1 + III_2).
\end{aligned}$$

To conclude the proof we only need to estimate $III_2$.

$$\begin{aligned}
III_2 &= \sum_j \int_{Q_j} |f_{Q_j}| M(w\chi_{\tilde{\Omega}^c})(y) d\mu(y) \\
&\leqslant \sum_j \int_{Q_j} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)| d\mu(x) M(w\chi_{\tilde{\Omega}^c})(y) d\mu(y) \\
&\leqslant \sum_j \int_{Q_j} |f(x)| d\mu(x) \inf_{Q_j} M(w\chi_{\tilde{\Omega}^c}) \\
&\leqslant \sum_j \int_{Q_j} |f(x)| M(w\chi_{\tilde{\Omega}^c})(x) d\mu(x) \\
&\leqslant C \int_X |f(x)| Mw(x) d\mu(x),
\end{aligned}$$

Combining the three estimates proves (4.18) and this concludes the proof of the theorem.

□

### 4.3.3 Proofs of Theorem 17 and Corollary 5

Now, we turn to the proof of Theorem 17. This extrapolation proof is reminiscent of the style of proof for Theorem 15. We first need two lemmas, originally in [25]. The first is presented here in detail for SHT.

**Lemma 38** (Factorization)**.** *If $w \in A_p$ and $u \in A_1$, we have that*

1. *For $1 \leqslant p < p_0 < \infty$, then $wu^{p-p_0}$ is in $A_{p_0}$ and*

$$[wu^{p-p_0}]_{A_{p_0}} \leqslant [w]_{A_p}[u]_{A_1}^{p_0-p}.$$

2. *For $1 < p_0 < p < \infty$, then $(w^{p_0-1p}u^{p-p_0})^{\frac{1}{p-1}}$ is in $A_{p_0}$ and*

$$[(w^{p_0-1}u^{p-p_0})^{\frac{1}{p-1}}]_{A_{p_0}} \leqslant [w]_{A_p}^{\frac{p_0-1}{p-1}}[u]_{A_1}^{\frac{p-p_0}{p-1}}.$$

*Proof of Lemma 38.* To prove (1) we observe since $p - p_0 < 0$, using the definition of the $A_1$ constant we obtain

$$\begin{aligned}\left(\fint_Q u\right)^{p_0-p} u^{p-p_0} &\leqslant \left(\fint_Q u\right)^{p_0-p}\left(\operatorname*{ess\,sup}_Q u^{-1}\right)^{p_0-p}\\ &\leqslant [u]_{A_1}^{p_0-p}.\end{aligned}$$

So that

$$u^{p-p_0} \leqslant \left(\fint_Q u\right)^{p-p_0}[u]_{A_1}^{p_0-p}. \tag{4.20}$$

On the other hand, using Hölder inequality with exponents $\frac{1-p'}{1-p_0'} = \frac{p_0-1}{p-1}$ and $\frac{1-p'}{p_0'-p'} = \frac{p_0-1}{p_0-p}$, we have

$$\begin{aligned}
\left(\fint_Q w^{1-p_0'} u^{(p-p_0)(1-p_0')}\right)^{p_0-1} &\leqslant \\
&\leqslant \left(\fint_Q w^{1-p'}\right)^{\frac{(p_0-1)(1-p_0')}{1-p'}} \left(\fint_Q u^{\frac{(p-p_0)(1-p_0')(1-p')}{(p_0'-p')}}\right)^{\frac{(p_0'-p')(p_0-1)}{1-p'}} \qquad (4.21)\\
&= \left(\fint_Q w^{1-p'}\right)^{p-1} \left(\fint u\right)^{p_0-p}
\end{aligned}$$

Thus combining (4.20) and (4.21) we obtain

$$\left(\fint_Q w u^{p-p_0}\right)\left(\fint_Q w^{1-p_0'} u^{(p-p_0)(1-p_0')}\right)^{p_0-1} \leqslant [w]_{A_p}[u]_{A_1}^{p_0-p}.$$

Finally taking supremum over all cubes $Q$ on the left hand side we get the desired result.

To show (2) we use a dualization argument on the weights and apply (1). We have that

$$\begin{aligned}
\left[w^{\frac{p_0-1}{p-1}} u^{\frac{p-p_0}{p-1}}\right]_{A_{p_0}} &= \left[\left(w^{\frac{p_0-1}{p-1}} u^{\frac{p-p_0}{p-1}}\right)^{1-p_0'}\right]_{A_{p_0'}}^{\frac{1}{p_0'-1}} \\
&= \left[w^{\frac{-1}{p-1}} u^{p'-p_0'}\right]_{A_{p_0'}}^{\frac{1}{p_0'-1}},
\end{aligned}$$

since $\frac{p-p_0}{p-1}(1-p_0') = p'-p_0'$. Next, using (1) since $w^{1-p'} \in A_{p'}$, $u \in A_1$ and $p_0' > p'$, we get the desired result

$$\begin{aligned}
\left[w^{\frac{p_0-1}{p-1}} u^{\frac{p-p_0}{p-1}}\right]_{A_{p_0}} &\leqslant [w]_{A_p}^{\frac{p'-1}{p_0'-1}} [u]_{A_1}^{\frac{p_0'-p'}{p_0'-1}} \\
&\leqslant [w]_{A_p}^{\frac{p_0-1}{p-1}} [u]_{A_1}^{\frac{p-p_0}{p-1}},
\end{aligned}$$

where we have used in the next to last inequality that $[w]_{A_p}^{p'-1} = [w^{1-p'}]_{A_{p'}}$ and in the last inequality the fact that $\frac{p_0'-p'}{p_0'-1} = \frac{p-p_0}{p-1}$ and $\frac{p'-1}{p_0'-1} = \frac{p_0-1}{p-1}$.

□

The second lemma is a Rubio de Francia algorithm for building $A_1$ weights. We omit the proof of this result since it follows exactly in the same manner as in [25].

**Lemma 39** (Rubio de Francia's algorithm)**.** *Let $p > 1$. Let $f$ be a nonnegative function in $L^p(w)$ and $w \in A_p$. Let $M^k$ be the $k^{th}$ iterate of $M$, $M^0 = f$, and $||M||_{L^p(w)}$ be the norm of $M$ as a bounded operator on $L^p(w)$. Define*

$$Rf(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2||M||^k_{L^p(w)}}.$$

*Then $f(x) \leqslant Rf(x)$ a.e., $||Rf||_{L^p(w)} \leqslant 2||f||_{L^p(w)}$, and $Rf$ is an $A_1$ weight with constant $[Rf]_{A_1} \leqslant 2||M||_{L^p(w)}$.*

*Proof of Theorem 17.* The proof of this result follows from Duoandikoetxea's proof except for the fact that we have to replace the sharp bound for the Hardy–Littlewood maximal function by the new Buckley's theorem in Hytönen and Kairema [35], so the constant $C_2$ in the proof now depends on $p$ and $X$. □

*Proof of Corollary 5.* This follows directly from the proof in [25]. □

### 4.3.4 Proof of Theorem 18 and Corollary 6

To prove the main result in this section we first need the following lemmas. The first is a sharp reverse Hölder inequality for $A_\infty$ weights in SHT adapted from an argument due to Hytönen, Pérez, and Rela whose proof can be found in [37]. Before we state and prove this, we note that in this same paper there is a weak version of this inequality stated below. They call this result a weak inequality since on the right hand side we have the dilation $2C_0B$ of the ball $B$ (recall $C_0$ is the quasitraingular constant).

**Lemma 40** ([37]). *Let $w \in A_\infty$ and define*

$$r = r_w = 1 + \frac{1}{\tau[w]_{A_\infty}} = 1 + \frac{1}{6(32C_0^2(4C_0^2 + C_0)^2)^{C_d}[w]_{A_\infty}}$$

*where $\tau$ depends on $C_0$, the quasimetric constant of $X$. Then*

$$\left(\fint_B w^r d\mu\right)^{1/r} \leqslant \left(2(4C_0)^{C_d}\fint_{2C_0B} w d\mu\right),$$

*for any ball $B \in X$.*

However, this lemma is not sufficient for our purposes. The difficulty lies in the fact that the Fujii-Wilson $A_\infty$ constant is comparable when it is defined with respect to cubes or balls, but the constant of comparison depends on the weight $w$. This provides a difficulty in converting between the constants, and since cubes are essential in the following lemmas, we need a sharp reverse Hölder inequality with cubes. Here is the lemma that we use with respect to cubes.

**Lemma 41.** *Let $w \in A_\infty$ and let*

$$0 < r \leqslant \frac{1}{\tau[w]_{A_\infty} - 1} = \frac{1}{2D[w]_{A_\infty} - 1},$$

*with $D = 1/\varepsilon$, where $\varepsilon$ is the absolute constant appearing in the dyadic decomposition of $X$. Then*

$$\fint_Q w^{1+r} d\mu \leqslant 2\left(\fint_Q w d\mu\right)^{1+r},$$

*for any cube $Q \subset X$.*

The proof will use the following sublemma.

