Sobolev Estimates for Fractional Partial Differential Equations

by

Yanze Liu

B.Sc., University of British Columbia; Vancouver, BC, Canada, 2018

A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in Department of Mathematics at Brown University

PROVIDENCE, RHODE ISLAND

May 2023

This dissertation by Yanze Liu is accepted in its present form
by Department of Mathematics as satisfying the
dissertation requirement for the degree of Doctor of Philosophy.

Date________________ ____________________________________

Hongjie Dong, Ph.D., Advisor

Recommended to the Graduate Council

Date________________ ____________________________________

Justin Holmer, Ph.D., Reader

Date________________ ____________________________________

Benoît Pausader, Ph.D., Reader

Approved by the Graduate Council

Date________________ ____________________________________

Thomas A. Lewis, Dean of the Graduate School

Vita

Yanze Liu was born in Beijing, China in 1996. In September 2014, he began his undergraduate studies in Mathematics at the University of British Columbia, and graduated in May 2018.

In September 2018, Yanze began his Ph.D. studies at Brown University in the Mathematics department. During his time at Brown, he has been conducting his research under Professor Hongjie Dong's supervision.

Acknowledgments

First and foremost, I would like to express my deep gratitude to my thesis advisor, Professor Hongjie Dong, for his invaluable mentorship and support throughout my research journey. Professor Dong taught my first math class at Brown, guided me in reading my first PDE book, taught me how to write academic papers, and took me to my first academic conference. His expertise and insights played an instrumental role in shaping the direction of my thesis, and his commitment to my academic success has been truly exceptional. I am deeply grateful for the time and effort he devoted to refining my ideas, reviewing my work, and providing constructive feedback. His role as a mentor and role model has been life-changing for me, and I feel fortunate to have had the opportunity to collaborate with such an esteemed academic.

I extend my heartfelt thanks to Professor Justin Holmer and Professor Benoît Pausader as well for their valuable contributions as members of my thesis defense committee.

Furthermore, I am deeply appreciative of Professor Malabika's contribution in sparking my interest in research through my first math research topic, which opened up a world of possibilities for me. Additionally, I would like to acknowledge Professor Feldman's commitment to his students and the field of math, his commitment has been a source of inspiration to me.

I would also like to express my gratitude to all of my teachers and friends for their encouragement and support throughout my academic journey. In particular, I am grateful to my friend Jiahua Zou, who provided me with invaluable support during the challenging moments of my projects.

Lastly, I would like to express my heartfelt appreciation to my parents for their constant love and support. Their presence in my life has given me a sense of security, protection, and confidence.

Abstract of “Sobolev Estimates for Fractional Partial Differential Equations ”, by Yanze Liu, Ph.D., Brown University, May 2023

In this thesis, we study the non-local differential equations.

The first part is a comprehensive study of L_p estimates for time fractional wave equations of order $\alpha \in (1, 2)$ in the whole space, a half space, or a cylindrical domain. We obtain weighted mixed-norm estimates and solvability of the equations in both non-divergence form and divergence form when the leading coefficients have small mean oscillation in small cylinders.

In the second part, we obtain L_p estimates for fractional parabolic equations with space-time non-local operators. We also derive a weighted mixed-norm estimate for the equations with operators that are local in time, i.e., $\alpha = 1$, which extend the previous results by using a quite different method.

Contents

Chapter One

Introduction

1.1 Introduction to Chapter 2

Chapter 2 is devoted to the study of L_p (and $L_{p,q}$) estimates for non-divergence form fractional wave equations (1.1.1) and divergence form fractional wave equations (1.1.2) with a non-local type time derivative term in nondivergence form

$$\partial_t^\alpha u - a^{ij}(t,x)D_{ij}u - b^i(t,x)D_iu - c(t,x)u = f(t,x) \tag{1.1.1}$$

as well as in divergence form

$$\partial_t^\alpha u - D_i(a^{ij}D_ju + a^iu) - b^iD_iu - cu = D_ig_i + f, \tag{1.1.2}$$

with zero initial conditions $u(0,\cdot) = 0$ and $\partial_t u(0,\cdot) = 0$ in a cylindrical domain $(0,T)\times\Omega$, where $\partial_t^\alpha u$ is the Caputo fractional derivative of order $\alpha \in (1,2)$ and Ω is the whole space $\mathbb{R}^d$, a half space $\mathbb{R}^d_+ := \{(x^1,\ldots,x^d) \in \mathbb{R}^d : x^1 > 0\}$, or a domain in $\mathbb{R}^d$. See the precise definition of $\partial_t^\alpha u$ in Section 2.1. When $\Omega = \mathbb{R}^d_+$ or a domain, we also impose the zero Dirichlet boundary condition: $u(t,x) = 0$ for $t \in (0,T)$ and $x \in \partial\Omega$.

Fractional wave equations have many applications in mechanics and probability. For example, they govern the propagation of mechanical diffusive waves in viscoelastic media Mainardi (1995), and they also appear in the probability theory related to non-Markovian diffusion processes with a memory Metzler and Klafter (2000). There is a vast literature about fractional parabolic equations (when $\alpha \in (0,1)$) and fractional wave equations (when $\alpha \in (1,2)$).

Such equations are also closely related to a type of Volterra equation Prüss (1993):

$$u(t,x) + \int_0^t a(t-s)Au(s,x)\,ds = f(t,x),$$

where A is a time-independent $\mathcal{R}$-sectorial operator. Using this interpretation, an operator-theoretic approach was applied in Zacher (2005) to prove the existence of unique solutions under the assumptions that $a^{ij} = \delta^{ij}$, $\alpha \in (1,2)$, $p = q$, and that α and p satisfy the algebraic condition

$$\alpha \notin \big\{2/(2p-1), 2/(p-1), 1/p+1, 3/(2p-1)+1\big\}.$$

In Kochubei (2014), for $\alpha \in (1,2)$, assuming that the coefficients depend only on x, and are bounded and uniformly Hölder continuous with a Hölder exponent $\gamma \in (2-2/\alpha, 1]$, the author established the unique solvability of (1.1.1) by using Levi's parametrix method together with the fundamental solution of $\partial_t^\alpha - \Delta$ constructed earlier in Pskhu (2009).

There are also many work on L_p and $L_{p,q}$ estimates for fractional parabolic and wave equations. In Kim et al. (2017), under the assumptions that $\alpha \in (0,2)$, and $a^{ij}(t,x)$ are piecewise continuous in t and uniformly continuous in x, the authors proved a priori $L_{p,q}$ estimates and the existence of unique solutions to (1.1.1) in the whole space. Their proofs are based on estimates of the fundamental solution of $\partial_t^\alpha - \Delta$ and classical tools from the PDE theory, in particular the Fefferman-Stein sharp function theorem, the Hardy-Littlewood maximal function theorem, and the Calderón-Zygmund theorem. Quite recently, the results were extended in Han et al. (2020), where mixed norms with Muckenhoupt weights were considered under the assumption that the leading coefficients are constants. In these work, the proofs rely on various estimates of the fundamental solutions.

In Dong and Kim (2019), the authors derived L_p estimates and solvability of (1.1.1) when $\alpha \in (0,1)$ under the weaker assumption that the leading coefficients have small mean oscillations with respect to the spatial variables only. See also Dong and Kim (2020) for results about divergence form equations with $\alpha \in (0,1)$. Later in Dong and Kim (2021), they further obtained weighted mixed-norm estimates for (1.1.1) when $\alpha \in (0,1)$ under the same assumption on the coefficients. In contrast to the aforementioned previous works, the proofs in Dong and Kim (2019, 2020, 2021) do not rely on the estimates of fundamental solutions. Instead, the argument is based on a level set argument or a mean oscillation argument, a bootstrap argument, and Sobolev type embeddings with fractional derivatives. The starting point is the L_2 estimate, which is obtained by testing the equation with u, noting that the integral

$$\int_0^t (\partial_t^\alpha u(s,\cdot))|u(s,\cdot)|^{p-2}u(s,\cdot)\,ds \tag{1.1.3}$$

is nonnegative when $\alpha \in (0,1]$ and $p \in (1,\infty)$.

In Chapter 2, we consider the equations (1.1.1) and (1.1.2) when $\alpha \in (1,2)$ and the leading coefficients $a^{ij} = a^{ij}(t,x)$ locally have small mean oscillations as functions of (t,x). See Assumption 2.1.2 (γ_0) for details. Our main theorem for non-divergence form equations reads that for any $p,q \in (1,\infty)$, Muckenhoupt weights $w_1 \in A_p(\mathbb{R}), w_2(x) \in A_q(\Omega)$, and a^{ij}'s with small mean oscillations, if u satisfies (1.1.1) with the zero initial conditions (and the zero Dirichlet boundary condition in the cases of the half space and $C^{1,1}$ domains), then we have

$$\sum_{i=0}^{2} \|D^i u\|_{L_{p,q,w}(\Omega_T)} + \|\partial_t^\alpha u\|_{L_{p,q,w}(\Omega_T)} \le N\|f\|_{L_{p,q,w}(\Omega_T)},$$

where $w(t,x) = w_1(t)w_2(x)$, $\Omega_T = (0,T)\times\Omega$, N is independent of u and f, and the definition of the $L_{p,q,w}$ norm is given in Section 2.1. Furthermore, for any $f \in$

$L_{p,q,w}((0,T)\times\Omega)$, there exists a unique solution u to (1.1.1) in the appropriate weighted mixed-norm Sobolev space defined in Section 2.1 with the zero initial (and boundary) conditions. This, in particular, extends the previous results in Kim et al. (2017), Han et al. (2020), Kochubei (2014) by relaxing the regularity assumption on the coefficients. Moreover, we obtain the a priori estimate and unique existence of solutions in the appropriate weighted mixed-norm Sobolev space to equations with non-initial conditions

$$\begin{cases} \partial_t^\alpha u - (a^{ij}D_{ij}u + b^iD_iu + cu) = f & \text{in} \quad \mathbb{R}^d_T \\ u(0,\cdot) = u_0(\cdot) & \text{in} \quad \mathbb{R}^d \\ u_t(0,\cdot) = u_1(\cdot) & \text{in} \quad \mathbb{R}^d, \end{cases} \tag{1.1.4}$$

where u_0 and u_1 are taken from appropriate trace space. See Corollary 2.1.12 for details. Furthermore, we also obtain the results to equations in the half space and domains with the Dirichlet boundary condition.

Our main theorem for divergence form equations reads that for any $p,q\in(1,\infty)$, Muckenhoupt weights $w_1\in A_p(\mathbb{R}), w_2(x)\in A_q(\Omega)$, and a^{ij}'s with small mean oscillations, if u satisfies (1.1.2) with the zero initial conditions (and the zero boundary condition in the cases of the half space and Lipschitz domains), then it holds that

$$\sum_{i=0}^{1}\|D^iu\|_{L_{p,q,w}(\Omega_T)} + \|\partial_t^\alpha u\|_{\mathbb{H}^{-1}_{p,q,w}(\Omega_T)} \le N\|f\|_{L_{p,q,w}(\Omega_T)} + N\|g\|_{L_{p,q,w}(\Omega_T)},$$

where N is independent of u, f and g, and the definitions of the norms are given in Section 2.1. Furthermore, for any $f,g\in L_{p,q,w}((0,T)\times\Omega)$, there exists a unique solution u to (1.1.2) in the appropriate space defined in Section 2.1 with the zero initial (and boundary) conditions. Moreover, the a priori estimate and unique existence of solutions for divergence form equations with non-zero initial conditions are obtained in Corollary 2.1.13.

For the proof of the a priori estimate for (1.1.1) in the whole space, we adapt the argument in Dong and Kim (2021) by establishing a mean oscillation estimate of D^2u (see Proposition 2.3.1) together with the Fefferman-Stein theorem for sharp functions and the Hardy-Littlewood theorem for (strong) maximal functions in weighted mixed-norm spaces, where u is a strong solution to (1.1.1). For this, we first consider the case when $a^{ij} = \delta^{ij}$ without lower-order terms, and decompose $u = w + v$, where w is constructed using Fourier series and solving an ODE in the time direction in cylindrical domains (see Lemma 2.2.6), and $v := u - w$ satisfies a homogeneous equation. To estimate v and w, we consider the infinite cylinder $(-\infty, t_0) \times B_r(x_0)$, instead of the usual parabolic cylinder $Q_r(t_0, x_0)$. In particular, we obtain an L_p estimate of w by taking periodic extensions in the spatial directions and applying the Poincaré inequality so that the constant in the estimate is independent of the length of the time interval. See Lemma 2.2.7 for details. In contrast to the case $\alpha \in (1, 2)$, when $\alpha > 1$, the time integral given in (1.1.3) may not be nonnegative. here instead we use the results from Kim et al. (2017), which are derived from the fundamental solution mentioned above, as well as a suitable extension argument. For v, we derive a Hölder estimate of D^2v by adapting a delicate bootstrap argument in Dong and Kim (2021): locally improve the regularity of v using Sobolev embedding together with suitable cutoff functions. Compared to the case considered in Dong and Kim (2021), where $\alpha \in (0, 1)$, and only the first-order derivative in time appears in the definition of ∂_t^α, the estimate of v for $\alpha \in (1, 2)$ is more involved due to the presence of an extra first-order term in the fractional derivative when we take the cutoff function in time to apply the bootstrap argument. To overcome this difficulty, we take a sequence of cutoff functions in time and apply an interpolation inequality for the time derivatives. The mean oscillation estimate enables us to treat VMO coefficients in a natural way by using a frozen coefficient argument. For non-divergence form equations in the half space, the results follow from taking suitable extensions, and in smooth domains by a partition of unity argument together with S. Agmon's idea, adapted from the proofs in (Krylov, 2008, Ch. 8).

The results for divergence form equations are obtained by applying the results for non-divergence form equations and taking weak derivatives on both sides of the non-divergence form equations (see Proposition 2.5.1) together with a similar perturbation argument.

Finally, to deal with equations with nonzero initial conditions, we reduce the problem to equation with the zero initial conditions by constructing auxiliary solutions explicitly using the fundamental solutions of $\partial_t^\alpha - \Delta$. See Lemma 2.8.1.

Chapter 2 is organized as follows. In Section 2.1, we introduce notation, definitions, and the main results of the Chapter 2. Equations in non-divergence form (1.1.1) are studied in Sections 2.2, 2.3, and 2.4. Particularly, in Section 2.2, we estimate the mean oscillation by using a decomposition argument mentioned above. In Section 2.3, we complete the proof of the theorem for non-divergence form equations in the whole space. In Section 2.4, we prove the results for non-divergence form equations in the half space and smooth domains. Finally, the results for divergence form equations (1.1.2) are proved in Section 2.5. In Section 2.6, we prove miscellaneous lemmas used in the main proofs. In Section 2.7, we give the details about the proofs for equations in domains, such as the extension of weights. In Section 2.8, we prove the results for equations in the whole space with non-zero initial conditions.

1.2 Introduction to Chapter 3

Chapter 3 is devoted to a study of L_p (and $L_{p,q}$) estimates for non-divergence form equations with space-time non-local operators of the form

$$\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } (0,T) \times \mathbb{R}^d \tag{1.2.1}$$

with the zero initial condition $u(0,\cdot) = 0$, where $T \in (0,\infty)$, $\partial_t^\alpha u$ is the Caputo fractional derivative of order $\alpha \in (0,1]$, and L is an integro-differential operator

$$Lu(t,x) := \int_{\mathbb{R}^d} \Big(u(t,x+y) - u(t,x) - y \cdot \nabla_x u(t,x) \chi^{(\sigma)}(y) \Big) K(t,x,y)\, dy, \tag{1.2.2}$$

where $\chi^{(\sigma)}(y) = 1_{\sigma \in (1,2)} + 1_{\sigma=1} 1_{|y| \le 1}$, and K is nonnegative. A simple example of L is the fractional Laplacian $-(-\Delta)^{\sigma/2}$ for $\sigma \in (0,2)$. We refer the reader to Section 3.1 for the precise definitions of $\partial_t^\alpha u$ and L.

Fractional parabolic equations of the form (1.2.1) have applications in various fields, including physics and probability theory. For example, by viewing the fundamental solution of (1.2.1) as the probability density of a peculiar self-similar stochastic process evolving in time, we can see that the equation is related to the continuous-time random walk Gorenflo and Mainardi (2003).

There are many works regarding L_p (or $L_{p,q}$) estimates for parabolic equations with either fractional derivatives in time or integro-differential operators in space or both. Equations with non-local time derivatives of the form

$$\partial_t^\alpha u - a^{ij}(t,x) D_{ij} u = f(t,x), \tag{1.2.3}$$

were studied in Kim et al. (2017), Han et al. (2020), Dong and Kim (2019, 2021), Dong and Liu (2022), Park (2021). In Kim et al. (2017), the authors derived mixed-norm estimates for (1.2.3) with $\alpha \in (0, 2)$ under the assumption that a^{ij} are piecewise continuous in t and uniformly continuous in x. In Han et al. (2020), a weighted mixed-norm estimate was obtained when $\alpha \in (0, 2)$ and $a^{ij} = \delta_{ij}$. More general coefficients a^{ij} were studied in Dong and Kim (2019, 2021), where the L_p and the weighted mixed-norm estimates were obtained respectively for $\alpha \in (0, 1)$ under the assumption that a^{ij} have small mean oscillations with respect to the spatial variables. Quite recently, in Dong and Liu (2022) the weighted mixed-norm estimates are obtained for $\alpha \in (1, 2)$ under the assumption that a^{ij} have small oscillations in (t, x). Furthermore, a weighted mixed-norm regularity theory for (1.2.3) was obtained in Park (2021) when the coefficients are at least uniformly continuous in both t and x.

Equations with non-local operators in x of the form

$$\partial_t u(t, x) - Lu(t, x) = f(t, x) \tag{1.2.4}$$

with L defined in (1.2.2) were studied in Mikulevičius and Pragarauskas (1992, 2014, 2019), Zhang (2013), Dong et al. (2021). In Mikulevičius and Pragarauskas (1992), using the Fourier multiplier theorem, the authors derived an L_p estimate for (1.2.4) under the conditions that p is sufficiently large, and the operator satisfies an ellipticity condition, and $K(t, x, y)|y|^{d+\sigma}$ is homogeneous of order zero and smooth in y, and its derivatives in y are continuous in x uniformly in (t, y). In Mikulevičius and Pragarauskas (2014), an L_p estimate for (1.2.4) was obtained under the conditions that p is sufficiently large, and $K(t, x, y)|y|^{d+\sigma}$ is bounded from above and below and Hölder continuous in x uniformly in (t, y). No regularity assumption is assumed for K in t and y. In particular, when $K = K(t, y)$, i.e., independent of x, and f is sufficiently smooth, using a probabilistic method, they constructed the solution for

(1.2.4) explicitly as the expected value involving some stochastic processes, and when $K = K(t,x,y)$, they applied the frozen coefficient argument. For time-independent equations with $K = K(y)$, such result was obtained earlier in Dong and Kim (2012) by using a purely analytic method. It has been an open question whether in the case when $K = K(t,x,y)$, the restriction of p in Mikulevičius and Pragarauskas (2014) can be removed. This problem was resolved in a recent paper Dong et al. (2021) by using a weighted estimate and an extrapolation argument. In Zhang (2013), by a probabilistic representation of the solution, the authors investigated the L_p-maximal regularity of (1.2.4) as well as more general equations with singular and non-symmetric Lévy operators.

Sobolev type estimates for equations with space-time non-local operators similar to (1.2.1) were studied in Kim et al. (2021) by deriving fundamental solutions using probability density functions of some stochastic processes. Particularly, the equation

$$\partial_t^\alpha u - \phi(\Delta) u = f \tag{1.2.5}$$

is considered, where $\alpha \in (0,1)$ and ϕ is a Bernstein function satisfying a certain growth condition. The authors obtained a mixed-norm estimate of (1.2.5) based on estimates of the fundamental solution from Bogdan et al. (2009) together with the Calderón-Zygmund theorem.

In Chapter 3, we obtain the a priori estimate and the unique solvability for (1.2.1) with space-time non-local operators. Compared to Kim et al. (2021), our estimates do not directly rely on the representation of the fundamental solution. However, to apply the method of continuity, we need the results in Kim et al. (2021) or Fackler et al. (2020). Indeed, for $K = K(t,y)$, we only require that $K(t,y)|y|^{d+\sigma}$ is bounded from above and below and measurable, an assumption for which the fundamental solution is unavailable. Furthermore, when $\alpha = 1$, we obtain the weighted mixed-norm estimate for (1.2.1), where the weights are taken in both spatial and time variables. The results are further extended to the case when $K = K(t, x, y)$, under the assumption that K is Hölder continuous in x uniformly in (t, y). See Assumptions 3.1.1 and 3.1.3 for details. We also consider equations which contain lower-order terms. These, in particular, extend the aforementioned results in Mikulevičius and Pragarauskas (2014), Zhang (2013), Dong et al. (2021) to the weighted mixed-norm setting by using a quite different method. Our main theorem reads that for any $p \in (1, \infty)$ and the kernel K such that $K(t, y)|y|^{d+\sigma}$ is bounded from above and below, if u satisfies (1.2.1) with the zero initial condition, then we have

$$\|(1-\Delta)^{\sigma/2}u\|_{L_p(0,T)\times\mathbb{R}^d)} + \|\partial_t^\alpha u\|_{L_p((0,T)\times\mathbb{R}^d)} \leq N\|f\|_{L_p((0,T)\times\mathbb{R}^d)},$$

where N is independent of u and f. Furthermore, for any $f \in L_p((0, T) \times \mathbb{R}^d)$, there exists a unique solution u to (1.2.1) with the zero initial condition in the appropriate Sobolev spaces defined in Section 3.1. Moreover, when $\alpha = 1$, the a priori estimate and the unique solvability hold for (1.2.4) in the appropriate weighted mixed-norm Sobolev spaces with Muckenhoupt weights.

For the proof of the L_p estimates for (1.2.1) when $\alpha \in (0, 1]$, we apply a level set argument together with Lemma 3.4.5 ("crawling of ink spots lemma") by adapting the argument in Dong and Kim (2019). More precisely, we first prove the theorem for $p = 2$ by using the Fourier transform. Then, we apply a bootstrap argument: assuming the theorem holds for p_0 and estimating the solution by a decomposition, we show that the theorem holds for $p \in (p_0, p_1)$ for some $p_1 > p_0$ such that the increment $p_1 - p_0$ is independent of p_0. Indeed, for $(t_0, x_0) \in (0, T] \times \mathbb{R}^d$, we decompose the solution u to (1.2.1) into $u = w + v$ such that

$$\partial_t^\alpha w - Lw + \lambda w = \zeta f \quad \text{in } (t_0 - 1, t_0) \times \mathbb{R}^d \tag{1.2.6}$$

with the zero initial condition at $t = t_0 - 1$, where ζ is a suitable cutoff function satisfying $\zeta = 1$ in $Q_1(t_0, x_0)$. Then v satisfies the homogeneous equation in $Q_1(t_0, x_0)$. We bound w by the maximal function of f using the global L_{p_0} estimate and prove a higher integrability result for v. See Proposition 3.2.6. Compared to Dong and Kim (2019), our proof is more involved since we need to deal with the nonlocality in both space and time. For the proof of the weighted mixed-norm estimates of (1.2.1) when $\alpha = 1$, we derive a mean oscillation estimate of $(-\Delta)^{\sigma/2}u$, where u is a solution to (1.2.1), by using an iteration argument. To this end, for $t_0 \in (0, T]$, we decompose $u = w + v$, where

$$\partial_t w - Lw + \lambda w = f \quad \text{in } (t_0 - 1, t_0) \times \mathbb{R}^d$$

so that v satisfies the homogeneous equation in the strip $(t_0 - 1, t_0) \times \mathbb{R}^d$. Compared to the estimate of w in the previous case (cf. (1.2.6)), here we cannot directly apply the global L_{p_0} estimate since the right-hand side is not compactly supported in x. Our idea is to use a localization and iteration argument. See Proposition 3.3.3. We also establish a Hölder estimate of v by using a delicate bootstrap argument: first derive a local estimate by using an estimate of the commutator (Lemma 3.4.3) and then apply the Sobolev embedding theorems (Lemmas 3.4.6 and 3.4.7). For this proof, it is crucial that v satisfies the homogeneous equation in the strip $(t_0 - 1, t_0) \times \mathbb{R}^d$ instead of just in a cylinder. The weighted mixed-norm estimates then follow by using the Fefferman-Stein theorem for sharp functions and the Hardy-Littlewood theorem for (strong) maximal functions in weighted mixed-norm spaces. For the general cases when $K = K(t, x, y)$, we use a boundedness result in Dong et al. (2021) with a perturbation argument and an extrapolation theorem. Finally, we extend our results to equations in the whole space with non-zero initial conditions, in which we focus on equations in weighted mixed-norm spaces with $\alpha = 1$ and power weights in time.

Chapter 3 is organized as follows. In Section 3.1, we introduce notation, definitions, and the main results of the Chapter 3. In Section 3.2, we obtain the L_p estimates using the level set argument. In Section 3.3, we first estimate the mean oscillation of solutions by using the decomposition mentioned above, and then prove the weighted mixed-norm estimates. In Section 3.4, we prove miscellaneous lemmas used in the main proofs. In Section 3.5, we consider equations with non-zero initial conditions.

Chapter Two

The Fractional Wave Equations

2.1 Notation and Main Results

We first introduce some notation used throughout the Chapter 2. For $\alpha \in (0,2)$ and $S \in \mathbb{R}$, we denote

$$I_S^\alpha u = \frac{1}{\Gamma(\alpha)} \int_S^t (t-s)^{\alpha-1} u(s,x)\, ds,$$

and for $\alpha \in (1,2)$

$$\partial_t^\alpha u = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \partial_t^2 u(s,x)\, ds.$$

Note that we have $\partial_t^\alpha u = \partial_t^2 I_0^{2-\alpha} u$ for a sufficiently smooth u with $u(0,x) = 0$ and $\partial_t u(0,x) = 0$. In some places, without assuming $S = 0$, we still use $\partial_t^\alpha u$ to indicate $\partial_t^2 I_S^{2-\alpha} u$ when $u(S,x) = 0$ and $\partial_t u(S,x) = 0$. Moreover, for $\beta \in (0,1)$, we use $\partial_t^\beta u$ to indicate $\partial_t I_S^{1-\beta} u$ when $u(S,x) = 0$.

For $\alpha \in (1,2)$, $r_1, r_2 > 0$, and $(t,x) \in \mathbb{R}^{d+1}$, we denote the parabolic cylinder by

$$Q_{r_1,r_2}(t,x) = (t - r_1^{2/\alpha}, t) \times B_{r_2}(x) \quad \text{and} \quad Q_r(t,x) = Q_{r,r}(t,x),$$

where $B_{r_2}(x) = \{y \in \mathbb{R}^d : |y-x| < r_2\}$. We often write B_r and Q_r for $B_r(0)$ and $Q_r(0,0)$. Furthermore, for $x \in \mathbb{R}^d$ and $r > 0$, we denote $x' = (x^2, \ldots, x^d) \in \mathbb{R}^{d-1}$ and $B_r'(x') := \{y' \in \mathbb{R}^{d-1} : |x'-y'| < r\}$. Next, for $p \in (1,\infty)$, $k \in \{1,2,\ldots\}$ and a domain $\Omega \subset \mathbb{R}^k$, let $A_p(\Omega, dx)$ be the set of all non-negative functions w on Ω such that

$$[w]_{A_p(\Omega)} := \sup_{x_0 \in \mathbb{R}^k \cap \Omega, r>0} \left(\fint_{B_r(x_0)\cap\Omega} w(x)\, dx \right) \left(\fint_{B_r(x_0)\cap\Omega} (w(x))^{\frac{-1}{p-1}}\, dx \right)^{p-1} < \infty,$$

where $B_r(x_0) = \{x \in \mathbb{R}^k : |x - x_0| < r\}$. Recall that $[w]_{A_p} \geq 1$. Furthermore, for a constant $K_1 > 0$, we write $[w]_{p,q,\Omega} \leq K_1$ and $[w]_{p,q} \leq K_1$ when $\Omega = \mathbb{R}^d$ if

$w = w_1(t)w_2(x)$ for some w_1 and w_2 satisfying

$$w_1(t) \in A_p(\mathbb{R}, dt), \quad w_2(x) \in A_q(\Omega, dx), \quad [w_1]_{A_p} \leq K_1, \quad \text{and} \quad [w_2]_{A_q(\Omega)} \leq K_1.$$

For a domain $\Omega \subset \mathbb{R}^d$, let $(0,T) \times \Omega := \Omega_T$ and particularly, $(0,T) \times \mathbb{R}^d := \mathbb{R}^d_T$. For a given $w(t,x) = w_1(t)w_2(x)$, where $(t,x) \in \mathbb{R} \times \Omega$, $w_1 \in A_p(\mathbb{R}, dt)$, and $w_2 \in A_q(\Omega, dx)$, we denote $L_{p,q,w}(\Omega_T)$ to be the set of all measurable functions f defined on Ω_T satisfying

$$\|f\|_{L_{p,q,w}(\Omega_T)} := \left(\int_0^T \left(\int_\Omega |f(t,x)|^q w_2(x)\, dx \right)^{p/q} w_1(t)\, dt \right)^{1/p} < \infty.$$

When $p = q$ and $w = 1$, $L_{p,q,w}(\Omega_T)$ becomes the usual Lebesgue space $L_p(\Omega_T)$. We write $u \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ if there exists a sequence of smooth functions $\{u_n\}$ such that $u_n \in C^\infty\big([0,T] \times \Omega\big)$ vanishing for large $|x|$, $u_n(0,x) = 0, \partial_t u_n(0,x) = 0$, and

$$\|u_n - u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}(\Omega_T)} := \sum_{i=0}^{2} \|D^i u_n - D^i u\|_{p,q,w} + \|\partial_t^\alpha u_n - \partial_t^\alpha u\|_{p,q,w} \to 0$$

as $n \to \infty$, where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\Omega_T)}$. We often write $\mathbb{H}^{\alpha,2}_{p,w,0}(\Omega_T)$ if $p = q$ and use the notation $u \in \mathbb{H}^{\alpha,2}_{p,0,\mathrm{loc}}(\mathbb{R}^d_T)$ to indicate the function satisfying

$$u \in \mathbb{H}^{\alpha,2}_{p,0}((0,T) \times B_R) := \mathbb{H}^{\alpha,2}_{p,1,0}((0,T) \times B_R)$$

for any $R > 0$. Furthermore, we denote $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ if $u \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ satisfies $u = 0$ on $(0,T) \times \partial\Omega$. In other words, the defining sequence $u_n \to u$ in $\mathbb{H}^{\alpha,2}_{p,q,w}(\Omega_T)$ can be taken to satisfy the following conditions:

$$u_n \in C_0^\infty([0,T] \times \overline{\Omega}), \quad u_n(0,\cdot) = 0, \quad \partial_t u_n(0,\cdot) = 0, \quad u_n(\cdot,x) = 0 \quad \text{for} \quad x \in \partial\Omega.$$

In Section 2.5, we consider equations in divergence form. Therefore, we define the following. For $u \in L_{p,q,w}((S,T)\times\Omega)$, we say $\partial_t^\alpha u \in \mathbb{H}^{-1}_{p,q,w}((S,T)\times\Omega)$ if there exist $f, g_i \in L_{p,q,w}((S,T)\times\Omega)$, $i=1,\ldots,d$, such that for any $\phi \in C_0^\infty([S,T)\times\Omega)$,

$$\int_S^T \int_\Omega \partial_t^\alpha u \phi \, dx dt = \int_S^T \int_\Omega f\phi \, dx dt - \int_S^T \int_\Omega g_i D_i \phi \, dx dt, \tag{2.1.1}$$

and we write

$$\begin{aligned}
&\|\partial_t^\alpha u\|_{\mathbb{H}^{-1}_{p,q,w}((S,T)\times\Omega)} \\
&= \inf\Big\{ \sum_{i=1}^d \|g_i\|_{L_{p,q,w}((S,T)\times\Omega)} + \|f\|_{L_{p,q,w}((S,T)\times\Omega)} : (2.1.1) \text{ is satisfied}\Big\}.
\end{aligned}$$

Let $\mathcal{H}^{\alpha,1}_{p,q,w}((S,T)\times\Omega)$ be the collection of functions $u \in L_{p,q,w}((S,T)\times\Omega)$ such that $Du \in L_{p,q,w}((S,T)\times\Omega)$ and $\partial_t^\alpha u \in \mathbb{H}^{-1}_{p,q,w}((S,T)\times\Omega)$, and denote

$$\|u\|_{\mathcal{H}^{\alpha,1}_{p,q,w}((S,T)\times\Omega)} = \|\partial_t^\alpha u\|_{\mathbb{H}^{-1}_{p,q,w}((S,T)\times\Omega)} + \sum_{i=0}^1 \|D^i u\|_{L_{p,q,w}((S,T)\times\Omega)}.$$

Then, we define $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}((S,T)\times\Omega)$ if $u \in \mathcal{H}^{\alpha,1}_{p,q,w}((S,T)\times\Omega)$, and there exists a sequence of $\{u_n\} \subset C^\infty([S,T]\times\Omega)$ vanishing for large $|x|$, $u_n(S,x)=0$, $\partial_t u_n(S,x) = 0$ and

$$\|u_n - u\|_{\mathcal{H}^{\alpha,1}_{p,q,w}((S,T)\times\Omega)} \to 0 \quad \text{as} \quad n \to \infty.$$

We denote $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}((S,T)\times\Omega)$ if $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}((S,T)\times\Omega)$ satisfies $u=0$ on $(0,T)\times\partial\Omega$. In other words, the defining sequence $u_n \to u$ in $\mathcal{H}^{\alpha,1}_{p,q,w}((S,T)\times\Omega)$ can be taken to satisfy the following conditions:

$$u_n \in C_0^\infty([S,T]\times\overline{\Omega}), \quad u_n(S,\cdot)=0, \quad \partial_t u_n(S,\cdot)=0, \quad u_n(\cdot,x)=0 \quad \text{for} \quad x\in\partial\Omega.$$

Moreover, we say that $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}((S,T)\times\Omega)$ satisfies the divergence form equation

$$\partial_t^\alpha u - D_i(a^{ij}D_j u + a^i u) - b^i D_i u - cu = D_i g_i + f$$

for $f, g_i \in L_{p,q,w}((S,T)\times\Omega)$ if

$$\begin{aligned}&\int_S^T \int_\Omega I_S^{2-\alpha} u \partial_t^2 \phi \, dxdt + \int_S^T \int_\Omega (a^{ij} D_j u D_i \phi + a^i u D_i \phi - b^i D_i u \phi - cu\phi) \, dxdt \\ &= \int_S^T \int_\Omega (f\phi - g_i D_i \phi) \, dxdt\end{aligned}$$

for any $\phi \in C_0^\infty([S,T)\times\Omega)$.

In this paper, we use maximal functions and strong maximal functions defined, respectively, by

$$\mathcal{M} f(t,x) = \sup_{Q_r(s,y)\ni(t,x)} \fint_{Q_r(s,y)} |f(r,z)| \chi_{\mathcal{D}} \, dz \, dr$$

and

$$(\mathcal{SM} f)(t,x) = \sup_{Q_{r_1,r_2}(s,y)\ni(t,x)} \fint_{Q_{r_1,r_2}(s,y)} |f(r,z)| \chi_{\mathcal{D}} \, dz \, dr$$

for $f \in L_{1,\mathrm{loc}}$ defined on $\mathcal{D} \subset \mathbb{R}^{d+1}$ and $(t,x) \in \mathcal{D}$.

In order to consider equations with non-zero initial conditions, we introduce the trace space as follows. For $p, q \in (1,\infty)$, a non-integer $\theta > 0$ with k being the integer part of θ, and $w_2 \in A_q(\mathbb{R}^d)$, we define

$$\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d) := \{u \in L_{q,w_2}(\mathbb{R}^d) : \|u\|_{\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d)} < \infty\},$$

where

$$\|u\|_{\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d)} = [u]_{\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d)} + \sum_{|\alpha|\le k} \|D^\alpha u\|_{L_{q,w_2}(\mathbb{R}^d)}$$

and

$$[u]_{\mathcal{B}^{\theta}_{p,q,w_2}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \|D^k u(\cdot + y) - D^k u(\cdot)\|^p_{L_{q,w_2}} |y|^{-d-p(\theta-k)}\, dy \right)^{1/p}.$$

Note that when $p = q$ and $w_2 \equiv 1$, $\mathcal{B}^{\theta}_{p,q,w_2}(\mathbb{R}^d) = \mathcal{B}^{\theta}_{p,p}(\mathbb{R}^d)$, the usual Besov space. For $\theta_0 \in \mathbb{R}$, we denote $\mathcal{X}_0 = L_{q,w_2}$ when $\theta_0 < 0, \mathcal{X}_0 = \mathcal{B}^{\theta_0}_{p,q,w_2}(\mathbb{R}^d)$ when $\theta_0 > 0$ is a non-integer, and $\mathcal{X}_0 = \mathcal{B}^{\theta_0+\varepsilon}_{p,q,w_2}(\mathbb{R}^d)$ for arbitrary $\varepsilon > 0$ when θ_0 is a nonnegative integer. For a different $\theta_1 \in \mathbb{R}$, we define $\mathcal{X}_1$ similarly.

Finally, we write $N = N(\dots)$ if the constant N depends only on the parameters in the parentheses.

Next, we present the assumptions for the coefficients and domains. Assumption 2.1.1 for a fixed δ is imposed throughout the chapter 2.

Assumption 2.1.1. There exists $\delta \in (0, 1)$ such that for any $(t, x) \in \mathbb{R}^{d+1}$ and $\xi \in \mathbb{R}^d$,

$$a^{ij}(t, x)\xi^i\xi^j \geq \delta|\xi|^2 \quad \text{and} \quad |a^{ij}|, |b^i|, |c| \leq \delta^{-1}.$$

Assumption 2.1.2 (γ_0)**.** There exists $R_0 \in (0, 1]$ such that for any $(t_0, x_0) \in \mathbb{R}^{d+1}$ and $0 < r \leq R_0$,

$$\sup_{i,j} \fint_{Q_r(t_0,x_0)} |a^{ij} - \overline{a}^{ij}| \leq \gamma_0,$$

where

$$\overline{a}^{ij} = \fint_{Q_r(t_0,x_0)} a^{ij}\, dxdt.$$

Remark 2.1.3. If we take $x_0 \in \Omega = \mathbb{R}^d_+$ and define

$$\overline{a}^{ij} = \fint_{Q_r(t_0,x_0)\cap\Omega_T} a^{ij}\, dxdt,$$

then by Assumption 2.1.2 (γ_0), we have

$$\sup_{i,j} \fint_{Q_r(t_0,x_0)\cap\Omega_T} |a^{ij} - \overline{a}^{ij}|\, dxdt \le 4\gamma_0.$$

Assumption 2.1.4 (θ)**.** There exists $R_2 \in (0,1]$ such that for any $z \in \partial\Omega$, there exists a Lipschitz function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$\Omega \cap B_{R_2}(z) = \{x \in B_{R_2}(z) : x^1 > \phi(x')\}$$

and

$$\sup_{x',y' \in B'_{R_2}(z'), x' \neq y'} \frac{|\phi(x') - \phi(y')|}{|x' - y'|} \le \theta$$

in a new coordinate system.

We need a stronger assumption on the domains when non-divergence form equations are considered.

Assumption 2.1.5. There exists $R_2 \in (0,1]$ such that for any $z \in \partial\Omega$, there exists a $C^{1,1}$ function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$\Omega \cap B_{R_2}(z) = \{x \in B_{R_2}(z) : x^1 > \phi(x')\},$$

$$\phi(0) = 0, \quad \nabla\phi(0) = 0, \quad \text{and} \quad \sup_{B'_{R_2}(z')} |D^2\phi| \le \delta^{-1}$$

in a new coordinate system.

With the assumptions above, we are ready to state the main theorems of chapter 2.

Theorem 2.1.6. *Let $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \leq K_1$. There exists $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ such that under Assumption 2.1.2 (γ_0), the following holds. For any $u \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ satisfying*

$$\partial_t^\alpha u - a^{ij} D_{ij} u - b^i D_i u - cu = f \quad \text{in} \quad \mathbb{R}^d_T, \tag{2.1.2}$$

we have

$$\|\partial_t^\alpha u\|_{p,q,w} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} + \|D^2 u\|_{p,q,w} \leq N \|f\|_{p,q,w}, \tag{2.1.3}$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$ and $N = N(d,\delta,\alpha,p,q,K_1,R_0,T)$. Moreover, for any $f \in L_{p,q,w}(\mathbb{R}^d_T)$, there exists a unique solution $u \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ to (2.1.2).

Next, we introduce the results for the half space and smooth domains under the zero initial conditions and the zero boundary condition.

Theorem 2.1.7. *Let $\Omega = \mathbb{R}^d_+$, $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$ and $[w]_{p,q,\Omega} \leq K_1$. There exists $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ such that under Assumption 2.1.2 (γ_0), the following holds. For any $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ satisfying*

$$\partial_t^\alpha u - a^{ij} D_{ij} u - b^i D_i u - cu = f \quad \text{in} \quad \Omega_T, \tag{2.1.4}$$

we have

$$\|\partial_t^\alpha u\|_{p,q,w} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} + \|D^2 u\|_{p,q,w} \leq N \|f\|_{p,q,w},$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\Omega_T)}$ and $N = N(d,\delta,\alpha,p,q,K_1,R_0,T)$. Moreover, for any $f \in L_{p,q,w}(\Omega_T)$, there exists a unique solution $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ to (2.1.4).

Theorem 2.1.8. *Let Ω be a $C^{1,1}$ domain, $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$,*

$K_1 \in [1,\infty)$ and $[w]_{p,q,\Omega} \leq K_1$. There exist a constant $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ and $\theta = \theta(d,\delta,\alpha,p,q,K_1) \in (0,1/3)$ such that under Assumption 2.1.2 (γ_0) and Assumption 2.1.5(θ), the following holds. For any $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ satisfying

$$\partial_t^\alpha u - a^{ij}D_{ij}u - b^iD_iu - cu = f \quad in \quad \Omega_T, \tag{2.1.5}$$

we have

$$\|\partial_t^\alpha u\|_{p,q,w} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} + \|D^2u\|_{p,q,w} \leq N\|f\|_{p,q,w}, \tag{2.1.6}$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\Omega_T)}$ and $N = N(d,\delta,\alpha,p,q,K_1,R_0,R_2,T)$. Moreover, for any $f \in L_{p,q,w}(\Omega_T)$, there exists a unique solution $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ to (2.1.5).

Next we state the results for equations in divergence form under the zero initial conditions and the zero boundary condition.

Theorem 2.1.9. *Let $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \leq K_1$. There exists $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ such that under Assumption 2.1.2 (γ_0), the following holds. For any $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ satisfying*

$$\partial_t^\alpha u - D_i(a^{ij}D_ju + a^iu) - b^iD_iu - cu = D_ig_i + f \quad in \quad \mathbb{R}^d_T, \tag{2.1.7}$$

we have

$$\|\partial_t^\alpha u\|_{\mathbb{H}^{-1}_{p,q,w}} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} \leq N(\|f\|_{p,q,w} + \|g\|_{p,q,w}), \tag{2.1.8}$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$ and $N = N(d,\delta,\alpha,p,q,K_1,R_0,T)$. Moreover, for any $f, g_i \in L_{p,q,w}(\mathbb{R}^d_T)$, there exists a unique solution $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ to (2.1.7).

Theorem 2.1.10. *Let $\Omega = \mathbb{R}^d_+$, $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$,* *and* $[w]_{p,q,\Omega} \leq K_1$. *There exists* $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ *such that under Assumption 2.1.2* (γ_0), *the following holds. For any* $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$ *satisfying*

$$\partial_t^\alpha u - D_i(a^{ij}D_ju + a^iu) - b^iD_iu - cu = D_ig_i + f \quad \text{in} \quad \Omega_T, \tag{2.1.9}$$

we have

$$\|\partial_t^\alpha u\|_{\mathbb{H}^{-1}_{p,q,w}} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} \leq N(\|f\|_{p,q,w} + \|g\|_{p,q,w}), \tag{2.1.10}$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\Omega_T)}$ *and* $N = N(d,\delta,\alpha,p,q,K_1,R_0,R_2,T)$. *Moreover, for any* $f, g_i \in L_{p,q,w}(\Omega_T)$, *there exists a unique solution* $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$ *to (2.1.9).*

Theorem 2.1.11. *Let* Ω *be a Lipschitz domain,* $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, *and* $[w]_{p,q,\Omega} \leq K_1$. *There exist* $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ *and* $\theta = \theta(d,\delta,\alpha,p,q,K_1) \in (0,1/3)$ *such that under Assumption 2.1.2* (γ_0) *and Assumption 2.1.4* (θ), *the following holds. For any* $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$ *satisfying*

$$\partial_t^\alpha u - D_i(a^{ij}D_ju + a^iu) - b^iD_iu - cu = D_ig_i + f \quad \text{in} \quad \Omega_T, \tag{2.1.11}$$

we have

$$\|\partial_t^\alpha u\|_{\mathbb{H}^{-1}_{p,q,w}} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} \leq N(\|f\|_{p,q,w} + \|g\|_{p,q,w}),$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\Omega_T)}$ *and* $N = N(d,\delta,\alpha,p,q,K_1,R_0,R_2,T)$. *Also, for any* $f, g_i \in L_{p,q,w}(\Omega_T)$, *there exists a unique solution* $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$ *to (2.1.11).*

Finally, we state the results for equations in the whole space with nonzero initial conditions,

$$D_t^\alpha(u - u_0 - tu_1) - (a^{ij}D_{ij}u + b^iD_iu + cu) = f \quad \text{in} \quad \mathbb{R}^d_T,$$

which is equivalent to (1.1.4).

Corollary 2.1.12. *Let* $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, $\mu \in (-1, p-1)$, *and* $w = w_1(t)w_2(x)$, *where* $w_1(t) = t^\mu$ *and* $w_2(x) \in A_q(\mathbb{R}^d, dx)$ *with* $[w_2]_{A_q} \le K_1$. *Let* $\theta_0 = 2-2(1+\mu)/(\alpha p)$, $\theta_1 = 2-2/\alpha-2(1+\mu)/(\alpha p)$, *and recall the definitions of the spaces* $\mathcal{X}_0$ *and* $\mathcal{X}_1$ *above. There exists* $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1,\mu) \in (0,1)$ *such that, under Assumption 2.1.2* (γ_0), *the following holds. For any* $f \in L_{p,q,w}(\mathbb{R}^d_T)$, $u_0 \in \mathcal{X}_0$, *and* $u_1 \in \mathcal{X}_1$, *there exists a unique function* u *in* $\mathbb{R}^d_T$ *satisfying* (1.1.4) *with the estimate*

$$
\begin{aligned}
&\|\partial_t^\alpha u\|_{p,q,w} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} + \|D^2 u\|_{p,q,w} \\
&\le N\|f\|_{p,q,w} + N\|u_0\|_{\mathcal{X}_0} + N\|u_1\|_{\mathcal{X}_1},
\end{aligned}
\tag{2.1.12}
$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$ *and* $N = N(d,\delta,\alpha,p,q,K_1,K_0,R_0,T,\mu)$.

For equations in divergence form

$$
\begin{cases}
\partial_t^\alpha u - D_i(a^{ij}D_j u + a^i u) - b^i D_i u - cu = D_i g_i + f & \text{in} \quad \mathbb{R}^d_T \\
u(0,\cdot) = u_0 := D u_{0,1}(\cdot) + u_{0,2}(\cdot) & \text{in} \quad \mathbb{R}^d \\
u_t(0,\cdot) = u_1 := D u_{1,1}(\cdot) + u_{1,2}(\cdot) & \text{in} \quad \mathbb{R}^d
\end{cases}
\tag{2.1.13}
$$

that is,

$$
D_t^\alpha(u - u_0 - tu_1) - D_i(a^{ij}D_j u + a^i u) - b^i D_i u - cu = f \quad \text{in} \quad \mathbb{R}^d_T,
$$

we have the following results.

Corollary 2.1.13. *Let* $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, $\mu \in (-1, p-1)$, *and* $w = w_1(t)w_2(x)$, *where* $w_1(t) = t^\mu$ *and* $w_2(x) \in A_q(\mathbb{R}^d, dx)$ *with*

$[w_2]_{A_q} \le K_1$. *Let* $\theta_0 = 2-2(1+\mu)/(\alpha p)$, $\theta_1 = 2-2/\alpha-2(1+\mu)/(\alpha p)$, *and recall the definitions of the spaces* $\mathcal{X}_0$ *and* $\mathcal{X}_1$ *above. There exists* $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1,\mu) \in (0,1)$ *such that, under Assumption 2.1.2* (γ_0)*, the following holds. For any* $f, g \in L_{p,q,w}(\mathbb{R}^d_T)$, $u_{0,1}, u_{0,2} \in \mathcal{X}_0$, *and* $u_{1,1}, u_{1,2} \in \mathcal{X}_1$, *there exists a unique function* u *in* $\mathbb{R}^d_T$ *satisfying* (2.1.13) *with the estimate*

$$
\begin{aligned}
&\|\partial_t^\alpha u\|_{\mathbb{H}^{-1}_{p,q,w}} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} \\
&\le N\||f|+|g|\|_{p,q,w} + N\||u_{0,1}|+|u_{0,2}|\|_{\mathcal{X}_0} + N\||u_{1,1}|+|u_{1,2}|\|_{\mathcal{X}_1},
\end{aligned}
$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$ *and* $N = N(d,\delta,\alpha,p,q,K_1,K_0,R_0,T,\mu)$.

At the end of this section, we derive an useful property of functions satisfying Assumption 2.1.2 (γ_0). The following lemma implies that Assumption 2.1.2 (γ_0) gives a control over the mean oscillation on sets that are large in the time direction and small in the spatial directions.

Lemma 2.1.14. *If* a *satisfies Assumption 2.1.2* (γ_0), $h \ge r^{2/\alpha} = \widetilde{r}$, *and* $r \le R_1 \le R_0$, *where* $R_1 = 2^{-\alpha/2}R_0$. *Then*

$$
⨍_{(t_0-h,t_0)} ⨍_{B_r(x_0)} |a-\overline{a}|\,dxdt < Nhr^{-2/\alpha}\gamma_0,
$$

where $\overline{a} = ⨍_{Q_r(t_0,x_0)} a$ *and* $N = N(\alpha,d)$.

Proof. Suppose that $(k-1)\widetilde{r} \le h < k\widetilde{r}$ for some integer $k \ge 2$. Then,

$$
\begin{aligned}
⨍_{(t_0-h,t_0)} ⨍_{B_r(x_0)} |a-\overline{a}|\,dxdt &\le \frac{1}{h}\int_{t_0-k\widetilde{r}}^{t_0} ⨍_{B_r(x_0)} |a-\overline{a}|\,dxdt \\
&\le \frac{\widetilde{r}}{h}\sum_{j=0}^{k-1} ⨍_{t_0-(j+1)\widetilde{r}}^{t_0-j\widetilde{r}} ⨍_{B_r(x_0)} |a-\overline{a}|\,dxdt. \qquad (2.1.14)
\end{aligned}
$$

For each term in the sum of the right-hand side of (2.1.14), we have

$$\begin{aligned}
\fint_{t_0-(j+1)\widetilde{r}}^{t_0-j\widetilde{r}}\fint_{B_r(x_0)} |a-\overline{a}|\,dxdt &\le \fint_{t_0-(j+1)\widetilde{r}}^{t_0-j\widetilde{r}}\fint_{B_r(x_0)} (|a-\overline{b}_j|+|\overline{a}-\overline{b}_j|)\,dxdt \\
&\le \gamma_0 + |\overline{a}-\overline{b}_j|, \qquad (2.1.15)
\end{aligned}$$

where

$$\overline{b}_j = \fint_{t_0-(j+1)\widetilde{r}}^{t_0-j\widetilde{r}}\fint_{B_r(x_0)} a\,dxdt \quad \text{for} \quad j=0,1,2,\ldots.$$

Note that $\overline{b}_0 = \overline{a}$. By the triangle inequality,

$$|\overline{a}-\overline{b}_j| \le \sum_{i=0}^{j-1} |\overline{b}_i - \overline{b}_{i+1}|. \qquad (2.1.16)$$

Also, we have

$$\begin{aligned}
|\overline{b}_i-\overline{b}_{i+1}| &= \left| \fint_{t_0-(i+1)\widetilde{r}}^{t_0-i\widetilde{r}}\fint_{B_r(x_0)} a\,dxdt - \fint_{t_0-(i+2)\widetilde{r}}^{t_0-(i+1)\widetilde{r}}\fint_{B_r(x_0)} a\,dxdt \right| \\
&\le \left| \fint_{t_0-(i+1)\widetilde{r}}^{t_0-i\widetilde{r}}\fint_{B_r(x_0)} a\,dxdt - \fint_{t_0-(i+2)\widetilde{r}}^{t_0-i\widetilde{r}}\fint_{B_{2^{\alpha/2}r}(x_0)} a\,dxdt \right| \\
&\quad + \left| \fint_{t_0-(i+2)\widetilde{r}}^{t_0-(i+1)\widetilde{r}}\fint_{B_r(x_0)} a\,dxdt - \fint_{t_0-(i+2)\widetilde{r}}^{t_0-i\widetilde{r}}\fint_{B_{2^{\alpha/2}r}(x_0)} a\,dxdt \right| \\
&\le 2\frac{2\widetilde{r}\left(2^{\alpha/2}r\right)^d}{\widetilde{r}r^d}\gamma_0 = 4 2^{d\alpha/2}\gamma_0, \qquad (2.1.17)
\end{aligned}$$

where for the last inequality, we used that for $A \subset B$,

$$\left| \fint_A f - \fint_B f \right| \le \frac{|B|}{|A|}\fint_B |f-(f)_B|, \qquad (2.1.18)$$

and the oscillation of a on $Q_{2^{\alpha/2}r}(t_0-i\widetilde{r},x_0)$ can be controlled by γ_0. Therefore, by

(2.1.16) and (2.1.17),

$$|\bar{a} - \bar{b}_j| \le \sum_{i=0}^{j-1} |\bar{b}_i - \bar{b}_{i+1}| \le 4j2^{d\alpha/2}\gamma_0,$$

which together with (2.1.15) implies that

$$\begin{aligned}
&\fint_{(t_0-h,t_0)} \fint_{B_r(x_0)} |a - \bar{a}| \, dxdt \le \frac{1}{k-1} \sum_{j=0}^{k-1} [r_0 + |\bar{a} - \bar{b}_j|] \\
&\le \frac{1}{k-1} \sum_{j=0}^{k-1} [42^{d\alpha/2} j + 1]\gamma_0 \le N(\alpha, d) k\gamma_0 \le N h r^{-2/\alpha} \gamma_0.
\end{aligned}$$

The lemma is proved. □

2.2 Mean Oscillation Estimates

Throughout the section, we assume that a^{ij}'s are constants, $b^i \equiv c \equiv 0$, $T \in (0, \infty)$, and $p = q$, i.e., the space with unmixed norm. The main goal of this section is to derive a mean oscillation estimate of $u \in \mathbb{H}^{\alpha,2}_{p_0,0,\mathrm{loc}}(\mathbb{R}^d_T)$ satisfying

$$\partial_t^\alpha u - a^{ij} D_{ij} u = f \quad \text{in} \quad \mathbb{R}^d_T$$

when $p_0 \in (1, 2)$. For $t_0 \in (0, T]$, $r > 0$, and $\Omega = (-r, r)^d \supset B_r$, we are going to decompose $u = v + w$ in $(0, t_0) \times \Omega$, where $w, v \in \mathbb{H}^{\alpha,2}_{p_0,0}((0, t_0) \times \Omega)$ satisfy

$$\partial_t^\alpha w - a^{ij} D_{ij} w = f \tag{2.2.1}$$

in $(0, t_0) \times \Omega$, $w = 0$ on $(0, t_0) \times \partial\Omega$, and

$$\partial_t^\alpha v - a^{ij} D_{ij} v = 0 \tag{2.2.2}$$

in $(0, t_0) \times \Omega$.

In Subsections 2.2.1 and 2.2.2, we estimate v and w separately. Then, in Subsection 2.2.3, we combine the results to get the estimate of u.

2.2.1 Estimates of v, the homogeneous part

We begin with a formula for computing the fractional derivative of the product with a cutoff function in time.

