Stability of certain geophysical structures in fluid dynamics via dispersive analysis

By

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B.S., Seoul National University, 2021

Thesis

Submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Department of Mathematics at Brown University

PROVIDENCE, RHODE ISLAND

May 2026

This dissertation by Haram Ko is accepted in its present form by the Department of Mathematics as satisfying the dissertation requirement for the degree of Doctor of Philosophy.

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3. *The effect of stratification on the stability of a rest state in the 2D inviscid Boussinesq system*, C. Jurja and **H. Ko**, Proc. Amer. Math. Soc., to appear.

2. *Global axisymmetric solutions for Navier-Stokes equation with rotation uniformly in the inviscid limit*, **H. Ko**, preprint https://arxiv.org/abs/2409.17528, Submitted.

1. *An introduction to the distorted Fourier transform*, **H. Ko** and W. Schlag, Advanced Nonlinear Studies **25** (2025), no. 2, 298-312.

Acknowledgments

I want to thank Benoît, my advisor, who led me to a whole new world of mathematics. Not only is he one of the most brilliant people I met and well-versed in several different areas of mathematics, he has connected me to great people all over the world, supported my academic growth, and given me invaluable advice for research and career. I am indebted to him a lot throughout my 5 years of graduate school.

In terms of the support I received, I must not forget to mention the staff at Brown Mathematics department, Doreen, Audrey, and Lori. They have been so kind and supportive with any issues I have faced, starting from my first day at Kassar, all the way through the times around the recent shooting incident. I feel so lucky to have them around during my years in an entirely new country.

Brown University has many more people than them that I have benefited from. Conversations with Javi, about topics on math or not, were always interesting, and I also appreciate all the advice and help J.J. has provided to me. Constantine Dafermos and Yan Guo in the Applied Mathematics department and Jill are models of senior professors who have helped me with math and life. I also want to thank graduate students and postdocs, in the (applied) math department and beyond, and visitors and soccer players, whom I have met here and enriched my life.

Outside Brown, Wilhelm Schlag is one of the first people I would like to acknowledge. My first collaborator except Benoît is not an enough phrase to capture his presence in my

Ph.d years. He truly feels like the epitome of brilliance and I have learned immensely from him through a number of interactions. Klaus and Ryo, my two most recent collaborators, are terrific mathematicians whom I enjoyed working with so much and hope to continue working with. Although the physical distance made us communicate mostly online, the times I spent with them when I visited were memorable moments, one of which led me to meet another collaborator, Catalina. She is such a bright and elegant person, and I wish both of us can grow into great mathematicians together in the future.

Although not on a research paper, I was fortune to work with Catherine Sulem and Tiên-Tài Nguyễn. The time I spent to talk and learn about the water wave system was an invaluable experience. I would like to thank Emmanuel Dormy and Christophe Lacave for giving me such a chance, which was just one out of several valuable opportunities, and for the discussions with them which broadened my view on fluid dynamics. A lot of these discussions, meetings, and collaborations might not have been possible without financial assistance as well. For this reason, I also appreciate NSF which funded me through my advisor, and the Brown Graduate School which also partially funded a lot of my travels.

Lastly, but most importantly, my parents should receive due credit for helping me become what I am now, and I want to conclude this acknowledgment in Korean for them. 아들이 지구 반대편으로 떠나고 5년이 되어 가는데, 그래도 걱정했던 것보다는 잘 하고 있는 것 같아요. 인생에서 처음으로 집 나가서 혼자 사는 생활을 대학원 생활 하면서, 그것도 외국에서 겪으니까 집에서 살았던 시간의 소중함이 느껴지네요. 여기 와서 배운 것도 알게 된 것들도 많은데, 그 중에 어떤 것들은 그걸 모르던 시절이, 그걸 모를 수 있었던 때가 좋은 시기였다는 것을 깨닫게 되는 것은 뭔가 슬프다. 내가 잘할 거라고 믿고 그렇게 말해줘서 고마워요. 어느 정도는 사실 희망에 기반한 말이었겠지만, 그래도 여기 근거가 될 만한 성과가 하나 더 생겼어요.

Abstract of "Stability of certain geophysical structures in fluid dynamics via dispersive analysis"

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Ph.D., Brown University, May 2026

This thesis establishes the stability of rotation in the 3D compressible Euler equation and incompressible Navier-Stokes equation, and the stability of linear stratification in the 2D Boussinesq equation. Using the common factor that these steady states induce dispersion, I establish a *systematic* method for linear analysis. This is explained through two different systems to highlight that the method is versatile enough to work for different dispersion relations in different systems. Combined with the energy estimates, this produces long-time stabilities of the respective steady states. Finally, I prove the global stability for one of these systems using a mix of paradifferential calculus, the spacetime resonance method, and the aforementioned linear theory.

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CHAPTER 1

Introduction

1.1 Fluid dynamics and dispersive PDEs

The mathematical definition of (constant coefficient) dispersive PDE is a PDE (partial differential equation) whose linearized equation can be written as $\partial_t u = Lu$, $u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^n$ where L has the form $\mathcal{F}(Lu) = ih(\xi)\mathcal{F}(u)$ for some real-valued diagonal matrix $h : \mathbb{R}^d \to \mathbb{R}^{n\times n}$ and the Fourier transform $\mathcal{F}$ (Tao06). The prototypical examples are Schrödinger equation, wave equation, or Maxwell's equation, although these will not be the focus of the thesis.

A less rigorous definition may be to say that a dispersive PDE is a PDE whose solution exhibits a wave-like behavior. In a loose sense, and at least in the linear level, this matches the mathematical definition, where the solution will be formed out of 'wave packets' $e^{ix\cdot\xi + ith_j(\xi)}$ for any j-th diagonal component of $h(\xi)$, $h_j(\xi)$. Indeed, this is true for all of the aforementioned examples; solutions of the linear Maxwell's equation are electromagnetic waves, solutions of the Schrödinger equation are named wavefunctions, and solutions of the wave equations are different types of waves like acoustic waves.

This raises a natural question whether or not the fluid dynamics is dispersive and can portray the waves we can observe most easily in nature, waves in the oceans, lakes, or rivers. This is a complicated matter that induces the question about what is the correct modeling of these phenomena, and indeed, there are dispersive PDEs, Korteweg-de Vries equation for example, that are designed to model these waves directly. However, it is true that the most common choices of model for fluids, Euler and Navier-Stokes equations, are not dispersive, at least not by themselves. The question only becomes more intriguing as one realizes that there are more waves in larger scales that are not easily seen by a human eye, such as equatorial waves, Kelvin waves, or Rossby waves.

The fascinating fact is that even though Euler equation or other common fluid models are not dispersive, they *become* dispersive under some specific settings or around certain steady state solutions. One example of the former fact is a free boundary incompressible Euler equation, now more commonly called as water wave system. (Zak68) proved that the system has a Hamiltonian structure and derived that certain types of solution obey nonlinear Schrödinger equation in the leading order. Numerous studies have followed using this method, now known as Craig-Sulem-Zakharov formulation, either showing the extended lifetime of solutions or establishing rigorous approximations of solutions to other dispersive systems in different regimes, (Ovs74; Cra85; ASL08; CGS21; TW12; IT19), to name a few.

This thesis is based on the latter fact that fluid dynamics become dispersive equations around some steady states, and in particular ones that emerge from geophysical considerations. Admittedly, the author is not the first to consider such topics, but follows a stream of works on geophysical fluids, such as (BMN97; CDGG02; EW15; GPW23). Two types of problem will be discussed. The first type is rotating fluids modeled by the incompressible Navier-Stokes-Coriolis equation:

$$\begin{aligned} \partial_t u - \nu\nabla u + (u\cdot\nabla)u + \varepsilon^{-1}e_3 \times u + \nabla p = 0, \\ \mathrm{div}(u) = 0, \end{aligned} \tag{NSC}$$

or the compressible Euler-Coriolis equation:

$$\begin{aligned} \partial_t n + \operatorname{div}(nu) &= 0, \\ \partial_t u + (u\cdot\nabla)u + \varepsilon^{-1} e_3 \times u + \frac{1}{n}\nabla p(n) &= 0. \end{aligned} \tag{1.1}$$

In both systems, $u : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ refers to the fluid velocity, ε is the inverse of the angular speed of rotation, $e_3 = (0,0,1)$ and p is the pressure term. p is an independent function $p : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ to be solved in (NSC), whereas in (1.1), we assume it follows a polytropic pressure law

$$p(n) = \frac{1}{\gamma} n^\gamma, \quad \gamma > 1,$$

and depends on the density $n : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}_+$ with a fixed number γ. Lastly, $\nu \geq 0$ in (NSC) represents the strength of viscosity of the fluid, where the system reduces to the Euler equation when $\nu = 0$.

The second type takes stratification into account in the Boussinesq equation in $\mathbb{R}^2$:

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= \kappa u_2, \\ \partial_t u + (u\cdot\nabla)u + \nabla p &= -\kappa \rho e_2, \\ \operatorname{div}(u) &= 0. \end{aligned} \tag{sBQ}$$

u refers to the velocity of the fluid again, and ρ is the density perturbation from the stratified profile $-\kappa x_2$. p is the pressure, $e_2 = (0,1)$ and κ is the strength of stratification. To be fully exact, one should view (sBQ) as the system satisfied by the perturbation from $(n,u,p) = (-\kappa x_2, 0, \kappa^2 x_2^2/2)$ of the standard 2D Boussinesq equation.

In all cases, the solutions represent the quantities modulo large structures that are natural steady states observed in the Earth or some astrophysical objects, and studying the existence and lifetime of solutions is equivalent to examining the stability of these structures. In geophysics, the sizes of parameters involved in these equations play important roles, which can be very small or large according to the type of fluids or the type of phenomena under investigation. In planetary scale, ε^{-1} is (after proper non-dimensionalization) often a large parameter, and hence the first studies of the models focused on the fast rotating limit of the system (NSC) (BMN97; CDGG02; KLT14b; KLT14a). When $\nu > 0$, these works have obtained global solutions, although the results implicitly depend on the other parameter ν, in the sense that the initial data are required to have a size bounded by some positive power of ν.

The analysis is more involved for (1.1), and, to the best of my knowledge, there are fewer results. Here, when considering the system around a steady state $(n, u) = (n_0, 0)$, another natural parameter is the ambient sound speed $c = p'(n_0) = n_0^{\frac{\gamma-1}{2}}$. Indeed, we study the stability of the quiescent state via the speed variables (ρ, u) where $\rho = \frac{2}{\gamma-1}(\sqrt{p'(n)} - c) = \frac{1}{\alpha}(n^\alpha - c)$, $\alpha = \frac{\gamma-1}{2}$, which satisfy

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho + (c + \alpha\rho)\mathrm{div}(u) &= 0, \\ \partial_t u + u \cdot \nabla u + \varepsilon^{-1}\vec{e}_3 \times u + (c + \alpha\rho)\nabla\rho &= 0. \end{aligned} \tag{CER}$$

1.2 Stability and blowup

Although these equations all arise from similar physical considerations, their long-time behaviors or even the existence of solutions themselves could differ a lot. Indeed, while the compressible Euler's equation, associated with (CER), is known to be able to blow up in multiple ways (Chr07; MRRS22a; MRRS22b; BSV23b; BSV23a; BCLGS25) and even for small data (Sid85), the global well-posedness of the incompressible Navier-Stokes equation, associated with (NSC), remains one of the most famous and long-standing open problems.

While the focus of the thesis is on proving the stability of certain steady states, this is not to say that a blowup cannot happen. In fact, adapting one of the earliest blowup mechanisms for compressible Euler's equation (Sid85), one can prove that (CER) can also generate singularity in finite time.

Theorem 1.2.1. *Suppose* (n,u) *is a* C^1 *solution to* (1.1) *for* $0 \le t \le T$ *with*

$$n(x,0) = \bar{n}(x) > 0; \quad \bar{n}(x) \equiv n_0, |x| \ge R,$$
$$u(x,0) = \bar{u}(x); \quad \bar{u}(x) \equiv u_0 = 0, |x| \ge R.$$

Let $c = n_0^{\frac{\gamma-1}{2}}$ *as in* (CER) *and define*

$$m(t) = \int_{\mathbb{R}^3} (n(x,t) - n_0)dx,$$
$$F(t) = \int_{\mathbb{R}^3} ze_3 \cdot (nu)dx.$$

Then, if

$$m(0) \ge 0, \quad F(0) > \frac{16\pi}{3} c \|\bar{n}\|_{L^\infty} R^4, \tag{1.2}$$

the lifespan of C^1 *solution is finite.*

Remark 1.2.2. *Note that the condition* (1.2) *does not involve* ε*. This alludes to the fact that the rotation in the planar direction cannot avoid blowup in* z*-direction due to the initial compression and outgoing velocity profile in* z*-direction.*

The theorem will follow easily by proving the finite speed of propagation of (1.1).

Proposition 1.2.3. *Let* $c = n_0^{\frac{\gamma-1}{2}}$, $v \in \mathbb{S}^2$ *and* $s \in \mathbb{R}$*. If* $U = (n,u)$ *is a* C^1 *solution of* (CER) *in the region*

$$\Omega(s;T) = \{(x,t) : x \cdot v \ge s + ct, 0 \le t \le T\}$$

and $U_0(x) \equiv (n_0, 0)$ *for* $x \cdot v \ge s$*. Then,* $U(x,t) \equiv (n_0, 0)$ *for* $(x,t) \in \Omega(s;T)$*.*

Proof. The proof adapts that of the original version for hyperbolic conservation laws in (Sid84). As was given in (MUK86), the isentropic Euler's equation can be expressed as a symmetric hyperbolic system using the change of variable

$$(n,u) \to (w := n^{\frac{\gamma-1}{2}}, u)$$

with matrices

$$A_0 := \begin{pmatrix} 1 & 0 \\ 0 & \frac{(\gamma-1)^2}{4} I_3 \end{pmatrix}, \quad A_k := \begin{pmatrix} u_k & \frac{\gamma-1}{2} w e_k^T \\ \frac{\gamma-1}{2} w e_k & \frac{(\gamma-1)^2}{4} u_k I_3 \end{pmatrix}.$$

Note that this variable w is what we denoted as $c + \alpha\rho$. The pressure in this thesis is defined by $p(n) = \frac{1}{\gamma} n^\gamma$ which differs from that in (MUK86) by a factor of γ^{-1}, so that it requires $\frac{(\gamma-1)^2}{4\gamma}$ to be $\frac{(\gamma-1)^2}{4}$ here instead, but this does not create any qualitative changes. Straightforward calculations give that the eigenvalues of $\sum \xi_k A_0^{-1} A_k$ is given by $\xi \cdot u$ with multiplicity 2, and $\xi \cdot u \pm w|\xi|$. With the same A_0 and A_k's, (1.1) is equivalent to

$$A_0 \partial_t W + \sum_{k=1}^{3} A_k \partial_k W + RW = 0, \quad R = \frac{(\gamma-1)^2}{4} \varepsilon^{-1} \begin{pmatrix} 0 & \\ & R_z \end{pmatrix}, R_z x = e_3 \times x, \quad W = \begin{pmatrix} w \\ u \end{pmatrix}.$$

Put $P = \bar{A}_0 \partial_t + \sum \bar{A}_k \partial_k$ where $\bar{A}_i = A_i(c, 0)$, Since the largest and smallest eigenvalues of $\sum\limits_{k=1,2,3} v_k A_0^{-1} A_k(n_0, 0)$ are $\pm c$ independent of v, modifying the energy estimate in (Sid84) gives

$$\int_{E(t)} |W - (c,0)^\top|^2 dx \lesssim \int_0^t \int_{E(\tau)} \langle P(W - (c,0)^\top), W - (c,0)^\top \rangle dx d\tau,$$

where $E(t) = \{x : -c(t_0 - t) \le v \cdot (x - x_0) \le c(t_0 - t)\}$ for any (x_0, t_0) in the supersonic region $\Omega(s;T)$ and $t \in [0, t_0]$. From the fact that $\langle RV, V \rangle = 0$ for any $V \in \mathbb{R}^4$ and that $W - (c,0)^\top$ satisfies the equation, we get $\langle P(W - (c,0)^\top), W - (c,0)^\top \rangle \le C_E |W - (c,0)^\top|^2$ in $E := \bigcup\limits_{0 \le t \le T} E(t)$. Since $W - (c,0)^\top \equiv 0$ on $E(0)$, Duhamel's formula gives $W - (c,0)^\top \equiv 0$ for any $E(t)$.

Since the point (x_0, t_0) is irrelevant in the calculations as long as it lies in the supersonic region, we obtain $U(x,t) \equiv (n_0, 0)^\top$ for every $(x,t) \in \Omega(s;T)$. $\square$

Since the direction v is also irrelevant, we also get a conclusion that if $U(x,0) \equiv (n_0, 0)^\top$ outside a ball B_R, then $U(x,t) \equiv (n_0, 0)^\top$ outside a ball $B(t) := B_{R+ct}$. This enables the proof of Theorem 1.2.1.

Proof of Theorem 1.2.1. We build an inequality for $F(t)$ to derive contradiction, which differs from one in (Sid85) to reflect the anisotropy of (1.1). By the mass conservation,

$$F'(t) = -\int_{B(t)} ze_3 \cdot [\mathrm{div}(nu \otimes u) + \varepsilon^{-1} e_3 \times (nu) + \nabla p] dx = \int_{B(t)} [nu_3^2 + (p - p_0)] dx,$$

where $p_0 = p(n_0)$. Jensen's inequality and the condition that $m(0) \geq 0$ gives,

$$\begin{aligned}
\int_{B(t)} p dx = \int_{B(t)} \frac{1}{\gamma} n^\gamma dx &\geq \frac{|B(t)|}{\gamma} \left(\frac{1}{|B(t)|} \int_{B(t)} n dx \right)^\gamma \\
&\geq \frac{|B(t)|}{\gamma} \left(\frac{1}{|B(t)|} \int_{B(t)} n_0 dx \right)^\gamma \\
&= \frac{|B(t)|}{\gamma} n_0^\gamma = \int_{B(t)} p_0 dx,
\end{aligned}$$

where Proposition 1.2.3 is also used in the second inequality. Note that $m(t) \equiv m(0)$ by mass conservation. Hence, we get $F'(t) \geq \int_{B(t)} nu_3^2 dx$, and in particular F is positive since $F(0) > 0$. From

$$\begin{aligned}
\int_{B(t)} nz^2 dx &\leq \int_{B(t)} n dx \cdot |R + ct|^2 \\
&= \int_{B(t)} \bar{n} dx \cdot |R + ct|^2 \\
&\leq \|\bar{n}\|_\infty \cdot \frac{4\pi}{3} |R + ct|^5
\end{aligned}$$

and the Cauchy-Schwarz inequality applied to $F(t)$, we obtain

$$F'(t) \geq \frac{3}{4\pi \|\bar{n}\|_\infty} \frac{1}{(R + ct)^5} F(t)^2,$$

which leads to

$$F(0)^{-1} \geq F(0)^{-1} - F(t)^{-1} \geq \frac{3}{16\pi \left\|\bar{n}\right\|_\infty} \frac{1}{c} \left(\frac{1}{R^4} - \frac{1}{(R+ct)^4} \right).$$

Therefore, if $F(0)$ satisfies (1.2), t cannot be arbitrarily large. $\square$

Even though the usual conclusion of compressible Euler's equation, the fact that large initial data can lead to singularity formation applies here, too, this also shows two points on the other side. Firstly, the small data blowup of (Sid85) does not apply here. There is no direct analogue of the scenario used by Sideris. In addition, condition (1.2) depends on the parameter c. This is something that Sideris himself had already noted in (Sid85), leaving the remark that his conclusion may not lead to a direct conclusion for the incompressible model.

I will pay extra attention to this parameter dependence. In the equations (NSC), (sBQ), and (CER) considered in this thesis, there are always some parameters included, and some emphasis will be given to how the result depends on the parameters of the equations. In some cases, the problem is trivialized by changing the parameter. For example, as $\nu \to \infty$, any initial data can lead to a global solution for (NSC), once we establish small data global well-posedness. Thus, the meaningful direction of research is the inviscid limit $\nu \to 0$, and how the result depends or does not depend on ν when it becomes small. Not only these viewpoints relevant to geophysics, the origin of these equations, but this is not new even in the mathematical literature. For example, (BGM17) considered a similar concept of stability threshold, where they paid attention to how the threshold for the size of the initial data depends on the viscosity in investigating the stability of the Couette flow. In some sense, this can be seen as a generalization of their concept.

1.3 Notations

This section includes standard and nonstandard notations that will be used throughout. Unless specified otherwise, $\varphi(\cdot)$ is used to denote a fixed smooth bump function that is used for a partition of unity. In other words,

$$0 \leq \varphi \leq 1, \quad \mathrm{supp}\varphi \subset [1/2, 2], \quad \sum_{n\in\mathbb{N}} \varphi(2^{-n}x) \equiv 1, x > 0.$$

We often use φ for $\varphi(x) + \varphi(-x)$ when we also localize in symmetric negative numbers. This is a more customary way of defining φ as it is usually defined by $\psi(x) - \psi(2x)$ for a function ψ that is positive, smooth, and supported on $[-2, 2]$. We will need a similar function as ψ for localization around the origin and indeed define

$$\bar{\varphi}(x) = 1 - \sum_{n\geq 0} \varphi(2^{-n}x).$$

Lastly, a tilde will be used to mean $\tilde{\varphi}(x) = \varphi(x)/x$. Double tildes can be understood as composition, i.e. $\tilde{\tilde{\varphi}}(x) = \varphi(x)/x^2$, and so on. Except that they do not sum up to 1, they are known to have qualitatively similar properties when used for localization. We use these functions to define the Littlewood-Paley localizations:

$$P_k f(x) = \int e^{ix\cdot\xi} \hat{f}(\xi)\varphi(2^{-k}|\xi|)d\xi, \quad k \in \mathbb{Z}.$$

It is well known that the exact shape of φ is not important, as long as we use a fixed function throughout. For more on the properties of these projections, I refer the readers to books on harmonic analysis, such as (Ste93) or (Gra14).

Although not as widely used as the standard Littlewood-Paley projectors, I define two more projections $P_{k,p}$ and $P_{k,p,q}$ that will be used often, which are

$$\mathcal{F}(P_{k,p}f)(\xi) = \varphi_{k,p}(\xi)\hat{f}(\xi), \quad \mathcal{F}(P_{k,p,q}f)(\xi) = \varphi_{k,p,q}(\xi)\hat{f}(\xi),$$

for $k \in \mathbb{Z}$, $p, q \in \mathbb{Z}_- = \{n \in \mathbb{Z}, n \leq 0\}$ where

$$\varphi_k(\xi) = \varphi(2^{-k}|\xi|), \quad \varphi_{k,p}(\xi) = \varphi_k(\xi)\varphi(2^{-k-p}|\xi_h|), \quad \varphi_{k,p,q}(\xi) = \varphi_{k,p}(\xi)\varphi(2^{-k-q}\xi_3).$$

These will be used to precisely determine the linear decay associated with some of the problems above. These decay rates often come with constants whose exact values are not important, as is the case for many calculations using harmonic analysis tools. In order to be free of these insignificant constants, I will use $\sim$ to mean that there exists some fixed constant $C > 1$ such that

$$A \lesssim B \Leftrightarrow A \leq CB, \quad A \sim B \text{ or } A \simeq B \Leftrightarrow A \lesssim B \text{ and } B \lesssim A,$$

where C is fixed in the sense that it is independent of the parameters, variables, or functions involved in A and B. For example, when $P_k f$ is involved in this type of inequality, C may depend on the choice of the partition of unity function φ, but will not depend on the parameter k or the function f whose size is under estimation. The value of C might change from line to line for convenience, such as $A_1 \lesssim B$ and $A_2 \lesssim B \Rightarrow A_1 + A_2 \lesssim B$, but in such a way that, if needed, the constants can be found and are fixed, independent of the variables and parameters, so that these do not affect the final estimate. Using a similar principle, we define

$$A \ll B \Leftrightarrow CA < B.$$

Whenever the polar coordinate or cylindrical coordinate is involved, r will denote the radius and θ will denote the angle measured from the x-axis in the counterclockwise direction. Whenever the spherical coordinate is used, the name of the variables will be (ρ, θ, ϕ), and ϕ will denote the angle measured from the z-axis, unless specified otherwise. For example, the volume element in the spherical coordinate is $\rho^2 \sin\phi$.

$\langle \cdot \rangle$, usually referred to as the Japanese bracket, will be used in the standard way as a shorthand notation of $\langle a \rangle = \sqrt{a^2 + 1}$.

CHAPTER 2

Linear Analysis

As mentioned in Chapter 1, dispersive equations are characterized by the fact that their linearized equation can be written as

$$\partial_t \hat{u} = ih(\xi)\hat{u},$$

for some real-valued diagonal matrix h. For scalar equations, Schrödinger equations, for example, this will usually be automatic. Denote n eigenvectors by $E_1, \cdots, E_n$ with corresponding eigenvalues $\Lambda_1, \cdots, \Lambda_n$. These eigenvectors and eigenvalues will generally be functions of ξ, but nevertheless, this enables us to write the linear solution as

$$\mathcal{F}(e^{tL}u_0) = \sum_{j=1}^{n} e^{it\Lambda_j} \frac{\hat{u_0} \cdot E_j}{|E_j|^2} E_j.$$

This naturally induces two preliminary goals. First, we should study the properties of linear evolution for each dispersion relation, $e^{it\Lambda_j}$. Unless Λ_j is a linear function in ξ, which would be the transport equation, dispersion gives rise to decay, and the canonical tool to derive this decay is the stationary phase method. In later sections, I present a general framework to use the stationary phase method to obtain decay in 2- or 3-dimensions.

Second, since these decays of $e^{it\Lambda_j}$ apply to projections defined in Fourier space, we need to piece the estimates back together. This means that we need to know how the norms of functions in the physical space are related to those in the Fourier space. This is done by finding a good change of variables and is problem-dependent. We discuss this point in the following section.

2.1 Dispersive unknowns and norm equivalence

We will show how this works explicitly for (NSC) and (CER). The process for (sBQ) will be skipped, as it is similar to the case (NSC). The linearized equation of (NSC) is

$$\partial_t u - \nu\Delta u + \varepsilon^{-1} e_3 \times u + \nabla p = 0 \tag{2.1}$$

We drop the dissipation term for the time being, since the dispersion is already present in the inviscid model. Due to the vanishing divergence condition, (NSC) effectively has two unknowns. Considering the anisotropy introduced by the Coriolis term, we reduce the system to system for $w_3 := \mathrm{curl}_h u = \partial_1 u_2 - \partial_2 u_1$ and u_3:

$$\begin{aligned} \partial_t w_3 - \varepsilon^{-1}\partial_3 u_3 &= 0, \\ \partial_t u_3 + \varepsilon^{-1}\partial_3 \Delta^{-1} w_3 &= 0, \end{aligned}$$

where the zero divergence condition was used to simplify in both lines. In Fourier space,

$$\partial_t \begin{pmatrix} \hat{w}_3 \\ \hat{u}_3 \end{pmatrix} = \begin{pmatrix} 0 & i\varepsilon^{-1}\xi_3 \\ i\varepsilon^{-1}\frac{\xi_3}{|\xi|^2} & 0 \end{pmatrix} \begin{pmatrix} \hat{w}_3 \\ \hat{u}_3 \end{pmatrix}.$$

One can diagonalize the system directly, but a simple observation already shows that, up to sign, the dispersion relation is

$$\Lambda_\varepsilon(\xi) = \varepsilon^{-1}\frac{\xi_3}{|\xi|}$$

by rewriting the system using w_3 and $|\nabla|u_3$

$$\begin{aligned} \partial_t w_3 - i\Lambda_\varepsilon(|\nabla|u_3) &= 0, \\ \partial_t|\nabla|u_3 - i\Lambda_\varepsilon w_3 &= 0, \end{aligned}$$

where I also used Λ_ε to denote the corresponding Fourier multiplier. Diagonalization is particularly simple in this case. The eigenvectors are $(1, \pm 1)^\top$ for the eigenvalues $\pm i\Lambda_\varepsilon$. Hence, in hindsight, one might want to define dispersive unknowns as $w_3 \pm |\nabla|u_3$. However, both terms have an extra derivative compared to u, and the better choice is

$$U_\pm := |\nabla_h|^{-1}(w_3 \pm |\nabla|u_3), \qquad \nabla_h = (\partial_1, \partial_2, 0), \tag{2.2}$$

which, regardless of the normalization $|\nabla_h|^{-1}$, still satisfy

$$\partial_t U_\pm = \pm i\Lambda_\varepsilon U_\pm.$$

Using elementary facts from harmonic analysis, one can check that $2\,\|u\|_{L^2(\mathbb{R}^3)} = \|U_+\|_{L^2(\mathbb{R}^3)} + \|U_-\|_{L^2(\mathbb{R}^3)}$. In fact, one can check that for any Fourier multiplier m, we have $2\,\|\mathrm{m}u\|_{L^2(\mathbb{R}^3)} = \|\mathrm{m}U_+\|_{L^2(\mathbb{R}^3)} + \|\mathrm{m}U_-\|_{L^2(\mathbb{R}^3)}$. (GPW23)

The process works similarly for (sBQ), and thus I refer to (EW15) or (JW24) for the derivation and leave here that the dispersive unknowns and dispersion are given by

$$\begin{aligned} Z_\pm &:= |\nabla|^{-1}\omega \pm \rho = |\nabla|^{-1}\nabla^\perp \cdot u \pm \rho, \\ \lambda_\kappa(\xi) &:= \kappa\frac{\xi_1}{|\xi|}. \end{aligned} \tag{2.3}$$

Linearization of (sBQ) becomes $\partial_t Z_\pm = \mp i\lambda_\kappa Z_\pm$ by abusing λ_κ as a Fourier multiplier again, and the variables satisfy

$$\|Z_+\|^2_{\dot{H}^m} + \|Z_-\|^2_{\dot{H}^m} = 2(\|u\|^2_{\dot{H}^m} + \|\rho\|^2_{\dot{H}^m}), \quad m \geq 0. \tag{2.4}$$

The calculations are rather simple for the above problems, as we only have two variables and the eigenvectors are constant. To illustrate that this is rather an ideal than a general case, we give another example using (CER). Linearizing (CER), we get

$$\begin{aligned} \partial_t \rho + c\ \mathrm{div}(u) = 0, \\ \partial_t u + \varepsilon^{-1}\vec{e}_3 \times u + c\nabla\rho = 0. \end{aligned} \tag{2.5}$$

We use the rescaling

$$(\rho, u)(t, x) \mapsto (\rho, u)(c^{-1}t, x),$$

and work with (for $\kappa = c\varepsilon > 0$)

$$\begin{aligned} \partial_t \rho + \mathrm{div}(u) = 0, \\ \partial_t u + \kappa^{-1}\vec{e}_3 \times u + \nabla\rho = 0. \end{aligned} \tag{2.6}$$

Even without prior knowledge in fluid dynamics, the previous example could serve as a motivation to use the (still physical) variables $(\rho, \mathrm{div}(u), \mathrm{curl}_h u, u_3)^\top$. Again, having the homogeneity in mind, we choose $V := (\rho, |\nabla|^{-1}\mathrm{div}(u), |\nabla|^{-1}\mathrm{curl}_h u, u_3)^\intercal$, which transforms (2.6) into

$$\partial_t \widehat{V}(\xi, t) = M(\xi)\widehat{V}(\xi, t),$$

where

$$M(\xi) := \begin{pmatrix} 0 & -|\xi| & 0 & 0 \\ |\xi| & 0 & \kappa^{-1} & 0 \\ 0 & -\kappa^{-1} & 0 & \kappa^{-1}\frac{i\xi_3}{|\xi|} \\ -i\xi_3 & 0 & 0 & 0 \end{pmatrix}.$$

The diagonalization is no longer obvious. One can compute that its characteristic polynomial is

$$\tau^4 + (\kappa^{-2} + |\xi|^2)\tau^2 + \kappa^{-2}\xi_3^2,$$

whose roots are given by

$$\tau^2 = \frac{1}{2}\left(-(\kappa^{-2} + |\xi|^2) \pm \sqrt{(\kappa^{-2} + |\xi|^2)^2 - 4\kappa^{-2}\xi_3^2}\right) = -\frac{1}{4}\left(d_1^2 + d_2^2 \mp 2d_1 d_2\right), \qquad (2.7)$$

where d_j's are defined by

$$d_1(\xi) := \sqrt{|\xi_h|^2 + (\xi_3 - \kappa^{-1})^2}, \qquad d_2(\xi) := \sqrt{|\xi_h|^2 + (\xi_3 + \kappa^{-1})^2}, \qquad \xi_h = (\xi_1, \xi_2, 0).$$

Thus, we have a short representation of the dispersion relations of (2.6) in terms of these distances, and we name them as

$$\Sigma := \frac{d_2 + d_1}{2}, \qquad \Omega := \frac{d_2 - d_1}{2}.$$

The eigenvalues are absolute, but we have a freedom to choose the eigenvectors. As was observed above, this determines the dispersive unknowns and is directly related to the norm equivalence between physical variables and the dispersive unknowns. I will give the answer in the similar pattern as before. First, the eigenvectors whose first component is normalized to 1 are

$$E_\tau = \begin{pmatrix} 1 & -i\tau|\xi|^{-1} & \kappa(\tau^2 - |\xi|^2)|\xi|^{-1} & -\tau^{-1}\xi_3 \end{pmatrix}^\top, \qquad \tau \in \{\Sigma, -\Sigma, \Omega, -\Omega\}.$$

It turns out that correct normalization uses

$$\mathcal{T}_2(\xi) := \begin{pmatrix} a_\Sigma E_\Omega & a_\Sigma E_{-\Omega} & a_\Omega E_\Sigma & a_\Omega E_{-\Sigma} \end{pmatrix}, \quad a_\Lambda = \frac{\kappa|\xi_h|}{[(\kappa^2\Lambda^2-1)^2+(\kappa|\xi_h|)^2]^{1/2}},$$

for $\Lambda \in \{\Sigma, \Omega\}$ in the following sense.

Proposition 2.1.1. *(KPTW25) Define the linear transform $\mathcal{T}$ as*

$$\mathcal{T}(\rho, u) = \mathcal{F}^{-1}\left(\mathcal{T}_2(\xi)\widehat{V}\right) =: (U_\Omega, U_{-\Omega}, U_\Sigma, U_{-\Sigma})^\intercal,$$

and the associated projections as

$$\mathbb{P}_{\pm\Lambda}(\rho, u) := U_{\pm\Lambda}, \quad \Lambda \in \{\Sigma, \Omega\}. \tag{2.8}$$

Then $\mathcal{T}$ is well-defined and invertible. Moreover, $\mathcal{T}$ is L^p-bounded for any $p \in (1, \infty)$, i.e. there exist $c_p, C_p > 0$ such that

$$\|\mathbb{P}_{\pm\Lambda}(\rho, u)\|_{L^p} = \|U_{\pm\Lambda}\|_{L^p} \leq c_p \|(\rho, u)\|_{L^p} \leq C_p \|(U_\Omega, U_{-\Omega}, U_\Sigma, U_{-\Sigma})\|_{L^p}, \qquad \Lambda \in \{\Sigma, \Omega\}.$$

Remark 2.1.2. *Note that the similar L^p equivalence is also true for the transformation $u \to (U_+, U_-)$ from the inviscid case of (2.1). This follows easily using the Calderon-Zygmund theory, but will not be used in the rest of the paper.*

To prove Proposition 2.1.1, we use the following representation of the transform.

Lemma 2.1.3. *(KPTW25) There holds that*

$$\begin{pmatrix} \rho \\ |\nabla_h|^{-1}\mathrm{curl}_h u \end{pmatrix} = 2 \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \frac{U_\Omega + U_{-\Omega}}{2} \\ \Re(U_\Sigma) \end{pmatrix}, \qquad \theta_1 \in [-\frac{\pi}{2}, 0], \tag{2.9}$$

$$\begin{pmatrix} |\nabla_h|^{-1}(\nabla_h \cdot u_h) \\ u_3 \end{pmatrix} = 2 \begin{pmatrix} i\sin\theta_2 & \cos\theta_2 \\ -\cos\theta_2 & -i\sin\theta_2 \end{pmatrix} \begin{pmatrix} \frac{U_\Omega - U_{-\Omega}}{2} \\ \Im(U_\Sigma) \end{pmatrix}, \quad \theta_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{2.10}$$

$\cos\theta_j = \cos(\theta_j(\xi)), \sin\theta_j = \sin(\theta_j(\xi)), j = 1, 2$ *are smooth except at the singular points* $(0, \pm\kappa^{-1})$.

Proof. Writing the multiplication by $\mathcal{T}_2$ in detail, it follows that

$$\begin{aligned}
\rho &= \frac{\kappa|\nabla_h|}{[(\kappa^2\Sigma^2-1)^2+(\kappa|\nabla_h|)^2]^{1/2}}(U_\Omega + U_{-\Omega}) + \frac{\kappa|\nabla_h|}{[(\kappa^2\Omega^2-1)^2+(\kappa|\nabla_h|)^2]^{1/2}} 2\Re(U_\Sigma), \\
|\nabla|^{-1}\mathrm{div}(u) &= \frac{-i\Omega\kappa|\nabla_h|/|\nabla|}{[(\kappa^2\Sigma^2-1)^2+(\kappa|\nabla_h|)^2]^{1/2}}(U_\Omega - U_{-\Omega}) + \frac{\Sigma\kappa|\nabla_h|/|\nabla|}{[(\kappa^2\Omega^2-1)^2+(\kappa|\nabla_h|)^2]^{1/2}} 2\Im(U_\Sigma), \\
|\nabla_h|^{-1}\mathrm{curl}_h u &= \frac{-(\kappa^2\Sigma^2-1)}{[(\kappa^2\Sigma^2-1)^2+(\kappa|\nabla_h|)^2]^{1/2}}(U_\Omega + U_{-\Omega}) + \frac{-(\kappa^2\Omega^2-1)}{[(\kappa^2\Omega^2-1)^2+(\kappa|\nabla_h|)^2]^{1/2}} 2\Re(U_\Sigma), \\
u_3 &= \frac{-\kappa^2\Sigma|\nabla_h|}{[(\kappa^2\Sigma^2-1)^2+(\kappa|\nabla_h|)^2]^{1/2}}(U_\Omega - U_{-\Omega}) + \frac{-i\kappa^2\Omega|\nabla_h|}{[(\kappa^2\Omega^2-1)^2+(\kappa|\nabla_h|)^2]^{1/2}} 2\Im(U_\Sigma).
\end{aligned} \tag{2.11}$$

We write the transformation concisely as

$$\begin{pmatrix} \rho \\ \frac{\nabla_h}{|\nabla_h|} \cdot u_h \\ |\nabla_h|^{-1}\mathrm{curl}_h u \\ u_3 \end{pmatrix} = 2 \begin{pmatrix} b_{\rho,1} & & b_{\rho,2} & \\ & b_{\alpha,1} & & b_{\alpha,2} \\ b_{\beta,1} & & b_{\beta,2} & \\ & b_{\gamma,1} & & b_{\gamma,2} \end{pmatrix} \begin{pmatrix} \frac{U_\Omega + U_{-\Omega}}{2} \\ \frac{U_\Omega - U_{-\Omega}}{2} \\ \Re(U_\Sigma) \\ \Im(U_\Sigma) \end{pmatrix}. \tag{2.12}$$

where $b_{i,j}$'s are Fourier multipliers. By abusing the notations where we use $b_{i,j}$ for $b_{i,j}(\xi)$, it is immediate that $b_{\rho,j}^2 + b_{\beta,j}^2 \equiv 1,\ j = 1, 2$. Put $r = |\xi_h| = (\xi_1^2 + \xi_2^2)^{1/2}, z = \xi_3$ and

$\mathcal{D} = d_1^2 d_2^2$. From

$$
\begin{aligned}
(\kappa^2\Sigma^2 - 1)^2 + \kappa^2 r^2 &= \frac{1}{4}(\kappa^2|\xi|^2 - 1 + \kappa^2\sqrt{\mathcal{D}})^2 + \kappa^2 r^2 \\
&= \frac{1}{2}(\kappa^4\mathcal{D} + (\kappa^2|\xi|^2 - 1)\kappa^2\sqrt{\mathcal{D}}) = \frac{\kappa^2\sqrt{\mathcal{D}}}{2}(\kappa^2\sqrt{\mathcal{D}} + \kappa^2|\xi|^2 - 1),
\end{aligned}
$$

and similarly $(\kappa^2\Omega^2 - 1)^2 + \kappa^2 r^2 = \frac{\kappa^2\sqrt{\mathcal{D}}}{2}(\kappa^2\sqrt{\mathcal{D}} - (\kappa^2|\xi|^2 - 1))$, so we have

$$
\begin{gathered}
(\kappa^2\Sigma^2 - 1)^2 + \kappa^2 r^2 + (\kappa^2\Omega^2 - 1)^2 + \kappa^2 r^2 = \kappa^4\mathcal{D}, \\
[(\kappa^2\Sigma^2 - 1)^2 + \kappa^2 r^2] \cdot [(\kappa^2\Omega^2 - 1)^2 + \kappa^2 r^2] = \frac{\kappa^4\mathcal{D}}{4}(\kappa^4\mathcal{D} - (\kappa^2|\xi|^2 - 1)^2) = \kappa^4\mathcal{D} \cdot \kappa^2 r^2.
\end{gathered}
$$

Therefore,

$$
b_{\rho,1}^2 + b_{\rho,2}^2 = \frac{\kappa^2 r^2}{(\kappa^2\Sigma^2 - 1)^2 + \kappa^2 r^2} + \frac{\kappa^2 r^2}{(\kappa^2\Omega^2 - 1)^2 + \kappa^2 r^2} = \frac{\kappa^2 r^2 \cdot \kappa^4\mathcal{D}}{\kappa^4\mathcal{D} \cdot \kappa^2 r^2} = 1.
$$

$b_{\beta,1}^2 + b_{\beta,2}^2 \equiv 1$ follows immediately from the above equalities.

We now show similar equalities for $b_{\alpha,j}$ and $b_{\gamma,j}$'s. First,

$$
\begin{aligned}
&|\xi_h|^2 \left(\left| \frac{-i\Omega + i\kappa\xi_3\Sigma}{|\xi_h|} \right|^2 + [-\kappa\Sigma]^2 \right) \\
&= [\Omega - \kappa\xi_3\Sigma]^2 + \kappa^2|\xi_h|^2\Sigma^2 \\
&= \Omega^2 - 2\kappa\xi_3\Omega\Sigma + \kappa^2|\xi|^2\Sigma^2 \\
&= \Omega^2 - 2\kappa\xi_3\Omega\Sigma + [\kappa^2\Omega^2 + \kappa^2\Sigma^2 - 1]\Sigma^2 \\
&= \kappa^2\Sigma^4 + \kappa^2\Omega^2\Sigma^2 - \Sigma^2 + \Omega^2 - 2\kappa\xi_3\Omega\Sigma \quad (\text{using } \Omega\Sigma = \kappa^{-1}\xi_3,) \\
&= \kappa^2\Sigma^4 - \Sigma^2 + \Omega^2 - \xi_3^2 \qquad (\text{using } \Omega^2 = \kappa^{-2} + |\xi|^2 - \Sigma^2,) \\
&= \kappa^2\Sigma^4 - 2\Sigma^2 + \kappa^{-2} + |\xi|^2 - \xi_3^2 \\
&= \kappa^{-2}((\kappa^2\Sigma^2 - 1)^2 + \kappa^2|\xi_h|^2) \\
&= |\xi_h|^2 \frac{(\kappa^2\Sigma^2 - 1)^2 + (\kappa|\xi_h|)^2}{(\kappa|\xi_h|)^2},
\end{aligned}
$$

and the other one can be computed symmetrically by exchanging the roles of Ω and Σ. These prove that $|b_{\alpha,j}|^2 + |b_{\gamma,j}|^2 \equiv 1$. Moreover, since $\kappa^2\Sigma^2 = \frac{1}{2}(\kappa^2|\xi|^2 + 1 + \kappa^2\sqrt{\mathcal{D}})$, $\kappa^2\Omega^2 = \frac{1}{2}(\kappa^2|\xi|^2 + 1 - \kappa^2\sqrt{\mathcal{D}})$,

$$\begin{aligned}
|b_{\gamma,1}|^2 + |b_{\gamma,2}|^2 &= \frac{\kappa^2 r^2}{\kappa^4\mathcal{D}\cdot\kappa^2 r^2}([(\kappa^2\Omega^2-1)^2 + \kappa^2 r^2]\kappa^2\Sigma^2 + [(\kappa^2\Sigma^2-1)^2 + \kappa^2 r^2]\kappa^2\Omega^2) \\
&= \frac{1}{\kappa^4\mathcal{D}}[\frac{\kappa^2\sqrt{\mathcal{D}}}{2}(\kappa^2\sqrt{\mathcal{D}} - (\kappa^2|\xi|^2 - 1))\cdot\frac{1}{2}(\kappa^2|\xi|^2 + 1 + \kappa^2\sqrt{\mathcal{D}}) \\
&\quad + \frac{\kappa^2\sqrt{\mathcal{D}}}{2}(\kappa^2\sqrt{\mathcal{D}} + \kappa^2|\xi|^2 - 1)\cdot\frac{1}{2}(\kappa^2|\xi|^2 + 1 - \kappa^2\sqrt{\mathcal{D}})] \\
&= \frac{1}{4\kappa^2\sqrt{\mathcal{D}}}[(\kappa^2\sqrt{\mathcal{D}}+1)^2 - (\kappa^2|\xi|^2)^2 + (\kappa^2|\xi|^2)^2 - (\kappa^2\sqrt{\mathcal{D}}-1)^2] = 1.
\end{aligned}$$

$|b_{\alpha,1}|^2 + |b_{\alpha,2}|^2 \equiv 1$ follows directly. Finally, note that $b_{\gamma,1} = -\kappa\Sigma\cdot b_{\rho,1}$ and $b_{\gamma,2} = -i\kappa\Omega\cdot b_{\rho,2}$ (see (2.11)). Therefore, to justify (2.9) and (2.10), we only need to check the signs. $b_{\beta,1} \le 0 \le b_{\beta,2}$ follows from the fact that $\kappa\Sigma \ge 1 \ge \kappa\Omega \ge -1$, and $b_{\rho,1}, b_{\rho,2}$ are obviously nonnegative. Hence, (2.9) follows.

(2.10) is proved similarly. From

$$(\kappa\Omega - \kappa^2 z\Sigma)\cdot\kappa\Sigma = \kappa z(1 - \kappa^2\Sigma^2),$$

$$\Sigma^2 \ge z^2 = \kappa z\cdot\Omega\cdot\Sigma,$$

$\Sigma - \kappa z\Omega$ is always positive and $\Omega - \kappa z\Sigma$ has the opposite sign to z. Hence, $b_{\alpha,2}$ is always positive, but $b_{\alpha,1}$ is purely imaginary with the imaginary part having the same sign as the sign of z. Also, $b_{\gamma,1}$ is clearly always negative, and $b_{\gamma,2}$ is purely imaginary, with the imaginary part having the opposite sign to the sign of z. This proves (2.10). $\square$

Proof of Proposition 2.1.1. The invertibility and boundedness of change of variable from (ρ, u) to the left-hand side of (2.12), $(\rho, \frac{\nabla_h}{|\nabla_h|}\cdot u_h, |\nabla_h|^{-1}\mathrm{curl}_h u, u_3)$, follows from direct computations and the boundedness of Riesz transforms, since it is given by $(\rho, u) \mapsto$

$\mathcal{F}^{-1}\left(\mathcal{T}_1(\xi)(\widehat{\rho},\widehat{u})\right)$, where

$$\mathcal{T}_1(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\xi_1|\xi_h|^{-1} & i\xi_2|\xi_h|^{-1} & 0 \\ 0 & -i\xi_2|\xi_h|^{-1} & i\xi_1|\xi_h|^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \xi_h = (\xi_1, \xi_2).$$

To prove Proposition 2.1.1 it thus suffices to study the matrices in (2.9) and (2.10). We can further reduce the problem to showing that $b_{\rho,1} = \cos\theta_1, b_{\rho,2} = -\sin\theta_1, b_{\gamma,1} = -\cos\theta_2$, and $b_{\gamma,2} = -i\sin\theta_2$ are L^p Fourier multipliers, because the inverse transformations are again rotation matrices using the same angles, so that this will prove the norm equivalence. The following elementary facts will be used to ensure that the equivalence does not depend on κ.

- If $m(\xi)$ is a Hörmander-Mikhlin multiplier, then so is $m(a\xi)$ for any $a \in \mathbb{R}$, and the boundedness is independent of a.
- If $f(\xi)$ is a smooth function in a compact set K, then it is a Hörmander-Mikhlin multiplier. Furthermore, the boundedness is independent of $a > 0$ for $f(a\xi)$ in $a^{-1}K$.

1. $b_{\rho,1}$ **and** $b_{\rho,2}$**.** For the matrix in (2.9), it is easier to investigate $b_{\rho,1}$ and $b_{\rho,2}$. We decompose $b_{\rho,1}$ into

$$\begin{aligned} b_{\rho,1} = {} & b_{\rho,1}\bar{\varphi}(2\kappa d_1)\phi(\frac{\xi_3-\kappa^{-1}}{r}) + b_{\rho,1}\bar{\varphi}(2\kappa d_1)(1-\phi(\frac{\xi_3-\kappa^{-1}}{r})) \\ & + b_{\rho,1}\bar{\varphi}(2\kappa d_2)\phi(\frac{\xi_3+\kappa^{-1}}{r}) + b_{\rho,1}\bar{\varphi}(2\kappa d_2)(1-\phi(\frac{\xi_3+\kappa^{-1}}{r})) \\ & + b_{\rho,1}(1-\bar{\varphi}(2\kappa d_1))(1-\bar{\varphi}(2\kappa d_2)), \end{aligned}$$

where $\bar{\varphi}$ was defined as the usual smooth bump function with supp $\bar{\varphi} \subset [-1,1]$ and $\bar{\varphi} \in C^\infty(\mathbb{R})$ is a smoothed version of the Heaviside function such that $\phi(x) = 0$ for

$x \leq -\frac{1}{10}$, $\phi(x) = 1$ for $x \geq \frac{1}{10}$ and monotone. We also decompose $b_{\rho,2}$ in the same way.

① $d_1 \leq \frac{\kappa^{-1}}{2}, \xi_3 - \kappa^{-1} \geq -\frac{1}{10}r$: It will be convenient to write $\mathbf{x} = (x, y, z) = (\xi_1, \xi_2, \xi_3 - \kappa^{-1})$ in this region, which makes $d_1 = |\mathbf{x}|$. Using (2.7), $b_{\rho,1}$ can be written as

$$b_{\rho,1} = \frac{r}{\{[z + \kappa\frac{r^2+z^2}{2} + \sqrt{(z + \kappa\frac{r^2+z^2}{2})^2 + r^2}]^2 + r^2\}^{1/2}}.$$

By expanding the squares in the denominator and collecting terms of same order,

$$\begin{aligned} b_{\rho,1}^2 &= \frac{r^2}{2d_1} \frac{1}{d_1 + z + d_1(\kappa z + \frac{\kappa^2}{4}d_1^2) + z(\sqrt{1 + \kappa z + \frac{\kappa^2}{4}d_1^2} - 1) + \frac{\kappa}{2}d_1^2\sqrt{1 + \kappa z + \frac{\kappa^2}{4}d_1^2}} \\ &= \frac{r^2}{2d_1(d_1 + z)} \frac{1}{1 + p_1(\kappa d_1, \kappa z)}. \end{aligned}$$

p_1 is continuous at the origin as a sum of product of 0th order rational function and a smooth function, and is greater than -1 under the assumed localization.(crude estimate gives a lower bound $-\frac{11}{100}$) Therefore, the problem reduces to proving that $\frac{r}{\sqrt{d_1(d_1+z)}} = \frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}$ is a L^p Fourier multiplier. By the Marcinkiewicz multiplier theorem,

$$\begin{aligned} \nabla_{x,y} \frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} &= \frac{1}{r\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{r}{|\mathbf{x}|\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} \frac{1}{|\mathbf{x}|} \begin{pmatrix} x \\ y \end{pmatrix} \\ &\quad - \frac{r}{(|\mathbf{x}| + z)\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} \frac{1}{|\mathbf{x}|} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \lesssim \frac{1}{|\mathbf{x}|}, \\ \partial_z \frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} &= -\frac{r}{|\mathbf{x}|\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} \frac{z}{|\mathbf{x}|} - \frac{r}{(|\mathbf{x}| + z)\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} \frac{|\mathbf{x}| + z}{|\mathbf{x}|} \lesssim \frac{1}{|z|}, \\ \partial_{xy} \frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} &= -\frac{xy}{r^3} \frac{1}{\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} \\ &\quad - \frac{2xy}{r|\mathbf{x}|} \left(\frac{1}{|\mathbf{x}|\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} + \frac{1}{(|\mathbf{x}| + z)\sqrt{|\mathbf{x}|(|\mathbf{x}| + z)}} \right) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{xy}{|\mathbf{x}|^3}\left(\frac{r}{|\mathbf{x}|\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}} + \frac{r}{(|\mathbf{x}|+z)\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right) \\
&- \frac{x}{|\mathbf{x}|}\left(\frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x}|+z}\partial_y\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right)\right) \\
&+ \frac{x}{|\mathbf{x}|}\left(\frac{y}{|\mathbf{x}|^3} + \frac{y}{|\mathbf{x}|(|\mathbf{x}|+z)^2}\right)\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}} \quad \lesssim \frac{1}{|xy|}
\end{aligned}
$$

$$
\begin{aligned}
\partial_z \nabla_{x,y} \frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}} &= \partial_z\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right)\frac{1}{r^2}\begin{pmatrix} x \\ y \end{pmatrix} \\
&+ \left(\frac{z}{|\mathbf{x}|^3} + \frac{1}{|\mathbf{x}|(|\mathbf{x}|+z)}\right)\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\frac{1}{|\mathbf{x}|}\begin{pmatrix} x \\ y \end{pmatrix} \\
&- \left(\frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x}|+z}\right)\partial_z\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right)\frac{1}{|\mathbf{x}|}\begin{pmatrix} x \\ y \end{pmatrix} \quad \lesssim \frac{1}{r|z|}
\end{aligned}
$$

$$
\begin{aligned}
\partial_{xyz} \frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}} &= -\frac{2xy}{r^4}\partial_z\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right) + \frac{x}{r^2}\partial_{yz}\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right) \\
&- \frac{xy}{|\mathbf{x}|^3}\left(\frac{z}{|\mathbf{x}|^3} + \frac{1}{|\mathbf{x}|(|\mathbf{x}|+z)}\right)\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}} \\
&- \frac{x}{|\mathbf{x}|}\left(\frac{3yz}{|\mathbf{x}|^5} + \frac{y}{|\mathbf{x}|^3(|\mathbf{x}|+z)} + \frac{y}{|\mathbf{x}|^2(|\mathbf{x}|+z)^2}\right)\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}} \\
&+ \frac{x}{|\mathbf{x}|}\left(\frac{z}{|\mathbf{x}|^3} + \frac{1}{|\mathbf{x}|(|\mathbf{x}|+z)}\right)\partial_y\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right) \\
&+ \frac{xy}{|\mathbf{x}|^3}\left(\frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x}|+z}\right)\partial_z\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right) \\
&+ \frac{x}{|\mathbf{x}|}\left(\frac{y}{|\mathbf{x}|^3} + \frac{y}{|\mathbf{x}|(|\mathbf{x}+z)^2}\right)\partial_z\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right) \\
&- \frac{x}{|\mathbf{x}|}\left(\frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x}|+z}\right)\partial_{yz}\left(\frac{r}{\sqrt{|\mathbf{x}|(|\mathbf{x}|+z)}}\right) \\
\lesssim \frac{1}{|xyz|},
\end{aligned}
$$

and these give us the desired conclusion.

To show that $b_{\rho,2}$ is also an operator bounded in $L^p - L^p$, we use the identity

$$\left[\kappa^2\left(\Sigma^2-\kappa^{-2}\right)^2+r^2\right]\cdot\left[\kappa^2\left(\Omega^2-\kappa^{-2}\right)^2+r^2\right]=(d_1d_2\kappa r)^2.$$

Then,

$$b_{\rho,2}=\frac{1}{\kappa d_1d_2}\left[\kappa^2\left(\Sigma^2-\kappa^{-2}\right)^2+r^2\right]^{1/2}=\frac{\sqrt{2}}{\kappa d_2}\left[1+\kappa z+\frac{\kappa^2}{4}d_1^2+(\frac{z}{d_1}+\frac{\kappa}{2}d_1)\sqrt{1+\kappa z+\frac{\kappa^2}{4}d_1^2}\right]^{1/2}.$$

Both κd_2 and the term inside the square root are smooth and nonzero in the area under consideration.(a crude estimate gives a lower bound $\frac{\sqrt{7}}{4}-\frac{7}{20}$) Hence, it satisfies the Hörmander-Mihklin condition, and therefore a L^p Fourier multiplier. Note that the right-hand side can be written as a function in $\kappa\mathbf{x}$ so that the boundedness is uniform.

② $d_1 \le \frac{\kappa^{-1}}{2}, \xi_3-\kappa^{-1}\le\frac{1}{10}r$: The formula for $b_{\rho,2}$ gives

$$\begin{aligned}b_{\rho,2}^2&=\frac{r^2}{2d_1}\frac{1}{d_1-z+d_1(\kappa z+\frac{\kappa^2}{4}d_1)-z(\sqrt{1+\kappa z+\frac{\kappa^2}{4}d_1^2}-1)-\frac{\kappa^2}{2}d_1^2\sqrt{1+\kappa z+\frac{\kappa^2}{4}d_1^2}}\\&=\frac{r^2}{2d_1(d_1-z)}\frac{1}{1+p_2(\kappa d_1,\kappa z)}.\end{aligned}$$

Again, p_2 is always greater than -1, and $(1+p_2(s,t))^{-1}$ has a removable singularity at the origin, making it a bounded Fourier multiplier. The different region $z\le\frac{1}{10}r$ instead of $z\ge-\frac{1}{10}r$ makes $b_{\rho,2}$ the favorable one to apply this formula, and it is straightforward to adjust the computations in ① to see that $\frac{r}{\sqrt{d_1(d_1-z)}}$ is a L^p Fourier multiplier in this region.

The boundedness of $b_{\rho,1}$ follows from a similar reasoning how we deduced $b_{\rho,2}$ is bounded in the region ①. By the same identity $\left[\kappa^2\left(\Sigma^2-\kappa^{-2}\right)^2+r^2\right]\cdot\left[\kappa^2\left(\Omega^2-\kappa^{-2}\right)^2+r^2\right]=$

$(d_1 d_2 \kappa r)^2$,

$$b_{\rho,1} = \frac{1}{\kappa d_1 d_2}\left[\kappa^2\left(\Omega^2 - \kappa^{-2}\right)^2 + r^2\right]^{1/2} = \frac{\sqrt{2}}{\kappa d_2}\left[1 + \kappa z + \frac{\kappa^2}{4}d_1^2 - (\frac{z}{d_1} + \frac{\kappa}{2}d_1)\sqrt{1 + \kappa z + \frac{\kappa^2}{4}d_1^2}\right]^{1/2}.$$

Again, the term inside the square root is nonzero under the current localization, and hence the Hörmander-Mihklin theorem applies directly.

③, ④ : These are the cases where $d_2 \leq \frac{\kappa^{-1}}{2}$ and $\xi_3 + \kappa^{-1} \geq -\frac{1}{10}r$ or $\xi_3 + \kappa^{-1} \leq \frac{1}{10}r$, which are 'upside-down' cases of ① and ②, respectively. Exchanging the roles of d_1 and d_2, one gets the same results.

