

HOLOGRAPHIC TECHNIQUES FOR STRONGLY COUPLED SCATTERING

by

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- “Strong Coupling Expansion for the Conformal Pomeron/Odderon Trajectories”, R. Brower (Boston U.), M. Costa (Porto U.), M. Djurić (Porto U.), T. Raben (Brown U.), C. Tan (Brown U.) arXiv:1409.2730 - JHEP02(2015)104
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Chapter 1

Introduction

Current Problems in the Standard Model and Beyond A commonly repeated motto is that elementary particle physics is understood up to the TeV scale via the Standard Model [3–5] via $U(1) \times SU(2) \times SU(3)$ unification. This statement has many caveats hiding a wealth of unexplored physics which we break down into three categories¹. First, it is assumed that the discussion is about sufficiently perturbative phenomena of elementary probes. This is to rule out looking at strange quark-gluon matter phases, plasmas, and a veritable zoo of exotic states. For many applications this is a safe assumption to be made, but even some relatively simple processes involving quarks and gluons the physics is strongly coupled. The second issue might seem like a purely technical issue, but crucially depends on better understanding of high energy physics: how can one compute physical quantities via scattering amplitudes? For some problems, like the calculation of the fine structure constant², there is a procedure to calculate up to arbitrary accuracy. For other problems, like the scattering of an arbitrary number of gluons at high loop order, the problem quickly becomes intractable; the number of perturbative diagrams needed grows factorially.³ The third proviso is the absence of a complete description of confinement. This thesis is a step towards better understanding solutions to the second and third issues. We look at deep inelastic scattering and inclusive processes from the perspective of a strongly coupled and confining model. We also discuss progress in the ability to calculate physical scaling dimensions using integrability methods.

¹Although one cannot rule out that these issues are related

²Calculated up to 10-th order in perturbation theory [6] and compared to experimental measurement

³For a brief introduction to the modern approach to this problem see [7]

History of Strong Force Physics By the late 1960's electroweak unification had firmly cemented itself as the theory of electromagnetic and weak interactions. At the same time, deep inelastic scattering experiments at SLAC had begun to show that the structure of the proton was composite and seemingly much more complicated than other elementary particles. An early attempt at the description of the proton was to use S-matrix techniques. In some limits, hadron scattering showed Regge behavior. *Dual Resonance Models* were developed to describe this S-matrix with this behavior. Here, the fundamental objects were two "resonances" that were connected via some interaction. This description was an early forerunner of string theory and the first attempt at a "QCD String". Although popular and promising, much of the string/dual resonance research was set aside by the mid-1970's with the formulation of QCD as a theory of quarks and gluons. With the experimental confirmation of quarks at SLAC and the gluon at DESY [8] QCD became the main description of the strong interaction.

Unfortunately, QCD is hard! At low energies the coupling is large and makes calculations difficult. Even at high energies confinement causes issues which we focus on here. Quarks and gluons that are scattered are forced to undergo *hadronization* and form color neutral objects. At high energies, the hadronization of a single quark can lead to an angularly confined shower of resultant color neutral objects referred to as jets. In fact, the quark/gluon content of the proton itself *depends* on the energy scale of the scattering. The probability of finding a hadron constituent with a fraction of the total energy changes as a function of that energy. This led to the development of parton models of QCD. Here the density and energy dependence of the hadron constituents are represented by parton distribution functions (pdfs). Non-perturbatively one can put QCD onto a lattice to great success⁴, but the calculations can involve extremely large computational times and there is a fermionic sign problem.⁵

While these perturbative and lattice approaches to QCD have been very successful, there is still a lot to be understood for the physics of the strong interaction. Many modern approaches to the strong force involve simplifying limits and symmetries to simplify QCD. Many supersymmetric models have been written down which contain QCD. Although it already contains conformal symmetry in particular limits [9], fully conformal gauge theories can be used to describe QCD once a scale has been introduced. Finally, the realization that non-Abelian gauge theories become simplified for large

⁴The proton mass has been calculated on the lattice to high accuracy.

⁵The path integral for fermionic quantities can involve *complex* densities. This leads to highly oscillatory integrations and standard Monte-Carlo techniques cannot be used.

gauge groups has lead to great progress. The recognition that large N gauge theory becomes planar aided the conjecture that some SUSY Yang-Mills theories have well defined string duals containing quantum gravity.

Outline of Thesis This thesis is organized as follows. In chapter (2) we present a very brief pedagogical review of scattering in field theory, Regge physics, experimental processes, confinement, and the conjectured AdS/CFT correspondence. This section is by no means complete, but will provide a cursory explanation of the various subjects to unfamiliar readers, present notation to be used in the rest of the text, and provide reference to more thorough treatments of the subject. Readers familiar with these topics are encouraged to skip or skim the section. Chapter (3) dsicusses scattering from the perspective of the AdS/CFT duality. Much of this material is found in the literature, but we will stress points relative to the analysis in this thesis and discuss issues which are often opaquely explained elsewhere. Chapters (4, 5,& 6) are the bulk of this writing and represent the primary thesis research. Chapter (4) discusses a holographic approach to deep inelastic scattering using a soft wall model. Analytic properties of the model are examined and then used to fit DIS data from HERA. In chapter (5) we use results from integrability to look at the Pomeron and Odderon intercepts via an expansion of the operator dimension $\Delta(\lambda, J)$. Next, in chapter (6) we discuss a preliminary analysis where we motivate a model of inclusive scattering again using the AdS/CFT and apply it to 7 Tev LHC data.

Chapter 2

Background

We review here some of the basic ingredients used in the main analysis. This section can be skipped by readers familiar with any of these subjects without loss of context.

2.1 Field Theory

Asymptotic States This thesis deals in detail about the description of quantum scattering processes. To describe scattering events one needs to define *asymptotic states*. In other words, what states get “smashed together” and what states emerge. The caveat here is the one needs to be able to somehow clearly differentiate the times when states are free and non-interacting, vs the actual scattering process.

$$\psi(t \rightarrow -\infty) \rightarrow |p_1, p_2, \dots\rangle_{in} \quad (2.1.1)$$

$$\psi(t \rightarrow -\infty) \rightarrow |p_k, p_{k+1}, \dots\rangle_{out} \quad (2.1.2)$$

S-matrix We imagine processes where k particles come together from the past, interact, and produce $n-k$ particles. This n -particle process is a scattering event. The effect of the interaction, the physical scattering, is described by an operator which we call the S-matrix. This relates in and out states to each other so that

$$|p_k, p_{k+1}, \dots\rangle_{out} = S^{-1} |p_1, p_2, \dots\rangle_{in} \quad (2.1.3)$$

So that the scattering matrix going from some initial asymptotic state to a final asymptotic state is

$$\langle p_k, p_{k+1}, \dots \text{out} | p_1, p_1, \dots \text{in} \rangle = \langle p_k, p_{k+1}, \dots | S | p_1, p_1, \dots \rangle \quad (2.1.4)$$

The S-matrix includes the trivial non-scattering case. Separating this out one can conveniently define a transition operator $\mathcal{T}$

$$\begin{aligned} S = \mathbb{1} + i\mathcal{T} &\Rightarrow \langle p_k, p_{k+1}, \dots | p_1, p_2, \dots \rangle + \langle p_k, p_{k+1}, \dots | i\mathcal{T} | p_1, p_2, \dots \rangle \\ &= \delta_{out}^{in} + (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{k-1} p_i - \sum_{j=k}^n p_j \right) i\mathcal{M}(\{p_i\} \rightarrow \{p_j\}) \end{aligned} \quad (2.1.5)$$

and a canonical matrix element $\mathcal{M}$. Our intuition from quantum mechanics tells us that the probability to go from the initial to final state is proportional to the square of the previous equation, $\mathcal{P} \sim |\langle out | in \rangle|^2$.

At this point one is usually taxed with calculating S-matrix elements. This can be done by more thoroughly specifying quantum *fields* via canonical quantization. A scalar field, $\varphi(x)$, for example is described by a complete set of creation and annihilation operators¹

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E}} (a_p(t) e^{-ipx} + a_p^\dagger(t) e^{ipx}) \quad (2.1.6)$$

These creation/annihilation operators are rotated into each other via an interaction Hamiltonian ($e^{iH\Delta t}$) which we can use to define their asymptotic states. For free fields this is simply $\varphi \sim e^{i\vec{x}\cdot\vec{p}}$, plane wave states as one might expect from quantum mechanics. At this point, in order to calculate physical probabilities, one would like to interpret the S-matrix elements as correlations of time ordered products of these fields; this is the LSZ reduction formalism. With this approach one can write down their favorite fields—composite, spinning, massless, etc.—and computes quantities in quantum field theory. This is the topic of an innumerable number of texts on QFT and not our main focus here.²

¹In quantum mechanics one often defines creation and annihilation operators as eigenstates of energy and/or momentum operators. The natural generalization to quantum fields is then done via fourier transform of these comfortable objects.

²We are however interested in the specific structure of the LSZ formalism and simple correlators. Some of these details are reviewed in in a future publication in the context of gapped vs conformal theories.

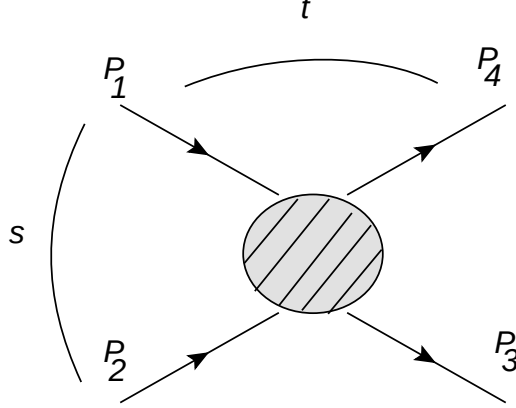


Figure 2.1.1: 2 to 2 scattering process.

S-matrix Theory Does it matter that the asymptotic states defined before are fundamental or composite particles? Does it matter if they are fermion or bosonic? Confined? Asymptotically free? In the 60's and 70's physicists developed an *analytic S-matrix*³ approach to quantum field theory to try and subvert these questions. The basic idea being that even if one doesn't know the underlying field theory, while making some safe assumptions, much can still be said about scattering processes. The assumptions are that amplitudes are functions of complex momenta p_i , that the S-matrix is Lorentz invariant, and unitarity holds ($SS^\dagger = 1$).

This approach with seemingly few assumptions can go a long way. Consider the illustrative 2-to-2 process in figure (2.1.1). The scattering amplitude will be a function of momenta via Mandelstam variables $\mathcal{A} = \mathcal{A}(s, t)$ ⁴. This amplitude represents a physical scatter in the s-channel when $s > m_{initial/final}^2$ and $t < 0$. However, as a complex function we can continue to another region, t positive and $s < 0$, and see that functionally our amplitude looks the same. That is

$$\mathcal{A}_{1,2 \rightarrow 3,4}(s, t) = \mathcal{A}_{1,4 \rightarrow 2,3}(t, u) \quad (2.1.7)$$

This is a specific example of a much more general phenomena of *crossing relations*; if we analytically continue a physical scattering amplitude to *unphysical* regions, we can learn about a different physical scattering process. A general $2 \rightarrow n$ scattering amplitude will be a more complicated object. It will contain poles, corresponding to bound states, and cuts, corresponding to multiparticle production. Complex analysis gives one great power to manipulate these amplitudes via (subtracted)

³The author has always loathed some of the historical naming conventions. While there are analytic functions in physics, most of the interesting applications involve functions with *poles* and *cuts*.

⁴see Appendix (A) for a more thorough discussion on Mandelstam variables

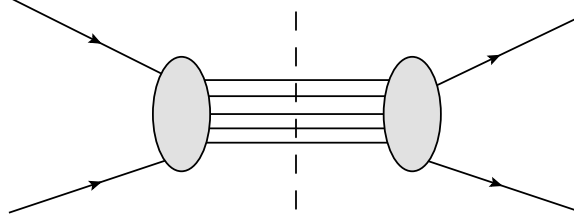


Figure 2.1.2: 2-2 cut for optical theorem

dispersion relations. For example, amplitudes with poles can be constructed via the residue theorem, cuts relate to integrated discontinuities, etc. For example

$$\mathcal{A}(s, t) = \frac{1}{2\pi i} \oint ds' \frac{\mathcal{A}(s', t)}{s' - s} \Big|_{\text{poles}} = \frac{1}{2\pi i} \int_{s_0}^{\infty} \frac{\text{Disc}\mathcal{A}(s', t)}{s' - s} \quad (2.1.8)$$

Next, let us take a closer look at a crucial aspect of S-matrix unitarity⁵: Cutkosky rules and the *optical theorem*. Let's take the matrix element of our unitary relation and look at what it says about the transition amplitude. Dropping momentum conserving delta functions we have

$$\mathbb{1} = \langle f | S^\dagger S | i \rangle = \langle f | \mathbb{1} + i(\mathcal{T} - \mathcal{T}^\dagger - \mathcal{T}\mathcal{T}^\dagger) | i \rangle \quad (2.1.9)$$

$$\Rightarrow -i \langle f | \mathcal{T}\mathcal{T}^\dagger | i \rangle = \langle f | \mathcal{T} | i \rangle - \langle f | \mathcal{T}^\dagger | i \rangle \quad (2.1.10)$$

$$\Rightarrow 2\text{Im}[\mathcal{A}_{i \rightarrow f}] = -i \sum_X \langle f | \mathcal{T} | X \rangle \langle X | \mathcal{T}^\dagger | i \rangle \text{ for } f=i \text{ in the forward limit and } s \text{ large}$$

$$\rightarrow 2\text{Im}[\mathcal{A}_{i \rightarrow i}]|_{t=0} = \sum_X |\mathcal{A}_{i \rightarrow X}|^2 \approx 2s \sigma_{\text{total}} \quad (2.1.11)$$

These unitarity relations tell us how the imaginary part of an amplitude is related to incoming and outgoing states mediated by an infinite sum of on-shell intermediate states (see figure 2.1.2). The much simpler optical theorem tells us that for 2-to-2 scattering the imaginary part of an amplitude (i.e. its discontinuity across a cut via the Cutkosky rules) is related to a total cross section. There are two extensions of the optical theorem. The first generalized optical theorem is just a restatement of the 3rd equation above that relate matrix elements at some loop order to a sum over elements at the next order. In chapter 6 we will also review *Mueller's* generalized optical theorem which states that for the 6 point amplitude, a differential cross section can be related to a discontinuity of the forward amplitude as well.

Next we turn to the use of the Sommerfeld-Watson transformation for the S-matrix: the partial

⁵Actually we also need to have a well defined completeness relation for our state/theories in question.

wave decomposition. In a 2-to-2 process the momentum transfer t varies as the angle between particles 1 and 4: $z_s = \cos(\theta_2)$. In terms of angular momentum J , the amplitude admits an expansion in terms of a complete set of Legendre polynomials given by

$$\mathcal{A}(s, z_s) = 16\pi \sum_{J=0}^{\infty} (2J+1) \mathcal{A}_J(s) P_J(z_s) \quad (2.1.12)$$

which is an infinite sum over reduced amplitudes $\mathcal{A}_J$ which are indexed by the angular momentum. Here the reduced amplitudes can be written in terms of a scattering phase shift. Froissart noticed for large s the reduced amplitudes are bounded. Using this bound and the optical theorem he came up with the *Froissart bound* (up to later determined constants)

$$\sigma_{total} \leq \frac{\pi}{m_\pi^2} \log^2 \left(\frac{s}{s_0} \right) \quad (2.1.13)$$

Finally we emphasize that the spirit of S-matrix theory is alive and well in many modern fields of high energy physics. Two examples are amplitude techniques and the conformal bootstrap. The method of on shell BCFW recursion relations [10] relating loop diagrams to a sum over trees relies on the simple pole structure of the tree level gluon amplitudes. The recent Conformal bootstrap program, begun in the 1970's and numerically advanced in the 2000s, relies on a conformal block decomposition of correlation functions⁶. This can be thought of as a generalization of a partial wave decomposition.

2.2 Regge Physics

If we go back to examine our Sommerfeld-Watson transformation 2.1.12 we can rewrite our amplitude decomposition as a sum of two pieces, one coming from a sum over even spins, $\mathcal{A}^+$, and one from odd spins, $\mathcal{A}^-$. We can re-write the sum by extending J to complex values and performing a contour integration.⁷

$$\mathcal{A}^\pm(s, t) = 8\pi i \int dJ (2J+1) \mathcal{A}_J^\pm \frac{P_J(-z_s) \pm P_J(z_s)}{\sin(\pi J)} \quad (2.2.1)$$

We now can take a *Regge limit* by taking s to be large with s/t fixed. After doing this we can try to close the J -plane contour in the above equation. This will reproduce the sum in the Sommerfeld-

⁶Briefly discussed in chapter 3.

⁷Depending on the form of $\mathcal{A}_J$ this is just an inverse Mellin transformation as described in appendix B

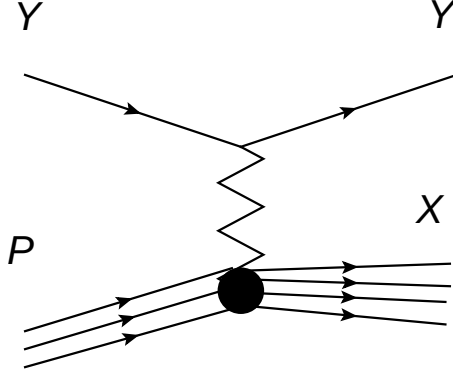


Figure 2.2.1: An example of Pomeron exchange in the virtual compton sub process of DIS

Watson transformation, but now it will sum over the contributions from poles, $J = \alpha_i(t)$, in the J -plane. In the Regge limit, the amplitude becomes

$$\mathcal{A}^{\pm}(s, t) \sim (1 \pm e^{-i\pi\alpha_i(t)}) \left(\frac{s}{s_0} \right)^{\alpha_i(t)} \quad (2.2.2)$$

where the poles, $\alpha_i(t)$, are called trajectories and are functions of t . *Regge behavior* is defined as amplitudes that behave like 2.2.2 in the Regge limit.

2.2.1 Reggeons

In the Regge limit we can interpret the poles, $J = \alpha_i(t)$, of the scattering amplitude as the exchange of an entire trajectory, or family, of particles. This is referred to as Reggeon exchange. (See figure 2.2.1) Experimentally it has been well established that many families of mesons appear to lie on linear trajectories. Regge scattering can well fit many of these experimental results via the description of an exchange of an entire trajectory. In other words, experiments tend to show Regge behavior and imply

$$J = \alpha_i(t) = J_0 + \alpha' t + \dots \quad (2.2.3)$$

where⁸ there might be small corrections at higher order in t . The intercept, J_0 , which describes the intercept of the trajectory in the t -plane, is responsible for the leading asymptotic Regge behavior in the forward limit. This is especially relevant for the calculation of total cross sections via the optical theorem.

⁸The slope labeling is conventional, be careful not to confuse it with the string worldsheet coupling when we discuss the Pomeron in string theory

Pomeron When one asks “What is a Pomeron?” the response is usually muddled and confusing. We stress that while the *role of the Pomeron is extremely important*, the experimental evidence for the Pomeron is itself muddled and at times confusing. The most straight forward definition of the Pomeron is that it is the Regge trajectory with the largest $Re[\alpha(t)]$. In other words it is the dominant Reggeon contribution to scattering amplitudes. The Pomeron represents exchange of a trajectory with the quantum numbers of the vacuum: isospin 0 and parity even.

There is no rigorous description of the Pomeron in terms of fundamental particles. The confusion surrounding Pomerons comes from their application to experiments and simulations. Fits to different experiments yield slightly differing values for the Pomeron intercepts, slope, amplitude scaling, etc. We review here a few of the common weak coupling descriptions of the Pomeron.

- Soft Pomeron

The soft Pomeron is generally described as leading to a trajectory with intercept just above 1, i.e. $j_0 \approx 1$. This corresponds to “soft” $\mathcal{A} \sim s$ behavior as is seen for Reggeon exchange like the rho-meson trajectory. Soft Pomerons should dominate interactions at large $t > 0$.

- Glueball Pomeron

Lattice simulations strongly suggest that there is a lightest 2^{++} glueball (spin 2, isospin 0, and parity even) which lies on a trajectory corresponding to a soft Pomeron. Experimentally, the identification of glueballs themselves has been a challenge.

- Donnachie - Landshoff Pomeron

Taking into account the “shadowing” of partons (partons close together are less likely to both interact with a probe) Donnachie and Landshoff wrote down a soft Pomeron description in the parton model and used it to fit total cross sections. They found that $\alpha_P(t) \approx 1.08 + 0.25t$.

- Hard Pomeron

Hard Pomerons generally refer to leading trajectories that have specific Q^2 dependence via DGLAP evolution⁹(unlike the soft Pomeron). Hard Pomeron’s are usually described via perturbation theory in α_s . The hard Pomeron should dominate for large $t < 0$.

- BFKL Pomeron

An example of a hard Pomeron is the BFKL (Balitsky, Fadin, Kuraev, Lipatov) Pomeron. This is a perturbative calculation at leading order in α_2 but including a resummation of terms of order $(\alpha_s \log(1/x))$. Here the intercept is much larger $\alpha_P(t) \approx 1.4 + 0.15t$.

⁹See the following section on DIS

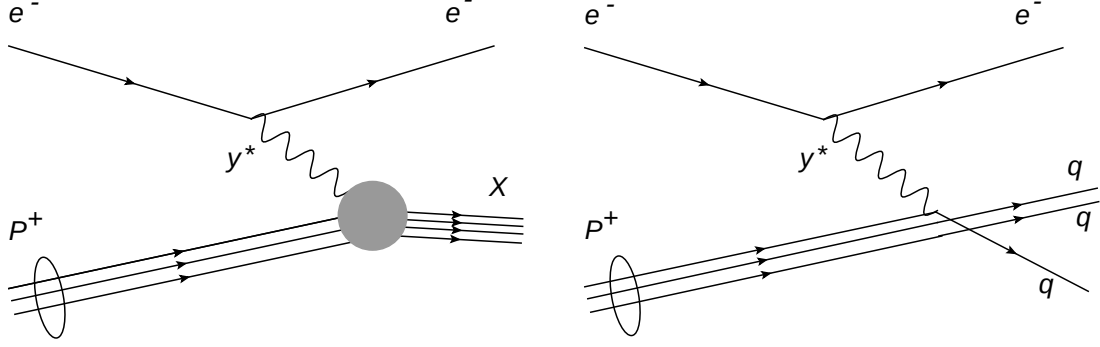


Figure 2.3.1: (a) General DIS Process. The electron has enough energy that it will scatter off the quark/gluon content of the proton. The result can be many more quarks and gluons being emitted and hadronizing into final states. (b) At first order in the QCD coupling the DIS process has the electron scattering off a quark in the proton.

Odderon The odderon is the charge parity odd counterpart to the Pomeron. For example, the difference in total cross sections between P^+P^+ scattering and P^+P^- scattering is dominated by the exchange of the odderon trajectory. A description of the odderon in terms of fundamental particles is even sparser than the Pomeron. For a nice review see [11]. Even at weak coupling there is disagreement whether there is a single dominant Odderon or two Odderons. From the strongly coupled perspective we present evidence for the existence of two Odderons in chapter 5

2.3 Deep Inelastic Scattering

We now turn our attention to one of the most important physical processes for understanding the physics of the strong interaction: deep inelastic scattering (DIS). The advantage of DIS is that we use a relatively simple probe, an electron, to probe a common hadron.¹⁰ We are interested in $e^-p^+ \rightarrow e^-X$ scattering, see figure 2.3.1, where the electron has enough energy to penetrate deeply into the proton and affect the strong dynamics.

The differential cross-section can be written in terms of a leptonic piece for the electron and a hadronic term for the proton.

$$\frac{d\sigma}{d\Omega dP^0} \sim L^{\mu\nu} W_{\mu\nu} \quad (2.3.1)$$

Where we will focus on the hadronic tensor $W^{\mu\nu}$. This piece gives the total inclusive amplitude for the sub virtual compton scattering process. In other words, up to momentum conservation,

¹⁰Electron meson scattering might be more simple in terms of quark content, but in terms of abundance and stability, the proton is an excellent target to use.

$\varepsilon_\mu \varepsilon_\nu W^{\mu\nu} \sim \text{sum}_X |\mathcal{A}_{\gamma^* p^+ \rightarrow X}|^2$. Extracting out factors of momentum so that the hadronic tensor explicitly satisfies the Ward identity we find that we can write out $W^{\mu\nu}$ in terms of structure functions. Consider an electron of momentum k^μ scattering off a proton of momentum P^μ via the exchange of a virtual photon with momentum q^μ . We have

$$W^{\mu\nu} = F_1(P, q) \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + F_2(P, q) \left(P^\mu - \frac{P \cdot q}{q^2} q^\mu \right) \left(P^\nu - \frac{P \cdot q}{q^2} q^\nu \right) \quad (2.3.2)$$

A convenient set of variables to switch to are magnitude of momentum transfer $Q^2 \equiv \sqrt{-q^2} > 0$ and Bjorken x , $x \equiv Q^2/(2P \cdot q)$. By using the parton model (i.e. assigning momentum densities to the possible proton parts) one can explicitly calculate these structure functions. An important result being that for large $Q \gg m_p$ the Callan-Gross relation says that the structure functions are not independent: $F_1(x, Q^2) = (Q^2/(4x^2))F_2(x, Q^2)$. This means that

for large Q^2 the unknown physics is all described by a single structure function: F_2

DGLAP Over a large range of x (x not very small) DIS amplitudes exhibit Bjorken scaling, that is the structure functions are independent of Q^2 up to some logarithmic corrections. This logarithmic dependence can be calculated via Dokshitzer, Gribov, Lipatov, Altarelli, and Parisi (DGLAP) evolution. The derivation can be done by starting with the parton model, where partons with a fraction of Proton momentum, ξ , are distributed according to a parton distribution function, $f(\xi)$.¹¹ The key to deriving DGLAP evolution is to work at *next-to-leading order* in the parton model. The leading order result implied Bjorken scaling. The NLO result involves gluon corrections, $\gamma^* q \xrightarrow{\text{tree}} q \mapsto \gamma^* q \xrightarrow{1\text{-loop}} q \& \gamma^* q \xrightarrow{\text{tree}} qg$. The resulting structure functions now have divergences that need to be cancelled via renormalization. The result is that

$$\Delta_Q W \sim \int_x^1 \frac{d\xi}{\xi} f(\xi) \left[\frac{\alpha_s}{2\pi} P_{qq} \left(\frac{x}{\xi} \right) \log \frac{Q^2}{Q_0^2} \right] \quad (2.3.3)$$

where we see the logarithmic correction in Q^2 . The P_{qq} are DGLAP splitting functions. The change in Q^2 can be reinterpreted as the renormalization of the parton distribution function. This is the customary explanation of DGLAP evolution.

