Resolution of singularities via logarithmic stacks

with a view toward the monodromy conjecture

by

Ming Hao Quek

B.Sc., National University of Singapore, 2018

A dissertation submitted in partial fulfillment of the

requirements for the degree of Doctor of Philosophy

in the Department of Mathematics at Brown University

Providence, Rhode Island

May 2023

This dissertation by Ming Hao Quek is accepted in its present form by the Department of Mathematics as satisfying the dissertation requirement for the degree of Doctor of Philosophy.

Date ______________ ____

Dan Abramovich, Advisor

Recommended to the Graduate Council

Date ______________ ____

Melody Chan, Reader

Date ______________ ____

Brendan Hassett, Reader

Approved by the Graduate Council

Date ______________ ____

Thomas A. Lewis
Dean of the Graduate School

Curriculum Vitae

Ming Hao Quek was born and raised in Singapore, and he completed his education in Singapore. He earned his Bachelor of Science degree in Mathematics from National University of Singapore (NUS) in 2018, where he also completed his undergraduate thesis under the guidance of Gan Wee Teck. From September 2018 to May 2023, he pursued his Ph.D. in Mathematics at Brown University under the supervision of Dan Abramovich. He has spent the Fall semester of 2021 as a pre-doctoral fellow at Institut Mittag-Leffler, and was also an academic guest under Johannes Nicaise at Katholieke Universiteit Leuven in the Spring semester of 2023.

Publication List

4. *Around the motivic monodromy conjecture for non-degenerate hypersurfaces.*
Submitted. arXiv: 2112.06361.

3. *Weighted blow-ups*, with David Rydh.
Draft available: https://people.kth.se/∼dary/weighted-blowups20220329.pdf.

2. *Logarithmic resolution via multi-weighted blow-ups*, with Dan Abramovich.
Submitted. arXiv: 2112.06361.

1. *Logarithmic resolution via weighted toroidal blow-ups.*
Algebraic Geometry, **9**(3), May 2022, 311–363.

Acknowledgements

First and foremost, I am deeply grateful for my amazing advisor Dan Abramovich. I would not have been able to produce the research in this thesis without your generosity with time, your commitment to your students, and your receptiveness to my ideas and suggestions. In particular, I thank you for introducing me to the world of resolution of singularities during my second year of graduate school, and I thank you for connecting me with so many inspiring mathematicians, e.g. Johannes Nicaise and David Rydh. I also cannot thank you enough for my overseas research experiences in Europe, which were instrumental in my development as a mathematician, and which I will be cherishing for years to come. They would not have been possible without your encouragements and generosity.

Secondly, I thank my friends at Brown, for the daily conversations, for the mathematics that they shared with me, and for making my life at Brown brighter. Thank you Veronica Arena, Shamil Asgarli, Megan Chang-Lee, Changjie Chen, Sam Freedman, Tangli Ge, Sarah Griffith-Siqueira, Siddarth Kannan, Troy (Yanze) Liu, Stephen Obinna, Claudia Yun, Jiahua Zhou, and Eric Zhu. In particular, I would like to address a very special thank you to Troy, for being such an awesome roommate over the last three years, for always looking out for me, and for working out with me in the gym during my second year at Brown (until the pandemic unfortunately struck and foiled a good routine). I would also like to thank my Singaporean friends Yan Sheng Ang and Jit Wu Yap, as well as my friends Rosie Shen and Anh Duc Vo, for the time we spent when I am over at Boston. I also thank Dori Bejleri, Elden Elmanto, Yu-Wen Hsu, and Jeremy Usatine for their advice, their guidance, as well as the interesting conversations.

During my time at the Mittag-Leffler Institute in Sweden and the Oberwolfach Research Institute for Mathematics in Germany, I also had the privilege of interacting with the following people: Toni Annala, Samir Canning, Neeraj Deshmukh, Louis Hainaut, Nick Kuhn, Oliver Leigh, Siddharth Mathur, Alex Motzkin, Matthias Paulsen, Michele Pernice, Martin Raum, Sebastian Schlegel-Mejia, and Tuomas Tajakka. I thank them for the memorable times, and for embracing my shy personality. In addition, I am especially thankful to Andrew Kresch for always striking a conversation with me in the library of Mittag-Leffler Institute. I also thank John Christian Ottem, Dan Petersen, and David Rydh for organizing the Moduli and Algbraic Cycles program at Mittag-Leffler Institute, and giving me the opportunity to participate.

Thirdly, my time in graduate school would have been much tougher without the support and the wisdom shared by the following professors: Melody Chan, Brendan Hassett, Andrew Kresch, Johannes Nicaise, Sam Payne, Mihnea Popa, David Rydh, Matt Satriano, Michael Temkin, and Ravi Vakil. In particular, I would like to express my gratitude to David Rydh for our collaboration, for patiently explaining many concepts to me, and for his insights. Chapter 2 of this thesis definitely would not have been possible without his insights. I also thank Johannes Nicaise for his suggestions (which led to Chapter 5 of this thesis), and for hosting my visit at Katholieke Universiteit Leuven during the Spring semester of 2023. I am grateful to Mihnea Popa for his course on Hodge ideals in the Spring semester of 2021, and to Brendan Hassett for the reading course on Hodge theory during my first year of graduate school. Finally, I am thankful to Ravi Vakil for always seeing something in me, and his generosity.

Last but not definitely not least, I thank my parents and my brother for their love and support. I dedicate this thesis to them. Thank you as well to the wonderful staff of the Brown mathematics department: Audrey Aguiar, Lori Nascimento, and Doreen Pappas, for making the department a wonderful work environment, and for having made my graduate student life at Brown so smooth sailing.

Table of Contents

CHAPTER 1

Introduction

This thesis focuses on the study of singularities in algebraic geometry, and involve two broad topics:

(i) logarithmic resolution of singularities,

(ii) and the monodromy conjecture of Denef–Loeser.

This thesis is based on four papers either authored or co-authored by the author. In chronological order, they are:

(i) [**Que22a**], which is the content of Chapter 3,

(ii) [**QR21**] co-authored with David Rydh, which is the content of Chapter 2,

(iii) [**AQ21**] co-authored with Dan Abramovich, which is the content of Chapter 4,

(iv) and finally, [**Que22b**], which is the content of Chapter 5.

In this thesis, all ideals considered on schemes (or more generally algebraic stacks) are always *quasi-coherent*, *finitely generated* ideals. In addition, since toric geometry is prevalent throughout this thesis, we shall fix, once and for all, the following:

Convention 1.0.1 (Conventions on lattices, fans, and their cones)**.** For $m \in \mathbf{N}$, we set $[m] = \{1, 2, \ldots, m\}$ (i.e. $[0] = \varnothing$). We reserve the letter $N = \mathbf{Z}^n$ (and sometimes L) for the lattice of one-parameter subgroups of a torus $\mathbb{G}_m^n$, and the letter $M = N^\vee := \mathrm{Hom}(N, \mathbf{Z})$ for its dual lattice of characters of $\mathbb{G}_m^n$. The standard basis vectors of N will be denoted $\mathbf{e}_i$ for $i \in [n]$, while the standard basis vectors of M will be denoted $\mathbf{e}_i^\vee$ for $i \in [n]$.

For a subring R of $\mathbf{R}$, we set $N_R = N \otimes_{\mathbf{Z}} R$ with positive half-space $N_R^+ = R_{\geq 0}^n \subset N_R$ (where $R_{\geq 0} = R \cap \mathbf{R}_{\geq 0}$). Likewise we also set $M_R = M \otimes_{\mathbf{Z}} R$ with positive half-space $M_R^+ = \mathrm{Hom}_{\mathbf{N}}(R_{\geq 0}^n, R_{\geq 0}) \subset M_R$. We write N^+ for $N_{\mathbf{Z}}^+$ and M^+ for $M_{\mathbf{Z}}^+$.

Given a fan Σ in $N_{\mathbf{R}}$, X_Σ will denote the toric variety of Σ. We will follow any conventions in [CLS11] pertaining to toric varieties. For a convex rational polyhedral cone σ in $N_{\mathbf{R}}$, the dimension of its $\mathbf{R}$-span will be denoted $\dim(\sigma)$. Its relative interior (= the interior of σ in its $\mathbf{R}$-span in $N_{\mathbf{R}}$) will be denoted as $\mathrm{relint}(\sigma)$. Given another rational convex polyhedral cone σ' in $N_{\mathbf{R}}$, we write $\sigma' \prec \sigma$ to mean that σ' is a face of σ. For any $d \in \mathbf{N}$, we will denote $\Sigma[d]$ (resp. $\sigma[d]$) to be the d-dimensional cones in Σ (resp. d-dimensonal faces $\sigma' \prec \sigma$). The cones in $\Sigma[1]$ will be called *rays*. We usually denote rays by the letter ρ instead of σ. The first lattice point on each ray $\rho \in \Sigma[1]$ will be written as $\mathbf{u}_\rho = (\mathrm{u}_{\rho,i})_{i=1}^n$. The cones in $\Sigma[\dim(N)]$ will be called full-dimensional. We also write $\Sigma[\max]$ to denote the set of maximal cones in Σ. All fans considered in this thesis fulfill the condition that $\Sigma[\max] = \Sigma[\dim(N)]$. The support of Σ (= the union of all cones in Σ) will be denoted by $|\Sigma|$. For $S \subset N_{\mathbf{R}}$, we write $\langle S \rangle$ for the cone in $N_{\mathbf{R}}^+$ generated by S.

Finally, σ_{std} denotes the *standard cone* in $N_{\mathbf{R}}$, that is, $\sigma_{\mathrm{std}} = \sum_{i=1}^n \mathbf{R}_{\geq 0}\mathbf{e}_i$. We also set Σ_{std} for the *standard fan* in $N_{\mathbf{R}}$, that is, the fan generated by the standard cone σ_{std}.

1.1. Weighted blow-ups

Scheme-theoretic weighted blow-ups appear naturally in the context of toric varieties and, more generally, in locally toric situations. Given a lattice N, a fan Σ on $N_{\mathbf{R}}$, a cone σ, and a lattice point $\mathbf{v} \in \mathrm{relint}(\sigma)$, the star subdivision $\Sigma^*(\mathbf{v})$ of Σ at $\mathbf{v}$ induces a map of toric varieties $X_{\Sigma^*(\mathbf{v})} \to X_\Sigma$ [CLS11, §11.1].

If σ is simplicial, then there is a unique way to write $\mathbf{v} = \sum_{i=1}^n d_i \mathbf{u}_{\rho_i}$ where $\rho_1, \rho_2, \ldots, \rho_n$ are the rays of σ with first lattice points $\mathbf{u}_{\rho_1}, \mathbf{u}_{\rho_2}, \ldots, \mathbf{u}_{\rho_n}$, and $d_1, d_2, \ldots, d_n \in \mathbf{N}_{>0}$. If $D_1, D_2, \ldots, D_n$ are the torus-invariant divisors on X_Σ corresponding to the rays $\rho_1, \rho_2, \ldots, \rho_n$, then $X_{\Sigma^*(\mathbf{v})} \to X_\Sigma$ is the scheme-theoretic weighted blow-up along the "weighted center"

$$\frac{1}{d_1}D_1 \cap \cdots \cap \frac{1}{d_n}D_n.$$

One way to make this precise is to take the usual blow-up of X_Σ along the integral closure of the ideal

$$I_{\frac{N}{d_1}D_1} + \cdots + I_{\frac{N}{d_n}D_n}$$

where N is a positive integer that is sufficiently divisible, and I_D is the ideal sheaf of a divisor D on X_Σ. If every cone containing σ is smooth, then the D_i are Cartier divisors and if all the multiplicities d_i are equal, then $X_{\Sigma^*(\mathbf{v})} \to X_\Sigma$ is the usual smooth blow-up of X_Σ along the smooth closed subscheme $D_1 \cap D_2 \cap \cdots \cap D_n$. If Σ is smooth, then the weighted blow-up $X_{\Sigma^*(\mathbf{v})} \to X_\Sigma$ of the smooth toric variety X_Σ is always singular unless the multiplicites d_i are equal. Fortunately, in that case, $X_{\Sigma^*(\mathbf{v})}$ is still a simplicial toric variety, and it is known that a simplicial toric variety is always the coarse space of a smooth toric stack, cf. [**BCS05**, Proposition 3.7] or [**FMN10**, Theorem II]. Therefore, if one wants to work with weighted blow-ups of smooth objects, the above discussion suggests that there is more promise in considering weighted blow-ups as stacks instead of schemes.

1.1.1 (Weighted blow-ups as stacks)**.** As before, consider a smooth fan Σ. It turns out that the toric stack $\mathscr{X}$, corresponding to the star subdivision $\Sigma^*(\mathbf{v})$ of Σ at $\mathbf{v}$, is what we will refer to as the *stack-theoretic* weighted blow-up of $X = X_\Sigma$ along the smallest $\mathbf{Z}$-graded $\mathscr{O}_X$-subalgbra $I_\bullet \subset \mathscr{O}_X[t]$ containing $\mathscr{O}_X$ in degree 0 and I_{D_i} in degree d_i for $1 \leq i \leq n$. By definition, this means that $\mathscr{X}$ is the *stack-theoretic Proj*:

$$\mathscr{P}\mathrm{roj}_X(I_\bullet) = \big[\mathrm{Spec}_X(I_\bullet) \smallsetminus V(I_+) \,/\, \mathbb{G}_m\big].$$

We call $I_\bullet$ a *Rees algebra* on X (Definition 2.3.1), and we express it as:

$$I_\bullet = (I_{D_1}, d_1) + (I_{D_2}, d_2) + \cdots + (I_{D_n}, d_n), \qquad \text{cf. Definition 2.3.5.} \tag{1.1}$$

The exceptional divisor of this weighted blow-up is the stack-theoretic Proj of $\bigoplus_{n\in\mathbf{N}} I_n/I_{n+1}$ which is a weighted projective stack — a smooth stack whose coarse space is a weighted projective space.

Rees algebras provide a common framework for stack-theoretic weighted blow-ups of arbitrary algebraic stacks, that is independent of any toric connections. This framework includes the following examples:

(i) the usual blow-up in the ideal I, which corresponds to the usual Rees algebra $I_\bullet = (I, 1) = \bigoplus_{n\geq 0} I^n$,

(ii) the dth root stack in the Cartier divisor D, which corresponds to the Rees algebra $I_\bullet = (I_D, d)$, and

(iii) the Cartierification of a $\mathbf{Q}$-Cartier divisor D, which corresponds to the Rees algebra $I_\bullet = (I_D, 1) + (I_{2D}, 2) + \cdots$.

Whereas usual blow-ups modify the space without introducing any stackiness, root stacks and Cartierifications leave the coarse space unmodified and introduces stackiness in codimension 1 and codimension ≥ 2 respectively. Toric stacks, with trivial generic stabilizer, can be obtained from their coarse toric variety by taking Cartierifications and root stacks (Example 2.2.21). Up to normalization, every weighted blow-up is a usual blow-up followed by a root stack (Proposition 2.3.44).

Finally, we would like to point out that the Rees algebra in (1.1) is an example of a smooth weighted center. These are the Rees algebras that can be locally written as $(x_1, d_1) + (x_2, d_2) + \cdots + (x_n, d_n)$ where $x_1, x_2, \ldots, x_n$ is a regular sequence and $V(x_1, x_2, \ldots, x_n)$ is smooth. Smooth weighted centers can also be characterized as the Rees algebras $I_\bullet$ whose "weighted" normal cone $\mathrm{Spec}_X\big(\bigoplus_{n\geq 0} I_n/I_{n+1}\big)$ is a *twisted weighted vector bundle* (Definition 2.2.3) and whose co-support $V(I_1)$ is smooth, cf. Proposition 2.5.4.

1.1.2 (Applications)**.** Besides arising naturally in the toric setting, weighted blow-ups have also been an indispensible tool in algebraic geometry. For example, weighted blow-ups have been used to study certain aspects of certain moduli stacks, e.g. in [**Inc22**]. More relevant to this thesis is the recent prominent example of weighted resolution of singularities by Abramovich, Temkin and Włodarczyk [**ATW19**], which uses stack-theoretic weighted blow-ups in smooth centers and is far more effective than Hironaka's classical algorithm but at the expense of using smooth stacks. In Chapters 3 and 4 of this thesis, we supply an enhancement of the results in [**ATW19**] from which one can recover the key features in Hironaka's resolution of singularities. All these demonstrate that as opposed to usual blow-ups, weighted blow-ups provide a significant amount of flexibility and simplification to many algorithms and arguments in algebraic geometry.

It is also understood among experts that weighted blow-ups is a more valuable and convenient tool than usual blow-ups in recovering the necessary data needed to understand a singularity. We mention a few examples illustrating this. For example, weighted blow-ups are used to compute the log canonical threshold of a plane curve singularity, cf. [**KSC04**, §6.5]. Weighted blow-ups also appear in the proof of Denef–Loeser's motivic monodromy conjecture for semi-quasihomogeneous singularities as in [**BBV21**]. Not surprisingly, this monodromy conjecture happens to be the subject of Chapter 5 in this thesis, although we approach it using the more efficient technique of *multi-weighted blow-ups*, which is the content of Chapter 4.

1.1.3 (Overview of Chapter 2)**.** Chapter 2 forms the first portion of this thesis. In §2.1, we study the stack-theoretic Proj. In particular, we describe its local charts (§2.1.C), its tautological line bundles (§2.1.D), its universal property (Proposition 2.1.4), and its properties (§2.1.E).

In §2.2, we give four examples of stack-theoretic Proj. The first example is weighted projective stacks, which includes root stacks of line bundles, and more generally twisted weighted projective bundles (§2.2.A). The second example is root stacks of generalized Cartier divisors (§2.2.B). The third example is a construction which transforms $\mathbf{Q}$-invertible sheaf into an invertible sheaf (§2.2.C), which generalizes Cartierification. This was prominently used by Abramovich and Hassett [**AH10**] to treat families of $\mathbf{Q}$-Gorenstein varieties. The fourth example is a stack-theoretic amplification of GIT quotients (§2.2.D).

In §2.3, we finally introduce Rees algebras and weighted blow-ups (§2.3.A and §2.3.B). In particular, we describe their exceptional divisor (Definition 2.3.13) and their universal property (Theorem 2.3.20). We also give a simplified description of their universal property in the case of normalized weighted blow-ups (Theorem 2.3.43). In §2.4, we treat weighted normal cones. In particular, we describe how the *extended Rees algebra* $I_\bullet^{\text{ext}}$ (Definition 2.3.3) gives rise to the *deformation to the weighted normal cone* $\operatorname{Spec}(I_\bullet^{\text{ext}})$ (§2.4.B).

In §2.5, we consider weighted blow-ups in regular and smooth centers. Firstly, we show that various notions of quasi-regularity coincide, and are equivalent to the weighted normal cone being a twisted weighted vector bundle (§2.5.A). Secondly, in §2.5.B, we show that when a Rees algebra is locally generated by a weighted regular sequence, then its extended Rees algebra has a very simple description (Proposition 2.5.9), which culminates in simple equations for its weighted blow-up. Thirdly, we discuss weighted blow-ups along *regular* (resp. *smooth*) *centers* (2.5.C, resp. §2.5.D), i.e. Rees algebras $I_\bullet$ that are locally generated by a weighted regular sequence and whose co-support $V(I_1)$ is regular (resp. smooth).

In §2.6, we expand on some of the toric connections alluded to in the beginning of this section. This naturally motivates a discussion of weighted blow-ups in the locally toric situation, or even more generally, in the setting of logarithmic schemes. This is the content of §2.7. In §2.7.A, we define the monomial part of a Rees algebra (2.7.4), before using it to define what it means for a Rees algebra to be fs = fine and saturated (Definition 2.7.6). We also endow logarithmic structures on weighted blow-ups in the logarithmic setting (4.2.18), so that the fs category of logarithmic schemes is closed under the operation of weighted blow-ups along fs Rees algebras. In §2.7.B, we work with schemes that are "étale locally toric", namely toroidal (= logarithmically regular) schemes. We define and discuss weighted blow-ups of toroidal schemes along *toroidal* (= *logarithmically regular*) *centers*. Similar to §2.5.B, we give simple equations for the weighted blow-up of a toroidal scheme along a toroidal center (Proposition 2.7.17).

1.1.4 (What's missing from Chapter 2)**.** Unlike the classical case where we treated both *regular* and *smooth* centers (cf. 2.5.C and 2.5.D), we did not define and treat weighted blow-ups of logarithmically smooth schemes over a base scheme S along *logarithmically smooth* centers. These do not play a part in this thesis, and have been omitted given the length of this thesis. Also excluded from Chapter 2 is a section that demonstrates a GIT wall-crossing between smooth Deligne–Mumford stacks is given by a stack-theoretic weighted blow-up followed by a stack-theoretic weighted blow-down, both in regular centers. These will eventually appear in [**QR21**] in the near future.

1.2. Resolution of singularities via logarithmic stacks

The middle portion of this thesis re-visits the celebrated theorem of Hironaka [**Hir64**] that one can resolve the singularities of a reduced, closed, singular subscheme X of a smooth scheme Y over a field $\mathbf{k}$ of characteristic zero, in a way that is functorial with respect to smooth morphisms of such pairs $(X \subset Y)$. Over the years, the proof of this theorem has seen simplifications, for example by Bierstone–Milman [**BM97**], by Encinas–Villamayor [**EV03**], and by Włodarczyk [**Wło05**]. Most recently, it was shown independently by Abramovich–Temkin–Włodarczyk [**ATW19**] and by McQuillan [**McQ20**] that one can do this by iteratively blowing up the "worst singular locus" and immediately witnessing a visible improvement in singularities, although one has to instead work with *weighted blow-ups along smooth centers*, and admit *smooth Deligne–Mumford stacks* as ambient spaces.

In addition, one typically requires, for the sake of applications, that the singular locus of X is transformed under the resolution into a simple normal crossings divisor. This was a feature of Hironaka's theorem in [**Hir64**], although it was only recently in a different paper of Abramovich–Temkin–Włodarczyk [**ATW20a**] that logarithmic geometry was first accessed as a tool to account for this requirement, by encoding exceptional divisors as logarithmic structures.

We remark, however, that the resolution algorithm in [**ATW19**] does not address the aforementioned requirement. It is thus natural to ask for an amalgamation of the two aforementioned techniques, which we mark by $\boxed{\times}$ below:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{logarithmic geometry in}\\ \text{the service of resolution}\end{array}} \arrow[rr, dotted] & & \boxed{\times} \\
\boxed{\begin{array}{c}\text{resolution in the sense of}\\ \text{Hironaka [Hir64],}\\ \text{Bierstone–Milman [BM97],}\\ \text{Encinas–Villamayor [EV03]}\end{array}} \arrow[u, squiggly, "\text{[ATW20a]}"] \arrow[rr, squiggly, "\text{[ATW19, McQ20]}"] & & \boxed{\begin{array}{c}\text{weighted blow-ups}\\ \text{in the service}\\ \text{of resolution}\end{array}} \arrow[u, dotted]
\end{tikzcd}$$

In §1.2.A and Chapter 3, we realize $\boxed{\times}$ as follows: the weighted blow-ups along smooth centers in [**ATW19**] are replaced by their logarithmic counterpart — *weighted blow-ups along toroidal centers*. It is essential to note that even if one takes a pair $(X \subset Y)$ from before as input for the algorithm in §1.2.A, one is inevitably led to admit *toroidal Deligne Mumford stacks* as ambient spaces — these are logarithmically smooth over $\mathbf{k}$, but not necessarily smooth over $\mathbf{k}$. As a consequence, one cannot expect to resolve the singularities of X solely via weighted blow-ups along toroidal centers, and the best one can hope for at the end is *toroidal singularities* (= logarithmic embedded resolution in the sense of Theorem A), where the singular locus of X is now transformed into a divisor with *toroidal support*. Nonetheless this is not a concern, since one can then apply resolution of toroidal singularities, cf. [**KKMSD73**, Theorem 11*] or [**Wło20a**, Theorem 6.5.1].

In §1.2.B and Chapter 4, we propose a different candidate for $\boxed{\times}$, although it is built on the same backbone of ideas as the previous candidate. Namely, we use a construction of Satriano in [**Sat13**, §3] to upgrade the aforementioned weighted blow-ups along toroidal centers to *multi-weighted blow-ups*. This is carried out in §4.1.B, where certain multi-weighted blow-ups are realized as canonical Artin stacks over weighted toroidal blow-ups. The reader can also find, in §4.1.A, an account of local aspects of multi-weighted blow-ups. The key advantage of using multi-weighted blow-ups over weighted toroidal blow-ups is that we remain in the ideal realm of *smooth ambient spaces*, and hence we can do without resolution of toroidal singularities at the end. However, the trade-off is that one has to work more broadly with *Artin stacks* as ambient spaces.

1.2.A. Logarithmic resolution via weighted blow-ups in toroidal centers. Consider a fs (Definition 2.7.1) logarithmic scheme Y which is logarithmically smooth over $\mathbf{k}$, or equivalently a *toroidal* $\mathbf{k}$-scheme Y (Definition 2.7.10). More generally, we consider a *toroidal Deligne–Mumford stack Y over* $\mathbf{k}$ [**ATW20b**, §3.3.3]. We also consider a reduced, closed subscheme $X \subset Y$, always endowed with the logarithmic structure $\mathscr{M}_X$ that is the pullback of the logarithmic structure $\mathscr{M}_Y$ on Y under $X \hookrightarrow Y$ (so that the inclusion $X \hookrightarrow Y$ of logarithmic algebraic stacks is strict). Additionally, we will always assume X is *generically toroidal*, that is, there is a dense open $U \subset X$ such that $(U, \mathscr{M}_Y|_U)$ is toroidal. Such pairs $(X \subset Y)$ form the objects of a category, where a morphism between pairs $(\widetilde{X} \subset \widetilde{Y}) \to (X \subset Y)$ is a cartesian square

$$\begin{tikzcd} \widetilde{X} = X \times_Y \widetilde{Y} \arrow[r, hook] \arrow[d] & \widetilde{Y} \arrow[d, "f"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

where $f\colon \widetilde{Y} \to Y$ is logarithmically smooth and surjective. We refer to such morphisms as logarithmically smooth, surjective morphisms of pairs. Note, however, that in certain situations below, we do not demand surjectivity in our morphisms of pairs.

The goal of Chapter 3 is to define a "*logarithmic embedded resolution*" functor on the aforementioned category, which assigns to each pair $X \subset Y$ as above, a proper, birational morphism $\Pi\colon Y_N \to Y$ such that both Y_N and the proper transform $X_N \subset X \times_Y Y_N$ are toroidal. More precisely:

Theorem A (Logarithmic embedded resolution via weighted blow-ups in toroidal centers)**.** *Given a reduced, generically toroidal, closed substack X of a toroidal Deligne–Mumford stack Y over $\mathbf{k}$, there exists a canonical sequence of weighted blow-ups along toroidal centers*

$$\Pi\colon Y^+ := Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_1} Y_0 = Y$$

together with proper transforms $X_i \subset Y_i$ of X, such that:

(i) *$X^+ := X_N$ is a toroidal Deligne–Mumford stack over $\mathbf{k}$.*

(ii) *Π is an isomorphism over the toroidal (= logarithmically smooth) locus $X^{\log-\mathrm{sm}} \subset X$.*

(iii) *$\Pi^{-1}(X \smallsetminus X^{\log-\mathrm{sm}})$ is set-theoretically contained in the toroidal divisor of X^+.*

This procedure $(X \subset Y) \mapsto (\widetilde{X} \subset \widetilde{Y})$ is functorial with respect to every logarithmically smooth morphism of such pairs $(\widetilde{X} \subset \widetilde{Y}) \to (X \subset Y)$, whether or not surjective. In fact, $\widetilde{Y}^+ = \widetilde{Y} \times_Y Y^+$.

Here, (i) is logarithmic embedded resolution of X, while (ii) and (iii) are the logarithmic analogues of two essential features in Hironaka's resolution of singularities [**Hir64**, Main Theorem I], cf. properties (ii) and (iii) in Corollary D below. Note that the Y_i's in Theorem A will be toroidal but not necessarily smooth over $\mathbf{k}$ (cf. the example in §3.4.C), so it does not make sense to ask for $\Pi^{-1}(X \smallsetminus X^{\log-\mathrm{sm}})$ to be a simple normal crossings divisor on X^+. Nevertheless, (iii) gives us some control over the exceptional locus of Π. As one might already notice in Corollary D, this control over the exceptional locus is sufficient for us to deduce Hironaka's resolution of singularities.

1.2.1 (On the role of idealistic exponents in Chapter 3)**.** A convenient *numerical* tool for representing and keeping track of toroidal centers is the notion of *idealistic exponents*, which we detail in §2.3.G. Unlike the original papers [**ATW19**] and [**Que22a**] on which Chapter 3 is based on, we have made the stylistic choice in this thesis to avoid any use of idealistic exponents in our proofs, with the exception of the proof of Theorem 3.3.9(iii), cf. Remark 3.3.12. Nevertheless, we still retain the language of idealistic exponents in our formulation of results throughout Chapter 3, cf. Convention 2.3.79 and 3.1.6.

Returning back to Theorem A, one obtains Theorem A by taking at the $(i+1)^{\mathrm{th}}$ step the weighted blow-up of Y_i along the "worst singular locus of $X_i \subset Y_i$". We give a formal statement of this procedure in Theorem B below. In the upcoming paragraphs, we will express the "worst singular locus of $X_i \subset Y_i$" in terms of a local singularity invariant.

1.2.2 (A local singularity invariant)**.** For a point $y \in |Y|$, we will associate, in §3.3.A, a local singularity invariant $\mathrm{inv}_y(X \subset Y)$ of $X \subset Y$ at y (which is an amalgamation of the invariants in [**ATW20a**] and [**ATW19**]), which is, very simply put, a non-decreasing truncated sequence of non-negative rational numbers, where the last entry is allowed to be ∞. We can well-order the set consisting of all local invariants of such pairs $X \subset Y$ at points $y \in Y$ by the lexicographic order $<$ (cf. last sentence of Definition 3.3.4), but with a caveat: our lexicographic order considers the truncation (from the end) of a sequence to be strictly larger than the sequence itself, cf. an example in 3.3.1. Letting $J \subset \mathscr{O}_Y$ denote the ideal underlying $X \subset Y$, the invariant satisfies the following properties, cf. 3.3.5 and Lemma 3.3.6:

(a) $\mathrm{inv}_y(X \subset Y) = (0)$ if and only if $y \notin |X|$. For $y \in |X|$, the invariant detects logarithmic smoothness of X at y. Namely, $\mathrm{inv}_y(X \subset Y)$ is bounded below (via the lexicographic order $<$ above) by the constant sequence $(1, 1, \ldots, 1)$ of length equal to the height of J_y, and equality holds if and only if X is toroidal at y.

(b) It is upper semi-continuous on Y.

(c) It is functorial with respect to logarithmically smooth morphisms of pairs $X \subset Y$, whether or not surjective.

(d) The first term of $\mathrm{inv}_y(X \subset Y)$ is the logarithmic order (Definition 3.1.3) of J at y. In particular, it lies in $\mathbf{N} \cup \{\infty\}$.

This invariant will be constructed via logarithmic analogues of the classical notions of maximal contact elements and coefficient ideals, which we detail in §3.2.

1.2.3 ("Worst singular locus")**.** Next, we set $\max \mathrm{inv}(X \subset Y) := \max_{y \in |X|} \mathrm{inv}_y(X \subset Y)$. This global invariant is functorial with respect to logarithmically smooth, surjective morphisms of pairs $X \subset Y$, cf. Corollary 3.3.7. Property (a) above suggests that the aforementioned "worst singular locus of $X \subset Y$" can be loosely interpreted as the closed substack of X consisting of points $y \in |X|$ such that $\mathrm{inv}_y(X \subset Y) = \max \mathrm{inv}(X \subset Y)$. More precisely, this "worst singular locus of $X \subset Y$" will be the co-support of a toroidal center $I_\bullet$ on Y, cf. Definitions 3.3.8 and 3.3.15, as well as the proof of Theorem A. Our next theorem says that the weighted blow-up of Y along this "worst singular locus" $I_\bullet$ improves the singularities on X immediately in a visible way:

Theorem B (Maximum invariant drops after each weighted blow-up)**.** *Given a reduced, logarithmically singular, closed substack X of a toroidal Deligne–Mumford stack Y over $\mathbf{k}$, there exists a canonical toroidal center $I_\bullet$ on Y, with weighted blow-up*

$$Y' := \mathrm{Bl}_{I_\bullet} Y \xrightarrow{\ \pi\ } Y$$

and proper transform $X' \subset Y'$ of X', such that:

(i) *Y' is a toroidal Deligne–Mumford stack over $\mathbf{k}$, and the exceptional divisor of π is set-theoretically contained in the toroidal divisor of Y'.*

(ii) *π is an isomorphism over the open complement in Y of the closed locus of points $y \in |Y|$ with $\mathrm{inv}_y(X \subset Y) = \max \mathrm{inv}(X \subset Y)$.*

(iii) $\max \mathrm{inv}(X' \subset Y') < \max \mathrm{inv}(X \subset Y)$.

This procedure $(X \subset Y) \mapsto (X' \subset Y')$ is functorial with respect to every logarithmically smooth, surjective morphism of such pairs $(\widetilde{X} \subset \widetilde{Y}) \to (X \subset Y)$. In fact, $\widetilde{Y}' = \widetilde{Y} \times_Y Y'$.

By iterating Theorem B, one can achieve the logarithmic analogue of *principalization* [Kol07, Therem 3.35] for the underlying ideal of $X \subset Y$. In contrast, to achieve logarithmic embedded resolution of $X \subset Y$ in the sense of Theorem A, some care needs to be taken while iterating Theorem B. We give these details in §3.4.B. In §3.4.B, we will also prove the following corollaries of Theorem A:

Corollary C (Logarithmic resolution)**.** *Given a pure-dimensional, reduced, generically toroidal, fs logarithmic Deligne–Mumford stack X of finite type over $\mathbf{k}$, there exists a proper, birational morphism $\Pi \colon X^+ \to X$ where:*

(i) *X^+ is a pure-dimensional, toroidal Deligne–Mumford stack over $\mathbf{k}$.*

(ii) *Π is an isomorphism over $X^{\mathrm{log-sm}} \subset X$.*

(iii) *$\Pi^{-1}(X \smallsetminus X^{\mathrm{log-sm}})$ is set-theoretically contained in the toroidal divisor of X^+.*

This procedure $X \mapsto X^+$ is functorial with respect to every logarithmically smooth morphism $\widetilde{X} \to X$, whether or not surjective.

We will prove Corollary C by first embedding X, locally in the étale topology, as a closed subscheme of pure codimension in a toroidal $\mathbf{k}$-scheme Y, before applying Theorem A to obtain local logrithmic resolutions of X. It then remains to show that one can patch these local logarithmic resolutions. This is a consequence of the functoriality in Theorem A and a standard "re-embedding principle" (Lemma 3.4.8). Finally, as promised earlier, we can also deduce, from the above corollary, Hironaka's resolution of singularities.

Corollary D (Resolution of singularities, à la Hironaka)**.** *Given a pure-dimensional, reduced scheme X of finite type over $\mathbf{k}$, there exists a projective, birational morphism $\Phi\colon X^{++} \to X$ where:*

(i) *X^{++} is a pure-dimensional smooth $\mathbf{k}$-scheme.*

(ii) *Φ is an isomorphism over the smooth locus $X^{\mathrm{sm}} \subset X$.*

(iii) *$\Phi^{-1}(X \smallsetminus X^{\mathrm{sm}})$ is a simple normal crossings divisor on X^{++}.*

This procedure $X \mapsto X^{++}$ is functorial with respect to every smooth morphism $\widetilde{X} \to X$, whether or not surjective.

Note that Φ is not just proper, but *projective.* We prove Corollary D by first proving a weaker version where Φ is possibly only proper and X^{++} is possibly a Deligne-Mumford stack over $\mathbf{k}$. This follows by applying resolution of toroidal singularities after the procedure in Corollary C, as noted at the very start of this section. To complete the proof of Corollary D, it remains to apply the destackification procedure of Bergh–Rydh [**BR19**].

1.2.B. Resolution of singularities via multi-weighted blow-ups. As hinted at the start of §1.2, we will provide, in this subsection, an adaptation of the main theorems in §1.2.A, in terms of multi-weighted blow-ups.

Instead of toroidal Deligne–Mumford stacks over $\mathbf{k}$, the "smallest" category of ambient spaces that is closed under the operation of multi-weighted blow-ups is the category of smooth, toroidal Artin stacks over $\mathbf{k}$. Unlike toroidal Deligne–Mumford stacks over $\mathbf{k}$, these objects can be described without the language of logarithmic geometry: they can be thought of as pairs (Y, E), where Y is a smooth Artin stack over $\mathbf{k}$, and $E \subset Y$ is a normal crossings divisor (potentially $\varnothing$). For such a pair (Y, E), if we set $U = Y \smallsetminus E$ and j to be $U \hookrightarrow Y$, then Y would be logarithmically smooth over $\mathbf{k}$, under the logarithmic structure $\alpha_Y \colon \mathscr{M}_Y := j_*(\mathscr{O}_U^*) \cap \mathscr{O}_Y \hookrightarrow \mathscr{O}_Y$ induced by E. Moreover, note that $\mathscr{M}_Y$ is a sheaf in the Zariski topology if and only if $E \subset Y$ is a simple normal crossings divisor. Therefore, smooth, strict toroidal Artin stacks over $\mathbf{k}$ are simply pairs (Y, E) where Y is a smooth Artin stack over $\mathbf{k}$, and E is a simple normal crossings divisor on Y (potentially $\varnothing$).

Let us therefore consider a reduced, closed substack X in a smooth, toroidal Artin stack Y over $\mathbf{k}$. We always endow X with the logarithmic structure $\mathscr{M}_X$ that is the pullback of the logarithmic structure $\mathscr{M}_Y$ under the inclusion $X \hookrightarrow Y$. Such pairs $(X \subset Y)$ form the objects of a category, where a morphism between pairs $(\widetilde{X} \subset \widetilde{Y}) \to (X \subset Y)$ is a cartesian square

$$\begin{tikzcd} \widetilde{X} = X \times_Y \widetilde{Y} \arrow[r, hook] \arrow[d] & \widetilde{Y} \arrow[d, "f"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

for a strict, smooth, and surjective morphism $f \colon \widetilde{Y} \to Y$. We refer to such a morphism as a *strict, smooth, and surjective morphism of pairs*. At times we might drop surjectivity as a condition, in which case we say f is a *strict, smooth morphism of pairs*. Chapter 4 concerns following adaptation of Theorem A:

Theorem E (Embedded resolution of singularities via multi-weighted blow-ups)**.** *Given a reduced, generically toroidal, closed substack X of a smooth, toroidal Artin stack Y over $\mathbf{k}$, there exists a canonical sequence of multi-weighted blow-ups*

$$\Pi\colon Y^+ := Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_1} Y_0 = Y$$

together with proper transforms $X_i \subset Y_i$ of X, such that:

(i) *$X^+ := X_N$ is a smooth, toroidal Artin stack over $\mathbf{k}$.*

(ii) *Π is an isomorphism over $X^{\log-\mathrm{sm}} \subset X$.*

(iii) *$\Pi^{-1}(X \smallsetminus X^{\log-\mathrm{sm}})$ is a simple normal crossings divisor on X^+.*

(iv) *Each π_i is birational, surjective, universally closed, and factors as $Y_i \to \mathbf{Y}_i \to Y_{i-1}$, where $Y_i \to \mathbf{Y}_i$ is a good moduli space of Y_i relative to Y_{i-1}, $\mathbf{Y}_i$ is normal, and $\mathbf{Y}_i \to Y_{i-1}$ is a schematic blow-up (whence birational and projective).*

This procedure $(X \subset Y) \mapsto (X^+ \subset Y^+)$ is functorial with respect to strict, smooth morphisms of such pairs $(\widetilde{X} \subset \widetilde{Y}) \to (X \subset Y)$, whether or not surjective. In fact, $\widetilde{Y}^+ = \widetilde{Y} \times_Y Y^+$.

Similar to Theorem A, one obtains Theorem E by taking at the $(i+1)^{\text{th}}$ step the multi-weighted blow-up of Y_i along the same "worst singular locus of $X_i \subset Y_i$" in §1.2.3, constructed using the local singularity invariant inv in 1.2.2. We formalize this in the following adaptation of Theorem B:

Theorem F (Maximum invariant drops after each multi-weighted blow-up)**.** *Given a reduced, logarithmically singular, closed substack X of a smooth, toroidal Artin stack Y over $\mathbf{k}$, there exists a canonical multi-weighted blow-up*

$$Y' \xrightarrow{\pi} Y$$

with proper transform $X' \subset Y'$ of X, so that:

(i) *Y' is a smooth, toroidal Artin stack over $\mathbf{k}$, and the exceptional divisor of π is a union of a subset of irreducible components of the simple normal crossings toroidal divisor on Y'.*

(ii) *π is an isomorphism away from the closed substack of X consisting of points $y \in |X|$ such that $\mathrm{inv}_y(X \subset Y) = \max \mathrm{inv}(X \subset Y)$.*

(iii) $\max \mathrm{inv}(X' \subset Y') < \max \mathrm{inv}(X \subset Y)$.

(iv) *π is birational, surjective, universally closed, and factors as $Y' \to \mathbf{Y}' \to Y$, where $Y' \to \mathbf{Y}'$ is a good moduli space relative to Y, $\mathbf{Y}'$ is normal, and $\mathbf{Y}' \to Y$ is a schematic blow-up (whence birational and projective).*

This procedure $(X \subset Y) \mapsto (X' \subset Y')$ is functorial with respect to strict, smooth, and surjective morphisms of such pairs $(\widetilde{X} \subset \widetilde{Y}) \to (X \subset Y)$. In fact, $\widetilde{Y}' = \widetilde{Y} \times_Y Y'$.

As hinted at the start of §1.2, various properties of Satriano's construction [**Sat13**, §3] will allow us to deduce Theorems E and F from Theorems A and B respectively. This will be done in §4.3.B. In §4.3.B, we will also establish the following adaptation of Corollary C:

Corollary G (Resolution of singularities)**.** *Given a pure-dimensional, reduced, generically toroidal, fs logarithmic Artin stack X of finite type over $\mathbf{k}$, there exists a proper, birational morphism $\Pi\colon X^+ \to X$ where:*

(i) *X^+ is a pure-dimensional, smooth, toroidal Artin stack over $\mathbf{k}$.*

(ii) *Π is an isomorphism over $X^{\log-\mathrm{sm}} \subset X$.*

(iii) *$\Pi^{-1}(X \smallsetminus X^{\log-\mathrm{sm}})$ is a simple normal crossings divisor on X^+.*

This procedure $X \mapsto X^+$ is functorial with respect to every strict, smooth morphism $\widetilde{X} \to X$, whether or not surjective.

The proof of the above corollary proceeds in a similar way as the proof of Corollary C, once one makes the relevant modifications. Finally, from Corollary G, one can deduce Hironaka's resolution of singularities in the sense of Corollary D, although Bergh–Rydh's destackification is not the only ingredient. Prior to destackification, one needs to apply reduction of stabilizers in the sense of Edidin–Rydh [**ER20**]. We sketch this toward the end of §4.3.B.

1.3. Monodromy conjecture of Denef–Loeser

For the final portion of the thesis, we consider a subfield $\mathbf{k} \subset \mathbf{C}$, and fix $0 \neq n \in \mathbf{N}$. For every $\mathbf{a} := (a_1, \ldots, a_n) \in \mathbf{N}^n$, let $x^{\mathbf{a}}$ denote the monomial $x_1^{a_1} \cdots x_n^{a_n}$ in $\mathbf{k}[x_1, \ldots, x_n]$. Let

$$f = \sum_{\mathbf{a} \in \mathbf{N}^n} c_{\mathbf{a}} \cdot x^{\mathbf{a}} \; \in \; \mathbf{k}[x_1, \ldots, x_n]$$

be a non-constant polynomial satisfying $c_{\mathbf{0}} = f(\mathbf{0}) = 0$, and let $V(f)$ be the hypersurface defined by $f = 0$ in $\mathbf{A}^n := \operatorname{Spec}(\mathbf{k}[x_1, \ldots, x_n])$. Let $\Gamma_+(f)$ denote the Newton polyhedron of f, defined as the convex hull in $\mathbf{R}^n$ of the finite union

$$\bigcup \big\{ \mathbf{a} + \mathbf{R}^n_{\geq 0} \colon \mathbf{a} \in \mathbf{N}^n, \; c_{\mathbf{a}} \neq 0 \big\}.$$

In this section and Chapter 5, we impose the following condition on f:

Definition 1.3.1. For a face ς of $\Gamma_+(f)$, let us set

$$f_\varsigma \; := \sum_{\mathbf{a} \in \mathbf{N}^n \cap \varsigma} c_{\mathbf{a}} \cdot x^{\mathbf{a}}. \tag{1.2}$$

We say that f is *non-degenerate with respect to* ς if the closed subscheme $V(f_\varsigma) \subset \mathbf{A}^n$ is non-singular in the torus $\mathbb{G}_m^n \subset \mathbf{A}^n$. We also say f is *non-degenerate*, if f is non-degenerate with respect to all *compact* faces of $\Gamma_+(f)$.

This non-degeneracy condition was first introduced in [**Kou76**], and it guarantees that the singularity theory of $V(f) \subset \mathbf{A}^n$ at the origin $\mathbf{0} \in \mathbf{A}^n$ is, to a certain extent, governed by $\Gamma_+(f)$. The extent to which the former is governed by the latter is the main interest of Chapter 5.

More precisely, Chapter 5 provides a *geometric* explanation (Theorem I) for the proposition in [**ELT22**, Proposition 3.8] that any pole of the topological zeta function of f at $\mathbf{0} \in \mathbf{A}^n$ [**DL92a**] cannot arise exclusively from a set of B_1-facets of $\Gamma_+(f)$ with consistent base directions. In the process, we obtain a *smaller* set of candidate poles for the motivic zeta function of f at $\mathbf{0} \in \mathbf{A}^n$ [**DL01**] than what was previously known in general (Theorem H), and in particular we also deduce (via Theorem J) a new, geometric proof of:

Theorem (= [**BV16**, Theorem 10.3])**.** *The motivic monodromy conjecture holds for non-degenerate polynomials in $n = 3$ variables.*

1.3.A. Statement of objectives, motivations, and results. We assume throughout this subsection that $f \in \mathbf{k}[x_1, \dots, x_n]$ is non-degenerate.

Convention 1.3.2 (Conventions on the Newton polyhedron of f)**.** Let $N = \mathbf{Z}^n$, and M be the dual lattice $N^\vee$. For reasons related to toric geometry (cf. Conventions 1.0.1), we view $\Gamma_+(f)$ as a polyhedron in $M^+_{\mathbf{R}}$ (instead of $N^+_{\mathbf{R}}$). For a face ς of $\Gamma_+(f)$,

$$\text{we write} \quad \begin{cases} \varsigma' \prec \varsigma \\ \varsigma' \prec^1 \varsigma \\ \mathrm{vert}(\varsigma) \\ \dim(\varsigma) \end{cases} \quad \text{for} \quad \begin{cases} \text{a face } \varsigma' \text{ of } \varsigma. \\ \text{a facet (= codimension 1 face) } \varsigma' \text{ of } \varsigma. \\ \text{the set of vertices of } \varsigma. \\ \text{the dimension of the affine span of } \varsigma. \end{cases}$$

Whenever $\varsigma \prec^1 \Gamma_+(f)$, we say ς is a facet of $\Gamma_+(f)$, and we usually use the letter τ instead of ς to denote facets of $\Gamma_+(f)$. If two facets τ_1 and τ_2 of $\Gamma_+(f)$ intersect in a common facet (i.e. $\tau_1 \cap \tau_2 \prec^1 \tau_1, \tau_2$), we say that τ_1 and τ_2 are *adjacent*, and write

$$\tau_1 \frown \tau_2.$$

Finally, for $i \in [n]$, let H_i denote the coordinate hyperplane in $M_{\mathbf{R}}$ defined by $\mathbf{e}_i = 0$. For $\tau \prec^1 \Gamma_+(f)$, let H_τ be its affine span in $M_{\mathbf{R}}$, with equation

$$(1.3) \qquad \{\mathbf{a} \in M_{\mathbf{R}} : \mathbf{a} \cdot \mathbf{u}_\tau = N_\tau\}$$

where the vector $\mathbf{u}_\tau := (u_{\tau,i})_{i=1}^n$ is the unique primitive vector in N^+ that is normal to H_τ. If $N_\tau > 0$ (i.e. τ is not contained in any coordinate hyperplane H_i in $M_{\mathbf{R}}$), the *numerical datum* of τ is defined as:

$$(1.4) \qquad \eta_\tau := (N_\tau, |\mathbf{u}_\tau|) := \big(N_\tau, u_{\tau,1} + u_{\tau,2} + \cdots + u_{\tau,n}\big)$$

and the *candidate pole* s_τ of τ is defined as the root of the polynomial $N_\tau s + |\mathbf{u}_\tau|$:

$$(1.5) \qquad s_\tau := -\frac{|\mathbf{u}_\tau|}{N_\tau}.$$

Finally, for $s_\circ \in \mathbf{Q}_{<0}$, we let $\mathcal{F}(f; s_\circ) := \{\tau \prec^1 \Gamma_+(f) \colon N_\tau > 0 \text{ and } s_\tau = s_\circ\}$.

1.3.3. The main theorem of Chapter 5 concerns the naïve motivic zeta function of f at $\mathbf{0} \in \mathbf{A}^n$ (cf. [**DL01**, Definition 3.2.1], and [**CLNS10**, Chapter 7, §3.3.1]), which we shall denote by $Z_{\mathrm{mot},\mathbf{0}}(f; s)$, and is tied to the singularity theory of $V(f) \subset \mathbf{A}^n$ at $\mathbf{0} \in \mathbf{A}^n$ via the motivic monodromy conjecture of Denef–Loeser.

In our setting, their conjecture states that there should exist a set of candidate poles Θ for $Z_{\mathrm{mot},\mathbf{0}}(f; s)$ (in the sense of [**BN20**, Definition 5.4.1]) such that every $s_\circ \in \Theta$ *induces a monodromy eigenvalue of f near $\mathbf{0} \in \mathbf{C}^n$* in the following sense: given any neighbourhood U of $\mathbf{0}$ in $f^{-1}(0) \subset \mathbf{C}^n$, there exists $x \in U$ such that $\exp\big(2\pi\sqrt{-1}s_\circ\big)$ is an eigenvalue of the monodromy transformation acting on the singular cohomology $\bigoplus_{i \geq 0} H^i_{\mathrm{sing}}(F_{f,x}, \mathbf{Z})$ of the Milnor fiber $F_{f,x}$ of f at x, cf. [**Mil68**] and [**CLNS10**, Chapter 1, §3.4.1].

1.3.4. To start, it has been established in the literature (cf. [**BV16**, Theorem 10.5] or [**BN20**, Theorem 8.3.5]) that

$$\Theta(f) := \{-1\} \cup \left\{ s_\tau \colon \tau \prec^1 \Gamma_+(f) \text{ with } N_\tau > 0 \right\} \tag{1.6}$$

is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$. More precisely, the preceding statement can be explicated as follows:

$$Z_{\mathrm{mot},\mathbf{0}}(f;s) \in \mathscr{M}_{\mathbf{k}}\left[\mathbf{L}^{-s}\right]\left[\frac{1}{1-\mathbf{L}^{-(Ns+\nu)}} \colon (N,\nu) \in \eta(f)\right] \tag{1.7}$$

where

$$\eta(f) := \{(1,1)\} \cup \left\{ \eta_\tau \colon \tau \prec^1 \Gamma_+(f) \text{ with } N_\tau > 0 \right\} \tag{1.8}$$

and $\mathscr{M}_{\mathbf{k}}$ denotes the localization of the Grothendieck ring $K_0(\mathsf{Var}_{\mathbf{k}})$ of $\mathbf{k}$-varieties (= finite-type $\mathbf{k}$-schemes) with respect to the class $\mathbf{L}$ of $\mathbf{A}^1$. Note that the letter T is sometimes used in place of the indeterminate $\mathbf{L}^{-s}$.

1.3.5. Unfortunately, the main difficulty in establishing the motivic monodromy conjecture for a non-degenerate polynomial f lies in the fact that *not every* candidate pole in $\Theta(f)$ induces a monodromy eigenvalue of f near $\mathbf{0} \in \mathbf{A}^n$. Therefore, one desires for a smaller set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$. This chapter gives a partial answer to the question of when a strictly smaller set of candidate poles than $\Theta(f)$ exists for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$, which can be seen as a *motivic upgrade* of some existing general results in the literature pertaining to a "close relative" of $Z_{\mathrm{mot},\mathbf{0}}(f;s)$, namely the topological zeta function $Z_{\mathrm{top},\mathbf{0}}(f;s)$ of f at $\mathbf{0} \in \mathbf{A}^n$, cf. [**DL92a**] and [**CLNS10**, Chapter 1, §3.3.1, equation (3.3.1.3)].

Remark 1.3.6. Indeed $Z_{\mathrm{top},\mathbf{0}}(f;s)$ is a "close relative" of $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ in the sense that $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ *specializes* to $Z_{\mathrm{top},\mathbf{0}}(f;s)$ via the motivic measure:

$$\mathrm{Eu}\colon \mathscr{M}_{\mathbf{k}} \to \mathbf{Z}$$

which sends a $\mathbf{k}$-variety X to the topological Euler characteristic of $X \otimes_{\mathbf{k}} \mathbf{C}$, cf. [**DL01**, Section 3.4] for details. In particular, one recovers in this way an analogue of (1.7) for $Z_{\mathrm{top},\mathbf{0}}(f;s)$ (which was observed earlier in [**DL92a**, Theorem 5.3(ii)]), namely that every pole of $Z_{\mathrm{top},\mathbf{0}}(f;s)$ lies in $\Theta(f)$.

1.3.7. To segue into the main results of Chapter 5, it is useful (as hinted in 1.3.5) to first recall some existing results in the literature which demonstrate that occasionally some candidate poles s_τ in $\Theta(f) \smallsetminus \{-1\}$ are *not* actual poles of $Z_{\mathrm{top},\mathbf{0}}(f;s)$. Few of these results are known for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ prior to [**Que22b**], especially for general n. We start with the following definition:

Definition 1.3.8 (B_1-facets, cf. [**ELT22**, Definition 3.1], [**LPS22**, Definition 1.4.1])**.** A facet τ of $\Gamma_+(f)$ is called a B_1-*facet* if there exists $\mathbf{v} \in \mathrm{vert}(\tau)$ and $i \in [n]$ such that:

(a) The i^{th} coordinate of $\mathbf{v}$ is 1.

(b) $\varnothing \neq \mathrm{vert}(\tau) \smallsetminus \{\mathbf{v}\} \subset H_i$.

(c) τ is *compact in the i^{th} coordinate*, i.e. $\tau + \mathbf{R}_{\geq 0}\mathbf{e}_i^\vee \not\subset \tau$ (cf. 5.1.8(iii)).

Note that in particular, (b) and (c) imply that $H_i \cap \tau \prec^1 \tau$. In this case, we call $\mathbf{v}$ an *apex* of τ with corresponding *base direction* $i \in [n]$. Note that the apex $\mathbf{v}$ and the base direction i uniquely determine each other.

1.3.9. Fix $-1 \neq s_\circ \in \mathbf{Q}_{<0}$. It is known that if $\mathcal{F}(f;s_\circ)$ only consists of one B_1-facet, then $s_\circ$ is not a pole of $Z_{\mathrm{top},\mathbf{0}}(f;s)$, cf. [**ELT22**, Proposition 3.7]. More generally one might guess that conclusion is true whenever $\mathcal{F}(f;s_\circ)$ comprises of only B_1-facets. However, this is *false*, cf. Example 5.2.13 and Remark 5.2.14 for a simple counterexample. One rectifies that guess (cf. [**ELT22**, Proposition 3.8]) by further imposing the following condition on $\mathcal{F}(f;s_\circ)$:

Definition 1.3.10. A set $\mathbb{B}$ of B_1-facets of $\Gamma_+(f)$ has *consistent base directions* if there exists, for each facet $\tau \in \mathbb{B}$, a choice of a distinguished base direction $b(\tau) \in [n]$, such that $b(\tau_1) = b(\tau_2)$ for every pair of adjacent facets $\tau_1, \tau_2 \in \mathbb{B}$. In this case we call $\{b(\tau) \colon \tau \in \mathbb{B}\}$ a *set of consistent base directions* for $\mathbb{B}$.

The main contribution of Chapter 5 can now be stated as follows:

Theorem H. *Let $\mathbb{B}$ be a set of B_1-facets of $\Gamma_+(f)$ with consistent base directions. Then*

$$\Theta^{\dagger,\mathbb{B}}(f) := \{-1\} \cup \big\{ s_\tau \colon \tau \prec^1 \Gamma_+(f) \text{ with } N_\tau > 0 \text{ and } \tau \notin \mathbb{B} \big\}$$

is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$.

1.3.11. We prove Theorem H towards the end of §5.3.C. The centerpiece of our proof (= Theorem I below) is perhaps more satisfying than Theorem H itself, especially given that previous attempts to understand the topological zeta function analogue of Theorem H, or even special cases of Theorem H, used roundabout methods: namely, they typically involve a *manipulation* of some explicit formula for $Z_{\mathrm{top},\mathbf{0}}(f;s)$ or $Z_{\mathrm{mot},\mathbf{0}}(f;s)$, cf. formulae in [**DL92a**, Theorem 5.3(iii)], [**DH01**, Theorem 4.2], and [**BV16**, Theorem 10.5]. In contrast, our proof is *geometric* in nature, in the sense that we construct an appropriate embedded desingularization of $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$ that bears witness to Theorem H.

1.3.12. To put our approach to Theorem H into perspective, we temporarily shift our attention to our approach towards its weaker counterpart (1.6), i.e. (1.7). Given that there is a motivic change of variables formula for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ under any proper, birational morphism $\pi\colon X \to \mathbf{A}^n$ (cf. [**CLNS10**, Chapter 6, §4.3]), one natural hope towards proving (1.7) would be to apply the change of variables to an appropriate embedded desingularization $\pi\colon X \to \mathbf{A}^n$ of $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$. A natural candidate for π would be the toric modification $\pi_{\Sigma'}\colon X_{\Sigma'} \to \mathbf{A}^n$ induced by any smooth subdivision Σ' of the normal fan $\Sigma(f)$ of $\Gamma_+(f)$. Indeed, one can show that the non-degeneracy condition on f implies that $\pi_{\Sigma'}$ desingularizes $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$, cf. [**Var76**, Sections 9 and 10]. Unfortunately, subdividing $\Sigma(f)$ into Σ' usually *introduces new rays* to $\Sigma(f)$. One can show this process of adding new rays cannot show in general the existence of a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ as small as $\Theta(f)$.

1.3.13. The above discussion suggests that one should perhaps *avoid* the process of adding new rays, and instead work directly on $\Sigma(f)$ and its associated toric modification $\pi_{\Sigma(f)}\colon X_{\Sigma(f)} \to \mathbf{A}^n$, despite the fact that $\pi_{\Sigma(f)}$ is usually not an embedded desingularization for $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$ (as $X_{\Sigma(f)}$ is usually singular).

Nevertheless, this was the approach in a recent paper of Bultot–Nicaise [**BN20**], where they instead showed that if one endows $X_{\Sigma(f)}$ with the divisorial logarithmic structure $\mathscr{M}$ associated to the reduction of

$$\pi_{\Sigma(f)}^{-1}\big(V(f) \cup V(x_1) \cup V(x_2) \cup \cdots \cup V(x_n)\big) \subset X_{\Sigma(f)}$$

the resulting logarithmic scheme $(X_{\Sigma(f)}, \mathscr{M})$ is logarithmically smooth. They then related $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ to a different motivic zeta function associated to $X_{\Sigma(f)}$ and the Gelfand–Leray form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n/df$ (cf. Loeser–Sebag [**LS03**] and [**BN20**, Definition 5.2.2]). Finally, the logarithmic smoothness of $(X_{\Sigma(f)}, \mathscr{M})$ enables them to deduce an explicit formula for the latter zeta function, from which (1.7) follows.

1.3.14. In contrast, our approach towards (1.7) is a *stack-theoretic* re-interpretation of Bultot–Nicaise's approach, and allows one to work directly on $\Sigma(f)$ while still remaining in the realm of *smooth ambient spaces*. The point here is that one can associate, to the potentially singular toric variety $X_{\Sigma(f)}$, a smooth toric Artin stack $\mathscr{X}_{\Sigma(f)}$ whose good moduli space (in the sense of [**Alp13**]) is $X_{\Sigma(f)}$, cf. §5.2.A. One can then show that the composition

$$\vartheta_{\Sigma(f)}\colon \mathscr{X}_{\Sigma(f)} \xrightarrow{\text{good moduli space}} X_{\Sigma(f)} \xrightarrow{\pi_{\Sigma(f)}} \mathbf{A}^n$$

desingularizes $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$ in the following sense:

Definition 1.3.15. A *stack-theoretic embedded desingularization of* $V(f) \subset \mathbf{A}^n$ *above* $\mathbf{0} \in \mathbf{A}^n$ is a morphism $\Pi\colon \mathscr{X} \to \mathbf{A}^n$ where:

(i) $\mathscr{X}$ is a smooth Artin stack over $\mathbf{k}$ admitting a good moduli space $\mathscr{X} \to \mathbf{X}$, and the induced morphism $\pi\colon \mathbf{X} \to \mathbf{A}^n$ is proper and birational.

(ii) $\Pi^{-1}\big(V(f)\big)$ is a simple normal crossings divisor at every point in $\Pi^{-1}(\mathbf{0})$ (in the stack-theoretic sense, cf. [**BR19**, Definition 3.1]).

1.3.16. In §5.2.A we also discuss a motivic change of variables for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ that is applicable to $\vartheta_{\Sigma(f)}$, although indirectly. By this we mean that one has to first take a *simplicial* subdivision $\boldsymbol{\Sigma}(f)$ of $\Sigma(f)$ *without adding new rays*. The effect of doing so is that the corresponding toric stack $\mathscr{X}_{\boldsymbol{\Sigma}(f)}$ is Deligne–Mumford, and the morphism $\vartheta_{\boldsymbol{\Sigma}(f)}\colon \mathscr{X}_{\boldsymbol{\Sigma}(f)} \to \mathbf{A}^n$ factors through $\vartheta_{\Sigma(f)}\colon \mathscr{X}_{\Sigma(f)} \to \mathbf{A}^n$ as an open substack, i.e. $\vartheta_{\boldsymbol{\Sigma}(f)}$ also desingularizes $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$. Finally we compute the set of numerical data associated to $(f, \vartheta_{\boldsymbol{\Sigma}(f)})$ (in the sense of Definition 1.3.17 below), and show that it is the set $\eta(f)$ in (1.8). Applying the aforementioned motivic change of variables to $\vartheta_{\boldsymbol{\Sigma}(f)}$, the preceding sentence then implies (1.7).

Definition 1.3.17. Let $\Pi\colon \mathscr{X} \to \mathbf{A}^n$ be a stack-theoretic embedded desingularization of $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$, such that $\mathscr{X}$ is a Deligne–Mumford stack. Let $\{E_i\colon i \in I\}$ denote the set of irreducible components of $\Pi^{-1}\big(V(f)\big)$. For each $i \in I$, let N_i (resp. $\nu_i - 1$) denote the multiplicity of E_i in the divisor $\Pi^{-1}\big(V(f)\big)$ (resp. the relative canonical divisor K_Π of Π). Then the *set of numerical data* associated to the pair (f, Π) is:

$$\eta(f, \Pi) := \big\{(N_i, \nu_i)\colon i \in I\big\}$$

where each (N_i, ν_i) is referred to as the *numerical datum* of the corresponding irreducible component $E_i \subset \Pi^{-1}\big(V(f)\big)$.

Similar to how the motivic change of variables in 1.3.16 reduces (1.6) to the existence of a stack-theoretic desingularization of $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$ whose set of numerical data equal to $\eta(f)$, that same change of variables would also reduce Theorem H to the following:

Theorem I ($\Longleftarrow$ Theorem 5.3.26)**.** *Given a set $\mathbb{B}$ of B_1-facets of $\Gamma_+(f)$ with consistent base directions, there exists a stack-theoretic embedded desingularization $\Pi\colon \mathscr{X} \to \mathbf{A}^n$ of $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$, such that $\mathscr{X}$ is a Deligne–Mumford stack, and whose set of numerical data is:*

$$\eta^{\dagger,\mathbb{B}}(f) := \{(1,1)\} \cup \big\{\eta_\tau \colon \tau \prec^1 \Gamma_+(f) \textit{ with } N_\tau > 0 \textit{ and } \tau \notin \mathbb{B}\big\}.$$

1.3.18. Our proof of Theorem I occupies the entirety of §5.3. As one might expect from the discussion in 1.3.12 and 1.3.14, the proof should involve the construction of a fan $\Sigma^\dagger$ that subdivides $N^+_{\mathbf{R}}$ and satisfies the following:

(i) The set of rays in $\Sigma^\dagger$ comprises of rays in $\Sigma(f)$ *except* those that are dual to facets in $\mathbb{B}$.

(ii) The induced toric modification $\vartheta_{\Sigma^\dagger}\colon \mathscr{X}_{\Sigma^\dagger} \to \mathbf{A}^n$ is a stack-theoretic embedded desingularization of $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$.

In the first two paragraphs of §5.2.B, we give a brief sketch as to how one could accomplish this construction, and in §5.3.A and §5.3.B, we provide the details of the construction. In addition, in §5.2.B we also verify our methods for three non-degenerate polynomials in $n = 3$ variables. We hope to highlight, through these examples, various aspects of Theorems H and I.

1.3.19. Finally, we indicate in §5.4 the various aspects in which Theorem H is incomplete for the motivic monodromy conjecture for non-degenerate polynomials (1.3.3), most of which we are pursuing separately in a sequel, using methods that are motivated by and similar to the ones in this chapter.

Nevertheless, Theorem H in particular recovers the motivic monodromy conjecture for non-degenerate polynomials in $n = 3$ variables, which was proven previously by Bories–Veys [**BV16**, Theorem 10.3], although (as hinted in 1.3.11) via an approach different from Theorem I. Indeed, in §5.4.A, we first show that Theorem H implies:

Theorem J (= Theorem 5.4.9)**.** *Let $n = 3$, and let $\mathcal{S}_\circ \subset \Theta(f) \smallsetminus \{-1\}$. If $\mathcal{F}(f; s_\circ)$ is a set of B_1-facets of $\Gamma_+(f)$ with consistent base directions for each $s_\circ \in \mathcal{S}_\circ$, then $\Theta(f) \smallsetminus \mathcal{S}_\circ$ is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f; s)$.*

Note that by specializing $Z_{\mathrm{mot},\mathbf{0}}(f; s)$ to $Z_{\mathrm{top},\mathbf{0}}(f; s)$ (cf. Remark 1.3.6), Theorem J in particular recovers [**LVP11**, Proposition 14]. Moreover, the authors in loc. cit. showed that $s_\circ \in \Theta(f) \smallsetminus \{-1\}$ induces a monodromy eigenvalue of f near $\mathbf{0} \in \mathbf{C}^n$ (in the sense indicated in 1.3.3) whenever $\mathcal{F}(f; s_\circ)$ satisfies either of the following hypotheses:

(i) $\mathcal{F}(f; s_\circ)$ contains a *non–B_1*-facet of $\Gamma_+(f)$ [**LVP11**, Theorem 10].

(ii) $\mathcal{F}(f; s_\circ)$ is a set of B_1-facets of $\Gamma_+(f)$, but *without* consistent base directions [**LVP11**, Theorem 15].

Therefore, we conclude from Theorem J and the preceding sentence that:

Corollary K. *The motivic monodromy conjecture holds for non-degenerate polynomials in $n = 3$ variables.*

CHAPTER 2

Weighted blow-ups

In this chapter, unless otherwise specified, we consider schemes (or more generally algebraic stacks) X over an arbitrary base scheme S. Occasionally we take S to be the final object $\mathrm{Spec}(\mathbf{Z})$ in the category of schemes, or $\mathrm{Spec}(\mathbf{k})$ for a field $\mathbf{k}$.

2.1. Stack-theoretic Proj

Let $R = \bigoplus_{n \in \mathbf{N}} R_n$ be a quasi-coherent graded $\mathscr{O}_X$-algebra. The grading on R corresponds to a co-action of $\mathscr{O}_X[t^{\pm 1}]$ on R:

$$\beta \colon R \to R \otimes_{\mathscr{O}_X} \mathscr{O}_X[t^{\pm 1}] = R[t^{\pm 1}]$$

mapping a section r of R_n to rt^n, or equivalently, an action $\alpha \colon \mathbb{G}_m \times \mathrm{Spec}_X(R) \to \mathrm{Spec}_X(R)$ of $\mathbb{G}_m = \mathrm{Spec}(\mathbf{Z}[t^{\pm 1}])$ on $\mathrm{Spec}_X(R)$. We denote by R_+ the ideal $\bigoplus_{n \geq 1} R_n \subset R$, and we denote the d^{th} Veronese subalgebra $\bigoplus_{n \geq 0} R_{dn}$ by $R^{(d)}$.

2.1.A. Stabilizers of $\mathbb{G}_m$-action. Let $x \colon \mathrm{Spec}(\kappa) \to \mathrm{Spec}_X(R)$ be a point. The stabilizer group scheme of x, denoted G_x, is a closed subgroup of $\mathbb{G}_m \otimes \kappa = \mathrm{Spec}(\kappa[t^{\pm 1}])$, and sits in the cartesian diagram

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \mathrm{Spec}(\kappa) \arrow[d, "{(x,x)}"] \\ \mathbb{G}_m \times \mathrm{Spec}_X(R) \arrow[r, "{(\alpha,\pi_2)}"] & \mathrm{Spec}_X(R) \times_X \mathrm{Spec}_X(R). \end{tikzcd}$$

Thus, either $G_x = \boldsymbol{\mu}_d \otimes \kappa = \mathrm{Spec}\big(\kappa[t]/(t^d - 1)\big)$ for some $d \geq 1$ or $G_x = \mathbb{G}_m \otimes \kappa$. Equivalently, the Cartier dual of G_x is either $\mathbf{Z}/d\mathbf{Z}$ or $\mathbf{Z}$.

Lemma 2.1.1. *The Cartier dual of G_x is $\mathbf{Z}/(d\colon x \notin V(R_d))$, that is,*

(i) *$G_x = \mathbb{G}_m$ if and only if $x \in V(R_+)$, and*

(ii) *$\boldsymbol{\mu}_d \subset G_x$ if and only if $x \in V(R_n)$ for all $n \in \mathbf{N}$ such that $d \nmid n$.*

Proof. The question is local on X so we may assume that X is affine. Let $\varphi_x\colon R \to \kappa$ be the corresponding ring homomorphism. Then $\boldsymbol{\mu}_d \subset G_x$ if and only if x is $\boldsymbol{\mu}_d$-equivariant, or equivalently, if and only if φ_x is $\mathbf{Z}/d\mathbf{Z}$-graded. This happens precisely when the kernel of φ_x contains R_n for all n such that $d \nmid n$. $\square$

In particular, $V(R_+)$ precisely contains the points fixed by $\mathbb{G}_m$, whence the action of $\mathbb{G}_m$ on $\operatorname{Spec}_X(R)$ restricts to an action of $\mathbb{G}_m$ on $W := \operatorname{Spec}_X(R) \smallsetminus V(R_+)$. Moreover, if R is generated in degree 1, then this action of $\mathbb{G}_m$ on W is free, i.e. $G_x = \{1\}$ for all points $x \in W$. This is because in that case $(R_n\colon d \nmid n) = R_+$ whenever $d > 1$.

2.1.B. Definition of the stack-theoretic Proj. Let $W := \operatorname{Spec}_X(R) \smallsetminus V(R_+)$. The *stack-theoretic Proj* of R is the stack quotient

$$\mathscr{P}\mathrm{roj}_X(R) := [W \,/\, \mathbb{G}_m].$$

By Lemma 2.1.1, $\mathscr{P}\mathrm{roj}_X(R)$ is a *tame* algebraic stack [**AOV08**]. If the orders of the stabilizer groups of the points of $\mathscr{P}\mathrm{roj}_X(R)$ are invertible on X, then $\mathscr{P}\mathrm{roj}_X(R)$ is a Deligne–Mumford stack. In particular, this holds in characteristic zero.

The $\mathbb{G}_m$-equivariant map $W \to X$, where X is equipped with the trivial action, gives a map $\mathscr{P}\mathrm{roj}_X(R) \to X \times \mathrm{B}\mathbb{G}_m$. We let $\pi\colon \mathscr{P}\mathrm{roj}_X(R) \to X$ and $p\colon \mathscr{P}\mathrm{roj}_X(R) \to \mathrm{B}\mathbb{G}_m$ be the induced maps. Then p fits in the following cartesian square:

$$\begin{tikzcd} W \arrow[r] \arrow[d, "q"] & * \arrow[d] \\ \mathscr{P}\mathrm{roj}_X(R) \arrow[r, "p"] & \mathrm{B}\mathbb{G}_m \end{tikzcd}$$

where $q\colon W \to [W \,/\, \mathbb{G}_m]$ is the quotient map. Note that since R is also an R_0-algebra, π factors through $\operatorname{Spec}_X(R_0)$, and $\mathscr{P}\mathrm{roj}_X(R) \to \operatorname{Spec}_X(R_0)$ is the stack-theoretic Proj of R as an R_0-algebra. It is thus harmless to assume that $R_0 = \mathscr{O}_X$.

If X is more generally an algebraic stack, the above definition still makes sense. In this case W is an algebraic stack with an action of $\mathbb{G}_m$ [**Rom05**].

2.1.C. Local charts. We can give local charts of $\mathscr{P}\mathrm{roj}_X(R)$ as follows. Let $f_i \in R_+$ be homogeneous elements of degrees $d_i \geq 1$, indexed by some indexing set I, such that $R_+ \subset \sqrt{(f_i : i \in I)}$. Then $W = \operatorname{Spec}_X(R) \smallsetminus V(R_+) = \bigcup_{i\in I} \operatorname{Spec}_X(R_{f_i})$, so we have an open covering $\mathscr{P}\mathrm{roj}_X(R) = \bigcup_{i\in I} D_+(f_i)$, where

$$D_+(f_i) := \Big[\operatorname{Spec}_X(R_{f_i}) \,/\, \mathbb{G}_m\Big] = \big[\operatorname{Spec}_X\big(R_{f_i}/(f_i - 1)\big) \,/\, \boldsymbol{\mu}_{d_i}\big]\,. \tag{2.1}$$

and is called the f_i-*chart*. The second equality in (2.1) follows from Lemma 2.1.2, with $A = \mathbf{Z}$, $a = d_i$, $R = R_{f_i}$ and $r = f_i$. The intersection of charts works as usual: $D_+(f_i) \cap D_+(f_j) = D_+(f_i f_j)$ and the open inclusion $D_+(f_i) \cap D_+(f_j) \subset D_+(f_i)$ is given by $f_j \neq 0$.

Lemma 2.1.2 (Slicing)**.** *Let A be a finitely generated abelian group, with corresponding diagonalizable algebraic group $D(A)$. Let $R = \bigoplus_{\alpha\in A} R_\alpha$ be an A-graded algebra, and let $r \in R$ be a homogeneous element of degree $a \in A$. Then $R/(r-1)$ is an $A/\langle a\rangle$-graded algebra and the $A/\langle a\rangle$-graded homomorphism $R \to R/(r-1)$ induces a morphism of algebraic stacks*

$$\big[\operatorname{Spec}(R/(r-1)\big) \,/\, D\big(A/\langle a\rangle\big)\big] \to \big[\operatorname{Spec}(R) \,/\, D(A)\big].$$

This is an isomorphism if r is invertible and a has infinite order.

Note that as A-graded modules $R \simeq R/(r-1)[r, r^{-1}]$ but the algebra structures do not coincide. Similarly, $R/(r-1) \simeq \bigoplus_{[\alpha]\in A/\langle a\rangle} R_\alpha$ but only as $A/\langle a\rangle$-graded modules.

Proof. Only the last statement requires proof. It suffices to prove that the natural $D(A)$-equivariant map

$$\operatorname{Spec}\bigl(R/(r-1)\bigr) \times^{D(A/\langle a\rangle)} D(A) \to \operatorname{Spec}(R)$$

is an isomorphism. Let us elaborate on the left hand side. We have two commuting actions on $\operatorname{Spec}\bigl(R/(r-1)\bigr) \times D(A) = \operatorname{Spec}(R/(r-1)[v^A]) := \operatorname{Spec}(R/(r-1)[v^\alpha \colon \alpha \in A])$:

(i) the diagonal $D(A/\langle a\rangle)$-action, given by $(y,t)\cdot s = (ys, s^{-1}t)$, where in the first factor the action corresponds to the induced $A/\langle a\rangle$-grading on $R/(r-1)$, and

(ii) the $D(A)$-action on the second factor given by $(y,t)\cdot s = (y, ts)$.

The $D(A/\langle a\rangle)$-action is free with quotient $\operatorname{Spec}\bigl(R/(r-1)\bigr) \times^{D(A/\langle a\rangle)} D(A) = \operatorname{Spec}(R^\circ)$ where R° is the degree 0 part of $R/(r-1)[v^A]$ with the $A/\langle a\rangle$-grading. The $D(A)$-action endows R° with the following A-grading

$$R^\circ = \bigoplus_{\alpha\in A} \bigl(R/(r-1)\bigr)_{[\alpha]} v^\alpha.$$

The natural A-graded algebra homomorphism $R \to R^\circ$ is thus an isomorphism. □

Remark 2.1.3. The full sub-category of algebraic stacks, whose objects are Zariski-locally of the form $[\operatorname{Spec}(B) \,/\, D(A)]$ for a finitely generated abelian group A with diagonalizable group scheme $D(A)$ and an A-graded ring B, is closed under taking stacky Proj. Indeed, let $X = [\operatorname{Spec}(B) \,/\, D(A)]$ as above, and let R be a quasi-coherent graded $\mathscr{O}_X$-algebra, i.e. a quasi-coherent $(A \times \mathbf{Z})$-graded B-algebra. For a collection of $(A \times \mathbf{Z})$-homogeneous elements $f_i \in R_+$ of degrees (a_i, d_i) such that $R_+ \subset \sqrt{(f_i \colon i \in I)}$, then $\mathscr{P}\mathrm{roj}_X(R)$ is then covered by the charts

$$D_+(f_i) = [\operatorname{Spec}(R_{f_i}) \,/\, D(A \times \mathbf{Z})] = \bigl[\operatorname{Spec}\bigl(R_{f_i}/(f_i - 1)\bigr) \,/\, D\bigl((A \times \mathbf{Z})/\langle (a_i, d_i)\rangle\bigr)\bigr]\,.$$

2.1.D. Tautological line bundles $\mathscr{O}(d)$. Let as before $p\colon W \to \mathscr{P}\mathrm{roj}_X(R)$ denote the presentation. pullback of line bundles induces an isomorphism

$$\rho^*\colon \mathrm{Pic}\big(\mathscr{P}\mathrm{roj}_X(R)\big) \xrightarrow{\simeq} \mathrm{Pic}^{\mathbb{G}_m}(W)$$

where the right hand side denotes the $\mathbb{G}_m$-equivariant Picard group of W. For each $d \in \mathbf{Z}$, there are tautological line bundles $\mathscr{O}(d)$ on the stack-theoretic Proj, arising from the shifts $R(d)$, as well as natural maps $\pi^* R_d \to \mathscr{O}(d)$ induced from the multiplication maps $R \otimes_{\mathscr{O}_X} R_d \to R(d)$. Note that $\mathscr{O}(1)$ is invertible and that $\mathscr{O}(d) = \mathscr{O}(1)^{\otimes d}$.

For each $d \in \mathbf{Z}$, we let $q_d\colon \mathscr{P}\mathrm{roj}_X(R) \to \mathrm{B}\mathbb{G}_m = [* \,/\, \mathbb{G}_m]$ be the morphism classifying the line bundle $\mathscr{O}(d)$. Then $q_1 = q$, that is, $\mathscr{O}(1)$ corresponds to the $\mathbb{G}_m$-torsor $W \to \mathscr{P}\mathrm{roj}_X(R)$. In particular, $q_d = (\cdot)^d \circ q$ where $(\cdot)^d\colon \mathrm{B}\mathbb{G}_m \to \mathrm{B}\mathbb{G}_m$ is induced by the d^{th} power morphism. Equivalently, $q_d\colon [W \,/\, \mathbb{G}_m] \to [* \,/\, \mathbb{G}_m]$ is induced by the structure morphism $W \to *$ and $(\cdot)^d\colon \mathbb{G}_m \to \mathbb{G}_m$. Therefore, the morphism also fits in the following cartesian square [**Ols16**, Exercise 10.F]:

$$\begin{tikzcd}
{[W \,/\, \boldsymbol{\mu}_d]} \arrow[r] \arrow[d] & * \arrow[d] \\
\mathscr{P}\mathrm{roj}_X(R) \arrow[r, "q_d"] & \mathrm{B}\mathbb{G}_m.
\end{tikzcd}$$

2.1.E. Properties. The stack-theoretic Proj enjoys the following universal property:

Proposition 2.1.4 (Universal property)**.** *Let R be a graded quasi-coherent $\mathscr{O}_X$-algebra. Given a scheme T with a morphism $f\colon T \to X$, a lift of f to $\mathscr{P}\mathrm{roj}_X(R)$ corresponds to the data of a line bundle $\mathscr{L}$ on T and a graded homomorphism $\varphi\colon f^*R \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$ of sheaves of algebras on T such that locally on T, $\varphi_n\colon f^*R_n \to \mathscr{L}^{\otimes n}$ is surjective for all sufficiently divisible n.*

Proof. To lift f to $\mathscr{P}\mathrm{roj}_X(R)$, one needs to supply a $\mathbb{G}_m$-torsor P over T mapping $\mathbb{G}_m$-equivariantly to W making the following diagram commute:

$$\begin{tikzcd} P \arrow[r, dotted] \arrow[d, dotted] & W \arrow[r, hook] & \mathrm{Spec}_X(R) \arrow[d] \\ T \arrow[rr, "f"] & & X \end{tikzcd}$$

Every line bundle $\mathscr{L}$ on T gives rise to a $\mathbb{G}_m$-torsor $P = \mathrm{Spec}_T\big(\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n}\big)$ over T, and conversely any $\mathbb{G}_m$-torsor P over T arises from some line bundle $\mathscr{L}$ on T. Then a $\mathbb{G}_m$-equivariant morphism

$$P = \mathrm{Spec}_T\left(\bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}\right) \smallsetminus V\left(\bigoplus_{n\geq 1} \mathscr{L}^{\otimes n}\right) \longrightarrow \mathrm{Spec}_X(R) \smallsetminus V(R_+) = W$$

making the diagram above commute, is equivalent to a graded homomorphism $\varphi\colon f^*R \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$ such that $\bigoplus_{n\geq 1} \mathscr{L}^{\otimes n} \subset \sqrt{\varphi(f^*R_+)}$, i.e. $\mathscr{L} \subset \sqrt{\varphi(f^*R_+)}$. Given a trivialization t of $\mathscr{L}$ over an open subset $U \subset T$, there therefore exists a positive integer N such that the trivializing section $t^{\otimes N}$ of $\mathscr{L}^{\otimes N}$ over U lifts to f^*R_N. Thus, whenever N divides n, $\varphi_n\colon f^*R_n \to \mathscr{L}^{\otimes n}$ is surjective over U. $\square$

Proposition 2.1.5. *Let R be a graded quasi-coherent $\mathscr{O}_X$-algebra.*

(i) *$\mathscr{P}\mathrm{roj}_X(R)$ has finite diagonal relative to X. In particular, $\mathscr{P}\mathrm{roj}_X(R)$ is separated over X.*

(ii) *If R is finitely generated, then $\mathscr{P}\mathrm{roj}_X(R)$ is proper over X.*

(iii) *The coarse space of $\mathscr{P}\mathrm{roj}_X(R)$, relative to X, is the usual* $\mathrm{Proj}_X(R)$.

(iv) *The morphism $q_1\colon \mathscr{P}\mathrm{roj}_X(R) \to X \times \mathrm{B}\mathbb{G}_m$ corresponding to $\mathscr{O}(1)$ is quasi-affine.*

(v) *The relative coarse space of $q_d\colon \mathscr{P}\mathrm{roj}_X(R) \to X \times \mathrm{B}\mathbb{G}_m$ is $\mathscr{P}\mathrm{roj}_X(R^{(d)})$, where $R^{(d)} = \bigoplus_{n\geq 0} R_{dn}$.*

(vi) *If S is another graded $\mathscr{O}_X$-algebra and $\varphi\colon R \to S$ is a graded homomorphism such that $S_+ \subset \sqrt{\varphi(R_+)}$, then there is an induced affine morphism $f\colon \mathscr{P}\mathrm{roj}_X(S) \to \mathscr{P}\mathrm{roj}_X(R)$*

such that $f^*\mathscr{O}(1) = \mathscr{O}(1)$. If R and S are of finite type and $R_0 \to S_0$ is finite, then f is finite.

Here, (iv) is saying that $\mathscr{O}(1)$ alone is *generating* for $\mathscr{P}\mathrm{roj}_X(R)$ in the sense of [**Gro17**], cf. [**Gro17**, Corollary 6.7].

PROOF. The questions are local on X, so we may assume that $X = \operatorname{Spec}(A)$ is affine. For (i), we need to show that $D_+(f_if_j) \to D_+(f_i) \times_X D_+(f_j)$ is finite. Over the tautological $\mathbb{G}_m^2$-torsor this map is $\operatorname{Spec}(R_{f_if_j}[u, u^{-1}]) \to \operatorname{Spec}(R_{f_i}) \times_X \operatorname{Spec}(R_{f_j})$, where $\varphi\colon R_{f_i} \otimes_A R_{f_j} \to R_{f_if_j}[u, u^{-1}]$ is given by $\varphi(r \otimes s) = rsu^d$ if $s \in R_d$. It is finite since $u^{d_i} = \varphi(f_i^{-1} \otimes f_i)$ and $u^{-d_j} = \varphi(f_j \otimes f_j^{-1})$.

For (iii), note that $\operatorname{Proj}_X(R) = \big(\operatorname{Spec}(R) \smallsetminus V(R_+)\big) \,/\, \mathbb{G}_m$. Indeed, the coarse space of $D_+(f_i) = [\operatorname{Spec}(R_{f_i}) \,/\, \mathbb{G}_m]$ is the spectrum of the invariant ring of R_{f_i}, that is, $R_{(f_i)}$. For (ii), if R is finitely generated, then $\operatorname{Proj}_X(R) \to X$ is proper. Coupled with the fact that the coarse space morphism $\mathscr{P}\mathrm{roj}_S(R) \to \operatorname{Proj}_X(R)$ is also proper, we conclude that $\mathscr{P}\mathrm{roj}_X(R)$ is proper over X. For (iv), it suffices to note that the total space of the $\mathbb{G}_m$-bundle corresponding to $\mathscr{O}(1)$ is the quasi-affine scheme $W = \operatorname{Spec}(R) \smallsetminus V(R_+)$, which we saw in §2.1.D.

For (v), the question can be checked flat-locally, so we may pass to the total space of the $\mathbb{G}_m$-bundle corresponding to $\mathscr{O}(d)$, which is $[W \,/\, \boldsymbol{\mu}_d]$ (by §2.1.D again), and which has coarse space $\operatorname{Spec}(R^{(d)}) \smallsetminus V(R_+^{(d)})$. Consequently, the relative coarse space of q_d is $[\operatorname{Spec}(R^{(d)}) \smallsetminus V(R_+^{(d)}) \,/\, \mathbb{G}_m] = \mathscr{P}\mathrm{roj}_X(R^{(d)})$.

Finally, for (vi), we obtain an affine $\mathbb{G}_m$-equivariant morphism $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ such that the inverse image of $V(R_+)$ contains $V(S_+)$, whence we obtain an affine morphism $f\colon \mathscr{P}\mathrm{roj}_X(S) \to \mathscr{P}\mathrm{roj}_X(R)$ over $\mathrm{B}\mathbb{G}_m$. If R and S are of finite type and $R_0 \to S_0$ is finite, then both stacky Proj are proper over $\operatorname{Spec}(R_0)$, and hence f is finite (as f is proper and affine). $\square$

Note that (vi) with $S = \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$ retrieves the universal property (Proposition 2.1.4). In the terminology of [**AH10**, §2.3], the stack-theoretic Proj is a *cyclotomic stack*, i.e. has stabilizers $\boldsymbol{\mu}_d$ (Lemma 2.1.1), which is *uniformized by* $\mathscr{O}(1)$, i.e. $q_1 \colon \mathscr{P}\mathrm{roj}_X(R) \to \mathrm{B}\mathbb{G}_m$ is representable (Proposition 2.1.5(iv)).

Corollary 2.1.6. *Let R be a graded $\mathscr{O}_X$-algebra. $\mathscr{P}\mathrm{roj}_X(R)$ coincides with the usual $\mathrm{Proj}_X(R)$ if and only if the action of $\mathbb{G}_m$ on W is free, in which case $\mathscr{O}(1)$ is very ample relative to X. In particular, this happens when R is generated in degree* 1.

Proof. $\mathscr{P}\mathrm{roj}_X(R)$ coincides with the coarse space $\mathrm{Proj}_X(R)$ if and only if $\mathscr{P}\mathrm{roj}_X(R)$ is an algebraic space, if and only if the action of $\mathbb{G}_m$ on W is free. □

Recall that the shift $R(d)$ also gives rise to a coherent sheaf $\widetilde{R(d)}$ on $\mathrm{Proj}_X(R)$, but this sheaf is not always invertible if R is not generated in degree 1. There is a canonical morphism $\widetilde{R(d)} \otimes \widetilde{R(e)} \to \widetilde{R(d+e)}$ but this is also not an isomorphism in general. If $p \colon \mathscr{P}\mathrm{roj}_X(R) \to \mathrm{Proj}_X(R)$ denotes the coarsening morphism, then $\widetilde{R(d)} = p_*\mathscr{O}(d)$.

Proposition 2.1.7. *Let R be a graded $\mathscr{O}_X$-algebra. If X is quasi-compact and R is finitely generated, then $R^{(d)} = \bigoplus_{n \geq 0} R_{dn}$ is generated in degree* 1 *for all sufficiently divisible d. In particular, the usual $\mathrm{Proj}_X(R)$ agrees with the stack-theoretic $\mathscr{P}\mathrm{roj}_X(R^{(d)})$ and is the relative coarse space of q_d.*

Proof. This can be verified locally on X so we can assume that $X = \mathrm{Spec}(A)$ is affine. If R has generators $f_1, \dots, f_m$ with degrees $d_1, d_2, \dots, d_m$, then we claim that choosing $d = m\ell$ suffices, where ℓ is the least common multiple of the d_i. Indeed, for every $n \geq 0$, R_n is generated by $f_1^{a_1} \cdots f_m^{a_m}$ with $\sum_{i=1}^m a_i d_i = n$. If $n \geq m\ell$, then for each such generator $f_1^{a_1} \cdots f_m^{a_m}$, there exists some $1 \leq i \leq m$ such that $a_i d_i \geq \ell$, i.e. $f_1^{a_1} \cdots f_m^{a_m}$ is divisible by $f_i^{\ell/d_i} \in R_\ell$. This shows that $R_n = R_{n-\ell} R_\ell$ whenever $n \geq m\ell$, which implies the claim. □

Remark 2.1.8. If R has generators of degrees $d_1, d_2, \dots, d_m$, then it is not sufficient to take d as the least common multiple ℓ of $d_1, d_2, \dots, d_m$ in Proposition 2.1.7.

2.1.F. Embeddings into the stack-theoretic Proj.

2.1.9. In this section, let $f\colon X \to S$ be a qcqs morphism of algebraic stacks and $\mathscr{L}$ a line bundle on X. If there exists for every $x \in X$ a positive integer N such that $f^*f_*\mathscr{L}^{\otimes N} \to \mathscr{L}^{\otimes N}$ is surjective at x, then the homomorphism $\bigoplus_{n\geq 0} f^*f_*\mathscr{L}^{\otimes n} \to \bigoplus_{n\geq 0}\mathscr{L}^{\otimes n}$ induces, via the universal property (Proposition 2.1.4), a morphism

$$\varphi_{\mathscr{L}}\colon X \to \mathscr{P}\mathrm{roj}_S\Big(\bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n}\Big) \tag{2.2}$$

such that $\varphi_{\mathscr{L}}^*\mathscr{O}(1) = \mathscr{L}$. In particular, if $\mathscr{L}$ is *uniformizing* relative to S, that is, the induced morphism $X \to S \times \mathrm{B}\mathbb{G}_m$ is representable, then so is the induced morphism $\varphi_{\mathscr{L}}\colon X \to \mathscr{P}\mathrm{roj}_S(\bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n})$.

Setup 2.1.10. Let $f\colon X \to S$ be a morphism of quasi-compact algebraic stacks with finite diagonal. Then there is a relative coarse space $p\colon X \to X_{\mathrm{cs}/S}$ and $f_{\mathrm{cs}}\colon X_{\mathrm{cs}/S} \to S$ is separated. Let $\mathscr{L}$ be a line bundle on X. Then for sufficiently divisible k, the line bundle $\mathscr{L}^{\otimes k}$ *descends to* $X_{\mathrm{cs}/S}$ [**Ryd15**]. To be precise, $p_*\mathscr{L}^{\otimes k}$ is a line bundle and $p^*p_*\mathscr{L}^{\otimes k} \to \mathscr{L}^{\otimes k}$ is an isomorphism.

Definition 2.1.11 (Ampleness)**.** In Setup 2.1.10, we say that $\mathscr{L}$ is *ample* relative to S if the line bundle $p_*\mathscr{L}^{\otimes k}$ is ample relative to S.

Lemma 2.1.12. *Keep the assumptions of Setup 2.1.10. If S is an affine scheme, then the following statements are equivalent:*

(i) *$\mathscr{L}$ is ample.*

(ii) *The open subsets X_g, for $g \in \Gamma(X, \mathscr{L}^{\otimes d})$ where d is a positive integer, form a basis for the topology on X.*

(iii) *There exists a positive integer d and finitely many sections $f_i \in \Gamma(X, \mathscr{L}^{\otimes d})$ such that $(X_{\mathrm{cs}})_{f_i}$ is affine for all i, and such that $X = \bigcup_i X_{f_i}$.*

(iii′) *There exists a positive integer d and finitely many sections $f_i \in \Gamma(X, \mathscr{L}^{\otimes d})$ such that $(X_{\mathrm{cs}})_{f_i}$ is quasi-affine for all i, and such that $X = \bigcup_i X_{f_i}$.*

This can be verified via passage to the coarse space X_{cs}, and applying the analogous classical result for $p_*\mathscr{L}^{\otimes k}$. The following example shows that some caution is warranted though.

Remark 2.1.13. Retain the situation of Setup 2.1.10. If $\mathscr{L}$ is ample, and F is a quasi-coherent $\mathscr{O}_X$-module of finite type, there does *not* always exist a positive integer n_0 such that $F \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ is globally generated over S for all $n \geq n_0$. It is also *not* true that $F \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ is globally generated over S for sufficiently divisible n. For example, take $X = \mathrm{B}\boldsymbol{\mu}_d$, $\mathscr{L} = \mathscr{O}_X$, and F to be the universal torsion line bundle on X. Then $F \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n} = F$ for every integer n, and F has no global sections.

Proposition 2.1.14. *In Setup 2.1.10, the following holds.*

(i) *If $\mathscr{L}$ is ample, then $f^*f_*\mathscr{L}^{\otimes N} \to \mathscr{L}^{\otimes N}$ is surjective for all sufficiently divisible N and thus induces a morphism*

$$\varphi_{\mathscr{L}} \colon X \to \mathscr{P}\mathrm{roj}_S\big(\textstyle\bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n}\big)$$

as in (2.2).

(ii) *If $\mathscr{L}$ is ample and uniformizing, the induced morphism $\varphi_{\mathscr{L}}$ is a quasi-compact, scheme-theoretically dominant, open immersion (so in particular, quasi-affine). If in addition f is proper, $\varphi_{\mathscr{L}}$ is an isomorphism.*

(iii) *Assume there exists a positive integer N such that $f^*f_*\mathscr{L}^{\otimes N} \to \mathscr{L}^{\otimes N}$ is surjective. If the induced morphism $\varphi_{\mathscr{L}}$ is quasi-affine, then $\mathscr{L}$ is ample and uniformizing.*

Proof. For (i), denote the induced morphism $X_{\mathrm{cs}/S} \to S$ by f_{cs}. Fix a positive integer k such that $\mathscr{L}^{\otimes k}$ descends to $X_{\mathrm{cs}/S}$. Since $p_*\mathscr{L}^{\otimes k}$ is ample over S, there exists a positive integer N such that for all $n \geq N$, $p_*\mathscr{L}^{\otimes kn} = (p_*\mathscr{L}^{\otimes k})^{\otimes n}$ is globally generated over S, that is, $f_{\mathrm{cs}}^*(f_{\mathrm{cs}})_*p_*\mathscr{L}^{\otimes kn} \to p_*\mathscr{L}^{\otimes kn}$ is surjective. Applying p^* and noting $f = f_{\mathrm{cs}} \circ p$, we see that $f^*f_*\mathscr{L}^{\otimes kn} \to p^*p_*\mathscr{L}^{\otimes kn} \xrightarrow{\simeq} \mathscr{L}^{\otimes kn}$ is surjective, as desired.

For (ii), the question is local and so we may assume that $S = \mathrm{Spec}(A)$ is affine. Set $R = \bigoplus_{n\geq 0}\Gamma(X, \mathscr{L}^{\otimes n})$. Since $\mathscr{L}$ is ample, we may apply Lemma 2.1.12(iii) above and thus $X = \bigcup_i X_{f_i}$ for some homogeneous $f_i \in R$ with $(X_{f_i})_{\mathrm{cs}}$ affine. Since each $[\mathrm{Spec}(R_{f_i}) \;/\; \mathbb{G}_m]$ is an open substack of $\mathscr{P}\mathrm{roj}_X(R)$, it suffices to prove that the induced morphism $X_{f_i} \to [\mathrm{Spec}(R_{f_i}) \;/\; \mathbb{G}_m]$ is an isomorphism for every i. Therefore, we set up the following diagram (where the reader should first disregard the dotted arrows and fill them in as the argument progresses):

$$\begin{tikzcd}
Z' \arrow[r, dotted, two heads, "\text{finite}"] \arrow[d, dotted, "\text{affine}"] & \mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbf{Z}}\mathscr{L}^{\otimes n}\right) \arrow[r] \arrow[d] & \mathrm{Spec}(R_{f_i}) \arrow[r] \arrow[d] & S \arrow[d] \\
Z \arrow[r, dotted, two heads, "\text{finite}"] \arrow[dr, dotted, "\text{integral}"'] & X_{f_i} \arrow[r, "\varphi_{\mathscr{L}}|_{X_{f_i}}"] \arrow[d] & \left[\mathrm{Spec}(R_{f_i}) \;/\; \mathbb{G}_m\right] \arrow[r] \arrow[d] & S \times \mathrm{B}\mathbb{G}_m \\
 & (X_{\mathrm{cs}})_{f_i} \arrow[r] & \mathrm{Spec}\big((R_{f_i})_0\big) &
\end{tikzcd} \tag{2.3}$$

Since X_{f_i} has finite diagonal over S (by assumption of Setup 2.1.10), there exists a finite cover $Z \to X_{f_i}$ from a scheme Z. Then Z is affine, since the composition $Z \to X_{f_i} \to (X_{\mathrm{cs}})_{f_i}$ is integral, and $(X_{\mathrm{cs}})_{f_i}$ is affine. Set

$$Z' := Z \times_{X_{f_i}} \mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbf{Z}}\mathscr{L}^{\otimes n}\right),$$

which is also an affine scheme. Since $\mathscr{L}$ is uniformizing, X_{f_i} is representable over $S \times \mathrm{B}\mathbb{G}_m$, so that $\mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbf{Z}}\mathscr{L}^{\otimes n}\right)$ is an algebraic space (over S). Moreover, it admits a finite surjection from the affine scheme Z', so Chevalley's Theorem implies that it is also an affine scheme. Hence, the morphism $\mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbf{Z}}\mathscr{L}^{\otimes n}\right) \to \mathrm{Spec}(R_{f_i})$, which induces an isomorphism on global sections

$$R_{f_i} = \left(\bigoplus_{n\geq 0}\Gamma(X,\mathscr{L}^{\otimes n})\right)_{f_i} \xrightarrow{\simeq} \bigoplus_{n\in\mathbf{Z}}\Gamma(X_{f_i},\mathscr{L}^{\otimes n}), \tag{2.4}$$

must be an isomorphism, and hence so is $X_{f_i} \to [\mathrm{Spec}(R_{f_i}) \,/\, \mathbb{G}_m]$, as desired.

For (iii), the question is again local, so we assume that $S = \mathrm{Spec}(A)$ is affine. Evidently, $\varphi_{\mathscr{L}}$ is representable, whence so is the morphism $X \to \mathrm{B}\mathbb{G}_m$ induced by $\mathscr{L}$. Thus, $\mathscr{L}$ is uniformizing. To show that $\mathscr{L}$ is ample, let d be a positive integer, and let $f \in \Gamma(X,\mathscr{L}^{\otimes d})$. Then $\varphi_{\mathscr{L}}|_{X_f}\colon X_f \to D_+(f)$ is quasi-affine, whence $(X_{\mathrm{cs}})_f$ is quasi-affine over $\mathrm{Spec}\left((R_f)_0\right)$. In particular, (iii′) of Lemma 2.1.12 is satisfied, so $\mathscr{L}$ is ample. $\square$

Remark 2.1.15. Let $f\colon X \to S$ and $\mathscr{L}$ be as in §2.1.9. In general $R = \bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n}$ is *not* finitely generated, so the corresponding stack-theoretic Proj is in general *not* proper over S. However, if S *is qcqs*, then R is the union of its finitely generated, quasi-coherent graded $\mathscr{O}_S$-subalgebras R_λ. Since X is quasi-compact, there exists an index λ_0 such that for all $\lambda \geq \lambda_0$, the composition $f^*(R_\lambda)_n \to f^*f_*\mathscr{L}^{\otimes n} \to \mathscr{L}^{\otimes n}$ is surjective for all sufficiently divisible n. For $\lambda \geq \lambda_0$, we get a morphism $\varphi_\lambda\colon X \to \mathscr{P}\mathrm{roj}_S(R_\lambda)$, where the stack-theoretic Proj is now proper over S, and such that φ_λ factors (rationally) through $\varphi_{\mathscr{L}}\colon X \to \mathscr{P}\mathrm{roj}_S(R)$.

Now assume that Setup 2.1.10 holds, assume that $\mathscr{L}$ is ample and uniformizing, and that X is *of finite type* over S. By Proposition 2.1.14(ii), $\varphi_{\mathscr{L}}\colon X \to \mathscr{P}\mathrm{roj}_S(R)$ is an open immersion. In this case, we will now show that, after increasing λ_0, the induced morphism $\varphi_\lambda\colon X \to \mathscr{P}\mathrm{roj}_S(R_\lambda)$ is an immersion for every $\lambda \geq \lambda_0$.

This is a local question on S, so we may assume that $S = \mathrm{Spec}(A)$ is affine and that $R = \bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n} = \bigoplus_{n\geq 0}\Gamma(X,\mathscr{L}^{\otimes n})$. We apply Lemma 2.1.12: there exists a positive integer d, as well as finitely many $f_i \in \Gamma(X,\mathscr{L}^{\otimes d})$ such that $X = \bigcup_i X_{f_i}$ and each $(X_{\mathrm{cs}})_{f_i}$ is affine (and each $\varphi_{\mathscr{L}}|_{X_{f_i}}\colon X_{f_i} \to D_+(f_i)$ is an isomorphism).

Since X is of finite type over S, the $\mathscr{O}_S$-algebra $R_{f_i} \simeq \bigoplus_{n\in\mathbf{Z}} \Gamma(X_{f_i}, \mathscr{L}^{\otimes n})$ is generated by finitely many homogeneous elements b_{ij}. Using the isomorphism in (2.4), we may, for a positive integer m, lift each b_{ij} to some $s_{ij}/f_i^m \in (R_{f_i})_{\deg(b_{ij})}$ for some $s_{ij} \in \Gamma(X, \mathscr{L}^{\otimes md+\deg(b_{ij})})$. Thus, for sufficiently large $\lambda \geq \lambda_0$, there exist homogeneous elements s_{ij}^λ and f_i^λ of R_λ which respectively lift each s_{ij} and each f_i to R_λ. Therefore, for all such λ, φ_λ maps each X_{f_i} into $D_+(f_i^\lambda) \subset \mathscr{P}\mathrm{roj}_S(R_\lambda)$. To complete the proof, it suffices to show each $\varphi_\lambda|_{X_{f_i}}$ is a closed immersion. To this end, recall each $\varphi_\lambda|_{X_{f_i}}$ fits into the following cartesian square:

$$\begin{tikzcd} \mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n}\right) \arrow[r] \arrow[d] & \mathrm{Spec}_S\left((R_\lambda)_{f_i^\lambda}\right) \arrow[d] \\ X_{f_i} \arrow[r, "\varphi_\lambda|_{X_{f_i}}"] & D_+(f_i^\lambda) \end{tikzcd} \tag{2.5}$$

and it suffices to show that the top row is a closed immersion. Since each X_{f_i} has finite diagonal and each $(X_{f_i})_{\mathrm{cs}}$ is affine, we may argue, as in (2.3), that the algebraic space $\mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbf{Z}} \mathscr{L}^{\otimes n}\right)$ is an affine scheme. Then the top row of (2.5) is a morphism of affine schemes, which induces a surjection on global sections

$$(R_\lambda)_{f_i^\lambda} \twoheadrightarrow R_{f_i} \xrightarrow{\simeq} \bigoplus_{n\in\mathbf{Z}} \Gamma(X_{f_i}, \mathscr{L}^{\otimes n}),$$

and hence is necessarily a closed immersion, as desired.

2.1.G. Sequences of stack-theoretic Proj. A sequence of stack-theoretic Proj

$$X' := X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X$$

is not a stack-theoretic Proj because it need not be cyclotomic: the stabilizers are subgroups of $\mathbb{G}_m^n$, not of $\mathbb{G}_m$. Instead of a single uniformizing line bundle $\mathscr{L} = \mathscr{O}(1)$, we have a uniformizing collection of line bundles $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n$ — the corresponding map to $\mathrm{B}\mathbb{G}_m^n$ is representable.

In fact, as we shall see next, this collection is even *generating* in the sense of [**Gro17**]: locally on X, every quasi-coherent sheaf on X' of finite type is a quotient of a direct sum of line bundles of the form $\mathscr{L}_1^{\otimes d_1} \otimes \cdots \otimes \mathscr{L}_n^{\otimes d_n}$, $d_i \in \mathbf{Z}$. Equivalently, the corresponding map to $\mathrm{B}\mathbb{G}_m^n$ is quasi-affine [**Gro17**, Corollary 6.7]. That is, X' is *divisorial*.

Proposition 2.1.16. *Let $X' := X_n \to X_{n-1} \to \cdots \to X_1 \to X$ be a sequence of n stack-theoretic Proj. Let $\mathscr{O}_{X_i}(1) \in \mathrm{Pic}(X_i)$ denote the corresponding ample uniformizing line bundle and let $\mathscr{L}_i$ be the pullback of $\mathscr{O}_{X_i}(1)$ to X'.*

(i) *$(\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n)$ is generating, i.e. the induced morphism $X' \to \mathrm{B}\mathbb{G}_m^n$ is quasi-affine.*
(ii) *If X is quasi-compact, then the line bundle $\mathscr{L}_1^{\otimes N_1} \otimes \mathscr{L}_2^{\otimes N_2} \otimes \cdots \otimes \mathscr{L}_N^{\otimes N_n}$ is ample relative to X for every $N_1 \gg N_2 \gg \cdots \gg N_n$.*

Proof. The first part is immediate and the second part follows from the following lemma and the classical result for compositions of projective morphisms [**EGA$_{\mathbf{II}}$**, Prop. 4.6.13 (ii)]. □

Lemma 2.1.17. *Let $f\colon X \to Y$ be a qcqs morphism of algebraic stacks with finite inertia and let $f_{\mathrm{cs}}\colon X_{\mathrm{cs}} \to Y_{\mathrm{cs}}$ be the induced morphism on coarse spaces. Let $\mathscr{L}$ be an invertible sheaf on X_{cs}. If $\mathscr{L}|_X$ is f-ample, then $\mathscr{L}$ is f_{cs}-ample.*

Proof. By definition of ample (Setup 2.1.10), we may replace X with $X_{\mathrm{cs}/Y}$ so that f becomes representable. The question is also local on Y_{cs} so we may assume that Y_{cs} is affine and that Y admits a finite flat presentation $Y' \to Y$ of constant rank d. Let $X' = X \times_Y Y'$ and note that $Y' \to Y_{\mathrm{cs}}$ and $X' \to X_{\mathrm{cs}}$ are affine. Let $x \in |X|$ be a point. We need to find an affine open neighborhood X_g of x for some $g \in \Gamma(X, \mathscr{L}^{\otimes m})$ such that $(X_g)_{\mathrm{cs}}$ is affine, or equivalently, such that X'_g is affine.