Lemma 2.2.1. *Let $p \in [1, \infty)$, $\alpha \in (1, 2)$, $-\infty < S < t_0 < T < \infty$, and $v \in \mathbb{H}_{p,0}^{\alpha,2}((S, T) \times \Omega)$. Then, for any infinitely differentiable function η defined on $\mathbb{R}$ such that $\eta(t) = 0$ for $t \leq t_0$, we have $\eta v \in \mathbb{H}_{p,0}^{\alpha,2}((t_0, T) \times \Omega)$ and*

$$\partial_t^\alpha (\eta v)(t, x) = \partial_t^2 I_{t_0}^{2-\alpha} (\eta v)(t, x) = \eta(t) \partial_t^2 I_S^{2-\alpha} v(t, x) + g(t, x)$$

for $(t, x) \in (t_0, T) \times \Omega$, where

$$\begin{aligned} g(t, x) = {} & \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} \int_S^t (t - s)^{-\alpha - 1} [\eta(s) - \eta(t) - \eta'(t)(s - t)] v(s, x) \, ds \\ & + \frac{\alpha}{\Gamma(2 - \alpha)} \eta'(t) \partial_t \int_S^t (t - s)^{1 - \alpha} v(s, x) \, ds. \end{aligned} \tag{2.2.3}$$

Proof. The fact that $\eta v \in \mathbb{H}_{p,0}^{\alpha,2}((t_0, T) \times \Omega)$ follows from the density of smooth functions in the space. The proof for the case of $\alpha \in (0, 1)$ is given in (Dong and Kim, 2019, Lemma 3.6), and it works here with minor modification. We skip the details. It remains to show (2.2.3). Without loss of generality, we assume $t_0 = 0$. Also, recall that $\partial_t^\alpha = \partial_t^2 I_S^{2-\alpha}$. Therefore,

$$\begin{aligned}
&g(t,x)\Gamma(2-\alpha) \\
&= \partial_t^2\Big[\int_0^t (t-s)^{1-\alpha}\eta(s)v(s,x)\,ds\Big] - \partial_t^2\Big[\eta(t)\int_S^t (t-s)^{1-\alpha}v(s,x)\,ds\Big] \\
&\quad + 2\eta'(t)\partial_t \int_S^t (t-s)^{1-\alpha}v(s,x)\,ds + \eta''(t)\int_S^t (t-s)^{1-\alpha}v(s,x)\,ds.
\end{aligned} \tag{2.2.4}$$

The first two terms of the right-hand side of (2.2.4) equal

$$\begin{aligned}
&\partial_t^2\Big[\int_S^t (t-s)^{1-\alpha}[\eta(s)-\eta(t)]v(s,x)\,ds\Big] \\
&= \partial_t\Big[\int_S^t (1-\alpha)(t-s)^{-\alpha}[\eta(s)-\eta(t)]v(s,x)\,ds - \int_S^t (t-s)^{1-\alpha}\eta'(t)v(s,x)\,ds\Big] \\
&= \partial_t\Big[\int_S^t (1-\alpha)(t-s)^{-\alpha}[\eta(s)-\eta(t)-\eta'(t)(s-t)]v(s,x)\,ds \\
&\quad + \int_S^t (\alpha-2)(t-s)^{1-\alpha}\eta'(t)v(s,x)\,ds\Big] \\
&= \int_S^t (-\alpha)(1-\alpha)(t-s)^{-\alpha-1}[\eta(s)-\eta(t)-\eta'(t)(s-t)]v(s,x)\,ds \\
&\quad + \int_S^t (1-\alpha)(t-s)^{1-\alpha}\eta''(t)v(s,x)\,ds + (\alpha-2)\eta''(t)\int_S^t (t-s)^{1-\alpha}v(s,x)\,ds \\
&\quad + (\alpha-2)\eta'(t)\partial_t\Big[\int_S^t (t-s)^{1-\alpha}v(s,x)\,ds\Big].
\end{aligned}$$

Thus, by (2.2.4),

$$\begin{aligned}
g(t,x)\Gamma(2-\alpha) = \alpha(\alpha-1)\int_S^t (t-s)^{-\alpha-1}[\eta(s)-\eta(t)-\eta'(t)(s-t)]v(s,x)\,ds \\
+ \alpha\eta'(t)\partial_t\int_S^t (t-s)^{1-\alpha}v(s,x)\,ds.
\end{aligned}$$

The lemma is proved. □

Remark 2.2.2. If there exists a function $\xi \in C^\infty(\mathbb{R})$ such that $\xi(t) = 0$ for $t \le t_0'$ for some $t_0' \in (S, t_0)$, and $\xi \equiv 1$ in $\operatorname{supp}\eta$, then we can rewrite the second term of g in

(2.2.3) as follows. With $\beta = \alpha - 1$,

$$\begin{aligned}
&\eta'(t)\partial_t \int_S^t (t-s)^{1-\alpha} v(s,x)\,ds = \frac{\eta'(t)}{\xi(t)}\left(\Gamma(1-\beta)\xi(t,x)\partial_t^\beta v(t,x)\right) \\
&= \frac{\eta'(t)}{\xi(t)}\left(\Gamma(1-\beta)\Big[\xi(t,x)\partial_t^\beta v(t,x) - \partial_t^\beta(\xi v)(t,x) + \partial_t^\beta(\xi v)(t,x)\Big]\right) \\
&= \frac{\eta'(t)}{\xi(t)}\left(\frac{\Gamma(1-\beta)\beta}{\Gamma(1-\beta)}\int_S^t (t-s)^{-\beta-1}[\xi(s)-\xi(t)]v(s,x)\,ds + \Gamma(1-\beta)\partial_t^\beta(\xi v)(t,x)\right) \\
&= \frac{\eta'(t)}{\xi(t)}\left(\Big[(\alpha-1)\int_S^t (t-s)^{-\alpha}[\xi(s)-\xi(t)]v(s,x)\,ds\Big] + \Gamma(2-\alpha)\partial_t^\beta(\xi v)(t,x)\right),
\end{aligned}$$

where we used (Dong and Kim, 2019, Lemma 3.6) in the second last equality.

The following interpolation inequality for the time derivatives will be used for the estimate of g in Lemma 2.2.1.

Lemma 2.2.3. *For any $\varepsilon > 0$, $\beta = \alpha - 1$, $p_0 \in (1,\infty)$, any open set $G \subset \mathbb{R}^d$, and $v \in \mathbb{H}^{\alpha,2}_{p_0,0}((0,T)\times G)$, we have*

$$\|\partial_t^\beta v\|_{L_{p_0}((0,T)\times G)} \le \varepsilon \|\partial_t^\alpha v\|_{L_{p_0}((0,T)\times G)} + N(\alpha,p_0)T^{\alpha-2}\varepsilon^{-1}\|v\|_{L_{p_0}((0,T)\times G).}$$

Proof. Using 2.6.1 and (Dong and Kim, 2021, Lemma 5.5), we conclude

$$\begin{aligned}
\|\partial_t^\beta v\|_{L_{p_0}((0,T)\times G)} &\le \varepsilon \|\partial_t^\alpha v\|_{L_{p_0}((0,T)\times G)} + N\varepsilon^{-1}\|I^{2-\alpha} v\|_{L_{p_0}((0,T)\times G)} \\
&\le \varepsilon \|\partial_t^\alpha v\|_{L_{p_0}((0,T)\times G)} + N(\alpha,p_0)T^{2-\alpha}\varepsilon^{-1}\|v\|_{L_{p_0}((0,T)\times G)}.
\end{aligned}$$

Note that in (Dong and Kim, 2021, Lemma 5.5), the author assumes $\alpha \in (0,1)$, but the proof works for all $\alpha \in (0,\infty)$. ∎

Next, we estimate v by using cutoff functions together with the Sobolev embedding.

Proposition 2.2.4. *Let $p_0 \in (1,\infty)$, $t_0 \in (0,\infty)$, $r > 0$, and $v \in \mathbb{H}^{\alpha,2}_{p_0,0}((0,t_0)\times B_r)$ satisfy (2.2.2) in $(0,t_0)\times B_r$. Then, for any $p_1 \in (p_0,\infty)$ satisfying*

$$1/p_1 \le 1/p_0 - 2/(d+2),$$

we have $v \in \mathbb{H}^{\alpha,2}_{p_1,0}((0,t_0)\times B_{r/2})$ and

$$(|D^2v|^{p_1})^{1/p_1}_{Q_{r/2}(t_1,0)} \le N \sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2v|^{p_0})^{1/p_0}_{Q_r(t_1-(j-1)r^{2/\alpha},0)} \tag{2.2.5}$$

for any $t_1 \le t_0$, where $N = N(d,\delta,\alpha,p_1,p_0)$. If $p_0 > d/2+1$, then

$$[D^2v]_{C^{\sigma\alpha/2,\sigma}(Q_{r/2}(t_1,0))} \le N r^{-\sigma} \sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2v|^{p_0})^{1/p_0}_{Q_r(t_1-(j-1)r^{2/\alpha},0)} \tag{2.2.6}$$

for any $t_1 \le t_0$, where $\sigma = \sigma(d,\alpha,p_0) \in (0,1)$ and $N = N(d,\delta,\alpha,p_0)$. Moreover, for any $p_1 \in (p_0,\infty)$, we have

$$(|D^2v|^{p_1})^{1/p_1}_{Q_{r/2}(t_1,0)} \le N \sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2v|^{p_0})^{1/p_0}_{Q_r(t_1-(j-1)r^{2/\alpha},0)} \tag{2.2.7}$$

for any $t_1 \le t_0$, where $N = N(d,\delta,\alpha,p_1,p_0)$. Furthermore,

$$[D^2v]_{C^{\sigma\alpha/2,\sigma}(Q_{r/2}(t_1,0))} \le N r^{-\sigma} \sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2v|^{p_0})^{1/p_0}_{Q_r(t_1-(j-1)r^{2/\alpha},0)} \tag{2.2.8}$$

for any $t_1 \le t_0$, where $\sigma = \sigma(d,\alpha,p_0) \in (0,1)$ and $N = N(d,\delta,\alpha,p_0)$.

Proof. Step 0. We first note that (2.2.7) and (2.2.8) follow from (2.2.5) and (2.2.6) by a finite iteration argument. See (Dong and Kim, 2021, Proposition 4.3) for details.

Step 1. By scaling, we assume $r = 1$. For $k = 0, 1, \dots$, let $\delta = \delta_0 = (1/2)^{2/\alpha}$ and

$$\delta_k = \delta + (1-\delta)\sum_{j=1}^{k} 2^{-j},$$

and take $\eta_k \in C^\infty(\mathbb{R})$ such that

$$\eta_k(t) = \begin{cases} 1 & t \in (t_1 - \delta_k, t_1), \\ 0 & t \in (t_1 - 1, t_1 - \delta_{k+1}), \end{cases}$$

$$|\eta_k'| \le (1-\delta)^{-1} 2^{k+2}, \quad |\eta_k''| \le (1-\delta)^{-2} 2^{2k+4}. \tag{2.2.9}$$

Note that $\eta_{k+1} \equiv 1$ in $\operatorname{supp} \eta_k$, and v can be extended as zero for $t \le 0$. Therefore, by Lemma 2.2.1, for any k, we have

$$\eta_k v \in \mathbb{H}^{\alpha,2}_{p_0,0}((t_1 - 1, t_1) \times B_{3/4}).$$

Furthermore, let

$$\begin{aligned} g_k &= \partial_t^\alpha(\eta_k v) - \eta_k \partial_t^\alpha v \quad \text{in} \quad (t_1 - 1, t_1) \times B_{3/4}, \\ r_k &= \frac{3}{4} - \frac{1}{4}\frac{1}{2^k}, \quad \text{and} \quad \Omega_k := (t_1 - 1, t_1) \times B_{r_k} \end{aligned} \tag{2.2.10}$$

with $r_0 = 1/2$ and $r_\infty = 3/4$. Then,

$$\begin{aligned} \|\eta_k v\|_{\mathbb{H}^{\alpha,2}_{p_0}(\Omega_k)} &\le N\|\eta_k v\|_{L_{p_0}(\Omega_k)} + \|D^2(\eta_k v)\|_{L_{p_0}(\Omega_k)} + \|\partial_t^\alpha(\eta_k v)\|_{L_{p_0}(\Omega_k)} \\ &\le N\|\eta_k v\|_{L_{p_0}(\Omega_k)} + N\|g_k\|_{L_{p_0}(\Omega_{k+1})} + \frac{N}{(r_{k+1} - r_k)^2}\|\eta_k v\|_{L_{p_0}(\Omega_{k+1})} \\ &\le N 2^{k+2}\|v\|_{L_{p_0}(\Omega_\infty)} + N\|g_k\|_{L_{p_0}(\Omega_{k+1})}, \end{aligned} \tag{2.2.11}$$

where $N = N(d, \delta, \alpha, p_0)$, and the second inequality is by a local estimate which can be proved using the estimate in whole space in Kim et al. (2017) together with a localization argument as in (Dong and Kim, 2019, Lemma 4.2).

To estimate g_k, we take $\zeta \in C^\infty(\mathbb{R})$ such that

$$|\zeta| \le 1 \quad \text{and} \quad \zeta(t) = \begin{cases} 1 & \text{when} \quad t \in (t_1 - 2, t_1), \\ 0 & \text{when} \quad t \in (-\infty, t_1 - 3). \end{cases} \tag{2.2.12}$$

Furthermore, let $v_1 = \zeta v$ and $v_2 = v - v_1$. Lemma 2.2.1 with $S = -\infty$ yields

$$|g_k| \le N(\alpha)\big[|g_{k,1}| + |g_{k,2}|\big],$$

where

$$\begin{aligned} g_{k,1}(t,x) = \alpha(\alpha-1)\int_{t_1-3}^{t} (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)]v_1(s,x)\,ds \\ + \alpha\eta_k'(t)\partial_t \int_{t_1-3}^{t} (t-s)^{1-\alpha} v_1(s,x)\,ds, \end{aligned} \tag{2.2.13}$$

and

$$\begin{aligned} g_{k,2}(t,x) &= \alpha(\alpha-1)\int_{-\infty}^{t_1-2} (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)]v_2(s,x)\,ds \\ &\quad + \alpha\eta_k'(t)\partial_t \int_{-\infty}^{t_1-2} (t-s)^{1-\alpha} v_2(s,x)\,ds \\ &= \alpha(\alpha-1)\int_{-\infty}^{t_1-2} (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t)]v_2(s,x)\,ds. \end{aligned} \tag{2.2.14}$$

Step 2. We further decompose $g_{k,1}$ as

$$g_{k,1} = \alpha(\alpha-1)g_{k,1,1} + \alpha g_{k,1,2},$$

where

$$g_{k,1,1}(t,x) = \int_{t_1-3}^{t} (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)]v_1(s,x)\,ds.$$

For $t \in (t_1 - 1, t_1)$,

$$\begin{aligned}
|g_{k,1,1}(t,x)| &\le \|\eta_k''\|_{L_\infty} \cdot \int_{t_1-3}^{t} |(t-s)^{-\alpha+1} v(s,x)|\,ds \\
&\le \|\eta_k''\|_{L_\infty} \int_{t-3}^{t} |(t-s)^{-\alpha+1} v(s,x)|\,ds \\
&= \|\eta_k''\|_{L_\infty} \int_{0}^{3} \tau^{-\alpha+1} |v(t-\tau,x)|\,d\tau,
\end{aligned}$$

and by the Minkowski inequality,

$$\|g_{k,1,1}\|_{L_{p_0}(\Omega_{k+1})} \le N\|\eta_k''\|_{L_\infty}\|v\|_{L_{p_0}((t_1-4,t_1)\times B_{3/4})}. \tag{2.2.15}$$

For $\beta := \alpha - 1 \in (0,1)$, using the facts that $v_1 \in \mathbb{H}^{\alpha,2}_{p_0,0}((t_1-3,t_1)\times B_{3/4}$ and $\eta_{k+1} \equiv 1$ in $\operatorname{supp}\eta_k$ together with Remark 2.2.2, we have

$$\begin{aligned}
g_{k,1,2}(t,x) = \frac{\eta_k'(t)}{\eta_{k+1}(t)} \Big[&(\alpha-1)\int_{t_1-3}^{t} (t-s)^{-\alpha}[\eta_{k+1}(s) - \eta_{k+1}(t)]v_1(s,x)\,ds \\
&+ \Gamma(2-\alpha)\partial_t^{\beta}(\eta_{k+1}v_1)\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
|g_{k,1,2}(t,x)| \le{} & N(\alpha)\|\eta_k'\|_{L_\infty} \int_{t_1-3}^{t} (t-s)^{-\alpha}|\eta_{k+1}(s) - \eta_{k+1}(t)||v(s,x)|\,ds \\
&+ N(\alpha)\|\eta_k'\|_{L_\infty}|\partial_t^{\beta}(\eta_{k+1}v_1)| \\
\le{} & N(\alpha)\|\eta_k'\|_{L_\infty}\|\eta_{k+1}'\|_{L_\infty} \int_{t_1-3}^{t} (t-s)^{1-\alpha}|v(s,x)|\,ds \\
&+ N(\alpha)\|\eta_k'\|_{L_\infty}|\partial_t^{\beta}(\eta_{k+1}v_1)|.
\end{aligned}$$

By applying Lemma 2.2.3 to $\eta_{k+1}v_1 = \eta_{k+1}v \in \mathbb{H}^{\alpha,2}_{p_0,0}(\Omega_{k+1})$ and Young's inequality, for any $\varepsilon > 0$, we have

$$\begin{aligned} \|g_{k,1,2}\|_{L_{p_0}(\Omega_{k+1})} &\leq N\|\eta_k'\|_{L_\infty}\|\eta_{k+1}'\|_{L_\infty}\|v\|_{L_{p_0}(t_1-4,t_1)\times B_{3/4}(x_0))} \\ &\quad + \varepsilon\|\partial_t^\alpha(\eta_{k+1}v)\|_{L_{p_0}(\Omega_{k+1})} + N\varepsilon^{-1}\|\eta_k'\|^2_{L_\infty}\|\eta_{k+1}v\|_{L_p(\Omega_{k+1})}, \end{aligned} \tag{2.2.16}$$

where $N = N(\alpha, d, p_0)$.

Recall the definition of $g_{k,2}$ in (2.2.14). For $j \geq 2$, $t \in (t_1 - 1, t_1)$, and $s \in (t_1 - j - 1, t_1 - j)$, it is easily seen that $t - s > (j - 1) > (j + 1)/4$. Thus,

$$\begin{aligned} |g_{k,2}(t,x)| &\leq \int_{-\infty}^{t_1-2}(t-s)^{-\alpha-1}|v(s,x)|\,ds \leq \sum_{j=2}^{\infty}\int_{t_1-j-1}^{t_1-j}(t-s)^{-\alpha-1}|v(s,x)|\,ds \\ &\leq N(\alpha)\sum_{j=2}^{\infty}\int_{t_1-j-1}^{t_1-j}(j+1)^{-\alpha-1}|v(s,x)|\,ds. \end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned} \|g_{k,2}\|_{L_{p_0}(\Omega_{k+1})} &\leq N\sum_{j=2}^{\infty}(j+1)^{-(\alpha+1)}\|v\|_{L_{p_0}((t_1-j-1,t_1-j)\times B_{3/4})} \\ &\leq N\sum_{j=2}^{\infty}(j+1)^{-(\alpha+1)}\|v\|_{L_{p_0}((t_1-j-1,t_1-j)\times B_1)}, \end{aligned} \tag{2.2.17}$$

where $N = N(\alpha, d, p_0)$. Combining (2.2.15), (2.2.16), and (2.2.17), we get

$$
\begin{aligned}
&\|g_k\|_{L_{p_0}(\Omega_{k+1})} \le N(\alpha)\|g_{k,2}\|_{L_{p_0}(\Omega_{k+1})} + N(\alpha)\|g_{k,1}\|_{L_{p_0}(\Omega_{k+1})} \\
&\le N \sum_{j=2}^{\infty} (j+1)^{-\alpha-1} \|v\|_{L_{p_0}((t_1-j-1,t_1-j)\times B_1)} \\
&\quad + N\|\eta_k''\|_{L_\infty} \sum_{j=0}^{3} \|v\|_{L_{p_0}((t_1-j-1,t_1-j)\times B_1)} \\
&\quad + N\|\eta_k'\|_{L_\infty}\|\eta_{k+1}'\|_{L_\infty} \sum_{j=0}^{3} \|v\|_{L_{p_0}((t_1-j-1,t_1-j)\times B_1)} \\
&\quad + \varepsilon\|\partial_t^\alpha(\eta_{k+1}v)\|_{L_{p_0}(\Omega_{k+1})} + N\varepsilon^{-1}\|\eta_k'\|_{L_\infty}^2 \|\eta_{k+1}v\|_{L_{p_0}(\Omega_{k+1})},
\end{aligned} \tag{2.2.18}
$$

where $N = N(\alpha, d, p_0)$.

Step 3. Multiplying both sides of (2.2.11) by ε^k, summing over $k = 0, 1, \ldots$, and using (2.2.18), we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \varepsilon^k \|\eta_k v\|_{\mathbb{H}_{p_0}^{\alpha,2}(\Omega_k)} \le N\|v\|_{L_{p_0}(\Omega_\infty)} \sum_{k=0}^{\infty} 2^k \varepsilon^k + \sum_{k=0}^{\infty} \varepsilon^k \|g_k\|_{L_{p_0}(\Omega_{k+1})} \\
&\le N\|v\|_{L_{p_0}(\Omega_\infty)} \sum_{k=0}^{\infty} 2^k \varepsilon^k + N \sum_{j=2}^{\infty} (j+1)^{-\alpha-1} \|v\|_{L_{p_0}(t_1-j-1,t_1-j)\times B_1} \sum_{k=0}^{\infty} \varepsilon^k \\
&\quad + N \sum_{j=0}^{3} \|v\|_{L_{p_0}(t_1-j-1,t_1-j)\times B_1} \sum_{k=0}^{\infty} \varepsilon^k 2^{2k} \\
&\quad + \sum_{k=0}^{\infty} \varepsilon^{k+1} \|\partial_t^\alpha(\eta_{k+1}v)\|_{L_{p_0}(\Omega_{k+1})} + N\|v\|_{L_{p_0}(\Omega_\infty)} \varepsilon^{-1} \sum_{k=0}^{\infty} \varepsilon^k 2^{2k},
\end{aligned} \tag{2.2.19}
$$

where $N = N(d, \delta, \alpha, p_0)$. Picking $\varepsilon < 1/8$ such that $\sum_{k=0}^{\infty} \varepsilon^k 2^{2k} < \infty$, and absorbing the second last summation to the left-hand side of (2.2.19), we derive

$$
\|\eta_0 v\|_{\mathbb{H}_{p_0}^{\alpha,2}(\Omega_0)} \le N \sum_{j=1}^{\infty} j^{-\alpha-1} \|v\|_{L_{p_0}((t_1-j,t_1-j+1)\times B_1)}.
$$

Therefore, by using the Sobolev embedding

$$\mathbb{H}^{\alpha,2}_{p_0,0}((t_1-1,t_1)\times B_{1/2}) \hookrightarrow \mathbf{W}^{1,2}_{p_0,0}((t_1-1,t_1)\times B_{1/2}) \hookrightarrow L_{p_1}((t_1-1,t_1)\times B_{1/2}),$$

we conclude

$$\begin{aligned} &\|v\|_{L_{p_1}(Q_{1/2}(t_1,0))} = \|\eta_0 v\|_{L_{p_1}(Q_{1/2}((t_1,0))} \le \|\eta_0 v\|_{L_{p_1}((t_1-1,t_1)\times B_{1/2})} \\ &\le N\|\eta_0 v\|_{\mathbb{H}^{\alpha,2}_{p_0}((t_1-1,t_1)\times B_{1/2})} \le N\sum_{j=1}^{\infty} j^{-\alpha-1}\|v\|_{L_{p_0}((t_1-j,t_1-j+1)\times B_1)}, \end{aligned} \tag{2.2.20}$$

where $N = N(d,\delta,\alpha,p_0,p_1)$.

For the proof of (2.2.6), we pick $\widetilde{p}_0 \le p_0$ such that $\widetilde{p}_0 \in (d/2+1, d+2)$, and let $\sigma = 2-(d+2)/\widetilde{p}_0 \in (0,1)$. By using the Sobolev embedding

$$\mathbb{H}^{\alpha,2}_{\widetilde{p}_0,0}((t_1-1,t_1)\times B_{1/2}) \hookrightarrow \mathbf{W}^{1,2}_{\widetilde{p}_0,0}((t_1-1,t_1)\times B_{1/2}) \hookrightarrow C^{\sigma/2,\sigma}((t_1-1,t_1)\times B_{1/2}),$$

we replace $\|v\|_{L_{p_1}(Q_{1/2}(t_1,0))}$ with $[v]_{C^{\sigma\alpha/2,\sigma}(Q_{r/2}(t_1,0))}$ in (2.2.20) to conclude

$$[v]_{C^{\sigma\alpha/2,\sigma}(Q_{1/2}(t_1,0))} \le N\sum_{j=1}^{\infty} j^{-\alpha-1}\|v\|_{L_{p_0}((t_1-j,t_1-j+1)\times B_1)}, \tag{2.2.21}$$

where $N = N(d,\delta,\alpha,p_0)$.

Step 4. It remains to prove $D^2 v \in \mathbb{H}^{\alpha,2}_{p_0,0}((0,t_1)\times B_{3/4})$. If so, (2.2.5) and (2.2.6) follow from replacing v with $D^2 v$ into (2.2.20) and (2.2.21). Note that by (Dong and Kim, 2019, Lemma 4.3), $(Dv)^\delta = D(v^\delta) \in \mathbb{H}^{\alpha,2}_{p_0,0}((0,t_1)\times B_{5/6})$ for δ sufficiently small, where $(Dv)^\delta$ and v^δ denote the mollification in the spatial variables. It follows that

$$(Dv)^\delta \to Dv \quad \text{in} \quad L_{p_0}((0,t_1)\times B_{5/6}).$$

By repeating the argument in Step 3, we have

$$\begin{aligned}\|(Dv)^\delta\|_{\mathbb{H}^{\alpha,2}_{p_0,0}((0,t_1)\times B_{5/6})} &\le N\sum_{j=1}^{\infty} j^{-\alpha-1}\|(Dv)^\delta\|_{L_{p_0}((t_1-j,t_1-j+1)\times B_{7/8})}\\ &\le N\sum_{j=1}^{\infty} j^{-\alpha-1}\|(Dv)^\delta\|_{L_{p_0}((0,t_1)\times B_{7/8})},\end{aligned}$$

where $N = N(d,\delta,\alpha,p_0)$. Therefore, $(Dv)^\delta$ is Cauchy in $\mathbb{H}^{\alpha,2}_{p_0,0}((0,t_1)\times B_{5/6})$, which implies

$$Dv \in \mathbb{H}^{\alpha,2}_{p_0,0}((0,t_1)\times B_{5/6}).$$

Replacing Dv with D^2v and repeating the above process, we conclude

$$D^2v \in \mathbb{H}^{\alpha,2}_{p_0,0}((0,t_1)\times B_{3/4}).$$

Finally, the statement $v \in \mathbb{H}^{\alpha,2}_{p_1,0}((0,t_1)\times B_{1/2})$ follows from a mollification argument. See (Dong and Kim, 2021, Lemma 4.1) for details. The proposition is proved. □

Remark 2.2.5. By slightly modifying the proof of (2.2.20), we derive an estimate that will be used later in Proposition 2.2.8. For $w \in \mathbb{H}^{\alpha,2}_{p_0,0}((0,t_0)\times B_1)$ satisfying (2.2.1), if we take cutoff functions η_k defined in (2.2.9) and g_k, Ω_k defined in (2.2.10) with w in place of v, then $\partial_t^\alpha(w\eta_k) - \Delta(w\eta_k) = f\eta_k + g_k$. Similar to (2.2.11), we have

$$\|\eta_k w\|_{\mathbb{H}^{\alpha,2}_{p_0}(\Omega_k)} \le N2^{k+2}\|w\|_{L_{p_0}(\Omega_\infty)} + N\|f\|_{L_{p_0}(\Omega_\infty)} + N\|g_k\|_{L_{p_0}(\Omega_{k+1})}. \qquad (2.2.22)$$

Furthermore, by redefining ζ in (2.2.12) by

$$\zeta(t) = \begin{cases} 1 & \text{when} \quad t\in(t_1-3,t_1),\\ 0 & \text{when} \quad t\in(-\infty,t_1-4),\end{cases}$$

the lower bounds of the integrals in (2.2.13) become $t_1 - 4$, and the upper bound of the integral in (2.2.14) becomes $t_1 - 3 = s_2$, where $s_j = t_1 - 2^j + 1$ for $j = 0, 1, \ldots$. In this case, we estimate $g_{k,1}$ similarly as in Step 2 above. However, we estimate $g_{k,2}$ differently:

$$\begin{aligned}|g_{k,2}(t,x)| &\le \int_{-\infty}^{t_1-3} (t-s)^{-\alpha-1}|v(s,x)|\,ds \le \sum_{j=2}^{\infty}\int_{s_{j+1}}^{s_j}(t-s)^{-\alpha-1}|v(s,x)|\,ds \\ &\le N(\alpha)\sum_{j=2}^{\infty}\int_{s_{j+1}}^{s_j} 2^{j(-\alpha-1)}|v(s,x)|\,ds,\end{aligned}$$

where the last inequality is because $t \in (t_1 - 1, t_1)$. Therefore, by repeating the argument in Step 3 above with (2.2.22), and applying the Minkowski inequality and Hölder's inequality, we conclude

$$(|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0,0)} \le N(|f|^{p_0})^{1/p_0}_{Q_1(t_0,0)} + N\sum_{k=0}^{\infty} 2^{-k\alpha}(|w|^{p_0})^{1/p_0}_{(s_{k+1},s_k)\times B_1},$$

where $N = N(d, \delta, \alpha, p_0)$.

2.2.2 Estimates of w, the non-homogeneous part

In this subsection, we prove the existence and the estimates of w satisfying (2.2.1) when $a^{ij} = \delta^{ij}$.

Lemma 2.2.6. *For $\Omega = (0,1)^d$ and $f \in L_2(\Omega_T)$, there exists a unique $w \in \mathbb{H}^{\alpha,2}_{2,0}(\Omega_T)$ to*

$$\partial_t^\alpha w - 4\Delta w = f \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad (0,T)\times\partial\Omega,$$

and w satisfies

$$\|w\|_{\mathbb{H}^{\alpha,2}_2(\Omega_T)} \le N\|f\|_{L_2(\Omega_T)}.$$

Proof. We use the method of separation of variables to construct w explicitly. Since

$f \in L_2((0,t_0)\times\Omega)$, we write

$$f(t,x^1,\dots,x^d) = \sum_{n_1,n_2,\dots,n_d=1}^{\infty} f_{n_1,\dots,n_d}(t)\sin(n_1\pi x^1)\cdots\sin(n_d\pi x^d).$$

By orthogonality,

$$f_{n_1,\dots,n_d}(t) = 2^d\int_0^1\cdots\int_0^1 \sin(n_1\pi x^1)\cdots\sin(n_d\pi x^d) f(t,x)\,dx.$$

We take

$$w(t,x^1,\dots,x^d) = \sum_{n_1,n_2,\dots,n_d=1}^{\infty} w_{n_1,\dots,n_d}(t)\sin(n_1\pi x^1)\cdots\sin(n_d\pi x^d). \tag{2.2.23}$$

Then,

$$\begin{aligned}\partial_t^\alpha w - 4\Delta w &= \sum_{n_1,\dots,n_d=1}^{\infty}\left[\partial_t^\alpha w_{n_1,\dots,n_d}(t) + 4(n_1\dots n_d)^2\pi^{2d}w_{n_1,\dots,n_d}(t)\right]\\ &\qquad\cdot\sin(n_1\pi x^1)\cdots\sin(n_d\pi x^d)\\ &= \sum_{n_1,n_2,\dots,n_d=1}^{\infty} f_{n_1,\dots,n_d}(t)\sin(n_1\pi x^1)\cdots\sin(n_d\pi x^d).\end{aligned}$$

To determine $w_{n_1,\dots,n_d}$, it is sufficient to solve the following ODE: for $\lambda>0$,

$$\partial_t^\alpha\phi(t) + \lambda\phi(t) = f(t), \quad \phi(0)=0, \quad \phi'(0)=0.$$

Let

$$\phi(t) = \int_0^t H_{\alpha,\lambda}(t-s)f(s)\,ds,$$

where

$$H_{\alpha,\lambda}(t) = I^{\alpha-1}E_\alpha(-\lambda t^\alpha), \quad \text{and} \quad E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}$$

is the Mittag-Leffler function. We check that this ϕ solves the differential equation.

It suffices to check that

$$L\phi = L(H_{\alpha,\lambda}) \cdot L(f) = \frac{Lf(s)}{s^\alpha + \lambda}, \tag{2.2.24}$$

where L denotes the Laplace transformation, and in the second equality of (2.2.24) we used the following proved in (Kim et al., 2017, Lemma 3.4)

$$L(H_{\alpha,\lambda})(s) = \frac{1}{s^\alpha + \lambda}.$$

Recall that $\partial_t^\alpha \phi = \partial_t I^{2-\alpha}(\phi')$. Therefore, $L[\partial_t^\alpha \phi](s) = s^\alpha L\phi$, which implies

$$s^\alpha L\phi(s) + \lambda L\phi(s) = Lf(s),$$

and we arrive at (2.2.24).

By Pollard (1948), $E_\alpha(z) \in (0,1)$ for $z \in (-\infty, 0)$, and thus,

$$\begin{aligned}
&\|H_{\alpha,\lambda} \cdot \chi_{(0,T)}\|_{L_1(\mathbb{R})} = I^\alpha E_\alpha(-\lambda \cdot^\alpha)(T) = \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} E_\alpha(-\lambda s^\alpha)\, ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \sum_{k=0}^\infty \frac{(-\lambda)^k s^{\alpha k}}{\Gamma(\alpha k+1)}\, ds \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{(-\lambda)^k}{\Gamma(\alpha k+1)} \int_0^T (T-s)^{\alpha-1} s^{\alpha k}\, ds \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{(-\lambda)^k}{\Gamma(\alpha k+1)} T^{\alpha(k+1)} \frac{\Gamma(\alpha)\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} = \lambda^{-1}(1 - E_\alpha(-\lambda T^\alpha)).
\end{aligned}$$

Also, note that

$$|\phi(t)| \le \int_{-\infty}^\infty |H_{\alpha,\lambda}(t-s) \cdot \chi_{t-s\in(0,T)}| \cdot |f(s) \cdot \chi_{s\in(0,T)}|\, ds.$$

By Young's inequality,

$$\begin{aligned}\|\phi\|_{L_2(0,T)} &\le \|f\cdot\chi_{s\in(0,T)}\|_{L_2(\mathbb{R})}\cdot\|\mathbb{H}_{\alpha,\lambda}\cdot\chi_{(0,T)}\|_{L_1(\mathbb{R})}\\ &\le \lambda^{-1}\|f\|_{L_2(0,T)},\end{aligned}$$

and

$$\|\partial_t^\alpha\phi\|_{L_2(0,T)} \le \|\lambda\phi\|_{L_2(0,T)} + \|f\|_{L_2(0,T)} \le 2\|f\|_{L_2(0,T)}.$$

Thus, we have

$$\|w_{n_1,\dots,n_d}\|_{L_2(0,T)} \le (n_1\pi\cdots n_d\pi)^{-2}\|f_{n_1,\dots,n_d}\|_{L_2(0,T)},$$

and

$$\|\partial_t^\alpha w_{n_1,\dots,n_d}\|_{L_2(0,T)} \le 2\|f_{n_1,\dots,n_d}\|_{L_2(0,T)}.$$

It follows that $w\in\mathbb{H}_{2,0}^{\alpha,2}(\Omega_T)$ and

$$\|w\|_{\mathbb{H}_2^{\alpha,2}(\Omega_T)} \le N\|f\|_{L_2(\Omega_T)}.$$

Thus, the lemma is proved. □

Lemma 2.2.7. *Let $p_0\in(1,2)$, $\Omega=(-1,1)^d$, and $f\in L_2(\Omega_T)$. There exists a unique $w\in\mathbb{H}_{2,0}^{\alpha,2}(\Omega_T)$ to*

$$\partial_t^\alpha w-\Delta w=f \quad in \quad \Omega_T, \quad w=0 \quad on \quad (0,T)\times\partial\Omega, \tag{2.2.25}$$

and w satisfies

$$\|w\|_{\mathbb{H}_{p_0}^{\alpha,2}(\Omega_T)} \le N\|f\|_{L_{p_0}(\Omega_T)}, \tag{2.2.26}$$

where $N=N(d,p_0,\alpha)$.

Proof. Let $\widetilde{f}(t, y^1, \ldots, y^d) = f(t, 2y^1 - 1, \ldots, 2y^d - 1)$ for $(y^1, \ldots, y^d) \in (0,1)^d$. Clearly, $\widetilde{f} \in L_2((0,T) \times (0,1)^d)$, and by Lemma 2.2.6, there exists a unique $\widetilde{w} \in \mathbb{H}^{\alpha,2}_{2,0}((0,T) \times (0,1)^d)$ such that

$$\partial_t^\alpha \widetilde{w} - 4\Delta \widetilde{w} = \widetilde{f} \quad \text{in} \quad (0,T) \times (0,1)^d, \quad \widetilde{w} = 0 \quad \text{on} \quad (0,T) \times \partial(0,1)^d.$$

Then, let $w(t, x^1, \ldots, x^d) = \widetilde{w}(t, (x^1+1)/2, \ldots, (x^d+1)/2)$ for $(x^1, \ldots, x^d) \in \Omega$. It follows that w satisfies (2.2.25). Next, we show (2.2.26) by extending w to the whole space. First, we take the odd extensions of w and f to $(-1,3)^d$, i.e., for $x^i \in (1,3)$,

$$f(t, x^1, \ldots, x^d) := (-1)^d f(t, 2 - x^1, \ldots, 2 - x^d).$$

Then, we take the periodic extensions of w and f to the whole space by extending along each spatial direction with a period of 4. Note that in order to smoothly extend the function up to the boundary, it is necessary to take the odd extension first. Let M be a positive integer and $\eta \in C^\infty(\mathbb{R}^d), \eta \in [0,1]$, be a cutoff function in the spatial variables so that

$$\eta(x) = \begin{cases} 1 & \text{when} \quad x \in (-M, M)^d, \\ 0 & \text{when} \quad x \notin (-2M, 2M)^d, \end{cases}$$

$|D\eta| \le (2M)^{-1}$, and $|D^2\eta| \le (2M)^{-2}$. Then, it follows from (2.2.23) that $\eta w \in \mathbb{H}^{\alpha,2}_{p_0,0}(\mathbb{R}^d_T)$ and

$$\partial_t^\alpha(\eta w) - \Delta(\eta w) = \eta f - \Delta\eta w - 2D_i\eta D_i w.$$

Using the result for equations with constant leading coefficients in the whole space (Kim et al., 2017, Theorem 2.9), we have

$$\|D^2(\eta w)\|_{L_{p_0}(\mathbb{R}^d_T)} \le N\Big[\|\eta f\|_{L_{p_0}(\mathbb{R}^d_T)} + \|wD_{ij}\eta\|_{L_{p_0}(\mathbb{R}^d_T)} + \|D_i\eta D_j w\|_{L_{p_0}(\mathbb{R}^d_T)}\Big],$$

where $N = N(d, p_0, \alpha)$. Therefore, since w is periodic,

$$M^{d/p_0} \|D^2 w\|_{L_{p_0}(\Omega_T)} \le N \Big[(2M)^{d/p_0} \|f\|_{L_{p_0}(\Omega_T)} + (2M)^{-2} (2M)^{d/p_0} \|w\|_{L_{p_0}(\Omega_T)} + (2M)^{-1} (2M)^{d/p_0} \|Dw\|_{L_{p_0}(\Omega_T)} \Big],$$

which implies

$$\|D^2 w\|_{L_{p_0}(\Omega_T)} \le N \Big[\|f\|_{L_{p_0}(\Omega_T)} + M^{-2} \|w\|_{L_{p_0}(\Omega_T)} + M^{-1} \|Dw\|_{L_{p_0}(\Omega_T)} \Big].$$

By sending $M \to \infty$, we get

$$\|D^2 w\|_{L_{p_0}(\Omega_T)} \le N \|f\|_{L_{p_0}(\Omega_T)}.$$

Moreover, we have

$$\|w\|_{L_{p_0}(\Omega_T)} \le N \|Dw\|_{L_{p_0}(\Omega_T)} \le N \|D^2 w\|_{L_{p_0}(\Omega_T)},$$

where the first inequality is by the boundary Poincaré inequality, and the second inequality is also by a version of the Poincaré inequality using

$$\int_\Omega Dw(t,x)\, dx = 0.$$

Therefore, (2.2.26) is proved. □

Proposition 2.2.8. *Let* $\Omega = (-1,1)^d$*,* $p_0 \in (1,2)$*,* $f \in L_2((0,t_0) \times B_{\sqrt{d}})$*, and* $w \in \mathbb{H}^{\alpha,2}_{p_0,0}((0,t_0) \times \Omega)$ *with* $w = 0$ *on* $(0,t_0) \times \partial\Omega$ *such that*

$$\partial_t^\alpha w - \Delta w = f \quad \text{in} \quad (0,t_0) \times \Omega.$$

Then, for any $t_1 \le t_0$, we have

$$(|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_1,0)} \le N \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+1},s_k)\times\Omega}$$
$$\le N \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+1},s_k)\times B_{\sqrt{d}}}, \tag{2.2.27}$$

where $s_k = t_1 - 2^k + 1$ and $N = N(\alpha, d, p_0, \delta)$.

Proof. Without loss of generality, we assume that $t_1 = t_0$. By Remark 2.2.5, we have

$$(|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0,0)} \le N (|f|^{p_0})^{1/p_0}_{Q_1(t_0,0)} + N \sum_{k=0}^{\infty} 2^{-k\alpha} (|w|^{p_0})^{1/p_0}_{(s_{k+1},s_k)\times B_1}$$
$$\le N (|f|^{p_0})^{1/p_0}_{Q_1(t_0,0)} + N \sum_{k=0}^{\infty} 2^{-k\alpha} (|w|^{p_0})^{1/p_0}_{(s_{k+1},s_k)\times\Omega}, \tag{2.2.28}$$

where $N = N(d, \delta, \alpha, p_0)$. It remains to estimate

$$A_k := (|w|^{p_0})^{1/p_0}_{(s_{k+1},s_k)\times\Omega}.$$

Take cutoff functions $\eta_k, \zeta \in C^\infty(\mathbb{R})$ such that

$$\eta_k(t) = \begin{cases} 1 & \text{when} \quad t \ge s_{k+1}, \\ 0 & \text{when} \quad t \le s_{k+2}, \end{cases} \qquad \zeta(t) = \begin{cases} 1 & \text{when} \quad t \in (s_{k+3}, t_0), \\ 0 & \text{when} \quad t \in (-\infty, s_{k+4}), \end{cases}$$

$|\eta_k'| \le 2^{-k}$, $|\eta_k''| \le 2^{-2k}$, and $|\zeta| \le 1$. Furthermore, let $w_1 = \zeta w$ and $w_2 = w - w_1$. By Lemma 2.2.1, it follows that $w\eta_k \in \mathbb{H}^{\alpha,2}_{p_0,0}((s_{k+2}, t_0) \times \Omega)$, and

$$\partial_t^\alpha (w\eta_k) - \Delta(w\eta_k) = g_k \quad \text{in} \quad (s_{k+2}, t_0) \times \Omega, \tag{2.2.29}$$

where

$$
\begin{aligned}
g_k &= f\eta_k + g_{1,k} + g_{2,k}, \\
g_{k,1}(t,x) &= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_{s_{k+4}}^{t} (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)] w_1(s,x)\, ds \\
&\quad + \frac{\alpha}{\Gamma(2-\alpha)} \eta_k'(t) \partial_t \int_{s_{k+4}}^{t} (t-s)^{1-\alpha} w_1(s,x)\, ds,
\end{aligned}
$$

and

$$
\begin{aligned}
g_{k,2}(t,x) &= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_{-\infty}^{s_{k+3}} (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)] w_2(s,x)\, ds \\
&\quad + \frac{\alpha}{\Gamma(2-\alpha)} \eta_k'(t) \partial_t \int_{-\infty}^{s_{k+3}} (t-s)^{1-\alpha} w_2(s,x)\, ds \\
&= N \int_{-\infty}^{s_{k+3}} (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t)] w_2(s,x)\, ds.
\end{aligned}
$$

Using the fact that $|\eta_k(s) - \eta_k(t)| = |\eta_k(t)|$ for $s < s_{k+3}$, we have

$$
|g_{k,2}(t,x)| \leq \sum_{j=k+3}^{\infty} \int_{s_{j+1}}^{s_j} (t-s)^{-\alpha-1} |w(s,x)|\, ds.
$$

Moreover, note that for $t \in (s_{k+2}, s_k)$, $s \in (s_{j+1}, s_j)$, and $j \geq k+3$, we have

$$
|t-s| \geq s_{k+2} - s_j = 2^j - 2^{k+2} \geq 2^j - 2^{j-1} = 2^{j-1},
$$

and thus,

$$
|t-s|^{-\alpha-1} \leq N(\alpha) 2^{-j(\alpha+1)}.
$$

Therefore, by the Minkowski inequality and Hölder's inequality

$$
\|g_{k,2}\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)} \leq N \sum_{j=k+3}^{\infty} 2^{-\alpha j} (3 \cdot 2^k)^{1/p_0} \left(\fint_{s_{j+1}}^{s_j} \fint_{\Omega} |w|^{p_0}\, dx ds \right)^{1/p_0}, \quad (2.2.30)
$$

where $N = N(\alpha, d)$.

For $g_{k,1}$, we decompose

$$g_{k,1} = g_{k,1,1} + g_{k,1,2} + g_{k,1,3},$$

where

$$g_{k,1,1}(t,x) = N_1(\alpha)\int_{s_{k+4}}^{t} (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)]w_1(s,x)\,ds,$$
$$g_{k,1,2}(t,x) = N_1(\alpha)\frac{\eta_k'(t)}{\eta_{k+1}(t)}\int_{s_{k+4}}^{t} (t-s)^{-\alpha}[\eta_{k+1}(s) - \eta_{k+1}(t)]w_1(s,x)\,ds,$$

and

$$g_{k,1,3}(t,x) = N_2(\alpha)\frac{\eta_k'(t)}{\eta_{k+1}(t)}\partial_t^{\alpha-1}(\eta_{k+1}w_1)(t,x) = N_2(\alpha)\frac{\eta_k'(t)}{\eta_{k+1}(t)}\partial_t^{\alpha-1}(\eta_{k+1}w)(t,x).$$

When $t \geq s > s_{k+1}$, we have $\eta_k(s) = \eta_k(t) = 1$ and $\eta_k'(t) = 0$, which imply

$$\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t) = 0.$$

Therefore, it follows

$$\begin{aligned}|g_{k,1,1}(t,x)| &\leq N\|\eta_k''\|_{L_\infty}\int_{s_{k+4}}^{t} (t-s)^{-\alpha+1}|w(s,x)|1_{s\leq s_{k+1}}\,ds \\ &\leq N2^{-2k}\int_0^{152^k} s^{-\alpha+1}|w(t-s,x)|1_{s\geq t-s_{k+1}}\,ds,\end{aligned}$$

which implies

$$
\begin{aligned}
&\|g_{k,1,1}\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)}\\
&\leq N2^{-2k}\int_0^{15\cdot 2^k} s^{-\alpha+1}\|w(\cdot-s,\cdot)1_{\cdot-s\leq s_{k+1}}\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)}\,ds\\
&\leq N2^{-\alpha k}\|w\|_{L_{p_0}((s_{k+5},s_{k+1})\times\Omega)} \leq N2^{-\alpha k}\sum_{j=1}^{4}\|w\|_{L_{p_0}((s_{k+j+1},s_{k+j})\times\Omega)},
\end{aligned}
\tag{2.2.31}
$$

where $N = N(\alpha, d, p_0)$. Furthermore,

$$
\begin{aligned}
|g_{k,1,2}(t,x)| &\leq N\|\eta_k'\|_{L_\infty}1_{t\in(s_{k+2},s_{k+1})}\|\eta_{k+1}'\|_{L_\infty}\int_{s_{k+4}}^t (t-s)^{-\alpha+1}|w(s,x)|1_{s\leq s_{k+2}}\,ds\\
&\leq N2^{-k}2^{-k}1_{t\in(s_{k+2},s_{k+1})}\int_{s_{k+4}}^t (t-s)^{-\alpha+1}|w(s,x)|1_{s\leq s_{k+1}}\,ds\\
&\leq N2^{-2k}1_{t\in(s_{k+2},s_{k+1})}\int_0^{152^k} s^{-\alpha+1}|w(t-s,x)|1_{s\geq t-s_{k+1}}\,ds,
\end{aligned}
$$

which implies that

$$
\|g_{k,1,2}\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)} \leq N2^{-\alpha k}\sum_{j=1}^{4}\|w\|_{L_{p_0}((s_{k+j+1},s_{k+j})\times\Omega)},
\tag{2.2.32}
$$

where $N = N(\alpha, d, p_0)$. For any $\varepsilon > 0$, using Lemma 2.2.3, we have

$$
\begin{aligned}
&\|g_{k,1,3}\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)} = N\|\eta_k'(t)\partial_t^\beta(\eta_{k+1}w)\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)}\\
&= N\|\eta_k'(t)\partial_t^\beta(\eta_{k+1}w)\|_{L_{p_0}((s_{k+2},s_{k+1})\times\Omega)}\\
&\leq N\|\eta_k'\|_{L_\infty}\|\partial_t^\beta(\eta_{k+1}w)\|_{L_{p_0}((s_{k+3},s_{k+1})\times\Omega)}\\
&\leq \varepsilon\ \|\partial_t^\alpha(\eta_{k+1}w)\|_{L_{p_0}((s_{k+3},s_{k+1})\times\Omega)}\\
&\quad + N\varepsilon^{-1}\|\eta_k'\|_{L_\infty}^2\|w\|_{L_{p_0}((s_{k+3},s_{k+1})\times\Omega)}(2^k)^{2-\alpha}\\
&\leq \varepsilon\|\eta_{k+1}w\|_{\mathbb{H}_{p_0}^{\alpha,2}((s_{k+3},s_{k+1})\times\Omega)} + N\varepsilon^{-1}2^{-\alpha k}\|w\|_{L_{p_0}((s_{k+3},s_{k+1})\times\Omega)}.
\end{aligned}
\tag{2.2.33}
$$

Thus, using Lemma 2.2.7, and combining (2.2.30), (2.2.31), (2.2.32), and (2.2.33), we derive

$$
\begin{aligned}
&\|w\eta_k\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+2},s_k)\times\Omega)} \le N\|g_k\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)}\\
&\le N\|f\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)} + N\|g_{k,1}\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)} + N\|g_{k,2}\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)}\\
&\le N\|f\|_{L_{p_0}((s_{k+2},s_k)\times\Omega)} + N2^{-\alpha k}\sum_{j=1}^{4}\|w\|_{L_{p_0}((s_{k+j+1},s_{k+j})\times\Omega)}\\
&\quad + \varepsilon\|\eta_{k+1}w\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+3},s_{k+1})\times\Omega)} + N\varepsilon^{-1}2^{-\alpha k}\|w\|_{L_{p_0}((s_{k+3},s_{k+1})\times\Omega)}\\
&\quad + N\sum_{j=k+4}^{\infty}2^{-\alpha j}(3\cdot 2^k)^{1/p_0}\left(\fint_{s_{j+1}}^{s_j}\fint_{\Omega}|w|^{p_0}\,dxds\right)^{1/p_0},
\end{aligned}
$$

where $N = N(\alpha, d, p_0, \delta)$. Dividing both sides by the measure, we have

$$
\begin{aligned}
&\|\eta_k w\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+2},s_k)\times\Omega)}\big(|\Omega||s_k - s_{k+2}|\big)^{-1/p_0}\\
&\le N(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega} + N\sum_{j=k+1}^{\infty}2^{-\alpha j}A_j + N\varepsilon^{-1}\sum_{j=k+1}^{k+2}2^{-\alpha j}A_j\\
&\quad + 6^{1/p_0}\varepsilon\|\eta_{k+1}w\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+3},s_{k+1})\times\Omega)}\big(|\Omega||s_{k+3} - s_{k+1}|\big)^{-1/p_0}. \qquad (2.2.34)
\end{aligned}
$$

For any fixed k_0, multiplying both sides of (2.2.34) by $2^{-\alpha k}$, and summing over $k = k_0, k_0+1, \ldots$, we get

$$
\begin{aligned}
&\sum_{k=k_0}^{\infty}2^{-\alpha k}\|\eta_k w\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+2},s_k)\times\Omega)}\big(|\Omega||s_k - s_{k+2}|\big)^{-1/p_0}\\
&\le N\sum_{k=k_0}^{\infty}2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega} + N\sum_{k=k_0}^{\infty}2^{-\alpha k}\sum_{j=k+1}^{\infty}2^{-\alpha j}A_j\\
&\quad + 6^{1/p_0}\varepsilon\sum_{k=k_0}^{\infty}2^{-\alpha k}\|\eta_{k+1}w\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+3},s_{k+1})\times\Omega)}\big(|\Omega||s_{k+3} - s_{k+1}|\big)^{-1/p_0}\\
&\quad + N\varepsilon^{-1}\sum_{k=k_0}^{\infty}2^{-\alpha k}\sum_{j=k+1}^{k+2}2^{-\alpha j}A_j. \qquad (2.2.35)
\end{aligned}
$$

By picking ε sufficiently small so that $6^{1/p_0}\varepsilon 2^{\alpha} \leq 1/2$, (2.2.35) becomes

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} 2^{-\alpha k}\|\eta_k w\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+2},s_k)\times\Omega)}\big(|\Omega||s_k - s_{k+2}|\big)^{-1/p_0} \\
&\leq N\sum_{k=k_0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega} + N\sum_{k=k_0}^{\infty} 2^{-\alpha k}\sum_{j=k+1}^{\infty} 2^{-\alpha j}A_j \\
&\quad + \frac{1}{2}\sum_{k=k_0}^{\infty} 2^{-\alpha(k+1)}\|\eta_{k+1} w\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+3},s_{k+1})\times\Omega)}\big(|\Omega||s_{k+3} - s_{k+1}|\big)^{-1/p_0}. \qquad (2.2.36)
\end{aligned}$$

By absorbing the third term to the left-hand side of (2.2.36), we conclude

$$\begin{aligned}
\sum_{k=k_0}^{\infty} 2^{-\alpha k}A_k &\leq \sum_{k=k_0}^{\infty} 2^{-\alpha k}\|\eta_k w\|_{\mathbb{H}^{\alpha,2}_{p_0}((s_{k+2},s_k)\times\Omega)}\cdot\big(|\Omega|\cdot|s_k - s_{k+2}|\big)^{-1/p_0} \\
&\leq N\sum_{k=k_0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega} + N\sum_{k=k_0}^{\infty} 2^{-\alpha k}\sum_{j=k+1}^{\infty} 2^{-\alpha j}A_j \\
&\leq N\sum_{k=k_0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega} + N\sum_{j=k_0+1}^{\infty} 2^{-\alpha j}A_j\sum_{k=k_0}^{j-1} 2^{-\alpha k} \\
&\leq N\sum_{k=k_0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega} + \frac{N2^{-\alpha k_0}}{1-2^{-\alpha}}\sum_{j=k_0+1}^{\infty} 2^{-\alpha j}A_j, \qquad (2.2.37)
\end{aligned}$$

where $N = N(\alpha, d, p_0, \delta)$. By picking k_0 sufficiently large so that $\frac{N2^{-\alpha k_0}}{1-2^{-\alpha}} \leq 1/2$, we have

$$\sum_{k=k_0}^{\infty} 2^{-\alpha k}A_k \leq N\sum_{k=k_0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega}.$$

Therefore, by induction,

$$\sum_{k=0}^{\infty} 2^{-\alpha k}A_k \leq N(\alpha, d, p_0, \delta)\sum_{k=0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega}. \qquad (2.2.38)$$

Indeed, if there exists $N = N(\alpha, d, p_0, \delta)$ such that

$$\sum_{k=j}^{\infty} 2^{-\alpha k}A_k \leq N\sum_{k=j}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(s_{k+2},s_k)\times\Omega},$$

for $j = 1, 2, \ldots, k_0$, then, by (2.2.37),

$$\begin{aligned}\sum_{k=j-1}^{\infty} 2^{-\alpha k} A_k &\leq N \sum_{k=j-1}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+2}, s_k)\times\Omega} + \big(\frac{N 2^{-\alpha k_0}}{1-2^{-\alpha}}\big) \sum_{k=j}^{\infty} 2^{-\alpha k} A_k \\ &\leq N \sum_{k=j-1}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+2}, s_k)\times\Omega} = N \sum_{k=j-1}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+2}, s_k)\times\Omega}.\end{aligned}$$

Finally, by (2.2.28) and (2.2.38),

$$\begin{aligned}(|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_1,0)} &\leq N \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+1}, s_k)\times\Omega} \\ &\leq N \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+1}, s_k)\times B_{\sqrt{d}}},\end{aligned}$$

where $N = N(\alpha, d, p_0, \delta)$. The proposition is proved. □

2.2.3 Estimates of u

By combining the estimates of v and w, we derive the mean oscillation estimates for u in this subsection.