¹¹Again, these PDFs have proven to be powerful tools. For many processes involving the same probes it has been shown experimentally that they have the same PDFs. I.e. the parton pdf can often be safely assumed to represent the actual probe and not be process specific.

$$\mu \frac{d}{d\mu} f_i(x, \mu) = \frac{\alpha_s}{\pi} \int_x^1 \frac{d\xi}{\xi} f(\xi, \mu) P_{qq} \left(\frac{x}{\xi} \right) \quad (2.3.4)$$

Operator Product Expansion We detail here our first look at the operator product expansion (OPE). The OPE is an operator statement that says two operators that are close to one another can be described by a sum over *local* operators. This expansion is rigorously well defined in a conformal field theory and extremely useful in QCD. We can write the statement of an OPE as

$$\lim_{x \rightarrow y} \mathcal{O}_1(x) \mathcal{O}_2(y) = \sum_n C_n(x-y) \mathcal{O}_n \xrightarrow{\text{momentum space}} \int d^4x e^{iqx} \mathcal{O}(x) \mathcal{O}(0) = \sum_n C_n(q) \mathcal{O}_n(0) \quad (2.3.5)$$

where the C_n are OPE coefficients. On the right hand side we have taken the position space formulation of the OPE and, via a Fourier transform, given it a momentum space interpretation. Let's try and use the OPE to learn something about inclusive and DIS processes. We can start again with our hadronic tensor $W_{\mu\nu}$. This describes the inclusive cross section for the virtual compton scatter, $\gamma^* p^+ \rightarrow X$. Specifically we have

$$W_{\mu\nu} = \sum_X \int_{X-P.S.} (2\pi)^4 \delta^{(4)}(q^\mu + P^\mu - p_X^\mu) \langle P | J_\mu(0) | X \rangle \langle X | J_\nu(0) | P \rangle \quad (2.3.6)$$

$$= \sum_X \int_{X-P.S.} \int d^4x e^{i(q^\mu + P^\mu - p_X^\mu)x_\mu} \left(e^{-i(P^\mu - p_X^\mu)x_\mu} \langle P | J_\mu(x) | X \rangle \right) \langle X | J_\nu(0) | P \rangle \quad (2.3.7)$$

$$= \sum_X \int_{X-P.S.} \int d^4x e^{iq \cdot x} \langle P | J_\mu(x) | X \rangle \langle X | J_\nu(0) | P \rangle \quad (2.3.8)$$

$$= \int d^4x e^{iq \cdot x} \langle P | J_\mu(x) J_\nu(0) | P \rangle \quad (2.3.9)$$

In the second line we have Fourier transformed the momentum conserving delta function and translated one of the correlators with the momentum operator. At this point we see that the hadronic tensor can be written as a sum over a product of three point functions $\langle 0 | P^\dagger J_\mu X | 0 \rangle$ or as the two point correlator of a current on excited states, $\langle P | J_\mu(x) J_\nu(0) | P \rangle$. This formulation of $W_{\mu\nu}$ will be important when we calculate inclusive cross-sections using CFT methods in 6. Note that none of the correlators thus far are time-ordered.

Usually in field theory we have tools to deal with correlations of *time-ordered* products. We can relate our hadronic tensor to forward ($t = 0$) virtual Compton scattering. I.e. we can take the imaginary part of

$$T_{\mu\nu} = i \int d^4x e^{iq \cdot x} \langle P | T J_\mu(x) J_\nu(0) | P \rangle \quad (2.3.10)$$

Now using standard perturbation theory we can evaluate this correlator and identify operators and OPE coefficients order-by-order in perturbation theory. An important result of the OPE is that the dominant contributions will come from operators with minimal twist, $\tau = \Delta - J$. (dimension minus spin). For QCD the minimal twist for any gauge invariant operator is $\tau = 2$. It will be important to remember in our later analysis that the dominant contribution from an OPE in DIS comes from twist 2 operators.

One last note is that we can try to match the operator product expansion with the parton model. The expectation values of the local operators in the OPE can be matched with PDFs in the parton model, giving them a more rigorous definition. We can then classify these PDFs as Mellin distribution moments. Reversing the argument, we see that calculating the moment of sub-amplitudes appearing in an OPE can tell us about PDFs.

Chapter 3

Regge Scattering in the AdS/CFT

Early descriptions of scattering in the AdS/CFT were greatly motivated by the observation that gauge theory kinematics, when viewed from the string perspective, are rescaled (or gravitationally redshifted) by the additional AdS dimension [12,13]. I.e. $p_\mu^{gauge} \sim (z/R)p_\mu^{gravity}$. Scaling along this extra radial direction can connect UV and IR limits of a single process. A historic disagreement of general amplitude behavior between flat space string theory and QCD, exponential vs power law suppression of large angle scattering, and the disconnect between two weak coupling Pomeron's can be rectified via this type of holographic scattering.

The ability of this extra dimension to mediate interactions from the gauge theory perspective is a subtle one. Let us first review the identification of a holographic Pomeron and Odderon and then discuss some details about the analytic behavior of holographic Regge scattering from the CFT point of view.

3.1 AdS/CFT

To discuss holographic Reggeons we must first introduce the AdS/CFT conjecture. At this point this subject has become borderline textbook material. The literature is vast and we refer the reader to any one of the multitude of published reviews [14–19] for more details. We aim to present here just enough information to get the flavor of the duality without discussing any of the myriad of subtleties.

The AdS/CFT is a conjectured duality (equivalence) of two seemingly disparate theories. The first theory is that of 10 dimensional type IIB string theory. Although a lot can be said about string theory whilst assuming little, it is still *hard*. To compute string amplitudes one usually resorts to

calculating a space-time effective string action perturbed about a curved background. The effective action effectively lets one compute supergravity amplitudes plus perturbative world sheet corrections and perturbative string corrections. The type IIB string theory contains D3 branes. When there are N coincident D3 branes, fluctuations near the branes feel the metric of branes which becomes $AdS_5 \times S^5$

$$ds^2 = \frac{R^2}{z^2} (dz^2 + dx^\mu dx_\mu) + R^2 d\Omega_5^2 \quad (3.1.1)$$

Here R is the AdS radius *and* the radius of the transverse S^5 . The 4 x coordinates are flat minkowski directions, and the final radial z coordinate we will see parameterizes our transition from the IR ($z \rightarrow \infty$) to the UV ($z \rightarrow 0$). We will be interested below in deforming away from this near horizon geometry to introduce confinement.

The second theory in the duality is $SU(N)$ $\mathcal{N} = 4$ Super Yang-Mills theory. This is the maximally symmetry (read: simplest!) conformal theory in four dimensions. We will comment more thoroughly on the operator content in chapter 5. Taking the large N limit one can use 't Hooft's double index notation to draw diagrams and observe that *planar* diagrams dominate. In terms of correlation functions we can consider operators as the trace (to be gauge invariant) of fields in the Yang-Mills theory and the planarity dominance is restated as the dominance of correlations of single-trace operators. Non-planar contributions correspond to finding the correlation of multi-trace operators.

The so-called AdS/CFT dictionary states that in the large 't Hooft coupling, λ , limit there is an equivalence of partition functions. Specifically, string fields, when taken to the boundary of the AdS theory source correlators in the gauge theory¹

$$Z_{AdS}[\varphi(\vec{x}, z \rightarrow 0) = \varphi_o(\vec{x})] = \langle e^{\int d^4x \varphi_o(\vec{x}) \mathcal{O}(\vec{x})} \rangle \quad (3.1.2)$$

The limit in which this statement is derived is that $\lambda = g_{ym}^2 N = g_s N = R^4/\alpha'^2$ is large. A closer inspection of the forms of the large 't Hooft coupling reveal a remarkable statement about the AdS/CFT duality. While λ is large, the Yang-Mills coupling is large as well. The CFT is strongly coupled. However, the string theory is *weakly* coupled (or at least weakly curved). That is the AdS/CFT is a *strong/weak* duality.

¹We say “=” because this equivalence is derived in the strict large λ limit. While there have been many non-trivial checks, matching of symmetries, protected correlation functions, appearance of integrability, loop level calculations, etc. there is still no formal all coupling proof of the duality.

3.2 Gauge/Gravity Reggeons

The gauge/gravity pomeron was exhaustively identified in [1]², where several derivations were examined³. Although most difficult, the easiest result to understand is from the inspection of a four point closed string amplitude $\mathcal{A} \sim \langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \rangle$. Here, world sheet symmetry will fix the locations of three of the positions of the vertex operators.⁴ To identify the Pomeron one can take the Regge limit of this amplitude and perform an OPE as the remaining vertex becomes close to one of the other vertex operators. The subtlety comes from the fact that in a usual OPE, say as $z \rightarrow 0$, one can expand for small z , but here the Regge limit dictates that terms like (sz) are order 1. The result is that one can unambiguously identify a Pomeron vertex operator

$$\mathcal{V}_P^\pm = \left(\frac{2}{\alpha'}\partial X^\pm \bar{\partial} X^\pm\right)^{1+\alpha' t/4} e^{\mp k \cdot X} \quad (3.2.1)$$

By inspection one realizes that the Pomeron vertex operator is the *Regge trajectory* associated with the graviton vertex operator. In search of a gauge/gravity Odderon a similar analysis performed in [20] shows that for the C=-1 sector one can identify a vertex operator corresponding to the trajectory of the Kalb-Ramond field.

$$\mathcal{V}_O^\pm = \left(\frac{2}{\alpha'}\right)(\partial X^\pm \bar{\partial} X^\perp - \partial X^\perp \bar{\partial} X^\pm) \left(\frac{2}{\alpha'}\partial X^\pm \bar{\partial} X^\pm\right)^{\alpha' t/4} e^{\mp k \cdot X} \quad (3.2.2)$$

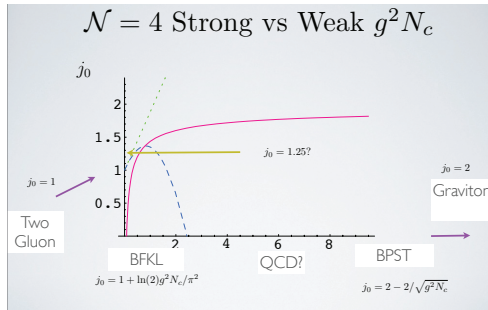


Figure 3.2.1: Intercept as a function of λ for the BPST Pomeron (solid red) and for BFKL (dotted and dashed to first and second order in λ respectively).

²In this context, “Pomeron” means the leading charge parity even object exchanged in a t-channel process.

³A heuristic derivation using locality and scaling approximations mentioned in the previous paragraph, an analysis of light cone string theory, and the detailed inspection of the Regge limit of general string vertex operators in curved space.

⁴The argument can easily be extended to n-particle scattering

3.3 Reggeon Kernels

Let's focus on the AdS Pomeron Kernel. The Odderon kernel will behave similarly and a detailed understanding of the differences will not be necessary to address the Odderon intercept in chapter 5. To set the proper stage, let us first review the situation in the conformal limit. Recall that, at finite λ , it has been shown in Ref. [1] that, due to curvature of AdS, the effective spin of a graviton exchange is lowered from 2 to $j_0 = 2 - 2/\sqrt{\lambda}$.⁵ As such it is necessary to adopt a J -plane formalism where the Pomeron kernel $\tilde{\mathcal{K}}_P$ is given by an inverse Mellin transform,

$$\tilde{\mathcal{K}}_P(s, t, z, z') = - \int_{-i\infty}^{i\infty} \frac{dj}{2\pi i} (\alpha' \hat{s})^j \frac{1 + e^{-i\pi j}}{\sin \pi j} \tilde{G}_j(t, z, z') . \quad (3.3.1)$$

with $\hat{s} = zz's/R^2$. Note that the $J = 2$ contribution is

$$(2/\pi)(\alpha' \hat{s})^2 \tilde{G}_2(t, z, z') . \quad (3.3.2)$$

This is simply the graviton Kernel, up to a constant factor, which can be absorbed into the coupling. When conformal invariance is maintained, this J -dependent propagator $\tilde{G}_j(t, z, z')$ satisfies the standard AdS_5 differential equation

$$\left(-z\partial_z z\partial_z + (2\sqrt{\lambda})(j - j_0) - z^2 t \right) \tilde{G}_j(z, z'; t) = z \delta(z - z') \quad (3.3.3)$$

with $j_0 = 2 - 2/\sqrt{\lambda}$. This equation can also be expressed in a standard Schrodinger form. It can be solved either by a spectral resolution in t or in j . Holding $j > j_0$ first and real, the spectrum in t can be seen to be continuous, along its positive real axis, leading to

$$\tilde{G}_j(z, z'; t) = \int_0^\infty k dk \frac{J_{(\Delta(j)-2)}(kz) J_{(\Delta(j)-2)}(kz')}{k^2 - t} . \quad (3.3.4)$$

where $\Delta(j) = 2 + \sqrt{2\sqrt{\lambda}(j - j_0)}$ and the $J_{(\Delta(j)-2)}(kz)$ are bessel functions.

3.4 Analytic Regge Limit

The previous section motivated the description of a holographic Pomeron and Odderon via scattering exchange. Here we broaden the discussion to reveal details about gauge/gravity amplitudes that

⁵The derivation involves applying the Virasoro on-shell condition $L_0 = \tilde{L}_0 = 1$ to the above Pomeron vertex operator and determining the dimension of the Pomeron.

can be realized purely by examining the Regge limit itself. While a conformal Regge analysis can be presented entirely in a CFT language [21–24], it is often useful to follow the earlier derivations in invoking a scattering process in *AdS* [1, 25–28]. Although both approaches are equivalent, they offer separate intuitive frameworks.

Regge Theory from CFT Partial Wave Expansion: The t-channel OPE conformal partial-wave expansion for the connected component of a 4-point function is given by a sum over conformal blocks,

$$F(u, v) = \sum_j \sum_\alpha C_{\alpha, j}^{(12), (34)} G(j, \Delta_\alpha(j); u, v). \quad (3.4.1)$$

For the Regge limit (on the light cone), these are Minkowski conformal blocks, defined with appropriate boundary conditions, or equivalently, as analytic continuation from Euclidean space as explained in Refs. [21, 22, 29]. Although, for planar $\mathcal{N} = 4$ SYM in the Regge limit, we shall restrict the sum to single-trace conformal primary operators, a completely general representation can be found by introducing basis function for the principle unitary conformal representation and expanding the amplitude as

$$F(u, v) = \sum_j \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} a(j, \nu) \mathcal{G}(j, \nu; u, v). \quad (3.4.2)$$

This group representation combines a discrete sum in the spin j and a Mellin transform in a complex Δ -plane, with $\Delta = 2 + i\nu$, as explained by Mack [30] and others⁶. The conformal harmonics, $\mathcal{G}(j, \nu; u, v)$, are eigen-functions of the quadratic Casimir operator of $SO(4, 2)$. To recover the standard conformal block expansion, it is conventional to close the contour in the lower-half ν -plane⁷, (equivalently, closing the contour in Δ -plane to the right), picking up only dynamical poles in $a(j, \nu)$, at $\nu(j) = -i(\Delta(j) - 2)$. After summing over these pole contributions, one arrives at the conformal partial wave expansion (3.4.1), which also serves as an OPE. These dynamical poles correspond to all allowed conformal primaries, $\mathcal{O}_{\Delta(j)}$, with spin j and dimension $\Delta(j)$.

We are now faced with a discrete sum over spin j . A distinguishing feature of the Regge limit is

⁶We have absorbed factors coming from Plancherel measure, etc., into the partial-wave amplitude $a(j, \nu)$ and will also normalize the conformal harmonics, $\mathcal{G}(j, \nu; u, v)$, so it eventually leads to conformal blocks with conventional normalization. It has also been demonstrated in [30] how the CFT “Mellin-representation” can be expressed in this group-theoretic form.

⁷Due to conformal invariance, the integrand is even in ν , or, equivalently, symmetric in $\Delta \leftrightarrow 4 - \Delta$. The contour can be closed either in the upper or the lower half ν -plane. The poles in the upper half ν -plane corresponds to “shadow” operators.

the fact that the conformal harmonics,

$$\mathcal{G}(j, \nu; u, v) \sim \sigma^{1-j} \Omega_{i\nu}(\rho), \quad \Omega_{i\nu}(\rho) = \frac{1}{4\pi^2} \frac{\nu \sin(\nu\rho)}{\sinh \rho}, \quad (3.4.3)$$

are more and more divergent for increasing $j > 1$ as $\sigma \rightarrow 0$. Therefore one cannot take the Regge limit term by term in (3.4.2). The traditional Regge hypothesis is that this sum can be evaluated by representing the partial wave expansion by the Sommerfeld-Watson transform in the analytic j -plane. For conformal Regge theory, this step leads to a double-Mellin representation [21–23],

$$F(u, v) = - \int_{-i\infty}^{i\infty} \frac{dj}{2\pi i} \frac{1 \pm e^{-i\pi j}}{\sin \pi j} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} a(j, \nu) \mathcal{G}(j, \nu; u, v), \quad (3.4.4)$$

where the contour in j is to stay to the right of singularities of $a(j, \nu)$. Note that in (3.4.4) we must consider separate expressions for even or odd spin values, which will correspond to $C = \pm 1$ contributions respectively. While a direct proof of this j -plane representation is lacking for CFT's in general, it has been shown to hold at strong coupling on the basis of the AdS/CFT correspondence for $\mathcal{N} = 4$ SYM [1] and at weak coupling in the BFKL limit [21, 22]. Moreover it is a natural assumption in order that non-conformal deformation give back the traditional Regge representation. This double-Mellin representation for conformal Regge theory leads to a meromorphic representation in the $\nu^2 - j$ plane, with poles specified by the collection of allowed spectral curves, $\Delta_\alpha(j)$. Still we should emphasize that this conformal “Regge pole hypothesis” is similar but is neither identical or a consequence of the conventional Regge theory. The conventional Regge j -plane analyticity with moving singularities in the $j-t$ plane, is replaced by analyticity and moving singularities in this $j-\nu^2$ plane. In Ref. [1] this distinction is clearly delineated by introducing a confining deformations of the Poincare patch of AdS space which interpolate smoothly between conformal and non-conformal Regge theory.

As an illustration, let us re-consider the $C = +1$ case and focus on the contribution from a single conformal Pomeron pole in ν^2 ,

$$a(j, \nu) = \frac{r(j)}{\nu^2 + (\Delta(j) - 2)^2}, \quad (3.4.5)$$

characterized by a spectral curve $\Delta(j)$, with the residue $r(j)$ which vanishes at $j = 0, -2, -4, \dots$. Closing the contour first in the lower-half ν -plane, (3.4.4) leads to a single-Mellin representation

$$F(u, v) \approx - \int \frac{dj}{2\pi i} \frac{1 + e^{-i\pi j}}{\sin \pi j} r(j) \sigma^{1-j} \frac{e^{(2-\Delta(j))\rho}}{\sinh \rho}, \quad (3.4.6)$$

where we have taken the limit of $\sqrt{u} = \sigma$ small, with $(1 - v)/\sqrt{u} \approx 2 \cosh \rho$ fixed. In this limit, the dominant contribution comes from the right-most singularity in the j -plane, which enters through $\Delta(j)$. Consider next the spectral curve $\Delta(j)$ for Pomeron exchange and focus on the strong coupling limit where one has $\Delta(j) = 2 + \sqrt{2}\lambda^{1/4}\sqrt{j - j_0}$, with j_0 given by (5.0.1). Observe that this gives a fixed branch-cut in the j -plane. In the limit $\sigma \rightarrow 0$, by pushing the j -contour to the left and picking up the contribution from the branch-cut at $j = j_0$, (3.4.6) leads directly to the singular behavior (5.0.7), as promised. A similar result is also obtained if one considers the weak coupling expansion of the spectral curve $\Delta(j)$, but now j_0 in (5.0.7) is given by the BFKL expansion (5.0.2).

AdS Impact parameter representation: Now let us turn to a momentum space treatment for the Regge limit in CFT. Consider the Fourier transform of the connected correlation function defined in (5.0.4),

$$(2\pi)^4 \delta^{(4)}\left(\sum p_j\right) iT(p_1, p_2, p_3, p_4) = \langle \mathcal{O}_1(p_1) \mathcal{O}_2(p_2) \mathcal{O}_3(p_3) \mathcal{O}_4(p_4) \rangle_c. \quad (3.4.7)$$

The amplitude $T(p_j)$ can be expressed as a function of Mandelstam invariants s , t , and p_j^2 . The Regge limit corresponds to s large, which defines a light-cone direction, with $t < 0$ and p_j^2 fixed. In this limit, the momentum transfer is asymptotically transverse, with $t = (p_1 + p_2)^2 \approx -q_\perp^2$. Using conformal symmetry, it is possible to express the amplitude $T(p_j)$ in an *AdS* impact parameter representation, which in the Regge limit takes the form [21, 22, 27, 28]

$$\begin{aligned} T(s, t, p_i^2) &\approx \int \frac{dz}{z^5} \frac{dz'}{z'^5} \Phi_1(z, p_1^2) \Phi_2(z, p_2^2) \mathcal{G}(s, t, z, z') \Phi_3(z', p_3^2) \Phi_4(z', p_4^2), \\ \mathcal{G}(s, t, z, z') &= (zz')^2 s \int \frac{d^2 b_\perp}{4\pi^2} e^{iq_\perp \cdot b_\perp} \mathcal{T}(S, L), \end{aligned} \quad (3.4.8)$$

with $b_\perp$ the two-dimensional impact parameter introduced earlier (5.0.6). The amplitude $\mathcal{T} = \mathcal{T}(S, L)$ encodes all dynamical information and, due to conformal symmetry, depends only on the variables

$$S = zz' s, \quad \cosh L = \frac{z^2 + z'^2 + b_\perp^2}{2zz'}. \quad (3.4.9)$$

The same representation was obtained through direct *AdS/CFT* considerations [1, 25, 26], via generalized Witten diagrams, string vertex operators, etc., leading to a Regge kernel, $\mathcal{K}(s, b_\perp, z, z')$. Up to irrelevant constants, this kernel is related to the amplitude $\mathcal{T}(S, L)$ by ⁸

$$\mathcal{K}(s, b_\perp, z, z') \sim N^2 (zz')^2 s \mathcal{T}(S, L). \quad (3.4.10)$$

⁸ The $1/N^2$ dependence in $\mathcal{T}(S, L)$, expected for *AdS* gravitational interactions, is normally removed from $\mathcal{K}$. In [25, 26], a reduced variable $\tilde{s} = S = zz' s$ was used extensively. One occasionally also used η or ξ , instead of L .

The Regge limit is now $S \rightarrow \infty$ with fixed L . It is important to note that the conformal representation (3.4.8) of the amplitude is valid for any value of the coupling constant, since it relies only on conformal invariance. However, it is quite natural from the view point of the dual AdS scattering process, where transverse space is precisely a three-dimensional hyperbolic space H_3 , whose boundary is conformal to the physical transverse space $\mathbb{R}^2$. The cross ratio L is then identified with the geodesic distance between two points in H_3 that are separated by $b_\perp$ along $\mathbb{R}^2$ and have radial coordinates z and z' . The other cross ratio $S = zz's$ measures the local energy squared of the scattering process in AdS , since z and z' define the local AdS scales for each incoming particle. Moreover, the functions $\Phi_i(z, p_i^2)$ are AdS bulk to boundary propagators with a plane wave source of momentum p_i created by the gauge theory operator at the boundary $z \rightarrow 0$. To define on-shell scattering for non-conformal amplitudes, $T(s, t)$, as introduced in Ref. [1], one deforms the dual AdS space in the IR, breaking conformal symmetry, and replaces $\Phi_i(z, p_i^2)$ by normalizable wave functions for “hadronic” (or glueball) eigenstates ⁹.