Consider the preimage $Z \subset |X'|$ of x. Since Z is finite and $\mathscr{L}|_{X'}$ is ample, there exists a section $f \in \Gamma(X', \mathscr{L}^{\otimes n})$ such that $Z \subset X'_f$ [**EGA$_{\mathbf{II}}$**, Cor. 4.5.4]. The norm of f along

$X' \to X$ gives a section $g \in \Gamma(X, \mathscr{L}^{dn})$ such that $Z \subset X'_g \subset X'_f$ and X'_g is affine, cf. [**EGA$_{\mathbf{II}}$**, Cor. 6.5.7]. $\square$

2.2. Examples of stack-theoretic Proj

In this section we give four examples of stack-theoretic Proj:

(2.2.A) twisted weighted projective stacks, which include root stacks of line bundles,

(2.2.B) root stacks of generalized Cartier divisors,

(2.2.C) a construction that transforms **Q**-invertible sheaves to invertible sheaves,

(2.2.D) and stack-theoretic GIT quotients.

In the next section, we will also present weighted blow-ups as another example of the stack-theoretic Proj construction.

2.2.A. Weighted projective stacks, root stacks of line bundles and twisted weighted vector bundles.

Example 2.2.1 (Weighted projective stacks [**AH10**, §2.1])**.** An important class of examples of stack-theoretic Proj is *weighted projective stacks.* Given weights $d_0, d_1, \dots, d_n \in \mathbf{Z}_{>0}$ we have the smooth stack

$$\mathscr{P}(d_0, d_1, \dots, d_n) = \mathscr{P}\mathrm{roj}_X\big(\mathscr{O}_X[x_0, x_1, \dots, x_n]\big)$$

where the degree of x_i is d_i. The generic stabilizer is $\boldsymbol{\mu}_d$, where $d = \gcd(d_0, d_1, \dots, d_n)$, and the coarse space is the usual, singular, weighted projective space $\mathbf{P}(d_0, d_1, d_2, \dots, d_n)$. Slightly more general, given vector bundles $\mathscr{E}_1, \mathscr{E}_2, \dots, \mathscr{E}_r$ on X and weights $d_1, \dots, d_r \in \mathbf{Z}_{>0}$, the *weighted* (or *graded*) vector bundle $\mathscr{E} = \mathscr{E}_1(-d_1) \oplus \cdots \oplus \mathscr{E}_r(-d_r)$ gives the smooth stack

$$\mathscr{P}(\mathscr{E}) = \mathscr{P}\big(\mathscr{E}_1(-d_1) \oplus \cdots \oplus \mathscr{E}_r(-d_r)\big) := \mathscr{P}\mathrm{roj}_X\left(\bigotimes_{i=1}^{r} \mathrm{Sym}_{\mathscr{O}_X}\big(\mathscr{E}_i(-d_i)\big)\right).$$

The universal property of this stack is as follows (cf. Proposition 2.1.4): given a morphism $f\colon T \to X$, a lift to $\mathscr{P}(\mathscr{E})$ corresponds to the data of a line bundle $\mathscr{L}$ on T and homomorphisms $\varphi_i\colon f^*\mathscr{E}_i \to \mathscr{L}^{\otimes d_i}$ such that locally on T at least one of the φ_i is surjective. An isomorphism between two lifts $(\mathscr{L}, \{\varphi_i\})$ and $(\mathscr{L}', \{\varphi_i'\})$ is an isomorphism $\mathscr{L} \simeq \mathscr{L}'$ compatible with the φ_i and φ_i'.

Example 2.2.2 (Root stacks of line bundles [**Cad07a**, Def. 2.2.6])**.** A special case of the previous example is roots of line bundles. Given a line bundle $\mathscr{E}$ on X and a positive integer d, the stack $\mathscr{P}\big(\mathscr{E}(-d)\big)$ parameterizes, for a morphism of schemes $f\colon T \to X$, a line bundle $\mathscr{L}$ on T together with an isomorphism $f^*\mathscr{E} \xrightarrow{\simeq} \mathscr{L}^{\otimes d}$. The corresponding graded algebra is $R = \mathrm{Sym}_{\mathscr{O}_X}\big(\mathscr{E}(-d)\big)$. We call the corresponding stack-theoretic Proj the d^{th} *root stack of the line bundle* $\mathscr{E}$, and denote it by $X_{(\mathscr{E},d)}$ or $X\big(\sqrt[d]{\mathscr{E}}\big)$.

We will need the following generalization of weighted vector bundles:

Definition 2.2.3. A *twisted weighted vector bundle* on X is a smooth affine morphism $E \to X$ with a $\mathbb{G}_m$-action such that E is smooth-locally $\mathbb{G}_m$-equivariantly isomorphic to $X \times \mathbf{A}^n$ where $\mathbb{G}_m$ acts linearly with some weights $d_1, d_2, \ldots, d_n \in \mathbf{Z}$.

The morphism $E \to X$ is called a $\mathbb{G}_m$-fibration in [**BB73**, §3]. Equivalently, $E = \mathrm{Spec}_X(R)$ where R is a quasi-coherent graded $\mathscr{O}_X$-algebra that smooth-locally looks like the symmetric algebra over $\mathscr{O}_X$ of a graded vector bundle.

We will only need the case where all weights are positive (called fully definite in [**BB73**, §2]). If X is a scheme and all weights are positive, then smooth-locally can be replaced with Zariski-locally, see Remark 2.2.8.

Example 2.2.4 (Białynicki-Birula decomposition [**BB73**, Thm. 4.1])**.** Let X be a smooth quasi-projective variety with an action of $\mathbb{G}_m$. Let $F \subset X^{\mathbb{G}_m}$ be a connected component of the fixed locus. Let $F^+ = \{x \in X \ : \ \lim_{t\to 0} t.x \in F\}$. Then F and F^+ are $\mathbb{G}_m$-equivariant, F is closed, F^+ is locally closed, F and F^+ are smooth, and the natural map $F^+ \to F$ is a twisted weighted vector bundle with strictly positive weights.

Definition 2.2.5. A *twisted weighted projective stack* over X is the stack-theoretic Proj of a graded algebra corresponding to a twisted weighted bundle on X with strictly positive weights.

In what follows, we always assume that $E = \mathrm{Spec}_X(R)$ is a twisted weighted bundle over a *connected* scheme X. Then there exist a Zariski open cover U_i of X, weights $\mathbf{d} = (d_1 < d_2 < \cdots < d_r)$, and dimensions $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbf{Z}^r_{>0}$, such that for every i,

$$R|_{U_i} \simeq \mathrm{Sym}\left(\bigoplus_{i=1}^{r} \mathscr{O}_{U_i}^{\oplus n_i}(-d_i)\right) = \bigotimes_{i=1}^{r} \mathrm{Sym}\big(\mathscr{O}_{U_i}^{\oplus n_i}(-d_i)\big).$$

Example 2.2.6. For a non-trivial example of a twisted weighted bundle, let us consider the weights $\mathbf{d} = (d_1, d_2, d_3) = (1, 2, 4)$ and the dimensions $\mathbf{n} = (n_1, n_2, n_3) = (1, 1, 1)$. Over each U_i, we have a graded isomorphism:

$$\alpha_i \colon R|_{U_i} \xrightarrow{\simeq} \mathscr{O}_{U_i}[x_i, y_i, z_i],$$

where x_i has weight 1, y_i has weight 2, and z_i has weight 4. Over pairwise intersections $U_{ij} := U_i \cap U_j$, we then have the graded isomorphism on U_{ij}:

$$\alpha_{ij} = \alpha_j|_{U_{ij}} \circ \alpha_i|_{U_{ij}}^{-1} \colon \mathscr{O}_{U_{ij}}[x_i, y_i, z_i] \xrightarrow{\simeq} R|_{U_{ij}} \xrightarrow{\simeq} \mathscr{O}_{U_{ij}}[x_j, y_j, z_j].$$

By considering the linear relations among the weights 1, 2, and 4, we deduce that these α_{ij}'s have the following form in general:

$$x_i \mapsto a_{ij} \cdot x_j$$

$$y_i \mapsto b_{ij} \cdot \big(y_j + d_{ij} \cdot x_j^2\big)$$

$$z_i \mapsto c_{ij} \cdot \big(z_j + e_{ij} \cdot x_j^4 + f_{ij} \cdot x_j^2 y_j + g_{ij} \cdot y_j^2\big)$$

where $a_{ij}, b_{ij}, c_{ij} \in \Gamma(U_{ij}, \mathscr{O}_{U_{ij}}^{\times})$ and $d_{ij}, e_{ij}, f_{ij}, g_{ij} \in \Gamma(U_{ij}, \mathscr{O}_{U_{ij}})$. Note that the data $d_{ij}, e_{ij}, f_{ij}, g_{ij}$ are precisely the "twists" in the twisted weighted bundle R. Over triple intersections $U_{ijk} := U_i \cap U_j \cap U_k$, we have the following cocycle conditions:

$$a_{ik} = a_{ij}a_{jk} \qquad b_{ik} = b_{ij}b_{jk} \qquad c_{ik} = c_{ij}c_{jk} \qquad d_{ik} = d_{jk} + \frac{a_{jk}^2}{b_{jk}} d_{ij}$$

and

$$\begin{pmatrix} e_{ik} \\ f_{ik} \\ g_{ik} \end{pmatrix} = \begin{pmatrix} e_{jk} \\ f_{jk} \\ g_{jk} \end{pmatrix} + \frac{1}{c_{jk}} \begin{pmatrix} a_{jk}^4 & a_{jk}^2 b_{jk} \cdot d_{jk} & b_{jk}^2 \cdot d_{jk}^2 \\ 0 & a_{jk}^2 b_{jk} \cdot 1 & b_{jk}^2 \cdot 2d_{jk} \\ 0 & 0 & b_{jk}^2 \cdot 1 \end{pmatrix} \begin{pmatrix} e_{ij} \\ f_{ij} \\ g_{ij} \end{pmatrix}$$

Therefore, twisted weighted bundles R on X with weights $\mathbf{d} = (1,2,4)$ and dimensions $\mathbf{n} = (1,1,1)$ are globally characterized by the Čech cocycles in $\check{\mathrm{H}}^1\big(X, G\big)$, where

$$G = \big(\mathbb{G}_m \times ((\mathbb{G}_m \times \mathbb{G}_m) \ltimes \mathbb{G}_a)\big) \ltimes \mathbb{G}_a^3,$$

and the semidirect product are given by the actions:

(i) $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \to \mathrm{Aut}(\mathbb{G}_a)$ where $(a,b) \mapsto \frac{a^2}{b}$,

(ii) $\mathbb{G}_m \times ((\mathbb{G}_m \times \mathbb{G}_m) \ltimes \mathbb{G}_a) \to \mathrm{GL}_3 \to \mathrm{Aut}(\mathbb{G}_a^3)$ where

$$(c,a,b,d) \longmapsto \frac{1}{c} \begin{pmatrix} a^4 & a^2bd & b^2d^2 \\ 0 & a^2b & 2b^2d \\ 0 & 0 & b^2 \end{pmatrix}.$$

2.2.7 (General description of twisted weighted bundles)**.** In general, twisted weighted bundles R on X with weights $\mathbf{d}$ and dimensions $\mathbf{n}$ are globally characterized by their respective

Čech cocycles in $\check{\mathrm{H}}^1(X, G_{\mathbf{d},\mathbf{n}})$, where the group $G_{\mathbf{d},\mathbf{n}}$ can be described as follows. If all weights are equal, that is, $r = 1$, then $G_{d_1,n_1} = \mathrm{GL}_{n_1}$ and twisted weighted vector bundles are just weighted vector bundles. If not, that is $r > 1$, set $\mathbf{d}' = (d_1, \ldots, d_{r-1})$, $\mathbf{n}' = (n_1, \ldots, n_{r-1})$, and $G_{\mathbf{d},\mathbf{n}}$ is a semidirect product of the form

$$G_{\mathbf{d},\mathbf{n}} = (\mathrm{GL}_{n_r} \times G_{\mathbf{d}',\mathbf{n}'}) \ltimes \mathbb{G}_a^{n_r N_r},$$

where N_r is the dimension of the $d_r{}^{\text{th}}$ degree piece of a graded polynomial algebra with free variables $\{x_{i,j} \colon 1 \leq i \leq r-1,\ 1 \leq j \leq n_i\}$, where $x_{i,j}$ is given weight d_i. That is,

$$N_r = \sum_{d_r = \sum_{m_i \geq 0} m_i d_i} \prod_{i=0}^{r-1} \binom{m_i + n_i - 1}{m_i}.$$

Remark 2.2.8. From the description of $G_{\mathbf{d},\mathbf{n}}$, we obtain an exact sequence

$$1 \longrightarrow U_{\mathbf{d},\mathbf{n}} \longrightarrow G_{\mathbf{d},\mathbf{n}} \longrightarrow \mathrm{GL}_{\mathbf{n}} \longrightarrow 1$$

where $U_{\mathbf{d},\mathbf{n}}$ is a smooth connected unipotent group scheme of dimension $N(\mathbf{d}, \mathbf{n}) = \sum_{i=2}^{r} n_i N_i$ and $\mathrm{GL}_{\mathbf{n}} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. In particular, $G_{\mathbf{d},\mathbf{n}}$ is *special* in the sense of Serre, that is, the Čech cohomology $\check{\mathrm{H}}^1(X, G_{\mathbf{d},\mathbf{n}})$ can be calculated in the Zariski topology if X is a scheme.

In particular, if d_i is not in the $\mathbf{Z}_{\geq 0}$-linear span of $d_1, \ldots, d_{i-1}$, for every $1 \leq i \leq r$, then $N(\mathbf{d}, \mathbf{n}) = 0$ and any twisted weighted bundle E with weights $(d_1, \ldots, d_r)$ *is a weighted bundle*, i.e. E "*splits*" as

$$E \simeq \mathrm{Spec}_X\big(\mathrm{Sym}_{\mathscr{O}_X}(\mathscr{E}_1(-d_1) \oplus \cdots \oplus \mathscr{E}_r(-d_r))\big)$$

for vector bundles $\mathscr{E}_1, \ldots, \mathscr{E}_r$ on X with respective dimensions $n_1, \ldots, n_r$.

2.2.9 (Associated weighted vector bundle)**.** To a twisted weighted vector bundle $E = \mathrm{Spec}_X(R)$ we can associate a weighted vector bundle $\mathscr{E} := R_+/R_+^2$ on X. This is nothing but the vector bundle corresponding to the image of E under $\check{\mathrm{H}}^1(X, G_{\mathbf{d},\mathbf{n}}) \to \check{\mathrm{H}}^1(X, \mathrm{GL}_{\mathbf{n}})$.

Since $\mathscr{E}$ is locally free, the quotient morphism $R_+ \to R_+/R_+^2 = \mathscr{E}$ *locally* admits a graded section, which *locally* induces a graded isomorphism $\mathrm{Sym}_{\mathscr{O}_X}(\mathscr{E}) \xrightarrow{\simeq} R$. One can interpret the presence of the "twists" in the twisted weighted bundle R as the obstructions to patching these local isomorphisms to a global isomorphism. Note that the weights $\mathbf{d}$ and the dimension $\mathbf{n}$ can be read off from $\mathscr{E}$.

2.2.B. Root stacks of (generalized) Cartier divisors.

Definition 2.2.10. A *generalized effective Cartier divisor* on X is a pair $(\mathscr{L}, s)$ where $\mathscr{L}$ is a line bundle and $s \in \Gamma(X, \mathscr{L})$ is a global section. Equivalently, s gives a homomorphism $s^\vee : \mathscr{L}^\vee \to \mathscr{O}_X$. We say that $(\mathscr{L}, s)$ is *ordinary* if $s^\vee$ is injective, or equivalently, if $(\mathscr{L}, s) = (\mathscr{O}_X(D), s_D)$ for an effective Cartier divisor D. Here, $s_D^\vee$ is the inclusion of the ideal $I_D = \mathscr{O}_X(-D)$.

Example 2.2.11 (Root stacks of generalized divisors [**Cad07a**, Def. 2.2.1])**.** Given a generalized Cartier divisor $(\mathscr{L}, s)$ on X and a positive integer d, we consider the following graded $\mathscr{O}_X$-algebra

$$R = \bigoplus_{n \geq 0} \mathscr{L}^{\otimes -\lceil n/d \rceil} \tag{2.6}$$

where the multiplication in this algebra makes sense by using the homomorphism $s^\vee : \mathscr{L}^\vee \to \mathscr{O}_X$ whenever applicable. For example, for $0 < k, \ell \leq d$, the multiplication $R_k \otimes R_\ell \to R_{k+\ell}$ is the canonical $\mathscr{L}^\vee \otimes \mathscr{L}^\vee \to (\mathscr{L}^\vee)^{\otimes 2}$ if $k + \ell > d$, and is given by $\mathscr{L}^\vee \otimes \mathscr{L}^\vee \to (\mathscr{L}^\vee)^{\otimes 2} \xrightarrow{\mathbb{1} \otimes s^\vee} \mathscr{L}^\vee$ if $k + \ell \leq d$. We denote the corresponding stack-theoretic Proj by $X_{(\mathscr{L}, s, d)}$, and call it the d^{th} *root stack of* $(\mathscr{L}, s)$.

The root stack $X_{(\mathscr{L}, s, d)}$ has the following universal property (cf. Proposition 2.1.4): if $f : T \to X$ is a morphism then a lift to the root stack is equivalent to giving a generalized Cartier divisor $(\mathscr{E}, t)$ on T together with an isomorphism $\varphi\colon f^*(\mathscr{L}, s) \xrightarrow{\simeq} (\mathscr{E}, t)^d$. The corresponding universal generalized Cartier divisor on the root stack is $(\mathscr{O}(-1), t)$ where $t^\vee$ is given by the natural map $R(1) \to R$. An isomorphism between two lifts $(\mathscr{E}, t, \varphi)$ and $(\mathscr{E}', t', \varphi')$ is an isomorphism $\mathscr{E} \to \mathscr{E}'$ compatible with t, t', φ and φ'.

Forgetting the section induces a morphism $X_{(\mathscr{L},s,d)} \to X_{(\mathscr{L},d)}$ to the root stack of Example 2.2.2. Note that $X_{(\mathscr{L},d)} \simeq X_{(\mathscr{L}^\vee,d)}$.

Since $R^{(d)} = \mathrm{Sym}_{\mathscr{O}_X}(\mathscr{L}^\vee)$, it follows that $\pi\colon X_{(\mathscr{L},s,d)} \to X$ is a coarse space. Since R is flat, π is also flat.

Remark 2.2.12. When $\mathscr{L}$ is trivial, then $\mathscr{P}\mathrm{roj}_X(R)$ is covered by a single chart as follows. Let $f \in \Gamma(X, \mathscr{L})$ be an everywhere non-vanishing section. Then $\mathscr{P}\mathrm{roj}_X(R) = D_+(f) = \big[\mathrm{Spec}_X\big(R/(f-1)\big) \,/\, \boldsymbol{\mu}_d\big]$. Note that $R/(f-1) \simeq \mathscr{O}_X[x]/(x^d - \frac{s}{f})$ where $\deg(x) = -1$.

More generally, if there exists a line bundle $\mathscr{E}$ such that $\mathscr{L} \simeq \mathscr{E}^d$, then we can write the root stack as a global quotient by $\boldsymbol{\mu}_d$ by first twisting R with $\mathscr{E}$ so that

$$\mathscr{P}\mathrm{roj}_X(R) = \left[\mathrm{Spec}_X\left(\bigoplus_{n=0}^{d-1} \mathscr{E}^{-n}\right) \,/\, \boldsymbol{\mu}_d\right]$$

where $\mathscr{E}^{-n}$ has degree $-n$ and the multiplication is induced by $s^\vee\colon \mathscr{E}^{-d} = \mathscr{L}^\vee \to \mathscr{O}_X$.

Example 2.2.13 (Root stacks of ordinary divisors)**.** If D is an effective Cartier divisor on X, the previous construction gives the following graded $\mathscr{O}_X$-graded algebra:

$$R = \bigoplus_{n \geq 0} I_D^{\lceil n/d \rceil}.$$

We sometimes denote $X_{(\mathscr{O}_X(D), s_D, d)}$ or $X\big(\sqrt[d]{D}\big)$ instead, and call it the d^{th} root stack of D. For morphisms $f\colon T \to X$ such that $f^{-1}(D)$ is a Cartier divisor, the root stack has the following universal property: a lift to the root stack is equivalent to giving an effective Cartier divisor E

on T such that $dE = f^{-1}(D)$. In particular, the groupoid $X_{(D,d)}(T \to X)$ is equivalent to a set in this case, i.e. there are no non-trivial isomorphisms between lifts.

The morphism $\pi\colon X_{(D,d)} \to X$ is a flat coarse space which is an isomorphism outside D. The morphism $E \to D$ is a gerbe isomorphic to the d^{th} root stack of the line bundle $\mathscr{O}_D(D)$.

2.2.C. Inverting Q-invertible sheaves.

Setup 2.2.14. Let X be a noetherian scheme satisfying Serre's condition S_2 (for example, a normal scheme). Let F be a coherent $\mathscr{O}_X$-module that is *generically locally free*, that is, there exists a dense open $j_V\colon V \hookrightarrow X$ on which F is locally free. We say that F has *rank* r if $F|_V$ is locally free of rank r. Let $\operatorname{tor}(F)$ be the torsion submodule of F, i.e.

$$\operatorname{tor}(F) = \ker\bigl(F \to j_{V*}(F|_V)\bigr)$$

and set $F_{\text{tf}} := F/\operatorname{tor}(F)$. Note that $\operatorname{tor}(F)$ is independent of V since X has no embedded points.

Suppose that $j\colon U \hookrightarrow X$ is an open subset whose complement *has codimension* ≥ 2 *in* X, and on which F_{tf} is locally free. If X is normal, then this can always be arranged for since F_{tf} is free at every point of codimension 1.

Lemma 2.2.15. *The canonical morphism $F \to F^{\vee\vee}$ can be identified with the canonical morphism $F \to F_{\text{tf}} \to j_*(F_{\text{tf}}|_U)$.*

We call $F^{\vee\vee} = j_*(F_{\text{tf}}|_U)$ the *reflexive hull* of F. We say F is *reflexive*, if the canonical morphism $F \to F^{\vee\vee} = j_*(F_{\text{tf}}|_U)$ is an isomorphism.

Proof. Firstly, since X is S_2, $F^{\vee\vee}$ is also S_2, i.e. $j_*(F^{\vee\vee}|_U) = F^{\vee\vee}$. Next, since $F^{\vee\vee}$ is torsion-free, $F \to F^{\vee\vee}$ factors through F_{tf}, and the resulting morphism $F_{\text{tf}} \to F^{\vee\vee}$ induces a morphism $j_*(F_{\mathrm{tf}}|_U) \to j_*(F^{\vee\vee}|_U) = F^{\vee\vee}$ of S_2-sheaves on X that is an isomorphism on U, and hence is an isomorphism on X. $\square$

For every integer $n \geq 0$, we define the *saturated n^{th} power* of F to be $F^{[n]} := (F^{\otimes n})^{\vee\vee}$. Note that $F^{[n]} = j_*(F_{\mathrm{tf}}^{\otimes n}|_U)$ since $F_{\mathrm{tf}}|_U$ is locally free.

We say that F is $\mathbf{Q}$-*invertible* if $F^{[N]}$ is invertible for some positive integer N, locally on X. Note that a $\mathbf{Q}$-invertible sheaf has rank 1. In what follows, we consider the graded $\mathscr{O}_X$-algebra

$$F^{[\bullet]} := \bigoplus_{n \geq 0} F^{[n]}.$$

Proposition 2.2.16 (cf. [**AH10**, Prop. 5.3.2])**.** *Let $X' = \mathscr{P}\mathrm{roj}_X(F^{[\bullet]})$ with structure morphism $\pi \colon X' \to X$. If F is $\mathbf{Q}$–invertible, then:*

(i) *$F^{[\bullet]}$ is finitely generated, and hence, X' is proper over X.*

(ii) *$\pi \colon X' \to X$ is a coarse space and an isomorphism over U. In particular, π is quasi-finite.*

(iii) *If $F^{[N]} \simeq \mathscr{O}_X$ for some $N \geq 1$, then $X' = \left[\mathrm{Spec}_X\left(\bigoplus_{n=0}^{N-1} F^{[n]}\right) \ / \ \boldsymbol{\mu}_N\right]$.*

(iv) *X' satisfies S_2. Moreover, if X is normal, so is X'.*

(v) *For every positive integer n, the canonical morphism $F^{[n]} \to \pi_*\mathscr{O}_{X'}(n)$ is an isomorphism, and the canonical morphism $\pi^*F^{[n]} \to \mathscr{O}_{X'}(n)$ is a reflexive hull.*

(vi) *$\pi \colon X' \to X$ satisfies the following universal property: if T is a scheme satisfying S_2 (resp. T is a normal scheme) which admits a morphism $f \colon T \to X$ such that $\mathrm{codim}_T\big(T \smallsetminus f^{-1}(U)\big) \geq 2$ (resp. $f^{-1}(U)$ is dense in T), then there is a lift of f to X', unique up to a unique 2-isomorphism, if and only if $(f^*F)^{\vee\vee}$ is invertible.*

Before proving the proposition, we note that in (vi), the hypothesis that $\mathrm{codim}_T(T \smallsetminus f^{-1}(U)) \geq 2$ is satisfied whenever f satisfies one of the following conditions:

(a) f is flat;

(b) f is dominant and integral and T is integral; or

(c) f is dominant and quasi-finite and T is integral.

Proof of Proposition 2.2.16. All statements, except (iii), are local on X so we may assume that $F^{[N]}$ is invertible for some integer N. For (i), note that the multiplication $F^{[kN]} \otimes F^{[n]} \to F^{[kN+n]}$ is an isomorphism for all integers $k, n \geq 0$. Thus $F^{[\bullet]}$ is generated in degrees $\leq N$. Since $F^{[n]}$ is coherent for every n, we deduce that $F^{[\bullet]}$ is finitely generated. Thus, X' is proper over X by Proposition 2.1.5(ii).

For (ii), we note that $F^{[N\bullet]}$ is generated in degree 1 and thus that the coarse space of X' is $\mathrm{Proj}_X(F^{[\bullet]}) = \mathrm{Proj}_X(F^{[N\bullet]}) = \mathbf{P}(F^{[N]}) = X$ (Proposition 2.1.5(iii)). Moreover, since $F_{\mathrm{tf}}|_U$ is invertible, π is an isomorphism over U. For (iii), this follows from $X' = D_+(f)$ where f is a nowhere vanishing section of $F^{[N]}$ (§2.1.C). For (iv), the question is local so we can assume that $F^{[N]} \simeq \mathscr{O}_X$ and hence that we have a faithfully flat presentation $\mathrm{Spec}_X\big(\bigoplus_{n=0}^{N-1} F^{[n]}\big) \to X'$. The result follows since $\bigoplus_{n=0}^{N-1} F^{[n]}$ is a coherent S_2-sheaf.

For (v), let $U' := \pi^{-1}(U)$, and consider the cartesian square:

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "\simeq"'] & X' \arrow[d, "\pi"] \\ U \arrow[r, hook, "j"] & X. \end{tikzcd}$$

Since $F^{[n]}|_U = F_{\mathrm{tf}}^{\otimes n}|_U$ is invertible, the canonical morphism $\pi^* F^{[n]} \to \mathscr{O}_{X'}(n)$ is an isomorphism when restricted to U'. Moreover, since $\mathscr{O}_{X'}(n)$ is invertible, it is S_2 and thus $\mathscr{O}_{X'}(n) = j'_*(\pi^* F^{[n]}|_{U'})$, i.e. $\mathscr{O}_{X'}(n)$ is the reflexive hull of $\pi^* F^{[n]}$. We also have that:

$$F^{[n]} \simeq j_* j^* F^{[n]} \simeq \pi_* j'_* j'^* \pi^* F^{[n]} \simeq \pi_* \mathscr{O}_{X'}(n).$$

Finally, for (vi), by Proposition 2.1.4 a morphism $T \to X'$ corresponds to a line bundle $\mathscr{L}$ on T and a graded homomorphism $\varphi\colon f^* F^{[\bullet]} \to \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$ of sheaves on T such that $\varphi_n\colon f^* F^{[n]} \to \mathscr{L}^{\otimes n}$ is surjective for sufficiently divisible n, or equivalently, such that the induced

$(\varphi_n)_{\mathrm{tf}}\colon (f^*F^{[n]})_{\mathrm{tf}} \to \mathscr{L}^{\otimes n}$ is surjective for sufficiently divisible n. Since $F^{[n]}|_U$ is invertible, $f^*F^{[n]}$ is invertible over $f^{-1}(U)$, so $(\varphi_n)_{\mathrm{tf}}|_{f^{-1}(U)}$ is an isomorphism for sufficiently divisible n, and hence an isomorphism for all n. This means the following:

(a) If T is S_2, then by hypothesis, $(\varphi_n)_{\mathrm{tf}}$ is an isomorphism away from codimension ≥ 2 for all n. Since $\mathscr{L}^{\otimes n}$ is invertible, $(\varphi_n)_{\mathrm{tf}}$ is the reflexive hull for all n, and thus so is φ_n.

(b) If T is normal, then by hypothesis, $(\varphi_n)_{\mathrm{tf}}$ is generically an isomorphism for all n. In addition, Serre's condition R_1 implies that $(f^*F^{[n]})_{\mathrm{tf}}$ is invertible in codimension 1, so $(\varphi_n)_{\mathrm{tf}}$ is an isomorphism in codimension 1 for sufficiently divisible n, and hence an isomorphism in codimension 1 for all n. In conclusion, $(\varphi_n)_{\mathrm{tf}}$ is an isomorphism away from codimension ≥ 2, and the same argument as in (a) shows that φ_n is the reflexive hull for all n.

In either case, we conclude that such an $\mathscr{L}$ and φ exist precisely when $(f^*F)^{\vee\vee}$ is invertible and then $\mathscr{L} = (f^*F)^{\vee\vee}$. Finally, note that if $F^{[N]}$ is invertible, then φ_N is an isomorphism and in particular surjective. □

Remark 2.2.17. When $F^{[N]}$ is invertible, the construction $X' \to X$ that makes F invertible is closely related to taking the N^{th} root of the invertible sheaf $F^{[N]}$ (Example 2.2.2). Since $\mathscr{O}_{X'}(1)^{\otimes n} = \mathscr{O}_{X'}(n) = \pi^*F^{[N]}$, there is a canonical map $\varphi\colon X' \to X\big(\sqrt[N]{F^{[N]}}\big)$ over X. This map is representable, hence finite, since φ^* is compatible with tautological line bundles. That is, φ is also induced, via Proposition 2.1.5(vi), by the graded homomorphism

$$\operatorname{Sym}_{\mathscr{O}_X}\big(F^{[N]}(-N)\big) \to \bigoplus_{n\geq 0} F^{[N]}.$$

The finite morphism φ is *not* an isomorphism. In fact, the root stack is a gerbe, whereas $X' \to X$ is generically an isomorphism (a *stacky modification*, i.e. proper and generically an isomorphism). On the root stack, $F^{[N]}$ has an N^{th} root, but it does not coincide with the reflexive hull of the pullback of F because it does not agree over U. This is explained by the presence of torsion in the Picard group in the root stack over U.

Example 2.2.18 ($\mathbf{Q}$-Gorenstein varieties)**.** We now apply Proposition 2.2.16 to the canonical sheaf. Let X be a $\mathbf{Q}$-*Gorenstein variety of index* N, that is, a normal variety of such that the N^{th} pluricanonical divisor NK_X is Cartier. Let $\omega_X^{[n]} = \mathscr{O}_X(nK_X)$ denote the n^{th} pluricanonical sheaf, or equivalently, $(\omega_X^{\otimes n})^{\vee\vee}$. Then $\omega_X^{[n]}$ is a reflexive sheaf of rank 1, which is invertible whenever N divides n. Let $X' = \mathscr{P}\mathrm{roj}_X(\omega_X^{[\bullet]})$. Then $\pi\colon X' \to X$ is an isomorphism over the locus where ω_X is invertible, i.e. where X is quasi-Gorenstein, that is, $\mathbf{Q}$-Gorenstein of index 1. The coarse moduli space of X' is $\mathrm{Proj}_X(\omega_X^{[N\bullet]})$ which equals X, and $\pi_*\mathscr{O}(n) = \omega_X^{[n]}$ for every positive integer n. The morphism $\pi\colon X' \to X$ only adds some stackiness in codimension ≥ 2. Finally, the canonical sheaf $\omega_{X'}$ is $(\pi^*\omega_X)^{\vee\vee}$, and hence is equal to $\mathscr{O}_{X'}(1)$.

Example 2.2.19 (Cartierification)**.** More generally, fix a normal noetherian scheme X, with an effective $\mathbf{Q}$-Cartier divisor $D \subset X$, say ND is Cartier. Let $I_D = \mathscr{O}_X(-D)$ be the ideal of $D \subset X$, which is a reflexive $\mathscr{O}_X$-submodule of $\mathscr{O}_X$ of rank one. Let U denote the largest open subset of X on which $D|_U$ is Cartier (i.e. $I_D|_U = I_{D|_U}$ is an invertible $\mathscr{O}_U$-submodule of $\mathscr{O}_U$), so that $I_D = j_*(I_{D|_U})$. Recall that $U \supset \mathrm{Reg}(X)$ (the latter has complement of codimension ≥ 2 in X), and moreover note that $U \supset Y \smallsetminus D$.

We apply Proposition 2.2.16 with $F = I_D$. Note that $F^{[n]} = I_D^{[n]}$ is precisely the n^{th} *symbolic power* I_{nD} of I_D, since all the associated points of I_D are non-embedded, and hence, are contained in U. The $\mathscr{O}_X$-algebra $F^{[\bullet]} = I_{\bullet D} = \bigoplus_{n\geq 0} I_{nD}$ is called the *symbolic Rees algebra* of D (or I_D). Let $X' = \mathscr{P}\mathrm{roj}_X(I_{\bullet D})$ with structure morphism $\pi\colon X' \to X$. The inverse image of D under π is a Cartier divisor (v), and π satisfies the following universal property (vi): if $f\colon T \to X$ is a morphism from a normal scheme T such that $f^{-1}(D)$ is nowhere dense in T, then f factors, uniquely, through π if and only the *inverse image* f^*D of D under f is an effective Cartier divisor on T. Here, f^*D is the Weil divisor on T whose underlying ideal sheaf is $(f^*I_D)^{\vee\vee}$.

As a final note, we will see later in Example 2.3.38 that $\bigoplus_{n\geq 0} I_{nD}$ is the integral closure of the $\mathscr{O}_X$-subalgebra $\bigoplus_{n\geq 0} I_{ND}^{\lceil n/N\rceil}$ of Example 2.2.13. In other words, there is a canonical finite morphism $X' = \mathscr{P}\mathrm{roj}_X(I_{\bullet D}) \to X\bigl(\sqrt[N]{ND}\bigr)$, which presents X' as the *normalization* of $X\bigl(\sqrt[N]{ND}\bigr)$.

Example 2.2.20 (Stacky modifications given by inverting **Q**-invertible sheaves)**.** Suppose that $\pi\colon X' \to X$ is a proper quasi-finite morphism of noetherian stacks satisfying S_2, that $\mathscr{L} \in \mathrm{Pic}(X')$ is an ample and uniformizing line bundle relative to X, and that π is an isomorphism over an open substack $U \subset X$ and that U and $\pi^{-1}(U)$ have complements of codimension at least 2. Then $X' = \mathscr{P}\mathrm{roj}_X(F^{[\bullet]})$ where $F = \pi_*\mathscr{L}$. Indeed, first note that $X' \to X$ is a relative coarse space since $X'_{\mathrm{cs}/X} \to X$ is a finite morphism between S_2-stacks that is an isomorphism outside codimension 2, hence an isomorphism. It follows that $\pi_*\mathscr{L}^{\otimes N}$ is a line bundle for sufficiently divisible N (2.1.10). Moreover, $X' = \mathscr{P}\mathrm{roj}_X\bigl(\bigoplus_{n\geq 0}\pi_*\mathscr{L}^{\otimes n}\bigr)$ by Proposition 2.1.14(ii) so it suffices to note that $\pi_*\mathscr{L}^{\otimes n} = \pi_* j'_* j'^* \mathscr{L}^{\otimes n} = j_*(\pi|_U)_* j'^*\mathscr{L}^{\otimes n} = j_* j^* F^{\otimes n} = F^{[n]}$.

Example 2.2.21 (Toric varieties and toric stacks)**.** Let Σ be a simplicial fan. Then the associated toric variety X_Σ is normal and the torus-invariant divisors $D_1, D_2, \dots, D_n$ are **Q**-Cartier. The corresponding toric stack $\mathscr{X}_\Sigma$ is smooth with smooth toric divisors. We thus get a map $\mathscr{X}_\Sigma \to X'$ where $X' \to X_\Sigma$ is the iterated stack-theoretic Proj that makes all the torus-invariant divisors Cartier (Example 2.2.19). Since $(\mathscr{O}(D_1), \mathscr{O}(D_2), \dots, \mathscr{O}(D_n))$ is uniformizing on the toric stack $\mathscr{X}_\Sigma$ (by the Cox construction) as well as on X' (see Proposition 2.1.16), it follows that $\mathscr{X}_\Sigma \to X'$ is a representable birational homeomorphism between normal stacks, hence an isomorphism.

The toric stack $\mathscr{X}_\Sigma$ is the *canonical stack* associated to the variety X_Σ with finite quotient singularities [**FMN10**, §4]. The Cartierification thus gives a different description of the canonical stack for a toric variety. If Σ is a *stacky fan*, then the associated toric stack can be described as the Cartierification of the torus-invariant divisors of the associated toric variety followed by taking root stacks of these divisors and then root stacks of line bundles [**FMN10**, Thm. 1].

2.2.D. Stack-theoretic amplification of GIT quotients. Let X be a projective variety with an action of a reductive group G and let $\mathscr{L}$ be an ample line bundle with a G-action. Then we can form the GIT quotient $X /\!\!/ G = \operatorname{Proj}\bigl(\bigoplus_{n\geq 0} \Gamma(X, \mathscr{L}^{\otimes n})^G\bigr)$, where for each $n \geq 0$, $\Gamma(X, \mathscr{L}^{\otimes n})^G$ denotes the G-invariant global sections of $\mathscr{L}^{\otimes n}$. If $X^{\mathrm{ss}} \subset X$ denotes the semi-stable locus of X, then $X^{\mathrm{ss}} \to X /\!\!/ G$ is a good quotient. This can also be phrased as saying that $[X^{\mathrm{ss}} \,/\, G] \to X /\!\!/ G$ is a good moduli space.

It is very natural to also consider the "stack-theoretic GIT quotient":

$$[X /\!\!/ G] := \mathscr{P}\mathrm{roj}\left(\bigoplus_{n\geq 0} \Gamma(X, \mathscr{L}^{\otimes n})^G\right).$$

Whereas $[X^{\mathrm{ss}} \,/\, G]$ is typically an Artin stack with infinite stabilizers, the stack $[X /\!\!/ G]$ is a tame Artin stack with finite cyclic stabilizers. We have

$$[X^{\mathrm{ss}} \,/\, G] \xrightarrow{\text{rel. good mod. space}} [X /\!\!/ G] \xrightarrow{\text{tame coarse space}} X /\!\!/ G.$$

The stack-theoretic GIT quotient $[X /\!\!/ G]$ was studied by Hassett [**Has05**, §3.1] and Gulbrandsen [**Gul11**] when X is the projective space of hypersurfaces in $\mathbf{P}^n$ of degree d and $G = \mathrm{SL}(n+1)$ for small d and n.

2.3. Rees algebras and weighted blow-ups

2.3.A. Rees algebras.

Definition 2.3.1 (Rees algebras)**.** A *Rees algebra* on X is a quasi-coherent, finitely generated, graded $\mathscr{O}_X$-subalgebra $R = \bigoplus_{n\geq 0} I_n \cdot t^n$ of $\mathscr{O}_X[t]$ such that $I_0 = \mathscr{O}_X$ and $I_n \supset I_{n+1}$ for every $n \in \mathbf{N}$. Equivalently, a Rees algebra is a descending filtration

$$I_\bullet = (I_0 \supset I_1 \supset I_2 \supset \dots)$$

of ideals of $\mathscr{O}_X$, satisfying the following conditions:

(i) $I_0 = \mathscr{O}_X$;

(ii) $I_n I_m \subset I_{n+m}$ for every n, m;

(iii) locally on X, there exists a sufficiently large positive integer d such that for all integers $n \geq 1$,

$$I_n = \left(I_1^{\ell_1} I_2^{\ell_2} \cdots I_d^{\ell_d} : \ell_i \in \mathbf{N}, \; \sum_{i=1}^{d} i\ell_i = n \right),$$

in which case, we say that $I_\bullet$ is *generated in degrees* $\leq d$. Equivalently, the graded module R_+/R_+^2 is concentrated in degrees $\leq d$.

Rees algebras are partially ordered by inclusion. The initial object is the *zero Rees algebra* which is $\mathscr{O}_X$ in degree 0 and zero in positive degrees. For any positive integer d, we write $I_{d\bullet}$ for the d^{th} Veronese subalgebra of $I_\bullet$.

Remark 2.3.2.

(i) That $I_\bullet$ is generated in degrees $\leq d$ does not imply that the Veronese subalgebra $I_{\ell\bullet}$ is generated in degree 1 for ℓ the least common multiple of $1, 2, \dots, d$, cf. Remark 2.1.8. But it does imply that the Veronese subalgebra $I_{d\bullet}$ is generated in degree 1 for sufficiently divisible d, see Proposition 2.3.7 below.

(ii) Rees algebras are called *idealistic filtrations* by Kawanoue [**Kaw07**]. Moreover, the element $gt^n \in I_n t^n$ is also written there as (g, n); however, we shall reserve that notation for the smallest Rees algebra containing gt^n.

(iii) Encinas–Villamayor [**EV07a**] do not require their Rees algebras to satisfy $I_i \supset I_{i+1}$. This condition is, however, essential for the purpose of weighted blow-ups (Definition 2.3.12): without this condition, the exceptional divisor (Definition 2.3.13) of a weighted blow-up would not make sense.

It is also convenient to account for the condition that $I_\bullet$ is a descending filtration by extending Rees algebras trivially in negative degrees:

Definition 2.3.3 (Extended Rees algebras)**.** An *extended Rees algebra* on X is a quasi-coherent, finitely generated $\mathbf{Z}$-graded $\mathscr{O}_X[t^{-1}]$-subalgebra $I_\bullet^{\mathrm{ext}} = \bigoplus_{n \in \mathbf{Z}} I_n^{\mathrm{ext}} \cdot t^n$ of $\mathscr{O}_X[t^{\pm 1}]$ such that $I_0^{\mathrm{ext}} = \mathscr{O}_X$.

2.3.4. For an extended Rees algebra $I_\bullet^{\mathrm{ext}}$ on X, $I_\bullet := \bigoplus_{n \geq 0} I_n^{\mathrm{ext}} \cdot t^n$ is a Rees algebra on X in the sense of Definition 2.3.1. Conversely, every Rees algebra $I_\bullet$ on X can uniquely be extended to an extended Rees algebra $I_\bullet^{\mathrm{ext}}$ on X by setting

$$I_n^{\mathrm{ext}} := \begin{cases} \mathscr{O}_X, & \text{if } n < 0; \\ I_n, & \text{if } n \geq 0. \end{cases}$$

Definition 2.3.5. Given an ideal $J \subset \mathscr{O}_X$ and $d \geq 1$, we let (J, d) denote the smallest Rees algebra containing Jt^d. Given a finite collection of Rees algebras $I_{k,\bullet}$ we let $\sum_k I_{k,\bullet}$ denote the smallest Rees algebra containing all the $I_{k,\bullet}$.

The *marked ideal* (J, d), used in resolution of singularities, can be identified with the Rees algebra (J, d). Explicitly, we have that $(J, d)_n = J^{\lceil n/d \rceil}$, that is:

$$(J, d) = \mathscr{O}_X \oplus Jt \oplus Jt^2 \oplus \cdots \oplus Jt^d \oplus J^2 t^{d+1} \oplus J^2 t^{d+2} \oplus \cdots$$

and that

$$(I_{1,\bullet} + \cdots + I_{r,\bullet})_n = \sum_{n=n_1+\cdots+n_r} I_{1,n_1} I_{2,n_2} \cdots I_{r,n_r}.$$

In particular, $I_\bullet$ is generated in degree $\leq d$ if and only if $I_\bullet = (I_1, 1) + (I_2, 2) + \cdots + (I_d, d)$.

Example 2.3.6 (Ordinary Rees algebras)**.** If a Rees algebra $I_\bullet$ is *generated in degree* 1, i.e. $I_n = I_1^n$ for all $n \geq 1$, then we say that $I_\bullet = I_1^\bullet = (I_1^n) = (I_1, 1)$ is the *Rees algebra of the ideal* $I_1 \subset \mathscr{O}_X$.

The next proposition is a direct translation of Proposition 2.1.7:

Proposition 2.3.7. *Let $I_\bullet$ be a Rees algebra on X and suppose that X is quasi-compact. Then for all sufficiently divisible d, we have $I_{rd} = (I_d)^r$ for all integers $r \geq 1$, i.e. $I_{d\bullet}$ is the Rees algebra of the ideal I_d.* □

Example 2.3.8. The graded $\mathscr{O}_X$-algebra of the d^{th} root stack of the divisor D (Example 2.2.13) is the Rees algebra (I_D, d). The symbolic Rees algebra $I_{\bullet D}$ of the Cartierification of D (Example 2.2.19) is also a Rees algebra.

We can also see (I_D, d) as a *dilation* of the ordinary Rees algebra $(I_D, 1)$:

Example 2.3.9 (Dilation)**.** Given a Rees algebra $I_\bullet$ on X, and a positive integer d, the d^{th} *dilation* of $I_\bullet$ is the Rees algebra $D_\bullet := I_{\lceil \bullet/d \rceil}$, i.e. $D_n := I_{\lceil n/d \rceil}$ for every integer $n \geq 0$. Note that the d^{th} Veronese subalgebra of $D_\bullet$ is $I_\bullet$, and $I_\bullet \subset D_\bullet$.

Taking the d^{th} dilation of the d^{th} Veronese subalgebra of a Rees algebra $I_\bullet$ on X gives:

Example 2.3.10 (Truncation)**.** Given a Rees algebra $I_\bullet$ on X, and a positive integer d, the d^{th} *truncation* of $I_\bullet$ is the Rees algebra $T_\bullet := I_{d\lceil \bullet/d \rceil} = \sum_{d|n} (I_n, n)$, i.e. $T_n = I_{d\lceil n/d \rceil}$ for every integer $n \geq 0$. Note that $T_{d\bullet} = I_{d\bullet}$, and $T_\bullet \subset I_\bullet$.

Remark 2.3.11. The Rees algebra $\bigoplus_{n\geq 0} I_{ND}^{\lceil n/N\rceil}$ mentioned at the end of Example 2.2.19 is precisely (I_{ND}, N), i.e. the N^{th} truncation of $I_{\bullet D}$, since ND is a Cartier divisor.

2.3.B. Weighted blow-ups.

Definition 2.3.12 (Weighted blow-ups)**.** If $I_\bullet$ is a Rees algebra on X, the *(stack-theoretic) weighted blow-up* of X along $I_\bullet$ is defined as the stack-theoretic Proj

$$\operatorname{Bl}_{I_\bullet} X := \mathscr{P}\mathrm{roj}_X(I_\bullet) \longrightarrow X.$$

This morphism is proper (Proposition 2.1.5(ii)). Note that $\sqrt{I_n} = \sqrt{I_1}$ for any positive integer n. We call $V(I_1)$ the *co-support* of the weighted blow-up (or the Rees algebra).

Definition 2.3.13 (Exceptional divisor)**.** Let $X' := \operatorname{Bl}_{I_\bullet} X \xrightarrow{\pi} X$. The natural inclusion $I_{\bullet+1} \hookrightarrow I_\bullet$ corresponds to the inclusion $\mathscr{O}_{X'}(1) \hookrightarrow \mathscr{O}_{X'}(0) = \mathscr{O}_{X'}$ of invertible sheaves, and defines an effective Cartier divisor E on X' such that $\mathscr{O}_{X'}(1) = \mathscr{O}_{X'}(-E)$. We call E the *exceptional divisor* of $\operatorname{Bl}_{I_\bullet} X$.

Remark 2.3.14. Explicitly, if we write $I_\bullet$ locally as $(f_1, d_1) + \cdots + (f_m, d_m)$, then the ideal sheaf I_E of E can be described locally on $\operatorname{Bl}_{I_\bullet} X$ as follows. On the chart

$$D_+(f_i \cdot t^{d_i}) = \left[\operatorname{Spec}_X\big(I_\bullet[(f_i \cdot t^{d_i})^{-1}]\big) \;/\; \mathbb{G}_m\right]$$

of $\operatorname{Bl}_{I_\bullet} X$, the ideal sheaf I_E is generated by $t^{-1} = \frac{f_i \cdot t^{d_i - 1}}{f_i \cdot t^{d_i}} \in I_\bullet[(f_i \cdot t^{d_i})^{-1}]$. In particular, the Cartier divisor $d_i E$ is principal and given by the vanishing of $\pi^{-1}(f_i)$ on this chart. Thus, for all N divisible by $d_1, d_2, \ldots, d_m$, the Cartier divisor NE has ideal sheaf

$$I_{NE} = \big(\pi^{-1}(f_i)^{N/d_i} \;:\; i = 1, 2, \ldots, m\big).$$

Remark 2.3.15 (Weighted blow-ups in terms of extended Rees algebras)**.** Note that if $I_\bullet^{\mathrm{ext}}$ denotes the extended Rees algebra of $I_\bullet$ (2.3.4), then:

$$\mathrm{Bl}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}}) := \left[\mathrm{Spec}_X(I_\bullet^{\mathrm{ext}}) \smallsetminus V(I_+^{\mathrm{ext}}) \,/\, \mathbb{G}_m\right].$$

Indeed, if we write $I_\bullet$ locally as $(f_1, d_1) + \cdots + (f_m, d_m)$, then one has, for each $1 \leq i \leq m$, that $I_\bullet^{\mathrm{ext}}[(f_i \cdot t^{d_i})^{-1}] = I_\bullet[(f_i \cdot t^{d_i})^{-1}]$, and thus

$$D_+(f_i \cdot t^{d_i}) = \left[\mathrm{Spec}_X\left(I_\bullet^{\mathrm{ext}}[(f_i \cdot t^{d_i})^{-1}]\right) \,/\, \mathbb{G}_m\right].$$

Evidently these identifications are compatible with each other.

Note too that the exceptional divisor E of $\mathrm{Bl}_{I_\bullet} X$ is induced by the principal divisor given by $t^{-1} = 0$ on $\mathrm{Spec}_X(I_\bullet^{\mathrm{ext}})$ whereas the ideal sheaf $I_{\bullet+1}$ on $\mathrm{Spec}_X(I_\bullet)$, which is not even invertible, only becomes principal over the localizations $I_\bullet[(f_i \cdot t^{d_i})^{-1}]$ (Remark 2.3.14).

The next proposition, like Proposition 2.3.7, is a direct translation of Proposition 2.1.7:

Proposition 2.3.16. *Let $I_\bullet$ be a Rees algebra on X. The coarse space of $\mathrm{Bl}_{I_\bullet} X$, relative to X, is the ordinary blow-up $\mathrm{Bl}_{I_d} X$ for any positive integer d such that $I_{d\bullet}$ is generated in degree 1. Such a d always exists if X is quasi-compact (Proposition 2.3.7).* ☐

Example 2.3.17 (Ordinary blow-ups)**.** Let $I \subset \mathscr{O}_X$ be an ideal. The weighted blow-up $\mathrm{Bl}_{I^\bullet} X$ of X along the Rees algebra $I^\bullet = (I, 1)$ of I is the usual blow-up $\mathrm{Bl}_I X$ of X along I.

Example 2.3.18 (Root stack of a divisor)**.** Given an effective Cartier divisor D on X, and a positive integer d, the root stack $X\left(\sqrt[d]{D}\right)$ is the weighted blow-up $\mathrm{Bl}_{(I_D,d)} X$ (see Examples 2.2.13 and 2.3.8).

Example 2.3.19 (Cartierification)**.** If X is normal and noetherian and D is an effective $\mathbf{Q}$-Cartier divisor, then the *Cartierification* of D in X is $\mathrm{Bl}_{I_{\bullet D}} X$ (see Examples 2.2.19 and 2.3.8).

Theorem 2.3.20 (Universal property of weighted blow-ups)**.** *Let $I_\bullet$ be a Rees algebra, and let $\pi\colon X' = \mathrm{Bl}_{I_\bullet} X \to X$ be the corresponding weighted blow-up.*

(i) *For every $n \in \mathbf{N}$ we have an inclusion of ideals $\pi^{-1}(I_n)\mathscr{O}_{X'} \subset I_E^n$, which is an equality for all sufficiently divisible n (locally on X).*

(ii) *Let $f\colon T \to X$ be a morphism such that $U := T \setminus f^{-1}\big(V(I_1)\big)$ is scheme-theoretically dense. The groupoid of factorizations through π is equivalent to the set of effective Cartier divisors D on T such that $f^{-1}(I_n)\mathscr{O}_T \subset I_D^n$ for all n with equality for all sufficiently divisible n (locally on T). If $f = \pi \circ g$, then $D = g^{-1}(E)$.*

Proof. By Proposition 2.1.4, a factorization of f through π corresponds to a line bundle $\mathscr{L}$ on T together with a graded algebra homomorphism $\varphi\colon \bigoplus_{n\geq 0} f^*I_n \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$ which is surjective for all sufficiently divisible n. The case $T = \mathrm{Bl}_{I_\bullet} X$ corresponds to $\mathscr{L} = \mathscr{O}_{X'}(1) = \mathscr{O}_{X'}(-E)$ with the canonical map φ.

Let N be a sufficiently divisible integer. We begin by noting that $\varphi_N|_U$ is an isomorphism and hence that $\varphi|_U$ is an isomorphism. Since $j\colon U \to T$ is scheme-theoretically dominant, we have that $\mathscr{L}^n \to j_*j^*\mathscr{L}^n = j_*j^*\mathscr{O}_T$ is injective whereas the image of $f^*I_n \to j_*j^*F^*I_n = j_*j^*\mathscr{O}_T$ is $f^{-1}(I_n)\mathscr{O}_T$. It follows that φ factors through an injective graded homomorphism

$$\psi\colon \bigoplus_{n\geq 0} f^{-1}(I_n)\mathscr{O}_T \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}.$$

In particular, ψ_N is an isomorphism. The composition of ψ_N^{-1}, the inclusion $f^{-1}(I_N)\mathscr{O}_T \subset f^{-1}(I_{N-1})\mathscr{O}_T$ and ψ_{N-1} gives an injective homomorphism $\mathscr{L}^N \hookrightarrow \mathscr{L}^{N-1}$, or equivalently, an injective homomorphism $s\colon \mathscr{L} \hookrightarrow \mathscr{O}_T$. This defines the Cartier divisor D. Note that $s|_U =$

$(\psi_1|_U)^{-1}$ so ψ_n together with s^n gives the inclusion of ideals $f^{-1}(I_n)\mathscr{O}_T \hookrightarrow \mathscr{L}^n = \mathscr{O}_T(-nD) \hookrightarrow \mathscr{O}_T$. $\square$

Remark 2.3.21. For all n, we have a commutative diagram

$$\begin{tikzcd} nE \arrow[r] \arrow[d, hook] & V(I_n) \arrow[d, hook] \\ \mathrm{Bl}_{I_\bullet} X \arrow[r, "\pi"] & X \end{tikzcd}$$

which is cartesian for sufficiently divisible n. Unlike the usual blow-up, the diagram is *not always cartesian* for $n = 1$. Nevertheless:

(i) π is an isomorphism away from $V(I_1)$.

(ii) $E_{\mathrm{red}} = \pi^{-1}(V(I_1))_{\mathrm{red}}$.

(iii) $\pi^{-1}(V(I_1))$ is of codimension 1 in $\mathrm{Bl}_{I_\bullet} X$ and its complement is scheme-theoretically dense in $\mathrm{Bl}_{I_\bullet} X$.

Remark 2.3.22. If one locally writes $I_\bullet$ as $(f_1, d_1) + \cdots + (f_m, d_m)$, then the condition in Theorem 2.3.20(ii) that $f^{-1}(I_n) \subset I_D^n$ for all n with equality for sufficiently divisible n (locally on T) can be explicated as the following equivalent condition: $f^{-1}(f_i) \in I_D^{d_i}$ for all $1 \leq i \leq m$ and locally on T there exists an i such that $I_D^{d_i} = \big(f^{-1}(f_i)\big)$. The latter occurs on the preimage of the chart $D_+(f_i \cdot t^{d_i})$ of $\mathrm{Bl}_{I_\bullet} X$ (Remark 2.3.14). Thus, $f^{-1}(I_n)\mathscr{O}_T = I_D^n$ for every n divisible by the $d_1, d_2, \ldots, d_m$.

The next corollary generalizes Example 2.3.18.

Corollary 2.3.23. *Let $I \subset \mathscr{O}_X$ be an ideal, and fix a positive integer d. Then $\mathrm{Bl}_{(I,d)} X$ coincides with the d^{th} root stack (Example 2.2.13) of the exceptional divisor of the usual blow-up $\mathrm{Bl}_I X$ of X along I.*

Proof. Let X' denote the d^{th} root stack of the exceptional divisor of $\mathrm{Bl}_I X$. For a morphism $f\colon T \to X$:

(a) The groupoid of factorizations through $X' \to X$ is equivalent to the set of effective Cartier divisors D on T such that $f^{-1}(I)\mathscr{O}_T = I_D^d$ (Example 2.2.13).

(b) The groupoid of factorizations through $\mathrm{Bl}_{(I,d)} X \to X$ is equivalent to the set of effective Cartier divisors D on T such that $f^{-1}(I^{\lceil n/d \rceil})\mathscr{O}_T \subset I_D^n$ for every n with equality whenever $d \mid n$ (Remark 2.3.22).

The groupoids in (a) and (b) are equivalent, and the corollary follows. $\square$

Remark 2.3.24. The corollary shows that our definition of $\mathrm{Bl}_{(I,d)} X$ agrees with the definition of $\mathrm{Bl}_{(I,d)} X$ as the d^{th} root stack of the usual blow-up in [**Ryd09**]. The weighted blow-up $\mathrm{Bl}_{(I,d)} X$ is called the *d^{th} Kummer blow-up of X along I* in [**Ryd09**].

Corollary 2.3.25 (Functoriality for weighted blow-ups)**.** *Let $f\colon Y \to X$ be a morphism of schemes, and let $I_\bullet$ be a Rees algebra on X. There is a unique morphism $g\colon \mathrm{Bl}_{f^{-1}(I_\bullet)\mathscr{O}_Y} Y \to \mathrm{Bl}_{I_\bullet} X$ making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Bl}_{f^{-1}(I_\bullet)\mathscr{O}_Y} Y \arrow[r, hook, dotted, "\iota"] \arrow[dr, "\pi_Y"'] \arrow[rr, bend left, dotted, "g"] & (\mathrm{Bl}_{I_\bullet} X) \times_X Y \arrow[r] \arrow[d] & \mathrm{Bl}_{I_\bullet} X \arrow[d, "\pi"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd} \tag{2.7}$$

and the induced morphism ι is a closed immersion. Hence:

(i) *$\mathrm{Bl}_{f^{-1}(I_\bullet)\mathscr{O}_Y} Y$ is the scheme-theoretic closure of the complement of $E \times_X Y$ in $(\mathrm{Bl}_{I_\bullet} X) \times_X Y$.*

(ii) *If f is a closed immersion, then so is $g\colon \mathrm{Bl}_{f^{-1}(I_\bullet)\mathscr{O}_Y} Y \to \mathrm{Bl}_{I_\bullet} X$.*

(iii) *If f is flat (e.g. f is an open immersion), then ι is an isomorphism.*

Proof. The existence and uniqueness of g follow from Theorem 2.3.20. To see that ι is a closed immersion, note that ι is induced by the natural surjective morphism $f^* I_\bullet \to f^{-1}(I_\bullet)\mathscr{O}_Y$

of graded $\mathscr{O}_Y$-algebras, i.e.

$$\iota\colon \operatorname{Bl}_{f^{-1}(I_\bullet)\mathscr{O}_Y} Y = \mathscr{P}\mathrm{roj}_Y(f^{-1}(I_\bullet)\mathscr{O}_Y) \hookrightarrow \mathscr{P}\mathrm{roj}_Y(f^* I_\bullet) = (\operatorname{Bl}_{I_\bullet} X) \times_X Y.$$

Then parts (i) and (ii) are immediate. For part (iii), note that if f is flat, then $f^* I_\bullet \to f^{-1}(I_\bullet)\mathscr{O}_Y$ is an isomorphism. $\square$

Definition 2.3.26. In the above corollary, $\operatorname{Bl}_{f^{-1}(I_\bullet)\mathscr{O}_Y} Y$ is known as the *proper* (or *strict*) *transform* of $Y \to X$ under the weighted blow-up $\operatorname{Bl}_{I_\bullet} X \xrightarrow{\pi} X$, while $\pi^{-1}(Y) = (\operatorname{Bl}_{I_\bullet} X) \times_X Y$ is the *total transform* of $Y \to X$ under the weighted blow-up $\operatorname{Bl}_{I_\bullet} X \xrightarrow{\pi} X$.

For the remainder of this section, we specialize to the case where $Y = V(J) \hookrightarrow X$ is a closed subscheme, where $J \subset \mathscr{O}_Y$ is an ideal.

Definition 2.3.27 (Admissibility)**.** For $x \in X$, we say that $I_\bullet$ is *J-admissible at x* if $(I_\bullet)_x$ contains the Rees algebra $J_x^\bullet$ of J_x. Equivalently, $(I_1)_x \supset J_x$.

We also say $I_\bullet$ is *J-admissible* if $I_\bullet$ is J-admissible at every $x \in X$. That is, $I_\bullet \supset J^\bullet$. Equivalently, $I_1 \supset J$, i.e. the co-support (Definition 2.3.12) of $I_\bullet$ is contained in $V(J) \subset X$.

We collate some straightforward properties:

Lemma 2.3.28 (Properties)**.** *Let $d_1, \dots, d_m \in \mathbf{N}_{>0}$, and $J_1, \dots, J_m$ be ideals on X. For $x \in X$:*

(i) *$I_\bullet$ is $\sum_i J_i$-admissible at x if and only if $I_\bullet$ is J_i-admissible at x for every $1 \le i \le m$.*

(ii) *If $I_{d_i \bullet}$ is J_i-admissible at x for every $1 \le i \le m$, then $I_{d\bullet}$ is $\prod_i J_i$-admissible at x, where $d := \sum_i d_i$.*

(iii) *For any $d \in \mathbf{N}_{>0}$, $I_\bullet$ is J-admissible at x if and only if $I_{d\bullet}$ is J^d-admissible at x.*

The following lemma says that we can check admissibility by passing to completions, and is a consequence of faithful flatness of $\mathscr{O}_{X,x} \to \widehat{\mathscr{O}}_{X,x}$, cf. [**Mat89**, Theorem 7.5].

Lemma 2.3.29. *For $x \in X$, $I_\bullet$ is J-admissible at x if and only if $I_1\widehat{\mathscr{O}}_{X,x} \supset J\widehat{\mathscr{O}}_{X,x}$.*

The notion of admissibility is related to weighted blow-ups as follows:

Lemma 2.3.30 (Relationship with weighted blow-ups)**.** *Let $\pi\colon X' := \mathrm{Bl}_{I_\bullet} X \to X$, and E be the exceptional divisor on X' with ideal $I_E \subset \mathscr{O}_{X'}$. For $m \in \mathbf{N}_{>0}$, the following statements are equivalent:*

(i) *$I_{m\bullet}$ is J-admissible.*

(ii) *$\pi^{-1}(J)\mathscr{O}_{X'} \subset I_E^m$, i.e. $\big(\pi^{-1}(J)\mathscr{O}_{X'}\big) = I_E^m \cdot J'$ for an ideal $J' \subset \mathscr{O}_{X'}$.*

Proof. For (i) $\Longrightarrow$ (ii), we use Theorem 2.3.20(i) to deduce from $J \subset I_m$ that $\pi^{-1}(J)\mathscr{O}_{X'} \subset \pi^{-1}(I_n)\mathscr{O}_{X'} \subset I_E^m$, as desired. For (ii) $\Longrightarrow$ (i), recall that $X' = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}})$, with $I_E = (t^{-1}) \subset I_\bullet^{\mathrm{ext}}$. Then (ii) implies $J \cdot t^m \subset I_E^{-m} \cdot \big(\pi^{-1}(J)\mathscr{O}_{X'}\big) \subset I_\bullet$, so that $J \subset I_m$. $\square$

We conclude this section by defining one additional transform:

Definition 2.3.31 (Weak transform)**.** Assume that X is noetherian. Let $\pi\colon X' := \mathrm{Bl}_{I_\bullet} X \to X$, and E be the exceptional divisor on X' with ideal $I_E \subset \mathscr{O}_{X'}$. Let

$$\ell := \max\{m \in \mathbf{N}_{>0} \colon I_{m\bullet} \text{ is } J\text{-admissible}\} = \max\{m \in \mathbf{N}_{>0} \colon \pi^{-1}(J)\mathscr{O}_{X'} \subset I_E^m\}$$

which exists by Krull's intersection theorem. Then we call the ideal $\pi_*^{-1}(J) := \pi^{-1}(J)\mathscr{O}_{X'} \cdot I_E^{-\ell}$ the *weak* transform (also known as the *birational* or *controlled* transform) of J under π, cf. [**Kol07**, Definition 3.60].

Note that $\pi_*^{-1}(J)$ is always contained in the proper transform $\widetilde{J}$ of J under π (Definition 2.3.26), with equality if J is locally principal. This follows from the fact that $\widetilde{J} = (\pi^{-1}(J)\mathscr{O}_{X'} \colon I_E^\infty) := \bigcup_{m\in\mathbf{N}} (\pi^{-1}(J)\mathscr{O}_{X'} \colon I_E^m)$, cf. [**Lee20**, Proposition A.2.2].

2.3.C. Integral closure of Rees algebras.

Definition 2.3.32 (Integral closure)**.** For a Rees algebra $I_\bullet$ (or more generally, a quasi-coherent $\mathscr{O}_X$-subalgebra of $\mathscr{O}_X[t]$) on X, we denote by $\mathrm{IC}(I_\bullet)$ the *integral closure* of $I_\bullet$ in $\mathscr{O}_X[t]$. We say $I_\bullet$ is *integrally closed* if $\mathrm{IC}(I_\bullet) = I_\bullet$.

Note that if X is normal, then $I_\bullet$ is integrally closed if and only if $\mathrm{Spec}_X(I_\bullet)$ is normal.

Remark 2.3.33. For a Rees algebra $I_\bullet$ on X, note that (by definition) the integral closure of $I_{d\bullet}$ is the d^{th} Veronese subalgebra of $\mathrm{IC}(I_\bullet)$.

Remark 2.3.34. Given a Rees algebra $I_\bullet$, the integral closure $\mathrm{IC}(I_\bullet)$ is always a quasi-coherent graded $\mathscr{O}_X$-subalgebra of $\mathscr{O}_X[t]$ but not necessarily of finite type. However, if X is *integral and Nagata*, then we claim that $\mathrm{IC}(I_\bullet)$ is of finite type over $\mathscr{O}_X$, and hence a Rees algebra on X.

Indeed, the question is local, so we may assume that $X = \mathrm{Spec}(A)$ is affine. Let K be the fraction field of A. Let $\overline{\mathrm{IC}}(I_\bullet)$ be the integral closure of $I_\bullet$ in $K(t)$. Since A is Nagata, so is $I_\bullet$ (being a finitely generated A-subalgebra of $A[t]$). Since $K(t)$ is also the field of fractions of $I_\bullet$[1], we conclude that $\overline{\mathrm{IC}}(I_\bullet)$ is finite over $I_\bullet$. In particular, it is a noetherian $I_\bullet$-module, so its $I_\bullet$-submodules (e.g. $\mathrm{IC}(I_\bullet)$) are finitely generated $I_\bullet$-modules, and hence, finitely generated A-algebras.

2.3.35 (Integral closure of ideals)**.** For an ideal I on X, the t^1-graded piece of $\mathrm{IC}(I^\bullet)$ is known in the literature (e.g. [**Laz04**, 9.6.A]) as the *integral closure* $\mathrm{IC}(I)$ of the ideal $I \subset \mathscr{O}_X$. Note that $I \subset \mathrm{IC}(I) \subset \sqrt{I}$.

Example 2.3.36. Let X be a normal scheme. If E is an effective Cartier divisor on X, then the ordinary Rees algebra $(I_E, 1)$ of $I_E = \mathscr{O}_X(-E)$ on X is integrally closed. Indeed, locally on X, we have that $I_E^\bullet \simeq \mathscr{O}_X[t]$ which is integrally closed.

[1] If $I_n = 0$ for $n > 0$, then $\mathrm{IC}(I_\bullet) = I_\bullet$ and there is nothing to prove.

Example 2.3.37 (Integral closure of truncations)**.** Let $I_\bullet$ be a Rees algebra on X, and for any positive integer d, let $T_\bullet$ be the d^{th} truncation of $I_\bullet$ (Example 2.3.10). Then $\mathrm{IC}(I_\bullet) = \mathrm{IC}(T_\bullet)$. For this, it suffices to observe that the d^{th} Veronese subalgebra of $\mathrm{IC}(I_\bullet)$ coincides with that of $\mathrm{IC}(T_\bullet)$.

Example 2.3.38 (Cartierification, II)**.** Adopt the setup in Example 2.2.19. We shall now show that $I_{\bullet D} = \bigoplus_{n\geq 0} I_{nD}$ is the integral closure in $\mathscr{O}_X[t]$ of $(I_{ND}, N) = \bigoplus_{n\geq 0} I_{ND}^{\lceil n/N\rceil}$. Since (I_{ND}, N) is the N^{th} truncation of $I_{\bullet D}$ (Remark 2.3.11(i)), it remains to show (because of Example 2.3.37) that $I_{\bullet D}$ is integrally closed.

For this, let U be as in Example 2.2.19. On U, we have $I_{\bullet D}|_U = (I_D, 1)|_U$, and hence, by Example 2.3.36, $I_{\bullet D}|_U$ is integrally closed. Thus, so is $I_{\bullet D} = j_*(I_{\bullet D}|_U)$.

For integrally closed Rees algebras, we can generalize the operation of taking Veronese subalgebra as follows:

Definition 2.3.39 (Veronese translate)**.** For an integrally closed Rees algebra $I_\bullet$ on X and $q = \frac{a}{b} \in \mathbf{Q}_{>0}$, the *Veronese q-translate* of $I_\bullet$ is defined as

$$I_{q\bullet} := \mathrm{IC}\left(\bigoplus_{d\in\mathbf{N}} I_{ad}\cdot t^{bd}\right) \subset \mathscr{O}_X[t].$$

Since we are passing to integral closures, $I_{q\bullet}$ is well-defined, independent of the presentation of q as a quotient of two positive integers $\frac{a}{b}$. For $d \in \mathbf{N}_{>0}$, note that the Veronese d-translate of $I_\bullet$ is the same as the d^{th} Veronese subalgebra of $I_\bullet$. By default, we shall also set $I_\bullet^0 := \mathscr{O}_X[t]$. Finally, by a *Veronese translate* of $I_\bullet$ we always mean $I_{q\bullet}$ for some $q \in \mathbf{Q}_{>0}$.

With respect to the above definition, we can extend Lemma 2.3.28(ii) as follows:

Corollary 2.3.40. *Let $x \in X$, and let $I_\bullet$ be an integrally closed Rees algebra on X. Let $q_1, \ldots, q_m \in \mathbf{Q}_{>0}$, and $J_1, \ldots, J_m$ be ideals on X. If $I_{q_i\bullet}$ is J_i-admissible at x for every $1 \leq i \leq m$, then $I_{q\bullet}$ is $\prod_i J_i$-admissible at x, where $q := \sum_i q_i$.*

Proof. Choose a common $b \in \mathbf{N}_{>0}$ so that for every $1 \leq i \leq m$, $q_i = \frac{a_i}{b}$ for some $a_i \in \mathbf{N}_{>0}$. By Lemma 2.3.28(iii), the hypothesis implies that $I_\bullet^{a_i} = I_{a_i\bullet}$ is J_i^b-admissible at x for every $1 \leq i \leq m$. Let $a := \sum_i a_i$. By Lemma 2.3.28(ii), $I_\bullet^a = I_{a\bullet}$ is $(\prod_i J_i)^b$-admissible at x. By Lemma 2.3.28(iii) again, $I_\bullet^q$ is $\prod_i J_i$-admissible at x. $\square$

2.3.D. Normalized weighted blow-ups. In this subsection we assume that X is normal.

Definition 2.3.41 (Normalized weighted blow-ups)**.** The *normalized weighted blow-up* of X along a Rees algebra $I_\bullet$ on X is the normalization of $\mathrm{Bl}_{I_\bullet} X$ (denoted $\mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X$), or equivalently, $\mathscr{P}\mathrm{roj}_X(\mathrm{IC}(I_\bullet))$.

Note that $\mathrm{IC}(I_\bullet)$ is not always a Rees algebra (Remark 2.3.34), and thus the normalized weighted blow-up $\mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X$ of X is not always proper (not always of finite type but always separated, quasi-compact and universally closed) over X.

Example 2.3.42. Adopt the setup in Example 2.2.19. Then the Cartierification (Example 2.3.19) of D in X is the normalized weighted blow-up of X along (I_{ND}, N) by Example 2.3.38. In other words,

$$\mathrm{Bl}_{I_{\bullet D}} X = \mathrm{Bl}^{\mathrm{norm}}_{(I_{ND},N)} X = \Big(X\big(\sqrt[N]{ND}\big)\Big)^{\mathrm{norm}}.$$

The integral closure $I_{\bullet D} = \mathrm{IC}\big((I_{ND}, N)\big)$ is always finitely generated and hence a Rees algebra by Proposition 2.2.16(i).

The universal property of weighted blow-ups in Theorem 2.3.20 has a neater re-interpretation after passage to normalizations:

Theorem 2.3.43 (Universal property of normalized weighted blow-ups)**.** *For a Rees algebra $I_\bullet$ on X, the normalized weighted blow-up $\pi\colon \mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X \to X$ satisfies the following universal* *property. Let $f\colon T \to X$ be a morphism, where T is normal and such that $f^{-1}(V(I_1))$ is nowhere dense. Then there exists at most one lift $g\colon T \to \operatorname{Bl}^{\mathrm{norm}}_{I_\bullet} X$ of f, and such a lift exists if and only if $\operatorname{IC}(f^{-1}(I_\bullet)\mathscr{O}_T) = (I_D, 1)$ for some effective Cartier divisor D on T. If this is the case, then $D = g^{-1}(E)$.*

Proof. By Theorem 2.3.20, the lifts $T \to \operatorname{Bl}^{\mathrm{norm}}_{I_\bullet} X$ are equivalent to the set of Cartier divisors D such that $f^{-1}(\operatorname{IC}(I_n))\mathscr{O}_T \subset I_D^n$ for every $n \geq 1$ with equality for sufficiently divisible n (locally on T). Since T is normal, the Rees algebra $(I_D, 1)$ is integrally closed and the condition is equivalent to $\operatorname{IC}\big(f^{-1}(I_\bullet)\mathscr{O}_T\big) \subset (I_D, 1)$ with equality for sufficiently divisible n. This means that we have an equality of Rees algebras (Examples 2.3.36 and 2.3.37). In particular, D is unique. $\square$

A partial converse to Corollary 2.3.23 is:

Proposition 2.3.44. *Let X be a normal, quasi-compact scheme. Every normalized weighted blow-up of X is a normalized Kummer blow-up $\operatorname{Bl}^{\mathrm{norm}}_{(I,d)} X$ of X.*

Proof. Let $I_\bullet$ be a Rees algebra on X, and let $X' = \operatorname{Bl}_{\operatorname{IC}(I_\bullet)} X$. By Proposition 2.3.7, there exists a positive integer d such that $I_{d\bullet}$ is generated in degree 1. Let $T_\bullet$ be the d^{th} truncation of $I_\bullet$. By Example 2.3.37, $\operatorname{IC}(I_\bullet) = \operatorname{IC}(T_\bullet)$, so $X' = \operatorname{Bl}_{\operatorname{IC}(T_\bullet)} X$. By definition, $T_\bullet$ is the d^{th} dilation of $I_{d\bullet}$. Thus, $X' = \operatorname{Bl}_{\operatorname{IC}(T_\bullet)} X = \operatorname{Bl}_{\operatorname{IC}(I_d,d)} X = \operatorname{Bl}^{\mathrm{norm}}_{(I_d,d)} X$. $\square$

Under an additional hypothesis on stackiness, we can also describe a Kummer blow-up as an ordinary blow-up followed by a Cartierification.

Proposition 2.3.45. *Let X be a normal scheme, $I \subset \mathscr{O}_X$ be an ideal, and d be a positive integer. If the normalized Kummer blow-up $\operatorname{Bl}^{\mathrm{norm}}_{(I,d)} X \to X$ is representable over an open subset $U \subset \operatorname{Bl}^{\mathrm{norm}}_{(I,d)} X$ with complement of codimension ≥ 2, then:*

(i) *$\operatorname{Bl}^{\mathrm{norm}}_{I} X$ has an effective $\mathbf{Q}$-Cartier divisor D such that dD is the exceptional divisor on $\operatorname{Bl}^{\mathrm{norm}}_{I} X$.*

(ii) $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \to X$ *can be factored as follows:*

$$\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \xrightarrow{p} \mathrm{Bl}^{\mathrm{norm}}_{I} X \xrightarrow{q} X$$

where q is the normalized blow-up of X along I, and p is the Cartierification of D in $\mathrm{Bl}^{\mathrm{norm}}_{I} X$.

Proof. Let E (resp. E') denote the exceptional divisor on $\mathrm{Bl}^{\mathrm{norm}}_{I} X$ (resp. $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X$). By the universal property of $\mathrm{Bl}^{\mathrm{norm}}_{I} X$, the map $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \to X$ factors uniquely through $\mathrm{Bl}^{\mathrm{norm}}_{I} X$ as follows:

$$\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \xrightarrow{p} \mathrm{Bl}^{\mathrm{norm}}_{I} X \xrightarrow{q} X.$$

Since $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X$ has no relative stackiness over X in codimension 1, and since p is a coarse moduli space (Proposition 2.1.5(iii)), p is an isomorphism in codimension 1, and thus p induces an identification of Weil divisors of both sides. Since $p^{-1}(E) = dE'$ with E' a Cartier divisor on $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X$, there exists a Weil divisor D on $\mathrm{Bl}^{\mathrm{norm}}_{I} X$ such that $dD = E$.

Next, set $Y := \mathrm{Bl}^{\mathrm{norm}}_{I} X$, and we show that p can be identified with the Cartierification of D, i.e. $\pi\colon \mathrm{Bl}_{I_{\bullet D}} Y \to Y$. We do so by comparing the universal properties of π and $q \circ p$. Let $f\colon T \to Y$ be a morphism from a normal scheme T, where $f^{-1}(D)$ is nowhere dense in T, i.e. $(q \circ f)^{-1}(V(I))$ is nowhere dense in T (for example, $f = \pi$ or $f = p$). Then:

(a) f factors uniquely through π if and only if f^*D is an effective Cartier divisor on T (Example 2.2.19).

(b) f factors uniquely through p if and only if

$$\begin{aligned} \mathrm{IC}\big((q \circ f)^{-1}(I,d)\mathscr{O}_T\big) &= \mathrm{IC}(f^{-1}(I_E, d)\mathscr{O}_T) \\ &= \mathrm{IC}(f^{-1}(I_{\bullet D})\mathscr{O}_T) = I_{\bullet f^*D} \end{aligned}$$

is generated in degree 1 by the underlying ideal of an effective Cartier divisor on T (Theorem 2.3.43). Note that the last equality above holds since both sides are integrally closed Rees algebras whose d^{th} Veronese subalgebras coincide.

By inspection, the universal properties in (a) and (b) coincide, as desired. $\square$

2.3.E. Zariski-Riemann spaces. For the remainder of §2.3, we will give a convenient reformulation of integrally closed Rees algebras on varieties in terms of certain objects on their Zariski-Riemann spaces. In this subsection, we begin by first recalling the notion of *Zariski-Riemann spaces*, originally referred to as "Riemann manifolds" by Zariski [**Zar39**], in his proof of resolution of singularities in dimensions 2 and 3. We fix a variety X over a field $\mathbf{k}$ (= integral, separated scheme over $\mathbf{k}$), and let $K = K(X)$ be its field of fractions.

Definition 2.3.46 (Zariski-Riemann space of $K/\mathbf{k}$)**.** We define $\mathrm{ZR}(K/\mathbf{k})$ in steps:

(i) As a set,

$$\mathrm{ZR}(K/\mathbf{k}) := \{\text{valuation rings } R \text{ of } K \text{ containing } \mathbf{k}\}.$$

We usually denote an element R of $\mathrm{ZR}(K/\mathbf{k})$ by its corresponding valuation $\nu\colon K^* \twoheadrightarrow G$ instead, where $G = \{aR\colon a \in K^*\}$ is the value group of ν. In that case, we write R_ν for R, and G_ν for G. We denote the unique maximal ideal of R_ν by $\mathfrak{m}_\nu$, and its residue field by $\kappa_\nu = R_\nu/\mathfrak{m}_\nu$.

(ii) As a topological space, $\mathrm{ZR}(K/\mathbf{k})$ has a basis of open sets of the form

$$U(a_1, a_2, \ldots, a_m) := \bigl\{\nu \in \mathrm{ZR}(K/\mathbf{k})\colon R_\nu \supset \mathbf{k}[a_1, a_2, \ldots, a_m]\bigr\}$$

where $m \in \mathbf{N}$ and $a_1, a_2, \ldots, a_m \in K^*$.

(iii) Finally, as a locally ringed space, $\mathrm{ZR}(K/\mathbf{k})$ is equipped with the sheaf of rings:

$$\mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}(U) := \bigcap_{\nu\in U} R_\nu \qquad \text{where } U \subset \mathrm{ZR}(K/\mathbf{k}) \text{ is open.}$$

In particular, $\mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}(U(a_1, a_2, \dots, a_m))$ is the integral closure of $\mathbf{k}[a_1, a_2, \dots, a_m]$ in K [**Mat89**, Theorem 10.4]. Then $\mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}$ is a subsheaf of the constant sheaf K on $\mathrm{ZR}(K/\mathbf{k})$, and the stalk of $\mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}$ at ν is R_ν.

(iv) $\mathrm{ZR}(K/\mathbf{k})$ also carries a sheaf of ordered groups $\Gamma = K^*/\mathscr{O}^*_{\mathrm{ZR}(K/\mathbf{k})}$, whose sections over an open set U are:

$$\left\{ (s_\nu)_{\nu\in U} \in \prod_{\nu\in U} G_\nu \, : \, \begin{array}{l} \text{for every } \nu \in U \text{, there exists an open neighbourhood} \\ V \subset U \text{ of } \nu \text{ and some } a \in K^* \text{ such that for all } \nu' \in V, \\ s_{\nu'} = \nu'(a) \end{array} \right\}$$

and whose stalk at ν is G_ν, with a morphism of sheaves of ordered groups $\mathrm{val}\colon K^* \twoheadrightarrow \Gamma$. The image $\mathrm{val}(\mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})} \setminus \{0\}) \subset \Gamma$ is the sheaf of monoids consisting of non-negative sections of Γ, denoted Γ_+. Its sections over an open set U are:

$$\left\{ (s_\nu)_{\nu\in U} \in \prod_{\nu\in U} G_\nu \, : \, \begin{array}{l} \text{for every } \nu \in U \text{, there exists an open neighbourhood} \\ V \subset U \text{ of } \nu \text{ and some } 0 \neq a \in \mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}(V) \text{ such that} \\ \text{for all } \nu' \in V,\ s_{\nu'} = \nu'(a) \end{array} \right\}.$$

Convention 2.3.47. If $\mathbf{k} = \mathbf{Q}$ or $\mathbf{Z}/p$ (depending on its characteristic), we can neglect the base $\mathbf{k}$. That is, a $\mathbf{k}$-variety will be called a variety, and we write $\mathrm{ZR}(K/\mathbf{k})$ as $\mathrm{ZR}(K)$.

We now focus on two properties of $\mathrm{ZR}(K/\mathbf{k})$. Firstly:

Lemma 2.3.48. $\mathrm{ZR}(K/\mathbf{k})$ *is quasi-compact.*

This is well-known, cf. [**Mat89**, Theorem 10.5]. Secondly and more importantly:

Lemma 2.3.49. $\mathrm{ZR}(K/\mathbf{k})$ *can be identified with the inverse limit of projective models of* $K/\mathbf{k}$ *in the category of locally ringed spaces.*

2.3.50. Let us expound on Lemma 2.3.49 further. A *projective model* of $K/\mathbf{k}$ is a projective $\mathbf{k}$-variety X whose field of functions $K(X)$ is isomorphic to K. For every $\nu \in \mathrm{ZR}(K/\mathbf{k})$, there exists a unique dotted arrow making the triangles in the diagram below commute:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "\text{generic pt.}"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(R_\nu) \arrow[r] \arrow[ur, dotted, "f_\nu"] & \mathrm{Spec}(\mathbf{k}) \end{tikzcd}$$

The composition $\mathrm{Spec}(\kappa_\nu) = \mathrm{Spec}(R_\nu/\mathfrak{m}_\nu) \to \mathrm{Spec}(R_\nu) \xrightarrow{f_\nu} X$ designates a point x_ν on X, which is called the *center* of R_ν on X [**Har77**, Exercise II.4.5]. This gives an injective local $\mathbf{k}$-homomorphism $f_\nu^\#\colon \mathscr{O}_{X,x_\nu} \to R_\nu$ of local rings whose field of fractions is K, in which case we say R_ν *dominates* $\mathscr{O}_{X,x_\nu}$ [**Har77**, Lemma II.4.4].

The projective models of $K/\mathbf{k}$ form an inverse system as follows. An arrow from a projective model X_a to another X_b is a birational morphism $\varphi_{a\to b}\colon X_b \to X_a$. For every ν in $\mathrm{ZR}(K/\mathbf{k})$, $\varphi_{a\to b}$ necessarily maps the center $(x_\nu)_b$ of R_ν on X_b to the center $(x_\nu)_a$ of R_ν on X_a. In other words, φ_a^b induces a local homomorphism $\varphi_{a\to b}^\#$ of local rings with field of fractions K, which makes the diagram below commute:

$$\begin{tikzcd} \mathscr{O}_{X_a,(x_\nu)_a} \arrow[rr, "\varphi^\#_{a\to b}"] \arrow[dr, swap, "(f_\nu)^\#_a"] & & \mathscr{O}_{X_b,(x_\nu)_b} \arrow[dl, "(f_\nu)^\#_b"] \\ & R_\nu & \end{tikzcd}$$

The join X of two projective models X_a and X_b admits birational morphisms $X \to X_a$ and $X \to X_b$, whence this is indeed an inverse system.

Proof of Lemma 2.3.49. We prove this in steps.

2.3.51 (*Step 1*: $\mathrm{ZR}(K/\mathbf{k})$ is the set-theoretic inverse limit)**.** As shown above, a point $\nu \in \mathrm{ZR}(K/\mathbf{k})$ determines a collection of points $\{x_\nu \in X \colon X \text{ is a projective model of } K/\mathbf{k}\}$, which is compatible with the arrows in the inverse system, and hence determines a point in the inverse limit.

Conversely, a point in the inverse limit is a collection of points $\Sigma = \{y_\Sigma \in X \colon X$ is a projective model of $K/\mathbf{k}\}$ which is compatible with the arrows in the inverse system. Let R be the direct limit of the system whose objects are the local rings $\mathscr{O}_{X,x_\Sigma}$, and whose arrows are given by the local $\mathbf{k}$-homomorphisms of local rings. Note that R is a local ring. By [**Har77**, Theorem I.6.1A], R is a valuation ring of K containing $\mathbf{k}$, and hence determines a point $\nu \in \mathrm{ZR}(K/\mathbf{k})$. For each projective model X of $K/\mathbf{k}$, $\mathscr{O}_{X,x_\Sigma}$ is the unique local ring of X dominated by R, i.e. the center of R on X is x_Σ. On the other hand, given a point $\nu \in \mathrm{ZR}(K/\mathbf{k})$, R_ν is the direct limit of the system of local rings $\mathscr{O}_{X,x_\nu}$ [**Har77**, Theorem I.6.1A]. This establishes the desired one-to-one correspondence of points.

2.3.52 (*Step 2*: $\mathrm{ZR}(K/\mathbf{k})$ is the topological inverse limit)**.** The inverse limit topology on $\mathrm{ZR}(K/\mathbf{k})$ is the coarsest topology such that for every projective model X of $K/\mathbf{k}$, the map $\pi_X \colon \mathrm{ZR}(K/\mathbf{k}) \to X$, which maps each $\nu \in \mathrm{ZR}(K/\mathbf{k})$ to the center x_ν of R_ν on X, is continuous. To see this, fix a projective model X of $K/\mathbf{k}$. Let $U = \mathrm{Spec}(A) \subset X$ be an open affine subset. Then $\pi_X^{-1}(\mathrm{Spec}(A))$ consists of those $\nu \in \mathrm{ZR}(K/\mathbf{k})$ for which there exists a dotted arrow filling in the diagram below:

$$\begin{tikzcd} K & A \arrow[l] \arrow[dl, dotted] \\ R_\nu \arrow[u] & \mathbf{k} \arrow[l] \arrow[u] \end{tikzcd}$$

Since A is a finitely generated $\mathbf{k}$-algebra, write $A = \mathbf{k}[a_1, a_2, \ldots, a_n]$ for $a_i \in K^*$. Then $U(a_1, a_2, \ldots, a_n) = \pi_X^{-1}(\mathrm{Spec}(A))$. Conversely, given $a_1, a_2, \ldots, a_n \in K^*$, there are $a_{n+1}, \ldots, a_m \in K^*$ such that $A = \mathbf{k}[a_1, a_2, \ldots, a_m]$ has fraction field K. The projection $T_i \mapsto a_i$ gives a presentation of A as $A \simeq \mathbf{k}[T_1, T_2, \ldots, T_m]/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset \mathbf{k}[T_1, T_2, \ldots, T_m]$. Homogenize

$\mathfrak{p} \subset \mathbf{k}[T_1, \dots, T_m]$ to a homogeneous prime ideal $\mathfrak{P} \subset \mathbf{k}[T_1, \dots, T_m, T_{m+1}]$. Then $U = \operatorname{Spec}(A)$ is an open affine subset of $X = \operatorname{Proj}(\mathbf{k}[T_1, \dots, T_{m+1}]/\mathfrak{P})$, which is a projective model of $K/\mathbf{k}$ with $\pi_X^{-1}(\operatorname{Spec}(A)) = U(a_1, \dots, a_m) \subset U(a_1, \dots, a_n)$. Since open affines form a basis for Zariski topology, this step is complete.

2.3.53 (*Step 3*: $\mathrm{ZR}(K/\mathbf{k})$ is the inverse limit in the category of locally ringed spaces)**.** Set $\mathscr{O} := \varinjlim_X \pi_X^{-1}\mathscr{O}_X$, where the direct limit is taken over projective models X of $K/\mathbf{k}$. In other words, $\mathscr{O}$ is the sheaf of rings on $\mathrm{ZR}(K/\mathbf{k})$ as the inverse limit in the category of locally ringed spaces [**Gil11**, Theorem 4 and Corollary 5]. It remains to show that $\mathscr{O} = \mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}$. For this, observe there are morphisms $\pi_X^{-1}\mathscr{O}_X \to \mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}$, adjoint to the canonical morphisms $\mathscr{O}_X \to (\pi_X)_*\mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}$, for each projective model X of $K/\mathbf{k}$. Together they culminate in a morphism $\mathscr{O}' \to \mathscr{O}_{\mathrm{ZR}(K/\mathbf{k})}$ which is an isomorphism on stalks, cf. second-to-last sentence in 2.3.51. □

Definition 2.3.54 (Zariski-Riemann space of a $\mathbf{k}$-variety)**.** More generally, we can define the Zariski-Riemann space of the $\mathbf{k}$-variety X. Let $K(X)$ denote the field of fractions of X. Note that not every $\nu \in \mathrm{ZR}(K(X)/\mathbf{k})$ possesses a center x_ν on X, but if it does, the center x_ν is unique. Therefore, we set

$$\mathrm{ZR}(X/\mathbf{k}) := \{\nu \in \mathrm{ZR}(K(X)/\mathbf{k}) \colon \nu \text{ has a center on } X\} \subset \mathrm{ZR}(K/\mathbf{k}).$$

Note that if $X/\mathbf{k}$ is proper, $\mathrm{ZR}(X/\mathbf{k})$ is simply $\mathrm{ZR}(K(X)/\mathbf{k})$. We let $\mathrm{ZR}(X/\mathbf{k})$ inherit its topology, sheaf of rings $\mathscr{O}_{\mathrm{ZR}(X/\mathbf{k})}$, and sheaf of ordered groups Γ_X from $\mathrm{ZR}(K(X)/\mathbf{k})$. As before, one can identify $\mathrm{ZR}(X/\mathbf{k})$ with the inverse limit of the system of modifications $X' \to X$ in the category of morphisms of locally ringed spaces with target X. We write π_X for the morphism $\mathrm{ZR}(X/\mathbf{k}) \to X$ sending each $\nu \in \mathrm{ZR}(X/\mathbf{k})$ to the center of ν on X. As in Convention 2.3.47, if $k = \mathbf{Q}$ or $\mathbf{Z}/p$, we will write $\mathrm{ZR}(X/\mathbf{k})$ as $\mathrm{ZR}(X)$.

2.3.55. $\mathrm{ZR}(X/\mathbf{k})$ is quasi-compact and open in $\mathrm{ZR}(K(X)/\mathbf{k})$. This can be seen as follows. First suppose $X = \mathrm{Spec}(A)$ is an affine $\mathbf{k}$-variety, with A generated as a $\mathbf{k}$-algebra by $x_1, \ldots, x_n \in K$. In this case, we have seen earlier that $\mathrm{ZR}(X/\mathbf{k})$ is $U(x_1, \ldots, x_n)$, and is quasi-compact [**Mat89**, Theorem 10.5]. In general, since X is covered by finitely many affine opens, one deduces that $\mathrm{ZR}(X/\mathbf{k})$ is quasi-compact and open in $\mathrm{ZR}(K(X)/\mathbf{k})$.

2.3.56 (Functoriality with respect to dominant morphisms)**.** If $f\colon X' \to X$ is a dominant morphism of $\mathbf{k}$-varieties, f induces a morphism $\mathrm{ZR}(f)\colon \mathrm{ZR}(X'/\mathbf{k}) \to \mathrm{ZR}(X/\mathbf{k})$ of locally ringed spaces, which maps $R_\nu \mapsto R_\nu \cap K(X)$. The morphism $\mathscr{O}_{\mathrm{ZR}(X/\mathbf{k})} \to \mathrm{ZR}(f)_*\mathscr{O}_{\mathrm{ZR}(X'/\mathbf{k})}$ is given by the inclusion $\bigcap_{\nu \in U} R_\nu \hookrightarrow \bigcap_{\eta \in \mathrm{ZR}(f)^{-1}(U)} R_\eta$ over an open set U, and is stalk-wise given by the local homomorphism $R_\nu \cap K(X) \hookrightarrow R_\nu$. This morphism $\mathscr{O}_{\mathrm{ZR}(X/\mathbf{k})} \to \mathrm{ZR}(f)_*\mathscr{O}_{\mathrm{ZR}(X'/\mathbf{k})}$ descends to a morphism of sheaves of ordered groups $\Gamma_X \to \mathrm{ZR}(f)_*\Gamma_{X'}$, as well as a morphism of sheaves of monoids $\Gamma_{X,+} \to \mathrm{ZR}(f)_*\Gamma_{X',+}$.

We conclude this section with one more noteworthy fact:

Lemma 2.3.57. *Let X be a $\mathbf{k}$-variety, with $\pi_X\colon \mathrm{ZR}(X/\mathbf{k}) \to X$. Then X is normal if and only if the morphism $\pi_X^\#\colon \mathscr{O}_X \to (\pi_X)_*\mathscr{O}_{\mathrm{ZR}(X/\mathbf{k})}$ is an isomorphism of sheaves on X.*

Proof. Only the forward implication is not clear. Since open affines form a basis for the Zariski topology on X, it suffices to check this isomorphism on open affines $U = \mathrm{Spec}(A) \subset X$. Since X is normal, $\mathscr{O}_X(U) = A$ is normal, whence by [**Mat89**, Theorem 10.4],

$$\mathscr{O}_X(U) = \bigcap \bigl\{ R_\nu \colon \nu \in \mathrm{ZR}(K(X)/\mathbf{k}),\ R_\nu \supset A \bigr\}$$

But the set of $\nu \in \mathrm{ZR}(K(X)/\mathbf{k})$ such that $R_\nu \supset A$ is precisely the set of $\nu \in \mathrm{ZR}(X/\mathbf{k})$ which has a center on $U = \mathrm{Spec}(A) \subset X$. Thus, $\mathscr{O}_X(U) = \bigcap_{\nu \in \pi_X^{-1}(U)} R_\nu = \mathscr{O}_{\mathrm{ZR}(K(X)/\mathbf{k})}(\pi_X^{-1}(U))$. □

2.3.F. Valuative ideals and idealistic classes. As before, let X be a variety over a field $\mathbf{k}$. In this subsection and the next, we introduce the relevant objects on $\mathrm{ZR}(X)$ which would allow us to re-formulate the notion of integrally closed Rees algebras on X.

Definition 2.3.58 (Valuative ideals, cf. [**ATW19**, §2.2]). A *valuative ideal* over X is defined to be a section $\gamma \in \Gamma(\mathrm{ZR}(X/\mathbf{k}), \Gamma_{X,+})$.

2.3.59 (Ideals $\rightsquigarrow$ Valuative ideals). A non-zero ideal $I \subset \mathscr{O}_X$ determines a valuative ideal γ_I over X as follows. For every $\nu \in \mathrm{ZR}(X/\mathbf{k})$, recall that x_ν denotes the center of ν on X, and $f_\nu^\#\colon \mathscr{O}_{X,x_\nu} \to R_\nu$ denotes the corresponding local $\mathbf{k}$-homomorphism. We set

$$\gamma_{I,\nu} := \min\{\nu(g)\colon g \in I_{x_\nu} \subset \mathscr{O}_{X,x_\nu}\},$$

where $\nu(g)$ will always be an abbreviation for $\nu(f_\nu^\#(g))$. This minimum exists in $(G_\nu)_+$. Indeed, if I_{x_ν} is generated by $g_1, \ldots, g_r \in \mathscr{O}_{X,x_\nu}$, then $\gamma_{I,\nu} = \min\{\nu(g_i)\colon 1 \leq i \leq r\}$. Moreover, if we let $1 \leq j \leq r$ be such that $\nu(g_j) = \gamma_{I,\nu}$, then $f_\nu^\#(I_{x_\nu})R_\nu$ is the principal ideal $(f_\nu^\#(g_j))R_\nu$ of R_ν. We next claim that

$$(\gamma_{I,\nu})_{\nu\in\mathrm{ZR}(X/\mathbf{k})} \in \prod_{\nu\in\mathrm{ZR}(X/\mathbf{k})} (G_\nu)_+$$

defines the desired valuative ideal γ_I over X. For this we need to check that it is a compatible collection of germs of $\Gamma_{X,+}$. Indeed, fix an arbitrary $\nu \in \mathrm{ZR}(X/\mathbf{k})$, and assume that $g_1, \ldots, g_r \in \mathscr{O}_{X,x_\nu}$ generate I_{x_ν}, with $\nu(g_j) = \min\{\nu(g_i)\colon 1 \leq i \leq r\}$. There exists an affine open neighbourhood V_ν of x_ν in X such that $g_1, \ldots, g_r$ extend to sections of I over V_ν which generate the stalk of I at every point in V_ν. Then $U_\nu = \pi_X^{-1}(V_\nu) \cap U\big(\frac{g_i}{g_j}\colon i \neq j\big)$ is an open neighbourhood of ν in $\mathrm{ZR}(X/\mathbf{k})$ such that for all $\nu' \in U_\nu$, $\gamma_{\nu'} = \nu'(g_j)$.

In fact, the same argument shows that any valuative ideal over X arising from an ideal on X is locally represented by generators of that ideal:

Lemma 2.3.60. *For a non-zero ideal $I \subset \mathscr{O}_X$, there exist:*

(i) *a finite open affine cover $\mathcal{V} = \{V_\ell \colon 1 \le \ell \le m\}$ of X;*

(ii) *for each $1 \le \ell \le m$, a finite open cover $\mathcal{U}_\ell = \{U_{\ell,j} \colon 1 \le j \le r_\ell\}$ of $\pi_X^{-1}(V_\ell)$;*

(iii) *for each $1 \le \ell \le m$, sections $\{g_{\ell,j} \colon 1 \le j \le r_\ell\}$ of I over V_ℓ, which generate I at every point of V_ℓ,*

such that for each $1 \le \ell \le m$, each $1 \le j \le r_\ell$, and every $\nu \in U_{\ell,j}$, we have $\gamma_{I,\nu} = \nu(g_{\ell,j})$.

Proof. For every $x \in X$, there exists $g_1, \dots, g_r \in I_x$ and an open affine neighbourhood $x \in V_x \subset X$ such that $g_1, \dots, g_r$ extend to sections of I over V_x generating I at every point of V_x. Since X is quasi-compact, there exists a finite open subcover of $\{V_x \colon x \in X\}$, say $\mathcal{V} = \{V_\ell \colon 1 \le \ell \le m\}$. For each ℓ, let $g_{\ell,1}, \dots, g_{\ell,r_\ell} \in I(V_\ell)$ be the sections chosen earlier.

For each $1 \le j \le r_\ell$, let $U_{\ell,j} = \pi_X^{-1}(V_\ell) \cap U\big(\frac{g_{\ell,i}}{g_{\ell,j}} \colon i \neq j\big)$. For all $\nu \in \pi_X^{-1}(V_\ell)$, we have $x_\nu \in V_\ell$, whence I_{x_ν} is generated by $\{g_{\ell,j} \colon 1 \le j \le r_\ell\}$, so $\gamma_{I,\nu} = \min\{\nu(g_{\ell,j}) \colon 1 \le j \le r_\ell\}$. From this, it is immediate that $\pi_X^{-1}(V_\ell) = \bigcup_{j=1}^{r_\ell} U_{\ell,j}$. The conclusion is also immediate. $\square$

Definition 2.3.61 (Idealistic classes)**.** Any valuative ideal $\gamma = \gamma_I$ over X arising from a non-zero ideal I on X will be called an *idealistic class* over X.

2.3.62 (Valuative ideals $\rightsquigarrow$ Ideals)**.** Conversely, every valuative ideal γ over X determines an ideal I_γ on X, whose sections over an open set $U \subset X$ are:

$$I_\gamma(U) = \{g \in \mathscr{O}_X(U) \colon \nu(g) \ge \gamma_\nu \text{ for every } \nu \in \pi_X^{-1}(U)\}$$

where we remind the reader that $\pi_X^{-1}(U) = \{\nu \in \mathrm{ZR}(X/\mathbf{k}) \colon x_\nu \in U\}$.

2.3.63. If I is an ideal of a ring A, the *integral closure* $\mathrm{IC}(I)$ of I [**Laz04**, §9.6.A] in A consists of elements $a \in A$ which satisfy a "weighted integral equation":

$$a^n + c_1 a^{n-1} + \cdots + c_{n-1} a + c_n = 0, \qquad \text{where } c_i \in I^i.$$

We say I is *integrally closed* in A if $I = \mathrm{IC}(I)$. Observe that $I \subset \mathrm{IC}(I) \subset \sqrt{I}$, where $\sqrt{I}$ is the radical of I. In the next section, we will prove that:

(a) $\mathrm{IC}(I)$ is an ideal of A,

(b) if I is an ideal on X, the presheaf $\mathrm{IC}(I)$ on X given by $U \mapsto \mathrm{IC}(I(U))$ is a sheaf,

(c) and the following lemma, which was essentially noted in [**Hir77**]:

Lemma 2.3.64.

(i) *If γ is a valuative ideal over X, then I_γ is integrally closed in $\mathscr{O}_X$.*

(ii) *Let $I \subset \mathscr{O}_X$ be a non-zero ideal, with associated idealistic class $\gamma = \gamma_I$ over X. Then $I_\gamma = \mathrm{IC}(I)$.*

In particular, 2.3.59 and 2.3.62 define a one-to-one correspondence between non-zero, integrally closed ideals of $\mathscr{O}_X$ and idealistic classes over X.

2.3.G. Valuative Q-ideals and idealistic exponents. Fix a variety X over a field $\mathbf{k}$ as in §2.3.F.

Definition 2.3.65 (Valuative $\mathbf{Q}$-ideals)**.** Let $\Gamma_{X,\mathbf{Q}}$ denote the sheaf of ordered groups $\Gamma_X \otimes \mathbf{Q}$ on $\mathrm{ZR}(X/\mathbf{k})$, and let $\Gamma_{X,\mathbf{Q}+}$ denote the sheaf of monoids on $\mathrm{ZR}(X/\mathbf{k})$ consisting of non-negative sections of $\Gamma_{X,\mathbf{Q}+}$. Then a *valuative* $\mathbf{Q}$*-ideal over* X is a section γ in $\Gamma(\mathrm{ZR}(X/\mathbf{k}), \Gamma_{X,\mathbf{Q}+})$.

Remark 2.3.66. Since γ is "locally constant" and $\mathrm{ZR}(X/\mathbf{k})$ is quasi-compact, there exists a sufficiently large natural number $N \geq 1$ such that $N \cdot \gamma$ is a valuative ideal over X.

2.3.67 (Rees algebras $\rightsquigarrow$ Valuative $\mathbf{Q}$-ideals)**.** A non-zero Rees algebra $I_\bullet$ (Definition 2.3.1) on X determines a valuative $\mathbf{Q}$-ideal $\gamma_{I_\bullet}$ over X by:

$$\gamma_{I_\bullet} := (\gamma_{I_\bullet,\nu})_{\nu \in \mathrm{ZR}(X/\mathbf{k})} \in \prod_{\nu \in \mathrm{ZR}(X/\mathbf{k})} (\mathbf{Q} \otimes G_\nu)_+$$

where

$$\gamma_{I_\bullet,\nu} := \min\left\{\frac{1}{n}\cdot\nu(g)\ :\ 0\neq gt^n\in (I_n)_{x_\nu},\ n\geq 1\right\}.$$

As before in 2.3.59, we have to show:

(i) this minimum exists in $(\mathbf{Q}\otimes G_\nu)_+$,

(ii) $(\gamma_{I_\bullet,\nu})_{\nu\in \mathrm{ZR}(X/\mathbf{k})}$ defines a compatible collection of germs, and hence, defines a valuative $\mathbf{Q}$-ideal over X.

