Low-temperature renormalization-group study of the random-axis model

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Momentum-shell recursion relations valid for low temperatures and small anisotropy are generated for the random-axis model of amorphous magnetism. The fixed-point structure of these relations suggests that ferromagnetism is absent below four dimensions. The critical behavior along the ferromagnetic—spin-glass phase boundary above four dimensions is explored, and, at least to first order in $\epsilon = d - 4$, the exponents, hyperscaling law, and behavior of the longitudinal susceptibility are identical to a nonrandom model in two dimensions less. We also present an attempt at a Mermin-Wagner proof of the absence of ferromagnetism below four dimensions, utilizing the replica method.

I. INTRODUCTION

The random-axis model was introduced by Harris, Plischke, and Zuckermann[1] to describe the magnetic properties of amorphous alloys containing rare-earth elements with asymmetric charge distributions (e.g., Tb or Dy). The asymmetry leads to a local easy axis of magnetization when the rare-earth ion interacts with the crystal field. Given the amorphous nature of the system one would expect this local easy axis to vary randomly from site to site. Harris *et al.* proposed the following Hamiltonian to incorporate the random-axis of magnetization:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j - D \sum_i (\hat{x}_i \cdot \vec{s}_i)^2 \quad , \tag{1.1}$$

where $\vec{s}$ is an n-component unit spin. The nearest-neighbor coupling J is assumed to be uniform and the randomness is introduced in the second term, where $\hat{x}_i$ is a random direction of site i. The directions $\hat{x}_i$ are assumed to be uncorrelated from site to site, and distributed uniformly over a unit sphere. Fluctuations in the anisotropy strength D are ignored. Numerical simulations[2] support the assumption of no correlations and a uniform distribution.

Previous work on this model has involved mean-field theory, numerical simulations, and renormalization-group calculations. A mean-field calculation by Harris and Zobin[3] predicts a phase diagram with both spin-glass and ferromagnetic states, depending on the degree of disorder. However, the classical zero-temperature mean-field analysis of Callen, Liu, and Cullen[4] indicates that the spin-glass state is metastable and the ground state is ferromagnetic for any value of anistropy. Patterson *et al.*[5] have also found only a ferromagnetic ground state by analyzing a local-mean-field approximation.

Monte Carlo calculations by Harris and Zobin[6] showed that the ferromagnetic state has lower energy than the spin-glass state for all ratios of D/J, but the difference in energy is comparable to round-off errors. Chi and Alben[7] have also numerically simulated the model and found that the spin-glass state is metastable, while the ground state is ferromagnetic even at large anisotropy.

The critical properties of the paramagnetic to ferromagnetic transition have been investigated within a $4-\epsilon$ expansion by Aharony.[8] Using renormalization-group techniques, he generated recursion relations and discovered that all flows from physically realizable initial Hamiltonians were "to infinity," or, more precisely, out of the range of validity of the ϵ expansion. Chen and Lubensky[9] assumed there is no magnetization for sufficiently large anisotropy, and derived an effective free energy of the same form as the random-bond Ising spin-glass. They speculated that the runaway seen by Aharony leads to the spin-glass fixed point studied by Harris, Lubensky, and Chen.[10]

This paper presents the details of a study of this problem using a low-temperature renormalization group and an expansion in powers of $\epsilon = d - 4$. The results were summarized in a recent Letter by Pelcovits, Pytte and Rudnick,[11] who found that in a quenched system, random anisotropy with an isotropic distribution of axes destroys ferromagnetism in fewer than four dimensions. The low-temperature phase may instead be an Edwards-Anderson (EA) spin-glass.[12] In more than four dimensions, either ferromagnetic or spin-glass ordering is found, depending on the degree of disorder.

A schematic phase diagram above four dimensions which summarizes our conclusions is shown in Fig. 1, where we plot D^2 vs T (the results are independent of the sign of D). The $\epsilon = d - 4$ expansion locates a fixed point at $T^* = 0$, $D^{2*} = O(\epsilon)$ that governs the behavior at the phase boundary separating the ferromagnetic state from a low-temperature large-

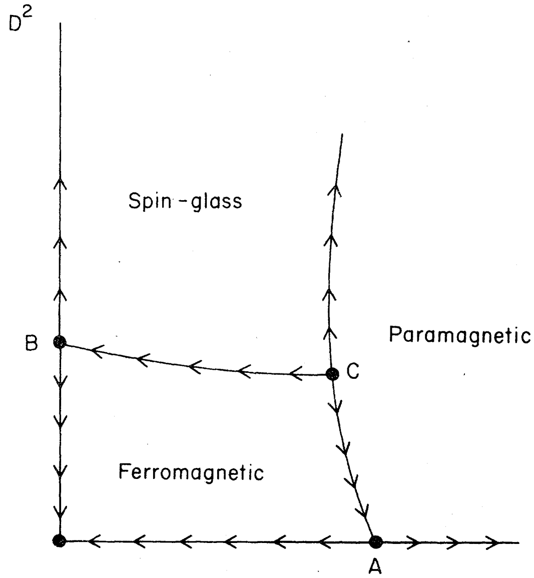

FIG. 1. Schematic phase diagram and renormalization-group flows for the random-axis model above four dimensions. Point A is the Gaussian fixed point, while point B is a fixed point located at $D^2=O(d-4)$, $T=0$. Point C is a multicritical point.

anisotropy state. An exact $n=\infty$ solution[13] explicitly demonstrates that the latter state is an EA spin-glass. The ferromagnetic-paramagnetic transition has been studied by an expansion in powers of $\epsilon=6-d$.[14] The exponents for the ferromagnetic–spin-glass transition coincide at least to first order in $\epsilon=d-4$ with the exponents for nonrandom n-vector models[15,18,19] in two fewer dimensions. As d approaches four, this phase boundary drops to $D=0$, and the ferromagnetic phase disappears.