Proposition 2.2.9. *Let $p_0 \in (1,2)$, $T \in (0,\infty)$, and a^{ij} be constant. If $u \in \mathbb{H}^{\alpha,2}_{p_0,0,\mathrm{loc}}(\mathbb{R}^d_T)$ satisfies*

$$\partial_t^\alpha u - a^{ij} D_{ij} u = f \quad \text{in} \quad \mathbb{R}^d_T.$$

Then, for any $(t_0, x_0) \in (0,T] \times \mathbb{R}^d$, $r \in (0,\infty)$, and $\kappa \in (0, 1/4)$, we have

$$\begin{aligned}&(|D^2 u - (D^2 u)_{(t_0-(\kappa r)^{2/\alpha}, t_0)\times B_{\delta\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha}, t_0)\times B_{\delta\kappa r}(x_0)} \\ &\quad \leq N\kappa^{\sigma} \sum_{k=0}^{\infty} 2^{-k\alpha} (|D^2 u|^{p_0})^{1/p_0}_{(t_0 - 2^k (r/2)^{2/\alpha} r, t_0)\times B_{\delta^{-1} r/2}(x_0)} \\ &\quad\quad + N\kappa^{-(d+2/\alpha)/p_0} \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(t_0 - 2^k r^{2/\alpha}, t_0)\times B_{\delta^{-1}\sqrt{d} r}(x_0)},\end{aligned} \tag{2.2.39}$$

where $\sigma = \sigma(d, \alpha, p_0)$ *and* $N = N(d, \delta, \alpha, p_0)$.

Proof. The proof is similar to that of (Dong and Kim, 2021, Proposition 4.7), where α is taken to be smaller than 1.

Step 1. We start with the case when $a^{ij} = \delta^{ij}$. By shifting and scaling, we also assume $x_0 = 0$ and $r = 1$. Let $\Omega = (-1, 1)^d$ be a cube in $\mathbb{R}^d$ such that $B_1 \subseteq \Omega \subseteq B_{\sqrt{d}}$. Then, by Lemma 2.2.7, there exists $w \in \mathbb{H}^{\alpha,2}_{p_0,0}((0, t_0) \times \Omega)$ with $w = 0$ on $(0, t_0) \times \partial\Omega$ satisfying

$$\partial_t^\alpha w - \Delta w = f \quad \text{in} \quad (0, t_0) \times \Omega.$$

By taking $v = u - w \in \mathbb{H}^{\alpha,2}_{p_0,0}((0, t_0) \times \Omega)$, we have

$$\partial_t^\alpha v - \Delta v = 0 \quad \text{in} \quad (0, t_0) \times \Omega.$$

Next, by Hölder's inequality,

$$\begin{aligned}&(|D^2u - (D^2u)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} \\ &\quad \le (|D^2v - (D^2v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} + N\kappa^{-(d+2/\alpha)/p_0}(|D^2w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0,0)}.\end{aligned} \tag{2.2.40}$$

Step 2. We focus on the estimate of the first term in the right-hand side of (2.2.40) since the estimate of the second term in given by (2.2.27).

It is easily seen that

$$(|D^2v - (D^2v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} \le N\kappa^\sigma [D^2v]_{C^{\sigma\alpha/2,\sigma}(Q_{1/4}(t_0,0))}, \tag{2.2.41}$$

and by applying Proposition 2.2.8 with $r = 1/2$ together with the triangle inequality, we have

$$
\begin{aligned}
[D^2 v]_{C^{\sigma\alpha/2,\sigma}(Q_{1/4}(t_0,0))} &\le N \sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2 v|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)} \\
&\le N \sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2 u|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)} \\
&\quad + N \sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)}. \qquad (2.2.42)
\end{aligned}
$$

It remains to estimate the summation involving $D^2 w$ in (2.2.42). For $j = 1, 2, \ldots$ and $s_k^j = t_0 - (j-1)2^{-2/\alpha} - 2^k + 1$, by (2.2.27), we have

$$
(|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)} \le N \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s^j_{k+1}, s^j_k)\times B_{\sqrt{d}}}.
$$

It follows that

$$
\begin{aligned}
&\sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)} \\
&\le N \sum_{j=1}^{\infty} j^{-(1+\alpha)} \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s^j_{k+1}, s^j_k)\times B_{\sqrt{d}}} \\
&= N \sum_{m=1}^{\infty} \sum_{\substack{j\in\mathbb{N} \\ m-1 \le (j-1)2^{-2/\alpha} < m}} j^{-(1+\alpha)} \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s^j_{k+1}, s^j_k)\times B_{\sqrt{d}}}.
\end{aligned}
$$

Moreover, observe that for $m - 1 \le (j-1)2^{-2/\alpha} < m$,

$$
(s^j_{k+1}, s^j_k) \subset (t_0 - 2^{k+1} + 1 - m, t_0 - m + 1).
$$

Then, for $s_k = t_0 - 2^k + 1$,

$$
(|f|^{p_0})^{1/p_0}_{(s^j_{k+1}, s^j_k)\times B_{\sqrt{d}}} \le 2^{1/p_0} (|f|^{p_0})^{1/p_0}_{(s_{k+1}-m, t_0-m+1)\times B_{\sqrt{d}}}. \qquad (2.2.43)
$$

Thus, by using (2.2.43) and Hölder's inequality,

$$\begin{aligned}
&\sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)} \\
&\le N \sum_{m=1}^{\infty} \sum_{\substack{j\in\mathbb{N} \\ m-1\le (j-1)2^{-2/\alpha}<m}} j^{-(1+\alpha)} \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+1}-m,t_0-m+1)\times B_{\sqrt{d}}} \\
&\le N \sum_{m=1}^{\infty} m^{-(1+\alpha)} \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+1}-m,t_0-m+1)\times B_{\sqrt{d}}} \\
&\le N \sum_{l=0}^{\infty} \sum_{m=2^l}^{2^{l+1}-1} 2^{-l(1+\alpha)} \sum_{k=0}^{\infty} 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(s_{k+1}-m,t_0-m+1)\times B_{\sqrt{d}}} \\
&= N \sum_{l=0}^{\infty} 2^{-l(1+\alpha)} \sum_{k=0}^{\infty} 2^{-\alpha k} \sum_{m=2^l}^{2^{l+1}-1} (|f|^{p_0})^{1/p_0}_{(s_{k+1}-m,t_0-m+1)\times B_{\sqrt{d}}} \\
&\le N \sum_{l=0}^{\infty} 2^{-l\alpha} \sum_{k=0}^{\infty} 2^{-\alpha k} \left[\sum_{m=2^l}^{2^{l+1}-1} 2^{-l} (|f|^{p_0})_{(s_{k+1}-m,t_0-m+1)\times B_{\sqrt{d}}} \right]^{1/p_0}.
\end{aligned} \tag{2.2.44}$$

Furthermore, for the most inner sum of (2.2.44), we have

$$\begin{aligned}
&\sum_{m=2^l}^{2^{l+1}-1} 2^{-l} (|f|^{p_0})_{(s_{k+1}-m,t_0-m+1)\times B_{\sqrt{d}}} \\
&= 2^{-l} 2^{-(k+1)} \sum_{m=2^l}^{2^{l+1}-1} \int_{s_{k+1}-m}^{t_0-m+1} \fint_{B_{\sqrt{d}}} |f|^{p_0}\,dx\,dt \\
&= 2^{-l} 2^{-(k+1)} \sum_{m=2^l}^{2^{l+1}-1} \sum_{i=0}^{2^{k+1}-1} \int_{t_0-m-i}^{t_0-m-i+1} \fint_{B_{\sqrt{d}}} |f|^{p_0}\,dx\,dt \\
&\le 2^{-l-k} 2^{\min\{l,k\}} \sum_{m=2^l}^{2^{l+1}+2^{k+1}-2} \int_{t_0-m}^{t_0-m+1} \fint_{B_{\sqrt{d}}} |f|^{p_0}\,dx\,dt \\
&\le \begin{cases} 4(|f|^{p_0})_{(t_0-2^{l+2}+2,t_0)\times B_{\sqrt{d}}} & \text{for} \quad l \ge k, \\ 4(|f|^{p_0})_{(t_0-2^{k+2}+2,t_0)\times B_{\sqrt{d}}} & \text{for} \quad k > l. \end{cases}
\end{aligned} \tag{2.2.45}$$

From (2.2.44) and (2.2.45), it follows that

$$\begin{aligned}&\sum_{j=1}^{\infty} j^{-(1+\alpha)}(|D^2 w|^{p_0})_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)}\\&\le N\sum_{k=0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(t_0-2^{k+2}+2,t_0)\times B_{\sqrt{d}}}.\end{aligned} \tag{2.2.46}$$

Thus, by (2.2.41), (2.2.42), and (2.2.46),

$$\begin{aligned}(|D^2 v-(D^2 v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} &\le N\kappa^{\sigma}\sum_{j=1}^{\infty} j^{-(1+\alpha)}(|D^2 u|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)}\\&\quad+N\kappa^{\sigma}\sum_{k=1}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(t_0-2^{k+2}+2,t_0)\times B_{\sqrt{d}}}.\end{aligned} \tag{2.2.47}$$

Step 3. By (2.2.40), (2.2.47), and (2.2.27), we conclude

$$\begin{aligned}&(|D^2 u-(D^2 u)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)}\\&\le (|D^2 v-(D^2 v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)}+N\kappa^{-(d+2/\alpha)/p_0}(|D^2 w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0,0)}\\&\le N\kappa^{\sigma}\sum_{j=1}^{\infty} j^{-(1+\alpha)}(|D^2 u|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)}\\&\quad+N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(t_0-2^{k},t_0)\times B_{\sqrt{d}}},\end{aligned} \tag{2.2.48}$$

where $N=N(d,\delta,\alpha,p_0)$. For the first term on the right-hand side of (2.2.48), by

Hölder's inequality for l_{p_0},

$$
\begin{aligned}
&\sum_{j=1}^{\infty} j^{-(1+\alpha)} (|D^2 u|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)} \\
&\le \sum_{k=0}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} 2^{-k(1+\alpha)} (|D^2 u|^{p_0})^{1/p_0}_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)} \\
&\le \sum_{k=0}^{\infty} 2^{-k\alpha} \left(2^{-k} \sum_{j=2^k}^{2^{k+1}-1} (|D^2 u|^{p_0})_{Q_{1/2}(t_0-(j-1)2^{-2/\alpha},0)} \right)^{1/p_0} \\
&\le N \sum_{k=0}^{\infty} 2^{-k\alpha} (|D^2 u|^{p_0})^{1/p_0}_{(t_0-2^k 2^{-2/\alpha},t_0)\times B_{1/2}}. \qquad (2.2.49)
\end{aligned}
$$

Therefore, (2.2.39) follows from (2.2.48) and (2.2.49) when $a^{ij} = \delta^{ij}$ and $r = 1$.

Step 4. For general a^{ij}'s, we apply a change of variables. Replacing a^{ij} with $(a^{ij} + a^{ji})/2$, without loss of generality, we may assume that $A = (a^{ij})$ is symmetric and positive definite. Then, there exists a symmetric and invertible $A^{1/2}$. Let

$$
\widetilde{u}(t,y) = u(t, A^{1/2} y) \quad \text{in} \quad \mathbb{R}^d_T \quad \text{and} \quad y_0 = A^{-1/2} x_0.
$$

Therefore,

$$
\partial_t^\alpha \widetilde{u} - \Delta \widetilde{u} = \widetilde{f} \quad \text{in} \quad \mathbb{R}^d_T
$$

and

$$
\begin{aligned}
&(|D^2 \widetilde{u} - (D^2 \widetilde{u})_{Q_{\kappa r}(t_0,y_0)}|)_{Q_{\kappa r}(t_0,y_0)} \\
&\le N \kappa^\sigma \sum_{k=0}^{\infty} 2^{-k\alpha} (|D^2 \widetilde{u}|^{p_0})^{1/p_0}_{(t_0-2^k (r/2)^{2/\alpha} r, t_0)\times B_{r/2}(y_0)} \\
&\quad + N \kappa^{-(d+2/\alpha)/p_0} \sum_{k=0}^{\infty} 2^{-\alpha k} (|\widetilde{f}|^{p_0})^{1/p_0}_{(t_0-2^k r^{2/\alpha}, t_0)\times B_{\sqrt{d} r}(y_0)}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(|D^2u-(D^2u)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}\\
&\le N\kappa^{\sigma}\sum_{k=0}^{\infty}2^{-k\alpha}(|D^2u|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B_{\delta^{-1}r/2}(x_0)}\\
&\quad+N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty}2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)},
\end{aligned}
$$

where we used the ellipticity condition of a^{ij}'s to derive

$$B_{\delta r}(x_0)\subset A^{1/2}(B_r(y_0))\subset B_{\delta^{-1}r}(x_0),$$

and

$$|B_{\delta^{-1}r}(x_0)|\le N(d,\delta)|A^{1/2}(B_r(y_0))|\le N(d,\delta)|B_{\delta r}(x_0)|$$

for any $r>0$. The proposition is proved. $\square$

2.3 Non-divergence form equations in the whole space

In this section, with the mean oscillation estimate (2.2.39), we will prove Theorem 2.1.6 by using the sharp function theorem and the maximal function theorem.

Proposition 2.3.1. *Let $\alpha\in(1,2)$, $T\in(0,\infty)$, $p,q\in(1,\infty)$, $K_1\in[1,\infty)$, and $[w]_{p,q}\le K_1$. There exist $p_0=p_0(d,p,q,K_1)\in(1,2)$ and $\mu=\mu(d,p,q,K_1)\in(1,\infty)$ such that*

$$p_0<p_0\mu<\min\{p,q\}$$

and the following holds. If $u\in\mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ is supported on $(0,T)\times B_{\xi R_0}(x_1)$ for *some* $x_1 \in \mathbb{R}^d$ *and* $\xi > 0$*, and satisfies*

$$\partial_t^\alpha u - a^{ij}(t,x) D_{ij} u = f \quad in \quad \mathbb{R}^d_T, \tag{2.3.1}$$

where the coefficients $a^{ij}(t,x)$*'s satisfy Assumption 2.1.2* (γ_0)*, then for any* $r \in (0,\infty)$*,* $\kappa \in (0,1/4)$*, and* $(t_0,x_0) \in (0,T] \times \mathbb{R}^d$*, we have*

$$\begin{aligned}
&(|D^2u - (D^2u)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\le N\xi^{d/q_0}\kappa^{-d/q_0}(|D^2u|^{p_0})^{1/p_0}_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\quad + N\kappa^{\sigma}\sum_{k=0}^{\infty} 2^{-k\alpha}(|D^2u|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B_{\delta^{-1}r/2}(x_0)} \\
&\quad + N\kappa^{-(d+2/\alpha)/p_0}\gamma_0^{1/(\nu p_0)}\sum_{k=0}^{\infty} 2^{-\alpha k}2^{k/(\nu p_0)}(|D^2u|^{\mu p_0})^{1/(\mu p_0)}_{(t_0-2^k r^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \\
&\quad + N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(t_0-2^k r^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)},
\end{aligned} \tag{2.3.2}$$

where $\nu = \mu/(\mu-1)$*,* $q_0 = p_0/(p_0-1)$*, and* $N = N(d,\delta,\alpha,p,q,K_1)$*. The functions* u *and* f *are defined to be zero whenever* $t \le 0$*.*

Proof. For the given $w_1 \in A_p(\mathbb{R},dt)$ and $w_2 \in A_q(\mathbb{R}^d,dx)$, using reverse Hölder's inequality for A_p weights, we pick $\sigma_1 = \sigma_1(d,p,K_1)$ and $\sigma_2 = \sigma_2(d,q,K_1)$ such that $p-\sigma_1 > 1$, $q - \sigma_2 > 1$, and

$$w_1 \in A_{p-\sigma_1}(\mathbb{R},dt), \quad w_2 \in A_{q-\sigma_2}(\mathbb{R}^d,dx).$$

Take $p_0, \mu \in (1,\infty)$ so that

$$p_0\mu = \min\left\{\frac{p}{p-\sigma_1}, \frac{q}{q-\sigma_2}\right\} > 1.$$

Note that

$$w_1 \in A_{p-\sigma_1} \subset A_{\frac{p}{p_0\mu}} \subset A_{\frac{p}{p_0}}(\mathbb{R}, dt)$$

and

$$w_2 \in A_{q-\sigma_2} \subset A_{\frac{q}{p_0\mu}} \subset A_{\frac{q}{p_0}}(\mathbb{R}^d, dx).$$

From these inclusions and the fact that $u \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$, it follows that $u \in \mathbb{H}^{\alpha,2}_{p_0\mu,0,\mathrm{loc}}(\mathbb{R}^d_T)$. See the proof of (Dong and Kim, 2018, Lemma 5.10).

We first consider the case when $r \geq R_1\delta/\sqrt{d}$, where $R_1 = 2^{-\alpha/2}R_0$. By Hölder's inequality,

$$\begin{aligned}
&(|D^2u - (D^2u)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\leq 2(|D^2u|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&= 2(|D^2u|\chi_{(0,T)\times B_{\xi R_0}(x_1)})_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\leq 2(|D^2u|^{p_0})^{1/p_0}_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}\Big(\frac{|B_{\xi R_0}|}{|B_{\delta\kappa r}|}\Big)^{1/q_0} \\
&\leq N(d,\delta,p,q)\xi^{d/q_0}\kappa^{-d/q_0}(|D^2u|^{p_0})^{1/p_0}_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}.
\end{aligned} \tag{2.3.3}$$

Then, we consider the case when $r < R_1\delta/\sqrt{d}$. Let

$$\overline{a}^{ij} = \fint_{Q_{\delta^{-1}2r}(t_0,x_0)} a^{ij}\,dtdy$$

and

$$\partial_t^\alpha u - \overline{a}^{ij}D_{ij}u = f + (a^{ij} - \overline{a}^{ij})D_{ij}u := \widetilde{f}.$$

By Proposition 2.2.9,

$$
\begin{aligned}
&(|D^2u-(D^2u)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\le N\kappa^{\sigma}\sum_{k=0}^{\infty}2^{-k\alpha}(|D^2u|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B_{\delta^{-1}r/2}(x_0)} \\
&\quad + N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty}2^{-\alpha k}(|\widetilde{f}|^{p_0})^{1/p_0}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)},
\end{aligned}
$$

where $N=N(d,\delta,\alpha,p,q,K_1)$. By the definition of $\widetilde{f}$ and Hölder's inequality,

$$
\begin{aligned}
&(|\widetilde{f}|^{p_0})^{1/p_0}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \\
&\le (|f|^{p_0})^{1/p_0}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} + (|\overline{a}^{ij}-a^{ij}|^{\nu p_0})^{1/(\nu p_0)}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \\
&\quad \cdot (|D^2u|^{\mu p_0})^{1/(\mu p_0)}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)}.
\end{aligned} \tag{2.3.4}
$$

Moreover, by Lemma 2.1.14,

$$
(|\overline{a}^{ij}-a^{ij}|)_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \le N(\alpha,d,\delta)\gamma_0 2^k. \tag{2.3.5}
$$

Therefore, by (2.3.4) and (2.3.5),

$$
\begin{aligned}
&\sum_{k=0}^{\infty}2^{-\alpha k}(|\widetilde{f}|^{p_0})^{1/p_0}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \\
&\le \sum_{k=0}^{\infty}2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \\
&\quad + N\gamma_0^{1/(\nu p_0)}\sum_{k=0}^{\infty}2^{-\alpha k}2^{k/(\nu p_0)}(|D^2u|^{\mu p_0})^{1/(\mu p_0)}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)}.
\end{aligned} \tag{2.3.6}
$$

Therefore, by combining both cases (2.3.3) and (2.3.6), we arrive at the estimate (2.3.2). The proposition is proved. □

Next, we introduce the dyadic cubes as follows: for each $n\in\mathbb{Z}$, pick $k(n)\in\mathbb{Z}$

such that

$$k(n) \le \frac{2n}{\alpha} < k(n) + 1,$$

and let

$$Q^n_{\vec{i}} = \Big[\frac{i_0}{2^{k(n)}} + T, \frac{i_0 + 1}{2^{k(n)}} + T\Big) \times \Big[\frac{i_1}{2^n}, \frac{i_1 + 1}{2^n}\Big) \times \cdots \times \Big[\frac{i_d}{2^n}, \frac{i_d + 1}{2^n}\Big) \tag{2.3.7}$$

and

$$\mathbb{C}_n := \big\{Q^n_{\vec{i}} = Q^n_{(i_0,\ldots,i_d)} : \vec{i} = (i_0, \ldots, i_d) \in \mathbb{Z}^{d+1}, i_0 \le -1\big\}.$$

Furthermore, denote the dyadic sharp function of g by

$$g^{\#}_{dy}(t,x) = \sup_{n<\infty} ⨍_{Q^n_{\vec{i}} \ni (t,x)} |g(s,y) - g_{|n}(t,x)| \, dy \, ds,$$

where

$$g_{|n}(t,x) = ⨍_{Q^n_{\vec{i}}} g(s,y) \, dy \, ds \quad \text{for} \quad (t,x) \in Q^n_{\vec{i}}.$$

Proposition 2.3.2. *Let $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \le K_1$. There exist $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) > 0$ and $\xi(d,\delta,\alpha,p,q,K_1) > 0$ such that under Assumption 2.1.2 (γ_0), for any $u \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ supported on $(0,T) \times B_{\xi R_0}(x_1)$ for some $x_1 \in \mathbb{R}^d$, satisfying (2.3.1), we have*

$$\|\partial_t^\alpha u\|_{L_{p,q,w}} + \|D^2 u\|_{L_{p,q,w}} \le N \|f\|_{L_{p,q,w}}, \tag{2.3.8}$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$ and $N = N(d,\delta,\alpha,p,q,K_1)$.

Proof. For any $(t_0,x_0) \in \mathbb{R}^d_T$, $n \in \mathbb{Z}$, and $Q^n_{\vec{i}}$ containing (t_0,x_0), there exist $r = r(n,\alpha,d,\delta) > 0$ and $t_1 = \min(T, t_0 + r^{2/\alpha}/2) \in (-\infty, T]$ such that

$$Q^n_{\vec{i}} \subset (t_1 - r^{2/\alpha}, t_1) \times B_{\delta r}(x_0) \quad \text{and} \quad |(t_1 - r^{2/\alpha}, t_1) \times B_{\delta r}(x_0)| \le N(\alpha,d,\delta)|Q^n_{\vec{i}}|.$$

Thus, by (2.3.2), we have

$$
\begin{aligned}
&\fint_{Q_{\tilde{i}}^n \ni (t_0,x_0)} |D^2u(s,y) - (D^2u)_{|n}(t_0,x_0)| \, dy \, ds \\
&\leq \left(|D^2u - (D^2u)_{(t_1-r^{2/\alpha},t_1)\times B_{\delta r}(x_0)}|\right)_{(t_1-r^{2/\alpha},t_1)\times B_{\delta r}(x_0)} \\
&\leq N\xi^{d/q_0}\kappa^{-d/q_0}(|D^2u|^{p_0})^{1/p_0}_{(t_1-r^{2/\alpha},t_1)\times B_{\delta r}(x_0)} \\
&+ N\kappa^{\sigma}\sum_{k=0}^{\infty} 2^{-k\alpha}(|D^2u|^{p_0})^{1/p_0}_{(t_1-2^k(r/(2\kappa))^{2/\alpha}r,t_1)\times B_{\delta^{-1}r/(2\kappa)}(x_0)} \\
&+ N\kappa^{-(d+2/\alpha)/p_0}\gamma_0^{1/(\nu p_0)}\sum_{k=0}^{\infty} 2^{-\alpha k}2^{k/(\nu p_0)}(|D^2u|^{\mu p_0})^{1/(\mu p_0)}_{(t_1-2^k(r/\kappa)^{2/\alpha},t_1)\times B_{\delta^{-1}\sqrt{d}r/\kappa}(x_0)} \\
&+ N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty} 2^{-\alpha k}(|f|^{p_0})^{1/p_0}_{(t_1-2^k(r/\kappa)^{2/\alpha},t_1)\times B_{\delta^{-1}\sqrt{d}r/\kappa}(x_0)},
\end{aligned}
$$

where N is independent of n and κ, and for the first inequality, we used (2.1.18). Therefore, since $\alpha > 1 > 1/(\nu p_0)$, we conclude that

$$
\begin{aligned}
(D^2u)^{\#}_{dy}(t_0,x_0) \leq\ & N\xi^{d/q_0}\kappa^{-d/q_0}(\mathcal{SM}|D^2u|^{p_0})^{1/p_0}(t_0,x_0) \\
&+ N\kappa^{\sigma}(\mathcal{SM}|D^2u|^{p_0})^{1/p_0}(t_0,x_0) \\
&+ N\kappa^{-(d+2/\alpha)/p_0}\gamma_0^{1/(\nu p_0)}(\mathcal{SM}|D^2u|^{\mu p_0})^{1/(\mu p_0)}(t_0,x_0) \\
&+ N\kappa^{-(d+2/\alpha)/p_0}(\mathcal{SM}|f|^{p_0})^{1/p_0}(t_0,x_0).
\end{aligned}
$$

Then, by the weighted sharp function theorem (Dong and Kim, 2018, Corollary 2.7) and the weighted maximal function theorem for strong maximal functions (Dong and Kim, 2021, Theorem 5.2),

$$
\begin{aligned}
\|D^2u\|_{p,q,w} \leq\ & N(\xi^{d/q_0}\kappa^{-d/q_0} + \kappa^{\sigma} + \kappa^{-(d+2/\alpha)/p_0}\gamma_0^{1/(\nu p_0)})\|D^2u\|_{p,q,w} \\
&+ N\kappa^{-(d+2/\alpha)/p_0}\|f\|_{p,q,w},
\end{aligned}
$$

where $N = N(d,\delta,\alpha,p,q,K_1)$. Therefore, by first taking a sufficiently small $\kappa < 1/4$, and then taking sufficiently small $\xi, \gamma_0 > 0$ such that

$$N(\xi^{d/q_0}\kappa^{-d/q_0} + \kappa^{\sigma} + \kappa^{-(d+2/\alpha)/p_0}\gamma_0^{1/(\nu p_0)}) < 1/2,$$

we arrive at

$$\|D^2 u\|_{L_{p,q,w}} \leq N\|f\|_{L_{p,q,w}}. \tag{2.3.9}$$

Finally, the equation (2.3.1) together with (2.3.9) yields (2.3.8). The proposition is proved. □

Next, we get rid of the restriction on the function in Proposition 2.3.2 by a partition of unity argument.

Corollary 2.3.3. *Let $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \leq K_1$. There exists $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) > 0$ such that under Assumption 2.1.2 (γ_0), for any $u \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ satisfying (2.1.2) in $\mathbb{R}^d_T$, we have*

$$\|\partial_t^\alpha u\|_{p,q,w} + \|D^2 u\|_{p,q,w} \leq N_0\|f\|_{p,q,w} + N_0 R_0^{-2}\|u\|_{p,q,w}, \tag{2.3.10}$$

where $N_0 = N_0(d,\delta,\alpha,p,q,K_1)$ and $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$.

Proof. Case 1. We first consider the case when $p = q$ and $b^i = c = 0$. We take γ_0 and ξ from Proposition 2.3.2, and use a partition of unity argument in the spatial variables, and to this end we find $\{x_k\} \subset \mathbb{R}^d$, $\zeta^k \geq 0$, $\zeta^k \in C_0^\infty(\mathbb{R}^d)$, $\text{supp}(\zeta^k) \subset B_{\xi R_0/\sqrt{2}}(x_k)$ and

$$1 \leq \sum_{k=1}^\infty |\zeta^k|^p \leq N(p), \quad \sum_{k=1}^\infty |D_x\zeta^k|^p \leq N R_0^{-p}, \quad \sum_{k=1}^\infty |D_x^2\zeta^k|^p \leq N R_0^{-2p}$$

for some $N = N(d, \delta, \alpha, p, K_1)$. For $u_k(t,x) := u(t,x)\zeta^k(x)$, we have

$$\begin{aligned}\partial_t^\alpha u_k - a^{ij}D_{ij}u_k &= (\partial_t^\alpha u)\zeta^k - (a^{ij}D_{ij}u)\zeta^k - 2a^{ij}D_iuD_j\zeta^k - a^{ij}uD_{ij}\zeta^k \\ &= f\zeta^k - 2a^{ij}D_iuD_j\zeta^k - a^{ij}uD_{ij}\zeta^k.\end{aligned}$$

Therefore, by Proposition 2.3.2,

$$\begin{aligned}&\|\partial_t^\alpha u_k\|_{p,w} + \|D^2u_k\|_{p,w} \\ &\le N\|f\zeta^k\|_{p,w} + N\sum_{i,j=1}^d \|D_i\zeta^kD_ju\|_{p,w} + N\sum_{i,j=1}^d \|uD_{ij}\zeta^k\|_{p,w},\end{aligned}$$

where $N = N(d, \delta, \alpha, p, K_1)$. By raising to the p-th power and summing in k, we get

$$\|\partial_t^\alpha u\|_{p,w}^p + \|D^2u\|_{p,w}^p \le N_0\|f\|_{p,w}^p + N_0R_0^{-2p}\|u\|_{p,w}^p + N_0R_0^{-p}\|Du\|_{p,w}^p,$$

where $N_0 = N_0(d, \delta, \alpha, p, K_1)$. Therefore, (2.3.10) follows from the weighted interpolation inequality for the spatial variables (see for instance, (Dong and Krylov, 2019, Lemma 3.8 (iii))).

Case 2. Then, we consider the case when $p = q$ without assuming $b^i, c = 0$. In this case, we have

$$\partial_t^\alpha u - a^{ij}D_{ij}u = f + b^iD_iu + cu.$$

Thus, the result follows from Case 1 and the interpolation inequality for the spatial variables.

Case 3. Finally, for the general case when $p \neq q$, we use the extrapolation theorem in (Dong and Kim, 2018, Theorem 2.5). See also (Cruz-Uribe et al., 2011, Theorem 1.4). The Corollary is proved. $\square$

To prove (2.1.3), we need to get rid of the u term on the right-hand side of (2.3.10). We start with the following Lemma.

Lemma 2.3.4. *Let $\alpha \in [1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \leq K_1$. For any $u \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$, we have*

$$\|u\|_{p,q,w} \leq NT^\alpha \|\partial_t^\alpha u\|_{p,q,w},$$

where $N = N(\alpha,p,q,K_1)$ and $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$.

Proof. Note that $u = I^\alpha \partial_t^\alpha u$. Therefore, the result follows from (Dong and Kim, 2021, Lemma 5.5). □

We are ready to prove Theorem 2.1.6. In the following proof, we use the notation $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$ and $\|\cdot\|_{p,q,w,(\tau_1,\tau_2)} = \|\cdot\|_{L_{p,q,w}((\tau_1,\tau_2)\times\mathbb{R}^d)}$.

Proof of Theorem 2.1.6 . We first prove the a priori estimate (2.1.3). By Corollary 2.3.3, it suffices to prove

$$\|u\|_{p,q,w} \leq N\|f\|_{p,q,w}. \tag{2.3.11}$$

By extending u and f to be zero for $t < 0$, we have

$$\partial_t^\alpha u - a^{ij}D_{ij}u - b^iD_iu - cu = f \quad \text{in} \quad (S,T)\times\mathbb{R}^d$$

for any $S \leq 0$.

Take a positive integer m to be specified below. For $j = -1, 0, \ldots, m-1$, set

$s_j = jT/m$, and for $j \geq 0$, let $\eta_j = \eta_j(t)$ be smooth functions such that

$$\eta_j(t) = \begin{cases} 1 & \text{when} \quad t \geq s_j, \\ 0 & \text{when} \quad t \leq s_{j-1}, \end{cases} \qquad |\eta_j'| \leq \frac{2m}{T}, \quad |\eta_j''| \leq 4\frac{m^2}{T^2}. \tag{2.3.12}$$

It follows from Lemma 2.2.1, $u\eta_j \in \mathbb{H}^{\alpha,2}_{p,q,w,0}((s_{j-1}, s_{j+1}) \times \mathbb{R}^d)$. Furthermore, similar to the decomposition right below (2.2.29), we have

$$\partial_t^\alpha(u\eta_j) - a^{ij}D_{ij}(u\eta_j) - b^iD_i(u\eta_j) - c(u\eta_j) = f\eta_j + h_j$$

in $(s_{j-1}, s_{j+1}) \times \mathbb{R}^d$, where for $t \in (s_{j-1}, s_{j+1})$,

$$\begin{aligned} h_j(t,x) &= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^t (t-s)^{-\alpha-1}[\eta_j(s) - \eta_j(t) - \eta_j'(t)(s-t)]u(s,x)\,ds \\ &\quad + \frac{\alpha}{\Gamma(2-\alpha)}\eta_j'(t)\partial_t \int_0^t (t-s)^{-\alpha+1}u(s,x)\,ds \\ &= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^t (t-s)^{-\alpha-1}[\eta_j(s) - \eta_j(t) - \eta_j'(t)(s-t)]u(s,x)\chi_{s\leq s_j}\,ds \\ &\quad + \frac{\alpha}{\Gamma(2-\alpha)}\eta_j'(t)\partial_t \int_0^t (t-s)^{-\alpha+1}u(s,x)\,ds. \end{aligned}$$

For $j = 0$, since $\eta_0'(t) = 0$ for $t \in (s_0, s_1)$, we have $h_0 = 0$. For $j = 1, 2, \ldots, m-1$, similar to the computation in Proposition 2.2.4, we have

$$\begin{aligned} h_j(t,x) = N_1(\alpha) &\int_{-\infty}^t (t-s)^{-\alpha-1}[\eta_j(s) - \eta_j(t) - \eta_j'(t)(s-t)]u(s,x)\chi_{s\leq s_j}\,ds \\ &+ N_1(\alpha)\frac{\eta_j'(t)}{\eta_{j-1}(t)} \int_{-\infty}^t (t-s)^{-\alpha}[\eta_{j-1}(s) - \eta_{j-1}(t)]u(s,x)\chi_{s\leq s_j}\,ds \\ &+ N_2(\alpha)\frac{\eta_j'(t)}{\eta_{j-1}(t)}\partial_t^\beta(\eta_{j-1}u)(t,x), \end{aligned}$$

where $\beta = \alpha - 1$. Thus, by the Minkowski inequality and Lemma 2.3.4 with the observation that $\partial_t \partial_t^\beta(\eta_{j-1}u) = \partial_t^\alpha(\eta_{j-1}u)$ in (s_{j-2}, s_j),

$$\begin{aligned}
&\|h_j\|_{p,q,w,(s_{j-1},s_{j+1})} \\
&\leq N_3\|\eta_j''\|_{L_\infty}\|I^{2-\alpha}[|u(\cdot,x)|\chi_{\cdot\leq s_j}]\|_{p,q,w,(s_{j-1},s_{j+1})} \\
&\quad + N_3\|\eta_j'\|_{L_\infty}\|\eta_{j-1}'\|_{L_\infty}\|I^{2-\alpha}[|u(\cdot,x)|\chi_{\cdot\leq s_j}]\|_{p,q,w,(s_{j-1},s_{j+1})} \\
&\quad + N_3\|\eta_j'\|_{L_\infty}\|\partial_t^\beta(\eta_{j-1}u)\|_{p,q,w,(s_{j-1},s_j)} \\
&\leq N_3 m^2 T^{-\alpha}\|u\|_{p,q,w,(0,s_j)} + N_3\|\partial_t^\alpha(\eta_{j-1}u)\|_{p,q,w,(s_{j-2},s_j)},
\end{aligned} \tag{2.3.13}$$

where N_3 is independent of j. By (2.3.10), for $j = 0, 1, \ldots, m-1$,

$$\begin{aligned}
&\|\partial_t^\alpha(u\eta_j)\|_{p,q,w,(s_{j-1},s_{j+1})} \leq N_0\|f\eta_j\|_{p,q,w,(s_{j-1},s_{j+1})} \\
&\qquad + N_0 R_0^{-2}\|u\eta_j\|_{p,q,w,(s_{j-1},s_{j+1})} + N_0\|h_j\|_{p,q,w,(s_{j-1},s_{j+1})},
\end{aligned}$$

where $N_0 = N_0(d, \delta, \alpha, p, q, K_1)$. In particular, for $j = 0$,

$$\|\partial_t^\alpha(u\eta_0)\|_{p,q,w,(s_{-1},s_1)} \leq N_0\|f\eta_0\|_{p,q,w,(s_{-1},s_1)} + N_0 R_0^{-2}\|u\eta_0\|_{p,q,w,(s_{-1},s_1)}. \tag{2.3.14}$$

For $j = 1$, by (2.3.13) and (2.3.14),

$$\begin{aligned}
&\|\partial_t^\alpha(u\eta_1)\|_{p,q,w,(s_0,s_2)} \\
&\leq N_0\|f\eta_1\|_{p,q,w,(s_0,s_2)} + N_0 R_0^{-2}\|u\eta_1\|_{p,q,w,(s_0,s_2)} + N_0\|h_1\|_{p,q,w,(s_{-1},s_1)} \\
&\leq N_0\|f\eta_1\|_{p,q,w,(s_0,s_2)} + N_0 R_0^{-2}\|u\eta_1\|_{p,q,w,(s_0,s_2)} + N_0\big[N_3 m^2 T^{-\alpha}\|u\|_{p,q,w,(0,s_1)} \\
&\quad + N_3\|\partial_t^\alpha(u\eta_0)\|_{(s_{-1},s_1)}\big] \\
&\leq N_0\|f\eta_1\|_{p,q,w,(s_0,s_2)} + N_0 R_0^{-2}\|u\eta_1\|_{p,q,w,(s_0,s_2)} + N_0 N_3 m^2 T^{-\alpha}\|u\|_{p,q,w,(0,s_1)} \\
&\quad + N_0^2 N_3\|f\eta_0\|_{p,q,w,(s_{-1},s_1)} + N_0^2 N_3 R_0^{-2}\|u\eta_0\|_{p,q,w,(s_{-1},s_1)} \\
&\leq N\|f\|_{p,q,w,(0,s_2)} + N\|u\|_{p,q,w,(0,s_1)} + N_1\|u\|_{p,q,w,(s_1,s_2)},
\end{aligned}$$

where N_1 is independent of m, and N depends on m, but both of them are independent of j. Repeating this process for $j = 2, 3, \ldots$, we conclude

$$\begin{aligned} &\|\partial_t^\alpha (u\eta_j)\|_{p,q,w,(s_{j-1},s_{j+1})} \\ &\leq N\|f\|_{p,q,w,(0,s_{j+1})} + N\|u\|_{p,q,w,(0,s_j)} + N_1\|u\|_{p,q,w,(s_j,s_{j+1})}, \end{aligned} \tag{2.3.15}$$

where $N_1 = N_1(d, \delta, \alpha, p, q, K_1, R_0)$ and $N = N(d, \delta, \alpha, p, q, K_1, R_0, T, m)$. Using Lemma 2.3.4 and (2.3.15), we obtain

$$\begin{aligned} &\|u\|_{p,q,w,(s_j,s_{j+1})} \leq \|u\eta_j\|_{p,q,w,(s_{j-1},s_{j+1})} \\ &\leq N(T/m)^\alpha \|\partial_t^\alpha (u\eta_j)\|_{p,q,w,(s_{j-1},s_{j+1})} \\ &\leq N\|f\|_{p,q,w,(0,s_{j+1})} + N\|u\|_{p,q,w,(0,s_j)} + N_1(T/m)^\alpha \|u\|_{p,q,w,(s_{j-1},s_{j+1})}, \end{aligned}$$

where $N = N(d, \delta, \alpha, p, q, K_1, R_0, T, m)$. By taking a sufficiently large m satisfying

$$N_1(T/m)^\alpha < 1/2,$$

we arrive at

$$\|u\|_{p,q,w,(s_j,s_{j+1})} \leq N\|f\|_{p,q,w,(0,s_{j+1})} + N\|u\|_{p,q,w,(0,s_j)},$$

where $N = N(d, \delta, \alpha, p, q, K_1, R_0, T)$. Therefore, (2.3.11) follows from induction. Thus, the a priori estimate (2.1.3) is proved. The existence of solutions follows the solvability of a simple equation with constant coefficients presented in (Kim et al., 2017, Theorem 2.9) or (Park et al., 2020, Theorem 2.8) together with the method of continuity. □

2.4 Non-divergence form equations in the half space or a smooth domain

In this section, we first prove Theorem 2.1.7 for the half space by taking extensions. Then, we prove Theorem 2.1.8 for smooth domains by using S. Agmon's idea and a partition of unity argument.

2.4.1 The half space case

In this subsection, we consider the case when $\Omega = \mathbb{R}^d_+$. Let

$$(\widetilde{\mathcal{SM}}f)(t,x) = \sup_{\substack{Q_{r_1,r_2}(s,y)\cap\Omega_T \ni (t,x)\\ (s,y)\in\Omega_T}} \fint_{Q_{r_1,r_2}(s,y)\cap\Omega_T} |f(r,z)|\,dz\,dr \tag{2.4.1}$$

and $B_r^+(x_0) := B_r(x_0)\cap\Omega$. Moreover, for $w(t,x) = w_1(t)w_2(x)$, where $w_1 \in A_p(\mathbb{R})$ and $w_2 \in A_q(\Omega)$, we take the even extension of w_2 to the whole space by setting $\overline{w_2}(x) = w_2(|x^1|, x')$. Furthermore, we denote $\overline{w}(t,x) = w_1(t)\overline{w_2}(x)$.

Remark 2.4.1. Note that if $[w_2]_{A_q(\Omega)} \le K_1$, then $[\overline{w_2}]_{A_q(\mathbb{R}^d)} \le 2^q K_1$. Indeed, for any $x \in \mathbb{R}^d, x^1 \ge 0$ and $r > 0$, we have

$$\begin{aligned}
&\left(\fint_{B_r(x_0)} w(x)\,dx\right)\left(\fint_{B_r(x_0)} (w(x))^{\frac{-1}{q-1}}\,dx\right)^{q-1}\\
&\le \left(2\fint_{B_r(x_0)\cap\Omega} w(x)\,dx\right)\left(2\fint_{B_r(x_0)\cap\Omega} (w(x))^{\frac{-1}{q-1}}\,dx\right)^{q-1} \le 2^q K_1.
\end{aligned}$$

The same inequality holds for $x^1 < 0$ by symmetry.

Lemma 2.4.2. *For any $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$, let $\overline{u}(t,x) := u(t,|x^1|,x')\operatorname{sgn} x^1$. Then,*

$\overline{u} \in \mathbb{H}^{\alpha,2}_{p,q,\overline{w},0}(\mathbb{R}^d_T)$ *and*

$$\|u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}(\Omega_T)} \le \|\overline{u}\|_{\mathbb{H}^{\alpha,2}_{p,q,\overline{w}}(\mathbb{R}^d_T)} \le 2\|u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}(\Omega_T)}.$$

Proof. When $w = 1$, the proof follows from (Krylov, 2008, Lemma 8.2.1). For general w, the estimates follows from the same proof by using the definitions of $\overline{u}$ and $\overline{w}$. $\square$

Proposition 2.4.3. *Let $\alpha \in (1,2)$, $p_0 \in (1,2)$, $T \in (0,\infty)$, a^{ij} be constants, and $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p_0,0,\mathrm{loc}}(\Omega_T)$ satisfy*

$$\partial_t^\alpha u - a^{ij} D_{ij} u = f \quad in \quad \Omega_T.$$

Then, for any $(t_0, x_0) \in (0,T] \times \Omega$, $r \in (0,\infty)$, and $\kappa \in (0, 1/4)$, we have

$$\begin{aligned}
&(|D^2u - (D^2u)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B^+_{\delta\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B^+_{\delta\kappa r}(x_0)} \\
&\quad \le N\kappa^\sigma \sum_{k=0}^\infty 2^{-k\alpha} (|D^2 u|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B^+_{\delta^{-1}r/2}(x_0)} \\
&\qquad + N\kappa^{-(d+2/\alpha)/p_0} \sum_{k=0}^\infty 2^{-\alpha k} (|f|^{p_0})^{1/p_0}_{(t_0-2^k r^{2/\alpha},t_0)\times B^+_{\delta^{-1}\sqrt{d}r}(x_0)},
\end{aligned} \tag{2.4.2}$$

where $N = N(d,\delta,\alpha,p_0)$ and $\sigma = \sigma(d,\alpha,p_0)$.

Proof. Case 1. We first consider the case when $a^{ij} = \delta^{ij}$. By taking the odd extensions of u and f in the x^1 direction, denoted by $\overline{u}, \overline{f}$, we see that $\overline{u}$ satisfies the equation

$$\partial_t^\alpha \overline{u} - \Delta \overline{u} = \overline{f}.$$

Using Lemma 2.4.2 and Proposition 2.2.9, we have

$$
\begin{aligned}
&(|D^2\overline{u} - (D^2\overline{u})_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\kappa r}(x_0)} \\
&\quad \le N\kappa^{\sigma}\sum_{k=0}^{\infty} 2^{-k\alpha}(|D^2\overline{u}|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B_{\delta^{-1}r/2}(x_0)} \\
&\qquad + N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty} 2^{-\alpha k}(|\overline{f}|^{p_0})^{1/p_0}_{(t_0-2^k r^{2/\alpha},t_0)\times B_{\sqrt{d}r}(x_0)}.
\end{aligned}
$$

Then, note that

$$
\begin{aligned}
&(|D^2u - (D^2u)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B^+_{\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B^+_{\kappa r}(x_0)} \\
&= (|D^2\overline{u} - (D^2\overline{u})_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B^+_{\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B^+_{\kappa r}(x_0)} \\
&\le 4(|D^2\overline{u} - (D^2\overline{u})_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\kappa r}(x_0)},
\end{aligned}
$$

and

$$
\fint_{B_r(x_0)} |\overline{f}|\,dx \le \frac{1}{|B^+_r|}\int_{B_r(x_0)} |\overline{f}|\,dx \le \frac{2}{|B^+_r|}\int_{B^+_r(x_0)} |f|\,dx
$$

for any $r > 0$ and $x_0 \in \Omega$. Therefore, (2.4.2) follows when $a^{ij} = \delta^{ij}$.

Case 2. For general a^{ij}'s, without loss of generality, we assume that $A = (a^{ij})$ is symmetric, and let $\widetilde{u}(t,x) = u(t, A^{1/2}x)$. Furthermore, we take an orthogonal matrix O such that $OA^{-1/2}(\Omega) = \Omega$, and let

$$
\widehat{u}(t,x) = \widetilde{u}(t, O^{-1}x) = u(t, A^{1/2}O^{-1}x) \quad \text{for} \quad x \in \Omega.
$$

It follows that

$$
\partial_t^{\alpha}\widehat{u} - \Delta\widehat{u} = \widehat{f} \quad \text{in} \quad \Omega_T.
$$

For $y_0 := OA^{-1/2}(x_0)$, we have

$$B^+_{\delta r}(x_0) \subset A^{1/2}O^{-1}(B^+_r(y_0)) \subset B^+_{\delta^{-1}r}(x_0)$$

for any $r > 0$. Therefore, (2.4.2) for general a^{ij}'s follows by applying Case 1 to $\widehat{u}$. $\square$

Proof of Theorem 2.1.7. Lemma 2.4.2 and Theorem 2.1.6 yield the existence and uniqueness of solutions to the simple equation $\partial_t^\alpha u - \Delta u = f$ in the half space. Then, with Proposition 2.4.3 and Remark 2.1.3, we follow the process in Section 4 with minor modifications by applying the Hardy-Littlewood and Fefferman-Stein theorems in the half space. We omit the details. $\square$

2.4.2 The domain case

In this section, we denote the operator

$$L = \partial_t^\alpha - a^{ij}D_{ij} - b^iD_i - c.$$

Most of this subsection is devoted to the study of $L + \lambda$ for λ sufficiently large.

Lemma 2.4.4. *Let $G = \mathbb{R}^d_+$ or $\mathbb{R}^d$, $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q,\Omega} \le K_1$. There exist $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ and $\lambda_0 = \lambda_0(d,\delta,\alpha,p,q,K_1,R_0) \ge 1$ such that under Assumption 2.1.2 (γ_0), the following holds. For any $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}((0,T)\times\mathbb{R}^d_+)$ or $\mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ and any $\lambda \ge \lambda_0$, we have*

$$\|u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}((0,T)\times G)} + \lambda\|u\|_{L_{p,q,w}} \le N\|Lu + \lambda u\|_{L_{p,q,w}},$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}((0,T)\times G)}$ and $N = N(d,\delta,\alpha,p,q,K_1)$.

Proof. We use an idea originally due to S. Agmon. Denote

$$(0,T)\times G\times\mathbb{R}=\big\{(t,x,y):t\in(0,T),x\in G,y\in\mathbb{R}\big\},$$

and let

$$v(t,x,y)=u(t,x)\zeta(y)\cos(\sqrt{\lambda}y),$$

where $\zeta\in C_0^\infty(\mathbb{R})$ and $\zeta\not\equiv 0$. Then, let

$$\widehat{f}=\partial_t^\alpha v-\sum_{i,j=1}^d a^{ij}D_{ij}v-D_y^2 v-\sum_{i=1}^d b^i D_i v-cv,$$

and note that

$$D_y^2 v=\big(\zeta''(y)\cos(\sqrt{\lambda}y)-2\sqrt{\lambda}\zeta'(y)\sin(\sqrt{\lambda}y)-\lambda\zeta(y)\cos(\sqrt{\lambda}y)\big)u. \tag{2.4.3}$$

Thus,

$$\widehat{f}=\zeta(y)\cos(\sqrt{\lambda}y)(Lu+\lambda u)+2\sqrt{\lambda}\zeta'(y)\sin(\sqrt{\lambda}y)u-\zeta''(y)\cos(\sqrt{\lambda}y)u.$$

By applying Corollary 2.3.3 or a version of it in the half space to v in $(0,T)\times G\times\mathbb{R}$ with the weight $w(t,x,y):=w(t,x)$ for all $y\in\mathbb{R}$, there exists $\gamma_0=$

$\gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ such that, if a^{ij}'s satisfy Assumption 2.1.2 (γ_0), then

$$\begin{aligned}
&\|D_y^2 v\|_{L_{p,q,w}((0,T)\times G\times\mathbb{R})} \le \|v\|_{\mathbb{H}^{\alpha,2}_{p,q,w}((0,T)\times G\times\mathbb{R})}\\
&\le N_0\|\widehat{f}\|_{L_{p,q,w}((0,T)\times G\times\mathbb{R})} + N_1\|v\|_{L_{p,q,w}((0,T)\times G\times\mathbb{R})}\\
&\le N_0\{\|\zeta(y)\cos(\sqrt{\lambda}y)(Lu+\lambda u)\|_{L_{p,q,w}((0,T)\times G\times\mathbb{R})}\\
&\quad + \sqrt{\lambda}\|\zeta'(y)\sin(\sqrt{\lambda}y)u\|_{L_{p,q,w}((0,T)\times G\times\mathbb{R})}\\
&\quad + \|\zeta''(y)\cos(\sqrt{\lambda}y)u\|)_{L_{p,q,w}((0,T)\times G\times\mathbb{R})}\} + N_1\|v\|_{L_{p,q,w}((0,T)\times G\times\mathbb{R})}\\
&\le N_0\|Lu+\lambda u\|_{p,q,w} + N_0(1+\sqrt{\lambda})\|u\|_{p,q,w} + N_1\|u\|_{p,q,w},
\end{aligned} \tag{2.4.4}$$

where $N_0 = N_0(d,\delta,\alpha,p,q,K_1)$ and $N_1 = N_1(d,\delta,\alpha,p,q,K_1,R_0)$. Moreover, by (2.4.3) and the fact that $\int_{\mathbb{R}} |\zeta(y)\cos(\sqrt{\lambda}y)|^p dy$ is bounded away from 0 independently of λ,

$$\begin{aligned}
\|\lambda u\|_{p,q,w} &\le \|D_y^2 v\|_{L_{p,q,w}((0,T)\times G\times\mathbb{R})} + N(1+\sqrt{\lambda})\|u\|_{p,q,w}\\
&\le N_0\|Lu+\lambda u\|_{p,q,w} + N_0(1+\sqrt{\lambda})\|u\|_{p,q,w} + N_1\|u\|_{p,q,w}.
\end{aligned} \tag{2.4.5}$$

At this point, we take a sufficiently large $\lambda_0(d,\delta,\alpha,p,q,K_1,R_0) \ge 1$ such that

$$N_0(1+\sqrt{\lambda_0}) + N_1 < \lambda_0/2.$$

Then, by (2.4.4) and (2.4.5), for any $\lambda \ge \lambda_0$, we have

$$\|\lambda u\|_{p,q,w} \le N_0\|Lu+\lambda u\|_{p,q,w},$$

which implies

$$
\begin{aligned}
&\|u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}((0,T)\times G)} + \lambda\|u\|_{p,q,w} \le N\|v\|_{\mathbb{H}^{\alpha,2}_{p,q,w}((0,T)\times G\times\mathbb{R})} + \lambda\|u\|_{p,q,w} \\
&\le N_0\|Lu+\lambda u\|_{p,q,w} + N_0(1+\sqrt{\lambda})\|u\|_{p,q,w} + N_1\|u\|_{p,q,w} + \lambda\|u\|_{p,q,w} \\
&\le N_0\|Lu+\lambda u\|_{p,q,w} + \lambda\|u\|_{p,q,w} \le N_0\|Lu+\lambda u\|_{p,q,w}.
\end{aligned}
$$

The lemma is proved. □

Proposition 2.4.5. *Let Ω be a $C^{1,1}$ domain, $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$ and $[w]_{p,q,\Omega} \le K_1$. There exist $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ such that under Assumption 2.1.2 (γ_0) and Assumption 2.1.5, the following holds. We can find $\lambda_1 = \lambda_1(d,\delta,\alpha,p,q,K_1,R_0,R_2) \ge 1$ such that for any $\lambda \ge \lambda_1$ and $g \in L_{p,q,w}(\Omega_T)$, there is a unique solution $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ to*

$$
Lu + \lambda u = g \quad in \quad \Omega_T, \tag{2.4.6}
$$

and u satisfies

$$
\|u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}(\Omega_T)} \le N\|g\|_{L_{p,q,w}(\Omega_T)},
$$

where $N = N(d,\delta,\alpha,p,q,K_1,R_0,R_2)$. Moreover, for any constant $\lambda \ge 0$ and $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ satisfying (2.4.6), we have

$$
\|u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}(\Omega_T)} \le N(\|Lu+\lambda u\|_{L_{p,q,w}(\Omega_T)} + \|u\|_{L_{p,q,w}(\Omega_T)}), \tag{2.4.7}
$$

where $N = N(d,\delta,\alpha,p,q,K_1,R_0,R_2)$. In particular when $\lambda = 0$, we have

$$
\|u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}(\Omega_T)} \le N(\|Lu\|_{L_{p,q,w}(\Omega_T)} + \|u\|_{L_{p,q,w}(\Omega_T)}). \tag{2.4.8}
$$

Proof. The proof is based on a partition of unity argument in the spatial variables together with the results for equations in the whole space and the half space. When $p = q$, we construct a weak solution to (2.4.6) in $\mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T)$, and prove that the weak solution is a strong solution in $\mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ when λ is sufficiently large. See Lemma 2.7.2-2.7.4 for details.

When $p \neq q$, we use the extrapolation theorem in (Dong and Kim, 2018, Theorem 2.5). □

Proof of Theorem 2.1.8. By Proposition 2.4.5 together with the method of continuity, it suffices to prove the a priori estimate (2.1.6). Thus, it remains to get rid of the u term on the right-hand side of (2.4.8). We follow the proof of (2.3.11) to conclude

$$\|u\|_{L_{p,q,w}(\Omega_T)} \leq N \|Lu\|_{L_{p,q,w}(\Omega_T)},$$

which leads to

$$\|u\|_{\mathbb{H}^{\alpha,2}_{p,q,w}(\Omega_T)} \leq N \|Lu\|_{L_{p,q,w}(\Omega_T)},$$

where $N = N(d, \delta, \alpha, p, q, K_1, R_0, R_2, T)$. The theorem is proved. □

2.5 Equations in divergence form

In this section, we are going to prove Theorems 2.1.9, 2.1.10, and 2.1.11 using the results for non-divergence form equations.

We pick $\zeta \in C_0^\infty(\mathbb{R}^d)$ satisfying $\mathrm{supp}(\zeta) \subset B_1$ and $\int_{\mathbb{R}^d} \zeta = 1$, and let $\zeta^\varepsilon(\cdot) = \varepsilon^{-d}\zeta(\cdot/\varepsilon)$. Furthermore, we denote $g = (g^1, \ldots, g^d)$, where g^i are real valued functions for all i, and denote the mollifier of $u = u(t,x)$ in the spatial variables by

$$u^\varepsilon(t,x) = u *_x \zeta^\varepsilon = \int_{\mathbb{R}^d} u(t,y)\zeta^\varepsilon(x-y)\,dy.$$

2.5.1 The whole space case

In this subsection, we prove Theorem 2.1.9 by following a similar procedure in Section 2.3, where we first consider the case when the leading coefficients are constants and then the general case using a partition of unity argument. Moreover, we denote $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$.