⑤ $d_1, d_2 \geq \frac{\kappa^{-1}}{4}$: First, in the intersection with $|\xi| \leq 2\kappa^{-1}$, the multipliers do not have singularities, so the multipliers and their derivatives are bounded by some constants simply by compactness. Hence, we assume $|\xi| \geq 2\kappa^{-1}$ instead. By writing $b_{\rho,1}, b_{\rho,2}$ as functions of $\frac{\xi}{|\xi|^2}$, we can invert the region into a compact area and proceed similarly. To illustrate, if we translate the origin to $(0, 0, \kappa^{-1})$ again,

$$b_{\rho,1} = \frac{d_1}{d_2}\left[\frac{2\sqrt{\frac{1}{4} + \frac{z}{\kappa d_1^2} + \frac{1}{\kappa^2 d_1^2}}}{\sqrt{\frac{1}{4} + \frac{z}{\kappa d_1^2} + \frac{1}{\kappa^2 d_1^2}} + \frac{1}{2} + \frac{z}{\kappa d_1^2}}\right]^{1/2} \frac{r}{\kappa d_1^2} \tag{2.13}$$

can be written as $\frac{|\mathbf{x}|}{d_2}F(\frac{\mathbf{x}}{\kappa|\mathbf{x}|^2})$ for some smooth function F, if we put $\mathbf{x} = \xi - \kappa^{-1}e_3$ again for the translated variable. Since $\frac{\mathbf{x}}{\kappa|\mathbf{x}|^2} = f(\kappa\mathbf{x})$ and $f(\mathbf{y}) = \frac{\mathbf{y}}{|\mathbf{y}|^2}$ satisfies the Hörmander-Mihklin condition in the exterior domain B_2^c, so does $F(\frac{\mathbf{x}}{\kappa|\mathbf{x}|^2})$ in $(B_{2\kappa^{-1}})^c$. Hence, we only need to check $\frac{|\mathbf{x}|}{d_2}$.

$$\begin{aligned}
\nabla\frac{|\mathbf{x}|}{d_2} &= \frac{(z+2\kappa^{-1})^2}{d_2^3|\mathbf{x}|}\mathbf{x} - 2\kappa^{-1}\frac{|\mathbf{x}|}{d_2^3}e_3 \quad \lesssim \frac{1}{|\mathbf{x}|},\\
\nabla^2\frac{|\mathbf{x}|}{d_2} &= 2\frac{z+2\kappa^{-1}}{d_2^3|\mathbf{x}|}\mathbf{x}e_3^\intercal - 3\frac{(z+2\kappa^{-1})^2}{d_2^5|\mathbf{x}|}\mathbf{x}(x,y,z+2\kappa^{-1})^\intercal - \frac{(z+2\kappa^{-1})^2}{d_2^3|\mathbf{x}|^3}\mathbf{x}\mathbf{x}^\intercal\\
&\quad + \frac{(z+2\kappa^{-1})^2}{d_2^3|\mathbf{x}|}\mathrm{Id} + 6\kappa^{-1}\frac{|\mathbf{x}|}{d_2^5}e_3(x,y,z+2\kappa^{-1})^\intercal - \frac{2\kappa^{-1}}{d_2^3|\mathbf{x}|}e_3\mathbf{x}^\intercal \quad \lesssim \frac{1}{|\mathbf{x}|^2}
\end{aligned}$$

hold on $d_1, d_2 \geq \frac{\kappa^{-1}}{4}$ since $d_1 \sim d_2$. Therefore, by the Hörmander-Mihklin theorem, $\frac{d_1}{d_2}$ is a L^p Fourier multiplier, and this finishes the proof.

2. $b_{\gamma,1}$ **and** $b_{\gamma,2}$**.** For the matrix in (2.10), it is easier to use formulas for $b_{\gamma,1}$ and $b_{\gamma,2}$ because $b_{\gamma,1} = -\kappa\Sigma b_{\rho,1}$ and $b_{\gamma,2} = -i\kappa\Omega b_{\rho,2}$. This time, we decompose them into

$$b_{\gamma,1} = b_{\gamma,1}\bar{\varphi}(2\kappa d_1) + b_{\gamma,1}\bar{\varphi}(2\kappa d_2) + b_{\gamma,1}(1-\bar{\varphi}(2\kappa d_1))(1-\bar{\varphi}(2\kappa d_2)),$$
$$b_{\gamma,2} = b_{\gamma,2}\phi(2\kappa\xi_3) + b_{\gamma,2}(1-\phi(2\kappa\xi_3)).$$

However, $b_{\gamma,1}$ and $b_{\gamma,2}$ are not as symmetric as $b_{\rho,1}$ and $b_{\rho,2}$. Hence, we investigate these two separately.

① L^p boundedness of $b_{\gamma,2}$: It is enough to prove that $\kappa\Omega$ is a L^p Fourier multiplier. From the definition,

$$\kappa\Omega = \frac{\kappa}{2}(d_2 - d_1) = \frac{2\xi_3}{d_1 + d_2}.$$

For $b_{\gamma,2}\phi(2\kappa\xi_3)$, we again use (2.13) $\mathbf{x} = \xi - \kappa^{-1}e_3$ to center on the singularity $(0,0,\kappa^{-1})$.

Then,

$$
\begin{aligned}
\nabla_{x,y}\kappa\Omega &= -\frac{2(z+\kappa^{-1})}{d_1d_2(d_1+d_2)}\begin{pmatrix} x \\ \\ y \end{pmatrix} \lesssim \frac{1}{d_1} = \frac{1}{|\mathbf{x}|}, \\
\partial_z\kappa\Omega &= -\frac{2(z+\kappa^{-1})^2}{d_1d_2(d_1+d_2)} + \frac{d_1+d_2}{2d_1d_2} \lesssim \frac{1}{|z|}, \\
\partial_{xy}\kappa\Omega &= \frac{d_1^2+d_1d_2+d_2^2}{d_1^3d_2^3}\frac{2xy(z+\kappa^{-1})}{d_1+d_2} \lesssim \frac{1}{d_1d_2} \le \frac{1}{|xy|}, \\
\partial_z\nabla_{x,y}\kappa\Omega &= \frac{d_1^2+d_1d_2+d_2^2}{d_1^3d_2^3}\frac{2(z+\kappa^{-1})^2}{d_1+d_2}\begin{pmatrix} x \\ \\ y \end{pmatrix} - \frac{d_1^3+d_2^3}{d_1^3d_2^3}\begin{pmatrix} x \\ \\ y \end{pmatrix} \lesssim \frac{1}{d_1^2}, \\
\partial_{xyz}\kappa\Omega &= \frac{d_1^4+d_1^3d_2+d_1^2d_2^2+d_1d_2^3+d_2^4}{d_1^5d_2^5}\frac{6xy(z+\kappa^{-1})^2}{d_1+d_2} + \frac{3xy(d_1^5+d_2^5)}{2d_1^5d_2^5} \lesssim \frac{1}{d_1^3},
\end{aligned}
$$

and hence the Marcinkiewicz multiplier theorem can be applied. For $b_{\gamma,2}(1-\phi(2\kappa\xi_3))$, by setting $z = \xi_3 + \kappa^{-1}$, the derivatives are of the same form in $z - \kappa^{-1}$ instead of $z + \kappa^{-1}$, and one can apply the Marcinkiewicz theorem again.

② $b_{\gamma,1}$ when $d_1, d_2 \ge \frac{\kappa^{-1}}{4}$: Since $\kappa\Sigma$ itself is unbounded, $b_{\gamma,1}$ cannot be treated in the same way as $b_{\gamma,2}$. We first look at the boundedness of the piece away from the singularities. Using the formula (2.13), the variable $\mathbf{x} = \xi - \kappa^{-1}e_3$ again,

$$
\begin{aligned}
-b_{\gamma,1} = b_{\rho,1}\kappa\Sigma &= \frac{d_1}{d_2}\left[\frac{2\sqrt{\frac{1}{4}+\frac{z}{\kappa d_1^2}+\frac{1}{\kappa^2 d_1^2}}}{\sqrt{\frac{1}{4}+\frac{z}{\kappa d_1^2}+\frac{1}{\kappa^2 d_1^2}}+\frac{1}{2}+\frac{z}{\kappa d_1^2}}\right]^{1/2}\frac{r}{\kappa d_1^2}\cdot\kappa\frac{d_1+d_2}{2} \\
&= \left[\frac{2\sqrt{\frac{1}{4}+\frac{z}{\kappa d_1^2}+\frac{1}{\kappa^2 d_1^2}}}{\sqrt{\frac{1}{4}+\frac{z}{\kappa d_1^2}+\frac{1}{\kappa^2 d_1^2}}+\frac{1}{2}+\frac{z}{\kappa d_1^2}}\right]^{1/2}\frac{r}{2d_1}\left(\frac{d_1}{d_2}+1\right).
\end{aligned}
$$

We have already seen that the outermost square root term and $\frac{d_1}{d_2}$ are L^p Fourier multipliers. Hence, it is enough to prove the same for $\frac{r}{d_1} = \frac{r}{|\mathbf{x}|}$. We can use the Marcinkiewicz multiplier theorem again:

$$
\begin{aligned}
\nabla_{x,y}\frac{r}{|\mathbf{x}|} &= \frac{1}{r|\mathbf{x}|}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{r}{|\mathbf{x}|^3}\begin{pmatrix} x \\ y \end{pmatrix} &&\lesssim \frac{1}{|\mathbf{x}|},\\
\partial_z\frac{r}{|\mathbf{x}|} &= -\frac{rz}{|\mathbf{x}|^3} &&\lesssim \frac{1}{|\mathbf{x}|},\\
\partial_{xy}\frac{r}{|\mathbf{x}|} &= -\frac{xy}{r^3|\mathbf{x}|} - \frac{2xy}{r|\mathbf{x}|^3} + \frac{3rxy}{|\mathbf{x}|^5} &&\lesssim \frac{1}{|xy|},\\
\partial_z\nabla_{x,y}\frac{r}{|\mathbf{x}|} &= \frac{3rz}{|\mathbf{x}|^5}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{z}{r|\mathbf{x}|^3}\begin{pmatrix} x \\ y \end{pmatrix} &&\lesssim \frac{1}{r|\mathbf{x}|},\\
\partial_{xyz}\frac{r}{|\mathbf{x}|} &= -\frac{15rxyz}{|\mathbf{x}|^7} + \frac{6xyz}{r|\mathbf{x}|^5} + \frac{xyz}{r^3|\mathbf{x}|^3} &&\lesssim \frac{1}{|xyz|}.
\end{aligned}
$$

This finishes the proof.

③ $b_{\gamma,1}$ when $d_1 \leq \frac{\kappa^{-1}}{2}$ or $d_2 \leq \frac{\kappa^{-1}}{2}$: Except around infinity, $\kappa\Sigma$ is also bounded, so now we can treat it separately. d_2 is a smooth function in $d_1 \leq \frac{\kappa^{-1}}{2}$, and it is well known that $d_1 = |\mathbf{x}|$ is also a L^p Fourier multiplier in a compact region. These make $\kappa\Sigma = \frac{\kappa}{2}(d_1 + d_2)$ a bounded Fourier multiplier in $d_1 \leq \frac{\kappa^{-1}}{2}$, independent of κ. By swapping the roles of d_1 and d_2 in $d_2 \leq \frac{\kappa^{-1}}{2}$, this proves that $b_{\gamma,1}$ is a L^p Fourier multiplier. □

2.2 Dispersive decay

One of the common features of dispersive PDEs is the dispersive decay. By their very nature, at least in the linear level, the L^2 norm of their evolution is always conserved:

$$\left\|e^{tL}u_0\right\|_{L^2(\mathbb{R}^n)} \equiv \|u_0\|_{L^2(\mathbb{R}^n)}.$$

The nice fact about dispersive equations is that, although the detailed norm involved and the exponent may vary, the amplitude of linear evolution often decays over time:

$$\left\|e^{tL}u_0\right\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\alpha} \left\|u_0\right\|_Y .$$

For the Schrödinger equation, $L = i\Delta$, this can be shown by an explicit formula with $\alpha = \frac{n}{2}$ and $Y = L^1(\mathbb{R}^n)$. This is in some sense regarded as the ideal case; in particular, $\alpha = \frac{n}{2}$ is considered the best exponent. For example, for the wave equation, $L = i|\nabla|$, $\alpha = \frac{n-1}{2}$. While the exact norm Y will depend on the dispersion given by the problem, the method used to obtain the decay is usually the stationary phase method. This method hinges on the fact that the linear evolution, for example for (2.1), satisfies

$$e^{it\Lambda_\varepsilon}(U_+)_0(\mathbf{x}) = \int e^{i(t\varepsilon^{-1}\frac{\xi_3}{|\xi|}+\mathbf{x}\cdot\xi)}\widehat{(U_+)_0}(\xi)d\xi.$$

Since we have an oscillatory term inside the integral on the right-hand side, if the exponent and its derivative are not simultaneously zero, we may be able to get a decay through integration by parts. As simple as it sounds, the process is highly problem-dependent and numerous different norms have been devised in the literature to capture this decay with the largest possible α. Here, I give a systematic way to derive the $L^1 - L^\infty$ type decay, only using relatively mild and natural conditions.

2.2.1 2-dimensional stationary phase theorem

Recall that φ is a fixed bump function around 1 used for the partition of unity, and $\bar{\varphi}$ is used for the chunk at the origin, i.e. $\bar{\varphi}(x) = 1 - \sum_{n\geq 0}\varphi(2^{-n}x)$. A tilde is used for $\tilde{\varphi}(x) = \varphi(x)/x$ which does not create qualitative differences, and multiple tildes are used for multiple divisions by x.

Theorem 2.2.1. *((KPTW25) 2-dimensional Stationary phase theorem) Let $L \in C^2(\mathbb{R}^2)$,* *and assume that there exist $M > 0$ and $f_j \in C^0(\mathbb{R}^2)$, $L_j \in C^0(\mathbb{R})$, $j \in \{1, 2\}$, such that*

$$|L| \leq M, \quad \iint |\partial_1 \partial_2 L| dx \leq M,$$
$$|\partial_j L|, \frac{|L|}{|f_j|}(x_1, x_2) \leq L_j(x_j), \quad \int L_j(x_j) dx_j \leq M.$$

Given $\Phi \in C^3(\mathrm{supp}(L))$, assume moreover that there exist $a, A > 0$, such that on the support of each $L(x_1, x_2) \cdot g_1(\partial_1 \Phi) \cdot g_2(\partial_2 \Phi)$, where $g_1(\cdot) \in \{\bar{\varphi}(a^{-1} \cdot)\} \cup \{\varphi(a^{-1} 2^{-n} \cdot) : n \geq 0\}$ and $g_2(\cdot) \in \{\bar{\varphi}(A^{-1} \cdot)\} \cup \{\varphi(A^{-1} 2^{-m} \cdot) : m \geq 0\}$, one of the following two conditions holds for $\{j, k\} = \{1, 2\}$:

1. *(Diagonal-dominant case)*

$$|\det \nabla^2 \Phi| \sim |\partial_1^2 \Phi| |\partial_2^2 \Phi|, \quad |\partial_k \partial_j^2 \Phi| \lesssim \frac{1}{|f_k|} |\partial_j^2 \Phi|, \quad |\partial_1^2 \Phi|^{\frac{1}{2}} \sim a, \quad |\partial_2^2 \Phi|^{\frac{1}{2}} \sim A, \tag{2.14}$$

   Note that $|\partial_1 \partial_2 \Phi| \lesssim aA$ follows directly.

2. *(Off-diagonal-dominant case)*

$$|\det \nabla^2 \Phi| \sim |\partial_1 \partial_2 \Phi|^2, \quad |\partial_j^2 \Phi| \lesssim |\partial_1 \partial_2 \Phi| \simeq a^2 = A^2, \quad |\partial_k \partial_j^2 \Phi| \lesssim \frac{1}{|f_j|} |\partial_1 \partial_2 \Phi|. \tag{2.15}$$

   In addition,

$$|\partial_i^2 L| \lesssim \frac{|L|}{|f_i|^2}, \qquad \frac{1}{|f_i|} \lesssim a, \quad i \in \{1, 2\} \tag{2.16}$$

   hold in the support of $L(x) \cdot \bar{\varphi}(a^{-1} \partial_{3-i} \Phi) \cdot \varphi(a^{-1} 2^{-n} \partial_i \Phi)$ for each $n \geq 0$.

Then, if $x \mapsto (\nabla \Phi)(x)$ is injective on the support of $\Phi \cdot L$,

$$\left| \iint e^{i\Phi} L(x_1, x_2) dx_1 dx_2 \right| \lesssim (aA)^{-1} M. \tag{2.17}$$

Remark 2.2.2. *It should be stressed that the constant hidden in the symbol $\lesssim$ in (2.17) depends only on the constants used in (2.14) or (2.15) and (2.16), and the choice of φ.*

In particular, it does not depend on the derivatives of L or Φ in any other implicit way. It should also be noted that the only mild conditions are used in the sense that they only assume a type of homogeneity.

The proof works by decomposing the integral into several pieces. To make the proof more readable, I treat the hardest piece separately by the following proposition.

Proposition 2.2.3. *Let $L \in C^2(\mathbb{R}^2)$, and assume that there exist $M > 0$ and $f_j \in C^0(\mathbb{R}^2)$, $L_j \in C^0(\mathbb{R})$, $j \in \{1, 2\}$, such that*

$$|L| \leq M, \qquad \iint |\partial_1 \partial_2 L| dx \leq M,$$
$$|\partial_j L|, \frac{|L|}{|f_j|}(x_1, x_2) \leq L_j(x_j), \qquad \int L_j(x_j) dx_j \leq M.$$

Given $\Phi \in C^3(\operatorname{supp}(L))$, assume moreover that there exist $a, A > 0$, such that on the support of $L(x_1, x_2) \cdot \varphi_1(a^{-1} 2^{-n} \partial_1 \Phi) \cdot \varphi_2(A^{-1} 2^{-m} \partial_2 \Phi)$ for any $n, m \geq 0$, one of the following two conditions holds:

(1) $|\det \nabla^2 \Phi| \sim |\partial_1^2 \Phi||\partial_2^2 \Phi|, \quad |\partial_k \partial_j^2 \Phi| \lesssim \frac{1}{|f_k|} |\partial_j^2 \Phi|, \quad |\partial_1^2 \Phi|^{\frac{1}{2}} \sim a, \quad |\partial_2^2 \Phi|^{\frac{1}{2}} \sim A,$

(2) $|\det \nabla^2 \Phi| \sim |\partial_1 \partial_2 \Phi|^2 \sim a^2 = A^2, \quad |\partial_1^2 \Phi|, |\partial_2^2 \Phi| \lesssim |\partial_1 \partial_2 \Phi|, \quad |\partial_k \partial_j^2 \Phi| \lesssim \frac{1}{|f_j|} |\partial_k \partial_j \Phi|,$

where $\{j, k\} = \{1, 2\}$. Then, if $x \mapsto (\nabla \Phi)(x)$ is injective on the support of $\Phi \cdot L$,

$$\left| \iint e^{i\Phi} \varphi_1(a^{-1} 2^{-n} \partial_1 \Phi) \varphi_2(A^{-1} 2^{-m} \partial_2 \Phi) L(x_1, x_2) dx_1 dx_2 \right| \lesssim (aA)^{-1} M 2^{-n-m}. \tag{2.18}$$

Proof of Theorem 2.2.1. Decompose the integral into $I + \sum_{n\geq 0} I_n + \sum_{m\geq 0} I^m + \sum_{n,m\geq 0} I_{n,m}$ where

$$\begin{aligned} I &= \iint e^{i\Phi}\bar{\varphi}(a^{-1}\partial_1\Phi)\bar{\varphi}(A^{-1}\partial_2\Phi)Ldx, \\ I_n &= \iint e^{i\Phi}\varphi(a^{-1}2^{-n}\partial_1\Phi)\bar{\varphi}(A^{-1}\partial_2\Phi)Ldx, \\ I^m &= \iint e^{i\Phi}\bar{\varphi}(a^{-1}\partial_1\Phi)\varphi(A^{-1}2^{-m}\partial_2\Phi)Ldx, \\ I_{n,m} &= \iint e^{i\Phi}\varphi(a^{-1}2^{-n}\partial_1\Phi)\varphi(A^{-1}2^{-m}\partial_2\Phi)Ldx. \end{aligned}$$

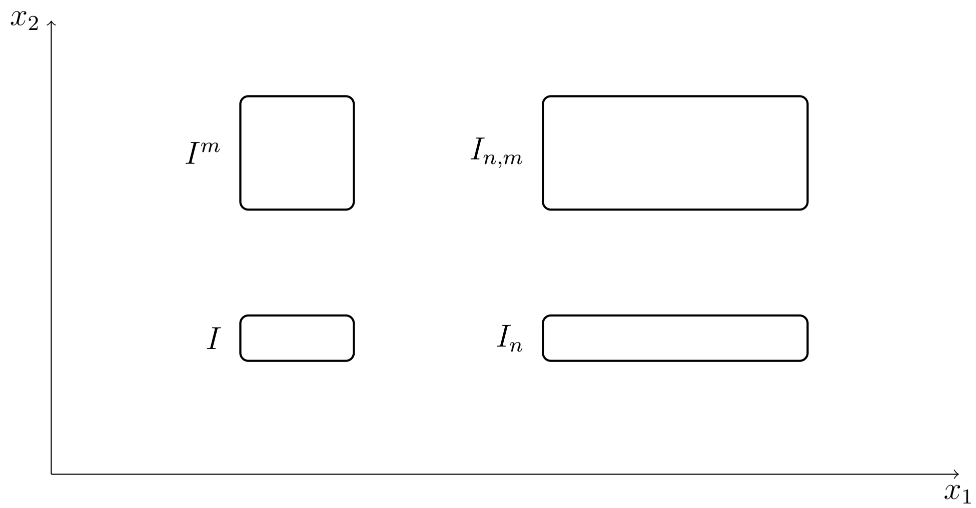

Figure 2.1: Schematic picture of the decomposition in the diagonal-dominant case.

For notational convenience, from hereon we will write single integral sign instead of two. Then,

$$|I| = \left|\int e^{i\Phi}\bar{\varphi}(a^{-1}\partial_1\Phi)\bar{\varphi}(A^{-1}\partial_2\Phi)L(x_1,x_2)dx_1dx_2\right| \lesssim (\det\nabla^2\Phi)^{-\frac{1}{2}}M,$$

by changing the variables. $I_{n,m}$'s are dealt with by Proposition 2.2.3. For I^m, we showcase the harder case (2.15) which involves using condition (2.16) (the case (2.14) does not need this extra assumption). Via integration by parts,

$$|I^m| = a^{-1}2^{-m}\left|\int e^{i\Phi}\partial_2\left[\bar{\varphi}(a^{-1}\partial_1\Phi)\tilde{\varphi}(a^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)\right]dx\right|.$$

Note that $a = A$ in case (2.15). The easiest term is when the derivative falls on $\varphi(a^{-1}2^{-m}\partial_2\Phi)$:

$$\begin{aligned}
&a^{-1}2^{-m}\left|\int e^{i\Phi}\bar{\varphi}(a^{-1}\partial_1\Phi)\cdot a^{-1}2^{-m}\partial_2^2\Phi\cdot\tilde{\varphi}'(a^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)dx\right|\\
&\lesssim a^{-1}2^{-m}\int\bar{\varphi}(a^{-1}\partial_1\Phi)\cdot a^{-1}2^{-m}a^2\cdot|\tilde{\varphi}'|(a^{-1}2^{-m}\partial_2\Phi)dx\cdot M\lesssim a^{-2}2^{-m}M,
\end{aligned}$$

by changing the variable from (x_1, x_2) to $(\partial_1\Phi, \partial_2\Phi)$. When it falls on L, we do another integration by parts and get

$$\begin{aligned}
&a^{-1}2^{-m}\left|\int e^{i\Phi}\bar{\varphi}(a^{-1}\partial_1\Phi)\tilde{\varphi}(a^{-1}2^{-m}\partial_2\Phi)\partial_2L(x_1,x_2)dx\right|\\
&=a^{-2}2^{-2m}\left|\int e^{i\Phi}\partial_2[\bar{\varphi}(a^{-1}\partial_1\Phi)\tilde{\tilde{\varphi}}(a^{-1}2^{-m}\partial_2\Phi)\partial_2L(x_1,x_2)]dx\right|\\
&\lesssim a^{-2}2^{-2m}\left[\int a^{-1}|\partial_1\partial_2\Phi||\bar{\varphi}'|(a^{-1}\partial_1\Phi)\tilde{\tilde{\varphi}}(a^{-1}2^{-m}\partial_2\Phi)L_2(x_2)dx\right.\\
&\qquad+\int a^{-1}2^{-m}|\partial_2^2\Phi||\tilde{\tilde{\varphi}}'|(a^{-1}2^{-m}\partial_2\Phi)L_2(x_2)dx\\
&\qquad\left.+\int\bar{\varphi}(a^{-1}\partial_1\Phi)\tilde{\tilde{\varphi}}(a^{-1}2^{-m}\partial_2\Phi)\frac{M}{|f_2|^2}dx\right]\lesssim a^{-2}2^{-m}M.
\end{aligned}$$

Finally, if the derivative falls on $\bar{\varphi}(a^{-1}\partial_1\Phi)$,

$$\begin{aligned}
&a^{-1}2^{-m}\left|\int e^{i\Phi}a^{-1}\partial_1\partial_2\Phi\bar{\varphi}'(a^{-1}\partial_1\Phi)\tilde{\varphi}(a^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)dx\right|\\
&=a^{-2}2^{-2m}\left|\int e^{i\Phi}\partial_2[a^{-1}\partial_1\partial_2\Phi\bar{\varphi}'(a^{-1}\partial_1\Phi)\tilde{\tilde{\varphi}}(a^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)]dx\right|\\
&\lesssim a^{-2}2^{-2m}\left[\int a^{-1}\frac{1}{|f_2|}|\partial_1\partial_2\Phi||\bar{\varphi}'|(a^{-1}\partial_1\Phi)\tilde{\tilde{\varphi}}(a^{-1}2^{-m}\partial_2\Phi)\cdot Mdx\right.\\
&\qquad+\int a^{-2}|\partial_1\partial_2\Phi|^2|\bar{\varphi}''|(a^{-1}\partial_1\Phi)\tilde{\tilde{\varphi}}(a^{-1}2^{-m}\partial_2\Phi)\cdot Mdx\\
&\qquad+\int a^{-2}2^{-m}|\partial_1\partial_2\Phi||\partial_2^2\Phi||\bar{\varphi}'|(a^{-1}\partial_1\Phi)|\tilde{\tilde{\varphi}}'|(a^{-1}2^{-m}\partial_2\Phi)\cdot Mdx\\
&\qquad\left.+\int a^{-1}|\partial_1\partial_2\Phi|\tilde{\tilde{\varphi}}(a^{-1}2^{-m}\partial_2\Phi)L_2(x_2)dx\right]\lesssim a^{-2}2^{-m}M,
\end{aligned}$$

which are all summable in m. I_n's can be treated similarly. □

Now we prove Proposition 2.2.3.

Proof of Proposition 2.2.3. Noting that the sizes of $\partial_1\Phi, \partial_2\Phi$ are specified, we integrate by parts in both directions. Then, the integral in (2.18) becomes

$$-a^{-1}2^{-n} \cdot A^{-1}2^{-m} \cdot \int e^{i\Phi}\partial_1\partial_2\left[\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\varphi}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)\right]dx.$$

Hence, from hereon, we only need to show that the integral above is bounded by M. First,

$$\left|\int e^{i\Phi}\partial_j\left[\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\varphi}_2(A^{-1}2^{-m}\partial_2\Phi)\right]\partial_k[L(x_1,x_2)]dx\right| \lesssim M,$$
$$\left|\int e^{i\Phi}\partial_j\left[\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\right]\partial_k[\tilde{\varphi}_2(A^{-1}2^{-m}\partial_2\Phi)]L(x_1,x_2)dx\right| \lesssim M,$$
$$\left|\int e^{i\Phi}\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\varphi}_2(A^{-1}2^{-m}\partial_2\Phi)\,\partial_j\partial_k[L(x_1,x_2)]dx\right| \lesssim M,$$

for $\{j,k\}=\{1,2\}$, by changing the variables from (x_1,x_2) to $(\partial_1\Phi,x_k),(\partial_2\Phi,x_k),(\partial_1\Phi,\partial_2\Phi)$ or (x_1,x_2) appropriately, and applying the assumptions on L and Φ. When the derivative from second integration by parts falls on the second order derivative of the phase from the first integration by parts,

$$\left|\int e^{i\Phi}\partial_2(a^{-1}2^{-n}\partial_1^2\Phi)\tilde{\varphi}_1'(a^{-1}2^{-n}\partial_1\Phi)\tilde{\varphi}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)dx\right| \lesssim M$$

follows by changing the variables from (x_1,x_2) to $(\partial_1\Phi,x_2)$ in the case (2.14), or from (x_1,x_2) to $(x_1,\partial_1\Phi)$ in the case (2.15). One can check that the analogous claim with the roles of ∂_1 and ∂_2 swapped holds as well.

For the only remaining case when both derivatives fall on the same phase localization term, we do another integration by parts. We will show

$$\left|\int e^{i\Phi}a^{-1}2^{-n}\partial_1^2\Phi\cdot a^{-1}2^{-n}\partial_2\partial_1\Phi\cdot\tilde{\varphi}_1''(a^{-1}2^{-n}\partial_1\Phi)\tilde{\varphi}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)dx\right|$$
$$\lesssim M(2^{-n-m}+2^{-2n}),$$

and one can check that the similar result for swapping the role of ∂_1 and ∂_2 holds as well. For notational convenience, we write $\tilde{\varphi}_1$ instead of $\tilde{\varphi}_1''$ as there is no qualitative difference. We integrate by parts once more in ∂_2 and deal with

$$A^{-1}2^{-m}\int \left|\partial_2\left[a^{-2}2^{-2n}\partial_1^2\Phi\cdot\partial_2\partial_1\Phi\cdot\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)\right]\right| dx.$$

1) When ∂_2 falls on L,

$$\begin{aligned}
&A^{-1}2^{-m}\int \left|a^{-2}2^{-2n}\partial_1^2\Phi\cdot\partial_2\partial_1\Phi\cdot\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)\partial_2 L(x_1,x_2)\right| dx\\
&\lesssim A^{-1}2^{-m}\int a^{-1}A2^{-2n}|\partial_1^2\Phi||\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)|L_2(x_2)dx\\
&\lesssim A^{-1}2^{-m}\cdot a^{-1}A2^{-2n}\cdot a2^nM = M2^{-n-m}.
\end{aligned}$$

2) When ∂_2 falls on $\tilde{\tilde{\varphi}}_2$,

$$\begin{aligned}
&A^{-1}2^{-m}\int \left|a^{-2}2^{-2n}\partial_1^2\Phi\cdot\partial_2\partial_1\Phi\cdot\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\partial_2[\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)]L(x_1,x_2)\right| dx\\
&\lesssim A^{-1}2^{-m}\int aA2^{-2n}|\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)|\cdot A2^{-m}|\tilde{\tilde{\varphi}}_2'(A^{-1}2^{-m}\partial_2\Phi)|dx\cdot M\\
&\lesssim aA2^{-2m-2n}\cdot a2^nA2^m\cdot(aA)^{-2}\cdot A^{-1}2^{-m}\cdot A2^m\cdot M = M2^{-n-m}.
\end{aligned}$$

3) When ∂_2 falls on $\tilde{\varphi}_1$,

$$\begin{aligned}
&A^{-1}2^{-m}\int \left|a^{-2}2^{-2n}\partial_1^2\Phi\cdot\partial_2\partial_1\Phi\cdot\partial_2[\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)]\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)\right| dx\\
&\lesssim A^{-1}2^{-m}\int aA2^{-2n}\cdot A2^{-n}|\tilde{\varphi}_1'(a^{-1}2^{-n}\partial_1\Phi)|\cdot|\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)|dx\cdot M\\
&\lesssim aA2^{-m-3n}\cdot a2^n\cdot A2^m\cdot(aA)^{-2}\cdot M = M2^{-2n}.
\end{aligned}$$

4) When ∂_2 falls on $\partial_2\partial_1\Phi$, in the case (2.14)

$$
\begin{aligned}
&A^{-1}2^{-m}\int \left|a^{-2}2^{-2n}\partial_1^2\Phi\cdot\partial_2^2\partial_1\Phi\cdot\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)\right| dx\\
&\lesssim A^{-1}2^{-m}\int 2^{-2n}\cdot\frac{1}{|f_1|}|\partial_2^2\Phi|\cdot|\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)||L(x_1,x_2)|dx\\
&\lesssim A^{-1}2^{-m}\cdot 2^{-2n}\cdot A2^m\cdot M = M2^{-2n}.
\end{aligned}
$$

In the case (2.15),

$$
\begin{aligned}
&A^{-1}2^{-m}\int \left|a^{-2}2^{-2n}\partial_1^2\Phi\cdot\partial_2^2\partial_1\Phi\cdot\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)\right| dx\\
&\lesssim A^{-1}2^{-m}\int 2^{-2n}\cdot\frac{1}{|f_2|}|\partial_1\partial_2\Phi|\cdot|\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)||L(x_1,x_2)|dx\\
&\lesssim A^{-1}2^{-m}\cdot 2^{-2n}\cdot A2^m\cdot M = M2^{-2n}.
\end{aligned}
$$

5) Finally, when ∂_2 falls on $\partial_1^2\Phi$,

$$
\begin{aligned}
&A^{-1}2^{-m}\int \left|a^{-2}2^{-2n}\partial_2\partial_1^2\Phi\cdot\partial_2\partial_1\Phi\cdot\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)\right| dx\\
&\lesssim A^{-1}2^{-m}\int a^{-1}A2^{-2n}\cdot\frac{1}{|f_2|}|\partial_1^2\Phi||\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)||L(x_1,x_2)|dx\\
&\lesssim A^{-1}2^{-m}\cdot a^{-1}A2^{-2n}\cdot a2^n\cdot M = M2^{-n-m},
\end{aligned}
$$

in the case (2.14). With (2.15),

$$
\begin{aligned}
&A^{-1}2^{-m}\int \left|a^{-2}2^{-2n}\partial_2\partial_1^2\Phi\cdot\partial_2\partial_1\Phi\cdot\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)\tilde{\tilde{\varphi}}_2(A^{-1}2^{-m}\partial_2\Phi)L(x_1,x_2)\right| dx\\
&\lesssim A^{-1}2^{-m}\int 2^{-2n}\cdot\frac{1}{|f_1|}|\partial_1\partial_2\Phi||\tilde{\varphi}_1(a^{-1}2^{-n}\partial_1\Phi)||L(x_1,x_2)|dx\\
&\lesssim A^{-1}2^{-m-2n}\cdot a2^n\cdot M = M2^{-n-m}.
\end{aligned}
$$

□

2.2.2 3-dimensional stationary phase theorem

While the above result is enough for the scope of this thesis, the proof can be generalized to higher dimensions in a suitable way. As an example, I give here the statement and the proof of theorem for 3-dimension and diagonal-dominant case. In general, I conjecture that up to $d+2$ derivative conditions on Φ and up to d many derivative conditions on L will be enough to derive a corresponding stationary phase theorem in d-dimension, but since I am mostly interested in problems originating from physics, I have not striven to generalize over dimension 3.

Theorem 2.2.4. *(3-dimensional stationary phase theorem) Let $L \in C^3(\mathbb{R}^3)$, and assume that there exist $M > 0$ and $0 < f_j \in C^0(\mathbb{R}^3)$, $L_j \in C^0(\mathbb{R}^2)$, $j \in \{1,2,3\}$, $L_{j,k} \in C^0(\mathbb{R})$, $1 \leq j < k \leq 3$, such that*

$$\begin{aligned} |L| \leq M, \qquad \iiint |\partial_{123} L| dx \leq M, \\ |\partial_j L|, \frac{|L|}{f_j}(x_1,x_2,x_3) \leq L_j(x_j), \qquad \int L_j(x_j) dx_j \leq M, \quad j \in \{1,2,3\} \\ |\partial_{j,k} L|, \frac{|L|}{f_j f_k}(x_1,x_2,x_3) \leq L_{j,k}(x_j,x_k), \qquad \iint L_{j,k} dx_j dx_k \leq M, \quad 1 \leq j < k \leq 3. \end{aligned}$$

Given $\Phi \in C^5(\mathrm{supp}(L))$, assume moreover that there exist $a_1, a_2, a_3 > 0$, such that on the support of each $L(x) \cdot \prod_{j=1}^3 g_j(\partial_j \Phi)$, where $g_j(\cdot) \in \{\bar{\varphi}(a_j^{-1} \cdot)\} \cup \{\varphi(a_j^{-1} 2^{-n} \cdot) : n \geq 0\}$, and the following two conditions hold:

1. *(Diagonal-dominance) For $j, k \in \{1,2,3\}$,*

$$|\det \nabla^2 \Phi| \sim |\partial_1^2 \Phi||\partial_2^2 \Phi||\partial_3^2 \Phi|, \quad |\det \nabla^2_{j,k} \Phi| \sim |\partial_j^2 \Phi||\partial_k^2 \Phi|, \quad |\partial_j^2 \Phi|^{\frac{1}{2}} \sim a_j. \tag{2.19}$$

2. *(Homogeneity of high-order derivatives)*

$$|\partial_{j_1, j_2, \cdots, j_l} \Phi| \lesssim f_{j_3}^{-1} \cdots f_{j_l}^{-1} a_{j_1} a_{j_2}, \quad j_1, \cdots, j_l \in \{1,2,3\}, \quad 3 \leq l \leq 5, \tag{2.20}$$

where (if exist) j_3, j_4, j_5 should be all distinct. Note that when $l = 2$, and as such drop the f_j^{-1}'s, this follows directly from (2.19) *(and hence excluded).*

Then, if $x \mapsto (\nabla\Phi)(x)$ is injective on the support of $\Phi \cdot L$,

$$\left| \iiint e^{i\Phi} L(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right| \lesssim (a_1 a_2 a_3)^{-1} M. \tag{2.21}$$

In particular, the constant in $\lesssim$ only depends on the choice of function φ used for the partition of unity and the constants implicit in conditions (2.19) *and* (2.20)*.*

Proof. For notational convenience, we will write all integrals in just single integral sign $\int$, as the number of integrals should be clear from the context.

Denote the integral on the left-hand side of (2.21) as I. We decompose it into

$$\begin{aligned} I_{0,0,0} &:= \int e^{i\Phi} \prod_{i=1}^{3} \bar{\varphi}(a_i^{-1} \partial_i \Phi) L(x) dx, \\ I_{n,0,0} &:= \int e^{i\Phi} \varphi(2^{-n} a_1^{-1} \partial_1 \Phi) \prod_{i=2}^{3} \bar{\varphi}(a_i^{-1} \partial_i \Phi) L(x) dx, \\ I_{n,m,0} &:= \int e^{i\Phi} \varphi(2^{-n} a_1^{-1} \partial_1 \Phi) \varphi(2^{-m} a_2^{-1} \partial_2 \Phi) \bar{\varphi}(a_3^{-1} \partial_3 \Phi) L(x) dx, \\ I_{n,m,l} &:= \int e^{i\Phi} \varphi(2^{-n} a_1^{-1} \partial_1 \Phi) \varphi(2^{-m} a_2^{-1} \partial_2 \Phi) \varphi(2^{-l} a_3^{-1} \partial_3 \Phi) L(x) dx, \end{aligned}$$

for $n, m, l \in \mathbb{N}$. Indeed, there are symmetric terms $I_{0,m,0}, I_{0,0,l}, I_{0,m,l}, I_{n,0,l}$ that can be analyzed similarly and hence will not be discussed. $I_{0,0,0}$ is the simplest to estimate, which, through the change of variable $x \to \nabla\Phi$, gives

$$|I_{0,0,0}| \lesssim a_1 a_2 a_3 \cdot M \cdot (\det \nabla^2 \Phi|_{\mathrm{supp}\ L \cdot \nabla\Phi})^{-1} \simeq (a_1 a_2 a_3)^{-1} M. \tag{2.22}$$

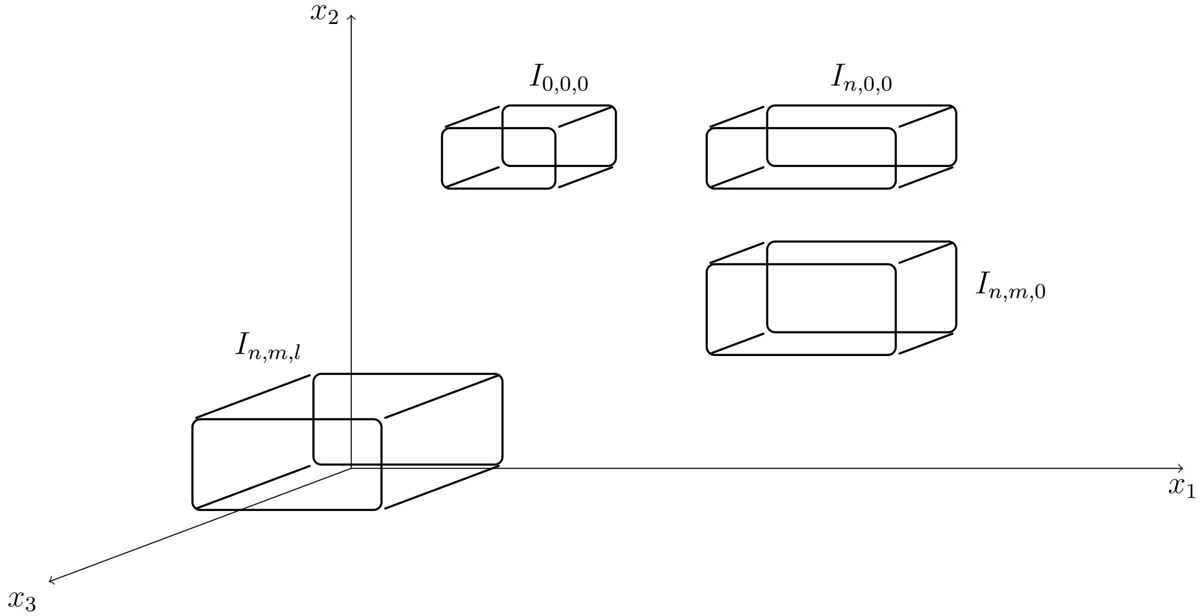

Figure 2.2: Schematic picture of the decomposition.

① $I_{n,0,0}$: Here, we first integrate by parts in ∂_1 once to obtain

$$\begin{aligned}
I_{n,0,0} = (2^n a_1)^{-1} i \int e^{i\Phi} \Bigg[& \tilde{\varphi}(2^{-n} a_1^{-1} \partial_1 \Phi) \prod_{j=2}^{3} \bar{\varphi}(a_j^{-1} \partial_j \Phi) \partial_1 L(x) \\
& + 2^{-n} a^{-1} \partial_1^2 \Phi \tilde{\varphi}'(2^{-n} a_1^{-1} \partial_1 \Phi) \prod_{j=2}^{3} \bar{\varphi}(a_j^{-1} \partial_j \Phi) L(x) \\
& + \tilde{\varphi}(2^{-n} a_1^{-1} \partial_1 \Phi) a_2^{-1} \partial_{1,2} \Phi \bar{\varphi}'(a_2^{-1} \partial_2 \Phi) \bar{\varphi}(a_3^{-1} \partial_3 \Phi) L(x) \qquad (2.23) \\
& + \tilde{\varphi}(2^{-n} a_1^{-1} \partial_1 \Phi)) \bar{\varphi}(a_2^{-1} \partial_2 \Phi) a_3^{-1} \partial_{1,3} \Phi \bar{\varphi}'(a_3^{-1} \partial_3 \Phi) L(x) \Big] \, dx.
\end{aligned}$$

The second line gives the desired bound of $2^{-n}(a_1 a_2 a_3)^{-1} M$ by changing the variable $x \to \nabla\Phi$ and the first line by $x \to (x_1, \partial_2\Phi, \partial_3\Phi)$ after using $|\partial_1 L| \lesssim L_1$. The third and fourth lines are symmetric and hence we only deal with (2.23). Here, we perform another integration by parts in ∂_1 (and from hereon, we will ignore all i's coming from integration by parts)

$$
\begin{aligned}
(2.23) = (2^n a_1)^{-2} \int e^{i\Phi} \Big[& \tilde{\tilde{\varphi}}(2^{-n}a_1^{-1}\partial_1\Phi)a_2^{-1}\partial_{1,2}\Phi\bar{\varphi}'(a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)\partial_1 L(x) \\
& + 2^{-n}a_1^{-1}\partial_1^2\Phi\tilde{\tilde{\varphi}}'(2^{-n}a_1^{-1}\partial_1\Phi)a_2^{-1}\partial_{1,2}\Phi\bar{\varphi}'(a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)L(x) \\
& + \tilde{\tilde{\varphi}}(2^{-n}a_1^{-1}\partial_1\Phi)a_2^{-1}\partial_1^2\partial_2\Phi\bar{\varphi}'(a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)L(x) \\
& + \tilde{\tilde{\varphi}}(2^{-n}a_1^{-1}\partial_1\Phi)a_2^{-2}(\partial_{1,2}\Phi)^2\bar{\varphi}''(a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)L(x) \\
& + \tilde{\tilde{\varphi}}(2^{-n}a_1^{-1}\partial_1\Phi)a_2^{-1}\partial_{1,2}\Phi\bar{\varphi}'(a_2^{-1}\partial_2\Phi)a_3^{-1}\partial_{1,3}\Phi\bar{\varphi}(a_3^{-1}\partial_3\Phi)L(x)\Big] dx
\end{aligned}
$$

On every line, we use $|\partial_{1,2}\Phi| \lesssim a_1a_2$, except in the third line where we instead use $|\partial_1^2\partial_2\Phi| \lesssim f_1^{-1}a_1a_2$. The first and third lines give the desired estimate by changing the variable to $(x_1, \partial_2\Phi, \partial_3\Phi)$, together with $|\partial_1 L|, f_1^{-1}|L| \lesssim L_1$. For all other lines, we change the variable $x \to \nabla\Phi$, and also $|\partial_{1,3}\Phi| \lesssim a_1a_3$ in the last line. These give an estimate of (at least) $2^{-n}(a_1a_2a_3)^{-1}M$, and therefore

$$\sum_{n\geq 1} |I_{n,0,0}| \lesssim (a_1a_2a_3)^{-1}M. \tag{2.24}$$

② $I_{n,m,0}$: As can be seen above, $\tilde{\varphi}, \tilde{\varphi}', \tilde{\tilde{\varphi}}, \tilde{\tilde{\varphi}}'$ do not have any qualitative difference in the estimates, and hence they will be loosely distinguished. For example, we will not strive to distinguish $\widetilde{(\tilde{\varphi}')}$ from $(\tilde{\tilde{\varphi}})'$ (or even the number of tildes) for notational convenience.

Integrating by parts in both ∂_1 and ∂_2, we have

$$\begin{aligned}
I_{n,m,0} = (2^{n+m}a_1a_2)^{-1}\int e^{i\Phi}\Big[&\partial_{1,2}L\,\tilde{\varphi}(2^{-n}a_1^{-1}\partial_1\Phi)\tilde{\varphi}(2^{-m}a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)\\
&+\partial_2L\cdot\Big\{2^{-n}a_1^{-1}\partial_1^2\Phi\tilde{\varphi}'(2^{-n}a_1^{-1}\partial_1\Phi)\tilde{\varphi}(2^{-m}a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)\\
&\qquad+\tilde{\varphi}(2^{-n}a_1^{-1}\partial_1\Phi)2^{-m}\partial_{1,2}\Phi\tilde{\varphi}'(2^{-m}a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)\\
&\qquad+\tilde{\varphi}(2^{-n}a_1^{-1}\partial_1\Phi)\tilde{\varphi}(2^{-m}a_2^{-1}\partial_2\Phi)a_3^{-1}\partial_{1,3}\Phi\bar{\varphi}'(a_3^{-1}\partial_3\Phi)\Big\}\\
&+\partial_1L\cdot\partial_2\Big(\tilde{\varphi}(2^{-n}a_1^{-1}\partial_1\Phi)\tilde{\varphi}(2^{-m}a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)\Big)\\
&+L\cdot\partial_{1,2}\left(\tilde{\varphi}(2^{-n}a_1^{-1}\partial_1\Phi)\tilde{\varphi}(2^{-m}a_2^{-1}\partial_2\Phi)\bar{\varphi}(a_3^{-1}\partial_3\Phi)\right)\Big]dx.
\end{aligned} \tag{2.25}$$

The first line is easily handled by changing the variable to $(x_1, x_2, \partial_3\Phi)$. The lines inside the curly brackets and the line with the round brackets are symmetric, and notice that second to fourth lines can be dealt with exactly the same way as we dealt with second to fourth lines for $I_{n,0,0}$ where (2.23) is, except that whenever we don't change the second variable to $\partial_2\Phi$, but use x_2. For example, the second line will use the variable $(\partial_1\Phi, x_2, \partial_3\Phi)$, and the third line will involve another integration by parts in ∂_1 as was done with (2.23), which all give $2^{-n-m}(a_1a_2a_3)^{-1}M$. (2.25) is genuinely a new term. From hereon, we make another notational abbreviation, and simply write $\tilde{\varphi}_1$ for $\tilde{\varphi}(2^{-n}a_1^{-1}\partial_1\Phi)$, and $\bar{\varphi}_3'$ for $\bar{\varphi}'(a_3^{-1}\partial_3\Phi)$, etc.

$$\begin{aligned}
(2.25) = (2^{n+m}a_1a_2)^{-1}\int e^{i\Phi}L\Big[&\{\partial_1(\tilde{\varphi}_1)\partial_2(\tilde{\varphi}_2)+\partial_2(\tilde{\varphi}_1)\partial_1(\tilde{\varphi}_1)\}\bar{\varphi}_3\\
&+(2^na_1)^{-1}\partial_1^2\partial_2\Phi\tilde{\varphi}_1'\tilde{\varphi}_2\bar{\varphi}_3+(2^ma_2)^{-1}\partial_1\partial_2^2\Phi\tilde{\varphi}_1\tilde{\varphi}_2'\bar{\varphi}\\
&+\partial_1\Big(\tilde{\varphi}_1\tilde{\varphi}_2\Big)a_3^{-1}\partial_{2,3}\Phi\bar{\varphi}_3'+\partial_2\Big(\tilde{\varphi}_1\tilde{\varphi}_2\Big)a_3^{-1}\partial_{1,3}\Phi\bar{\varphi}_3'\\
&+\tilde{\varphi}_1\tilde{\varphi}_2a_3^{-1}\partial_{123}\Phi\bar{\varphi}_3' && (2.26)\\
&+\tilde{\varphi}_1\tilde{\varphi}_2a_3^{-2}\partial_{1,3}\Phi\partial_{2,3}\Phi\bar{\varphi}_3''\Big]dx. && (2.27)
\end{aligned}$$

The first line generates $2^{-n-m}a_1a_2\partial_1^2\Phi\partial_2^2\Phi$ or $2^{-n-m}a_1a_2(\partial_{1,2}\Phi)^2$, both of which are bounded by $2^{-n-m}a_1a_2$ that can be absorbed by integration in $(\partial_1\Phi, \partial_2\Phi)$. Hence, using the variable $\nabla\Phi$, the first line gives the acceptable bound of $2^{-n-m}(a_1a_2a_3)^{-1}M$.

The second line has symmetric terms. The first one can be dealt with by using $|\partial_1^2\partial_2\Phi| \lesssim f_2^{-1}a_1^2$, and changing the variable to $(\partial_1\Phi, x_2, \partial_3\Phi)$. The second one follows symmetrically.

Starting from the third line, we need another integration by parts. The third line has two symmetric objects, so we again only deal with the first one, which is

$$(2^{n+m}a_1a_2)^{-1}\int e^{i\Phi}L\Big[2^{-n}a_1^{-1}\partial_1^2\Phi\tilde{\varphi}_1'\tilde{\varphi}_2\cdot a_3^{-1}\partial_{2,3}\Phi\bar{\varphi}_3' \tag{2.28}$$

$$+\,2^{-m}a_2^{-1}\partial_{1,2}\Phi\tilde{\varphi}_1\tilde{\varphi}_2'\cdot a_3^{-1}\partial_{2,3}\Phi\bar{\varphi}_3'\Big]\,dx. \tag{2.29}$$

Both of these will need another integration by parts. For (2.28) we integrate by parts in ∂_2:

$$\begin{aligned}(2.28) = (2^{n+m}a_1a_2)^{-1}\int e^{i\Phi}2^{-n-m}a_1^{-1}a_2^{-1}\Big[&\partial_2L\,\partial_1^2\Phi\,\tilde{\varphi}_1'\,\tilde{\tilde{\varphi}}_2\cdot a_3^{-1}\partial_{2,3}\Phi\,\bar{\varphi}_3'\\
&+L\,\partial_1^2\partial_2\Phi\,\tilde{\varphi}_1'\,\tilde{\tilde{\varphi}}_2\cdot a_3^{-1}\partial_{2,3}\Phi\,\bar{\varphi}_3'\\
&+L\,2^{-n}\partial_1^2\Phi\cdot a_1^{-1}\partial_{1,2}\Phi\,\tilde{\varphi}_1''\,\tilde{\tilde{\varphi}}_2\cdot a_3^{-1}\partial_{2,3}\Phi\,\bar{\varphi}_3'\\
&+L\,\partial_1^2\Phi\,\tilde{\varphi}_1'\,2^{-m}a_2^{-1}\partial_2^2\Phi\tilde{\tilde{\varphi}}_2'\cdot a_3^{-1}\partial_{2,3}\Phi\,\bar{\varphi}_3'\\
&+L\,\partial_1^2\Phi\,\tilde{\varphi}_1'\,\tilde{\tilde{\varphi}}_2\cdot a_3^{-1}\partial_2^2\partial_3\Phi\,\bar{\varphi}_3'\\
&+L\,\partial_1^2\Phi\,\tilde{\varphi}_1'\,\tilde{\tilde{\varphi}}_2\cdot a_3^{-2}(\partial_{2,3}\Phi)^2\,\bar{\varphi}_3''\Big]\,dx.\end{aligned}$$

The first line is fine by changing the variable to $(\partial_1\Phi, x_2, \partial_3\Phi)$ after using $|\partial_1^2\Phi\cdot\partial_{2,3}\Phi| \lesssim a_1^2a_2a_3$ and $|\partial_2L| \lesssim L_2$. The second line uses the same change of variable, but with $|\partial_1^2\partial_2\Phi| \lesssim f_2^{-1}a_1^2$. The third, fourth, and last lines all use $\nabla\Phi$ as the integration variable after applying $|\partial_{j,k}\Phi| \lesssim a_ja_k$ everywhere. Finally, the fifth line can be dealt with by

$|\partial_2^2\partial_3\Phi| \lesssim f_2^{-1}a_2a_3$, and changing the variable to $(\partial_1\Phi, x_2, \partial_3\Phi)$. These all give the desired estimate of $(2^{n+m}a_1a_2)^{-1}\cdot a_3^{-1}M$ or smaller (such as $(2^{n+2m}a_1a_2)^{-1}\cdot a_3^{-1}M$, which has no essential difference after summation in n and m). Next,

$$\begin{aligned}(2.29) = (2^{n+m}a_1a_2)^{-1}\int e^{i\Phi}2^{-n-m}a_1^{-1}a_2^{-1}\Big[&\partial_1 L\,\partial_{1,2}\Phi\,\tilde{\tilde{\varphi}}_1\,\tilde{\varphi}_2'\cdot a_3^{-1}\partial_{2,3}\Phi\,\bar{\varphi}_3' \\ &+ L\,\partial_1^2\partial_2\Phi\,\tilde{\tilde{\varphi}}_1\,\tilde{\varphi}_2'\cdot a_3^{-1}\partial_{2,3}\Phi\,\bar{\varphi}_3' \\ &+ L\,2^{-n}a_1^{-1}\partial_1^2\Phi\cdot\partial_{1,2}\Phi\,\tilde{\tilde{\varphi}}_1'\,\tilde{\varphi}_2'\cdot a_3^{-1}\partial_{2,3}\Phi\,\bar{\varphi}_3' \\ &+ L\,2^{-m}a_2^{-1}(\partial_{1,2}\Phi)^2\,\tilde{\tilde{\varphi}}_1\,\tilde{\varphi}_2''\cdot a_3^{-1}\partial_{2,3}\Phi\,\bar{\varphi}_3' \\ &+ L\,\partial_{1,2}\Phi\,\tilde{\tilde{\varphi}}_1\,\tilde{\varphi}_2'\cdot a_3^{-1}\partial_{123}\Phi\,\bar{\varphi}_3' \\ &+ L\;\tilde{\tilde{\varphi}}_1\,\tilde{\varphi}_2'\cdot a_3^{-2}\partial_{1,3}\Phi\partial_{2,3}\Phi\,\bar{\varphi}_3''\Big]\,dx,\end{aligned}$$

by integration by parts in ∂_1. The analysis is similar. The first line uses $(x_1, \partial_2\Phi, \partial_3\Phi)$, the second and fifth lines use $(\partial_1\Phi, x_2, \partial_3\Phi)$ after using $|\partial_1^2\partial_2\Phi| \lesssim f_2^{-1}a_1^2$, or $|\partial_{123}\Phi| \lesssim f_2^{-1}a_1a_3$, and all the other lines change variables to $\nabla\Phi$ after applying appropriate bounds of $|\partial_j\partial_k\Phi| \lesssim a_ja_k$ again.

Now we move onto (2.26). Integrating by parts again in ∂_2 gives

$$\begin{aligned}(2.26) = (2^{n+m}a_1a_2a_3)^{-1}\int e^{i\Phi}2^{-m}a_2^{-1}\Big[&\partial_2 L\,\tilde{\varphi}_1\,\tilde{\tilde{\varphi}}_2\,\partial_{123}\Phi\,\bar{\varphi}_3' \\ &+ L\,2^{-n}a_1^{-1}\partial_{2,1}\Phi\,\tilde{\varphi}_1'\,\tilde{\tilde{\varphi}}_2\,\partial_{123}\Phi\,\bar{\varphi}_3' \\ &+ L\,\tilde{\varphi}_1\,2^{-m}a_2^{-1}\partial_2^2\Phi\,\tilde{\tilde{\varphi}}_2'\,\partial_{123}\Phi\,\bar{\varphi}_3' \\ &+ L\,\tilde{\varphi}_1\,\tilde{\tilde{\varphi}}_2\,\partial_1\partial_2^2\partial_3\Phi\,\bar{\varphi}_3' \\ &+ L\,\tilde{\varphi}_1\,\tilde{\tilde{\varphi}}_2\,a_3^{-1}\partial_{2,3}\Phi\partial_{123}\Phi\,\bar{\varphi}_3''\Big]\,dx.\end{aligned}$$

The first line can be integrated in $(x_1, x_2, \partial_3\Phi)$ after using $|\partial_{123}\Phi| \lesssim f_1^{-1}a_2a_3$, and the second, third, and fifth lines use $(x_1, \partial_2\Phi, \partial_3\Phi)$ after using the same bound. The fourth line uses $|\partial_1\partial_2^2\partial_3\Phi| \lesssim (f_1f_2)^{-1}a_2a_3$ and changes the variable to $(x_1, x_2, \partial_3\Phi)$, all of which give either $(2^{n+m}a_1a_2a_3)^{-1}M$ or $(2^{n+2m}a_1a_2a_3)^{-1}M$.

Now we are only left with (2.27). This needs two more integration by parts, both in ∂_1 and ∂_2. As mentioned above, ignoring nonessential differences, for example between $\widetilde{(\tilde{\varphi}')}$ and $(\tilde{\tilde{\varphi}})'$,

$$(2.27) = (2^{n+m}a_1a_2a_3)^{-1}\int 2^{-n-m}a_1^{-1}a_2^{-1}a_3^{-1}e^{i\Phi}\partial_1\partial_2\Big[L\,\tilde{\tilde{\varphi}}_1\tilde{\tilde{\varphi}}_2\partial_{1,3}\Phi\partial_{2,3}\Phi\bar{\varphi}_3''\Big]\,dx.$$

When $\partial_{1,2}$ falls on L, changing the variable to $(x_1, x_2, \partial_3\Phi)$ suffices. When one of the derivatives, ∂_2 without loss of generality, falls on L and the other one falls on the square bracket, the conclusion follows from similar computations as in the very beginning (see the paragraph following (2.25)). Since we appeal to analogy with the analysis in ① there, and since a bit involved terms of this type will arise in ③, we give explicit computation here. The integral under consideration, without $(2^{n+m}a_1a_2a_3)^{-1}$ in the very front, is

$$\begin{aligned}\int e^{i\Phi}2^{-n-m}a_1^{-1}a_2^{-1}a_3^{-1}\partial_2L\cdot\Big[&2^{-n}a_1^{-1}\partial_1^2\Phi\,\tilde{\tilde{\varphi}}_1'\,\tilde{\tilde{\varphi}}_2\,\partial_{1,3}\Phi\partial_{2,3}\Phi\,\bar{\varphi}_3''\\ &+\tilde{\tilde{\varphi}}_1\,2^{-m}a_2^{-1}\partial_{1,2}\Phi\,\tilde{\tilde{\varphi}}_2'\,\partial_{1,3}\Phi\partial_{2,3}\Phi\,\bar{\varphi}_3''\\ &+\tilde{\tilde{\varphi}}_1\,\tilde{\tilde{\varphi}}_2\,a_3^{-1}(\partial_{1,3}\Phi)^2\partial_{2,3}\Phi\,\bar{\varphi}_3'''\\ &+\tilde{\tilde{\varphi}}_1\,\tilde{\tilde{\varphi}}_2\,\partial_1^2\partial_3\Phi\partial_{2,3}\Phi\,\bar{\varphi}_3''\\ &+\tilde{\tilde{\varphi}}_1\,\tilde{\tilde{\varphi}}_2\,\partial_{1,3}\Phi\partial_{123}\Phi\,\bar{\varphi}_3''\Big]\,dx.\end{aligned}$$

The first three lines simply apply $|\partial_{j,k}\Phi| \lesssim a_ja_k$ and integrate by parts in $(\partial_1\Phi, x_2, \partial_3\Phi)$. The last two lines can be handled by applying $|\partial_1^2\partial_3\Phi| \lesssim f_1^{-1}a_1a_3$ or $|\partial_{123}\Phi| \lesssim f_1^{-1}a_2a_3$ and integrating in $(x_1, x_2, \partial_3\Phi)$.