This paper is organized as follows: in Sec. II we construct momentum-shell recursion relations for the random-axis model in d dimensions valid for low temperatures and small values of the anisotropy. In Sec. III we use these recursion equations to calculate the critical exponents and hyperscaling relation along the ferromagnetic–spin-glass phase boundary, and to derive the asymptotic behavior of the longitudinal susceptibility in the ferromagnetic phase. We offer some concluding remarks in Sec. IV.

In the Appendix we present an attempt at a Mermin-Wagner[16,17] type proof of the absence of ferromagnetism below four dimensions in this model. However, our demonstration utilizes the replica trick, and thus cannot be considered an ironclad proof.

II. MOMENTUM-SHELL RECURSION RELATIONS

We implement the Migdal-Polyakov low-temperature renormalization group[17–19] by constructing momentum-shell recursion relations, following an adaption of the method used by Nelson and Pelcovits.[20] Because the approach is based on spin-wave ideas,[20] we can only treat small values of the temperature and anisotropy, i.e., T/J, $D/J \ll 1$.

Amorphous magnetic alloys made by sputtering are quenched (versus annealed) random systems, where the impurities are fixed and cannot immediately reach thermal equilibrium with the host system. Computation of a thermodynamic quantity in a quenched system requires first a calculation of the quantity for a fixed configuration of impurities, followed by a configurational averaging.[21] Thus in the quenched random-axis model, the magnetization is given by

$$M=\langle\langle \vec{s}\rangle\rangle_c=\prod_i \int d\hat{x}_i\, M_{\{\hat{x}\}} =\prod_i \int d\hat{x}_i\, Z^{-1}_{\{\hat{x}\}}\,\mathrm{Tr}\,\vec{s}\,e^{-\beta\mathcal{H}} \;. \tag{2.1}$$

Ideally we would like to integrate over the random parameters first and remain with an effective translationally invariant system. This integration is, in general, very difficult because of the Z^{-1} factor in Eq. (2.1), which is a function of the axis configuration. Usually replicas are introduced to accomplish this integration.[22] (The replica method will be described in the Appendix.) Recursion relations, however, are straightforward to generate,[23] without the use of the replica trick. We characterize the system by the probability distribution of its couplings, and generate renormalization-group equations for the distribution.[24] This procedure does indeed correspond to the quenched problem and is usually implemented by developing recursion relations for the cumulants of the distribution. Specifically, we will calculate the new couplings after integrating out short-wavelength degrees of freedom, and then construct equations governing the renormalization of the cumulants.

The reduced Hamiltonian arising from the continuum version of Eq. (1.1) is

$$\mathcal{H}=-\frac{\mathcal{H}}{k_BT}=\int d^dx\left[-\frac{1}{2T}(\partial_\mu\vec{s})^2-\frac{D}{2T}(\hat{x}\cdot\vec{s})^2\right] \;, \tag{2.2}$$

where $\vec{s}^{\,2}=1$. Following Refs. 15 and 20, we write $\vec{s}=(\sigma,\vec{\pi})$ and assume that the mean magnetization is along the σ direction with small fluctuations about this direction, i.e., $\langle\pi^2\rangle \ll 1$. Using the fixed-length constraint

$$\sigma^2+\vec{\pi}^{\,2}=1 \;, \tag{2.3}$$

we integrate out σ, and expand $\mathcal{H}$ in powers of π. We then obtain

$$\bar{\mathcal{H}} = \int d^dx \left[-\frac{1}{2T}(\partial_\mu \vec{\pi})^2 - \frac{1}{2T}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + \cdots \text{ (nonrandom terms)} \right.$$
$$\left. -\frac{D}{2T}\left(\sum_i^{n-1} (x_i^2 - x_\sigma^2)\pi_i^2 + 2\sum_{i>j} x_i x_j \pi_i \pi_j + 2\sum_i^{n-1} x_i x_\sigma \pi_i - x_\sigma \vec{\pi}^2 \sum_i x_i \pi_i + \cdots \right) \right] , \tag{2.4}$$

where x_i is the ith component of $\hat{x}$, and x_σ is the component of $\hat{x}$ along the σ direction.

We observe two important features of Eq. (2.4). First, upon averaging over the random directions, the theory is massless. This can be shown to all orders,[13] and is easily seen at lowest order since

$$\langle x_i^2 - x_\sigma^2 \rangle_c = 0 \; , \; \langle x_i x_\sigma \rangle_c = 0 \; , \tag{2.5}$$

where $\langle \; \rangle_c$ denotes angular averaging. Physically, this statement means that the system is macroscopically isotropic. Additionally, we note the presence of the term linear in π, which corresponds to a transverse random-field with zero mean, and links this problem to the random-*field* model, where it has also been shown that the lower critical dimensionality is 4.[25] Indeed, we shall see that this term plays the most important role in the theory.