Proposition 2.5.1. *Let $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \le K_1$. For any $g_i \in L_{p,q,w}(\mathbb{R}^d_T)$, there exists a unique $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ to*

$$\partial_t^\alpha u - \Delta u = D_i g_i \quad \text{in} \quad \mathbb{R}^d_T, \tag{2.5.1}$$

and u satisfies

$$\|\partial_t^\alpha u\|_{\mathbb{H}^{-1}_{p,q,w}(\mathbb{R}^d_T)} + \|u\|_{p,q,w} + \|Du\|_{p,q,w} \le N\|g\|_{p,q,w}, \tag{2.5.2}$$

where $N = N(d,\delta,\alpha,p,q,K_1,T)$.

Proof. By Theorem 2.1.6, for $i = 1,\dots,d$, there exist $v_i \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ satisfying

$$\partial_t^\alpha v_i - \Delta v_i = g_i,$$

and

$$\|Dv_i\|_{p,q,w} + \|D^2 v_i\|_{p,q,w} \le N\|g_i\|_{p,q,w},$$

where $N = N(d, \delta, \alpha, p, q, K_1, T)$. It follows that

$$u := \sum_{i=1}^{d} D_i v_i \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$$

is a solution of (2.5.1). Moreover,

$$\|u\|_{p,q,w} + \|Du\|_{p,q,w} \leq \sum_{i=1}^{d} \|Dv_i\|_{p,q,w} + \sum_{i=1}^{d} \|D^2 v_i\|_{p,q,w} \leq N \|g\|_{p,q,w},$$

which together with the equation (2.5.1) implies (2.5.2).

To prove the uniqueness of the solution, we consider $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ satisfying

$$\partial_t^\alpha u - \Delta u = 0.$$

It suffices to prove $u = 0$. We have $u^\varepsilon = u *_x \zeta^\varepsilon \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ and

$$\partial_t^\alpha u^\varepsilon - \Delta u^\varepsilon = 0. \tag{2.5.3}$$

Indeed, by the Fubini theorem,

$$I^{2-\alpha} u^\varepsilon(t,x) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} u^\varepsilon(s,x)\, ds = \int_{\mathbb{R}^d} I^{2-\alpha} u(t,y) \zeta^\varepsilon(x-y)\, dy.$$

Thus, for any $\phi \in C_0^\infty([0,T)\times\mathbb{R}^d)$ and $\widetilde{\phi}(t,x) := \int_{\mathbb{R}^d} \phi(t,y)\zeta^\varepsilon(y-x)\,dy$, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} I^{2-\alpha}u^\varepsilon(t,x)\phi_{tt}(t,x) + \int_0^T\int_{\mathbb{R}^d} D_i u^\varepsilon D_i\phi \\
&= \int_0^T \int_{\mathbb{R}^d} I^{2-\alpha}u(t,y)\left(\int_{\mathbb{R}^d}\phi_{tt}\zeta^\varepsilon(x-y)\,dx\right)dydt \\
&\qquad + \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D_iu(t,y)\zeta^\varepsilon(x-y)\,dyD_i\phi(t,x)\,dxdt \\
&= \int_0^T \int_{\mathbb{R}^d} I^{2-\alpha}u(t,y)\left(\int_{\mathbb{R}^d}\phi_{tt}\zeta^\varepsilon(x-y)\,dx\right)dydt \\
&\qquad + \int_0^T\int_{\mathbb{R}^d} D_iu(t,y)\int_{\mathbb{R}^d} D_i\phi(t,x)\zeta^\varepsilon(x-y)\,dxdydt \\
&= \int_0^T\int_{\mathbb{R}^d} I^{2-\alpha}u\widetilde{\phi}_{tt} + \int_0^T\int_{\mathbb{R}^d} D_iuD_i\widetilde{\phi} = 0.
\end{aligned}
$$

We arrive at (2.5.3). Therefore, $u^\varepsilon = 0$ by (2.5.3) and Theorem 2.1.6. Finally, by sending $\varepsilon \to 0$, we arrive at $u = 0$. The proposition is proved. $\square$

The mean oscillation estimate follows immediately from the construction of the solution in Proposition 2.5.1.

Proposition 2.5.2. *Let $\alpha \in (1,2)$, $p_0 \in (1,2)$, $T \in (0,\infty)$, a^{ij} be constant and $u \in \mathcal{H}^{\alpha,1}_{p_0,0,\mathrm{loc}}(\mathbb{R}^d_T)$ satisfy*

$$
\partial_t^\alpha u - D_i(a^{ij}D_ju) = D_ig_i \quad in \quad \mathbb{R}^d_T.
$$

Then, for any $(t_0,x_0) \in (0,T]\times\mathbb{R}^d, r\in(0,\infty)$, and $\kappa \in (0,1/4)$, we have

$$
\begin{aligned}
&(|Du - (Du)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\quad \le N\kappa^\sigma \sum_{k=0}^\infty 2^{-k\alpha}(|Du|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B_{\delta^{-1}r/2}(x_0)} \\
&\qquad + N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^\infty 2^{-\alpha k}(|g|^{p_0})^{1/p_0}_{(t_0-2^kr^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)},
\end{aligned}
$$

where $N = N(d,\delta,\alpha,p_0)$ and $\sigma = \sigma(d,\alpha,p_0)$.

Proof. When $a^{ij} = \delta^{ij}$, the result follows from the uniqueness of solution in Propositions 2.5.1 and 2.2.9. For general a^{ij}'s, we make a change of variables as in the proof of Proposition 2.2.9. □

Proposition 2.5.3. *Let $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \le K_1$. There exist $p_0 = p_0(d,p,q,K_1) \in (1,2)$ and $\mu = \mu(d,p,q,K_1) \in (1,\infty)$ such that*

$$p_0 < p_0\mu < \min\{p,q\},$$

and the following holds. If $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ is supported on $(0,T) \times B_{\xi R_0}(x_1)$ for some $x_1 \in \mathbb{R}^d$, $\xi > 0$, and satisfies

$$\partial_t^\alpha u - D_i(a^{ij}(t,x)D_j u) = D_i g_i \quad \text{in} \quad \mathbb{R}^d_T, \tag{2.5.4}$$

where the coefficients a^{ij} satisfy Assumption 2.1.2 (γ_0), then for any $r \in (0,\infty)$, $\kappa \in (0,1/4)$, and $(t_0,x_0) \in (0,T] \times \mathbb{R}^d$, we have

$$\begin{aligned}
&(|Du - (Du)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\quad \le N\xi^{d/q_0}\kappa^{-d/q_0}(|Du|^{p_0})^{1/p_0}_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\qquad + N\kappa^{\sigma}\sum_{k=0}^{\infty} 2^{-k\alpha}(|Du|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B_{\delta^{-1}r/2}(x_0)} \\
&\qquad + N\kappa^{-(d+2/\alpha)/p_0}\gamma_0^{1/(\nu p_0)}\sum_{k=0}^{\infty} 2^{-\alpha k}2^{k/(\nu p_0)}(|Du|^{\mu p_0})^{1/(\mu p_0)}_{(t_0-2^k r^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \\
&\qquad + N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty} 2^{-\alpha k}(|g|^{p_0})^{1/p_0}_{(t_0-2^k r^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)},
\end{aligned} \tag{2.5.5}$$

where $\nu = \mu/(\mu-1)$, $q_0 = p_0/(p_0-1)$, and $N = N(d,\delta,\alpha,p,q,K_1)$. The functions u and g are defined to be zero whenever $t \le 0$.

Proof. Note that this proposition is similar to Proposition 2.3.1. Indeed, there exist $p_0 = p_0(d,p,q,K_1) \in (1,2)$ and $\mu = \mu(d,p,q,K_1) \in (1,\infty)$ such that

$$p_0 < p_0\mu < \min\{p,q\}$$

and $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T) \subset \mathcal{H}^{\alpha,1}_{p_0\mu,0,\mathrm{loc}}(\mathbb{R}^d_T)$.

If $r < R_1\delta/\sqrt{d}$, where $R_1 = 2^{-\alpha/2}R_0$, let

$$\overline{a}^{ij} = \fint_{Q_{\delta^{-1}2r}(t_0,x_0)} a^{ij}\,dydt$$

and $\overline{g}_i = g_i + (a^{ij} - \overline{a}^{ij})D_j u$. By using Lemma 2.1.14 and Proposition 2.5.2, we have

$$\begin{aligned}
&(|Du - (Du)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\le N\kappa^{\sigma}\sum_{k=0}^{\infty} 2^{-k\alpha}(|Du|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B_{\delta^{-1}r/2}(x_0)} \\
&\quad + N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty} 2^{-\alpha k}(|\overline{g}|^{p_0})^{1/p_0}_{(t_0-2^k r^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \\
&\le N\kappa^{\sigma}\sum_{k=0}^{\infty} 2^{-k\alpha}(|Du|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B_{\delta^{-1}r/2}(x_0)} \\
&\quad + N\kappa^{-(d+2/\alpha)/p_0}\gamma_0^{1/(\nu p_0)}\sum_{k=0}^{\infty} 2^{-\alpha k}2^{k/(\nu p_0)}(|Du|^{\mu p_0})^{1/(\mu p_0)}_{(t_0-2^k r^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)} \\
&\quad + N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^{\infty} 2^{-\alpha k}(|g|^{p_0})^{1/p_0}_{(t_0-2^k r^{2/\alpha},t_0)\times B_{\delta^{-1}\sqrt{d}r}(x_0)},.
\end{aligned}$$

If $r \geq R_1\delta/\sqrt{d}$, then by Hölder's inequality,

$$
\begin{aligned}
&(|Du - (Du)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\leq 2(|Du|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&= 2(|Du|\chi_{(0,T)\times B_{\xi R_0}(x_1)})_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \\
&\leq 2(|Du|^{p_0})^{1/p_0}_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)} \left(\frac{|B_{\xi R_0}|}{|B_{\delta\kappa r}|}\right)^{1/q_0} \\
&\leq N(d,\delta,p,q)\xi^{d/q_0}\kappa^{-d/q_0}(|Du|^{p_0})^{1/p_0}_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B_{\delta\kappa r}(x_0)}.
\end{aligned}
$$

Therefore, by combining both cases, we have the estimate (2.5.5), and the proposition is proved. □

Proposition 2.5.4. *Let $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \leq K_1$. There exist $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) > 0$ and $\xi(d,\delta,\alpha,p,q,K_1) > 0$ such that under Assumption 2.1.2 (γ_0), for any $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ supported on $(0,T) \times B_{\xi R_0}(x_1)$ for some $x_1 \in \mathbb{R}^d$ and satisfying (2.5.4) in $\mathbb{R}^d_T$, we have*

$$
\|Du\|_{L_{p,q,w}} \leq N\|g\|_{L_{p,q,w}},
$$

where $N = N(d,\delta,\alpha,p,q,K_1)$.

Proof. With (2.5.5), the proof is similar to that of Proposition 2.3.2 using the sharp function theorem with respect to the same dyadic cubes defined in (2.3.7) and the weight sharp and maximal function theorems. □

Different from the non-divergence form case, we introduce the parameter λ in order to extend the result to more general functions.

Corollary 2.5.5. *Let $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \leq K_1$. There exist $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) > 0$ and $\xi(d,\delta,\alpha,p,q,K_1) > 0$*

such that under Assumption 2.1.2 (γ_0), the following holds. There exists $\lambda_0 = \lambda_0(d, \delta, \alpha, p, q, K_1, R_0)$ *such that for any* $\lambda \geq \lambda_0$ *and* $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ *supported on* $(0,T) \times B_{\xi R_0/\sqrt{2}}(x_1)$ *for some* $x_1 \in \mathbb{R}^d$ *and satisfying*

$$\partial_t^\alpha u - D_i(a^{ij} D_j u) + \lambda u = D_i g_i + f \quad \text{in} \quad \mathbb{R}^d_T,$$

where $f, g_i \in L_{p,q,w}(\mathbb{R}^d_T)$, *we have*

$$\sqrt{\lambda}\|u\|_{L_{p,q,w}} + \|Du\|_{L_{p,q,w}} \leq N\|g\|_{L_{p,q,w}} + \frac{N}{\sqrt{\lambda}}\|f\|_{L_{p,q,w}},$$

where $N = N(d, \delta, \alpha, p, q, K_1)$.

Proof. With Proposition 2.5.4, the proof is similar to that of Proposition 2.4.4 using S. Agmon's idea. See (Krylov, 2007, Lemma 5.5) for details. □

Corollary 2.5.6. *Let* $\alpha \in (1,2)$, $T \in (0,\infty)$, $p, q \in (1,\infty)$, $K_1 \in [1,\infty)$, *and* $[w]_{p,q} \leq K_1$. *There exists* $\gamma_0 = \gamma_0(d, \delta, \alpha, p, q, K_1) > 0$ *such that under Assumption 2.1.2 (γ_0), the following holds. There exists* $\lambda_0 = \lambda_0(d, \delta, \alpha, p, q, K_1, R_0)$ *such that for any* $\lambda \geq \lambda_0$ *and* $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ *satisfying*

$$\partial_t^\alpha u - D_i(a^{ij} D_j u + a^i u) - b^i D_i u - cu + \lambda u = D_i g_i + f \quad \text{in} \quad \mathbb{R}^d_T,$$

where $f, g_i \in L_{p,q,w}(\mathbb{R}^d_T)$, *we have*

$$\sqrt{\lambda}\|u\|_{L_{p,q,w}} + \|Du\|_{L_{p,q,w}} \leq N\|g\|_{L_{p,q,w}} + \frac{N}{\sqrt{\lambda}}\|f\|_{L_{p,q,w}},$$

where $N = N(d, \delta, \alpha, p, q, K_1)$.

Proof. Case 1. We first consider the case when $p = q$ and $a^i = b^i = c = 0$. We take

γ_0, ξ, and λ_0 from Corollary 2.5.5, and use a partition of unity argument in the spatial variables by picking $\{x_k\} \subset \mathbb{R}^d$, $\zeta^k \geq 0$, $\zeta^k \in C_0^\infty(\mathbb{R}^d)$, $\mathrm{supp}(\zeta^k) \subset B_{\xi R_0/\sqrt{2}}(x_k)$ and

$$1 \leq \sum_{k=1}^{\infty} |\zeta^k|^p \leq N(p), \quad \sum_{k=1}^{\infty} |D_x \zeta^k|^p \leq N(d, \alpha, p, K_1, R_0).$$

For $u_k(t,x) := u(t,x)\zeta^k(x)$, we have

$$\begin{aligned}
&\partial_t^\alpha u_k - D_i(a^{ij} D_j u_k) + \lambda u_k \\
&\quad = (\partial_t^\alpha u)\zeta^k - D_i(a^{ij}\zeta^k D_j u) + \lambda u \zeta^k - D_i(a^{ij} u D_j \zeta^k) \\
&\quad = D_i(g_i \zeta^k - a^{ij} u D_j \zeta^k) + f\zeta^k - a^{ij} D_i \zeta^k D_j u - g_i D_i \zeta^k.
\end{aligned}$$

Therefore, by Corollary 2.5.5 for $\lambda \geq \lambda_0$,

$$\begin{aligned}
&\sqrt{\lambda}\|u_k\|_{p,w} + \|Du_k\|_{p,w} \\
&\leq N\big[\|\zeta^k g\|_{p,w} + \|u D_j \zeta^k\|_{p,w}\big] + \frac{N}{\sqrt{\lambda}}\big[\|f\zeta^k\|_{p,w} + \|D_i\zeta^k D_j u\|_{p,w} + \|D_i \zeta^k g_i\|_{p,w}\big],
\end{aligned}$$

where $N = N(d, \delta, \alpha, p, K_1)$. By raising to the p-th power and summing in k, we get

$$\begin{aligned}
&\lambda^{p/2}\|u\|_{p,w}^p + \|Du\|_{p,w}^p \\
&\leq N_0\|g\|_{p,w}^p + N_1\|u\|_{p,w}^p + \frac{N_0}{\lambda^{p/2}}\|f\|_{p,w}^p + \frac{N_1}{\lambda^{p/2}}\|Du\|_{p,w}^p + \frac{N_1}{\lambda^{p/2}}\|g\|_{p,w}^p,
\end{aligned}$$

where $N_0(d, \delta, \alpha, p, K_1)$ and $N_1 = (d, \delta, \alpha, p, K_1, R_0)$. We further pick λ_0 sufficiently large such that for all $\lambda \geq \lambda_0(d, \delta, \alpha, p, K_1, R_0)$,

$$\lambda^{p/2} - N_1 > \frac{\lambda^{p/2}}{2} \quad \text{and} \quad 1 - \frac{N_1}{\lambda^{p/2}} > \frac{1}{2},$$

which implies

$$\sqrt{\lambda}\|u\|_{p,w} + \|Du\|_{p,w} \leq N\|g\|_{p,w} + \frac{N}{\sqrt{\lambda}}\|f\|_{p,w},$$

where $N = N(d, \delta, \alpha, p, K_1)$.

Case 2. Then, we consider the case when $p = q$ without assuming $a^i, b^i, c = 0$. In this case, we have

$$\partial_t^\alpha u - D_i(a^{ij} D_j u) + \lambda u = D_i(g_i + a^i u) + f + b^i D_i u + cu.$$

Therefore, the result follows from Case 1 by picking a possibly larger λ_0 depending on the upper bound of a^i, b^i, and c.

Case 3. Finally, for the general case when $p \neq q$, we use the extrapolation theorem in (Dong and Kim, 2018, Theorem 2.5). The corollary is proved. □

Before we prove Theorem 2.1.9, we introduce the following lemma, which implies that if $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$, then $\partial_t^{\alpha-1} u \in \mathbb{H}^{-1}_{p,q,w}(\mathbb{R}^d_T)$, and its norm can be controlled by the norm of $\partial_t^\alpha u$.

Lemma 2.5.7. *For $[w]_{p,q} \le K_1$, if there exist $f, g \in L_{p,q,w}(\Omega_T)$ such that*

$$\int_{\Omega_T} u\phi_{tt} \, dxdt = -\int_{\Omega_T} g_i D_i \phi \, dxdt + \int_{\Omega_T} f\phi \, dxdt$$

for any $\phi \in C_0^\infty([0,T) \times \Omega)$, then there exist $\widetilde{g}, \widetilde{f} \in L_{p,q,w}(\Omega_T)$ such that

$$\int_{\Omega_T} u\phi_t \, dxdt = \int_{\Omega_T} \widetilde{g}_i D_i \phi \, dxdt - \int_{\Omega_T} f\phi \, dxdt$$

for any $\phi \in C_0^\infty([0,T) \times \Omega)$. Furthermore,

$$\|\widetilde{g}\|_{L_{p,q,w}} \le NT\|g\|_{L_{p,q,w}} \quad \text{and} \qquad \|\widetilde{f}\|_{L_{p,q,w}} \le NT\|f\|_{L_{p,q,w}}, \tag{2.5.6}$$

where $N = N(p, K_1)$.

Proof. Let

$$\widetilde{g}_i(t,x) = \int_0^t g_i(s,x)\,ds \quad \text{and} \quad \widetilde{f}(t,x) = \int_0^t f(s,x)\,ds.$$

For any $\phi \in C_0^\infty([0,T)\times\Omega)$, we check that

$$\int_{\Omega_T} u\phi_t\,dxdt = \int_{\Omega_T} \widetilde{g}_i D_i\phi\,dxdt - \int_{\Omega_T} \widetilde{f}\phi\,dxdt. \tag{2.5.7}$$

Note that the right-hand side of (2.5.7) equals

$$\begin{aligned}
&\int_{\Omega_T}\int_0^t g_i(s,x)\,ds\,D_i\phi(t,x)\,dxdt - \int_{\Omega_T}\int_0^t f(s,x)\,ds\,\phi(t,x)\,dxdt \\
&= \int_{\Omega_T}\int_t^T D_i\phi(s,x)\,ds\,g_i(t,x)\,dxdt - \int_{\Omega_T}\int_t^T \phi(s,x)\,ds\,f(t,x)\,dxdt \\
&= -\int_{\Omega_T} u\Phi_{tt}\,dxdt = \int_{\Omega_T} u\phi_t\,dxdt,
\end{aligned}$$

where

$$\Phi(t,x) = \int_t^T \phi(s,x)\,ds \quad \text{and} \quad \Phi \in C_0^\infty([0,T)\times\Omega).$$

To prove (2.5.6), note that

$$\int_0^t \|g_i(s,\cdot)\|_{L_{q,w_2}}\,ds \le t \fint_0^t \|g_i(s,\cdot)\|_{L_{q,w_2}}\,ds \le T\mathcal{M}_t\left(\|g_i(\cdot,\cdot)\|_{L_{q,w_2}}\right)(t),$$

where $\mathcal{M}_t$ denotes the maximal function in t. Thus, by the Minkowski inequality and the weighted maximal function theorem, we have

$$\|\widetilde{g}_i\|_{p,q,w} \le \Big\| \int_0^t \|g_i(s,\cdot)\|_{L_{q,w_2}}\,ds \Big\|_{L_{p,w}(0,T)} \le NT\|g_i\|_{p,q,w},$$

where $N = N(p, K_1)$. The lemma is proved. $\square$

To prove the a priori estimate (2.1.8), we are going to take cutoff functions in time (cf. the proof of (2.1.3)). To that end, we compute the fractional derivative of the product in the following lemma (cf. Lemma 2.2.1).

Lemma 2.5.8. *Let $p \in [1, \infty)$, $\alpha \in (1, 2)$, $k \in \{1, 2, \dots\}$, $-\infty < S < t_0 < T < \infty$, and $v \in \mathcal{H}^{\alpha,1}_{p,q,w,0}((S,T) \times \Omega)$. Then, for any infinitely differentiable function η defined on $\mathbb{R}$ such that $\eta(t) = 0$ for $t \le t_0$, we have $\eta v \in \mathcal{H}^{\alpha,1}_{p,q,w,0}((t_0, T) \times \Omega)$ and*

$$\partial_t^\alpha (\eta v) = \partial_t^2 I_{t_0}^{2-\alpha}(\eta v) = \eta \partial_t^2 I_S^{2-\alpha} v + h,$$

where

$$\begin{aligned} h(t,x) = {} & \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_S^t (t-s)^{-\alpha-1} [\eta(s) - \eta(t) - \eta'(t)(s-t)] v(s,x) \, ds \\ & + \frac{\alpha}{\Gamma(2-\alpha)} \eta'(t) \partial_t I_{t_0}^{2-\alpha} v \end{aligned}$$

in the sense of distribution.

Proof. Let $f, g \in L_{p,q,w}((S,T) \times \Omega)$ such that

$$\partial_t^\alpha v = \partial_t^2 I_S^{2-\alpha} v = \operatorname{div} g + f.$$

By taking the mollification in the spatial variables and using the Fubini theorem, we have

$$\partial_t^\alpha v^\varepsilon = \partial_t^2 I_S^{2-\alpha} v^\varepsilon = \operatorname{div} g^\varepsilon + f^\varepsilon \in L_{p,q,w}((S,T) \times \Omega).$$

Therefore, $v^\varepsilon \in \mathbb{H}^{\alpha,2}_{p,q,w,0}((t_0,T)\times\Omega)$, and by Lemma 2.2.1,

$$\begin{aligned}\partial_t^\alpha(\eta v^\varepsilon) &= \eta\partial_t^2 I_S^{2-\alpha} v^\varepsilon \\ &\quad + \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_S^t (t-s)^{-\alpha-1}[\eta(s)-\eta(t)-\eta'(t)(s-t)]v^\varepsilon(s,x)\,ds \\ &\quad + \frac{\alpha}{\Gamma(2-\alpha)}\eta'(t)\partial_t I_{t_0}^{2-\alpha} v^\varepsilon\end{aligned}$$

in the strong sense. Then, by sending $\varepsilon \to 0$, and using the dominated convergence theorem and Lemma 2.5.7, we conclude

$$\int_S^t (t-s)^{-\alpha-1}[\eta(s)-\eta(t)-\eta'(t)(s-t)]v^\varepsilon(s,x)\,ds$$

converges to

$$\int_S^t (t-s)^{-\alpha-1}[\eta(s)-\eta(t)-\eta'(t)(s-t)]v(s,x)\,ds \quad \text{in} \quad L_{p,q,w}((S,T)\times\Omega),$$

$$\partial_t^2 I_S^{2-\alpha} v^\varepsilon \to \partial_t^2 I_S^{2-\alpha} v \quad \text{in} \quad \mathbb{H}^{-1}_{p,q,w}((S,T)\times\Omega),$$

and

$$\partial_t I_{t_0}^{2-\alpha} v^\varepsilon \to \partial_t I_{t_0}^{2-\alpha} v \quad \text{in} \quad \mathbb{H}^{-1}_{p,q,w}((t_0,T)\times\Omega).$$

Therefore, the lemma is proved. □

We are ready to prove Theorem 2.1.9. Recall that $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$, and $\|\cdot\|_{p,q,w,(\tau_1,\tau_2)} := \|\cdot\|_{L_{p,q,w}((\tau_1,\tau_2)\times\mathbb{R}^d)}$.

Proof of Theorem 2.1.9. We first prove the a priori estimate (2.1.8). By adding λu to both sides of (2.1.7), we have

$$\partial_t^\alpha u - D_i(a^{ij}D_j u + a^i u) - b^i D_i u - cu + \lambda u = D_i g_i + f + \lambda u.$$

By Corollary 2.5.6, there exists $\gamma_0 = \gamma_0(d, \delta, \alpha, p, q, K_1) > 0$ such that under Assumption 2.1.2 (γ_0), the following holds. There exists $\lambda_0 = \lambda_0(d, \delta, \alpha, p, K_1, R_0)$ such that for all $\lambda \geq \lambda_0$,

$$\sqrt{\lambda}\|u\|_{p,q,w} + \|Du\|_{p,q,w} \leq N\|g\|_{p,q,w} + \frac{N}{\sqrt{\lambda}}\|f\|_{p,q,w} + N\sqrt{\lambda}\|u\|_{p,q,w},$$

and by picking $\lambda = \max(1, \lambda_0)$, we have

$$\|u\|_{p,q,w} + \|Du\|_{p,q,w} \leq N\|g\|_{p,q,w} + N\|f\|_{p,q,w} + N_1\|u\|_{p,q,w}, \tag{2.5.8}$$

where $N = N(d, \delta, \alpha, p, K_1)$ and $N_1 = N_1(d, \delta, \alpha, p, K_1, R_0)$. Therefore, in order to prove (2.1.8), it is sufficient to prove

$$\|u\|_{p,q,w} \leq N\|g\|_{p,q,w} + N\|f\|_{p,q,w} \tag{2.5.9}$$

for some $N = N(d, \delta, \alpha, p, K_1, R_0, T)$.

Step 1. Take a positive integer m to be specified below. Set $s_k = kT/m$ for $k = -1, 0, 1, \ldots, m-1$, and define $\eta_k = \eta_k(t)$ as in (2.3.12) for $k \geq 0$. Then, by Lemma 2.5.8, $u\eta_k \in \mathcal{H}^{\alpha,1}_{p,q,w,0}((s_{k-1}, s_{k+1}) \times \mathbb{R}^d)$ and

$$\begin{aligned}
&\partial_t^\alpha(u\eta_k) - D_i(a^{ij}D_j(u\eta_k) + a^i(u\eta_k)) - b^iD_i(u\eta_k) - c(u\eta_k) \\
&= D_i(g_i\eta_k) + f\eta_k - \eta_k\partial_t^\alpha u + \partial_t^\alpha(u\eta_k) \\
&= D_i(g_i\eta_k) + f\eta_k + \tilde{N}\int_{-\infty}^t (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)]u(s,x)\,ds \\
&\quad + N\eta_k'\partial_t I_{t_0}^{2-\alpha}u \\
&= D_i(g_i\eta_k) + ND_i(\widetilde{g}_i\eta_k') + f\eta_k + N\widetilde{f}\eta_k'(t) \\
&\qquad + \tilde{N}\int_{-\infty}^t (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)]\chi_{s\leq s_k}u(s,x)\,ds,
\end{aligned} \tag{2.5.10}$$

where N and $\tilde{N}$ depend only on α, and

$$\partial_t I_0^{2-\alpha} u =: \operatorname{div} \widetilde{g} + \widetilde{f}.$$

By using the equation (2.1.7), Lemma 2.5.7, and (2.5.8), we have

$$\begin{aligned}
&\|\widetilde{g}\eta_k'\|_{p,q,w,(s_{k-1},s_{k+1})} \le \|\eta_k'\|_{L_\infty} \|\widetilde{g}\|_{p,q,w,(0,s_k)} \\
&\le N\|\eta_k'\|_{L_\infty} \frac{kT}{m} \{\|g\|_{p,q,w,(0,s_k)} + \|Du\|_{p,q,w,(0,s_k)} + \|u\|_{p,q,w,(0,s_k)}\} \\
&\le Nk(\|g\|_{p,q,w,(0,s_k)} + \|Du\|_{p,q,w,(0,s_k)} + \|u\|_{p,q,w,(0,s_k)}) \\
&\le Nk(\|g\|_{p,q,w,(0,s_k)} + \|f\|_{p,q,w,(0,s_k)} + \|u\|_{p,q,w,(0,s_k)}), \qquad (2.5.11)
\end{aligned}$$

where $N = N(d, \delta, \alpha, p, q, K_1, R_0)$, and similarly,

$$\|\widetilde{f}\eta_k'\|_{p,q,w,(s_{k-1},s_{k+1})} \le Nk(\|g\|_{p,q,w,(0,s_k)} + \|f\|_{p,q,w,(0,s_k)} + \|u\|_{p,q,w,(0,s_k)}). \qquad (2.5.12)$$

Step 2. By the triangle inequality,

$$\begin{aligned}
&\|u\|_{p,q,w,(s_k,s_{k+1})} \le \|u\eta_k\|_{p,q,w,(s_{k-1},s_{k+1})} \\
&\le \|(u\eta_k)^\varepsilon - u\eta_k\|_{p,q,w,(s_{k-1},s_{k+1})} + \|(u\eta_k)^\varepsilon\|_{p,q,w,(s_{k-1},s_{k+1})}. \qquad (2.5.13)
\end{aligned}$$

For the first term in the right-hand side of (2.5.13), note for each t, by Lemma 2.6.2, we have

$$\|(u\eta_k)^\varepsilon(t,\cdot) - (u\eta_k)(t,\cdot)\|_{q,w_2} \le N\varepsilon\|D(u\eta_k)(t,\cdot)\|_{q,w_2},$$

where $N = N(d, q, K_1)$, which implies

$$\|(u\eta_k)^\varepsilon - u\eta_k\|_{p,q,w,(s_{k-1},s_{k+1})} \leq N\varepsilon\|D(u\eta_k)\|_{p,q,w,(s_{k-1},s_{k+1})}. \tag{2.5.14}$$

To estimate $\|(u\eta_k)^\varepsilon\|_{p,q,w,(s_{k-1},s_{k+1})}$ in (2.5.13), we take the mollification in the spatial variables on both sides of (2.5.10), and then move all the terms except $\partial_t^\alpha(u\eta_k)^\varepsilon$ to the right-hand side of the equation. Using Lemma 2.6.3, (2.5.11), and (2.5.12), for $\varepsilon < 1$, we have

$$\begin{aligned}
&\|\partial_t^\alpha(u\eta_k)^\varepsilon\|_{p,q,w,(s_{k-1},s_{k+1})} \\
&\leq \frac{N}{\varepsilon}\big\{\|g\eta_k\|_{p,q,w,(s_{k-1},s_{k+1})} + \|\widetilde{g}\eta_k'\|_{p,q,w,(s_{k-1},s_{k+1})} \\
&\qquad + \|D(u\eta_k)\|_{p,q,w,(s_{k-1},s_{k+1})} + \|u\eta_k\|_{p,q,w,(s_{k-1},s_{k+1})}\big\} \\
&\quad + N\big\{\|f\eta_k\|_{p,q,w,(s_{k-1},s_{k+1})} + \|\widetilde{f}\eta_k'\|_{p,q,w,(s_{k-1},s_{k+1})} + m^2T^{-\alpha}\|u\|_{p,q,w,(0,s_k)}\big\} \\
&\leq \frac{N}{\varepsilon}\big\{m\|f\|_{p,q,w,(0,s_{k+1})} + m\|g\|_{p,q,w,(0,s_{k+1})} + m\|u\|_{p,q,w,(0,s_k)} \\
&\qquad + \|u\|_{p,q,w,(s_{k-1},s_{k+1})}\big\} + N\big\{m\|f\|_{p,q,w,(0,s_{k+1})} + m\|g\|_{p,q,w,(0,s_{k+1})} \\
&\qquad + m\|u\|_{p,q,w,(0,s_k)} + m^2T^{-\alpha}\|u\|_{p,q,w,(0,s_k)}\big\} \\
&\leq \frac{N}{\varepsilon}\big\{m\|f\|_{p,q,w,(0,T)} + m\|g\|_{p,q,w,(0,T)} \\
&\qquad + m\|u\|_{p,q,w,(0,s_k)} + \|u\|_{p,q,w,(s_{k-1},s_{k+1})}\big\} + Nm^2T^{-\alpha}\|u\|_{p,q,w,(0,s_k)},
\end{aligned} \tag{2.5.15}$$

where $N = N(d, \delta, \alpha, p, K_1, R_0)$. For the first inequality (2.5.15), we used

$$\begin{aligned}
&\|\int_0^\cdot (\cdot - s)^{-\alpha-1}[\eta_k(s) - \eta_k(\cdot) - \eta_k'(\cdot)(s - \cdot)]\chi_{s\leq s_k}u(s,x)\,ds\|_{p,q,w,(s_{k-1},s_{k+1})} \\
&\leq \|\eta_k''\|_{L_\infty}\|I^{2-\alpha}[|u(\cdot,x)|\chi_{\cdot\leq s_k}]\|_{p,q,w,(s_{k-1},s_{k+1})} \\
&\leq N(\alpha, p, q, K_1)m^2T^{-\alpha}\|u\|_{p,q,w,(0,s_k)},
\end{aligned}$$

and for the second inequality, we used (2.5.8) to derive

$$\begin{aligned}
&\|D(u\eta_k)\|_{p,q,w,(s_{k-1},s_{k+1})} \le \|Du\|_{p,q,w,(0,s_{k+1})}\\
&\le N\big\{\|f\|_{p,q,w,(0,T)} + \|g\|_{p,q,w,(0,T)} + \|u\|_{p,q,w,(0,s_{k+1})}\big\},
\end{aligned}$$

where $N = N(d, \delta, \alpha, p, K_1, R_0)$. Combining (2.5.14), (2.5.15), and (2.5.13), we have

$$\begin{aligned}
&\|u\|_{p,q,w,(s_k,s_{k+1})}\\
&\le \|(u\eta_k)^\varepsilon - u\eta_k\|_{p,q,w,(s_{k-1},s_{k+1})} + \|(u\eta_k)^\varepsilon\|_{p,q,w,(s_{k-1},s_{k+1})}\\
&\le N\varepsilon\|D(u\eta_k)\|_{p,q,w,(s_{k-1},s_{k+1})} + N\big(\frac{T}{m}\big)^\alpha\|\partial_t^\alpha(u\eta_k)^\varepsilon\|_{p,q,w,(s_{k-1},s_{k+1})}\\
&\le N\varepsilon\big(\|f\|_{p,q,w,(0,T)} + \|g\|_{p,q,w,(0,T)} + \|u\|_{p,q,w,(0,s_k)} + \|u\|_{p,q,w,(s_k,s_{k+1})}\big)\\
&\quad + \big(\frac{T}{m}\big)^\alpha\frac{N}{\varepsilon}\{m\|f\|_{p,q,w,(0,T)} + m\|g\|_{p,q,w,(0,T)} + m\|u\|_{p,q,w,(0,s_k)}\\
&\quad + \|u\|_{p,q,w,(s_{k-1},s_{k+1})}\} + \big(\frac{T}{m}\big)^\alpha Nm^2T^{-\alpha}\|u\|_{p,q,w,(0,s_k)},
\end{aligned} \tag{2.5.16}$$

where $N = N(d, \delta, \alpha, p, q, K_1, R_0)$. By first picking ε sufficiently small, and then picking m sufficiently large so that $\|u\|_{p,q,w,(s_k,s_{k+1})}$ can be absorbed to the left-hand side of (2.5.16), we arrive at

$$\|u\|_{p,q,w,(s_k,s_{k+1})} \le N\|f\|_{p,q,w,(0,T)} + N\|g\|_{p,q,w,(0,T)} + N\|u\|_{p,q,w,(0,s_k)},$$

where $N = N(d, \delta, \alpha, p, q, K_1, R_0, T)$. Therefore, (2.5.9) follows from an induction on k, and the a priori estimate (2.1.8) is proved. Finally, the existence of solutions follows from the solvability of the simple equation with leading coefficients $a^{ij} = \delta^{ij}$ given in Proposition 2.5.1 together with the method of continuity. □

Next, we prove Theorem 2.1.10 by following a similar procedure as in Section 2.4.

2.5.2 The half space case

When $\Omega = \mathbb{R}^d_+$, i.e., the half space, we derive the mean oscillation estimate by taking odd and even extensions to the whole space. Recall the definition of the strong maximal function for the half space in (2.4.1). Moreover, for $w(t,x) = w_1(t)w_2(x)$ where $w_1 \in A_p(\mathbb{R})$ and $w_2 \in A_q(\Omega)$, we denote the even extension of w_2 to the whole space by $\overline{w_2}(x)$, and let $\overline{w}(t,x) = w_1(t)\overline{w_2}(x)$. Similar to Lemma 2.4.2, we have the following lemma.

Lemma 2.5.9. *For any $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$, let $\overline{u}(t,x) = u(t,|x^1|,x')\operatorname{sgn} x^1$. Then, $\overline{u} \in \mathcal{H}^{\alpha,1}_{p,q,\overline{w},0}(\mathbb{R}^d_T)$ and*

$$\|u\|_{\mathcal{H}^{\alpha,1}_{p,q,w}(\Omega_T)} \le \|\overline{u}\|_{\mathcal{H}^{\alpha,1}_{p,q,\overline{w}}(\mathbb{R}^d_T)} \le 2\|u\|_{\mathcal{H}^{\alpha,1}_{p,q,w}(\Omega_T)}.$$

Proposition 2.5.10. *Let $\alpha \in (1,2)$, $p_0 \in (1,2)$, $T \in (0,\infty)$, a^{ij} be constants, and $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p_0,0,\mathrm{loc}}(\Omega_T)$ satisfy*

$$\partial_t^\alpha u - D_i(a^{ij}D_j u) = D_i g_i \quad \text{in} \quad \Omega_T. \tag{2.5.17}$$

Then, for any $(t_0,x_0) \in (0,T] \times \Omega, r \in (0,\infty)$, and $\kappa \in (0,1/4)$, we have

$$\begin{aligned}
&(|Du - (Du)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B^+_{\delta\kappa r}(x_0))}|)_{(t_0-(\kappa r)^{2/\alpha},t_0)\times B^+_{\delta\kappa r}(x_0)} \\
&\quad \le N\kappa^\sigma \sum_{k=0}^\infty 2^{-k\alpha}(|Du|^{p_0})^{1/p_0}_{(t_0-2^k(r/2)^{2/\alpha}r,t_0)\times B^+_{\delta^{-1}r/2}(x_0)} \\
&\qquad + N\kappa^{-(d+2/\alpha)/p_0}\sum_{k=0}^\infty 2^{-\alpha k}(|g|^{p_0})^{1/p_0}_{(t_0-2^k r^{2/\alpha},t_0)\times B^+_{\delta^{-1}\sqrt{d}r}(x_0)},
\end{aligned} \tag{2.5.18}$$

where $N = N(d,\delta,\alpha,p_0)$ and $\sigma = \sigma(d,\alpha,p_0)$.

Proof. The proof is similar to that of Proposition 2.4.3. Indeed, when $a_{ij} = \delta_{ij}$, we extend g by letting

$$g_1(x) = g_1(|x^1|, x'), \quad g_i(x) = g_i(|x^1|, x')\operatorname{sgn} x^1 \quad \text{for} \quad i = 2, \ldots, d$$

so that the odd extension of u is a solution of (2.5.17) in the whole space. Therefore, (2.5.18) follows from Proposition 2.5.2, Lemma 2.5.9, and a change of variables. □

To prove Theorem 2.1.10, we only need the following proposition for the case $G = \mathbb{R}^d_+$. However, for convenience, here we include the result for the whole space, since we will need both of them later.

Lemma 2.5.11. *Let $G = \mathbb{R}^d_+$ or $\mathbb{R}^d$, $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q} \le K_1$. There exists $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) > 0$ such that under Assumption 2.1.2 (γ_0), the following holds. There exists $\lambda_0 = \lambda_0(d,\delta,\alpha,p,q,K_1,R_0)$ such that for any $\lambda \ge \lambda_0$ and $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}((0,T)\times\mathbb{R}^d_+)$ or $\mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ satisfying*

$$Lu + \lambda u = D_i g_i + f \quad \text{in} \quad (0,T)\times G,$$

we have

$$\begin{aligned}&\sqrt{\lambda}\|u\|_{L_{p,q,w}((0,T)\times G)} + \|Du\|_{L_{p,q,w}((0,T)\times G)}\\ &\le N\|g\|_{L_{p,q,w}((0,T)\times G)} + \frac{N}{\sqrt{\lambda}}\|f\|_{L_{p,q,w}((0,T)\times G)},\end{aligned}$$

where $N = N(d,\delta,\alpha,p,q,K_1)$.

Proof. The case when $G = \mathbb{R}^d$ was proved in Corollary 2.5.6. When $G = \mathbb{R}^d_+$, we can easily derive results analogous to Propositions 2.5.3, 2.5.4, and Corollary 2.5.5 by using Proposition 2.5.10, the weighted Hardy-Littlewood and Fefferman-Stein theorems in the half space, and S. Agmon's idea, respectively. Then, Lemma 2.5.11 follows from a partition of unity argument in the half space. □

In the following proof, for $\Omega = \mathbb{R}^d_+$, we denote $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\Omega_T)}$ and $\|\cdot\|_{p,q,w,(\tau_1,\tau_2)} = \|\cdot\|_{L_{p,q,w}((\tau_1,\tau_2)\times\Omega)}$.

Proof of Theorem 2.1.10. First, note that the solvability of the simple equation

$$\partial_t^\alpha u - \Delta u = D_i g_i + f$$

can be proved by extending f, g as

$$g_1(x) = g_1(|x^1|, x'), \quad g_i(x) = g_i(|x^1|, x')\operatorname{sgn} x^1 \quad \text{for} \quad i = 2, \ldots, d,$$

$$f(x) = f(|x^1|, x')\operatorname{sgn} x^1,$$

and then applying Theorem 2.1.9. Therefore, by the method of continuity it suffices to prove the a priori estimate (2.1.10) under Assumption 2.1.2 (γ_0), where γ_0 is taken from Lemma 2.5.11. By adding $\max(1, \lambda_0)u$ to both sides of the equation (2.1.9), where $\lambda_0 = \lambda_0(d, \delta, \alpha, p, q, K_1, R_0)$ is derived from Lemma 2.5.11 in the half space, we have

$$\|u\|_{p,q,w} + \|Du\|_{p,q,w} \le N\|g\|_{p,q,w} + N\|f\|_{p,q,w} + N_1\|u\|_{p,q,w}, \tag{2.5.19}$$

where $N = N(d, \delta, \alpha, p, K_1)$ and $N_1 = N_1(d, \delta, \alpha, p, K_1, R_0)$. Therefore, in order to prove (2.1.10), it remains to prove

$$\|u\|_{p,q,w} \le N\|g\|_{p,q,w} + N\|f\|_{p,q,w} \tag{2.5.20}$$

for some $N = N(d, \delta, \alpha, p, K_1, R_0, T)$. We follow the proof of (2.5.9). The difference is that we need to take extensions before taking the mollification. For a positive integer m to be specified below, we take $s_k = kT/m$ for $k = -1, 0, 1, \ldots, m-1$, and define the cutoff functions η_k as in (2.3.12) for $k \geq 0$. Then (2.5.10) is satisfied in the half space, i.e.,

$$\begin{aligned}\partial_t^\alpha(u\eta_k) = D_i(g_i\eta_k + a^{ij}D_j(u\eta_k) + a^i(u\eta_k))) + N_4 D_i(\widetilde{g}_i\eta_k') + c(u\eta_k)\\ +b^i D_i(u\eta_k) + f\eta_k + N_4\widetilde{f}\eta_k'(t)\\ + N_3\int_{-\infty}^t (t-s)^{-\alpha-1}[\eta_k(s) - \eta_k(t) - \eta_k'(t)(s-t)]\chi_{s\leq s_k}u(s,x)\,ds,\end{aligned}$$

where N_3 and N_4 depend only on α. Therefore, by Lemma 2.6.3 and (2.5.19), we conclude

$$\begin{aligned}&\|\partial_t^\alpha \overline{u\eta_k}^{\varepsilon}\|_{L_{p,q,\overline{w}}((s_{k-1},s_{k+1})\times\mathbb{R}^d)}\\ &\leq \frac{N}{\varepsilon}\big\{m\|f\|_{p,q,w,(0,T)} + m\|g\|_{p,q,w,(0,T)} + m\|u\|_{p,q,w,(0,s_k)}\\ &\qquad + \|u\|_{p,q,w,(s_{k-1},s_{k+1})}\big\} + Nm^2T^{-\alpha}\|u\|_{p,q,w,(0,s_k)},\end{aligned} \tag{2.5.21}$$

where $N = N(d, \delta, \alpha, p, q, K_1, R_0)$, and $\overline{u\eta_k} \in \mathcal{H}^{\alpha,1}_{p,q,\overline{w},0}(\mathbb{R}^d_T)$ denotes the odd extension of $u\eta_k$ in the x^1 direction. Then, by (2.5.19) and (2.5.21),

$$
\begin{aligned}
&\|u\|_{p,q,w,(s_k,s_{k+1})} \le \|\overline{u\eta_k}\|_{L_{p,q,\overline{w}}((s_{k-1},s_{k+1})\times\mathbb{R}^d)}\\
&\le \|\overline{u\eta_k}^{\varepsilon} - \overline{u\eta_k}\|_{L_{p,q,\overline{w}}((s_{k-1},s_{k+1})\times\mathbb{R}^d)} + \|\overline{u\eta_k}^{\varepsilon}\|_{L_{p,q,\overline{w}}((s_{k-1},s_{k+1})\times\mathbb{R}^d)}\\
&\le N_1\varepsilon\|D(\overline{u\eta_k})\|_{L_{p,q,\overline{w}}((s_{k-1},s_{k+1})\times\mathbb{R}^d)} + N_2(\frac{T}{m})^{\alpha}\|\partial_t^{\alpha}\overline{u\eta_k}^{\varepsilon}\|_{L_{p,q,\overline{w}}((s_{k-1},s_{k+1})\times\mathbb{R}^d)}\\
&\le N_1\varepsilon\|D(u\eta_k)\|_{p,q,w,(s_{k-1},s_{k+1})}\\
&\qquad + (\frac{T}{m})^{\alpha}\frac{N}{\varepsilon}\{m\|f\|_{p,q,w,(0,T)} + m\|g\|_{p,q,w,(0,T)} + m\|u\|_{p,q,w,(0,s_k)}\\
&\qquad\quad + \|u\|_{p,q,w,(s_{k-1},s_{k+1})}\} + (\frac{T}{m})^{\alpha}Nm^2T^{-\alpha}\|u\|_{p,q,w,(0,s_k)}\\
&\le N\varepsilon(\|g\|_{p,q,w,(0,T)} + \|f\|_{p,q,w,(0,T)} + \|u\|_{p,q,w,(0,s_k)} + \|u\|_{p,q,w,(s_k,s_{k+1})})\\
&\qquad + (\frac{T}{m})^{\alpha}\frac{N}{\varepsilon}\{m\|f\|_{p,q,w,(0,T)} + m\|g\|_{p,q,w,(0,T)} + m\|u\|_{p,q,w,(0,s_k)}\\
&\qquad\quad + \|u\|_{p,q,w,(s_{k-1},s_{k+1})}\} + (\frac{T}{m})^{\alpha}Nm^2T^{-\alpha}\|u\|_{p,q,w,(0,s_k)},
\end{aligned}
\tag{2.5.22}
$$

where $N_1 = N_1(d,q,K_1)$, $N_2 = N_2(\alpha,p,q,K_1)$, $N = N(d,\delta,\alpha,p,q,K_1,R_0)$, and $\overline{w}$ denotes the even extension of w. By first picking ε sufficiently small, and then picking m sufficiently large so that $\|u\|_{p,q,w,(s_k,s_{k+1})}$ can be absorbed to the left-hand side of (2.5.22), we conclude

$$
\|u\|_{p,q,w,(s_k,s_{k+1})} \le N\|f\|_{p,q,w,(0,T)} + N\|g\|_{p,q,w,(0,T)} + N\|u\|_{p,q,w,(0,s_k)},
$$

where $N = N(d,\delta,\alpha,p,q,K_1,R_0,T)$. Therefore, (2.5.20) follows from an induction on k, and Theorem 2.1.10 is proved. $\square$

2.5.3 The domain case

In this subsection, we denote

$$Lu = \partial_t^\alpha u - D_i(a^{ij} D_j u + a^i u) - b^i D_i u - cu.$$

To prove Theorem 2.1.11, we first deal with $L + \lambda$ for λ sufficiently large (cf. Proposition 2.4.5).

Proposition 2.5.12. *Let Ω be a Lipschitz domain, $\alpha \in (1,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $K_1 \in [1,\infty)$, and $[w]_{p,q,\Omega} \le K_1$. There exist $\gamma_0 = \gamma_0(d,\delta,\alpha,p,q,K_1) \in (0,1)$ and $\theta = \theta(d,\delta,\alpha,p,q,K_1) \in (0,1/3)$ such that under Assumption 2.1.2 (γ_0) and Assumption 2.1.4 (θ), the following holds. There exists $\lambda_1(d,\delta,\alpha,p,q,K_1,R_0,R_2) \ge 1$ such that for any $\lambda \ge \lambda_1$ and $f, g_i \in L_{p,q,w}(\Omega_T)$, there exists a unique solution $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$ to*

$$Lu + \lambda u = D_i g_i + f \quad \text{in} \quad \Omega_T, \tag{2.5.23}$$

and u satisfies

$$\|u\|_{\mathcal{H}^{\alpha,1}_{p,q,w}(\Omega_T)} \le N(\|g\|_{p,q,w} + \|f\|_{p,q,w}),$$

where $N = N(d,\delta,\alpha,p,q,K_1,R_0,R_2)$ and $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\Omega_T)}$. Moreover, for any $\lambda \ge 0$, $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$ satisfying (2.5.23), we have

$$\|u\|_{\mathcal{H}^{\alpha,1}_{p,q,w}(\Omega_T)} \le N(\|f\|_{p,q,w} + \|g\|_{p,q,w} + \|u\|_{p,q,w}), \tag{2.5.24}$$

where $N = N(d,\delta,\alpha,p,q,K_1,R_0,R_2)$.

Proof. The proof is similar to that of Proposition 2.4.5. When $p = q$ and λ is sufficiently large, we construct a weak solution to (2.5.23). See Lemma 2.7.5 - 2.7.7 for details. When $p \neq q$, we use the extrapolation theorem in (Dong and Kim, 2018, Theorem 2.5). □

We denote

$$\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\Omega_T)} \quad \text{and} \quad \|\cdot\|_{p,q,w,(\tau_1,\tau_2)} = \|\cdot\|_{L_{p,q,w}((\tau_1,\tau_2)\times\Omega)}$$

in the following proof.

Proof of Theorem 2.1.10. By the solvability of

$$Lu + \lambda u = D_i g_i + f$$

for λ sufficiently large given in Proposition 2.5.12 together with the method of continuity, it remains to prove the a priori estimate (2.1.10) under Assumption 2.1.2 (γ_0) and Assumption 2.1.4 (θ), where γ_0 and θ are taken from Proposition 2.5.12. By (2.5.24), it suffices to prove

$$\|u\|_{p,q,w} \le N\|g\|_{p,q,w} + N\|f\|_{p,q,w}. \tag{2.5.25}$$

We follow the proof of (2.5.9) together with a partition of unity argument in the spatial variables. For a positive integer m to be specified below, we take $s_k = kT/m$ for $k = -1, 0, 1, \ldots, m-1$, and define the cutoff functions η_k as in (2.3.12) for $k \ge 0$. Furthermore, we take $\{\xi^l\}$ to be a partition of unity in Ω defined in (2.7.4). Note that ξ^l is either supported in $B_{r_0/40}(x_l)$ for some $x_l \in \Omega$ or supported in $B_{r_0/12}(x_l)$ for some $x_l \in \partial\Omega$, where $r_0 = \min(R_0, R_2)$. Then, for all l, $\xi^l u$ satisfies

$$L(\xi^l u) := D_i g_i^l + f^l \quad \text{in} \quad \Omega \cap \operatorname{supp}(\xi^l),$$

where

$$g_i^l = \xi^l g_i - a^{ij} u D_j \xi^l \quad \text{and} \quad f^l = \xi^l f - g_i D_i \xi^l - (a^{ij} D_j u + a^i u) D_i \xi^l - b^i u D_i \xi^l$$

such that

$$|g^l| + |f^l| \le N(|\xi^l g| + |uD\xi^l| + |gD\xi^l| + |\xi^l f| + |DuD\xi^l|).$$

It follows that for any fixed k and l,

$$\begin{aligned}
&\|\eta_k \xi^l u\|^p_{p,w,(s_{k-1},s_{k+1})} \\
&\le N\varepsilon\big(\|f^l\|_{p,w,(0,T)} + \|g^l\|_{p,w,(0,T)} + \|\xi^l u\|_{p,w,(0,s_k)} + \|\xi^l u\|_{p,w,(s_k,s_{k+1})}\big) \\
&\quad + (\frac{T}{m})^\alpha \frac{N}{\varepsilon}\big\{m\|f^l\|_{p,w,(0,T)} + m\|g^l\|_{p,w,(0,T)} + m\|\xi^l u\|_{p,w,(0,s_k)} \\
&\quad + \|\xi^l u\|_{p,w,(s_{k-1},s_{k+1})}\big\} + (\frac{T}{m})^\alpha N m^2 T^{-\alpha} \|\xi^l u\|_{p,w,(0,s_k)},
\end{aligned} \tag{2.5.26}$$

where $N = N(d, \delta, \alpha, p, K_1, R_0, R_2)$. Indeed, to prove (2.5.26), when $x_l \in \Omega$, we take the extension of weights to the whole space, and then apply (2.5.16); when $x_l \in \partial\Omega$, we apply a change of coordinates followed by the extension of weights to the half space, and then apply (2.5.22). By using (2.5.26), (2.7.5) and (2.5.24), we have

$$
\begin{aligned}
&\|u\|^p_{p,w,(s_k,s_{k+1})} \le \|u\eta_k\|^p_{p,w,(s_{k-1},s_{k+1})} \le \sum_l \|\eta_k \xi^l u\|^p_{p,w,(s_{k-1},s_{k+1})} \\
&\le N\varepsilon^p \Big(\sum_l \||g^l| + |f^l|\|^p_{p,w,(0,s_{k+1})} + \|\xi^l u\|^p_{p,w,(0,s_k)} + \|\xi^l u\|^p_{p,w,(s_k,s_{k+1})} \Big) \\
&\qquad + \Big[(\frac{T}{m})^\alpha \frac{1}{\varepsilon} \Big]^p N \Big\{ \sum_l m\||g^l| + |f^l|\|^p_{p,w,(0,s_{k+1})} + m\|\xi^l u\|^p_{p,w,(0,s_k)} \\
&\qquad\quad + \|\xi^l u\|^p_{p,w,(s_{k-1},s_{k+1})} \Big\} + \Big[(\frac{T}{m})^\alpha m^2 T^{-\alpha} \Big]^p N \sum_l \|\xi^l u\|^p_{p,w,(0,s_k)} \\
&\le N\varepsilon^p \Big(\|g\|^p_{p,w,(0,s_{k+1})} + \|f\|^p_{p,w,(0,s_{k+1})} + \|u\|^p_{p,w,(0,s_{k+1})} + \|Du\|^p_{p,w,(0,s_{k+1})} \Big) \\
&\qquad + \Big[(\frac{T}{m})^\alpha \frac{1}{\varepsilon} \Big]^p N \Big\{ m\|f\|^p_{p,w,(0,s_{k+1})} + m\|g\|^p_{p,w,(0,s_{k+1})} + m\|u\|^p_{p,w,(0,s_{k+1})} \\
&\qquad\quad + m\|Du\|^p_{p,w,(0,s_{k+1})} \Big\} + \Big[(\frac{T}{m})^\alpha m^2 T^{-\alpha} \Big]^p N \|u\|^p_{p,w,(0,s_k)} \\
&\le N\varepsilon^p \Big(\|g\|^p_{p,w,(0,s_{k+1})} + \|f\|^p_{p,w,(0,s_{k+1})} + \|u\|^p_{p,w,(0,s_k)} + \|u\|^p_{p,w,(s_k,s_{k+1})} \Big) \\
&\qquad + \Big[(\frac{T}{m})^\alpha \frac{1}{\varepsilon} \Big]^p N \Big\{ m\|f\|^p_{p,w,(0,s_{k+1})} + m\|g\|^p_{p,w,(0,s_{k+1})} + m\|u\|^p_{p,w,(0,s_k)} \\
&\qquad\quad + m\|u\|^p_{p,w,(s_k,s_{k+1})} \Big\} + \Big[(\frac{T}{m})^\alpha m^2 T^{-\alpha} \Big]^p N \|u\|^p_{p,w,(0,s_k)},
\end{aligned} \tag{2.5.27}
$$

where $N = N(d, \delta, \alpha, p, K_1, R_0, R_2)$. Therefore, using (2.5.24) and the fact that $\alpha \in (1, 2)$ and by first picking ε sufficiently small, and then picking m sufficiently large so that we can absorb $\|u\|^p_{p,w,(s_k,s_{k+1})}$ to the left-hand side of (2.5.27), we have

$$
\|u\|_{p,w,(s_k,s_{k+1})} \le N\|f\|_{p,w,(0,T)} + N\|g\|_{p,w,(0,T)} + N\|u\|_{p,w,(0,s_k)},
$$

where $N = N(d, \delta, \alpha, p, K_1, R_0, R_2, T)$. Thus, when $p = q$, (2.5.25) follows from an induction on k. When $p \neq q$, we use the extrapolation theorem in (Dong and Kim, 2018, Theorem 2.5) to conclude (2.5.25). The theorem is proved. □

2.6 Miscellaneous results

In the first part of this section, we prove some results mentioned in previous sections.