As a result, we are left with

$$(2^{n+m}a_1a_2a_3)^{-1}\int 2^{-n-m}a_1^{-1}a_2^{-1}a_3^{-1}e^{i\Phi}L\,\partial_1\partial_2\Big[\tilde{\tilde{\varphi}}_1\tilde{\tilde{\varphi}}_2\partial_{1,3}\Phi\partial_{2,3}\Phi\bar{\varphi}_3''\Big]\,dx.$$

We break this down into easier pieces. Using the fact

$$|\partial_j(\varphi_k)| \leq a_k^{-1}|\partial_{j,k}\Phi||\varphi_k'| \lesssim a_j|\varphi_k'|,$$

we see that ∂_1, ∂_2 will essentially have the effect of multiplication by a_1 or a_2 when acted on $\tilde{\tilde{\varphi}}_1, \tilde{\tilde{\varphi}}_2$, or $\bar{\varphi}_3''$. Also, from $|\partial_k(\partial_{j_1,j_2}\Phi)| \lesssim f_k^{-1}a_{j_1}a_{j_2}$, ∂_1, ∂_2 can instead have an effect of multiplication by f_1^{-1} or f_2^{-1}. In summary,

$$\begin{aligned}
&\left|\int 2^{-n-m}a_1^{-1}a_2^{-1}a_3^{-1}e^{i\Phi}L\,\partial_1\partial_2\left[\tilde{\tilde{\varphi}}_1\tilde{\tilde{\varphi}}_2\partial_{1,3}\Phi\partial_{2,3}\Phi\bar{\varphi}_3''\right]dx\right| \\
\lesssim &\int 2^{-n-m}a_1^{-1}a_2^{-1}a_3^{-1}\cdot L\,\phi_1\phi_2\phi_3\cdot\left[a_1^2a_2^2a_3^2 + f_1^{-1}a_1a_2^2a_3^2 + f_2^{-1}a_1^2a_2a_3^2 + f_1^{-1}f_2^{-1}a_1a_2a_3^2\right]dx,
\end{aligned}$$

where ϕ_j's are some functions that have qualitatively the same property as $\tilde{\tilde{\varphi}}_1, \tilde{\tilde{\varphi}}_2, \bar{\varphi}_3$ or their derivatives. Choosing the right variable to integrate between $\nabla\Phi$, $(x_1, \partial_2\Phi, \partial_3\Phi)$, $(\partial_1\Phi, x_2, \partial_3\Phi)$, or $(x_1, x_2, \partial_3\Phi)$, we arrive at the conclusion that the integral above is bounded by 1. Therefore, we have

$$|I_{n,m,0}| \lesssim 2^{-n-m}(a_1a_2a_3)^{-1}M. \tag{2.30}$$

Summing up in $n, m \geq 1$, we have the desired conclusion.

③ $I_{n,m,l}$. We keep the shorthand notation of φ_1, φ_2, and φ_3. Indeed, we will abuse the notation and even drop all tildes and derivatives of these as we have seen that they do not incur any qualitative difference. In addition, we will use I_{n_1,n_2,n_3} instead of $I_{n,m,l}$ for notational purposes. Integrating by parts in all variables,

$$I_{n_1,n_2,n_3} = (2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}\int e^{i\Phi}\partial_{123}\Big\{L\,\varphi_1\,\varphi_2\,\varphi_3\Big\}dx.$$

Hence, the goal is to show that the integral is bounded by M. The general strategy is not different from above, only difference being that there are more terms to estimate and more integration by parts needed. The most trivial case is when all derivatives hit L, in which case $\int |\partial_{123} L| dx \lesssim M$ finishes the job.

Case 3-1. We first consider when L gets two derivatives, and without loss of generality, it's enough to consider

$$\begin{aligned}
&(2^{n_1+n_2+n_3} a_1 a_2 a_3)^{-1} \int e^{i\Phi} \partial_{1,2} L \, \partial_3 \Big\{ \varphi_1 \, \varphi_2 \, \varphi_3 \Big\} dx \\
&= (2^{n_1+n_2+n_3} a_1 a_2 a_3)^{-1} \int e^{i\Phi} \partial_{1,2} L \, \partial_3 \Big\{ 2^{-n_1} a_1^{-1} \partial_{3,1} \Phi \varphi_1 \, \varphi_2 \, \varphi_3 \\
&\qquad\qquad + \varphi_1 \, 2^{-n_2} a_2^{-1} \partial_{3,2} \varphi_2 \, \varphi_3 \\
&\qquad\qquad + \varphi_1 \, \varphi_2 \, 2^{-n_3} a_3^{-1} \partial_3^2 \Phi \varphi_3 \Big\} dx.
\end{aligned}$$

The last line is easily handled by integrating in $(x_1, x_2, \partial_3 \Phi)$. Remaining two lines are symmetric, so we only look at the first one. Integrating by parts in ∂_3 again, we get

$$\begin{aligned}
(2^{n_1+n_2+n_3} a_1 a_2 a_3)^{-1} \int e^{i\Phi} 2^{-n_1-n_3} a_1^{-1} a_3^{-1} \Big\{ &\partial_{123} L \partial_{3,1} \Phi \varphi_1 \, \varphi_2 \, \varphi_3 \\
&+ \partial_{1,2} L \Big[\partial_1 \partial_3^2 \Phi \, \varphi_1 \, \varphi_2 \, \varphi_3 \\
&\qquad + \partial_{3,1} \Phi \, \partial_3 \Big(\varphi_1 \, \varphi_2 \, \varphi_3 \Big) \Big] \Big\} dx.
\end{aligned}$$

First two lines don't require changing variables but only use $|\partial_{3,1}\Phi| \lesssim a_1 a_3$ and $\int |\partial_{123} L| dx \lesssim M$ for the first line, and $|\partial_1 \partial_3^2 \Phi| \lesssim f_3^{-1} a_1 a_3$ and $\int f_3^{-1} |\partial_{1,2} L| dx \lesssim M$ for the second line. For the last line, similar to the analysis in the last part of ② preceding line (2.30), note that ∂_3 creates $2^{-n_j} a_j^{-1} \partial_{3,j} \Phi \lesssim a_3$. Since $a_1^{-1} a_3^{-1} \partial_{3,1} \Phi$ outside is less than a fixed constant and there's also 2^{-n_3} outside, changing the variable to $(x_1, x_2, \partial_3 \Phi)$ gives a bound of $(2^{n_1+n_2+n_3} a_1 a_2 a_3)^{-1} M$ in all cases.

Case 3-2. Now we consider when L gets one derivative and the later part gets two derivatives. By symmetry, we only deal with

$$(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}\int e^{i\Phi}\partial_3 L\,\partial_{1,2}\Big\{\varphi_1\,\varphi_2\,\varphi_3\Big\}dx.$$

This is same in spirit to (2.25), except that we always use variable x_3 instead of using $\partial_3\Phi$ as we did for (2.25). The lack of a_3^{-1} in front of (2.25) is compensated by the fact that we have $\partial_3 L$ here, forcing us to use dx_3. One can check from ② that we only used variable $\partial_3\Phi$ as the third variable when dealing with (2.25).

Case 3-3. Now we consider when all three derivatives hit our phase delimiters. Namely,

$$(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}\int e^{i\Phi}L\,\partial_{123}\Big\{\varphi_1\,\varphi_2\,\varphi_3\Big\}dx.$$

Case 3-3-1. The easiest case is when the three derivatives hit each one of them. Then,

$$\begin{aligned}
&(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}\int e^{i\Phi}L\,\partial_{j_1}(\varphi_1)\,\partial_{j_2}(\varphi_2)\,\partial_{j_3}(\varphi_3)dx\\
&=(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}\int e^{i\Phi}L\prod_{k=1}^{3}2^{-n_k}\partial_{j_k,k}\Phi\,\varphi_k dx\\
&\lesssim(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}\int 2^{-n_1-n_2-n_3}a_1a_2a_3\varphi_1\,\varphi_2\,\varphi_3 dx\cdot M\quad\lesssim(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}M,
\end{aligned}$$

as desired by changing the variable to $\nabla\Phi$.

Case 3-3-2. Next, we look at when one of the φ_k receives two derivatives, and without loss of generality, choose that to be φ_3. Up to symmetry, there are two cases:

$$(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}\int e^{i\Phi}L\,\partial_1\Big\{\varphi_1\,\varphi_2\Big\}\,\partial_{2,3}(\varphi_3)dx, \tag{2.31}$$

$$(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}\int e^{i\Phi}L\,\partial_3\Big\{\varphi_1\,\varphi_2\Big\}\,\partial_{1,2}(\varphi_3)dx. \tag{2.32}$$

We omit the $(2^{n_1+n_2+n_3}a_1a_2a_3)^{-1}$ in front from here, which is already the desired result except M. (2.31) has four different cases: two according to where ∂_1 falls on, and two for where the second derivative falls on in $\partial_{2,3}(\varphi_3)$, independently. First,

$$\int e^{i\Phi} L\, 2^{-n_1}a_1^{-1}\partial_1^2\Phi\, \varphi_1\varphi_2 \cdot 2^{-n_3}a_3^{-1}\partial_2\partial_3^2\Phi\, \varphi_3 dx \lesssim \int f_2^{-1}L \cdot 2^{-n_1-n_3}a_1a_3\varphi_1\varphi_3 dx \lesssim M.$$

Secondly, when ∂_1 still falls on φ_1 but both ∂_2 and ∂_3 falls on φ_3, we integrate by parts again in ∂_2.

$$\begin{aligned}
&\int e^{i\Phi} L\, 2^{-n_1}a_1^{-1}\partial_1^2\Phi\, \varphi_1\varphi_2 \cdot 2^{-2n_3}a_3^{-2}\partial_{2,3}\Phi\, \partial_3^2\Phi\, \varphi_3 dx \\
=&\int e^{i\Phi}\partial_2 L\, 2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\partial_1^2\Phi\, \partial_{2,3}\Phi\, \partial_3^2\Phi\, \varphi_1\varphi_2\, \varphi_3 dx \\
+&\int e^{i}\Phi L\, 2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\partial_2\Big(\partial_1^2\Phi\, \partial_{2,3}\Phi\, \partial_3^2\Phi\Big)\, \varphi_1\varphi_2\, \varphi_3 dx \\
+&\int e^{i\Phi} L\, 2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\partial_1^2\Phi\, \partial_{2,3}\Phi\, \partial_3^2\Phi\, \partial_2\Big(\varphi_1\varphi_2\, \varphi_3\Big) dx
\end{aligned}$$

Second line is handled by integrating in $(\partial_1\Phi, x_2, \partial_3\Phi)$ after applying $|\partial_{j,k}\Phi| \lesssim a_ja_k$. Third line uses same change of variable by noticing $|\partial_2\partial_{j,k}\Phi| \lesssim f_2^{-1}a_ja_k$. Finally, the last line uses that $|\partial_2\varphi_j| \lesssim a_2\varphi_j$ to change the variable to $\nabla\Phi$ there by obtaining the estimate of M for all of these.

Third case is when ∂_1 falls on φ_2 and $\partial_{2,3}$ generates $\partial_2\partial_3^2\Phi$. Here, we integrate by parts in ∂_1.

$$\begin{aligned}
&\int e^{i\Phi} L\, \varphi_1 2^{-n_2}a_2^{-1}\partial_{1,2}\Phi\, \varphi_2 \cdot 2^{-n_3}a_3^{-1}\partial_2\partial_3^2\Phi\, \varphi_3 dx \\
=&\int e^{i\Phi}\partial_1 L 2^{-n_1-n_2-n_3}a_1^{-1}a_2^{-1}a_3^{-1}\, \partial_{1,2}\Phi\partial_2\partial_3^2\Phi\, \varphi_1\varphi_2\varphi_3 dx \\
+&\int e^{i\Phi} L 2^{-n_1-n_2-n_3}a_1^{-1}a_2^{-1}a_3^{-1}\, \partial_1\Big(\partial_{1,2}\Phi\partial_2\partial_3^2\Phi\Big)\, \varphi_1\varphi_2\varphi_3 dx \\
+&\int e^{i\Phi} L 2^{-n_1-n_2-n_3}a_1^{-1}a_2^{-1}a_3^{-1}\, \partial_{1,2}\Phi\partial_2\partial_3^2\Phi\, \partial_1\Big(\varphi_1\varphi_2\varphi_3\Big) dx
\end{aligned}$$

Second line uses $|\partial_2\partial_3^2\Phi| \lesssim f_2^{-1}a_3^2$ and integration in $(x_1, x_2, \partial_3\Phi)$. Second line also use the same change of variable using the fact that $|\partial_1^2\partial_2\Phi\partial_2\partial_3^2\Phi| \lesssim f_1^{-1}a_2a_3 \cdot f_2^{-1}a_3^2$ and $|\partial_{1,2}\Phi\partial_1\partial_2\partial_3^2\Phi| \lesssim a_1a_2 \cdot f_1^{-1}f_2^{-1}a_3^2$. Last line is similar to last line above which uses $|\partial_1\varphi_j| \lesssim a_1\varphi_j$. Together with $|\partial_2\partial_3^2\Phi| \lesssim f_3^{-1}a_2a_3$, changing the variable to $(\partial_1\Phi, \partial_2\Phi, x_3)$ gives the bound of M.

Last case is when ∂_1 still falls on φ_2, but ∂_2 and ∂_3 both fall on φ_3. Here, we integrate by parts in ∂_1 again.

$$
\begin{aligned}
&\int e^{i\Phi}L\,\varphi_1 2^{-n_2}a_2^{-1}\partial_{1,2}\Phi\,\varphi_2 \cdot 2^{-2n_3}a_3^{-2}\partial_{2,3}\Phi\,\partial_3^2\Phi\,\varphi_3 dx \\
=&\int e^{i\Phi}\partial_1 L\,2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\partial_{1,2}\Phi\,\partial_{2,3}\Phi\,\partial_3^2\Phi\,\varphi_1\varphi_2\,\varphi_3 dx \\
+&\int e^i\Phi L\,2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\partial_1\Big(\partial_{1,2}\Phi\,\partial_{2,3}\Phi\,\partial_3^2\Phi\Big)\,\varphi_1\varphi_2\,\varphi_3 dx \\
+&\int e^{i\Phi}L\,2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\partial_{1,2}\Phi\,\partial_{2,3}\Phi\,\partial_3^2\Phi\,\partial_1\Big(\varphi_1\varphi_2\,\varphi_3\Big)dx
\end{aligned}
$$

Although not exactly symmetric, the estimates work essentially the same as in the second case, and we don't repeat here.

Now we look at (2.32). Since this is symmetric in 1 and 2, we only consider when ∂_3 falls on φ_2. If $\partial_{1,2}$ creates $\partial_{123}\Phi$,

$$
\left|\int e^{i\Phi}L \cdot \varphi_1\,2^{-n_2}a_2^{-1}\partial_{3,2}\Phi\,\varphi_2 \cdot 2^{-n_3}a_3^{-1}\partial_{123}\Phi\,\varphi_3 dx\right| \lesssim \int f_1^{-1}L \cdot 2^{-n_2-n_3}a_2a_3\,\varphi_2\varphi_3 dx \lesssim M.
$$

For the remaining term, we integrate in ∂_1.

$$\begin{aligned}
&\int e^{i\Phi} L\cdot\varphi_1\, 2^{-n_2}a_2^{-1}\partial_{3,2}\Phi\,\varphi_2\cdot 2^{-2n_3}a_3^{-2}\partial_{1,3}\Phi\partial_{2,3}\Phi\,\varphi_3 dx\\
=&\int e^{i\Phi}\partial_1 L\, 2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\,\partial_{1,3}\Phi(\partial_{2,3}\Phi)^2\,\varphi_1\varphi_2\,\varphi_3 dx\\
+&\int e^{i\Phi} L\, 2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\,\partial_1\Big(\partial_{1,3}\Phi(\partial_{2,3}\Phi)^2\Big)\,\varphi_1\varphi_2\,\varphi_3 dx\\
+&\int e^{i\Phi} L\, 2^{-n_1-n_2-2n_3}a_1^{-1}a_2^{-1}a_3^{-2}\,\partial_{1,3}\Phi(\partial_{2,3}\Phi)^2\,\partial_1\Big(\varphi_1\varphi_2\,\varphi_3\Big) dx
\end{aligned}$$

The second line uses variable $(x_1,\partial_2\Phi,\partial_3\Phi)$ to integrate, and so does the third line after pulling out f_1^{-1} from ∂_1 in one of the third derivatives. Last line uses change of variable to $\nabla\Phi$ after noting $|\partial_1\varphi_j|\lesssim a_1\varphi_j$ again.

Case 3-3-3. Finally, we look at the case when all three derivatives falls on the same term, which will choose to be φ_3 by symmetry. When the triple derivative creates fourth order derivative of Φ,

$$\begin{aligned}
&\int e^{i\Phi} L\,\varphi_1\,\varphi_2\, 2^{-n_3}a_3^{-1}\partial_{1,2}\partial_3^2\Phi\varphi_3 dx\\
=&\int e^{i\Phi}2^{-n_2-n_3}a_2^{-1}a_3^{-1}\partial_2 L\,\partial_{1,2}\partial_3^2\Phi\,\varphi_1\,\varphi_2\,\varphi_3 dx\\
+&\int e^{i\Phi}2^{-n_2-n_3}a_2^{-1}a_3^{-1}L\,\varphi_1\,\varphi_2\,\partial_1\partial_2^2\partial_3^2\Phi\varphi_3 dx\\
+&\int e^{i\Phi}2^{-n_2-n_3}a_2^{-1}a_3^{-1}L\,\partial_{1,2}\partial_3^2\Phi\cdot\partial_2\Big(\varphi_1\,\varphi_2\,\varphi_3\Big) dx.
\end{aligned}$$

The second line is handled by $|\partial_{1,2}\partial_3^2\Phi|\lesssim f_1^{-1}f_3^{-1}a_2a_3$, and the third line by $|\partial_1\partial_2^2\partial_3^2\Phi|\lesssim (f_1f_2f_3)^{-1}a_2a_3$ and integrating. Last line again uses $\partial_2\varphi_j\lesssim a_2\varphi_j$.

When the triple derivative creates third order derivative of Φ, up to symmetry between 1 and 2, there are two cases. First one is when ∂_3 creates the third order derivative.

$$\int e^{i\Phi} L\,\varphi_1\,\varphi_2\, 2^{-2n_3}a_3^{-2}\partial_3\Big[\partial_{1,3}\Phi\cdot\partial_{2,3}\Phi\Big]\varphi_3 dx$$

Without loss of generality, we assume the derivative falls on $\partial_{2,3}\Phi$. In such a case, we need to integrate by parts again in ∂_1.

$$\begin{aligned}
&\int e^{i\Phi} L\, \varphi_1\, \varphi_2\, 2^{-2n_3} a_3^{-2} \partial_{1,3}\Phi\, \partial_2\partial_3^2\Phi\, \varphi_3 dx \\
=& \int e^{i\Phi} \partial_1 L\, 2^{-n_1-2n_3} a_1^{-1} a_3^{-2} \partial_{1,3}\Phi\, \partial_2\partial_3^2\Phi\, \varphi_1\, \varphi_2\, \varphi_3 dx \\
+& \int e^{i\Phi} L\, 2^{-n_1-2n_3} a_1^{-1} a_3^{-2} \partial_1 \Big(\partial_{1,3}\Phi\, \partial_2\partial_3^2\Phi \Big)\, \varphi_1\, \varphi_2\, \varphi_3 dx \\
+& \int e^{i\Phi} L\, 2^{-n_1-2n_3} a_1^{-1} a_3^{-2} \partial_{1,3}\Phi\, \partial_2\partial_3^2\Phi\, \partial_1 \Big(\varphi_1\, \varphi_2\, \varphi_3 \Big) dx.
\end{aligned}$$

Second line uses $|\partial_{1,3}\Phi\partial_2\partial_3^2\Phi| \lesssim f_2^{-1} a_1 a_3^3$, and the third line uses $|\partial_1(\partial_{1,3}\Phi\, \partial_2\partial_3^2\Phi)| \lesssim f_1^{-1} f_2^{-1} a_1 a_3^3$. Last line uses $|\partial_1 \varphi_j| \lesssim a_1 \varphi_j$ and all of them can be bounded by M with change of variable $x \to (x_1, x_2, \partial_3\Phi)$.

For the next case, we use the symmetry to only consider when ∂_2 creates the third order derivative, which comes from

$$\int e^{i\Phi} L\, \varphi_1\, \varphi_2\, 2^{-2n_3} a_3^{-2} \partial_2 \Big[\partial_{1,3}\Phi \cdot \partial_3^2\Phi \Big] \varphi_3 dx$$

If ∂_2 falls on $\partial_3^2\Phi$, this is exactly what we considered above. Hence, we assume it generates $\partial_{123}\Phi$ and integrate by parts in ∂_2.

$$\begin{aligned}
&\int e^{i\Phi} L\, \varphi_1\, \varphi_2\, 2^{-2n_3} a_3^{-2} \partial_{123}\Phi\, \partial_3^2\Phi\, \varphi_3 dx \\
=& \int e^{i\Phi} \partial_2 L\, 2^{-n_2-2n_3} a_2^{-1} a_3^{-2} \partial_{123}\Phi\, \partial_3^2\Phi\, \varphi_1\, \varphi_2\, \varphi_3 dx \\
+& \int e^{i\Phi} L\, 2^{-n_2-2n_3} a_2^{-1} a_3^{-2} \partial_2 \Big(\partial_{123}\Phi\, \partial_3^2\Phi \Big)\, \varphi_1\, \varphi_2\, \varphi_3 dx \\
+& \int e^{i\Phi} L\, 2^{-n_2-2n_3} a_2^{-1} a_3^{-2} \partial_{123}\Phi\, \partial_3^2\Phi\, \partial_2 \Big(\varphi_1\, \varphi_2\, \varphi_3 \Big) dx.
\end{aligned}$$

Second line uses $|\partial_{123}\Phi\partial_3^2\Phi| \lesssim f_1^{-1} a_2 a_3^3$ and the third line uses $|\partial_2(\partial_{123}\Phi\partial_3^2\Phi)| \lesssim f_1^{-1} f_2^{-1} a_2 a_3^3$ to integrate in $(x_1, x_2, \partial_3\Phi)$. Last line uses $(x_1, \partial_2\Phi, \partial_3\Phi)$ after similar treatment as before.

Thus, we are now left with only one term which is

$$\int e^{i\Phi} L\, \varphi_1\, \varphi_2\, 2^{-3n_3} a_3^{-3} \partial_{1,3}\Phi \cdot \partial_{2,3}\Phi \cdot \partial_3^2\Phi\, \varphi_3 dx.$$

This is similar in spirit to (2.27), and indeed proceeds in the same way of integrating by parts twice more in ∂_1 and ∂_2. One can check that the estimates will work in the exact same way since the extra terms here are $2^{-3n_3} a_3^{-2} \partial_3^2\Phi$ which is bounded by 1, and provide integrability in $\partial_3\Phi$, and thus does not hurt. The only difference arise when this extra term is hit by derivative(s). However, note that in the analysis of (2.27), when $\partial_{1,3}\Phi\partial_{2,3}\Phi$ received derivative(s), we used $|\partial_1(\partial_{1,3}\Phi\partial_{2,3}\Phi)| \lesssim f_1^{-1} a_1 a_2 a_3^2$, which can be replaced here by $|\partial_1(\partial_{1,3}\Phi\partial_{2,3}\Phi\partial_3^2\Phi)| \lesssim f_1^{-1} a_1 a_2 a_3^4$. When $\partial_{1,3}\Phi\partial_{2,3}\Phi$ was hit by both derivatives, we used $|\partial_1\partial_2(\partial_{1,3}\Phi\partial_{2,3}\Phi)| \lesssim f_1^{-1} f_2^{-1} a_1 a_2 a_3^2$, which can be replaced by $|\partial_{1,2}(\partial_{1,3}\Phi\partial_{2,3}\Phi\partial_3^2\Phi) \lesssim f_1^{-1} f_2^{-1} a_1 a_2 a_3^4$ and the estimates will remain consistent to give a bound of M (or less). In summary, this finishes the proof of

$$|I_{n,m,l}| \lesssim (2^{n+m+l} a_1 a_2 a_3)^{-1} M. \tag{2.33}$$

In conclusion, summing up (2.22), (2.24), (2.30) and (2.33) and also in their appropriate indices, we have proved

$$|I| \lesssim (a_1 a_2 a_3)^{-1} M.$$

□

2.3 Linear decay of (sBQ) and (CER)

The dispersion relation of (sBQ), $\kappa\frac{\xi_1}{|\xi|}$, is $0-$homogeneous and much simpler compared to those of (CER). Thus, in order to demonstrate the full usage of Theorem 2.2.1, I prove the $L^1 - L^\infty$ decay of (CER), and refer to (EW15; JW24; JK25) for the proof of the linear decay of (sBQ), but only state the results.

Proposition 2.3.1. *((EW15)) Let $f \in C_c^\infty(\mathbb{R}^2)$ and λ_κ be given by* (2.3)*. Then there holds*

$$\left\| e^{\pm it\lambda_\kappa} f \right\|_{L^\infty} \lesssim (\kappa t)^{-1/2} \left\| f \right\|_{\dot{B}^2_{1,1}} .$$

((JK25)) Equivalently, using Littlewood-Paley projectors,

$$\left\| e^{it\lambda_\kappa} P_k f \right\|_{L^\infty} \leq C(\kappa t)^{-1/2} 2^{2k} \left\| P_k f \right\|_{L^1} . \tag{2.34}$$

Theorem 2.3.2. *Let $f \in \mathcal{S}(\mathbb{R}^2)$, $F \in \mathcal{S}(\mathbb{R}^3)$ and λ_κ be the dispersion relation* (2.3)*. Moreover, let (q, r) be a $\frac{1}{2}$-admissible pair, that is*

$$\frac{1}{q} + \frac{1}{2r} = \frac{1}{4}, \qquad 4 \leq q \leq \infty,\ 2 \leq r \leq \infty.$$

Then, from (2.34)*, there holds*

1. $\left\| e^{it\lambda_\kappa} P_k f \right\|_{L^q_t L^r_x} \lesssim_q \kappa^{-\frac{1}{q}} 2^{\frac{4k}{q}} \left\| P_k f \right\|_{L^2}$;
2. $\left\| \int_{\mathbb{R}} e^{-is\lambda_\kappa} P_k F(s, x) ds \right\|_{L^2_x} \lesssim_q \kappa^{-\frac{1}{q}} 2^{\frac{4}{q}k} \left\| P_k F \right\|_{L^{q'}_t L^{r'}_x}$*, where* $\frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1$;
3. *for any (q_i, r_i) $\frac{1}{2}-$admissible pairs, $i = 1, 2$ there holds*

$$\left\| \int_0^t e^{i(t-s)\lambda_\kappa} P_k F(s, x) ds \right\|_{L^{q_1}_t L^{r_1}_x} \lesssim_{q_1, q_2} \kappa^{-\frac{1}{q_1} - \frac{1}{q_2}} 2^{\frac{4k}{q_1} + \frac{4k}{q_2}} \left\| P_k F \right\|_{L^{q_2'}_t L^{r_2'}_x} ,$$

   where $\frac{1}{q_i} + \frac{1}{q_i'} = \frac{1}{r_i} + \frac{1}{r_i'} = 1$.

Now we focus on (CER), or its linearized equation (2.5). It will be more convenient to work with (2.6) and rescale to write results for (2.5). The first step is to know the size of the determinant of Hessian of dispersion relations.

Lemma 2.3.3. *((KPTW25)) There holds that*

$$\det \nabla^2 \Lambda = \kappa^{-2} \frac{r^2 \Lambda}{(d_1 d_2)^4}, \quad \Lambda \in \{\Sigma, \Omega\}. \tag{2.35}$$

Proof. We start from the formulas:

$$d_1 \nabla d_1 = r e_r + (z - \kappa^{-1}) e_z, \quad d_2 \nabla d_2 = r e_r + (z + \kappa^{-1}) e_z, \qquad \nabla^2 d = \frac{1}{d} \left[Id - \nabla d \otimes \nabla d \right],$$

and we can evaluate on the orthonormal basis (e_r, e_θ, e_z):

$$\begin{aligned}
&d_1 \nabla^2 d_1(e_r, e_r) = 1 - r^2/d_1^2 = \frac{(z - \kappa^{-1})^2}{d_1^2}, \qquad d_1 \nabla^2 d_1(e_r, e_z) = -\frac{r(z - \kappa^{-1})}{d_1^2} \\
&d_1 \nabla^2 d_1(e_r, e_\theta) = 0 = d_1 \nabla^2 d_1(e_z, e_\theta), \qquad d_1 \nabla^2 d_1(e_\theta, e_\theta) = 1 \\
&d_1 \nabla^2 d_1(e_z, e_z) = 1 - (z - \kappa^{-1})^2/d_1^2 = \frac{r^2}{d_1^2}.
\end{aligned}$$

As a consequence, we can write the matrix

$$\nabla^2 \Omega = \frac{1}{2} \begin{pmatrix} \frac{(z+\kappa^{-1})^2}{d_2^3} - \frac{(z-\kappa^{-1})^2}{d_1^3} & 0 & -r\left(\frac{z+\kappa^{-1}}{d_2^3} - \frac{z-\kappa^{-1}}{d_1^3}\right) \\ 0 & \frac{1}{d_2} - \frac{1}{d_1} & 0 \\ -r\left(\frac{z+\kappa^{-1}}{d_2^3} - \frac{z-\kappa^{-1}}{d_1^3}\right) & 0 & r^2\left(\frac{1}{d_2^3} - \frac{1}{d_1^3}\right) \end{pmatrix}.$$

and similarly for Σ. (2.35) is now immediate. □

In addition to the $P_{k,p}$ and $P_{k,p,q}$ defined in the Notation section, I define $P_{k,p;l}$ and $P_{k,p,q;l}$ as

$$\mathcal{F}(P_{k,p;l} f)(\xi) = \varphi_{k,p;l}(\xi) \hat{f}(\xi), \quad \mathcal{F}(P_{k,p,q;l} f)(\xi) = \varphi_{k,p,q;l}(\xi) \hat{f}(\xi),$$

for $k \in \mathbb{Z}, p, q \in \mathbb{Z}_-$ and $p \leq l \leq 0$ where

$$\varphi_{k,p;l}(\xi) = \varphi_{k,p}(\xi) \varphi(2^{-k-l}(\xi_3 - \kappa^{-1})), \quad \varphi_{k,p,q;l}(\xi) = \varphi_{k,p,q}(\xi) \varphi(2^{-k-l}(\xi_3 - \kappa^{-1})).$$

Lastly, we define $P_{k,p;\leq p}$ and $P_{k,p,q;\leq p}$ by

$$\mathcal{F}(P_{k,p;\leq p}f) = \varphi_{k,p;\leq p}(\xi)\hat{f}(\xi), \quad \mathcal{F}(P_{k,p,q;\leq p}f)(\xi) = \varphi_{k,p,q;\leq p}(\xi)\hat{f}(\xi),$$

for k, p, q as above where

$$\varphi_{k,p;\leq p}(\xi) = \varphi_{k,p}(\xi)\bar{\varphi}(2^{-k-p}(\xi_3 - \kappa^{-1})), \quad \varphi_{k,p,q;\leq p}(\xi) = \varphi_{k,p,q}(\xi)\bar{\varphi}(2^{-k-p}(\xi_3 - \kappa^{-1})).$$

Now I can state the decay of linear evolutions.

Proposition 2.3.4. *(KPTW25) For any $t > 0$, $k \in \mathbb{Z}$ and $p, q \in \mathbb{Z}_-$,*

$$\|e^{it\Sigma_\kappa}P_{k,p}f\|_{L^\infty} \lesssim t^{-\frac{3}{2}}\|P_{k,p}f\|_{L^1}\begin{cases} \kappa 2^{\frac{5}{2}k-p} & \kappa 2^k \gg 1, \\ 2^{\frac{3}{2}k-p} & \kappa 2^k \sim 1, \\ \kappa^{-\frac{5}{2}}2^{-k-p} & \kappa 2^k \ll 1, \end{cases}$$

$$\|e^{it\Omega_\kappa}P_{k,p,q}f\|_{L^\infty} \lesssim t^{-\frac{3}{2}}\|P_{k,p,q}f\|_{L^1}\begin{cases} \kappa^{\frac{3}{2}} 2^{3k-p-\frac{q}{2}} & \kappa 2^k \gg 1, \\ 2^{\frac{3}{2}k-p-\frac{q}{2}} & \kappa 2^k \sim 1, \\ \kappa^{-3}2^{-\frac{3}{2}k-p-\frac{q}{2}} & \kappa 2^k \ll 1. \end{cases}$$

For the medium frequencies, we have finer (in terms of coefficients) decay,

$$\|e^{it\Sigma_\kappa}P_{k,p;l}f\|_{L^\infty} \lesssim 2^{\frac{3}{2}k-p+2l}t^{-\frac{3}{2}}\|P_{k,p;l}f\|_{L^1}, \qquad \kappa 2^k \sim 1, p \leq l \leq 0,$$
$$\|e^{it\Sigma_\kappa}P_{k,p;\leq p}f\|_{L^\infty} \lesssim 2^{\frac{3}{2}k+p}t^{-\frac{3}{2}}\|P_{k,p;\leq p}f\|_{L^1},$$

$$\|e^{it\Omega_\kappa}P_{k,p,q;l}f\|_{L^\infty} \lesssim 2^{\frac{3}{2}k-p+2l-\frac{q}{2}}t^{-\frac{3}{2}}\|P_{k,p,q;l}f\|_{L^1}, \quad \kappa 2^k \sim 1, p \leq l \leq 0,$$
$$\|e^{it\Omega_\kappa}P_{k,p,q;\leq p}f\|_{L^\infty} \lesssim 2^{\frac{3}{2}k+p-\frac{q}{2}}t^{-\frac{3}{2}}\|P_{k,p,q;\leq p}f\|_{L^1}.$$

Scaling back, the following is direct for (2.5).

Corollary 2.3.5. *With same t and k, p, q as in 2.3.4, there holds that*

$$\left\|e^{itc\Omega_{c\varepsilon}} P_k f\right\|_{L^\infty} \lesssim c^{-3}\varepsilon^{-2}t^{-1}\langle 2^k c\varepsilon\rangle^3 \left\|P_k f\right\|_{L^1},$$
$$\left\|e^{itc\Sigma_{c\varepsilon}} P_k f\right\|_{L^\infty} \lesssim c^{-\frac{8}{3}}\varepsilon^{-\frac{5}{3}}t^{-1}\langle 2^k c\varepsilon\rangle^{\frac{7}{3}} 2^{\frac{k}{3}} \left\|P_k f\right\|_{L^1}.$$

Since it can be easily seen that L^2 norms are preserved, we obtain the following Strichartz estimates directly.

Theorem 2.3.6. *Let $(a, b), (q, r) \in [2, \infty]^2$ be admissible in the sense that*

$$\frac{2}{a} + \frac{2}{b} = 1 = \frac{2}{q} + \frac{2}{r}, \qquad (a, b), (q, r) \neq (2, \infty).$$

Then there holds for $f \in \mathcal{S}(\mathbb{R}^3)$ that

$$\begin{aligned} \|e^{itc\Omega_{c\varepsilon}} P_k f\|_{L^q_t L^r_x} &\lesssim (\langle 2^k c\varepsilon\rangle^3 \epsilon^{-2} c^{-3})^{\frac{1}{q}} \|P_k f\|_{L^2_x(\mathbb{R}^3)}, \\ \|e^{itc\Sigma_{c\varepsilon}} P_k f\|_{L^q_t L^r_x} &\lesssim (\langle 2^k c\varepsilon\rangle^{\frac{7}{3}} 2^{\frac{1}{3}k} \varepsilon^{-\frac{5}{3}} c^{-\frac{8}{3}})^{\frac{1}{q}} \|P_k f\|_{L^2_x(\mathbb{R}^3)}, \end{aligned} \tag{2.36}$$

and for $F \in C_t \mathcal{S}_x(([0, T], \mathbb{R}^3)$ that

$$\begin{aligned} \left\|\int_0^t e^{i(t-s)c\Omega_{c\varepsilon}} P_k F(s, x) ds\right\|_{L^q_t L^r_x} &\lesssim (\langle 2^k c\varepsilon\rangle^3 \epsilon^{-2} c^{-3})^{\frac{1}{q}+\frac{1}{a}} \|P_k F\|_{L^{a'}_t L^{b'}_x}, \\ \left\|\int_0^t e^{i(t-s)c\Sigma_{c\varepsilon}} P_k F(s, x) ds\right\|_{L^q_t L^r_x} &\lesssim (\langle 2^k c\varepsilon\rangle^{\frac{7}{3}} 2^{\frac{1}{3}k} \varepsilon^{-\frac{5}{3}} c^{-\frac{8}{3}})^{\frac{1}{q}+\frac{1}{a}} \|P_k F\|_{L^{a'}_t L^{b'}_x}, \end{aligned} \tag{2.37}$$

where a', b' denote the Hölder conjugates of a, b, repectively.

Proof. This follows by combining Corollary 2.3.5 with a standard TT' argument – see e.g. (KT98). $\square$

A fascinating fact is that this recovers the $L^1 - L^\infty$ decay and the Strichartz estimates for incompressible Euler's equation in the incompressible limit $c \to \infty$. Indeed, it's easy to see that $\Sigma \to |\xi|$ as $c \to \infty$. Thus, using the identity $\Sigma_\kappa \Omega_\kappa = \kappa^{-1}\xi_3$,

$$c\Omega_{c\varepsilon} \to \Lambda_\varepsilon = \varepsilon^{-1}\frac{\xi_3}{|\xi|}, \qquad c \to \infty,$$

where Λ_ε is the dispersion relation of (NSC). Therefore, from the weak convergence $e^{itc\Omega_{c\varepsilon}}P_k f \rightharpoonup e^{it\Lambda_\varepsilon}P_k f$, we have

$$\|e^{it\Lambda_\varepsilon}P_k f\|_{L_t^q L_x^r} \lesssim (\varepsilon 2^{3k})^{\frac{1}{q}} \|P_k f\|_{L^2}. \tag{2.38}$$

The Strichartz estimates again follow directly, which matches the result of (KLT14b).

2.4 Proof of Proposition 2.3.4

This section is dedicated to the proof of Proposition 2.3.4. Using Young's inequality, it is enough to prove the decay of

$$\int e^{it\Sigma + i\mathbf{x}\cdot\xi}\varphi(2^{-k}|\xi|)\varphi(2^{-k-p}|(\xi_1,\xi_2)|)d\xi,$$
$$\int e^{it\Omega + i\mathbf{x}\cdot\xi}\varphi(2^{-k}|\xi|)\varphi(2^{-k-p}|(\xi_1,\xi_2)|)\varphi(2^{-k-q}\xi_3)d\xi.$$

Both Σ and Ω are invariant under rotation around the ξ_3-axis, and so are the localization terms with φ's. By using this axisymmetry we can assume that $\mathbf{x} = (x_1, 0, -x_3)$, and it is natural to use cylindrical coordinates $\xi \to (r,\theta,z)$. Integrating in θ and using standard results on Bessel functions as in (GPW23, p.194), the problem boils down to estimating

$$I_{\Sigma,\pm} := \int_{\mathbb{R}}\int_0^\infty e^{it\Sigma - ix_3 z \pm ix_1 r}\varphi(2^{-k}|(r,z)|)\varphi(2^{-k-p}r)H_\pm(x_1 r)rdrdz,$$
$$I_{\Omega,\pm} := \int_{\mathbb{R}}\int_0^\infty e^{it\Omega - ix_3 z \pm ix_1 r}\varphi(2^{-k}|(r,z)|)\varphi(2^{-k-p}r)\varphi(2^{-k-q}z)H_\pm(x_1 r)rdrdz,$$

where

$$\left|\left(\frac{d}{dx}\right)^n H_\pm(x)\right| \lesssim \langle x\rangle^{-\frac{1}{2}-n}. \tag{2.39}$$

We will localize further when we deal with the medium frequencies.

Throughout the section, it will be helpful to distinguish the following sizes in three main subcases:

$$\begin{aligned}
\text{low frequencies } \kappa 2^k \ll 1 \quad &\Rightarrow \quad \Sigma \sim \kappa^{-1}, \quad \Omega \sim \xi_3, \quad \sqrt{\mathcal{D}} \sim \kappa^{-2}, \quad d_1, d_2 \sim \kappa^{-1} \\
\text{medium frequencies } \kappa 2^k \sim 1 \quad &\Rightarrow \quad \Sigma \sim \kappa^{-1}, \quad \Omega \sim \xi_3, \quad \sqrt{\mathcal{D}} \lesssim \kappa^{-2}, \quad \max\{d_1, d_2\} \sim \kappa^{-1} \\
\text{high frequencies } \kappa 2^k \gg 1 \quad &\Rightarrow \quad \Sigma \sim |\xi|, \quad \Omega \sim \frac{\xi_3}{\kappa|\xi|}, \quad \sqrt{\mathcal{D}} \sim |\xi|^2, \quad d_1, d_2 \sim 2^k.
\end{aligned}$$

2.4.1 Decay of Σ evolution

We prove $t^{-\frac{3}{2}}$ decay of the L^∞ norm of the following integral:

$$I_\Sigma := \iint e^{i\Phi} \varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)H(x_1 r) r\,dr\,dz.$$

Here, $\Phi = t\Sigma - (r,z)\cdot(x_1,x_3)$. In general, we need to consider $t\Sigma - x_3 z \pm x_1 r$, but we only deal with the negative sign case as there's no qualitative difference. Also, whenever needed, we will assume $z \geq 0$ using the symmetry without loss of generality. We write here some derivatives of Φ that will appear often:

$$\begin{aligned}
\partial_r \Phi &= t\frac{r\Sigma}{d_1 d_2} - x_1, \\
\partial_z \Phi &= \frac{t}{2}\left(\frac{z-\kappa^{-1}}{d_1} + \frac{z+\kappa^{-1}}{d_2}\right) - x_3, \\
\partial_r^2 \Phi &= \frac{t}{2}\left(\frac{(z-\kappa^{-1})^2}{d_1^3} + \frac{(z+\kappa^{-1})^2}{d_2^3}\right), \\
\partial_z^2 \Phi &= \frac{t}{2}\left(\frac{r^2}{d_1^3} + \frac{r^2}{d_2^3}\right), \\
\partial_r^3 \Phi &= -\frac{3tr}{2}\left(\frac{(z-\kappa^{-1})^2}{d_1^5} + \frac{(z+\kappa^{-1})^2}{d_2^5}\right), \\
\partial_r \partial_z^2 \Phi &= \frac{tr}{2}\left(\frac{2(z-\kappa^{-1})^2 - r^2}{d_1^5} + \frac{2(z+\kappa^{-1})^2 - r^2}{d_2^5}\right), \\
\partial_z \partial_r^2 \Phi &= t\frac{z-\kappa^{-1}}{2}\frac{2r^2 - (z-\kappa^{-1})^2}{d_1^5} + t\frac{z+\kappa^{-1}}{2}\frac{2r^2 - (z+\kappa^{-1})^2}{d_2^5},
\end{aligned} \tag{2.40}$$

whose sizes will vary according to the subcases. We also note here

$$
\begin{aligned}
|\partial_r^n (H_\pm(x_1 r))| \lesssim \frac{x_1^n}{\langle x_1 r\rangle^{n+\frac{1}{2}}} \leq r^{-n}\langle x_1 r\rangle^{-\frac{1}{2}} \sim (2^{-k-p})^n \langle x_1 r\rangle^{-\frac{1}{2}}, \\
\partial_r^n (r) \lesssim r \cdot (2^{-k-p})^n, \\
|\partial_r^n (\varphi(2^{-k-p} r))| \lesssim (2^{-k-p})^n, \\
|\partial_r^n (\varphi(2^{-k}|(r,z)|))| \lesssim (2^{-k})^n,
\end{aligned}
\tag{2.41}
$$

where the first line follows from (2.39).

Low frequency, $\kappa 2^k \ll 1$.

If $x_1 \nsim t\kappa 2^{k+p}$, we integrate by parts in ∂_r first. When the derivative falls on terms not related to the phase, (2.41) shows that there is at most 2^{-k-p} loss from each ∂_r derivative. Also, using (2.40), we get $|\partial_r \Phi| \gtrsim t\kappa 2^{k+p}$ from $\partial_r \Sigma \sim 2^{k+p}\frac{\kappa^{-1}}{\kappa^{-2}} = \kappa 2^{k+p}$, $\partial_r^2 \Phi \sim t\kappa$, and $\partial_r^3 \Phi \sim t 2^{k+p}\kappa^3 \ll t\kappa 2^{-k+p}$, so that, in an averaged sense, ∂_r derivative always yields 2^{-k-p} loss, i.e.

$$
\begin{aligned}
\left|\frac{\partial_r^2 \Phi}{(\partial_r \Phi)^2}\right| \lesssim t^{-1}\kappa^{-1} 2^{-2k-2p}, \\
\left|\frac{\partial_r^3 \Phi}{(\partial_r \Phi)^3}\right| \lesssim t^{-2}\kappa^{-2} 2^{-4k-2p} \leq (t^{-1}\kappa^{-1}2^{-2k-2p})^2.
\end{aligned}
$$

Thus,

$$
|I_\Sigma| \lesssim 2^{3k+2p} \cdot \min\{1, (t\kappa 2^{2k+2p})^{-2}\} \leq \kappa^{-\frac{3}{2}} 2^{-p} t^{-\frac{3}{2}} \ll \kappa^{-\frac{5}{2}} 2^{-k-p} t^{-\frac{3}{2}}.
$$

Hence, now we assume $x_1 \sim t\kappa 2^{k+p}$. Now, from (2.40),

$$
\begin{aligned}
\partial_r^2\Phi &= \frac{t}{2}\left(\frac{(z-\kappa^{-1})^2}{d_1^3} + \frac{(z+\kappa^{-1})^2}{d_2^3}\right) \sim t\kappa,\\
\partial_z^2\Phi &= \frac{t}{2}\left(\frac{r^2}{d_1^3} + \frac{r^2}{d_2^3}\right) \sim t\kappa^3 2^{2k+2p},\\
\partial_r\partial_z^2\Phi &= t\frac{r}{2}\left(\frac{2(z-\kappa^{-1})^2 - r^2}{d_1^5} + \frac{2(z+\kappa^{-1})^2 - r^2}{d_2^5}\right) \sim t\kappa^3 2^{k+p},\\
\partial_z\partial_r^2\Phi &= t\frac{z-\kappa^{-1}}{2}\frac{2r^2 - (z-\kappa^{-1})^2}{d_1^5} + t\frac{z+\kappa^{-1}}{2}\frac{2r^2 - (z+\kappa^{-1})^2}{d_2^5} \lesssim t\kappa^2,\\
\det \nabla_{r,z}^2\Phi &= t^2\kappa^{-2}\frac{r^2}{d_1^3 d_2^3} \sim t^2\kappa^4 2^{2k+2p},
\end{aligned}
$$

where the last line follows from (2.35), but by ignoring the contribution from $\partial_\theta^2\Sigma$. Together with $L(r,z) = \varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)H(x_1 r)r$, $L_r(r) = \varphi(2^{-k-p}r)H(x_1 r)$, $L_z(z) = 2^{-k}M\bar{\varphi}(2^{-k}z)$, and $M = 2^{k+p}\langle t\kappa 2^{2k+2p}\rangle^{-1/2}$, the conditions of Proposition **??** are all satisfied by using $f_r = 2^{k+p}$, $f_z = 2^k$, $a = (t\kappa)^{1/2}$, and $A = (t\kappa^3 2^{2k+2p})^{1/2}$, and we have

$$
|I_\Sigma| \lesssim (t^2\kappa^4 2^{2k+2p})^{-\frac{1}{2}} 2^{k+p}\langle t\kappa 2^{2k+2p}\rangle^{-1/2} \le \kappa^{-\frac{5}{2}} 2^{-k-p} t^{-\frac{3}{2}}.
$$

High frequency, $\kappa 2^k \gg 1$.

If $x_1 \nsim t2^p$, since $|\partial_r\Sigma| \sim 2^p$, we have $|\partial_r\Phi| \gtrsim t2^p$, and thus can integrate by parts along ∂_r. Along with the common losses (2.41), $|\partial_r^2\Phi| \lesssim t\cdot 2^{-k}$, and $\partial_r^3\Phi \lesssim t\cdot 2^{-2k+p}$ by (2.40) which shows that the derivatives yield 2^{-k-p} loss at worst. Thus, two integration by parts give us

$$
|I_\Sigma| \lesssim 2^{3k+2p}\cdot \min\{1, (t^{-1}2^{-k-2p})^2\} \le 2^{\frac{3}{2}k-p} t^{-\frac{3}{2}} \ll \kappa 2^{\frac{5}{2}k-p} t^{-\frac{3}{2}}.
$$

Hence, from hereon, we assume $x_1 \sim t2^p$. We now change the variable to the spherical variables (ρ,ϕ)(but since we are in dimension 2 with r and z, one can also think of this as the polar coordinates), and use the derivatives $\partial_\rho = \frac{r}{\rho}\partial_r + \frac{z}{\rho}\partial_z$, $\partial_\phi = z\partial_r - r\partial_z$. From integration by parts in ρ, using

$$I_{\Sigma,n} := \int e^{i\Phi}\varphi(t^{-1}\kappa 2^{k-n}\partial_\rho\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\, d\rho d\phi,$$

we get

$$\begin{aligned}
|I_{\Sigma,n}| &= t^{-1}\kappa 2^{k-n}\left|\int e^{i\Phi}\partial_\rho\left\{\tilde\varphi(t^{-1}\kappa 2^{k-n}\partial_\rho\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right\}d\rho d\phi\right| \\
&\le (t^{-1}\kappa 2^{k-n})^2\int|\partial_\rho^2\Phi||\tilde\varphi'(t^{-1}\kappa 2^{k-n}\partial_\rho\Phi)||\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi|d\rho d\phi \\
&\quad + t^{-1}\kappa 2^{k-n}\int\left|\partial_\rho\left\{\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right\}\right|d\rho d\phi \\
&\lesssim t^{-1}\kappa 2^{k-n}2^{2k+2p}(t2^{k+2p})^{-\frac{1}{2}} \le 2^{-n}\kappa 2^{\frac{5}{2}k-p}t^{-\frac{3}{2}}.
\end{aligned}$$

Summing over n for $2^n \gtrsim 1$, we obtain the desired result. Hence, it remains to deal with the region where $|\partial_\rho\Phi| \ll t\kappa^{-1}2^{-k}$, which in particular makes

$$|(r,z)\cdot(x_1,x_3)| = \left|\rho\partial_\rho\Phi - t\Sigma - \frac{t}{2\kappa}\left(\frac{z-\kappa^{-1}}{d_1} - \frac{z+\kappa^{-1}}{d_2}\right)\right| \sim t2^k.$$

We additionally localize according to the size of $\partial_\phi\Phi$ to remove the region where its size is large, using

$$I_\Sigma^m = \int e^{i\Phi}\varphi(t^{-1}\kappa 2^{-p-m}\partial_\phi\Phi)\bar\varphi(t^{-1}\kappa 2^k\partial_\rho\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\, d\rho d\phi.$$

The integration by parts give

$$
\begin{aligned}
|I_\Sigma^m| = \ & t^{-1}\kappa 2^{-p-m} \\
& \cdot \left| \int e^{i\Phi}\partial_\phi \left\{ \tilde{\varphi}(t^{-1}\kappa 2^{-p-m}\partial_\phi\Phi)\bar{\varphi}(t^{-1}\kappa 2^k\partial_\rho\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi \right\} d\rho d\phi \right| \\
\leq \ & (t^{-1}\kappa 2^{-p-m})^2 \int |\partial_\phi^2\Phi||\tilde{\varphi}'(t^{-1}\kappa 2^{-p-m}\partial_\phi\Phi)||\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi| d\rho d\phi \\
& + t^{-1}\kappa 2^{-p-m} \int t^{-1}\kappa 2^k |\partial_\phi\partial_\rho\Phi||\bar{\varphi}'(t^{-1}\kappa 2^k\partial_\rho\Phi)||\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi| d\rho d\phi \\
& + t^{-1}\kappa 2^{-p-m} \int |\partial_\phi \left\{\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right\}| d\rho d\phi \\
\lesssim \ & t^{-1}\kappa 2^{-p-m}2^{3k+p}(t2^{k+2p})^{-\frac{1}{2}} = 2^{-m}\kappa 2^{\frac{5}{2}k-p}t^{-\frac{3}{2}}.
\end{aligned}
$$

Summing over m for $2^m \gtrsim 1$, we obtain another acceptable bound, and can then assume that $|\partial_\phi\Phi| \ll t\kappa^{-1}2^p$. From

$$
\begin{aligned}
|\partial_\rho\partial_\phi\Phi - \frac{1}{\rho}\partial_\phi\Phi| &= \left| t\frac{r}{2\kappa\rho}\left(\frac{r^2}{d_2^3} + \frac{z(z+\kappa^{-1})}{d_2^3} - \frac{r^2}{d_1^3} - \frac{z(z-\kappa^{-1})}{d_1^3} \right) \right| \\
&\lesssim t\kappa^{-2}2^{-2k+p} \ll t\kappa^{-1}2^{-k+p},
\end{aligned} \tag{2.42}
$$

we also have $|\partial_\rho\partial_\phi\Phi| \ll t\kappa^{-1}2^{-k+p}$. Moreover,

$$
\begin{aligned}
\partial_\rho^2\Phi &= \frac{t}{(\kappa|\xi|)^2}\frac{1}{2}\left(\frac{r^2}{d_2^3} + \frac{r^2}{d_1^3}\right) \sim t\kappa^{-2}2^{-3k+2p}, \\
\partial_\phi^2\Phi &= t\kappa^{-1}z\frac{\Omega}{d_1d_2} - (\kappa^{-1}r)^2\frac{t}{2}\left(\frac{1}{d_2^3} + \frac{1}{d_1^3}\right) + (r,z)\cdot(x_1,x_3) \sim t2^k.
\end{aligned}
$$

We are now left with redefined I,

$$
\tilde{I}_\Sigma = \int e^{i\Phi}\bar{\varphi}(t^{-1}\kappa 2^k\partial_\rho\Phi)\bar{\varphi}(t^{-1}\kappa 2^{-p}\partial_\phi\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi \, d\rho d\phi.
$$

In practice, thanks to the precise localizations that determine the sizes of the second and third order derivatives, we can start applying Theorem 2.2.1. We elaborate why we can proceed as such in full detail for this case (high frequency of Σ), and let the reader check how they are repeated for other cases, as these are essentially repetitions of the proof of Theorem 2.2.1 as detailed below. Put $a = (t\kappa^{-2}2^{-3k+2p})^{1/2}, A = (t2^k)^{1/2}$, $b = t\kappa^{-1}2^{-k}, B = t\kappa^{-1}2^p$ and consider four different cases.

① $a \geq b, A \geq B$: This means the above steps already yield small enough region. Simply

$$\begin{aligned}|\tilde{I}_\Sigma| &\lesssim \int \bar{\varphi}(b^{-1}\partial_\rho\Phi)\bar{\varphi}(B^{-1}\partial_\phi\Phi)|L(\rho,\phi)|d\rho d\phi \\ &\lesssim bB\cdot(\det\nabla^2_{\rho,\phi}\Phi)^{-1}M \leq aA\cdot(aA)^{-2}M \leq \kappa 2^{\frac{5}{2}k-p}t^{-\frac{3}{2}}.\end{aligned}$$

Here, $L(\rho,\phi) = \varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi$ and $M = 2^{2k+p}\langle t2^{k+2p}\rangle^{-1/2}$. L, M, a, and A are intended to resemble the notations in Proposition 2.2.3 and Theorem 2.2.1 as these will be used as such in step ④.

② $a \geq b, A \leq B$: Solving this in terms of t gives

$$\kappa 2^{\frac{1}{2}k-p} \leq t^{\frac{1}{2}} \leq 2^{-\frac{1}{2}k+p} \quad \Rightarrow \quad \kappa 2^{k-2p} \leq 1,$$

which is against the assumption.

③ $a \leq b, A \geq B$: Since a is smaller, we decompose $\tilde{I}_\Sigma = \tilde{I}_{\Sigma,-1} + \sum_{n\geq 0}\tilde{I}_{\Sigma,n}$ where

$$\tilde{I}_{\Sigma,n} = \int e^{i\Phi}\varphi(a^{-1}2^{-n}\partial_\rho\Phi)\bar{\varphi}(B^{-1}\partial_\phi\Phi)L(\rho,\phi)d\rho d\phi,$$

and integrate by parts in ∂_ρ. If ∂_ρ falls on L,

$$a^{-1}2^{-n}\int e^{i\Phi}\tilde{\varphi}(a^{-1}2^{-n}\partial_\rho\Phi)\bar{\varphi}(B^{-1}\partial_\phi\Phi)\partial_\rho L\, d\rho d\phi \lesssim a^{-1}2^{-n}\cdot B\cdot A^{-2}\cdot M \leq (aA)^{-1}M2^{-n}.$$

If ∂_ρ falls on $\bar{\varphi}(B^{-1}\partial_\phi\Phi)$,

$$\begin{aligned}
&\left|a^{-1}2^{-n}\int e^{i\Phi}\tilde{\varphi}(a^{-1}2^{-n}\partial_\rho\Phi)B^{-1}\partial_\rho\partial_\phi\Phi\cdot\bar{\varphi}'(B^{-1}\partial_\phi\Phi)L(\rho,\phi)d\rho d\phi\right| \\
&\quad\lesssim a^{-1}2^{-n}\int\tilde{\varphi}(a^{-1}2^{-n}\partial_\rho\Phi)2^{-k}|\bar{\varphi}'|(B^{-1}\partial_\phi\Phi)|L(\rho,\phi)|d\rho d\phi \\
&\quad\lesssim a^{-1}2^{-n}\int|\bar{\varphi}'|(B^{-1}\partial_\phi\Phi)L_\rho(\rho)d\rho d\phi\lesssim a^{-1}2^{-n}\cdot B\cdot A^{-2}\cdot M\leq(aA)^{-1}M2^{-n}.
\end{aligned}$$

Here, (2.42) was used in the first inequality and $L_\rho = 2^{-k}\cdot M\varphi(2^{-k}\rho)$ is a corresponding choice of L_1 in Proposition 2.2.3 where one can check that it indeed satisfies the required condition. Lastly, if ∂_ρ falls on $\varphi(a^{-1}2^{-n}\partial_\rho\Phi)$,

$$\begin{aligned}
&\left|a^{-1}2^{-n}\int e^{i\Phi}a^{-1}2^{-n}\partial_\rho^2\Phi\tilde{\varphi}'(a^{-1}2^{-n}\partial_\rho\Phi)\bar{\varphi}(B^{-1}\partial_\phi\Phi)Ld\rho d\phi\right| \\
&\quad\lesssim a^{-1}2^{-n}\int a^{-1}2^{-n}|\tilde{\varphi}'|(a^{-1}2^{-n}\partial_\rho\Phi)\bar{\varphi}(B^{-1}\partial_\phi\Phi)\,a^2d\rho d\phi\cdot M \\
&\quad\lesssim a^{-1}2^{-n}\cdot B\cdot A^{-2}\cdot M\leq(aA)^{-1}M\cdot 2^{-n}.
\end{aligned}$$

All of these are summable in n, resulting in

$$\sum_{n\geq 0}|\tilde{I}_{\Sigma,n}|\lesssim(t^2\kappa^{-2}2^{-2k+2p})^{-1/2}2^{2k+p}\langle t2^{k+2p}\rangle^{-1/2}\leq\kappa 2^{\frac{5}{2}k-p}t^{-\frac{3}{2}}.$$

$\tilde{I}_{\Sigma,-1}$ can be treated the same way as in ①.