By considering the radial Fourier decomposition in the AdS impact parameter space H_3 , one can derive a double Mellin representation of the kernel, as done earlier for CFT analysis in a coordinate representation, Eq. (3.4.4). This radial decomposition simply considers harmonic functions $\Omega_{i\nu}$ on H_3 , given by (3.4.3), which satisfy $(\nabla^2 + 1 + \nu^2)\Omega_{i\nu} = 0$, where ∇^2 is the H_3 Laplacian. This is equivalent to introducing $\mathbb{R}^2$ harmonic functions, $e^{iq_\perp b_\perp}$, in the standard impact parameter decomposition. The only difference is that now we have a scattering process in AdS space. From a CFT point of view, this representation can also be derived by writing the conformal partial wave decomposition of the amplitude $\mathcal{T}(S, L)$, and then taking the Regge limit. Thus, as before, the ν -integral reflects conformal invariance due to dilatation, and the j -integral represents a coherent sum of t -channel spin fields, as was done earlier for the OPE sum via a Sommerfeld-Watson transform. To be more explicit, since the Pomeron/Odderon kernels receive contributions respectively from all even/odd spins, these kernels can be expressed as

$$\mathcal{K}_\pm(s, b_\perp^2, z, z') = -(zz') \int \frac{dj}{2\pi i} \frac{1 \pm e^{-i\pi j}}{\sin \pi j} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} S^j G_\pm(j, \nu) \Omega_{i\nu}(L). \quad (3.4.11)$$

This representation is a consequence of conformal invariance, which must next be supplemented by dynamics, i.e., specifying the Pomeron/Odderon propagator $G(j, \nu)$.

By examining Witten diagrams for exchanging spin- j fields in the Regge limit and also their

⁹ Note that at weak coupling the product of wave functions $\Phi_1\Phi_2$ and $\Phi_3\Phi_4$ are similarly replaced by the dipole parton distributions for the external particles [22], so that this impact parameter representation (3.4.8) is maintained.

string duals, one is led to $G(j, \nu)$ having a simple pole in the ν^2 -plane, determined by the spectral curve $\Delta(j)$ associated with these fields, exactly as in coordinate treatment (3.4.5). Here $G(j, \nu)$ can be related to the ν -transform of a transverse scalar bulk-bulk propagator with an effective j -dependent AdS mass. The residue at this pole can be related to the local AdS coupling of the exchanged fields to the external states. Upon closing the ν -contour, one picks up a factor $G(j, b_\perp, z, z') \sim e^{-(\Delta(j)-2)L}/\sinh L$. Finally, for Pomeron(Odderon) exchange, by identifying the j -plane branch-point at j_0 associated with the Pomeron(Odderon) spectral curve, from (3.4.11) one has for S large, by pulling the contour to the left,

$$\mathcal{K}(s, b_\perp^2, z, z') \approx (zz')\tilde{f}(L) \frac{S^{j_0}}{|\ln S|^{3/2}}, \quad (3.4.12)$$

just as in the coordinate representation (5.0.7).

Regge Dictionary for CFT: We have therefore two representations of the correlation function in the Regge limit. One derived from the CFT analysis in position space $F(u, v)$, given by (3.4.4), and another from a computation in momentum space with a clear geometrical interpretation as a scattering process in AdS , given by (3.4.11). This establishes a dictionary, where, in the Regge limit,

$$\begin{aligned} F(u, v) &\leftrightarrow \mathcal{T}(S, L) = N^{-2} (zz')^{-2} s^{-1} \mathcal{K}(s, b_\perp^2, z, z'), \\ \sigma = \sqrt{u} &\leftrightarrow S^{-1} = (zz's)^{-1}, \\ \cosh \rho \approx \frac{1-v}{2\sqrt{u}} &\leftrightarrow \cosh L = \frac{b_\perp^2 + z^2 + z'^2}{2zz'}. \end{aligned} \quad (3.4.13)$$

Although it is possible to carry out a more formal analysis in establishing this equivalence, we will not pursue this here [23]. It suffices to emphasize the exact equivalence of the two approaches to identify the spectral curve, $\Delta(j)$ in Fig. 5.0.1, which serves as the common link between them.

Chapter 4

Confining AdS

4.1 Confinement in AdS/CFT

Let us begin by first considering confinement deformations keeping the ultraviolet region conformal. If confinement sets in at a scale Λ in the gauge theory, this leads to a change in the metric away from AdS_5 in the region near $z = R^2/r \sim 1/\Lambda = z_0$. One can think of the space being “cut-off” or “rounded off”, in some natural way at $z = z_0$, leading to a “wave-guide” effect. This in general leads to a theory with a discrete hadron spectrum, with mass splitting of the order Λ among hadrons of spin ≤ 2 . As discussed in [1], the differential operator determining the J -plane spectrum remains approximately unaffected for $-t \gg \Lambda^2$, while the effect of confinement becomes important as $t \rightarrow 0^-$, and for any $t > 0$. To gain a qualitative understanding, it is instructive to treat below the “hard-wall” model, so that Eq. (3.3.3) remains valid, with z cutoff in the range $[0, z_0]$. While this model is not a fully consistent theory, it does capture key features of confining theories with string theoretic dual descriptions.

Confinement is casually described by the growth in the strength of interaction between particles as they are separated. A more rigorous definition is given in terms of Wilson loops. The Wilson loop operator is the trace of the path ordered exponential of a gauge connection around a closed loop

$$W_C = \frac{1}{N} \text{Tr} P \exp \left(i \oint_C A \right) \quad (4.1.1)$$

If this gauge field is associated with the quark potential then its expectation value, in the limit of large times T , is given by

$$W_C \sim \exp(-TE(L)) \quad (4.1.2)$$

where E is the energy of a quark anti-quark pair separated by a distance L . *Confinement* can be characterized by an area law dependence of the Wilson loop. In other words, $E \sim L$, leads to confined quarks.

Just after the proposal of the AdS/CFT duality Maldacena [31] proposed that Wilson loops in $\mathcal{N} = 4$ SYM should be dual to minimal string surfaces in the bulk theory.¹ Using this idea he calculated that for a rectangular Wilson loop, like that above, the energy of the dual string surface behaves like $E \sim 1/L$. As expected, a conformal theory, without a natural scale, does not exhibit confining behavior. The question remains, can the AdS/CFT duality be perturbed to include confinement?

4.1.1 Hard Wall Model

The simplest form of confinement deformation is that of a sharp IR hard-wall cutoff. When introduced, the spectrum in t becomes discrete. The propagator is now given by a discrete sum,

$$\tilde{G}_j(z, z'; t) = \sum_n \frac{\tilde{\varphi}_n(z, j) \tilde{\varphi}_n(z', j)}{m_n^2(j) - t}. \quad (4.1.3)$$

The wave-functions $\tilde{\varphi}_n(z, j)$ can again be expressed in terms of Bessel functions, with $m_n(j)$ fixed by a Neumann condition at z_0 , $[z^j \tilde{\varphi}_n(z, j)]'|_{z=z_0} = 0$. This discrete structure is what is to be expected when confinement deformation is introduced.

Alternatively, one finds that

$$\tilde{G}_j(z, z'; t) = \int_{-\infty}^{\infty} \frac{d\nu}{\pi^2} (\nu \sinh \pi \nu) \frac{K_{i\nu}(qz) K_{i\nu}(qz')}{2\sqrt{\lambda}(j - j_0) + \nu^2} \quad (4.1.4)$$

where $t = -q^2 < 0$. (4.1.4) provides a different representation for the Pomeron propagator in the conformal limit. One observes more directly the presence of the branch cut at $j = j_0$, which corresponds to the minimum of the denominator in (4.1.4). From (3.3.4), the presence of this cut has to be inferred from the dependence through $\Delta(j)$. Equivalently, one can work directly with a spectrum analysis in the J -plane, as carried out in Ref. [26]. The analysis can best be thought of as working with the boost operator, $\hat{H} = M_{+-}$, which is conjugate to J .

¹One can naively see why Wilson loops of the gauge theory might correspond to surfaces extending into AdS since

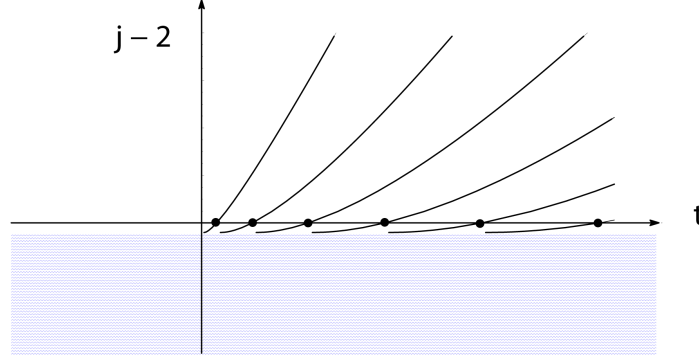


Figure 4.1.1: Reproduced from [1], the analytic behavior of Regge trajectories in the hard-wall model, showing the location of the bound-state poles at $j = 2$ and the t -independent continuum cut (shaded) at $j = j_0 = 2 - 2/\sqrt{\lambda}$ into which the Regge trajectories disappear. The lowest Regge trajectory intersects the cut at a small positive value of t . At sufficiently large t each trajectory attains a fixed slope, corresponding to the tension of the model's confining flux tubes.

The Mellin transformation of the previous equation can be carried out explicitly. [32] Re-writing the Bessel functions in terms of their associated Bessel functions we find a surprisingly informative result. This can best be seen in the forward limit where the kernel then becomes

$$\tilde{\mathcal{K}}(z, z'; \tau(s), t = 0) = \int_{-\infty}^{\infty} \frac{d\nu}{4\pi} (e^{-i\nu \log z} + S(\nu, 0)e^{i\nu \log z})^* (e^{-i\nu \log z'} + S(\nu, 0)e^{i\nu \log z'}) e^{-\tau(s)\nu^2} \quad (4.1.5)$$

Where $S(\nu, 0) = (\nu - 2i)/(\nu + 2i)$ and $\tau \sim \log(s)/2\sqrt{\lambda}$. One can think of this equation representing a plane wave and a scattered plane wave at both z and z' . This integral can be evaluated exactly giving

$$\tilde{\mathcal{K}}(z, z'; \tau(s), t = 0) = \frac{1}{2\pi\tau} \left(e^{-(\log z - \log z')/4\tau} + F(z, z, \tau) e^{-(\log z + \log z')/4\tau} \right) \quad (4.1.6)$$

The first term is just the exact conformal result, that is the Mellin transform of eq 3.3.4 in the forward limit. The second term is new and due to the hard wall confinement. The function F is bounded $|F| \leq 1$ and its magnitude can characterize the strength of the confinement.

4.2 Soft Wall Model

Soft wall confinement was born out of the question: can one produce linear trajectories via a confined AdS/CFT? The relationship between resonances and masses (i.e. the trajectory) essentially comes

distances in the bulk theory diverge near the conformal boundary.

from a mass-shell condition. In the AdS/CFT the energy of states depends on the operator's² dimension, Δ , and spin, J^3 . If one considers the exchange of some state φ governed by a kernel $\mathcal{K}(\Delta, J, \lambda)$ satisfying

$$\mathcal{K}(\Delta, J, \lambda)\varphi(\tilde{x}) = \delta^{(10)}(\tilde{x} - \tilde{x}') \quad (4.2.1)$$

Then one is able to use spectral methods to find m_n . The standard approach to do this is to rewrite the above kernel

$$\mathcal{K}(\Delta, J, \lambda)\varphi(\tilde{x}) = (\mathcal{K}'(\Delta, J, \lambda) + m^2) \varphi(\tilde{x}) \quad (4.2.2)$$

By solving the above one finds quantized values of $m = m_n$. *We caution greatly that this is a naive approximation.* Via the AdS/CFT dictionary the mass is related to the exchanged operator dimension, in turn a function of the operator spin. In other words, *the functional dependence of $\mathcal{K}'$ on Δ and J will affect the functional behavior of $m(n)$.*

The first attempts at producing confinement via a hard cutoff (cite polchinski strassler, BPST, Others?) as described in the previous section. Using the above approach, and working at leading order in $1/\sqrt{\lambda}$, this leads to trajectories $m \sim n^2$ and $m \sim J^2$ for $n, J \gg 1$. To improve upon this, in [33] a model was proposed to give linear trajectories. Here, they considered an effective 5-d theory of matter with a dynamical background dilaton field, $\Phi(z)$, and a deformed metric Weyl factor $A(z)$.⁴

$$I = \int d^5x \sqrt{g} e^{-\Phi} \mathcal{L}_{matter} \quad \text{with} \quad g_{MN} dx^M dx^N = e^A (dz^2 + dx_\mu dx^\mu) \quad (4.2.3)$$

In this case, using the above first order spectral approximation, it was shown that $\Phi - A \sim \log(z)$ at small z and $\Phi - A \sim z^2$ at large z leads to the desired behavior: $m^2 = 4(n + J)$. The main drawback of this model was the lacking of an *ab initio* string framework. This was improved upon in the following years [34,35] where an explicit toy model consisting of a 5-dimensional theory with a dynamical dilaton and tachyon was realized and in [36–38] where a generic class of 5-d dilaton

²Since we are dealing with a conformal theory, the operator-state correspondence allows us to switch back and forth our language.

³Using the AdS/CFT and results from integrability (see chapter 5) the dimension and spin are actually related via a $\Delta(J, \lambda)$ curve.

⁴Clearly these two factors are related via the backreaction of Φ on the metric, but this case was just a proof of concept.

theories was considered in great detail. The backreaction dictated the expected regge behavior and the models were able to incorporate other phenomenology like matter fields, asymptotic freedom, and a higgs mechanism.

An important consequence of these works was that confinement, as defined by the appropriate Wilson loop scaling and appropriate higher spin regge trajectories, is only achievable in these theories for $A \sim -Cz^2$ for some $C > 0$. [39] Furthermore, solutions involving $C < 0$ lead to the appearance of an unphysical zero mode in the meson spectrum. However, several authors have argued that this “unphysical” sign leads to interesting phenomenology [40–42] and naive scaling behavior due *purely* to the metric also lead to the proper Wilson loop scaling [43]. These proposals lack a motivating effective theory, or even toy model, nonetheless we will aim to keep the sign of the dilaton arbitrary through most of our analysis. However, we will conform to the (-) sign to preform the numerical analysis.

4.2.1 Model

We are now ready to describe our soft wall model. As we are ultimately interested in looking at Regge limits we focus on the Pomeron exchange. As described above, this corresponds to the Regge trajectory of the graviton. We first work here with a more heuristic approach: using the supergravity limit as a guide, we look at the transverse traceless perturbations of the metric and include stringy effects. This approach follows that of [1,12] where in [1] a more formal string worldsheet calculation was done to order $1/\sqrt{\lambda}$. We defer the latter approach to the future.

Soft Wall Model We begin with our deformed metric

$$ds^2 = e^{2A(z)} (dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu) + d\Omega_5^2 \quad (4.2.4)$$

Here x^μ are flat coordinates which will also correspond to the conformal theory. z is the extra dimension in the string space with $z \rightarrow 0$ being the conformal boundary in the UV and $z \rightarrow \infty$ the IR. The conformal AdS corresponds to $A(z) = -\log(z/R)$. Using the results from the previous section we take $A(z)$ to be of the form

$$A(z) = \frac{c(\Lambda z)^2}{d-1} - \log\left(\frac{z}{R}\right) \quad (4.2.5)$$

where the factor $(d-1)$ is just for convenience. The coefficient $c = \pm 1$ is the aforementioned sign term.

We previously described the conformal Pomeron kernel as the exchange of a trajectory of particles with a spin dependent mass, i.e.

$$\mathcal{K}(J) = \frac{1}{(J-2) - (\alpha'/2)\Delta_J} \quad (4.2.6)$$

The final term is the spin-J laplace operator. This is related to the scalar laplace operator, up to a curvature term⁵, via $\Delta_J = e^{JA}\Delta_0 e^{-JA}$.⁶ The final term, with $\alpha' = R^2/\sqrt{\lambda}$, contains the stringy effects. In other words, we are interested in solving

$$\left[\frac{2\sqrt{\lambda}}{R^2}(J-2) - \left(\frac{1}{\sqrt{-g}} \partial_M \sqrt{-g} g^{MN} \partial_N \right) \right] \tilde{h}_{tt} = 0 \quad (4.2.7)$$

where we have done a similarity transform from our original wavefunction $\tilde{h} = e^{-JA}h$. We can safely assume that our wavefunction h does not fluctuate in the transverse space, W . This is assumed in [1], but we will see in chapter 5 that this just means we are considering string states that are not rotating in s^5 , or correspondingly, CFT operators without R charge. After safely dropping the transverse space, we see the metric is diagonal and only depends on z . The previous equation then becomes

$$\begin{aligned} & \left[\frac{2\sqrt{\lambda}}{R^2}(J-2) - e^{-A(d+1)} \left(\partial_z (e^{A(d-1)} \partial_z + e^{A(d-1)} \nabla_\mu^2) \right) \right] \tilde{h}_{tt} = 0 \\ & \left[\left(-\partial_z^2 + c^2 \Lambda^4 z^2 - (t + c(d-2)\Lambda^2) + \frac{d^2-1}{4z^2} \right) + \alpha^2(J) \frac{e^{2A}}{R^2} \right] \tilde{h}_{tt} = 0 \end{aligned} \quad (4.2.8)$$

where in the second equation we have performed a fourier transformation in the μ coordinates and labeled the momenta in a suggestive way: $\nabla_\mu^2 \rightarrow -q^2 = t$. We have also introduced $\alpha^2(J) = 2\sqrt{\lambda}(J-2)$ and organized things into two terms: the first, derivative term, is a kinetic term reminiscent of a harmonic oscillator. This term also corresponds to the supergravity description of the graviton propagation. This term most agrees with the methods described in the previous section. The second term is essentially a mass term. Most of the previous softwall analyses have used the conformal holographic dictionary to argue that $m \sim J$. We argue that *a softwall description must contain a*

⁵The curvature term is actually subtracted off by requiring the metric perturbations to scale like the background metric at long wavelengths

⁶For a conformally flat metric, like the AdS part above, the spin-laplace operator is obtained via $\partial \rightarrow \bar{\partial} = -(\sqrt{-g}^{s/d})\partial(\sqrt{-g}^{-s/d})$

deformed mass term of the form above.

The deformed mass can be interpreted in three ways. Firstly, one can argue that as one flows from the UV to the IR the effects of the softwall should be felt on AdS masses. I.e. since lengths go from $l|_{UV} \sim 1/z \rightarrow l|_{IR} e^{2A}$ then masses should also be deformed $m_{AdS} \rightarrow m_{AdS} e^{2A}$.⁷ The second argument comes from examination of the exchanged operator dimension in the conformal limit.⁸ For the Pomeron, there is a convergent expansion

$$(\Delta - 2)^2 = 4 + \beta_1(\lambda)(j - 2) + \beta_2(\lambda)(j - 2)^2 + \dots = 4 + B(\lambda, J) \quad (4.2.9)$$

The scalar laplacian can be expressed as an operator with eigenvalue $\Delta(\Delta - 4) = B(\lambda, J)$. Before deforming this to the softwall we make note of two things: (1) for large λ , near $J \rightarrow 2$ (appropriate for strongly coupled Regge limit) we have $B(J, \lambda) \approx 2\sqrt{\lambda}(J - 2) = \alpha^2(J)$. (2) For large λ and large J , $B(J, \lambda) \approx J^2 + J \log J$ in agreement with calculations of the cusp anomalous dimension.

Now we can use this expansion and deform it by e^{2A} as above. The resulting mass term will be that found above in eqn (4.2.8). Since we are interested in the physics near $J \rightarrow 2$ we take the first non-trivial term in the $B(J, \lambda)$ expansion. To find a solution to our problem we will also expand the softwall deformation term *to the first non-trivial order*: $e^{2(\Lambda^2 z)^2/(d-1)} \approx 1 + 2(\Lambda^2 z)^2/(d-1) + \mathcal{O}(z^4)$. Putting this all together we have the full form of our Pomeron exchange kernel

$$\left[-\partial_z^2 + \Lambda^4 z^2 - (t + \Lambda^2(2c - \frac{2}{3}\alpha^2(J) - \frac{3}{8})) + \frac{\alpha^2(J) - \frac{1}{4}}{z^2} \right] h_{tt} = 0 \quad (4.2.10)$$

Finally, the third motivation is that if one wants to move away from the infinite coupling supergravity limit then one must perform a string worldsheet calculation as in [1]. Here, keeping all terms of order $1/\sqrt{\lambda}$ leads to a spin dependence like that in the above equation.

4.2.2 Spectral Softwall Solution

We are now ready to find solutions to our softwall kernel 4.2.10. This equation has the form of a radial harmonic oscillator. We will look for Green's function solutions of the form

$$\left[-\partial_z^2 + \Lambda^4 z^2 - (t + \Lambda^2 \tilde{\alpha}) + \frac{\alpha^2(J) - \frac{1}{4}}{z^2} \right] \mathcal{K}(J, t, z, z') = \delta(z - z') \quad (4.2.11)$$

⁷This first argument is similar in spirit to that of [40]. However, there the focus was on discussing external state wavefunctions and here we are interested in an exchange kernel.

⁸See section 5 for the details. There we use this expansion and results from integrability to calculate the Pomeron and Odderon intercepts to high order in $1/\sqrt{\lambda}$.

We first look for a spectral resolution. If we change coordinates to $y = \Lambda^2 z^{29}$ we can see that the higher derivative terms look like that of Laguerre's equation. We will therefore look for solutions of the form $\psi(J, t, z) = A(z, J, t(n)) L_n^\alpha(\Lambda^2 z^2)$. Feeding this back into eqn (4.2.11) we find, up to an overall normalization, a solution with quantized $t = t_n$.

$$\psi_n(J, t, z) = \beta(n, J) (\Lambda z)^{\alpha'} e^{-\Lambda^2 z^2} L_n^{\alpha'}(\Lambda^2 z^2) \quad (4.2.12)$$

$$\text{with } t_n = \Lambda^2(4n + 2\alpha(J) + 2 + \tilde{\alpha}) = \Lambda^2 \left(4n + \left(2\alpha(J) - \frac{2}{3}\alpha^2(J) \right) + \left(\frac{13}{8} + 2c \right) \right) \quad (4.2.13)$$

where $\alpha' = \alpha^2(J) + 5/4$. We can fix the normalization via the orthogonality of the Laguerre polynomials

$$\int_0^\infty dy \psi_n(J, t, y) \psi_m(J, t, y) = \delta_{mn} \Rightarrow \beta(n, J) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha'+1)}} \quad (4.2.14)$$

The functions ψ also form a complete basis, $\sum_n \psi_n(z) \psi_n(z') = \delta(z - z')$, which we can use to form a solution to our original Green's function equation as

$$\mathcal{K}(J, t, z, z') = \sum_n \frac{\psi_n(\Lambda^2 z^2) \psi_n(\Lambda^2 z'^2)}{t_n - t} \quad (4.2.15)$$

$$= 2\Lambda^{2(\alpha'+1)} (z z')^{\alpha' + \frac{1}{2}} e^{-\Lambda^2(z^2 + z'^2)} \sum_{n=0}^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha'+1)} \frac{L_n^{\alpha'}(\Lambda^2 z^2) L_n^{\alpha'}(\Lambda^2 z'^2)}{t_n - t} \quad (4.2.16)$$

This form of the solution is similar in form to that found in (insert citations). Here is where analysis usually ends as one can examine the functional dependence of the exchanged momentum, t_n . For our case we see that $t \sim n$ and, for large spin, $t \sim J$ as expected.

4.2.3 General Solution

The previous spectral resolution, eqn (4.2.15), is rather unwieldy. Although it correctly produces the exchanged momentum behavior, since it is in the form of an infinite sum it is difficult to use numerically or understand the full analytic dependence on J and t . To begin with let's go back to eqn (4.2.11) and again perform the coordinate transformation $y = \Lambda^2 z^2$. This time, if we rescale our kernel $\mathcal{K}(J, t, y, y') = (yy')^{-1/4} \tilde{\mathcal{K}}(J, t, y, y')$ then we can get rid of the linear derivative and are left with.

⁹This is a one-to-one mapping as we consider $z > 0$ only

$$\left[\partial_y^2 + \frac{\kappa}{y} + \frac{\frac{1}{4} - \mu^2}{y^2} - \frac{1}{4} \right] \tilde{\mathcal{K}}(J, t, y, y') = \delta(y - y') \quad (4.2.17)$$

This is Whittaker's equation with parameters $\kappa = \tilde{\alpha}/4 + t/(4\Lambda^2)$ and $\mu^2 = \alpha^2(J)/4$. The solutions to the homogeneous equation can be written down

$$f_1(y) = M_{\kappa, \mu}(y) = e^{-y/2} y^{\mu+1/2} M\left(\mu + \frac{1}{2} - \kappa, 2\mu + 1, y\right) \quad (4.2.18)$$

$$f_2(y) = W_{\kappa, \mu}(y) = e^{-y/2} y^{\mu+1/2} U\left(\mu + \frac{1}{2} - \kappa, 2\mu + 1, y\right) \quad (4.2.19)$$

$$(4.2.20)$$

and a general solution via variation of parameters can be constructed as

$$\tilde{\mathcal{K}}(J, t, y, y') = (yy')^{-1/4} \frac{f_1(y_<) f_2(y_>)}{2W(f_1, f_2)} \quad (4.2.21)$$

Where $y_< < y_>$ and $W(f_1, f_2)$ is the Wronskain as defined in appendix (C). Finally, we can expand the previous equation into a solution in terms of the confluent hypergeometric function M

$$\mathcal{K}(J, t, y, y') = \frac{e^{-(y+y')/2} (yy')^{1/4}}{4\mu} M(\mu + 1/2 - \kappa, 2\mu + 1, y_<) \quad (4.2.22)$$

$$\times \left(\left(\frac{y_<}{y_>} \right)^\mu M(-\mu + 1/2 - \kappa, -2\mu + 1, y_>) - \mathcal{S}(\mu, \kappa) (y_< y_>)^\mu M(\mu + 1/2 - \kappa, 2\mu + 1, y_>) \right) \\ \mathcal{S}(\mu, \kappa) = \frac{\Gamma(-2\mu + 1) \Gamma(\mu + 1/2 - \kappa)}{\Gamma(2\mu + 1) \Gamma(-\mu + 1/2 - \kappa)} \quad (4.2.23)$$

This is the form of the solution that is used for numerical fits and analytic evaluation.