Lemma 2.6.1. *If $p \in [1,\infty)$ and $u \in C^2[0,\infty)$ satisfies $u(0) = u'(0) = 0$, then for any $\varepsilon > 0$ and $T > 0$,*

$$\|u'\|_{L_p(0,T)} \le N\varepsilon^{-1}\|u\|_{L_p(0,T)} + \varepsilon\|u''\|_{L_p(0,T)},$$

where N is independent of u, p, ε, T.

Proof. Take $\xi \in C_0^\infty(\mathbb{R})$ such that $\xi(0) = 1, \xi'(0) = 0$, and take the zero extension of u on $\mathbb{R}$. Then, by the fundamental theorem of calculus,

$$(u'(t+s)\xi(s))_s = u''(t+s)\xi(s) + (u(t+s)\xi'(s))_s - u(t+s)\xi''(s),$$

and

$$u'(t) = \int_{-\infty}^0 (u'(t+s)\xi(s))_s\,ds = \int_{-\infty}^0 u''(t+s)\xi(s)\,ds - \int_{-\infty}^0 u(t+s)\xi''(s)\,ds.$$

By the Minkowski inequality and the fact that u is supported in $[0,\infty)$,

$$\begin{aligned}\|u'\|_{L_p(0,T)} &\le \int_{-\infty}^0 (|\xi(s)| + |\xi''(s)|)(\|u''(\cdot+s)\|_{L_p(0,T)} + \|u(\cdot+s)\|_{L_p(0,T)})\\ &\le N\big[\|u''\|_{L_p(0,T)} + \|u\|_{L_p(0,T)}\big]. \qquad (2.6.1)\end{aligned}$$

For any $\varepsilon > 0$, let $v(\cdot) := u(\varepsilon\cdot)$. Because N in (2.6.1) is independent of T, we replace u with v to get

$$\|v'\|_{L_p(0,T/\varepsilon)} \le N\big[\|v''\|_{L_p(0,T/\varepsilon)} + \|v\|_{L_p(0,T/\varepsilon)}\big],$$

which implies

$$\varepsilon\|u'\|_{L_p(0,T)} \le N\big[\varepsilon^2\|u''\|_{L_p(0,T)} + \|u\|_{L_p(0,T)}\big].$$

Therefore, the desired result follows. □

In Lemmas 2.6.2 and 2.6.3 below, let $\zeta \in C_0^\infty(\mathbb{R}^d)$ satisfy $\mathrm{supp}(\zeta) \subset B_1$ and $\int_{\mathbb{R}^d} \zeta = 1$, and let $\zeta^\varepsilon(\cdot) := \varepsilon^{-d}\zeta(\cdot/\varepsilon)$.

Lemma 2.6.2. *For $q \in [1,\infty)$, $w_2 \in A_q$ with $[w_2]_{A_q} \le K_1$, $v \in W^1_{q,w_2}$, and $v^\varepsilon := v * \zeta^\varepsilon$, we have*

$$\|v^\varepsilon - v\|_{q,w_2} \le N\varepsilon\|Dv\|_{q,w_2},$$

where $N = N(d,q,K_1)$.

Proof. By the definition of v^ε and the Fubini theorem,

$$\begin{aligned}
|v^\varepsilon(x) - v(x)| &= |\int_{B_1} [v(x-\varepsilon y) - v(x)]\zeta(y)\,dy| \\
&= |\int_{B_1} \varepsilon \int_0^1 y\cdot Dv(x - s\varepsilon y)\,ds\,\zeta(y)\,dy| \\
&= \varepsilon|\int_0^1 \int_{B_1} \zeta(y)[y\cdot Dv(x-s\varepsilon y)]\,dy\,ds| \\
&\le \varepsilon\cdot N\int_0^1 \fint_{B_1} |Dv(x-s\varepsilon y)|\,dy\,ds \\
&\le N\varepsilon\mathcal{M}Dv(x),
\end{aligned}$$

where $\mathcal{M}$ denotes the maximal function. Therefore, by the weighted maximal function theorem, we have

$$\|v^\varepsilon - v\|_{q,w_2} \le N\varepsilon\|\mathcal{M}Dv(x)\|_{q,w_2} \le N\varepsilon\|Dv\|_{q,w_2}.$$

The lemma is proved. □

Lemma 2.6.3. *If $[w]_{p,q} \le K_1$ and $u \in \mathcal{H}^{\alpha,1}_{p,q,w,0}(\mathbb{R}^d_T)$ satisfies*

$$\partial_t^\alpha u = D_i g_i + f \quad in \quad \mathbb{R}^d_T \tag{2.6.2}$$

for some $f, g_i \in L_{p,q,w}(\Omega_T)$. Then $u^\varepsilon := u *_x \zeta^\varepsilon \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$, and for any $\varepsilon < 1$, we have

$$\|\partial_t^\alpha u^\varepsilon\|_{L_{p,q,w}(\mathbb{R}^d_T)} \le \frac{N}{\varepsilon}\|g\|_{L_{p,q,w}(\mathbb{R}^d_T)} + N\|f\|_{L_{p,q,w}(\mathbb{R}^d_T)}, \tag{2.6.3}$$

where $N = N(d, \delta, q, K_1)$. Moreover, let $\Omega = \mathbb{R}^d_+$, $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$, $[w]_{p,q,\Omega} \le K_1$, $\overline{u}$ be the odd extension of u in the x^1 direction, and $\overline{w}$ be the even extension of w in the x^1 direction. If (2.6.2) is satisfied in Ω_T for some $f, g_i \in L_{p,q,w}(\Omega_T)$, then

$$\|\partial_t^\alpha \overline{u}^\varepsilon\|_{L_{p,q,w}(\mathbb{R}^d_T)} \le \frac{N}{\varepsilon}\|g\|_{L_{p,q,w}(\Omega_T)} + N\|f\|_{L_{p,q,w}(\Omega_T)}, \tag{2.6.4}$$

where $N = N(d, \delta, q, K_1)$.

Proof. We first consider the whole space case. Taking the convolution in the spatial variables in both sides of (2.6.2) yields

$$\partial_t^\alpha u^\varepsilon = g_i *_x D_i\zeta^\varepsilon + f *_x \zeta^\varepsilon. \tag{2.6.5}$$

Note that

$$\begin{aligned}|f * \zeta^\varepsilon(t,x)| &= |\int_{B_1} f(t, x - \varepsilon y)\zeta(y)\,dy| \\ &\le \|\zeta\|_{L^\infty}|B_1||\fint_{B_1} f(t, x - \varepsilon y)\,dy| \\ &\le \|\zeta\|_{L^\infty}|B_1|\mathcal{M}_x f(t,x),\end{aligned}$$

where $\mathcal{M}_x$ denotes the maximal function in the spatial variables for fixed t. Thus, by the weighted maximal function theorem,

$$\|f * \zeta^\varepsilon(t,\cdot)\|_{q,w_2} \le \|\zeta\|_{L^\infty} N \|f(t,\cdot)\|_{q,w_2}, \tag{2.6.6}$$

where $N = N(d,q,K_1)$, and similarly,

$$\|g_i * D_i\zeta^\varepsilon(t,\cdot)\|_{q,w_2} \le \|D_i\zeta\|_{L^\infty} \cdot \frac{N}{\varepsilon} \|g_i(t,\cdot)\|_{q,w_2}. \tag{2.6.7}$$

Therefore, (2.6.3) follows from (2.6.5), (2.6.6), and (2.6.7).

To prove (2.6.4), we note that $\overline{u} \in \mathcal{H}^{\alpha,1}_{p,q,\overline{w},0}(\mathbb{R}^d_T)$ by Lemma 2.5.9. By taking proper odd and even extensions of g_i and f, the extensions satisfy (2.6.2) in the whole space, and (2.6.4) follows from (2.6.3). The lemma is proved. $\square$

2.7 Results for equations in a smooth domain

In this section, we prove some results for equations in a smooth domain Ω. This section is organized as follows. First, we consider the restrictions and extensions of the weights of the spatial variables. Then, we prove the results for non-divergence form equations in Lemmas 2.7.2-2.7.4. Finally, the results for equations in divergence form are proved in Lemmas 2.7.5-2.7.7.

Let a^{ij}'s satisfy Assumption 2.1.2 (γ_0) and let Ω satisfy Assumption 2.1.4 (θ) for divergence form equations or Assumption 2.1.5 (θ) for non-divergence form equations with $\gamma_0 \in (0,1)$ and $\theta \in (0,1/3)$ to be specified. We locally flatten the boundary of $\partial\Omega$ as follows. For $r_0 = \min(R_0, R_2)$ and any $z \in \partial\Omega$, there exists a coordinate system and a mapping ϕ such that

$$\Omega \cap B_{r_0}(z) = \{x \in B_{r_0}(z) : x^1 > \phi(x')\}.$$

Let

$$\Phi^1(x) = x^1 - \phi(x'), \quad \Phi^j(x) = x^j \quad \text{for} \quad j \geq 2, \quad \Psi = \Phi^{-1}, \tag{2.7.1}$$

and without loss of generality, assume that $\Phi(z) = 0$. Moreover, let

$$\eta \in C_0^\infty(B_{3r_0/32}(z)),\ \eta = 1 \text{ in } B_{r_0/12}(z),\ \eta^z(\cdot) = \eta(\cdot - z),\ \text{and } \hat{\eta}^z(\cdot) = \eta^z(\Psi(\cdot)). \tag{2.7.2}$$

Since $\theta \leq 1/3$, $\hat{\eta}^z \in C_0^\infty(B_{r_0/8})$ and $\hat{\eta} = 1$ in $B_{r_0/16}$, where we used

$$B_{3r_0/32}(z) \subset B_{\frac{r_0}{8(1+\theta)}}(z) \subset \Psi(B_{r_0/8}), \quad \Psi(B_{r_0/16}) \subset B_{r_0(1+\theta)/16}(z) \subset B_{r_0/12}(z).$$

Furthermore, for $r_1 := r_0/2 > 0$, we have

$$B_{r_1} \subset \Phi(B_{r_0}(z)) \quad \text{and} \quad B_{r_1}^+ \subset \Phi(\Omega \cap B_{r_0}(z)).$$

Indeed, for $(y^1, y') \in B_{r_1}$, we can easily check that $(y^1 + \phi(y'), y') \in B_{r_0}(z)$ and $\Phi(y^1 + \phi(y'), y') = (y^1, y')$.

Lemma 2.7.1. *(1) If $w \in A_p(\Omega)$, then $w \in A_p(\Omega \cap B_{r_0}(z))$ and $w(\Psi) \in A_p(\Phi(\Omega \cap B_{r_0}(z)))$.*

(2) Let $\delta > 0$ and $C_\delta^+ = [0, \delta] \times [-\delta/2, \delta/2]^{d-1}$. If $w \in A_p(C_\delta^+)$, then there exists $\overline{w} \in A_p(\mathbb{R}^d_+)$ such that $\overline{w} = w$ in C_δ^+.

Proof. For (1), for any $x \in \Omega \cap B_{r_0}(z), r > 0$, we have

$$
\begin{aligned}
&\left[\fint_{B_r(x)\cap\Omega\cap B_{r_0}(z)} w\,dx\right]\left[\fint_{B_r(x)\cap\Omega\cap B_{r_0}(z)} w^{-\frac{1}{p-1}}\,dx\right]^{p-1} \\
&\le \left(\frac{|B_r(x)\cap\Omega|}{|B_r(x)\cap\Omega\cap B_{r_0}(z)|}\fint_{B_r(x)\cap\Omega} w\,dx\right)\left(\frac{|B_r(x)\cap\Omega|}{|B_r(x)\cap\Omega\cap B_{r_0}(z)|}\fint_{B_r(x)\cap\Omega} w^{-\frac{1}{p-1}}\,dx\right)^{p-1} \\
&\le N(r_0,d,p)[w]_{p,\Omega},
\end{aligned}
$$

where when r is small, we used the smoothness of Ω, and when $r > 2r_0$, we used the fact that $B_r(x) \cap \Omega \cap B_{r_0}(z) = \Omega \cap B_{r_0}(z)$. Thus, $w \in A_p(\Omega \cap B_{r_0}(z))$.

Next, for any $\overline{y} \in \Phi(\Omega \cap B_{r_0}(z))$ and $r > 0$, since $\det D\Psi = 1$, we have

$$
\begin{aligned}
&\left[\fint_{B_r(\overline{y})\cap\Phi(\Omega\cap B_{r_0}(z))} w(\Psi(y))\,dy\right]\left[\fint_{B_r(\overline{y})\cap\Phi(\Omega\cap B_{r_0}(z))} [w(\Psi(y))]^{-\frac{1}{p-1}}\,dy\right]^{p-1} \\
&= \left[\fint_{\Psi(B_r(\overline{y}))\cap\Omega\cap B_{r_0}(z)} w(x)\,dx\right]\left[\fint_{\Psi(B_r(\overline{y}))\cap\Omega\cap B_{r_0}(z)} w(x)^{-\frac{1}{p-1}}\cdot\,dx\right]^{p-1} \\
&\le \left(\frac{|B_{2r}(\Psi(\overline{y}))\cap\Omega\cap B_{r_0}(z)|}{|\Psi(B_r(\overline{y}))\cap\Omega\cap B_{r_0}(z)|}\right)^p\left(\fint_{B_{2r}(\Psi(\overline{y}))\cap\Omega\cap B_{r_0}(z)} w\,dx\right) \\
&\quad\cdot\left(\fint_{B_{2r}(\Psi(\overline{y}))\cap\Omega\cap B_{r_0}(z)} w^{-\frac{1}{p-1}}\,dx\right)^{p-1} \\
&\le \left(\frac{|B_{2r}(\Psi(\overline{y}))\cap\Omega\cap B_{r_0}(z)|}{|B_{r/2}(\Psi(\overline{y}))\cap\Omega\cap B_{r_0}(z)|}\right)^p\left(\fint_{B_{2r}(\Psi(\overline{y}))\cap\Omega\cap B_{r_0}(z)} w\,dx\right) \\
&\quad\cdot\left(\fint_{B_{2r}(\Psi(\overline{y}))\cap\Omega\cap B_{r_0}(z)} w^{-\frac{1}{p-1}}\,dx\right)^{p-1} \\
&\le N(r_0,d,p)[w]_{p,\Omega\cap B_{r_0}(z)}.
\end{aligned}
$$

Thus, $w(\Psi) \in A_p(\Phi(\Omega \cap B_{r_0}(z)))$, and (1) is proved.

For (2), we define $\overline{w}$ by taking the even extension of w along each side as follows.

First, for $x^1 \in (\delta, 2\delta]$ and $x' \in [-\delta/2, \delta/2]^{d-1}$, let

$$\overline{w}(x^1, x') = w(2\delta - x^1, x'),$$

and after defining $\overline{w}$ on $(0, 2\delta] \times [-\delta/2, \delta/2]^{d-1}$, we take the periodic extension, along the x^1 direction, of $\overline{w}$ to $(0, \infty) \times [-\delta/2, \delta/2]^{d-1}$ with period 2δ. Next, for any $x^1 \in (0, \infty)$, $x^2 \in (\delta/2, 3\delta/2]$, and $x'' = (x^3, \ldots, x^d) \in [-\delta/2, \delta/2]^{d-2}$, let

$$\overline{w}(x^1, x^2, x') = \overline{w}(x^2, \delta - x^2, x'),$$

and extend $\overline{w}$, along the x^2 direction, to $(0, \infty) \times \mathbb{R} \times [-\delta/2, \delta/2]^{d-2}$ with period 2δ. By repeating this process, we extend $\overline{w}$ to $\mathbb{R}^d_+$. For any $x = (x^1, \ldots, x^d) \in \mathbb{R}^d_+$, we find its "preimage" $x_0 \in C^+_\delta$, i.e., the point at which $w(x)$ is defined by. More precisely, if $x_1 - \lfloor x_1/2\delta \rfloor 2\delta > \delta$, we take $x_0^1 = (\lfloor x_1/2\delta \rfloor + 1)2\delta - x_1$ and if $x_1 - \lfloor x_1/2\delta \rfloor 2\delta \le \delta$, we take $x_0^1 = x_1 - \lfloor x_1/2\delta \rfloor 2\delta$. Similarly, we find $(x_0^2, \ldots, x_0^d)$. When $r < \delta/2$, $B_r(x)$ intersects l cubes $C_1, \ldots, C_l$ for some $l \le 2^d$. Thus, by the symmetry of $\overline{w}$,

$$\int_{C^+_\delta \cap B_r(x_0)} w^\beta = \int_{C(x) \cap B_r(x)} \overline{w}^\beta \ge \int_{C_i \cap B_r(x)} \overline{w}^\beta, \quad i = 1, \ldots, l,$$

where $\beta = 1$ or $-1/(p-1)$, and $C(x)$ is the cube containing x. Therefore, when $r < \delta/2$,

$$\begin{aligned}
&\left[\fint_{B_r(x) \cap \mathbb{R}^d_+} \overline{w}\, dx \right] \left[\fint_{B_r(x) \cap \mathbb{R}^d_+} \overline{w}^{-\frac{1}{p-1}}\, dx \right]^{p-1} \\
&\le \left[\frac{|C^+_\delta \cap B_r(x_0)|}{|B_r(x) \cap \mathbb{R}^d_+|} 2^d \fint_{C^+_\delta \cap B_r(x_0)} w\, dx \right] \left[\frac{|C^+_\delta \cap B_r(x_0)|}{|B_r(x) \cap \mathbb{R}^d_+|} 2^d \fint_{C^+_\delta \cap B_r(x_0)} w^{-\frac{1}{p-1}}\, dx \right]^{p-1} \\
&\le N(d, p) [w]_{p, C^+_\delta}.
\end{aligned}$$

On the other hand, when $r \ge \delta/2$, $B_r(x) \cap \mathbb{R}^d_+$ can be covered by $N(r, d, \delta)$ cubes, and $N(r,d,\delta)|C_\delta^+|/|B_r(x)\cap\mathbb{R}^d_+|$ is bounded above by some constant independent of r and δ. Therefore, we still have

$$\begin{aligned}
&\left[\fint_{B_r(x)\cap\mathbb{R}^d_+} \overline{w}\,dx \right] \left[\fint_{B_r(x)\cap\mathbb{R}^d_+} \overline{w}^{-\frac{1}{p-1}}\,dx \right]^{p-1} \\
&\le \left[\frac{N(r,d,\delta)|C_\delta^+|}{|B_r(x)\cap\mathbb{R}^d_+|} \fint_{C_\delta^+} w\,dx \right] \left[\frac{N(r,d,\delta)|C_\delta^+|}{|B_r(x)\cap\mathbb{R}^d_+|} \fint_{C_\delta^+} w^{-\frac{1}{p-1}}\,dx \right]^{p-1} \\
&\le N(d,p)[w]_{p,C_\delta^+}.
\end{aligned}$$

The lemma is proved. □

2.7.1 Results for non-divergence form equations

For $z\in\partial\Omega$, we denote

$$L=\partial_t^\alpha - a^{ij}D_{ij} - b^iD_i - c \quad \text{and} \quad \hat{L}^z = \partial_t^\alpha - \hat{a}^{ij}D_{ij} - \hat{b}^iD_i - \hat{c},$$

where

$$\begin{aligned}
&\hat{a}^{ij}(t,y) = a^{rl}(t,\Psi(y))\Phi^i_{x^r}(\Psi(y))\Phi^j_{x^l}(\Psi(y)), \quad \hat{c}(t,y)=c(t,\Psi(y)),\\
&\hat{b}^i(t,y) = a^{rl}(t,\Psi(y))\Phi^i_{x^rx^l}(\Psi(y)) + b^r(t,\Psi(y))\Phi^i_{x^r}(\Psi(y)).
\end{aligned}$$

For $u\in\mathring{\mathbb{H}}^{\alpha,2}_{p,q,w,0}(\Omega_T)$ satisfying $Lu=f$, we have

$$\hat{L}^z\hat{u}=\hat{f} \quad \text{in} \quad (0,T)\times B^+_{r_1},$$

where $\hat{u}(t,\cdot)=u(t,\Psi(\cdot))$ vanishing on $B_{r_1}\cap\partial\mathbb{R}^d_+$ and $\hat{f}(t,y)=f(t,\Psi(y))$. Note that the coefficients $\hat{b}^i$'s contain the second-order derivatives of Φ. Therefore, we need a stronger assumption on the smoothness of Ω for non-divergence form equations. Next, recall the definitions of η and $\hat{\eta}^z$ in (2.7.2). We extend $\hat{L}^z$ to the half space by letting

$$\widetilde{L}^z := \hat{L}^z \eta^z + \Delta(1-\hat{\eta}^z)$$

with the leading coefficients

$$\widetilde{a}^{ij} := \hat{a}^{ij}\hat{\eta}^z + \delta^{ij}(1-\hat{\eta}^z).$$

Using Assumption 2.1.2 (γ_0) and Assumption 2.1.5, it is easily seen that we can find a sufficiently small $r_2 \in (0, R_0)$ depending on d, δ, and r_1 such that for any $r \le r_2$, and $(t_0, x_0) \in \mathbb{R}^d_T$, we have

$$\sup_{i,j} \fint_{Q_r(t_0,x_0)} |\widetilde{a}^{ij} - (\widetilde{a}^{ij})_{Q_r(t_0,x_0)}| \le 2\gamma_0. \tag{2.7.3}$$

Therefore, we can apply Lemma 2.4.4 in the half space.

Then, we take a partition of unity on $\overline{\Omega}$ as follows. Let

$$\Omega^{r_0/20} = \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) \ge r_0/20\},$$

and take

$$\{x_k\}_{k=1}^\infty \subset \Omega^{r_0/20} \cup \partial\Omega, \quad \xi^k \ge 0, \quad \xi^k \in C_0^\infty(\mathbb{R}^d)$$

such that

$$\operatorname{supp}(\xi^k) \subset \begin{cases} B_{r_0/40}(x_k) & x_k \in \Omega^{r_0/20}, \\ B_{r_0/12}(x_k) & x_k \in \partial\Omega, \end{cases} \qquad \sum_k \xi^k = 1 \quad \text{in } \Omega, \tag{2.7.4}$$

and let

$$\zeta^k = \frac{\xi^k}{(\sum_k (\xi^k)^2)^{1/2}},$$

such that there exists an upper bound on the number of non-vanishing ζ^k's at each $x \in \Omega$. It follows by Hölder's inequality that

$$\varepsilon(d,p) \le \sum_{k=1}^{\infty} |\zeta^k|^p, \sum_{k=1}^{\infty} |\xi^k|^p \le 1, \quad \sum_{k=1}^{\infty} |D_x \xi^k|^p, \sum_{k=1}^{\infty} |D_x \zeta^k|^p \le N(d,p,r_0). \quad (2.7.5)$$

Furthermore, by (2.7.3), we pick θ and γ_0 sufficiently small such that the assumptions of Lemma 2.4.4 are satisfied. Then, for λ_0 defined in Lemma 2.4.4 and any $\lambda \ge \lambda_0$, let

$$R_\lambda^k(\cdot) = \begin{cases} \Phi(\widetilde{L}^{x_k} + \lambda)^{-1}\Phi^{-1}[\eta^{x_k} \cdot] & \text{for} \quad x_k \in \partial\Omega, \\ (L+\lambda)^{-1}(\cdot) & \text{for} \quad x_k \in \Omega^{r_0/20}, \end{cases}$$

which is essentially the inverse of operator $L + \lambda$. See (Krylov, 2008, Section 8.3) for more details. Note that in (Krylov, 2008, Section 8.3), the weight is taken to be 1. For general weights, in order to use the results in the whole space and the half space, we need to take restrictions and extensions of the weight given in Lemma 2.7.1.

Lemma 2.7.2. *For $\lambda \ge \lambda_0$, $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,w,0}(\Omega_T)$, and $f = Lu + \lambda u$, we have*

$$u = \sum_k \zeta^k R_\lambda^k(\zeta^k f - L^k u),$$

where

$$L^k u := \zeta^k (Lu) - L(\zeta^k u) = u(a^{ij} D_{ij}\zeta^k + b^i D_i \zeta^k) + 2a^{ij} D_i \zeta^k D_j u.$$

Proof. Similar to (Krylov, 2008, Theorem 8.3.7), we have $\zeta^k u = R_\lambda^k((L+\lambda)(\zeta^k u))$

for any k. Thus,

$$u = \sum_k \zeta^k \zeta^k u = \sum_k \zeta^k R^k_\lambda((L+\lambda)(\zeta^k u)) = \sum_k \zeta^k R^k_\lambda(\zeta^k f - L^k u).$$

We omit the details. □

Lemma 2.7.3. *Let $\lambda \geq \lambda_0$, $f \in L_{p,w}(\Omega_T)$, and $u \in \mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T)$ such that*

$$u = \sum_k \zeta^k R^k_\lambda(\zeta^k f - L^k u).$$

Then, $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,w,0}(\Omega_T)$, and there exists $\lambda_1(d,\delta,\alpha,p,q,K_1,R_0,R_2) \geq 1$ such that if $\lambda \geq \lambda_1$, then

$$Lu + \lambda u = f.$$

Proof. The first assertion follows from the definition of R^k_λ. For the second assertion, we use the fact that the mapping

$$\sum_k L^k R^k_\lambda(\zeta^k \cdot) : L_{p,w}(\Omega_T) \longrightarrow L_{p,w}(\Omega_T)$$

is a contraction when λ is sufficiently large, which follows from the a priori estimate in Lemma 2.4.4 and the observation that L^k contains only the zeroth and the first order terms of u. See (Krylov, 2008, Lemma 8.5.1) for details. A similar proof for equations in divergence form is given below in Lemma 2.7.6. □

Lemma 2.7.4. *There exists a constant $\lambda_1(d,\delta,\alpha,p,q,K_1,R_0,R_2) \geq 1$ such that for any $\lambda \geq \lambda_1$ and $f \in L_{p,w}(\Omega_T)$, there exists a unique $u \in \mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T)$ to*

$$u = \sum_k \zeta^k R^k_\lambda(\zeta^k f - L^k u),$$

and u satisfies

$$\lambda\|u\|_{L_{p,w}(\Omega_T)} + \lambda^{1/2}\|Du\|_{L_{p,w}(\Omega_T)} \le N\|f\|_{L_{p,w}(\Omega_T)}. \tag{2.7.6}$$

Proof. It suffices to prove that

$$\sum_k \zeta^k R^k_\lambda(\zeta^k f - L^k\cdot) : \mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T) \longrightarrow \mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T)$$

is a contraction. For this, we prove

$$\lambda\|u\|_{L_{p,w}(\Omega_T)} + \lambda^{1/2}\|Du\|_{L_{p,w}(\Omega_T)} \le N(\|f\|_{L_{p,w}(\Omega_T)} + \|v\|_{\mathbb{H}^{0,1}_{p,w}(\Omega_T)}), \tag{2.7.7}$$

where

$$u = \sum_k \zeta^k R^k_\lambda(\zeta^k f - L^k v).$$

Indeed, by the definition of R^k_λ and Lemma 2.4.4, we have

$$\begin{aligned}
\lambda^p\|u\|^p_{L_{p,w}(\Omega_T)} &\le N\sum_k \lambda^p\|\zeta^k R^k_\lambda(\zeta^k f - L^k v)\|^p_{L_{p,w}(\Omega_T)} \\
&\le N\sum_k \|\zeta^k f - L^k v\|^p_{L_{p,w}(\Omega_T)} \\
&\le N(\|f\|^p_{L_{p,w}(\Omega_T)} + \|v\|^p_{\mathbb{H}^{0,1}_{p,w}(\Omega_T)}),
\end{aligned}$$

where we used (2.7.5) in the last inequality. Similarly,

$$\lambda^{1/2}\|Du\|_{L_{p,w}(\Omega_T)} \le N(\|f\|_{L_{p,w}(\Omega_T)} + \|v\|_{\mathbb{H}^{0,1}_{p,w}(\Omega_T)}).$$

Thus, (2.7.7) follows, and the lemma is proved. □

Combining Lemmas 2.7.3 and 2.7.4, we conclude that there exists λ_1 as in Lemma

2.7.4 such that for any $\lambda \geq \lambda_1$ and $g \in L_{p,w}(\Omega_T)$, there exists a unique solution $u \in \mathring{\mathbb{H}}^{\alpha,2}_{p,w,0}(\Omega_T)$ satisfying

$$\partial_t^\alpha u - a^{ij} D_{ij} u - b^i D_i u - cu + \lambda u = g.$$

If $\lambda \geq \lambda_1$, (2.4.7) follows from (2.7.6). If $\lambda \in [0, \lambda_1)$, let $h := Lu + \lambda_1 u$. We have

$$\begin{aligned}
\|u\|^p_{\mathbb{H}^{\alpha,2}_{p,w}(\Omega_T)} &\leq N \sum_k \|\zeta^k R^k_{\lambda_1}(\zeta^k h - L^k u)\|^p_{\mathbb{H}^{\alpha,2}_{p,w}(\Omega_T)} \\
&\leq N \sum_k \|\zeta^k h - L^k u\|^p_{L_{p,w}(\Omega_T)} \\
&\leq N(\|h\|^p_{L_{p,w}(\Omega_T)} + \|u\|^p_{L_{p,w}(\Omega_T)} + \|Du\|^p_{L_{p,w}(\Omega_T)}).
\end{aligned}$$

Thus, by the interpolation inequality,

$$\begin{aligned}
\|u\|_{\mathbb{H}^{\alpha,2}_{p,w}(\Omega_T)} &\leq N(\|Lu + \lambda u\|_{L_{p,w}(\Omega_T)} + |\lambda_1 - \lambda| \|u\|_{L_{p,w}(\Omega_T)} + \|u\|_{L_{p,w}(\Omega_T)}) \\
&\leq N(\|Lu + \lambda u\|_{L_{p,w}(\Omega_T)} + \|u\|_{L_{p,w}(\Omega_T)}).
\end{aligned}$$

2.7.2 Results for divergence form equations

Next, we use a similar process to prove the results for equations in divergence form. For $z \in \partial\Omega$, we denote

$$\begin{aligned}
Lu &= \partial_t^\alpha u - D_i(a^{ij} D_j u + a^i u) - b^i D_i u - cu, \\
\hat{L}^z \hat{u} &= \partial_t^\alpha \hat{u} - D_i(\hat{a}^{ij} D_j \hat{u} - \hat{a}^i \hat{u}) - \hat{b}^i D_i \hat{u} - \hat{c}\hat{u},
\end{aligned}$$

where

$$\hat{a}^{ij}(t,y) = a^{rl}(t,\Psi(y))\Phi^i_{x^r}(\Psi(y))\Phi^j_{x^l}(\Psi(y)), \quad \hat{c}(t,y) = c(t,\Psi(y)),$$
$$\hat{b}^i(t,y) = b^r(t,\Psi(y))\Phi^i_{x^r}(\Psi(y)), \quad \hat{a}^i(t,y) = a^r(t,\Psi(y))\Phi^i_{x^r}(\Psi(y)),$$

where Φ and Ψ are defined in (2.7.1). For $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,q,w,0}(\Omega_T)$ satisfying $Lu = D_i g_i + f$, we have

$$\hat{L}^z \hat{u} = D_i \hat{g}_i + \hat{f} \quad \text{in} \quad (0,T) \times B^+_{r_1},$$

where $\hat{u}(t,\cdot) = u(t,\Psi(\cdot))$ vanishing on $B_{r_1} \cap \partial\mathbb{R}^d_+$,

$$\hat{g}_i = g^r(t,\Psi(y))\Phi^i_{x^r}(\Psi(y)), \quad \text{and} \quad \hat{f}(t,y) = f(t,\Psi(y)).$$

Next, similar to Section B.1, we define $\widetilde{L}^z$ and $\widetilde{a}^{ij}$ with the same r_2 so that (2.7.3) still holds. Furthermore, we pick θ and γ_0 sufficiently small such that the assumptions of Lemma 2.5.11 are satisfied. Then, for λ_0 defined in Lemma 2.4.4 and any $\lambda \geq \lambda_0$, let

$$R^k_\lambda(\cdot) = \begin{cases} \Phi(\widetilde{L}^{x_k} + \lambda)^{-1}\Phi^{-1}[\eta^{x_k}\cdot] & \text{for} \quad x_k \in \partial\Omega, \\ (L+\lambda)^{-1}(\cdot) & \text{for} \quad x_k \in \Omega^{r_0/20}, \end{cases}$$

which is essentially the inverse of operator $L + \lambda$.

Lemma 2.7.5. *For $\lambda \geq \lambda_0$, $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,w,0}(\Omega_T)$, and $D_i g_i + f = Lu + \lambda u$, we have*

$$u = \sum_k \zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i \zeta^k + \zeta^k f - L^k u),$$

where

$$L^k u := \zeta^k Lu - L(\zeta^k u) = D_i(a^{ij} u D_j \zeta^k) + (a^{ij} D_j u + a^i u + b^i u) D_i \zeta^k. \tag{2.7.8}$$

Proof. Since $\zeta^k u = R^k_\lambda((L+\lambda)(\zeta^k u))$ for any k,

$$\begin{aligned} u &= \sum_k \zeta^k \zeta^k u = \sum_k \zeta^k R^k_\lambda((L+\lambda)(\zeta^k u)) \\ &= \sum_k \zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i \zeta^k + \zeta^k f - L^k u). \end{aligned}$$

The lemma is proved. □

Lemma 2.7.6. *Let $\lambda \geq \lambda_0$, $f, g_i \in L_{p,w}(\Omega_T)$, and $u \in \mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T)$ such that*

$$u = \sum_k \zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i \zeta^k + \zeta^k f - L^k u).$$

Then, $u \in \mathring{\mathcal{H}}^{\alpha,1}_{p,w,0}(\Omega_T)$, and there exists $\lambda_1(d, \delta, \alpha, p, q, K_1, R_0, R_2)$ such that if $\lambda \geq \lambda_1$, then

$$Lu + \lambda u = D_i g_i + f.$$

Proof. The first assertion follows from the definition of R^k_λ. For the second assertion, let

$$u := \sum_k \zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i \zeta^k + \zeta^k f - L^k u)$$

and

$$Lu + \lambda u =: D_i \overline{g_i} + \overline{f}.$$

Then, by Lemma 2.7.5,

$$u = \sum_k \zeta^k R^k_\lambda(D_i(\zeta^k \overline{g_i}) - \overline{g_i} D_i \zeta^k + \zeta^k \overline{f} - L^k u).$$

It is sufficient to prove if λ is sufficiently large, and

$$T_\lambda(D_i g_i + f) := \sum_k \zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i \zeta^k + \zeta^k f) = 0,$$

then $D_i g_i + f = 0$. Given $T_\lambda(D_i g_i + f) = 0$, we have

$$0 = (\lambda + L) T_\lambda(D_i g_i + f) = D_i g_i + f - \sum_k L^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i \zeta^k + \zeta^k f).$$

To finish the proof, it suffices to prove the mapping

$$\sum_k L^k R^k_\lambda(\zeta^k \cdot) : \mathbb{H}^{-1}_{p,w}(\Omega_T) \longrightarrow \mathbb{H}^{-1}_{p,w}(\Omega_T) \tag{2.7.9}$$

is a contraction when λ is sufficiently large, where for $h = D_i g_i + f \in \mathbb{H}^{-1}_{p,w}(\Omega_T)$ with some $f, g_i \in L_{p,w}(\Omega_T)$ in the distribution sense, we write

$$\zeta^k h := D_i(\zeta^k g_i) - g_i D_i \zeta^k + \zeta^k f \in \mathbb{H}^{-1}_{p,w}(\Omega_T).$$

To this end, we take the following equivalent norm in $\mathbb{H}^{-1}_{p,w}(\Omega_T)$:

$$\|h\|_{\mathbb{H}^{-1}_{p,w}(\Omega_T)} := \inf \Big\{ \sum_{i=1}^d \|g_i\|_{L_{p,w}} + \frac{1}{\sqrt{\lambda}} \|f\|_{L_{p,w}} : h = D_i g_i + f \Big\},$$

where $\|\cdot\|_{p,w} = \|\cdot\|_{L_{p,w}(\Omega_T)}$.

For any $f, g_i \in L_{p,w}(\Omega_T)$ and $h = D_i g_i + f$, by (2.7.8), we have

$$L^k U_k = D_i(a^{ij} U_k D_j \zeta^k) + (a^{ij} D_j U_k + a^i U_k + b^i U_k) D_i \zeta^k,$$

where $U_k = R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i\zeta^k + \zeta^k f)$. Then, by (2.7.5) and Lemma 2.5.11,

$$\begin{aligned}
&\sum_k \|a^{ij}U_k D_j\zeta^k)\|^p_{p,w} + \frac{1}{\lambda^{p/2}}\|(a^{ij}D_jU_k + a^iU_k + b^iU_k)D_i\zeta^k\|^p_{p,w} \\
&\leq N\sum_k \big\{\|R^k_\lambda(D_i(\zeta^k g_i) - g_iD_i\zeta^k + \zeta^k f)\|^p_{p,w} \\
&\qquad + \frac{1}{\lambda^{p/2}}\|DR^k_\lambda(D_i(\zeta^k g_i) - g_iD_i\zeta^k + \zeta^k f)\|^p_{p,w}\big\} \\
&\leq N\sum_k \frac{1}{\lambda^{p/2}}\|\zeta^k g_i\|^p_{p,w} + \frac{1}{\lambda^p}\|\zeta^k f\|^p_{p,w} + \frac{1}{\lambda^p}\|g_iD_i\zeta^k\|^p_{p,w} \\
&\leq N\Big(\frac{1}{\lambda^{p/2}}\|g_i\|^p_{p,w} + \frac{1}{\lambda^p}\|f\|^p_{p,w}\Big) = \frac{N}{\lambda^{p/2}}\Big(\|g_i\|^p_{p,w} + \frac{1}{\lambda^{p/2}}\|f\|^p_{p,w}\Big),
\end{aligned}$$

where $N = N(d, \delta, p, K_1, R_0, R_2)$. Thus, by the Minkowski inequality,

$$\|\sum_k L^kR^k_\lambda(\zeta^k h)\|^p_{\mathbb{H}^{-1}_{p,w}(\Omega_T)} \leq \frac{N}{\lambda^{p/2}}\|h\|^p_{\mathbb{H}^{-1}_{p,w}(\Omega_T)}.$$

Therefore, by picking λ sufficiently large, this mapping (2.7.9) is a contraction, and the lemma is proved. $\square$

Lemma 2.7.7. *There exists a constant $\lambda_1(d, \delta, \alpha, p, q, K_1, R_0, R_2) \geq 1$ such that for any $\lambda \geq \lambda_1$ and $f, g_i \in L_{p,w}(\Omega_T)$, there exists a unique $u \in \mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T)$ to*

$$u = \sum_k \zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_iD_i\zeta^k + \zeta^k f - L^k u),$$

and u satisfies

$$\sqrt{\lambda}\|u\|_{p,w} + \|Du\|_{p,w} \leq N\|g\|_{p,w} + \frac{N}{\sqrt{\lambda}}\|f\|_{p,w},$$

where $\|\cdot\|_{p,w} = \|\cdot\|_{L_{p,w}(\Omega_T)}$.

Proof. In order to prove the existence of solutions, it suffices show

$$\sum_k \zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i\zeta^k + \zeta^k f - L^k \cdot) : \mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T) \longrightarrow \mathring{\mathbb{H}}^{0,1}_{p,w,0}(\Omega_T)$$

is a contraction when λ is sufficiently large. To this end, we prove

$$\lambda\|u\|_{p,w} + \sqrt{\lambda}\|Du\|_{p,w} \le N(\sqrt{\lambda}\|g\|_{L_{p,w}} + \|f\|_{L_{p,w}} + \|v\|_{\mathbb{H}^{0,1}_{p,w}(\Omega_T)}), \tag{2.7.10}$$

where

$$u = \sum_k \zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i\zeta^k + \zeta^k f - L^k v),$$

and N is independent of λ. Indeed, by the definition of R^k_λ, (2.7.5), and Lemma 2.5.11, we have

$$\begin{aligned}
\lambda^{p/2}\|u\|^p_{p,w} &\le N\sum_k \lambda^{p/2}\|\zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i\zeta^k + \zeta^k f - L^k v)\|^p_{p,w} \\
&\le N\sum_k \|\zeta^k g\|^p_{p,w} + \frac{1}{\lambda^{p/2}}\|\zeta^k f\|^p_{p,w} + \frac{1}{\lambda^{p/2}}\|L^k v\|^p_{p,w} \\
&\le N\Big(\|g\|^p_{p,w} + \frac{1}{\lambda^{p/2}}\|f\|^p_{p,w} + \frac{1}{\lambda^{p/2}}\|v\|^p_{\mathbb{H}^{0,1}_{p,w}(\Omega_T)}\Big),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\|Du\|^p_{p,w} &\le N\sum_k \|D(\zeta^k R^k_\lambda(D_i(\zeta^k g_i) - g_i D_i\zeta^k + \zeta^k f - L^k v))\|^p_{p,w} \\
&\le N\Big(\|g\|^p_{p,w} + \frac{1}{\lambda^{p/2}}\|f\|^p_{p,w} + \frac{1}{\lambda^{p/2}}\|v\|^p_{\mathbb{H}^{0,1}_{p,w}(\Omega_T)}\Big),
\end{aligned}$$

which imply (2.7.10). The lemma is proved. □

2.8 Proofs of results for equations with non-zero initial conditions

In this section, we prove Corollary 2.1.12 by applying Theorem 2.1.6 to

$$\tilde{f} := f + (a^{ij}D_{ij}U + b^i D_i U + cU) - D_t^\alpha(U - u_0 - tu_1),$$

where U is given in Lemma 2.8.1. Indeed, by Theorem 2.1.6 there exists a unique $w \in \mathbb{H}^{\alpha,2}_{p,q,w,0}(\mathbb{R}^d_T)$ satisfying

$$\partial_t^\alpha w - (a^{ij}D_{ij}w + b^i D_i w + cw) = \tilde{f} \quad \text{in} \quad \mathbb{R}^d_T.$$

Then, it follows that $u := w + U$ is a solution to the equation (1.1.4), and the estimates (2.1.12) follows from (2.1.3) and (2.8.1).

Similarly, Corollary 2.1.13 follows by taking the derivatives of U in the spatial variables and repeating the above process together with Theorem 2.1.9. Indeed, by the following lemma, we have $D_t^\alpha(U - u_0 - tu_1) \in L_{p,q,w}(\mathbb{R}^d_T)$, which implies $D_t^\alpha(DU - Du_0 - tDu_1) \in \mathbb{H}^{-1}_{p,q,w}(\mathbb{R}^d_T)$.

Lemma 2.8.1. *Under the assumptions of Theorem 2.1.12, for any $u_0 \in \mathcal{X}_0$ and $u_1 \in \mathcal{X}_1$, there exists a function U in $\mathbb{R}^d_T$ such that $U(0,\cdot) = u_0(\cdot)$ and $U_t(0,\cdot) = u_1(\cdot)$ in the trace sense and*

$$\|\partial_t^\alpha U\|_{p,q,w} + \|U\|_{p,q,w} + \|DU\|_{p,q,w} + \|D^2U\|_{p,q,w} \le N\|u_0\|_{\mathcal{X}_0} + N\|u_1\|_{\mathcal{X}_1}, \quad (2.8.1)$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$ and $N = N(d, \alpha, p, q, K_1, T, \mu)$.

Proof. We can assume that $u_0(x)$ and $u_1(x)$ are infinitely differentiable with compact support. Then, let $p = p(t,x)$ be the fundamental solution of $\partial_t^\alpha - \Delta$, and let $\tilde{p}(t,x) := \int_0^t p(s,x)\,ds$. See (Kim et al., 2017, Section 6.2) for the construction of p. For the existence of U, we define

$$U(t,\cdot) := p(t,\cdot) * u_0(\cdot) + \tilde{p}(t,\cdot) * u_1(\cdot).$$

Then by (Kochubei, 2014, Section 1), U is a solution to

$$\begin{cases} \partial_t^\alpha U - \Delta U = 0 & \text{in} \quad \mathbb{R}^d_T \\ U(0,\cdot) = u_0(\cdot) & \text{on} \quad \mathbb{R}^d \\ U_t(0,\cdot) = u_1(\cdot) & \text{on} \quad \mathbb{R}^d \end{cases}.$$

Next, we prove the estimate (2.8.1) following the method in (Dong and Kim, 2021, Lemma 5.7). Without loss of generality, we assume that $u_0 = 0$, i.e., $U = \tilde{p}(t,\cdot) * u_1(\cdot)$. Indeed, in order to estimate U, we estimate $p(t,\cdot) * u_0(\cdot)$ and $\tilde{p}(t,\cdot) * u_1(\cdot)$ separately. The estimate of the first term is similar to that of the second term, and it is given in (Dong and Kim, 2021, Lemma 5.7).

By (Kim et al., 2017, Section 6.2) and (Kochubei, 2014, Section 2.2), we have the following normalization property and estimates of $\tilde{p}$

$$\tilde{p}(t,x) = t^{-\alpha d/2+1}\tilde{p}(1, t^{-\alpha/2}x), \tag{2.8.2}$$

for any $m = 0, 1, 2, 3$ and $x \neq 0$,

$$|D^m \tilde{p}(1, x)| \leq N 1_{|x|\geq 1} e^{-\sigma |x|^{\frac{2}{2-\alpha}}} + N 1_{|x|<1} |x|^{-d-m} (|x|^2 + |x|^2 |\log |x|| 1_{d=2,m=0} + |x| 1_{d=1,m=0}), \tag{2.8.3}$$

and

$$|\Delta \tilde{p}(1, x)| \leq N 1_{|x|\geq 1} e^{-\sigma |x|^{\frac{2}{2-\alpha}}} + N 1_{|x|<1} (|x|^{-d+2} 1_{d\geq 3} + |\log |x|| 1_{d=2} + 1), \tag{2.8.4}$$

where $N = N(d, \alpha, m)$ and $\sigma = \sigma(d, \alpha, m) > 0$.

To prove (2.8.1), we start with the estimate of U. From (2.8.2) and (2.8.3) together with a suitable dyadic decomposition of $\mathbb{R}^d$, it follows that for any $t > 0$ and $x \in \mathbb{R}^d$,

$$|U(t, x)| \leq t N \mathcal{M}_x u_1(x),$$

where $\mathcal{M}_x$ is the maximal operator with respect to x and N is independent of t. Therefore, by the Hardy-Littlewood maximal function theorem with A_p weights,

$$\|U(t, \cdot)\|_{L_{q,w_2}(\mathbb{R}^d)} \leq t N \|u_1\|_{L_{q,w_2}(\mathbb{R}^d)},$$

where $N = N(d, \alpha, q, K_1)$, and

$$\|U\|_{L_{p,q,w}(\mathbb{R}^d_T)} \leq N \|u_1\|_{L_{q,w_2}(\mathbb{R}^d)}, \tag{2.8.5}$$

where $N = N(d, \alpha, q, K_1, T, \mu)$.

Next, we estimate $D^2 U$. By (2.8.2) and (2.8.3), we have $\tilde{p}(t, \cdot), D\tilde{p}(t, \cdot) \in L^1(\mathbb{R}^d)$,

which implies

$$\int_{\mathbb{R}^d} D\tilde{p}(t,y)\,dy = 0.$$

Then, by integration by parts,

$$\begin{aligned} D^2U(t,x) &= \int_{\mathbb{R}^d} \tilde{p}(t,y)D^2u_1(x-y)\,dy = \int_{\mathbb{R}^d} D\tilde{p}(t,y)Du_1(x-y)\,dy \\ &= \int_{\mathbb{R}^d} D\tilde{p}(t,y)(t,y)(Du_1(x-y) - Du_1(x))\,dy. \end{aligned} \tag{2.8.6}$$

Moreover, we observe that

$$\Delta U(t,x) = \int_{\mathbb{R}^d} \Delta\tilde{p}(t,y)(u_1(x-y) - u_1(x))\,dy. \tag{2.8.7}$$

Indeed, from (2.8.6) we have

$$\begin{aligned} \Delta U(1,x) &= -\int_{\mathbb{R}^d} \nabla\tilde{p}(1,y)\cdot\nabla_y(u_1(x-y) - u_1(x))\,dy \\ &= \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^d\setminus B_\varepsilon} \nabla\tilde{p}(1,y)\cdot\nabla_y(u_1(x) - u_1(x-y))\,dy \\ &=: \lim_{\varepsilon\searrow 0} L_\varepsilon. \end{aligned}$$

An integration by parts yields

$$L_\varepsilon = \int_{\mathbb{R}^d\setminus B_\varepsilon} \Delta\tilde{p}(1,y)(u_1(x-y) - u_1(x))\,dy - \int_{\partial B_\varepsilon} \frac{\partial\tilde{p}}{\partial\nu}(1,y)(u_1(x-y) - u_1(x))\,dS.$$

By the continuity of u_1 and the estimates in (2.8.3), it is seen that the second term in the above expression vanishes as $\varepsilon \to 0$. Furthermore, (2.8.4) implies that $\Delta\tilde{p}(1,\cdot) \in L_1(\mathbb{R}^d)$. Thus, using the dominated convergence theorem, we have

$$\Delta U(1,x) = \int_{\mathbb{R}^d} \Delta\tilde{p}(1,y)(u_1(x-y) - u_1(x))\,dy. \tag{2.8.8}$$

Therefore, (2.8.7) follows from (2.8.2).

Then, we consider the following four cases.

Case 1. $\theta_1 \in (0,1)$. In this case,

$$\alpha p \in (1+\mu+p, 2(1+\mu+p)). \tag{2.8.9}$$

By the weighted Calderón-Zygmund estimate and the Minkowski inequality applied to (2.8.7), we have

$$\begin{aligned} \|D^2 U(t,\cdot)\|_{L_{q,w_2}} &\le N\|\Delta U(t,\cdot)\|_{L_{q,w_2}} \\ &\le \int_{\mathbb{R}^d} |\Delta\tilde{p}(t,y)| \|u_1(\cdot - y) - u_1(\cdot)\|_{L_{q,w_2}}\, dy. \end{aligned}$$

Therefore, by Hölder's inequality, for any $\beta \in (0,1)$, we have

$$\begin{aligned} &\|D^2 U\|^p_{L_{p,q,w}} \le \int_0^T \left(\int_{\mathbb{R}^d} |\Delta\tilde{p}(t,y)| \|u_1(\cdot - y) - u_1(\cdot)\|_{L_{q,w_2}}\, dy \right)^p t^\mu \, dt \\ &\le N \int_0^T \left(\int_{\mathbb{R}^d} |\Delta\tilde{p}(t,y)|^{p(1-\beta)} \|u_1(\cdot - y) - u_1(\cdot)\|^p_{L_{q,w_2}}\, dy \right) \cdot J_1 t^\mu \, dt, \end{aligned} \tag{2.8.10}$$

where

$$J_1 := \left(\int_{\mathbb{R}^d} |\Delta\tilde{p}(t,y)|^{\beta p/(p-1)}\, dy \right)^{p-1}.$$

By taking $\beta < (p-1)/p$, (2.8.2) and (2.8.3) imply that

$$J_1 \le N t^{-\alpha\beta(d+2)p/2 + \alpha d(p-1)/2 + \beta p}.$$

By (2.8.10) with the Fubini theorem, we have

$$\|D^2U\|^p_{L_{p,q,w}} \leq N \int_{\mathbb{R}^d} J_2 \|u_1(\cdot - y) - u_1(\cdot)\|^p_{L_{q,w_2}}\, dy,$$

where

$$J_2 := \int_0^T |\Delta\tilde{p}(t,y)|^{p(1-\beta)} t^{-\alpha\beta(d+2)p/2+\alpha d(p-1)/2+\mu+\beta p}\, dt.$$

To estimate J_2, we first observe that by using the assumption (2.8.9) and picking $\beta = (p-1)/p - \varepsilon$ for a sufficiently small $\varepsilon > 0$, the following inequality holds,

$$\mu + 1 + p + \alpha d(p(1-\beta) - 1)/2 < \alpha p.$$

Then, by (2.8.2), (2.8.3), and splitting the integral J_2 into two integrals over $t \leq |y|^{2/\alpha}$ and $t > |y|^{2/\alpha}$, we see that

$$J_2 \leq N|y|^{-d-p\theta_1},$$

where N is independent of T. Therefore, it follows that

$$\|D^2U\|_{L_{p,q,w}((0,T)\times\mathbb{R}^d} \leq N\|u_1\|_{\mathcal{X}_1}. \tag{2.8.11}$$

Case 2. $\theta_1 \in (1,2)$. In this case,

$$\alpha p \in (2(1+\mu+p), \infty). \tag{2.8.12}$$

By using (2.8.6) and Hölder's inequality, for any $\beta \in (0,1)$, we have

$$\begin{aligned} &\|D^2U\|^p_{L_{p,q,w}} \\ &\leq \int_0^T \left(\int_{\mathbb{R}^d} |D\tilde{p}(t,y)|^{p(1-\beta)} \|Du_1(\cdot - y) - Du_1(\cdot)\|^p_{L_{q,w_2}}\, dy \right) \cdot J_1 t^{\mu}\, dt, \end{aligned} \tag{2.8.13}$$

where

$$J_1 := \left(\int_{\mathbb{R}^d} |D\tilde{p}(t,y)|^{\beta p/(p-1)}\,dy\right)^{p-1}.$$

By taking $\beta < (p-1)d/(p(d-1))$, from (2.8.2) and (2.8.3) we get

$$J_1 \le N t^{\alpha\beta(d+1)p/2+\alpha d(p-1)/2+\beta p}.$$

This together with (2.8.13) and the Fubini theorem yields

$$\|D^2U\|_{L_{p,q,w}}^p \le N\int_{\mathbb{R}^d} J_2 \|Du_1(\cdot-y)-Du_1(\cdot)\|_{L_{q,w_2}}^p\,dy,$$

where

$$J_2 := \int_0^T |D\tilde{p}(t,y)|^{p(1-\beta)} t^{-\alpha\beta(d+1)p/2+\alpha d(p-1)/2+\mu+\beta p}\,dt.$$

To estimate J_2, we observe that by using the assumption (2.8.12) and picking $\beta = \min\{1,(p-1)d/(p(d-1))\}-\varepsilon$ for a sufficiently small $\varepsilon>0$, the following inequality holds

$$\mu+1+p+\alpha((d-1)p(1-\beta)-d)/2 < \alpha p/2.$$

Then, by (2.8.2), (2.8.3), and splitting the integral J_2 into two integrals as in Case 1, we see that

$$J_2 \le N|y|^{-d-p(\theta_1-1)},$$

which implies (2.8.11).

Case 3. $\theta_1 < 0$. In this case,

$$\alpha p \in (0, 1+\mu+p). \tag{2.8.14}$$

By (2.8.8), (2.8.2), and (2.8.3), we have

$$|\Delta U(t,x)| \le N t^{-\alpha+1}\mathcal{M}_x u_1(x) + N t^{-\alpha+1}|u_1(x)|.$$

Then, by applying the weighted Calderón-Zygmund estimate and the weighted Hardy-Littlewood maximal function theorem,

$$\|D^2 U(t,\cdot)\|_{L_{q,w_2}} \le N\|\Delta U(t,\cdot)\|_{L_{q,w_2}} \le N t^{-\alpha+1}\|u_1\|_{L_{q,w_2}}.$$

Therefore, with the assumption (2.8.14),

$$\|D^2 U\|^p_{L_{p,q,w}} \le \int_0^T \|u_1\|^p_{L_{q,w_2}} t^{-\alpha p+p+\mu}\,dt \le N(T)\|u_1\|^p_{L_{q,w_2}},$$

which implies (2.8.11).

Case 4. $\theta_1 = 0$ or 1. We only consider the case when $\theta_1 = 1$ since the other case is similar. Observe that in this case $\alpha p = 2(\mu+1+p)$. We take $\mu' \in (-1,\mu)$ such that $\alpha p \in ((\mu'+1+p), 2(\mu'+1+p))$, and let $\tilde{w} = \tilde{w}_1(t)w_2(x)$, where $\tilde{w}_1(t) = t^{\mu'}$. By Case 2, we have

$$\|D^2 U\|_{L_{p,q,\tilde{w}}(\mathbb{R}^d_T)} \le N\|u_1\|_{\mathcal{X}_1}.$$

Since $t^{\mu} \le N t^{\mu'}$ for $t \in (0,T)$, (2.8.11) follows.

