From Kinetic Theory to Fluid Mechanics: Viscous Surface Wave and Hydrodynamic Limit

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Abstract of "From Kinetic Theory to Fluid Mechanics: Viscous Surface Wave and Hydrodynamic Limit" by Lei Wu, Ph.D., Brown University, May 2015

In this dissertation, we mainly discuss two topics of partial differential equations in fluid dynamics and kinetic theory: viscous surface wave and diffusive limit.

With respect to viscous surface wave, we consider an incompressible viscous flow without surface tension in a finite-depth domain of three dimensions, with a free top boundary and a fixed bottom boundary. The system is governed by the Navier-Stokes equations in this moving domain and the transport equation on the moving boundary. Following the framework of geometric mapping by Y. Guo and I. Tice, we further prove the local well-posedness with general smooth data and give a simpler proof of global well-posedness. Also, we construct a stable numerical scheme to simulate the evolution of this system by discontinuous Galerkin method.

With respect to diffusive limit, we revisit the asymptotic analysis of a steady neutron transport equation in a two-dimensional unit disk with one-speed velocity. We disprove the classical boundary layer theory by a concrete counterexample with a different boundary layer expansion with geometric correction. Also, we provide the correct boundary layer construction with both in-flow and diffusive boundary.

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Chapter One

Introduction

In classical mechanics, matter can be described under different scales: for microscopic scale, Newton's laws dominate the movement of atoms and molecules; for macroscopic objects, fluid mechanics and solid mechanics provide effective tools to predict the behavior; in between, from mesoscopic viewpoint, kinetic theory states the statistical rules of the pattern of particles.

Hilbert's Six Problem in axiomatization of physics attempts to connect the fundamental laws behind these theories, i.e. to derive the mesoscopic and macroscopic rules from basic Newton's laws in microscopic scale. Following the path Hilbert himself proposed, the first step from microscope to mesoscope is the justification of kinetic theory, which involves the unfinished project of rigorous derivation of Boltzmann equation. The second step from mesoscope to macroscope is the hydrodynamic limit, which is the bridge between fluid equations and kinetic theory. Our research focuses on the second step and we will mainly discuss the decay of viscous surface wave and diffusive limit of neutron transport equation.

In Chapter 2, we will first discuss the local and global well-posedness and decay of viscous surface wave under the framework of geometric mapping. In Chapter 3, we will provide a stable numerical scheme to viscous surface wave with discontinuous Galerkin method and analyze its convergence. In Chapter 4, we turn to the boundary layer effect in steady neutron transport equation and provide a detailed analysis of Milne problem with geometric correction.

Chapter Two

Well-Posedness and Decay of Viscous Surface Wave

2.1 Introduction

2.1.1 Problem Presentation

We consider an incompressible viscous flow in the moving domain

$$\Omega(t) = \{y \in \Sigma \times \mathbb{R} : -b(y_1, y_2) < y_3 < \eta(y_1, y_2, t)\}. \tag{2.1.1}$$

Here we can take either $\Sigma = \mathbb{R}^2$ or $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$ for which $\mathbb{T}$ denotes the 1-torus and $L^1, L^2 > 0$ the periodicity lengths. The lower boundary b is fixed and given satisfying

$$0 < b(y_1, y_2) \le \bar{b} = \text{constant} \quad \text{and} \quad b \in C^\infty(\Sigma). \tag{2.1.2}$$

When $\Sigma = \mathbb{R}^2$, we further require $b(y_1, y_2) \to b_0 =$ positive constant as $|y_1|+|y_2| \to \infty$ and $b - b_0 \in H^s(\Sigma)$ for any $s \ge 0$. When $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$, we just denote $b_0 = 1/2(\max\{b(y_1, y_2)\} + \min\{b(y_1, y_2)\})$. It is easy to see this also implies $b - b_0 \in H^s(\Sigma)$ for any $s \ge 0$ since Σ is bounded. We denote the initial domain $\Omega(0) = \Omega_0$. For each t, the flow is described by its velocity and pressure $(u, p) : \Omega(t) \mapsto \mathbb{R}^3 \times \mathbb{R}$ which satisfies the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u & \text{in} \quad \Omega(t), \\ \nabla \cdot u = 0 & \text{in} \quad \Omega(t), \\ (pI - \mu\mathbb{D}(u))\nu = g\eta\nu & \text{on} \quad \{y_3 = \eta(y_1, y_2, t)\}, \\ u = 0 & \text{on} \quad \{y_3 = -b(y_1, y_2)\}, \\ \partial_t \eta = u_3 - u_1 \partial_{y_1}\eta - u_2 \partial_{y_2}\eta & \text{on} \quad \{y_3 = \eta(y_1, y_2, t)\}, \\ u(t = 0) = u_0 & \text{in} \quad \Omega_0, \\ \eta(t = 0) = \eta_0 & \text{on} \quad \Sigma, \end{cases} \tag{2.1.3}$$

for ν the outward-pointing unit normal vector on $\{y_3 = \eta\}$, I the 3×3 identity matrix, $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u, g the gravitational constant and $\mu > 0$ the viscosity. The fifth equation in (2.1.3) implies the free surface is convected with the fluid. Note that in (2.1.3), we have shifted the actual pressure $\bar{p}$ by the constant atmosphere pressure p_{atm} according to $p = \bar{p} + gy_3 - p_{atm}$.

We always assume a natural condition that there exists a positive real number ρ such that $\eta_0 + b \geq \rho > 0$ on Σ, which means the initial free surface is strictly separated from the bottom. Also without loss of generality, we may assume $\mu = g = 1$. In the following, we will use the term "infinite case" when $\Sigma = \mathbb{R}^2$ and "periodic case" when $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$.

The problem (2.1.3) possesses a natural energy structure that for the solution triple (u, p, η) which is sufficiently regular, we have

$$\frac{1}{2}\int_{\Omega(t)} |u(t)|^2 + \frac{1}{2}\int_\Sigma |\eta(t)|^2 + \frac{1}{2}\int_0^t \int_{\Omega(s)} |\mathbb{D}u(s)|^2 \,\mathrm{ds} = \frac{1}{2}\int_{\Omega_0} |u_0|^2 + \frac{1}{2}\int_\Sigma |\eta_0|^2 . \tag{2.1.4}$$

The first two integrals represent the kinetic energy and potential energy, while the third represents the dissipation.

2.1.2 Previous Results

Historically, this problem has been studied in several different settings according to whether we consider the viscosity and surface tension.

In the inviscid case, traditionally we replace the non-slip condition $u = 0$ with non-penetration condition $u \cdot n = 0$ on the bottom boundary. It is often assumed that the initial fluid is irrotational and then this curl-free property is preserved in the later time. This allows to reformulate the problem to one only on the free surface. Local well-posedness in this framework was proved by Wu [51, 52] and Lannes [31]. On the other hand, local well-posedness without this irrotational assumption was proved by Zhang-Zhang [55], Christodoulou-Lindblad [12], Lindblad [42], Coutand-Shkoller [16] and Shatah-Zheng [44]. In the viscous case, vorticity is naturally introduced at the free surface, so the surface formulation is not valid. Also, in the inviscid irrotational case, global well-posedness was shown by Wu [53] and Germain-Masmoudi-Shatah [20].

In the viscous case without surface tension, local well-posedness was proved by Solonnikov and Beale. Solonnikov [46] employed the framework of Hölder spaces to study the problem in a bounded domain, all of whose boundaries are free. Beale [8] utilized the framework of L^2 spaces to study the domain like ours. Both of them took this system as a perturbation of parabolic equations and made use of the regularity results for linear equations. Abel [1] extended this result to the framework of L^p spaces. Also, Hataya [29] proved global well-posedness in the periodic case. Many authors have also considered the effects of surface tension, which can help stabilize the problem and gain regularity. e.g. global well-posedness in the small data case was shown by Bae [7].

The idea of mapping the moving domain into a fixed domain can date back to Beale's work in [8] for graph-like domains and was also utilized by Lannes in [31]. A recent progress in this field is by Shatah-Zeng [45], which can deal with more general domains by maps with harmonic extensions.

Most of above results are built in the Lagrangian coordinates and do not utilize the natural energy structure of the problem in a direct manner. Hence, showing decay becomes very difficult. In [26], [25] and [24], Y. Guo and I. Tice introduced a geometric energy framework and proved the solutions with small data are globally well-posed and decay in time. The main idea in their proof was that, instead of considering the perturbation of parabolic equations, they showed the regularity under a time-dependent basis. In our paper, we follow this path and employ a similar argument.

In [26], it is assumed the initial data u_0 and η_0 are sufficiently small to guarantee that: (1) the transform between the fixed domain and the moving domain is a diffeomorphism; (2) the elliptic estimates in the linear Navier-Stokes problem achieve the optimal regularity; (3) in proving boundedness and contraction of the iteration sequence, the constant is small enough to obtain desired estimates. In Section 2.2 of this dissertation, we drop this smallness assumption and prove local well-posedness for general initial data. Basically, we should use different techniques to recover above three properties.

Also, in proving global well-posedness and decay in [25], the authors introduced a quite complicated interpolation argument to improve the decaying rate of energy with minimal counts. In Section 2.3 of this dissertation, through redefining the energy and dissipation, we greatly simplify the proof in: (1) avoiding the complicated bootstrapping argument in the interpolation estimates; (2) avoiding the bootstrapping argument in the comparison estimates; (3) handling the energy and dissipation with 1-minimal count and 2-minimal count simultaneously instead of separately.

2.1.3 Geometric Formulation

In order to work in a fixed domain, we need to flatten the free surface via a coordinates transform. We follow the idea in [8] and [26], and use a slightly modified version as the geometric transform. Define a fixed domain

$$\Omega = \{x \in \Sigma \times R \mid -b_0 < x_3 < 0\}, \tag{2.1.5}$$

for which we write the coordinates $x \in \Omega$. In this slab, we take $\Sigma : \{x_3 = 0\}$ as the upper boundary and $\Sigma_b : \{x_3 = -b_0\}$ as the lower boundary. Simply denote $x' = (x_1, x_2)$ and $x = (x_1, x_2, x_3)$. We need to utilize slightly different techniques to construct the extension of free surface in the local and global cases.

Geometric Formulation in the Local Case

In the local case, we cannot apply the idea in [26] directly. We introduce a parameterized transform instead. Define the ϵ-Poisson integral for η:

$$\bar{\eta}^\epsilon = \mathcal{P}^\epsilon \eta = \begin{cases} \int_{\mathbb{R}^2} \hat{\eta}(\xi) e^{\epsilon|\xi| x_3} e^{2\pi i x' \cdot \xi} \mathrm{d}\xi & \text{in the infinite case,} \\ \\ \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} e^{2\pi i n \cdot x'} e^{\epsilon|n| x_3} \hat{\eta}(n) & \text{in the periodic case,} \end{cases} \tag{2.1.6}$$

where $\hat{\eta}(\xi)$ and $\hat{\eta}(n)$ denote the Fourier transform of $\eta(x')$ in either continuous or discrete form and $0 < \epsilon < 1$ the parameter. The detailed definitions are shown in (2.2.1) and (2.2.16).

Consider the geometric transform from Ω to $\Omega(t)$:

$$\Phi^\epsilon : (x_1, x_2, x_3) \mapsto (x_1, x_2, \frac{b}{b_0} x_3 + \bar{\eta}^\epsilon (1 + \frac{x_3}{b_0})) = (y_1, y_2, y_3). \tag{2.1.7}$$

This transform maps Ω into $\Omega(t)$ with the Jacobi matrix

$$\nabla\Phi^\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A^\epsilon & B^\epsilon & J^\epsilon \end{pmatrix} \tag{2.1.8}$$

and the transform matrix

$$\mathscr{A}^\epsilon = ((\nabla\Phi^\epsilon)^{-1})^T = \begin{pmatrix} 1 & 0 & -A^\epsilon K^\epsilon \\ 0 & 1 & -B^\epsilon K^\epsilon \\ 0 & 0 & K^\epsilon \end{pmatrix} \tag{2.1.9}$$

where

$$\begin{aligned} \tilde{b} &= 1 + \frac{x_3}{b_0}, \\ A^\epsilon &= \frac{\partial_1 b}{b_0} x_3 + \partial_1 \bar{\eta}^\epsilon \tilde{b}, \qquad B^\epsilon = \frac{\partial_2 b}{b_0} x_3 + \partial_2 \bar{\eta}^\epsilon \tilde{b}, \\ J^\epsilon &= \frac{b}{b_0} + \frac{\bar{\eta}^\epsilon}{b_0} + \partial_3 \bar{\eta}^\epsilon \tilde{b}, \qquad K^\epsilon = \frac{1}{J^\epsilon}. \end{aligned} \tag{2.1.10}$$

Notice that this transform is not necessarily a homeomorphism for arbitrary $\epsilon > 0$. By our assumption $\eta_0(x') + b(x') \geq \rho > 0$ and Theorem 2.2.7, we can always choose a sufficiently small ϵ depending on $\|\eta_0\|_{H^{5/2}}$ such that there exists a $\delta > 0$ satisfying $J^\epsilon(0) > \delta > 0$. Then this positive Jacobi matrix implies the transform $\Phi^\epsilon(t)$ is a homeomorphism at $t = 0$. Furthermore, based on Lemma 2.2.2, Lemma 2.2.3, Lemma 2.2.5 and Lemma 2.2.6, we can deduce $\Phi^\epsilon(0)$ is actually a C^1 diffeomorphism. In the following, we just write $\bar{\eta}$ instead of $\bar{\eta}^\epsilon$ for simplicity, and the same convention applies to $\mathscr{A}$, Φ, A, B, J and K.

Define the transformed operators as follows:

$$
\begin{aligned}
&(\nabla_{\mathscr{A}} f)_i = \mathscr{A}_{ij}\partial_j f,\\
&\nabla_{\mathscr{A}} \cdot \vec{g} = \mathscr{A}_{ij}\partial_j g_i,\\
&\Delta_{\mathscr{A}} f = \nabla_{\mathscr{A}} \cdot \nabla_{\mathscr{A}} f,\\
&\mathscr{N} = (-\partial_1\eta, -\partial_2\eta, 1),\\
&(\mathbb{D}_{\mathscr{A}} u)_{ij} = \mathscr{A}_{ik}\partial_k u_j + \mathscr{A}_{jk}\partial_k u_i,\\
&S_{\mathscr{A}}(p,u) = pI - \mathbb{D}_{\mathscr{A}} u,
\end{aligned}
\tag{2.1.11}
$$

where the summation should be understood in the Einstein convention. If we extend the divergence $\nabla_{\mathscr{A}}\cdot$ to act on symmetric tensors in the natural way, then a straightforward computation reveals $\nabla_{\mathscr{A}} \cdot S_{\mathscr{A}}(p,u) = \nabla_{\mathscr{A}} p - \Delta_{\mathscr{A}} u$ for vector fields satisfying $\nabla_{\mathscr{A}} \cdot u = 0$.

In our new coordinates, the original system (2.1.3) becomes

$$
\begin{cases}
\partial_t u - \partial_t\bar{\eta}\tilde{b}K\partial_3 u + u\cdot\nabla_{\mathscr{A}} u - \Delta_{\mathscr{A}} u + \nabla_{\mathscr{A}} p = 0 & \text{in} \quad \Omega,\\
\nabla_{\mathscr{A}} \cdot u = 0 & \text{in} \quad \Omega,\\
S_{\mathscr{A}}(p,u)\mathscr{N} = \eta\mathscr{N} & \text{on} \quad \Sigma,\\
u = 0 & \text{on} \quad \Sigma_b,\\
u(x,0) = u_0(x) & \text{in} \quad \Omega,\\
\\
\partial_t\eta + u_1\partial_1\eta + u_2\partial_2\eta = u_3 & \text{on} \quad \Sigma,\\
\eta(x',0) = \eta_0(x') & \text{on} \quad \Sigma.
\end{cases}
\tag{2.1.12}
$$

Since $\mathscr{A}$ is determined by η through the transform, all the quantities above are related to η, i.e. the geometric structure of the free surface. This is the central idea of our proof. It is noticeable that in proving local well-posedness, we need to verify $\Phi(t)$ is a C^1 diffeomorphism for any $t \in [0,T]$, where the theorem holds.

Geometric Formulation in the Global Case

In the global case, we focus on the infinite domain which always possess a flat bottom, i.e. $b =$ constant. Since our theorem only holds for sufficiently small data, we may directly define $\bar{\eta}$ as the Poisson integral of η which is shown in [26, (A-8)] and [26, (A-14)], i.e. $\bar{\eta} = \mathcal{P}\eta$. Then the map can be taken as

$$\Phi : (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + \bar{\eta}(1 + \frac{x_3}{b})) = (y_1, y_2, y_3), \tag{2.1.13}$$

which also transforms Ω to $\Omega(t)$. It is obvious that the Jacobi matrix and the transform matrix are identical to those in the local case except that we need to redefine the quantities

$$\begin{aligned} \tilde{b} &= 1 + x_3/b, \\ A &= \partial_1\bar{\eta}\tilde{b}, \qquad B = \partial_2\bar{\eta}\tilde{b}, \\ J = 1 + \bar{\eta}/b &+ \partial_3\bar{\eta}\tilde{b}, \qquad K = 1/J. \end{aligned} \tag{2.1.14}$$

Naturally, the definitions of the weighted operators in (3.1.7) and the transformed equations (2.1.12) are the same as those in the local case with replacing the related quantities with above definitions. We can easily see as long as $\|\eta_0\|_{H^{5/2}(\Sigma)}$ is sufficiently small, Φ is a C^1 diffeomorphism. Hence, the smallness itself can guarantee the validity of the transform.

2.1.4 Main Theorems

Theorem 2.1.1. *Let $N \geq 2$ be an integer. Assume the initial data $\eta_0 + b \geq 2b_0\delta > 0$ for some $\delta > 0$. Suppose (u_0, η_0) satisfies the estimate $\mathcal{K}_0 = \|u_0\|^2_{H^{2N}} +$*

$\|\eta_0\|^2_{H^{2N+1/2}(\Sigma)} < \infty$ *as well as the* N^{th} *compatibility conditions (2.2.159). Then there exists a* $0 < T_0 < 1$ *such that for any* $0 < T < T_0$*, there exists a solution* (u, p, η) *to the system (2.1.12) on* $[0, T]$ *that satisfies the initial data. Furthermore, the solution obeys the estimate*

$$\begin{aligned}
&\left(\sum_{j=0}^{N} \sup_{0\le t\le T} \left\|\partial_t^j u\right\|^2_{H^{2N-2j}} + \sum_{j=0}^{N} \int_0^T \left\|\partial_t^j u\right\|^2_{H^{2N-2j+1}}\right) \qquad (2.1.15)\\
&+\left(\sum_{j=0}^{N-1} \sup_{0\le t\le T} \left\|\partial_t^j p\right\|^2_{H^{2N-2j-1}} + \sum_{j=0}^{N-1} \int_0^T \left\|\partial_t^j p\right\|^2_{H^{2N-2j}}\right)\\
&+\Bigg(\sup_{0\le t\le T} \|\eta\|^2_{H^{2N+1/2}(\Sigma)} + \sum_{j=1}^{N} \sup_{0\le t\le T} \left\|\partial_t^j \eta\right\|^2_{H^{2N-2j+3/2}(\Sigma)}\\
&\qquad + \sum_{j=2}^{N+1} \int_0^T \left\|\partial_t^j \eta\right\|^2_{H^{2N-2j+5/2}(\Sigma)}\Bigg) \le C(\Omega_0, \delta) P(K_0),
\end{aligned}$$

where $C(\Omega_0, \delta)$ *is a positive constant depending on the initial domain* Ω_0 *and the separation quantity* δ*, and* $P(\cdot)$ *is a polynomial in one variable satisfying* $P(0) = 0$*. The solution is unique among functions that satisfy the initial data and for which the left-hand side of the estimate (2.1.15) is finite. Moreover,* η *is such that the mapping* $\Phi(t)$ *defined by (2.1.7) is a* C^{2N-1} *diffeomorphism for each* $t \in [0, T]$.

Remark 2.1.2. *Since the map* $\Phi(t)$ *is a* C^{2N-1} *diffeomorphism, we may change the variables to* $y \in \Omega(t)$ *to produce the solution of (2.1.3).*

Remark 2.1.3. *Define quantities $\mathcal{E}_{2N}$, $\mathcal{F}_{2N}$ and $\mathcal{G}_N$ as follows:*

$$\begin{aligned}\mathcal{E}_{n,2} &= \left\|D^2 u\right\|^2_{H^{2n-2}} + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j}} + \left\|D^2 p\right\|^2_{H^{2n-3}} + \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|^2_{H^{2n-2j-1}} \\ &\quad + \left\|D^2 \eta\right\|^2_{H^{2n-2}(\Sigma)} + \sum_{j=1}^{n} \left\|\partial_t^j \eta\right\|^2_{H^{2n-2j}(\Sigma)},\end{aligned} \tag{2.1.16}$$

$$\begin{aligned}\mathcal{E}_{n,1} &= \|Du\|^2_{H^{2n-1}} + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j}} + \|Dp\|^2_{H^{2n-2}} + \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|^2_{H^{2n-2j-1}} \\ &\quad + \|D\eta\|^2_{H^{2n-1}(\Sigma)} + \sum_{j=1}^{n} \left\|\partial_t^j \eta\right\|^2_{H^{2n-2j}(\Sigma)},\end{aligned} \tag{2.1.17}$$

$$\begin{aligned}\mathcal{E}_{n} &= \|I_\lambda u\|^2_{H^0} + \|I_\lambda \eta\|^2_{H^0(\Sigma)} + \sum_{j=0}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j}} + \sum_{j=0}^{n-1} \left\|\partial_t^j p\right\|^2_{H^{2n-2j-1}} \\ &\quad + \sum_{j=0}^{n} \left\|\partial_t^j \eta\right\|^2_{H^{2n-2j}(\Sigma)},\end{aligned} \tag{2.1.18}$$

where $I_\lambda u$ and $I_\lambda \eta$ denote the Riesz potential of u and η as defined in (A.1.10) and (A.1.11).

$$\mathcal{F}_{2N} = \|\eta\|^2_{H^{4N+1/2}(\Sigma)}. \tag{2.1.19}$$

The total energy is defined as follows:

$$\mathcal{G}_N(t) = \sup_{0\le r\le t} \mathcal{E}_{2N}(r) + \sum_{m=1}^{2} \sup_{0\le r\le t} (1+r)^{m+\lambda} \mathcal{E}_{N+2,m}(r) + \sup_{0\le r\le t} \frac{\mathcal{F}_{2N}(r)}{(1+r)}. \tag{2.1.20}$$

Theorem 2.1.4. *Let $N \ge 4$ be an integer. Suppose the initial data (u_0, η_0) satisfies the compatibility conditions (2.2.159). Then there exists a $\kappa > 0$ such that if*

$$\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \le \kappa, \tag{2.1.21}$$

there exists a unique solution (u, p, η) to the system (2.1.12) on $[0, \infty)$ that satisfies the initial data. Also, the solution obeys the estimate

$$\mathcal{G}_N(\infty) \leq C(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) \leq C\kappa, \tag{2.1.22}$$

where $C > 0$ is a universal constant.

Remark 2.1.5. *Based on above theorem, naturally global well-posedness implies decay of $\mathcal{E}_{n,2}$ and $\mathcal{E}_{n,1}$.*

2.1.5 Convention and Terminology

We now mention some of the definitions, notation and conventions we use throughout this chapter.

1. Unless specified, we use exactly the same notation as in [26] and [25].

2. Throughout this chapter, $C > 0$ denotes a constant that only depends on the parameters N and Ω of the problem, but does not depend on the data. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on quantity z.

   There are two exceptions to above rule in Section 2.2. The first one is in the elliptic estimates and Korn's inequality, there are constants depending on the initial domain Ω_0. The second one is the constant $C(\delta)$ depending on the separation quantity δ. Though they are related to the initial domain Ω_0, we still call them universal since they are given implicitly. Apart from these, all the other constants related to the initial data Ω_0, u_0 and η_0 should be specified in detail.

$a \lesssim b$ denotes $a \leq Cb$, where C is a universal constant as defined above.

3. We write $P(\cdot)$ to denote polynomials in one variable. The polynomials always satisfy $P(0) = 1$ unless specified as exceptions. They may change from one inequality or equality to another.

4. We write the multi-indices $\mathbb{N}^{1+m} = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)\}$ to emphasize the 0-index term is related to the temporal variable. For just spatial variables, we write $\mathbb{N}^m$ without the 0-index. We define the parabolic counting of such multi-indices as $|\alpha| = 2\alpha_0 + \alpha_1 + \ldots + \alpha_m$. We always write Df to denote the horizontal derivatives of f and ∇f for the full derivatives.

   For a given norm $\|\cdot\|$ and integers $k, m \geq 0$, we introduce the following notation for the sums of derivatives:

$$\left\|D^k_m f\right\|^2 = \sum_{\alpha \in \mathbb{N}^2 \ m \leq |\alpha| \leq k} \|\partial^\alpha f\|^2 , \qquad \left\|\nabla^k_m f\right\|^2 = \sum_{\alpha \in \mathbb{N}^3 \ m \leq |\alpha| \leq k} \|\partial^\alpha f\|^2 ,$$
$$\left\|\bar{D}^k_m f\right\|^2 = \sum_{\alpha \in \mathbb{N}^{1+2} \ m \leq |\alpha| \leq k} \|\partial^\alpha f\|^2 , \qquad \left\|\bar{\nabla}^k_m f\right\|^2 = \sum_{\alpha \in \mathbb{N}^{1+3} \ m \leq |\alpha| \leq k} \|\partial^\alpha f\|^2 ,$$

   where D and ∇ denote spacial derivatives and $\bar{D}$ and $\bar{\nabla}$ denote full derivatives. Also we define

$$\left\|D^k f\right\|^2 = \left\|D^k_k f\right\|^2 , \ \left\|\nabla^k f\right\|^2 = \left\|\nabla^k_k f\right\|^2 ,$$
$$\left\|\bar{D}^k f\right\|^2 = \left\|\bar{D}^k_k f\right\|^2 , \ \left\|\bar{\nabla}^k f\right\|^2 = \left\|\bar{\nabla}^k_k f\right\|^2 .$$

2.2 Local Well-posedness for General Data

2.2.1 ϵ-Poisson Integral

In this section, we define the ϵ-Poisson integral which is utilized to construct the geometric transform (2.1.7), and discuss its fundamental properties.

Poisson Integral in the Infinite Case

For a function f defined on $\Sigma = \mathbb{R}^2$, the ϵ-Poisson integral is defined by

$$\mathcal{P}^\epsilon f(x', x_3) = \int_{\mathbb{R}^2} \hat{f}(\xi) e^{\epsilon|\xi| x_3} e^{2\pi i x' \cdot \xi} \mathrm{d}\xi \tag{2.2.1}$$

where $\hat{f}(\xi)$ denotes the Fourier transform of $f(x')$ in $\mathbb{R}^2$ and $0 < \epsilon < 1$ is a parameter. Although $\mathcal{P}^\epsilon f$ is defined on $\mathbb{R}^2 \times (-\infty, 0)$, we only consider the part $\mathbb{R}^2 \times (-b_0, 0)$ here.

Lemma 2.2.1. *Let $\mathcal{P}^\epsilon f$ be the ϵ-Poisson integral of function f which is in the homogeneous Sobolev space $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$. Then we have*

$$\|\nabla^q \mathcal{P}^\epsilon f\|_{H^0}^2 \leq \frac{C}{\epsilon} \|f\|_{\dot{H}^{q-1/2}}^2 , \tag{2.2.2}$$

where $C > 0$ is a constant independent of ϵ. In particular, we have

$$\|\mathcal{P}^\epsilon f\|_{H^q}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q-1/2}(\Sigma)} . \tag{2.2.3}$$

Proof. By the definition of the Fourier transform, Fubini's theorem and Parseval's identity, we may bound

$$
\begin{aligned}
\|\nabla^q \mathcal{P}^\epsilon f\|_{H^0}^2 &\leq (2\pi)^{2q} \int_{\mathbb{R}^2} \int_{-b_0}^{0} |\xi|^{2q} \left|\hat{f}(\xi)\right|^2 e^{2\epsilon|\xi|x_3} \mathrm{d}\xi \mathrm{dx}_3 \\
&\leq (2\pi)^{2q} \int_{\mathbb{R}^2} |\xi|^{2q} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \int_{-b_0}^{0} e^{2\epsilon|\xi|x_3} \mathrm{dx}_3 \\
&= (2\pi)^{2q} \int_{\mathbb{R}^2} |\xi|^{2q} \left|\hat{f}(\xi)\right|^2 \left(\frac{1-e^{-2\epsilon b_0|\xi|}}{2\epsilon\,|\xi|}\right) \mathrm{d}\xi \\
&\leq \frac{(2\pi)^{2q}}{2\epsilon} \int_{\mathbb{R}^2} |\xi|^{2q-1} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \\
&\leq \frac{\pi}{\epsilon} \|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2 .
\end{aligned} \tag{2.2.4}
$$

We can simply take $C = \pi$ to achieve the estimate. Furthermore, considering

$$
\frac{1-e^{-2\epsilon b_0|\xi|}}{2\epsilon\,|\xi|} \leq \min\left\{\frac{1}{2\epsilon\,|\xi|}, b_0\right\} \tag{2.2.5}
$$

and $0 < \epsilon < 1$, we have

$$
\|\mathcal{P}^\epsilon f\|_{H^0}^2 \leq \frac{\pi b_0}{\epsilon} \|f\|_{H^0(\Sigma)}^2 \tag{2.2.6}
$$

Hence, (2.2.3) easily follows. □

Lemma 2.2.2. *Let $\mathcal{P}^\epsilon f$ be the ϵ-Poisson integral of function f. Then for $q \in \mathbb{N}$ and $s > 1$, we have*

$$
\|\nabla^q \mathcal{P}^\epsilon f\|_{L^\infty}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q+s}(\Sigma)}^2 . \tag{2.2.7}
$$

Proof. A simple application of the Sobolev embedding theorem and Lemma 2.2.1 reveals

$$
\|\nabla^q \mathcal{P}^\epsilon f\|_{L^\infty}^2 \leq C \|\nabla^q \mathcal{P}^\epsilon f\|_{H^{s+1/2}}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q+s}(\Sigma)}^2 . \tag{2.2.8}
$$

□

The following lemma illustrates the special properties of the vertical derivative.

Lemma 2.2.3. *Let $\mathcal{P}^\epsilon f$ be the ϵ-Poisson integral of function f. Then we have*

$$\|\partial_3 \mathcal{P}^\epsilon f\|_{L^\infty}^2 \le C\epsilon \|f\|_{H^{5/2}(\Sigma)}^2 . \tag{2.2.9}$$

where $C > 0$ is a constant independent of ϵ.

Proof. We can simply bound

$$\|\partial_3 \mathcal{P}^\epsilon f\|_{L^\infty}^2 \le C \|\partial_3 \mathcal{P}^\epsilon f\|_{H^2}^2 \le C\epsilon^2 \|\mathcal{P}^\epsilon f\|_{H^3}^2 \le C\epsilon \|f\|_{H^{5/2}(\Sigma)}^2 . \tag{2.2.10}$$

Note the first inequality is based on the Sobolev embedding theorem and the third one is a straightforward application of Lemma 2.2.1, so we only need to verify the second inequality in detail.

By definition, it is easy to see

$$\partial_3 \mathcal{P}^\epsilon f = \epsilon \int_{R^2} |\xi| \hat{f}(\xi) e^{\epsilon|\xi|x_3} e^{2\pi i x'\cdot\xi} \mathrm{d}\xi. \tag{2.2.11}$$

Hence, we have

$$\begin{aligned} \|\partial_3 \mathcal{P}^\epsilon f\|_{H^0}^2 &= \epsilon^2 \int_{\mathbb{R}^2} \int_{-b_0}^0 |\xi|^2 \left|\hat{f}(\xi)\right|^2 e^{2\epsilon|\xi|x_3} \mathrm{d}\xi \mathrm{dx}_3 \\ &= \frac{\epsilon^2}{(2\pi)^2} \left((2\pi)^2 \int_{\mathbb{R}^2} \int_{-b_0}^0 |\xi|^2 \left|\hat{f}(\xi)\right|^2 e^{2\epsilon|\xi|x_3} \mathrm{d}\xi \mathrm{dx}_3 \right) \\ &\le \frac{\epsilon^2}{(2\pi)^2} \|\partial_1 \mathcal{P}^\epsilon f\|_{H^0}^2 . \end{aligned} \tag{2.2.12}$$

Similarly, we can show for $i, j = 1, 2, 3$

$$\left\|\partial_{3i}\mathcal{P}^{\epsilon}f\right\|_{H^0}^2 \leq \frac{\epsilon^2}{(2\pi)^4}\left\|\partial_{11}\mathcal{P}^{\epsilon}f\right\|_{H^0}^2, \tag{2.2.13}$$

$$\left\|\partial_{3ij}\mathcal{P}^{\epsilon}f\right\|_{H^0}^2 \leq \frac{\epsilon^2}{(2\pi)^6}\left\|\partial_{111}\mathcal{P}^{\epsilon}f\right\|_{H^0}^2. \tag{2.2.14}$$

Thus we have

$$\left\|\partial_3\mathcal{P}^{\epsilon}f\right\|_{H^2}^2 \leq C\epsilon^2\left\|\mathcal{P}^{\epsilon}f\right\|_{H^3}^2. \tag{2.2.15}$$

□

Poisson Integral in the Periodic Case

Suppose $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$, we define the ϵ-Poisson integral as follows:

$$\mathcal{P}^{\epsilon}f(x', x_3) = \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} e^{2\pi i n\cdot x'}e^{\epsilon|n|x_3}\hat{f}(n) \tag{2.2.16}$$

where

$$\hat{f}(n) = \int_{\Sigma} f(x')\frac{e^{-2\pi i n\cdot x'}}{L_1L_2}\mathrm{dx}'. \tag{2.2.17}$$

Now we need to show the same types of estimates as in the infinite case. Since the main ideas and the basic techniques are quite similar, we omit the detailed proofs here and only present the results.

Lemma 2.2.4. *Let $\mathcal{P}^{\epsilon}f$ be the ϵ-Poisson integral of function f which is in the homogeneous Sobolev space $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$. Then we have*

$$\left\|\nabla^q\mathcal{P}^{\epsilon}f\right\|_{H^0}^2 \leq \frac{C}{\epsilon}\left\|f\right\|_{\dot{H}^{q-1/2}}^2, \tag{2.2.18}$$

where $C > 0$ is a constant independent of ϵ. In particular, we have

$$\|\mathcal{P}^\epsilon f\|_{H^q}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q-1/2}(\Sigma)}^2 . \tag{2.2.19}$$

Lemma 2.2.5. *Let $\mathcal{P}^\epsilon f$ be the ϵ-Poisson integral of function f. Then for $q \in \mathbb{N}$ and $s > 1$, we have*

$$\|\nabla^q \mathcal{P}^\epsilon f\|_{L^\infty}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q+s}(\Sigma)}^2 . \tag{2.2.20}$$

Lemma 2.2.6. *Let $\mathcal{P}^\epsilon f$ be the ϵ-Poisson integral of function f. Then we have*

$$\|\partial_3 \mathcal{P}^\epsilon f\|_{L^\infty}^2 \leq C\epsilon \|f\|_{H^{5/2}(\Sigma)}^2 , \tag{2.2.21}$$

where $C > 0$ is a constant independent of ϵ.

Homeomorphism of the Geometric Transform

Theorem 2.2.7. *If the initial data satisfies $\eta_0 + b > \rho > 0$ for all $x' \in \Sigma$, then we can choose a sufficiently small $\epsilon > 0$ such that there exists a $\delta > 0$ satisfying $J^\epsilon(0) \geq \delta > 0$.*

Proof. Naturally there always exists a $\delta > 0$, such that $\eta_0(x') + b(x') \geq 2b_0\delta > 0$ for any $x' \in \Sigma$. Based on Lemma 2.2.3 and Lemma 2.2.6, if taking $\epsilon \leq \delta^2/(4C^2 \|\eta_0\|_{H^{5/2}}^2)$, we have

$$\|\partial_3 \bar{\eta}_0^\epsilon\|_{L^\infty} \leq \delta/2, \tag{2.2.22}$$

where $\bar{\eta}_0^\epsilon$ denotes $\bar{\eta}^\epsilon$ at $t = 0$. Furthermore, by the fundamental theorem of calculus, we can estimate for any $x_3 \in [-b_0, 0]$ and $x' \in \Sigma$

$$|\bar{\eta}_0^\epsilon(x', x_3) - \bar{\eta}_0^\epsilon(x', 0)| \leq \int_{-b_0}^0 |\partial_3 \bar{\eta}_0^\epsilon(x', z)| \,\mathrm{dz} \leq b_0 \left\| \partial_3 \bar{\eta}_0^\epsilon \right\|_{L^\infty} \leq b_0 \delta/2. \tag{2.2.23}$$

Since $\bar{\eta}_0^\epsilon(x', 0) = \eta_0(x')$, we have

$$\begin{aligned} J^\epsilon(0) &= \frac{b}{b_0} + \frac{\bar{\eta}_0^\epsilon}{b_0} + \partial_3 \bar{\eta}_0^\epsilon \tilde{b} = \frac{b + \eta_0}{b_0} + \frac{\bar{\eta}_0^\epsilon - \eta_0}{b_0} + \partial_3 \bar{\eta}_0^\epsilon \tilde{b} \\ &\geq 2\delta - \delta/2 - \delta/2 = \delta > 0. \end{aligned} \tag{2.2.24}$$

Therefore, for given initial data η_0, by taking $\epsilon = \delta^2 / (4C^2 \left\| \eta_0 \right\|_{H^{5/2}}^2)$, we can guarantee $J^\epsilon \geq \delta > 0$. □

2.2.2 Preliminaries

Preliminary Lemmas

Throughout this chapter, we employ the functional spaces defined in [26, Section 2]. We write $H^k(\Omega)$ and $H^k(\Sigma)$ for usual Sobolev space of either scalar or vector functions. Define

$$\begin{aligned} &\mathcal{H}^k(t) = \{u(t) \in H^k(\Omega) : \ u(t)|_{\Sigma_b} = 0\}, \\ &\mathcal{X}(t) = \{u(t) \in H^1(\Omega) : \ u(t)|_{\Sigma_b} = 0, \nabla_{\mathcal{A}(t)} \cdot u(t) = 0\}. \end{aligned} \tag{2.2.25}$$

Moreover, let $L^2([0,T]; H^k(\Omega))$ and $L^2([0,T]; H^k(\Sigma))$ denotes the usual time-involved Sobolev space. Define

$$\begin{aligned} &\mathcal{H}_T^k = \{u \in L^2([0,T]; H^k(\Omega)) : \ u|_{\Sigma_b} = 0\}, \\ &\mathcal{X}_T = \{u \in L^2([0,T]; H^1(\Omega)) : \ u|_{\Sigma_b} = 0, \nabla_{\mathcal{A}} \cdot u = 0\}. \end{aligned} \tag{2.2.26}$$

Define the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^k}$ to denote the normalized inner product in $\mathcal{H}^k$ as

$$\langle u, v \rangle_{\mathcal{H}^0} = \int_\Omega J u \cdot v, \tag{2.2.27}$$

$$\langle u, v \rangle_{\mathcal{H}^1} = \int_\Omega J \mathbb{D}_{\mathscr{A}} u : \mathbb{D}_{\mathscr{A}} v, \tag{2.2.28}$$

and $\langle \cdot, \cdot \rangle_{\mathcal{H}^k_t}$ to denote the normalized inner product in $L^2([0,T]; \mathcal{H}^k_T)$, i.e.

$$\langle u, v \rangle_{\mathcal{H}^k_t} = \int_0^t \langle u, v \rangle_{\mathcal{H}^k}. \tag{2.2.29}$$

It is easy to see $\mathcal{H}^k(t)$, $\mathcal{X}(t)$, $\mathcal{H}^k_T$ and $\mathcal{X}_T$ are all Hilbert spaces with above inner products.

Also we will need an orthogonal decomposition $\mathcal{H}^0 = Y(t) \oplus Y^\perp(t)$, where

$$Y^\perp(t) = \{\nabla_{\mathcal{A}(t)} \varphi(t) : \ \varphi(t) \in H^1(t), \varphi(t)|_\Sigma = 0\}. \tag{2.2.30}$$

In our use of these spaces, we will often drop the (t) when there is no potential of confusion.

Finally we will define the regular divergence-free space in $H^1(\Omega)$.

$${}_0H^1_\sigma = \{u \in H^1(\Omega) : \ u|_{\Sigma_b} = 0, \nabla \cdot u = 0\}. \tag{2.2.31}$$

Basically, following the path of [26], we need to recover all the necessary lemmas in the general data case. Since most of the results can be generalized to our case with obvious modifications, we omit the details and only present some important propositions. Only when key different techniques are employed, are all the details of the lemmas and proofs given.

The following result gives a version of Korn's type inequality for the initial domain

Ω_0. Here we apply Beale's idea in [8].

Lemma 2.2.8. *Suppose Ω_0 is the initial domain and $\eta_0 \in H^{5/2}(\Sigma)$. Then*

$$\|v\|^2_{H^1(\Omega_0)} \lesssim \|\mathbb{D}v\|^2_{H^0(\Omega_0)}, \tag{2.2.32}$$

for any $v \in H^1(\Omega_0)$ and $v = 0$ on $\{y_3 = -b\}$.

Proof. In the periodic case, Σ is bounded and $v|_{\Sigma_b} = 0$, so naturally Korn's inequality is valid (see the proof of [8, Lemma 2.7]). Then we consider the infinite case. First we prove decay of the initial surface η_0, i.e. for $|\alpha| \leq 1$, $|\partial^\alpha \bar{\eta}_0| \to 0$ as $|x| \to \infty$ in the slab. For the horizontal derivative ∂^α, we have

$$\partial^\alpha \bar{\eta}_0 = \int_{\mathbb{R}^2} (2\pi i\xi)^\alpha e^{\epsilon x_3|\xi|} e^{2\pi i x'\cdot\xi} \hat{\eta}_0(\xi) \mathrm{d}\xi, \tag{2.2.33}$$

where $\hat{\eta_0}$ is the Fourier transform of η_0 in $\mathbb{R}^2$. Then

$$\begin{aligned} & \int_{\mathbb{R}^2} \left|(2\pi i\xi)^\alpha e^{\epsilon x_3|\xi|} \hat{\eta}_0(\xi)\right| \mathrm{d}\xi \qquad (2.2.34) \\ \lesssim & \int_{\mathbb{R}^2} |\xi|^\alpha e^{\epsilon x_3|\xi|} |\hat{\eta}_0(\xi)| (1+|\xi|^{5/4}) \left(\frac{1}{1+|\xi|^{5/4}}\right) \mathrm{d}\xi \\ \leq & \left(\int_{\mathbb{R}^2} |\xi|^{2\alpha} |\hat{\eta}_0(\xi)|^2 (1+|\xi|^{5/4})^2 \mathrm{d}\xi\right)^{1/2} \left(\int_{\mathbb{R}^2} \frac{e^{2\epsilon x_3|\xi|}\mathrm{d}\xi}{(1+|\xi|^{5/4})^2}\right)^{1/2} \\ \lesssim & \|\eta_0\|_{H^{\alpha+5/4}} \left(\int_{\mathbb{R}^2} \frac{e^{2\epsilon x_3|\xi|}\mathrm{d}\xi}{(1+|\xi|^{5/4})^2}\right)^{1/2} < \infty. \end{aligned}$$

The last inequality is valid since $x_3 < 0$. Hence, $(2\pi i\xi)^\alpha e^{\epsilon x_3|\xi|} \hat{\eta}_0(\xi) \in L^1(\mathbb{R}^2)$. A similar proof can justify the result if $\partial^\alpha = \partial_3$ the vertical derivative. Since $|x| \to \infty$ naturally implies $|x'| \to \infty$ in the slab, the Riemann-Lebesgue lemma implies $|\partial^\alpha \bar{\eta}_0| \to 0$ as $|x| \to \infty$.

We construct a mapping $\bar{\sigma} = \Phi(0)$ as defined in (2.1.7), which maps $\Omega = \{\mathbb{R}^3 :$

$-b_0 < x_3 < 0\}$ to Ω_0. Denote $\bar{\sigma}(x) = x + \sigma(x)$. The above decaying property and our assumption on b lead to $\partial^\alpha \sigma(x) \to 0$ as $|x| \to \infty$ for $|\alpha| \leq 1$.

We partition the slab Ω into cubes

$$Q_{j,k} = \{x : j < x_1 < j+1, k < x_2 < k+1, -b_0 < x_3 < 0\}. \tag{2.2.35}$$

In each cube, we have Korn's inequality

$$\|v\|^2_{H^1(Q)} \leq C(\|\mathbb{D}v\|^2_{H^0(Q)} + \|v\|^2_{H^0(Q)}). \tag{2.2.36}$$

Employing the compactness argument as Beale [8] did, under the condition $v = 0$ on $\{x_3 = -b_0\}$, we can strengthen this result to

$$\|v\|^2_{H^1(Q)} \leq C \|\mathbb{D}v\|^2_{H^0(Q)}. \tag{2.2.37}$$

This argument relies on the fact that for such v

$$\|v\|^2_{H^0(Q)} \leq C \|\mathbb{D}v\|^2_{H^0(Q)}. \tag{2.2.38}$$

Now suppose $D \subseteq \Omega_0$ is the image of the union of such cubes contained in $\{|x| \geq R\}$ for some R. Applying last estimate to $v = u \circ \bar{\sigma}$ and then transforming to Ω_0, we have

$$\|u\|_{H^1(D)} \leq C \|\mathbb{D}u\|_{H^0(D)} + \theta \|u\|_{H^1(D)}, \tag{2.2.39}$$

where $\theta < 1$ when R is large enough based on the decaying property. Then we use Korn's inequality in the bounded domain $\Omega_0 - D$ and combine with above estimates to obtain the desired result. $\square$

Next, we show the weights introduced by J and $\mathscr{A}$ do not affect the Sobolev norms. Although this result is quite obvious for the small data (see [26, Lemma 2.1]), now we need to utilize above Korn's inequality to show it for the general data.

Lemma 2.2.9. *There exists a* $0 < \epsilon_0 < 1$*, such that if* $\|\eta - \eta_0\|_{H^{5/2}} < \epsilon_0$*, then the following relations*

$$\|u\|^2_{H^0(\Omega)} \lesssim \int_\Omega J\,|u|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})\,\|u\|^2_{H^0(\Omega)}\,, \tag{2.2.40}$$

$$\frac{1}{(1+\|\eta_0\|_{H^{5/2}(\Sigma)})^3}\,\|u\|^2_{H^1(\Omega)} \lesssim \int_\Omega J\,|\mathbb{D}_{\mathscr{A}} u|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3\,\|u\|^2_{H^1(\Omega)}\,, \tag{2.2.41}$$

hold for all $u \in \mathcal{H}^1$*.*

Proof. (2.2.40) is just a natural corollary of the relation $\delta \lesssim \|J\|_{L^\infty} \lesssim (1+\|\nabla\bar{\eta}\|_{L^\infty}) \lesssim (1+\|\eta\|_{H^{5/2}}) \lesssim (1+\|\eta_0\|_{H^{5/2}})$ based on our assumptions and the Sobolev embedding theorem.

To derive (2.2.41), notice

$$\int_\Omega J\,|\mathbb{D}_{\mathscr{A}} u|^2 = \int_\Omega J\,|\mathbb{D}_{\mathscr{A}_0} u|^2 + \int_\Omega J(\mathbb{D}_{\mathscr{A}} u + \mathbb{D}_{\mathscr{A}_0} u) : (\mathbb{D}_{\mathscr{A}} u - \mathbb{D}_{\mathscr{A}_0} u). \tag{2.2.42}$$

For the first term on the right-hand side of (2.2.42), based on the integral substitution and Korn's inequality we proved above, we have

$$\begin{aligned}
&\int_\Omega J\,|\mathbb{D}_{\mathscr{A}_0} u|^2\,\mathrm{dx} \\
\gtrsim\;& \frac{1}{1+\|\eta_0\|_{H^{5/2}(\Sigma)}} \int_\Omega J_0\,|\mathbb{D}_{\mathscr{A}_0} u|^2\,\mathrm{dx} = \frac{1}{1+\|\eta_0\|_{H^{5/2}(\Sigma)}} \int_{\Omega_0} |\mathbb{D} v|^2\,\mathrm{dy} \\
\gtrsim\;& \frac{1}{1+\|\eta_0\|_{H^{5/2}(\Sigma)}}\,\|v\|^2_{H^1(\Omega_0)} \gtrsim \frac{1}{(1+\|\eta_0\|_{H^{5/2}(\Sigma)})^3}\,\|u\|^2_{H^1(\Omega)}\,.
\end{aligned} \tag{2.2.43}$$

For the second term on the right-hand side of (2.2.42), using Korn's inequality in slab Ω, we can naturally estimate

$$\begin{aligned}
& \left| \int_\Omega J(\mathbb{D}_{\mathscr{A}} u + \mathbb{D}_{\mathscr{A}_0} u) : (\mathbb{D}_{\mathscr{A}} u - \mathbb{D}_{\mathscr{A}_0} u) \right| && (2.2.44) \\
\lesssim \; & \|J\|_{L^\infty} \|\mathscr{A} + \mathscr{A}_0\|_{L^\infty} \|\mathscr{A} - \mathscr{A}_0\|_{L^\infty} \int_\Omega |\nabla u|^2 \\
\lesssim \; & \epsilon_0 (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3 \int_\Omega |\mathbb{D} u|^2 \\
\lesssim \; & \epsilon_0 (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3 \|u\|_{H^1(\Omega)}^2 .
\end{aligned}$$

For given initial data, we can always take ϵ_0 sufficiently small to absorb (2.2.44) into (2.2.43). Then we have

$$\int_\Omega J |\mathbb{D}_{\mathscr{A}} u|^2 \gtrsim \frac{1}{(1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3} \|u\|_{H^1(\Omega)}^2 . \tag{2.2.45}$$

The first part of (2.2.41) is verified. For the second part, notice

$$\begin{aligned}
\int_\Omega J |\mathbb{D}_{\mathscr{A}} u|^2 \; & \lesssim \; (1 + \|\eta\|_{H^{5/2}}) \int_\Omega |\mathbb{D}_{\mathscr{A}} u|^2 && (2.2.46) \\
& \lesssim \; (1 + \|\eta_0\|_{H^{5/2}}) \max\{1, \|AK\|_{L^\infty}^2 , \|BK\|_{L^\infty}^2 , \|K\|_{L^\infty}^2\} \|u\|_{H^1(\Omega)}^2 .
\end{aligned}$$

Since it is easy to see

$$\begin{aligned}
\max\{1, \|AK\|_{L^\infty}^2 , \|BK\|_{L^\infty}^2 , \|K\|_{L^\infty}^2\} \; & \lesssim \; 1 + (1 + \|\nabla \bar{\eta}\|_{L^\infty}^2) \|K\|_{L^\infty}^2 && (2.2.47) \\
& \lesssim \; (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^2,
\end{aligned}$$

then the second part of (2.2.41) follows.

$\square$

Remark 2.2.10. *Throughout this section, we can always assume the restriction of η depending on ϵ_0 is justified, and finally we verify this closeness condition for $t \in [0, T]$*

in the iteration.

It is easy to see [26, Lemma 2.4] still holds in our settings. In the following, we give an updated version of [26, Lemma 2.5]. Define

$$M = M(t) = K\nabla\Phi = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ AK & BK & 1 \end{pmatrix}. \tag{2.2.48}$$

Then M induces a linear operator $\mathcal{M}_t : u \to \mathcal{M}_t(u) = M(t)u$.

Lemma 2.2.11. *For each $t \in [0,T]$, $k = 0,1,2$, $\mathcal{M}_t$ is a bounded linear isomorphism from $H^k(\Omega)$ to $H^k(\Omega)$ and from ${}_0H^1_\sigma(\Omega)$ to $\mathcal{X}(t)$. The constant is given by $P(\|\eta(t)\|_{H^{7/2}(\Sigma)})$. If we further define $\mathcal{M}$ by $\mathcal{M}u(t) = \mathcal{M}_t u(t)$, then it is a bounded linear isomorphism from $L^2([0,T];H^k(\Omega))$ to $L^2([0,T];H^k(\Omega))$ and from $L^2([0,T];_0 H^1_\sigma(\Omega))$ to $\mathcal{X}_T$. The constant is given by $P(\sup_{0\le t\le T}\|\eta(t)\|_{H^{7/2}(\Sigma)})$.*

Proof. It is easy to see for each $t \in [0,T]$,

$$\|\mathcal{M}_t u\|_{H^0} \lesssim \|\mathcal{M}_t\|_{C^0}\|u\|_{H^0} \lesssim P(\|\eta(t)\|_{H^{5/2}(\Sigma)})\|u\|_{H^0}, \tag{2.2.49}$$

$$\|\mathcal{M}_t u\|_{H^1} \lesssim \|\mathcal{M}_t\|_{C^1}\|u\|_{H^1} \lesssim P(\|\eta(t)\|_{H^{7/2}(\Sigma)})\|u\|_{H^1}, \tag{2.2.50}$$

$$\|\mathcal{M}_t u\|_{H^2} \lesssim \|\mathcal{M}_t\|_{C^1}\|u\|_{H^2} + \|u\|_{C^0}\|\mathcal{M}_t\|_{H^2} \lesssim P(\|\eta(t)\|_{H^{7/2}(\Sigma)})\|u\|_{H^2}. \tag{2.2.51}$$

This implies $\mathcal{M}_t$ is a bounded operator from H^k to H^k. Since $\mathcal{M}_t$ is invertible, we can estimate $\|\mathcal{M}^{-1}v\|_{H^k} \lesssim P(\|\eta(t)\|_{H^{7/2}(\Sigma)})\|v\|_{H^k}$. Hence, $\mathcal{M}_t$ is an isomorphism between H^k. Also [26, (2-18)] implies $\mathcal{M}_t$ maps divergence-free functions to divergence-$\mathscr{A}$-free functions. Then it is also an isomorphism from ${}_0H^1_\sigma(\Omega)$ to $\mathcal{X}(t)$.

A similar argument can justify the case of $L^2([0,T];H^k)$ and $\mathcal{X}_T$. $\square$

Pressure as a Multiplier

It is well-known that in the Navier-Stokes equations, pressure can be taken as a multiplier. With the free surface involved, [26, Proposition 2.9] gives a construction of pressure from the transformed equation (2.1.12), which is valid for small data. With obvious modifications, we can achieve the following result.

Proposition 2.2.12. *If $\Lambda_t \in (\mathcal{H}_1(t))^*$ such that $\Lambda_t(v) = 0$ for all $v \in \mathcal{X}(t)$, then there exists a unique $p(t) \in \mathcal{H}^0(t)$ such that*

$$\langle p(t), \nabla_{\mathscr{A}} \cdot v\rangle_{\mathcal{H}^0} = \Lambda_t(v) \ for\ all\ v \in \mathcal{H}^1(t) \tag{2.2.52}$$

and $\|p(t)\|_{\mathcal{H}^0} \lesssim P(\|\eta(t)\|_{H^{7/2}(\Sigma)})\|\Lambda_t\|_{(\mathcal{H}_1(t))^*}$.

If $\Lambda \in (\mathcal{H}^1_T)^*$ such that $\Lambda(v) = 0$ for all $v \in \mathcal{X}_T$, then there exists a unique $p \in \mathcal{H}^0_T$ such that

$$\langle p, \nabla_{\mathscr{A}} \cdot v\rangle_{\mathcal{H}^0_T} = \Lambda(v) \ for\ all\ v \in \mathcal{H}^1_T \tag{2.2.53}$$

and $\|p\|_{\mathcal{H}^0_T} \lesssim P(\sup_{0\le t\le T}\|\eta(t)\|_{H^{7/2}(\Sigma)})\|\Lambda\|_{(\mathcal{H}^1_T)^*}$.

Proof. Similar to the proof of Lemma 2.2.11, we can easily recover [26, Lemma 2.5 - Lemma 2.8] with only replacing the universal constant C by $P(\|\eta(t)\|_{H^{7/2}(\Sigma)})$. Hence, our results naturally follow. $\square$

2.2.3 Elliptic Equations

In this section, we study two types of elliptic problems, which are employed in deriving the linear estimates. For both equations, we prove well-posedness to the higher order regularity.

$\mathscr{A}$-Stokes Equation

Let us consider the stationary Navier-Stokes problem.

$$\begin{cases} \nabla_{\mathscr{A}} \cdot S_{\mathscr{A}}(p,u) = F & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}} \cdot u = G & \text{in} \quad \Omega, \\ S_{\mathscr{A}}(p,u)\mathscr{N} = V & \text{on} \quad \Sigma, \\ u = 0 & \text{on} \quad \Sigma_b. \end{cases} \tag{2.2.54}$$

Since this problem is stationary, we temporarily ignore the time dependence of η, $\mathscr{A}$, etc. The existence and uniqueness are shown in [26, Lemma 3.5 - Lemma 3.6]. Notice the estimate in [26, Lemma 3.6] does not achieve the optimal regularity of (u,p). Hence, we employ an approximation argument to improve this estimate. This is a similar result as in [26, Lemma 3.7], but with different techniques. For clarity, we divide it into two steps. In the next lemma, we first prove that the constant can actually only depend on the initial free surface.

Lemma 2.2.13. *Let $k \geq 3$ be an integer and suppose $\eta \in H^{k+1/2}(\Sigma)$ and $\eta_0 \in H^{k+1/2}(\Sigma)$. Then there exists $\epsilon_0 > 0$ such that if $\|\eta - \eta_0\|_{H^{k-3/2}(\Sigma)} \leq \epsilon_0$, the solution (u,p) to the system (2.2.54) satisfies*

$$\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta_0)\bigg(\|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|V\|_{H^{r-3/2}(\Sigma)} \bigg), \tag{2.2.55}$$

for $r = 2, \dots, k-1$, whenever the right-hand side is finite, where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$.

Proof. Based on [26, Lemma 3.6], we have the estimate

$$\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta)\bigg(\|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|V\|_{H^{r-3/2}(\Sigma)} \bigg), \tag{2.2.56}$$

for $r = 2, \dots, k-1$, whenever the right-hand side is finite, where $C(\eta)$ is a constant depending on $\|\eta\|_{H^{k+1/2}(\Sigma)}$. Define $\xi = \eta - \eta_0$, then $\xi \in H^{k+1/2}(\Sigma)$.
Let us denote $\mathscr{A}_0$ and $\mathscr{N}_0$ as quantities in terms of η_0. We rewrite the system (2.2.54) as a perturbation of the initial status

$$\begin{cases} \nabla_{\mathscr{A}_0} \cdot \mathcal{S}_{\mathscr{A}_0}(p,u) = F + F^0 & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}_0} \cdot u = G + G^0 & \text{in} \quad \Omega, \\ \mathcal{S}_{\mathscr{A}_0}(p,u)\mathscr{N}_0 = V + V^0 & \text{on} \quad \Sigma, \\ u = 0 & \text{on} \quad \Sigma_b, \end{cases} \tag{2.2.57}$$

where

$$F^0 = \nabla_{\mathscr{A}_0 - \mathscr{A}} \cdot S_{\mathscr{A}}(p,u) + \nabla_{\mathscr{A}_0} \cdot \mathcal{S}_{\mathscr{A}_0 - \mathscr{A}}(p,u), \tag{2.2.58}$$

$$G^0 = \nabla_{\mathscr{A}_0 - \mathscr{A}} \cdot u, \tag{2.2.59}$$

$$V^0 = \mathcal{S}_{\mathscr{A}_0}(p,u)(\mathscr{N}_0 - \mathscr{N}) + \mathcal{S}_{\mathscr{A}_0 - \mathscr{A}}(p,u). \tag{2.2.60}$$

Suppose $\|\xi\|_{H^{k-3/2}(\Sigma)} \le 1$, which implies $\|\xi\|^l_{H^{k-3/2}(\Sigma)} \le \|\xi\|_{H^{k-3/2}(\Sigma)} < 1$ for any $l > 1$.

A straightforward calculation reveals

$$\left\|F^0\right\|_{H^{r-2}} \leq C(1+\|\eta_0\|_{H^{k+1/2}(\Sigma)})^4 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^r} + \|p\|_{H^{r-1}}), \tag{2.2.61}$$

$$\left\|G^0\right\|_{H^{r-1}} \leq C(1+\|\eta_0\|_{H^{k+1/2}(\Sigma)})^2 \|\xi\|_{H^{k-3/2}(\Sigma)} \|u\|_{H^r}, \tag{2.2.62}$$

$$\|V\|_{H^{r-3/2}(\Sigma)} \leq C(1+\|\eta_0\|_{H^{k+1/2}(\Sigma)})^2 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^r} + \|p\|_{H^{r-1}}), \tag{2.2.63}$$

for $r = 2, \ldots, k-1$.

Since the initial surface function η_0 satisfies all the requirements of [26, Lemma 3.6], we arrive at the estimate that for $r = 2, \ldots, k-1$

$$\|u\|_{H^r}+\|p\|_{H^{r-1}} \lesssim C(\eta_0)\bigg(\left\|F+F^0\right\|_{H^{r-2}}+\left\|G+G^0\right\|_{H^{r-1}}+\left\|V+V^0\right\|_{H^{r-3/2}(\Sigma)} \bigg), \tag{2.2.64}$$

where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$. Combining all above, we obtain

$$\begin{aligned} & \|u\|_{H^r} + \|p\|_{H^{r-1}} \\ \lesssim\ & C(\eta_0)\bigg(\|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|V\|_{H^{r-3/2}(\Sigma)} \bigg) \\ & + C(\eta_0)(1+\|\eta_0\|_{H^{k+1/2}(\Sigma)})^4 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^r} + \|p\|_{H^{r-1}}). \end{aligned} \tag{2.2.65}$$

So if

$$\|\xi\|_{H^{k-3/2}(\Sigma)} \leq \min\left\{\frac{1}{2}, \frac{1}{4CC(\eta_0)(1+\|\eta_0\|_{H^{k+1/2}(\Sigma)})^4}\right\}, \tag{2.2.66}$$

we can absorb the extra terms on the right-hand side of (2.2.65) into the left-hand side and get a succinct form

$$\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta_0)\bigg(\|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|V\|_{H^{r-3/2}(\Sigma)} \bigg), \tag{2.2.67}$$

for $r = 2, \dots, k-1$. $\square$

The next result allows us to achieve the optimal regularity.

Proposition 2.2.14. *Let $k \geq 3$ be an integer. Suppose $\eta \in H^{k+1/2}(\Sigma)$ and $\eta_0 \in H^{k+1/2}(\Sigma)$ satisfying $\|\eta - \eta_0\|_{H^{k+1/2}(\Sigma)} \leq \epsilon_0$. Then the solution (u,p) to the system (2.2.54) satisfies*

$$\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta_0)\Big(\|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|V\|_{H^{r-3/2}(\Sigma)} \Big), \tag{2.2.68}$$

for $r = 2, \dots, k+1$, whenever the right-hand side is finite, where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$.

Proof. If $r \leq k-1$, then this is just the conclusion of Lemma 2.2.13, so our main goal is to gain more regularity for $r = k$ and $r = k+1$. The main ideas here is utilizing the equations to control the higher order vertical derivatives by the horizontal derivatives. The weight $\mathscr{A}$ makes this procedure quite complicated.

In the following, we first define an approximating sequence for η. In the infinite case, we let $\rho \in C_0^\infty(\mathbb{R}^2)$ be such that *supp*$(\rho) \subset B(0,2)$ and $\rho = 1$ for $B(0,1)$. For $m \in \mathbb{N}$, define η^m by $\hat{\eta}^m(\xi) = \rho(\xi/m)\hat{\eta}(\xi)$ where $\hat{}$ denotes the Fourier transform. For each m, $\eta^m \in H^j(\Sigma)$ for arbitrary $j \geq 0$ and also $\eta^m \to \eta$ in $H^{k+1/2}(\Sigma)$ as $m \to \infty$. In the periodic case, we define $\hat{\eta}^m$ by throwing away the higher frequencies: $\hat{\eta}^m(n) = 0$ for $|n| \geq m$. Then η^m has the same convergence property as above. Let $\mathscr{A}^m$ and $\mathscr{N}^m$ be defined in terms of η^m.

Consider the problem (2.2.54) with $\mathscr{A}$ and $\mathscr{N}$ replaced by $\mathscr{A}^m$ and $\mathscr{N}^m$. Since $\eta^m \in H^{k+5/2}$, we can apply [26, Lemma 3.6] to deduce the existence of (u^m, p^m) that solves

$$\begin{cases} \nabla_{\mathscr{A}^m} \cdot \mathcal{S}_{\mathscr{A}^m}(p^m, u^m) = F & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}} \cdot u^m = G & \text{in} \quad \Omega, \\ \mathcal{S}_{\mathscr{A}^m}(p^m, u^m)\mathscr{N}_0 = V & \text{on} \quad \Sigma, \\ u^m = 0 & \text{on} \quad \Sigma_b, \end{cases} \tag{2.2.69}$$

and such that

$$\|u^m\|_{H^r} + \|p^m\|_{H^{r-1}} \lesssim C(\|\eta^m\|_{H^{k+5/2}(\Sigma)})\Bigg(\|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|V\|_{H^{r-3/2}(\Sigma)} \Bigg), \tag{2.2.70}$$

for $r = 2, \ldots, k+1$. We can rewrite above equations in the following shape:

$$\begin{cases} -\Delta_{\mathscr{A}^m} u^m + \nabla_{\mathscr{A}^m} p^m = F + \nabla_{\mathscr{A}^m} G & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}^m} \cdot u^m = G & \text{in} \quad \Omega, \\ (p^m I - \mathbb{D}_{\mathscr{A}^m} u^m)\mathscr{N}^m = V & \text{on} \quad \Sigma, \\ u^m = 0 & \text{on} \quad \Sigma_b. \end{cases} \tag{2.2.71}$$

In the following, we prove an improved estimate for (u^m, p^m) in terms of $\|\eta^m\|_{H^{k+1/2}(\Sigma)}$. We divide the proof into several steps.

Step 1: Preliminaries.

To abuse the notation, within the following procedure, we always use (u, p, η) instead of (u^m, p^m, η^m) to make the expressions succinct. Also it is easy to see the term $\nabla_{\mathscr{A}^m} G$ does not affect the estimates because $\|\nabla_{\mathscr{A}^m} G\|_{H^{k-1}} \lesssim \|\eta^m\|_{H^{k+1/2}(\Sigma)} \|G\|_{H^k}$. Hence, we still write F to indicate the forcing terms.

We write explicitly each term in above equations,

$$\partial_{11}u_1 + \partial_{22}u_1 + (1 + A^2 + B^2)K^2\partial_{33}u_1 - 2AK\partial_{13}u_1 - 2BK\partial_{23}u_1 \\ +(AK\partial_3(AK) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3K)\partial_3u_1 + \partial_1p - AK\partial_3p = F_1, \tag{2.2.72}$$

$$\partial_{11}u_2 + \partial_{22}u_2 + (1 + A^2 + B^2)K^2\partial_{33}u_2 - 2AK\partial_{13}u_2 - 2BK\partial_{23}u_2 \\ +(AK\partial_3(AK) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3K)\partial_3u_2 + \partial_2p - BK\partial_3p = F_2, \tag{2.2.73}$$

$$\partial_{11}u_3 + \partial_{22}u_3 + (1 + A^2 + B^2)K^2\partial_{33}u_3 - 2AK\partial_{13}u_3 - 2BK\partial_{23}u_3 \\ +(AK\partial_3(AK) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3K)\partial_3u_3 + K\partial_3p = F_3, \tag{2.2.74}$$

$$\partial_1u_1 - AK\partial_3u_1 + \partial_2u_2 - BK\partial_3u_2 + K\partial_3u_3 = G, \tag{2.2.75}$$

For convenience, we define

$$\mathcal{Z} = C(\eta_0)P(\eta)\left(\|F\|^2_{H^{k-1}} + \|G\|^2_{H^k} + \|V\|^2_{H^{k-1/2}(\Sigma)} \right), \tag{2.2.76}$$

where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$ and $P(\eta)$ is a polynomial of $\|\eta\|_{H^{k+1/2}(\Sigma)}$.

Step 2: $r = k$ case.

With the $(k-1)^{th}$ order elliptic estimate, we have

$$\|u\|^2_{H^{k-1}} + \|p\|^2_{H^{k-2}} \leq C(\eta_0)\left(\|F\|^2_{H^{k-3}} + \|G\|^2_{H^{k-2}} + \|V\|^2_{H^{k-5/2}(\Sigma)} \right) \lesssim \mathcal{Z}. \tag{2.2.77}$$

By Lemma 2.2.13, the constant $C(\eta_0)$ only depends on $\|\eta_0\|_{H^{k+1/2}}$. For $i = 1, 2$,

$(\partial_i u, \partial_i p)$ satisfies the equations

$$\begin{cases} -\Delta_{\mathscr{A}}(\partial_i u) + \nabla_{\mathscr{A}}(\partial_i p) = \bar{F} & in \quad \Omega, \\ \nabla_{\mathscr{A}} \cdot (\partial_i u) = \bar{G} & in \quad \Omega, \\ ((\partial_i p)I - \mathbb{D}_{\mathscr{A}}(\partial_i u))\mathscr{N} = \bar{V} & on \quad \Sigma, \\ \partial_i u = 0 & on \quad \Sigma_b, \end{cases} \tag{2.2.78}$$

where

$$\bar{F} = \partial_i F + \nabla_{\partial_i \mathscr{A}} \cdot \nabla_{\mathscr{A}} u + \nabla_{\mathscr{A}} \cdot \nabla_{\partial_i \mathscr{A}} u - \nabla_{\partial_i \mathscr{A}} p, \tag{2.2.79}$$

$$\bar{G} = \partial_i G - \nabla_{\partial_i \mathscr{A}} \cdot u, \tag{2.2.80}$$

$$\bar{V} = \partial_i V - (pI - \mathbb{D}_{\mathscr{A}} u)\partial_i \mathscr{N} + \mathbb{D}_{\partial_i \mathscr{A}} u \mathscr{N}. \tag{2.2.81}$$

Employing the $(k-1)^{th}$ order elliptic estimate, we have

$$\|\partial_i u\|_{H^{k-1}}^2 + \|\partial_i p\|_{H^{k-2}}^2 \lesssim C(\eta_0)\bigg(\big\|\bar{F}\big\|_{H^{k-3}}^2 + \big\|\bar{G}\big\|_{H^{k-2}}^2 + \big\|\bar{V}\big\|_{H^{k-5/2}(\Sigma)}^2 \bigg). \tag{2.2.82}$$

Except for the derivatives of F, G and V, all the other terms on the right-hand side have the form $\|A \cdot B\|_{H^r}$, in which $A = \partial^\alpha \bar{\eta}$ and $B = \partial^\beta u$ or $\partial^\beta p$. These kinds of estimates can be achieved by [26, Lemma A.1 - Lemma A.2]. Hence, we have the forcing estimate

$$\begin{aligned} & \big\|\bar{F}\big\|_{H^{k-3}}^2 + \big\|\bar{G}\big\|_{H^{k-2}}^2 + \big\|\bar{V}\big\|_{H^{k-5/2}}^2 \\ \lesssim \; & \|F\|_{H^{k-2}}^2 + \|G\|_{H^{k-1}}^2 + \|V\|_{H^{k-3/2}(\Sigma)}^2 + C(\eta_0)P(\eta)\bigg(\|u\|_{H^{k-1}}^2 + \|p\|_{H^{k-2}}^2 \bigg) \\ \lesssim \; & \mathcal{Z}. \end{aligned} \tag{2.2.83}$$

In detail, this means

$$\|\partial_1 u\|_{H^{k-1}}^2 + \|\partial_1 p\|_{H^{k-2}}^2 + \|\partial_2 u\|_{H^{k-1}}^2 + \|\partial_2 p\|_{H^{k-2}}^2 \lesssim \mathcal{Z}. \tag{2.2.84}$$

Hence, most parts in $\|u\|_{H^k} + \|p\|_{H^{k-1}}$ have been covered by this estimate, except those with the highest order derivative of ∂_3. Multiplying (2.2.74) by A and adding it to (2.2.72) eliminate the $\partial_3 p$ term and lead to

$$\begin{aligned}
&(\partial_{11}u_1 + A\partial_{11}u_3) + (\partial_{22}u_1 + A\partial_{22}u_3) + (1 + A^2 + B^2)K^2(\partial_{33}u_1 + A\partial_{33}u_3) \\
&-2AK(\partial_{13}u_1 + A\partial_{13}u_3) - 2BK(\partial_{23}u_1 + A\partial_{23}u_3) \\
&+(AK\partial_3(AK) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3 K)(\partial_1 u_1 + A\partial_1 u_3) \\
&+\partial_1 p = F_1 + AF_3.
\end{aligned} \tag{2.2.85}$$

Then taking derivative ∂_3^{k-2} on both sides, we can focus on the term $(1 + A^2 + B^2)K^2(\partial_3^k u_1 + A\partial_3^k u_3)$. The estimates of all the other terms in H^0 norm imply

$$\left\|\partial_3^k u_1 + A\partial_3^k u_3\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.86}$$

Similarly, we have

$$\left\|\partial_3^k u_2 + B\partial_3^k u_3\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.87}$$

Rearranging the terms in (2.2.75), we get

$$K(1 + A^2 + B^2)\partial_3 u_3 = G - \partial_1 u_1 - \partial_2 u_2 + AK(\partial_3 u_1 + A\partial_3 u_3) + BK(\partial_3 u_2 + B\partial_3 u_3). \tag{2.2.88}$$

Taking derivative ∂_3^{k-1} on both sides, we obtain

$$\left\|\partial_3^k u_3\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.89}$$

Combining (2.2.86), (2.2.87) and (2.2.89), it is easy to see

$$\left\|\partial_3^k u_1\right\|_{H^0}^2 + \left\|\partial_3^k u_2\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.90}$$

Plugging this into (2.2.74) and taking derivative ∂_3^{k-2} on both sides, we get

$$\left\|\partial_3^{k-1} p\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.91}$$

Combining this with all above estimates, we have proved

$$\|u\|_{H^k}^2 + \|p\|_{H^{k-1}}^2 \lesssim \mathcal{Z}. \tag{2.2.92}$$

This is the result for the case $r = k$.

Step 3: $r = k + 1$ case: horizontal derivatives.

We can take ∂_j derivative for $i, j = 1, 2$ on both sides of (2.2.78). Then by a similar argument as in Step 2, we obtain

$$\|\partial_{11} u\|_{H^{k-1}}^2 + \|\partial_{12} u\|_{H^{k-1}}^2 + \|\partial_{22} u\|_{H^{k-1}}^2 + \|\partial_{11} p\|_{H^{k-2}}^2 + \|\partial_{12} p\|_{H^{k-2}}^2 + \|\partial_{22} p\|_{H^{k-2}}^2 \lesssim \mathcal{Z}. \tag{2.2.93}$$

Step 4: $r = k + 1$ case: mixed derivatives.

Similar to the $r = k$ case argument, for $i = 1, 2$, taking derivative $\partial_3^{k-2}\partial_i$ on both sides of (2.2.85) and focusing on the term $\partial_3^k \partial_i u_1 + A\partial_3^k \partial_i u_3$, we get

$$\left\|\partial_3^k \partial_i u_1 + A\partial_3^k \partial_i u_3\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.94}$$

Similarly, we have

$$\left\|\partial_3^k \partial_i u_2 + B\partial_3^k \partial_i u_3\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.95}$$

Plugging this result into (2.2.85) and taking derivative $\partial_3^{k-2}\partial_i$ on both sides, we get

$$\left\|\partial_3^k \partial_i u_3\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.96}$$

The direct estimates for these three terms imply

$$\left\|\partial_3^k \partial_i u\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.97}$$

Plugging this into (2.2.74) and taking derivative $\partial_3^{k-2}\partial_i$ on both sides, we get

$$\left\|\partial_3^{k-1} \partial_i p\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.98}$$

Combining all above estimates, we have shown

$$\|\partial_{3i} u\|_{H^{k-1}}^2 + \|\partial_{3i} p\|_{H^{k-2}}^2 \lesssim \mathcal{Z}. \tag{2.2.99}$$

This is actually

$$\|\partial_{13} u\|_{H^{k-1}}^2 + \|\partial_{23} u\|_{H^{k-1}}^2 + \|\partial_{13} p\|_{H^{k-2}}^2 + \|\partial_{23} p\|_{H^{k-2}}^2 \lesssim \mathcal{Z}. \tag{2.2.100}$$

Step 5: $r = k+1$ case: vertical derivatives.

Again we use the same trick as above. Taking derivative ∂_3^{k-1} on both sides of (2.2.85) and focusing on the term $\partial_3^{k+1}u_1 + A\partial_3^{k+1}u_3$, we can bound $\partial_3^{k+1}u_1 - A\partial_3^{k+1}u_3$. Similarly, we control $\partial_3^{k+1}u_2 - B\partial_3^{k+1}u_3$. Plugging this result into (2.2.75) and taking derivative

∂_3^{k-1} on both sides, we can estimate $\partial_3^{k+1}u_3$. Then we have

$$\left\|\partial_3^{k+1}u\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.101}$$

Plugging this into (2.2.74) and taking derivative ∂_3^{k-1} on both sides, we get

$$\left\|\partial_3^{k}p\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.2.102}$$

Combining all above estimate, we get

$$\|\partial_{33}u\|_{H^{k-1}}^2 + \|\partial_{33}p\|_{H^{k-2}}^2 \lesssim \mathcal{Z}. \tag{2.2.103}$$

Step 6: $r = k+1$ case: conclusion.

To synthesize, (2.2.93), (2.2.100) and (2.2.103) imply all the second order derivatives of u in the $(k-1)^{th}$ order Sobolev norm and p in the $(k-2)^{th}$ order Sobolev norm are bounded, so we naturally have the estimate

$$\|u\|_{H^{k+1}}^2 + \|p\|_{H^k}^2 \lesssim C(\eta_0)P(\eta)\bigg(\|F\|_{H^{k-1}}^2 + \|G\|_{H^k}^2 + \|V\|_{H^{k-1/2}}^2\bigg). \tag{2.2.104}$$

Now let us go back to the original notation. Since $P(\cdot)$ is a fixed polynomial, this gives the estimate

$$\begin{aligned} & \|u^m\|^2_{H^{k+1}} + \|p^m\|^2_{H^k} \\ \lesssim\ & C(\eta_0)P(\eta^m)\Big(\|F\|^2_{H^{k-1}} + \|G\|^2_{H^k} + \|V\|^2_{H^{k-1/2}(\Sigma)} \Big) \\ \lesssim\ & C(\eta_0)P(\eta)\Big(\|F\|^2_{H^{k-1}} + \|G\|^2_{H^k} + \|V\|^2_{H^{k-1/2}(\Sigma)} \Big) \\ \lesssim\ & C(\eta_0)P(\eta_0)\Big(\|F\|^2_{H^{k-1}} + \|G\|^2_{H^k} + \|V\|^2_{H^{k-1/2}(\Sigma)} \Big) \\ \lesssim\ & C(\eta_0)\Big(\|F\|^2_{H^{k-1}} + \|G\|^2_{H^k} + \|V\|^2_{H^{k-1/2}(\Sigma)} \Big), \end{aligned} \tag{2.2.105}$$

where $C(\eta_0)$ depends on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$. This bound implies the sequence (u^m, p^m) is uniformly bounded in $H^{k+1} \times H^k$, so we can extract weakly convergent subsequence $u^m \rightharpoonup u^0$ and $p^m \rightharpoonup p^0$.

In the second equation of (2.2.69), we multiply both sides by $J^m w$ for $w \in C_0^\infty$ to achieve

$$\begin{aligned} \int_\Omega G w J^m &= \int_\Omega (\nabla_{\mathscr{A}^m} \cdot u^m) w J^m = -\int_\Omega u^m \cdot (\nabla_{\mathscr{A}^m} w) J^m \to -\int_\Omega u^0 \cdot (\nabla_{\mathscr{A}} w) J \\ &= \int_\Omega (\nabla_{\mathscr{A}} \cdot u^0) w J, \end{aligned} \tag{2.2.106}$$

which implies $\nabla_{\mathscr{A}} \cdot u^0 = G$. Then we multiply the first equation of (2.2.69) by wJ^m for $w \in \mathcal{H}^1$ and integrate by parts to get

$$\frac{1}{2}\int_\Omega \mathbb{D}_{\mathscr{A}^m} u^m : \mathbb{D}_{\mathscr{A}^m} w J^m - \int_\Omega p^m \nabla_{\mathscr{A}^m} \cdot w J^m = \int_\Omega F \cdot w J^m - \int_\Sigma V \cdot w. \tag{2.2.107}$$

Passing to the limit $m \to \infty$, we deduce

$$\frac{1}{2}\int_\Omega \mathbb{D}_{\mathscr{A}} u^0 : \mathbb{D}_{\mathscr{A}} w J - \int_\Omega p^0 \nabla_{\mathscr{A}} \cdot w J = \int_\Omega F \cdot w J - \int_\Sigma V \cdot w, \tag{2.2.108}$$

which reveals, upon integrating by parts again, (u^0, p^0) satisfies (2.2.54). Since (u, p) is the unique strong solution to (2.2.54), we have $u^0 = u$ and $p^0 = p$. The weak lower semi-continuity and the estimate (2.2.105) imply (2.2.68). $\square$

$\mathscr{A}$-Poisson Equation

We consider the elliptic problem

$$\begin{cases} \Delta_{\mathscr{A}} p = f & \text{in} \quad \Omega, \\ p = g & \text{on} \quad \Sigma, \\ \nabla_{\mathscr{A}} p \cdot \nu = v & \text{on} \quad \Sigma_b. \end{cases} \tag{2.2.109}$$

The existence and uniqueness of this equation can be found in [26, Lemma 3.8]. A similar argument as in Proposition 2.2.14 shows the following result.

Proposition 2.2.15. *Let $k \geq 3$ be an integer. Suppose $\eta \in H^{k+1/2}(\Sigma)$ and $\eta_0 \in H^{k+1/2}(\Sigma)$. Then there exists a $\epsilon_0 > 0$ such that if $\|\eta - \eta_0\|_{H^{k-3/2}} \leq \epsilon_0$, solution p to the system (2.2.109) satisfies*

$$\|p\|_{H^r} \lesssim C(\eta_0)\bigg(\|f\|_{H^{r-2}} + \|g\|_{H^{r-1/2}(\Sigma)} + \|v\|_{H^{r-3/2}(\Sigma_b)} \bigg), \tag{2.2.110}$$

for $r = 2, \ldots, k-1$, whenever the right-hand side is finite, where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$.

2.2.4 Linear Navier-Stokes Equations

In this section, we show the well-posedness of the linear Navier-Stokes problem

$$\begin{cases} \partial_t u - \Delta_{\mathscr{A}} u + \nabla_{\mathscr{A}} u = F & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}} \cdot u = 0 & \text{in} \quad \Omega, \\ (pI - \mathbb{D}_{\mathscr{A}} u)\mathscr{N} = V & \text{on} \quad \Sigma, \\ u = 0 & \text{on} \quad \Sigma_b. \end{cases} \tag{2.2.111}$$

Following the path of I. Tice and Y. Guo in [26], the result is quite obvious and we only present the main propositions.