For compact notation we define

$$A_i \equiv x_i^2 - x_\sigma^2 \; ,$$
$$B_{ij} \equiv x_i x_j \; , \quad i \neq j$$
$$B_{i\sigma} \equiv x_i x_\sigma \; . \tag{2.6}$$

The angular averages of the above quantities are all zero. The following averages are also zero:

$$\langle A_i B_{ij} \rangle_c = \langle A_i B_{i\sigma} \rangle_c = \langle B_{ij} B_{i\sigma} \rangle_c = 0 \; . \tag{2.7}$$

We also note that

$$D^2 \langle A_i^2 \rangle_c = 4\Delta \; ,$$
$$D^2 \langle B_{ij} B_{kl} \rangle_c = \Delta(\delta_{ij}\delta_{jl} + \delta_{il}\delta_{jk}) \; ,$$
$$D^2 \langle B_{i\sigma} B_{j\sigma} \rangle_c = \Delta \delta_{ij} \; , \tag{2.8}$$

where

$$\Delta = D^2/n(n+2) \tag{2.9}$$

and the quantities in angular brackets are both at the same *site*, since axes at different sites are uncorrelated.

We arrive at a recursion relation for T by considering the renormalization of $(\partial_\mu \vec{\pi})^2$. This procedure yields, after averaging over the axes,

$$\frac{1}{T'} = \zeta^2 b^{-d-2} \left(\frac{1}{T} + \ln b \, K_d + \ln b \, K_d \Delta + O(T, D^4) \right) , \tag{2.10}$$

where $K_d = 2^{-d+1}\pi^{-d/2}/\Gamma(d/2)$ and the graphs have been evaluated in d dimensions. The graphs entering Eq. (2.10) are shown in Fig. 2. The graph proportional to D^2 arises from the contraction of $(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2$ with *two* terms linear in $\vec{\pi}$, and thus the integrand is proportional to $1/q^4$, where q is the momentum carried by the internal line. This double propagator graphically illustrates why the lower critical dimensionality is 4 instead of 2.

To determine the spin-rescaling factor ζ, we add a magnetic field h to $\bar{\mathcal{H}}$,

$$\bar{\mathcal{H}} \to \bar{\mathcal{H}} + \frac{h}{T}\int d^dx$$
$$= \bar{\mathcal{H}} + \frac{h}{T}\int d^dx \left(1 - \tfrac{1}{2}\vec{\pi}^2 - \tfrac{1}{8}\vec{\pi}^4 + \cdots \right) . \tag{2.11}$$

We generate a recursion relation for h/T by considering the renormalization of $(h/2T)\vec{\pi}^2$, obtaining after averaging over the randomness,

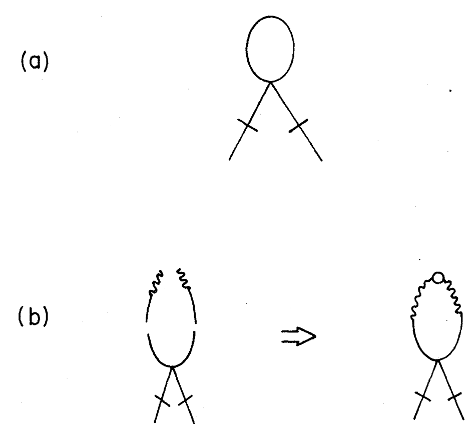

FIG. 2. Graphs entering Eq. (2.10). Slashes denote derivatives. (a) Graph also appearing in nonrandom system. (b) Graph proportional to D^2, formed by contracting a four-point coupling term with two random transverse-field terms and averaging. The wavy lines denote $B_{i\sigma}$ [see Eq. (2.6)], while the open circle joining them corresponds to angular averaging.

$$-\frac{h'}{2T'}=\zeta^2 b^{-d}\Bigg[-\frac{h}{2T}-\frac{h}{4}\frac{n-1}{1+h}K_d\ln b-\frac{hD^2}{4T}\frac{(n+1)\langle B_{i\sigma}^2\rangle_c}{(1+h)^2}K_d\ln b-\frac{D^2}{2T}\frac{\langle B_{i\sigma}^2\rangle_c}{(1+h)^2}K_d\ln b$$
$$+\frac{1}{2}\frac{D^2}{T}\frac{\langle A_i^2\rangle_c}{1+h}K_d\ln b+\frac{1}{2}\frac{D^2}{T}(n-2)\frac{\langle B_{ij}^2\rangle_c}{1+h}K_d\ln b$$
$$-\frac{1}{2}\frac{D^2}{T}\frac{K_d\ln b}{1+h}[(n-1)\langle B_{i\sigma}^2\rangle_c+2\langle B_{i\sigma}^2\rangle_c]\Bigg] \quad . \qquad (2.12)$$

The graphs corresponding to the last five terms in the right-hand side of the above equation appear in Figs. 3(a)–3(e), respectively. The first two terms in the right-hand side appear even when $D=0$.[20]

Combining terms in Eq. (2.12), we find

$$\frac{h'}{T'}=\zeta^2 b^{-d}\Bigg[\frac{h}{T}+\frac{h}{2T}\frac{(n-1)(\Delta+T)}{(1+h)^2}K_d\ln b+hO(\Delta^2,T^2,T\Delta)\Bigg] \quad . \qquad (2.13)$$

By rotational symmetry (which is preserved in the configurationally averaged system) the magnetic field renormalizes trivially as[26]

$$h'/T'=\zeta h/T \quad . \qquad (2.14)$$

Combining Eqs. (2.13) and (2.14), we then obtain for the spin-rescaling factor as $h\to 0$,

$$\zeta=b^d[1-\tfrac{1}{2}(n-1)(T+\Delta)K_d\ln b+O(T^2,\Delta^2,T\Delta)] \quad . \qquad (2.15)$$

To complete the renormalization-group transformation, we derive recursion equations for the cumulants of the probability distribution of $\hat{x}$. First, consider the renormalization of $(D/T)B_{i\sigma}\pi_i$, which yields the equation

$$\frac{D'}{T'}B_{i\sigma}'=\zeta\Bigg[\frac{D}{T}B_{i\sigma}-\frac{D^2}{T}A_iB_{i\sigma}-\frac{D^2}{T}B_{ij}B_{i\sigma}+\frac{D^3}{T}B_{i\sigma}(B_{j\sigma}^2+2B_{j\sigma}B_{i\sigma})\Bigg] \qquad (2.16)$$

for a *fixed* configuration of random-axes. The graphs entering Eq. (2.16) appear in Fig. 4(a).