To prove (2.8.1), we estimate DU using the interpolation inequality (Dong and Krylov, 2019, Lemma 3.8) together with (2.8.11) and (2.8.5). Furthermore, the estimate of $\partial_t^\alpha U$ follows from the equation $\partial_t^\alpha U = \Delta U$. The lemma is proved. $\square$

Chapter Three

Equations with Non-local Operators in Spacial Variables

3.1 Notation and Main Results

To start, we introduce some notation and recall some concepts that were defined in Chapter 2, as these will be used throughout Chapter 3. For $\alpha \in (0,1)$ and $S \in \mathbb{R}$, we denote

$$I_S^\alpha u(t,x) = \frac{1}{\Gamma(\alpha)} \int_S^t (t-s)^{\alpha-1} u(s,x)\, ds, \quad t \geq S,\, x \in \mathbb{R}^d,$$

and

$$\partial_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_t u(s,x)\, ds, \quad t \geq 0,\, x \in \mathbb{R}^d.$$

It is easily seen that $\partial_t^\alpha u = \partial_t I_0^{1-\alpha} u$ for a sufficiently smooth u with $u(0,x) = 0$.

We use $\mathcal{F}(u)$ and $\hat{u}$ to denote the Fourier transform of u. If $u = u(t,x)$, then $\mathcal{F}(u)(t,\xi)$ and $\hat{u}(t,\xi)$ denote the Fourier transform of u in x for a fixed time t. For $p \in (0,\infty)$ and $\sigma \in (0,2)$, recall the definition of the Bessel potential space

$$H_p^\sigma(\mathbb{R}^d) = \{u \in L_p(\mathbb{R}^d) : (1-\Delta)^{\sigma/2} u \in L_p(\mathbb{R}^d)\}$$

and

$$\|u\|_{H_p^\sigma(\mathbb{R}^d)} := \|(1-\Delta)^{\sigma/2} u\|_{L_p(\mathbb{R}^d)},$$

where

$$(1-\Delta)^{\sigma/2} u(x) = \mathcal{F}^{-1}\Big[(1+|\xi|^2)^{\sigma/2} \mathcal{F}(u)(\xi)\Big](x).$$

For $p, q \in (1,\infty)$ and $-\infty < S < T < \infty$, we define $L_{p,q}\big((S,T) \times \mathbb{R}^d\big)$ to be the set of all measurable functions f defined on $(S,T)\times\mathbb{R}^d$ satisfying

$$\|f\|_{L_{p,q}((S,T)\times\mathbb{R}^d)} := \left(\int_S^T \big(\int_{\mathbb{R}^d} |f(t,x)|^q\,dx\big)^{p/q}\,dt\right)^{1/p} < \infty.$$

When $p=q$, we write $L_p\big((S,T)\times\mathbb{R}^d\big) = L_{p,p}\big((S,T)\times\mathbb{R}^d\big)$. Furthermore, for $\alpha\in(0,1]$ and $\sigma\in(0,2)$, we denote $\mathbb{H}_{p,q}^{\alpha,\sigma}(S,T)$ to be the collection of functions such that $u\in L_p\big((S,T);H_q^\sigma(\mathbb{R}^d)\big)$, $\partial_t^\alpha u\in L_{p,q}\big((S,T)\times\mathbb{R}^d\big)$, and

$$\|u\|_{\mathbb{H}_{p,q}^{\alpha,\sigma}(S,T)} := \left(\int_S^T \big(\|u(t,\cdot)\|_{H_q^\sigma(\mathbb{R}^d)}\big)^{p/q}\,dt\right)^{1/p} + \|\partial_t^\alpha u\|_{L_{p,q}((S,T)\times\mathbb{R}^d)}.$$

We write $u\in\mathbb{H}_{p,q,0}^{\alpha,\sigma}(S,T) := \mathbb{H}_{p,q,0}^{\alpha,\sigma}\big((S,T)\times\mathbb{R}^d\big)$ if there exists a sequence of functions $\{u_n\}$ such that $u_n\in C^\infty([S,T]\times\mathbb{R}^d)$ with $u_n(S,x)=0$ vanishing for large $|x|$, and

$$\|u_n-u\|_{\mathbb{H}_{p,q}^{\alpha,\sigma}(S,T)}\to 0 \quad \text{as } n\to\infty.$$

Moreover, for a domain $\Omega\subset\mathbb{R}^d$ and u defined on $(S,T)\times\Omega$, we write $u\in\mathbb{H}_{p,q,0}^{\alpha,\sigma}\big((S,T)\times\Omega\big)$ if there exists a extension of u to $(S,T)\times\mathbb{R}^d$, i.e.,

$$\|u\|_{\mathbb{H}_{p,q}^{\alpha,\sigma}((S,T)\times\Omega)} := \inf\{\|\overline{u}\|_{\mathbb{H}_{p,q}^{\alpha,\sigma}(S,T)} : \overline{u}\in\mathbb{H}_{p,q,0}^{\alpha,\sigma}(S,T) \quad \text{and} \quad \overline{u}|_{(S,T)\times\Omega}=u\}<\infty.$$

We denote $\mathbb{H}_{p,0}^{\alpha,\sigma}((S,T)\times\Omega) := \mathbb{H}_{p,p,0}^{\alpha,\sigma}((S,T)\times\Omega)$ if $p=q$. Furthermore, we take $\psi(x)=1/(1+|x|^{d+\sigma})$, and denote

$$\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))} := \|\psi u\|_{L_p((0,T);L_1(\mathbb{R}^d))}. \tag{3.1.1}$$

For $T\in(0,\infty)$, we denote $(0,T)\times\mathbb{R}^d := \mathbb{R}_T^d$ and we often denote $\mathbb{H}_{p,q}^{\alpha,\sigma}(T) := \mathbb{H}_{p,q}^{\alpha,\sigma}(0,T)$ and $\mathbb{H}_{p,q,0}^{\alpha,\sigma}(T) := \mathbb{H}_{p,q,0}^{\alpha,\sigma}(0,T)$. We use the notation $u\in\mathbb{H}_{p,0,\mathrm{loc}}^{\alpha,\sigma}(\mathbb{R}_T^d)$ to indicate a function satisfying $u\in\mathbb{H}_{p,0}^{\alpha,\sigma}((0,T)\times B_R)$ for all $R>0$.

For $\alpha \in (0,1)$, $\sigma \in (0,2)$, $r_1, r_2 > 0$, and $(t,x) \in \mathbb{R}^{d+1}$, we denote the parabolic cylinder by

$$Q_{r_1,r_2}(t,x) = (t - r_1^{\sigma/\alpha}, t) \times B_{r_2}(x) \quad \text{and} \quad Q_r(t,x) = Q_{r,r}(t,x),$$

where $B_{r_2}(x) = \{y \in \mathbb{R}^d : |y - x| < r_2\}$. We write B_r and Q_r for $B_r(0)$ and $Q_r(0,0)$. Furthermore, for $f \in L_{1,\text{loc}}$ defined on $\mathcal{D} \subset \mathbb{R}^{d+1}$ and $(t,x) \in \mathcal{D}$, we define its maximal function and strong maximal function, respectively, by

$$\mathcal{M}f(t,x) = \sup_{Q_r(s,y) \ni (t,x)} \fint_{Q_r(s,y)} |f(r,z)| \chi_{\mathcal{D}} \, dz \, dr$$

and

$$(\mathcal{SM}f)(t,x) = \sup_{Q_{r_1,r_2}(s,y) \ni (t,x)} \fint_{Q_{r_1,r_2}(s,y)} |f(r,z)| \chi_{\mathcal{D}} \, dz \, dr.$$

Next, for $p \in (1,\infty)$, $k \in \{1,2,\ldots\}$, let $A_p(\mathbb{R}^k, dx)$ be the set of all non-negative functions w on $\mathbb{R}^k$ such that

$$[w]_{A_p(\mathbb{R}^k)} := \sup_{x_0 \in \mathbb{R}^k, r > 0} \left(\fint_{B_r(x_0)} w(x) \, dx \right) \left(\fint_{B_r(x_0)} (w(x))^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty,$$

where $B_r(x_0) = \{x \in \mathbb{R}^k : |x - x_0| < r\}$. Furthermore, for a constant $M_1 > 0$, we write $[w]_{p,q} \le M_1$ if $w = w_1(t) w_2(x)$ for some w_1 and w_2 satisfying

$$w_1(t) \in A_p(\mathbb{R}, dt), \quad w_2(x) \in A_q(\mathbb{R}^d, dx), \quad \text{and } [w_1]_{A_p(\mathbb{R})}, [w_2]_{A_q(\mathbb{R}^k)} \le M_1.$$

We denote $L_{p,q,w}(\mathbb{R}^d_T)$ to be the set of all measurable functions f defined on $\mathbb{R}^d_T$ satisfying

$$\|f\|_{L_{p,q,w}(\mathbb{R}^d_T)} := \left(\int_0^T \Big(\int_{\mathbb{R}^d} |f(t,x)|^q w_2(x) \, dx \Big)^{p/q} w_1(t) \, dt \right)^{1/p} < \infty.$$

When $p = q$ and $w \equiv 1$, $L_{p,q,w}(\mathbb{R}^d_T)$ becomes the usual Lebesgue space $L_p(\mathbb{R}^d_T)$.

We write $N = N(\cdots)$ if the constant N depends only on the parameters in the parentheses.

Next, we present the assumptions for the operators. In chapter 3, we consider equations which are non-local in both time and space:

$$\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T,$$

where L is defined in (1.2.2). We impose the following assumptions on the kernel $K = K(t,x,y) > 0$.

Assumption 3.1.1. 1. There exist some positive constants $\nu, \Lambda > 0$ and a function $m_0 = m_0(t,y) \geq 0$ which is measurable and homogeneous in y with index 0 such that

$$\inf_{\xi \in S^{d-1}} \int_{S^{d-1}} |\xi \cdot y|^\sigma m_0(t,y) \mu_{d-1}(dy) \geq \nu \quad \text{for all } t,$$

and

$$(2-\sigma)\frac{m_0(t,y)}{|y|^{d+\sigma}} \leq K(t,x,y) \leq (2-\sigma)\frac{\Lambda}{|y|^{d+\sigma}} \quad \text{for all } t, x \text{ and } y, \tag{3.1.2}$$

where μ_{d-1} is the standard Lebesgue measure on the unit sphere S^{d-1}.

2. When $\sigma = 1$, for any $r > 0$,

$$\int_{S^{d-1}} yK(t,x,ry)\, \mu_{d-1}(dy) = 0, \tag{3.1.3}$$

where μ_{d-1} is the standard Lebesgue measure on the unit sphere S^{d-1}.

Remark 3.1.2. Assumption 3.1.1 (1) with additional homogeneity, cancellation, and smoothness conditions was introduced in Mikulevičius and Pragarauskas (2014) (see also Zhang (2013)), and it is satisfied, for instance, when

$$(2-\sigma)\nu|y|^{-d-\sigma} \le K(t,x,y) \le (2-\sigma)\Lambda|y|^{-d-\sigma}.$$

Also, note that (3.1.3) implies that χ^1 in (1.2.2) can be replaced by $1_{y\in B_r}$ for any $r>0$.

Assumption 3.1.3. There exist $\beta \in (0,1)$ and a continuous increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|y|^{d+\sigma}|K(t,x_1,y) - K(t,x_2,y)| \le (2-\sigma)\omega(|x_1-x_2|) \tag{3.1.4}$$

and

$$\int_{|y|\le 1} \omega(|y|)|y|^{-d-\beta}\, dy < \infty. \tag{3.1.5}$$

Remark 3.1.4. On one hand, it is easily seen that if $|y|^{d+\sigma}K(t,\cdot,y)$ is Hölder continuous in x with exponent $\beta+\varepsilon$ for any $\varepsilon>0$ uniformly in (t,y), then (3.1.5) is satisfied. On the other hand, since ω is increasing, from (3.1.5) we see that $\omega(r)\le Nr^\beta$ for any $r\le 1/2$, which implies that $|y|^{d+\sigma}K(t,\cdot,y)$ is Hölder continuous in x with exponent β uniformly in (t,y).

Remark 3.1.5. It is well known that for $\sigma\in(0,2)$, if $K(t,x,y) = K(y) := c|y|^{-d-\sigma}$, where

$$c = c(d,\sigma) = \frac{\sigma(2-\sigma)\Gamma(\frac{d+\sigma}{2})}{\pi^{d/2}2^{2-\sigma}\Gamma(2-\frac{\sigma}{2})},$$

then

$$Lu = \frac{c}{2}\int_{\mathbb{R}^d}\Big(u(t,x+y) - u(t,x) - 2u(x)\Big)|y|^{-d-\sigma}\, dy = -(-\Delta)^{\sigma/2}u.$$

In some cases, the lower bound of the operator is not necessary. Thus, we use L^1 to denote operators with the kernel

$$K_1(t,x,y) = K_1(t,y) \le (2-\sigma)\frac{\Lambda}{|y|^{d+\sigma}}, \tag{3.1.6}$$

and we assume that (3.1.3) is satisfied when $\sigma = 1$.

With the assumptions above, we are ready to state the main theorems of chapter 3.

Theorem 3.1.6. *Let $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $\lambda \ge 0$, and $p \in (1,\infty)$. Suppose that the kernel $K = K(t,y)$ satisfies Assumption 3.1.1. Then $\partial_t^\alpha - L$ is a continuous operator from $\mathbb{H}_{p,0}^{\alpha,\sigma}(T)$ to $L_p(\mathbb{R}^d_T)$. Also, for any $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$ satisfying*

$$\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T, \tag{3.1.7}$$

and any L^1 satisfying (3.1.3) when $\sigma = 1$ and (3.1.6), we have

$$\|\partial_t^\alpha u\|_{L_p(\mathbb{R}^d_T)} + \|L^1 u\|_{L_p(\mathbb{R}^d_T)} + \lambda\|u\|_{L_p(\mathbb{R}^d_T)} \le N\|f\|_{L_p(\mathbb{R}^d_T)} \tag{3.1.8}$$

and

$$\|u\|_{\mathbb{H}_p^{\alpha,\sigma}(T)} \le N\min(T^\alpha, \lambda^{-1})\|f\|_{L_p(\mathbb{R}^d_T)}, \tag{3.1.9}$$

where $N = N(d,\nu,\Lambda,\alpha,\sigma,p)$. Moreover, for any $f \in L_p(\mathbb{R}^d_T)$, there exists a unique solution $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$ to (3.1.7).

Remark 3.1.7. Indeed, (3.1.9) follows from (3.1.8) upon setting $L^1 = (-\Delta)^{\sigma/2}$, $L^1 = L$, and using (3.1.7) and Lemma 3.4.2. Furthermore, by (3.1.8) with $\lambda = 0$, for any $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$ if

$$\partial_t^\alpha u + (-\Delta)^{\sigma/2} u = g \quad \text{in } \mathbb{R}^d_T,$$

then

$$\|L^1 u\|_{L_p(\mathbb{R}^d_T)} \leq N\|g\|_{L_p(\mathbb{R}^d_T)}. \tag{3.1.10}$$

For any $v \in H_p^\sigma(\mathbb{R}^d)$, by first taking $T > 0$ and a nonzero function $\eta \in C_0^\infty(\mathbb{R})$ with $\eta(0) = 0$, then applying (3.1.10) to $u(t,x) := \eta(t/T)v(x)$, and finally sending $T \to \infty$, we conclude that

$$\|L^1 v\|_{L_p(\mathbb{R}^d)} \leq N\|(-\Delta)^{\sigma/2} v\|_{L_p(\mathbb{R}^d)}. \tag{3.1.11}$$

Thus, for $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$ and $t \in (0,T)$

$$\|L^1 u(t,\cdot)\|_{L_p(\mathbb{R}^d)} \leq N\|(-\Delta)^{\sigma/2} u(t,\cdot)\|_{L_p(\mathbb{R}^d)},$$

which implies

$$\|L^1 u\|_{L_p(\mathbb{R}^d_T)} \leq N\|(-\Delta)^{\sigma/2} u\|_{L_p(\mathbb{R}^d_T)} \tag{3.1.12}$$

and the continuity of the operator $\partial_t^\alpha - L$. We refer the reader to Dong and Kim (2012) for a different proof of (3.1.11).

Next, we have the following results when $K = K(t,x,y)$ and for a constant $M > 0$,

$$|b| = \left|\left(b^1(t,x), \ldots, b^d(t,x)\right)\right| \leq M, \quad |c| = |c(t,x)| \leq M. \tag{3.1.13}$$

Corollary 3.1.8. *Let $\beta \in (0,1)$, $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, and $p \in (1,\infty)$. Suppose that the kernel $K = K(t,x,y)$ satisfies Assumptions 3.1.1 and 3.1.3, and (3.1.13) holds. Then $\partial_t^\alpha - L$ is a continuous operator from $\mathbb{H}_{p,0}^{\alpha,\sigma}(T)$ to $L_p(\mathbb{R}^d_T)$.*

1. *There exists $\lambda_0 = \lambda_0(d,\nu,\Lambda,\alpha,\sigma,p,M,\beta,\omega) \geq 1$ such that for any $\lambda \geq \lambda_0$ and*

$u \in \mathbb{H}^{\alpha,\sigma}_{p,0}(T)$ satisfying

$$\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T,$$

we have

$$\|u\|_{\mathbb{H}^{\alpha,\sigma}_p(T)} \leq N\|f\|_{L_p(\mathbb{R}^d_T)},$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p, M, \beta, \omega)$ is independent of T.

2. *Also, for any $u \in \mathbb{H}^{\alpha,\sigma}_{p,0}(T)$ satisfying*

$$\partial_t^\alpha u - Lu + b^i D_i u 1_{\sigma>1} + cu = f \quad \text{in } \mathbb{R}^d_T, \tag{3.1.14}$$

we have

$$\|u\|_{\mathbb{H}^{\alpha,\sigma}_p(T)} \leq N\|f\|_{L_p(\mathbb{R}^d_T)}, \tag{3.1.15}$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p, T, M, \beta, \omega)$. Moreover, for any $f \in L_p(\mathbb{R}^d_T)$, there exists a unique solution $u \in \mathbb{H}^{\alpha,\sigma}_{p,0}(T)$ to (3.1.14). Furthermore, when $\sigma = 1$, if $\|b\|_\infty$ is sufficiently small, then the a priori estimate and the unique solvability hold for

$$\partial_t^\alpha u - Lu + b^i D_i u + cu = f \quad \text{in } \mathbb{R}^d_T. \tag{3.1.16}$$

When $\alpha = 1$ and $\sigma = 1$, if b is uniformly continuous, then the a priori estimate and the unique solvability also hold for (3.1.16).

In the case of $\alpha = 1$, i.e., the operator is local in time, we have the following results regarding the weighted mixed-norm.

Theorem 3.1.9. *Let $\sigma \in (0, 2)$, $T \in (0, \infty)$, $\lambda \geq 0$, $p, q \in (1, \infty)$, $M_1 \in [1, \infty)$, and $[w]_{p,q} \leq M_1$. Suppose the kernel $K = K(t, y)$ satisfies Assumption 3.1.1. Then $\partial_t - L$ is a continuous operator from $\mathbb{H}^{1,\sigma}_{p,q,w,0}(T)$ to $L_{p,q,w}(\mathbb{R}^d_T)$. Also, for any $u \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(T)$*

satisfying

$$\partial_t u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T \tag{3.1.17}$$

and any L^1 *satisfying* (3.1.3) *when* $\sigma = 1$ *and* (3.1.6), *we have*

$$\|\partial_t u\|_{L_{p,q,w}(\mathbb{R}^d_T)} + \|L^1 u\|_{L_{p,q,w}(\mathbb{R}^d_T)} + \lambda\|u\|_{L_{p,q,w}(\mathbb{R}^d_T)} \leq N\|f\|_{L_{p,q,w}(\mathbb{R}^d_T)} \tag{3.1.18}$$

and

$$\|u\|_{\mathbb{H}^{1,\sigma}_{p,q,w}(T)} \leq N \min(T, \lambda^{-1})\|f\|_{L_{p,q,w}(\mathbb{R}^d_T)}, \tag{3.1.19}$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p, q, M_1)$. *Moreover, for any* $f \in L_{p,q,w}(T)$, *there exists a unique solution* $u \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(T)$ *to* (3.1.17).

With bounded b and c satisfying (3.1.13), we have the following result.

Corollary 3.1.10. *Let* $\beta \in (0, 1)$, $\sigma \in (0, 2)$, $T \in (0, \infty)$, $p, q \in (1, \infty)$, $M_1 \in [1, \infty)$, *and* $[w]_{p,q} \leq M_1$. *Suppose that the kernel* $K = K(t, x, y)$ *satisfies Assumptions 3.1.1 and 3.1.3. Then* $\partial_t^\alpha - L$ *is a continuous operator from* $\mathbb{H}^{1,\sigma}_{p,q,w,0}(T)$ *to* $L_{p,q,w}(\mathbb{R}^d_T)$.

1. *There exists* $\lambda_0 = \lambda_0(d, \nu, \Lambda, \sigma, p, q, M_1, M, \beta, \omega) \geq 1$ *such that for any* $\lambda \geq \lambda_0$ *and* $u \in \mathbb{H}^{\alpha,\sigma}_{p,0}(T)$ *satisfying*

   $$\partial_t u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T,$$

   we have

   $$\|u\|_{\mathbb{H}^{1,\sigma}_{p,q,w}(T)} \leq N\|f\|_{L_{p,q,w}(\mathbb{R}^d_T)}, \tag{3.1.20}$$

   where $N = N(d, \nu, \Lambda, \sigma, p, q, M_1, M, \beta, \omega)$ *is independent of* T.

2. *Also, for any* $u \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(T)$ *satisfying*

$$\partial_t u - Lu + b^i D_i u 1_{\sigma>1} + cu = f \quad \text{in } \mathbb{R}^d_T, \tag{3.1.21}$$

we have

$$\|u\|_{\mathbb{H}^{1,\sigma}_{p,q,w}(T)} \leq N\|f\|_{L_{p,q,w}(\mathbb{R}^d_T)}, \tag{3.1.22}$$

where $N = N(d, \nu, \Lambda, \sigma, p, q, T, M_1, M, \beta, \omega)$. *Moreover, for any* $f \in L_{p,q,w}(T)$, *there exists a unique solution* $u \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(T)$ *to* (3.1.21). *Furthermore, when* $\sigma = 1$, *if* $\|b\|_\infty$ *is sufficiently small or* b *is uniformly continuous, then the a priori estimate and the unique solvability hold for*

$$\partial_t u - Lu + b^i D_i u + cu = f \quad \text{in } \mathbb{R}^d_T. \tag{3.1.23}$$

Remark 3.1.11. For $\sigma \in (0, 2)$, due to the presence of $(2 - \sigma)$ in the bounds (3.1.2) and (3.1.4) in Assumption 3.1.1 and 3.1.3 respectively, the constant N in above theorems and corollaries can be chosen to be dependent on $\tilde{\sigma}$ instead of on σ itself, where

$$\tilde{\sigma} = \begin{cases} (\sigma_0, \sigma_1) & \text{when } 0 < \sigma_0 \leq \sigma \leq \sigma_1 < 1, \\ 1 & \text{when } \sigma = 1, \\ \sigma_0 & \text{when } 1 < \sigma_0 \leq \sigma < 2. \end{cases} \tag{3.1.24}$$

Thus, when $\sigma > 1$, N does not blow up when $\sigma \nearrow 2$. To see this, we keep track of the dependence of constants on σ in Lemmas 3.4.3 and the proof of Theorem 3.1.6 when $p = 2$. See also Remark 3.3.7. Moreover, note that Lemmas 3.4.5 and 3.4.7 hold for σ_0, and

$$\|u\|_{\mathbb{H}^{\alpha,\sigma_0}_p(T)} \leq N\|u\|_{\mathbb{H}^{\alpha,\sigma}_p(T)},$$

where N can be chosen to be independent of σ by the Mikhlin multiplier theorem (see, for instance, Kurtz and Wheeden (1979)) and (Dong et al., 2021, Lemma 3.4).

Thus, we can choose the increment of p in the iteration arguments in the proofs of Theorems 3.1.6 and 3.1.9 using σ_0 instead of σ. Similar phenomena were observed before, for example in Caffarelli and Silvestre (2009) and Dong and Kim (2012).

3.2 Equations in L_p

In this section, we prove Theorem 3.1.6 and Corollary 3.1.8. To prove Theorem 3.1.6, we use a level set argument and a bootstrap argument with the Sobolev embedding.

3.2.1 The case of $p = 2$ and auxiliary results

In order to apply the bootstrap argument, we start with the case when $p = 2$.

Proof of Theorem 3.1.6 when $p = 2$. We first prove the continuity of $\partial_t^\alpha - L$. By taking the Fourier transform,

$$\widehat{Lu}(t,\xi) = \hat{u}(t,\xi) \int_{\mathbb{R}^d} (e^{i\xi\cdot y} - 1 - iy\cdot\xi\chi^{(\sigma)}(y))K(t,y)\,dy := \hat{u}(t,\xi)m(\xi), \tag{3.2.1}$$

and in particular, by Remark 3.1.5,

$$\widehat{(-\Delta)^{\sigma/2}u}(t,\xi) = \hat{u}(t,\xi) \int_{\mathbb{R}^d} \big(1 - \cos(\xi\cdot y)\big)c|y|^{-d-\sigma}\,dy. \tag{3.2.2}$$

By a change of variables $y \to y/|\xi|$ and the upper bound of K, it is seen that

$$|m(\xi)| \le N(d,\Lambda,\tilde{\sigma})|\xi|^\sigma,$$

where $\tilde{\sigma}$ is defined in Remark 3.1.11. Thus, (3.2.1) leads to

$$\|Lu\|_{L_2(\mathbb{R}^d_T)} = \|\widehat{Lu}\|_{L_2(\mathbb{R}^d_T)} \leq N\|\widehat{u}(\cdot,\xi)|\xi|^{\sigma}\|_{L_2(\mathbb{R}^d_T)} = N\|(-\Delta)^{\sigma/2}u\|_{L_2(\mathbb{R}^d_T)}, \tag{3.2.3}$$

where $N = N(d, \Lambda, \tilde{\sigma})$, which implies the continuity of $\partial_t^\alpha - L$ from $\mathbb{H}_2^{\alpha,\sigma}(T)$ to $L_2(\mathbb{R}^d_T)$.

Next, we prove the a priori estimate (3.1.8). With the continuity of the operator and the density of smooth functions in $\mathbb{H}_{2,0}^{\alpha;\sigma}(T)$, without loss of generality, we assume that $u \in C_0^\infty([0,T] \times \mathbb{R}^d)$ with $u(0,\cdot) = 0$. Multiplying $(-\Delta)^{\sigma/2}u$ to both sides of (3.1.7) and integrating over $\mathbb{R}^d_T$, we arrive at

$$\int_{\mathbb{R}^d_T} \partial_t^\alpha u(-\Delta)^{\sigma/2}u - \int_{\mathbb{R}^d_T} Lu(-\Delta)^{\sigma/2}u + \lambda \int_{\mathbb{R}^d_T} u(-\Delta)^{\sigma/2}u = \int_{\mathbb{R}^d_T} f(-\Delta)^{\sigma/2}u. \tag{3.2.4}$$

For the first term on the left-hand side of (3.2.4), due to (3.2.2) and the fact that it is real,

$$\begin{aligned}
&\int_{\mathbb{R}^d_T} \partial_t^\alpha u(t,x)(-\Delta)^{\sigma/2}u(t,x)\,dx\,dt = \int_{\mathbb{R}^d_T} \widehat{(-\Delta)^{\sigma/2}}u(t,\xi)\overline{\partial_t^\alpha \widehat{u}(t,\xi)}\,d\xi\,dt \\
&= c\int_{\mathbb{R}^d_T} \overline{\partial_t^\alpha \widehat{u}(t,\xi)}\hat{u}(t,\xi)\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)|y|^{-d-\sigma}\,dy\,d\xi\,dt \\
&= c\int_{\mathbb{R}^d_T} \partial_t^\alpha \mathrm{Re}(\widehat{u})(t,\xi)\mathrm{Re}(\hat{u})(t,\xi)\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)|y|^{-d-\sigma}\,dy\,d\xi\,dt \\
&\quad + c\int_{\mathbb{R}^d_T} \partial_t^\alpha \mathrm{Im}(\widehat{u})(t,\xi)\mathrm{Im}(\hat{u})(t,\xi)\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)|y|^{-d-\sigma}\,dy\,d\xi\,dt.
\end{aligned} \tag{3.2.5}$$

Also, by (Dong and Kim, 2019, Proposition 4.1), we have

$$\begin{aligned}&\partial_t^\alpha \mathrm{Re}(\widehat{u})(t,\xi)\mathrm{Re}(\hat{u})(t,\xi)+\partial_t^\alpha \mathrm{Im}(\widehat{u})(t,\xi)\mathrm{Im}(\hat{u})(t,\xi)\\ &\geq \frac{1}{2}\partial_t^\alpha|\mathrm{Re}(\widehat{u})|^2(t,\xi)+\frac{1}{2}\partial_t^\alpha|\mathrm{Im}(\widehat{u})|^2(t,\xi)=\frac{1}{2}\partial_t^\alpha|\hat{u}|^2(t,\xi).\end{aligned}\tag{3.2.6}$$

Thus, by (3.2.5), (3.2.6), and the definition of ∂_t^α together with the fundamental theorem of calculus,

$$\begin{aligned}&\int_{\mathbb{R}^d_T}\partial_t^\alpha u(t,x)(-\Delta)^{\sigma/2}u(t,x)\,dx\,dt\\ &\geq c\int_{\mathbb{R}^d_T}\frac{1}{2}\partial_t^\alpha|\hat{u}|^2(t,\xi)\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)|y|^{-d-\sigma}\,dy\,d\xi\,dt\\ &=N(d,\sigma)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)|y|^{-d-\sigma}\int_0^T\partial_t^\alpha|\hat{u}|^2(t,\xi)\,dt\,dy\,d\xi\\ &=N(d,\alpha,\sigma)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)|y|^{-d-\sigma}\int_0^T(T-s)^{-\alpha}|\hat{u}(s,\xi)|^2\,ds\,dy\,d\xi\geq 0.\end{aligned}$$

For the second term on the left-hand side of (3.2.4), by (3.2.2) and (3.2.1),

$$\begin{aligned}&-\int_{\mathbb{R}^d_T}Lu(t,x)(-\Delta)^{\sigma/2}u(t,x)\,dx\,dt=-\int_{\mathbb{R}^d_T}\overline{\widehat{Lu}(t,\xi)}\widehat{(-\Delta)^{\sigma/2}}u(t,\xi)\,d\xi\,dt\\ &=\int_{\mathbb{R}^d_T}|\hat{u}(t,\xi)|^2\Big(\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)K(t,y)\,dy\Big)\int_{\mathbb{R}^d}\big(1-\cos(\xi\cdot y)\big)c|y|^{-d-\sigma}\,dy\,d\xi\,dt\end{aligned}\tag{3.2.7}$$

where in the second equality we used the fact that the left-hand side is real. Furthermore, by Assumption 3.1.1, a change of variables, and an integration in polar coordinates, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big(1-\cos(\xi\cdot y)\big)K(t,y)\,dy \geq \int_{\mathbb{R}^d} \big(1-\cos(\xi\cdot y)\big)(2-\sigma)m_0(t,y)|y|^{-d-\sigma}\,dy \\
&= |\xi|^{\sigma}\int_{\mathbb{R}^d} \big(1-\cos(|\xi|^{-1}\xi\cdot y)\big)(2-\sigma)m_0(t,y)|y|^{-d-\sigma}\,dy \\
&= |\xi|^{\sigma}\int_{S^{d-1}} \left(\int_0^\infty \big(1-\cos(r|\xi|^{-1}\xi\cdot y)\big)(2-\sigma)r^{-\sigma-1}\,dr\right)m_0(t,y)\,\mu_{d-1}(dy) \\
&= |\xi|^{\sigma}\int_{S^{d-1}} \left(\int_0^\infty \big(1-\cos(r)\big)(2-\sigma)r^{-\sigma-1}\,dr\right)||\xi|^{-1}\xi\cdot y|^{\sigma}m_0(t,y)\,\mu_{d-1}(dy) \\
&\geq N(\tilde{\sigma})\nu|\xi|^{\sigma}. \qquad (3.2.8)
\end{aligned}$$

Therefore, (3.2.7) and (3.2.8) lead to

$$\begin{aligned}
&-\int_{\mathbb{R}^d_T} Lu(t,x)(-\Delta)^{\sigma/2}u(t,x)\,dx\,dt \\
&\geq N(d,\tilde{\sigma})\nu\int_{\mathbb{R}^d_T} |\hat{u}(t,\xi)|^2|\xi|^{2\sigma}\,d\xi\,dt = N(d,\tilde{\sigma},\nu)\|(-\Delta)^{\sigma/2}u\|^2_{L_2(\mathbb{R}^d_T)}. \qquad (3.2.9)
\end{aligned}$$

Similarly, multiplying λu to both sides of (3.1.7) and integrating over $\mathbb{R}^d_T$, we get

$$\lambda^2\int_{\mathbb{R}^d_T} u^2 \leq \int_{\mathbb{R}^d_T} f^2. \qquad (3.2.10)$$

Thus, (3.2.4), (3.2.5), (3.2.9), and (3.2.10) together with Young's inequality yield

$$\|(-\Delta)^{\sigma/2}u\|_{L_2(\mathbb{R}^d_T)} + \lambda\|u\|_{L_2(\mathbb{R}^d_T)} \leq N(d,\tilde{\sigma},\nu)\|f\|_{L_2(\mathbb{R}^d_T)}.$$

Together with (3.2.3), we arrive at

$$\begin{aligned}
\||L^1u| + \lambda|u|\|_{L_2(\mathbb{R}^d_T)} &\leq N(d,\Lambda,\tilde{\sigma})\||(-\Delta)^{\sigma/2}u| + \lambda|u|\|_{L_2(\mathbb{R}^d_T)} \\
&\leq N(d,\nu,\Lambda,\tilde{\sigma})\|f\|_{L_2(\mathbb{R}^d_T)},
\end{aligned}$$

where L^1 is defined in (3.1.6). Finally, by (Kim et al., 2021, Theorem 2.8) together with the method of continuity, we have the existence of solutions. Theorem 3.1.6 is proved when $p = 2$. $\square$

Remark 3.2.1. Indeed, (3.2.8) is the only place where we use the lower bound, m_0 defined in Assumption 3.1.1, of the kernel K.

Next, assuming that Theorem 3.1.6 holds, we derive a local estimate. In the following lemma, we denote $\|\cdot\|_{p,r} := \|\cdot\|_{L_p((0,T)\times B_r)}$ for any $r > 0$. Also, recall the definition of $\|\cdot\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}$ in (3.1.1).

Lemma 3.2.2. *Let $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $p \in (1,\infty)$, and $0 < r < R < \infty$. Also, let ζ_0 be a cutoff function such that*

$$\zeta_0 \in C_0^\infty(B_{(R+r)/2}), \quad \zeta_0 = 1 \quad \text{in } B_r, \quad \text{and } |D\zeta_0| \leq 4/(R-r).$$

Suppose that Theorem 3.1.6 holds for this p. If $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$, and L^1 satisfies (3.1.3) when $\sigma = 1$ and (3.1.6), and

$$\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T, \tag{3.2.11}$$

then

$$\begin{aligned}
&\|\partial_t^\alpha(\zeta_0 u)\|_{L_p(\mathbb{R}^d_T)} + \|L^1(\zeta_0 u)\|_{L_p(\mathbb{R}^d_T)} + \lambda\|\zeta_0 u\|_{L_p(\mathbb{R}^d_T)} \\
&\leq N\|f\|_{p,R} + N\frac{\|u\|_{p,R}}{(R-r)^\sigma} + N\frac{R^{d/p}(1+R^{d+\sigma})}{(R-r)^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))},
\end{aligned} \tag{3.2.12}$$

and

$$\begin{aligned}&\|\partial_t^\alpha u\|_{p,r}+\|L^1u\|_{p,r}+\lambda\|u\|_{p,r}\\&\le N\|f\|_{p,R}+N\frac{\|u\|_{p,R}}{(R-r)^\sigma}+N\frac{R^{d/p}(1+R^{d+\sigma})}{(R-r)^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))},\end{aligned}\tag{3.2.13}$$

where $N=N(d,\nu,\Lambda,\alpha,\sigma,p)$.

Proof. We first take cutoff functions in the spatial variables as follows. For $k=1,2,\ldots$, let

$$r_k=r+(R-r)\sum_{j=1}^k 2^{-j},\quad \zeta_k\in C_0^\infty(B_{r_{k+1}}),$$

$$\zeta_k\in[0,1],\quad \zeta_k=1\quad\text{in } B_{r_k},\quad |D\zeta_k|\le\frac{4\cdot 2^k}{R-r},\quad\text{and } |D^2\zeta_k|\le\frac{16\cdot 2^{2k}}{(R-r)^2}.\tag{3.2.14}$$

It follows that $\zeta_k u\in\mathbb{H}_{p,0}^{\alpha,\sigma}(T)$ and

$$\partial_t^\alpha(\zeta_k u)-L(\zeta_k u)+\lambda(\zeta_k u)=\zeta_k f+\zeta_k Lu-L(\zeta_k u)\quad\text{in }\mathbb{R}^d_T.\tag{3.2.15}$$

By (3.1.12),

$$\begin{aligned}&\|\partial_t^\alpha u\|_{p,r}+\|L^1u\|_{p,r_k}+\lambda\|u\|_{p,r}\\&\le\|\partial_t^\alpha(\zeta_k u)\|_{p,r}+\|\zeta_k L^1u\|_{p,r_k}+\lambda\|\zeta_k u\|_{p,r}\\&\le\|L^1(\zeta_k u)-\zeta_k L^1u\|_{L_p(\mathbb{R}^d_T)}+\|\partial_t^\alpha(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)}\\&\quad+\|L^1(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)}+\lambda\|\zeta_k u\|_{L_p(\mathbb{R}^d_T)}\\&\le\|L^1(\zeta_k u)-\zeta_k L^1u\|_{L_p(\mathbb{R}^d_T)}+N\|\partial_t^\alpha(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)}\\&\quad+\|(-\Delta)^{\sigma/2}(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)}+\lambda\|\zeta_k u\|_{L_p(\mathbb{R}^d_T)}.\end{aligned}\tag{3.2.16}$$

Moreover, by applying Theorem 3.1.6 to (3.2.15), we have

$$\begin{aligned}
&\|\partial_t^\alpha(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2}(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)} + \lambda\|\zeta_k u\|_{L_p(\mathbb{R}^d_T)} \\
&\leq N\|\zeta_k f + \zeta_k Lu - L(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)} \\
&\leq N\|f\|_{p,R} + N\|\zeta_k Lu - L(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)},
\end{aligned} \tag{3.2.17}$$

where $N = N(d,\nu,\Lambda,\alpha,\sigma,p)$. We refer the reader to Lemma 3.4.3 for the details about the estimates of the commutator term $\|\zeta_k Lu - L(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)}$. Note that since we only used the upper bound of the kernel of the operator in Lemma 3.4.3, the estimate can be applied to $\|L^1(\zeta_k u) - \zeta_k L^1 u\|_{L_p(\mathbb{R}^d_T)}$.

Case 1: $\sigma \in (0,1)$. In this case, (3.2.13) and (3.2.12) follow directly from (3.2.16), (3.2.17) and (3.4.3) with $k = 0$.

Case 2: $\sigma \in (1,2)$. In this case, by (3.2.17) and (3.4.4), we have

$$\begin{aligned}
&\|\partial_t^\alpha(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2}(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)} + \lambda\|\zeta_k u\|_{L_p(\mathbb{R}^d_T)} \\
&\leq N\|f\|_{p,R} + N\frac{2^{(\sigma-1)k}}{(R-r)^{\sigma-1}}\|Du\|_{p,r_{k+3}} + N\frac{2^{\sigma k}}{(R-r)^{\sigma}}\|u\|_{p,R} \\
&\quad + N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}}R^{d/p}(1+R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\end{aligned} \tag{3.2.18}$$

Also, note that by Lemma 3.4.1, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
&N\frac{2^{(\sigma-1)k}}{(R-r)^{\sigma-1}}\|Du\|_{p,r_{k+3}} \leq N\frac{2^{(\sigma-1)k}}{(R-r)^{\sigma-1}}\|D(\zeta_{k+3}u)\|_{L_p(\mathbb{R}^d_T)} \\
&\leq N\frac{2^{\sigma k}}{(R-r)^{\sigma}}\varepsilon^{3/(1-\sigma)}\|\zeta_{k+3}u\|_{L_p(\mathbb{R}^d_T)} + \varepsilon^3\|(-\Delta)^{\sigma/2}(\zeta_{k+3}u)\|_{L_p(\mathbb{R}^d_T)} \\
&\leq N\frac{2^{\sigma k}}{(R-r)^{\sigma}}\varepsilon^{3/(1-\sigma)}\|u\|_{p,R} + \varepsilon^3\|(-\Delta)^{\sigma/2}(\zeta_{k+3}u)\|_{L_p(\mathbb{R}^d_T)}.
\end{aligned} \tag{3.2.19}$$

Therefore, (3.2.18) and (3.2.19) lead to

$$
\begin{aligned}
&\|\partial_t^\alpha(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2}(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)} + \lambda\|\zeta_k u\|_{L_p(\mathbb{R}^d_T)}\\
&\leq N\|f\|_{p,R} + N\frac{2^{\sigma k}}{(R-r)^\sigma}\varepsilon^{3/(1-\sigma)}\|u\|_{p,R} + \varepsilon^3\|(-\Delta)^{\sigma/2}(\zeta_{k+3}u)\|_{L_p(\mathbb{R}^d_T)}\\
&\quad + N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}}R^{d/p}(1+R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\end{aligned}
\tag{3.2.20}
$$

Multiplying ε^k to both sides of (3.2.20) and taking the sum over $k = 0, 1, \ldots$ lead to

$$
\begin{aligned}
&\||\partial_t^\alpha(\zeta_0 u)| + |\lambda(\zeta_0 u)|\|_{L_p(\mathbb{R}^d_T)}\sum_{k=0}^\infty \varepsilon^k + \sum_{k=0}^\infty \varepsilon^k\|(-\Delta)^{\sigma/2}(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)}\\
&\leq N\frac{\varepsilon^{3/(1-\sigma)}}{(R-r)^\sigma}\|u\|_{p,R}\sum_{k=0}^\infty(\varepsilon 2^\sigma)^k + \sum_{k=0}^\infty \varepsilon^{k+3}\|(-\Delta)^{\sigma/2}(\zeta_{k+3}u)\|_{L_p(\mathbb{R}^d_T)}\\
&\quad + N\|f\|_{p,R}\sum_{k=0}^\infty \varepsilon^k + N\frac{R^{d/p}(1+R^{d+\sigma})}{(R-r)^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}\sum_{k=0}^\infty(\varepsilon 2^{d+\sigma})^k.
\end{aligned}
\tag{3.2.21}
$$

Therefore, by first taking ε to be sufficiently small so that $\varepsilon 2^{d+\sigma} < 1$, and then absorbing the second term on the right-hand side of (3.2.21) to the left-hand side, we arrive at (3.2.12) with $(-\Delta)^{\sigma/2}$ in place of L^1 on the left-hand side. The estimates for general L^1 follow from (3.1.12). Similarly, it is easily seen that if we replace $\zeta_0 u$ with $\zeta_3 u$ on the left-hand side of (3.2.12), the inequality still holds. Thus, by (3.4.4), (3.1.12), and (3.2.19) with $k = 0$,

$$
\begin{aligned}
&\|L^1 u\|_{p,r} \leq \|L^1(\zeta_0 u)\|_{L_p(\mathbb{R}^d_T)} + \|L^1(\zeta_0 u) - \zeta_0 L^1 u\|_{L_p(\mathbb{R}^d_T)}\\
&\leq \|(-\Delta)^{\sigma/2}(\zeta_0 u)\|_{L_p(\mathbb{R}^d_T)} + N\frac{\|u\|_{p,R}}{(R-r)^\sigma} + \|(-\Delta)^{\sigma/2}(\zeta_3 u)\|_{L_p(\mathbb{R}^d_T)}\\
&\quad + N\frac{R^{d/p}(1+R^{d+\sigma})}{(R-r)^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}\\
&\leq N\|f\|_{p,R} + N\frac{\|u\|_{p,R}}{(R-r)^\sigma} + N\frac{R^{d/p}(1+R^{d+\sigma})}{(R-r)^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\end{aligned}
$$

Combining this with (3.2.12), we infer (3.2.13) when $\sigma \in (1, 2)$.

Case 3: $\sigma = 1$. In this case, for any $\varepsilon \in (0,1)$ and $k \geq 1$, by (3.2.16), (3.2.17), and (3.4.5), we have

$$\||\partial_t^\alpha u| + |\lambda u|\|_{p,r_1} + \|Du\|_{p,r_k} \leq N\|f\|_{p,R} + \varepsilon^3\|Du\|_{p,r_{k+3}} + N\frac{2^k}{(R-r)}\varepsilon^{-3}\|u\|_{p,R}$$
$$+ NR^{d/p}\left(1 + \frac{2^{(d+1)k}\varepsilon^{-3(d+1)}}{(R-r)^{d+1}}(1+R^{d+1})\right)\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}. \tag{3.2.22}$$

Multiplying ε^{k-1} to both sides of (3.2.22) and taking the sum over $k = 1, \ldots$ lead to

$$\||\partial_t^\alpha u| + |\lambda u|\|_{p,r_1}\sum_{k=1}^\infty \varepsilon^{k-1} + \sum_{k=1}^\infty \varepsilon^{k-1}\|Du\|_{p,r_k}$$
$$\leq N\|f\|_{p,R}\sum_{k=1}^\infty \varepsilon^{k-1} + \sum_{k=1}^\infty \varepsilon^{k+2}\|Du\|_{p,r_{k+3}} + N\frac{\varepsilon^{-4}\|u\|_{p,R}}{(R-r)}\sum_{k=1}^\infty(\varepsilon 2)^k$$
$$+ NR^{d/p}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}\sum_{k=1}^\infty \varepsilon^{k-1}\left(1 + \frac{2^{(d+1)k}\varepsilon^{-3(d+1)}}{(R-r)^{d+1}}(1+R^{d+1})\right). \tag{3.2.23}$$

Therefore, by first taking ε to be sufficiently small so that $\varepsilon 2^{d+1} < 1$, and then absorbing the second term on the right-hand side of (3.2.23) to the left-hand side, we obtain

$$\|\partial_t^\alpha u\|_{p,r_1} + \|Du\|_{p,r_1} + \lambda\|u\|_{p,r_1}$$
$$\leq N\|f\|_{p,R} + N\frac{\|u\|_{p,R}}{R-r} + N\frac{R^{d/p}(1+R^{d+1})}{(R-r)^{d+1}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.$$

Also, by (3.1.12) and

$$\|\partial_t^\alpha(\zeta_0 u)\|_{L_p(\mathbb{R}^d_T)} + \|D(\zeta_0 u)\|_{L_p(\mathbb{R}^d_T)} + \lambda\|\zeta_0 u\|_{L_p(\mathbb{R}^d_T)}$$
$$\leq \|\partial_t^\alpha u\|_{p,r_1} + \|Du\|_{p,r_1} + N\frac{\|u\|_{p,r_1}}{R-r},$$

we obtain (3.2.12) when $\sigma = 1$. As before, replacing $\zeta_0 u$ with $\zeta_3 u$ on the left-hand side of (3.2.12) together with (3.2.16) and (3.4.5), we obtain (3.2.13). The lemma is proved. $\square$

Remark 3.2.3. In Lemma 3.2.2, we assume that $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$. However, in the proof we only used the fact that $\zeta_k u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$. Thus, it suffices to assume that $u|_{B_R} \in \mathbb{H}_{p,0}^{\alpha,\sigma}((0,T)\times B_R)$ for u defined on $(0,T)\times\mathbb{R}^d$ satisfying (3.2.11) in $(0,T)\times B_R$.

Lemma 3.2.2 and the embeddings in Lemmas 3.4.6 and 3.4.7 lead to the following corollary.

Corollary 3.2.4. *Let $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $p \in (1,\infty)$, $0 < r < R < \infty$, $\lambda \geq 0$, $u \in L_p((0,T);L_1(\mathbb{R}^d,\psi))$ be such that $u|_{(0,T)\times B_R} \in \mathbb{H}_{p,0}^{\alpha,\sigma}((0,T)\times B_R)$, and $q \in (p,\infty)$ satisfy*

$$1/q = 1/p - \alpha\sigma/(\alpha d + \sigma).$$

Let $f = \partial_t^\alpha u - Lu + \lambda u$ in $(0,T)\times B_R$.

1. *If $p \leq d/\sigma + 1/\alpha$, then for any $l \in [p,q]$,*

$$\begin{aligned}\|u\|_{L_l((0,T)\times B_r)} &\leq N\|f\|_{L_p((0,T)\times B_R)} + N\frac{\|u\|_{L_p((0,T)\times B_R)}}{(R-r)^\sigma}\\ &\quad + N\frac{R^{d/p}(1+R^{d+\sigma})}{(R-r)^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))},\end{aligned}$$

   where $N = N(d,\alpha,\sigma,p,l,T)$.

2. *If $p > d/\sigma + 1/\alpha$, then there exists $\tau = \sigma - (d+\sigma/\alpha)/p \in (0,1)$ such that*

$$\begin{aligned}\|u\|_{C^{\tau\alpha/\sigma,\tau}((0,T)\times B_r)} &\leq N\|f\|_{L_p((0,T)\times B_R)} + N\frac{\|u\|_{L_p((0,T)\times B_R)}}{(R-r)^\sigma}\\ &\quad + N\frac{R^{d/p}(1+R^{d+\sigma})}{(R-r)^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))},\end{aligned}$$

where $N = N(d, \alpha, \sigma, p, T)$.

Proof. (1) If $p \le d/\sigma + 1/\alpha$, recall the definition of ζ_0 in Lemma 3.2.2. By the Sobolev embeddings in Lemma 3.4.6 and Hölder's inequality, we have

$$
\begin{aligned}
&\|u\|_{L_l((0,T)\times B_r)} \le N\|\zeta_0 u\|_{L_l(\mathbb{R}^d_T)} \le N\|\zeta_0 u\|_{\mathbb{H}^{\alpha,\sigma}_p(T)} \\
&\le N\big(\|f\|_{L_p((0,T)\times B_R)} + \frac{\|u\|_{L_p((0,T)\times B_R)}}{(R-r)^\sigma} + \frac{R^{d/p}(1+R^{d+\sigma})}{(R-r)^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}\big),
\end{aligned}
$$

where for the last inequality, we used (3.2.12) and Remark 3.2.3.

(2) If $p > d/\sigma + 1/\alpha$, the proof is similar to that of (1) by using Lemma 3.4.7. □

By Lemma 3.2.2 with a scaling in the spatial coordinates, we derive an estimate that will be used later in Section 3.3.

Corollary 3.2.5. *Let* $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $p \in (1,\infty)$, *and* $R \in (0,\infty)$. *Suppose that Theorem 3.1.6 holds for this* p. *If* $u \in \mathbb{H}^{\alpha,\sigma}_{p,0}(T)$, *and* L^1 *satisfies* (3.1.3) *when* $\sigma = 1$ *and* (3.1.6), *and*

$$
\partial_t^\alpha u - Lu + \lambda u = f \quad \text{in } \mathbb{R}^d_T,
$$

then

$$
\begin{aligned}
&\big(|\partial_t^\alpha u|^p\big)^{1/p}_{(0,T)\times B_{R/2}(x_0)} + \big(|L^1 u|^p\big)^{1/p}_{(0,T)\times B_{R/2}(x_0)} + \lambda\big(|u|^p\big)^{1/p}_{(0,T)\times B_{R/2}(x_0)} \\
&\le N\big(|f|^p\big)^{1/p}_{(0,T)\times B_R(x_0)} + NR^{-\sigma}\sum_{k=0}^\infty 2^{-k\sigma}\big(|u|^p\big)^{1/p}_{(0,T)\times B_{2^k R}(x_0)},
\end{aligned}
\tag{3.2.24}
$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p)$.

Proof. By shifting the coordinates, without loss of generality, we assume that $x_0 = 0$.

When $R = 1$, denoting $p' = p/(p-1)$, by the Minkowski inequality and Hölder's inequality, we have

$$\begin{aligned}
&\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}\\
&= \Big(\int_0^T \big(\int_{\mathbb{R}^d} |u(t,x)| \frac{1}{1+|x|^{d+\sigma}}\, dx \big)^p \, dt \Big)^{1/p}\\
&\le \int_{\mathbb{R}^d} \big(\int_0^T |u(t,x)|^p \, dt \big)^{1/p} \frac{1}{1+|x|^{d+\sigma}}\, dx\\
&\le \sum_{k=0}^{\infty} \int_{\widehat{B}_{2^k} \setminus \widehat{B}_{2^{k-1}}} \big(\int_0^T |u(t,x)|^p \, dt \big)^{1/p} \frac{1}{1+|x|^{d+\sigma}}\, dx\\
&\le \sum_{k=0}^{\infty} \Big(\int_{\widehat{B}_{2^k} \setminus \widehat{B}_{2^{k-1}}} \int_0^T \frac{|u(t,x)|^p}{1+|x|^{d+\sigma}} \, dt\, dx \Big)^{1/p} \Big(\int_{\widehat{B}_{2^k} \setminus \widehat{B}_{2^{k-1}}} \frac{1}{1+|x|^{d+\sigma}}\, dx \Big)^{1/p'}\\
&\le T^{1/p} \sum_{k=0}^{\infty} \Big(2^{-k\sigma} \fint_0^T \fint_{B_{2^k}} |u(t,x)|^p \, dx\, dt \Big)^{1/p} 2^{-k\sigma/p'}\\
&\le N T^{1/p} \sum_{k=0}^{\infty} 2^{-k\sigma} \big(|u|^p \big)^{1/p}_{(0,T)\times B_{2^k}},
\end{aligned} \tag{3.2.25}$$

where $\widehat{B}_{2^k} = B_{2^k}$ for $k \ge 0$ and $\widehat{B}_{2^{-1}} = \emptyset$. Thus, when $R = 1$, (3.2.24) follows from (3.2.13) and (3.2.25).

For general $R > 0$, we take

$$\tilde{u}(t,x) = R^{-\sigma} u(R^{\sigma/\alpha} t, Rx) \quad \text{and } \tilde{f}(t,x) = f(R^{\sigma/\alpha} t, Rx). \tag{3.2.26}$$

It follows that

$$\partial_t^\alpha \tilde{u} - \tilde{L}\tilde{u} + \lambda \tilde{u} = \tilde{f} \quad \text{in } (0,\tilde{T}) \times \mathbb{R}^d,$$

where $\tilde{T} = R^{-\sigma/\alpha} T$, and $\tilde{L}$ is the operator with the kernel $R^{d+\sigma} K(R^{\sigma/\alpha} t, Ry)$ satisfying (3.1.2) with the same ν and Λ as K, the kernel of L. Since

$$
\begin{aligned}
&\left(|\partial_t^\alpha \tilde{u}|^p\right)_{(0,\tilde{T})\times B_{1/2}}^{1/p} + \left(|L^1 \tilde{u}|^p\right)_{(0,\tilde{T})\times B_{1/2}}^{1/p} + \lambda\left(|\tilde{u}|^p\right)_{(0,\tilde{T})\times B_{1/2}}^{1/p} \\
&\leq N\left(|\tilde{f}|^p\right)_{(0,\tilde{T})\times B_1}^{1/p} + N\sum_{k=0}^{\infty} 2^{-k\sigma}\left(|\tilde{u}|^p\right)_{(0,\tilde{T})\times B_{2^k}}^{1/p},
\end{aligned}
$$

we arrive at (3.2.24) by a change of variables. □

3.2.2 The level set argument

In this subsection, we prove Theorem 3.1.6 for general $p \in (1,\infty)$. Recall the equation

$$
\partial_t^\alpha u - Lu + \lambda u = f. \tag{3.2.27}
$$

We start with a decomposition of the solution.