Initial Data and Compatibility Conditions

We need to define the energy and dissipation in our estimates. It is noticeable that our definitions are slightly different from those in [26]. Here are some quantities related to the initial data.

$$\mathcal{H}_0(u,p) = \sum_{j=0}^{N} \left\|\partial_t^j u(0)\right\|_{H^{2N-2j}}^2 + \sum_{j=0}^{N-1} \left\|\partial_t^j p(0)\right\|_{H^{2N-2j-1}}^2, \tag{2.2.112}$$

$$\mathcal{K}_0(u_0) = \|u_0\|_{H^{2N}}^2, \tag{2.2.113}$$

$$\mathcal{H}_0(\eta) = \|\eta(0)\|_{H^{2N+1/2}(\Sigma)}^2 + \sum_{j=1}^{N} \left\|\partial_t^j \eta(0)\right\|_{H^{2N-2j+3/2}(\Sigma)}^2, \tag{2.2.114}$$

$$\mathcal{K}_0(\eta_0) = \|\eta_0\|_{H^{2N+1/2}(\Sigma)}^2, \tag{2.2.115}$$

$$\mathcal{H}_0(F,V) = \sum_{j=0}^{N-1} \left\|\partial_t^j F(0)\right\|_{H^{2N-2j-2}}^2 + \sum_{j=0}^{N-1} \left\|\partial_t^j V(0)\right\|_{H^{2N-2j-3/2}(\Sigma)}^2. \tag{2.2.116}$$

Furthermore, we need to define mappings $\mathcal{G}^1$ on Ω and $\mathcal{G}^2$ on Σ.

$$\begin{aligned} \mathcal{G}^1(v,q) &= -(R+\partial_t JK)\Delta_{\mathscr{A}} v - \partial_t R v + (\partial_t JK + R + R^T)\nabla_{\mathscr{A}} q \quad (2.2.117) \\ &\quad +\nabla_{\mathscr{A}} \cdot (\mathbb{D}_{\mathscr{A}}(Rv) - R\mathbb{D}_{\mathscr{A}} v + \mathbb{D}_{\partial_t \mathscr{A}} v), \\ \mathcal{G}^2(v,q) &= \mathbb{D}_{\mathscr{A}}(Rv)\mathscr{N} - (qI - \mathbb{D}_{\mathscr{A}} v)\partial_t \mathscr{N} + \mathbb{D}_{\partial_t \mathscr{A}} v \mathscr{N}. \quad (2.2.118) \end{aligned}$$

The mappings above allow us to define the higher order forcing terms. For $j = 1, \ldots, N$, we define

$$\begin{aligned} F^0 &= F, && (2.2.119) \\ V^0 &= V, && (2.2.120) \\ F^j &= D_t F^{j-1} + \mathcal{G}^1(D_t^{j-1} u, \partial_t^{j-1} p) && (2.2.121) \\ &= D_t^j F + \sum_{l=0}^{j-1} D_t^l \mathcal{G}_1(D_t^{j-l-1} u, \partial_t^{j-l-1} p), \\ V^j &= D_t V^{j-1} + \mathcal{G}^2(D_t^{j-1} u, \partial_t^{j-1} p) && (2.2.122) \\ &= D_t^j V + \sum_{l=0}^{j-1} D_t^l \mathcal{G}_2(D_t^{j-l-1} u, \partial_t^{j-l-1} p). \end{aligned}$$

Finally, we define the general quantities we need to estimate as follows:

$$\begin{aligned} \mathcal{E}(u,p) &= \sum_{j=0}^{N} \left\|\partial_t^j u\right\|^2_{L^\infty H^{2N-2j}} + \sum_{j=0}^{N-1} \left\|\partial_t^j p\right\|^2_{L^\infty H^{2N-2j-1}}, && (2.2.123) \\ \mathcal{D}(u,p) &= \sum_{j=0}^{N} \left\|\partial_t^j u\right\|^2_{L^2 H^{2N-2j+1}} + \sum_{j=0}^{N-1} \left\|\partial_t^j p\right\|^2_{L^2 H^{2N-2j}}, && (2.2.124) \\ \mathcal{K}(u,p) &= \mathcal{E}(u,p) + \mathcal{D}(u,p), && (2.2.125) \end{aligned}$$

$$\mathcal{E}(\eta) = \|\eta\|^2_{L^\infty H^{2N+1/2}(\Sigma)} + \sum_{j=1}^{N} \left\|\partial_t^j \eta\right\|^2_{L^\infty H^{2N-2j+3/2}(\Sigma)}, \tag{2.2.126}$$

$$\mathcal{D}(\eta) = \sum_{j=2}^{N+1} \left\|\partial_t^j \eta\right\|^2_{L^2 H^{2N-2j+5/2}(\Sigma)}, \tag{2.2.127}$$

$$\mathcal{K}(\eta) = \mathcal{E}(\eta) + \mathcal{D}(\eta), \tag{2.2.128}$$

$$\begin{aligned}\mathcal{K}(F,V) = & \sum_{j=0}^{N-1} \left\|\partial_t^j F\right\|^2_{L^2 H^{2N-2j-1}} + \left\|\partial_t^N F\right\|^2_{\mathcal{X}_T^*} + \sum_{j=0}^{N} \left\|\partial_t^j V\right\|^2_{L^2 H^{2N-2j-1/2}(\Sigma)}, \\ & + \sum_{j=0}^{N-1} \left\|\partial_t^j F\right\|^2_{L^\infty H^{2N-2j-2}} + \sum_{j=0}^{N-1} \left\|\partial_t^j V\right\|^2_{L^\infty H^{2N-2j-3/2}(\Sigma)}.\end{aligned} \tag{2.2.129}$$

For $j = 0, \ldots, N-1$, we say that the j^{th} compatibility conditions are satisfied if

$$\begin{cases} D_t^j u(0) \in H^2(\Omega) \cap \mathcal{X}(0), \\ \Pi_0(V^j(0) + \mathbb{D}_{\mathscr{A}_0} D_t^j u(0) \mathscr{N}_0) = 0, \end{cases} \tag{2.2.130}$$

where

$$D_t u = \partial_t u - Ru \quad for \quad R = \partial_t M M^{-1}, \tag{2.2.131}$$

M is defined in (2.2.48) and

$$\Pi_0(v) = v - (v \cdot \mathscr{N}_0)\mathscr{N}_0 \left|\mathscr{N}_0\right|^{-2} \quad for \quad \mathscr{N}_0 = (-\partial_1\eta_0, -\partial_2\eta_0, 1). \tag{2.2.132}$$

The construction of the higher order initial data satisfying these conditions are discussed carefully in [26, pp.327-328]. Under this construction, we have the estimate

$$\mathcal{H}_0(u,p) \lesssim P(\mathcal{H}_0(\eta)) + P(\mathcal{K}_0(u_0)) + P(\mathcal{H}_0(F,V)). \tag{2.2.133}$$

Local Well-posedness of Linear System

Theorem 2.2.16. *Suppose $\mathcal{K}_0(u_0) < \infty$, $\mathcal{K}(\eta) < \infty$ and the initial conditions are constructed to satisfy the j^{th} compatibility conditions (2.2.130) for $j = 0, 1, \ldots, N-1$, with the forcing functions satisfying $\mathcal{K}(F, V) + \mathcal{H}_0(F, V) < \infty$. Then there exists a universal constant $T_0(\eta) > 0$, which depends on $\mathcal{L}(\eta)$ as defined in the following, such that if $0 < T < T_0(\eta)$, then there exists a unique strong solution (u, p) to the system (2.2.111) on $[0, T]$ such that*

$$\begin{aligned} &\partial_t^j u \in L^2([0,T]; H^{2N-2j+1}) \cap C^0([0,T]; H^{2N-2j}) \quad for \quad j = 0, \ldots, N, \\ &\partial_t^j p \in L^2([0,T]; H^{2N-2j}) \cap C^0([0,T]; H^{2N-2j-1}) \quad for \quad j = 0, \ldots, N-1, \\ &\partial_t^{N+1} u \in \mathcal{X}_T^*. \end{aligned} \tag{2.2.134}$$

Also, the pair (u, p) satisfies the estimate

$$\mathcal{K}(u,p) \lesssim \mathcal{L}(\eta)\Big(P(\mathcal{K}_0(u_0)) + P(\mathcal{H}_0(F,V)) + P(\mathcal{K}(F,V))\Big), \tag{2.2.135}$$

where

$$\mathcal{L}(\eta) = P(\mathcal{K}(\eta)) \exp\Big(T P(\mathcal{K}(\eta))\Big). \tag{2.2.136}$$

Furthermore, the pair $(D_t^j u, \partial_t^j p)$ satisfies the system

$$\begin{cases} \partial_t(D_t^j u) - \Delta_{\mathscr{A}}(D_t^j u) + \nabla_{\mathscr{A}}(\partial_t^j p) = F^j & in \quad \Omega, \\ \nabla_{\mathscr{A}} \cdot (D_t^j u) = 0 & in \quad \Omega, \\ S_{\mathscr{A}}(\partial_t^j p, D_t^j u)\mathscr{N} = V^j & on \quad \Sigma, \\ D_t^j u = 0 & on \quad \Sigma_b, \end{cases} \tag{2.2.137}$$

in the strong sense for $j = 0, \ldots, N-1$ and in the weak sense for $j = N$.

Proof. This theorem is almost identical to [26, Theorem 4.8]. Since our free surface η can be general, the only difference here is we keep the polynomials of $\mathcal{K}(\eta)$ and $\mathcal{H}_0(\eta)$ in the final estimate instead of further simplifying them. $\square$

2.2.5 Transport Equation

In this section, we prove the well-posedness of the transport equation

$$\begin{cases} \partial_t\eta + u_1\partial_1\eta + u_2\partial_2\eta = u_3 & \text{on} \quad \Sigma, \\ \eta(t=0) = \eta_0 & \text{on} \quad \Sigma. \end{cases} \tag{2.2.138}$$

Local Well-posedness of Transport Equation

Define the quantity

$$\mathcal{Q}(u) = \sum_{j=0}^{N} \left\|\partial_t^j u\right\|_{L^2H^{2N-2j+1}}^2 + \sum_{j=0}^{N-1} \left\|\partial_t^j u\right\|_{L^\infty H^{2N-2j}}^2 . \tag{2.2.139}$$

Obviously, we have the relation $\mathcal{Q}(u) \leq \mathcal{K}(u,p)$. Simply taking temporal derivative on both sides of (4.1.1), we can iteratively define the higher order initial data, which satisfy the estimate

$$\mathcal{H}_0(\eta_0) \lesssim P(\mathcal{K}_0(\eta_0)) + P(\mathcal{H}_0(u,p)). \tag{2.2.140}$$

Theorem 2.2.17. *Suppose $\mathcal{K}_0(\eta_0) < \infty$ and $\mathcal{Q}(u) < \infty$. Then the problem (4.1.1) admits a unique solution η obeying $\mathcal{K}(\eta) < \infty$ and satisfying the initial data $\partial_t^j\eta(0)$ for $j = 0, \ldots, N$. Moreover, there exists a $0 < \bar{T}(u) < 1$, depending on $\mathcal{Q}(u)$, such* *that if* $0 < T < \bar{T}$, *then we have the estimate*

$$\mathcal{K}(\eta) \lesssim P(\mathcal{K}_0(\eta_0)) + P(\mathcal{Q}(u)), \tag{2.2.141}$$

where $P(\cdot)$ *is a polynomial satisfying* $P(0) = 0$.

Proof. This can be done following a similar path as in [26, Lemma 5.4], so we omit it here. □

Next, we introduce a lemma to describe the difference between η and η_0 in a small time period.

Lemma 2.2.18. *If* $\mathcal{Q}(u) + \mathcal{K}_0(\eta_0) < \infty$ *and* $\mathcal{H}_0(u,p) < \infty$, *then for any* $\theta > 0$, *there exists a* $\tilde{T}(\theta) > 0$ *depending on* $\mathcal{Q}(u)$ *and* $\mathcal{K}_0(\eta_0)$, *such that for any* $0 < T < \tilde{T}$, *we have the estimate*

$$\|\eta - \eta_0\|^2_{L^\infty H^{2N+1/2}(\Sigma)} \leq \theta. \tag{2.2.142}$$

Proof. $\xi = \eta - \eta_0$ satisfies the transport equation

$$\begin{cases} \partial_t \xi + u_1 \partial_1 \xi + u_2 \partial_2 \xi = u_3 - u_1 \partial_1 \eta_0 - u_2 \partial_2 \eta_0 \quad on \quad \Sigma, \\ \xi(0) = 0. \end{cases} \tag{2.2.143}$$

By the transport estimate in [26, Lemma A.11], we have

$$
\begin{aligned}
&\|\xi\|_{L^\infty H^{2N+1/2}(\Sigma)} && (2.2.144)\\
\leq\ & \exp\left(C\int_0^T \|u(t)\|_{H^{2N+1/2}(\Sigma)}\,dt\right)\\
&\times\left(\int_0^T \|u_3(t)-u_1(t)\partial_1\eta_0-u_2(t)\partial_2\eta_0\|_{H^{2N+1/2}(\Sigma)}\,dt\right)\\
\lesssim\ & \sqrt{T}\exp\left(\sqrt{T\mathcal{Q}(u)}\right)\left(P(\mathcal{K}_0(\eta_0))+P(\mathcal{H}_0(u,p))+P(\mathcal{Q}(u))\right).
\end{aligned}
$$

Hence, when T is small enough, our result naturally follows. □

Forcing Estimates

Now we need to estimate the forcing terms which appear on the right-hand sides of the linear Navier-Stokes equations. Compared to the system (2.1.12), the forcing terms are as follows:

$$F = \partial_t\bar{\eta}\tilde{b}K\partial_3 u - u\cdot\nabla_{\mathscr{A}}u, \tag{2.2.145}$$

$$V = \eta\mathscr{N}. \tag{2.2.146}$$

Recall we define the forcing quantities as follows:

$$
\begin{aligned}
\mathcal{K}(F,V) &= \sum_{j=0}^{N-1}\left\|\partial_t^j F\right\|^2_{L^2H^{2N-2j-1}} + \left\|\partial_t^N F\right\|^2_{\mathcal{X}_T^*} + \sum_{j=0}^{N}\left\|\partial_t^j V\right\|^2_{L^2H^{2N-2j-1/2}(\Sigma)}\\
&\quad + \sum_{j=0}^{N-1}\left\|\partial_t^j F\right\|^2_{L^\infty H^{2N-2j-2}} + \sum_{j=0}^{N-1}\left\|\partial_t^j V\right\|^2_{L^\infty H^{2N-2j-3/2}(\Sigma)},
\end{aligned} \tag{2.2.147}
$$

$$\mathcal{H}_0(F,V) = \sum_{j=0}^{N-1}\left\|\partial_t^j F(0)\right\|^2_{H^{2N-2j-2}} + \sum_{j=0}^{N-1}\left\|\partial_t^j V(0)\right\|^2_{H^{2N-2j-3/2}(\Sigma)}. \tag{2.2.148}$$

In the estimates of the Navier-Stokes-transport system, we also need some other forcing quantities.

$$\mathcal{F}(F,V) = \sum_{j=0}^{N-1} \left\|\partial_t^j F\right\|^2_{L^2H^{2N-2j-1}} + \left\|\partial_t^N F\right\|^2_{L^2H^0} + \sum_{j=0}^{N} \left\|\partial_t^j V\right\|^2_{L^\infty H^{2N-2j-1/2}(\Sigma)}, \tag{2.2.149}$$

$$\mathcal{H}(F,V) = \sum_{j=0}^{N-1} \left\|\partial_t^j F\right\|^2_{L^2H^{2N-2j-1}} + \sum_{j=0}^{N-1} \left\|\partial_t^j V\right\|^2_{L^2H^{2N-2j-1/2}(\Sigma)}. \tag{2.2.150}$$

Theorem 2.2.19. *The forcing terms satisfy the estimates*

$$\mathcal{K}(F,V) \lesssim P(\mathcal{K}(\eta)) + P(\mathcal{Q}(u)), \tag{2.2.151}$$

$$\mathcal{H}_0(F,V) \lesssim P(\mathcal{K}_0(\eta_0)) + P(\mathcal{K}_0(u_0),) \tag{2.2.152}$$

$$\mathcal{F}(F,V) \lesssim P(\mathcal{K}(\eta)) + P(\mathcal{Q}(u)), \tag{2.2.153}$$

$$\mathcal{H}(F,V) \lesssim T\Big(P(\mathcal{K}(\eta)) + P(\mathcal{Q}(u))\Big), \tag{2.2.154}$$

where $P(\cdot)$ is a polynomial with $P(0) = 0$.

Proof. The estimates follow from simple but lengthy computations, involving the standard argument, so we only present a sketch of the proof here.

After we expand all the terms with the Leibniz rule and plug into the definitions of F^j and V^j, it is easy to see the basic form of estimate is $\|XY\|^2_{L^2H^k}$ or $\|XY\|^2_{L^\infty H^k}$ where X includes the terms only involving $\bar\eta$ and Y includes the terms involving u, p, F or V. Employing [26, Lemma A.1 - Lemma A.3], we can always bound $\|XY\|^2_{L^2H^k} \lesssim \|X\|^2_{L^2H^{k_1}} \|Y\|^2_{L^\infty H^{k_2}}$ or $\|XY\|^2_{L^2H^k} \lesssim \|X\|^2_{L^\infty H^{k_1}} \|Y\|^2_{L^2H^{k_2}}$, and also $\|XY\|^2_{L^\infty H^k} \lesssim \|X\|^2_{L^\infty H^{k_1}} \|Y\|^2_{L^\infty H^{k_2}}$. The principle of choice in these two options for L^2H^k is not to exceed the highest order of derivatives in the right-hand sides of the estimates. A detailed analysis shows this goal can always be achieved. In (2.2.151), note the trivial bound

$$\left\|\partial_t^N F\right\|_{\mathcal{X}_T^*}^2 \lesssim \left\|\partial_t^N F\right\|_{L^2H^0}^2. \tag{2.2.155}$$

Also in (2.2.154), the key part is the appearance of the constant T. We first trivially bound it as follows:

$$\begin{aligned}
&\sum_{j=0}^{N-1}\left\|\partial_t^j F\right\|_{L^2H^{2N-2j-1}}^2 + \sum_{j=0}^{N-1}\left\|\partial_t^j V\right\|_{L^2H^{2N-2j-1/2}(\Sigma)}^2 \\
&\quad \leq T\left(\sum_{j=0}^{N-1}\left\|\partial_t^j F\right\|_{L^\infty H^{2N-2j-1}}^2 + \sum_{j=0}^{N-1}\left\|\partial_t^j V\right\|_{L^\infty H^{2N-2j-1/2}(\Sigma)}^2\right).
\end{aligned} \tag{2.2.156}$$

Then an application of the inequality $2ab \leq a^2 + b^2$ implies the desired estimate. $\square$

Remark 2.2.20. *The reason why we can get the constant T lies in that, in the nonlinear Navier-Stokes equations, velocity in the nonlinear terms actually has one less derivative than the linear terms, which makes it possible to use Lemma A.1.15. The appearance of T in (2.2.154) plays a key role in the following nonlinear iteration argument.*

2.2.6 Navier-Stokes-Transport System

In this section, we study the nonlinear system

$$\begin{cases} \partial_t u - \partial_t \bar{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathscr{A}} u - \Delta_{\mathscr{A}} u + \nabla_{\mathscr{A}} p = 0 & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}} \cdot u = 0 & \text{in} \quad \Omega, \\ S_{\mathscr{A}}(p,u)\mathscr{N} = \bar{\eta}\mathscr{N} & \text{on} \quad \Sigma, \\ u = 0 & \text{on} \quad \Sigma_b, \\ u(x,0) = u_0(x), & \\ & \\ \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on} \quad \Sigma, \\ \eta(x',0) = \eta_0(x'). & \end{cases} \tag{2.2.157}$$

We first show the compatibility conditions and then construct an iteration to show the well-posedness.

Initial Data and Compatibility Conditions

Define

$$\mathcal{K}_0 = \mathcal{K}_0(u_0) + \mathcal{K}_0(\eta_0). \tag{2.2.158}$$

We say (u_0, η_0) satisfies the N^{th} order compatibility conditions if

$$\begin{cases} \nabla_{\mathscr{A}_0} \cdot (D_t^j u(0)) = 0 & \text{in} \quad \Omega, \\ D_t^j u(0) = 0 & \text{on} \quad \Sigma_b, \\ \Pi_0(V^j(0) + \mathbb{D}_{\mathscr{A}_0} D_t^j u(0) \mathscr{N}_0) = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.2.159}$$

for $j = 0, \ldots, N-1$. The details can be found in [26, pp.338-339]. The following is a direct estimate of this construction.

Theorem 2.2.21. *Suppose (u_0, η_0) satisfies $\mathcal{K}_0 < \infty$. Let $\partial_t^j u(0)$, $\partial_t^j \eta(0)$ for $j = 0, \ldots, N$ and $\partial_t^j p(0)$ for $j = 0, \ldots, N-1$ be defined to satisfy the N^{th} compatibility conditions. Then*

$$\mathcal{K}_0 \leq \mathcal{H}_0(u,p) + \mathcal{H}_0(\eta) \lesssim P(\mathcal{K}_0) \tag{2.2.160}$$

for a given polynomial $P(\cdot)$ with $P(0) = 0$.

Construction of Iteration

For given initial data (u_0, η_0), by the Sobolev extension theorem in [26, Lemma A.5], there exists a u^0 defined in $\Omega \times [0, \infty)$ such that $\mathcal{Q}(u^0) \lesssim \mathcal{K}_0(u_0)$ coinciding with the initial data to the N^{th} order. By solving the transport equation with respect to u^0, Theorem 2.2.17 implies there exists a η^0 defined in $\Omega \times [0, T_0)$ such that $\mathcal{K}(\eta^0) \lesssim (1 + \mathcal{K}_0(\eta_0))P(\mathcal{Q}(u^0)) \lesssim P(\mathcal{K}_0) < \infty$. This is our starting point.

For any integer $m \geq 1$, define the approximating sequence (u^m, p^m, η^m) on the existence interval $[0, T_m)$ by solving the system

$$\begin{cases} \partial_t u^m - \Delta_{\mathscr{A}^{m-1}} u^m + \nabla_{\mathscr{A}^{m-1}} p^m = \partial_t \bar{\eta}^{m-1} \tilde{b} K^{m-1} \partial_3 u^{m-1} & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}^{m-1}} \cdot u^m = 0 & \text{in} \quad \Omega, \\ (p^m I - \mathbb{D}_{\mathscr{A}^{m-1}} u^m) \mathscr{N}^{m-1} = \eta^{m-1} \mathscr{N}^{m-1} & \text{on} \quad \Sigma, \\ u^m = 0 & \text{on} \quad \Sigma_b, \\ \\ \partial_t \eta^m + u_1^m \partial_1 \eta^m + u_2^m \partial_2 \eta^m = u_3^m & \text{on} \quad \Sigma, \end{cases} \tag{2.2.161}$$

where (u^m, η^m) satisfies the common initial data (u_0, η_0), and $\mathscr{A}^m$, $\mathscr{N}^m$, K^m are given in terms of η^m.

This is only a formal definition of iteration. In the following theorems, we finally prove this approximating sequence can be defined for any $m \in \mathbb{N}$ and the existence intervals T_m do not shrink to 0 as $m \to \infty$.

Boundedness Theorem

Before we start to prove the boundedness result, it is useful to notice that, based on the linear estimate (2.2.135) and the forcing estimates in Lemma 2.2.19, this sequence can always be constructed and we can directly derive an estimate

$$\mathcal{K}(u^{m+1}, p^{m+1}) + \mathcal{K}(\eta^{m+1}) \lesssim P(\mathcal{Q}(u^m) + \mathcal{K}(\eta^m) + \mathcal{K}_0), \tag{2.2.162}$$

when T is sufficiently small, where $P(\cdot)$ is a polynomial satisfying $P(0) = 0$. Since the initial data may not be small, this estimate cannot meet our requirements. Hence, we have to go back to the energy structure and derive a stronger estimate.

Theorem 2.2.22. *Assume $J^0 > \delta > 0$ and the initial data (u_0, η_0) satisfies the N^{th} compatibility condition. Then there exists a constant $0 < \mathcal{Z} < \infty$ and $0 < \bar{T} < 1$ depending on $\mathcal{K}_0$, such that if $0 < T \leq \bar{T}$ and $\mathcal{K}_0 < \infty$, then there exists an infinite sequence $(u^m, p^m, \eta^m)_{m=0}^{\infty}$ satisfying the iteration equation (2.2.161) within the existence interval $[0, T)$ and the following properties.*

1. *The iteration sequence satisfies the estimate*

$$\mathcal{Q}(u^m) + \mathcal{K}(\eta^m) \leq \mathcal{Z} \tag{2.2.163}$$

for arbitrary m, where the temporal norm is taken with respect to $[0, T)$.

2. *For any m, $J^m(t) > \delta/2$ for $0 \leq t \leq T$.*

Proof. This theorem is basically an updated version of [26, Theorem 6.1]. However, this proof is much more complicated than the original one. The original proof directly applies the estimates in the Navier-Stokes equations and the transport equation. In our case, without the smallness assumption, the iteration will not be bounded with the same method. Hence, we have to go back to the natural energy structure.

Let's denote the above two assertions related to m as statement $\mathbb{P}_m$. We use an induction to prove this theorem.

Step 1: $\mathbb{P}_0$ case.
This is only related to the initial data. Obviously, the construction of u^0 leads to $\mathcal{Q}(u^0) \lesssim \mathcal{K}_0$. By the transport estimate in Lemma 2.2.17, we have $\mathcal{K}(\eta^0) \lesssim P(\mathcal{K}_0)$ for $P(0) = 0$. We can choose $\mathcal{Z} \geq P(\mathcal{K}_0)$, so the first assertion is verified.

Define $\xi^0 = \eta^0 - \eta_0$ the difference between the free surface at later time and its initial data. Then the estimate in Lemma 2.2.18 implies $\|\xi^0\|_{L^\infty H^{5/2}} \lesssim T\mathcal{Z}$. Naturally, $\sup_{t\in[0,T]} |J^0(t) - J^0(0)| \lesssim \|\xi^0\|_{L^\infty H^{5/2}} \lesssim T\mathcal{Z}$. Thus when we take $T \leq \delta/(2\mathcal{Z})$, then $J^0(t) \geq \delta/2$. So the second assertion is also verified. In a similar fashion, we can verify the closeness assumption in Remark 2.2.10. Hence, $\mathbb{P}_0$ is true.

In the following, we assume $\mathbb{P}_{m-1}$ is true for $m \geq 1$ and prove $\mathbb{P}_m$ is also true. As long as we can show this, by induction, $\mathbb{P}_n$ is valid for arbitrary $n \in \mathbb{N}$. Certainly, the induction hypothesis and (2.2.162) imply $\mathcal{Q}(u^{m-1}) + \mathcal{K}(\eta^{m-1}) \leq \mathcal{Z}$ and $\mathcal{K}(u^m, p^m) + \mathcal{K}(\eta^m) \leq CP(\mathcal{K}_0 + \mathcal{Z})$.

Step 2: $\mathbb{P}_m$ case: estimates of u^m via energy structure.

By Theorem 2.2.16, the pair $(D_t^N u^m, \partial_t^N u^m)$ satisfies the equations (2.2.137) in the weak sense, i.e.

$$\begin{cases} \partial_t(D_t^N u^m) - \Delta_{\mathscr{A}^{m-1}}(D_t^N u^m) + \nabla_{\mathscr{A}^{m-1}}(\partial_t^N p^m) = F^N & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}^{m-1}} \cdot (D_t^N u^m) = 0 & \text{in} \quad \Omega, \\ \mathcal{S}_{\mathscr{A}^{m-1}}(D_t^N u^m, \partial_t^N p^m)\mathscr{N} = V^N & \text{on} \quad \Sigma, \\ D_t^N u^m = 0 & \text{on} \quad \Sigma_b, \end{cases} \tag{2.2.164}$$

where F^N and V^N are given in terms of u^m and η^{m-1}. Hence, we have the standard weak formulation, i.e. for any test function $\psi \in \mathcal{X}_T^{m-1}$, the following holds

$$\langle \partial_t D_t^N u^m, \psi \rangle_{\mathcal{H}_T^0} + \frac{1}{2}\langle \mathbb{D}_{\mathscr{A}^{m-1}} D_t^N u^m, \mathbb{D}_{\mathscr{A}^{m-1}} \psi \rangle_{\mathcal{H}_T^0} = \langle D_t^N u^m, F^N \rangle_{\mathcal{H}_T^0} - \langle D_t^N u^m, V^N \rangle_{0,\Sigma,T}. \tag{2.2.165}$$

Therefore, when plugging into the test function $\psi = D_t^N u^m$, we have the natural energy structure

$$\begin{aligned} &\frac{1}{2}\int_\Omega J \left| D_t^N u^m \right|^2 + \frac{1}{2}\int_0^t \int_\Omega J \left| \mathbb{D}_{\mathscr{A}^{m-1}} D_t^N u^m \right|^2 = \\ &\quad \frac{1}{2}\int_\Omega J(0) \left| D_t^N u^m(0) \right|^2 + \frac{1}{2}\int_0^t \int_\Omega \partial_t J \left| D_t^N u^m \right|^2 \\ &\qquad + \int_0^t \int_\Omega J F^N \cdot D_t^N u^m - \int_0^t \int_\Omega V^N \cdot D_t^N u^m. \end{aligned} \tag{2.2.166}$$

We can easily give the preliminary estimates for the left-hand side (LHS) and the right-hand side (RHS).

$$LHS \gtrsim \left\|D_t^N u^m\right\|_{L^\infty H^0}^2 + \left\|D_t^N u^m\right\|_{L^2 H^1}^2, \tag{2.2.167}$$

$$\begin{aligned} RHS &\lesssim P(\mathcal{K}_0) + T\mathcal{Z}\left\|D_t^N u^m\right\|_{L^\infty H^0}^2 \\ &\quad + \sqrt{T}\mathcal{Z}\left\|F^N\right\|_{L^2 H^0}\left\|D_t^N u^m\right\|_{L^\infty H^0} + \sqrt{T}\left\|V^N\right\|_{L^\infty H^{-1/2}(\Sigma)}\left\|D_t^N u^m\right\|_{L^2 H^{1/2}(\Sigma)} \\ &\lesssim P(\mathcal{K}_0) + T\mathcal{Z}\left\|D_t^N u^m\right\|_{L^\infty H^0}^2 + \sqrt{T}\left\|F^N\right\|_{L^2 H^0}^2 \\ &\quad + \sqrt{T}\mathcal{Z}^2\left\|D_t^N u^m\right\|_{L^\infty H^0}^2 + \sqrt{T}\left\|V^N\right\|_{L^\infty H^{-1/2}(\Sigma)}^2 + \sqrt{T}\left\|D_t^N u^m\right\|_{L^2 H^1}^2, \end{aligned} \tag{2.2.168}$$

for a polynomial $P(0) = 0$. Taking $T \leq 1/(16\mathcal{Z}^4)$ and absorbing the extra terms on RHS into LHS imply

$$\left\|D_t^N u^m\right\|_{L^\infty H^0}^2 + \left\|D_t^N u^m\right\|_{L^2 H^1}^2 \lesssim P(\mathcal{K}_0) + \sqrt{T}\left\|F^N\right\|_{L^2 H^0}^2 + \sqrt{T}\left\|V^N\right\|_{L^\infty H^{-1/2}(\Sigma)}^2. \tag{2.2.169}$$

Dropping the $\left\|D_t^N u^m\right\|_{L^\infty H^0}^2$ term, we derive the further estimate for $\partial_t^N u^m$

$$\begin{aligned} \left\|\partial_t^N u^m\right\|_{L^2 H^1}^2 &\lesssim P(\mathcal{K}_0) + \sqrt{T}\left\|F^N\right\|_{L^2 H^0}^2 + \sqrt{T}\left\|V^N\right\|_{L^\infty H^{-1/2}(\Sigma)}^2 \\ &\quad + \left\|D_t^N u^m - \partial_t^N u^m\right\|_{L^2 H^1}^2. \end{aligned} \tag{2.2.170}$$

Then we need to estimate each term on RHS. For the middle two terms, it suffices to show they are bounded, However, for the last term, we need a temporal constant T within the estimate, which can be done by Lemma A.1.15.

$$\begin{aligned} \left\|F^N\right\|_{L^2 H^0}^2 &\lesssim P(\mathcal{K}(\eta^{m-1}))\left(\sum_{j=0}^{N-1}\left\|\partial_t^j u^m\right\|_{L^2 H^2}^2 + \sum_{j=0}^{N-1}\left\|\partial_t^j p^m\right\|_{L^2 H^1}^2\right) + \mathcal{F}(F,V) \\ &\lesssim P(\mathcal{K}_0 + \mathcal{Z}) + \mathcal{F}(F,V), \end{aligned} \tag{2.2.171}$$

$$\begin{aligned}\left\|V^N\right\|^2_{L^\infty H^{-1/2}(\Sigma)} &\lesssim P(\mathcal{K}(\eta^{m-1}))\left(\sum_{j=0}^{N-1}\left\|\partial_t^j u^m\right\|^2_{L^\infty H^2}+\sum_{j=0}^{N-1}\left\|\partial_t^j p^m\right\|^2_{L^\infty H^1}\right)+\mathcal{F}(F,V)\\ &\lesssim P(\mathcal{K}_0+\mathcal{Z})+\mathcal{F}(F,V),\end{aligned} \tag{2.2.172}$$

$$\begin{aligned}\left\|D_t^N u^m-\partial_t^N u^m\right\|_{L^2H^1} &\lesssim \sqrt{T}\left\|D_t^N u^m-\partial_t^N u^m\right\|_{L^\infty H^1}\\ &\lesssim \sqrt{T}P(\mathcal{K}(\eta^{m-1}))\left(\sum_{j=0}^{N-1}\left\|\partial_t^j u^m\right\|^2_{L^\infty H^1}\right)\\ &\lesssim \sqrt{T}P(\mathcal{K}_0+\mathcal{Z}).\end{aligned} \tag{2.2.173}$$

Therefore, to sum up, we have

$$\left\|\partial_t^N u^m\right\|^2_{L^2H^1} \lesssim P(\mathcal{K}_0)+\sqrt{T}P(\mathcal{K}_0+\mathcal{Z})+\sqrt{T}\mathcal{F}(F,V). \tag{2.2.174}$$

Step 3: $\mathbb{P}_m$ case: estimates of u^m via elliptic estimates.

For $0\leq n\leq N-1$, the n^{th} order Navier-Stokes system is as follows.

$$\begin{cases}\partial_t(D_t^n u^m)-\Delta_{\mathscr{A}}(D_t^n u^m)+\nabla_{\mathscr{A}}(\partial_t^n p^m)=F^n & \text{in}\quad \Omega,\\ \nabla_{\mathscr{A}}\cdot(D_t^n u^m)=0 & \text{in}\quad \Omega,\\ S_{\mathscr{A}}(D_t^n u^m,\partial_t^n p^m)\mathscr{N}=V^n & \text{on}\quad \Sigma,\\ D_t^n u^m=0 & \text{on}\quad \Sigma_b,\end{cases} \tag{2.2.175}$$

where F^n and V^n are given in terms of u^m and η^{m-1}.

A straightforward application of the elliptic estimates reveals

$$\|D_t^n u^m\|^2_{L^2H^{2N-2n+1}} \lesssim \|\partial_t D_t^n u^m\|^2_{L^2H^{2N-2n-1}} + \|F^n\|^2_{L^2H^{2N-2n-1}} + \|V^n\|^2_{L^2H^{2N-2n-1/2}} . \tag{2.2.176}$$

By triangle's inequality, we have

$$\begin{aligned} \|\partial_t^n u^m\|^2_{L^2H^{2N-2n+1}} \lesssim\ & \|F^n\|^2_{L^2H^{2N-2n-1}} + \|V^n\|^2_{L^2H^{2N-2n-1/2}} + \left\|\partial_t^{n+1} u^m\right\|^2_{L^2H^{2N-2n-1}} \\ & + \|\partial_t(D_t^n u^m - \partial_t^n u^m)\|^2_{L^2H^{2N-2n-1}} + \|D_t^n u^m - \partial_t^n u^m\|^2_{L^2H^{2N-2n+1}} . \end{aligned} \tag{2.2.177}$$

Then we give a detailed estimates for the terms on RHS. These estimates can be easily obtained as what we did before, so we omit the details here.

$$\begin{aligned} & \|F^n\|^2_{L^2H^{2N-2n-1}} \\ \lesssim\ & TP(\mathcal{K}(\eta^{m-1}))\bigg(\sum_{j=0}^{N-2} \left\|\partial_t^j u^m\right\|^2_{L^\infty H^{2N-2j-1}} + \sum_{j=0}^{N-2} \left\|\partial_t^j p^m\right\|^2_{L^\infty H^{2N-2j-2}} \bigg) + \mathcal{H}(F,V) \\ \lesssim\ & TP(\mathcal{K}_0 + \mathcal{Z}) + \mathcal{H}(F,V), \end{aligned} \tag{2.2.178}$$

$$\begin{aligned} & \|V^n\|^2_{L^2H^{2N-2n-1/2}(\Sigma)} \\ \lesssim\ & TP(\mathcal{K}(\eta^{m-1}))\bigg(\sum_{j=0}^{N-2} \left\|\partial_t^j u^m\right\|^2_{L^\infty H^{2N-2j-1}} + \sum_{j=0}^{N-2} \left\|\partial_t^j p^m\right\|^2_{L^\infty H^{2N-2j-2}} \bigg) \\ & + \mathcal{H}(F,V) \\ \lesssim\ & TP(\mathcal{K}_0 + \mathcal{Z}) + \mathcal{H}(F,V), \end{aligned} \tag{2.2.179}$$

$$\begin{aligned}
&\|\partial_t(D_t^n u^m - \partial_t^n u^m)\|^2_{L^2H^{2N-2n-1}} \\
\lesssim\ & TP(\mathcal{K}(\eta^{m-1}))\bigg(\sum_{j=0}^{N-1}\|\partial_t^j u^m\|^2_{L^\infty H^{2N-2j-1}}\bigg) \\
\lesssim\ & TP(\mathcal{K}_0+\mathcal{Z}),
\end{aligned} \tag{2.2.180}$$

$$\begin{aligned}
&\|D_t^n u^m - \partial_t^n u^m\|^2_{L^2H^{2N-2n+1}} \\
\lesssim\ & TP(\mathcal{K}(\eta^{m-1}))\bigg(\sum_{j=0}^{N-2}\|\partial_t^j u^m\|^2_{L^\infty H^{2N-2j+1}}\bigg) \\
\lesssim\ & TP(\mathcal{K}_0+\mathcal{Z}).
\end{aligned} \tag{2.2.181}$$

Therefore, to sum up, we have

$$\sum_{n=0}^{N-1}\|\partial_t^n u^m\|^2_{L^2H^{2N-2n+1}} \lesssim TP(\mathcal{K}_0+\mathcal{Z})+\mathcal{H}(F,V). \tag{2.2.182}$$

Step 4: $\mathbb{P}_m$ case: synthesis of estimates for u^m.

Combining (2.2.174), (2.2.182) and Lemma A.1.14, we can deduce

$$\mathcal{Q}(u^m) \lesssim P(\mathcal{K}_0)+\sqrt{T}P(1+\mathcal{K}_0+\mathcal{Z})+\sqrt{T}\mathcal{F}(F,V)+\mathcal{H}(F,V). \tag{2.2.183}$$

Then, by the forcing estimates in Lemma 2.2.19, we have

$$\mathcal{F}(F,V) \lesssim P(\mathcal{K}(\eta^{m-1}))+P(\mathcal{Q}(u^{m-1})) \lesssim P(\mathcal{Z}) \tag{2.2.184}$$

$$\mathcal{H}(F,V) \lesssim T\bigg(P(\mathcal{K}(\eta^{m-1}))+P(\mathcal{Q}(u^{m-1}))\bigg) \lesssim TP(\mathcal{Z}). \tag{2.2.185}$$

Hence, to sum up, we achieve the estimate

$$\mathcal{Q}(u^m) \lesssim P(\mathcal{K}_0) + \sqrt{T}P(\mathcal{K}_0 + \mathcal{Z}). \tag{2.2.186}$$

This is actually

$$\mathcal{Q}(u^m) \leq CP(\mathcal{K}_0) + \sqrt{T}CP(\mathcal{K}_0 + \mathcal{Z}) \tag{2.2.187}$$

for some universal constant $C > 0$. So we can take $\mathcal{Z} \geq 2CP(\mathcal{K}_0)$. Taking T sufficiently small depending on $\mathcal{Z}$, we can bound $\mathcal{Q}(u^m) \lesssim 2CP(\mathcal{K}_0) \leq \mathcal{Z}$.

Step 5: $\mathbb{P}_m$ case: estimate of η^m via transport estimate.

The transport estimate in Lemma 2.2.17 leads to

$$\mathcal{K}(\eta^m) \lesssim P(\mathcal{K}_0) + P(\mathcal{Q}(u^m)) \tag{2.2.188}$$

for $P(0) = 0$. Hence, we can take $\mathcal{Z} \geq P(\mathcal{K}_0) + P(2CP(\mathcal{K}_0))$ to obtain

$$\mathcal{K}(\eta^m) \lesssim \mathcal{Z}. \tag{2.2.189}$$

Step 6: $\mathbb{P}_m$ case: estimate of $J^m(t)$.

For $\xi^m = \eta^m - \eta_0$, the transport estimate in Lemma 2.2.18 implies $\|\xi^m\|_{L^\infty H^{5/2}} \lesssim T\mathcal{Z}$. Naturally, $\sup_{t\in[0,T]} |J^m(t) - J^m(0)| \lesssim \|\xi^m\|_{L^\infty H^{5/2}} \lesssim T\mathcal{Z}$. Thus taking $T \leq \delta/(2\mathcal{Z})$, we have $J^m(t) \geq \delta/2$. A similar argument can justify the closeness assumption in Remark 2.2.10.

Synthesis.

Above estimates show we can take $\mathcal{Z} = CP(\mathcal{K}_0)$ and T small enough depending on $\mathcal{Z}$ to obtain

$$\mathcal{Q}(u^m) + \mathcal{K}(\eta^m) \le \mathcal{Z}, \tag{2.2.190}$$

and

$$J^m(t) \ge \delta/2 \quad for \quad t \in [0, T]. \tag{2.2.191}$$

Therefore, $\mathbb{P}_m$ is verified. By induction, we conclude $\mathbb{P}_n$ is valid for any $n \in \mathbb{N}$. It is noticeable we can take $\mathcal{Z} = P(\mathcal{K}_0)$ where $P(\cdot)$ satisfies $P(0) = 0$. □

Theorem 2.2.23. *Assume exactly the same conditions as Theorem 2.2.22. Then we have the estimate*

$$\mathcal{K}(u^m, p^m) + \mathcal{K}(\eta^m) \lesssim P(\mathcal{K}_0) \tag{2.2.192}$$

for a polynomial satisfying $P(0) = 0$.

Proof. This is a natural corollary of above theorem. Since we know $\mathcal{Q}(u^{m-1}) \lesssim \mathcal{Z}$, by relation (2.2.162), it implies $\mathcal{K}(u^m, p^m)$ is bounded for any $m \in \mathbb{N}$. □

Contraction Theorem

Define

$$\begin{aligned}
\mathfrak{N}(v, q; T) &= \|v\|^2_{L^\infty H^2} + \|v\|^2_{L^2 H^3} + \|\partial_t v\|^2_{L^\infty H^0} + \|\partial_t v\|^2_{L^2 H^1} + \|q\|^2_{L^\infty H^1} + \|q\|^2_{L^2 H^2}\,, \\
\mathfrak{M}(\zeta; T) &= \|\zeta\|^2_{L^\infty H^{5/2}(\Sigma)} + \|\partial_t \zeta\|^2_{L^\infty H^{3/2}(\Sigma)} + \|\partial_t^2 \zeta\|^2_{L^2 H^{1/2}(\Sigma)}\,.
\end{aligned} \tag{2.2.193}$$

Theorem 2.2.24. *For $j = 1, 2$, suppose v^j, q^j, w^j and ζ^j satisfy the same initial* *conditions for different j and the system*

$$\begin{cases} \partial_t v^j - \Delta_{\mathscr{A}^j} v^j + \nabla_{\mathscr{A}^j} q^j = \partial_t \bar{\zeta}^j \tilde{b} K^j \partial_3 w^j - w^j \cdot \nabla_{\mathscr{A}^j} w^j & in \quad \Omega, \\ \nabla_{\mathscr{A}^j} \cdot v^j = 0 & in \quad \Omega, \\ \mathcal{S}_{\mathscr{A}^j}(q^j, v^j)\mathscr{N}^j = \zeta^j \mathscr{N}^j & on \quad \Sigma, \\ v^j = 0 & on \quad \Sigma_b, \\ \\ \partial_t \zeta^j = w^j \cdot \mathscr{N}^j & in \quad \Omega, \end{cases} \tag{2.2.194}$$

where $\mathscr{A}^j$, $\mathscr{N}^j$ and K^j are in terms of ζ^j. Assume $\mathcal{K}(w^j, 0)$, $\mathcal{K}(v^j, q^j)$ and $\mathcal{K}(\zeta^j)$ is bounded by $\mathcal{Z}$. Then there exists $0 < \tilde{T} < 1$ such that for any $0 < T < \tilde{T}$, we have the contraction relations

$$\mathfrak{N}(v^1 - v^2, q^1 - q^2; T) \leq \frac{1}{2}\mathfrak{N}(w^1 - w^2, 0; T), \tag{2.2.195}$$

$$\mathfrak{M}(\zeta^1 - \zeta^2; T) \lesssim \mathfrak{N}(w^1 - w^2, 0; T). \tag{2.2.196}$$

Proof. We divide this proof into several steps.

Step 1: Lower order equations.

Define $v = v^1 - v^2$, $q = q^1 - q^2$, $w = w^1 - w^2$ and $\zeta = \zeta^1 - \zeta^2$, which has trivial initial condition. Then they satisfy the equation as follows:

$$\begin{cases} \partial_t v + \nabla_{\mathcal{A}^1} \cdot \mathcal{S}_{\mathcal{A}^1}(q, v) = H^1 + \nabla_{\mathscr{A}^1} \cdot (\mathbb{D}_{(\mathscr{A}^1 - \mathscr{A}^2)} v^2) & \text{in} \quad \Omega, \\ \nabla_{\mathcal{A}_1} \cdot v = H^2 & \text{in} \quad \Omega, \\ \mathcal{S}_{\mathcal{A}_1}(q, v)\mathscr{N}^1 = H^3 + \mathbb{D}_{(\mathscr{A}^1 - \mathscr{A}^2)} v^2 \mathscr{N}^1 & \text{on} \quad \Sigma, \\ v = 0 & \text{on} \quad \Sigma_b, \end{cases} \tag{2.2.197}$$

where

$$
\begin{aligned}
H^1 &= \nabla_{(\mathscr{A}^1-\mathscr{A}^2)}(\mathbb{D}_{\mathscr{A}^2}v^2) - (\mathscr{A}^1-\mathscr{A}^2)\nabla p^2 + \partial_t\zeta^1\tilde{b}K^1\partial_3 w + \partial_t\zeta\tilde{b}K^1\partial_3 w^2 \\
&\quad +\partial_t\zeta^1\tilde{b}(K^1-K^2)\partial_3 w^2 - w\cdot\nabla_{\mathscr{A}^1}w^1 - w^2\cdot\nabla_{\mathscr{A}^1}w - w^2\cdot\nabla_{(\mathscr{A}^1-\mathscr{A}^2)}w^2,
\end{aligned} \tag{2.2.198}
$$

$$
H^2 = -\nabla_{(\mathscr{A}^1-\mathscr{A}^2)}\cdot v^2, \tag{2.2.199}
$$

$$
\begin{aligned}
H^3 &= -q^2(\mathscr{N}^1-\mathscr{N}^2) + \mathbb{D}_{\mathscr{A}^1}v^2(\mathscr{N}^1-\mathscr{N}^2) - \mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2(\mathscr{N}^1-\mathscr{N}^2) \\
&\quad +\zeta\mathscr{N}^1 + \zeta^2(\mathscr{N}^1-\mathscr{N}^2).
\end{aligned} \tag{2.2.200}
$$

The solutions are sufficiently regular for us to differentiate in time, which are the following:

$$
\begin{cases}
\partial_t(\partial_t v) + \nabla_{\mathcal{A}^1}\cdot\mathcal{S}_{\mathcal{A}^1}(\partial_t q,\partial_t v) = \tilde{H}^1 + \nabla_{\mathscr{A}^1}\cdot(\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2) & \text{in} \quad \Omega, \\
\nabla_{\mathcal{A}_1}\cdot\partial_t v = \tilde{H}^2 & \text{in} \quad \Omega, \\
\mathcal{S}_{\mathcal{A}_1}(\partial_t q,\partial_t v)\mathscr{N}^1 = \tilde{H}^3 + \mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2\mathscr{N}^1 & \text{on} \quad \Sigma, \\
\partial_t v = 0 & \text{on} \quad \Sigma_b,
\end{cases} \tag{2.2.201}
$$

where

$$
\begin{aligned}
\tilde{H}^1 &= \partial_t H^1 + \nabla_{\partial_t\mathscr{A}^1}\cdot(\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2) + \nabla_{\mathscr{A}^1}\cdot(\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}\partial_t v^2) + \nabla_{\partial_t\mathscr{A}^1}\cdot(\mathbb{D}_{\mathscr{A}^1}v) \\
&\quad +\nabla_{\mathscr{A}^1}(\mathbb{D}_{\partial_t\mathscr{A}^1}v) - \nabla_{\partial_t\mathscr{A}^1}q,
\end{aligned} \tag{2.2.202}
$$

$$
\tilde{H}^2 = \partial_t H^2 - \nabla_{\partial_t\mathscr{A}^1}\cdot v, \tag{2.2.203}
$$

$$
\tilde{H}^3 = \partial_t H^3 + \mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}\partial_t v^2\mathscr{N}^1 + \mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2\partial_t\mathscr{N}^1 - \mathcal{S}_{\mathscr{A}^1}(q,v)\partial_t\mathscr{N}^1 + \mathbb{D}_{\partial_t\mathscr{A}^1}v\mathscr{N}^1. \tag{2.2.204}
$$

Step 2: Energy evolution for $\partial_t v$.

Multiply $J^1\partial_t v$ on both sides to get the natural energy structure:

$$\begin{aligned} & \frac{1}{2}\int_\Omega |\partial_t v|^2 J^1 + \frac{1}{2}\int_0^t\int_\Omega |\mathbb{D}_{\mathscr{A}^1}\partial_t v|^2 J^1 \\ = & \int_0^t\int_\Omega J^1\tilde{H}^1\cdot\partial_t v + \int_0^t\int_\Omega J^1\tilde{H}^2\partial_t q - \int_0^t\int_\Sigma \tilde{H}^3\cdot\partial_t v \\ + & \frac{1}{2}\int_0^t\int_\Omega |\partial_t v|^2 \partial_t J^1 - \frac{1}{2}\int_0^t\int_\Omega J^1\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2 : \mathbb{D}_{\mathscr{A}^1}\partial_t v. \end{aligned} \tag{2.2.205}$$

LHS is simply the energy and dissipation term, so we focus on estimate of RHS.

$$\begin{aligned} \int_0^t\int_\Omega J^1\tilde{H}^1\cdot\partial_t v &\lesssim \mathcal{Z}\int_0^t \left\|\tilde{H}^1\right\|_{H^0}\|\partial_t v\|_{H^0} \lesssim \mathcal{Z}\|\partial_t v\|_{L^\infty H^0}\sqrt{T}\left\|\tilde{H}^1\right\|_{L^2H^0} \\ &\lesssim \sqrt{T}\mathcal{Z}\sqrt{\mathfrak{N}^v}\left\|\tilde{H}^1\right\|_{L^2H^0}, \end{aligned} \tag{2.2.206}$$

$$\begin{aligned} \int_0^t\int_\Omega J^1\tilde{H}^2\partial_t q &\lesssim \int_\Omega J^1 q\tilde{H}^2 - \int_0^t\int_\Omega(\partial_t J^1 q\tilde{H}^2 + J^1 q\partial_t\tilde{H}^2) \\ &\lesssim \int_\Omega J^1 q\tilde{H}^2 + \|q\|_{L^\infty H^0}\sqrt{T}\mathcal{Z}\left\|\tilde{H}^2\right\|_{L^2H^0} + \|q\|_{L^\infty H^0}\sqrt{T}\mathcal{Z}\left\|\partial_t\tilde{H}^2\right\|_{L^2H^0}, \\ &\lesssim \int_\Omega J^1 q\tilde{H}^2 + \sqrt{T}\mathcal{Z}\sqrt{\mathfrak{N}^v}\left(\left\|\tilde{H}^2\right\|_{L^2H^0} + \left\|\partial_t\tilde{H}^2\right\|_{L^2H^0}\right) \end{aligned} \tag{2.2.207}$$

$$\begin{aligned} \int_0^t\int_\Sigma \tilde{H}^3\cdot\partial_t v &\lesssim \int_0^t\left\|\tilde{H}^3\right\|_{H^{-1/2}(\Sigma)}\|\partial_t v\|_{H^{1/2}(\Sigma)} \lesssim \sqrt{T}\left\|\tilde{H}^3\right\|_{L^\infty H^{-1/2}(\Sigma)}\|\partial_t v\|_{L^2H^{1/2}(\Sigma)} \\ &\lesssim \sqrt{T}\sqrt{\mathfrak{N}^v}\left\|\tilde{H}^3\right\|_{L^\infty H^{-1/2}(\Sigma)}, \end{aligned} \tag{2.2.208}$$

$$\begin{aligned} \int_0^t\int_\Omega |\partial_t v|^2\partial_t J^1 &\lesssim \left\|\partial_t J^1\right\|_{L^\infty H^2} T\|\partial_t v\|^2_{L^\infty H^0} \\ &\lesssim T\mathcal{Z}\mathfrak{N}^v. \end{aligned} \tag{2.2.209}$$

So we need several estimates on $\tilde{H}^i$. For the following terms, it suffices to show they are bounded. We use the usual way to estimate these quadratic terms. Since we have repeatedly used this method, we will not give the details here.

$$\begin{aligned}\left\|\tilde{H}^1\right\|_{L^2H^0} &\lesssim \mathcal{Z}\bigg(\|\zeta\|_{L^2H^{3/2}} + \|\partial_t\zeta\|_{L^2H^{1/2}} + \left\|\partial_t^2\zeta\right\|_{L^2H^{-1/2}} + \|w\|_{L^2H^1} + \|\partial_t w\|_{L^2H^1} \\ &\quad + \|v\|_{L^2H^2} + \|q\|_{L^2H^1} \bigg) \\ &\lesssim \mathcal{Z}\bigg(\sqrt{\mathfrak{N}^v} + \sqrt{\mathfrak{N}^w} + \sqrt{\mathfrak{M}}\bigg),\end{aligned} \tag{2.2.210}$$

$$\left\|\tilde{H}^2\right\|_{L^2H^0} \lesssim \mathcal{Z}\bigg(\|\partial_t\zeta\|_{L^2H^{1/2}} + \|\zeta\|_{L^2H^{1/2}} + \|v\|_{L^2H^1} \bigg) \lesssim \mathcal{Z}\bigg(\sqrt{\mathfrak{N}^v} + \sqrt{\mathfrak{M}}\bigg), \tag{2.2.211}$$

$$\begin{aligned}\left\|\partial_t\tilde{H}^2\right\|_{L^2H^0} &\lesssim \mathcal{Z}\bigg(\left\|\partial_t^2\zeta\right\|_{L^2H^{1/2}} + \|\partial_t\zeta\|_{L^2H^{1/2}} + \|\zeta\|_{L^2H^{1/2}} + \|\partial_t v\|_{L^2H^1} + \|v\|_{L^2H^1} \bigg) \\ &\lesssim \mathcal{Z}\bigg(\sqrt{\mathfrak{N}^v} + \sqrt{\mathfrak{M}}\bigg),\end{aligned} \tag{2.2.212}$$

$$\begin{aligned}\left\|\tilde{H}^3\right\|_{L^\infty H^{-1/2}(\Sigma)} &\lesssim \mathcal{Z}\bigg(\|\zeta\|_{L^\infty H^{1/2}} + \|\partial_t\zeta\|_{L^\infty H^{1/2}} + \|q\|_{L^\infty H^1} + \|v\|_{L^\infty H^2} \bigg) \\ &\lesssim \mathcal{Z}\bigg(\sqrt{\mathfrak{N}^v} + \sqrt{\mathfrak{M}}\bigg).\end{aligned} \tag{2.2.213}$$

Also we have the following estimate.

$$
\begin{aligned}
\int_\Omega J^1 q\tilde{H}^2 &\lesssim \mathcal{Z}\,\|q\|_{L^\infty H^0}\left\|\tilde{H}^2\right\|_{L^\infty H^0} \\
&\lesssim \mathcal{Z}\,\|q\|_{L^\infty H^0}\bigg(\|\zeta\|_{L^\infty H^{1/2}}+\|\partial_t\zeta\|_{L^\infty H^{1/2}}+\|v\|_{L^\infty H^1}\bigg) \\
&\lesssim \mathcal{Z}\sqrt{\mathfrak{N}^v}\bigg(\|\zeta\|_{L^\infty H^{1/2}}+\|\partial_t\zeta\|_{L^\infty H^{1/2}}+\|v\|_{L^\infty H^1}\bigg).
\end{aligned}
\tag{2.2.214}
$$

Since ζ satisfies the equation

$$
\begin{cases}
\partial_t\zeta + w_1^1\partial_1\zeta + w_2^1\partial_2\zeta = -\mathscr{N}^2\cdot w, \\
\zeta(0)=0,
\end{cases}
$$

employing transport estimate and the boundedness of higher order norms, we have

$$
\|\zeta\|_{L^\infty H^{5/2}} \lesssim \sqrt{T}\mathcal{Z}\,\|w\|_{L^2H^{5/2}} \lesssim \sqrt{T}\mathcal{Z}\sqrt{\mathfrak{N}^w}. \tag{2.2.215}
$$

$\partial_t\zeta$ satisfies the equation

$$
\begin{cases}
\partial_t(\partial_t\zeta)+w_1^1\partial_1(\partial_t\zeta)+w_2^1\partial_2(\partial_t\zeta) = -\mathscr{N}^2\cdot\partial_t w-\partial_t\mathscr{N}^2\cdot w-\partial_t w_1^1\partial_1\zeta-\partial_t w_2^1\partial_2\zeta, \\
\partial_t\zeta(0)=0.
\end{cases}
\tag{2.2.216}
$$

Similar argument as above shows that

$$
\|\partial_t\zeta\|_{L^\infty H^{1/2}} \lesssim \sqrt{T}\mathcal{Z}\bigg(\|\partial_t w\|_{L^2H^{1/2}}+\|w\|_{L^2H^{1/2}}\bigg) \lesssim \sqrt{T}\mathcal{Z}\sqrt{\mathfrak{N}^w}. \tag{2.2.217}
$$

Combining above transport estimate and lemma A.1.14, we have the final version

$$\begin{aligned}\int_\Omega J^1 q\tilde{H}^2 &\lesssim \sqrt{T}\mathcal{Z}\sqrt{\mathfrak{N}^v}\sqrt{\mathfrak{N}^w} + \mathcal{Z}\sqrt{\mathfrak{N}^v}\left\|v\right\|_{L^\infty H^1} \\ &\lesssim \sqrt{T}\mathcal{Z}\sqrt{\mathfrak{N}^v}\sqrt{\mathfrak{N}^w} + \mathcal{Z}\sqrt{\mathfrak{N}^v}\Big(\left\|v\right\|_{L^2 H^2} + \left\|\partial_t v\right\|_{L^2 H^0}\Big) \\ &\lesssim \sqrt{T}\mathcal{Z}\sqrt{\mathfrak{N}^v}\sqrt{\mathfrak{N}^w} + \sqrt{T}\mathcal{Z}\sqrt{\mathfrak{N}^v}\Big(\left\|v\right\|_{L^\infty H^2} + \left\|\partial_t v\right\|_{L^\infty H^0}\Big) \\ &\lesssim \sqrt{T}\mathcal{Z}\Big(\sqrt{\mathfrak{N}^v}\sqrt{\mathfrak{N}^w} + \mathfrak{N}^v\Big).\end{aligned} \tag{2.2.218}$$

Then we consider simplify the last term in RHS of energy structure:

$$\int_0^t\int_\Omega J^1\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2 : \mathbb{D}_{\mathscr{A}^1}\partial_t v \lesssim \frac{1}{4\epsilon}\left\|J^1\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2\right\|_{L^2H^0}^2 + \epsilon\left\|\mathbb{D}_{\mathscr{A}^1}\partial_t v\right\|_{L^2H^0}^2, \tag{2.2.219}$$

where ϵ should be determined in order for the second term in RHS to be absorbed in LHS.

$$\left\|J^1\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2\right\|_{L^2H^0}^2 \lesssim \sqrt{T}\left\|J^1\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2\right\|_{L^\infty H^0}^2 \lesssim \sqrt{T}\mathcal{Z}\mathfrak{M}. \tag{2.2.220}$$

To summarize, using Cauchy inequality, we have

$$\left\|\partial_t v\right\|_{L^\infty H^0}^2 + \left\|\partial_t v\right\|_{L^2H^1}^2 \lesssim \sqrt{T}\mathcal{Z}(\mathfrak{N}^w + \mathfrak{N}^v + \mathfrak{M}). \tag{2.2.221}$$

Step 3: Elliptic estimate for v.

Based on standard elliptic regularity theory, we have the estimate

$$\begin{aligned}\left\|v\right\|_{H^{r+2}}^2 + \left\|q\right\|_{H^{r+1}}^2 &\lesssim \left\|\partial_t v\right\|_{H^r}^2 + \left\|H^1\right\|_{H^r}^2 + \left\|H^2\right\|_{H^{r+1}}^2 + \left\|H^3\right\|_{H^{r+\frac{1}{2}}(\Sigma)}^2 \\ &\quad + \left\|\nabla_{\mathscr{A}^1}\cdot(\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2)\right\|_{H^r}^2 + \left\|\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2\mathscr{N}^1\right\|_{H^{r+\frac{1}{2}}(\Sigma)}^2.\end{aligned} \tag{2.2.222}$$

Setting $r = 0$ and taking L^∞ on both sides implies

$$
\begin{aligned}
\|v\|_{L^\infty H^2}^2 + \|q\|_{L^\infty H^1}^2 \lesssim\; & \|\partial_t v\|_{L^\infty H^0}^2 + \left\|H^1\right\|_{L^\infty H^0}^2 + \left\|H^2\right\|_{L^\infty H^1}^2 + \left\|H^3\right\|_{L^\infty H^{\frac{1}{2}}(\Sigma)}^2 \\
& + \left\|\nabla_{\mathscr{A}^1} \cdot (\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} v^2)\right\|_{L^\infty H^0}^2 + \left\|\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} v^2 \mathscr{N}^1\right\|_{L^\infty H^{\frac{1}{2}}(\Sigma)}^2 .
\end{aligned} \tag{2.2.223}
$$

Setting $r = 1$ and taking L^2 on both sides yields

$$
\begin{aligned}
\|v\|_{L^2 H^3}^2 + \|q\|_{L^2 H^2}^2 \lesssim\; & \|\partial_t v\|_{L^2 H^1}^2 + \left\|H^1\right\|_{L^2 H^1}^2 + \left\|H^2\right\|_{L^2 H^2}^2 + \left\|H^3\right\|_{L^2 H^{\frac{3}{2}}(\Sigma)}^2 \\
& + \left\|\nabla_{\mathscr{A}^1} \cdot (\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} v^2)\right\|_{L^2 H^1}^2 + \left\|\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} v^2 \mathscr{N}^1\right\|_{L^2 H^{\frac{3}{2}}(\Sigma)}^2 .
\end{aligned} \tag{2.2.224}
$$

We need to estimate all the RHS terms. We can employ the usual way to estimate quadratic terms in $L^2 H^k$ norm, however, for $L^\infty H^k$ norm, we use lemma A.1.14. Then we have the estimate

$$
RHS \lesssim \|\partial_t v\|_{L^2 H^1}^2 + \|\partial_t v\|_{L^\infty H^0}^2 + T\mathcal{Z}\Bigg(\mathfrak{N}^w + \mathfrak{M}\Bigg). \tag{2.2.225}
$$

To summarize, we have

$$
\begin{aligned}
& \|v\|_{L^\infty H^2}^2 + \|q\|_{L^\infty H^1}^2 + \|v\|_{L^2 H^3}^2 + \|q\|_{L^2 H^2}^2 \\
\lesssim\; & \|\partial_t v\|_{L^2 H^1}^2 + \|\partial_t v\|_{L^\infty H^0}^2 + T\mathcal{Z}\Bigg(\mathfrak{N}^w + \mathfrak{M}\Bigg).
\end{aligned} \tag{2.2.226}
$$

In total of (2.2.221) and (2.2.226), we get the succinct form of estimate

$$
\mathfrak{N}^v \lesssim \sqrt{T}\mathcal{Z}(\mathfrak{N}^w + \mathfrak{N}^v + \mathfrak{M}). \tag{2.2.227}
$$

Step 4: Transport estimate for ζ.

ζ satisfies the equation

$$\begin{cases} \partial_t \zeta + w_1^1 \partial_1 \zeta + w_2^1 \partial_2 \zeta = -\mathscr{N}^2 \cdot w \\ \zeta(0) = 0 \end{cases} \tag{2.2.228}$$

A straightforward application of transport estimate can show

$$\begin{aligned} \|\zeta\|^2_{L^\infty H^{5/2}(\Sigma)} &\lesssim \exp\left(C \int_0^t \|w_1\|_{H^3}\right) \int_0^t \left\|\mathscr{N}^2 \cdot w\right\|_{H^{5/2}(\Sigma)} \\ &\lesssim \sqrt{T} \mathcal{Z} \|w\|^2_{L^2 H^{5/2}(\Sigma)} \lesssim \sqrt{T} \mathcal{Z} \mathfrak{N}^w. \end{aligned} \tag{2.2.229}$$

Similar to the proof of transport estimate in theorem 2.2.17, we can use the transport equation to estimate the higher order derivatives.

$$\|\partial_t \zeta\|^2_{L^\infty H^{3/2}(\Sigma)} \lesssim \left\|\zeta^2\right\|^2_{L^\infty H^{5/2}(\Sigma)} \|w\|^2_{L^\infty H^{3/2}(\Sigma)} + \|\zeta\|^2_{L^\infty H^{5/2}(\Sigma)} \|w_1\|^2_{L^\infty H^{3/2}(\Sigma)} \lesssim \mathcal{Z} \mathfrak{N}^w. \tag{2.2.230}$$

In the same fashion, we can easily show

$$\left\|\partial_t^2 \zeta\right\|^2_{L^2 H^{1/2}} \lesssim \mathcal{Z} \mathfrak{N}^w. \tag{2.2.231}$$

To sum up, we have the estimate

$$\mathfrak{M} \lesssim \mathfrak{N}^w. \tag{2.2.232}$$

Step 5: Synthesis:

In (2.2.227), for T sufficiently small, we can easily absorbed all $\mathfrak{N}^v$ term from RHS to LHS and replace all $\mathfrak{M}$ with $\mathfrak{N}^w$ to achieve

$$\mathfrak{N}^v \lesssim \sqrt{T}\mathcal{Z}\mathfrak{N}^w. \tag{2.2.233}$$

Certainly, the smallness of T can guarantee the contraction. □

Local Well-posedness Theorem

Now we can combine Theorem 2.2.23 and Theorem 2.2.24 to produce a unique strong solution to the system (2.1.12).

Theorem 2.2.25. *Assume (u^0, η^0) satisfies $\mathcal{K}_0 < \infty$. Then there exists a $0 < T_0 < 1$ such that for any $0 < T < T_0$, there exists a solution triple (u, p, η) to the system (2.1.12) on $[0, T]$ that satisfies the initial data and obeys*

$$\mathcal{K}(u, p) + \mathcal{K}(\eta) \leq CP(\mathcal{K}_0) \tag{2.2.234}$$

for a universal constant $C > 0$ and a polynomial $P(\cdot)$ satisfying $P(0) = 0$. The solution is unique among functions that satisfy the initial data and $\mathcal{K}(u, p) + \mathcal{K}(\eta) < \infty$. Moreover, η is such that the mapping Φ is a C^{2N-1} diffeomorphism for each $t \in [0, T]$.

Proof. See [26, Theorem 6.3]. □

2.3 Global Well-posedness for Horizontally Infinite Domain

2.3.1 Preliminaries

In this section, we present two forms of system (2.1.12) and describe the corresponding energy structures. More details can be found in [25, Section 2].

Geometric Structure Form

Suppose η, u are known and $\mathscr{A}$, $\mathscr{N}$, etc. are given in terms of η as usual. Then we consider the linear equations for (v, q, ξ)

$$\begin{cases} \partial_t v - \partial_t \bar{\eta}\tilde{b}K\partial_3 v + u\cdot\nabla_{\mathscr{A}} v + \nabla_{\mathscr{A}}\cdot S_{\mathscr{A}}(q,v) = F^1 & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}}\cdot v = F^2 & \text{in} \quad \Omega, \\ S_{\mathscr{A}}(q,v)\mathscr{N} = \xi\mathscr{N} + F^3 & \text{on} \quad \Sigma, \\ \partial_t \xi - v\cdot\mathscr{N} = F^4 & \text{on} \quad \Sigma, \\ v = 0 & \text{on} \quad \Sigma_b. \end{cases} \tag{2.3.1}$$

Lemma 2.3.1. *Suppose (u, p, η) satisfies the system (2.1.12) and (v, q, ξ) is the solution of the system (2.3.1). Then*

$$\begin{aligned} &\partial_t\left(\frac{1}{2}\int_\Omega J\left|v\right|^2 + \frac{1}{2}\int_\Sigma \left|\xi\right|^2\right) + \frac{1}{2}\int_\Omega J\left|\mathbb{D}_{\mathscr{A}} v\right|^2 \\ &\quad = \int_\Omega J(v\cdot F^2 + qF^2) + \int_\Sigma -v\cdot F^3 + \xi F^4. \end{aligned} \tag{2.3.2}$$

The geometric structure form is utilized to estimate the temporal derivatives.

Hence, we may apply differential operator $\partial^\alpha = \partial_t^{\alpha_0}$ to the system (2.1.12) with the resulting equations being (2.3.1) for $v = \partial^\alpha u$, $q = \partial^\alpha p$ and $\xi = \partial^\alpha \eta$, where

$$
\begin{aligned}
F^1 &= F^{1,1} + F^{1,2} + F^{1,3} + F^{1,4} + F^{1,5} + F^{1,6}, && (2.3.3)\\
F_i^{1,1} &= \sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta (\partial_t \bar{\eta} \tilde{b} K) \partial^{\alpha-\beta} \partial_3 u_i + \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} \partial_t \bar{\eta} \partial^\beta (\tilde{b} K) \partial_3 u_i, && (2.3.4)\\
F_i^{1,2} &= -\sum_{0<\beta\le\alpha} C_{\alpha,\beta} \left(\partial^\beta (u_j \mathscr{A}_{jk}) \partial^{\alpha-\beta} \partial_k u_i + \partial^\beta \mathscr{A}_{jk} \partial^{\alpha-\beta} \partial_k p \right), && (2.3.5)\\
F_i^{1,3} &= \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \partial^\beta \mathscr{A}_{jl} \partial^{\alpha-\beta} \partial_l (\mathscr{A}_{im} \partial_m u_j + \mathscr{A}_{jm} \partial_m u_i), && (2.3.6)\\
F_i^{1,4} &= \sum_{0<\beta<\alpha} C_{\alpha,\beta} \mathscr{A}_{jk} \partial_k (\partial^\beta \mathscr{A}_{il} \partial^{\alpha-\beta} \partial_l u_j + \partial^\beta \mathscr{A}_{jl} \partial^{\alpha-\beta} \partial_l u_i), && (2.3.7)\\
F_i^{1,5} &= \partial^\alpha \partial_t \bar{\eta} \tilde{b} K \partial_3 u_i, && (2.3.8)\\
F_i^{1,6} &= \mathscr{A}_{jk} \partial_k (\partial^\alpha \mathscr{A}_{il} \partial_l u_j + \partial^\alpha \mathscr{A}_{jl} \partial_l u_i), && (2.3.9)
\end{aligned}
$$

$$
\begin{aligned}
F^2 &= F^{2,1} + F^{2,2}, && (2.3.10)\\
F^{2,1} &= -\sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta \mathscr{A}_{ij} \partial^{\alpha-\beta} \partial_j u_i, && (2.3.11)\\
F^{2,2} &= -\partial^\alpha \mathscr{A}_{ij} \partial_j u_i, && (2.3.12)
\end{aligned}
$$

$$
\begin{aligned}
F^3 &= F^{3,1} + F^{3,2}, && (2.3.13)\\
F^{3,1} &= \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \partial^\beta D\eta (\partial^{\alpha-\beta} \eta - \partial^{\alpha-\beta} p), && (2.3.14)\\
F^{3,2} &= \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \left(\partial^\beta (\mathscr{N}_j \mathscr{A}_{im}) \partial^{\alpha-\beta} \partial_m u_j + \partial^\beta (\mathscr{N}_j \mathscr{A}_{jm}) \partial^{\alpha-\beta} \partial_m u_i \right), && (2.3.15)
\end{aligned}
$$

$$
F^4 = \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \partial^\beta D\eta \cdot \partial^{\alpha-\beta} u. \qquad (2.3.16)
$$

In all above, $C_{\alpha,\beta}$ represents constants depending on α and β.

Perturbed Linear Form

We may also take the system (2.1.12) as a perturbation of the linear system

$$\begin{cases} \partial_t u - \Delta u + \nabla p = G^1 & \text{in} \quad \Omega, \\ \nabla \cdot u = G^2 & \text{in} \quad \Omega, \\ (pI - \mathbb{D}u - \eta I)e_3 = G^3 & \text{on} \quad \Sigma, \\ \partial_t \eta - u_3 = G^4 & \text{on} \quad \Sigma, \\ u = 0 & \text{on} \quad \Sigma_b. \end{cases} \tag{2.3.17}$$

Lemma 2.3.2. *Suppose* (u, p, η) *satisfies the system (2.3.17). Then*

$$\partial_t \left(\frac{1}{2} \int_\Omega |u|^2 + \frac{1}{2} \int_\Sigma |\eta|^2 \right) + \frac{1}{2} \int_\Omega J \, |\mathbb{D}u|^2 = \int_\Omega u \cdot G^2 + pG^2 + \int_\Sigma -u \cdot G^3 + \eta G^4. \tag{2.3.18}$$

Compared to the system (2.1.12), we can write the detailed forms of the perturbation terms.

$$\begin{aligned} G^1 &= G^{1,1} + G^{1,2} + G^{1,3} + G^{1,4} + G^{1,5}, && (2.3.19) \\ G_i^{1,1} &= (\delta_{ij} - \mathscr{A}_{ij}) \partial_j p, && (2.3.20) \\ G_i^{1,2} &= u_j \mathscr{A}_{jk} \partial_k u_i, && (2.3.21) \\ G_i^{1,3} &= [K^2(1 + A^2 + B^2) - 1] \partial_{33} u_i - 2AK \partial_{13} u_i - 2BK \partial_{23} u_i, && (2.3.22) \\ G_i^{1,4} &= [-K^3(1 + A^2 + B^2) \partial_3 J + AK^2(\partial_1 J + \partial_3 A) && (2.3.23) \\ &\quad + BK^2(\partial_2 J + \partial_3 B) - K(\partial_1 A + \partial_2 B)] \partial_3 u_i, \\ G_i^{1,5} &= \partial_t \bar{\eta}(1 + x_3/b) K \partial_3 u_3, && (2.3.24) \end{aligned}$$

$$G^2 = (1 - J)(\partial_1 u_1 + \partial_2 u_2) + A \partial_2 u_1 + B \partial_3 u_2, \tag{2.3.25}$$

$$G^3 = \partial_1\eta\begin{pmatrix} p-\eta-2(\partial_1 u_1 - AK\partial_3 u_1) \\ -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ -\partial_1 u_3 - K\partial_3 u_1 + AK\partial_3 u_3 \end{pmatrix} + \partial_2\eta\begin{pmatrix} -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ p-\eta-2(\partial_2 u_2 - BK\partial_3 u_2) \\ -\partial_2 u_3 - K\partial_3 u_2 + BK\partial_3 u_3 \end{pmatrix} + \begin{pmatrix} (K-1)\partial_3 u_1 + AK\partial_3 u_3 \\ (K-1)\partial_3 u_2 + BK\partial_3 u_3 \\ 2(K-1)\partial_3 u_3 \end{pmatrix}, \tag{2.3.26}$$

where based on the equations we have

$$\begin{aligned} p-\eta &= \partial_1\eta(-\partial_1 u_3 - K\partial_3 u_1 + AK\partial_3 u_3) \\ &\quad +\partial_2\eta(-\partial_2 u_3 - K\partial_3 u_2 + BK\partial_3 u_3) + 2K\partial_3 u_3, \end{aligned} \tag{2.3.27}$$

$$G^4 = -D\eta\cdot u. \tag{2.3.28}$$

2.3.2 Definitions of Energy and Dissipation

In this section we give the definitions of energy and dissipation with or without minimal counts. Compared to those in [25, (2.46) - (2.55)], the main differences here lie in the dissipation definitions, where we consider at most two vertical derivatives for velocity and one vertical derivative for pressure. It is this key improvement that significantly simplifies the whole proof.

We define the energy and dissipation with 2-minimal count as follows:

$$\bar{\mathcal{E}}_{n,2} = \left\|\bar{D}_2^{2n-1}\right\|_{H^0}^2 + \left\|D\bar{D}^{2n-1}\right\|_{H^0}^2 + \left\|\sqrt{J}\partial_t^n u\right\|_{H^0}^2 + \left\|\bar{D}_2^{2n}\eta\right\|_{H^0(\Sigma)}^2, \tag{2.3.29}$$

$$\bar{\mathcal{D}}_{n,2} = \left\|\bar{D}_2^{2n}\mathbb{D}u\right\|_{H^0}^2. \tag{2.3.30}$$

$$\begin{aligned}\mathcal{E}_{n,2} = {} & \left\|D^2 u\right\|_{H^{2n-2}}^2 + \sum_{j=1}^{n}\left\|\partial_t^j u\right\|_{H^{2n-2j}}^2 + \left\|D^2 p\right\|_{H^{2n-3}}^2 + \sum_{j=1}^{n-1}\left\|\partial_t^j p\right\|_{H^{2n-2j-1}}^2 \\ & + \left\|D^2\eta\right\|_{H^{2n-2}(\Sigma)}^2 + \sum_{j=1}^{n}\left\|\partial_t^j\eta\right\|_{H^{2n-2j}(\Sigma)}^2,\end{aligned} \tag{2.3.31}$$

$$\begin{aligned}\mathcal{D}_{n,2} = {} & \left\|\bar{D}_2^{2n-1}u\right\|_{H^2}^2 + \left\|\nabla\bar{D}_2^{2n-1}p\right\|_{H^0}^2 \\ & + \left\|D_3^{2n-1}\eta\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\partial_t D_1^{2n-1}\eta\right\|_{H^{1/2}(\Sigma)}^2 + \sum_{j=2}^{n+1}\left\|\partial_t^j D_0^{2n-2j+2}\eta\right\|_{H^{1/2}(\Sigma)}^2.\end{aligned} \tag{2.3.32}$$

We define the energy and dissipation without minimal count as follows, where $I_\lambda u$ and $I_\lambda\eta$ denote the Riesz potential of u and η as defined in [25, (A.7) - (A.8)]:

$$\bar{\mathcal{E}}_n = \|I_\lambda u\|_{H^0}^2 + \|I_\lambda\eta\|_{H^0(\Sigma)}^2 + \left\|\bar{D}_0^{2n}u\right\|_{H^0}^2 + \left\|\bar{D}_0^{2n}\eta\right\|_{H^0(\Sigma)}^2, \tag{2.3.33}$$

$$\bar{\mathcal{D}}_n = \|\mathbb{D}I_\lambda u\|_{H^0}^2 + \left\|\bar{D}_0^{2n}\mathbb{D}u\right\|_{H^0}^2. \tag{2.3.34}$$

$$\begin{aligned}\mathcal{E}_n &= \|I_\lambda u\|_{H^0}^2 + \|I_\lambda \eta\|_{H^0(\Sigma)}^2 + \sum_{j=0}^{n} \left\|\partial_t^j u\right\|_{H^{2n-2j}}^2 + \sum_{j=0}^{n-1} \left\|\partial_t^j p\right\|_{H^{2n-2j-1}}^2 \\ &\quad + \sum_{j=0}^{n} \left\|\partial_t^j \eta\right\|_{H^{2n-2j}(\Sigma)}^2 ,\end{aligned} \tag{2.3.35}$$

$$\begin{aligned}\mathcal{D}_n &= \|I_\lambda u\|_{H^1}^2 + \left\|\bar{D}_0^{2n-1} u\right\|_{H^2}^2 + \left\|\nabla \bar{D}_0^{2n-1} p\right\|_{H^0}^2 \\ &\quad + \left\|D_0^{2n-1}\eta\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\partial_t D_0^{2n-2}\eta\right\|_{H^{1/2}(\Sigma)}^2 + \sum_{j=2}^{n+1} \left\|\partial_t^j D_0^{2n-2j+1}\eta\right\|_{H^{1/2}(\Sigma)}^2 .\end{aligned} \tag{2.3.36}$$

Finally, we define the energy with 1-minimal count as follows:

$$\begin{aligned}\mathcal{E}_{n,1} &= \|Du\|_{H^{2n-1}}^2 + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|_{H^{2n-2j}}^2 + \|Dp\|_{H^{2n-2}}^2 + \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|_{H^{2n-2j-1}}^2 \\ &\quad + \|D\eta\|_{H^{2n-1}(\Sigma)}^2 + \sum_{j=1}^{n} \left\|\partial_t^j \eta\right\|_{H^{2n-2j}(\Sigma)}^2 .\end{aligned} \tag{2.3.37}$$

Since we only consider the decay of energy and utilize the interpolation relation between $\mathcal{E}_{n,2}$ and $\mathcal{E}_{n,1}$ to estimate it, it is not necessary to define the dissipation with 1-minimum count and other horizontal quantities.

Some other useful quantities are as follows:

$$\mathcal{K} = \|\nabla u\|_{L^\infty}^2 + \left\|\nabla^2 u\right\|_{L^\infty}^2 + \sum_{i=1}^{2} \|Du_i\|_{H^2(\Sigma)}^2 . \tag{2.3.38}$$

$$\mathcal{F}_{2N} = \|\eta\|_{H^{4N+1/2}(\Sigma)}^2 . \tag{2.3.39}$$

The total energy is defined as follows:

$$\mathcal{G}_N(t) = \sup_{0\le r\le t} \mathcal{E}_{2N}(r) + \sum_{m=1}^{2} \sup_{0\le r\le t} (1+r)^{m+\lambda} \mathcal{E}_{N+2,m}(r) + \sup_{0\le r\le t} \frac{\mathcal{F}_{2N}(r)}{(1+r)}. \tag{2.3.40}$$

Note that our definition of total energy is different from [25, (2.58)], since we do not include the integral of dissipation and merely concentrate on the instantaneous quantities.

2.3.3 Interpolation Estimates

In the following, we provide several interpolation estimates in the form of

$$\|X\|^2 \lesssim (\mathcal{E}_{N+2,2})^\theta (\mathcal{E}_{2N})^{1-\theta}, \tag{2.3.41}$$

$$\|X\|^2 \lesssim (\mathcal{D}_{N+2,2})^\theta (\mathcal{E}_{2N})^{1-\theta}, \tag{2.3.42}$$

where $\theta \in [0,1]$, X is some quantity, and $\|\cdot\|$ is some norm.

For brevity, we employ the notation in [25] to write the interpolation power in the tables below. For example, the following part of the table

L^2	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
X	θ	κ

(2.3.43)

should be understood as

$$\|X\|_{L^2}^2 \lesssim (\mathcal{E}_{N+2,2})^\theta (\mathcal{E}_{2N})^{1-\theta}, \tag{2.3.44}$$

$$\|X\|_{L^2}^2 \lesssim (\mathcal{D}_{N+2,2})^\kappa (\mathcal{E}_{2N})^{1-\kappa}. \tag{2.3.45}$$

When we write $\mathcal{E}_{N+2,2} \sim \mathcal{D}_{N+2,2}$ in a table, it means θ is the same when interpolating between $\mathcal{E}_{N+2,2}$ and $\mathcal{E}_{2N}$, and between $\mathcal{D}_{N+2,2}$ and $\mathcal{E}_{2N}$. When we write multiple entries, it means the same interpolation estimates hold for each item listed. Sometimes, an interpolation index r appears. Then we can always take arbitrary $r \in (0,1)$. More detailed properties of these tables can be found in [25, pp.20].

Lemma 2.3.3. *The following tables encode the power in the L^∞ or L^2 interpolation estimates for u with its derivatives either in Ω or Σ.*

L^∞	$\mathcal{E}_{N+2,2} \sim \mathcal{D}_{N+2,2}$
u	$1/2$
Du	$2/(2+r)$
∇u	$1/2$
$D\nabla u$	$2/(2+r)$
$\nabla^2 u$	$1/2$
$D\nabla^2 u$	$2/(2+r)$

L^2	$\mathcal{E}_{N+2,2} \sim \mathcal{D}_{N+2,2}$
u	$\lambda/(\lambda+2)$
Du	$(\lambda+1)/(\lambda+2)$
∇u	$\lambda/(\lambda+2)$
$D\nabla u$	$(\lambda+1)/(\lambda+2)$
$\nabla^2 u$	$\lambda/(\lambda+2)$
$D\nabla^2 u$	$(\lambda+1)/(\lambda+2)$

(2.3.46)

The following tables encode the power in the L^∞ or L^2 interpolation estimates for η and $\bar{\eta}$ with their derivatives either in Ω or Σ.

L^∞	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\eta, \bar{\eta}$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D\eta, \nabla\bar{\eta}$	$(\lambda+2)/(\lambda+2+r)$	$(\lambda+2)/(\lambda+3)$
$D^2\eta, \nabla^2\bar{\eta}$	1	$(\lambda+3)/(\lambda+3+r)$
$\partial_t\eta, \partial_t\bar{\eta}$	1	$2/(2+r)$

(2.3.47)

L^2	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\eta, \bar{\eta}$	$\lambda/(\lambda+2)$	$\lambda/(\lambda+3)$
$D\eta, \nabla\bar{\eta}$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D^2\eta, \nabla^2\bar{\eta}$	1	$(\lambda+2)/(\lambda+3)$
$\partial_t\eta, \partial_t\bar{\eta}$	1	$1/2$

(2.3.48)

The following tables encode the power in the L^∞ or L^2 interpolation estimates for p

with its derivatives either in Ω *or* Σ.

L^∞	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
∇p	$1/2$	$1/2$
Dp	$2/(2+r)$	$2/3$
$D\nabla p$	$2/(2+r)$	$2/(2+r)$
$D^2 p$	1	$2/(2+r)$

L^2	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
∇p	0	0
Dp	$1/2$	$1/3$
$D\nabla p$	$1/2$	$1/2$
$D^2 p$	1	$2/3$

(2.3.49)

The following tables encode the power in the L^∞ *or* L^2 *interpolation estimates for* G^i,.

L^∞	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^1	1	$(3\lambda+5)/(2\lambda+6)$
DG^1	1	1
G^2	1	1
DG^2	1	1
∇G^2	1	1
G^3	1	$(3\lambda+5)/(2\lambda+6)$
DG^3	1	1
G^4	1	1

(2.3.50)

L^2	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^1	$(3\lambda+2)/(2\lambda+4))$	$(3\lambda+3)/(2\lambda+6)$
DG^1	1	$(3\lambda+5)/(2\lambda+6)$
G^2	1	m_2
DG^2	1	1
∇G^2	1	m_2
G^3	$(2\lambda+1)/(\lambda+2)$	m_1
DG^3	1	m_2
D^2G^3	1	1
G^4	1	m_2

(2.3.51)

for

$$m_1 = \min\{(2\lambda^2+6\lambda+2)/(\lambda^2+5\lambda+6), (3\lambda+3)/(2\lambda+6)\}, \tag{2.3.52}$$

$$m_2 = \min\{(2\lambda^2+7\lambda+4)/(\lambda^2+5\lambda+6), (3\lambda+5)/(2\lambda+6)\}. \tag{2.3.53}$$

Proof. The estimates follow directly from the Sobolev embedding theorem and [25, Lemma A.6 - Lemma A.8], using the bounds $\|I_\lambda \eta\|_{H^0}^2 \lesssim \mathcal{E}_{2N}$ and $\|I_\lambda \partial_t \eta\|_{H^0}^2 \lesssim \mathcal{E}_{2N}$, which can be directly deduced from [25, Lemma 2.8]. For G^i, we may utilize the product rules

$$\|XY\|_{L^\infty}^2 \lesssim \|X\|_{L^\infty}^2 \|Y\|_{L^\infty}^2, \tag{2.3.54}$$

$$\|XY\|_{L^2}^2 \lesssim \|X\|_{L^\infty}^2 \|Y\|_{L^2}^2, \tag{2.3.55}$$

$$\|XY\|_{L^2}^2 \lesssim \|X\|_{L^2}^2 \|Y\|_{L^\infty}^2, \tag{2.3.56}$$

where in the L^2 bound, we may take the larger value of θ produced by the two options. □

Then we can show several important lemmas to improve above estimates.