Power-counting arguments indicate that cumulants involving D to a power greater than 2 are irrelevant below three dimensions. Though at the moment we have not established an ϵ expansion about any particular dimension, we will disregard all but the cumu-

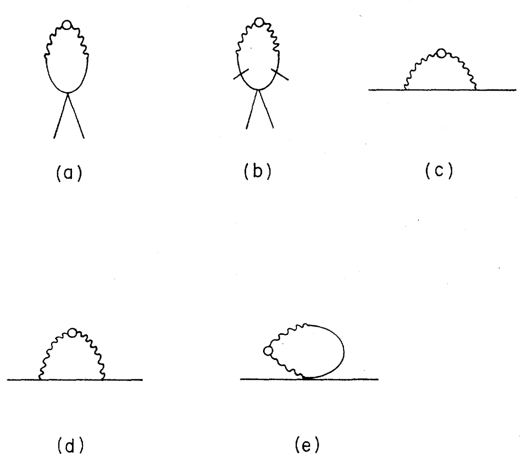

FIG. 3. Graphs entering Eq. (2.12). Wavy lines denote the random couplings A_i, $B_{i\sigma}$, B_{ij}.

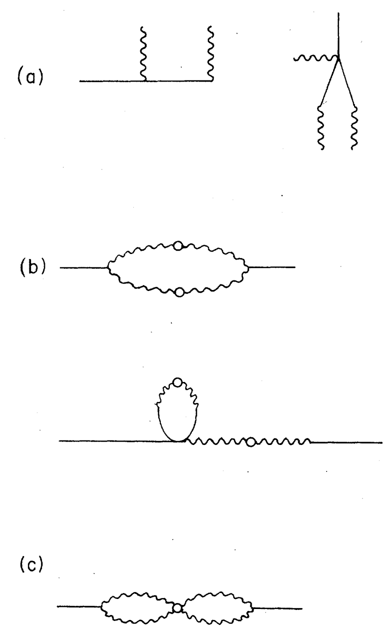

FIG. 4. (a) Graphs contributing to Eq. (2.16). (b) The two relevant classes of graphs found by squaring Eq. (2.16) and averaging. (c) Graph corresponding to higher-order irrelevant cumulant and thus not contributing to Eq. (2.17).

lant proportional to D^2. We shall finally see that 4 is the critical dimensionality, above which we will find a nontrivial fixed point.

The recursion relation for the cumulant Δ defined by Eq. (2.9) is derived by squaring Eq. (2.16) and averaging over the random-axes. This procedure yields the recursion relation

$$\left(\frac{\Delta}{T^2}\right)' = \zeta^2 b^{-d}\left(\frac{\Delta}{T} + \frac{\Delta^2}{T^2} K_d \ln b\right) , \quad (2.17)$$

and is illustrated diagrammatically in Fig. 4(b). A diagram, which does not enter Eq. (2.17) because of the irrelevancy of higher-order cumulants, is shown in Fig. 4(c).

Combining Eqs. (2.10), (2.15), and (2.17), we derive differential recursion relations for T and Δ by taking the limit $b \to 1$,

$$\frac{dT}{dl} = -(d-2)T + (n-2)(T+\Delta)TK_d + O(T^3, T\Delta^2, \Delta T^2) , \quad (2.18a)$$

$$\frac{d\Delta}{dl} = -(d-4)\Delta + \Delta[(n-2)(T+\Delta) - T]K_d + O(\Delta T^2, \Delta^2 T) . \quad (2.18b)$$

There are two fixed points for the above equations,

$$T^* = 0 , \quad \Delta^* = \frac{d-4}{K_d(n-2)} , \quad (2.19)$$

$$T^* = \Delta^* = 0 . \quad (2.20)$$

The flows generated by Eq. (2.18) are illustrated in Fig. 5. Above four dimensions the ferromagnetic state is stable, as Aharony's calculation[8] also demonstrated. The fixed point Eq. (2.19) controls a phase boundary separating the ferromagnetic phase from the large anisotropy region. Below four dimensions, the aligned state is unstable to spin-wave fluctuations. However, the recursion relations cannot rule out a state with nonzero magnetization and large transverse fluctuations. Whereas in a uniform system we would not expect such a state, it could exist in a random system. Thus a rigorous proof of the absence of any ferromagnetism below four dimensions would be very helpful here (see the Appendix).

III. CRITICAL BEHAVIOR

We now explore the critical behavior at the boundary separating the ferromagnetic phase from the large anisotropy region using the recursion relations in Eq. (2.18). Linearizing Eq. (2.18b) about the fixed point, Eq. (2.19), we determine the exponent ν. To first order in $\epsilon = d-4$ we find

$$1/\nu = \lambda_\Delta = \epsilon + O(\epsilon^2) . \quad (3.1)$$

The determination of η illustrates the mechanism of the "dangerous irrelevant variable,"[27] namely, the temperature, prevalent in this model. Consider the behavior of the connected correlation function $G_c(x,x')$,

$$G_c(x,x') = \langle\langle \vec{s}(x) \cdot \vec{s}(x') \rangle\rangle_c - \langle\langle \vec{s}(x) \rangle \cdot \langle \vec{s}(x') \rangle\rangle_c , \quad (3.2)$$

which in Fourier space obeys the following scaling law near the fixed point, Eq. (2.19),

$$G_c(k,T,\Delta^*) \sim b^{d-[(n-1)/(n-2)]\epsilon} G_c(kb, Tb^{\lambda_T}, \Delta^*) , \quad (3.3)$$

where we have used Eq. (2.15). The eigenvalue λ_T about the fixed point Eq. (2.19) is found to be