**Lemma 42.** *Let $w \in A_\infty$ and $Q_0$ a cube. Then for all*

$$0 < r \leqslant \frac{1}{2D[w]_{A_\infty} - 1}$$

*we have*

$$⨍_{Q_0} (Mw)^{1+r} d\mu \leqslant 2[w]_\infty \left( ⨍_{Q_0} w d\mu \right)^{1+r}.$$

*Proof of Lemma 42.* Assume without loss of generality that $w = w\chi_{Q_0}$. Let $\Omega_\lambda = Q_o \cap \{Mw > \lambda\}$. Then

$$\int_{Q_0} (Mw)^{1+r} d\mu = \int_0^\infty r\lambda^{r-1} Mw(\Omega_\lambda) d\mu d\lambda$$

$$= \int_0^{w_{Q_0}} r\lambda^{r-1} \int_{Q_0} Mw d\lambda + \int_{w_{Q_0}}^\infty r\lambda^{r-1} Mw(\Omega_\lambda) d\lambda.$$

Now select a dyadic cube $Q_j$ if it is maximal with respect to the following condition: $\lambda < w_{Q_j}$. Then $\Omega_\lambda = \cup_j Q_j$ where $\lambda < ⨍_{Q_j} w \leqslant \frac{1}{\varepsilon}\lambda$ and $\varepsilon$ is the absolute constant from Theorem 10. Hence we have

$$\int_{Q_0} (Mw)^{1+r} \leqslant w_{Q_0}^r [w]_\infty w(Q_0) + \int_{w_{Q_0}}^\infty r\lambda^{r-1} \sum_j \int_{Q_j} Mw d\mu d\lambda.$$

Now we can localize

$$Mw(x) = M(w\chi_{Q_j})(x)$$

by the maximality of the $Q_j's$ for any $x \in Q_j$. Then,

$$\int_{Q_j} Mw d\mu = \int_{Q_j} M(w\chi_{Q_j}) \leqslant [w]_\infty w(Q_j) \leqslant [w]_\infty w(\hat{Q}_j)$$

$$= [w]_\infty w_{\hat{Q}_j} \mu(\hat{Q}_j) \leqslant [w]_\infty \lambda \frac{1}{\varepsilon} \mu(Q_j),$$

where $\hat{Q}_j$ is the parent of the cube $Q_j$ and we have used the definition of $A_\infty$ and the maximality and containment properties of the cubes. Call $\frac{1}{\varepsilon} = D$. Hence

$$\sum_j \int_{Q_j} Mw d\mu \leqslant \sum_j [w]_\infty \lambda D \mu(Q_j) \leqslant [w]_\infty \lambda D \mu(\Omega_\lambda)$$

so

$$\fint_{Q_0} (Mw)^{1+r} \leqslant w_{Q_0}^r [w]_\infty w(Q_0) + r[w]_\infty D \int_{w_{Q_0}}^\infty \lambda^r \mu(\Omega_\lambda) d\lambda.$$

Dividing by $\mu(Q_0)$, we obtain

$$\fint_{Q_0} (Mw)^{1+r} \leqslant w_{Q_0}^{1+r} [w]_\infty + \frac{rD[w]_\infty}{1+r} \fint_{Q_0} (Mw)^{1+r},$$

and by subtracting the last term on the left hand side from both sides of the equation, to get the desired constant of 2 we must have that

$$1 - \frac{rD[w]_\infty}{1+r} \geqslant \frac{1}{2}.$$

Afte some calculation, this results in choosing $0 < r \leqslant \frac{1}{2D[w]_\infty - 1}$ as stated. $\square$

*Proof of Lemma 41.* Without loss of generality let $w = w\chi_{Q_0}$. Then

$$\int_{Q_0} w^{1+r} \leqslant \int_{Q_0} (Mw)^r w = \int_0^\infty r\lambda^{r-1} w(\Omega_\lambda) d\lambda,$$

where $\Omega_\lambda = Q_0 \cap \{Mw > \lambda\}$. Note that as in the previous lemma we can decompose $\Omega_\lambda = \cup_j Q_j$ where the $Q_j$ are the Calderón-Zygumnd cubes. Then splitting up the integral we get

$$\begin{aligned} &\int_0^{w_{Q_0}} r\lambda^{r-1} d\lambda + \int_{w_{Q_0}}^\infty r\lambda^{r-1} w(\Omega_\lambda) d\lambda \\ &\leqslant w_{Q_0}^r w(Q_0) + \int_{w_{Q_0}}^\infty r\lambda^{r-1} \sum_j w(Q_j) d\lambda. \end{aligned}$$

Now by the decomposition, we have that $w_{Q_j} \leqslant C_X \lambda \mu(Q_j)$, where $C_X = \frac{1}{\varepsilon}$ since the decomposition is with respect to dyadic cubes, so we get

$$\int_{Q_0} (Mw)^r w d\mu \leqslant w_{Q_0}^r w(Q_0) + rC_X \int_{w_{Q_0}}^{\infty} r\lambda^r \sum_j \mu(Q_j) d\lambda$$

$$\leqslant w_{Q_0}^r w(Q_0) + rC_X \int_{w_{Q_0}}^{\infty} \lambda^r \mu(\Omega_\lambda) d\lambda \leqslant w_{Q_0}^r w(Q_0) + \frac{rC_X}{1+r} \int_{Q_0} (Mw)^{1+r}.$$

Hence, dividing by $w(Q_0)$ and using Lemma 42, we arrive at

$$\begin{aligned} \fint_{Q_0} w^{1+r} &\leqslant w_{Q_0}^{1+r} + \frac{rC_X)2[w]_\infty}{1+r} \left( \fint_{Q_0} w \right)^{1+r} \\ &\leqslant \frac{rC_X 2[w]_\infty + 1 + r}{1+r} \left( \fint_{Q_0} w \right)^{1+r}. \end{aligned}$$

Therefore, choosing $\varepsilon$ in the mentioned range, we can make the constant on the right hand side less than or equal to 2.

$\square$

The next lemma is a precise version of John-Nirenberg inequality in SHT that will be very useful in the following results.

**Lemma 43** (John-Nirenberg inequality)**.** *There are absolute constants* $0 \leqslant \alpha_X < 1 < \beta_X$ *such that*

$$\sup_Q \frac{1}{\mu(Q)} \int_Q \exp \frac{\alpha_X}{||b||_{BMO}} |b(y) - b_Q| d\mu(y) \leqslant \beta_X. \tag{4.22}$$

*In fact, we can take* $\alpha_X = \ln \sqrt[3]{2^\varepsilon}$*, where* $0 < \varepsilon < 1$ *is an absolute constant.*

*Proof of Lemma 43.* For the proof of this lemma we follow the scheme of proof showed in [43]. Suppose that $b$ is bounded, so that the above supremum makes sense for all $\alpha$. Then we will prove (4.22) with a bound independent of $||b||_\infty$.

Fix a cube $Q_0$ and a dyadic cube $Q \in D(Q_0)$. Denote by $\tilde{Q}$ the parent of $Q$.

Then we can show that

$$\left| b_Q - b_{\tilde{Q}} \right| \leqslant \frac{1}{\varepsilon} ||b||_{BMO}, \tag{4.23}$$

where $0 < \varepsilon < 1$ is an absolute constant from the dyadic decomposition.

$$\begin{aligned} \left| b_Q - b_{\tilde{Q}} \right| &\leqslant \frac{1}{\mu(Q)} \int_Q \left| b - b_{\tilde{Q}} \right| \\ &\leqslant \frac{\mu(\tilde{Q})}{\mu(Q)} \frac{1}{\mu(\tilde{Q})} \int_{\tilde{Q}} \left| b - b_{\tilde{Q}} \right| \\ &\leqslant \frac{1}{\varepsilon} ||b||_{BMO}. \end{aligned}$$

Next, consider the Calderón–Zygmund decomposition of $(b - b_{Q_0})\, \chi_{Q_0}$ for the level $2||b||_{BMO}$. Then there exists a collection of pairwise disjoint cubes $\{Q_i\} \subset D$, maximal with respect to inclusion, satisfying

$$2||b||_{BMO} < \frac{1}{\mu(Q_i)} \int_{Q_i} |(b - b_{Q_0})\, \chi_{Q_0}| < 2C_X ||b||_{BMO}$$

and

$$|(b - b_{Q_0})\, \chi_{Q_0}| < 2||b||_{BMO}, \text{ on } (\cup Q_i)^c.$$

Clearly, $Q_i \subset Q_0$ for each $j$, and

$$\mu(\cup Q_i) \leqslant \frac{||(b-b_{Q_0})\chi_{Q_0}||_{L^1}}{2||b||_{BMO}} \leqslant \frac{\mu(Q_0)}{2}.$$