For (i), fix $\nu\in \mathrm{ZR}(X/\mathbf{k})$, and suppose $g_1t^{n_1},\ldots,g_rt^{n_r}$ generate $(I_\bullet)_{x_\nu}$ as a $\mathscr{O}_{X,x_\nu}$-algebra. Then we claim:

$$\gamma_{I_\bullet,\nu} = \min\left\{\frac{1}{n_i}\cdot\nu(g_i)\colon 1\leq i\leq r\right\}$$

from which (i) is immediate. Indeed, suppose $gt^n\in I_{x_\nu}$. Then we can write

$$gt^n = \sum_{\substack{\mathbf{d}=(d_1,\ldots,d_r)\\ d_1n_1+\cdots+d_rn_r=n}} a_{\mathbf{d}}\cdot\prod_{i=1}^{r}(g_it^{n_i})^{d_i} \qquad \text{in } \mathscr{O}_{X,x_\nu}[t]$$

which means

$$g = \sum_{\substack{\mathbf{d}=(d_1,\ldots,d_r)\\ d_1n_1+\cdots+d_rn_r=n}} a_{\mathbf{d}}\cdot\prod_{i=1}^{r}g_i^{d_i} \qquad \text{in } \mathscr{O}_{X,x_\nu}.$$

Consequently,

$$\begin{aligned}\frac{1}{n}\cdot\nu(g) &\geq \min\left\{\frac{1}{n}\sum_{i=1}^{r}d_in_i\cdot\left(\frac{1}{n_i}\cdot\nu(g_i)\right)\colon d_1n_1+\cdots+d_rn_r=n\right\}\\ &\geq \min\left\{\left(\frac{1}{n}\sum_{i=1}^{r}d_in_i\right)\cdot\min\left\{\frac{1}{n_i}\cdot\nu(g_i)\colon 1\leq i\leq r\right\}\ :\ d_1n_1+\cdots+d_rn_r=n\right\}\\ &= \min\left\{\frac{1}{n_i}\cdot\nu(g_i)\colon 1\leq i\leq r\right\}\end{aligned}$$

as desired.

For (ii), there exist an affine open neighbourhood V_ν of x_ν in X such that $g_1t^{n_1},\dots,g_rt^{n_r}$ extend to sections of $I_\bullet$ over V_ν which generate the stalk of $I_\bullet$ at every point in V_ν. Let $1\leq j\leq r$ such that $\gamma_{I_\bullet,\nu}=\frac{1}{n_j}\cdot\nu(g_j)$. Then $U_\nu=\pi_X^{-1}(V_\nu)\cap U\bigl(g_i^{n_j}/g_j^{n_i}\colon i\neq j\bigr)$ is an open neighbourhood of ν in $\mathrm{ZR}(X/\mathbf{k})$ such that for all $\nu'\in U_\nu$, $\gamma_{\nu'}=\frac{1}{n_j}\cdot\nu'(g_j)$.

Remark 2.3.68. Let I be a non-zero ideal on X, with Rees algebra $I^\bullet$ (Example 2.3.6). Then $\gamma_{I^\bullet}=\gamma_I$.

Moreover, imitating the proof of Lemma 2.3.60 yields the analogous lemma:

Lemma 2.3.69. *For a non-zero Rees algebra $I_\bullet$ on X, there exist:*

- (i) *a finite open affine cover $\mathcal{V}=\{V_\ell\colon 1\leq\ell\leq m\}$ of X;*
- (ii) *for each $1\leq\ell\leq m$, a finite open cover $\mathcal{U}_\ell=\{U_{\ell,j}\colon 1\leq j\leq r_\ell\}$ of $\pi_X^{-1}(V_\ell)$;*
- (iii) *for each $1\leq\ell\leq m$, sections $\{g_{\ell,j}T^{n_{\ell,j}}\colon 1\leq j\leq r_\ell\}$ of $I_\bullet$ over V_ℓ, which generate $I_\bullet$ at every point of V_ℓ (as a $\mathscr{O}_{X,y}$-algebra),*

such that for each $1\leq\ell\leq m$, each $1\leq j\leq r_\ell$, and every $\nu\in U_{\ell,j}$, we have $\gamma_{I_\bullet,\nu}=\frac{1}{n_{\ell,j}}\cdot\nu(g_{\ell,j})$.

Definition 2.3.70 (Idealistic exponents, cf. [**Hir77**, Definition 3])**.** A valuative $\mathbf{Q}$-ideal $\gamma=\gamma_{I_\bullet}$ over X arising from a non-zero Rees algebra $I_\bullet$ on X will be called an *idealistic exponent* over X.

2.3.71 (Valuative $\mathbf{Q}$-ideals $\rightsquigarrow$ Rees algebras)**.** Conversely, let γ be a valuative $\mathbf{Q}$-ideal over X. As in 2.3.62, γ also determines an ideal I_γ on X whose sections g over an open set U satisfy $\nu(g)\geq\gamma_\nu$ for every $\nu\in\pi_X^{-1}(U)=\{\nu\in\mathrm{ZR}(X/\mathbf{k})\colon x_\nu\in U\}$. Slightly more generally, given any $m\in\mathbf{N}$, we have the valuative $\mathbf{Q}$-ideal $m\gamma$, and similarly the ideal $I_{m\gamma}$ on X. Together these $I_{m\gamma}$ form an $\mathscr{O}_X$-subalgebra of $\mathscr{O}_X[t]$:

$$I_{\bullet\gamma}:=\bigoplus_{m\in\mathbf{N}}I_{m\gamma}\cdot t^m\subset\mathscr{O}_X[t]$$

In general, $I_{\bullet\gamma}$ is not a Rees algebra on X, but Proposition 2.3.73(ii) below says $I_{\bullet\gamma}$ is a Rees algebra on X whenever γ is an idealistic exponent over X. Note that $I_{\bullet\gamma}$ contains the Rees algebra of I_γ, but these are rarely equal, cf. Corollary 2.3.75.

Lemma 2.3.72. *Let γ be a valuative $\mathbf{Q}$-ideal over X. Then $I_{\bullet\gamma}$ is integrally closed in $\mathscr{O}_X[t]$.*

Proof. It suffices to show that whenever a non-zero homogeneous section gt^r of $\mathscr{O}_X[t]$ over an open set $U \subset X$ satisfies an equation of the form

$$(gT^r)^n + a_1(gt^r)^{n-1} + \cdots + a_{n-1}(gt^r) + a_n = 0, \qquad a_i \in I_{\bullet\gamma}(U),$$

then gt^r is a section of $I_{\bullet\gamma}$ over U. By writing each a_i as a sum of homogeneous sections in $I_{\bullet\gamma}(U)$ and comparing degrees, we may assume that each a_i is $\alpha_i t^{ir}$ for some $\alpha_i \in I_{ir\gamma}(U)$. If $r = 0$, there is nothing to show. If $r > 0$, we have

$$g^n + \alpha_1 g^{n-1} + \cdots + \alpha_{n-1} g + \alpha_n = 0 \qquad \text{in } \mathscr{O}_X(U).$$

Let $\nu \in \pi_X^{-1}(U) \subset \mathrm{ZR}(X/\mathbf{k})$. We claim that there must exist some $1 \leq j \leq n$ such that $j\nu(g) \geq \nu(\alpha_j)$. Indeed, if not, then $i\nu(g) < \nu(\alpha_i)$ for all $1 \leq i \leq n$, so $\nu(g^n) < \nu(\alpha_i g^{n-i})$ for all $1 \leq i \leq n$. This implies $g^n + \alpha_1 g^{n-1} + \cdots + \alpha_{n-1} g + \alpha_n \neq 0$, a contradiction. Now our claim implies $\nu(g) \geq \frac{1}{j}\nu(\alpha_j) \geq r\gamma_\nu$, so $g \in I_{r\gamma}(U)$, as desired. $\square$

Proposition 2.3.73. *Let $\gamma = \gamma_{I_\bullet}$ be the idealistic exponent over X arising from a non-zero Rees algebra $I_\bullet$ on X. Then:*

(i) $I_{\bullet\gamma} = \mathrm{IC}(I_\bullet)$.

(ii) *In particular, $I_{\bullet\gamma}$ is a finite $I_\bullet$-module, and hence is a Rees algebra on X.*

Recall that $\mathrm{IC}(I_{\bullet\gamma})$ denotes the integral closure of $I_\bullet$ in $\mathscr{O}_X[t]$ (Definition 2.3.32).

PROOF. By Lemma 2.3.72, $I_{\bullet\gamma}$ contains $\mathrm{IC}(I_\bullet)$. We check the converse on stalks: let $x \in X$, and it suffices to show that whenever a homogeneous element gt^n of $\mathscr{O}_{X,y}[t]$ is not integral over $(I_\bullet)_y$, then gt^n is not in $(I_{\bullet\gamma})_y$. Fixing generators $g_1t^{n_1}, \ldots, g_rt^{n_r}$ for $(I_\bullet)_y$ as a $\mathscr{O}_{X,x}$-algebra, our goal is to find $\nu \in \mathrm{ZR}(X/\mathbf{k})$ whose center x_ν on X is x, and such that

$$\frac{1}{n}\nu(g) < \min\left\{\frac{1}{n_i}\nu(g_i)\colon 1 \le i \le r\right\}.$$

Let

$$A = \mathscr{O}_{X,x}\left[\frac{g_i^n}{g^{n_i}}\colon 1 \le i \le r\right]$$

and let I be the ideal of A generated by $\{g_i^n/g^{n_i}\colon 1 \le i \le r\}$ and the maximal ideal $\mathfrak{m}_{X,x}$ of $\mathscr{O}_{X,x}$. We claim that $1 \notin I$. If not,

$$1 = \alpha + \sum_{\substack{\mathbf{j}=(j_1,\ldots,j_r)\\ j_1+\cdots+j_r \ge 1}} \beta_{\mathbf{j}} \prod_{i=1}^{r} \left(\frac{g_i^n}{g^{n_i}}\right)^{j_i} \tag{2.8}$$

where $\alpha \in \mathfrak{m}_{X,x}$ and only finitely many $\beta_{\mathbf{j}} \in \mathscr{O}_{X,x}$ are non-zero. Since $1-\alpha$ is a unit in $\mathscr{O}_{X,x}$, we may assume $\alpha = 0$. For each $1 \le i \le r$, let $s_i = \max\bigl\{j_i\colon \text{there exists } \mathbf{j} = (j_1,\ldots,j_r) \text{ such that } \beta_{\mathbf{j}} \neq 0\bigr\}$. Multiplying (2.8) throughout by $g^{\sum_{i=1}^r n_is_i} = \prod_{i=1}^r (g^{n_i})^{s_i}$, we get

$$g^{\sum_{i=1}^r n_is_i} = \sum_{\substack{\mathbf{j}=(j_1,\ldots,j_r)\\ j_1+\cdots+j_r \ge 1}} \beta_{\mathbf{j}} \prod_{i=1}^{r} \left(g_i^{nj_i} \cdot g^{n_i(s_i-j_i)}\right) = \sum_{\substack{\mathbf{j}=(j_1,\ldots,j_r)\\ j_1+\cdots+j_r \ge 1}} \left(\beta_{\mathbf{j}} \prod_{i=1}^{r} g_i^{nj_i}\right) \cdot g^{\sum_{i=1}^r n_i(s_i-j_i)}$$

which implies

$$(gt^n)^{\sum_{i=1}^r n_is_i} - \sum_{\substack{\mathbf{j}=(j_1,\ldots,j_r)\\ j_1+\cdots+j_r \ge 1}} \left(\beta_{\mathbf{j}} \prod_{i=1}^{r} (g_it^{n_i})^{nj_i}\right) \cdot (gt^n)^{\sum_{i=1}^r n_i(s_i-j_i)} = 0$$

which is an integral equation for gt^n over $(I_\bullet)_x = \mathscr{O}_{X,x}[g_it^{n_i}\colon 1 \le i \le r]$, a contradiction. Therefore, I is a proper ideal of A, so there exists a maximal ideal $\mathfrak{p}$ of A containing I. By [**Mat89**, Theorem 10.2], there exists $\nu \in \mathrm{ZR}(K(X), \mathbf{k})$ such that $R_\nu \supset A$ and $\mathfrak{m}_\nu \cap A = \mathfrak{p}$.

Consequently, $\{g_i^n/g^{n_i} : 1 \le i \le r\} \subset \mathfrak{p} \subset \mathfrak{m}_\nu$, whence for each $1 \le i \le r$, $g^{n_i}/g_i^n \notin R_\nu$. This means that for each $1 \le i \le r$,

$$\nu\left(\frac{g^{n_i}}{g_i^n}\right) < 0, \qquad \text{which implies} \qquad \frac{1}{n}\nu(g) < \frac{1}{n_i}\nu(g_i),$$

as desired. Moreover, $\mathfrak{p} \cap \mathscr{O}_{X,x} = \mathfrak{m}_{X,x}$, so $\mathfrak{m}_\nu \cap \mathscr{O}_{X,x} = \mathfrak{m}_{X,x}$. Thus, the center of ν on X is necessarily x. Note that in particular, $\nu \in \mathrm{ZR}(X/\mathbf{k})$. This proves (i). Then (ii) is a consequence of Remark 2.3.34. $\square$

Corollary 2.3.74. *Together 2.3.67 and 2.3.71 describe a one-to-one correspondence between non-zero, integrally closed Rees algebras on X and idealistic exponents over X.*

This follows from Proposition 2.3.73, and so does the next corollary (cf. Remark 2.3.68):

Corollary 2.3.75. *Let I be a non-zero ideal on X, with associated idealistic class $\gamma = \gamma_I$ over X. Then the Rees algebra $I_{\bullet\gamma}$ associated to γ is the integral closure of the Rees algebra $I^\bullet$ of I in $\mathscr{O}_X[t]$.*

We can now tie some loose ends from the end of §2.3.F. If I is an ideal of a ring A, note that the Rees algebra $I^\bullet$ of I is integrally closed in $A[t]$ if and only if I^r is integrally closed in A for all $r \ge 1$. In the same vein, $\mathrm{IC}(I^r)$ is the t^r-graded piece of $\mathrm{IC}(I^\bullet)$ for all $r \ge 1$. In particular, $\mathrm{IC}(I)$ must be an ideal of A, i.e. we get assertion (a) in 2.3.63. Assertion (b) in 2.3.63 is proven similarly. We also deduce Lemma 2.3.64 from results in this section:

Proof of Lemma 2.3.64. Let γ be a valuative ideal over X. By Lemma 2.3.72, $I_{\bullet\gamma}$ is integrally closed in $\mathscr{O}_X[t]$. Using the preceding paragraph, we get (i). For (ii), Corollary 2.3.75 says that I_γ, being the t^1-graded piece of $I_{\bullet\gamma}$, is equal to the t^1-graded piece of $\mathrm{IC}(I^\bullet)$, i.e. equal to $\mathrm{IC}(I)$. $\square$

We conclude this subsection with some miscellaneous remarks. Firstly:

2.3.76 (Functoriality with respect to dominant morphisms)**.** Let $f\colon X' \to X$ be a dominant morphism of $\mathbf{k}$-varieties. Recall from 2.3.56 that f induces a morphism $\mathrm{ZR}(f)\colon \mathrm{ZR}(X'/\mathbf{k}) \to \mathrm{ZR}(X/\mathbf{k})$ of locally ringed spaces, which induces a morphism of ordered groups $\Gamma_X \to \mathrm{ZR}(f)_*\Gamma_{X'}$, as well as a morphism of sheaves of monoids $\Gamma_{X,+} \to \mathrm{ZR}(f)_*\Gamma_{X',+}$. Tensoring with $\mathbf{Q}$, we also get a morphism of ordered groups $\Gamma_{X,\mathbf{Q}} \to \mathrm{ZR}(f)_*\Gamma_{X',\mathbf{Q}}$, and a morphism of sheaves of monoids $\Gamma_{X,\mathbf{Q}+} \to \mathrm{ZR}(f)_*\Gamma_{X',\mathbf{Q}+}$. In particular, for every valuative $\mathbf{Q}$-ideal γ over X, we can consider the pullback of γ to X', denoted $f^{-1}(\gamma)\mathscr{O}_{X'}$. If γ is an idealistic exponent $\gamma_{I_\bullet}$ where $I_\bullet$ is some non-zero Rees algebra on X, then $f^{-1}(\gamma)\mathscr{O}_{X'}$ is simply $\gamma_{f^{-1}(I_\bullet)\mathscr{O}_{X'}}$. More generally, whenever $f\colon X' \to X$ is a morphism of $\mathbf{k}$-varieties satisfying the condition that $I_\bullet\mathscr{O}_{X'}$ is non-zero, we can define the pullback $f^{-1}(\gamma)\mathscr{O}_X$ of $\gamma = \gamma_I$ (resp. $\gamma = \gamma_{I_\bullet}$) as above.

Secondly, we give a re-characterization of admissibility (Definition 2.3.27) in terms of valuative $\mathbf{Q}$-ideals:

Lemma 2.3.77 (Valuative criterion for admissibility)**.** *If X is integral and $x \in X$, then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii), where:*

(i) *$I_\bullet$ is J-admissible at x.*

(ii) *For every open affine neighbourhood* $\mathrm{Spec}(A) \subset X$ *of x, $\gamma_{I_\bullet|_A} \leq \gamma_{J|_A}$.*

(iii) *There exists an open affine neighbourhood* $\mathrm{Spec}(A) \subset X$ *of x such that $\gamma_{I_\bullet|_A} \leq \gamma_{J|_A}$.*

If $I_\bullet$ is integrally closed, (iii) $\Longrightarrow$ (i). In particular, if X is a variety, $I_\bullet$ is J-admissible implies that $\gamma_{I_\bullet} \leq \gamma_J$, with the converse holding if $I_\bullet$ is integrally closed.

Proof. There is nothing to show for (ii) $\Longrightarrow$ (iii), and (i) $\Longrightarrow$ (ii) follows from 2.3.59 and 2.3.67. If $I_\bullet$ is integrally closed, (i) is equivalent to the statement that $I_\bullet$ is $\mathrm{IC}(J)$-admissible. By 2.3.71 and Corollary 2.3.74, the latter is clearly implied by (iii). $\square$

Thirdly, we can also translate the universal property of the normalized weighted blow-up (Theorem 2.3.43) in terms of idealistic exponents:

Theorem 2.3.78. *Let γ be an idealistic exponent on an normal $\mathbf{k}$-variety X. The normalized weighted blow-up*

$$\mathrm{Bl}_{I_{\gamma,\bullet}} X = \mathscr{P}\mathrm{roj}_X(I_{\gamma,\bullet}) \xrightarrow{\pi} X$$

satisfies the following universal property. Let $f\colon T \to X$ be a dominant morphism, where T is a normal and integral scheme. Then there exists at most one lift $g\colon T \to \mathrm{Bl}_\gamma X$ of f and such a lift exists if and only if $\gamma\mathscr{O}_T$ is equal to the idealistic exponent over T arising from an effective Cartier divisor D on T. If this is the case, then $D = g^{-1}(E)$, where E is the exceptional divisor on $\mathrm{Bl}_\gamma X$.

We remark that the morphism π in the above theorem is known as the weighted blow-up of X along the idealistic exponent γ in [**ATW19**, Section 3.3]. Finally, we introduce a convenient notation that we shall adopt moving ahead, especially in Chapter 3:

Convention 2.3.79 (Convention on integrally closed Rees algebras)**.** In light of the the discussion in this subsection, we adopt the following conventions in this thesis. For $g_i \in \Gamma(X, \mathscr{O}_X)$ and $q_i = \frac{a_i}{b_i} \in \mathbf{Q}_{>0}$, the notation $I_\bullet = (g_1^{q_1}, \ldots, g_r^{q_r})$ refers to $\mathrm{IC}(\underline{I}_\bullet)$ or its corresponding idealistic exponent, where $\underline{I}_\bullet = (g_1^{a_1}, b_1) + \cdots + (g_r^{a_r}, b_r)$. It is important to note that because of the passage to integral closure, this notation is well-defined, independent of the presentation of q_i as a quotient of two positive integers $\frac{a_i}{b_i}$. The inspiration behind this notation is as follows. Interpreting $(g_1^{q_1}, \ldots, g_r^{q_r})$ as an idealistic exponent, its stalk at $\nu \in \mathrm{ZR}(X/\mathbf{k})$ can be easily remembered as:

$$(g_1^{q_1}, \ldots, g_r^{q_r})_\nu = \min\bigl\{q_1\nu(g_1), \ldots, q_r\nu(g_r)\bigr\}.$$

Slightly more generally, for non-zero ideals $I_i \subset \mathscr{O}_X$ and $q_i = \frac{a_i}{b_i} \in \mathbf{Q}_{>0}$, the notation $I_\bullet = (I_1^{q_1}, \ldots, I_r^{q_r})$ refers to $\mathrm{IC}(\underline{I}_\bullet)$ or its corresponding idealistic exponent, where $\underline{I}_\bullet = (I_1^{a_1}, b_1) + \cdots + (I_r^{a_r}, b_r) + (I^a, b)$. Note that for $I_\bullet = (I_1^{q_1}, \ldots, I_r^{q_r})$ as in the preceding sentence and $q \in \mathbf{Q}_{>0}$, then the Veronese q-translate $I_{q\bullet}$ (Definition 2.3.39) is

$$I_{q\bullet} = (I_1^{q_1 q}, \dots, I_r^{q_r q})$$

so we occasionally write $I_{q\bullet}$ as $(I_1^{q_1}, \dots, I_r^{q_r})^q$.

2.4. Weighted normal cones

2.4.A. Weighted closed embeddings. A *weighted closed embedding* $Z_\bullet \hookrightarrow X$ is a sequence of closed subschemes $(Z_1 \subset Z_2 \subset Z_3 \subset \cdots) \hookrightarrow X$ whose corresponding ideal sheaves $(\cdots \subset I_3 \subset I_2 \subset I_1 \subset \mathscr{O}_X)$ define a Rees algebra $I_\bullet$ on X. We call Z_1 the *co-support* of $Z_\bullet$.

2.4.1. Given $Z_\bullet \hookrightarrow X$ as above, we can forget the weighting and form

(i) the conormal algebra $C_{Z_1/X} := \bigoplus_{n \geq 0} (I_1)^n/(I_1)^{n+1}$, and

(ii) the conormal sheaf $\mathscr{N}^\vee_{Z_1/X} := I_1/I_1^2$,

(iii) the normal cone $C_{Z_1}X := \mathrm{Spec}_X(C_{Z_1/X})$.

For every n, the weighting defines a decreasing filtration F on $(I_1)^n/(I_1)^{n+1}$ where for $d \in \mathbf{N}$, the d^{th} filtered piece is $F^d = \big((I_1)^n \cap I_d + (I_1)^{n+1}\big)/(I_1)^{n+1}$.

We next introduce the weighted analogues of the above constructions:

Definition 2.4.2. Associated to a weighted closed embedding $Z_\bullet \hookrightarrow X$ are

(i) its *weighted conormal algebra* $C_{Z_\bullet/X} := \bigoplus_{n \geq 0} I_n/I_{n+1} = I_\bullet^{\text{ext}}/(t^{-1})$,

(ii) its *weighted conormal sheaf* $\mathscr{N}^\vee_{Z_\bullet/X} := (C_{Z_\bullet/X})_+/(C_{Z_\bullet/X})_+^2$, and

(iii) its *weighted normal cone* $C_{Z_\bullet}X := \mathrm{Spec}_X(C_{Z_\bullet/X})$.

Note that the sheaf $\mathscr{N}^\vee_{Z_\bullet/X}$ has a natural $\mathbf{N}$-grading induced from that of the weighted conormal algebra. Recall too from Definition 2.3.13 that:

(iv) the *projectivized* weighted normal cone $\mathscr{P}\mathrm{roj}_X(C_{Z_\bullet/X})$ of $Z_\bullet \hookrightarrow X$ is the *exceptional divisor* E of the weighted blow-up $\mathrm{Bl}_{Z_\bullet} X \to X$.

In the event that Z_1 is a point, (iv) can also be considered as the *projectivized weighted tangent cone* of $Z_\bullet \hookrightarrow X$.

2.4.B. Deformation to the weighted normal cone. We will now generalize the classical deformation to the normal cone [**Ful84**, §5.2] to the weighted case[2]. Recall that in the classical case, given a closed embedding $i\colon Z \hookrightarrow X$, there is a flat morphism $\pi\colon D_Z X \to \mathbf{A}^1$ such that $\pi^{-1}(0) = C_Z X$ is the normal cone and $\pi^{-1}(p) = X$ for all $p \neq 0$. More precisely, there is a closed embedding $k\colon Z \times \mathbf{A}^1 \hookrightarrow D_Z X$ such that outside $0 \in \mathbf{A}^1$, the embedding k is identified with $k|_{\mathbf{A}^1 \smallsetminus 0} = i \times \mathbb{1}\colon Z \times \mathbb{G}_m \hookrightarrow X \times \mathbb{G}_m$ and over 0, the embedding k is the zero section $k|_0 = s\colon Z \hookrightarrow C_Z X$ of the normal cone. One can also replace $\mathbf{A}^1$ with $\mathbf{P}^1$.

Let us first begin with some generalities.

Definition 2.4.3 (Weighted cones)**.** For a quasi-coherent, finitely generated graded $\mathscr{O}_X$-algebra R with $R_0 = \mathscr{O}_X$, we call $C = \mathrm{Spec}_X(R)$ the *weighted cone* of R.

If R is generated in degree 1, we simply call C the *cone* of R. Note that every twisted weighted bundle (Definition 2.2.3) is a weighted cone.

2.4.4 (Zero section)**.** Every weighted cone $C = \mathrm{Spec}_X(R)$ over X admits a *zero section* $s\colon X \hookrightarrow C$, induced by the surjection $R \twoheadrightarrow R_0 = \mathscr{O}_X$. This closed embedding $s\colon X \hookrightarrow C$ fits into a weighted closed embedding $s_\bullet\colon X_\bullet \hookrightarrow C$ with $s_1 = s$. This weighted embedding $s_\bullet\colon X_\bullet \hookrightarrow C$ is defined by the Rees algebra $I_\bullet$ on C given by

$$I_d := \bigoplus_{n \geq d} R_n \qquad \text{for } d \in \mathbf{N}.$$

Note that the weighted normal cone $C_{X_\bullet} C$ recovers C. Finally, $C \to X$ is a twisted weighted bundle, if and only if $s_\bullet$ is a regular weighted embedding.

[2]We use the notation $D_Z X \subset \overline{D_Z X}$ instead of $M_Z^\circ X \subset M_Z X$ and the special point 0 instead of ∞.

Definition 2.4.5. Let $i_\bullet \colon Z_\bullet \hookrightarrow X$ be a weighted embedding. The *deformation to the weighted normal cone* is $D_{Z_\bullet}X = \operatorname{Spec}_X(I_\bullet^{\mathrm{ext}})$.

We make the following observations:

(i) The grading equips $D_{Z_\bullet}X$ with a $\mathbb{G}_m$-action.

(ii) The inclusion $\mathscr{O}_X[t^{-1}] \subset I_\bullet^{\mathrm{ext}}$ induces a $\mathbb{G}_m$-equivariant map $p \colon D_{Z_\bullet}X \to X \times \mathbf{A}^1$.

(iii) The induced morphism $\pi \colon D_{Z_\bullet}X \to \mathbf{A}^1$ is flat since t^{-1} is regular on $I_\bullet^{\mathrm{ext}}$.

(iv) The filtration $(I_{\geq d}^{\mathrm{ext}})$ induces a weighted closed $\mathbb{G}_m$-equivariant embedding $k_\bullet \colon (Z \times \mathbf{A}^1)_\bullet \hookrightarrow D_{Z_\bullet}X$.

Note that k_1 is induced by the surjection $I_\bullet^{\mathrm{ext}} \to I_\bullet^{\mathrm{ext}}/(I_+^{\mathrm{ext}}) = (\mathscr{O}_X/I_1)[t^{-1}]$ so $(Z \times \mathbf{A}^1)_1 = Z_1 \times \mathbf{A}^1$. In general, however, $(Z \times \mathbf{A}^1)_n \neq Z_n \times \mathbf{A}^1$, e.g. $I_\bullet^{\mathrm{ext}}/(I_{\geq 2}^{\mathrm{ext}})$ is $\mathscr{O}_X/I_2$ in degrees ≤ 0 and I_1/I_2 in degree 1.

Proposition 2.4.6 (Deformation to the weighted normal cone)**.** *Consider the sequence* $(Z \times \mathbf{A}^1)_\bullet \xrightarrow{k_\bullet} D_{Z_\bullet}X \xrightarrow{p} X \times \mathbf{A}^1 \to \mathbf{A}^1$.

(i) *Outside* $0 \in \mathbf{A}^1$, *this restricts to*

$$Z_\bullet \times \mathbb{G}_m \xhookrightarrow{i_\bullet \times \mathbb{1}} X \times \mathbb{G}_m \xrightarrow{\mathbb{1}} X \times \mathbb{G}_m \to \mathbb{G}_m$$

(ii) *Over* $0 \in \mathbf{A}^1$, *this restricts to*

$$Z_\bullet \xhookrightarrow{s_\bullet \times \mathbb{1}} C_{Z_\bullet}X \to X \to \{0\}$$

where $s_\bullet$ *is the weighted zero section.*

Proof. Outside 0, we are inverting t^{-1}. This gives

$$I_\bullet^{\mathrm{ext}}[t] = \mathscr{O}_X[t^{\pm 1}] \qquad \text{and} \qquad (I_{\geq d}^{\mathrm{ext}}) = I_d[t^{\pm 1}].$$

Over 0, we are taking the quotient with t^{-1} which gives

$$I_\bullet^{\text{ext}}/(t^{-1}) = \bigoplus_{n\geq 0} I_n/I_{n+1} = C_{Z_\bullet/X} \qquad \text{and} \qquad (I_{\geq d}^{\text{ext}}) = \bigoplus_{n\geq d} I_n/I_{n+1}. \qquad \square$$

2.4.C. Compactified version. There is also a compactified version of the deformation to the normal cone which is projective over $X \times \mathbf{A}^1$ (or $X \times \mathbf{P}^1$). This is constructed as $\overline{D_Z X} = \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbf{A}^1)$. Similarly, given a weighted embedding $i_\bullet \colon Z_\bullet \to X$, we let

$$\overline{D_{Z_\bullet} X} = \mathrm{Bl}_{Z_\bullet\times\{0\}}(X \times \mathbf{A}^1)$$

where $Z_n \times \{0\} \hookrightarrow X \times \mathbf{A}^1$ is the natural map. To describe $\overline{D_{Z_\bullet} X}$, we first need to introduce projective completions of weighted cones.

Definition 2.4.7 (Projective completion)**.** Fix a quasi-coherent, finitely generated graded $\mathscr{O}_X$-algebra R with $R_0 = \mathscr{O}_X$, with corresponding weighted cone C. Let y be an indeterminate (with degree 1), and let $R[y]$ denote the quasi-coherent graded $\mathscr{O}_X$-algebra whose degree d piece is:

$$R[y]_d := R_d \oplus R_{d-1}\cdot y \oplus \cdots \oplus R_1 \cdot y^{d-1} \oplus R_0 \cdot y^d.$$

Then the *projective completion* of C, denoted $\mathscr{P}(C \oplus \mathbb{1})$, is defined as

$$\mathscr{P}(C \oplus \mathbb{1}) := \mathscr{P}\mathrm{roj}_X(R[y]).$$

The *projectivized weighted cone* $\mathscr{P}(C) := \mathscr{P}\mathrm{roj}_X(R)$ sits inside $\mathscr{P}\mathrm{roj}_X(R[y])$ as the "hyperplane at infinity" cut out by $y = 0$, whose complement is the y-chart $D_+(y) := [\mathrm{Spec}_X(R[y]_y) \,/\, \mathbb{G}_m] = \mathrm{Spec}_X(R)$, which coincides with the cone C, cf. §2.1.C.

Proposition 2.4.8. *Let $Z_\bullet \hookrightarrow X$ be a weighted embedding, and let π denote the composition*

$$\overline{D_{Z_\bullet} X} := \mathrm{Bl}_{Z_\bullet\times\{0\}}(X \times \mathbf{A}^1) \xrightarrow{p} X \times \mathbf{A}^1 \xrightarrow{\mathrm{pr}_2} \mathbf{A}^1.$$

Then:

(i) *The exceptional divisor of* $\mathrm{Bl}_{Z_\bullet \times \{0\}}(X \times \mathbf{A}^1)$ *is canonically identified with the projective completion* $\mathscr{P}(C_{Z_\bullet}X \oplus \mathbb{1})$ *of the weighted normal cone* $C_{Z_\bullet}X$*.*

(ii) $\pi^{-1}(0) = p^{-1}(X \times \{0\})$ *is canonically identified with*

$$\mathrm{Bl}_{Z_\bullet} X \cup_E \mathscr{P}(C_{Z_\bullet}X \oplus \mathbb{1})$$

where $E = \mathscr{P}(C_{Z_\bullet}X)$ *is the exceptional divisor of* $\mathrm{Bl}_{Z_\bullet} X$*.*

(iii) *The deformation to the weighted normal cone* $D_{Z_\bullet}X$ *is naturally identified as the open substack* $\overline{D_{Z_\bullet}X} \smallsetminus \mathrm{Bl}_{Z_\bullet} X$*.*

Proof. Let $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[u])$. Let $J_\bullet$ be the Rees algebra of the embedding $Z_\bullet \times \{0\} \hookrightarrow X \times \mathbf{A}^1$. Then $J_\bullet^{\mathrm{ext}} = I_\bullet^{\mathrm{ext}}[u, U]/(t^{-1}U - u)$. In particular, $C_{Z_\bullet \times \{0\}/X \times \mathbf{A}^1} = J_\bullet^{\mathrm{ext}}/(t^{-1}) = C_{Z_\bullet/X}[U]$ and the part (i) follows.

For part (ii), the fiber $\pi^{-1}(0)$ corresponds to $t^{-1}U = u = 0$ and thus splits up in two components. The first, $t^{-1} = 0$, is the exceptional divisor $\mathscr{P}(C_{Z_\bullet}X \oplus \mathbb{1})$, the second, $U = 0$, is $\mathrm{Bl}_{Z_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}})$ and their intersection $t^{-1} = U = 0$ is the exceptional divisor E.

For part (iii), the open set in question is

$$D_+(U) = \left[\mathrm{Spec}_X(I_\bullet^{\mathrm{ext}}[U, U^{-1}]) \,/\, \mathbb{G}_m\right] = \mathrm{Spec}_X(I_\bullet^{\mathrm{ext}}) = D_{Z_\bullet}X. \qquad \square$$

2.5. Weighted blow-ups along regular and smooth centers

Recall that if $Z \hookrightarrow X$ is a regular embedding, then it is a quasi-regular embedding [**Stacks**, 00LN], i.e. the conormal sheaf $\mathscr{N}_{Z/X}^\vee = I/I^2$ is locally free and the canonical surjection $\mathrm{Sym}_{\mathscr{O}_Z}(\mathscr{N}_{Z/X}^\vee) = \mathrm{Sym}_{\mathscr{O}_Z}(I/I^2) \twoheadrightarrow \bigoplus_{n \geq 0} I^n/I^{n+1} = C_{Z/X}$ is an isomorphism. In this section we first introduce and discuss the notions of *quasi-regular weighted embeddings* and *regular weighted embeddings*. However, we forewarn the reader that even if $Z_\bullet \hookrightarrow X$ is a quasi-regular weighted embedding, there are only local isomorphisms between $\mathrm{Sym}_{\mathscr{O}_{Z_1}}(\mathscr{N}^{\vee}_{Z_\bullet/X})$ and $C_{Z_\bullet/X}$, which in general are not compatible with each other. The former is a weighted (or graded) vector bundle (2.2.9) while the latter in general is a *twisted* weighted vector bundle (Proposition 2.5.4).

In the noetherian case, a weighted embedding is regular, if and only if it is quasi-regular (Corollary 2.5.11). In this case there is a simple description of the extended Rees algebra $I_\bullet^{\mathrm{ext}}$ (Proposition 2.5.9) and the charts of the weighted blow-up (Corollary 2.5.12).

2.5.A. Quasi-regular weighted embeddings. Let $x_1, x_2, \dots, x_m$ be global sections of $\mathscr{O}_X$ and $d_1, d_2, \dots, d_m$ be positive integers such that $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$. We consider the graded polynomial ring $(\mathscr{O}_X/I_1)[X_1, X_2, \dots, X_m]$ where X_i has degree d_i. We have a natural graded homomorphism

$$\alpha\colon (\mathscr{O}_X/I_1)[X_1, X_2, \dots, X_m] \to C_{Z_\bullet/X} = \bigoplus_{n\geq 0} I_n/I_{n+1}$$

taking X_i to $x_i \in I_{d_i}/I_{d_i+1}$. This map is evidently surjective.

Definition 2.5.1. We say that $(x_1, d_1), \dots, (x_m, d_m)$ is a *quasi-regular* sequence if α is bijective.

When $d_1 = d_2 = \cdots = d_m = 1$, then $I_n = I_1^n$ and this is the usual notion of a quasi-regular sequence $x_1, x_2, \dots, x_m$.

Proposition 2.5.2. *The following are equivalent:*

(i) $x_1, x_2, \dots, x_m$ *is quasi-regular.*

(ii) $(x_1, d_1), \dots, (x_m, d_m)$ *is quasi-regular for any* $d_1, d_2, \dots, d_m \in \mathbf{Z}_{>0}$.

Proof. We prove that if $(x_1, d_1), \dots, (x_m, d_m)$ is quasi-regular for some $d_1, d_2, \dots, d_m \in \mathbf{Z}_{>0}$, then so is $(x_1, e_1), \dots, (x_m, e_m)$ for every sequence $e_1, e_2, \dots, e_m \in \mathbf{Z}_{>0}$. Let $J_\bullet = (x_1, e_1) +$

$\cdots + (x_m, e_m)$ and note that $I_1 = J_1$. For a multi-index $\alpha \in \mathbf{N}^m$, let $|\alpha|_d = d_1\alpha_1 + \cdots + d_m\alpha_m$ and $|\alpha|_e = e_1\alpha_1 + \cdots + e_m\alpha_m$.

By quasi-regularity of $(x_1, d_1), \ldots, (x_m, d_m)$, we have that I_a/I_{a+1} is a free $\mathscr{O}_X/I_1$-module with generators x^α such that $|\alpha|_d = a$. We have a filtration

$$I_a/I_{a+1} = I_a \cap J_1/I_{a+1} \supset (I_a \cap J_2 + I_{a+1})/I_{a+1} \supset \cdots$$

By definition, we have the inclusion

$$\bigoplus_{\substack{|\alpha|_d = a \\ |\alpha|_e \geq b}} (\mathscr{O}_X/I_1)x^\alpha \subset (I_a \cap J_b + I_{a+1})/I_{a+1}. \tag{2.9}$$

Claim. The inclusion (2.9) is an equality.

Proof of Claim. To see this, let $z \in I_a \cap J_b$. Then by definition of $J_\bullet$, we have that $z = \sum_{|\alpha|_e \geq b} z_\alpha x^\alpha$ for some (non-unique) $z_\alpha \in \mathscr{O}_X$. Let $c = \min\{|\alpha|_d \ : \ z_\alpha \neq 0\}$. If $c \geq a$, then $z \bmod I_{a+1}$ is in the left hand side and we are done.

If $c < a$ and there exists α with $|\alpha|_d = c$ and $z_\alpha \notin I_1$, then it follows by quasi-regularity of $(x_1, d_1), \ldots, (x_m, d_m)$ that $z \in I_c \smallsetminus I_{c+1}$. This contradicts that $z \in I_a$.

Finally, suppose that $c < a$ and $z_\alpha \in I_1 = (x_1, x_2, \ldots, x_m)$ for all α such that $|\alpha|_d = c$. Then z can be written as a polynomial $\sum_\alpha \tilde{z}_\alpha x^\alpha$ where the non-zero terms have $|\alpha|_d > c$ and $|\alpha|_e \geq b$. Repeating the argument with this expression of z increases c and eventually $c \geq a$ and we get the desired equality. $\triangle$

Next, we show that the equality (2.9) implies that $(x_1, e_1), \ldots, (x_m, e_m)$ is quasi-regular. Indeed, let $f(X) = \sum_{|\alpha|_e = b} f_\alpha X^\alpha \in \mathscr{O}_X[X_1, X_2, \ldots, X_m]$ be a polynomial that is homogeneous of degree b with respect to the e_i-grading, and such that the image of f in $(\mathscr{O}_X/J_1)[X_1, X_2, \ldots, X_m]$ is non-zero. We want to show that $f(x) \notin J_{b+1}$. Set $a = \min\{|\alpha|_d \colon f_\alpha \notin I_1\}$, so that by the quasi-regularity of $(x_1, d_1), \ldots, (x_m, d_m)$, $f(x) \in I_a \smallsetminus I_{a+1}$. Then the image of $f(x)$ in

$(I_a \cap J_b + I_{a+1})/I_{a+1}$ is non-zero, and is not contained in $(I_a \cap J_{b+1} + I_{a+1})/I_{a+1}$, by the claim and the hypothesis that $f(X)$ is homogeneous of degree b. In other words, $f(x) \notin J_{b+1}$, as desired. $\square$

Definition 2.5.3. A weighted closed embedding $Z_\bullet \hookrightarrow X$ is *quasi-regular* if at every point $p \in |Z_1|$, there exists a smooth neighborhood $U \to X$ of p, a quasi-regular sequence $x_1, x_2, \ldots, x_m$ on U, and $d_1, d_2, \ldots, d_m \in \mathbf{Z}_{>0}$, such that $I_\bullet|_U = (x_1, d_1) + \cdots + (x_m, d_m)$.

Proposition 2.5.4. *A weighted closed embedding $Z_\bullet \hookrightarrow X$ is quasi-regular if and only if the weighted normal cone $C_{Z_\bullet/X} \to Z_1$ is a twisted weighted vector bundle.*

Proof. If $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$ for a quasi-regular sequence, then we have seen that $C_{Z_\bullet/X}$ is a graded polynomial ring. Conversely, if $C_{Z_\bullet/X} \to Z_1$ is a twisted weighted vector bundle, then locally on X we have that $C_{Z_\bullet/X}$ is a graded polynomial ring $\mathscr{O}_{Z_1}[X_1, X_2, \ldots, X_m]$. If we take any preimages $x_i \in I_{d_i}$ of $X_i \in I_{d_i}/I_{d_i+1}$, then $(x_1, d_1), \ldots, (x_m, d_m)$ is quasi-regular. $\square$

Remark 2.5.5. Recall that the weighted conormal sheaf $\mathscr{N}^\vee_{Z_\bullet/X}$ is a graded vector bundle whereas the *unweighted* conormal sheaf $\mathscr{N}^\vee_{Z_1/X}$ is equipped with a filtration (§2.4). There is a canonical surjection $\mathscr{N}^\vee_{Z_\bullet/X} \to \mathrm{Gr}_F(\mathscr{N}^\vee_{Z_1/X})$ but it is not an isomorphism of $\mathscr{O}_{Z_1}$-modules in general, see Example 2.5.7. For $d \geq 1$, the d^{th} graded pieces are as follows:

$$\begin{aligned}
(\mathscr{N}^\vee_{Z_\bullet/X})_d &\simeq \frac{I_d/I_{d+1}}{(I_d/I_{d+1}) \cap (C_{Z_\bullet/X})^2_+} \\
\mathrm{Gr}^d_F(\mathscr{N}^\vee_{Z_1/X}) &\simeq \frac{I_d}{(I_1^2 + I_{d+1}) \cap I_d} \simeq \frac{I_d/I_{d+1}}{\big((I_1^2 \cap I_d) + I_{d+1}\big)/I_{d+1}}.
\end{aligned}$$

Remark 2.5.6. If $Z_\bullet \hookrightarrow X$ is quasi-regular, then the canonical surjection $\mathscr{N}^\vee_{Z_\bullet/X} \twoheadrightarrow \mathrm{Gr}_F(\mathscr{N}^\vee_{Z_1/X})$ in Remark 2.5.5 is an isomorphism. Indeed, we may assume we are in the local situation where $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$ for a quasi-regular sequence $x_1, x_2, \ldots, x_m \in \mathscr{O}_X$, and positive integers $d_1, d_2, \ldots, d_m$. By Definition 2.5.1, $(\mathscr{N}^\vee_{Z_\bullet/X})_d$ is only non-zero (in which case it is locally free as an $\mathscr{O}_{Z_1}$-module) in degrees $d \in \{d_1, d_2, \ldots, d_m\}$, and the same holds for $\mathrm{Gr}^d_F(\mathscr{N}^\vee_{Z_1/X})$. It remains to note that the canonical surjection carries the free generators x_i in each non-zero degree of $\mathscr{N}^\vee_{Z_\bullet/X}$ to free generators in the corresponding degree of $\mathrm{Gr}^d_F(\mathscr{N}^\vee_{Z_1/X})$.

Example 2.5.7. For a counterexample, consider on $X = \mathbf{A}^1_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[x])$ the Rees algebra $I_\bullet = (x, 1) + (x^2, 3)$. Then $(\mathscr{N}^\vee_{Z_\bullet/X})_d \neq 0$ if and only if $d \in \{1, 3\}$, where $(\mathscr{N}^\vee_{Z_\bullet/X})_1 = (x)/(x^2)$ and $(\mathscr{N}^\vee_{Z_\bullet/X})_3 = (x^2)/(x^3)$. On the other hand, $\mathrm{Gr}^1_F(\mathscr{N}^\vee_{Z_1/X}) = (x)/(x^2)$, and $\mathrm{Gr}^d_F(\mathscr{N}^\vee_{Z_1/X}) = 0$ for $d \neq 1$.

2.5.B. Regular weighted embeddings. As before, suppose first that we have global sections $x_1, x_2, \ldots, x_m$ of $\mathscr{O}_X$ and positive integers $d_1, d_2, \ldots, d_m$ such that $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$. We have the graded polynomial ring

$$\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m]$$

where we let $\deg(t^{-1}) = -1$ and $\deg(X_i) = d_i$. There is a natural map

$$\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m] \to I^{\mathrm{ext}}_\bullet \subset \mathscr{O}_X[t^{\pm 1}]$$

of graded $\mathscr{O}_X$-algebras that takes each X_i to $x_i t^{d_i}$ and t^{-1} to t^{-1}. We note that $t^{-d_i}X_i - x_i$ is in the kernel of β, so that we obtain a map

$$B := \mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m]/(t^{-d_i}X_i - x_i \colon 1 \leq i \leq m) \xrightarrow{\beta} I^{\mathrm{ext}}_\bullet.$$

Note that β is surjective.

The next lemma generalizes [**Stacks**, 0G8S].

Lemma 2.5.8. *The kernel of β equals the kernel of $B \to B_{t^{-1}}$, i.e.* $\ker \beta = \bigcup_{n \geq 1} \operatorname{Ann}_B(t^{-n})$. *In particular, β is bijective if and only if $t^{-1} \in B$ is a non-zero divisor.*

PROOF. $B \xrightarrow{\beta} I_\bullet^{\mathrm{ext}} \hookrightarrow \mathscr{O}_X[t^{\pm 1}]$ has the same kernel as β, and factors through the localization $B \to B_{t^{-1}}$. The result follows since $B_{t^{-1}} \to \mathscr{O}_X[t^{\pm 1}]$ is an isomorphism. $\square$

The following proposition generalizes [**Stacks**, 0BIQ]. For the definition of H_1-regular sequences, see [**Stacks**, 062D]. For noetherian stacks, H_1–regular is equivalent to regular.

Proposition 2.5.9. *If $x_1, x_2, \ldots, x_m$ is an H_1-regular sequence, then β is bijective, so that*

$$\mathrm{Bl}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}}) \xrightarrow{\;\simeq\;} \mathscr{P}\mathrm{roj}_X \left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m]}{(t^{-d_i} X_i - x_i \colon 1 \leq i \leq m)} \right).$$

PROOF. If $x_1, x_2, \ldots, x_m \in \mathscr{O}_X$ is an H_1-regular sequence, the sequence $x_1, x_2, \ldots, x_m, t^{-1}$ in $\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m]$ is also H_1-regular [**Stacks**, 0668]. At the same time, the sequence $t^{-d_1} X_1 - x_1, t^{-d_2} X_2 - x_2, \ldots, t^{-d_m} X_m - x_m, t^{-1}$ generates the same ideal, and hence is also H_1-regular [**Stacks**, 066A]. Thus, $t^{-1} \in B$ is a non-zero divisor [**Stacks**, 068L]. $\square$

Question 2.5.10. *Is $x_1, x_2, \ldots, x_m$ always an H_1-regular sequence if β is bijective?*

The following corollary shows that the answer is 'yes' in the noetherian case.

Corollary 2.5.11. *Let $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$ as before. Consider the conditions*

(i) *$x_1, x_2, \ldots, x_m$ is an H_1-regular sequence.*

(ii) *β is bijective.*

(iii) *$x_1, x_2, \ldots, x_m$ is a quasi-regular sequence.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). If X is locally noetherian, then the three conditions are equivalent.

Proof. We have seen that (i) $\implies$ (ii) and if X is locally noetherian then (iii) $\implies$ (i). It thus remains to prove (ii) $\implies$ (iii). But if β is bijective, then

$$\begin{aligned} C_{Z_\bullet/X} = I_\bullet^{\text{ext}}/(t^{-1}) &= \mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m]/(t^{-1}, x_1, x_2, \ldots, x_m) \\ &= \mathscr{O}_{Z_1}[X_1, X_2, \ldots, X_m] \end{aligned}$$

so the sequence $x_1, x_2, \ldots, x_m$ is quasi-regular. $\square$

Note $\mathrm{Bl}_{I_\bullet} X$ is covered by the charts $D_+(x_i \cdot t^{d_i})$ for $1 \leq i \leq m$. As a consequence of Proposition 2.5.9 and Lemma 2.1.2 (with $A = \mathbf{Z}$, $a = d_i$, and $r = x_i \cdot t^{d_i}$), we can explicate these charts:

Corollary 2.5.12 (Charts for blow-ups along regular weighted embeddings). *If $x_1, x_2, \ldots, x_m$ is an H_1-regular sequence, then for each $1 \leq i \leq m$, the chart $D_+(x_i \cdot t^{d_i})$ of $\mathrm{Bl}_{I_\bullet} X$ is:*

$$\begin{aligned} &\left[\mathrm{Spec}_X \left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m][X_i^{-1}]}{(t^{-d_j} X_j - x_j \colon 1 \leq j \leq m)}\right) \bigm/ \mathbb{G}_m\right] \\ &= \left[\mathrm{Spec}_X \left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots \widehat{X}_i, \ldots, X_m]}{(t^{-d_i} - x_i) + (t^{-d_j} X_j - x_j \colon 1 \leq j \leq m,\ j \neq i)}\right) \bigm/ \boldsymbol{\mu}_{d_i}\right] \end{aligned}$$

where $\widehat{X}_i$ means X_i omitted, and the action of $\boldsymbol{\mu}_{d_i}$ corresponds to the weights $\mathrm{wt}_{\mathbf{Z}/d_i}(X_j) = d_j$ for $j \neq i$, $\mathrm{wt}_{\mathbf{Z}/d_i}(t^{-1}) = -1$.

Slightly more generally, let A be a finitely generated abelian group, $D(A)$ be the corresponding diagonalizable algebraic group, and let $D(A)$ act on X. Assume that $x_1, x_2, \ldots, x_m \in \mathscr{O}_X$ is an A-homogeneous H_1-regular sequence, with weights $\mathrm{wt}_A(x_i) = a_i$ for $1 \leq i \leq m$. Let $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$. Then $I_\bullet$ is an A-graded Rees algebra on X, so $I_\bullet$ descends to a Rees algebra $\mathscr{I}_\bullet$ on $[X \mathbin{/} D(A)]$.

Consider the diagram:

$$\begin{tikzcd}
D_+(x_i \cdot t^{d_i}) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Bl}_{I_\bullet} X \arrow[r] \arrow[d, two heads] & X \arrow[d, two heads] \\
{[D_+(x_i \cdot t^{d_i}) \ / \ D(A)]} \arrow[r, hook] & {[\mathrm{Bl}_{I_\bullet} X \ / \ D(A)]} \arrow[r] & {[X \ / \ D(A)]}
\end{tikzcd}$$

where the right morphism in the bottom row is also the weighted blow-up $\mathrm{Bl}_{\mathscr{I}_\bullet}[X \ / \ D(A)] \to [X \ / \ D(A)]$. The next corollary obtains a description for $[D_+(x_i \cdot t^{d_i}) \ / \ D(A)]$ which is analogous to that for $D_+(x_i \cdot t^{d_i})$ in the previous corollary:

Corollary 2.5.13 (Charts for blow-ups along regular weighted embeddings, with respect to a $D(A)$-action)**.** *For $1 \leq i \leq m$, the chart $[D_+(x_i \cdot t^{d_i}) \ / \ D(A)]$ of $\mathrm{Bl}_{\mathscr{I}_\bullet}[X \ / \ D(A)]$ is:*

$$\left[\mathrm{Spec}_X\left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots \widehat{X}_i, \ldots, X_m]}{(t^{-d_i} - x_i) + (t^{-d_j}X_j - x_j \colon 1 \leq j \leq m, \ j \neq i)}\right) \ / \ D(A')\right]$$

where $A' = A\langle -\frac{a_i}{d_i}\rangle := (A \oplus \mathbf{Z})/\langle (a_i, d_i)\rangle$ and the action of $D(A')$ corresponds to the weights $\mathrm{wt}_{A'}(X_j) = a_j - d_j\frac{a_i}{d_i}$ for $j \neq i$, $\mathrm{wt}_{A'}(t^{-1}) = \frac{a_i}{d_i}$.

Proof. Note that the chart is, by Proposition 2.5.9:

$$\left[\mathrm{Spec}_X\left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m][X_i^{-1}]}{(t^{-d_j}X_j - x_j \colon 1 \leq j \leq m)}\right) \ / \ D(A) \times \mathbb{G}_m\right]$$

where the action of $D(A) \times \mathbb{G}_m$ is expressed via the weights $\mathrm{wt}_{A \oplus \mathbf{Z}}(X_j) = (a_j, d_j)$ for $1 \leq j \leq m$, and $\mathrm{wt}_{A \oplus \mathbf{Z}}(t^{-1}) = (0, -1)$. The corollary thus follows from Lemma 2.1.2, with A there replaced by $A \oplus \mathbf{Z}$ here, a there replaced by (a_i, d_i) here, and $r = X_i$. $\square$

Motivated by the results above, we conclude the subsection with:

Definition 2.5.14 (Regular weighted closed embeddings)**.** A weighted closed embedding $Z_\bullet \hookrightarrow X$ is *regular* if at every point $p \in |Z_1|$, there exists a smooth neighborhood $U \to X$ of

p, a regular sequence $x_1, x_2, \ldots, x_m$ on U, and positive integers $d_1, d_2, \ldots, d_m$, so that $I_\bullet|_U = (x_1, d_1) + \cdots + (x_m, d_m)$.

2.5.C. Weighted blow-ups along regular centers. In this subsection we assume that X is noetherian and regular, and we let $Z_\bullet \hookrightarrow X$ be a weighted closed embedding.

Definition 2.5.15 (Regular centers)**.** A weighted closed embedding $Z_\bullet \hookrightarrow X$ is called a *regular center*, if its co-support Z_1 is regular and $Z_\bullet \hookrightarrow X$ is a regular weighted embedding.

Recall that the latter condition is equivalent to the weighted normal cone $C_{Z_\bullet/X} \to Z_1$ being a twisted weighted vector bundle (Proposition 2.5.4). Note that while Z_1 being regular ensures that $Z_1 \hookrightarrow X$ is a regular embedding, it does not imply that $Z_\bullet \hookrightarrow X$ is a regular weighted embedding (Example 2.5.7).

The above definition means that smooth locally around each point $p \in |Z_1|$, $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$, where $x_1, x_2, \ldots, x_m$ is a regular sequence that can be extended to a regular system of parameters at p, and $d_1, d_2, \ldots, d_m$ are positive integers. Note that Definition 2.5.15 coincides with the notion of "centers" in [**ATW19**, Section 2.4].

Corollary 2.5.16. *If $Z_\bullet \hookrightarrow X$ is a regular center, then the deformation to the weighted normal cone $D_{Z_\bullet}X = \operatorname{Spec}_X(I_\bullet^{\mathrm{ext}})$ is regular. In particular, the weighted blow-up $\mathrm{Bl}_{Z_\bullet} X \subset \left[D_{Z_\bullet}X \,/\, \mathbb{G}_m\right]$ is regular.*

Proof. By Proposition 2.5.4, the weighted normal cone $C_{Z_\bullet}X$ is a twisted weighted vector bundle over Z_1, hence regular if Z_1 is regular. Since $C_{Z_\bullet}X$ is the Cartier divisor $t^{-1} = 0$ in $D_{Z_\bullet}X$ and its complement is $X \times \mathbb{G}_m$, it follows that $D_{Z_\bullet}X$ is regular. $\square$

2.5.D. Weighted blow-ups along smooth centers.

Definition 2.5.17 (Smooth centers)**.** Let $X \to S$ be a smooth morphism of algebraic stacks. A weighted closed embedding $Z_\bullet \hookrightarrow X$ is called a *smooth center* if $Z_1 \to S$ is smooth and $Z_\bullet \hookrightarrow X$ is a regular weighted embedding.

Let $X \to S$ be a smooth morphism which is relatively Deligne–Mumford. We say that a sequence $x_1, x_2, \ldots, x_r \in \Gamma(X, \mathscr{O}_X)$ is a *coordinate system* if $dx_1, dx_2, \ldots, dx_r$ gives a basis for $\Omega_{X/S}$. Equivalently, the sections induce an étale morphism $X \to \mathbf{A}^n_S$. Then smooth-locally on X, a coordinate system always exists and given a smooth closed substack Z, a subset of the coordinate system can be chosen to generate I_Z.

Thus, $Z_\bullet \hookrightarrow X$ is a smooth center if and only if we can find, smooth-locally on X, a coordinate system $x_1, x_2, \ldots, x_r$ such that $I_\bullet = (x_1, d_1) + (x_2, d_2) + \cdots + (x_m, d_m)$ for $m \leq r$. For the remainder of this subsection, we focus on the local case where this holds on X itself. Adopting the notation in Corollary 2.5.12, Corollary 2.5.16 can be made more precise as follows:

Proposition 2.5.18. *Let $Z_\bullet \hookrightarrow X$ be a smooth center. On $D_{Z_\bullet} X$, the sequence*

$$t^{-1}, X_1, X_2, \ldots, X_m, x_{m+1}, \ldots, x_r$$

is a $\mathbf{Z}$-homogeneous coordinate system. In particular, it is a $\mathbf{Z}$-homogeneous coordinate system on the total space of the $\mathbb{G}_m$-torsor on $\mathrm{Bl}_{Z_\bullet} X$. On the chart $D_+(x_i \cdot t^{d_i})$ of $\mathrm{Bl}_{I_\bullet} X$, we also have a $\boldsymbol{\mu}_{d_i}$-torsor as in Corollary 2.5.12 and on its total space, we have the $\mathbf{Z}/d_i\mathbf{Z}$-homogeneous coordinate system

$$t^{-1}, X_1, X_2, \ldots, \widehat{X}_i, \ldots, X_m, x_{m+1}, \ldots, x_r$$

In both cases, $\mathrm{wt}_{\mathbf{Z}/d_i}(t^{-1}) = -1$ *and* $\mathrm{wt}_{\mathbf{Z}/d_i}(X_j) = d_j$.

Proof. Since $X \to \mathbf{A}^r_S$ is étale, we can assume that $X = \operatorname{Spec} \mathscr{O}_S[x_1, x_2, \ldots, x_r]$. Then $D_{Z_\bullet} X = \operatorname{Spec} \mathscr{O}_S[t^{-1}, X_1, X_2, \ldots, X_m, x_{m+1}, \ldots, x_r] = \mathbf{A}^{r+1}_S$ by Proposition 2.5.9 and the total space of the $\boldsymbol{\mu}_{d_i}$-torsor is the closed subscheme defined by $X_i = 1$, hence isomorphic to $\mathbf{A}^r_S$. The proposition follows. $\square$

Slightly more generally, adopt the hypotheses and notations of Corollary 2.5.13, and moreover assume that $x_1, x_2 \dots, x_m, x_{m+1} \dots, x_r$ is an A-homogeneous coordinate system. Equivalently, the induced map $X \to [\mathbf{A}^r/D(A)]$ is étale. Then it is immediate that:

Corollary 2.5.19. *Let $Z_\bullet \hookrightarrow X$ be a smooth center, and let $\mathscr{I}_\bullet$ be as in Corollary 2.5.13. On the chart $D_+(x_i \cdot t^{d_i})$ of $\mathrm{Bl}_{\mathscr{I}_\bullet}[X \,/\, D(A)]$, the sequence*

$$t^{-1}, X_1, X_2, \dots, \widehat{X}_i, \dots, X_m, x_{m+1}, \dots, x_r$$

is an A'-homogeneous coordinate system on the total space of the $D(A')$-torsor with $\mathrm{wt}_{A'}(t^{-1}) = \frac{a_i}{d_i}$ and $\mathrm{wt}_{A'}(X_j) = a_j - d_j \frac{a_i}{d_i}$.

The homogeneous coordinate systems above can also be interpreted as sequences of sections of line bundles on the stack $D_+(x_i \cdot t^{d_i})$ itself. Similarly, in the A-graded case $x_i \cdot t^{d_i}$ is also only a section of a line bundle.

2.6. Toric weighted blow-ups

In this section, we consider toric varieties and stacks over $S = \mathrm{Spec}(\mathbf{Z})$. Let N be a lattice and let Σ be a fan in $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$. Then associated to Σ is the toric variety X_Σ.

2.6.1 (Stacky fans)**.** More generally, let (Σ, β) be a stacky fan [**GS15**, Definition 2.4]. That is, Σ is a fan in $L_{\mathbf{R}}$ where L is a lattice, and $\beta \colon L \to N$ is a homomorphism of lattices with finite cokernel. Associated to (Σ, β) is the toric stack $X_{\Sigma,\beta} := [X_\Sigma/G_\beta]$, where G_β is the kernel of the homomorphism of tori $T_\beta \colon T_L = \mathrm{Hom}(L^\vee, \mathbb{G}_m) \to \mathrm{Hom}(N^\vee, \mathbb{G}_m) = T_N$ induced by β, and G_β acts on X_Σ via the inclusion of tori $G_\beta \subset T_L \curvearrowright X_\Sigma$. Every fan Σ in $N_{\mathbf{R}}$ can be considered as a stacky fan, by taking $L = N$ and β to be the identity $\mathbb{1}$ on N, in which case $X_\Sigma = X_{\Sigma,\mathbb{1}}$. In addition, if $\Sigma[1]$ denotes the set of rays of Σ, we may also associate to Σ the stacky fan $(\widehat{\Sigma}, \beta)$, where:

(i) $\beta\colon \mathbf{Z}^{\Sigma[1]} \to N$ sends each standard basis vector $\mathbf{e}_\rho$ of $\mathbf{Z}^{\Sigma[1]}$ (for $\rho \in \Sigma[1]$) to the first lattice point on ρ, and

(ii) $\widehat{\Sigma}$ is the fan in $\mathbf{Z}^{\Sigma[1]}$ generated by $\{\widehat{\sigma}\colon \sigma \in \Sigma\}$, where for every cone σ in Σ, $\widehat{\sigma} = \sum_{\rho\in\sigma[1]} \mathbf{e}_\rho$.

Since $\widehat{\Sigma}$ is smooth, the associated toric stack $\mathscr{X}_\Sigma := X_{\widehat{\Sigma},\beta}$ is smooth, and moreover has good moduli space X_Σ. If Σ is simplicial, $\mathscr{X}_\Sigma$ has finite stabilizers and coarse space X_Σ. We refer to this as Cox's quotient construction of toric varieties [**CLS11**, Section 5.1]. For more details, see [**BCS05**, **FMN10**, **GS15**] and §4.1.A.

2.6.2 (Coarse toric weighted blow-up)**.** Let N be a lattice, and let Σ be a smooth[3] fan in $N_{\mathbf{R}} = N \otimes \mathbf{R}$. Let γ be a maximal cone in Σ with rays $\rho_1, \rho_2, \ldots, \rho_n$ generated by primitive lattice points $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let $\mathbf{v}$ be a lattice point contained in $\mathrm{relint}(\gamma)$. Then it can be uniquely expressed as $\mathbf{v} = \sum_{i=1}^n d_i \mathbf{u}_i$ for some $d_i \in \mathbf{N}_{>0}$.

Consider the subdivision $\Sigma^*(\mathbf{v})$ of Σ at $\mathbf{v}$, i.e. this is the set of all cones in $\Sigma \smallsetminus \{\gamma\}$, as well as the cones generated by subsets of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{v}\}$ not containing $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. Letting $X_{\Sigma^*(\mathbf{v})}$ be the corresponding toric variety, the identity map on N is compatible with the fans $\Sigma^*(\mathbf{v})$ and Σ, and thus induces a toric morphism

$$X_{\Sigma^*(\mathbf{v})} \longrightarrow X_\Sigma$$

which is the *coarse space* of the weighted blow-up of X_Σ along $I_\bullet := (I_{D_1}, d_1) + \cdots + (I_{D_n}, d_n)$ (cf. 2.6.3), where each I_{D_i} is the ideal of the torus-invariant divisor $D_i \subset X_\Sigma$ corresponding to the ray ρ_i. The new ray $\rho_E = \mathbf{R}_{\geq 0}\mathbf{v}$ in $\Sigma^*(\mathbf{v})$ corresponds to the exceptional divisor E.

[3]More generally, we only require every cone containing σ to be smooth: this ensures that the D_i are Cartier divisors in a neighborhood of $D_1 \cap \cdots \cap D_n$. Alternatively, if Σ is not smooth, we could work with Cox's construction, namely the stacky fan $(\widehat{\Sigma}, \beta)$ as in 2.6.1 and the associated smooth toric stack $X_{\widehat{\Sigma},\beta}$, cf. Remark 2.6.4.

2.6.3 (Toric weighted blow-up)**.** In fact, we can recover the stack-theoretic weighted blow-up in 2.6.2. To do so we deploy a "partial" Cox construction. Retaining the notation and hypotheses in 2.6.2, consider the lattice $N' = N \oplus \mathbf{Z}$ and the homomorphism $\beta\colon N' \to N$ defined by $\beta(\mathbf{u}, 0) = \mathbf{u}$ and $\beta(0, 1) = \mathbf{v}$. This gives the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\ \alpha\ } N' \xrightarrow{\ \beta\ } N \longrightarrow 0 \qquad \text{where } \alpha(1) = (\mathbf{v}, -1).$$

We lift the fan $\Sigma^*(\mathbf{v})$ in $N_{\mathbf{R}}$ to a fan $\Sigma'(\mathbf{v}) := \{\sigma' \colon \sigma \in \Sigma^*(\mathbf{v})\}$ in $N'_{\mathbf{R}}$, where:

(a) we lift every ray $\rho \in \Sigma[1]$ with generator $\mathbf{u}$ to the ray ρ' with generator $\mathbf{u}' := (\mathbf{u}, 0)$, and the ray $\rho_E = \langle \mathbf{v} \rangle$ is lifted to the ray ρ'_E generated by $\mathbf{v}' := (0, 1)$,

(b) and generally, every $\sigma \in \Sigma^*(\mathbf{v})$ is lifted to the cone $\sigma' = \sum_{\rho \in \sigma[1]} \rho'$.

Then β induces $\Sigma'(\mathbf{v}) \to \Sigma^*(\mathbf{v})$, which is a bijection on cones. It is also natural to augment $\Sigma'(\mathbf{v})$ to $\overline{\Sigma'(\mathbf{v})}$ by adding the cones $\rho'_1 + \cdots + \rho'_n$ and $\gamma' := \rho'_1 + \cdots + \rho'_n + \rho'_E$. Likewise β also induces $\overline{\Sigma'(\mathbf{v})} \to \Sigma$, although it is only a bijection on maximal cones. We now claim that:

(i) $X_{\Sigma'(\mathbf{v})} = \operatorname{Spec}(I_\bullet^{\mathrm{ext}}) \smallsetminus V(I_+) \hookrightarrow X_{\overline{\Sigma'(\mathbf{v})}} = \operatorname{Spec}(I_\bullet^{\mathrm{ext}})$,

(ii) α induces the $\mathbb{G}_m$-action on both $X_{\Sigma'(\mathbf{v})} \hookrightarrow X_{\overline{\Sigma'(\mathbf{v})}}$ and β induces a toric morphism $X_{\overline{\Sigma'(\mathbf{v})}} \to X_\Sigma$.

(iii) $X_{\Sigma'(\mathbf{v})} \hookrightarrow X_{\overline{\Sigma'(\mathbf{v})}} \to X_\Sigma$ descends to $\left[X_{\Sigma'(\mathbf{v})} \,/\, \mathbb{G}_m\right] \hookrightarrow \left[X_{\overline{\Sigma'(\mathbf{v})}} \,/\, \mathbb{G}_m\right] \to X_\Sigma$, which is the weighted blow-up $\mathrm{Bl}_{I_\bullet} X_\Sigma \to X_\Sigma$, with $I_\bullet = (I_{D_1}, d_1) + \cdots + (I_{D_n}, d_n)$ as in 2.6.2. In particular, $\mathrm{Bl}_{I_\bullet} X_\Sigma$ is the toric stack $X_{\Sigma'(\mathbf{v}),\beta}$ associated to the stacky fan $(\Sigma'(\mathbf{v}), \beta)$.

To see these, let $\sigma \in \Sigma$ be a maximal cone and let $\sigma' \in \overline{\Sigma'(\mathbf{v})}$ be the corresponding maximal cone. This induces a toric morphism $U_{\sigma'} \to U_\sigma$ given by the monoid homomorphism

$$M_\sigma := \sigma^\vee \cap M \xrightarrow{\ \beta^\vee|_{M_\sigma}\ } (\sigma')^\vee \cap M' = M_{\sigma'}$$

induced by $\beta^\vee \colon M := N^\vee \to (N')^\vee =: M' = M \oplus \mathbf{Z}$ where $\beta^\vee(m) = \big(m, m(\mathbf{v})\big)$. Let $\iota\colon M \to M'$ be given by $\iota(m) = (m, 0)$.

Case 1: If $\sigma \neq \gamma$, then $M_{\sigma'} = \iota(M_\sigma) \oplus \mathbf{Z} = \beta^\vee(M_\sigma) \oplus \mathbf{Z}$, so that $U_{\sigma'} = U_\sigma \times \mathbb{G}_m \xrightarrow{\text{projection}} U_\sigma$, and the $\mathbb{G}_m$-action on $U_{\sigma'}$ is trivial on the first factor, and is given by multiplication on the second factor.

Case 2: If $\sigma = \gamma$, then $M_{\gamma'} = \iota(M_\gamma) \oplus \mathbf{N}$. Letting $\mathbf{u}_1^\vee, \mathbf{u}_2^\vee, \ldots, \mathbf{u}_n^\vee$ be the dual basis to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, the $\mathbb{G}_m$-action on $U_{\gamma'}$ corresponds to the weights $\alpha^\vee(\mathbf{u}_i^\vee, 0) = \mathbf{u}_i^\vee(\mathbf{v}) = d_i$ and $\alpha^\vee(0,1) = -1$. Moreover, since $\beta^\vee(\mathbf{u}_i^\vee) = (\mathbf{u}_i^\vee, d_i)$ for every $1 \leq i \leq n$, the morphism $U_{\gamma'} \to U_\gamma$ can be written as

$$\operatorname{Spec}(\mathbf{Z}[x_1', x_2', \ldots, x_n', t^{-1}]) \xrightarrow{t^{-d_i} x_i' \leftarrow\!\shortmid x_i} \operatorname{Spec}(\mathbf{Z}[x_1, x_2, \ldots, x_n])$$

where $V(x_i) = D_i$, $V(x_i') \subset U_{\gamma'}$ is the torus-invariant divisor corresponding to ρ_i', and $V(t^{-1}) \subset U_{\gamma'}$ is the torus-invariant divisor corresponding to ρ_E'.

In addition, if Z is the union of the two torus orbits in $X_{\overline{\Sigma'(\mathbf{v})}}$ corresponding to the two cones $\rho_1' + \cdots + \rho_n'$ and γ', then $Z = V(x_1', \ldots, x_n') \subset \operatorname{Spec}(\mathbf{Z}[x_1', x_2', \ldots, x_n', t^{-1}]) = U_{\gamma'}$. These descriptions agree with Proposition 2.5.9.

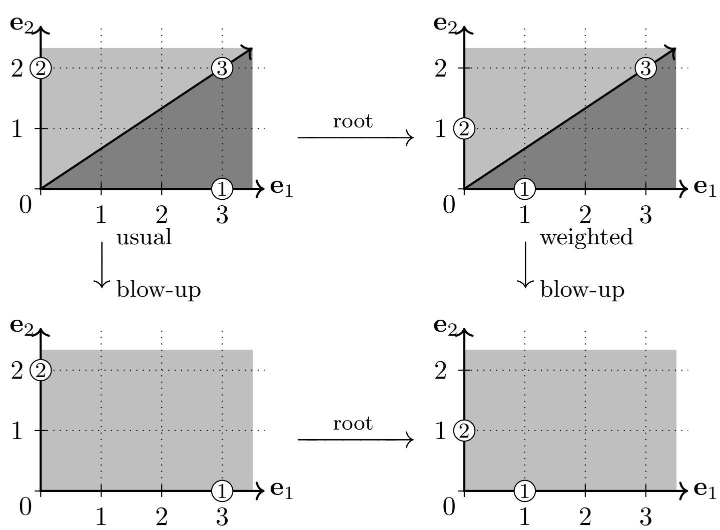
Remark 2.6.4. More generally, let $(\Sigma, L \xrightarrow{\beta} N)$ be a stacky fan where Σ is smooth. Let $\sigma \in \Sigma$ be a maximal cone generated by rays $\rho_1, \rho_2, \ldots, \rho_n$ with lattice generators $\mathbf{u}_i$. Pick a lattice point $\mathbf{v}$ contained in $\operatorname{relint}(\sigma)$ and write $\mathbf{v}$ as $\mathbf{v} = \sum_{i=1}^n d_i b_i$ for unique $d_i \in \mathbf{N}_{>0}$. As in 2.6.2, let $\Sigma^*(\mathbf{v})$ be the subdivision of Σ at $\mathbf{v}$. As in 2.6.3, define a homomorphism $\beta' \colon L' = L \oplus \mathbf{Z} \to L$ in the same manner, and lift $\Sigma^*(\mathbf{v})$ in $L_\mathbf{R}$ to a fan $\Sigma'(\mathbf{v})$ in $L'_\mathbf{R}$ in the same manner. Set $\beta'(\mathbf{v}) := \beta \circ \beta' \colon L' \to N$, and let $I_\bullet = (I_{D_1}, d_1) + \cdots + (I_{D_n}, d_n)$, where each I_{D_i} is the ideal of the torus-invariant divisor $D_i \subset X_{\Sigma,\beta}$. Then the weighted blow-up $\operatorname{Bl}_{I_\bullet} X_{\Sigma,\beta} \to X_{\Sigma,\beta}$ can be identified with the toric morphism $X_{\Sigma'(\mathbf{v}),\beta'(\mathbf{v})} \to X_{\Sigma,\beta}$ induced by β'.

Example 2.6.5 (A local description of weighted blow-ups via root stacks and ordinary blow-ups)**.** The above toric interpretation also motivates a *local* description of weighted blow-ups along regular centers via a sequence of *root stacks* (Example 2.2.13), followed by a *usual blow-up*, and finally a sequence of *"de-root-ings"*. For convenience, let us illustrate this via an example, which is no less informative than the general case.

Let $X = \mathbf{A}^2_{\mathbf{C}} = \operatorname{Spec}(\mathbf{C}[x_1, x_2])$, i.e. the toric variety associated to the standard fan Σ_{std} in $\mathbf{R}^2$. Consider the weighted blow-up of X along $I_\bullet = (x_1, 3) + (x_2, 2)$. Let $D_i = V(x_i)$ for $i = 1, 2$. Then there exists a canonical identification as shown in the dotted arrow below:

$$\begin{tikzcd}
\mathrm{Bl}_{\left(x_1^{1/3}, x_2^{1/2}\right)} X\left(\sqrt[3]{D_1}, \sqrt[2]{D_2}\right) \arrow[r, dotted, leftrightarrow, "\simeq"] \arrow[d, "\text{usual blow-up}"] & \left(\mathrm{Bl}_{I_\bullet} X\right)\left(\sqrt[3]{D_1'}, \sqrt[2]{D_2'}\right) \arrow[r, "\text{sequence of}", "\text{root stacks}"'] & \mathrm{Bl}_{I_\bullet} X \arrow[d, "\text{weighted blow-up}"'] \\
X\left(\sqrt[3]{D_1}, \sqrt[2]{D_2}\right) \arrow[rr, "\text{sequence of root stacks}"'] & & X
\end{tikzcd}$$

where D_i' is the proper transform of D_i in $\mathrm{Bl}_{I_\bullet} X$. The corresponding stacky fans are shown below:



In the diagram above, each corner illustrates a fan Σ in the usual lattice $N = \mathbf{Z}^2$, as well as some markings which define a homomorphism $\beta\colon \mathbf{Z}^k \to N$, where $\beta(\mathbf{e}_i)$ is marked with the circle that is labeled i. The data (Σ, β) at each corner defines a stacky fan (Σ, β).

In addition, the diagram suggests that there are compatible canonical identifications between the ($x_1' := x_1 \cdot t^3$)-chart (resp. ($x_2' := x_2 \cdot t^2$)-chart) of $\mathrm{Bl}_{I_\bullet} X$ and the $(x_1^{1/3})'$-chart (resp. $(x_2^{1/2})'$-chart) of $\mathrm{Bl}_{(x^{1/3},y^{1/2})} X(\sqrt[3]{D_1}, \sqrt[2]{D_2})$. This can be visualized as follows:

$$(x_1, x_2) \overset{\text{root}}{\rightsquigarrow} \left(x_1^{1/3}, x_2^{1/2}\right) \overset{\text{blow-up}}{\rightsquigarrow} \begin{cases} \left(x_1^{1/3}, \frac{x_2^{1/2}}{x_1^{1/3}}\right) & \text{on the } (x_1^{1/3})'\text{-chart} \\ \left(\frac{x_1^{1/3}}{x_2^{1/2}}, x_2^{1/2}\right) & \text{on the } (x_2^{1/2})'\text{-chart} \end{cases}$$

$$\overset{\text{de-root}}{\rightsquigarrow} \begin{cases} \left(x_1^{1/3}, \frac{x_2}{\left(x_1^{1/3}\right)^2}\right) & \text{on the } x_1'\text{-chart} \\ \left(\frac{x_1}{\left(x_2^{1/2}\right)^3}, x_2^{1/2}\right) & \text{on the } x_2'\text{-chart.} \end{cases}$$

We leave it to readers to convince themselves of the general case.

2.7. Weighted blow-ups along toroidal centers

In this section, we require the language of logarithmic geometry, and we follow the conventions in [**Ogu18**]. Throughout this section, fix a fs logarithmic scheme X. We use the same letter X to refer to its underlying scheme, and we always denote the logarithmic structure on X by $\alpha = \alpha_X \colon \mathscr{M}_X \to \mathscr{O}_X$, where $\mathscr{M}_X$ is a sheaf of monoids. We also remind the reader that $\overline{\mathscr{M}}_X$ denotes the characteristic $\mathscr{M}_X/\mathscr{O}_X^*$ of $\mathscr{M}_X$.

2.7.A. Fs logarithmic structures on weighted blow-ups.

For most applications, it usually suffices to work within the *fs* category of logarithmic schemes:

Definition 2.7.1. We say a logarithmic scheme X is *fs*, if X admits a covering $\mathcal{U}$ (in the Zariski or étale topology, depending if $\mathscr{M}_X$ is a sheaf in the Zariski or étale topology) such that the pullback of $\mathscr{M}_X$ to each $U \in \mathcal{U}$ admits a chart subordinate to a fs (= *fine* and *saturated*) monoid M. Equivalently, each $U \in \mathcal{U}$ admits a strict morphism $U \to \mathrm{Spec}(M \to \mathbf{Z}[M])$ for a fs monoid M. See [**Ogu18**, Proposition II.1.1.3, Definition II.2.1.5, and Corollary II.2.3.6].

2.7.2. In what follows, x always denotes a point in X, while $\overline{x}$ denotes a geometric point over x. If X is fs, one can show the following desirable properties:

(i) $\overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}}$ is a free abelian group of finite rank $r(x)$ [**Ogu18**, Proposition I.1.3.5(2)]. (Note $r(x)$ is independent of choice of $\overline{x}$ over x.)

(ii) $r(x) = \operatorname{rank}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}})$ is upper semi-continuous on X, i.e. for each $n \in \mathbf{N}$,

$$X(\leq n) := \{x \in X : \operatorname{rank}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}}) \leq n\}$$

is Zariski open in X [**Ogu18**, Corollary II.2.16]. In particular, $X^* := \{x \in X : \mathscr{M}_{X,\overline{x}} := \mathscr{O}^*_{X,\overline{x}}\}$ is a Zariski open in X, called the *locus of triviality* of X.

(iii) For each $n \in \mathbf{N}$,

$$X(n) := \{x \in X : \operatorname{rank}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}}) = n\} \subset X(\leq n)$$

is a Zariski closed subscheme of $X(\leq n)$, and has the following étale-local description: for all $\overline{x} \in X(n)$, $\mathscr{O}_{X(n),\overline{x}} = \mathscr{O}_{X,\overline{x}}/I(\overline{x})$, where $I(\overline{x})$ is the ideal of $\mathscr{O}_{X,\overline{x}}$ generated by the image of the unique maximal ideal $\mathscr{M}^+_{X,\overline{x}}$ of $\mathscr{M}_{X,\overline{x}}$ under $\mathscr{M}_{X,\overline{x}} \xrightarrow{\alpha_{X,\overline{x}}} \mathscr{O}_{X,\overline{x}}$ [**AT17**, 2.2.10].

(iv) After replacing X by an étale neighbourhood of $\overline{x}$, X admits a fine chart $M \to \Gamma(X, \mathscr{M}_X)$ that is neat at $\overline{x}$, i.e. the composition $M \to \Gamma(X, \mathscr{M}_X) \to \mathscr{M}_{X,\overline{x}} \to \overline{\mathscr{M}}_{X,\overline{x}}$ is an isomorphism [**Ogu18**, Proposition III.1.2.7].

If $\mathscr{M}_X$ is Zariski, all statements apply with $\overline{x}$ replaced by the scheme-theoretic point $x \in X$, and (iv) holds after replacing X by a Zariski neighbourhood of x.

Following [**AT17**, 2.2.10], we make the following

Definition 2.7.3. Let X be a fs logarithmic scheme. The *logarithmic stratification* of X is the stratification given by $\{X(n) : n \in \mathbf{N}\}$ in 2.7.2. For each $x \in X$, we set $\mathfrak{s}_x = X(n)$ for $n = \operatorname{rank}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}})$, and $\mathfrak{s}_x$ is called the *logarithmic stratum* through x.

From now on, X denotes a fs logarithmic scheme. Let $I_\bullet$ be a Rees algebra on X.

2.7.4 (Monomial part). The *monomial part* $\mathscr{M}_{I_\bullet}$ of $I_\bullet$ is the sheaf of monoids on X defined as the cartesian product of the right square in the diagram:

$$\begin{tikzcd} \mathscr{M}_X \arrow[r, hook, dotted] \arrow[d, equal] & \mathscr{M}_{I_\bullet} \arrow[r] \arrow[d, hook] & I_\bullet \arrow[d, hook] \\ \mathscr{M}_X \arrow[r, hook, "(\mathbb{1},0)"] & \mathscr{M}_X \oplus \mathbf{N} \arrow[r] & \mathscr{O}_X[t] \end{tikzcd} \tag{2.10}$$

where $\mathscr{M}_X \oplus \mathbf{N} \to \mathscr{O}_X[t]$ is $(m,k) \mapsto \alpha_X(m)t^k$ for local sections m of $\mathscr{M}_X$ and k of $\mathbf{N}$, and $I_\bullet$ and $\mathscr{O}_X[t]$ are considered as sheaves of monoids under multiplication. Note that via the map $\mathscr{M}_X \xrightarrow{\alpha_X} \mathscr{O}_X = I_0 \subset I_\bullet$, the injection $\mathscr{M}_X \hookrightarrow \mathscr{M}_X \oplus \mathbf{N}$ factors through $\mathscr{M}_{I_\bullet} \hookrightarrow \mathscr{M}_X \oplus \mathbf{N}$. In fact, the left square of (2.10) is also cartesian.

We can now define two properties:

Definition 2.7.5 (Monomial ideals and Rees algebras). We say $I_\bullet$ is *monomial* if the image of $\mathscr{M}_{I_\bullet} \to I_\bullet$ in (2.10) generates $I_\bullet$ as a $\mathscr{O}_X$-algebra. We also say an ideal $I \subset \mathscr{O}_X$ is *monomial* if its associated Rees algebra $I^\bullet$ (Example 2.3.6) is *monomial*. Equivalently, there exists an ideal $Q \subset \mathscr{M}_X$ such that the image of Q under $\mathscr{M}_X \xrightarrow{\alpha_X} \mathscr{O}_X$ generates I as an ideal.

Definition 2.7.6 (Fs Rees algebras). We say that $I_\bullet$ is *fs* if its monomial part $\mathscr{M}_{I_\bullet}$ is a fs sheaf of monoids. Equivalently, $\mathscr{M}_{I_\bullet}$ is saturated and finitely generated over $\mathscr{M}_X$.

2.7.7 (Logarithmic structure on $\mathrm{Bl}_{I_\bullet} X$). Likewise, for the extended Rees algebra $I^{\mathrm{ext}}_\bullet$ on X, the *monomial part* $\mathscr{M}_{I^{\mathrm{ext}}_\bullet}$ of $I^{\mathrm{ext}}_\bullet$ is the sheaf of monoids given by the cartesian product of the right square of the following diagram:

$$\begin{tikzcd} \mathscr{M}_X \oplus (-\mathbf{N}) \arrow[r, hook, dotted] \arrow[d, equal] & \mathscr{M}_{I_\bullet^{\mathrm{ext}}} \arrow[r] \arrow[d, hook] & I_\bullet^{\mathrm{ext}} \arrow[d, hook] \\ \mathscr{M}_X \oplus (-\mathbf{N}) \arrow[r, hook] & \mathscr{M}_X \oplus \mathbf{Z} \arrow[r] & \mathscr{O}_X[t^{\pm 1}] \end{tikzcd} \tag{2.11}$$

where $\mathscr{M}_X \oplus \mathbf{Z} \to \mathscr{O}_X[t^{\pm 1}]$ is similarly defined, $\mathscr{M}_X \oplus (-\mathbf{N}) \hookrightarrow \mathscr{M}_X \oplus \mathbf{Z}$ is the canonical injection which factors through $\mathscr{M}_{I_\bullet^{\mathrm{ext}}}$, and likewise the left square of (2.11) is cartesian.

The top row $\mathscr{M}_{I_\bullet^{\mathrm{ext}}} \to I_\bullet^{\mathrm{ext}}$ of (2.11) defines a logarithmic structure (and *not* just a pre-logarithmic sructure, cf. Lemma 2.7.8 below) on the deformation to the normal cone $\mathrm{Spec}_X(I_\bullet^{\mathrm{ext}})$, as well as the weighted blow-up $\mathrm{Bl}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}}) = [\mathrm{Spec}_X(I_\bullet^{\mathrm{ext}}) \smallsetminus V(I_+^{\mathrm{ext}}) \ / \ \mathbb{G}_m]$. Note that this logarithmic structure on $\mathrm{Bl}_{I_\bullet} X$ can also be defined via the top row $\mathscr{M}_{I_\bullet} \to I_\bullet$ of (2.10) and the usual presentation $\mathrm{Bl}_{I_\bullet} = [\mathrm{Spec}_X(I_\bullet) \smallsetminus V(I_+) \ / \ \mathbb{G}_m]$. This can be checked by passage to charts. For various reasons (e.g. the deformation to the normal cone, and Proposition 2.7.17), we usually work with $I_\bullet^{\mathrm{ext}}$ rather than $I_\bullet$.

If $I_\bullet$ is fs, then $\mathscr{M}_{\bullet}^{\mathrm{ext}}$ is saturated and finitely generated over $\mathscr{M}_X \oplus (-\mathbf{N})$ (and vice versa), in which case the aforementioned logarithmic structures are fs. Likewise, note too that $I_\bullet$ is monomial if and only if the image of $\mathscr{M}_{I_\bullet^{\mathrm{ext}}} \to I_\bullet^{\mathrm{ext}}$ in (2.11) generates $I_\bullet^{\mathrm{ext}}$ as a $\mathscr{O}_X$-algebra.

Lemma 2.7.8. *Let $I_\bullet$ be a Rees algebra on X. Then $\mathscr{M}_{I_\bullet} \to I_\bullet$ and $\mathscr{M}_{I_\bullet^{\mathrm{ext}}} \to I_\bullet^{\mathrm{ext}}$ are both logarithmic morphisms of sheaves of monoids.*

Proof. A local section $\sum_{i=d}^{e} c_i t^i$ of $\mathscr{O}_X[t^{\pm}]$ is invertible if and only if $d = e$ and c_d is invertible in $\mathscr{O}_X$. Therefore, the bottom row $\mathscr{M}_X \oplus \mathbf{Z} \to \mathscr{O}_X[t^{\pm 1}]$ in (2.11) is logarithmic. Consequently, since the square (2.11) is cartesian, so is the top row $\mathscr{M}_{I_\bullet^{\mathrm{ext}}} \to I_\bullet^{\mathrm{ext}}$ in (2.11). The same argument works to show the top row $\mathscr{M}_{I_\bullet} \to I_\bullet$ in (2.10) is logarithmic. ☐

Remark 2.7.9. Note that under the above logarithmic structure on $\mathrm{Bl}_{I_\bullet} X$ in 2.7.7, the ideal $I_E = (t^{-1})$ underlying the exceptional divisor E on $\mathrm{Bl}_{I_\bullet} X$ is a monomial ideal in the sense of 2.7.5.

2.7.B. Weighted blow-ups along toroidal centers. In this subsection, we assume further that X is *toroidal*, that is:

Definition 2.7.10. We say that a fs logarithmic scheme X is *toroidal* (or *logarithmically regular*) at a point $x \in X$ [**Kat94**, Definition 2.1], if for some (and hence any) geometric point $\overline{x}$ over x,

$$\mathfrak{s}_y \text{ is regular at } \overline{x} \text{ and the equality } \dim(\mathscr{O}_{X,\overline{x}}) = \mathrm{rank}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}}) + \dim(\mathscr{O}_{\mathfrak{s}_x,\overline{x}}) \text{ holds.}$$

If X is a fs Zariski logarithmic scheme, we say X is logarithmically regular at $x \in X$, if the same statement holds with $\overline{y}$ replaced by the scheme-theoretic point x throughout. We say X is toroidal (or logarithmically regular) if X is toroidal at every point $x \in X$. We also say X is *strict toroidal* if X is toroidal and $\mathscr{M}_X$ is a sheaf in the Zariski topology.

Let us first review some facts pertaining to toroidal schemes:

2.7.11. Let X be a fs logarithmic scheme.

(i) In general, for every $x \in X$, $\dim(\mathscr{O}_{X,\overline{x}}) \leq \mathrm{rank}(\overline{\mathscr{M}}^{\mathrm{gp}}_{X,\overline{x}}) + \dim(\mathscr{O}_{\mathfrak{s}_x,\overline{x}})$ [**Kat94**, Lemma 2.3].

(ii) Let $U = X^*$ be the triviality locus of X, with open embedding $j \colon X^* \hookrightarrow X$. If X is logarithmically regular, then $\alpha_X \colon \mathscr{M}_X \to \mathscr{O}_X$ is injective, and the image of α_X is $j_*(\mathscr{O}_U^*) \cap \mathscr{O}_X$. If $X \neq U$, then D is a divisor on X, sometimes called the *toroidal divisor* of X. See [**Kat94**, Theorem 3.2.4] and [**Niz06**, Proposition 2.6].

If $\mathscr{M}_X$ is Zariski, then the above statements hold with $\overline{x}$ replaced by the scheme-theoretic points $x \in X$. In addition,

(iii) If X is toroidal, $\underline{X}$ is Cohen-Macaulay and normal [**Kat94**, Theorem 4.1]. In particular, $\underline{X}$ is reduced, and if X is locally noetherian, $\underline{X}$ is a disjoint union of its irreducible components. Moreover, $\underline{X}$ is catenary, so each non-empty logarithmic stratum $X(n)$ of X has pure codimension n.

(iv) If X is toroidal at all its closed points, X is toroidal [**Kat94**, Proposition 7.1].

(v) If $f\colon \widetilde{X} \to X$ is a logarithmically smooth morphism of fs logarithmic schemes with X toroidal, then $\widetilde{X}$ is also toroidal [**Kat94**, Theorem 8.2].

2.7.12. Let $\mathbf{k}$ be a field, and endow $\mathrm{Spec}(\mathbf{k})$ the trivial logarithmic structure. Then:

(i) Every regular $\mathbf{k}$-scheme is a toroidal $\mathbf{k}$-scheme by equipping it the trivial logarithmic structure. If $\mathbf{k} = \overline{\mathbf{k}}$, strict toroidal $\mathbf{k}$-varieties correspond to the toroidal embeddings without self-intersections in [**KKMSD73**]. More generally, toroidal $\mathbf{k}$-varieties are general toroidal embeddings, i.e. possibly with self-intersections.

(ii) If X is toroidal, X is logarithmically smooth over $\mathbf{k}$. The converse is true if $\mathbf{k}$ is perfect [**Kat94**, Proposition 8.3].

(iii) Assume X is strict toroidal. For $x \in X$, fix $x_1, x_2, \ldots, x_n \in \mathscr{O}_{X,x}$ which reduce to a regular system of parameters $x_1, x_2, \ldots, x_n$ of $\mathscr{O}_{\mathfrak{s}_x,x}$, fix a local fs chart $\beta\colon M \to \Gamma(U, \mathscr{M}_X)$ that is neat at x, and fix a coefficient field κ for $\widehat{\mathscr{O}}_{X,x}$. Then the induced surjective homomorphism

$$\kappa[\![X_1, X_2, \ldots, X_n, M]\!] \xrightarrow{X_i \mapsto x_i} \widehat{\mathscr{O}}_{X,x}$$

is an isomorphism [**Kat94**, Theorem 3.2(1)]. For this reason, we call $x_1, x_2, \ldots, x_n$ a system of *ordinary parameters* at x, and we call any element of $\beta(M \smallsetminus \{0\})$ a *monomial parameter* at x.

We are now ready to introduce the key notion of this subsection:

Definition 2.7.13 (Toroidal centers)**.** A fs weighted closed embedding $Z_\bullet \hookrightarrow X$ is *toroidal* or *logarithmically regular* (resp. is called a *toroidal center*) if $I_\bullet = \mathscr{M}_{I_\bullet} + I'_\bullet$ for a Rees algebra $I'_\bullet$ on X such that $V(I'_\bullet \mathscr{O}_{X(n)}) \hookrightarrow X(n)$ is a regular weighted closed embedding (resp. a regular center) for every logarithmic stratum $X(n) \subset X$.

Setup 2.7.14. Let X be a toroidal logarithmic scheme, and $Z_\bullet \hookrightarrow X$ be a toroidal weighted closed embedding. Our next goal is to give a local description of the weighted blow-up $\mathrm{Bl}_{I_\bullet} X \to X$, similar to the one for regular weighted blow-ups in Proposition 2.5.9. Fix $x \in X$. After replacing X by an étale neighbourhood of x in X, there is a fs chart $M \to \Gamma(X, \mathscr{M}_X)$ that is neat at x.

We set up the following diagram, where $M \oplus (-\mathbf{N}) \hookrightarrow M \oplus \mathbf{Z}$ factors through $M_{I_\bullet^{\mathrm{ext}}} \hookrightarrow M \oplus \mathbf{Z}$, and the squares are cartesian:

$$\begin{tikzcd} M \oplus (-\mathbf{N}) \arrow[r, hook, dotted] \arrow[d, equal] & M_{I_\bullet^{\mathrm{ext}}} \arrow[r, hook, "\text{chart}"] \arrow[d, hook] & \mathscr{M}_{I_\bullet^{\mathrm{ext}}} \arrow[r, hook] \arrow[d, hook] & I_\bullet^{\mathrm{ext}} \arrow[d, hook] \\ M \oplus (-\mathbf{N}) \arrow[r, hook] & M \oplus \mathbf{Z} \arrow[r, hook, "\text{chart}"] & \mathscr{M}_X \oplus \mathbf{Z} \arrow[r, hook] & \mathscr{O}_X[t^{\pm 1}] \end{tikzcd} \tag{2.12}$$

Then $M_{I_\bullet^{\mathrm{ext}}} \to \mathscr{M}_{I_\bullet^{\mathrm{ext}}}$ is a chart. Moreover, the induced injection $M \xhookrightarrow{(\mathbb{1},0)} M \oplus (-\mathbf{N}) \hookrightarrow M_{I_\bullet^{\mathrm{ext}}}$ gives $\mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}]$ the structure of a $\mathbf{Z}[M]$-algebra, and the chart $M \to \mathscr{M}_X$ also gives $\mathscr{O}_X$ the structure of a $\mathbf{Z}[M]$-algebra. Note too that because of the leftmost cartesian square in (2.12), $(M_{I_\bullet^{\mathrm{ext}}})^{\mathrm{gp}}/M^{\mathrm{gp}} = \mathbf{Z}$.

In addition, after possibly shrinking X further, there are global sections $x_1, x_2, \ldots, x_k$ of $\mathscr{O}_X$ and positive integers $d_1, d_2, \ldots, d_k$ such that

$$I_\bullet^{\mathrm{ext}} = M_{I_\bullet^{\mathrm{ext}}} + (x_1, d_1) + (x_2, d_2) + \cdots + (x_k, d_k).$$

Let $\mathfrak{a}_X$ denote the ideal in $\mathscr{O}_X$ generated by the image of $M \smallsetminus \{1\}$ under $M \xrightarrow{\text{chart}} \mathscr{M}_X \xrightarrow{\alpha_X} \mathscr{O}_X$, and let $\mathfrak{a}$ denote the ideal in $\mathbf{Z}[M]$ generated by $M \smallsetminus \{1\}$. Then $\mathfrak{a}_X = \mathfrak{a}\mathscr{O}_X$, and $V(\mathfrak{a}_X) \subset X$

is the logarithmic stratum $\mathfrak{s}_x$ of X passing through x, and is also the logarithmic stratum with the highest rank in X. Since $I_\bullet$ is toroidal, we may also choose $x_1, x_2, \ldots, x_k$ so that their images $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$ under $\mathscr{O}_X \twoheadrightarrow \mathscr{O}_{\mathfrak{s}_x} = \mathscr{O}_X/\mathfrak{a}_X = \mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M]/\mathfrak{a}$ form a regular sequence in $\mathscr{O}_{\mathfrak{s}_x}$. Furthermore, if $I_\bullet$ is a toroidal center, we may also choose $x_1, x_2, \ldots, x_k$ so that the subscheme $V(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k) \subset \mathfrak{s}_x$ is regular.

Lemma 2.7.15 (Kato's Lemma). *Assume Setup 2.7.14. Then $x_1, x_2, \ldots, x_k$ is also a regular sequence in $\mathscr{O}_X$. In fact, for every monoid homomorphism $M \to P$ with P a fine monoid, and any ideal J of P, the images of $x_1, x_2, \ldots, x_k$ in $\mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[P]/(J)$ remain a regular sequence.*

Proof. Let $R := \mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M]/\mathfrak{a} = \mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}$. Note that since the short exact sequence $0 \to \mathfrak{a} \to \mathbf{Z}[M] \to \mathbf{Z}[M]/\mathfrak{a} = \mathbf{Z} \to 0$ can be split by the canonical inclusion $\mathbf{Z} \hookrightarrow \mathbf{Z}[M]$, we have $R[M] = \mathscr{O}_X$. Since the image $\overline{x}_1$ of x_1 under $\mathscr{O}_X \twoheadrightarrow R$ is a non-zero divisor, it follows that $x_1 \in R[P] = \mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[P]$ satisfies the hypotheses in [**Kat94**, Lemma 6.3], so the image of x_1 in $R[P]/(J) = \mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[P]/(J)$ is a non-zero divisor.

Next, suppose by induction that we have shown for some $1 \leq \ell < k$ that the images of $x_1, \ldots, x_\ell$ in $R[P]/(J)$ is a regular sequence. Set $R_\ell := \mathscr{O}_X/(x_1, x_2, \ldots, x_\ell) \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M]/\mathfrak{a} = \mathscr{O}_X/(x_1, x_2, \ldots, x_\ell) \otimes_{\mathbf{Z}[M]} \mathbf{Z}$. Since the images $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_\ell, \overline{x}_{\ell+1}$ of $x_1, x_2, \ldots, x_\ell, x_{\ell+1}$ under $\mathscr{O}_X \twoheadrightarrow R$ form a regular sequence, it follows that $x_{\ell+1} \in R_\ell[P] = \mathscr{O}_X/(x_1, x_2, \ldots, x_\ell) \otimes_{\mathbf{Z}[M]} \mathbf{Z}[P]$ satisfies the hypotheses in [**Kat94**, Lemma 6.3] (with respect to R_ℓ instead of R), so the image of $x_{\ell+1}$ in $R_\ell[P]/(J) = \mathscr{O}_X/(x_1, x_2, \ldots, x_\ell) \otimes_{\mathbf{Z}[M]} \mathbf{Z}[P]/(J) = R[P]/(J, x_1, x_2, \ldots, x_\ell)$ is a non-zero divisor. □

2.7.16. We return back to Setup 2.7.14. For every $(m, e) \in M_{I_\bullet^{\mathrm{ext}}} \subset M \oplus \mathbf{Z}$, let mt^e be the corresponding monomial in $\mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}]$, where $m \in M$ is regarded as an element in $\mathscr{O}_X$ under $M \xrightarrow{\text{chart}} \mathscr{M}_X \xrightarrow{\alpha_X} \mathscr{O}_X$. In particular, $t^{-1} \in \mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}]$. Consider the graded polynomial ring

$$\mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][X_1, X_2, \ldots, X_k]$$

where $\deg(X_i) = d_i$, and $\deg(mt^e) = e$ for $(m,d) \in M_{I_\bullet^{\mathrm{ext}}} \subset M \oplus \mathbf{Z}$. There is a natural map

$$\mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][X_1, X_2, \ldots, X_k] \to I_\bullet^{\mathrm{ext}} \subset \mathscr{O}_X[t^{\pm 1}]$$

of graded $\mathscr{O}_X$-algebras induced by $M_{I_\bullet^{\mathrm{ext}}} \xrightarrow{\text{chart}} \mathscr{M}_{I_\bullet^{\mathrm{ext}}} \hookrightarrow I_\bullet^{\mathrm{ext}}$ that takes each X_i to $x_i t^{d_i}$. Note that $t^{-d_i}X_i - x_i$ is in the kernel of β for every $1 \le i \le k$, so that we obtain a map

$$B := \mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][X_1, X_2, \ldots, X_m]/(t^{-d_i}X_i - x_i \colon 1 \le i \le k) \xrightarrow{\ \beta\ } I_\bullet^{\mathrm{ext}}.$$

Note that β is surjective. Similar to Proposition 2.5.9, we have:

Proposition 2.7.17. *β is bijective, so that*

$$\mathrm{Bl}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}}) \xrightarrow{\ \simeq\ } \mathscr{P}\mathrm{roj}_X\left(\frac{\mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][X_1, X_2, \ldots, X_k]}{(t^{-d_i}X_i - x_i \colon 1 \le i \le k)}\right).$$

Proof. By the same argument as in Lemma 2.5.8, it suffices to show that $t^{-1} \in B$ is a non-zero divisor. By Kato's Lemma 2.7.15, $x_1, x_2, \ldots, x_k$ is a H_1-regular sequence in $\mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}]/(t^{-1})$. Thus, $x_1, x_2, \ldots, x_k, t^{-1} \in \mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][X_1, X_2, \ldots, X_k]$ is a H_1-regular sequence [**Stacks**, 0668]. At the same time, the sequence $t^{d_1}X_1 - x_1, t^{-d_2}X_2 - x_2, \ldots, t^{-d_k}X_k - x_k, t^{-1}$ generates the same ideal, and hence, is also H_1-regular [**Stacks**, 066A]. Consequently, $t^{-1} \in B$ is a non-zero divisor [**Stacks**, 068L]. □

Corollary 2.7.18. *If $I_\bullet$ is monomial, then $\pi\colon \mathrm{Bl}_{I_\bullet} X \to X$ is logarithmically smooth.*

Proof. This is a local question, so we may assume that we are in Setup 2.7.14. Then Proposition 2.7.17 says π is $\mathscr{P}\mathrm{roj}_X(\mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}]) \to X$, with $M^{\mathrm{gp}}_{I_\bullet^{\mathrm{ext}}}/M^{\mathrm{gp}} = \mathbf{Z}$, whence the corollary follows. □

For the next corollary, we fix $(m_1,e_1),(m_2,e_2),\dots,(m_r,e_r)\in M_{I_\bullet}$ so that $M_{I_\bullet^{\mathrm{ext}}}=\big\langle(m_1,e_1),(m_2,e_2),\dots,(m_r,e_r),M\oplus(-\mathbf{N})\big\rangle$, and hence

$$I_\bullet=(x_1,d_1)+(x_2,d_2)+\cdots+(x_k,d_k)+(m_1,e_1)+(m_2,e_2)+\cdots+(m_r,e_r)$$

where each m_i is regarded as an element in $\mathscr{O}_X$ under $M\xrightarrow{\text{chart}}\mathscr{M}_X\xrightarrow{\alpha_X}\mathscr{O}_X$. Then $\mathrm{Bl}_{I_\bullet}X$ is covered by the charts $D_+(x_i\cdot t^{d_i})$ for $1\le i\le k$ and $D_+(m_i\cdot t^{e_i})$ for $1\le i\le r$. As a consequence of Proposition 2.7.17 and Lemma 2.1.2 (with $A=\mathbf{Z}$, $a=d_i$ or e_i, and $r=x_i\cdot t^{d_i}$ or $m_i\cdot t^{e_i}$), we can explicate these charts:

Corollary 2.7.19 (Charts for blow-ups along toroidal weighted embeddings)**.**

(i) *For each $1\le i\le k$, the chart $D_+(x_i\cdot t^{d_i})$ of $\mathrm{Bl}_{I_\bullet}X$ is:*

$$\begin{aligned}&\left[\mathrm{Spec}_X\left(\frac{\mathscr{O}_X\otimes_{\mathbf{Z}[M]}\mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][X_1,X_2,\dots,X_k][X_i^{-1}]}{(t^{-d_j}X_j-x_j\colon 1\le j\le k)}\right)\Big/\,\mathbb{G}_m\right]\\ &=\left[\mathrm{Spec}_X\left(\frac{\mathscr{O}_X[t^{-1},X_1,X_2,\dots\widehat{X}_i,\dots,X_k]}{(t^{-d_i}-x_i)+(t^{-d_j}X_j-x_j\colon 1\le j\le k,\ j\ne i)}\right)\Big/\,\boldsymbol{\mu}_{d_i}\right]\end{aligned}$$

where $\widehat{X}_i$ means X_i omitted, and the action of $\boldsymbol{\mu}_{d_i}$ corresponds to the weights $\mathrm{wt}_{\mathbf{Z}/d_i}(X_j)=d_j$ for $j\ne i$, $\mathrm{wt}(mt^e)=e$ for $(m,e)\in M_{I_\bullet^{\mathrm{ext}}}$.

(ii) *For each $1\le i\le r$, the chart $D_+(m_i\cdot t^{e_i})$ of $\mathrm{Bl}_{I_\bullet}X$ is:*

$$\begin{aligned}&\left[\mathrm{Spec}_X\left(\frac{\mathscr{O}_X\otimes_{\mathbf{Z}[M]}\mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][(m_it^{e_i})^{-1}][X_1,X_2,\dots,X_k]}{(t^{-d_j}X_j-x_j\colon 1\le j\le k)}\right)\Big/\,\mathbb{G}_m\right]\\ &=\left[\mathrm{Spec}_X\left(\frac{\mathscr{O}_X\otimes_{\mathbf{Z}[M]}\mathbf{Z}\big[M_{I_\bullet^{\mathrm{ext}}}/\langle(m_i,0)\sim(0,-e_i)\rangle\big][X_1,X_2,\dots,X_k]}{(t^{-d_j}X_j-x_j\colon 1\le j\le k)}\right)\Big/\,\boldsymbol{\mu}_{e_i}\right]\end{aligned}$$

where $M_{I_\bullet^{\mathrm{ext}}}/\langle(m_i,0)\sim(0,-e_i)\rangle$ is the quotient of $M_{I_\bullet^{\mathrm{ext}}}$ by the congruence relation generated by $(m_i,0)\sim(0,-e_i)$, and the action of $\boldsymbol{\mu}_{e_i}$ corresponds to the weights $\mathrm{wt}_{\mathbf{Z}/e_i}(X_j)=d_j$ for $j\ne i$, $\mathrm{wt}(mt^e)=e$ for $(m,e)\in M_{I_\bullet^{\mathrm{ext}}}/\langle(m_i,0)\sim(0,-e_i)\rangle$.