4.3 Confining DIS & Phenomenology

In the previous section, in order to find specific functional solutions to our Pomeron kernel—in either a spectral representation and as an analytic function—we had to work in mixed representation where the kernel was a function of spin, transverse momentum transfer, and AdS radial-coordinate. To describe physical scattering it is often necessary to understand the behavior of the kernel as a function of center-of-mass energy, $\sqrt{s}$, and impact parameter b . In fact, to see Regge behavior we have to perform an inverse Mellin transformation back to s -space

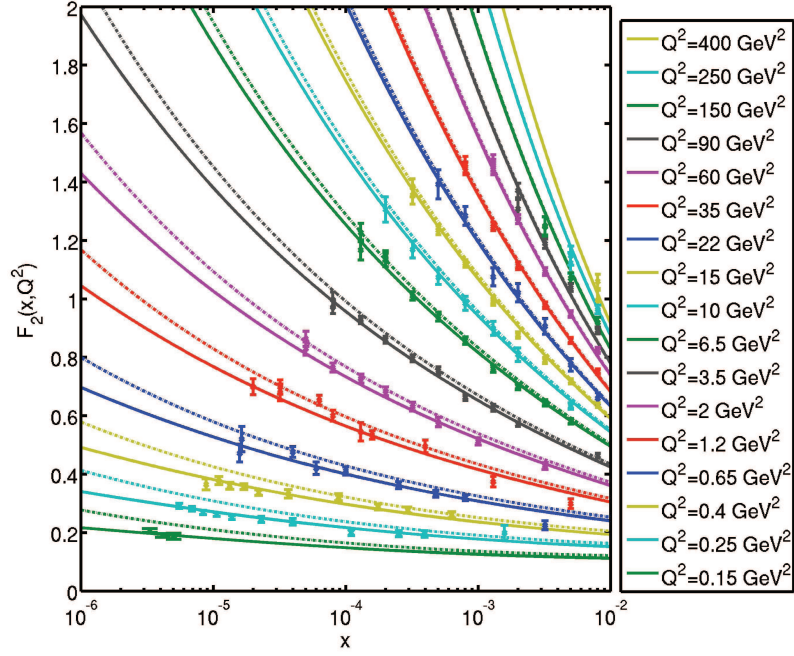


Figure 4.3.1: Structure function F_2 fits for the hardwall model.

$$\mathcal{K}(s, t, z, z') = \int \frac{(\hat{s}^J + (-\hat{s})^J)}{\sin(\pi J)} \quad (4.3.1)$$

Pomeron-Graviton Duality and Holographic Treatment of DIS: It can be shown for a wide range of scattering processes that the amplitude in the Regge limit, $s \gg t$, is dominated by Pomeron exchange, together with the associated s-channel screening correction, e.g., via eikonalization. In this representation, the near-forward amplitude can be expressed in terms of a Fourier transform over the 2-d transverse (impact parameter) space,

$$A(s, t) = 2is \int d^2b e^{i\vec{q} \cdot \vec{b}} \{1 - e^{i\chi(s, \vec{b})}\} \quad (4.3.2)$$

where $\chi(s, \vec{b})$ is the eikonal. To first order in χ , one has

$$A(s, 0) \simeq 2s \int d^2b \chi(s, b) \sim s^{j_0} \quad (4.3.3)$$

where j_0 is the Pomeron intercept. Traditionally, Pomeron has been modeled at weak coupling using perturbative QCD; in lowest order, a bare Pomeron was first identified as a two gluon exchange,

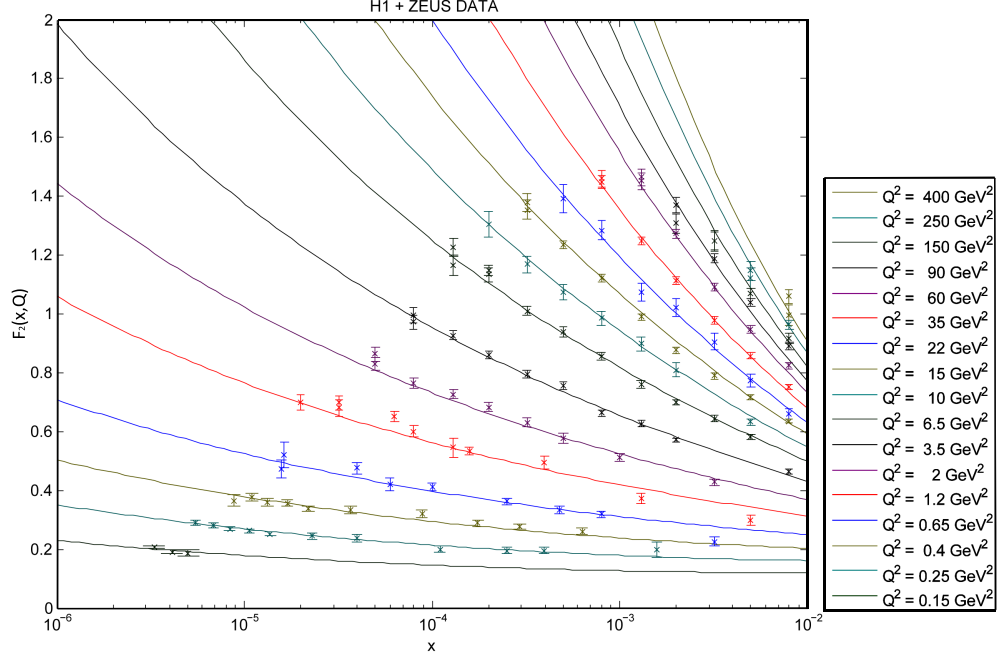


Figure 4.3.2: Structure function F_2 fits for the softwall model.

corresponding to a Regge cut in the J -plane at $j_0 = 1$.¹⁰ Going beyond the leading order by summing generalized two gluon exchange diagrams, led to the so-called BFKL Pomeron. The position of this J -plane cut is at $j_0 = 1 + \log(2)\lambda/\pi^2$, recovering the two-gluon cut in the $\lambda \rightarrow 0$ limit. In a holographic approach, the weak coupling Pomeron is replaced by the “Regge graviton” in AdS space, as formulated by Brower, Polchinski, Strassler and Tan (BPST) [1, 26]. The BPST Pomeron contains both the hard component due to near conformality in the UV and the soft Regge component in the IR. To first order in $1/\sqrt{\lambda}$, the intercept moves from $j = 2$, appropriate for a graviton, down to

$$j_0 = 2 - 2/\sqrt{\lambda} \quad (4.3.4)$$

In Fig. 3.2.1 a, we compare the BPST Pomeron intercept with the weak coupling BFKL intercept for $\mathcal{N} = 4$ YM as a function of 't Hooft coupling λ . A typical phenomenological estimate for this parameter for QCD is about $j_0 \simeq 1.25$, which suggests that the physics of diffractive scattering is in the cross over region between strong and weak coupling. A corresponding treatment for Odderons has also been carried out [44, 45].

In a holographic approach, the transverse space $(\vec{b}, z)$ is 3 dimensional, where $z \geq 0$ is the warped radial 5th dimension. The near-forward elastic amplitude again has the eikonal form [25–27],

¹⁰See chapter 2

$$A(s, t) = 2is \int d^2b \, e^{i\vec{q}\vec{b}} \int dz dz' \, P_{13}(z) \{1 - e^{i\chi(s, b, z, z')}\} P_{24}(z'). \quad (4.3.5)$$

where $t = -q_\perp^2$. For hadron-hadron scattering, $P_{ij}(z) = \sqrt{-g(z)}(z/R)^2 \varphi_i(z) \varphi_j(z)$ involves a product of two external normalizable wave functions for the projectile and the target respectively. Expanding in $\chi(s, b, z, z')$, to first order, it is seen that the eikonal function is related to a BPST Pomeron kernel in a transverse AdS_3 representation, $\mathcal{K}(s, b, z, z')$, with $\chi(s, b, z, z') = \frac{g_0^2}{2s} (\frac{R^2}{zz'})^2 \mathcal{K}(s, b, z, z')$.

An important unifying features for the holographic map is factorization in the AdS space. This approach can also be applied to all DIS cross sections since they can be related to the Pomeron exchange amplitude via the optical theorem, $\sigma = s^{-1} \text{Im} A(s, t = 0)$. For DIS, states 1 and 3 are replaced by currents, and we can simply replace P_{13} by product of the appropriate unnormalized wave-functions. In the conformal limit, P_{13} was calculated in [46] in terms of Bessel functions, so that, to obtain F_2 , we simply replace in (4.3.5),

$$P_{13}(z) \rightarrow P_{13}(z, Q^2) = (Q^2 z) [K_0^2(Qz) + K_1^2(Qz)]. \quad (4.3.6)$$

(Similarly, for F_1 , one has $P_{13}(z, Q^2) = (Q^2 z) K_1^2(Qz)$.) With this substitution, one has, e.g.,

$$F_2 = \frac{Q^2}{4\pi\alpha_{em}} (\sigma_T + \sigma_L) = \frac{Q^2}{4\pi\alpha_{em}s} [\text{Im} A(s, 0)_T + \text{Im} A(s, 0)_L] \quad (4.3.7)$$

When expanded to first order in χ , Eq. (4.3.5) provides the contribution to F_2 from exchanging a single Pomeron, i.e., the BPST kernel, $\mathcal{K}(s, b, z, z')$.

The momentum-space BPST kernel in the J -plane, $G_j(t, z, z')$, obeys a Schrödinger equation on AdS_3 space, with j serving as eigenvalue for the Lorentz boost operators M_{+-} .

In the conformal limit, it takes on a simple form, eq 3.3.3, where

$$\Delta(j) = 2 + \lambda^{1/4} \sqrt{2(j - j_0)} \quad (4.3.8)$$

is the conformal $\Delta - j$ curve shown in Fig. 5.0.1 b. The full Pomeron kernel can then be obtained via an inverse Mellin transform. In the mixed-representation, one has

$$K(s, b, z, z') \sim - \int \frac{dj}{2\pi i} \, \bar{s}^j \frac{e^{-i\pi j} + 1}{\sin \pi j} \frac{e^{(2-\Delta(j))\eta}}{\sinh \eta} \quad (4.3.9)$$

where $\cosh \eta$ is the chordal distance in AdS_3 . By integrating over $\vec{b}$, one obtains for the imaginary part of the Pomeron kernel at $t = 0$,

$$\text{Im } \mathcal{K}(s, t = 0, z, z') \sim \frac{s^{j_0}}{\sqrt{\pi \mathcal{D} \log s}} e^{-(\log z - \log z')^2 / \mathcal{D} \log s} \quad (4.3.10)$$

which exhibits diffusion in the “size” parameter $\log z$ for the exchanged closed string, analogous to the BFKL kernel at weak coupling, with diffusing taking place in $\log(k_\perp)$, the virtuality of the off-shell gluon dipole. The diffusion constant takes on $\mathcal{D} = 2/\sqrt{g^2 N_c}$ at strong coupling compared to $\mathcal{D} = 7\zeta(3)g^2 N_c/2\pi^2$ in weak coupling.

4.4 Fit to HERA Data:

To confront data at HERA, it is necessary to face the issue of confinement and saturation. Confinement can be addressed via a hardwall cutoff, $z < z_0$, or via a softwall model, which we shall return to shortly. The effect of saturation can next be included via the AdS_3 eikonal representation (4.3.5).

To extract the key feature of holographic treatment, we shall first adopt a simplifying assumption. We note that both integrals in z and z' in (4.3.5), are sharply peaked, the first around $z \sim 1/Q$ and the second around the inverse proton mass, $z' \equiv 1/Q' \sim 1/m_p$. To gain an understanding on the key features of dual approach, it is sufficient to approximate both integrand by delta functions. Under such an “ultra-local” approximation, all structure functions take on very simple form, e.g.,

$$F_2(x, Q^2) = \frac{g_0^2}{8\pi^2 \lambda} \frac{Q}{Q'} \frac{e^{(j_0 - 1) \tau}}{\sqrt{\pi \mathcal{D} \tau}} e^{-(\log Q - \log Q')^2 / \mathcal{D} \tau} + \text{Confining Images}, \quad (4.4.1)$$

with diffusion time given more precisely as $\tau = \log(s/QQ'\sqrt{\lambda}) = \log(1/x) - \log(\sqrt{\lambda}Q'/Q)$. Here the first term is conformal. To incorporate confinement, we consider first the hardwall model where the confining effect can be expressed in terms of image charges [1]. It is important to note, with or without confinement, the amplitude corresponding to (4.4.1) grows asymptotically as $(1/x)^{j_0 - 1} \sim s^{j_0 - 1}$, thus violating the Froissart unitarity bound at very high energies. The eikonal approximation in AdS space, (4.3.5), restores unitarity via multi-Pomeron shadowing [25–27].

We have shown various comparisons of our results [47] to the small- x DIS data from the combined H1 and ZEUS experiments at HERA [48] in Fig. 4.4.1. Both the conformal, the hard-wall model, soft-wall, as well as the eikonalized hard-wall model can fit the data reasonably well. This can best

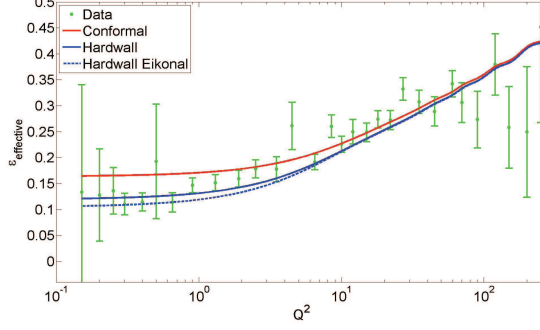


Figure 4.4.1: The BPST Pomeron intercept at 1.22, Q^2 dependence for “effective intercept” is shown for conformal, hardwall and hardwall eikonal model.

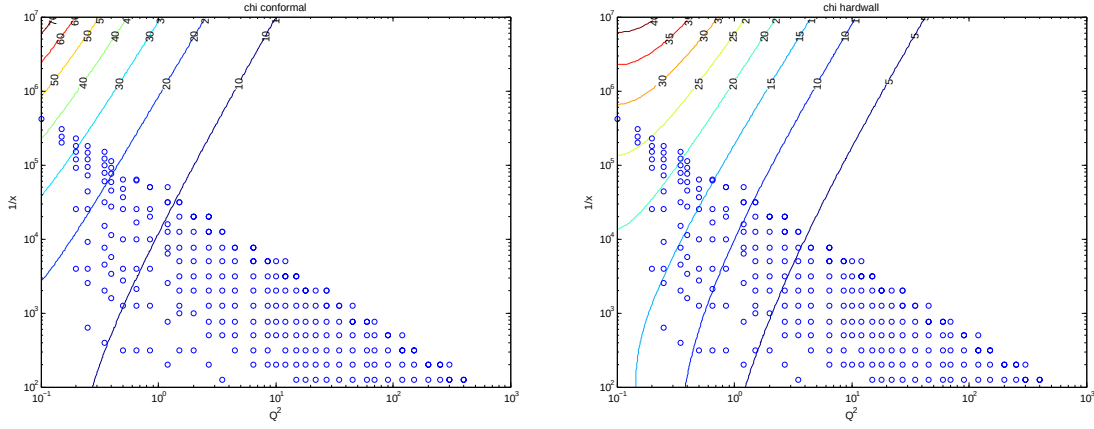


Figure 4.4.2: Contour plots of $\text{Im}(\chi)$ for the conformal (left) and hardwall (right) models.

be seen in Fig. 4.4.1a to the left which exhibits the Q^2 dependence of an effective Pomeron intercept. This can be understood as a consequence of diffusion. However, it is important to observe that the hard-wall model provides a much better fit than the conformal result for Q^2 less than the transition scale, $Q_c \sim 2 \sim 3 \text{ GeV}^2$. The best fit to data is obtained using the hard-wall eikonal model, with a $\chi^2 = 1.04$. This is clearly shown by Fig. 4.4.1b, where we present a comparison of the relative importance of confinement versus eikonal at the current energies. We observe that the transition scale $Q_c^2(x)$ from conformal to confinement increases with $1/x$, and it comes before saturation effect becomes important. For more details, see Ref. [47].

Confinement and Softwall: It is clear that, for Q^2 small, confinement is important. We find that confinement effect persists at an increasingly large value of Q^2 as $1/x$ increases. Equally important is the fact that the transition scale $Q_c^2(x)$ from conformal to confinement increases with $1/x$, and it comes before saturation effect becomes important. Therefore the physics of saturation should

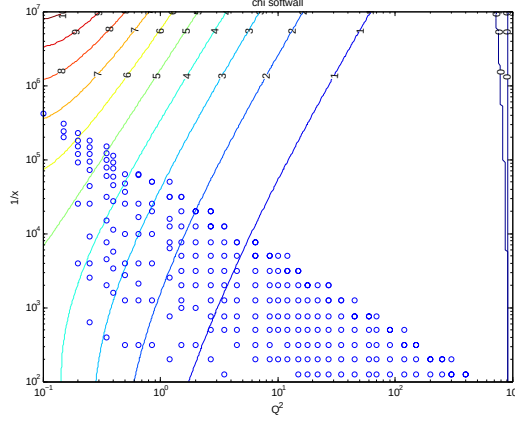


Figure 4.4.3: Contour plot of $\text{Im}(\chi)$ for the softwall model.

be discussed in a confining setting ¹¹.

The soft wall model was originally proposed in [49], showing what type of AdS confinement would lead to linear meson trajectories. Several dynamical softwall toy models, where the confinement is due to a non-trivial dilaton field, have subsequently been described. ¹² There has even been some success in using the softwall model to fit QCD mesons. [50] ¹³ Significant effort has been put forth to develop standard model and QCD features in these softwall models. [35, 50, 51] For our purpose, as emphasized in [1, 26], it is sufficient to consider traceless transverse fluctuations, which do *not* couple to the dilaton field, and thus a purely geometric confinement model is sufficient ¹⁴

In the softwall model, the graviton dynamics involves a spin dependent mass-like term $\alpha^2(j) = 2\sqrt{\lambda}(j - j_0)$, leading to a scalar-like equation for Pomeron propagator, $[-\partial_z^2 - (t + 2c\Lambda^2) + (\Lambda^4 z^2 + 15/4z^2) + \alpha^2(j)e^{2A(z)}/R^2]\chi_P(j, z, z', t) = \delta(z - z')$. More precisely, for soft-wall model, each Regge contribution, label by n , corresponds to a coherent sum of exchanging t-channel modes with $t \rightarrow t_n(j) \simeq \Lambda^2(4n + d(j))$, with $n = 0$ for the leading Pomeron. In the previous sections we saw that the corresponding propagator can be written as combination of Whittaker's functions and their

¹¹This has been stressed in [47]. In contrast, conventional treatment, e.g., color-glass condensate, assumes that saturation scale can be understood perturbatively.

¹²See various references cited in [49]. For a more general discussion of confining potentials in holographic theories and their relationship to QCD see [36–38] and the references therein.

¹³These models involve an additional dynamical dilaton and tachyon field, but there is still debate about some of the signs of some parameters [39]

¹⁴For our present purpose, we replace A in our metric Weyl factor by $c(\Lambda z)^2/3 - \log(z/R)$, with $c = \pm 1$. The choice for c remains a source of debate [39]. For this analysis, we shall keep $c = -1$, as originally done in [33]. It is interesting to note that the choice of the sign $c = +1$ leads to potentially interesting physics in the diffraction region of $t \leq 0$, and this and other details will be treated in a forthcoming publication.

Wronskian

$$\chi_P(j, z, z', t) = \frac{M_{\kappa, \mu}(z_{<})W_{\kappa, \mu}(z_{>})}{W(M_{\kappa, \mu}, W_{\kappa, \mu})} \quad (4.4.2)$$

for $\kappa = \kappa(t)$ and $\mu = \mu(j)$. Λ controls the strength of the soft wall and, in the limit $\Lambda \rightarrow 0$, one recovers the conformal solution¹⁵

$$\text{Im}(\chi_P^{\text{conformal}}(t=0)) = \frac{g_0^2}{16} \sqrt{\frac{\rho^3}{\pi}} (zz')^{\frac{e^{(1-\rho)\tau}}{\tau^{1/2}}} \exp\left(\frac{-(\ln(z) - \ln(z'))^2}{\rho\tau}\right) \quad (4.4.3)$$

If we look at the energy dependence of the pomeron propagator, we can see a softened behavior in the regge limit. In the forward limit, $t = 0$, the conformal amplitude scales as $-s^{\alpha_0} \log^{-1/2}(s)$, but this behavior is softened to $-s^{\alpha_0} \log^{-3/2}(s)$ in the hardwall and softwall models. This corresponds to the softening of a j -plane singularity from $1/\sqrt{j - j_0} \rightarrow \sqrt{j - j_0}$.

We provide below a comparative analysis for various options. The data examined comes from the combined H1 and ZEUS experiments at HERA. [48] A fit using softwall treatment was done with the same methods used previously for the conformal and hardwall models in [47], making the results directly comparable.

Model	ρ	g_0^2	z_0 (GeV ⁻¹)	Q' (GeV)	χ_{dof}^2
conformal	$0.774^* \pm 0.0103$	$110.13^* \pm 1.93$	–	$0.5575^* \pm 0.0432$	11.7 (0.75*)
hard wall	0.7792 ± 0.0034	103.14 ± 1.68	4.96 ± 0.14	0.4333 ± 0.0243	1.07 (0.69*)
softwall	0.7774	108.3616	8.1798	0.4014	1.1035
softwall*	0.6741	154.6671	8.3271	0.4467	1.1245

Table 4.1: Comparison of the best fit (including a χ sieve) values for the conformal, hard wall, and soft wall AdS models. The final row includes the soft wall with improved intercept, calculated up to third order in λ using Eq. (5.0.1).

The softwall* row describes indicates that the fit was run using a pomeron intercept (which determines λ) up to order $\mathcal{O}(\lambda^{-5/2})$ [44, 45]. This has been made possible due to integrability and Regge techniques in $\mathcal{N} = 4$ SYM. (See chapter 5) It is interesting to examine higher order effects further, particularly on the confining structure in the b -space, eikonalization, and saturation. The will be addressed elsewhere.

For the purely conformal case, one can include saturation effects by observing when $\chi_{b=0} \sim \mathcal{O}(1)$, at this point an eikonal approach can be used to govern multiple pomeron exchange [25]. In the original hard wall approximation, it was shown that the χ contours are shifted to larger Q^2 indicating that the leading effects of confinement set in before saturation occurs. Fig 4.4.3 shows that this

¹⁵This has a similar behavior to the weak coupling BFKL solution where $\text{Im}(\chi(p_{\perp}, p'_{\perp}, s)) \sim \frac{s^{j_0}}{\sqrt{\pi \mathcal{D} \ln(s)}} \exp(-(\ln(p'_{\perp}) - \ln(p_{\perp}))^2 / \mathcal{D} \ln(s))$

similar shift happens for the softwall indicating that one must address the issue of confinement before saturation sets in.

Chapter 5

Anomalous Dimensions

motivation, $\Delta(j)$ curves, integrability, pomeron and odderon anomalous dimension to high order

AdS/CFT correspondence [52–55], has provided a useful perspective on several domains of non-perturbative QCD, such as heavy ion collisions, low-energy meson dynamics, and high energy scattering. In particular, at strong coupling the leading Pomeron exchange has been identified as a *Reggeized AdS Graviton*, in the planar approximation to $\mathcal{N} = 4$ Super Yang Mills (SYM) Theory [1].

A crucial observation made in [1] is the role played by the analytic continuation in the $\Delta - j$ plane for anomalous dimensions, $\gamma(j) = \Delta(j) - j - \tau$, for the leading twist operators as a function of j and the 't Hooft coupling $\lambda = g^2 N_c$. In a conformal field theory, the inverse curve in the “Dimension- j ” plane, $j(\Delta)$, plays a central role analogous to the traditional Regge pole trajectory $\alpha(t)$ in “Energy- j ” plane. As a spectral curve, $\Delta(j)$ has a remarkable symmetry due to conformal invariance: the inverse curve, $j(\Delta)$, is symmetric under $\Delta \leftrightarrow 4 - \Delta$,¹ with a minimum at $\Delta = 2$, as shown in Fig. 5.0.1 for the leading twist-2 spectral curve. At integer j this symmetry relates operators to shadow operators in the conformal field theory. The value of j at this minimum, $j_0(\lambda)$, corresponds to the location of the conformal Regge intercept associated with a given spectral curve $\Delta(j)$. In strong coupling, the Pomeron intercept [1, 56],

$$\alpha_P = j_0(\lambda) = 2 - \frac{2}{\lambda^{1/2}} + O(1/\lambda), \quad (5.0.1)$$

¹More succinctly stated, conformal symmetry implies a j -plane trajectory as a function of $\Delta(\Delta - 4) = M_{ads}^2 R_{ads}^2$ just as Lorentz invariance implies Regge j -plane trajectories as a function of $\alpha' t$. We will occasionally refer to $j(\Delta)$ as the “conformal Reggeon spin”, or simply “Reggeon spin”.

was determined by perturbing the spectral curve $\Delta_P(j)$ about the graviton at $j = 2$ in the super-gravity limit. The same principle can be applied to weak coupling, with the minimum identified with the BFKL Pomeron intercept [57–61]

$$\alpha_P = j_0(\lambda) = 1 + \frac{4 \ln 2}{\pi} \lambda + O(\lambda^2). \quad (5.0.2)$$

A major challenge in $\mathcal{N} = 4$ SYM is to determine this intercept $j_0(\lambda)$ for all λ in the large N_c approximation and to apply this analysis to other trajectories [62].