Lemma 2.3.4. *We have the estimate*

$$\mathcal{K} \lesssim \mathcal{E}_{2N}^{r/(2+r)} \mathcal{E}_{N+2}^{2/(2+r)}. \tag{2.3.57}$$

Proof.

$$\mathcal{K} = \|\nabla u\|_{L^\infty}^2 + \left\|\nabla^2 u\right\|_{L^\infty}^2 + \sum_{i=1}^{2} \|Du_i\|_{H^2(\Sigma)}^2 . \tag{2.3.58}$$

We need to estimate each term above.

Step 1: $\|\nabla u\|_{L^\infty}^2$.

By definition, we have

$$\|\nabla u\|_{L^\infty}^2 \lesssim \|Du\|_{L^\infty}^2 + \|\partial_3 u\|_{L^\infty}^2 . \tag{2.3.59}$$

By the divergence equation in (2.3.17), we have

$$\|\partial_3 u_3\|_{L^\infty}^2 \lesssim \|Du\|_{L^\infty}^2 + \left\|G^2\right\|_{L^\infty}^2 . \tag{2.3.60}$$

For $i = 1, 2$, by the Poincaré inequality we can naturally estimate

$$\|\partial_3 u_i\|_{L^\infty}^2 \lesssim \left\|\partial_3^2 u_i\right\|_{L^\infty}^2 + \|\partial_3 u_i\|_{L^\infty(\Sigma)}^2 . \tag{2.3.61}$$

The upper boundary condition in (2.3.17) implies

$$\|\partial_3 u_i\|_{L^\infty(\Sigma)}^2 \lesssim \|Du_3\|_{L^\infty(\Sigma)}^2 + \left\|G^3\right\|_{L^\infty(\Sigma)}^2 \lesssim \|D\nabla u_3\|_{L^\infty}^2 + \left\|G^3\right\|_{L^\infty(\Sigma)}^2 . \tag{2.3.62}$$

Considering the Navier-Stokes equations in (2.3.17), we have

$$\left\|\partial_3^2 u_i\right\|_{L^\infty}^2 \lesssim \|\partial_t u_i\|_{L^\infty}^2 + \left\|D^2 u_i\right\|_{L^\infty}^2 + \|Dp\|_{L^\infty}^2 + \left\|G^1\right\|_{L^\infty}^2 . \tag{2.3.63}$$

Therefore, we obtain

$$\begin{aligned}\|\nabla u\|_{L^\infty}^2 \lesssim{}& \|Du\|_{L^\infty}^2 + \left\|G^2\right\|_{L^\infty}^2 + \|D\nabla u_3\|_{L^\infty}^2 + \left\|G^3\right\|_{L^\infty(\Sigma)}^2 + \|\partial_t u_i\|_{L^\infty}^2 + \left\|D^2 u_i\right\|_{L^\infty}^2 \\ &+ \left\|G^1\right\|_{L^\infty}^2 + \|Dp\|_{L^\infty}^2 .\end{aligned} \tag{2.3.64}$$

The estimate of each term above with Lemma 2.3.3 satisfies our requirement.

Step 2: $\|\nabla^2 u\|_{L^\infty}^2$.
By definition, we have

$$\left\|\nabla^2 u\right\|_{L^\infty}^2 \lesssim \|D\nabla u\|_{L^\infty}^2 + \left\|\partial_3^2 u_i\right\|_{L^\infty}^2 + \left\|\partial_3^2 u_3\right\|_{L^\infty}^2 . \tag{2.3.65}$$

The first term naturally satisfies our requirement while the second term has been successfully estimated in the first step above. Hence, we only need to estimate the third one. Taking ∂_3 in the divergence equation of (2.3.17) reveals

$$\left\|\partial_3^2 u_3\right\|_{L^\infty}^2 \lesssim \|D\nabla u_i\|_{L^\infty}^2 + \left\|\nabla G^2\right\|_{L^\infty}^2 . \tag{2.3.66}$$

where all the terms can be easily estimated via Lemma 2.3.3.

Step 3: $\sum_{i=1}^2 \|Du_i\|_{H^2(\Sigma)}^2$.

By the trace theorem and the Poincaré inequality, we have

$$\sum_{i=1}^{2} \|Du_i\|_{H^2(\Sigma)}^2 \lesssim \left\|D^3\nabla u_i\right\|_{H^0}^2 + \left\|D^2\nabla u_i\right\|_{H^0}^2 + \|Du_i\|_{H^1}^2 . \tag{2.3.67}$$

The first and second term are naturally estimated, so we only need to focus on the last one. By the Poincaré inequality, we have

$$\|Du_i\|_{H^1}^2 \lesssim \|D\partial_3 u_i\|_{H^0}^2 + \left\|D^2 u_i\right\|_{H^0}^2 , \tag{2.3.68}$$

and also

$$\|D\partial_3 u_i\|_{H^0}^2 \lesssim \left\|D\partial_3^2 u_i\right\|_{H^0}^2 + \|D\partial_3 u_i\|_{H^0(\Sigma)}^2 . \tag{2.3.69}$$

The boundary condition in (2.3.17) shows

$$\|D\partial_3 u_i\|_{H^0(\Sigma)}^2 \lesssim \left\|D^2 u_3\right\|_{H^0(\Sigma)}^2 + \left\|DG^3\right\|_{H^0(\Sigma)}^2 \lesssim \left\|D^2\partial_3 u_3\right\|_{H^0}^2 + \left\|DG^3\right\|_{H^0(\Sigma)}^2 . \tag{2.3.70}$$

By the Navier-Stokes equations in (2.3.17), we have

$$\left\|D\partial_3^2 u_i\right\|_{H^0}^2 \lesssim \|D\partial_t u_i\|_{H^0}^2 + \left\|D^3 u_i\right\|_{H^0}^2 + \left\|D^2 p\right\|_{H^0}^2 + \left\|DG^1\right\|_{H^0}^2 . \tag{2.3.71}$$

Hence, in total, we achieve

$$\begin{aligned} &\sum_{i=1}^{2} \|Du_i\|_{H^2}^2 \\ \lesssim\ & \left\|D^2\nabla u_i\right\|_{H^0}^2 + \left\|D^2 u\right\|_{H^0}^2 + \left\|D^2\partial_3 u_3\right\|_{H^0}^2 + \left\|DG^3\right\|_{H^0(\Sigma)}^2 + \|D\partial_t u_i\|_{H^0}^2 \\ & + \left\|D^2 p\right\|_{H^0}^2 + \left\|DG^1\right\|_{H^0}^2 . \end{aligned} \tag{2.3.72}$$

The estimate of each term above with Lemma 2.3.3 satisfies our requirement, so we finish this proof. $\square$

Lemma 2.3.5. *We have the estimate*

$$\bar{\mathcal{E}}_{N+2,2} \lesssim \mathcal{E}_{2N}^{1/(\lambda+3)} \mathcal{D}_{N+2,2}^{(\lambda+2)/(\lambda+3)}. \tag{2.3.73}$$

Proof. The dissipation $\mathcal{D}_{N+2,2}$ can control all the terms in $\bar{\mathcal{E}}_{N+2,2}$ except $\partial_t\eta$, $D^2\eta$ and $D^{2(N+2)}\eta$. The standard Sobolev interpolation estimate reveals

$$\left\|D^{2(N+2)}\eta\right\|_{H^0}^2 \lesssim \mathcal{E}_{2N}^{1/(4N-7)} \mathcal{D}_{N+2,2}^{(4N-8)/(4N-7)}. \tag{2.3.74}$$

Since $N \geq 3$, above estimate satisfies our requirement. Also Lemma 2.3.3 reveals the estimate of $D^2\eta$ is not an obstacle. Hence, the dominant term is $\partial_t\eta$. By the transport equation in (2.3.17), we have

$$\|\partial_t\eta\|_{H^0(\Sigma)}^2 \lesssim \|u_3\|_{H^0(\Sigma)}^2 + \left\|G^4\right\|_{H^0(\Sigma)}^2. \tag{2.3.75}$$

By the Poincaré-type inequality in Lemma A.1.4, we can estimate

$$\|u_3\|_{H^0(\Sigma)}^2 \lesssim \|\partial_3 u_3\|_{H^0}^2, \tag{2.3.76}$$

Then the divergence equation in (2.3.17) and the Poincaré inequality imply

$$\|\partial_3 u_3\|_{H^0}^2 \lesssim \|Du_i\|_{H^0}^2 + \left\|G^2\right\|_{H^0}^2 \lesssim \|D\partial_3 u_i\|_{H^0}^2 + \left\|G^2\right\|_{H^0}^2. \tag{2.3.77}$$

Similar to the third step in the proof of Lemma 2.3.4, by the Poincaré inequality, we have

$$\|D\partial_3 u_i\|_{H^0}^2 \lesssim \left\|D\partial_3^2 u_i\right\|_{H^0}^2 + \|D\partial_3 u_i\|_{H^0(\Sigma)}^2. \tag{2.3.78}$$

The upper boundary condition in (2.3.17) shows

$$\|D\partial_3 u_i\|^2_{H^0(\Sigma)} \lesssim \left\|D^2 u_3\right\|^2_{H^0(\Sigma)} + \left\|DG^3\right\|^2_{H^0(\Sigma)} \lesssim \left\|D^2\partial_3 u_3\right\|^2_{H^0} + \left\|DG^3\right\|^2_{H^0(\Sigma)}. \tag{2.3.79}$$

By the Navier-Stokes equations in (2.3.17), we have

$$\left\|D\partial_3^2 u_i\right\|^2_{H^0} \lesssim \|D\partial_t u_i\|^2_{H^0} + \left\|D^3 u_i\right\|^2_{H^0} + \left\|D^2 p\right\|^2_{H^0} + \left\|DG^1\right\|^2_{H^0}, \tag{2.3.80}$$

where $D^2 p$ dominates. Utilizing the Poincaré inequality, we have

$$\left\|D^2 p\right\|^2_{H^0} \lesssim \left\|D^2\partial_3 p\right\|^2_{H^0} + \left\|D^2 p\right\|^2_{H^0(\Sigma)}, \tag{2.3.81}$$

in which we may again use the upper boundary condition in (2.3.17) to estimate

$$\left\|D^2 p\right\|^2_{H^0(\Sigma)} \lesssim \left\|D^2\eta\right\|^2_{H^0(\Sigma)} + \left\|D^2\partial_3 u_3\right\|^2_{H^0(\Sigma)} + \left\|D^2 G^3\right\|^2_{H^0(\Sigma)}. \tag{2.3.82}$$

Hence, in total, we achieve

$$\begin{aligned} &\|\partial_t \eta\|^2_{H^0} \\ \lesssim\ & \left\|D^2\partial_3 u_3\right\|^2_{H^0} + \left\|DG^3\right\|^2_{H^0(\Sigma)} + \|D\partial_t u_i\|^2_{H^0} + \left\|D^3 u_i\right\|^2_{H^0} + \left\|DG^1\right\|^2_{H^0} \\ & + \left\|D^2\partial_3 p\right\|^2_{H^0} + \left\|D^2\eta\right\|^2_{H^0(\Sigma)} + \left\|D^2\partial_3 u_3\right\|^2_{H^0(\Sigma)} + \left\|G^2\right\|^2_{H^0} + \left\|G^4\right\|^2_{H^0(\Sigma)} \\ & + \left\|D^2 G^3\right\|^2_{H^0(\Sigma)}. \end{aligned} \tag{2.3.83}$$

We may easily check via Lemma 2.3.3 that each term above satisfies our requirement, so we finish this proof. □

Lemma 2.3.6. *We have the estimate*

$$\mathcal{E}_{N+2,1} \lesssim \mathcal{E}_{N+2,2}^{(\lambda+1)/(\lambda+2)} \mathcal{E}_{2N}^{1/(\lambda+2)}. \tag{2.3.84}$$

Proof. We only need to estimate the lowest order terms that only appear in $\mathcal{E}_{N+2,1}$, which are Du, Dp and $D\eta$. Based on Lemma 2.3.3, the estimates of Du and $D\eta$ satisfy our requirement, so we focus on Dp. By a similar argument as in the proof of Lemma 2.3.5, we have

$$\begin{aligned} \|Dp\|_{H^0}^2 &\lesssim \|Dp\|_{H^0(\Sigma)}^2 + \|D\partial_3 p\|_{H^0}^2 \\ &\lesssim \|D\eta\|_{H^0(\Sigma)}^2 + \left\|D\partial_3^2 u\right\|_{H^0}^2 + \left\|G^3\right\|_{H^0(\Sigma)}^2 + \|D\Delta u\|_{H^0}^2 + \left\|DG^1\right\|_{H^0}^2, \end{aligned} \tag{2.3.85}$$

where each term can be estimated directly through Lemma 2.3.3. Then our result easily follows. □

2.3.4 Nonlinear Estimates

Now we give several estimates for the nonlinear terms in the perturbed linear form. Similar to [25, Theorem 4.1 - Theorem 4.2], the methods are quite standard, so we simply give the basic rules in estimating:

1. Since all the terms are quadratic or of higher order, we may expand the differential operators with the Leibniz rule and estimate the resulting quadratic terms by the basic interpolation estimates in Lemma 2.3.3 either in L^∞ or L^2 norm, combining with the definitions of energy and dissipation, the trace theorem and the Sobolev embedding theorem. For the options of form (2.3.55) or (2.3.56), we should always take the higher interpolation index for $\mathcal{E}_{N+2,2}$ and $\mathcal{D}_{N+2,2}$.

2. For the $2N$ level, there is no minimal count for the derivatives, so we only need to refer to the definitions of energy and dissipation; however, in the $N+2$ level, we have to resort to Lemma 2.3.3 for terms that cannot be estimated directly.

3. Note the most important difference between our proofs and those of [25, Theorem 4.1 - Theorem 4.2] is we concentrate on the horizontal derivatives instead of the full derivatives. Thereafter, in the estimates related to $\mathcal{D}_{N+2,2}$, we should try to avoid the introduction of vertical derivatives via the Sobolev embedding theorem, especially for pressure p.

4. For the term in the form of $\|X\|_{H^2(\Sigma)}$, instead of using the trace theorem directly, we should first write it into $\|X\|_{H^0(\Sigma)}+\|DX\|_{H^0(\Sigma)}+\|D^2X\|_{H^0(\Sigma)}$ and then apply the trace theorem, which only produces one vertical derivative rather than three.

Lemma 2.3.7. *There exists a $\theta > 0$ such that*

$$\begin{aligned} \left\|\bar{D}^2\bar{\nabla}_0^{2(N+2)-4}G^1\right\|_{H^0}^2 + \left\|\bar{D}^2\bar{\nabla}_0^{2(N+2)-3}G^2\right\|_{H^0}^2 & \\ + \left\|\bar{D}^2\bar{\nabla}_0^{2(N+2)-4}G^3\right\|_{H^{1/2}(\Sigma)}^2 &\lesssim \mathcal{E}_{2N}^{\theta}\mathcal{E}_{N+2,2}, \end{aligned} \tag{2.3.86}$$

and

$$\begin{aligned} \left\|\bar{D}_2^{2(N+2)-1}G^1\right\|_{H^0}^2 + \left\|\bar{D}_1^{2(N+2)-1}G^2\right\|_{H^1}^2 & \\ + \left\|\bar{D}_2^{2(N+2)-1}G^3\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\bar{D}_1^{2(N+2)-1}G^4\right\|_{H^{1/2}(\Sigma)}^2 &\lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{N+2,2}. \end{aligned} \tag{2.3.87}$$

Lemma 2.3.8. *There exists a $\theta > 0$ such that*

$$\left\|\bar{\nabla}_0^{4N-2}G^1\right\|_{H^0}^2 + \left\|\bar{\nabla}_0^{4N-1}G^2\right\|_{H^0}^2 + \left\|\bar{\nabla}_0^{4N-2}G^3\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^{1+\theta}, \tag{2.3.88}$$

and

$$\left\|\bar{D}_0^{4N-1}G^1\right\|_{H^0}^2 + \left\|\bar{D}_0^{4N-1}G^2\right\|_{H^1}^2 + \left\|\bar{D}_0^{4N-1}G^3\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\bar{D}_0^{4N-1}G^4\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{2.3.89}$$

The other nonlinear terms have been proved in [25, Proposition 4.3 - Proposition 4.5]. Although our definitions of energy and dissipation are a little different, we do not even need to revise the proofs there.

2.3.5 Energy Estimates

In this section, we present the energy estimates for the horizontal derivatives. Since this part has been carefully studied in [25, Section 5] and is not the main improvement of our new proof, it is not necessary to show all the details. The following is an example to demonstrate how we can obtain the same types of results as before.

Lemma 2.3.9. *Let $\partial^\alpha = \partial_t^{N+2}$ and let F^i be defined as in the geometric structure form. Then*

$$\left\|F^1\right\|_{H^0}^2 + \left\|\partial_t(JF^2)\right\|_{H^0}^2 + \left\|F^3\right\|_{H^0(\Sigma)}^2 + \left\|F^4\right\|_{H^0(\Sigma)}^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{N+2,2}. \tag{2.3.90}$$

Also, if $N \geq 3$, then there exists a $\theta > 0$ such that

$$\left\|F^2\right\|_{H^0}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{E}_{N+2,2}. \tag{2.3.91}$$

Proof. This is a revised version of [25, Theorem 5.2]. The proof of the first part is exactly the same as that of the original theorem. However, we utilize different techniques in the second part. By definition, $F^2 = F^{2,1} + F^{2,2}$. By the same argument as [25, (5.5)], we have

$$\left\| F^{2,1} \right\|_{H^0}^2 \lesssim \mathcal{E}_{2N}^{\theta} \mathcal{E}_{N+2,2}. \tag{2.3.92}$$

For $F^{2,2}$, a calculation reveals

$$F^{2,2} = \partial_t^{N+2}(\partial_1 \bar{\eta} \tilde{b} K) \partial_3 u_1 + \partial_t^{N+2}(\partial_2 \bar{\eta} \tilde{b} K) \partial_3 u_2 - \partial_t^{N+2} K \partial_3 u_3. \tag{2.3.93}$$

We can estimate

$$\left\| \partial_t^{N+2}(\partial_1 \bar{\eta} \tilde{b} K) \partial_3 u_1 \right\|_{H^0}^2 \lesssim \left\| \partial_t^{N+2}(\partial_1 \bar{\eta} \tilde{b} K) \right\|_{H^0}^2 \left\| \partial_3 u_1 \right\|_{L^\infty}^2. \tag{2.3.94}$$

Lemma 2.3.4 implies

$$\left\| \partial_3 u_1 \right\|_{L^\infty}^2 \lesssim \left\| \nabla u \right\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}^{r/(2+r)} \mathcal{E}_{N+2,2}^{2/(2+r)}. \tag{2.3.95}$$

Also, for $0 \leq |\alpha| \leq N+2$, there exists a $\theta > 0$ such that

$$\left\| \partial_t^{\alpha} \partial_1 \bar{\eta} \right\|_{H^0}^2 \lesssim \left\| \partial_t^{\alpha} \eta \right\|_{H^{1/2}(\Sigma)}^2 \lesssim \left\| \partial_t^{\alpha} \eta \right\|_{H^0(\Sigma)} \left\| \partial_t^{\alpha} \eta \right\|_{H^1(\Sigma)} \lesssim \mathcal{E}_{2N}^{1-\theta} \mathcal{E}_{N+2,2}^{\theta}. \tag{2.3.96}$$

Hence we have

$$\left\| \partial_t^{N+2}(\partial_1 \bar{\eta} \tilde{b} K) \right\|_{H^0}^2 \lesssim \sum_{\alpha=0}^{N+2} \left\| \partial_t^{\alpha} \partial_1 \bar{\eta} \right\|_{H^0}^2 \left\| \partial_t^{N+2-\alpha} (\tilde{b} K) \right\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}^{1-\theta} \mathcal{E}_{N+2,2}^{\theta}. \tag{2.3.97}$$

Then when r is sufficiently small, we can naturally estimate

$$\left\| \partial_t^{N+2}(\partial_1 \bar{\eta} \tilde{b} K) \partial_3 u_1 \right\|_{H^0}^2 \lesssim \mathcal{E}_{2N}^{\theta} \mathcal{E}_{N+2,2}. \tag{2.3.98}$$

Similarly, we have

$$\left\|\partial_t^{N+2}(\partial_2\bar{\eta}\tilde{b}K)\partial_3 u_2\right\|_{H^0}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{E}_{N+2,2}. \tag{2.3.99}$$

The same argument as [25, (5.11)] shows

$$\left\|\partial_t^{N+2}K\partial_3 u_3\right\|_{H^0}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{E}_{N+2,2}. \tag{2.3.100}$$

Therefore, our result easily follows. □

In a similar fashion, we can recover all the other lemmas in [25, Section 5]. Finally, we can show the same energy estimates.

Proposition 2.3.10. *Let F^2 be given with ∂_t^{N+2}. Then there exists a $\theta > 0$ such that*

$$\partial_t\left(\bar{\mathcal{E}}_{N+2,2} - 2\int_\Omega K\partial_t^{N+1}pF^2\right) + \bar{\mathcal{D}}_{N+2,2} \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{N+2,2}. \tag{2.3.101}$$

Proposition 2.3.11. *There exists a $\theta > 0$ such that*

$$\begin{aligned}\bar{\mathcal{E}}_{2N}(t) + \int_0^t \bar{\mathcal{D}}_{2N}(r)\mathrm{dr} \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t (\mathcal{E}_{2N}(r))^{\theta}\mathcal{D}_{2N}\mathrm{dr}\\ + \int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\mathrm{dr}.\end{aligned} \tag{2.3.102}$$

2.3.6 Comparison Estimates

In this section, we prove several comparison estimates between the horizontal derivatives and the full derivatives, both in energy and dissipation. This is the key improvement in our proof, so we present all the necessary details.

Dissipation Comparison Estimates

Proposition 2.3.12. *There exists a $\theta > 0$ such that*

$$\mathcal{D}_{N+2,2} \lesssim \bar{\mathcal{D}}_{N+2,2} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{N+2,2}. \tag{2.3.103}$$

Proof. We define

$$\begin{aligned}\mathcal{Y}_{N+2,2} \;=\;& \left\|\bar{D}_2^{2(N+2)-1}G^1\right\|_{H^0}^2 + \left\|\bar{D}_2^{2(N+2)-1}G^2\right\|_{H^1}^2 \\ &+ \left\|\bar{D}_2^{2(N+2)-1}G^3\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\bar{D}_1^{2(N+2)-1}G^4\right\|_{H^{1/2}(\Sigma)}^2\end{aligned} \tag{2.3.104}$$

and

$$\mathcal{Z} = \bar{\mathcal{D}}_{N+2,2} + \mathcal{Y}_{N+2,2}. \tag{2.3.105}$$

Then the proof is divided into several steps.

Step 1: Application of Korn's inequality.

Since any horizontal derivatives of velocity vanishes on the lower boundary, we can directly apply Korn's inequality to achieve

$$\left\|\bar{D}_2^{2(N+2)}u\right\|_{H^1}^2 \lesssim \left\|\bar{D}_2^{2(N+2)}\mathbb{D}u\right\|_{H^0}^2 = \bar{\mathcal{D}}_{N+2,2} \lesssim \mathcal{Z}. \tag{2.3.106}$$

Step 2: Initial estimates of velocity and pressure.

Let $0 \le j \le (N+2)-1$ and $\alpha \in \mathbb{N}^2$ be such that $2 \le 2j + |\alpha| \le 2(N+2)-1$. Hence, we naturally have

$$\left\|\partial^\alpha \partial_t^{j+1} u\right\|_{H^0}^2 \leq \left\|\bar{D}_2^{2(N+2)} u\right\|_{H^1}^2 \lesssim \mathcal{Z}. \tag{2.3.107}$$

We apply the operator $\partial^\alpha \partial_t^j \partial_3$ to the divergence equation in (2.3.17) and rearrange the terms as follows:

$$\partial^\alpha \partial_t^j \partial_3(\partial_3 u_3) = \partial^\alpha \partial_t^j \partial_3(-\partial_1 u_1 - \partial_2 u_2) + \partial^\alpha \partial_t^j \partial_3 G^2. \tag{2.3.108}$$

Thus we have

$$\left\|\partial^\alpha \partial_t^j \partial_3^2 u_3\right\|_{H^0}^2 \lesssim \left\|\bar{D}_2^{2(N+2)} u\right\|_{H^1}^2 + \left\|\partial^\alpha \partial_t^j G^2\right\|_{H^1}^2 \lesssim \mathcal{Z}. \tag{2.3.109}$$

Hence, it easily implies

$$\left\|\Delta \partial^\alpha \partial_t^j u_3\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.3.110}$$

Applying the operator $\partial^\alpha \partial_t^j$ to the third equation in (2.3.17), we have

$$\left\|\partial^\alpha \partial_t^j \partial_3 p\right\|_{H^0}^2 \lesssim \left\|\partial^\alpha \partial_t^j \Delta u\right\|_{H^0}^2 + \left\|\partial^\alpha \partial_t^j G^1\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.3.111}$$

So we only need to estimate the terms related to $\partial_3^2 u_i$ and $\partial_i p$ for $i = 1, 2$.

Step 3: Estimates of velocity and pressure in the upper domain.

In order to achieve above estimates, we need to employ a localization argument. Define a cutoff function $\chi_1 = \chi_1(x_3)$ given by $\chi_1 \in C_c^\infty(\mathbb{R})$ with $\chi(x_3) = 1$ for $x_3 \in \Omega_1 = [-2b/3, 0]$ and $\chi_1(x_3) = 0$ for $x_3 \notin (-3b/4, 1/2)$.

Define the curl of the velocity $w = \nabla \times u$, which means $w_1 = \partial_2 u_3 - \partial_3 u_2$ and $w_2 = \partial_3 u_1 - \partial_1 u_3$. Therefore, $\chi_1 w_i$ for $i = 1, 2$ satisfies the equations

$$\begin{aligned} \Delta \partial^\alpha \partial_t^j (\chi_1 w_i) &= \chi_1 \partial^\alpha \partial_t^{j+1} w_i + 2(\partial_3 \chi_1)(\partial^\alpha \partial_t^j \partial_3 w_i) \\ &\quad + (\partial_3^2 \chi_1)(\partial^\alpha \partial_t^j w_i) - \chi_1 \nabla \times (\partial^\alpha \partial_t^j G^1) \end{aligned} \tag{2.3.112}$$

in Ω as well as the boundary conditions

$$\begin{cases} \partial^\alpha \partial_t^j (\chi_1 w_1) = 2\partial_2 \partial^\alpha \partial_t^j u_3 + \partial^\alpha \partial_t^j G^3 \cdot e_2 & \text{on } \Sigma, \\ \partial^\alpha \partial_t^j (\chi_1 w_2) = -2\partial_1 \partial^\alpha \partial_t^j u_3 - \partial^\alpha \partial_t^j G^3 \cdot e_1 & \text{on } \Sigma, \\ \partial^\alpha \partial_t^j (\chi_1 w_1) = \partial^\alpha \partial_t^j (\chi_1 w_2) = 0 & \text{on } \Sigma_b. \end{cases} \tag{2.3.113}$$

Based on the standard elliptic estimate of the Poisson equation, we have

$$\left\| \partial^\alpha \partial_t^j (\chi_1 w_i) \right\|_{H^1}^2 \lesssim \left\| \Delta \partial^\alpha \partial_t^j (\chi_1 w_i) \right\|_{H^{-1}}^2 + \left\| \partial^\alpha \partial_t^j (\chi_1 w_i) \right\|_{H^{1/2}(\Sigma)}^2 . \tag{2.3.114}$$

Hence, we have to estimate each term on the right-hand side.

Let $\phi \in H_0^1(\Omega)$. For $\alpha \neq 0$ we may write $\alpha = \beta + (\alpha - \beta)$ with $|\beta| = 1$. Then an integration by parts reveals

$$\left| \int_\Omega \phi \chi_1 \partial^\alpha \partial_t^{j+1} w_i \right| = \left| \int_\Omega \partial^\beta \phi \chi_1 \partial^{\alpha-\beta} \partial_t^{j+1} w_i \right| \leq \|\phi\|_{H^1} \left\| \chi_1 \bar{D}_2^{2(N+2)} w_i \right\|_{H^0} . \tag{2.3.115}$$

Since $2(j+1) + |\alpha - \beta| \in [3, 2(N+2)]$, we naturally have

$$\left\| \chi_1 \bar{D}_2^{2(N+2)} w_i \right\|_{H^0}^2 \lesssim \left\| \bar{D}_2^{2(N+2)} u \right\|_{H^1}^2 \lesssim \mathcal{Z}. \tag{2.3.116}$$

Then if we take the supremum over all ϕ such that $\|\phi\|_{H^1} \leq 1$, then we may get

$$\left\| \chi_1 \partial^\alpha \partial_t^{j+1} w_i \right\|_{H^{-1}}^2 \lesssim \mathcal{Z}. \tag{2.3.117}$$

A similar argument without integration by parts shows above result is also true for $\alpha = 0$ since in this case $j \leq (N+2) - 1$ implies $4 \leq 2(j+1) \leq 2(N+2)$. In a similar fashion, we may apply integration by parts for ∂_3 for the other terms in H^{-1} norm and achieve

$$\begin{aligned} \left\|2(\partial_3\chi_1)(\partial^\alpha\partial_t^j\partial_3 w_i)\right\|_{H^{-1}}^2 &\lesssim (\|\partial_3\chi_1\|_{L^\infty}^2 + \left\|\partial_3^2\chi_1\right\|_{L^\infty}^2)\left\|\bar{D}_2^{2(N+2)}w_i\right\|_{H^0}^2 \\ &\lesssim \left\|D_2^{2(N+2)}u\right\|_{H^1}^2 \lesssim \mathcal{Z}, \end{aligned} \tag{2.3.118}$$

$$\begin{aligned} \left\|(\partial_3^2\chi_1)(\partial^\alpha\partial_t^j w_i)\right\|_{H^{-1}}^2 &\lesssim \left\|\partial_3^2\chi_1\right\|_{L^\infty}^2\left\|\bar{D}_2^{2(N+2)}w_i\right\|_{H^0}^2 \\ &\lesssim \left\|D_2^{2(N+2)}u\right\|_{H^1}^2 \lesssim \mathcal{Z}, \end{aligned} \tag{2.3.119}$$

$$\left\|\chi_1\nabla\times(\partial^\alpha\partial_t^j G^1)\right\|_{H^{-1}}^2 \lesssim (\|\chi_1\|_{L^\infty}^2 + \|\partial_3\chi_1\|_{L^\infty}^2)\left\|\bar{D}_2^{2(N+2)-1}G^1\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.3.120}$$

In total, above estimates together yield

$$\left\|\Delta\partial^\alpha\partial_t^j(\chi_1 w_i)\right\|_{H^{-1}}^2 \lesssim \mathcal{Z}. \tag{2.3.121}$$

Then for the boundary terms, a direct application of the trace theorem shows

$$\left\|\partial^\alpha\partial_t^j\partial_1 u_3\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\partial^\alpha\partial_t^j\partial_2 u_3\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \left\|\bar{D}_2^{2(N+2)}u\right\|_{H^1}^2 \lesssim \mathcal{Z}, \tag{2.3.122}$$

$$\left\|\partial^\alpha\partial_t^j G^3\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \left\|\bar{D}_2^{2(N+2)-1}G^3\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{Z}. \tag{2.3.123}$$

Then above implies

$$\left\|\partial^\alpha \partial_t^j (\chi_1 w_i)\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{Z}. \tag{2.3.124}$$

Hence, the elliptic estimate gives us the final form

$$\left\|\partial^\alpha \partial_t^j w_i\right\|_{H^1(\Omega_1)}^2 \lesssim \left\|\partial^\alpha \partial_t^j (\chi_1 w_i)\right\|_{H^1}^2 \lesssim \mathcal{Z}. \tag{2.3.125}$$

Since $\partial_3^2 \partial^\alpha \partial_t^j u_1 = \partial_3 \partial^\alpha \partial_t^j (w_2 + \partial_1 u_3)$ and $\partial_3^2 \partial^\alpha \partial_t^j u_2 = \partial_3 \partial^\alpha \partial_t^j (\partial_2 u_3 - w_1)$, we can deduce from above that for $i = 1, 2$,

$$\left\|\partial_3^2 \partial^\alpha \partial_t^j u_i\right\|_{H^0(\Omega_1)}^2 \lesssim \mathcal{Z}, \tag{2.3.126}$$

which, by considering the Navier Stokes equations in (2.3.17), further implies

$$\left\|\partial_i \partial^\alpha \partial_t^j p\right\|_{H^0(\Omega_1)}^2 \lesssim \mathcal{Z}. \tag{2.3.127}$$

Combining all above, we have achieved

$$\left\|\bar{D}_2^{2(N+2)-1} u\right\|_{H^2(\Omega_1)}^2 + \left\|\bar{D}_2^{2(N+2)-1} \nabla p\right\|_{H^0(\Omega_1)}^2 \lesssim \mathcal{Z}. \tag{2.3.128}$$

Step 4: Estimates of velocity and pressure in the lower domain.

Now we extend above estimates to the lower domain $\Omega_2 = [-b, -b/3]$. Define a cutoff function $\chi_2 \in C_c^\infty(\mathbb{R})$ such that $\chi_2(x_3) = 1$ for $x_3 \in \Omega_2$ and $\chi_2(x_3) = 0$ for $x_3 \notin (-2b, -b/6)$. Notice this cutoff function satisfies $supp(\nabla \chi_2) \in \Omega_1$, which will play a key role in the following.

Then the localized variables satisfy the equations

$$\begin{cases} -\Delta(\chi_2 u) + \nabla(\chi_2 p) = -\partial_t(\chi_2 u) + \chi_2 G^1 + H^1 & \text{in } \Omega, \\ \nabla\cdot(\chi_2 u) = \chi_2 G^2 + H^2 & \text{in } \Omega, \\ \chi_2 u = 0 & \text{on } \Sigma, \\ \chi_2 u = 0 & \text{on } \Sigma_b, \end{cases} \tag{2.3.129}$$

where

$$H^1 = \partial_3\chi_2(pe_3 - 2\partial_3 u) - (\partial_3^2\chi_2)u \qquad H^2 = \partial_3\chi_2 u_3. \tag{2.3.130}$$

Let $0 \le j \le (N+2)-1$ and $\alpha \in \mathbb{N}^2$ be such that $2 \le 2j + |\alpha| \le 2(N+2)-1$. Then we may apply the differential operator $\partial^\alpha\partial_t^j$ on (2.3.129) and the elliptic estimates of the steady Navier-Stokes equations imply

$$\begin{aligned} \left\|\partial^\alpha\partial_t^j\partial_i(\chi_2 u)\right\|_{H^1}^2 + \left\|\partial^\alpha\partial_t^j\partial_i(\chi_2 p)\right\|_{H^0}^2 &\lesssim \left\|\partial^\alpha\partial_t^{j+1}\partial_i(\chi_2 u)\right\|_{H^{-1}}^2 \\ &+ \left\|\partial^\alpha\partial_t^j\partial_i(\chi_2 G^1 + H^1)\right\|_{H^{-1}}^2 + \left\|\partial^\alpha\partial_t^j\partial_i(\chi_2 G^2 + H^2)\right\|_{H^0}^2. \end{aligned} \tag{2.3.131}$$

Similar to the argument in the upper domain, via integration by parts, it is easy to deduce the H^{-1} estimate that

$$\left\|\partial^\alpha\partial_t^{j+1}\partial_i(\chi_2 u)\right\|_{H^{-1}}^2 + \left\|\partial^\alpha\partial_t^j\partial_i(\chi_2 G^1)\right\|_{H^{-1}}^2 + \left\|\partial^\alpha\partial_t^j\partial_i(\chi_2 G^2)\right\|_{H^0}^2 \lesssim \mathcal{Z}. \tag{2.3.132}$$

For the remaining part, we may directly estimate

$$\begin{aligned} \left\|\partial^\alpha\partial_t^j\partial_i H^1\right\|_{H^{-1}}^2 + \left\|\partial^\alpha\partial_t^j\partial_i H^2\right\|_{H^0}^2 &\lesssim \left\|\partial^\alpha\partial_t^j\partial_i H^1\right\|_{H^0}^2 + \left\|\partial^\alpha\partial_t^j\partial_i H^2\right\|_{H^0}^2 \\ &\lesssim \left\|\bar{D}_2^{2(N+2)-1} u\right\|_{H^2(\Omega_1)}^2 + \left\|\bar{D}_2^{2(N+2)-1}\nabla p\right\|_{H^0(\Omega_1)}^2 \lesssim \mathcal{Z}, \end{aligned} \tag{2.3.133}$$

where we utilize the property for $\nabla\chi_2$ stated above. Thus we have

$$\left\|\partial_i\partial^\alpha\partial_t^j p\right\|_{H^0(\Omega_2)}^2 \lesssim \left\|\partial^\alpha\partial_t^j\partial_i(\chi_2 p)\right\|_{H^0}^2 \lesssim \mathcal{Z}, \tag{2.3.134}$$

which, based on the Navier-Stokes equations in (2.3.17), further implies

$$\left\|\partial_3^2\partial^\alpha\partial_t^j u\right\|_{H^0(\Omega_2)}^2 \lesssim \mathcal{Z}. \tag{2.3.135}$$

Combining all above, we have

$$\left\|\bar{D}_2^{2(N+2)-1}u\right\|_{H^2(\Omega_2)}^2 + \left\|\bar{D}_2^{2(N+2)-1}\nabla p\right\|_{H^0(\Omega_2)}^2 \lesssim \mathcal{Z}. \tag{2.3.136}$$

Step 5: Estimates of free surface.

With all above estimates in hand, we may derive the estimates for η by employing the boundary condition of (2.3.17), i.e.

$$\eta = p - 3\partial_3 u_3 - G_3^3. \tag{2.3.137}$$

We may take horizontal derivatives on both sides of above equation, which shows for $\alpha \in \mathbb{N}^2$ and $3 \le |\alpha| \le 2(N+2)-1$, we have

$$\begin{aligned}
\left\|\partial^\alpha\eta\right\|_{H^{1/2}(\Sigma)}^2 &\lesssim \left\|\partial^\alpha p\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\partial^\alpha\partial_3 u_3\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\partial^\alpha G_3^3\right\|_{H^{1/2}(\Sigma)}^2 \\
&\lesssim \left\|\nabla\bar{D}_2^{2(N+2)-1}p\right\|_{H^0}^2 + \left\|\bar{D}_2^{2(N+2)-1}u\right\|_{H^2}^2 + \left\|\partial^\alpha G_3^3\right\|_{H^{1/2}(\Sigma)}^2 \\
&\lesssim \mathcal{Z}.
\end{aligned} \tag{2.3.138}$$

For the temporal derivatives of η, we need to resort to the transport equation in

(2.3.17), i.e.

$$\partial_t \eta = u_3 + G^4. \tag{2.3.139}$$

For $\alpha \in \mathbb{N}^2$ and $1 \leq |\alpha| \leq 2(N+2)-1$, by the trace theorem, the Poincaré inequality and the divergence equation $\nabla \cdot u = G^2$, we have

$$\begin{aligned} \|\partial^\alpha \partial_t \eta\|^2_{H^{1/2}(\Sigma)} &\lesssim \|\partial^\alpha u_3\|^2_{H^{1/2}(\Sigma)} + \left\|\partial^\alpha G^4\right\|^2_{H^{1/2}(\Sigma)} \\ &\lesssim \|\partial^\alpha u_3\|^2_{H^1} + \mathcal{Z} \\ &\lesssim \|\partial^\alpha u_3\|^2_{H^0} + \|\partial^\alpha D u_3\|^2_{H^0} + \|\partial^\alpha \partial_3 u_3\|^2_{H^0} + \mathcal{Z} \\ &\lesssim \|\partial^\alpha \partial_3 u_3\|^2_{H^0} + \mathcal{Z} \\ &\lesssim \|\partial^\alpha D u\|^2_{H^0} + \left\|\partial^\alpha G^2\right\|^2_{H^0} + \mathcal{Z} \\ &\lesssim \mathcal{Z}. \end{aligned} \tag{2.3.140}$$

Similarly, for $j = 2, \ldots, (N+2)+1$, we may apply ∂_t^{j-1} to the transport equation to obtain that for $\alpha \in \mathbb{N}^2$ with $0 \leq |\alpha| \leq 2(N+2) - 2j + 2$, we have

$$\begin{aligned} \left\|\partial^\alpha \partial_t^j \eta\right\|^2_{H^{1/2}(\Sigma)} &\lesssim \left\|\partial^\alpha \partial_t^{j-1} u_3\right\|^2_{H^{1/2}(\Sigma)} + \left\|\partial^\alpha \partial_t^{j-1} G^4\right\|^2_{H^{1/2}(\Sigma)} \\ &\lesssim \left\|\partial^\alpha \partial_t^{j-1} u_3\right\|^2_{H^1} + \mathcal{Z} \\ &\lesssim \mathcal{Z}. \end{aligned} \tag{2.3.141}$$

Step 6: Synthesis.

To summarize all above, since $\Omega = \Omega_1 \cup \Omega_2$ we have the estimate

$$\mathcal{D}_{N+2,2} \lesssim \bar{\mathcal{D}}_{N+2,2} + \mathcal{Y}_{N+2,2}. \tag{2.3.142}$$

By Lemma 2.3.7, it is obvious that there exists a $\theta > 0$ such that

$$\mathcal{Y}_{N+2,2} \lesssim \mathcal{E}_{2N}^{\theta} \mathcal{D}_{N+2,2}. \tag{2.3.143}$$

Therefore, our result naturally follows. $\square$

Proposition 2.3.13. *There exists a $\theta > 0$ such that*

$$\mathcal{D}_{2N} \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta} \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{2.3.144}$$

Proof. It is easy to see most of the proof is identical to that of Proposition 2.3.12, except that we do not need the minimal counts and interpolation now. Also we utilize Lemma 2.3.8 instead to estimate the nonlinear terms in the perturbed linear form. Finally, the direct estimate for Riesz potential term

$$\|I_\lambda u\|_{H^1}^2 \lesssim \|\mathbb{D} I_\lambda u\|_{H^0}^2 \tag{2.3.145}$$

can close the proof. $\square$

Energy Comparison Estimates

Proposition 2.3.14. *There exists a $\theta > 0$ such that*

$$\mathcal{E}_{N+2,2} \lesssim \bar{\mathcal{E}}_{N+2,2} + \mathcal{E}_{2N}^{\theta} \mathcal{E}_{N+2,2}. \tag{2.3.146}$$

Proof. We define

$$\begin{aligned}\mathcal{W}_{N+2,2} &= \left\| \bar{D}^2 \bar{\nabla}_0^{2(N+2)-4} G^1 \right\|_{H^0}^2 + \left\| \bar{D}^2 \bar{\nabla}_0^{2(N+2)-3} G^2 \right\|_{H^0}^2 \\ &\quad + \left\| \bar{D}^2 \bar{\nabla}_0^{2(N+2)-4} G^3 \right\|_{H^{1/2}(\Sigma)}^2\end{aligned} \tag{2.3.147}$$

and

$$\mathcal{Z} = \bar{\mathcal{E}}_{N+2,2} + \mathcal{W}_{N+2,2}. \tag{2.3.148}$$

Then we can directly apply the elliptic estimate in [26, Lemma A.15] to the perturbed linear form. For $\alpha \in \mathbb{N}^2$ and $|\alpha| = 2$, we have

$$\begin{aligned}\|\partial^\alpha u\|_{H^{2(N+2)-2}}^2 + \|\partial^\alpha p\|_{H^{2(N+2)-3}}^2 &\lesssim \|\partial^\alpha \partial_t u\|_{H^{2(N+2)-4}}^2 + \left\|\partial^\alpha G^1\right\|_{H^{2(N+2)-4}}^2 \\ &+ \left\|\partial^\alpha G^2\right\|_{H^{2(N+2)-3}}^2 + \|\partial^\alpha \eta\|_{H^{2(N+2)-7/2}(\Sigma)}^2 + \left\|\partial^\alpha G^3\right\|_{H^{2(N+2)-7/2}(\Sigma)}^2 .\end{aligned} \tag{2.3.149}$$

Naturally, we have

$$\left\|\partial^\alpha G^1\right\|_{H^{2(N+2)-4}}^2 + \left\|\partial^\alpha G^2\right\|_{H^{2(N+2)-3}}^2 + \left\|\partial^\alpha G^3\right\|_{H^{2(N+2)-7/2}(\Sigma)}^2 \lesssim \mathcal{W}_{N+2,2} \lesssim \mathcal{Z}, \tag{2.3.150}$$

and

$$\|\partial^\alpha \eta\|_{H^{2(N+2)-7/2}(\Sigma)}^2 \lesssim \bar{\mathcal{E}}_{N+2,2} \lesssim \mathcal{Z}. \tag{2.3.151}$$

So the only remaining term is

$$\|\partial^\alpha \partial_t u\|_{H^{2(N+2)-4}}^2 \lesssim \|\partial_t u\|_{H^{2(N+2)-2}}^2 , \tag{2.3.152}$$

which should be estimated with a finite induction in the following.

For $j = 1, \ldots, (N+2)-1$, we have the elliptic estimate

$$\begin{aligned}\left\|\partial_t^j u\right\|_{H^{2(N+2)-2j}}^2 + \left\|\partial_t^j p\right\|_{H^{2(N+2)-2j-1}}^2 \lesssim \left\|\partial_t^{j+1} u\right\|_{H^{2(N+2)-2j-2}}^2 + \left\|\partial_t^j G^1\right\|_{H^{2(N+2)-2j-2}}^2 \\ + \left\|\partial_t^j G^2\right\|_{H^{2(N+2)-2j-1}}^2 + \left\|\partial_t^j \eta\right\|_{H^{2(N+2)-2j-3/2}(\Sigma)}^2 + \left\|\partial_t^j G^3\right\|_{H^{2(N+2)-2j-3/2}(\Sigma)}^2 .\end{aligned} \tag{2.3.153}$$

By definition, it is easy to see

$$\left\|\partial_t^j G^1\right\|_{H^{2(N+2)-2j-2}}^2 + \left\|\partial_t^j G^2\right\|_{H^{2(N+2)-2j-1}}^2 + \left\|\partial_t^j G^3\right\|_{H^{2(N+2)-2j-3/2}(\Sigma)}^2 \lesssim \mathcal{W}_{N+2,2} \lesssim \mathcal{Z}, \tag{2.3.154}$$

and

$$\left\|\partial_t^j \eta\right\|_{H^{2(N+2)-2j-3/2}(\Sigma)}^2 \lesssim \bar{\mathcal{E}}_{N+2,2} \lesssim \mathcal{Z}. \tag{2.3.155}$$

When $j = (N+2)-1$, we have

$$\left\|\partial_t^{j+1} u\right\|_{H^{2(N+2)-2j-2}}^2 = \left\|\partial_t^{N+2} u\right\|_{H^0}^2 \lesssim \bar{\mathcal{E}}_{N+2,2} \lesssim \mathcal{Z}. \tag{2.3.156}$$

Hence, the elliptic estimate shows

$$\left\|\partial_t^{N+1} u\right\|_{H^2}^2 + \left\|\partial_t^{N+1} p\right\|_{H^1}^2 \lesssim \mathcal{Z}. \tag{2.3.157}$$

Inductively, we can estimate all the temporal derivatives of velocity and pressure for $j = 1, \ldots, N$ to close the proof.

Therefore, we have shown

$$\mathcal{E}_{N+2,2} \lesssim \bar{\mathcal{E}}_{N+2,2} + \mathcal{W}_{N+2,2}. \tag{2.3.158}$$

Based on Lemma 2.3.7, there exists a $\theta > 0$ such that

$$\mathcal{W}_{N+2,2} \lesssim \mathcal{E}_{2N}^{\theta} \mathcal{E}_{N+2,2}. \tag{2.3.159}$$

Hence, our result naturally follows. □

Proposition 2.3.15. *There exists a $\theta > 0$ such that*

$$\mathcal{E}_{2N} \lesssim \bar{\mathcal{E}}_{2N} + \mathcal{E}_{2N}^{1+\theta}. \tag{2.3.160}$$

Proof. It is easy to see the proof is almost the same as that of Proposition 2.3.14 except that there is no minimal counts now and we should utilize Lemma 2.3.8 instead. So we omit the proof here. □

2.3.7 A Priori Estimates

We may check [25, Lemma 9.1 - Corollary 9.3] can be recovered with obvious modifications, which is the following result.

Proposition 2.3.16. *There exists a universal constant $0 < \delta < 1$ such that if $\mathcal{G}_N(T) \lesssim \delta$, then*

$$\sup_{0 \leq r \leq t} \mathcal{F}_{2N}(r) \lesssim \mathcal{F}_{2N}(0) + t \int_0^t \mathcal{D}_{2N}(r) \mathrm{d}r, \tag{2.3.161}$$

and

$$\int_0^t \mathcal{K}(r) \mathcal{F}_{2N}(r) \mathrm{d}r \lesssim \delta^{(8+2\lambda)/(8+4\lambda)} \mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(8+4\lambda)} \int_0^t \mathcal{D}_{2N}(r) \mathrm{d}r, \tag{2.3.162}$$

$$\int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\mathrm{dr} \lesssim \mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(16+6\lambda)} \int_0^t \mathcal{D}_{2N}(r)\mathrm{dr} \quad (2.3.163)$$

for all $0 \leq t \leq T$*.*

Hence, we naturally have the a priori estimate for the $2N$ level.

Theorem 2.3.17. *There exists a universal constant* $0 < \delta < 1$ *such that if* $\mathcal{G}_N(T) \lesssim \delta$*, then*

$$\sup_{0\leq r\leq t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N} + \sup_{0\leq r\leq t} \frac{\mathcal{F}_{2N}(r)}{1+r} \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \quad (2.3.164)$$

for all $0 \leq t \leq T$*.*

Proof. This is identical to [25, Theorem 9.4]. Notice that in the section of energy estimates and comparison estimates, we have recovered all the necessary results for the $2N$ level a priori estimate here. □

Then we consider the a priori estimate for the $N+2$ level.

Theorem 2.3.18. *There exists a universal constant* $0 < \delta < 1$ *such that if* $\mathcal{G}_N(T) \lesssim \delta$*, then*

$$\sup_{0\leq r\leq t} (1+r)^{2+\lambda}\mathcal{E}_{N+2,2}(r) \lesssim \mathcal{E}_{2N}(0) \quad (2.3.165)$$

for all $0 \leq t \leq T$*.*

Proof. Utilizing Lemma 2.3.5, we may employ the same argument as in [25, theorem 9.7] to show this. □

Next, we give an interpolation argument to achieve the decay of energy with different minimal counts.

Theorem 2.3.19. *There exists a universal constant* $0 < \delta < 1$ *such that if* $\mathcal{G}_N(T) \lesssim \delta$*, then*

$$\sup_{0 \le r \le t} (1+r)^{1+\lambda} \mathcal{E}_{N+2,1}(r) \lesssim \mathcal{E}_{2N}(0) \tag{2.3.166}$$

for all $0 \le t \le T$.

Proof. By Lemma 2.3.6, we have

$$\mathcal{E}_{N+2,1} \lesssim \mathcal{E}_{N+2,2}^{(1+\lambda)/(2+\lambda)} \mathcal{E}_{2N}^{1/(2+\lambda)}. \tag{2.3.167}$$

Combining with Theorem 2.3.18 and Theorem 2.3.17, we can easily see

$$\begin{aligned} (1+r)^{1+\lambda} \mathcal{E}_{N+2,1}(r) &\lesssim \Big((1+r)^{2+\lambda} \mathcal{E}_{N+2,2}(r) \Big)^{(1+\lambda)/(2+\lambda)} \Big(\mathcal{E}_{2N}(r) \Big)^{1/(2+\lambda)} \\ &\lesssim \Big(\mathcal{E}_{2N}(0) \Big)^{(1+\lambda)/(2+\lambda)} \Big(\mathcal{E}_{2N}(0) \Big)^{1/(2+\lambda)} = \mathcal{E}_{2N}(0). \end{aligned} \tag{2.3.168}$$

Hence, our result easily follows. ∎

Finally, we can sum up all above to achieve the a priori estimate.

Theorem 2.3.20. *There exists a universal constant* $0 < \delta < 1$ *such that if* $\mathcal{G}_N(T) \lesssim \delta$*, then*

$$\mathcal{G}_N(t) \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \tag{2.3.169}$$

for all $0 \leq t \leq T$.

2.3.8 Global Well-posedness

Finally, we come to the conclusion of global well-posedness. With the a priori estimate in hand, we may directly refer to [25, Theorem 11.2], which completes the whole proof.

Theorem 2.3.21. *Suppose the initial data* (u_0, η_0) *satisfies the* $2N^{th}$ *compatibility condition. There exists a* $\kappa > 0$ *such that if* $\|u_0\|^2_{H^{4N}} + \|\eta_0\|^2_{H^{4N+1/2}(\Sigma)} \leq \kappa$, *then there exists a unique solution triple* (u, p, η) *on the interval* $[0, \infty)$ *that satisfies the initial data. The solution obeys the estimate*

$$\mathcal{G}_N(\infty) \leq C(\|u_0\|^2_{H^{4N}} + \|\eta_0\|^2_{H^{4N+1/2}(\Sigma)}) < C\kappa. \tag{2.3.170}$$

Chapter Three

Numerical Solution to Viscous Surface Wave

3.1 Introduction

We consider a two-dimensional incompressible viscous flow in the moving domain

$$\Omega(t) = \{y = (y_1, y_2) \in \Sigma \times \mathbb{R} \ : \ -1 < y_2 < \eta(y_1, t)\}, \tag{3.1.1}$$

where $\Sigma = \mathbb{T}$ for which $\mathbb{T}$ denotes the 1-torus. We denote the initial domain $\Omega(0) = \Omega_0$. For each t, the flow is described by its velocity and pressure $(u, p) : \Omega(t) \mapsto \mathbb{R}^2 \times \mathbb{R}$ which satisfies the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + S & \text{in} \quad \Omega(t), \\ \nabla \cdot u = 0 & \text{in} \quad \Omega(t), \\ (pI - \nu \mathbb{D}(u))\mu = g\eta\mu & \text{on} \quad \{y_2 = \eta(y_1, t)\}, \\ u = 0 & \text{on} \quad \{y_2 = -1\}, \\ \partial_t \eta = u_2 - u_1 \partial_{y_1} \eta & \text{on} \quad \{y_2 = \eta(y_1, t)\}, \\ u(t = 0) = u_0 & \text{in} \quad \Omega_0, \\ \eta(t = 0) = \eta_0 & \text{on} \quad \Sigma, \end{cases} \tag{3.1.2}$$

for μ the outward-pointing unit normal vector on $\{y_2 = \eta\}$, I the 2×2 identity matrix, $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u, g the gravitational constant, $\nu > 0$ the viscosity and $S = S(t, y_1, y_2)$ an external source term. The fifth equation in (3.1.2) implies the free surface is convected with the fluid. Note in (3.1.2), we have shifted the actual pressure $\bar{p}$ by the constant atmosphere pressure p_{atm} according to $p = \bar{p} + gy_2 - p_{atm}$.

We always assume the natural condition that there exists a positive number δ such that $\eta_0 + 1 \geq \delta > 0$ on Σ, which means the initial free surface is strictly separated from the bottom.

Traditionally, based on the handling of the free surface, this type of problems can be solved via moving-grid technique as in [47], marker-and-cell method as in [28], volume-of-fluid method as in [30] and level-set method as in [23]. In each case, finite difference method, finite volume method and finite element method can be applied to solve the Navier-Stokes equation in $\Omega(t)$. However, to the best of authors' knowledge, there is very little in the literature on solving this problem with discontinuous Galerkin method other than [23]. On the other hand, in spite of many computational tests presented in the literature, there is very little discussion on the stability and convergence of the numerical scheme. In this chapter, we employ the idea from [26], [24] and [50], to construct a stable numerical scheme and give detailed analysis.

Our central idea is to flatten the free surface via a coordinates transform. First, we define a fixed domain

$$\Omega = \{x = (x_1, x_2) \in \Sigma \times \mathbb{R} \mid -1 < x_2 < 0\}, \tag{3.1.3}$$

for which we write the coordinates $x \in \Omega$. In this slab, we take $\Sigma : \{x_2 = 0\}$ as the upper boundary and $\Sigma_b : \{x_2 = -1\}$ as the lower boundary.

Consider the geometric transform from Ω to $\Omega(t)$, which is first introduced in [8] and further extended in [26] and [50]:

$$\Phi : (x_1, x_2) \mapsto (x_1, x_2 + \eta(1 + x_2)) = (y_1, y_2). \tag{3.1.4}$$

We may directly verify this transform maps Ω into $\Omega(t)$ with the Jacobi matrix

$$\nabla\Phi = \begin{pmatrix} 1 & 0 \\ A & J \end{pmatrix}, \tag{3.1.5}$$

and the transform matrix

$$\mathscr{A} = ((\nabla\Phi)^{-1})^T = \begin{pmatrix} 1 & -AK \\ 0 & K \end{pmatrix}, \tag{3.1.6}$$

where

$$\tilde{b} = 1 + x_2, \quad A = \partial_1 \eta \tilde{b},$$
$$J = 1 + \eta, \quad K = 1/J.$$

Here we denote the derivative with respect to x_1 as ∂_1 and with respect to x_2 as ∂_2. Define the transformed operators as follows:

$$\begin{aligned}
&(\nabla_{\mathscr{A}} f)_i = \mathscr{A}_{ij}\partial_j f,\\
&\nabla_{\mathscr{A}} \cdot \vec{g} = \mathscr{A}_{ij}\partial_j g_i,\\
&\Delta_{\mathscr{A}} f = \nabla_{\mathscr{A}} \cdot \nabla_{\mathscr{A}} f,\\
&\mathscr{N} = (-\partial_1\eta, 1),\\
&\chi = \partial_1 \eta,\\
&(\mathbb{D}_{\mathscr{A}} u)_{ij} = \mathscr{A}_{ik}\partial_k u_j + \mathscr{A}_{jk}\partial_k u_i,\\
&S_{\mathscr{A}}(p, u) = pI - \mathbb{D}_{\mathscr{A}} u,
\end{aligned} \tag{3.1.7}$$

where the summation index should be understood in the Einstein convention. If we extend the divergence $\nabla_{\mathscr{A}}\cdot$ to act on symmetric tensors in the natural way, then a straightforward computation reveals $\nabla_{\mathscr{A}} \cdot S_{\mathscr{A}}(p, u) = \nabla_{\mathscr{A}} p - \Delta_{\mathscr{A}} u$ for vector fields satisfying $\nabla_{\mathscr{A}} \cdot u = 0$.

In our new coordinates, the original system (3.1.2) becomes

$$\begin{cases} \partial_t u - \partial_t \eta \tilde{b} K \partial_2 u + u \cdot \nabla_{\mathscr{A}} u - \nu \Delta_{\mathscr{A}} u + \nabla_{\mathscr{A}} p = S & \text{in} \quad \Omega, \\ \nabla_{\mathscr{A}} \cdot u = 0 & \text{in} \quad \Omega, \\ S_{\mathscr{A}}(p,u)\mathscr{N} = g\eta\mathscr{N} & \text{on} \quad \Sigma, \\ u = 0 & \text{on} \quad \Sigma_b, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ & \\ \partial_t \eta = u \cdot \mathscr{N} & \text{on} \quad \Sigma, \\ \eta(x',0) = \eta_0(x') & \text{on} \quad \Sigma. \end{cases} \tag{3.1.8}$$

Since $\mathscr{A}$ depends on η through the transform, most of the operators in the Navier-Stokes equations are related to the free surface η. Hence, the Navier-Stokes equations and the transport equation are essentially coupled.

Based on [26], the equation (3.1.8) with $S = 0$ possesses a natural energy equality as follows:

$$\begin{aligned} & \int_\Omega J(t) \left| u(t) \right|^2 + g \int_\Sigma \left| \eta(t) \right|^2 + \nu \int_0^t \int_\Omega J(s) \left| \mathbb{D}_{\mathscr{A}} u(s) \right|^2 \mathrm{ds} \\ = & \int_\Omega J(0) \left| u(0) \right|^2 + g \int_\Sigma \left| \eta(0) \right|^2 . \end{aligned} \tag{3.1.9}$$

Hence, we try to construct a numerical scheme to recover this energy stability for the numerical solution.

In the following, we refer to the term “continuous case” when we consider the exact solution triple (u, p, η) which is sufficiently smooth. On the other hand, we refer to the term “discrete case” when we consider the numerical solution triple (u_h, p_h, η_h).

Throughout this chapter, $C > 0$ denotes a positive constant that only depends on the parameters Ω, g and ν of the problem, and the exact solution (u, p, η). It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a positive constant depending on the quantity z. $a \lesssim b$ denotes $a \leq Cb$, where C is a universal constant as defined above.

The method we discuss in this paper belongs to the class of discontinuous Galerkin (DG) methods. DG methods are finite element methods which use completely discontinuous piecewise polynomial solution and test spaces. They are particularly useful for convection dominated wave problems, and have the advantage of flexibility in adaptivity and efficient parallel implementation. We refer to the references [14, 13, 15, 49] for more details.

3.2 Numerical Scheme

3.2.1 Basic Settings

In our construction and analysis of the numerical scheme, we focus on the semi-discrete form of the system. The time discretization is based on the Runge-Kutta method for differential-algebraic equations of index 2, as presented in [27]. We do not discuss the fully-discrete scheme in this paper.

We choose the bulk domain as $\Omega : (x_1, x_2) \in [0, 1] \times [-1, 0]$ and the surface domain as $\Sigma : x_1 \in [0, 1]$, which are 1-periodic in the x_1 direction. The surface mesh is defined by dividing Σ into N uniform elements with length $h = 1/N$. The bulk mesh construction can be divided into two steps: First we divide Ω into $N \times N$ uniform squares, where each element is sized $h \times h$. Then, we divide each square into two right triangles by cutting along the diagonal from the left-up corner to the right-down corner. The bulk mesh is shown in Figure 3.1.

Remark 3.2.1. *Since the discretization of the weighted differential operators* $\nabla_{\mathscr{A}}$*,* $\nabla_{\mathscr{A}}\cdot$ *and* $\Delta_{\mathscr{A}}$ *depends on* η*, i.e. the surface variable, we need the bulk discretization to match the surface discretization in the vertical direction. Hence, we need to first choose the strip-shape mesh as in Figure 3.2, whose projection on the upper boundary is exactly the surface mesh. Then in each strip, any triangulation is permitted. For convenience, we made the choice as in Figure 3.1.*

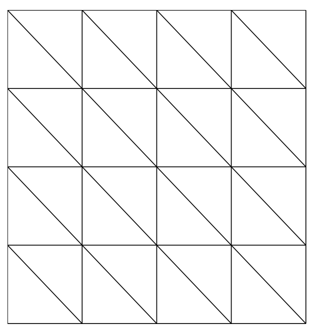

Figure 3.1: Mesh Distribution in $N = 4$

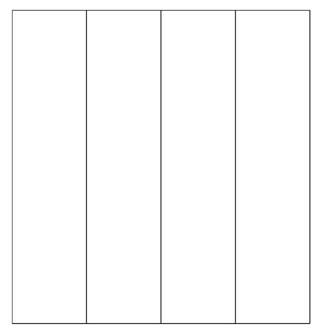

Figure 3.2: Strip-Shape Mesh Distribution in $N = 4$

Let $\mathcal{E}_h$ be the set consisting of all the triangular elements in the bulk mesh and $\partial\mathcal{E}_h$ be the set of all the sides of the triangular cells. Let $\mathcal{F}_h$ be the set consisting of all the interval elements in the surface mesh and $\partial\mathcal{F}_h$ be the set of all the boundary points of intervals.

Define the usual Sobolev space in Ω as $H^k(\Omega)$ and in Σ as $H^k(\Sigma)$. We define the space $P_h^k(\Omega)$ where $f \in P_h^k(\Omega)$ if and only if $f|_E \in P^k(E)$ for all $E \in \mathcal{E}_h$ ($P^k(E)$ denotes the set of polynomials of degree at most k defined on E). Similarly, we can define the space $P_h^k(\Sigma)$.

We define the solution spaces as

$$X_h^k = \{u_h \in (L^2(\Omega))^2 : \ u_h \in (P_h^k(\Omega))^2\}, \tag{3.2.1}$$

$$M_h^k = \{p_h \in L^2(\Omega) : \ p_h \in P_h^k(\Omega)\}, \tag{3.2.2}$$

$$S_h^k = \{\eta_h \in L^2(\Sigma) : \ \eta_h \in P_h^k(\Sigma)\}, \tag{3.2.3}$$

for $k \geq 1$.

For discrete functions $f_h \in X_h^k$ and $g_h \in S_h^k$, we may define the H_h norms as follows:

$$\|f_h\|_{H_h}^2 = \sum_{E\in\mathcal{E}_h} \|f_h\|_{H^1(E)}^2 , \tag{3.2.4}$$

$$\|g_h\|_{H_h}^2 = \sum_{F\in\mathcal{F}_h} \|g_h\|_{H^1(F)}^2 . \tag{3.2.5}$$

Define the X_h norm for $v_h \in X_h^k$ as

$$\|v_h\|_{X_h}^2 = \sum_{E\in\mathcal{E}_h} \left(\|\nabla v_h\|_{L^2(E)}^2 + \int_{\partial E} \frac{1}{h}[v_h]^2 \right). \tag{3.2.6}$$

Before stating the scheme, we announce the notation used below. For a given boundary belonging to a reference cell $E \in \mathcal{E}_h$ and a function f defined in Ω, $f^{ext(E)}$ denotes the value of f read from the exterior direction on the boundary and $f^{int(E)}$ denotes that read from the interior direction. Also for the boundary shared by two cells, we define

$$\{f\} = \frac{1}{2}(f^{ext(E)} + f^{int(E)}), \tag{3.2.7}$$

$$[f] = f^{int(E)} - f^{ext(E)}. \tag{3.2.8}$$

For the side on the boundary of Ω, we make a special definition that on Σ_b, $f^{int(E)} =$

$\{f\} = [f]$ and $f^{ext(E)} = 0$, while on Σ, $f^{int(E)} = f^{ext(E)} = \{f\}$ and $[f] = 0$. The definitions are reasonable, since the system (3.1.8) for u gives Dirichlet-type condition on Σ_b and Neumann-type condition on Σ.

The numerical scheme is based on the weak formulation of the system (3.1.8). Taking test functions $v_h \in X_h^k$ and $q_h \in M_h^{k-1}$. We multiply Jv_h and Jq_h respectively on both sides of the Navier-Stokes equations and integrate over $E \in \mathcal{E}_h$ to obtain

$$\begin{aligned} &\int_E J\partial_t u \cdot v_h - \int_E J(\partial_t \eta \tilde{b} K \partial_2 u) \cdot v_h \qquad (3.2.9)\\ &+ \int_E J(u \cdot \nabla_{\mathscr{A}} u) \cdot v_h - \nu \int_E J\Delta_{\mathscr{A}} u \cdot v_h + \int_E J\nabla_{\mathscr{A}} p \cdot v_h = \int_E J(S \cdot v_h), \\ &\int_E J(\nabla_{\mathscr{A}} \cdot u) q_h = 0. \qquad (3.2.10) \end{aligned}$$

Considering the transport equation in (3.1.8), we make an addition and subtraction to modify two terms in above formulation as

$$\begin{aligned} \Big(\int_E J\partial_t u \cdot v_h - \int_E J(\partial_t \eta \tilde{b} K \partial_2 u) \cdot v_h + \frac{1}{2} \int_{\partial E \cap \Sigma} \partial_t \eta (u \cdot v_h) \Big) \qquad (3.2.11)\\ + \Big(\int_E J(u \cdot \nabla_{\mathscr{A}} u) \cdot v_h - \frac{1}{2} \int_{\partial E \cap \Sigma} (u \cdot \mathscr{N})(u \cdot v_h) \Big) \\ -\nu \int_E J\Delta_{\mathscr{A}} u \cdot v_h + \int_E J\nabla_{\mathscr{A}} p \cdot v_h &= \int_E J(S \cdot v_h), \\ \int_E J(\nabla_{\mathscr{A}} \cdot u) q_h &= 0, \qquad (3.2.12) \end{aligned}$$

where ∂E denotes the boundary of the triangular element E. This trick is to enforce the transport relation and shows its power in the stability proof.

Multiplying a test functions $\phi_h \in S_h^k$ on both sides of the transport equation and integrating over $F \in \mathcal{F}_h$, we obtain

$$\int_F \partial_t \eta \phi_h + \int_F \bar{u}_1 \partial_1 \eta \phi_h - \int_F \bar{u}_2 \phi_h = 0, \qquad (3.2.13)$$

where $\bar{u}_1$ and $\bar{u}_2$ denote the traces of u_1 and u_2 on Σ.

The weak formulations of the system (3.1.8) are based on the integration by parts of (3.2.11), (3.2.12) and (4.1.1). In the following, we define and analyze the numerical scheme term by term.

3.2.2 Discretization

We define the semi-discrete scheme as follows:

$$\left\{\begin{array}{rcl} \kappa_F(\eta_h, u_h, \phi_h) & = & 0, \\ \lambda_F(\chi_h, \psi_h) & = & 0, \\ & & \\ \zeta_E(\eta_h, u_h, v_h) + \gamma_E(u_h, \eta_h, u_h, v_h) + \nu\alpha_E(\eta_h, u_h, \eta_h, v_h) & & \\ +\beta_E(p_h, \eta_h, v_h) + g\mu_E(\eta_h, \eta_h, v_h) & = & \omega_E(\eta_h, v_h), \\ \rho_E(u_h, \eta_h, q_h) & = & 0, \end{array}\right. \tag{3.2.14}$$

for any test functions $\phi_h \in S_h^k$, $\psi_h \in S_h^k$, $v_h \in X_h^k$ and $q_h \in M_h^{k-1}$, where the first two equations denote the transport discretization and the last two equations denote the fluid discretization. This forms a complete differential-algebraic system for $\eta_h(t) \in S_h^k$, $\chi_h(t) \in S_h^k$, $u_h(t) \in X_h^k$ and $p_h(t) \in M_h^{k-1}$. The detailed definitions of above multi-linear terms κ, λ, ζ, γ, α, β, μ, ω, and ρ are in the following.

Discretization of Transport Terms as κ_F and λ_F

We define

$$
\begin{aligned}
& \kappa_F(\eta_h, u_h, \phi_h) && (3.2.15)\\
= \; & \int_F \partial_t \eta_h \phi_h - \int_F (\bar{u}_2)_h \phi_h - \int_F \partial_1 (\bar{u}_1)_h \eta_h \phi_h - \int_F (\bar{u}_1)_h \eta_h \partial_1 \phi_h \\
& +(\hat{u}_1)_h \hat{\eta}_h \phi_h^- |_{F+1/2} - (\hat{u}_1)_h \hat{\eta}_h \phi_h^+ |_{F-1/2} + \frac{1}{2}\Big(\eta_h^- \phi_h^- [u_h]|_{F+1/2} - \eta_h^+ \phi_h^+ [u_h]|_{F-1/2} \Big) \\
& \lambda_F(\chi_h, \psi_h) && (3.2.16)\\
= \; & \int_F \chi_h \psi_h + \int_F \eta_h \partial_1 \psi_h - \eta_h^+ \psi_h^- |_{F+1/2} + \eta_h^+ \psi_h^+ |_{F-1/2},
\end{aligned}
$$

where $F - 1/2$ and $F + 1/2$ denote the left and right boundary points of F. Here, for each boundary point, "$-$" means the value read from the left and "+" means the value read from the right. Also, $(\hat{u}_1)_h \; \hat{\eta}_h$ denotes the numerical flux on the boundary. We always take

$$(\hat{u}_1)_h = \{(\bar{u}_1)_h\}, \tag{3.2.17}$$

and $\hat{\eta}_h$ should be determined from the sign of $\{(\bar{u}_1)_h\}$ following the upwinding rule as

$$\hat{\eta}_h = \begin{cases} \eta_h^- & \text{if } \; \{(\bar{u}_1)_h\} \geq 0, \\ \eta_h^+ & \text{if } \; \{(\bar{u}_1)_h\} < 0, \end{cases} \tag{3.2.18}$$

where $(\bar{u}_1)_h$ is the trace of $(u_1)_h$ on Σ. Note $\frac{1}{2}\Big(\eta_h^- \phi_h^- [u_h]|_{F+1/2} - \eta_h^+ \phi_h^+ [u_h]|_{F-1/2} \Big)$ are extra penalty terms, which helps to show the stability. Since u_h is also discontinuous across the boundary point, these penalty terms are used to transform u_h into $\{u_h\}$ on the boundary after integrating by parts in (3.2.15).

In all the applications below, we use χ_h to discretize $\partial_1\eta$, and η_h to discretize η. Hence, we denote $\mathscr{N}_h$ for the vector $(-\chi_h, 1)$. Also, A_h, J_h, K_h and $\mathscr{A}_h$ can be defined in the same convention.

Spacial Discretization of Temporal Terms as ζ_E

We define

$$\begin{aligned} &\zeta_E(\eta_h, u_h, v_h) \\ = \; & \int_E \partial_t(J_h u_h)\cdot v_h + \int_E \partial_t\eta_h\tilde{b}u_h\cdot\partial_2 v_h - \int_{\partial E}\partial_t\eta_h\tilde{b}(\hat{u}_h\cdot v_h^{int(E)})n_2 \\ & +\frac{1}{2}\int_{\partial E\cap\Sigma}\partial_t\eta_h(u_h\cdot v_h), \end{aligned} \tag{3.2.19}$$

where $n_E = (n_1, n_2)$ denotes the outward normal vector on ∂E. The last term in (3.2.19) is the newly-added penalty term in the formulation (3.2.11). The argument η_h in $\zeta_E(\eta_h, u_h, v_h)$ denotes η_h and its derivatives. We utilize the upwinding flux in this scheme, i.e.

$$\hat{u}_h = \begin{cases} u_h^+ & \text{if } \partial_t\eta_h\tilde{b}\geq 0, \\ u_h^- & \text{if } \partial_t\eta_h\tilde{b} < 0, \end{cases} \tag{3.2.20}$$

where “+” denotes the value read from the up direction and “−” denote that from the down direction. If a side is vertical, then its contribution is zero.

Discretization of Convection Terms as γ_E

We define

$$
\begin{aligned}
& \gamma_E(u_h, \eta_h, u_h, v_h) \qquad (3.2.21)\\
= \ & -\frac{1}{2}\int_{\partial E\cap\Sigma}(u_h\cdot\mathscr{N}_h)(u_h\cdot v_h) + \int_E J_h(u_h\cdot\nabla_{\mathscr{A}_h}u_h)\cdot v_h \\
& +\frac{1}{2}\int_E J_h(\nabla_{\mathscr{A}_h}\cdot u_h)(u_h\cdot v_h) - \frac{1}{2}\int_{\partial E\setminus\partial\Omega}[u_h\cdot J_h\mathscr{A}_h]\cdot n_E\frac{u_h^{int(E)}\cdot v_h^{int(E)}}{2} \\
& +\int_{\partial E^-}\left|\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E\right|(u_h^{int(E)} - u_h^{ext(E)})\cdot v_h^{int(E)},
\end{aligned}
$$

where $\nabla_{\mathscr{A}_h}$ is understood as in (3.1.7) with $\mathscr{A}$ replaced by $\mathscr{A}_h$ and

$$\partial E^- = \{x\in\partial E : \{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E < 0\}. \qquad (3.2.22)$$

The first term in (3.2.21) is the newly added penalty term in the formulation (3.2.11). Since

$$J_h\mathscr{A}_h = \begin{pmatrix} J_h & -A_h \\ 0 & 1 \end{pmatrix}, \qquad (3.2.23)$$

we only need one η_h argument in $\gamma_E(u_h, \eta_h, u_h, v_h)$.

Discretization of Diffusion Terms as α_E

We utilize the Symmetric Internal Penalty Galerkin Method (SIPG) or the Nonsymmetric Internal Penalty Galerkin Method (NIPG) to define

$$\begin{aligned} & \alpha_E(\eta_h, u_h, \eta_h, v_h) \qquad (3.2.24) \\ = \; & \frac{1}{2}\int_E J_h \mathbb{D}_{\mathscr{A}_h} u_h : \mathbb{D}_{\mathscr{A}_h} v_h - \int_{\partial E\setminus\Sigma} \{\mathbb{D}_{\mathscr{A}_h} u_h J_h \mathscr{A}_h\} \cdot n_E \cdot v_h^{int(E)} \\ & \pm \frac{1}{2}\int_{\partial E\setminus\Sigma} \mathbb{D}_{\mathscr{A}_h^{int(E)}} v_h^{int(E)} J_h^{int(E)} \mathscr{A}_h^{int(E)} \cdot n_E \cdot [u_h] + \frac{\sigma}{h}\int_{\partial E} [u_h] \cdot v_h^{int(E)}. \end{aligned}$$

For the third term, if we take $+$, it is NIPG and if we take $-$, it is SIPG. The penalty constant σ can always be taken as 1 for NIPG and a sufficiently large number for SIPG. Note that we need two η_h arguments in $\alpha_E(\eta_h, u_h, \eta_h, v_h)$ to denote the dependence of $\mathscr{A}_h$.

Discretization of Pressure Terms as β_E

We define

$$\beta_E(p_h, \eta_h, v_h) \;=\; -\int_E J_h p_h \nabla_{\mathscr{A}_h} \cdot v_h + \int_{\partial E\setminus\Sigma} v_h^{int(E)} \cdot \{p_h J_h \mathscr{A}_h\} \cdot n_E. \qquad (3.2.25)$$

Discretization of Forcing Terms as μ_E

We define

$$\begin{aligned} & \mu_E(\eta_h, \eta_h, v_h) \qquad (3.2.26) \\ = \; & \int_{\partial E\cap\Sigma} \eta_h (v_2)_h - \int_{\partial E\cap\Sigma} \eta_h \partial_1 \eta_h (\bar{v}_1)_h + \frac{1}{4}[\eta_h^2]\bar{v}_h^-|_{\partial E\cap\Sigma+1/2} - \frac{1}{4}[\eta_h^2]\bar{v}_h^+|_{\partial E\cap\Sigma-1/2}, \end{aligned}$$

where $\partial E \cap \Sigma + 1/2$ and $\partial E \cap \Sigma - 1/2$ denote the left and right boundary points of $\partial E \cap \Sigma$ respectively, i.e. similar to $F - 1/2$ and $F + 1/2$. Note that the forcing term is nontrivial only for the top cells and for all other cells it is zero, $\mu_E(\eta_h, \eta_h, v_h) = 0$. $\frac{1}{4}[\eta_h^2]\bar{v}_h^-|_{\partial E\cap\Sigma+1/2} - \frac{1}{4}[\eta_h^2]\bar{v}_h^+|_{\partial E\cap\Sigma-1/2}$ are extra penalty terms, which help to show the stability. Since η_h is discontinuous across the boundary, these penalty terms are used to transform η_h^2 into $\{\eta_h^2\}$ on the boundary after integrating by parts in (3.4.60).

Discretization of Source Terms as ω_E

We define

$$\omega_E(\eta_h, v_h) = \int_E J_h S \cdot v_h. \tag{3.2.27}$$

Discretization of Divergence Terms as ρ_E

We define

$$\begin{aligned} &\rho_E(u_h, \eta_h, q_h) \\ = \; & -\int_E J_h q_h \nabla_{\mathscr{A}_h} \cdot u_h + \frac{1}{2}\int_{\partial E\setminus\Sigma} [u_h] \cdot q_h^{int(E)} J_h^{int(E)} \mathscr{A}_h^{int(E)} \cdot n_E. \end{aligned} \tag{3.2.28}$$

3.2.3 Properties and Estimates of Discretization

Estimate of Temporal Terms

We consider the temporal terms, i.e.

$$\left(\int_E J \partial_t u \cdot v_h - \int_E J(\partial_t \eta \tilde{b} K \partial_2 u) \cdot v_h + \frac{1}{2} \int_{\partial E \cap \Sigma} \partial_t \eta (u \cdot v_h) \right). \tag{3.2.29}$$

In the continuous case, a direct integration by parts reveals

$$\begin{aligned} \int_E J \partial_t u \cdot v_h &= \int_E \partial_t (Ju) \cdot v_h - \int_E \partial_t J (u \cdot v_h), && (3.2.30) \\ - \int_E J(\partial_t \eta \tilde{b} K \partial_2 u) \cdot v_h &= - \int_E \partial_t \eta \tilde{b} \partial_2 u \cdot v_h && (3.2.31) \\ &= \int_E \partial_t \eta \tilde{b} u \cdot \partial_2 v_h + \int_E \partial_2 (\partial_t \eta \tilde{b})(u \cdot v_h) - \int_{\partial E} \partial_t \eta \tilde{b} (u \cdot v_h) n_2. \end{aligned}$$

Since $\partial_t J = \partial_2 (\partial_t \eta \tilde{b})$, we can simplify above terms into

$$\begin{aligned} & \int_E J \partial_t u \cdot v_h - \int_E J(\partial_t \eta \tilde{b} K \partial_2 u) \cdot v_h && (3.2.32) \\ = & \int_E \partial_t (Ju) \cdot v_h + \int_E \partial_t \eta \tilde{b} u \cdot \partial_2 v_h - \int_{\partial E} \partial_t \eta \tilde{b} (u \cdot v_h) n_2. \end{aligned}$$

Hence, we may define the discretization of the temporal term as (3.2.19). Since in our bulk mesh, there are only two effective boundary integrals for each triangle, we may call them the upper boundary $\partial E + 1/2$ and lower boundary $\partial E - 1/2$ without ambiguity. If we denote $f(u_h) = -\partial_t \eta_h \tilde{b} u_h$, then (3.2.19) is actually

$$\begin{aligned} & \zeta_E(\eta_h, u_h, v_h) - \frac{1}{2} \int_{\partial E \cap \Sigma} \partial_t \eta_h (u_h \cdot v_h) && (3.2.33) \\ = & \int_E \partial_t (J_h u_h) \cdot v_h - \int_E f(u_h) \cdot \partial_2 v_h + \int_{\partial E + 1/2} \hat{f}(u_h) \cdot v_h^- - \int_{\partial E - 1/2} \hat{f}(u_h) \cdot v_h^+, \end{aligned}$$

where the flux reduces to

$$\hat{f}(u_h) = f(\hat{u}_h) = \begin{cases} f(u_h^+) & \text{if } \partial_t \eta_h \tilde{b} \geq 0, \\ f(u_h^-) & \text{if } \partial_t \eta_h \tilde{b} < 0. \end{cases} \tag{3.2.34}$$

Lemma 3.2.2. *If we take $v_h = u_h$, the discretized temporal term satisfies*

$$\sum_{E \in \mathcal{E}_h} \zeta_E(\eta_h, u_h, u_h) \geq \frac{1}{2} \partial_t \int_\Omega J_h \left|u_h\right|^2. \tag{3.2.35}$$

Proof. Note the fact that $\partial_t J_h = \partial_2(\partial_t \eta_h \tilde{b})$. We can directly compute

$$\begin{aligned}
& \zeta_E(\eta_h, u_h, u_h) - \frac{1}{2} \int_{\partial E \cap \Sigma} \partial_t \eta_h \left|u_h\right|^2 && (3.2.36) \\
= & \int_E \partial_t (J_h u_h) \cdot u_h - \int_E f(u_h) \cdot \partial_2 u_h + \int_{\partial E+1/2} \hat{f}(u_h) \cdot u_h^- - \int_{\partial E-1/2} \hat{f}(u_h) \cdot u_h^+ \\
= & \frac{1}{2} \partial_t \int_E J_h \left|u_h\right|^2 + \frac{1}{2} \int_E \partial_t J_h \left|u_h\right|^2 - \frac{1}{2} \int_E \partial_2(\partial_t \eta_h \tilde{b}) \left|u_h\right|^2 \\
& - \frac{1}{2} \int_{\partial E+1/2} f(u_h^-) \cdot u_h^- + \frac{1}{2} \int_{\partial E-1/2} f(u_h^+) \cdot u_h^+ \\
& + \int_{\partial E+1/2} \hat{f}(u_h) \cdot u_h^- - \int_{\partial E-1/2} \hat{f}(u_h) \cdot u_h^+ \\
= & \frac{1}{2} \partial_t \int_E J_h \left|u_h\right|^2 \\
& - \frac{1}{2} \int_{\partial E+1/2} f(u_h^-) \cdot u_h^- + \frac{1}{2} \int_{\partial E-1/2} f(u_h^+) \cdot u_h^+ \\
& + \int_{\partial E+1/2} \hat{f}(u_h) \cdot u_h^- - \int_{\partial E-1/2} \hat{f}(u_h) \cdot u_h^+ \\
= & \frac{1}{2} \partial_t \int_E J_h \left|u_h\right|^2 + \hat{F}_{\partial E+1/2} - \hat{F}_{\partial E-1/2} + \Theta_E,
\end{aligned}$$

where

$$\hat{F}_{\partial E+1/2} = -\frac{1}{2} \int_{\partial E+1/2} f(u_h^-) \cdot u_h^- + \int_{\partial E+1/2} \hat{f}(u_h) \cdot u_h^-, \tag{3.2.37}$$

$$\hat{F}_{\partial E-1/2} = -\frac{1}{2} \int_{\partial E-1/2} f(u_h^-) \cdot u_h^- + \int_{\partial E-1/2} \hat{f}(u_h) \cdot u_h^-, \tag{3.2.38}$$

and

$$\Theta_E = \frac{1}{2}\int_{\partial E-1/2}\Big(f(u_h^+)\cdot u_h^+ - f(u_h^-)\cdot u_h^-\Big) + \int_{\partial E-1/2}\Big(\hat{f}(u_h)\cdot u_h^- - \hat{f}(u_h)\cdot u_h^+\Big). \tag{3.2.39}$$

Based on the flux definition (3.2.34), we have the estimate

$$\Theta_E = \int_{\partial E-1/2}\Big(\partial_t\eta_h\tilde{b}(u_h^+ - u_h^-)\cdot(\hat{u}_h - \frac{u_h^+ + u_h^-}{2})\Big) \geq 0. \tag{3.2.40}$$

Hence, combining with (3.2.36), we obtain

$$\zeta_E(\eta_h, u_h, u_h) - \frac{1}{2}\int_{\partial E\cap\Sigma}\partial_t\eta_h\left|u_h\right|^2 \geq \frac{1}{2}\partial_t\int_E J_h\left|u_h\right|^2 + \hat{F}_{\partial E+1/2} - \hat{F}_{\partial E-1/2}. \tag{3.2.41}$$

When summing up over all $E \in \mathcal{E}_h$, we can easily see when $\partial E \subset \Omega$, all the terms involving $\hat{F}$ are canceled out. Therefore, only the terms on $\partial\Omega$ remain, i.e.

$$\begin{aligned} &\sum_{E\in\mathcal{E}_h}\zeta_E(\eta_h, u_h, u_h) - \frac{1}{2}\int_\Sigma \partial_t\eta_h\left|u_h\right|^2 \\ \geq\ &\sum_{E\in\mathcal{E}_h}\frac{1}{2}\partial_t\int_E J_h\left|u_h\right|^2 + \sum_{\partial E\cap\Sigma\neq\emptyset}\hat{F}_{\partial E+1/2} - \sum_{\partial E\cap\Sigma_b\neq\emptyset}\hat{F}_{\partial E-1/2}. \end{aligned} \tag{3.2.42}$$

On Σ_b we have $u_h^- = 0$, so $-\hat{F}_{\partial E-1/2} = 0$, i.e.

$$\sum_{\partial E\cap\Sigma_b\neq\emptyset}\hat{F}_{\partial E-1/2} = 0. \tag{3.2.43}$$

On Σ we have $u_h^+ = u_h^- = u_h|_\Sigma$ and we always take $\hat{u}_h = u_h$, so it implies

$$\hat{F}_{\partial E+1/2} = -\frac{1}{2}\int_{\partial E+1/2} f(u_h^-)u_h^- + \int_{\partial E+1/2}\hat{f}(u_h)u_h^- = -\frac{1}{2}\int_{\partial E\cap\Sigma}\partial_t\eta_h\left|u_h\right|^2. \tag{3.2.44}$$

Then summing up over $E \in \mathcal{E}_h$ gives a full integration over Σ, i.e.

$$\sum_{\partial E \cap \Sigma \neq \emptyset} \hat{F}_{\partial E+1/2} = -\frac{1}{2}\int_\Sigma \partial_t \eta_h \left|u_h\right|^2. \tag{3.2.45}$$

Therefore, combining (3.2.42), (3.2.43) and (3.2.45), we deduce

$$\sum_{E\in\mathcal{E}_h} \zeta_E(\eta_h, u_h, u_h) - \frac{1}{2}\int_\Sigma \partial_t\eta_h \left|u_h\right|^2 \geq \frac{1}{2}\partial_t \int_E J_h \left|u_h\right|^2 - \frac{1}{2}\int_\Sigma \partial_t \eta_h \left|u_h\right|^2. \tag{3.2.46}$$

Hence, our result easily follows. □

Remark 3.2.3. *This proof is based on the mesh as in Figure 1. For general triangulation, it is possible to have three effective boundaries for each element. However, it is easy to see Lemma 3.2.2 still holds.*

Estimate of Convection Terms

We consider the convection term

$$\int_E J(u\cdot\nabla_{\mathscr{A}} u)\cdot v_h - \frac{1}{2}\int_F (u\cdot\mathscr{N})(u\cdot v_h). \tag{3.2.47}$$

This is the key nonlinear term in the Navier-Stokes equations. Our discretization is inspired by the idea in [22]. In the continuous case, when summing up over all $E \in \mathcal{E}_h$, we can see

$$\begin{aligned} &\sum_{E\in\mathcal{E}_h} \gamma_E(u,\eta,u,v_h) \qquad (3.2.48)\\ =\ &\int_\Omega J(u\cdot\nabla_{\mathscr{A}} u)\cdot v_h + \frac{1}{2}\int_\Omega J(\nabla_{\mathscr{A}}\cdot u)(u\cdot v_h) - \frac{1}{2}\int_\Sigma (u\cdot\mathscr{N})(u\cdot v_h), \end{aligned}$$

where all the other boundary terms vanish. Hence, considering the continuous $\mathscr{A}$-divergence-free condition for u, this discretization is consistent.