Finally, let us specialize our discussion to the case of toroidal centers.

Corollary 2.7.20. *Let X be a toroidal logarithmic scheme. If $Z_\bullet \hookrightarrow X$ is a toroidal center, then the deformation to the weighted normal cone $D_{Z_\bullet}X = \mathrm{Spec}_X(I_\bullet^{\mathrm{ext}})$ is toroidal. In particular, the weighted blow-up $\mathrm{Bl}_{Z_\bullet} X \subset \left[D_{Z_\bullet}X \;/\; \mathbb{G}_m\right]$ is toroidal.*

Proof. This is a local question, so we may assume that we are in Setup 2.7.14. By Proposition 2.7.17, we have

$$I_\bullet^{\mathrm{ext}}/(t^{-1}) \xleftarrow{\ \simeq\ } \mathscr{O}_{V(x_1,x_2,\ldots,x_k)} \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][X_1, X_2, \ldots, X_k]/(t^{-1})$$

with $(M_{I_\bullet^{\mathrm{ext}}})^{\mathrm{gp}}/M^{\mathrm{gp}} = \mathbf{Z}$. This implies the weighted normal cone $C_{Z_\bullet}X = \mathrm{Spec}_X\big(I_\bullet^{\mathrm{ext}}/(t^{-1})\big)$, endowed with the idealized logarithmic structure induced by $M_{I_\bullet^{\mathrm{ext}}} \hookrightarrow I_\bullet^{\mathrm{ext}} \twoheadrightarrow I_\bullet^{\mathrm{ext}}/(t^{-1})$ and the ideal sheaf $\big((0,-1)\big) \subset M_{I_\bullet^{\mathrm{ext}}}$, is idealized logarithmically smooth over $V(x_1, x_2, \ldots, x_k) \subset X$. Since $V(x_1, x_2, \ldots, x_k) \subset X$ is toroidal [**ATW20b**, Lemma 5.1.2], $C_{Z_\bullet}X$ is therefore idealized toroidal. It then follows that $D_{Z_\bullet}X = \mathrm{Spec}_X(M_{I_\bullet^{\mathrm{ext}}} \to I_\bullet^{\mathrm{ext}})$ is toroidal at points in $C_{Z_\bullet}X = V(t^{-1}) \subset D_{Z_\bullet}X$. On the other hand, the open complement of $C_{Z_\bullet}X$ in $D_{Z_\bullet}X$ is the toroidal scheme $X \times \mathbb{G}_m$. This completes the proof. $\square$

Corollary 2.7.21. *Any toroidal center $I_\bullet$ on a toroidal scheme X is integrally closed in $\mathscr{O}_X[t]$.*

Proof. Since X and $D_{Z_\bullet}X = \mathrm{Spec}_X(I_\bullet^{\mathrm{ext}})$ are toroidal, they are in particular normal (2.7.11). Consequently, $I_\bullet^{\mathrm{ext}}$ must be integrally closed. $\square$

CHAPTER 3

Logarithmic resolution via weighted blow-ups along toroidal centers

3.1. Toroidal centers in characteristic zero

3.1.A. Preliminaries. This chapter concerns the theorems outlined in §1.2.A. Throughout this chapter, we fix a field $\mathbf{k}$ of characteristic zero. Although ideals on toroidal Deligne-Mumford stacks over $\mathbf{k}$ [**ATW20a**, §3.3.3] are the main objects of interest in the results of §1.2.A, a significant portion of the paper instead deals with ideals on strict toroidal $\mathbf{k}$-schemes (Definition 2.7.10). There are two reasons for this:

(a) Étale locally a toroidal Deligne-Mumford stack over $\mathbf{k}$ is a strict toroidal $\mathbf{k}$-scheme, cf. [**GR18**, Proposition 12.5.46].

(b) The constructions and discussions in this paper are étale-local. This is indeed a feature of Theorem A in §1.2.A.

Henceforth, we shall assume Y is a strict toroidal $\mathbf{k}$-scheme (with the exception of Corollary 3.3.17), and denote its logarithmic structure by $\alpha_Y : \mathscr{M}_Y \to \mathscr{O}_Y$. We follow any conventions and notations outlined in §2.7. We also fix the following notations:

$\mathscr{D}_Y^1$ — logarithmic tangent sheaf of Y

$\mathscr{D}_Y^\infty$ — sheaf of logarithmic differential operators on Y [**ATW20a**, §3.3]

$\mathscr{D}_Y^{\leq n}$ — sheaf of logarithmic differential operators on Y of order $\leq n$

At a closed point $y \in Y$, the stalk of $\mathscr{D}_Y^1$ at y can be described as follows, cf. [**ATW20a**, Lemma 3.3.4] or [**Que22a**, Lemma B.9]. Fix a system of ordinary parameters $x_1, x_2, \ldots, x_n$ at y, and fix a basis $m_1, m_2, \ldots, m_{r(y)} \in \overline{\mathscr{M}}_{Y,y} =: M$ for the free abelian group $\overline{\mathscr{M}}_{Y,y}^{\mathrm{gp}}$ of rank $r(y)$. With respect to these data are the following differential operators of order 1:

(i) ∂_{x_i} which vanishes on M and satisfies $\partial_{x_i}(x_j) = \delta_{ij}$,

(ii) $m_i\partial_{m_i}$ which vanishes on every x_i and satisfies $m_i\partial_{m_i}(m_j) = \delta_{ij}m_i$.

Then $\mathscr{D}^1_{Y,y}$ is a free $\mathscr{O}_{Y,y}$-module with basis $\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \partial_{m_1}, \partial_{m_2}, \ldots, \partial_{m_{r(y)}}$. Since we are in characteristic zero, $\mathscr{D}^\infty_Y$ is generated as a $\mathscr{O}_Y$-algebra by $\mathscr{D}^1_Y$ (where the ring structure on $\mathscr{D}^\infty_Y$ is given by composition). In addition, with respect to an ideal J on Y, we adopt the following notations:

$\mathscr{D}^{\leq n}_Y(J)$ — ideal on Y generated by the image of J under $\mathscr{D}^{\leq n}_Y$

$\mathscr{D}^{\infty}_Y(J)$ — ideal on Y generated by the image of J under $\mathscr{D}^{\infty}_Y$

It turns out that $\mathscr{D}^\infty_Y(J)$ coincides with the following notion, cf. the subsequent lemma:

Definition 3.1.1. The *monomial saturation* $\mathscr{M}_Y(J)$ of J is the intersection of all monomial ideals (Definition 2.7.5) on Y containing J.

When Y is clear from context, we usually drop Y from the subscripts in the aforementioned notations, e.g. we sometimes just write $\mathscr{M}(J)$ instead of $\mathscr{M}_Y(J)$.

Lemma 3.1.2 (cf. [**ATW20a**, Corollary 3.3.12, Theorem 3.4.2 and Lemma 3.5.2])**.**

(i) $\mathscr{D}^\infty_Y(J) = \mathscr{M}_Y(J)$.

(ii) *J is monomial if and only if* $\mathscr{D}^{\leq 1}_Y(J) = J$.

(iii) *If $f\colon \widetilde{Y} \to Y$ is a logarithmically smooth morphism of strict toroidal* $\mathbf{k}$*-schemes, then* $\mathscr{D}^{\leq n}_{\widetilde{Y}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}}) = f^{-1}(\mathscr{D}^{\leq n}_Y(J))\mathscr{O}_{\widetilde{Y}}$ *for all* $n \in \mathbf{N}$*, and* $\mathscr{M}_{\widetilde{Y}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}}) = f^{-1}(\mathscr{M}_Y(J))\mathscr{O}_{\widetilde{Y}}$.

(iv) *If Q is a monomial ideal on Y, then* $\mathscr{D}^{\leq n}_Y(Q \cdot J) = Q \cdot \mathscr{D}^{\leq n}_Y(J)$ *for all* $n \in \mathbf{N}$.

Additionally, we require the following notion, which is the logarithmic analogue of the classical order of ideals at points:

Definition 3.1.3. The *logarithmic order* of J at y is defined as $\text{log-ord}_y(J) := \text{ord}_y(J|_{\mathfrak{s}_y}) \in \mathbf{N} \sqcup \{\infty\}$, where $\mathfrak{s}_y$ is the logarithmic stratum through y, cf. Definition 2.7.3.

Lemma 3.1.4 (cf. [**ATW20a**, Lemma 3.6.3, Lemma 3.6.5, Corollary 3.66 and Lemma 3.6.8])**.**

(i) $\operatorname{log-ord}_y(J) = \inf\{m \in \mathbf{N} \colon \mathscr{D}_Y^{\leq m}(J)_y = \mathscr{O}_{Y,y}\}$, *where* $\inf(\varnothing) = \infty$.

(ii) $\operatorname{log-ord}_y(J) = \infty$ *if and only if* $y \in V(\mathscr{M}_Y(J))$.

(iii) $\mathscr{M}_Y(J)_y = \mathscr{O}_{Y,y}$ *if and only if* $\operatorname{log-ord}_y(J) < \infty$.

(iv) *If* $f \colon \widetilde{Y} \to Y$ *is a logarithmically smooth morphism of strict toroidal* $\mathbf{k}$*-schemes, and* $\widetilde{y} \in \widetilde{Y}$ *maps to* $y \in Y$, *then* $\operatorname{log-ord}_{\widetilde{y}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}}) = \operatorname{log-ord}_y(J)$.

Together the first two parts of the lemma say that $\operatorname{log-ord}_y(J)$ is upper semi-continuous on Y. Indeed:

(i) For $n \in \mathbf{N}$, $V(\mathscr{D}_Y^{\leq n}(J))$ is the locus of points $y \in Y$ satisfying $\operatorname{log-ord}_y(J) > n$,

(ii) $V(\mathscr{M}_Y(J))$ is the locus of points $y \in Y$ satisfying $\operatorname{log-ord}_y(J) = \infty$.

Therefore, since Y is noetherian, the *maximum logarithmic invariant*

$$\max \operatorname{log-ord}(J) := \max_{y \in Y} \operatorname{log-ord}_y(J)$$

of J exists. Finally, as specified in Theorem A, weighted blow-ups along toroidal centers (Definition 2.7.13) play a big role in this chapter, and the next section studies them in the setting of characteristic zero. To better formulate various results in this chapter, we make the following definition:

Definition 3.1.5 ($\mathbf{Q}$-toroidal centers)**.** A $\mathbf{Q}$*-toroidal center* on Y is any Veronese translate (Definition 2.3.39) of a toroidal center on Y. We usually reserve the notation $\mathscr{I}_\bullet$ for $\mathbf{Q}$-toroidal centers (as opposed to $I_\bullet$).

3.1.B. Weighted blow-ups along toroidal centers in characteristic zero. We start by recalling and fixing some conventions for the remainder of this thesis:

3.1.6 (Local presentation of toroidal centers)**.** For the remainder of this subsection, let $I_\bullet$ be a toroidal center on Y. For each $y \in Y$, fix a local fs chart $\beta\colon M \to \Gamma(U, \mathscr{M}_Y)$ that is neat at y (so $M \simeq \overline{\mathscr{M}}_{Y,y}$). Replacing Y by a neighbourhood of y, we may assume Setup 2.7.14 for $I_\bullet$. Fix $(m_1, e_1), (m_2, e_2), \ldots, (m_r, e_r) \in M_{I_\bullet}$ so that $M_{I_\bullet^{\mathrm{ext}}} = \big\langle (m_1, e_1), (m_2, e_2), \ldots, (m_r, e_r), M \oplus (-\mathbf{N}) \big\rangle$, as in the paragraph before Corollary 2.7.19. Then

$$I_\bullet = (x_1, d_1) + (x_2, d_2) + \cdots + (x_k, d_k) + (m_1, e_1) + (m_2, e_2) + \cdots + (m_r, e_r)$$

in the sense of Definition 2.3.5, where each m_i is regarded in $\mathscr{O}_X$ via $M \xrightarrow{\beta} \mathscr{M}_X \xrightarrow{\alpha_X} \mathscr{O}_X$.

Next, since $I_\bullet$ is integrally closed (Corollary 2.7.21), we find it more convenient in this chapter to follow Convention 2.3.79 and express $I_\bullet$ as

$$I_\bullet = \left(x_1^{1/d_1}, x_2^{1/d_2}, \ldots, x_k^{1/d_k}, m_1^{1/e_1}, m_2^{1/e_2}, \ldots, m_r^{1/e_r}\right). \tag{3.1}$$

Without loss of generality, we always assume $d_1 \geq d_2 \geq \cdots \geq d_k$. Letting $e := \operatorname{lcm}(e_i \colon 1 \leq i \leq r)$ and Q be the ideal

$$Q := \left(m_1^{e/e_1}, m_2^{e/e_2}, \ldots, m_r^{e/e_r}\right) \subset M,$$

we can further simplify $I_\bullet$ as

$$I_\bullet = \left(x_1^{1/d_1}, x_2^{1/d_2}, \ldots, x_k^{1/d_k}, Q^{1/e}\right) \tag{3.2}$$

Likewise, any $\mathbf{Q}$-toroidal center $\mathscr{I}_\bullet$ can be written in a neighbourhood of any $y \in Y$ as

$$\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a) \tag{3.3}$$

for some positive rational numbers $a_1 \leq a_2 \leq \ldots \leq a_k$ and a, some ordinary parameters $x_1, x_2, \ldots, x_k$ at y (cf. 2.7.12(iii)), and an ideal $Q \subset \overline{\mathscr{M}}_{Y,y}$. The expression in (3.2) (resp. (3.3)) will be referred to as a *local presentation* of a toroidal center $I_\bullet$ (resp. a $\mathbf{Q}$-toroidal center $\mathscr{I}_\bullet$) at y.

3.1.7. Next, let us explicate the weighted blow-up of Y along $I_\bullet$. For the purposes of this chapter, we will always work locally around some $y \in Y$, and assume that $I_\bullet$ has the local presentation (3.1) at y (with $d_1 \geq d_2 \geq \cdots \geq d_k$). Replacing Y by the neighbourhood of y on which that local presentation is defined, we recall from Proposition 2.7.17 that the weighted blow-up of Y along $I_\bullet$ can then be written as follows:

$$Y' := \mathrm{Bl}_{I_\bullet} Y = \mathscr{P}\mathrm{roj}_Y(I_\bullet^{\mathrm{ext}}) = \mathscr{P}\mathrm{roj}_Y\left(\frac{\mathscr{O}_Y \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}][x_1', x_2', \ldots, x_k']}{(u^{d_i}x_i' - x_i \colon 1 \leq i \leq k)}\right) \xrightarrow{\pi} Y \tag{3.4}$$

where $u := t^{-1}$ is the monomial in $\mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}]$ given by $(0,-1) \in M_{I_\bullet^{\mathrm{ext}}}$, and

$$x_i' := x_i \cdot t^{d_i} \in I_\bullet \qquad \text{for every } 1 \leq i \leq k.$$

Recall that $E = V(u)$ is the exceptional divisor on Y', cf. Remark 2.3.15. Let I_E denote the ideal sheaf of $E \subset Y'$. By Corollary 2.7.20, Y' is toroidal under the logarithmic structure induced by $M_{I_\bullet^{\mathrm{ext}}} \to I_\bullet^{\mathrm{ext}}$ in (2.12). In addition, for each $1 \leq i \leq r$, let

$$m_i' := m_i \cdot t^{e_i} \in I_\bullet$$

be the monomial in $\mathbf{Z}[M_{I_\bullet^{\mathrm{ext}}}]$ given by $(m_i, e_i) \in M_{I_\bullet^{\mathrm{ext}}}$. Then the map $M \to M_{I_\bullet^{\mathrm{ext}}}$, which is part of the data of (3.4), maps

$$m_i \mapsto u^{e_i} m_i' \qquad \text{for every } 1 \leq i \leq r.$$

Finally, recall that Y' is covered by the charts $D_+(x_i')$ for $1 \leq i \leq k$, and the charts $D_+(m_i')$ for $1 \leq i \leq r$, cf. Corollary 2.7.19 for a description of these charts. Then we have the following fundamental lemma:

Lemma 3.1.8. *For any ideal $J \subset \mathscr{O}_Y$, we have*

$$\pi^{-1}\left(\mathscr{D}_Y^{\leq 1}(J)\right)\mathscr{O}_{Y'} \subset I_E^{-d_1} \cdot \mathscr{D}_{Y'}^{\leq 1}\left(\pi^{-1}(J)\mathscr{O}_{Y'}\right).$$

PROOF. Let $W = \mathrm{Spec}_Y(I_\bullet^{\mathrm{ext}}) \smallsetminus V(I_+^{\mathrm{ext}})$ be the total space of the $\mathbb{G}_m$-torsor on Y', which is toroidal (Corollary 2.7.20). Letting π' denote $W \to Y' \to Y$, the map $\pi'^*\Omega_Y \to \Omega_W$ is defined by the following rules:

$$\begin{cases} \frac{dx_i}{x_i} = \frac{dx_i'}{x_i'} + d_i\frac{du}{u} & \text{for } 1 \leq i \leq k \\ \frac{dm_i}{m_i} = \frac{dm_i'}{m_i'} + e_i\frac{du}{u} & \text{for } 1 \leq i \leq r \end{cases}$$

so that $T_W \to \pi'^*T_Y$ is induced by

$$\begin{cases} x_i'\partial_{x_i'} = x_i\partial_{x_i} & \text{for } 1 \leq i \leq k \\ m_i'\partial_{m_i'} = m_i\partial_{m_i} & \text{for } 1 \leq i \leq r \\ u\partial_u = \sum_{i=1}^k d_ix_i\partial_{x_i} + \sum_{i=1}^r e_im_i\partial_{m_i} & \end{cases}$$

For $1 \leq i \leq k$, the total space of the $\boldsymbol{\mu}_{d_i}$-torsor on the chart $D_+(x_i')$ of $\mathrm{Bl}_{I_\bullet} X$ is $V(x_i'-1) \subset W$. Letting $\pi_{x_i'}\colon V(x_i'-1) \subset W \to Y' \to Y$, the map $T_{V(x_i'-1)} \to \pi_{x_i'}^*T_Y$ is then induced by

$$\begin{cases} \partial_{x_j} = u^{-d_j}\partial_{x_j'} & \text{for } 1 \leq j \leq k,\ j \neq i \\ m_j\partial_{m_j} = m_j'\partial_{m_j'} & \text{for } 1 \leq j \leq r \\ \partial_{x_i} = d_i^{-1}u^{-d_i}\left(u\partial_u - \sum_{\substack{1\leq j\leq k \\ j\neq i}} x_j'\partial_{x_j'} - \sum_{1\leq j\leq r} e_im_j'\partial_{m_j'}\right) & \end{cases}$$

Since $d_1 \geq d_i$, the lemma holds on the chart $D_+(x_i')$ of $\mathrm{Bl}_{I_\bullet} Y$. On the other hand, for $1 \leq i \leq r$, the total space of the $\boldsymbol{\mu}_{e_i}$-torsor on the chart $D_+(m_i')$ of $\mathrm{Bl}_{I_\bullet} X$ is $V(m_i'-1) \subset W$. Letting

$\pi_{m'_i}\colon V(m'_i - 1) \subset W \to Y' \to Y$, the map $T_{V(m'_i-1)} \to \pi^*_{m'_i} T_Y$ is similarly induced by

$$\begin{cases} \partial_{x_j} = u^{-d_j} \partial_{x'_j} & \text{for } 1 \leq j \leq k \\ m_j \partial_{m_j} = m'_j \partial_{m'_j} & \text{for } 1 \leq j \leq r,\ j \neq i \\ m_i \partial_{m_i} = e_i^{-1}\big(u\partial_u - \sum_{1\leq j\leq k} x'_j \partial_{x'_j} - \sum_{\substack{1\leq j\leq r \\ j\neq i}} e_i m'_j \partial_{m'_j}\big) \end{cases}$$

Since $d_1 \geq 0$, the lemma likewise holds on the charts $D_+(m'_i)$ of $\mathrm{Bl}_{I_\bullet} Y$ for $1 \leq i \leq r$. $\square$

The above lemma will allow us to extend Lemma 2.3.30, which says that for any $\ell \in \mathbf{N}_{>0}$, $I_{\ell\bullet}$ is J-admissible if and only if $\pi^{-1}(J)\mathscr{O}_{Y'} = I_E^\ell \cdot J'$ for an ideal $J' \subset \mathscr{O}_{Y'}$.

Corollary 3.1.9. *Assume that $I_{\ell\bullet}$ is J-admissible for an integer $\ell \geq d_1$, so that $\pi^{-1}(J)\mathscr{O}_{Y'} = I_E^\ell \cdot J'$ for an ideal $J' \subset \mathscr{O}_{Y'}$. Then for any integer $0 \leq j \leq \lfloor \ell/d_1 \rfloor$, we have:*

$$\pi^{-1}\left(\mathscr{D}_Y^{\leq j}(J)\right)\mathscr{O}_{Y'} \subset I_E^{\ell - jd_1} \cdot \mathscr{D}_{Y'}^{\leq j}(J'). \tag{3.5}$$

That is, $\pi^{-1}\big(\mathscr{D}_{Y'}^{\leq j}(J)\big)\mathscr{O}_{Y'} = I_E^{\ell-jd_1} \cdot J'_j$ for an ideal $J'_j \subset \mathscr{D}_{Y'}^{\leq j}(J') \subset \mathscr{O}_{Y'}$.

Proof. We proceed by induction. We already noted the base case $j = 0$. Assume (3.5) for some $1 \leq j < \lfloor \ell/d_1 \rfloor$, and we shall prove (3.5) for $j+1$. Since we are in characteristic zero, we have $\mathscr{D}_Y^{\leq m}(J) = \mathscr{D}_Y^{\leq 1}(\mathscr{D}_Y^{\leq m-1}(J))$ for every $m \in \mathbf{N}_{>0}$, and thus:

$$\begin{aligned} I_E^{\ell-jd_1} \cdot \mathscr{D}_{Y'}^{\leq j+1}(J') = \mathscr{D}_{Y'}^{\leq 1}\left(I_E^{\ell-jd_1} \cdot \mathscr{D}_{Y'}^{\leq j}(J')\right) &\supset \mathscr{D}_{Y'}^{\leq 1}\left(\pi^{-1}\big(\mathscr{D}_Y^{\leq j}(J)\big)\mathscr{O}_{Y'}\right) && \text{by (3.5)} \\ &\supset I_E^{d_1} \cdot \pi^{-1}\left(\mathscr{D}_Y^{\leq j+1}(J)\right)\mathscr{O}_{Y'} && \text{by Lemma 3.1.8.} \ \square \end{aligned}$$

The above corollary can in turn be combined with Lemma 2.3.30 to study the behaviour of admissibility of $\mathbf{Q}$-toroidal centers under the operation $\mathscr{D}_Y^{\leq j}$, cf. part (i) of the next lemma. For the remainder of this subsection, let $\mathscr{I}_\bullet$ be a $\mathbf{Q}$-toroidal center on Y. Fixing any $y \in Y$ and replacing Y by a neighbourhood of y in Y, we may also assume that $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a)$ as in (3.3).

Lemma 3.1.10. *Assume $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a)$ is J-admissible, and let $j \in \mathbf{N}_{>0}$. Then:*

(i) *If $a_1 \geq j$, then $\mathscr{I}_{\left(\frac{a_1-j}{a_1}\right)\bullet}$ is $\mathscr{D}_Y^{\leq j}(J)$-admissible.*

(ii) *$\mathscr{I}_{\left(\frac{a_1+j}{a_1}\right)\bullet}$ is $(x_1^j \cdot J)$-admissible.*

Proof. Fix a sufficiently divisible $\ell \in \mathbf{N}_{>0}$ such that $\mathscr{I}_\bullet = I_{\ell\bullet}$ for some toroidal center $I_\bullet = (x_1^{1/d_1}, x_2^{1/d_2}, \ldots, x_k^{1/d_k}, Q^{a/\ell})$ on Y. For (i), we have $\ell = a_1 d_1 \geq j d_1$, so Corollary 3.1.9 implies that $\pi^{-1}\big(\mathscr{D}_Y^{\leq j}(J)\big)\mathscr{O}_{Y'} \subset I_E^{\ell - j d_1}$. By Lemma 2.3.30, $\mathscr{I}_{\left(\frac{a_1-j}{a_1}\right)\bullet} = I_{(\ell - j d_1)\bullet}$ is $\mathscr{D}_Y^{\leq j}(J)$-admissible. For (ii), since $x_1 = x_1' u^{d_1}$ on $Y' = \mathrm{Bl}_{I_\bullet} Y$, we have $\pi^{-1}(x_1^j \cdot J)\mathscr{O}_{Y'} \subset \pi^{-1}(J)\mathscr{O}_{Y'} \cdot I_E^{j d_1} \subset I_E^{\ell + j d_1}$, where the last inclusion is provided by Corollary 3.1.9 again. Thus, by Lemma 2.3.30 again, $\mathscr{I}_{\left(\frac{a_1+j}{a_1}\right)\bullet} = I_{(\ell + j d_1)\bullet}$ is $(x_1^j \cdot J)$-admissible. $\square$

As a consequence, we obtain our first constraint on $\mathbf{Q}$-toroidal centers that are admissible at y. This constraint will be refined further in §3.2.B.

Corollary 3.1.11. *Assume that $\operatorname{log-ord}_y(J) = b_1 < \infty$. If $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a)$ is J-admissible at y with $k \geq 1$, then $a_1 \leq b_1$.*

Proof. By replacing Y by an open neighbourhood of y, we may assume that $\mathscr{I}_\bullet$ is J-admissible and $\mathscr{D}_Y^{\leq b_1}(J) = (1)$. Assume for a contradiction that $a_1 > b_1$. By Lemma 3.1.10(i), we then have $\mathscr{I}_{\left(\frac{a_1-b_1}{a_1}\right)\bullet}$ is $\mathscr{D}_Y^{\leq b_1}(J)$-admissible. Since $\mathscr{D}_Y^{\leq b_1}(J) = (1)$, this forces $\mathscr{I}_{\left(\frac{a_1-b_1}{a_1}\right)\bullet} = \mathscr{O}_Y[t]$. But this is not possible, since

$$\mathscr{I}_{\left(\frac{a_1-b_1}{a_1}\right)\bullet} = \left(x_1^{a_1-b_1}, x_2^{a_2(a_1-b_1)/a_1}, \ldots, x_k^{a_k(a_1-b_1)/a_1}, Q^{a(a_1-b_1)/a_1}\right) \quad \text{with } a_1 - b_1 > 0. \qquad \square$$

3.1.C. Local invariants of Q-toroidal centers in characteristic zero. The preceding Corollary 3.1.11 suggests that it is perhaps meaningful to consider the following local invariant of a $\mathbf{Q}$-toroidal center $\mathscr{I}_\bullet$ on a strict toroidal $\mathbf{k}$-scheme Y:

Definition 3.1.12 (Local invariant of a $\mathbf{Q}$-toroidal center)**.** For $y \in Y$, we define the *invariant* of $\mathscr{I}_\bullet$ at y as

$$\operatorname{inv}_y(\mathscr{I}_\bullet) := \begin{cases} (a_1, a_2, \ldots, a_k, \infty) & \text{if } Q \neq \varnothing \\ (a_1, a_2, \ldots, a_k) & \text{if } Q = \varnothing \end{cases}$$

where $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a)$ is a local presentation of $\mathscr{I}_\bullet$ at y, as in (3.3). Note that if $(\mathscr{I}_\bullet)_y = \mathscr{O}_{Y,y}[t]$, then the local presentation of $\mathscr{I}_\bullet$ at y is $\mathscr{I}_\bullet = (Q = \varnothing)$, so that $\operatorname{inv}_y(\mathscr{I}_\bullet) = ()$. The objective of this subsection is to show inv_y is well-defined:

Theorem 3.1.13. $\operatorname{inv}_y(\mathscr{I}_\bullet)$ *and the number k of finite entries in* $\operatorname{inv}_y(\mathscr{I}_\bullet)$ *are both independent of choice of local presentation of $\mathscr{I}_\bullet$ at y.*

To prove this theorem, fix $y \in Y$. After replacing Y by a neighbourhood of y, we may assume that *globally* on Y, $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a)$, where $x_1, x_2, \ldots, x_k$ are ordinary parameters at y. For the next two lemmas, let us extend $x_1, x_2, \ldots, x_k$ to a system of ordinary parameters $x_1, x_2, \ldots, x_n$ at y.

Lemma 3.1.14 (Exchange)**.** *Suppose that $k \geq 1$. Assume that $x_1', x_2, \ldots, x_n$ is a system of ordinary parameters at y for which $\mathscr{I}_\bullet$ is $(x_1'^{a_1})$-admissible. After possibly replacing Y by a neighbourhood of y, we have $\mathscr{I}_\bullet = \big((x_1')^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a\big)$.*

Proof. The hypothesis says that $\mathscr{I}_\bullet \supset \big((x_1')^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a\big)$. This is necessarily an equality, by passing to

$$\widehat{\mathscr{O}}_{\mathfrak{s}_y,y} = \kappa(y)[\![x_1, x_2, \ldots, x_n]\!] = \kappa(y)[\![x_1', x_2, \ldots, x_n]\!]$$

and observing that the $\kappa(y)$-dimensions of each t^n-graded piece of both sides match. $\square$

Lemma 3.1.15. *Suppose that $k \geq 1$. Let $f \cdot t^\ell \in \mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a)$ be a homogeneous section, and write its image $\overline{f}$ of f under $\mathscr{O}_Y \to \mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{\mathfrak{s}_y,y} \to \widehat{\mathscr{O}}_{\mathfrak{s}_y,y} = \kappa(y)[\![x_1, x_2, \ldots, x_n]\!]$ as*

$$\overline{f} = \sum_{\mathbf{v} \in \mathbf{N}^n} c_{\mathbf{v}} \cdot x^{\mathbf{v}} \quad := \sum_{\mathbf{v}=(v_1,v_2,\ldots,v_n) \in \mathbf{N}^n} c_{\mathbf{v}} \cdot x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} \qquad \text{for } c_{\mathbf{v}} \in \kappa(y).$$

Then $\sum_{i=1}^k \frac{v_i}{a_i} \geq \ell$ for every $\mathbf{v} \in \mathbf{N}^n$ with $c_{\mathbf{v}} \neq 0$.

Proof. We may replace Y with $\mathfrak{s}_y$, and reduce to the case where Y is a smooth $\mathbf{k}$-scheme with trivial logarithmic structure, and $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k})$. By replacing $\mathscr{I}_\bullet$ with $\mathscr{I}_{\ell\bullet}$, we may assume $\ell = 1$, i.e. $\mathscr{I}_\bullet$ is (f)-admissible. Next, note that:

(i) there are only finitely many $\mathbf{v} \in \mathbf{N}^n$ for which $c_{\mathbf{v}} \neq 0$ and $\sum_{i=1}^k v_i/a_i = \nu(\overline{f}) := \min\bigl\{\sum_{i=1}^k v_i/a_i \colon \mathbf{v} \in \mathbf{N}^n,\ c_{\mathbf{v}} \neq 0\bigr\}$, and

(ii) $\mathscr{I}_\bullet$ is $(x^{\mathbf{v}})$-admissible for any monomial $x^{\mathbf{v}}$ for which $c_{\mathbf{v}} \neq 0$, because $\mathscr{I}_\bullet$ is the integral closure of a Rees algebra generated by monomials.

Given these two reasons, we may simply consider $R := \kappa(y)[x_1, x_2, \ldots, x_n]$, replace $\overline{f}$ by

$$\sum_{\substack{\mathbf{v} \in \mathbf{N}^n \\ \sum_{i=1}^k v_i/a_i = \nu(\overline{f})}} c_{\mathbf{v}} \cdot x^{\mathbf{v}} \in R$$

and replace $\mathscr{I}_\bullet$ by $\mathscr{I}_\bullet \cap R = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}) \subset R$. Letting K be the field of fractions of R, consider the following valuation $\nu \in \mathrm{ZR}(K/\kappa(y))$ defined by:

$$\nu\left(\sum_{\mathbf{v} \in \mathbf{N}^n} c_{\mathbf{a}} \cdot x^{\mathbf{v}}\right) = \min\left\{\sum_{i=1}^k \frac{v_i}{a_i} \colon \mathbf{v} \in \mathbf{N}^n,\ c_{\mathbf{v}} \neq 0\right\}.$$

Since $\mathscr{I}_\bullet$ is $(\overline{f})$-admissible, Lemma 2.3.77 implies

$$\nu(\overline{f}) = \gamma_{(\overline{f}),\nu} \geq \gamma_{\mathscr{I}_\bullet,\nu} = \min\{a_i \colon \nu(x_i) \colon 1 \leq i \leq k\} = 1 = \ell. \qquad \square$$

We can now return back to our original goal:

Proof of Theorem 3.1.13. Suppose $\mathscr{I}_\bullet$ admits two local presentations at y:

$$(x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a) = \mathscr{I}_\bullet = \bigl((x_1')^{b_1}, (x_2')^{b_2}, \ldots, (x_k')^{b_k}, (Q')^a\bigr)$$

By considering the image $\overline{\mathscr{I}_\bullet}$ of $\mathscr{I}_\bullet$ under $\mathscr{O}_Y \to \mathscr{O}_{Y,y} \to \mathscr{O}_{\mathfrak{s}_y,y}$, note that $k = 0$ if and only if $\overline{\mathscr{I}_\bullet} = 0$ if and only if $\ell = 0$, in which case it is immediate that $Q \neq \varnothing$ if and only if $Q' \neq \varnothing$, i.e. the lemma is immediate. Henceforth, we may assume $k, \ell \geq 1$. By replacing $\mathscr{I}_\bullet$ by $\mathscr{I}_{\ell\bullet}$ for some sufficiently divisible $\ell \in \mathbf{N}$, we may assume $a_1, b_1 \in \mathbf{N}_{>0}$. In particular, $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a)$ is $(x_1')^{b_1}$-admissible. By Corollary 3.1.11, we have $a_1 \leq b_1$. By reversing roles, we also have $a_1 \geq b_1$, whence $a_1 = b_1$. Applying Lemma 3.1.10(i) repeatedly, we have $\mathscr{I}_{(1/a_1)\bullet} = (x_1, x_2^{a_2/a_1}, \ldots, x_k^{a_k/a_1}, Q^{a/a_1})$ is (x_1')-admissible. Extending $x_1, x_2, \ldots, x_k$ to a system of ordinary parameters $x_1, x_2, \ldots, x_n$ at y, let us write the image of x_1' under $\mathscr{O}_Y \to \mathscr{O}_{Y,y} \to \mathscr{O}_{\mathfrak{s}_y,y} \to \widehat{\mathscr{O}}_{Y,y} = \kappa(y)[\![x_1, x_2, \ldots, x_n]\!]$ as $\sum_{\mathbf{v}\in\mathbf{N}^n} c_{\mathbf{v}} \cdot x^{\mathbf{v}}$ for some $c_{\mathbf{a}} \in \kappa(y)$. By Lemma 3.1.15, we see that for $\mathbf{v} \in \mathbf{N}^n$ with $c_{\mathbf{v}} \neq 0$, we have

$$v_1 + \sum_{i=2}^{k} v_i a_1/a_i \geq 1.$$

Let $k_0 := \max\{1 \leq i \leq k \colon a_i = a_1\} \geq 1$. It follows from the above inequality that $x_1' \in (x_1, x_2, \ldots, x_{k_0}) + \mathfrak{m}_{\mathfrak{s}_y,y}^2 \subset \mathscr{O}_{\mathfrak{s}_y,y}$, where $\mathfrak{m}_{\mathfrak{s}_y,y}$ is the maximal ideal of $\mathscr{O}_{\mathfrak{s}_y,y}$. Thus, after possibly re-ordering $x_1, x_2, \ldots, x_{k_0}$, we may replace x_1' so that $(x_1', x_2, \ldots, x_n)$ is a system of ordinary parameters at y. Applying Lemma 3.1.14, we obtain

$$\bigl((x_1')^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^a\bigr) = \mathscr{I}_\bullet = \bigl((x_1')^{b_1}, (x_2')^{b_2}, \ldots, (x_k')^{a_k}, (Q')^a\bigr)$$

Let us now restrict to the hypersurface $H = V(x_1') \subset Y$, where

$$\left(x_2^{a_2}, \dots, x_k^{a_k}, Q^a\right) = \mathscr{I}_\bullet \mathscr{O}_H = \left((x_2')^{b_2}, \dots, (x_k')^{a_k}, (Q')^a\right).$$

We now apply induction hypothesis to conclude $k = \ell$, $a_i = b_i$ for $2 \le i \le k = \ell$, and $Q \ne \varnothing$ if and only if $Q' \ne \varnothing$ (so $\mathrm{inv}_y(\mathscr{I}_\bullet)$ is well-defined). $\square$

Because of Theorem 3.1.13, we can make the following definition:

Definition 3.1.16 (Reducedness)**.** Let $\mathscr{I}_\bullet$ be a $\mathbf{Q}$-toroidal center on Y. We say $\mathscr{I}_\bullet$ is *reduced at* $y \in Y$ if

$$\mathrm{inv}_y(\mathscr{I}_\bullet) = (1/d_1, 1/d_2, \dots, 1/d_k, *) \qquad (* = \text{empty or } \infty)$$

where either $k = 0$, or $d_1, \dots, d_k \in \mathbf{N}_{>0}$ with $\gcd(d_1, d_2, \dots, d_k) = 1$. We also say $\mathscr{I}_\bullet$ is *reduced* if it is reduced at every $y \in Y$, in which case $\mathscr{I}_\bullet$ is a toroidal center on Y.

3.1.17 (Reduction)**.** Given any $\mathbf{Q}$-toroidal center $\mathscr{I}_\bullet$ on Y and $y \in Y$, let $\mathrm{inv}_y(\mathscr{I}_\bullet) = (a_1, a_2, \dots, a_k, *)$, where the final entry $*$ is either empty or ∞. If $k \ge 1$, there always exists $q \in \mathbf{Q}_{>0}$ such that:

(i) for every $1 \le i \le k$, $a_i q = 1/d_i$ for $d_i \in \mathbf{N}_{>0}$,

(ii) and moreover, $\gcd(d_1, d_2, \dots, d_k) = 1$.

If $k = 0$, set $q := 1$. Then $\mathscr{I}_{q\bullet}$ is reduced at y, and is called the *reduction* of $\mathscr{I}_\bullet$ at y.

3.2. Maximal contact elements and coefficient ideals

In this section, let $J \subset \mathscr{O}_Y$ be an ideal on a strict toroidal $\mathbf{k}$-scheme Y. In this section, we extend various classical notions in the theory of resolution of singularities to the logarithmic geometric setting.

3.2.A. Maximal contact elements. Let $y \in V(J) \subset Y$ and assume $a := \text{log-ord}_y(J) < \infty$.

Definition 3.2.1 (Maximal contact elements, cf. [Kol07, Definition 3.79])**.** A *maximal contact element* of J at y is an element of $\mathscr{D}^{\leq a-1}(J)_y$ that has logarithmic order 1, i.e. can be extended to a system of ordinary parameters at y. The vanishing locus of a maximal contact element of J at y, defined locally around y, is called a *hypersurface of maximal contact* for J at y. The ideal $\mathscr{D}^{\leq a-1}(J)_y \subset \mathscr{O}_{Y,y}$ is called the *maximal contact ideal* of J at y.

It is well-known that hypersurfaces of maximal contact play a crucial role in resolution of singularities in characteristic zero, in the sense that they always exist and they allow for arguments to proceed via induction on dimension: namely, one passes to a hypersurface of maximal contact in the induction step.

Lemma 3.2.2. *If* $\text{char}(\mathbf{k}) = 0$*, a maximal contact element of* J *at* y *always exists.*

Proof. If $\text{char}(\mathbf{k}) = 0$, then $\mathscr{D}^{\leq a}(J) = \mathscr{D}^{\leq 1}(\mathscr{D}^{\leq a-1}(J))$ for any $a \in \mathbf{N}_{>0}$. If $a = \text{log-ord}_y(J)$, we have $\mathscr{D}^{\leq 1}(\mathscr{D}^{\leq a-1}(J))_y = (1)$, from which the lemma follows. $\square$

Definition 3.2.3 (MC-invariant, cf. [Kol07, Definition 3.53])**.** We say J is *MC-invariant* at y if $\mathscr{D}^{\leq a-1}(J)_y \cdot \mathscr{D}^{\leq 1}(J)_y \subset J_y$.

The reason why we care about such a property is reflected in the following

Theorem 3.2.4 (Invariance of maximal contact for MC-invariant ideals)**.** *Assume* J *is MC-invariant at* y*. For every pair of maximal contact elements* x *and* x' *of* J *at* y*, there exist strict and étale morphisms*

$$\widetilde{Y} \underset{\phi_{x'}}{\overset{\phi_x}{\rightrightarrows}} Y$$

from a strict toroidal $\mathbf{k}$*-scheme* $\widetilde{Y}$ *into* Y*, and a point* $\widetilde{y}$ *of* $\widetilde{Y}$ *such that* $\phi_x(\widetilde{y}) = y = \phi_{x'}(\widetilde{y})$*, satisfying the following properties:*

(i) $\phi_x^*(J) = \phi_{x'}^*(J)$;

(ii) $\phi_x^*(x) = \phi_{x'}^*(x')$ *in* $\mathscr{D}^{\leq a-1}(\widetilde{J})$, *where* $\widetilde{J}$ *denotes the ideal in* (i).

Our proof of Theorem 3.2.4 follows the proofs of [**Wło05**, Lemma 3.5.5], [**ATW20a**, Lemma 5.3.3] and [**Kol07**, Theorem 3.92] very closely. For completeness, we provide it here. Let us first fix some notation: let $\kappa/\mathbf{k}$ be a field extension, M be a sharp monoid, and consider the logarithmic κ-algebra $M \to \kappa[\![x_1, \ldots, x_n, M]\!] =: R$, with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n, M \setminus \{1\})$ and sheaf $\mathscr{D} = \mathscr{D}_{M \to R}$ of logarithmic differential operators with order filtration $\mathscr{D}^{\leq m}$. For an ideal $\mathfrak{b} \subset \mathfrak{m}$ of R, we say an automorphism ψ of R is *of the form* $\mathbb{1} + \mathfrak{b}$, if ψ fixes M and maps each x_i to $x_i + f_i$ for some $f_i \in \mathfrak{b}$. Let $J \subset R$ be an ideal.

Lemma 3.2.5 (cf. [**Kol07**, Proposition 3.94])**.** *The following statements are equivalent:*

(i) $\psi(J) = J$ *for every automorphism* ψ *of the form* $\mathbb{1} + \mathfrak{b}$.

(ii) $\mathfrak{b} \cdot \mathscr{D}^{\leq 1}(J) \subset J$.

(iii) $\mathfrak{b}^m \cdot \mathscr{D}^{\leq m}(J) \subset J$ *for every* $m \in \mathbf{N}_{>0}$.

Proof. Assume (iii). Let ψ be an automorphism of the form $\mathbb{1} + \mathfrak{b}$, and for all $1 \leq i \leq n$, let $b_i \in \mathfrak{b}$ such that $\psi(x_i) = x_i + b_i$. Similar to Taylor's expansion, we have:

$$\psi(f) = f + \sum_{i=1}^{n} b_i \cdot \partial_{x_i} f + \frac{1}{2} \sum_{i,j=1}^{n} b_i b_j \cdot \partial_{x_i} \partial_{x_j} f + \cdots$$

i.e. for any $\ell \geq 1$, we get

$$\psi(f) \in J + \mathfrak{b} \cdot \mathscr{D}^{\leq 1}(J) + \cdots + \mathfrak{b}^{\ell} \cdot \mathscr{D}^{\leq \ell}(J) + \mathfrak{m}^{\ell+1} \subset J + \mathfrak{m}^{\ell+1}.$$

By Krull's Intersection Theorem, this implies $\psi(f) \in J$, so we get (i).

Next, assume (i). Let $b \in \mathfrak{b}$, and let $1 \leq i \leq n$. For general $\lambda \in \mathbf{k}$, the endomorphism on R, which maps $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_i + \lambda b, x_{i+1}, \ldots, x_n)$ and fixes M, is an automorphism of R of the form $\mathbb{1} + \mathfrak{b}$. Therefore, for every $f \in J$, and every $\ell \geq 1$,

$$f + \lambda b \cdot \partial_{x_i} f + \cdots + (\lambda b)^\ell \cdot \partial_{x_i}^\ell f \in \psi(f) + \mathfrak{m}^{\ell+1} \subset I + \mathfrak{m}^{\ell+1}.$$

For $\ell + 1$ general elements $\lambda = \lambda_0, \ldots, \lambda_\ell$ in $\mathbf{k}$, the column vector obtained from

$$\begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^\ell \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^\ell \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_\ell & \lambda_\ell^2 & \cdots & \lambda_\ell^\ell \end{pmatrix} \cdot \begin{pmatrix} f \\ b\partial_{x_i} f \\ \vdots \\ b^\ell \partial_{x_i}^\ell f \end{pmatrix}$$

has entries in $J + \mathfrak{m}^{\ell+1}$, and the Vandermonde matrix (λ_i^j) is invertible. In particular, $b \cdot \partial_{x_i} f \in J + \mathfrak{m}^{\ell+1}$. By Krull's intersection theorem again, $b \cdot \partial_{x_i} f \in J$. Since $\mathfrak{b} \cdot \mathscr{D}^{\leq 1}(J)$ is generated by elements of the form $b \cdot f$ or $b \cdot \partial_{x_i} f$ for $b \in \mathfrak{b}$, $f \in J$ and $1 \leq i \leq n$, we get (ii).

Finally, assume (ii). We prove by induction that $\mathfrak{b}^m \cdot \mathscr{D}^{\leq m}(J) \subset J$ for every $m \in \mathbf{N}_{>0}$. The ideal $\mathfrak{b}^{m+1} \cdot \mathscr{D}^{\leq m+1}(J)$ is generated by elements of the form $b_0 \cdots b_m \cdot \mathscr{D}^{\leq 1}(g)$ for $b_0, \ldots, b_m \in \mathfrak{b}$ and $g \in \mathscr{D}^{\leq m}(J)$. Then:

$$\begin{aligned} b_0 \cdots b_m \cdot \mathscr{D}^{\leq 1}(g) &= b_0 \cdot \mathscr{D}^{\leq 1}(b_1 \cdots b_m \cdot g) - \sum_{i=1}^m \mathscr{D}^{\leq 1}(b_i) \cdot (b_0 \cdots \widehat{b_i} \cdots b_m \cdot g) \\ &\in \mathfrak{b} \cdot \mathscr{D}^{\leq 1}(\mathfrak{b}^m \cdot \mathscr{D}^{\leq m}(J)) + \mathfrak{b}^m \cdot \mathscr{D}^{\leq m}(J) \subset \mathfrak{b} \cdot \mathscr{D}^{\leq 1}(J) + \mathfrak{b}^m \cdot \mathscr{D}^{\leq m}(J) \subset J, \end{aligned}$$

where the last two inclusions hold by the induction hypothesis. This proves (iii). □

We can now return back to our original goal:

Proof of Theorem 3.2.4. Let $x_2, \ldots, x_n \in \mathscr{O}_{\mathfrak{s}_y, y}$ such that $x, x_2, \ldots, x_n$ and $x', x_2, \ldots, x_n$ are both regular systems of parameters of $\mathscr{O}_{\mathfrak{s}_y, y}$. We have

$$\kappa[\![x, x_2 \ldots, x_n, M]\!] = \widehat{\mathscr{O}}_{Y,y} = \kappa[\![x', x_2, \ldots, x_n, M]\!], \quad \text{where } \kappa = \kappa(y) \text{ and } M = \overline{\mathscr{M}}_{Y,y}.$$

Consider the endomorphism ψ of $\widehat{\mathscr{O}}_{Y,y}$, which maps $(x, x_2, \ldots, x_n) \mapsto (x' = x+(x'-x), x_2, \ldots, x_n)$ and fixes M. Since $x', x_2, \ldots, x_n$ are linearly independent modulo $\mathfrak{m}_{Y,y}^2$ (where $\mathfrak{m}_{Y,y}$ is the maximal ideal of $\mathscr{O}_{Y,y}$), ψ is an automorphism of $\widehat{\mathscr{O}}_{Y,y}$. Moreover, since x and x' are maximal contact elements at y, we have $x' - x \in \mathscr{D}^{\leq a-1}(\widehat{J}_y)$, whence ψ is an automorphism of $\widehat{\mathscr{O}}_{Y,y}$ of the form $\mathbb{1} + \mathscr{D}^{\leq a-1}(\widehat{J}_y)$. Finally, since J is MC-invariant at y, we have $\mathscr{D}^{\leq a-1}(\widehat{J}_y) \cdot \mathscr{D}^{\leq 1}(\widehat{J}_y) \subset \widehat{J}_y$, whence Lemma 3.2.5 implies $\psi(\widehat{J}_y) = \widehat{J}_y$.

Our goal now is to "realize" this automorphism ψ on $\widehat{\mathscr{O}}_{Y,y}$ on some strict, étale neighbourhood $\widetilde{Y}$ of y. We first extend both $(x, x_2, \ldots, x_n)$ and $(x', x_2, \ldots, x_n)$ to systems of logarithmic coordinates at y:

$$\big(x, x_2 \ldots, x_N, M \xrightarrow{\beta} \Gamma(U, \mathscr{M}_Y|_U)\big) \quad \text{and} \quad \big(x', x_2 \ldots, x_N, M \xrightarrow{\beta} \Gamma(U, \mathscr{M}_Y|_U)\big).$$

After shrinking Y if necessary, Y admits strict and étale morphisms

$$Y \underset{\tau_{x'}}{\overset{\tau_x}{\rightrightarrows}} \operatorname{Spec}(M \to \mathbf{k}[X_1, \ldots, X_N, M])$$

induced by the ring morphisms $\tau_x^{\#}, \tau_{x'}^{\#} \colon \mathbf{k}[X_1, \ldots, X_n] \rightrightarrows \Gamma(Y, \mathscr{O}_Y)$ mapping $(X_1, X_2, \ldots, X_N) \mapsto (x, x_2, \ldots, x_N)$ and $(X_1, X_2, \ldots, X_N) \mapsto (x', x_2, \ldots, x_N)$ respectively, as well as the chart $\beta \colon M \to \Gamma(Y, \mathscr{M}_Y)$. Finally, we obtain the desired $\widetilde{Y}$, by forming the following cartesian square (in the category of fs logarithmic schemes):

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\phi_{x'}"] \arrow[d, "\phi_x"] & Y \arrow[d, "\tau_{x'}"] \\
Y \arrow[r, "\tau_x"] & \operatorname{Spec}(M \to \mathbf{k}[X_1, \ldots, X_N, M])
\end{tikzcd}$$

Since both τ_x and $\tau_{x'}$ are strict and étale, ϕ_x and $\phi_{x'}$ are also strict and étale. Moreover, $\phi_x^*(x) = \phi_x^*(\tau_x^*(X_1)) = \phi_{x'}^*(\tau_{x'}^*(X_1)) = \phi_{x'}^*(x')$. Note that τ_x and $\tau_{x'}$ maps y to the same point in $\operatorname{Spec}(M \to \mathbf{k}[X_1, \ldots, X_N, M])$, so there is a unique point $\widetilde{y} \in \widetilde{Y}$ which is mapped to y via

ϕ_x and $\phi_{x'}$. Finally, the completion of $\widetilde{Y}$ at $\widetilde{y}$ is the graph of the automorphism ψ on $\widehat{\mathscr{O}}_{Y,y}$, and since $\psi(\widehat{J}_y) = \widehat{J}_y$, we may shrink $\widetilde{Y}$ if necessary to arrange for $\phi_x^*(J) = \phi_{x'}^*(J)$. ☐

3.2.B. Coefficient ideals. In this section, we recall the method of taking coefficient ideals. This method originates from Hironaka [**Hir64**], and has been studied extensively in the papers of Bierstone-Milman ([**BM08**], etc), Encinas-Villamayor ([**EV00**], etc), Włodarczyk [**Wło05**], and many others. Our treatment closely follows [**ATW19**], which studies coefficient ideals from the Rees algebra approach of [**EV07b**]. At the end of this section, we briefly indicate why they are necessary for the purposes of this chapter.

Definition 3.2.6 (Coefficient Rees algebras)**.** For $a \in \mathbf{N}_{>0}$, the a^{th} *coefficient Rees algebra* of J is the Rees algebra $G_\bullet(J,a) \subset \mathscr{O}_Y[t]$ on Y generated by $\mathscr{D}^{\leq j}(J) \cdot t^{a-j}$ for $0 \leq j < a$. In other words, its graded pieces are

$$G_m(J,a) := \left(\prod_{j=0}^{a-1} (\mathscr{D}^{\leq j}(J))^{c_j} \colon c_j \in \mathbf{N},\ \sum_{j=0}^{a-1} (a-j)c_j \geq m \right) \subset \mathscr{O}_Y \qquad \text{for } m \in \mathbf{N}.$$

The main reason for putting each $\mathscr{D}^{\leq j}(J)$ in degree $a-j$ is the following easy lemma:

Lemma 3.2.7. *Let $y \in V(J) \subset Y$, $a \in \mathbf{N}_{>0}$.*

(i) *If* $\text{log-ord}_y(J) \geq a$*, then* $\text{log-ord}_y(G_m(J,a)) \geq m$ *for every* $m \in \mathbf{N}$*.*

(ii) *If* $\text{log-ord}_y(J) < a$*, then* $G_m(J,a)_y = (1)$*.*

Proof. For (i), each term $\prod_{j=0}^{a-1} (\mathscr{D}^{\leq j}(J))^{c_j}$ in $G_m(J,a)$ has logarithmic order at y:

$$\sum_{j=0}^{a-1} c_j \cdot \text{log-ord}_y(\mathscr{D}^{\leq j}(J)) = \sum_{j=0}^{a-1} c_j(\text{log-ord}_y(J) - j) \geq \sum_{j=0}^{a-1} c_j(a-j) \geq m$$

whence $\text{log-ord}_y(G_m(J,a)) \geq m$. (ii) follows from the inclusion $\mathscr{D}^{\leq a-1}(J)^a \subset G_m(J,a)$. ☐

Remark 3.2.8. The formation of $G_\bullet(J,a)$ is functorial with respect to logarithmically smooth morphisms, i.e. if $f\colon \widetilde{Y} \to Y$ is a logarithmically smooth morphism of toroidal $\mathbf{k}$-schemes, then $f^{-1}(G_\bullet(J,a))\mathscr{O}_{\widetilde{Y}} = G_\bullet(f^{-1}(J)\mathscr{O}_{\widetilde{Y}}, a)$. This is because the formation of $\mathscr{D}^{\leq j}(J)$ is likewise functorial with respect to logarithmically smooth morphisms.

Lemma 3.2.9. *Let $a \in \mathbf{N}_{>0}$, and $G_\bullet = G_\bullet(J,a)$. Fix $y \in V(J) \subset Y$ such that* $\text{log-ord}_y(J) = a$. *Then:*

(i) $\mathscr{D}^{\leq 1}(G_{m+1}) = G_m$ *for every* $m \in \mathbf{N}$.

(ii) $\mathscr{D}^{\leq m-1}(G_m) = G_1 = \mathscr{D}^{\leq a-1}(J)$ *for every* $m \in \mathbf{N}$. *In particular,* $m = \text{log-ord}_y(G_m)$, *and any maximal contact element of J at y is also a maximal contact element for $G_s(J,a)$ at y.*

(iii) *For every $m \in \mathbf{N}$, G_m is MC-invariant at y.*

(iv) $G_\ell \cdot G_m = G_{\ell+m}$ *whenever* $m \geq (a-1)\cdot\mathrm{lcm}(2,\dots,a)$ *and ℓ is a multiple of* $\mathrm{lcm}(2,\dots,a)$. *In particular, this holds if $m \geq a!$.*

(v) $(G_m)^j = G_{jm}$ *whenever* $m = r\cdot\mathrm{lcm}(2,\dots,a)$ *for some $r \geq a-1$. In particular, this holds for $m = a!$.*

(vi) $(\mathscr{D}^{\leq i}(G_m))^m \subset G_m^{m-i}$ *whenever* $m = r\cdot\mathrm{lcm}(2,\dots,a)$ *for some $r \geq a-1$, and $0 \leq i < m$. In particular, this holds for $m = a!$.*

Proof. This is the "logarithmic" analogue of [**Kol07**, Proposition 3.99]. The proof there works verbatim, although we should point out an inconsequential difference: for the inclusion $G_m \subset \mathscr{D}^{\leq 1}(G_{m+1})$ in (i), the proof utilizes a maximal contact element x of J at a point, which in the logarithmic case is an ordinary parameter, and hence the corresponding logarithmic derivation is still $\frac{\partial}{\partial x}$. ☐

With the exception of property (vi), all the properties in Lemma 3.2.9 are self-explanatory. For example, property (v) says that the $a!$-Veronese subalgebra $G_{a!\bullet}(J,a)$ of $G_\bullet(J,a)$ is generated in degree 1, i.e. it is the Rees algebra of:

Definition 3.2.10 (Coefficient ideals)**.** For $a \in \mathbf{N}_{>0}$, the a^{th} *coefficient ideal* of J is $C(J,a) := G_{a!}(J,a) \subset \mathscr{O}_Y$.

It is well-established in the literature that the coefficient ideal (and its variants) provides a method to enrich an ideal with its higher derivatives, which retains information that would otherwise be lost when one restricts the original ideal (as opposed to the coefficient ideal) to a hypersurface of maximal contact. Next, let us explicate property (vi) in Lemma 3.2.9. Following [Kol07, Definition 3.83], we first formalize it into a definition:

Definition 3.2.11 ($\mathscr{D}$-balanced)**.** Let $y \in V(J) \subset Y$ such that $a := \text{log-ord}_y(J) < \infty$. We say J is *$\mathscr{D}$-balanced* at y if

$$\mathscr{D}^{\leq i}(J)^a_y \subset I_y^{a-i} \qquad \text{for every} \quad 0 \leq i < a.$$

In other words, property (vi) of Lemma 3.2.9 says that the a^{th} coefficient ideal $C(J,a)$ is $\mathscr{D}$-balanced at points $y \in V(J) \subset Y$ satisfying $\text{log-ord}_y(J) = a$.

3.2.12 (What does the "$\mathscr{D}$-balanced" property achieve?)**.** The "$\mathscr{D}$-balanced" property plays a subtle role in this chapter. To start, let $y \in Y$, let x be a maximal contact element of J at $y \in V(J) \subset Y$. For simplicity, let us replace Y by a neighbourhood of y in Y so that the hypersurface $H = V(x) \subset Y$ of maximal contact for J at y is globally defined. If one extends x to a system of ordinary parameters at y, it is not hard to see that

$$\mathscr{D}^{\leq 1}(J\mathscr{O}_H) \subset \mathscr{D}^{\leq 1}(J)\mathscr{O}_H. \tag{3.6}$$

Note, however, that the reverse inclusion does not hold in general. To quote [Kol07, Definition 3.83], the "$\mathscr{D}$-balanced" property provides a partial remedy to this issue.

To explain why the lack of equality in (3.6) poses an issue, consider the following setup. Let $\mathscr{I}_\bullet$ be a $\mathbf{Q}$-toroidal center on Y, and assume the restriction $\mathscr{I}_\bullet\mathscr{O}_H$ of $\mathscr{I}_\bullet$ to H is $J\mathscr{O}_H$-admissible at y. Then a repeated application of Lemma 3.1.10(i) tells us that after replacing $\mathscr{I}_\bullet\mathscr{O}_H$ by some power of itself, $\mathscr{I}_\bullet\mathscr{O}_H$ is $\mathscr{D}^{\leq i}(J\mathscr{O}_H)$-admissible at y. Unfortunately, since the reverse inclusion in (3.6) does not hold,

$$\mathscr{I}_\bullet\mathscr{O}_H \text{ is } \mathscr{D}^{\leq i}(J\mathscr{O}_H)\text{-admissible at } y \quad \not\Rightarrow \quad \mathscr{I}_\bullet\mathscr{O}_H \text{ is } \mathscr{D}^{\leq i}(J)\mathscr{O}_H\text{-admissible at } y.$$

However, if J is $\mathscr{D}$-balanced at y, i.e. $\mathscr{D}^{\leq i}(J)_y^a \subset J_y^{a-i}$, then we obtain the following chain of implications:

$$\begin{aligned}
\mathscr{I}_\bullet\mathscr{O}_H \text{ is } J\mathscr{O}_H\text{-admissible at } y \quad &\Longrightarrow \quad \mathscr{I}_{(a-i)\bullet}\mathscr{O}_H \text{ is } J^{a-i}\mathscr{O}_H\text{-admissible at } y \\
&\Longrightarrow \quad \mathscr{I}_{(a-i)\bullet}\mathscr{O}_H \text{ is } \mathscr{D}^{\leq i}(J)^a\mathscr{O}_H\text{-admissible at } y \\
&\Longrightarrow \quad \mathscr{I}_{\left(\frac{a-i}{a}\right)\bullet}\mathscr{O}_H \text{ is } \mathscr{D}^{\leq i}(J)\mathscr{O}_H\text{-admissible at } y.
\end{aligned}$$

The first and last implications follow from Lemma 2.3.28(iii). It turns out that this strategy is crucial in §3.3.B, cf. the proof of Theorem 3.3.9.

3.2.C. Formal decomposition of coefficient ideal. Let $y \in V(J) \subset Y$, and assume $a := \text{log-ord}_y(J) < \infty$. Let x_1 be a maximal contact element of J at $y \in Y$. Extending it to a system of ordinary parameters $x_1, \ldots, x_n$ at y, we have $\widehat{\mathscr{O}}_{Y,y} = \kappa[\![x_1, x_2 \ldots, x_n, M]\!]$, where $\kappa = \kappa(y)$ and $M = \overline{\mathscr{M}}_{Y,y}$. For $m \in \mathbf{N}_{>0}$, we set

(i) $\widehat{G}_m(J,a) := G_m(J,a)\widehat{\mathscr{O}}_{Y,y}$,

(ii) $\overline{G}_m(J,a) := \widehat{G}_m(J,a)/(x_1) \subset \kappa[\![x_2, \ldots, x_n, M]\!]$,

(iii) and $\widetilde{G}_m(J,a) = \overline{G}_m(J,a)\kappa[\![x_1, x_2, \ldots, x_n, M]\!]$.

Lemma 3.2.13 (Formal decomposition)**.** *For $m \in \mathbf{N}_{>0}$, we have:*

$$\widehat{G}_m(J,a) = (x_1^m) + (x_1^{m-1})\widetilde{G}_1(J,a) + \cdots + (x_1)\widetilde{G}_{m-1}(J,a) + \widetilde{G}_m(J,a).$$

In particular, if $\widehat{C}(J,a) := C(J,a)\widehat{\mathscr{O}}_{Y,y}$, we have:

$$\widehat{C}(J,a) = (x_1^{a!}) + (x_1^{a!-1})\widetilde{G}_1(J,a) + \cdots + (x_1)\widetilde{G}_{a!-1}(J,a) + \widetilde{G}_{a!}(J,a).$$

Proof. We shall prove by induction on m. The base case $m=1$ is clear from the definition of $\widehat{G}_1(J,a)$. For integers $N \geq m$, we have the ideals $(x_1^{N+1}) \subset \widehat{G}_m(J,a)$, which are stable under the linear operator $x_1\partial_{x_1}$. Thus, $x_1\partial_{x_1}$ descends to a linear operator on $\widehat{G}_m(J,a)/(x_1^{N+1})$, and decomposes it into a direct sum of ℓ-eigenspaces for integers $0 \leq \ell \leq N$. These ℓ-eigenspaces are independent of choice of $N \geq m$. More precisely, the ℓ-eigenspace is of the form $x_1^\ell \cdot \widehat{G}_m^{(\ell)}(J,a)$ for a fixed subspace $\widehat{G}_m^{(\ell)}(J,a) \subset \kappa[\![x_2,\ldots,x_n,M]\!]$ independent of $N \geq m$. That is,

$$\widehat{G}_m(J,a)/(x_1^{N+1}) = \bigoplus_{\ell=0}^{N} x_1^\ell \cdot \widehat{G}_m^{(\ell)}(J,a).$$

Here, $\widehat{G}_m^{(0)}(J,a) = \overline{G}_m(J,a)$, and $\widehat{G}_m^{(\ell)}(J,a) = \kappa[\![x_2,\ldots,x_n,M]\!]$ for $\ell \geq m$. For $0 < \ell < m$,

$$\begin{aligned}\widehat{G}_m^{(\ell)}(J,a) = \partial_{x_1}^\ell\big(x_1^\ell \cdot \widehat{G}_m^{(\ell)}(J,a)\big) &\subset \mathscr{D}^{\leq \ell}(\widehat{G}_m(J,a)) \cap \kappa[\![x_2,\ldots,x_n,M]\!] \\ &= \widehat{G}_{m-\ell}(J,a) \cap \kappa[\![x_2,\ldots,x_n,M]\!] \subset \overline{G}_{m-\ell}(J,a),\end{aligned}$$

where the equality in the second line follows from Lemma 3.2.9(i). Thus, we get:

$$\widehat{G}_m(J,a) \subset \widetilde{G}_m(J,a) + (x_1)\widetilde{G}_{m-1}(J,a) + \cdots + (x_1^{m-1})\widetilde{G}_1(J,a) + (x_1^m).$$

The induction hypothesis gives:

$$(x_1)\widetilde{G}_{m-1}(J,a) + \cdots + (x_1^{m-1})\widetilde{G}_1(J,a) + (x_1^m) = (x_1)\widehat{G}_{m-1}(J,a) \subset \widehat{G}_m(J,a).$$

Since $\widetilde{G}_m(J,a) \subset \widehat{G}_m(J,a)$ as well, the lemma follows. $\square$

3.3. A local singularity invariant in characteristic zero

In this section, we outline the main constructions and ideas involved in the proof of Theorem A. As before, J denotes an ideal on a strict toroidal $\mathbf{k}$-scheme Y.

3.3.A. Definition of invariant. Before defining a local singularity invariant for $V(J) \subset Y$, let us first fix the following:

3.3.1 (A well-ordered set)**.** For $k \in \mathbf{N}_{>0}$, we define:

$$\mathbf{N}_{>0}^{k,!} := \left\{ (b_i)_{i=1}^k \in \mathbf{N}_{>0}^k \colon b_1 \leq \frac{b_2}{(b_1-1)!} \leq \frac{b_3}{\prod_{j=1}^{2}(b_j-1)!} \leq \cdots \leq \frac{b_k}{\prod_{j=1}^{k-1}(b_j-1)!} \right\}$$

$$\mathbf{N}_{\infty}^{k,!} := \left(\mathbf{N}_{>0}^{k-1,!} \times \{\infty\}\right) \sqcup \mathbf{N}_{>0}^{k,!}$$

and for $d \in \mathbf{N}_{>0}$, we set:

$$\mathbf{N}_{>0}^{\leq d,!} := \big\{(0)\big\} \sqcup \left(\bigsqcup_{k=0}^{d} \mathbf{N}_{>0}^{k,!} \right) \qquad \text{and} \qquad \mathbf{N}_{\infty}^{\leq d,!} := \big\{(0)\big\} \sqcup \left(\bigsqcup_{k=0}^{d} \mathbf{N}_{\infty}^{k,!} \right).$$

We well-order the set $\mathbf{N}_{\infty}^{\leq d,!}$ by the lexicographic order $<$, with a *caveat*: our lexicographic order considers truncations of sequences to be strictly larger, e.g. in $\mathbf{N}_{\infty}^{\leq 3,!}$, we have:

$$(0) < (1,2,8) < (1,3,6) < (1,3) < (1,4,24) < (1,\infty) < (1) < (\infty) < ().$$

3.3.2. Next, we associate the following preliminary data to the ideal J at a point $y \in Y$:

(i) a sequence $(b_1, \ldots, b_k) \in \mathbf{N}_{>0}^{\leq n,!}$ where $k \leq n := \operatorname{codim}_{\mathfrak{s}_y}(\overline{\{y\}})$,

(ii) a finite sequence of ordinary parameters $x_1, \ldots, x_k$ at y,

(iii) and an ideal $Q \subset M := \overline{\mathscr{M}}_{Y,y}$.

We define these in steps. To begin, let us consider cases:

$\boxed{\textbf{Case 1a}}$ If $\text{log-ord}_y(J) = 0$ (i.e. $y \notin V(J) \subset Y$), we set $k := 1$, $b_1 := 0$, and $Q := \varnothing$. Let x_1 be any ordinary parameter at p.

$\boxed{\textbf{Case 1b}}$ If $\text{log-ord}_y(J) = \infty$ (i.e. $\mathscr{M}(J)_y \neq (1)$), we set $k := 0$, that is, we do not define any b_i or x_i. Define Q to be the image of $\mathscr{M}(J)_y$ under $\mathscr{M}_{Y,y} \to \overline{\mathscr{M}}_{Y,y}$, which we denote by $\overline{\mathscr{M}(J)}_y$. Note that if $J_y = \mathscr{M}(J)_y = 0$, then $Q = \varnothing$.

$\boxed{\textbf{Case 2}}$ If not, set $b_1 := \text{log-ord}_y(J) \in \mathbf{N}_{>0}$ and let x_1 be a maximal contact element of J at y (Definition 3.2.1).

In $\boxed{\textbf{Case 2}}$, set $J[1] = J$, and we shall define the remaining b_i, x_i and Q by means of induction. Assuming that $J[i], b_i, x_i$ are defined for $i \leq \ell$, we set

$$J[\ell+1] := C(J[\ell], b_\ell)\mathscr{O}_{V(x_1,\ldots,x_\ell)} \subset \mathscr{O}_{V(x_1,\ldots,x_\ell)}.$$

In what follows, we pull back the logarithmic structure $\mathscr{M}_Y$ on Y back to define a logarithmic structure $\alpha_{V(x_1,\ldots,x_\ell)} \colon \mathscr{M}_{V(x_1,\ldots,x_\ell)} \to \mathscr{O}_{V(x_1,\ldots,x_k)}$ on $V(x_1,\ldots,x_\ell)$. Note that since $x_1,\ldots,x_\ell$ are ordinary parameters at y, $V(x_1,\ldots,x_\ell)$ is a strict toroidal $\mathbf{k}$-scheme under this logarithmic structure [**ATW20b**, Lemma 5.1.2].

$\boxed{\textbf{Case A}}$ If $\text{log-ord}_y(J[\ell+1]) = \infty$ (i.e. $\mathscr{M}(J[\ell+1])_y \neq (1)$), we set $k := \ell$, that is, no further b_i or x_i are defined. Define Q to be the preimage of $\overline{\mathscr{M}(J[\ell+1])}_y$ under the canonical isomorphism $\overline{\mathscr{M}}_{Y,y} \xrightarrow{\simeq} \overline{\mathscr{M}}_{V(x_1,\ldots,x_k),y}$.

$\boxed{\textbf{Case B}}$ If not, set $b_{\ell+1} := \text{log-ord}_y(J[\ell+1]) \in \mathbf{N}_{>0}$, and define $x_{\ell+1}$ to be a lifting to $\mathscr{O}_Y$ of the maximal contact element of $J[\ell+1]$ at y.

This concludes the induction. Although different choices of ordinary parameters x_i can be made above, the next lemma shows that the b_i and Q are well-defined:

Lemma 3.3.3. *$b_1, b_2, \ldots, b_k$ and Q are independent of the choices of ordinary parameters x_i. In particular, the number of b_i's is also independent of choices of ordinary parameters x_i.*

Proof. Fix a choice of ordinary parameters $x_1, x_2, \ldots, x_k$ as in 3.3.2, and we proceed by induction on k. The case $k = 0$ is $\boxed{\textbf{Case 1b}}$ and it occurs if and only if $\mathscr{M}(J)_y \neq (1)$, in which case there are no b_i and the definition of Q does not require choices. Henceforth, consider $k \geq 1$ (i.e. $\text{log-ord}_y(J) < \infty$). Evidently the integer $b_1 = \text{log-ord}_y(J)$ requires no choices. If $b_1 = 0$, we are in $\boxed{\textbf{Case 1a}}$ and there is nothing to show as well. Thus, we assume $b_1 > 0$. Let x_1' be another maximal contact element of J at y. Because $C(J, b_1)$ is MC-invariant (Lemma 3.2.9(iii)), and x_1, x_1' are still maximal contact elements of $C(J, b_1)$ at y (Lemma 3.2.9(ii)), we can apply Theorem 3.2.4 to $C(J, b_1)$: we get strict and étale morphisms $\phi_{x_1}, \phi_{x_1'} : \widetilde{Y} \rightrightarrows Y$, and a point $\widetilde{y} \in \widetilde{Y}$ such that $\phi_{x_1}(\widetilde{y}) = y = \phi_{x_1'}(\widetilde{y})$. Moreover, $\phi_{x_1}^*(C(J, b_1)) = \phi_{x_1'}^*(C(J, b_1)) =: \widetilde{J}$ and $\phi_{x_1}^*(x_1) = \phi_{x_1'}^*(x_1') =: z \in \widetilde{I}$. Letting $J[2] = C(J, b_1)\mathscr{O}_{V(x_1)}$ and $J[2'] = C(J, b_1)\mathscr{O}_{V(x_1')}$, we have:

$$\phi_{x_1}^*(J[2]) = \widetilde{J}\mathscr{O}_{V(z)} = \phi_{x_1'}^*(J[2']). \tag{3.7}$$

If $k = 1$, we are in $\boxed{\textbf{Case A}}$ above. By [Ogu18, Proposition IV.3.1.6] and Lemma 3.1.2(iii),

$$\phi_{x_1}^*\big(\mathscr{M}(J[2])\big) = \mathscr{M}\left(\widetilde{J}\mathscr{O}_{V(z)}\right) = \phi_{x_1'}^*\big(\mathscr{M}(J[2'])\big). \tag{3.8}$$

Since ϕ_{x_1} is strict, $\phi_{x_1}^\flat : \phi_{x_1}^*(\mathscr{M}_Y) \to \mathscr{M}_{\widetilde{Y}}$ is an isomorphism. We therefore get isomorphisms

$$\overline{\mathscr{M}}_{V(x_1),y} \xleftarrow{\simeq} \overline{\mathscr{M}}_{Y,y} \xrightarrow{\simeq} \overline{\phi_{x_1}^*(\mathscr{M}_Y)}_{\widetilde{y}} \xrightarrow{\simeq} \overline{\mathscr{M}}_{\widetilde{Y},\widetilde{y}}$$

which maps $\overline{\mathscr{M}(J[2])}_y$ on the left, isomorphically, onto $\overline{\phi_{x_1}^*\big(\mathscr{M}(J[2])\big)}_{\widetilde{y}}$ on the right. The same statement holds with $V(x_1)$ replaced by $V(x_1')$, ϕ_{x_1} replaced by $\phi_{x_1'}$, and $J[2]$ replaced by $J[2']$. Combining this with (3.8), one concludes that Q is also independent of choices (and there can be no more b_i's).

On the other hand, if $k \geq 2$, we are in $\boxed{\textbf{Case B}}$ above. Then (3.7) implies

$$\operatorname{log-ord}_y(J[2]) = \operatorname{log-ord}_{\widetilde{y}}(\widetilde{J}\mathscr{O}_{V(z)}) = \operatorname{log-ord}_y(J[2']).$$

Thus, b_2 is independent of choices. By induction hypothesis, the remaining $b_3, b_4, \ldots, b_k$ and Q are independent of choices. $\square$

We are now ready to define the key invariant associated to an ideal at a point:

Definition 3.3.4 (Invariant)**.** The *invariant* of J at $y \in Y$ is defined as:

$$\operatorname{inv}_y(J) = \begin{cases} \left(b_1, \frac{b_2}{(b_1-1)!}, \frac{b_3}{(b_1-1)!(b_2-1)!}, \cdots, \frac{b_k}{\prod_{i=1}^{k-1}(b_i-1)!}\right) & \text{if } Q = \varnothing \\ \left(b_1, \frac{b_2}{(b_1-1)!}, \frac{b_3}{(b_1-1)!(b_2-1)!}, \cdots, \frac{b_k}{\prod_{i=1}^{k-1}(b_i-1)!}, \infty\right) & \text{if } Q \neq \varnothing \end{cases}$$

where $(b_1, b_2, \ldots, b_k)$ and Q are defined for J at y as in 3.3.2. For the remainder of this chapter, we denote the finite entries of $\operatorname{inv}_y(J)$ by a_i. In particular, $a_1 = b_1$. Note that the set of all possible invariants of ideals at points in Y can be well-ordered by the same lexicographic order in 3.3.1, since it is order-isomorphic to $\mathbf{N}_\infty^{\leq \dim(Y),!}$ in 3.3.1.

3.3.5. Note that $\operatorname{inv}_y(J)$ is:

(i) the empty sequence () if and only if $J_y = 0$, i.e. $y \notin \operatorname{Supp}(J)$ (support of J).

(ii) (0) if and only if $J_y = (1)$, i.e. $y \notin V(J) \subset Y$.

(iii) (a_1) for an integer $a_1 \geq 1$ if and only if $J_y = (x_1^{a_1})$ for an ordinary parameter x_1 at y.

(iv) (∞) if and only if $\mathscr{M}(J_y) \neq (1)$, i.e. $y \in V(\mathscr{M}(J)) \subset Y$.

Lemma 3.3.6. inv_y *satisfies the following properties:*

(i) *If* $1 \leq \operatorname{log-ord}_y(J) = a_1 < \infty$*, and* x_1 *is a maximal contact element of* J *at* y*, then* $\operatorname{inv}_y(J) = \left(a_1, \frac{\operatorname{inv}_y\left(C(J,a_1)\mathscr{O}_{V(x_1)}\right)}{(a_1-1)!}\right)$.

(ii) $\operatorname{inv}_y(J)$ *is upper semi-continuous on* Y*, with respect to the lexicographic order in 3.3.1.*

(iii) *Let $V(J)$ inherit its logarithmic structure from Y via the inclusion $V(J) \hookrightarrow Y$. Then $V(J)$ is toroidal at y if and only if $\mathrm{inv}_y(J)$ is the constant sequence $(1, 1, \ldots, 1)$ of length equal to the height of $J_y \subset \mathscr{O}_{Y,y}$.*

(iv) *If $f\colon \widetilde{Y} \to Y$ is a logarithmically smooth morphism of strict toroidal $\mathbf{k}$-schemes which maps $\widetilde{y} \in \widetilde{Y}$ to $y \in Y$, then $\mathrm{inv}_{\widetilde{y}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}}) = \mathrm{inv}_y(J)$.*

PROOF. Part (i) is evident from Definition 3.3.4, while part (iv) follows from Lemma 3.1.4(iv) and Remark 3.2.8. Part (iii) is a consequence of [**ATW20b**, Theorem 5.1.2]. For part (ii), fix some $p \in Y$, and we need to show the locus $Z := \{y \in Y\colon \mathrm{inv}_y(J) \geq \mathrm{inv}_p(J)\}$ is closed in Y. We do so by induction on $k = \mathrm{length}(J, y)$. If $k = 0$, $Z = Y \setminus \mathrm{Supp}(J)$. Since Y is a disjoint union of its irreducible components (2.7.11(iii)), $\mathrm{Supp}(J)$ is a union of some of the irreducible components of Y, whence it is open (and closed) in Y, so Z is closed in Y. Now assume $k \geq 1$. If $a_1 = 0$, $Z = Y$. If $a_1 = \infty$, then $Z = V(\mathscr{M}(J))$ by Lemma 3.1.4(ii). Finally, if $a_1 \in \mathbf{N}_{>0}$, then we first recall that the locus W of points $y \in Y$ with $\text{log-ord}_y(J) > a_1$ is $V(\mathscr{D}^{\leq a_1}(J))$, cf. Lemma 3.1.4(i). Using part (i) of this lemma and induction hypothesis, the locus W' of points y in $V(x_1)$ such that $\mathrm{inv}_y(C(J, a_1)\mathscr{O}_{V(x_1)}) \geq (a_1 - 1)! \cdot (a_2, \ldots, a_k)$ is closed in $V(x_1)$ (and hence, in Y). Note that if $y \in W'$, then $\text{log-ord}_y(J) \geq a_1$; if not, $C(J, a_1)_y = (1)$ (Lemma 3.2.7(ii)), whence $\mathrm{inv}_y(C(J, a_1)\mathscr{O}_{V(x_1)}) = (0) < (a_1 - 1)! \cdot (a_2, \ldots, a_k)$, a contradiction. By part (i) of this lemma again, $Z = W \cup W'$, so Z is closed in Y. $\square$

Because Y is noetherian, part (ii) of the preceding lemma implies part (i) of

Corollary 3.3.7 (Maximum invariant)**.**

(i) $\max \mathrm{inv}(J) := \max_{y \in Y} \mathrm{inv}_y(J)$ *exists.*

(ii) *If $f\colon \widetilde{Y} \to Y$ is a logarithmically smooth and surjective morphism of strict toroidal $\mathbf{k}$-schemes, then $\max \mathrm{inv}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}}) = \max \mathrm{inv}(J)$.*

Definition 3.3.8. Let $y \in Y$. For a choice of ordinary parameters $x_1, x_2, \ldots, x_k$ associated to J at y as in 3.3.2, we define a **Q**-toroidal center $\mathscr{I}(J,y)_\bullet$ on a neighbourhood of y in Y:

$$\mathscr{I}(J,y)_\bullet := \begin{cases} \left(x_1^{b_1}, x_2^{\frac{b_2}{(b_1-1)!}}, x_3^{\frac{b_3}{(b_1-1)!\cdot(b_2-1)!}}, \ldots, x_k^{\frac{b_k}{\prod_{i=1}^{k-1}(b_i-1)!}}\right) & \text{if } Q = \varnothing \\ \left(x_1^{b_1}, x_2^{\frac{b_2}{(b_1-1)!}}, x_3^{\frac{b_3}{(b_1-1)!\cdot(b_2-1)!}}, \ldots, x_k^{\frac{b_k}{\prod_{i=1}^{k-1}(b_i-1)!}}, Q^{\frac{1}{\prod_{i=1}^{k}(b_i-1)!}}\right) & \text{if } Q \neq \varnothing \end{cases}$$

where $(b_1, \ldots, b_k)$ and Q are defined for J at y as in 3.3.2. We call $\mathscr{I}(J,y)$ the **Q**-*toroidal center associated to J at y*. We take a moment to elucidate two "peculiar" cases:

(i) If $k = 0$ and $Q = \varnothing$ (i.e. $J_y = 0$), then $\mathscr{I}(J,y)_\bullet = ()$ is the zero Rees algebra.

(ii) If $k = 1$ and $b_1 = 0$ (i.e. $J_y = (1)$), we use the convention that $x_1^0 := 1$, i.e. $\mathscr{I}(J,y)_\bullet = \mathscr{O}_{Y,y}[t]$.

For the remainder of this chapter, we denote $\mathscr{I}(J,y)_\bullet$ by $(x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d})$, where $Q \subset M := \overline{\mathscr{M}}_{Y,y}$ is an ideal, a_i were similarly defined in Definition 3.3.4, and d is always the positive integer $\prod_{i=1}^{k}(b_i-1)!$. Observe from definition that $\operatorname{inv}_y(\mathscr{I}(J,y)_\bullet) = \operatorname{inv}_y(J)$.

We will soon observe in Corollary 3.3.10 that the stalk of $\mathscr{I}(J,y)_\bullet$ at y does not actually depend on the choice of ordinary parameters $x_1, x_2, \ldots, x_k$ associated to J at y, which justifies the omission of $x_1, x_2, \ldots, x_k$ from the notation.

3.3.B. Unique admissibility of associated Q-toroidal centers.

Theorem 3.3.9 (Unique admissibility)**.** *Let $y \in Y$, and let $\mathscr{I}_\bullet$ be a* **Q***-toroidal center on a neighbourhood U of $y \in Y$ that is J-admissible at y. Then:*

(i) *For any choice of ordinary parameters x_i at y as in 3.3.2, $\mathscr{I}(J,y)_\bullet$ is J-admissible at y.*

(ii) *We have* $\mathrm{inv}_y(\mathscr{I}_\bullet) \leq \mathrm{inv}_y(J)$. *Consequently, we have the characterization:*

$$\mathrm{inv}_y(J) = \max\left\{ \mathrm{inv}_y(\mathscr{I}_\bullet) : \begin{array}{l} \mathscr{I}_\bullet \textit{ is a } \mathbf{Q}\textit{-toroidal center on a neigh-} \\ \textit{bourhood of } y \textit{ that is } J\textit{-admissible at} \\ y \end{array} \right\}.$$

(iii) *Assume* $\mathrm{inv}_y(\mathscr{I}_\bullet) = \mathrm{inv}_y(J)$, *and let* $\mathscr{I}_\bullet = ((x_1')^{a_1}, (x_2')^{a_2}, \ldots, (x_k')^{a_k}, Q'^a)$ *be a local presentation of* $\mathscr{I}_\bullet$ *at* y. *For any choice of ordinary parameters* $x_1, x_2, \ldots, x_k$ *associated to* J *at* y *as in 3.3.2, we have* $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q'^a)$ *after possibly shrinking* U.

Before proving the theorem, let us note an immediate consequence of Theorem 3.3.9:

Corollary 3.3.10. *The stalk of* $\mathscr{I}(J,y)_\bullet$ *at* y *does not depend on the choice of ordinary parameters* $x_1, x_2, \ldots, x_k$ *associated to* J *at* y.

The following lemma will be crucial in the induction step of the proof of Theorem 3.3.9:

Lemma 3.3.11. *Let* $\mathscr{I}_\bullet = ((x_1')^{a_1}, (x_2')^{a_2}, \ldots, (x_k')^{a_k}, (Q')^a)$ *be a* $\mathbf{Q}$*-toroidal center on a neighbourhood of a point* $y \in Y$, *where* $k \geq 1$, *and* $a_1 \in \mathbf{N}_{>0}$. *Then:*

(i) *Suppose* $\mathscr{I}_\bullet$ *is* J*-admissible at* y. *Then for any integer* $1 \leq a \leq a_1$ *and* $s \geq 1$, $\mathscr{I}_{(m/a)\bullet}$ *is* $G_m(J,a)$*-admissible at* y.

(ii) *Conversely, if* $\mathscr{I}_{(a-1)!\bullet}$ *is* $C(J,a)$*-admissible at* y *for some integer* $1 \leq a \leq a_1$, *then* $\mathscr{I}_\bullet$ *is* J*-admissible at* y.

In particular, for any integer $1 \leq a \leq a_1$, $\mathscr{I}_\bullet$ *is* J*-admissible at* y *if and only if* $\mathscr{I}_{(a-1)!\bullet}$ *is* $C(J,a)$*-admissible at* y.

Proof. If $\mathscr{I}_\bullet$ is J-admissible at y, re-iterating Lemma 3.1.10(i) tells us that for all $0 \leq j \leq a-1$, $\mathscr{I}_{\left(\frac{a_1-j}{a_1}\right)\bullet}$ is $\mathscr{D}^{\leq j}(J)$-admissible at y. For $c_0, \ldots, c_{a-1} \in \mathbf{N}$, Lemma 2.3.28(ii) implies that $\mathscr{I}_{\left(\sum_{j=0}^{a-1} \frac{a_1-j}{a_1} c_j\right)\bullet}$ is $\left(\prod_{j=0}^{a-1} (\mathscr{D}^{\leq j}(J))^{c_j}\right)$-admissible at y. Since $a \leq a_1$, we have $\frac{a-j}{a} \leq$

$\frac{a_1-j}{a_1}$ for $0 \leq j \leq a-1$, whence $\mathscr{I}_{(\sum_{j=0}^{a-1} \frac{a-j}{a} c_j)\bullet}$ is $\left(\prod_{j=0}^{a-1} (\mathscr{D}^{\leq j}(J))^{c_j}\right)$-admissible at y. For $(c_0, \ldots, c_{a-1}) \in \mathbf{N}^a$ satisfying $\sum_{j=0}^{a-1} (a-j)c_j \geq m$, we have $\sum_{j=0}^{a-1} \frac{a-j}{a} c_j \geq \frac{m}{a}$, and hence, $\mathscr{I}_{(m/a)\bullet}$ is $\left(\prod_{j=0}^{a-1} (\mathscr{D}^{\leq j}(J))^{c_j}\right)$-admissible at y. By Lemma 2.3.28(i) and Definition 3.2.6, $(\mathscr{I}_\bullet)^{m/a}$ is $G_m(J,a)$-admissible at y. This proves (i).

Conversely, if $\mathscr{I}_{(a-1)!\bullet}$ is $C(J,a)$-admissible at y, then in particular $\mathscr{I}_{(a-1)!\bullet}$ is $J^{(a-1)!}$-admissible at y. By Lemma 2.3.28(iii), $\mathscr{I}_\bullet$ is J-admissible at y. This proves (ii). $\square$

Proof of Theorem 3.3.9(i). Throughout this proof, let us write $\mathscr{I}_\bullet := \mathscr{I}(J,y)_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d})$. We proceed by induction on the length L of $\operatorname{inv}_y(J) = \operatorname{inv}_y(\mathscr{I}_\bullet)$. There is nothing to show if $L = 0$. For $L = 1$, the case $\operatorname{inv}_y(\mathscr{I}_\bullet) = (a_1)$, with $a_1 \in \mathbf{N}$, is evident. If $\operatorname{inv}_y(\mathscr{I}_\bullet) = (\infty)$, then $\mathscr{I}_\bullet$ is J-admissible at y because $\mathscr{M}(J)_y \supset J_y$.

Henceforth, assume $L \geq 2$, so in particular, the first entry in $\operatorname{inv}_y(\mathscr{I}_\bullet)$ is an integer $a_1 \geq 1$. By Lemma 3.3.11(i), we may replace J by $C := C(J, a_1)$ and replace $\mathscr{I}_\bullet$ by $\mathscr{I}_{(a_1-1)!\bullet}$. By Lemma 2.3.29, we may pass to completion at y, and instead show that $\widehat{\mathscr{I}}_{(a_1-1)!\bullet} := \mathscr{I}_{(a_1-1)!\bullet}\widehat{\mathscr{O}}_{Y,y} \supset C^\bullet \widehat{\mathscr{O}}_{Y,y} =: \widehat{C}^\bullet$. By Lemma 3.2.13 we can decompose

$$\widehat{C} = (x_1^{a_1!}) + (x_1^{a_1!-1})\widetilde{G}_1 + \cdots + (x_1)\widetilde{G}_{a_1!-1} + \widetilde{G}_{a_1!}, \qquad \text{where } \widetilde{G}_{a_1!-i} := \widetilde{G}_{a_1!-i}(J, a_1),$$

and therefore, it remains to show $\widehat{\mathscr{I}}_{(a_1-1)!} \supset (x_1^i \cdot \widetilde{G}_{a_1!-i})$ for every $0 \leq i \leq a_1!$. The case $i = a_1!$ is immediate from the definition of $\widehat{\mathscr{I}}$.

For the remaining $0 \leq i < a_1!$, let us replace Y by a neighbourhood of y so that the hypersurface $H := V(x_1) \subset Y$ of maximal contact for J at y is globally defined. By Lemma 3.3.11(i), as well as the induction hypothesis (applied to $C\mathscr{O}_H$ at $y \in H$), $\mathscr{I}_{(a_1-1)!\bullet}\mathscr{O}_H$ is $C\mathscr{O}_H$-admissible at y. By Lemma 2.3.28(iii), we see that

$$\mathscr{I}_{(a_1-1)!(a_1!-i)\bullet}\mathscr{O}_H = \left(x_2^{a_2(a_1-1)!(a_1!-i)}, \ldots, x_k^{a_k(a_1-1)!(a_1!-i)}, Q^{(a_1-1)!(a_1!-i)/d}\right)$$

is $C^{(a_1!-i)}\mathscr{O}_H$-admissible at y. (In the above expression, note that each x_i is more precisely the reduction of x_i modulo $x_1 = 0$.) By parts (i) and (vi) ($=C$ is $\mathscr{D}$-balanced at y) of Lemma 3.2.9, we have $(G_{a_1!-i})_y^{a_1!} = \mathscr{D}^{\leq i}(C)_y^{a_1!} \subset C_y^{a_1!-i}$. Thus, $\mathscr{I}_{(a_1-1)!(a_1!-i)\bullet}\mathscr{O}_H$ is also $G_{a_1!-i}^{a_1!}\mathscr{O}_H$-admissible at y. Consequently, by passing to the completion $\widehat{\mathscr{O}}_{H,y}$ and then extending to $\widehat{\mathscr{O}}_{Y,y} = \widehat{\mathscr{O}}_{H,y}[\![x_1]\!]$, the t^1-graded piece of

$$\widehat{\mathscr{I}_{(a_1-1)!(a_1!-i)\bullet}} = \left(x_1^{a_1!(a_1!-i)}, x_2^{a_2(a_1-1)!(a_1!-i)}, \ldots, x_k^{a_k(a_1-1)!(a_1!-i)}, Q^{(a_1-1)!(a_1!-i)/d}\right)$$

contains $\widetilde{G}_{a_1!-i}^{a_1!}$. Next, by Lemma 3.1.10(ii), the t^1-graded piece of

$$\widehat{\mathscr{I}_{(a_1-1)!a_1!\bullet}} = \left(x_1^{a_1!a_1!}, x_2^{a_2(a_1-1)!a_1!}, \ldots, x_k^{a_k(a_1-1)!a_1!}, Q^{(a_1-1)!a_1!/d}\right)$$

contains $(x_1^{ia_1!} \cdot \widetilde{G}_{a_1!-i}^{a_1!})$, so Lemma 2.3.28(iii) implies that the t^1-graded piece of $\widehat{\mathscr{I}_{(a_1-1)!\bullet}}$ contains $x_1^i \cdot \widetilde{G}_{a_1!-i}$, as desired. □

The proof of the remaining parts is similar to the earlier proof of Theorem 3.1.13. We will prove both parts simultaneously.

Proof of Theorem 3.3.9(ii) & (iii). Again, we prove both parts by induction on the length L of $\mathrm{inv}_y(J)$. There is nothing to show if $L = 0$. For $L = 1$, there is also nothing to show for the cases $\mathrm{inv}_y(J) = (0)$ and $\mathrm{inv}_y(J) = (\infty)$. If $\mathrm{inv}_y(J) = (a_1)$ with $a_1 \in \mathbf{N}_{>0}$, then $J_y = (x_1^{a_1})$ for an ordinary parameter x_1 at y, and both parts are immediate.

Henceforth, assume $L \geq 2$. Let

$$\mathscr{I}_\bullet = ((x_1')^{\alpha_1}, (x_2')^{\alpha_2}, \ldots, (x_\ell')^{\alpha_\ell}, (Q')^{\alpha})$$

be a local presentation of $\mathscr{I}_\bullet$ at y. Since $L \geq 2$, the first entry in $\mathrm{inv}_y(J)$ is the integer $1 \leq a_1 = \text{log-ord}_y(J) < \infty$. Consequently, $\ell \geq 1$. Applying Corollary 3.1.11, $\alpha_1 \leq a_1$. If $\alpha_1 < a_1$, $\mathrm{inv}_y(\mathscr{I}_\bullet) \leq \mathrm{inv}_y(J)$ follows. Thus, assume $\alpha_1 = a_1$ for the remainder of this proof.

Let x_1 be any maximal contact element for J at y. Applying Lemma 3.1.10(i) repeatedly, $\mathscr{I}_{(1/a_1)\bullet} = (x_1', (x_2')^{\alpha_2/a_1}, \ldots, (x_\ell')^{\alpha_\ell/a_1}, (Q' \subset M)^{\alpha/a_1})$ is $\mathscr{D}^{\leq a_1-1}(J)$-admissible at y, and hence, (x_1)-admissible at y. Extending $x_1', x_2', \ldots, x_\ell'$ to a system of ordinary parameters $x_1', x_2', \ldots, x_n'$ at y, and passing to completion at y, we can write the image of x_1 under $\mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{\mathfrak{s}_y,y} \to \widehat{\mathscr{O}}_{\mathfrak{s}_y,y} = \kappa(y)[\![x_1', \ldots, x_n']\!]$ as $\sum_{\mathbf{v}\in\mathbf{N}^n} c_{\mathbf{v}}(x_1')^{v_1}\cdots(x_n')^{v_n}$ for some $c_{\mathbf{v}} \in \kappa(y)$. By Lemma 3.1.15, for any $\mathbf{v} \in \mathbf{N}^n$ such that $c_{\mathbf{v}} \neq 0$, we have:

$$v_1 + \sum_{i=2}^{k} v_i a_1/\alpha_i \geq 1.$$

Consequently, if we let $\ell_0 = \max\{1 \leq i \leq \ell \colon \alpha_i = a_1\} \geq 1$, then the image of x_1 in $\mathscr{O}_{\mathfrak{s}_y,y}$ lies in $(x_1', \ldots, x_{\ell_0}') + \mathfrak{m}_{\mathfrak{s}_y,y}^2$, where $\mathfrak{m}_{\mathfrak{s}_y,y}$ is the maximal ideal of $\mathscr{O}_{\mathfrak{s}_y,y}$. Therefore, after possibly reordering $x_1', \ldots, x_{\ell_0}'$, we may replace x_1' by x_1 so that $(x_1, x_2', \ldots, x_n')$ is a system of ordinary parameters at y. By Lemma 3.1.14, we obtain

$$\mathscr{I}_\bullet = (x_1^{a_1}, (x_2')^{\alpha_2}, \ldots, (x_k')^{\alpha_\ell}, (Q')^{\alpha}).$$

The next natural step is to pass to the induction step.

We may replace Y by a neighbourhood of y so that the hypersurface $H := V(x_1)$ of maximal contact for J at y is globally defined. Let $C := C(J, a_1)$. By Lemma 3.3.11(i), $\mathscr{I}_{(a_1-1)!\bullet}$ is C-admissible at y. In particular,

$$\mathscr{I}_{(a_1-1)!\bullet}\mathscr{O}_H = ((x_2')^{\alpha_2(a_1-1)!}, \ldots, (x_k')^{\alpha_k(a_1-1)!}, (Q')^{\alpha(a_1-1)!})$$

is $C\mathscr{O}_H$-admissible at $y \in H$. By the induction hypothesis for part (ii) of the theorem (applied to $C\mathscr{O}_H$ at $y \in H$), we see that

$$\mathrm{inv}_y(\mathscr{I}_{(a_1-1)!\bullet}\mathscr{O}_H) = (a_2(a_1-1)!, \ldots, a_k(a_1-1)!, *) \leq \mathrm{inv}_y(C\mathscr{O}_H)$$

where the final entry $*$ is either empty or ∞, so that

$$\mathrm{inv}_y(\mathscr{I}_\bullet \mathscr{O}_H) = (a_2, \ldots, a_k, *) \leq \frac{1}{(a_1-1)!} \cdot \mathrm{inv}_y(C\mathscr{O}_H).$$

Applying Lemma 3.3.6(i), we obtain

$$\mathrm{inv}_y(\mathscr{I}_\bullet) = (a_1, a_2, \ldots, a_k, *) = \big(a_1, \mathrm{inv}_y(\mathscr{I}_\bullet \mathscr{O}_H)\big) \leq \left(a_1, \frac{\mathrm{inv}_y(C\mathscr{O}_H)}{(a_1-1)!}\right) = \mathrm{inv}_y(J)$$

which completes the induction step for part (ii) of the theorem. On the other hand, in the event that $\mathrm{inv}_y(\mathscr{I}_\bullet) = \mathrm{inv}_y(J)$ (i.e. $\mathscr{I}_\bullet = (x_1^{a_1}, (x_2')^{a_2}, \ldots, (x_k')^{a_k}, (Q')^{\alpha})$), then by Lemma 3.3.6(i), we have

$$\mathrm{inv}_y(\mathscr{I}_{\bullet(a_1-1)!}\mathscr{O}_H) = (a_2(a_1-1)!, \ldots, a_k(a_1-1)!, *) = \mathrm{inv}_y(C\mathscr{O}_H)$$

where the final entry $*$ is either empty or ∞. For any extension of x_1 to a choice of ordinary parameters $x_1, x_2, \ldots, x_k$ associated to J at y (3.3.2), we may therefore apply the induction hypothesis for part (iii) of the theorem (to $\mathscr{I}_{(a_1-1)!\bullet}\mathscr{O}_H$ which is $C\mathscr{O}_H$-admissible at $y \in H$) to obtain:

$$\mathscr{I}_\bullet \mathscr{O}_H = ((x_2')^{a_2}, \ldots, (x_k')^{a_k}, (Q')^{a}) = (x_2^{a_2}, \ldots, x_k^{a_k}, (Q')^{a}).$$

In the above expression, each x_i' is more precisely the reduction of x_i' modulo $x_1 = 0$, and similarly each x_i is the reduction of x_i modulo $x_1 = 0$. We claim that this implies

$$\mathscr{I}_\bullet = (x_1^{a_1}, (x_2')^{a_2}, \ldots, (x_k')^{a_k}, (Q')^{a}) = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, (Q')^{a}).$$

This equality of integrally closed Rees algebras can be checked directly, but it is more straightforward to check it using idealistic exponents, cf. the next remark. This completes the induction step for part (iii) of the theorem. $\square$

Remark 3.3.12. Let $\mathscr{I}_\bullet$ be a $\mathbf{Q}$-toroidal center, and assume that *globally on Y* we have $\mathscr{I}_\bullet = (x_1^{a_1}, x_2^{a_2}, \dots, x_k^{a_k}, Q^a)$. For each $1 \le i \le k$, consider

$$x_i' = (\lambda_{i,1}x_1 + \cdots + \lambda_{i,i-1}x_{i-1}) + x_i,$$

where $\lambda_{i,j}$ are sections of $\mathscr{O}_Y$. Then we claim that $\mathscr{I}_\bullet = ((x_1')^{a_1}, (x_2')^{a_2}, \dots, (x_k')^{a_k}, (Q')^a)$. To check this painlessly, we simply pass to idealistic exponents: it then suffices to check that for every $\nu \in \mathrm{ZR}(Y/\mathbf{k})$, we have the following equality:

$$\min\bigl(\{a_i\nu(x_i) \colon 1 \le i \le k\} \cup \{a\nu(q) \colon q \in Q\}\bigr) = \min\bigl(\{a_i\nu(x_i') \colon 1 \le i \le k\} \cup \{a\nu(q) \colon q \in Q\}\bigr)$$

which is not a difficult exercise. More generally, note that one can replace each x_i by

$$x_i' = (\lambda_{i,1}x_1 + \cdots + \lambda_{i,i-1}x_{i-1}) + x_i + (\lambda_{i,i+1}x_{i+1} + \cdots + \lambda_{i,\ell}x_\ell),$$

where $\ell = \max\{1 \le j \le k \colon a_j = a_i\}$, and once again $\lambda_{i,j}$ are sections of $\mathscr{O}_Y$.

Combining Theorem 3.3.9(ii) with Lemma 2.3.28 and Lemma 3.3.11, we obtain:

Corollary 3.3.13. *Let $y \in Y$. Then:*

(i) $\mathrm{inv}_y(J^d) = d \cdot \mathrm{inv}_y(J)$ *for any* $d \in \mathbf{N}_{>0}$.

(ii) *If* $\text{log-ord}_y(J) = a_1 \in \mathbf{N}_{>0}$, *then* $\mathrm{inv}_y(C(J,a)) = (a-1)! \cdot \mathrm{inv}_y(J)$ *for any integer* $1 \le a \le a_1$.

3.3.C. The associated toroidal center. Recall from Corollary 3.3.7 that $\max \mathrm{inv}(J) := \max_{y \in Y} \mathrm{inv}_y(J)$ exists. In this subsection, we use unique admissibility from the previous subsection to prove:

Theorem 3.3.14. *There exists a unique J-admissible* $\mathbf{Q}$*-toroidal center* $\mathscr{I}(J)_\bullet$ *on* Y *such that for all* $y \in Y$*, the stalk of* $\mathscr{I}(J)_\bullet$ *at* y *is:*

$$(\mathscr{I}(J)_\bullet)_y = \begin{cases} \mathscr{I}(J,y)_\bullet & \text{if } \mathrm{inv}_y(J) = \max \mathrm{inv}(J) \\ \mathscr{O}_{Y,y}[t] & \text{if } \mathrm{inv}_y(J) < \max \mathrm{inv}(J). \end{cases}$$

The above theorem says that set-theoretically, the co-support of $\mathscr{I}(J)_\bullet$ consists of points $y \in Y$ for which $\mathrm{inv}_y(J) = \max \mathrm{inv}(J)$, and therefore by definition can be interpreted as the *"worst singular" locus* of $V(J) \subset Y$. We make the following

Definition 3.3.15 (Associated toroidal center)**.** The $\mathbf{Q}$*-toroidal center associated to* J is $\mathscr{I}(J)_\bullet$ in Theorem 3.3.14. We also define the *toroidal center* $\underline{\mathscr{I}}(J)_\bullet$ *associated to* J to be the reduction (3.1.17) of $\mathscr{I}(J)_\bullet$ at any point $y \in Y$ satisfying $\mathrm{inv}_y(J) = \max \mathrm{inv}(J)$. This is well-defined, independent of the choice of y, by the previous theorem. Note too that $\underline{\mathscr{I}}(J)_\bullet$ is a reduced toroidal center on Y in the sense of Definition 3.1.16.

Proof of Theorem 3.3.14. Let V denote the open locus in Y whose points are those $y \in Y$ where $\mathrm{inv}_y(J) < \max \mathrm{inv}(J)$. We need to show that we can glue the following:

(i) $\mathscr{O}_V[t]$, and

(ii) for each $y \in Y$ with $\mathrm{inv}_y(J) = \max \mathrm{inv}(J)$, the $\mathbf{Q}$-toroidal center $\mathscr{I}(J,y)_\bullet$ restricted to an sufficiently small open affine neighbourhood U of y.

To this end, fix $y \in Y$ with $\mathrm{inv}_y(J) = \max \mathrm{inv}(J)$. Let

$$\mathscr{I}(J,y)_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d}) \tag{3.9}$$

be a local presentation of $\mathscr{I}(J,y)_\bullet$ at y as in Definition 3.3.8, defined on an open affine neighbourhood U of y in Y. Recall that $x_1, x_2, \ldots, x_k$ are choices of ordinary parameters associated to J at y, and $Q = \overline{\mathscr{M}(J[k+1])}_y \subset \overline{\mathscr{M}}_{V(x_1,\ldots,x_k),y} = \overline{\mathscr{M}}_{Y,y} =: M$ as in 3.3.2. We need to show that after possibly shrinking U, the following are true for any $y' \in U$:

(a) If $\mathrm{inv}_{y'}(J) = \max \mathrm{inv}(J)$, then the stalks of $\mathscr{I}(J,y)_\bullet$ and $\mathscr{I}(J,y')_\bullet$ at y' coincide.

(b) If $\mathrm{inv}_{y'}(J) < \max \mathrm{inv}(J)$, then the stalk of $\mathscr{I}(J,y)_\bullet$ at y' is $\mathscr{O}_{Y,y'}[t]$.

For (a), $x_1,\ldots,x_k$ are also ordinary parameters associated to J at y'. By Theorem 3.3.9(iii), we can therefore express

$$\mathscr{I}(J,y')_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, (Q')^{1/d})$$

where $Q' = \overline{\mathscr{M}(J[k+1])}_{y'} \subset \overline{\mathscr{M}}_{V(x_1,\ldots,x_k),y'} \xleftarrow{\simeq} \overline{\mathscr{M}}_{Y,y'} =: M'$. It remains to note that the ideal of $\mathscr{M}_Y|_U$ generated by the image of $Q = \overline{\mathscr{M}(J[k+1])}_y$ under a chart $\beta\colon M \to \Gamma(U,\mathscr{M}_Y)$ is $\mathscr{M}(J[k+1])$), from which (a) follows.