④ $a\leq b, A\leq B$: This is the case where we need stationary phase estimates in both directions, which will be handled by Theorem 2.2.1. Direct calculations show that

$$\begin{aligned}
\partial_\phi\partial_\rho^2\Phi &= \frac{tr}{2(\kappa|\xi|)^2}\left(\frac{2z}{d_2^3}+\frac{2z}{d_1^3}+\frac{3\kappa^{-1}r^2}{d_2^5}-\frac{3\kappa^{-1}r^2}{d_1^5}\right)\lesssim t\kappa^{-2}2^{-3k+p}, \\
\partial_\rho\partial_\phi^2\Phi &= -\frac{tz}{2\kappa^2|\xi|}\left[\frac{z+\kappa^{-1}}{d_2^3}+\frac{z-\kappa^{-1}}{d_1^3}\right]+\frac{r^2}{2\kappa^2|\xi|}\left(\frac{1}{d_2^3}+\frac{1}{d_1^3}\right) \\
&\quad-\frac{3tr^2}{2\kappa^2|\xi|}\left(\frac{\kappa^{-1}(z+\kappa^{-1})}{d_2^5}-\frac{\kappa^{-1}(z-\kappa^{-1})}{d_1^5}\right)+\frac{1}{|\xi|}(r,z)\cdot(x_1,x_3)\sim t.
\end{aligned}$$

In particular, $|\partial_\rho\partial_\phi\Phi|^2\ll|\partial_\rho^2\Phi||\partial_\phi^2\Phi|$, $|\partial_\phi\partial_\rho^2\Phi|\lesssim 2^{-p}|\partial_\rho^2\Phi|$, and $|\partial_\rho\partial_\phi^2\Phi|\lesssim 2^{-k}|\partial_\phi^2\Phi|$.

Applying Theorem 2.2.1 with (2.14) for $L(\rho,\phi) = \varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi$ by observing that it satisfies all conditions with $M = 2^{2k+p}\langle t2^{k+2p}\rangle^{-1/2}$, we have

$$|I| \lesssim (t^2\kappa^{-2}2^{-2k+2p})^{-1/2}2^{2k+p}\langle t2^{k+2p}\rangle^{-1/2} \leq \kappa 2^{\frac{5}{2}k-p}t^{-\frac{3}{2}},$$

which is consistent with other estimates, and thus the final bound.

Medium frequency, $\kappa 2^k \sim 1$.

In this regime we are faced with the lack of smoothness of the dispersion relations at $r = 0$, $z = \pm\kappa^{-1}$, see also (2.35). In order to treat these singularities one at a time, without loss of generality we consider the stationary phase integral only on the upper half-plane and localize in the size of $z - \kappa^{-1}$ by 2^{k+l}, i.e. we consider

$$\begin{aligned}
I_{\Sigma,l} &:= \int_{z=0}^{\infty}\int_{r=0}^{\infty} e^{i\Phi}\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k-l}(z-\kappa^{-1}))H(x_1 r)r\,dr\,dz, \quad p \leq l \leq 0,\\
I_{\Sigma,\leq p} &:= \int_{z=0}^{\infty}\int_{r=0}^{\infty} e^{i\Phi}\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\bar{\varphi}(2^{-k-p}(z-\kappa^{-1}))H(x_1 r)r\,dr\,dz, \quad l \leq p.
\end{aligned}$$

Here, we will consider the cases $p \leq l \leq 0$ and $l \leq p$ separately, as that is the turning point of relative size between r and $z - \kappa^{-1}$. In this section, we will alternatively use (ρ,ϕ) as the spherical coordinates centered at $(0,\kappa^{-1})$. Note that d_1 and ρ indicate exactly same quantities and will be used interchangeably. In terms of the above cylindrical coordinates, the spherical derivatives take the form

$$\partial_\rho = \frac{r}{d_1}\partial_r + \frac{z-\kappa^{-1}}{d_1}\partial_z, \quad \partial_\phi = (z-\kappa^{-1})\partial_r - r\partial_z.$$

It is convenient to record the following derivatives of the phase before separating the cases:

$$\begin{aligned}
\partial_\rho\Phi &= \frac{t}{2}\left(1+\frac{r^2+z^2-\kappa^{-2}}{d_1d_2}\right)-\frac{1}{d_1}(r,z-\kappa^{-1})\cdot(x_1,x_3), \quad \partial_\rho^2\Phi = 2t\frac{\kappa^{-2}r^2}{d_1^2d_2^3},\\
\partial_\phi\Phi &= -t\frac{\kappa^{-1}r}{d_2}-(z-\kappa^{-1},-r)\cdot(x_1,x_3),\\
\partial_\phi^2\Phi + d_1\partial_\rho\Phi &= t\frac{(r^2+z^2-\kappa^{-2})^2+d_1d_2^3}{d_2^3}, \quad d_1\partial_\rho\partial_\phi\Phi-\partial_\phi\Phi = t\frac{\kappa^{-1}r}{d_2^2}\frac{r^2+z^2-\kappa^{-2}}{d_2}.
\end{aligned} \tag{2.43}$$

3-1. $p\le l\le 0$. This is when $d_1\sim z-\kappa^{-1}\sim 2^{k+l}$. When $x_1\not\sim t2^{p-l}$, from $\partial_r\Sigma\sim 2^{p-l}$ we obtain that $|\partial_r\Phi|\gtrsim t2^{p-l}$. Since (see (2.40))

$$\begin{aligned}
\frac{\partial_r^2\Phi}{|\partial_r\Phi|} &\lesssim \frac{t2^{-k-l}}{t2^{p-l}} = 2^{-k-p},\\
\left|\frac{\partial_r^3\Phi}{\partial_r\Phi}\right| &\lesssim t^{-1}2^{l-p}\cdot\frac{t}{2}\left(\frac{r(z-\kappa^{-1})^2}{d_1^5}+\frac{r(z+\kappa^{-1})^2}{d_2^5}\right)\lesssim 2^{-2k-2l}\le(2^{-k-p})^2,
\end{aligned}$$

the gain from one integration by parts in ∂_r is $(t2^{p-l})^{-1}\cdot 2^{-k-p}=t^{-1}2^{-k-2p+l}$, up to twice. Hence, integration by parts in ∂_r gives

$$|I_{\Sigma,l}|\lesssim 2^{3k+2p+l}\cdot\min\{1,(t2^{k+2p-l})^{-2}\}\le 2^{\frac{3}{2}k-p+\frac{5}{2}l}t^{-\frac{3}{2}},$$

so we now assume $x_1\sim t2^{p-l}$ and restrict the size of first order derivatives. Let

$$\begin{aligned}
I_{\Sigma,l,n} &:= \int e^{i\Phi}\varphi((t2^n)^{-1}\partial_\rho\Phi)\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k-l}(z-\kappa^{-1}))H(x_1r)rdrdz\\
&= i(t2^n)^{-1}\int e^{i\Phi}\partial_\rho[\tilde\varphi((t2^n)^{-1}\partial_\rho\Phi)\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k-l}(z-\kappa^{-1}))H(x_1r)r]drdz.
\end{aligned}$$

If the derivative falls on anything else than $\tilde\varphi((t2^n)^{-1}\partial_\rho\Phi)$, we can simply integrate to gain the set size, and this gives $(t2^n)^{-1}2^{-k-l}2^{3k+2p+l}\langle t2^{k+2p-l}\rangle^{-\frac{1}{2}}\le 2^{\frac{3}{2}k+p+\frac{1}{2}l}2^{-n}t^{-\frac{3}{2}}$. If the derivative falls on $\tilde{\varphi}$, we change variables to (ρ, ϕ) first and then to $(\partial_\rho \Phi, \phi)$ to get

$$
\begin{aligned}
&(t2^n)^{-1} \int (t2^n)^{-1} \partial_\rho^2 \Phi \tilde{\varphi}'((t2^n)^{-1}\partial_\rho \Phi)\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k-l}(z-\kappa^{-1}))H(x_1 r) r dr dz \\
&\lesssim (t2^n)^{-1} \int (t2^n)^{-1} \partial_\rho^2 \Phi \tilde{\varphi}'((t2^n)^{-1}\partial_\rho \Phi)\varphi(2^{-k-p}\rho \sin\phi)\varphi(2^{-k-l}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2 \sin\phi d\rho d\phi \\
&\lesssim (t2^n)^{-1} \int (t2^n)^{-1} \tilde{\varphi}'((t2^n)^{-1} y)\varphi(2^{l-p}\sin\phi) d\rho d\phi \cdot 2^{2k+p+l} \langle t2^{k+2p-l}\rangle^{-\frac{1}{2}} dy d\phi \\
&\lesssim (t2^n)^{-1} 2^{p-l} 2^{2k+p+l} (t2^{k+2p-l})^{-\frac{1}{2}} = 2^{\frac{3}{2}k+p+\frac{1}{2}l} 2^{-n} t^{-\frac{3}{2}}.
\end{aligned}
$$

Hence, summing up in n for $2^n \gtrsim 1$, we get acceptable bound of $\lesssim 2^{\frac{3}{2}k+p+\frac{1}{2}l} t^{-\frac{3}{2}}$, and can assume $\partial_\rho \Phi \ll t$ from hereon. We repeat a similar process with

$$
\begin{aligned}
I_{\Sigma,l,m} &:= \int e^{i\Phi} \varphi((t2^{k+p+\frac{1}{2}l+m})^{-1}\partial_\phi \Phi)\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k-l}(z-\kappa^{-1}))H(x_1 r) r dr dz \\
&= i(t2^{k+p+\frac{1}{2}l+m})^{-1} \\
&\quad \cdot \int e^{i\Phi}\partial_\phi[\tilde{\varphi}((t2^{k+p+\frac{1}{2}l+m})^{-1}\partial_\phi\Phi)\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k-l}(z-\kappa^{-1}))H(x_1 r) r] dr dz,
\end{aligned}
$$

to obtain $|I_{\Sigma,l,m}| \lesssim 2^{\frac{3}{2}k-p+2l-m} t^{-\frac{3}{2}}$. Thus we are left with the case where $|\partial_\rho \Phi| \ll t$ and $|\partial_\phi \Phi| \ll t2^{k+p+\frac{1}{2}l}$, and from (2.43)

$$
\partial_\rho^2 \Phi \sim t2^{-k+2p-2l}, \quad \partial_\phi^2 \Phi \sim t2^{k+l}, \quad \rho\partial_\rho\partial_\phi\Phi - \partial_\phi \Phi \lesssim t2^{k+p+l}.
$$

The formula

$$
\rho^2 |\nabla^2_{\rho,\phi}\Phi| = \rho^4 |\nabla^2_{r,z}\Phi|^2 - \rho^3 \partial_\rho^2 \Phi \cdot \partial_\rho \Phi - 2(\rho\partial_\rho\partial_\phi \Phi - \partial_\phi\Phi)\cdot\partial_\phi\Phi - (\partial_\phi \Phi)^2 \tag{2.44}
$$

ensures us that we are now in the case (2.14) since the third derivative conditions are met by

$$\partial_\phi\partial_\rho^2\Phi = t\frac{4\kappa^{-2}r}{d_1^2d_2^5}[d_2^2(z-\kappa^{-1})+3\kappa^{-1}r^2] \lesssim |\partial_\rho^2\Phi|\cdot\frac{2^{3k+l}}{2^{3k+p}} \sim \frac{1}{\phi}|\partial_\rho^2\Phi|,$$

$$d_1\partial_\rho\partial_\phi^2\Phi - \partial_\phi^2\Phi = t\left(\frac{\kappa^{-1}(z-\kappa^{-1})}{d_2^2}+\frac{6\kappa^{-2}r^2}{d_2^4}\right)\frac{r^2+z^2-\kappa^{-2}}{d_2} - 2t\frac{\kappa^{-2}r^2}{d_2^3} \lesssim t2^{k+2l} \lesssim \partial_\phi^2\Phi,$$

so that $|\partial_\rho\partial_\phi^2\Phi| \lesssim \frac{1}{d_1}|\partial_\phi^2\Phi|$. One can also check that

$$L(\rho,\phi) = \varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-l}\rho\cos\phi)\rho^2\sin\phi H(x_1\rho\sin\phi)$$

satisfies the needed property with $M = 2^{2k+p+l}(t2^{k+2p-l})^{-\frac{1}{2}}$. Therefore, using Theorem 2.2.1, we obtain

$$|I_{\Sigma,l}| \lesssim 2^{\frac{3}{2}k-p+2l}t^{-\frac{3}{2}}.$$

Using $p \le l \le 0$, one can see that the previous bounds are at least as strong as this, hence this is the single estimate that works for all $I_{\Sigma,l}$.

3-2. $\boldsymbol{l \le p}$. We treat all l's with a single localization $\bar{\varphi}(2^{-k-p}(z-\kappa^{-1})) := \sum_{l\le p}\varphi(2^{-k-l}(z-\kappa^{-1}))$. When $x_1 \nsim t$, from $\frac{|\partial_r^2\Phi|}{|\partial_r\Phi|} \lesssim 2^{-k-p}$, $\frac{|\partial_r^3\Phi|}{|\partial_r\Phi|} \lesssim 2^{-2k-2p}$, we have $|I_{\Sigma,\le p}| \lesssim 2^{3k+3p}\cdot\min\{1,(t2^{k+p})^{-2}\} \le 2^{\frac{3}{2}k+\frac{3}{2}p}t^{-\frac{3}{2}}$. Hence, we now assume $x_1 \sim t$ and start using the first derivatives first. Repeating what we did in the previous case with

$$\begin{aligned}I_{\Sigma,\le p,n} :=& \int e^{i\Phi}\varphi((t2^n)^{-1}\partial_\rho\Phi)\varphi(2^{-k-p}r)\bar{\varphi}(2^{-k-p}(z-\kappa^{-1}))H(x_1r)rdrdz\\ =& i(t2^n)^{-1}\int e^{i\Phi}\partial_\rho[\tilde{\varphi}((t2^n)^{-1}\partial_\rho\Phi)\varphi(2^{-k-p}r)\bar{\varphi}(2^{-k-p}(z-\kappa^{-1}))H(x_1r)r]drdz,\end{aligned}$$

gives $|I_{\Sigma,\leq p,n}| \lesssim 2^{\frac{3}{2}k+\frac{3}{2}p}2^{-n}t^{-\frac{3}{2}}$. Also,

$$\begin{aligned}
I^m_{\Sigma,\leq p} &:= \int e^{i\Phi}\varphi((t2^{k+\frac{3}{2}p+m})^{-1}\partial_\phi\Phi)\varphi(2^{-k-p}r)\bar{\varphi}(2^{-k-p}(z-\kappa^{-1}))H(x_1r)rdrdz \\
&= i(t2^{k+\frac{3}{2}p+m})^{-1}\int e^{i\Phi}\partial_\phi[\tilde{\varphi}((t2^{k+\frac{3}{2}p+m})^{-1}\partial_\phi\Phi)\varphi(2^{-k-p}r)\bar{\varphi}(2^{-k-p}(z-\kappa^{-1}))H(x_1r)r]drdz \\
&\lesssim (t2^{k+\frac{3}{2}p+m})^{-1}\cdot 2^{3k+3p}(t2^{k+p})^{-\frac{1}{2}} = 2^{\frac{3}{2}k+p}2^{-m}t^{-\frac{3}{2}}.
\end{aligned}$$

Summing up in $n, m \geq 0$ gives acceptable results, and hence, now we are in a region where $\partial_\rho\Phi \ll t$ and $\partial_\phi\Phi \ll t2^{k+\frac{3}{2}p}$. Thus, from (2.44) and

$$\partial_\rho^2\Phi \sim t2^{-k}, \quad \partial_\phi^2\Phi \sim t2^{k+p}, \quad \rho\partial_\rho\partial_\phi\Phi - \partial_\phi\Phi \lesssim t2^{k+2p},$$

which follows from (2.43), one can check that the first condition of (2.14) holds. The function

$$L(\rho,\phi) = \varphi(2^{-k-p}r)\bar{\varphi}(2^{-k-p}(z-\kappa^{-1}))H(x_1\rho\sin\phi)\rho^2\sin\phi$$

satisfies the conditions with size $2^{\frac{3}{2}k+\frac{3}{2}p}t^{-\frac{1}{2}}$ this time, and

$$\begin{aligned}
\partial_\phi\partial_\rho^2\Phi &= \partial_\rho^2\Phi\cdot\frac{2}{rd_2^2}[d_2^2(z-\kappa^{-1})+3\kappa^{-1}r^2] \lesssim |\partial_\rho^2\Phi|, \\
d_1\partial_\rho\partial_\phi^2\Phi - \partial_\phi^2\Phi &= t\left(\frac{\kappa^{-1}(z-\kappa^{-1})}{d_2^2}+\frac{6\kappa^{-2}r^2}{d_2^4}\right)\frac{r^2+z^2-\kappa^{-2}}{d_2} - 2t\frac{\kappa^{-2}r^2}{d_2^3} \lesssim t2^{k+2p} \lesssim |\partial_\phi^2\Phi|,
\end{aligned}$$

hence $|\partial_\rho\partial_\phi^2\Phi| \lesssim \frac{1}{\rho}|\partial_\phi^2\Phi|$. Therefore, all the conditions of Theorem 2.2.1 are satisfied and this gives

$$|I_{\Sigma,\leq p}| \lesssim (t2^{\frac{1}{2}p})^{-1}2^{\frac{3}{2}k+\frac{3}{2}p}t^{-\frac{1}{2}} = 2^{\frac{3}{2}k+p}t^{-\frac{3}{2}}.$$

Again, this is the weakest among the estimates, and hence the final bound for $I_{\Sigma,\leq p}$.

2.4.2 Decay of Ω evolution

We will now localize in z in addition by default, and investigate the decay of

$$I_\Omega := \int e^{i\Phi}\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k-q}z)H(x_1 r)r\, drdz.$$

$\Phi = t\Omega - (r,z)\cdot(x_1,x_3)$, and when needed, we will assume $z \geq 0$ using the symmetry without loss of generality. We again give some derivatives of Φ that will be used several times:

$$\begin{aligned}
\partial_r\Phi &= -t\frac{r\Omega}{d_1 d_2} - x_1,\\
\partial_z\Phi &= \frac{t}{2}\left(\frac{z+\kappa^{-1}}{d_2} - \frac{z-\kappa^{-1}}{d_1}\right) - x_3,\\
\partial_r^2\Phi &= \frac{t}{2}\left(\frac{(z+\kappa^{-1})^2}{d_2^3} - \frac{(z-\kappa^{-1})^2}{d_1^3}\right),\\
\partial_z^2\Phi &= \frac{t}{2}\left(\frac{r^2}{d_2^3} - \frac{r^2}{d_1^3}\right),\\
\partial_r^3\Phi &= -\frac{3tr}{2}\left(\frac{(z+\kappa^{-1})^2}{d_2^5} - \frac{(z-\kappa^{-1})^2}{d_1^5}\right),\\
\partial_r\partial_z^2\Phi &= \frac{tr}{2}\left(\frac{2(z+\kappa^{-1})^2 - r^2}{d_2^5} - \frac{2(z-\kappa^{-1})^2 - r^2}{d_1^5}\right),\\
\partial_z\partial_r^2\Phi &= t\frac{z+\kappa^{-1}}{2}\frac{2r^2-(z+\kappa^{-1})^2}{d_2^5} - t\frac{z-\kappa^{-1}}{2}\frac{2r^2-(z-\kappa^{-1})^2}{d_1^5}.
\end{aligned} \tag{2.45}$$

Low frequency, $\kappa 2^k \ll 1$.

When $x_1 \nsim t\kappa^2 2^{2k+p+q}$, we have $|\partial_r\Phi| \gtrsim t\kappa^2 2^{2k+p+q}$ from $\partial_r\Omega = -\frac{r\Omega}{d_1 d_2} \sim \kappa^2 2^{2k+p+q}$. Integration by parts along ∂_r gains $(t\kappa^2 2^{2k+p+q})^{-1}2^{-k-p}$ from (2.41) and

$$\begin{aligned}
\left|\frac{\partial_r^2\Phi}{\partial_r\Phi}\right| &\lesssim \frac{t\kappa^2 2^{k+q}}{t\kappa^2 2^{2k+p+q}} = 2^{-k-p},\\
\left|\frac{\partial_r^3\Phi}{\partial_r\Phi}\right| &\lesssim \frac{t\kappa^4 2^{2k+p+q}}{t\kappa^2 2^{2k+p+q}} = \kappa^2 \ll 2^{-2k} \leq 2^{-2k-2p},
\end{aligned}$$

since by (2.45)

$$
\begin{aligned}
\partial_r^2\Phi &= t\frac{z^2+\kappa^{-2}}{2}\left(\frac{1}{d_2^3}-\frac{1}{d_1^3}\right)+2t\kappa^{-1}z\left(\frac{1}{d_2^3}+\frac{1}{d_1^3}\right)\lesssim t(\kappa^{-2}\cdot\kappa^4 2^{k+q}+\kappa^{-1}2^{k+q}\kappa^3)\sim t\kappa^2 2^{k+q},\\
|\partial_r^3\Phi| &= \frac{3tr}{2}\left|\frac{(z-\kappa^{-1})^2}{d_1^5}-\frac{(z+\kappa^{-1})^2}{d_2^5}\right|\le\frac{3tr}{2}(z^2+\kappa^{-2})\left|\frac{1}{d_1^5}-\frac{1}{d_2^5}\right|+tr\kappa^{-1}|z|\left(\frac{1}{d_1^5}+\frac{1}{d_2^5}\right)\\
&\lesssim t\kappa^4 2^{2k+p+q}.
\end{aligned}
$$

Thus, integration by parts along ∂_r gives

$$
|I_\Omega|\lesssim 2^{3k+2p+q}\cdot\min\{1,(t\kappa^2 2^{3k+2p+q})^{-2}\}\le\kappa^{-3}2^{-\frac{3}{2}k-p-\frac{q}{2}}t^{-\frac{3}{2}}.
$$

Now we can assume $x_1\sim t\kappa^2 2^{2k+p+q}$. From the size of $\partial_z^2\Phi$

$$
\partial_z^2\Phi=t\frac{r^2}{2}\frac{(d_1-d_2)(d_1^2+d_1d_2+d_2^2)}{d_1^3d_2^3}=-tr^2\frac{\Omega(d_1^2+d_1d_2+d_2^2)}{d_1^3d_2^3}\sim t\kappa^4 2^{3k+2p+q},
$$

we can also remove the regions with large $\partial_z\Phi$ by setting

$$
\begin{aligned}
L(r,z) &:= \varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k-q}z)H(x_1r)r,\\
I_{\Omega,n} &:= \int e^{i\Phi}\varphi\left(\frac{\partial_z\Phi}{t\kappa^2 2^{2k+2p+n}}\right)L(r,z)\,drdz,
\end{aligned}
$$

since $I_{\Omega,n}$ give the desired estimate after summation in n from

$$
\begin{aligned}
|I_{\Omega,n}| &= \left|\frac{i}{t\kappa^2 2^{2k+2p+n}}\int e^{i\Phi}\partial_z\left[\tilde\varphi\left(\frac{\partial_z\Phi}{t\kappa^2 2^{2k+2p+n}}\right)L(r,z)\right]\,drdz\right|\\
&\le\left|\frac{i}{t\kappa^2 2^{2k+2p+n}}\int e^{i\Phi}\tilde\varphi'\left(\frac{\partial_z\Phi}{t\kappa^2 2^{2k+2p+n}}\right)\cdot\frac{\partial_z^2\Phi}{t\kappa^2 2^{2k+2p+n}}\cdot L(r,z)\,drdz\right|\\
&\quad+\left|\frac{i}{t\kappa^2 2^{2k+2p+n}}\int e^{i\Phi}\tilde\varphi\left(\frac{\partial_z\Phi}{t\kappa^2 2^{2k+2p+n}}\right)\partial_zL(r,z)\,drdz\right|\\
&\lesssim t^{-\frac{3}{2}}\kappa^{-3}2^{-\frac{3}{2}k-p-\frac{1}{2}q}2^{-n}.
\end{aligned}
$$

Now we only need to estimate $\tilde{I}_\Omega = \int e^{i\Phi}\bar{\varphi}\left((t\kappa^2 2^{2k+2p})^{-1}\partial_z\Phi\right) L(r,z)\, dr dz$. From the fact that $r(\partial_r^2\Omega + \partial_z^2\Omega) = \partial_r\Omega \sim \kappa^2 2^{2k+p+q}$, one can conclude

$$\begin{aligned}
\partial_r^2\Omega &\sim \kappa^2 2^{k+q},\\
\partial_z^2\Omega &\sim \kappa^4 2^{3k+2p+q},\\
\partial_r\partial_z\Omega &\sim \kappa^2 2^{k+p},
\end{aligned}$$

which means we are now in a position to use Theorem 2.2.1 in the case with (2.15). Since this is the first usage of off-diagonal-dominant case, we again explain in full detail here.

The 3rd order derivative conditions are satisfied since

$$2\partial_z\partial_r^2\Omega = \frac{2(z+\kappa^{-1})}{d_2^3} - \frac{2(z-\kappa^{-1})}{d_1^3} - 3\frac{(z+\kappa^{-1})^3}{d_2^5} + \frac{3(z-\kappa^{-1})^3}{d_1^5} \lesssim \kappa^2 \sim 2^{-k-p}|\partial_r\partial_z\Omega|,$$
$$2\partial_r\partial_z^2\Omega = \frac{4}{r}\partial_z^2\Omega - \frac{3r^3}{d_2^5} + \frac{3r^3}{d_1^5} \lesssim \kappa^4 2^{2k+p} \ll 2^{-k}|\partial_r\partial_z\Omega|.$$

Having said that, we will use $(2^p\partial_r, 2^q\partial_z)$ instead of (∂_r, ∂_z). The sizes change into

$$\begin{gathered}
|2^p\partial_r\Phi| \lesssim t\kappa^2 2^{2k+2p+q}, \quad |2^q\partial_z\Phi| \lesssim t\kappa^2 2^{2k+2p+q},\\
|2^{2p}\partial_r^2\Phi| \sim t\kappa^2 2^{k+2p+q}, \quad |2^{2q}\partial_z^2\Phi| \sim t\kappa^4 2^{3k+2p+3q}, \quad |2^p 2^q\partial_r\partial_z\Phi| \sim t\kappa^2 2^{k+2p+q},\\
|\det \nabla^2_{2^p\partial_r, 2^q\partial_z}\Phi| \sim t^2\kappa^4 2^{2k+4p+2q},\\
|2^{2p}2^q\partial_z\partial_r^2\Phi| \lesssim 2^{-k}|2^p 2^q\partial_r\partial_z\Phi|, \quad |2^p 2^{2q}\partial_r\partial_z^2\Phi| \lesssim 2^{-k+q}|2^p 2^q\partial_r\partial_z\Phi| \leqslant 2^{-k}|2^p 2^q\partial_r\partial_z\Phi|,
\end{gathered}$$

which still satisfies (2.15), but now with $f_r = f_z = 2^k$. We set

$$a := t^{\frac{1}{2}}\kappa 2^{\frac{1}{2}k+p+\frac{q}{2}} \sim |2^p 2^q\partial_r\partial_z\Phi|^{\frac{1}{2}}, \quad b := t\kappa^2 2^{2k+2p+q}.$$

① $a \geq b$: This means we are already localized enough. Change of variable brings

$$
\begin{aligned}
|\tilde{I}_\Omega| &= \left| \int e^{i\Phi} \bar{\varphi}(b^{-1}2^p \partial_r \Phi) \bar{\varphi}(b^{-1}2^q \partial_z \Phi) L(r,z) dr dz \right| \\
&\lesssim b2^{-p} b2^{-q} \cdot (a^4 2^{-2p} 2^{-2q})^{-1} \cdot M \\
&\leq a^2 2^{-p-q} \cdot a^{-4} 2^{2p+2q} \cdot M = t^{-\frac{3}{2}} \kappa^{-3} 2^{-\frac{3}{2}k - p - \frac{q}{2}},
\end{aligned}
$$

since we can choose $|L| \leq \langle t\kappa^2 2^{2k+p+q} \cdot 2^{k+p} \rangle^{-1/2} \cdot 2^{k+p} = t^{-\frac{1}{2}} \kappa^{-1} 2^{-\frac{1}{2}k - \frac{q}{2}} =: M$, and we are done.

② $a \leq b$: Solving this in terms of t, we have $t^{\frac{1}{2}} \geqslant \kappa^{-1} 2^{-\frac{3}{2}k - p - \frac{q}{2}}$, which then gives $f_r^{-1} = f_z^{-1} = 2^{-k} \leq a$. First condition of (2.16) is straightforward from the homogeneity of L, and hence, applying Theorem 2.2.1 with L, M the same as above, we get

$$
|\tilde{I}_\Omega| \lesssim a^{-2} 2^{p+q} \cdot M = \kappa^{-3} 2^{-\frac{3}{2}k - p - \frac{q}{2}} t^{-\frac{3}{2}}.
$$

High frequency, $\kappa 2^k \gg 1$.

When $x_1 \not\sim t\kappa^{-1} 2^{-k+p+q}$, from

$$
\partial_r \Omega = \frac{1}{2}\left(\frac{r}{d_2} - \frac{r}{d_1}\right) = \frac{r}{2}\frac{d_1 - d_2}{d_1 d_2} = -\kappa^{-1} \frac{rz}{d_1 d_2 \Sigma} \sim \kappa^{-1} \frac{2^{2k+p+q}}{2^{3k}} = \kappa^{-1} 2^{-k+p+q},
$$

we have $|\partial_r \Phi| \gtrsim t\kappa^{-1} 2^{-k+p+q}$ in this case. Also, along with (2.41),

$$
\begin{aligned}
\partial_r^2 \Omega &= \frac{t}{2}\left(\frac{(z+\kappa^{-1})^2}{d_2^3} - \frac{(z-\kappa^{-1})^2}{d_1^3}\right) = \frac{t}{2}\left[(z^2+\kappa^{-2})\left(\frac{1}{d_2^3} - \frac{1}{d_1^3}\right) + 2\kappa^{-1} z \left(\frac{1}{d_2^3} + \frac{1}{d_1^3}\right)\right] \\
&\lesssim t\kappa^{-1} 2^{-2k+q}
\end{aligned}
$$

since

$$\frac{1}{d_2^3} - \frac{1}{d_1^3} = \frac{(d_1 - d_2)(d_1^2 + d_1 d_2 + d_2^2)}{d_1^3 d_2^3} = -\frac{2\Omega(d_1^2 + d_1 d_2 + d_2^2)}{d_1^3 d_2^3} \sim \kappa^{-1} 2^{-4k+q},$$

so that $\left|\frac{\partial_r^2 \Phi}{\partial_r \Phi}\right| \lesssim \frac{t\kappa^{-1}2^{-2k+q}}{t\kappa^{-1}2^{-k+p+q}} = 2^{-k-p}$, and

$$\begin{aligned}
|\partial_r^3 \Phi| &= \frac{t}{2}\left|\frac{r(z-\kappa^{-1})^2}{d_1^5} - \frac{r(z+\kappa^{-1})^2}{d_2^5}\right| \\
&= tr\left|(z^2 + \kappa^{-2})\frac{\kappa^{-1} z(d_1^4 + d_1^3 d_2 + d_1^2 d_2^2 + d_1 d_2^3 + d_2^4)}{d_1^5 d_2^5 \Sigma} - \kappa^{-1} z\left(\frac{1}{d_1^5} + \frac{1}{d_2^5}\right)\right| \\
&\lesssim t2^{k+p}[(2^{2k+2q} + \kappa^{-2})\frac{\kappa^{-1}2^{k+q}2^{4k}}{2^{11k}}\kappa^{-1}2^{k+q}2^{-5k}] \lesssim t\kappa^{-1}2^{-3k+p+q},
\end{aligned}$$

so that one can check that from each integration by parts we gain $(t\kappa^{-1}2^{-k+p+q})^{-1}2^{-k-p} = t^{-1}\kappa 2^{-2p-q}$ up to twice. Therefore,

$$|I_\Omega| \lesssim 2^{3k+2p+q} \cdot \min\{1, (t^{-1}\kappa 2^{-2p-q})^2\} \leq \kappa^{\frac{3}{2}} 2^{3k-p-\frac{q}{2}} t^{-\frac{3}{2}}.$$

Hence, now we assume $x_1 \sim t\kappa^{-1}2^{-k+p+q}$. When $q \ll 0$, we have the sizes

$$\begin{aligned}
\partial_r^2 \Phi &\sim t\kappa^{-1}2^{-2k+q}, \\
\partial_z^2 \Phi &= t\frac{r^2}{2}\left(\frac{1}{d_2^3} - \frac{1}{d_1^3}\right) \sim t\kappa^{-1}2^{-2k+q}, \\
\partial_r\partial_z \Phi &= -\frac{3tr}{2}\left(\frac{z+\kappa^{-1}}{d_2^3} - \frac{z-\kappa^{-1}}{d_1^3}\right) = -\frac{3tr}{2}\left(z\left[\frac{1}{d_2^3} - \frac{1}{d_1^3}\right] + \kappa^{-1}\left[\frac{1}{d_2^3} + \frac{1}{d_1^3}\right]\right) \sim t\kappa^{-1}2^{-2k},
\end{aligned}$$

which satisfy (2.15) as the 3rd order derivatives also have bounds

$$\partial_r \partial_z^2 \Phi = \frac{2}{r}\partial_z^2 \Phi + \frac{3tr^3}{2}\left(\frac{1}{d_1^5} - \frac{1}{d_2^5}\right) \lesssim 2^{-k}|\partial_z^2 \Phi| \ll 2^{-k-q}|\partial_r \partial_z \Phi|,$$

$$
\begin{aligned}
\partial_z\partial_r^2\Omega &= \frac{1}{2}\left(\frac{2(z+\kappa^{-1})}{d_2^3} - \frac{2(z-\kappa^{-1})}{d_1^3} - \frac{3(z+\kappa^{-1})^3}{d_2^5} + \frac{3(z-\kappa^{-1})^3}{d_1^5}\right) \\
&= z\left(\frac{1}{d_2^3} - \frac{1}{d_1^3}\right) + \kappa^{-1}\left(\frac{1}{d_2^3} + \frac{1}{d_1^3}\right) - 3z(z^2+3\kappa^{-2})\left(\frac{1}{d_2^5} - \frac{1}{d_1^5}\right) \\
&\quad - 3\kappa^{-1}(3z^2+\kappa^{-2})\left(\frac{1}{d_2^5} + \frac{1}{d_1^5}\right) \\
&\lesssim \kappa^{-1}2^{-3k} \lesssim 2^{-k}|\partial_r\partial_z\Omega|.
\end{aligned}
$$

As in the low frequency case, the estimates above enables us to use $(\partial_r, 2^q\partial_z)$. When $(t\kappa^{-1}2^{-2k+q})^{1/2} \leq 2^{-k}$, simple set size estimate is already enough to give $|I_\Omega| \lesssim 2^{3k+q} \leq \kappa^{\frac{3}{2}}2^{3k-p-\frac{q}{2}}t^{-\frac{3}{2}}$. Hence, assuming When $(t\kappa^{-1}2^{-2k+q})^{1/2} \leq 2^{-k} = f_r^{-1} = f_z^{-1}$, condition (2.16) has been met, hence with $L(r,z) = \varphi(2^{-k}|r,z|)\varphi(2^{-k}r)\varphi(2^{-k-q}z)H(x_1r)r$ and $M = \kappa^{\frac{1}{2}}2^{k-\frac{q}{2}}t^{-\frac{1}{2}}$, we can use Theorem 2.2.1 which gives

$$
|I_\Omega| \lesssim (t\kappa^{-1}2^{-2k+p})^{-1}\kappa^{\frac{1}{2}}2^{k-\frac{q}{2}}t^{-\frac{1}{2}} = \kappa^{\frac{3}{2}}2^{3k-p-\frac{q}{2}}t^{-\frac{3}{2}}.
$$

When $q \sim 0$, we use the spherical coordinates. We first consider

$$
I_{\Omega,n} := \int e^{i\Phi}\varphi(t^{-1}\kappa 2^{k-2p-q-n}\partial_\rho\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)\varphi(2^{-q}\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\, d\rho d\phi.
$$

By integration by parts along ∂_ρ,

$$
\begin{aligned}
|I_{\Omega,n}| =\ & t^{-1}\kappa 2^{k-2p-q-n} \\
&\cdot\left|\int e^{i\Phi}\partial_\rho\left[\tilde\varphi(t^{-1}\kappa 2^{k-2p-q-n}\partial_\rho\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)\varphi(2^{-q}\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right]d\rho d\phi\right| \\
\leq\ & (t^{-1}\kappa 2^{k-2p-q-n})^2\int|\partial_\rho^2\Phi|\tilde\varphi'(t^{-1}\kappa 2^{k-2p-q-n}\partial_\rho\Phi)\varphi(2^{-p}\sin\phi)\varphi(2^{-q}\cos\phi)\langle x_1 2^{k+p}\rangle^{-\frac{1}{2}}2^{2k+p}d\rho d\phi \\
&+ t^{-1}\kappa 2^{k-2p-q-n}\int\left|\partial_\rho\left[\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)\varphi(2^{-q}\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right]\right|d\rho d\phi \\
\lesssim\ & \kappa^{\frac{3}{2}}2^{3k-p-\frac{q}{2}}2^{-n}t^{-\frac{3}{2}},
\end{aligned}
$$

giving acceptable contribution by summing over n for $2^n \gtrsim 1$. Similarly,

$$
\begin{aligned}
|I_\Omega^m| &:= \left| \int e^{i\Phi}\varphi(t^{-1}\kappa 2^{-p-m}\partial_\phi\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)\varphi(2^{-q}\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi d\rho d\phi \right| \\
&\le t^{-1}\kappa 2^{-p-m} \\
&\quad \cdot \int \left|\partial_\phi\left[\varphi(t^{-1}\kappa 2^{-p-m}\partial_\phi\Phi)\varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)\varphi(2^{-q}\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right]\right| d\rho d\phi \\
&\lesssim t^{-1}\kappa 2^{-p-m}\langle t\kappa^{-1}2^{2p+q}\rangle^{-\frac{1}{2}}2^{3k+p} \le \kappa^{\frac{3}{2}}2^{3k-p-\frac{q}{2}}t^{-\frac{3}{2}}2^{-m},
\end{aligned}
$$

and again we have acceptable contribution for $2^m \gtrsim 1$. Therefore, now we have the sizes

$$
\begin{aligned}
\partial_\rho^2\Phi &= t\frac{r^2}{2\kappa^2|\xi|^2}\left(\frac{1}{d_2^3}-\frac{1}{d_1^3}\right) \sim t\kappa^{-3}2^{-4k+2p+q}, \\
\partial_\phi^2\Phi &= -\frac{t}{2}\left[\frac{\rho^2\Omega}{d_1d_2}+\kappa^{-2}r^2\left(\frac{1}{d_2^3}-\frac{1}{d_1^3}\right)\right]-\rho\partial_\rho\Phi \sim t\kappa^{-1}2^q \\
\partial_\rho\partial_\phi\Phi &= \frac{t}{\rho}\frac{\kappa^{-1}r}{2}\left(\frac{1}{d_2}+\frac{1}{d_1}-\frac{\kappa^{-1}(z+\kappa^{-1})}{d_2^3}+\frac{\kappa^{-1}(z-\kappa^{-1})}{d_1^3}\right)+\frac{1}{\rho}\partial_\phi\Phi \sim t\kappa^{-1}2^{-k+p},
\end{aligned}
$$

where the second line holds due to $q \sim 0$ (so actually the size is simply $t\kappa^{-1}$). Using $(2^k\partial_\rho, 2^p\partial_\phi)$ instead of $(\partial_\rho, \partial_\phi)$ similarly as previous cases shows that this satisfy (2.15) along with

$$
2^k\partial_\phi\partial_\rho^2\Phi = 2^k\frac{2z}{r}\partial_\rho^2\Phi + 2^k\frac{tr^2}{\kappa^2\rho^2}\left(\frac{3\kappa^{-1}r}{d_2^5}+\frac{3\kappa^{-1}r}{d_1^5}\right) \lesssim 2^{k-p}|\partial_\rho^2\Phi| \lesssim |\partial_\rho\partial_\phi\Phi|,
$$

$$
\begin{aligned}
2^p\partial_\rho\partial_\phi^2\Phi = -\frac{t\kappa^{-1}2^p}{2\rho}\Bigg(&-\frac{\kappa^{-1}r^2}{d_2^3}+3\frac{\kappa^{-1}r^2(z+\kappa^{-1})^2}{d_2^5}-z\kappa^{-1}\frac{2(z+\kappa^{-1})d_2^2-3(z+\kappa^{-1})^3}{d_2^5} \\
&+\frac{\kappa^{-1}r^2}{d_1^3}-3\frac{\kappa^{-1}r^2(z-\kappa^{-1})^2}{d_1^5}+z\kappa^{-1}\frac{2(z-\kappa^{-1})d_1^2-3(z-\kappa^{-1})^3}{d_1^5}\Bigg) \\
\lesssim t\kappa^{-1}2^{-k+p} \lesssim 2^{-k+p}&|\partial_\phi^2\Phi| \lesssim |\partial_\rho\partial_\phi\Phi|.
\end{aligned}
$$

Thus, proceeding similarly as in the low frequency case to use Theorem 2.2.1 with $M = 2^{2k-\frac{q}{2}}\kappa^{\frac{1}{2}}t^{-\frac{1}{2}}$ and $L(\rho,\phi) = \varphi(2^{-k}\rho)\varphi(2^{-p}\sin\phi)\varphi(2^{-q}\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi$, we have

$$|I_\Omega| \lesssim (t\kappa^{-1}2^{-k+p})^{-1} \cdot 2^{2k-\frac{q}{2}}\kappa^{\frac{1}{2}}t^{-\frac{1}{2}} = 2^{3k-p-\frac{q}{2}}\kappa^{\frac{3}{2}}t^{-\frac{3}{2}}.$$

Medium frequency

The situations are subtly different according to the regions, and we work in three different regimes : $q \sim 0, p \le l \le 0$; $q \sim 0, l \le p$; and $q \ll 0$, where l is introduced again for $z - \kappa^{-1} \sim 2^{k+l}$ as in case 5.1.3. This implies we will only be working on frequencies on the upper half plane.

① $q \sim 0, p \le l \le 0$: Here, we work with the integral

$$I_{\Omega,l} := \int e^{i\Phi}\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k}z)\varphi(2^{-k-l}(z-\kappa^{-1}))H(x_1 r)rdrdz.$$

When $x_1 \nsim t2^{p-l}$, $|\partial_r\Phi| \gtrsim t2^{p-l}$ since $\partial_r\Omega = \frac{r}{2}\left(\frac{1}{d_2} - \frac{1}{d_1}\right) = -\frac{r\Omega}{d_1 d_2} \sim 2^{p+q-l}$, and

$$\begin{aligned}
|\partial_r^2\Phi| &= \frac{t}{2}\left|\frac{(z+\kappa^{-1})^2}{d_2^3} - \frac{(z-\kappa^{-1})^2}{d_1^3}\right| \lesssim t2^{-k-l}, \\
|\partial_r^3\Phi| &= \frac{3t}{2}\left|\frac{r(z+\kappa^{-1})^2}{d_2^5} - \frac{r(z-\kappa^{-1})^2}{d_1^5}\right| \lesssim t2^{-2k-2l},
\end{aligned}$$

we have $\frac{|\partial_r^2\Phi|}{|\partial_r\Phi|} \lesssim 2^{-k-p}$ and $\frac{|\partial_r^3\Phi|}{|\partial_r\Phi|} \lesssim 2^{-2k-p-l} \le 2^{-2k-2p}$. Hence,

$$|I_{\Omega,l}| \lesssim 2^{3k+2p+l} \cdot \min\{1, (t2^{p-l} \cdot 2^{k+p})^{-2}\} \le 2^{\frac{3}{2}k-p+\frac{5}{2}l}t^{-\frac{3}{2}}.$$

So we assume $x_1 \sim t2^{p-l}$ and now change to spherical coordinates centered at $(0, \kappa^{-1})$, denoted by (ρ,ϕ) again as in the analysis of medium frequency of Σ. From

$$
\begin{aligned}
|I_{\Omega,l,n}| &:= \left|\int e^{i\Phi}\varphi(t^{-1}2^{-n}\partial_\rho\Phi)\varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-l}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi d\rho d\phi\right| \\
&\le t^{-1}2^{-n}\int\left|\partial_\rho\left[\varphi(t^{-1}2^{-n}\partial_\rho\Phi)\varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-l}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right]d\rho d\phi\right| \\
&\lesssim t^{-1}2^{-n}(2^{-k-l}2^{3k+2p+l}+2^{2k+2l})\langle t2^{k+2p-l}\rangle^{-\frac{1}{2}} \le 2^{\frac{3}{2}k-p+\frac{5}{2}l}2^{-n}t^{-\frac{3}{2}},
\end{aligned}
$$

we get acceptable contributions for summations over n. $\varphi(2^{-k}|r,z|)$ and $\varphi(2^{-k}z)$ were omitted for notational convenience and will be dropped from hereon, because

$$
\begin{aligned}
\left|\partial_\rho[\varphi(2^{-k}|r,z|),\varphi(2^{-k}z)]\right| &\lesssim \frac{z}{\rho}2^{-k},\frac{|z-\kappa^{-1}|}{\rho}2^{-k}\le 2^{-k}, \\
\left|\partial_\phi[\varphi(2^{-k}|r,z|),\varphi(2^{-k}z)]\right| &\lesssim 0, r2^{-k}\lesssim 2^p,
\end{aligned}
$$

and hence are terms of smaller sizes. Also from

$$
\begin{aligned}
|I^m_{\Omega,l}| &:= \left|\int e^{i\Phi}\varphi(t^{-1}2^{-k-p-\frac{1}{2}l-m}\partial_\phi\Phi)\varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-l}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi d\rho d\phi\right| \\
&\le t^{-1}2^{-k-p-\frac{1}{2}l-m} \\
&\quad\cdot\int\left|\partial_\phi\left[\varphi(t^{-1}2^{-k-p-\frac{1}{2}l-m}\partial_\phi\Phi)\varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-l}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right]\right|d\rho d\phi \\
&\lesssim t^{-1}2^{-k-p-\frac{1}{2}l-m}(2^{l-p}2^{3k+2p+l}+2^{3k+p+2l})\langle t2^{k+2p-l}\rangle^{-\frac{1}{2}} \le 2^{\frac{3}{2}k-p+2l}2^{-m}t^{-\frac{3}{2}},
\end{aligned}
$$

we again get acceptable contributions for summations over m. Therefore, now we are in a region where $\partial_\rho\Phi \ll t$ and $\partial_\phi\Phi \ll t2^{k+p+\frac{1}{2}l}$, which yields

$$
\partial_\rho^2\Phi \sim t2^{-k+2p-2l}, \quad \partial_\phi^2\Phi \sim t2^{k+l},
$$

from

$$
\begin{aligned}
\partial_\rho^2\Phi &= \frac{t}{2}\left(\frac{1}{d_2} - \frac{(r^2+z^2-\kappa^{-2})^2}{d_1^2 d_2^3}\right) = 2t\frac{\kappa^{-2}r^2}{d_1^2 d_2^3},\\
\rho\partial_\rho\partial_\phi\Phi - \partial_\phi\Phi &= t\kappa^{-1}r\frac{r^2+(z+\kappa^{-1})(z-\kappa^{-1})}{d_2^3},\\
\partial_\phi^2\Phi + \rho\partial_\rho\Phi &= \frac{t}{2}\left(-d_1 + \frac{r^2+z^2+\kappa^{-2}}{d_2} - \frac{2\kappa^{-1}(z-\kappa^{-1})}{d_2} - \frac{4(\kappa^{-1}r)^2}{d_2^3}\right) \\
&= \frac{t}{2}\left(-d_1 + \frac{d_1^2}{d_2} - 4\frac{(\kappa^{-1}r)^2}{d_2^3}\right)\\
&= -t\left(\frac{d_1}{d_2}\Omega + 2\frac{(\kappa^{-1}r)^2}{d_2^3}\right),
\end{aligned} \tag{2.46}
$$

since this shows that $|\rho\partial_\rho\partial_\phi\Phi - \partial_\phi\Phi| \lesssim t2^{k+p+l}$ and $|\partial_\phi^2\Phi + \rho\partial_\rho\Phi| \sim t2^{k+l}$. Last conditions of (2.14) can be checked by

$$
\begin{aligned}
\partial_\phi\partial_\rho^2\Phi &= \partial_\rho^2\Phi\cdot\frac{2}{rd_2^2}[d_2^2(z-\kappa^{-1}) + 3\kappa^{-1}r^2] \lesssim 2^{l-p}|\partial_\rho^2\Phi|,\\
\frac{1}{t}[\rho\partial_\rho\partial_\phi^2\Phi - \partial_\phi^2\Phi] &= \left(\frac{\kappa^{-1}(z-\kappa^{-1})}{d_2^2} + 6\frac{\kappa^{-2}r^2}{d_2^4}\right)\frac{r^2+z^2-\kappa^{-2}}{d_2} - \frac{2\kappa^{-2}r^2}{d_2^3} \lesssim 2^{k+2l} \lesssim \frac{1}{t}|\partial_\phi^2\Phi|
\end{aligned}
$$

Therefore, with $L(\rho,\phi) = \varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-l}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi$ and $M = 2^{\frac{3}{2}k+\frac{3}{2}l}t^{-\frac{1}{2}}$, we have

$$
|I_{\Omega,l}| \lesssim (t2^{p-\frac{1}{2}l})^{-1}\cdot 2^{\frac{3}{2}k+\frac{3}{2}l}t^{-\frac{1}{2}} = 2^{\frac{3}{2}k-p+2l}t^{-\frac{3}{2}}.
$$

This is the weakest estimate among the bounds obtained above, and therefore the final estimate.

② $q\sim 0, l\le p$: As in the analysis of Σ, we don't localize in l individually anymore, but consider

$$
I_{\Omega,\le p} := \int e^{i\Phi}\varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k}z)\bar{\varphi}(2^{-k-p}(z-\kappa^{-1}))H(x_1r)rdrdz.
$$

An important change from the case ① is that now $d_1 \sim 2^{k+p}$. If $x_1 \nsim t$,

$$|\partial_r^2 \Phi| = \frac{t}{2}\left|\frac{(z+\kappa^{-1})^2}{d_2^3} - \frac{(z-\kappa^{-1})^2}{d_1^3}\right| \lesssim t2^{-k-p},$$
$$|\partial_r^3 \Phi| = \frac{3t}{2}\left|\frac{r(z+\kappa^{-1})^2}{d_2^5} - \frac{r(z-\kappa^{-1})^2}{d_1^5}\right| \lesssim t2^{-2k-2p},$$

and hence $\frac{|\partial_r^2\Phi|}{|\partial_r\Phi|} \lesssim 2^{-k-p}$, $\frac{|\partial_r^3\Phi|}{|\partial_r\Phi|} \lesssim 2^{-2k-2p}$. With the other usual loss from ∂_r derivative, this gives

$$|I_{\Omega,\leq p}| \lesssim 2^{3k+3p} \cdot \min\{1, (t2^{k+p})^{-2}\} \leq 2^{\frac{3}{2}k+\frac{3}{2}p}t^{-\frac{3}{2}}.$$

Now we assume $x_1 \sim t$ and start using polar coordinates around $(0, \kappa^{-1})$ again. From

$$\begin{aligned}
|I_{\Omega,\leq p,n}| &= \left|\int e^{i\Phi}\varphi(t^{-1}2^{-n}\partial_\rho\Phi)\varphi(2^{-k-p}\rho\sin\phi)\bar{\varphi}(2^{-k-p}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi d\rho d\phi\right| \\
&= t^{-1}2^{-n}\int\left|\partial_\rho\left[\varphi(t^{-1}2^{-n}\partial_\rho\Phi)\varphi(2^{-k-p}\rho\sin\phi)\bar{\varphi}(2^{-k-p}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right]d\rho d\phi\right| \\
&\lesssim t^{-1}2^{-n}(2^{-k-p}2^{3k+3p} + 2^{2k+2p})\langle t2^{k+p}\rangle^{-\frac{1}{2}} \leq 2^{\frac{3}{2}k+\frac{3}{2}p}2^{-n}t^{-\frac{3}{2}},
\end{aligned}$$

summation over $2^n \gtrsim 1$ gives acceptable bound, and hence we assume $\partial_\rho\Phi \ll t$ now. Also,

$$\begin{aligned}
|I^m_{\Omega,\leq p}| &= \left|\int e^{i\Phi}\varphi(t^{-1}2^{-k-\frac{3}{2}p-m}\partial_\phi\Phi)\varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-p}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi d\rho d\phi\right| \\
&\leq t^{-1}2^{-k-\frac{3}{2}p-m} \\
&\quad \cdot \int\left|\partial_\phi\left[\varphi(t^{-1}2^{-k-\frac{3}{2}p-m}\partial_\phi\Phi)\varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-p}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi\right]\right|d\rho d\phi \\
&\lesssim t^{-1}2^{-k-\frac{3}{2}p-m} \cdot 2^{3k+3p}\langle t2^{k+p}\rangle^{-\frac{1}{2}} \leq 2^{\frac{3}{2}k+p}2^{-m}t^{-\frac{3}{2}},
\end{aligned}$$

so by the same reasoning, we now assume $\partial_\phi\Phi \ll t2^{k+\frac{3}{2}p}$. These, together with (2.46) gives

$$\partial_\rho^2\Phi \sim t2^{-k}, \quad \partial_\phi^2\Phi \sim t2^{k+p}.$$

Also, from

$$\partial_\phi \partial_\rho^2 \Phi = \partial_\rho^2 \Phi \cdot \frac{2}{rd_2^2}[d_2^2(z-\kappa^{-1}) + 3\kappa^{-1}r^2] \lesssim |\partial_\rho^2 \Phi|,$$
$$\frac{1}{t}[\rho\partial_\rho\partial_\phi^2\Phi - \partial_\phi^2\Phi] = \left(\frac{\kappa^{-1}(z-\kappa^{-1})}{d_2^2} + 6\frac{\kappa^{-2}r^2}{d_2^4}\right)\frac{r^2+z^2-\kappa^{-2}}{d_2} - \frac{2\kappa^{-2}r^2}{d_2^3} \lesssim 2^{k+2p} \lesssim \frac{1}{t}|\partial_\phi^2\Phi|,$$

the phase and $L(\rho,\phi) = \varphi(2^{-k-p}\rho\sin\phi)\varphi(2^{-k-p}\rho\cos\phi)H(x_1\rho\sin\phi)\rho^2\sin\phi$ with $M = 2^{\frac{3}{2}k+\frac{3}{2}p}t^{-\frac{1}{2}}$ satisfy (2.14), and thus

$$|I_{\Omega,\le p}| \lesssim t^{-1}2^{-\frac{1}{2}p} \cdot 2^{\frac{3}{2}k+\frac{3}{2}p}t^{-\frac{1}{2}} = 2^{\frac{3}{2}k+p}t^{-\frac{3}{2}}.$$

Again, other bounds are as strong as this last one, and hence this is the final estimate for case ②.

③ $q \ll 0$: When $x_1 \nsim t2^q$, we have $|\partial_r\Phi| \gtrsim t2^q$,

$$|\partial_r^2\Phi| = \frac{t}{2}\left|(z^2+\kappa^{-2})\left(\frac{1}{d_2^3}-\frac{1}{d_1^3}\right) + 2\kappa^{-1}z\left(\frac{1}{d_2^3}+\frac{1}{d_1^3}\right)\right| \lesssim t2^{-k+q-3l},$$
$$|\partial_r^3\Phi| = \frac{3rt}{2}\left|(z^2+\kappa^{-2})\left(\frac{1}{d_2^5}-\frac{1}{d_1^5}\right) + 2\kappa^{-1}z\left(\frac{1}{d_2^5}+\frac{1}{d_1^5}\right)\right| \lesssim 2^{-2k+p+q-5l}.$$

Noting that $p = l = 0$ when $q \ll 0$, this means ∂_r derivative causes 2^{-k} loss, and therefore, $|I_\Omega| \lesssim 2^{\frac{3}{2}k-\frac{1}{2}q}t^{-\frac{3}{2}}$ via the usual reasoning. Now we assume $x_1 \sim t2^q$. Second order derivatives satisfy

$$\partial_r^2\Phi \lesssim t2^{-k+q},$$
$$\partial_z^2\Phi = t\frac{r^2}{2}\left(\frac{1}{d_2^3}-\frac{1}{d_1^3}\right) = t\frac{r^2\Omega(d_1^2+d_1d_2+d_2^2)}{d_1^3d_2^3} \sim t2^{-k+q},$$
$$\partial_r\partial_z\Phi = -t\frac{r}{2}\left[\frac{z+\kappa^{-1}}{d_2} - \frac{z-\kappa^{-1}}{d_1}\right] = -t\frac{r}{2}\left[z\left(\frac{1}{d_2^3}-\frac{1}{d_1^3}\right) + \kappa^{-1}\left(\frac{1}{d_2^3}+\frac{1}{d_1^3}\right)\right] \sim t2^{-k},$$

when $q \ll 0$. Along with

$$\partial_r \partial_z^2 \Phi = tr\left(\frac{1}{d_2^3} - \frac{1}{d_1^3}\right) + \frac{3tr^3}{2}\left(\frac{1}{d_1^5} - \frac{1}{d_2^5}\right) \lesssim t2^{-2k} \lesssim 2^{-k}|\partial_r\partial_z\Phi|,$$

$$\partial_z \partial_r^2 \Phi = t\left(\frac{z+\kappa^{-1}}{d_2^3} - \frac{z-\kappa^{-1}}{d_1^3}\right) + \frac{3t}{2}\left(\frac{(z-\kappa^{-1})^3}{d_1^5} - \frac{(z+\kappa^{-1})^3}{d_2^5}\right) \lesssim t2^{-2k} \lesssim 2^{-k}|\partial_r\partial_z\Phi|,$$

the phase satisfy (2.15) and can be made to satisfy (2.16) by using $(\partial_r, 2^q\partial_z)$ and removing small time region similarly as in the low frequency case. Therefore, together with $L(r,z) = \varphi(2^{-k}|r,z|)\varphi(2^{-k-p}r)\varphi(2^{-k}z)H(x_1 r)r$ and $M = 2^{\frac{1}{2}k-\frac{1}{2}q}t^{-\frac{1}{2}}$, we obtain

$$|I| \lesssim t^{-1}2^k \cdot 2^{\frac{1}{2}k-\frac{1}{2}q}t^{-\frac{1}{2}} = 2^{\frac{3}{2}k-\frac{1}{2}q}t^{-\frac{3}{2}}$$

from Theorem 2.2.1.

2.5 Linear decay of (NSC)

Linearized equation of (NSC) has the dissipation term, and this will lead to exponential decay. Indeed, this is how, in the literature, the global solutions of (NSC) were obtained. The difference here is that we will look at how stability is affected as $\nu \to 0$, which means that excessive reliance on dissipation should be avoided. On the other hand, the linear decay of $e^{it\Lambda_\varepsilon}$ obtained as 'the incompressible limit' will be too weak to derive the global stability. This is why I will use the following estimate for (NSC).

Proposition 2.5.1. *(GHPW23; GPW23) There exists a norm D and a decomposition*

$$P_{k,p,q}e^{it\Lambda_1}f = I_{k,p,q}(f) + II_{k,p,q}f$$

such that for any $0 < \beta' < \frac{1}{60}$,

$$||I_{k,p,q}(f)||_\infty \lesssim 2^{\frac{3}{2}k-3k^+} \cdot \min\{2^{2p+q}, 2^{-p-\frac{q}{2}}t^{-\frac{3}{2}}\}||f||_D,$$
$$||II_{k,p,q}(f)||_2 \lesssim 2^{-3k^+}t^{-1-\beta'}2^{(-1-2\beta')p} \cdot \mathbf{1}_{2^{2p+q}\gtrsim t^{-1}}||f||_D.$$

Proof. See (GPW23, Proposition 4.1). □

I defer the precise definition of the D-norm since it requires introducing other related settings, and for a similar reason, refer to the original paper for the proof of the proposition.