Lemma 3.2.4. *If we take $v_h = u_h$, the discretized convection term satisfies*

$$\sum_{E \in \mathcal{E}_h} \gamma_E(u_h, \eta_h, u_h, u_h) \geq 0. \tag{3.2.49}$$

Proof. We divide the proof into several steps:

Step 1: Direct integration by parts.

We plug the test function $v_h = u_h$ into (3.2.21) to obtain

$$\begin{aligned} & \gamma_E(u_h, \eta_h, u_h, u_h) + \frac{1}{2}\int_F (u_h \cdot \mathscr{N}_h)\, |u_h|^2 \\ = \; & \int_E J_h(u_h \cdot \nabla_{\mathscr{A}_h} u_h) \cdot u_h + \frac{1}{2}\int_E J_h(\nabla_{\mathscr{A}_h} \cdot u_h)(u_h \cdot u_h) \\ & -\frac{1}{2}\int_{\partial E \setminus \partial\Omega} [u_h \cdot J_h \mathscr{A}_h] \cdot n_E \frac{u_h^{int(E)} \cdot u_h^{int(E)}}{2} \\ & + \int_{\partial E^-} |\{u_h \cdot J_h \mathscr{A}_h\} \cdot n_E|\, (u_h^{int(E)} - u_h^{ext(E)}) \cdot u_h^{int(E)}, \end{aligned} \tag{3.2.50}$$

A direct integration by parts yields

$$\begin{aligned} & \int_E J_h(u_h \cdot \nabla_{\mathscr{A}_h} u_h) \cdot u_h \\ = \; & -\int_E J_h(u_h \cdot \nabla_{\mathscr{A}_h} u_h) \cdot u_h - \int_E J_h(\nabla_{\mathscr{A}_h} \cdot u_h)(u_h \cdot u_h) \\ & + \int_{\partial E} (u_h^{int(E)} \cdot J_h^{int(E)} \mathscr{A}_h^{int(E)} \cdot n_E)(u_h^{int(E)} \cdot u_h^{int(E)}). \end{aligned} \tag{3.2.51}$$

Therefore, (3.2.50) can be simplified as

$$
\begin{aligned}
& \gamma_E(u_h, \eta_h, u_h, u_h) + \frac{1}{2}\int_{\partial E\cap\Sigma} (u_h \cdot \mathscr{N}_h)\,|u_h|^2 \qquad (3.2.52)\\
= \; & -\Big(\int_E J_h(u_h \cdot \nabla_{\mathscr{A}_h} u_h)\cdot u_h + \frac{1}{2}\int_E J_h(\nabla_{\mathscr{A}_h}\cdot u_h)(u_h\cdot u_h)\Big) \\
& -\frac{1}{2}\int_{\partial E\backslash\partial\Omega} [u_h \cdot J_h\mathscr{A}_h]\cdot n_E \frac{u_h^{int(E)}\cdot u_h^{int(E)}}{2} \\
& +\int_{\partial E^-} |\{u_h \cdot J_h\mathscr{A}_h\}\cdot n_E|\,(u_h^{int(E)} - u_h^{ext(E)})\cdot u_h^{int(E)} \\
& +\int_{\partial E} (u_h^{int(E)} \cdot J_h^{int(E)}\mathscr{A}_h^{int(E)}\cdot n_E)(u_h^{int(E)}\cdot u_h^{int(E)}) \\
= \; & I + II + III + IV,
\end{aligned}
$$

where

$$
I = -\Big(\int_E J_h(u_h \cdot \nabla_{\mathscr{A}_h} u_h)\cdot u_h + \frac{1}{2}\int_E J_h(\nabla_{\mathscr{A}_h}\cdot u_h)(u_h\cdot u_h)\Big), \qquad (3.2.53)
$$

and II, III and IV can be understood respectively.

Step 2: Estimates of II and IV.

In IV, for $e \in \partial E\backslash\partial\Omega$, we have the decomposition

$$
\begin{aligned}
& u_h^{int(E)} \cdot J_h^{int(E)}\mathscr{A}_h^{int(E)} \qquad (3.2.54)\\
= \; & \{u_h \cdot J_h\mathscr{A}_h\} + \frac{1}{2}(u_h^{int(E)}\cdot J_h^{int(E)}\mathscr{A}_h^{int(E)} - u_h^{ext(E)}\cdot J_h^{ext(E)}\mathscr{A}_h^{ext(E)}).
\end{aligned}
$$

For $e \subset \partial E\backslash\partial\Omega$, we can sum up the second term on the right-hand side of (3.2.54) over E and its neighboring cells, which have the opposition outward normal vectors, to show

$$
\begin{aligned}
&\int_e \frac{1}{2}(u_h^{int(E)} \cdot J_h^{int(E)} \mathscr{A}_h^{int(E)} - u_h^{ext(E)} \cdot J_h^{ext(E)} \mathscr{A}_h^{ext(E)}) \cdot n_E (u_h^{int(E)} \cdot u_h^{int(E)}) \\
&+ \int_e \frac{1}{2}(u_h^{ext(E)} \cdot J_h^{ext(E)} \mathscr{A}_h^{ext(E)} - u_h^{int(E)} \cdot J_h^{int(E)} \mathscr{A}_h^{int(E)}) \cdot (-n_E)(u_h^{ext(E)} \cdot u_h^{ext(E)}) \\
= \ & \int_e [u_h \cdot J_h \mathscr{A}_h] \cdot n_e \{u_h \cdot u_h\},
\end{aligned} \tag{3.2.55}
$$

where n_e denotes the normal vector read from the same reference cell as $[u_h \cdot J_h \mathscr{A}_h]$. Note for $e \subset \partial E \cap \partial\Omega$, $u_h^{int(E)} \cdot J_h^{int(E)} \mathscr{A}_h^{int(E)} = \{u_h \cdot J_h \mathscr{A}_h\}$. Therefore, we obtain

$$
\begin{aligned}
\sum_{E \in \mathcal{E}_h} IV = & \sum_{e \in \partial\mathcal{E}_h} \int_e \{u_h \cdot J_h \mathscr{A}_h\} \cdot n_E (u_h^{int(E)} \cdot u_h^{int(E)}) \\
& + \sum_{e \in \partial\mathcal{E}_h \backslash \partial\Omega} \int_e [u_h \cdot J_h \mathscr{A}_h] \cdot n_e \{u_h \cdot u_h\}.
\end{aligned} \tag{3.2.56}
$$

Also, in II, for $e \subset \partial E \backslash \partial\Omega$, we can sum up over E and its neighboring cells to achieve

$$
\begin{aligned}
&-\frac{1}{2}\int_e [u_h \cdot J_h \mathscr{A}_h] \cdot n_E \frac{u_h^{int(E)} \cdot u_h^{int(E)}}{2} - \frac{1}{2}\int_e (-[u_h \cdot J_h \mathscr{A}_h]) \cdot (-n_E) \frac{u_h^{ext(E)} \cdot u_h^{ext(E)}}{2} \\
= \ & -\frac{1}{2}\int_e [u_h \cdot J_h \mathscr{A}_h] \cdot n_e \{u_h \cdot u_h\},
\end{aligned} \tag{3.2.57}
$$

which further leads to

$$
\sum_{E \in \mathcal{E}_h} II = -\sum_{e \in \partial\mathcal{E}_h \backslash \partial\Omega} \frac{1}{2} \int_e [u_h \cdot J_h \mathscr{A}_h] \cdot n_e \{u_h \cdot u_h\}. \tag{3.2.58}
$$

Step 3: Further estimates in (3.2.52).

Hence, summing over all $E \in \mathcal{E}_h$ in (3.2.52) and combining (3.2.56) and (3.2.58), we deduce

$$
\begin{aligned}
&\sum_{E\in\mathcal{E}_h}\gamma_E(u_h,\eta_h,u_h,u_h)+\frac{1}{2}\int_{\Sigma}(u_h\cdot\mathscr{N}_h)\,|u_h|^2 \\
= \; & -\Big(\int_{\Omega}J_h(u_h\cdot\nabla_{\mathscr{A}_h}u_h)\cdot u_h+\frac{1}{2}\int_{\Omega}J_h(\nabla_{\mathscr{A}_h}\cdot u_h)(u_h\cdot u_h) \\
& -\sum_{e\in\partial\mathcal{E}_h\backslash\partial\Omega}\frac{1}{2}\int_e[u_h\cdot J_h\mathscr{A}_h]\cdot n_E\{u_h\cdot u_h\}\Big) \\
& +\sum_{E\in\mathcal{E}_h}\Big(\int_{\partial E^-}|\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E|\,(u_h^{int(E)}-u_h^{ext(E)})\cdot u_h^{int(E)} \\
& +\int_{\partial E}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)})\Big).
\end{aligned}
\tag{3.2.59}
$$

We can further estimate the last two terms in (3.2.59) as follows:

$$
\begin{aligned}
&\sum_{E\in\mathcal{E}_h}\Big(\int_{\partial E^-}|\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E|\,(u_h^{int(E)}-u_h^{ext(E)})\cdot u_h^{int(E)} \\
& +\int_{\partial E}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)})\Big) \\
= \; & \sum_{E\in\mathcal{E}_h}\int_{\partial E^-}|\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E|\,u_h^{ext(E)}\cdot(u_h^{ext(E)}-u_h^{int(E)}) \\
& +\sum_{E\in\mathcal{E}_h}\int_{\partial E^+\cap\partial\Omega}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)}) \\
& +\sum_{E\in\mathcal{E}_h}\int_{\partial E^-\cap\Sigma}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)}).
\end{aligned}
\tag{3.2.60}
$$

Then combining (3.2.59) and (3.2.60), we have the complete form

$$\begin{aligned}
&\sum_{E\in\mathcal{E}_h}\gamma_E(u_h,\eta_h,u_h,u_h)+\frac{1}{2}\int_\Sigma (u_h\cdot\mathscr{N}_h)\,|u_h|^2 \qquad (3.2.61)\\
= \; & -\Big(\int_\Omega J_h(u_h\cdot\nabla_{\mathscr{A}_h}u_h)\cdot u_h+\frac{1}{2}\int_\Omega J_h(\nabla_{\mathscr{A}_h}\cdot u_h)(u_h\cdot u_h)\\
& -\sum_{e\in\partial\mathcal{E}_h}\frac{1}{2}\int_{e\notin\partial\Omega}[u_h\cdot J_h\mathscr{A}_h]\cdot n_E\{u_h\cdot u_h\}\Big)\\
& +\sum_{E\in\mathcal{E}_h}\int_{\partial E^-}|\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E|\,u_h^{ext(E)}\cdot(u_h^{ext(E)}-u_h^{int(E)})\\
& +\sum_{E\in\mathcal{E}_h}\int_{\partial E^+\cap\partial\Omega}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)})\\
& +\sum_{E\in\mathcal{E}_h}\int_{\partial E^-\cap\Sigma}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)}).
\end{aligned}$$

Step 4: Synthesis.

Summing up (3.2.50) over $E\in\mathcal{E}_h$ and adding it to (3.2.61) imply

$$\begin{aligned}
&\sum_{E\in\mathcal{E}_h}\gamma_E(u_h,\eta_h,u_h,u_h)+\frac{1}{2}\int_\Sigma (u_h\cdot\mathscr{N}_h)\,|u_h|^2 \qquad (3.2.62)\\
= \; & \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_{\partial E^-}|\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E|\left|u_h^{ext(E)}-u_h^{int(E)}\right|^2\\
& +\sum_{E\in\mathcal{E}_h}\frac{1}{2}\int_{\partial E^+\cap\partial\Omega}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)})\\
& +\sum_{E\in\mathcal{E}_h}\frac{1}{2}\int_{\partial E^-\cap\Sigma}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)})\\
\geq \; & \sum_{E\in\mathcal{E}_h}\frac{1}{2}\int_{\partial E^+\cap\Sigma}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)})\\
& +\sum_{E\in\mathcal{E}_h}\frac{1}{2}\int_{\partial E^-\cap\Sigma}\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)})\\
= \; & \frac{1}{2}\int_\Sigma\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E(u_h^{int(E)}\cdot u_h^{int(E)})\\
= \; & \frac{1}{2}\int_\Sigma (u_h\cdot\mathscr{N}_h)\,|u_h|^2,
\end{aligned}$$

where the last equality can be directly verified by the definitions of J_h, $\mathscr{A}_h$ and $\mathscr{N}_h$. Then our result naturally follows. □

Estimate of Diffusion Terms

In order to show the coercivity of the diffusion term, we need the discrete form of Korn's inequality. The proof here is based on [48].

Lemma 3.2.5. *Assume $f_h \in X_h^k$ satisfies for $e \subset \partial\mathcal{E}_h \cap \Sigma_b$, it holds that $f_h^{ext}|_e = 0$. Then for sufficiently large $\sigma_0 > 0$, we have*

$$\sum_{E\in\mathcal{E}_h} \int_E |\mathbb{D} f_h|^2 + \sum_{e\in\partial\mathcal{E}_h} \frac{\sigma_0}{h} \int_e [f_h]^2 \gtrsim \|f_h\|_{X_h}^2 \gtrsim \|f_h\|_{H_h}^2 . \tag{3.2.63}$$

Proof. The second inequality has been shown in [22], so we turn to the first one. It is easy to see the key part is to show the derivatives of f_h can be controlled. Hence, we only need to show

$$\sum_{E\in\mathcal{E}_h} \int_E |\mathbb{D} f_h|^2 + \sum_{e\in\partial\mathcal{E}_h} \frac{\sigma_0}{h} \int_e [f_h]^2 \gtrsim \|f_h\|_{H_h}^2 . \tag{3.2.64}$$

If this is not true, then we can construct a sequence $f_h^n \in X_h^k$ satisfying

$$\|f_h^n\|_{H_h} = 1, \tag{3.2.65}$$

and

$$\sum_{E\in\mathcal{E}_h} \|\mathbb{D} f_h^n\|_{L^2(E)} + \sum_{e\in\partial\mathcal{E}_h} \frac{\sigma_0}{h} \int_e [f_h^n]^2 \leq \frac{1}{n}. \tag{3.2.66}$$

To abuse the notation, we can extract the weakly convergent subsequence

$$f_h^n \rightharpoonup f_h \ \text{ in } \ H^1(E) \ \text{ for } \ \forall E \in \mathcal{E}_h. \tag{3.2.67}$$

By the compact embedding theorem in each cell E and the weak lower semi-continuity of $H^1(E)$ norm, we have the strongly convergent subsequence

$$f_h^n \to f_h \ \text{ in } \ L^2(\Omega). \tag{3.2.68}$$

By (3.2.66), we also have

$$\sum_{E \in \mathcal{E}_h} \|\mathbb{D} f_h^n\|_{L^2(\Omega)} \ \to \ 0. \tag{3.2.69}$$

Notice the fact that in each cell E, we still have the continuous Korn's inequality

$$\|f_h^m - f_h^n\|_{H^1(E)} \lesssim \|\mathbb{D} f_h^m - \mathbb{D} f_h^n)\|_{L^2(E)} + \|f_h^m - f_h^n\|_{L^2(E)} \ \text{ for } \ \forall m, n \in \mathbb{N}. \tag{3.2.70}$$

Hence, combining all above, we know f_h^n is a Cauchy's sequence under the norm $H^1(E)$, which means it is a Cauchy's sequence under the norm H_h. Then

$$f_h^n \to f_h \ \text{ in } \ H_h. \tag{3.2.71}$$

Thus this means

$$\|f_h\|_{H_h} = 1, \tag{3.2.72}$$

which further implies

$$\mathbb{D} f_h = 0 \ \text{ in } \ \forall E \in \mathcal{E}_h, \tag{3.2.73}$$

and

$$\sum_{e\in\partial\mathcal{E}_h}\frac{\sigma_0}{h}\int_e [f_h]^2 = 0. \tag{3.2.74}$$

This means

$$f_h = (a,b) + c(-x_1, x_2), \tag{3.2.75}$$

for some constant a, b, c and f_h is continuous in $\bar{\Omega}$. Certainly, the zero bottom implies $f_h = 0$, which contradicts $\|f_h\|_{H_h} = 1$. Therefore, our hypothesis is invalid and (3.2.64) holds. $\square$

Lemma 3.2.6. *Assume f_h satisfies for $e \subset \partial\mathcal{E}_h \cap \Sigma_b$, it holds that $f_h^{ext}|_e = 0$. Also, η_h satisfies $\eta_h + 1 \geq \delta > 0$ and*

$$Q = \sup_{F\in\mathcal{F}_h} \|\eta_h\|_{W^{1,\infty}(F)} < \infty. \tag{3.2.76}$$

Then for sufficiently large $\sigma_0 > 0$, we have

$$\sum_{E\in\mathcal{E}_h}\int_E J_h \left|\mathbb{D}_{\mathscr{A}_h} f_h\right|^2 + \sum_{e\in\partial\mathcal{E}_h}\frac{\sigma_0}{h}\int_e [f_h]^2 \gtrsim C(Q)\,\|f_h\|_{X_h}^2 \gtrsim C(Q)\,\|f_h\|_{H_h}^2. \tag{3.2.77}$$

Proof. Note that in each cell E, the free surface η_h is smooth. Hence, we can change it back by the transform Φ_h^{-1} to a curved cell $\Phi_h^{-1}E$ which satisfies

$$\int_E J_h(x)\left|\mathbb{D}_{\mathscr{A}_h(x)} f_h(x)\right|^2 \mathrm{dx} = \int_{\Phi_h^{-1}E} \left|\mathbb{D} f_h(y)\right|^2 \mathrm{dy}. \tag{3.2.78}$$

Hence, in this curved cell, we have the Korn's inequality

$$\|f_h\|_{H^1(\Phi_h^{-1}E)} \lesssim \|\mathbb{D} f_h\|_{L^2(\Phi_h^{-1}E)} + \|f_h\|_{L^2(\Phi_h^{-1}E)}. \tag{3.2.79}$$

Since the transform Φ_h^{-1} is a diffeomorphism between E and $\Phi_h^{-1}(E)$, then the H^1 norms in these two spaces are comparable. Hence, we have

$$\|f_h\|_{H^1(E)} \lesssim C(Q)\bigg(\int_E J_h \left|\mathbb{D}_{\mathscr{A}_h} f_h\right|^2 + \|f_h\|_{L^2(E)}\bigg). \tag{3.2.80}$$

Then a similar proof as that of Lemma 3.2.5 naturally yields the desired result. □

Lemma 3.2.7. *Assume f_h satisfies for $e \subset \partial\mathcal{E}_h \cap \Sigma_b$, it holds that $f_h^{ext}|_e = 0$. Also, η_h satisfies $\eta_h + 1 \geq \delta > 0$ and*

$$Q = \sup_{F\in\mathcal{F}_h} \|\eta_h\|_{W^{1,\infty}(F)} < \infty. \tag{3.2.81}$$

Then there exists a sufficiently small constant $\delta' > 0$ such that for η' satisfying

$$\sup_{F\in\mathcal{F}_h} \|\eta_h' - \eta_h\|_{W^{1,\infty}(F)} \leq \delta', \tag{3.2.82}$$

and for sufficiently large $\sigma_0' > 0$, we have

$$\sum_{E\in\mathcal{E}_h} \int_E J_h' \left|\mathbb{D}_{\mathscr{A}_h'} f_h\right|^2 + \sum_{e\in\partial\mathcal{E}_h} \frac{\sigma_0'}{h} \int_e [f_h]^2 \gtrsim C(Q)\, \|f_h\|_{X_h}^2 \gtrsim C(Q)\, \|f_h\|_{H_h}^2. \tag{3.2.83}$$

Proof. By Lemma 3.2.6, we know

$$\sum_{E\in\mathcal{E}_h} \int_E J_h \left|\mathbb{D}_{\mathscr{A}_h} f_h\right|^2 + \sum_{e\in\partial\mathcal{E}_h} \frac{\sigma_0}{h} \int_e [f_h]^2 \gtrsim C(Q)\, \|f_h\|_{X_h}^2 \gtrsim C(Q)\, \|f_h\|_{H_h}^2. \tag{3.2.84}$$

We can rewrite our formula in a perturbed form as

$$\begin{aligned} &\sum_{E\in\mathcal{E}_h} \int_E J_h' \left|\mathbb{D}_{\mathscr{A}_h'} f_h\right|^2 - \sum_{E\in\mathcal{E}_h} \int_E J_h \left|\mathbb{D}_{\mathscr{A}_h} f_h\right|^2 \\ =\; &\sum_{E\in\mathcal{E}_h} \int_E J_h' \Big(\mathbb{D}_{\mathscr{A}_h'-\mathscr{A}_h} f_h\Big)\Big(2\mathbb{D}_{\mathscr{A}_h} f_h + \mathbb{D}_{\mathscr{A}_h'-\mathscr{A}_h} f_h\Big) + \sum_{E\in\mathcal{E}_h} \int_E (J_h' - J_h) \left|\mathbb{D}_{\mathscr{A}_h} f_h\right|^2 \\ \lesssim\; &\delta' \, \|f_h\|_{H_h}^2 . \end{aligned} \tag{3.2.85}$$

Naturally, when taking δ' sufficiently small, we can absorb the perturbation into the principle part. Then our result easily follows. $\square$

For the diffusion term

$$-\int_E J\Delta_{\mathscr{A}}u\cdot v_h, \tag{3.2.86}$$

a direct integration by parts implies

$$\begin{aligned} & -\sum_{E\in\mathcal{E}_h}\int_E J\Delta_{\mathscr{A}}u\cdot v_h \qquad (3.2.87)\\ = \ & \frac{1}{2}\int_\Omega J\mathbb{D}_{\mathscr{A}}u:\mathbb{D}_{\mathscr{A}}v_h-\sum_{e\in\partial\mathcal{E}_h\backslash\Sigma}\int_e\{\mathbb{D}_{\mathscr{A}}uJ\mathscr{A}\cdot n_E\}\cdot[v_h]-\int_\Sigma(\mathbb{D}_{\mathscr{A}}u)\mathscr{N}\cdot v_h^{int(E)}. \end{aligned}$$

Also in the continuous case, our discretization yields

$$\begin{aligned} & \sum_{E\in\mathcal{E}_h}\alpha_E(\eta,u,\eta,v_h) \qquad (3.2.88)\\ = \ & \frac{1}{2}\int_\Omega J\mathbb{D}_{\mathscr{A}}u:\mathbb{D}_{\mathscr{A}}v_h-\sum_{e\in\partial\mathcal{E}_h\backslash\Sigma}\int_e\{\mathbb{D}_{\mathscr{A}}uJ\mathscr{A}\cdot n_E\}\cdot[v_h]. \end{aligned}$$

We can notice the difference is the extra physical boundary term $-\int_\Sigma(\mathbb{D}_{\mathscr{A}}u)\mathscr{N}\cdot v_h^{int(E)}$, which contributes to the boundary condition on Σ. This is separately added to the scheme as a forcing term later combined with the pressure contribution. Hence, our scheme is consistent.

Lemma 3.2.8. *Assume η_h satisfies $\eta_h+1\geq\delta>0$. If we take $v_h=u_h$ and choose the penalty constant σ properly, the discretized diffusion term with NIPG satisfies*

$$\sum_{E\in\mathcal{E}_h}\alpha_E(\eta_h,u_h,\eta_h,u_h)\geq\int_\Omega J_h\left|\mathbb{D}_{\mathscr{A}_h}u_h\right|^2. \tag{3.2.89}$$

If we further assume η_h satisfies

$$Q = \sup_{F \in \mathcal{F}_h} \|\eta_h\|_{W^{1,\infty}(F)} < \infty, \tag{3.2.90}$$

then the discretized diffusion term with SIPG or NIPG satisfies

$$\sum_{E \in \mathcal{E}_h} \alpha_E(\eta_h, u_h, \eta_h, u_h) \gtrsim C(Q) \|u_h\|_{X_h}^2 . \tag{3.2.91}$$

Proof. We can directly compute

$$\begin{aligned} &\sum_{E \in \mathcal{E}_h} \alpha_E(\eta_h, u_h, \eta_h, v_h) \qquad (3.2.92) \\ = \;& \frac{1}{2} \sum_{E \in \mathcal{E}_h} \int_E J_h \mathbb{D}_{\mathscr{A}_h} u_h : \mathbb{D}_{\mathscr{A}_h} v_h - \sum_{e \in \partial\mathcal{E}_h \setminus \Sigma} \int_e \{\mathbb{D}_{\mathscr{A}_h} u_h J_h \mathscr{A}_h \cdot n_E\} \cdot [v_h] \\ & \pm \sum_{e \in \partial\mathcal{E}_h \setminus \Sigma} \int_e \{\mathbb{D}_{\mathscr{A}_h} v_h J_h \mathscr{A}_h \cdot n_E\} \cdot [u_h] + \frac{\sigma}{h} \sum_{e \in \partial\mathcal{E}_h} [u_h][v_h]. \end{aligned}$$

For NIPG, when $v_h = u_h$, (3.2.92) reduces to

$$\sum_{E \in \mathcal{E}_h} \alpha_E(\eta_h, u_h, \eta_h, u_h) = \frac{1}{2} \int_\Omega J_h \left|\mathbb{D}_{\mathscr{A}_h} u_h\right|^2 + \frac{\sigma}{h} \sum_e [u_h]^2. \tag{3.2.93}$$

Hence, we may simply take the penalty $\sigma = 1$ and by the discrete Korn's inequality in Lemma 3.2.6, our result naturally follows.

For SIPG, when $v_h = u_h$, (3.2.92) reduces to

$$\begin{aligned} &\sum_{E \in \mathcal{E}_h} \alpha_E(\eta_h, u_h, \eta_h, u_h) \qquad (3.2.94) \\ = \;& \frac{1}{2} \int_\Omega J_h \left|\mathbb{D}_{\mathscr{A}_h} u_h\right|^2 - 2 \sum_{e \in \partial\mathcal{E}_h \setminus \Sigma} \int_e \{\mathbb{D}_{\mathscr{A}_h} u_h J_h \mathscr{A}_h \cdot n_E\} \cdot [u_h] + \frac{\sigma}{h} \sum_{e \in \partial\mathcal{E}_h} [u_h]^2. \end{aligned}$$

By the discrete Korn's inequality in Lemma 3.2.6, we have

$$\sum_{E\in\mathcal{E}_h}\int_E J_h\left|\mathbb{D}_{\mathscr{A}_h}u_h\right|^2+\sum_{e\in\partial\mathcal{E}_h}\frac{\sigma_0}{h}\int_e[u_h]^2\gtrsim C(Q)\left\|u_h\right\|_{X_h}^2\gtrsim C(Q)\left\|u_h\right\|_{H_h}^2. \quad (3.2.95)$$

Then we utilize Hölder's inequality, the trace theorem and Cauchy's inequality to estimate

$$\begin{aligned}
&\sum_{e\in\partial\mathcal{E}_h\backslash\partial\Omega}\int_e\{\mathbb{D}_{\mathscr{A}_h}u_hJ_h\mathscr{A}_h\cdot n_E\}\cdot[u_h] && (3.2.96)\\
\leq\ & C(Q)\sum_{e\in\partial\mathcal{E}_h\backslash\partial\Omega}\left\|\nabla u_h\right\|_{L^2(e)}\left\|[u_h]\right\|_{L^2(e)}\\
\leq\ & C(Q)\sum_{e\in\partial\mathcal{E}_h\backslash\partial\Omega}\left\|u_h\right\|_{H^1(e)}\left\|[u_h]\right\|_{L^2(e)}\\
\leq\ & CC(Q)\sum_{e\in\partial\mathcal{E}_h\backslash\partial\Omega}h\left\|u_h\right\|_{H^1(e)}^2+\frac{C(Q)}{4C}\frac{1}{h}\sum_{e\in\partial\mathcal{E}_h\backslash\partial\Omega}\left\|[u_h]\right\|_{L^2(e)}^2\\
\leq\ & CC(Q)\sum_{E\in\mathcal{E}_h}h\left\|u_h\right\|_{H^{3/2}(E)}^2+\frac{C(Q)}{4C}\frac{1}{h}\sum_{e\in\partial\mathcal{E}_h\backslash\partial\Omega}\left\|[u_h]\right\|_{L^2(e)}^2\\
\leq\ & CC(Q)\left\|u_h\right\|_{H_h}^2+\frac{C(Q)}{4C}\frac{1}{h}\sum_{e\in\partial\mathcal{E}_h\backslash\partial\Omega}\left\|[u_h]\right\|_{L^2(e)}^2.
\end{aligned}$$

The last inequality is valid since $u_h\in P_h^k$. Then we take C sufficiently small, and $\sigma\geq\sigma+C(Q)/(4C)$ to absorb (3.2.96) into (3.2.95). Hence, our result easily follows. □

Estimate of Pressure Terms

For the pressure term

$$\int_E J\nabla_{\mathscr{A}}p\cdot v_h, \quad (3.2.97)$$

a direct integration by parts reveals the following equality in the continuous case

$$\begin{aligned} &\sum_{E\in\mathcal{E}_h}\int_E J\nabla_{\mathscr{A}} p\cdot v_h \\ =\ & -\int_\Omega Jp\nabla_{\mathscr{A}}\cdot v_h + \sum_{e\in\partial\mathcal{E}_h\backslash\Sigma}\int_e [v_h]\cdot(pJ\mathscr{A})\cdot n_e) + \int_\Sigma p\mathscr{N}\cdot v_h^{int(E)}. \end{aligned} \tag{3.2.98}$$

It is easy to see in the continuous case, our discretization reduces to

$$\sum_{E\in\mathcal{E}_h}\beta_E(p,\eta,v_h) = -\int_\Omega Jp\nabla_{\mathscr{A}}\cdot v_h + \sum_{e\in\partial\mathcal{E}_h\backslash\Sigma}\int_e [v_h]\cdot(pJ\mathscr{A})\cdot n_e. \tag{3.2.99}$$

We can notice the difference is the extra physical boundary term $\int_\Sigma p\mathscr{N}\cdot v^{int(E)}$, which contributes to the boundary condition on Σ. This is separately added to the scheme as a forcing term later combined with the diffusion contribution. Hence, our scheme is consistent.

Lemma 3.2.9. *When we take $v_h = u_h$, the discretized pressure term satisfies*

$$\sum_{E\in\mathcal{E}_h}\beta_E(p_h,\eta_h,u_h) = -\int_\Omega J_h p_h\nabla_{\mathscr{A}_h}\cdot u_h + \sum_{e\in\partial\mathcal{E}_h\backslash\Sigma}\int_e [u_h]\cdot\{p_h J_h\mathscr{A}_h\}\cdot n_e, \tag{3.2.100}$$

where n_e denotes the outward normal vector read from the same direction as $[u_h]$.

Proof. This is a direct corollary of the definition, so we omit the proof here. □

Estimate of Forcing Terms

Considering the physical boundary condition on Σ of the system (3.1.8), we need to take the upper boundary condition as a forcing term, i.e.

$$(pI - \mathbb{D}_{\mathscr{A}} u)\mathscr{N} = \eta \mathscr{N} \quad \text{on} \quad \Sigma. \tag{3.2.101}$$

Multiplying the test function v_h and integrating over F yield

$$\int_F \eta \mathscr{N} v_h \quad = \quad \int_F \eta (\bar{v}_2)_h - \int_F \eta \partial_1 \eta (\bar{v}_1)_h. \tag{3.2.102}$$

In the continuous case, our discretization reduces to

$$\sum_{E \in \mathcal{E}_h} \mu_E(\eta, \eta, v_h) \quad = \quad \int_\Sigma \eta (\bar{v}_2)_h - \int_\Sigma \eta \partial_1 \eta (\bar{v}_1)_h, \tag{3.2.103}$$

and the penalty terms vanish. Hence, this discretization is consistent.

Lemma 3.2.10. *When we take $v_h = u_h$, the discretized forcing term satisfies*

$$\sum_{E \in \mathcal{E}_h} \mu_E(\eta_h, \eta_h, u_h) \geq \frac{1}{2} \int_\Sigma \partial_t \left| \eta_h \right|^2. \tag{3.2.104}$$

Proof. We may sum up over $E \in \mathcal{E}_h$ to obtain

$$\begin{aligned} & \sum_{E \in \mathcal{E}_h} \mu_E(\eta_h, \eta_h, v_h) \\ = \quad & \int_\Sigma \eta_h (\bar{v}_2)_h + \int_\Sigma \eta_h \partial_1 \eta_h (\bar{v}_1)_h \\ & + \frac{1}{2} \sum_{F \in \mathcal{F}_h} \left(\{(\bar{u}_1)_h\} (\eta_h^-)^2 |_{F+1/2} - \{(\bar{u}_1)_h\} (\eta_h^+)^2 |_{F-1/2} \right). \end{aligned} \tag{3.2.105}$$

Taking $\phi_h = \eta_h$ in the transport discretization (3.2.15) and integrating by parts imply

$$\begin{aligned}&\frac{1}{2}\int_F \partial_t |\eta_h|^2 - \int_F (\bar{u}_2)_h\eta_h + \int_F (\bar{u}_1)_h\eta_h\partial_1\eta_h - (\bar{u}_1)_h^-(\eta_h^-)^2|_{F+1/2} + (\bar{u}_1)_h^+(\eta_h^+)^2|_{F-1/2}\\&+(\hat{u}_1)_h\hat{\eta}_h\eta_h^-|_{F+1/2} - (\hat{u}_1)_h\hat{\eta}_h\eta_h^+|_{F-1/2} + \frac{1}{2}\Big((\eta_h^-)^2[u_h]|_{F+1/2} - (\eta_h^+)^2[u_h]|_{F-1/2}\Big) = 0,\end{aligned} \tag{3.2.106}$$

which further leads to

$$\begin{aligned}&\frac{1}{2}\int_F \partial_t |\eta_h|^2 - \int_F (\bar{u}_2)_h\eta_h + \int_F (\bar{u}_1)_h\eta_h\partial_1\eta_h\\&-\{(\bar{u}_1)_h\}(\eta_h^-)^2|_{F+1/2} + \{(\bar{u}_1)_h\}(\eta_h^+)^2|_{F-1/2}\\&+(\hat{u}_1)_h\hat{\eta}_h\eta_h^-|_{F+1/2} - (\hat{u}_1)_h\hat{\eta}_h\eta_h^+|_{F-1/2} = 0.\end{aligned} \tag{3.2.107}$$

Summing over $F \in \mathcal{F}_h$ implies

$$\begin{aligned}&\frac{1}{2}\int_\Sigma \partial_t |\eta_h|^2 - \int_\Sigma (\bar{u}_2)_h\eta_h + \int_\Sigma (\bar{u}_1)_h\eta_h\partial_1\eta_h - \sum_{F\in\mathcal{F}_h}\{(\bar{u}_1)_h\}(\eta_h^-)^2|_{F+1/2}\\&+\sum_{F\in\mathcal{F}_h}\{(\bar{u}_1)_h\}(\eta_h^+)^2|_{F-1/2} - \sum_{F\in\mathcal{F}_h}(\hat{u}_1)_h\hat{\eta}_h\eta_h^-|_{F+1/2} + \sum_{F\in\mathcal{F}_h}(\hat{u}_1)_h\hat{\eta}_h\eta_h^+|_{F-1/2} = 0.\end{aligned} \tag{3.2.108}$$

Since we always take $\hat{u}_1 = \{(\bar{u}_1)_h\}$, combining (3.2.105) and (3.2.108) yields

$$\begin{aligned}&\sum_{E\in\mathcal{E}_h}\mu_E(\eta_h,\eta_h,v_h)\\=\;&\frac{1}{2}\int_\Sigma \partial_t |\eta_h|^2 + \sum_{F\in\mathcal{F}_h}\Big(-\frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^-)^2|_{F+1/2} + \frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^+)^2|_{F-1/2}\\&+\{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^-|_{F+1/2} - \{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^+|_{F-1/2}\Big).\end{aligned} \tag{3.2.109}$$

We can decompose the boundary terms in (3.2.109) to obtain

$$\begin{aligned}
& -\frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^-)^2|_{F+1/2} + \frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^+)^2|_{F-1/2} \\
& +\{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^-|_{F+1/2} - \{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^+|_{F-1/2} \\
= \quad & \Psi_{F+1/2} - \Psi_{F-1/2} + \Theta_F,
\end{aligned} \tag{3.2.110}$$

where

$$\Psi_{F+1/2} = -\frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^-)^2|_{F+1/2} + \{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^-|_{F+1/2}, \tag{3.2.111}$$

$$\Psi_{F-1/2} = -\frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^-)^2|_{F-1/2} + \{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^-|_{F-1/2}, \tag{3.2.112}$$

and

$$\begin{aligned}
\Theta_F = \; & \frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^+)^2|_{F-1/2} - \frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^-)^2|_{F-1/2} \\
& -\{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^+|_{F-1/2} + \{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^-|_{F-1/2}.
\end{aligned} \tag{3.2.113}$$

By the flux definition (3.2.18), we can derive

$$\Theta_F = \{(\bar{u}_1)_h\}\left(\eta_h^+ - \eta_h^-\right)\left(\frac{\eta_h^+ + \eta_h^-}{2} - \hat{\eta}_h\right) \geq 0. \tag{3.2.114}$$

When we sum up (3.2.110) over $F \in \mathcal{F}_h$, all $\Psi_{F+1/2}$ are canceled out due to the periodicity. Then we have

$$\begin{aligned}
\sum_{F\in\mathcal{F}_h} \Bigg(& -\frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^-)^2|_{F+1/2} + \frac{1}{2}\{(\bar{u}_1)_h\}(\eta_h^+)^2|_{F-1/2} \\
& +\{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^-|_{F+1/2} - \{(\bar{u}_1)_h\}\hat{\eta}_h\eta_h^+|_{F-1/2}\Bigg) \geq 0.
\end{aligned} \tag{3.2.115}$$

Hence, combining (3.2.109) and (3.2.115), we can obtain the desired result. $\square$

Estimate of Divergence Terms

In the continuous case, our discretization reduces to

$$\sum_{E\in\mathcal{E}_h} \rho_E(u,\eta,q_h) \quad = \quad -\int_\Omega J(\nabla_{\mathscr{A}}\cdot u)q_h. \tag{3.2.116}$$

Hence, this discretization is consistent.

Lemma 3.2.11. *When we take $q_h = p_h$, the discretized divergence term satisfies*

$$\sum_{E\in\mathcal{E}_h} \rho_E(u_h,\eta_h,p_h) \quad = \quad -\int_\Omega J_h p_h \nabla_{\mathscr{A}_h}\cdot u_h + \sum_{e\in\partial\mathcal{E}_h\backslash\Sigma}\int_e [u_h]\cdot\{p_h J_h \mathscr{A}_h\}\cdot n_e. \tag{3.2.117}$$

Proof. This is a natural corollary of the definition, so we omit the proof here. $\square$

3.3 Stability Analysis

Condition 3.3.1. *The free surface $\eta_h(t)$ satisfies $1+\eta_h(t)\geq\delta>0$ for some $\delta>0$ independent of h and t.*

Condition 3.3.2. *The free surface $\eta_h(t)$ satisfies $\sup_{F\in\mathcal{F}_h}\|\eta_h(t)\|_{W^{1,\infty}(F)}\leq Q$ for some $Q>0$ independent of h and t.*

Theorem 3.3.3. *If $S=0$, suppose Condition 3.3.1 is valid in $t\in[0,T]$ for some $T>0$. Then there exists a unique numerical solution triple $(u_h,p_h,\eta_h)\in X_h^k\times M_h^{k-1}\times S_h^k$*

to the scheme (3.2.14) with NIPG, which satisfies the estimate

$$\left\|\sqrt{J_h(t)}u_h(t)\right\|^2_{L^2(\Omega)} + g\,\|\eta_h(t)\|^2_{L^2(\Sigma)} + \nu \int_0^t \int_\Omega J_h(s)\left|\mathbb{D}_{\mathscr{A}_h(s)}u_h(s)\right|^2 \mathrm{ds} \qquad (3.3.1)$$
$$\leq \left\|\sqrt{J_h(0)}u_h(0)\right\|^2_{L^2(\Omega)} + g\,\|\eta_h(0)\|^2_{L^2(\Sigma)}\,,$$

for any $t \in [0,T]$.

For general S, *suppose Condition 3.3.1 and Condition 3.3.2 are valid in* $t \in [0,T]$ *for some* $T > 0$. *Then there exists a unique numerical solution triple* $(u_h, p_h, \eta_h) \in X_h^k \times M_h^{k-1} \times S_h^k$ *to the scheme (3.2.14) with SIPG or NIPG, which satisfies the estimate*

$$\left\|\sqrt{J_h(t)}u_h(t)\right\|^2_{L^2(\Omega)} + \|\eta_h(t)\|^2_{L^2(\Sigma)} + \int_0^t \|u_h(s)\|^2_{X_h} \mathrm{ds} \qquad (3.3.2)$$
$$\lesssim C(Q)\bigg(\left\|\sqrt{J_h(0)}u_h(0)\right\|^2_{L^2(\Omega)} + \|\eta_h(0)\|^2_{L^2(\Sigma)} + \int_0^t \int_\Omega |S(r)|^2 \mathrm{dr}\bigg),$$

for any $t \in [0,T]$.

Proof. The existence and uniqueness follow from a standard argument for the differential-algebraic equations, so we omit it here and focus on the energy estimate. In the system (3.2.14), we take the test function $v_h = u_h$ and $q_h = p_h$, and sum up over $E \in \mathcal{E}_h$. Then it yields

$$\sum_{E\in\mathcal{E}_h} \zeta_E(\eta_h, u_h, u_h) + \sum_{E\in\mathcal{E}_h} \gamma_E(u_h, \eta_h, u_h, u_h) \qquad (3.3.3)$$
$$+\nu \sum_{E\in\mathcal{E}_h} \alpha_E(\eta_h, u_h, \eta_h, u_h) + \sum_{E\in\mathcal{E}_h} \beta_E(p_h, \eta_h, u_h)$$
$$+g \sum_{E\in\mathcal{E}_h} \mu_E(\eta_h, \eta_h, u_h) = \sum_{E\in\mathcal{E}_h} \omega_E(\eta_h, u_h),$$
$$\sum_{E\in\mathcal{E}_h} \rho_E(u_h, \eta_h, p_h) = 0. \qquad (3.3.4)$$

By Lemma 3.2.9 and Lemma 3.2.11, we have

$$\sum_{E\in\mathcal{E}_h} \beta_E(p_h,\eta_h,u_h) = \sum_{E\in\mathcal{E}_h} \rho_E(u_h,\eta_h,p_h) = 0. \tag{3.3.5}$$

Hence, we can simplify (3.3.3) into

$$\begin{aligned} &\sum_{E\in\mathcal{E}_h} \zeta_E(\eta_h,u_h,u_h) + \sum_{E\in\mathcal{E}_h} \gamma_E(u_h,\eta_h,u_h,u_h) \\ +\nu &\sum_{E\in\mathcal{E}_h} \alpha_E(\eta_h,u_h,\eta_h,u_h) + g\sum_{E\in\mathcal{E}_h} \mu_E(\eta_h,\eta_h,u_h) \quad = \quad \sum_{E\in\mathcal{E}_h} \omega_E(\eta_h,u_h). \end{aligned} \tag{3.3.6}$$

Then by Lemma 3.2.2, Lemma 3.2.4, Lemma 3.2.8 and Lemma 3.2.10, we have

$$\frac{1}{2}\partial_t \int_\Omega J_h \left|u_h\right|^2 + \frac{g}{2}\partial_t \int_\Sigma \left|\eta_h\right|^2 + CC(Q)\left\|u_h\right\|_{X_h}^2 \le \sum_{E\in\mathcal{E}_h} \omega_E(u_h). \tag{3.3.7}$$

Note Condition 3.3.1 and Condition 3.3.2 guarantee the existence of $C(Q) > 0$ which is independent of t. An application of Cauchy's inequality implies

$$\sum_{E\in\mathcal{E}_h} \omega_E(u_h) \le C' \int_\Omega J_h^2 \left|u_h\right|^2 + \frac{1}{4C'}\int_\Omega \left|S\right|^2 \le Q^2 C' \left\|u_h\right\|_{X_h}^2 + \frac{1}{4C'}\int_\Omega \left|S\right|^2. \tag{3.3.8}$$

In (3.3.8), when taking C' sufficiently small, we can always absorb it into $CC(Q)\left\|u_h\right\|_{X_h}^2$ in (3.3.7). Hence, we have

$$\frac{1}{2}\partial_t \int_\Omega J_h \left|u_h\right|^2 + \frac{g}{2}\partial_t \int_\Sigma \left|\eta_h\right|^2 + \frac{CC(Q)}{2}\left\|u_h\right\|_{X_h}^2 \le C(Q)\int_\Omega \left|S\right|^2. \tag{3.3.9}$$

Then integrating over $[0,t]$ leads to the desired result. The $S = 0$ case is easily derived from the general case without discussion of the source term. □

Remark 3.3.4. *Condition 3.3.1 and Condition 3.3.2 are not always satisfied a priori. In [41], [3] and [43], this type of assumptions were also introduced in the numerical analysis.*

3.4 Discussion on the Error Analysis

For the continuous solution (u, η), the analysis in [50] reveals in the Navier-Stokes equations of (3.1.8), we need H^1 norm of $\eta(t)$ to bound L^2 norm of $u(t)$. However, the result in [17] implies in the transport equation of (3.1.8), we need H^2 norm of $u(t)$ to control H^1 norm of $\eta(t)$. This type of inconsistent coupling cannot be improved even if we go to higher order derivatives. Hence, this implies the coupled system in (3.1.8) is not closed in the usual Sobolev norms, which means we cannot expect to obtain the error estimates for the whole system (3.1.8).

In the following, we mainly analyze the error in the Navier-Stokes equations provided we have the error estimates of the free surface.

Condition 3.4.1. *The free surface η_h satisfies the error estimates*

$$\|\eta_h - \eta\|_{L^2} \lesssim h^{k+1}, \tag{3.4.1}$$

$$\|\eta_h - \eta\|_{L^\infty} \lesssim h^{k}, \tag{3.4.2}$$

$$\|\chi_h - \chi\|_{L^2} \lesssim h^{k}, \tag{3.4.3}$$

$$\|\chi_h - \chi\|_{L^\infty} \lesssim h^{k-1}, \tag{3.4.4}$$

$$\|\partial_t(\eta_h - \eta)\|_{L^2} \lesssim h^{k}, \tag{3.4.5}$$

$$\|\partial_t(\eta_h - \eta)\|_{L^\infty} \lesssim h^{k-1}. \tag{3.4.6}$$

3.4.1 Velocity Error Estimates

In this section, we give the complete error estimates of the Navier-Stokes equations. Although the expressions look rather complicated, the basic ideas behind are relatively simple. Since most of the terms have been clearly estimated in [22], here we mainly illustrate the basic estimating rules.

Rule 1:

For the estimates of the product terms as

$$\int F_1 F_2 \dots F_k, \tag{3.4.7}$$

we usually apply the Hölder's inequality to obtain two terms in the L^2 norm and several terms in the L^∞ norm. The priority of taking the L^∞ norm is as follows:

1. F_i only containing the exact solution (u, p, η) has the highest priority.
2. F_i containing the free surface error $\eta_h - \eta$, $\chi_h - \chi$ or $\partial_t(\eta_h - \eta)$, or the numerical solution η_h, χ_h or $\partial_t \eta_h$ has the second priority.
3. F_i containing the projection error $u - \mathcal{P}u$ or $p - \mathcal{P}p$ for some $\mathcal{P}$ has the third priority.
4. F_i only containing the velocity error $u_h - \mathcal{P}u$ or the pressure error $p_h - \mathcal{P}p$ for some projection $\mathcal{P}$, or the numerical solution u_h or p_h, has the lowest priority.

In this fashion, we can make the best use of the regularity of the exact solutions and the known error estimates in the free surface to avoid the estimates for numerical solution in undesired norm due to the embedding theorem.

Rule 2:

For the boundary term, we always apply the trace theorem

$$\|F\|_{L^2(\partial E)} \lesssim \|F\|_{H^r(E)}, \tag{3.4.8}$$

for $r > 1/2$. This introduces more regularity in the estimates, so we should always try to avoid to use it directly and apply certain projection property to eliminate the boundary terms.

Rule 3:

Note the simple fact that for $r > 0$, we have

$$h^r \|F_h\|_{H^r(E)} \lesssim \|F_h\|_{L^2(E)}, \tag{3.4.9}$$

where h is the mesh size and $F_h \in P_h^k$. This can bound the higher order Sobolev norm by the lower norm at the price of sacrificing some orders of h.

Since in the following estimates, above rules can be applied in a very direct fashion, we do not give all the details, but just remark when necessary.

We decompose the velocity error and the pressure error as follows:

$$u_h - u = (u_h - \mathcal{P}u) + (\mathcal{P}u - u) = \epsilon_u + \delta_u, \tag{3.4.10}$$

$$p_h - p = (p_h - \mathcal{P}p) + (\mathcal{P}p - p) = \epsilon_p + \delta_p, \tag{3.4.11}$$

where $\mathcal{P}$ is some projection such that $\mathcal{P}u \in P_h^k$ and $\mathcal{P}p \in P_h^{k-1}$ achieving the optimal accuracy, i.e.

$$\|\delta_u\|_{L^2} \lesssim h^{k+1}, \tag{3.4.12}$$

$$\|\delta_p\|_{L^2} \lesssim h^k. \tag{3.4.13}$$

Hence, the key part of the the error estimates is ϵ_u and ϵ_p.

Lemma 3.4.2. *Suppose Condition 3.3.1, Condition 3.3.2 and Conditions 3.4.1 are valid. Assume the exact solution satisfies* $u \in C^{k+1}(\Omega)$, $p \in C^k(\Omega)$ *and* $\eta \in C^{k+1}(\Sigma)$. *Then for* h *sufficiently small, the numerical solution to the scheme (3.2.14) satisfies the estimate*

$$\begin{aligned}
&\partial_t \int_\Omega J_h \left|\epsilon_u\right|^2 + \left\|\epsilon_u\right\|_{H_h}^2 && (3.4.14)\\
\lesssim\ & C(Q)\bigg(h^{k-1-r}\left\|\epsilon_u\right\|_{H_h} + h^{k-r}\left\|\epsilon_p\right\|_{L^2} + \left\|\epsilon_u\right\|_{L^2}^2\bigg) \ \ \textit{for}\ \ 1/2 < r < 1.
\end{aligned}$$

Proof. The discretization of the Navier-Stokes equation is

$$\begin{aligned}
\zeta_E(\eta_h, u_h, v_h) + \gamma_E(u_h, \eta_h, u_h, v_h) + \nu\alpha_E(\eta_h, u_h, \eta_h, v_h) & & (3.4.15)\\
+\beta_E(p_h, \eta_h, v_h) + \mu_E(\eta_h, \eta_h, v_h) &= \omega_E(\eta_h, v_h),\\
\rho_E(u_h, \eta_h, q_h) &= 0. & (3.4.16)
\end{aligned}$$

The consistency implies the exact solution also satisfies this scheme, i.e.

$$\begin{aligned}
\zeta_E(\eta, u, v_h) + \gamma_E(u, \eta, u, v_h) + \nu\alpha_E(\eta, u, \eta, v_h) & & (3.4.17)\\
+\beta_E(p, \eta, v_h) + \mu_E(\eta, \eta, v_h) &= \omega_E(\eta, v_h),\\
\rho_E(u, \eta, q_h) &= 0. & (3.4.18)
\end{aligned}$$

Therefore, taking the difference of above two sets of equations, we obtain the error equations

$$
\begin{aligned}
\Big(\zeta_E(\eta_h,u_h,v_h)-\zeta_E(\eta,u,v_h)\Big) & & \quad (3.4.19)\\
+\Big(\gamma_E(u_h,\eta_h,u_h,v_h)-\gamma_E(u,\eta,u,v_h)\Big) & & \\
+\nu\Big(\alpha_E(\eta_h,u_h,\eta_h,v_h)-\alpha_E(\eta,u,\eta,v_h)\Big) & & \\
+\Big(\beta_E(p_h,\eta_h,v_h)-\beta_E(p,\eta,v_h)\Big) & & \\
+\Big(\mu_E(\eta_h,\eta_h,v_h)-\mu_E(\eta,\eta,v_h)\Big) &= \Big(\omega_E(\eta_h,v_h)-\omega_E(\eta,v_h)\Big), & \\
\Big(\rho_E(u_h,\eta_h,q_h)-\rho_E(u,\eta,q_h)\Big) &= 0. & \quad (3.4.20)
\end{aligned}
$$

Now we need to analyze each term in the error equations (3.4.19) and (3.4.20). We always decompose the difference of the bilinear forms into two parts: the energy part E_* and the remaining part R_*, such that

$$\text{Difference} = E_* + R_*, \tag{3.4.21}$$

where E_* builds the main body of the error equations and is put in the left-hand side (LHS), and R_* is moved into the right-hand side (RHS) and can be taken as the perturbed source term. The $*$ can be ζ, γ or α, etc. In the following, we give details on how to make this decomposition in our settings and define each E_* and R_*. In all the forms, we take the test functions $v_h = \epsilon_u$ and $q_h = \epsilon_p$.

Step 1: Pressure Term and Divergence Term.

We define the extended pressure form as

$$\beta(p_h,\eta_h,v_h) \;=\; -\sum_{E\in\mathcal{E}_h}\int_E J_h p_h \nabla_{\mathscr{A}_h}\cdot v_h + \sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e [v_h]\cdot\{p_h J_h \mathscr{A}_h\}\cdot n_e, \tag{3.4.22}$$

and define the extended divergence form as

$$\rho(q_h, \eta_h, u_h) \;=\; -\sum_{E\in\mathcal{E}_h}\int_E J_h q_h \nabla_{\mathscr{A}_h}\cdot u_h + \sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e [u_h]\cdot\{q_h J_h \mathscr{A}_h\}\cdot n_e. \tag{3.4.23}$$

The pressure error term can be decomposed as follows:

$$\begin{aligned}
\beta(p_h, \eta_h, v_h) - \beta(p, \eta, v_h) &= \beta(p_h - p, \eta_h, v_h) + \beta(p, \eta_h - \eta, v_h) \\
&= \beta(p_h - p, \eta_h, \epsilon_u) + \beta(p, \eta_h - \eta, \epsilon_u) \\
&= \beta(\epsilon_p, \eta_h, \epsilon_u) + \Big(\beta(\delta_p, \eta_h, \epsilon_u) + \beta(p, \eta_h - \eta, \epsilon_u)\Big) \\
&= E_\beta + R_\beta.
\end{aligned} \tag{3.4.24}$$

The divergence error term can be decomposed as follows:

$$\begin{aligned}
0 = \rho(q_h, \eta_h, u_h) - \rho(q_h, \eta, u) &= \rho(q_h, \eta_h, u_h - u) + \rho(q_h, \eta_h - \eta, u) \\
&= \rho(\epsilon_p, \eta_h, u_h - u) + \rho(\epsilon_p, \eta_h - \eta, u) \\
&= \rho(\epsilon_p, \eta_h, \epsilon_u) + \Big(\rho(\epsilon_p, \eta_h, \delta_u) + \rho(\epsilon_p, \eta_h - \eta, u)\Big) \\
&= E_\rho + R_\rho.
\end{aligned} \tag{3.4.25}$$

As in the proof of stability, the energy part satisfies

$$\beta(\epsilon_p, \eta_h, \epsilon_u) = \rho(\epsilon_p, \eta_h, \epsilon_u), \tag{3.4.26}$$

which means they can be canceled out, so we only need to estimate the remaining part. We can directly obtain

$$\beta(\delta_p, \eta_h, \epsilon_u) \;\lesssim\; h^{k-1}\left\|\epsilon_u\right\|_{H_h}, \tag{3.4.27}$$

$$\beta(p, \eta_h - \eta, \epsilon_u) \lesssim h^{k-1} \|\epsilon_u\|_{H_h}. \tag{3.4.28}$$

By applying the trace theorem for $1/2 < r < 1$, we can obtain

$$\rho(\epsilon_p, \eta_h, \delta_u) \lesssim h^{k-r} \|\epsilon_p\|_{L^2}. \tag{3.4.29}$$

Note the boundary terms vanish in the following term, so we have

$$\rho(\epsilon_p, \eta_h - \eta, u) \lesssim h^{k-r} \|\epsilon_p\|_{L^2}. \tag{3.4.30}$$

Hence, in total we achieve

$$\begin{aligned} E_\beta &= E_\rho, && (3.4.31)\\ |R_\beta| &\lesssim h^{k-1} \|\epsilon_u\|_{H_h}, && (3.4.32)\\ |R_\rho| &\lesssim h^{k-r} \|\epsilon_p\|_{L^2} \quad \text{for} \quad 1/2 < r < 1. && (3.4.33) \end{aligned}$$

Step 2: Temporal Term.

We define the extended temporal form as

$$\begin{aligned} &\zeta(\eta_h, u_h, v_h) && (3.4.34)\\ = &\sum_{E\in\mathcal{E}_h} \int_E \partial_t (J_h u_h) \cdot v_h + \sum_{E\in\mathcal{E}_h} \int_E \partial_t \eta_h \tilde{b} u_h \cdot \partial_2 v_h - \sum_{E\in\mathcal{E}_h} \int_{\partial E} \partial_t \eta_h \tilde{b}(\hat{u}_h \cdot v_h^{int}) n_2 \\ &+ \sum_{E\in\mathcal{E}_h} \frac{1}{2} \int_{\partial E\cap\Sigma} \partial_t \eta_h (u_h \cdot v_h). \end{aligned}$$

The temporal error term can be decomposed as follows:

$$\begin{aligned}\zeta(\eta_h, u_h, v_h) - \zeta(\eta, u, v_h) &= \zeta(\eta_h, u_h - u, v_h) + \zeta(\eta_h - \eta, u, v_h) \\ &= \zeta(\eta_h, \epsilon_u, v_h) + \zeta(\eta_h, \delta_u, v_h) + \zeta(\eta_h - \eta, u, v_h) \\ &= \zeta(\eta_h, \epsilon_u, \epsilon_u) + \Big(\zeta(\eta_h, \delta_u, \epsilon_u) + \zeta(\eta_h - \eta, u, \epsilon_u)\Big) \\ &= E_\zeta + R_\zeta.\end{aligned} \tag{3.4.35}$$

In the stability proof, we have shown

$$E_\zeta = \zeta(\eta_h, \epsilon_u, \epsilon_u) \geq \frac{1}{2}\partial_t \int_\Omega J_h \left|\epsilon_u\right|^2. \tag{3.4.36}$$

Hence, we only need to estimate the remaining part. A direct estimate shows

$$\zeta(\eta_h, \delta_u, \epsilon_u) \lesssim h^k \left\|\epsilon_u\right\|_{H_h}. \tag{3.4.37}$$

Applying the trace theorem for $1/2 < r < 1$, we can obtain

$$\zeta(\eta_h - \eta, u, \epsilon_u) \lesssim h^{k-1-r} \left\|\epsilon_u\right\|_{H_h}. \tag{3.4.38}$$

Hence, in total we achieve

$$E_\zeta \geq \frac{1}{2}\partial_t \int_\Omega J_h \left|\epsilon_u\right|^2, \tag{3.4.39}$$

$$\left|R_\zeta\right| \lesssim h^{k-1-r} \left\|\epsilon_u\right\|_{H_h} \quad \text{for} \quad 1/2 < r < 1. \tag{3.4.40}$$

Step 3: Convection Term.

We define the extended convection form as

$$
\begin{aligned}
& \sum_{E\in\mathcal{E}_h} \gamma(u_h,\eta_h,u_h,v_h) && (3.4.41)\\
= \; & -\sum_{E\in\mathcal{E}_h} \frac{1}{2}\int_{\partial I_E\cap\Sigma} (u_h\cdot\mathscr{N}_h)(u_h\cdot v_h) \\
& + \sum_{E\in\mathcal{E}_h} \int_E J_h(u_h\cdot\nabla_{\mathscr{A}_h} u_h)\cdot v_h + \sum_{E\in\mathcal{E}_h} \frac{1}{2}\int_E J_h(\nabla_{\mathscr{A}_h}\cdot u_h)(u_h\cdot v_h) \\
& - \sum_{E\in\mathcal{E}_h} \frac{1}{2}\int_{\partial E\setminus\partial\Omega} [u_h\cdot J_h\mathscr{A}_h]\cdot n_E \frac{u_h^{int}\cdot v_h^{int}}{2} \\
& + \sum_{E\in\mathcal{E}_h} \int_{\partial E^-} \left|\{u_h\cdot J_h\mathscr{A}_h\}\cdot n_E\right| (u_h^{int}-u_h^{ext})\cdot v_h^{int}.
\end{aligned}
$$

The convection error term can be decomposed as follows:

$$
\begin{aligned}
& \gamma(u_h,\eta_h,u_h,v_h)-\gamma(u,\eta,u,v_h) && (3.4.42)\\
= \; & \gamma(u_h,\eta_h,u_h-u,v_h)+\gamma(u_h,\eta_h-\eta,u,v_h)+\gamma(u_h-u,\eta,u,v_h) \\
= \; & \gamma(u_h,\eta_h,\epsilon_u,v_h)+\gamma(u_h,\eta_h,\delta_u,v_h)+\gamma(u_h,\eta_h-\eta,u,v_h)+\gamma(u_h-u,\eta,u,v_h) \\
= \; & \gamma(u_h,\eta_h,\epsilon_u,\epsilon_u)+\Big(\gamma(u_h,\eta_h,\delta_u,\epsilon_u)+\gamma(u_h,\eta_h-\eta,u,\epsilon_u)+\gamma(u_h-u,\eta,u,\epsilon_u)\Big) \\
= \; & E_\gamma+R_\gamma.
\end{aligned}
$$

Note the last term in (3.4.41) vanishes for the exact solution, so we do not need to worry about E^- definition. In the stability proof, we have shown

$$
E_\gamma=\gamma(u_h,\eta_h,\epsilon_u,\epsilon_u)\geq 0 \tag{3.4.43}
$$

Hence, we only need to estimate the remaining part. Here we have more tools to utilize other than the rules listed before. Consider the results from [22, Proposition 4.1]. Although the norm there is $\|\cdot\|_{X_h}$, based on the proof, it is easy to see we can further extend it to the norm $\|\cdot\|_{H_h}$, i.e. for $r>2$, w sufficiently small and

$u_h, w_h, v_h \in P_h^k$, we have

$$\gamma(u_h, \eta_h, w_h, v_h) \lesssim \|\eta_h\|_{W^{1,\infty}} \|u_h\|_{H_h} \|w_h\|_{H_h} \|v_h\|_{H_h}, \tag{3.4.44}$$

$$\gamma(u_h, \eta_h, w, v_h) \lesssim \|\eta_h\|_{W^{1,\infty}} \|u_h\|_{L^2} \|w\|_{H^2} \|v_h\|_{H_h}, \tag{3.4.45}$$

$$\gamma(u_h, \eta_h, w - \mathcal{P}w, v_h) \lesssim \|\eta_h\|_{W^{1,\infty}} \left(\sum_{E \in \mathcal{E}_h} \|w - \mathcal{P}w\|_{H^2(E)} \right) \|u_h\|_{L^2} \|v_h\|_{H_h}. \tag{3.4.46}$$

Combining this with the rules before, we can get the estimate

$$\gamma(u_h, \eta_h, \delta_u, \epsilon_u) \lesssim h^{k-1}(1 + \|\epsilon_u\|_{L^2}) \|\epsilon_u\|_{H_h}, \tag{3.4.47}$$

$$\gamma(u_h, \eta_h - \eta, u, \epsilon_u) \lesssim h^{k-1} \|\epsilon_u\|_{H_h}, \tag{3.4.48}$$

$$\gamma(u_h - u, \eta, u, \epsilon_u) \lesssim h^{k+1} \|\epsilon_u\|_{H_h} + \|\epsilon_u\|_{L^2} \|\epsilon_u\|_{H_h}. \tag{3.4.49}$$

Hence, in total we achieve

$$E_\gamma \geq 0, \tag{3.4.50}$$

$$|R_\gamma| \lesssim h^{k-1}(1 + \|\epsilon_u\|_{L^2}) \|\epsilon_u\|_{H_h} + \|\epsilon_u\|_{L^2} \|\epsilon_u\|_{H_h}. \tag{3.4.51}$$

Step 4: Diffusion Term.

We define the extended diffusion form as

$$\begin{aligned} &\alpha(\eta_h, u_h, \eta_h, v_h) && (3.4.52)\\ = \;& \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E J_h \mathbb{D}_{\mathscr{A}_h} u_h : \mathbb{D}_{\mathscr{A}_h} v_h - \sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e \{\mathbb{D}_{\mathscr{A}_h} u_h J_h \mathscr{A}_h \cdot n_E\}\cdot[v_h] \\ & \pm \sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e \{\mathbb{D}_{\mathscr{A}_h} v_h J_h \mathscr{A}_h \cdot n_E\}\cdot[u_h] + \frac{\sigma}{h}\sum_{e\in\partial\mathcal{E}_h}[u_h][v_h]. \end{aligned}$$

The diffusion error term can be decomposed as follows:

$$\begin{aligned} &\alpha(\eta_h, u_h, \eta_h, v_h) - \alpha(\eta, u, \eta, v_h) && (3.4.53)\\ = \;& \alpha(\eta_h, u_h - u, \eta_h, v_h) + \alpha(\eta_h - \eta, u, \eta_h, v_h) + \alpha(\eta, u, \eta_h - \eta, v_h) \\ = \;& \alpha(\eta_h, \epsilon_u, \eta_h, v_h) + \alpha(\eta_h, \delta_u, \eta_h, v_h) + \alpha(\eta_h - \eta, u, \eta_h, v_h) + \alpha(\eta, u, \eta_h - \eta, v_h) \\ = \;& \alpha(\eta_h, \epsilon_u, \eta_h, \epsilon_u) + \Big(\alpha(\eta_h, \delta_u, \eta_h, \epsilon_u) + \alpha(\eta_h - \eta, u, \eta_h, \epsilon_u) + \alpha(\eta, u, \eta_h - \eta, \epsilon_u)\Big) \\ = \;& E_\alpha + R_\alpha. \end{aligned}$$

In the stability proof, we have shown

$$E_\alpha = \alpha(\eta_h, \epsilon_u, \eta_h, \epsilon_u) \geq K \left\|\epsilon_u\right\|_{H_h}^2. \tag{3.4.54}$$

Hence, we only need to estimate the remaining part. It can be directly shown that

$$\alpha(\eta_h, \delta_u, \eta_h, \epsilon_u) \;\lesssim\; h^k \left\|\epsilon_u\right\|_{H_h}, \tag{3.4.55}$$

$$\alpha(\eta_h - \eta, u, \eta_h, \epsilon_u) \;\lesssim\; h^{k-1} \left\|\epsilon_u\right\|_{H_h}, \tag{3.4.56}$$

$$\alpha(\eta, u, \eta_h - \eta, \epsilon_u) \;\lesssim\; h^{k-1} \left\|\epsilon_u\right\|_{H_h}. \tag{3.4.57}$$

Hence, in total we achieve

$$
\begin{aligned}
E_\alpha &\geq K \left\|\epsilon_u\right\|_{H_h}^2, && (3.4.58)\\
|R_\alpha| &\lesssim h^{k-1} \left\|\epsilon_u\right\|_{H_h}. && (3.4.59)
\end{aligned}
$$

Step 5: Forcing Term.

We define the extended forcing form as

$$
\begin{aligned}
\mu(\eta_h, \eta_h, v_h) &= \sum_{E\in\mathcal{E}_h} \int_{\partial E\cap\Sigma} \eta_h \cdot (v_2)_h - \sum_{E\in\mathcal{E}_h} \int_{\partial E\cap\Sigma} \eta_h \partial_1 \eta_h \cdot (v_1)_h && (3.4.60)\\
&+ \sum_{E\in\mathcal{E}_h} \frac{1}{4}[\eta_h^2] \cdot v_h^-|_{\partial E\cap\Sigma+1/2} - \sum_{E\in\mathcal{E}_h} \frac{1}{4}[\eta_h^2] \cdot v_h^+|_{\partial E\cap\Sigma-1/2}.
\end{aligned}
$$

The forcing error term can be decomposed as follows:

$$
\begin{aligned}
\mu(\eta_h, \eta_h, v_h) - \mu(\eta, \eta, v_h) &= \mu(\eta_h - \eta, \eta_h, v_h) + \mu(\eta, \eta_h - \eta, v_h) && (3.4.61)\\
&= \mu(\eta_h - \eta, \eta_h, \epsilon_u) + \mu(\eta, \eta_h - \eta, \epsilon_u)\\
&= R_\mu.
\end{aligned}
$$

There is no energy part here, so we only need to estimate the remaining part.

$$
\mu(\eta_h - \eta, \eta_h, \epsilon_u) \lesssim h^k \left\|\epsilon_u\right\|_{H_h}, \qquad (3.4.62)
$$

$$
\mu(\eta, \eta_h - \eta, \epsilon_u) \lesssim h^k \left\|\epsilon_u\right\|_{H_h}. \qquad (3.4.63)
$$

Hence, in total we achieve

$$
|R_\mu| \lesssim h^k \left\|\epsilon_u\right\|_{H_h}. \qquad (3.4.64)
$$

Step 6: Source Term.

We define the extended source form as

$$\omega_E(\eta_h, v_h) = \sum_{E \in \mathcal{E}_h} \int_E J_h S \cdot v_h. \tag{3.4.65}$$

Then we can directly estimate

$$\begin{aligned} \omega(\eta_h, S, v_h) - \epsilon(\eta, S, v_h) &= \omega(\eta_h - \eta, S, v_h) \qquad (3.4.66) \\ &= \omega(\eta_h - \eta, S, \epsilon_u) \\ &= R_\omega. \end{aligned}$$

There is no energy part here, so we only need to estimate the remaining part.

$$\omega(\eta_h - \eta, S, \epsilon_u) \lesssim h^{k+1} \|\epsilon_u\|_{H_h}. \tag{3.4.67}$$

Hence, it is easy to see that Hence, in total we achieve

$$|R_\omega| \lesssim h^{k+1} \|\epsilon_u\|_{H_h}. \tag{3.4.68}$$

Step 7: Synthesis:

We can summarize all above to simplify the error equations (3.4.19) as

$$E_\zeta + E_\gamma + E_\alpha + E_\beta = -R_\zeta - R_\gamma - R_\alpha - R_\beta - R_\mu - R_\omega, \tag{3.4.69}$$

$$E_\rho = -R_\rho. \tag{3.4.70}$$

Then we can utilize the known facts about the energy part to simplify it as

$$\frac{1}{2}\partial_t \int_\Omega J_h \left|\epsilon_u\right|^2 + K \left\|\epsilon_u\right\|_{H_h}^2 \lesssim R_\rho - R_\zeta - R_\gamma - R_\alpha - R_\beta - R_\mu - R_\omega. \tag{3.4.71}$$

Utilizing the known facts about the remaining part, we can further obtain

$$\begin{aligned}\frac{1}{2}\partial_t \int_\Omega J_h \left|\epsilon_u\right|^2 + K \left\|\epsilon_u\right\|_{H_h}^2 &\lesssim h^{k-1-r} \left\|\epsilon_u\right\|_{H_h} + h^{k-r} \left\|\epsilon_p\right\|_{L^2} \\ &+ h^{k-1}(1 + \left\|\epsilon_u\right\|_{L^2}) \left\|\epsilon_u\right\|_{H_h} + \left\|\epsilon_u\right\|_{L^2} \left\|\epsilon_u\right\|_{H_h} \qquad \text{for} \;\; 1/2 < r < 1.\end{aligned} \tag{3.4.72}$$

The stability shows both u_h and u are bounded in L^2, so ϵ_u is also bounded in L^2. Then we get

$$\begin{aligned}&\frac{1}{2}\partial_t \int_\Omega J_h \left|\epsilon_u\right|^2 + K \left\|\epsilon_u\right\|_{H_h}^2 \\ &\lesssim h^{k-1-r} \left\|\epsilon_u\right\|_{H_h} + h^{k-r} \left\|\epsilon_p\right\|_{L^2} + \left\|\epsilon_u\right\|_{L^2} \left\|\epsilon_u\right\|_{H_h} \qquad \text{for} \;\; 1/2 < r < 1.\end{aligned} \tag{3.4.73}$$

Applying Cauchy's inequality to the last term implies

$$\begin{aligned}&\frac{1}{2}\partial_t \int_\Omega J_h \left|\epsilon_u\right|^2 + K \left\|\epsilon_u\right\|_{H_h}^2 \\ &\lesssim h^{k-1-r} \left\|\epsilon_u\right\|_{H_h} + h^{k-r} \left\|\epsilon_p\right\|_{L^2} + \frac{1}{4C} \left\|\epsilon_u\right\|_{L^2}^2 + C \left\|\epsilon_u\right\|_{H_h}^2 \qquad \text{for} \;\; 1/2 < r < 1.\end{aligned} \tag{3.4.74}$$

Taking C sufficiently small, we can absorb it into the left-hand side to achieve

$$\begin{aligned}&\frac{1}{2}\partial_t \int_\Omega J_h \left|\epsilon_u\right|^2 + \frac{K}{2} \left\|\epsilon_u\right\|_{H_h}^2 \lesssim \\ &h^{k-1-r} \left\|\epsilon_u\right\|_{H_h} + h^{k-r} \left\|\epsilon_p\right\|_{L^2} + \left\|\epsilon_u\right\|_{L^2}^2 \quad \text{for} \;\; 1/2 < r < 1.\end{aligned} \tag{3.4.75}$$

This is the desired result.

□

3.4.2 Pressure Error Estimates

In order for further analysis, we first need to show the inf-sup condition for our pressure discretization.

Lemma 3.4.3. *In the pressure discretization $\beta(p_h, \eta_h, v_h) = \sum_{E\in\mathcal{E}_h} \beta_E(p_h, \eta_h, v_h)$, suppose Condition 3.3.1, Condition 3.3.2 and Conditions 3.4.1 are valid. Assume the exact solution satisfies $\eta(t) \in C^{k+1}(\Sigma)$. Then for sufficiently small time T which is independent of h, there exists a constant $\Xi > 0$ such that*

$$\inf_{p_h\in M_h^{k-1}} \sup_{v_h\in X_h^k} \frac{\beta(p_h, \eta_h, v_h)}{\|p_h\|_{L^2} \|v_h\|_{H_h}} \geq \Xi, \tag{3.4.76}$$

for $t \in [0, T]$, where Ξ is independent of h.

Proof. The pressure discretization is as follows.

$$\beta(p_h, \eta_h, v_h) \;=\; -\sum_{E\in\mathcal{E}_h} \int_E J_h p_h(\nabla_{\mathscr{A}_h} \cdot v_h) + \sum_{e\in\partial\mathcal{E}_h\setminus\Sigma} \int_e [v_h] \cdot \{p_h J_h \mathscr{A}_h\} \cdot n_e. \tag{3.4.77}$$

We divide our proof into several steps:

Step 1: Classical Inf-sup condition.

In [21], it is shown that for any $v \in H^1(\Omega)$, there exists a projection $R_h v \in X_h^k$ such that

$$\int_E r_h \nabla \cdot (R_h v - v) \;=\; 0 \;\text{ for }\; \forall r_h \in P_h^{k-1}, \tag{3.4.78}$$

$$\int_{\partial E} r_h [R_h v] \;=\; 0 \;\text{ for }\; \forall r_h \in P_h^{k-1}, \tag{3.4.79}$$

$$\|R_h v - v\|_{H_h} \;\leq\; C \|v\|_{H^1}. \tag{3.4.80}$$

Also, it is a classical result that for $q_h \in M_h^{k-1}(\Omega)$ and $v \in H_0^1(\Omega)$, there exists $\Xi_1 > 0$ such that

$$\sup_{v \in H_0^1} \frac{\int_\Omega q_h(\nabla \cdot v)}{\|v\|_{H^1}} \geq \Xi_1 \|q_h\|_{L^2}. \tag{3.4.81}$$

Combining above results, taking $v_h = R_h v$, we can naturally show

$$\sup_{v_h \in X_h^k} \frac{\sum_{E \in \mathcal{E}_h} \int_E q_h(\nabla \cdot v_h)}{\|v_h\|_{H_h}} \geq \Xi_2 \|q_h\|_{L^2}. \tag{3.4.82}$$

Step 2: Inf-sup condition without perturbations.

Let $J_0\mathscr{A}_0$ denote the values of $J\mathscr{A}$ at $t = 0$, and $\bar{J}_0\bar{\mathscr{A}_0}$ denote the average of $J_0\mathscr{A}_0$ in each cell E which is a piece-wise constant. Then considering the mapping between Ω_0 and Ω, we can obtain

$$\sup_{v \in H_0^1} \frac{\int_\Omega J_0 q_h(\nabla_{\mathscr{A}_0} \cdot v)}{\|v\|_{H^1}} \geq \Xi_3 \|q_h\|_{L^2}. \tag{3.4.83}$$

Since we can naturally obtain

$$\left\| J_0\mathscr{A}_0 - \bar{J}_0\bar{\mathscr{A}_0} \right\|_{L^\infty} \leq C_0 h, \tag{3.4.84}$$

which implies

$$\sup_{v \in H_0^1} \frac{\int_\Omega \bar{J}_0 q_h(\nabla_{\bar{\mathscr{A}_0}} \cdot v)}{\|v\|_{H^1}} \geq (\Xi_3 - C_0 h) \|q_h\|_{L^2}. \tag{3.4.85}$$

Furthermore, from a lengthy but not difficult computation, we can easily see property (3.4.78), (3.4.79) and (3.4.80) are still valid if we replace the usual gradient operator ∇ by weighted gradient $\bar{J}_0\nabla_{\bar{\mathscr{A}_0}}$ This means we can still find the projection $R_h v$ for each $v \in H_0^1$ such that

$$\int_E r_h \bar{J}_0 \nabla_{\bar{\mathscr{A}}_0} \cdot (R_h v - v) = 0 \quad \text{for} \quad \forall r_h \in P_h^{k-1}, \tag{3.4.86}$$

$$\int_{\partial E} r_h [R_h v] = 0 \quad \text{for} \quad \forall r_h \in P_h^{k-1}, \tag{3.4.87}$$

$$\|R_h v - v\|_{H_h} \leq C \|v\|_{H^1}. \tag{3.4.88}$$

Hence, by taking $v_h = R_h v$, based on the property (3.4.86) and (3.4.88), we can show

$$\sup_{v_h \in X_h^k} \frac{\sum_{E \in \mathcal{E}_h} \int_E \bar{J}_0 q_h (\nabla_{\bar{\mathscr{A}}_0} \cdot v_h)}{\|v_h\|_{H_h}} \geq \frac{\Xi_3 - C_0 h}{1 + C} \|q_h\|_{L^2}. \tag{3.4.89}$$

In the following, we consider a perturbation of this inf-sup condition.

Step 3: Time perturbations.

We turn to our pressure discretization. Since

$$J \nabla_{\mathscr{A}} v = J \partial_1 v_1 - A \partial_2 v_1 + \partial_2 v_2. \tag{3.4.90}$$

We can separate the bulk integral as follows.

$$-\sum_{E \in \mathcal{E}_h} \int_E J_h p_h \nabla_{\mathscr{A}_h} \cdot v_h = -\sum_{E \in \mathcal{E}_h} \int_E J p_h \nabla_{\mathscr{A}} \cdot v_h - \sum_{E \in \mathcal{E}_h} \int_E p_h (J_h \nabla_{\mathscr{A}_h} - J \nabla_{\mathscr{A}}) \cdot v_h. \tag{3.4.91}$$

we can directly estimate

$$\begin{aligned}
& -\sum_{E\in\mathcal{E}_h}\int_E Jp_h\nabla_{\mathscr{A}}\cdot v_h \\
= & -\sum_{E\in\mathcal{E}_h}\int_E J_0p_h\nabla_{\mathscr{A}_0}\cdot v_h-\sum_{E\in\mathcal{E}_h}\int_E p_h(J\nabla_{\mathscr{A}}-J_0\nabla_{\mathscr{A}_0})\cdot v_h \\
= & -\sum_{E\in\mathcal{E}_h}\int_E p_h\bar{J}_0\nabla_{\bar{\mathscr{A}}_0}\cdot v_h-\sum_{E\in\mathcal{E}_h}\int_E p_h(J_0\nabla_{\mathscr{A}_0}-\bar{J}_0\nabla_{\bar{\mathscr{A}}_0})\cdot v_h \\
& -\sum_{E\in\mathcal{E}_h}\int_E p_h(J\nabla_{\mathscr{A}}-J_0\nabla_{\mathscr{A}_0})\cdot v_h.
\end{aligned} \tag{3.4.92}$$

By the analysis in Step 2, we have

$$\sup_{v_h\in X_h^k}\frac{\sum_{E\in\mathcal{E}_h}\int_E p_h\bar{J}_0\nabla_{\bar{\mathscr{A}}_0}\cdot v_h}{\|v_h\|_{H_h}}\geq\frac{\Xi_3-C_0h}{1+C}\|p_h\|_{L^2}. \tag{3.4.93}$$

By Taylor's expansion around the average and continuity of exact solution, we have

$$\sum_{E\in\mathcal{E}_h}\int_E p_h(J_0\nabla_{\mathscr{A}_0}-\bar{J}_0\nabla_{\bar{\mathscr{A}}_0})\cdot v_h\leq C_1h\|p_h\|_{L^2}\|v_h\|_{H_h}, \tag{3.4.94}$$

for some $C_1>0$. Furthermore, for sufficiently small time T, [2] shows $\|\eta(t)-\eta_0\|_{H^1}$ can be arbitrarily small. Then we can obtain

$$-\sum_{E\in\mathcal{E}_h}\int_E p_h(J\nabla_{\mathscr{A}}-J_0\nabla_{\mathscr{A}_0})\cdot v_h\leq C_2(T)\|p_h\|_{L^2}\|v_h\|_{H_h}. \tag{3.4.95}$$

Also, by the error estimates for the free surface, there exists constants $C_3>0$ satisfying

$$-\sum_{E\in\mathcal{E}_h}\int_E p_h(J_h\nabla_{\mathscr{A}_h}-J\nabla_{\mathscr{A}})\cdot v_h\ \leq\ C_3h^{k-1}\|p_h\|_{L^2}\|v_h\|_{H_h}. \tag{3.4.96}$$

In summary, we have

$$
-\sum_{E\in\mathcal{E}_h}\int_E J_h p_h \nabla_{\mathscr{A}_h}\cdot v_h \geq \left(\frac{\Xi_3 - C_0 h}{1+C} - C_1 h - C_2(T) - C_3 h^{k-1}\right)\|p_h\|_{L^2}\|v_h\|_{H_h}. \tag{3.4.97}
$$

Step 4: Boundary perturbations.