Since the cubes $Q_i$ are maximal, we have that $(|b-b_{Q_0}|)_{\widetilde{Q_i}} \leqslant 2||b||_{BMO}$. Using the last inequality together with (4.23) we get

$$|b_{Q_i}-b_{Q_0}| \leqslant |b_{Q_i}-b_{\widetilde{Q_i}}| + |b_{\widetilde{Q_i}}-b_{Q_0}| \leqslant \left(\frac{1}{\varepsilon}+2\right)||b||_{BMO}.$$

Denote $X(\alpha) = \sup_Q \frac{1}{\mu(Q)} \int_Q \exp \frac{\alpha}{||b||_{BMO}} |b-b_Q| d\mu(x)$, which is finite because $b$ is bounded. From the properties of the cubes $Q_i$ we get

$$\begin{aligned}
&\frac{1}{\mu(Q_0)} \int_{Q_0} \exp\left(\frac{\alpha}{||b||_{BMO}}|b-b_{Q_0}|\right) d\mu(x) \\
&\leqslant \frac{1}{\mu(Q_0)} \int_{Q_0 \setminus \cup Q_i} e^{2\alpha} d\mu(x) \\
&+ \sum_j \frac{\mu(Q_i)}{\mu(Q_0)} \frac{1}{\mu(Q_i)} \left(\int_{Q_i} \exp\left(\frac{\alpha}{||b||_{BMO}}|b-b_{Q_i}| d\mu(x)\right) e^{(\frac{1}{\varepsilon}+2)\alpha}\right) \\
&\leqslant e^{2\alpha} + \frac{1}{2} e^{(\frac{1}{\varepsilon}+2)\alpha} X(\alpha).
\end{aligned}$$

Taking the supremum over all cubes $Q_0$, we get the bound

$$X(\alpha)\left(1-\frac{1}{2}e^{(\frac{1}{\varepsilon}+2)\alpha}\right) \leqslant e^{2\alpha},$$

which implies that $X(\alpha) \leqslant C$, if $\alpha$ is small enough.

Since $0<\varepsilon<1$, if we impose that $\frac{1}{2}e^{(\frac{1}{\varepsilon}+2)\alpha} < 1$, then $\alpha < \frac{\varepsilon \ln 2}{2\varepsilon+1}$. Therefore we can choose a smaller parameter $\alpha$, such as $\alpha_X = \ln \sqrt[3]{2^\varepsilon}$. $\square$

Now we will prove two lemmas related to the $A_2$ and $A_\infty$ constants of a particular weight that we will need in the following, which are extensions of a result from [36].

**Lemma 44.** *There are absolute constants $\gamma$ and $c$ such that*

$$[we^{2Rezb}]_{A_2} \leqslant c[w]_{A_2}$$

*for all*

$$|z| \leqslant \frac{\gamma}{\|b\|_{BMO}([w]_{A_\infty} + [\sigma]_{A_\infty})},$$

*where $\gamma = \max\{C_1\alpha_X, C_2\alpha_X\}$ with $C_1$ and $C_2$ absolute constants.*

*Proof of Lemma 44.* We will use the sharp reverse Hölder inequality twice, first for $r = 1 + \frac{1}{\tau[w]_{A_\infty}}$ and then for $r = 1 + \frac{1}{\tau[\sigma]_{A_\infty}}$. With the sharp reverse Hölder inequality for the first choice of $r$, Hölder's inequality and the sharp John-Nirenberg inequality (4.22), we have

$$\begin{aligned} \fint_Q we^{Rezb} &\leqslant \left(\fint_Q w^r\right)^{1/r} \left(\fint_Q e^{r'Rez(b-b_Q)}\right)^{1/r'} e^{Rezb_Q} \\ &\leqslant \left(2\fint_Q w\right) \cdot \beta_X \cdot e^{Rezb_Q}, \end{aligned}$$

for $|z| \leqslant \frac{C_1\alpha_X}{\|b\|_{BMO}[w]_{A_\infty}}$. Note that the constant $\alpha_X$ comes from (4.22) and $C_1$ is an absolute constant from the sharp reverse Hölder inequality since by our choice of $r$, $r' = C_1[w]_{A_\infty}$ (we can even calculate that $\tau[w]_{A_\infty} < r' \leqslant (\tau+1)[w]_{A_\infty}$. We can also get a similar bound as above for the second choice of $r = 1 + \frac{1}{\tau[\sigma]_{A_\infty}}$, giving us

$$\fint_Q w^{-1}e^{-Rezb} \leqslant \left(2\fint_Q w^{-1}\right) \cdot \beta_X \cdot e^{-Rezb_Q}$$

for $|z| \leqslant \frac{C_2\alpha_X}{\|b\|_{BMO}[\sigma]_{A_\infty}}$. Multiplying these two estimates and taking supremum, we finish the proof by showing that for all $z$ as in the assumption

$$\left(\fint_Q we^{Rezb}\right)\left(\fint_Q w^{-1}e^{-Rezb}\right) \leqslant 4\beta_X^2[w]_{A_2}.$$

$\square$

We also have a similar lemma for the $A_\infty$ weight constant.

**Lemma 45.** *There are absolute constants $\gamma$ and $c$ such that*

$$[we^{2Rezb}]_{A_\infty} \leqslant c[w]_{A_\infty}$$

*for all*

$$|z| \leqslant \frac{\gamma'}{\|b\|_{BMO}([w]_{A_\infty})},$$

*where we can take*

$$\gamma' = \frac{\alpha_X}{4\tau}$$

*being $\tau$ an absolute constant from Lemma 41.*

*Proof of Lemma 45.* The proof follows in a similar way as in [11], substituting the appropriate constants from the sharp John Nirenberg inequality in 4.22 and the sharp reverse Hölder inequality in Lemma 41. $\square$

Next we will prove Theorem 18 obtaining a mixed $A_2 - A_\infty$ bound for Calderón–Zygmund operators in SHT. Due to Lerner's decomposition in SHT from section 3, we can fairly easily prove the mixed result. The proof essentially follows from [39]. Only a brief sketch is given below.

*Proof of Theorem 18.* As stated in [39, Sect. 2D], Theorem 18 follows from verifying the following testing conditions:

1. $\|S_Q(\sigma \cdot \chi_Q)\|_{L^2(w)} \leqslant C_1 \|\chi_Q\|_{L^2(\sigma)}$
2. $\|S_Q(w \cdot \chi_Q)\|_{L^2(\sigma)} \leqslant C_2 \|\chi_Q\|_{L^2(w)}$

where

$$S_Q f = \sum_{L \in S, L \subseteq Q} \left( \fint_L f \right) \chi_L,$$

and $S$ is a sparse family (this is a sparse operator). This is still the case in SHT due to the mentioned result of Treil [80], which holds in SHT.

To verify the testing conditions, one simply follows the argument outlined in [39, Sect. 5A]. □

Finally, we can prove the results involving the commutator and its iterates. These proofs follow the scheme from [11].

*Proof of Corollary 6.* First, we prove (4.13). We will "conjugate" the operator $T$ as follows, that is, for any complex number $z$ we define

$$T_z(f) = e^{zb} T(e^{-zb} f). \tag{4.24}$$

By using the Cauchy integral theorem, we get for appropriate functions,

$$T_b(f) = \frac{d}{dz} T_z(f)|_{z=0} = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{T_z(f)}{z^2} d\mu(z), \quad \varepsilon > 0,$$

Therefore, we can write

$$
\begin{aligned}
\|T_b(f)\|_{L^2(w)} &= \left\| (2\pi i)^{-1} \int_{|z|=\varepsilon} \frac{T(fe^{-zb})}{z^2} e^{zb} \right\|_{L^2(w)} \\
&\leqslant \frac{1}{C\varepsilon^2} \int_{|z|=\varepsilon} \left( \int_X |T(fe^{-zb})e^{zb}|^2 w d\mu(x) \right)^{1/2} |d\mu(z)| \\
&= \frac{C}{\varepsilon} \|T(fe^{-zb})\|_{L^2(we^{2Rezb})} \\
&\leqslant \frac{C}{\varepsilon} [we^{2Rezb}]_{A_2}^{1/2} ([we^{2Rezb}]_{A_\infty} + [\sigma e^{2Rezb}]_{A_\infty})^{1/2} \\
&\times \left( \int_X |fe^{-zb}|^2 we^{2Rezb} d\mu \right)^{1/2} \\
&= \frac{C}{\varepsilon} [we^{2Rezb}]_{A_2}^{1/2} ([we^{2Rezb}]_{A_\infty} + [\sigma e^{2Rezb}]_{A_\infty})^{1/2} \\
&\times \left( \int_X |f|^2 w d\mu(x) \right)^{1/2},
\end{aligned}
$$

where we have used the Minkowski inequality for integrals and the $A_2$ theorem for SHT. We also have

$$
[we^{2Rezb}]_{A_2} \leqslant c[w]_{A_2} \tag{4.25}
$$

and similarly for the $A_\infty$ constants, for all $|z| \leqslant \frac{\delta}{\|b\|_{BMO}([w]_{A_\infty}+[\sigma]_{A_\infty})}$ where the $\delta$ is the minimum of the absolute constants from the corresponding lemmas.