Proposition 3.2.6. *Let $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $\lambda \geq 0$ and $p \in (1,\infty)$. Suppose that Theorem 3.1.6 holds for this p, and $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$ satisfies Equation (3.2.27). Then there exists $p_1 = p_1(d,\alpha,\sigma,p) \in (p,\infty]$ satisfying*

$$
p_1 - p > \delta(d,\alpha,\sigma) > 0, \tag{3.2.28}
$$

and the following holds. For any $(t_0,x_0) \in [0,T]\times\mathbb{R}^d$, $R>0$, and $S = \min\{0, t_0 - R^{\sigma/\alpha}\}$, there exist

$$
w \in \mathbb{H}_{p,0}^{\alpha,\sigma}((t_0 - R^{\sigma/\alpha}, t_0)\times\mathbb{R}^d) \quad \text{and } v \in \mathbb{H}_{p,0}^{\alpha,\sigma}((S,t_0)\times\mathbb{R}^d),
$$

such that $u = w + v$ *in* $Q_R(t_0, x_0)$,

$$(|L^1 w|^p)^{1/p}_{Q_R(t_0,x_0)} + (|\lambda w|^p)^{1/p}_{Q_R(t_0,x_0)} \le N(|f|^p)^{1/p}_{Q_{2R}(t_0,x_0)}, \tag{3.2.29}$$

$$\begin{aligned}\big(|L^1 v|^{p_1}\big)^{1/p_1}_{Q_{R/2}(t_0,x_0)} \le{}& N\big(|f|^p\big)^{1/p}_{Q_{2R}(t_0,x_0)} + N\sum_{k=0}^{\infty} 2^{-k\sigma}\big(|f|^p\big)^{1/p}_{(t_0-R^{\sigma/\alpha},t_0)\times B_{2^kR}(x_0)}\\ &+ N\sum_{k=0}^{\infty} 2^{-k\sigma}\big(|L^1 u|^p\big)^{1/p}_{(t_0-R^{\sigma/\alpha},t_0)\times B_{2^kR}(x_0)}\\ &+ N\sum_{k=0}^{\infty} 2^{-k\alpha}\big(|L^1 u|^p\big)^{1/p}_{(t_0-(2^{k+1}+1)R^{\sigma/\alpha},t_0)\times B_R(x_0)},\end{aligned} \tag{3.2.30}$$

and

$$\begin{aligned}\big(|\lambda v|^{p_1}\big)^{1/p_1}_{Q_{R/2}(t_0,x_0)} \le{}& N\big(|f|^p\big)^{1/p}_{Q_{2R}(t_0,x_0)} + N\sum_{k=0}^{\infty} 2^{-k\sigma}\big(|f|^p\big)^{1/p}_{(t_0-R^{\sigma/\alpha},t_0)\times B_{2^kR}(x_0)}\\ &+ N\sum_{k=0}^{\infty} 2^{-k\sigma}\big(|\lambda u|^p\big)^{1/p}_{(t_0-R^{\sigma/\alpha},t_0)\times B_{2^kR}(x_0)}\\ &+ N\sum_{k=0}^{\infty} 2^{-k\alpha}\big(|\lambda u|^p\big)^{1/p}_{(t_0-(2^{k+1}+1)R^{\sigma/\alpha},t_0)\times B_R(x_0)},\end{aligned} \tag{3.2.31}$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p)$,

$$\big(|\cdot|^{p_1}\big)^{1/p_1}_{Q_{R/2}(t_0,x_0)} := \|\cdot\|_{L_\infty(Q_{R/2}(t_0,x_0))} \quad \textit{when } p_1 = \infty,$$

and u, f *are extended to be zero for* $t < 0$.

Proof. The proof of (3.2.31) would be similar to the proof of (3.2.30) by replacing $L^1 v$ with λv. Thus, we focus on (3.2.30).

By a shift of the coordinates and a scaling similar to (3.2.26), without loss of generality, we assume that $x_0 = 0$ and $R = 1$.

For any $t_0 \in [0, T]$, we take a cutoff function $\zeta \in C_0^\infty((t_0 - 2^{\sigma/\alpha}, t_0 + 2^{\sigma/\alpha}) \times B_2)$ satisfying

$$\zeta \in [0, 1] \quad \text{and} \quad \zeta = 1 \quad \text{in } (t_0 - 1, t_0) \times B_1.$$

By Theorem 3.1.6, there exists $w \in \mathbb{H}_{p,0}^{\alpha,\sigma}((t_0 - 1, t_0) \times \mathbb{R}^d)$ satisfying

$$\partial_t^\alpha w - Lw + \lambda w = \zeta f \quad \text{in } (t_0 - 1, t_0) \times \mathbb{R}^d,$$

and

$$\||L^1 w| + |\lambda w|\|_{L_p((t_0-1,t_0)\times\mathbb{R}^d)} \le N\|\zeta f\|_{L_p((t_0-1,t_0)\times\mathbb{R}^d)} \le N\|f\|_{L_p(Q_2(t_0,0))}, \tag{3.2.32}$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p)$. We then obtain the estimate (3.2.29).

Next, we take $S = \min(0, t_0 - 1)$ and $v = u - w$. Indeed, by taking the zero extension of w for $t < t_0 - 1$, we have $w \in \mathbb{H}_{p,0}^{\alpha,\sigma}((S, t_0) \times \mathbb{R}^d)$. See (Dong and Kim, 2019, Lemma 3.5) for details. Thus, it follows that $v \in \mathbb{H}_{p,0}^{\alpha,\sigma}((S, t_0) \times \mathbb{R}^d)$ and

$$\partial_t^\alpha v - Lv + \lambda v = (1 - \zeta) f \quad \text{in } (S, t_0) \times \mathbb{R}^d.$$

Furthermore, we take $\eta \in C^\infty(\mathbb{R})$ such that

$$\eta(t) = \begin{cases} 1 & \text{when } t \in (t_0 - (1/2)^{\sigma/\alpha}, t_0), \\ 0 & \text{when } t \in (t_0 - 1, t_0 + 1)^c, \end{cases} \quad \text{and} \quad |\eta'| \le N(\alpha, \sigma). \tag{3.2.33}$$

It follows from (Dong and Kim, 2019, Lemma 3.6) that $h := \eta v \in \mathbb{H}_{p,0}^{\alpha,\sigma}((t_0 - 1, t_0) \times$

$\mathbb{R}^d$) and

$$\partial_t^\alpha h - Lh + \lambda h = \begin{cases} g + \eta(1-\zeta)f \quad \text{in } (t_0-1,t_0)\times\mathbb{R}^d & \text{when } \alpha\in(0,1), \\ \eta' v + \eta(1-\zeta)f \quad \text{in } (t_0-1,t_0)\times\mathbb{R}^d & \text{when } \alpha = 1, \end{cases} \tag{3.2.34}$$

where

$$g(t,x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_S^t (t-s)^{-\alpha-1}(\eta(t)-\eta(s))v(s,x)\,ds.$$

Moreover, we take

$$\xi\in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}(\xi)\subset B_1, \quad \int_{\mathbb{R}^d}\xi = 1, \quad \text{and } \xi^\varepsilon(\cdot) := \varepsilon^{-d}\xi(\cdot/\varepsilon).$$

To estimate $L^1 v$, we first mollify Equation (3.2.34) and then take L^1 on both sides to get

$$\partial_t^\alpha(L^1h^\varepsilon) - L(L^1h^\varepsilon) + \lambda L^1h^\varepsilon = \tilde{g}^\varepsilon + L^1(\eta[(1-\zeta)f]^\varepsilon) \quad \text{in } (t_0-1,t_0)\times\mathbb{R}^d, \tag{3.2.35}$$

where $v^\varepsilon := v *_x \xi^\varepsilon$ and

$$\tilde{g}^\varepsilon(t,x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_S^t (t-s)^{-\alpha-1}(\eta(t)-\eta(s))L^1v^\varepsilon(s,x)\,ds,$$

when $\alpha\in(0,1)$ and $\tilde{g}(t,x) = \eta' L v^\varepsilon$ when $\alpha = 1$.

Next, we take p_1 such that

$$1/p_1 = 1/p - \alpha\sigma/(2\alpha d + 2\sigma) \quad \text{if } \ p \le d/\sigma + 1/\alpha,$$

and

$$p_1 = \infty \quad \text{if } \ p > d/\sigma + 1/\alpha.$$

Note that p_1 satisfies (3.2.28), i.e., the increment is independent of p. By the fact that $L^1 h^\varepsilon \in \mathbb{H}^{\alpha;\sigma}_{p,0}((t_0-1,t_0)\times\mathbb{R}^d)$ and Corollary 3.2.4, we have

$$\begin{aligned}\|L^1 h^\varepsilon\|_{L_{p_1}(Q_{1/2}(t_0,0))} &\le \|L^1 h^\varepsilon\|_{L_{p_1}((t_0-1,t_0)\times B_{1/2})}\\ &\le N\||L^1 h^\varepsilon| + |\tilde{g}^\varepsilon| + |L^1(\eta[(1-\zeta)f]^\varepsilon)|\|_{L_p((t_0-1,t_0)\times B_{3/4})}\\ &\quad + N\|L^1 h^\varepsilon\|_{L_p((t_0-1,t_0);L_1(\mathbb{R}^d,\psi))},\end{aligned}$$

which, by taking the limit of $\varepsilon \to 0$, implies that

$$\begin{aligned}\|L^1 h\|_{L_{p_1}(Q_{1/2}(t_0,0))} &\le N\||L^1 h| + |\tilde{g}| + |L^1(\eta[(1-\zeta)f])|\|_{L_p((t_0-1,t_0)\times B_{3/4})}\\ &\quad + N\|L^1 h\|_{L_p((t_0-1,t_0);L_1(\mathbb{R}^d,\psi))},\end{aligned} \tag{3.2.36}$$

where

$$\tilde{g}(t,x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_S^t (t-s)^{-\alpha-1}(\eta(t)-\eta(s))L^1 v(s,x)\,ds$$

when $\alpha \in (0,1)$ and $\tilde{g}(t,x) = \eta' L v$ when $\alpha = 1$. Then, by the definition of h and (3.2.36), we obtain

$$\begin{aligned}&\|L^1 v\|_{L_{p_1}(Q_{1/2}(t_0,0))} = \|L^1 h\|_{L_{p_1}(Q_{1/2}(t_0,0))}\\ &\le N\||L^1 v| + |\tilde{g}|\|_{L_p((t_0-1,t_0)\times B_1)} + N\|L^1((1-\zeta)f)\|_{L_p((t_0-1,t_0)\times B_{3/4})}\\ &\quad + N\|L^1 v\|_{L_p((t_0-1,t_0);L_1(\mathbb{R}^d,\psi))},\end{aligned} \tag{3.2.37}$$

where $N = N(d,\nu,\Lambda,\alpha,\sigma,p,p_1)$. We estimate the terms on the right-hand side of (3.2.37) separately as follows.

When $\alpha \in (0,1)$, a similar computation as in (Dong and Kim, 2019, Proposition

5.1) leads to

$$
\begin{aligned}
\|\tilde{g}\|_{L_p(Q_1(t_0,0))} &\le N\sum_{k=0}^{\infty} 2^{-k\alpha}\big(|L^1 u|^p\big)^{1/p}_{(t_0-(2^{k+1}+1)^{\sigma/\alpha},t_0)\times B_1}\\
&\quad + N\sum_{k=0}^{\infty} 2^{-k\alpha}\big(|L^1 w|^p\big)^{1/p}_{(t_0-(2^{k+1}+1)^{\sigma/\alpha},t_0)\times B_1}\\
&\le N\sum_{k=0}^{\infty} 2^{-k\alpha}\big(|L^1 u|^p\big)^{1/p}_{(t_0-(2^{k+1}+1)^{\sigma/\alpha},t_0)\times B_1} + N(|L^1 w|^p)^{1/p}_{Q_1(t_0,0)}\\
&\le N\sum_{k=0}^{\infty} 2^{-k\alpha}\big(|L^1 u|^p\big)^{1/p}_{(t_0-(2^{k+1}+1)^{\sigma/\alpha},t_0)\times B_1} + N(|f|^p)^{1/p}_{Q_2(t_0,0)}, \quad (3.2.38)
\end{aligned}
$$

where in the second inequality we used the fact that w vanishes when $t < t_0 - 1$, and in the last inequality, we used (3.2.32). Recall that when $\alpha = 1$, $\tilde{g} = \eta' L^1 v$. Then it is easily seen that

$$
\|\tilde{g}\|_{L_p(Q_1(t_0,0))} \le N\big(|L^1 u|^p\big)^{1/p}_{(t_0-3^\sigma,t_0)\times B_1} + N(|f|^p)^{1/p}_{Q_2(t_0,0)}.
$$

Also, by the fact that $1-\zeta$ vanishes in $Q_1(t_0,0)$ together with the Minkowski inequality and Hölder's inequality, we have

$$
\begin{aligned}
&\|L^1((1-\zeta)f)\|_{L_p((t_0-1,t_0)\times B_{3/4}))}\\
&\le N\|\int_{\mathbb{R}^d} |(1-\zeta(\cdot,\cdot+y))f(\cdot,\cdot+y)||y|^{-d-\sigma}\,dy\|_{L_p((t_0-1,t_0)\times B_{3/4}))}\\
&\le N\|\int_{|y|>1/4} |(1-\zeta(\cdot,\cdot+y))f(\cdot,\cdot+y)||y|^{-d-\sigma}\,dy\|_{L_p((t_0-1,t_0)\times B_{3/4}))}\\
&\le N\|\sum_{k=-1}^{\infty} 2^{-kd-k\sigma}\int_{B_{2^k}\setminus B_{2^{k-1}}} |f(\cdot,\cdot+y)|\,dy\|_{L_p((t_0-1,t_0)\times B_{3/4}))}\\
&\le N\sum_{k=0}^{\infty} 2^{-k\sigma}\left(\fint_{t_0-1}^{t_0}\fint_{B_{2^k}} |f(t,x)|^p\,dx\,dt\right)^{1/p}. \quad (3.2.39)
\end{aligned}
$$

Finally, using an estimate similar to (3.2.25) together with the fact that $v = u-w$

and (3.2.32), we obtain

$$
\begin{aligned}
&\|L^1 v\|_{L_p((t_0-1,t_0);L_1(\mathbb{R}^d,\psi))} \\
&\le N \sum_{k=0}^{\infty} 2^{-k\sigma} \left(|L^1 v|^p\right)^{1/p}_{(t_0-1,t_0)\times B_{2^k}} \\
&\le N \sum_{k=0}^{\infty} 2^{-k\sigma} \left(\left(|L^1 u|^p\right)^{1/p}_{(t_0-1,t_0)\times B_{2^k}} + \left(|L^1 w|^p\right)^{1/p}_{(t_0-1,t_0)\times B_{2^k}}\right) \\
&\le N \sum_{k=0}^{\infty} 2^{-k\sigma} \left(|L^1 u|^p\right)^{1/p}_{(t_0-1,t_0)\times B_{2^k}} + N(|f|^p)^{1/p}_{Q_2(t_0,0)}.
\end{aligned}
\tag{3.2.40}
$$

Thus, combining (3.2.37), (3.2.38), (3.2.39), and (3.2.40), we arrive at (3.2.30) when $R = 1$. The proof of (3.2.31) is similar (and actually simpler) by using

$$
\|h\|_{L_{p_1}(Q_{1/2}(t_0,0))} \le N\||h| + |g|\|_{L_p((t_0-1,t_0)\times B_{3/4})} + N\|h\|_{L_p((t_0-1,t_0);L_1(\mathbb{R}^d,\psi))}
$$

instead of (3.2.36). We omit the details. The proposition is proved. □

Next, with the estimates above, we verify Assumption 3.4.4 of Lemma 3.4.5, which is the key to the level set argument. For $R > 0$, $p \in (1,\infty)$, and $p_1 = p_1(d,\alpha,\sigma,p)$ from Proposition 3.2.6, denote

$$
\mathcal{A}(s) = \left\{(t,x) \in (-\infty,T) \times \mathbb{R}^d : |L^1 u(t,x)| > s\right\},
$$

and

$$
\begin{aligned}
\mathcal{B}_\gamma(s) = \big\{&(t,x) \in (-\infty,T) \times \mathbb{R}^d : \\
&\gamma^{-1/p}(\mathcal{M}|f|^p(t,x))^{1/p} + \gamma^{-1/p_1}(\mathcal{S}\mathcal{M}|L^1 u|^p(t,x))^{1/p} > s\big\}.
\end{aligned}
\tag{3.2.41}
$$

Also, we write

$$\mathcal{C}_R(t,x) = (t - R^{\sigma/\alpha}, t + R^{\sigma/\alpha}) \times B_R(x) \quad \text{and } \widehat{\mathcal{C}}_R(t,x) = \mathcal{C}_R(t,x) \cap \{t \le T\}. \tag{3.2.42}$$

Lemma 3.2.7. *Let $\gamma \in (0,1)$, $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $R > 0$, $\lambda \ge 0$ and $p \in (1,\infty)$. Suppose that Theorem 3.1.6 holds for this p, and $u \in \mathbb{H}^{\alpha,\sigma}_{p,0}(T)$ satisfies Equation (3.2.27). Then, there exists a sufficiently large constant $\kappa = \kappa(d,\nu,\Lambda,\alpha,\sigma,p) > 1$ such that for any $(t_0,x_0) \in (-\infty,T] \times \mathbb{R}^d$ and $s > 0$, if*

$$|\mathcal{C}_{\tilde{R}}(t_0,x_0) \cap \mathcal{A}(\kappa s)| \ge \gamma |\mathcal{C}_{\tilde{R}}(t_0,x_0)|, \tag{3.2.43}$$

then we have

$$\widehat{\mathcal{C}}_{\tilde{R}}(t_0,x_0) \subset \mathcal{B}_\gamma(s),$$

where $\tilde{R} = 2^{-1-\alpha/\sigma}R$.

Proof. By dividing Equation (3.2.27) by s, we assume that $s = 1$.

First note that since u vanishes when $t < 0$, if $t_0 + \tilde{R}^{\sigma/\alpha} < 0$, then

$$\mathcal{C}_{\tilde{R}}(t_0,x_0) \cap \mathcal{A}(\kappa) \subset \left\{(t,x) \in (-\infty,0) \times \mathbb{R}^d : |L^1 u(t,x)| > 1\right\} = \emptyset,$$

and (3.2.43) is not satisfied. Thus, it suffices to consider the case when $t_0 + \tilde{R}^{\sigma/\alpha} \ge 0$.

We argue by contradiction. Suppose that there exist some $(s,y) \in \widehat{\mathcal{C}}_{\tilde{R}}(t_0,x_0)$ such that

$$\gamma^{-1/p}(\mathcal{M}|f|^p(s,y))^{1/p} + \gamma^{-1/p_1}(\mathcal{S}\mathcal{M}|L^1 u|^p(s,y))^{1/p} \le 1. \tag{3.2.44}$$

Let $t_1 := \min\{t_0 + \tilde{R}^{\sigma/\alpha}, T\}$. Then by Proposition 3.2.6, for $S = \min\{0, t_1 - R^{\sigma/\alpha}\}$, there exist $w \in \mathbb{H}^{\alpha,\sigma}_{p,0}((t_1 - R^{\sigma/\alpha}, t_1) \times \mathbb{R}^d)$ and $v \in \mathbb{H}^{\alpha,\sigma}_{p,0}((S, t_1) \times \mathbb{R}^d)$ such that $u = w + v$ in $Q_R(t_1, x_0)$,

$$(|L^1 w|^p)^{1/p}_{Q_R(t_1,x_0)} \le N(|f|^p)^{1/p}_{Q_{2R}(t_1,x_0)}, \tag{3.2.45}$$

and

$$\begin{aligned}\big(|L^1 v|^{p_1}\big)^{1/p_1}_{Q_{R/2}(t_1,x_0)} &\le N\big(|f|^p\big)^{1/p}_{Q_{2R}(t_1,x_0)} \\ &+ N\sum_{k=0}^{\infty} 2^{-k\sigma}\bigg(\fint_{t_1-R^{\sigma/\alpha}}^{t_1} \fint_{B_{2^kR}(x_0)} |f(t,x)|^p \, dx \, dt\bigg)^{1/p} \\ &+ N\sum_{k=0}^{\infty} 2^{-k\alpha}\bigg(\fint_{t_1-(2^{k+1}+1)R^{\sigma/\alpha}}^{t_1} \fint_{B_R(x_0)} |L^1 u(t,y)|^p \, dy \, dt\bigg)^{1/p} \\ &+ N\sum_{k=0}^{\infty} 2^{-k\sigma}\bigg(\fint_{t_1-R^{\sigma/\alpha}}^{t_1} \fint_{B_{2^kR}(x_0)} |L^1 u(t,y)|^p \, dx \, dt\bigg)^{1/p}, \end{aligned} \tag{3.2.46}$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p)$. We have

$$(s, y) \in \widehat{\mathcal{C}}_{\tilde{R}}(t_0, x_0) \subset Q_{R/2}(t_1, x_0) \subset Q_{2R}(t_1, x_0),$$

$$(s, y) \in \widehat{\mathcal{C}}_{\tilde{R}}(t_0, x_0) \subset (t_1 - (2^{k+1} + 1)R^{\sigma/\alpha}, t_1) \times B_R(x_0),$$

and

$$(s, y) \in \widehat{\mathcal{C}}_{\tilde{R}}(t_0, x_0) \subset (t_1 - R^{\sigma/\alpha}, t_1) \times B_{2^kR}(x_0)$$

for all $k = 0, 1, \ldots$. By these set inclusions together with (3.2.44), (3.2.45), and (3.2.46), we infer

$$(|L^1 v|^{p_1})^{1/p_1}_{Q_{R/2}(t_1,x_0)} \le N\gamma^{1/p_1} \quad \text{and } (|L^1 w|^p)^{1/p}_{Q_R(t_1,x_0)} \le N\gamma^{1/p},$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p)$. Then for a constant $C_1 > 0$ to be determined, by the

Chebyshev inequality,

$$
\begin{aligned}
&|\mathcal{C}_{\tilde{R}}(t_0,x_0)\cap\mathcal{A}(\kappa)| \\
&= |\{(t,x)\in\mathcal{C}_{\tilde{R}}(t_0,x_0), t\in(-\infty,T) : |L^1u(t,x)|>\kappa\}| \\
&\le \big|\{(t,x)\in Q_{R/2}(t_1,x_0) : |L^1u(t,x)|>\kappa\}\big| \\
&\le \big|\{(t,x)\in Q_{R/2}(t_1,x_0) : |L^1w(t,x)|>\kappa-C_1\}\big| \\
&\quad + \big|\{(t,x)\in Q_{R/2}(t_1,x_0) : |L^1v(t,x)|>C_1\}\big| \\
&\le \int_{Q_{R/2}(t_1,x_0)} (\kappa-C_1)^{-p}|L^1w|^p + C_1^{-p_1}|L^1v|^{p_1}\,dx\,dt \\
&\le \frac{N^p\gamma|Q_{R/2}|}{(\kappa-C_1)^p} + \frac{N^{p_1}\gamma|Q_{R/2}|}{C_1^{p_1}}1_{p_1\neq\infty} \\
&\le N_0(d,\alpha,\sigma)|\mathcal{C}_{\tilde{R}}(t_0,x_0)|\gamma\bigg(\frac{N^p}{(\kappa-C_1)^p}+\Big(\frac{N}{C_1}\Big)^p 1_{p_1\neq\infty}\bigg).
\end{aligned}
$$

By first taking a sufficiently large $C_1 = C_1(d,\nu,\Lambda,\alpha,\sigma,p)$ such that

$$N_0(N/C_1)^p < 1/2,$$

and then taking a large $\kappa = \kappa(d,\nu,\Lambda,\alpha,\sigma,p)$ so that

$$N_0N^p/(\kappa-C_1)^p < 1/2,$$

we obtain

$$|\mathcal{C}_{\tilde{R}}(t_0,x_0)\cap\mathcal{A}(\kappa)| < \gamma|\mathcal{C}_{\tilde{R}}(t_0,x_0)|. \tag{3.2.47}$$

However, (3.2.47) contradicts (3.2.43). The lemma is proved. □

Remark 3.2.8. By replacing L^1u with λu in the above proof together with (3.2.31), we conclude that Lemma 3.2.7 holds for

$$\mathcal{A}'(s) = \left\{(t,x) \in (-\infty, T) \times \mathbb{R}^d : |\lambda u(t,x)| > s\right\},$$

and

$$\mathcal{B}'_\gamma(s) = \big\{(t,x) \in (-\infty, T) \times \mathbb{R}^d : \\ \gamma^{-1/p}(\mathcal{M}|f|^p(t,x))^{1/p} + \gamma^{-1/p_1}(\mathcal{S}\mathcal{M}|\lambda u|^p(t,x))^{1/p} > s\big\}.$$

We are ready to prove Theorem 3.1.6 for general $p \in (1, \infty)$.

Proof of Theorem 3.1.6. Since the existence of solutions for the equation

$$\partial_t^\alpha u + (-\Delta)^{\sigma/2} u = f$$

was derived in (Kim et al., 2021, Theorem 2.8), by the method of continuity, it suffices to prove the a priori estimate (3.1.8). Note that by the argument in Remark 3.1.7, if (3.1.8) holds for smooth functions, then (3.1.12) holds for smooth functions. Thus, by the density of smooth functions in $\mathbb{H}_{p,0}^{\alpha,\sigma}(T)$, without loss of generality, we assume that $u \in C_0^\infty([0,T] \times \mathbb{R}^d)$ with $u(0,\cdot) = 0$.

We first consider the case when $p \in [2, \infty)$ by using an iterative argument to successively increase the exponent p, which is referred to as the bootstrap argument. Recall that we have proved the base case when $p = 2$. Now we assume that the theorem holds for some $p_0 \in [2, \infty)$, and we prove (3.1.8) for $p \in (p_0, p_1)$, where

$p_1 = p_1(d, \alpha, \sigma, p_0)$ is from Proposition 3.2.6. Note that

$$\|L^1 u\|^p_{L_p(\mathbb{R}^d_T)} = p \int_0^\infty |\mathcal{A}(s)| s^{p-1}\, ds = p\kappa^p \int_0^\infty |\mathcal{A}(\kappa s)| s^{p-1}\, ds,$$

and Lemma 3.2.7 together with Lemma 3.4.5 leads to

$$|\mathcal{A}(\kappa s)| \le N(d, \alpha)\gamma |\mathcal{B}_\gamma(s)|$$

for all $s \in (0, \infty)$, where $\kappa = \kappa(d, \nu, \Lambda, \alpha, \sigma, p)$ is from Lemma 3.2.7 and $\mathcal{A}, \mathcal{B}_\gamma$ are defined in (3.2.41). Thus, by the Hardy-Littlewood theorem for strong maximal functions,

$$\begin{aligned}
&\|L^1 u\|^p_{L_p(\mathbb{R}^d_T)} \le N p \kappa^p \gamma \int_0^\infty |\mathcal{B}_\gamma(s)| s^{p-1}\, ds \\
&\le N\gamma \int_0^\infty \left| \left\{ (t,x) \in (-\infty, T) \times \mathbb{R}^d : \gamma^{-\frac{1}{p_1}} (\mathcal{S}\mathcal{M}|L^1 u|^{p_0}(t,x))^{\frac{1}{p_0}} > s/2 \right\} \right| s^{p-1}\, ds \\
&\quad + N\gamma \int_0^\infty \left| \left\{ (t,x) \in (-\infty, T) \times \mathbb{R}^d : \gamma^{-\frac{1}{p_0}} (\mathcal{M}|f|^{p_0}(t,x))^{\frac{1}{p_0}} > s/2 \right\} \right| s^{p-1}\, ds \\
&\le N\gamma^{1-p/p_1} \|L^1 u\|^p_{L_p(\mathbb{R}^d_T)} + N\gamma^{1-p/p_0} \|f\|^p_{L_p(\mathbb{R}^d_T)},
\end{aligned}$$

where $N = N(d, \nu, \Lambda, \alpha, \sigma, p)$. Similarly, by Remark (3.2.8), we have

$$|\mathcal{A}'(\kappa s)| \le N(d, \alpha)\gamma |\mathcal{B}'_\gamma(s)|,$$

which implies that

$$\|\lambda u\|^p_{L_p(\mathbb{R}^d_T)} \le N\gamma^{1-p/p_1} \|\lambda u\|^p_{L_p(\mathbb{R}^d_T)} + N\gamma^{1-p/p_0} \|f\|^p_{L_p(\mathbb{R}^d_T)}.$$

By the fact that $p < p_1$, we take a sufficiently small $\gamma \in (0,1)$ so that

$$N\gamma^{1-p/p_1} < 1/2,$$

which implies (3.1.8) for $p \in (p_0, p_1)$. We repeat this procedure. Recall that $p_1 - p$ depends only on d, α, σ. Thus in finite steps, we get a p_0 which is larger than $d/2 + 1/\alpha$, so that $p_1 = p_1(d, \alpha, p_0) = \infty$. Therefore, the theorem is proved for any $p \in [2, \infty)$.

For $p \in (1, 2)$, we use a duality argument. Again, we assume that $u \in C_0^\infty([0, T] \times \mathbb{R}^d)$ with $u(0, x) = 0$ and prove (3.1.8). Let L^* be the operator with the kernel $K(-t, -y)$, where K is the kernel of L. For $p' = p/(p-1)$ and $\phi \in L_{p'}(\mathbb{R}^d_T)$, there exist $w \in \mathbb{H}^{\alpha,\sigma}_{p',0}((-T, 0) \times \mathbb{R}^d)$ satisfying

$$\partial_t^\alpha w - L^* w + \lambda w = \phi(-t, x) \quad \text{in } (-T, 0) \times \mathbb{R}^d$$

and

$$\||L^1 w| + |\lambda w|\|_{L_{p'}((-T,0)\times\mathbb{R}^d)} \leq N\|\phi(-t, x)\|_{L_{p'}((-T,0)\times\mathbb{R}^d)} = N\|\phi\|_{L_{p'}(\mathbb{R}^d_T)},$$

where $\partial_t^\alpha w = \partial_t I_{-T}^{1-\alpha} w$ and $N = N = N(d, \nu, \Lambda, \alpha, \sigma, p)$. It follows that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \phi L^1 u \, dx \, dt = \int_{-T}^0 \int_{\mathbb{R}^d} \phi(-t, x) L^1 u(-t, x) \, dx \, dt \\
&= \int_{-T}^0 \int_{\mathbb{R}^d} (\partial_t^\alpha w(t, x) - L^* w(t, x)) L^1 u(-t, x) \, dx \, dt \\
&= \int_0^T \int_{\mathbb{R}^d} (\partial_t^\alpha u(t, x) - L u(t, x)) (L^1)^* w(-t, x) \, dx \, dt \\
&= \int_0^T \int_{\mathbb{R}^d} f(t, x) (L^1)^* w(-t, x) \, dx \, dt \leq N\|f\|_{L_p(\mathbb{R}^d_T)} \|\phi\|_{L_{p'}(\mathbb{R}^d_T)}.
\end{aligned} \tag{3.2.48}$$

Note that for the third equality, if w is smooth, then we apply the Plancherel theorem to the integral of $L^* w(t, x) L^1 u(-t, x)$ and apply an integration by parts to the integral of $\partial_t^\alpha w(t, x) L^1 u(-t, x)$ using the zero initial conditions of w and u. If w is not necessarily smooth, we take $w_k \in C_0^\infty([-T, 0] \times \mathbb{R}^d)$ with $w_k(-T, 0) = 0$ such that

$$w_k \to w \quad \text{in } \mathbb{H}^{\alpha,\sigma}_{p',0}((-T,0)\times\mathbb{R}^d).$$

Similarly, we have

$$\int_0^T \int_{\mathbb{R}^d} \phi\lambda u \, dx\, dt = \int_0^T \int_{\mathbb{R}^d} f(t,x)\lambda w(-t,x)\, dx\, dt \le N\|f\|_{L_p(\mathbb{R}^d_T)}\|\phi\|_{L_{p'}(\mathbb{R}^d_T)}. \tag{3.2.49}$$

Thus, by (3.2.48) and (3.2.49), we arrive at (3.1.8). As before, by the method of continuity, the solvability and thus theorem are proved for the remaining case when $p \in (1,2)$. $\square$

Proof of Corollary 3.1.8. With Theorem 3.1.6, the proof of Corollary 3.1.8 proceeds similar to that of Corollary 3.1.10, using the frozen coefficient argument and a partition of unity. We refer the reader to the proof of Corollary 3.1.8 at the end of Section 3.3. $\square$

3.3 Equations in $L_{p,q,w}$ when $\alpha = 1$

In this section, we prove Theorem 3.1.9 by deriving a mean oscillation estimate of $u \in \mathbb{H}^{1,\sigma}_{p_0}((-\infty,T)\times\mathbb{R}^d)$ satisfying

$$\partial_t u - Lu + \lambda u = f \quad \text{in } (-\infty,T)\times\mathbb{R}^d.$$

In particular, for $t_0 \in (-\infty, T]$, we are going to decompose $u = w + v$ in $(t_0-1, t_0) \times \mathbb{R}^d$ and estimate them separately, where $w \in \mathbb{H}^{1,\sigma}_{p_0,0}((t_0-1,t_0)\times\mathbb{R}^d)$ satisfies

$$\partial_t w - Lw + \lambda w = f \quad \text{in } (t_0-1, t_0) \times \mathbb{R}^d \tag{3.3.1}$$

and

$$\partial_t v - Lv + \lambda v = 0 \qquad \text{in } (t_0-1, t_0) \times \mathbb{R}^d. \tag{3.3.2}$$

Note that the decomposition above differs from the one in the previous section.

Throughout this section, We denote L^1 to be an operator satisfying (3.1.3) when $\sigma = 1$ and (3.1.6). Furthermore, for $r > 0$ and $z = (z^1, \ldots, z^d) \in \mathbb{R}^d$, the cube centered at z with radius r is defined as

$$C_r(z) := (z^1 - r/2, z^1 + r/2) \times (z^2 - r/2, z^2 + r/2) \times \ldots \times (z^d - r/2, z^d + r/2).$$

We start with the estimate of v. In the following proposition, using the Sobolev embedding and an iteration argument, we bound the Hölder semi-norm of $L^1 v$, in particular $(-\Delta)^{\sigma/2} v$, by a sum of its L_{p_0} norms taken over cubes of fixed size centered at integer points. For the convenience of computation, we use cubes instead of balls in the spatial coordinates. Note that since $B_{r/2} \subset C_r \subset B_{\sqrt{d}r/2}$, we can replace the norms taken over balls with norms taken over cubes in Corollary 3.2.4.

Proposition 3.3.1. *Let $\sigma \in (0,2)$, $p_0 \in (1,\infty)$, $t_0 \in \mathbb{R}$, and $v \in \mathbb{H}^{1,\sigma}_{p_0}((-\infty,t_0)\times\mathbb{R}^d)$ satisfy (3.3.2). Then for any $r > 0$,*

1. *for any $p \in (p_0, \infty)$ and $x_0 \in \mathbb{R}^d$, we have*

$$(|L^1 v|^p)^{1/p}_{Q_{r/2}(t_0,x_0)} \leq N \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \big(|L^1 v|^{p_0}\big)^{1/p_0}_{(t_0-r^\sigma,t_0)\times C_r(z+x_0)} \tag{3.3.3}$$

and

$$(|\lambda v|^p)^{1/p}_{Q_{r/2}(t_0,x_0)} \leq N \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \big(|\lambda v|^{p_0}\big)^{1/p_0}_{(t_0-r^\sigma,t_0)\times C_r(z+x_0)}, \tag{3.3.4}$$

where $N = N(d, \nu, \Lambda, \sigma, p_0, p)$.

2. *For any $x_0 \in \mathbb{R}^d$, there exists $\tau = \tau(d, \sigma) \in (0, 1)$ such that*

$$[L^1 v]_{C^{\tau/\sigma,\tau}(Q_{r/2}(t_0,x_0))} \leq N r^{-\tau} \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \big(|L^1 v|^{p_0}\big)^{1/p_0}_{(t_0-r^\sigma,t_0)\times C_r(z+x_0)} \tag{3.3.5}$$

and

$$[\lambda v]_{C^{\tau/\sigma,\tau}(Q_{r/2}(t_0,x_0))} \leq N r^{-\tau} \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \big(|\lambda v|^{p_0}\big)^{1/p_0}_{(t_0-r^\sigma,t_0)\times C_r(z+x_0)}, \tag{3.3.6}$$

where $N = N(d, \nu, \Lambda, \sigma, p_0)$.

Proof. The proofs of (3.3.4) and (3.3.6) are similar to the proofs of (3.3.3) and (3.3.5) by replacing $L^1 v$ with λv. Thus, we focus on (3.3.3) and (3.3.5).

By scaling and shifting the coordinates, we assume that $r = 1$ and $x_0 = 0$. Moreover, for $m = 1, 2, \ldots$, we take $p_m = p_m(d, \sigma, p_0)$ satisfying

$$1/p_m = 1/p_0 - m\sigma/(d + \sigma) \tag{3.3.7}$$

whenever the right-hand side is positive.

We first prove (3.3.3). We take cutoff function $\eta \in C^\infty(\mathbb{R})$ satisfying (3.2.33). It follows that $\eta v \in \mathbb{H}^{1,\sigma}_{p_0,0}((t_0-1,t_0)\times\mathbb{R}^d)$ and

$$\partial_t(\eta v) - L(\eta v) + \lambda(\eta v) = \partial_t(\eta v) - \eta\partial_t v = v\eta' \quad \text{in } (t_0-1,t_0)\times\mathbb{R}^d. \tag{3.3.8}$$

Taking L^1 on both sides of (3.3.8) leads to

$$\partial_t(\eta L^1 v) - L(\eta L^1 v) + \lambda(\eta L^1 v) = L^1 v\eta' \quad \text{in } (t_0-1,t_0)\times\mathbb{R}^d. \tag{3.3.9}$$

Note that if v is not regular enough, we can first mollify the equation and then take the limit of the estimate as in (3.2.35). Moreover, note that by the Minkowski inequality

$$\begin{aligned}
&\|L^1 v\|_{L_{p_0}((t_0-1,t_0);L_1(\mathbb{R}^d,\psi))}\\
&= \left(\int_{t_0-1}^{t_0}\Big(\int_{\mathbb{R}^d}|L^1 v(t,x)|\frac{1}{1+|x|^{d+\sigma}}\,dx\Big)^{p_0}dt\right)^{1/p_0}\\
&\le \int_{\mathbb{R}^d}\Big(\int_{t_0-1}^{t_0}|L^1 v(t,x)|^{p_0}\,dt\Big)^{1/p_0}\frac{1}{1+|x|^{d+\sigma}}\,dx\\
&\le \sum_{z\in\mathbb{Z}^d}\int_{C_1(z)}\Big(\int_{t_0-1}^{t_0}|L^1 v(t,x)|^{p_0}\,dt\Big)^{1/p_0}\frac{1}{1+|x|^{d+\sigma}}\,dx\\
&\le N(d,\sigma,p_0)\sum_{z\in\mathbb{Z}^d}(1+|z|^{d+\sigma})^{-1}\left(\fint_{t_0-1}^{t_0}\fint_{C_1(z)}|L^1 v(t,x)|^{p_0}\,dx\,dt\right)^{1/p_0}\\
&\le N\sum_{z\in\mathbb{Z}^d}(1+|z|^{d+\sigma})^{-1}\big(|L^1 v|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)}.
\end{aligned}$$

Hence, by applying Corollary 3.2.4 to (3.3.9), we have

$$\begin{aligned}
&\|L^1 v\|_{L_{p_1}(Q_{1/2}(t_0,0))}\\
&\le \|\eta L^1 v\|_{L_{p_1}((t_0-1,t_0)\times B_{1/2})}\\
&\le N\|L^1 v\|_{L_{p_0}((t_0-1,t_0)\times B_1)} + N\|L^1 v\|_{L_{p_0}((t_0-1,t_0);L_1(\mathbb{R}^d,\psi))}\\
&\le N\sum_{z\in\mathbb{Z}^d}(1+|z|^{d+\sigma})^{-1}\big(|L^1 v|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)},
\end{aligned} \tag{3.3.10}$$

where $N = N(d,\nu,\Lambda,\sigma,p_0)$. If $p \le p_1$, then we arrive at (3.3.3) by Hölder's inequality.

Otherwise, note that with a minor modification of the above estimates, we conclude that

$$v,\ (-\Delta)^{\sigma/2}v,\ \partial_t v \in L_{p_1}((t_0 - 1/2^\sigma, t_0)\times B_{3/4}).$$

Furthermore, we take a cutoff function

$$\zeta \in C_0^\infty(\mathbb{R}^d), \quad \zeta = 1 \quad \text{in } B_{1/2}, \quad \text{and} \ \ \zeta = 0 \quad \text{in } B^c_{3/4}.$$

By taking the mollification (in both the time and the spatial variables) of ζv as the defining sequence, we conclude that

$$\zeta v \in \mathbb{H}^{1,\sigma}_{p_1}((t_0 - 1/2^\sigma, t_0)\times\mathbb{R}^d). \tag{3.3.11}$$

Moreover, similar to (3.3.10) with a shift of the coordinates and some minor modifications, we have

$$\begin{aligned}
&\|L^1 v\|_{L_{p_1}((t_0-(1/2)^\sigma,t_0)\times B_{\sqrt{d}/2}(x_0))}\\
&\le N\sum_{z\in\mathbb{Z}^d}(1+|z|^{d+\sigma})^{-1}\big(|L^1 v|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z+x_0)}.
\end{aligned} \tag{3.3.12}$$

Therefore, by taking anther cutoff function in time together with (3.3.11), Corollary 3.2.4, and (3.3.12), we obtain

$$
\begin{aligned}
&\|L^1 v\|_{L_{p_2}(Q_{1/4}(t_0,0))} \\
&\leq N \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \big(|L^1 v|^{p_1}\big)^{1/p_1}_{(t_0-(1/2)^\sigma,t_0)\times C_{1/2}(z)} \\
&\leq N \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \sum_{y\in\mathbb{Z}^d} (1+|y-z|^{d+\sigma})^{-1} \big(|L^1 v|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(y)} \\
&\leq N \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \big(|L^1 v|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)},
\end{aligned}
$$

where for the last inequality, we used

$$
\begin{aligned}
&\sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1}(1+|y-z|^{d+\sigma})^{-1} \\
&\leq N(d,\sigma)(1+|y|^{d+\sigma})^{-1} \sum_{z\in\mathbb{Z}^d} \Big[(1+|z|^{d+\sigma})^{-1} + (1+|y-z|^{d+\sigma})^{-1}\Big] \\
&\leq N(d,\sigma)(1+|y|^{d+\sigma})^{-1}.
\end{aligned}
$$

We repeat the above process with p_m, $m = 3,4,5,\ldots$, finite many times until $p_i \geq p$ or $p_i > d/\sigma + 1$. Note that the number of iterations depends only on d, σ, and p by (3.3.7). Thus, by a covering argument, we obtain (3.3.3).

For (3.3.5), if $p_0 \geq 2(d/\sigma+1)$, then by Corollary 3.2.4, we can replace the left-hand side of the inequality (3.3.10) with $[L^1 v]_{C^{\tau/\sigma,\tau}(Q_{1/2}(t_0,0))}$ to get (3.3.5). Otherwise, we apply the iteration argument as above until $p_k \geq 2(d/\sigma+1)$, and by a covering argument, we arrive at (3.3.5). The proposition is proved. $\square$

Remark 3.3.2. It is worth noting that the operator we considered in Proposition 3.3.1 was local in time, i.e., $\alpha = 1$. For general $\alpha \neq 1$, it is not clear to us whether one can derive a similar estimate. In particular, we cannot get an expression as simple as (3.3.9) (see the extra $\tilde{g}$ term on (3.2.35)), and the second inequality of (3.3.10) no longer holds.

Next, we estimate the non-homogeneous part, which satisfies the zero initial condition at $t = t_0 - 1$. In the following proposition, we denote $\|\cdot\|_{p_0,\Omega} := \|\cdot\|_{L_{p_0}((t_0-1,t_0)\times\Omega)}$ for any $\Omega \subset \mathbb{R}^d$ and $\|\cdot\|_{p_0} := \|\cdot\|_{L_{p_0}((t_0-1,t_0)\times\mathbb{R}^d)}$.

Proposition 3.3.3. *Let $t_0 \in \mathbb{R}$ and $w \in \mathbb{H}^{1,\sigma}_{p_0,0}((t_0-1,t_0)\times\mathbb{R}^d)$ be such that*

$$\partial_t w - Lw + \lambda w = f \quad in \quad (t_0-1,t_0)\times\mathbb{R}^d.$$

Then, for any $x_0 \in \mathbb{R}^d$ and $R \geq 1$, we have

$$\begin{aligned}
&(|L^1 w|^{p_0})^{1/p_0}_{(t_0-1,t_0)\times B_{R/2}(x_0)} + \lambda\big(|w|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{R/2}} \\
&\leq N\sum_{k=0}^{\infty} 2^{-k\sigma}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^kR}(x_0)},
\end{aligned} \tag{3.3.13}$$

where $N = N(d,\nu,\Lambda,\sigma,p_0)$.

Proof. By shifting the coordinates, we assume that $x_0 = 0$. It follows from Corollary 3.2.5 that

$$\begin{aligned}
&\big(|L^1 w|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{R/2}} + \lambda\big(|w|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{R/2}} \\
&\leq N\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_R} + N\sum_{k=0}^{\infty} 2^{-k\sigma}\big(|w|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^kR}} \\
&:= N\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_R} + N\sum_{k=0}^{\infty} 2^{-k\sigma} A_k.
\end{aligned} \tag{3.3.14}$$

It remains to estimate A_k. By Lemma 3.4.2 and Corollary 3.2.5, for each $k = 0, 1, \ldots$,

$$\begin{aligned}
A_k &\le N\big(|\partial_t w|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^k R}} \\
&\le N\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^{k+1}R}} + N\sum_{j=0}^{\infty} 2^{-(j+k+1)\sigma}\big(|w|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^{j+k+1}R}} \\
&\le N\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^{k+1}R}} + N\sum_{j=k+1}^{\infty} 2^{-j\sigma} A_j, \qquad (3.3.15)
\end{aligned}$$

where N is independent of k. By first multiplying both sides of (3.3.15) by $2^{-\sigma k}$ and then summing over $k = k_0, k_0+1, \ldots$, for some integer k_0 to be determined, we obtain

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} 2^{-\sigma k} A_k \\
&\le N\sum_{k=k_0}^{\infty} 2^{-\sigma(k+1)}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^{k+1}R}} + N\sum_{k=k_0}^{\infty} 2^{-\sigma k}\sum_{j=k+1}^{\infty} 2^{-\sigma j} A_j \\
&\le N\sum_{k=k_0}^{\infty} 2^{-\sigma(k+1)}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^{k+1}R}} + \frac{N2^{-\sigma k_0}}{1-2^{-\sigma}}\sum_{j=k_0+1}^{\infty} 2^{-\sigma j} A_j.
\end{aligned}$$

Picking k_0 sufficiently large so that $\frac{N2^{-\sigma k_0}}{1-2^{-\sigma}} \le 1/2$, we have

$$\sum_{k=k_0}^{\infty} 2^{-\sigma k} A_k \le N\sum_{k=k_0}^{\infty} 2^{-\sigma k}(|f|^{p_0})^{1/p_0}_{(t_0-1,t_0)\times B_{2^k R}}.$$

Therefore, by induction,

$$\sum_{k=0}^{\infty} 2^{-\sigma k} A_k \le N\sum_{k=0}^{\infty} 2^{-\sigma k}(|f|^{p_0})^{1/p_0}_{(t_0-1,t_0)\times B_{2^k R}}. \qquad (3.3.16)$$

Indeed, if there exists $N = N(d, \nu, \Lambda, \sigma, p_0)$ such that

$$\sum_{k=j}^{\infty} 2^{-\sigma k} A_k \le N\sum_{k=j}^{\infty} 2^{-\sigma k}(|f|^{p_0})^{1/p_0}_{(t_0-1,t_0)\times B_{2^k R}}$$

for $j = 1, 2, \ldots, k_0$, then, by (3.3.15),

$$
\begin{aligned}
\sum_{k=j-1}^{\infty} 2^{-\sigma k} A_k &\le 2^{-\sigma(j-1)} A_{j-1} + \sum_{k=j}^{\infty} 2^{-\sigma k} A_k \\
&\le N 2^{-\sigma j} \big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^j R}} + N \sum_{k=j}^{\infty} 2^{-k\sigma} A_k \\
&\le N \sum_{k=j-1}^{\infty} 2^{-\sigma k} (|f|^{p_0})^{1/p_0}_{(t_0-1,t_0)\times B_{2^k R}}.
\end{aligned}
$$

Therefore, by (3.3.14) and (3.3.16), we arrive at (3.3.13), and the lemma is proved. □

Remark 3.3.4. It is worth noting that although the operator we considered in Proposition 3.3.3 was local in time, i.e., $\alpha = 1$, the proof still works for any $\alpha \in (0, 1]$ since the embedding used in (3.3.15) still holds in this case.

With the estimates of v and w, we are ready to prove a mean oscillation estimate for u.

Proposition 3.3.5. *Let $\sigma \in (0, 2)$, $T \in (0, \infty)$, $p_0 \in (1, \infty)$, and $u \in \mathbb{H}^{1,\sigma}_{p_0}((-\infty, T) \times \mathbb{R}^d)$ satisfy*

$$
\partial_t u - Lu + \lambda u = f \quad \text{in } (-\infty, T) \times \mathbb{R}^d.
$$

Then, for any $(t_0, x_0) \in (-\infty, T] \times \mathbb{R}^d$, $r \in (0, \infty)$, and $\kappa \in (0, 1/4)$, we have

$$
\begin{aligned}
&(|L^1 u - (L^1 u)_{Q_{\kappa r}(t_0,x_0)}|)_{Q_{\kappa r}(t_0,x_0)} + (|\lambda u - (\lambda u)_{Q_{\kappa r}(t_0,x_0)}|)_{Q_{\kappa r}(t_0,x_0)} \\
&\quad \le N\kappa^{\tau} \sum_{k=0}^{\infty} 2^{-k\sigma} \Big[\big(|L^1 u|^{p_0}\big)^{1/p_0}_{(t_0-r^\sigma,t_0)\times B_{2^k r}(x_0)} + \big(|\lambda u|^{p_0}\big)^{1/p_0}_{(t_0-r^\sigma,t_0)\times B_{2^k r}(x_0)} \Big] \\
&\quad + N\kappa^{-(d+\sigma)/p_0} \sum_{k=0}^{\infty} 2^{-k\sigma} \big(|f|^{p_0}\big)^{1/p_0}_{(t_0-r^\sigma,t_0)\times B_{2^k r}(x_0)},
\end{aligned}
\tag{3.3.17}
$$

where $\tau = \tau(d,\sigma,p_0)$ *and* $N = N(d,\nu,\Lambda,\sigma,p_0)$.

Proof. By shifting the coordinates and scaling, we assume $x_0 = 0$ and $r = 1$. Moreover, by Theorem 3.1.6, there exists $w \in \mathbb{H}^{1,\sigma}_{p_0,0}((t_0-1,t_0)\times\mathbb{R}^d)$ satisfying (3.3.1). Also, $v := u - w \in \mathbb{H}^{1,\sigma}_{p_0}((-\infty,t_0)\times\mathbb{R}^d)$ satisfies (3.3.2).

Next, by Hölder's inequality and Proposition 3.3.3,

$$
\begin{aligned}
&(|L^1u-(L^1u)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} + (|\lambda u-(\lambda u)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)}\\
&\le (|L^1v-(L^1v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} + N\kappa^{-(d+\sigma)/p_0}(|L^1w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0,0)}\\
&\quad + (|\lambda v-(\lambda v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} + N\kappa^{-(d+\sigma)/p_0}(|\lambda w|^{p_0})^{1/p_0}_{Q_{1/2}(t_0,0)}\\
&\le (|L^1v-(L^1v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} + (|\lambda v-(\lambda v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)}\\
&\quad + N\kappa^{-(d+\sigma)/p_0}\sum_{k=0}^{\infty}2^{-k\sigma}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^k}}.
\end{aligned}
\tag{3.3.18}
$$

For the first two terms on the right-hand side of (3.3.18), by Proposition 3.3.1, there exists $\tau \in (0,1)$ such that

$$
\begin{aligned}
&(|L^1v-(L^1v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)} + (|\lambda v-(\lambda v)_{Q_\kappa(t_0,0)}|)_{Q_\kappa(t_0,0)}\\
&\le \kappa^\tau[L^1v]_{C^{\tau/\sigma,\tau}(Q_{1/4}(t_0,0))} + \kappa^\tau[\lambda v]_{C^{\tau/\sigma,\tau}(Q_{1/4}(t_0,0))}\\
&\le N\kappa^\tau\sum_{z\in\mathbb{Z}^d}(1+|z|^{d+\sigma})^{-1}\Big[\big(|L^1v|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)} + \big(|\lambda v|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)}\Big]\\
&\le N\kappa^\tau\sum_{z\in\mathbb{Z}^d}(1+|z|^{d+\sigma})^{-1}\big(|L^1u|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)}\\
&\quad + N\kappa^\tau\sum_{z\in\mathbb{Z}^d}(1+|z|^{d+\sigma})^{-1}\big(|\lambda u|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)}\\
&\quad + N\kappa^\tau\sum_{z\in\mathbb{Z}^d}(1+|z|^{d+\sigma})^{-1}\Big[\big(|L^1w|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)} + \big(|\lambda w|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)}\Big]\\
&=: N\kappa^\tau I_1 + N\kappa^\tau I_2 + N\kappa^\tau I_3.
\end{aligned}
\tag{3.3.19}
$$

To estimate I_1, I_2, and I_3, for $R > r \geq 1$, we denote N_R to be the number of $z \in \mathbb{Z}^d$ lying in B_R, and $N_{r,R}$ to be the number of $z \in \mathbb{Z}^d$ lying in $B_R \setminus B_r^o$. Note that

$$N_{r,R} \leq N_R \quad \text{and } |B_{R-\sqrt{d}/2}| \leq N_R \leq |B_{R+\sqrt{d}/2}| \leq N(d)R^d, \tag{3.3.20}$$

and for $k = 0, 1, 2, 3, \ldots$, we have

$$N_{2^k,2^{k+1}} \geq N(d)2^{kd}. \tag{3.3.21}$$

Then, by Hölder's inequality for l_{p_0}, (3.3.20), and (3.3.21), we have

$$\begin{aligned}
&I_1 - \left(|L^1 u|^{p_0}\right)^{1/p_0}_{(t_0-1,t_0)\times C_1(0)} \\
&\leq N \sum_{k=0}^{\infty} \sum_{\substack{|z|=2^k \\ z\in\mathbb{Z}^d}}^{2^{k+1}} 2^{-kd-k\sigma} \left(|L^1 u|^{p_0}\right)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)} \\
&\leq N \sum_{k=0}^{\infty} 2^{-kd-k\sigma} N_{2^k,2^{k+1}} \sum_{\substack{|z|=2^k \\ z\in\mathbb{Z}^d}}^{2^{k+1}} N^{-1}_{2^k,2^{k+1}} \left(|L^1 u|^{p_0}\right)^{1/p_0}_{(t_0-1,t_0)\times C_1(z)} \\
&\leq N \sum_{k=0}^{\infty} 2^{-kd-k\sigma} N_{2^k,2^{k+1}} \left(\sum_{\substack{|z|=2^k \\ z\in\mathbb{Z}^d}}^{2^{k+1}} N^{-1}_{2^k,2^{k+1}} \left(|L^1 u|^{p_0}\right)_{(t_0-1,t_0)\times C_1(z)} \right)^{1/p_0} \\
&\leq N \sum_{k=0}^{\infty} 2^{-k\sigma} \left(|L^1 u|^{p_0}\right)^{1/p_0}_{(t_0-1,t_0)\times B_{2^{k+1}+\sqrt{d}/2}}.
\end{aligned}$$

Thus,

$$I_1 \leq N \sum_{k=0}^{\infty} 2^{-k\sigma} \left(|L^1 u|^{p_0}\right)^{1/p_0}_{(t_0-1,t_0)\times B_{2^k}}. \tag{3.3.22}$$

Similarly, by replacing $L^1 u$ with λu, we conclude that

$$I_2 \leq N \sum_{k=0}^{\infty} 2^{-k\sigma} \left(|\lambda u|^{p_0}\right)^{1/p_0}_{(t_0-1,t_0)\times B_{2^k}}. \tag{3.3.23}$$

For I_3, by Proposition 3.3.3, we have

$$
\begin{aligned}
I_3 &\le N \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \sum_{j=0}^{\infty} 2^{-j\sigma} \left(|f|^{p_0}\right)^{1/p_0}_{(t_0-1,t_0)\times B_{2^j\sqrt{d}}(z)} \\
&\le N \sum_{z\in\mathbb{Z}^d} (1+|z|^{d+\sigma})^{-1} \sum_{j=0}^{\infty} 2^{-j\sigma} \left(|f|^{p_0}\right)^{1/p_0}_{(t_0-1,t_0)\times C_{2^j}(z)}.
\end{aligned} \tag{3.3.24}
$$

For $j \ge 0$, $z = (z^1, \dots, z^d), i = (i^1, \dots, i^d) \in \mathbb{Z}^d$ and each component of i takes value in $\{0, 1, \dots 2^j - 1\}$, we denote

$$
z_{j,i} = (z^1 - (2^j-1)/2 + i^1, \dots, z^d - (2^j-1)/2 + i^d).
$$

Since $C_1(z_{j,i})$ is disjoint with respect to i for fixed z and j, and it is also disjoint with respect to z for fixed i and j, we have

$$
\begin{aligned}
\sum_{\substack{|z|=2^k\\ z\in\mathbb{Z}^d}}^{2^{k+1}} \int_{(t_0-1,t_0)} \int_{C_{2^j}(z)} |f|^{p_0} &= \sum_{\substack{|z|=2^k\\ z\in\mathbb{Z}^d}}^{2^{k+1}} \sum_i \int_{(t_0-1,t_0)} \int_{C_1(z_{j,i})} |f|^{p_0} \\
&\le 2^{\min\{k,j\}d} \int_{(t_0-1,t_0)} \int_{B_{2^{j-1}\sqrt{d}+2^{k+1}}} |f|^{p_0}.
\end{aligned} \tag{3.3.25}
$$

By (3.3.24), (3.3.21), Hölder's inequality, and (3.3.25),

$$
\begin{aligned}
& I_3 - N\sum_{j=0}^{\infty} 2^{-j\sigma}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_{2^j}(0)} \\
& \le N\sum_{k=0}^{\infty}\sum_{\substack{|z|=2^k\\ z\in\mathbb{Z}^d}}^{2^{k+1}} 2^{-kd-k\sigma}\sum_{j=0}^{\infty}2^{-j\sigma}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_{2^j}(z)} \\
& \le N\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{-k\sigma-j\sigma}\sum_{\substack{|z|=2^k\\ z\in\mathbb{Z}^d}}^{2^{k+1}} N^{-1}_{2^k,2^{k+1}}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times C_{2^j}(z)} \\
& \le N\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{-k\sigma-j\sigma}\Bigg(2^{-kd-jd}\sum_{\substack{|z|=2^k\\ z\in\mathbb{Z}^d}}^{2^{k+1}}\int_{(t_0-1,t_0)}\int_{C_{2^j}(z)}|f|^{p_0}\Bigg)^{1/p_0} \\
& \le N\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{-k\sigma-j\sigma}\Bigg(2^{-kd-jd}2^{\min\{k,j\}d}\int_{(t_0-1,t_0)}\int_{B_{2^{j-1}\sqrt{d}+2^{k+1}}}|f|^{p_0}\Bigg)^{1/p_0} \\
& \le N\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{-k\sigma-j\sigma}\Bigg(1_{k\ge j}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^k}} + 1_{k<j}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^j}}\Bigg) \\
& \le N\sum_{k=0}^{\infty}2^{-k\sigma}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^k}},
\end{aligned}
$$

which implies that

$$
I_3 \le N\sum_{k=0}^{\infty}2^{-k\sigma}\big(|f|^{p_0}\big)^{1/p_0}_{(t_0-1,t_0)\times B_{2^k}}. \tag{3.3.26}
$$

Thus, combining (3.3.18), (3.3.19), (3.3.22), (3.3.23), and (3.3.26), we arrive at (3.3.17), and the proposition is proved. □

Next, we introduce the dyadic cubes as follows: for each $n \in \mathbb{Z}$, pick $k(n)\in \mathbb{Z}$ such that

$$
k(n) \le \sigma n < k(n)+1,
$$

and let

$$Q^n_{\vec{i}} = \Big[\frac{i_0}{2^{k(n)}} + T, \frac{i_0+1}{2^{k(n)}} + T\Big) \times \Big[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\Big) \times \cdots \times \Big[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\Big)$$

and

$$\mathbb{C}_n := \big\{Q^n_{\vec{i}} = Q^n_{(i_0,\dots,i_d)} : \vec{i} = (i_0,\dots,i_d) \in \mathbb{Z}^{d+1}, i_0 \le -1\big\}.$$

Furthermore, denote the dyadic sharp function of g by

$$g^{\#}_{dy}(t,x) = \sup_{n<\infty} \fint_{Q^n_{\vec{i}} \ni (t,x)} |g(s,y) - g_{|n}(t,x)|\, dy\, ds,$$

where

$$g_{|n}(t,x) = \fint_{Q^n_{\vec{i}}} g(s,y)\, dy\, ds \quad \text{for} \quad (t,x) \in Q^n_{\vec{i}}.$$

Proof of Theorem 3.1.9. Step 1. The a priori estimate.