On each boundary, taking E as a reference cell, denoting p^* as the value read from outside and p read from inside, we have

$$
\begin{aligned}
\{p_h J_h \mathscr{A}_h\} &= \frac{1}{2}p_h J_h \mathscr{A}_h + \frac{1}{2}p_h^* J_h^* \mathscr{A}_h^* \\
&= \frac{1}{2}(Cp_h + C^* p_h^*) + \Big(\{p_h J_h \mathscr{A}_h\} - (Cp_h + C^* p_h^*)\Big).
\end{aligned} \tag{3.4.98}
$$

where $J^*\mathscr{A}^*$ is taken in the same convention of p^*, and matrix C is the average of $J_h\mathscr{A}_h$ and C^* of $J^*\mathscr{A}^*$ on that boundary. By Taylor's expansion around the average and the continuity of the numerical solution in each cell, we have the natural estimate

$$
\begin{aligned}
&\{p_h J_h \mathscr{A}_h\} - \frac{1}{2}(Cp_h + C^* p_h^*) \\
= \;& \frac{1}{2}\Big(J_h\mathscr{A}_h - C\Big)p_h + \frac{1}{2}\Big(J_h^*\mathscr{A}_h^* - C^*\Big)p_h^* \lesssim h\Big(|p_h| + |p_h^*|\Big).
\end{aligned} \tag{3.4.99}
$$

Then we can decompose the boundary integral in the same fashion as

$$
\begin{aligned}
\sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e [v_h]\cdot\{p_h J_h \mathscr{A}_h\}\cdot n_e &= \sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e [v_h]\cdot\frac{1}{2}(Cp_h + C^* p_h^*)\cdot n_e \\
&+ \sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e [v_h]\cdot\Big(\{p_h J_h \mathscr{A}_h\} - \frac{1}{2}(Cp_h + C^* p_h^*)\Big)\cdot n_e.
\end{aligned} \tag{3.4.100}
$$

Since we define $v_h = R_h v$, due to $p_h, p_h^* \in P_h^{k-1}$, we can utilize the orthogonal property

(3.4.87) to show

$$\sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e [v_h]\cdot(Cp_h + C^*p_h^*)\cdot n_e = 0. \tag{3.4.101}$$

For the remaining term in (3.4.100), we can apply Hölder's inequality and the trace theorem to obtain

$$\begin{aligned}
&\sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e [v_h]\cdot\left(\{p_h J_h \mathscr{A}_h\} - (Cp_h + C^*p_h^*)\right)\cdot n_e \qquad (3.4.102)\\
\lesssim\ & h\sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\|p_h\|_{L^2(e)}\|v_h\|_{L^2(e)}\\
\lesssim\ & h\sum_{E\in\mathcal{E}_h}\|p_h\|_{H^{3/4}(E)}\|v_h\|_{H_h}\\
\lesssim\ & h^{1/4}\|p_h\|_{L^2}\|v_h\|_{H_h}.
\end{aligned}$$

In summary, we have

$$\sum_{e\in\partial\mathcal{E}_h\setminus\Sigma}\int_e [v_h]\cdot\{p_h J_h\mathscr{A}_h\}\cdot n_e \le C_4 h^{1/4}\|p_h\|_{L^2}\|v_h\|_{H_h}. \tag{3.4.103}$$

Step 5: Synthesis.

In total, summarizing (3.4.97) and (3.4.103), we can obtain

$$\sup_{v_h\in X_h^k}\frac{\beta(p_h,\eta_h,v_h)}{\|v_h\|_{H_h}} \ge \left(\frac{\Xi_3 - C_0h}{1+C} - C_1h - C_2(T) - C_3h^{k-1} - C_4h^{1/4}\right)\|p_h\|_{L^2}. \tag{3.4.104}$$

C_2 can be arbitrarily small as long as we take T sufficiently small. Hence, if h is also sufficiently small, we can obtain

$$\sup_{v_h\in X_h^k}\frac{\beta(p_h,\eta_h,v_h)}{\|v_h\|_{H_h}} \ge \frac{\Xi_3}{2C}\|p_h\|_{L^2}. \tag{3.4.105}$$

Hence, taking $\Xi = \Xi_3/(2C)$, we have shown

$$\inf_{p_h \in M_h^{k-1}} \sup_{v_h \in X_h^k} \frac{\beta(p_h, \eta_h, v_h)}{\|p_h\|_{L^2} \|v_h\|_{H^1}} \geq \Xi, \tag{3.4.106}$$

which is the inf-sup condition desired. □

Lemma 3.4.4. *Suppose Condition 3.3.1, Condition 3.3.2 and Conditions 3.4.1 are valid. Assume the exact solution satisfies $u \in C^{k+1}(\Omega)$, $p \in C^K(\Omega)$ and $\eta \in C^{k+1}(\Sigma)$. Then for sufficiently small T and h, the numerical solution to the scheme (3.2.14) satisfies the estimate*

$$\begin{aligned} &\int_0^t \|\epsilon_p\|_{L^2} \\ \lesssim\ & C(Q)\bigg(h^{k-1-r} + \int_0^t \|\epsilon_u\|_{L^2} + \|\epsilon_u(t)\|_{L^2} + \frac{1}{h}\int_0^t \|\epsilon_u\|_{H_h}\bigg) \quad \text{for} \quad 1/2 < r < 1, \end{aligned} \tag{3.4.107}$$

within $t \in [0, T]$.

Proof. We can take any test function $v_h \in X_h^k$ in the error equation (3.4.19) and integrate over time $[0, t]$. Here we do not distinguish between the energy part and remaining part, but only concentrate on

$$\int_0^T \beta(p_h, \eta_h, v_h). \tag{3.4.108}$$

All the other terms can be moved into the right-hand side and estimated in terms of $\|v_h\|_{H_h}$. Basically, the estimates are similar to the velocity error estimates and the only exception is the energy part, where we cannot obtain the perfect estimates as in the stability proof now. Hence, we need to give some explanations for their estimates. For the temporal term, since v_h is independent of time, we can integrate by part for the temporal derivative and obtain

$$\int_0^t \zeta(\eta_h, \epsilon_u, v_h) \lesssim \|\epsilon_u(t)\|_{L^2} \|v_h\|_{H_h} + \|v_h\|_{H_h} \int_0^t \|\epsilon_u\|_{H_h}. \tag{3.4.109}$$

We may directly estimate the convection term as

$$\begin{aligned} & \int_0^t \gamma(u_h, \eta_h, \epsilon_u, v_h) \qquad (3.4.110)\\ \lesssim\ & \|v_h\|_{H_h} \int_0^t \|u_h\|_{H_h} \|\epsilon_u\|_{H_h} \lesssim \|v_h\|_{H_h} \int_0^t \frac{1}{h} \|u_h\|_{L^2} \|\epsilon_u\|_{H_h} \\ \lesssim\ & \frac{1}{h} \|v_h\|_{H_h} \int_0^t \|\epsilon_u\|_{H_h}. \end{aligned}$$

Also, we can estimate the diffusion term as

$$\int_0^t \alpha(\eta_h, \epsilon_u, \eta_h, v_h) \ \lesssim\ \|v_h\|_{H_h} \int_0^t \|\epsilon_u\|_{H_h}. \tag{3.4.111}$$

Based on the inf-sup condition in Lemma 3.4.3, we can deduce the desired result. □

3.4.3 Error Analysis of the Fluid

Theorem 3.4.5. *Suppose Condition 3.3.1, Condition 3.3.2 and Conditions 3.4.1 are valid. Assume the exact solution satisfies $u \in C^{k+1}(\Omega)$, $p \in C^k(\Omega)$ and $\eta \in C^{k+1}(\Sigma)$. Then for sufficiently small T and h, the numerical solution to the scheme (3.2.14) satisfies the estimates*

$$\|u_h - u\|_{L^2} \ \lesssim\ C(Q)h^{k-1-r}, \tag{3.4.112}$$

$$\left(\int_0^t \|u_h - u\|_{H_h}^2\right)^{1/2} \ \lesssim\ C(Q)h^{k-1-r}, \tag{3.4.113}$$

$$\int_0^t \|p_h - p\|_{L^2} \ \lesssim\ C(Q)h^{k-1-r} \quad \textit{for} \quad 1/2 < r < 1, \tag{3.4.114}$$

in $t \in [0, T]$.

Proof. In Lemma 3.4.2 and Lemma 3.4.4, we have shown

$$\begin{aligned} & \partial_t \int_\Omega J_h \left|\epsilon_u\right|^2 + \left\|\epsilon_u\right\|_{H_h}^2 \\ \lesssim\ & C(Q)\Big(h^{k-1-r}\left\|\epsilon_u\right\|_{H_h} + h^{k-r}\left\|\epsilon_p\right\|_{L^2} + \left\|\epsilon_u\right\|_{L^2}^2\Big) \ \text{ for } \ 1/2 < r < 1, \end{aligned} \tag{3.4.115}$$

and

$$\begin{aligned} & \int_0^t \left\|\epsilon_p\right\|_{L^2} \\ \lesssim\ & C(Q)\Big(h^{k-1-r} + \int_0^t \left\|\epsilon_u\right\|_{L^2} + \left\|\epsilon_u(t)\right\|_{L^2} + \frac{1}{h}\int_0^t \left\|\epsilon_u\right\|_{H_h}\Big) \ \text{ for } \ 1/2 < r < 1. \end{aligned} \tag{3.4.116}$$

Integrating over $[0, t]$ for any $t \in [0, T]$ in (3.4.115), we obtain

$$\begin{aligned} & \int_\Omega J_h(t)\left|\epsilon_u(t)\right|^2 + \int_0^t \left\|\epsilon_u\right\|_{H_h}^2 \\ \lesssim\ & C(Q)\Big(h^{k+2} + h^{k-1-r}\int_0^t \left\|\epsilon_u\right\|_{H_h} + h^{k-r}\int_0^t \left\|\epsilon_p\right\|_{L^2} + \int_0^t \left\|\epsilon_u\right\|_{L^2}^2\Big). \end{aligned} \tag{3.4.117}$$

Then we plug (3.4.116) into (3.4.117) to eliminate ϵ_p and obtain

$$\begin{aligned} & \int_\Omega J_h(t)\left|\epsilon_u(t)\right|^2 + \int_0^t \left\|\epsilon_u\right\|_{H_h}^2 \\ \lesssim\ & C(Q)\Big(h^{2k-1-2r} + h^{k-1-r}\int_0^t \left\|\epsilon_u\right\|_{H_h} + h^{k-r}\left\|\epsilon_u(t)\right\|_{L^2} + \int_0^t \left\|\epsilon_u\right\|_{L^2}^2\Big) \\ \lesssim\ & C(Q)\Big(h^{2k-1-2r} + h^{2k-2-2r} + \Big(\int_0^t \left\|\epsilon_u\right\|_{H_h}\Big)^2 + h^{k-r}\left\|\epsilon_u(t)\right\|_{L^2} + \int_0^t \left\|\epsilon_u\right\|_{L^2}^2\Big) \\ \lesssim\ & C(Q)\Big(h^{2k-2-2r} + \sqrt{t}\int_0^t \left\|\epsilon_u\right\|_{H_h}^2 + h^{k-r}\left\|\epsilon_u(t)\right\|_{L^2} + \int_0^t \left\|\epsilon_u\right\|_{L^2}^2\Big). \end{aligned} \tag{3.4.118}$$

When T is sufficiently small, we can absorb $C(Q)\sqrt{t}\int_0^t \left\|\epsilon_u\right\|_{H_h}^2$ into the left-hand side of (3.4.118) to achieve

$$\int_\Omega J_h(t)\,|\epsilon_u(t)|^2 + \int_0^t \|\epsilon_u\|_{H_h}^2 \lesssim C(Q)\left(h^{2k-2-2r} + h^{k-r}\,\|\epsilon_u(t)\|_{L^2} + \int_0^t \|\epsilon_u\|_{L^2}^2\right). \tag{3.4.119}$$

Since (3.4.119) holds for any $t \in [0, T]$, we can take the maximum to get

$$\max_{t\in[0,T]} \|\epsilon_u(t)\|_{L^2}^2 \lesssim C(Q)\left(h^{2k-2-2r} + h^{k-r}\max_{t\in[0,T]} \|\epsilon_u(t)\|_{L^2} + T\max_{t\in[0,T]} \|\epsilon_u(t)\|_{L^2}^2\right). \tag{3.4.120}$$

When T and h are sufficiently small, we can absorb $C(Q)(h^{k-r}+T)\max_{t\in[0,T]} \|\epsilon_u\|_{L^2}^2$ into the left-hand side of (3.4.120) to obtain

$$\max_{t\in[0,T]} \|\epsilon_u(t)\|_{L^2}^2 \lesssim C(Q)h^{2k-2-2r}. \tag{3.4.121}$$

Then this leads to

$$\max_{t\in[0,T]} \|\epsilon_u(t)\|_{L^2} \lesssim C(Q)h^{k-1-r}. \tag{3.4.122}$$

Hence, we plug (3.4.122) into (3.4.119) and (3.4.116) to achieve

$$\int_0^t \|\epsilon_u\|_{H_h}^2 \lesssim C(Q)h^{2k-2-2r}, \tag{3.4.123}$$

$$\int_0^t \|\epsilon_p\|_{L^2} \lesssim C(Q)h^{k-1-r}. \tag{3.4.124}$$

Combining with the projection errors δ_u and δ_p, we show the desired result. □

Remark 3.4.6. *The convergent rate $k-1-r$ is not optimal. If we can obtain a nicer error estimate of the temporal term, then this rate can be further improved.*

3.5 Numerical Tests

We perform the accuracy tests for our numerical scheme. Our tests are based on a set of exact solutions

$$\eta = Z\sin(2\pi(x_1 - t)), \tag{3.5.1}$$

$$u_1 = Z(6x_2^5 + 15x_2^4 + 12x_2^3 + 3x_2^2)\sin(2\pi(x_1 - t)) + (6x_2^5 + 5x_2^4 + 1), \tag{3.5.2}$$

$$\begin{aligned} u_2 = {} & \pi Z^2(4x_2^6 + 15x_2^5 + 21x_2^4 + 13x_2^3 + 3x_2^2)\sin(4\pi(x_1 - t)) \\ & + 2\pi Z(4x_2^6 + 7x_2^5 + 2x_2^4 - x_2^3)\cos(2\pi(x_1 - t)), \end{aligned} \tag{3.5.3}$$

$$p = Z\sin(2\pi(x_1 - t)), \tag{3.5.4}$$

with the source term

$$\begin{aligned} S_1 = {} & (u_1)_t - \eta_t(1 + x_2)K\partial_2 u_1 + \Big(u_1(\partial_1 u_1 - AK\partial_2 u_1) + Ku_2\partial_2 u_1\Big) \\ & -\nu\Big(\partial_{11}u_1 + (1 + A^2)K^2\partial_{22}u_1 - 2AK\partial_{12}u_1 \\ & +(AK^2\partial_2 A - A\partial_1 K - K\partial_1 A)\partial_2 u_1\Big) + (\partial_1 p - AK\partial_2 p), \end{aligned} \tag{3.5.5}$$

$$\begin{aligned} S_2 = {} & (u_2)_t - \eta_t(1 + x_2)K\partial_2 u_2 + \Big(u_1(\partial_1 u_2 - AK\partial_2 u_2) + Ku_2\partial_2 u_2\Big) \\ & -\nu\Big(\partial_{11}u_2 + (1 + A^2)K^2\partial_{22}u_2 - 2AK\partial_{12}u_2 \\ & +(AK^2\partial_2 A - A\partial_1 K - K\partial_1 A)\partial_2 u_2\Big) + K\partial_2 p, \end{aligned} \tag{3.5.6}$$

where $0 < Z < 1$ can be any fixed constant, A and K are defined as in (3.1.7), and differential operators ∂_t, ∂_i and ∂_{ij} are defined in the usual sense. In the following test, we always take $Z = 0.1$, $\nu = 0.05$ and $T = 1/8$.

3.5.1 Accuracy Tests with SIPG

The following are the error tables for our numerical scheme with SIPG:

Table 3.1: L^2 Error Table for $X_h^1 - M_h^0 - S_h^1$ formulation with SIPG

(a) Free Surface Error $\eta - \eta_h$

N	Error	Order
4	9.4028E-4	-
8	2.5493E-4	1.8830
16	6.6612E-5	1.9362
32	1.7409E-5	1.9359
64	4.9397E-6	1.8173

(b) Velocity Error $u - u_h$

N	Error	Order
4	4.8233E-2	-
8	1.3863E-2	1.7988
16	3.6182E-3	1.9378
32	9.4871E-4	1.9312
64	2.4593E-4	1.9477

(c) Pressure Error $p - p_h$

N	Error	Order
4	3.1028E-3	-
8	1.6255E-3	0.9327
16	8.2709E-4	0.9748
32	4.1570E-4	0.9925
64	2.0818E-4	0.9977

Table 3.2: L^2 Error Table for $X_h^2 - M_h^1 - S_h^2$ formulation with SIPG

(a) Free Surface Error $\eta - \eta_h$

N	Error	Order
4	1.4435E-4	-
8	1.6283E-5	3.1481
16	2.1495E-6	2.9213
32	2.9961E-7	2.8428

(b) Velocity Error $u - u_h$

N	Error	Order
4	5.8590E-3	-
8	7.7408E-4	2.9201
16	9.8674E-5	2.9717
32	1.2586E-5	2.9709

(c) Pressure Error $p - p_h$

N	Error	Order
4	9.6076E-4	-
8	2.3182E-4	2.0512
16	5.4527E-5	2.0880
32	1.2942E-5	2.0749

3.5.2 Accuracy Tests with NIPG

The following are the error tables for our numerical scheme with NIPG:

Table 3.3: L^2 Error Table for $X_h^1 - M_h^0 - S_h^1$ formulation with NIPG

(a) Free Surface Error $\eta - \eta_h$

N	Error	Order
4	9.3809E-4	-
8	2.5179E-4	1.8975
16	6.5861E-5	1.9347
32	1.7258E-5	1.9322
64	4.9108E-6	1.8132

(b) Velocity Error $u - u_h$

N	Error	Order
4	4.5428E-2	-
8	1.8218E-2	1.3182
16	6.7565E-3	1.4310
32	2.1758E-3	1.6347
64	7.5074E-4	1.5352

(c) Pressure Error $p - p_h$

N	Error	Order
4	3.2035E-3	-
8	1.6481E-3	0.9588
16	8.4487E-4	0.9640
32	4.2453E-4	0.9929
64	2.1239E-4	0.9992

Table 3.4: L^2 Error Table for $X_h^2 - M_h^1 - S_h^2$ formulation with NIPG

(a) Free Surface Error $\eta - \eta_h$

N	Error	Order
4	1.4810E-4	-
8	1.7895E-5	3.0489
16	3.3218E-6	2.4295
32	8.5392E-7	1.9598

(b) Velocity Error $u - u_h$

N	Error	Order
4	6.0580E-3	-
8	1.2693E-3	2.2548
16	1.9305E-4	2.7170
32	2.7534E-5	2.8097

(c) Pressure Error $p - p_h$

N	Error	Order
4	9.2305E-4	-
8	2.8213E-4	1.7100
16	7.5701E-5	1.8980
32	2.0531E-5	1.8825

3.5.3 Discussion on the Numerical Tests

In above accuracy tests, we can obtain the optimal order of convergence when applying scheme (3.2.14) with SIPG, and the sub-optimal order with NIPG. However, in both cases, our numerical results are much better than the analytical result obtained in Theorem 3.4.5.

3.6 Conclusions and Remarks

In this chapter, we construct a stable numerical scheme to solve the system (3.1.8) with discontinuous Galerkin method, and discuss the error analysis in the fluid with certain assumptions.

Although we focus on the 2-D fluid throughout this chapter, it is easy to see this scheme can be naturally extended to the 3-D case with periodic settings for two horizontal directions. The main restriction to our scheme is that we require the free surface to be a single-valued function of the horizontal variables, which is not always true in practice, especially when the topological structure of the free surface varies during the evolution. Our scheme is non-conservative. Hence, it might not give the qualitatively correct simulation if the exact solution possesses singularities. However, for smooth exact solutions, our scheme can give a quite good approximation. Here we use the piecewise polynomial P^k due to the connection between the bulk discretization and surface discretization. Note that Q^k, which denotes the piecewise polynomials of degree at most k for each variables, is acceptable in the scheme, but not suitable in analysis since it cannot satisfy the desired inf-sup condition for pressure term in the error analysis.

Chapter Four

Asymptotic Analysis of Neutron Transport Equation

4.1 Introduction and Notation

4.1.1 Problem Formulation

We consider a homogeneous isotropic steady neutron transport equation in a two-dimensional unit disk $\Omega = \{\vec{x} = (x_1, x_2) : \ |\vec{x}| \leq 1\}$ with one-speed velocity $\Sigma = \{\vec{w} = (w_1, w_2) : \ \vec{w} \in \mathcal{S}^1\}$ as

$$\left\{\begin{array}{rcl} \epsilon\vec{w}\cdot\nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon &=& 0 \quad \text{in} \quad \Omega, \\ u^\epsilon(\vec{x}_0, \vec{w}) &=& g(\vec{x}_0, \vec{w}) \quad \text{for} \quad \vec{w}\cdot\vec{n} < 0 \quad \text{and} \quad \vec{x}_0 \in \partial\Omega, \end{array}\right. \tag{4.1.1}$$

where

$$\bar{u}^\epsilon(\vec{x}) = \frac{1}{2\pi}\int_{\mathcal{S}^1} u^\epsilon(\vec{x}, \vec{w})\mathrm{d}\tilde{\mathrm{w}}. \tag{4.1.2}$$

and $\vec{n}$ is the outward normal vector on $\partial\Omega$, with the Knudsen number $0 < \epsilon << 1$. We will study the diffusive limit of u^ϵ as $\epsilon \to 0$.

Based on the flow direction, we can divide the boundary $\Gamma = \{(\vec{x}, \vec{w}) : \ \vec{x} \in \partial\Omega\}$ into the in-flow boundary Γ^-, the out-flow boundary Γ^+, and the grazing set Γ^0 as

$$\Gamma^- = \{(\vec{x}, \vec{w}) : \ \vec{x} \in \partial\Omega, \ \vec{w}\cdot\vec{n} < 0\}, \tag{4.1.3}$$

$$\Gamma^+ = \{(\vec{x}, \vec{w}) : \ \vec{x} \in \partial\Omega, \ \vec{w}\cdot\vec{n} > 0\}, \tag{4.1.4}$$

$$\Gamma^0 = \{(\vec{x}, \vec{w}) : \ \vec{x} \in \partial\Omega, \ \vec{w}\cdot\vec{n} = 0\}. \tag{4.1.5}$$

It is easy to see $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$. Hence, the boundary condition is only given on Γ^-.

The study of neutron transport equation dates back to 1950s. The main methods include the explicit formula and spectral analysis of the transport operators(see [33, 32, 35, 34, 36, 37, 38, 40, 39]). A classical result in [9] states that

$$\|u^{\epsilon} - U_0 - \mathscr{U}_0\|_{L^\infty} = O(\epsilon) \tag{4.1.6}$$

where $\mathscr{U}_0$ is the Knudsen layer to the Milne problem (4.2.25) while U_0 is the corresponding interior solution to the Laplace equation (4.2.26). The goal of this paper is to construct a counterexample to such a result in a disk.

4.1.2 Main Results

Theorem 4.1.1. *Assume $g(\vec{x}_0, \vec{w}) \in C^3(\Gamma^-)$. Then for the steady neutron transport equation (4.1.1), the unique solution $u^{\epsilon}(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$ satisfies*

$$\|u^{\epsilon} - U_0^{\epsilon} - \mathscr{U}_0^{\epsilon}\|_{L^\infty} = O(\epsilon) \tag{4.1.7}$$

where the interior solution U_0^{ϵ} and boundary layer $\mathscr{U}_0^{\epsilon}$ are defined in (4.2.51) and (4.2.50). Moreover, if $g(\theta, \phi) = \cos\phi$, then there exists a $C > 0$ such that

$$\|u^{\epsilon} - U_0 - \mathscr{U}_0\|_{L^\infty} \geq C > 0 \tag{4.1.8}$$

when ϵ is sufficiently small, where U_0 and $\mathscr{U}_0$ are defined in (4.2.26) and (4.2.25).

Remark 4.1.2. *θ and ϕ are defined in (4.2.13) and (4.2.37). For the diffusive boundary case, the zeroth order classical Knudsen layer is always absent. Our method leads to a counterexample to the classical Knudsen layer expansion at first order(see Theorem 4.6.9).*

Our results demonstrates that the classical Knudsen layer expansion in Theorem 4.2.1 breaks down in a unit disk in L^∞. Even though the ϵ-Milne equation with geometric correction (4.2.50) only modifies the Milne equation (4.2.25) with an ϵ order term, the difference between the solutions can be order 1 via a contradiction argument in Step 5 of Section 5. We remark that L^∞ is a natural space to characterize boundary layer contributions, whose L^p norm is $O(\epsilon^{1/p})$ for $1 \leq p < \infty$. Unfortunately, our new expansion can only be established in a disk. A new mathematical theory is needed to characterize such a diffusive limit in a general domain. Our analysis is based on a careful study of the ϵ-Milne problem with geometric correction. Our proof is self-contained and without any use of probabilistic techniques as in [9]. Some recent results on stationary Boltzmann equation with a similar type of techniques can be found in [5, 4, 6].

4.1.3 Notation and Conventions

Throughout this chapter, $C > 0$ denotes a constant that only depends on the parameter Ω, but does not depend on the data. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z.

4.2 Asymptotic Analysis

4.2.1 Interior Expansion

We define the interior expansion as follows:

$$U(\vec{x},\vec{w}) \sim \sum_{k=0}^{\infty} \epsilon^k U_k(\vec{x},\vec{w}), \tag{4.2.1}$$

where U_k can be defined by comparing the order of ϵ via plugging (4.2.1) into the equation (4.1.1). Thus, we have

$$\begin{aligned} U_0 - \bar{U}_0 &= 0, && (4.2.2)\\ U_1 - \bar{U}_1 &= -\vec{w}\cdot\nabla_x U_0, && (4.2.3)\\ U_2 - \bar{U}_2 &= -\vec{w}\cdot\nabla_x U_1, && (4.2.4)\\ &\cdots \\ U_k - \bar{U}_k &= -\vec{w}\cdot\nabla_x U_{k-1}. && (4.2.5)\end{aligned}$$

The following analysis reveals the equation satisfied by U_k:

Plugging (4.2.2) into (4.2.3), we obtain

$$U_1 = \bar{U}_1 - \vec{w}\cdot\nabla_x \bar{U}_0. \tag{4.2.6}$$

Plugging (4.2.6) into (4.2.4), we get

$$U_2 - \bar{U}_2 = -\vec{w}\cdot\nabla_x(\bar{U}_1 - \vec{w}\cdot\nabla_x\bar{U}_0) = -\vec{w}\cdot\nabla_x\bar{U}_1 + \vec{w}^2\Delta_x\bar{U}_0 + 2w_1w_2\partial_{x_1x_2}\bar{U}_0 \tag{4.2.7}$$

Integrating (4.2.7) over $\vec{w} \in \mathcal{S}^1$, we achieve the final form

$$\Delta_x \bar{U}_0 = 0, \tag{4.2.8}$$

which further implies $U_0(\vec{x}, \vec{w})$ satisfies the equation

$$\left\{ \begin{array}{rcl} U_0 &=& \bar{U}_0 \\ \Delta_x U_0 &=& 0 \end{array} \right. \tag{4.2.9}$$

Similarly, we can derive $U_k(\vec{x}, \vec{w})$ for $k \geq 1$ satisfies

$$\left\{ \begin{array}{rcl} U_k &=& \bar{U}_k - \vec{w} \cdot \nabla_x U_{k-1} \\ \Delta_x \bar{U}_k &=& 0 \end{array} \right. \tag{4.2.10}$$

4.2.2 Milne Expansion

In order to determine the boundary condition fir U_k, it is well known that we need to define the boundary layer expansion. Hence, we need several substitutions:

Substitution 1:

We consider the substitution into quasi-polar coordinates $u^\epsilon(x_1, x_2, w_1, w_2) \rightarrow u^\epsilon(\mu, \theta, w_1, w_2)$ with $(\mu, \theta, w_1, w_2) \in [0, 1) \times [-\pi, \pi) \times \mathcal{S}^1$ defined as

$$\left\{ \begin{array}{rcl} x_1 &=& (1-\mu)\cos\theta, \\ x_2 &=& (1-\mu)\sin\theta. \end{array} \right. \tag{4.2.11}$$

Here μ denotes the distance to the boundary $\partial\Omega$ and θ is the space angular variable. In these new variables, equation (4.1.1) can be rewritten as

$$\left\{\begin{array}{l} -\epsilon\bigg(w_1\cos\theta+w_2\sin\theta\bigg)\dfrac{\partial u^\epsilon}{\partial\mu}-\dfrac{\epsilon}{1-\mu}\bigg(w_1\sin\theta-w_2\cos\theta\bigg)\dfrac{\partial u^\epsilon}{\partial\theta}\\ +u^\epsilon-\dfrac{1}{2\pi}\displaystyle\int_{\mathcal{S}^1}u^\epsilon \mathrm{d}\tilde{\mathrm{w}}=0,\\ u^\epsilon(0,\theta,w_1,w_2)=g(\theta,w_1,w_2)\ \ \text{for}\ \ w_1\cos\theta+w_2\sin\theta<0. \end{array}\right. \tag{4.2.12}$$

Substitution 2:

We further define the stretched variable η by making the scaling transform for $u^\epsilon(\mu,\theta,w_1,w_2)\rightarrow u^\epsilon(\eta,\theta,w_1,w_2)$ with $(\eta,\theta,w_1,w_2)\in[0,1/\epsilon)\times[-\pi,\pi)\times\mathcal{S}^1$ as

$$\left\{\begin{array}{rcl} \eta &=& \mu/\epsilon,\\ \theta &=& \theta, \end{array}\right. \tag{4.2.13}$$

which implies

$$\frac{\partial u^\epsilon}{\partial\mu}=\frac{1}{\epsilon}\frac{\partial u^\epsilon}{\partial\eta}. \tag{4.2.14}$$

Then equation (4.1.1) is transformed into

$$\left\{\begin{array}{l} -\bigg(w_1\cos\theta+w_2\sin\theta\bigg)\dfrac{\partial u^\epsilon}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}\bigg(w_1\sin\theta-w_2\cos\theta\bigg)\dfrac{\partial u^\epsilon}{\partial\theta}\\ +u^\epsilon-\dfrac{1}{2\pi}\displaystyle\int_{\mathcal{S}^1}u^\epsilon \mathrm{d}\tilde{\mathrm{w}}=0,\\ u^\epsilon(0,\theta,w_1,w_2)=g(\theta,w_1,w_2)\ \ \text{for}\ \ w_1\cos\theta+w_2\sin\theta<0. \end{array}\right. \tag{4.2.15}$$

Substitution 3:

Define the velocity substitution for $u^\epsilon(\eta,\theta,w_1,w_2)\rightarrow u^\epsilon(\eta,\theta,\xi)$ with $(\eta,\theta,\xi)\in[0,1/\epsilon)\times$

$[-\pi, \pi) \times [-\pi, \pi)$ as

$$\begin{cases} w_1 = -\sin\xi, \\ w_2 = -\cos\xi. \end{cases} \tag{4.2.16}$$

Here ξ denotes the velocity angular variable. We have the succinct form for (4.1.1) as

$$\begin{cases} \sin(\theta+\xi)\dfrac{\partial u^\epsilon}{\partial\eta} - \dfrac{\epsilon}{1-\epsilon\eta}\cos(\theta+\xi)\dfrac{\partial u^\epsilon}{\partial\theta} + u^\epsilon - \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} u^\epsilon \mathrm{d}\xi = 0, \\ u^\epsilon(0,\theta,\xi) = g(\theta,\xi) \quad \text{for} \quad \sin(\theta+\xi) > 0. \end{cases} \tag{4.2.17}$$

We now define the Milne expansion of boundary layer as follows:

$$\mathscr{U}(\eta,\theta,\phi) \sim \sum_{k=0}^{\infty} \epsilon^k \mathscr{U}_k(\eta,\theta,\phi), \tag{4.2.18}$$

where $\mathscr{U}_k$ can be determined by comparing the order of ϵ via plugging (4.2.18) into the equation (4.2.17). Thus, in a neighborhood of the boundary, we have

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_0}{\partial\eta} + \mathscr{U}_0 - \bar{\mathscr{U}}_0 = 0, \tag{4.2.19}$$

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_1}{\partial\eta} + \mathscr{U}_1 - \bar{\mathscr{U}}_1 = \frac{1}{1-\epsilon\eta}\cos(\theta+\xi)\frac{\partial \mathscr{U}_0}{\partial\theta}, \tag{4.2.20}$$

$$\ldots$$

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_k}{\partial\eta} + \mathscr{U}_k - \bar{\mathscr{U}}_k = \frac{1}{1-\epsilon\eta}\cos(\theta+\xi)\frac{\partial \mathscr{U}_{k-1}}{\partial\theta}, \tag{4.2.21}$$

where

$$\bar{\mathscr{U}}_k(\eta,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{U}_k(\eta,\theta,\xi)\mathrm{d}\xi. \tag{4.2.22}$$

The construction of U_k and $\mathscr{U}_k$ in [9] can be summarized as follows:

Step 1: Construction of $\mathscr{U}_0$ and U_0.

Assume the cut-off function ψ and ψ_0 are defined as

$$\psi(\mu) = \begin{cases} 1 & 0 \leq \mu \leq 1/2, \\ 0 & 3/4 \leq \mu \leq \infty. \end{cases} \tag{4.2.23}$$

$$\psi_0(\mu) = \begin{cases} 1 & 0 \leq \mu \leq 1/4, \\ 0 & 3/8 \leq \mu \leq \infty. \end{cases} \tag{4.2.24}$$

Then the zeroth order boundary layer solution is defined as

$$\begin{cases} \mathscr{U}_0(\eta, \theta, \xi) &=& \psi_0(\epsilon\eta)\Big(f_0(\eta, \theta, \xi) - f_0(\infty, \theta)\Big), \\ \sin(\theta + \xi)\dfrac{\partial f_0}{\partial \eta} + f_0 - \bar{f}_0 &=& 0, \\ f_0(0, \theta, \xi) &=& g(\theta, \xi) \quad \text{for} \quad \sin(\theta + \xi) > 0, \\ \lim_{\eta\to\infty} f_0(\eta, \theta, \xi) &=& f_0(\infty, \theta). \end{cases} \tag{4.2.25}$$

Assuming $g \in L^\infty(\Gamma^-)$, by Theorem 4.4.12, we can show there exists a unique solution $f_0(\eta, \theta, \xi) \in L^\infty([0, \infty) \times [-\pi, \pi) \times [-\pi, \pi))$. Hence, $\mathscr{U}_0$ is well-defined. Then we can define the zeroth order interior solution as

$$\begin{cases} U_0 &=& \bar{U}_0, \\ \Delta_x \bar{U}_0 &=& 0 \quad \text{in} \quad \Omega, \\ \bar{U}_0 &=& f_0(\infty, \theta) \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.2.26}$$

Step 2: Construction of $\mathscr{U}_1$ and U_1.

Define the first order boundary layer solution as

$$
\begin{cases}
\mathscr{U}_1(\eta,\theta,\xi) &= \psi_0(\epsilon\eta)\bigg(f_1(\eta,\theta,\xi)-f_1(\infty,\theta)\bigg),\\
\sin(\theta+\xi)\dfrac{\partial f_1}{\partial\eta}+f_1-\bar{f}_1 &= \cos(\theta+\xi)\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}_0}{\partial\theta},\\
f_1(0,\theta,\xi) &= \vec{w}\cdot\nabla_x U_0(\vec{x}_0,\vec{w}) \ \text{ for } \ \sin(\theta+\xi)>0,\\
\lim_{\eta\to\infty} f_1(\eta,\theta,\xi) &= f_1(\infty,\theta).
\end{cases}
\tag{4.2.27}
$$

where $(\vec{x}_0,\vec{w})$ is the same point as $(0,\theta,\xi)$. Define the first order interior solution as

$$
\begin{cases}
U_1 &= \bar{U}_1-\vec{w}\cdot\nabla_x U_0,\\
\Delta_x\bar{U}_1 &= 0 \ \text{ in } \ \Omega,\\
\bar{U}_1 &= f_1(\infty,\theta) \ \text{ on } \ \partial\Omega.
\end{cases}
\tag{4.2.28}
$$

Step 3: Generalization to arbitrary k.

Similar to above procedure, we can define the k^{th} order boundary layer solution as

$$
\begin{cases}
\mathscr{U}_k(\eta,\theta,\xi) &= \psi_0(\epsilon\eta)\bigg(f_k(\eta,\theta,\xi)-f_k(\infty,\theta)\bigg),\\
\sin(\theta+\xi)\dfrac{\partial f_k}{\partial\eta}+f_k-\bar{f}_k &= \cos(\theta+\xi)\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}_{k-1}}{\partial\theta},\\
f_k(0,\theta,\xi) &= \vec{w}\cdot\nabla_x U_{k-1}(\vec{x}_0,\vec{w}) \ \text{ for } \ \sin(\theta+\xi)>0,\\
\lim_{\eta\to\infty} f_k(\eta,\theta,\xi) &= f_k(\infty,\theta).
\end{cases}
\tag{4.2.29}
$$

Define the k^{th} order interior solution as

$$
\begin{cases}
U_k &= \bar{U}_k-\vec{w}\cdot\nabla_x U_{k-1},\\
\Delta_x\bar{U}_k &= 0 \ \text{ in } \ \Omega,\\
\bar{U}_k &= f_k(\infty,\theta) \ \text{ on } \ \partial\Omega.
\end{cases}
\tag{4.2.30}
$$

In [9, pp.136], the author proved the following result:

Theorem 4.2.1. *Assume $g(\vec{x}_0,\vec{w})$ is sufficiently smooth. Then for the steady neutron* *transport equation (4.1.1), the unique solution* $u^\epsilon(\vec{x},\vec{w}) \in L^\infty(\Omega\times\mathcal{S}^1)$ *satisfies*

$$\|u^\epsilon - U_0 - \mathscr{U}_0\|_{L^\infty} = O(\epsilon). \tag{4.2.31}$$

Our work begins with a crucial observation that based on Remark 4.4.15, the existence of solution f_1 requires the source term

$$\cos(\theta+\xi)\frac{\psi}{1-\epsilon\eta}\frac{\partial\mathscr{U}_0}{\partial\theta} \in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)). \tag{4.2.32}$$

Since the support of $\psi(\epsilon\eta)$ depends on ϵ, by (4.2.25), this in turn requires

$$\frac{\partial}{\partial\theta}\bigg(f_0(\eta,\theta,\xi) - f_0(\infty,\theta)\bigg) \in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)) \tag{4.2.33}$$

Note that $Z = \partial_\theta(f_0 - f_0(\infty,\theta))$ satisfies the equation

$$\left\{\begin{array}{rcl}\sin(\theta+\xi)\dfrac{\partial Z}{\partial\eta} + Z - \bar{Z} &=& -\cos(\theta+\xi)\dfrac{\partial f_0}{\partial\eta},\\ Z(0,\theta,\xi) &=& \dfrac{\partial g(\theta,\xi)}{\partial\theta} - \dfrac{\partial f_0(\infty,\theta)}{\partial\theta} \quad\text{for}\quad \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty} Z(\eta,\theta,\xi) &=& Z(\infty,\theta).\end{array}\right. \tag{4.2.34}$$

In order for $Z\in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$, assuming the boundary data $\partial_\theta g \in L^\infty(\Gamma^-)$, we require the source term

$$-\cos(\theta+\xi)\frac{\partial f_0}{\partial\eta} \in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)). \tag{4.2.35}$$

On the other hand, as shown by Lemma B.1.1, we can show for specific g, it holds that $\partial_\eta f_0 \notin L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$. Due to intrinsic singularity for (4.2.25), the construction in [9] breaks down.

In fact, in general geometry with curved boundary, we need to control the normal derivative of the boundary layer solution for the Milne expansion.

4.2.3 ϵ-Milne Expansion with Geometric Correction

Our main goal is to overcome the difficulty in estimating

$$\cos(\theta+\xi)\frac{\psi}{1-\epsilon\eta}\frac{\partial\mathscr{U}_k}{\partial\theta}. \tag{4.2.36}$$

We introduce one more substitution to decompose the term (4.2.36).

Substitution 4:

We make the rotation substitution for $u^{\epsilon}(\eta,\theta,\xi)\rightarrow u^{\epsilon}(\eta,\theta,\phi)$ with $(\eta,\theta,\phi)\in[0,1/\epsilon)\times[-\pi,\pi)\times[-\pi,\pi)$ as

$$\left\{\begin{array}{rcl}\eta&=&\eta,\\ \theta&=&\theta,\\ \phi&=&\theta+\xi,\end{array}\right. \tag{4.2.37}$$

and transform the equation (4.1.1) into

$$\left\{\begin{array}{l}\sin\phi\dfrac{\partial u^{\epsilon}}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}\cos\phi\bigg(\dfrac{\partial u^{\epsilon}}{\partial\phi}+\dfrac{\partial u^{\epsilon}}{\partial\theta}\bigg)+u^{\epsilon}-\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}u^{\epsilon}\mathrm{d}\phi=0,\\ u^{\epsilon}(0,\theta,\phi)=g(\theta,\phi)\ \ \text{for}\ \ \sin\phi>0.\end{array}\right. \tag{4.2.38}$$

Inspired by [11], [54] and [5], the most important idea is to include the most singular term

$$-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial u^{\epsilon}}{\partial\phi} \tag{4.2.39}$$

in the Milne problem.

We define the ϵ-Milne expansion with geometric correction of boundary layer as follows:

$$\mathscr{U}^{\epsilon}(\eta,\theta,\phi)\sim\sum_{k=0}^{\infty}\epsilon^k\mathscr{U}_k^{\epsilon}(\eta,\theta,\phi), \tag{4.2.40}$$

where $\mathscr{U}_k^{\epsilon}$ can be determined by comparing the order of ϵ via plugging (4.2.40) into the equation (4.2.38). Thus, in a neighborhood of the boundary, we have

$$\begin{aligned}
\sin\phi\frac{\partial\mathscr{U}_0^{\epsilon}}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0^{\epsilon}}{\partial\phi}+\mathscr{U}_0^{\epsilon}-\bar{\mathscr{U}}_0^{\epsilon} &= 0, && (4.2.41)\\
\sin\phi\frac{\partial\mathscr{U}_1^{\epsilon}}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1^{\epsilon}}{\partial\phi}+\mathscr{U}_1^{\epsilon}-\bar{\mathscr{U}}_1^{\epsilon} &= \frac{1}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0^{\epsilon}}{\partial\theta}, && (4.2.42)\\
&\ldots\\
\sin\phi\frac{\partial\mathscr{U}_k^{\epsilon}}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_k^{\epsilon}}{\partial\phi}+\mathscr{U}_k^{\epsilon}-\bar{\mathscr{U}}_k^{\epsilon} &= \frac{1}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{k-1}^{\epsilon}}{\partial\theta}. && (4.2.43)
\end{aligned}$$

where

$$\bar{\mathscr{U}}_k^{\epsilon}(\eta,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{U}_k^{\epsilon}(\eta,\theta,\phi)\mathrm{d}\phi. \tag{4.2.44}$$

It is important to note the solution $\mathscr{U}_k^{\epsilon}$ depends on ϵ.

We refer to the cut-off function ψ and ψ_0 as (4.2.23) and (4.2.24), and define the force as

$$F(\epsilon;\eta)=-\frac{\epsilon\psi(\epsilon\eta)}{1-\epsilon\eta}, \tag{4.2.45}$$

Define the interior expansion as follows:

$$U^{\epsilon}(\vec{x},\vec{w})\sim\sum_{k=0}^{\infty}\epsilon^{k}U_{k}^{\epsilon}(\vec{x},\vec{w}) \tag{4.2.46}$$

where U_k^{ϵ} satisfies the same equations as U_k in (4.2.9) and (4.2.10). Here, to highlight its dependence on ϵ via the ϵ-Milne problem and boundary data, we add the superscript ϵ.

The bridge between the interior solution and the boundary layer solution is the boundary condition of (4.1.1), so we first consider the boundary condition expansion

$$U_0^{\epsilon}+\mathscr{U}_0^{\epsilon} = g, \tag{4.2.47}$$

$$U_1^{\epsilon}+\mathscr{U}_1^{\epsilon} = 0, \tag{4.2.48}$$

$$\ldots$$

$$U_k^{\epsilon}+\mathscr{U}_k^{\epsilon} = 0. \tag{4.2.49}$$

The construction of U_k^{ϵ} and $\mathscr{U}_k^{\epsilon}$ are as follows:

Step 1: Construction of $\mathscr{U}_0^{\epsilon}$ and U_0^{ϵ}.

Define the zeroth order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_0^{\epsilon}(\eta,\theta,\phi) &=& \psi_0(\epsilon\eta)\bigg(f_0^{\epsilon}(\eta,\theta,\phi)-f_0^{\epsilon}(\infty,\theta)\bigg),\\ \sin\phi\dfrac{\partial f_0^{\epsilon}}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f_0^{\epsilon}}{\partial\phi}+f_0^{\epsilon}-\bar{f}_0^{\epsilon} &=& 0,\\ f_0^{\epsilon}(0,\theta,\phi) &=& g(\theta,\phi)\ \ \text{for}\ \ \sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f_0^{\epsilon}(\eta,\theta,\phi) &=& f_0^{\epsilon}(\infty,\theta).\end{array}\right. \tag{4.2.50}$$

In contrast to the classical Milne problem (4.2.25), the key advantage is, due to the geometry, $\dfrac{\partial F(\epsilon;\eta)}{\partial\theta}=0$, such that (4.2.50) is invariant in θ.

Then we define the zeroth order interior solution $U_0^\epsilon(\vec{x})$ as

$$\left\{\begin{array}{rcl} U_0^\epsilon &=& \bar{U}_0^\epsilon,\\ \Delta_x\bar{U}_0^\epsilon &=& 0 \ \text{ in } \ \Omega,\\ \bar{U}_0^\epsilon &=& f_0^\epsilon(\infty,\theta) \ \text{ on } \ \partial\Omega. \end{array}\right. \tag{4.2.51}$$

Step 2: Estimates of $\dfrac{\partial\mathscr{U}_0^\epsilon}{\partial\theta}$.

By Theorem 4.4.13, we can easily see f_0^ϵ is well-defined in $L^\infty(\Omega\times\mathcal{S}^1)$ and approaches $f_0^\epsilon(\infty)$ exponentially fast as $\eta\rightarrow\infty$. Then we can naturally derive $Z=\partial_\theta(f_0^\epsilon-f_0^\epsilon(\infty))$ also satisfies the same type of ϵ-Milne problem

$$\left\{\begin{array}{rcl} \sin\phi\dfrac{\partial Z}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial Z}{\partial\phi}+Z-\bar{Z} &=& 0,\\ Z(0,\theta,\phi) &=& \dfrac{\partial g}{\partial\theta}-\dfrac{\partial f_0^\epsilon(\infty)}{\partial\theta} \ \text{ for } \ \sin\phi>0,\\ \lim_{\eta\rightarrow\infty}Z(\eta,\phi) &=& C. \end{array}\right. \tag{4.2.52}$$

By Theorem 4.4.13, we can see $Z\rightarrow C$ exponentially fast as $\eta\rightarrow\infty$. It is natural to obtain this constant C must be zero. Hence, if $g\in C^r(\Gamma^-)$, it is obvious to check $f_0^\epsilon(\infty)\in C^r(\partial\Omega)$. By the standard elliptic estimate in (4.2.51), there exists a unique solution $\bar{U}_0^\epsilon\in W^{r,p}(\Omega)$ for arbitrary $p\geq2$ satisfying

$$\left\|\bar{U}_0^\epsilon\right\|_{W^{r,p}(\Omega)}\leq C(\Omega)\left\|f_0^\epsilon(\infty)\right\|_{W^{r-1/p,p}(\partial\Omega)}, \tag{4.2.53}$$

which implies $\nabla_x\bar{U}_0^\epsilon\in W^{r-1,p}(\Omega)$, $\nabla_x\bar{U}_0^\epsilon\in W^{r-1-1/p,p}(\partial\Omega)$ and $\bar{U}_0^\epsilon\in C^{r-1,1-2/p}(\Omega)$.

Step 3: Construction of $\mathscr{U}_1^\epsilon$ and U_1^ϵ.

Define the first order boundary layer solution as

$$\begin{cases} \mathscr{U}_1^\epsilon(\eta,\theta,\phi) &=& \psi_0(\epsilon\eta)\bigg(f_1^\epsilon(\eta,\theta,\phi)-f_1^\epsilon(\infty,\theta)\bigg),\\ \sin\phi\dfrac{\partial f_1^\epsilon}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f_1^\epsilon}{\partial\phi}+f_1^\epsilon-\bar{f}_1^\epsilon &=& \dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_0^\epsilon}{\partial\theta},\\ f_1^\epsilon(0,\theta,\phi) &=& \vec{w}\cdot\nabla_xU_0^\epsilon(\vec{x}_0,\vec{w}) \ \text{ for } \ \sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f_1^\epsilon(\eta,\theta,\phi) &=& f_1^\epsilon(\infty,\theta). \end{cases} \tag{4.2.54}$$

where $(\vec{x}_0,\vec{w})$ is the same point as $(0,\theta,\phi)$. Then we define the first order interior solution $U_1^\epsilon(\vec{x})$ as

$$\begin{cases} U_1^\epsilon &=& \bar{U}_1^\epsilon-\vec{w}\cdot\nabla_xU_0^\epsilon,\\ \Delta_x\bar{U}_1^\epsilon &=& 0 \ \text{ in } \ \Omega,\\ \bar{U}_1^\epsilon &=& f_1^\epsilon(\infty,\theta) \ \text{ on } \ \partial\Omega. \end{cases} \tag{4.2.55}$$

Step 4: Estimates of $\dfrac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}$.

By Theorem 4.4.13, we can easily see f_1^ϵ is well-defined in $L^\infty(\Omega\times\mathcal{S}^1)$ and approaches $f_1^\epsilon(\infty)$ exponentially fast as $\eta\rightarrow\infty$. Also, since $\vec{w}\cdot\nabla_xU_0^\epsilon\in W^{r-1-1/p,p}(\partial\Omega)$, $\partial_\theta f_1^\epsilon$ is well-defined and decays exponentially fast. Hence, $f_1^\epsilon(\infty,\theta)\in W^{r-1-1/p,p}(\partial\Omega)$. By the standard elliptic estimate in (4.2.55), there exists a unique solution $\bar{U}_1^\epsilon\in W^{r-1,p}(\Omega)$ and satisfies

$$\left\|\bar{U}_1^\epsilon\right\|_{W^{r-1,p}(\Omega)}\leq C(\Omega)\left\|f_1^\epsilon(\infty)\right\|_{W^{r-1-1/p,p}(\partial\Omega)}, \tag{4.2.56}$$

which implies $\nabla_x\bar{U}_1^\epsilon\in W^{r-2,p}(\Omega)$, $\nabla_x\bar{U}_1^\epsilon\in W^{r-2-1/p,p}(\partial\Omega)$ and $\bar{U}_1^\epsilon\in C^{r-2,1-2/p}(\Omega)$.

Step 5: Generalization to arbitrary k.

In a similar fashion, as long as g is sufficiently smooth, above process can go on. We construct the k^{th} order boundary layer solution as

$$
\begin{cases}
\mathscr{U}_k^\epsilon(\eta,\theta,\phi) &= \psi_0(\epsilon\eta)\bigg(f_k^\epsilon(\eta,\theta,\phi)-f_k^\epsilon(\infty,\theta)\bigg),\\
\sin\phi\dfrac{\partial f_k^\epsilon}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f_k^\epsilon}{\partial\phi}+f_k^\epsilon-\bar f_k^\epsilon &= \dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_{k-1}^\epsilon}{\partial\theta},\\
f_k^\epsilon(0,\theta,\phi) &= \vec w\cdot\nabla_x U_{k-1}^\epsilon(\vec x_0,\vec w) \ \text{ for } \ \sin\phi>0,\\
\lim_{\eta\to\infty} f_k^\epsilon(\eta,\theta,\phi) &= f_k^\epsilon(\infty,\theta).
\end{cases}
\tag{4.2.57}
$$

Then we define the k^{th} order interior solution as

$$
\begin{cases}
U_k^\epsilon &= \bar U_k^\epsilon-\vec w\cdot\nabla_x U_{k-1}^\epsilon,\\
\Delta_x\bar U_k^\epsilon &= 0 \ \text{ in } \ \Omega,\\
\bar U_k^\epsilon &= f_k^\epsilon(\infty,\theta) \ \text{ on } \ \partial\Omega.
\end{cases}
\tag{4.2.58}
$$

For $g\in C^{k+1}(\Gamma^-)$, the interior solution and boundary layer solution can be well-defined up to k^{th} order, i.e. up to U_k^ϵ and $\mathscr{U}_k^\epsilon$.

4.3 Well-posedness of Steady Neutron Transport Equation

In this section, we consider the well-posedness of the steady neutron transport equation

$$
\begin{cases}
\epsilon\vec w\cdot\nabla_x u+u-\bar u &= f(\vec x,\vec w) \ \text{ in } \ \Omega,\\
u(\vec x_0,\vec w) &= g(\vec x_0,\vec w) \ \text{ for } \ \vec x_0\in\partial\Omega \ \text{ and } \vec w\cdot\vec n<0.
\end{cases}
\tag{4.3.1}
$$

We define the L^2 and L^∞ norms in $\Omega\times\mathcal{S}^1$ as usual:

$$
\begin{aligned}
\|f\|_{L^2(\Omega\times\mathcal{S}^1)} &= \left(\int_\Omega\int_{\mathcal{S}^1}|f(\vec x,\vec w)|^2\,\mathrm{d}\tilde{\mathrm{w}}\mathrm{d}\tilde{\mathrm{x}}\right)^{1/2}, && (4.3.2)\\
\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} &= \sup_{(\vec x,\vec w)\in\Omega\times\mathcal{S}^1}|f(\vec x,\vec w)|. && (4.3.3)
\end{aligned}
$$

Define the L^2 and L^∞ norms on the boundary as follows:

$$
\begin{aligned}
\|f\|_{L^2(\Gamma)} &= \left(\iint_\Gamma|f(\vec x,\vec w)|^2\,|\vec w\cdot\vec n|\,\mathrm{d}\tilde{\mathrm{w}}\mathrm{d}\tilde{\mathrm{x}}\right)^{1/2}, && (4.3.4)\\
\|f\|_{L^2(\Gamma^\pm)} &= \left(\iint_{\Gamma^\pm}|f(\vec x,\vec w)|^2\,|\vec w\cdot\vec n|\,\mathrm{d}\tilde{\mathrm{w}}\mathrm{d}\tilde{\mathrm{x}}\right)^{1/2}, && (4.3.5)\\
\|f\|_{L^\infty(\Gamma)} &= \sup_{(\vec x,\vec w)\in\Gamma}|f(\vec x,\vec w)|, && (4.3.6)\\
\|f\|_{L^\infty(\Gamma^\pm)} &= \sup_{(\vec x,\vec w)\in\Gamma^\pm}|f(\vec x,\vec w)|. && (4.3.7)
\end{aligned}
$$

4.3.1 Preliminaries

In order to show the L^2 and L^∞ well-posedness of the equation (4.3.1), we start with some preparations with the penalized neutron transport equation.

Lemma 4.3.1. *Assume $f(\vec x,\vec w)\in L^\infty(\Omega\times\mathcal{S}^1)$ and $g(x_0,\vec w)\in L^\infty(\Gamma^-)$. Then for the penalized transport equation*

$$
\left\{
\begin{array}{rcl}
\lambda u_\lambda+\epsilon\vec w\cdot\nabla_x u_\lambda+u_\lambda &=& f(\vec x,\vec w)\quad in\quad \Omega,\\
u_\lambda(\vec x_0,\vec w) &=& g(\vec x_0,\vec w)\quad for\quad \vec x_0\in\partial\Omega\quad and\ \vec w\cdot\vec n<0.
\end{array}
\right. \tag{4.3.8}
$$

with $\lambda>0$ as a penalty parameter, there exists a solution $u_\lambda(\vec x,\vec w)\in L^\infty(\Omega\times\mathcal{S}^1)$ satisfying

$$
\|u_\lambda\|_{L^\infty(\Omega\times\mathcal{S}^1)}\leq\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\|g\|_{L^\infty(\Gamma^-)}. \tag{4.3.9}
$$

Proof. The characteristics $(X(s), W(s))$ of the equation (4.3.8) which goes through $(\vec{x}, \vec{w})$ is defined by

$$\begin{cases} (X(0), W(0)) &= (\vec{x}, \vec{w}) \\ \dfrac{\mathrm{dX(s)}}{\mathrm{ds}} &= \epsilon W(s), \\ \dfrac{\mathrm{dW(s)}}{\mathrm{ds}} &= 0. \end{cases} \tag{4.3.10}$$

which implies

$$\begin{cases} X(s) &= \vec{x} + (\epsilon\vec{w})s \\ W(s) &= \vec{w} \end{cases} \tag{4.3.11}$$

Hence, we can rewrite the equation (4.3.8) along the characteristics as

$$u_\lambda(\vec{x}, \vec{w}) = g(\vec{x} - \epsilon t_b\vec{w}, \vec{w})e^{-(1+\lambda)t_b} + \int_0^{t_b} f(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w})e^{-(1+\lambda)(t_b-s)}\mathrm{ds}, \tag{4.3.12}$$

where the backward exit time t_b is defined as

$$t_b(\vec{x}, \vec{w}) = \inf\{t \geq 0 : (\vec{x} - \epsilon t\vec{w}, \vec{w}) \in \Gamma^-\}. \tag{4.3.13}$$

Then we can naturally estimate

$$\begin{aligned} \|u_\lambda\|_{L^\infty(\Omega\times\mathcal{S}^1)} &\leq e^{-(1+\lambda)t_b}\|g\|_{L^\infty(\Gamma^-)} + \frac{1 - e^{-(1+\lambda)t_b}}{1+\lambda}\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} \\ &\leq \|g\|_{L^\infty(\Gamma^-)} + \|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \end{aligned} \tag{4.3.14}$$

Since u_λ can be explicitly traced back to the boundary data, the existence naturally follows from above estimate. □

Lemma 4.3.2. *Assume $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$ and $g(x_0, \vec{w}) \in L^\infty(\Gamma^-)$. Then for the* *penalized neutron transport equation*

$$\left\{\begin{array}{rcl}\lambda u_{\lambda}+\epsilon\vec{w}\cdot\nabla_x u_{\lambda}+u_{\lambda}-\bar{u}_{\lambda} &=& f(\vec{x},\vec{w}) \quad in \quad \Omega,\\ u_{\lambda}(\vec{x}_0,\vec{w}) &=& g(\vec{x}_0,\vec{w}) \quad for \quad \vec{x}_0\in\partial\Omega \quad and\ \vec{w}\cdot\vec{n}<0.\end{array}\right. \tag{4.3.15}$$

with $\lambda > 0$, there exists a solution $u_{\lambda}(\vec{x},\vec{w}) \in L^{\infty}(\Omega\times\mathcal{S}^1)$ satisfying

$$\|u_{\lambda}\|_{L^{\infty}(\Omega\times\mathcal{S}^1)} \leq \frac{1+\lambda}{\lambda}\bigg(\|f\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\|g\|_{L^{\infty}(\Gamma^-)}\bigg). \tag{4.3.16}$$

Proof. We define an approximating sequence $\{u_{\lambda}^k\}_{k=0}^{\infty}$, where $u_{\lambda}^0 = 0$ and

$$\left\{\begin{array}{rcl}\lambda u_{\lambda}^k+\epsilon\vec{w}\cdot\nabla_x u_{\lambda}^k+u_{\lambda}^k-\bar{u}_{\lambda}^{k-1} &=& f(\vec{x},\vec{w}) \ \text{ in } \ \Omega,\\ u_{\lambda}^k(\vec{x}_0,\vec{w}) &=& g(\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{x}_0\in\partial\Omega \ \text{ and } \vec{w}\cdot\vec{n}<0.\end{array}\right. \tag{4.3.17}$$

By Lemma 4.3.1, this sequence is well-defined and $\left\|u_{\lambda}^k\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)} < \infty$.

The characteristics and the backward exit time are defined as (4.3.10) and (4.3.13), so we rewrite equation (4.3.17) along the characteristics as

$$\begin{aligned}&u_{\lambda}^k(\vec{x},\vec{w}) \qquad\qquad (4.3.18)\\ =\ & g(\vec{x}-\epsilon t_b\vec{w},\vec{w})e^{-(1+\lambda)t_b}+\int_0^{t_b}(f+\bar{u}_{\lambda}^{k-1})(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})e^{-(1+\lambda)(t_b-s)}\mathrm{ds}.\end{aligned}$$

We define the difference $v^k = u_{\lambda}^k - u_{\lambda}^{k-1}$ for $k \geq 1$. Recall (4.1.2) for $\bar{v}^k$, then v^k satisfies

$$v^{k+1}(\vec{x},\vec{w}) = \int_0^{t_b}\bar{v}^k(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})e^{-(1+\lambda)(t_b-s)}\mathrm{ds}. \tag{4.3.19}$$

Since $\left\|\bar{v}^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}$, we can directly estimate

$$\left\|v^{k+1}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \int_0^{t_b} e^{-(1+\lambda)(t_b-s)}\mathrm{ds} \leq \frac{1-e^{-(1+\lambda)t_b}}{1+\lambda} \left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.3.20}$$

Hence, we naturally have

$$\left\|v^{k+1}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \frac{1}{1+\lambda} \left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.3.21}$$

Thus, this is a contraction sequence for $\lambda > 0$. Considering $v^1 = u_\lambda^1$, we have

$$\left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \left(\frac{1}{1+\lambda}\right)^{k-1} \left\|u_\lambda^1\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}, \tag{4.3.22}$$

for $k \geq 1$. Therefore, u_λ^k converges strongly in L^∞ to a limit solution u_λ satisfying

$$\left\|u_\lambda\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \sum_{k=1}^{\infty} \left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \frac{1+\lambda}{\lambda} \left\|u_\lambda^1\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.3.23}$$

Since u_λ^1 can be rewritten along the characteristics as

$$u_\lambda^1(\vec{x},\vec{w}) = g(\vec{x}-\epsilon t_b\vec{w},\vec{w})e^{-(1+\lambda)t_b} + \int_0^{t_b} f(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})e^{-(1+\lambda)(t_b-s)}\mathrm{ds}, \tag{4.3.24}$$

based on Lemma 4.3.1, we can directly estimate

$$\left\|u_\lambda^1\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \left\|g\right\|_{L^\infty(\Gamma^-)}. \tag{4.3.25}$$

Combining (4.3.23) and (4.3.25), we can easily deduce the lemma. $\square$

4.3.2 L^2 Estimate

It is easy to see when $\lambda \to 0$, the estimate in Lemma 4.3.2 blows up. Hence, we need to show a uniform estimate of the solution to the penalized neutron transport equation (4.3.15).

Lemma 4.3.3. *(Green's Identity) Assume $f(\vec{x},\vec{w}),\ g(\vec{x},\vec{w}) \in L^2(\Omega\times\mathcal{S}^1)$ and $\vec{w}\cdot\nabla_x f,\ \vec{w}\cdot\nabla_x g \in L^2(\Omega\times\mathcal{S}^1)$ with $f,\ g\in L^2(\Gamma)$. Then*

$$\iint_{\Omega\times\mathcal{S}^1}\Big((\vec{w}\cdot\nabla_x f)g + (\vec{w}\cdot\nabla_x g)f\Big)\mathrm{d}\tilde{\mathrm{x}}\mathrm{d}\tilde{\mathrm{w}} = \int_\Gamma fg\mathrm{d}\gamma, \tag{4.3.26}$$

where $\mathrm{d}\gamma = (\vec{w}\cdot\vec{n})\mathrm{ds}$ on the boundary.

Proof. See [10, Chapter 9] and [19]. □

Lemma 4.3.4. *The solution u_λ to the equation (4.3.15) satisfies the uniform estimate*

$$\begin{aligned} & \epsilon\left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} \qquad (4.3.27)\\ \leq\ & C(\Omega)\bigg(\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon\left\|u_\lambda\right\|_{L^2(\Gamma^+)} + \epsilon\left\|g\right\|_{L^2(\Gamma^-)}\bigg), \end{aligned}$$

for $0\leq\lambda<<1$ and $0<\epsilon<<1$.

Proof. Applying Lemma 4.3.3 to the solution of the equation (4.3.15). Then for any $\phi\in L^2(\Omega\times\mathcal{S}^1)$ satisfying $\vec{w}\cdot\nabla_x\phi\in L^2(\Omega\times\mathcal{S}^1)$ and $\phi\in L^2(\Gamma)$, we have

$$\begin{aligned} & \lambda\iint_{\Omega\times\mathcal{S}^1}u_\lambda\phi + \epsilon\int_\Gamma u_\lambda\phi\mathrm{d}\gamma - \epsilon\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)u_\lambda + \iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi \\ =\ & \iint_{\Omega\times\mathcal{S}^1} f\phi. \end{aligned} \tag{4.3.28}$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function ζ. Since $u_\lambda \in L^\infty(\Omega \times \mathcal{S}^1)$, it naturally implies $\bar{u}_\lambda \in L^\infty(\Omega)$ which further leads to $\bar{u}_\lambda \in L^2(\Omega)$. We define $\zeta(\vec{x})$ on Ω satisfying

$$\begin{cases} \Delta\zeta &= \bar{u}_\lambda \ \text{ in } \ \Omega, \\ \zeta &= 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{4.3.29}$$

In the bounded domain Ω, based on the standard elliptic estimate, we have

$$\|\zeta\|_{H^2(\Omega)} \leq C(\Omega) \|\bar{u}_\lambda\|_{L^2(\Omega)}. \tag{4.3.30}$$

We plug the test function

$$\phi = -\vec{w} \cdot \nabla_x \zeta \tag{4.3.31}$$

into the weak formulation (4.3.28) and estimate each term there. Naturally, we have

$$\|\phi\|_{L^2(\Omega)} \leq C \|\zeta\|_{H^1(\Omega)} \leq C(\Omega) \|\bar{u}_\lambda\|_{L^2(\Omega)}. \tag{4.3.32}$$

Easily we can decompose

$$-\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi) u_\lambda \ = \ -\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi) \bar{u}_\lambda - \epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi)(u_\lambda - \bar{u}_\lambda). \tag{4.3.33}$$

We estimate the two term on the right-hand side separately. By (4.3.29) and (4.3.31), we have

$$
\begin{aligned}
& -\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w}\cdot\nabla_x\phi)\bar{u}_\lambda && (4.3.34)\\
= \ & \epsilon \iint_{\Omega\times\mathcal{S}^1} \bar{u}_\lambda\bigg(w_1(w_1\partial_{11}\zeta + w_2\partial_{12}\zeta) + w_2(w_1\partial_{12}\zeta + w_2\partial_{22}\zeta)\bigg)\\
= \ & \epsilon \iint_{\Omega\times\mathcal{S}^1} \bar{u}_\lambda\bigg(w_1^2\partial_{11}\zeta + w_2^2\partial_{22}\zeta\bigg)\\
= \ & \epsilon\pi \int_{\Omega} \bar{u}_\lambda(\partial_{11}\zeta + \partial_{22}\zeta)\\
= \ & \epsilon\pi \left\|\bar{u}_\lambda\right\|_{L^2(\Omega)}^2\\
= \ & \frac{1}{2}\epsilon \left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)}^2 .
\end{aligned}
$$

In the second equality, above cross terms vanish due to the symmetry of the integral over $\mathcal{S}^1$. On the other hand, for the second term in (4.3.33), Hölder's inequality and the elliptic estimate imply

$$
\begin{aligned}
-\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w}\cdot\nabla_x\phi)(u_\lambda - \bar{u}_\lambda) \ &\leq \ C(\Omega)\epsilon \left\|u_\lambda - \bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} \left\|\zeta\right\|_{H^2(\Omega)} && (4.3.35)\\
&\leq \ C(\Omega)\epsilon \left\|u_\lambda - \bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} \left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} .
\end{aligned}
$$

Based on (4.3.30), (4.3.32), the boundary condition of the penalized neutron transport equation (4.3.15), the trace theorem, Hölder's inequality and the elliptic estimate, we have

$$
\begin{aligned}
\epsilon \int_{\Gamma} u_\lambda\phi \mathrm{d}\gamma \ &= \ \epsilon \int_{\Gamma^+} u_\lambda\phi \mathrm{d}\gamma + \epsilon \int_{\Gamma^-} u_\lambda\phi \mathrm{d}\gamma && (4.3.36)\\
&\leq \ C(\Omega)\bigg(\epsilon \left\|u_\lambda\right\|_{L^2(\Gamma^+)} \left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon \left\|g\right\|_{L^2(\Gamma^-)} \left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)}\bigg),
\end{aligned}
$$

$$\begin{aligned}
\lambda\iint_{\Omega\times\mathcal{S}^1}u_\lambda\phi &= \lambda\iint_{\Omega\times\mathcal{S}^1}\bar{u}_\lambda\phi+\lambda\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi \qquad (4.3.37)\\
&= \lambda\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi\\
&\leq C(\Omega)\lambda\left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)}\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)},
\end{aligned}$$

$$\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi\leq C(\Omega)\left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)}\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)}, \qquad (4.3.38)$$

$$\iint_{\Omega\times\mathcal{S}^1}f\phi\leq C(\Omega)\left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)}\left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)}. \qquad (4.3.39)$$

Collecting terms in (4.3.34), (4.3.35), (4.3.36), (4.3.37), (4.3.38) and (4.3.39), we obtain

$$\begin{aligned}
&\epsilon\left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} \qquad (4.3.40)\\
\leq\ & C(\Omega)\bigg((1+\epsilon+\lambda)\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)}+\epsilon\left\|u_\lambda\right\|_{L^2(\Gamma^+)}+\left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)}+\epsilon\left\|g\right\|_{L^2(\Gamma^-)}\bigg),
\end{aligned}$$

When $0\leq\lambda<1$ and $0<\epsilon<1$, we get the desired uniform estimate with respect to λ. $\square$

Theorem 4.3.5. *Assume $f(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $g(x_0,\vec{w})\in L^\infty(\Gamma^-)$. Then for the steady neutron transport equation (4.3.1), there exists a unique solution $u(\vec{x},\vec{w})\in L^2(\Omega\times\mathcal{S}^1)$ satisfying*

$$\left\|u\right\|_{L^2(\Omega\times\mathcal{S}^1)}\leq C(\Omega)\bigg(\frac{1}{\epsilon^2}\left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|g\right\|_{L^2(\Gamma^-)}\bigg). \qquad (4.3.41)$$

Proof. In the weak formulation (4.3.28), we may take the test function $\phi=u_\lambda$ to get the energy estimate

$$\lambda \left\| u_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{2}\epsilon \int_\Gamma \left| u_\lambda \right|^2 \mathrm{d}\gamma + \left\| u_\lambda - \bar{u}_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} = \iint_{\Omega\times\mathcal{S}^1} f u_\lambda. \tag{4.3.42}$$

Hence, this naturally implies

$$\frac{1}{2}\epsilon \left\| u_\lambda \right\|^2_{L^2(\Gamma^+)} + \left\| u_\lambda - \bar{u}_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \leq \iint_{\Omega\times\mathcal{S}^1} f u_\lambda + \frac{1}{2}\epsilon \left\| g \right\|^2_{L^2(\Gamma^-)}. \tag{4.3.43}$$

On the other hand, we can square on both sides of (4.3.27) to obtain

$$\begin{aligned} & \epsilon^2 \left\| \bar{u}_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \\ \leq\ & C(\Omega)\bigg(\left\| u_\lambda - \bar{u}_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \left\| f \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon^2 \left\| u_\lambda \right\|^2_{L^2(\Gamma^+)} + \epsilon^2 \left\| g \right\|^2_{L^2(\Gamma^-)} \bigg). \end{aligned} \tag{4.3.44}$$

Multiplying a sufficiently small constant on both sides of (4.3.44) and adding it to (4.3.43) to absorb $\left\| u_\lambda \right\|^2_{L^2(\Gamma^+)}$ and $\left\| u_\lambda - \bar{u}_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)}$, we deduce

$$\begin{aligned} & \epsilon \left\| u_\lambda \right\|^2_{L^2(\Gamma^+)} + \epsilon^2 \left\| \bar{u}_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \left\| u_\lambda - \bar{u}_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \\ & \qquad \leq C(\Omega)\bigg(\left\| f \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \iint_{\Omega\times\mathcal{S}^1} f u_\lambda + \epsilon \left\| g \right\|^2_{L^2(\Gamma^-)} \bigg). \end{aligned} \tag{4.3.45}$$

Hence, we have

$$\epsilon \left\| u_\lambda \right\|^2_{L^2(\Gamma^+)} + \epsilon^2 \left\| u_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \leq C(\Omega)\bigg(\left\| f \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \iint_{\Omega\times\mathcal{S}^1} f u_\lambda + \epsilon \left\| g \right\|^2_{L^2(\Gamma^-)} \bigg). \tag{4.3.46}$$

A simple application of Cauchy's inequality leads to

$$\iint_{\Omega\times\mathcal{S}^1} f u_\lambda \leq \frac{1}{4C\epsilon^2} \left\| f \right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + C\epsilon^2 \left\| u_\lambda \right\|^2_{L^2(\Omega\times\mathcal{S}^1)}. \tag{4.3.47}$$

Taking C sufficiently small, we can divide (4.3.46) by ϵ^2 to obtain

$$\frac{1}{\epsilon}\left\|u_\lambda\right\|^2_{L^2(\Gamma^+)}+\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^2)}\leq C(\Omega)\bigg(\frac{1}{\epsilon^4}\left\|f\right\|^2_{L^2(\Omega\times\mathcal{S}^2)}+\frac{1}{\epsilon}\left\|g\right\|^2_{L^2(\Gamma^-)}\bigg). \tag{4.3.48}$$

Since above estimate does not depend on λ, it gives a uniform estimate for the penalized neutron transport equation (4.3.15). Thus, we can extract a weakly convergent subsequence $u_\lambda\rightarrow u$ as $\lambda\rightarrow 0$. The weak lower semi-continuity of norms $\left\|\cdot\right\|_{L^2(\Omega\times\mathcal{S}^2)}$ and $\left\|\cdot\right\|_{L^2(\Gamma^+)}$ implies u also satisfies the estimate (4.3.48). Hence, in the weak formulation (4.3.28), we can take $\lambda\rightarrow 0$ to deduce that u satisfies equation (4.3.1). Also $u_\lambda-u$ satisfies the equation

$$\left\{\begin{array}{rcl}\epsilon\vec{w}\cdot\nabla_x(u_\lambda-u)+(u_\lambda-u)-(\bar{u}_\lambda-\bar{u})&=&-\lambda u_\lambda\ \ \text{in}\ \ \Omega,\\ (u_\lambda-u)(\vec{x}_0,\vec{w})&=&0\ \ \text{for}\ \ \vec{x}_0\in\partial\Omega\ \ \mathit{and}\ \vec{w}\cdot\vec{n}<0.\end{array}\right. \tag{4.3.49}$$

By a similar argument as above, we can achieve

$$\left\|u_\lambda-u\right\|^2_{L^2(\Omega\times\mathcal{S}^2)}\leq C(\Omega)\bigg(\frac{\lambda}{\epsilon^4}\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^2)}\bigg). \tag{4.3.50}$$

When $\lambda\rightarrow 0$, the right-hand side approaches zero, which implies the convergence is actually in the strong sense. The uniqueness easily follows from the energy estimates.

□

4.3.3 L^∞ Estimate

Theorem 4.3.6. *Assume $f(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $g(x_0,\vec{w})\in L^\infty(\Gamma^-)$. Then for the neutron transport equation (4.3.1), there exists a unique solution $u(\vec{x},\vec{w})\in$*

$L^\infty(\Omega\times\mathcal{S}^1)$ *satisfying*

$$\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}\le C(\Omega)\bigg(\frac{1}{\epsilon^3}\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{3/2}}\|g\|_{L^\infty(\Gamma^-)}\bigg). \tag{4.3.51}$$

Proof. We divide the proof into several steps to bootstrap an L^2 solution to an L^∞ solution:

Step 1: Double Duhamel iterations.

The characteristics of the equation (4.3.1) is given by (4.3.10). Hence, we can rewrite the equation (4.3.1) along the characteristics as

$$\begin{aligned} u(\vec{x},\vec{w}) &= g(\vec{x}-\epsilon t_b\vec{w},\vec{w})e^{-t_b}+\int_0^{t_b}f(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})e^{-(t_b-s)}\mathrm{ds} \\ &\quad+\frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathcal{S}^1}u(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}_t)\mathrm{d}\tilde{\mathrm{w}}_\mathrm{t}\bigg)e^{-(t_b-s)}\mathrm{ds}. \end{aligned} \tag{4.3.52}$$

where the backward exit time t_b is defined as (4.3.13). Note we have replaced $\bar{u}$ by the integral of u over the dummy velocity variable $\vec{w}_t$. For the last term in this formulation, we apply the Duhamel's principle again to $u(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}_t)$ and obtain

$$\begin{aligned} &u(\vec{x},\vec{w}) \\ =\;& g(\vec{x}-\epsilon t_b\vec{w},\vec{w})e^{-t_b}+\int_0^{t_b}f(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})e^{-(t_b-s)}\mathrm{ds} \\ &+\frac{1}{2\pi}\int_0^{t_b}\int_{\mathcal{S}^1}g(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon s_b\vec{w}_t,\vec{w}_t)e^{-s_b}\mathrm{d}\tilde{\mathrm{w}}_\mathrm{t}e^{-(t_b-s)}\mathrm{ds} \\ &+\frac{1}{2\pi}\int_0^{t_b}\int_{\mathcal{S}^1}\bigg(\int_0^{s_b}f(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t,\vec{w}_t)e^{-(s_b-r)}\mathrm{dr}\bigg)\mathrm{d}\tilde{\mathrm{w}}_\mathrm{t}e^{-(t_b-s)}\mathrm{ds} \\ &+\bigg(\frac{1}{2\pi}\bigg)^2\int_0^{t_b}\int_{\mathcal{S}^1}e^{-(t_b-s)} \\ &\times\bigg(\int_0^{s_b}\int_{\mathcal{S}^1}e^{-(s_b-r)}u(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t,\vec{w}_s)\mathrm{d}\tilde{\mathrm{w}}_\mathrm{s}\mathrm{dr}\bigg)\mathrm{d}\tilde{\mathrm{w}}_\mathrm{t}\mathrm{ds}, \end{aligned} \tag{4.3.53}$$

where we introduce another dummy velocity variable $\vec{w}_s$ and

$$s_b(\vec{x},\vec{w},s,\vec{w}_t)=\inf\{r\geq 0:(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon r\vec{w}_t,\vec{w}_t)\in\Gamma^-\}. \tag{4.3.54}$$

Step 2: Estimates of all but the last term in (4.3.53).

We can directly estimate as follows:

$$\left|g(\vec{x}-\epsilon t_b\vec{w},\vec{w})e^{-t_b}\right|\leq \|g\|_{L^\infty(\Gamma^-)}\,, \tag{4.3.55}$$

$$\left|\frac{1}{2\pi}\int_0^{t_b}\int_{\mathcal{S}^1}g(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon s_b\vec{w}_t,\vec{w}_t)e^{-s_b}\mathrm{d}\tilde{\mathrm{w}}_\mathrm{t}e^{-(t_b-s)}\mathrm{ds}\right|\leq \|g\|_{L^\infty(\Gamma^-)}\,, \tag{4.3.56}$$

$$\left|\int_0^{t_b}f(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})e^{-(t_b-s)}\mathrm{ds}\right|\leq \|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}\,, \tag{4.3.57}$$

$$\begin{aligned}&\left|\frac{1}{2\pi}\int_0^{t_b}\int_{\mathcal{S}^1}\left(\int_0^{s_b}f(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t,\vec{w}_t)e^{-(s_b-r)}\mathrm{dr}\right)\mathrm{d}\tilde{\mathrm{w}}_\mathrm{t}e^{-(t_b-s)}\mathrm{ds}\right|\\&\qquad\leq \|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}\,.\end{aligned} \tag{4.3.58}$$

Step 3: Estimates of the last term in (4.3.53).

Now we decompose the last term in (4.3.53) as

$$\int_0^{t_b}\int_{\mathcal{S}^1}\int_0^{s_b}\int_{\mathcal{S}^1}=\int_0^{t_b}\int_{\mathcal{S}^1}\int_{s_b-r\leq\delta}\int_{\mathcal{S}^1}+\int_0^{t_b}\int_{\mathcal{S}^1}\int_{s_b-r\geq\delta}\int_{\mathcal{S}^1}=I_1+I_2, \tag{4.3.59}$$

for some $\delta > 0$. We can estimate I_1 directly as

$$I_1 \leq \int_0^{t_b} e^{-(t_b-s)}\bigg(\int_{\max(0,s_b-\delta)}^{s_b} \|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}\,\mathrm{dr}\bigg)\mathrm{ds} \leq \delta\,\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}\,. \tag{4.3.60}$$

Then we can bound I_2 as

$$\begin{aligned} I_2 \leq\ & C\int_0^{t_b}\int_{\mathcal{S}^1}\int_0^{\max(0,s_b-\delta)} \\ & \times\int_{\mathcal{S}^1}\left|u(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t,\vec{w}_s)\right| e^{-(t_b-s)}\mathrm{d\tilde{w}_s dr d\tilde{w}_t ds}. \end{aligned} \tag{4.3.61}$$

By the definition of t_b and s_b, we always have $\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t \in \bar{\Omega}$. Hence, we may interchange the order of integration and apply Hölder's inequality to obtain

$$\begin{aligned} I_2 \leq\ & C\int_0^{t_b}\int_0^{\max(0,s_b-\delta)}\int_{\mathcal{S}^1}\int_{\mathcal{S}^1}\mathbf{1}_\Omega(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t) \\ & \left|u(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t,\vec{w}_s)\right| e^{-(t_b-s)}\mathrm{d\tilde{w}_t d\tilde{w}_s dr ds} \\ \leq\ & C\int_0^{t_b}\int_{\mathcal{S}^1}\bigg(\int_0^{\max(0,s_b-\delta)}\int_{\mathcal{S}^1}\mathbf{1}_\Omega(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t) \\ & \left|u(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t,\vec{w}_s)\right|^2\mathrm{d\tilde{w}_t dr}\bigg)^{1/2} e^{-(t_b-s)}\mathrm{d\tilde{w}_s ds}. \end{aligned} \tag{4.3.62}$$

Note $\vec{w}_t \in \mathcal{S}^1$, which is essentially a one-dimensional variable. Thus, we may write it in a new variable ψ as $\vec{w}_t = (\cos\psi, \sin\psi)$. Then we define the change of variable $[-\pi,\pi)\times\mathbb{R}\to\Omega : (\psi, r)\to(y_1,y_2) = \vec{y} = \vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t$, i.e.

$$\left\{\begin{aligned} y_1 &= x_1-\epsilon(t_b-s)w_1-\epsilon(s_b-r)\cos\psi, \\ y_2 &= x_2-\epsilon(t_b-s)w_2-\epsilon(s_b-r)\sin\psi. \end{aligned}\right. \tag{4.3.63}$$

Therefore, for $s_b - r \geq \delta$, we can directly compute the Jacobian

$$\left|\frac{\partial(y_1,y_2)}{\partial(\psi,r)}\right| = \left\|\begin{array}{cc} -\epsilon(s_b-r)\sin\psi & \epsilon\cos\psi \\ \epsilon(s_b-r)\cos\psi & \epsilon\sin\psi \end{array}\right\| = \epsilon^2(s_b-r) \geq \epsilon^2\delta. \tag{4.3.64}$$

Hence, we may simplify (4.3.62) as

$$I_2 \le \frac{C}{\epsilon\sqrt{\delta}}\int_0^{t_b}\int_{\mathcal{S}^1}\left(\int_\Omega |u(\vec{y},\vec{w}_s)|^2\,\mathrm{d}\tilde{\mathrm{y}}\right)^{1/2} e^{-(t_b-s)}\mathrm{d}\tilde{\mathrm{w}}_\mathrm{s}\mathrm{ds}. \tag{4.3.65}$$

Then we may further utilize Cauchy's inequality and the L^2 estimate of u in Theorem 4.3.5 to obtain

$$\begin{aligned}
I_2 &\le \frac{C}{\epsilon\sqrt{\delta}}\int_0^{t_b}\left(\int_{\mathcal{S}^1}\int_\Omega |u(\vec{y},\vec{w}_s)|^2\,\mathrm{d}\tilde{\mathrm{y}}\mathrm{d}\tilde{\mathrm{w}}_\mathrm{s}\right)^{1/2} e^{-(t_b-s)}\mathrm{ds} \qquad (4.3.66)\\
&= \frac{C}{\epsilon\sqrt{\delta}}\int_0^{t_b} e^{-(t_b-s)}\left\|u\right\|_{L^2(\Omega\times\mathcal{S}^1)}\mathrm{ds}\\
&\le \frac{C}{\epsilon\sqrt{\delta}}\left\|u\right\|_{L^2(\Omega\times\mathcal{S}^1)}\\
&\le \frac{C(\Omega)}{\sqrt{\delta}}\left(\frac{1}{\epsilon^3}\left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{3/2}}\left\|g\right\|_{L^2(\Gamma^-)}\right)\\
&\le \frac{C(\Omega)}{\sqrt{\delta}}\left(\frac{1}{\epsilon^3}\left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{3/2}}\left\|g\right\|_{L^\infty(\Gamma^-)}\right).
\end{aligned}$$

In summary, collecting (4.3.55), (4.3.56), (4.3.57), (4.3.58), (4.3.60) and (4.3.66), for fixed $0<\delta<1$, we have

$$|u(\vec{x},\vec{w})| \le \delta\left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{C(\Omega)}{\sqrt{\delta}}\left(\frac{1}{\epsilon^3}\left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{3/2}}\left\|g\right\|_{L^\infty(\Gamma^-)}\right). \tag{4.3.67}$$

Then we may take $0<\delta\le 1/2$ to obtain

$$|u(\vec{x},\vec{w})| \le \frac{1}{2}\left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{C(\Omega)}{\sqrt{\delta}}\left(\frac{1}{\epsilon^3}\left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{3/2}}\left\|g\right\|_{L^\infty(\Gamma^-)}\right). \tag{4.3.68}$$

Taking supremum of u over all $(\vec{x},\vec{w})$, we have

$$\left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \le \frac{1}{2}\left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{C(\Omega)}{\sqrt{\delta}}\left(\frac{1}{\epsilon^3}\left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{3/2}}\left\|g\right\|_{L^\infty(\Gamma^-)}\right). \tag{4.3.69}$$

Finally, absorbing $\|u\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}$, for fixed $0<\delta\leq 1/2$, we get

$$\|u\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}\leq C(\Omega)\bigg(\frac{1}{\epsilon^3}\|f\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{3/2}}\|g\|_{L^{\infty}(\Gamma^-)}\bigg). \tag{4.3.70}$$

□

4.3.4 Well-posedness of Transport Equation

Theorem 4.3.7. *Assume $g(x_0,\vec{w})\in L^{\infty}(\Gamma^-)$. Then for the steady neutron transport equation (4.1.1), there exists a unique solution $u^{\epsilon}(\vec{x},\vec{w})\in L^{\infty}(\Omega\times\mathcal{S}^1)$ satisfying*

$$\|u^{\epsilon}\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}\leq C(\Omega)\frac{1}{\epsilon^{3/2}}\|g\|_{L^{\infty}(\Gamma^-)}. \tag{4.3.71}$$

Proof. We can apply Theorem 4.3.6 to the equation (4.1.1). The result naturally follows. □

4.4 ϵ-Milne Problem

We consider the ϵ-Milne problem for $f^{\epsilon}(\eta,\theta,\phi)$ in the domain $(\eta,\theta,\phi)\in[0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)$

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f^{\epsilon}}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f^{\epsilon}}{\partial\phi}+f^{\epsilon}-\bar{f}^{\epsilon}&=&S^{\epsilon}(\eta,\theta,\phi),\\ f^{\epsilon}(0,\theta,\phi)&=&h^{\epsilon}(\theta,\phi)\ \text{ for }\ \sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f^{\epsilon}(\eta,\theta,\phi)&=&f^{\epsilon}_{\infty}(\theta),\end{array}\right. \tag{4.4.1}$$

where

$$\bar{f}^\epsilon(\eta,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^\epsilon(\eta,\theta,\phi)\mathrm{d}\phi, \tag{4.4.2}$$

$F(\epsilon;\eta)$ is defined as (4.2.45),

$$|h^\epsilon(\theta,\phi)| \le M, \tag{4.4.3}$$

and

$$|S^\epsilon(\eta,\theta,\phi)| \le Me^{-K\eta}, \tag{4.4.4}$$

for $M > 0$ and $K > 0$ uniform in ϵ and θ.