All that remains is to bound

$$
\frac{C}{\varepsilon}[w]_{A_2}\|f\|_{L^2(w)}.
$$

Since $|z| = \varepsilon$ we are restricted to certain $\varepsilon$ by (4.25), so we choose $\varepsilon = \frac{\gamma}{\|b\|_{BMO}([w]_{A_\infty}+[\sigma]_{A_\infty})}$, so that

$$
\frac{1}{\varepsilon} = \frac{1}{\delta}([w]_{A_\infty} + [\sigma]_{A_\infty})\|b\|_{BMO}
$$

as wanted.

Putting everything together gives us the desired bound

$$||T_b(f)||_{L^2(w)} \leqslant C[w]_{A_2}^{1/2}([w]_{A_\infty} + [\sigma]_{A_\infty})^{3/2}\|b\|_{BMO}||f||_{L^2(w)}.$$

Finally, to prove the general estimate (4.12), we use again the Cauchy integral theorem to write the k-th commutator for appropriate functions as

$$T_b^k(f) = \frac{d^k}{dz^k}T_z(f)|_{z=0} = \frac{k!}{2\pi i}\int_{|z|=\varepsilon} \frac{T_z(f)}{z^{k+1}}d\mu(z), \quad \varepsilon > 0,$$

where $T_z$ is defined as in (4.24). Then, following the computation for $T_b$ we can arrive at the desired bound for $T_b^k$. □

*Remark* 18. Corollary 6 can be proved under the weaker assumption that $T$ is a linear operator that satisfies the sharp weak mixed $A_2 - A_\infty$ in spaces of homogeneous type.

CHAPTER **Five**

# Lerner techniques and extrapolation in the two-weighted setting

An active area of harmonic analysis lately has been discovery of the precise dependence of the constants in weighted norm inequalities for Calderón-Zygmund singular integral operators (CZOs). In searching for a proof for the $A_2$ theorem, analysts further developed the new theory of non-homogeneous analysis. A natural follow-up to the one-weighted results are related questions about two-weighted norm inequalities for CZOs, but in most cases basic bounds are not even known. For example, necessary and sufficient conditions for $L^p$ boundedness are only known for a few operators, such as the maximal function [76] and Hilbert transform [46], [44]. The two-weighted theory is a developing field, with relatively little known compared to the one-weighted theory. We hope that the background and results discussed further propel development in these areas.

We seek to determine sufficient conditions on the weights $(w, v)$ that give us a two-weighted bound for most CZOs, that is, a bound of the form:

$$\|Tf\|_{L^p(w)} \leqslant C\|f\|_{L^p(v)}.$$

The bump condition theory that we will discuss was introduced to help answer this question. We will relate some history of determining necessary and/or sufficient conditions for $L^p$ boundedness in the classical case of $\mathbb{R}^n$ with Lebesgue measure. We assume $1 < p < \infty$ unless otherwise stated.

An initial idea to find sufficient conditions for $L^p$ boundedness in the two-weight case was to form a two-weighted analogue of the Muckenhoupt $A_p$ condition, that is, for a pair of weights $(w, v)$:

$$\sup_Q \fint_Q w d\mu \left( \fint_Q v^{-p'/p} d\mu \right)^{p-1} < \infty, \tag{5.1}$$

where $Q$ is a cube. However, 5.1 was only necessary and sufficient for the maximal function to be bounded on weak $L^p$, and only necessary for boundedness of some "interesting operators" under consideration (see [61] and [76] for a discussion of this). Therefore, other conditions were then proposed on $(w, v)$ to give sufficient boundedness conditions for CZOs. Neugebauer helped to pioneer this effort [63], by "bumping up" the power on the weights into a condition like:

$$\sup_Q \left( \fint_Q w^{pr} d\mu \right)^{1/pr} \left( \fint_Q v^{-p'r} d\mu \right)^{1/p'r} < \infty, \tag{5.2}$$

for $r > 1$. This condition was strong enough to ensure that both the maximal function and CZOs were bounded from $L^p(v^p)$ to $L^p(w^p)$. A further generalization of 5.1 came from Orlicz space theory. Orlicz norms provide a finer scale of norms "in between" the $L^p$ ones, which we will make precise later. Andrei Lerner used Orlicz $B_p$ norms on both weights in his assumption to prove sufficiency for strong $L^p$ boundedness of all CZOs, answering a question of Cruz-Uribe and Pérez [48].

This section contains two main results. The first is a *Coifman-Fefferman inequality*: bounding a singular integral operator by a corresponding maximal function, which in many cases, leads to a two-weighted bound for CZOs. Our assumptions are related to those in the still open "separated bump conjecture" (see [18], [19], [2], [62]), but not exactly the same since they represent a hybrid of both Neugebauer's and Lerner's assumptions. The proof is a twist on an extrapolation technique from one-weighted theory. As far as we are aware, the use of the reverse Hölder extrapolation in the two-weighted theory is new.

To state this result, we recall that Lerner proved that every CZO can be bounded above in norm by a supremun of sparse operators in a space of homogeneous type (see Chapter 3, [2] and [5]).

**Theorem 19.** *Let $Y$ be a Banach function space (if $Y = L^B$ this is the Orlicz norm), $T^S$ be a sparse operator on $\mathbb{R}^n$ with Lebesgue measure and $1 < p < \infty$. Let $(w, v)$ be weights such that $\sup_Q \|w\|_{q,Q}\|v^{-1}\|_{Y,Q} \leqslant K$ for some $q > p$ and all $Q \in S$. Then*

$$\int_{\mathbb{R}^n} (T^S fw)^p dy \leqslant C^p \int_{\mathbb{R}^n} M_{Y'}(fv)(y)^p dy, \tag{5.3}$$

*where $M_{Y'}$ is a generalization of the Orlicz maximal function.*

If the maximal function $M_{Y'}$ is bounded on weighted $L^p$, then we obtain a two-weighted $L^p$ bound for CZOs. For $\mathbb{R}^n$, broader results are known – see [50] and [21]. The proof of this theorem is general enough that it extends to spaces of homogeneous type.

Secondly, we prove that the Orlicz maximal function, an extension of the usual Hardy-Littlewood maximal function, is bounded on $L^p$ in spaces of homogeneous type without assuming that the associated Young function is doubling. This generalizes a result of Pérez [67]. The importance of the Orlicz maximal function is underscored by its appearance in the proof of the first main result. Moreover, this maximal function has become an integral part of the bump theory and the search for sufficient conditions for two-weighted bounds.

This Chapter is organized in the following fashion. We introduce some background in section 2 and mention a few preliminary results in section 3, including the result on the Orlicz maximal function. Section 4 contains some background for the two-weighted theorem, proved in section 5. In section 6, we give some corollaries, including a strong two-weighted bound for Calderón-Zygmund operators, and examples.

## 5.1 $B_p$ classes and maximal function results

The following class of Young functions is useful in both boundedness of associated maximal functions and in two-weight norm inequalities.

*Definition* 24. A Young function $A$ belongs to $B_p$ (we write $A \in B_p$) for some $1 < p < \infty$ if

$$\int_c^\infty \frac{A(t)}{t^p}\frac{dt}{t} < \infty,$$

for some $c > 0$.

It would be desirable to determine when the Orlicz maximal function is bounded. We start by mentioning the $L^\infty$ bound, whose proof is included for completeness.

**Lemma 46.** *If $A$ is a Young function then $\|M_A f\|_{L^\infty} \leqslant \|f\|_{L^\infty}$.*

*Proof.* Let $Q$ be a cube and let $\lambda = \|f\|_\infty$, then $\frac{|f(y)|}{\lambda} \leqslant 1$ a.e on $Q$. Since $A$ is increasing and $A(1) = 1$, it follows that

$$A\left(\frac{|f(y)|}{\lambda}\right) \leqslant 1$$

a.e. on $Q$. Hence

$$\fint_Q A\left(\frac{|f(y)|}{\lambda}\right) \leqslant 1.$$

Therefore

$$\inf_{\lambda>0}\{\lambda : \fint_Q A\left(\frac{|f(y)|}{\lambda}\right) \leqslant 1\} \leqslant \|f\|_\infty.$$

This is true for all $Q$, so by taking supremums, the result follows. $\square$

Many $L^p$ results concerning the maximal function can be found in [64], [67], [69]. One of the most useful criteria is an $L^p$ boundedness result on $\mathbb{R}^n$ explicitly tied to the $B_p$ condition.

**Theorem 20.** *[64] Let $1 < p < \infty$ and $A$ be a doubling Young function with $\frac{A(t)}{t}$ increasing, $A(t) \to \infty$ as $t \to \infty$, and $A(1) = 1$. Then the following are equivalent:*

*1.* $A \in B_p$

*2. There is a constant* $c$ *such that* $\|M_A f\|_p \leqslant c\|f\|_p$ *for all nonnegative* $f$*.*

Though doubling was originally assumed, it is actually not needed (see below 21).