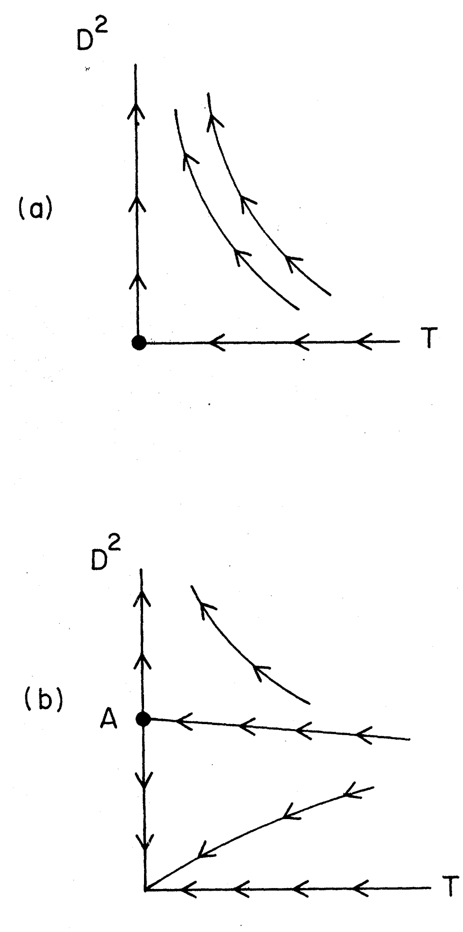

FIG. 5. Renormalization-group flows generated by Eq. (2.18): (a) below four dimensions and (b) above four dimensions. Point A is the fixed point Eq. (2.19).

$$\lambda_T = -2 + O(\epsilon^2) \ . \tag{3.4}$$

Choosing $kb = 1$, Eq. (3.3) becomes

$$G_c(k, T, \Delta^*) \sim k^{-d+[(n-1)/(n-2)]\epsilon} G_c(1, Tk^2, \Delta^*) \ . \tag{3.5}$$

However, a direct calculation shows that $G_c(1, T', \Delta^*) \sim T'$ for small T'. Thus the scaling behavior of $G_c(k, T, \Delta^*)$ is given by

$$G_c(k, T, \Delta^*) \sim k^{-2+\eta} \ , \tag{3.6}$$

where

$$\eta = \epsilon/(n-2) + O(\epsilon^2) \ . \tag{3.7}$$

Comparing these results for ν and η with those obtained in Refs. 15, 18, and 19, we see that, at least to first order in ϵ, the exponents here are identical to those of the nonrandom n vector in two dimensions less. We have not established an exact correspondence to all orders in ϵ, but the same correspondence has been shown exactly in the random-field problem.[28] The non-Gaussian exponents along this phase boundary suggest that the large anisotropy phase is not a continuation of the paramagnetic phase discussed in Aharony's work.[8] Indeed, an exact $n = \infty$ solution[13] suggests that the large anisotropy phase is an EA spin glass.[12]

We have also found that hyperscaling is violated along this phase boundary as in the magnetic transition in the random-field model,[29] again arising from the presence of the dangerous irrelevant temperature variable. The singular part of the averaged free energy near the fixed point, Eq. (2.19), obeys

$$F(T, \Delta_r) \sim b^{-d} F(Tb^{\lambda_T}, \Delta_r b^{1/\nu}) \ , \tag{3.8}$$

where Δ_r measures the deviation from the fixed point, namely,

$$\Delta_r = \Delta - \Delta^* \ , \tag{3.9}$$

and ν and λ_T are given by Eqs. (3.1) and (3.4), respectively. With the choice $\Delta_r b^{1/\nu} = 1$, Eq. (3.8) becomes

$$F(T, \Delta_r) \sim \Delta_r^{d\nu} F(T\Delta_r^{2\nu}, 1) \ . \tag{3.10}$$

The usual hyperscaling law $d\nu = 2 - \alpha$ would result if $F(T, 1)$ approached a finite constant as $T \to 0$. However, a graphical expansion of $F(T, 1)$ shows that it diverges as $1/T$ as $T \to 0$, whereupon the scaling law Eq. (3.10) becomes

$$F(T, \Delta_r) \sim \Delta_r^{(d-2)\nu} f(T) \ , \tag{3.11}$$

where $f(0)$ is a finite constant. Thus we arrive at the modified hyperscaling relation,

$$(d-2)\nu = 2 - \alpha \ . \tag{3.12}$$

Another example of the shifting of the effective dimensionality by 2 appears in the behavior of the longitudinal susceptibility χ_L. In a nonrandom isotropic n-vector model, χ_L diverges between two and four dimensions as the applied field $h \to 0$, specifically

$$\chi_L \sim h^{-(4-d)/2} \ , \tag{3.13}$$

as can be shown from spin-wave[30,31] and renormalization-group calculations.[32,33] We now consider the scaling behavior of $\overline{\chi}_L = \chi_L/T$ in the random-axis model near the fixed point $h^* = T^* = \Delta^* = 0$, namely,

$$\overline{\chi}_L(T, h, \Delta) \sim b^{4-d} \overline{\chi}_L(Tb^{-(d-2)}, hb^2, \Delta b^{-(d-4)}) \ , \tag{3.14}$$

which can be derived from the scaling expression for the free energy. The eigenvalues for T, Δ, and h are found from linearizing Eqs. (2.18a), (2.18b), and (2.14), respectively. Choosing $hb^2 = 1$, Eq. (3.14) becomes

$$\overline{\chi}_L(T, h, \Delta) \sim h^{d/2-2} \overline{\chi}_L(Th^{(d-2)/2}, 1, \Delta h^{(d-4)/2}) \ . \tag{3.15}$$