We may further define a potential function $V(\epsilon;\eta)$ satisfying $V(\epsilon;0) = 0$ and $F(\epsilon;\eta) = -\partial_\eta V(\epsilon,\eta)$. The following lemma illustrates the main properties of F and V.

Lemma 4.4.1. *$V(\epsilon;\eta)$ is monotonically increasing with respect to η and satisfies*

$$0 \le V(\epsilon;\eta) \le \ln 4. \tag{4.4.5}$$

Define

$$V_\infty(\epsilon) = \lim_{\eta\to\infty} V(\epsilon;\eta). \tag{4.4.6}$$

Then $V_\infty(\epsilon) = V_\infty$ independent of ϵ. For any $0 \le y \le z \le \infty$

$$-\ln 4 \le \int_y^z F(\epsilon;\eta)\mathrm{d}\eta \le 0. \tag{4.4.7}$$

For any $\sigma > 0$ and $\eta \geq 0$, we have

$$e^{V(\eta+\sigma)-V(\eta)} \leq 1 + 4\epsilon\sigma. \tag{4.4.8}$$

Moreover, we have

$$\int_0^\infty \int_\eta^\infty |F(y)|^2 \,\mathrm{d}y\mathrm{d}\eta \leq 3 - \ln 4. \tag{4.4.9}$$

$$\int_0^\infty |F(\epsilon;\eta)|^2 \,\mathrm{d}\eta \leq 3\epsilon. \tag{4.4.10}$$

$$\|F(\epsilon;\eta)\|_{L^\infty} \leq 4\epsilon. \tag{4.4.11}$$

Proof. Since $F(\epsilon;\eta) \leq 0$, by definition, we know $V(\epsilon;\eta)$ is monotonically increasing with respect to η. Since $V(\epsilon;0) = 0$, $V(\epsilon;\eta) \geq 0$. Then based on (4.2.23), we can direct estimate

$$\begin{aligned} V(\epsilon;\eta) &= V(\epsilon;0) - \int_0^\eta F(\epsilon;y)\mathrm{d}y = \int_0^\eta \frac{\epsilon\psi(\epsilon y)}{1-\epsilon y}\mathrm{d}y \\ &\leq \int_0^{\frac{3}{4\epsilon}} \frac{\epsilon}{1-\epsilon y}\mathrm{d}y = -\ln(1-\epsilon y)\Big|_{y=0}^{y=\frac{3}{4\epsilon}} = \ln 4. \end{aligned} \tag{4.4.12}$$

This verifies (4.4.5).

Also, letting $\mu = \epsilon\eta$, we have

$$V_\infty(\epsilon) = -\int_0^\infty \frac{\epsilon\psi(\epsilon\eta)}{1-\epsilon\eta}\mathrm{d}\eta = -\int_0^\infty \frac{\psi(\epsilon\eta)}{1-\epsilon\eta}\mathrm{d}(\epsilon\eta) = -\int_0^\infty \frac{\psi(\mu)}{1-\mu}\mathrm{d}\mu \leq \ln 4. \tag{4.4.13}$$

Hence, we have $V_\infty(\epsilon) = V_\infty$ independent of ϵ. Since

$$\int_y^z F(\epsilon;\eta)\mathrm{d}\eta = V(\epsilon;y) - V(\epsilon;z), \tag{4.4.14}$$

we can naturally obtain (4.4.7).

Moreover, we have

$$\begin{aligned} V(\eta+\sigma) - V(\eta) &= -\int_\eta^{\eta+\sigma} F(\epsilon;y)\mathrm{dy} = \int_\eta^{\eta+\sigma} \frac{\epsilon\psi(\epsilon y)}{1-\epsilon y}\mathrm{dy} & (4.4.15) \\ &\leq \int_\eta^{\min\{(\eta+\sigma),\frac{3}{4\epsilon}\}} \frac{\epsilon\psi(\epsilon y)}{1-\epsilon y}\mathrm{dy} \leq -\ln(1-\epsilon y)\Big|_{y=\eta}^{y=\min\{(\eta+\sigma),\frac{3}{4\epsilon}\}} & (4.4.16) \end{aligned}$$

If $\eta+\sigma \leq \frac{3}{4\epsilon}$, we have

$$e^{V(\eta+\sigma)-V(\eta)} \leq \frac{1-\epsilon\eta}{1-\epsilon(\eta+\sigma)} = 1 + \frac{\epsilon\sigma}{1-\epsilon(\eta+\sigma)} \leq 1 + 4\epsilon\sigma. \tag{4.4.17}$$

On the other hand, if $\eta+\sigma \geq \frac{3}{4\epsilon}$ and $\eta \leq \frac{3}{4\epsilon}$, we define $\sigma' = \frac{3}{4\epsilon} - \eta$ to obtain

$$e^{V(\eta+\sigma)-V(\eta)} \leq e^{V(\eta+\sigma')-V(\eta)} \leq 1 + 4\epsilon\sigma' \leq 1 + 4\epsilon\sigma. \tag{4.4.18}$$

Finally, if $\eta+\sigma \geq \frac{3}{4\epsilon}$ and $\eta \geq \frac{3}{4\epsilon}$, we have

$$e^{V(\eta+\sigma)-V(\eta)} = e^0 = 1. \tag{4.4.19}$$

Therefore, (4.4.8) follows.

Furthermore, considering the cut-off function (4.2.23), we have

$$
\begin{aligned}
&\int_0^{\infty}\int_{\eta}^{\infty}\left|F(y)\right|^2\mathrm{d}y\mathrm{d}\eta && (4.4.20)\\
\leq\ & \int_0^{\infty}\int_{\eta}^{\infty}\left|\frac{\epsilon\psi(\epsilon y)}{1-\epsilon y}\right|^2\mathrm{d}y\mathrm{d}\eta\leq\int_0^{\frac{3}{4\epsilon}}\int_{\eta}^{\frac{3}{4\epsilon}}\frac{\epsilon^2}{(1-\epsilon y)^2}\mathrm{d}y\mathrm{d}\eta\\
=\ & \int_0^{\frac{3}{4\epsilon}}\left(\frac{\epsilon}{1-\epsilon y}\bigg|_{y=\eta}^{y=\frac{3}{4\epsilon}}\right)\mathrm{d}\eta=\int_0^{\frac{3}{4\epsilon}}\left(4\epsilon-\frac{\epsilon}{1-\epsilon\eta}\right)\mathrm{d}\eta\\
=\ & \bigg(4\epsilon\eta+\ln(1-\epsilon\eta)\bigg)\bigg|_{\eta=0}^{\eta=\frac{3}{4\epsilon}}=3-\ln 4.
\end{aligned}
$$

Also, we can directly estimate

$$
\begin{aligned}
\int_0^{\infty}\left|F(\epsilon;\eta)\right|^2\mathrm{d}\eta &= \int_0^{\infty}\frac{\epsilon^2\psi^2(\epsilon\eta)}{(1-\epsilon\eta)^2}\mathrm{d}\eta\leq\int_0^{\frac{3}{4\epsilon}}\frac{\epsilon^2}{(1-\epsilon\eta)^2}\mathrm{d}\eta && (4.4.21)\\
&= \frac{\epsilon}{1-\epsilon\eta}\bigg|_{\eta=0}^{\eta=\frac{3}{4\epsilon}}=3\epsilon.
\end{aligned}
$$

Finally, based on the definition of F and ψ, we obtain

$$
|F(\epsilon;\eta)|=\left|\frac{\epsilon\psi(\epsilon\eta)}{1-\epsilon\eta}\right|\leq\frac{\epsilon}{1-\epsilon\frac{3}{4\epsilon}}=4\epsilon. \tag{4.4.22}
$$

$\square$

For notational simplicity, we omit ϵ and θ dependence in f^{ϵ} in this section. The same convention also applies to $F(\epsilon;\eta)$, $V(\epsilon;\eta)$, $S^{\epsilon}(\eta,\theta,\phi)$ and $h^{\epsilon}(\theta,\phi)$. However, our estimates are independent of ϵ and θ.

In this section, we introduce some special notation to describe the norms in the space $(\eta,\phi)\in[0,\infty)\times[-\pi,\pi)$. Define the L^2 norm as follows:

$$\|f(\eta)\|_{L^2} = \left(\int_{-\pi}^{\pi}|f(\eta,\phi)|^2\,\mathrm{d}\phi\right)^{1/2}, \tag{4.4.23}$$

$$\|f\|_{L^2L^2} = \left(\int_0^{\infty}\int_{-\pi}^{\pi}|f(\eta,\phi)|^2\,\mathrm{d}\phi\mathrm{d}\eta\right)^{1/2}. \tag{4.4.24}$$

Define the inner product in ϕ space

$$\langle f,g\rangle_\phi(\eta) = \int_{-\pi}^{\pi} f(\eta,\phi)g(\eta,\phi)\mathrm{d}\phi. \tag{4.4.25}$$

Define the L^∞ norm as follows:

$$\|f(\eta)\|_{L^\infty} = \sup_{\phi\in[-\pi,\pi)}|f(\eta,\phi)|, \tag{4.4.26}$$

$$\|f\|_{L^\infty L^\infty} = \sup_{(\eta,\phi)\in[0,\infty)\times[-\pi,\pi)}|f(\eta,\phi)|, \tag{4.4.27}$$

$$\|f\|_{L^\infty L^2} = \sup_{\eta\in[0,\infty)}\left(\int_{-\pi}^{\pi}|f(\eta,\phi)|^2\,\mathrm{d}\phi\right)^{1/2}. \tag{4.4.28}$$

Since the boundary data $h(\phi)$ is only defined on $\sin\phi>0$, we naturally extend above definitions on this half-domain as follows:

$$\|h\|_{L^2} = \left(\int_{\sin\phi>0}|h(\phi)|^2\,\mathrm{d}\phi\right)^{1/2}, \tag{4.4.29}$$

$$\|h\|_{L^\infty} = \sup_{\sin\phi>0}|h(\phi)|. \tag{4.4.30}$$

Lemma 4.4.2. *We have*

$$\|h\|_{L^2} \leq C\|h\|_{L^\infty} \leq CM \tag{4.4.31}$$

$$\|S\|_{L^2L^2} \leq C\frac{M}{K} \tag{4.4.32}$$

$$\|S\|_{L^\infty L^2} \leq C\|S\|_{L^\infty L^\infty} \leq CM \tag{4.4.33}$$

Proof. They can be verified via direct computation, so we omit the proofs here. $\square$

4.4.1 L^2 Estimates

Finite Slab with $\bar{S} = 0$

Consider the ϵ-Milne problem for $f^L(\eta, \phi)$ in a finite slab $(\eta, \phi) \in [0, L] \times [-\pi, \pi)$

$$\left\{ \begin{array}{rcl} \sin\phi \dfrac{\partial f^L}{\partial \eta} + F(\eta) \cos\phi \dfrac{\partial f^L}{\partial \phi} + f^L - \bar{f}^L &=& S(\eta, \phi), \\ f^L(0, \phi) &=& h(\phi) \ \text{ for } \ \sin\phi < 0, \\ f^L(L, \phi) &=& f^L(L, R\phi), \end{array} \right. \tag{4.4.34}$$

where $R\phi = -\phi$ and S satisfies $\bar{S}(\eta) = 0$ for any η. We may decompose the solution

$$f^L(\eta, \phi) = q_f^L(\eta) + r_f^L(\eta, \phi), \tag{4.4.35}$$

where the hydrodynamical part q_f^L is in the null space of the operator $f^L - \bar{f}^L$, and the microscopic part r_f^L is the orthogonal complement, i.e.

$$q_f^L(\eta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^L(\eta, \phi) \mathrm{d}\phi \quad r_f^L(\eta, \phi) = f^L(\eta, \phi) - q_f^L(\eta). \tag{4.4.36}$$

In the following, when there is no confusion, we simply write $f^L = q^L + r^L$.

Lemma 4.4.3. *Assume $\bar{S}(\eta) = 0$ for any $\eta \in [0, L]$ with (4.4.3) and (4.4.4). Then* *there exists a solution $f(\eta,\phi)$ to the finite slab problem (4.4.34) satisfying*

$$\begin{aligned}\int_0^L \left\|r^L(\eta)\right\|_{L^2}^2 \mathrm{d}\eta &\leq C\left(M+\frac{M}{K}\right)^2 < \infty, &(4.4.37)\\ \left\|q^L(\eta)\right\|_{L^2}^2 &\leq C\left(1+M+\frac{M}{K}\right)^2\left(1+\eta^{1/2}+\left\|r^L(\eta)\right\|_{L^2}\right), &(4.4.38)\\ \langle\sin\phi, r^L\rangle_\phi(\eta) &= 0, &(4.4.39)\end{aligned}$$

for arbitrary $\eta\in[0,L]$.

Proof. We divide the proof into several steps:

Step 1: Assume $\|H\|_{L^\infty L^\infty}<\infty$ and $\|h\|_{L^\infty}<\infty$, then the solution $f_\lambda(\eta,\phi)$ to the penalized ϵ-transport equation

$$\left\{\begin{array}{rcl}\lambda f_\lambda+\sin\phi\dfrac{\partial f_\lambda}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f_\lambda}{\partial\phi}+f_\lambda &=& H(\eta,\phi),\\ f_\lambda(0,\phi) &=& h(\phi)\ \text{ for }\ \sin\phi<0,\\ f_\lambda(L,\phi) &=& f_\lambda(L,R\phi).\end{array}\right. \tag{4.4.40}$$

satisfies

$$\|f_\lambda\|_{L^\infty L^\infty}\leq\|h\|_{L^\infty}+\|H\|_{L^\infty L^\infty}. \tag{4.4.41}$$

The proof of (4.4.41): To construct the solution of (4.4.40), we define the energy as

$$E(\eta,\phi)=\cos\phi e^{-V(\eta)}. \tag{4.4.42}$$

This curve with constant energy is the characteristics of the equation (4.4.40). Hence, on this curve the equation can be simplified as follows:

$$\lambda f_\lambda + \sin\phi \frac{\mathrm{d}f_\lambda}{\mathrm{d}\eta} + f_\lambda = H. \tag{4.4.43}$$

An implicit function $\eta^+(\eta, \phi)$ can be determined through

$$|E(\eta, \phi)| = e^{-V(\eta^+)}. \tag{4.4.44}$$

which means (η^+, ϕ_0) with $\sin\phi_0 = 0$ is on the same characteristics as (η, ϕ). Define the quantities for $0 \leq \eta' \leq \eta^+$ as follows:

$$\phi'(\phi, \eta, \eta') = \cos^{-1}(\cos\phi e^{V(\eta')-V(\eta)}), \tag{4.4.45}$$

$$R\phi'(\phi, \eta, \eta') = -\cos^{-1}(\cos\phi e^{V(\eta')-V(\eta)}) = -\phi'(\phi, \eta, \eta'), \tag{4.4.46}$$

where the inverse trigonometric function can be defined single-valued in the domain $[0, \pi)$ and the quantities are always well-defined due to the monotonicity of V. Finally we put

$$G^\lambda_{\eta,\eta'}(\phi) = \int_{\eta'}^{\eta} \frac{1+\lambda}{\sin(\phi'(\phi, \eta, \xi))} \mathrm{d}\xi. \tag{4.4.47}$$

We can define the solution to (4.4.40) along the characteristics as follows:

Case I:

For $\sin\phi > 0$,

$$f_\lambda(\eta, \phi) = h(\phi'(\phi, \eta, 0)) \exp(-G^\lambda_{\eta,0}) + \int_0^\eta \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G^\lambda_{\eta,\eta'}) \mathrm{d}\eta'. \tag{4.4.48}$$

Case II:

For $\sin\phi < 0$ and $|E(\eta,\phi)| \leq e^{-V(L)}$,

$$\begin{aligned} f_\lambda(\eta,\phi) &= h(\phi'(\phi,\eta,0))\exp(-G^\lambda_{L,0}-G^\lambda_{L,\eta}) \\ &+ \bigg(\int_0^L \frac{H(\eta',\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G^\lambda_{L,\eta'}-G^\lambda_{L,\eta})\mathrm{d}\eta' \\ &\quad + \int_\eta^L \frac{H(\eta',R\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G^\lambda_{\eta,\eta'})\mathrm{d}\eta'\bigg). \end{aligned} \tag{4.4.49}$$

Case III:

For $\sin\phi < 0$ and $|E(\eta,\phi)| \geq e^{-V(L)}$,

$$\begin{aligned} f_\lambda(\eta,\phi) &= h(\phi'(\phi,\eta,0))\exp(-G^\lambda_{\eta^+,0}-G^\lambda_{\eta^+,\eta}) \\ &+ \bigg(\int_0^{\eta^+} \frac{H(\eta',\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G^\lambda_{\eta^+,\eta'}-G^\lambda_{\eta^+,\eta})\mathrm{d}\eta' \\ &\quad + \int_\eta^{\eta^+} \frac{H(\eta',R\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G^\lambda_{\eta,\eta'})\mathrm{d}\eta'\bigg). \end{aligned} \tag{4.4.50}$$

Note the fact

$$\frac{\mathrm{d}}{\mathrm{d}\eta'}G^\lambda_{\eta,\eta'}(\phi) = -\frac{1+\lambda}{\sin(\phi'(\phi,\eta,\eta'))}. \tag{4.4.51}$$

Hence, we can directly estimate as follows:

In Case I:

$$\begin{aligned} &|f_\lambda(\eta,\phi)| \qquad (4.4.52) \\ \leq\ & \exp(-G^\lambda_{\eta,0})\left\|h\right\|_{L^\infty} + \left\|H\right\|_{L^\infty L^\infty}\int_0^\eta \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G^\lambda_{\eta,\eta'})\mathrm{d}\eta' \\ =\ & \exp(-G^\lambda_{\eta,0})\left\|h\right\|_{L^\infty} + \frac{1}{1+\lambda}\left\|H\right\|_{L^\infty L^\infty}\exp(-G^\lambda_{\eta,\eta'})\bigg|_0^\eta \\ =\ & \exp(-G^\lambda_{\eta,0})\left\|h\right\|_{L^\infty} + \frac{1}{1+\lambda}\bigg(1-\exp(-G^\lambda_{\eta,0})\bigg)\left\|H\right\|_{L^\infty L^\infty} \leq \left\|h\right\|_{L^\infty} + \left\|H\right\|_{L^\infty L^\infty}. \end{aligned}$$

In Case II:

$$\begin{aligned}
|f_\lambda(\eta,\phi)| &\leq \exp(-G_{L,0}^\lambda - G_{L,\eta}^\lambda)\,\|h\|_{L^\infty} \\
&\quad+\frac{1}{1+\lambda}\bigg\{\exp(-G_{L,\eta}^\lambda)\Big(1-\exp(-G_{L,0}^\lambda)\Big)\|H\|_{L^\infty L^\infty} \\
&\quad+\Big(1-\exp(-G_{L,\eta}^\lambda)\Big)\|H\|_{L^\infty L^\infty}\bigg\} \\
&\leq \|h\|_{L^\infty}+\|H\|_{L^\infty L^\infty}.
\end{aligned} \tag{4.4.53}$$

In Case III:

$$\begin{aligned}
|f_\lambda(\eta,\phi)| &\leq \exp(-G_{\eta^+,0}^\lambda - G_{\eta^+,\eta}^\lambda)\,\|h\|_{L^\infty} \\
&\quad+\frac{1}{1+\lambda}\bigg\{\exp(-G_{\eta^+,\eta}^\lambda)\Big(1-\exp(-G_{\eta^+,0}^\lambda)\Big)\|H\|_{L^\infty L^\infty} \\
&\quad+\Big(1-\exp(-G_{\eta^+,\eta}^\lambda)\Big)\|H\|_{L^\infty L^\infty}\bigg\} \\
&\leq \|h\|_{L^\infty}+\|H\|_{L^\infty L^\infty}.
\end{aligned}$$

This completes the proof of (4.4.41).

Step 2: Assume $\|S\|_{L^\infty L^\infty} < \infty$ and $\|h\|_{L^\infty} < \infty$, then the solution $f_\lambda^L(\eta,\phi)$ to the penalized ϵ-Milne equation

$$\left\{\begin{array}{rcl}
\lambda f_\lambda^L+\sin\phi\dfrac{\partial f_\lambda^L}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f_\lambda^L}{\partial\phi}+f_\lambda^L-\bar{f}_\lambda^L &=& S(\eta,\phi),\\
f_\lambda^L(0,\phi) &=& h(\phi)\ \ \text{for}\ \ \sin\phi<0,\\
f_\lambda^L(L,\phi) &=& f_\lambda^L(L,R\phi).
\end{array}\right. \tag{4.4.54}$$

where $\bar{f}_\lambda^L$ is defined as (4.4.2), satisfies

$$\left\|f_\lambda^L\right\|_{L^\infty L^\infty}\leq\frac{1+\lambda}{\lambda}\bigg(\|h\|_{L^\infty}+\|S\|_{L^\infty L^\infty}\bigg). \tag{4.4.55}$$

The proof of (4.4.55): In order to construct the solution of (4.4.54), we iteratively define the sequence $\{f_m^L\}_{m=0}^{\infty}$ as $f_0^L = 0$ and

$$\left\{\begin{array}{rcl} \lambda f_m^L + \sin\phi\dfrac{\partial f_m^L}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial f_m^L}{\partial\phi} + f_m^L - \bar{f}_{m-1}^L &=& S(\eta,\phi),\\ f_m^L(0,\phi) &=& h(\phi)\ \text{ for }\ \sin\phi<0,\\ f_m^L(L,\phi) &=& f_m^L(L,R\phi). \end{array}\right. \tag{4.4.56}$$

Based on the analysis in Step 1 with $H = S + \bar{f}_{m-1}^L$, we know f_m^L is well-defined and $\left\|f_m^L\right\|_{L^\infty L^\infty} < \infty$. We further define $g_m^L = f_m^L - f_{m-1}^L$ for $m \geq 1$. Then g_m^L can be rewritten along the characteristics as follows: Case I:

For $\sin\phi > 0$,

$$g_{m+1}^L(\eta,\phi) = \int_0^\eta \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{4.4.57}$$

Case II:

For $\sin\phi < 0$ and $|E(\eta,\phi)| \leq e^{-V(L)}$,

$$\begin{aligned} g_{m+1}^L(\eta,\phi) = &\bigg(\int_0^L \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{L,\eta'} - G_{L,\eta})\mathrm{d}\eta' \\ &+ \int_\eta^L \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta'\bigg). \end{aligned} \tag{4.4.58}$$

Case III:

For $\sin\phi < 0$ and $|E(\eta,\phi)| \geq e^{-V(L)}$,

$$\begin{aligned} g_{m+1}^L(\eta,\phi) = &\bigg(\int_0^{\eta^+} \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta^+,\eta'} - G_{\eta^+,\eta})\mathrm{d}\eta' \\ &+ \int_\eta^{\eta^+} \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta'\bigg). \end{aligned} \tag{4.4.59}$$

In all three cases, we can always obtain

$$\left\|g_{m+1}^L\right\|_{L^\infty L^\infty} \leq \frac{1}{1+\lambda}\left\|g_m^L\right\|_{L^\infty L^\infty}. \tag{4.4.60}$$

For $\lambda > 0$, this is a contraction sequence. Also, we have

$$\left\|g_1^L\right\|_{L^\infty L^\infty} = \left\|f_1^L\right\|_{L^\infty L^\infty} \leq \|h\|_{L^\infty} + \|H\|_{L^\infty L^\infty}. \tag{4.4.61}$$

Hence, f_m^L converges strongly in $L^\infty([0,L]\times[-\pi,\pi))$ to f_λ^L which satisfies (4.4.54). Also, f_λ^L satisfies

$$\left\|f_\lambda^L\right\|_{L^\infty L^\infty} \leq \frac{1+\lambda}{\lambda}\bigg(\|h\|_{L^\infty} + \|S\|_{L^\infty L^\infty}\bigg). \tag{4.4.62}$$

Naturally, we obtain $f_\lambda^L \in L^2([0,L]\times[-\pi,\pi))$.

However, we can see the estimate blows up when $\lambda \to 0$. Therefore, we need to show the uniform estimate of f_λ^L with respect to λ. Similarly, we can define r_λ^L and q_λ^L for f_λ^L as in (4.4.36).

Step 3: r_λ^L satisfies

$$\int_0^L \left\|r_\lambda^L(\eta)\right\|_{L^2}^2 \mathrm{d}\eta \;\;\leq\;\; 4\|h\|_{L^2}^2 + 8\int_0^L \|S(\eta)\|_{L^2}^2 \mathrm{d}\eta. \tag{4.4.63}$$

The proof of (4.4.63): Multiplying f_λ^L on both sides of (4.4.54) and integrating over $\phi\in[-\pi,\pi)$, we get the energy estimate

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f_\lambda^L, f_\lambda^L\sin\phi\rangle_\phi(\eta) \qquad (4.4.64)\\ =\;\;& -\lambda\left\|f_\lambda^L(\eta)\right\|_{L^2}^2 - \left\|r_\lambda^L(\eta)\right\|_{L^2}^2 - F(\eta)\langle\frac{\partial f_\lambda^L}{\partial\phi}, f_\lambda^L\cos\phi\rangle_\phi(\eta) + \langle S, f_\lambda^L\rangle_\phi(\eta). \end{aligned}$$

A further integration by parts reveals

$$-F(\eta)\langle\frac{\partial f_\lambda^L}{\partial\phi}, f_\lambda^L\cos\phi\rangle_\phi(\eta) \quad = \quad -\frac{1}{2}F(\eta)\langle f_\lambda^L, f_\lambda^L\sin\phi\rangle_\phi(\eta). \tag{4.4.65}$$

Also, the assumption $\bar{S}(\eta) = 0$ leads to

$$\langle S, f_\lambda^L\rangle_\phi(\eta) \quad = \quad \langle S, q_\lambda^L\rangle_\phi(\eta) + \langle S, r_\lambda^L\rangle_\phi(\eta) = \langle S, r_\lambda^L\rangle_\phi(\eta). \tag{4.4.66}$$

Hence, we have the simplified form of (4.4.64) as follows:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f_\lambda^L, f_\lambda^L\sin\phi\rangle_\phi(\eta) \qquad (4.4.67)\\ = \quad & -\lambda\left\|f_\lambda^L(\eta)\right\|_{L^2}^2 - \left\|r_\lambda^L(\eta)\right\|_{L^2}^2 - \frac{1}{2}F(\eta)\langle f_\lambda^L, f_\lambda^L\sin\phi\rangle_\phi(\eta) + \langle S, r_\lambda^L\rangle_\phi(\eta). \end{aligned}$$

Define

$$\alpha(\eta) = \frac{1}{2}\langle f_\lambda^L, f_\lambda^L\sin\phi\rangle_\phi(\eta). \tag{4.4.68}$$

Then (4.4.67) can be rewritten as follows:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\eta} = -\lambda\left\|f_\lambda^L(\eta)\right\|_{L^2}^2 - \left\|r_\lambda^L(\eta)\right\|_{L^2}^2 - F(\eta)\alpha(\eta) + \langle S, r_\lambda^L\rangle_\phi(\eta). \tag{4.4.69}$$

We can integrate above on $[\eta, L]$ and $[0, \eta]$ respectively to obtain

$$\begin{aligned} \alpha(\eta) \quad = \quad & \alpha(L)\exp\left(\int_\eta^L F(y)\mathrm{d}(y)\right) \qquad (4.4.70)\\ & + \int_\eta^L \exp\left(\int_\eta^y F(z)\mathrm{d}z\right)\left(\lambda\left\|f_\lambda^L(y)\right\|_{L^2}^2 + \left\|r_\lambda^L(y)\right\|_{L^2}^2 - \langle S, r_\lambda^L\rangle_\phi(y)\right)\mathrm{d}y,\\ \alpha(\eta) \quad = \quad & \alpha(0)\exp\left(-\int_0^\eta F(y)\mathrm{d}(y)\right) \qquad (4.4.71)\\ & + \int_0^\eta \exp\left(-\int_y^\eta F(z)\mathrm{d}z\right)\left(-\lambda\left\|f_\lambda^L(y)\right\|_{L^2}^2 - \left\|r_\lambda^L(y)\right\|_{L^2}^2 + \langle S, r_\lambda^L\rangle_\phi(y)\right)\mathrm{d}y. \end{aligned}$$

The specular reflexive boundary $f_\lambda^L(L,\phi) = f_\lambda^L(L,R\phi)$ ensures $\alpha(L) = 0$. Hence, based on (4.4.70), we have

$$\alpha(\eta) \geq \int_\eta^L \exp\bigg(\int_\eta^y F(z)\mathrm{dz}\bigg)\bigg(-\langle S, r_\lambda^L\rangle_\phi(y)\bigg)\mathrm{dy}. \tag{4.4.72}$$

Also, (4.4.71) implies

$$\begin{aligned}\alpha(\eta) &\leq \alpha(0)\exp[V(\eta)] + \int_0^\eta \exp\bigg(-\int_y^\eta F(z)\mathrm{dz}\bigg)\bigg(\langle S, r_\lambda^L\rangle_\phi(y)\bigg)\mathrm{dy} \quad (4.4.73)\\ &\leq 2\left\|h\right\|_{L^2}^2 + \int_0^\eta \exp\bigg(-\int_y^\eta F(z)\mathrm{dz}\bigg)\bigg(\langle S, r_\lambda^L\rangle_\phi(y)\bigg)\mathrm{dy},\end{aligned}$$

due to the fact

$$\alpha(0) = \frac{1}{2}\langle \sin\phi f_\lambda^L, f_\lambda^L\rangle_\phi(0) \leq \frac{1}{2}\bigg(\int_{\sin\phi>0} h(\phi)^2\sin\phi\mathrm{d}\phi\bigg) \leq \frac{1}{2}\left\|h\right\|_{L^2}^2. \tag{4.4.74}$$

Then in (4.4.71) taking $\eta = L$, from $\alpha(L) = 0$, we have

$$\begin{aligned}&\int_0^L \exp\bigg(\int_0^y F(z)\mathrm{dz}\bigg)\left\|r_\lambda^L(y)\right\|_{L^2}^2\mathrm{dy} \quad (4.4.75)\\ \leq\;& \alpha(0) + \int_0^L \exp\bigg(\int_0^y F(z)\mathrm{dz}\bigg)\langle S, r_\lambda^L\rangle_\phi(y)\mathrm{dy}\\ \leq\;& \frac{1}{2}\left\|h\right\|_{L^2}^2 + \int_0^L \exp\bigg(\int_0^y F(z)\mathrm{dz}\bigg)\langle S, r_\lambda^L\rangle_\phi(y)\mathrm{dy}.\end{aligned}$$

On the other hand, by (4.4.7), we can directly estimate as follows:

$$\int_0^L \exp\bigg(\int_0^y F(z)\mathrm{dz}\bigg)\left\|r_\lambda^L(y)\right\|_{L^2}^2\mathrm{dy} \geq \frac{1}{4}\int_0^L \left\|r_\lambda^L(y)\right\|_{L^2}^2\mathrm{dy}. \tag{4.4.76}$$

Combining (4.4.75) and (4.4.76) yields

$$\int_0^L \left\|r_\lambda^L(\eta)\right\|_{L^2}^2\mathrm{d}\eta \leq 2\left\|h\right\|_{L^2}^2 + 4\int_0^L \exp\bigg(\int_0^y F(z)\mathrm{dz}\bigg)\langle S, r_\lambda^L\rangle_\phi(y)\mathrm{dy}. \tag{4.4.77}$$

By Cauchy's inequality and (4.4.7), we have

$$\left|\int_0^L \exp\left(\int_0^y F(z)\mathrm{dz}\right)\langle S, r_\lambda^L\rangle_\phi(y)\mathrm{dy}\right| \leq \left|\int_0^L \langle S, r_\lambda^L\rangle_\phi(y)\mathrm{dy}\right| \tag{4.4.78}$$
$$\leq \frac{1}{8}\int_0^L \left\|r_\lambda^L(\eta)\right\|_{L^2}^2 \mathrm{d}\eta + 2\int_0^L \|S(\eta)\|_{L^2}^2 \mathrm{d}\eta.$$

Therefore, summarizing (4.4.77) and (4.4.78), we deduce (4.4.63).

Step 4: q_λ^L satisfies

$$\left\|q_\lambda^L(\eta)\right\|_{L^2}^2 \leq 256\pi^2(1+\lambda)\left(1+\eta^{1/2}+\left\|r_\lambda^L(\eta)\right\|_{L^2}\right) \tag{4.4.79}$$
$$\times\left(1+\|h\|_{L^2}^2+\int_0^L \|S(\eta)\|_{L^2}^2 \mathrm{d}\eta+\int_0^\eta \|S(y)\|_{L^\infty} \mathrm{dy}\right).$$

The proof of (4.4.79): Multiplying $\sin\phi$ on both sides of (4.4.54) and integrating over $\phi \in [-\pi,\pi)$ lead to

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\langle \sin^2\phi, f_\lambda^L\rangle_\phi(\eta) = -\lambda\langle\sin\phi, f_\lambda^L\rangle_\phi(\eta) - \langle\sin\phi, r_\lambda^L\rangle_\phi(\eta) \tag{4.4.80}$$
$$-F(\eta)\langle\sin\phi\cos\phi, \frac{\partial f_\lambda^L}{\partial\phi}\rangle_\phi(\eta) + \langle\sin\phi, S\rangle_\phi(\eta).$$

We can further integrate by parts as follows:

$$-F(\eta)\langle\sin\phi\cos\phi, \frac{\partial f_\lambda^L}{\partial\phi}\rangle_\phi(\eta) = F(\eta)\langle\cos(2\phi), f_\lambda^L\rangle_\phi(\eta) = F(\eta)\langle\cos(2\phi), r_\lambda^L\rangle_\phi(\eta), \tag{4.4.81}$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\langle \sin^2\phi, f_\lambda^L\rangle_\phi(\eta) = -\lambda\langle\sin\phi, f_\lambda^L\rangle_\phi(\eta) - \langle\sin\phi, r_\lambda^L\rangle_\phi(\eta) \tag{4.4.82}$$
$$+F(\eta)\langle\cos(2\phi), r_\lambda^L\rangle_\phi(\eta) + \langle\sin\phi, S\rangle_\phi(\eta).$$

Define

$$\beta^L_\lambda(\eta) = \langle \sin^2\phi, f^L_\lambda\rangle_\phi(\eta). \tag{4.4.83}$$

Then we can simplify (4.4.80) as follows:

$$\frac{\mathrm{d}\beta^{\mathrm{L}}_\lambda}{\mathrm{d}\eta} = D^L_\lambda(\eta,\phi), \tag{4.4.84}$$

where

$$\begin{aligned} & D^L_\lambda(\eta,\phi) && (4.4.85)\\ = \;& -\lambda\langle \sin\phi, f^L_\lambda\rangle_\phi(\eta) - \langle \sin\phi, r^L_\lambda\rangle_\phi(\eta) + F(\eta)\langle \cos(2\phi), r^L_\lambda\rangle_\phi(\eta) + \langle \sin\phi, S\rangle_\phi(\eta). \end{aligned}$$

Since

$$-\lambda\langle \sin\phi, f^L_\lambda\rangle_\phi(\eta) = -\lambda\langle \sin\phi, r^L_\lambda\rangle_\phi(\eta) - \lambda\langle \sin\phi, q^L_\lambda\rangle_\phi(\eta) = -\lambda\langle \sin\phi, r^L_\lambda\rangle_\phi(\eta). \tag{4.4.86}$$

we can further get

$$\begin{aligned} & D^L_\lambda(\eta,\phi) && (4.4.87)\\ = \;& -\lambda\langle \sin\phi, r^L_\lambda\rangle_\phi(\eta) - \langle \sin\phi, r^L_\lambda\rangle_\phi(\eta) + F(\eta)\langle \cos(2\phi), r^L_\lambda\rangle_\phi(\eta) + \langle \sin\phi, S\rangle_\phi(\eta). \end{aligned}$$

We can integrate over $[0,\eta]$ in (4.4.84) to obtain

$$\beta^L_\lambda(\eta) = \beta^L_\lambda(0) + \int_0^\eta D^L_\lambda(y)\mathrm{d}y. \tag{4.4.88}$$

It is important to note that D^L_λ only depends on r^L_λ and is independent of q^L_λ. Then we can directly estimate

$$\begin{aligned}
& \left\|D_\lambda^L(\eta)\right\|_{L^\infty} \\
\leq\ & 2\pi\bigg(1+\lambda+|F(\eta)|\bigg)\left\|r_\lambda^L(\eta)\right\|_{L^2}+\|S(\eta)\|_{L^\infty}\leq 4\pi(1+\lambda)\left\|r_\lambda^L(\eta)\right\|_{L^2}+\|S(\eta)\|_{L^\infty}.
\end{aligned} \tag{4.4.89}$$

Also, for the initial data

$$\begin{aligned}
\beta_\lambda^L(0) &= \langle\sin^2\phi, f_\lambda^L\rangle_\phi(0) \\
&\leq \bigg(\langle f_\lambda^L, f_\lambda^L\,|\sin\phi|\rangle_\phi(0)\bigg)^{1/2}\|\sin\phi\|_{L^2}^{3/2}\leq 8\bigg(\langle f_\lambda^L, f_\lambda^L\,|\sin\phi|\rangle_\phi(0)\bigg)^{1/2}.
\end{aligned} \tag{4.4.90}$$

Obviously, we have

$$\langle f_\lambda^L, f_\lambda^L\,|\sin\phi|\rangle_\phi(0)=\int_{\sin\phi>0}h^2(\phi)\sin\phi\mathrm{d}\phi-\int_{\sin\phi<0}\bigg(f_\lambda^L(0,\phi)\bigg)^2\sin\phi\mathrm{d}\phi. \tag{4.4.91}$$

However, based on the definition of $\alpha(\eta)$ and (4.4.72), we can obtain

$$\begin{aligned}
& \int_{\sin\phi>0}h^2(\phi)\sin\phi\mathrm{d}\phi+\int_{\sin\phi<0}\bigg(f_\lambda^L(0,\phi)\bigg)^2\sin\phi\mathrm{d}\phi \\
=\ & 2\alpha(0) \\
\geq\ & 2\int_0^L\exp\bigg(\int_0^y F(z)\mathrm{dz}\bigg)\bigg(-\langle S, r_\lambda^L\rangle_\phi(y)\bigg)\mathrm{dy} \\
\geq\ & -\frac{1}{2}\int_0^L\langle S, r_\lambda^L\rangle_\phi(y)\mathrm{dy}.
\end{aligned} \tag{4.4.92}$$

Hence, we can deduce

$$\begin{aligned}
& -\int_{\sin\phi<0}\bigg(f_\lambda^L(0,\phi)\bigg)^2\sin\phi\mathrm{d}\phi \\
\leq\ & \int_{\sin\phi>0}h^2(\phi)\sin\phi\mathrm{d}\phi+\frac{1}{2}\int_0^L\langle S, r_\lambda^L\rangle_\phi(y)\mathrm{dy} \\
\leq\ & \|h\|_{L^2}^2+\frac{1}{4}\int_0^L\left\|r_\lambda^L(\eta)\right\|_{L^2}^2\mathrm{d}\eta+\frac{1}{4}\int_0^L\|S(\eta)\|_{L^2}^2\mathrm{d}\eta.
\end{aligned} \tag{4.4.93}$$

From (4.4.63), we can deduce

$$
\begin{aligned}
& \beta_\lambda^L(0)^2 && (4.4.94)\\
\leq\ & 64\langle f_\lambda^L, f_\lambda^L\,|\sin\phi|\rangle_\phi(0) \leq 128\,\|h\|_{L^2}^2 + 16\int_0^L \left\|r_\lambda^L(\eta)\right\|_{L^2}^2 \mathrm{d}\eta + 16\int_0^L \|S(\eta)\|_{L^2}^2 \mathrm{d}\eta\\
\leq\ & 192\,\|h\|_{L^2}^2 + 192\int_0^L \|S(\eta)\|_{L^2}^2 \mathrm{d}\eta.
\end{aligned}
$$

From (4.4.63), (4.4.88), (4.4.89) and (4.4.94), we have

$$
\begin{aligned}
& \left\|\beta_\lambda^L(\eta)\right\|_{L^\infty} && (4.4.95)\\
\leq\ & 8 + 192\,\|h\|_{L^2}^2 + 192\int_0^L \|S(\eta)\|_{L^2}^2 \mathrm{d}\eta\\
& +2\pi(1+\lambda)\int_0^\eta \left\|r_\lambda^L(y)\right\|_{L^2} \mathrm{dy} + \int_0^\eta \|S(y)\|_{L^\infty} \mathrm{dy}\\
\leq\ & 8 + 192\,\|h\|_{L^2}^2 + 192\int_0^L \|S(\eta)\|_{L^2}^2 \mathrm{d}\eta + \int_0^\eta \|S(y)\|_{L^\infty} \mathrm{dy}\\
& +2\pi(1+\lambda)\eta^{1/2}\bigg(\int_0^\eta \left\|r_\lambda^L(y)\right\|_{L^2}^2 \mathrm{dy}\bigg)^{1/2}\\
\leq\ & 64\pi(1+\lambda)(1+\eta^{1/2})\bigg(\|h\|_{L^2}^2 + \int_0^L \|S(\eta)\|_{L^2}^2 \mathrm{d}\eta + \int_0^\eta \|S(y)\|_{L^\infty} \mathrm{dy}\bigg).
\end{aligned}
$$

By (4.4.83) this implies

$$
\left\|q_\lambda^L(\eta)\right\|_{L^2}^2 \ \leq\ 2\left\|\beta_\lambda^L(\eta)\right\|_{L^2} + 2\left\|r_\lambda^L(\eta)\right\|_{L^2} \leq 4\pi\left\|\beta_\lambda^L(\eta)\right\|_{L^\infty} + 2\left\|r_\lambda^L(\eta)\right\|_{L^2}, \tag{4.4.96}
$$

which completes the proof of (4.4.79).

Step 5: Passing to the limit.

Since estimates (4.4.63) and (4.4.79) are uniform in λ, we can take weakly convergent subsequence $f_\lambda^L \to f^L \in L^2([0,L]\times[-\pi,\pi))$ as $\lambda \to 0$. Hence, f^L is the solution of (4.4.34) and satisfies the estimates (4.4.37) and (4.4.38).

Step 6: Orthogonality relation (4.4.39).

A direct integration over $\phi\in[-\pi,\pi)$ in (4.4.34) implies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\langle\sin\phi, f^L\rangle_{\phi}(\eta)=-F\langle\cos\phi,\frac{\mathrm{d}f^{\mathrm{L}}}{\mathrm{d}\phi}\rangle_{\phi}(\eta)+\bar{S}(\eta)=-F\langle\sin\phi, f^L\rangle_{\phi}(\eta). \tag{4.4.97}$$

thanks to $\bar{S}=0$. The specular reflexive boundary $f^L(L,\phi)=f^L(L,R\phi)$ implies $\langle\sin\phi, f^L\rangle_{\phi}(L)=0$. Then we have

$$\langle\sin\phi, f^L\rangle_{\phi}(\eta)=0. \tag{4.4.98}$$

It is easy to see

$$\langle\sin\phi, q^L\rangle_{\phi}(\eta)=0. \tag{4.4.99}$$

Hence, we may derive

$$\langle\sin\phi, r^L\rangle_{\phi}(\eta)=0. \tag{4.4.100}$$

This leads (4.4.39) and completes the proof of (4.4.3). $\square$

Infinite Slab with $\bar{S}=0$

We turn to the ϵ-Milne problem for $f(\eta,\phi)$ in the infinite slab $(\eta,\phi)\in[0,\infty)\times[-\pi,\pi)$

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi}+f-\bar{f}&=&S(\eta,\phi),\\ f(0,\phi)&=&h(\phi)\ \ \text{for}\ \ \sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f(\eta,\phi)&=&f_{\infty}.\end{array}\right. \tag{4.4.101}$$

Define r and q for f as r^L and q^L for f^L.

Lemma 4.4.4. *Assume $\bar{S}(\eta) = 0$ for any $\eta \in [0, \infty)$ with (4.4.3) and (4.4.4). Then there exists a solution $f(\eta, \phi)$ of the infinite slab problem (4.4.101), satisfying*

$$\begin{aligned} \|r\|_{L^2L^2} &\le C\left(M + \frac{M}{K}\right)^2 < \infty, && (4.4.102) \\ \langle \sin\phi, r\rangle_\phi(\eta) &= 0, && (4.4.103) \end{aligned}$$

for any $\eta \in [0, \infty)$. Also there exists a constant $q_\infty = f_\infty \in \mathbb{R}$ such that the following estimates hold,

$$\begin{aligned} |q_\infty| &\le C\left(1 + M + \frac{M}{K}\right)^2 < \infty, && (4.4.104) \\ & && (4.4.105) \\ \|q(\eta) - q_\infty\|_{L^2} &\le C\left(\|r(\eta)\|_{L^2} + \int_\eta^\infty |F(y)|\,\|r(y)\|_{L^2}\,\mathrm{dy} + \int_\eta^\infty \|S(y)\|_{L^\infty}\,\mathrm{dy}\right), \\ \|q - q_\infty\|_{L^2L^2} &\le C\left(M + \frac{M}{K}\right)^2 < \infty. && (4.4.106) \end{aligned}$$

The solution is unique among functions such that (4.4.102), (4.4.104)and (4.4.106) hold.

Proof.

Step 1: Existence and estimates (4.4.102) and (4.4.103).

By the uniform estimates from Lemma 4.4.4, the solution f^L of the finite problem (4.4.34) in the slab $[0, L]$ is uniformly bounded in $L^2_{loc}([0, \infty); L^2[-\pi, \pi))$. Then there exists a subsequence such that

$$\begin{aligned} q^L &\rightharpoonup q, && (4.4.107) \\ r^L &\rightharpoonup r, && (4.4.108) \end{aligned}$$

weakly in $L^2_{loc}([0,\infty);L^2[-\pi,\pi))$. Also, $f = q + r$ satisfies the boundary condition at $\eta = 0$. This shows the existence of the solution. Then property (4.4.102) naturally holds due to the weak lower semi-continuity of norm $\|\cdot\|_{L^2L^2}$. Also, the orthogonal relation (4.4.103) is preserved.

Step 2: Estimates (4.4.104), (4.4.105) and (4.4.106).

We continue using the notation in Step 5 of the proof of Lemma 4.4.3. Recall (4.4.83) to (4.4.85) with $\lambda = 0$ and $L = \infty$. We have

$$\beta(\eta) = \langle \sin^2\phi, f\rangle_\phi(\eta) \tag{4.4.109}$$

and

$$\frac{\mathrm{d}\beta}{\mathrm{d}\eta} = D(\eta,\phi), \tag{4.4.110}$$

where

$$D(\eta,\phi) = -\langle \sin\phi, r\rangle_\phi + F(\eta)\langle \cos(2\phi), r\rangle_\phi + \langle \sin\phi, S\rangle_\phi(\eta). \tag{4.4.111}$$

The orthogonal relation (4.4.103) implies

$$D(\eta,\phi) = F(\eta)\langle \cos(2\phi), r\rangle_\phi + \langle \sin\phi, S\rangle_\phi(\eta). \tag{4.4.112}$$

Hence, we can integrate (4.4.110) over $[0,\eta]$ to show

$$\beta(\eta) - \beta(0) = \int_0^\eta F(y)\langle \cos(2\phi), r\rangle_\phi(y)\mathrm{d}y + \int_0^\eta \langle \sin\phi, S\rangle_\phi(y)\mathrm{d}y. \tag{4.4.113}$$

Based on Lemma 4.4.1, since $F \in L^1[0,\infty) \cap L^2[0,\infty)$, $r \in L^2([0,\infty)\times[-\pi,\pi))$, and S exponentially decays, by (4.4.94) and (4.4.102), there exists some constant β_∞ such that $\beta_\infty = \lim_{\eta\to\infty}\beta(\eta)$ satisfying

$$\begin{aligned}|\beta_\infty| &\le |\beta(0)| + \left|\int_0^\infty F(y)\langle\cos(2\phi), r\rangle_\phi(y)\mathrm{dy}\right| + \left|\int_0^\infty \langle\sin\phi, S\rangle_\phi(y)\mathrm{dy}\right| \\ &\le 8 + 192\,\|h\|_{L^2}^2 + 200\int_0^\infty \|S(\eta)\|_{L^2}^2\,\mathrm{d}\eta + 2\pi\|F\|_{L^2L^2}\|r\|_{L^2L^2} \\ &\le C\left(1 + M + \frac{M}{K}\right)^2.\end{aligned} \tag{4.4.114}$$

We define $q_\infty = \beta_\infty/\|\sin\phi\|_{L^2}^2$. Hence, the estimate of $|q_\infty|$ in (4.4.104) is valid. Moreover,

$$\beta_\infty - \beta(\eta) = \int_\eta^\infty D(y)\mathrm{dy} = \int_\eta^\infty F(y)\langle\cos(2\phi), r\rangle_\phi(y)\mathrm{dy} + \int_\eta^\infty \langle\sin\phi, S\rangle_\phi(y)\mathrm{dy}. \tag{4.4.115}$$

Note

$$\begin{aligned}\beta(\eta) &= \langle\sin^2\phi, f\rangle_\phi(\eta) = \langle\sin^2\phi, q\rangle_\phi(\eta) + \langle\sin^2\phi, r\rangle_\phi(\eta) \qquad (4.4.116)\\ &= q(\eta)\,\|\sin\phi\|_{L^2}^2 + \langle\sin^2\phi, r\rangle_\phi(\eta).\end{aligned}$$

Thus we can estimate

$$\begin{aligned}&\|\sin\phi\|_{L^2}^2\,\|q(\eta) - q_\infty\|_{L^2} \qquad (4.4.117)\\ =\ & \sqrt{2\pi}\,\|\sin\phi\|_{L^2}^2\,\|q(\eta) - q_\infty\|_{L^\infty} = \sqrt{2\pi}\Big(\beta(\eta) - \langle\sin^2\phi, r\rangle_\phi(\eta) - \beta_\infty\Big) \\ \le\ & \sqrt{2\pi}\Big(\left|\langle\sin^2\phi, r\rangle_\phi(\eta)\right| + \int_\eta^\infty |F(y)\langle\cos(2\phi), r(y)\rangle_\phi\mathrm{dy}|\,\mathrm{d}\eta \\ &+ \int_\eta^\infty |\langle\sin\phi, S\rangle_\phi(y)|\,\mathrm{dy}\Big) \\ \le\ & 2\pi^2\Big(\|r(\eta)\|_{L^2} + \int_\eta^\infty |F(y)|\,\|r(y)\|_{L^2}\,\mathrm{dy} + \int_\eta^\infty \|S(y)\|_{L^\infty}\,\mathrm{dy}\Big).\end{aligned}$$

This implies (4.4.105). Furthermore, we integrate (4.4.117) over $\eta \in [0, \infty)$. Cauchy's inequality and (4.4.9) imply

$$\begin{aligned} \int_0^\infty \left(\int_\eta^\infty |F(y)| \, \|r(y)\|_{L^2} \, \mathrm{dy} \right)^2 \mathrm{d}\eta &\leq \|r\|_{L^2L^2}^2 \int_0^\infty \int_\eta^\infty |F(y)|^2 \, \mathrm{dy}\mathrm{d}\eta \\ &\leq C. \end{aligned} \tag{4.4.118}$$

The exponential decays shows

$$\int_0^\infty \left(\int_\eta^\infty \|S(y)\|_{L^\infty} \, \mathrm{dy} \right)^2 \mathrm{d}\eta \leq C. \tag{4.4.119}$$

Hence, the estimate of $\|q - q_\infty\|_{L^2L^2}$ in (4.4.106) naturally follows.

Step 3: Uniqueness

In order to show the uniqueness of the solution, we assume there are two solutions f_1 and f_2 to the equation (4.4.101) satisfying (4.4.102) and (4.4.103). Then $f' = f_1 - f_2$ satisfies the equation

$$\left\{ \begin{aligned} \sin\phi \frac{\partial f'}{\partial \eta} + F(\eta)\cos\phi \frac{\partial f'}{\partial \phi} + f' - \bar{f}' &= 0, \\ f'(0, \phi) &= 0 \quad \text{for} \quad \sin\phi > 0, \\ \lim_{\eta\to\infty} f'(\eta, \phi) &= f'_\infty. \end{aligned} \right. \tag{4.4.120}$$

Similarly, we can define r' and q'. Multiplying $e^{-V(\eta)} f'$ on both sides of (4.4.120) and integrating over $\phi \in [-\pi, \pi)$ yields

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\langle f', f' \sin\phi \rangle_\phi(\eta) e^{-V(\eta)} \right) = -\left(\|r'(\eta)\|_{L^2}^2 \, e^{-V(\eta)} \right) \leq 0. \tag{4.4.121}$$

This is due to the fact

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\Big(\langle f', f'\sin\phi\rangle_\phi(\eta)e^{-V(\eta)}\Big) \qquad (4.4.122)\\
=\;& \Big(\langle f', \frac{\mathrm{df}'}{\mathrm{d}\eta}\sin\phi\rangle_\phi(\eta)e^{-V(\eta)}\Big) + \frac{1}{2}\Big(F(\eta)\langle f', f'\sin\phi\rangle_\phi(\eta)e^{-V(\eta)}\Big)\\
=\;& \Big(\langle f', \frac{\mathrm{df}'}{\mathrm{d}\eta}\sin\phi\rangle_\phi(\eta)e^{-V(\eta)}\Big) + \Big(F(\eta)\langle f', \frac{\mathrm{df}'}{\mathrm{d}\phi}\cos\phi\rangle_\phi(\eta)e^{-V(\eta)}\Big).
\end{aligned}
$$

Thus, we have

$$
\gamma(\eta) = \frac{1}{2}\langle f', f'\sin\phi\rangle_\phi(\eta)e^{-V(\eta)}. \qquad (4.4.123)
$$

is decreasing. Since $r' \in L^2([0,\infty)\times[-\pi,\pi))$ and $q'-q'_\infty \in L^2([0,\infty)\times[-\pi,\pi))$, there exists a convergent subsequence $\eta_k \to \infty$ satisfying $\|r'(\eta_k)\|_{L^2} \to 0$ and $q'(\eta_k)-q'_\infty \to 0$. Hence, this implies

$$
\frac{1}{2}\langle r', r'\sin\phi\rangle_\phi(\eta_k)e^{-V(\eta_k)} \to 0. \qquad (4.4.124)
$$

Also, due to the fact that $q'(\eta_k)$ is independent of ϕ and it is finite, we have

$$
\gamma(\eta_k) \to 0. \qquad (4.4.125)
$$

By the monotonicity, $\gamma(\eta)$ decreases to zero and $\gamma(\eta) \geq 0$. Then we can integrate (4.4.121) over $\eta \in [0,\infty)$ to obtain

$$
\gamma(\infty) - \gamma(0) = -2\int_0^\infty \|r'(y)\|_{L^2}^2\, e^{-V(y)}\mathrm{dy}, \qquad (4.4.126)
$$

which implies

$$
\gamma(0) = \langle f', f'\sin\phi\rangle_\phi(0)e^{-V(0)} = 2\int_0^\infty \|r'(y)\|_{L^2}^2\, e^{-V(y)}\mathrm{dy}. \qquad (4.4.127)
$$

Also, we know

$$0 \leq \frac{1}{2}\langle f', f'\sin\phi\rangle_{\phi}(0)e^{-V(0)} = \frac{1}{2}\langle f', f'\sin\phi\rangle_{\phi}(0) \leq \int_{\sin\phi>0}(f')^2(\phi)\sin\phi \mathrm{d}\phi = 0. \tag{4.4.128}$$

Naturally, we have

$$\langle f', f'\sin\phi\rangle_{\phi}(0)e^{-V(0)} = 2\int_0^{\infty}\left\lVert r'(y)\right\rVert_{L^2}^2 e^{-V(y)}\mathrm{dy} = 0. \tag{4.4.129}$$

Hence, we have $r' = 0$ and $f'(\eta, \phi) = q'(\eta)$. Plugging this into the equation (4.4.120) reveals $\partial_\eta q' = 0$. Therefore, $f'(\eta, \phi) = C$ for some constant C. Naturally the boundary data leads to $C = 0$. In conclusion, we must have $f' = 0$, which means $f_1 = f_2$, and the uniqueness follows. $\square$

$\bar{S} \neq 0$ Case

Consider the ϵ-Milne problem for $f(\eta, \phi)$ in $(\eta, \phi) \in [0, \infty) \times [-\pi, \pi)$ with a general source term

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi} + f - \bar{f} &=& S(\eta, \phi),\\ f(0, \phi) &=& h(\phi) \ \text{ for } \ \sin\phi > 0,\\ \lim_{\eta\to\infty} f(\eta, \phi) &=& f_{\infty}.\end{array}\right. \tag{4.4.130}$$

Lemma 4.4.5. *Assume (4.4.3) and (4.4.4) hold. Then there exists a solution $f(\eta, \phi)$ of the problem (4.4.130), satisfying*

$$\lVert r\rVert_{L^2L^2} < C\left(1 + M + \frac{M}{K}\right)^2 \leq \infty, \tag{4.4.131}$$

$$\langle\sin\phi, r\rangle_{\phi}(\eta) = -\int_{\eta}^{\infty} e^{V(\eta)-V(y)}\bar{S}(y)\mathrm{dy}. \tag{4.4.132}$$

Also there exists a constant $q_\infty = f_\infty \in \mathbb{R}$ such that the following estimates hold,

$$
\begin{aligned}
|q_\infty| &\leq C\Big(1+M+\frac{M}{K}\Big)^2 < \infty, && (4.4.133)\\
& && (4.4.134)\\
\|q(\eta)-q_\infty\|_{L^2} &\leq C\Big(\|r(\eta)\|_{L^2} + \int_\eta^\infty |F(y)|\,\|r(y)\|_{L^2}\,\mathrm{dy} + \int_\eta^\infty \|S(y)\|_{L^\infty}\,\mathrm{dy}\Big),\\
\|q-q_\infty\|_{L^2L^2} &\leq C\Big(1+M+\frac{M}{K}\Big)^2 < \infty. && (4.4.135)
\end{aligned}
$$

The solution is unique among functions satisfying $\|f-f_\infty\|_{L^2L^2} < \infty$.

Proof. We can apply superposition property for this linear problem, i.e. write $S = \bar{S} + (S-\bar{S}) = S_Q + S_R$. Then we solve the problem by the following steps. For simplicity, we just call the estimates (4.4.131), (4.4.133), (4.4.134) and (4.4.135) as the L^2 estimates.

Step 1: Construction of auxiliary function f^1.

We first solve f^1 as the solution to

$$
\left\{
\begin{aligned}
\sin\phi\frac{\partial f^1}{\partial\eta} + F(\eta)\cos\phi\frac{\partial f^1}{\partial\phi} + f^1 - \bar{f}^1 &= S_R(\eta,\phi),\\
f^1(0,\phi) &= h(\phi) \quad \text{for} \quad \sin\phi > 0,\\
\lim_{\eta\to\infty} f^1(\eta,\phi) &= f^1_\infty.
\end{aligned}
\right.
\tag{4.4.136}
$$

Since $\bar{S}_R = 0$, by Lemma 4.4.4, we know there exists a unique solution f^1 satisfying the L^2 estimate.

Step 2: Construction of auxiliary function f^2.

We seek a function f^2 satisfying

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sin\phi\frac{\partial f^2}{\partial\eta}+F(\eta)\cos\phi\frac{\partial f^2}{\partial\phi}\right)\mathrm{d}\phi+S_Q=0. \tag{4.4.137}$$

The following analysis shows this type of function can always be found. An integration by parts transforms the equation (4.4.137) into

$$-\int_{-\pi}^{\pi}\sin\phi\frac{\partial f^2}{\partial\eta}\mathrm{d}\phi-\int_{-\pi}^{\pi}F(\eta)\sin\phi f^2\mathrm{d}\phi+2\pi S_Q=0. \tag{4.4.138}$$

Setting

$$f^2(\phi,\eta)=a(\eta)\sin\phi. \tag{4.4.139}$$

and plugging this ansatz into (4.4.138), we have

$$-\frac{\mathrm{d}a}{\mathrm{d}\eta}\int_{-\pi}^{\pi}\sin^2\phi\mathrm{d}\phi-F(\eta)a(\eta)\int_{-\pi}^{\pi}\sin^2\phi\mathrm{d}\phi+2\pi S_Q=0. \tag{4.4.140}$$

Hence, we have

$$-\frac{\mathrm{d}a}{\mathrm{d}\eta}-F(\eta)a(\eta)+2S_Q=0. \tag{4.4.141}$$

This is a first order linear ordinary differential equation, which possesses infinitely many solutions. We can directly solve it to obtain

$$a(\eta)=e^{-\int_0^\eta F(y)\mathrm{dy}}\left(a(0)+\int_0^\eta e^{\int_0^y F(z)\mathrm{dz}}2S_Q(y)\mathrm{dy}\right). \tag{4.4.142}$$

We may take

$$a(0)=-\int_0^\infty e^{\int_0^y F(z)\mathrm{dz}}2S_Q(y)\mathrm{dy}. \tag{4.4.143}$$

Based on the exponential decay of S_Q, we can directly verify $a(\eta)$ decays exponentially to zero as $\eta\rightarrow\infty$ and f^2 satisfies the L^2 estimate.

Step 3: Construction of auxiliary function f^3.

Based on above construction, we can directly verify

$$\int_{-\pi}^{\pi}\bigg(-\sin\phi\frac{\partial f^2}{\partial\eta}-F(\eta)\cos\phi\frac{\partial f^2}{\partial\phi}-f^2+\bar{f}^2+S_Q\bigg)\mathrm{d}\phi=0. \tag{4.4.144}$$

Then we can solve f^3 as the solution to

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f^3}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^3}{\partial\phi}+f^3-\bar{f}^3&=&-\sin\phi\dfrac{\partial f^2}{\partial\eta}-F(\eta)\cos\phi\dfrac{\partial f^2}{\partial\phi}\\ &&-f^2+\bar{f}^2+S_Q,\\ f^3(0,\phi)&=&-a(0)\sin\phi\quad\text{for}\quad\sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f^3(\eta,\phi)&=&f^3_{\infty}.\end{array}\right. \tag{4.4.145}$$

By (4.4.144), we can apply Lemma 4.4.4 to obtain a unique solution f^3 satisfying the L^2 estimate.

Step 4: Construction of auxiliary function f^4.

We now define $f^4=f^2+f^3$ and an explicit verification shows

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f^4}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^4}{\partial\phi}+f^4-\bar{f}^4&=&S_Q(\eta,\phi),\\ f^4(0,\phi)&=&0\quad\text{for}\quad\sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f^4(\eta,\phi)&=&f^4_{\infty},\end{array}\right. \tag{4.4.146}$$

and f^4 satisfies the L^2 estimate.

In summary, we deduce that $f^1 + f^4$ is the solution of (4.4.130) and satisfies the L^2 estimate. A direct computation of $\langle \sin\phi, f^i\rangle_\phi(\eta)$ for $i = 1, 2, 3, 4$ leads to (4.4.132). From $\|f - f_\infty\|_{L^2L^2} < \infty$, we deduce $\left\|\bar{f} - f_\infty\right\|_{L^2L^2} < \infty$, a similar argument as in Lemma 4.4.4 shows the uniqueness of solution. $\square$

Combining all above, we have the following theorem.

Theorem 4.4.6. *For the ϵ-Milne problem (4.4.1), there exists a unique solution $f(\eta,\phi)$ satisfying the estimates*

$$\|f - f_\infty\|_{L^2L^2} \leq C\left(1 + M + \frac{M}{K}\right) < \infty, \tag{4.4.147}$$

for some real number f_∞ satisfying

$$|f_\infty| \leq C\left(1 + M + \frac{M}{K}\right)^2 < \infty. \tag{4.4.148}$$

4.4.2 L^∞ Estimates

Finite Slab

Consider the ϵ-transport problem for $f(\eta,\phi)$ in a finite slab $(\eta,\phi) \in [0, L] \times [-\pi, \pi)$

$$\begin{cases} \sin\phi\dfrac{\partial f}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi} + f &= H(\eta,\phi), \\ f(0,\phi) &= h(\phi) \quad \text{for} \quad \sin\phi > 0, \\ f(L,\phi) &= f(L, R\phi). \end{cases} \tag{4.4.149}$$

Define the energy as follows:

$$E(\eta, \phi) = \cos\phi e^{-V(\eta)}. \tag{4.4.150}$$

In the plane $(\eta, \phi) \in [0, \infty) \times [-\pi, \pi)$, on the curve $\phi = \phi(\eta)$ with constant energy, we can see

$$\frac{\mathrm{dE}}{\mathrm{d}\eta} = \frac{\partial E}{\partial \eta} + \frac{\partial E}{\partial \phi}\frac{\partial \phi}{\partial \eta} = \cos\phi F(\eta) e^{-V(\eta)} - \sin\phi e^{-V(\eta)} \frac{\partial \phi}{\partial \eta} = 0, \tag{4.4.151}$$

which further implies

$$\frac{\partial \phi}{\partial \eta} = \frac{\cos\phi F(\eta)}{\sin\phi}. \tag{4.4.152}$$

Plugging this into the equation (4.4.149), on this curve, we deduce

$$\frac{\mathrm{df}}{\mathrm{d}\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial \eta} = \frac{1}{\sin\phi}\bigg(\sin\phi \frac{\partial f}{\partial \eta} + \cos\phi F(\eta) \frac{\partial f}{\partial \phi}\bigg). \tag{4.4.153}$$

Hence, this curve with constant energy is exactly the characteristics of the equation (4.4.149). Also, on this curve the equation can be simplified as follows:

$$\sin\phi \frac{\mathrm{df}}{\mathrm{d}\eta} + f = H. \tag{4.4.154}$$

An implicit function $\eta^+(\eta, \phi)$ can be determined through

$$|E(\eta, \phi)| = e^{-V(\eta^+)}. \tag{4.4.155}$$

which means (η^+, ϕ_0) with $\sin\phi_0 = 0$ is on the same characteristics as (η, ϕ). Define the quantities for $0 \le \eta' \le \eta^+$ as follows:

$$\phi'(\phi, \eta, \eta') = \cos^{-1}(\cos\phi e^{V(\eta')-V(\eta)}), \tag{4.4.156}$$

$$R\phi'(\phi, \eta, \eta') = -\cos^{-1}(\cos\phi e^{V(\eta')-V(\eta)}) = -\phi'(\phi, \eta, \eta'), \tag{4.4.157}$$

where the inverse trigonometric function can be defined single-valued in the domain $[0, \pi)$ and the quantities are always well-defined due to the monotonicity of V. We can see $(\eta', \phi'(\phi, \eta, \eta'))$ and (η, ϕ) are on the same characteristics. Finally we put

$$G_{\eta,\eta'}(\phi) = \int_{\eta'}^{\eta} \frac{1}{\sin(\phi'(\phi, \eta, \xi))} \mathrm{d}\xi. \tag{4.4.158}$$

We can rewrite the solution to the equation (4.4.149) along the characteristics as follows:

Case I:

For $\sin\phi > 0$,

$$f(\eta, \phi) = h(\phi'(\phi, \eta, 0)) \exp(-G_{\eta,0}) + \int_0^{\eta} \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta,\eta'}) \mathrm{d}\eta'. \tag{4.4.159}$$

Case II:

For $\sin\phi < 0$ and $|E(\eta,\phi)| \leq e^{-V(L)}$,

$$
\begin{aligned}
f(\eta,\phi) &= h(\phi'(\phi,\eta,0))\exp(-G_{L,0}-G_{L,\eta}) \\
&+ \bigg(\int_0^L \frac{H(\eta',\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta' \\
&\quad + \int_\eta^L \frac{H(\eta',R\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta'\bigg).
\end{aligned} \tag{4.4.160}
$$

Case III:

For $\sin\phi < 0$ and $|E(\eta,\phi)| \geq e^{-V(L)}$,

$$
\begin{aligned}
f(\eta,\phi) &= h(\phi'(\phi,\eta,0))\exp(-G_{\eta^+,0}-G_{\eta^+,\eta}) \\
&+ \bigg(\int_0^{\eta^+} \frac{H(\eta',\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta' \\
&\quad + \int_\eta^{\eta^+} \frac{H(\eta',R\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta'\bigg).
\end{aligned} \tag{4.4.161}
$$

Infinite Slab

Consider the ϵ-transport problem for $f(\eta,\phi)$ in an infinite slab $(\eta,\phi) \in [0,\infty)\times[-\pi,\pi)$

$$
\left\{
\begin{aligned}
\sin\phi\frac{\partial f}{\partial\eta} + F(\eta)\cos\phi\frac{\partial f}{\partial\phi} + f &= H(\eta,\phi), \\
f(0,\phi) &= h(\phi) \quad \text{for} \quad \sin\phi > 0, \\
\lim_{\eta\to\infty} f(\eta,\phi) &= f_\infty.
\end{aligned}
\right. \tag{4.4.162}
$$

We can define the solution via taking limit $L \to \infty$ in (4.4.159), (4.4.160) and (4.4.161) as follows:

$$
f(\eta,\phi) = \mathcal{A}h(\phi) + \mathcal{T}H(\eta,\phi), \tag{4.4.163}
$$

where

Case I:

For $\sin\phi > 0$,

$$\mathcal{A}h(\phi) = h(\phi'(\phi,\eta,0))\exp(-G_{\eta,0}) \tag{4.4.164}$$

$$\mathcal{T}H(\eta,\phi) = \int_0^{\eta} \frac{H(\eta',\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{4.4.165}$$

Case II:

For $\sin\phi < 0$ and $|E(\eta,\phi)| \le e^{-V_\infty}$,

$$\mathcal{A}h(\phi) = 0 \tag{4.4.166}$$

$$\mathcal{T}H(\eta,\phi) = \int_{\eta}^{\infty} \frac{H(\eta',R\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta'. \tag{4.4.167}$$

Case III:

For $\sin\phi < 0$ and $|E(\eta,\phi)| \ge e^{-V_\infty}$,

$$\mathcal{A}h(\phi) = h(\phi'(\phi,\eta,0))\exp(-G_{\eta^+,0} - G_{\eta^+,\eta}) \tag{4.4.168}$$

$$\mathcal{T}H(\eta,\phi) = \bigg(\int_0^{\eta^+} \frac{H(\eta',\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta^+,\eta'} - G_{\eta^+,\eta})\mathrm{d}\eta' + \int_{\eta}^{\eta^+} \frac{H(\eta',R\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta'\bigg). \tag{4.4.169}$$

Notice that

$$\lim_{L\to\infty}\exp(-G_{L,\eta}) = 0, \tag{4.4.170}$$

for $\sin\phi < 0$ and $|E(\eta,\phi)| \leq e^{-V_\infty}$. Hence, above derivation is valid. In order to achieve the estimate of f, we need to control $\mathcal{T}H$ and $\mathcal{A}h$.

Preliminaries

We first give several technical lemmas to be used for proving L^∞ estimates of f.

Lemma 4.4.7. *For any $0 \leq \beta \leq 1$, we have*

$$\left\| e^{\beta\eta}\mathcal{A}h \right\|_{L^\infty} \leq \|h\|_{L^\infty}. \tag{4.4.171}$$

In particular,

$$\|\mathcal{A}h\|_{L^\infty} \leq \|h\|_{L^\infty}. \tag{4.4.172}$$

Proof. Since ϕ' is always in the domain $[0,\pi)$, we naturally have

$$0 \leq \sin(\phi'(\phi,\eta,\xi)) \leq 1, \tag{4.4.173}$$

which further implies

$$\frac{1}{\sin(\phi'(\phi,\eta,\xi))} \geq 1. \tag{4.4.174}$$

Combined with the fact $\eta^+ \geq \eta$, we deduce

$$\begin{aligned} \exp(-G_{\eta,0}) &\leq e^{-\eta} && (4.4.175)\\ \exp(-G_{\eta^+,0} - G_{\eta^+,\eta}) &\leq \exp(-G_{\eta^+,0}) \leq \exp(-G_{\eta,0}) \leq e^{-\eta}. && (4.4.176) \end{aligned}$$

Hence, our result easily follows. $\square$

Lemma 4.4.8. *The integral operator $\mathcal{T}$ satisfies*

$$\|\mathcal{T}H\|_{L^\infty L^\infty} \leq C\|H\|_{L^\infty L^\infty}, \tag{4.4.177}$$

and for any $0 \leq \beta \leq 1/2$

$$\left\|e^{\beta\eta}\mathcal{T}H\right\|_{L^\infty L^\infty} \leq C\left\|e^{\beta\eta}H\right\|_{L^\infty L^\infty}, \tag{4.4.178}$$

where C is a universal constant independent of data.

Proof. For (4.4.177), when $\sin\phi > 0$

$$\begin{aligned} |\mathcal{T}H| &\leq \int_0^\eta |H(\eta',\phi'(\phi,\eta,\eta'))| \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \qquad (4.4.179)\\ &\leq \|H\|_{L^\infty L^\infty}\int_0^\eta \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \end{aligned}$$

We can directly estimate

$$\int_0^\eta \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \leq \int_0^\infty e^{-z}\mathrm{d}z = 1, \tag{4.4.180}$$

and (4.4.177) naturally follows. For $\sin\phi < 0$ and $|E(\eta,\phi)| \leq e^{-V_\infty}$,

$$\begin{aligned} |\mathcal{T}H| &\leq \int_\eta^\infty |H(\eta',\phi'(\phi,\eta,\eta'))| \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta' \qquad (4.4.181)\\ &\leq \|H\|_{L^\infty L^\infty}\int_\eta^\infty \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta'. \end{aligned}$$

we have

$$\int_\eta^\infty \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta' \leq \int_{-\infty}^0 e^{z}\mathrm{d}z = 1, \tag{4.4.182}$$

and (4.4.177) easily follows. The case $\sin\phi < 0$ and $|E(\eta,\phi)| \geq e^{-V_\infty}$ can be proved combining above two techniques, so we omit it here.

For (4.4.178), when $\sin\phi > 0$, $\eta \geq \eta'$ and $\beta < 1/2$, since $G_{\eta,\eta'} \geq \eta - \eta'$, we have

$$\beta(\eta-\eta') - G_{\eta,\eta'} \leq \beta(\eta-\eta') - \frac{1}{2}(\eta-\eta') - \frac{1}{2}G_{\eta,\eta'} \leq -\frac{1}{2}G_{\eta,\eta'}. \tag{4.4.183}$$

Then it is natural that

$$\begin{aligned} & \int_0^\eta \frac{1}{\sin(\phi'(\phi,\eta,\eta'))} \exp(\beta(\eta-\eta') - G_{\eta,\eta'})\mathrm{d}\eta' \qquad (4.4.184) \\ \leq & \int_0^\eta \frac{1}{\sin(\phi'(\phi,\eta,\eta'))} \exp(-G_{\eta,\eta'}/2)\mathrm{d}\eta' \\ \leq & \int_0^\infty e^{-z/2}\mathrm{dz} = 2. \end{aligned}$$

This leads to

$$\begin{aligned} & \left|e^{\beta\eta}\mathcal{T}H\right| \qquad (4.4.185) \\ \leq & \; e^{\beta\eta}\int_0^\eta \left|H(\eta',\phi'(\phi,\eta,\eta'))\right| \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\ \leq & \; \left\|e^{\beta\eta}H\right\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(\beta(\eta-\eta') - G_{\eta,\eta'})\mathrm{d}\eta' \\ \leq & \; C\left\|e^{\beta\eta}H\right\|_{L^\infty L^\infty}, \end{aligned}$$

and (4.4.178) naturally follows. For $\sin\phi < 0$ and $|E(\eta,\phi)| \leq e^{-V_\infty}$, note for $\eta' \geq \eta$

$$\beta(\eta-\eta') + G_{\eta,\eta'} \leq \beta(\eta-\eta') + \frac{1}{2}(\eta-\eta') + \frac{1}{2}G_{\eta,\eta'} \leq \frac{1}{2}G_{\eta,\eta'}. \tag{4.4.186}$$

Then (4.4.178) holds by obvious modifications of $\sin\phi > 0$ case. The case $\sin\phi < 0$ and $|E(\eta,\phi)| \geq e^{-V_\infty}$ can be shown by combining above two cases, so we omit it here. $\square$

Lemma 4.4.9. *For any $\delta > 0$ there is a constant $C(\delta) > 0$ independent of data such* *that*

$$\|\mathcal{T}H\|_{L^\infty L^2} \leq C(\delta)\|H\|_{L^2L^2} + \delta\|H\|_{L^\infty L^\infty}. \tag{4.4.187}$$

Proof. We divide the proof into several steps.

Step 1: The case of $\sin\phi > 0$.

We consider

$$\begin{aligned} & \int_{\sin\phi>0} |\mathcal{T}H(\eta)|^2 \,\mathrm{d}\phi \qquad (4.4.188) \\ = & \int_{\sin\phi>0} \left(\int_0^\eta \frac{H(\eta',\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\right)^2 \mathrm{d}\phi \\ = & \int\left(\int \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))>m\}}\cdots\right)^2 + \int\left(\int \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))\leq m\}}\cdots\right)^2 \\ = & I_1 + I_2. \end{aligned}$$

By Cauchy's inequality and (4.4.180), we get

$$\begin{aligned} I_1 \leq & \int_{\sin\phi>0}\left(\int_0^\eta |H(\eta',\phi'(\phi,\eta,\eta'))|^2 \,\mathrm{d}\eta'\right) \qquad (4.4.189) \\ & \times\left(\int_0^\eta \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))>m\}}\frac{\exp(-2G_{\eta,\eta'})}{\sin^2(\phi'(\phi,\eta,\eta'))}\mathrm{d}\eta'\right)\mathrm{d}\phi \\ \leq & \frac{1}{m}\|H\|_{L^2L^2}^2 \int_{\sin\phi>0}\left(\int_0^\eta \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))>m\}}\frac{\exp(-2G_{\eta,\eta'})}{\sin(\phi'(\phi,\eta,\eta'))}\mathrm{d}\eta'\right)\mathrm{d}\phi \\ \leq & \frac{\pi}{m}\|H\|_{L^2L^2}^2. \end{aligned}$$

On the other hand, for $\eta' \leq \eta$, we can directly estimate $\phi'(\phi,\eta,\eta') \geq \phi$. Hence, we have the relation

$$\sin\phi \leq \sin(\phi'(\phi,\eta,\eta')). \tag{4.4.190}$$

Therefore, we can directly estimate I_2 as follows:

$$
\begin{aligned}
I_2 &\leq \|H\|^2_{L^\infty L^\infty}\int_{\sin\phi>0}\bigg(\int_0^\eta \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))\leq m\}}\frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi \\
&\leq \|H\|^2_{L^\infty L^\infty}\int_{\sin\phi>0}\bigg(\int_0^\eta \mathbf{1}_{\{\sin\phi\leq m\}}\frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi \\
&= \|H\|^2_{L^\infty L^\infty}\int_{\sin\phi>0}\mathbf{1}_{\{\sin\phi\leq m\}}\bigg(\int_0^\eta \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi.
\end{aligned}
\tag{4.4.191}
$$

Note

$$
\int_0^\eta \frac{1}{\sin(\phi'(\phi,\eta,\eta'))}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \leq \int_0^\infty e^{-z}\mathrm{dz} = 1. \tag{4.4.192}
$$

Therefore, for m sufficiently small, we have

$$
I_2 \leq \|H\|^2_{L^\infty L^\infty}\int_{\sin\phi>0}\mathbf{1}_{\{\sin\phi\leq m\}}\mathrm{d}\phi \leq 4m\|H\|^2_{L^\infty L^\infty}. \tag{4.4.193}
$$

Summing up (4.4.189) and (4.4.193), for m sufficiently small, we deduce (4.4.187).

Step 2: The case of $\sin\phi<0$ and $|E(\eta,\phi)|\leq e^{-V_\infty}$.