CHAPTER 3

Energy estimates

Energy estimates are standard and are already needed to prove local well-posedness. However, since they are of crucial importance both for long-time and global stability, and more involved for nonlinear stability, I record here their concrete forms.

3.1 Energy estimates for (sBQ) and (CER)

For these two problems, I will establish long-time stability, and hence we need no more than the standard estimates.

Proposition 3.1.1. *(EW15; JK25) Let $s \geq 0$. There exists $K = K(s) \geq 0$ such that if $(\rho, u) \in C_t([0,T], H^s_x(\mathbb{R}^2))$ solve* (sBQ) *with initial data $(\rho_0, u_0) \in H^s(\mathbb{R}^2)$ for some $T \geq 0$, then for $t \in [0,T]$,*

$$\|(\rho, u)(t)\|_{H^s} \leq \|(\rho_0, u_0)\|_{H^s} \exp\left(K(s) \int_0^t \|(\nabla\rho, \nabla u)(\tau)\|_{L^\infty} \, d\tau\right).$$

Proposition 3.1.2. *(KPTW25) Let $s \geq 0$. There exists $K = K(s) \geq 0$ such that if $(\rho, u) \in C_t([0,T], H^s_x(\mathbb{R}^3))$ solve* (CER) *with initial data $(\rho_0, u_0) \in H^s(\mathbb{R}^3)$ for some $T \geq 0$,* *then for* $t \in [0,T]$,

$$\|(\rho,u)(t)\|_{H^s} \le \|(\rho_0,u_0)\|_{H^s} \exp\left(K(s)\int_0^t \|(\nabla\rho,\nabla u)(\tau)\|_{L^\infty}\, d\tau\right). \tag{3.1}$$

For dispersive PDEs, the linear operator is usually anti-symmetric and does not contribute to the L^2 norm. In practice, this means that the energy estimates work as in the usual Euler's equation and the Boussinesq equation, respectively. The standard way to deal with the Euler-type quasilinear quadratic nonlinearity is to use the following, often referred to as commutator estimates.

Lemma 3.1.3. *(KP88) For any* $s \ge 0$ *and* $f \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$, $g \in L^\infty(\mathbb{R}^3) \cap H^{s-1}(\mathbb{R}^3)$,

$$||(1-\Delta)^{s/2}(fg) - f(1-\Delta)^{s/2}g||_{L^2} \lesssim ||\nabla f||_\infty ||g||_{H^{s-1}} + ||f||_{H^s}||g||_\infty.$$

Proof of Proposition 3.1.1 and 3.1.2. The proof follows the standard energy estimates, and hence we only explain the harder one, 3.1. Define $W = (\rho,u)^\intercal$, and take the H^s dot product with (CER). The linear terms except the time derivatives cancel out since

$$\langle \mathrm{div}(cu), \rho\rangle_{H^s} + \langle c\nabla\rho, u\rangle_{H^s} + \langle \varepsilon^{-1} u \times e_3, u\rangle_{H^s} = 0,$$

from integration by parts and the property of cross product. Hence, we get

$$\frac{1}{2}\frac{d}{dt}\|W(t)\|^2_{H^s} = -\langle u\cdot\nabla\rho,\rho\rangle_{H^s} - \langle \alpha\rho\mathrm{div}(u),\rho\rangle_{H^s} - \langle u\cdot\nabla u, u\rangle_{H^s} - \langle \alpha\rho\nabla\rho, u\rangle_{H^s}.$$

We look at the third term first. Using Lemma 3.1.3,

$$\begin{aligned}\langle u\cdot\nabla u, u\rangle_{H^s} &= \int (u\cdot\nabla)(1-\Delta)^{\frac{s}{2}}u\cdot(1-\Delta)^{\frac{s}{2}}udx\\ &\quad+\int[(1-\Delta)^{\frac{s}{2}}(u\cdot\nabla)u-u\cdot(1-\Delta)^{\frac{s}{2}}\nabla u]\cdot(1-\Delta)^{\frac{s}{2}}udx\\ &= -\frac{1}{2}\int \operatorname{div}(u)|(1-\Delta)^{\frac{s}{2}}u|^2dx\\ &\quad+\int[(1-\Delta)^{\frac{s}{2}}(u\cdot\nabla)u-u\cdot(1-\Delta)^{\frac{s}{2}}\nabla u]\cdot(1-\Delta)^{\frac{s}{2}}udx\\ &\lesssim \|\operatorname{div}(u)\|_\infty\|u\|_{H^s}^2+(\|\nabla u\|_\infty\|\nabla u\|_{H^{s-1}}+\|u\|_{H^s}\|\nabla u\|_\infty)\|u\|_{H^s}\\ &\lesssim \|\nabla u\|_\infty\|u\|_{H^s}^2.\end{aligned}$$

The first term can be dealt similarly:

$$\begin{aligned}\langle u\cdot\nabla\rho,\rho\rangle_{H^s} &= \int[(1-\Delta)^{\frac{s}{2}}u\cdot\nabla\rho-u\cdot\nabla(1-\Delta)^{\frac{s}{2}}\rho](1-\Delta)^{\frac{s}{2}}\rho dx\\ &\quad+\int u\cdot\frac{1}{2}\nabla|(1-\Delta)^{\frac{s}{2}}\rho|^2dx\\ &\lesssim (||\nabla u||_\infty||\nabla\rho||_{H^{s-1}}+||u||_{H^s}||\nabla\rho||_\infty)||\rho||_{H^s}+\left|\int \operatorname{div}(u)|(1-\Delta)^{\frac{s}{2}}\rho|^2dx\right|\\ &\lesssim ||\nabla u||_\infty||\rho||_{H^s}^2+||\nabla\rho||_\infty||\rho||_{H^s}||u||_{H^s}.\end{aligned}$$

Lastly, we deal with the remaining two terms together after pulling out α:

$$\begin{aligned}\langle\rho\ \operatorname{div}(u),\rho\rangle_{H^s}+\langle\rho\nabla\rho,u\rangle_{H^s} &= \int \left[(1-\Delta)^{\frac{s}{2}}(\rho\operatorname{div}(u))-\rho(1-\Delta)^{\frac{s}{2}}\operatorname{div}(u)\right](1-\Delta)^{\frac{s}{2}}\rho dx\\ &\quad+\int[(1-\Delta)^{\frac{s}{2}}\operatorname{div}(u)]\rho(1-\Delta)^{\frac{s}{2}}\rho dx+\int\rho\nabla(1-\Delta)^{\frac{s}{2}}\rho\cdot(1-\Delta)^{\frac{s}{2}}udx\\ &\quad+\int[(1-\Delta)^{\frac{s}{2}}(\rho\nabla\rho)-\rho(1-\Delta)^{\frac{s}{2}}\nabla\rho](1-\Delta)^{\frac{s}{2}}udx.\end{aligned}$$

Integration by parts reduces the second line to $-\int[(1-\Delta)^{\frac{s}{2}}\rho]\nabla\rho\cdot(1-\Delta)^{\frac{s}{2}}udx$. Therefore,

$$\begin{aligned}|\langle\rho\ \mathrm{div}(u),\rho\rangle_{H^s}+\langle\rho\nabla\rho,u\rangle_{H^s}| &\lesssim (||\nabla\rho||_\infty||\mathrm{div}(u)||_{H^{s-1}}+||\rho||_{H^s}||\mathrm{div}(u)||_\infty)||\rho||_{H^s}\\ &\quad+||\nabla\rho||_\infty||\rho||_{H^s}||u||_{H^s}\\ &\quad+(||\nabla\rho||_\infty||\nabla\rho||_{H^{s-1}}+||\rho||_{H^s}||\nabla\rho||_\infty)||u||_{H^s}\\ &\lesssim ||\nabla u||_\infty||\rho||_{H^s}^2+||\nabla\rho||_\infty||\rho||_{H^s}||u||_{H^s}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{d}{dt}\|W(t)\|_{H^s}^2 &\lesssim ||\nabla u||_\infty(||\rho||_{H^s}^2+||u||_{H^s}^2)+||\nabla\rho||_\infty||\rho||_{H^s}||u||_{H^s}\\ &\leq (||\nabla\rho||_\infty+||\nabla u||_\infty)(||\rho||_{H^s}^2+||u||_{H^s}^2),\end{aligned}$$

and hence,

$$\frac{d}{dt}||W(t)||_{H^s}\lesssim(||\nabla\rho(t)||_\infty+||\nabla u(t)||_\infty)W(t).$$

From Gronwall's inequality, we obtain (3.1). Proposition 3.1.1 can be obtained similarly, noting that

$$\langle\rho,\kappa u_2\rangle_{H^s}+\langle u,-\kappa\rho e_2\rangle_{H^s}=0,\quad \langle u,\nabla p\rangle_{H^s}=0$$

due to the zero divergence condition, and that this condition leaves us with the same nonlinearity $u\cdot\nabla\rho$ in the equation for ρ. □

3.2 Energy estimates for (NSC)

The results in this section are based on (Ko24). First, we derive the usual energy estimates.

Lemma 3.2.1. *If u is a solution to* (NSC)*, it satisfies*

$$||u(t)||_{H^m}^2-||u_0||_{H^m}^2+2\nu\int_0^t||\nabla u(s)||_{H^m}^2ds\lesssim\int_0^t||\nabla u(s)||_{L^\infty}||u(s)||_{H^m}^2ds$$

Proof. This follows from the fact that

$$\int (1-\Delta)^{m/2}u \cdot (u \cdot \nabla (1-\Delta)^{m/2}u)dx = 0$$

due to $\mathrm{div}(u) = 0$ after integration by parts. Hence, combined with the Lemma 3.1.3, if we take derivatives and then take a dot product to (NSC)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||u||_{H^m}^2 - \nu \int \Delta(1-\Delta)^{m/2}u \cdot (1-\Delta)^{m/2}u \, dx \\
&= -\int (1-\Delta)^{m/2}u \cdot (1-\Delta)^{m/2}(u\cdot\nabla)u \, dx \\
&\lesssim ||\nabla u||_{L^\infty}||u||_{H^m}^2.
\end{aligned}$$

Integrating by parts for the integral on the left-hand side in x and then integrating in time gives the desired result. □

For the nonlinear analysis, I also derive energy estimates for $S^n u$ where S is defined as $Su = (x\cdot\nabla - 1)u = \rho\partial_\rho u - u$ for vector fields u, and $Sp = (x\cdot\nabla - 2)p$ for scalar p. S commutes with the equation (NSC) up to 'lower order' terms, since it comes from the symmetry of the equation with $\nu = 0$ and $[S, \Delta] = -2\Delta$.

Lemma 3.2.2. *Put $c_{n,k}$ as the kth coefficient of $(x+\frac{3}{2})(-1)^{n-k}(x+3)^{n-k-1}(x-2)^n$ when $k < n$, and $c_{n,n} = 1$. Then,*

$$-\langle S^n u, S^n \Delta u\rangle = \sum_{k=0}^{n} c_{n,k}||\nabla S^k u||_{L^2}^2.$$

Proof. From $S\Delta u = \Delta(S-2)u$, we can first put $\langle S^n u, S^n\Delta u\rangle = \langle S^n u, \Delta(S-2)^n u\rangle$. Using the fact that $S^* = -S-5$ for vectors,

$$\langle S^{n+1}u, \Delta S^k u\rangle = \langle S^n u, (-S-5)\Delta S^k u\rangle = -\langle S^n u, \Delta(S+3)S^k u\rangle.$$

Also, we have $\langle Sv, \Delta Sv\rangle = -||\nabla Sv||_{L^2}^2$ and

$$\begin{aligned}\langle Sv, \Delta v\rangle &= \sum_{i,j,k}\int x_j\partial_j v_i\partial_k\partial_k v_i dx - \int v\cdot\Delta v dx\\ &= -\sum_{i,j,k}\int [x_j\partial_j\partial_k v_i\partial_k v_i + \partial_k v_i\partial_k v_i]dx - \int v\cdot\Delta v dx\\ &= -\sum_{i,j}\int x_j\partial_j|\nabla v_i|^2/2dx = \frac{3}{2}||\nabla v||_{L^2}^2.\end{aligned}$$

Combining these 3 facts, we can write $\langle S^n u, \Delta(S-2)^n u\rangle$ as a linear combination of $||\nabla S^k u||_{L^2}^2, 0 \le k \le n$. Except for $||\nabla S^n u||_{L^2}^2$ and $||\nabla S^{n-1}u||_{L^2}^2$, the coefficients will be determined by $\langle S^k u, \Delta S^k u\rangle = -||\nabla S^k u||_{L^2}^2$ and $\langle S^{k+1}u, \Delta S^k u\rangle = \frac{3}{2}||\nabla S^k u||_{L^2}^2$ after sending S several times from left to right in the inner product $\langle S^n u, \Delta S^a u\rangle$. The lemma follows from these 3 rules and the fact that the initial coefficients are $\binom{n}{k}(-2)^k$. $\square$

Similarly, $\langle S^n u, (S-2)^n u\rangle$ can also be written as $\sum_{k=0}^n c'_{n,k}||S^k u||_2^2$ for some $c'_{n,k}$'s where $c'_{n,n} = 1$. Using these facts, we can get bounds for $S^n u$'s.

Lemma 3.2.3. *There exist $a_{n,k}, a'_{n,k} \ge 0, a_{n,n} = a'_{n,n} = 1$ such that*

$$\begin{aligned}&\sum_{k=0}^n a_{n,k}(||S^k u(t)||_2^2 - ||S^k u_0||_2^2) + 2\nu\int_0^t ||\nabla S^n u(s)||_2^2 ds\\ &\lesssim \int_0^t (||u(s)||_\infty + ||\nabla u(s)||_\infty)(\sum_{k=0}^n ||S^k u(s)||_2)^2 ds,\end{aligned}$$

$$\begin{aligned}&\sum_{k=0}^n a'_{n,k}(|||\nabla|^{-1}S^k u(t)||_2^2 - |||\nabla|^{-1}S^k u_0||_2^2) + 2\nu\int_0^t ||S^n u(s)||_2^2 ds\\ &\lesssim \int_0^t ||u(s)||_\infty |||\nabla|^{-1}S^n u(s)||_2(\sum_{k=0}^n ||S^k u(s)||_2)ds.\end{aligned}$$

Proof. Since S commutes with the Euler-Coriolis equation, iterating S, we obtain

$$\partial_t S^n u - \nu S^n\Delta u + e_3\times S^n u + \partial_j\{S^n\{u_i u_j\}\} + \nabla S^n p = 0. \tag{3.2}$$

Taking an inner product with $S^n u$ and arranging in the same way as in (GHPW23),

$$\frac{1}{2}\frac{d}{dt}||S^n u||_{L^2}^2 - \nu\langle S^n u, S^n \Delta u\rangle \lesssim ||\nabla u||_{L^\infty}||S^n u||_{L^2}^2 + ||S^n u||_{L^2}||u||_{L^\infty}(\sum_{a=0}^{n}||S^a u||_{L^2}).$$

We now focus on the dissipation term. In particular, the highest order term will always be $\kappa||\nabla S^n u||_{L^2}^2$. So we induct on n. For $n = 0$, the usual energy estimate gives,

$$||u(t)||_{L^2}^2 - ||u_0||_{L^2}^2 + 2\nu\int_0^t ||\nabla u(s)||_{L^2}^2 ds = 0$$

Suppose that the lemma holds until $n-1$. Apply Lemma 4.2 and put $d_{n,k} = \min\{0, c_{n,k}\}$ for $k < n$ and $d_{n,n} = 1$. Then,

$$\begin{aligned}
&||S^n u(t)||_2^2 - ||S^n u_0||_2^2 - 2\nu\int_0^t \langle S^n u, S^n \Delta u\rangle ds \\
&= ||S^n u(t)||_2^2 - ||S^n u_0||_2^2 + 2\nu\int_0^t \sum_{k=0}^{n} c_{n,k}||\nabla S^k u(s)||_2^2 ds \\
&\geq ||S^n u(t)||_2^2 - ||S^n u_0||_2^2 + 2\nu\sum_{k=0}^{n} d_{n,k}\int_0^t ||\nabla S^k u(s)||_2^2 ds \\
&= ||S^n u(t)||_2^2 - ||S^n u_0||_2^2 + 2\nu\int_0^t ||\nabla S^n u(s)||_2^2 ds \\
&\quad - \sum_{k=0}^{n-1} d_{n,k}\left[\sum_{j=0}^{k} a_{k,j}(||S^j u(t)||_2^2 - ||S^k u_0||_2^2) + \int_0^t \langle S^k u(s), S^k((u\cdot\nabla)u(s))\rangle ds\right] \\
&= ||S^n u(t)||_2^2 - ||S^n u_0||_2^2 + 2\nu\int_0^t ||\nabla S^n u(s)||_2^2 ds \\
&\quad + \sum_{j=0}^{n-1}\sum_{k=j}^{n-1} |d_{n,k}|a_{k,j}(||S^j u(t)||_2^2 - ||S^k u_0||_2^2) - \sum_{k=0}^{n-1} d_{n,k}\int_0^t \langle S^k u(s), S^k((u\cdot\nabla)u(s))\rangle ds.
\end{aligned}$$

Hence, by defining $a_{n,j} = \sum_{k=j}^{n-1}|d_{n,k}|a_{k,j}$ for $j < n$ and dealing with the nonlinearity as in (GHPW23), we obtain the first inequality for n, completing the induction.

The proof for the $\dot{H}^{-1}$ norms works the same, since from (3.2), we can also get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||\nabla|^{-1}S^n u||_{L^2}^2 + \kappa\langle S^n u, (S-2)^n u\rangle &\lesssim |||\nabla|^{-1}S^n u||_{L^2}||S^n(u\otimes u)||_2 \\ &\lesssim |||\nabla|^{-1}S^n u||_{L^2}||u||_{L^\infty}\sum_{a=0}^{n}||S^a u||_{L^2}.\end{aligned}$$

Hence, using the similar induction process as before, we get the second inequality by putting $a'_{n,k} = \sum_{j=k}^{n-1}|\min\{c'_{n,j}, 0\}|a'_{k,j}$. □

CHAPTER 4

Long-time stability

The analysis from above - dispersive unknown formulation, $L^1 - L^\infty$ dispersive decay, and energy estimates - is already enough to go beyond the local well-posedness timescale of $\|\rho_0, u_0\|^{-1}$ and prove longer stability. The bootstrap framework I will use below is very straightforward, and thus I believe this can be adapted to a lot of different dispersive PDEs, but is still enough to improve upon existing literatures. Instead of formalizing in words, I will explain through two examples - (sBQ) and (CER).

4.1 Stability of linear stratification in the Boussinesq equation in $\mathbb{R}^2$

Theorem 4.1.1. *Let $m > 3$. There exists $C_m > 0$ such that if $\rho_0, u_0 \in H^m(\mathbb{R}^2)$, then there exists*

$$T \geq C_m \kappa^{\frac{1}{3}} \|\rho_0, u_0\|_{H^m(\mathbb{R}^2)}^{-\frac{4}{3}},$$

and a unique solution $(\rho, u) \in C([0,T], H^m(\mathbb{R}^2))$ to (sBQ) *with initial data (ρ_0, u_0). In particular, the corresponding unique solution of the standard 2D Boussinesq equation with initial data $(\rho_0 - \kappa x_2, u_0)$ exists on the same time interval.*

The theorem and the following proof are adapted from (JK25). The proof works by bootstrap on the energy norm and a norm in space $B := \dot{B}^0_{\infty,1} \cap \dot{B}^1_{\infty,1}$, which is motivated by the following product lemma.

Lemma 4.1.2. *For any $m \geq 0$ and $f \in \dot{B}^1_{\infty,1}(\mathbb{R}^d) \cap H^{m+1}(\mathbb{R}^d)$, $g \in \dot{B}^0_{\infty,1}(\mathbb{R}^d) \cap H^m(\mathbb{R}^d)$, if $u = \nabla^\perp(-\Delta)^{-1/2} g$, the following holds:*

$$\|u \cdot \nabla f\|_{H^m} + \|u \cdot (|\nabla| f)\|_{H^m} \lesssim \|g\|_{H^m} \|f\|_{\dot{B}^1_{\infty,1}} + \|g\|_{\dot{B}^0_{\infty,1}} \|f\|_{H^{m+1}} .$$

Proof. The result is direct when $m = 0$, hence we only consider $m > 0$. Using the standard paraproduct estimates, (BCD11, Corollary 2.86) for example, we have

$$\|u \cdot \nabla f\|_{H^m} \lesssim_m \|u\|_{L^\infty} \|\nabla f\|_{H^m} + \|\nabla f\|_{L^\infty} \|u\|_{H^m}.$$

To deal with the unboundedness of the Riesz transform in L^∞, we use the embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$ and the fact that this operator is bounded in the smaller space. This results in

$$\begin{aligned}\|u \cdot \nabla f\|_{H^m} &\lesssim \sum_k \|P_k u\|_{L^\infty} \|\nabla f\|_{H^m} + \sum_k 2^k \|P_k f\|_{L^\infty} \|u\|_{H^m} \\ &\lesssim \|g\|_{\dot{B}^0_{\infty,1}} \|f\|_{H^{m+1}} + \|f\|_{\dot{B}^1_{\infty,1}} \|g\|_{H^m} .\end{aligned}$$

The same estimates follow for $u \cdot |\nabla| f$ and this yields the claim of the lemma. □

The proof of Theorem 4.1.1 is powered by the following bootstrap argument.

Proposition 4.1.3. *Let $m > 3$ and $4 \leq q \leq \infty$, and put $\delta = \|\rho_0, u_0\|_{H^m}$. Then there exists $C_{q,m} > 0$ such that if $T \leq C_{q,m} \kappa^{\frac{1}{q-1}} \delta^{-\frac{q}{q-1}}$, and $\rho, u \in C([0,T], H^m(\mathbb{R}^2))$ solve* (sBQ) *and if for any $\tau \in [0,T]$,*

$$\|(Z_+, Z_-)\|_{L^\infty_t([0,\tau], H^m_x)} \leq 100\delta, \quad \|(Z_+, Z_-)\|_{L^1_t([0,\tau], B_x)} \leq 2, \tag{4.1}$$

hold for the dispersive unknowns $Z_\pm$ of (ρ, u), then in fact, the improved bounds hold:

$$\|(Z_+, Z_-)\|_{L_t^\infty([0,\tau],H_x^m)} \leq 10\delta, \quad \|(Z_+, Z_-)\|_{L_t^1([0,\tau],B_x)} \leq 1. \tag{4.2}$$

Proof. By the energy estimates Proposition 3.1.1 and the isometry (2.4), we have

$$\|Z_+(\tau), Z_-(\tau)\|_{H^m}^2 \leq \|Z_{+,0}, Z_{-,0}\|_{H^m}^2 \cdot \exp\{K_m(\|\nabla\rho, \nabla u\|_{L_t^1([0,\tau],L_x^\infty)})\},$$

where $Z_{\pm,0}$ are the dispersive unknowns for (ρ_0, u_0). The term on the right-hand side is controlled by the bootstrap norm as follows:

$$\begin{aligned}
\|\nabla(\rho, u)\|_{L_t^1 L_x^\infty} &= \frac{1}{2}\|\nabla\nabla^\perp|\nabla|^{-1}(Z_+ + Z_-)\|_{L_t^1 L_x^\infty} + \frac{1}{2}\|\nabla(Z_+ - Z_-)\|_{L_t^1 L_x^\infty} \\
&\leq K\sum_k 2^k \|P_k\nabla^\perp|\nabla|^{-1}(Z_+ + Z_-)\|_{L_t^1 L_x^\infty} + \sum_k 2^k\|P_k(Z_+ - Z_-)\|_{L_t^1 L_x^\infty} \\
&\leq K\|(Z_+, Z_-)\|_{L_t^1 B_x}.
\end{aligned}$$

Now, from the Duhamel's formula for $Z_\pm$ (JW24; JK25),

$$Z_\pm(t) = e^{\mp it\lambda_\kappa} Z_{\pm,0} + \int_0^t e^{\mp i(t-s)\lambda_\kappa}\mathcal{NL}_\pm(Z_+, Z_-)ds,$$

$$\begin{aligned}
\mathcal{NL}_\pm(Z_+, Z_-) = \frac{1}{4}|\nabla|^{-1}\mathrm{div}\Big(\nabla^\perp|\nabla|^{-1}(Z_+ + Z_-) \cdot |\nabla|(Z_+ + Z_-)\Big) \\
\pm \frac{1}{4}\nabla^\perp|\nabla|^{-1}(Z_+ + Z_-) \cdot \nabla(Z_+ - Z_-).
\end{aligned}$$

Bootstrap will be completed if these can be estimated in $L_t^1 B_x$ using what we can control. For the linear term, using Theorem 2.3.2(1) and the Bernstein inequality:

$$\|e^{\mp it\lambda_\kappa} Z_{\pm,0}\|_{L_t^1 B_x} \lesssim_q \tau^{1-\frac{1}{q}}\kappa^{-\frac{1}{q}} \sum_k (2^k + 1)2^{(\frac{2}{r}+\frac{4}{q})k} \|P_k Z_{\pm,0}\|_{L_x^2} \lesssim_{q,m} \tau^{1-\frac{1}{q}}\kappa^{-\frac{1}{q}} \|Z_{\pm,0}\|_{H_x^m}.$$

For the nonlinear term, using Theorem 2.3.2(3) with $(q_2, r_2) = (\infty, 2)$ and Lemma 4.1.2

in the last inequality:

$$
\begin{aligned}
\|\int_0^t e^{\mp i(t-s)\lambda_\kappa}\mathcal{NL}_\pm ds\|_{L^1_t([0,\tau],B_x)} &\lesssim_q \tau^{1-\frac{1}{q}}\kappa^{-\frac{1}{q}}\sum_k (2^k+1)2^{(\frac{2}{r}+\frac{4}{q})k}\|P_k\mathcal{NL}_\pm\|_{L^1_tL^2_x} \\
&\lesssim_{q,m} \tau^{1-\frac{1}{q}}\kappa^{-\frac{1}{q}}\sum_{\mu,\nu\in\{+,-\}}\Big[\|\nabla^\perp|\nabla|^{-1}Z_\mu\cdot|\nabla|Z_\nu\|_{L^1_tH^{m-1}_x} \\
&\qquad\qquad +\|\nabla^\perp|\nabla|^{-1}Z_\mu\cdot\nabla Z_\nu\|_{L^1_tH^{m-1}_x}\Big] \\
&\lesssim_{q,m} \tau^{1-\frac{1}{q}}\kappa^{-\frac{1}{q}}\sum_{\mu,\nu\in\{+,-\}}\|Z_\mu\|_{L^1_tB_x}\|Z_\nu\|_{L^\infty_tH^m_x}.
\end{aligned}
$$

Altogether, we obtain the claim

$$
\|Z\|_{L^1_t([0,\tau],B_x)} \le K_{q,m}\tau^{1-\frac{1}{q}}\kappa^{-\frac{1}{q}}(\|Z_0\|_{H^m_x}+\|Z\|_{L^1_t([0,\tau],B_x)}\|Z\|_{H^m_x}),
$$

for some constant $K_{q,m}$ that depends only on q and m and $Z=(Z_+,Z_-)$. Hence, if $\tau\le T$, together with the bootstrap assumption (4.1) we obtain the improved bounds (4.2) by choosing $C_{q,m}>0$ such that

$$
\begin{cases}
\|Z_+,Z_-\|_{L^1_t([0,\tau],B_x)} \le 300K_{q,m}C_{q,m}^{\frac{q-1}{q}} \le 1, \\
\exp\{300K_mKK_{q,m}C_{q,m}^{\frac{q-1}{q}}\} \le 100.
\end{cases}
$$

□

Proof of Theorem 4.1.1. Due to (2.4), we have $\|Z_{\pm,0}\|_{H^m}=\sqrt{2}\delta$. The theorem now follows from the local well-posedness theory for the Boussinesq equation when $m>2$ and a continuity argument provided by Proposition 4.1.3. Existence of some time $T>0$ that satisfies (4.1) comes from the local well-posedness theory. The first inequality is standard, and the second inequality comes from

$$
\|Z_\pm(t)\|_B \simeq \sum_{k\in\mathbb{Z}}(2^k+1)\|P_kZ_\pm(t)\|_{L^\infty} \lesssim \sum_{k\in\mathbb{Z}}(2^k+1)\cdot 2^k\|P_kZ_\pm(t)\|_{L^2} \lesssim_m \|Z_\pm(t)\|_{H^m},
$$

for any $m > 2$. $\square$

4.2 Stability of rotation in the compressible Euler's equation in $\mathbb{R}^3$

Although the bootstrap framework above can be applied analogously, we add one more substance here and cover the 'lower regularity' up to the local well-posedness regularity $H^{\frac{5}{2}+}$. This section and the next one are adapted from (KPTW25).

Theorem 4.2.1. *Let $q > 2$. For $m > \frac{7}{2}$ there exists $M = M(q,m)$ such that the solutions to the Cauchy problem to* (CER) *with initial data $(\rho_0, u_0) \in H^m(\mathbb{R}^3)$ exist at least up to time*

$$T \geq M\varepsilon^{-\frac{1}{q-1}} \min\{1, (c\varepsilon)^{\frac{3}{q-1}}\} \|(\rho_0, u_0)\|_{H^m}^{-\frac{q}{q-1}}. \tag{4.3}$$

When $m \in (\frac{5}{2}, \frac{7}{2})$, the solutions exist at least up to time

$$T \geq M\varepsilon^{-\frac{m-\frac{5}{2}}{q-(m-\frac{5}{2})}} \min\{1, (c\varepsilon)^{\frac{3(m-\frac{5}{2})}{q-(m-\frac{5}{2})}}\} \|(\rho_0, u_0)\|_{H^m}^{-\frac{q}{q-(m-\frac{5}{2})}}. \tag{4.4}$$

Proof of Theorem 4.2.1. We recall from (2.8) the projection operators $\mathbb{P}_{\mu\Lambda}$, $\mu \in \{+,-\}$, $\Lambda \in \{\Sigma, \Omega\}$, onto the eigenspaces corresponding to $i\mu\Lambda$. From (CER) we obtain

$$\partial_t \mathbb{P}_{\mu\Lambda} \begin{pmatrix} \rho \\ u \end{pmatrix} = i\mu\Lambda \mathbb{P}_{\mu\Lambda} \begin{pmatrix} \rho \\ u \end{pmatrix} - \mathbb{P}_{\mu\Lambda} \begin{pmatrix} u \cdot \nabla\rho + \alpha\rho \operatorname{div}(u) \\ u \cdot \nabla u + \alpha\rho\nabla\rho \end{pmatrix}.$$

Employing the shorthand $W := (\rho, u)^\intercal$, Duhamel's formula gives

$$\begin{aligned} \mathbb{P}_{\mu\Lambda} W(t) &= e^{it\mu\Lambda} \mathbb{P}_{\mu\Lambda} W(0) - \int_0^t e^{i(t-\tau)\mu\Lambda} \mathbb{P}_{\mu\Lambda} \mathcal{N}(W,W)(\tau) d\tau, \\ \mathcal{N}(W,W) &= \begin{pmatrix} u \cdot \nabla\rho + \alpha\rho \operatorname{div}(u) \\ u \cdot \nabla u + \alpha\rho\nabla\rho \end{pmatrix}. \end{aligned} \tag{4.5}$$

The remainder of this proof gives a lower bound on the time of existence by closing a bootstrap for the energy estimates (3.1). More precisely, by standard local well-posedness and (3.1), it suffices to establish an a priori bound on $\|\nabla W\|_{L^1_t(0,T;L^\infty_x)}$, although we will bound $\|(W,\nabla W)\|_{L^1_t(0,T;L^\infty_x)}$ for technical reasons. For notational simplicity, in the remaining part of the proof, we omit the time-region of integration and simply write, e.g. $\|\nabla W\|_{L^1_t L^\infty_x}$

① High regularity case $m > \frac{7}{2}$: In order to put us in a position where we can invoke the Strichartz estimates from Theorem 2.3.6, we observe that by Hölder in time and Sobolev embedding in space, for $(s,R) \in (0,1)\times(2,\infty)$ satisfying $\frac{s}{3} - \frac{1}{R} > 0$ and $q > 2$ with $\frac{1}{q} + \frac{1}{R} = \frac{1}{2}$, there holds

$$\|\nabla W\|_{L^1_t L^\infty_x} \lesssim \|\nabla W\|_{L^1_t W^{s,R}_x} \leq T^{1-\frac{1}{q}} \|\nabla W\|_{L^q_t W^{s,R}_x} .$$

Moreover, by Proposition 2.1.1 and (4.5) we have

$$\begin{aligned}\|\nabla W\|_{L^q_t W^{s,R}_x} &\lesssim \sum_{\substack{\mu\in\{-,+\},\\ \Lambda\in\{\Sigma,\Omega\}}} \|\mathbb{P}_{\mu\Lambda}\nabla W\|_{L^q_t W^{s,R}_x} \\ &\lesssim \sum_{\substack{\mu\in\{-,+\},\\ \Lambda\in\{\Sigma,\Omega\}}} \left\|e^{it\mu\Lambda}\mathbb{P}_{\mu\Lambda}\nabla W(0)\right\|_{L^q_t W^{s,R}_x} + \left\|\int_0^t e^{i(t-\tau)\mu\Lambda}\nabla\mathbb{P}_{\mu\Lambda}\mathcal{N}(W,W)(\tau)d\tau\right\|_{L^q_t W^{s,R}_x}\end{aligned} \tag{4.6}$$

To bound the linear terms in (4.6), we note that by the Strichartz estimates (2.36) we have that for $k \in \mathbb{Z}$

$$\left\|P_k e^{it\mu\Lambda}\mathbb{P}_{\mu\Lambda}\nabla W(0)\right\|_{L^q_t W^{s,R}_x} \lesssim 2^{(1+s)k^+}\left(\langle 2^k\kappa\rangle^3 \varepsilon\kappa^{-3}\right)^{\frac{1}{q}} \|P_k\mathbb{P}_{\mu\Lambda}W(0)\|_{L^2} ,$$

where $k^+ = \max\{0,k\}$, and thus

$$\begin{aligned}\left\|e^{it\mu\Lambda}\mathbb{P}_{\mu\Lambda}\nabla W(0)\right\|_{L_t^q W_x^{s,R}} &\lesssim \left\{\sum_{k\in\mathbb{Z}}\left[2^{(1+s)k^+}\varepsilon^{\frac{1}{q}}\max\{2^{3k},\kappa^{-3}\}^{\frac{1}{q}}\left\|P_k\mathbb{P}_{\mu\Lambda}W(0)\right\|_{L^2}\right]^2\right\}^{1/2}\\ &\lesssim \varepsilon^{\frac{1}{q}}\left\|\mathbb{P}_{\mu\Lambda}W(0)\right\|_{H^{1+s+\frac{3}{q}}} + \left(\frac{\varepsilon}{\kappa^3}\right)^{\frac{1}{q}}\left\|\mathbb{P}_{\mu\Lambda}W(0)\right\|_{H^{1+s}}.\end{aligned} \tag{4.7}$$

Likewise,

$$\left\|P_k e^{it\mu\Lambda}\mathbb{P}_{\mu\Lambda}W(0)\right\|_{L_t^q W_x^{s,R}} \lesssim 2^{sk^+}(\langle 2^k\kappa\rangle^3\varepsilon\kappa^{-3})^{\frac{1}{q}}\left\|P_k\mathbb{P}_{\mu\Lambda}W(0)\right\|_{L^2}, \tag{4.8}$$

which results in

$$\begin{aligned}\left\|e^{it\mu\Lambda}\mathbb{P}_{\mu\Lambda}W(0)\right\|_{L_t^q W_x^{s,R}} &\lesssim \left\{\sum_{k\in\mathbb{Z}}\left[2^{sk^+}\varepsilon^{\frac{1}{q}}\max\{2^{3k},\kappa^{-3}\}^{\frac{1}{q}}\left\|P_k\mathbb{P}_{\mu\Lambda}W(0)\right\|_{L^2}\right]^2\right\}^{1/2}\\ &\lesssim \varepsilon^{\frac{1}{q}}\left\|\mathbb{P}_{\mu\Lambda}W(0)\right\|_{H^{s+\frac{3}{q}}} + \left(\frac{\varepsilon}{\kappa^3}\right)^{\frac{1}{q}}\left\|\mathbb{P}_{\mu\Lambda}W(0)\right\|_{H^{s}}.\end{aligned} \tag{4.9}$$

For the nonlinear terms in (4.6), by the Strichartz estimates (2.37) we have

$$\begin{aligned}&\left\|\int_0^t e^{i(t-\tau)\mu\Lambda}\nabla\,\mathbb{P}_{\mu\Lambda}\mathcal{N}(W,W)(\tau)d\tau\right\|_{L_t^q W_x^{s,R}}\\ &\lesssim \left\{\sum_{k\in\mathbb{Z}}\left[2^{(1+s)k^+}\varepsilon^{\frac{1}{q}}\max\{2^{3k},\kappa^{-3}\}^{\frac{1}{q}}\|P_k\mathbb{P}_{\mu\Lambda}\mathcal{N}(W,W)\|_{L_t^1L_x^2}\right]^2\right\}^{1/2}\\ &\lesssim \varepsilon^{\frac{1}{q}}\|\mathcal{N}(W,W)\|_{L_t^1H_x^{1+s+\frac{3}{q}}} + \left(\frac{\varepsilon}{\kappa^3}\right)^{\frac{1}{q}}\|\mathcal{N}(W,W)\|_{L_t^1H_x^{1+s}}\\ &\lesssim \varepsilon^{\frac{1}{q}}[\|W\|_{L_t^1L_x^\infty}\|\nabla W\|_{L_t^\infty H_x^{1+s+\frac{3}{q}}} + \|W\|_{L_t^\infty H_x^{1+s+\frac{3}{q}}}\|\nabla W\|_{L_t^1L_x^\infty}]\\ &\qquad + \left(\frac{\varepsilon}{\kappa^3}\right)^{\frac{1}{q}}[\|W\|_{L_t^1L_x^\infty}\|\nabla W\|_{L_t^\infty H_x^{1+s}} + \|W\|_{L_t^\infty H_x^{1+s}}\|\nabla W\|_{L_t^1L_x^\infty}],\end{aligned}$$

since the nonlinear terms are all of the form $f\cdot\nabla g$ with $f,g\in\{\rho,u\}$, where ∇ and multiplication are understood in a suitable sense according to their dimensions. Similarly,

$$
\begin{aligned}
&\left\| \int_0^t e^{i(t-\tau)\mu\Lambda}\, \mathbb{P}_{\mu\Lambda}\mathcal{N}(W,W)(\tau)d\tau \right\|_{L^q_t W^{s,R}_x} \\
&\lesssim \left\{ \sum_{k\in\mathbb{Z}} \left[2^{sk^+} \varepsilon^{\frac{1}{q}} \max\{2^{3k}, \kappa^{-3}\}^{\frac{1}{q}} \|P_k \mathbb{P}_{\mu\Lambda}\mathcal{N}(W,W)\|_{L^1_t L^2_x} \right]^2 \right\}^{1/2} \\
&\lesssim \varepsilon^{\frac{1}{q}} \|\mathcal{N}(W,W)\|_{L^1_t H_x^{s+\frac{3}{q}}} + \left(\frac{\varepsilon}{\kappa^3}\right)^{\frac{1}{q}} \|\mathcal{N}(W,W)\|_{L^1_t H^s_x} \\
&\lesssim \varepsilon^{\frac{1}{q}} [\|W\|_{L^1_t L^\infty_x} \|\nabla W\|_{L^\infty_t H_x^{s+\frac{3}{q}}} + \|W\|_{L^\infty_t H_x^{s+\frac{3}{q}}} \|\nabla W\|_{L^1_t L^\infty_x}] \\
&\quad + \left(\frac{\varepsilon}{\kappa^3}\right)^{\frac{1}{q}} [\|W\|_{L^1_t L^\infty_x} \|\nabla W\|_{L^\infty_t H^s_x} + \|W\|_{L^\infty_t H^s_x} \|\nabla W\|_{L^1_t L^\infty_x}].
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&\left\| \int_0^t e^{i(t-\tau)\mu\Lambda} \mathbb{P}_{\mu\Lambda}\mathcal{N}(W,W)(\tau)d\tau \right\|_{L^q_t W^{s,R}_x} + \left\| \int_0^t e^{i(t-\tau)\mu\Lambda} \nabla \mathbb{P}_{\mu\Lambda}\mathcal{N}(W,W)(\tau)d\tau \right\|_{L^q_t W^{s,R}_x} \\
&\quad \lesssim \varepsilon^{\frac{1}{q}} \|W\|_{L^1_t W^{1,\infty}_x} \|W\|_{L^\infty_t H_x^{2+s+\frac{3}{q}}} + (\varepsilon\kappa^{-3})^{\frac{1}{q}} \|W\|_{L^1_t W^{1,\infty}_x} \|W\|_{L^\infty_t H^{2+s}_x} .
\end{aligned}
\tag{4.10}
$$

In conclusion, by combining the bounds (4.7), (4.9), and (4.10) with the energy estimates (3.1), we thus have for any $m > 0$ that

$$
\begin{aligned}
\|W\|_{L^\infty_t H^m_x} \leq& \|W(0)\|_{H^m_x} \exp\left(K \|\nabla W\|_{L^1_t L^\infty_x} \right), \\
\|W\|_{L^1_t W^{1,\infty}_x} \leq& C\Big[(T^{q-1}\varepsilon)^{\frac{1}{q}} \|W(0)\|_{H_x^{1+s+\frac{3}{q}}} + (T^{q-1}\varepsilon\kappa^{-3})^{\frac{1}{q}} \|W(0)\|_{H^{1+s}_x} \\
&\qquad + (T^{q-1}\varepsilon)^{\frac{1}{q}} \|W\|_{L^1_t W^{1,\infty}_x} \|W\|_{L^\infty_t H_x^{2+s+\frac{3}{q}}} \\
&\qquad + (T^{q-1}\varepsilon\kappa^{-3})^{\frac{1}{q}} \|W\|_{L^1_t W^{1,\infty}_x} \|W\|_{L^\infty_t H^{2+s}_x} \Big],
\end{aligned}
\tag{4.11}
$$

where $K = K(m)$ and $C = C(q,s)$.

For $m > \frac{7}{2}$ given, we set $s = m - 2 - \frac{3}{q}$. Note that since we can choose s to be any number greater than $\frac{3}{R} = \frac{3}{2} - \frac{3}{q}$, this choice is consistent. Then, until the time given by the assumption (4.3) of Theorem 4.2.1

$$T = M\varepsilon^{-\frac{1}{q-1}} \min\{1, (c\varepsilon)^{\frac{3}{q-1}}\} \|W(0)\|_{H^m}^{-\frac{q}{q-1}}.$$

we obtain from (4.11) that

$$\|W\|_{L_t^\infty H_x^m} \leq 2e^K \|W(0)\|_{H^m}, \qquad \|W\|_{L_t^1 W_x^{1,\infty}} \leq 1,$$

provided that $M = M(q,m) > 0$ is small enough. This closes the bootstrap and finishes the proof of Theorem 4.2.1 for $m > \frac{7}{2}$.

② Low regularity case $\frac{5}{2} < m < \frac{7}{2}$: Here, we control the high frequency of W differently to prevent loss of derivatives. We first note that we may assume $T \gtrsim \|W(0)\|_{H^m}^{-1}$ from the local well-posedness. Hence, if $\|W(0)\|_{H^m} \gtrsim \varepsilon^{-1}$, we are done when $\kappa \gtrsim 1$ since

$$\varepsilon^{-\frac{m-\frac{5}{2}}{q-(m-\frac{5}{2})}} \|W(0)\|_{H^m}^{-\frac{q}{q-(m-\frac{5}{2})}} \lesssim \|W(0)\|_{H^m}^{-1},$$

so that the theorem is true by simple local theory. Likewise, if $\|W(0)\|_{H^m} \gtrsim \varepsilon^2 c^3$ when $\kappa \lesssim 1$,

$$\varepsilon^{\frac{2(m-\frac{5}{2})}{q-(m-\frac{5}{2})}} c^{\frac{3(m-\frac{5}{2})}{q-(m-\frac{5}{2})}} \|W(0)\|_{H^m}^{-\frac{q}{q-(m-\frac{5}{2})}} \lesssim \|W(0)\|_{H^m}^{-1},$$

so that again, we are already done. Thus, we assume $\|W(0)\|_{H^m} \lesssim \varepsilon^{-1}(1+\kappa^{-3})^{-1}$, and in particular assume that $T \geq \varepsilon(1+\kappa^{-3})$. For $k_0 \geq 0$ to be determined, we control the blow-up norm by

$$\begin{aligned}
\|W\|_{L_t^1 W_x^{1,\infty}} &\lesssim \|P_{\geq k_0} W\|_{L_t^1 \dot{W}^{1,\infty}} + \|P_{\leq k_0} W\|_{L_t^1 W_x^{1,\infty}} \\
&\lesssim \sum_{k \geq k_0} 2^{\frac{5}{2}k} \|P_k W\|_{L_t^1 L_x^2} + T^{1-\frac{1}{q}} \sum_{k \leq k_0} \|P_k W\|_{L_t^q W_x^{1,\infty}} \\
&\lesssim T 2^{(\frac{5}{2}-m)k_0} \|W\|_{L_t^\infty H_x^m} + T^{1-\frac{1}{q}} \sum_{k \leq k_0} 2^{\frac{3}{R}k} \|P_k W\|_{L_t^q W_x^{1,R}}.
\end{aligned}$$

The low frequency part can be analyzed in the similar way as in the high regularity case. Using (4.8) with $s=1$,

$$\begin{aligned}\sum_{k\le k_0} 2^{\frac{3}{R}k}\|P_k e^{it\mu\Lambda}\mathbb{P}_{\mu\Lambda}W(0)\|_{L_t^qW_x^{1,R}} &\lesssim \sum_{k\le k_0} 2^{\frac{3}{R}k}2^{k^+}(\langle 2^k\kappa\rangle^3\varepsilon\kappa^{-3})^{\frac{1}{q}}\left\|P_k\mathbb{P}_{\mu\Lambda}W(0)\right\|_{L^2}\\
&\lesssim \varepsilon^{\frac{1}{q}}\sum_{k\le k_0}(2^{\frac{3}{2}k+k^+}+2^{\frac{3}{R}k+k^+}\kappa^{-\frac{3}{q}})\left\|P_kW(0)\right\|_{L^2}\\
&= \varepsilon^{\frac{1}{q}}\sum_{k\le k_0}(2^{\frac{3}{2}k+k^++(1-m)k^+}\left\|W(0)\right\|_{H^{m-1}}+2^{\frac{3}{R}k+k^++(1-m)k^+}\kappa^{-\frac{3}{q}}\left\|W(0)\right\|_{H^{m-1}})\\
&\lesssim \varepsilon^{\frac{1}{q}}(1+\kappa^{-\frac{3}{q}})2^{(\frac{7}{2}-m)k_0}\left\|W(0)\right\|_{H^{m-1}},\end{aligned}$$

and similarly

$$\begin{aligned}&\sum_{k\le k_0} 2^{\frac{3}{R}k}\left\|P_k\int_0^t e^{i(t-\tau)\mu\Lambda}\,\mathbb{P}_{\mu\Lambda}\mathcal{N}(W,W)(\tau)d\tau\right\|_{L_t^qW_x^{1,R}}\\
&\lesssim \sum_{k\le k_0}2^{\frac{3}{R}k+k^+}(\langle 2^k\kappa\rangle^3\varepsilon\kappa^{-3})^{\frac{1}{q}}\left\|\mathcal{N}(W,W)\right\|_{L_t^1L_x^2}\\
&\lesssim \varepsilon^{\frac{1}{q}}(1+\kappa^{-\frac{3}{q}})2^{(\frac{7}{2}-m)k_0}\left\|\mathcal{N}(W,W)\right\|_{H^{m-1}}\\
&\lesssim \varepsilon^{\frac{1}{q}}(1+\kappa^{-\frac{3}{q}})2^{(\frac{7}{2}-m)k_0}\left\|W\right\|_{L_t^1W_x^{1,\infty}}\left\|W\right\|_{L_t^\infty H^m}.\end{aligned}$$

Putting together, we obtain

$$\begin{aligned}\|W\|_{L_t^1W_x^{1,\infty}} &\lesssim T(2^{k_0})^{(\frac{5}{2}-m)}\left\|W\right\|_{L_t^\infty H_x^m}\\
&\quad+T^{1-\frac{1}{q}}\varepsilon^{\frac{1}{q}}(1+\kappa^{-\frac{3}{q}})(2^{k_0})^{(\frac{7}{2}-m)}(\left\|W(0)\right\|_{H^{m-1}}+\left\|W\right\|_{L_t^1W_x^{1,\infty}}\left\|W\right\|_{L_t^\infty H_x^m}).\end{aligned}$$

Therefore, choosing $2^{k_0}=(T\varepsilon^{-1})^{\frac{1}{q}}(1+\kappa^{-\frac{3}{q}})^{-1}$, we get the bound

$$\|W\|_{L_t^1W_x^{1,\infty}} \lesssim (T^{q-(m-\frac{5}{2})}\varepsilon^{m-\frac{5}{2}}(1+\kappa^{-3})^{(m-\frac{5}{2})})^{\frac{1}{q}}(\left\|W\right\|_{L_t^\infty H_x^m}+\left\|W\right\|_{L_t^1W_x^{1,\infty}}\left\|W\right\|_{L_t^\infty H_x^m}). \tag{4.12}$$

Hence, until the time given by the assumption (4.4)

$$T = M\varepsilon^{-\frac{m-\frac{5}{2}}{q-(m-\frac{5}{2})}} \min\{1, \kappa^{\frac{3(m-\frac{5}{2})}{q-(m-\frac{5}{2})}}\} \|W(0)\|_{H^m}^{-\frac{q}{q-(m-\frac{5}{2})}},$$

we obtain from (4.12) and the energy estimates that

$$\|W\|_{L_t^\infty H_x^m} \leq 2e^K \|W(0)\|_{H^m}, \qquad \|W\|_{L_t^1 W_x^{1,\infty}} \leq 1,$$

so that the bootstrap will work until such time. This finishes the proof of Theorem 4.2.1.

□

4.3 Stability of rotation in the incompressible Euler's equation in $\mathbb{R}^3$

The 'incompressible limit' (2.38) of the estimate (2.36) enables us to obtain the corresponding 'incompressible limit' of Theorem 4.2.1. As a manifestation of how robust the bootstrap framework is, I apply it to the incompressible Euler-Coriolis equation,

$$\begin{aligned} \partial_t u + \varepsilon^{-1} e_3 \times u + (u \cdot \nabla)u + \nabla p = 0, \\ \operatorname{div}(u) = 0, \end{aligned}$$

and prove that the result indeed improves the previous result in (KLT14b). Take their energy estimates

$$\|u(t)\|_{H^m} \leq \|u_0\|_{H^m} \exp(K_m \|\nabla u(\tau)\|_{L_\tau^1(0,t;L_x^\infty)}), \quad m \geq 0,$$

and Duhamel's formula for granted:

$$\mathbb{P}_\pm u(t) = e^{\pm it\Lambda_\varepsilon}\mathbb{P}_\pm u_0 - \int_0^t e^{\pm i\Lambda_\varepsilon(t-\tau)}\mathbb{P}_\pm(u\cdot\nabla)u(\tau)d\tau, \quad \widehat{\mathbb{P}_\pm u} = \frac{1}{2}\left((I - \frac{\xi\xi^\top}{|\xi|^2})\hat{u} \pm i\frac{\xi}{|\xi|}\times\hat{u}\right).$$

Note that $\mathbb{P}_\pm$ is a standard Calderón-Zygmund operator and is bounded on L^p, $1 < p < \infty$. Using the same strategy of bootstrapping $\|u\|_{L_t^\infty H_x^m}$ and $\|u\|_{L_t^1 W_x^{1,\infty}}$ together, when $m > \frac{7}{2}$, the blowup norm of the linear term is treated by

$$\begin{aligned}
\left\|e^{\pm it\Lambda_\varepsilon}\mathbb{P}_\pm u_0\right\|_{L_t^1 W_x^{1,\infty}} &\lesssim T^{1-\frac{1}{q}}\left\|e^{\pm it\Lambda_\varepsilon}u_0\right\|_{L_t^q W_x^{1+s,R}} \\
&\lesssim T^{1-\frac{1}{q}}\left\{\sum_{k\in\mathbb{Z}}\left[2^{(1+s)k^+}\varepsilon^{\frac{1}{q}}2^{\frac{3}{q}k}\left\|P_k u_0\right\|_{L^2}\right]^2\right\}^{1/2} \\
&\lesssim T^{1-\frac{1}{q}}\varepsilon^{\frac{1}{q}}\left\|u_0\right\|_{H^{1+s+\frac{3}{q}}}.
\end{aligned}$$

The nonlinear term follows similarly:

$$\begin{aligned}
&\left\|\int_0^t e^{\pm i\Lambda_\varepsilon(t-\tau)}\mathbb{P}_\pm(u\cdot\nabla)u(\tau)d\tau\right\|_{L_t^1 W_x^{1,\infty}} \\
&\lesssim T^{1-\frac{1}{q}}\left\{\sum_{k\in\mathbb{Z}}\left[2^{(1+s)k^+}\varepsilon^{\frac{1}{q}}2^{\frac{3}{q}k}\|P_k(u\cdot\nabla)u\|_{L_t^1 L_x^2}\right]^2\right\}^{1/2} \\
&\lesssim T^{1-\frac{1}{q}}\varepsilon^{\frac{1}{q}}\left\|(u\cdot\nabla)u\right\|_{H^{1+s+\frac{3}{q}}} \\
&\lesssim T^{1-\frac{1}{q}}\varepsilon^{\frac{1}{q}}[\|u\|_{L_t^\infty H^{1+s+\frac{3}{q}}}\|\nabla u\|_{L_t^1 L_x^\infty} + \|\nabla u\|_{L_t^\infty H^{1+s+\frac{3}{q}}}\|u\|_{L_t^1 L_x^\infty}].
\end{aligned}$$

We set $s = m - 2 - \frac{3}{q}$, which gives us,

$$\begin{aligned}
\|u\|_{L_t^\infty H^m} &\leq \|u_0\|_{H^m}\exp(K_m\|u\|_{L_t^1 W_x^{1,\infty}}), \\
\|u\|_{L_t^1 W_x^{1,\infty}} &\leq C_{q,m}T^{1-\frac{1}{q}}\varepsilon^{\frac{1}{q}}(\|u_0\|_{H^m} + \|u\|_{L_t^\infty H^m}\|u\|_{L_t^1 W_x^{1,\infty}}).
\end{aligned}$$

By the same argument as before, this shows that the solution exists at least up to

$$T \gtrsim \varepsilon^{-\frac{1}{q-1}}\|u_0\|_{H^m}^{-\frac{q}{q-1}}.$$

The case $m \in (\frac{5}{2}, \frac{7}{2})$ can be shown similarly.

4.4 Remark on the role of parameters

One can see that the above result when $m > \frac{7}{2}$ improves the result of (KLT14b), which proved

$$T \gtrsim \log(\varepsilon^{-1}) \|u_0\|_{H^m}^{-1},$$

from a gain of $\log(\varepsilon^{-1})$ to $\varepsilon^{-\frac{1}{q-1}}$ for any $q > 2$. Even in (KLT14b), the point was that accounting for the dispersive effect can ensure a longer lifetime of the solution, and the improvement in this thesis is a good point to emphasize that it is important to pay attention to the parameters in the final results. Not only do these theorems confirm that faster rotation and stronger stratification are more stable, but they also quantify how much exactly the added stability is, in terms of the least amount of time for the solution to remain regular and bounded. To achieve this, one can see that the estimates are done carefully to track the dependence on the parameters. This effort is most salient in the analysis of compressible Euler's equation, (CER), where we have two parameters which are involved in a nontrivial way, in contrast to (sBQ) and (NSC, $\nu = 0$), where one can 'scale out' the parameters. I hope the readers can see the value of the new stationary phase theorems 2.2.1 and 2.2.4 that make this possible, unlike the usual ones.

Having said that, although we have better results in the limit where people are usually interested, $\varepsilon \to 0$ or $\kappa \to \infty$, this is not the case for (NSC) where people are more interested in the inviscid limit $\nu \to 0$. Indeed, if one uses naive scaling, any result one can obtain with $\nu = 1$ will degenerate in this direction. In the next section, I prove that a "uniform" global stability with respect to ν can be obtained by incorporating nonlinear analysis.

CHAPTER 5

Global stability of rotation in the incompressible Navier-Stokes equation in $\mathbb{R}^3$

Although useful and versatile, the bootstrap framework used in the previous chapter usually cannot produce a global solution. This requires a more meticulous approach than using Strichartz estimates, which treats nonlinearity in a rough way. To be precise, I will decompose the functions into tinier pieces to use finer linear decay and to exploit how they interact at smaller scales. As such, more tools are needed before proving the main theorem, and more concepts should be introduced to use them. These tools and methods stem from the history of work on nonlinear dispersive PDEs, which developed the vector field method, normal form, spacetime resonance, and the design of adapted topology, through works such as (Kla85; Sha85; GNT09; Ger10; GMS12; GMP13; IP14; GMS15; GIP16; GPW23). The analysis in this chapter incorporates the achievements of these literature and is heavily influenced by (GPW23), which investigates inviscid version of (NSC). Unless specified otherwise, the content of this chapter is based on (Ko24).