If $(X, \rho, \mu)$ is an SHT and $\mu(X) = \infty$ then this theorem is still true [74]. However, if $\mu(X) < \infty$, this theorem fails (see the counterexample given in [74]).

Inspired by a question from Cruz-Uribe and recent result from [55] where a similar proof is done in $\mathbb{R}^n$, we present the following which extends 20 to the setting of infinite measure spaces of homogeneous type, without requiring the doubling of the Young function.

**Theorem 21.** *Let* $(X, \rho, \mu)$ *be a space of homogeneous type with doubling constant* $C_d$ *and* $\mu(X) = \infty$*. Let the assumptions of 20 hold, except for doubling of* $A$*. Then* $M_A$ *is bounded on* $L^p$ *if and only if* $A \in B_p$*.*

*Proof.* Since the same argument of 20 applies in showing that if $A \in B_p$ then $M_A$ is bounded, we will only show the converse. Assume that $M_A$ is bounded. Then using the definition, one can show for any measurable set $B$ that

$$M_A(\chi_B)(x) = \sup_{B' \ni x} \frac{1}{A^{-1}(\frac{\mu(B')}{\mu(B' \cap B)})}$$

where $B'$ are balls (see for example, [69]). Note that if $x \in B$, $M_A(\chi_B)(x) = 1$ and for $x \notin B$,

$$1 \geqslant M_A(\chi_B)(x) > \frac{1}{A^{-1}\left(\frac{\mu(B')}{\mu(B)}\right)},$$

where $B' \ni x$. Assume for the moment that $X$ is nonatomic and fix $x_0 \in X$. Later we will lift the nonatomic assumption. Define $r := r_x = \rho(x, x_0)$ for any $x \in X$. Let

$$F(r) = \mu(B(x_0, r)) \tag{5.4}$$

Clearly $F$ is non-decreasing. Since $\mu(X) = \infty$, we can define an inverse for $F$ as

$$F^{-1}(s) = \begin{cases} \inf\{r : F(r) = s\} & : F(r) = s \text{ for some } r \\ \sup\{r : F(r) < s\} & \text{otherwise} \end{cases}$$

Note that since $x_0$ is not an atom, then every $s$ has a well-defined inverse. Hence, if $B' = B(x_0, r)$, we have that

$$\mu\left(x \in X : M_A(\chi_B)(x) > t\right) \geqslant \mu\left(x \in X : \frac{1}{A^{-1}\left(\frac{\mu(B')}{\mu(B)}\right)} > t\right) = \mu\left(x \in X : \frac{1}{A^{-1}\left(\frac{F(r)}{\mu(B)}\right)} > t\right)$$

by 5.4.

Denote $\mu(B) = b$. Since $M_A$ is bounded on $L^p$, then surely it is bounded for the function $f = \chi_B$, so using the above estimate:

$$\infty > \int_X M_A(\chi_B)(x)^p d\mu \geqslant p\int_0^\infty t^p \mu\left(x \in X : \frac{1}{A^{-1}\left(\frac{F(r)}{b}\right)} > t\right)\frac{dt}{t}$$

$$\geqslant p\int_0^\infty t^p \mu\left(A\left(\frac{1}{t}\right)b > F(r)\right)\frac{dt}{t} \geqslant p\int_0^\infty t^p \mu\left(F^{-1}A\left(\frac{1}{t}\right)b > r\right)\frac{dt}{t}$$

since $\{r : F^{-1}(s) > r\} \subseteq \{r : s > F(r)\}$. Continuing, the above

$$= p\int_0^\infty t^p FF^{-1}\left(A\left(\frac{1}{t}\right)b\right)\frac{dt}{t} \geqslant p\int_0^\infty t^p A\left(\frac{1}{t}\right)\frac{b}{C_d}\frac{dt}{t}.$$

Here we have used an important fact from the definition of $F^{-1}$: if $F(r_1) = FF^{-1}(s)$ then

$$\frac{1}{C_d}s \leqslant \frac{1}{C_d}F(r_1 + \varepsilon) \leqslant F\left(\frac{r_1 + \varepsilon}{2}\right) \leqslant F(r_1) = FF^{-1}(s),$$

for some $\varepsilon > 0$ such that $F(\frac{r_1+\varepsilon}{2}) \leqslant F(r_1)$. This is where doubling of the measure $\mu$ is used. For many $s$, we have $FF^{-1}(s) = s$ and do not need to use the above fact. However, for certain $s$ that fall in the range on the $s$-axis where $F$ has a jump discontinuity; doubling allows us to replace

$FF^{-1}\left(A(\frac{1}{t})b\right)$ by $A\left(\frac{1}{t}\right)\frac{b}{C_d}$.

Call $\frac{b}{C_d} = C'$. We can finish the argument by recalling that

$$\infty > C'p\int_0^\infty t^{p-1}A\left(\frac{1}{t}\right)dt > C'p\int_0^{t_0} t^{p-1}A\left(\frac{1}{t}\right)dt = C'p\int_{1/t_0}^\infty \frac{A(t)}{t^p}\frac{dt}{t}.$$

Thus we have actually found that

$$\frac{p}{C_d}\int_{1/t_0}^\infty \frac{A(t)}{t^p}\frac{dt}{t} \leqslant \|M_A\|_{L^p}^p$$

since $\|\chi_B\|_{L^p}^p = b$.

Now, if $x_0$ is an atom, then $\mu(x_0) = \mu(B(x_0,\varepsilon)) = \delta$ for some $\varepsilon$, $\delta > 0$, so $F^{-1}(s)$ is defined for $s \geqslant \delta$. For atoms, the only issue that arises in the proof is when we take $F^{-1}\left(A\left(\frac{1}{t}\right)b\right)$, so as long as $A\left(\frac{1}{t}\right) \geqslant \frac{\delta}{b}$, no adjustment to our argument is needed.

In other words, $F^{-1}\left(A\left(\frac{1}{t}\right)b\right)$ only fails to be defined when $A\left(\frac{1}{t}\right) < \frac{\delta}{b}$. Therefore, if $\delta \leqslant b$, $F^{-1}\left(A\left(\frac{1}{t}\right)b\right)$ can only fail to be defined when $t > 1$ since $A$ is increasing and $A(1) = 1$.

However, for $t > 1$, $\mu(x \in X : M_A(\chi_B)(x) > t) = 0$. We only need to consider level sets where $t < 1$ in our argument, and $F^{-1}$ exists for all points in these sets.

Since from the beginning, we were free to choose any measurable $B$, let $B = B(x_0,\varepsilon)$ for some atom $x_0$. Then

$$b = \mu(B) = \mu(B(x_0,\varepsilon)) = \delta,$$

as wanted. Therefore, the above proof holds for spaces where every point is an atom as well.

□

*Remark* 19. If $\mu(X) < \infty$, then there exists some $R$ such that $B(x_0,R) = X$. Then in the above proof, $\sup\{r : F(r) < F(R)+1\} = \infty$, so $F^{-1}$ would not be well-defined. Since $\mu(X) = \infty$, then for all $s$ there exists $r$ such that $F(r) \geqslant s$.

Note that we could also have defined $F^{-1}$ using $\inf\{r : F(r) > s\}$ instead of $\sup\{r : F(r) < s\}$. This definition would have led to the same result.

We now give some background so we can state an earlier theorem of Pérez about maximal function bounds. Let $Y$ be a Banach function space normed in the following manner: $\|v\|_{Y,Q} = \sup\{\int_X |vg| d\mu : \|g\|_{Y'} = 1\}$, where $Y'$ is the associate space:

$$Y' = \left\{ f \text{ measurable} : \|f\|_{Y'} := \sup_{\|g\|_Y \leqslant 1} \int_X |fg| d\mu < \infty \right\}.$$

Banach function spaces include Orlicz spaces and $L^p$ spaces. For example, if $Y = L^A$ with $A$ a Young function, then $\|\cdot\|_Y = \|\cdot\|_A$ and $Y' = L^{\bar{A}}$. For more information, see [67] or [50]. Let

$$M_{Y'} f(x) = \sup_{Q \ni x} \|f(x)\|_{Y',Q}$$

for cubes Q, be the associated maximal function.

Now we can state the theorem of Pérez [67].

**Theorem 22.** *Assume $M_{Y'}$ is bounded on $L^p(\mathbb{R}^n)$ where $Y$ is a Banach function space and let $(w, v)$ be a pair of weights such that*

$$\sup_Q \|w\|_{p,Q} \|v^{-1}\|_{Y,Q} \leqslant K. \tag{5.5}$$

*Then*

$$\int_{\mathbb{R}^n} (M_{Y'} fw)^p dx \leqslant c \int_{\mathbb{R}^n} (fv)^p dx$$

*for all nonnegative $f$.*

This theorem is also true in an SHT (see [69]).

## 5.2 Preliminary weighted results

To prove our two-weighted extrapolation result, we need the Reverse Hölder classes ([22]).