In this paper we apply an expansion procedure for spectral curves, that describes short strings in AdS [63–65], to improve the strong coupling expansion for both the leading conformal Pomeron ($C = +1$) and Odderon ($C = -1$) trajectories. For the Pomeron this is a simple extension of earlier results [24, 66–68], while the application to the Odderon trajectory [20, 44] is new. For $\mathcal{N} = 4$ SYM, the $C = +1$ exchange is associated with exchange of local operators with $\pm$ light-cone components,

$$\mathcal{O}_P^\pm(j, k) = Tr[F^{\pm\perp}(D^\pm)^{j-2}F_\perp^\pm Z^k] + \dots, \quad (5.0.3)$$

with $j = 2, 4, 6, \dots$ and $k = 0, 1, \dots$. Here, Z is a scalar field with $SU(4)$ R-charge. The leading Pomeron has $k = 0$, but we extend known results to the case of R-charge exchange with $k \neq 0$. Due to super-symmetry, the analysis can be simplified by relating the relevant spectral curves to that for single-trace operators in the $sl(2)$ sector, symbolically expressed as $Tr[(D^\pm)^{j-2}Z^{k+2}] + \dots$ [19, 69]. For the case of the $C = -1$ Odderon trajectory, we shall show how similar techniques can also be applied ².

²Preliminary result was first reported at the “Low-x Workshop”, 2013, Rehovot and Eilat, Israel [44].

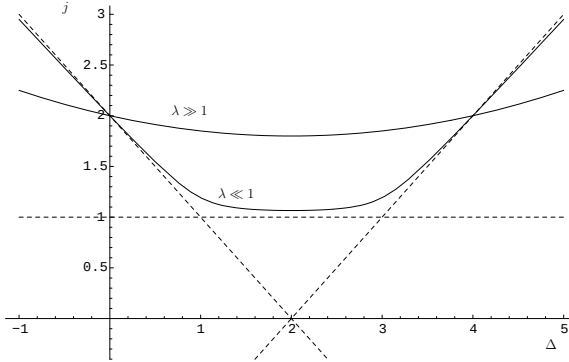


Figure 5.0.1: The conformal Regge trajectory reproduced from Ref. [1] : Schematic form of the $\Delta - j$ relation for twist-2 spectral curve at weak ($\lambda \ll 1$) and strong coupling ($\lambda \gg 1$).

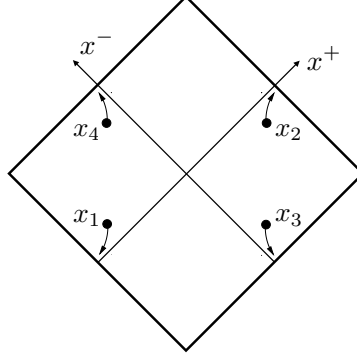


Figure 5.0.2: Conformal compactification of the light-cone showing points taken to null infinity in the Regge limit. In light-cone coordinates $(x^+, x^-, x_\perp)$ we take $-x_1^+ \sim x_2^+ \rightarrow \infty$ and $-x_3^- \sim x_4^- \rightarrow \infty$, keeping $x_\perp^i$ fixed.

We shall restrict our attention to the pure conformal limit and begin here by briefly describing the Regge limit in the context of conformal field theories [21–24]. Consider the connected component for a four point correlation function of primary operators $\mathcal{O}_i$ of dimension Δ_i . Defining $x_{ij} = x_i - x_j$, we have

$$A(x_i) = \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle_c = \frac{1}{(x_{12}^2)^{\Delta_1} (x_{34}^2)^{\Delta_3}} F(u, v), \quad (5.0.4)$$

where

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (5.0.5)$$

are the cross ratios and for simplicity we have assumed $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4$. We need to examine the double light-cone limit of vanishing x_{12}^2 and x_{34}^2 , which corresponds to $u \rightarrow 0$ and $v \rightarrow 1$ in a Minkowski setting. From the perspective of light-cone OPE, this limit can be reached by scaling $x_1^+ \rightarrow \lambda x_1^+$, $x_2^+ \rightarrow \lambda x_2^+$, $x_3^- \rightarrow \lambda x_3^-$, $x_4^- \rightarrow \lambda x_4^-$ with $\lambda \rightarrow \infty$, keeping the causal relations $x_{14}^2, x_{23}^2 < 0$, as illustrated in Fig. 5.0.2. In a frame where $x_{1\perp} = x_{2\perp}$ and $x_{3\perp} = x_{4\perp}$, this corresponds to approaching the respective null infinity while keeping the relative impact parameter

$$b_\perp = x_{1\perp} - x_{3\perp}, \quad (5.0.6)$$

fixed. In terms of the cross ratios, this corresponds to $u \rightarrow 0$ and $v \rightarrow 1$, with $(1-v)/\sqrt{u}$ fixed. Alternatively, defining $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$ with $z = \sigma e^\rho$ and $\bar{z} = \sigma e^{-\rho}$, the precise Regge limit can be specified by $\sigma \rightarrow 0$ for fixed ρ .

Using Regge theory the leading behavior of the reduced correlation functions F can be determined by the intercept j_0 , computed from the corresponding leading spectral curve $\Delta(j)$ described earlier,

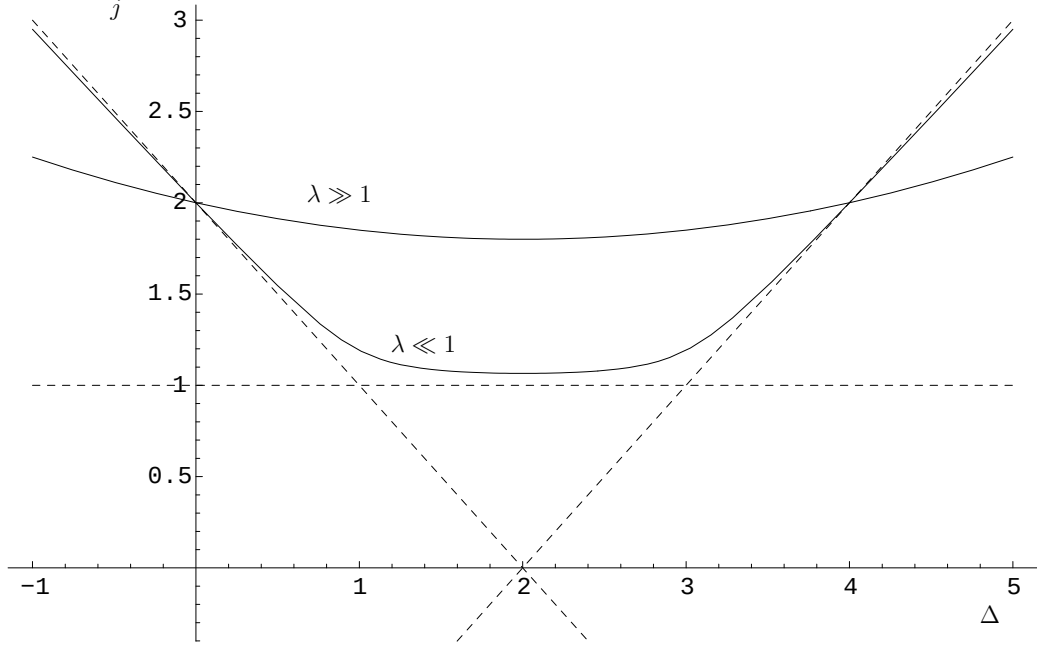


Figure 5.0.3: $\Delta(J)$ curve.

with a general form

$$F(\sigma, \rho) \approx f(\rho) \frac{\sigma^{1-j_0}}{|\ln \sigma|^{3/2}}. \quad (5.0.7)$$

This is entirely analogous to conventional Regge theory where the spectrum of Regge trajectories determines high energy scattering amplitudes.

The paper is organized as follows. In Sec. 3.4, we will review the dictionary that translates the Regge description from the language of CFT and OPE into scattering amplitudes in AdS space. This will allow us to proceed directly within the framework of CFT's without explicit recourse to the AdS/CFT correspondence. Nonetheless reference to AdS space can and will be made to provide additional intuition. In Sec. 5.1, we analyze the Pomeron intercept beyond the “diffusion limit”, (5.0.1), leading to a systematic expansion for the Pomeron intercept in $\lambda^{-1/2}$. A similar analysis is done for Odderons in Sec. 5.2. In particular, we clarify how one of the Odderon solutions has intercept that remains fixed at 1 to all orders in $1/\sqrt{\lambda}$. These results are summarized and discussed in Sec. 5.3 where we also clarify further the all-coupling formula proposed by Basso in [65], and its possible generalization. We also provide additional comments relating to the weak coupling limit and other issues.

5.1 Conformal Pomeron

The Pomeron spectral curve $\Delta_P(j)$ in the strong coupling limit, Fig. 5.0.1, can be obtained by an intuitive derivation based on a flat-space leading closed-string linear trajectory. Through AdS/CFT, this simple result can be understood as a perturbation about the traceless-transverse graviton mode, $\nabla^2 h_{MN} = 0$, with $j = 2$ and $\Delta = 4$ in the $\lambda = \infty$ super-gravity limit. Here ∇^2 is the tensor Laplacian on AdS_5 . Let us now consider the limit of $j \rightarrow 2$ and $\lambda \rightarrow \infty$ with $\sqrt{\lambda}(j - 2)$ fixed. This limit can be understood by introducing a Reggeon vertex operator, $\mathcal{V}^\pm$, on the string world sheet in a weakly curved target $AdS_5 \times S^5$ space [1]. This Reggeon vertex operator depends on (j, ν, t) for the $O(4, 2)$ Casimir, and on $k = \tau - 2$ in case we wish to consider exchange of $SO(6)$ R -charge. The effect of the Reggeon operator is to resum the exchange of all modes in the leading Regge trajectory with even positive integral spins, which leads to the effective Regge spin

$$j = 2 - \frac{\tau^2 + \nu^2}{2\sqrt{\lambda}}. \quad (5.1.1)$$

Then the world-sheet Virasoro on-shell condition, $L_0 \mathcal{V}^\pm = \bar{L}_0 \mathcal{V}^\pm = \mathcal{V}^\pm$, establishes the relation

$$\Delta_P(j, \tau) = 2 + i\nu \quad (5.1.2)$$

where $\Delta_P(j, \tau)$ is the continuation of the anomalous dimension curve for the exchanged gauge theory operators in the leading Regge trajectory $\mathcal{O}_P^\pm(j, \tau)$ given in (5.0.3). By following this procedure, one relates the exchange of AdS higher spin fields to the dual gauge theory operators with spectral curve $\Delta_P(j, \tau)$.

In the above double limit the Regge spin has a quadratic dependence in the dimension ν , also known as the diffusion limit (in a weak coupling expansion we may also consider such a diffusion limit). The terminology stems from the fact that the kernel in momentum space takes on a diffusion form at $t = 0$, with diffusion time $\ln(zz's)$ fixed by the AdS energy $S = zz's$ [1]. In this limit we can already observe that the spectral curve $\Delta_P(j, \tau)$ has a branch point at j_0 ,

$$\Delta_P(j) = 2 + \sqrt{2\lambda^{1/2}} \sqrt{j - j_0}. \quad (5.1.3)$$

where

$$j_0 = 2 - \frac{\tau^2}{2\sqrt{\lambda}}, \quad (5.1.4)$$

which generalizes (5.0.1) for the case of twist τ .

Beyond the diffusion limit (5.1.1) the Reggeon spin $j = j(\nu, \tau)$ admits the strong coupling expansion

$$j(\nu, \tau) = 2 - \frac{\tau^2 + \nu^2}{2\sqrt{\lambda}} \left(1 + \sum_{n=2}^{\infty} \frac{\tilde{j}_n(\nu^2, \tau)}{\lambda^{(n-1)/2}} \right), \quad (5.1.5)$$

which is a simple generalisation to arbitrary twist τ of the results presented in [21, 24] (such that at infinite coupling, for $j = 2$, the dimension of the operator is given by the protected value of $2 + \tau$). Notice that $j(\nu, \tau)$ must be an even function of ν to implement the symmetry property $\Delta_P(j, \tau) \leftrightarrow 4 - \Delta_P(j, \tau)$. The function $\tilde{j}_n(\nu^2, \tau)$, defined for $n \geq 2$, is a polynomial of degree $n - 2$,

$$\tilde{j}_n(\nu^2, \tau) = \sum_{k=0}^{n-2} c_{n,k} \nu^{2k}, \quad (5.1.6)$$

with τ -dependent coefficients $c_{n,k}$. This follows from the requirement that the *AdS* amplitude has a well defined flat space limit [21]. Consistency with the strong coupling expansion of the spectral curve $\Delta_P(j, \tau)$ further restricts this polynomials to have smaller degree [24], more precisely, for $n \geq 4$

$$c_{n,k} = 0 \quad \text{for} \quad \left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n - 2, \quad (5.1.7)$$

as also confirmed in [67].

Eq. (5.1.5) corresponds to an expansion for the Reggeon spin about the symmetry point $\Delta = 2$, ($\nu^2 = 0$), in the strong coupling limit, subject to the constraint that $j = 2$ at $\nu^2 = -\tau^2$. In the next subsection we review recent results for the strong coupling expansion of the spectral curve $\Delta_P(j, \tau)$ about $j = 2$ that will allow us to compute the pomeron spin $j(\nu)$, i.e., the inverse of the spectral curve $\Delta(j)$, with Δ and ν related by (5.1.2), beyond the diffusion limit at arbitrary twist τ . The discussion leads to a unified picture that can also be applied to the Odderon Regge trajectories, in Sec. 5.2.

5.1.1 Pomeron spin versus anomalous dimensions at strong coupling

Much attention has been paid in recent years to the study of anomalous dimensions for composite operators of $\mathcal{N} = 4$ SYM. Because of supersymmetry, many related operators share the same anomalous dimensions. It is generally believed that, due to integrability [70], scaling dimensions for gauge invariant operators can be efficiently calculated for all 't Hooft coupling, in the large- N planar limit, via the so-called TBA/Y-system and its generalizations [15]. These operators and their cousins can be treated as generalized Heisenberg spin chains. For instance, the weak-coupling one-loop

anomalous dimension $\gamma(S, \tau)$ of single-trace operators in the $sl(2)$ sector, symbolically expressed as $Tr[D_{\pm}^S Z^{\tau}] + \dots$, can be calculated explicitly, with $\Delta(S, \tau) = S + \tau + \gamma(S, \tau)$ their scaling dimension. However, beyond one-loop, and particularly for short operators (S and τ small), analytic solutions have been difficult to obtain. For strong coupling, conformal dimensions can be calculated semi-classically in a world-sheet sigma model approach around soliton solutions [71–73], as semi-classical treatment for GKP strings. This leads to a strong coupling loop-expansion, with $1/\sqrt{\lambda}$ playing the role of $\hbar$,

$$\Delta = \lambda^{1/4} \left(\delta_0 + \frac{\delta_1}{\lambda^{1/2}} + \frac{\delta_2}{\lambda} + \frac{\delta_3}{\lambda^{3/2}} + \frac{\delta_4}{\lambda^2} + \dots \right), \quad (5.1.8)$$

with S and τ dependence entering the ℓ -loop contribution δ_{ℓ} through a scaling hypothesis³. However, calculation beyond 1-loop is impractical. In most approaches of this type, the emphasis has been on long strings.

One analysis of particular interest to us is the expansion for the spectral curve about the point $S = 0$,

$$\Delta_Z(S, \tau) = \tau + \alpha_1(\tau, \lambda)S + \alpha_2(\tau, \lambda)S^2 + \alpha_3(\tau, \lambda)S^3 + \dots \quad (5.1.9)$$

where $\Delta_Z(0, \tau) = \tau$, since the operator is 1/2-BPS and its dimension is protected. The form of the “slope function”, $\alpha_1(\lambda, \tau)$, has recently been conjectured by Basso for all ’t Hooft coupling [65] (see also [76, 77]), which can be expressed in a compact form in terms of Bessel functions,

$$\alpha_1(\lambda, \tau) = \frac{\sqrt{\lambda}}{\tau} Y_{\tau}(\sqrt{\lambda}), \quad (5.1.10)$$

for all λ , with $Y_{\tau}(x) = I'_{\tau}(x)/I_{\tau}(x)$, where $I_{\tau}(x)$ is the τ -th modified bessel function. At weak coupling, $\alpha_1(\lambda, \tau) = 1 + O(\lambda)$, and at the strong coupling, $\alpha_1(\lambda, \tau) = \sqrt{\lambda}/\tau + O(1/\sqrt{\lambda})$. This result was first derived for $\tau = 2$ as a solution to the “asymptotic Bethe ansatz” (ABA) equations. It has been argued in [65], with further support in [76, 77], that this holds for all $\tau > 2$, for the configurations with “minimum mode numbers”⁴. More recently, the second coefficient, $\alpha_2(\lambda, \tau)$, has also been calculated numerically, but it is not possible at this moment to express it in a closed form in terms of elementary functions [68, 78]. Due to super-symmetry, it is known that the Pomeron

³ δ_{ℓ} is assumed to be a function of $S = S/\sqrt{\lambda}$ and $\mathcal{T} = \tau/\sqrt{\lambda}$. Occasional discussions for small S and/or small τ are typically based on extrapolation under this scaling hypothesis. As such, less attention has been paid in the past to the symmetry property in Δ . For an alternative but related study, see [74, 75].

⁴We will return to a discussion on this and related issues in Secs. 5.2 and 5.3.

spectral curve ⁵ is directly related to $\Delta_Z(S, \tau)$ at $\tau = 2$ by [19, 69]

$$\Delta_P(j) = 2 + \Delta_Z(j - 2, 2). \quad (5.1.11)$$

Therefore, these recent analyses, appropriately generalized, can be applied to our study of conformal Pomeron and Odderon, particularly in the large λ limit.

For planar $\mathcal{N} = 4$ SYM, it is possible to generalize our discussion for the Pomeron spectral curve to include CFT operators with R charges, which through AdS/CFT, amounts to allow fluctuations in S^5 . For our purpose, as already described in (5.0.3), the relevant CFT operators are

$$\mathcal{O}_P(j, \tau) = Tr [F_{\mu\sigma} D_{\rho_1} \cdots D_{\rho_s} F_{\sigma\nu} Z^{\tau-2}] + \cdots, \quad (5.1.12)$$

with $\tau \equiv 2 + k \geq 2$. For the leading Regge singularity, we will be dealing with the light-cone components $Tr [F_{\pm\pm} D_{\pm} \cdots D_{\pm} F_{\pm\pm} Z^{\tau-2}] + \cdots$. The generalized Pomeron spectral curve $\Delta_P(j, \tau)$ can be expanded using (5.1.11) and (5.1.9) around $j = 2$, leading to

$$\Delta_P(j, \tau) = 2 + \tau + \alpha_1(\lambda, \tau)(j - 2) + \alpha_2(\lambda, \tau)(j - 2)^2 + \alpha_3(\lambda, \tau)(j - 2)^3 + \cdots. \quad (5.1.13)$$

where (5.1.10) applies.

Because of the symmetry under $\Delta_P(j, \tau) \leftrightarrow 4 - \Delta_P(j, \tau)$, we again require the function $\Delta_P(j, \tau)$ to have a square-root singularity at $j_0(\tau)$, with $j_0(\tau) = 2 - O(\lambda^{-1/2})$. This branch point renders the expansion (5.1.13) with a radius of convergence which vanishes as $\lambda^{-1/2}$, leading to expansion coefficients which grow as $\alpha_n \sim O(\lambda^{n/2})$. Nevertheless, a convergent expansion can be achieved by considering the symmetric combination $(\Delta_P(j, \tau) - 2)^2 = -\nu^2$, for which this square-root branch point is absent. This in turn leads to a convergent expansion in the strong coupling limit,

$$(\Delta_P(j, \tau) - 2)^2 = \tau^2 + \beta_1(\lambda, \tau)(j - 2) + \beta_2(\lambda, \tau)(j - 2)^2 + \beta_3(\lambda, \tau)(j - 2)^3 + \cdots, \quad (5.1.14)$$

where $\beta_1(\lambda, \tau) = 2\tau\alpha_1(\lambda, \tau)$, $\beta_2(\lambda, \tau) = \alpha_1(\lambda, \tau)^2 + 2\tau\alpha_2(\lambda, \tau)$, etc. Based partly on semi-classical analysis of GKP strings [71–73], one expects a radius of convergence of the order $O(\lambda^{1/2})$. Consistency with (5.1.3) and the existence of a smooth super-gravity limit then require that $\beta_n \sim$

⁵In order to avoid notational confusion, in what follows, instead of S , we shall switch to $j = S + 2$, e.g., for the $sl(2)$ sector, we have $Tr[D_{\pm}^{j-2} Z^{\tau}]$, instead of $Tr[D_{\pm}^S Z^{\tau}]$.

$O(\lambda^{(2-n)/2})$, so that each coefficient $\beta_n(\lambda)$ in turn admits an expansion,

$$\beta_n = 2\lambda^{\frac{2-n}{2}} \left(b_{n,0} + \frac{b_{n,1}}{\lambda^{1/2}} + \frac{b_{n,2}}{\lambda} + \frac{b_{n,3}}{\lambda^{3/2}} + \dots \right), \quad (5.1.15)$$

where we have taken the factor of 2 out so that later on the expansion will be normalized with $b_{1,0} = 1$, and the coefficients $b_{n,m}$ are in general τ -dependent. The viability of the strong coupling treatment done by Basso in [65] relies on this rapidly convergent expansion⁶. It is now a simple exercise to check that, since β_n starts at order $\lambda^{(2-n)/2}$, we have

$$(\Delta_P(j, \tau) - 2)^2 = \tau^2 + 2\sqrt{\lambda}(j-2) \left(1 + \sum_{k=1}^{\infty} \lambda^{-\frac{k}{2}} H_k(j-2, \tau) \right), \quad (5.1.16)$$

with

$$H_k(j-2, \tau) = \sum_{n=0}^k b_{n+1, k-n} (j-2)^n. \quad (5.1.17)$$

a polynomial of degree k in $(j-2)$. The form of this expansion should be compared with (5.1.5) for the function $j(\nu, \tau)$. Both expansions make explicit that we are doing a strong coupling expansion around the diffusion limit $j \rightarrow 2$ and $\lambda \rightarrow \infty$ with $\sqrt{\lambda}(j-2)$ fixed.

Let us now relate the expansions (5.1.14) and (5.1.16) for the dimension $\Delta_P(j, \tau)$, to the expansion (5.1.5) for the Reggeon spin $j(\nu, \tau)$. For simplicity we consider first the computation of the intercept. Since $\beta_1(\lambda, \tau) > 0$ the function $j(\nu, \tau)$ has a minimum at $\nu = 0$ ($\Delta = 2$). This minimum determines the Pomeron intercept $j_0 = j(0, \tau)$. Thus, setting $\Delta_P(j_0, \tau) = 2$ in (5.1.14) we learn that the intercept is fixed by

$$0 = (\Delta_P(j_0, \tau) - 2)^2 = \tau^2 + \beta_1(\lambda, \tau)(j_0 - 2) + \beta_2(\lambda, \tau)(j_0 - 2)^2 + \beta_3(\lambda, \tau)(j_0 - 2)^3 + \dots, \quad (5.1.18)$$

with β_n given by the expansion (5.1.15). Replacing the expansion for the intercept, as defined by (5.1.5), it is now possible to find j_0 iteratively in an expansion

$$j_0 = 2 + \frac{c_1}{\lambda^{1/2}} + \frac{c_2}{\lambda} + \frac{c_3}{\lambda^{3/2}} + \frac{c_4}{\lambda^2} + \dots. \quad (5.1.19)$$

Note that the coefficients c_i are already defined in the expansion (5.1.5) of the Reggeon spin function, more precisely we have $c_1 = -\tau^2/2$ and $c_n = c_1 c_{n,0}$ for $n \geq 2$. Substituting (5.1.19) and (5.1.15) into

⁶It is worth noting that the expansion for β_1 in $1/\sqrt{\lambda}$ can be identified with the GKP-loop expansion, i.e., the coefficient $b_{1,m}$ is a m -loop contribution. The same no longer holds for β_n , $n > 1$. In general, each coefficient $b_{n,m}$ mixes contributions from different loop orders.