5.1 Preliminaries

5.1.1 Dispersive variables

As was mentioned at the end of the last chapter, the meaningful question is whether the system remains stable or not and how the stability becomes affected as $\nu \to 0$. This will lead to two actions. First, we scale the variables using

$$\tilde{u}(t,x) = \varepsilon \cdot u(\varepsilon t, x), \quad \tilde{p}(x) = \varepsilon^2 \cdot p(x), \tag{5.1}$$

to fix the speed of rotation to 1. Now we have only one small parameter to consider, and the result for (NSC) can be recovered from rescaling. Hence, now we work on

$$\begin{cases} \partial_t u - \kappa \Delta u + (u \cdot \nabla) u + e_3 \times u + \nabla p & = 0, \\ \operatorname{div}(u) & = 0, \end{cases} \tag{5.2}$$

where $\kappa = \varepsilon\nu$ and I dropped the tildes for notational convenience. As I have essentially fixed $\varepsilon = 1$, I define

$$\Lambda = \frac{\xi_3}{|\xi|}$$

without the subscript ε accordingly. Secondly, since we are interested in the case when κ is small, we rely more on the dispersive structure of the system in order to depend less on the dissipation effect whose strength decreases with κ. Hence, we again use the dispersive unknowns $U_\pm := |\nabla_h|^{-1}(w_3 \pm |\nabla| u_3)$ (2.2), even though the equation is dispersive-dissipative. In fact, I will actually work on the dispersive variables, usually referred to as profiles, of $U_\pm$ defined by

$$\mathcal{U}_\pm := e^{\mp it\Lambda} U_\pm(t).$$

One of the key findings of (GHPW23) was that for axisymmetric functions, the part of the nonlinearity that has 2D Euler structure vanishes. I adapt the same assumption, which makes the equation for $\mathcal{U}_\pm$ into

$$\begin{aligned}\partial_t\mathcal{U}_+ - \kappa\Delta\mathcal{U}_+ =&\mathcal{Q}_{\mathfrak{m}_+^{++}}(\mathcal{U}_+,\mathcal{U}_+) + \mathcal{Q}_{\mathfrak{m}_+^{+-}}(\mathcal{U}_+,\mathcal{U}_-) + \mathcal{Q}_{\mathfrak{m}_+^{--}}(\mathcal{U}_-,\mathcal{U}_-),\\ \partial_t\mathcal{U}_- - \kappa\Delta\mathcal{U}_- =&\mathcal{Q}_{\mathfrak{m}_-^{++}}(\mathcal{U}_+,\mathcal{U}_+) + \mathcal{Q}_{\mathfrak{m}_-^{+-}}(\mathcal{U}_+,\mathcal{U}_-) + \mathcal{Q}_{\mathfrak{m}_-^{--}}(\mathcal{U}_-,\mathcal{U}_-),\end{aligned} \tag{5.3}$$

where

$$\mathcal{Q}_{\mathfrak{m}_\mu^{\mu_1,\mu_2}}(\mathcal{U}_{\mu_1},\mathcal{U}_{\mu_2})(t) := \mathcal{F}^{-1}\left(\int_{\mathbb{R}^3} e^{it\Phi_\mu^{\mu_1,\mu_2}}\mathfrak{m}_\mu^{\mu_1,\mu_2}(\xi,\eta)\widehat{\mathcal{U}_{\mu_1}}(\xi-\eta)\widehat{\mathcal{U}_{\mu_2}}(\eta)d\eta\right), \quad \mu,\mu_1,\mu_2\in\{+,-\},$$

$$\Phi_\mu^{\mu_1,\mu_2}(\xi,\eta) = \mu\Lambda(\xi) + \mu_1\Lambda(\xi-\eta) + \mu_2\Lambda(\eta), \quad \mu,\mu_1,\mu_2\in\{+,-\}, \tag{5.4}$$

and the multipliers are of the form

$$\begin{aligned}&\mathfrak{m}_\mu^{\mu_1,\mu_2}(\xi,\eta) = |\xi|\mathfrak{n}_\mu^{\mu_1,\mu_2}(\xi,\eta),\\ &\mathfrak{n}_\mu^{\mu_1,\mu_2}(\xi,\eta) \in \operatorname{span}\left\{\Lambda(\zeta_1)\sqrt{1-\Lambda^2(\zeta_2)}\cdot\bar{\mathfrak{n}}(\xi,\eta), \zeta_j\in\{\xi,\xi-\eta,\eta\}, \bar{\mathfrak{n}}\in\bar{E}\right\}, \mu,\mu_1,\mu_2\in\{+,-\},\\ &\bar{E} := \left\{\Lambda(\zeta), \sqrt{1-\Lambda^2(\zeta)}, \frac{\xi_h\cdot\theta_h}{|\xi_h||\theta_h|}, \frac{\xi_h^\perp\cdot\theta_h}{|\xi_h||\theta_h|} : \zeta\in\{\xi,\xi-\eta,\eta\}, \theta\in\{\xi-\eta,\eta\}\right\}.\end{aligned}$$

I refer to (GHPW23)[Lemma 2.1] for the derivation, where the only difference is the dissipation term. Since this term is linear, one can easily derive (5.3) by adapting the proof in an obvious way. In practice, when the signs do not matter, I will omit μ,μ_1,μ_2 and simply write $\Phi,\mathfrak{m}$ and $\mathcal{U}$ or $\mathcal{U}_\pm$. For example, the Duhamel's formula becomes

$$\begin{aligned}\mathcal{U}_\pm(t) &= e^{t\kappa\Delta}\mathcal{U}_\pm(0) + \sum_{\mathfrak{m}}\mathcal{B}_{\mathfrak{m}}(\mathcal{U}_\pm,\mathcal{U}_\pm)(t),\\ \mathcal{B}_{\mathfrak{m}}(f_1,f_2)(t) &:= \int_0^t e^{(t-s)\kappa\Delta}\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)(s)ds.\end{aligned}$$

5.1.2 Localizations

Recall from the notations that I have defined P_k, $P_{k,p}$ and $P_{k,p,q}$ for the Littlewood-Paley type projections. I make another localization to measure the smoothness in the angular direction, $\bar{R}_{\leq l}$ and $\bar{R}_l$, by

$$\bar{R}_{\leq l} f(x) = \sum_{n\geq 0} \psi(2^{-l}n) \int_{\mathbb{S}^2} f(|x|\theta) \mathfrak{Z}_n(\langle \theta, \frac{x}{|x|} \rangle) dS(\theta),$$
$$\bar{R}_l f(x) = \sum_{n\geq 0} \varphi(2^{-l}n) \int_{\mathbb{S}^2} f(|x|\theta) \mathfrak{Z}_n(\langle \theta, \frac{x}{|x|} \rangle) dS(\theta),$$

for $l \geq 0$ and the usual measure on the sphere dS, using the Legendre polynomial

$$\mathfrak{Z}_n(x) = \frac{2n+1}{4\pi} L_n(x), \quad L_n(x) = \frac{1}{2^n(n!)} \frac{d^n}{dz^n}[(z^2-1)^n].$$

Proposition 5.1.1. *For any $l \in \mathbb{N}$, the angular localization operator $\bar{R}_l$ satisfies the following.*

(1) $\bar{R}_l$ commutes with the regular Littlewood-Paley projectors, vector fields S and $\Omega_{ab} = x_z \partial_{x_b} - x_b \partial_{x_a}$, the Fourier transform, and the Laplacian. In other words,

$$[P_k, \bar{R}_l] = [S, \bar{R}_l] = [\Omega_{ab}, \bar{R}_l] = [\mathcal{F}, \bar{R}_l] = [\Delta, \bar{R}_l] = 0.$$

(2) $\bar{R}_l$ constitutes an orthogonal partition of unity in the sense that

$$f = \sum_{l\geq 0} \bar{R}_l f, \quad ||f||_2^2 \simeq \sum_{l\geq 0} ||\bar{R}_l f||_2^2, \quad \bar{R}_l \bar{R}_{l'} = 0 \ \textit{if} \ |l-l'| > 3.$$

(3) Bernstein property holds, i.e.

$$\sum_{1\leq a<b\leq 3} ||\Omega_{ab} \bar{R}_l f||_{L^r} \simeq 2^l ||\bar{R}_l f||_{L^r}. \tag{5.5}$$

Proof. Everything is proved in (GPW23, Proposition 3.1) except $[\Delta, \bar{R}_l] = 0$. However, this is evident since $\bar{R}_l$ commutes with the Fourier transform, hence,

$$\begin{aligned}\mathcal{F}\{\bar{R}_l(-\Delta f)\}(\xi) &= \sum_{n\geq 0}\varphi(2^{-l}n)\int_{\mathbb{S}^2}||\xi|\theta|^2 f(|\xi|\theta)\mathfrak{Z}_n(\langle\theta,\frac{x}{|x|}\rangle)dS(\theta)\\ &= |\xi|^2(\bar{R}_l\hat{f})(\xi) = \mathcal{F}\{(-\Delta\bar{R}_l f)\}(\xi).\end{aligned}$$

□

Note that they measure the size of the angular derivatives in the sense of (5.5). I again refer to (GPW23) to state that due to the 'uncertainty principle'

$$||[\Omega_{a3}, P_{k,p}]||_{L^r\to L^r} \lesssim 2^{-p},$$

we only need to consider the case $l+p\geq 0$, and hence define

$$R_l = \begin{cases} 0, & p+l<0,\\ \bar{R}_{\leq l}, & p+l=0,\\ \bar{R}_l, & p+l>0.\end{cases}$$

The dependence on p exists, but will be obvious in practice, and hence we neglect it in the notation.

From Duhamel's formula, one can observe that there are two input functions from the quadratic nonlinearity, and one output function, which will require us to make three different localizations. The localization indices of the output functions will be denoted by k, p, q, and l, while those of the input functions will be denoted by k_j, p_j, q_j, and l_j,

$j = 1, 2$. For example, typical localized bilinear term under analysis will take the form

$$
\begin{aligned}
&P_{k,p}\mathcal{Q}_{\mathfrak{m}}(P_{k_1,p_1}f_1, P_{k_2,p_2}f_2) \\
&= \mathcal{F}^{-1}\left\{\int_{\mathbb{R}^3} e^{is\Phi(\xi,\eta)}\mathfrak{m}(\xi,\eta)\varphi_{k,p}(\xi)\varphi_{k_1,p_1}(\xi-\eta)\varphi_{k_2,p_2}(\eta)\hat{f}_1(\xi-\eta)\hat{f}_2(\eta)d\eta\right\},
\end{aligned}
$$

with possibly additional localizations in q, q_j or l, l_j's. For this reason, it will be convenient to introduce

$$\chi_h(\xi,\eta) := \varphi_{k,p}(\xi)\varphi_{k_1,p_1}(\xi-\eta)\varphi_{k_2,p_2}(\eta), \quad \chi(\xi,\eta) = \varphi_{k,p,q}(\xi)\varphi_{k_1,p_1,q_1}(\xi-\eta)\varphi_{k_2,p_2,q_2}(\eta).$$

It is important to note that the localizations are not independent. From the definition of R_l, we implicitly assumed that $p + l$ will be nonnegative whenever we localize in both sizes, but even earlier than that, one can check that $2^{2p} + 2^q \simeq 1$ by definition. The same holds for p_j, q_j, l_j's, $j = 1, 2$. Another fact of great importance is that the indices of the input and output functions also cannot be independent, simply due to $\xi = (\xi - \eta) + \eta$. This will be investigated in more detail in Section 5.3. As such, we use

$$k_{\max} := \max\{k, k_1, k_2\}, \quad k_{\min} := \min\{k, k_1, k_2\},$$

since we will frequently need to compare their sizes. The same type of notations will be used for p, q, and l's.

5.2 Main Theorem

For some $\beta > 0$, I define the B and X norms as follows.

$$
\begin{aligned}
||f||_B &:= \sup_{k,p,q\in\mathbb{Z},\ p,q\leq 0} 2^{3k^+-\frac{1}{2}k^-}2^{-p-\frac{q}{2}}||P_{k,p,q}f||_2, \\
||f||_X &:= \sup_{\substack{k,p\in\mathbb{Z},\ p\leq 0\leq l, \\ p+l\geq 0}} 2^{3k^+}2^{(1+\beta)l+\beta p}||P_{k,p}R_l f||_2,
\end{aligned}
$$

where $k^+ = \max\{k, 0\}$ and $k^- = \min\{k, 0\}$. In the main theorem and onward, we use ϵ for the size of functions. This should not confuse the reader with a different symbol ε used for the inverse speed of rotation, especially now that we have scaled out this parameter and absorbed it into κ.

Theorem 5.2.1. *(a) There exist $M, N, N_0, \beta, \epsilon > 0$ with $\beta^{-1}, N \ll M \ll N_0$ and $\beta^{-1} \geq 60$, all of which are independent of $\kappa \in [0, \infty)$, such that if u_0 is axisymmetric and its dispersive unknowns $U_0^{\pm}$ satisfy*

$$\begin{aligned} ||U_0^{\pm}||_{H^{2N_0} \cap \dot{H}^{-1}} + ||S^a U_0^{\pm}||_{L^2 \cap \dot{H}^{-1}} \leq \epsilon, \quad 0 \leq a \leq M, \\ ||S^b U_0^{\pm}||_B + ||S^b U_0^{\pm}||_X \leq \epsilon, \quad 0 \leq b \leq N, \end{aligned} \tag{5.6}$$

then there exists a global solution to the Cauchy problem of (5.2), $u \in C([0, \infty); H^{2N_0})$.

In particular, the solution $U(x, y, z) = (-y, x, 0)$ of the Navier-Stokes equation

$$\begin{cases} \partial_t U - \nu \Delta U + (U \cdot \nabla) U + \nabla P &= 0, \\ \operatorname{div}(U) &= 0, \end{cases} \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3,$$

is globally stable under axisymmetric perturbation, uniformly for all $0 \leq \nu < \infty$.
(b) Two solutions u_1, u_2 to (NSC) of viscosities ν_1, ν_2, respectively, with the same axisymmetric initial data u_0 whose rescaling by (5.1) satisfies the assumptions in (a) above, satisfy

$$||u_1(t) - u_2(t)||_{L^2}^2 \lesssim |\nu_1 - \nu_2| \cdot \varepsilon^{C\epsilon^*} (\epsilon^*)^2 t^{1 + C\epsilon^*} \tag{5.7}$$

for some C and small enough ν_1, ν_2.

When $\kappa \gg 1$, scaling together with the small data global well-posedness for (5.2) with $\kappa = 1$ trivially gives the result. Hence, from hereon, I consider $\kappa \in [0, 1]$, where I prove the main theorem by the following bootstrap argument.

Theorem 5.2.2. *Suppose that u is a solution to (5.2) up to some time $T > 0$ with initial* *data satisfying* (5.6)*. There exists* $c > c' > 1$ *such that if*

$$||S^n \mathcal{U}_{\pm}(t)||_B + ||S^n \mathcal{U}_{\pm}(t)||_X \leq c\epsilon, \quad 0 \leq n \leq N, \tag{5.8}$$

holds for all $t \in [0, T]$*, then actually*

$$||S^n \mathcal{U}_{\pm}(t)||_B + ||S^n \mathcal{U}_{\pm}(t)||_X \leq c'\epsilon, \quad 0 \leq n \leq N,$$

for $t \in [0, T]$*. Moreover, the estimates are uniform for* $0 \leq \kappa \lesssim 1$*.*

The proof makes use of the fact that the dispersive decay Proposition 2.5.1 holds for

$$||f||_D := \sup_{\substack{0 \leq a \leq 3 \\ 0 \leq b \leq 2}} (||S^a f||_B + ||S^b f||_X).$$

.

Proof of Theorem 5.2.1. (a) Using the notation in the Proposition 2.5.1, one has

$$\sum_{p,q \leq 0} \|I_{k,p,q} f\|_{L^\infty} + \sum_{2^{2p+q} \gtrsim t^{-1}} \|II_{k,p,q} f\|_{L^\infty} \lesssim 2^{\frac{3}{2}k - 3k^+} t^{-1} \|f\|_D ,$$

and hence, under the assumption (5.8), we have $\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty} \lesssim t^{-1}\epsilon$. From the energy estimates, this implies that

$$||U_{\pm}(t)||_{H^{2N_0} \cap \dot{H}^{-1}} + ||S^a U_{\pm}(t)||_{L^2 \cap \dot{H}^{-1}} \leq \epsilon \langle t \rangle^{C\epsilon}, \quad 0 \leq a \leq M.$$

Hence, the bootstrap Proposition 5.2.2 is enough to obtain global solution.

(b) We prove the inequality for solutions u_1, u_2 of (5.2) with viscosities κ_1, κ_2, respectively. Subtracting two equations and taking inner product with $u_1 - u_2$, we have

$$\frac{1}{2}\frac{d}{dt}||u_1 - u_2||_2^2 + \kappa_2 ||u_1 - u_2||_{\dot{H}^1}^2 \leq |\kappa_1 - \kappa_2| ||u_1 - u_2||_{\dot{H}^1} ||u_1||_{\dot{H}^1} + ||\nabla u_1||_\infty ||u_1 - u_2||_2^2.$$

Putting $v = u_1 - u_2$ and assuming $\kappa_1 \le \kappa_2$ WLOG, we can absorb the $\dot{H}^1$ norm of v on the right-hand side to deduce

$$\frac{d}{dt}||v||_2^2 \le |\kappa_1 - \kappa_2|||u_1||_{\dot{H}^1}^2 + 2||\nabla u_1||_\infty ||v||_2^2.$$

Since $v(0) \equiv 0$, Gronwall's inequality gives us

$$||v(t)||_2^2 \le \int_0^t |\kappa_1 - \kappa_2|||u_1||_{\dot{H}^1}^2 \cdot e^{\int_s^t 2||\nabla u_1(\tau)||_\infty d\tau} ds.$$

From the decay of L^∞ norms and the slow growth of energy norms noted above, we get

$$||v(t)||_2^2 \lesssim |\kappa_1 - \kappa_2| \int_0^t \epsilon^2 \langle s\rangle^{2C\epsilon} \left(\frac{t}{s}\right)^{C\epsilon} ds \lesssim |\kappa_1 - \kappa_2|\epsilon^2 t^{1+C\epsilon}.$$

Scaling back, we obtain (5.7). $\square$

Therefore, the rest of this chapter is devoted to proving Proposition 5.2.2. Looking at the linear terms $e^{t\kappa\Delta}\mathcal{U}_\pm(0)$ in Duhamel's formulas, although these do not commute with the S-vector field since it does not commute with the equation, these are harmless, as

$$[S, e^{t\kappa\Delta}] = 2t\kappa\Delta e^{t\kappa\Delta} \tag{5.9}$$

has an operator norm bounded by an absolute constant. One can check that more iterations of the S-vector fields do not generate qualitative change, so that we only need to choose c and c' larger than the sum of these types.

Hence, the challenge is to deal with the bilinear term. Details of the estimates will be provided later, but we describe here how we reduce the problem to estimating atoms. First, we cite (GPW23) to claim that

$$S_\xi \mathcal{F}\left(\mathcal{Q}_\mathfrak{m}(\mathcal{U}_\pm, \mathcal{U}_\pm)\right)(\xi) = \mathcal{F}\left(-2\,\mathcal{Q}_\mathfrak{m}(\mathcal{U}_\pm, \mathcal{U}_\pm) + \mathcal{Q}_\mathfrak{m}(S\mathcal{U}_\pm, \mathcal{U}_\pm) + \mathcal{Q}_\mathfrak{m}(\mathcal{U}_\pm, S\mathcal{U}_\pm)\right)(\xi).$$

Together with (5.9), the bilinear terms with some copies of the S-vector field have the form

$$S^n \mathcal{B}_{\mathfrak{m}}(\mathcal{U}_\pm, \mathcal{U}_\pm) = \sum_{a+b+c \le n} \int_0^t h_a((t-s)\kappa\Delta) e^{(t-s)\kappa\Delta} \mathcal{Q}_{\mathfrak{m}}(S^b \mathcal{U}_\pm, S^c \mathcal{U}_\pm)(s) ds,$$

where $h_a(\cdot)$ is a polynomial of order a. After writing this as a sum of localized pieces, we use energy estimates together with the bootstrap assumptions to reduce the problem to estimating finitely many localized pieces. In addition to the localizations in space introduced already, we also use the time localizations $\tau_m(s), m = 0, 1, \cdots, L+1,\ L = \lceil \log_2(t+2) \rceil$ where

$$\begin{gathered} \text{supp } \tau_m \subset [2^{m-1}, 2^{m+1}], \quad \int_0^t \tau_m'(s) ds \lesssim 1, \qquad m = 1, \cdots, L, \\ \text{supp } \tau_0 \subset [0, 2], \quad \text{supp } \tau_{L+1} \subset [t-2, t], \qquad \sum_{m=0}^{L+1} \tau_m(s) \equiv 1. \end{gathered}$$

After these reductions, most of the time we will be working on $(\log\langle t \rangle)$-many pieces of

$$P_{k,p(,q)}(R_l) \int_0^t \tau_m(s) h_a((t-s)\kappa\Delta) e^{(t-s)\kappa\Delta} \mathcal{Q}_{\mathfrak{m}}(P_{k_1,p_1(,q_1)}(R_{l_1}) S^b \mathcal{U}_\pm, P_{k_2,p_2(,q_2)}(R_{l_2}) S^c \mathcal{U}_\pm)(s) ds,$$

and show that their sum is bounded by $c'\epsilon$. Parentheses around q, q_j, R_l, R_{l_j}s denote that we will use finer localizations only when they are needed, not from the outset. As was previously mentioned, the localizations are not independent. As such, we will divide the cases according to their relations explained in the following section, and each configuration will require different methods.

5.3 Tools for bootstrap

Two major ways of controlling the bilinear terms are integration by parts and normal form. We first set the framework for integration by parts. The crucial finding in (GPW23) showing that there is no spacetime resonance was through the quantity

$$\bar{\sigma}(\xi,\eta) := \xi_3\eta_h - \eta_3\xi_h = -(\xi\times\eta)_h^{\perp}$$

defined in the Fourier space.

Proposition 5.3.1. *Assume that* $|\Phi| \le 2^{q_{\max}-10}$*. Then* $2^{p_{\max}} \sim 1$ *and* $|\bar{\sigma}| \gtrsim 2^{q_{\max}} 2^{k_{\max}+k_{\min}}$*. Also,*

$$|S_\eta\Phi| + |\partial_{\theta,\eta}\Phi| \sim \frac{|\xi_h - \eta_h|}{|\xi-\eta|^3}|\bar{\sigma}(\xi,\eta)|.$$

$\Phi = \Phi(\xi,\eta)$ is defined at (5.4) and $\partial_{\theta,\eta}$ refers to the angular derivative of the spherical coordinates in η, in other words, $\partial_{\theta,\eta} = \eta_1\partial_{\eta_2} - \eta_2\partial_{\eta_1}$. Recall the comment following the definition that we will not specify the signs when they are irrelevant.

Proof. See (GPW23, Lemma 5.1, Proposition 5.2). □

This proposition implies that when we cannot perform normal form due to the lack of a lower bound on the phase, we can integrate by parts using either S or ∂_θ thanks to the following lemma.

5.3.1 Bounds for repeated integration by parts

The following lemmas provide bounds for repeated integration by parts, which will be used frequently.

Lemma 5.3.2. *By symmetry, the following estimates also hold when the role of* f_1 *and* f_2*, and thus the indices of them are swapped.*

1. *Assume the localization parameters are such that* $|\bar{\sigma}|\cdot\chi_h \gtrsim 2^{k_{\max}+k_{\min}}2^{p_{\max}} =: L_1$.

Then for any $N \in \mathbb{N}$,

$$||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}\chi_h}(R_{l_1}f_1, R_{l_2}f_2)\}||_\infty \lesssim ||\mathfrak{m}\chi_h||_\infty \cdot (s^{-1}2^{-p_1+2k_1}L_1^{-1}[1+2^{k_2-k_1+l_1}])^N$$
$$\cdot ||P_{k_1,p_1}R_{l_1}(1,S)^N f_1||_2 \cdot ||P_{k_2,p_2}R_{l_2}(1,S)^N f_2||_2$$

2. *Assume the localization parameters are such that $|\bar{\sigma}| \cdot \chi \gtrsim 2^{k_{\max}+k_{\min}}2^{p_{\max}+q_{\max}} =: L_2$. Then for any $N \in \mathbb{N}$,*

$$||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}\chi}(R_{l_1}f_1, R_{l_2}f_2)\}||_\infty \lesssim ||\mathfrak{m}\chi||_\infty \cdot (s^{-1}2^{-p_1+2k_1}L_2^{-1}[1+2^{k_2-k_1}(2^{q_2-q_1}+2^{l_1})])^N$$
$$\cdot ||P_{k_1,p_1,q_1}R_{l_1}(1,S)^N f_1||_2 \cdot ||P_{k_2,p_2,q_2}R_{l_2}(1,S)^N f_2||_2.$$

Proof. See (GPW23, Lemma 5.7). □

In addition to this main tool for integration by parts, we will have occasions where we will need to integrate by parts along the vertical direction, using

$$D_3^\eta := |\eta|\partial_{\eta_3}.$$

Lemma 5.3.3. *Assume that the localization parameters are such that $p_1, p_2 \ll 0$, $k_1 \sim k_2$, and $|D_3^\eta \Phi| \cdot \chi_h \gtrsim L_3$. Then, for any $N \in \mathbb{N}$,*

$$||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}\chi_h}(R_{l_1}f_1, R_{l_2}f_2)\}||_\infty \lesssim ||\mathfrak{m}\chi_h||_\infty \cdot (s^{-1}L_3^{-1}[1+2^{l_1+p_1}+2^{l_2+p_2}])^N$$
$$\cdot ||P_{k_1,p_1}R_{l_1}(1,S)^N f_1||_2 \cdot ||P_{k_2,p_2}R_{l_2}(1,S)^N f_2||_2.$$

Proof. See (GPW23, Section 5.3.3). □

5.3.2 Normal Form

When we have a good lower bound on the phase, which we can realize using Proposition 5.3.1, we have another way to estimate the quadratic nonlinearities.

Lemma 5.3.4. *For the solution* $\mathcal{U}$ *of* (5.2) *and* $0 \le m \le L$*, define* $\mathcal{U}_+^m(t), \mathcal{U}_-^m(t)$ *as*

$$\widehat{\mathcal{U}_\mu^m}(t) = \sum_{\mu_1,\mu_2\in\{+,-\}} \int_0^t \tau_m(s) e^{-(t-s)\kappa|\xi|^2} \int_{\mathbb{R}^3} e^{is\Phi_\mu^{\mu_1,\mu_2}} \mathfrak{m}_\mu^{\mu_1,\mu_2}(\xi,\eta) \widehat{\mathcal{U}}_{\mu_1}(s,\eta) \widehat{\mathcal{U}}_{\mu_2}(s,\xi-\eta) d\eta ds.$$

Then, there holds

$$\begin{aligned}
\mathcal{U}_+^m(t) = \sum_{\mu_1,\mu_2\in\{+,-\}} \Big\{ &-\delta_{m,L} \mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2}}(\mathcal{U}_{\mu_1},\mathcal{U}_{\mu_2})(t) + \delta_{m,0} e^{t\kappa\Delta}[\mathcal{U}_+^m(0) + \mathcal{Q}_{\mathfrak{m}_+^{\mu_1,\mu_2}}(\mathcal{U}_{\mu_1},\mathcal{U}_{\mu_2})(0)] \\
&+ \int_0^t e^{(t-s)\kappa\Delta} \tau_m(s) \mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2}}((\partial_t - \kappa\Delta)\mathcal{U}_{\mu_1},\mathcal{U}_{\mu_2})(s) ds \\
&+ \int_0^t e^{(t-s)\kappa\Delta} \tau_m(s) \mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2}}(\mathcal{U}_{\mu_1},(\partial_t - \kappa\Delta)\mathcal{U}_{\mu_2})(s) ds \\
&+ \int_0^t e^{(t-s)\kappa\Delta} \tau_m'(s) \mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2}}(\mathcal{U}_{\mu_1},\mathcal{U}_{\mu_2})(s) ds \Big\},
\end{aligned}$$

$$\mathfrak{n}_+^{\mu_1,\mu_2} = -\frac{\mathfrak{m}_+^{\mu_1,\mu_2}}{i\Phi_\mu^{\mu_1,\mu_2} + 2\kappa(\xi-\eta)\cdot\eta},$$

where $\delta_{j,k}$ *is the Kronecker delta. The similar formula also holds for* $\mathcal{U}_-^m$*.*

Proof. By computation,

$$\begin{aligned}
&(\partial_t + \kappa|\xi|^2) \int \tau_m(t) e^{it\Phi_\mu^{\mu_1,\mu_2}} \mathfrak{n}_+^{\mu_1,\mu_2}(\xi,\eta) \widehat{\mathcal{U}}_{\mu_1}(t,\eta) \widehat{\mathcal{U}}_{\mu_2}(t,\xi-\eta) d\eta \\
&= \int \tau_m'(t) e^{it\Phi_\mu^{\mu_1,\mu_2}} \mathfrak{n}_+^{\mu_1,\mu_2}(\xi,\eta) \widehat{\mathcal{U}}_{\mu_1}(t,\eta) \widehat{\mathcal{U}}_{\mu_2}(t,\xi-\eta) d\eta \\
&+ \int \tau_m(t) e^{it\Phi_\mu^{\mu_1,\mu_2}} \mathfrak{n}_+^{\mu_1,\mu_2}(\xi,\eta)\, (\partial_t + \kappa|\eta|^2) \widehat{\mathcal{U}}_{\mu_1}(t,\eta)\, \widehat{\mathcal{U}}_{\mu_2}(t,\xi-\eta) d\eta \\
&+ \int \tau_m(t) e^{it\Phi_\mu^{\mu_1,\mu_2}} \mathfrak{n}_+^{\mu_1,\mu_2}(\xi,\eta)\, \widehat{\mathcal{U}}_{\mu_1}(t,\eta)\, (\partial_t + \kappa|\xi-\eta|^2) \widehat{\mathcal{U}}_{\mu_2}(t,\xi-\eta) d\eta \\
&+ \int [i\Phi_\mu^{\mu_1,\mu_2} + \kappa|\xi|^2 - \kappa|\eta|^2 - \kappa|\xi-\eta|^2] \tau_m(t) e^{it\Phi_\mu^{\mu_1,\mu_2}} \mathfrak{n}_+^{\mu_1,\mu_2}(\xi,\eta) \widehat{\mathcal{U}}_{\mu_1}(t,\eta) \widehat{\mathcal{U}}_{\mu_2}(t,\xi-\eta) d\eta
\end{aligned}$$

where the last line is simply the Fourier transform of $-\mathcal{Q}^{(m)}_{\mathfrak{m}_+^{\mu_1,\mu_2}}(\mathcal{U}_{\mu_1},\mathcal{U}_{\mu_2})(t) = -(\partial_t - \kappa\Delta)\mathcal{U}_+^m(t)$. Hence, combining this term with the left-hand side, we get a heat equation for $\mathcal{U}_+^m(t) + \mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2}}(\mathcal{U}_{\mu_1},\mathcal{U}_{\mu_2})$, and solving the heat equation gives the desired result. □

The normal form will be helpful when we can use the discrepancy between the sizes of Λ in $\xi, \xi-\eta$, and η. Since the normal form produces "time derivative terms", $(\partial_t - \kappa\Delta)f$, the following lemma presents a way to bound these using the quantities we can control.

Lemma 5.3.5. *Let $\mathcal{U}_\pm$ be a solution of* (5.2)*, satisfying the bootstrap assumptions* (5.8) *in Theorem 5.2.2. For $m \in \mathbb{N}$ and $t \in [2^m, 2^{m+1}]$, we have*

$$\|P_k S^n (\partial_t - \kappa\Delta)\mathcal{U}_\pm(t)\|_{L^2} \lesssim 2^{\frac{k}{2}-k^+} \cdot 2^{-\frac{3}{2}m+\gamma m} \cdot \epsilon^2, \tag{5.10}$$

for some $0 < \gamma \ll \beta$ and any $0 \leq n \leq N$.

Proof. The proof modifies that of (GPW23, Lemma 6.1). The same method works since applying $\partial_t - \kappa\Delta$ removes the main flow in the Duhamel's formula, and only leaves the dispersive bilinear term, i.e. the same terms as in the time derivative of Euler-Coriolis flow. In other words,

$$\begin{aligned}(\partial_t - \kappa\Delta)\mathcal{U}_\pm =& (\partial_t - \kappa\Delta)e^{t\kappa\Delta}\mathcal{U}_\pm(0) + (\partial_t - \kappa\Delta)\sum_{\mathfrak{m}} \int_0^t e^{(t-s)\kappa\Delta}\mathcal{Q}_{\mathfrak{m}}(\mathcal{U}_\pm, \mathcal{U}_\pm)(s)ds \\ =& \sum_{\mathfrak{m}} \mathcal{Q}_{\mathfrak{m}}(\mathcal{U}_\pm, \mathcal{U}_\pm)(t).\end{aligned}$$

Hence, (5.10) follows in the same way, as we have the same dispersive decay and proved similar energy estimates as in (GPW23). □

Corollary 5.3.6. *For the solution $\mathcal{U}$ of* (5.2) *and $0 \leq m \leq L$, $a, b \in \mathbb{N}$, define $\mathcal{U}_+^{m,a,b}(t), \mathcal{U}_-^{m,a,b}(t)$ as*

$$\widehat{\mathcal{U}}_\mu^{m,a,b}(t) = \sum_{\mu_1,\mu_2 \in \{+,-\}} \int_0^t \tau_m(s) e^{(t-s)\kappa\Delta} \int_{\mathbb{R}^3} e^{is\Phi_\mu^{\mu_1,\mu_2}} \mathfrak{m}_\mu^{\mu_1,\mu_2}(\xi,\eta) \widehat{S^a\mathcal{U}}_{\mu_1}(s,\eta) \widehat{S^b\mathcal{U}}_{\mu_2}(s,\xi-\eta) d\eta ds.$$

Then, there holds

$$\begin{aligned}
\mathcal{U}_+^{m,a,b}(t) = \sum_{\mu_1,\mu_2\in\{+,-\}} \Big\{ & -\delta_{m,L}\mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2,a,b}}(S^a\mathcal{U}_{\mu_1}, S^b\mathcal{U}_{\mu_2})(t) \\
& + \delta_{m,0}e^{t\kappa\Delta}[\mathcal{U}_+^{m,a,b}(0) + \mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2,a,b}}(S^a\mathcal{U}_{\mu_1}, S^b\mathcal{U}_{\mu_2})(0)] \\
& + \int_0^t e^{(t-s)\kappa\Delta}\tau_m(s)\mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2,a,b}}(S^a(\partial_t - \kappa\Delta)\mathcal{U}_{\mu_1}, S^b\mathcal{U}_{\mu_2})(s)ds \qquad (5.11) \\
& + \int_0^t e^{(t-s)\kappa\Delta}\tau_m(s)\mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2,a,b}}(S^a\mathcal{U}_{\mu_1}, S^b(\partial_t - \kappa\Delta)\mathcal{U}_{\mu_2})(s)ds \\
& + \int_0^t e^{(t-s)\kappa\Delta}\tau_m'(s)\mathcal{Q}_{\mathfrak{n}_+^{\mu_1,\mu_2,a,b}}(S^a\mathcal{U}_{\mu_1}, S^b\mathcal{U}_{\mu_2})(s)ds \Big\},
\end{aligned}$$

$$\mathfrak{n}_+^{\mu_1,\mu_2,a,b} = -\frac{\mathfrak{m}_+^{\mu_1,\mu_2}}{i\Phi_\mu^{\mu_1,\mu_2} + \kappa|\xi|^2 - (2a+1)\kappa|\eta|^2 - (2b+1)\kappa|\xi-\eta|^2},$$

and similarly for $\mathcal{U}_-^{m,a,b}$.

The terms multiplied by κ in the denominator of $\mathfrak{n}$ will be ignored in the estimates using the normal form, because we want to show estimates that are uniform with respect to κ, in particular as $\kappa \to 0$. In other words, qualitatively, this corollary will be used in the same way as Lemma 5.3.4, as we will only use the imaginary part of the denominator, $i\Phi_\mu^{\mu_1,\mu_2}$, and ignore terms with a and b, whenever we have changed the multipliers. It suffices to note that we can use the normal form even when the S-vector fields are applied to $\mathcal{U}_\pm$, even though (5.2) does not commute with the S vector field. Together with Lemma 5.3.5, this gives another way to estimate bilinear terms.

Proof. We modify the proof of Lemma 5.3.4. Again, by computation,

$$\begin{aligned}
& (\partial_t + \kappa|\xi|^2)\int \tau_m(t)e^{it\Phi_\mu^{\mu_1,\mu_2}}\mathfrak{n}_+^{\mu_1,\mu_2,a,b}(\xi,\eta)\widehat{S^a\mathcal{U}}_{\mu_1}(t,\eta)\widehat{S^b\mathcal{U}}_{\mu_2}(t,\xi-\eta)d\eta \\
= & \int \tau_m'(t)e^{it\Phi_\mu^{\mu_1,\mu_2}}\mathfrak{n}_+^{\mu_1,\mu_2,a,b}(\xi,\eta)\widehat{S^a\mathcal{U}}_{\mu_1}(t,\eta)\widehat{S^b\mathcal{U}}_{\mu_2}(t,\xi-\eta)d\eta \\
& + \int \tau_m(t)e^{it\Phi_\mu^{\mu_1,\mu_2}}\mathfrak{n}_+^{\mu_1,\mu_2,a,b}(\xi,\eta)\,(\partial_t + \kappa|\eta|^2 + 2a\kappa|\eta|^2)\widehat{S^a\mathcal{U}}_{\mu_1}(t,\eta)\,\widehat{S^b\mathcal{U}}_{\mu_2}(t,\xi-\eta)d\eta \\
& + \int \tau_m(t)e^{it\Phi_\mu^{\mu_1,\mu_2}}\mathfrak{n}_+^{\mu_1,\mu_2,a,b}(\xi,\eta)\,\widehat{S^a\mathcal{U}}_{\mu_1}(t,\eta)\,(\partial_t + \kappa|\xi-\eta|^2 + 2b\kappa|\xi-\eta|^2)\widehat{S^b\mathcal{U}}_{\mu_2}(t,\xi-\eta)d\eta
\end{aligned}$$

$$
\begin{aligned}
&+\int [i\Phi_{\mu}^{\mu_1,\mu_2}+\kappa|\xi|^2-(2a+1)\kappa|\eta|^2-(2b+1)\kappa|\xi-\eta|^2]\\
&\qquad\cdot \tau_m(t)e^{it\Phi_{\mu}^{\mu_1,\mu_2}}\mathfrak{n}_{+}^{\mu_1,\mu_2,a,b}(\xi,\eta)\widehat{S^a\mathcal{U}}_{\mu_1}(t,\eta)\widehat{S^b\mathcal{U}}_{\mu_2}(t,\xi-\eta)d\eta\\
&=\int \tau_m'(t)e^{it\Phi_{\mu}^{\mu_1,\mu_2}}\mathfrak{n}_{+}^{\mu_1,\mu_2,a,b}(\xi,\eta)\widehat{S^a\mathcal{U}}_{\mu_1}(t,\eta)\widehat{S^b\mathcal{U}}_{\mu_2}(t,\xi-\eta)d\eta\\
&+\int \tau_m(t)e^{it\Phi_{\mu}^{\mu_1,\mu_2}}\mathfrak{n}_{+}^{\mu_1,\mu_2,a,b}(\xi,\eta)\,\mathcal{F}\{S^a(\partial_t-\kappa\Delta)\mathcal{U}_{\mu_1}\}(t,\eta)\,\widehat{S^b\mathcal{U}}_{\mu_2}(t,\xi-\eta)d\eta\\
&+\int \tau_m(t)e^{it\Phi_{\mu}^{\mu_1,\mu_2}}\mathfrak{n}_{+}^{\mu_1,\mu_2,a,b}(\xi,\eta)\,\widehat{S^a\mathcal{U}}_{\mu_1}(t,\eta)\,\mathcal{F}\{S^b(\partial_t-\kappa\Delta)\mathcal{U}_{\mu_2}\}(t,\xi-\eta)d\eta\\
&-\int \tau_m(t)e^{it\Phi_{\mu}^{\mu_1,\mu_2}}\mathfrak{m}_{+}^{\mu_1,\mu_2}(\xi,\eta)\widehat{S^a\mathcal{U}}_{\mu_1}(t,\eta)\widehat{S^b\mathcal{U}}_{\mu_2}(t,\xi-\eta)d\eta
\end{aligned}
$$

Observing that we recovered $\mathcal{U}_{+}^{m,a,b}$ in the last line, the same argument as in the proof of Lemma 5.3.4 gives us the desired result. □

Remark 5.3.7. *Since we now know how to use the normal form and Lemma 5.3.5 together to obtain additional decay, for notational convenience, when we are already in the framework of considering functions $f_j = S^{n_j}\mathcal{U}_j$, we will write $(\partial_t - \kappa\Delta)f_j$'s instead of reverting to $S^{n_j}(\partial_t - \kappa\Delta)\mathcal{U}_j$, and declare we have used normal form.*

Also, whenever it is clear what we are dealing with, we will simply write $\mathfrak{n}$ for the altered multiplier instead of writing the full notation $\mathfrak{n}_{\pm}^{\mu_1,\mu_2,a,b}$ to keep the notations simpler.

As can be seen from (5.10), when the normal form can be applied, we can obtain a fast decay from one of the terms. Hence, slower decay for another one can be tolerated at times, and the following lemma may be used.

Lemma 5.3.8. *Under the bootstrap assumption* (5.8)*, for some $0 < \gamma \ll \beta$, we have*

$$||P_k e^{it\Lambda} S^b \mathcal{U}_{\pm}||_\infty \lesssim 2^{\frac{3}{2}k-3k^+} t^{-1+\gamma}\epsilon, \quad 0 \le b \le N.$$

Proof. See (GPW23, Corollary A.7). □

There are occasions where we need to perform normal form using an arbitrary cutoff

$\lambda > 0$. In these cases, we use the notation

$$\mathfrak{m} = \bar{\varphi}(\lambda^{-1}\Phi)\mathfrak{m} + (1 - \bar{\varphi}(\lambda^{-1}\Phi))\mathfrak{m} =: \mathfrak{m}^{res} + \mathfrak{m}^{nr}$$

to denote the resonant and nonresonant decomposition via multiplier. The bilinear terms can be expressed similarly, by writing

$$\mathcal{Q}_{\mathfrak{m}}(f_1, f_2) = \mathcal{Q}_{\mathfrak{m}^{res}}(f_1, f_2) + \mathcal{Q}_{\mathfrak{m}^{nr}}(f_1, f_2), \quad \mathcal{B}_{\mathfrak{m}}(g_1, g_2) = \mathcal{B}_{\mathfrak{m}^{res}}(g_1, g_2) + \mathcal{B}_{\mathfrak{m}^{nr}}(g_1, g_2),$$

and we have the following estimates.

Lemma 5.3.9. *Let $\lambda > 0$ and $F_j = P_{k_j,p_j}F_j$, $j = 1, 2$.*

1. *The non-resonant part satisfies*

$$||P_{k,p}\mathcal{Q}_{\Phi^{-1}\mathfrak{m}^{nr}}(F_1, F_2)||_2 \lesssim 2^{k+p_{\max}}\lambda^{-1} \cdot \min\{2^{k+p}, 2^{k_1+p_1}, 2^{k_2+p_2}\}||F_1||_2||F_2||_2. \tag{5.12}$$

2. *If we can choose $\lambda > 0$ such that $|\Phi\chi_h| \geq \lambda \gtrsim 1$, then we have $\mathfrak{m}^{res} = 0$ and thus $\mathfrak{m} = \mathfrak{m}^{nr}$, and in addition to (5.12), we also have*

$$||P_{k,p}\mathcal{Q}_{\Phi^{-1}\mathfrak{m}^{nr}}(F_1, F_2)||_2 \lesssim 2^{k+p_{\max}} \cdot \min\{||e^{it\Lambda}F_1||_\infty||F_2||_2,\ ||F_1||_2||e^{it\Lambda}F_2||_\infty\}.$$

3. *If $|\partial_{\eta_3}\Phi\chi_h| \gtrsim L > 0$ holds, then we can use the set size estimates for both resonant and nonresonant terms:*

$$||P_{k,p}\mathcal{Q}_{\mathfrak{m}^{res}}(F_1, F_2)||_2 \lesssim 2^{k+p_{\max}}\lambda^{1/2}L^{-1/2} \cdot \min\{2^{k_1+p_1}, 2^{k_2+p_2}\}||F_1||_2||F_2||_2,$$
$$||P_{k,p}\mathcal{Q}_{\Phi^{-1}\mathfrak{m}^{nr}}(F_1, F_2)||_2 \lesssim 2^{k+p_{\max}}|\log\lambda|\lambda^{-1/2}L^{-1/2} \cdot \min\{2^{k_1+p_1}, 2^{k_2+p_2}\}||F_1||_2||F_2||_2.$$

Proof. See (GPW23, Lemma 5.10). □

Set size estimates can be used more generally and will be used a lot in the analysis. In fact, the following estimate will be used frequently.

Lemma 5.3.10. *(1) For a bilinear term $\mathcal{Q}_{\mathfrak{m}}$ with a multiplier $\mathfrak{m}$ and localized by $P_{k,p,q}$, i.e.*

$$P_{k,p,q}\mathcal{Q}_{\mathfrak{m}}(P_{k_1,p_1,q_1}f_1, P_{k_2,p_2,q_2}f_2) = \mathcal{F}^{-1}\left\{\int_{\mathbb{R}^3_\eta} e^{is\Phi}\chi(\xi,\eta)\mathfrak{m}(\xi,\eta)\hat{f}_1(\xi-\eta)\hat{f}_2(\eta)d\eta\right\},$$

then with

$$|S| := \min\{2^{k+p}, 2^{k_1+p_1}, 2^{k_2+p_2}\}\cdot\min\{2^{\frac{k+q}{2}}, 2^{\frac{k_1+q_1}{2}}, 2^{\frac{k_2+q_2}{2}}\},$$

the following bound holds:

$$||P_{k,p,q}\mathcal{Q}_{\mathfrak{m}}(P_{k_1,p_1,q_1}f_1, P_{k_2,p_2,q_2}f_2)||_2 \lesssim |S|\cdot||\mathfrak{m}\chi||_\infty\cdot||P_{k_1,p_1,q_1}f_1||_2\cdot||P_{k_2,p_2,q_2}f_2)||_2.$$

(2) If the bilinear term is localized where the phase is small in addition, i.e.

$$\mathcal{Q}_{\mathfrak{m}}(f_1,f_2) = \mathcal{F}^{-1}\left\{\int_{\mathbb{R}^3_\eta} e^{is\Phi}\varphi(\lambda^{-1}\Phi)\chi(\xi,\eta)\mathfrak{m}(\xi,\eta)\hat{f}_1(\xi-\eta)\hat{f}_2(\eta)d\eta\right\},$$

for some $\lambda>0$, and we have a lower bound $|\nabla_{\eta_h}\Phi|\gtrsim L>0$ on the support of χ, then

$$||\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)||_2 \lesssim \min\{2^{\frac{k_1+q_1}{2}}, 2^{\frac{k_2+q_2}{2}}\}\cdot 2^{\frac{k_2+p_2}{2}}\cdot(\lambda L^{-1})^{\frac{1}{2}}\cdot||\mathfrak{m}\chi||_\infty\cdot||P_{k_1,p_1,q_1}f_1||_2\cdot||P_{k_2,p_2,q_2}f_2)||_2.$$

Proof. The estimates use the fact that the localizations, as well as the conditions on Φ in case (2), give set size related terms. See (GPW23, Lemma A.3, A.4). $\square$

5.3.3 Geometry of vectors

The configuration of vectors $\xi, \xi-\eta, \eta$ will determine whether we can use integration by parts or the normal form method. As one can see, the three vectors are related, and hence we can group the configurations into some cases amenable for our analysis.

Lemma 5.3.11. *Assume that $p\le\min\{p_1,p_2\}-10$. Then on the support of χ_h, there* *holds that $p+k < p_1+k_1-4$, and thus $p_2+k_2-2 \le p_1+k_1 \le p_2+k_2+2$. Moreover, one of the following relations holds:*

1. *$|k_1-k_2| \le 4$, and thus also $|p_1-p_2| \le 6$,*
2. *$k_2 < k_1-4$ then $|k-k_1| \le 2$ and $p_1 \le p_2-2$, so that $p \le p_1-10 \le p_2-12$,*
3. *$k_1 < k_2-4$ then $|k-k_2| \le 2$ and $p_2 \le p_1-2$, so that $p \le p_2-10 \le p_1-12$.*

Proof. See (GPW23, Lemma 5.8). □

Similar results hold for any permutation of p, p_1, p_2, and also for q, q_1, q_2 instead of p, p_1, p_2, as the proof only uses the configuration of vectors. I add one more type of case where we have two small p's and one large p (or similarly with q's).

Lemma 5.3.12. *Assume that $p_1, p_2 \le p-15$ and $|p_1-p_2| \le 2$. Then on the support of χ_h, there holds that $k_1, k_2 \ge k+6$, $|k_1-k_2| \le 2$, and $|k_1+p_1-k_2-p_2| \le 4$, $k_2+p_2 \ge k+p-2$.*

Similarly, assume that $q_1, q_2 \le q-15$ and $|q_1-q_2| \le 2$. Then on the support of χ, there holds that $k_1, k_2 \ge k+6$, $|k_1-k_2| \le 2$, and $|k_1+q_1-k_2-q_2| \le 4$, $k_2+q_2 \ge k+q-2$.

Proof. Suppose, for the sake of contradiction, that $k_1 < k+6$. From $\xi_h = (\xi-\eta)_h + \eta_h$, we have $2^{k+p-4} - 2^{k_1+p_1} \le 2^{k_2+p_2}$. Since $(k+p)-(k_1+p_1) > 9$, this gives $2^{k+p-5} \le 2^{k_2+p_2}$, hence $k-k_2 \le p_2-p+5 \le -10$. However, this leads to $k \le k_2-10$ and $k_1 \le k_2-4$, which contradicts $\xi = (\xi-\eta)+\eta$. Therefore, $k_1 \ge k+6$. Symmetric arguments give $k_2 \ge k+6$, and as the two largest indices, k_1 and k_2 should have similar sizes.

$|k_1+p_1-k_2-p_2| \le 4$ is now immediate and hence it should be the case $(\xi-\eta)_h \sim \eta_h \gtrsim \xi_h$, which gives $k_2+p_2 \ge k+p-2$.

Replacing (p, p_1, p_2) with (q, q_1, q_2) and $(\xi_h, (\xi-\eta)_h, \eta_h)$ with $(\xi_3, (\xi-\eta)_3, \eta_3)$ and applying the same arguments gives the results between the sizes of q's and k's. □

Remark 5.3.13. *Again, permuting the roles of p, p_j's gives analogous results. We will* *now use the notation $\ll$ instead of, for example, saying $p_1 \leq p - 15$, since the results follow from the configuration of vectors.*

5.4 B-norm bounds

In this section, we prove the first half of the bootstrap argument.

Theorem 5.4.1. *If u is the solution to* (5.2) *with $\kappa \lesssim 1$ and the initial data satisfying* (5.6)*, then assuming* (5.8)*, we have*

$$||S^n \mathcal{U}_\pm(t)||_B \lesssim \epsilon + \epsilon^2, \qquad 0 \leq n \leq N.$$

Proof. Control of the linear term $||S^n(e^{t\kappa\Delta}\mathcal{U}_\pm(0))||_B = ||\sum_{a=0}^{n} h_a(t\kappa\Delta)e^{t\kappa\Delta}S^{n-a}\mathcal{U}_\pm(0)||_B \lesssim \epsilon$ was already mentioned after the main theorem, and so were the steps to deal with the nonlinear term. We localize in time $t \in [2^m, 2^{m+1}]$ first, then in space by k, p, l, and in q, reducing the indices every time by the energy estimates or bootstrap assumptions. After we are restricted to certain ranges of indices, we use repeated integration by parts using vector fields or perform normal forms, depending on the geometry of vectors in the restricted localized regions. Recall that

$$S^n \mathcal{B}_{\mathfrak{m}}(\mathcal{U}_\pm, \mathcal{U}_\pm) = \sum_{a+b+c \leq n} \int_0^t \tau_m(s) h_a((t-s)\kappa\Delta) e^{(t-s)\kappa\Delta} \mathcal{Q}_{\mathfrak{m}}(S^b \mathcal{U}_\pm, S^c \mathcal{U}_\pm)(s) ds.$$

Since $||h_a(-(t-s)\kappa|\xi|^2)e^{-(t-s)\kappa|\xi|^2}||_\infty \lesssim 1$, it's enough to show

$$||\mathcal{B}_{\mathfrak{m}}(F_1, F_2)||_B \lesssim 2^{-\delta m}\epsilon^2$$

for $F_j = S^{b_j}\mathcal{U}_\pm$. Also, we will often prove a stronger bound

$$2^{k+4k^+}||\mathcal{F}\{\mathcal{B}_{\mathfrak{m}}(F_1, F_2)\}||_\infty \lesssim 2^{-\delta m}\epsilon^2.$$

First of all, using the energy bounds,

$$2^{k+4k^+}||\mathcal{F}\{P_k\mathcal{B}_{\mathfrak{m}}(P_{k_1}F_1, P_{k_2}F_2)\}||_\infty \lesssim 2^{2k+4k^+}\sum_{k_1,k_2} 2^m \cdot \min\{2^{-N_0k_1}||F_1||_{H^{N_0}}, 2^{k_1}||F_1||_{\dot{H}^{-1}}\}$$
$$\cdot \min\{2^{-N_0k_2}||F_2||_{H^{N_0}}, 2^{k_2}||F_2||_{\dot{H}^{-1}}\}.$$

Since the two biggest indices of $\{k, k_1, k_2\}$ should have similar sizes, the summations to either infinity give some trivial decay:

$$\begin{aligned}
&\sum_{k_1\geq\delta_0 m, k_2\in\mathbb{Z}} 2^{2k+4k^+}\cdot \min\{2^{-N_0k_1}||F_1||_{H^{N_0}}, 2^{k_1}||F_1||_{\dot{H}^{-1}}\}\cdot\min\{2^{-N_0k_2}||F_2||_{H^{N_0}}, 2^{k_2}||F_2||_{\dot{H}^{-1}}\}\\
\leq &\sum_{k_1\geq\delta_0 m} 2^{-(N_0-6)k_1}\epsilon\cdot\sum_{k_2}\min\{2^{-N_0k_2}, 2^{k_2}\}\epsilon \qquad \text{(when } k\simeq k_1\gg k_2)\\
+ &\sum_{k_1\geq\delta_0 m} 2^{-N_0k_1}\epsilon\cdot\sum_{k_2}\min\{2^{-(N_0-6)k_2}, 2^{3k_2}\}\epsilon \qquad \text{(when } k\simeq k_2\gg k_1)\\
+ &\sum_{k_1\geq\delta_0 m} 2^{-(2N_0-6)k_1}\epsilon^2 \qquad \text{(when } k_1\simeq k_2\gg k)\\
\simeq\ & 2^{-(N_0-6)\delta_0 m}\epsilon^2 = 2^{(-2+\frac{12}{N_0})m}\epsilon^2.
\end{aligned}$$

For $\delta_0 = 2/N_0$, this gives an acceptable contribution as long as $N_0 > 12$. Also,

$$\begin{aligned}
&\sum_{k_1\leq -2m, k_2\in\mathbb{Z}} 2^{2k+4k^+}\cdot \min\{2^{-N_0k_1}||F_1||_{H^{N_0}}, 2^{k_1}||F_1||_{\dot{H}^{-1}}\}\cdot\min\{2^{-N_0k_2}||F_2||_{H^{N_0}}, 2^{k_2}||F_2||_{\dot{H}^{-1}}\}\\
\leq &\sum_{k_1\leq -2m} 2^{3k_1}\epsilon\cdot\sum_{k_2}\min\{2^{-N_0k_2}, 2^{k_2}\}\epsilon \qquad \text{(when } k\simeq k_1\gg k_2)\\
+ &\sum_{k_1\leq -2m} 2^{k_1}\epsilon\cdot\sum_{k_2}\min\{2^{-(N_0-6)k_2}, 2^{3k_2}\}\epsilon \qquad \text{(when } k\simeq k_2\gg k_1)\\
+ &\sum_{k_1\leq -2m} 2^{2k_1}\epsilon^2 \qquad \text{(when } k_1\simeq k_2\gg k)\\
\simeq\ & 2^{-2m}\epsilon^2,
\end{aligned}$$

which is again an acceptable summation. Similar calculations apply for summations over $k_1\in\mathbb{Z}, k_2\notin(-2m,\delta_0 m)$, and from $2^k\lesssim 2^{k_1}+2^{k_2}$, these also restrict k to be less than

$\delta_0 m$. Lastly, when $k \leq -2m$,

$$\sum_{k_1,k_2\in\mathbb{Z}} 2^{2k+4k^+} \cdot \min\{2^{-N_0k_1}||F_1||_{H^{N_0}}, 2^{k_1}||F_1||_{\dot{H}^{-1}}\} \cdot \min\{2^{-N_0k_2}||F_2||_{H^{N_0}}, 2^{k_2}||F_2||_{\dot{H}^{-1}}\}$$
$$\leq 2^{-2m}\sum_{k_1\in\mathbb{Z}} \min\{2^{-N_0k_1}, 2^{k_1}\}\epsilon \cdot \sum_{k_2\in\mathbb{Z}} \min\{2^{-N_0k_2}, 2^{k_2}\}\epsilon \simeq 2^{-2m}\epsilon^2,$$

so we can restrict to the case where $-2m \leq k, k_1, k_2 \leq \delta_0 m$. Now localizing further in $p_j, l_j, j = 1, 2$, with denoting $f_j = P_{k_j,p_j}R_{l_j}F_j$,

$$\begin{aligned} 2^{k+4k^+}||\mathcal{F}\{P_k\mathcal{B}_{\mathfrak{m}}(f_1, f_2)\}||_\infty &\lesssim 2^{k+4k^+} \cdot 2^m \cdot 2^{k+p_{\max}} \cdot 2^{-3k_1^+-l_1}||f_1||_X \cdot 2^{-3k_2^+-l_2}||f_2||_X \\ &\lesssim 2^{(1+3\delta_0)m-l_1-l_2}\epsilon^2, \end{aligned}$$

so that summation over either $l_1 \geq 2m$ or $l_2 \geq 2m$ gives acceptable bounds. Along with the natural assumption made before that $p_j + l_j \geq 0$, it's enough to only consider

$$-2m \leq k, k_1, k_2 \leq \delta_0 m, \quad -2m \leq p_1, p_2 \leq 0, \quad -p_1 \leq l_1 \leq 2m, -p_2 \leq l_2 \leq 2m.$$

Since these are only $(\log\langle t\rangle)^6$-many cases (k is fixed), now it's enough to show

$$2^{3k^+-\frac{1}{2}k^--p-\frac{q}{2}}||P_{k,p,q}\mathcal{B}_{\mathfrak{m}}(f_1, f_2)||_2 \lesssim 2^{-\delta m}\epsilon^2$$

for fixed $k, k_j, p_j, q_j, j = 1, 2$ in the above ranges. Also, in many cases, we will show instead

$$2^{k+4k^+}||\mathcal{F}\{P_{k,p,q}\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty \lesssim 2^{-(1+\delta)m}\epsilon^2.$$

1. Gap in p with $p_{\max} \sim 0$.

From $|\bar{\sigma}| \sim 2^{k_{\min}+k_{\max}+p_{\max}}$, we can conduct integration by parts in a controllable way in this region. We use Lemma 5.3.2 to conduct repeated integration by parts using either S or ∂_θ, and assume without loss of generality that $p_1 \leq p_2$.

① $p \ll p_1, p_2$: By Lemma 5.3.11, we have three possible cases.

1-1 : $2^{k_1} \sim 2^{k_2},\ 2^{p_1} \sim 2^{p_2} \sim 1$. First, integration by parts gives

$$\begin{aligned} 2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty &\lesssim 2^{k+4k^+} \cdot 2^{k+p_{\max}} \cdot (s^{-1} \cdot 2^{-p_1+2k_1-k_{\max}-k_{\min}}[1 + 2^{k_2-k_1+l_1}])^N \\ &\quad \cdot ||(1, S^N)f_1||_2 \cdot ||(1, S^N)f_2||_2 \\ &\lesssim 2^{2k+4k^+} \cdot (s^{-1} \cdot 2^{k_1-k+l_1})^N \epsilon^2, \end{aligned}$$

so that when $l_1 \leq k - k_1 + (1-\delta)m$, we are done. By symmetry, we are okay when $l_2 \leq k - k_2 + (1-\delta)m$. Else,

$$\begin{aligned} 2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty &\lesssim 2^{2k+4k^+} 2^{-3k_1^+ - l_1}||f_1||_X 2^{-3k_2^+ - l_2}||f_2||_X \\ &\leq 2^{k_1+k_2-2(1-\delta)m}\epsilon^2 \leq 2^{-(1+\delta)m}\epsilon^2. \end{aligned}$$

1-2 : $2^{k_2} \ll 2^{k_1} \sim 2^k,\ p \ll p_1 \ll p_2 \sim 0$. In this case, having 2^{-p_2} is favorable. Hence, integrating by parts in either S_η or $\partial_{\theta,\eta}$,

$$\begin{aligned} 2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty &\lesssim 2^{k+4k^+} \cdot 2^k \cdot (s^{-1} \cdot 2^{-p_2+2k_2-k_{\max}-k_{\min}}[1 + 2^{k_1-k_2+l_2}])^N \\ &\quad \cdot ||(1, S^N)f_1||_2||(1, S^N)f_2||_2 \\ &\lesssim 2^{2k+4k^+} \cdot (s^{-1} \cdot 2^{l_2})^N \epsilon^2, \end{aligned}$$

so that if $l_2 \leq (1-\delta)m$ we are done. Else,

$$\begin{aligned} 2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty &\lesssim 2^{2k+4k^+}||f_1||_2 \cdot 2^{-3k_2^+ - (1+\beta)l_2 - \beta p_2}||f_2||_X \\ &\lesssim 2^{2k+4k^+} 2^{-(1+\beta)(1-\delta)m}\epsilon^2 \leq 2^{-(1+\delta)m}\epsilon^2. \end{aligned}$$

1-3 : $2^{k_1} \ll 2^{k_2} \sim 2^k,\ p \ll p_2 \ll p_1 \sim 0$. This is against the assumption $p_1 \leq p_2$, hence is excluded.

② $p_1 \ll p_2, p$: By Lemma 5.3.11, we again have three possible cases.