For (b), let $\mathrm{inv}_{y'}(J) = (\alpha_1,\alpha_2,\ldots,\alpha_\ell,*) < \max \mathrm{inv}(J)$, where the final entry $*$ is either empty or ∞. Note that we must have $k \le \ell$. Let us first consider the case when there exists $1 \le j \le k$ such that $\alpha_j < a_j$. Let $j_0 = \min\{1 \le j \le k \colon \alpha_j < a_j\}$, so that $x_1, x_2, \ldots, x_{j_0-1}$ are ordinary parameters associated to J at y'. Let $J[j-1] \subset \mathscr{O}_{V(x_1,\ldots,x_{j_0-1})}$ be the ideal defined inductively as in 3.3.2. Then $\alpha_{j_0} < a_{j_0}$ implies that $\text{log-ord}_{y'}(J[j-1]) < \text{log-ord}_y(J[j-1]) =: s$, so $\mathscr{D}^{\le s-1}(J)_y \ne (1)$ but $\mathscr{D}^{\le s-1}(J)_{y'} = (1)$. Thus, there exists some section $\delta \in \mathscr{D}^{\le s-1}(J)$ over U such that δ is in the maximal ideal $\mathfrak{m}_{Y,y}$ of $\mathscr{O}_{Y,y}$, but is also an unit in $\mathscr{O}_{Y,y'}$. (If $U = \mathrm{Spec}(A)$ and $y, y' \in U$ correspond to prime ideals $\mathfrak{p}, \mathfrak{p}' \subset A$, then any $\delta \in \mathscr{D}^{\le s-1}(J)|_U \smallsetminus \mathfrak{p}'$ suffices.) By replacing δ by δ^2, we may assume that $\delta \in \mathfrak{m}_{Y,y}^2$. If x_{j_0} is already a unit in $\mathscr{O}_{Y,y'}$, (b) is evident from (3.9). If not, $x_{j_0} + \delta$ is a unit in $\mathscr{O}_{Y,y'}$ and is also an ordinary parameter associated to J at y. Because of Theorem 3.3.9(iii), we are allowed to replace x_{j_0} by $x_{j_0} + \delta$ in (3.9), and (b) follows.

The second case occurs when $\alpha_i = a_i$ for all $1 \le i \le k$. In this case, $x_1, x_2, \ldots, x_k$ are also ordinary parameters associated to J at y'. We always rule out the case $Q = \varnothing$ (which occurs if and only if $J[k+1]_y = 0$, or equivalently, $\mathrm{inv}_y(J) = (a_1, \ldots, a_k)$), *by shrinking* U *so that* $U \cap \mathrm{Supp}(J[k+1]) = \varnothing$. We then have $Q \neq \varnothing$, so that $k < \ell$ with $\alpha'_{k+1} < \infty$, and hence $\mathscr{M}(J[k+1])_{y'} = (1)$. Arguing as in (a) completes the proof for (b) in the second case. $\square$

3.3.D. Functoriality of associated toroidal center.

Lemma 3.3.16. *Let* $f \colon \widetilde{Y} \to Y$ *be a logarithmically smooth morphism of strict toroidal* $\mathbf{k}$*-schemes, which maps* $\widetilde{y} \in Y$ *to* $y \in Y$*. Then*

$$f^{-1}(\mathscr{I}(J,y)_\bullet)\mathscr{O}_{\widetilde{Y}} = \mathscr{I}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}}, \widetilde{y})_\bullet$$

on an open neighbourhood of $\widetilde{y} \in \widetilde{Y}$*. If* f *is moreover surjective,* $f^{-1}(\mathscr{I}(J)_\bullet)\mathscr{O}_{\widetilde{Y}} = \mathscr{I}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}})$*.*

Proof. We may replace Y with an open neighbourhood of y on which a presentation $\mathscr{I}(J,y)_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d})$ as in Definition 3.3.8 is defined. We proceed by induction on k. There is nothing to show if $\text{log-ord}_y(J) = 0$, i.e. $J_y = (1)$. If $\text{log-ord}_y(J) = \infty$, then $f^{-1}(\mathscr{M}(J))\mathscr{O}_{\widetilde{Y}} = \mathscr{M}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}})$ by Lemma 3.1.2(iii), and the lemma is immediate. On the other hand, if $a_1 := \text{log-ord}_y(J) \in \mathbf{N}_{>0}$, then any maximal contact element x_1 of J at y is also a maximal contact element of $f^{-1}(J)\mathscr{O}_{\widetilde{Y}}$ at $\widetilde{y}$. Let H (resp. $\widetilde{H}$) be the hypersurface $V(x_1) \subset Y$ (resp. $V(x_1) \subset \widetilde{Y}$) with the induced logarithmic structure from Y (resp. $\widetilde{Y}$). Then $f \colon \widetilde{H} = f^{-1}(H) \to H$ is logarithmically smooth, so Remark 3.2.8 gives:

$$C(f^{-1}(J)\mathscr{O}_{\widetilde{Y}}, a_1)\mathscr{O}_{\widetilde{H}} = \left(f^{-1}\big(C(J,a_1)\big)\mathscr{O}_{\widetilde{Y}}\right)\mathscr{O}_{\widetilde{H}} = f^{-1}(C(J,a_1)\mathscr{O}_H)\mathscr{O}_{\widetilde{H}}.$$

The proof concludes by applying the induction hypothesis to the ideal $C(J,a_1)\mathscr{O}_H$ on H, and the morphism $f \colon \widetilde{H} = f^{-1}(H) \to H$. $\square$

Corollary 3.3.17. *Let Y be a toroidal Deligne–Mumford stack over $\mathbf{k}$, and fix an atlas*

$$r\colon Y_1 := Y_0 \times_Y Y_0 \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y_0 \xrightarrow{q} Y$$

for Y, where Y_0 and Y_1 are strict toroidal $\mathbf{k}$-schemes. Let $y \in |Y|$.

(i) *If $y_0, y_0' \in Y_0$ are points over y, then $\operatorname{inv}_{y_0}(q^{-1}(J)\mathscr{O}_{Y_0}) = \operatorname{inv}_{y_0'}(q^{-1}(J)\mathscr{O}_{Y_0})$.*

(ii) *If $y_0 \in Y_0$ is a point over y, the $\mathbf{Q}$-toroidal center $\mathscr{I}(q^{-1}(J)\mathscr{O}_{Y_0}, y_1)_\bullet$ descends to a $\mathbf{Q}$-toroidal center $\mathscr{I}(J,y)_\bullet$ on an open substack of Y containing y.*

In particular, the $\mathbf{Q}$-toroidal center $\mathscr{I}(q^{-1}(J)\mathscr{O}_{Y_0})_\bullet$ descends to a $\mathbf{Q}$-toroidal center $\mathscr{I}(J)_\bullet$ on Y.

Proof. Let $(y_0, y_0') \in Y_1$ denote the point mapping to y_i via p_i for $i = 1, 2$. Since $p_{1,2}$ are both strict and étale, Lemma 3.3.6(iii) implies $\operatorname{inv}_{y_0}(p^{-1}(J)\mathscr{O}_{Y_0}) = \operatorname{inv}_{(y_0,y_0')}(r^{-1}(J)\mathscr{O}_{Y_1}) = \operatorname{inv}_{y_0'}(p^{-1}(J)\mathscr{O}_{Y_0})$, so part (i) follows. If that invariant is equal to $\max \operatorname{inv}(p^{-1}(J)\mathscr{O}_{Y_0})$, then Lemma 3.3.16 implies $p_1^*\mathscr{I}(p^{-1}(J)\mathscr{O}_{Y_0}, y_0)_\bullet = \mathscr{I}\big(r^{-1}(J)\mathscr{O}_{Y_1}, (y_0, y_0')\big)_\bullet = p_2^*\mathscr{I}(p^{-1}(J)\mathscr{O}_{Y_0}, y_0')_\bullet$. If not, evidently the same equality holds (they are all equal to $\mathscr{O}_{Y_1,(y_0,y_0')}[t]$). Therefore, we obtain the desired descent in the final statement. Part (ii) is a consequence of the final statement by replacing Y with an invariant open neighbourhood of y_1 on which $\mathscr{I}(J\mathscr{O}_{Y_0}, y_1)_\bullet$ is defined. $\square$

3.4. Iterative logarithmic resolution in characteristic zero

3.4.A. Invariant drops with each weighted blow-up along associated toroidal center.

Let J be a *non-zero* ideal on a strict toroidal $\mathbf{k}$-scheme Y. Write $\max \operatorname{inv}(J) = (a_1, a_2, \ldots, a_k, *)$ as in Definition 3.3.4, and let $\mathscr{I}(J)_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d})$ be a presentation of $\mathscr{I}(J)_\bullet$ at any $y \in Y$ with $\operatorname{inv}_y(J) = \max \operatorname{inv}(J)$ as in Definition 3.3.8. As always, $*$ is either empty or ∞, depending on whether $Q = \varnothing$ or not.

Next, if $k \geq 1$, since $a_1 \in \mathbf{N}_{>0}$, there exists a unique $\ell \in \mathbf{N}_{>0}$ so that $a_i/\ell = 1/d_i$ for every $1 \leq i \leq k$, and $\gcd(d_1, d_2, \ldots, d_k) = 1$ (cf. 3.1.17). If $k = 0$, we set $\ell = 1$. Then the toroidal center $\underline{\mathscr{I}}(J)_\bullet$ associated to J (Definition 3.3.15) has the following local presentation at every $y \in Y$ with $\operatorname{inv}_y(J) = \max \operatorname{inv}(J)$:

$$\underline{\mathscr{I}}(J)_\bullet = \mathscr{I}(J)_{(1/\ell)\bullet} = \left(x_1^{1/d_1}, x_2^{1/d_2}, \ldots, x_k^{1/d_k}, Q^{1/d\ell}\right). \tag{3.10}$$

Recall that if $k = 0$, $d = 1$. The main goal of this subsection is to demonstrate the following

Theorem 3.4.1 (Invariant drops). *Let $\pi \colon Y' := \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet}(Y) \to Y$ be the weighted blow-up of Y along the toroidal center $\underline{\mathscr{I}}(J)_\bullet$ associated to J. Let E be the exceptional divisor of π, with ideal sheaf $I_E \subset \mathscr{O}_{Y'}$. Then:*

(i) *Y' is a toroidal Deligne–Mumford stack over $\mathbf{k}$.*

(ii) *π is an isomorphism away from the closed locus of points $y \in Y$ satisfying $\operatorname{inv}_y(J) < \max \operatorname{inv}(J)$.*

(iii) *We have $\ell = \max\{m \in \mathbf{N} \colon \pi^{-1}(J)\mathscr{O}_{Y'} \subset I_E^m\}$. In particular, if we write $\pi^{-1}(J)\mathscr{O}_{Y'} = I_E^\ell \cdot J'$, then $J' \subset \mathscr{O}_{Y'}$ is the weak transform $\pi_*^{-1}(J)$ of J under π.*

(iv) *$\max \operatorname{inv}(J') < \max \operatorname{inv}(J)$.*

For the definition of weak transform, see Definition 2.3.31. In fact, for the purposes of induction, we must consider a slight generalization:

Theorem 3.4.2. *More generally, for any $c \in \mathbf{N}_{>0}$, consider the weighted blow-up $\pi \colon Y' := \mathrm{Bl}_{\underline{\mathscr{I}}(J)_{(1/c)\bullet}}(Y) \to Y$ along the toroidal center $\underline{\mathscr{I}}(J)_{(1/c)\bullet}$. Let E be the exceptional divisor of π, with ideal sheaf $I_E \subset \mathscr{O}_{Y'}$. Then:*

(i) *Y' is a toroidal Deligne–Mumford stack over $\mathbf{k}$.*

(ii) *π is an isomorphism away from the closed locus of points $y \in Y$ satisfying $\operatorname{inv}_y(J) < \max \operatorname{inv}(J)$.*

(iii) *We have $\ell c = \max\{m \in \mathbf{N} \colon \pi^{-1}(J)\mathscr{O}_{Y'} \subset I_E^m\}$. In particular, if we write $\pi^{-1}(J)\mathscr{O}_{Y'} = I_E^{\ell c} \cdot J'$, then $J' \subset \mathscr{O}_{Y'}$ is the weak transform $\pi_*^{-1}(J)$ of J under π.*

(iv) $\max \mathrm{inv}(J') < \max \mathrm{inv}(J)$.

We first prove the first three parts:

Proof of Theorem 3.4.2(i), (ii) & (ii). Part (i) follows from Corollary 2.7.20, while part (ii) follows from the definition of $\underline{\mathscr{I}}(J)_\bullet$, cf. Theorem 3.3.14. For part (iii), we have $\max\{m \in \mathbf{N} \colon \pi^{-1}(J)\mathscr{O}_{Y'} \subset I_E^m\} = \max\{m \in \mathbf{N} \colon \underline{\mathscr{I}}(J)_{(m/c)\bullet} \text{ is } J\text{-admissible}\}$, by Lemma 2.3.30. Since $\underline{\mathscr{I}}(J)_{\ell\bullet} = \mathscr{I}(J)_\bullet$, the latter is equal to ℓc by Theorem 3.3.9(i). $\square$

We now work towards part (iv) of the theorem, starting with the case $k = 0$, i.e. $\max \mathrm{inv}(J) = ()$ or (∞). If $\max \mathrm{inv}(J) = ()$, J vanishes on at least one connected component of Y. Since $J \neq 0$, J is also non-zero on at least one component of Y. Then $\underline{\mathscr{I}}(J)_\bullet$ is the zero Rees algebra on the components Y_α of Y on which J is zero, and is $\mathscr{O}_{Y_\beta}[t]$ on the remaining components Y_β of Y. Then the weighted blow-up $\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet}$ is simply the inclusion of the components $\bigsqcup_\beta Y_\beta \hookrightarrow Y$ (i.e. it "blows the Y_α's out of existence"), and Theorem 3.4.2(iv) is clear. On the other hand:

Lemma 3.4.3 ("Cleaning up")**.** *If* $\max \mathrm{inv}(J) = (\infty)$, *Theorem 3.4.2(iv) holds.*

Proof. Fix $y \in Y$ such that $\mathrm{inv}_y(J) = \max \mathrm{inv}(J) = (\infty)$. It suffices to show that $\mathrm{inv}_{y'}(J') < \mathrm{inv}_y(J)$ for every $y' \in |Y'|$ mapping to y under π. For that reason we replace Y by a neighbourhood of y so that the presentation $\underline{\mathscr{I}}(J)_\bullet = (Q)$ at y (3.10) is defined on Y, i.e. we have a chart $M \to \Gamma(Y, \mathscr{M}_Y)$ that is neat at y, with $Q \subset M = \overline{\mathscr{M}}_{Y,y}$. As in 3.1.7, we can then write

$$Y' = \mathscr{P}\mathrm{roj}_Y(\mathscr{O}_X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M']) \xrightarrow{\pi} Y \qquad \text{where } M' = M_{\underline{\mathscr{I}}(J)^{\mathrm{ext}}_{(1/c)\bullet}}$$

where $M \to M'$ maps $m \mapsto u^c m'$ for every $m \in Q$. Here, $u = t^{-1}$ is the monomial in $\mathbf{Z}[M']$ corresponding to $(0, -1) \in M'$, and $m' = m \cdot t^c$ is the monomial in $\mathbf{Z}[M']$ corresponding to $(m, c) \in M'$. Since $\pi \colon Y' \to Y$ is logarithmically smooth (Corollary 2.7.18), Lemma 3.1.2(iii) implies that $\mathscr{M}(\pi^{-1}(J)\mathscr{O}_{Y'}) = \pi^{-1}(\mathscr{M}(J))\mathscr{O}_{Y'}$, so that

$$\mathscr{M}(\pi^{-1}(J)\mathscr{O}_{Y'})_{y'} = \pi^{-1}(\mathscr{M}(J)_y)\mathscr{O}_{Y',y'} = \pi^{-1}(Q)\mathscr{O}_{Y',y'}. \tag{3.11}$$

Since every $m \in Q \subset M$ factors as $u^c m'$ in M', and Y' is covered by (m')-charts where m varies over a finite set of generators for Q, we have:

$$\pi^{-1}(Q)\mathscr{O}_{Y',y'} = (u^c) = I_E^c. \tag{3.12}$$

Combining Theorem 3.4.2(iii) with (3.11) and (3.12), we obtain

$$\pi^{-1}(J)\mathscr{O}_{Y',y'} = I_{E,y'}^c \cdot J'_{y'} = \mathscr{M}(\pi^{-1}(J)\mathscr{O}_{Y'})_{y'} \cdot J'_{y'}.$$

Taking $\mathscr{M}(-)$ on both sides and applying Lemma 3.1.2(iv), we obtain

$$\mathscr{M}(\pi^{-1}(J)\mathscr{O}_{Y'})_{y'} = \mathscr{M}(\pi^{-1}(J)\mathscr{O}_{Y'})_{y'} \cdot \mathscr{M}(J')_{y'}.$$

so that by Nakayama's lemma, $\mathscr{M}(J')_{y'} = \mathscr{O}_{Y',y'}$, i.e. $a'_1 := \text{log-ord}_{y'}(J') < \infty$. Thus, $\text{inv}_{y'}(J') = (a'_1, \dots) < (\infty) = \text{inv}_y(J)$. $\square$

For the case $k \geq 1$, the next lemma (and its corollary) shows we can replace J by the coefficient ideal $C := C(J, a_1)$:

Lemma 3.4.4 (cf. [**BM08**, Lemma 3.3])**.** *Assume $k \geq 1$, so that $a_1 := \text{log-ord}_y(J) \in \mathbf{N}_{>0}$. Let $C := C(J, a_1)$. Then:*

(i) $\pi^{-1}(C)\mathscr{O}_{Y'} \subset I_E^{a_1! d_1 c}$, *i.e.* $\pi^{-1}(C)\mathscr{O}_{Y'} = I_E^{a_1! d_1 c} \cdot C'$ *for some ideal* $C' \subset \mathscr{O}_{Y'}$.

(ii) *We have the inclusions:* $(J')^{(a_1-1)!} \subset C' \subset C(J', a_1)$.

Proof. By definition, we have:

$$\pi^{-1}(C)\mathscr{O}_{Y'} = \left(\prod_{j=0}^{a_1-1} (\pi^{-1}(\mathscr{D}_Y^{\leq j}(J)^{c_j})\mathscr{O}_{Y'} \ : \ c_j \in \mathbf{N}, \ \sum_{j=0}^{a_1-1} (a_1 - j)c_j \geq a_1!\right).$$

By Theorem 3.4.2(iii), $\pi^{-1}(J)\mathscr{O}_{Y'} = I_E^{a_1 d_1 c} \cdot J'$. Therefore, by Corollary 3.1.9, we see that for every $1 \leq j < a_1$,

$$\pi^{-1}(\mathscr{D}^{\leq j}(J))\mathscr{O}_{Y'} \subset I_E^{(a_1-j)d_1 c} \cdot \mathscr{D}^{\leq j}(J').$$

Plugging this into the first equation yields:

$$\pi^{-1}(C)\mathscr{O}_{Y'} \subset I_E^{a_1! d_1 c} \cdot \left(\prod_{j=0}^{a_1-1} (\mathscr{D}^{\leq j}(J'))^{c_j} : c_j \in \mathbf{N}, \ \sum_{j=0}^{a_1-1} (a_1 - j)c_j \geq a_1!\right) = I_E^{a_1! d_1 c} \cdot C(J', a_1).$$

From this, we obtain (i) and the second inclusion in (ii). Thus, we get the second inclusion $C' \subset C(J', a_1)$. The first inclusion in (ii) follows from the inclusion

$$\pi^{-1}(C)\mathscr{O}_{Y'} \supset \pi^{-1}(J^{(a_1-1)!})\mathscr{O}_{Y'} = I_E^{a_1! d_1 c} \cdot (J')^{(a_1-1)!}$$

where the inclusion follows from the definition of C, and the equality follows from Theorem 3.4.2(iii). □

Corollary 3.4.5. *For every point $y' \in |Y'|$ over y, we have:*

(i) $\mathrm{inv}_{y'}(C') = (a_1 - 1)! \cdot \mathrm{inv}_{y'}(J')$.

(ii) $\mathrm{inv}_{y'}(J') < \mathrm{inv}_y(J)$ *if and only if* $\mathrm{inv}_{y'}(C') < \mathrm{inv}_y(C)$.

Proof. By Lemma 3.4.4(ii), we have:

$$\mathrm{inv}_{y'}\left((J')^{(a_1-1)!}\right) \geq \mathrm{inv}_{y'}(C') \geq \mathrm{inv}_{y'}(C(J', a_1 - 1)),$$

but Corollary 3.3.13(ii) implies $\mathrm{inv}_{y'}(C(J', a_1)) = (a_1 - 1)! \cdot \mathrm{inv}_{y'}(J') = \mathrm{inv}_{y'}\left((J')^{(a_1-1)!}\right)$. This forces equality throughout, yielding part (i). Part (ii) follows from part (i) and Corollary 3.3.13(ii). $\square$

Proof of Theorem 3.4.2(iv). Fix $y \in Y$ such that $\mathrm{inv}_y(J) = \max \mathrm{inv}(J)$, and it suffices to show that $\mathrm{inv}_{y'}(J') < \mathrm{inv}_y(J)$ for any $y' \in Y'$ mapping to y under π. For that reason we replace Y by a neighbourhood of y so that the presentation $\mathscr{I}(J)_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d})$ at y (3.10) is defined on Y.

We induct on the length L of $\mathrm{inv}_y(J)$. First consider the base case $L \leq 1$. The cases $\mathrm{inv}(J) = (\infty)$ and $\mathrm{inv}_y(J) = ()$ have already been settled (resp. before) in Lemma 3.4.3. On the other hand, if $\mathrm{inv}_y(J) = (a_1)$ with $a_1 < \infty$, then $J_y = (x_1^{a_1})$, and π is the weighted blow-up of $(x_1^{1/c})$ with weak transform $J'_{y'} = (1)$. Henceforth, assume $L \geq 2$. In particular, $k \geq 1$. Then we may and can replace J by C, because of Corollary 3.4.5, as well as the fact that $\underline{\mathscr{I}}(J)_\bullet = \underline{\mathscr{I}}(C)_\bullet$, which is implied by $\mathscr{I}(C)_\bullet = \mathscr{I}(J)_{(a_1-1)!\bullet} = \underline{\mathscr{I}}(J)_{a_1! d_1 \bullet}$.

Let us outline the setup for induction. Let $H := V(x_1) \subset Y$, i.e. a hypersurface of maximal contact for J at y. Let $\underline{\mathscr{I}_H}(J)_\bullet$ denote the reduction of $\mathscr{I}(J)_\bullet \mathscr{O}_H = (x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d})$ at y, i.e.

$$\underline{\mathscr{I}_H}(J)_\bullet = \mathscr{I}(J)_{(c'/\ell)\bullet} \mathscr{O}_H = (x_2^{c'/d_2}, \ldots, x_k^{c'/d_k}, Q^{c'/d\ell})$$

where $c' := \gcd(d_2, \ldots, d_k)$. Then

$$\mathscr{I}(C\mathscr{O}_H)_\bullet = \left(x_2^{a_2(a_1-1)!}, \ldots, x_k^{a_k(a_1-1)!}, Q^{(a_1-1)!/d}\right) = \mathscr{I}^H_{(a_1-1)!\bullet} = \underline{\mathscr{I}_H}(J)_{\left(\frac{\ell(a_1-1)!}{c'}\right)\bullet}$$

so $\underline{\mathscr{I}_H}(J)_\bullet = \underline{\mathscr{I}}(C\mathscr{O}_H)_\bullet$. Since the length of $\mathrm{inv}_y(C\mathscr{O}_H)$ is $< L$, the induction hypothesis implies in particular that the invariant of $C\mathscr{O}_H$ at y drops after the weighted blow-up of H along

$$\underline{\mathscr{I}}(C\mathscr{O}_H)_{\left(\frac{1}{cc'}\right)\bullet} = \underline{\mathscr{I}_H}(J)_{\left(\frac{1}{cc'}\right)\bullet} = \left(x_2^{1/d_2 c}, \ldots, x_k^{1/d_k c}, Q^{1/d\ell c}\right)$$

which coincides with the the proper transform $H' := V(x_1') \to V(x_1) := H$ of H under the weighted blow-up along $\underline{\mathscr{I}}(J)_{(1/c)\bullet} = (x_1^{1/d_1c}, x_2^{1/cd_2}, \ldots, x_k^{1/d_kc}, Q^{1/d\ell c})$. For the remainder of the proof, we shall adopt all the conventions introduced in 3.1.7 for $\pi\colon Y' = \mathrm{Bl}_{\underline{\mathscr{I}}(J)_{(1/c)\bullet}} Y \to Y$. To leverage on the preceding setup, we consider the following cases for y':

(a) y' is in the x_1'-chart of Y';

(b) otherwise, y' is in the proper transform $H' = V(x_1')$.

For case (a), the local section $x_1^{a_1!}$ of C factors as $x_1^{a_1!} = u^{a_1!d_1c} \cdot 1$ in $C\mathscr{O}_{Y'} = I_E^{a_1!d_1c} \cdot C' = (u^{a_1!d_1c} \cdot C')$. Therefore, $C'_{y'} = (1)$, i.e. $\mathrm{inv}_{y'}(C') = (0) < \mathrm{inv}_y(C)$, as desired. For case (b), we saw earlier that the induction hypothesis implies:

$$\mathrm{inv}_{y'}(C'\mathscr{O}_{H'}) < \mathrm{inv}_y(C\mathscr{O}_H). \tag{3.13}$$

Moreover, the local section $x_1^{a_1!}$ of C now factors as $x_1^{a_1!} = u^{a_1!d_1c} \cdot (x_1')^{a_1!}$ in $C\mathscr{O}_{Y'} = I_E^{a_1!d_1c} \cdot C' = (u^{a_1!d_1c} \cdot C')$. Thus, $(x_1')^{a_1!} \subset C'_{y'}$, so that $\text{log-ord}_{y'}(C') \leq a_1!$. We now split (b) into two further sub-cases:

(bi) If $\text{log-ord}_{y'}(C') < a_1!$, then a fortiori $\mathrm{inv}_{y'}(C') < \mathrm{inv}_y(C)$.

(bii) On the other hand, if $\text{log-ord}_{y'}(C') = a_1!$, then x_1' is a maximal contact element for C' at y', whence:

$$\begin{aligned}
\mathrm{inv}_{y'}(C') &= \left(a_1!, \frac{\mathrm{inv}_{y'}(C(C', a_1!)\mathscr{O}_{H'})}{(a_1!-1)!}\right) && \text{by Lemma 3.3.6(i)} \\
&\leq \left(a_1!, \frac{\mathrm{inv}_{y'}(C(C'\mathscr{O}_{H'}, a_1!))}{(a_1!-1)!}\right) && \text{since } C(C', a_1!)\mathscr{O}_{H'} \supset C(C'\mathscr{O}_{H'}, a_1!) \\
&= \big(a_1!, \mathrm{inv}_{y'}(C'\mathscr{O}_{H'})\big) && \text{by Corollary 3.3.13(ii)} \\
&< \big(a_1!, \mathrm{inv}_y(C\mathscr{O}_H)\big) && \text{by (3.13)} \\
&= (a_1-1)! \cdot \mathrm{inv}_y(J) && \text{by Lemma 3.3.6(i)} \\
&= \mathrm{inv}_y(C) && \text{by Corollary 3.3.13(ii).}
\end{aligned}$$

This completes the proof of the induction step. $\square$

3.4.B. Proof of main theorems in §1.2.A.

Proof of Theorem B. By hypothesis, $X \neq Y$. Let J be the underlying non-zero ideal of $X \subset Y$. We take $I_\bullet = \underline{\mathscr{I}}(J)_\bullet$ (Definition 3.3.15), with weighted blow-up $\pi\colon Y' \to Y$ to be $\pi_{\underline{\mathscr{I}}(J)_\bullet}\colon \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$. Then part (i) follows from Theorem 3.4.2(i) and Remark 2.7.9, while part (ii) is Theorem 3.4.2(ii). For part (iii), the proper transform $\widetilde{J}$ of J under π always contains the weak transform J' of J under π (cf. Definition 2.3.31), and thus by definition, $\max\mathrm{inv}(\widetilde{J}) \leq \max\mathrm{inv}(J')$. Since $\max\mathrm{inv}(J') < \max\mathrm{inv}(J)$ by Theorem 3.4.2(iv), we are done. Finally, functoriality with respect to logarithmically smooth, surjective morphisms of pairs follows from Lemma 3.3.16. $\square$

To deduce Theorem A from Theorem B, we require an additional observation:

Lemma 3.4.6. *Let $J \subset \mathscr{O}_Y$ be an ideal. If $\max\mathrm{inv}(J)$ is the constant sequence $(1, 1, \ldots, 1)$ of some length c, then the locus C consisting of points $y \in Y$ such that $\mathrm{inv}_y(J) = \max\mathrm{inv}(J)$ is both open and closed in X.*

Proof. We may assume that Y is a smooth, strict toroidal $\mathbf{k}$-scheme. We already know from Lemma 3.3.6(ii) that C is also closed in X. To show C is open in X, let $y \in C$, and let $x_1, \ldots, x_c$ be ordinary parameters associated to J at y (3.3.2), defined on some open $U \subset Y$. Then $\mathscr{I}(J, y)_\bullet$ is simply the Rees algebra associated to the ideal $(x_1, \ldots, x_c)$, so that:

(a) $J_y \subset (x_1, \ldots, x_c)$, by Theorem 3.3.9(i).

(b) By the definition of $<$ in 3.3.1, note that for $p \in U \cap X$, we have $\mathrm{inv}_p(J) = (a_1, \ldots, a_\ell) < \max\mathrm{inv}(J) = (1, 1, \ldots, 1)$ (of length c), if and only if

$$\mathrm{inv}_p(J) = (\overbrace{1, 1, \ldots, 1}^{\text{length } c}, a_{c+1}, \ldots, a_\ell) \qquad \text{with } \ell > c.$$

By 3.3.2, that happens if and only if $J_p|_{V(x_1,\dots,x_c)} \neq 0$, i.e. $J_p \not\subset (x_1, \dots, x_c)$.

Set $U' := (U \cap X) \smallsetminus V\big((x_1, \dots, x_c) : J\big)$. Then U' is open in X, contains the point p (by (a)), and is moreover contained in C (by (b)). Since $p \in C$ was arbitrary, we conclude that C is open in X. □

Proof of Theorem A. If $X = Y$, there is nothing to show. If not, we define Π inductively. After the k^{th} step of the algorithm (i.e. we have defined $Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} Y_0 = Y$ with proper transforms $X_i \subset Y_i$ of X), we undertake the following steps for the $(k+1)^{\text{th}}$ step:

(i) If $\max \operatorname{inv}(X_k \subset Y_k) = (1, 1, \dots, 1)$ of some length c, then Lemma 3.4.6 says that the locus C_k consisting of points $y \in |Y_k|$ such that $\operatorname{inv}_y(X_k \subset Y_k) = \max \operatorname{inv}(X_k \subset Y_k) = (1, 1, \dots, 1)$ (of length c) is both open and closed in X_k, and hence is a smooth connected component of X_k. We consider two cases:

 (ia) If $C_k = X_k$, we then stop at the k^{th} step.

 (ib) If $C_k \neq X_k$ and $\max \operatorname{inv}(X_k \smallsetminus C_k \subset Y_k) = (1, 1, \dots, 1)$ of some length $c' > c$, we repeat step (i) with $X_k \subset Y_k$ replaced by $X_k \smallsetminus C_k \subset Y_k$. Otherwise, we proceed to step (ii) with $X_k \subset Y_k$ replaced by $X_k \smallsetminus C_k \subset Y_k$.

(ii) If $\max \operatorname{inv}(X_k \subset Y_k) \neq (1, 1, \dots, 1)$ of any length c, we apply Theorem B to $X_k \subset Y_k$, which gives us $\pi_{k+1}\colon Y_{k+1} \to Y_k$ and proper transform $X_{k+1} \subset Y_{k+1}$ of X_k which satisfies $\max \operatorname{inv}(X_{k+1} \subset Y_{k+1}) < \max \operatorname{inv}(X_k \subset Y_k)$.

Under this procedure, observe that at every point y of X, the invariant of proper transforms $X_i \subset Y_i$ at points $y' \in X_i$ above y must eventually drop to $(1, 1, \dots, 1)$ of some length, and moreover, cannot drop to (0) without first dropping to $(1, 1, \dots, 1)$ of some length. This is because X is reduced and generically toroidal, and therefore so are the proper transforms X_i of X. Since the lengths of these invariants are bounded above by $\dim(Y) = \dim(Y')$ (cf. Definition 3.3.4), this procedure eventually terminates to the desired Π.

Finally, if $f\colon \widetilde{Y} \to Y$ is a logarithmically smooth morphism of toroidal Deligne–Mumford stacks over $\mathbf{k}$, and the logarithmic embedded resolution of $X \subset Y$ is $\Pi\colon Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_1} Y_0 = Y$, then it follows from the functoriality in Theorem B that the logarithmic embedded resolution of $X \times_Y \widetilde{Y} \subset \widetilde{Y}$ agrees step-by-step with the pullback of Π along $f\colon \widetilde{Y} \to Y$:

$$Y_N \times_Y \widetilde{Y} \xrightarrow{f^*\pi_N} Y_{N-1} \times_Y \widetilde{Y} \xrightarrow{f^*\pi_{N-1}} \cdots \xrightarrow{f^*\pi_1} \widetilde{Y}$$

after removing any $f^*\pi_i$ which are empty blow-ups, which may occur whenever f is not surjective. $\square$

Remark 3.4.7. Note that the proof of Theorem A simplifies if one assumes $X \subset Y$ is of pure codimension c. In that case, one iterates Theorem B until $\max \operatorname{inv}(X_k \subset Y_k) = (1, 1, \ldots, 1)$ of length c, and the procedure terminates. Indeed, $C_k = X_k$ in part (i) of the proof of Theorem A, because they both contain the dense open $X^{\log-\mathrm{sm}} \subset X$, and are both of pure codimension c in Y_k.

Next, to deduce Corollary C from Theorem A, we require the following:

Lemma 3.4.8 (Re-embedding principle for Theorem B)**.** *Let X be a reduced, closed substack of a toroidal Deligne–Mumford stack Y over $\mathbf{k}$. Let Y_1 be the fiber product $Y \otimes_{\mathbf{k}} \mathbf{A}^1$ in the category of fs logarithmic Deligne-Mumford stacks, where $\mathbf{A}^1 := \operatorname{Spec}(\mathbf{k}[x_0])$ and $\mathbf{k}$ are given the trivial logarithmic structure. Then:*

(i) *For every $y \in |X|$, $\operatorname{inv}_y(X \subset Y_1)$ is the concatenation $(1, \operatorname{inv}_y(X \subset Y))$.*

(ii) *Let $(X \subset Y) \mapsto (X' \subset Y')$ be the procedure in Theorem B. Then Y' is canonically identified with the proper transform of $Y = Y \times \{0\} \subset Y_1$ under the weighted blow-up $Y_1' \to Y_1$. Under this identification, we have $X' = X_1'$.*

Proof. We may assume Y is a strict toroidal $\mathbf{k}$-scheme. Letting J denote the ideal of X in Y, the ideal J_1 of X in Y_1 is $(x_0) + J$. For any $y \in |X|$, $\mathscr{D}_Y^{\leq 1}(J_1)_p = (1)$, with x_0 being a maximal contact element for J_1. As in 3.3.2, we then have $J[2] = J_1\mathscr{O}_{V(x_0)=Y} = J$. Therefore, part (i) follows by definition of the invariant (Definition 3.3.4).

For part (ii), part (i) implies that there is a canonical identification

$$\{y \in Y\colon\ \mathrm{inv}_y(J) = \max \mathrm{inv}(J)\} \xleftrightarrow{\simeq} \{p_1 \in Y_1\colon\ \mathrm{inv}_{p_1}(J_1) = \max \mathrm{inv}(J_1)\}$$

which sends $y \mapsto (y, 0)$. Moreover, for any $y \in Y$ with $\mathrm{inv}_y(J) = \max \mathrm{inv}(J)$, if

$$\mathscr{I}(J, y)_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d})$$

as in Definition 3.3.8, then

$$\mathscr{I}\big(J_1, (y, 0)\big)_\bullet = (x_0, x_1^{a_1}, \ldots, x_k^{a_k}, Q^{1/d})$$

from which the first assertion of part (ii) follows. Moreover, if J' (resp. J_1') denotes the underlying ideal of $X' \subset Y'$ (resp. $X_1' \subset Y_1'$), then $I_1' = (x_0') + I'$ where (x_0') is the proper transform of (x_0) under the weighted blow-up $Y_1' \to Y_1$, whence $X' = X_1'$. □

Proof of Corollary C. Since X can be embedded, locally in the étale topology, as a closed subscheme of pure codimension in a pure-dimensional, toroidal $\mathbf{k}$-scheme, the corollary follows once we show the following. Given two strict closed embeddings of X into pure-dimensional, toroidal Deligne–Mumford stacks Y_i over $\mathbf{k}$ (for $i = 1, 2$), the logarithmic resolutions of X obtained from the logarithmic embedded resolutions of $X \subset Y_1$ and $X \subset Y_2$ in Theorem A coincide. First assume that $\dim(Y_1) = \dim(Y_2)$: in this case, the two embeddings are étale locally isomorphic. By the functoriality of Theorem A, the logarithmic embedded resolutions of $X \subset Y_1$ and $X \subset Y_2$ are étale locally isomorphic, whence the resulting logarithmic resolutions of X coincide. In general, this reduces to the earlier case, by a repeated application of Lemma 3.4.8. □

Proof of Corollary D. We give X the trivial logarithmic structure, in which case $X^{\log-\mathrm{sm}} = X^{\mathrm{sm}}$. We apply Corollary C to produce a logarithmic resolution $\Pi\colon X^+ \to X$. Since X^+ is toroidal with toroidal divisor $D^+ := \Pi^{-1}(X \smallsetminus X^{\mathrm{sm}})$, we can apply [**Wło20b**, Theorem 6.5.1] to resolve its toroidal singularities: it gives us a projective, birational morphism $\Psi\colon X^{++}_w \to X^+$ where:

(i) X^{++}_w is a smooth Deligne–Mumford stack over $\mathbf{k}$.

(ii) Ψ is an isomorphism over $(X^+)^{\mathrm{sm}} \subset X^+$.

(iii) $\Psi^{-1}(D^+)$ is a simple normal crossings divisor.

Moreover, the procedure $X^+ \mapsto X^{++}_w$ is functorial with respect to smooth morphisms $\widetilde{X}^+ \to X^+$. Noting that $X^{\mathrm{sm}} \simeq \Pi^{-1}(X^{\mathrm{sm}}) \subset (X^+)^{\mathrm{sm}}$, the composition

$$\Phi_w\colon X^{++}_w \xrightarrow{\ \Psi\ } X^+ \xrightarrow{\ \Pi\ } X$$

supplies a weak version of Corollary D, namely: Φ_w is proper, but possibly not projective, and X^{++}_w is not necessarily a scheme.

To deduce the corollary from its aforementioned weaker version, we apply destackification in the sense of Bergh–Rydh [**BR19**, Theorem 7.1]. In the language of loc. cit, we apply destackification to the standard pair (X^{++}_w, D^{++}_w) where D^{++}_w is the simple normal crossings divisor $\Pi^{-1}(X \smallsetminus X^{\mathrm{sm}})$, and the morphism $\Phi_w\colon X^{++}_w \to X$. We obtain a projective, birational morphism $\varphi\colon \mathscr{X}^{++} \to X^{++}_w$ such that we have

$$\begin{tikzcd}
(\mathscr{X}^{++}, \varphi^{-1}(D^{++}_w)) \arrow[r, "\varphi"] \arrow[d, "\text{relative coarse space over } X"'] & (X^{++}_w, D^{++}_w) \arrow[r, "\Phi_w"] & X \\
(X^{++}, D^{++}) \arrow[urr, bend right, "\text{projective}"] & &
\end{tikzcd}$$

and such that both $(\mathscr{X}^{++}, \varphi^{-1}(D^{++}_w))$ and (X^{++}, D^{++}) are standard pairs. Moreover, this procedure is functorial with respect to smooth morphisms that are stabilizer preserving. Consequently, the diagonal arrow $X^{++} \to X$ supplies the desired Φ. $\square$

3.4.C. An example. We show, by way of example, that the toroidal Deligne-Mumford stacks Y_i in Theorem A is not necessarily smooth over $\mathbf{k}$, and the proper transform $X_N = Y_N \times_Y X_N$ is also not necessarily smooth over $\mathbf{k}$. This necessitates the need for resolution of toroidal singularities, as outlined in the proof of Corollary D above.

We revisit the following singular surface in [**ATW19**, §8.3]:

$$X = V(J) = V(x^2yz + y^4z) \subset Y = \mathbf{A}^3_{\mathbf{k}}.$$

While Y_1 and Y_2 for this example are smooth over $\mathbf{k}$, we will see below that Y_3 is not. We do this by focusing on a particular chart at each step of our logarithmic resolution.

3.4.9 (*Step 1*)**.** Since $\mathscr{D}^{\leq 4}(J) = (x,y,z)$, we have $\max\operatorname{inv}(X \subset Y) = \operatorname{inv}_{(0,0,0)}(X \subset Y) = (4,4,4)$, and $\mathscr{I}(J)_\bullet = (x^4, y^4, z^4)$. Thus, the first step in our logarithmic resolution involves the blow-up $Y_1 \to Y$ along $\underline{\mathscr{I}}(J)_\bullet = (x,y,z)$, which is not only a strict toroidal $\mathbf{k}$-scheme, but also smooth over $\mathbf{k}$.

For convenience, let us replace Y_1 by one chart on this blow-up, namely:

$$Y_1 := \operatorname{Spec}(\mathbf{N}^1 \to \mathbf{k}[x_1, y_1, \underline{u}_1]) \xrightarrow{\pi_1} Y = \operatorname{Spec}(\mathbf{k}[x,y,z])$$

where $x_1\underline{u}_1 \leftarrow\!\shortmid x$, $y_1\underline{u}_1 \leftarrow\!\shortmid y$, $\underline{u}_1 \leftarrow\!\shortmid z$, and $V(\underline{u}_1) \subset Y_1$ is the equation of the exceptional divisor. We underline $\underline{u}_1$ because its vanishing locus also defines the toroidal divisor on Y_1. On this chart, the total transform of $X \subset Y$ is $\underline{u}_1^4(x_1^2y_1 + y_1^4\underline{u}_1) = 0$, with proper transform

$$X_1 := V(J_1) := V(x_1^2y_1 + y_1^4\underline{u}_1) \subset Y_1 = \operatorname{Spec}(\mathbf{N}^1 \to \mathbf{k}[x_1, y_1, \underline{u}_1]).$$

3.4.10 (*Step 2*)**.** Next, $\mathscr{D}^{\leq 1}(J_1) = (x_1y_1, x_1^2 + 4y_1^3\underline{u}_1, y_1^4\underline{u}_1)$ and $\mathscr{D}^{\leq 2}(J_1) = (x_1, y_1)$, whence $C(J_1, 3)|_{y_1=0} = (x_1^6)$. Therefore, $\max\operatorname{inv}(X_1 \subset Y_1) = (3,3) < (4,4,4) = \max\operatorname{inv}(X \subset Y)$, and

$\mathscr{I}(J_1)_\bullet = (x_1^3, y_1^3)$. The second step in our logarithmic resolution involves the blow-up $Y_2 \to Y_1$ along $\underline{\mathscr{I}}(J_1)_\bullet = (x_1, y_1)$. Similar to the earlier step, Y_2 is a smooth, strict toroidal $\mathbf{k}$-scheme.

Again, we replace Y_2 by one chart on this blow-up, namely:

$$Y_2 := \mathrm{Spec}(\mathbf{N}^2 \to \mathbf{k}[x_2, \underline{u}_2, \underline{v}_2]) \xrightarrow{\pi_2} Y_1 = \mathrm{Spec}(\mathbf{N}^1 \to \mathbf{k}[x_1, y_1, \underline{u}_1])$$

where $x_2\underline{v}_2 \leftarrow\!\shortmid x_1$, $\underline{u}_2\underline{v}_2 \leftarrow\!\shortmid \underline{u}_1$, $\underline{v}_2 \leftarrow\!\shortmid y_1$, and $V(\underline{v}_2) \subset Y_2$ is the equation of the exceptional divisor. As before, we underline $\underline{u}_2$ and $\underline{v}_2$ because the union of their vanishing loci defines the toroidal divisor on Y_2. On this chart, the total transform of $X_1 \subset Y_1$ is $\underline{v}_2^3(x_2^2 + \underline{u}_2\underline{v}_2) = 0$, with proper transform

$$X_2 := V(J_2) := V(x_2^2 + \underline{u}_2\underline{v}_2) \subset Y_2 = \mathrm{Spec}(\mathbf{N}^2 \to \mathbf{k}[x_2, \underline{u}_2, \underline{v}_2]).$$

3.4.11 (*Step 3*)**.** Finally, $\mathscr{D}^{\leq 1}(J_2) = (x_2, \underline{u}_2\underline{v}_2)$, whence $\max\mathrm{inv}(X_2 \subset Y_2) = (2, \infty) < (3, 3) = \max\mathrm{inv}(X_1 \subset Y_1)$. Since $C(J_2, 2)|_{x_2=0} = (\underline{u}_2\underline{v}_2)$, we have $\mathscr{I}(J_2)_\bullet = (x_2^2, \underline{u}_2\underline{v}_2)$. Thus, the third step in our logarithmic resolution involves the weighted blow-up $Y_3 \to Y_2$ along $\underline{\mathscr{I}}(J_2)_\bullet = (x_2, (\underline{u}_2\underline{v}_2)^{1/2})$. We know Y_3 is a toroidal Deligne-Mumford stack over $\mathbf{k}$, but this time the following chart of Y_3 is a scheme that is not smooth over $\mathbf{k}$:

$$U_3 := \mathrm{Spec}(M \to \mathbf{k}[\underline{u}_2, \underline{v}_2, \underline{w}, \underline{u}_2\underline{v}_2\underline{w}^{-2}]) \subset Y_3 \xrightarrow{\pi_3} Y_2 = \mathrm{Spec}(\mathbf{N}^2 \to \mathbf{k}[x_2, \underline{u}_2, \underline{v}_2]).$$

Here, $\underline{w} \leftarrow\!\shortmid x_2$, $V(\underline{w})$ is the equation of the exceptional divisor on U_3, and M is the (saturated) submonoid of $\mathbf{N}^3$ generated by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$. As before, we underline $\underline{u}_2$, $\underline{v}_2$, and $\underline{w}$ to indicate that the union of their vanishing loci defines the toroidal divisor on the aforementioned chart. Moreover, on this chart, the total transform of $X_2 \subset Y_2$ is $\underline{w}^2(1 + \underline{u}_2\underline{v}_2\underline{w}^{-2}) = 0$, with proper transform

$$X_3 \cap U_3 := V(1 + \underline{u}_2\underline{v}_2\underline{w}^{-2}) \hookrightarrow U_3 = \mathrm{Spec}(M \to \mathbf{k}[\underline{u}_2, \underline{v}_2, \underline{w}, \underline{u}_2\underline{v}_2\underline{w}^{-2}]).$$

Note that $\max \mathrm{inv}(U_3 \cap Y_3 \subset U_3) = (1) < (2, \infty) = \max \mathrm{inv}(X_2 \subset Y_2)$, so our logarithmic embedded resolution algorithm stops here (for this chart). In other words, X_3 is toroidal on U_3. However, as a scheme,

$$X_3 \cap U_3 \simeq \operatorname{Spec}\left(\frac{\mathbf{k}[u, v, w]}{(uv + w^2)}\right)$$

is not smooth over $\mathbf{k}$.

CHAPTER 4

Resolution of singularities via multi-weighted blow-ups

4.1. Multi-weighted blow-ups: local aspects

4.1.A. Multi-weighted blow-ups on affine spaces. Throughout this chapter, we follow the conventions on fans in Convention 1.0.1. We first review the notion of fantastacks in [**GS15**]. In the process, we will also fix some notation.

Definition 4.1.1 (Fantastacks)**.** Given a lattice N with dual lattice $M = N^\vee$, let Σ be a fan in $N_{\mathbf{R}}$, and $\beta\colon \mathbf{Z}^r \to N$ a homomorphism with finite cokernel, satisfying the conditions:

(a) Every ray in $\Sigma[1]$ contains some $\beta(\mathbf{e}_i)$.

(b) Every $\beta(\mathbf{e}_i)$ lies in the support of Σ.

Let $\overline{\Sigma}$ denote the set of all sub-cones σ' of cones σ in $N_{\mathbf{R}}$ such that $\sigma'[1] \subset \sigma[1]$, in which case we say σ' can be *inscribed in* σ, and write $\sigma' \sqsubset \sigma$. We call $\overline{\Sigma}$ the *augmentation* of Σ. For each cone σ in $\overline{\Sigma}$, we associate to it the cone $\widehat{\sigma}$ in $\mathbf{R}^r = \mathbf{Z}^r \otimes \mathbf{R}$ generated by those $\mathbf{e}_i$ such that $\beta(\mathbf{e}_i) \in \sigma$. Then $\widehat{\Sigma} := \{\widehat{\sigma} \colon \sigma \in \overline{\Sigma}\}$ is a fan in $\mathbf{R}^r$, that is generated by cones in $\{\widehat{\sigma} \colon \sigma \in \Sigma\}$. The *fantastack* $\mathscr{X}_{\Sigma,\beta}$ associated to (Σ, β) is the toric stack $X_{\widehat{\Sigma},\beta}$ associated to the stacky fan $(\widehat{\Sigma}, \beta)$. That is:

$$\mathscr{X}_{\Sigma,\beta} := \left[X_{\widehat{\Sigma}} \,/\, G_\beta\right], \qquad \text{where} \quad G_\beta := \mathrm{Ker}\left(\mathbb{G}_m^r = T_{\mathbf{Z}^r} \xrightarrow{T_\beta} T_N\right)$$

where:

(i) $X_{\widehat{\Sigma}}$ is the toric variety associated to the fan $\widehat{\Sigma}$ on $\mathbf{Z}^r$.

(ii) $T_N = \mathrm{Hom}_{\mathrm{Grp-Sch}}(N^\vee, \mathbb{G}_m)$ (resp. $T_{\mathbf{Z}^r}$) is the torus of N (resp. $\mathbf{Z}^r$).

(iii) T_β is the homomorphism of tori induced by β.

(iv) G_β acts on $X_{\widehat{\Sigma}}$ as a subgroup of $\mathbb{G}_m^r = T_{\mathbf{Z}^r}$.

4.1.2. Given a fan Σ in $N_{\mathbf{R}}$, we saw in 2.6.1 that there is a canonical fantastack $\mathscr{X}_\Sigma$ associated to the toric variety X_Σ. Slightly more generally, let $r = \#\Sigma[1]$, and define β so that it sends each standard basis vector $\mathbf{e}_\rho$ indexed by a ray ρ in Σ to some lattice point on ρ. For each $\rho \in \Sigma[1]$, write $\beta(\mathbf{e}_\rho)$ as $b_\rho \cdot \mathbf{u}_\rho$ for some $b_\rho \in \mathbf{N}_{>0}$. Then we usually write $\mathscr{X}_{\Sigma,\beta}$ as $\mathscr{X}_{\Sigma,\mathbf{b}}$, where $\mathbf{b} = (b_\rho)_{\rho\in\Sigma[1]} \in \mathbf{N}_{>0}^{\Sigma[1]}$. If $\mathbf{b}$ is the unit vector $(1, 1, \dots, 1)$, then $\mathscr{X}_{\Sigma,\mathbf{b}} = \mathscr{X}_\Sigma$.

4.1.3 (Cox presentation for fantastacks)**.** By [**CLS11**, §5.1], we can express

$$X_{\widehat{\Sigma}} = \mathbf{A}^r \smallsetminus V(\mathbb{J}_\Sigma) = \operatorname{Spec}(\mathbf{k}[x_1, x_2, \dots, x_r]) \smallsetminus V(\mathbb{J}_\Sigma) = \bigcup\{U_\sigma \colon \sigma \in \Sigma[\max]\}$$

where

$$\mathbb{J}_\Sigma = \left(x_\sigma := \prod_{\substack{1\le i\le r\\ \beta(\mathbf{e}_i)\notin\sigma}} x_i \colon \sigma \in \Sigma[\max] \right)$$

is sometimes known as the *irrelevant ideal*, and $U_\sigma := \operatorname{Spec}(\mathbf{k}[x_1, x_2, \dots, x_r][x_\sigma^{-1}])$ for each maximal cone $\sigma \in \Sigma$. Then $\mathscr{X}_{\Sigma,\beta}$ admits a covering by principal open substacks $D_+(\sigma) := [U_\sigma \,/\, G_\beta]$, as σ varies over all maximal cones of σ. We call $D_+(\sigma)$ the x_σ*-chart* of $\mathscr{X}_{\Sigma,\beta}$, and sometimes denote it by $D_+(x_\sigma)$. We cana also define $D_+(\sigma)$ for every cone σ in the augmentation $\overline{\Sigma}$ of Σ.

4.1.4. As already noted in 2.6.1, $\widehat{\Sigma}$ is a smooth fan. Moreover, the torus of $\mathscr{X}_{\Sigma,\beta}$ is $\mathbb{G}_m^r/G_\beta$, which is isomorphic to T_N via T_β. In other words, fantastacks are smooth toric Artin stacks with trivial generic stabilizer.

4.1.5 (Good moduli space of fantastacks)**.** By definition, the morphism β is compatible with the fans $\widehat{\Sigma}$ and Σ, and therefore induces a toric morphism $X_{\widehat{\Sigma}} \to X_\Sigma$, which descends to the good moduli space $\mathscr{X}_{\Sigma,\beta} \to X_\Sigma$, cf. [**GS15**, Example 6.24].

We can now make the key definition of this chapter:

Definition 4.1.6 (Multi-weighted blow-ups on affine spaces)**.** A *multi-weighted blow-up* of the n-dimensional affine space $\mathbf{A}^n$ is the composition

$$\vartheta\colon \mathscr{X}_{\Sigma,\mathbf{b}} \xrightarrow{\text{good moduli space}} X_\Sigma \to \mathbf{A}^n$$

where:

(i) Σ is a fan in $\mathbf{R}^n = (\mathbf{Z}^n)_{\mathbf{R}}$ with $|\Sigma| = (\mathbf{Z}^n)^+_{\mathbf{R}} = \mathbf{R}^n_{\geq 0}$, i.e. Σ subdivides the standard fan Σ_{std} in $\mathbf{R}^n$ and hence induces a proper, birational morphism $X_\Sigma \to X_{\Sigma_{\text{std}}} = \mathbf{A}^n$,

(ii) $\mathbf{b} \in \mathbf{N}^{\Sigma[1]}_{>0}$ as in 4.1.2,

(iii) and $\mathscr{X}_{\Sigma,\mathbf{b}} \to X_\Sigma$ is the good moduli space in the preceding 4.1.4.

4.1.7. Every multi-weighted blow-up ϑ is birational, as explained in 4.1.4. By [**Alp13**, Theorem 4.16], the good moduli space morphism $\mathscr{X}_\Sigma \to X_{\Sigma_{\mathfrak{a}}}$ is universally closed and surjective. Therefore, so is ϑ.

Convention 4.1.8. For the remainder of this thesis, set $N = \mathbf{Z}^n$, and $M = (\mathbf{Z}^n)^\vee$. We will make the obvious identification $\Sigma_{\text{std}}[1] = [n]$. Given a fan Σ in $N_{\mathbf{R}}$ whose support is $N^+_{\mathbf{R}} = \mathbf{R}^n_{\geq 0}$, we always view $[n] = \Sigma_{\text{std}}[1]$ as a subset of $\Sigma[1]$, and we denote the complement $\Sigma[1] \smallsetminus [n]$ by $\Sigma[\text{ex}]$. We call the rays in $[n] \subset \Sigma[1]$ *standard rays*, and the rays in $\Sigma[\text{ex}]$ *exceptional rays*. This terminology will be justified later, cf. 4.1.19. In addition, for a set S, we write $\mathbf{A}^S$ (resp. $\mathbf{Z}^S$) to mean $\mathbf{k}[x_s\colon s \in S]$ (resp. the free $\mathbf{Z}$-module with basis $\{\mathbf{e}_s\colon s \in S\}$. Moving ahead, for $n \in \mathbf{N}_{>0}$, $\mathbf{A}^n$ shall mean $\mathbf{A}^{[n]}$, and $\mathbf{Z}^n$ means $\mathbf{Z}^{[n]}$.

4.1.9 (Explicating multi-weighted blow-ups)**.** Let us further assume that

$$b_i = 1 \qquad \text{for every } i \in [n] = \Sigma_{\text{std}}[1] \subset \Sigma[1]. \tag{4.1}$$

Then the homomorphism $\beta\colon \mathbf{Z}^{\Sigma[1]} \to \mathbf{Z}^n$ induced by $\mathbf{b}$ (as in 4.1.2) fits nicely into the short exact sequence:

$$0 \to \mathbf{Z}^{\Sigma[\mathrm{ex}]} \xrightarrow{\alpha = \begin{bmatrix} \mathbf{B} \\ -\mathbf{I} \end{bmatrix}} \mathbf{Z}^{\Sigma[1]} \xrightarrow{\beta = \begin{bmatrix} \mathbf{I} & \mathbf{B} \end{bmatrix}} \mathbf{Z}^n \to 0$$

where $\mathbf{I}$ denotes the identity matrix of order $\#\Sigma[\mathrm{ex}]$ and $\mathbf{B} = (B_{i,\rho})_{1\le i\le n,\ \rho\in\Sigma[\mathrm{ex}]}$ is the matrix whose ρ^{th}-indexed column is $\mathbf{u}_\rho$ for each $\rho \in \Sigma[\mathrm{ex}]$. Unravelling the definitions, let us highlight some key details:

(i) The matrix $\beta = \begin{bmatrix} I_k & \mathbf{B} \end{bmatrix}$ induces the following commutative diagram:

$$\begin{tikzcd}
X_{\widehat{\Sigma}} = \mathbf{A}^{\Sigma[1]} \smallsetminus V(\mathbb{J}_\Sigma) \arrow[r, hook, "\text{open}"] \arrow[d, two heads, "\text{stack-theoretic quotient}"'] & \mathbf{A}^{\Sigma_{\mathfrak{a}}[1]} \arrow[r] & \mathbf{A}^n = \operatorname{Spec}(\mathbf{k}[x_1, x_2, \ldots, x_n]) \\
\mathscr{X}_{\Sigma,\mathbf{b}} = \left[X_{\widehat{\Sigma}} \,/\, \mathbb{G}_m^{\Sigma[\mathrm{ex}]}\right] \arrow[urr, bend right, "\vartheta"'] & &
\end{tikzcd}$$

where we follow the preceding convention and write

$$\mathbf{A}^{\Sigma[1]} = \operatorname{Spec}\big(\mathbf{k}[x'_1, x'_2, \ldots, x'_n]\,[x'_\rho\colon \rho \in \mathrm{ex}(\mathfrak{a})]\big)$$

so that the morphism $\mathbf{A}^{\Sigma[1]} \to \mathbf{A}^n$ is induced by the homomorphism $\mathbf{k}[x_1, x_2, \ldots, x_n] \to \mathbf{k}[x'_1, x'_2, \ldots, x'_n]\big[x'_\rho\colon \rho \in \Sigma[\mathrm{ex}]\big]$ which maps

$$x_i \mapsto \left(x'_i \cdot \prod_{\rho\in\Sigma[\mathrm{ex}]} (x'_\rho)^{\mathrm{B}_{i,\rho}} \right) = \left(x'_i \cdot \prod_{\rho\in\Sigma[\mathrm{ex}]} (x'_\rho)^{\mathrm{b}_\rho\cdot \mathrm{u}_{\rho,i}} \right)$$

for every $1 \le i \le n$. For $\rho \in \Sigma[\mathrm{ex}]$, the corresponding coordinate x'_ρ of $\mathbf{A}^{\Sigma[1]}$ will occasionally be written as u_ρ during examples (e.g. §4.1.B and §4.4.A).

(ii) On the other hand, the matrix $\alpha = \begin{bmatrix} \mathbf{B} \\ -I_k \end{bmatrix}$ determines the action of $\mathbb{G}_m^{\Sigma[\mathrm{ex}]}$ on $X_{\widehat{\Sigma}} \subset \mathbf{A}^{\Sigma[1]} = \operatorname{Spec}\big(\mathbf{k}[x'_1, x'_2, \ldots, x'_n]\big[x'_\rho\colon \rho \in \Sigma[\mathrm{ex}]\big]\big)$ as follows:

(a) For every $1 \le i \le n$, x'_i has $\mathbf{Z}^{\Sigma[\mathrm{ex}]}$-grading $(\mathrm{B}_{i,\rho})_{\rho\in\Sigma[\mathrm{ex}]} = (\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i})_{\rho\in\Sigma[\mathrm{ex}]}$.

(b) For every $\rho \in \Sigma[\mathrm{ex}]$, x'_ρ has $\mathbf{Z}^{\Sigma[\mathrm{ex}]}$-grading $-\mathbf{e}_\rho = (-\delta_{\rho,\widetilde{\rho}})_{\widetilde{\rho}\in\Sigma[\mathrm{ex}]}$.

(iii) By Definition 4.1.1(i), $\mathscr{X}_{\Sigma,\mathbf{b}}$ admits an open cover by x'_σ*-charts* $D_+(\sigma) = D_+(x'_\sigma) := \left[U_\sigma \,/\, \mathbb{G}_m^{\Sigma[\mathrm{ex}]}\right]$, where σ varies over all maximal cones σ of Σ. Recall from 4.1.3 that for every cone σ in $\overline{\Sigma}$, we have:

$$U_\sigma = \operatorname{Spec}\left(\mathbf{k}[x'_1, x'_2, \ldots, x'_n]\left[x'_\rho \colon \rho \in \mathrm{ex}(\mathfrak{a})\right]\left[(x'_\sigma)^{-1}\right]\right)$$

where

$$x'_\sigma := \prod_{\rho\in\Sigma[1]\smallsetminus\sigma[1]} x'_\rho.$$

(iv) The orbit-cone correspondence for $X_{\widehat{\Sigma}}$ descends to an orbit-cone correspondence for $\mathscr{X}_{\Sigma,\mathbf{b}}$. More precisely, there is a correspondence between the torus orbits of $\mathscr{X}_{\Sigma,\mathbf{b}}$ and the cones in the augmentation $\overline{\Sigma}$ of Σ as follows. For every cone σ in $\overline{\Sigma}$, its corresponding $\mathbb{G}_m^{\Sigma[1]}$-orbit O_σ of $X_{\widehat{\Sigma}}$:

$$O_\sigma := U_\sigma \smallsetminus \bigcup\left\{U_{\sigma'} \colon \sigma' \sqsubset \sigma,\ \sigma' \neq \sigma\right\} \xhookrightarrow{\text{closed}} U_\sigma$$

descends to its corresponding $\left(\mathbb{G}_m^{\Sigma_{\mathfrak{a}}[1]} \,/\, \mathbb{G}_m^{\mathrm{ex}(\mathfrak{a})}\right)$-orbit $O(\sigma)$ of $\mathscr{X}_\Sigma$:

$$\begin{aligned} O(\sigma) &:= \left[O_\sigma \,/\, \mathbb{G}_m^{\mathrm{ex}(\mathfrak{a})}\right] \\ &= D_+(\sigma) \smallsetminus \bigcup\left\{D_+(\sigma') \colon \sigma' \sqsubset \sigma,\ \sigma' \neq \sigma\right\} = V\left(x'_\rho \colon \rho \in \sigma[1]\right) \xhookrightarrow{\text{closed}} D_+(\sigma). \end{aligned}$$

Note that since $U_\sigma = \bigsqcup\{O_\sigma \colon \sigma' \sqsubset \sigma\}$, we also have

$$D_+(\sigma) \ = \ \bigsqcup\left\{O(\sigma) \colon \sigma' \sqsubset \sigma\right\}.$$

Convention 4.1.10. For most parts in this chapter (with the exception of §4.2.A), we usually only consider $\mathbf{b} = (\mathrm{b}_\rho)_{\rho\in\Sigma_{\mathfrak{a}}[1]} \in \mathbf{N}_{>0}^{\Sigma[1]}$ satisfying the hypothesis in (4.1), i.e. $\mathrm{b}_i = 1$ for every $i \in [n] \subset \Sigma[1]$. In this case, we usually view $\mathbf{b}$ as a vector in $\mathbf{N}_{>0}^{\Sigma[\mathrm{ex}]}$. Vice versa, any $\mathbf{b} = (\mathrm{b}_\rho)_{\rho\in\Sigma[\mathrm{ex}]} \in \mathbf{N}_{>0}^{\Sigma[\mathrm{ex}]}$ will always be considered as a vector $(\mathrm{b}_\rho)_{\rho\in\Sigma[1]}$ in $\mathbf{N}_{>0}^{\Sigma[1]}$ by setting $\mathrm{b}_i = 1$ for every $i \in [n]$.

Remark 4.1.11. In the event that $b_\rho \neq 1$ for some $\rho \in \Sigma[\mathrm{ex}]$, one can still explicate the multi-weighted blow-up $\mathscr{X}_{\Sigma,\mathbf{b}}$ in the same way as 4.1.9, although partially. Namely:

$$\mathscr{X}_{\Sigma,\mathbf{b}} = \big[\big(\mathbf{A}^{\Sigma[1]} \smallsetminus V(\mathrm{J}_\Sigma)\big) \;/\; D\big(\mathrm{Coker}\big(\beta^\vee\big)\big)\big] \xrightarrow{\;\vartheta\;} \mathbf{A}^n$$

where ϑ is induced by $\mathbf{k}[x_1, x_2, \ldots, x_n] \to \mathbf{k}[x_1', x_2', \ldots, x_n']\big[x_\rho' \colon \rho \in \Sigma[\mathrm{ex}]\big]$ which maps

$$x_i \mapsto (x_i')^{\mathrm{b}_i} \cdot \prod_{\rho\in\Sigma[\mathrm{ex}]} (x_\rho')^{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}}$$

and where $D(\mathrm{Coker}(\beta^\vee))$ acts via the morphism of diagonalizable groups obtained from $(\mathbf{Z}^{\Sigma[1]})^\vee \twoheadrightarrow \mathrm{Coker}(\beta^\vee)$ by applying $D(-) := \mathrm{Hom}_{\mathrm{Grp-Sch}}(-, \mathbb{G}_m)$. Alternatively, one can resort to the next remark. Finally, $\mathscr{X}_{\Sigma,\mathbf{b}}$ likewise admits an open cover by x_σ'-charts $D_+(\sigma)$ with the same description, and the same orbit-cone corrrespondence for $\mathscr{X}_{\Sigma,\mathbf{b}}$ persists.

Remark 4.1.12. For $i = 1, 2$, let $\mathbf{b}_i = (\mathrm{b}_{i,\rho})_{\rho\in\Sigma[1]} \in \mathbf{N}_{>0}^{\Sigma[1]}$ be such that for each $\rho \in \Sigma[1]$,

$$\mathrm{b}_{2,\rho} = c_\rho \cdot \mathrm{b}_{1,\rho} \qquad \text{for some } c_\rho \in \mathbf{N}_{>0}.$$

Then $\mathscr{X}_{\Sigma,\mathbf{b}_2}$ can be obtained from $\mathscr{X}_{\Sigma,\mathbf{b}_1}\mathbf{A}^n$ by iteratively taking ${c_\rho}^{\mathrm{th}}$ root stacks (cf. Example 2.2.13, [**Cad07b**, Definition 2.3.1], or [**AGV08**, Appendix B]) along each coordinate hyperplane $V(x_\rho')$ of $\mathscr{B}\mathrm{l}_{\mathfrak{a},\mathbf{b}_1} \mathbf{A}^n$.

Most multi-weighted blow-ups on affine spaces $\mathbf{A}^n$ used in applications are "canonically associated" to a monomial ideal $\mathfrak{a} \subset \mathbf{k}[x_1, x_2, \ldots, x_n]$. We focus on these next.

Setup 4.1.13. Let $N = \mathbf{Z}^n$ and $M = (\mathbf{Z}^n)^\vee$. Consider a monomial ideal $\mathfrak{a}$ of $\mathbf{k}[x_1, x_2, \ldots, x_n]$ under the logarithmic structure induced by the chart $\mathbf{N}^n \to \mathbf{k}[x_1, x_2, \ldots, x_n]$ sending each standard basis vector $\mathbf{e}_i$ of $\mathbf{N}^n$ to x_i (cf. Definition 2.7.5). That is, a monomial in $\mathbf{k}[x_1, x_2, \ldots, x_n]$ is $x^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for some $\mathbf{a} \in \mathbf{N}^n$, and $\mathfrak{a}$ is generated by a finite set of monomials. Then associated to $\mathfrak{a}$ are the following classical notions:

(i) the submonoid $\Gamma_{\mathfrak{a}} = \{\mathbf{a} \in M^+ \colon x^{\mathbf{a}} \in \mathfrak{a}\} \subset M^+$,

(ii) the Newton polyhedron $\Gamma_+(\mathfrak{a})$ of $\mathfrak{a}$ given by the convex hull of $\Gamma_{\mathfrak{a}}$ in $M^+_{\mathbf{R}}$,

(iii) and the normal fan $\Sigma_{\mathfrak{a}}$ of $\Gamma_+(\mathfrak{a})$, which is a subdivision of Σ_{std} in N, and hence induces a proper, birational toric morphism $X_{\Sigma_{\mathfrak{a}}} \to \mathbf{A}^n$.

These are reviewed in slightly greater generality in the next chapter, cf. §5.1.A and §5.1.B. We also follow the conventions pertaining to Newton polyhedra there. For example, we shall denote faces of $\Gamma_+(\mathfrak{a})$ by ς, and denote facets of $\Gamma_+(\mathfrak{a})$ by τ instead of ς. Additionally, for every integer $0 \leq k \leq n$, there is an inclusion-reversing "dual" correspondence between k-dimensional cones σ of $\Sigma_{\mathfrak{a}}$ and $(n-k)$-dimensional faces ς of $\Gamma_+(\mathfrak{a})$, and we notate this "duality" as follows:

(a) ς_σ is the face of $\Gamma_+(\mathfrak{a})$ dual to a cone σ in $\Sigma_{\mathfrak{a}}$, while σ_ς is the cone in $\Sigma_{\mathfrak{a}}$ dual to a face ς of $\Gamma_+(\mathfrak{a})$.

(b) If σ is a maximal cone in $\Sigma_{\mathfrak{a}}$, we denote the vertex ς_σ of $\Gamma_+(f)$ by $\mathbf{v}_\sigma = (\mathrm{v}_{\sigma,i})_{i=1}^n$ instead. If ς is a facet τ of $\Gamma_+(\mathfrak{a})$, then we denote the ray σ_ς as ρ_τ instead.

(c) Given a ray ρ in Σ with dual facet τ of $\Gamma_+(\mathfrak{a})$, we denote the affine span of τ_ρ by H_ρ or H_τ. We also let $N_\rho(\mathfrak{a}) = N_\tau(\mathfrak{a})$ be the natural number so that H_ρ has the equation $\{\mathbf{a} \in M_{\mathbf{R}} \colon \mathbf{a} \cdot \mathbf{u}_\rho = N_\rho(\mathfrak{a})\}$ in $M_{\mathbf{R}}$.

Additionally, in this chapter, we usually write $\mathrm{ex}(\mathfrak{a}) := \Sigma_{\mathfrak{a}}[\mathrm{ex}]$, and we also set $\mathrm{ex}^+(\mathfrak{a}) := \{\rho \in \Sigma_{\mathfrak{a}}[1] \colon N_\rho(\mathfrak{a}) > 0\}$. Note that $\mathrm{ex}^+(\mathfrak{a}) \supset \mathrm{ex}(\mathfrak{a})$, since the set of rays in $\mathrm{ex}^+(\mathfrak{a})$ correspond to the set of facets of $\Gamma_+(\mathfrak{a})$ that are not contained in any coordinate hyperplane in N.

Remark 4.1.14. The Newton polyhedron $\Gamma_+(\mathfrak{a})$ is "invariant" under the lens of logarithmic geometry. More precisely, $\Gamma_+(\mathfrak{a})$ is independent, up to symmetry, of any change in local coordinates at $\mathbf{0} \in \mathbf{A}^n$ which respects the logarithmic structure on $\mathbf{A}^n$ induced by the chart $\mathbf{N}^n \to \mathbf{k}[x_1, x_2, \ldots, x_n]$ sending each $\mathbf{e}_i \mapsto x_i$. Indeed, any such coordinate change corresponds to a monoid automorphism of $\mathbf{N}^n \oplus \mathscr{O}^*_{\mathbf{A}^n,0}$, which must map each $(\mathbf{e}_i, 1)$ to $(\mathbf{e}_j, \mu)$ for some $j \in [n]$ and $\mu \in \mathscr{O}^*_{\mathbf{A}^n,0}$.

Definition 4.1.15 (Multi-weighted blow-ups along monomial ideals)**.** For $\mathbf{b} \in \mathbf{N}_{>0}^{\Sigma_{\mathfrak{a}}[1]}$, the *multi-weighted blow-up* of $\mathbf{A}^n$ along $\mathfrak{a}$ and $\mathbf{b}$ is the composition

$$\vartheta_{\mathfrak{a},\mathbf{b}} \colon \mathscr{B}l_{\mathfrak{a},\mathbf{b}}\, \mathbf{A}^n := \mathscr{X}_{\Sigma_{\mathfrak{a}},\mathbf{b}} \xrightarrow{\text{good moduli space}} X_{\Sigma_{\mathfrak{a}}} \to \mathbf{A}^n$$

as in Definition 4.1.6. When $\mathbf{b}$ is the unit vector $(1, 1, \ldots, 1)$ in $\mathbf{N}_{>0}^{\mathrm{ex}(\mathfrak{a})}$, we instead write the above expression as $\vartheta_{\mathfrak{a}} \colon \mathscr{B}l_{\mathfrak{a}}\, \mathbf{A}^n \to \mathbf{A}^n$. This should not be confused with the usual blow-up of $\mathbf{A}^n$ along $\mathfrak{a}$, which we have denoted by $\mathrm{Bl}_{\mathfrak{a}}\, \mathbf{A}^n$. In fact, we will see later that the normalization of $\mathrm{Bl}_{\mathfrak{a}}\, \mathbf{A}^n$ is precisely $X_{\Sigma_{\mathfrak{a}}}$, cf. Remark 4.2.15.

As hinted before Setup 4.1.13, $\mathscr{B}l_{\mathfrak{a}}\, \mathbf{A}^n$ is, in a specific sense that will be spelled out later in §4.2.A, the "canonical" multi-weighted blow-up of $\mathbf{A}^n$ associated to $\mathfrak{a}$. In fact, in §4.2.A, we define more generally the "canonical" multi-weighted blow-up $\mathscr{B}l_{\mathfrak{a}_\bullet}\, \mathbf{A}^n$ of $\mathbf{A}^n$ associated a monomial Rees algebra $\mathfrak{a}_\bullet$ on $\mathbf{A}^n$.

4.1.16. Two monomial ideals can possess the same normal fan, and thus yield the same multi-weighted blow-up. Below we list some essential examples:

(i) $\Gamma_+(\mathfrak{a}) = \Gamma_+(\mathrm{IC}(\mathfrak{a}))$, cf. 2.3.35 for definition of $\mathrm{IC}(\mathfrak{a})$. Indeed, $\mathrm{IC}(\mathfrak{a}) = \{x^{\mathbf{a}} \colon \mathbf{a} \in \Gamma_+(\mathfrak{a})\}$.

(ii) Let $f_1, \ldots, f_r$ be monomials generating $\mathfrak{a}$. For any $\ell \in \mathbf{N}_{>0}$, $\mathrm{IC}(\mathfrak{a}^\ell) = \mathrm{IC}(f_1^\ell, \ldots, f_r^\ell)$, so (i) says $\mathfrak{a}^\ell$ and $(f_1^\ell, \ldots, f_r^\ell)$ have the same Newton polyhedra.

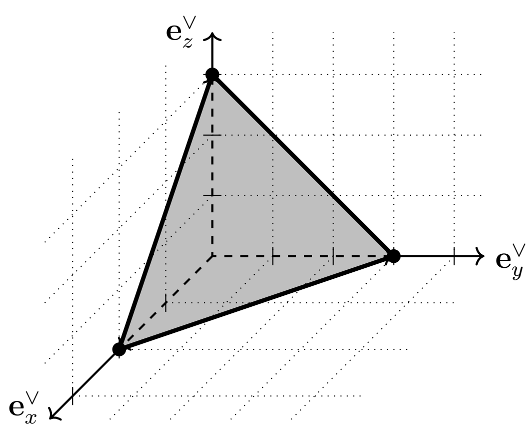
(iii) For any $\mathbf{a} \in \mathbf{N}^n$, $\Sigma_{x^{\mathbf{a}} \cdot \mathfrak{a}} = \Sigma_{\mathfrak{a}}$. However, while the multi-weighted blow-up along $x^{\mathbf{a}} \cdot \mathfrak{a}$ is the same as that along $\mathfrak{a}$, there might be a subtle difference in their "exceptional" divisors (cf. Remark 4.1.24).

(iv) Lastly, as ℓ varies, although the Newton polyhedron of $\mathfrak{a}^\ell$ varies, the normal fan of $\mathfrak{a}^\ell$ remains the same, and so does the multi-weighted blow-up of $\mathfrak{a}^\ell$.

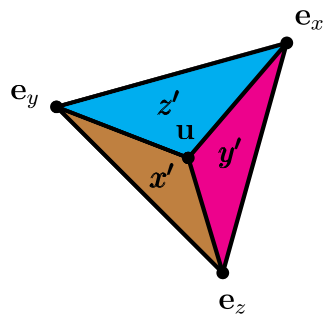
4.1.B. Examples. We illustrate 4.1.9 via two examples. Both examples are multi-weighted blow-ups along monomial ideals. The first example is old, i.e. we already encountered its kind back in Chapter 2:

Example 4.1.17 (Weighted blow-ups)**.** Let $d_1, d_2, \ldots, d_n \in \mathbf{N}_{>0}$, and $\ell := \operatorname{lcm}(d_1, d_2, \ldots, d_n)$. The weighted blow-up of $\mathbf{A}^n = \operatorname{Spec}(\mathbf{k}[x_1, x_2, \ldots, x_n])$ along the smooth center $(x_1, d_1) + (x_2, d_2) + \cdots + (x_n, d_n) = \left(x_1^{1/d_1}, x_2^{1/d_2}, \ldots, x_n^{1/d_n}\right)$ (cf. Conventions 2.3.79) is also the multi-weighted blow-up $\mathscr{B}l_{\mathfrak{a},\mathbf{b}}\, \mathbf{A}^n$, where $\mathfrak{a} = (x_1^{\ell/d_1}, x_2^{\ell/d_2}, \ldots, x_n^{\ell/d_n})$ and $\mathbf{b} = \gcd(d_1, d_2, \ldots, d_n) \in \mathbf{N}_{>0} = \mathbf{N}_{>0}^{\mathrm{ex}(\mathfrak{a})}$.

Instead of showing this in full generality, it is more direct to demonstrate this claim via the following explicit example. Namely, consider $\mathfrak{a} = (x^2, y^3, z^3) \subset \mathbf{k}[x, y, z]$, and $\mathbf{b} = \gcd(2, 3, 3) = 1$. We draw the Newton polyhedron $\Gamma_+(\mathfrak{a})$, which has four facets:



Taking a cross-section of the normal fan $\Sigma_{\mathfrak{a}}$, we obtain:



where $\mathbf{u} = (3,2,2)$ is the normal vector to the shaded facet of $\Gamma_+(\mathfrak{a})$ above. The vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,3)$ of $\Gamma_+(\mathfrak{a})$ correspond to the maximal cones of $\Sigma_{\mathfrak{a}}$ represented above by brown, magenta and cyan-coloured triangles, which also correspond to the x', y' and z'-charts on $\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3 = [X_{\widehat{\Sigma}_{\mathfrak{a}}} \,/\, \mathbb{G}_m]$, where $X_{\widehat{\Sigma}_{\mathfrak{a}}} = \mathbf{A}^4 \smallsetminus V(\mathbb{J}_{\Sigma_{\mathfrak{a}}}) = \mathbf{A}^4 \smallsetminus V(x', y', z')$.

As in 4.1.9, the morphism $\vartheta_{\mathfrak{a}} \colon \mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3 \to \mathbf{A}^3$ can be read off the following matrix:

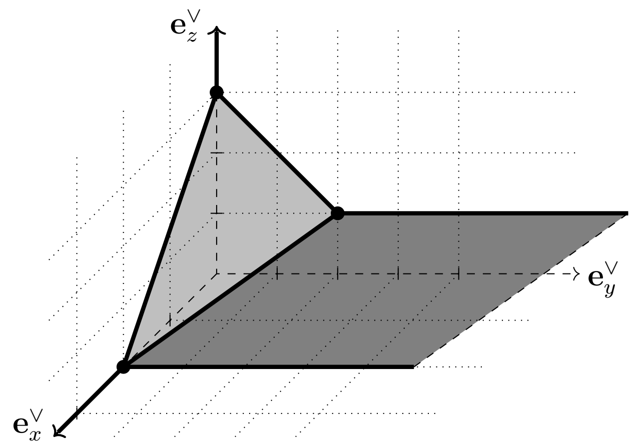
$$\begin{bmatrix} \mathbf{I}_3 & \mathbf{u} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \rightsquigarrow \qquad \begin{cases} x = x'u^3 \\ y = y'u^2 \\ z = z'u^2 \end{cases}$$

and the $\mathbb{G}_m$-action on $X_{\widehat{\Sigma}_{\mathfrak{a}}}$ can be read off from the matrix:

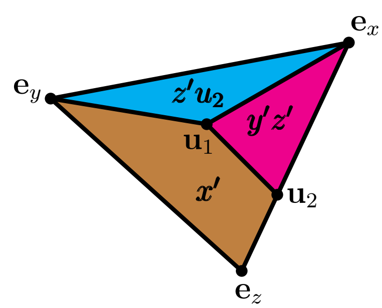
$$\begin{bmatrix} \mathbf{u} \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} \qquad \rightsquigarrow \qquad \begin{cases} x' \text{ has } \mathbf{Z}\text{-weight } 3 \\ y' \text{ has } \mathbf{Z}\text{-weight } 2 \\ z' \text{ has } \mathbf{Z}\text{-weight } 2 \\ u \text{ has } \mathbf{Z}\text{-weight } -1 \end{cases}$$

This is precisely the description of the weighted blow-up of $\mathbf{A}^3$ along $(x,3)+(y,2)+(z,2) = \left(x^{1/3}, y^{1/2}, z^{1/2}\right)$ as in Proposition 2.5.9.

Example 4.1.18 (A new example)**.** The Newton polyhedron $\Gamma_+(\mathfrak{a})$ of the monomial ideal $\mathfrak{a} = (x^2, y^2z, z^3) \subset \mathbf{k}[x, y, z]$ has five facets:



We sketch a cross-section of the normal fan $\Sigma_{\mathfrak{a}}$:



where $\mathbf{u}_1 = (3, 2, 2)$ and $\mathbf{u}_2 = (1, 0, 2)$ are the normal vectors to the shaded facets of $\Gamma_+(\mathfrak{a})$ above. The vertices $(2, 0, 0)$, $(0, 2, 1)$ and $(0, 0, 3)$ of $\Gamma_+(\mathfrak{a})$ correspond to the maximal cones of $\Sigma_{\mathfrak{a}}$ represented above by the brown, magenta and cyan-coloured regions, and these also correspond to the x', $y'z'$, and $z'u_1$-charts on $\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3 = [X_{\widehat{\Sigma}_{\mathfrak{a}}} \,/\, \mathbb{G}_m]$ respectively, where $X_{\widehat{\Sigma}_{\mathfrak{a}}} = \mathbf{A}^5 \smallsetminus V(\mathrm{J}_{\Sigma_{\mathfrak{a}}}) = \mathbf{A}^5 \smallsetminus V(x', y'z', z'u_2)$.

The morphism $\vartheta_{\mathfrak{a}}\colon \mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3 \to \mathbf{A}^3$ can then be determined from the following matrix:

$$\begin{bmatrix} \mathbf{I}_3 & \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right] \qquad \rightsquigarrow \qquad \begin{cases} x = x' u_1^3 u_2 \\ y = y' u_1^2 \\ z = z' u_1^2 u_2^2 \end{cases}$$

and the $\mathbb{G}_m^2$-action on $X_{\widehat{\Sigma}_{\mathfrak{a}}}$ can be determined from the matrix:

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 2 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rightsquigarrow \qquad \begin{cases} x' \text{ has } \mathbf{Z}^2\text{-weight } (3,1) \\ y' \text{ has } \mathbf{Z}^2\text{-weight } (2,0) \\ z' \text{ has } \mathbf{Z}^2\text{-weight } (2,2) \\ u_1 \text{ has } \mathbf{Z}^2\text{-weight } (-1,0) \\ u_2 \text{ has } \mathbf{Z}^2\text{-weight } (0,-1) \end{cases}$$

4.1.C. Exceptional divisors and transforms. The presentation $\vartheta\colon \mathscr{X}_{\Sigma,\mathbf{b}} = \left[X_{\widehat{\Sigma}} \,/\, \mathbb{G}_m^{\Sigma[\mathrm{ex}]}\right] \to \mathbf{A}^n$ induces an isomorphism

$$\mathrm{Pic}\left(\mathscr{X}_{\Sigma,\mathbf{b}}\right) \xrightarrow{\ \simeq\ } \mathrm{Pic}^{\mathbb{G}_m^{\Sigma[\mathrm{ex}]}}\left(X_{\widehat{\Sigma}}\right)$$

where the right hand side denotes the $\mathbb{G}_m^{\Sigma[\mathrm{ex}]}$-equivariant Picard group of $X_{\widehat{\Sigma}}$. In particular, for each $\mathbf{d} \in \mathbf{Z}^{\Sigma[\mathrm{ex}]}$, there are tautological line bundles $\mathscr{O}(\mathbf{d}) := \mathscr{O}_{\mathscr{X}_{\Sigma,\mathbf{b}}}(\mathbf{d})$ on $\mathscr{X}_{\Sigma,\mathbf{b}}$, which correspond to the trivial line bundle $\mathscr{O}_{X_{\widehat{\Sigma}}}$ on $X_{\widehat{\Sigma}}$ endowed with the $\mathbb{G}_m^{\Sigma[\mathrm{ex}]}$-linearization given by the "$\mathbf{d}$-shift", i.e.

$$\vartheta^*\mathscr{O}(\mathbf{d}) = \mathscr{O}_{X_{\widehat{\Sigma}}}(\mathbf{d}) := \left(\mathbf{k}[x_1', \ldots, \widetilde{x_n'] \left[x_\rho'\right.}\colon \rho \in \Sigma[\mathrm{ex}]]\right)(\mathbf{d})\,\Big|_{X_{\widehat{\Sigma}_{\mathfrak{a}}}}$$

where $\widetilde{(\cdot)}$ refers to the passage to the associated sheaf of $\mathscr{O}_{X_{\widehat{\Sigma}}}$-modules [**Har77**, Chapter II.5], and the $\mathbf{Z}^{\Sigma[\mathrm{ex}]}$-grading of x_i' and x_ρ' in the right hand side can be obtained by subtracting $\mathbf{d}$ from their respective $\mathbf{Z}^{\Sigma[\mathrm{ex}]}$-gradings in 4.1.9(ii).

4.1.19 (Exceptional divisors on a multi-weighted blow-up)**.** For every $\rho \in \Sigma[\mathrm{ex}]$, recall that the corresponding coordinate x_ρ' on $X_{\widehat{\Sigma}}$ has $\mathbf{Z}^{\Sigma[\mathrm{ex}]}$-weight $-\mathbf{e}_\rho$ (4.1.9(iib)), and hence there is an injection

$$\mathscr{O}(\mathbf{e}_\rho) \hookrightarrow \mathscr{O}(\mathbf{0}) = \mathscr{O}$$

induced by multiplication by x_ρ', which embeds $\mathscr{O}(\mathbf{e}_\rho)$ as an ideal sheaf on $\mathscr{X}_{\Sigma,\mathbf{b}}$ cutting out the divisor

$$E_\rho := V(x_\rho') \subset \mathscr{X}_{\Sigma,\mathbf{b}}$$

which we claim is an irreducible exceptional divisor of the multi-weighted blow-up $\vartheta\colon \mathscr{X}_{\Sigma,\mathbf{b}} \to \mathbf{A}^n$ (which explains the terminology "exceptional rays" in Convention 4.1.8). Indeed, using Lemma 2.1.2 and the description of $\mathbb{J}_\Sigma$ in 4.1.3, observe that ϑ maps the complement in $\mathscr{X}_{\Sigma,\mathbf{b}}$ of $\bigcup\{E_\rho \colon \rho \in \Sigma[\mathrm{ex}]\}$, i.e.

$$\mathscr{U} = \left[\operatorname{Spec}\left(\mathbf{k}[x_1', x_2', \ldots, x_n']\left[(x_\rho')^{\pm 1} \colon \rho \in \Sigma[\mathrm{ex}]\right]\right) \smallsetminus V(\mathbb{J}_\Sigma) \;/\; \mathbb{G}_m^{\Sigma[\mathrm{ex}]}\right] \xrightarrow{\text{open substack}} \mathscr{X}_{\Sigma,\mathbf{b}}$$

isomorphically onto the complement U in $\mathbf{A}^n$ of the closed subscheme

$$V\left(\prod_{i \in [n] \smallsetminus \sigma[1]} x_i \;:\; \sigma \in \Sigma[\max]\right) \xhookrightarrow{\text{codimension} \geq 2, \text{ if } \#\Sigma[\mathrm{ex}] \geq 1} \mathbf{A}^n.$$

Remark 4.1.20. If $\Sigma = \Sigma_{\mathfrak{a}}$ for a monomial ideal $\mathfrak{a} \subset \mathbf{k}[x_1, x_2, \ldots, x_n]$, we have

$$U = \mathbf{A}^n \smallsetminus V\left(x^{\mathbf{v}} \Big/ \prod_{i \in [n]} x_i^{N_i(\mathfrak{a})} \colon \mathbf{v} \in \operatorname{vert}(\Gamma_+(\mathfrak{a}))\right) = \mathbf{A}^n \smallsetminus V\left(\mathfrak{a} : \prod_{i \in [n]} x_i^{N_i(\mathfrak{a})}\right).$$

where the first equality follows from the inclusion-reversing correspondence between cones σ in Σ and faces ς of $\Gamma_+(\mathfrak{a})$. Noting that $\mathfrak{a}$ is the integral closure $\mathrm{IC}\big(x^{\mathbf{v}} \colon \mathbf{v} \in \mathrm{vert}(\Gamma_+(\mathfrak{a}))\big)$, we have $V(\mathfrak{a}) = V(x^{\mathbf{v}} \colon \mathbf{v} \in \mathrm{vert}(\Gamma_+(\mathfrak{a})))$, and therefore the second equality follows.

The remainder of this section is devoted to the various transforms of ideals under a multi-weighted blow-up along a monomial ideal. We first make the following definition, which is part of the subsequent proposition.

Definition 4.1.21. Given an ideal $J \subset \mathbf{k}[x_1, x_2, \ldots, x_n]$, the Newton polyhedron $\Gamma_+(J)$ of J is defined as the Newton polyhedron of the monomial saturation of J (Definition 3.1.1) under the logarithmic structure on $\mathbf{A}^n$ induced by the chart $\mathbf{N}^n \to \mathbf{k}[x_1, x_2, \ldots, x_n]$ sending each $\mathbf{e}_i$ to x_i, cf. Setup 4.1.13. Therefore, for $\mathbf{u} = (\mathrm{u}_i)_{i=1}^n \in \mathbf{N}^n$ and $m \in \mathbf{N}$, we will say that $\Gamma_+(J)$ is *bounded below* by the hyperplane $\sum_{i=1}^n \mathrm{u}_i \cdot \mathbf{e}_i = m$ if for every $g = \sum_{\mathbf{a} \in \mathbf{N}^n} c_{\mathbf{a}} \cdot x^{\mathbf{a}} \in J$, we have $\mathbf{u} \cdot \mathbf{a} \geq m$ whenever $c_{\mathbf{a}} \neq 0$.

Lemma 4.1.22. *Let $\vartheta \colon \mathscr{X}_{\Sigma,\mathbf{b}} \to \mathbf{A}^n$ be a multi-weighted blow-up of $\mathbf{A}^n$, and $\mathscr{O} := \mathscr{O}_{\mathscr{X}_{\Sigma,\mathbf{b}}}$. Then we have:*

(i) *Let $J \subset \mathbf{k}[x_1, x_2, \ldots, x_n]$ be an ideal. For $(n_\rho)_{\rho \in \Sigma[1]} \in \mathbf{N}^{\Sigma[1]}$, the ideal sheaf $\vartheta^{-1}(J)\mathscr{O}$ underlying the total transform $V(J) \times_{\mathbf{A}^n, \vartheta} \mathscr{X}_{\Sigma,\mathbf{b}}$ satisfies the inclusion*

$$\vartheta^{-1}(J)\mathscr{O} \subset \prod_{\rho \in \Sigma[1]} (x'_\rho)^{\mathrm{b}_\rho \cdot n_\rho}$$

if and only if for every $\rho \in \Sigma[1]$, the Newton polyhedron $\Gamma_+(J)$ of J is bounded below by the hyperplane $\sum_{i=1}^n \mathrm{u}_{\rho,i} \cdot \mathbf{e}_i = n_\rho$.

(ii) *Moreover, if $\vartheta = \vartheta_{\mathfrak{a},\mathbf{b}} \colon \mathscr{B}l_{\mathfrak{a},\mathbf{b}}\, \mathbf{A}^n \to \mathbf{A}^n$ and $J = \mathfrak{a}$, we have equality in (i):*

$$\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(\mathfrak{a})\mathscr{O} = \prod_{\rho \in \mathrm{ex}^+(\mathfrak{a})} (x'_\rho)^{\mathrm{b}_\rho \cdot N_\rho(\mathfrak{a})} \tag{4.2}$$

(cf. Setup 4.1.13 for the definitions of $\mathrm{ex}^+(\mathfrak{a})$, $N_i(\mathfrak{a})$, and $N_\rho(\mathfrak{a})$).

Before giving the proof, let us illustrate it by re-visiting Example 4.1.18:

Example 4.1.23. Let $\mathfrak{a} = (x^2, y^2z, z^3)$, and consider $\vartheta_{\mathfrak{a}}\colon \mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3 \to \mathbf{A}^3$. Using the equations in Example 4.1.18, one computes the following:

(i) Let $J = (x^2 + y^2 + z^2) \subset \mathbf{k}[x,y,z]$. Its total transform $\vartheta_{\mathfrak{a}}^{-1}(J)\mathscr{O}_{\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3}$ is $(x'^2u_1^6u_2^2 + y'^2u_1^4 + z'^2u_1^4u_2^4)$, which is contained in (u_1^4), but is contained neither in (u_1^i) for $i \geq 5$, nor in (u_2^j) for $j \geq 1$. This agrees with Lemma 4.1.22(i): the Newton polyhedron of J has the vertices $\mathbf{v}_1 = (2,0,0)$, $\mathbf{v}_2 = (0,2,0)$ and $\mathbf{v}_3 = (0,0,2)$, and we have $\min_{1\leq i\leq 3}\mathbf{u}_1\cdot\mathbf{v}_i = 4$ and $\min_{1\leq i\leq 3}\mathbf{u}_2\cdot\mathbf{v}_i = 0$.

(ii) The total transform $\vartheta_{\mathfrak{a}}^{-1}(\mathfrak{a})\mathscr{O}_{\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3}$ of $\mathfrak{a}$ equals

$$(x'^2u_1^6u_2^2, y'^2z'u_1^6u_2^2, z'^3u_1^6u_2^6) = (u_1^6u_2^2)\cdot(x'^2, y'^2z', z'^3).$$

The ideal (x'^2, y'^2z', z'^3) is the unit ideal on each chart of $\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3$, so $\vartheta_{\mathfrak{a}}^{-1}(\mathfrak{a})\mathscr{O}_{\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3} = (u_1^6u_2^2)$. This agrees with Lemma 4.1.22(ii): note that $N_{\mathbf{e}_i}(\mathfrak{a}) = 0$ for $1 \leq i \leq 3$, $N_{\mathbf{u}_1}(\mathfrak{a}) = 6$, and $N_{\mathbf{u}_2}(\mathfrak{a}) = 2$.

Proof of Lemma 4.1.22. It suffices to compute on the smooth cover $X_{\widehat{\Sigma}}$ of $\mathscr{X}_{\Sigma,\mathbf{b}}$. By replacing J in (i) by its monomial saturation, we may assume J is monomial. Recall from 4.1.9(i) that for every monomial $x^{\mathbf{a}} \in J$, we have:

$$x^{\mathbf{a}} = \prod_{i=1}^{n}(x_i')^{\mathrm{b}_i\cdot a_i}\cdot\prod_{\rho\in\Sigma[\mathrm{ex}]}(x_\rho')^{\mathrm{b}_\rho\cdot(\mathbf{u}_\rho\cdot\mathbf{a})}\qquad\text{in } \vartheta^{-1}(J)\mathscr{O}. \tag{4.3}$$

Then part (i) follows from (4.3), since it says that for every $\rho \in \Sigma[\mathrm{ex}]$ (resp. $i \in [n]$) and $\mathbf{a} \in \Gamma_+(J)$, we have $\mathbf{u}_\rho\cdot\mathbf{a} \geq n_\rho$ (resp. $a_i \geq n_i$), if and only if $(x_\rho')^{\mathrm{b}_\rho\cdot n_\rho}$ divides $x^{\mathbf{a}}$ (resp. $(x_i')^{\mathrm{b}_i\cdot n_i}$ divides $x^{\mathbf{a}}$).

The forward inclusion in (ii) follows from (i). For the reverse inclusion, it suffices to compute locally on the open charts $U_\sigma \subset X_{\widehat{\Sigma}_{\mathfrak{a}}}$ as σ varies over all maximal cones of $\Sigma_{\mathfrak{a}}$. Therefore, fix a maximal cone σ of $\Sigma_{\mathfrak{a}}$, and, as in Setup 4.1.13, let $\mathbf{v}_\sigma = (\mathrm{v}_{\sigma,i})_{i=1}^n$ be the corresponding vertex of $\Gamma_+(\mathfrak{a})$. Setting $\mathbf{a} = \mathbf{v}_\sigma$ in (4.3), we have

$$x^{\mathbf{v}_\sigma} = \prod_{i=1}^{n} (x_i')^{\mathrm{b}_i \cdot \mathrm{v}_{\sigma,i}} \cdot \prod_{\rho \in \mathrm{ex}(\mathfrak{a})} (x_\rho')^{\mathrm{b}_\rho \cdot (\mathbf{u}_\rho \cdot \mathbf{v}_\sigma)} \qquad \text{in } \vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(\mathfrak{a})\mathscr{O}.$$

Recalling that x_ρ' is invertible on U_σ for any $\rho \in \Sigma_{\mathfrak{a}}[1] \smallsetminus \sigma[1]$, we obtain:

$$\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(\mathfrak{a})\mathscr{O} \supset (x^{\mathbf{v}_\sigma}) = \prod_{i \in [n] \cap \sigma[1]} (x_i')^{\mathrm{b}_i \cdot \mathrm{v}_{\sigma,i}} \cdot \prod_{\rho \in \mathrm{ex}(\mathfrak{a}) \cap \sigma[1]} (x_\rho')^{\mathrm{b}_\rho \cdot (\mathbf{u}_\rho \cdot \mathbf{v}_\sigma)} \qquad \text{on } U_\sigma,$$

To complete the proof of (ii), it remains to note the following. For each $\rho \in \mathrm{ex}(\mathfrak{a}) \cap \sigma[1]$, we have $\mathbf{v}_\sigma \in \tau_\rho$, which implies $\mathbf{u}_\rho \cdot \mathbf{v}_\sigma = N_\rho(\mathfrak{a})$. Likewise, if $\rho = i \in [n] \cap \sigma[1]$, we have $\mathrm{v}_{\sigma,i} = N_i(\mathfrak{a})$. ☐

Remark 4.1.24. In Lemma 4.1.22(ii), note that for every $i \in [n]$, $N_i(\mathfrak{a}) > 0$ if and only if $\mathfrak{a} \subset (x_i)$, i.e. $V(\mathfrak{a}) \supset V(x_i)$. Analogous to how the usual blow-up of $\mathbf{A}^n$ along a divisor D does nothing except declare D to be "exceptional", the multi-weighted blow-up $\vartheta_{\mathfrak{a},\mathbf{b}}$ similarly declares the divisor $V(x_i') \subset \mathscr{B}l_{\mathfrak{a},\mathbf{b}}\,\mathbf{A}^n$ to be "exceptional" for every $i \in [n] \cap \mathrm{ex}^+(\mathfrak{a})$, i.e. every $i \in [n]$ with $\mathfrak{a} \subset (x_i)$. In this sense, Lemma 4.1.22(ii) expresses the total transform of $\mathfrak{a}$ as a sum of "exceptional" divisors $V(x_\rho')$ for $\rho \in \mathrm{ex}^+(\mathfrak{a})$. This discussion should be compared to the observation in Remark 4.1.20.

Next, we explicate two classical transforms for multi-weighted blow-ups along monomial ideals:

Definition 4.1.25 (Proper transform)**.** Set $\mathscr{O} := \mathscr{O}_{\mathscr{B}l_{\mathfrak{a},\mathbf{b}}\,\mathbf{A}^n}$. The *proper (or strict) transform* of an ideal $J \subset \mathbf{k}[x_1, x_2, \ldots, x_n]$ under the multi-weighted blow-up $\vartheta_{\mathfrak{a},\mathbf{b}} \colon \mathscr{B}l_{\mathfrak{a},\mathbf{b}}\,\mathbf{A}^n \to \mathbf{A}^n$ is

$$\widetilde{\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(J)}\mathscr{O} := \left(\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(J)\mathscr{O} : \prod_{\rho \in \mathrm{ex}^+(\mathfrak{a})} (x_\rho')^\infty\right) := \bigcup_{(n_\rho) \in \mathbf{N}^{\mathrm{ex}^+(\mathfrak{a})}} \left(\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(J)\mathscr{O} : \prod_{\rho \in \mathrm{ex}^+(\mathfrak{a})} (x_\rho')^{n_\rho}\right).$$

Equivalently, by Lemma 4.1.22(ii), the *proper transform* of $V(J) \subset \mathbf{A}^n$ under $\vartheta_{\mathfrak{a},\mathbf{b}}$ is the scheme-theoretic closure of $V(\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(J)\mathscr{O}) \smallsetminus V(\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(\mathfrak{a})\mathscr{O})$ in $\mathscr{B}l_{\mathfrak{a},\mathbf{b}} \mathbf{A}^n$.

Definition 4.1.26 (Weak transform)**.** Set $\mathscr{O} := \mathscr{O}_{\mathscr{B}l_{\mathfrak{a},\mathbf{b}} \mathbf{A}^n}$. The *weak (or birational, or controlled) transform* of an ideal $J \subset \mathbf{k}[x_1, \ldots, x_n]$ under the multi-weighted blow-up $\vartheta_{\mathfrak{a},\mathbf{b}} \colon \mathscr{B}l_{\mathfrak{a},\mathbf{b}} \mathbf{A}^n \to \mathbf{A}^n$ is

$$(\vartheta_{\mathfrak{a},\mathbf{b}})_*^{-1}(J) := \left(\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(J)\mathscr{O}\right) \cdot \prod_{\rho \in \mathrm{ex}^+(\mathfrak{a})} (x'_\rho)^{-\mathrm{b}_\rho \cdot N_\rho(J)}$$

where for each $\rho \in \mathrm{ex}^+(\mathfrak{a})$, $N_\rho(J)$ is the largest natural number n_ρ such that:

(i) $\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(J)\mathscr{O} \subset (x'_\rho)^{\mathrm{b}_\rho \cdot n_\rho}$, i.e. $\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(J)\mathscr{O} = (x'_\rho)^{b_\rho n_\rho} \cdot J'$ for some ideal $J' \subset \mathscr{O}$;

(ii) or equivalently, by Lemma 4.1.22(i), the Newton polyhedron $\Gamma_+(J)$ of J is bounded below by the hyperplane $\sum_{i=1}^n \mathrm{u}_{\rho,i} \cdot \mathbf{e}_i = n_\rho$.

Remark 4.1.27. Similar to what was noted in Definition 2.3.31, we always have

$$(\vartheta_{\mathfrak{a},\mathbf{b}})_*^{-1}(J) \subset \vartheta_{\mathfrak{a},\mathbf{b}}^{\widetilde{-1}(J)}\mathscr{O}$$

with equality if J is a principal ideal. As with usual blow-ups, if J is radical, so is the proper transform $\vartheta_{\mathfrak{a},\mathbf{b}}^{\widetilde{-1}(J)}\mathscr{O}$. In other words, if $V(J) \subset \mathbf{A}^n$ is reduced, so is its proper transform in $\mathscr{B}l_{\mathfrak{a},\mathbf{b}} \mathbf{A}^n$.

As with the case of usual blow-ups, it is a more intricate issue to identify generators of proper transforms, as opposed to generators for weak transforms. For example:

Example 4.1.28. Consider the non-principal ideal $J = (x^2 + y^2, z - y^2) \subset \mathbf{k}[x, y, z]$ and the multi-weighted blow-up $\vartheta_{\mathfrak{a}}$ in Example 4.1.18.

(i) The total transform $\vartheta_{\mathfrak{a}}^{-1}(J)\mathscr{O}_{\mathscr{B}l_{\mathfrak{a}} \mathbf{A}^3}$ of J under $\pi_{\mathfrak{a}}$ is

$$(x'^2 u_1^6 u_2^2 + y'^2 u_1^4, z' u_1^2 u_2^2 - y'^2 u_1^4) = (u_1^2) \cdot (x'^2 u_1^4 u_2^2 + y'^2 u_1^2, z' u_2^2 - y'^2 u_1^2).$$

Hence, the weak transform $(\pi_{\mathfrak{a}})_*^{-1}(J)$ of J under $\vartheta_{\mathfrak{a}}$ is

$$u_1^{-2} \cdot \vartheta_{\mathfrak{a}}^{-1}(J)\mathscr{O}_{\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3} = (x'^2 u_1^4 u_2^2 + y'^2 u_1^2, z' u_2^2 - y'^2 u_1^2).$$

(ii) On the other hand, while we have

$$x^2 + y^2 = u_1^4 \cdot (x' u_1^2 u_2^2 + y'^2) \;\text{ and }\; z - y^2 = u_1^2 \cdot (z' u_2^2 - y'^2 u_1^2),$$

the proper transform $\vartheta_{\mathfrak{a}}^{-1}\widetilde{(J)\mathscr{O}}_{\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3}$ of J under $\vartheta_{\mathfrak{a}}$ is not generated by the elements $x' u_1^2 u_2^2 + y'^2$ and $z' u_2^2 - y'^2 u_1^2$. Indeed, note that $x^2 + z = (x^2 + y^2) + (z - y^2) \in J$, and

$$x^2 + z = x'^2 u_1^6 u_2^2 + z' u_1^2 u_2^2 = u_1^2 u_2^2 \cdot (x'^2 u_1^4 + z').$$

Thus, $x'^2 u_1^4 + z' \in \vartheta_{\mathfrak{a}}^{-1}\widetilde{(J)\mathscr{O}}_{\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^3}$, but $x'^2 u_1^4 + z' \notin (x' u_1^2 u_2^2 + y'^2, z' u_2^2 - y'^2 u_1^2)$.

4.1.D. Multi-graded Rees algebras and idealistic exponents. In this subsection, we reinterpret some of the earlier discussions, in terms of multi-graded Rees algebras and idealistic exponents.

4.1.29 (Multi-graded Rees algebras)**.** The discussion in 4.1.9 can be summarized by the compact, but notation-heavy, statement that $\mathscr{B}l_{\mathfrak{a},\mathbf{b}}\,\mathbf{A}^n$ equals:

$$\left[(\mathrm{Spec}_{\mathbf{A}^n}(\mathscr{R}) \smallsetminus V(\mathrm{J}_{\Sigma_{\mathfrak{a}}})) \;\Big/ \begin{bmatrix} \mathbf{B} \\ -I_k \end{bmatrix} \mathbb{G}_m^{\mathrm{ex}(\mathfrak{a})} \right] \tag{4.4}$$

where

$$\mathscr{R} = \mathscr{R}(\mathfrak{a}, \mathbf{b}) := \frac{\mathscr{O}_{\mathbf{A}^n}[x_1', \ldots, x_n']\left[x_\rho' \colon \rho \in \mathrm{ex}(\mathfrak{a})\right]}{\left(x_i' \cdot \prod_{\rho \in \mathrm{ex}(\mathfrak{a})} (x_\rho')^{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}} - x_i \colon 1 \leq i \leq n\right)} \tag{4.5}$$

and the matrix

$$\begin{bmatrix} \mathbf{B} \\ \\ -I_k \end{bmatrix}$$

records the $\mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}$-grading of each x'_i and each x'_ρ as in 4.1.9(ii), and hence, describes the $\mathbb{G}_m^{\mathrm{ex}(\mathfrak{a})}$-action displayed above.