(5.1.18), and collecting all terms in powers of $1/\sqrt{\lambda}$, one can determine c_n iteratively. To illustrate how this goes, we list here the first few coefficients (note that we set $b_{10} = 1$),

$$\begin{aligned}
c_1 &= -\tau^2/2, \\
c_2 &= -b_{1,1}c_1, \\
c_3 &= -[b_{1,1}c_2 + b_{1,2}c_1 + b_{2,0}c_1^2], \\
c_4 &= -[b_{1,1}c_3 + b_{1,2}c_2 + b_{1,3}c_1 + 2b_{2,0}c_1c_2 + b_{2,1}c_1^2], \\
c_5 &= -[b_{1,1}c_4 + b_{1,2}c_3 + b_{1,3}c_2 + b_{1,4}c_1 + b_{2,0}(c_2^2 + 2c_1c_3) + 2b_{2,1}c_1c_2 + b_{2,2}c_1^2 + b_{3,0}c_1^3], \\
c_6 &= -[b_{3,1}c_1^3 + b_{2,3}c_1^2 + 3b_{3,0}c_2c_1^2 + b_{1,5}c_1 + 2b_{2,2}c_2c_1 + 2b_{2,1}c_3c_1 + 2b_{2,0}c_4c_1 + b_{2,1}c_2^2 \\
&\quad + b_{1,4}c_2 + b_{1,3}c_3 + 2b_{2,0}c_2c_3 + b_{1,2}c_4 + b_{1,1}c_5].
\end{aligned} \tag{5.1.20}$$

In the diffusion limit, the intercept reduces to $\alpha_P(\tau) \equiv j_0(\tau) = 2 - \tau^2/2\sqrt{\lambda}$, as stated above. As for the leading twist case, $\tau = 2$, this intercept corresponds to the location of a square-root branch point for the spectral curve $\Delta_P(j, \tau)$, and it approaches $j = 2$ in the limit of $\lambda \rightarrow \infty$.

More generally, we can relate the coefficients of the polynomials $H_k(j-2, \tau)$ entering the expansion (5.1.16) of $\Delta_P(j, \tau)$, with the coefficients of the polynomials $\tilde{j}_n(\nu^2, \tau)$ entering the expansion (5.1.5) of $j(\nu, \tau)$. These functions are simply related by the inversion formula $\Delta_P(j(\nu, \tau), \tau) = 2 + i\nu$ given in (5.1.2). This is a mechanical computation, so we only give here the relation between the first coefficients without further explanations (excluding the coefficients $c_{n,0}$ already given above)

$$\begin{aligned}
c_{3,1} &= b_{2,0}/2, & c_{4,1} &= [-3b_{1,1}b_{2,0} + b_{2,1}]/2, & c_{4,2} &= 0, \\
c_{5,1} &= [6b_{1,1}^2b_{2,0} - 3b_{1,2}b_{2,0} + 2\tau^2b_{2,0}^2 - 3b_{1,1}b_{2,1} + b_{2,2} - \tau^2b_{3,0}]/2, \\
c_{5,2} &= [2b_{2,0}^2 - b_{3,0}]/4, & c_{5,3} &= 0, \\
c_{6,1} &= [-10b_{1,1}^3b_{2,0} + 12b_{1,1}b_{1,2}b_{2,0} - 3b_{1,3}b_{2,0} - 10\tau^2b_{1,1}b_{2,0}^2 + 6b_{1,1}^2b_{2,1} - 3b_{1,2}b_{2,1} + 4\tau^2b_{2,0}b_{2,1} \\
&\quad - 3b_{1,1}b_{2,2} + b_{2,3} + 4\tau^2b_{1,1}b_{3,0} - \tau^2b_{3,1}]/2, \\
c_{6,2} &= [-10b_{1,1}b_{2,0}^2 + 4b_{2,0}b_{2,1} + 4b_{1,1}b_{3,0} - b_{3,1}]/4, & c_{6,3} &= c_{6,4} = 0.
\end{aligned} \tag{5.1.21}$$

5.1.2 Explicit results for $\mathcal{N} = 4$ SYM

It is now clear that we can use the known results for the spectral curve $\Delta_P(j, \tau)$ for $j \sim 2$ to extract information about the strong coupling expansion Pomeron spin $j(\nu, \tau)$, and in particular to compute the Pomeron intercept for $\tau \geq 2$. We begin by noting that, since $\beta_1(\lambda, \tau) = 2\tau\alpha_1(\lambda, \tau)$ is known

analytically, it can be easily expanded in powers of $1/\sqrt{\lambda}$, with the result [65]

$$\beta_1(\lambda, \tau) = 2\lambda^{\frac{1}{2}} \left(1 - \frac{1}{2\lambda^{1/2}} + \frac{4\tau^2 - 1}{8\lambda} + \frac{4\tau^2 - 1}{8\lambda^{3/2}} + \frac{-16\tau^4 + 104\tau^2 - 25}{128\lambda^2} + \frac{-16\tau^4 + 56\tau^2 - 13}{32\lambda^{5/2}} + \dots \right). \quad (5.1.22)$$

From this expansion it is straightforward to extract the coefficients $b_{1,m}$. This is enough to fix c_1 and c_2 in the computation of the intercept (5.1.20), and in particular agrees with the diffusion limit result $c_1 = -\tau^2/2$.⁷

To find the intercept coefficients c_n for $n > 2$, knowledge of the coefficients $b_{n,m}$ for higher β_n are required. As mentioned earlier, (5.1.13) is an expansion with increasingly divergent coefficients, i.e. $\alpha_n = O(\lambda^{n/2})$. Clearly, very special cancellations must take place in moving from (5.1.13) to (5.1.14) for convergence. Recently, explicit expressions for $\alpha_2(\lambda, \tau)$, $\tau = 2$ and 3 , have been obtained [68]. With the aid of numerical analysis, together with consistency matching with (5.1.22), [68] also gives strong coupling expressions for arbitrary τ , up to order $\lambda^{-3/2}$,

$$\begin{aligned} \alpha_2(\lambda, \tau) = & -\frac{\lambda}{2\tau^3} + \frac{\lambda^{1/2}}{2\tau^3} + \frac{1}{4\tau} + \frac{1 - \tau^2(24\zeta(3) + 1)}{16\tau^3\lambda^{1/2}} - \frac{8\tau^4 + \tau^2(72\zeta(3) + 11) - 4}{32\tau^3\lambda} \\ & + \frac{24\tau^4(16\zeta(3) + 20\zeta(5) - 7) - 48\tau^2(31\zeta(3) + 20\zeta(5) + 7) + 75}{256\tau^3\lambda^{3/2}} + O(\lambda^{-2}). \end{aligned} \quad (5.1.23)$$

It is then possible to calculate expansions for β_2 , with relevant low order coefficients $b_{2,m}$ extracted from the expansions of α_1 and α_2 ,

$$\begin{aligned} \beta_2(\lambda, \tau) = & 2 \left(\frac{3}{4} - \frac{3\zeta(3) - 3/8}{2\lambda^{1/2}} - \frac{\tau^2 + 9\zeta(3) - 5/8}{4\lambda^{3/2}} \right. \\ & \left. + \frac{\tau^2(3\zeta(3) + 15\zeta(5)/4 - 27/16) - 15\zeta(5)/2 - 93\zeta(3)/8 - 3/16}{2\lambda^2} + \dots \right). \end{aligned} \quad (5.1.24)$$

The expansions of β_n for $n > 2$ are currently not known to high order. However, as discussed in section 6.3 of [68], from an analysis of classical energy, with semi-classical corrections, it is in principle possible to extract the τ -independent coefficients $b_{n,0}$ and $b_{n,1}$ for all n . For our calculation of the intercepts we use $b_{3,0} = -3/16$ and $b_{3,1} = (60\zeta(3) + 60\zeta(5) - 17)/32$. Note that all coefficients $b_{n,m}$ are polynomials in τ , regular at $\tau = 0$. We have also identified, for each coefficient, its order in a string loop-expansion [71–73]. With these coefficients it is possible to fix the c_n up to $n = 6$. Thus,

⁷The all coupling expansion carries more information about the coefficients $c_{n,k}$ of the $j(\nu, \tau)$ expansion (5.1.5). Indeed, it is simple to check that the combination $\sum_{k=0}^{n-2} (-1)^k \tau^{2k} c_{n,k}$ is entirely fixed by the coefficients $b_{1,m}$ with $m < n$.

from (5.1.20), we find that the generalized Pomeron intercept is given by

$$\begin{aligned} \alpha_P(\lambda, \tau) = j_0(\lambda, \tau) = 2 - \frac{\tau^2}{2\lambda^{1/2}} - \frac{\tau^2}{4\lambda} + \frac{\tau^2(-3 + \tau^2)}{16\lambda^{3/2}} - \frac{\tau^2[-12 + \tau^2(11 + 24\zeta(3))]}{64\lambda^2} \\ + \frac{\tau^2[-63 + 6\tau^2(19 + 48\zeta(3)) - 2\tau^4]}{256\lambda^{5/2}} \\ + \frac{\tau^2[-216 + \tau^2(637 + 1536\zeta(3) + 480\zeta(5)) - 2\tau^4(17 + 36\zeta(3) + 60\zeta(5))]}{512\lambda^3} + \dots \end{aligned} \quad (5.1.25)$$

For $\tau = 2$ notice that the $\zeta(5)$ term is absent from c_6 due to cancellation, but it is in general present. We will show in the next section that just by varying τ , we can use this equation to calculate the Odderon intercept to the same order as above.

We may also compute, with the above information, the remaining coefficients $c_{n,k}$ up to $n = 6$, by using (5.1.21). Such non-vanishing coefficients are ⁸

$$\begin{aligned} c_{3,1} = \frac{3}{8}, \quad c_{4,1} = 3 \frac{7 - 8\zeta(3)}{32}, \quad c_{5,1} = \frac{59 - 144\zeta(3) - 2\tau^2}{64}, \quad c_{5,2} = \frac{21}{64}, \\ c_{6,1} = \frac{291 - 480\zeta(5) - 76\tau^2 - 48\zeta(3)(32 + 7\tau^2)}{256}, \quad c_{6,2} = \frac{137 - 204\zeta(3) - 60\zeta(5)}{128}. \end{aligned} \quad (5.1.26)$$

5.2 Conformal Odderon

It is appropriate to begin by first mentioning that the importance of Pomeron lies partly in the fact that all high energy hadron-hadron total cross sections σ_T continue to rise from collider to cosmic ray energies. This universal behavior can be understood as driven initially by the leading $1/N_c$ power law growth, $\sigma_T \sim s^{\alpha_P - 1}$ for the Pomeron intercept $\alpha_P > 1$. Eventual agreement with the Froissart bound $\sigma_T \sim \log^2 s$ requires a re-summation of higher order terms in $1/N_c^2$ expansion ⁹. The importance of the leading $C = -1$ component, generically referred to as the Odderon [79–82], lies in the fact that it contributes to the difference of the antiparticle-particle and particle-particle total cross sections, $\Delta\sigma_T(s) \sim s^{\alpha_O - 1}$.

In the weak coupling limit, the Pomeron [57–61] can be associated with 2-gluon exchange whereas Odderons can be thought as a $C = -1$ composite of a three-gluon system [83–89]. Two leading Odderons have been identified. One has an intercept slightly below one [83–85], with $\alpha_{O,a} \approx$

⁸Actually, since [68] also computes $b_{4,0} = 31/128$ and $b_{4,1} = (901 - 5520\zeta(3) - 5120\zeta(5) - 3640\zeta(7))/1024$, we can determine $c_{7,3} = 391/1024$ and $c_{8,4} = (15081 - 27120\zeta(3) - 12320\zeta(5) - 3640\zeta(7))/8192$.

⁹One often adopts an eikonal sum. Alternatively, the data is sometimes fitted directly by $\sigma_T \sim \log^2 s$, the maximally allowed asymptotic term consistent with saturating the Froissart unitarity bound. A more thorough discussion can be found in [20] and references therein.

$1 - O(\lambda)$, and the second has an intercept exactly at one, $\alpha_{O,b} \approx 1$, up to third order in the 't Hooft coupling [86]. It has also been suggested recently, for $\mathcal{N} = 4$ SYM, that the latter remains exactly at $j = 1$, to all orders in weak coupling [90, 91]. Interestingly, these correspond nicely with strong coupling analysis in the diffusion limit [20, 92].

Recall that, in a weak coupling BFKL treatment, the spectral curve is obtained by an expansion about $j = 1$ in the weak coupling α_s , i.e., $j \approx 1 - \alpha_s E_N(\Delta; \{\ell_N\})$, where E_N can be identified with the spectrum for a system of N -reggeon states [93–95], labelled by additional indices $\{\ell_N\}$. For $N = 2$, this leads to

$$j = 1 + \alpha_s \left\{ 2\Psi(1) - \Psi\left(\frac{(n+3) - \Delta}{2}\right) - \Psi\left(\frac{\Delta - (n+1)}{2}\right) \right\} \quad (5.2.1)$$

where there is a single index, $n = 0, 1, \dots$, labelling the principal series representation of $SL(2, \mathcal{C})$. The leading Pomeron corresponds to $n = 0$. Note that this representation is a perturbation about $j = 1$, and the right-hand side develops singularities due to poles of the Ψ -function. This representation therefore cannot be extended to the region of large j and Δ . For $N = 3$, appropriate for the Odderons, one solution coincides with that for $N = 2$, with $n = 1$ and $\Delta = 2$, leading to an Odderon intercept $j = 1$, as indicated above.

In this Section, we examine these strong coupling results, going beyond the diffusion limit. In particular, we clarify how for the special Odderon solution, $\alpha_{O,b} = 1$ at $k = 0$, can hold to all order in $1/\sqrt{\lambda}$.

For the Odderon, the large λ diffusion limit corresponds to setting $\lambda \rightarrow \infty$ and $j \rightarrow 1$, with $\sqrt{\lambda}(j - 1)$ fixed. In this limit the Odderon propagator can be obtained by perturbing about the EOM for the anti-symmetric Kalb-Ramond field, B_{MN} in AdS : $(-\square_{Maxwell} + m_{ads}^2)B_{MN} = 0$. Here $\square_{Maxwell}$ stands for the Maxwell operator. Its exact form can be found in [20], and it can again be diagonalized in terms of $O(4, 2)$ Casimir. The result is that all modes with odd positive integral j contribute to the exchange, and one arrives at an effective propagator in the $\nu^2 - j$ plane of the form

$$G_O(j, \nu) \sim \frac{1}{\nu^2 + m_{AdS}^2 + 2\sqrt{\lambda}(j - 1)}. \quad (5.2.2)$$

As for the Pomeron, the relevant string modes can be represented by on-shell world-sheet Reggeon vertex operators $\mathcal{V}_O^\pm$ in AdS . The on-shell condition, $L_0 \mathcal{V}_O^\pm = \bar{L}_0 \mathcal{V}_O^\pm = \mathcal{V}_O^\pm$, in analogy with (5.1.2), leads to $\Delta_O(j) = 2 + i\nu$. It follows that the Odderon spectral curve in the strong coupling diffusion

limit, given by the pole locations of $G_O(j, \nu)$, is

$$(\Delta_O(j) - 2)^2 = m_{AdS}^2 + 2\sqrt{\lambda}(j - 1). \quad (5.2.3)$$

Denoting $\alpha_O(\lambda)$ for the Odderon intercept, it follows that

$$\Delta_O(j) = 2 + \sqrt{2}\lambda^{1/4}\sqrt{j - \alpha_O}, \quad (5.2.4)$$

where in the diffusion limit

$$\alpha_O(\lambda) = 1 - \frac{m_{AdS}^2}{2\sqrt{\lambda}}. \quad (5.2.5)$$

We also stress that because of super-symmetry the anomalous dimension at $j = 1$ is zero, more precisely, $\Delta_O(1) = 2 + m_{AdS}$, for any value of the coupling. In the diffusion limit, the spectral curve $\Delta_O(j)$ is again parabolic.

From the SUGRA modes, we see that there are two sets of solutions, a set with $m_{AdS,a}^2 = (4+k)^2$ which we dub as set (a), and a set (b) for which $m_{AdS,b}^2 = k^2$ with $k = 0, 1, 2, \dots$. For $k \neq 0$, these modes can be associated with fluctuations in S^5 . For the $k = 0$ mode of set (b), at $j = 1$, it is known that it can be gauged away since its coupling is through the field-strength which vanishes [96–98]. However, in the diffusion limit of λ large but finite, one moves away from $j = 1$ with the perturbation introducing an effective AdS mass so that the field-strength no longer vanishes and the mode is now physical. We will return later to discuss this mode further. For each mode k of the S^5 , the two distinct AdS masses directly lead to two distinct Odderon trajectories with intercept given by (5.2.5) with the associated mass,

$$\alpha_O(\lambda) = 1 - \frac{m_{AdS,a/b}^2}{2\sqrt{\lambda}}. \quad (5.2.6)$$

From the perspective of OPE, at $j = 1$, there are two sets of conformal primaries, each indexed by integer k , with *protected conformal dimensions*¹⁰,

$$\Delta_O^{(a)}(1) = 2 + \tau_a = 2 + m_{AdS,a} = 6 + k, \quad (5.2.7)$$

$$\Delta_O^{(b)}(1) = 2 + \tau_b = 2 + m_{AdS,b} = 2 + k. \quad (5.2.8)$$

Candidate dual CFT operators dual to these protected string modes are $Tr(F_{\perp\pm} F^2 Z^k) + \dots$ and $Tr(F_{\perp\pm} Z^k)$, respectively. As usual, one expects that higher spin operators in the leading Regge

¹⁰Of course, each contributes only to correlation functions with appropriate R-charge.

trajectory can be obtained by acting with derivatives $D^\pm$, leading to operator dimensions $\Delta_O^{(a)}(2n+1)$ and $\Delta_O^{(b)}(2n+1)$ respectively, at $j = 2n+1$, $n = 1, 2, \dots$. However, these can only be obtained meaningfully beyond the diffusion limit.

5.2.1 Odderon Intercepts in Strong Coupling

To go beyond the diffusion limit, let us return to (5.1.14). In analogy to that equation and (5.1.16), we expand $(\Delta_O(j, \tau) - 2)^2$ about $j = 1$,

$$(\Delta_O(j, \tau) - 2)^2 = \tau_O^2 + \beta_1^{(-)}(j-1) + \beta_2^{(-)}(j-1)^2 + \beta_3^{(-)}(j-1)^3 + \dots \quad (5.2.9)$$

We have also added a superscript to the expansion coefficients, $\beta_n^{(-)}$, to remind ourselves that we are dealing with the $C = -1$ sector. Recall that (5.2.9) properly reflects the symmetry in $\Delta_O \leftrightarrow 4 - \Delta_O$. To match the diffusion limit, we require

$$\beta_1^{(-)}(\lambda, \tau_O) = 2\sqrt{\lambda} + O(1). \quad (5.2.10)$$

As is the case with the Pomeron, $j(\Delta_O)$ has a minimum at $\Delta_O(\alpha_O, \tau) = 2$, which defines the Odderon intercept. This, of course, is also equivalent to the existence a square-root singularity, (5.2.4). We further assume that, as the case for the Pomeron, in the strong coupling limit, the radius of convergence for (5.2.9) is $O(\lambda^{1/2})$, and, $\beta_n^{(-)} = O(\lambda^{(2-n)/2})$. Correspondingly, we can develop a systematic expansion for $\beta_n^{(-)}(\lambda)$ in $1/\sqrt{\lambda}$,

$$\beta_n^{(-)} = 2\lambda^{\frac{2-n}{2}} \left(b_{n,0}^{(-)} + \frac{b_{n,1}^{(-)}}{\lambda^{1/2}} + \frac{b_{n,2}^{(-)}}{\lambda} + \frac{b_{n,3}^{(-)}}{\lambda^{3/2}} + \dots \right). \quad (5.2.11)$$

We are now in the position to carry out a similar analysis for Odderon intercepts beyond the diffusion limit. Consider the expansion for the intercept

$$\alpha_0(\lambda) = 1 + \frac{c_1^{(-)}}{\lambda^{1/2}} + \frac{c_2^{(-)}}{\lambda} + \frac{c_3^{(-)}}{\lambda^{3/2}} + \frac{c_4^{(-)}}{\lambda^2} + \frac{c_5^{(-)}}{\lambda^{5/2}} + \dots \quad (5.2.12)$$

The coefficients in this expansion can be found by solving $\Delta_O(\alpha_O, \tau) = 2$ iteratively. Observe that the situation is nearly identical to that for the Pomeron. It follows that $c_i^{(-)}$ are given exactly by the corresponding coefficients c_i for the Pomeron intercept, Eq. (5.1.20), with the replacements of τ by m_{AdS} , c_i by $c_i^{(-)}$ and $b_{n,i}$ by $b_{n,i}^{(-)}$. For simplicity, we shall drop the superscript in what follows.

In the diffusion limit, our two sets of Odderon solutions are structurally similar. However, there

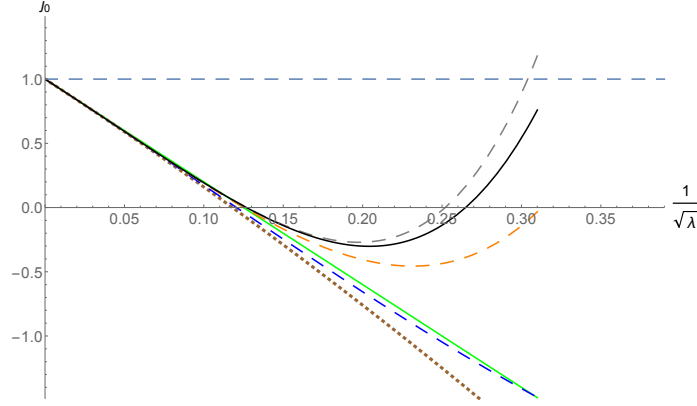


Figure 5.2.1: Odderon-a intercept at strong coupling. The solid green-line is to first order in $1/\sqrt{\lambda}$, the dotted brown-line is to second order, the dashed blue-, orange- and grey-line are to third order, fourth order and fifth order respectively. Finally the solid black-line is the intercept up to sixth order.

is no particular reason why these two sets remain similar in higher orders and we shall treat them separately in what follows. We shall first consider type-(a), characterized by $\tau_a = m_{AdS} = 4 + k$, $k = 0, 1, 2, \dots$, before treating the case for type-(b). It is worth mentioning again that the all-coupling formula (5.1.10) was derived from ABA equations, without the so-called wrapping corrections. It is surprising that they do not appear in the present context as one would expect them especially in the small spin limit. The physical motivation for their absence is not fully understood¹¹. Furthermore, it is supposed to hold only for the $sl(2)$ sector for the configuration with “minimum mode numbers”, which should correspond to the spectral curve with minimum scaling dimension. It is indeed possible to generalize the solution of ABA for “non-minimum string modes”. However, no systematic attempt has been made [76, 77]. Further discussion will be provided in Sec. 5.3.

I – Type-(a) Odderons:

As pointed out earlier, the $j = 1$ mode survives in the supergravity limit, and it can be identified with the protected CFT operator of the type $Tr(F_{\pm\pm}F^2Z^k) + \dots$. This conformal primary, just as the case for the Pomeron, can formally be considered as a descendent of super-conformal primary in the $sl(2)$ sector. However, there can be many spectral curves emanating from this protected configuration at $j = 1$, for λ finite. We assume that the all-coupling formula (5.1.10) for the slope function applies to the set $\Delta_{O,a}$, corresponding to the “minimum twist” set, and we shall proceed to calculate the Odderon intercept to higher orders in $1/\sqrt{\lambda}$ under this assumption. The validity of this assumption will be examined in the next section.

That is, we assume that, for type-(a) Odderons, $\beta_n(\tau_{(a)}) = \beta_n(4 + k)$, with β_n given by the

¹¹See [76, 77] for a discussion and comparison to ABJM theory.

same functions used for the case of Pomeron. With this we can find the intercept for any k , but for simplicity, and to study the case most relevant for QCD, we write the result in the limit $k \rightarrow 0$. Hence $\tau_{(a)} = 4$, and we take advantage of the expansion (5.2.12). Under the assumptions made above, c_i can be found by solving the equation

$$0 = 4^2 + \beta_1(\lambda, 4)(j_0 - 1) + \beta_2(\lambda, 4)(j_0 - 1)^2 + \beta_3(\lambda, 4)(j_0 - 1)^3 + \beta_4(\lambda, 4)(j_0 - 1)^4 + \dots,$$

iteratively. It is possible to directly adopt the calculation previously done for the Pomeron intercept in Sec. 3, with c_n given by (5.1.20), by evaluating equation (5.1.25) at $\tau = 4$, after shifting the spin j by 1. One finds

$$\alpha_{O,a} = 1 - \frac{8}{\lambda^{1/2}} - \frac{4}{\lambda} + \frac{13}{\lambda^{3/2}} + \frac{96\zeta(3) + 41}{\lambda^2} + \frac{288\zeta(3) + \frac{1249}{16}}{\lambda^{5/2}} + \frac{-720\zeta(5) + 192\zeta(3) + \frac{159}{4}}{\lambda^3} + \dots \quad (5.2.13)$$

This intercept $\alpha_{O,a}$ is illustrated in Fig. 5.2.1. Note that coefficients $\{c_n\}$ change signs, with $c_1, c_2 < 0$, $c_3, c_4, c_5 > 0$, and $c_6 < 0$. We also note that, in the range $0 < 1/\sqrt{\lambda} < 0.3$, where strong coupling is expected to be useful, the intercept $\alpha_{O,a}$ is below $j = 1$. As one increases $1/\sqrt{\lambda}$ beyond 0.2, interestingly, it begins to turn around and move towards $j = 1$, as it should, only after c_4 and higher terms are kept. Note as well that the intercept does not continue to blow up, but begins to flatten out as higher orders are taken into account, e.g., with $c_6 < 0$. This behavior fits nicely with the expected matching behavior to first order weak coupling calculation, at $1/\sqrt{\lambda} \approx 0.3$, as shown in Fig. 5.2.2

Notice that it is a simple exercise to determine the coefficients in an expansion of the type (5.1.5) for the Odderon spin function $j(\nu, \tau)$.