2-1 : $2^k \sim 2^{k_2}$, $2^p \sim 2^{p_2} \sim 1$. Via the same integration by parts as in the case 1-1,

$$\begin{aligned} 2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+}\cdot(s^{-1}2^{-p_1+l_1})^N\epsilon^2, \\ \text{or, } &\lesssim 2^{2k+4k^+}\cdot(s^{-1}[2^{k_2-k_1}+2^{l_2}])^N\epsilon^2, \end{aligned}$$

so that if $l_1 \le p_1 + (1-\delta)m$ or $\max\{k_2-k_1, l_2\} \le (1-\delta)m$, we are done. Else, we use the set size estimate Lemma 5.3.10:

$$\begin{aligned} 2^{3k^+-\frac{1}{2}k^--p-\frac{q}{2}}||P_{k,p,q}\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)||_2 &\lesssim 2^{3k^+-\frac{1}{2}k^--\frac{q}{2}}2^k2^{k_1+p_1+\frac{k+q}{2}}2^{-l_1}||f_1||_X\cdot 2^{-3k_2^+-l_2}||f_2||_X \\ &\lesssim 2^{\frac{k^+}{2}+2k+(k_1-k_2)-l_2+(p_1-l_1)}\epsilon^2 \le 2^{\frac{k^+}{2}+2k-2(1-\delta)m}\epsilon^2 \le 2^{-\delta m}\epsilon^2. \end{aligned}$$

2-2 : $2^{k_2} \ll 2^k \sim 2^{k_1}$, $p_1 \ll p \ll p_2 \sim 0$. The relations between k's and the size of p_2 are the same as those in the case 1-2, and thus can be dealt with exactly the same way as those were enough to close the estimate.

2-3 : $2^k \ll 2^{k_2} \sim 2^{k_1}$, $p_1 \ll p_2 \ll p \sim 0$. Integrating by parts using S or ∂_θ,

$$\begin{aligned} 2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+}\cdot(s^{-1}\cdot 2^{-p_2+2k_2-k_{\max}-k_{\min}}[1+2^{k_1-k_2+l_2}])^N \\ &\quad\cdot||(1,S^N)f_1||_2||(1,S^N)f_2||_2 \\ &\lesssim 2^{2k+4k^+}(s^{-1}2^{-p_2+k_2-k+l_2})^N\epsilon^2, \end{aligned}$$

so that when $l_2 - p_2 + k_2 - k \le (1-\delta)m$, we are done. Else, we also use $2^{k+p} \sim 2^{k_2+p_2}$, so that $2^{\beta(k_2-k)} \sim 2^{-\beta p_2}$, and

$$
\begin{aligned}
2^{3k^+-\frac{1}{2}k^--p-\frac{q}{2}}||P_{k,p,q}\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)||_2 &\lesssim 2^{3k^+-\frac{1}{2}k^--\frac{q}{2}}2^k 2^{k_2+p_2+\frac{k+q}{2}} \cdot 2^{p_1}||f_1||_2 \cdot 2^{-3k_2^+-(1+\beta)l_2-\beta p_2}||f_2||_X \\
&\lesssim 2^{\frac{1}{2}k^++k+k_2}2^{p_1+p_2}2^{-\beta p_2}2^{-(1+\beta)p_2+(1+\beta)(k_2-k)-(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{\frac{1}{2}k^++2k_2}2^{p_1-3\beta p_2}2^{-(1+\beta/2)m}\epsilon^2 \leq 2^{-(1+\delta)m}\epsilon^2
\end{aligned}
$$

from $p_1 \leq p_2$ and $\beta \leq 1/3$.

③ $p_2 \ll p_1, p$: This case is excluded from our assumption $p_1 \leq p_2$.

④ $p_1 \sim p_2 \ll p \sim 0$: By Lemma 5.3.12, $k_1 \sim k_2 \gg k$ holds, and using integration by parts first,

$$
\begin{aligned}
2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+} \cdot (s^{-1}2^{-p_1+2k_1-k-k_1} \cdot [1+2^{k_2-k_1+l_1}])^N \\
&\quad \cdot ||(1,S)^N f_1||_2||(1,S)^N f_2||_2 \\
&\lesssim 2^{2k+4k^+} \cdot (s^{-1}2^{-p_1+k_1-k+l_1})^N\epsilon^2,
\end{aligned}
$$

so that if $l_1 - p_1 + k_1 - k \leq (1-\delta)m$, we are done. Else, we also use $2^{k_1+p_1} \sim 2^{k_2+p_2} \gtrsim 2^{k+p} \sim 2^k \leftrightarrow -p_1 - k_1 + k \lesssim 0$, so that

$$
\begin{aligned}
2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+} \cdot 2^{-3k_1^+-(1+\beta)l_1-\beta p_1}||f_1||_X \cdot 2^{-3k_2^++p_2}||f_2||_B \\
&\lesssim 2^{2k}2^{(1+\beta)(-p_1+k_1-k)-\beta p_1}2^{-(1+\beta)(1-\delta)m} \cdot 2^{p_2}\epsilon^2 \\
&\lesssim 2^{2k+(1+3\beta)(k_1-k)}2^{p_2-p_1}2^{-(1+\beta/2)m}\epsilon^2 \lesssim 2^{-(1+\delta)m}\epsilon^2,
\end{aligned}
$$

since $p_1 \sim p_2$ and $\beta \leq 1/3$.

⑤ $p \sim p_1 \ll p_2 \sim 0:$ Lemma 5.3.12 gives $k \sim k_1 \gg k_2$. From integration by parts,

$$\begin{aligned}2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+}\cdot(s^{-1}2^{-p_2+2k_2-k_1-k_2}[1+2^{k_1-k_2+l_2}])^N \\ &\quad\cdot||(1,S)^N f_1||_2||(1,S)^N f_2||_2 \\ &\lesssim 2^{2k+4k^+}\cdot(s^{-1}2^{l_2})^N\epsilon^2,\end{aligned}$$

so that when $l_2 \leq (1-\delta)m$, we are done. Else,

$$\begin{aligned}2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+}||f_1||_2\cdot 2^{-3k_2^+-(1+\beta)l_2-\beta p_2}||f_2||_X \\ &\lesssim 2^{2k+4k^+}2^{-(1+\beta)(1-\delta)m}\epsilon^2 \lesssim 2^{(-(1+\delta)m)}\epsilon^2.\end{aligned}$$

⑥ $p \sim p_2 \ll p_1 \sim 0:$ This case is excluded by our assumption $p_1 \leq p_2$.

2. Gap in p with $p_{\max} \ll 0$. In this case, every $\Lambda(\zeta) \geq \frac{1}{2}, \zeta \in \{\xi, \xi-\eta, \eta\}$, so that we have $|\Phi| \geq \frac{1}{2}$, which allows us to perform the normal form with $\mathfrak{n} \lesssim 2^k$ every time. Compared to (5.11), we have $h_a((t-s)\kappa\Delta)$'s in addition, but one can immediately observe that this can be absorbed by the exponential again. The first two terms with time integration are easily bounded from

$$\begin{aligned}2^{k+4k^+}||\mathcal{F}\{\mathcal{B}_{\mathfrak{n}}((\partial_t-\kappa\Delta)f_1,f_2)\}||_\infty &\lesssim 2^{k+4k^+}2^m2^{k+p_{\max}}||(\partial_t-\kappa\Delta)f_1||_2||f_2||_2 \\ &\lesssim 2^{2k+4k^+}2^m2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot\epsilon \lesssim 2^{-\frac{1}{4}m}\epsilon^3,\end{aligned} \tag{5.13}$$

and the bound for the other term works symmetrically.

We now turn to the first boundary term. This term is evaluated exactly at time t, so that we can use the linear decay estimates. When $\min\{p_1,p_2\} = p_1 \lesssim p$, the set size estimate Lemma 5.3.10 gives

$$2^{3k^+-\frac{1}{2}k^-}2^{-p-\frac{q}{2}}||\mathcal{Q}_{\mathfrak{n}}(f_1,f_2)(t)||_2 \lesssim 2^{3k^+-\frac{1}{2}k^- -p}2^{k+p_{\max}}2^{p_1}||f_1||_B||e^{it\Lambda}f_2||_\infty$$
$$\lesssim 2^{4k^++\frac{1}{2}k}2^{-(1-\delta)m}\epsilon^2 \lesssim 2^{-\delta m}\epsilon^2,$$

and similarly when $\min\{p_1,p_2\} = p_2 \lesssim p$. Hence, we can now assume $p_{\min} = p$. If furthermore, $p_{\max} \leq -\delta m$, then

$$2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{n}}(f_1,f_2)\}||_\infty \lesssim 2^{2k+4k^+}2^{p_{\max}}\cdot 2^{p_1}||f_1||_B\cdot 2^{p_2}||f_2||_B \lesssim 2^{-\delta m}\epsilon^2.$$

Lastly, if $p_{\max} \geq -\delta m$, then from $|\bar{\sigma}| \gtrsim 2^{-\delta m}2^{k_{\max}+k_{\min}}$ and

$$\left|\frac{1}{s\cdot V_\eta\Phi}V_\eta\left(\frac{1}{i\Phi+2\kappa(\xi-\eta)\cdot\eta}\right)\right| = \left|\frac{1}{s\cdot V_\eta\Phi}\frac{iV_\eta\Phi+2\kappa(\xi-\eta)\cdot\eta-2\kappa|\eta|^2}{[i\Phi+2\kappa(\xi-\eta)\cdot\eta]^2}\right|$$
$$\lesssim s^{-1}|V_\eta\Phi|^{-1}(1+\kappa 2^{2k_2}),$$

we can use repeated integration by parts again as in the Case 1.

The second boundary term only exists when $m=0$, and hence the restrictions on the indices become fixed numbers. Also, $||e^{t\kappa\Delta}\mathcal{Q}_{\mathfrak{n}}(\mathcal{U}_\pm,\mathcal{U}_\pm)(0)||_2 \lesssim \epsilon^2$, so it's acceptable.

Lastly, from the above results,

$$2^{3k^+-\frac{1}{2}k^-}2^{-p-\frac{q}{2}}\left\|\int_0^t e^{(t-s)\kappa\Delta}\tau_m'(s)\mathcal{Q}_{\mathfrak{n}}(f_1,f_2)(s)ds\right\|_2$$
$$\leq \int_0^t |\tau_m'(s)|\cdot 2^{3k^+-\frac{1}{2}k^-}2^{-p-\frac{q}{2}}||\mathcal{Q}_{\mathfrak{n}}(f_1,f_2)(s)||_2 ds$$
$$\lesssim 2^{-\delta m}\epsilon^2\int_0^t|\tau_m'(s)|ds \lesssim 2^{-\delta m}\epsilon^2,$$

hence we are done.

The proof above shows that the boundary term for $m=0(t=0)$ and the term with $\tau_m'(s)$ will never be an issue when the other terms can be bounded. Hence, we omit them from the future use of normal forms.

From hereon, we assume $p \sim p_1 \sim p_2$.

3. Gap in q. We localize further in q_1, q_2, and reduce the cases first. For notational convenience, we keep using f_j's for $P_{k_j,p_j,q_j} R_{l_j} F_j$'s. From

$$2^{k+4k^+} ||\mathcal{F}\{\mathcal{B}_{\mathfrak{m}}(f_1, f_2)\}||_\infty \lesssim 2^{2k+4k^+} 2^m \cdot 2^{\frac{q_1}{2}} ||f_1||_B \cdot 2^{\frac{q_2}{2}} ||f_2||_B,$$

we can see that it's enough to consider the cases when $-3m \le q_1, q_2 \le 0$. Also, the assumption $q_{\min} \ll q_{\max} \le 0$ makes $p_{\max} \sim 0$, so that all $2^p, 2^{p_1}, 2^{p_2}$ are comparable to 0 now. Without loss of generality, we assume $q_1 \le q_2$.

① $q \ll q_1, q_2$: By Lemma 5.3.11, we have three possible cases.

3-1 : $2^{k_1} \sim 2^{k_2}$, $2^{q_1} \sim 2^{q_2}$. ntegration by parts gives

$$\begin{aligned} 2^{k+4k^+} ||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty &\lesssim 2^{2k+4k^+} (s^{-1} 2^{2k_1 - k_{\max} - k_{\min} - q_{\max}} \cdot [1 + 2^{k_2 - k_1}(2^{q_2 - q_1} + 2^{l_1})])^N \\ &\quad \cdot ||(1, S)^N f_1||_2 ||(1, S)^N f_2||_2 \\ &\lesssim 2^{2k+4k^+} \cdot (s^{-1} 2^{k_1 - k - q_{\max} + l_1})^N \epsilon^2, \end{aligned}$$

so that if $k_1 - k - q_{\max} + l_1 \le (1-\delta)m$, we are done. Else,

$$\begin{aligned} 2^{k+4k^+} ||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty &\lesssim 2^{k+4k^+} 2^{k+q_{\max}} \cdot 2^{-3k_1^+ + \frac{q_1}{2} - (1+\beta) l_1} ||f_1||_X ||f_2||_2 \\ &\lesssim 2^{2k+k^+} 2^{\frac{3}{2} q_1} 2^{(1+\beta)(k_2 - k) - (1+\beta) q_1 - (1+\beta)(1-\delta) m} \epsilon^2 \\ &\lesssim 2^{-\delta m} \epsilon^2. \end{aligned}$$

3-2 : $2^{k_2} \ll 2^{k_1}$, $2^{q_1 - q_2} \sim 2^{k_2 - k_1} \ll 1$. Here, it's advantageous to see $-q_2$ from integration by parts, so

$$\begin{aligned}2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+}\cdot(s^{-1}2^{2k_2-k_1-k_2-q_2}\cdot[1+2^{k_1-k_2}(2^{q_1-q_2}+2^{l_2})])^N\epsilon^2\\ &\lesssim 2^{2k+4k^+}\cdot(s^{-1}2^{-q_2+l_2})^N\epsilon^2,\end{aligned}$$

so that if $-q_2+l_2\le(1-\delta)m$, we are done. Else,

$$\begin{aligned}2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{k+4k^+}2^{k+q_{\max}}2^{\frac{q_1}{2}}||f_1||_B2^{-3k_2-(1+\beta)l_2}||f_2||_X\\ &\lesssim 2^{2k+k^+}2^{q_2+\frac{1}{2}q_1}2^{-(1+\beta)q_2-(1+\beta)(1-\delta)m}\epsilon^2\lesssim 2^{-\delta m}\epsilon^2.\end{aligned}$$

3-3 : $2^{k_1}\ll 2^{k_2},\ \ 2^{q_2-q_1}\ll 1$. This is against assumption $q_1\le q_2$, so we will not consider this.

② $q_1\ll q,q_2$: Again, by Lemma 5.3.11, we have three scenarios.

4-1 : $2^k\sim 2^{k_2},\ \ 2^q\sim 2^{q_2}$. Integration by parts gives

$$\begin{aligned}2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+}\cdot(s^{-1}2^{2k_2-k_{\max}-k_{\min}-q_{\max}}[1+2^{k_1-k_2}(2^{q_1-q_2}+2^{l_2})])^N\epsilon^2\\ &\lesssim 2^{2k+4k^+}\cdot(s^{-1}[2^{k_2-k_1-q_2}+2^{-q_2+l_2}])^N\epsilon^2,\end{aligned}$$

so that if $k_2-k_1-q_2\le(1-\delta)m$ and $-q_2+l_2\le(1-\delta)m$, we are done. Also,

$$\begin{aligned}2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_\infty &\lesssim 2^{2k+4k^+}\cdot(s^{-1}2^{2k_1-k_{\max}-k_{\min}-q_{\max}}[1+2^{k_2-k_1}(2^{q_2-q_1}+2^{l_1})])^N\epsilon^2\\ &\lesssim 2^{2k+4k^+}\cdot(s^{-1}[2^{-q_1}+2^{-q_2+l_1}])^N\epsilon^2,\end{aligned}$$

so that we are also okay if $-q_1, -q_2 + l_1 \leq (1-\delta)m$. If $q_1 \leq -(1-\delta)m$,

$$2^{3k^+ - \frac{1}{2}k^- - \frac{q}{2}}||\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)||_2 \lesssim 2^{3k^+ - \frac{1}{2}k^- - \frac{q}{2}} 2^{k + \frac{k_1+q_1}{2}} 2^{k+q_2} \cdot 2^{\frac{q_1}{2}} ||f_1||_B \cdot 2^{-(1+\beta)l_2} ||f_2||_X$$
$$\lesssim 2^{\frac{3}{2}k + 4k^+} 2^{q_1 + \frac{q_2+k_1}{2}} 2^{-(1+\beta)l_2} \epsilon^2,$$

and since either $q_2 + k_1 \leq k_2 - (1-\delta)m$ or $q_2 - l_2 \leq -(1-\delta)m$, we have $-\frac{3}{2}(1-\delta)m$ combined with the bound from q_1 and we are done. Lastly, if $q_2 - l_1 \leq -(1-\delta)m$,

$$2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty \lesssim 2^{k+4k^+} 2^{k+q_2} 2^{-(1+\beta)l_1} ||f_1||_X 2^{\frac{q_2}{2}} ||f_2||_B \lesssim 2^{-(1+\delta)m} \epsilon^2.$$

4-2 : $2^k \sim 2^{k_1}$, $2^{q-q_2} \sim 2^{k_2-k} \ll 1$. From

$$2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty \lesssim 2^{2k+4k^+} \cdot (s^{-1} 2^{2k_2 - k_{\max} - k_{\min} - q_{\max}} [1 + 2^{k_1-k_2}(2^{q_1-q_2} + 2^{l_2})])^N \epsilon^2$$
$$\lesssim 2^{2k+4k^+} \cdot (s^{-1} 2^{-q_2+l_2})^N \epsilon^2,$$

so that if $-q_2 + l_2 \leq (1-\delta)m$, we are done. Else,

$$2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty \lesssim 2^{k+4k^+} \cdot 2^{k+q_2} \cdot 2^{q_1} ||f_1||_B \cdot 2^{-(1+\beta)l_2} ||f_2||_B$$
$$\lesssim 2^{2k+4k^+} 2^{(1-\beta)q_1} 2^{(1+\beta)(q_2 - l_2)} \epsilon^2 \lesssim 2^{-\delta m} \epsilon^2,$$

since $q_1 \leq q_2$.

4-3 : $2^{k_1} \sim 2^{k_2}$, $2^{q_2-q} \sim 2^{k-k_2} \ll 1$. Again, integration by parts gives,

$$2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty \lesssim 2^{2k+4k^+} \cdot (s^{-1} 2^{2k_2 - k_{\max} - k_{\min} - q_{\max}} [1 + 2^{k_1-k_2}(2^{q_1-q_2} + 2^{l_2})])^N \epsilon^2$$
$$\lesssim 2^{2k+4k^+} \cdot (s^{-1} 2^{k_2 - k - q + l_2})^N \epsilon^2,$$

so that if $k_2 - k - q + l_2 \le (1-\delta)m$, we are done. Else, from $q_1 \le q$,

$$2^{k+4k^+}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1, f_2)\}||_\infty \lesssim 2^{k+4k^+} \cdot 2^{k+q} \cdot 2^{\frac{q_1}{2}}||f_1||_B \cdot 2^{-(1+\beta)l_2}||f_2||_X$$
$$\lesssim 2^{(1-\beta)k+4k^+ + (\frac{1}{2}-\beta)q_1} 2^{(1+\beta)(k+q-l_2)}\epsilon^2 \lesssim 2^{-\delta m}\epsilon^2.$$

③ $q_2 \ll q, q_1$: This case is against our assumption $q_1 \le q_2$, hence is excluded.

④ $q_1 \sim q_2 \ll q$: By Lemma 5.3.12, we get $k_1 \sim k_2 \gg k$. In this case, we have $|\Phi| \gtrsim 2^q$, so that we can use the normal form. One can see that the terms with $\partial_t - \kappa\Delta$ can be treated in the same way as in (5.13). The boundary term is even easier to bound by

$$2^{3k^+ - \frac{1}{2}k^- - \frac{q}{2}}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{n}}(f_1, f_2)\}||_2 \lesssim 2^{3k^+ - \frac{1}{2}k^- - \frac{q}{2}}||\mathcal{F}\mathfrak{n}||_1||e^{it\Lambda}f_1||_\infty 2^{\frac{q_2}{2}}||f_2||_B \lesssim 2^{-\delta m}\epsilon^3.$$

⑤ $q \sim q_1 \ll q_2$: Lemma 5.3.12 gives $k \sim k_1 \gg k_2$, and $|\Phi| \gtrsim 2^{q_2}$, so similarly as above,

$$2^{3k^+ - \frac{1}{2}k^- - \frac{q}{2}}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{n}}(f_1, f_2)\}||_2 \lesssim 2^{3k^+ - \frac{1}{2}k^- - \frac{q}{2}}||\mathcal{F}\mathfrak{n}||_1 2^{\frac{q_1}{2}}||f_1||_B||e^{it\Lambda}f_2||_\infty \lesssim 2^{-\delta m}\epsilon^3,$$

and the other terms follow by the same reasoning.

⑥ $q \sim q_2 \ll q_1$: This case is excluded from our assumption $q_1 \le q_2$.

Hence, from hereon, we further assume $q \sim q_1 \sim q_2$.

4. No gaps. Assume without loss of generality that f_1 has at most $\frac{N}{2}$ copies of the vector field S. Using Proposition 2.5.1, we can decompose $P_{k_1,p_1,q_1}e^{it\Lambda}f_1$ into two terms such that

$$||I||_\infty \lesssim 2^{-p_1 - \frac{q_1}{2}} t^{-\frac{3}{2}} 2^{\frac{3}{2}k_1 - 3k_1^+}\epsilon, \qquad ||II||_2 \lesssim 2^{3k_1^+}t^{-1/2}\epsilon,$$

for each k_1, p_1, q_1 that are leftover, and we have

$$2^{3k^+-\frac{1}{2}k^--\frac{q}{2}}||\mathcal{F}\{\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)\}||_2 \lesssim 2^{3k^+-\frac{1}{2}k^--\frac{q}{2}}2^{k+p+q}[||I||_\infty 2^{p+\frac{q}{2}}||f_2||_B + ||II||_2||e^{it\Lambda}f_2||_\infty]$$
$$\lesssim 2^{\frac{k}{2}+4k^++\frac{q}{2}}[2^{-\frac{3}{2}m}\epsilon^2 + 2^{3k_1^+}2^{-\frac{1}{2}m}2^{-\frac{2}{3}m}\epsilon^2] \lesssim 2^{-\frac{13}{12}m}\epsilon^2,$$

hence we are finished.

□

5.5 X-norm bounds

To bound the X-norm, first note that again the bound of the linear term is straightforward from the initial condition and $S^n(e^{t\kappa\Delta}\mathcal{U}_\pm(0)) = \sum_{0\le a\le n} h_a(t\kappa\Delta)e^{t\kappa\Delta}S^{n-a}\mathcal{U}_\pm(0)$. Hence, this section is devoted to controlling the nonlinear term in the Duhamel's formula.

5.5.1 Large l case

We first bound the X norm in the case when the parameter l is larger than m. To be precise, we prove the following.

Proposition 5.5.1. *Assume that the bootstrap assumption* (5.8) *holds, and let* $\delta = 2M^{-1/2} \ll \beta = \frac{1}{60}$. *Then, for* $F_j = S^{b_j}\mathcal{U}_\pm, 0 \le b_1 + b_2 \le N, j = 1,2$, *we have*

$$\sup_{k\in\mathbb{Z}, l+p\ge 0, l\ge(1+\delta)m} 2^{3k^+}2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(F_1,F_2)\|_{L^2} \lesssim 2^{-\delta^2 m}\epsilon^2. \quad (5.14)$$

The rest of this subsection will be the proof of this proposition. As in the previous section, $\mathcal{B}_{\mathfrak{m}}$ should be modified so that we have $h_n((t-s)\kappa\Delta)e^{(t-s)\kappa\Delta}$'s instead of $e^{(t-s)\kappa\Delta}$, but it will be immaterial as they mostly have same qualitative effect, and in particular have bounded operator norms. They will manifest their presence only when we perform normal forms, where the effects were already captured in Corollary 5.3.6.

5.5.1.1 When $l + p \le \delta m$

We first restrict the ranges of indices involved as in the previous section. By the energy estimates,

$$
\begin{aligned}
&2^{3k^+}2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l S^n \mathcal{B}_{\mathfrak{m}}(f_1, f_2)\|_{L^2} \\
\lesssim &\sum_{\substack{a+b+c\le n\\ k_1,k_2}} 2^{3k^+}2^{(1+\beta)l}2^{\beta p}\cdot 2^m \cdot ||h_a((t-s)\kappa|\xi|^2)e^{-(t-s)\kappa|\xi|^2}||_{L^\infty_\xi}\cdot 2^{\frac{3}{2}k+p} \\
&\qquad \cdot \min\{2^{-N_0k_1}||S^b f_1||_{H^{N_0}}, 2^{k_1}||S^b f_1||_{\dot{H}^{-1}}\}\cdot \min\{2^{-N_0k_2}||S^c f_2||_{H^{N_0}}, 2^{k_2}||S^c f_2||_{\dot{H}^{-1}}\} \\
\lesssim &\ 2^{(1+\beta)\delta m+m}2^{3k^+ + \frac{3}{2}k}\cdot \min\{2^{-N_0k_1}, 2^{k_1}\}\cdot \min\{2^{-N_0k_2}, 2^{k_2}\}\epsilon^2.
\end{aligned}
$$

Recalling that the biggest parameters in k, k_1, k_2 always come at least as a pair, we see that the summation gives (5.14) unless $-2m \le k, k_1, k_2 \le \delta_0 m$, where $\delta_0 = 2N_0^{-1}$ again. Similarly,

$$
\begin{aligned}
&2^{3k^+}2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l S^n \mathcal{B}_{\mathfrak{m}}(f_1, f_2)\|_{L^2} \\
\lesssim &\sum_{\substack{a+b+c\le n\\ p_1,p_2\le 0\le p_1+l_1,p_2+l_2}} 2^{3k^+}2^{(1+\beta)l}2^{\beta p}\cdot 2^m \cdot ||h_a((t-s)\kappa|\xi|^2)e^{-(t-s)\kappa|\xi|^2}||_{L^\infty_\xi} \\
&\qquad \cdot 2^{\frac{3}{2}k+p}\cdot \min\{2^{p_1}||S^b f_1||_B, 2^{-l_1}||S^b f_1||_X\}\cdot \min\{2^{p_2}||S^c f_2||_B, 2^{-l_2}||S^c f_2||_X\} \\
\lesssim &\ 2^{(1+\beta)\delta m+m+5\delta_0 m}\cdot \min\{2^{p_1}, 2^{-l_1}\}\cdot \min\{2^{p_2}, 2^{-l_2}\}\epsilon^2
\end{aligned}
$$

gives (5.14) if we limit the summation to the complement of $-2m \le p_1, p_2 \le 0$ and $-p_j \le l_j \le 2m, j = 1, 2$. Hence, from hereon, we assume

$$
-2m \le k, k_j \le \delta_0 m, \quad -2m \le p_j \le 0, \quad -p_j \le l_j \le 2m, \quad j = 1, 2,
$$

and hence ignore 2^{3k^+} in front. We divide into a few cases according to the indices.

① $p_1, p_2 \ll 0$: Since we already have $p \le \delta m - l \le -m$, this means $q \sim q_1 \sim q_2 \sim 0$, and hence we can use the normal form. Terms with the time derivative can be bounded by

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}((\partial_t-\kappa\Delta)f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta)l}2^{\beta p}\cdot 2^m\cdot 2^k\cdot 2^{\frac{3}{2}k+p}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot\epsilon\\
&\lesssim 2^{(-\frac{1}{2}+\gamma+2\delta+3\delta_0)m}\epsilon^3,
\end{aligned}$$

and symmetrically for the term with $(\partial_t-\kappa\Delta)f_2$. The boundary term needs more care. WLOG, assume $p_2\le p_1$. When $p_2\lesssim p$,

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta)(l+p)}2^{-p}\cdot 2^k\cdot ||e^{it\Lambda}f_1||_\infty\cdot 2^{p_2}||f_2||_B\\
&\lesssim 2^{\delta m+\delta_0 m}\cdot 2^{-m+\gamma m}\epsilon\cdot 2^{p_2-p}\epsilon\\
&\lesssim 2^{(-1+\gamma+\delta+\delta_0)m}\epsilon^2.
\end{aligned}$$

So now, we assume $p\ll p_2\le p_1$. We integrate by parts using the fact that $|\bar{\sigma}|\sim 2^{p_1+k_{\max}+k_{\min}}$ in this region. This will be enough if

$$2^{-2p_1+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1+l_1})\le 2^{(1-\delta)m},$$

so we look at the complement cases. If $k=k_{\min}$, then $k_1\sim k_2$, so this means we can assume $2p_1+k-k_1-l_1\le-(1-\delta)m$. Hence,

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta)l}2^{\beta p}\cdot 2^k\cdot 2^{\frac{3}{2}k+p}\cdot 2^{-l_1}||f_1||_X\cdot 2^{p_2}||f_2||_B\\
&\lesssim 2^{(1+\beta)\delta m}\cdot 2^{\frac{1}{2}(2p_1+k-l_1)}2^{2k-\frac{l_1}{2}}\epsilon^2\\
&\lesssim 2^{(-\frac{1}{2}+2\delta+2\delta_0)m}\epsilon^2.
\end{aligned}$$

If $k_{\min}=k_1$ or k_2, this means that we can assume $2p_1-l_1\le-(1-\delta)m$ or $2p_1+k_2-k_1\le-(1-\delta)m$ depending on the minimum k_j. By changing the set size estimate from $2^{\frac{3}{2}k+p}$ to $2^{k+p+\frac{k_{\min}}{2}}$, we can close the case in exactly the same way as above.

② $p_1 \sim 0$: The other case $p_2 \sim 0$ will follow by a symmetric argument. Here, we can benefit from integration by parts as now we have $|\bar{\sigma}| \sim 2^{k_{\max}+k_{\min}}$. Hence, integration by parts will be enough if

$$2^{-p_1+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1+l_1}) \leq 2^{(1-\delta)m},$$

so we look at the complement cases again. If $k = k_{\min}$, then $k_1 \sim k_2$, so this means that we can assume $k - k_1 - l_1 \leq -(1-\delta)m$. Hence,

$$\begin{aligned} 2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta)l}2^{\beta p}\cdot 2^k \cdot 2^{\frac{3}{2}k+p}\cdot 2^{-l_1}||f_1||_X \cdot ||f_2||_2 \\ &\lesssim 2^{(1+\beta)\delta m}\cdot 2^{k-l_1}2^{\frac{3}{2}k}\epsilon^2 \\ &\lesssim 2^{(-1+2\delta+2\delta_0)m}\epsilon^2. \end{aligned}$$

If $k_{\min} = k_1$ or k_2, we can assume $-l_1 \leq -(1-\delta)m$ or (depending on the minimum k_j) $k_2 - k_1 \leq -(1-\delta)m$. The additional k_2 (if needed) can be obtained by using $\dot{H}^{-1}$ norm for f_2, and we can close the case in the same way as above.

5.5.1.2 When $l + p \geq \delta m$

In this case, we have to deal with the largest parameter l directly. Hence, the reduction of the indices will be done in terms of l. From the energy estimates, we have

$$\begin{aligned} &2^{3k^+}2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2} \\ &\lesssim \sum_{k_1,k_2} 2^{3k^+}2^{(1+\beta)(l+p)}\cdot 2^m \\ &\qquad \cdot 2^{\frac{5}{2}k}\cdot \min\{2^{-N_0k_1}||f_1||_{H^{N_0}}, 2^{k_1}||f_1||_{\dot{H}^{-1}}\}\cdot \min\{2^{-N_0k_2}||f_2||_{H^{N_0}}, 2^{k_2}||f_2||_{\dot{H}^{-1}}\} \\ &\lesssim\ 2^{(1+\beta)(l+p)+m}2^{3k^++\frac{3}{2}k}\cdot \min\{2^{-N_0k_1},2^{k_1}\}\cdot \min\{2^{-N_0k_2},2^{k_2}\}\epsilon^2. \end{aligned}$$

By the same reasoning as in the case **5.5.1.1**, this already gives the bound (5.14) unless $-2l \leq k, k_1, k_2 \leq \delta_0 l$. Similar repetitions with the use of the B and X norms enable us to reduce the ranges to

$$-2l \le k, k_j \le \delta_0 l, \quad -2l \le p_j \le 0, \quad -p_j \le l_j \le 2l, \quad j = 1, 2.$$

From a crude estimate

$$\begin{aligned}
&2^{3k^+} 2^{(1+\beta)l} 2^{\beta p} \| P_{k,p} R_l \mathcal{B}_{\mathfrak{m}}(f_1, f_2) \|_{L^2} \\
&\lesssim 2^{3k^+} 2^{(1+\beta)l} 2^{\beta p} \cdot 2^m \cdot 2^{k+p_{\max}} \cdot 2^{\frac{k_1+p_1}{3} + \frac{2(k_2+p_2)}{3} + \frac{1}{2}k_2} \cdot ||f_1||_2 ||f_2||_2 \\
&\lesssim 2^m 2^{(1+\beta)(l-l_1-l_2)} 2^{\beta p + (\frac{1}{3}-\beta)p_1 + (\frac{2}{3}-\beta)p_2 + p_{\max}} 2^{\frac{4}{3}k_{\max} + \frac{7}{6}k_2} \epsilon^2,
\end{aligned}$$

if

$$l_1 + l_2 \ge (1+\delta^2)l + (1 - \frac{\beta}{2})m + k_2, \tag{5.15}$$

we have the desired bound. Now we have to establish a subtly different reasoning to extract the larger parameter l instead of m from the integration by parts. Fortunately, the estimates are not so different from the Euler-Coriolis case, since we will use

$$R_l^{(j)} \mathcal{Q}_{\mathfrak{m}}(f_1, f_2) = 2^{-2l} R_l^{(j+1)} \Omega_{a3}^2 \mathcal{Q}_{\mathfrak{m}}(f_1, f_2).$$

Recall that $\Omega_{a3} = x_a \partial_3 - x_3 \partial_a$. Ω_{a3} acts on the bilinear term in the same way as in the Euler-Coriolis case, thanks to $\Omega_{a3}^{\xi} |\xi| = 0$, so that $\Omega_{a3}^{\xi}(h_n(-(t-s)\kappa|\xi|^2) e^{-(t-s)\kappa|\xi|^2}) = 0$. Here, $R_l^{(j)}$s are all qualitatively similar localizations of $R_l = R_l^{(0)}$. Hence, we only need to estimate the terms where Ω_{a3}^{ξ} hits the phase, multiplier, or input functions, which are the same terms as in the Euler-Coriolis case. To be more explicit, we will have terms of the form

$$\begin{aligned}
\int_0^t (h_n(-(t-s)\kappa|\xi|^2) e^{-(t-s)\kappa|\xi|^2} \int_{\mathbb{R}^3_\eta} &(\Omega_{a3}^{\xi})^{n_1} P_{k,p}(\xi) \cdot (\Omega_{a3}^{\xi})^{n_2} \mathfrak{m} \cdot (\Omega_{a3}^{\xi})^{n_3} e^{is\Phi} \\
&\cdot (\Omega_{a3}^{\xi})^{n_4} P_{k_1,p_1}(\xi-\eta) \cdot (\Omega_{a3}^{\xi})^{n_5} \hat{f}_1(\xi-\eta) \cdot \hat{f}_2(\eta) d\eta ds,
\end{aligned} \tag{5.16}$$

where $n_1+n_2+n_3+n_4+n_5 \le K$ for some K. To change the derivatives in ξ to those of $\xi-\eta$, we introduce

$$\Omega^{\xi}_{a3} = \Gamma_S S^{\xi-\eta} + \Gamma_\Omega \Omega^{\xi-\eta}_{a3},$$
$$\Gamma_S := \frac{\xi_a(\xi-\eta)_3 - \xi_3(\xi-\eta)_a}{|\xi-\eta|^2}, \quad \Gamma_\Omega := \frac{(\xi-\eta)_a\xi_a + (\xi-\eta)_3\xi_3}{|\xi-\eta|^2}$$

which satisfy $|\Gamma_S|, |\Gamma_\Omega| \lesssim 2^{k-k_1}$. Direct computations to the 'elementary components' give

$$\begin{aligned}
\Omega^{\xi}_{a3}\frac{(\xi-\eta)_a}{|\xi-\eta|} &= -\Gamma_\Omega\frac{(\xi-\eta)_3}{|\xi-\eta|},\\
\Omega^{\xi}_{a3}\frac{(\xi-\eta)_3}{|\xi-\eta|} &= \Gamma_\Omega\frac{(\xi-\eta)_a}{|\xi-\eta|},\\
\Omega^{\xi}_{a3}\frac{\xi_a}{|\xi-\eta|} &= -\frac{\xi_3}{|\xi-\eta|} - \Gamma_S\frac{\xi_a}{|\xi-\eta|},\\
\Omega^{\xi}_{a3}\frac{\xi_3}{|\xi-\eta|} &= \frac{\xi_a}{|\xi-\eta|} - \Gamma_S\frac{\xi_3}{|\xi-\eta|},
\end{aligned}$$

and hence,

$$\begin{aligned}
\Omega^{\xi}_{a3}\Gamma_S &= -\Gamma_\Omega - \Gamma_S^2 + \Gamma_\Omega^2,\\
\Omega^{\xi}_{a3}\Gamma_\Omega &= \Gamma_S(1-2\Gamma_\Omega),
\end{aligned}$$

so that they form an algebra under the operation Ω^{ξ}_{a3}, where each operation increases the order by 1. We also need to know how $\Lambda(\xi)$, $\sqrt{1-\Lambda^2(\xi)}$, $\frac{\xi_h\cdot\theta_h}{|\xi_h||\theta_h|}$, $\frac{\xi_h^{\perp}\cdot\theta_h}{|\xi_h||\theta_h|}$, $\theta \in \{\xi-\eta, \eta\}$

change under Ω^{ξ}_{a3}. Again, the action of Ω_{a3} on 'elementary components' yields

$$
\begin{aligned}
\Omega^{\xi}_{a3}\Lambda(\xi) &= \frac{\xi_a}{|\xi|},\\
\Omega^{\xi}_{a3}\frac{\xi_a}{|\xi|} &= -\Lambda(\xi),\\
\Omega^{\xi}_{a3}\sqrt{1-\Lambda^2(\xi)} &= -(1-\Lambda^2(\xi))^{-\frac{1}{2}}\frac{\xi_a}{|\xi|},\\
\Omega^{\xi}_{a3}\frac{\xi_a}{|\xi_h|} &= -\frac{\xi_3}{|\xi_h|}+\frac{\xi_3\xi_a^2}{|\xi_h|^3},\\
\Omega^{\xi}_{a3}\frac{\xi_b}{|\xi_h|} &= \frac{\xi_a\xi_b\xi_3}{|\xi_h|^3},\\
\Omega^{\xi}_{a3}\frac{\xi_3}{|\xi_h|} &= \frac{\xi_a}{|\xi_h|}+\frac{\xi_a\xi_3^2}{|\xi_h|^3},
\end{aligned}
$$

so that these form an algebra again. Using these to estimate the terms in (5.16), we have

$$
\begin{aligned}
|(\Omega^{\xi}_{a3})^{n_1}P_{k,p}(\xi)| &\lesssim (2^{-p})^{n_1},\\
|(\Omega^{\xi}_{a3})^{n_2}\mathfrak{m}| &\lesssim 2^{k+p_{\max}}(1+2^{-p}+2^{k-k_1-p_1})^{n_2}\\
|(\Omega^{\xi}_{a3})^{n_4}P_{k_1,p_1}(\xi-\eta)| &\lesssim (2^{k-k_1-p_1})^{n_4}\\
|(\Omega^{\xi}_{a3})^{n_5}\hat{f}_1(\xi-\eta)| &\lesssim (2^{k-k_1})^{n_5}S^{j_1}\Omega^{j_2}_{a3}\hat{f}_1(\xi-\eta), \quad j_1+j_2\le n_5.
\end{aligned}
$$

The phase term is more complicated, as it changes in different patterns depending on whether Ω^{ξ}_{a3} acts on $e^{is\Phi}$ or on $(\Omega^{\xi}_{a3})^n\Phi$. Instead of an almost exact estimate, we use

$$
|\Omega^{\xi}_{a3}\Phi| \lesssim 2^p+2^{k-k_1+p_1}, \quad |(\Omega^{\xi}_{a3})^n\Phi| \lesssim 1+(2^{k-k_1})^n
$$

to obtain

$$
\begin{aligned}
|(\Omega^{\xi}_{a3})^{n_3}e^{is\Phi}| &\lesssim \sum_{j=1}^{n_3}[s(2^p+2^{k-k_1+p_1})]^j(1+(2^{k-k_1})^{n_3-j})+\sum_{j=1}^{n_3/2}s^j(1+(2^{k-k_1})^{n_3})\\
&\le [s(2^p+2^{k-k_1+p_1})+1+2^{k-k_1}]^{n_3}+s^{n_3/2}(1+2^{k-k_1})^{n_3}.
\end{aligned}
$$

Combining all of these, modulo lower order terms, we finally achieve

$$
\begin{aligned}
&||R_l^{(n)}\mathcal{Q}_{\mathfrak{m}}(f_1,f_2)||_2 \\
&\lesssim 2^{\frac{3}{2}k_{\min}}2^{k+p_{\max}}\sum_{j=0}^{1}2^{-Kl}[2^{-p}+2^m(2^p+2^{k-k_1+p_1})+2^{m/2}(1+2^{k-k_1})+2^{k-k_1+l_1}]^K||S^jf_1||_2||f_2||_2 \\
&+2^{\frac{3}{2}k_{\min}}2^{k+p_{\max}}2^{-2l}(2^{-l}[2^{-p}+2^m(2^p+2^{k-k_1+p_1})+2^{m/2}(1+2^{k-k_1})+2^{k-k_1+l_1}])^{K'-2}||S^2f_1||_2||f_2||_2,
\end{aligned}
$$

by stopping integration by parts if f_1 gets two S vector fields applied.

This shows that integration by parts along Ω_{a3} is enough when $2^{-p}+2^m(2^p+2^{k-k_1+p_1})+2^{m/2}(1+2^{k-k_1})+2^{k-k_1+l_1}<(1-\frac{\delta}{2})l$. In the complement range, we first remove the case where $-p$ is the dominant index. In this case,

$$
\begin{aligned}
&2^{3k^+}2^{(1+\beta)l}2^{\beta p}||P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)||_{L^2} \\
&\lesssim \sum_{\substack{k_1,p_1,l_1,\\ k_2,p_2,l_2}} 2^{3k^++\frac{5}{2}k}2^{(1+\beta-K)(l+p)}\cdot 2^m\cdot\min\{2^{-N_0k_2},2^{k_2},2^{p_2},2^{-l_1}\}\cdot\min\{2^{-N_0k_2},2^{k_2},2^{p_2},2^{-l_2}\}\epsilon^2 \\
&\lesssim 2^{-(1+\beta-K)\delta m+m}\epsilon^2,
\end{aligned}
$$

which can give an acceptable contribution when K is taken large enough. Hence, we drop 2^{-p} in the front from hereon.

After obtaining the analogous result for both $\xi-\eta$ and η, and hence assuming $k_1\geq k_2$ without loss of generality, we are left with the cases where

$$
l_1\geq(1-\delta^2)l,\quad k-k_2+\max\{m+p_2,\frac{m}{2},l_2\}\geq(1-\delta^2)l.
$$

If l_2 is the biggest, then (5.15) is satisfied, and we are done. If $m+p_2$ is the biggest, we have $k-k_2+p_2\geq\delta m/2$, and integration by parts is enough if

$$
2^{-p_2+2k_2-p_{\max}-k_{\max}-k_{\min}}(1+2^{k_1-k_2+l_2})\sim 2^{-p_2-p_{\max}+k_2-k_{\min}+l_2}\leq 2^{(1-3\delta)m},
$$

since we now have a bound on l in terms of m, $l \leq (1+2\delta^2)(m+p_2+k-k_2)$, so that we can bound $2^{(1+\beta)l+m}$ through iterated integration by parts. Finally, we assume $p_2+p_{\max}+k_{\min}-k_2-l_2 \leq -(1-3\delta)m$. The harder case is when $k_{\min}=k$ so that $k_2-k_{\min}$ does not vanish. In this case,

$$\begin{aligned}
&2^{3k^+}2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2}\\
&\lesssim 2^{3k^+}2^{(1+\beta)l}2^{\beta p}\cdot 2^m\cdot 2^{k+p_{\max}}\cdot 2^{k_1+p_1+\frac{1}{2}k}\cdot 2^{-3k_1^+-(1+\beta)l_1-\beta p_1}||f_1||_X\cdot 2^{-(1+\beta)l_2-\beta p_2}||f_2||_X\\
&\lesssim 2^m\cdot 2^{k_1+\frac{3}{2}k}2^{(1+\beta)(l-l_1)}2^{(1+\beta)(-p_2-p_{\max}+k_2-k-(1-3\delta)m)}2^{\beta(p-p_1-p_2)+p_1+p_{\max}}\epsilon^2\\
&\lesssim 2^m\cdot 2^{k_2+\frac{3}{2}k}2^{\delta^2 l}2^{(1+\beta)(k_2-k)-(1+\beta)m+6\delta m}2^{\beta(p-p_{\max})-(1+2\beta)p_2}\epsilon^2\\
&\lesssim 2^{(-\beta+6\delta)m}\cdot 2^{2\delta^2(m+k-k_2)}\cdot 2^{\frac{5}{2}k_2}2^{(1+2\beta)(k-k_2-\delta m/2)}\epsilon^2\\
&\lesssim 2^{(-\beta+6\delta)m}\epsilon^2.
\end{aligned}$$

Lastly, when $\frac{m}{2}$ is the biggest, we have $k-k_2 \geq \frac{1}{2}l$. Thus,

$$\begin{aligned}
&2^{3k^+}2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2}\\
&\lesssim 2^{3k^+}2^{(1+\beta)l}2^{\beta p}\cdot 2^m\cdot 2^{k+p_{\max}}\cdot 2^{\frac{3}{2}k_2+p_2}\cdot 2^{-3k_1^+-l_1}||f_1||_X\cdot 2^{k_2}||f_2||_{\dot{H}^{-1}}\\
&\lesssim 2^m\cdot 2^{\beta l+(l-l_1)}2^{k+\frac{5}{2}k_2}\epsilon^2\\
&\lesssim 2^m\cdot 2^{\beta l+\delta^2 l}\cdot 2^{\delta_0 l-\frac{5}{4}l+\frac{5}{2}k}\epsilon^2\\
&\lesssim 2^{(-\frac{1}{4}+\beta+\delta^2+4\delta_0)l}\epsilon^2.
\end{aligned}$$

This completes the case with large l.

5.5.2 Small l case

We investigate the case where $l \leq (1+\delta)m$. Now m plays the role of large parameter again, and we can always use $(1+\beta)l \leq (1+\beta+2\delta)m$ when estimating the X norm.

Proposition 5.5.2. *Assume that the bootstrap condition* (5.8) *holds, and let* $\delta =$

$2M^{-1/2} \ll \beta \leq \frac{1}{60}$. *Then for* $F_j = S^{b_j}\mathcal{U}_\pm$, $0 \leq b_1 + b_2 \leq N, j = 1, 2$, *we have*

$$\sup_{k\in\mathbb{Z}, l+p\geq 0, l\leq(1+\delta)m} 2^{3k_+}2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(F_1, F_2)\|_{L^2} \lesssim 2^{-\delta^2 m}\epsilon^2. \tag{5.17}$$

The remainder of the subsection will be the proof of this proposition, and the proof will follow a pattern similar to that of the proof of B-norm bounds. The difference is that the estimates have to be more delicate as the $2^{(1+\beta)l}$ weight is harder to control.

Through the usual reduction process, we can assume that

$$-2m \leq k, k_j \leq \delta_0 m, \quad -2m \leq p_j \leq 0, \quad -p_j \leq l_j \leq 2m, \quad j = 1, 2$$

Now we subdivide the cases according to the sizes of and relations between the indices as in the previous section.

5.5.2.1 Gap in p with $p_{\max} \sim 0$

We consider the case where $p_{\min} \ll p_{\max} \sim 0$. WLOG, we may assume $p_1 \leq p_2$, which leaves us with 4 different cases.

① $p \ll p_1, p_2$: We invoke Lemma 5.3.11 again.

1-1) $k_1 \sim k_2$ and $p_1 \sim p_2 \sim 0$: Iterated integration by parts gives the result if $2^{k_1-k+l_1} \lesssim 2^{(1-\delta)m}$, or $2^{k_2-k+l_2} \lesssim 2^{(1-\delta)m}$. In other cases,

$$\begin{aligned} 2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1, f_2)\|_{L^2} &\lesssim 2^{3k_+}2^{(1+\beta)l}2^{\beta p} \cdot 2^m \cdot 2^k \cdot 2^{\frac{3}{2}k+p} \cdot 2^{-(1+\beta)l_1}\|f_1\|_X \cdot 2^{-(1+\beta)l_2}\|_X \\ &\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{\frac{5}{2}k} \cdot 2^{-2(1+\beta)(k_1-k-(1-\delta)m)}\epsilon^2 \\ &\lesssim 2^{-(\beta-6\delta)m}\epsilon^2, \end{aligned}$$

which is acceptable since $\delta \ll \beta$.

1-2) $k_2 \ll k_1 \sim k$, and $p \ll p_1 \ll p_2 \sim 0$: Iterated integration by parts is enough if either $l_2 \le (1-\delta)m$ or $-p_1 + \max\{k_1 - k_2, l_1\} \le (1-\delta)m$. In the opposite case, if $k_1 - k_2 \ge l_1$,

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta)l}2^{\beta p}\cdot 2^m \cdot 2^k \cdot 2^{\frac{3}{2}k_2+p_2}\cdot 2^{p_1}||f_1||_B \cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+\frac{3}{2}k_2}\cdot 2^{(1+\beta)p_1}\cdot 2^{-(1+\beta)l_2}\epsilon^2 \\
&\lesssim 2^{(2+\beta+2\delta)m}2^{k+(1+\beta)k_1+(\frac{1}{2}-\beta)k_2}2^{-(1+\beta)(1-\delta)m}2^{-(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{-(\beta-6\delta)m}\epsilon^2.
\end{aligned}$$

If $k_1 - k_2 \le l_1$, using $2^{k_2} \sim 2^{k_1+p_1}$ in addition,

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(1+\beta)l}2^{\beta p}\cdot 2^m\cdot 2^k\cdot 2^{\frac{1}{2}k_2+k_1+p_1}\cdot 2^{-(1+\beta)l_1-\beta p_1}||f_1||_X\cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+\frac{3}{2}k_1+\frac{3}{2}p_1}\cdot 2^{-(1+\beta)l_1}2^{-(1+\beta)l_2}\epsilon^2 \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{3\delta_0 m}2^{(\frac{3}{2}-\beta)p_1}2^{-2(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{-(\beta-6\delta)m}\epsilon^2.
\end{aligned}$$

1-3) $p \ll p_2 \ll p_1$: This case is excluded from the assumption $p_1 \le p_2$.

② $p_1 \ll p_2, p$. Again, we use Lemma 5.3.11 to subdivide cases.

2-1) $k \sim k_2$, and $p \sim p_2 \sim 0$: Integration by parts is enough if $-p_1 + l_1 \le (1-\delta)m$ or $\max\{k_2 - k_1, l_2\} \le (1-\delta)m$. In other cases, when $k_2 - k_1 \ge l_2$,

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta)l}\cdot 2^m\cdot 2^k\cdot 2^{\frac{3}{2}k_1+p_1}\cdot 2^{-(1+\beta)l_1-\beta p_1}||f_1||_X\cdot\|f_2\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}2^{\frac{5}{2}k_2-2\beta l_1}2^{-(1-\beta)(1-\delta)m}2^{-\frac{3}{2}(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{2}-2\beta-5\delta)m}\epsilon^2,
\end{aligned}$$

which gives an acceptable bound. When $k_2 - k_1 \le l_2$, we need to divide the cases even further. If $2p_1 + k \gtrsim k_1$, we can get the extra p_1 from k_1:

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta)l}\cdot 2^m \cdot 2^k \cdot 2^{\frac{3}{2}k_1+p_1}\cdot 2^{-(1+\beta)l_1-\beta p_1}||f_1||_X \cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}2^{\frac{3}{2}k+k_1}\cdot 2^{-(1+\beta)l_1+(2-\beta)p_1}2^{-(1+\beta)l_2}\epsilon^2 \\
&\lesssim 2^{(2+\beta+2\delta+3\delta_0)m}2^{-2(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{-(\beta-6\delta)m}\epsilon^2.
\end{aligned}$$

Otherwise, we use a normal form. From $\partial_{\eta_3}\Phi = \pm\frac{|\eta_h|^2}{|\eta|^3} \pm \frac{|\xi_h-\eta_h|^2}{|\xi-\eta|^3}$ and $\frac{|\eta_h|^2}{|\eta|^3} \sim 2^{2p_2-k_2} \sim 2^{-k_2}$, $\frac{|\xi_h-\eta_h|^2}{|\xi-\eta|^3} \sim 2^{2p_1-k_1}$, we have $|\partial_{\eta_3}\Phi| \gtrsim 2^{-k_2}$. We choose $\lambda = 2^{-10\delta m}$ as the cutoff for the resonant term and use Lemma 5.3.9:

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}^{res}}(f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(1+\beta)l}\cdot 2^m\cdot 2^k\cdot 2^{k_1+p_1}\cdot(2^{-10\delta m}2^{k_2})^{1/2}\cdot 2^{-l_1}||f_1||_X\cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}2^{\frac{3}{2}k+k_1}2^{-5\delta m}2^{p_1-l_1}2^{-(1+\beta)l_2}\epsilon^2 \\
&\le 2^{[(4+\beta)\delta+3\delta_0-5\delta]m}\epsilon^2,
\end{aligned}$$

which gives an acceptable contribution. For the nonresonant term, we can bound each term after using the normal form. We use another version of Lemma 5.3.9 for the boundary term:

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(1+\beta)l}|\log(2^{-10\delta m})|\cdot 2^k\cdot 2^{k_1+p_1}\cdot(2^{-10\delta m}2^{-k_2})^{-1/2}\cdot 2^{-l_1}||f_1||_X\cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 10\delta m\cdot 2^{\frac{3}{2}k+k_1}2^{5\delta m}2^{p_1-l_1}2^{-(1+\beta)l_2}\epsilon^2 \\
&\lesssim 2^{-(1-20\delta)m}\epsilon^2.
\end{aligned}$$

The other terms can be treated using Lemma 5.3.5 :

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}((\partial_t-\kappa\Delta)f_1,f_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+10\delta m}\cdot 2^{\frac{3}{2}k_1+p_1}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot 2^{-(1+\beta)l_2}||f_2||_X\\
&\lesssim 2^{-(\frac{1}{2}-\gamma-16\delta)m}\epsilon^3,
\end{aligned}$$

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}(f_1,(\partial_t-\kappa\Delta)f_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+10\delta m}\cdot 2^{\frac{3}{2}k_1+p_1}\cdot 2^{-l_1}||f_1||_X\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\\
&\lesssim 2^{-(\frac{1}{2}-\beta-\gamma-16\delta)m}\epsilon^3,
\end{aligned}$$

which is enough since $\frac{1}{3}>\beta\gg\gamma$.

2-2) $k_2\ll k_1\sim k$, and $p_1\ll p\ll p_2\sim 0$: Again, integration by parts is enough when

$$2^{-p_1+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1+l_1})\sim 2^{-p_1}(2^{k_1-k_2}+2^{l_1})\leq 2^{(1-\delta)m},\quad \text{or,}$$
$$2^{-p_2+2k_2-k_{\max}-k_{\min}}(1+2^{k_1-k_2+l_2})\sim 2^{-p_2+l_2}\leq 2^{(1-\delta)m},$$

i.e. when $l_2\leq(1-\delta)m$ or $-p_1+\max\{k_1-k_2,l_1\}\leq(1-\delta)m$. Hence, we look at the opposite case.

First, we check the case $k_1-k_2\geq l_1$. If $k_2\leq -20\delta m$ in addition,

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2}&\lesssim 2^{(2+\beta+2\delta)m}2^{\beta p}\cdot 2^k\cdot 2^{\frac{3}{2}k_2+p_2}\cdot 2^{p_1}||f_1||_B\cdot 2^{-(1+\beta)l_2}||f_2||_X\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+\frac{k_2}{2}}\cdot 2^{k_2+p_1}\cdot 2^{-(1+\beta)l_2}\epsilon^2\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{2k_1-10\delta m}\cdot 2^{-(1-\delta)m}\cdot 2^{-(1+\beta)(1-\delta)m}\epsilon^2\\
&\leq 2^{-5\delta m+2\delta_0 m}\epsilon^2,
\end{aligned}$$

which is acceptable since $\delta_0 \ll \delta$. Otherwise, if $k_2 \geq -20\delta m$,

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}2^{\beta p}\cdot 2^k\cdot 2^{\frac{3}{2}k_1+p_1}\cdot 2^{p_1}||f_1||_B\cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+\frac{3}{2}k_1+2(k_2+20\delta m)}\cdot 2^{2p_1-(1+\beta)l_2}\epsilon^2 \\
&\lesssim 2^{(2+\beta+42\delta)m}\cdot 2^{\frac{9}{2}k_1}\cdot 2^{-2(1-\delta)m-(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{(-1+46\delta+5\delta_0)m}\epsilon^2,
\end{aligned}$$

which also gives a small enough contribution.

Now, we assume $l_1 \geq k_1 - k_2$, which gives us $p_1 - l_1 \leq -(1-\delta)m$. We use the set size estimate Lemma 5.3.9(3) along with the normal form using the fact that $|\partial_3\Phi| \sim |2^{2p_1-k_1} \pm 2^{2p_2-k_2}| \gtrsim 2^{2p_2-k_2} \sim 2^{-k_2}$ is bounded below by $2^{-\delta_0 m}$. Dividing the terms into resonant and nonresonant terms according to $\lambda = 2^{-10\delta m}$,

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}^{res}}(f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot(\lambda 2^{\delta_0 m})^{\frac{1}{2}}\cdot 2^{k_1+p_1}\cdot 2^{-l_1}||f_1||_X\cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+k_1}\cdot 2^{(-5\delta+\frac{\delta_0}{2})m}\cdot 2^{-(1-\delta)m-(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{(-(1-\beta)\delta+3\delta_0)m}\epsilon^2,
\end{aligned}$$

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\lambda^{-1}\cdot 2^{k_1+p_1}\cdot 2^{-l_1}||f_1||_X\cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^{k+k_1+10\delta m}\cdot 2^{-(1-\delta)m-(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{(-1+15\delta+2\delta_0)m}\epsilon^2,
\end{aligned}$$

$$
\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}((\partial_t-\kappa\Delta)f_1,f_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\lambda^{-1}2^{k_1+p_1}||(\partial_t-\kappa\Delta)f_1||_2\cdot 2^{-(1+\beta)l_2}||f_2||_X\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+k_1+10\delta m}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot 2^{-(1+\beta)(1-\delta)m}\epsilon\\
&\lesssim 2^{(-\frac{1}{2}+\gamma+15\delta+2\delta_0)m}\epsilon^3,
\end{aligned}
$$

$$
\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}(f_1,(\partial_t-\kappa\Delta)f_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\lambda^{-1}2^{k_1+p_1}\cdot 2^{-l_1}||f_1||_X||\cdot(\partial_t-\kappa\Delta)f_2||_2\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+k_1+10\delta m}\cdot 2^{-(1-\delta)m}\epsilon\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\\
&\lesssim 2^{(-\frac{1}{2}+\beta+\gamma+15\delta+2\delta_0)m}\epsilon^3.
\end{aligned}
$$

Hence, we are done.