We first prove (3.1.18) under the assumption that u is compactly supported in the spacial variables. For the given $w_1 \in A_p(\mathbb{R}, dt)$ and $w_2 \in A_q(\mathbb{R}^d, dx)$, using reverse Hölder's inequality for A_p weights, we pick $\gamma_1 = \gamma_1(d,p,M_1)$ and $\gamma_2 = \gamma_2(d,q,M_1)$ such that $p - \gamma_1 > 1$, $q - \gamma_2 > 1$, and

$$w_1 \in A_{p-\gamma_1}(\mathbb{R}, dt), \quad w_2 \in A_{q-\gamma_2}(\mathbb{R}^d, dx).$$

Take $p_0 = p_0(d,p,q,M_1) \in (1,\infty)$ so that

$$p_0 = \min\left\{\frac{p}{p-\gamma_1}, \frac{q}{q-\gamma_2}\right\} > 1.$$

Note that

$$w_1 \in A_{p-\gamma_1}(\mathbb{R}, dt) \subset A_{\frac{p}{p_0}}(\mathbb{R}, dt)$$

and

$$w_2 \in A_{q-\gamma_2}(\mathbb{R}^d, dx) \subset A_{\frac{q}{p_0}}(\mathbb{R}^d, dx).$$

From these inclusions and the fact that $u \in \mathbb{H}^{1,\sigma}_{p,q,w,0}(\mathbb{R}^d_T)$, it follows that $u \in \mathbb{H}^{1,\sigma}_{p_0,0,\mathrm{loc}}(\mathbb{R}^d_T)$. See the proof of (Dong and Kim, 2018, Lemma 5.10) for details. Furthermore, since u is compactly supported, we have $u \in \mathbb{H}^{1,\sigma}_{p_0,0}(\mathbb{R}^d_T)$, and $u \in \mathbb{H}^{1,\sigma}_{p_0}((-\infty,0)\times\mathbb{R}^d)$ by taking the zero extension for $t<0$. Therefore, by Proposition 3.3.5, for any $r\in(0,\infty)$, $\kappa\in(0,1/4)$, and $(t_1,x_0)\in(0,T]\times\mathbb{R}^d$, we have

$$\begin{aligned}
&(|L^1u-(L^1u)_{Q_{\kappa r}(t_1,x_0)}|)_{Q_{\kappa r}(t_1,x_0)} + (|\lambda u-(\lambda u)_{Q_{\kappa r}(t_1,x_0)}|)_{Q_{\kappa r}(t_1,x_0)} \\
&\quad \le N\kappa^\tau \sum_{k=0}^\infty 2^{-k\sigma}\left[\left(|L^1u|^{p_0}\right)^{1/p_0}_{(t_1-r^\sigma,t_1)\times B_{2^k r}(x_0)} + \left(|\lambda u|^{p_0}\right)^{1/p_0}_{(t_1-r^\sigma,t_1)\times B_{2^k r}(x_0)}\right] \\
&\quad + N\kappa^{-(d+\sigma)/p_0}\sum_{k=0}^\infty 2^{-k\sigma}\left(|f|^{p_0}\right)^{1/p_0}_{(t_1-r^\sigma,t_1)\times B_{2^k r}(x_0)},
\end{aligned} \tag{3.3.27}$$

where $\tau=\tau(d,\sigma,p,q,M_1)$ and $N=N(d,\nu,\Lambda,\sigma,p,q,M_1)$.

Note that for any $(t_0,x_0)\in\mathbb{R}^d_T$, $n\in\mathbb{Z}$, and $Q^n_{\vec{i}}$ containing (t_0,x_0), there exist $r=r(d,\sigma,n)>0$ and $t_1=\min(T,t_0+r^\sigma/2)\in(-\infty,T]$ such that

$$Q^n_{\vec{i}} \subset Q_r(t_1,x_0) \quad \text{and} \quad |Q_r(t_1,x_0)| \le N(d,\sigma)|Q^n_{\vec{i}}|.$$

Thus, by (3.3.27), we have

$$\begin{aligned}
&⨍_{Q^n_{\vec{i}}\ni(t_0,x_0)} |L^1u(s,y)-(L^1u)_{|n}(t_0,x_0)| + |\lambda u(s,y)-(\lambda u)_{|n}(t_0,x_0)|\,dy\,ds \\
&\le N\kappa^\tau \sum_{k=0}^\infty 2^{-k\sigma}\left[\left(|L^1u|^{p_0}\right)^{1/p_0}_{(t_1-(r/\kappa)^\sigma,t_1)\times B_{2^k r/\kappa}(x_0)} + \left(|\lambda u|^{p_0}\right)^{1/p_0}_{(t_1-(r/\kappa)^\sigma,t_1)\times B_{2^k r/\kappa}(x_0)}\right] \\
&\quad + N\kappa^{-(d+\sigma)/p_0}\sum_{k=0}^\infty 2^{-k\sigma}\left(|f|^{p_0}\right)^{1/p_0}_{(t_1-r^\sigma,t_1)\times B_{2^k r/\kappa}(x_0)} \\
&\le N\left(\kappa^\tau(\mathcal{SM}|L^1u|^{p_0})^{\frac{1}{p_0}} + \kappa^\tau(\mathcal{SM}|\lambda u|^{p_0})^{\frac{1}{p_0}} + \kappa^{-(d+\sigma)/p_0}(\mathcal{SM}|f|^{p_0})^{\frac{1}{p_0}}\right)(t_0,x_0),
\end{aligned}$$

where N is independent of n and κ, and for the last inequality, we used that for any function g and $A \subset B$,

$$\left| \fint_A g - \fint_B g \right| \leq \frac{|B|}{|A|} \fint_B |g - (g)_B|.$$

Therefore,

$$\begin{aligned}
&(L^1 u)^{\#}_{dy}(t_0, x_0) + (\lambda u)^{\#}_{dy}(t_0, x_0) \\
&\leq N \left(\kappa^\tau (\mathcal{SM}|L^1 u|^{p_0})^{\frac{1}{p_0}} + \kappa^\tau (\mathcal{SM}|\lambda u|^{p_0})^{\frac{1}{p_0}} + \kappa^{-(d+\sigma)/p_0} (\mathcal{SM}|f|^{p_0})^{\frac{1}{p_0}} \right)(t_0, x_0).
\end{aligned}$$

Then, by the weighted sharp function theorem (Dong and Kim, 2018, Corollary 2.7) and the weighted maximal function theorem for strong maximal functions (Dong and Kim, 2021, Theorem 5.2),

$$\|L^1 u\|_{p,q,w} + \|\lambda u\|_{p,q,w} \leq N\kappa^\tau \|L^1 u\|_{p,q,w} + N\kappa^\tau \|\lambda u\|_{p,q,w} + N\kappa^{-(d+\sigma)/p_0} \|f\|_{p,q,w},$$

where $N = N(d, \nu, \Lambda, \sigma, p, q, M_1)$. Therefore, by taking a sufficiently small $\kappa < 1/4$ such that

$$N\kappa^\tau < 1/2,$$

we get

$$\|L^1 u\|_{L_{p,q,w}} + \lambda \|u\|_{L_{p,q,w}} \leq N \|f\|_{L_{p,q,w}}.$$

Since the space of functions that are compactly supported in the spacial variables is dense in $\mathbb{H}^{1,\sigma}_{p,q,w,0}(T)$, we obtain (3.1.18) and the continuity of $\partial_t - L$. Then (3.1.19) follows from the Equation (3.1.17) together with the triangle in equality.

Step 2. Existence of solutions.

By the method of continuity, it suffices to prove the existence of solutions for the simple equation with $L = -(-\Delta)^{\sigma/2}$ and $\lambda = 0$. By the a priori estimate proved in Step 1 and the density of smooth functions in Lebesgue spaces, without loss of generality, we assume that $f \in C_0^\infty((0,\infty) \times \mathbb{R}^d)$. Let

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \zeta(t-s, x-y) f(s,y)\, dy\, ds, \tag{3.3.28}$$

where $\hat{\zeta}(t,\xi) = e^{-t|\xi|^\sigma}$. Then by (Kim et al., 2021, Lemma 4.1),

$$\partial_t u + (-\Delta)^{\sigma/2} u = f.$$

When $p = q$, it follows from (Fackler et al., 2020, Theorem 5.14), a Fourier multiplier theorem, that

$$\|\partial_t u\|_{L_{p,w}(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2} u\|_{L_{p,w}(\mathbb{R}^d_T)} \le N \|f\|_{L_{p,w}(\mathbb{R}^d_T)}.$$

For general p and q, using the same definition of u as in (3.3.28) together with the extrapolation theorem in (Dong and Kim, 2018, Theorem 2.5) or (Cruz-Uribe et al., 2011, Theorem 1.4), we have

$$\|\partial_t u\|_{L_{p,q,w}(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2} u\|_{L_{p,q,w}(\mathbb{R}^d_T)} \le N \|f\|_{L_{p,q,w}(\mathbb{R}^d_T)},$$

which implies that $u \in H^{1,\sigma}_{p,q,w,0}(\mathbb{R}^d_T)$. The theorem is proved. $\square$

Next, we prove Corollary 3.1.10 using the frozen coefficient argument and a partition of unity.

Corollary 3.3.6. *Let $\beta \in (0,1)$, $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $p \in (1,\infty)$, $\lambda \ge 0$, $M_1 \in [1,\infty)$, and $[w]_{p,p} \le M_1$. There exists $r_0 = r_0(d,\nu,\Lambda,\alpha,\sigma,p,\beta,\omega,M_1) > 0$ such that under Assumptions 3.1.1 and 3.1.3, for any $u \in \mathbb{H}^{\alpha,\sigma}_{p,w,0}(\mathbb{R}^d_T)$ supported on*

$(0,T)\times B_{r_0}(x_1)$ for some $x_1\in\mathbb{R}^d$ and satisfies

$$\partial_t u - Lu + \lambda u = f \quad \text{in} \;\; \mathbb{R}^d_T,$$

we have

$$\begin{aligned} &\|\partial_t u\|_{L_{p,w}(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2}u\|_{L_{p,w}(\mathbb{R}^d_T)} + \lambda\|u\|_{L_{p,w}(\mathbb{R}^d_T)} \\ &\le N\|f\|_{L_{p,w}(\mathbb{R}^d_T)} + N\|u\|_{L_{p,w}(\mathbb{R}^d_T)}, \end{aligned}$$

where $N=N(d,\nu,\Lambda,\sigma,p,M_1,\beta,\omega)$.

Proof. This follows from (Dong et al., 2021, Lemma 5.1) and Theorem 3.1.9 with the frozen coefficient argument. See the proof of (Dong et al., 2021, Lemma 5.2) for details. □

Proof of Corollary 3.1.10. First, the continuity of $\partial_t - L$ follows by (Dong et al., 2021, Corollary 2.5). By Theorem 3.1.6 together with the method of continuity, it remains to prove the a priori estimate (3.1.15).

We first consider the case when $b=c=0$ and $p=q$ with general weight $w(t,x)=w_1(t)w_2(x)$. We take r_0 from Corollary 3.3.6, and use a partition of unity argument with respect to the spatial variables. Let $\zeta\in C_0^\infty(B_{r_0})$ satisfy

$$\|\zeta\|_{L_p}=1, \quad \|D_x\zeta\|_{L_p}\le N, \quad \|D_x^2\zeta\|_{L_p}\le N, \quad \text{and } \zeta_z(x)=\zeta(x-z),$$

where $z\in\mathbb{R}^d$ and $N=N(d,\nu,\Lambda,\sigma,p,\beta,\omega)$. It follows that

$$\partial_t(\zeta_z u) - L(\zeta_z u) + \lambda\zeta_z u = \zeta_z f + \zeta_z Lu - L(\zeta_z u).$$

By Corollary 3.3.6, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\|\partial_t u\|^p_{L_{p,w}(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2}u\|^p_{L_{p,w}(\mathbb{R}^d_T)} + \|\lambda u\|^p_{L_{p,w}(\mathbb{R}^d_T)} \\
&= \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(|\zeta_z(x)\partial_t u(t,x)|^p + |\zeta_z(x)\lambda u(t,x)|^p\big) w(t,x)w(t,x)\,dz\,dx\,dt \\
&\quad + \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)(-\Delta)^{\sigma/2}u(t,x)|^p w(t,x)\,dz\,dx\,dt \\
&\le \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(|\partial_t\big(\zeta_z(x)u(t,x)\big)|^p + |\lambda\zeta_z(x)u(t,x)|^p\big) w(t,x)\,dz\,dx\,dt \\
&\quad + N\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(-\Delta)^{\sigma/2}(\zeta_z u)(t,x)|^p w(t,x)\,dx\,dz\,dt \\
&\quad + N\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)(-\Delta)^{\sigma/2}u(t,x) - (-\Delta)^{\sigma/2}(\zeta_z u)(t,x)|^p w(t,x)\,dx\,dz\,dt \\
&\le N\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(|\zeta_z(x)f(t,x)|^p + |\zeta_z(x)u(t,x)|^p\big) w(t,x)\,dx\,dz\,dt \\
&\quad + N\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)Lu(t,x) - L(\zeta_z u)(t,x)|^p w(t,x)\,dx\,dz\,dt \\
&\quad + N\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)(-\Delta)^{\sigma/2}u(t,x) - (-\Delta)^{\sigma/2}(\zeta_z u)(t,x)|^p w(t,x)\,dx\,dz\,dt \\
&\le N\|f\|^p_{L_{p,w}(\mathbb{R}^d_T)} + N\|u\|^p_{L_{p,w}(\mathbb{R}^d_T)} + 1_{\sigma>1}N\|Du\|^p_{L_{p,w}(\mathbb{R}^d_T)} + 1_{\sigma=1}\varepsilon\|Du\|^p_{L_{p,w}(\mathbb{R}^d_T)}.
\end{aligned} \tag{3.3.29}$$

Indeed, for the last inequality, to estimate

$$I := \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\zeta_z(x)Lu(t,x) - L(\zeta_z u)(t,x)|^p w_2(x)w_1(t)\,dx\,dz\,dt,$$

we consider the following three cases depending on the value of σ. When $\sigma \in (0,1)$,

$$\begin{aligned}
&|\zeta_z(x)Lu(t,x) - L(\zeta_z u)(t,x)| \\
&\le N\bigg(\int_{B_1} + \int_{B_1^c}\bigg)|\zeta_z(x+y) - \zeta_z(x)||u(t,x+y)||y|^{-d-\sigma}\,dy =: I_1^1 + I_2^1.
\end{aligned}$$

Since

$$I_1^1 \leq \int_{B_1} |u(t,x+y)||y|^{-d-\sigma+1} \int_0^1 |D\zeta_z(x+sy)|\,ds\,dy,$$

by the Minkowski inequality, (Dong et al., 2021, Lemma 3.2), and the weighted Hardy-Littlewood maximal function theorem, we conclude that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (I_1^1(t,x,z))^p w_2(x) w_1(t)\,dx\,dz\,dt \\
&\leq \|D\zeta\|^p_{L_p(\mathbb{R}^d)} \int_0^T \int_{\mathbb{R}^d} \big(\int_{B_1} |u(t,x+y)||y|^{-d-\sigma+1}\,dy\big)^p w_2(x) w_1(t)\,dx\,dt, \\
&\leq N \int_0^T \int_{\mathbb{R}^d} \mathcal{M}_x u(t,x)^p w_2(x) w_1(t)\,dx\,dt \leq N\|u\|^p_{L_{p,w}(\mathbb{R}^d_T)},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (I_2^1(t,x,z))^p w_2(x) w_1(t)\,dx\,dz\,dt, \\
&\leq N\|\zeta\|^p_{L_p(\mathbb{R}^d)} \int_0^T \int_{\mathbb{R}^d} \big(\int_{B_1^c} |u(t,x+y)||y|^{-d-\sigma}\,dy\big)^p w_2(x) w_1(t)\,dy\,dt \\
&\leq N\|\zeta\|^p_{L_p(\mathbb{R}^d)} \int_0^T \int_{\mathbb{R}^d} \mathcal{M}_x u(t,x)^p w_2(x) w_1(t)\,dy\,dt \\
&\leq N\|u\|^p_{L_{p,w}(\mathbb{R}^d_T)}.
\end{aligned}$$

Here $\mathcal{M}_x u(t,\cdot)$ stands for the maximal function of u with respect to the x variable. Furthermore, using a similar decomposition as in the proof of Lemma 3.4.3 together with the maximal function as above, when $\sigma \in (1,2)$,

$$I \leq N\|u\|^p_{L_{p,w}(\mathbb{R}^d_T)} + 1_{\sigma>1} N\|Du\|^p_{L_{p,w}(\mathbb{R}^d_T)},$$

and when $\sigma = 1$,

$$I \leq N(\varepsilon)\|u\|^p_{L_{p,w}(\mathbb{R}^d_T)} + \varepsilon\|Du\|^p_{L_{p,w}(\mathbb{R}^d_T)}.$$

Therefore, by (3.3.29) together with Lemma 3.4.1 or picking a sufficiently small ε when $\sigma = 1$, we obtain

$$\begin{aligned}&\|\partial_t u\|_{L_{p,w}(\mathbb{R}^d_T)} + \|(-\Delta)^{\sigma/2}u\|_{L_{p,w}(\mathbb{R}^d_T)} + \lambda\|u\|_{L_{p,w}(\mathbb{R}^d_T)}\\ &\leq N\|f\|_{L_{p,w}(\mathbb{R}^d_T)} + N\|u\|_{L_{p,w}(\mathbb{R}^d_T)}.\end{aligned} \tag{3.3.30}$$

By taking $\lambda_0 = \max(2N, 1)$, for all $\lambda \geq \lambda_0$, we have (3.1.20). On the other hand, when $\lambda = 0$, by dividing $(0, T)$ into sufficiently small sub-intervals, taking cutoff functions in time, and applying an induction argument as in (Dong and Kim, 2021, Theorem 2.2), we get

$$\|u\|_{L_{p,w}(\mathbb{R}^d_T)} \leq N(T)\|f\|_{L_{p,w}(\mathbb{R}^d_T)},$$

which together with (3.3.30) implies (3.1.22).

For general b and c and $p = q$, we have

$$\partial_t u - Lu = f - b^i D_i u 1_{\sigma\geq 1} - cu.$$

Thus, when $\sigma > 1$ (3.3.30) follows from the case when $b = c = 0$ and Lemma 3.4.1. If $\sigma = 1$, then

$$\begin{aligned}&\|\partial_t u\|_{L_{p,w}(\mathbb{R}^d_T)} + \|Du\|_{L_{p,w}(\mathbb{R}^d_T)}\\ &\leq N\|f\|_{L_{p,w}(\mathbb{R}^d_T)} + N\|u\|_{L_{p,w}(\mathbb{R}^d_T)} + N\|bDu\|_{L_{p,w}(\mathbb{R}^d_T)}.\end{aligned} \tag{3.3.31}$$

If b is sufficiently small such that $N\|b\|_{L_\infty(\mathbb{R}^d_T)} \le 1/2$, then by absorbing the last term on the right-hand side of (3.3.31) to the left, we obtain (3.3.30). On the other hand, if $\alpha = 1$ and b is uniformly continuous, we take

$$B(t) = \int_0^t b(s,0)\,ds, \quad \tilde{b}(t,x) = b(t,x-B(t)), \quad \tilde{c}(t,x) = c(t,x-B(t)),$$

and

$$\tilde{u}(t,x) = u(t,x-B(t)), \quad \tilde{f}(t,x) = f(t,x-B(t)).$$

It follows that

$$\partial_t\tilde{u}(t,x) - \tilde{L}\tilde{u}(t,x) + (\tilde{b}^i(t,x) - b^i(t,0))D_i\tilde{u}(t,x) + \tilde{c}\tilde{u}(t,x) = \tilde{f}(t,x), \tag{3.3.32}$$

where $\tilde{L}$ is the operator with the kernel $\tilde{K}(t,x,y) = K(t,x-B(t),y)$. Then, by moving the last two terms on the left-hand side of (3.3.32) to the right-hand side and applying the estimate to $\tilde{u}$ together with a change of variables, we have

$$\|Du\|_{L_{p,w}(\mathbb{R}^d_T)} \le N\|f\|_{L_{p,w}(\mathbb{R}^d_T)} + N\|u\|_{L_{p,w}(\mathbb{R}^d_T)} + N\|(b - b(\cdot,0))Du\|_{L_{p,w}(\mathbb{R}^d_T)},$$

which implies that there exists $r_0 > 0$ depending on the modulus of continuity of b such that if u is supported on $(0,T) \times B_{r_0}$, then

$$\|Du\|_{L_{p,w}(\mathbb{R}^d_T)} \le N\|f\|_{L_{p,w}(\mathbb{R}^d_T)} + N\|u\|_{L_{p,w}(\mathbb{R}^d_T)}.$$

Therefore, by a partition of unity and (3.3.31), we arrive at (3.3.30), and (3.1.22) follows.

Finally, for general p, q, we use the extrapolation theorems in (Dong and Kim, 2018, Theorem 2.5) or (Cruz-Uribe et al., 2011, Theorem 1.4). The Corollary is proved. □

Remark 3.3.7. Indeed, r_0 and N in Corollary 3.3.6 can be taken to be dependent on $\tilde{\sigma}$ instead of σ due to the presence of $(2-\sigma)$ in the bound (3.1.4) in Assumption 3.1.3. Thus, based on the proof above, in Corollary 3.1.10 the dependence of N on σ can be reduced to the dependence on $\tilde{\sigma}$.

3.4 Miscellaneous Lemmas

Lemma 3.4.1 (An interpolation inequality in the spatial variables)**.** *Let $\sigma \in (1,2)$, $p \in (1,\infty)$, and $u \in H_p^\sigma(\mathbb{R}^d)$. For any $\varepsilon > 0$,*

$$\|Du\|_{L_p(\mathbb{R}^d)} \le N(d,\sigma,p)\varepsilon^{1/(1-\sigma)}\|u\|_{L_p(\mathbb{R}^d)} + \varepsilon\|(-\Delta)^{\sigma/2}u\|_{L_p(\mathbb{R}^d)}. \tag{3.4.1}$$

Proof. By taking $f := u + (-\Delta)^{\sigma/2}u$, we have

$$\mathcal{F}(Du)(\xi) = \frac{i\xi}{1+|\xi|^\sigma}\mathcal{F}(f)(\xi) =: m(\xi)\mathcal{F}(f)(\xi).$$

Since $\sigma > 1$, the multiplier $m(\xi)$ satisfies

$$\sup_{\xi\in\mathbb{R}^d} |\xi|^{|\gamma|}|\partial^\gamma m| \le N(d,\sigma,p),$$

for any multi-index γ of length $|\gamma| \le d/2+1$. Thus, by the Mikhlin multiplier theorem, we have

$$\|Du\|_{L_p(\mathbb{R}^d)} \le N\|f\|_{L_p(\mathbb{R}^d)} \le N\|u\|_{L_p(\mathbb{R}^d)} + N\|(-\Delta)^{\sigma/2}u\|_{L_p(\mathbb{R}^d)}, \tag{3.4.2}$$

where $N = N(d, \sigma, p)$. Then taking $v(\cdot) := u(\varepsilon \cdot)$ and applying (3.4.2) to v yield

$$\varepsilon \|Du\|_{L_p(\mathbb{R}^d)} \le N\|u\|_{L_p(\mathbb{R}^d)} + N\varepsilon^\sigma \|(-\Delta)^{\sigma/2} u\|_{L_p(\mathbb{R}^d)},$$

which implies (3.4.1). The lemma is proved. □

Lemma 3.4.2. *Let $\alpha \in (0, 1]$, $T \in (0, \infty)$, $p, q \in (1, \infty)$. For any $u \in \mathbb{H}^{\alpha,\sigma}_{p,q,0}(T)$, we have*

$$\|u\|_{L_{p,q}(\mathbb{R}^d_T)} \le NT^\alpha \|\partial_t^\alpha u\|_{L_{p,q}(\mathbb{R}^d_T)},$$

where $N = N(\alpha, p, q)$.

Proof. Note that $u = I^\alpha \partial_t^\alpha u$. Therefore, the result follows from (Dong and Kim, 2021, Lemma 5.5). □

Lemma 3.4.3. *Let L satisfy (3.1.3) when $\sigma = 1$ and (3.1.6). Recall the definition of $\tilde{\sigma}$ in (3.1.24). For $k = 0, 1, \ldots$, let ζ_k be the cutoff functions defined in (3.2.14). Under the same assumptions and the same notation as in Lemma 3.2.2, we have the following estimates for $I_k := \|\zeta_k Lu - L(\zeta_k u)\|_{L_p(\mathbb{R}^d_T)}$.*

1. *When $\sigma \in (0, 1)$,*

$$I_k \le N\frac{2^{\sigma k}}{(R-r)^\sigma}\|u\|_{p,R} + N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}} R^{d/p}(1 + R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}; \tag{3.4.3}$$

2. *when $\sigma \in (1, 2)$,*

$$\begin{aligned} I_k \le{}& N\frac{2^{(\sigma-1)k}}{(R-r)^{\sigma-1}}\|Du\|_{p,r_{k+3}} + N\frac{2^{\sigma k}}{(R-r)^\sigma}\|u\|_{p,R} \\ &+ N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}} R^{d/p}(1 + R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}; \end{aligned} \tag{3.4.4}$$

3. when $\sigma = 1$, for any $\varepsilon \in (0,1)$,

$$I_k \le \varepsilon^3 \|Du\|_{p,r_{k+3}} + N\frac{2^k}{R-r}\varepsilon^{-3}\|u\|_{p,R}$$
$$+ NR^{d/p}\left(1 + \frac{2^{(d+1)k}\varepsilon^{-3(d+1)}}{(R-r)^{d+1}}(1+R^{d+1})\right)\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}, \quad (3.4.5)$$

where $N = N(d,\Lambda,\tilde{\sigma})$ and $\|\cdot\|_{p,r} := \|\cdot\|_{L_p((0,T)\times B_r)}$ for any $r > 0$.

Proof. In this proof, we use N to denote a constant which may depend on d, Λ, and $\tilde{\sigma}$. Note that N does not depend on ν, the lower bound of the operator.

Next, let $\tilde{r}_k = r_{k+3} - r_{k+2}$, and note that

$$L(\zeta_k u)(t,x) - \zeta_k Lu(t,x)$$
$$= \int_{\mathbb{R}^d} \left((\zeta_k(x+y) - \zeta_k(x))u(t,x+y) - y\cdot\nabla\zeta_k(x)u(t,x)\chi^{(\sigma)}(y)\right)K(t,y)\,dy. \quad (3.4.6)$$

We are going to split the proof into 3 cases depending on the value of σ.

Case 1: $\sigma \in (0,1)$. In this case, by (3.4.6),

$$|L(\zeta_k u)(t,x) - \zeta_k Lu(t,x)|$$
$$\le N\int_{\mathbb{R}^d} |\zeta_k(x+y) - \zeta_k(x)||u(t,x+y)||y|^{-d-\sigma}\,dy$$
$$\le N\left(\int_{B_{\tilde{r}_k}} + \int_{B^c_{\tilde{r}_k}}\right)|\zeta_k(x+y) - \zeta_k(x)||u(t,x+y)||y|^{-d-\sigma}\,dy =: I^1_{k,1} + I^1_{k,2}.$$

For $I^1_{k,1}$, since $y \in B_{\tilde{r}_k}$, we have

$$|\zeta_k(x+y) - \zeta_k(x)| \le \|D\zeta_k\|_{L_\infty}|y|1_{|x|<r_{k+2}} \le N\frac{2^k}{R-r}|y|1_{|x|<r_{k+2}}, \quad (3.4.7)$$

which together with the Minkowski inequality implies that

$$\begin{aligned}
\|I^1_{k,1}\|_{L_p(\mathbb{R}^d_T)} &\le N\frac{2^k}{R-r}\int_{B_{\tilde r_k}} \|u(\cdot,\cdot+y)\|_{p,r_{k+2}}|y|^{1-d-\sigma}\,dy \\
&\le N\frac{2^k}{R-r}\|u\|_{p,R}\int_{B_{\tilde r_k}} |y|^{1-d-\sigma}\,dy \\
&\le N(d,\sigma_1)\frac{2^k}{R-r}\tilde r_k^{1-\sigma}\|u\|_{p,R} \le N\frac{2^{\sigma k}}{(R-r)^\sigma}\|u\|_{p,R}. \qquad (3.4.8)
\end{aligned}$$

On the other hand,

$$I^1_{k,2} \le \int_{B^c_{\tilde r_k}} \left(1_{|x+y|<r_{k+1}} + 1_{|x|<r_{k+1}}\right)|u(t,x+y)||y|^{-d-\sigma}\,dy := I^1_{k,2,1} + I^1_{k,2,2},$$

$$\|I^1_{k,2,1}\|_{L_p(\mathbb{R}^d_T)} \le \int_{B^c_{\tilde r_k}} |y|^{-d-\sigma}\,dy\|u\|_{p,R} \le N(d,\sigma_0)\tilde r_k^{-\sigma}\|u\|_{p,R} \le N\frac{2^{\sigma k}}{(R-r)^\sigma}\|u\|_{p,R}. \qquad (3.4.9)$$

By Hölder's inequality and the Minkowski inequality,

$$\begin{aligned}
\|I^1_{k,2,2}\|_{L_p(\mathbb{R}^d_T)} &\le N(d)r_{k+1}^{d/p}\left(\int_0^T \|I^1_{k,2,2}(t,\cdot)\|^p_{L_\infty(B_{r_{k+1}})}\,dt\right)^{1/p} \\
&\le Nr_{k+1}^{d/p}\frac{1+r_{k+3}^{d+\sigma}}{\tilde r_k^{d+\sigma}}\left(\int_0^T \|u(t,\cdot)\|^p_{L_1(\mathbb{R}^d,\psi)}\,dt\right)^{1/p} \\
&= Nr_{k+1}^{d/p}\frac{1+r_{k+3}^{d+\sigma}}{\tilde r_k^{d+\sigma}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))} \\
&\le N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}}R^{d/p}(1+R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}, \qquad (3.4.10)
\end{aligned}$$

where for the second inequality, we used the fact that if $|y| \ge \tilde r_k$ and $|x| < r_{k+1}$ then

$$\frac{1+|x+y|^{d+\sigma}}{|y|^{d+\sigma}} \le \frac{1+(r_{k+1}+\tilde r_k)^{d+\sigma}}{\tilde r_k^{d+\sigma}} \le \frac{1+r_{k+3}^{d+\sigma}}{\tilde r_k^{d+\sigma}}.$$

Combining (3.4.8), (3.4.9), and (3.4.10), we conclude

$$
\begin{aligned}
I_k &\le \|I^1_{k,1}\|_{L_p(\mathbb{R}^d_T)} + \|I^1_{k,2}\|_{L_p(\mathbb{R}^d_T)} \\
&\le N\frac{2^{\sigma k}}{(R-r)^\sigma}\|u\|_{p,R} + N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}}R^{d/p}(1+R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\end{aligned}
$$

Case 2: $\sigma \in (1,2)$. In this case, by (3.4.6),

$$
\begin{aligned}
&|L(\zeta_k u)(t,x) - \zeta_k Lu(t,x)| \\
&\le N(\Lambda)(2-\sigma)\int_{\mathbb{R}^d}|\zeta_k(x+y)-\zeta_k(x)||u(t,x+y)-u(t,x)||y|^{-d-\sigma}\,dy \\
&\quad + N(\Lambda)(2-\sigma)\int_{\mathbb{R}^d}|\zeta_k(x+y)-\zeta_k(x)-y\cdot\nabla\zeta_k(x)||u(t,x)||y|^{-d-\sigma}\,dy \\
&=: I^2_{k,1} + I^2_{k,2}.
\end{aligned}
$$

By (3.4.7) and the fundamental theorem of calculus,

$$
\begin{aligned}
I^2_{k,1} &\le N(2-\sigma)\left(\int_{B_{\tilde r_k}} + \int_{B^c_{\tilde r_k}}\right)|\zeta_k(x+y)-\zeta_k(x)||u(t,x+y)-u(t,x)||y|^{-d-\sigma}\,dy \\
&\le N(2-\sigma)\frac{2^k}{R-r}\int_{B_{\tilde r_k}}\int_0^1 1_{|x|<r_{k+2}}|\nabla u(t,x+sy)||y|^{2-d-\sigma}\,ds\,dy \\
&\quad + N\int_{B^c_{\tilde r_k}}\left(1_{|x+y|<r_{k+1}} + 1_{|x|<r_{k+1}}\right)|u(t,x+y)-u(t,x)||y|^{-d-\sigma}\,dy.
\end{aligned}
$$

Similar to (3.4.8), (3.4.9), and (3.4.10), we have

$$
\begin{aligned}
\|I^2_{k,1}\|_{L_p(\mathbb{R}^d_T)} &\le N\frac{2^{(\sigma-1)k}}{(R-r)^{\sigma-1}}\|Du\|_{p,r_{k+3}} + N\frac{2^{\sigma k}}{(R-r)^\sigma}\|u\|_{p,R} \\
&\quad + N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}}R^{d/p}(1+R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\end{aligned}
\tag{3.4.11}
$$

For $I_{k,2}^2$, when $y \in B_{\tilde{r}_k}$, the mean value theorem leads to

$$\begin{aligned}
&|\zeta_k(x+y) - \zeta_k(x) - y \cdot \nabla\zeta_k(x)| \\
&\leq \|D^2\zeta_k\|_{L_\infty}|y|^2 1_{|x|<r_{k+2}} \leq N\frac{2^{2k}}{(R-r)^2}|y|^2 1_{|x|<r_{k+2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_{k,2}^2 &\leq N(2-\sigma)\bigg(\int_{B_{\tilde{r}_k}} + \int_{B_{\tilde{r}_k}^c}\bigg)|\zeta_k(x+y) - \zeta_k(x) - y\cdot\nabla\zeta_k(x)||u(t,x)||y|^{-d-\sigma}\,dy \\
&\leq N(2-\sigma)\frac{2^{2k}}{(R-r)^2}|u(t,x)|1_{|x|<r_{k+2}}\int_{B_{\tilde{r}_k}}|y|^{2-d-\sigma}\,dy \\
&\quad + N\int_{B_{\tilde{r}_k}^c}\bigg(1_{|x+y|<r_{k+1}} + 1_{|x|<r_{k+1}}(1+\frac{2^k}{(R-r)}|y|)\bigg)|u(t,x)||y|^{-d-\sigma}\,dy,
\end{aligned}$$

which, similar to (3.4.8), (3.4.9), and (3.4.10), implies that

$$\begin{aligned}
\|I_{k,2}^2\|_{L_p(\mathbb{R}_T^d)} &\leq N\frac{2^{\sigma k}}{(R-r)^\sigma}\|u\|_{p,R} \\
&\qquad + N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}}R^{d/p}(1+R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\end{aligned} \tag{3.4.12}$$

By (3.4.11) and (3.4.12),

$$\begin{aligned}
I_k &\leq \|I_{k,1}^2\|_{L_p(\mathbb{R}_T^d)} + \|I_{k,2}^2\|_{L_p(\mathbb{R}_T^d)} \\
&\leq N\frac{2^{(\sigma-1)k}}{(R-r)^{\sigma-1}}\|Du\|_{p,r_{k+3}} + N\frac{2^{\sigma k}}{(R-r)^\sigma}\|u\|_{p,R} \\
&\quad + N\frac{2^{(d+\sigma)k}}{(R-r)^{d+\sigma}}R^{d/p}(1+R^{d+\sigma})\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\end{aligned}$$

Case 3: $\sigma = 1$. In this case, by (3.4.6) and (3.1.3), for any $\delta_k > 0$ to be determined,

$$
\begin{aligned}
&|L(\zeta_k u)(t,x) - \zeta_k Lu(t,x)| \\
&\leq N \int_{B_{\delta_k}} |\zeta_k(x+y) - \zeta_k(x)||u(t,x+y) - u(t,x)||y|^{-d-1}\,dy \\
&\quad + N \int_{B_{\delta_k}} |\zeta_k(x+y) - \zeta_k(x) - y\cdot\nabla\zeta_k(x)||u(t,x)||y|^{-d-1}\,dy \\
&\quad + N \int_{B_{\delta_k}^c} |\zeta_k(x+y) - \zeta_k(x)||u(t,x+y)||y|^{-d-1}\,dy \\
&=: I_{k,1}^3 + I_{k,2}^3 + I_{k,3}^3.
\end{aligned}
$$

The estimates of $I_{k,1}^3$, $I_{k,2}^3$, and $I_{k,3}^3$ are similar to that of $I_{k,1}^2$, $I_{k,2}^2$, and $I_{k,2}^1$ respectively. It follows that

$$
\|I_{k,1}^3\|_{L_p(\mathbb{R}^d_T)} \leq N\frac{2^k}{(R-r)}\delta_k\|Du\|_{p,r_{k+3}}, \quad \|I_{k,2}^3\|_{L_p(\mathbb{R}^d_T)} \leq N\frac{2^{2k}}{(R-r)^2}\delta_k\|u\|_{p,R},
$$

and

$$
\|I_{k,3}^3\|_{L_p(\mathbb{R}^d_T)} \leq N\delta_k^{-1}\|u\|_{p,R} + NR^{d/p}\frac{1+(R+\delta_k)^{d+1}}{\delta_k^{d+1}}\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))},
$$

where N is independent of the choice of δ_k. For any $\varepsilon \in (0,1)$, let $\delta_k = \varepsilon^3\tilde{r}_k/N$. Then,

$$
\begin{aligned}
I_k &\leq \|I_{k,1}^3\|_{L_p(\mathbb{R}^d_T)} + \|I_{k,2}^3\|_{L_p(\mathbb{R}^d_T)} + \|I_{k,2}^3\|_{L_p(\mathbb{R}^d_T)} \\
&\leq \varepsilon^3\|Du\|_{p,r_{k+3}} + N\frac{2^k}{R-r}\varepsilon^{-3}\|u\|_{p,R} \\
&\quad + NR^{d/p}\left(1 + \frac{2^{(d+1)k}\varepsilon^{-3(d+1)}}{(R-r)^{d+1}}(1+R^{d+1})\right)\|u\|_{L_p((0,T);L_1(\mathbb{R}^d,\psi))}.
\end{aligned}
$$

The lemma is proved. □

Assumption 3.4.4. For given $E \subset F \subset (-\infty, T)$, if

$$|\mathcal{C}_R(t,x) \cap E| \geq \gamma |\mathcal{C}_R(t,x)|$$

for $(t,x) \in (-\infty, T] \times \mathbb{R}^d$ and $R > 0$, then

$$\widehat{\mathcal{C}}_R(t,x) \subset F,$$

where $\mathcal{C}_R(t,x)$ and $\widehat{\mathcal{C}}_R(t,x)$ are defined in (3.2.42).

Lemma 3.4.5 (Crawling of ink spots)**.** *Let $\gamma \in (0,1)$ and $|E| < \infty$. Suppose that $E \subset F \subset (-\infty, T) \times \mathbb{R}^d$ satisfy Assumption 3.4.4. Then*

$$|E| \leq N(d,\alpha)\gamma |F|.$$

Proof. See (Dong and Kim, 2019, Lemma A.20). □

Lemma 3.4.6 (Sobolev embeddings)**.** *Let $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $p \in (1,\infty)$, and $u \in \mathbb{H}^{\alpha,\sigma}_{p,0}(T)$.*

1. *If $p < d/\sigma + 1/\alpha$, then there exists $q \in (p, \infty)$ satisfying*

$$1/q = 1/p - \alpha\sigma/(\alpha d + \sigma)$$

such that for all $l \in [p,q]$,

$$\|u\|_{L_l(\mathbb{R}^d_T)} \leq N \|u\|_{\mathbb{H}^{\alpha,\sigma}_p(T)}, \tag{3.4.13}$$

where $N = N(d,\alpha,\sigma,p,l,T)$.

2. *If $p = d/\sigma + 1/\alpha$, then the same estimate holds for $l \in [p, \infty)$.*

3. If $p > d/\sigma + 1/\alpha$, then

$$\|u\|_{L_\infty(\mathbb{R}^d_T)} \leq N\|u\|_{\mathbb{H}^{\alpha,\sigma}_p(T)},$$

where $N = N(d, \alpha, \sigma, p, T)$.

Proof. (1) By the mixed derivative theorem Sobolevskii (1964), we have

$$\mathbb{H}^{\alpha,\sigma}_{p,0}(T) \hookrightarrow H^{\alpha(1-\theta)}_p([0,T]; H^{\sigma\theta}_p(\mathbb{R}^d)) \quad \text{for all } \theta \in [0,1].$$

Let $\theta = \alpha d/(\alpha d + \sigma)$ and

$$1/q = 1/p - \alpha(1-\theta) = 1/p - \sigma\theta/d = 1/p - \alpha\sigma/(\alpha d + \sigma).$$

The Sobolev embeddings

$$H^{\alpha(1-\theta)}_p([0,T]) \hookrightarrow L_q([0,T]) \quad \text{and } H^{\sigma\theta}_p(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d),$$

imply that

$$\begin{aligned}\|u\|_{L_q(\mathbb{R}^d_T)} &\leq N\Big(\int_0^T \|u(t,\cdot)\|^q_{H^{\sigma\theta}_p(\mathbb{R}^d)} dt\Big)^{1/q} \\ &\leq N(T)\|u\|_{H^{\alpha(1-\theta)}_p([0,T];H^{\sigma\theta}_p(\mathbb{R}^d))} \leq N\|u\|_{\mathbb{H}^{\alpha,\sigma}_p(T)}.\end{aligned}$$

Since (3.4.13) holds for $l = p$, it holds for all $l \in [p, q]$ by Hölder's inequality.

(2) Using the embedding

$$H^{\alpha(1-\theta)}_p([0,T]) \hookrightarrow L_l([0,T]) \quad \text{and } H^{\sigma\theta}_p(\mathbb{R}^d) \hookrightarrow L_l(\mathbb{R}^d),$$

the proof is similar to that of (1).

(3) Due to the facts that $\alpha(1-\theta)p > 1$ and $\sigma\theta p > d$, and the Sobolev embeddings

$$H_p^{\alpha(1-\theta)}([0,T]) \hookrightarrow L_\infty([0,T]) \quad \text{and } H_p^{\sigma\theta}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d),$$

we have

$$\|u\|_{L_\infty(\mathbb{R}^d_T)} \leq N(T)\|u\|_{H_p^{\alpha(1-\theta)}([0,T];H_p^{\sigma\theta}(\mathbb{R}^d))} \leq N\|u\|_{\mathbb{H}_p^{\alpha,\sigma}(T)}.$$

□

Next, we prove an embedding into Hölder spaces. Note that we only use the result when $\alpha = 1$ for chapter 3.

Lemma 3.4.7 (A Hölder estimate)**.** *Let* $\alpha \in (0,1]$, $\sigma \in (0,2)$, $T \in (0,\infty)$, $p \in (1,\infty)$, $p \in (d/\sigma + 1/\alpha, 2d/\sigma + 2/\alpha)$, *and* $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}(T)$. *There exists* $\tau = \sigma - (d + \sigma/\alpha)/p \in (0,1)$ *such that*

$$\|u\|_{C^{\tau\alpha/\sigma,\tau}(\mathbb{R}^d_T)} \leq N\|u\|_{\mathbb{H}_p^{\alpha,\sigma}(T)},$$

where $N = N(d,\alpha,\sigma,p,T)$.

Proof. Without loss of generality, we assume that $u \in C_0^\infty([0,T]\times\mathbb{R}^d)$ with $u(0,\cdot) = 0$. Denote

$$K = \sup\Big\{\frac{|u(t_1,x) - u(t_2,y)|}{|t_1 - t_2|^{\tau\alpha/\sigma} + |x-y|^\tau} : (t_1,x),(t_2,y) \in \mathbb{R}^d_T, 0 < |t_1 - t_2|^{\alpha/\sigma} + |x-y| \leq 1\Big\}$$

and

$$\rho = \varepsilon(|t_1 - t_2|^{\alpha/\sigma} + |x - y|),$$

where $\varepsilon \in (0, 1)$ is a constant to be determined.

Note that by Lemma 3.4.6 and the fact that $p > d/\sigma + 1/\alpha$, it suffices to prove

$$K \leq N\|u\|_{\mathbb{H}_p^{\alpha,\sigma}(T)}. \tag{3.4.14}$$

To this end, we write

$$|u(t_1, x) - u(t_2, y)| \leq |u(t_1, x) - u(t_2, x)| + |u(t_2, x) - u(t_2, y)| := J_1 + J_2. \tag{3.4.15}$$

For J_1, by taking $z \in B_\rho(x)$ and the triangle inequality, we have

$$\begin{aligned} J_1 &\leq |u(t_1, x) - u(t_1, z)| + |u(t_2, x) - u(t_2, z)| + |u(t_1, z) - u(t_2, z)| \\ &\leq 2K\rho^\tau + |u(t_1, z) - u(t_2, z)| \\ &\leq 2K\rho^\tau + N(\alpha, p)|t_1 - t_2|^{\alpha - 1/p}\|\partial_t^\alpha u(\cdot, z)\|_{L_p(0,T)}, \end{aligned} \tag{3.4.16}$$

where for the last inequality, we used (Dong and Kim, 2019, Lemma A.14) together with the fact that $\alpha > 1/p$. Then by taking the average over $B_\rho(x)$ on both sides of (3.4.16) together with Hölder's inequality, we get

$$\begin{aligned} J_1 &\leq 2K\rho^\tau + N(d, \alpha, p)|t_1 - t_2|^{\alpha - 1/p}\rho^{-d/p}\|\partial_t^\alpha u\|_{L_p(\mathbb{R}^d_T)} \\ &\leq 2K\rho^\tau + N(d, \alpha, p)\varepsilon^{\sigma/(\alpha p) - \sigma}\rho^\tau\|u\|_{\mathbb{H}_p^{\alpha,\sigma}(T)}. \end{aligned} \tag{3.4.17}$$

On the other hand, for $s \in I := (t_2 - \rho^{\sigma/\alpha}, t_2 + \rho^{\sigma/\alpha}) \cap (0,T)$,

$$\begin{aligned} J_2 &\le |u(t_2,x) - u(s,x)| + |u(t_2,y) - u(s,y)| + |u(s,x) - u(s,y)| \\ &\le 2K\rho^\tau + |u(s,x) - u(s,y)| \\ &\le 2K\rho^\tau + N(d,\sigma,p)|x-y|^{\sigma - d/p}\|u(s,\cdot)\|_{H_p^\sigma(\mathbb{R}^d)}, \end{aligned} \tag{3.4.18}$$

where for the last inequality, we used the embedding $H_p^\sigma(\mathbb{R}^d) \hookrightarrow C^{\sigma - d/p}(\mathbb{R}^d)$. Then by taking the average over I on both sides of (3.4.18) together with Hölder's inequality, we obtain

$$\begin{aligned} J_2 &\le 2K\rho^\tau + N(d,\sigma,p)|x-y|^{\sigma-d/p}\rho^{-\sigma/(\alpha p)}\|u\|_{L_p((0,T);H_p^\sigma(\mathbb{R}^d))} \\ &\le 2K\rho^\tau + N(d,\sigma,p)\varepsilon^{d/p-\sigma}\rho^\tau\|u\|_{\mathbb{H}_p^{\alpha,\sigma}(T)}. \end{aligned} \tag{3.4.19}$$

Note that N is independent of T because we can assume that $|I| \ge \rho^{\sigma/\alpha}$ by extending $u(t,\cdot)$ to be zero for $t < 0$.

Combining (3.4.15), (3.4.17), and (3.4.19), we have

$$|u(t_1,x) - u(t_2,y)| \le 4K\rho^\tau + N(\varepsilon^{\sigma/(\alpha p)-\sigma} + \varepsilon^{d/p-\sigma})\rho^\tau\|u\|_{\mathbb{H}_p^{\alpha,\sigma}(T)},$$

where $N = N(d,\alpha,\sigma,p)$, which implies that

$$K \le 4\varepsilon^\tau K + N(\varepsilon^{-d/p} + \varepsilon^{-\sigma/(\alpha p)})\rho^\tau\|u\|_{\mathbb{H}_p^{\alpha,\sigma}(T)}.$$

Thus, by picking sufficiently small ε so that $4\varepsilon^\tau < 1/2$, we arrive at (3.4.14), and the lemma is proved. $\square$

3.5 Equations with Non-zero Initial Conditions

In this section, we consider equations with non-zero initial conditions. In particular, we focus on the case when $\alpha = 1$, i.e., the order of the derivative in time is 1, with A_p weights (c.f. Corollary 3.1.10),

$$\begin{cases} \partial_t u - Lu + b^i D_i u 1_{\sigma>1} + cu = f & \text{in} \quad \mathbb{R}^d_T \\ u(0,\cdot) = u_0(\cdot) & \text{in} \quad \mathbb{R}^d, \end{cases} \tag{3.5.1}$$

where u_0 is taken from an appropriate trace space defined below and L, b^i, c satisfy the assumptions in Corollary 3.1.10. When $\sigma = 1$ and either $\|b\|_\infty$ is sufficiently small or b is uniformly continuous, a similar result holds for (3.1.23) by the same proof. For general $\alpha \in (0,1)$ when there are no A_p weights, see (Kim et al., 2021, Lemma 5.4).

Before we introduce the results, we define the trace space as follows. For $p, q \in (1,\infty)$, a non-integer $\theta > 0$ with k being the integer part of θ, and $w_2 \in A_q(\mathbb{R}^d)$, we define

$$\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d) := \{u \in L_{q,w_2}(\mathbb{R}^d) : \|u\|_{\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d)} < \infty\},$$

where

$$\|u\|_{\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d)} = [u]_{\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d)} + \sum_{|\alpha|\le k} \|D^\alpha u\|_{L_{q,w_2}(\mathbb{R}^d)}$$

and

$$[u]_{\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \|D^k u(\cdot + y) - D^k u(\cdot)\|^p_{L_{q,w_2}} |y|^{-d-p(\theta-k)}\, dy\right)^{1/p}.$$

Note that when $w_2 \equiv 1$, $\mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d) = \mathcal{B}^\theta_{p,q}(\mathbb{R}^d)$, the usual Besov space. When $\theta > 0$ is a non-integer, we denote $\mathcal{X}_0 = \mathcal{B}^\theta_{p,q,w_2}(\mathbb{R}^d)$, and when θ_0 is a nonnegative integer, we denote $\mathcal{X}_0 = \mathcal{B}^{\theta+\varepsilon}_{p,q,w_2}(\mathbb{R}^d)$ for arbitrary $\varepsilon > 0$.

Our result for equations with non-zero initial conditions is

Theorem 3.5.1. *Let $\sigma \in (0,2)$, $T \in (0,\infty)$, $p,q \in (1,\infty)$, $M_1 \in [1,\infty)$, $\mu \in (-1,p-1)$, and $w = w_1(t)w_2(x)$, where $w_1(t) = t^\mu$ and $w_2(x) \in A_q(\mathbb{R}^d, dx)$ with $[w_2]_{A_q} \le M_1$. Suppose that the kernel $K = K(t,x,y)$ satisfies Assumptions 3.1.1 and 3.1.3, and (3.1.13) holds. Let $\theta = \sigma - \sigma(1+\mu)/p (\in (0,\sigma))$, and recall the definitions of the spaces $\mathcal{X}_0$ above. For any $f \in L_{p,q,w}(\mathbb{R}^d_T)$ and $u_0 \in \mathcal{X}_0$, there exists a unique function u in $\mathbb{R}^d_T$ satisfying (3.5.1) with the estimate*

$$\|\partial_t u\|_{p,q,w} + \|u\|_{p,q,w} + \|(-\Delta)^{\sigma/2} u\|_{p,q,w} \le N\|f\|_{p,q,w} + N\|u_0\|_{\mathcal{X}_0}, \tag{3.5.2}$$

where $\|\cdot\|_{p,q,w} = \|\cdot\|_{L_{p,q,w}(\mathbb{R}^d_T)}$ and $N = N(d,\nu,\Lambda,\sigma,p,q,T,M_1,M,\beta,\omega,\mu)$.

The proof of Theorem 3.5.1 is similar to the proof of Theorem 2.1.6. For details, see Section 2.8 or (Dong and Liu, 2023, Appendix B).

Chapter Four

Future Problems

In the future, it is possible to extend the results by considering equations in domains and divergence form equations. Indeed, some interesting work has been done in those directions. For example, solutions for equations in domains were studied in Baeumer et al. (2018) using the probabilistic method. Divergence form equations were also studied in Allen et al. (2016), Mengesha et al. (2021) with various assumptions on the kernel K. More precisely, Hölder estimates and L_p estimates were obtained for the divergence form equations in Allen et al. (2016) and Mengesha et al. (2021), respectively. Furthermore, there are results about divergence form non-local elliptic equations in Kuusi et al. (2015), Nowak (2020, 2021), where more general operators were considered. For example, in Nowak (2021), the authors obtained some regularity results for operators with VMO kernels and non-linearity. In order to prove the corresponding results for the parabolic equation (3.1.17) with VMO kernels, the main difficult is to prove a version of Gehring's Lemma. We tried to adapt the exiting time argument in Kuusi et al. (2015), but was unsuccessful.

Another interesting question is whether the weighted mixed-norm estimates still hold for (1.2.1) when $\alpha \in (0, 1)$, where the weights are taken over the time or the spatial variables. See Remarks 3.3.2 and 3.3.4 for more information.

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