We provide a re-interpretation of the $\mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}$-graded $\mathscr{O}_{\mathbf{A}^n}$-algebra $\mathscr{R} = \mathscr{R}(\mathfrak{a}, \mathbf{b})$ as a *multi-graded Rees algebra* on $\mathbf{A}^n$. Consider the homomorphism of $\mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}$-graded $\mathscr{O}_{\mathbf{A}^n}$-algebras $\mathscr{R} \to \mathscr{O}_{\mathbf{A}^n}[t_\rho^{\pm} \colon \rho \in \mathrm{ex}(\mathfrak{a})]$ defined by

$$\begin{aligned} x'_i &\mapsto \left(x_i \cdot \prod_{\rho \in \mathrm{ex}(\mathfrak{a})} t_\rho^{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}} \right) && \text{for } 1 \le i \le n \\ x'_\rho &\mapsto (1 \cdot t_\rho^{-1}) && \text{for } \rho \in \mathrm{ex}(\mathfrak{a}). \end{aligned} \tag{4.6}$$

This is an isomorphism of $\mathscr{R}$ onto its image

$$\begin{aligned} \mathscr{R}_\bullet &:= \mathscr{O}_{\mathbf{A}^n}\left[t_\rho^{-1} \colon \rho \in \mathrm{ex}(\mathfrak{a})\right] \left[x_i \cdot \prod_{\rho \in \mathrm{ex}(\mathfrak{a})} t_\rho^{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}} \colon 1 \le i \le n \right] \\ &\subset \ \mathscr{O}_{\mathbf{A}^n}\left[t_\rho^{\pm} \colon \rho \in \mathrm{ex}(\mathfrak{a})\right]. \end{aligned} \tag{4.7}$$

The image $\mathscr{R}_\bullet$ is a $\mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}$*-graded Rees algebra* on $\mathbf{A}^n$, i.e. it is a finitely generated, quasi-coherent $\mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}$-graded $\mathscr{O}_{\mathbf{A}^n}$-subalgebra

$$\mathscr{R}_\bullet = \bigoplus_{\mathbf{m} \in \mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}} \mathscr{R}_{\mathbf{m}} \ \subset \ \mathscr{O}_{\mathbf{A}^n}[t_\rho^{\pm} \colon \rho \in \mathrm{ex}(\mathfrak{a})]$$

satisfying the following three conditions:

(i) $\mathscr{R}_{\mathbf{0}} = \mathscr{O}_{\mathbf{A}^n}$.

(ii) $1 \cdot t_\rho^{-1} \in \mathscr{R}_\bullet$ for all $\rho \in \mathrm{ex}(\mathfrak{a})$.

(iii) For every $\mathbf{m} \in \mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}$, we have $\mathscr{R}_{\mathbf{m}} = \bigcap_{\rho \in \mathrm{ex}(\mathfrak{a})} \mathscr{R}_{m_\rho \cdot \mathbf{e}_\rho}$.

Note that under (i), (ii) is equivalent to:

(ii′) $\mathscr{R}_{\mathbf{m}+\mathbf{e}_\rho} \subset \mathscr{R}_{\mathbf{m}}$ for every $\mathbf{m} \in \mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}$ and $\rho \in \mathrm{ex}(\mathfrak{a})$.

In particular, (ii) already implies the forward inclusion in (iii). Moreover, note that (iii) is redundant if $\#\mathrm{ex}(\mathfrak{a}) = 1$.

We usually make the identification $\mathscr{R} = \mathscr{R}_\bullet$, and hence, will not make any distinction between both sides of (4.6). Occasionally we neglect the negative degrees, and only work with the $\mathbf{N}^{\mathrm{ex}(\mathfrak{a})}$-graded part of $\mathscr{R}_\bullet$, which is a $\mathbf{N}^{\mathrm{ex}(\mathfrak{a})}$*-graded Rees algebra* on $\mathbf{A}^n$, i.e. a finitely generated, quasi-coherent $\mathbf{N}^{\mathrm{ex}(\mathfrak{a})}$-graded $\mathscr{O}_{\mathbf{A}^n}$-subalgebra $\bigoplus_{\mathbf{m}\in\mathbf{N}^{\mathrm{ex}(\mathfrak{a})}} \mathscr{R}_{\mathbf{m}} \subset \mathscr{O}_{\mathbf{A}^n}[t_\rho \colon \rho \in \mathrm{ex}(\mathfrak{a})]$ satisfying (i), (ii′) and (iii) with the phrase "$\mathbf{m} \in \mathbf{Z}^{\mathrm{ex}(\mathfrak{a})}$" in the last two conditions are replaced by "$\mathbf{m} \in \mathbf{N}^{\mathrm{ex}(\mathfrak{a})}$".

Remark 4.1.30 (Alternative description of the proper transform)**.** It is also under the interpretation in 4.1.29 that the proper transform of a closed subscheme $V(J) \subset \mathbf{A}^n$ under $\vartheta_{\mathfrak{a},\mathbf{b}} \colon \mathscr{B}l_{\mathfrak{a},\mathbf{b}}\,\mathbf{A}^n \to \mathbf{A}^n$ has a natural description. Namely, it is given by the similar-looking expression:

$$\left[\left(\mathrm{Spec}_{V(J)}\big(\mathscr{R}\mathscr{O}_{V(J)}\big) \smallsetminus V(\mathbb{J}_{\Sigma_\mathfrak{a}})\right) \Big/ _{\begin{bmatrix}\mathbf{B}\\ -I_k\end{bmatrix}} \mathbb{G}_m^{\mathrm{ex}(\mathfrak{a})}\right].$$

If one interprets this as the "*multi-weighted blow-up of $V(J)$ along $\mathscr{R}\mathscr{O}_{V(J)}$*", this description parallels that in Corollary 2.3.25 and [**Har77**, Corollary II.7.15].

For 4.1.31 below only, let Y denote a $\mathbf{k}$-variety, e.g. $Y = \mathbf{A}^n$. In §2.3.G, we defined a one-to-one correspondence between non-zero, integrally closed, $\mathbf{N}$-graded Rees algebras on Y and idealistic exponents over Y. This can be immediately promoted to an one-to-one correspondence between non-zero, integrally closed, $\mathbf{N}^k$-graded Rees algebras $\mathscr{R}_\bullet$ on Y and k-tuples $\gamma = (\gamma^{[\rho]} \colon \rho \in [1,k])$ of idealistic exponents over Y.

4.1.31 (Tuples of idealistic exponents)**.** Let $\mathrm{ZR}(Y)$ denote the Zariski-Riemann space of Y, and let $\pi_Y\colon \mathrm{ZR}(Y) \to Y$ denote the morphism of locally ringed spaces which maps $\nu \in \mathrm{ZR}(Y)$ to the center x_ν of ν on Y (Definition 2.3.54). Then:

(a) Given a k-tuple $\gamma := (\gamma^{[\rho]}\colon \rho \in [1,k])$ of idealistic exponents over Y, let $\mathscr{R}_\bullet^{[\rho]}$ be the integrally closed, $\mathbf{N}$-graded Rees algebra on Y associated to each $\gamma^{[\rho]}$. Then the integrally closed $\mathbf{N}^k$-graded Rees algebra $\mathscr{R}_\bullet := \bigoplus_{\mathbf{m}\in\mathbf{N}^k} \mathscr{R}_{\mathbf{m}} \cdot \boldsymbol{t}^{\mathbf{m}}$ on Y associated to γ is defined by $\mathscr{R}_{\mathbf{m}} := \bigcap_{\rho\in[1,k]} \mathscr{R}^{[\rho]}_{m_\rho}$ for every $\mathbf{m} \in \mathbf{N}^k$. In other words, for any open $U \subset Y$,

$$\mathscr{R}_{\mathbf{m}}(U) := \left\{ g \in \mathscr{O}_Y(U)\colon \begin{array}{l} \nu(g) \geq m_\rho \cdot (\gamma^{[\rho]})_\nu \text{ for every} \\ \nu \in \pi_Y^{-1}(U) \text{ and } \rho \in [1,k] \end{array} \right\}.$$

(b) Conversely, given an non-zero, integrally closed, $\mathbf{N}^k$-graded Rees algebra $\mathscr{R}_\bullet$ on Y, we associate a k-tuple $\gamma := (\gamma^{[\rho]}\colon \rho \in [1,k])$ of idealistic exponents over Y, where each $\gamma^{[\rho]}$ is the idealistic exponent over Y associated to the non-zero, integrally closed, $\mathbf{N}$-graded Rees algebra $\mathscr{R}_\bullet^{[\rho]} := \bigoplus_{m\in\mathbf{N}} \mathscr{R}_{m\cdot\mathbf{e}_\rho} \cdot t^m$. In other words, the stalk of $\gamma^{[\rho]}$ at each $\nu \in \mathrm{ZR}(Y)$ is:

$$(\gamma^{[\rho]})_\nu := \min \left\{ \frac{1}{m_\rho} \cdot \nu(g)\colon 0 \neq g \cdot \boldsymbol{t}^{\mathbf{m}} \in (\mathscr{R}_\bullet)_{x_\nu} \text{ with } m_\rho \geq 1 \right\}.$$

Together, (a) and (b) give the desired one-to-one correspondence.

4.1.32. Under the above one-to-one correspondence, the $\mathbf{N}^{\mathrm{ex}(\mathfrak{a})}$-graded part of $\mathscr{R}(\mathfrak{a},\mathbf{b})_\bullet$ in (4.7) then corresponds to the tuple $\gamma(\mathfrak{a},\mathbf{b}) := \big(\gamma(\mathfrak{a},\mathbf{b})^{[\rho]}\colon \rho \in \mathrm{ex}(\mathfrak{a})\big)$ of $\#\mathrm{ex}(\mathfrak{a})$ idealistic exponents over $\mathbf{A}^n$, where each $\gamma(\mathfrak{a},\mathbf{b})^{[\rho]}$ is defined stalk-wise at each $\nu \in \mathrm{ZR}(\mathbf{A}^n)$ by:

$$\big(\gamma(\mathfrak{a},\mathbf{b})^{[\rho]}\big)_\nu := \min_{\substack{i\in[n] \\ \mathrm{u}_{\rho,i}\neq 0}} \left(\frac{1}{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}} \cdot \nu(x_i) \right). \tag{4.8}$$

Following Convention 2.3.79, for each $\rho \in [1,k]$, we shall use the suggestive notation

$$\left(x_i^{\frac{1}{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}}} : i \in [n],\ \mathrm{u}_{\rho,i} \neq 0\right)$$

to denote the corresponding integrally closed, $\mathbf{N}$-graded Rees algebra $\mathscr{R}(\mathfrak{a}, \mathbf{b})^{[\rho]}$.

Our next objective is to give a coordinate-free interpretation of the weak transform (Definition 4.1.26) in terms of the idealistic exponents in $\gamma(\mathfrak{a}, \mathbf{b})$.

Setup 4.1.33. For the remainder of this subsection, fix a monomial ideal $\mathfrak{a}$ on $\mathbf{A}^n$ and $\mathbf{b} \in \mathbf{N}_{>0}^{\mathrm{ex}(\mathfrak{a})}$. Let $\widetilde{\vartheta} := \widetilde{\vartheta}_{\mathfrak{a},\mathbf{b}}$ denote the composition

$$X_{\widehat{\Sigma}_{\mathfrak{a}}} \xrightarrow[\text{quotient}]{\text{stack-theoretic}} \mathscr{B}l_{\mathfrak{a},\mathbf{b}}\,\mathbf{A}^n \xrightarrow{\vartheta_{\mathfrak{a},\mathbf{b}}} \mathbf{A}^n.$$

aFor an ideal J on $\mathbf{A}^n$ (or $X_{\widehat{\Sigma}_{\mathfrak{a}}}$), let γ_J denote the idealistic exponent over $\mathbf{A}^n$ (or $X_{\widehat{\Sigma}_{\mathfrak{a}}}$) associated to J (cf. 2.3.59). Let $\widetilde{\vartheta}^{-1}(\gamma)\mathscr{O}$ denote the pullback of γ to $X_{\widehat{\Sigma}_{\mathfrak{a}}}$ via $\widetilde{\vartheta}$. Unless otherwise mentioned, set $\mathscr{O} := \mathscr{O}_{X_{\Sigma_{\mathfrak{a}}}}$, and set $\gamma^{[\rho]} := \gamma(\mathfrak{a}, \mathbf{b})^{[\rho]}$ for $\rho \in \mathrm{ex}(\mathfrak{a})$. For $i \in [n]$, we also set $\gamma^{[i]} := \gamma_{(x_i)}$.

To re-interpret the weak transform, we begin with the elementary lemma:

Lemma 4.1.34. *For $(k_\rho)_{\rho \in \Sigma_{\mathfrak{a}}[1]} \in \mathbf{N}^{\Sigma_{\mathfrak{a}}[1]}$, the following statements are equivalent:*

(i) $\widetilde{\vartheta}^{-1}(J)\mathscr{O} \subset \prod_{\rho \in \Sigma_{\mathfrak{a}}[1]} (x'_\rho)^{k_\rho}$.

(ii) $\widetilde{\vartheta}^{-1}(\gamma_J)\mathscr{O} \geq \sum_{\rho \in \Sigma_{\mathfrak{a}}[1]} k_\rho \cdot \gamma_{(x'_\rho)}$.

(i′) $\widetilde{\vartheta}^{-1}(J)\mathscr{O} \subset (x'_\rho)^{k_\rho}$ *for every* $\rho \in \Sigma_{\mathfrak{a}}[1]$.

(ii′) $\widetilde{\vartheta}^{-1}(\gamma_J)\mathscr{O} \geq k_\rho \cdot \gamma_{(x'_\rho)}$ *for every* $\rho \in \Sigma_{\mathfrak{a}}[1]$.

Proof. (i) $\iff$ (i′) is evident. Both (i) $\iff$ (ii) and (i′) $\iff$ (ii′) follow from the fact that if X is a normal variety, J is an ideal on X, and D is a divisor on X with underlying ideal I_D, then $\gamma_D \leq \gamma_J$ if and only if $I_D \supset J$. Indeed, by Lemma 2.3.57, both statements are equivalent to $J \cdot I_D^{-1} \subset \mathscr{O}_X$. $\square$

Our next goal is to provide a re-characterization of statement (ii′) in Lemma 4.1.34 in terms of idealistic exponents over $\mathbf{A}^n$. Before doing that, we need the following lemma:

Lemma 4.1.35. *For each $\rho \in \Sigma_{\mathfrak{a}}[1]$, we have:*

$$\widetilde{\vartheta}^{-1}(\gamma^{[\rho]})\mathscr{O} \geq \gamma_{(x'_\rho)}.$$

Proof. Let $\nu \in \mathrm{ZR}(X_{\widehat{\Sigma}_{\mathfrak{a}}})$ be arbitrary. If $\rho = i \in [n]$, the lemma then follows from:

$$\gamma_{(x'_i),\nu} = \nu(x'_i) = \nu(x_i) - \sum_{\rho \in \mathrm{ex}(\mathfrak{a})} (\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}) \cdot \nu(x'_\rho) \leq \nu(x_i) = \left(\widetilde{\vartheta}^{-1}(\gamma^{[i]})\mathscr{O}\right)_\nu.$$

If instead $\rho \in \mathrm{ex}(\mathfrak{a})$, we have, for every $1 \leq i \leq n$ such that $\mathrm{u}_{\rho,i} \neq 0$:

$$\begin{aligned}\gamma_{(x'_\rho),\nu} = \nu(x'_\rho) &= \frac{1}{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}} \cdot \left(\nu(x_i) - \nu(x'_i) - \sum_{\widetilde{\rho} \in \mathrm{ex}(\mathfrak{a}) \smallsetminus \{\rho\}} (\mathrm{b}_{\widetilde{\rho}} \cdot \mathrm{u}_{\widetilde{\rho},i}) \cdot \nu(x'_{\widetilde{\rho}})\right) \\ &\leq \frac{1}{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}} \cdot \nu(x_i).\end{aligned}$$

Taking the minimum over all such $1 \leq i \leq n$, the lemma follows. $\square$

Let $\mathrm{ZR}(\widetilde{\vartheta})\colon \mathrm{ZR}(X_{\widehat{\Sigma}_{\mathfrak{a}}}) \to \mathrm{ZR}(\mathbf{A}^n)$ denote the morphism of Zariski-Riemann spaces induced by $\widetilde{\vartheta}$ (cf. 2.3.56). For each $\rho \in \Sigma_{\mathfrak{a}}[1]$, let ν'_ρ be the divisorial valuation on $X_{\widehat{\Sigma}_{\mathfrak{a}}}$ induced by $V(x'_\rho) \subset X_{\widehat{\Sigma}_{\mathfrak{a}}}$, and let $\nu_\rho = \mathrm{ZR}(\widetilde{\vartheta})(\nu'_\rho)$. By 4.1.9(i), $\nu_\rho(x_i) = \mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}$ for every $\rho \in \Sigma_{\mathfrak{a}}[1]$ and $1 \leq i \leq n$.

Proposition 4.1.36. *For $\rho \in \Sigma_{\mathfrak{a}}[1]$ and $k \in \mathbf{Q}_{>0}$, the following statements are equivalent:*

(i) $\gamma_J \geq k \cdot \gamma^{[\rho]}$.

(ii) $\widetilde{\vartheta}^{-1}(\gamma_J)\mathscr{O} \geq k \cdot \gamma_{(x'_\rho)}$.

(iii) $\gamma_{J,\nu_\rho} \geq k \ \big(= k \cdot (\gamma^{[\rho]})_{\nu_\rho} = k \cdot \gamma_{(x'_\rho),\nu'_\rho}\big)$.

Proof. (i) $\Longrightarrow$ (ii) follows from Lemma 4.1.35. For (ii) $\Longrightarrow$ (iii), we localize the inequality in (ii) at $\nu := \nu'_\rho$ to obtain $\gamma_{J,\nu_\rho} = \big(\widetilde{\vartheta}^{-1}(\gamma_J)\mathscr{O}\big)_\nu \geq k \cdot \gamma_{E_\rho,\nu} = k$. For (iii) $\Longrightarrow$ (i), (iii) says that for $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \cdot x^{\mathbf{a}} \in J$, we have

$$\min_{c_{\mathbf{a}}\neq 0}\left\{\sum_{i=1}^{n} a_i \cdot (\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i})\right\} = \nu_\rho(f) \geq \gamma_{J,\nu_\rho} \geq k. \tag{4.9}$$

Then for arbitrary $\nu \in \mathrm{ZR}(\mathbf{A}^n)$ and $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \cdot x^{\mathbf{a}} \in J$, we have

$$\begin{aligned}
\nu(f) \geq \min_{c_{\mathbf{a}}\neq 0}\left\{\sum_{i=1}^{n} a_i \cdot \nu(x_i)\right\} &\geq \min_{c_{\mathbf{a}}\neq 0}\left\{\sum_{\substack{i=1\\ \mathrm{u}_{\rho,i}\neq 0}}^{n} a_i \cdot (\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i})\left(\frac{1}{\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}} \cdot \nu(x_i)\right)\right\}\\
&\geq (\gamma^{[\rho]})_\nu \cdot \min_{c_{\mathbf{a}}\neq 0}\left\{\sum_{i=1}^{n} a_i \cdot (\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i})\right\}\\
&\geq k \cdot (\gamma^{[\rho]})_\nu
\end{aligned}$$

where the last inequality follows from (4.9). Therefore, $\gamma_{J,\nu} = \min\{\nu(f) \colon f \in J\} \geq k \cdot (\gamma^{[\rho]})_\nu$. $\square$

Remark 4.1.37. The above discussion suggests that we can interpret γ_J as the "Newton polyhedron $\Gamma_+(J)$ of J", and $\gamma^{[\rho]}$ as the "hyperplane $\sum_{i=1}^{n} (\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}) \cdot \mathbf{e}_i = k$". Then Proposition 4.1.36(i) translates to the statement that "$\Gamma_+(J)$ is bounded below by the hyperplane $\sum_{i=1}^{n} (\mathrm{b}_\rho \cdot \mathrm{u}_{\rho,i}) \cdot \mathbf{e}_i = k$" (Definition 4.1.21). Combining Lemma 4.1.34 and Proposition 4.1.36, we get a re-interpretation of Lemma 4.1.22(i) in terms of idealistic exponents. This re-interpretation is justified by (4.9), which says that for every $\mathbf{a} \in \Gamma_+(J)$, $(\mathrm{b}_\rho \cdot \mathbf{u}_\rho) \cdot \mathbf{a} \geq k$.

Remark 4.1.38. Let us apply Proposition 4.1.36 to $J = \mathfrak{a}$. For every $\rho \in \Sigma_{\mathfrak{a}}[1]$, one can compute that $\gamma_{\mathfrak{a},\nu_\rho} = \mathrm{b}_\rho \cdot N_\rho(\mathfrak{a})$, so the proposition says $\gamma_{\mathfrak{a}} \geq (\mathrm{b}_\rho \cdot N_\rho(\mathfrak{a})) \cdot \gamma^{[\rho]}$. In fact, $\sup\{k \in \mathbf{Q}_{>0} \colon \gamma_{\mathfrak{a}} \geq k \cdot \gamma^{[\rho]}\} = \mathrm{b}_\rho \cdot N_\rho(\mathfrak{a})$, because whenever $\gamma_{\mathfrak{a}} \geq k \cdot \gamma^{[\rho]}$, then $k \leq \gamma_{\mathfrak{a},\nu_\rho} =$

$\mathrm{b}_\rho \cdot N_\rho(\mathfrak{a})$. By Lemma 4.1.34, we therefore have $\widetilde{\vartheta}^{-1}(\gamma_{\mathfrak{a}})\mathscr{O} \geq \sum_{\rho\in \mathrm{ex}^+(\mathfrak{a})} (\mathrm{b}_\rho \cdot N_\rho(\mathfrak{a})) \cdot \gamma_{(x'_\rho)}$. In fact, Lemma 4.1.22(ii) says more: this inequality is an equality!

By the equivalences in Lemma 4.1.34 and Proposition 4.1.36, the following definition of the weak transform is equivalent to Definition 4.1.26, and should be compared to Lemma 2.3.30 and Lemma 2.3.77:

Definition 4.1.39 (Weak transform, re-visited)**.** Set $\mathscr{O} := \mathscr{O}_{\mathscr{B}l_{\mathfrak{a},\mathbf{b}}\,\mathbf{A}^n}$. The weak transform of an ideal $J \subset \mathbf{k}[x_1,\ldots,x_n]$ under the multi-weighted blow-up $\vartheta_{\mathfrak{a},\mathbf{b}} \colon \mathscr{B}l_{\mathfrak{a},\mathbf{b}}\,\mathbf{A}^n \to \mathbf{A}^n$ can be also be defined as

$$(\vartheta_{\mathfrak{a},\mathbf{b}})_*^{-1}(J) := \left(\vartheta_{\mathfrak{a},\mathbf{b}}^{-1}(J)\mathscr{O}\right) \cdot \prod_{\rho\in\mathrm{ex}^+(\mathfrak{a})} (x'_\rho)^{-K_\rho(J)}$$

where for each $\rho \in \mathrm{ex}^+(\mathfrak{a})$, $K_\rho(J)$ is the largest natural number k_ρ such that $\gamma_J \geq k_\rho \cdot \gamma^{[\rho]}$ (or equivalently, $\gamma_{I,\nu_\rho} \geq k_\rho$).

4.2. Multi-weighted blow-ups: canonical aspects

4.2.A. Canonicity of multi-weighted blow-ups, I. In this section, we continue to follow the conventions in Setup 4.1.13, and we endow $\mathbf{A}^n$ with the toroidal logarithmic structure induced by $\mathbf{N}^n \xrightarrow{\mathbf{e}_i \mapsto x_i} \mathbf{k}[x_1, x_2, \ldots, x_n]$. Let $\mathfrak{a}_\bullet$ be a monomial Rees algebra on $\mathbf{A}^n$ under the above logarithmic scheme (Definition 2.7.5), i.e. a finitely generated, $\mathbf{N}$-graded $\mathscr{O}_{\mathbf{A}^n}$-subalgebra $\mathfrak{a}_\bullet = \bigoplus_{m\in\mathbf{N}} \mathfrak{a}_m \cdot t^m \subset \mathscr{O}_{\mathbf{A}^n}[t]$ such that $\mathfrak{a}_0 = \mathscr{O}_{\mathbf{A}^n}$, $\mathfrak{a}_m \supset \mathfrak{a}_{m+1}$ for every $m \in \mathbf{N}$, and each $\mathfrak{a}_m$ is a monomial ideal of $\mathbf{k}[x_1, x_2, \ldots, x_n]$ in the sense of Setup 4.1.13.

We give a definition of $\mathscr{B}l_{\mathfrak{a}_\bullet}\,\mathbf{A}^n$, which generalizes the notion of $\mathscr{B}l_{\mathfrak{a}}\,\mathbf{A}^n$ (Definition 4.1.15) for a monomial ideal $\mathfrak{a}$ on $\mathbf{A}^n$, before demonstrating that this notion is canonically associated to $\mathfrak{a}_\bullet$.

Definition 4.2.1 (Multi-weighted blow-ups along monomial Rees algebras)**.** Fix a sufficiently large $\ell \in N_{>0}$ such that the ℓ^{th} Veronese subalgebra $\mathfrak{a}_{\ell\bullet}$ of $\mathfrak{a}_\bullet$ is generated in degree 1.

The multi-weighted blow-up of $\mathbf{A}^n$ along $\mathfrak{a}_\bullet$ is then defined as:

$$\vartheta_{\mathfrak{a}_\bullet}\colon \mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet}\mathbf{A}^n := \mathscr{B}\mathrm{l}_{\mathfrak{a}_\ell,\widetilde{\mathbf{b}}}\mathbf{A}^n \xrightarrow{\vartheta_{\mathfrak{a}_\ell,\widetilde{\mathbf{b}}}} \mathbf{A}^n$$

where

$$\widetilde{\mathbf{b}} := \left(\frac{\ell}{\gcd\bigl(\ell, N_\rho(\mathfrak{a}_\ell)\bigr)} : \rho \in \Sigma_{\mathfrak{a}_\ell}[1]\right) \in \mathbf{N}_{>0}^{\Sigma_{\mathfrak{a}_\ell}[1]}$$

(cf. Remark 4.1.11 and Convention 4.1.10). We endow $\mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet}\mathbf{A}^n$ with the toroidal logarithmic structure "dictated by that of $\mathbf{A}^n = \mathrm{Spec}(\mathbf{N}^n \xrightarrow{\mathbf{e}_i \mapsto x_i} \mathbf{k}[x_1,\ldots,x_n])$ and the exceptional divisors on $\mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet}\mathbf{A}^n$". Namely, it is obtained by descent from the following toroidal logarithmic structure on $\mathbf{A}^{\Sigma_{\mathfrak{a}}[1]} \smallsetminus V(\mathbb{J}_{\Sigma_{\mathfrak{a}}})$:

$$\mathbf{N}^{\Sigma_{\mathfrak{a}}[1]} \to \mathbf{k}[x_1',\ldots,x_n']\left[x_\rho' \colon \rho \in \mathrm{ex}(\mathfrak{a})\right]$$

which sends $\mathbf{e}_\rho \mapsto x_\rho'$ for every $\rho \in \Sigma_{\mathfrak{a}}[1]$.

Note that if $\mathfrak{a}_\bullet$ is generated in degree 1, then $\mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet}\mathbf{A}^n$ equals $\mathscr{B}\mathrm{l}_{\mathfrak{a}_1}\mathbf{A}^n$ in Definition 4.1.15. It is also simple but essential to verify that:

Lemma 4.2.2. *The definition of $\mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet}\mathbf{A}^n$ does not depend on the choice of $\ell \in \mathbf{N}_{>0}$ such that $\mathfrak{a}_{\ell\bullet}$ is generated in degree 1.*

Proof. Let $\ell, L \in \mathbf{N}_{>0}$ be such that both $\mathfrak{a}_{\ell\bullet}$ and $\mathfrak{a}_{L\bullet}$ are generated in degree 1. By comparing $\mathfrak{a}_{\ell\bullet}$ and $\mathfrak{a}_{L\bullet}$ with $\mathfrak{a}_{\ell L\bullet}$, we reduce to the case where $L = r\ell$ for some $r \in \mathbf{N}_{>0}$. Then $\mathfrak{a}_L = (\mathfrak{a}_\ell)^r$, and thus the normal fans of $\mathfrak{a}_\ell$ and of $\mathfrak{a}_L$ are identical. In particular, $\Sigma_{\mathfrak{a}_\ell}[1] = \Sigma_{\mathfrak{a}_L}[1]$. Lastly, note that $N_\rho(\mathfrak{a}_L) = r \cdot N_\rho(\mathfrak{a}_\ell)$ for every $\rho \in \Sigma_{\mathfrak{a}_\ell}[1] = \Sigma_{\mathfrak{a}_L}[1]$, so $\ell/\gcd(\ell, N_\rho(\mathfrak{a}_\ell)) = L/\gcd(L, N_\rho(\mathfrak{a}_L))$. □

Remark 4.2.3.

(i) Note that $\mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet}\mathbf{A}^n = \mathscr{B}\mathrm{l}_{\mathrm{IC}(\mathfrak{a}_\bullet)}\mathbf{A}^n$, cf. 4.1.16.

(ii) For any $\ell \in \mathbf{N}_{>0}$ such that $\mathfrak{a}_{\ell\bullet}$ is generated in degree 1, consider the Veronese $(1/\ell)$-translate of the Rees algebra $\mathfrak{a}_\ell^\bullet$ associated to the monomial ideal $\mathfrak{a}_\ell$, i.e.

$$\mathfrak{a}_\ell^{(1/\ell)\bullet} = \mathrm{IC}\left(\bigoplus_{d\in\mathbf{N}} \mathfrak{a}_{d\ell} \cdot t^{d\ell}\right)$$

cf. Convention 2.3.79. Then $\mathfrak{a}_\ell^{(1/\ell)\bullet}$ is simply the integral closure $\mathrm{IC}(\mathfrak{a}_\bullet)$ of $\mathfrak{a}_\bullet$ in $\mathscr{O}_{\mathbf{A}^n}[t]$, since they are both integrally closed and their ℓ^{th} Veronese subalgebras coincide. In particular, $\mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet} \mathbf{A}^n = \mathscr{B}\mathrm{l}_{\mathfrak{a}_\ell^{(1/\ell)\bullet}} \mathbf{A}^n$.

4.2.4. Before stating the key proposition of this subsection, let us temporarily assume $\mathfrak{a}_\bullet$ is integrally closed in $\mathscr{O}_{\mathbf{A}^n}[t]$, and let us consider the weighted blow-up of $\mathbf{A}^n$ along $\mathfrak{a}_\bullet$ (Definition 2.3.12), i.e.

$$\mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n := \mathscr{P}\mathrm{roj}_{\mathbf{A}^n}(\mathfrak{a}_\bullet) \xrightarrow{\pi_{\mathfrak{a}_\bullet}} \mathbf{A}^n.$$

Since $\mathbf{A}^n$ is toroidal under the logarithmic structure induced by $\mathbf{N}^n \xrightarrow{\mathbf{e}_i \mapsto x_i} \mathbf{k}[x_1, x_2, \ldots, x_n]$, and $\mathfrak{a}_\bullet$ is monomial and integrally closed (and hence a toroidal center on $\mathbf{A}^n$), we know from Corollary 2.7.20 that $\mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n$ carries a logarithmic structure under which it is toroidal. As in 2.7.7, this logarithmic structure is induced by the chart $\Gamma \hookrightarrow \mathfrak{a}_\bullet^{\text{ext}}$ defined by the following cartesian square

(4.10)
$$\begin{tikzcd}
\Gamma \arrow[r, hook] \arrow[d, hook] & \mathfrak{a}_\bullet^{\text{ext}} \arrow[d, hook] \\
\mathbf{N}^n \oplus \mathbf{Z} \arrow[r, hook] & \mathbf{k}[x_1, x_2, \ldots, x_n][t^\pm]
\end{tikzcd}$$

where the bottom row sends each $(\mathbf{a}, m) \in \mathbf{N}^n \oplus \mathbf{Z}$ to $x^{\mathbf{a}} t^{-m}$, and $\mathfrak{a}_\bullet^{\text{ext}}$ is the extended Rees algebra of $\mathfrak{a}_\bullet$, cf. 2.3.4. Recall that $t^{-1} \in \mathfrak{a}_\bullet^{\text{ext}}$ cuts out the exceptional divisor $E = V(t^{-1})$ on $\mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n = \mathscr{P}\mathrm{roj}_{\mathbf{A}^n}(\mathfrak{a}_\bullet^{\text{ext}})$.

If we write $\mathfrak{a}_\bullet = \mathfrak{a}^{(1/\ell)\bullet}$ for a monomial ideal $\mathfrak{a}$ on $\mathbf{A}^n$ and $\ell \in \mathbf{N}_{>0}$ as in Remark 4.2.3(ii), then Γ can be explicated as the saturation of the submonoid of $\mathbf{N}^n \oplus \mathbf{Z}$ generated by $\mathbf{N}^{n+1}$ and

$(\mathbf{v}, -\ell)$ for vertices $\mathbf{v}$ of the Newton polyhedron $\Gamma_+(\mathfrak{a})$, cf. 4.1.16(i). Moreover, as $\mathbf{v}$ varies over the vertices of $\Gamma_+(\mathfrak{a})$, the $(x^{\mathbf{v}} \cdot t^{\ell})$-charts of $\mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n$

$$D_+\big(x^{\mathbf{v}} \cdot t^{\ell}\big) := \big[\mathrm{Spec}_{\mathbf{A}^n}\big(\mathfrak{a}_\bullet\big[(x^{\mathbf{v}} \cdot t^{\ell})^{-1}\big]\big) \;/\; \mathbb{G}_m\big]\,.$$

form an open cover of $\mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n$, cf. §2.1.C and 4.1.16(i).

Then the "canonicity" implied in the title of this subsection is the following result:

Proposition 4.2.5. *$\mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet} \mathbf{A}^n$ is the canonical smooth, toroidal Artin stack over $\mathrm{Bl}_{\mathrm{IC}(\mathfrak{a}_\bullet)} \mathbf{A}^n$.*

The canonicity asserted in the above proposition is in the sense of Satriano in [**Sat13**], which we will now recall in detail.

4.2.6. Given a toroidal $\mathbf{k}$-scheme Y, Satriano demonstrates in [**Sat13**, §3] that there is a smooth, toroidal Artin stack $\mathscr{Y}$ over Y, which satisfies the following universal property. Any *sliced resolution* [**Sat13**, Definition 2.6] from a fs logarithmic scheme $(T, \mathscr{M}_T)$ to $(Y, \mathscr{M}_Y)$ factors uniquely as a strict morphism $(T, \mathscr{M}_T) \to (\mathscr{Y}, \mathscr{M}_{\mathscr{Y}})$ followed by $(\mathscr{Y}, \mathscr{M}_{\mathscr{Y}}) \to (Y, \mathscr{M}_Y)$. We call $\mathscr{Y} \to Y$ the *canonical smooth, toroidal Artin stack over Y.*

In [**Sat13**, Proposition 3.1], Satriano gives the following local description of $\mathscr{Y} \to Y$. Let $Y = \mathrm{Spec}(\Gamma \hookrightarrow \mathbf{k}[\Gamma])$ for a sharp, toric monoid Γ. Let $C(\Gamma)$ denote the rational cone generated by Γ in $M_{\mathbf{R}} := \Gamma^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathbf{R}$, and $C(\Gamma)^\vee$ be the dual cone in $N_{\mathbf{R}} := M_{\mathbf{R}}^\vee$. For an extremal ray ρ of $C(\Gamma)^\vee$, we denote by $\mathbf{u}_\rho$ the first lattice point on ρ.

Let F denote the free monoid on the set S of extremal rays ρ of $C(\Gamma)^\vee$, and consider $\iota \colon \Gamma \hookrightarrow F$ which sends

$$\mathbf{v} \mapsto (\mathbf{u}_\rho \cdot \mathbf{v})_{\rho \in S} \qquad \text{for } \mathbf{v} \in \Gamma \subset C(\Gamma). \tag{4.11}$$

Then ι is a *minimal free resolution*, in the sense of [**Sat13**, Definition 2.3]. Setting $D(-) := \mathrm{Hom}_{\mathrm{Grp-Sch}}(-, \mathbb{G}_m)$, ι induces the morphism

$$[\mathrm{Spec}(F \hookrightarrow \mathbf{k}[F]) \;/\; D(F^{\mathrm{gp}}/\Gamma^{\mathrm{gp}})] \to \mathrm{Spec}(\Gamma \hookrightarrow \mathbf{k}[\Gamma]) \qquad \text{which is } \mathscr{Y} \to Y.$$

Remark 4.2.7. By descent, Satriano's demonstration immediately generalizes for a toroidal Artin stack Y over $\mathbf{k}$. We may appeal to descent, because Satriano's construction *commutes with strict, smooth morphisms*. More precisely, given a strict morphism $f\colon \widetilde{Y} \to Y$ between toroidal Artin stacks, the canonical smooth, toroidal Artin stack over $\widetilde{Y}$ is the cartesian product $\widetilde{Y} \times_Y \mathscr{Y}$ in the category of fs logarithmic Artin stacks, where $\mathscr{Y} \to Y$ is the canonical smooth, toroidal Artin stack over Y. This can be seen using the universal property in 4.2.6. Indeed, it suffices to note that given any sliced resolution $g\colon T \to \widetilde{Y}$, the composition $g \circ f\colon T \to Y$ is still a sliced resolution, because the induced morphism $\overline{\mathscr{M}}_{Y,f(p)} \to \overline{\mathscr{M}}_{\widetilde{Y},p}$ is an isomorphism for all geometric points p of $\widetilde{Y}$. Thus, $g \circ f$ factors uniquely as $T \xrightarrow{\text{strict}} \mathscr{Y} \to Y$, and hence g factors uniquely as $T \xrightarrow{\text{strict}} \mathscr{Y} \times_Y \widetilde{Y} \to \widetilde{Y}$.

In particular, Satriano's construction can be explicated for a toric Artin stack Y arising from a *stacky cone* (σ, β) [**GS15**, Definition 2.4]. Here, β is a homomorphism of lattices $N \to L$ with finite cokernel, and we assume σ is a strongly convex, rational cone in $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$. The dual morphism $\beta^\vee\colon L^\vee \to N^\vee$ is injective, and the dual cone $\sigma^\vee$ in $N^\vee_{\mathbf{R}}$ yields the sharp, toric monoid $\Gamma := \sigma^\vee \cap N^\vee$, and hence gives rise to the affine toric variety $\mathrm{Spec}(\Gamma \hookrightarrow \mathbf{k}[\Gamma])$. We then get the toric stack $Y := [\mathrm{Spec}(\Gamma \hookrightarrow \mathbf{k}[\Gamma]) \;/\; G]$, where $G := D(\mathrm{Coker}(\beta^\vee))$ acts as a subgroup of the torus $T_N := D(N)$ (with $D(-) := \mathrm{Hom}_{\mathrm{Grp-Sch}}(-, \mathbb{G}_m)$).

4.2.8. With the above notation, the canonical smooth, toroidal Artin stack $\mathscr{Y}$ over Y can be constructed as follows. Let F denote the monoid on the set S of extremal rays ρ of $\sigma = C(\Gamma)^\vee$, and set $N_F^\vee := F^{\mathrm{gp}}$. The same rule in (4.11) defines an embedding of lattices $\eta^\vee\colon N^\vee \hookrightarrow N_F^\vee$, which restricts to a minimal free resolution $\iota\colon \Gamma \hookrightarrow F$, and fits in the commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & L^\vee \arrow[r, hook, "\beta_F^\vee"] & N_F^\vee \arrow[r] & \mathrm{Coker}(\beta_F^\vee) \arrow[r] & 0 \\
0 \arrow[r] & L^\vee \arrow[r, hook, "\beta^\vee"] \arrow[u, equal] & N^\vee \arrow[r] \arrow[u, hook, "\eta^\vee"'] & \mathrm{Coker}(\beta^\vee) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}
\tag{4.12}
$$

The stacky cone $(\sigma_{\mathrm{std}}, \beta_F)$, where σ_{std} is the standard cone on N_F and $\beta_F\colon N_F \to L$ is the dual of $\beta_F^\vee$, then induces the corresponding smooth toric stack $[\mathrm{Spec}(F \hookrightarrow \mathbf{k}[F]) \,/\, G_F]$, where $G_F := D(\mathrm{Coker}(\beta_F))$ acts as a subgroup of the torus $T_{N_F} := D(N_F)$ (with $D(-) := \mathrm{Hom}_{\mathrm{Grp-Sch}}(-, \mathbb{G}_m)$). Finally the above commutative diagram induces the toric morphism

$$[\mathrm{Spec}(F \hookrightarrow \mathbf{k}[F]) \,/\, G_F] \to [\mathrm{Spec}(\Gamma \hookrightarrow \mathbf{k}[\Gamma]) \,/\, G] \qquad \text{which is } \mathscr{Y} \to Y.$$

Proof of Proposition 4.2.5. Without loss of generality (cf. Remark 4.2.3(i)), we may replace $\mathfrak{a}_\bullet$ by $\mathrm{IC}(\mathfrak{a}_\bullet)$. Write $\mathfrak{a}_\bullet = \mathfrak{a}^{(1/\ell)\bullet}$ for a monomial ideal $\mathfrak{a}$ on $\mathbf{A}^n$, and $\ell \in \mathbf{N}_{>0}$. Our approach is to first compute 4.2.8 for the toric Artin stack

$$\mathfrak{M} := \left[\mathrm{Spec}_{\mathbf{A}^n}(\Gamma \xhookrightarrow{(4.10)} \mathfrak{a}_\bullet^{\mathrm{ext}}) \,/\, \mathbb{G}_m\right]$$

before doing the same for $\mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n$, which is a strict, open substack of $\mathfrak{M}$.

4.2.9 (Step 1)**.** By definition, the toric Artin stack $\mathfrak{M}$ arises from the stacky cone (σ, β), whose dual is given by

$$\beta^\vee = (\mathbb{1}_{\mathbf{Z}^n}, 0)\colon \mathbf{Z}^n \hookrightarrow \mathbf{Z}^{n+1} \qquad \text{and} \qquad \sigma^\vee \cap \mathbf{Z}^{n+1} = \Gamma$$

i.e. $\sigma := C(\Gamma)^\vee \subset \mathbf{R}^{n+1}$, and $\beta\colon \mathbf{Z}^{n+1} \to \mathbf{Z}^n$ is the projection onto the first n factors. Next, the extremal rays of σ are the normal rays to the facets of $C(\Gamma)$, and so the set $\mathbf{S}$ of their first lattice points is the disjoint union of:

(i) $\{\mathbf{e}_i\colon 1 \le i \le n\}$, and

(ii) $\left\{\widetilde{\mathbf{u}}_\rho := \left(\frac{\ell}{\gcd(\ell, N_\rho(\mathfrak{a}))} \cdot \mathbf{u}_\rho, \frac{N_\rho(\mathfrak{a})}{\gcd(\ell, N_\rho(\mathfrak{a}))}\right) : \rho \in \Sigma_{\mathfrak{a}}[1] \text{ with } N_\rho(\mathfrak{a}) > 0\right\}$.

Indeed, (i) is evident since the coordinate hyperplanes $\mathbf{e}_i = 0$ $(1 \leq i \leq n)$ intersect $C(\Gamma)$ in facets. For (ii), note that the intersection of $C(\Gamma)$ with the hyperplane $\mathbf{e}_{n+1} = -\ell$ is canonically identified with $\Gamma_+(\mathfrak{a})$, and the (non-empty) intersection of every other facet of $C(\Gamma)$ with this hyperplane $\mathbf{e}_{n+1} = -\ell$ corresponds to a unique facet τ_ρ of $\Gamma_+(\mathfrak{a})$ satisfying $N_\rho(\mathfrak{a}) > 0$.

4.2.10 (Step 2)**.** Let us re-write $\mathbf{S}$ as the following disjoint union:

(i) $\mathbf{S}_1 := \{\mathbf{e}_i \colon 1 \leq i \leq n \text{ with } N_i(\mathfrak{a}) > 0\}$, and

(ii) $\mathbf{S}_2 := \left\{\widetilde{\mathbf{u}}_\rho := \left(\frac{\ell}{\gcd(\ell, N_\rho(\mathfrak{a}))} \cdot \mathbf{u}_\rho, \frac{N_\rho(\mathfrak{a})}{\gcd(\ell, N_\rho(\mathfrak{a}))}\right) : \rho \in \Sigma_{\mathfrak{a}}[1]\right\}$.

We take the indexing set of $\mathbf{S}_2$ to be $\Sigma_{\mathfrak{a}}[1]$, and we denote the indexing set of $\mathbf{S}_1$ by $I := \{1 \leq i \leq n \colon N_i(\mathfrak{a}) > 0\}$.

By 4.2.8, the canonical smooth, toroidal Artin stack $\mathscr{M}$ over $\mathfrak{M}$ arises from the stacky cone $(\sigma_{\mathrm{std}}, \beta_F)$, where σ_{std} is the standard cone on $\mathbf{Z}^I \oplus \mathbf{Z}^{\Sigma_{\mathfrak{a}}[1]}$, and the dual of β_F fits in the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z}^n \arrow[r, hook, "\beta_F^\vee"] & \mathbf{Z}^I \oplus \mathbf{Z}^{\Sigma_{\mathfrak{a}}[1]} \arrow[r] & \mathrm{Coker}(\beta_F^\vee) \arrow[r] & 0 \\ 0 \arrow[r] & \mathbf{Z}^n \arrow[u, equal] \arrow[r, hook, "\beta^\vee"] & \mathbf{Z}^{n+1} \arrow[u, hook, "\eta^\vee"] \arrow[r] & \mathbf{Z} \arrow[u, hook] \arrow[r] & 0 \end{tikzcd} \tag{4.13}$$

Here, the matrix of $\eta^\vee$ has rows given by $\mathbf{e}_i$ for $i \in I$ and $\widetilde{\mathbf{u}}_\rho$ for $\rho \in \Sigma_{\mathfrak{a}}[1]$, and matrix of $\beta_F^\vee$ is obtained by deleting the last column of the matrix of $\eta^\vee$. Recall that $\eta^\vee$ restricts to a minimal free resolution $\iota\colon \Gamma \hookrightarrow \mathbf{N}^I \oplus \mathbf{N}^{\Sigma_{\mathfrak{a}}[1]}$. Explicitly:

$$\mathscr{M} = \left[\mathrm{Spec}\left(\mathbf{N}^I \oplus \mathbf{N}^{\Sigma_{\mathfrak{a}}[1]} \hookrightarrow \mathbf{k}[\chi_i \colon i \in I][x'_\rho \colon \rho \in \Sigma_{\mathfrak{a}}[1]]\right) \,/\, D\left(\mathrm{Coker}(\beta_F^\vee)\right)\right]$$

and $\mathscr{M} \to \mathfrak{M}$ is induced by

$$x_i \mapsto \chi_i \cdot \prod_{\rho \in \Sigma_{\mathfrak{a}}[1]} (x'_\rho)^{\frac{\ell}{\gcd(\ell, N_\rho(\mathfrak{a}))} \cdot \mathrm{u}_{\rho,i}} \quad \text{and} \quad t^{-1} \mapsto \prod_{\rho \in \Sigma_{\mathfrak{a}}[1]} (x'_\rho)^{\frac{N_\rho(\mathfrak{a})}{\gcd(\ell, N_\rho(\mathfrak{a}))}} \tag{4.14}$$

where $\chi_i := 1$ whenever $i \in [n] \smallsetminus I$.

4.2.11 (Step 3)**.** We show, in this step, the following strengthening of Proposition 4.2.5:

Proposition 4.2.12. *Let $\mathfrak{a}_\bullet = \mathfrak{a}^{(1/\ell)\bullet}$ for some monomial ideal $\mathfrak{a}$ on $\mathbf{A}^n$ and $\ell \in \mathbf{N}_{>0}$. For every maximal cone σ of $\Sigma_{\mathfrak{a}}$, $D_+(\sigma) \subset \mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet} \mathbf{A}^n$ is the canonical smooth, toroidal Artin stack over $D_+(x^{\mathbf{v}_\sigma} \cdot t^\ell) \subset \mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n$.*

Proof of Proposition 4.2.12. Since $D_+(x^{\mathbf{v}_\sigma} \cdot t^\ell)$ is a strict, open substack of $\mathfrak{M}$, Remark 4.2.7 says that the canonical smooth, toroidal Artin stack over $D_+(x^{\mathbf{v}_\sigma} \cdot t^\ell)$ is $\mathscr{M}_\sigma := D_+(x_\sigma^{\mathbf{v}} \cdot t^\ell) \times_{\mathfrak{M}} \mathscr{M} \xhookrightarrow{\text{strict, open}} \mathscr{M}$. To explicate $\mathscr{M}_\sigma$, note that (4.14) maps

$$x^{\mathbf{v}_\sigma} \cdot t^\ell \quad \mapsto \quad \prod_{i \in I} \chi_i^{\mathrm{v}_{\sigma,i}} \cdot \prod_{\rho \in \Sigma_{\mathfrak{a}}[1]} (x'_\rho)^{\frac{\ell}{\gcd(\ell, N_\rho(\sigma))} \cdot \left(\sum_{i=1}^n \mathrm{v}_i \cdot \mathrm{u}_{\rho,i} - N_\rho(\mathfrak{a})\right)}$$

where $\mathrm{v}_{\sigma,i} \geq N_i(\mathfrak{a}) > 0$ for all $i \in I$, and $\sum_{i=1}^n \mathrm{v}_i \cdot \mathrm{u}_{\rho,i} - N_\rho(\mathfrak{a}) > 0$ if and only if $\mathbf{v}_\sigma \notin \tau_\rho$, i.e. $\rho \not\subset \sigma$. Therefore, $\mathscr{M}_\sigma$ is equal to:

$$\left[\mathrm{Spec}\left(\mathbf{N}^I \oplus \mathbf{N}^{\Sigma_{\mathfrak{a}}[1]} \to \mathbf{k}\left[\chi_i^{\pm} \colon i \in I\right]\left[x'_\rho \colon \rho \in \Sigma_{\mathfrak{a}}[1]\right]\left[(x'_\sigma)^{-1}\right]\right) \,/\, D\left(\mathrm{Coker}(\beta_F^\vee)\right)\right].$$

For every $i \in I$, note that the image of $\mathbf{e}_i$ in $\mathrm{Coker}(\beta_F^\vee)$ (= the weight of χ_i under the $\mathrm{Coker}(\beta_F^\vee)$-grading) has infinite order. Therefore, by Lemma 2.1.2, we have:

$$\mathscr{M}_\sigma = \left[\mathrm{Spec}\left(\mathbf{N}^{\Sigma_{\mathfrak{a}}[1]} \to \mathbf{k}\left[x'_\rho \colon \rho \in \Sigma_{\mathfrak{a}}[1]\right]\left[(x'_\sigma)^{-1}\right]\right) \,/\, D\left(\mathrm{Coker}\left(\widetilde{\beta}^\vee\right)\right)\right]$$

where $\widetilde{\beta}^\vee$ is the composition $\mathbf{Z}^n \xhookrightarrow{\beta_F^\vee} \mathbf{Z}^I \oplus \mathbf{Z}^{\Sigma_{\mathfrak{a}}[1]} \xrightarrow{\text{projection}} \mathbf{Z}^{\Sigma_{\mathfrak{a}}[1]}$, and $D(\mathrm{Coker}(\widetilde{\beta}^\vee))$ acts as a subgroup of the torus $D(\mathbf{Z}^{\Sigma_{\mathfrak{a}}[1]}) = T_{\mathbf{Z}^{\Sigma_{\mathfrak{a}}[1]}}$.

Since the matrix of $\widetilde{\beta}^\vee$ has rows given by $\frac{\ell}{\gcd(\ell, N_\rho(\mathfrak{a}))} \cdot \mathbf{u}_\rho$ for $\rho \in \Sigma_{\mathfrak{a}}[1]$, it follows by definition that $\mathscr{M}_\sigma = D_+(\sigma) \subset \mathscr{B}\mathrm{l}_{\mathfrak{a}_\bullet} \mathbf{A}^n$ (this also means that their logarithmic structures coincide). $\square$

Finally, as σ varies over all maximal cones of $\Sigma_{\mathfrak{a}}$, the charts $D_+(\sigma)$ cover $\mathscr{B}l_{\mathfrak{a}_\bullet}\mathbf{A}^n$, and the charts $D_+(x^{\mathbf{v}_\sigma}\cdot t^\ell)$ cover $\mathrm{Bl}_{\mathfrak{a}_\bullet}\mathbf{A}^n$. Since Satriano's construction is canonical, this completes the proof of Proposition 4.2.5. □

Remark 4.2.13. Let $\mathfrak{a}_\bullet = \mathfrak{a}^{(1/\ell)\bullet}$ be as before. Then the morphism

$$\begin{array}{c}\mathscr{B}l_{\mathfrak{a}_\bullet}\mathbf{A}^n = \Big[\mathrm{Spec}\big(\mathbf{k}[x_1',\dots,x_n'][x_\rho' \colon \rho\in\mathrm{ex}(\mathfrak{a})]\big)\smallsetminus V(\mathbb{J}_{\Sigma_{\mathfrak{a}}}) \;/\; D\big(\mathrm{Coker}\big(\widetilde{\beta}^\vee\big)\big)\Big]\\ \Big\downarrow\\ \mathrm{Bl}_{\mathfrak{a}_\bullet}\mathbf{A}^n = \big[\mathrm{Spec}\big(\mathfrak{a}_\bullet^{\mathrm{ext}}\big)\smallsetminus V(\mathfrak{a}_+^{\mathrm{ext}}) \;/\; \mathbb{G}_m\big]\end{array}$$

is induced by:

$$\begin{aligned} x_i &\mapsto (x_i')^{\frac{\ell}{\gcd(\ell,N_\rho(\mathfrak{a}))}}\cdot\prod_{\rho\in\mathrm{ex}(\mathfrak{a})}(x_\rho')^{\frac{\ell}{\gcd(\ell,N_\rho(\mathfrak{a}))}\cdot \mathrm{u}_{\rho,i}} \qquad \text{for } 1\le i\le n\\ t^{-1} &\mapsto \prod_{\rho\in\Sigma_{\mathfrak{a}}[1]}(x_\rho')^{\frac{N_\rho(\mathfrak{a})}{\gcd(\ell,N_\rho(\mathfrak{a}))}}. \end{aligned} \tag{4.15}$$

Remark 4.2.14. The morphism $\mathscr{B}l_{\mathfrak{a}_\bullet}\mathbf{A}^n\to\mathrm{Bl}_{\mathfrak{a}_\bullet}\mathbf{A}^n$ is evidently toric (in particular, logarithmically smooth), and birational. Since $\vartheta_{\mathfrak{a}_\bullet}\colon\mathscr{B}l_{\mathfrak{a}_\bullet}\mathbf{A}^n\to\mathbf{A}^n$ is universally closed (Remark 4.1.7), and $\pi_{\mathfrak{a}_\bullet}\colon\mathrm{Bl}_{\mathfrak{a}_\bullet}\mathbf{A}^n\to\mathbf{A}^n$ is proper (Proposition 2.1.5(ii)), we deduce that $\mathscr{B}l_{\mathfrak{a}_\bullet}\mathbf{A}^n\to\mathrm{Bl}_{\mathfrak{a}_\bullet}\mathbf{A}^n$ is universally closed. Therefore, it is also surjective, since it is both dominant and closed. Finally, as a birational morphism it is *small*, i.e. it has no exceptional divisors. This can be seen from Remark 4.2.13, or directly from the fact that $\mathrm{Bl}_{\mathfrak{a}_\bullet}\mathbf{A}^n$ is normal and therefore smooth in codimension 1.

Remark 4.2.15. If $\mathfrak{a}_\bullet$ is generated in degree ℓ, the coarse moduli space of $\mathrm{Bl}_{\mathfrak{a}_\bullet}\mathbf{A}^n$ is the usual blow-up $\mathrm{Bl}_{\mathrm{IC}(\mathfrak{a}_\ell)}\mathbf{A}^n$ of $\mathbf{A}^n$ along $\mathrm{IC}(\mathfrak{a}_\ell)$ (Proposition 2.1.5(iii)). We claim that this coincides with the good moduli space $X_{\Sigma_{\mathfrak{a}_\ell}}$ of $\mathscr{B}l_{\mathfrak{a}_\bullet}\mathbf{A}^n$. The reader can check this computationally, but we propose a more direct approach. By [**Alp13**, Theorem 6.6], there exists a unique morphism $\iota\colon X_{\Sigma_{\mathfrak{a}_\ell}} \to \mathrm{Bl}_{\mathrm{IC}(\mathfrak{a}_\ell)} \mathbf{A}^n$ making the following diagram commute:

$$\begin{tikzcd}
\mathscr{B}l_{\mathfrak{a}_\bullet} \mathbf{A}^n \arrow[r, "\text{good moduli space}"] \arrow[d] & X_{\Sigma_{\mathfrak{a}_\ell}} \arrow[d, dotted, "\iota"'] \\
\mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n \arrow[r, "\text{coarse moduli space}"] & \mathrm{Bl}_{\mathrm{IC}(\mathfrak{a}_\ell)} \mathbf{A}^n
\end{tikzcd}$$

It remains to note that ι is a birational and integral morphism between normal schemes, and hence an isomorphism [**Stacks**, 0AB1]. Indeed, ι is birational, because $\mathscr{B}l_{\mathfrak{a}_\bullet} \mathbf{A}^n \to \mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n$ is an isomorphism above $\mathbf{A}^n \smallsetminus V(\mathfrak{a}_1) \subset \mathbf{A}^n$. To see that ι is integral, it suffices, by [**Stacks**, 01WM], to observe that:

(i) ι is affine. Indeed, for every vertex $\mathbf{v}$ of $\Gamma_+(\mathfrak{a}_\ell)$, the preimage of the coarse space of $D_+(x^{\mathbf{v}} \cdot t^\ell) \subset \mathrm{Bl}_{\mathfrak{a}_\bullet} \mathbf{A}^n$ is the good moduli space of $D_+(\mathbf{v}) \subset \mathscr{B}l_{\mathfrak{a}_\bullet} \mathbf{A}^n$ (cf. Proposition 4.2.12), and they are both affine.

(ii) ι is universally closed, since $\mathscr{B}l_{\mathfrak{a}_\bullet} \mathbf{A}^n \to \mathrm{Bl}_{\mathrm{IC}(\mathfrak{a}_\ell)} \mathbf{A}^n$ is universally closed by Remark 4.2.14 and [**Ols16**, Theorem 11.1.2(ii)], and good moduli spaces remain surjective after any base change [**Alp13**, Propositions 4.7(i) and 4.16(i)].

4.2.B. Canonicity of multi-weighted blow-ups, II. We consider in this section a slightly more general setting than §4.2.A. Given $0 \leq r \leq n$, we instead endow $\mathbf{A}^n$ with the following logarithmic structure:

$$\mathbf{A}^{n;r} := \mathrm{Spec}(\mathbf{N}^r \xrightarrow{\mathbf{e}_i \mapsto \underline{x}_{n-r+i}} \mathbf{k}[x_1, x_2, \ldots, x_{n-r}, \underline{x}_{n-r+1}, \ldots, \underline{x}_n]). \tag{4.16}$$

Here, we underline $\underline{x}_i$ for $i > n-r$ to emphasize that the union of their vanishing loci defines the toroidal divisor on $\mathbf{A}^{n;r}$. Note that the case $r = n$ was considered in §4.2.A. For $\mathbf{a} \in \mathbf{N}^r$, we write $\underline{x}^{\mathbf{a}}$ for the image of $\mathbf{a}$ under $\mathbf{N}^r \to \mathbf{k}[x_1, x_2, \ldots, x_{n-r}, \underline{x}_{n-r+1}, \ldots, \underline{x}_n]$.

Setup 4.2.16. Let $\mathfrak{j}$ be an ideal on $\mathbf{A}^{n;r}$ of the form $(x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, \mathfrak{a})$, where $0 \leq k \leq n-r$, $a_i \in \mathbf{N}_{>0}$, and $\mathfrak{a}$ is a monomial ideal on $\mathbf{A}^{n;r}$ in the sense of Definition 2.7.5, i.e. $\mathfrak{a}$ is generated by monomials in $\underline{x}_{n-r+1}, \ldots, \underline{x}_n$. We set $\ell := \mathrm{lcm}(a_1, a_2, \ldots, a_k)$ ($:= 1$ if $k = 0$), set $d_i := \ell/a_i$ for $1 \leq i \leq k$, and set $\mathfrak{j}_\bullet := \mathfrak{j}^{1/\ell\bullet} := (x_1^{1/d_1}, x_2^{1/d_2}, \ldots, x_k^{1/d_k}, \mathfrak{a}^{1/\ell})$, i.e. the integral closure in $\mathscr{O}_{\mathbf{A}^n}[t]$ of the $\mathscr{O}_{\mathbf{A}^n}$-subalgebra generated by $\{x_i \cdot t^{d_i} \colon 1 \leq i \leq k\}$ and $\mathfrak{a} \cdot t^\ell$, cf. Convention 2.3.79.

4.2.17. Analogous to 4.2.4, before stating our main objective of this subsection, let us consider the weighted blow-up of $\mathbf{A}^n$ along $\mathfrak{j}_\bullet$ (Definition 2.3.12), i.e.

$$\mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^n := \mathscr{P}\mathrm{roj}_{\mathbf{A}^n}(\mathfrak{j}_\bullet) \xrightarrow{\pi_{\mathfrak{j}_\bullet}} \mathbf{A}^n.$$

Similar to before, we know from Corollary 2.7.20 that $\mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^n$ carries a logarithmic structure under which it is toroidal. As in 2.7.7, this logarithmic structure is induced by the chart $\Gamma_2 \hookrightarrow \mathfrak{j}_\bullet^{\mathrm{ext}}$ defined by the following cartesian squares

$$\begin{tikzcd} \Gamma_2 \arrow[r, hook] \arrow[d, hook] & \Gamma \arrow[r, hook] \arrow[d, hook] & \mathfrak{j}_\bullet^{\mathrm{ext}} \arrow[d, hook] \\ \mathbf{N}^r \oplus \mathbf{Z} \arrow[r, hook] & \mathbf{N}^n \oplus \mathbf{Z} \arrow[r, hook] & \mathbf{k}[x_1, x_2, \ldots, x_n][t^\pm] \end{tikzcd} \tag{4.17}$$

where $\mathbf{N}^r \oplus \mathbf{Z} \hookrightarrow \mathbf{N}^n \oplus \mathbf{Z}$ is induced by the canonical injection of $\mathbf{N}^r$ into the last r coordinates of $\mathbf{N}^n$ and the identity on $\mathbf{Z}$, $\mathbf{N}^n \oplus \mathbf{Z}$ maps each $(\mathbf{a}, m) \in \mathbf{N}^n \oplus \mathbf{Z}$ to $x^{\mathbf{a}} t^{-m}$, and $\mathfrak{j}_\bullet^{\mathrm{ext}}$ is the extended Rees algebra of $\mathfrak{j}_\bullet$, cf. 2.3.4. Moreover, via the canonical splitting $\mathbf{N}^n \oplus \mathbf{Z} = \mathbf{N}^{n-r} \oplus (\mathbf{N}^r \oplus \mathbf{Z})$ of $\mathbf{N}^r \oplus \mathbf{Z} \hookrightarrow \mathbf{N}^n \oplus \mathbf{Z}$ in (4.17), we can write $\Gamma = \Gamma_1 \oplus \Gamma_2$, where:

(i) Γ_1 denotes the submonoid of $\mathbf{N}^{n-r}$ generated by $(\mathbf{e}_i, -d_i)$ for $1 \leq i \leq k$ and $(\mathbf{e}_i, 0)$ for $k < i \leq n-r$.

(ii) Γ_2 denotes the saturation of the submonoid of $\mathbf{N}^r \oplus \mathbf{Z}$ generated by $\mathbf{N}^{r+1}$ and $(\mathbf{v}, -\ell)$ for every vertex $\mathbf{v}$ of $\Gamma_+(\mathfrak{a})$.

Recall again that $t^{-1} \in \mathfrak{j}_\bullet^{\mathrm{ext}}$ cuts out the exceptional divisor $E = V(t^{-1})$ on $\mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^n = \mathscr{P}\mathrm{roj}_{\mathbf{A}^n}(\mathfrak{j}_\bullet^{\mathrm{ext}})$. Finally, $\mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^{n;r}$ is covered by the following charts:

(i) $D_+\big(x_i \cdot t^{d_i}\big) = \big[\mathrm{Spec}_{\mathbf{A}^n}\big(\mathfrak{j}_\bullet\big[(x_i \cdot t^{d_i})^{-1}\big]\big) \;/\; \mathbb{G}_m\big]$ for $1 \leq i \leq k$, and

(ii) $D_+\big(\underline{x}^{\mathbf{v}} \cdot t^{\ell}\big) = \big[\mathrm{Spec}_{\mathbf{A}^n}\big(\mathfrak{j}_\bullet\big[(\underline{x}^{\mathbf{v}} \cdot t^{\ell})^{-1}\big]\big) \;/\; \mathbb{G}_m\big]$ for vertices $\mathbf{v}$ of $\Gamma_+(\mathfrak{a})$.

4.2.18 (Logarithmic structure on $\mathscr{B}\mathrm{l}_{\mathfrak{j}} \mathbf{A}^n$)**.** On the other hand, we may also consider $\mathscr{B}\mathrm{l}_{\mathfrak{j}} \mathbf{A}^n$ as defined in Definition 4.1.15. For the next proposition, we endow $\mathscr{B}\mathrm{l}_{\mathfrak{j}} \mathbf{A}^n$ with the toroidal logarithmic structure obtained by descent from the following toroidal logarithmic structure on $\mathbf{A}^{\Sigma_{\mathfrak{j}}[1]} \smallsetminus V(\mathbb{J}_{\Sigma_{\mathfrak{j}}})$:

$$\mathbf{N}^r \oplus \mathbf{N}^{\mathrm{ex}(\mathfrak{j})} \to \mathbf{k}[x_1', \ldots, x_{n-r}', \underline{x}_{n-r+1}', \ldots, \underline{x}_n']\left[\underline{x}_\rho' \colon \rho \in \mathrm{ex}(\mathfrak{j})\right]$$

which sends $\mathbf{e}_i \mapsto \underline{x}_{n-r+i}'$ for $1 \leq i \leq r$, and $\mathbf{e}_\rho \mapsto \underline{x}_\rho'$ for $\rho \in \mathrm{ex}(\mathfrak{j})$. We denote by $\mathscr{B}\mathrm{l}_{\mathfrak{j}} \mathbf{A}^{n;r}$ the resulting logarithmic Artin stack. Using the language introduced in 4.2.6, we can now state the main objective of this section:

Proposition 4.2.19. *$\mathscr{B}\mathrm{l}_{\mathfrak{j}} \mathbf{A}^{n;r}$ is the canonical smooth, toroidal Artin stack over $\mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^{n;r}$.*

We prove this via Proposition 4.2.5, and the following digression:

4.2.20. We return to the discussion in 4.2.8. Adopting the notation there, we suppose further that $N^\vee = N_1^\vee \oplus N_2^\vee$ for sublattices $N_i^\vee \subset N^\vee$, and hence $\Gamma = \Gamma_1 \oplus \Gamma_2$ for the submonoids $\Gamma_i := \Gamma \cap N_i^\vee \subset \Gamma$, such that Γ_1 is a free monoid of finite rank satisfying $\Gamma_1^{\mathrm{gp}} = N_1^\vee$, i.e. its free generators form a basis of $N_1^\vee$. We re-define Y as

$$Y := [\mathrm{Spec}(\Gamma_2 \hookrightarrow \mathbf{k}[\Gamma]) \;/\; G]$$

which is a toroidal Artin stack by hypothesis. As before, our goal here is to explicate the canonical smooth, toroidal Artin stack $\mathscr{Y}$ over Y.

Recall that in 4.2.8, F denotes the free monoid on the set S of extremal rays ρ of σ, $N_F^\vee$ denotes F^{gp}, and we defined an embedding of lattices $\eta^\vee \colon N^\vee \hookrightarrow N_F^\vee$ which restricts to a minimal free resolution $\iota \colon \Gamma \hookrightarrow F$. We claim that there exists free submonoids $F_i \subset F$ such that:

(i) $F = F_1 \oplus F_2$.

(ii) Set $N_{F_i}^\vee := F_i^{\mathrm{gp}}$. Then $\eta^\vee \colon N^\vee \xrightarrow{\simeq} N_F^\vee$ decomposes as $\eta_1^\vee \oplus \eta_2^\vee$, where $\eta_1^\vee = \eta^\vee|_{N_1^\vee} \colon N_1^\vee \xrightarrow{\simeq} N_{F_1}^\vee$ which restricts to $\iota_1 \colon \Gamma_1 \xrightarrow{\simeq} F_1$, and $\eta_2^\vee = \eta^\vee|_{N_2^\vee} \colon N_2^\vee \hookrightarrow N_{F_2}^\vee$ which restricts to a minimal free resolution $\iota_2 \colon \Gamma_2 \hookrightarrow F_2$. Moreover, $\iota = \iota_1 \oplus \iota_2$.

Combining this claim with (4.12) yields the logarithmically smooth morphism

$$[\mathrm{Spec}(F_2 \hookrightarrow \mathbf{k}[F]) \;/\; G_F] \to [\mathrm{Spec}(\Gamma_2 \hookrightarrow \mathbf{k}[\Gamma]) \;/\; G]\,.$$

and moreover shows that it is $\mathscr{Y} \to Y$.

Proof of claim. For $i = 1, 2$, let σ_i denote the dual cone in N of $C(\Gamma_i) \subset N_i^\vee \subset N^\vee$, and let σ_i' denote the dual cone in N_i of $C(\Gamma_i) \subset N_i^\vee$. Since $\Gamma = \Gamma_1 \oplus \Gamma_2 \subset N_1^\vee \oplus N_2^\vee = N^\vee$, we have $\sigma = \sigma_1 \cap \sigma_2$, with:

$$\sigma_1 = \sigma_1' \oplus N_2^\vee \qquad \text{and} \qquad \sigma_2 = N_1^\vee \oplus \sigma_2'.$$

Thus, we may decompose $S = S_1 \sqcup S_2$, where S_i is the set of extremal rays of σ_i. For $i = 1, 2$, let F_i denote the free monoid on S_i. Then part (i) is immediate, while part (ii) follows from the definition of $\eta^\vee$ (in 4.2.8), together with the following pair of observations:

(i) $\{\mathbf{u}_\rho \colon \rho \in S_1\} = \{(\mathbf{u}_{\overline{\rho}}, 0) \colon \overline{\rho} \text{ extremal ray of } \sigma_1'\}$

(ii) $\{\mathbf{u}_\rho \colon \rho \in S_2\} = \{(0, \mathbf{u}_{\overline{\rho}}) \colon \overline{\rho} \text{ extremal ray of } \sigma_2'\}$

where $\mathbf{u}_{\overline{\rho}}$ denotes the first lattice point on $\overline{\rho}$. △

Proof of Proposition 4.2.19. It suffices to assume $k \geq 1$, or else we are in the situation of Proposition 4.2.5. We may also assume $\mathfrak{a} \neq 0$, or else $\mathrm{Bl}_{\mathfrak{i}_\bullet} \mathbf{A}^n$ is already smooth over $\mathbf{k}$ and is equal to $\mathscr{B}\mathrm{l}_{\mathfrak{j}}\,\mathbf{A}^n$, cf. Example 4.1.17. Our approach is to first re-visit 4.2.10 for the toric Artin stack

$$\mathfrak{M} := \left[\mathrm{Spec}_{\mathbf{A}^n}(\Gamma \xhookrightarrow{(4.17)} \mathfrak{j}_\bullet^{\mathrm{ext}}) \;/\; \mathbb{G}_m\right]$$

before using 4.2.20 to deduce the canonical smooth, toroidal Artin stack over

$$\mathfrak{M}' := \left[\mathrm{Spec}_{\mathbf{A}^n}(\Gamma_2 \xhookrightarrow{(4.17)} \mathfrak{j}_\bullet^{\mathrm{ext}}) \;/\; \mathbb{G}_m\right]$$

which contains $\mathrm{Bl}_{\mathfrak{j}_\bullet}\,\mathbf{A}^{n;r}$ as a strict, open substack.

4.2.21 (Step 1)**.** Before re-visiting 4.2.10 for $\mathfrak{M}$, let us first establish the following lemma:

Lemma 4.2.22. *Assume $k \geq 1$ and $\mathfrak{a} \neq 0$. For any $\rho \in \mathrm{ex}(\mathfrak{j})$:*

(i) *ℓ divides $N_\rho(\mathfrak{j})$.*

(ii) *The corresponding facet τ_ρ of $\Gamma_+(\mathfrak{j})$ contains the vertices $\{a_i \cdot \mathbf{e}_i^\vee : 1 \leq i \leq k\}$. In other words, $a_i \cdot \mathrm{u}_{\rho,i} = N_\rho(\mathfrak{j})$ for every $1 \leq i \leq k$.*

(iii) *$\mathrm{u}_{\rho,i} = 0$ for every $k < i \leq n-r$.*

Proof of Lemma 4.2.22. Let $\rho \in \mathrm{ex}(\mathfrak{j})$. Let τ_ρ denote the corresponding facet of $\Gamma_+(\mathfrak{j})$, whose affine span is given by $\sum_{i=1}^{n} \mathrm{u}_{\rho,i} \cdot \mathbf{e}_i = N_\rho(\mathfrak{j})$.

On one hand, note that $\Gamma_+(\mathfrak{j}) \cap \{\mathbf{e}_{n-r+1} = \cdots = \mathbf{e}_n = 0\}$ is the Newton polyhedron $\Gamma_+(\mathfrak{x})$ of $\mathfrak{x} := (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}) \subset \mathbf{k}[x_1, x_2, \ldots, x_{n-r}]$, and that there is only one ray $\widetilde{\rho} \in \mathrm{ex}^+(\mathfrak{x})$, whose corresponding facet $\tau_{\widetilde{\rho}}$ of $\Gamma_+(\mathfrak{x})$ has the affine span $\sum_{i=1}^{k} \frac{\ell}{a_i} \cdot \mathbf{e}_i = \ell$.

On the other hand, $\tau_\rho \cap \{\mathbf{e}_{n-r+1} = \cdots = \mathbf{e}_n = 0\}$ is a facet of $\Gamma_+(\mathfrak{x})$ whose affine span is $\sum_{i=1}^{n-r} \mathrm{u}_{\rho,i} \cdot \mathbf{e}_i = N_\rho(\mathfrak{j})$. Since $\rho \in \mathrm{ex}(\mathfrak{j})$, we must have $N_\rho(\mathfrak{j}) > 0$, so the facet of $\Gamma_+(\mathfrak{x})$ in the preceding sentence must be $\tau_{\widetilde{\rho}}$ in the preceding paragraph. By comparing equations, and noting that $\gcd\left(\frac{\ell}{a_i} : 1 \leq i \leq k\right) = 1$, part (i) follows. Parts (ii) and (iii) are also now immediate. ☐

Let us note in addition that since $k \geq 1$ and $\mathfrak{a} \neq 0$, then $\mathrm{ex}^+(\mathfrak{j}) = \mathrm{ex}(\mathfrak{j})$, i.e. $N_i(\mathfrak{j}) = 0$ for all $i \in [n]$. Therefore, combining 4.2.10 with the above observations, the canonical smooth, toroidal Artin stack $\mathscr{M}$ over $\mathfrak{M}$ arises from the stacky cone $(\sigma_{\mathrm{std}}, \beta_F)$, where σ_{std} is the standard cone on $\mathbf{Z}^{\Sigma_{\mathrm{j}}[1]}$, and the dual of β_F fits in the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z}^n \arrow[r, hook, "\beta_F^\vee"] & \mathbf{Z}^{\Sigma_{\mathrm{j}}[1]} \arrow[r] & \mathrm{Coker}(\beta_F^\vee) \arrow[r] & 0 \\ 0 \arrow[r] & \mathbf{Z}^n \arrow[r, hook, "\beta^\vee"] \arrow[u, equal] & \mathbf{Z}^{n+1} \arrow[r] \arrow[u, hook, "\eta^\vee"'] & \mathbf{Z} \arrow[r] \arrow[u, hook] & 0 \end{tikzcd} \tag{4.18}$$

Here, the matrix of $\eta^\vee$ has rows given by $\left(\mathbf{u}_\rho, \frac{N_\rho(\mathrm{j})}{\ell}\right)$ for $\rho \in \Sigma_{\mathrm{j}}[1]$, and the matrix of $\beta_F^\vee$ has rows given by $\mathbf{u}_\rho$ for $\rho \in \Sigma_{\mathrm{j}}[1]$. Explicitly:

$$\mathscr{M} = \left[\mathrm{Spec}\left(\mathbf{N}^{\Sigma_{\mathrm{j}}[1]} \to \mathbf{k}[\underline{x}'_\rho \colon \rho \in \Sigma_{\mathrm{j}}[1]]\right) \,/\, D(\mathrm{Coker}(\beta_F^\vee))\right]$$

and $\mathscr{M} \to \mathfrak{M}$ is induced by

$$x_i \mapsto x_i \cdot \prod_{\rho \in \mathrm{ex}(\mathrm{j})} (x'_\rho)^{\mathrm{u}_{\rho,i}} \qquad \text{and} \qquad t^{-1} \mapsto \prod_{\rho \in \mathrm{ex}(\mathrm{j})} (x'_\rho)^{\frac{N_\rho(\mathrm{j})}{\ell}}. \tag{4.19}$$

4.2.23 (Step 2)**.** By Lemma 4.2.22(ii), we have, for every $1 \leq i \leq k$:

$$\eta^\vee(\mathbf{e}_i, -d_i) \;=\; \mathbf{e}_i + \sum_{\rho \in \mathrm{ex}(\mathrm{j})} \left(\mathrm{u}_{\rho,i} - \frac{N_\rho(\mathrm{j})}{a_i}\right) \cdot \mathbf{e}_\rho \;=\; \mathbf{e}_i$$

i.e. (4.19) maps $x_i \cdot t^{d_i} \mapsto x'_i$. By part (iii) of the same lemma, we have, for every $k < i \leq n-r$, $\eta^\vee(\mathbf{e}_i, 0) = \mathbf{e}_i$, i.e. (4.19) maps $x_i \mapsto x'_i$. Therefore, $\eta^\vee$ maps Γ_1 isomorphically onto $\mathbf{N}^{[1,n-r]} \subset \mathbf{Z}^{\Sigma_{\mathrm{j}}[1]}$.

On the other hand, it is plain that $\eta^\vee$ maps Γ_2 into $\mathbf{N}^{[n-r+1,n]} \oplus \mathbf{N}^{\mathrm{ex}(\mathrm{j})} \subset \mathbf{Z}^{\Sigma_{\mathrm{j}}[1]}$. By 4.2.20, we know $\eta^\vee|_{\Gamma_2}$ is a minimal free resolution of Γ_2, and the canonical smooth, toroidal Artin stack $\mathscr{M}'$ over $\mathfrak{M}'$ is the stack quotient of

$$\mathrm{Spec}\left(\mathbf{N}^{[n-r+1,n]} \oplus \mathbf{N}^{\mathrm{ex}(\mathrm{j})} \to \mathbf{k}[x'_1, x'_2, \ldots, x'_{n-r}, \underline{x}'_{n-r+1}, \ldots, \underline{x}'_n][\underline{x}'_\rho \colon \rho \in \mathrm{ex}(\mathrm{j})]\right)$$

by the action of $D(\mathrm{Coker}(\beta)^\vee) \subset D(\mathbf{Z}^{\Sigma_{\mathrm{j}}[1]}) = T_{\mathbf{Z}^{\Sigma_{\mathrm{j}}[1]}}$.

4.2.24 (Step 3)**.** By Remark 4.2.7, the canonical smooth, toroidal Artin stack over $\mathrm{Bl}_{\mathfrak{j}_\bullet}\mathbf{A}^{n;r}$ is

$$\mathrm{Bl}_{\mathfrak{j}_\bullet}\mathbf{A}^{n;r}\times_{\mathfrak{M}'}\mathscr{M}'\xhookrightarrow{\text{strict, open}}\mathscr{M}'$$

which is scheme-theoretically identical to $\mathrm{Bl}_{\mathfrak{j}_\bullet}\mathbf{A}^{n;n}\times_{\mathfrak{M}}\mathscr{M}$, i.e. the canonical smooth, toroidal Artin stack over $\mathrm{Bl}_{\mathfrak{j}_\bullet}\mathbf{A}^{n;n}$. By Proposition 4.2.5 and Lemma 4.2.22(i), the latter is $\mathscr{B}\mathrm{l}_{\mathfrak{j}}\mathbf{A}^{n;n}$. Therefore, the former is $\mathscr{B}\mathrm{l}_{\mathfrak{j}}\mathbf{A}^{n}$ with the logarithmic structure induced from that of $\mathscr{M}'$, i.e. $\mathscr{B}\mathrm{l}_{\mathfrak{j}}\mathbf{A}^{n;r}$. $\square$

Remark 4.2.25. The morphism $\mathscr{B}\mathrm{l}_{\mathfrak{j}}\mathbf{A}^{n;r}\to\mathrm{Bl}_{\mathfrak{j}_\bullet}\mathbf{A}^{n;r}$ is logarithmically smooth. Of course, it also satisfies all the properties listed in Remark 4.2.14 and Remark 4.2.15.

Remark 4.2.26. It is possible to say more in Lemma 4.2.22. First we note that $\Gamma_+(\mathfrak{j})\cap\{\mathbf{e}_1=\cdots=\mathbf{e}_{n-r}=0\}$ is the Newton polyhedron $\Gamma_+(\mathfrak{a})$ of $\mathfrak{a}\subset\mathbf{k}[x_{n-r+1},\dots,x_n]$. Correspondingly, for $\rho\in\mathrm{ex}(\mathfrak{j})$, $\tau_\rho\cap\{\mathbf{e}_1=\cdots=\mathbf{e}_{n-r}=0\}$ must be a facet $\tau_{\overline{\rho}}$ of $\Gamma_+(\mathfrak{a})$ (for some $\overline{\rho}\in\Sigma_{\mathfrak{a}}[1]$). Moreover, since $N_\rho(\mathfrak{j})>0$, we must have $N_{\overline{\rho}}(\mathfrak{a})>0$. Then $\rho\mapsto\overline{\rho}$ sets up a one-to-one correspondence $\mathrm{ex}(\mathfrak{j})=\mathrm{ex}^+(\mathfrak{j})\xrightarrow{\simeq}\mathrm{ex}^+(\mathfrak{a})$. Through this correspondence, Lemma 4.2.22 can be supplemented as follows:

$$\begin{aligned}\mathrm{u}_{\rho,i}&=\frac{\ell}{\gcd(\ell,N_{\overline{\rho}}(\mathfrak{a}))}\cdot\mathrm{u}_{\overline{\rho},i}\qquad\text{for }n-r<i\leq n\\ N_\rho(\mathfrak{j})&=\frac{\ell N_{\overline{\rho}}(\mathfrak{j})}{\gcd(\ell,N_{\overline{\rho}}(\mathfrak{a}))}.\end{aligned}$$

In particular, the number $\frac{N_\rho(\mathfrak{j})}{\ell}$ in (4.19) is equal to $\frac{N_{\overline{\rho}}(\mathfrak{a})}{\gcd(\ell,N_{\overline{\rho}}(\mathfrak{a}))}$.

Corollary 4.2.27. *Suppose that a_1 divides $\mathrm{lcm}(a_2,\dots,a_k)$ ($:=1$ if $k=1$). Let $\mathbf{A}^{n-1;r}=V(x_1)\subset\mathbf{A}^{n;r}$, and set $\mathfrak{j}_1:=\mathfrak{j}|_{V(x_1)}=(x_2^{a_2},\dots,x_k^{a_k},\mathfrak{a})\subset\mathbf{k}[x_2,\dots,x_{n-r},\underline{x}_{n-r+1},\underline{x}_n]$. Then the proper transform $V(x_1')$ of $V(x_1)\subset\mathbf{A}^{n;r}$ under $\vartheta_{\mathfrak{j}}\colon\mathscr{B}\mathrm{l}_{\mathfrak{j}}\mathbf{A}^{n;r}\to\mathbf{A}^{n;r}$ is canonically identified with $\vartheta_{\mathfrak{j}_1}\colon\mathscr{B}\mathrm{l}_{\mathfrak{j}_1}\mathbf{A}^{n-1;r}\to\mathbf{A}^{n-1;r}$.*

PROOF. We saw at the start of 4.2.23 that $V(x_1 \cdot t^{d_1}) \times_{\mathrm{Bl}_{\mathfrak{J}_\bullet} \mathbf{A}^{n;r}} \mathscr{B}l_{\mathfrak{J}} \mathbf{A}^{n;r} = V(x_1')$. Thus, by Remark 4.2.7, $V(x_1')$ is the canonical smooth, toroidal Artin stack over $V(x_1 \cdot t^{d_1})$.

On the other hand, $V(x_1 \cdot t^{d_1})$, being the proper transform of $\mathbf{A}^{n-1;r} = V(x_1) \subset \mathbf{A}^{n;r}$ under $\mathrm{Bl}_{\mathfrak{J}_\bullet} \mathbf{A}^{n;r} \to \mathbf{A}^{n;r}$, is equal to $\mathrm{Bl}_{\mathfrak{J}_{1\bullet}} \mathbf{A}^{n-1;r}$, where $\mathfrak{J}_{1\bullet} = \mathfrak{J}_\bullet|_{V(x_1)} = \mathfrak{J}_1^{1/\ell}$. By hypothesis, $\ell = \mathrm{lcm}(a_2, \ldots, a_k)$, whence Proposition 4.2.19 implies that the canonical smooth, toroidal Artin stack over $V(x_1 \cdot t^{d_1})$ is also $\mathscr{B}l_{\mathfrak{J}_1} \mathbf{A}^{n-1;r}$. Combining this with the preceding paragraph, the corollary follows. $\square$

To lift the hypothesis in Corollary 4.2.27, one needs to consider a natural extension of the discussion in this section, which we do not need for purposes of this chapter. Nevertheless we treat this briefly below:

4.2.28. Slightly more generally, for any $c \in \mathbf{N}_{>0}$, we may consider

$$\mathfrak{J}_\bullet^{1/c} := \mathfrak{J}^{1/\ell c} := (x_1^{1/d_1 c}, \ldots, x_k^{1/d_k c}, \mathfrak{a}^{1/\ell c})$$

i.e. the integral closure in $\mathscr{O}_{\mathbf{A}^n}[t]$ of the $\mathscr{O}_{\mathbf{A}^n}$-subalgebra generated by $\{x_i \cdot t^{d_i c} \colon 1 \leq i \leq k\}$ and $\mathfrak{a} \cdot t^{\ell c}$. Scheme-theoretically, the *multi-weighted blow-up of* $\mathbf{A}^{n;r}$ *along* $\mathfrak{J}_\bullet^{1/c}$ is defined as:

$$\vartheta_{\mathfrak{J}_\bullet^{1/c}} \colon \mathscr{B}l_{\mathfrak{J}_\bullet^{1/c}} \mathbf{A}^n := \mathscr{B}l_{\mathfrak{J},\mathbf{b}} \mathbf{A}^n \xrightarrow{\vartheta_{\mathfrak{J},\mathbf{b}}} \mathbf{A}^n$$

where

$$\mathbf{b} := \left(\frac{\ell c}{\gcd\big(\ell c, N_\rho(\mathfrak{J})\big)} \colon \rho \in \mathrm{ex}(\mathfrak{J}) \right) = \left(\frac{c}{\gcd\left(c, \frac{N_\rho(\mathfrak{J})}{\ell}\right)} \colon \rho \in \mathrm{ex}(\mathfrak{J}) \right).$$

The same toroidal logarithmic structure on $\mathbf{A}^{\Sigma_{\mathfrak{J}}[1]} \smallsetminus V(\mathbb{J}_{\Sigma_{\mathfrak{J}}})$ in 4.2.18 descends to a toroidal logarithmic structure on $\mathscr{B}l_{\mathfrak{J}_\bullet^{1/c}} \mathbf{A}^n$. We denote by $\mathscr{B}l_{\mathfrak{J}_\bullet^{1/c}} \mathbf{A}^{n;r}$ the resulting logarithmic Artin stack. If $c = 1$, note that $\mathscr{B}l_{\mathfrak{J}_\bullet} \mathbf{A}^{n;r} = \mathscr{B}l_{\mathfrak{J}} \mathbf{A}^{n;r}$. Then:

(i) By the same method of proof as Proposition 4.2.19, it is the canonical smooth, toroidal Artin stack over $\mathrm{Bl}_{\mathfrak{J}_\bullet^{1/c}} \mathbf{A}^{n;r} := \mathscr{P}\mathrm{roj}_{\mathbf{A}^n}\big(\mathfrak{J}_\bullet^{1/c}\big)$.

(ii) Corollary 4.2.27 has the following natural generalization, with a similar proof. Let $\mathbf{A}^{n-1;r} = V(x_1) \subset \mathbf{A}^{n;r}$ and $\mathfrak{j}_1 = \mathfrak{j}|_{V(x_1)}$, so that $\mathfrak{j}_{1\bullet} = \mathfrak{j}_1^{1/\ell_1}$, where $\ell_1 := \mathrm{lcm}(a_2, \ldots, a_k)$ ($:= 1$ if $k = 1$). Then the proper transform $V(x_1')$ of $V(x_1) \subset \mathbf{A}^{n;r}$ under $\vartheta_{\mathfrak{j}_\bullet^{1/c}}$ is $\vartheta_{\mathfrak{j}_{1\bullet}^{1/cc'}} \colon \mathscr{B}\mathrm{l}_{\mathfrak{j}_{1\bullet}^{1/cc'}} \mathbf{A}^{n-1;r} \to \mathbf{A}^{n-1;r}$, where $c' := \ell/\ell_1$.

4.3. Iterative resolution of singularities in characteristic zero

This section concerns the results outlined in §1.2.B. We adopt the same notations and conventions outlined in §3.1.A. Similar to §3.1.A, while ideals on smooth, toroidal Artin stacks over $\mathbf{k}$ are the main objects of interest in the results of §1.2.B, it suffices to consider ideals on smooth, strict toroidal $\mathbf{k}$-schemes.

Henceforth, fix a smooth, strict toroidal $\mathbf{k}$-scheme Y, and an ideal $J \subset \mathscr{O}_Y$. With respect to Y and J, the constructions and definitions in §3.3 apply, and we freely assume them and their accompanying notation in this section. These include:

(i) the local invariant $\mathrm{inv}_y(J)$ of J at $y \in Y$ as in §3.3.A, together with the preliminary data $x_1, x_2, \ldots, x_k$, $b_1, b_2, \ldots, b_k$, and $Q \subset M = \overline{\mathscr{M}}_{Y,y}$ as in §3.3.2,

(ii) the local $\mathbf{Q}$-toroidal center $\mathscr{I}(J,y)_\bullet = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d})$ associated to J at $y \in Y$ (Definition 3.3.8),

(iii) and the toroidal center $\underline{\mathscr{I}}(J)_\bullet$ associated to J (Definition 3.3.15).

4.3.A. Multi-weighted blow-up along the associated toroidal center. Motivated by the discussion §4.2.B, we make the following:

Definition 4.3.1. The *multi-weighted blow-up* $\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y$ *of* Y *along* $\underline{\mathscr{I}}(J)_\bullet$ is the composition

$$\vartheta \colon \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y \xrightarrow{\zeta} \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y \xrightarrow{\pi} Y,$$

where π is the weighted blow-up of Y along $\underline{\mathscr{I}}(J)_\bullet$ (Definition 2.3.12), and

$$\mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y \xrightarrow{\zeta} \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y \tag{4.20}$$

is the canonical smooth, toroidal Artin stack, in the sense of 4.2.6 (i.e. [**Sat13**, §3]), over the toroidal Deligne–Mumford stack $\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y$.

Remark 4.3.2. Since π is an isomorphism away from the closed locus of points $y \in Y$ such that $\mathrm{inv}_y(J) = \max \mathrm{inv}(J)$ (Theorem 3.4.2(ii)), the same holds for ϑ.

While $\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y$ is a global quotient stack, we warn that $\mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y$ is typically not. Nevertheless, $\mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y$ is locally a quotient stack, cf. 4.3.3 and 4.3.5 below.

4.3.3 (Local description of the multi-weighted blow-up in Definition 4.3.1)**.** Fix $y \in Y$ such that $\mathrm{inv}_y(J) = \max \mathrm{inv}(J)$. Let us replace Y by a neighbourhood U of y on which a presentation $\mathscr{I}(J,y)_\bullet = (x_1^{a_1}, x_2^{a_2}, \dots, x_k^{a_k}, Q^{1/d})$ as in Definition 3.3.8 is defined. In particular, $\mathscr{I}(J)_\bullet = (x_1^{a_1}, x_2^{a_2}, \dots, x_k^{a_k}, Q^{1/d})$, cf. Theorem 3.3.14. Since Y is smooth and toroidal, we can extend $x_1, x_2, \dots, x_k$ to a system of ordinary coordinates $(x_1, x_2, \dots, x_{n-r})$ at y (with $n-r \geq k$) and a system of mononial parameters $(\underline{x}_{n-r+1}, \dots, \underline{x}_n)$ at y. After possibly shrinking U, this local system of logarithmic coordinates at y then induces an étale, strict morphism

$$U \xrightarrow{\boldsymbol{x}} \mathbf{A}^{n;r} = \mathrm{Spec}(\mathbf{k}[x_1, x_2, \dots, x_{n-r}, \underline{x}_{n-r+1}, \dots, \underline{x}_n]).$$

We set

$$\begin{aligned} \mathfrak{j} &:= \mathscr{I}(J)_d \cap \mathscr{O}_{\mathbf{A}^n} = \text{integral closure in } \mathscr{O}_{\mathbf{A}^n} \text{ of } \left(x_1^{a_1 d}, x_2^{a_2 d}, \dots, x_k^{a_k d}, \mathfrak{a}\right) \\ \underline{\mathfrak{j}}_\bullet &:= \underline{\mathscr{I}}(J)_\bullet \cap \mathscr{O}_{\mathbf{A}^{n;r}}[t] = \left(x_1^{1/d_1}, x_2^{1/d_2}, \dots, x_k^{1/d_k}, \mathfrak{a}^{1/d\ell}\right) \end{aligned}$$

where $\mathfrak{a} := Q \cap \mathbf{k}[\underline{x}_{n-r+1}, \dots, \underline{x}_n]$ is a monomial ideal on $\mathbf{A}^{n;r}$ that generates Q. Moreover, $\underline{\mathfrak{j}}_\bullet = \mathfrak{j}^{1/\ell d}$, with $\ell := \mathrm{lcm}(a_i \colon 1 \leq i \leq k)$ $(:= 1$ if $k = 0)$, cf. §4.2.B. Recall too that $d = 1$ if

$k = 0$. Then we have the commutative diagram with cartesian squares:

$$\begin{tikzcd}
\mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} U \arrow[r, "\text{étale, strict}"] \arrow[d] \arrow[dd, bend right=60, "\vartheta"'] & \mathscr{B}l_{\mathfrak{j}} \mathbf{A}^{n;r} \arrow[d] \arrow[dd, bend left=60, "\vartheta_{\mathfrak{j}} \,=\, \vartheta_{\mathfrak{j}_\bullet}"] \\
\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U \arrow[r, "\text{étale, strict}"] \arrow[d, "\pi"] & \mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^{n;r} \arrow[d, "\pi_{\mathfrak{j}_\bullet}"] \\
U \arrow[r, "\boldsymbol{x}"] & \mathbf{A}^{n;r}
\end{tikzcd} \tag{4.21}$$

where $\pi_{\mathfrak{j}_\bullet}$ is the weighted blow-up in 4.2.17, and $\vartheta_{\mathfrak{j}}$ is the multi-weighted blow-up of $\mathbf{A}^n$ along $\mathfrak{j}$, cf. Definition 4.1.15 and 4.2.18. Next, since $\boldsymbol{x}^*\underline{\mathfrak{j}}_\bullet = \underline{\mathscr{I}}(J)_\bullet$, the bottom square of (4.21) is cartesian, i.e.

$$\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U = U \times_{\mathbf{A}^{n;r}} \mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^{n;r}.$$

Moreover, since $\mathscr{B}l_{\mathfrak{j}} \mathbf{A}^{n;r}$ is the canonical smooth, toroidal Artin stack over $\mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^{n;r}$ (Proposition 4.2.19), we deduce from Remark 4.2.7 that the top square of (4.21) is cartesian, i.e.

$$\mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} U = \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U \times_{\mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^{n;r}} \mathscr{B}l_{\mathfrak{j}} \mathbf{A}^{n;r}.$$

Finally, we make a few remarks:

(i) Since $\mathscr{B}l_{\mathfrak{j}} \mathbf{A}^{n;r} \to \mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^{n;r}$ is logarithmically smooth, birational, universally closed, surjective, and small (Remark 4.2.25), so is $\mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} U \to \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U$.

(ii) Since $\pi_{\mathfrak{j}}$ is birational, surjective, and universally closed (Remark 4.1.7), so is π.

(iii) If $k = 0$, ϑ is logarithmically smooth (Corollary 2.7.20), and thus by (i), so is π. This is not true if $k \geq 1$.

(iv) On the other hand, if $Q = 0$, then $\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U$ is smooth over $\mathbf{k}$, so $\mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} U = \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U$ (cf. beginning of proof of Proposition 4.2.19).

(v) $\mathscr{B}l_{\mathscr{I}(J)_\bullet} U$ admits a good moduli space, and it coincides with the coarse moduli space of $\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U$, which is equal to the usual blow-up $\mathrm{Bl}_{\mathscr{I}(J)_d} U$ (Proposition 2.1.5(iii)).

Indeed, because the bottom square of (4.21) is cartesian, we have

$$\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U = \mathrm{Bl}_{\mathscr{I}(J)_d} U \times_{\mathrm{Bl}_{\mathfrak{j}} \mathbf{A}^n} \mathrm{Bl}_{\mathfrak{j}_\bullet} \mathbf{A}^n.$$

Thus,

$$\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} U = \mathrm{Bl}_{\mathscr{I}(J)_d} U \times_{\mathrm{Bl}_{\mathfrak{j}} \mathbf{A}^n} \mathscr{B}\mathrm{l}_{\mathfrak{j}} \mathbf{A}^n.$$

Since $\mathrm{Bl}_{\mathfrak{j}} \mathbf{A}^n$ is the good moduli space of $\mathscr{B}\mathrm{l}_{\mathfrak{j}} \mathbf{A}^n$ (Remark 4.2.25), it follows from [**Alp13**, Proposition 4.7(i)] that $\mathrm{Bl}_{\mathscr{I}(J)_d} U$ is the good moduli space of $\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} U$.

Remark 4.3.4. Because of Remark 4.3.2, the remarks in 4.3.3(i)–(v) globalize immediately. For example, (v) implies that $\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y$ admits a good moduli space, and it coincides with the coarse moduli space $\mathrm{Bl}_{\mathscr{I}(J)_d} Y$ of $\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y$, cf. [**Alp13**, Proposition 4.7(ii)].

4.3.5 (Local description via multi-graded Rees algebras)**.** Let $y \in U \subset Y$ be as in 4.3.3. As in 4.1.29, one may express $\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} U$ as

$$\left[\mathrm{Spec}_U\left(\mathscr{R}_\bullet^U\right) \smallsetminus V(\mathbb{J}_{\Sigma_{\mathfrak{j}}}) \;/\; \mathbb{G}_m^{\mathrm{ex}(\mathfrak{j})}\right]$$

where $\mathscr{R}_\bullet^U$ is the $\mathbf{Z}^{\mathrm{ex}(\mathfrak{j})}$-graded Rees algebra

$$\mathscr{O}_U\left[t_\rho^{-1} \colon \rho \in \mathrm{ex}(\mathfrak{j})\right]\left[x_i \cdot \prod_{\rho \in \mathrm{ex}(\mathfrak{j})} t_\rho^{\mathrm{u}_{\rho,i}} \colon 1 \le i \le n\right] \subset \mathscr{O}_U\left[t_\rho^{\pm} \colon \rho \in \mathrm{ex}(\mathfrak{j})\right]$$

and the $\mathbb{G}_m^{\mathrm{ex}(\mathfrak{j})}$-action is induced by the $\mathbf{Z}^{\mathrm{ex}(\mathfrak{j})}$-grading on $\mathscr{R}_\bullet^U$. As in (4.6), we set $x_i' := x_i \cdot \prod_{\rho \in \mathrm{ex}(\mathfrak{j})} t_\rho^{\mathrm{u}_{\rho,i}}$ for $1 \le i \le n$, and $x_\rho' := t_\rho^{-1}$ for $\rho \in \mathrm{ex}(\mathfrak{j})$. Moreover, if $Q \neq 0$, the morphism

$$\begin{array}{c}
\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} U = \left[\mathrm{Spec}_U\left(\mathscr{R}_\bullet^U\right) \smallsetminus V(\mathbb{J}_{\Sigma_{\mathfrak{j}}}) \;/\; \mathbb{G}_m^{\mathrm{ex}(\mathfrak{j})}\right] \\
\downarrow \\
\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U = \left[\mathrm{Spec}_U(\underline{\mathscr{I}}(J)_\bullet) \smallsetminus V(\underline{\mathscr{I}}(J)_+) \;/\; \mathbb{G}_m\right]
\end{array}$$

is then induced by

$$x_i' \mapsto x_i \cdot \prod_{\rho \in \mathrm{ex}(\mathrm{j})} (x_\rho')^{\mathrm{u}_{\rho,i}}$$

$$t^{-1} \mapsto \prod_{\rho \in \mathrm{ex}^+(\mathrm{j})} (x_\rho')^{\frac{N_\rho(\mathrm{j})}{\ell d}} = \begin{cases} \prod_{\rho \in \mathrm{ex}^+(\mathfrak{a})} (x_\rho')^{N_\rho(\mathfrak{a})} & \text{if } k = 0 \\ \prod_{\rho \in \mathrm{ex}(\mathrm{j})} (x_\rho')^{\frac{N_\rho(\mathrm{j})}{\ell d}} & \text{if } k \geq 1 \end{cases} \tag{4.22}$$

cf. (4.15) and (4.19).

We next turn our attention to the various natural transforms of J under the multi-weighted blow-up of Y along $\underline{\mathscr{I}}(J)_\bullet$. We first recall the following:

4.3.6. Let $Y' := \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y$, and let d and ℓ be as in 4.3.3. By Theorem 3.4.2(iii),

$$\pi^{-1}(J)\mathscr{O}_{Y'} = (t^{-1})^\ell \cdot J' \tag{4.23}$$

where (t^{-1}) is the ideal sheaf $I_E \subset \mathscr{O}_{Y'}$ underlying the exceptional divisor $E \subset Y'$, and $J' \subset \mathscr{O}_{Y'}$ is the weak transform $\pi_*^{-1}(J)$ of J under π (Definition 2.3.31).

Definition 4.3.7. Let $\mathscr{Y}' := \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y$, $Y' := \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y$, and $\zeta \colon \mathscr{Y}' \to Y'$ as in (4.20). We define the *weak transform* of J under $\vartheta \colon \mathscr{Y}' \to Y$ as

$$\vartheta_*^{-1}(J) := \zeta^{-1}(J')\mathscr{O}_{\mathscr{Y}'}$$

where J' is the weak transform of J under π.

The next proposition shows the above definition agrees with Definition 4.1.26.

Proposition 4.3.8 (Local description of weak transform of J)**.** *Let $p \in U \subset Y$ be as in 4.3.3, and set $\mathscr{U}' := \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} U$. Then the restriction of $\vartheta_*^{-1}(J)$ to $\mathscr{U}' \subset \mathscr{Y}'$ is equal to*

$$\left(\vartheta^{-1}(J)\mathscr{O}_{\mathscr{U}'}\right) \cdot \prod_{\rho \in \mathrm{ex}^+(\mathfrak{j})} (x'_\rho)^{-N_\rho}$$

where for each $\rho \in \mathrm{ex}^+(\mathfrak{j})$, N_ρ is the largest natural number n_ρ such that $\vartheta^{-1}(J)\mathscr{O}_{\mathscr{U}'} \subset (x'_\rho)^{n_\rho}$.

Proof of Proposition 4.3.8. If $Q = 0$, then $\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} U = \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} U$, and there is nothing to show. Henceforth, assume $Q \neq 0$. Let $\mathscr{R}_\bullet^U$ be defined as in 4.3.5. Under the correspondence in 4.1.31, $\mathscr{R}_\bullet^U$ corresponds to a tuple $\gamma^U := (\gamma^{[\rho]} \colon \rho \in \mathrm{ex}(\mathfrak{j}))$ of $\#\mathrm{ex}(\mathfrak{j})$ idealistic exponents over U, where each $\gamma^{[\rho]}$ is the idealistic exponent over U associated to the following integrally closed, $\mathbf{N}$-graded Rees algebra on U:

$$\mathscr{R}_\bullet^{[\rho]} := \left(x_i^{\frac{1}{\mathrm{u}_{\rho,i}}} : i \in [n],\ \mathrm{u}_{\rho,i} \neq 0\right)$$

cf. 4.1.32. For each $i \in [n]$, we also set $\gamma^{[i]}$ to be the idealistic exponent over U associated to the ideal (x_i) on U. Finally, γ_J denotes the idealistic exponent over U associated to $J\mathscr{O}_U$. Then we have:

Proposition 4.3.9 (Local description of weak transform of J, explicated)**.** *With the above hypotheses and notations, we have, for every $\rho \in \mathrm{ex}^+(\mathfrak{j})$:*

$$N_\rho = K_\rho = \frac{N_\rho(\mathfrak{j})}{d}$$

where K_ρ is the largest natural number k_ρ such that $\gamma_J \geq k_\rho \cdot \gamma^{[\rho]}$. If $k \geq 1$, this number is also equal to $a_i \cdot \mathrm{u}_{\rho,i}$ for every $1 \leq i \leq k$.

Proof of Proposition 4.3.9. The equality $N_\rho = K_\rho$ can be shown by the same methods in the proofs of Lemma 4.1.34 and Proposition 4.1.36.

For the equality $N_\rho = \frac{N_\rho(\mathfrak{j})}{d}$, we prove the cases $k=0$ and $k\geq 1$ separately. If $k=0$, we make the canonical identification $\mathrm{ex}^+(\mathfrak{j}) = \mathrm{ex}^+(\mathfrak{a})$, and for every ρ in that set, note that $N_\rho(\mathfrak{j}) = N_\rho(\mathfrak{a})$. In this case, $\underline{\mathscr{I}}(J)_\bullet$ is the integral closure in $\mathscr{O}_Y[t]$ of the Rees algebra of $Q = \mathscr{M}(J)$. By Lemma 4.1.22(ii):

$$\vartheta^{-1}(Q)\mathscr{O}_{\mathscr{U}'} = \prod_{\rho\in \mathrm{ex}^+(\mathfrak{a})} (x'_\rho)^{N_\rho(\mathfrak{a})}.$$

Since $Q \supset J$, the left hand side contains $\vartheta^{-1}(J)\mathscr{O}_{\mathscr{U}'}$, so that $N_\rho \geq N_\rho(\mathfrak{a})$ for every $\rho \in \mathrm{ex}^+(\mathfrak{a})$. Conversely, by the definition of N_ρ, $\vartheta^{-1}(J)\mathscr{O}_{\mathscr{U}'} \subset \prod_{\rho\in\mathrm{ex}^+(\mathfrak{j})}(x'_\rho)^{N_\rho}$. Taking monomial saturation $\mathscr{M}(-)$, we get:

$$\vartheta^{-1}(Q)\mathscr{O}_{\mathscr{U}'} = \mathscr{M}\big(\vartheta^{-1}(J)\mathscr{O}_{\mathscr{U}'}\big) \subset \prod_{\rho\in\mathrm{ex}^+(\mathfrak{j})} (x'_\rho)^{N_\rho}$$

where the first equality follows from Lemma 3.1.2(iii), since ϑ is logarithmically smooth if $k=0$ (4.3.3(iii)). That inclusion shows the other inequality $N_\rho(\mathfrak{a}) \geq N_\rho$ for every $\rho\in\mathrm{ex}^+(\mathfrak{a})$, as desired.