The divergence of $\overline{\chi}_L$ between two and four dimensions given by Eq. (3.13) would arise if $\chi_L(0,1,0)$ were well behaved. However, a graphical expansion shows that $\overline{\chi}_L(T', 1, \Delta') \sim \Delta'/T'$ for small T', Δ'. Hence, the complete scaling behavior of $\overline{\chi}_L$ is given by

$$\overline{\chi}_L(T, h, \Delta) \sim h^{-(6-d)/2} \ , \tag{3.16}$$

which is equivalent to Eq. (3.13) if we let $d \to d+2$. This result can also be established from spin-wave arguments[34] and a direct calculation of $\overline{\chi}_L$ (Ref. 14) with the Nelson-Rudnick trajectory integral formalism.[35]

IV. CONCLUSIONS

The momentum-shell recursion relations constructed in Sec. II indicated that 4 is the lower critical dimensionality for ferromagnetic ordering in the random-axis model. A similar conclusion for the random-*field* model was established by Imry and Ma.[25] Recently, Aharony has shown that ferromagnetism is absent below four dimensions in a large class of systems including the random-axis model and the random dipolar magnet.[36]

Possible experimental realizations of the random-axis model exist in the alloys DyCu, TbAg, DyNi, and DyAl,[37] which comprise a rare earth with an asymmetric charge distribution and a nonmagnetic host. The experimental situation in these systems is still unclear; however, they may exhibit spin-glass behavior. The original example for the model was $TbFe_2$, which exhibits long-range order in three dimensions.[38] However, $TbFe_2$ is ferrimagnetic, and a realistic model would have to include iron-iron magnetic interactions and rare-earth–iron interactions, whose effects have not yet been studied.

As we noted earlier, the mean-field studies of the model, excluding the work of Harris and Zobin,[3] find only a ferromagnetic state even at large anisotropy. This result may not disagree with our phase diagram of Fig. 1 slightly above four dimensions, which exhibits a spin-glass phase. It has been suggested that the spin-glass state in this model does not exist above six dimensions.[39] Within our theory, this suggestion is very plausible, since the ferromagnetic-spin-glass boundary located at $D^2=O(d-4)$ could continue to rise as the dimensionality, wiping out the spin-glass phase at six dimensions. Also, if the exponents along this boundary are indeed exactly equal to their counterparts in the uniform system in two less dimensions, they will become Gaussian at six dimensions, and equal to the corresponding exponents along the ferromagnetic-paramagnetic boundary. The evidence for the existence of the spin-glass state in this model would then be less compelling above six dimensions. If the mean-field theory in this model corresponds to infinite dimensionality as it does in simple uniform systems, then its results could be understood in the context of our theory.

Note added in proof: Recent numerical work on this model by M. C. Chi and T. Egami[40] allowing for collective spin reorientation suggests that ferromagnetism is unstable in the limit of large anisotropy.

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APPENDIX: MERMIN-WAGNER "PROOF"

The $\epsilon=d-4$ expansion suggested that four dimensions is the lower critical dimensionality for ferromagnetism in the random-axis model. However, that calculation did not exclude the possibility of a partially aligned state below four dimensions, where $M\neq0$ but with large transverse fluctuations. To rule out that state, a rigorous proof of the absence of ferromagnetism is needed. Following Schuster's analogous proof for the random-field model,[41] we have constructed a demonstration of the absence of ferromagnetism below four dimensions, at the expense of using the replica trick,[22] which makes the proof less than ironclad.

We consider for simplicity the X-Y random-axis model, but presumably the proof can be generalized to higher n. Writing $\vec{s}_i$ and $\hat{x}_i$ in terms of polar angles θ_i and ψ_i respectively, Eq. (1.1) becomes, in the presence of a uniform applied field h,

$$\mathcal{H}=-J\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j)-D\sum_i\cos^2(\theta_i-\psi_i)-h\sum_i\cos\theta_i\ . \tag{A1}$$

For the quenched system, the free energy is given by

$$-\beta F=\prod_i\int_0^{2\pi}d\psi_i\ln Z\{\psi_i\}\ , \tag{A2}$$

where

$$Z\{\psi_i\}=\prod_i\int_0^{2\pi}d\theta_i\,e^{-\beta\mathcal{H}}\ . \tag{A3}$$

We now introduce the identity[22]

$$\ln Z=\frac{\partial}{\partial m}Z^m|_{m=0} \tag{A4}$$

and replicate the system m times, writing

$$(Z\{\psi_i\})^m=\prod_i\prod_{\alpha=1}^{m}\int d\theta_{i\alpha}\,e^{-\beta\mathcal{H}'}\ , \tag{A5}$$

where

$$\mathcal{H}'=\sum_{\alpha=1}^{m}\left[-J\sum_{\langle i,j\rangle}\cos(\theta_{i\alpha}-\theta_{j\alpha})-D\sum_i\cos^2(\theta_{i\alpha}-\psi_i)-h\sum_i\cos\theta_{i\alpha}\right]\ . \tag{A6}$$