*Definition* 25. Fix $1 < p < \infty$. We say that a weight $\rho$ belongs to the Reverse Hölder class $p$, denoted $RH_p$, if for all cubes $Q$,

$$\left(\fint_Q \rho^p d\mu\right)^{1/p} \leqslant \kappa \frac{\rho(Q)}{\mu(Q)}. \tag{5.6}$$

The smallest such $\kappa$ is called $[\rho]_{RH_p}$.

Though general cubes are not well-defined in an SHT, we have a dyadic cube construction (see [10], [34]). In our main theorem, we only work with dyadic cubes, so we can take 5.6 with respect to dyadic cubes. We remark that all $A_\infty$ weights satisfy the reverse Hölder property.

**Lemma 47.** *If $w \in A_\infty$, then*

$$w(Q) \leqslant \gamma w(E(Q))$$

*where $\gamma$ depends on $X$ and on $[w]_{A_\infty}$ only, and $E(Q)$ is the disjoint family created from a sparse one.*

*Proof.* Let $\bigcup_{Q' \subsetneq Q, Q' \in S} Q' = T$. Due to the condition of sparse family, we have

$$\mu(T) \leqslant \frac{\mu(Q)}{2}$$

so by the definition of $A_\infty$ we have that

$$w(T) \leqslant \beta w(Q)$$

for some $0 < \beta < 1$. Thus, $w(Q) = w(T) + w(E(Q)) \leqslant \beta w(Q) + w(E(Q))$ so $(1-\beta)w(Q) \leqslant w(E(Q))$ or

$$w(Q) \leqslant \frac{1}{1-\beta} w(E(Q)).$$

$\square$

We now come to the proof of 19.

### 5.3 Proof of two-weighted theorem

*Proof.* First, in Step 1, we show for $p = 1$ and any $\rho \in RH_{q'}$ where $q$ is in the above range that

$$\int_X T^S f w\rho d\mu(y) \leqslant C \int_X M_{Y'}(fv)(y)\rho d\mu(y).$$

We then move to the main statement via an extrapolation argument in Step 2.

*Step 1* For this step, we use the notation $w\rho(Q) = \int_Q w\rho d\mu$. Then

$$\int_X T^S f w\rho d\mu(y) = \sum_{Q\in S}\left(\fint_Q f\right) w\rho(Q) = \sum_{Q\in S}\left(\fint_Q f\cdot v\cdot v^{-1}\right) w\rho(Q)$$

$$\leqslant \sum_{Q\in S}\|fv\|_{Y',Q}\|v^{-1}\|_{Y,Q} w\rho(Q) = \sum_{Q\in S}\|fv\|_{Y',Q}\|v^{-1}\|_{Y,Q}\frac{w\rho(Q)}{\rho(Q)}\rho(Q) := I.$$

Now, estimating part of this expression, we get

$$\|v^{-1}\|_{Y,Q}\frac{w\rho(Q)}{\rho(Q)} \leqslant \|v^{-1}\|_{Y,Q}\left(\int_Q w^q d\mu\right)^{1/q}\left(\int_Q \rho^{q'} d\mu\right)^{1/q'}\cdot\frac{1}{\rho(Q)}$$

$$= \|v^{-1}\|_{Y,Q}\left(\fint_Q w^q d\mu\right)^{1/q}\left(\fint_Q \rho^{q'} d\mu\right)^{1/q'}\frac{\mu(Q)}{\rho(Q)} \leqslant K_q\kappa_q,$$

by 5.5 and 5.6. Then

$$I \leqslant K_q\kappa_q\sum_{Q\in S}\|fv\|_{Y',Q}\rho(Q) \leqslant \gamma K_q\kappa_q\sum_{Q\in S}\|fv\|_{Y',Q}\rho(E(Q))$$

due to 47. The above is bounded by

$$C\sum_{Q\in S}\int_{E(Q))} M_{Y'}(fv)d\rho(y) \leqslant C\int_X M_{Y'}(fv)\rho d\mu(y),$$

where $C = \gamma K_q \kappa_q$. This gives Step 1.

*Step 2:* For extrapolation, we'll follow the procedure of Rubio de Francia. However, we need to adapt this to Reverse Hölder classes since Step 1 is a statement for all $\rho \in RH_{q'}$.

Define the Rubio de Francia operator

$$Rh = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \|M\|^k_{L^{p'/q'}}}.$$

Let

$$\tilde{R}(h) = R(h^{q'})^{1/q'}.$$

Note that for $h \in L^{p'}(d\mu)$,

1. $h \leqslant \tilde{R}(h)$
2. $\|\tilde{R}(h)\|_{p'} \leqslant 2^{1/q'}\|h\|_{p'}$
3. $M(R(h^{q'})) \leqslant 2\|M\|_{L^{p'/q'}} R(h^{q'})$

Item 1 is clear since all terms are positive. Item 2 can be seen by the following argument:

$$\|\tilde{R}(h)\|_{p'} = \|R(h^{q'})\|^{1/q'}_{p'/q'} = \left(\|\sum_k \frac{M^k(h^{q'})}{2^k\|M\|^k_{p'/q'}}\|_{p'/q'}\right)^{1/q'} \leqslant$$

$$\left(\sum_k \frac{\|M^k(h^{q'})\|_{p'/q'}}{2^k\|M\|^k_{p'/q'}}\right)^{1/q'} \leqslant \left(\sum_k \frac{1}{(2\|M\|_{p'/q'})^k}\|M\|^k_{p'/q'}\|h^{q'}\|_{p'/q'}\right)^{1/q'}$$

$$= \|h^{q'}\|^{1/q'}_{p'/q'} 2^{1/q'} = 2^{1/q'}\|h\|_{p'}$$

Item 3 implies that $R(h^{q'}) = \tilde{R}(h)^{q'} \in A_1$, and since by a similar calculation, we also have $\tilde{R}(h) \in A_1$, then $\tilde{R}(h) \in RH_{q'}$. (All of these norms are with respect to d$\mu$.) Notice that we need $p' > q'$ (equivalently $p < q$) for the maximal function norm to be bounded.

Then we run the Rubio de Francia machine: by duality,

$$\|T^S fw\|_p = \int_X T^S fw \cdot h d\mu$$

for some $h \in L^{p'}(d\mu)$ with norm 1. Now by property 1 of $\tilde{R}h$, followed by the fact that $R(h^{q'})^{1/q'} \in RH_{q'}$,

$$\|T^S fw\|_p \leqslant \int_X T^S fw \cdot \tilde{R}(h) d\mu \leqslant C \int_X M_{Y'}(fv)\tilde{R}(h) \leqslant$$

$$C\left(\int_X M_{Y'}(fv)^p d\mu\right)^{1/p} \left(\int_X (\tilde{R}h)^{p'} d\mu\right)^{1/p'} \leqslant C\left(\int_X M_{Y'}(fv)^p d\mu\right)^{1/p} \left(\int_X h^{p'} d\mu\right)^{1/p'}$$

$$= C(\int_X M_{Y'}(fv)^p d\mu)^{1/p}$$

Taking $p$th powers, we get 5.3 as wanted.

Tracing the constant from Step 1, we see that $C = 2^{1/q'} K_q \cdot [\tilde{R}(h)]_{RH_{q'}} \cdot \gamma$, where $\gamma$ depends on $[\tilde{R}h]_{A_\infty}, SHT$. Since we can bound $[\tilde{R}(h)]_{RH_{q'}} \leqslant [\tilde{R}(h)]_{A_1} \leqslant (2\|M\|_{p'/q'})^{1/q'}$ we have that $C \leqslant 2^{2/q'} K_q \|M\|_{p'/q'}^{1/q'} \cdot \gamma$.

$\square$

## 5.4 Corollaries and examples

**Corollary 7.** *Let the conditions of 19 hold. Then*

$$\|Tf\|_{L^p(w^p)} \leqslant C\|M_{Y'}(fv)\|_p.$$

*Proof.* By the main decomposition in chapter 3,

$$\|T(f)\|_Z \leqslant C(SHT, T) \sup_{D,S} \|T^S(f)\|_Z \tag{5.7}$$

where $D$ is a dyadic grid and $Z$ is a Banach function space, such as $L^p(u)$ (note that $Z$ is not necessarily the same as the Banach function space $Y$ in the theorem). Recall that $L^p(u)$ is a Banach function space whenever $ud\mu$ is a locally integrable positive function, that is, when $u$ is any weight. $\square$

**Corollary 8.** *Let $M_{Y'}$ be bounded on $L^p(\mu)$, and the conditions of 19 hold. Then,*

$$\|Tf\|_{L^p(w^p)} \leqslant C\|f\|_{L^p(v^p)}.$$

*Remark* 20. We can apply 19 to the case $v = w = u^p$ and $Y' = L^p$. Then our assumption becomes the $A_p$ condition. However, 8 does not hold since $M_{Y'} = M_p$ is not bounded on $L^p$.

A version of the following Corollary appeared in [21].