If $k\geq 1$, we show instead that $K_\rho = \frac{N_\rho(\mathfrak{j})}{d}$. By Lemma 2.3.77,

$$K_\rho = \max\big\{k_\rho \in \mathbf{N}_{>0} \colon \mathscr{R}^{[\rho]}_{k_\rho \bullet} \text{ is } J\text{-admissible}\big\}.$$

By Lemma 4.2.22(ii), $N_\rho(\mathfrak{j}) = (a_i d)\cdot \mathrm{u}_{\rho,i}$ for every $1\leq i\leq k$ and $\mathrm{u}_{\rho,i} = 0$ for all $k < i \leq n-r$. Therefore, we have

$$\mathscr{R}^{[\rho]}_{\left(\frac{N_\rho(\mathfrak{j})}{d}\right)\bullet} = \left(x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, \left(\underline{x}_i^{N_\rho(\mathfrak{j})/\mathrm{u}_{\rho,i}} \colon n-r < i \leq n,\ \mathrm{u}_{\rho,i}\neq 0\right)^{1/d}\right).$$

Letting $\rho\mapsto\overline{\rho}$ be the one-to-one correspondence $\mathrm{ex}^+(\mathfrak{j}) \xrightarrow{\simeq} \mathrm{ex}^+(\mathfrak{a})$ in Remark 4.2.26, the same remark says that

$$\mathscr{R}^{[\rho]}_{\left(\frac{N_\rho(\mathfrak{j})}{d}\right)\bullet} = \left(x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, \left(\underline{x}_i^{N_{\overline{\rho}}(\mathfrak{a})/\mathrm{u}_{\overline{\rho},i}} \colon n-r < i \leq n,\ \mathrm{u}_{\rho,i}\neq 0\right)^{1/d}\right). \tag{4.24}$$

By Remark 4.1.38, we have

$$\gamma_Q \geq N_{\overline{\rho}}(\mathfrak{a}) \cdot \gamma^{[\overline{\rho}]} \text{ for every } \rho \in \operatorname{ex}^+(\mathfrak{j}) \tag{4.25}$$

where $\gamma^{[\overline{\rho}]}$ is the idealistic exponent over U associated to the integrally closed Rees algebra $\left(x_i^{1/u_{\overline{\rho},i}} \colon n-r < i \leq n, \mathrm{u}_{\rho,i} \neq 0\right)$ on U. From (4.24) and (4.25), we deduce that $\mathscr{R}^{[\rho]}_{\left(\frac{N_\rho(\mathfrak{j})}{d}\right)\bullet}$ contains $\mathscr{I}(J,y)_\bullet = \left(x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}, Q^{1/d}\right)$. Since $\mathscr{I}(J,y)_\bullet$ is J-admissible at y (Theorem 3.3.9(i)), so is $\mathscr{R}^{[\rho]}_{\left(\frac{N_\rho(\mathfrak{j})}{d}\right)\bullet}$, whence $K_\rho \geq \frac{N_\rho(\mathfrak{j})}{d}$. On the other hand, $a_1 = \operatorname{log-ord}_y(J)$, whence Corollary 3.1.11 implies that for any $k_\rho > \frac{N_\rho(\mathfrak{j})}{d}$, $\mathscr{R}^{[\rho]}_{k_\rho\bullet}$ cannot be J-admissible at y, as desired. □

We return back to the proof of Proposition 4.3.8. Applying $\zeta^{-1}(-) \cdot \mathscr{O}_{\mathscr{Y}'}$ to (4.23) and then applying (4.22) and Proposition 4.3.9, we obtain the desired equality:

$$\vartheta^{-1}(J)\mathscr{O}_{\mathscr{Y}'} = \left(\zeta^{-1}(J')\mathscr{O}_{\mathscr{Y}'}\right) \cdot \prod_{\rho \in \operatorname{ex}^+(\mathfrak{j})} (x'_\rho)^{\frac{N_\rho(\mathfrak{j})}{\ell d} \cdot \ell} = \vartheta_*^{-1}(J) \cdot \prod_{\rho \in \operatorname{ex}^+(\mathfrak{j})} (x'_\rho)^{N_\rho}. \qquad \square$$

The next proposition shows that the same equality in Definition 4.3.7 holds as well for proper transforms:

Proposition 4.3.10. *Let $\widetilde{J}$ (resp. $\widetilde{J}_\pi$) denote the proper transform of J under $\vartheta \colon \mathscr{Y}' := \mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$ (resp. $\pi \colon Y' := \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$). Then we have:*

$$\widetilde{J} = \zeta^{-1}(\widetilde{J}_\pi)\mathscr{O}_{\mathscr{Y}'}$$

where ζ was the morphism $\mathscr{Y}' \to Y'$ (4.20) in Definition 4.3.1.

Proof. Recall that $V(\widetilde{J})$ (resp. $V(\widetilde{J}_\pi)$) is the smallest closed substack of $\mathscr{Y}'$ (resp. Y') containing $V\left(\vartheta^{-1}(J)\mathscr{O}_{\mathscr{Y}'}\right) \smallsetminus V\left(\vartheta^{-1}(\underline{\mathscr{I}}(J)_1)\mathscr{O}_{\mathscr{Y}'}\right)$ (resp. $V\left(\pi^{-1}(J)\mathscr{O}_{Y'}\right) \smallsetminus V\left(\pi^{-1}(\underline{\mathscr{I}}(J)_1)\mathscr{O}_{Y'}\right)$). The proposition then follows from the following equalities:

$$\zeta^{-1}\left(V\left(\pi^{-1}(J)\mathscr{O}_{Y'}\right)\right) = V\left(\vartheta^{-1}(J)\mathscr{O}_{\mathscr{Y}'}\right)$$

$$\zeta^{-1}\left(V\big(\pi^{-1}(\underline{\mathscr{I}}(J)_1\big)\mathscr{O}_{Y'})\big)\right) = V\big(\vartheta^{-1}(\underline{\mathscr{I}}(J)_1)\mathscr{O}_{\mathscr{Y}'}\big)$$

and the fact that ζ is closed, cf. 4.3.3(i). $\square$

We can finally deduce:

Theorem 4.3.11 (Invariant drops in a well-ordered set). *Let $J \subset \mathscr{O}_Y$ be a non-zero ideal, and let $\widetilde{J}$ be its proper transform under $\vartheta \colon \mathscr{Y}' := \mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$. Then*

$$\max\mathrm{inv}(\widetilde{J}) \leq \max\mathrm{inv}(\vartheta_*^{-1}(J)) < \max\mathrm{inv}(J).$$

and all three maximum invariants are contained in the well-ordered set $\mathbf{N}_\infty^{\leq \dim(Y),!}$, cf. 3.3.1.

Proof. We adopt the notation in Definition 4.3.7, and Proposition 4.3.10. Since ζ is logarithmically smooth and surjective (4.3.3(i)), we have

$$\max\mathrm{inv}\big(\vartheta_*^{-1}(J)\big) = \max\mathrm{inv}\left(\zeta^{-1}(J')\mathscr{O}_{\mathscr{Y}'}\right) = \max\mathrm{inv}(J') < \max\mathrm{inv}(J)$$

where the middle equality is given by Lemma 3.3.6(iii), and the strict inequality is given by Theorem 3.4.2(iv). Recall from Definition 3.3.4 that the lengths of $\max\mathrm{inv}(J)$ and $\max\mathrm{inv}(J')$ are bounded above by $\dim(Y) = \dim(Y')$, and hence, so is the length of $\max\mathrm{inv}\big(\vartheta_*^{-1}(J)\big)$. Moreover, since $\widetilde{J} \supset \vartheta_*^{-1}(J)$ (Remark 4.1.27), we also have $\max\mathrm{inv}(\widetilde{J}) \leq \max\mathrm{inv}\big(\vartheta_*^{-1}(J)\big)$. Finally, the proposition, together with Lemma 3.3.6(iii), imply that

$$\max\mathrm{inv}\big(\widetilde{J}\big) = \max\mathrm{inv}\big(\zeta^{-1}(\widetilde{J}_\pi)\mathscr{O}_{\mathscr{Y}'}\big) = \max\mathrm{inv}\big(\widetilde{J}_\pi\big).$$

Since the length of $\max\mathrm{inv}\big(\widetilde{J}_\pi\big)$ is also bounded above by $\dim(Y) = \dim(Y')$, so is the length of $\max\mathrm{inv}\big(\widetilde{J}\big)$. $\square$

4.3.12 (Functoriality). Given a strict, smooth, and surjective morphism $f\colon \widetilde{Y} \to Y$ of smooth, strict toroidal $\mathbf{k}$-schemes, we have

$$\widetilde{Y} \times_Y \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y = \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}})} \widetilde{Y}$$

where as always, the fiber product is taken in the category of fs logarithmic Artin stacks. Indeed, since $\underline{\mathscr{I}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}})_\bullet = f^{-1}(\underline{\mathscr{I}}(J)_\bullet)\mathscr{O}_{\widetilde{Y}}$ (Lemma 3.3.16), we have $\widetilde{Y} \times_Y \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y = \mathrm{Bl}_{\underline{\mathscr{I}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}})_\bullet} \widetilde{Y}$. As a consequence,

$$\begin{aligned}\widetilde{Y} \times_Y \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y &= \big(\widetilde{Y} \times_Y \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y\big) \times_{\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y} \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y \\ &= \mathrm{Bl}_{\underline{\mathscr{I}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}})_\bullet} \times_{\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y} \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y.\end{aligned}$$

Since $\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y$ is by definition the canonical smooth, toroidal Artin stack over $\mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y$, Remark 4.2.7 implies that $\widetilde{Y} \times_Y \mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(J)_\bullet} Y$ is then the canonical smooth, toroidal Artin stack over $\mathrm{Bl}_{\underline{\mathscr{I}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}})_\bullet} \widetilde{Y}$, and is therefore by definition $\mathscr{B}\mathrm{l}_{\underline{\mathscr{I}}(f^{-1}(J)\mathscr{O}_{\widetilde{Y}})} \widetilde{Y}$.

4.3.B. Proof of main theorems in §1.2.B.

Proof of Theorem F. By hypothesis, $X \neq Y$. Let J be the underlying non-zero ideal of $X \subset Y$. We set $\pi\colon Y' \to Y$ to be $\pi\colon \mathrm{Bl}_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$ in Definition 4.3.1. Then part (i) is immediate, part (ii) follows from Remark 4.3.2, part (iii) follows from Theorem 4.3.11, and part (iv) follows from parts (ii) and (iv) of 4.3.3 (cf. Remark 4.3.4). Finally, functoriality with respect to strict, smooth, and surjective morphisms of pairs follows from 4.3.12. □

Proof of Theorem E. This can be deduced from Theorem F in the same way as how one deduces Theorem A from Theorem B, cf. §3.4.B. We leave this to the reader. □

As in §3.4.B, to deduce Corollary G from Theorem E, we require the following adaptation of Lemma 4.3.13:

Lemma 4.3.13 (Re-embedding principle for Theorem F). *Let X be a reduced, closed substack of a smooth, toroidal Artin stack Y over $\mathbf{k}$. Let Y_1 be the fiber product $Y \otimes_{\mathbf{k}} \mathbf{A}^1$ in the category of fs Artin stacks, where $\mathbf{A}^1 := \operatorname{Spec}(\mathbf{k}[x_0])$ and $\mathbf{k}$ are given the trivial logarithmic structure. Then:*

(i) *For every $y \in |X|$, $\operatorname{inv}_y(X \subset Y_1)$ is the concatenation $(1, \operatorname{inv}_y(X \subset Y))$.*

(ii) *Let $(X \subset Y) \mapsto (X' \subset Y')$ be the procedure in Theorem F. Then Y' is canonically identified with the proper transform of $Y = Y \times \{0\} \subset Y_1$ under the weighted blow-up $Y_1' \to Y_1$. Under this identification, we have $X' = X_1'$.*

Proof. We may assume Y is a strict toroidal $\mathbf{k}$-scheme. Then part (i) is Lemma 3.4.8(i). Part (ii) also follows the same way as Lemma 3.4.8(ii), except that one also needs Corollary 4.2.27. □

Proof of Corollary G. Since X can be embedded, locally in the smooth topology, as a closed subscheme of pure codimension in a pure-dimensional, smooth, toroidal $\mathbf{k}$-scheme, the corollary follows once we show the following. Given two strict closed embeddings of X into pure-dimensional, smooth, toroidal Artin stacks Y_i over $\mathbf{k}$ (for $i = 1, 2$), the resolutions of singularities of X obtained from the embedded resolutions of singularities of $X \subset Y_1$ and $X \subset Y_2$ in Theorem E coincide. First assume that $\dim(Y_1) = \dim(Y_2)$: in this case, the two embeddings are smooth locally isomorphic. By the functoriality of Theorem E, the embedded resolutions of singularities of $X \subset Y_1$ and $X \subset Y_2$ are smooth locally isomorphic, whence the resulting resolutions of singularities of X coincide. In general, this reduces to the earlier case, by a repeated application of Lemma 4.3.13. □

Finally, we sketch how one can obtain Corollary D from Corollary G. The first ingredient is a special case of [ER20, Theorem 2.11].

Theorem 4.3.14 (Reduction of stabilizers: smooth, toroidal case)**.** *Let X be a smooth Artin stack over $\mathbf{k}$ that admits a good moduli space $\mathbf{X}$, has affine diagonal, and has no generic stackiness. Let $E \subset X$ be a simple normal crossings divisor. Then there exists a canonical sequence of saturated blow-ups* [**ER20**, Definition 3.2] *of Artin stacks $\Phi\colon X_N \xrightarrow{\phi_N} X_{N-1} \xrightarrow{\phi_{N-1}} \cdots \xrightarrow{\phi_1} X_0 = X$ along smooth, closed substacks $C_i \subset X_i$, together with simple normal crossings divisors $E_i \subset X_i$ with $E_0 = E$, such that:*

(i) *Each X_i is a smooth Artin stack over $\mathbf{k}$ admitting a good moduli space $X_i \xrightarrow{\varphi_i} \mathbf{X}_i$.*

(ii) *Each $|C_i|$ is the locus in X_i of points of maximum dimensional stabilizer.*

(iii) *Each ϕ_i restricts to an isomorphism $X_i \smallsetminus \phi_i^{-1}(C_{i-1}) \xrightarrow{\simeq} X_{i-1} \smallsetminus \varphi_{i-1}^{-1}(\varphi_{i-1}(C_{i-1}))$.*

(iv) *Each E_i is the inverse image of $C_{i-1} \cup E_{i-1}$ under ϕ_i.*

(v) *The maximum dimension of the stabilizers of points of X_i is strictly smaller than that of the stabilizers of points of X_{i-1}.*

(vi) *The final stack X_N has finite inertia, with coarse moduli space $X_N \xrightarrow{\varphi_N} \mathbf{X}_N$.*

(vii) *Each ϕ_i induces a schematic blow-up of good moduli spaces $\mathbf{X}_i \to \mathbf{X}_{i-1}$, which is an isomorphism over $\mathbf{X}_{i-1} \smallsetminus \varphi_{i-1}(C_{i-1})$.*

The sequence Φ does not depend on E. This procedure $X \mapsto X_N$ is functorial with respect to strong morphisms [**ER20**, Definition 6.8]*.*

A SECOND PROOF OF COROLLARY D. We apply Corollary G to a pure-dimensional, reduced scheme X of finite type over $\mathbf{k}$, endowed with the trivial logarithmic structure. We obtain a proper, birational morphism $\Pi\colon X^+ \to X$, where X^+ is a pure-dimensional, smooth Artin stack over $\mathbf{k}$, Π is an isomorphism over $X^{\mathrm{sm}} \subset X$, and $E := \Pi^{-1}(X \smallsetminus X^{\mathrm{sm}})$ is a simple normal crossings divisor on X^+. Now apply the above theorem to the pair (X, E) to obtain a proper, birational morphism $\Phi\colon X_w^{++} \to X^+$ such that X_w^{++} is a smooth Artin stack over $\mathbf{k}$ with finite inertia, $\Phi_w := \Phi \circ \Pi$ is an isomorphism over $X^{\mathrm{sm}} \subset X$, and $\Phi_w^{-1}(X \smallsetminus X^{\mathrm{sm}})$ is a simple normal crossings divisor on X_w^{++}. The remainder of the proof now follows in the same way as outlined in the second paragraph of the first proof of Corollary D in §3.4.B. $\square$

4.4. Examples and further remarks

4.4.A. Examples. Throughout this section, we freely adopt the notation introduced in Chapters 3.3, 4.2 and 4.3.

Example 4.4.1. Let $Y = \mathbf{A}^{3;3} = \operatorname{Spec}(\mathbf{N}^3 \to \mathbf{k}[\underline{x}, \underline{y}, \underline{z}])$, and consider the following hypersurface:

$$X := V(J) := V\big(\underline{x}^2 + \underline{y}^2\underline{z} + \underline{z}^3\big) \subset Y.$$

Then $\max\operatorname{inv}(J) = (\infty)$, and $\underline{\mathscr{I}}(J)_\bullet$ is the integral closure in $\mathscr{O}_Y[t]$ of the Rees algebra of $\mathscr{M}(J) = (\underline{x}^2, \underline{y}^2\underline{z}, \underline{z}^3)$. Let $\pi\colon Y' := \mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$, which was explicated in Example 4.1.18. By the equations therein, the total transform of I is:

$$\pi^{-1}(J)\mathscr{O}_{Y'} = \underline{u}_1^6\underline{u}_2^2 \cdot \underbrace{\big(\underline{x}'^2 + \underline{y}'^2\underline{z}' + \underline{z}'^3\underline{u}_2^4\big)}_{\text{proper transform } J'}. \tag{4.26}$$

Finally, $\mathscr{D}^{\leq 1}(J') = (\underline{x}'^2, \underline{y}'^2\underline{z}', \underline{z}'^3\underline{u}_2^4)$ which is the unit ideal on the $\underline{x}'$-chart, $\underline{y}'\underline{z}'$-chart, and $\underline{z}'\underline{u}_2$-chart of Y'. Therefore, $\max\operatorname{inv}(J') = (1) < (\infty) = \max\operatorname{inv}(J)$, and we get resolution of singularities in one step.

The above is an example of a polynomial that is not just non-degenerate, but in fact non-degenerate *with respect to all faces of its Newton polyhedron*, cf. Definition 1.3.1. We bring this to the reader's attention because Example 4.4.1 is then manifested by a general phenomenon which was earlier observed in [**BN20**, Proposition 8.31] for all polynomials that are non-degenerate with respect to all faces of its Newton polyhedron:

Theorem 4.4.2. *Let $f \in \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial that is non-degenerate with respect to all faces of its Newton polyhedron, and assume x_i does not divide f for every $1 \leq i \leq n$. Then* *the multi-weighted blow-up of $\mathbf{A}^n$ along the monomial saturation $\mathfrak{a}_{(f)}$ of (f) (Definition 4.1.21) is an embedded resolution of singularities for $V(f) \subset \mathbf{A}^{n;n} = \operatorname{Spec}(\mathbf{N}^n \to \mathbf{k}[\underline{x}_1, \ldots, \underline{x}_n])$.*

In other words, the embedded resolution of singularities in Theorem E, applied to the pair $V(f) \subset \mathbf{A}^{n;n} = \operatorname{Spec}(\mathbf{N}^n \to \mathbf{k}[\underline{x}_1, \ldots, \underline{x}_n])$, terminates after one step.

We remark that we impose the condition that x_i does not divide f for every $1 \leq i \leq n$, so that $V(f) \subset \mathbf{A}^{n;n}$ is generically toroidal, and hence satisfies the hypotheses of Theorem E. The same proof below, with some minor modifications, continues to work if one drops that condition.

Proof. Let $\mathfrak{a} := \mathfrak{a}_{(f)}$, and $\pi_{\mathfrak{a}} \colon \operatorname{Bl}_{\mathfrak{a}} \mathbf{A}^n = \left[X_{\widehat{\Sigma}_{\mathfrak{a}}} \,/\, \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})}\right] \to \mathbf{A}^n$. Let $\overline{\sigma}$ be an arbitrary cone in $\widehat{\Sigma}_{\mathfrak{a}}$, and let σ denote its image under the morphism $\beta \colon \mathbf{Z}^{\Sigma_{\mathfrak{a}}(1)} \to \mathbf{Z}^n$ which sends $\mathbf{e}_\rho \mapsto \mathbf{u}_\rho$ for every $\rho \in \Sigma_{\mathfrak{a}}(1)$ (Definition 4.1.15). By definition of $\widehat{\Sigma}_{\mathfrak{a}}$, there is a *smallest* cone σ' in $\Sigma_{\mathfrak{a}}$ such that σ is a sub-cone of σ'. Let τ be the face of $P_{\mathfrak{a}} = P_f$ dual to σ'. If $O(\sigma)$ denotes the $\mathbb{G}_m^{\widehat{\Sigma}_{\mathfrak{a}}(1)}$-orbit of $X_{\widehat{\Sigma}_{\mathfrak{a}}}$ corresponding to $\overline{\sigma}$, we claim the proper transform of $V(f) \subset \mathbf{A}^n$ under π is non-singular on the $\left(\mathbb{G}_m^{\widehat{\Sigma}_{\mathfrak{a}}(1)} \,/\, \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})}\right)$-orbit $\left[O(\sigma) \,/\, \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})}\right] \subset \left[X_{\widehat{\Sigma}_{\mathfrak{a}}} \,/\, \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})}\right] = \operatorname{Bl}_{\mathfrak{a}} \mathbf{A}^n$. This claim proves the proposition, since $X_{\widehat{\Sigma}_{\mathfrak{a}}} = \bigsqcup_{\overline{\sigma} \in \widehat{\Sigma}_{\mathfrak{a}}} O(\sigma)$. We prove the claim in three steps.

4.4.3 (Step 1)**.** Let U_σ denote the affine toric variety associated to the cone $\overline{\sigma}$ in $\widehat{\Sigma}_{\mathfrak{a}}$. By (4.4) and (4.5), $D_+(\sigma) := \left[U_\sigma \,/\, \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})}\right] \subset \operatorname{Bl}_{\mathfrak{a}} \mathbf{A}^n$ is:

$$\left[\operatorname{Spec}_{\mathbf{A}^n}\left(\frac{\mathscr{O}_{\mathbf{A}^n}[x_1', \ldots, x_n']\left[x_\rho' \colon \rho \in \mathbf{E}(\mathfrak{a})\right]\left[(x_\sigma')^{-1}\right]}{\left(x_i' \cdot \prod_{\rho \in \mathbf{E}(\mathfrak{a})} (x_\rho')^{\mathrm{u}_{\rho,i}} - x_i \colon 1 \leq i \leq n\right)}\right) \,/\, \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})}\right]$$

where

$$x_\sigma' := \prod_{\rho \in \Sigma_{\mathfrak{a}}(1) \smallsetminus \sigma(1)} x_\rho' \qquad \text{with} \quad \sigma(1) := \{\rho \in \Sigma_{\mathfrak{a}}(1) \colon \rho \subset \sigma\}. \tag{4.27}$$

Next, by Lemma 2.1.2, the assignment $x'_\rho \mapsto 1$ for every $\rho \in \mathbf{E}(\mathfrak{a}) \smallsetminus \sigma(1)$ identifies $D_+(\sigma)$ with

$$\left[\operatorname{Spec}_{\mathbf{A}^n}\left(\frac{\mathscr{O}_{\mathbf{A}^n}[x'_1,\dots,x'_n]\left[x'_\rho\colon \rho \in \mathbf{E}(\mathfrak{a})\cap\sigma(1)\right]\left[(x'_\sigma)^{-1}\right]}{\left(x'_i\cdot\prod_{\rho\in\mathbf{E}(\mathfrak{a})\cap\sigma(1)}(x'_\rho)^{\mathrm{u}_{\rho,i}} - x_i\colon 1\le i\le n\right)}\right)\ /\ \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})\cap\sigma(1)}\right] \tag{4.28}$$

where we re-define x'_σ as

$$x'_\sigma := \prod_{i\in[1,n]\smallsetminus\sigma(1)} x'_i.$$

4.4.4 (Step 2)**.** Write $f = \sum_{\mathbf{a}\in\mathbf{N}^n} c_{\mathbf{a}}\cdot\boldsymbol{x}^{\mathbf{a}}$. By Lemma 4.1.22(i), the total transform of f on (4.28) is:

$$f = \prod_{\rho\in\mathbf{E}(\mathfrak{a})\cap\sigma(1)}(x'_\rho)^{N_\rho(\mathfrak{a})}\cdot\underbrace{\left(\sum_{\mathbf{a}\in\mathbf{N}^n} c_{\mathbf{a}}\cdot(\boldsymbol{x}')^{\mathbf{a}}\cdot\prod_{\rho\in\mathbf{E}(\mathfrak{a})\cap\sigma(1)}(x'_\rho)^{(\mathbf{a}\cdot\mathbf{u}_\rho)-N_\rho(\mathfrak{a})}\right)}_{\text{proper transform } f'}.$$

Let us record two essential observations about f':

(i) If $\mathbf{a}\in\mathbf{N}^n\cap\tau$, then for every $\rho\in\sigma(1)$, we have $\mathbf{a}\in\tau\subset H_\rho$, i.e. $\mathbf{a}\cdot\mathbf{u}_\rho = N_\rho(\mathfrak{a})$. In particular, if $\rho = i\in[1,n]\cap\sigma(1)$, we note separately that this means $a_i = 0$.

(ii) If $\mathbf{a}\in\mathbf{N}^n\smallsetminus\tau$, there exists $\rho\in\sigma(1)$ such that $\mathbf{a}\in\mathbf{N}^n\smallsetminus H_\rho$, i.e. $\mathbf{a}\cdot\mathbf{u}_\rho > N_\rho(\mathfrak{a})$. This is because $\tau = \bigcap_{\rho\in\sigma'(1)} H_\rho = \bigcap_{\rho\in\sigma(1)} H_\rho$.

4.4.5 (Step 3)**.** Finally,

$$\left[O(\sigma)\ /\ \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})}\right] = V\left(x'_\rho\colon \rho\in\sigma(1)\right) \xhookrightarrow{\text{closed}} D_+(\sigma).$$

Combining the above with (4.28), we get the identification

$$\left[O(\sigma)\ /\ \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})}\right] = \left[\operatorname{Spec}\left(\mathbf{k}\left[(x'_i)^{\pm}\colon i\in[1,n]\smallsetminus\sigma(1)\right]\right)\ /\ \mathbb{G}_m^{\mathbf{E}(\mathfrak{a})\cap\sigma(1)}\right]. \tag{4.29}$$

Moreover, by 4.4.4(i)–(ii), the restriction of f' to (4.29) is:

$$\sum_{\mathbf{a}\in\mathbf{N}^n\cap\tau} c_{\mathbf{a}} \cdot (\boldsymbol{x}')^{\mathbf{a}}.$$

Since the above expression matches that of f_τ, the claim follows. □

The next three examples (Examples 4.4.6, 4.4.7, 4.4.9) will re-visit the same hypersurface $X = V(I) = V(f) := V(x^2 + y^2 z + z^3) \subset \mathbf{A}^3$ from before, but we explore what happens if we vary the toroidal logarithmic structure on $\mathbf{A}^3$.

Example 4.4.6. Consider $Y = \mathbf{A}^{3;2} = \operatorname{Spec}(\mathbf{N}^2 \to \mathbf{k}[x, \underline{y}, \underline{z}])$. Then we have $\max\operatorname{inv}(J) = (2, \infty)$, and $\underline{\mathscr{I}}(J)_\bullet = (x, (\underline{y}^2\underline{z}, \underline{z}^3)^{1/2})$. The multi-weighted blow-up $\pi\colon Y' := \mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$ is schematically the same as the one in Example 4.4.1, and we still have (4.26) (but x' is no longer underlined) and resolution of singularities in one step.

Example 4.4.7. Next, consider $Y = \mathbf{A}^{3;0}$ (trivial logarithmic structure). Then $\max\operatorname{inv}(J) = (2, 3, 3)$, and $\underline{\mathscr{I}}(J)_\bullet = (x^{1/3}, y^{1/2}, z^{1/2})$. The multi-weighted blow-up $\pi\colon Y' := \mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$ is the weighted blow-up of Example 4.1.17. By the equations therein, the total transform of I is:

$$\pi^{-1}(J)\mathscr{O}_{Y'} = \underline{u}^6 \cdot \underbrace{\left(x'^2 + y'^2 z' + z'^3\right)}_{\text{proper transform } J'}.$$

We have $\mathscr{D}^{\leq 1}(J') = (x', y'z', y'^2 + 3z'^2, z'^3)$ which is the unit ideal on the x'-chart, y'-chart, and z'-chart of Y'. Thus, we have $\max\operatorname{inv}(J') = (1) < (\infty) = \max\operatorname{inv}(I)$, i.e. resolution of singularities in one step.

Remark 4.4.8. Example 4.4.7 is also part of a more general phenomenon: namely, $X = V(J)$ has a $(3, 2, 2)$-weighted homogeneous isolated singularity at $\mathbf{0} \in \mathbf{A}^3$, and hence, its singularities are resolved after the $(3, 2, 2)$-weighted blow-up of $\mathbf{A}^3$ in 0. From the viewpoint of the *monodromy conjecture* of Denef–Loeser [**DL92b**], this resolution is "more minimal" than the one in Example 4.4.1, since it has one less exceptional divisor, namely the one corresponding to the B_1-*facet* [**LVP11**, Definition 3] of the Newton polyhedron of (f). This begs the question of whether in general and to what extent Theorem 4.4.2 can be refined in this direction. This is the content of the following Chapter 5.

Example 4.4.9. Finally, consider $Y = \mathbf{A}^{3;1} = \operatorname{Spec}(\mathbf{N} \to \mathbf{k}[x, y, \underline{z}])$. Then $\max \operatorname{inv}(J) = (2, \infty)$ and $\underline{\mathscr{I}}(J)_\bullet = (x, \underline{z}^{1/2})$. Then $\pi \colon Y' := \mathscr{B}l_{\underline{\mathscr{I}}(J)_\bullet} Y \to Y$ is

$$\left[\operatorname{Spec}_{\mathbf{A}^3}\left(\frac{\mathscr{O}_{\mathbf{A}^3}[x', \underline{z}', \underline{u}]}{(x'u - x, \underline{z}'\underline{u}^2 - \underline{z})}\right) \smallsetminus V(x', z') \;/\; \mathbb{G}_m\right] \to \mathbf{A}^{3;1}$$

so the total transform of J under π is

$$\pi^{-1}(J)\mathscr{O}_{Y'} = \underline{u}^2 \cdot \underbrace{\left(x'^2 + y^2\underline{z}' + \underline{z}'^3\underline{u}^4\right)}_{\text{proper transform } J'}.$$

Note that $V(J') \subset Y'$ is non-singular in every chart except the $\underline{z}'$-chart of Y'. Nevertheless, we have $\max \operatorname{inv}(J') = (2, 2, \infty) < (2, \infty) = \max \operatorname{inv}(J)$, and $\underline{\mathscr{I}}(J)_\bullet = (x', y, \underline{u}^2)$. The composition $\pi' \colon Y'' := \mathscr{B}l_{\underline{\mathscr{I}}(J')_\bullet} Y' \to Y' \xrightarrow{\pi} Y$ is

$$\left[\operatorname{Spec}_{\mathbf{A}^3}\left(\frac{\mathscr{O}_{\mathbf{A}^3}[x'', y', \underline{z}', \underline{u}', \underline{v}]}{(x''\underline{u}'v^3 - x, y'\underline{v}^2 - y, z'\underline{u}'^2\underline{v}^2 - z)}\right) \smallsetminus V\big(x''\underline{v}, \underline{z}'(x'', y', \underline{u}')\big) \;/\; \mathbb{G}_m^2\right] \to \mathbf{A}^{3;1}$$

and the total transform of J under π' is

$$\pi'^{-1}(J)\mathscr{O}_{Y''} = \underline{u}'^2\underline{v}^6 \cdot \underbrace{\left(x''^2 + y'^2\underline{z}' + \underline{z}'^3\underline{u}'^4\right)}_{\text{proper transform } J''}.$$

We have $\mathscr{D}^{\leq 1}(J'') = (x'', y'\underline{z}', \underline{z}'^3\underline{u}_2^4)$ which is the unit ideal on every chart of Y''. Thus, $\max \operatorname{inv}(J'') = (1) < (2, 2, \infty) = \max \operatorname{inv}(J')$, and we achieve resolution of singularities in two steps.

Remark 4.4.10. Note that the Newton polyhedron of the first center $\underline{\mathscr{I}}(J)_{\bullet} = (x, \underline{z}^{1/2}) \subset \mathbf{k}[x, y, \underline{z}]$ in Example 4.4.9 contains the B_1-facet of the Newton polyhedron of (f). As mentioned in the preceding remark, B_1-facets are known to be "problematic" from the viewpoint of the monodromy conjecture, cf. [**LVP11**]. Indeed, we saw above that the first multi-weighted blow-up in Example 4.4.9 did not completely resolve the singularities of $X = V(J) \subset Y$.

CHAPTER 5

Around the monodromy conjecture of Denef–Loeser

5.1. Nuts and bolts

5.1.A. Newton $\mathbf{Q}$-polyhedra and piecewise-linear convex $\mathbf{Q}$-functions. We begin by reviewing some fundamentals in convex geometry in §5.1.A and §5.1.B. A reader who is familiar with convex geometry can skip to §5.1.C. Throughout this chapter, we adopt the conventions in 1.3.2. Along the way we also fix some conventions and notations for the remainder of this chapter.

Definition 5.1.1 (Newton $\mathbf{Q}$-polyhedra)**.** By a *rational, positive half-space* in $M^+_{\mathbf{R}}$, we mean any set of the form

$$H^+_{\mathbf{u},m} := \left\{\mathbf{a} \in M^+_{\mathbf{R}} \colon \mathbf{a}\cdot\mathbf{u} \geq m\right\} \subset M^+_{\mathbf{R}}$$

for some $\mathbf{0} \neq \mathbf{u} \in N^+$ and $m \in \mathbf{N}_{>0}$. We also set:

$$H_{\mathbf{u},m} := \left\{\mathbf{a} \in M^+_{\mathbf{R}} \colon \mathbf{a}\cdot\mathbf{u} = m\right\} \subset M^+_{\mathbf{R}}.$$

We call an intersection of finitely many rational, positive half-spaces in $M^+_{\mathbf{R}}$ a *Newton* $\mathbf{Q}$-*polyhedron* (with the empty intersection defined as $M^+_{\mathbf{R}}$), typically denoted by the letter Γ_+. Equivalently, a Newton $\mathbf{Q}$-polyhedron is the convex hull in $M_{\mathbf{R}}$ of $\bigcup\{\mathbf{a} + M^+_{\mathbf{R}} \colon \mathbf{a} \in S\}$ for a finite subset of points $S \subset M^+_{\mathbf{Q}}$.

Remark 5.1.2. If the vertices of a Newton $\mathbf{Q}$-polyhedron Γ_+ also lie in M^+, then Γ_+ is simply known as a Newton polyhedron.

Convention 5.1.3 (Conventions on Newton $\mathbf{Q}$-polyhedra)**.** In Convention 1.3.2, we outlined a few conventions on the Newton polyhedron $\Gamma_+(f)$ of a polynomial $f \in \mathbf{k}[x_1, \ldots, x_n]$. The same conventions make sense for a Newton $\mathbf{Q}$-polyhedron, and moving ahead we will also adopt them for Newton $\mathbf{Q}$-polyhedra.

5.1.4 (Piecewise-linear convex $\mathbf{Q}$-functions)**.** We may associate, to a Newton $\mathbf{Q}$-polyhedron Γ_+, a piecewise-linear, convex function $\varphi\colon N^+_{\mathbf{R}} \to \mathbf{R}_{\geq 0}$ defined as follows:

$$\varphi(\mathbf{u}) := \inf_{\mathbf{a}\in\Gamma_+} \mathbf{a}\cdot\mathbf{u} \qquad \text{for every } \mathbf{u} \in N^+_{\mathbf{R}}.$$

Recall this means that there exists a finite set $S \subset M^+_{\mathbf{R}}$ such that $\varphi(\mathbf{u}) = \min_{\mathbf{a}\in S} \mathbf{a}\cdot\mathbf{u}$ for every $\mathbf{u} \in N^+_{\mathbf{R}}$. Indeed, for the above φ, we may take $S = \operatorname{vert}(\Gamma_+)$. In fact, since $\operatorname{vert}(\Gamma_+) \subset M^+_{\mathbf{Q}}$, φ is a *piecewise-linear, convex* $\mathbf{Q}$*-function*, that is, either of the following equivalent conditions hold for φ:

(i) φ is a piecewise-linear, convex function such that $\varphi(N^+) \subset \mathbf{Q}_{\geq 0}$.

(ii) There exists a finite set $S \subset M^+_{\mathbf{Q}}$ such that $\varphi(\mathbf{u}) = \min_{\mathbf{a}\in S} \mathbf{a}\cdot\mathbf{u}$.

This sets up a one-to-one correspondence between:

$$\big\{\text{Newton } \mathbf{Q}\text{-polyhedra in } M^+_{\mathbf{R}}\big\} \longleftrightarrow \left\{\begin{matrix}\text{piecewise linear, convex, } \mathbf{Q}\text{-} \\ \text{functions } \varphi\colon N^+_{\mathbf{R}} \to \mathbf{R}_{\geq 0}\end{matrix}\right\}.$$

Indeed, we claim that every φ in the right hand side arises uniquely from the following Newton $\mathbf{Q}$-polyhedron:

$$\Gamma_+ = \big\{\mathbf{a} \in M^+_{\mathbf{R}} : \mathbf{a}\cdot\mathbf{u} \geq \varphi(\mathbf{u}) \text{ for all } \mathbf{u} \in N^+_{\mathbf{R}}\big\} = \bigcap_{\mathbf{u}\in N^+_{\mathbf{R}}} H^+_{\mathbf{u},\varphi(\mathbf{u})}. \tag{5.1}$$

Proof of claim. It remains to demonstrate Γ_+ is a Newton $\mathbf{Q}$-polyhedron. Let $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_r\} \subset M^+_{\mathbf{Q}}$ be such that $\varphi(\mathbf{u}) = \min_{i\in[r]} \mathbf{a}_i\cdot\mathbf{u}$ for every $\mathbf{u} \in N^+_{\mathbf{R}}$. Then Γ_+ is the intersection of all rational, positive half-spaces in $M^+_{\mathbf{R}}$ containing S:

(i) If $H^+_{\mathbf{u},m}$ contains S, then $\mathbf{a}_i \cdot \mathbf{u} \geq m$ for every $i \in [r]$, so that if $\mathbf{a} \in \Gamma_+$, $\mathbf{a} \cdot \mathbf{u} \geq \min_{i\in[r]} \mathbf{a}_i \cdot \mathbf{u} \geq m$. Thus, $\Gamma_+ \subset H^+_{\mathbf{u},m}$.

(ii) Conversely, let $\mathbf{a} \in \bigcap\{H^+_{\mathbf{u},m} \colon \mathbf{u} \in N^+,\ m \in \mathbf{N}_{>0},\ S \subset H^+_{\mathbf{u},m}\}$. To show $\mathbf{a} \in \Gamma_+$, it suffices to show for every $\mathbf{u} \in N^+$ that $\mathbf{a}\cdot\mathbf{u} \geq \min_{i\in[r]} \mathbf{a}_i \cdot \mathbf{u}$. To this end, fix $k \in \mathbf{N}_{>0}$ so that $k\mathbf{a}_i \in M^+$ for every $i \in [r]$, and set $m := k \min_{i\in[r]} \mathbf{a}_i \cdot \mathbf{u} \in \mathbf{N}_{>0}$. Then $S \subset H^+_{k\mathbf{u},m}$, so the hypothesis implies $\mathbf{a} \in H^+_{k\mathbf{u},m}$. This means that $k(\mathbf{a} \cdot \mathbf{u}) \geq m = k \min_{i\in[r]} \mathbf{a}_i \cdot \mathbf{u}$, i.e. $\mathbf{a} \cdot \mathbf{u} \geq \min_{i\in[r]} \mathbf{a}_i \cdot \mathbf{u}$ as desired.

Thus, Γ_+ is the convex hull in $M_{\mathbf{R}}$ of $\bigcup\{\mathbf{a} + M^+_{\mathbf{R}} \colon \mathbf{a} \in S\}$. Since $S \subset M^+_{\mathbf{Q}}$ is finite, so is $\operatorname{vert}(\Gamma_+) \subset S$. Therefore, Γ_+ only has finitely many faces. For each facet τ of Γ_+, let $\mathbf{u}_\tau$ be the unique primitive vector in N^+ that is normal to the affine hyperplane spanned by τ. Then Γ_+ is the following finite intersection of rational, positive half-spaces in $M^+_{\mathbf{R}}$:

$$\Gamma_+ = \bigcap_{\tau \prec^1 \Gamma_+} H^+_{\mathbf{u}_\tau,\varphi(\mathbf{u}_\tau)} \tag{5.2}$$

and hence a Newton $\mathbf{Q}$-polyhedron. △

5.1.5. As a consequence of (5.2), we obtain the following alternative description of φ in terms of facets of Γ_+ (as opposed to points in Γ_+):

$$\varphi = \min \mathscr{S}$$

where

$$\mathscr{S} := \begin{Bmatrix} \text{linear functions } \ell\colon N^+_{\mathbf{R}} \to \mathbf{R}_{\geq 0} \text{ such} \\ \text{that } \ell(\mathbf{u}_\tau) \geq N_\tau \text{ for every facet } \tau \prec^1 \Gamma_+ \end{Bmatrix}.$$

Recall from Convention 5.1.3 that for every $\tau \prec^1 \Gamma_+$, $N_\tau \in \mathbf{Q}_{>0}$ is defined via the equation $\{\mathbf{a} \in M_{\mathbf{R}} \colon \mathbf{a} \cdot \mathbf{u}_\tau = N_\tau\}$ of the affine span H_τ of τ in $M_{\mathbf{R}}$.

5.1.B. Newton Q-polyhedra and their normal fans. We continue to follow Convention 1.3.2, and in addition, we follow any conventions on fans in Convention 1.0.1.

5.1.6 (Normal fans)**.** Every Newton $\mathbf{Q}$-polyhedron Γ_+ also naturally induces a fan Σ in $N^+_{\mathbf{R}}$, called the *normal fan* of Γ_+, whose cones σ correspond in an inclusion-reversing manner with faces $\varsigma \prec \Gamma_+$. Namely, let φ be the piecewise linear, convex, rational function associated to Γ_+, and we define the normal fan Σ as follows:

$$\Sigma := \left\{\sigma_{\mathbf{a}} \colon \mathbf{a} \in M^+_{\mathbf{R}}\right\}$$

where for each $\mathbf{a} \in \Gamma_+$,

$$\sigma_{\mathbf{a}} := \left\{\mathbf{u} \in N^+_{\mathbf{R}} \colon \varphi(\mathbf{u}) = \mathbf{a}\cdot\mathbf{u}\right\}.$$

This is a closed convex cone in $N^+_{\mathbf{R}}$: indeed, if $\mathbf{u}_1$, $\mathbf{u}_2 \in \sigma_{\mathbf{a}}$, then

$$\mathbf{a}\cdot(\mathbf{u}_1+\mathbf{u}_2) = \mathbf{a}\cdot\mathbf{u}_1 + \mathbf{a}\cdot\mathbf{u}_2 = \varphi(\mathbf{u}_1)+\varphi(\mathbf{u}_2) \le \varphi(\mathbf{u}_1+\mathbf{u}_2) \le \mathbf{a}\cdot(\mathbf{u}_1+\mathbf{u}_2)$$

which forces equality throughout, i.e. $\mathbf{u}_1+\mathbf{u}_2 \in \sigma_{\mathbf{a}}$. In particular, we obtain an alternative characterization of $\sigma_{\mathbf{a}}$:

Corollary 5.1.7. *For $\mathbf{a} \in \Gamma_+$, $\sigma_{\mathbf{a}}$ is the largest closed convex cone in $N^+_{\mathbf{R}}$ on which φ is the linear function $\mathbf{u} \mapsto \mathbf{a}\cdot\mathbf{u}$.* □

5.1.8. Our next goal is to explicate $\sigma_{\mathbf{a}}$ further; in particular, we will see that $\sigma_{\mathbf{a}}$ is a convex rational polyhedral cone in $N^+_{\mathbf{R}}$. To do this, let us first introduce a notion dual to $\sigma_{\mathbf{a}}$. Namely, for each $\mathbf{u} \in N^+_{\mathbf{R}}$, the *first meet locus* of $\mathbf{u}$ is defined as:

$$\begin{aligned}\varsigma_{\mathbf{u}} \;&:=\; \left\{\mathbf{a} \in M^+_{\mathbf{R}} \colon \varphi(\mathbf{u}') \le \mathbf{a}\cdot\mathbf{u}' \text{ for all } \mathbf{u}' \in N^+_{\mathbf{R}} \text{ with equality if } \mathbf{u}' = \mathbf{u}\right\}\\ &=\; \{\mathbf{a} \in \Gamma_+ \colon \varphi(\mathbf{u}) = \mathbf{a}\cdot\mathbf{u}\} \;\prec\; \Gamma_+.\end{aligned}$$

Note that $\varsigma_{\mathbf{0}} = \Gamma_+$. Here are some other observations about $\varsigma_{\mathbf{u}}$:

(i) For $\mathbf{0} \neq \mathbf{u} \in N^+_{\mathbf{R}}$, $\varsigma_{\mathbf{u}} = H_{\mathbf{u},\varphi(\mathbf{u})} \cap \Gamma_+$, i.e. $H_{\mathbf{u},\varphi(\mathbf{u})}$ is a supporting hyperplane of Γ_+. Every proper face of Γ_+ is $\varsigma_{\mathbf{u}}$ for a $\mathbf{0} \neq \mathbf{u} \in N^+_{\mathbf{R}}$.

(ii) Every facet τ of Γ_+ is $\varsigma_{\mathbf{u}_\tau}$ for a unique primitive vector $\mathbf{u}_\tau \in N^+$, namely the one normal to the affine hyperplane spanned by τ.

(iii) For each $i \in [n]$, the following statements are equivalent:

  (a) $\mathbf{u}_i = 0$.

  (b) $\varsigma_{\mathbf{u}}$ is non-compact in the i^{th} coordinate, i.e. $\varsigma_{\mathbf{u}} + \mathbf{R}_{\geq 0}\mathbf{e}_i^\vee = \varsigma_{\mathbf{u}}$.

  (c) There exists $\mathbf{a} \in \varsigma_{\mathbf{u}}$ such that $\mathbf{a} + \mathbf{e}_i^\vee \in \varsigma_{\mathbf{u}}$.

In particular, (iii) says that $\varsigma_{\mathbf{u}}$ is compact if and only if all coordinates of $\mathbf{u}$ are non-zero. We will also need the following lemma:

Lemma 5.1.9.

(i) *Let* $\mathbf{u}_1, \mathbf{u}_2 \in N^+_{\mathbf{R}}$. *Then* $\varsigma_{\mathbf{u}_1} \cap \varsigma_{\mathbf{u}_2} \subset \varsigma_{\mathbf{u}_1+\mathbf{u}_2}$, *with equality if and only if* $\varsigma_{\mathbf{u}_1} \cap \varsigma_{\mathbf{u}_2} \neq \varnothing$.

(ii) *Let* $\mathbf{a}, \mathbf{a}' \in \Gamma_+$. *For every* $0 < t < 1$, $\sigma_{\mathbf{a}} \cap \sigma_{\mathbf{a}'} = \sigma_{t\mathbf{a}+(1-t)\mathbf{a}'}$.

Proof. We first prove (i). Let $\mathbf{a} \in \varsigma_{\mathbf{u}_1} \cap \varsigma_{\mathbf{u}_2}$. Then

$$\mathbf{a} \cdot \mathbf{u}_1 + \mathbf{a} \cdot \mathbf{u}_2 = \varphi(\mathbf{u}_1) + \varphi(\mathbf{u}_2) \leq \varphi(\mathbf{u}_1 + \mathbf{u}_2) \leq \mathbf{a} \cdot (\mathbf{u}_1 + \mathbf{u}_2)$$

which forces equality throughout, i.e. $\mathbf{a} \in \varsigma_{\mathbf{u}_1+\mathbf{u}_2}$, as desired. Conversely, if $\varsigma_{\mathbf{u}_1} \cap \varsigma_{\mathbf{u}_2} \neq \varnothing$, then $\mathbf{u}_1, \mathbf{u}_2 \in \sigma_{\mathbf{a}}$ for some $\mathbf{a} \in \Gamma_+$. For $\mathbf{a}' \in \varsigma_{\mathbf{u}_1+\mathbf{u}_2}$,

$$\begin{aligned}
\mathbf{a}' \cdot (\mathbf{u}_1 + \mathbf{u}_2) &= \varphi(\mathbf{u}_1 + \mathbf{u}_2) \\
&= \varphi(\mathbf{u}_1) + \varphi(\mathbf{u}_2) \qquad \text{by Corollary 5.1.7} \\
&\leq \mathbf{a}' \cdot \mathbf{u}_1 + \mathbf{a}' \cdot \mathbf{u}_2.
\end{aligned}$$

This forces equality throughout, so $\varphi(\mathbf{u}_i) = \mathbf{a}' \cdot \mathbf{u}_i$ for each $i = 1, 2$, i.e. $\mathbf{a}' \in \varsigma_{\mathbf{u}_1} \cap \varsigma_{\mathbf{u}_2}$. This settles (i). Next, let us prove (ii). Firstly, if $\mathbf{u} \in \sigma_{\mathbf{a}} \cap \sigma_{\mathbf{a}'}$, then $\mathbf{a}, \mathbf{a}' \in \varsigma_{\mathbf{u}}$. By convexity of $\varsigma_{\mathbf{u}}$, we have

$$t\mathbf{a} + (1-t)\mathbf{a}' \in \varsigma_{\mathbf{u}}$$

for every $0 \leq t \leq 1$, i.e. $\mathbf{u} \in \sigma_{t\mathbf{a}+(1-t)\mathbf{a}'}$ for every $0 \leq t \leq 1$. Secondly, if $\mathbf{u} \in \sigma_{t\mathbf{a}+(1-t)\mathbf{a}'}$ for $0 < t < 1$, then

$$\begin{aligned}\varphi(\mathbf{u}) = (t\mathbf{a} + (1-t)\mathbf{a}') \cdot \mathbf{u} &= t\mathbf{a} \cdot \mathbf{u} + (1-t)\mathbf{a}' \cdot \mathbf{u} \\ &\geq t\varphi(\mathbf{u}) + (1-t)\varphi(\mathbf{u}) = \varphi(\mathbf{u})\end{aligned}$$

which forces equality throughout. Since $t > 0$ and $1 - t > 0$, this means $\varphi(\mathbf{u}) = \mathbf{a} \cdot \mathbf{u}$ and $\varphi(\mathbf{u}) = \mathbf{a}' \cdot \mathbf{u}$, i.e. $\mathbf{u} \in \sigma_{\mathbf{a}} \cap \sigma_{\mathbf{a}'}$. $\square$

The next lemma is the key step towards explicating $\sigma_{\mathbf{a}}$:

Lemma 5.1.10. *For $\mathbf{a} \in \Gamma_+$ and $\mathbf{u} \in N_{\mathbf{R}}^+$, $\mathbf{u}$ generates an extremal ray of $\sigma_{\mathbf{a}}$ if and only if $\mathbf{a} \in \varsigma_{\mathbf{u}} \prec^1 \Gamma_+$.*

Proof. We may assume $\mathbf{u} \neq \mathbf{0}$. For the reverse implication, let $\mathbf{u}_1$, $\mathbf{u}_2 \in \sigma_{\mathbf{a}}$ such that $\mathbf{u}_1 + \mathbf{u}_2 \in \langle \mathbf{u} \rangle$. We want to show $\mathbf{u}_1$, $\mathbf{u}_2 \in \langle \mathbf{u} \rangle$. By Lemma 5.1.9(i), we have $\varsigma_{\mathbf{u}_1} \cap \varsigma_{\mathbf{u}_2} = \varsigma_{\mathbf{u}_1+\mathbf{u}_2} = \varsigma_{\mathbf{u}}$. By hypothesis, $\varsigma_{\mathbf{u}}$ is maximal among all faces of Γ_+ containing $\mathbf{a}$, so the above forces $\varsigma_{\mathbf{u}_i} = \varsigma_{\mathbf{u}}$ for $i = 1, 2$. Since the affine span of $\varsigma_{\mathbf{u}}$ is an affine hyperplane in $N_{\mathbf{R}}$, this means that $\mathbf{u}_i \in \langle \mathbf{u} \rangle$ for $i = 1, 2$, as desired.

Next, let us show the forward implication. Firstly, since $\mathbf{u} \in \sigma_{\mathbf{a}}$, $\mathbf{a} \in \varsigma_{\mathbf{u}}$. It remains to show that $\varsigma_{\mathbf{u}}$ is maximal among all proper faces $\varsigma \prec \Gamma_+$ containing $\mathbf{a}$. To this end, let ς be a proper face of Γ_+ that contains $\varsigma_{\mathbf{u}}$, and choose $\mathbf{0} \neq \mathbf{u}' \in \sigma_{\mathbf{a}}$ such that $\varsigma = \varsigma_{\mathbf{u}'}$. For every $i \in [n]$ such that $\mathbf{u}_i = 0$, we have

$$\varsigma_{\mathbf{u}} + \mathbf{R}_{\geq 0}\mathbf{e}_i^\vee = \varsigma_{\mathbf{u}} \subset \varsigma_{\mathbf{u}'}$$

which implies $\mathbf{u}'_i = 0$ (cf. 5.1.8(iii)). Therefore, for $N \gg 0$, $N\mathbf{u} - \mathbf{u}' \in N^+_{\mathbf{R}}$. In fact, we *claim* that for $N \gg 0$, we also have $N\mathbf{u}-\mathbf{u}' \in \sigma_{\mathbf{a}}$. If not, for every $N \gg 0$, we have $N\mathbf{u}-\mathbf{u}' \in N^+_{\mathbf{R}} \smallsetminus \sigma_{\mathbf{a}}$, i.e. there exists $\mathbf{a}'_N \in \mathrm{vert}(\Gamma_+)$ so that

$$\varphi\left(\mathbf{u} - \frac{1}{N}\mathbf{u}'\right) = \mathbf{a}'_N \cdot \left(\mathbf{u} - \frac{1}{N}\mathbf{u}'\right) < \mathbf{a} \cdot \left(\mathbf{u} - \frac{1}{N}\mathbf{u}'\right).$$

Since $\mathrm{vert}(\Gamma_+)$ is finite, there exists a constant subsequence $(\mathbf{a}'_{N_k})_{k\geq 1} = (\mathbf{a}', \mathbf{a}', \mathbf{a}', \cdots)$ of $(\mathbf{a}'_N)_{N\gg 0}$. For all $k \geq 1$, we have

$$\mathbf{a}' \cdot \left(\mathbf{u} - \frac{1}{N_k}\mathbf{u}'\right) < \mathbf{a} \cdot \left(\mathbf{u} - \frac{1}{N_k}\mathbf{u}'\right). \tag{5.3}$$

Letting $k \to \infty$, we obtain $\mathbf{a}' \cdot \mathbf{u} \leq \mathbf{a} \cdot \mathbf{u}$. But $\mathbf{a} \in \varsigma_{\mathbf{u}}$, so $\mathbf{a} \cdot \mathbf{u} = \varphi(\mathbf{u}) \leq \mathbf{a}' \cdot \mathbf{u}$. This forces $\mathbf{a}' \cdot \mathbf{u} = \mathbf{a} \cdot \mathbf{u}$, i.e. $\mathbf{a}' \in \varsigma_{\mathbf{u}}$. In addition, $\varsigma_{\mathbf{u}} \subset \varsigma_{\mathbf{u}'}$, so $\mathbf{a}, \mathbf{a}' \in \varsigma_{\mathbf{u}'}$, i.e. $\mathbf{a}' \cdot \mathbf{u}' = \varphi(\mathbf{u}') = \mathbf{a} \cdot \mathbf{u}'$. However, these conclusions that $\mathbf{a}' \cdot \mathbf{u} = \mathbf{a} \cdot \mathbf{u}$ and $\mathbf{a}' \cdot \mathbf{u}' = \mathbf{a} \cdot \mathbf{u}'$ would contradict (5.3). Thus, our earlier *claim* holds, i.e. by replacing $\mathbf{u}$ by a sufficiently large multiple of itself, we may assume $\mathbf{u} - \mathbf{u}' \in \sigma_{\mathbf{a}}$. Since $\mathbf{u}$ generates an extremal ray of $\sigma_{\mathbf{a}}$, one has that $\mathbf{u}'$ and $\mathbf{u} - \mathbf{u}'$ both lie in $\mathbf{R}_{\geq 0}\mathbf{u}$. In particular, $\varsigma = \varsigma_{\mathbf{u}'} = \varsigma_{\mathbf{u}}$, as desired. □

Corollary 5.1.11. *For $\mathbf{a} \in \Gamma_+$, $\sigma_{\mathbf{a}}$ is a convex rational polyhedral cone in $N^+_{\mathbf{R}}$. More precisely,*

$$\sigma_{\mathbf{a}} = \langle \mathbf{u}_\tau \colon \mathbf{a} \in \tau \prec^1 \Gamma_+ \rangle.$$

In particular, $\sigma_{\mathbf{a}} \neq \{\mathbf{0}\}$ if and only if $\mathbf{a}$ lies in the boundary of Γ_+.

Proof. Since $\sigma_{\mathbf{a}}$ is a closed convex cone inside $N^+_{\mathbf{R}}$, $\sigma_{\mathbf{a}}$ is generated by its extremal rays [**Roc70**, Theorem 18.5]. Moreover, since there are finitely many facets of Γ_+ containing $\mathbf{a}$, the preceding lemma says $\sigma_{\mathbf{a}}$ has finitely many extremal rays. □

For the next corollary, we recall that the relative interior $\mathrm{relint}(\varsigma)$ of a polyhedron ς in $M_{\mathbf{R}}$ is the interior of ς in its affine span in $M_{\mathbf{R}}$.

Corollary 5.1.12. *For $\mathbf{a} \in M^+_{\mathbf{R}}$ and $\mathbf{u} \in N^+_{\mathbf{R}}$, the following statements are equivalent:*

(i) $\mathbf{u} \in \operatorname{relint}(\sigma_{\mathbf{a}})$.

(ii) $\varsigma_{\mathbf{u}} = \bigcap\{\tau \prec^1 \Gamma_+ : \mathbf{a} \in \tau\}$, *where* $\bigcap \varnothing := \Gamma_+$

(iii) $\mathbf{a} \in \operatorname{relint}(\varsigma_{\mathbf{u}})$.

Moreover, for $\mathbf{u} \in \sigma_{\mathbf{a}}$, we have $\bigcap\{\tau \prec^1 \Gamma_+ : \mathbf{a} \in \tau\} \prec \varsigma_{\mathbf{u}}$.

Proof. We may assume $\mathbf{u} \neq \mathbf{0}$. Note that (ii) $\iff$ (iii), since $\bigcap\{\tau \prec^1 \Gamma_+ : \mathbf{a} \in \tau\}$ is the unique face ς of Γ_+ such that $\mathbf{a} \in \operatorname{relint}(\varsigma)$. For (i) $\iff$ (ii), it suffices to focus on the case $\mathbf{u} \in \sigma_{\mathbf{a}}$ (since otherwise, $\mathbf{a} \notin \varsigma_{\mathbf{u}}$), and by the preceding corollary $\mathbf{u} = \sum_{\mathbf{a} \in \tau \prec^1 \Gamma_+} \lambda_\tau \mathbf{u}_\tau$ for some $\lambda_\tau \in \mathbf{R}_{\geq 0}$. By repeatedly applying Lemma 5.1.9(i), we have

$$\varsigma_{\mathbf{u}} = \bigcap\left\{\varsigma_{(\lambda_\tau \mathbf{u}_\tau)} : \mathbf{a} \in \tau \prec^1 \Gamma_+\right\} = \bigcap\left\{\tau \prec^1 \Gamma_+ : \mathbf{a} \in \tau \text{ and } \lambda_\tau > 0\right\}$$

which contains $\bigcap\{\tau \prec^1 \Gamma_+ : \mathbf{a} \in \tau\}$ as a face. It remains to note that we can arrange $\{\lambda_\tau : \mathbf{a} \in \tau \prec^1 \Gamma_+\} \subset \mathbf{R}_{>0}$ if and only if $\mathbf{u} \in \operatorname{relint}(\sigma_{\mathbf{a}})$. □

5.1.13. The preceding corollary sets up a natural correspondence between:

$$\left\{\text{faces } \varsigma \prec \Gamma_+\right\} \longleftrightarrow \left\{\text{cones } \sigma \text{ in } \Sigma\right\}$$

$$\varsigma \longmapsto \sigma_\varsigma$$

$$\varsigma_\sigma \longleftarrow\!\shortmid\, \sigma$$

which is defined as follows. Given $\varsigma \prec \Gamma_+$, $\sigma_\varsigma := \sigma_{\mathbf{a}}$ for any $\mathbf{a} \in \operatorname{relint}(\varsigma)$. We call σ_ς the cone in Σ dual to ς. Conversely, given σ in Σ, $\varsigma_\sigma := \varsigma_{\mathbf{u}}$ for any $\mathbf{u} \in \operatorname{relint}(\sigma)$. We call ς_σ the face of Γ_+ dual to σ. Then:

(i) If faces $\varsigma, \varsigma' \prec \Gamma_+$ correspond to cones σ, σ' in Σ, then $\varsigma \prec \varsigma'$ if and only if $\sigma \succ \sigma'$. Indeed, the reverse implication is given by the preceding corollary. For the forward implication, Corollary 5.1.11 says that every extremal ray of σ' is an extremal ray of σ. We also note that if $\operatorname{relint}(\sigma) \cap \operatorname{relint}(\sigma') \neq \varnothing$, then $\varsigma = \varsigma_{\mathbf{u}} = \varsigma'$ for any $\mathbf{u} \in \operatorname{relint}(\sigma) \cap \operatorname{relint}(\sigma')$, i.e. $\sigma = \sigma'$. The preceding two sentences together imply that $\sigma' \prec \sigma$.

(ii) If a face $\varsigma \prec \Gamma_+$ corresponds to a cone σ in Σ, then $\dim(\varsigma) + \dim(\sigma) = n$. This follows by induction on $\dim(\sigma)$, where the induction step is supplied by (i).

(iii) If a facet $\tau \prec^1 \Gamma_+$ corresponds to a ray ρ in Σ, note that $\mathbf{u}_\tau = \mathbf{u}_\rho$.

Corollary 5.1.14. *Σ is a fan in $N_{\mathbf{R}}$ whose support $|\Sigma|$ equals $N_{\mathbf{R}}^+$.*

PROOF. That Σ is a fan in $N_{\mathbf{R}}$ follows from Lemma 5.1.9(ii), Corollary 5.1.11, and 5.1.13. It remains to see that $|\Sigma| \supset N_{\mathbf{R}}^+$. Indeed, if $\mathbf{u} \in N_{\mathbf{R}}^+$, fix any $\mathbf{a} \in \varsigma_{\mathbf{u}} \prec \Gamma_+$, and we have $\mathbf{u} \in \sigma_{\mathbf{a}}$, as desired. □

Convention 5.1.15. For $\rho \in \Sigma[1]$, we will denote the facet $\varsigma_\rho = \varsigma_{\mathbf{u}} \prec^1 \Gamma_+$ dual to ρ by τ_ρ or $\tau_{\mathbf{u}}$ instead, cf. Convention 1.3.2. Likewise, for $\tau \prec^1 \Gamma_+$, we denote the ray $\sigma_\tau \in \Sigma[1]$ dual to τ by ρ_τ instead, cf. Convention 1.0.1. Then the following corollary is immediate from Corollary 5.1.11 and Corollary 5.1.12:

Corollary 5.1.16. *For a face $\varsigma \prec \Gamma_+$, we have:*

$$\sigma_\varsigma = \big\langle \rho_\tau \colon \varsigma \prec \tau \prec^1 \Gamma_+ \big\rangle.$$

Dually, for a cone σ in Σ, we have:

$$\varsigma_\sigma = \bigcap \big\{ \tau_\rho \colon \rho \in \sigma[1] \big\}, \qquad \textit{where } \bigcap \varnothing := \Gamma_+.$$

The next corollary follows from the preceding corollary, and 5.1.8(iii).

Corollary 5.1.17. *Let ς be a face of Γ_+, and σ be the cone in Σ dual to ς. For $i \in [n]$, let $\{\mathbf{e}_i^\vee = 0\}$ denote the coordinate hyperplane in $N_{\mathbf{R}}$ defined by $\mathbf{e}_i^\vee = 0$. Then the following statements are equivalent:*

(i) $\sigma \subset \{\mathbf{e}_i^\vee = 0\}$, *i.e. for every $\rho \in \sigma[1]$, $u_{\rho,i} = 0$.*

(ii) ς *is non-compact in the i^{th} coordinate, i.e. $\varsigma + \mathbf{R}_{\geq 0}\mathbf{e}_i^\vee = \varsigma$.*

(iii) *There exists $\mathbf{a} \in \varsigma$ such that $\mathbf{a} + \mathbf{e}_i^\vee \in \varsigma$.*

In particular, ς is compact if and only if σ is not contained in any coordinate hyperplane $\{\mathbf{e}_i^\vee = 0\}$ in $N_{\mathbf{R}}$.

Remark 5.1.18. In this paragraph, we give an alternative argument for the inequality $\varphi \leq \min \mathscr{S}$ in 5.1.5.

Let $\ell \in \mathscr{S}$, fix any $\mathbf{u} \in N_{\mathbf{R}}^+$, and it suffices to show $\ell(\mathbf{u}) \geq \varphi(\mathbf{u})$. Indeed, let Σ be the fan in $N_{\mathbf{R}}$ arising from Γ_+ as above. Since the support of Σ is $N_{\mathbf{R}}^+$, $\mathbf{u}$ lies in $\mathrm{relint}(\sigma)$ for some cone $\sigma \in \Sigma$. By Corollary 5.1.11, $\mathbf{u} = \sum_{\tau \prec^1 \Gamma_+} \lambda_\tau \mathbf{u}_\tau$ for some $\lambda_\tau \in \mathbf{R}_{\geq 0}$, where $\lambda_\tau > 0$ if and only if $\mathbf{u}_\tau$ is an extremal ray of σ. Thus

$$\ell(\mathbf{u}) = \sum_{\tau \prec^1 \Gamma_+} \lambda_\tau \ell(\mathbf{u}_\tau) \geq \sum_{\tau \prec^1 \Gamma_+} \lambda_\tau N_\tau = \sum_{\tau \prec^1 \Gamma_+} \lambda_\tau \varphi(\mathbf{u}_\tau) = \varphi(\mathbf{u})$$

where the last equality follows from Corollary 5.1.7, as desired.

5.1.C. A quick lemma on multi-weighted blow-ups. Consider a fan Σ in $N_{\mathbf{R}}$ whose support $|\Sigma|$ is $N_{\mathbf{R}}^+$, e.g. the normal fan of a Newton $\mathbf{Q}$-polytope as in §5.1.B. We recall from Definition 4.1.6 that Σ induces multi-weighted blow-ups of $\mathbf{A}^n$:

$$\vartheta\colon \mathscr{X}_{\Sigma,\mathbf{b}} = \left[X_{\widehat{\Sigma}} \,/\, \mathbb{G}_m^{\Sigma[\mathrm{ex}]}\right] \to \mathbf{A}^n \qquad \text{for } \mathbf{b} = (\mathrm{b}_\rho)_{\rho \in \Sigma[1]} \in \mathbf{N}_{>0}^{\Sigma[1]}.$$

Starting from §5.2, our discussion will utilize multi-weighted blow-ups, and we will freely assume any conventions introduced in §4.1, e.g. Convention 4.1.8 and the notations in 4.1.9. Our discussion will also involve the next lemma, and more importantly, its corollary:

Lemma 5.1.19. *Let σ be a cone in the augmentation $\overline{\Sigma}$ of Σ. If σ satisfies the condition:*

$$D_+(\sigma) \cap \vartheta_\Sigma^{-1}(\mathbf{0}) \neq \varnothing$$

then σ is not contained in any coordinate hyperplane $\{\mathbf{e}_i^\vee = 0\}$ in $N_{\mathbf{R}}$.

Proof. Indeed, for $i \in [n]$, we have:

$$\begin{aligned} \sigma \subset \{\mathbf{e}_i^\vee = 0\} &\iff u_{\rho,i} = 0 \text{ for every } \rho \in \sigma[1] \\ &\iff \vartheta_\Sigma^\#(x_i) = \prod_{\rho \in \Sigma[1]} (x'_\rho)^{\mathrm{b}_\rho \cdot u_{\rho,i}} \text{ is invertible on } D_+(\sigma) \\ &\iff D_+(\sigma) \cap \vartheta_\Sigma^{-1}\big(V(x_i)\big) = \varnothing \\ &\implies D_+(\sigma) \cap \vartheta_\Sigma^{-1}(\mathbf{0}) = \varnothing. \end{aligned}$$

□

Corollary 5.1.20. *We have:*

$$\vartheta_\Sigma^{-1}(\mathbf{0}) \subset \bigsqcup \left\{ O(\sigma) : \begin{array}{l} \sigma \in \overline{\Sigma} \text{ not contained in} \\ \text{any coordinate hyperplane} \\ \text{in } N_{\mathbf{R}} \end{array} \right\}.$$

Additionally, we relate the above corollary to §5.1.B as follows:

5.1.21. If Σ is the normal fan of a Newton $\mathbf{Q}$-polytope Γ_+, note that a cone $\sigma \in \overline{\Sigma}$ is not contained in any coordinate hyperplane in $N_{\mathbf{R}}$ if and only if $\bigcap\{\tau_\rho \colon \rho \in \sigma[1]\} \prec \Gamma_+$ is compact. Indeed, if $\sigma \in \Sigma$, we have $\bigcap\{\tau_\rho \colon \rho \in \sigma[1]\} = \varsigma_\sigma$ (Corollary 5.1.16), so the assertion follows from Corollary 5.1.17. Otherwise, let σ' be the *smallest* cone in Σ such that $\sigma \sqsubset \sigma'$. Then

$\bigcap\{\tau_\rho \colon \rho \in \sigma[1]\} = \bigcap\{\tau_\rho \colon \rho \in \sigma'[1]\} = \varsigma_{\sigma'}$ (cf. 5.1.13 and Corollary 5.1.16), so the assertion still follows from Corollary 5.1.17.

5.2. Preliminaries and examples

5.2.A. A stack-theoretic re-interpretation of a classical embedded desingularization of non-degenerate polynomials. We return to the setting at the start of §1.3: namely, let $f = \sum_{\mathbf{a}\in\mathbf{N}^n} c_{\mathbf{a}} \cdot x^{\mathbf{a}} \in \mathbf{k}[x_1, \ldots, x_n]$ be a non-degenerate polynomial, and let $\Gamma_+(f)$ denote the Newton polyhedron of f. Let $\Sigma(f)$ denote the normal fan of $\Gamma_+(f)$, cf. §5.1.B.

It is known in the literature that one can construct, using $\Sigma(f)$, an embedded desingularization of $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$ (in fact, more is true, cf. the next theorem). This construction manifests in various equivalent forms in the literature, e.g. in Varchenko [**Var76**, §10] and more recently, in Bultot–Nicaise [**BN20**, Proposition 8.31] and Theorem 4.4.2 (= [**AQ21**, Theorem 5.1.2]). As motivated in §1.3 (cf. 1.3.12, 1.3.13, 1.3.14), we follow the last approach. Indeed, by following the description in 4.1.9, the proof of Theorem 4.4.2 shows:

Theorem 5.2.1. *The multi-weighted blow-up of* $\mathbf{A}^n$*:*

$$\vartheta_{\Sigma(f)} \colon \mathscr{X}_{\Sigma(f)} \to \mathbf{A}^n$$

is a stack-theoretic embedded desingularization of $V(f) \cup V(x_1x_2\cdots x_n) \subset \mathbf{A}^n$ *above the origin* $\mathbf{0} \in \mathbf{A}^n$*.*

5.2.2. This means $\vartheta_{\Sigma(f)}^{-1}\big(V(f) \cup V(x_1x_2\cdots x_n)\big)$ is a simple normal crossings divisor at every point in $\vartheta_{\Sigma(f)}^{-1}(\mathbf{0})$. To explicate this, we note from 4.1.9(i) that:

$$\vartheta_{\Sigma(f)}^{\#}(f) = \sum_{\mathbf{a}\in\mathbf{N}^n} c_{\mathbf{a}} \cdot (x')^{\mathbf{a}} \cdot \prod_{\rho\in\Sigma(f)[\mathrm{ex}]} (x'_\rho)^{\mathbf{a}\cdot\mathbf{u}_\rho}$$

where for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{N}^n$, $(x')^{\mathbf{a}} := (x'_1)^{a_1} \cdots (x'_n)^{a_n}$. Setting $N_\rho := N_{\tau_\rho} = \inf_{\mathbf{a}\in\Gamma_+(f)} \mathbf{a} \cdot \mathbf{u}_\rho$ for each $\rho \in \Sigma[1]$ (cf. Conventions 5.1.3 and 5.1.15, as well as 5.1.5), we define the proper transform of f under $\vartheta_{\Sigma(f)}$ as:

$$f' := \frac{\vartheta^{\#}_{\Sigma(f)}(f)}{\prod_{\rho \in \Sigma(f)[1]} (x'_\rho)^{N_\rho}} = \sum_{\mathbf{a} \in \mathbf{N}^n} c_{\mathbf{a}} \cdot (\boldsymbol{x'})^{\mathbf{a}-\mathbf{n}} \cdot \prod_{\rho \in \Sigma(f)[\mathrm{ex}]} (x'_\rho)^{\mathbf{a}\cdot \mathbf{u}_\rho - N_\rho} \tag{5.4}$$

where $\mathbf{n} := \big(N_i \colon i \in [n]\big)$. In other words, $V(f') \subset \mathscr{X}_{\Sigma(f)}$ is the proper transform of all irreducible components of $V(f) \subset \mathbf{A}^n$ that are not contained in $V(x_1 x_2 \cdots x_n) \subset \mathbf{A}^n$. Note that since f is non-degenerate, $V(f') \subset \mathscr{X}_{\Sigma(f)}$ is reduced. Then the preceding theorem is asserting that at every point in $\vartheta^{-1}_{\Sigma(f)}(\mathbf{0}) \subset \mathscr{X}_{\Sigma(f)}$, $V(f') \subset \mathscr{X}_{\Sigma(f)}$ is smooth, and intersects the smooth divisors $\{V(x'_\rho) \subset \mathscr{X}_{\Sigma(f)} \colon \rho \in \Sigma(f)[1]\}$ transversely.

5.2.3. We next claim that via an *appropriate* motivic change of variables formula, the desingularization $\vartheta_{\Sigma(f)}$ of $V(f) \subset \mathbf{A}^n$ supplies a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ given by

$$\Theta(f) := \{-1\} \cup \big\{ s_\tau \colon \tau \prec^1 \Gamma_+(f) \text{ with } N_\tau > 0 \big\} \qquad (\text{cf. } (1.5)).$$

To this end, we find it the most convenient to appeal to the formula in [**LCMMVVS20**, Theorem 4]. Other formulae that apply to our context include [**BN20**, Theorem 5.3.1], [**Yas06**, Theorem 3.41], or [**SU21**, Theorem 1.3], although the first demands some background on logarithmic geometry, and the latter two are less explicit. However, as it is, the embedded desingularization $\vartheta_{\Sigma(f)}$ of $V(f) \subset \mathbf{A}^n$ does not satisfy the key hypothesis of [**LCMMVVS20**, Theorem 4], since $\mathscr{X}_{\Sigma(f)}$ typically does *not* have finite stabilizers. Nevertheless this can be resolved by further subdividing the fan $\Sigma(f)$ to a *simplicial* fan $\boldsymbol{\Sigma}(f)$ *without adding new rays.*

5.2.4 (Frugal simplicial subdivisions)**.** From 5.2.4 to 5.2.6, let Σ be a fan in $N_{\mathbf{R}}$ whose support $|\Sigma|$ is $N^+_{\mathbf{R}}$, and we fix a subdivision $\boldsymbol{\Sigma}$ of Σ such that:

(i) $\boldsymbol{\Sigma}$ is a simplicial fan, i.e. every cone $\boldsymbol{\sigma}$ in $\boldsymbol{\Sigma}$ is a simplicial cone, i.e. $\boldsymbol{\sigma}[1]$ is a linearly independent set for every $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$.

(ii) Every cone $\boldsymbol{\sigma}$ in $\mathbf{\Sigma}$ can be *inscribed* in some cone σ in Σ (in which case one writes $\sigma \sqsubset \sigma'$), cf. Definition 4.1.1.

Such a $\mathbf{\Sigma}$ always exists by [**DH01**, Lemma 2.8], and we call any such $\mathbf{\Sigma}$ a *frugal simplicial subdivision* of Σ. Note too that $\mathbf{\Sigma}[1] = \Sigma[1]$.

5.2.5. Let $(\widehat{\Sigma}, \beta)$ denote the stacky fan associated to Σ in $N_{\mathbf{R}}$, cf. 2.6.1. Since $\mathbf{\Sigma}[1] = \Sigma[1]$, the stacky fan associated to $\widehat{\Sigma}$ is of the form $(\widehat{\mathbf{\Sigma}}, \beta)$ for the same homomorphism $\beta\colon \mathbf{Z}^{\Sigma[1]} = \widehat{N} \to N = \mathbf{Z}^n$ appearing in $(\widehat{\Sigma}, \beta)$. Moreover, $\widehat{\mathbf{\Sigma}}$ is a sub-fan of $\widehat{\Sigma}$. Indeed, recall from Definition 4.1.1 that $\widehat{\mathbf{\Sigma}}$ is generated by $\{\widehat{\boldsymbol{\sigma}}\colon \boldsymbol{\sigma} \in \mathbf{\Sigma}\}$, where for every cone $\boldsymbol{\sigma}$ in $\mathbf{\Sigma}$,

$$\widehat{\boldsymbol{\sigma}} = \big\langle \mathbf{e}_\rho\colon \rho \in \boldsymbol{\sigma}[1] \big\rangle \subset \mathbf{Z}^{\mathbf{\Sigma}[1]} = \widehat{N}.$$

If σ is a cone in Σ such that $\boldsymbol{\sigma} \sqsubset \sigma$, $\widehat{\boldsymbol{\sigma}}$ is then a face of the cone $\widehat{\sigma} = \langle \mathbf{e}_\rho\colon \rho \in \sigma[1] \rangle$ in $\widehat{\Sigma}$, and hence is in $\widehat{\Sigma}$, as desired. Consequently, the toric morphism induced by the inclusion $\widehat{\mathbf{\Sigma}} \subset \widehat{\Sigma}$ is a $\mathbf{G}_\beta$-equivariant open immersion $X_{\widehat{\mathbf{\Sigma}}} \hookrightarrow X_{\widehat{\Sigma}}$, which descends to the open immersion of stacks in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{X}_{\mathbf{\Sigma}} = \big[X_{\widehat{\mathbf{\Sigma}}} \,/\, \mathbf{G}_\beta\big] \arrow[rr, hook, "\text{open}"] \arrow[dr, "\vartheta_{\mathbf{\Sigma}}"'] & & \big[X_{\widehat{\Sigma}} \,/\, \mathbf{G}_\beta\big] = \mathscr{X}_\Sigma \arrow[dl, "\vartheta_\Sigma"] \\
& \mathbf{A}^n &
\end{tikzcd} \tag{5.5}$$

Explicitly, adopting the notations in the description of $\vartheta_\Sigma\colon \mathscr{X}_\Sigma \to \mathbf{A}^n$ in 4.1.9, the open immersion $\mathscr{X}_{\mathbf{\Sigma}} \hookrightarrow \mathscr{X}_\Sigma$ identifies the former with the following open substack of the latter:

$$\mathscr{X}_{\mathbf{\Sigma}} = \bigcup \big\{ D_+(\boldsymbol{\sigma})\colon \boldsymbol{\sigma} \in \mathbf{\Sigma}[\max] \big\} \subset \mathscr{X}_\Sigma$$

where for each $\boldsymbol{\sigma} \in \mathbf{\Sigma}[\max]$, we set

$$x'_{\boldsymbol{\sigma}} := \prod_{\rho \in \Sigma[1] \smallsetminus \boldsymbol{\sigma}[1]} x'_\rho.$$

and

$$D_+(\boldsymbol{\sigma}) := \left[\operatorname{Spec}\!\left(\mathbf{k}[x'_1,\dots,x'_n]\left[x'_\rho\colon \rho\in\Sigma[\mathrm{ex}]\right]\left[x'^{-1}_{\boldsymbol{\sigma}}\right]\right) \,/\, \mathbb{G}_m^{\Sigma[\mathrm{ex}]}\right] \;\subset\; \mathscr{X}_\Sigma \tag{5.6}$$

is also the $\boldsymbol{\sigma}$-chart of $\mathscr{X}_{\boldsymbol{\Sigma}}$ (4.1.9(iii)). Note too that for every σ in Σ and $\boldsymbol{\sigma}$ in $\boldsymbol{\Sigma}$ such that $\boldsymbol{\sigma}\sqsubset\sigma$, we have $D_+(\boldsymbol{\sigma})\subset D_+(\sigma)$, since x'_σ divides $x'_{\boldsymbol{\sigma}}$.

5.2.6. Since $\boldsymbol{\Sigma}$ is a simplicial fan, $\mathscr{X}_{\boldsymbol{\Sigma}}$ has finite stabilizers, i.e. $X_{\boldsymbol{\Sigma}}$ has finite quotient singularities. While this assertion is classical in toric geometry [**CLS11**, Theorem 11.4.8], we will need, for each $\boldsymbol{\sigma}\in\boldsymbol{\Sigma}[\max]$, an explicit presentation of the σ-chart $D_+(\boldsymbol{\sigma})\subset\mathscr{X}_{\boldsymbol{\Sigma}}$ as the stack quotient of a smooth $\mathbf{k}$-scheme by an action of a finite abelian group. This presentation will be used later in 5.2.7.

Let us start from the expression in (5.6). Firstly, since x'_ρ is invertible on $D_+(\boldsymbol{\sigma})$ for $\rho\in\Sigma[\mathrm{ex}]\smallsetminus\boldsymbol{\sigma}[1]$, and their $\mathbf{Z}^{\Sigma[\mathrm{ex}]}$-weights $\{-\mathbf{e}_\rho\colon \rho\in\Sigma[\mathrm{ex}]\smallsetminus\boldsymbol{\sigma}[1]\}$ are linearly independent over $\mathbf{Z}$, we observe from Lemma 2.1.2 that by setting

$$x'_\rho = 1 \qquad \text{for every } \rho\in\Sigma[\mathrm{ex}]\smallsetminus\boldsymbol{\sigma}[1]$$

we obtain an isomorphism

$$D_+(\boldsymbol{\sigma}) = \left[\operatorname{Spec}\!\left(\mathbf{k}[x'_1,\dots,x'_n]\left[x'_\rho\colon \rho\in\boldsymbol{\sigma}[\mathrm{ex}]\right]\left[x'^{-1}_{\boldsymbol{\sigma}}\right]\right) \,/\, \mathbb{G}_m^{\boldsymbol{\sigma}[\mathrm{ex}]}\right] \tag{5.7}$$

where:

(i) $\boldsymbol{\sigma}[\mathrm{ex}] := \Sigma[\mathrm{ex}]\cap\boldsymbol{\sigma}[1]$.

(ii) $x'_{\boldsymbol{\sigma}}$ becomes $\prod_{i\in[n]\smallsetminus\boldsymbol{\sigma}[1]} x'_i$.

(iii) The action $\mathbb{G}_m^{\boldsymbol{\sigma}[\mathrm{ex}]} \curvearrowright \operatorname{Spec}(\mathbf{k}[x'_1,\dots,x'_n][x'_\rho\colon \rho\in\boldsymbol{\sigma}[\mathrm{ex}]][x'^{-1}_{\boldsymbol{\sigma}}])$ is specified as follows. For each $i\in[n]$, the $\mathbf{Z}^{\boldsymbol{\sigma}[\mathrm{ex}]}$-weight of x'_i is $(u_{\rho,i})_{\rho\in\boldsymbol{\sigma}[\mathrm{ex}]}$, and for each $\rho\in\boldsymbol{\sigma}[\mathrm{ex}]$, the $\mathbf{Z}^{\boldsymbol{\sigma}[\mathrm{ex}]}$-weight of x'_ρ is $-\mathbf{e}_\rho$.

Secondly, since $\boldsymbol{\sigma}$ is simplicial,

$$\{\mathbf{u}_\rho \colon \rho \in \boldsymbol{\sigma}[1]\} = \{\mathbf{e}_i \colon i \in [n] \cap \boldsymbol{\sigma}[1]\} \sqcup \{\mathbf{u}_\rho \colon \rho \in \boldsymbol{\sigma}[\mathrm{ex}]\}$$

is linearly independent, and hence, so is

$$\left\{ (u_{\rho,i})_{i\in[n]\smallsetminus\boldsymbol{\sigma}[1]} = \mathbf{u}_\rho - \sum_{i\in[n]\cap\boldsymbol{\sigma}[1]} u_{\rho,i}\mathbf{e}_i \colon \rho \in \boldsymbol{\sigma}[\mathrm{ex}] \right\}. \tag{5.8}$$

Moreover, since $\dim(\boldsymbol{\sigma}) = n$, we have $\#\boldsymbol{\sigma}[1] = n$, so that:

$$\begin{aligned} \#\boldsymbol{\sigma}[\mathrm{ex}] + n &= \#\boldsymbol{\sigma}[\mathrm{ex}] + \#\big([n] \cap \boldsymbol{\sigma}[1]\big) + \#\big([n] \smallsetminus \boldsymbol{\sigma}[1]\big) \\ &= \#\boldsymbol{\sigma}[1] + \#\big([n] \smallsetminus \boldsymbol{\sigma}[1]\big) = n + \#\big([n] \smallsetminus \boldsymbol{\sigma}[1]\big) \end{aligned}$$

i.e. $\#([n] \smallsetminus \boldsymbol{\sigma}[1]) = \#\boldsymbol{\sigma}[\mathrm{ex}]$. Consequently, the vectors in (5.8) are the columns of an invertible square matrix $\widetilde{\mathbf{B}}$ of order $\#\boldsymbol{\sigma}[\mathrm{ex}]$, which implies that the set of rows of $\widetilde{\mathbf{B}}$:

$$\{(u_{\rho,i})_{\rho\in\boldsymbol{\sigma}[\mathrm{ex}]} \colon i \in [n] \smallsetminus \boldsymbol{\sigma}[1]\} = \{\mathbf{Z}^{\boldsymbol{\sigma}[\mathrm{ex}]}\text{-weights of } x_i' \colon i \in [n] \smallsetminus \boldsymbol{\sigma}[1]\}$$

is linearly independent. Together with the fact that x_i' is invertible on $D_+(\boldsymbol{\sigma})$ for $i \in [n] \smallsetminus \boldsymbol{\sigma}[1]$, we observe again from Lemma 2.1.2 that by setting

$$x_i' = 1 \qquad \text{for every } i \in [n] \smallsetminus \boldsymbol{\sigma}[1]$$

in (5.7), we obtain an isomorphism

$$D_+(\boldsymbol{\sigma}) = \left[\operatorname{Spec}\big(\mathbf{k}[x_\rho' \colon \rho \in \boldsymbol{\sigma}[1]]\big) \;/\; \boldsymbol{\mu}\right] \tag{5.9}$$

where:

(i) $\boldsymbol{\mu} := \mathrm{Hom}_{\mathrm{Grp-Sch}}(A, \mathbb{G}_m)$, where A is the finite abelian group

$$A := \frac{\mathbf{Z}^{\boldsymbol{\sigma}[\mathrm{ex}]}}{\big\langle (u_{\rho,i})_{\rho\in\boldsymbol{\sigma}[\mathrm{ex}]} \colon i \in [n] \smallsetminus \boldsymbol{\sigma}[1] \big\rangle}.$$

(ii) Letting $\overline{(-)}$ denote the quotient $\mathbf{Z}^{\boldsymbol{\sigma}[\mathrm{ex}]} \twoheadrightarrow A$, we specify the action $\boldsymbol{\mu} \curvearrowright \mathrm{Spec}(\mathbf{k}[x'_\rho \colon \rho \in \boldsymbol{\sigma}[1]])$ as follows. If $i \in [n] \cap \boldsymbol{\sigma}[1]$, the A-weight of x'_i is $\overline{(u_{\rho,i})}_{\rho\in\boldsymbol{\sigma}[\mathrm{ex}]}$. If $\rho \in \boldsymbol{\sigma}[\mathrm{ex}]$, the A-weight of x'_ρ is $-\overline{\mathbf{e}_\rho}$.

Since $\{D_+(\boldsymbol{\sigma}) \colon \boldsymbol{\sigma} \in \boldsymbol{\Sigma}[\max]\}$ covers $\mathscr{X}_{\boldsymbol{\Sigma}}$, the expression in (5.9) in particular shows that $\mathscr{X}_{\boldsymbol{\Sigma}}$ has finite stabilizers.

5.2.7. In this paragraph, we compute the relative canonical divisor $K_{\vartheta_{\boldsymbol{\Sigma}}}$ of $\vartheta_{\boldsymbol{\Sigma}}$. For each $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}[\max]$, recall that the composition

$$\vartheta_{\boldsymbol{\Sigma}(\boldsymbol{\sigma})} \colon D_+(\boldsymbol{\sigma}) \overset{(5.9)}{=\!=} \Big[\mathrm{Spec}\Big(\mathbf{k}\big[x'_\rho \colon \rho \in \boldsymbol{\sigma}[1]\big]\Big) \;/\; \boldsymbol{\mu}\Big] \xhookrightarrow{\text{open}} \mathscr{X}_{\boldsymbol{\Sigma}} \xrightarrow{\vartheta_{\boldsymbol{\Sigma}}} \mathbf{A}^n$$

is induced by the $\mathbf{k}$-algebra homomorphism

$$\vartheta_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})^{\#} \colon \mathbf{k}[x_1, \ldots, x_n] \to \mathbf{k}\big[x'_\rho \colon \rho \in \boldsymbol{\sigma}[1]\big]$$

which maps

$$x_i \mapsto \prod_{\rho\in\boldsymbol{\sigma}[1]} (x'_\rho)^{u_{\rho,i}} =: \alpha_i$$

for every $i \in [n]$. We then compute, for each $i \in [n]$:

$$\vartheta_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})^*\big(dx_i\big) = \sum_{\rho\in\boldsymbol{\sigma}[1]} u_{\rho,i}\alpha_i \cdot \frac{dx'_\rho}{x'_\rho}.$$

Letting $\mathfrak{S}([n], \boldsymbol{\sigma}[1])$ denote the set of bijections $\theta \colon [n] \xrightarrow{\simeq} \boldsymbol{\sigma}[1]$, we therefore have:

$$\vartheta_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})^*\big(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\big)$$

$$= \sum_{\theta \in \mathfrak{S}([n], \boldsymbol{\sigma}[1])} \prod_{i \in [n]} \frac{u_{\theta(i),i}\alpha_i}{x'_{\theta(i)}} \cdot dx'_{\theta(1)} \wedge dx'_{\theta(2)} \wedge \cdots \wedge dx'_{\theta(n)}$$

$$= \frac{\prod_{i \in [n]} \alpha_i}{\prod_{\rho \in \boldsymbol{\sigma}[1]} x'_\rho} \cdot \left(\sum_{\theta \in \mathfrak{S}([n], \boldsymbol{\sigma}[1])} \prod_{i \in [n]} u_{\theta(i),i} \cdot dx'_{\theta(1)} \wedge dx'_{\theta(2)} \wedge \cdots \wedge dx'_{\theta(n)} \right)$$

$$= \prod_{\rho \in \boldsymbol{\sigma}[1]} (x'_\rho)^{|\mathbf{u}_\rho| - 1} \cdot \big(\det(\mathbf{B}_{\boldsymbol{\sigma}}) \cdot \wedge_{\rho \in \boldsymbol{\sigma}[1]} dx'_\rho \big)$$

where:

(i) $|\mathbf{u}_\rho| := \mathrm{u}_{\rho,1} + \mathrm{u}_{\rho,2} + \cdots + \mathrm{u}_{\rho,n}$,

(ii) $\mathbf{B}_{\boldsymbol{\sigma}}$ denotes the square matrix of order n whose ρ^{th} column is the vector $\mathbf{u}_\rho$ for $\rho \in \boldsymbol{\sigma}[1]$, which is invertible since $\boldsymbol{\sigma}$ is simplicial,

(iii) $\wedge_{\rho \in \boldsymbol{\sigma}[1]} dx'_\rho := dx'_{\theta(1)} \wedge dx'_{\theta(2)} \wedge \cdots \wedge dx'_{\theta(n)}$ for a fixed $\theta \in \mathfrak{S}([n], \boldsymbol{\sigma}[1])$.

From the above computation, we obtain

$$K_{\vartheta_{\boldsymbol{\Sigma}}}|_{D_+(\boldsymbol{\sigma})} = \sum_{\rho \in \boldsymbol{\sigma}[1]} \big(|\mathbf{u}_\rho| - 1 \big) \cdot V(x'_\rho).$$

Finally, since $\{D_+(\boldsymbol{\sigma}) \colon \boldsymbol{\sigma} \in \boldsymbol{\Sigma}[\max]\}$ is an open cover of $\mathscr{X}_{\boldsymbol{\Sigma}}$, we deduce that

$$K_{\vartheta_{\boldsymbol{\Sigma}}} = \sum_{\rho \in \Sigma[1]} \big(|\mathbf{u}_\rho| - 1 \big) \cdot V(x'_\rho). \tag{5.10}$$

5.2.8. Returning to our claim in 5.2.3, fix a frugal simplicial subdivision $\boldsymbol{\Sigma}(f)$ of the normal fan $\Sigma(f)$. We then have:

$$\begin{tikzcd}
\vartheta^{-1}_{\boldsymbol{\Sigma}(f)}\big(V(f)\big) \arrow[r, hook, "\text{closed}"] \arrow[d, two heads, "\text{coarse space}"'] & \mathscr{X}_{\boldsymbol{\Sigma}(f)} \arrow[r, hook, "\text{open}"] \arrow[d, two heads, "\text{coarse space}"'] & \mathscr{X}_{\Sigma(f)} \arrow[r, "\vartheta_{\Sigma(f)}"] & \mathbf{A}^n \\
\pi^{-1}_{\boldsymbol{\Sigma}(f)}\big(V(f)\big) \arrow[r, hook, "\text{closed}"] & X_{\boldsymbol{\Sigma}(f)} \arrow[urr, bend right, "\pi_{\Sigma(f)}"'] & &
\end{tikzcd}$$

where:

(i) $\pi_{\boldsymbol{\Sigma}(f)}$ is proper and birational.

(ii) $X_{\mathbf{\Sigma}(f)}$ has finite quotient singularities (5.2.6).

(iii) $\pi^{-1}_{\mathbf{\Sigma}(f)}\big(V(f)\big)$ is a $\mathbf{Q}$-simple normal crossings divisor [**LCMMVVS20**, Definition 1.6] at every point in $\pi^{-1}_{\mathbf{\Sigma}(f)}(\mathbf{0}) \subset X_{\mathbf{\Sigma}(f)}$. Indeed, $\vartheta_{\mathbf{\Sigma}(f)}$ factors as $\mathscr{X}_{\mathbf{\Sigma}(f)} \xhookrightarrow{\text{open}} \mathscr{X}_{\Sigma(f)} \xrightarrow{\Pi_{\Sigma(f)}} \mathbf{A}^n$ in the above diagram. We therefore deduce, from (5.4), that:

$$\vartheta^{-1}_{\mathbf{\Sigma}(f)}\big(V(f)\big) = V(f') + \sum_{\rho\in\Sigma(f)[1]} N_\rho \cdot V(x'_\rho) \tag{5.11}$$

where each $V(x'_\rho)$, as well as $V(f')$, is now regarded as a divisor in $\mathscr{X}_{\mathbf{\Sigma}(f)} \xhookrightarrow{\text{open}} \mathscr{X}_{\Sigma(f)}$. By Theorem 5.2.1, $\vartheta^{-1}_{\mathbf{\Sigma}(f)}\big(V(f)\big)$ is a simple normal crossings divisor at every point in $\vartheta^{-1}_{\mathbf{\Sigma}(f)}(\mathbf{0}) \subset \mathscr{X}_{\mathbf{\Sigma}(f)}$. It remains to note that $\pi^{-1}_{\mathbf{\Sigma}(f)}\big(V(f)\big)$ is the coarse space of $\vartheta^{-1}_{\mathbf{\Sigma}(f)}\big(V(f)\big)$, since the coarse space morphism $\mathscr{X}_{\mathbf{\Sigma}(f)} \to X_{\mathbf{\Sigma}(f)}$ maps the latter onto the former.

In other words, $\pi_{\mathbf{\Sigma}(f)} \colon X_{\mathbf{\Sigma}(f)} \to \mathbf{A}^n$ is an *embedded* $\mathbf{Q}$*-desingularization of* $V(f) \subset \mathbf{A}^n$ *above the origin* $\mathbf{0} \in \mathbf{A}^n$, in the sense that it satisfies (i), (ii) and (iii) above.

We additionally note that the motivic change of variables formula in [**LCMMVVS20**, Theorem 4] applies more generally for any embedded $\mathbf{Q}$-desingularization $\pi\colon Y \to X$ of $D_1 + D_2 \subset X$ above a closed subscheme $W \subset X$: the proof in loc. cit. works verbatim, once one recognizes that:

(a) [**LCMMVVS20**, Theorem 2] is a general change of variables rule for the $\mathbf{Q}$-Gorenstein motivic zeta function via any proper and birational morphism of pure-dimensional $\mathbf{Q}$-Gorenstein varieties.

(b) After applying [**LCMMVVS20**, Theorem 2], note that the remainder of the proof of [**LCMMVVS20**, Theorem 4] only uses the fact that $\pi^{-1}(D_1+D_2) \subset Y$ is a $\mathbf{Q}$-simple normal crossings divisor at every point in $\pi^{-1}(W)$.

We can therefore apply [**LCMMVVS20**, Theorem 4] with $\pi := \pi_{\boldsymbol{\Sigma}(f)}$, $D_1 := V(f)$, $D_2 := 0$, and $W = \{\mathbf{0}\}$. Together with (5.10) and (5.11), we deduce that $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ lies in

$$\mathscr{M}_{\mathbf{k}}\left[\mathbf{L}^{-s}\right]\left[\frac{1}{1-\mathbf{L}^{-(s+1)}}\right]\left[\frac{1}{1-\mathbf{L}^{-(N_\rho s+|\mathbf{u}_\rho|)}} : \rho \in \Sigma(f)[1]\right]$$

i.e. $\Theta(f) = \{-1\} \cup \left\{-\frac{|\mathbf{u}_\rho|}{N_\rho} : \rho \in \Sigma(f)[1],\ N_\rho > 0\right\}$ is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$.

5.2.B. A case study for Theorem H.

5.2.9. In §5.2.A we explained why there is a set of candidate poles $\Theta(f)$ for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ whose elements, with the *possible* exception of -1, are naturally indexed by facets $\tau \prec^1 \Gamma_+(f)$ satisfying $N_\tau > 0$. Namely, the preimage of $V(f) \subset \mathbf{A}^n$ under the multi-weighted blow-up $\mathscr{X}_{\Sigma(f)} \to \mathbf{A}^n$ is a simple normal crossings divisor at points above $\mathbf{0} \in \mathbf{A}^n$, comprising of:

(i) the proper transform of the irreducible components of $V(f) \subset \mathbf{A}^n$ that are not contained in $V(x_1x_2\cdots x_n) \subset \mathbf{A}^n$,

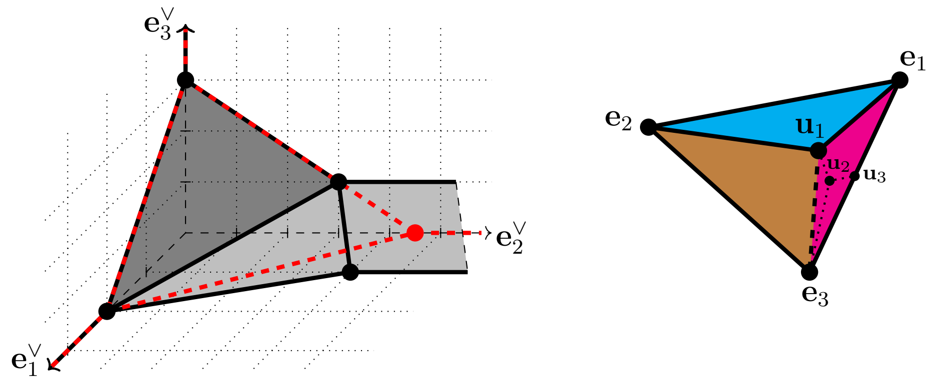
(ii) the proper transform of $V(x_i) \subset \mathbf{A}^n$ for every $i \in [n]$ with $x_i \mid f$,

(iii) and the irreducible exceptional divisors of $\Pi_{\Sigma(f)}$,

where the irreducible components in (ii) and (iii) are naturally indexed by the facets $\tau \prec^1 \Gamma_+(f)$ satisfying $N_\tau > 0$.

It is therefore natural to imagine that a proof of Theorem H would involve showing that $V(f) \subset \mathbf{A}^n$ is *also* desingularized by a multi-weighted blow-up $\Theta_{\Sigma^\dagger}$ of $\mathbf{A}^n$ where $\Sigma^\dagger$ is the normal fan of a Newton $\mathbf{Q}$-polyhedron $\Gamma_+^\dagger$ obtained from $\Gamma_+(f)$ by "dropping the facets in $\mathbb{B}$" (cf. Theorem I). Ideally, one hopes that every supporting hyperplane of $\Gamma_+(f)$, *except* those intersecting $\Gamma_+(f)$ in a face of some facet in $\mathbb{B}$, should also be a supporting hyperplane of $\Gamma_+^\dagger$. In this section we show that this idea works for three non-degenerate polynomials.

Example 5.2.10. Let $f = x_1^2 + x_1x_2^4 + x_2^3x_3 + x_3^3$. On the left side of the diagram below, we shaded the facets of $\Gamma_+(f)$ that are not contained in any coordinate hyperplane H_i in $M_{\mathbf{R}}$. For now the red vertex and dashed lines, and the right side of the diagram, should be ignored.



Among the shaded facets, we used a *darker* shade for the non–B_1-facet

$$\tau_1 := \{\mathbf{a} \in \Gamma_+(f) \colon \mathbf{a} \cdot \mathbf{u}_1 = 18\} \qquad \text{where } \mathbf{u}_1 := 9\mathbf{e}_1 + 4\mathbf{e}_2 + 6\mathbf{e}_3$$

with candidate pole $-\frac{19}{18}$, and used a *lighter* shade for the two B_1-facets

$$\tau_2 := \{\mathbf{a} \in \Gamma_+(f) \colon \mathbf{a} \cdot \mathbf{u}_2 = 8\} \qquad \text{where } \mathbf{u}_2 := 4\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3$$

$$\tau_3 := \{\mathbf{a} \in \Gamma_+(f) \colon \mathbf{a} \cdot \mathbf{u}_3 = 1\} \qquad \text{where } \mathbf{u}_3 := \mathbf{e}_1 + \mathbf{e}_3$$

with candidate poles $-\frac{5}{4}$ and -2 respectively. Together τ_2 and τ_3 form a pair $\mathbb{B}$ of adjacent B_1-facets of $\Gamma_+(f)$ with consistent base direction 3. Then Theorem H asserts that $\Theta^{\dagger,\mathbb{B}}(f) = \{-1, -\frac{19}{18}\} \subsetneq \{-1, -\frac{19}{18}, -\frac{5}{4}, -2\} = \Theta(f)$ is also a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f; s)$.

To show that we execute our idea in 5.2.9. Indeed, we first note that

$$\Gamma_+(f) = H^+_{\mathbf{u}_1,18} \cap H^+_{\mathbf{u}_2,8} \cap H^+_{\mathbf{u}_3,1} \qquad \text{(cf. 5.1.1 for notation).}$$

Since $H_{\mathbf{u}_2,8}$ and $H_{\mathbf{u}_3,1}$ intersect $\Gamma_+(f)$ in the two B_1-facets τ_2 and τ_3, we "drop" $H^+_{\mathbf{u}_2,8}$ and $H^+_{\mathbf{u}_3,1}$ from $\Gamma_+(f)$ to define the Newton $\mathbf{Q}$-polyhedron:

$$\Gamma_+^\dagger = H^+_{\mathbf{u}_1,18}$$

which we have outlined in *red* on the left side of the above diagram.

Illustrated on the right side of the diagram is a cross-section of the normal fan $\Sigma^\dagger$ of $\Gamma_+^\dagger$, except that the rays $\langle \mathbf{u}_2 \rangle$ and $\langle \mathbf{u}_3 \rangle$, as well as the 2-dimensional cones $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$, $\langle \mathbf{u}_2, \mathbf{u}_3 \rangle$ and $\langle \mathbf{e}_3, \mathbf{u}_2 \rangle$ which are outlined by *dotted line segments*, are not in $\Sigma^\dagger$ but originally in $\Sigma(f)$. In comparison, the 2-dimensional cone $\langle \mathbf{e}_3, \mathbf{u}_1 \rangle$ in $\Sigma^\dagger$, which is outlined by the *dashed thick line segment*, is originally not in $\Sigma(f)$.

Finally, we consider the following multi-weighted blow-up of $\mathbf{A}^3$:

$$\vartheta_{\Sigma^\dagger} \colon \mathscr{X}_{\Sigma^\dagger} = \Big[\mathrm{Spec}\Big(\mathbf{k}[x_1', x_2', x_3', u_1]\Big) \smallsetminus V(x_1', x_2', x_3') \;/\; \mathbb{G}_m\Big] \to \mathbf{A}^3$$

induced by the homomorphism $\vartheta^\#_{\Sigma^\dagger} \colon \mathbf{k}[x_1, x_2, x_3] \to \mathbf{k}[x_1', x_2', x_3', u_1]$ mapping $x_1 \mapsto x_1' u_1^9$, $x_2 \mapsto x_2' u_1^4$ and $x_3 \mapsto x_3' u_1^6$. We show next that $\vartheta_{\Sigma^\dagger}$ is a stack-theoretic embedded desingularization of $V(f) \subset \mathbf{A}^3$ above $\mathbf{0} \in \mathbf{A}^3$. We first compute that $\vartheta^\#_{\Sigma^\dagger}(f) = u_1^{18} \cdot f'$, where the proper transform of f under $\vartheta_{\Sigma^\dagger}$ is given by $f' := x_1'^2 + x_1' x_2'^4 u_1^7 + x_2'^3 x_3' + x_3'^3$. Since $\left|\vartheta^{-1}_{\Sigma^\dagger}(\mathbf{0})\right| = |V(u_1)| \subset |\mathscr{X}_{\Sigma^\dagger}|$, it suffices to show $V(f'|_{V(u_1)}) = V\big(x_1'^2 + x_2'^3 x_3' + x_3'^3\big) \subset V(u_1)$ is smooth. Indeed, if $J\big(f'|_{V(u_1)}\big)$ denotes the Jacobian ideal of $f'|_{V(u_1)}$, note

$$\sqrt{\big(f'|_{V(u_1)}\big) + J\big(f'|_{V(u_1)}\big)} = \sqrt{\big(x_1', x_2'^2 x_3', x_2'^3 + 3x_3'^2, x_3'^3\big)} = \big(x_1', x_2', x_3'\big)$$

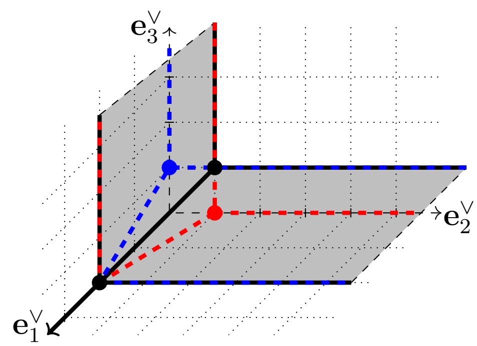
is the unit ideal on $\mathscr{X}_{\Sigma^\dagger}$, as desired.