From Eqs. (A2)–(A6), we obtain for the free energy

$$-\beta F=\frac{\partial}{\partial m}\left[\prod_{i,\alpha}\int d\theta_{i\alpha}\,e^{-\beta\mathcal{H}_m}\right]\Big|_{m=0}\ , \tag{A7}$$

where

$$H_m=\sum_{\alpha=1}^{m}\left[-J\sum_{\langle i,j\rangle}\cos(\theta_{i\alpha}-\theta_{j\alpha})-h\sum_i\cos\theta_{i\alpha}-\sum_i g(\theta_i)\right] \tag{A8}$$

and

$$e^{\beta g(\theta_{i\alpha})}=\int_0^{2\pi}d\psi_i\exp\left[D\beta\sum_{\alpha=1}^{m}\cos^2(\theta_{i\alpha}-\psi_{i\alpha})\right]\ . \tag{A9}$$

Defining

$$Z_{\text{eff}}=\prod_{i,\alpha}\int d\theta_{i\alpha}\, e^{-\beta\mathcal{H}_m} ,$$

we see from Eq. (A7) that

$$F=-\frac{1}{\beta}\frac{\partial}{\partial m}Z_{\text{eff}}|_{m=0}=-\frac{1}{\beta}\lim_{m\to 0}\frac{1}{m}\ln Z_{\text{eff}} , \tag{A10}$$

i.e., the free energy of the original system is equal in the limit $m\to 0$ to the free energy per degree of freedom of the system described by $\mathcal{H}_m$.

Evaluating Eq. (A9), we obtain

$$g(\theta_{i\alpha})=\frac{D^2}{16k_BT}\sum_{\beta}\cos 2(\theta_{i\alpha}-\theta_{i\beta})+O(D^4) . \tag{A11}$$

The higher-order terms in D involve sufficient numbers of replica summations so as to not contribute to our final result when $m\to 0$.

We use Bogoliubov's inequality in the form given by Mermin[17] for classical systems,

$$\langle AA^*\rangle_m > k_BT|\langle\{C^*,A\}\rangle_m|^2/\langle\{C,\{C^*,\mathcal{H}_m\}\}\rangle_m , \tag{A12}$$

where $\langle\cdots\rangle_m$ denotes the statistical average in the system described by $\mathcal{H}_m$, $\{C^*,A\}$ is the Poisson bracket defined by

$$\{C^*,A\}\equiv\sum_{\alpha=1}^{m}\sum_{i}\left[\frac{\partial C^*}{\partial\theta_{i\alpha}}\frac{\partial A}{\partial L_{i\alpha}}-\frac{\partial C^*}{\partial L_{i\alpha}}\frac{\partial A}{\partial\theta_{i\alpha}}\right] , \tag{A13}$$

and $L_{i\alpha}$ is the angular momentum conjugate to $\theta_{i\alpha}$. We choose the operators A and C as follows:

$$A=A_\mu(\vec{k})=\sum_{\alpha}^{m}\sum_{j}U_{\mu\alpha}\sin\theta_{j\alpha}e^{-i\vec{k}\cdot\vec{R}_j} ,$$
$$C=C_\mu(\vec{k})=\sum_{\alpha}^{m}\sum_{j}U_{\mu\alpha}L_{j\alpha}e^{-i\vec{k}\cdot\vec{R}_j} , \tag{A14}$$

where $\underline{U}$ is an orthogonal matrix which diagonalizes an $m\times m$ matrix of the form

$$\begin{bmatrix} A & B & \cdots & & B\\ B & A & & & \cdot\\ \cdot & & \cdot & & \cdot\\ \cdot & & & \cdot & \cdot\\ \cdot & & & \cdot & B\\ B & & \cdots & B & A\end{bmatrix} . \tag{A15}$$

Useful properties of $\underline{U}$ are

$$\sum_{\alpha}^{m}U_{\mu\alpha}=\delta_{\mu 1}\sqrt{m} ,\quad U_{1\alpha}=\frac{1}{\sqrt{m}} . \tag{A16}$$

The choice, Eq. (A14), is necessary to obtain a useful result. Dropping the factors $U_{\mu\alpha}$ in Eq. (A14) would yield from Eq. (A12) the trivial inequality $1>0$. Using Eqs. (A13), (A14), and (A16), we evaluate Eq. (A12) and obtain after summation over k and μ,

$$mN^2>k_BTNM^2(m)\sum_{\mu}^{m}\sum_{k}\left[\sigma k^2+\frac{D^2m}{8k_BT}(1-\delta_{\mu 1})+hM(m)\right]^{-1} \tag{A17}$$

for a system with N spins. $M(m)$ is defined from

$$M=\lim_{m\to 0}M(m)=\lim_{m\to 0}\frac{1}{mN}\sum_{\alpha}^{m}\sum_{i}\langle\cos\theta_{i\alpha}\rangle_m \tag{A18}$$

and σ is proportional to J and finite for the nearest-neighbor interaction.

After explicitly summing over μ in Eq. (A17), we find

$$mN^2>k_BTNM^2(m)\sum_{k}\left[[\sigma k^2+hM(m)]^{-1}+(m-1)\left[\sigma k^2+\frac{D^2m}{8k_BT}+hM(m)\right]^{-1}\right] . \tag{A19}$$

Dividing both sides of Eq. (A19) by mN^2, we let $m\to 0$ and $N\to\infty$ and obtain the following inequality:

$$1>\frac{k_BTM^2}{n}\int\frac{d^dk}{(2\pi)^d}\left[\frac{1}{\sigma k^2+hM}+\frac{D^2}{8k_BT}\frac{1}{(\sigma k^2+hM)^2}\right] , \tag{A20}$$

where n is the volume per spin.