II – Type-(b) Odderons and $\tau_b = k \rightarrow 0$:

As stated earlier, although there are no obvious structural differences between type-(a) and type-(b) Odderon spectral curves in the diffusion limit, one crucial difference is the absence of a protected local CFT operator at $j = 1$ at $\tau_b \rightarrow 0$, in the supergravity limit. For $k \neq 0$, the simplest set of local operators one can identify are $Tr[(D^\pm)^{j-1} F_{\pm\pm} Z^k]$. These operators decouple however in the $k \rightarrow 0$ limit. It is likely that the lowest physical mode on the $k = 0$ spectral curve occurs at $j = 3$ is $Tr[F_{\pm\pm} F_{\pm\pm} F_{\pm\pm}]$ and it is not protected. It has been suggested¹² that, for $j = 3, 5, \dots$, these type-(b) modes could be associated with local operators: $Tr[D_\pm^{j-3} F_{\pm\pm} F_{\pm\pm} F_{\pm\pm}] + \dots$. However, these lead to a different system of $sl(2)$ spin chain, and a separate analysis is required [99–102].

¹²B. Basso, private communication.

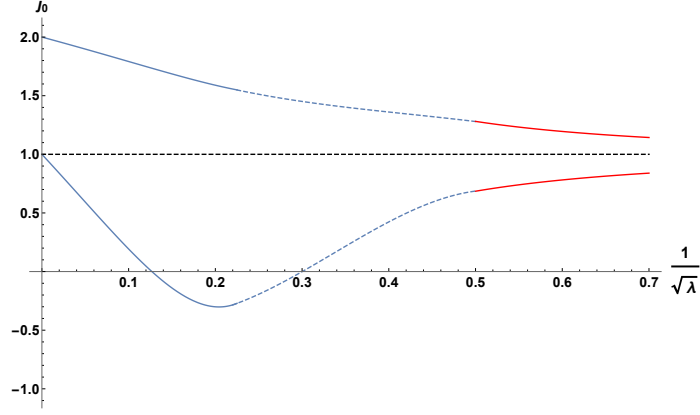


Figure 5.2.2: The Pomeron and Odderon-(a) intercepts from strong to weak coupling. The dark blue curves are the calculated strong coupling results, the red curves are the known weak coupling intercepts, and the dashed line is an interpolation. It is interesting to note that up to their current orders, both the Pomeron and Odderon intercepts appear consistent with weak coupling results in the transition region. Black dashed line is for the Odderon-(b) solution where $\alpha_{O,b} = 1$.

An equally interesting possibility is to consider the sequence $Tr[(F_{\pm\perp} F_{\pm\perp})^{(j-1)/2} F_{\pm\perp}]$. Neither sequence leads to well-defined local operators at $j = 1$. For this and other considerations, we do not expect the all-coupling formula (5.1.10) to work for type-(b) Odderon, specially for the $k = 0$ mode. Therefore, we shall proceed to carry a more general analysis *without* assuming the Basso formula for the slope function, α_1 . We shall search for, if any, universal behavior which might survive in the limit of $\tau_b \rightarrow 0$.

We proceed by making the expansion for β_n , (5.2.11), as before, and arrive at

$$\begin{aligned}
 (\Delta_{O,b}(j, \tau_b) - 2)^2 = & \tau_b^2 + 2\lambda^{1/2} \left(1 - \frac{b_{1,1}}{\lambda^{1/2}} + \frac{b_{1,2}}{\lambda^{3/2}} + \frac{b_{1,3}}{\lambda^2} + \dots \right) (j-1) \\
 & + 2 \left(b_{2,0} + \frac{b_{2,1}}{\sqrt{\lambda}} + \frac{b_{2,2}}{\lambda} + \frac{b_{2,3}}{\lambda^{3/2}} + \dots \right) (j-1)^2 \\
 & + 2\lambda^{-1/2} \left(b_{3,0} + \frac{b_{3,1}}{\lambda^{1/2}} + \frac{b_{3,2}}{\lambda} + \frac{b_{3,3}}{\lambda^{3/2}} + \dots \right) (j-1)^3 \\
 & + O((j-1)^4).
 \end{aligned} \tag{5.2.14}$$

Note, with the exception of $\tau_b = k$ and $b_{1,0} = 1$, other coefficients $b_{n,j}$ are *unspecified*. Expanding α_O as in (5.2.12), after substituting into (5.2.14), the coefficients c_i can be determined iteratively, e.g., leading to formulas essentially given by Eq. (5.1.20). To be more explicit, we can directly express

all coefficients c_i in terms of $b_{n,i}$. For the first few terms we have, from (5.1.20),

$$\begin{aligned}
c_1(\tau_b) &= -\frac{\tau_b^2}{2}, \\
c_2(\tau_b) &= \frac{\tau_b^2}{2} b_{1,1}, \\
c_3(\tau_b) &= \frac{\tau_b^2}{2} \left((-b_{1,1}^2 + b_{1,2}) - b_{2,0} \frac{\tau_b^2}{2} \right), \\
c_4(\tau_b) &= \frac{\tau_b^2}{2} \left((b_{1,1}^3 - 2b_{1,1}b_{1,2} + b_{1,3}) + (3b_{1,1}b_{2,1} - b_{2,1}) \frac{\tau_b^2}{2} \right), \\
c_5(\tau) &= \frac{\tau_b^2}{2} \left((-b_{1,1}^4 + \dots) + (-6b_{1,1}^2b_{2,0} + \dots) \frac{\tau_b^2}{2} + (-2b_{2,0}^2 + \dots) \frac{\tau_b^4}{2^2} \right), \\
c_6(\tau_b) &= \frac{\tau_b^2}{2} \left((b_{1,1}^5 + \dots) + (10b_{1,1}^3b_{2,0} + \dots) \frac{\tau_b^2}{2} + (10b_{1,1}b_{2,0}^2 + \dots) \frac{\tau_b^4}{2^2} \right).
\end{aligned} \tag{5.2.15}$$

From [68], we expect that $b_{n,j}$ to be polynomials of τ_b^2 . Note that in the limit $\tau_b \rightarrow 0$, all coefficients vanish as τ_b^2 , e.g., $c_1 \sim \tau_b^2$, $c_2 \sim \tau_b^2$, etc., for arbitrary values for coefficients $b_{n,i}$. It is also easy to verify that, $c_{n+1}/c_n \sim O(1)$, for $\tau_b \rightarrow 0$. We thus arrive at an important result where, $c_n = O(\tau_b^2) \rightarrow 0$, for all n , in the limit $\tau_b \rightarrow 0$. It follows that the leading Odderon intercept for the set-(b), for $\tau_b = 0$, remains at

$$\alpha_{O,b} = 1, \tag{5.2.16}$$

without higher order correction in an $1/\sqrt{\lambda}$ expansion. This is the long promised result. To state it more graphically, in the limit $\tau_b \rightarrow 0$, higher order corrections can change the shape of the spectral curve, without changing its minimum at $\Delta = 2$.

For $\tau_b \neq 0$, more information is required in order to determine the higher order expansion for their intercepts, e.g., adopting the all-coupling formula (5.1.10). We will not engage in this exercise here, but note that due to the generality of the above derivation, the $\tau_b = 0$ result would survive for all possible expansions of β_n .

5.3 Discussion

We have focused in this study on the leading $C = \pm 1$ Regge singularities, the Pomeron and Odderon respectively, in strong coupling. Central to our discussion is the notion of spectral curve $\Delta(j)$ for single-trace gauge invariance operators of $\mathcal{N} = 4$ SYM. Identifying $\Delta(j)$ in weak coupling remains involved due to possible operator mixings [99]. In strong coupling, spectral curves for leading twist can be identified with bulk degrees of freedom for $D = 10$ SUGRA on $AdS_5 \times S^5$ [98]. The Pomeron trajectory can be associated with the Reggeized Graviton, while the Odderon trajectories correspond

to Reggeized anti-symmetric AdS_5 Kalb-Ramond tensor-fields. With string-vertex operators [1, 20], the relevant single-trace CFT operators and their associated string modes for both Pomeron and Odderon sectors can be identified.

We began by first providing a general discussion on Regge limit in CFT from the perspective of light-cone OPE, and showed how the double-Mellin transforms (3.4.4) and (3.4.11) can be used directly in a Minkowski setting. A Regge dictionary (3.4.13) is established between CFT in coordinate representation and that based on AdS/CFT in a momentum treatment. An important and probably difficult theoretical problem left unresolved is to determine the conditions required in a conformal theory to allow for this representation in the double-Mellin plane. Is this conformal Regge representation a generic property of all 4D Lorentz invariant conformal theories, or is it restricted to a smaller class of theories?

Due to integrability, these spectral curves can in principle be determined [15], with their inverse $j(\Delta)$ being symmetric under $\Delta \leftrightarrow 4 - \Delta$, due to conformal invariance. In this study, we have focused on “short strings” where each spectral curve takes on a relatively simple form in the large λ limit while maintaining the $\Delta \leftrightarrow 4 - \Delta$ symmetry. In particular, by adopting the approach advocated in [64, 65, 76, 77], higher order expansion in $1/\sqrt{\lambda}$ for the Pomeron intercept has been carried out recently [24, 66, 68]. We have generalized this analysis for the Pomeron sector to include non-zero R charge, (5.1.25), and have also extended the treatment to the case of Odderons, (5.2.13) and (5.2.16). For the case of the Pomeron with large classical R charge, it would be interesting to see the appearance of this trajectory in the Regge limit of four-point functions of heavy operators computed in [103] at strong coupling.

It is important to emphasise that our analysis has been carried out in the context of AdS/CFT, appropriate for a strong coupling expansion in the large- N_c limit. Simplicity in the complex $\Delta - j$ plane is achieved partly due to the ability to identify modes of SUGRA with protected gauge-invariant YM operators in the limit of $\lambda \rightarrow \infty$, as discussed in Sec. 5 and also in Sec. 5.2.1. This in turn allows us to treat leading $\Delta(j, \tau)$ curves which dominate the Regge limit through the double-Mellin representation discussed in Sec. 3.4. It is expected [1] that additional sub-dominate spectral curves exist, leading to “fine-structure” to the complex $\Delta - j$ plane. It is interesting to note in this connection that anomalous dimensions of higher-twist Wilson operators in generic gauge theories have previously been investigated and a robust structure, particularly at large j , has been found, e.g., a band of trajectories of width growing logarithmically with spin- j [104]. At large- j , anomalous dimensions increase with spin as $\ln j$, with leading coefficients given by “cusp-anomalous dimensions”. The analysis in [104] was carried out for physical integral j -values, in the framework

of asymptotic Baxter equation and also based on semiclassical expansion. It is reasonable to expect that this “band-like” structure identified for higher-twist sectors should persist at low- j values, and it is interesting to ask how a smooth connection can be achieved¹³. Clearly, this can only be discussed meaningfully in the context of the large- N_c limit where the j -plane structure is expected to be simplified, e.g., adopting the approach of quantum spectral curves, advocated in [62,68]. More immediately, the all coupling analysis in [64,65,76,77], which focuses on the small spin region, and is also based on the asymptotic Bethe ansatz, can shed light on this issue.

Since our result depends crucially on the small spin expansion (5.1.9), it is worth first adding a brief comment on the slope function $\alpha_1(\lambda, \tau)$, (5.1.10), as promised earlier. We first note that the set of gauge invariant operators in the $sl(2)$ sector, designated symbolically by $Tr(D_+^S Z^\tau)$, should be interpreted as a collections of operators, $Tr(D_+^{s_1} Z D_+^{s_2} Z \cdots D_+^{s_\tau} Z)$, with $\sum_{j=1}^\tau s_j = S$. In the large N limit, the dilation operator closes on this subspace, leading to a set of spectral curves, $\Delta_{Z,k}(S, \tau)$, labelled by an index k . Our focus here is for strong coupling, where, in the diffusion limit, $\Delta_Z \approx \tau + (\sqrt{\lambda}/\tau)S$. Therefore, *all these curves are degenerate in this limit*. The degeneracy is lifted by going beyond the leading $1/\sqrt{\lambda}$ limit. Expanding $\Delta_{Z,k}(S, \tau)$ for S small, i.e., $\Delta_{Z,k}(S, \tau) = \tau + \alpha_{1,k}(\tau, \lambda)S + O(S^2)$, these slope functions, $\alpha_{1,k}(\tau, \lambda)$, can be found via ABA, without “wrapping corrections”, where the index k can be specified by a set of filling fractions [65,76,77]¹⁴. The “minimum filling”, (or “minimum mode” configuration), leads to the original Basso formula. Clearly, there are many additional lower spectral curves, corresponding to other allowed filling fractions. It would be interesting if these additional spectral curves can be identified with those found at high- j and at weak coupling [99–102,104]. It is also interesting to find out how these can be related to the mode number in the discussion of GKP strings. In our current treatment, the minimum filling solution has been used for both the Pomeron and the type-(a) Odderon sectors. Note that for both the Pomeron and the type-(a) Odderon keeping the $O(1/\sqrt{\lambda})$ result for the intercept stemming from the Basso formula agrees with the Pomeron and Odderon intercepts found independently by perturbing about the supergravity limit [1,20].

Let us turn next to the type-(b) Odderon. As we have stated earlier, in the super-gravity limit,

¹³It is also appropriate to point out that, in the case of a theory with a mass gap, the analytic continuation in j is unique for the partial-wave amplitudes, following what is known as the “Froissart-Gribov” procedure. We fully expect a similar procedure can be carried out for generic CFTs, e.g., by adopting, for instance, the Mack representation [30] as a starting point of discussion. As a consequence, there is a unique analytic continuation for $\Delta(j, \tau)$ away from integral j and τ values. This will be discussed in a future treatment.

¹⁴To be more explicit, $k \equiv \{\kappa_m\}$, where $\sum_{m \neq 0} \kappa_m = 1$. Since $\Delta_Z \approx \tau + (\sqrt{\lambda}/\tau)S$ in the diffusion limit, it follows that there exists another constraint $\sum_m |m| \kappa_m = 1$. Minimal filling corresponds to $\kappa_1 = \kappa_{-1} = 1/2$.

the $k = 0$ mode decouples at $j = 1$, and, for λ finite, its physical modes begin at $j = 3, 5, \dots$. In Sec. 5.2.1, we have carried out a more generally analysis without invoking the all-coupling formula (5.1.10). Our treatment is based on the structure of large λ expansion. Given the diffusion result of $\alpha_{O,b} = 1$, for $k = 0$, Eq. (5.2.16) follows to all orders. The phenomenon of decoupling also has its counter part in flat space string theory. For $d = 4$, Kalb-Ramond tensor field does not lead to a spin-1 massless particle since $B_{\mu\nu}$ has only one independent transverse component. As a 4-d Regge trajectory, its higher string recurrences at $j = 3, 5, \dots$ are physical. Therefore, the issue of decoupling can be accomplished by an appropriate vanishing of the coupling, while the dynamics of the whole trajectory remains. (Instead, at $t = 0$, $B_{\mu\nu}$ leads to a spin-0 state, the “axion”.) A more detailed analysis will be presented elsewhere.

In this study, we have not fully explored the consequence of super-symmetry, in particular the possibility of more general symmetry patterns for spectral curves $\Delta_Z(S, \tau)$. A useful study is a careful examination for the spectral curves for other modes of SUGRA, e.g., scalars, vectors, etc. A preliminary finding involves the possibility of having a more complex structure, e.g., the symmetry about $\Delta = 2$ is realized by a pairing of spectral curves, with one symmetric about $\Delta = 0$ and the other about $\Delta = 4$. Equally interesting is the question for the Odderon intercept in strong coupling from a brane construct alone, without imposing super-symmetry, and its relation to results obtained in weak coupling.

As stressed in [76], $\Delta_Z(S, \tau)$ is in general a complicated function of S , τ , λ , and also of other quantum numbers. It can in principle possess infinitely many branches, connected through the so-called “crossing-point singularities,” i.e., the phenomenon of level-crossings, leading to root-type branch point in $\Delta_Z(S, \tau)$. Indeed, in perturbing about the supergravity limit, the multi-valued property of Δ_Z seems to play an important in ensuring $\Delta \rightarrow \Delta - 4$ symmetry. It should be stressed that these crossing-point singularities do not lead to branch points for the analytically continued conformal partial-wave amplitude, Eq. (3.4.2). This type of crossing-point singularities has also been noted previously in a weak coupling treatment for Odderons via BFKL-like analysis. As mentioned earlier in Sec. 5.2, in such a treatment, the spectral curve is obtained by an expansion about $j = 1$, i.e., $j \approx 1 - \alpha_s E_N(\Delta; \{\ell\})$, where E_N can be identified with the spectrum for a system of N -reggeon states [93–95]. It can be shown that, for $N \geq 3$, level-crossing occurs, leading to crossing-point branch points. However, it is unclear if there is a correspondence for such singularities at weak and strong coupling. Making a precise connection between the strong and weak coupling Odderon

solutions remains a challenge ¹⁵. The approach of quantum spectral curve [62, 68] holds the promise of further progress in this direction.

We have focused in this study on the leading planar limit. Note that, in the planar limit, the conformal amplitude growth with a power of s , or, equivalently, $1/\sqrt{u}$, which would violate the flat-space Froissart bound. Clearly, in order to address the issue of Froissart bound for CFT's, one must consider the extension to higher orders in $1/N^2$. One such re-summation is given by the eikonal approximation. From the perspective of light-cone OPE, one must begin including multiple-trace primaries in order to carry out such analysis. It is also interesting to examine the effect of confinement deformation. Since scale invariance is broken in the IR, adopting Poincare-patch for AdS is most suitable for such a treatment. Instead of spectral curves $\Delta(j)$, one now has ordinary Regge singularities $\alpha(t)$ at positive t , leading to discrete physical states at integral j , e.g., glueballs. One also finds that our double-Mellin representation, (3.4.4) and (3.4.11), reduces to a single Mellin (Regge) representation, with a sum over Regge trajectories. These and other related issues will be addressed in future publications.

¹⁵There also exists a slight disagreement [93–95] on the proper interpretation for the Odderon solutions within this weak coupling approach.

Chapter 6

Inclusive AdS processes

6.1 Introduction

It has been a long-held belief that large- N QCD can be realized as a string theory. The germ of this conjecture began with the observation that, in the limit of large N_c with $\lambda = g_{YM}^2 N_c$ fixed, QCD perturbation sum can be organized into a 2-D string-like topological expansion. Until recently, however, a more precise relationship between “QCD string” and YM gauge theories was missing. With the advent of AdS/CFT correspondence [105], or equivalently Gauge-String duality, the theoretical landscape has taken a dramatic turn and a string realization of QCD has again become a serious goal for current theoretical studies.

Phenomenologically, most problematic for a string-like description for QCD has been the requirement of hard partonic behavior at short distance. With AdS/CFT, a framework in resolving various phenomenological difficulties now exists. Two applications are discussed in great detail in the previous two chapters. In a most interesting paper, Polchinski and Strassler identified a mechanism for power fall-off for wide-angle scattering due to warped geometry of dual theory [12]. AdS/CFT has also been extended to the Regge limit of small-angle (near-forward) scattering.

In this chapter, we discuss how AdS/CFT correspondence can be extended to inclusive production at high energy via conformal techniques. Recent inclusive studies in a CFT context began with works of Strassler [106], Hofman and Maldacena [107] and Belitsky *et al.* [108–110]. Instead of focussing on final state particle distribution, which is ill-defined in the strict CFT limit, the emphasis has been

on infrared safety [111, 112], e.g., the energy flows, leading to vacuum expectations

$$\sigma_w(p) = \int d^4x e^{-ipx} \langle 0 | \mathcal{O}^\dagger(x) \mathcal{D}[w] \mathcal{O}(0) | 0 \rangle. \quad (6.1.1)$$

Here $\mathcal{O}$ serves as the source for the initial state $|\mathcal{O}(p)\rangle$, e.g., carrying 4-momentum p_μ , and $\mathcal{D}[w]$ is a product of a set of local operators, measuring flows of conserved quantities, such as energy-momentum or other conserved charges, generically referred to as inclusive “event shape” distributions. Since $\mathcal{D}[w]$ is not time-ordered, it leads to a Wightman function. In [106–110], the focus has been on processes where the source involves a single local operator, $\mathcal{O}$, i.e., appropriate for processes such as $e^+e^- \rightarrow \gamma^* \rightarrow X$. It has been advocated in [107–110] that it is possible to explore conformal invariance for such distributions by examining appropriately continued Minkowski CFT correlation functions in *coordinate representation*, e.g., the one-point energy distribution by exploring the structure of a 3-point CFT correlation function.

In this chapter we will discuss inclusive production at high energy, with special emphasis on scattering processes where the source itself is non-local. Our discussion will be carried out mostly in a *momentum representation*, where inclusive cross sections can be identified as “discontinuities” of appropriate forward scattering amplitudes through the use of generalized optical theorems. Our focus will be primarily on non-perturbative issues. In particular, we explore consequence of conformal invariance, including stringy effects, by making use of t-channel OPE [1, 45]. With confining deformation in the IR, we test this framework by comparing central inclusive particle production with data at LHC.

The simplest inclusive scattering process, $ab \rightarrow X$, leads to the total cross section, $\sigma_{total}^{ab}(s)$. Inclusive processes unavoidably involve near-forward particle production. Since the relevant physics is intrinsically non-perturbative, and it cannot be reduced simply to pure partonic scatterings. By invoking AdS/CFT, one is able to begin addressing both perturbative and non-perturbative aspects of inclusive production at high energy in a unified setting, e.g., the BPST treatment for exclusive scattering in the near-forward limit [1]. We already saw one interesting inclusive study in chapter 4, the treatment of DIS at small x , which relies on identifying the DIS distribution with the total cross section for virtual photon-proton scattering, $\gamma^*p \rightarrow X$.

Generically, for any scattering process, after summing over contributions from all possible final states, the completeness relation, $\sum_X |X\rangle\langle X| = I$, leads to the usual optical theorem,

$$\sigma_{total}(s) \simeq \frac{1}{s} \text{Im} T(s, t=0). \quad (6.1.2)$$

as described in chapter 2. That is, $\sigma_{total}(s)$ is given by the imaginary part of the elastic amplitude $T(s, t)$ in the forward limit, $t = 0$. Indeed, the BPST treatment for DIS cross section at small $x = Q^2/s$ and large s , which can be understood as an OPE in the t-channel, has been highly successful in accounting for the HERA data [47, 113–116].

Beyond total cross sections, the next simplest inclusive process is the single-particle production, $ab \rightarrow c + X$. Kinematically, this can be treated as a 2-to-2 process, with X having a variable mass, often referred to as the “missing-mass”, $M^2 = (p_a + p_b - p_c)^2$. The invariant cross section, $d\sigma_{ab \rightarrow c+X}/(d^3\mathbf{p}_c/E_c)$, therefore depends on, instead of the usual two, now on three invariants. One can simply adopt the usual Mandelstam variables, s, t, u , or, more conveniently, s, t, M^2 , with $s + t + u = m_a^2 + m_b^2 + m_c^2 + M^2$.

This inclusive cross section can equally well be identified as the appropriate discontinuity of a 6-point amplitude for the process $abc' \rightarrow a'b'c$, in the invariant M^2 , symbolically expressed

$$\frac{d\sigma_{ab \rightarrow c+X}}{d^3\mathbf{p}_c/E_c} \simeq \frac{1}{2is} \text{Disc}_{M^2} T_{abc' \rightarrow a'b'c}, \quad (6.1.3)$$

The discontinuity is taken in the “forward” limit, with $p_{c'} = p_c$, $p_{a'} = p_a$, $p_{b'} = p_b$. This identification, also known as “Mueller’s formula”, can be generalized to other inclusive processes involving multi-particle final states. It has been known that all inclusive cross sections can be obtained via generalized optical theorems. As is the case for σ_{total} , these theorems [117–120] also follow from unitarity, i.e., $SS^\dagger = I$, with $S = I + iT$.

Our discussion in this chapter can be divided into two parts. We first focus on the more formal question of how to treat CFT inclusive shape distributions as weighted discontinuities of multiparticle amplitudes, $T_{ab1'2'\dots \rightarrow a'b'12\dots}$ in momentum representation. This treatment is carried out necessarily in a Minkowski setting. The discontinuity is taken in the missing mass, $M^2 = (p_a + p_b - \sum p_i)^2$, for the process, $ab1'2'\dots \rightarrow a'b'12\dots$, in the “forward limit”, as is the case for the more familiar optical theorem for a total cross section. This procedure applies to both events initially by a single local source, e.g., (6.1.1), and those initially in a scattering processes at high energy, e.g., (6.1.2) and (6.1.3).

By multiparticle amplitudes here we simply refer to the usual Euclidean CFT correlations functions, $\langle \varphi(x_1)\varphi(x_2)\dots \rangle$, continued to the Minkowski limit, leading to vacuum expectation values for time-ordered conformal primaries, $\langle 0|T\{\varphi(x_1)\varphi(x_2)\dots\}|0\rangle$. One assumes a standard Hilbert space structure, e.g., spanned by states associated with conformal primaries, and shall denote the associated completeness relation symbolically by $\sum_X |X\rangle\langle X| = I$. Any concern for formal manipulation

can always be alleviated by considering the large 't Hooft coupling limit, $\lambda \rightarrow \infty$, where by invoking AdS/CFT, these amplitudes can be enumerated perturbatively by the associated Witten diagrams. As such, the desired discontinuities can be obtained from scattering in the bulk. Furthermore, we shall also be mainly concerned with theories with IR confining deformation, leading to final mass gaps thus well-defined on-shell scattering amplitudes.