**Corollary 9.** *Let $A$ be a Young function such that $\bar{A} \in B_p$ and the conditions of 19 hold. Then*

$$\|Tf\|_{L^p(w^p)} \leqslant C\|f\|_{L^p(v^p)}.$$

*Proof.* In this case, $Y' = L^{\bar{A}}$, and $M_{\bar{A}}$ is bounded by 20. $\square$

Step 2 can be written as a general extrapolation theorem for Reverse Hölder classes:

**Theorem 23.** *Fix $1 < p < \infty$. Let $S_1(x)$ and $S_2(x)$ be two objects (operators, functions, etc) such that*

$$\int_X S_1(x)\rho d\mu \leqslant \int_X S_2(x)\rho d\mu$$

*for all $x \in X$ and for all $\rho \in RH_{q'}$, $q \in (p, p+\varepsilon)$ or $q \in (p, \infty)$. Then we have that*

$$\int_X (S_1(x))^p d\mu \leqslant \int_X (S_2(x))^p d\mu.$$

Now we discuss some examples of weights that satisfy the conditions of the extrapolation theorem.

It is interesting to note the following about power weights.

*Remark* 21. Let $w = |x|^{-\alpha}$ and $v = |x|^{-\beta}$ in $\mathbb{R}$. Then

$$\|w\|_{p,Q}\|v^{-1}\|_{p',Q} \leqslant K$$

if and only if $w = v$. This is due to the stringent requirements on the exponents when we integrate over a cube containing the origin. This eliminates certain power weight examples from our consideration.

(we can see this with $X = \mathbb{R}$ and $Q = [-c, c]$ where $c = \frac{l(Q)}{2}$). Then (as long as $\alpha \cdot p \neq 1$),

$$\left(\fint_Q w^p\right)^{1/p} = \left(\frac{|x|^{-\alpha \cdot p+1}}{D \cdot l(Q)} \,|_{-c}^{c}\right)^{1/p} = D' l(Q)^{-\alpha}.$$

Similarly,

$$\left(\fint_Q v^{-p'}\right)^{1/p'} = \left(\frac{|x|^{\beta \cdot p'+1}}{Dl(Q)} \,|_{-c}^{c}\right)^{1/p'} = D' l(Q)^{\beta},$$

where $D$ and $D'$ are constants. Hence since there are both arbitrarily small and large cubes centered at the origin, in order to have the suprenum over these cubes finite, we must have that $\alpha = \beta$. Even if we use the Young function $B(t) = t^r$ for $r < p'$, the calculation above for $\left(\fint_Q v^{-r}\right)^{1/r}$ is the same for cubes around the origin, and we still get that $\alpha = \beta$.

Note that the following examples are variations of the classic two-weight "$A_1$ pair" $(u, M_u)$ [18].

**Example 2.** Take $M_q$ to be the $L^q$ normalized Hardy-Littlewood maximal function with respect to (dyadic) cubes. We will show that the pair $(u, M_q u)$, where $u$ is a weight, satisfies our assumptions for 19 with the pairs: $(p, B)$ and $(q, B)$, where $B$ is a Young function.

We have then the following bound (adapted from [2]). For $q > p$

$$M_q u(x) \geqslant \|u\|_{q,Q}$$

for all $x \in Q$ so that

$$\|u\|_{q,Q}\|M_q(u)^{-1}\|_{B,Q} \leqslant \|u\|_{q,Q}\|u\|_{q,Q}^{-1}\|\chi_Q\|_{B,Q} = 1.$$

This is true since $\|\|u\|_{B,Q}^{-1}\|_{p',Q} = \|u\|_{B,Q}^{-1}\|1\|_{p',Q} = \|u\|_{B,Q}^{-1}\|\chi_Q\|_{p',Q} = \|u\|_{B,Q}^{-1}$. Here we have used the normalization $B(1) = 1$. We also get that

$$\|u\|_{p,Q}\|M_q(u)^{-1}\|_{B,Q} \leqslant \|u\|_{q,Q}\|M_q(u)^{-1}\|_{B,Q} = 1$$

by Holder's inequality. Hence, the pair $(u, M_q(u))$ satisfies the conditions of 19.

**Example 3.** Take the pair $(M_A u^{1-p}, w)$ where $A$ is a Young function and $u, w$ are weights. If we also assume that $\|u\|_{A,Q}^{1-p}\|w^{-1}\|_{B,Q} \leqslant K$ for all $Q$, then

$$\|M_A u^{1-p}\|_{p,Q}\|w^{-1}\|_{B,Q} \leqslant \|u\|_{A,Q}^{1-p}\|w^{-1}\|_{B,Q} < K.$$

Moreover, if we change the first norm to the $\|\cdot\|_{q,Q}$ norm, the bound remains the same. Thus $(M_A u^{1-p}, w)$ satisfies 19.

# BIBLIOGRAPHY


[1] Theresa C. Anderson. A new sufficient two-weighted bump assumption for $L^p$ boundedness of Calderón-Zygmund operators. Accepted. To appear in Proceedings of the AMS, preprint available on arXiv.

[2] Theresa C. Anderson, David Cruz-Uribe, and Kabe Moen. Logarithmic bump conditions for caldern-zygmund operators on spaces of homogeneous type. *Publicacions Mathemàtiques*, 59(1):17–43, 2015.

[3] Theresa C. Anderson and W. Damián. Calderón-Zygmund operators and commutators in spaces of homogeneous type: weighted inequalities. *Submitted, available at ArXiv e-prints*, January 2014.

[4] Theresa C. Anderson, Tuomas Hytönen, and Olli Tapiola. Weak a-infinity weights and weak reverse hölder property in a space of homogeneous type. 2014. Submitted.

[5] Theresa C. Anderson and Armen Vagharshakyan. A Simple Proof of the Sharp Weighted Estimate for Calderón–Zygmund Operators on Homogeneous Spaces. *J. Geom. Anal.*, 24(3):1276–1297, 2014.

[6] A. Bernardis, G. Pradolini, M. Lorente, and M. Riveros. Composition of fractional Orlicz maximal operators and $A_1$-weights on spaces of homogeneous type. *Acta Math. Sin. (Engl. Ser.)*, 26(8):1509–1518, 2010.

[7] Oleksandra V. Beznosova. Linear bound for the dyadic paraproduct on weighted Lebesgue space $L_2(w)$. *J. Funct. Anal.*, 255(4):994–1007, 2008.

[8] Stephen M. Buckley. Estimates for operator norms on weighted spaces and reverse Jensen inequalities. *Trans. Amer. Math. Soc.*, 340(1):253–272, 1993.

[9] A.-P. Calderón. Inequalities for the maximal function relative to a metric. *Studia Math.*, 57(3):297–306, 1976.

[10] M. Christ. *Lectures on singular integral operators*, volume 77 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1990.

[11] Daewon Chung, M. Cristina Pereyra, and Carlos Perez. Sharp bounds for general commutators on weighted Lebesgue spaces. *Trans. Amer. Math. Soc.*, 364(3):1163–1177, 2012.

[12] R. R. Coifman and C. Fefferman. Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.*, 51:241–250, 1974.

[13] R. R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes*. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin, 1971. Étude de certaines intégrales singulières.

[14] R. R. Coifman and G. Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.*, 83(4):569–645, 1977.

[15] Ronald R. Coifman and Guido Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.*, 83(4):569–645, 1977.

[16] Jose M. Conde. A note on dyadic coverings and nondoubling Calderón-Zygmund theory. *J. Math. Anal. Appl.*, 397(2):785–790, 2013.

[17] D. Cruz-Uribe, J. M. Martell, and C. Pérez. Sharp weighted estimates for classical operators. *Adv. Math.*, 229:408–441, 2011.

[18] D. Cruz-Uribe, J. M. Martell, and C. Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.

[19] D. Cruz-Uribe, A. Reznikov, and A. Volberg. Logarithmic bump conditions and the two-weight boundedness of Calderón-Zygmund operators. *Adv. Math*, (255):706–729, 2014. Preprint.

[20] David Cruz-Uribe, José María Martell, and Carlos Pérez. Sharp weighted estimates for approximating dyadic operators. *Electron. Res. Announc. Math. Sci.*, 17:12–19, 2010.

[21] David Cruz-Uribe and Carlos Pérez. On the two-weight problem for singular integral operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 1(4):821–849, 2002.

[22] Cruz-Uribe D. and C. Neugebauer. The structure of the reverse hölder classes. *Transactions of the AMS*, 347:2941–2960, 1995.

[23] Oliver Dragičević, Loukas Grafakos, María Cristina Pereyra, and Stefanie Petermichl. Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces. *Publ. Mat.*, 49(1):73–91, 2005.

[24] Javier Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[25] Javier Duoandikoetxea. Extrapolation of weights revisited: new proofs and sharp bounds. *J. Funct. Anal.*, 260(6):1886–1901, 2011.

[26] Fernando Farroni and Raffaella Giova. Change of variables for $A_\infty$ weights by means of quasiconformal mappings: sharp results. *Ann. Acad. Sci. Fenn. Math.*, 38(2):785–796, 2013.

[27] N. Fujii. Weighted bounded mean oscillation and singular integrals. *Math. Japon.*, 22(5):529–534, 1977/78.