Remark 5.2.11. In the above example, note that unlike $\Gamma_+(f)$, $\Gamma_+^\dagger$ has a vertex with non-integer coordinates, namely the *red vertex* $\frac{9}{2}\mathbf{e}_2^\vee$.

Remark 5.2.12. Moreover, the morphism $\vartheta_{\Sigma^\dagger}$ is the weighted blow-up of $\mathbf{A}^3$ along the center $(x_1, 9) + (x_2, 4) + (x_3, 6) = \left(x_1^{1/9}, x_2^{1/4}, x_3^{1/6}\right)$, cf. first paragraph of Example 4.1.17. We also remark $V(f)$ has a semi-quasihomogeneous singularity at $\mathbf{0} \in \mathbf{A}^3$ (with the same weights 9 on x_1, 4 on x_2 and 6 on x_3), and the strong monodromy conjecture is known for semi-quasihomogeneous hypersurfaces, cf. [**BBV21**]. In fact, the proof also uses weighted blow-ups.

The next example, together with its remark, shows that the hypothesis in Theorem H that "$\mathbb{B}$ has consistent base directions" *cannot be dropped*:

Example 5.2.13. Let $f = x_1^2 + x_2x_3$. In the diagram below we shaded only the facets of $\Gamma_+(f)$ that are not contained in any coordinate hyperplane H_i in $M_{\mathbf{R}}$. As with the previous example we ignore the red/blue vertices and dashed lines for now.



The two shaded facets of $\Gamma_+(f)$:

$$\tau_1 := \left\{\mathbf{a} \in \Gamma_+(f) \colon \mathbf{a} \cdot \mathbf{u}_1 = 2\right\} \qquad \text{where } \mathbf{u}_1 := \mathbf{e}_1 + 2\mathbf{e}_2$$

$$\tau_2 := \left\{\mathbf{a} \in \Gamma_+(f) \colon \mathbf{a} \cdot \mathbf{u}_2 = 2\right\} \qquad \text{where } \mathbf{u}_2 := \mathbf{e}_1 + 2\mathbf{e}_3$$

are adjacent B_1-facets with the same candidate pole $-\frac{3}{2}$, but together they form a set of B_1-facets with *inconsistent* base directions 2 and 3.

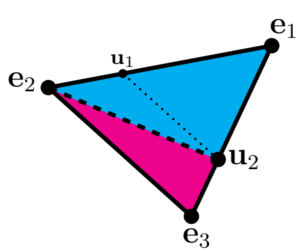
Thus, Theorem H does not apply to the set $\mathbb{B} = \{\tau_1, \tau_2\}$. In fact, our idea in 5.2.9 *fails* in this scenario. Indeed, "dropping" both $H^+_{\mathbf{u}_1,2}$ and $H^+_{\mathbf{u}_2,2}$ from $\Gamma_+(f) = H^+_{\mathbf{u}_1,2} \cap H^+_{\mathbf{u}_2,2}$ yields $\Gamma^\dagger_+ = M^+_{\mathbf{R}}$, but the multi-weighted blow-up of $\mathbf{A}^3$ along $M^+_{\mathbf{R}}$ is the identity morphism on $\mathbf{A}^3$!

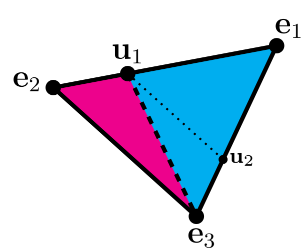
Nevertheless, in Theorem H one could take $\mathbb{B}$ to be either $\{\tau_1\}$ or $\{\tau_2\}$, although in either case $\Theta^{\dagger,\mathbb{B}}(f) = \{-1, -\frac{3}{2}\}$ is the same set as $\Theta(f)$. In spite of that, our idea in 5.2.9 should still say something of consequence. Namely, for $\mathbb{B} = \{\tau_1\}$ (resp. $\mathbb{B} = \{\tau_2\}$), we claim that the multi-weighted blow-up of $\mathbf{A}^3$ along the Newton polyhedron

$$\Gamma^{\dagger,\tau_1}_+ = H^+_{\mathbf{u}_2,2} \qquad (\text{resp. } \Gamma^{\dagger,\tau_2}_+ = H^+_{\mathbf{u}_1,2})$$

is a stack-theoretic embedded desingularization of $V(f) \subset \mathbf{A}^3$ above $\mathbf{0} \in \mathbf{A}^3$.

To verify this claim, let us first outline, in the diagram above, the Newton polyhedra $\Gamma^{\dagger,\tau_1}_+$ and $\Gamma^{\dagger,\tau_2}_+$ in *blue* and *red* respectively. On the left (resp. right) side of the diagram below, we also sketched a cross-section of the normal fan $\Sigma^{\dagger,\tau_1}$ (resp. $\Sigma^{\dagger,\tau_2}$) of $\Gamma^{\dagger,\tau_1}_+$ (resp. $\Gamma^{\dagger,\tau_2}_+$), keeping the same conventions as before in Example 5.2.10.





From this diagram we see that the following multi-weighted blow-up of $\mathbf{A}^3$:

$$\vartheta_{\Sigma^{\dagger,\tau_2}} \colon \mathscr{X}_{\Sigma^{\dagger,\tau_2}} = \left[\mathrm{Spec}\big(\mathbf{k}[x_1', x_2', x_3, u_1]\big) \smallsetminus V(x_1', x_2') \,/\, \mathbb{G}_m\right] \to \mathbf{A}^3$$

is induced by the homomorphism $\vartheta^\#_{\Sigma^{\dagger,\tau_2}} \colon \mathbf{k}[x_1, x_2, x_3] \to \mathbf{k}[x_1', x_2', x_3, u_1]$ mapping $x_1 \mapsto x_1' u_1$, $x_2 \mapsto x_2' u_1^2$ and $x_3 \mapsto x_3$. (This is the weighted blow-up of $\mathbf{A}^3$ along the center $(x_1, 1) + (x_2, 2) = \big(x_1, x_2^{1/2}\big)$.) Thus, $\vartheta^\#_{\Sigma^{\dagger,\tau_2}}(f) = u_1^2 \cdot f'$, where $f' := x_1'^2 + x_2' x_3$ defines the proper transform of f

under $\vartheta_{\Sigma^\dagger,\tau_2}$. It remains to note the Jacobian ideal $J(f')$ of f' is $\big(x_1', x_2', x_3\big)$, i.e. the unit ideal on $\mathscr{X}_{\Sigma^\dagger,\tau_2}$. The same can be shown with τ_2 replaced by τ_1.

Remark 5.2.14. In fact, for $f = x_1^2 + x_2x_3$, $\Theta(f) = \{-1, -\frac{3}{2}\}$ is the *smallest* set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$. To see this, it suffices to show that $-\frac{3}{2}$ is a pole of $Z_{\mathrm{top},\mathbf{0}}(f;s)$ (cf. Remark 1.3.6), which we compute via the embedded resolution of $V(f) \subset \mathbf{A}^3$ given by the blow-up of $\mathbf{A}^3$ in $\mathbf{0} \in \mathbf{A}^3$:

$$\pi\colon\ \mathrm{Bl}_{\mathbf{0}}\,\mathbf{A}^3 = \mathrm{Proj}_{\mathbf{A}^3}\left(\mathscr{O}_{\mathbf{A}^3}\big[x_1' := x_1t, x_2' := x_2t, x_3' := x_3t, t^{-1}\big]\right) \to \mathbf{A}^3.$$

Here, $\pi^{-1}\big(V(f)\big) = 2\cdot E_1 + E_2$, where $E_1 := V(t^{-1})$ is the exceptional divisor of π, and $E_2 := V\big(x_1'^2 + x_2'x_3'\big)$ is the proper transform of $V(f)$ under π. Moreover, the relative canonical divisor is $K_\pi = 2\cdot E_1$. Since $\pi^{-1}(\mathbf{0}) = E_1 \simeq \mathbf{P}^2$, we have by definition [CLNS10, Chapter 1, §3.3] that:

$$\begin{aligned}
Z_{\mathrm{top},\mathbf{0}}(f;s) &= \frac{\mathrm{Eu}(E_1 \smallsetminus E_2)}{2s+3} + \frac{\mathrm{Eu}(E_1 \cap E_2)}{(s+1)(2s+3)} \\
&= \frac{\mathrm{Eu}\big(\mathbf{P}^2 \smallsetminus V(x_1'^2 + x_2'x_3')\big)}{2s+3} + \frac{\mathrm{Eu}\big(V(x_1'^2 + x_2'x_3') \subset \mathbf{P}^2\big)}{(s+1)(2s+3)} \\
&= \frac{\mathrm{Eu}\big(\mathbf{P}^2 \smallsetminus V(x_1'^2 + x_2'x_3')\big)s + \mathrm{Eu}(\mathbf{P}^2)}{(s+1)(2s+3)} \overset{\text{[GS]}}{=\!=} \frac{s+3}{(s+1)(2s+3)}.
\end{aligned}$$

Example 5.2.15. Let $f = x_2x_3 + x_1^2x_2^2 + x_1^2x_3^2$. Depicted on the left side of the diagram below is $\Gamma_+(f)$, where a darker shade is used for the non–B_1-facet:

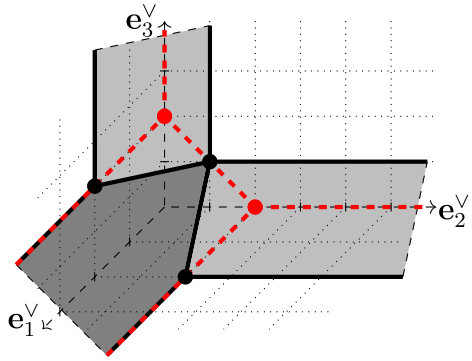
$$\tau_1 := \big\{\mathbf{a} \in \Gamma_+(f)\colon \mathbf{a}\cdot\mathbf{u}_1 = 2\big\} \qquad \text{where } \mathbf{u}_1 := \mathbf{e}_2 + \mathbf{e}_3$$

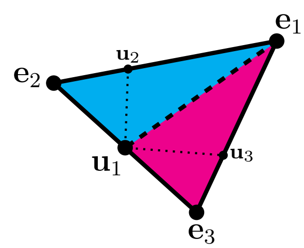
with candidate pole -1, and a lighter shade is used for the two B_1-facets:

$$\tau_2 := \big\{\mathbf{a} \in \Gamma_+(f)\colon \mathbf{a}\cdot\mathbf{u}_2 = 2\big\} \qquad \text{where } \mathbf{u}_2 := \mathbf{e}_1 + 2\mathbf{e}_2$$

$$\tau_3 := \{\mathbf{a} \in \Gamma_+(f) \colon \mathbf{a} \cdot \mathbf{u}_3 = 2\} \qquad \text{where } \mathbf{u}_3 := \mathbf{e}_1 + 2\mathbf{e}_3$$

each with candidate poles $-\frac{3}{2}$. Although τ_2 and τ_3 have *different* base directions 2 and 3, they are *non-adjacent* and hence still form a set $\mathbb{B}$ of B_1-facets of $\Gamma_+(f)$ with consistent base directions.





Consequently, Theorem H says that $\Theta^{\dagger,\mathbb{B}}(f) = \{-1\} \subsetneq \{-1, -\frac{3}{2}\} = \Theta(f)$ is also a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$. To see this, we proceed as with previous examples: we "drop" $H^+_{\mathbf{u}_2,2}$ and $H^+_{\mathbf{u}_3,2}$ from $\Gamma_+(f) = \bigcap\{H^+_{\mathbf{u}_i,2} \colon 1 \leq i \leq 3\}$ to define the Newton polyhedron $\Gamma^\dagger_+ = H^+_{\mathbf{u}_1,2}$, which we have outlined in red on the left side of the above diagram.

On the right side we sketched the normal fan $\Sigma^\dagger$ of $\Gamma^\dagger_+$, keeping the same conventions as before in Example 5.2.10. The multi-weighted blow-up of $\mathbf{A}^3$:

$$\vartheta_{\Sigma^\dagger} \colon \mathscr{X}_{\Sigma^\dagger} = \mathrm{Proj}_{\mathbf{A}^3}\left(\mathscr{O}_{\mathbf{A}^3}\left[x_2' := x_2 t, x_3' := x_3 t, u_1 := t^{-1}\right]\right) \to \mathbf{A}^3.$$

is then simply the blow-up of $\mathbf{A}^3$ along $V(x_2, x_3) \subset \mathbf{A}^3$. We have $\vartheta^\#_{\Sigma^\dagger}(f) = u_1^2 \cdot f'$, where $f' := x_2' x_3' + x_1^2 x_2'^2 + x_1^2 x_3'^2$ defines the proper transform of f under $\vartheta_{\Sigma^\dagger}$. If $J(f')$ denotes the Jacobian of f', we then have:

$$\begin{aligned}\sqrt{(f') + J(f')} &= \sqrt{\left(x_1 x_2'^2 + x_1 x_3'^2, x_3' + 2x_1^2 x_2', x_2' + 2x_1^2 x_3', x_2' x_3'\right)} \\ &= \sqrt{\left(x_3' + 2x_1^2 x_2', x_2' + 2x_1^2 x_3', x_2' x_3', x_1 x_2', x_1 x_3'\right)} = \left(x_2', x_3'\right)\end{aligned}$$

which is the unit ideal on $\mathscr{X}_{\Sigma^\dagger}$, i.e. $\vartheta_{\Sigma^\dagger}$ is a stack-theoretic embedded desingularization for $V(f) \subset \mathbf{A}^3$ as desired.

Remark 5.2.16. While $\Theta(f) \smallsetminus \{-\frac{3}{2}\} = \{-1\}$ is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$, $-\frac{3}{2}$ *still* induces a monodromy eigenvalue of f near $\mathbf{0} \in \mathbf{C}^3$.

5.3. Proof of main theorem

5.3.A. Dropping a set of facets from a Newton Q-polyhedron. In this section, we fix a Newton $\mathbf{Q}$-polyhedron Γ_+, with associated piecewise-linear, convex $\mathbf{Q}$-function φ (§5.1.A), and associated normal fan Σ in $N_{\mathbf{R}}$ (§5.1.B). We first fix the following conventions for the remainder of this paper:

Convention 5.3.1. If two rays $\rho_1, \rho_2 \in \Sigma[1]$ satisfy $\rho_1 + \rho_2 \in \Sigma[2]$, we say that ρ_1 and ρ_2 are *adjacent in* Σ, and write

$$\rho_1 \frown \rho_2 \qquad \text{in } \Sigma.$$

Given $\tau, \tau' \prec^1 \Gamma_+$, note $\tau \frown \tau'$ (Convention 5.1.3) if and only if $\rho_\tau \frown \rho_{\tau'}$ in Σ, cf. 5.1.13.

5.3.2. Throughout this section, consider a subset $\mathbb{B}$ of facets of Γ_+ that are *not* contained in any translate $m\mathbf{e}_i + H_i$ of any coordinate hyperplane H_i in $M^+_{\mathbf{R}}$. For any such $\mathbb{B}$, we set

$$\Sigma[1]|_{\mathbb{B}} \;:=\; \big\{\rho_\tau \in \Sigma[1] \colon \tau \in \mathbb{B}\big\} \;\subset\; \Sigma[1].$$

As motivated in §5.2.B, we study in this section the Newton $\mathbf{Q}$-polyhedron obtained from Γ_+ by "dropping the facets in $\mathbb{B}$":

Definition 5.3.3. Recalling from (5.1) that

$$\Gamma_+ = \bigcap_{\tau \prec^1 \Gamma_+} H^+_{\mathbf{u}_\tau, N_\tau}$$

we define the $\mathbb{B}$-*cut* of Γ_+ to be the following Newton **Q**-polyhedron:

$$\Gamma_+^{\dagger,\mathbb{B}} := \bigcap \left\{ H^+_{\mathbf{u}_\tau, N_\tau} : \tau \prec^1 \Gamma_+,\ \tau \notin \mathbb{B} \right\} \supset \Gamma_+.$$

We call its normal fan in $N_{\mathbf{R}}$ the $\mathbb{B}$-cut of Σ, and denote it by $\Sigma^{\dagger,\mathbb{B}}$. When $\mathbb{B}$ is unambiguous from context, we write $\Gamma_+^\dagger$ for $\Gamma_+^{\dagger,\mathbb{B}}$ and $\Sigma^\dagger$ for $\Sigma^{\dagger,\mathbb{B}}$.

Lemma 5.3.4. *Let $\tau \prec^1 \Gamma_+$ such that $\tau \notin \mathbb{B}$. Then:*

(i) *There exists a (unique) facet $\tau^\dagger \prec^1 \Gamma_+^\dagger$ such that $\tau^\dagger \cap \Gamma_+ = \tau$.*

(ii) *If moreover τ is not adjacent to any facet in $\mathbb{B}$, then $\tau^\dagger = \tau$. In other words, τ remains a facet of $\Gamma_+^\dagger$.*

Proof. For (i), note that

$$\tau = H_{\mathbf{u}_\tau, N_\tau} \cap \bigcap \left\{ H^+_{\mathbf{u}_{\tau'}, N_{\tau'}} : \tau' \prec^1 \Gamma_+ \right\}.$$

Set

$$\tau^\dagger := H_{\mathbf{u}_\tau, N_\tau} \cap \Gamma_+^\dagger = H_{\mathbf{u}_\tau, N_\tau} \cap \bigcap \left\{ H^+_{\mathbf{u}_\tau, N_\tau} : \tau' \prec \Gamma_+,\ \tau' \notin \mathbb{B} \right\}$$

from which it follows that $\tau^\dagger \cap \Gamma_+ = \tau$. Since $\tau \subset \tau^\dagger \subset H_{\mathbf{u}_\tau, N_\tau}$, $\dim(\tau^\dagger) = n-1$, i.e. $H_{\mathbf{u}_\tau, N_\tau}$ is a supporting hyperplane for $\Gamma_+^\dagger$, and $\tau^\dagger$ is a facet of Γ_+. For (ii), note that since every face of τ is the intersection of a subset of facets of τ, we have:

$$\tau = H_{\mathbf{u}_\tau, N_\tau} \cap \bigcap \left\{ H^+_{\mathbf{u}_{\tau'}, N_{\tau'}} : \tau' \prec^1 \Gamma_+,\ \tau' \frown \tau \right\}.$$

By hypothesis, $\{\tau' \prec \Gamma_+ : \tau' \frown \tau\} \subset \{\tau' \prec \Gamma_+, : \tau' \notin \mathbb{B}\}$. Therefore, $\tau \supset \tau^\dagger$, which proves (ii). $\square$

5.3.5 (A correspondence)**.** The preceding lemma sets up an injection

$$\begin{aligned} \{\text{facets of } \Gamma_+\} \smallsetminus \mathbb{B} &\hookrightarrow \{\text{facets of } \Gamma_+^\dagger\} \\ \tau &\longmapsto \tau^\dagger \end{aligned} \tag{5.12}$$

which is in fact a bijection, since we have assumed that each facet in $\mathbb{B}$ is not contained in $m\mathbf{e}_i + H_i$ for any $i \in [n]$ and $m \in \mathbf{Q}_{>0}$. We will freely adopt this correspondence for the remainder of this paper. Note that in particular, $\Sigma^\dagger[1] = \Sigma[1] \smallsetminus \Sigma[1]|_{\mathbb{B}}$. For $\rho \in \Sigma^\dagger[1]$, we may therefore consider ρ as a ray in $\Sigma[1]$: in that case, we continue to denote by τ_ρ the facet of Γ_+ dual to ρ in $\Sigma[1]$. On the other hand, we denote by $\tau_\rho^\dagger$ the facet of $\Gamma_+^\dagger$ dual to ρ in $\Sigma^\dagger[1]$. This does not contradict the notation in (5.12).

5.3.6. Let $\varphi^\dagger \colon N_{\mathbf{R}}^+ \to \mathbf{R}_{\geq 0}$ be the piecewise-linear, convex $\mathbf{Q}$-function corresponding to the Newton $\mathbf{Q}$-polyhedron $\Gamma_+^\dagger$, cf. 5.1.4. By 5.1.5 and 5.3.5, $\varphi^\dagger$ can be explicated as

$$\varphi^\dagger = \min \mathscr{S}^\dagger$$

where

$$\mathscr{S}^\dagger := \left\{ \begin{matrix} \text{linear functions } \ell\colon N_{\mathbf{R}}^+ \to \mathbf{R}_{\geq 0} \text{ such that} \\ \ell(\mathbf{u}_\tau) \geq N_\tau \text{ for every facet } \tau \prec^1 \Gamma_+ \text{ not in } \mathbb{B} \end{matrix} \right\}.$$

We also note that for every facet $\tau \prec^1 \Gamma_+$ not in $\mathbb{B}$,

$$\varphi(\mathbf{u}_\tau) = N_\tau = \varphi^\dagger(\mathbf{u}_\tau). \tag{5.13}$$

For the remainder of this section, we switch our focus to the cones in $\Sigma^\dagger$. For later purposes (e.g. in §5.3.C), we occasionally state some of our definitions and results for cones in the augmentation $\overline{\Sigma}^\dagger$ of $\Sigma^\dagger$.

Definition 5.3.7. We say a cone σ in $\overline{\Sigma}^\dagger$ is *old* if σ can be inscribed in some cone σ' in Σ (in which case one writes $\sigma \sqsubset \sigma'$). If not, we say σ is *new*.

Lemma 5.3.8.

(i) *For any cone σ in Σ, the cone $\sigma^\dagger$ in $N_{\mathbf{R}}$ generated by rays in*

$$\sigma[1] \smallsetminus \Sigma[1]|_{\mathbb{B}}$$

is a cone in $\Sigma^\dagger$ (hence, all its faces are old cones in $\Sigma^\dagger$). Moreover, for every $\mathbf{u} \in \sigma^\dagger$, $\varphi^\dagger(\mathbf{u}) = \varphi(\mathbf{u})$.

(ii) *For every facet $\tau \prec^1 \Gamma_+$ with $\tau \in \mathbb{B}$, we have:*

$$\varphi^\dagger(\mathbf{u}_\tau) < \varphi(\mathbf{u}_\tau).$$

Proof. For (i), let $\mathbf{a} \in \operatorname{relint}(\varsigma_\sigma)$, so that $\sigma = \sigma_{\mathbf{a}} = \{\mathbf{u} \in N_{\mathbf{R}}^+ \colon \varphi(\mathbf{u}) = \mathbf{a} \cdot \mathbf{u}\}$, cf. 5.1.13. Let $\sigma_{\mathbf{a}}^\dagger := \{\mathbf{u} \in N_{\mathbf{R}}^+ \colon \varphi^\dagger(\mathbf{u}) = \mathbf{a} \cdot \mathbf{u}\}$, which by definition is a cone in $\Sigma^\dagger$. We claim that $\sigma_{\mathbf{a}}^\dagger[1] = \sigma_{\mathbf{a}}[1] \smallsetminus \Sigma[1]|_{\mathbb{B}}$, which would prove (i). Indeed, given any $\tau \prec^1 \Gamma_+$ not in $\mathbb{B}$, we have $\varphi^\dagger(\mathbf{u}_\tau) = \varphi(\mathbf{u}_\tau)$, cf. (5.13), and hence we have $\varphi^\dagger(\mathbf{u}_\tau) = \mathbf{a} \cdot \mathbf{u}_\tau$ if and only if τ is dual to a ray in $\sigma_{\mathbf{a}}[1] \smallsetminus \Sigma[1]|_{\mathbb{B}}$. By Corollary 5.1.11 and 5.3.5, this proves our claim.

For (ii), we apply the above argument to the case where σ is the ray ρ_τ in Σ dual to $\tau \prec \Gamma_+$, and we obtain that for $\mathbf{a} \in \operatorname{relint}(\tau)$, we have $\{\mathbf{u} \in N_{\mathbf{R}}^+ \colon \varphi^\dagger(\mathbf{u}) = \mathbf{a} \cdot \mathbf{u}\} = \{\mathbf{0}\}$. Combining that with the fact that $\mathbf{a} \in \Gamma_+ \subset \Gamma_+^\dagger$, we must have $\varphi^\dagger(\mathbf{u}_\tau) < \mathbf{a} \cdot \mathbf{u}_\tau = \varphi(\mathbf{u}_\tau)$. ☐

Lemma 5.3.9. *Let σ be a cone in $\Sigma^\dagger$.*

(i) *If there is an extremal ray ρ of σ that is not adjacent in Σ to any ray in $\Sigma[1]|_{\mathbb{B}}$, then σ is old.*

(ii) *If moreover $\dim(\sigma) = 2$, then σ is a cone in Σ.*

Proof. By Lemma 5.3.4(ii), the facet $\tau_\rho \prec^1 \Gamma_+$ dual to $\rho \in \Sigma[1]$ remains a facet of $\Gamma_+^\dagger$. Therefore, the face $\varsigma \prec \Gamma_+^\dagger$ dual to σ, being a face of τ_ρ, remains a face of $\Gamma_+^\dagger$. Consequently, for every $\tau \prec^1 \Gamma_+$ such that $\tau \notin \mathbb{B}$, we have the following equivalences:

$$\varsigma \prec \tau \iff \varsigma \subset H_{\mathbf{u}_\tau, N_\tau} \iff \varsigma \prec \tau^\dagger. \tag{5.14}$$

The reverse implication in (5.14) means that σ is inscribed in the cone in Σ dual to the face $\varsigma \prec \Gamma_+$, as desired.

If $\dim(\sigma) = 2$, let ρ and ρ' be the extremal rays of σ. By Corollary 5.1.16, $\varsigma = \tau_\rho^\dagger \cap \tau_{\rho'}^\dagger$. By (5.14), ς is a face of both τ_ρ and $\tau_{\rho'}$. Since ς is a $(n-2)$-dimensional face of Γ_+, ς is a face of *exactly two* facets of Γ_+, which by the preceding sentence are necessarily τ_ρ and $\tau_{\rho'}$. This means that the cone in Σ dual to the face $\varsigma \prec \Gamma_+$ is generated by ρ and ρ', i.e. is equal to σ. In particular, σ is a cone in Σ. □

By part (i) of the preceding lemma, we see that if σ is a new cone in $\Sigma^\dagger$, then all its extremal rays must be adjacent in Σ to some ray in $\Sigma[1]|_{\mathbb{B}}$. The next proposition refines that observation. We first introduce some notation:

5.3.10 (An equivalence relation)**.** For any subset $\mathbb{B}$ of facets of Γ_+, we use the same symbol $\sim$ to denote the equivalence closure of $\frown$ (cf. Conventions 5.1.3 and 5.3.1) on *either* $\mathbb{B}$ *or* $\Sigma[1]|_{\mathbb{B}}$. We also let $\mathbb{B}_{/\sim}$ denote the set of equivalence classes of $\mathbb{B}$ under $\sim$.

Proposition 5.3.11. *Let $k := \#\mathbb{B}_{/\sim}$, and let $\mathbb{B}_{/\sim} = \{\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_k\}$ be a total order on $\mathbb{B}_{/\sim}$, and for each $\ell \in [k]$, let $\mathsf{T}_{\leq \ell} := \bigcup\{\mathsf{T}_j \colon j \leq \ell\}$. Then for any new cone σ in $\Sigma^{\dagger,\mathbb{B}}$, there exists a unique $\ell \in [k]$ such that:*

(i) *σ cannot be inscribed in any cone in $\Sigma^{\dagger,\mathsf{T}_{\leq \ell-1}}$.*

(ii) *σ is a cone in $\Sigma^{\dagger,\mathsf{T}_{\leq \ell}}$.*

Moreover, every extremal ray of σ is adjacent <u>in Σ</u> to some ray in $\Sigma[\mathsf{T}_\ell]$.

Remark 5.3.12. We remind the reader that for any $\ell \in [k]$, $\Sigma^{\dagger,\mathsf{T}_{\leq \ell}}$ is the $\mathsf{T}_{\leq \ell}$-cut of Σ, as in Definition 5.3.3. Note that if $\ell > 1$, $\Sigma^{\dagger,\mathsf{T}_{\leq \ell}}$ is also the T_ℓ-cut of $\Sigma^{\dagger,\mathsf{T}_{\leq \ell-1}}$. This observation will be used for the purposes of induction in the proof below. Finally, note that $\Sigma^{\dagger,\mathsf{T}_{\leq k}}$ is simply $\Sigma^{\dagger,\mathbb{B}}$.

Proof. Proceed by induction on $k = \#\mathbb{B}_{/\sim}$. If $k = 1$, this was Lemma 5.3.9(i). If $k > 1$, we consider two cases:

(a) If σ can be inscribed in some cone in $\Sigma^{\dagger,\mathsf{T}_{\leq k-1}}$, then let σ' be the *smallest* cone in $\Sigma^{\dagger,\mathsf{T}_{\leq k-1}}$ such that $\sigma \sqsubset \sigma'$. Then we claim $\sigma = \sigma'$. Indeed, by Lemma 5.3.9(i), all the extremal rays of σ' are adjacent in Σ to some ray in $\Sigma[1]|_{\mathsf{T}_{\leq k-1}}$, and hence, $\sigma'[1] \cap \Sigma[1]|_{\mathsf{T}_k} = \varnothing$. By Lemma 5.3.8(i), σ' is therefore a cone in $\Sigma^{\dagger,\mathsf{T}_{\leq k}} = \Sigma^{\dagger,\mathbb{B}}$. Given that $\sigma \subset \sigma'$ are both cones in $\Sigma^{\dagger,\mathbb{B}}$, and σ *does not lie in any proper face of* σ' but can be inscribed in σ', we must have $\sigma = \sigma'$, as desired. Therefore, σ was already a new cone in $\Sigma^{\dagger,\mathsf{T}_{\leq k-1}}$, and the proposition follows by induction hypothesis.

(b) Otherwise, only the last sentence of the proposition needs proof. By Lemma 5.3.9(i), every extremal ray of σ is adjacent $\underline{\text{in } \Sigma^{\dagger,\mathsf{T}_{\leq k-1}}}$ to some ray in $\Sigma[1]|_{\mathsf{T}_k}$. Since every ray in $\Sigma[1]|_{\mathsf{T}_k}$ is by definition not adjacent $\underline{\text{in } \Sigma}$ to any ray in $\Sigma[1]|_{\mathsf{T}_{\leq k-1}}$, Lemma 5.3.9(ii) says that every extremal ray of σ is in fact adjacent $\underline{\text{in } \Sigma}$ to some ray in $\Sigma[1]|_{\mathsf{T}_{\leq k-1}}$.

This completes the induction. $\square$

Remark 5.3.13. Given that the total order on $\mathbb{B}_{/\sim}$ plays an auxiliary role in the above proof, the following stronger assertion should be true. Namely, for any new cone σ in $\Sigma^{\dagger}$, there exists a unique $\mathsf{T} \in \mathbb{B}_{/\sim}$ such that σ was already a new cone in $\Sigma^{\dagger,\mathsf{T}}$ (so every extremal ray of σ is adjacent to some ray in $\Sigma[\mathsf{T}]$). However, this stronger assertion is not needed for this chapter.

We conclude this section with one more crucial observation:

Lemma 5.3.14. *For a cone σ in $\overline{\Sigma}^{\dagger}$, the following statements are equivalent:*

(i) *σ is new.*

(ii) $\bigcap\{\tau_\rho\colon \rho \in \sigma[1]\} = \varnothing$.

Moreover, if σ is old and not contained in any coordinate hyperplane $\{\mathbf{e}_i^\vee = 0\}$ in $N_{\mathbf{R}}$, then $\bigcap\{\tau_\rho\colon \rho \in \sigma[1]\}$ is a compact face of Γ_+.

Proof. For (ii) $\implies$ (i), suppose σ is inscribed in a cone σ' in Σ. By Corollary 5.1.16, the face $\varsigma' \prec \Gamma_+$ dual to σ' is $\varsigma' = \bigcap\{\tau_\rho\colon \rho \in \sigma'[1]\}$. Since $\sigma[1] \subset \sigma'[1]$, we have $\varsigma' = \bigcap\big\{\tau_\rho\colon \rho \in \sigma'[1]\big\} \subset \bigcap\big\{\tau_\rho\colon \rho \in \sigma[1]\big\}$, so that in particular, the latter must be non-empty.

For (i) $\implies$ (ii), set $\underline{\varsigma} := \bigcap\{\tau_\rho\colon \rho \in \sigma[1]\}$. If $\underline{\varsigma} \neq \varnothing$, then $\underline{\varsigma}$ is a (non-empty) face of Γ_+. In that case we claim that σ is inscribed in the cone $\overline{\sigma}$ in Σ dual to $\underline{\varsigma} \prec \Gamma_+$, a contradiction. Indeed, letting ς denote the face of $\Gamma_+^\dagger$ dual to $\sigma \in \Sigma^\dagger$, the claim amounts to the following implication for every $\tau \prec^1 \Gamma_+$:

$$\varsigma \prec \tau^\dagger \implies \underline{\varsigma} \prec \tau.$$

That implication follows from $\{\tau^\dagger \prec^1 \Gamma_+^\dagger\colon \varsigma \prec \tau^\dagger\} = \{\tau_\rho^\dagger\colon \rho \in \sigma[1]\}$ (Corollary 5.1.16) and the definition of $\underline{\varsigma}$. Finally, for the last statement, $\overline{\sigma}$ is also not contained in any coordinate hyperplane in $N_{\mathbf{R}}$. By Corollary 5.1.17, $\underline{\varsigma}$ is therefore compact. $\square$

5.3.B. Dropping a set of B_1-facets with consistent base directions. In this section, let $f \in \mathbf{k}[x_1, \ldots, x_n]$ be a non-degenerate polynomial. We specialize the earlier discussion in §5.3.A to the case when Γ_+ is the Newton polyhedron $\Gamma_+(f)$ of f, and $\mathbb{B}$ is a set of B_1-facets of $\Gamma_+(f)$ with consistent base directions, cf. Definition 1.3.10. As in §5.2.A, let $\Sigma(f)$ denote the normal fan of $\Gamma_+(f)$. Before that, we state (without proof) some easy observations:

5.3.15. Suppose $\Gamma_+(f)$ has a B_1-facet τ with apex $\mathbf{v}$ and corresponding base direction $i \in [n]$. Let $J(\tau) := \{j \in [n]\colon \tau \text{ is non-compact in the } j^{\text{th}} \text{ coordinate}\}$ (cf. Corollary 5.1.17), so that by definition $i \notin J(\tau)$. Then:

(i) Let τ^c denote the convex hull of $\mathrm{vert}(\tau) = \mathrm{vert}(H_i \cap \tau) \cup \{\mathbf{v}\}$ in $M^+_{\mathbf{R}}$. Then $\tau = \tau^c + \sum_{j \in J(\tau)} \mathbf{R}_{\geq 0}\mathbf{e}_j$.

(ii) $H_i \cap \tau \prec^1 \tau$.

(iii) τ is not contained in any translate $m\mathbf{e}_k + H_k$ of any coordinate hyperplane H_k in $M_{\mathbf{R}}$.

(iv) The facet τ_i of $\Gamma_+(f)$ dual to the ray $\langle \mathbf{e}_i \rangle$ in $\Sigma(f)$ is $H_i \cap \Gamma_+(f)$. In other words, $N_{\tau_i} = 0$ (recall 5.1.5 for definition of N_{τ_i}).

5.3.16. For a set $\mathbb{B}$ of B_1-facets of $\Gamma_+(f)$, the following are equivalent:

(i) $\mathbb{B}$ is a set of B_1-facets of $\Gamma_+(f)$ with consistent base directions.

(ii) For every $\mathsf{T} \in \mathbb{B}_{/\sim}$, there exists $\mathbf{v} \in \bigcap\{\mathrm{vert}(\tau) \colon \tau \in \mathsf{T}\}$ and $i \in [n]$ such that every τ in T is a B_1-facet with apex $\mathbf{v}$ and corresponding base direction i.

In (ii), we call $\mathbf{v}$ an *apex* of T with corresponding *base direction* $i \in [n]$.

Convention 5.3.17. Let $\Gamma^\dagger_+$ denote the $\mathbb{B}$-cut of $\Gamma_+(f)$, and let $\Sigma^\dagger$ denote its normal fan in $N_{\mathbf{R}}$. We also fix, for each $\mathsf{T} \in \mathbb{B}_{/\sim}$, an apex $\mathbf{v}_{\mathsf{T}}$ of T and denote the corresponding base direction by $b(\mathsf{T})$. For the remainder of this section, we fix a new cone σ in $\Sigma^\dagger$, and let ς denote the face of $\Gamma^\dagger_+$ dual to σ. With respect to an auxiliary total order $\mathbb{B}_{/\sim} = \{\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_k\}$ on $\mathbb{B}_{/\sim}$, let ℓ be the unique natural number in $[k]$ for which σ satisfies the properties stated in Proposition 5.3.11. We then set $\mathsf{T} := \mathsf{T}_\ell$.

Proposition 5.3.18. *For each $\rho \in \sigma[1]$, τ_ρ is adjacent to some facet in T. Moreover:*

(i) *$\langle \mathbf{e}_{b(\mathsf{T})} \rangle$ is an extremal ray of σ.*

(ii) *The cone σ° in $N_{\mathbf{R}}$ generated by the rays in*

$$\sigma[1] \smallsetminus \{\langle \mathbf{e}_{b(\mathsf{T})} \rangle\}$$

is a face of σ (and hence is a cone in $\Sigma^\dagger$) that can be inscribed in the maximal cone in $\Sigma(f)$ dual to the vertex $\mathbf{v}_{\mathsf{T}}$ of $\Gamma_+(f)$.

(iii) *The face*

$$\underline{\varsigma}^{\circ} := \bigcap\big\{\tau_\rho \colon \rho \in \sigma^{\circ}[1]\big\} \prec \Gamma_+(f)$$

has empty intersection with $H_{b(\mathsf{T})}$. *Moreover, for every* $\tau \in \mathsf{T}$, $\underline{\varsigma}^{\circ} \cap \tau$ *is either* $\{\mathbf{v}_{\mathsf{T}}\}$ *or a non-compact face of* τ *containing* $\mathbf{v}_{\mathsf{T}}$.

Proof. The first statement is a restatement of the last property in Proposition 5.3.11. For $\rho \in \sigma[1]$, let τ be a facet in T adjacent to τ_ρ. Then $\tau \cap \tau_\rho$ is a facet of τ, and if $\rho \neq \langle \mathbf{e}_{b(\mathsf{T})}\rangle$, $\tau \cap \tau_\rho$ cannot be equal to $H_{b(\mathsf{T})} \cap \tau$, and hence must contain $\mathbf{v}_{\mathsf{T}}$ (cf. 5.3.15(i)). In particular, $\mathbf{v}_{\mathsf{T}} \in \tau_\rho$. Therefore, we deduce that

$$\mathbf{v}_{\mathsf{T}} \in \bigcap\big\{\tau_\rho \colon \rho \in \sigma[1] \smallsetminus \{\langle \mathbf{e}_{b(\mathsf{T})}\rangle\}\big\}. \tag{5.15}$$

If $\langle \mathbf{e}_{b(\mathsf{T})}\rangle \notin \sigma[1]$, then (5.15) becomes $\mathbf{v}_{\mathsf{T}} \in \bigcap\big\{\tau_\rho \colon \rho \in \sigma[1]\big\}$, which contradicts Lemma 5.3.14. This proves (i).

For (ii), (5.15) already shows that σ° can be inscribed in the cone in $\Sigma(f)$ dual to the vertex $\mathbf{v}_{\mathsf{T}}$ of $\Gamma_+(f)$. It remains to show σ° is a cone in $\Sigma^{\dagger}$. More precisely, we show σ° is dual to the face

$$\varsigma^{\circ} := \bigcap\big\{\tau_\rho^{\dagger} \colon \rho \in \sigma^{\circ}[1]\big\} \prec \Gamma_+^{\dagger}. \tag{5.16}$$

By Corollary 5.1.16, this amounts to showing that $\big\{\tau_\rho^{\dagger} \colon \rho \in \sigma^{\circ}[1]\big\}$ are the only facets $\tau^{\dagger} \prec^1 \Gamma_+^{\dagger}$ containing ς°. Indeed, any facet $\tau^{\dagger} \prec^1 \Gamma_+^{\dagger}$ containing ς° must also contain the face $\varsigma \prec \Gamma_+^{\dagger}$ dual to $\sigma \in \Sigma^{\dagger}$, and hence, must be dual to an extremal ray ρ in $\sigma[1]$. It remains to observe that that ρ cannot be $\langle \mathbf{e}_{b(\mathsf{T})}\rangle$, since $\mathbf{v}_{\mathsf{T}} \in \varsigma^{\circ}$ (5.15) but $\mathbf{v}_{\mathsf{T}} \notin H_{b(\mathsf{T})} \cap \Gamma_+(f)$.

Finally, we prove (iii). By Lemma 5.3.14, we obtain:

$$\varnothing = \bigcap\big\{\tau_\rho \colon \rho \in \sigma[1]\big\} = \big(H_{b(\mathsf{T})} \cap \Gamma_+(f)\big) \cap \bigcap\big\{\tau_\rho \colon \rho \in \sigma^{\circ}[1]\big\}$$

$$= H_{b(\mathsf{T})} \cap \underline{\varsigma}^{\circ}.$$

In particular, for every $\tau \in \mathsf{T}$, $\underline{\varsigma}^{\circ} \cap \tau$ is a face of τ that does not intersect the facet $H_{b(\mathsf{T})} \cap \tau \prec^1 \tau$. By (5.15), $\underline{\varsigma}^{\circ} \cap \tau$ also contains $\mathbf{v}_{\mathsf{T}}$. Since the only compact face of τ satisfying those two conditions is $\{\mathbf{v}_{\mathsf{T}}\}$ (cf. 5.3.15(i)), this proves (iii). □

As an immediate consequence of the preceding proposition, we have:

Corollary 5.3.19. *Every* $\mathbf{a} \in \underline{\varsigma}^{\circ}$ *has* $b(\mathsf{T})^{th}$ *coordinate* ≥ 1.

Proof. Since $\underline{\varsigma}^{\circ} \cap H_{b(\mathsf{T})} = \varnothing$, all vertices of $\underline{\varsigma}^{\circ}$ have $b(\mathsf{T})^{\text{th}}$ coordinate > 0. On the other hand, since $\Gamma_+(f)$ is a Newton polyhedron, all vertices of $\underline{\varsigma}^{\circ}$ have integer coordinates, and hence, must have $b(\mathsf{T})^{\text{th}}$ coordinate ≥ 1. □

For later purposes, the preceding corollary is however not sufficient. We instead need the following refinement:

Proposition 5.3.20. *If the face* $\varsigma \prec \Gamma_+^{\dagger}$ *dual to* σ *is compact, then every* $\mathbf{a} \in \underline{\varsigma}^{\circ} \smallsetminus \{\mathbf{v}_{\mathsf{T}}\}$ *has* $b(\mathsf{T})^{th}$ *coordinate* > 1.

5.3.21. We prove the preceding proposition after a few observations and results. For the remainder of this section, let ς° denote the face of $\Gamma_+^{\dagger}$ dual to σ°, cf. (5.16). By Corollary 5.1.16, we have:

$$\begin{aligned} \varsigma &= \bigcap \{\tau_\rho^{\dagger} \colon \rho \in \sigma[1]\} = \left(H_{b(\mathsf{T})} \cap \Gamma_+^{\dagger}\right) \cap \bigcap \{\tau_\rho^{\dagger} \colon \rho \in \sigma^{\circ}[1]\} \\ &= H_{b(\mathsf{T})} \cap \varsigma^{\circ} \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} \underline{\varsigma}^{\circ} &= \bigcap \{\tau_\rho \colon \rho \in \sigma^{\circ}[1]\} = \bigcap \{\tau_\rho^{\dagger} \cap \Gamma_+(f) \colon \rho \in \sigma^{\circ}[1]\} \\ &= \varsigma^{\circ} \cap \Gamma_+(f). \end{aligned} \tag{5.18}$$

From these equalities we deduce the next lemma. In particular, note that part (ii) of the next lemma refines Proposition 5.3.18(iii).

Lemma 5.3.22. *If ς is compact, then:*

(i) *Both ς° and $\underline{\varsigma}^\circ$ are either non-compact in the $b(\mathsf{T})^{th}$ coordinate, or compact.*

(ii) *For any $\tau \in \mathsf{T}$, we have $\varsigma^\circ \cap \tau = \underline{\varsigma}^\circ \cap \tau = \{\mathbf{v}_\mathsf{T}\}$.*

Proof. (i) follows from (5.17) and (5.18), since $H_{b(\mathsf{T})}$ is non-compact in the i^{th} coordinate for $i \in [n] \smallsetminus \{b(\mathsf{T})\}$. For (ii), we note, from (i) and the fact that any $\tau \in \mathsf{T}$ cannot be non-compact in the $b(\mathsf{T})^{\text{th}}$ coordinate (Definition 1.3.8(ii)), that $\underline{\varsigma}^\circ \cap \tau$ is a compact face of τ, and hence is $\{\mathbf{v}_\mathsf{T}\}$ by Proposition 5.3.18(iii). Note finally that $\varsigma^\circ \cap \tau = \underline{\varsigma}^\circ \cap \tau$ by (5.18). $\square$

Proposition 5.3.23. *If ς is compact, then $\underline{\varsigma}^\circ$ is either $\{\mathbf{v}_\mathsf{T}\}$ or 1-dimensional. In the latter case, the affine span of $\underline{\varsigma}^\circ$ contains $\mathbf{v}_\mathsf{T}$, and intersects $H_{b(\mathsf{T})}$ at a point.*

Proof. By Lemma 5.3.22(ii), we have:

$$\varsigma^\circ \cap \bigcup\{\tau \colon \tau \in \mathsf{T}\} = \{\mathbf{v}_\mathsf{T}\}. \tag{5.19}$$

To exploit the above equation, we consider the $(\mathbb{B} \smallsetminus \mathsf{T})$-cut of $\Gamma_+(f)$, i.e.

$$\Gamma_+^\ddagger = \Gamma_+^\dagger \cap \bigcap_{\tau \in \mathsf{T}} H^+_{\mathbf{u}_\tau, N_\tau} \subset \Gamma_+^\dagger \tag{5.20}$$

and let $\Sigma^\ddagger$ be its normal fan in $N_\mathbf{R}$. For $\rho \in \Sigma^\ddagger[1] = \Sigma(f)[1] \smallsetminus \Sigma[1]|_{\mathbb{B}\smallsetminus\mathsf{T}}$, we also let $\tau_\rho^\ddagger$ denote the facet of $\Gamma_+^\ddagger$ dual to ρ. We make a few important observations:

(a) Firstly, by Lemma 5.3.4(ii), each $\tau \in \mathsf{T}$ is still a facet of $\Gamma_+^\ddagger$. That is, for $\rho \in \Sigma[\mathsf{T}]$, $\tau_\rho^\ddagger = \tau_\rho$.

(b) Secondly, by replacing $\mathbb{B}$ by $\mathbb{B} \smallsetminus \bigcup\{\mathsf{T}_j \colon j > \ell\}$ (recall from Convention 5.3.17 that $\mathbb{B}_{/\sim} = \{\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_k\}$ with $\mathsf{T} = \mathsf{T}_\ell$), we may assume that σ cannot be inscribed in any cone in $\Sigma^{\ddagger}$, cf. Proposition 5.3.11. We then have:

$$\begin{aligned}
\varnothing &= \bigcap\{\tau_\rho^{\ddagger}\colon \rho\in\sigma[1]\} && \text{by Lemma 5.3.14}\\
&= \left(H_{b(\mathsf{T})}\cap\Gamma_+^{\ddagger}\right)\cap\bigcap\{\tau_\rho^{\ddagger}\colon \rho\in\sigma^\circ[1]\}\\
&= \left(H_{b(\mathsf{T})}\cap\Gamma_+^{\ddagger}\right)\cap\bigcap\{\tau_\rho^{\dagger}\cap\Gamma_+^{\ddagger}\colon \rho\in\sigma^\circ[1]\} && \text{by Lemma 5.3.4(i)}\\
&= \Gamma_+^{\ddagger}\cap\left(H_{b(\mathsf{T})}\cap\Gamma_+^{\dagger}\right)\cap\bigcap\{\tau_\rho^{\dagger}\colon \rho\in\sigma^\circ[1]\}\\
&= \Gamma_+^{\ddagger}\cap\left(H_{b(\mathsf{T})}\cap\Gamma_+^{\dagger}\right)\cap\varsigma^\circ \overset{(5.17)}{=\!=} \varsigma\cap\Gamma_+^{\ddagger}
\end{aligned}$$

i.e. $\varsigma\subset\varsigma^\circ\smallsetminus\Gamma_+^{\ddagger}$. In particular, $\varsigma^\circ\smallsetminus\Gamma_+^{\ddagger}\neq\varnothing$. We also note that

$$\underline{\varsigma}^\circ \overset{(5.18)}{=\!=} \varsigma^\circ\cap\Gamma_+(f)\subset\varsigma^\circ\cap\Gamma_+^{\ddagger}$$

i.e. in particular, $\varsigma^\circ\cap\Gamma_+^{\ddagger}$ is a (non-empty) face of $\Gamma_+^{\ddagger}$.

(c) Thirdly, by (5.20), any line segment connecting a point in $\Gamma_+^{\dagger}\smallsetminus\Gamma_+^{\ddagger}$ to a point in $\Gamma_+^{\ddagger}$ must pass through a point in

$$\bigcup\{\Gamma_+^{\ddagger}\cap H_{\mathbf{u}_\tau,N_\tau}\colon \tau\in\mathsf{T}\}\overset{\text{(a)}}{=\!=}\bigcup\{\tau\colon \tau\in\mathsf{T}\}.$$

By (5.19), we therefore deduce that any line segment connecting a point in $\varsigma^\circ\smallsetminus\Gamma_+^{\ddagger}$ to a point in $\varsigma^\circ\cap\Gamma_+^{\ddagger}\prec\Gamma_+^{\ddagger}$ must pass through $\mathbf{v}_{\mathsf{T}}$.

We can now conclude the proof by considering two cases.

Case 1: Suppose that $\mathbf{v}_{\mathsf{T}}$ is always one of the two vertices of *every* line segment connecting a point in $\varsigma^\circ\smallsetminus\Gamma_+^{\ddagger}$ to a point in $\varsigma^\circ\cap\Gamma_+^{\ddagger}$. Then we claim $\varsigma^\circ\cap\Gamma_+^{\ddagger}=\{\mathbf{v}_{\mathsf{T}}\}$. If not, choose a point $\mathbf{a}_1\in(\varsigma^\circ\cap\Gamma_+^{\ddagger})\smallsetminus\{\mathbf{v}_{\mathsf{T}}\}$. By (b), we may also choose a point $\mathbf{a}_2$ in $\varsigma^\circ\smallsetminus\Gamma_+^{\ddagger}$. By (c), the line segment connecting $\mathbf{a}_1$ to $\mathbf{a}_2$ must contain $\mathbf{v}_{\mathsf{T}}$ in its relative interior, contradicting the hypothesis of this case. From our claim we obtain:

$$\{\mathbf{v}_{\mathsf{T}}\} = \varsigma^\circ \cap \Gamma_+^\ddagger \supset \varsigma^\circ \cap \Gamma_+(f) \overset{(5.18)}{=\!=} \underline{\varsigma}^\circ \supset \{\mathbf{v}_{\mathsf{T}}\}$$

which forces $\underline{\varsigma}^\circ = \{\mathbf{v}_{\mathsf{T}}\}$.

Case 2: Suppose *there exists* a line segment $\mathfrak{l}$ connecting some $\mathbf{a}_1 \in \varsigma^\circ \setminus \Gamma_+^\ddagger$ to some $\mathbf{a}_2 \in \varsigma^\circ \cap \Gamma_+^\ddagger$ that contains $\mathbf{v}_{\mathsf{T}}$ in its relative interior. In particular, note $\mathbf{a}_1 \neq \mathbf{v}_{\mathsf{T}} \neq \mathbf{a}_2$, so that $\dim(\varsigma^\circ) \geq \dim(\varsigma^\circ \cap \Gamma_+^\ddagger) \geq 1$. We claim that in fact

$$\dim(\varsigma^\circ) = \dim(\varsigma^\circ \cap \Gamma_+^\ddagger) = 1.$$

Indeed, given any $\mathbf{a}_1' \in \varsigma^\circ \setminus \Gamma_+^\ddagger$, (c) implies that the line segment connecting $\mathbf{a}_1'$ to $\mathbf{a}_2$ must contain $\mathbf{v}_{\mathsf{T}}$, and thus $\mathbf{a}_1'$ must lie on the affine span of $\mathfrak{l}$. Likewise, given any $\mathbf{a}_2' \in \varsigma^\circ \cap \Gamma_+^\ddagger$, the line segment connecting $\mathbf{a}_1$ to $\mathbf{a}_2'$ must contain $\mathbf{v}_{\mathsf{T}}$, and thus $\mathbf{a}_2'$ must lie on the affine span of $\mathfrak{l}$.

Finally, by (5.18), $\underline{\varsigma}^\circ = \varsigma^\circ \cap \Gamma_+(f) \subset \varsigma^\circ \cap \Gamma_+^\ddagger$, so $\dim(\underline{\varsigma}^\circ) \leq \dim(\varsigma^\circ \cap \Gamma_+^\ddagger) = 1$. Since $\underline{\varsigma}^\circ$ always contains $\mathbf{v}_{\mathsf{T}}$, we conclude that $\underline{\varsigma}^\circ$ is either $\{\mathbf{v}_{\mathsf{T}}\}$ or 1-dimensional.

Together these two cases prove the first statement of the proposition. For the second statement, first note that $\dim(\underline{\varsigma}^\circ) = 1$ only occurs in Case 2. In that case, we also have $\dim(\varsigma^\circ) = 1$ and $\varsigma^\circ \cap \Gamma_+(f) = \underline{\varsigma}^\circ$, so the affine span of $\underline{\varsigma}^\circ$ must be equal to the affine span of ς°. By (5.17), ς° has non-empty intersection with $H_{b(\mathsf{T})}$ (namely, the face $\varsigma \prec \Gamma_+^\dagger$). That intersection must be a point since $\mathbf{v}_{\mathsf{T}} \in \underline{\varsigma}^\circ \subset \varsigma^\circ$ has $b(\mathsf{T})^{\text{th}}$ coordinate 1. $\square$

Remark 5.3.24. From the proof above, one may supplement Proposition 5.3.23 as follows. If $\dim(\underline{\varsigma}^\circ) = 1$, then $\dim(\varsigma^\circ) = 1$ and $\dim(\varsigma) = 0$, i.e. $\sigma \in \Sigma^\dagger[\max]$. Note however that if $\underline{\varsigma}^\circ = \{\mathbf{v}_{\mathsf{T}}\}$, $\dim(\varsigma^\circ)$ and $\dim(\varsigma)$ are arbitrary.

Proof of Proposition 5.3.20. We saw that $\underline{\varsigma}^\circ$ is either $\{\mathbf{v}_\mathsf{T}\}$ or 1-dimensional. There is nothing to show in the former case. In the latter case, we saw that $\mathbf{v}_\mathsf{T}$ is the *only* point in $\underline{\varsigma}^\circ$ with $b(\mathsf{T})^{\text{th}}$ coordinate 1. Combining this with Corollary 5.3.19 finishes the proof. □

5.3.C. A refined desingularization of non-degenerate polynomials above the origin. In this section, let $f \in \mathbf{k}[x_1, \ldots, x_n]$ be a non-degenerate polynomial, and we continue adopting the conventions outlined at the start of §5.3.B and in Convention 5.3.17. We show next that the following multi-weighted blow-up of $\mathbf{A}^n$:

$$\vartheta_{\Sigma^\dagger} \colon \mathscr{X}_{\Sigma^\dagger} \to \mathbf{A}^n$$

supplies a stack-theoretic embedded desingularization of $V(f) \subset \mathbf{A}^n$ above the origin $\mathbf{0} \in \mathbf{A}^n$ (Definition 1.3.15). Let us first make this goal concrete.

5.3.25. For the remainder of this section, we write

$$f = \sum_{\mathbf{a} \in \mathbf{N}^n} c_{\mathbf{a}} \cdot x^{\mathbf{a}} \in \mathbf{k}[x_1, \ldots, x_n]$$

where $c_{\mathbf{0}} = f(\mathbf{0}) = 0$, and adopt the notations in 4.1.9 (but with Σ there replaced by $\Sigma^\dagger$ here). By 4.1.9(i), the total transform of f under $\vartheta_{\Sigma^\dagger}$ is:

$$\vartheta_{\Sigma^\dagger}^{\#}(f) = \sum_{\mathbf{a} \in \mathbf{N}^n} c_{\mathbf{a}} \cdot (x')^{\mathbf{a}} \cdot \prod_{\rho \in \Sigma^\dagger[\mathrm{ex}]} (x'_\rho)^{\mathbf{a} \cdot \mathbf{u}_\rho}$$

where for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{N}^n$, $(x')^{\mathbf{a}} := (x'_1)^{a_1} \cdots (x'_n)^{a_n}$. Next, for each $\rho \in \Sigma^\dagger[1] = [n] \sqcup \Sigma^\dagger[\mathrm{ex}]$ (cf. Convention 4.1.8), we set:

$$N_\rho := N_{\tau_\rho} = \inf_{\mathbf{a} \in \Gamma_+(f)} \mathbf{a} \cdot \mathbf{u}_\rho = \inf_{\mathbf{a} \in \Gamma_+^\dagger} \mathbf{a} \cdot \mathbf{u}_\rho \tag{5.21}$$

cf. Conventions 1.0.1 and 5.1.3, as well as 5.1.5 and 5.3.6. In the same way as 5.2.2, we define the proper transform of f under $\vartheta_{\Sigma^\dagger}$ as:

$$f' := \frac{\Pi^{\#}_{\Sigma^\dagger}(f)}{\prod_{\rho\in\Sigma^\dagger[1]}(x'_\rho)^{N_\rho}} = \sum_{\mathbf{a}\in\mathbf{N}^n} c_{\mathbf{a}}\cdot(x')^{\mathbf{a}-\mathbf{n}}\cdot\prod_{\rho\in\Sigma^\dagger[\mathrm{ex}]}(x'_\rho)^{\mathbf{a}\cdot\mathbf{u}_\rho - N_\rho} \tag{5.22}$$

where $\mathbf{n} := \big(N_i \colon i\in[n]\big)$. We can now state our goal more precisely in the following theorem:

Theorem 5.3.26. *At points in $\vartheta^{-1}_{\Sigma^\dagger}(\mathbf{0})\subset\mathscr{X}_{\Sigma^\dagger}$, the divisor*

$$V(f')\subset\mathscr{X}_{\Sigma^\dagger}$$

is smooth and intersects the divisors $\big\{V(x'_\rho)\subset\mathscr{X}_{\Sigma^\dagger}\colon \rho\in\Sigma^\dagger[1],\ N_\rho>0\big\}$ transversely. In other words,

$$\vartheta^{-1}_{\Sigma^\dagger}\big(V(f)\big)\subset\mathscr{X}_{\Sigma^\dagger}$$

is a simple normal crossings divisor at every point in $\vartheta^{-1}_{\Sigma}(\mathbf{0})\subset\mathscr{X}_{\Sigma^\dagger}$.

Proof. We prove this theorem in steps.

5.3.27. Let $\overline{\Sigma}^\dagger$ be the augmentation of $\Sigma^\dagger$. For an arbitrary cone σ in $\overline{\Sigma}^\dagger$, we will need a simplified presentation for the σ-chart $D_+(\sigma)$ of $\mathscr{X}_{\Sigma^\dagger}$. Let us first recall from 4.1.9(iii) that

$$D_+(\sigma) = \Big[\mathrm{Spec}\left(\mathbf{k}[x'_1,\dots,x'_n]\big[x'_\rho\colon\rho\in\Sigma^\dagger[\mathrm{ex}]\big]\big[x'^{-1}_\sigma\big]\right)\ /\ \mathbb{G}_m^{\Sigma^\dagger[\mathrm{ex}]}\Big]$$

where $x'_\sigma = \prod_{\rho\in\Sigma^\dagger[1]\smallsetminus\sigma[1]} x'_\rho$. Since x'_ρ is invertible on $D_+(\sigma)$ for $\rho\in\Sigma^\dagger[\mathrm{ex}]\smallsetminus\sigma[1]$, and their $\mathbf{Z}^{\Sigma^\dagger[\mathrm{ex}]}$-weights $\{-\mathbf{e}_\rho\colon\rho\in\Sigma^\dagger[\mathrm{ex}]\smallsetminus\sigma[1]\}$ are linearly independent over $\mathbf{Z}$ (4.1.9(ii)), we observe from Lemma 2.1.2 that by setting

$$x'_\rho = 1 \qquad \text{for every } \rho\in\Sigma^\dagger[\mathrm{ex}]\smallsetminus\sigma[1]$$

we obtain an isomorphism:

$$D_+(\sigma) = \left[\operatorname{Spec}\left(\mathbf{k}[x'_1, \ldots, x'_n]\left[x'_\rho \colon \rho \in \sigma[\mathrm{ex}]\right]\left[x'^{-1}_\sigma\right]\right) \ / \ \mathbb{G}_m^{\sigma[\mathrm{ex}]}\right] \tag{5.23}$$

where:

(i) $\sigma[\mathrm{ex}] := \Sigma^\dagger[\mathrm{ex}] \cap \sigma[1]$.

(ii) x'_σ becomes $\prod_{i \in [n] \smallsetminus \sigma[1]} x'_i$.

(iii) The action $\mathbb{G}_m^{\sigma[\mathrm{ex}]} \curvearrowright \operatorname{Spec}(\mathbf{k}[x'_1, \ldots, x'_n][x'_\rho \colon \rho \in \sigma[\mathrm{ex}][x'^{-1}_\sigma])$ is specified as follows. For each $i \in [n]$, the $\mathbf{Z}^{\sigma[\mathrm{ex}]}$-weight of x'_i is $(u_{\rho,i})_{\rho \in \sigma[\mathrm{ex}]}$, and for each $\rho \in \sigma[\mathrm{ex}]$, the $\mathbf{Z}^{\sigma[\mathrm{ex}]}$-weight of x'_ρ is $-\mathbf{e}_\rho \in \mathbf{Z}^{\sigma[\mathrm{ex}]}$.

On the right hand side of (5.23), the expression for the proper transform f' of f under $\vartheta_{\Sigma^\dagger}$ becomes:

$$f' = \sum_{\mathbf{a} \in \mathbf{N}^n} c_{\mathbf{a}} \cdot (x')^{\mathbf{a} - \mathbf{n}} \cdot \prod_{\rho \in \sigma[\mathrm{ex}]} (x'_\rho)^{\mathbf{a} \cdot \mathbf{u}_\rho - N_\rho}. \tag{5.24}$$

5.3.28. For a cone σ in $\overline{\Sigma}^\dagger$, we deduce from (5.23) an expression for the $(\mathbb{G}_m^{\Sigma^\dagger[1]} \ / \ \mathbb{G}_m^{\Sigma^\dagger[\mathrm{ex}]})$-orbit $O(\sigma)$ of $\mathscr{X}_{\Sigma^\dagger}$ corresponding to σ, cf. 4.1.9(iv):

$$\begin{aligned} O(\sigma) &= \left[\operatorname{Spec}\left(\mathbf{k}\left[x_i^\pm \colon i \in [n] \smallsetminus \sigma[1]\right]\right) \ / \ \mathbb{G}_m^{\sigma[\mathrm{ex}]}\right] \\ &= V\left(x'_\rho \colon \rho \in \sigma[1]\right) \xhookrightarrow{\text{closed}} D_+(\sigma). \end{aligned} \tag{5.25}$$

For $\sigma \in \overline{\Sigma}^\dagger$ not contained in any coordinate hyperplane $\{\mathbf{e}_i^\vee = 0\}$ in $N_{\mathbf{R}}$, we claim that at every point in $O(\sigma)$, the divisor $V(f') \subset \mathscr{X}_{\Sigma^\dagger}$ is smooth and intersects the divisors in $\{V(x'_\rho) \subset \mathscr{X}_{\Sigma^\dagger} \colon \rho \in \sigma[1],\ N_\rho > 0\}$ transversely. By Corollary 5.1.20, this claim proves the theorem. We consider two cases.

5.3.29 (*Case A*). Assume that σ is old. Using the simplified expression for $D_+(\sigma)$ in (5.23) and the corresponding expression for f' in (5.24), we claim:

$$
\begin{aligned}
f'|_{V(x'_\rho\colon \rho\in\sigma[1])} &= \sum_{\mathbf{a}\in\mathbf{N}^n\cap\bigcap\{\tau_\rho\colon \rho\in\sigma[1]\}} c_{\mathbf{a}}\cdot (x')^{\mathbf{a}-\mathbf{n}} \\
&= \sum_{\mathbf{a}\in\mathbf{N}^n\cap\bigcap\{\tau_\rho\colon \rho\in\sigma[1]\}} c_{\mathbf{a}}\cdot \prod_{i\in[n]\smallsetminus\sigma[1]} (x'_i)^{a_i-N_i}.
\end{aligned}
\tag{5.26}
$$

Indeed, the only $\mathbf{a}\in\mathbf{N}^n$, whose corresponding monomial

$$
(x')^{\mathbf{a}-\mathbf{n}}\cdot\prod_{\rho\in\sigma[\mathrm{ex}]}(x'_\rho)^{\mathbf{a}\cdot\mathbf{u}_\rho-N_\rho}
$$

in f' remains non-zero after setting $x'_\rho=0$ for all $\rho\in\sigma[1]$, must satisfy:

(i) $\mathbf{a}\cdot\mathbf{u}_\rho=N_\rho$ for every $\rho\in\sigma[\mathrm{ex}]$, i.e. $\mathbf{a}\in\tau_\rho$ for every $\rho\in\sigma[\mathrm{ex}]$;

(ii) $\mathbf{a}\cdot\mathbf{e}_i=a_i=N_i$ for every $i\in[n]\cap\sigma[1]$, i.e. $\mathbf{a}\in\tau_i$ for every $i\in[n]\cap\sigma[1]$.

Next, since σ is old, we know from 5.1.21 and Lemma 5.3.14 that $\bigcap\{\tau_\rho\colon\rho\in\sigma[1]\}$ is a (non-empty) compact face $\underline{\varsigma}\prec\Gamma_+$. Then the expression for $f'|_{V(x'_\rho\colon \rho\in\sigma[1])}$ in (5.26) matches the expression for $f_{\underline{\varsigma}}/x^{\mathbf{n}}$ (1.2), after replacing x'_i in the former with x_i for each $i\in[n]\smallsetminus\sigma[1]$. By the non-degeneracy assumption on f, $f_{\underline{\varsigma}}/x^{\mathbf{n}}$ is smooth on the torus $\mathbb{G}_m^n\subset\mathbf{A}^n$, which implies that

$$
V\big(f'|_{V(x'_\rho\colon \rho\in\sigma[1])}\big)\subset O(\sigma)
$$

is smooth, i.e. at every point in $O(\sigma)\subset\mathscr{X}_{\Sigma^\dagger}$, the divisor $V(f')\subset\mathscr{X}_{\Sigma^\dagger}$ is smooth and intersects the divisors in $\{V(x'_\rho)\subset\mathscr{X}_{\Sigma^\dagger}\colon\rho\in\sigma[1]\}$ transversely.

5.3.30 (*Case B*). Assume that σ is new. Let σ' be the *smallest* cone in $\Sigma^\dagger$ such that $\sigma\sqsubset\sigma'$. With respect to σ', we fix, as in Convention 5.3.17, a corresponding $\mathsf{T}\in\mathbb{B}_{/\sim}$ with apex $\mathbf{v}_\mathsf{T}$ and base direction $b(\mathsf{T})$, such that all the hypotheses, observations and results in §5.3.B hold. In particular, $\mathbf{R}_{\geq 0}\mathbf{e}_{b(\mathsf{T})}$ must be an extremal ray of σ, or else σ is old by Proposition 5.3.18(ii).

Letting σ° be the cone in $N_{\mathbf{R}}$ generated by the rays in $\sigma[1] \smallsetminus \{\langle \mathbf{e}_{b(\mathsf{T})} \rangle\}$, we consider the following factorization of (5.25):

$$
\begin{aligned}
O(\sigma) = V\big(x'_\rho \colon \rho \in \sigma[1]\big) &= \big[\mathrm{Spec}\big(\mathbf{k}\big[x_i^{\pm} \colon i \in [n] \smallsetminus \sigma[1]\big]\big) \ / \ \mathbb{G}_m^{\sigma[\mathrm{ex}]}\big] \\
&\hookrightarrow V\big(x'_\rho \colon \rho \in \sigma^\circ[1]\big) = \big[\mathrm{Spec}\big(\mathbf{k}[x'_{b(\mathsf{T})}]\big[x_i^{\pm} \colon i \in [n] \smallsetminus \sigma[1]\big]\big) \ / \ \mathbb{G}_m^{\sigma[\mathrm{ex}]}\big] \\
&\hookrightarrow D_+(\sigma)
\end{aligned}
$$

where the expression for $V(x'_\rho \colon \rho \in \sigma^\circ[1])$ is similarly deduced from (5.23). Next, set $\underline{\varsigma}^\circ := \bigcap\{\tau_\rho \colon \rho \in \sigma^\circ[1]\}$. Similar to Case A, we have:

$$
\begin{aligned}
f'|_{V(x'_\rho \colon \rho \in \sigma^\circ[1])} &= \sum_{\mathbf{a} \in \mathbf{N}^n \cap \underline{\varsigma}^\circ} c_{\mathbf{a}} \cdot (x')^{\mathbf{a}-\mathbf{n}} \\
&= \sum_{\mathbf{a} \in \mathbf{N}^n \cap \underline{\varsigma}^\circ} c_{\mathbf{a}} \cdot (x'_{b(\mathsf{T})})^{a_{b(\mathsf{T})}} \cdot \prod_{i \in [n] \smallsetminus \sigma[1]} (x'_i)^{a_i - N_i}.
\end{aligned} \tag{5.27}
$$

(Recall that $N_{b(\mathsf{T})} = 0$, cf. 5.3.15(iii).) We now claim that there exists $g \in \mathbf{k}[x'_{b(\mathsf{T})}]\big[x'_i \colon i \in [n] \smallsetminus \sigma[1]\big]$ such that

$$
f'|_{V(x'_\rho \colon \rho \in \sigma^\circ[1])} = c_{\mathbf{v}_\mathsf{T}} \cdot x'_{b(\mathsf{T})} \cdot \prod_{i \in [n] \smallsetminus \sigma[1]} (x'_i)^{v_i - N_i} + (x'_{b(\mathsf{T})})^2 \cdot g \tag{5.28}
$$

where each v_i is the i^{th} coordinate of $\mathbf{v}_\mathsf{T}$. The general case can be reduced to the aforementioned case where $\sigma \in \Sigma^\dagger$. This reduction is standard (similar to 5.1.21), so we have chosen to explicate this separately in Remark 5.3.31 below. Consequently, we deduce from (5.28) that

$$
\frac{\partial f'|_{V(x'_\rho \colon \rho \in \sigma^\circ[1])}}{\partial x'_{b(\mathsf{T})}} \Big|_{V(x'_{b(\mathsf{T})})} = c_{\mathbf{v}_\mathsf{T}} \cdot \prod_{i \in [n] \smallsetminus \sigma[1]} (x'_i)^{v_i - N_i}
$$

which is a unit on $O(\sigma) = V(x'_\rho \colon \rho \in \sigma^\circ[1]) \cap V(x'_{b(\mathsf{T})})$, since $c_{\mathbf{v}_\mathsf{T}} \neq 0$ ($\mathbf{v}_\mathsf{T}$ is a vertex of $\Gamma_+(f)$) and x'_i is invertible on $O(\sigma)$ for each $i \in [n] \smallsetminus \sigma[1]$ (5.25). Thus, $V\big(f'|_{V(x'_\rho \colon \rho \in \sigma^\circ[1])}\big)$ is smooth at every point in $O(\sigma) \subset V\big(x'_\rho \colon \rho \in \sigma^\circ[1]\big)$, i.e. at every point in $O(\sigma) \subset \mathscr{X}_{\Sigma^\dagger}$, the divisor

$V(f') \subset \mathscr{X}_{\Sigma^\dagger}$ is smooth and intersects the divisors in

$$\left\{V(x'_\rho) \subset \mathscr{X}_{\Sigma^\dagger} \colon \rho \in \sigma[1],\ N_\rho > 0\right\} \subset \left\{V(x'_\rho) \subset \mathscr{X}_{\Sigma^\dagger} \colon \rho \in \sigma^\circ[1]\right\}$$

transversely. This completes the proof.

$\square$

Remark 5.3.31. In this remark, we prove (5.28) for all cones $\sigma \in \overline{\Sigma}^\dagger$. Retain the notation in the above proof. Letting $(\sigma')^\circ$ be the cone in $N_{\mathbf{R}}$ generated by the rays in $\sigma'[1] \smallsetminus \{\langle \mathbf{e}_{b(\mathsf{T})} \rangle\}$ (as in Proposition 5.3.18(ii)), we have $\sigma^\circ \sqsubset (\sigma')^\circ$. In fact, $(\sigma')^\circ$ is also the *smallest* cone in $\Sigma^\dagger$ such that $\sigma^\circ \sqsubset (\sigma')^\circ$. If not, σ° lies in a proper face of $(\sigma')^\circ$. Since $(\sigma')^\circ \prec \sigma'$ (Proposition 5.3.18(ii)), $\sigma = \sigma^\circ + \langle \mathbf{e}_{b(\mathsf{T})} \rangle$ must also lie in a proper face of $\sigma' = (\sigma')^\circ + \langle \mathbf{e}_{b(\mathsf{T})} \rangle$, contradicting our choice of σ'. Consequently,

$$\bigcap \left\{\tau^\dagger_\rho \colon \rho \in \sigma^\circ[1]\right\} \ = \ \bigcap \left\{\tau^\dagger_\rho \colon \rho \in (\sigma')^\circ[1]\right\}$$

cf. 5.1.13 and Lemma 5.1.16. Intersecting both sides of the above equality by $\Gamma_+(f)$, we obtain $\underline{\varsigma}^\circ = \bigcap\{\tau_\rho \colon \rho \in (\sigma')^\circ[1]\}$. Then (5.28) follows from the preceding sentence together with (5.27), Proposition 5.3.20, and 5.1.21.

We conclude this section by proving the main theorems of this chapter:

Proof of Theorems H and I. After replacing $\Sigma(f)$ with $\Sigma^\dagger$, the argument in 5.2.8 works verbatim. Fixing a frugal simplicial subdivision $\boldsymbol{\Sigma}^\dagger$ of $\Sigma^\dagger$ (5.2.4), we have:

$$\begin{tikzcd}
\vartheta^{-1}_{\boldsymbol{\Sigma}^\dagger}\big(V(f)\big) \arrow[r, hook, "\text{closed}"] \arrow[d, two heads, "\text{coarse space}"'] & \mathscr{X}_{\boldsymbol{\Sigma}^\dagger} \arrow[r, hook, "\text{open}"] \arrow[d, two heads, "\text{coarse space}"'] & \mathscr{X}_{\Sigma^\dagger} \arrow[r, "\vartheta_{\Sigma^\dagger}"] & \mathbf{A}^n \\
\pi^{-1}_{\boldsymbol{\Sigma}^\dagger}\big(V(f)\big) \arrow[r, hook, "\text{closed}"] & X_{\boldsymbol{\Sigma}^\dagger} \arrow[urr, bend right, "\pi_{\Sigma^\dagger}"'] & &
\end{tikzcd}$$

where:

(i) $\pi_{\boldsymbol{\Sigma}^\dagger}$ is proper and birational.

(ii) $X_{\mathbf{\Sigma}^\dagger}$ has finite quotient singularities (5.2.6).

(iii) $\pi_{\mathbf{\Sigma}^\dagger}^{-1}\big(V(f)\big) \subset X_{\mathbf{\Sigma}^\dagger}$ is a $\mathbf{Q}$-simple normal crossings divisor [**LCMMVVS20**, Definition 1.6] at every point in $\pi_{\mathbf{\Sigma}^\dagger}^{-1}(\mathbf{0}) \subset X_{\mathbf{\Sigma}^\dagger}$. Indeed, $\vartheta_{\mathbf{\Sigma}^\dagger}$ factors as $\mathscr{X}_{\mathbf{\Sigma}^\dagger} \xhookrightarrow{\text{open}} \mathscr{X}_{\Sigma^\dagger} \xrightarrow{\Pi_{\Sigma^\dagger}} \mathbf{A}^n$ in the above diagram. We therefore deduce, from (5.22), that:

$$\vartheta_{\mathbf{\Sigma}^\dagger}^{-1}\big(V(f)\big) = V(f') + \sum_{\rho \in \Sigma^\dagger[1]} N_\rho \cdot V(x'_\rho) \tag{5.29}$$

where each $V(x'_\rho)$, as well as $V(f')$, is now regarded as a divisor in $\mathscr{X}_{\mathbf{\Sigma}^\dagger} \xhookrightarrow{\text{open}} \mathscr{X}_{\Sigma^\dagger}$. By Theorem 5.3.26, $\vartheta_{\mathbf{\Sigma}^\dagger}^{-1}\big(V(f)\big)$ is a simple normal crossings divisor at every point in $\vartheta_{\mathbf{\Sigma}^\dagger}^{-1}(\mathbf{0}) = \vartheta_{\Sigma^\dagger}^{-1}(\mathbf{0}) \cap \mathscr{X}_{\mathbf{\Sigma}^\dagger}$. It remains to note that $\pi_{\mathbf{\Sigma}^\dagger}^{-1}\big(V(f)\big)$ is the coarse space of $\vartheta_{\mathbf{\Sigma}^\dagger}^{-1}\big(V(f)\big)$, since the coarse space morphism $\mathscr{X}_{\mathbf{\Sigma}^\dagger} \to X_{\mathbf{\Sigma}^\dagger}$ maps the latter onto the former.

In other words, $\pi_{\mathbf{\Sigma}^\dagger} \colon X_{\mathbf{\Sigma}^\dagger} \to \mathbf{A}^n$ is an *embedded* $\mathbf{Q}$*-desingularization of* $V(f) \subset \mathbf{A}^n$ *above the origin* $\mathbf{0} \in \mathbf{A}^n$, in the sense that it satisfies (i), (ii) and (iii) above. As noted in 5.2.8, [**LCMMVVS20**, Theorem 4] applies more generally to our case of $\pi := \pi_{\mathbf{\Sigma}^\dagger}$, $D_1 := V(f)$, $D_2 := 0$, and $W = \{\mathbf{0}\}$. Together with (5.10) and (5.29), we deduce that $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ lies in

$$\mathscr{M}_{\mathbf{k}}\big[\mathbf{L}^{-s}\big]\left[\frac{1}{1-\mathbf{L}^{-(s+1)}}\right]\left[\frac{1}{1-\mathbf{L}^{-(N_\rho s+|\mathbf{u}_\rho|)}} : \rho \in \Sigma^\dagger[1] = \Sigma(f)[1] \smallsetminus \Sigma(f)[\mathbb{B}]\right]$$

i.e. $\Theta^{\dagger,\mathbb{B}}(f) = \{-1\} \cup \big\{-\frac{|\mathbf{u}_\rho|}{N_\rho} : \rho \in \Sigma(f)[1] \smallsetminus \Sigma(f)[\mathbb{B}] \text{ with } N_\rho > 0\big\}$ is indeed a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$. □

5.4. Further remarks and future directions

5.4.A. On a potential refinement of Theorem H in the case of B_1-facets. In this section, we revisit Theorem H and explain why the theorem does not seem to give a complete answer even in the case of B_1-facets. Recall that $Z_{\mathrm{top},\mathbf{0}}(f;s)$ denotes the topological zeta function of f at the origin $\mathbf{0} \in \mathbf{A}^n$, cf. 1.3.5 and Remark 1.3.6.

5.4.1. Using our conventions, [**ELT22**, Proposition 3.8] can be stated as follows. Let $\mathcal{S}_\circ \subset \Theta(f) \smallsetminus \{-1\}$. If $\mathcal{F}(f; s_\circ)$ is a set of B_1-facets with consistent base directions for every $s_\circ \in \mathcal{S}_\circ$, then every pole of $Z_{\mathrm{top},\mathbf{0}}(f;s)$ is contained in $\Theta(f) \smallsetminus \mathcal{S}_\circ$. This can be seen as a consequence of our Theorem H as follows. Indeed, we first note an immediate consequence of Theorem H:

Corollary 5.4.2. *Let $s_\circ \in \Theta(f) \smallsetminus \{-1\}$. If $\mathcal{F}(f; s_\circ)$ is a set of B_1-facets with consistent base directions, then $\Theta(f) \smallsetminus \{s_\circ\}$ is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$.*

Proof of statement in 5.4.1. In view of Remark 1.3.6, Corollary 5.4.2 in particular implies that for every $s_\circ \in \mathcal{S}_\circ$, every pole of $Z_{\mathrm{top},\mathbf{0}}(f;s)$ is contained in $\Theta(f) \smallsetminus \{s_\circ\}$. Thus, every pole of $Z_{\mathrm{top},\mathbf{0}}(f;s)$ is contained in $\Theta(f) \smallsetminus \mathcal{S}_\circ = \bigcap \big\{\Theta(f) \smallsetminus \{s_\circ\} \colon s_\circ \in \mathcal{S}_\circ\big\}$. $\square$

5.4.3. Unfortunately, it is not immediate that the motivic analogue of 5.4.1 is true. Namely, for $\mathcal{S}_\circ \subset \Theta(f) \smallsetminus \{-1\}$, one could pose the following question. If $\mathcal{F}(f; s_\circ)$ is a set of B_1-facets with consistent base directions for every $s_\circ \in \mathcal{S}_\circ$, then is $\Theta(f) \smallsetminus \mathcal{S}_\circ$ a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$?

One key difficulty behind this question lies in our current lack of understanding of the zero divisors in the localized Grothendieck ring of $\mathbf{k}$-varieties $\mathscr{M}_{\mathbf{k}} = K_0(\mathsf{Var}_{\mathbf{k}})[\mathbf{L}^{-1}]$. More precisely, while Corollary 5.4.2 says that $\Theta(f) \smallsetminus \{s_\circ\}$ is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$ for each $s_\circ \in \mathcal{S}_\circ$, it is not clear if that would imply that $\Theta(f) \smallsetminus \mathcal{S}_\circ = \bigcap \big\{\Theta(f) \smallsetminus \{s_\circ\} \colon s_\circ \in \mathcal{S}_\circ\big\}$ is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$.

5.4.4. Nevertheless, one could try the following different line of attack to the question posed in 5.4.3. For $\mathcal{S}_\circ \subset \Theta(f) \smallsetminus \{-1\}$, one can hope that if $\mathcal{F}(f; s_\circ)$ is a set of B_1-facets of $\Gamma_+(f)$ with consistent base directions for each $s_\circ \in \mathcal{S}_\circ$, then so is $\mathcal{F}(f; \mathcal{S}_\circ) := \bigsqcup \{\mathcal{F}(f; s_\circ) \colon s_\circ \in \mathcal{S}_\circ\}$. If this is true, Theorem H would give a positive answer to the question in 5.4.3. Unfortunately, in general this statement is just not true. For that reason among others, we believe that the notion of "consistent base directions" is *still incomplete* for the case of B_1-facets. In what follows, we present a broader notion that is motivated by [**ELT22**, Conjecture 1.3(i)], although for the case of B_1-facets, ours is slightly broader than theirs.

Definition 5.4.5. A set $\mathbb{B}$ of B_1-facets of $\Gamma_+(f)$ has *compatible apices* if there exists, for each facet $\tau \in \mathbb{B}$, a choice of a distinguished apex $\mathbf{v}_\tau$ with corresponding base direction $b(\tau)$, such that $b(\tau_1) = b(\tau_2)$ for every pair of adjacent facets $\tau_1, \tau_2 \in \mathbb{B}$ sharing the same distinguished apex $\mathbf{v}_{\tau_1} = \mathbf{v}_{\tau_2}$. In this case we call $\{\mathbf{v}_\tau \colon \tau \in \mathbb{B}\}$ a *set of compatible apices* for $\mathbb{B}$.

Remark 5.4.6. If $\mathbb{B}$ has consistent base directions, then $\mathbb{B}$ has compatible apices, cf. 5.3.16.

In view of 5.4.4, the next lemma supports the narrative that the notion of "compatible apices" is *possibly* the correct notion to consider:

Lemma 5.4.7. *Let $\mathcal{S}_\circ \subset \Theta(f) \setminus \{-1\}$. If $\mathcal{F}(f; s_\circ)$ is a set of B_1-facets of $\Gamma_+(f)$ with compatible apices for each $s_\circ \in \mathcal{S}_\circ$, then so is $\mathcal{F}(f; \mathcal{S}_\circ) := \bigsqcup\{\mathcal{F}(f; s_\circ) \colon s_\circ \in \mathcal{S}_\circ\}$.*

Proof. For every $s_\circ \in \mathcal{S}_\circ$, fix a compatible set of apices $\{\mathbf{v}_\tau \colon \tau \in \mathcal{F}(f; s_\circ)\}$ for $\mathcal{F}(f; s_\circ)$. We claim that $\{\mathbf{v}_\tau \colon \tau \in \mathcal{F}(f; \mathcal{S}_\circ)\}$ is a compatible set of apices for $\mathcal{F}(f; \mathcal{S}_\circ)$. Suppose not. Then there exists adjacent facets $\tau_1, \tau_2 \in \mathcal{F}(f; \mathcal{S}_\circ)$ such that $\mathbf{v}_{\tau_1} = \mathbf{v}_{\tau_2} =: \mathbf{v}$ but $b(\tau_1) \neq b(\tau_2)$. Letting $\varsigma := \tau_1 \cap \tau_2$, observe that:

(i) $\mathbf{v} \in \mathrm{vert}(\varsigma)$, and the $b(\tau_1)^{\mathrm{th}}$ and $b(\tau_2)^{\mathrm{th}}$ coordinates of $\mathbf{v}$ are both 1.

(ii) Any $\mathbf{w} \in \mathrm{vert}(\varsigma) \setminus \{\mathbf{v}\}$ lies in $H_{b(\tau_1)} \cap H_{b(\tau_2)}$.

(iii) ς is compact in the $b(\tau_1)^{\mathrm{st}}$ and $b(\tau_2)^{\mathrm{th}}$ coordinates.

Together, these imply that ς is contained in the hyperplane H in $M_{\mathbf{R}}$ defined by $\mathbf{e}_{b(\tau_1)} - \mathbf{e}_{b(\tau_2)} = 0$. In fact, since $\varsigma \prec^1 \tau_1, \tau_2$, we have $\varsigma = H \cap \tau_1 = H \cap \tau_2$. For $i = 1, 2$, s_{τ_i} is the unique positive rational number for which $s_{\tau_i}^{-1} \cdot (1, 1, \ldots, 1)$ lies on the affine span of τ_i, or equivalently, $s_{\tau_i}^{-1} \cdot (1, 1, \ldots, 1)$ lies on the affine span of $\tau_i \cap H = \varsigma$. Since that last condition is independent of i, we deduce $s_{\tau_1} = s_{\tau_2}$, a contradiction to the first sentence of this proof. $\square$

Motivated by [**ELT22**, Conjecture 1.3(i)], one could ask the following:

Question 5.4.8. *Are the following statements true?*

(i) *Let* $\mathbb{B}$ *be a set of* B_1*-facets of* $\Gamma_+(f)$ *with compatible apices. Then*

$$\Theta^{\dagger,\mathbb{B}}(f) := \{-1\} \cup \big\{ s_\tau \colon \tau \prec^1 \Gamma_+(f) \text{ with } N_\tau > 0 \text{ and } \tau \notin \mathbb{B} \big\}$$

is a set of candidate poles for $Z_{\mathrm{mot},\mathbf{0}}(f;s)$*.*

(ii) *Let* $\mathcal{S}_\circ \subset \Theta(f) \setminus \{-1\}$*. If* $\mathcal{F}(f;s_\circ)$ *is a set of* B_1*-facets of* $\Gamma_+(f)$ *with compatible apices for each* $s_\circ \in \mathcal{S}_\circ$*, then* $\Theta(f) \setminus \mathcal{S}_\circ$ *is a set of candidate poles for* $Z_{\mathrm{mot},\mathbf{0}}(f;s)$*.*

Note (i) *is a generalization of Theorem H,* (i) *implies* (ii) *by Lemma 5.4.7, and* (ii) *in particular gives a positive answer to the question posed in 5.4.3.*

Unfortunately, these are false, as indicated by a counterexample in [**LPS22**, Example 2.2.2]. Nevertheless, some refinement should be true, and this will be pursued in a separate sequel. For now, we have:

Theorem 5.4.9 (= Theorem J)**.** *If* $n = 3$*, Question 5.4.8(i) is positive.*

Indeed, this follows from Theorem H and the following lemma:

Lemma 5.4.10. *Let* $n = 3$*, and let* $\mathbb{B}$ *be a set of* B_1*-facets of* $\Gamma_+(f)$*. Then* $\mathbb{B}$ *has consistent base directions if and only if* $\mathbb{B}$ *has compatible apices.*

Proof. Suppose there exists a compatible set of apices $\{\mathbf{v}_\tau \colon \tau \in \mathbb{B}\}$ for $\mathbb{B}$. We then claim that whenever two facets $\tau_1, \tau_2 \in \mathbb{B}$ are adjacent and $b(\tau_1) \neq b(\tau_2)$, then one of τ_1 or τ_2, *say* τ_2, satisfies the following:

(a) τ_1 is the *only* facet in $\mathbb{B}$ adjacent to τ_2.

(b) $\mathbf{v}_{\tau_1}$ is also an apex for τ_2, with corresponding base direction $b(\tau_1)$.

Admitting this claim, we re-assign τ_2 with the base direction $b(\tau_1)$. Repeating this re-assignment of base direction for all such pairs (τ_1, τ_2) in $\mathbb{B}$ would then culminate in a set of consistent base directions for $\mathbb{B}$. To prove the claim, we make three successive observations:

(i) Firstly, every facet of τ_1, with the exception of $H_{b(\tau_1)} \cap \tau_1 \prec^1 \tau_1$, contains $\mathbf{v}_{\tau_1}$ (cf. 5.3.15(i)). Thus, $\mathbf{v}_{\tau_1}$ is a vertex of $\tau_1 \cap \tau_2 \prec^1 \tau_1$. Likewise, $\mathbf{v}_{\tau_2}$ is a vertex of $\tau_1 \cap \tau_2 \prec^1 \tau_2$. We conclude $\tau_1 \cap \tau_2$ is the line segment in $M^+_{\mathbf{R}}$ connecting the vertex $\mathbf{v}_{\tau_1}$ to the vertex $\mathbf{v}_{\tau_2}$.

(ii) Secondly, by re-ordering coordinates if necessary, we may assume $b(\tau_1) = 1$ and $b(\tau_2) = 2$. Since $\mathbf{v}_{\tau_1} \in \mathrm{vert}(\tau_1 \cap \tau_2) \smallsetminus \{\mathbf{v}_{\tau_2}\} \subset \mathrm{vert}(\tau_2) \smallsetminus \{\mathbf{v}_{\tau_2}\}$, the 2^{nd} coordinate of $\mathbf{v}_{\tau_1}$ is 0. Likewise, the 1^{st} coordinate of $\mathbf{v}_{\tau_2}$ is 0. Summing up, we have $\mathbf{v}_{\tau_1} = (1, 0, a)$ and $\mathbf{v}_{\tau_2} = (0, 1, b)$ for some $a, b \in \mathbf{N}$.

(iii) Thirdly, we claim that besides $\mathbf{v}_{\tau_1}$ and $\mathbf{v}_{\tau_2}$, there can only be at most one other $\mathbf{v} \in \mathrm{vert}(\Gamma_+(f))$ satisfying $\mathbf{v} \cdot (\mathbf{e}_1 + \mathbf{e}_2) \leq 1$, and moreover such a $\mathbf{v}$ must equal $(0, 0, c)$ for some $c \in \mathbf{N}$. Indeed, if $\mathbf{v} = (v_1, v_2, v_3) \in \mathrm{vert}(\Gamma_+(f))$ satisfies $\mathbf{v} \cdot (\mathbf{e}_1 + \mathbf{e}_2) = v_1 + v_2 \leq 1$, then $(v_1, v_2) = (0, 0)$, $(1, 0)$ or $(0, 1)$. The case $(v_1, v_2) = (1, 0)$ cannot happen since otherwise $\mathbf{v} - \mathbf{v}_{\tau_1} \in \mathbf{R}\mathbf{e}_3^\vee$, but no two distinct vertices of $\Gamma_+(f)$ can differ by a vector in $\sum_{i=1}^n \mathbf{R}_{\geq 0}\mathbf{e}_i^\vee$ or in $\sum_{i=1}^n \mathbf{R}_{\leq 0}\mathbf{e}_i^\vee$. Likewise, $(v_1, v_2) \neq (0, 1)$, or else $\mathbf{v} - \mathbf{v}_{\tau_2} \in \mathbf{R}\mathbf{e}_3^\vee$. Thus, $(v_1, v_2) = (0, 0)$. Note too that there cannot be two distinct $\mathbf{v}, \mathbf{v}' \in \mathrm{vert}(\Gamma_+(f))$ of the form $(0, 0, c)$ for $c \in \mathbf{N}$, or else $\mathbf{v} - \mathbf{v}' \in \mathbf{R}\mathbf{e}_3^\vee$.

Returning back to the claim, we deduce from (iii) that the hyperplane $H_{\mathbf{e}_1+\mathbf{e}_2,1} = \{\mathbf{a} \in M^+_{\mathbf{R}} \colon \mathbf{a} \cdot (\mathbf{e}_1 + \mathbf{e}_2) = 1\}$ intersects $\Gamma_+(f)$ in $(\tau_1 \cap \tau_2) + \mathbf{R}_{\geq 0}\mathbf{e}_3^\vee$. Thus, if $H_{\mathbf{e}_1+\mathbf{e}_2,1}$ is a supporting hyperplane for $\Gamma_+(f)$, either τ_1 or τ_2 is $(\tau_1 \cap \tau_2) + \mathbf{R}_{\geq 0}\mathbf{e}_3^\vee$. Otherwise, by (iii) there must exist a unique $\mathbf{v} \in \mathrm{vert}(\Gamma_+(f))$ such that $\mathbf{v} \cdot (\mathbf{e}_1 + \mathbf{e}_2) < 1$, and $\mathbf{v} = (0, 0, c)$ for some $c \in \mathbf{N}$. Then the convex hull of $(\tau_1 \cap \tau_2) \cup \{\mathbf{v}\}$ in $M^+_{\mathbf{R}}$ is a 2-dimensional face of $\Gamma_+(f)$ that contains $\tau_1 \cap \tau_2$ as a face, and hence, must be either τ_1 or τ_2. In either case one verifies from its respective conclusion that our claim holds. $\square$

5.4.B. Other remarks and directions.

5.4.11 (Looking beyond B_1-facets)**.** It is natural to ask if the consideration of B_1-facets is sufficient for the monodromy conjecture for non-degenerate polynomials in $n \geq 4$ variables. The answer is *no*: in [**ELT22**], the authors described what they call a B_2-*facet*, and showed that for the case $n = 4$, certain configurations of B_1 and B_2-facets of $\Gamma_+(f)$ contribute to fake poles of $Z_{\mathrm{top},\mathbf{0}}(f;s)$. For general n, the authors also gave, in [**ELT22**, Conjecture 1.3(i)], a conjectural description of when a configuration of facets of $\Gamma_+(f)$ could culminate in fake poles of $Z_{\mathrm{top},\mathbf{0}}(f;s)$. There does not seem to be a clear connection between their conjectural description and our methods. In fact, Larson–Payne–Stapledon recently supplied a counterexample [**LPS22**, Example 2.2.1] to that conjecture. Nevertheless we anticipate the case of B_2-facets, which we are pursuing in a sequel, would demystify matters.

5.4.12 (On Corollary K)**.** While half of the proof of Corollary K was input from this chapter, the other half uses observations that are proven separately in [**LVP11**]. Nevertheless, we expect that one can use the stack-theoretic embedded desingularization $\vartheta_{\Sigma^\dagger} : \mathscr{X}_{\Sigma^\dagger} \to \mathbf{A}^n$ of $V(f) \subset \mathbf{A}^n$ above $\mathbf{0} \in \mathbf{A}^n$ in §5.3.C to re-prove the other half of Corollary K, via a "stack-theoretic analogue" of A'Campo's formula [**A'C75**] for the monodromy zeta function, e.g. [**MM13**, Theorem 2.8]. For brevity, we omit pursuing this here.

Bibliography

[A'C75] Norbert A'Campo, *La fonction zeta d'une monodromie*, Comment. Math. Helv. **50** (1975), 233–248.

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398.

[AH10] Dan Abramovich and Brendan Hassett, *Stable varieties with a twist*, Classification of Algebraic Varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2010, pp. 1–38.

[Alp13] Jarod Alper, *Good moduli spaces for Artin stacks*, Annales de l'Institut Fourier **63** (2013), no. 6, 2349–2402.

[AOV08] Dan Abramovich, Martin Olsson, and Angelo Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091.

[AQ21] Dan Abramovich and Ming Hao Quek, *Logarithmic resolution via multi-weighted blow-ups*, Preprint, December 2021, arXiv:2112.06361.

[AT17] Dan Abramovich and Michael Temkin, *Torification of diagonalizable group actions on toroidal schemes*, J. Algebra **272** (2017), 279—338.

[ATW19] Dan Abramovich, Michael Temkin, and Jarosław Włodarczyk, *Functorial embedded resolution via weighted blowings up*, Preprint, June 2019, arXiv:1906.07106.

[ATW20a] Dan Abramovich, Michael Temkin, and Jarosław Włodarczyk, *Principalization of ideals on toroidal orbifolds*, JEMS **22** (2020), no. 12, 3805—-3866.

[ATW20b] Dan Abramovich, Michael Temkin, and Jarosław Włodarczyk, *Toroidal orbifolds, destackification, and Kummer blowings up*, Algebra and Number Theory **14** (2020), no. 8, 2001–2035.

[BB73] Andrzej Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[BBV21] Guillem Blanco, Nero Budur, and Robin van der Veer, *Monodromy conjecture for semi-quasihomogeneous hypersurfaces*, Preprint, September 2021, arXiv:2106.11015.

[BCS05] Lev Borisov, Linda Chen, and Gregory Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. **18** (2005), no. 1, 193–215.

[BM97] Edward Bierstone and Pierre Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302.

[BM08] Edward Bierstone and Pierre Milman, *Functoriality in resolution of singularities*, Publ. Res. Inst. Math. Sci. **44**, no. 2. MR 2426359 (2008), 609–639.

[BN20] Emmanuel Bultot and Johannes Nicaise, *Computing motivic zeta functions on log smooth models*, Mathematische Zeitschrift **295** (2020), 427—462.

[BR19] Daniel Bergh and David Rydh, *Functorial destackification and weak factorization of orbifolds*, Preprint, May 2019, arXiv:1905.00872.

[BV16] Bart Bories and Willem Veys, *Igusa's p-adic local zeta function and the monodromy conjecture for non-degenerate surface singularities*, vol. 242, Memoirs of the American Mathematical Society, no. 1145, American Mathematical Society, 2016.

[Cad07a] Charles Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), no. 2, 405–427.

[Cad07b] Charles Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), no. 2, 405–427.

[CLNS10] Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag, *Motivic integration*, Progress in Mathematics, vol. 325, Springer Science+Business Media, 2010.

[CLS11] David Cox, John Little, and Hal Schenck, *Toric varieties*, American Mathematical Society, 2011.

[DH01] Jan Denef and Kathleen Hoornaert, *Newton polyhedra and Igusa's local zeta function*, J. Number Theory **89** (2001), no. 1, 31–164.

[DL92a] Jan Denef and François Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zéta locales et modifications analytiques*, J. Amer. Math. Soc. **5** (1992), no. 4, 705–720.

[DL92b] Jan Denef and François Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zéta locales et modifications analytiques*, J. Amer. Math. Soc. **5** (1992), no. 4, 705–720.

[DL01] Jan Denef and François Loeser, *Geometry on arc spaces of algebraic varieties*, Progr. Math. **201** (2001), 327–348.

[EGA$_{\text{II}}$] Alexander Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8, 222.

[ELT22] Alexander Esterov, Ann Lemahieu, and Kiyoshi Takeuchi, *On the monodromy conjecture for non-degenerate hypersurfaces*, J. Eur. Math. Soc., Available at https://doi.org/10.4171/JEMS/1241, 2022.

[ER20] Dan Edidin and David Rydh, *Canonical reduction of stabilizers for Artin stacks with good moduli spaces*, Duke Math. J. (2020), 54.

[EV00] Santiago Encinas and Orlando Villamayor, *A course on constructive desingularization and equivariance*, Resolution of singularities (Obergurgl, 1997). Progr. Math. Birkhäuser, Basel **181** (2000), 147–227.

[EV03] Santiago Encinas and Orlando Villamayor, *A new proof of desingularization over fields of characteristic zero*, Proceedings of the International Conference on Algebraic Geometry and Singularities (Spanish) (Sevilla, 2001) **19** (2003), 339–353.

[EV07a] Santiago Encinas and Orlando Villamayor, *Rees algebras and resolution of singularities*, Preprint, Feb 2007, arXiv:math.AG/0702836v1.

[EV07b] Santiago Encinas and Orlando Villamayor, *Rees algebras and resolution of singularities*, Proceedings of the XVI-th Latin American Algebra Colloquium (Spanish), Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid. (2007), 63–85.

[FMN10] Barbara Fantechi, Etienne Mann, and Fabio Nironi, *Smooth toric Deligne-Mumford stacks*, J. Reine Angew. Math. **648** (2010), 201–244.

[Ful84] William Fulton, *Intersection theory*, Springer-Verlag Berlin Heidelberg, 1984.

[Gil11] William Daniel Gillam, *Localization of ringed spaces*, Adv. Pure Math. 1(5) (2011), 250–263.

[GR18] Ofer Gabber and Lorenzo Ramero, *Foundations for almost ring theory*, September 2018, arXiv:math/0409584v13.

[Gro17] Philipp Gross, *Tensor generators on schemes and stacks*, Algebr. Geom. **4** (2017), no. 4, 501–522.

[GS] Daniel Grayson and Michael Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[GS15] Anton Geraschenko and Matthew Satriano, *Toric stacks I: The theory of stacky fans*, Trans. Amer. Math. Soc. **367** (2015), no. 2, 1033–1071.

[Gul11] Martin Gulbrandsen, *Stack structures on GIT quotients parametrizing hypersurfaces*, Math. Nachr. **284** (2011), no. 7, 885–898.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer-Verlag, New York, 1977.

[Has05] Brendan Hassett, *Classical and minimal models of the moduli space of curves of genus two*, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 169–192.

[Hir64] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I & II*, Ann. of Math. **79** (1964), no. 2, 109–326.

[Hir77] Heisuke Hironaka, *Idealistic exponents of singularity*, Algebraic geometry (J. J. Sylvester Sympos., Johns Hopkins Univ., Baltimore, Md., 1976), Johns Hopkins Univ. Press, Baltimore, Md. MR 0498562 (1977), 52—-125.

[Inc22] Giovanni Inchiostro, *Moduli of genus one curves with two marked points as a weighted blow-up*, Mathematische Zeitschrift **302** (2022), no. 3, 1905—1925.

[Kat94] Kazuya Kato, *Toric singularities*, Amer. J. Math **116** (1994), no. 5, 1073–1099.

[Kaw07] Hiraku Kawanoue, *Toward resolution of singularities over a field of positive characteristic. I. Foundation; the language of the idealistic filtration*, Publ. Res. Inst. Math. Sci. **43** (2007), no. 3, 819–909.

[KKMSD73] George Kempf, Finn Faye Knudsen, David Mumford, and Bernard Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, vol. 339, Springer-Verlag, Berlin-New York, 1973.

[Kol07] János Kollár, *Lectures on resolution of singularities*, vol. 166, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2007.

[Kou76] Anatoly Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1–31.

[KSC04] János Kollár, Karen Smith, and Alessio Corti, *Rational and nearly rational varieties*, Cambridge studies in advanced mathematics, vol. 92, Cambridge University Press, 2004.

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry* II*: positivity for vector bundles, and multiplier ideals*, Springer-Verlag Berlin Heidelberg, 2004.

[LCMMVVS20] Edwin León-Cardenal, Jorge Martín-Morales, Willem Veys, and Juan Viu-Sos, *Motivic zeta functions on* $\mathbf{Q}$*-Gorenstein varieties*, Advances in Mathematics **370** (2020), 107–192.

[Lee20] Jonghyun Lee, *Algorithmic resolution via weighted blowings up*, Preprint, Aug 2020, arXiv:2008.02169.

[LPS22] Matt Larson, Sam Payne, and Alan Stapledon, *The local motivic monodromy conjecture for simplicial nondegenerate singularities*, Preprint, September 2022, arXiv:2209.03553.

[LS03] François Loeser and Julien Sebag, *Motivic integration on smooth rigid varieties and invariants of degenerations*, Duke Math. J. **119** (2003), 315–344.

[LVP11] Ann Lemahieu and Lise Van Proeyen, *Monodromy conjecture for surface singularities*, Trans. Amer. Math. Soc. **363** (2011), no. 9, 4801–4829.

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, vol. 8, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1989.

[McQ20] Michael McQuillan, *Very fast, very functorial, and very easy resolution of singularities*, GAFA **30** (2020), 858–909.

[Mil68] John Milnor, *Singular points of complex hypersurfaces*, Annals of Math. Studies, vol. 61, Princeton University Press, 1968.

[MM13] Jorge Martín-Morales, *Monodromy zeta function formula for embedded* $\mathbf{Q}$*-resolutions*, Rev. Mat. Iberoam. **29** (2013), no. 3, 939–967.

[Niz06] Wiesława Nizioł, *Toric singularities: log-blow-ups and global resolutions*, J. Algebraic Geom. **15** (2006), no. 1, 1–29.

[Ogu18] Arthur Ogus, *Lectures on logarithmic algebraic geometry*, Cambridge studies in advanced mathematics, 178, 2018.

[Ols16] Martin Olsson, *Algebraic spaces and stacks*, vol. 62, American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2016.

[QR21] Ming Hao Quek and David Rydh, *Weighted blow-ups*, In preparation: https://people.kth.se/~dary/weighted-blowups20220329.pdf, 2021.

[Que22a] Ming Hao Quek, *Logarithmic resolution via weighted toroidal blow-ups*, Algebraic Geometry **9** (2022), 311–363.

[Que22b] Ming Hao Quek, *On the motivic monodromy conjecture for non-degenerate hypersurfaces*, Preprint, August 2022, arXiv:2208.02566.

[Roc70] Ralph Tyrrell Rockafellar, *Convex analysis*, Princeton University Press, 1970.

[Rom05] Matthieu Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), no. 1, 209–236.

[Ryd09] David Rydh, *Compactification of tame Deligne–Mumford stacks*, Draft, available on https://people.kth.se/~dary/papers.html, May 2009.

[Ryd15] David Rydh, *Do line bundles descend to coarse moduli spaces of Artin stacks with finite inertia?*, MathOverflow answer at https://mathoverflow.net/a/206117/40, 2015.

[Sat13] Matthew Satriano, *Canonical Artin stacks over log smooth schemes*, Mathematische Zeitschrift **274** (2013), no. 3-4, 779–804.

[Stacks] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`.

[SU21] Matthew Satriano and Jeremy Usatine, *A motivic change of variable formula for Artin stacks*, Preprint, September 2021, arXiv:2109.09800.

[Var76] Alexander Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math **37** (1976), 253–262.

[Wło05] Jarosław Włodarczyk, *Simple Hironaka resolution in characteristic zero*, J. Amer. Math. Soc. **18** (2005), no. 4, 779–822.

[Wło20a] Jarosław Włodarczyk, *Functorial resolution except of log smooth locus. Toroidal compactification*, Preprint, July 2020, arXiv:2007.13846.

[Wło20b] Jarosław Włodarczyk, *Functorial resolution except of log smooth locus. Toroidal compactification*, Preprint, July 2020, arXiv:2007.13846.

[Yas06] Takehiko Yasuda, *Motivic integration over Deligne-Mumford stacks*, Adv. Math **207** (2006), no. 2, 707–761.

[Zar39] Oscar Zariski, *The reduction of singularities of an algebraic surface*, Annals of Mathematics **40** (1939), no. 3, 639–689.