We can decompose

$$
\begin{aligned}
&\int_{\sin\phi<0}\mathbf{1}_{\{|E(\eta,\phi)|\leq e^{-V_\infty}\}}\left|\mathcal{T}H\right|^2\mathrm{d}\phi \\
=& \int_{\sin\phi<0}\mathbf{1}_{\{|E(\eta,\phi)|\leq e^{-V_\infty}\}}\bigg(\int_\eta^\infty \frac{H(\eta',R\phi'(\phi,\eta,\eta'))}{\sin(\phi'(\phi,\eta,\eta'))}\exp(G_{\eta,\eta'})\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi \\
=& \int\bigg(\int \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))>m\}}\cdots\bigg)^2+\int\bigg(\int \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))\leq m\}}\mathbf{1}_{\{\eta'-\eta\geq\sigma\}}\cdots\bigg)^2 \\
&\int\bigg(\int \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))\leq m\}}\mathbf{1}_{\{\eta'-\eta\leq\sigma\}}\cdots\bigg)^2 \\
=& I_1+I_2+I_3.
\end{aligned}
\tag{4.4.194}
$$

We can directly estimate I_1 as follows:

$$
\begin{aligned}
I_1 &\leq \int_{\sin\phi<0}\bigg(\int_\eta^\infty \left|H(\eta',R\phi'(\phi,\eta,\eta'))\right|^2\mathrm{d}\eta'\bigg) \\
&\quad\times\bigg(\int_\eta^\infty \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))>m\}}\frac{\exp(2G_{\eta,\eta'})}{\sin^2(\phi'(\phi,\eta,\eta'))}\mathrm{d}\eta'\bigg)\mathrm{d}\phi \\
&\leq \frac{1}{m}\|H\|^2_{L^2L^2}\int_{\sin\phi<0}\bigg(\int_\eta^\infty \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))>m\}}\frac{\exp(2G_{\eta,\eta'})}{\sin(\phi'(\phi,\eta,\eta'))}\mathrm{d}\eta'\bigg)\mathrm{d}\phi \\
&\leq \frac{\pi}{m}\|H\|^2_{L^2L^2}.
\end{aligned} \tag{4.4.195}
$$

On the other hand, for I_2 we have

$$
\begin{aligned}
I_2 &\leq \|H\|^2_{L^\infty L^\infty}\int_{\sin\phi<0}\mathbf{1}_{\{|E(\eta,\phi)|\leq e^{-V_\infty}\}} \\
&\quad\times\bigg(\int_\eta^\infty \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))\leq m\}}\mathbf{1}_{\{\eta'-\eta\geq\sigma\}}\frac{\exp(G_{\eta,\eta'})}{\sin(\phi'(\phi,\eta,\eta'))}\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi.
\end{aligned} \tag{4.4.196}
$$

Note

$$
G_{\eta,\eta'} = \int_{\eta'}^{\eta}\frac{1}{\sin(\phi'(\phi,\eta,y))}\mathrm{dy} \leq -\frac{\eta'-\eta}{m} = -\frac{\sigma}{m}. \tag{4.4.197}
$$

Then we can obtain

$$
I_2 \leq \|H\|^2_{L^\infty L^\infty}\int_{\sin\phi<0}\bigg(\int_\infty^{-\sigma/m}e^z\mathrm{dz}\bigg)^2\mathrm{d}\phi = 4e^{-\frac{\sigma}{m}}\|H\|^2_{L^\infty L^\infty}. \tag{4.4.198}
$$

For I_3, we can estimate as follows:

$$
\begin{aligned}
I_3 &\leq \|H\|^2_{L^\infty L^\infty}\int_{\sin\phi<0}\mathbf{1}_{\{|E(\eta,\phi)|\leq e^{-V_\infty}\}} \\
&\quad\times\bigg(\int_\eta^\infty \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))\leq m\}}\mathbf{1}_{\{\eta'-\eta\leq\sigma\}}\frac{\exp(G_{\eta,\eta'})}{\sin(\phi'(\phi,\eta,\eta'))}\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi \\
&\leq \|H\|^2_{L^\infty L^\infty}\int_{\sin\phi<0}\mathbf{1}_{\{|E(\eta,\phi)|\leq e^{-V_\infty}\}} \\
&\quad\times\bigg(\int_\eta^{\eta+\sigma} \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))\leq m\}}\mathbf{1}_{\{\eta'-\eta\leq\sigma\}}\frac{\exp(G_{\eta,\eta'})}{\sin(\phi'(\phi,\eta,\eta'))}\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi.
\end{aligned} \tag{4.4.199}
$$

Note

$$\int_{\eta}^{\infty} \frac{1}{\sin(\phi'(\phi,\eta,\eta'))} \exp(G_{\eta,\eta'}) \mathrm{d}\eta' \leq \int_{-\infty}^{0} e^{z} \mathrm{dz} = 1. \tag{4.4.200}$$

Then $1 \leq \alpha = e^{V(\eta')-V(\eta)} \leq e^{V(\eta+\sigma)-V(\eta)} \leq 1+4\sigma$ due to (4.4.8), and for $\eta' \in [\eta, \eta+\sigma]$, $\sin\phi'(\phi,\eta,\eta')) = \sin\bigg(\cos^{-1}(\alpha\cos\phi)\bigg)$, $\sin(\phi'(\phi,\eta,\eta')) < m$ lead to

$$\begin{aligned} |\sin\phi| &= \sqrt{1-\cos^2\phi} = \sqrt{1-\frac{\cos^2\phi'(\phi,\eta,\eta')}{\alpha^2}} = \frac{\sqrt{\alpha^2-\bigg(1-\sin^2\phi'(\phi,\eta,\eta')\bigg)}}{\alpha} \\ &\leq \frac{\sqrt{\alpha^2-1+m^2}}{\alpha} \leq \frac{\sqrt{(1+4\sigma)^2-1+m^2}}{\alpha} \leq \sqrt{9\sigma+m^2}. \end{aligned} \tag{4.4.201}$$

Hence, we can obtain

$$\begin{aligned} I_3 &\leq \|H\|^2_{L^\infty L^\infty} \int_{\sin\phi<0} \mathbf{1}_{\{\sin(\phi'(\phi,\eta,\eta'))\leq m\}} \mathrm{d}\phi \leq \|H\|^2_{L^\infty L^\infty} \int_{\sin\phi<0} \mathbf{1}_{\{|\sin\phi|\leq\sqrt{9\sigma+m^2}\}} \mathrm{d}\phi \\ &\leq 4\sqrt{9\sigma+m^2}. \end{aligned} \tag{4.4.202}$$

Summarizing (4.4.195), (4.4.198) and (4.4.202), for sufficiently small σ, we can always choose m small enough to guarantee the relation (4.4.187).

Step 3: The case of $\sin\phi < 0$ and $|E(\eta,\phi)| \geq e^{-V_\infty}$.

We can decompose $\mathcal{T}H$ as follows:

$$
\begin{aligned}
& \mathcal{T}H(\eta,\phi) \qquad\qquad (4.4.203)\\
= & \left(\int_0^{\eta^+}\ldots\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta'+\int_\eta^{\eta^+}\ldots\exp(G_{\eta,\eta'})\mathrm{d}\eta'\right)\\
= & \int_0^{\eta}\ldots\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta'\\
& +\left(\int_\eta^{\eta^+}\ldots\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta'+\int_\eta^{\eta^+}\ldots\exp(G_{\eta,\eta'})\mathrm{d}\eta'\right)\\
= & I_1+I_2.
\end{aligned}
$$

For I_1, we can apply a similar argument as in Step 1 and for I_2, a similar argument as in Step 2 conclude the lemma. $\square$

Estimates of ϵ-Milne Equation

Consider the equation satisfied by $z=f-f_\infty$ as follows:

$$
\left\{
\begin{aligned}
\sin\phi\frac{\partial z}{\partial\eta}+F(\eta)\cos\phi\frac{\partial z}{\partial\phi}+z &= \bar{z}+S,\\
z(0,\phi) &= p(\phi)=h(\phi)-f_\infty \quad\text{for}\quad \sin\phi>0,\\
\lim_{\eta\rightarrow\infty}z(\eta,\phi) &= 0.
\end{aligned}
\right. \tag{4.4.204}
$$

Lemma 4.4.10. *Assume (4.4.3) and (4.4.4) hold. Then there exists a constant C depending on the data such that the solution of equation (4.4.204) verifies*

$$
\|z\|_{L^\infty L^\infty}\leq C\bigg(1+M+\frac{M}{K}+\|z\|_{L^2L^2}\bigg). \tag{4.4.205}
$$

Proof. We first show the following important facts:

$$
\begin{aligned}
\|\bar{z}\|_{L^2L^2} &\le \|z\|_{L^2L^2}, && (4.4.206)\\
\|\bar{z}\|_{L^\infty L^\infty} &\le \frac{1}{2\pi}\|z\|_{L^\infty L^2}. && (4.4.207)
\end{aligned}
$$

We can directly derive them by Cauchy's inequality as follows:

$$
\begin{aligned}
\|\bar{z}\|^2_{L^2L^2} &= \int_0^\infty \int_{-\pi}^{\pi} \left(\frac{1}{2\pi}\right)^2 \left(\int_{-\pi}^{\pi} z(\eta,\phi)\mathrm{d}\phi\right)^2 \mathrm{d}\phi\mathrm{d}\eta && (4.4.208)\\
&\le \int_0^\infty \int_{-\pi}^{\pi} \left(\frac{1}{2\pi}\right)\left(\int_{-\pi}^{\pi} z^2(\eta,\phi)\mathrm{d}\phi\right)\mathrm{d}\phi\mathrm{d}\eta \\
&= \int_0^\infty \left(\int_{-\pi}^{\pi} z^2(\eta,\phi)\mathrm{d}\phi\right)\mathrm{d}\eta = \|z\|^2_{L^2L^2}.\\
\|\bar{z}\|^2_{L^\infty L^\infty} &= \sup_\eta \bar{z}^2(\eta) = \sup_\eta \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} z(\eta,\phi)\mathrm{d}\phi\right)^2 && (4.4.209)\\
&\le \sup_\eta \left(\frac{1}{2\pi}\right)^2 \left(\int_{-\pi}^{\pi} z^2(\eta,\phi)\mathrm{d}\phi\right)\left(\int_{-\pi}^{\pi} 1^2\mathrm{d}\phi\right)\\
&= \frac{1}{2\pi}\sup_\eta \left(\int_{-\pi}^{\pi} z^2(\eta,\phi)\mathrm{d}\phi\right) = \|z\|^2_{L^\infty L^2}.
\end{aligned}
$$

By (4.4.204), $z = \mathcal{A}p + \mathcal{T}(\bar{z}+S)$ leads to

$$
\mathcal{T}(\bar{z}+S) = z - \mathcal{A}p, \tag{4.4.210}
$$

Then by Lemma 4.4.9, (4.4.206) and (4.4.207), we can show

$$
\begin{aligned}
&\|z-\mathcal{A}p\|_{L^\infty L^2} && (4.4.211)\\
\le\ & C(\delta)\Big(\|\bar{z}\|_{L^2L^2} + \|S\|_{L^2L^2}\Big) + \delta\Big(\|\bar{z}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\Big)\\
\le\ & C(\delta)\Big(\|z\|_{L^2L^2} + \|S\|_{L^2L^2}\Big) + \delta\Big(\|z\|_{L^\infty L^2} + \|S\|_{L^\infty L^\infty}\Big).
\end{aligned}
$$

Therefore, based on Lemma 4.4.7 and (4.4.211), we can directly estimate

$$
\begin{aligned}
&\|z\|_{L^\infty L^2} &(4.4.212)\\
\leq\ & \|\mathcal{A}p\|_{L^\infty}+C(\delta)\bigg(\|z\|_{L^2L^2}+\|S\|_{L^2L^2}\bigg)+\delta\bigg(\|z\|_{L^\infty L^2}+\|S\|_{L^\infty L^\infty}\bigg)\\
\leq\ & \|p\|_{L^\infty}+C(\delta)\bigg(\|z\|_{L^2L^2}+\|S\|_{L^2L^2}\bigg)+\delta\bigg(\|z\|_{L^\infty L^2}+\|S\|_{L^\infty L^\infty}\bigg).
\end{aligned}
$$

We can take $\delta = 1/2$ to obtain

$$
\|z\|_{L^\infty L^2}\leq C\bigg(\|z\|_{L^2L^2}+\|S\|_{L^2L^2}+\|S\|_{L^\infty L^\infty}+\|p\|_{L^\infty}\bigg). \tag{4.4.213}
$$

Therefore, based on Lemma 4.4.8, (4.4.213) and (4.4.207), we can achieve

$$
\begin{aligned}
&\|z\|_{L^\infty L^\infty} &(4.4.214)\\
\leq\ & \|\mathcal{A}p\|_{L^\infty L^\infty}+\|\mathcal{T}(\bar z+S)\|_{L^\infty L^\infty}\leq C\bigg(\|p\|_{L^\infty}+\|\bar z\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)\\
\leq\ & C\bigg(\|p\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\|z\|_{L^\infty L^2}\bigg)\\
\leq\ & C\bigg(\|p\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\|S\|_{L^2L^2}+\|z\|_{L^2L^2}\bigg).
\end{aligned}
$$

Since $\|p\|_{L^\infty}$, $\|S\|_{L^2L^2}$ and $\|S\|_{L^\infty L^\infty}$ are finite, our result easily follows. $\square$

Lemma 4.4.10 naturally implies the following.

Theorem 4.4.11. *The solution $f(\eta,\phi)$ to the Milne problem (4.4.1) satisfies*

$$
\|f-f_\infty\|_{L^\infty L^\infty}\leq C\bigg(1+M+\frac{M}{K}+\|f-f_\infty\|_{L^2L^2}\bigg). \tag{4.4.215}
$$

Combining Theorem 4.4.11 and Theorem 4.4.6, we deduce the main theorem.

Theorem 4.4.12. *There exists a unique solution $f(\eta,\phi)$ to the ϵ-Milne problem*

(4.4.1) satisfying

$$\|f-f_\infty\|_{L^\infty L^\infty}\leq C\bigg(1+M+\frac{M}{K}\bigg). \tag{4.4.216}$$

4.4.3 Exponential Decay

In this section, we prove the spatial decay of the solution to the Milne problem.

Theorem 4.4.13. *Assume (4.4.3) and (4.4.4) hold. For $K_0>0$ sufficiently small, the solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.4.1) satisfies*

$$\left\|e^{K_0\eta}(f-f_\infty)\right\|_{L^\infty L^\infty}\leq C\bigg(1+M+\frac{M}{K}\bigg), \tag{4.4.217}$$

Proof. Define $Z=e^{K_0\eta}g$ for $z=f-f_\infty$. We divide the analysis into several steps:

Step 1: We have

$$\|Z\|_{L^2L^2}^2=\int_0^\infty e^{2K_0\eta}\bigg(\int_{-\pi}^{\pi}(f(\eta,\phi)-f_\infty)^2\mathrm{d}\phi\bigg)\mathrm{d}\eta\leq C\bigg(1+M+\frac{M}{K}\bigg)^2. \tag{4.4.218}$$

The proof of (4.4.218): The orthogonal property (4.4.39) reveals

$$\langle f,f\sin\phi\rangle_\phi(\eta)=\langle r,r\sin\phi\rangle_\phi(\eta). \tag{4.4.219}$$

Multiplying $e^{2K_0\eta}f$ on both sides of equation (4.4.1) and integrating over $\phi\in[-\pi,\pi)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\bigg(e^{2K_0\eta}\langle r,r\sin\phi\rangle_\phi(\eta)\bigg)+\frac{1}{2}F(\eta)\bigg(e^{2K_0\eta}\langle r,r\sin\phi\rangle_\phi(\eta)\bigg)& \\ -e^{2K_0\eta}\bigg(K_0\langle r,r\sin\phi\rangle_\phi(\eta)-\langle r,r\rangle_\phi(\eta)\bigg) &= e^{2K_0\eta}\langle S,f\rangle_\phi(\eta).\end{aligned} \tag{4.4.220}$$

For $K_0 < \min\{1/2, K\}$, we have

$$\frac{3}{2}\left\|r(\eta)\right\|_{L^2}^2 \geq -K_0\langle r, r\sin\phi\rangle_\phi(\eta) + \langle r, r\rangle_\phi(\eta) \geq \frac{1}{2}\left\|r(\eta)\right\|_{L^2}^2. \tag{4.4.221}$$

Similar to the proof of Lemma 4.4.3 and Lemma 4.4.4, formula as (4.4.220) and (4.4.221) imply

$$\left\|e^{K_0\eta}r\right\|_{L^2L^2}^2 = \int_0^\infty e^{2K_0\eta}\langle r, r\rangle_\phi(\eta)\mathrm{d}\eta \leq C\Big(1 + M + \frac{M}{K}\Big)^2. \tag{4.4.222}$$

From (4.4.105), Cauchy's inequality and (4.4.9), we can deduce

$$\begin{aligned}
& \int_0^\infty e^{2K_0\eta}\Big(\int_{-\pi}^{\pi}(f(\eta,\phi) - f_\infty)^2\mathrm{d}\phi\Big)\mathrm{d}\eta \qquad (4.4.223)\\
\leq\ & \int_0^\infty e^{2K_0\eta}\Big(\int_{-\pi}^{\pi} r^2(\eta,\phi)\mathrm{d}\phi\Big)\mathrm{d}\eta + \int_0^\infty e^{2K_0\eta}\Big(\int_{-\pi}^{\pi}(q(\eta) - q_\infty)^2\mathrm{d}\phi\Big)\mathrm{d}\eta\\
\leq\ & \int_0^\infty e^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta\\
& + \int_0^\infty e^{2K_0\eta}\Big(\int_\eta^\infty |F(y)|\left\|r(y)\right\|_{L^2}\mathrm{dy}\Big)^2\mathrm{d}\eta + \int_0^\infty e^{2K_0\eta}\Big(\int_\eta^\infty \left\|S(y)\right\|_{L^\infty}\mathrm{dy}\Big)^2\mathrm{d}\eta\\
\leq\ & C\Big(1 + M + \frac{M}{K}\Big)^2\\
& + C\Big(\int_0^\infty e^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta\Big)\Big(\int_0^\infty\int_\eta^\infty e^{2K_0(\eta-y)}F^2(y)\mathrm{dy}\mathrm{d}\eta\Big)\\
& + \int_0^\infty e^{2K_0\eta}\Big(\int_\eta^\infty \left\|S(y)\right\|_{L^\infty}\mathrm{dy}\Big)^2\mathrm{d}\eta\\
\leq\ & C\Big(1 + M + \frac{M}{K}\Big)^2\\
& + C\Big(\int_0^\infty e^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta\Big)\Big(\int_0^\infty\int_\eta^\infty F^2(y)\mathrm{dy}\mathrm{d}\eta\Big)\\
& + \int_0^\infty e^{2K_0\eta}\Big(\int_\eta^\infty \left\|S(y)\right\|_{L^\infty}\mathrm{dy}\Big)^2\mathrm{d}\eta\\
\leq\ & C\Big(1 + M + \frac{M}{K}\Big)^2.
\end{aligned}$$

This completes the proof of (4.4.218) when $\bar{S} = 0$. By the method introduced in

Lemma 4.4.5, we can extend above L^2 estimates to the general S case. Note all the auxiliary functions constructed in Lemma 4.4.5 satisfy the estimates (4.4.218).

Step 2: We have

$$\|Z\|_{L^\infty L^\infty} \leq C\bigg(1 + M + \frac{M}{K} + \|Z\|_{L^2L^2}\bigg). \tag{4.4.224}$$

Proof of (4.4.224): Z satisfies the equation

$$\left\{ \begin{array}{rcl} \sin\phi\dfrac{\partial Z}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial Z}{\partial\phi} + Z &=& \bar{Z} + e^{K_0\eta}S + K_0\sin\phi Z, \\ Z(0,\phi) &=& p(\phi) = h(\phi) - f_\infty \ \text{ for } \ \sin\phi > 0. \end{array} \right. \tag{4.4.225}$$

Since we know $Z = \mathcal{A}p + \mathcal{T}(\bar{Z} + e^{K_0\eta}S + K_0\sin\phi Z)$ leads to

$$\mathcal{T}(\bar{Z} + e^{K_0\eta}S + K_0\sin\phi Z) = Z - \mathcal{A}p, \tag{4.4.226}$$

then by Lemma 4.4.9, (4.4.206) and (4.4.207), we can show

$$\begin{aligned} & \|Z - \mathcal{A}p\|_{L^\infty L^2} \\ \leq\ & C(\delta)\bigg(\left\|\bar{Z}\right\|_{L^2L^2} + \left\|e^{K_0\eta}S\right\|_{L^2L^2} + K_0\|Z\|_{L^2L^2}\bigg) \\ & +\delta\bigg(\left\|\bar{Z}\right\|_{L^\infty L^\infty} + \left\|e^{K_0\eta}S\right\|_{L^\infty L^\infty} + K_0\|Z\|_{L^\infty L^\infty}\bigg) \\ \leq\ & C(\delta)\bigg(\|Z\|_{L^2L^2} + \left\|e^{K_0\eta}S\right\|_{L^2L^2} + K_0\|Z\|_{L^2L^2}\bigg) \\ & +\delta\bigg(\|Z\|_{L^\infty L^2} + \left\|e^{K_0\eta}S\right\|_{L^\infty L^\infty} + K_0\|Z\|_{L^\infty L^\infty}\bigg). \end{aligned} \tag{4.4.227}$$

Therefore, based on Lemma 4.4.7 and (4.4.211), we can directly estimate

$$
\begin{aligned}
& \|Z\|_{L^\infty L^2} && (4.4.228)\\
\leq\ & \|\mathcal{A}p\|_{L^\infty} + C(\delta)\Big(\|Z\|_{L^2L^2} + \big\|e^{K_0\eta}S\big\|_{L^2L^2} + K_0\|Z\|_{L^2L^2}\Big) \\
& +\delta\Big(\|Z\|_{L^\infty L^2} + \big\|e^{K_0\eta}S\big\|_{L^\infty L^\infty} + K_0\|Z\|_{L^\infty L^\infty}\Big) \\
\leq\ & \|p\|_{L^\infty} + 2C(\delta)\Big(\|Z\|_{L^2L^2} + \big\|e^{K_0\eta}S\big\|_{L^2L^2}\Big) \\
& +\delta\Big(\|Z\|_{L^\infty L^2} + \big\|e^{K_0\eta}S\big\|_{L^\infty L^\infty} + K_0\|Z\|_{L^\infty L^\infty}\Big).
\end{aligned}
$$

We can take $\delta = 1/2$ to obtain

$$
\|Z\|_{L^\infty L^2} \leq C\Big(\|Z\|_{L^2L^2} + \|S\|_{L^2L^2} + \|S\|_{L^\infty L^\infty} + \|p\|_{L^\infty} + K_0\|Z\|_{L^\infty L^\infty}\Big). \tag{4.4.229}
$$

Then based on Lemma 4.4.7, Lemma 4.4.8 and Lemma 4.4.9, we can deduce

$$
\begin{aligned}
\|Z\|_{L^\infty L^\infty} \leq\ & \big\|e^{K_0\eta}\mathcal{A}p\big\|_{L^\infty} + \big\|e^{K_0\eta}\mathcal{T}S\big\|_{L^\infty L^\infty} + \big\|\bar{Z}\big\|_{L^\infty L^\infty} && (4.4.230)\\
\leq\ & \|p\|_{L^\infty} + \big\|e^{K_0\eta}S\big\|_{L^\infty L^\infty} + \big\|\bar{Z}\big\|_{L^\infty L^\infty} \\
\leq\ & \|p\|_{L^\infty} + \big\|e^{K_0\eta}S\big\|_{L^\infty L^\infty} + \|Z\|_{L^\infty L^2} \\
\leq\ & C\Big(\|Z\|_{L^2L^2} + \big\|e^{K_0\eta}S\big\|_{L^2L^2} + \big\|e^{K_0\eta}S\big\|_{L^\infty L^\infty} + \|p\|_{L^\infty} + K_0\|Z\|_{L^\infty L^\infty}\Big).
\end{aligned}
$$

Taking K_0 sufficiently small, this completes the proof of (4.4.224).

Combining (4.4.218) and (4.4.224), we deduce (4.4.217). $\square$

4.4.4 Maximum Principle

Theorem 4.4.14. *The solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.4.1) with $S=0$ satisfies the maximum principle, i.e.*

$$\min_{\sin\phi>0} h(\phi) \leq f(\eta,\phi) \leq \max_{\sin\phi>0} h(\phi). \tag{4.4.231}$$

Proof. We claim it is sufficient to show $f(\eta,\phi)\leq 0$ whenever $h(\phi)\leq 0$. Suppose this claim is justified. Denote $m=\min_{\sin\phi>0} h(\phi)$ and $M=\max_{\sin\phi>0} h(\phi)$. Then $f^1=f-M$ satisfies the equation

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f^1}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^1}{\partial\phi}+f^1-\bar{f}^1 &=& 0,\\ f^1(0,\phi) &=& h(\phi)-M \quad\text{for}\quad \sin\phi>0,\\ \lim_{\eta\rightarrow\infty} f^1(\eta,\phi) &=& f^1_{\infty}.\end{array}\right. \tag{4.4.232}$$

Hence, $h-M\leq 0$ implies $f^1\leq 0$ which is actually $f\leq M$. On the other hand, $f^2=m-f$ satisfies the equation

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f^2}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^2}{\partial\phi}+f^2-\bar{f}^2 &=& 0,\\ f^2(0,\phi) &=& m-h(\phi) \quad\text{for}\quad \sin\phi>0,\\ \lim_{\eta\rightarrow\infty} f^2(\eta,\phi) &=& f^2_{\infty}.\end{array}\right. \tag{4.4.233}$$

Thus, $m-h\leq 0$ implies $f^2\leq 0$ which further leads to $f\geq m$. Therefore, the maximum principle is established.

We now prove if $h(\phi)\leq 0$, we have $f(\eta,\phi)\leq 0$. We divide the proof into several steps:

Step 1: Penalized ϵ-Milne problem in a finite slab.

Assuming $h(\phi)\leq 0$, we then consider the penalized Milne problem for $f_{\lambda}^L(\eta,\phi)$ in the finite slab $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$

$$\left\{\begin{array}{rcl}\lambda f_{\lambda}^L+\sin\phi\dfrac{\partial f_{\lambda}^L}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f_{\lambda}^L}{\partial\phi}+f_{\lambda}^L-\bar f_{\lambda}^L&=&0,\\ f_{\lambda}^L(0,\phi)&=&h(\phi)\ \ \text{for}\ \ \sin\phi<0,\\ f_{\lambda}^L(L,\phi)&=&f_{\lambda}^L(L,R\phi).\end{array}\right. \tag{4.4.234}$$

In order to construct the solution of (4.4.234), we iteratively define the sequence $\{f_m^L\}_{m=1}^{\infty}$ as $f_0^L=0$ and

$$\left\{\begin{array}{rcl}\lambda f_{m}^L+\sin\phi\dfrac{\partial f_{m}^L}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f_{m}^L}{\partial\phi}+f_{m}^L-\bar f_{m-1}^L&=&0,\\ f_{m}^L(0,\phi)&=&h(\phi)\ \ \text{for}\ \ \sin\phi<0,\\ f_{m}^L(L,\phi)&=&f_{m}^L(L,R\phi).\end{array}\right. \tag{4.4.235}$$

Along the characteristics, it is easy to see we always have $f_m^L<0$. In the proof of Lemma 4.4.3, we have shown f_m^L converges strongly in $L^{\infty}([0,L]\times[-\pi,\pi))$ to f_{λ}^L which satisfies (4.4.234). Also, f_{λ}^L satisfies

$$\left\|f_{\lambda}^L\right\|_{L^{\infty}L^{\infty}}\leq\frac{1+\lambda}{\lambda}\left\|h\right\|_{L^{\infty}}. \tag{4.4.236}$$

Naturally, we obtain $f_{\lambda}^L\in L^2([0,L]\times[-\pi,\pi))$ and $f_{\lambda}^L\leq 0$.

Step 2: ϵ-Milne problem in a finite slab.

Consider the Milne problem for $f^L(\eta,\phi)$ in a finite slab $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f^L}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^L}{\partial\phi}+f^L-\bar f^L&=&0,\\ f^L(0,\phi)&=&h(\phi)\ \ \text{for}\ \ \sin\phi<0,\\ f^L(L,\phi)&=&f^L(L,R\phi).\end{array}\right. \tag{4.4.237}$$

In the proof of Lemma 4.4.3, we have shown f^L_λ is uniformly bounded in $L^2([0,L)\times[-\pi,\pi))$ with respect to λ, which implies we can take weakly convergent subsequence $f^L_\lambda \rightharpoonup f^L$ as $\lambda\to 0$ with $f^L\in L^2([0,L]\times[-\pi,\pi))$. Naturally, we have $f^L(\eta,\phi)\leq 0$.

Step 3: ϵ-Milne problem in an infinite slab.

Finally, in the proof of Lemma 4.4.4, by taking $L\to\infty$, we have

$$f^L \rightharpoonup f \quad in \quad L^2_{loc}([0,L)\times[-\pi,\pi), \tag{4.4.238}$$

where f satisfies (4.4.1). Certainly, we have $f(\eta,\phi)\leq 0$. This justifies the claim in Step 1. Hence, we complete the proof. □

Remark 4.4.15. *Note that when $F=0$, then all the previous proofs can be recovered and Theorem 4.4.12, Theorem 4.4.13 and Theorem 4.4.14 still hold. Hence, we can deduce the well-posedness, decay and maximum principle of the classical Milne problem*

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f}{\partial\eta}+f-\bar f &=& S(\eta,\phi),\\ f(0,\phi) &=& h(\phi)\quad for\quad \sin\phi>0,\\ \lim_{\eta\to\infty}f(\eta,\phi) &=& f_\infty.\end{array}\right. \tag{4.4.239}$$

4.5 Proof of Theorem 4.1.1

We divide the proof into several steps:

Step 1: Remainder definitions.

We may rewrite the asymptotic expansion as follows:

$$u^\epsilon \sim \sum_{k=0}^{\infty}\epsilon^k U_k^\epsilon + \sum_{k=0}^{\infty}\epsilon^k \mathscr{U}_k^\epsilon. \tag{4.5.1}$$

The remainder can be defined as

$$R_N = u^\epsilon - \sum_{k=0}^{N}\epsilon^k U_k^\epsilon - \sum_{k=0}^{N}\epsilon^k \mathscr{U}_k^\epsilon = u^\epsilon - Q_N - \mathscr{Q}_N, \tag{4.5.2}$$

where

$$Q_N = \sum_{k=0}^{N}\epsilon^k U_k^\epsilon, \tag{4.5.3}$$

$$\mathscr{Q}_N = \sum_{k=0}^{N}\epsilon^k \mathscr{U}_k^\epsilon. \tag{4.5.4}$$

Noting the equation (4.2.38) is equivalent to the equation (4.1.1), we write $\mathcal{L}$ to denote the neutron transport operator as follows:

$$\begin{aligned}\mathcal{L}u &= \epsilon\vec{w}\cdot\nabla_x u + u - \bar{u} \qquad (4.5.5)\\ &= \sin\phi\frac{\partial u}{\partial\eta} - \frac{\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial u}{\partial\phi}+\frac{\partial u}{\partial\theta}\right) + u - \bar{u}.\end{aligned}$$

Step 2: Estimates of $\mathcal{L}Q_N$.

The interior contribution can be estimated as

$$\mathcal{L}Q_0 = \epsilon\vec{w}\cdot\nabla_x Q_0 + Q_0 - \bar{Q}_0 = \epsilon\vec{w}\cdot\nabla_x U_0^\epsilon + (U_0^\epsilon - \bar{U}_0^\epsilon) = \epsilon\vec{w}\cdot\nabla_x U_0^\epsilon. \tag{4.5.6}$$

We have

$$\left|\epsilon\vec{w}\cdot\nabla_x U_0^\epsilon\right| \leq C\epsilon\left|\nabla_x U_0^\epsilon\right| \leq C\epsilon. \tag{4.5.7}$$

This implies

$$|\mathcal{L}Q_0| \leq C\epsilon. \tag{4.5.8}$$

Similarly, for higher order term, we can estimate

$$\mathcal{L}Q_N = \epsilon\vec{w}\cdot\nabla_x Q_N + Q_N - \bar{Q}_N \quad = \quad \epsilon^{N+1}\vec{w}\cdot\nabla_x U_N^{\epsilon}. \tag{4.5.9}$$

We have

$$\left|\epsilon^{N+1}\vec{w}\cdot\nabla_x U_N^{\epsilon}\right| \leq C\epsilon^{N+1}\left|\nabla_x U_N^{\epsilon}\right| \leq C\epsilon^{N+1}. \tag{4.5.10}$$

This implies

$$|\mathcal{L}Q_N| \leq C\epsilon^{N+1}. \tag{4.5.11}$$

Step 3: Estimates of $\mathcal{L}\mathscr{Q}_N$.

The boundary layer solution is $\mathscr{U}_k^{\epsilon} = (f_k^{\epsilon} - f_k^{\epsilon}(\infty))\cdot\psi_0 = \mathscr{V}_k\psi_0$ where $f_k^{\epsilon}(\eta,\theta,\phi)$ solves the ϵ-Milne problem and $\mathscr{V}_k = f_k^{\epsilon} - f_k^{\epsilon}(\infty)$. Notice $\psi_0\psi = \psi_0$, so the boundary layer contribution can be estimated as

$$
\begin{aligned}
& \mathcal{L}\mathscr{Q}_0 && (4.5.12)\\
= \;& \sin\phi\frac{\partial \mathscr{Q}_0}{\partial\eta} - \frac{\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial \mathscr{Q}_0}{\partial\phi} + \frac{\partial \mathscr{Q}_0}{\partial\theta}\right) + \mathscr{Q}_0 - \bar{\mathscr{Q}}_0\\
= \;& \sin\phi\left(\psi_0\frac{\partial \mathscr{V}_0}{\partial\eta} + \mathscr{V}_0\frac{\partial \psi_0}{\partial\eta}\right) - \frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial \mathscr{V}_0}{\partial\phi} + \frac{\partial \mathscr{V}_0}{\partial\theta}\right) + \psi_0\mathscr{V}_0 - \psi_0\bar{\mathscr{V}}_0\\
= \;& \sin\phi\left(\psi_0\frac{\partial \mathscr{V}_0}{\partial\eta} + \mathscr{V}_0\frac{\partial \psi_0}{\partial\eta}\right) - \frac{\psi_0\psi\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial \mathscr{V}_0}{\partial\phi} + \frac{\partial \mathscr{V}_0}{\partial\theta}\right) + \psi_0\mathscr{V}_0 - \psi_0\bar{\mathscr{V}}_0\\
= \;& \psi_0\left(\sin\phi\frac{\partial \mathscr{V}_0}{\partial\eta} - \frac{\epsilon\psi}{1-\epsilon\eta}\cos\phi\frac{\partial \mathscr{V}_0}{\partial\phi} + \mathscr{V}_0 - \bar{\mathscr{V}}_0\right) + \sin\phi\frac{\partial \psi_0}{\partial\eta}\mathscr{V}_0 - \frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial \mathscr{V}_0}{\partial\theta}\\
= \;& \sin\phi\frac{\partial \psi_0}{\partial\eta}\mathscr{V}_0 - \frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial \mathscr{V}_0}{\partial\theta}.
\end{aligned}
$$

Since $\psi_0 = 1$ when $\eta \leq 1/(4\epsilon)$, the effective region of $\partial_\eta\psi_0$ is $\eta \geq 1/(4\epsilon)$ which is further and further from the origin as $\epsilon \to 0$. By Theorem 4.4.13, the first term in (4.5.12) can be controlled as

$$
\left|\sin\phi\frac{\partial \psi_0}{\partial\eta}\mathscr{V}_0\right| \;\leq\; Ce^{-\frac{K_0}{\epsilon}} \leq C\epsilon. \tag{4.5.13}
$$

For the second term in (4.5.12), we have

$$
\left|-\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial \mathscr{V}_0}{\partial\theta}\right| \;\leq\; C\epsilon\left|\frac{\partial \mathscr{V}_0}{\partial\theta}\right| \leq C\epsilon. \tag{4.5.14}
$$

This implies

$$
|\mathcal{L}\mathscr{Q}_0| \leq C\epsilon. \tag{4.5.15}
$$

Similarly, for higher order term, we can estimate

$$\begin{aligned} \mathcal{L}\mathscr{Q}_N &= \sin\phi\frac{\partial\mathscr{Q}_N}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\bigg(\frac{\partial\mathscr{Q}_N}{\partial\phi}+\frac{\partial\mathscr{Q}_N}{\partial\theta}\bigg)+\mathscr{Q}_N-\bar{\mathscr{Q}}_N \\ &= \sum_{i=0}^{k}\epsilon^i\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_i-\frac{\psi_0\epsilon^{k+1}}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_k}{\partial\theta}. \end{aligned} \tag{4.5.16}$$

Away from the origin, the first term in (4.5.16) can be controlled as

$$\left|\sum_{i=0}^{k}\epsilon^i\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_i\right| \leq Ce^{-\frac{K_0}{\epsilon}}\leq C\epsilon^{k+1}. \tag{4.5.17}$$

For the second term in (4.5.16), we have

$$\left|-\frac{\psi_0\epsilon^{k+1}}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_k}{\partial\theta}\right| \leq C\epsilon^{k+1}\left|\frac{\partial\mathscr{V}_k}{\partial\theta}\right|\leq C\epsilon^{k+1}. \tag{4.5.18}$$

This implies

$$|\mathcal{L}\mathscr{Q}_N|\leq C\epsilon^{k+1}. \tag{4.5.19}$$

Step 4: Proof of (4.1.7).

In summary, since $\mathcal{L}u^\epsilon=0$, collecting (4.5.2), (4.5.11) and (4.5.19), we can prove

$$|\mathcal{L}R_N|\leq C\epsilon^{N+1}. \tag{4.5.20}$$

Consider the asymptotic expansion to $N=3$, then the remainder R_3 satisfies the equation

$$\left\{\begin{aligned} \epsilon\vec{w}\cdot\nabla_xR_3+R_3-\bar{R}_3 &= \mathcal{L}R_3 \ \text{ for } \ \vec{x}\in\Omega,\\ R_3(\vec{x}_0,\vec{w}) &= 0 \ \text{ for } \ \vec{w}\cdot\vec{n}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega. \end{aligned}\right. \tag{4.5.21}$$

By Theorem 4.3.6, we have

$$\|R_3\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \frac{C(\Omega)}{\epsilon^3}\|\mathcal{L}R_3\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \frac{C(\Omega)}{\epsilon^3}(C\epsilon^4) = C(\Omega)\epsilon. \tag{4.5.22}$$

Hence, we have

$$\left\| u^\epsilon - \sum_{k=0}^{3}\epsilon^k U_k^\epsilon - \sum_{k=0}^{3}\epsilon^k \mathscr{U}_k^\epsilon \right\|_{L^\infty(\Omega\times\mathcal{S}^1)} = O(\epsilon). \tag{4.5.23}$$

Since it is easy to see

$$\left\| \sum_{k=1}^{3}\epsilon^k U_k^\epsilon + \sum_{k=1}^{3}\epsilon^k \mathscr{U}_k^\epsilon \right\|_{L^\infty(\Omega\times\mathcal{S}^1)} = O(\epsilon), \tag{4.5.24}$$

our result naturally follows. This completes the proof of (4.1.7).

Step 5: Basic settings to show (4.1.8).

By (4.2.25), the solution f_0 satisfies the Milne problem

$$\left\{ \begin{array}{rcl} \sin(\theta+\xi)\dfrac{\partial f_0}{\partial\eta} + f_0 - \bar{f}_0 &=& 0,\\ f_0(0,\theta,\xi) &=& g(\theta,\xi) \quad \text{for} \quad \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty} f_0(\eta,\theta,\xi) &=& f_0(\infty,\theta). \end{array} \right. \tag{4.5.25}$$

For convenience of comparison, we make the substitution $\phi = \theta+\xi$ to obtain

$$\left\{ \begin{array}{rcl} \sin\phi\dfrac{\partial f_0}{\partial\eta} + f_0 - \bar{f}_0 &=& 0,\\ f_0(0,\theta,\phi) &=& g(\theta,\phi) \quad \text{for} \quad \sin\phi>0,\\ \lim_{\eta\to\infty} f_0(\eta,\theta,\phi) &=& f_0(\infty,\theta). \end{array} \right. \tag{4.5.26}$$

Assume (4.1.8) is incorrect, such that

$$\lim_{\epsilon\to 0} \|(U_0 + \mathscr{U}_0) - (U_0^\epsilon + \mathscr{U}_0^\epsilon)\|_{L^\infty} = 0. \tag{4.5.27}$$

Since the boundary $g(\phi) = \cos\phi$ independent of θ, by (4.2.25) and (4.2.50), it is obvious the limit of zeroth order boundary layer $f_0(\infty,\theta)$ and $f_0^\epsilon(\infty,\theta)$ satisfy $f_0(\infty,\theta) = C_1$ and $f_0^\epsilon(\infty,\theta) = C_2$ for some constant C_1 and C_2 independent of θ. By (4.2.26) and (4.2.51), we can derive the interior solutions are indeed constants $U_0 = C_1$ and $U_0^\epsilon = C_2$. Hence, we may further derive

$$\lim_{\epsilon\to 0} \|(f_0(\infty) + \mathscr{U}_0) - (f_0^\epsilon(\infty) + \mathscr{U}_0^\epsilon)\|_{L^\infty} = 0. \tag{4.5.28}$$

For $0 \le \eta \le 1/(2\epsilon)$, we have $\psi_0 = 1$, which means $f_0 = \mathscr{U}_0 + f_0(\infty)$ and $f_0^\epsilon = \mathscr{U}_0^\epsilon + f_0^\epsilon(\infty)$ on $[0, 1/(2\epsilon)]$. Define $u = f_0 + 2$, $U = f_0^\epsilon + 2$ and $G = g + 2 = \cos\phi + 2$, then $u(\eta,\phi)$ satisfies the equation

$$\left\{\begin{array}{rcl} \sin\phi\dfrac{\partial u}{\partial\eta} + u - \bar{u} &=& 0, \\ u(0,\phi) &=& G(\phi) \quad \text{for} \quad \sin\phi > 0, \\ \lim_{\eta\to\infty} u(\eta,\phi) &=& 2 + f_0(\infty), \end{array}\right. \tag{4.5.29}$$

and $U(\eta,\phi)$ satisfies the equation

$$\left\{\begin{array}{rcl} \sin\phi\dfrac{\partial U}{\partial\eta} + F(\epsilon;\eta)\cos\phi\dfrac{\partial U}{\partial\phi} + U - \bar{U} &=& 0, \\ U(0,\phi) &=& G(\phi) \quad \text{for} \quad \sin\phi > 0, \\ \lim_{\eta\to\infty} U(\eta,\phi) &=& 2 + f_0^\epsilon(\infty). \end{array}\right. \tag{4.5.30}$$

Based on (4.5.28), we have

$$\lim_{\epsilon\to 0} \|U(\eta,\phi) - u(\eta,\phi)\|_{L^\infty} = 0. \tag{4.5.31}$$

Then it naturally implies

$$\lim_{\epsilon\to 0}\left\|\bar{U}(\eta)-\bar{u}(\eta)\right\|_{L^\infty}=0. \tag{4.5.32}$$

Step 6: Continuity of $\bar{u}$ and $\bar{U}$ at $\eta=0$.

For the problem (4.5.29), we have for any $r_0>0$

$$|\bar{u}(\eta)-\bar{u}(0)| \leq \frac{1}{2\pi}\bigg(\int_{\sin\phi\leq r_0}|u(\eta,\phi)-u(0,\phi)|\,\mathrm{d}\phi+\int_{\sin\phi\geq r_0}|u(\eta,\phi)-u(0,\phi)|\,\mathrm{d}\phi\bigg). \tag{4.5.33}$$

Since we have shown $u\in L^\infty([0,\infty)\times[-\pi,\pi))$, then for any $\delta>0$, we can take r_0 sufficiently small such that

$$\frac{1}{2\pi}\int_{\sin\phi\leq r_0}|u(\eta,\phi)-u(0,\phi)|\,\mathrm{d}\phi \leq \frac{C}{2\pi}\arcsin r_0\leq\frac{\delta}{2}. \tag{4.5.34}$$

For fixed r_0 satisfying above requirement, we estimate the integral on $\sin\phi\geq r_0$. By Ukai's trace theorem, $u(0,\phi)$ is well-defined in the domain $\sin\phi\geq r_0$ and is continuous. Also, by consider the relation

$$\frac{\partial u}{\partial\eta}(0,\phi)=\frac{\bar{u}(0)-u(0,\phi)}{\sin\phi}, \tag{4.5.35}$$

we can obtain in this domain $\partial_\eta u$ is bounded, which further implies $u(\eta,\phi)$ is uniformly continuous at $\eta=0$. Then there exists $\delta_0>0$ sufficiently small, such that for any $0\leq\eta\leq\delta_0$, we have

$$\frac{1}{2\pi}\int_{\sin\phi\geq r_0}|u(\eta,\phi)-u(0,\phi)|\,\mathrm{d}\phi \leq \frac{1}{2\pi}\int_{\sin\phi\geq r_0}\frac{\delta}{2}\mathrm{d}\phi\leq\frac{\delta}{2}. \tag{4.5.36}$$

In summary, we have shown for any $\delta > 0$, there exists $\delta_0 > 0$ such that for any $0 \leq \eta \leq \delta_0$,

$$|\bar{u}(\eta) - \bar{u}(0)| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{4.5.37}$$

Hence, $\bar{u}(\eta)$ is continuous at $\eta = 0$. By a similar argument along the characteristics, we can show $\bar{U}(\eta, \phi)$ is also continuous at $\eta = 0$.

In the following, by the continuity, we assume for arbitrary $\delta > 0$, there exists a $\delta_0 > 0$ such that for any $0 \leq \eta \leq \delta_0$, we have

$$\begin{aligned} |\bar{u}(\eta) - \bar{u}(0)| &\leq \delta, & (4.5.38)\\ \left|\bar{U}(\eta) - \bar{U}(0)\right| &\leq \delta. & (4.5.39) \end{aligned}$$

Step 7: Milne formulation.

We consider the solution at a specific point $(\eta, \phi) = (n\epsilon, \epsilon)$ for some fixed $n > 0$. The solution along the characteristics can be rewritten as follows:

$$u(n\epsilon, \epsilon) = G(\epsilon)e^{-\frac{1}{\sin\epsilon}n\epsilon} + \int_0^{n\epsilon} e^{-\frac{1}{\sin\epsilon}(n\epsilon-\kappa)}\frac{1}{\sin\epsilon}\bar{u}(\kappa)\mathrm{d}\kappa, \tag{4.5.40}$$

$$U(n\epsilon, \epsilon) = G(\epsilon_0)e^{-\int_0^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta} + \int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta}\frac{1}{\sin\phi(\kappa)}\bar{U}(\kappa)\mathrm{d}\kappa, \tag{4.5.41}$$

where we have the conserved energy along the characteristics

$$E(\eta, \phi) = \cos\phi e^{-V(\eta)}, \tag{4.5.42}$$

in which $(0, \epsilon_0)$ and $(\zeta, \phi(\zeta))$ are in the same characteristics of $(n\epsilon, \epsilon)$.

Step 8: Estimates of (4.5.40).

We turn to the Milne problem for u. We have the natural estimate

$$\begin{aligned}\int_0^{n\epsilon} e^{-\frac{1}{\sin\epsilon}(n\epsilon-\kappa)}\frac{1}{\sin\epsilon}\mathrm{d}\kappa &= \int_0^{n\epsilon} e^{-\frac{1}{\epsilon}(n\epsilon-\kappa)}\frac{1}{\epsilon}\mathrm{d}\kappa + o(\epsilon) \\ &= e^{-n}\int_0^{n\epsilon} e^{\frac{\kappa}{\epsilon}}\frac{1}{\epsilon}\mathrm{d}\kappa + o(\epsilon) \\ &= e^{-n}\int_0^{n} e^{\zeta}\mathrm{d}\zeta + o(\epsilon) \\ &= (1-e^{-n}) + o(\epsilon).\end{aligned} \tag{4.5.43}$$

Then for $0 < \epsilon \leq \delta_0$, we have $|\bar{u}(0) - \bar{u}(\kappa)| \leq \delta$, which implies

$$\begin{aligned}\int_0^{n\epsilon} e^{-\frac{1}{\sin\epsilon}(n\epsilon-\kappa)}\frac{1}{\sin\epsilon}\bar{u}(\kappa)\mathrm{d}\kappa &= \int_0^{n\epsilon} e^{-\frac{1}{\sin\epsilon}(n\epsilon-\kappa)}\frac{1}{\sin\epsilon}\bar{u}(0)\mathrm{d}\kappa + O(\delta) \\ &= (1-e^{-n})\bar{u}(0) + o(\epsilon) + O(\delta).\end{aligned} \tag{4.5.44}$$

For the boundary data term, it is easy to see

$$G(\epsilon)e^{-\frac{1}{\sin\epsilon}n\epsilon} = e^{-n}G(\epsilon) + o(\epsilon) \tag{4.5.45}$$

In summary, we have

$$u(n\epsilon,\epsilon) = (1-e^{-n})\bar{u}(0) + e^{-n}G(\epsilon) + o(\epsilon) + O(\delta). \tag{4.5.46}$$

Step 9: Estimates of (4.5.41).

We consider the ϵ-Milne problem for U. For $\epsilon << 1$ sufficiently small, $\psi(\epsilon) = 1$. Then we may estimate

$$\cos\phi(\zeta)e^{-V(\zeta)} = \cos\epsilon e^{-V(n\epsilon)}, \tag{4.5.47}$$

which implies

$$\cos\phi(\zeta) = \frac{1 - n\epsilon^2}{1 - \epsilon\zeta}\cos\epsilon. \tag{4.5.48}$$

and hence

$$\sin\phi(\zeta) = \sqrt{1 - \cos^2\phi(\zeta)} = \sqrt{\frac{\epsilon(n\epsilon - \zeta)(2 - \epsilon\zeta - n\epsilon^2)}{(1 - \epsilon\zeta)^2}\cos^2\epsilon + \sin^2\epsilon}. \tag{4.5.49}$$

For $\zeta \in [0, \epsilon]$ and $n\epsilon$ sufficiently small, by Taylor's expansion, we have

$$\begin{aligned} 1 - \epsilon\zeta &= 1 + o(\epsilon), && (4.5.50)\\ 2 - \epsilon\zeta - n\epsilon^2 &= 2 + o(\epsilon), && (4.5.51)\\ \sin^2\epsilon &= \epsilon^2 + o(\epsilon^3), && (4.5.52)\\ \cos^2\epsilon &= 1 - \epsilon^2 + o(\epsilon^3). && (4.5.53) \end{aligned}$$

Hence, we have

$$\sin\phi(\zeta) = \sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)} + o(\epsilon^2). \tag{4.5.54}$$

Since $\sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)} = O(\epsilon)$, we can further estimate

$$\begin{aligned} \frac{1}{\sin\phi(\zeta)} &= \frac{1}{\sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)}} + o(1) && (4.5.55)\\ & && (4.5.56)\\ -\int_{\kappa}^{n\epsilon} \frac{1}{\sin\phi(\zeta)} \mathrm{d}\zeta &= \sqrt{\frac{\epsilon + 2n\epsilon - 2\zeta}{\epsilon}}\Bigg|_{\kappa}^{n\epsilon} + o(\epsilon) = 1 - \sqrt{\frac{\epsilon + 2n\epsilon - 2\kappa}{\epsilon}} + o(\epsilon). \end{aligned}$$

Then we can easily derive the integral estimate

$$\begin{aligned}
& \int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta}\frac{1}{\sin\phi(\kappa)}\mathrm{d}\kappa \\
= \; & e^1\int_0^{n\epsilon} e^{-\sqrt{\frac{\epsilon+2n\epsilon-2\kappa}{\epsilon}}}\frac{1}{\sqrt{\epsilon(\epsilon+2n\epsilon-2\kappa)}}\mathrm{d}\kappa + o(\epsilon) \\
= \; & \frac{1}{2}e^1\int_\epsilon^{(1+2n)\epsilon} e^{-\sqrt{\frac{\sigma}{\epsilon}}}\frac{1}{\sqrt{\epsilon\sigma}}\mathrm{d}\sigma + o(\epsilon) \\
= \; & \frac{1}{2}e^1\int_1^{1+2n} e^{-\sqrt{\rho}}\frac{1}{\sqrt{\rho}}\mathrm{d}\rho + o(\epsilon) \\
= \; & e^1\int_1^{\sqrt{1+2n}} e^{-t}\mathrm{dt} + o(\epsilon) \\
= \; & (1-e^{1-\sqrt{1+2n}}) + o(\epsilon).
\end{aligned} \tag{4.5.57}$$

Then for $0<\epsilon\leq\delta_0$, we have $\left|\bar{U}(0)-\bar{U}(\kappa)\right|\leq\delta$, which implies

$$\begin{aligned}
& \int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta}\frac{1}{\sin\phi(\kappa)}\bar{U}(\kappa)\mathrm{d}\kappa \\
= \; & \int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta}\frac{1}{\sin\phi(\kappa)}\bar{U}(0)\mathrm{d}\kappa + O(\delta) \\
= \; & (1-e^{1-\sqrt{1+2n}})\bar{U}(0) + o(\epsilon) + O(\delta).
\end{aligned} \tag{4.5.58}$$

For the boundary data term, since $G(\phi)$ is C^1, a similar argument shows

$$G(\epsilon_0)e^{-\int_0^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta} \;=\; e^{1-\sqrt{1+2n}}G(\sqrt{1+2n}\epsilon) + o(\epsilon). \tag{4.5.59}$$

Therefore, we have

$$U(n\epsilon,\epsilon) = (1-e^{1-\sqrt{1+2n}})\bar{U}(0) + e^{1-\sqrt{1+2n}}G(\sqrt{1+2n}\epsilon) + o(\epsilon) + O(\delta). \tag{4.5.60}$$

Step 10: Proof of (4.1.8).

In summary, we have the estimate

$$u(n\epsilon,\epsilon) = (1-e^{-n})\bar{u}(0) + e^{-n}G(\epsilon) + o(\epsilon) + O(\delta), \tag{4.5.61}$$

$$U(n\epsilon,\epsilon) = (1-e^{1-\sqrt{1+2n}})\bar{U}(0) + e^{1-\sqrt{1+2n}}G(\sqrt{1+2n}\epsilon) + o(\epsilon) + O(\delta). \tag{4.5.62}$$

The boundary data is $G = \cos\phi + 2$. Then by the maximum principle in Theorem 4.4.14, we can achieve $1 \leq u(0,\phi) \leq 3$ and $1 \leq U(0,\phi) \leq 3$. Since

$$\begin{aligned} \bar{u}(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(0,\phi)\mathrm{d}\phi = \frac{1}{2\pi}\int_{\sin\phi>0} u(0,\phi)\mathrm{d}\phi + \frac{1}{2\pi}\int_{\sin\phi<0} u(0,\phi)\mathrm{d}\phi \\ &= \frac{1}{2\pi}\int_{\sin\phi>0} (2+\cos\phi)\mathrm{d}\phi + \frac{1}{2\pi}\int_{\sin\phi<0} u(0,\phi)\mathrm{d}\phi \\ &= 1 + \frac{1}{2\pi}\int_{\sin\phi<0} u(0,\phi)\mathrm{d}\phi, \end{aligned} \tag{4.5.63}$$

we naturally obtain $3/2 \leq \bar{u}(0) \leq 5/2$. Similarly, we can obtain $3/2 \leq \bar{U}(0) \leq 5/2$. Furthermore, for ϵ sufficiently small, we have

$$G(\sqrt{1+2n}\epsilon) = 3 + o(\epsilon), \tag{4.5.64}$$

$$G(\epsilon) = 3 + o(\epsilon). \tag{4.5.65}$$

Hence, we can obtain

$$u(n\epsilon,\epsilon) = \bar{u}(0) + e^{-n}(-\bar{u}(0)+3) + o(\epsilon) + O(\delta), \tag{4.5.66}$$

$$U(n\epsilon,\epsilon) = \bar{U}(0) + e^{1-\sqrt{1+2n}}(-\bar{U}(0)+3) + o(\epsilon) + O(\delta). \tag{4.5.67}$$

Then we can see $\lim_{\epsilon\to 0}\left\|\bar{U}(0) - \bar{u}(0)\right\|_{L^\infty} = 0$ naturally leads to $\lim_{\epsilon\to 0}\left\|(-\bar{u}(0)+3) - (-\bar{U}(0)+3)\right\|_{L^\infty} = 0$. Also, we have $-\bar{u}(0)+3 = O(1)$ and

$-\bar{U}(0)+3=O(1)$. Due to the smallness of ϵ and δ, and also $e^{-n}\neq e^{1-\sqrt{1+2n}}$, we can obtain

$$|U(n\epsilon,\epsilon)-u(n\epsilon,\epsilon)|=O(1). \tag{4.5.68}$$

However, above result contradicts our assumption that $\lim_{\epsilon\to 0}\|U(\eta,\phi)-u(\eta,\phi)\|_{L^\infty}=0$ for any (η,ϕ). This completes the proof of (4.1.8).

4.6 Neutron Transport Equation with Diffusive Boundary

4.6.1 Problem Settings

We consider the steady neutron transport equation in a two-dimensional unit disk with diffusive boundary. In the space domain $\Omega=\{\vec{x}: \ |\vec{x}|\leq 1\}$ and the velocity domain $\Sigma=\{\vec{w}: \ \vec{w}\in\mathcal{S}^1\}$, the neutron density $u^\epsilon(\vec{x},\vec{w})$ satisfies

$$\left\{\begin{array}{rcl}\epsilon\vec{w}\cdot\nabla_x u^\epsilon+(1+\epsilon^2)u^\epsilon-\bar{u}^\epsilon &=& 0\ \text{ in }\ \Omega,\\ u^\epsilon(\vec{x}_0,\vec{w}) &=& \mathcal{P}u^\epsilon(\vec{x}_0)+\epsilon g(\vec{x}_0,\vec{w})\\ && \text{for }\ \vec{w}\cdot\vec{n}<0\ \text{ and }\ \vec{x}_0\in\partial\Omega,\end{array}\right. \tag{4.6.1}$$

where

$$\bar{u}^\epsilon(\vec{x})=\frac{1}{2\pi}\int_{\mathcal{S}^1}u^\epsilon(\vec{x},\vec{w})\mathrm{d}\tilde{\mathrm{w}}, \tag{4.6.2}$$

$$\mathcal{P}u^{\epsilon}(\vec{x}_0) = \frac{1}{2}\int_{\vec{w}\cdot\vec{n}>0} u^{\epsilon}(\vec{x}_0, \vec{w})(\vec{w}\cdot\vec{n})\mathrm{d}\tilde{\mathrm{w}}, \tag{4.6.3}$$

with the Knudsen number $0 < \epsilon << 1$.

4.6.2 Interior Expansion

We define the interior expansion as follows:

$$U(\vec{x}, \vec{w}) \sim \sum_{k=0}^{\infty} \epsilon^k U_k(\vec{x}, \vec{w}), \tag{4.6.4}$$

where U_k can be defined by comparing the order of ϵ by plugging (4.6.4) into the equation (4.6.1). Thus we have

$$\begin{aligned}
U_0 - \bar{U}_0 &= 0, && (4.6.5)\\
U_1 - \bar{U}_1 &= -\vec{w}\cdot\nabla_x U_0, && (4.6.6)\\
U_2 - \bar{U}_2 &= -\vec{w}\cdot\nabla_x U_1 - U_0, && (4.6.7)\\
&\cdots\\
U_k - \bar{U}_k &= -\vec{w}\cdot\nabla_x U_{k-1} - U_{k-2}. && (4.6.8)
\end{aligned}$$

The following analysis reveals the equation satisfied by U_k:

Plugging (4.6.5) into (4.6.6), we obtain

$$U_1 = \bar{U}_1 - \vec{w}\cdot\nabla_x \bar{U}_0. \tag{4.6.9}$$

Plugging (4.6.9) into (4.6.7), we get

$$\begin{aligned} U_2-\bar U_2 &= -\vec w\cdot\nabla_x(\bar U_1-\vec w\cdot\nabla_x\bar U_0) \\ &= -\vec w\cdot\nabla_x\bar U_1+\vec w^2\Delta_x\bar U_0+2w_xw_y\partial_{xy}\bar U_0-U_0. \end{aligned} \tag{4.6.10}$$

Integrating (4.6.10) over $\vec w\in\mathcal{S}^1$, we achieve the final form

$$\Delta_x\bar U_0-\bar U_0=0. \tag{4.6.11}$$

which further implies $U_0(\vec x,\vec w)$ satisfies the equation

$$\left\{\begin{aligned} U_0 &= \bar U_0, \\ \Delta_x\bar U_0-\bar U_0 &= 0. \end{aligned}\right. \tag{4.6.12}$$

In a similar fashion, U_1 satisfies

$$\left\{\begin{aligned} U_1 &= \bar U_k-\vec w\cdot\nabla_x U_0, \\ \Delta_x\bar U_1-\bar U_1 &= -\int_{\mathcal{S}^1}\vec w\cdot\nabla_x U_0\mathrm{d}\tilde{\mathrm{w}}. \end{aligned}\right. \tag{4.6.13}$$

Also, $U_k(\vec x,\vec w)$ for $k\geq 2$ satisfies

$$\left\{\begin{aligned} U_k &= \bar U_k-\vec w\cdot\nabla_x U_{k-1}-U_{k-2}, \\ \Delta_x\bar U_k-\bar U_k &= -\int_{\mathcal{S}^1}\vec w\cdot\nabla_x U_{k-1}\mathrm{d}\tilde{\mathrm{w}}-\int_{\mathcal{S}^1}U_{k-2}\mathrm{d}\tilde{\mathrm{w}}. \end{aligned}\right. \tag{4.6.14}$$

4.6.3 Milne Expansion

In order to determine the boundary condition for U_k, it is well known that we need to define the boundary layer expansion. We still perform the substitutions (4.2.11), (4.2.13) and (4.2.16) to the equation (4.6.1) to obtain the form

$$\begin{cases} \sin(\theta+\xi)\dfrac{\partial u^\epsilon}{\partial\eta} - \dfrac{\epsilon}{1-\epsilon\eta}\cos(\theta+\xi)\dfrac{\partial u^\epsilon}{\partial\theta} + (1+\epsilon^2)u^\epsilon - \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} u^\epsilon \mathrm{d}\xi = 0, \\ u^\epsilon(0,\theta,\xi) = \mathcal{P}u^\epsilon(0,\theta) + \epsilon g(\theta,\xi) \quad \text{for} \quad \sin(\theta+\xi) > 0, \end{cases} \tag{4.6.15}$$

where

$$\mathcal{P}u^\epsilon(0,\theta) = -\frac{1}{2}\int_{\sin(\theta+\xi)<0} u^\epsilon(0,\theta,\xi)\sin(\theta+\xi)\mathrm{d}\xi. \tag{4.6.16}$$

Define the Milne expansion of boundary layer as follows:

$$\mathscr{U}(\eta,\theta,\phi) \sim \sum_{k=0}^{\infty} \epsilon^k \mathscr{U}_k(\eta,\theta,\phi) \tag{4.6.17}$$

where $\mathscr{U}_k$ is defined by comparing the order of ϵ via plugging (4.6.17) into the equation (4.6.15). Thus, in a neighborhood of the boundary, we have

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_0}{\partial\eta} + \mathscr{U}_0 - \bar{\mathscr{U}}_0 = 0, \tag{4.6.18}$$

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_1}{\partial\eta} + \mathscr{U}_1 - \bar{\mathscr{U}}_1 = \frac{1}{1-\epsilon\eta}\cos(\theta+\xi)\frac{\partial \mathscr{U}_0}{\partial\theta}, \tag{4.6.19}$$

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_2}{\partial\eta} + \mathscr{U}_2 - \bar{\mathscr{U}}_2 = \frac{1}{1-\epsilon\eta}\cos(\theta+\xi)\frac{\partial \mathscr{U}_1}{\partial\theta} - \mathscr{U}_0, \tag{4.6.20}$$

$$\ldots$$

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_k}{\partial\eta} + \mathscr{U}_k - \bar{\mathscr{U}}_k = \frac{1}{1-\epsilon\eta}\cos(\theta+\xi)\frac{\partial \mathscr{U}_{k-1}}{\partial\theta} - \mathscr{U}_{k-2}, \tag{4.6.21}$$

where

$$\bar{\mathscr{U}}_k(\eta,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{U}_k(\eta,\theta,\xi)\mathrm{d}\xi. \tag{4.6.22}$$

The construction of U_k and $\mathscr{U}_k$ in [9] can be summarized as follows:

Step 1: Construction of $\mathscr{U}_0$.

Define ψ and ψ_0 as (4.2.23) and (4.2.24). Then the zeroth order boundary layer solution can be defined as

$$\left\{\begin{array}{rcl}\mathscr{U}_0 &=& \psi_0(\epsilon\eta)\bigg(f_0(\eta,\theta,\xi)-f_0(\infty,\theta)\bigg),\\ \sin(\theta+\xi)\dfrac{\partial f_0}{\partial\eta}+f_0-\bar{f}_0 &=& 0,\\ f_0(0,\theta,\xi) &=& \mathcal{P}f_0(0,\theta)\ \ \text{for}\ \ \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty}f_0(\eta,\theta,\xi) &=& f_0(\infty,\theta),\end{array}\right. \tag{4.6.23}$$

with

$$\mathcal{P}f_0(0,\theta)=0. \tag{4.6.24}$$

It is easy to see $\mathscr{U}_0=f_0=0$.

Step 2: Construction of $\mathscr{U}_1$ and U_0.

Define the first order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_1 &=& \psi_0(\epsilon\eta)\bigg(f_1(\eta,\theta,\xi)-f_1(\infty,\theta)\bigg),\\ \sin(\theta+\xi)\dfrac{\partial f_1}{\partial\eta}+f_1-\bar{f}_1 &=& \cos(\theta+\xi)\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}_0}{\partial\theta},\\ f_1(0,\theta,\xi) &=& \mathcal{P}f_1(0,\theta)+g_1(\theta,\xi)\ \ \text{for}\ \ \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty}f_1(\eta,\theta,\xi) &=& f_1(\infty,\theta),\end{array}\right. \tag{4.6.25}$$

with

$$\mathcal{P}f_1(0,\theta)=0, \tag{4.6.26}$$

where $g_1=\vec{w}\cdot\nabla_x U_0-\mathcal{P}(\vec{w}\cdot\nabla_x U_0)+g$ in which $(\vec{x}_0,\vec{w})$ and $(0,\theta,\xi)$ denote the same point. Since $\mathscr{U}_0=0$, we can obtain a unique $f_1(\eta,\theta,\xi)\in L^{\infty}([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$. Hence, $\mathscr{U}_1$ is well-defined. By the compatibility condition (4.6.99), we can define the zeroth order interior solution as

$$\left\{\begin{array}{rcl}U_0&=&\bar{U}_0,\\ \Delta_x\bar{U}_0-\bar{U}_0&=&0\ \text{ in }\ \Omega,\\ \dfrac{\partial U_0}{\partial\vec{n}}&=&\dfrac{1}{\pi}\displaystyle\int_{\sin(\theta+\xi)>0}g(\theta,\xi)\sin(\theta+\xi)\mathrm{d}\xi\ \text{ on }\ \partial\Omega.\end{array}\right.\tag{4.6.27}$$

Step 3: Construction of $\mathscr{U}_2$ and U_1.

Define the second order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_2&=&\psi_0(\epsilon\eta)\Big(f_2(\eta,\theta,\xi)-f_2(\infty,\theta)\Big),\\ \sin(\theta+\xi)\dfrac{\partial f_2}{\partial\eta}+f_2-\bar{f}_2&=&\cos(\theta+\xi)\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}_1}{\partial\theta}-\psi(\epsilon\eta)\mathscr{U}_0(\eta,\theta,\xi),\\ f_2(0,\theta,\xi)&=&\mathcal{P}f_2(0,\theta)+g_2(\theta,\xi)\ \text{ for }\ \sin(\theta+\xi)>0,\\ \lim_{\eta\rightarrow\infty}f_2(\eta,\theta,\xi)&=&f_2(\infty,\theta),\end{array}\right.\tag{4.6.28}$$

with

$$\mathcal{P}f_2(0,\theta)=0.\tag{4.6.29}$$

where $g_2=\vec{w}\cdot\nabla_x U_1-\mathcal{P}(\vec{w}\cdot\nabla_x U_1)+U_0-\mathcal{P}U_0$. By the compatibility condition (4.6.99), we define the first order interior solution as

$$\left\{\begin{array}{rcl}U_1&=&\bar{U}_1-\vec{w}\cdot\nabla_x U_0,\\ \Delta_x\bar{U}_1-\bar{U}_1&=&-\displaystyle\int_{\mathcal{S}^1}(\vec{w}\cdot\nabla_x U_0)\mathrm{d}\tilde{\mathrm{w}}\ \text{ in }\ \Omega,\\ \dfrac{\partial U_1}{\partial\vec{n}}&=&\dfrac{1}{\pi}\displaystyle\int_0^{\infty}\int_{-\pi}^{\pi}\dfrac{\psi(\epsilon s)}{1-\epsilon s}\cos(\theta+\xi)\dfrac{\partial\mathscr{U}_1}{\partial\theta}(s,\theta,\xi)\mathrm{d}\xi\mathrm{d}s\ \text{ on }\ \partial\Omega.\end{array}\right.\tag{4.6.30}$$

Step 4: Generalization to arbitrary k.

Define the k^{th} order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_k &=& \psi_0(\epsilon\eta)\bigg(f_k(\eta,\theta,\xi)-f_k(\infty,\theta)\bigg),\\ \sin(\theta+\xi)\dfrac{\partial f_k}{\partial\eta}+f_k-\bar{f}_k &=& \cos(\theta+\xi)\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}_{k-1}}{\partial\theta}-\psi(\epsilon\eta)\mathscr{U}_{k-2}(\eta,\theta,\xi),\\ f_k(0,\theta,\xi) &=& \mathcal{P}f_k(0,\theta)+g_k(\theta,\xi)\quad\text{for}\quad\sin(\theta+\xi)>0,\\ \lim_{\eta\rightarrow\infty}f_k(\eta,\theta,\xi) &=& f_k(\infty,\theta),\end{array}\right. \tag{4.6.31}$$

with

$$\mathcal{P}f_k(0,\theta)=0. \tag{4.6.32}$$

where $g_k=\vec{w}\cdot\nabla_xU_{k-1}-\mathcal{P}(\vec{w}\cdot\nabla_xU_{k-1})+U_{k-2}-\mathcal{P}U_{k-2}$. By the compatibility condition (4.6.99), we can define the $(k-1)^{th}$ order interior solution as

$$\left\{\begin{array}{rcl}U_{k-1} &=& \bar{U}_{k-1}-\vec{w}\cdot\nabla_xU_{k-2}-U_{k-3},\\ \Delta_x\bar{U}_{k-1}-\bar{U}_{k-1} &=& -\displaystyle\int_{\mathcal{S}^1}(\vec{w}\cdot\nabla_xU_{k-2})\mathrm{d}\tilde{\mathrm{w}}-\int_{\mathcal{S}^1}U_{k-3}\mathrm{d}\tilde{\mathrm{w}}\quad\text{in}\quad\Omega,\\ \dfrac{\partial\bar{U}_{k-1}}{\partial\vec{n}} &=& \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\bigg(\vec{w}\cdot\nabla_x(\vec{w}\cdot U_{k-2}+U_{k-3})\bigg)\sin\phi\mathrm{d}\phi\\ && +\dfrac{1}{\pi}\displaystyle\int_0^{\infty}\int_{-\pi}^{\pi}\bigg(\dfrac{\psi(\epsilon s)}{1-\epsilon s}\cos(\theta+\xi)\dfrac{\partial\mathscr{U}_{k-1}}{\partial\theta}\\ && -\psi(\epsilon s)\mathscr{U}_{k-2}\bigg)(s,\theta,\xi)\mathrm{d}\xi\mathrm{ds}\quad\text{on}\quad\partial\Omega.\end{array}\right. \tag{4.6.33}$$

In [9, pp.143], the author proved the expansion can be applied to the second order of ϵ. Based on Remark 4.6.8 to the Milne problem, in order to show the existence of a solution $f_2(\eta,\theta,\xi)\in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$, we at least require the source term

$$\cos(\theta+\xi)\frac{\psi}{1-\epsilon\eta}\frac{\partial\mathscr{U}_1}{\partial\theta}\in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)), \tag{4.6.34}$$

since $\mathscr{U}_0=0$. We thus need to require

$$\frac{\partial(f_1-f_1(\infty,\theta))}{\partial\theta}\in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)). \tag{4.6.35}$$

Since $Z=\partial_\theta(f_1-f_1(\infty))$ satisfies the equation

$$\begin{cases} \sin(\theta+\xi)\dfrac{\partial Z}{\partial\eta}+Z-\bar{Z} &= -\cos(\theta+\xi)\dfrac{\partial f_1}{\partial\eta},\\ Z(0,\theta,\xi) &= \mathcal{P}Z(0,\theta)+\dfrac{\partial g_1}{\partial\theta}(\theta,\xi)\ \text{ for }\ \sin(\theta+\xi)>0, \end{cases} \tag{4.6.36}$$

in order for $Z\in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$, assuming the boundary data $\partial_\theta g_1\in L^\infty(\Gamma^-)$, we require the source term

$$-\cos(\theta+\xi)\frac{\partial f_1}{\partial\eta}\in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)). \tag{4.6.37}$$

On the other hand, by Lemma B.2.1, we can show for specific g, it holds that $\partial_\eta f_1\notin L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$. Due to the intrinsic singularity in the solution to (4.6.25), the construction in [9] breaks down.

4.6.4 ϵ-Milne Expansion with Geometric Correction

In order to overcome the difficulty in estimating

$$\cos(\theta+\xi)\frac{\psi}{1-\epsilon\eta}\frac{\partial\mathscr{U}_k}{\partial\theta}, \tag{4.6.38}$$

we introduce one more substitution (4.2.37) to decompose this term and transform the equation (4.6.1) into

$$\begin{cases} \sin\phi\dfrac{\partial u^\epsilon}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}\cos\phi\left(\dfrac{\partial u^\epsilon}{\partial\phi}+\dfrac{\partial u^\epsilon}{\partial\theta}\right)+(1+\epsilon^2)u^\epsilon-\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}u^\epsilon\mathrm{d}\phi=0, \\ u^\epsilon(0,\theta,\phi)=\mathcal{P}u^\epsilon(0,\theta)+\epsilon g(\theta,\phi) \quad \text{for} \quad \sin\phi>0, \end{cases} \tag{4.6.39}$$

with

$$\mathcal{P}u^\epsilon(0,\theta)=-\frac{1}{2}\int_{\sin\phi<0}u^\epsilon(0,\theta,\xi)\sin\phi\mathrm{d}\phi. \tag{4.6.40}$$

We define the ϵ-Milne expansion of boundary layer as follows:

$$\mathscr{U}^\epsilon(\eta,\theta,\phi)\sim\sum_{k=0}^{\infty}\epsilon^k\mathscr{U}_k^\epsilon(\eta,\theta,\phi) \tag{4.6.41}$$

where $\mathscr{U}_k^\epsilon$ can be defined by comparing the order of ϵ via plugging (4.6.41) into the equation (4.6.39). Thus, in a neighborhood of the boundary, we have

$$\sin\phi\frac{\partial\mathscr{U}_0^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0^\epsilon}{\partial\phi}+\mathscr{U}_0^\epsilon-\bar{\mathscr{U}}_0^\epsilon = 0 \tag{4.6.42}$$

$$\sin\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\phi}+\mathscr{U}_1^\epsilon-\bar{\mathscr{U}}_1^\epsilon = \frac{1}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0^\epsilon}{\partial\theta} \tag{4.6.43}$$

$$\sin\phi\frac{\partial\mathscr{U}_2^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_2^\epsilon}{\partial\phi}+\mathscr{U}_2^\epsilon-\bar{\mathscr{U}}_2^\epsilon = \frac{1}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}-\mathscr{U}_0^\epsilon \tag{4.6.44}$$

$$\ldots$$

$$\sin\phi\frac{\partial\mathscr{U}_k^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_k^\epsilon}{\partial\phi}+\mathscr{U}_k^\epsilon-\bar{\mathscr{U}}_k^\epsilon = \frac{1}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{k-1}^\epsilon}{\partial\theta}-\mathscr{U}_{k-2}^\epsilon \tag{4.6.45}$$

where

$$\bar{\mathscr{U}}_k^\epsilon(\eta,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{U}_k^\epsilon(\eta,\theta,\phi)\mathrm{d}\phi. \tag{4.6.46}$$

We refer to the cut-off function ψ and ψ_0 as (4.2.23) and (4.2.24), and define the force as (4.2.45). Define the interior expansion as follows:

$$U^{\epsilon}(\vec{x},\vec{w}) \sim \sum_{k=0}^{\infty} \epsilon^k U_k^{\epsilon}(\vec{x},\vec{w}) \tag{4.6.47}$$

where U_k^{ϵ} satisfies the same equations as U_k in (4.6.12), (4.6.13) and (4.6.14). Here, to highlight its dependence on ϵ via the ϵ-Milne problem and boundary data, we add the superscript ϵ.

The bridge between the interior solution and the boundary layer solution is the boundary condition of (4.6.1), so we first consider the boundary condition expansion:

$$\begin{aligned}
(U_0^{\epsilon}+\mathscr{U}_0^{\epsilon}) &= \mathcal{P}(U_0^{\epsilon}+\mathscr{U}_0^{\epsilon}), && (4.6.48)\\
(U_1^{\epsilon}+\mathscr{U}_1^{\epsilon}) &= \mathcal{P}(U_1^{\epsilon}+\mathscr{U}_1^{\epsilon})+g, && (4.6.49)\\
(U_2^{\epsilon}+\mathscr{U}_2^{\epsilon}) &= \mathcal{P}(U_2^{\epsilon}+\mathscr{U}_2^{\epsilon}), && (4.6.50)\\
&\ldots\\
(U_k^{\epsilon}+\mathscr{U}_k^{\epsilon}) &= \mathcal{P}(U_k^{\epsilon}+\mathscr{U}_k^{\epsilon}). && (4.6.51)
\end{aligned}$$

Note the fact that $\bar{U}_k^{\epsilon}=\mathcal{P}\bar{U}_k^{\epsilon}$, we can simplify above conditions as follows:

$$\begin{aligned}
\mathscr{U}_0^{\epsilon} &= \mathcal{P}\mathscr{U}_0^{\epsilon}, && (4.6.52)\\
\mathscr{U}_1^{\epsilon} &= \mathcal{P}\mathscr{U}_1^{\epsilon}+(\vec{w}\cdot U_0^{\epsilon}-\mathcal{P}(\vec{w}\cdot U_0^{\epsilon}))+g, && (4.6.53)\\
\mathscr{U}_2^{\epsilon} &= \mathcal{P}\mathscr{U}_2^{\epsilon}+(\vec{w}\cdot U_1^{\epsilon}-\mathcal{P}(\vec{w}\cdot U_1^{\epsilon}))+(U_0^{\epsilon}-\mathcal{P}U_0^{\epsilon}), && (4.6.54)\\
&\ldots\\
\mathscr{U}_k^{\epsilon} &= \mathcal{P}\mathscr{U}_k^{\epsilon}+(\vec{w}\cdot U_{k-1}^{\epsilon}-\mathcal{P}(\vec{w}\cdot U_{k-1}^{\epsilon}))+(U_{k-2}^{\epsilon}-\mathcal{P}U_{k-2}^{\epsilon}). && (4.6.55)
\end{aligned}$$

The construction of U_k^{ϵ} and $\mathscr{U}_k^{\epsilon}$ are as follows:

Step 1: Construction of $\mathscr{U}_0^\epsilon$.

Define the zeroth order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_0^\epsilon(\eta,\theta,\phi) &=& \psi_0(\epsilon\eta)\bigg(f_0^\epsilon(\eta,\theta,\phi)-f_0^\epsilon(\infty,\theta)\bigg),\\ \sin\phi\dfrac{\partial f_0^\epsilon}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f_0^\epsilon}{\partial\phi}+f_0^\epsilon-\bar{f}_0^\epsilon &=& 0,\\ f_0^\epsilon(0,\theta,\phi) &=& \mathcal{P}f_0^\epsilon(0,\theta)\ \ \text{for}\ \ \sin\phi>0,\\ \lim_{\eta\to\infty}f_0^\epsilon(\eta,\theta,\phi) &=& f_0^\epsilon(\infty,\theta),\end{array}\right. \tag{4.6.56}$$

with

$$\mathcal{P}f_0^\epsilon(0,\theta)=0. \tag{4.6.57}$$

By Theorem 4.6.5, $\mathscr{U}_0^\epsilon$ is well-defined. It is obvious to see $f_0^\epsilon=f_0^\epsilon(\infty)=0$ is the only solution.

Step 2: Construction of $\mathscr{U}_1^\epsilon$ and U_0^ϵ.

Define the first order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_1^\epsilon(\eta,\theta,\phi) &=& \psi_0(\epsilon\eta)\bigg(f_1^\epsilon(\eta,\theta,\phi)-f_1^\epsilon(\infty,\theta)\bigg),\\ \sin\phi\dfrac{\partial f_1^\epsilon}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f_1^\epsilon}{\partial\phi}+f_1^\epsilon-\bar{f}_1^\epsilon &=& \dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_0^\epsilon}{\partial\theta},\\ f_1^\epsilon(0,\theta,\phi) &=& \mathcal{P}f_1^\epsilon(0,\theta)+g_1(\theta,\phi)\ \ \text{for}\ \ \sin\phi>0,\\ \lim_{\eta\to\infty}f_1^\epsilon(\eta,\theta,\phi) &=& f_1^\epsilon(\infty,\theta),\end{array}\right. \tag{4.6.58}$$

with

$$\mathcal{P}f_1^\epsilon(0,\theta)=0, \tag{4.6.59}$$

where

$$g_1 = (\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0) - \mathcal{P}(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0))) + g, \tag{4.6.60}$$

with $\vec{x}_0$ is the same boundary point as $(0,\theta)$ and

$$\vec{w} = (-\sin(\phi-\theta), -\cos(\phi-\theta)), \tag{4.6.61}$$

$$\vec{n} = (\cos\theta, \sin\theta). \tag{4.6.62}$$

To solve (4.6.58), we require the compatibility condition (4.6.99) for the boundary data

$$\begin{aligned}&\int_{\sin\phi>0}\bigg(g+\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0) - \mathcal{P}(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0))\bigg)\sin\phi\mathrm{d}\phi \\ &\quad+\int_0^{\infty}\int_{-\pi}^{\pi}e^{-V(s)}\frac{\psi}{1-\epsilon s}\cos\phi\frac{\partial\mathscr{U}_0^{\epsilon}}{\partial\theta}(s,\theta,\phi)\mathrm{d}\phi\mathrm{d}s = 0.\end{aligned} \tag{4.6.63}$$

Note the fact

$$\begin{aligned}&\int_{\sin\phi>0}\bigg(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0) - \mathcal{P}(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0))\bigg)\sin\phi\mathrm{d}\phi \\ =&\int_{\sin\phi>0}(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0))\sin\phi\mathrm{d}\phi - 2\mathcal{P}(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0)) \\ =&\int_{\sin\phi>0}(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0))\sin\phi\mathrm{d}\phi + \int_{\sin\phi<0}(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0))\sin\phi\mathrm{d}\phi \\ =&\int_{-\pi}^{\pi}(\vec{w}\cdot\nabla_x U_0^{\epsilon}(\vec{x}_0))\sin\phi\mathrm{d}\phi \\ =& -\pi\nabla_x\bar{U}_0^{\epsilon}(\vec{x}_0)\cdot\vec{n} = -\pi\frac{\partial\bar{U}_0^{\epsilon}(\vec{x}_0)}{\partial\vec{n}}.\end{aligned} \tag{4.6.64}$$

We can simplify the compatibility condition as follows:

$$\begin{aligned} & \int_{\sin\phi>0} g(\phi)\sin\phi \mathrm{d}\phi - \pi\frac{\partial \bar{U}_0^{\epsilon}(\vec{x}_0)}{\partial\vec{n}} \\ + & \int_0^{\infty}\int_{-\pi}^{\pi} e^{-V(s)}\frac{\psi}{1-\epsilon s}\cos\phi\frac{\partial \mathscr{U}_0^{\epsilon}}{\partial\theta}(s,\theta,\phi)\mathrm{d}\phi\mathrm{ds} = 0. \end{aligned} \tag{4.6.65}$$

Then we have

$$\begin{aligned} & \frac{\partial \bar{U}_0^{\epsilon}(\vec{x}_0)}{\partial\vec{n}} \\ = & \frac{1}{\pi}\int_{\sin\phi>0} g(\theta,\phi)\sin\phi \mathrm{d}\phi + \frac{1}{\pi}\int_0^{\infty}\int_{-\pi}^{\pi} e^{-V(s)}\frac{\psi}{1-\epsilon s}\cos\phi\frac{\partial \mathscr{U}_0^{\epsilon}}{\partial\theta}(s,\theta,\phi)\mathrm{d}\phi\mathrm{ds} \\ = & \frac{1}{\pi}\int_{\sin\phi>0} g(\theta,\phi)\sin\phi \mathrm{d}\phi. \end{aligned} \tag{4.6.66}$$

Hence, we define the zeroth order interior solution $U_0^{\epsilon}(\vec{x},\vec{w})$ as

$$\left\{\begin{array}{rcl} U_0^{\epsilon} & = & \bar{U}_0^{\epsilon}, \\ \Delta_x \bar{U}_0^{\epsilon} - \bar{U}_0^{\epsilon} & = & 0 \text{ in } \Omega, \\ \dfrac{\partial \bar{U}_0^{\epsilon}}{\partial\vec{n}} & = & \dfrac{1}{\pi}\displaystyle\int_{\sin\phi>0} g(\theta,\phi)\sin\phi \mathrm{d}\phi \text{ on } \partial\Omega. \end{array}\right. \tag{4.6.67}$$

Step 3: Analysis of U_0^{ϵ} and $\mathscr{U}_1^{\epsilon}$.

By Theorem 4.6.6, we can easily see f_1^{ϵ} is well-defined in $L^{\infty}(\Omega\times\mathcal{S}^1)$ and approaches $f_1^{\epsilon}(\infty)$ exponentially fast as $\eta\rightarrow\infty$. Since $f_0^{\epsilon}=0$, then if $g\in C^r(\Gamma^-)$, it is obvious to check $\partial_n U_0^{\epsilon}\in C^r(\partial\Omega)$. Hence, by the standard elliptic estimate, there exists a unique solution $U_0^{\epsilon}\in W^{r+1,p}(\Omega)$ for arbitrary $p\geq 2$ satisfying

$$\left\|\bar{U}_0^{\epsilon}\right\|_{W^{r+1,p}(\Omega)} \leq C(\Omega)\left\|\frac{\partial \bar{U}_0^{\epsilon}}{\partial\vec{n}}\right\|_{W^{r-1/p,p}(\partial\Omega)}, \tag{4.6.68}$$

which implies $\nabla_x U_0^{\epsilon}\in W^{r,p}(\Omega)$, $\nabla_x U_0^{\epsilon}\in W^{r-1/p,p}(\partial\Omega)$ and $U_0^{\epsilon}\in C^{r,1-2/p}(\Omega)$.

Step 4: Construction of $\mathscr{U}_2^\epsilon$ and U_1^ϵ.