[28] J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985.

[29] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec. *Weight theory for integral transforms on spaces of homogeneous type*, volume 92 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman, Harlow, 1998.

[30] L. Grafakos. *Classical Fourier Analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, 2009.

[31] L. Grafakos. *Modern Fourier Analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, 2009.

[32] Sergei V. Hruščev. A description of weights satisfying the $A_\infty$ condition of Muckenhoupt. *Proc. Amer. Math. Soc.*, 90(2):253–257, 1984.

[33] T. Hytönen. The sharp weighted bound for general Calderón-Zygmund operators. *Ann. of Math. (2)*, 175(3):1473–1506, 2012.

[34] T. Hytönen and A. Kairema. Systems of dyadic cubes in a doubling metric space. *Colloq. Math.*, 126(1):1–33, 2012.

[35] Tuomas Hytönen and Anna Kairema. Systems of dyadic cubes in a doubling metric space. *Colloq. Math.*, 126(1):1–33, 2012.

[36] Tuomas Hytönen and Carlos Pérez. Sharp weighted bounds involving $A_\infty$. *Anal. PDE*, 6(4):777–818, 2013.

[37] Tuomas Hytönen, Carlos Pérez, and Ezequiel Rela. Sharp reverse Hölder property for $A_\infty$ weights on spaces of homogeneous type. *J. Funct. Anal.*, 263(12):3883–3899, 2012.

[38] Tuomas Hytönen, Carlos Pérez, Sergei Treil, and Alexander Volberg. Sharp weighted estimates for dyadic shifts and the $A_2$ conjecture. *J. Reine Angew. Math.*, 687:43–86, 2014.

[39] Tuomas P. Hytönen. The $A_2$ theorem: remarks and complements. In *Harmonic analysis and partial differential equations*, volume 612 of *Contemp. Math.*, pages 91–106. Amer. Math. Soc., Providence, RI, 2014.

[40] Tuomas P. Hytönen, Michael T. Lacey, Henri Martikainen, Tuomas Orponen, Maria Carmen Reguera, Eric T. Sawyer, and Ignacio Uriarte-Tuero. Weak and strong type estimates for maximal truncations of Calderón-Zygmund operators on $A_p$ weighted spaces. *J. Anal. Math.*, 118(1):177–220, 2012.

[41] B. Jawerth and A. Torchinsky. Local sharp maximal functions. *J. Approx. Theory*, 43(3):231–270, 1985.

[42] F. John. Quasi-isometric mappings. Seminari 19621963 di Analisi, Algebra, Geometria e Topologia, Ist Nazarene Alta Matematica 2 (Edizioni Cremonese, Rome, 1965), 462473.

[43] Jean-Lin Journé. *Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón*, volume 994 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[44] M. T Lacey. Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization, II. *ArXiv e-prints, to appear in Duke Mathematical Journal*, January 2013.

[45] M. T. Lacey, E. T. Sawyer, C.-Y. Shen, and I. Uriarte-Tuero. Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization, I. *ArXiv e-prints, to appear in Duke Mathematical Journal*, January 2012.

[46] M. T. Lacey, E. T. Sawyer, C.-Y. Shen, and I. Uriarte-Tuero. Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization, I. *ArXiv e-prints*, January 2012.

[47] Michael T. Lacey, Stefanie Petermichl, and Maria Carmen Reguera. Sharp $A_2$ inequality for Haar shift operators. *Math. Ann.*, 348(1):127–141, 2010.

[48] A. Lerner. On an estimate of Calderón-Zygmund operators by dyadic positive operators. *J. Anal. Math.*, 2012.

[49] Andrei K. Lerner. A pointwise estimate for the local sharp maximal function with applications to singular integrals. *Bull. Lond. Math. Soc.*, 42(5):843–856, 2010.

[50] Andrei K. Lerner. On an estimate of Calderón-Zygmund operators by dyadic positive operators. *J. Anal. Math.*, 121:141–161, 2013.

[51] Andrei K. Lerner. A simple proof of the $A_2$ conjecture. *Int. Math. Res. Not. IMRN*, (14):3159–3170, 2013.

[52] Andrei K. Lerner, Sheldy Ombrosi, and Carlos Pérez. Sharp $A_1$ bounds for Calderón-Zygmund operators and the relationship with a problem of Muckenhoupt and Wheeden. *Int. Math. Res. Not. IMRN*, (6):Art. ID rnm161, 11, 2008.

[53] Andrei K. Lerner, Sheldy Ombrosi, and Carlos Pérez. $A_1$ bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Wheeden. *Math. Res. Lett.*, 16(1):149–156, 2009.

[54] J. Li, J. Pipher, and L. A. Ward. One-parameter and multiparameter function classes are intersections of finitely many dyadic classes. *ArXiv e-prints*, April 2012.

[55] L. Liu and T. Luque. A \$B_p\$ condition for the strong maximal function. *To appear in Transactions of the AMS. ArXiv e-prints*, November 2012.

[56] T. Luque, C. Pérez, and E. Rela. Optimal exponents in weighted estimates without examples. *To Appear im Math. Res. Lett.*

[57] R. Macías and C. Segovia. A well behaved quasi-distance for spaces of homogeneous type. In *Trabajos de Matemática*, volume 32, pages 1–18. Instituto Argentino de Matemática (IAM), 1980. Available at http://www.cimec.org.ar/ojs/index.php/cmm/article/viewFile/457/439.

[58] Roberto A. Macías and Carlos Segovia. Lipschitz functions on spaces of homogeneous type. *Adv. in Math.*, 33(3):257–270, 1979.

[59] Tao Mei. BMO is the intersection of two translates of dyadic BMO. *C. R. Math. Acad. Sci. Paris*, 336(12):1003–1006, 2003.

[60] Benjamin Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[61] F. Nazarov, A. Reznikov, S. Treil, and A. Volberg. A Bellman function proof of the \$L^2\$ bump conjecture. *ArXiv e-prints*, February 2012.

[62] Fedor Nazarov, Alexander Reznikov, and Alexander Volberg. The proof of $A_2$ conjecture in a geometrically doubling metric space. *Indiana Univ. Math. J.*, 62(5):1503–1533, 2013.

[63] C. J. Neugebauer. Inserting $A_p$-weights. *Proc. Amer. Math. Soc.*, 87(4):644–648, 1983.

[64] C. Pérez. Two weighted inequalities for potential and fractional type maximal operators. *Indiana Univ. Math. J.*, 43(2):663–683, 1994.

[65] C. Pérez. Weighted norm inequalities for singular integral operators. *J. London Math. Soc. (2)*, 49(2):296–308, 1994.

[66] C. Pérez. On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted $L^p$-spaces with different weights. *Proc. London Math. Soc. (3)*, 71(1):135–157, 1995.

[67] C. Pérez. On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted $L^p$-spaces with different weights. *Proc. London Math. Soc. (3)*, 71(1):135–157, 1995.

[68] C. Pérez, S. Treil, and A. Volberg. On A2 conjecture and corona decomposition of weights. *Preprint.*

[69] C. Pérez and R. Wheeden. Uncertainty principle estimates for vector fields. *J. Funct. Anal.*, (181):146–188, 2001.

[70] Carlos Pérez. To appear in Birkhäuser as part of the series *Advanced courses in Mathematics at the C.R.M., Barcelona*, Title = A course on Singular Integrals and weights, Year = 2009.

[71] S. Petermichl. The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical $A_p$ characteristic. *Amer. J. Math.*, 129(5):1355–1375, 2007.

[72] Stefanie Petermichl. The sharp weighted bound for the Riesz transforms. *Proc. Amer. Math. Soc.*, 136(4):1237–1249, 2008.

[73] Stefanie Petermichl and Alexander Volberg. Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular. *Duke Math. J.*, 112(2):281–305, 2002.

[74] G. Pradolini and O. Salinas. Maximal operators on spaces of homogeneous type. *Proc. Amer. Math. Soc.*, 132(2):435–441 (electronic), 2004.

[75] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[76] E. T. Sawyer. A characterization of a two-weight norm inequality for maximal operators. *Studia Math.*, 75(1):1–11, 1982.

[77] E. M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton NJ, 1993.

[78] Jan-Olov Strömberg. Bounded mean oscillation with Orlicz norms and duality of Hardy spaces. *Indiana Univ. Math. J.*, 28(3):511–544, 1979.

[79] R. Toledano. A note on the Lebesgue differentiation theorem in spaces of homogeneous type. *Real Anal. Exchange*, 29(1):335–339, 2003/04.

[80] S. Treil. A remark on two weight estimates for positive dyadic operators. *preprint*, 2012.

[81] Armen Vagharshakyan. Recovering singular integrals from Haar shifts. *Proc. Amer. Math. Soc.*, 138(12):4303–4309, 2010.

[82] J. M. Wilson. Weighted inequalities for the dyadic square function without dyadic $A_\infty$. *Duke Math. J.*, 55(1):19–50, 1987.