We let $h\to 0$ in Eq. (A20), and if $D=0$ we recover the Hohenberg-Mermin-Wagner result[16,17,42] that $M=0$ for $d\leqslant 2$, except at $T=0$. Nonzero D modifies this conclusion, forcing $M=0$ at $d\leqslant 4$ even at zero temperature.

The above proof cannot be considered rigorous because of its use of the replica trick. Analytically continuing $m\to 0$ is risky in an inequality which might change direction as m passes through one. However, Eq. (A20) is consistent with our other calculations, and we believe it is correct. A justification of the use of replicas would be desirable. Alternatively, we could avoid replicas completely by constructing an inequality for an arbitrary configuration of random-axes and then averaging over configurations. This approach, however, would require a clever choice (as yet undiscovered) of the operators A and C entering Eq. (A12) to facilitate the configurational averaging.

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[1]R. Harris, M. Plischke, and M. J. Zuckermann, Phys. Rev. Lett. 31, 160 (1973); see also R. W. Cochrane, R. Harris, M. Plischke, D. Zobin, and M. J. Zuckermann, J. Phys. F 5, 763 (1975).
[2]R. W. Cochrane, R. Harris, and M. Plischke, J. Non-Cryst. Solids, 15, 239 (1974).
[3]R. Harris and D. Zobin, AIP Conf. Proc. 29, 156 (1976).
[4]E. Callen, Y. J. Liu, and J. R. Cullen, Phys. Rev. B 16, 263 (1977).
[5]J. D. Patterson, G. Gurzalski, and D. Sellmyer, Phys. Rev. B 18, 1377 (1978); G. Gruzalski, Ph.D. thesis (U. of Nebraska, 1977) (unpublished).
[6]R. Harris and D. Zobin, J. Phys. F 7, 337 (1977).
[7]M. Chi and R. Alben, J. Appl. Phys. 48, 2987 (1977).
[8]A. Aharony, Phys. Rev. B 12, 1038 (1975).
[9]J. H. Chen and T. C. Lubensky, Phys. Rev. B 16, 2106 (1977).
[10]A. B. Harris, T. C. Lubensky, and J. H. Chen, Phys. Rev. Lett. 36, 415 (1976).
[11]R. A. Pelcovits, E. Pytte, and J. Rudnick, Phys. Rev. Lett. 40, 476 (1978).
[12]S. F. Edwards and P. W. Anderson, J. Phys. F 5, 965 (1975).
[13]J. Rudnick (unpublished).
[14]E. Pytte, Phys. Rev. B 18, 5046 (1978).
[15]E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976); Phys. Rev. B 14, 3110 (1976).
[16]N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).
[17]N. D. Mermin, J. Math. Phys. 8, 1061 (1967).
[18]A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 810 (1975); 69, 1457 (1975) [Sov. Phys. JETP 42, 413 (1976); 42, 743 (1976)]; see also L. P. Kadanoff, Ann. Phys. 100, 359 (1976); Rev. Mod. Phys. 49, 267 (1977).
[19]A. M. Polyakov, Phys. Lett. B 59, 79 (1975).
[20]D. R. Nelson and R. A. Pelcovits, Phys. Rev. B 16, 2191 (1977).
[21]R. Brout, Phys. Rev. 115, 824 (1959).
[22]V. J. Emery, Phys. Rev. B 11, 239 (1975).
[23]T. C. Lubensky, Phys. Rev. B 11, 3573 (1975).
[24]A. B. Harris and T. C. Lubensky, Phys. Rev. Lett. 33, 1540 (1974); see also S. -K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, Mass., 1976).
[25]Y. Imry and S. -K. Ma, Phys. Rev. Lett. 35, 1399 (1975).
[26]See, e.g., J. Hubbard, Phys. Lett. A 39, 365 (1972); 40, 111 (1972).
[27]M. E. Fisher, *Renormalization Group in Critical Phenomena and Quantum Fields: Proceedings of a Conference*, edited by J. D. Gunton and M. S. Green (Temple Univ., Philadelphia, Pa, 1974), p. 66.
[28]A. Aharony, Y. Imry, and S. -K. Ma, Phys. Rev. Lett. 37, 1364 (1976); A. P. Young, J. Phys. C 10, 1257 (1977).
[29]G. Grinstein, Phys. Rev. Lett. 37, 944 (1976).
[30]P. W. Anderson, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic, New York, 1963), Vol. 14.
[31]V. G. Vaks, A. J. Larkin, and S. A. Pikin, Zh. Eksp. Teor. Fiz. 53, 281 (1967); 53, 1089 (1967) [Sov. Phys. JETP 26, 188 (1968); 26, 647 (1968)].
[32]E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. B 7, 232 (1973); E. Brézin, and D. J. Wallace, *ibid.* 7, 1967 (1973).
[33]D. R. Nelson, Phys. Rev. B 13, 2222 (1976).
[34]B. I. Halperin (private communication).
[35]D. R. Nelson and J. Rudnick, Phys. Rev. Lett. 35, 178 (1975); J. Rudnick and D. R. Nelson, Phys. Rev. B 13, 2208 (1976).
[36]A. Aharony (unpublished).
[37]J. M. D. Coey, J. Appl. Phys. 49, 1646 (1978).
[38]J. J. Rhyne, J. H. Schelling, and N. C. Koon, Phys. Rev. B 10, 4672 (1974).
[39]S. Kirkpatrick (private communication).
[40]M. C. Chi and T. Egami, Proceedings of Magnetism and Magnetic Material Conference, Cleveland (1978) (unpublished).
[41]H. G. Schuster, Phys. Lett. A 60, 89 (1977).
[42]P. C. Hohenberg, Phys. Rev. 158, 383 (1967).