2-3) $k\ll k_1\sim k_2$, and $p_1\ll p_2\ll p\sim 0$: Again, integration by parts is enough if

$$
\begin{aligned}
&2^{-p_1+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1+l_1})\sim 2^{-p_1+k_1-k+l_1}\leq 2^{(1-\delta)m},\quad \text{or,}\\
&2^{-p_2+2k_2-k_{\max}-k_{\min}}(1+2^{k_1-k_2+l_2})\sim 2^{-p_2+k_2-k+l_2}\leq 2^{(1-\delta)m}.
\end{aligned}
$$

So we assume $p_1+k-l_1, p_2+k-l_2\leq k_1-(1-\delta)m$. The issue is that it's hard to obtain all $p_1,p_2,-l_1,-l_2$ at the same time. In the case where $p_1\geq -10\delta m$, this issue is resolved and we have

$$
\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k+p}\cdot 2^{(1+\beta)(p_1+p_2+20\delta m)}\cdot 2^{-(1+\beta)l_1}||f_1||_X\cdot 2^{-(1+\beta)l_2}||f_2||_X\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{(\frac{1}{2}-2\beta)k+(1+\beta)20\delta m}\cdot 2^{(1+\beta)(p_1+k-l_1+p_2+k-l_2)}\epsilon^2\\
&\lesssim 2^{(-\beta+30\delta+3\delta_0)m}\epsilon^2,
\end{aligned}
$$

which is enough. Hence, we now assume $p_1 \leq -10\delta m$ and use normal form with $\lambda = 2^{-20}(2^q + 2^{2p_2})$. First, the nonresonant terms are bounded by

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1, f_2)\|_{L^2} \\
&\lesssim 2^{(1+\beta+2\delta)m} \cdot 2^k(2^q + 2^{2p_2})^{-1} \cdot 2^{k_1+p_1+\frac{k+q}{2}} \cdot 2^{-l_1}||f_1||_X \cdot 2^{-(1+\beta)l_2-\beta p_2}||f_2||_X \\
&\lesssim 2^{(1+\beta+2\delta)m} \cdot \frac{2^{q/2}2^{p_2}}{2^q + 2^{2p_2}} \cdot 2^{\frac{3}{2}k+k_1} \cdot 2^{k_1-k-(1-\delta)m}2^{-(1+\beta)(l_2+p_2)}\epsilon^2 \\
&\leq 2^{(\beta+3\delta)m} \cdot 2^{2k_1+\frac{k}{2}}2^{-(1+\beta)(l_2+p_2)}\epsilon^2,
\end{aligned}$$

which gives enough contribution if either $k - k_1 \leq -3\beta m$ or $l_2 + p_2 \geq 2\beta m$. However, if none of them holds,

$$2\beta m \geq l_2 + p_2 \geq 2p_2 + k - k_1 + (1-\delta)m \gtrsim 3(k - k_1) + (1-\delta)m \geq (1-\delta-9\beta)m$$

gives a contradiction, where we used $2^{-p_2} \sim 2^{k_2-k} \sim 2^{k_1-k}$ in the current case. Hence, the boundary term is done, and

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}((\partial_t - \kappa\Delta)f_1, f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^k(2^q + 2^{2p_2})^{-1} \cdot 2^{k_2+p_2+\frac{k+q}{2}} \cdot ||(\partial_t - \kappa\Delta)f_1||_2 \cdot ||f_2||_2.
\end{aligned}$$

From here, if $p_2 \leq -\frac{m}{3}$,

$$\begin{aligned}
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{k+k_2} \cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon \cdot ||f_2||_2 \\
&\sim 2^{(\frac{1}{2}+\beta+\gamma+2\delta)m} \cdot 2^{2k}2^{2p_2}||f_2||_B\epsilon^2 \\
&\leq 2^{(-\frac{1}{6}+\beta+\gamma+2\delta+2\delta_0)m}\epsilon^3,
\end{aligned}$$

closes the bound using $2^{p_2} \sim 2^{k-k_2}$, and if $p_2 \geq -\frac{m}{3}$,

$$\begin{aligned}
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{k+k_2} \cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2 \cdot 2^{-l_2}||f_2||_X \cdot 2^{p_2+\frac{m}{3}} \\
&\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta+2\delta_0)m}\epsilon^3.
\end{aligned}$$

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1, (\partial_t - \kappa\Delta)f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^k(2^q + 2^{2p_2})^{-1}2^{k_1+p_1+\frac{k+q}{2}} \cdot ||f_1||_2 \cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2 \\
&\lesssim 2^{(\frac{1}{2}+\beta+\gamma+2\delta)m} \cdot 2^{2k_1+\frac{k}{2}} \cdot 2^{2p_1}||f_1||_B\epsilon^2
\end{aligned}$$

finishes this term if $p_1 \leq -\frac{m}{3}$, and else,

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1, (\partial_t - \kappa\Delta)f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^k(2^q + 2^{2p_2})^{-1}2^{k_2+p_2+\frac{k+q}{2}} \cdot ||f_1||_2 \cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2 \\
&\lesssim 2^{(\frac{1}{2}+\beta+\gamma+2\delta)m} \cdot 2^{k_2+\frac{3}{2}k} \cdot 2^{p_1+\frac{m}{3}}2^{-l_1}||f_1||_X\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3,
\end{aligned}$$

and this completes the nonresonant term.

For the resonant term, we use the estimate using the vertical variant Lemma 5.3.3, since $|\partial_3\Phi| \sim 2^{2p_2-k_2}$. Then, integration by parts in $|\eta|\partial_{\eta_3}$ gives enough result if

$$2^{-2p_2}[1 + 2^{l_1+p_1} + 2^{l_2+p_2}] \leq 2^{(1-\delta)m},$$

as $|\eta|\partial\eta_3$ commutes with $e^{(t-s)\kappa\Delta}$ so that we can use the same proof. Also, note that we have $q \sim 2p_2$ here, as

$$|\Lambda(\xi-\eta)| - |\Lambda(\eta)| = \frac{\sqrt{1-\Lambda^2(\eta)}^2 - \sqrt{1-\Lambda^2(\xi-\eta)}^2}{|\Lambda(\xi-\eta)| + |\Lambda(\eta)|} \in [2^{-6}2^{2p_2}, 2^3 2^{2p_2}],$$

so that $|\Phi|$ cannot be smaller than our claimed bound $\lambda = 2^{-20}(2^q + 2^{2p_2})$ unless $|\Lambda(\xi)| \sim 2^q$ matches the size of 2^{2p_2}.

We first look at the case where $l_1 + p_1 - 2p_2 \geq (1-\delta)m$. Then,

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{res}}(f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k \cdot 2^{\frac{3}{2}k+\frac{1}{2}q}\cdot 2^{-(1+\beta)l_1-\beta p_1}||f_1||_X \cdot 2^{-(1+\beta)l_2-\beta p_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{\frac{5}{2}k+p_2}\cdot 2^{p_1-2(1+\beta)p_2-(1+\beta)(1-\delta)m}2^{-(1+\beta)(p_2-k_1+k+(1-\delta)m)-\beta p_2}\epsilon^2 \\
&\lesssim 2^{(-\beta+(4+2\beta)\delta)m}\cdot 2^{\frac{5}{2}(k_2+p_2)}\cdot 2^{p_1-p_2}\cdot 2^{-(2+5\beta)p_2}\epsilon^2 \\
&\lesssim 2^{(-\beta+(4+2\beta)\delta+3\delta_0)m}2^{(\frac{1}{2}-5\beta)p_2}\epsilon^2
\end{aligned}$$

gives an acceptable bound since $\beta \leq \frac{1}{10}$. Here, we used $\frac{q}{2} \sim p_2$ and $-k_1 + k \sim p_2$.

Finally, we assume $l_2 - p_2 \geq (1-\delta)m$, in which case,

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{res}}(f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k 2^{k_1+p_1+\frac{k+q}{2}}\cdot 2^{-(1+\beta)l_1-\beta p_1}\cdot 2^{-(1+\beta)l_2-\beta p_2}\epsilon^2 \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k_1+\frac{3}{2}k+p_2}\cdot 2^{p_1-l_1}2^{-\beta(p_1+l_1)}\cdot 2^{-(1+2\beta)p_2-(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{5\delta m}\cdot 2^{2k_1+\frac{k}{2}}2^{-\beta(p_1+l_1)}2^{-2\beta p_2}\epsilon^2 \\
&\sim 2^{5\delta m}\cdot 2^{\frac{5}{2}k_1}2^{-\beta(p_1+l_1)}2^{(\frac{1}{2}-2\beta)p_2}\epsilon^2.
\end{aligned}$$

Thus, if $p_2 \leq -20\delta m$ or $p_1 + l_1 \geq 400\delta m$, we are done. If not, we perform another normal form using $L = 2^{-300\delta m}$. One can see that we are dividing by a smaller cutoff, as the former cutoff is of order $2^{2p_2} \geq 2^{-40\delta m}$ with the newest assumption. From $|\partial_3\Phi| \gtrsim$

$2^{2p_2-k_2} \geq 2^{-100\delta m}$ and the set size estimates,

$$
\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{rr}}(f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot (L2^{100\delta m})^{\frac{1}{2}}\cdot 2^{k_1+p_1}\cdot 2^{-l_1}||f_1||_X\cdot 2^{-(1+\beta)l_2-\beta p_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}2^{-100\delta m}2^{2k_1}2^{p_1+k-k_1-l_1}\cdot 2^{-(1+\beta)(p_2-l_2)-(1+2\beta)p_2}\epsilon^2 \\
&\lesssim 2^{(2+\beta+2\delta)m}2^{-100\delta m}2^{2\delta_0 m}2^{-(1-\delta)m}2^{-(1+\beta)(1-\delta)m+(1+2\beta)20\delta m}\epsilon^2 \\
&\leq 2^{-70\delta m}\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nrr}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^{k+300\delta m}2^{k_1+p_1+\frac{k+q}{2}}\cdot 2^{-l_1}||f_1||_X\cdot 2^{-l_2}||f_2||_X \\
&\lesssim 2^{(1+\beta+302\delta)m}\cdot 2^{2k_1}\cdot 2^{p_1+k-k_1-l_1}2^{p_2-l_2}\epsilon^2 \\
&\lesssim 2^{(-1+\beta+304\delta+2\delta_0)m}\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nrr}}((\partial_t-\kappa\Delta)f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+300\delta m}\cdot 2^{k_2+p_2}2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot 2^{-l_2}||f_2||_X \\
&\lesssim 2^{(-\frac{1}{2}+\beta+\gamma+303\delta+2\delta_0)m}\epsilon^3,
\end{aligned}
$$

$$
\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nrr}}(f_1,(\partial_t-\kappa\Delta)f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+300\delta m}\cdot 2^{k_1+p_1}\cdot 2^{-l_1}||f_1||_X\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{2}+\beta+\gamma+303\delta+2\delta_0)m}\epsilon^3.
\end{aligned}
$$

This finishes the case ②, $p_1 \ll p, p_2$.

③ $p \sim p_1 \ll p_2 \sim 0$: By Lemma 5.3.12, we get $k_2 \ll k_1 \sim k$ and $k_2 + p_2 \lesssim k_1 + p_1$.

Integration by parts are enough if

$$2^{-p_1+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1+l_1}) \sim 2^{-p_1}(2^{k_1-k_2}+2^{l_1}) \leq 2^{(1-\delta)m}, \quad \text{or},$$
$$2^{-p_2+2k_2-k_{\max}-k_{\min}}(1+2^{k_1-k_2+l_2}) \sim 2^{-p_2+l_2} \leq 2^{(1-\delta)m}.$$

Otherwise, if $k_1-k_2 \leq l_1$, we have $p_j - l_j \leq -(1-\delta)m$ for $j=1,2$, and hence

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}2^{\beta p}\cdot 2^k \cdot 2^{\frac{3}{2}k_2+p_2}\cdot 2^{-(1+\beta)l_1-\beta p_1}||f_1||_X \cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}2^k 2^{\frac{3}{2}(k_1+p_1)}2^{-(1+\beta)l_1}2^{-(1+\beta)(p_2-l_2)}\epsilon^2 \\
&\lesssim 2^{(2+\beta+2\delta)m}2^{\frac{5}{2}k+(\frac{1}{2}-\beta)p_1}2^{-2(1+\beta)(1-\delta)m}\epsilon^2 \\
&\lesssim 2^{(-\beta+5\delta+3\delta_0)m}\epsilon^2.
\end{aligned}$$

If $k_1-k_2 \geq l_1$, we have $p_1-k_1+k_2 \leq -(1-\delta)m$, and hence,

$$\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}2^{\beta p_1}\cdot 2^k\cdot 2^{\frac{3}{2}k_2+p_2}\cdot 2^{p_1}||f_1||_B \cdot 2^{-(1+\beta)l_2}||f_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+(1+\beta)k_1+(\frac{1}{2}-\beta)k_2}\cdot 2^{(1+\beta)(p_1-k_1+k_2)}2^{(1+\beta)(p_2-l_2)}\epsilon^2 \\
&\lesssim 2^{(-\beta+5\delta+3\delta_0)m}\epsilon^2.
\end{aligned}$$

④ $p_1 \sim p_2 \ll p \sim 0:$ Again by Lemma 5.3.12, we have $k \ll k_1 \sim k_2$ and $k+p \lesssim k_1+p_1 \sim k_2+p_2$. Integration by parts is sufficient if

$$2^{-p_1+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1+l_1}) \sim 2^{-p_1+k_1-k+l_1} \leq 2^{(1-\delta)m}, \quad \text{or}, \tag{5.18}$$
$$2^{-p_2+2k_2-k_{\max}-k_{\min}}(1+2^{k_1-k_2+l_2}) \sim 2^{-p_2+k_2-k+l_2} \leq 2^{(1-\delta)m}. \tag{5.19}$$

Hence, we assume $p_j+k-l_j \leq k_1-(1-\delta)m, j=1,2$ from now on. We do another integration by parts in vertical variant, using Lemma 5.3.3. From

$$|\partial_{\eta_3}\Phi| = |\partial_{\eta_3}\Lambda(\eta) \pm \partial_{\eta_3}\Lambda(\xi-\eta)| = \left| \frac{|\eta_h|^2}{|\eta|^3} \mp \frac{|\xi_h-\eta_h|^2}{|\xi-\eta|^3} \right|,$$

we can see that the smaller case comes from (5.18). In this case,

$$|\partial_{\eta_3}\Phi| = |\partial_{\eta_3}\Lambda(\eta) + \partial_{\eta_3}\Lambda(\xi-\eta)| = |\partial_{\eta_3}\Lambda(\eta) - \partial_{\eta_3}\Lambda(\eta-\xi)| \sim |\partial_3\nabla\Lambda(\eta)\cdot\xi| \sim 2^{p_2+k-2k_2}.$$

Hence, using the vertical variant of integration by parts with respect to $D_3^\eta = |\eta|\partial_{\eta_3}$, we get an acceptable contribution if

$$2^{-p_2-k+k_2}(1+2^{p_1+l_1}+2^{p_2+l_2}) \sim 2^{-k+k_1}(2^{l_1}+2^{l_2}) \leq 2^{(1-\delta)m}.$$

For the remaining regions, WLOG we now assume $k-k_1-l_2 \leq -(1-\delta)m$, because the relations on indices are symmetric in $\xi-\eta$ and η up to this point.

Now, since $|\partial_{\xi_3}\Phi| \sim |2^{2p-k} \pm 2^{2p_1-k_1}| \sim 2^{-k}$, we can use the set size estimate. Combining this with the normal form with $\lambda = 2^{-\frac{2}{3}m}$,

$$\begin{aligned}
&2^{(1+\beta)l}2^{\beta p}||P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{res}}(f_1,f_2)||_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot(\lambda 2^k)^{\frac{1}{2}}\cdot 2^{k+p}\cdot||f_1||_2||f_2||_2\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{\frac{5}{2}k}2^{-\frac{1}{3}m}\cdot(2^{p_1}||f_1||_B)^{\frac{1}{4}}(2^{-l_1}||f_1||_X)^{\frac{3}{4}}\cdot 2^{-l_2}||f_2||_X\\
&\lesssim 2^{(\frac{5}{3}+\beta+2\delta)m}\cdot 2^{\frac{3}{4}(k+k_1+p_1-l_1)}2^{\frac{1}{4}p_1}\cdot 2^{k-l_2}\epsilon^2\\
&\lesssim 2^{(-\frac{1}{12}+\beta+4\delta+3\delta_0)m}\epsilon^2,
\end{aligned}$$

where we used $k \lesssim k_1 + p_1$, and

$$\begin{aligned}2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot(\lambda^{-1}2^k)^{\frac{1}{2}}\cdot 2^{k+p}\cdot||f_1||_2||f_2||_2\\ &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^{\frac{5}{2}k}2^{\frac{1}{3}m}2^{-\frac{3}{4}l_1+\frac{1}{4}p_1}\cdot 2^{-l_2}\epsilon^2\\ &\lesssim 2^{(\frac{4}{3}+\beta+2\delta)m}\cdot 2^{\frac{3}{4}(k+k_1+p_1-l_1)}2^{\frac{1}{4}p_1}\cdot 2^{k-l_2}\epsilon^2\\ &\lesssim 2^{(-\frac{5}{12}+\beta+4\delta+3\delta_0)m}\epsilon^2,\end{aligned}$$

$$\begin{aligned}&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}((\partial_t-\kappa\Delta)f_1,f_2)\|_{L^2}\\ &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot(\lambda^{-1}2^k)^{\frac{1}{2}}\cdot 2^{k+p}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot||f_2||_2\\ &\lesssim 2^{(\frac{1}{2}+\beta+\gamma+2\delta)m}\cdot 2^{\frac{5}{2}k}2^{\frac{1}{3}m}2^{-l_2}||f_2||_X\epsilon^2\\ &\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3,\end{aligned}$$

$$\begin{aligned}&2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}(f_1,(\partial_t-\kappa\Delta)f_2)\|_{L^2}\\ &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot(\lambda^{-1}2^k)^{\frac{1}{2}}\cdot 2^{k+p}\cdot||f_1||_2\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\\ &\lesssim 2^{(\frac{1}{2}+\beta+\gamma+2\delta)m}\cdot 2^{\frac{5}{2}k}2^{\frac{1}{3}m}2^{-l_1}||f_1||_X\epsilon^2\\ &\lesssim 2^{(\frac{5}{6}+\beta+\gamma+2\delta)m}2^{\frac{1}{2}k}2^{k+k_1+p_1-l_1}\epsilon^3\\ &\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3.\end{aligned}$$

This finishes the final case for gap in p with $p_{\max} \sim 0$.

5.5.2.2 Gap in p with $p_{\max} \ll 0$

In this case, every $\Lambda(\zeta) \geq \frac{1}{2}, \zeta \in \{\xi, \xi-\eta, \eta\}$, so that we have $|\Phi| \geq \frac{1}{2}$, enabling us to perform a normal form with $|\mathfrak{n}| \lesssim 2^k$ every time. The terms with time derivative are easier to bound by

$$
\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}((\partial_t-\kappa\Delta)f_1,f_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot ||(\partial_t-\kappa\Delta)f_1||_2\cdot ||e^{it\Lambda}f_2||_\infty \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot 2^{-m+\gamma m}\epsilon \\
&\lesssim 2^{(-\frac{1}{2}+\beta+2\gamma+2\delta+\delta_0)m}\epsilon^3,
\end{aligned}
$$

where we used Lemma 5.3.8. The symmetric term works similarly, and the boundary term is the toughest. WLOG assume f_1 has less vector field applied than to f_2, and first consider the case where $p_1\gtrsim p_2$. If $p_2\le -\frac{m}{2}$,

$$
\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot ||e^{it\Lambda}f_1||_\infty\cdot 2^{p_2}||f_2||_B \\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k 2^{-m+\gamma m}\epsilon\cdot 2^{-\frac{m}{2}}\epsilon \\
&\le 2^{(-\frac{1}{2}+\beta+\gamma+2\delta+\delta_0)m}\epsilon^2,
\end{aligned}
$$

and when $p_2\ge -\frac{m}{2}$, we divide f_1 into f_1^I and f_1^{II} as in Proposition 2.5.1 to achieve

$$
\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1^I,f_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot ||e^{it\Lambda}f_1^I||_\infty\cdot ||f_2||_2 \\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot 2^{-p_1-\frac{3}{2}m}||f_1^I||_D\cdot 2^{p_2}||f_2|||_B \\
&\lesssim 2^{(-\frac{1}{2}+\beta+2\delta+\delta_0)m}\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1^{II},f_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot ||e^{it\Lambda}f_1^{II}||_2\cdot ||e^{it\Lambda}f_2||_\infty \\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot 2^{(-1-\beta')m-(1-2\beta')p_1}||f_1^I||_D\cdot 2^{-m+\gamma m}\epsilon \\
&\lesssim 2^{(\beta+\gamma+2\delta+\delta_0)m}2^{(-1-\beta')m+(\frac{1}{2}+\beta')m}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{2}+\beta+2\delta+\delta_0)m}\epsilon^2.
\end{aligned}
$$

Now when $p_1\ll p_2$, from the fact that $\bar{\sigma}\sim 2^{p_{\max}+k_{\max}+k_{\min}}$, integration by parts is enough if

$$2^{-p_1-p_{\max}+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1+l_1}) \leq 2^{(1-\delta)m}$$

Hence, we are left with the case where

$$p_1 + p_{\max} + k_{\max} + k_{\min} - 2k_1 \leq -(1-\delta)m, \quad \text{or,}$$
$$p_1 + p_{\max} + k_{\max} + k_{\min} - k_1 - k_2 - l_1 \leq -(1-\delta)m.$$

Since k appears from $\mathfrak{m}$ and k_2 can be obtained when using Lemma 5.3.8, we can always have one copy of $k_{\min}$.(If $k_1 = k_{\min}$, it will cancel out with $-k_1$) Therefore,

$$\begin{aligned} 2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(f_1,f_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^{k+p_{\max}}(2^{p_1}||f_1||_B)^{\frac{1}{2}}(2^{-l_1}||f_1||_X)^{\frac{1}{2}}\cdot ||e^{it\Lambda}f_2||_\infty \\ &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^{\frac{k+p_{\max}}{2}}2^{\frac{1}{2}(p_1+p_{\max}+k-l_1)}\cdot 2^{\frac{3}{2}k_2-3k_2^+-m+\gamma m}\epsilon^2 \\ &\lesssim 2^{(-\frac{1}{2}+\beta+\gamma+3\delta+\delta_0)m}\epsilon^2. \end{aligned}$$

This finishes the case with $p_{\max} \ll 0$.

5.5.2.3 Gap in q

We can assume $p \sim p_1 \sim p_2 \sim 0$ from now on. We further localize in q and write $g_i = P_{k_i,p_i,q_i}\mathcal{U}_i$, $i = 1,2$, and look at the case where we have a gap in q. Using the boostrap assumption on the B-norm,

$$2^{(1+\beta)l}2^{\beta p}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1,g_2)\|_{L^2} \lesssim 2^{(1+\beta+2\delta)m}\cdot 2^m\cdot 2^{k+q_{\max}}2^{\frac{3}{2}k_{\max}+\frac{1}{2}q_{\min}}\cdot 2^{\frac{1}{2}q_1}||g_1||_B\cdot 2^{\frac{1}{2}q_2}||g_2||_B,$$

we can reduce to the case where $q_{\max} \geq -\frac{14}{15}m$ and $q_{\min} \geq -5m$. This means we are again dealing with only $(\log\langle t\rangle)^6$ many cases, and hence it's enough to prove

$$2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(f_1,f_2)\|_{L^2}\lesssim 2^{-\delta m}.$$

① $q\ll q_1,q_2$: We divide the cases according to Lemma 5.3.11 again.

1-1) $k_1\sim k_2$: This implies $q_1\sim q_2$ and hence integration by parts is sufficient when $-q_1+k_1-k+l_i\le(1-\delta)m$, $i=1,2$. Now that we can assume $p_1,p_2\sim 0$, we actively use the precise decay estimates. Without loss of generality, we assume that g_2 has less copies of S, and denoting $g_2=g_2^I+g_2^{II}:=\ I+II$ as in Proposition 2.5.1,

$$\begin{aligned}2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1,g_2^I)\|_{L^2}&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_{\max}}||g_1||_2||e^{it\Lambda}g_2^I||_\infty\\&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}(2^{q_1/2}||g_1||_B)^{1/4}(2^{-(1+\beta)l_1}||g_1||_X)\cdot 2^{-\frac{q_2}{2}-\frac{3}{2}m}\epsilon\\&\lesssim 2^{(\frac12+\beta+2\delta+\delta_0)m}\cdot 2^{\frac34(q_2-l_1)}\epsilon^2\\&\lesssim 2^{(-\frac14+\beta+3\delta)m}\epsilon^2,\end{aligned}$$

$$\begin{aligned}2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1,g_2^{II})\|_{L^2}&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_1}\cdot 2^{k+\frac12k_1+\frac12q_1}\cdot 2^{-(1+\beta)l_1}||g_1||_X\cdot||e^{it\Lambda}g_2^{II}||_2\\&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{(1-\beta)k+\frac{k_1}{2}+\frac{1-2\beta}{2}q_1}\cdot 2^{(1+\beta)(q_1+k-l_1)}2^{-(1+\beta')m}||g_2||_D\\&\lesssim 2^{(-\beta'+4\delta)m}\epsilon^2,\end{aligned}$$

give satisfactory bounds.

1-2) $k_2\ll k_1$: From $2^{k_1+q_1}\sim 2^{k_2+q_2}$, we also have $2^{q_1-q_2}\sim 2^{k_2-k_1}\ll 1$, and $q\ll q_1\ll q_2$. Integration by parts is sufficient if

$$2^{-q_2+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1}(2^{q_2-q_1}+2^{l_1}))\sim 2^{-q_1}+2^{-q_2+l_1}\le 2^{(1-\delta)m},\quad\text{or,}$$
$$2^{-q_2+2k_2-k_{\max}-k_{\min}}(1+2^{k_1-k_2}(2^{q_1-q_2}+2^{l_2}))\sim 2^{-q_2+l_2}\le 2^{(1-\delta)m}.$$

So we look at the converse situation and when $q_1 \geq (1+2\beta)q_2$ in addition,

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1^I, g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}\cdot 2^{-\frac{3}{2}m-\frac{q_1}{2}}||e^{it\Lambda}g_1^I||_D\cdot 2^{-(1+\beta)l_2}||g_2||_X \\
&\lesssim 2^{(\frac{1}{2}+\beta+2\delta+\delta_0)m}2^{(1+\beta)(q_2-l_2)}2^{-\beta q_2-\frac{q_1}{2}}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{2}+4\delta)m}2^{(\frac{1}{2}+2\beta)\frac{14}{15}m}\epsilon^2
\end{aligned}
$$

gives a sufficient bound since $\beta \leq \frac{1}{60}$, and

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1^{II}, g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}\cdot 2^{\frac{3}{2}k_2+\frac{1}{2}q_2}\cdot 2^{-(1+\beta')m}||g_1||_D\cdot 2^{-(1+\beta)l_2}||g_2||_X \\
&\lesssim 2^{(1-\beta'+\beta+2\delta+3\delta_0)m}2^{(1+\beta)(q_2-l_2)+\frac{1-2\beta}{2}q_2}\epsilon^2 \\
&\lesssim 2^{(-\beta'+4\delta)m}\epsilon^2
\end{aligned}
$$

can be used when g_1 has less power of S. When g_2 has less power of S,

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1, g_2^I)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}\cdot 2^{-(1+\beta)l_1}||g_1||_X\cdot 2^{\frac{3}{2}k_2-\frac{3}{2}m-\frac{q_2}{2}}||g_2||_D \\
&\lesssim 2^{(\frac{1}{2}+\beta+2\delta+\delta_0)m}\cdot 2^{\frac{1-2\beta}{2}k_2-(\frac{1}{2}+\beta)q_2}2^{(1+\beta)(q_2+k_2-l_1)}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{2}+4\delta)m}2^{(\frac{1}{2}+\beta)\frac{14}{15}m}\epsilon^2
\end{aligned}
$$

gives a sufficient bound since $q_2+k_2 \sim q_1+k_1$ and $\beta \leq \frac{1}{30}$, and

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1, g_2^{II})\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}\cdot 2^{k_2+\frac{k_1+q_1}{2}}\cdot 2^{-(1+\beta)l_1}||g_1||_X\cdot 2^{-(1+\beta')m}||g_2||_D \\
&\lesssim 2^{(1-\beta'+\beta+2\delta)m}2^k 2^{\frac{3}{2}(k_2+q_2)-(1+\beta)l_1}\epsilon^2 \\
&\lesssim 2^{(-\beta'+4\delta)m}\epsilon^2,
\end{aligned}
$$

using $k_1+q_1 \sim k_2+q_2$.

Now we look at the case where $q_1 \leq (1+2\beta)q_2$. Here, we use $k_2 \sim k_1+q_1-q_2 \leq k_1+2\beta q_2$, so that we can generate q_2 using k_2. Using the normal form since $|\Phi| \gtrsim 2^{q_2}$,

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}}(g_1,g_2)\|_{L^2} \\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k_2+\frac{1}{2}q_2}(2^{\frac{1}{2}q_1}||g_1||_B)^{1-\frac{4\beta}{3}}(2^{-(1+\beta)l_1}||g_1||_X)^{\frac{4\beta}{3}}\cdot 2^{-(1+\beta)l_2}||g_2||_X \\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^{\frac{5}{2}k}\cdot 2^{(\frac{1}{2}+3\beta)q_2}\cdot 2^{(\frac{1}{2}-2\beta)q_2+\frac{4\beta}{3}q_1-\frac{4\beta}{3}(1+\beta)l_1}\cdot 2^{-(1+\beta)l_2}\epsilon^2 \\
&\lesssim 2^{(-\beta+4\delta+3\delta_0)m}\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}((\partial_t-\kappa\Delta)g_1,g_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k_2+\frac{1}{2}q_2}\cdot||(\partial_t-\kappa\Delta)g_1||_2\cdot 2^{-(1+\beta)l_2}||g_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k_1+(\frac{1}{2}+3\beta)q_2}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot 2^{-(1+\beta)l_2}||g_2||_X \\
&\lesssim 2^{(-2\beta+\gamma+3\delta)m}\epsilon^3
\end{aligned}
$$

are acceptable contributions since $\gamma \ll \beta$, and

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1,(\partial_t-\kappa\Delta)g_2)\|_{L^2} \\
&\qquad\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k_2+\frac{1}{2}q_2}\cdot(2^{\frac{q_1}{2}}||g_1||_B)^{\frac{1}{3}}(2^{-(1+\beta)l_1}||g_1||_X)^{\frac{2}{3}}\cdot||(\partial_t-\kappa\Delta)g_2||_2 \\
&\qquad\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{2k_2+\frac{1}{2}k_1+\frac{1}{2}q_1}\cdot 2^{\frac{1}{6}q_1-\frac{2}{3}l_1}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^3 \\
&\qquad\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta)m}\epsilon^3,
\end{aligned}
$$

hence we get acceptable contributions for all terms after the normal form transformation.

1-3) $k_1 \ll k_2$: This forces $q_2 \ll q_1$ which is against the assumption, and hence excluded.

② $q_1 \ll q_2, q$: Again, we divide the cases by Lemma 5.3.11.

2-1) $k \sim k_2$: This implies $q \sim q_2$ and hence integration by parts is enough if

$$2^{-q_2+k_1-k_2}(1+2^{k_2-k_1}(2^{q_2-q_1}+2^{l_1})) \sim 2^{-q_1}+s^{-q_2+l_1} \leq 2^{(1-\delta)m}, \quad \text{or,} \tag{5.20}$$

$$2^{-q_2+k_2-k_1}(1+2^{k_1-k_2}(2^{q_1-q_2}+2^{l_2})) \sim 2^{-q_2+k_2-k_1}+2^{-q_2+l_2} \leq 2^{(1-\delta)m}. \tag{5.21}$$

So we look at the other cases. Assume first that (5.21) does not hold because $q_2+k_1-k_2 \leq -(1-\delta)m$. Then,

$$\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1,g_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{k+q_2} \cdot 2^{\frac{3}{2}k_1+\frac{1}{2}q_1} \cdot (2^{k_1}\|g_1\|_{\dot{H}^{-1}})^{\frac{2}{3}}(2^{\frac{q_1}{2}}\|g_1\|_B)^{\frac{1}{3}} \cdot 2^{\frac{1}{2}q_2}||g_2||_B \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{k} \cdot 2^{(2+\frac{1}{6})(k_1+q_2)}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{6}+\beta+5\delta)m}\epsilon^2,
\end{aligned}$$

so from here, we assume $q_2 - l_2 \leq -(1-\delta)m \ll q_2 - k_2 + k_1$.

Assume that the inequality in (5.20) fails due to $q_1 \leq -(1-\delta)m$. Together with the above line, we have $q_1 \ll q_2 - k_2 + k_1$. Hence, from $|\nabla_{\eta_h}\Phi| \gtrsim |2^{q_2-k_2} \pm 2^{q_1-k_1}| \sim 2^{q_2-k_2}$, we can use the set size estimate Lemma 5.3.9 along with a normal form. For $\lambda = 2^{(1+2\beta)q_2-2\beta m}$, the resonant term can be bounded by

$$\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}^{res}}(g_1,g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{\frac{k_1+q_1}{2}} \cdot 2^{\frac{k_2}{2}} \cdot (\lambda 2^{k_2-q_2})^{\frac{1}{2}} \cdot 2^{k+q_2} \cdot ||g_1||_2 \cdot ||g_2||_2 \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{\frac{k_1}{2}+2k_2}\lambda^{\frac{1}{2}}2^{\frac{1}{2}(q_1+q_2)} \cdot 2^{\frac{q_1}{2}}||g_1||_B \cdot 2^{-(1+\beta)l_2}||g_2||_X \\
&\leq 2^{(2+2\delta)m} \cdot 2^{\frac{k_1}{2}+2k_2} \cdot 2^{q_1} \cdot 2^{(1+\beta)(q_2-l_2)}\epsilon^2 \\
&\leq 2^{(-\beta+5\delta+3\delta_0)m}\epsilon^2.
\end{aligned}$$

Now we turn to nonresonant terms. First, the boundary term can be bounded by

$$\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(g_1,g_2)\|_{L^2}\\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^{\frac{k_1+q_1}{2}}\cdot 2^{\frac{k_2}{2}}\cdot(\lambda 2^{k_2-q_2})^{\frac{1}{2}}\cdot 2^{k+q_2}\lambda^{-1}\cdot 2^{\frac{q_1}{2}}||g_1||_B\cdot 2^{-(1+\beta)l_2}||g_2||_X\\
&\lesssim 2^{(1+2\beta+2\delta)m}\cdot 2^{\frac{k_1}{2}+2k_2}\cdot 2^{q_1}2^{-\beta q_2}\cdot 2^{-(1+\beta)l_2}\epsilon^2\\
&\lesssim 2^{(1+2\beta+2\delta)m}\cdot 2^{3\delta_0 m}\cdot 2^{-(1-\delta)m}2^{\frac{14}{15}\beta m}\cdot 2^{-(1+\beta)(\frac{1}{15}-\delta)m}\epsilon^2\\
&\lesssim 2^{(-\frac{1}{15}+\frac{43}{15}\beta+4\delta+3\delta_0)m}\epsilon^2,
\end{aligned}$$

since $q_2\sim q_{\max}\gtrsim -\frac{14}{15}m$ and hence $-l_2\le -(1-\delta)m-q_2\le -(\frac{1}{15}-\delta)m$. As $\beta<\frac{1}{43}$, this gives an acceptable contribution. The rest are bounded by

$$\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}((\partial_t-\kappa\Delta)g_1,g_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{\frac{k_1+q_1}{2}}\cdot 2^{\frac{k_2}{2}}\cdot(\lambda 2^{k_2-q_2})^{\frac{1}{2}}\cdot 2^{k+q_2}\lambda^{-1}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot||g_2||_2\\
&\lesssim 2^{(\frac{1}{2}+2\beta+\gamma+2\delta)m}\cdot 2^{\frac{k_1}{2}+2k_2}\cdot 2^{-\beta q_2+\frac{q_1}{2}}\cdot(2^{\frac{q_2}{2}}||g_2||_B)^{\frac{2}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{1}{3}}\epsilon^2\\
&\lesssim 2^{(3\beta+\gamma+3\delta+3\delta_0)m}\cdot 2^{\frac{1}{3}q_2-\frac{1+\beta}{3}l_2}\epsilon^3\\
&\lesssim 2^{(-\frac{1}{3}+3\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3,
\end{aligned}$$

$$\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}(g_1,(\partial_t-\kappa\Delta)g_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{\frac{k_1+q_1}{2}}\cdot 2^{\frac{k_2}{2}}\cdot(\lambda 2^{k_2-q_2})^{\frac{1}{2}}\cdot 2^{k+q_2}\lambda^{-1}\cdot 2^{\frac{q_1}{2}}||g_1||_B\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\\
&\lesssim 2^{(\frac{1}{2}+2\beta+\gamma+2\delta)m}\cdot 2^{\frac{k_1}{2}+2k_2}\cdot 2^{-\beta q_2+q_1}\epsilon^3\\
&\lesssim 2^{(-\frac{1}{2}+3\beta+\gamma+3\delta_0)m}\epsilon^3.
\end{aligned}$$

So now we assume $q_2-l_1\le-(1-\delta)m\ll q_2$ for the remaining complement case of (5.20). We use the precise decay estimate and the g_j^I,g_j^{II} notations again. It's easier when g_1 has more S-vector field applied. In these cases,

$$\begin{aligned} 2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1,g_2^I)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}\cdot(2^{\frac{q_1}{2}}||g_1||_B)^{4\beta}(2^{-(1+\beta)l_1})^{1-4\beta}\cdot 2^{-\frac{3}{2}m-\frac{q_2}{2}}\epsilon \\ &\lesssim 2^{(\frac{1}{2}+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{1}{2}(1+4\beta)(q_2-l_2)}\epsilon^2 \\ &\lesssim 2^{(-\beta+3\delta+\delta_0)m}\epsilon^2, \end{aligned}$$

$$\begin{aligned} 2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1,g_2^{II})\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}\cdot 2^{\frac{3}{2}k_1+\frac{1}{2}q_1}\cdot 2^{-(1+\beta)l_1}||g_1||_B\cdot 2^{-(1+\beta')m}\epsilon \\ &\lesssim 2^{(-\beta'+4\delta)m}\epsilon^2. \end{aligned}$$

When g_2 has more copies of S,

$$\begin{aligned} 2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1^{II},g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}\cdot 2^{\frac{3}{2}k_1+\frac{1}{2}q_1}\cdot 2^{-(1+\beta')m}\epsilon\cdot 2^{-(1+\beta)l_2}||g_2||_B \\ &\lesssim 2^{(-\beta'+4\delta)m}\epsilon^2 \end{aligned}$$

follows in the same pattern. To deal with the term involving g_1^I, we have to subdivide the case, and we first consider when $q_1\geq-(1-\beta)m-2\beta q_2$. Then,

$$\begin{aligned} 2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1^I,g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q_2}\cdot 2^{-\frac{3}{2}m-\frac{q_1}{2}}\epsilon\cdot 2^{-(1+\beta)l_2}||g_2||_X \\ &\lesssim 2^{(\frac{1}{2}+\beta+2\delta)m}\cdot 2^k\cdot 2^{q_2+\frac{1-\beta}{2}m+\beta q_2}\cdot 2^{-(1+\beta)l_2}\epsilon^2 \\ &\lesssim 2^{(1+\frac{\beta}{2}+2\delta+\delta_0)m}2^{(1+\beta)(q_2-l_2)}\epsilon^2 \\ &\lesssim 2^{(-\frac{\beta}{2}+4\delta+\delta_0)m}\epsilon^2, \end{aligned}$$

and we are done. Hence, now we assume $q_1\leq-(1-\beta)m-2\beta q_2$. Here, we compare q_1

and $q_2 - k_2 + k_1$. When $q_1 \gtrsim q_2 - k_2 + k_1$, the same remaining term can be bounded by

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1^I, g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{k+q_2} \cdot 2^{\frac{3}{2}k_1 - \frac{3}{2}m - \frac{q_1}{2}}\epsilon \cdot ||g_2||_2 \\
&\lesssim 2^{(\frac{1}{2}+\beta+2\delta)m} \cdot 2^{k+k_1} \cdot 2^{q_2 + \frac{k_2-q_2}{2}}\epsilon \cdot (2^{\frac{q_2}{2}}||g_2||_B)^{\frac{1}{3}}(2^{-(1+\beta)l_2}||g_2||_X) \\
&\lesssim 2^{(\frac{1}{2}+\beta+2\delta+3\delta_0)m}2^{\frac{2}{3}(q_2-l_2)}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{6}+\beta+3\delta+3\delta_0)m}\epsilon^2,
\end{aligned}
$$

and hence we are finished again. The last case where $q_1 \ll q_2 - k_2 + k_1$, we can use the set size estimate again using the fact that in this case, $|\nabla_{\eta_h}\Phi| \gtrsim 2^{k_2-q_2}$. The difference comes from the fact that we are no longer in the range $q_1 \leq -(1-\delta)m$, and the additional decay comes from $q_1 \leq -(1-\beta)m - 2\beta q_2$. Setting $\lambda = 2^{(1+6\beta)q_2 - 4\beta m}$,

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{res}}(g_1, g_2)\|_{L^2} \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{\frac{k_1+q_1}{2}} \cdot 2^{\frac{k_2}{2}} \cdot (\lambda 2^{k_2-q_2})^{\frac{1}{2}} \cdot 2^{k+q_2} \cdot ||g_1||_2 \cdot ||g_2||_2 \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^{\frac{k_1}{2}+2k_2} \cdot 2^{(\frac{1}{2}+3\beta)q_2 - 2\beta m} \cdot 2^{\frac{1}{2}(q_1+q_2)} \cdot 2^{\frac{q_1}{2}}||g_1||_B \cdot 2^{-(1+\beta)l_2}||g_2||_X \\
&\leq 2^{(2-\beta+2\delta)m} \cdot 2^{\frac{k_1}{2}+2k_2} \cdot 2^{-(1-\beta)m-2\beta q_2} \cdot 2^{(1+3\beta)q_2} \cdot 2^{-(1+\beta)l_2}\epsilon^2 \\
&\leq 2^{(-\beta+4\delta+3\delta_0)m}\epsilon^2.
\end{aligned}
$$

Nonresonant terms use normal form along with the set size estimates. First, the boundary term can be bounded by

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}^{nr}}(g_1, g_2)\|_{L^2} \\
&\lesssim 2^{(1+\beta+2\delta)m} \cdot 2^{\frac{k_1+q_1}{2}} \cdot 2^{\frac{k_2}{2}} \cdot (\lambda 2^{k_2-q_2})^{\frac{1}{2}} \cdot 2^{k+q_2}\lambda^{-1} \cdot 2^{\frac{q_1}{2}}||g_1||_B \cdot ||g_2||_2 \\
&\lesssim 2^{(1+3\beta+2\delta)m} \cdot 2^{\frac{k_1}{2}+2k_2} \cdot 2^{q_1}2^{-3\beta q_2} \cdot (2^{\frac{1}{2}q_2}||g_2||_B)^{\frac{2}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{1}{3}}\epsilon \\
&\lesssim 2^{(4\beta+2\delta)m} \cdot 2^{3\delta_0 m} \cdot 2^{(\frac{1}{3}-5\beta)q_2 - \frac{1+\beta}{3}l_2}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{3}+9\beta+3\delta+3\delta_0)m}\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}((\partial_t-\kappa\Delta)g_1,g_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{\frac{k_1+q_1}{2}}\cdot 2^{\frac{k_2}{2}}\cdot(\lambda 2^{k_2-q_2})^{\frac{1}{2}}\cdot 2^{k+q_2}\lambda^{-1}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot||g_2||_2\\
&\lesssim 2^{(\frac{1}{2}+3\beta+\gamma+2\delta)m}\cdot 2^{\frac{k_1}{2}+2k_2}\cdot 2^{-3\beta q_2+\frac{q_1}{2}}\cdot(2^{\frac{q_2}{2}}||g_2||_B)^{\frac{2}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{1}{3}}\epsilon^2\\
&\lesssim 2^{(\frac{7}{2}\beta+\gamma+2\delta+3\delta_0)m}\cdot 2^{(\frac{1}{3}-4\beta)q_2-\frac{1+\beta}{3}l_2}\epsilon^3\\
&\lesssim 2^{(-\frac{1}{3}+\frac{15}{2}\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3,
\end{aligned}
$$

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}^{nr}}(g_1,(\partial_t-\kappa\Delta)g_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{\frac{k_1+q_1}{2}}\cdot 2^{\frac{k_2}{2}}\cdot(\lambda 2^{k_2-q_2})^{\frac{1}{2}}\cdot 2^{k+q_2}\lambda^{-1}\cdot||g_1||_2\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\\
&\lesssim 2^{(\frac{1}{2}+3\beta+\gamma+2\delta)m}\cdot 2^{\frac{k_1}{2}+2k_2}\cdot 2^{-3\beta q_2+\frac{q_1}{2}}\cdot(2^{\frac{q_1}{2}}||g_1||_B)^{\frac{2}{3}}(2^{-(1+\beta)l_1}||g_1||_X)^{\frac{1}{3}}\epsilon^2\\
&\lesssim 2^{(\frac{7}{2}\beta+\gamma+2\delta+3\delta_0)m}\cdot 2^{(\frac{1}{3}-4\beta)q_2-\frac{1+\beta}{3}l_1}\epsilon^3\\
&\lesssim 2^{(-\frac{1}{3}+\frac{15}{2}\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3.
\end{aligned}
$$

This finishes the case 2-1.

2-2) $k\ll k_1\sim k_2$: In this case, we have $k+q\sim k_2+q_2$ and $q_2-q\sim k-k_2\ll 0$, so that $q_1\ll q_2\ll q$. Integration by parts already gives enough result if

$$
\begin{gathered}
2^{-q+k_1-k}(1+(2^{q_2-q_1}+2^{l_1}))\sim 2^{-q_1}+2^{-q_2+l_1}\leq 2^{(1-\delta)m},\quad \text{or,}\\
2^{-q+k_2-k}(1+(2^{q_1-q_2}+2^{l_2}))\sim 2^{-q_2+l_2}\leq 2^{(1-\delta)m}.
\end{gathered}
$$

In the complement cases, we use the normal form using the fact that $|\Phi|\gtrsim 2^q=2^{q_{\max}}$. The boundary term can be bounded as

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}}(g_1,g_2)\|_{L^2}&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot||e^{it\Lambda}g_1||_\infty||g_2||_2\\
&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot 2^{-m+\gamma m}||g_1||_D\cdot(2^{\frac{q_2}{2}}||g_2||_B)^{\frac{2}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{1}{3}}\\
&\lesssim 2^{(-\frac{1}{3}+\beta+\gamma+3\delta+\delta_0)m}\epsilon^2,
\end{aligned}
$$

using Lemma 5.3.8. Lastly, the time derivative terms can be bounded easily as

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}((\partial_t-\kappa\Delta)g_1,g_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k_1+\frac{1}{2}q_1}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot(2^{\frac{q_2}{2}}||g_2||_B)^{\frac{1}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{2}{3}}\\
&\lesssim 2^{(\frac{1}{2}+\beta+\gamma+2\delta+3\delta_0)m}\cdot 2^{\frac{2}{3}(q_2-l_2)}\epsilon^3\\
&\leq 2^{(-\frac{1}{6}+\beta+\gamma+3\delta)m}\epsilon^3,
\end{aligned}
$$

$$
\begin{aligned}
&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}(g_1,(\partial_t-\kappa\Delta)g_2)\|_{L^2}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k_1+\frac{1}{2}q_1}\cdot(2^{\frac{q_1}{2}}||g_1||_B)^{\frac{1}{3}}(2^{-(1+\beta)l_1}||g_1||_X)^{\frac{2}{3}}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\\
&\lesssim 2^{(\frac{1}{2}+\beta+\gamma+2\delta+3\delta_0)m}\cdot 2^{\frac{2}{3}(q_1-l_2)}\epsilon^3\\
&\leq 2^{(-\frac{1}{6}+\beta+\gamma+3\delta)m}\epsilon^3,
\end{aligned}
$$

since $q_1-l_1\leq\min\{q_1,q_2-l_1\}$.

2-3) $k_2\ll k_1\sim k$. In this case, we have $k+q\sim k_2+q_2$ and $q-q_2\sim k_2-k\ll 0$, so that $q_1\ll q\ll q_2$. The case is symmetric to the case 2-2 and the estimates follow similarly. Integration by parts is enough if

$$
2^{-q_2+k_1-k_2}(1+2^{k_2-k_1}(2^{q_2-q_1}+2^{l_1}))\sim 2^{-q_1}+s^{-q_2+l_1}\leq 2^{(1-\delta)m},\quad\text{or,}
$$
$$
2^{-q_2+k_2-k_1}(1+2^{k_1-k_2}(2^{q_1-q_2}+2^{l_2}))\sim 2^{-q_2+l_2}\leq 2^{(1-\delta)m}.
$$

The conditions on the indices are the same as in 2-2, and we can still use the normal form since $|\Phi|\gtrsim 2^{q_{\max}}=2^{q_2}$. Hence, the result follows exactly the same way as in the case 2-2.

③ $q\sim q_1\ll q_2$: By Lemma 5.3.12, we have $k\sim k_1\gg k_2$, $k+q\sim k_1+q_1\gtrsim k_2+q_2$ in this case. Integration by parts is sufficient if

$$2^{-q_2+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1}(2^{q_2-q_1}+2^{l_1}))\sim 2^{-q_2}(2^{k_1-k_2}+2^{l_1})\le 2^{(1-\delta)m},\quad \text{or,}$$
$$2^{-q_2+2k_2-k_{\max}-k_{\min}}(1+2^{k_1-k_2}(2^{q_1-q_2}+2^{l_2}))\sim 2^{-q_2+l_2}\le 2^{(1-\delta)m}.$$

For the remaining case, we use the normal form since $|\Phi|\sim 2^{q_2}$.

$$\begin{aligned}2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}}(g_1,g_2)\|_{L^2}&\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot ||e^{it\Lambda}g_1||_\infty\cdot(2^{\frac{q_2}{2}}||g_2||_B)^{\frac{2}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{1}{3}}\\ &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot 2^{-m+\gamma m}\epsilon\cdot 2^{\frac{q_2-l_2}{3}}\epsilon\\ &\lesssim 2^{(-\frac{1}{3}+\beta+\gamma+3\delta)m}\epsilon^2,\end{aligned}$$

where the second inequality holds by Lemma 5.3.8. For the derivative terms,

$$\begin{aligned}&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}((\partial_t-\kappa\Delta)g_1,g_2)\|_{L^2}\\ &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k_2+\frac{1}{2}q_2}\cdot||(\partial_t-\kappa\Delta)g_1||_2\cdot(2^{\frac{q_2}{2}}||g_2||_B)^{\frac{1}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{2}{3}}\\ &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{\frac{5}{2}k}\dot{2}^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot 2^{\frac{2}{3}(q_2-l_2)}\epsilon\\ &\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3,\end{aligned}$$

$$\begin{aligned}&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}(g_1,(\partial_t-\kappa\Delta)g_2)\|_{L^2}\\ &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k_2+\frac{1}{2}q_2}\cdot(2^{\frac{q_1}{2}}||g_2||_B)^{\frac{1}{3}}(2^{-(1+\beta)l_1}||g_2||_X)^{\frac{2}{3}}\cdot||(\partial_t-\kappa\Delta)g_2||_2\\ &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{2}{3}(q_2-l_1)+\frac{3}{2}k_2}\epsilon\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\\ &\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3.\end{aligned}$$

4 $q_1\sim q_2\ll q:$ Again by Lemma 5.3.12, we have $k\ll k_1\sim k_2$ and $k+q\lesssim k_1+q_1\sim k_2+q_2$.

Integration by parts is sufficient if

$$2^{-q+2k_1-k_{\max}-k_{\min}}(1+2^{k_2-k_1}(2^{q_2-q_1}+2^{l_1}))\sim 2^{-q+k_1-k+l_1}\le 2^{(1-\delta)m},\quad \text{or,}$$
$$2^{-q+2k_2-k_{\max}-k_{\min}}(1+2^{k_1-k_2}(2^{q_1-q_2}+2^{l_2}))\sim 2^{-q+k_2-k+l_2}\le 2^{(1-\delta)m}.$$

Hence, now we assume $q+k-k_1-l_j\le -(1-\delta)m$, $j=1,2$. Since the conditions are symmetric on η and $\xi-\eta$, assume WLOG that g_1 has more copies of S. Using the normal form,

$$\begin{aligned}2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}}(g_1,g_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^k\cdot(2^{\frac{q_1}{2}}||g_1||_B)^{\frac{2}{3}}(2^{-(1+\beta)l_1}||g_1||_X)^{\frac{1}{3}}\cdot||e^{it\Lambda}g_2||_\infty\\ &\lesssim 2^{(1+\beta+2\delta)m}\cdot 2^{\frac{2}{3}k+\frac{1}{3}k_1}\cdot 2^{\frac{1}{3}(q_1+k-k_1-l_1)}\cdot 2^{-m}\epsilon^2\\ &\lesssim 2^{(-\frac{1}{3}+\beta+3\delta+\delta_0)m}\epsilon^2,\end{aligned}$$

$$\begin{aligned}&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}((\partial_t-\kappa\Delta)g_1,g_2)\|_{L^2}\\ &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k+\frac{1}{2}q}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\cdot(2^{\frac{q_2}{2}}||g_2||_B)^{\frac{1}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{2}{3}}\\ &\lesssim 2^{(\frac{1}{2}+\beta+\gamma+2\delta)m}\cdot 2^{\frac{11}{6}k+\frac{4}{6}k_2}\cdot 2^{\frac{2}{3}(q_2+k-k_2-l_2)}\epsilon^3\\ &\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3,\end{aligned}$$

$$\begin{aligned}&2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}(g_1,(\partial_t-\kappa\Delta)g_2)\|_{L^2}\\ &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^k\cdot 2^{\frac{3}{2}k+\frac{1}{2}q}\cdot(2^{\frac{q_1}{2}}||g_1||_B)^{\frac{2}{3}}(2^{-(1+\beta)l_1}||g_1||_X)^{\frac{1}{3}}\cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2\\ &\lesssim 2^{(\frac{1}{2}+\beta+\gamma+2\delta)m}\cdot 2^{\frac{11}{6}k+\frac{4}{6}k_1}\cdot 2^{\frac{2}{3}(q_1+k-k_1-l_1)}\epsilon^3\\ &\lesssim 2^{(-\frac{1}{6}+\beta+\gamma+3\delta+3\delta_0)m}\epsilon^3.\end{aligned}$$

This finishes the cases with gap in q.

5.5.2.4 No Gaps

Now we can assume $p \sim p_1 \sim p_2 \sim 0$, and $-\frac{14}{15}m \leq q_{\max} \sim q_{\min}$. WLOG we assume that g_2 has more copies of S applied and divide into resonant and nonresonant terms according to $\lambda = 2^{q-20}$.

We consider the nonresonant term first, and perform normal form. Then,

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}}(g_1^I, g_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m} \cdot 2^k \cdot ||e^{it\Lambda}g_1^I||_\infty \cdot 2^{\frac{q_2}{2}}||g_2||_B \\
&\lesssim 2^{(1+\beta+2\delta)m} \cdot 2^k \cdot 2^{-\frac{3}{2}m-\frac{q_1}{2}}||g_1||_D \cdot 2^{\frac{q_2}{2}}\epsilon \\
&\lesssim 2^{(-\frac{1}{2}+\beta+2\delta+\delta_0)m}\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{Q}_{\mathfrak{n}}(g_1^{II}, g_2)\|_{L^2} &\lesssim 2^{(1+\beta+2\delta)m} \cdot 2^k \cdot ||e^{it\Lambda}g_1^{II}||_2 \cdot ||e^{it\Lambda}g_2||_\infty \\
&\lesssim 2^{(1+\beta+2\delta)m} \cdot 2^k \cdot 2^{-m-\beta' m}||g_1||_D \cdot 2^{-m+\gamma m}\epsilon \\
&\lesssim 2^{(-\frac{1}{2}+\beta+\gamma-\beta'+2\delta+\delta_0)m}\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}((\partial_t-\kappa\Delta)g_1, g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^k \cdot ||(\partial_t-\kappa\Delta)g_1||_2 \cdot ||e^{it\Lambda}g_2||_2 \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^k \cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2 \cdot 2^{-m+\gamma m}\epsilon \\
&\lesssim 2^{(-\frac{1}{2}+\beta+2\gamma+\delta+\delta_0)m}\epsilon^3,
\end{aligned}
$$

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{n}}(g_1, (\partial_t-\kappa\Delta)g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^k \cdot ||e^{it\Lambda}g_1||_\infty \cdot ||(\partial_t-\kappa\Delta)g_2||_2 \\
&\lesssim 2^{(2+\beta+2\delta)m} \cdot 2^k \cdot 2^{-m}||g_1||_D \cdot 2^{-\frac{3}{2}m+\gamma m}\epsilon^2 \\
&\lesssim 2^{(-\frac{1}{2}+\beta+\gamma+\delta+\delta_0)m}\epsilon^3.
\end{aligned}
$$

For the resonant term, a trivial estimate gives

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1,g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q}\cdot 2^{\frac{3}{2}k_{\min}+\frac{1}{2}q}\cdot 2^{\min\{\frac{q_1}{2},k_1\}}\cdot 2^{\min\{\frac{q_2}{2},k_2\}}\epsilon^2\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k_{\max}}\cdot 2^{\frac{3}{2}k_{\min}+2q}\cdot 2^{\min\{\frac{q}{2},k_{\min}\}}\epsilon^2\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k_{\max}}\cdot 2^{\frac{13}{6}(k_{\min}+q)}\epsilon^2,
\end{aligned}
$$

which is enough if $k_{\min}+q\le -(1-\beta)m$. Hence, we assume $k_{\min}+q\ge -(1-\beta)m$. Now, since $|\Phi|\le 2^{q-20}$ for the resonant term, from Proposition 5.3.1 we can use integration by parts, and this is sufficient if

$$
\begin{aligned}
&2^{2k_1-q-k_{\max}-k_{\min}}(1+2^{k_2-k_1}(2^{q_2-q_1}+2^{l_1}))\sim 2^{2k_1-q-k_{\max}-k_{\min}}(1+2^{k_2-k_1+l_1})\le 2^{(1-\delta)m},\quad \text{or},\\
&2^{2k_2-q-k_{\max}-k_{\min}}(1+2^{k_1-k_2}(2^{q_1-q_2}+2^{l_2}))\sim 2^{2k_2-q-k_{\max}-k_{\min}}(1+2^{k_1-k_2+l_2})\le 2^{(1-\delta)m}.
\end{aligned}
$$

In other words, if $\max\{2k_j,k_1+k_2+l_j\}\le k_{\max}+k_{\min}+q+(1-\delta)m$, we are done. Thus, we are in the case

$$
\max\{2k_j,k_1+k_2+l_j\}\ge k_{\max}+k_{\min}+q+(1-\delta)m\ge k_{\max}-(1-\beta)m+(1-\delta)m=k_{\max}+(\beta-\delta)m.
$$

If $2k_j=\max\{2k_j,k_1+k_2+l_j\}$ for any $j=1,2$, we get a contradiction by $\delta_0 m\ge k_{\max}\ge 2k_j-k_{\max}\ge(\beta-\delta)m$. Therefore, $k_1+k_2+l_j$'s have to be the maximums. Plugging this back to get $q-l_j+k_{\max}+k_{\min}-k_1-k_2\le -(1-\delta)m$,

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1^I,g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q}\cdot ||e^{it\Lambda}g_1^I||_\infty\cdot(2^{\frac{q_2}{2}}||g_2||_B)^{\frac{1}{3}}(2^{-(1+\beta)l_2}||g_2||_X)^{\frac{2}{3}}\\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q}\cdot 2^{-\frac{3}{2}m-\frac{q}{2}}\epsilon\cdot 2^{\frac{1}{6}q-\frac{2}{3}(1+\beta)l_2}\epsilon\\
&\lesssim 2^{(\frac{1}{2}+\beta+2\delta)m}\cdot 2^{k+\frac{2}{3}(k_1+k_2-k_{\max}-k_{\min}-(1-\delta)m)}\epsilon^2\\
&\lesssim 2^{(-\frac{1}{2}+\beta+3\delta+\delta_0)m}\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
2^{(1+\beta)l}\|P_{k,p}R_l\mathcal{B}_{\mathfrak{m}}(g_1^{II},g_2)\|_{L^2} &\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{k+q}\cdot 2^{\frac{3}{2}k+\frac{1}{2}q}\cdot ||e^{it\Lambda}g_1^{II}||_2\cdot 2^{-(1+\beta)l_2}||g_2||_X \\
&\lesssim 2^{(2+\beta+2\delta)m}\cdot 2^{-(1+\beta')m}\epsilon\cdot 2^{\frac{5}{2}k+(1+\beta)(q-l_2)}\epsilon \\
&\lesssim 2^{(-\beta'+4\delta+3\delta_0)m}\epsilon^2,
\end{aligned}
$$

and this completes the proof.

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