Define the second order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_2^\epsilon(\eta,\theta,\phi) &=& \psi(\epsilon\eta)\bigg(f_2^\epsilon(\eta,\theta,\phi)-f_2^\epsilon(\infty,\theta)\bigg),\\ \sin\phi\dfrac{\partial f_2^\epsilon}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f_2^\epsilon}{\partial\phi}+f_2^\epsilon-\bar{f}_2^\epsilon &=& \dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}-\psi(\epsilon\eta)\mathscr{U}_0^\epsilon,\\ f_2^\epsilon(0,\theta,\phi) &=& \mathcal{P}f_2^\epsilon(0,\theta)+g_2(\theta,\phi)\ \text{ for }\ \sin\phi>0,\\ \lim_{\eta\to\infty}f_2^\epsilon(\eta,\theta,\phi) &=& f_2^\epsilon(\infty,\theta),\end{array}\right. \tag{4.6.69}$$

with

$$\mathcal{P}f_2^\epsilon(0,\theta)=0, \tag{4.6.70}$$

where

$$g_2 = (\vec{w}\cdot\nabla_x U_1^\epsilon(\vec{x}_0)-\mathcal{P}(\vec{w}\cdot\nabla_x U_1^\epsilon(\vec{x}_0)))+U_0^\epsilon(\vec{x}_0)-\mathcal{P}U_0^\epsilon(\vec{x}_0). \tag{4.6.71}$$

In order for equation (4.6.69) being well-posed, we require the compatibility condition (4.6.99) for the boundary data

$$\begin{aligned}&\int_{\sin\phi>0}\bigg(\vec{w}\cdot\nabla_x U_1^\epsilon(\vec{x}_0)-\mathcal{P}(\vec{w}\cdot\nabla_x U_1^\epsilon(\vec{x}_0))\bigg)\sin\phi\mathrm{d}\phi \\ &+\int_0^\infty\int_{-\pi}^{\pi}e^{-V(s)}\bigg(\frac{\psi}{1-\epsilon s}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}(s,\theta,\phi)-\psi\mathscr{U}_0^\epsilon(s,\theta,\phi)\bigg)\mathrm{d}\phi\mathrm{ds} = 0.\end{aligned} \tag{4.6.72}$$

Similarly, we can directly verify the relation

$$\int_{\sin\phi>0}\bigg(\vec{w}\cdot\nabla_x U_1^\epsilon(\vec{x}_0)-\mathcal{P}(\vec{w}\cdot\nabla_x U_1^\epsilon(\vec{x}_0))\bigg)\sin\phi\mathrm{d}\phi = -\pi\frac{\partial\bar{U}_1^\epsilon(\vec{x}_0)}{\partial\vec{n}}. \tag{4.6.73}$$

We can simplify the compatibility condition as follows:

$$-\pi\frac{\partial\bar U_1^\epsilon(\vec x_0)}{\partial\vec n}+\int_0^\infty\int_{-\pi}^{\pi}e^{-V(s)}\bigg(\frac{\psi}{1-\epsilon s}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}(s,\theta,\phi)-\psi\mathscr{U}_0^\epsilon(s,\theta,\phi)\bigg)\mathrm{d}\phi\mathrm{ds} \;=\; 0. \tag{4.6.74}$$

Then we have

$$\begin{aligned}-\pi\frac{\partial\bar U_1^\epsilon(\vec x_0)}{\partial\vec n} &= \frac{1}{\pi}\int_0^\infty\int_{-\pi}^{\pi}e^{-V(s)}\bigg(\frac{\psi}{1-\epsilon s}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}(s,\theta,\phi)-\psi\mathscr{U}_0^\epsilon(s,\theta,\phi)\bigg)\mathrm{d}\phi\mathrm{ds}\\ &= \frac{1}{\pi}\int_0^\eta\int_{-\pi}^{\pi}e^{-V(s)}\frac{\psi}{1-\epsilon s}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}(s,\theta,\phi)\mathrm{d}\phi\mathrm{ds}.\end{aligned} \tag{4.6.75}$$

Hence, we define the first order interior solution $U_1^\epsilon(\vec x)$ as

$$\left\{\begin{array}{rcl}U_1^\epsilon &=& \bar U_1^\epsilon-\vec w\cdot\nabla_xU_0^\epsilon,\\ \Delta_x\bar U_1^\epsilon-\bar U_1^\epsilon &=& -\displaystyle\int_{\mathcal{S}^1}(\vec w\cdot\nabla_xU_0^\epsilon)\mathrm{d}\tilde{\mathrm{w}}\ \text{ in }\ \Omega,\\ \dfrac{\partial\bar U_1^\epsilon}{\partial\vec n} &=& \dfrac{1}{\pi}\displaystyle\int_0^\infty\int_{-\pi}^{\pi}e^{-V(s)}\frac{\psi(\epsilon s)}{1-\epsilon s}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}(s,\theta,\phi)\mathrm{d}\phi\mathrm{ds}\ \text{ on }\ \partial\Omega.\end{array}\right. \tag{4.6.76}$$

Step 5: Analysis of U_1^ϵ and $\mathscr{U}_2^\epsilon$.

By Theorem 4.6.6, we can easily see f_2^ϵ is well-defined in $L^\infty(\Omega\times\mathcal{S}^1)$ and approaches $f_2^\epsilon(\infty)$ exponentially fast as $\eta\to\infty$. By above analysis, it is obvious to check $\partial_n\bar U_1^\epsilon\in C^{r-1}(\partial\Omega)$. Hence, by the standard elliptic estimate, there exists a unique solution $\bar U_1^\epsilon\in W^{r,p}(\Omega)$ for arbitrary $p\geq 2$ satisfying

$$\left\|\bar U_1^\epsilon\right\|_{W^{r,p}(\Omega)}\leq C(\Omega)\bigg(\left\|\frac{\partial\bar U_1^\epsilon}{\partial\vec n}\right\|_{W^{r-1-1/p,p}(\partial\Omega)}+\|\nabla_xU_0^\epsilon\|_{W^{r-2,p}(\Omega)}\bigg) \tag{4.6.77}$$

which implies $\nabla_xU_1^\epsilon\in W^{r-1,p}(\Omega)$, $\nabla_xU_1^\epsilon\in W^{r-1-1/p,p}(\partial\Omega)$ and $U_1^\epsilon\in C^{r-1,1-2/p}(\Omega)$.

Step 6: Generalization to arbitrary k.

Then this process can proceed to arbitrary k as long as g is sufficiently smooth. We can always determine $\mathscr{U}_k^\epsilon$ and U_{k-1}^ϵ simultaneously based on the compatibility condition. Define the k^{th} order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_k^\epsilon(\eta,\theta,\phi) &=& \psi(\epsilon\eta)\bigg(f_k^\epsilon(\eta,\theta,\phi)-f_k^\epsilon(\infty,\theta)\bigg),\\ \sin\phi\dfrac{\partial f_k^\epsilon}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f_k^\epsilon}{\partial\phi}+f_k^\epsilon-\bar{f}_k^\epsilon &=& \dfrac{\psi}{1-\epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_{k-1}^\epsilon}{\partial\theta}-\psi\mathscr{U}_{k-2}^\epsilon,\\ f_k^\epsilon(0,\theta,\phi) &=& \mathcal{P}f_k^\epsilon(0,\theta)+g_k(\theta,\phi)\ \ \text{for}\ \ \sin\phi>0,\\ \lim_{\eta\to\infty}f_k^\epsilon(\eta,\theta,\phi) &=& f_k^\epsilon(\infty,\theta),\end{array}\right. \tag{4.6.78}$$

with

$$\mathcal{P}f_k^\epsilon(0,\theta)=0, \tag{4.6.79}$$

where

$$g_k = (\vec{w}\cdot\nabla_x U_{k-1}^\epsilon(\vec{x}_0)-\mathcal{P}(\vec{w}\cdot\nabla_x U_{k-1}^\epsilon(\vec{x}_0)))+U_{k-2}^\epsilon(\vec{x}_0)-\mathcal{P}U_{k-2}^\epsilon(\vec{x}_0) \tag{4.6.80}$$

Hence, we define the $(k-1)^{th}$ order interior solution as

$$\left\{\begin{array}{rcl}U_{k-1}^\epsilon &=& \bar{U}_{k-1}^\epsilon-\vec{w}\cdot\nabla_x U_{k-2}^\epsilon-U_{k-3}^\epsilon,\\ \Delta_x\bar{U}_{k-1}^\epsilon-\bar{U}_{k-1}^\epsilon &=& -\displaystyle\int_{\mathcal{S}^1}(\vec{w}\cdot\nabla_x U_{k-2}^\epsilon)\mathrm{d}\tilde{\mathrm{w}}-\int_{\mathcal{S}^1}U_{k-3}^\epsilon\mathrm{d}\tilde{\mathrm{w}}\ \ \text{in}\ \ \Omega,\\ \dfrac{\partial\bar{U}_{k-1}^\epsilon}{\partial\vec{n}} &=& \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\bigg(\vec{w}\cdot\nabla_x(\vec{w}\cdot U_{k-2}^\epsilon+U_{k-3}^\epsilon)\bigg)\sin\phi\mathrm{d}\phi\\ && +\dfrac{1}{\pi}\displaystyle\int_0^\infty\int_{-\pi}^{\pi}e^{-V(s)}\bigg(\dfrac{\psi(\epsilon s)}{1-\epsilon s}\cos\phi\dfrac{\partial\mathscr{U}_{k-1}^\epsilon}{\partial\theta}\\ && -\psi(\epsilon s)\mathscr{U}_{k-2}^\epsilon\bigg)(s,\theta,\phi)\mathrm{d}\phi\mathrm{ds}\ \ \text{on}\ \ \partial\Omega.\end{array}\right. \tag{4.6.81}$$

In particular, for $g\in C^{k+1}(\Gamma^-)$, we can define the interior solution up to k^{th} order and the boundary layer solution up to $(k+1)^{th}$ order, i.e. up to U_k^ϵ and $\mathscr{U}_{k+1}^\epsilon$.

4.6.5 Well-Posedness of Steady Neutron Transport Equation

In this section, we consider the well-posedness of the steady neutron transport equation

$$\left\{\begin{array}{rcl}\epsilon\vec{w}\cdot\nabla_x u+(1+\epsilon^2)u-\bar{u} &=& f(\vec{x},\vec{w}) \ \text{ in } \ \Omega,\\ u(\vec{x}_0,\vec{w}) &=& \mathcal{P}u(\vec{x}_0)+\epsilon g(\vec{x}_0,\vec{w})\\ && \text{for } \ \vec{w}\cdot\vec{n}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega.\end{array}\right. \tag{4.6.82}$$

We define the L^∞ norms in $\Omega\times\mathcal{S}^1$ as usual:

$$\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} = \sup_{(\vec{x},\vec{w})\in\Omega\times\mathcal{S}^1}|f(\vec{x},\vec{w})|. \tag{4.6.83}$$

Theorem 4.6.1. *Assume $f(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $g(x_0,\vec{w})\in L^\infty(\Gamma^-)$. Then for the transport equation*

$$\left\{\begin{array}{rcl}\epsilon\vec{w}\cdot\nabla_x u+(1+\epsilon^2)u-\bar{u} &=& f(\vec{x},\vec{w}) \ \textit{ in } \ \Omega,\\ u(\vec{x}_0,\vec{w}) &=& \mathcal{P}u(\vec{x}_0)+\epsilon g(\vec{x}_0,\vec{w})\\ && \textit{for } \ \vec{x}_0\in\partial\Omega \ \textit{ and } \vec{w}\cdot\vec{n}<0,\end{array}\right. \tag{4.6.84}$$

there exists a unique solution $u(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ satisfying

$$\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}\leq C\bigg(\frac{1}{\epsilon^2}\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^2}\|g\|_{L^\infty(\Gamma^-)}\bigg). \tag{4.6.85}$$

Proof. We iteratively construct an approximating sequence $\{u^k\}_{k=0}^\infty$ where $u^0=0$ and

$$\left\{\begin{array}{rcl}\epsilon\vec{w}\cdot\nabla_x u^k_\lambda+(1+\epsilon^2)u^k-\bar{u}^{k-1} &=& f(\vec{x},\vec{w}) \ \text{ in } \ \Omega,\\ u^k(\vec{x}_0,\vec{w}) &=& \mathcal{P}u^{k-1}(\vec{x}_0)+\epsilon g(\vec{x}_0,\vec{w})\\ && \text{for } \ \vec{x}_0\in\partial\Omega \ \text{ and } \vec{w}\cdot\vec{n}<0.\end{array}\right. \tag{4.6.86}$$

By Lemma 4.3.1, this sequence is well-defined and $\left\|u^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} < \infty$. We rewrite equation (4.6.86) along the characteristics as

$$\begin{aligned} u^k(\vec{x},\vec{w}) &= (\epsilon g + \mathcal{P}u^{k-1})(\vec{x}-\epsilon t_b\vec{w},\vec{w})e^{-(1+\epsilon^2)t_b} \\ &\quad + \int_0^{t_b}(f+\bar{u}^{k-1})(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})e^{-(1+\epsilon^2)(t_b-s)}\mathrm{ds}. \end{aligned} \tag{4.6.87}$$

where the backward exit time t_b is defined as (4.3.13). We define the difference $v^k = u^k - u^{k-1}$ for $k \geq 1$. Then v^k satisfies

$$v^{k+1}(\vec{x},\vec{w}) = \mathcal{P}v^k(\vec{x}-\epsilon t_b\vec{w},\vec{w})e^{-(1+\epsilon^2)t_b} + \int_0^{t_b}\bar{v}^k(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})e^{-(1+\epsilon^2)(t_b-s)}\mathrm{ds}. \tag{4.6.88}$$

Since $\left\|\bar{v}^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}$ and $\left\|\mathcal{P}v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}$, we can directly estimate

$$\begin{aligned} \left\|v^{k+1}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} &\leq e^{-(1+\epsilon^2)t_b}\left\|v^{k+1}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}\int_0^{t_b}e^{-(1+\epsilon^2)(t_b-s)}\mathrm{ds} \\ &\leq e^{-(1+\epsilon^2)t_b}\left\|v^{k+1}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \frac{1}{1+\epsilon^2}(1-e^{-(1+\epsilon^2)t_b})\left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \end{aligned} \tag{4.6.89}$$

Hence, we naturally have

$$\left\|v^{k+1}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \frac{1}{1+\epsilon^2}\left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.6.90}$$

Thus, this is a contraction iteration. Considering $v^1 = u^1$, we have

$$\left\|v^k\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \left(\frac{1}{1+\epsilon^2}\right)^{k-1}\left\|u^1\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.6.91}$$

for $k\geq 1$. Therefore, u^k converges strongly in L^{∞} to the limiting solution u satisfying

$$\|u\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}\leq\sum_{k=1}^{\infty}\left\|v^k\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}\leq\frac{1+\epsilon^2}{\epsilon^2}\left\|u^1\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}\leq\frac{2}{\epsilon^2}\left\|u^1\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}. \tag{4.6.92}$$

Since u^1 satisfies the equation

$$u^1(\vec x,\vec w) \;=\; g(\vec x-\epsilon t_b\vec w,\vec w)e^{-(1+\epsilon^2)t_b}+\int_0^{t_b}f(\vec x-\epsilon(t_b-s)\vec w,\vec w)e^{-(1+\epsilon^2)(t_b-s)}\mathrm{ds}.$$

Based on Lemma 4.3.1, we can directly estimate

$$\left\|u^1\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}\leq\|f\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\|g\|_{L^{\infty}(\Gamma^-)}. \tag{4.6.93}$$

Combining (4.6.92) and (4.6.93), we can naturally obtain the existence and estimates. Also, the uniqueness easily follows from the energy estimates. □

Theorem 4.6.2. *Assume $g(x_0,\vec w)\in L^{\infty}(\Gamma^-)$. Then for the steady neutron transport equation (4.6.1), there exists a unique solution $u^{\epsilon}(\vec x,\vec w)\in L^{\infty}(\Omega\times\mathcal{S}^1)$ satisfying*

$$\|u^{\epsilon}\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}\leq\frac{C}{\epsilon^2}\|g\|_{L^{\infty}(\Gamma^-)}. \tag{4.6.94}$$

Proof. We can apply Theorem 4.6.1 to the equation (4.6.1). The result naturally follows. □

4.6.6 ϵ-Milne Problem

We consider the ϵ-Milne problem for $f^{\epsilon}(\eta,\theta,\phi)$ in the domain $(\eta,\theta,\phi)\in[0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)$

$$
\left\{
\begin{array}{rcl}
\sin\phi\dfrac{\partial f^{\epsilon}}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f^{\epsilon}}{\partial\phi}+f^{\epsilon}-\bar{f}^{\epsilon}&=&S^{\epsilon}(\eta,\theta,\phi),\\
f^{\epsilon}(0,\theta,\phi)&=&h^{\epsilon}(\theta,\phi)+\mathcal{P}f^{\epsilon}(0)\ \ for\ \ \sin\phi>0,\\
\lim_{\eta\rightarrow\infty}f^{\epsilon}(\eta,\theta,\phi)&=&f^{\epsilon}_{\infty}(\theta),
\end{array}
\right.
\tag{4.6.95}
$$

where

$$
\mathcal{P}f^{\epsilon}(\eta,\theta)=-\frac{1}{2}\int_{\sin\phi<0}f^{\epsilon}(\eta,\theta,\phi)\sin\phi\mathrm{d}\phi,\tag{4.6.96}
$$

$F(\epsilon;\eta)$ is defined as (4.2.45),

$$
\left|h^{\epsilon}(\theta,\phi)\right|\leq M,\tag{4.6.97}
$$

and

$$
\left|S^{\epsilon}(\eta,\theta,\phi)\right|\leq Me^{-K\eta},\tag{4.6.98}
$$

for M and K uniform in ϵ and θ.

For notational simplicity, we omit ϵ and θ dependence in f^{ϵ} in this section. The same convention also applies to $F(\epsilon;\eta)$, $V(\epsilon;\eta)$, $S^{\epsilon}(\eta,\theta,\phi)$ and $h^{\epsilon}(\theta,\phi)$. However, our estimates are independent of ϵ and θ.

Compatibility Condition

Lemma 4.6.3. *In order for the equation (4.6.95) to have a solution $f \in L^\infty([0,\infty) \times [-\pi,\pi))$, the boundary data h and the source term S must satisfy the compatibility condition*

$$\int_{\sin\phi>0} h(\phi)\sin\phi \mathrm{d}\phi + \int_0^\infty \int_{-\pi}^{\pi} e^{-V(s)} S(s,\phi)\mathrm{d}\phi\mathrm{ds} = 0. \tag{4.6.99}$$

In particular, if $S = 0$, then the compatibility condition reduces to

$$\int_{\sin\phi>0} h(\phi)\sin\phi \mathrm{d}\phi = 0. \tag{4.6.100}$$

Proof. We just integrate over $\phi \in [-\pi,\pi)$ on both sides of (4.6.95) and integrate by parts to achieve

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\int_{-\pi}^{\pi} f(\eta,\phi)\sin\phi\mathrm{d}\phi + F(\eta)\int_{-\pi}^{\pi} f(\eta,\phi)\sin\phi\mathrm{d}\phi = \int_{-\pi}^{\pi} S(\eta,\phi)\mathrm{d}\phi. \tag{4.6.101}$$

This is a first order ordinary differential equation for $\int_{-\pi}^{\pi} f(\eta,\phi)\sin\phi\mathrm{d}\phi$. The initial data is

$$\begin{aligned}\int_{-\pi}^{\pi} f(0,\phi)\sin\phi\mathrm{d}\phi &= \int_{\sin\phi>0} f(0,\phi)\sin\phi\mathrm{d}\phi + \int_{\sin\phi<0} f(0,\phi)\sin\phi\mathrm{d}\phi \qquad (4.6.102)\\ &= \int_{\sin\phi>0}\Big(h(\phi)+\mathcal{P}f(0)\Big)\sin\phi\mathrm{d}\phi + \int_{\sin\phi<0} f(0,\phi)\sin\phi\mathrm{d}\phi\\ &= \int_{\sin\phi>0} h(\phi)\sin\phi\mathrm{d}\phi + \int_{\sin\phi>0}\mathcal{P}f(0)\sin\phi\mathrm{d}\phi - 2\mathcal{P}f(0)\\ &= \int_{\sin\phi>0} h(\phi)\sin\phi\mathrm{d}\phi.\end{aligned}$$

Hence, we can directly solve (4.6.101) as

$$\int_{-\pi}^{\pi} f(\eta,\phi)\sin\phi \mathrm{d}\phi = e^{V(\eta)}\bigg(\int_{\sin\phi>0} h(\phi)\sin\phi \mathrm{d}\phi + \int_0^{\eta}\int_{-\pi}^{\pi} e^{-V(s)}S(s,\phi)\mathrm{d}\phi\mathrm{d}s\bigg). \tag{4.6.103}$$

The problem (4.6.95) requires $f(\eta,\phi)\to f_{\infty}$ as $\eta\to\infty$. Hence, we must have

$$\lim_{\eta\to\infty}\int_{-\pi}^{\pi} f(\eta,\phi)\sin\phi \mathrm{d}\phi = 0. \tag{4.6.104}$$

Therefore, the only possibility to justify above requirement is

$$\int_{\sin\phi>0} h(\phi)\sin\phi \mathrm{d}\phi + \int_0^{\infty}\int_{-\pi}^{\pi} e^{-V(s)}S(s,\phi)\mathrm{d}\phi\mathrm{d}s = 0. \tag{4.6.105}$$

This is the desired compatibility condition. If $S=0$, then above condition reduces to

$$\int_{\sin\phi>0} h(\phi)\sin\phi \mathrm{d}\phi = 0. \tag{4.6.106}$$

□

Reduction to In-flow ϵ-Milne Problem

It is easy to see if f is a solution to (4.6.95), then $f+C$ is also a solution for any constant C. Hence, in order to obtain a unique solution, we need a normalization condition

$$\mathcal{P}f(0) = 0. \tag{4.6.107}$$

The following lemma tells us the problem (4.6.95) can be reduced to the ϵ-Milne problem (4.4.1) with in-flow boundary.

Lemma 4.6.4. *If the boundary data h and S satisfy the compatibility condition (4.6.99), then the solution f to the ϵ-Milne problem (4.4.1) with in-flow boundary as $f = h$ on $\sin\phi > 0$ is also a solution to the ϵ-Milne problem (4.6.95) with diffusive boundary, which satisfies the normalization condition (4.6.107). Furthermore, this is the unique solution to (4.6.95) among the functions satisfying (4.6.107) and $\|f - f_\infty\|_{L^2L^2} < \infty$.*

Proof. Consider f satisfies the ϵ-Milne problem with in-flow boundary as follows:

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi}+f-\bar{f} &=& S(\eta,\phi),\\ f(0,\phi) &=& h(\phi)\ \ \text{for}\ \ \sin\phi>0,\\ \lim_{\eta\to\infty}f(\eta,\phi) &=& f_\infty.\end{array}\right. \tag{4.6.108}$$

Then there exists a f_∞ such that $\|f - f_\infty\|_{L^\infty L^\infty} < \infty$ and f decays to f_∞ exponentially. Therefore, $z = f - f_\infty$ satisfies the equation

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial z}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial z}{\partial\phi}+z-\bar{z} &=& S(\eta,\phi),\\ z(0,\phi) &=& h(\phi)-f_\infty\ \ \text{for}\ \ \sin\phi>0,\\ \lim_{\eta\to\infty}z(\eta,\phi) &=& 0.\end{array}\right. \tag{4.6.109}$$

Multiplying $e^{-V(\eta)}$ on both sides of (4.6.109) and integrating over $\phi \in [-\pi, \pi)$ imply

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\int_{-\pi}^{\pi}e^{-V(\eta)}z(\eta,\phi)\sin\phi\mathrm{d}\phi=\int_{-\pi}^{\pi}e^{-V(\eta)}S(\eta,\phi)\mathrm{d}\phi. \tag{4.6.110}$$

Since z decays exponentially with respect to η, we know $z \in L^1([0,\infty) \times [-\pi, \pi))$.

Hence, we can integrate (4.6.110) over $\eta \in [0,\infty)$ to obtain

$$0-\int_{-\pi}^{\pi}e^{-V(0)}z(0,\phi)\sin\phi\mathrm{d}\phi=\int_0^{\infty}\int_{-\pi}^{\pi}e^{-V(s)}S(s,\phi)\mathrm{d}\phi\mathrm{ds}, \tag{4.6.111}$$

which further implies

$$\int_{-\pi}^{\pi}z(0,\phi)\sin\phi\mathrm{d}\phi+\int_0^{\infty}\int_{-\pi}^{\pi}e^{-V(s)}S(s,\phi)\mathrm{d}\phi\mathrm{ds}=0. \tag{4.6.112}$$

Then we may compute

$$\begin{aligned}\int_{-\pi}^{\pi}z(0,\phi)\sin\phi\mathrm{d}\phi &= \int_{\sin\phi>0}z(0,\phi)\sin\phi\mathrm{d}\phi+\int_{\sin\phi<0}z(0,\phi)\sin\phi\mathrm{d}\phi\\ &= \int_{\sin\phi>0}(h(\phi)-f_\infty)\sin\phi\mathrm{d}\phi-2\mathcal{P}z(0)\\ &= \int_{\sin\phi>0}h(\phi)\sin\phi\mathrm{d}\phi-2\Big(\mathcal{P}z(0)+f_\infty\Big).\end{aligned} \tag{4.6.113}$$

Combining (4.6.112), (4.6.113) and the compatibility condition (4.6.99), we have

$$\mathcal{P}z(0)+f_\infty=0. \tag{4.6.114}$$

Since

$$\mathcal{P}z(0)=\mathcal{P}f(0)-\mathcal{P}f_\infty(0)=\mathcal{P}f(0)-f_\infty. \tag{4.6.115}$$

naturally, we have $\mathcal{P}f(0)=0$. Hence, f is a solution to the ϵ-Milne problem (4.6.95) with the normalization condition (4.6.107). By Cauchy's inequality, we can deduce the fact that $\|f-f_\infty\|_{L^2L^2}<\infty$ implies $\left\|\bar{f}-f_\infty\right\|_{L^2L^2}<\infty$. Then by a similar argument as Step 3 in the proof of Lemma 4.4.4, we can show the uniqueness. $\square$

In summary, based on above analysis, we can utilize the known result for ϵ-Milne problem with in-flow boundary to obtain the well-posedness, decay and maximum principle of the solution to the ϵ-Milne problem (4.6.95).

Theorem 4.6.5. *There exists a unique solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.6.95) with the normalization condition (4.6.107) satisfying*

$$\|f-f_{\infty}\|_{L^{\infty}L^{\infty}} \leq C\bigg(1+M+\frac{M}{K}\bigg). \tag{4.6.116}$$

Theorem 4.6.6. *For $K_0>0$ sufficiently small, the solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.6.95) with the normalization condition (4.6.107) satisfies*

$$\left\|e^{K_0\eta}(f-f_{\infty})\right\|_{L^{\infty}L^{\infty}} \leq C\bigg(1+M+\frac{M}{K}\bigg). \tag{4.6.117}$$

Theorem 4.6.7. *The solution to the ϵ-Milne problem (4.6.95) with $S=0$ and the normalization condition (4.6.107) satisfies the maximum principle, i.e.*

$$\min_{\sin\phi>0} h(\phi) \leq f(\eta,\phi) \leq \max_{\sin\phi>0} h(\phi). \tag{4.6.118}$$

Remark 4.6.8. *Note that when $F=0$, then all the previous proofs can be recovered and Theorem 4.6.5, Theorem 4.6.6 and Theorem 4.6.7 still hold. Hence, we can obtain the well-posedness, decay and maximum principle of the classical Milne problem*

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f}{\partial\eta}+f-\bar{f} &=& S(\eta,\phi),\\ f(0,\phi) &=& \mathcal{P}f(0)+h(\phi) \ \ \text{for} \ \ \sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f(\eta,\phi) &=& f_{\infty},\end{array}\right. \tag{4.6.119}$$

with the normalization condition (4.6.107). Note that now we always have $V(\eta)=0$.

4.6.7 Main Results

Theorem 4.6.9. *Assume $g(\vec{x}_0,\vec{w}) \in C^3(\Gamma^-)$. Then for the steady neutron transport equation (4.6.1), the unique solution $u^\epsilon(\vec{x},\vec{w}) \in L^\infty(\Omega\times\mathcal{S}^1)$ satisfies*

$$\|u^\epsilon - U_0^\epsilon - \mathscr{U}_0^\epsilon\|_{L^\infty} = O(\epsilon), \tag{4.6.120}$$

Moreover, if $g(\theta,\phi)=\cos\phi$, then there exists a $C>0$ such that

$$\|u^\epsilon - U_0 - \mathscr{U}_0 - \epsilon U_1 - \epsilon\mathscr{U}_1\|_{L^\infty} \geq C\epsilon > 0, \tag{4.6.121}$$

when ϵ is sufficiently small.

Proof. We can divide the proof into several steps:

Step 1: Remainder definitions.

Note that the boundary layer solution depends on ϵ due to the force and the interior solution also depends on ϵ due the boundary condition. We may rewrite the asymptotic expansion as follows:

$$u^\epsilon \sim \sum_{k=0}^{\infty}\epsilon^k U_k^\epsilon + \sum_{k=0}^{\infty}\epsilon^k\mathscr{U}_k^\epsilon. \tag{4.6.122}$$

The remainder can be defined as

$$R_N = u^\epsilon - \sum_{k=0}^{N}\epsilon^k U_k^\epsilon - \sum_{k=0}^{N}\epsilon^k\mathscr{U}_k^\epsilon = u - Q_N - \mathscr{Q}_N, \tag{4.6.123}$$

where

$$Q_N = \sum_{k=0}^{N} \epsilon^k U_k^\epsilon, \tag{4.6.124}$$

$$\mathscr{Q}_N = \sum_{k=0}^{N} \epsilon^k \mathscr{U}_k^\epsilon. \tag{4.6.125}$$

Noting the equation (4.6.39) is equivalent to the equation (4.6.1), we use $\mathcal{L}$ to denote the neutron transport operator as follows:

$$\begin{aligned} \mathcal{L}u &= \epsilon\vec{w}\cdot\nabla_x u + (1+\epsilon^2)u - \bar{u} \\ &= \sin\phi\frac{\partial u}{\partial\eta} - \frac{\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial u}{\partial\phi} + \frac{\partial u}{\partial\theta}\right) + (1+\epsilon^2)u - \bar{u} \end{aligned} \tag{4.6.126}$$

Step 2: Estimates of $\mathcal{L}Q_N$.

The interior contribution can be estimated as

$$\begin{aligned} \mathcal{L}Q_0 &= \epsilon\vec{w}\cdot\nabla_x Q_0 + (1+\epsilon^2)Q_0 - \bar{Q}_0 \\ &= (Q_0 - \bar{Q}_0) + \epsilon\vec{w}\cdot\nabla_x U_0^\epsilon + \epsilon^2 U_0^\epsilon = \epsilon\vec{w}\cdot\nabla_x U_0^\epsilon + \epsilon^2 U_0^\epsilon. \end{aligned} \tag{4.6.127}$$

We can directly estimate

$$|\epsilon\vec{w}\cdot\nabla_x U_0^\epsilon| \leq C\epsilon\,|\nabla_x U_0^\epsilon| \leq C\epsilon, \tag{4.6.128}$$

$$\left|\epsilon^2 U_0^\epsilon\right| \leq C\epsilon^2\,|U_0^\epsilon| \leq C\epsilon^2. \tag{4.6.129}$$

This implies

$$|\mathcal{L}Q_0| \leq C\epsilon. \tag{4.6.130}$$

For higher order term, we can estimate

$$\mathcal{L}Q_N = \epsilon\vec{w}\cdot\nabla_x Q_N + (1+\epsilon^2)Q_N - \bar{Q}_N \;=\; \epsilon^{N+1}\vec{w}\cdot\nabla_x U_N^{\epsilon} + \epsilon^{N+2}U_N^{\epsilon} + \epsilon^{N+1}U_{N-1}^{\epsilon}. \tag{4.6.131}$$

We have

$$\left|\epsilon^{N+1}\vec{w}\cdot\nabla_x U_N^{\epsilon}\right| \;\leq\; C\epsilon^{N+1}\left|\nabla_x U_N^{\epsilon}\right| \leq C\epsilon^{N+1}, \tag{4.6.132}$$

$$\left|\epsilon^{N+2}U_N^{\epsilon} + \epsilon^{N+1}U_{N-1}^{\epsilon}\right| \;\leq\; C\epsilon^{N+2}\left|U_N^{\epsilon}\right| + C\epsilon^{N+1}\left|U_{N-1}^{\epsilon}\right| \leq C\epsilon^{N+1}. \tag{4.6.133}$$

This implies

$$\left|\mathcal{L}Q_N\right| \leq C\epsilon^{N+1}. \tag{4.6.134}$$

Step 3: Estimates of $\mathcal{L}\mathscr{Q}_N$.

The boundary layer solution is $\mathscr{U}_k^{\epsilon} = (f_k^{\epsilon} - f_k^{\epsilon}(\infty))\cdot\psi_0 = \mathscr{V}_k\psi_0$ where $f_k^{\epsilon}(\eta,\theta,\phi)$ solves the ϵ-Milne problem and $\mathscr{V}_k = f_k^{\epsilon} - f_k^{\epsilon}(\infty)$. Notice $\psi_0\psi = \psi_0$, so the boundary layer contribution can be estimated as

$$
\begin{aligned}
& \mathcal{L}\mathscr{Q}_0 && (4.6.135)\\
= \ & \sin\phi\frac{\partial\mathscr{Q}_0}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial\mathscr{Q}_0}{\partial\phi}+\frac{\partial\mathscr{Q}_0}{\partial\theta}\right)+(1+\epsilon^2)\mathscr{Q}_0-\bar{\mathscr{Q}}_0\\
= \ & \sin\phi\left(\psi_0\frac{\partial\mathscr{V}_0}{\partial\eta}+\mathscr{V}_0\frac{\partial\psi_0}{\partial\eta}\right)-\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial\mathscr{V}_0}{\partial\phi}+\frac{\partial\mathscr{V}_0}{\partial\theta}\right)\\
& +(1+\epsilon^2)\psi_0\mathscr{V}_0-\psi_0\bar{\mathscr{V}}_0\\
= \ & \sin\phi\left(\psi_0\frac{\partial\mathscr{V}_0}{\partial\eta}+\mathscr{V}_0\frac{\partial\psi_0}{\partial\eta}\right)-\frac{\psi_0\psi\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial\mathscr{V}_0}{\partial\phi}+\frac{\partial\mathscr{V}_0}{\partial\theta}\right)\\
& +(1+\epsilon^2)\psi_0\mathscr{V}_0-\psi_0\bar{\mathscr{V}}_0\\
= \ & \psi_0\left(\sin\phi\frac{\partial\mathscr{V}_0}{\partial\eta}-\frac{\epsilon\psi}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_0}{\partial\phi}+\mathscr{V}_0-\bar{\mathscr{V}}_0\right)+\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_0\\
& -\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_0}{\partial\theta}+\epsilon^2\psi_0\mathscr{V}_0\\
= \ & \sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_0-\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_0}{\partial\theta}+\epsilon^2\psi_0\mathscr{V}_0.
\end{aligned}
$$

Since $\psi_0=1$ when $\eta\leq 1/(4\epsilon)$, the effective region of $\partial_\eta\psi_0$ is $\eta\geq 1/(4\epsilon)$ which is further and further from the origin as $\epsilon\to 0$. By Theorem 4.6.6, the first term in (4.6.135) can be controlled as

$$\left|\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_0\right| \ \leq \ Ce^{-\frac{K_0}{\epsilon}}\leq C\epsilon. \tag{4.6.136}$$

For the second term in (4.6.135), we have

$$\left|-\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_0}{\partial\theta}\right| \ \leq \ C\epsilon\left|\frac{\partial\mathscr{V}_0}{\partial\theta}\right|\leq C\epsilon. \tag{4.6.137}$$

For the third term in (4.6.135), we have

$$\left|\epsilon^2\psi_0\mathscr{V}_0\right|\leq C\epsilon. \tag{4.6.138}$$

This implies

$$|\mathcal{L}\mathscr{Q}_0| \leq C\epsilon. \tag{4.6.139}$$

For higher order term, we can estimate

$$\begin{aligned}\mathcal{L}\mathscr{Q}_N &= \sin\phi\frac{\partial\mathscr{Q}_N}{\partial\eta} - \frac{\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial\mathscr{Q}_N}{\partial\phi} + \frac{\partial\mathscr{Q}_N}{\partial\theta}\right) + (1+\epsilon^2)\mathscr{Q}_N - \bar{\mathscr{Q}}_N \\ &= \sum_{i=0}^{N}\epsilon^i\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_i - \frac{\psi_0\epsilon^{N+1}}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_N}{\partial\theta} + \epsilon^{N+2}\psi_0\mathscr{V}_N + \epsilon^{N+1}\psi_0\mathscr{V}_{N-1}.\end{aligned} \tag{4.6.140}$$

Away from the origin, the first term in (4.6.140) can be controlled as

$$\left|\sum_{i=0}^{N}\epsilon^i\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_i\right| \leq Ce^{-\frac{K_0}{\epsilon}} \leq C\epsilon^{N+1}. \tag{4.6.141}$$

For the second term in (4.6.140), we have

$$\left|-\frac{\psi_0\epsilon^{N+1}}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_N}{\partial\theta}\right| \leq C\epsilon^{N+1}\left|\frac{\partial\mathscr{V}_N}{\partial\theta}\right| \leq C\epsilon^{N+1}. \tag{4.6.142}$$

For the third term in (4.6.140), we have

$$\left|\epsilon^{N+2}\psi_0\mathscr{V}_N + \epsilon^{N+1}\psi_0\mathscr{V}_{N-1}\right| \leq C\epsilon^{N+1}. \tag{4.6.143}$$

This implies

$$|\mathcal{L}\mathscr{Q}_N| \leq C\epsilon^{N+1}. \tag{4.6.144}$$

Step 4: Proof of (4.6.120).

In summary, since $\mathcal{L}u^\epsilon = 0$, collecting (4.6.123), (4.6.134) and (4.6.144), we can prove

$$|\mathcal{L}R_N| \leq C\epsilon^{N+1}. \tag{4.6.145}$$

Consider the asymptotic expansion to $N = 2$, then the remainder R_2 satisfies the equation

$$\left\{ \begin{array}{rcl} \epsilon\vec{w}\cdot\nabla_x R_2 + R_2 - \bar{R}_2 &=& \mathcal{L}R_2 \ \text{ for } \ \vec{x}\in\Omega,\\ R_2 &=& \mathcal{P}R_2 \ \text{ for } \ \vec{w}\cdot\vec{n}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega. \end{array} \right. \tag{4.6.146}$$

By Theorem 4.6.1, we have

$$\|R_2\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \frac{C}{\epsilon^2}\|\mathcal{L}R_2\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \frac{C\epsilon^3}{\epsilon^2} \leq C\epsilon. \tag{4.6.147}$$

Hence, we have

$$\left\|u^\epsilon - \sum_{k=0}^{2}\epsilon^k U_k^\epsilon - \sum_{k=0}^{2}\epsilon^k \mathscr{U}_k^\epsilon\right\|_{L^\infty} = O(\epsilon). \tag{4.6.148}$$

Since it is easy to see

$$\left\|\sum_{k=1}^{2}\epsilon^k U_k^\epsilon + \sum_{k=1}^{2}\epsilon^k \mathscr{U}_k^\epsilon\right\|_{L^\infty} \leq C\epsilon, \tag{4.6.149}$$

our result naturally follows. This completes the proof of (4.6.120).

Step 5: Proof of (4.6.121).

By (4.6.25), the solution f_1 satisfies the Milne problem

$$\begin{cases} \sin(\theta+\xi)\dfrac{\partial f_1}{\partial\eta}+f_1-\bar{f}_1 &= 0,\\ f_1(0,\theta,\xi) &= \mathcal{P}f_1(0,\theta)+g_1(\theta,\xi) \quad\text{for}\quad \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty}f_1(\eta,\theta,\xi) &= f_1(\infty,\theta). \end{cases} \tag{4.6.150}$$

For convenience of comparison, we make the substitution $\phi=\theta+\xi$ to obtain

$$\begin{cases} \sin\phi\dfrac{\partial f_1}{\partial\eta}+f_1-\bar{f}_1 &= 0,\\ f_1(0,\phi) &= \mathcal{P}f_1(0)+g_1(\theta,\phi) \quad\text{for}\quad \sin\phi>0,\\ \lim_{\eta\to\infty}f_1(\eta,\theta,\xi) &= f_1(\infty,\theta). \end{cases} \tag{4.6.151}$$

Assume (4.6.121) is incorrect. For our $g(\phi)=\cos\phi$ which is independent of θ, since $\mathscr{U}_0=\mathscr{U}_0^\epsilon=0$ and $U_0=U_0^\epsilon=0$, we have

$$\lim_{\epsilon\to0}\left\|(U_1+\mathscr{U}_1)-(U_1^\epsilon+\mathscr{U}_1^\epsilon)\right\|_{L^\infty}=0. \tag{4.6.152}$$

Since now $\mathscr{U}_1$ and $\mathscr{U}_1^\epsilon$ are independent of θ, by (4.6.30) and (4.6.76), we can directly estimate

$$\frac{\partial\bar{U}_1^\epsilon}{\partial\vec{n}} = -\int_0^\infty\int_{-\pi}^{\pi}e^{-V(s)}\bigg(\frac{\psi(\epsilon s)}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\theta}(s,\phi)-\psi(\epsilon s)\mathscr{U}_0^\epsilon(s,\phi)\bigg)\mathrm{d}\phi\mathrm{ds}=0, \tag{4.6.153}$$

and also

$$\frac{\partial\bar{U}_1}{\partial\vec{n}} = -\int_0^\infty\int_{-\pi}^{\pi}e^{-V(s)}\bigg(\frac{\psi(\epsilon s)}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\theta}(s,\phi)-\psi(\epsilon s)\mathscr{U}_0(s,\phi)\bigg)\mathrm{d}\phi\mathrm{ds}=0. \tag{4.6.154}$$

Hence, we have $\bar{U}_1^\epsilon=\bar{U}_1$ in the domain, which further implies $U_1^\epsilon=U_1$. Therefore, we can obtain

$$\lim_{\epsilon\to 0}\|\mathscr{U}_1-\mathscr{U}_1^\epsilon\|_{L^\infty}=0. \tag{4.6.155}$$

Then on the boundary of $\sin\phi>0$, these two boundary layer solutions satisfy

$$\mathscr{U}_1 = g-f_1(\infty), \tag{4.6.156}$$

$$\mathscr{U}_1^\epsilon = g-f_1^\epsilon(\infty). \tag{4.6.157}$$

Naturally, we have the estimate $\lim_{\epsilon\to 0}\|f_1^\epsilon(\infty)-f_1(\infty)\|_{L^\infty}=0$ based on above assumptions. Hence, we may further derive

$$\lim_{\epsilon\to\infty}\|(f_1(\infty)+\mathscr{U}_1)-(f_1^\epsilon(\infty)+\mathscr{U}_1^\epsilon)\|_{L^\infty}=0. \tag{4.6.158}$$

For $0\leq\eta\leq 1/(2\epsilon)$, we have $\psi_0=1$, which means $f_1=\mathscr{U}_1+f_1(\infty)$ and $f_1^\epsilon=\mathscr{U}_1^\epsilon+f_1^\epsilon(\infty)$ on $[0,1/(2\epsilon)]$. Since we have $\mathcal{P}\mathscr{U}_1^\epsilon(0)=-f_1^\epsilon(\infty)$ and $\mathcal{P}\mathscr{U}_1(0)=-f_1(\infty)$, we have recovered the normalization condition, i.e. $\mathcal{P}f_1(0)=\mathcal{P}f_1^\epsilon(0)=0$. Note that g satisfies the compatibility condition (4.6.100). Therefore, the ϵ-Milne problem satisfied by f_1 and f_1^ϵ can be reduced to the ϵ-Milne problem with in-flow boundary. Hence, we can naturally obtain the desired result through the proof of Theorem 4.1.1. $\square$

Chapter Five

Future Work

In this dissertation, we mainly talk about the viscous surface flow and diffusive limit in neutron transport equation. In the future, we would like to extend these results to more general settings and get a deeper understanding of these phenomena.

For the viscous surface wave, in the domain with infinite cross-section and curved bottom, although the local well-posedness has been proved in Chapter 2, the global well-posedness is still open. The main difficulty is the curved bottom will absorb the derivatives aiming at interpolation estimates with minimal counts and lower the decaying rate. This requires us to delicately redesign the energy and dissipation and improve interpolation exponent. On the other hand, we can also consider the problem in more general models, either in an infinite-depth domain, or with surface tension involved.

For the hydrodynamic limit, although we disprove the classical boundary layer theory, the current technique can only tackle the two-dimensional steady problem in a unit plate. There are at least three directions to extend this result: two-dimensional steady problem in general smooth convex domain, which involves more complicated influence due to non-constant boundary curvature and requires further decomposition of the tangential component; three dimensional steady problem in a unit ball, which includes more tangential components in the Milne problem and natural singularity due to coordinate substitution; one-dimensional unsteady problem with general initial and boundary data, which requires a composite of initial layer, boundary layer, and initial-boundary layer whose analysis involves evolutionary Milne problem. Further, if we extend the result from neutron transport equation to Boltzmann equation, a new type of difficulty arising is the regularity requirement. Since we do not general know whether there exist smooth solutions to fluid equations and kinetic equations, the analysis of boundary effect has to be in a new framework with weaker assumptions.

As for the numerical analysis, it is a classical problem to treat multi-scale problem in simulation of kinetic equations. We intend to utilize the asymptotic-preserving scheme to construct a uniform scheme for these two domains. The main difficulty is the analyze of discrete boundary layer, especially discrete Milne problem. This requires us to recover the results for continuous Milne problem in the discrete settings with weaker assumptions.

Appendix A

Related Estimates in Viscous Surface Wave

A.1 Analytic Tools

A.1.1 Products in Sobolev Space

We will need some estimates of the products of functions in Sobolev spaces. Since these results have been proved in lemma A.1 and lemma A.2 of [**?**], we will present the statement of the lemmas here without proof.

Lemma A.1.1. *Let U denote either Σ or Ω.*

1. *Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_1 > n/2$. Let $f \in H^{s_1}(U)$, $g \in H^{s_2}(U)$. Then $fg \in H^r(U)$ and*
$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}} . \tag{A.1.1}$$

2. *Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{s_1}(U)$, $g \in H^{s_2}(U)$. Then $fg \in H^r(U)$ and*
$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}} . \tag{A.1.2}$$

3. *Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{-r}(\Sigma)$, $g \in H^{s_2}(\Sigma)$. Then $fg \in H^{-s_1}(\Sigma)$ and*
$$\|fg\|_{H^{-s_1}} \lesssim \|f\|_{H^{-r}} \|g\|_{H^{s_2}} . \tag{A.1.3}$$

Lemma A.1.2. *Suppose that $f \in C^1(\Sigma)$ and $g \in H^{1/2}(\Sigma)$. Then $fg \in H^{1/2}(\Sigma)$ and*
$$\|fg\|_{H^{1/2}} \lesssim \|f\|_{C^1} \|g\|_{H^{1/2}} . \tag{A.1.4}$$

A.1.2 Poincare-Type Inequality

We need several Poincare-type inequality in Ω. Since all these lemmas have be proved in lemma A.10-A.13 in [25], we will only give the statement here without proof.

Lemma A.1.3. *It holds that*

$$\|f\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^2(\Sigma)}^2 + \|\partial_3 f\|_{L^2(\Omega)}^2, \tag{A.1.5}$$

for all $f \in H^1(\Omega)$. Also, if $f \in W^{1,\infty}(\Omega)$, then

$$\|f\|_{L^\infty(\Omega)}^2 \lesssim \|f\|_{L^\infty(\Sigma)}^2 + \|\partial_3 f\|_{L^\infty(\Omega)}^2. \tag{A.1.6}$$

Lemma A.1.4. *It holds that $\|f\|_{H^0(\Sigma)} \lesssim \|\partial_3 f\|_{H^0}$ for $f \in H^1(\Omega)$ such that $f = 0$ on Σ_b. It also holds that $\|f\|_{L^\infty(\Sigma)} \lesssim \|\partial_3 f\|_{L^\infty(\Omega)}$ for $f \in W^{1,\infty}(\Omega)$ such that $f = 0$ on Σ_b.*

Lemma A.1.5. *It holds that $\|u\|_{H^1} \lesssim \|\mathbb{D}u\|_{H^0}$ for all $u \in H^1(\Omega; R^3)$ such that $f = 0$ on Σ_b.*

Lemma A.1.6. *It holds that $\|f\|_{H^1} \lesssim \|\nabla f\|_{H^0}$ for all $f \in H^1(\Omega)$ such that $f = 0$ on Σ_b. Also, $\|f\|_{W^{1,\infty}(\Omega)} \lesssim \|\nabla f\|_{L^\infty(\Omega)}$ for all $f \in W^{1,\infty}(\Omega)$ such that $f = 0$ on Σ_b.*

A.1.3 Poisson Integral

For a function f defined on $\Sigma = R^2$, the Poisson integral $\mathcal{P}f$ in $R^2 \times (-\infty, 0)$ is defined by

$$\mathcal{P}f(x', x_3) = \int_{R^2} \hat{f}(\xi) e^{2\pi|\xi|x_3} e^{2\pi i x'\cdot\xi} \mathrm{d}\xi, \tag{A.1.7}$$

where $\hat{f}(\xi)$ is the Fourier transform of $f(x')$ on R^2. Then within the slab $R^2 \times (-b, 0)$, we have the following estimate based on lemma A.5 in [25].

Lemma A.1.7. *The Poisson integral satisfies that for $q \in \mathbb{N}$,*

$$\|\nabla^q \mathcal{P} f\|_{H^0}^2 \lesssim \|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2 , \tag{A.1.8}$$

$$\|\nabla^q \mathcal{P} f\|_{H^0}^2 \lesssim \|f\|_{\dot{H}^{q}(\Sigma)}^2 , \tag{A.1.9}$$

where $\dot{H}^s$ denotes the usual homogeneous Sobolev space.

A.1.4 Riesz Potential

For a function f defined in Ω, we define the Riesz potential

$$I_\lambda f(x', x_3) = \int_{-b}^{0} \int_{R^2} \hat{f}(\xi, x_3) \, |\xi|^{-\lambda} \, e^{2\pi i x' \cdot \xi} \mathrm{d}\xi \mathrm{dx}_3. \tag{A.1.10}$$

Similarly, for f defined on Σ, we set

$$I_\lambda f(x') = \int_{R^2} \hat{f}(\xi) \, |\xi|^{-\lambda} \, e^{2\pi i x' \cdot \xi} \mathrm{d}\xi. \tag{A.1.11}$$

We have the following lemmas to describe the product of Riesz potential and its interaction with the horizontal derivatives as lemma A.3 and A.4 in [25].

Lemma A.1.8. *Let $\lambda \in (0, 1)$. If $f \in H^0(\Omega)$ and $g, Dg \in H^1(\Omega)$, then*

$$\|I_\lambda (fg)\|_{H^0} \lesssim \|f\|_{H^0} \, \|g\|_{H^1}^{\lambda} \, \|Dg\|_{H^1}^{1-\lambda} . \tag{A.1.12}$$

If $f \in H^0(\Sigma)$ *and* $g, Dg \in H^1(\Sigma)$*, then*

$$\|I_\lambda(fg)\|_{H^0(\Sigma)} \lesssim \|f\|_{H^0(\Sigma)} \|g\|_{H^1(\Sigma)}^{\lambda} \|Dg\|_{H^1(\Sigma)}^{1-\lambda}. \tag{A.1.13}$$

Lemma A.1.9. *Let* $\lambda \in (0,1)$*. If* $f \in H^k(\Omega)$ *for* $k \geq 1$ *an integer, then*

$$\left\|I_\lambda D^k f\right\|_{H^0} \lesssim \left\|D^{k-1} f\right\|_{H^0}^{\lambda} \left\|D^k f\right\|_{H^0}^{1-\lambda}. \tag{A.1.14}$$

A.1.5 Interpolation Estimates

Here we record several interpolation estimate utilized in our proof. Since they have been proved in lemma A.6-A.8 and lemma 3.18 in [25], we will omit the proof now.

Lemma A.1.10. *For* $s, q > 0$ *and* $0 \leq r \leq s$*, we have the estimate*

$$\|f\|_{H^s(\Sigma)} \lesssim \|f\|_{H^{s-r}(\Sigma)}^{q/(r+q)} \|f\|_{H^{s+q}(\Sigma)}^{r/(r+q)}, \tag{A.1.15}$$

whenever the right hand side is finite.

Lemma A.1.11. *Let* $\mathcal{P}f$ *be the Poisson integral of* f*, defined on* Σ*. Let* $\lambda \geq 0$*,* $q, s \in \mathbb{N}$*, and* $r \geq 0$*. Then the following estimates hold.*

1. *Let*

$$\theta = \frac{s}{q+s+\lambda}, \quad 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A.1.16}$$

Then

$$\|\nabla^q \mathcal{P}f\|_{H^0}^2 \lesssim \left(\|I_\lambda f\|_{H^0}^2\right)^{\theta} \left(\left\|D^{q+s} f\right\|_{H^0}^2\right)^{1-\theta}. \tag{A.1.17}$$

2. *Let* $r+s>1$,

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \quad 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A.1.18}$$

Then

$$\|\nabla^q \mathcal{P} f\|_{L^\infty}^2 \lesssim \left(\|I_\lambda f\|_{H^0}^2 \right)^\theta \left(\left\|D^{q+s} f\right\|_{H^r}^2 \right)^{1-\theta}. \tag{A.1.19}$$

3. *Let* $s>1$. *Then*

$$\|\nabla^q \mathcal{P} f\|_{L^\infty}^2 \lesssim \|D^q f\|_{H^s}^2. \tag{A.1.20}$$

Lemma A.1.12. *Let* f *be defined on* Σ. *Let* $\lambda \geq 0$. *Then we have the following estimates.*

1. *Let* $q, s \in (0,\infty)$ *and*

$$\theta = \frac{s}{q+s+\lambda}, \quad 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A.1.21}$$

Then

$$\|D^q f\|_{H^0}^2 \lesssim \left(\|I_\lambda f\|_{H^0}^2 \right)^\theta \left(\left\|D^{q+s} f\right\|_{H^0}^2 \right)^{1-\theta}. \tag{A.1.22}$$

2. *Let* $q, s \in \mathbb{N}$, $r \geq 0$, $r+s>1$,

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \quad 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A.1.23}$$

Then

$$\|D^q f\|_{L^\infty}^2 \lesssim \left(\|I_\lambda f\|_{H^0}^2 \right)^\theta \left(\left\|D^{q+s} f\right\|_{H^r}^2 \right)^{1-\theta}. \tag{A.1.24}$$

Lemma A.1.13. *Let f be defined in Ω. Let $\lambda \geq 0$, $q, s \in \mathbb{N}$, and $r \geq 0$. Then we have the following estimates.*

1. *Let*

$$\theta = \frac{s}{q+s+\lambda}, \quad 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A.1.25}$$

Then

$$\|D^q f\|_{H^0}^2 \lesssim \left(\|I_\lambda f\|_{H^0}^2 \right)^{\theta} \left(\left\|D^{q+s} f\right\|_{H^0}^2 \right)^{1-\theta}. \tag{A.1.26}$$

2. *Let $r+s>1$*

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \quad 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A.1.27}$$

Then

$$\|D^q f\|_{L^\infty}^2 \lesssim \left(\|I_\lambda f\|_{H^1}^2 \right)^{\theta} \left(\left\|D^{q+s} f\right\|_{H^{r+1}}^2 \right)^{1-\theta}, \tag{A.1.28}$$

and

$$\|D^q f\|_{L^\infty(\Sigma)}^2 \lesssim \left(\|I_\lambda f\|_{H^1}^2 \right)^{\theta} \left(\left\|D^{q+s} f\right\|_{H^{r+1}}^2 \right)^{1-\theta}. \tag{A.1.29}$$

A.1.6 Continuity and Temporal Derivative

In the following, we give two important lemmas to connect L^2H^k norm and $L^\infty H^k$ norm.

Lemma A.1.14. *suppose that $u \in L^2([0,T]; H^{s_1}(\Omega))$ and $\partial_t u \in L^2([0,T]; H^{s_2}(\Omega))$*

for $s_1 \geq s_2 \geq 0$ *and* $s = (s_1 + s_2)/2$. *Then* $u \in C^0([0,T]; H^s(\Omega))$ *and satisfies the estimate*

$$\|u\|^2_{L^\infty H^s} \leq \|u(0)\|^2_{H^s} + \|u\|^2_{L^2 H^{s_1}} + \|\partial_t u\|^2_{L^2 H^{s_2}}, \tag{A.1.30}$$

where the $L^2 H^k$ *norm and* $L^\infty H^k$ *norm are evaluated in* $[0,T]$.

Proof. Considering the extension theorem in Sobolev space, we only need to prove this result in the R^n case. The periodic case can be derived in a similar fashion. Using Fourier transform,

$$\begin{aligned} \partial_t \|u(t)\|^2_{H^s} &= 2\Re\left(\int_{R^n} \langle\xi\rangle^{2s} \hat{u}(\xi,t)\overline{\partial_t \hat{u}(\xi,t)} d\xi \right) \qquad \text{(A.1.31)} \\ &\leq 2\int_{R^n} \langle\xi\rangle^{2s} |\hat{u}(\xi,t)||\partial_t \hat{u}(\xi,t)| d\xi \\ &= 2\int_{R^n} \langle\xi\rangle^{s_1} |\hat{u}(\xi,t)| \langle\xi\rangle^{s_2} |\partial_t \hat{u}(\xi,t)| d\xi \\ &\leq \int_{R^n} \langle\xi\rangle^{2s_1} |\hat{u}(\xi,t)|^2 d\xi + \int_{R^n} \langle\xi\rangle^{2s_2} |\partial_t \hat{u}(\xi,t)|^2 d\xi \\ &= \|u(t)\|^2_{H^{s_1}} + \|\partial_t u(t)\|^2_{H^{s_2}}. \end{aligned}$$

So integrate with respect to time on $[0,t]$

$$\|u(t)\|^2_{H^s} \leq \|u(0)\|^2_{H^s} + \|u\|^2_{L^2 H^{s_1}} + \|\partial_t u\|^2_{L^2 H^{s_2}}. \tag{A.1.32}$$

□

The following lemma shows the estimate in another direction.

Lemma A.1.15. *For any* $u \in L^\infty([0,T]; H^k)$ *within* $[0,T]$, *we must have* $u \in L^2([0,T]; H^k)$ *and satisfies the estimate*

$$\|u\|^2_{L^2 H^k} \leq T\|u\|^2_{L^\infty H^k}. \tag{A.1.33}$$

Proof. The result is simply based on the definition of these two norms. □

A.1.7 Extension Theorem

The following are two extension theorems which will be used to construct start point of iteration from initial data in proving well-posedness of Naiver-Stokes-transport system. Since it is identical as lemma A.5 and A.6 in [26], we omit the proof here.

Lemma A.1.16. *Suppose that $\partial_t^j u(0) \in H^{2N-2j}(\Omega)$ for $j = 0, \ldots, N$, then there exists a extension u achieving the initial data, such that*

$$\partial_t^j u \in L^2([0,\infty); H^{2N-2j+1}(\Omega)) \cap L^\infty([0,\infty); H^{2N-2j}(\Omega))$$

for $j = 0, \ldots, N$. Moreover,

$$\sum_{j=0}^{N} \left\|\partial_t^j u\right\|^2_{L^2 H^{2N-2j+1}} + \left\|\partial_t^j u\right\|^2_{L^\infty H^{2N-2j}} \lesssim \sum_{j=0}^{N} \left\|\partial_t^j u(0)\right\|^2_{H^{2N-2j}}. \tag{A.1.34}$$

Lemma A.1.17. *Suppose that $\partial_t^j p(0) \in H^{2N-2j-1}(\Omega)$ for $j = 0, \ldots, N-1$, then there exists a extension p achieving the initial data, such that*

$$\partial_t^j p \in L^2([0,\infty); H^{2N-2j}(\Omega)) \cap L^\infty([0,\infty); H^{2N-2j-1}(\Omega))$$

for $j = 0, \ldots, N-1$. Moreover,

$$\sum_{j=0}^{N-1} \left\|\partial_t^j p\right\|^2_{L^2 H^{2N-2j}} + \left\|\partial_t^j p\right\|^2_{L^\infty H^{2N-2j-1}} \lesssim \sum_{j=0}^{N-1} \left\|\partial_t^j p(0)\right\|^2_{H^{2N-2j-1}}. \tag{A.1.35}$$

A.2 Estimates for Fundamental Equations

A.2.1 Transport Estimates

Let Σ be either infinite or periodic. Consider the equation

$$\begin{cases} \partial_t \eta + u \cdot D\eta = g, \quad \text{in} \quad \Sigma \times (0,T) \\ \eta(t=0) = \eta_0. \end{cases} \tag{A.2.1}$$

We have the following estimate of the regularity solution to this equation, which is a particular case of a more general result proved in proposition 2.1 of [17]. Note that the result in [17] is stated for $\Sigma = R^2$, but the same result holds in the periodic case as described in [18].

Lemma A.2.1. *Let η be a solution to equation (A.2.1). Then there exists a universal constant $C > 0$ such that for any $s \geq 3$*

$$\|\eta\|_{L^\infty H^s} \leq \exp\left(C \int_0^t \|u(r)\|_{H^s(\Sigma)} \,\mathrm{dr}\right)\left(\|\eta_0\|_{H^s} + \int_0^t \|g(r)\|_{H^s} \,\mathrm{dr}\right). \tag{A.2.2}$$

Proof. Use $p = p_2 = 2$, $r = 2$ and $\sigma = s$ in proposition 2.1 of [18]. See the proof of Theorem 5.4 in [25]. □

Lemma A.2.2. *Let η be a solution to (A.2.1). Then there is a universal constant $C > 0$ such that for any $0 \leq s < 2$*

$$\|\eta\|_{L^\infty H^s} \leq \exp\left(C \int_0^t \|Du(r)\|_{H^{3/2}(\Sigma)} \,\mathrm{dr}\right)\left(\|\eta_0\|_{H^s(\Sigma)} + \int_0^t \|g(r)\|_{H^s(\Sigma)} \,\mathrm{dr}\right). \tag{A.2.3}$$

Proof. The same as lemma A.9 in [25]. □

A.2.2 Elliptic Estimates

We need two different type of elliptic estimates in our proof either in weak sense or strong sense.

Lemma A.2.3. *(weak solution) Assume* $\int_\Omega \psi \mathrm{dx} = 0$*. We call* $(u,p) \in H_0^1(\Omega) \times H^0(\Omega)$ *a weak solution to the elliptic equation*

$$\begin{cases} -\Delta u + \nabla p = \phi \in H^{-1}(\Omega), & in \quad \Omega \\ \nabla \cdot u = \psi \in H^0(\Omega), & in \quad \Omega \\ u = 0, & on \quad \Sigma \\ u = 0, & on \quad \Sigma_b \end{cases} \tag{A.2.4}$$

if $\nabla \cdot u = \psi$ *in* Ω *and* (u,p) *satisfy the weak formulation that for arbitrary* $f \in H_0^1(\Omega)$*, it holds that*

$$\langle \mathbb{D}u : \mathbb{D}f \rangle_{H^0} + \langle p, \nabla \cdot f \rangle_{H^0} = \langle \phi, f \rangle_{H^{-1}}, \tag{A.2.5}$$

where $\langle \cdot, \cdot \rangle_{H^0}$ *represents the inner product in* $H^0(\Omega)$ *and* $\langle \cdot, \cdot \rangle_{H^{-1}}$ *denotes the dual pairing between* $H_0^1(\Omega)$ *and* $H^{-1}(\Omega)$*. Then we have the estimate*

$$\|u\|_{H^1}^2 + \|p\|_{H^0}^2 \lesssim \|\phi\|_{H^{-1}}^2 + \|\psi\|_{H^0}^2 . \tag{A.2.6}$$

Proof. An obvious modification for the proof of lemma 3.3 in [8] implies that for any $\psi \in H^0(\Omega)$, there exists a $v \in H_0^1(\Omega)$ such that $\nabla \cdot v = \psi$ in Ω and satisfies the estimate $\|v\|_{H^1}^2 \lesssim \|\psi\|_{H^0}^2$. Then we may change to the unknown $w = u - v$, which satisfies the system

$$\begin{cases} -\Delta w + \nabla p = \phi + \Delta v, & \text{in} \quad \Omega \\ \nabla \cdot w = 0, & \text{in} \quad \Omega \\ w = 0, & \text{on} \quad \Sigma \\ w = 0. & \text{on} \quad \Sigma_b \end{cases} \tag{A.2.7}$$

Then we have the new weak formulation that for arbitrary $f \in H_0^1(\Omega)$, it is valid that

$$\langle \mathbb{D}w : \mathbb{D}f \rangle_{H^0} + \langle p, \nabla \cdot f \rangle_{H^0} = \langle \phi, f \rangle_{H^{-1}} - \langle \mathbb{D}v : \mathbb{D}f \rangle_{H^0}. \tag{A.2.8}$$

We may further restrict that f satisfies $\nabla \cdot f = 0$ as the pressureless weak formulation that

$$\langle \mathbb{D}w : \mathbb{D}f \rangle_{H^0} = \langle \phi, f \rangle_{H^{-1}} - \langle \mathbb{D}v : \mathbb{D}f \rangle_{H^0}. \tag{A.2.9}$$

Riesz theorem implies that there exists a pressureless weak solution $w \in H_0^1(\Omega)$ satisfying $\nabla \cdot w = 0$ and the estimate

$$\|w\|_{H^1}^2 \lesssim \|\phi\|_{H^{-1}}^2 + \|v\|_{H^1}^2 \lesssim \|\phi\|_{H^{-1}}^2 + \|\psi\|_{H^0}^2. \tag{A.2.10}$$

Then it is wellknown that pressure can be taken as the multiplier for the Navier-Stokes system. Similar to proposition 2.9 in [26], we have that there exists a $p \in H^1(\Omega)$ such that the weak formulation (A.2.8) holds and it satisfies

$$\|p\|_{H^0}^2 \lesssim \|\phi\|_{H^{-1}}^2 + \|\psi\|_{H^0}^2 + \|w\|_{H^1}^2 \lesssim \|\phi\|_{H^{-1}}^2 + \|\psi\|_{H^0}^2. \tag{A.2.11}$$

Therefore, (u, p) satisfy the weak formulation (A.2.5) and our result easily follows. $\square$

Lemma A.2.4. *(strong solution) Suppose (u,p) solve the equation*

$$\begin{cases} -\Delta u + \nabla p = \phi \in H^{r-2}(\Omega), & in \quad \Omega \\ \nabla \cdot u = \psi \in H^{r-1}(\Omega), & in \quad \Omega \\ (pI - \mathbb{D}u)e_3 = \varphi \in H^{r-3/2}(\Sigma), & on \quad \Sigma \\ u = 0, & on \quad \Sigma_b \end{cases} \tag{A.2.12}$$

in the strong sense. Then for $r \geq 2$,

$$\|u\|_{H^r}^2 + \|p\|_{H^{r-1}}^2 \lesssim \|\phi\|_{H^{r-2}}^2 + \|\psi\|_{H^{r-1}}^2 + \|\varphi\|_{H^{r-3/2}}^2 . \tag{A.2.13}$$

whenever the right hand side is finite.

Proof. The same as lemma A.15 in [26]. □

Appendix B

Counterexamples in Neutron Transport Equation

B.1 Construction of the Counterexample with In-Flow Boundary

Lemma B.1.1. *For the Milne problem*

$$\begin{cases} \sin(\theta+\xi)\dfrac{\partial f}{\partial\eta}+f-\bar{f} &= 0,\\ f(0,\theta,\xi) &= g(\theta,\xi) \quad \text{for} \quad \sin(\theta+\xi)>0,\\ \lim_{\eta\rightarrow\infty} f(\eta,\theta,\xi) &= f(\infty,\theta), \end{cases} \tag{B.1.1}$$

if $g(\theta,\xi)=\cos(3(\theta+\xi))$*, then we have*

$$\frac{\partial f}{\partial\eta}\notin L^{\infty}([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)). \tag{B.1.2}$$

Proof. We divide the proof into several steps: we first assume $\partial_{\eta}f\in L^{\infty}([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$ and then show it can lead to a contradiction.

Step 1: Maximum principle

Theorem 4.4.14 implies the solution f to the problem (B.1.1) satisfies the maximum principle, i.e. for any (η,θ,ϕ)

$$\min_{\sin(\theta+\xi)>0} g(\theta,\xi)\leq f(\eta,\theta,\xi)\leq \max_{\sin(\theta+\xi)>0} g(\theta,\xi). \tag{B.1.3}$$

We can see the data $g(\theta,\xi)=\cos(3(\theta+\xi))$ satisfying $g(\theta,-\theta)=1$ and $|g|\leq 1$. Based on the maximum principle, we can derive for any (η,θ,ξ)

$$f(\eta,\theta,\xi)\leq 1. \tag{B.1.4}$$

Hence, certainly we have $f(0,\theta,\xi)\leq 1$ for $\sin(\theta+\xi)<0$.

Step 2: Estimates of $\bar f(0,\theta)$.

We can directly estimate

$$\begin{aligned}\bar f(0,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(0,\theta,\xi)\mathrm{d}\xi \\ &= \frac{1}{2\pi}\bigg(\int_{\sin(\theta+\xi)<0} f(0,\theta,\xi)\mathrm{d}\xi + \int_{\sin(\theta+\xi)>0} f(0,\theta,\xi)\mathrm{d}\xi\bigg) \\ &\leq \frac{1}{2\pi}\int_{\sin(\theta+\phi)>0} f(0,\theta,\xi)\mathrm{d}\xi + \frac{1}{2}.\end{aligned} \tag{B.1.5}$$

By the choice of g, we naturally have

$$\int_{\sin(\theta+\xi)>0} f(0,\theta,\xi)\mathrm{d}\xi = \int_{\sin(\theta+\xi)>0} g(\theta,\xi)\mathrm{d}\xi = 0. \tag{B.1.6}$$

Then this implies

$$\bar f(0,\theta) \leq \frac{1}{2}. \tag{B.1.7}$$

Step 3: Definition of trace.

It is easy to see $\partial_\eta f$ satisfies the Milne problem

$$\sin(\theta+\xi)\frac{\partial(\partial_\eta f)}{\partial\eta} + \partial_\eta f - \overline{\partial_\eta f} = 0. \tag{B.1.8}$$

Since we have $\partial_\eta f \in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$ which implies $\overline{\partial_\eta f} \in L^\infty([0,L]\times[-\pi,\pi))$, by Ukai's trace theorem, we may define the trace of $\partial_\eta f$ on $\eta = 0$ satisfying $\partial_\eta f(0,\theta,\phi) \in L^\infty[-\pi,\pi)\times[-\pi,\pi)$.

However, we can define the trace of $\partial_\eta f$ in another fashion. For any $\xi \neq -\theta$ and $\xi \neq \pi-\theta$, we have $\sin(\theta+\xi)\neq 0$. Since we have $f \in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$

as well as $\bar{f} \in L^\infty[0,\infty) \times [-\pi,\pi)$, by the Milne problem (B.1.1), it is naturally to define for $\eta > 0$

$$\partial_\eta f(\eta,\theta,\xi) = \frac{\bar{f}(\eta,\theta) - f(\eta,\theta,\xi)}{\sin(\theta+\xi)}. \tag{B.1.9}$$

Since $\partial_\eta f \in L^\infty([0,\infty) \times [-\pi,\pi) \times [-\pi,\pi))$, we know f is continuous with respect to η for a.e. (θ,ξ). Taking $\eta \to 0$ defines the trace for $\partial_\eta f$ at$(0,\theta,\xi)$

$$\partial_\eta f(0,\theta,\xi) = \frac{\bar{f}(0,\theta) - f(0,\theta,\xi)}{\sin(\theta+\xi)}. \tag{B.1.10}$$

Since the grazing set $\{(\theta,\xi) : \theta+\xi = 0 \quad \text{or} \quad \theta+\xi = \pi\}$ is zero-measured on the boundary $\eta = 0$, then we have the trace of $\partial_\eta f$ is a.e. well-defined.

By the uniqueness of trace of $\partial_\eta f$, above two types of traces must coincide with each other a.e.. Then we may combine them both and obtain $\partial_\eta f(0,\theta,\xi) \in L^\infty[-\pi,\pi) \times [-\pi,\pi)$ is a.e. well-defined and satisfies the formula

$$\partial_\eta f(0,\theta,\xi) = \frac{\bar{f}(0,\theta) - f(0,\theta,\xi)}{\sin(\theta+\xi)}. \tag{B.1.11}$$

Step 4: Contradiction.

Therefore, we may consider the limiting process

$$\lim_{\xi\to-\theta^+} \frac{\partial f}{\partial\eta}(0,\theta,\xi) = \lim_{\xi\to-\theta^+} \frac{\bar{f}(0,\theta) - f(0,\theta,\xi)}{\sin(\theta+\xi)}. \tag{B.1.12}$$

Since we know as $\xi \to -\theta^+$, it follows that

$$\sin(\theta+\xi) \quad \to \quad 0^+ \tag{B.1.13}$$

$$\bar{f}(0,\theta) - f(0,\theta,\xi) \quad \to \quad \bar{f}(0,\theta) - g(\theta,-\theta) = \bar{f}(0,\theta) - 1 < 0. \tag{B.1.14}$$

Then this leads to

$$\lim_{\xi \to -\theta^+} \frac{\partial f}{\partial \eta}(0, \theta, \xi) = -\infty. \tag{B.1.15}$$

which means $\partial_\eta f(0, \theta, \xi) \notin L^\infty[-\pi, \pi) \times [-\pi, \pi)$. This contradicts our result in the previous step. Hence, our assumption that $\partial_\eta f \in L^\infty([0, \infty) \times [-\pi, \pi) \times [-\pi, \pi))$ cannot be true.

Step 5: Another contradiction.

There is another way to show this fact. Since $\partial_\eta f \in L^\infty([0, L] \times [-\pi, \pi) \times [-\pi, \pi))$. Also, we have $f \in L^\infty([0, L] \times [-\pi, \pi) \times [-\pi, \pi))$. Then this implies f is Lipschitz continuous in $[0, \infty)$ with respect to η for a.e. (θ, ξ). Hence, this implies $\bar{f}$ is also Lipschitz continuous in $\eta \in [0, \infty)$. Without loss of generality, we may assume $\|\partial_\eta f\|_\infty \leq M$. Thus we have for a.e. $(\theta, \xi) \in \sin(\theta + \phi) > 0$

$$\begin{aligned} |f(\eta, \theta, \xi) - f(0, \theta, \xi)| &\leq M\eta & \text{(B.1.16)} \\ \left|\bar{f}(\eta, \theta) - \bar{f}(0, \theta)\right| &\leq M\eta. & \text{(B.1.17)} \end{aligned}$$

Then

$$\begin{aligned} & \left|f(\eta, \theta, \xi) - \bar{f}(\eta, \theta)\right| & \text{(B.1.18)} \\ \geq\ & \left|f(0, \theta, \xi) - \bar{f}(0, \theta)\right| - |f(\eta, \theta, \xi) - f(0, \theta, \xi)| - \left|\bar{f}(\eta, \theta) - \bar{f}(0, \theta)\right| \\ \geq\ & \left|\bar{f}(0, \theta) - g(\theta, -\theta)\right| - 2M\eta. \end{aligned}$$

Since we know $\left|\bar{f}(0, \theta) - g(\theta, -\theta)\right| \geq C > 0$ for some constant C. Then as long as $\eta \leq C/(4M)$, we have

$$\left|f(\eta, \theta, \xi) - \bar{f}(\eta, \theta)\right| \geq \frac{C}{2}. \tag{B.1.19}$$

Since for (θ,ξ) not in the grazing set, we always have

$$\partial_\eta f(\eta,\theta,\xi)=\frac{\bar{f}(\eta,\theta)-f(\eta,\theta,\xi)}{\sin(\theta+\xi)}. \tag{B.1.20}$$

then $|\partial_\eta f|$ can be arbitrarily large as long as $\sin(\theta+\xi)$ is sufficiently small, and also it possesses a positive measure. This implies $\partial_\eta f\in L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi))$ cannot be true. $\square$

B.2 Construction of the Counterexample with Diffusive Boundary

Lemma B.2.1. *For the Milne problem*

$$\left\{\begin{array}{rcl}\sin(\theta+\xi)\dfrac{\partial f}{\partial\eta}+f-\bar{f}&=&0,\\ f(0,\theta,\xi)&=&\mathcal{P}f(0,\theta)+g_1(\theta,\xi)\quad for\quad \sin(\theta+\xi)>0,\\ \lim_{\eta\rightarrow\infty}f(\eta,\theta,\xi)&=&f(\infty,\theta),\end{array}\right. \tag{B.2.1}$$

with

$$\mathcal{P}f(0,\theta)=0. \tag{B.2.2}$$

If $g(\theta,\xi)=\cos(3(\theta+\xi))$, then we have

$$\frac{\partial f}{\partial\eta}\notin L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)). \tag{B.2.3}$$

Proof. For $g(\theta,\xi)=\cos(3(\theta+\xi))$, we can easily derive $U_0=0$. Hence $g_1=g$. Note that g satisfies the compatibility condition (4.6.100). With the normalization condition $\mathcal{P}f(0,\theta)=0$, we can see this problem reduces to the Milne problem with in-flow boundary

$$\left\{\begin{array}{rcl}\sin(\theta+\xi)\dfrac{\partial f}{\partial\eta}+f-\bar{f} &=& 0,\\ f(0,\theta,\xi) &=& g(\theta,\xi)\ \ \text{for}\ \ \sin(\theta+\xi)>0,\\ \lim_{\eta\rightarrow\infty}f(\eta,\theta,\xi) &=& f_1(\infty,\theta).\end{array}\right. \tag{B.2.4}$$

Therefore, we can complete the proof by Theorem B.1.1. $\square$

Bibliography

[1] H. Abels, *The initial-value problem for the Navier-Stokes equations with a free surface in L^q-Sobolev spaces*, Adv. Differential Equations, 10 (2005), pp. 45–64.

[2] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions i.*, Comm. Pure Appl. Math., 12 (1959), pp. 623–727.

[3] M. J. Ahn, H. Y. Lee, and M. R. Ohm, *Error estimates for fully discrete approximation to a free boundary problem in polymer technology*, Appl. Math. Comput., 138 (2003), pp. 227–238.

[4] L. Arkeryd, R. Esposito, R. Marra, and A. Nouri, *Stability for Rayleigh-Benard convective solutions of the Boltzmann equation*, Arch. Ration. Mech. Anal., 198 (2010), pp. 125–187.

[5] ——, *Ghost effect by curvature in planar Couette flow*, Kinet. Relat. Models, 4 (2011), pp. 109–138.

[6] L. Arkeryd and A. Nouri, *On a Taylor-Couette type bifurcation for the stationary nonlinear Boltzmann equation*, J. Stat. Phys., 124 (2006), pp. 401–443.

[7] H. Bae, *Solvability of the free boundary value problem of the Navier-Stokes equations*, Discrete Contin. Dyn. Syst., 29 (2011), pp. 769–801.

[8] J. T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure Appl. Math., 34 (1981), pp. 359–392.

[9] A. Bensoussan, J.-L. Lions, and G. C. Papanicolaou, *Boundary layers and homogenization of transport processes*, Publ. Res. Inst. Math. Sci., 15 (1979), pp. 53–157.

[10] C. Cercignani, R. Illner, and M. Pulvirenti, *The mathematical theory of dilute gases*, Springer-Verlag, New York, 1994.

[11] C. Cercignani, R. Marra, and R. Esposito, *The Milne problem with a force term*, Transport Theory Statist. Phys., 27 (1998), pp. 1–33.

[12] D. Christodoulou and H. Lindblad, *On the motion of the free surface of a liquid*, Comm. Pure Appl. Math., 53 (2000), pp. 1536–1602.

[13] B. Cockburn, S. Hou, and C.-W. Shu, *The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV. the multidimensional case*, Math. Comp., 54 (1990), pp. 545–581.

[14] B. Cockburn and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws II. general framework*, Math. Comp., 52 (1989), pp. 411–435.

[15] ——, *Runge-Kutta discontinuous Galerkin methods for convection-dominated problems*, J. Sci. Comput., 16 (2001), pp. 173–261.

[16] D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc., 20 (2007), pp. 829–930.

[17] R. Danchin, *Estimates in Besov spaces for transport and transport-diffusion equations with almost Lipschitz coefficients*, Rev. Mat. Iberoamericana, 21 (2005), pp. 863–888.

[18] ——, *Fourier analysis methods for PDEs*, Preprint, http://perso-math.univ-mlv.fr/users/danchin.raphael/recherche.html (2005).

[19] R. Esposito, Y. Guo, C. Kim, and R. Marra, *Non-isothermal boundary in the Boltzmann theory and Fourier law*, Comm. Math. Phys., 323 (2013), pp. 177–239.

[20] P. Germain, N. Masmoudi, and J. Shatah, *Global solutions for the gravity water waves equation in dimension 3*, Ann. of Math., 175 (2012), pp. 691–754.

[21] V. Girault, B. Riviere, and M. F. Wheeler, *A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems*, Math. Comp., 74 (2005), pp. 53–84.

[22] ——, *A splitting method using discontinuous Galerkin for the transient incompressible Navier-Stokes equations*, ESAIM Math. Model. Numer. Anal., 39 (2005), pp. 1115–1147.

[23] J. Grooss and J. S. Hesthaven, *A level set discontinuous Galerkin method for free surface flows*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 3406–3429.

[24] Y. Guo and I. Tice, *Almost exponential decay of periodic viscous surface waves without surface tension*, Anal. PDE, 6 (2013), pp. 459–531.

[25] ——, *Decay of viscous surface waves without surface tension in horizontally infinite domains*, Arch. Ration. Mech. Anal., 207 (2013), pp. 1429–1533.

[26] ——, *Local well-posedness of the viscous surface wave problem without surface tension*, Anal. PDE, 6 (2013), pp. 287–369.

[27] E. Hairer, C. Lubich, and M. Roche, *The numerical solution of differential-algebraic systems by Runge-Kutta methods*, Springer-Verlag, Berlin, 1989.

[28] F. H. Harlow and J. E. Welch, *Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface*, Phys. Fluids, 8 (1965), pp. 2182–2189.

[29] Y. Hataya, *Decaying solution of a Navier-Stokes flow without surface tension*, J. Math. Kyoto Univ., 49 (2009), pp. 691–717.

[30] C. Hirt and B. Nichols, *Volume of fluid method (VOF) for the dynamics of free boundaries*, J. Comput. Phys., 39 (1981), pp. 201–225.

[31] D. Lannes, *Well-posedness of the water-waves equations*, J. Amer. Math. Soc., 18 (2005), pp. 605–654.

[32] E. W. Larsen, *A functional-analytic approach to the steady, one-speed neutron transport equation with anisotropic scattering*, Comm. Pure Appl. Math., 27 (1974), pp. 523–545.

[33] ——, *Solutions of the steady, one-speed neutron transport equation for small mean free paths*, J. Mathematical Phys., 15 (1974), pp. 299–305.

[34] ——, *Neutron transport and diffusion in inhomogeneous media I.*, J. Mathematical Phys., 16 (1975), pp. 1421–1427.

[35] ——, *Asymptotic theory of the linear transport equation for small mean free paths II.*, SIAM J. Appl. Math., 33 (1977), pp. 427–445.

[36] E. W. Larsen and J. D'Arruda, *Asymptotic theory of the linear transport equation for small mean free paths I.*, Phys. Rev., 13 (1976), pp. 1933–1939.

[37] E. W. Larsen and G. J. Habetler, *A functional-analytic derivation of Case's full and half-range formulas*, Comm. Pure Appl. Math., 26 (1973), p-p. 525–537.

[38] E. W. Larsen and J. B. Keller, *Asymptotic solution of neutron transport problems for small mean free paths*, J. Mathematical Phys., 15 (1974), pp. 75–81.

[39] E. W. Larsen and P. F. Zweifel, *On the spectrum of the linear transport operator*, J. Mathematical Phys., 15 (1974), pp. 1987–1997.

[40] ——, *Steady, one-dimensional multigroup neutron transport with anisotropic s-cattering*, J. Mathematical Phys., 17 (1976), pp. 1812–1820.

[41] H. Y. Lee, *Error analysis of finite element approximation of a Stefan problem with nonlinear free boundary condition*, J. Appl. Math. Comput., 22 (2006), pp. 223–235.

[42] H. Lindblad, *Well-posedness for the motion of an incompressible liquid with free surface boundary*, Ann. of Math., 162 (2005), pp. 109–194.

[43] R. H. Nochetto and C. Verdi, *An efficient linear scheme to approximate parabolic free boundary problems: error estimates and implementation*, Math. Comp., 51 (1988), pp. 27–53.

[44] J. Shatah and C. Zeng, *Geometry and a priori estimates for free boundary problems of the Euler equation*, Comm. Pure Appl. Math., 61 (2008), pp. 698–744.

[45] ——, *Local well-posedness for fluid interface problems*, Arch. Ration. Mech. Anal., 199 (2011), pp. 653–705.

[46] V. Solonnikov, *Solvability of a problem on the motion of a viscous incompressible fluid that is bounded by a free surface*, Math. USSR-Izv., 11 (1978), pp. 1323–1358.

[47] M. Sussman and M. Y. Hussaini, *A discontinuous spectral element method for the level set equation*, J. Sci. Comput., 19 (2003), pp. 479–500.

[48] L.-h. Wang, *On Korn's inequality*, J. Comput. Math., 21 (2003), pp. 321–324.

[49] M. F. Wheeler, *An elliptic collocation-finite element method with interior penalties*, SIAM J. Numer. Anal., 15 (1978), pp. 152–161.

[50] L. Wu, *Well-posedness and decay of the viscous surface wave*, SIAM J. Math. Anal., 46 (2014), pp. 2084–2135.

[51] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 2-D*, Invent. Math., 130 (1997), pp. 39–72.

[52] ——, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc., 12 (1999), pp. 445–495.

[53] ——, *Global well-posedness of the 3-D full water wave problem*, Invent. Math., 184 (2011), pp. 125–220.

[54] X. Yang, *Asymptotic behavior on the milne problem with a force term*, J. Differential Equations, 252 (2012), pp. 4656–4678.

[55] P. Zhang and Z. Zhang, *On the free boundary problem of three-dimensional incompressible Euler equations*, Comm. Pure Appl. Math., 61 (2008), pp. 877–940.