We turn next to inclusive production for scattering processes and explore, in particular, consequence of AdS/CFT and conformal invariance, for *finite 't Hooft coupling*, i.e., λ large but finite. In this limit, it is necessary to move beyond the super-gravity limit with the inclusion of contribution from string modes, i.e., one is effectively dealing with string amplitudes in the *AdS*. We shall focus on a single-particle inclusive production, and in particular, for the central region, where X can be separated into right- and left-moving groups, X_1 and X_2 respectively. We show, for such events, BPST procedure can be applied. In particular, the distribution is controlled by a matrix element, $\langle \mathcal{V}_P V_{c\bar{c}} \mathcal{V}_P \rangle$, involving two Pomeron vertex operators [1].

Just as the case of exclusive fixed-angle scattering, flat-space string theory [121–123] A leads to exponential cutoff in transverse momentum $\sim e^{-4\alpha' p_\perp^2}$. Due to warped AdS, we argue that the effect of confinement deformation becomes unimportant at high p_T , and the production cross sections again exhibit partonic power-law fall-off

$$\frac{d\sigma_{ab \rightarrow c+X}}{d^3\mathbf{p}_c/E_c} \sim \text{Disc}_{M^2} \langle \mathcal{V}_P V_{c\bar{c}} \mathcal{V}_P \rangle \sim p_\perp^{-\delta_c} \quad (6.1.4)$$

via a generalized Polchinski-Strassler mechanism [124, 125]. Through AdS/CFT, δ_c is fixed by conformal invariance, independent of initial state, ab , and is simply given by the conformal dimension Δ_c , associated with the produced hadron, e.g., for scalar glueballs ¹, $\delta_c = 2\Delta_c = 8$. This behavior is analogous to the dimensional counting rule, initially formulated for exclusive processes [126–128]. We test this finding by comparing with the recent Alice data from LHC ².

This chapter is organized as follows. In Sec. 6.2, we focus on inclusive distributions as discontinuities. We begin in Sec. 6.2 by reviewing the notion amplitude discontinuities for conventional field theories with mass-gap. We show next in Sec. 6.2 how this can be generalized in the limit of CFT. In Sec. 6.2, we turn to scattering process and illustrate with the example of DIS at small- x and

¹Strictly speaking, our formalism applies more directly to production of glueballs. For standard picture of a meson consisting of two quarks, production involves higher order $1/N_c$ mixings, leading to corrections for this power.

²Single-particle inclusive cross section in other kinematic regions has been addressed recently from a perturbative dimensional counting perspective.

the use of t-channel OPE. In Sec. 6.2, by invoking AdS/CFT, inclusive cross sections can next be expressed as discontinuities in terms of Witten diagrams. We also introduce confining deformation in the IR, leading to well-defined on-shell scattering amplitudes.

In Sec. 6.3, we reiterate our discussion of Gauge-String duality beyond the super-gravity limit appropriate for inclusive production. This serves as the step-stone for discussing inclusive distribution in the central production, which we turn to in Sec. 6.4. In [121–123], inclusive cross section was evaluated by applying the generalized optical theorem to the $3 \rightarrow 3$ flat-space string amplitude. We apply this same method to compute the curved-space string theory prediction for the inclusive cross section. Finally, we test this finding by comparing with the recent Alice data from LHC³. We contrast our finding with other attempts where a naive $\delta_c = 4$ value is expected. We will conclude with a brief discussion of our essential results in Sec. 5.3.

6.2 Inclusive Cross Sections as Discontinuities:

In this Section, we first review the use of discontinuities for inclusive cross sections in a conventional field theoretic setting before turning to the case of CFT. By introducing “inclusive density”, general inclusive flows for conserved quantities, e.g., energy, charge, etc., can be expressed as weighted discontinuities. This, in turn, allows one to deal with observables in CFT with IR safety. We also distinguish two types of inclusive processes, events with a single initial local source, e.g., $\gamma^* \rightarrow c_1 + c_2 + \dots + X$ and that with a non-local source, as in scattering, e.g., $a + b \rightarrow c_1 + c_2 + \dots + X$. We next turn to AdS/CFT, introducing confining deformation in IR, defining on-shell amplitudes and, finally, obtaining generalized Mueller formula for on-shell scattering, through the use of Witten diagrams for the AdS bulk.

Mueller Discontinuities: Whereas scattering amplitudes can be related to the Fourier transforms of *time-ordered* Green’s functions, $\langle 0|T(\varphi_1\varphi_2\cdots)|0\rangle$, inclusive cross sections involve matrix elements of products of local operators. This can best be illustrated for a 3-to-3 process, $abc' \rightarrow a'b'c$, where the scattering amplitude $T_{abc' \rightarrow a'b'c}(p_{a'}, p_{b'}, p_c; p_a, p_b, p_{c'})$ is the Fourier transform of a *T-product*⁴,

³Single-particle Inclusive cross section in other kinematic regions has been addressed recently from a perturbative dimensional counting perspective.

⁴Due to translational invariance, the amplitude contains an overall delta-function, i.e., $\delta^4((p_a + p_b + p_{c'}) - (p_{a'} + p_{b'} + p_c))$. Here, energy components for all 4-vectors are positive, e.g., $p_c^0 > 0$. For simplicity, the overall delta-function will be considered to have been removed for all scattering amplitudes in what follows. We also note that, strictly speaking, for the amputated on-shell amplitudes, φ_c should be replaced by a source function, $j_c(x) = (\square - m_c^2)\varphi_c(x)$. We will skip this step to avoid notational overload.

$\langle p_{a'}, p_{b'} | T(\varphi_c(x)\varphi_c(y)) | p_a, p_b \rangle$. In contrast, for inclusive production of a scalar particle, $ab \rightarrow c + X$, the invariant differential cross section can be expressed as

$$\begin{aligned} \frac{d\sigma_{ab \rightarrow c+X}}{d^3\mathbf{p}_c/E_c} &\propto \sum_X \delta^{(4)}(p_a + p_b - p_c - p_X) \left| \langle p_c, X | p_a, p_b \rangle \right|^2 \\ &= \sum_X \int \frac{d^4x}{(2\pi)^4} e^{-ip_c \cdot x} \langle p_a, p_b | \varphi_c(x) | X \rangle \langle X | \varphi_c(0) | p_a, p_b \rangle \\ &= \langle p_a, p_b | \tilde{\mathcal{O}}_c(p_c) | p_a, p_b \rangle \end{aligned} \quad (6.2.1)$$

where $\tilde{\mathcal{O}}_c$ is given in terms of the product of two local operators

$$\tilde{\mathcal{O}}_c(p_c) = \int \frac{d^4x}{(2\pi)^4} e^{-ip_c \cdot x} \varphi_c(x) \varphi_c(0) \quad (6.2.2)$$

Here, p_c is the four-momentum for the produced scalar, and we have made use of the completeness relation, $\sum_X |X\rangle\langle X| = I$, as well as translational invariance. Since the product $\varphi_c(x)\varphi_c(0)$ is not time-ordered, one is in general dealing with a ‘‘Wightman function’’.

The transition from a T-product to a Wightman function corresponds to taking a discontinuity. This discontinuity is taken in the missing-mass squared, $M^2 = (p_a + p_b - p_c)^2$, i.e.,

$$\frac{d\sigma_{ab \rightarrow c+X}}{d^3\mathbf{p}_c/E_c} \propto \frac{1}{2is} \text{Disc}_{M^2 > 0} T(p_{a'}, p_{b'}, p_c; p_a, p_b, p_{c'}), \quad (6.2.3)$$

in the forward limit, i.e., $p_{a'} = p_a, p_{b'} = p_b, p_{c'} = p_c$. In taking the discontinuity in (6.2.1), all particles in the intermediate state X are on-shell⁵. As mentioned in the Introduction, Eq. (6.2.3) corresponds to a generalized optical theorem [117–120], and is also known as the Mueller formula. It can be generalized for n -particle inclusive distributions, $n \geq 1$.

This on-shell discontinuity condition can best be illustrated by first comparing the Feynman propagator for a free scalar, $G_F = \langle 0 | T(\varphi(x)\varphi(0)) | 0 \rangle$, with that for a simple product, $G_W = \langle 0 | \varphi(x)\varphi(0) | 0 \rangle$. In a momentum representation, $G_F(p^2) = \frac{i}{p^2 - m^2 + i\varepsilon}$ and

$$G_W(p^2) = \int \frac{d^4x e^{ip \cdot x}}{(2\pi)^3} \langle 0 | \varphi(x)\varphi(0) | 0 \rangle = 2\pi\delta(p^2 - m^2)\theta(p^0), \quad (6.2.4)$$

Therefore, $G_W(p^2)$ corresponds to the discontinuity of $G_F(p^2)$ for $p^0 > 0$. With interactions, in addition to the pole, $G_F(p^2)$, acquires a branch cut for $p^2 > 4m^2$, and it can be represented in a

⁵From the perspective of $T(p_{a'}, p_{b'}, p_c; p_a, p_b, p_{c'})$, M^2 is a ‘‘cross-channel’’ invariant, as oppose to ‘‘direct-channel’’ invariant, e.g., $s_{abc'} = (p_a + p_b + p_{c'})^2$. Both singularities are present in the physical region.

spectral representation, again with $G_W(p^2)$ as its discontinuity.

Eq. (6.2.1) can be understood similarly, by replacing the 4-vector p in (6.2.4) by $(p_a + p_b - p_c)$, with $p_c^0 > 0$ and $p_a^0 + p_b^0 - p_c^0 > 0$. Therefore, the discontinuity in (6.2.3) yields a sum over all allowed multi-particle states X , multiplied by a delta function factor, $\delta((p_a + p_b - p_c)^2 - M_X^2)$. Each state, X , contributes a term which is the product of an on-shell amplitude for $ab \rightarrow cX$ with its conjugate, $T_{a'b' \rightarrow c'X}^* T_{ab \rightarrow cX}$. The total discontinuity involves carrying out an integral over the phase space for each multi-particle state X and sum over all allowed states.

Instead of inclusive cross sections, it is useful to introduce inclusive density, $\rho_{ab}^{(c)}$, normalized relative to the total cross section,

$$\rho_{ab}^{(c)}(p_c) = (\sigma_{ab}^{total})^{-1} \frac{d\sigma_{ab \rightarrow c+X}}{d^3p_c/E_c}. \quad (6.2.5)$$

The integral over the density directly leads to the multiplicity, $\langle n \rangle_c = \int \rho_{ab}^{(c)}(p_c) (d^3p_c/E_c)$, and, similarly, when weighted by energy E_c , it yields the average energy carries by the particle type-c, $\langle E \rangle_c = \int [E_c(p_c)] \rho_{ab}^{(c)}(d^3p_c/E_c)$. It also allows one to obtain "flow rates" for conserved quantities, e.g., energy density flowing into a solid angle $d^2\Omega$ about a direction $\hat{n}$, $\langle E \rangle_{\hat{n}} = \int [\sum_c E_c \delta^2(\hat{p}_c - \hat{n})] \rho_{ab}^{(c)}(p_c) (d^3p_c/E_c)$, where a sum over all particle types has also been performed. More generally, for every conserved charge Q , we can introduce local flow density $\langle Q \rangle_{\hat{n}}$ into a differential solid angle $d^2\Omega$, which, through (6.1.3), can be expressed as a discontinuity in the forward limit,

$$\langle Q \rangle_{\hat{n}} = \frac{1}{2i\sigma_{ab}^{total}} \sum_c \int \frac{d^3p_c}{E_c} [w_Q(p_c, \hat{n})] \text{Disc}_{M^2 > 0} T(p_{a'}, p_{b'}, p_c; p_a, p_b, p_{c'}) \quad (6.2.6)$$

where $w_Q(p_c, \hat{n}) = Q_c \delta^2(\hat{p}_c - \hat{n})$. In taking the discontinuity, one obtains a sum over states, X , over a product of Q_c with $\langle X | \varphi_c | ab \rangle$ and its conjugate. Formally, this can also be re-written as a matrix element of a product of local operator, $Q_c \langle a'b' | \tilde{\mathcal{O}}_c(\hat{n}) | ab \rangle$, where $\tilde{\mathcal{O}}_c(\hat{n}) = \int d^4p \delta^2(\hat{p} - \hat{n}) \tilde{\mathcal{O}}_c(p) = \int dx \delta^2(\hat{x} - \hat{n}) \varphi_c(x) \varphi_c(0)$. In what follows, we shall adopt the notation $\tilde{\mathcal{O}}_w(\hat{n}) = \sum_c w_c \tilde{\mathcal{O}}_c(\hat{n})$ for a general differential charge flow. We have thus succeeded in expressing observables such as the so-called "event shapes" as "weighted discontinuities". Incidentally, conservation of charge leads to a sum rule, $\int d^2\hat{n} \langle Q \rangle_{\hat{n}} = Q_{total} = Q_a + Q_b$.

Inclusive Distribution in CFT: In CFT, one normally begins with Euclidean correlation functions for conformal primaries, e.g., $\langle \varphi(x_1) \varphi(x_2) \cdots \rangle$. After continuing to Lorentzian signature, this leads to time-ordered vacuum expectation values, $G_n(x_1, x_2, \cdots) = \langle 0 | T(\varphi(x_1) \varphi(x_2) \cdots) | 0 \rangle$, i.e.,

off-shell Green's functions. Due to translational invariance, it is always possible to go to momentum space, $G_n(p_1, p_2, \dots)$ ⁶. We assume that, by keeping each p_i^2 euclidean, it is meaningful to address their analytic structure in all relevant Mandelstam invariants. This certainly can be carried for perturbation theories. It is also the case for theories admitting a large N_c limit. For our purpose, we will be primarily be interested in the strong coupling limit, i.e., AdS/CFT, with a confining deformation in the IR, thus avoiding the issue of massless singularities.

Interest in inclusive processes in CFT has been heightened with the work of Hofman and Maldacena [107]. It is useful to re-state (6.1.1) as a normalized distribution

$$\langle \tilde{O}_w \rangle = \frac{\sigma_w(p)}{\sigma(p)} = \frac{\int d^4x e^{ipx} \langle 0 | \mathcal{O}^\dagger(x) \tilde{O}_w \mathcal{O}(0) | 0 \rangle}{\int d^4x e^{ipx} \langle 0 | \mathcal{O}^\dagger(x) \mathcal{O}(0) | 0 \rangle} = \frac{\langle \mathcal{O}(p) | \tilde{O}_w | \mathcal{O}(p) \rangle}{\langle \mathcal{O}(p) | \mathcal{O}(p) \rangle}, \quad (6.2.7)$$

where $\mathcal{O}$ serves as the source for the initial state and $\tilde{O}_w$ is chosen for infrared safety, e.g., flow of conserved energy or charge. In general, $\tilde{O}_w$ is a product of a set of local operators and is not time-ordered, thus Wightman functions, e.g., for a scalar conformal field φ of dimension Δ , the two-point Wightman function, $\langle 0 | \varphi(x) \varphi(0) | 0 \rangle$, in a momentum representation, is a distribution⁷

$$\int d^4x e^{iq \cdot x} \frac{1}{[\bar{x}^2 - (t - i\varepsilon)^2]^\Delta} = c(\Delta) (q^2)^{\Delta-2} \theta(q^2) \theta(q^0). \quad (6.2.8)$$

where $c(\Delta)$ is a normalization constant. When used in (6.2.7), as done before for (6.2.1), the four-vector q is replaced by $p - p_c$, with $p^0 > p_c^0 > 0$, where p_c is the momentum associated with the flow. The relation between the continued time-ordered Green's functions, $G_n(x_1, x_2, \dots)$, and Wightman functions has been elaborated further in [108–110].

Eq. (6.2.7) is appropriate for processes such as $e^+e^- \rightarrow \gamma^* \rightarrow c_1 + c_2 + \dots + X$, where the source involves a single local operator, $\mathcal{O}$. The cross section for such process can be expressed generically as

$$\sigma_w(p, \hat{n}) = \sum_X (2\pi)^4 \delta^{(4)}(p - p_X) w(X) \left| \langle X | \gamma^*(p) \rangle \right|^2 \quad (6.2.9)$$

where the sum over states X involves integrating the phase space for each state, weighted by $w(X)$. For example, the simplest inclusive event involves a single measurement of charge Q , by a “colorimeter”, located at spatial “infinity”, with a differential solid angle $d^{(2)}\Omega$ around a direction $\hat{n}$. This

⁶Again, we should strip off the overall energy-momentum delta function, $\delta^{(4)}(\sum p_i)$. We will also, in what follows, consider only the connected part of G_n

⁷In contrast, for the ordinary Feynman propagator, $\int d^4x e^{iq \cdot x} [\bar{x}^2 - t^2 + i\varepsilon]^{-\Delta} \sim (-q^2)^{\Delta-2}$, with a branch cut across $0 < q^2 < \infty$. Since there is no mass-gap, instead of a delta-function as in (6.2.4), the Wightman function in momentum space is a continuum, for $q^2 > 0$.

corresponds to having $w_Q = \sum_c Q_c \delta^{(2)}(\hat{p}_c - \hat{n}_\Omega) \theta(p_c^0)$. Here the sum is over all particles in state X . The cross section can also be re-written as $\sigma_w(p, \hat{n}) \sim \sum_X \left[\langle \gamma^*(p) | X \rangle w_Q \langle X | \gamma^*(p) \rangle \right]$. If the factor w_Q is replaced by a delta-function of four-momentum, $\delta(q_c - p_c)$, using completeness, $\sum_{X'} |X'\rangle \langle X'| = I$ where X' stands for having removed a state φ_c from the original sum X , as shown in Eq. (6.2.9). This in turn simply leads to the discontinuity of a 4-point function, in analogy to (6.2.10). One can formally introduce $\tilde{O}_Q = \sum_c w_Q \int d^4x e^{-ip_c x} \varphi_c(x) \varphi_c(0)$, in (6.2.7), which leads to

$$\sigma_Q(p, \hat{n}) = \sum_c \int d^4p_c \frac{1}{2i} [w_Q(p_c)] \text{Disc}_s T_{\gamma^* c' \rightarrow \gamma'^* c}. \quad (6.2.10)$$

in the forward limit of $p_{\gamma'^*} = p_{\gamma^*}$ and $p_c = p_{c'}$. The discontinuity is taken in region $s = (p_{\gamma^*} - p_c)^2 > 0$. Clearly, this can be generalized to the case of other conserved quantities, e.g., energy and momentum. One can also treat the case of higher order correlations, $\langle \tilde{O}_w(1) \tilde{O}_w(2) \cdots \rangle$, where

$$\sigma_w(\hat{n}_1, \hat{n}_2, \cdots) = \sum_{c_1, c_2, \cdots} \int d^4p_{c_1} \int d^4p_{c_2} \cdots \frac{1}{2i} w(p_{c_1}, p_{c_2}, \cdots) \text{Disc}_s T_{\gamma^* c'_1 c'_2 \cdots \rightarrow \gamma'^* c_1 c_2 \cdots}, \quad (6.2.11)$$

again in the forward limit.

It should be pointed out that (6.2.10) is in a momentum space representation. Kinematically, it is possible to relate $\langle \mathcal{O}(p) | \tilde{O}_w | \mathcal{O}(p) \rangle$ to a three-point (Wightman) function, $\langle \varphi_1(x) \varphi_2(y) \varphi_3(z) \rangle$, appropriately continued from Euclidean to Minkowski region, and correlator $\langle \mathcal{O}(p) | \tilde{O}_w(1) \tilde{O}_w(2) | \mathcal{O}(p) \rangle$ to a four-point function, etc. It is therefore possible to explore directly conformal invariance for (6.2.7) in a coordinate representation, as done for ordinary CFT correlators. As such, non-trivial features emerge only at the four-point level, e.g., for energy-energy correlation, $\gamma^* \rightarrow E(\hat{n}_1) + E(\hat{n}_2) + X$, as stressed in [107–110].

We will, in what follows, adopt the momentum representation for analyzing CFT inclusive distributions as discontinuities. We will also be primarily be interested in scattering processes, where non-trivial dynamics already occurs at the lowest level, e.g., that for the $\gamma^* p$ total cross section. More importantly, for forward scattering at high energy, as we shall demonstrate below, inclusive discontinuity can be taken readily.

Scattering Processes, OPE and Anomalous Dimensions: For scattering processes at high energies, the source is in general non-local. It is nevertheless possible to exploit consequence of conformal invariance at high energy via AdS/CFT, e.g., the BPST program. We illustrate here first with the example of deep inelastic scattering (DIS), i.e., the inclusive process $\gamma^*(q) + \text{proton}(p) \rightarrow X$,

in the limit of $Q^2 = -q^2 \rightarrow \infty$.

For DIS, the hadronic tensor, $W^{\mu\nu}(p, q) = i \int d^4x e^{iq \cdot x} \langle p | [J^\mu(x), J^\nu(0)] | p \rangle$, is the Fourier transform of a commutator. In contrast, for virtual Compton scattering, $\gamma^*(q) + \text{proton}(p) \rightarrow \gamma^*(q') + \text{proton}(p')$, the scattering amplitude is given by the Fourier transform of a T-product, $T^{\mu\nu}(p', q'; p, q) = i \int d^4x e^{iq' \cdot x} \langle p' | T \{ J^\mu(x) J^\nu(0) \} | p \rangle$. In the forward limit of $p' = p$ and $q' = q$, the hadronic tensor is related to the forward amplitude, $W^{\mu\nu}(p, q) = \frac{1}{2i} \text{Disc}_{s>0} T^{\mu\nu}(p, q; p, q)$.

To be more explicit, one can express the hadronic tensor in terms of two structure functions, $W^{\mu\nu} = F_1(x, Q^2) \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + F_2(x, Q^2) \left(p_\mu + \frac{q_\mu}{2x} \right) \left(p_\nu + \frac{q_\nu}{2x} \right)$, where $x = Q^2/s$ and $s = (p+q)^2$. For forward scattering, one can also expand $T^{\mu\nu}$ as done for $W^{\mu\nu}$, with $\tilde{F}_s(x, Q^2)$ replacing $F_s(x, Q^2)$, $s = 1, 2$, which can be treated as analytic functions of $s \sim 1/x$. The discontinuity relation can be expressed more directly as

$$F_s(x, Q^2) = 2\pi \text{Im} \tilde{F}_s(x, Q^2), \quad (6.2.12)$$

for $s = 1, 2$. Note that imaginary part is taken over the physical region, $s = (p+q)^2 > 0$, where $|x| < 1$. This is the optical theorem, e.g., (6.1.2), for virtual Compton scattering. As explained in [46], in a large- N_c treatment for QCD, the s-channel discontinuity consists of an infinite sequence of delta-functions, coming from Regge recurrences for the proton. As a prelude to our study of AdS/CFT, we mention here that DIS structure functions are strongly peaked phenomenologically at $x \rightarrow 0$, e.g., $F_s \sim x^{-1-(2-s)-\varepsilon}$, $\varepsilon > 0$, which can be used to infer the dominance of gluon dynamics, consistent with the large N_c expectation.

On a more theoretical side, this singular small- x behavior allows a direct measurement for the anomalous dimensions for twist-two operators which enters through the t-channel OPE for the full amplitudes $T^{\mu\nu}$. A standard analysis leads to an expansion for $\tilde{F}_s$ in $1/x$, valid in the limit of large Q^2 . Through a dispersion relation, the coefficients of expansion in x^{-n} can be expressed in terms its discontinuity across $0 < |x| < 1$, i.e., in terms of structure functions F_s , $s = 1, 2$. These "moments" can be expressed as $M_n^{(s)}(Q^2) = \int dx x^{n-s} F_s(x, Q^2)$, $s = 1, 2$, and, in the large Q^2 limit,

$$M_n^{(s)}(Q^2) \sim (Q^2)^{-\gamma_n} \quad (6.2.13)$$

where γ_n are anomalous dimensions for twist-2 operators with spin $j = n$. For $j = 2$, we have $\gamma_2 = 0$ due to energy momentum conservation. For $j \neq 2$, anomalous dimensions do not vanish, which leads us directly to CFT dynamics at large 't Hooft coupling λ . That is, by treating the structure functions as discontinuities, one can explore anomalous dimensions through a t-channel OPE, which

can serve as a spring-board for introducing stringy effects via AdS/CFT ⁸.

Witten Diagrams, Confinement and Inclusive Cross Sections: With gauge-string duality, CFT correlation functions in strong coupling can be expressed in terms of dual amplitudes defined in the AdS bulk. In the limit of large 't Hooft coupling, they can be given as perturbative sums of Witten-like diagrams. For our QCD study, we shall consider the *AdS* dual of $\mathcal{N} = 4$ SUSY. We shall work in the Poincare patch where the metric is $ds^2 = \frac{z^2}{R^2} \left\{ -dt^2 + d\vec{x}^2 + dz^2 \right\}$, with $z \rightarrow 0$ and $z \rightarrow \infty$ for the UV and IR respectively. Due to translational invariance, one can always go to a momentum representation, through which continuation from Euclidean to Minkowski is well-defined perturbatively. A connected Green's function, G_n , can now be expressed in terms of an amplitude in the *AdS*-bulk, i.e.,

$$G_n(p_1, p_2, \dots) = \int \cdots \int \Pi_i \{ d\mu(z_i) \Phi_i(p_i, z_i) \} \mathcal{T}_n(p_1, z_1, p_2, z_2, \dots) \quad (6.2.14)$$

where $d\mu(z) = dz\sqrt{-g}$ and $g = \det g$, and $\Phi(p, z)$ is a bulk-to-boundary propagator. The reduced amplitude, $\mathcal{T}_n$, can be expanded in terms of bulk-to-bulk propagators and various allowed vertices. For example, the scalar bulk-to-bulk propagator, $G(x', z'; x, z)$, with conformal dimension Δ , can be written down easily in a momentum representation, $G(z, z', p_\mu)$, in terms of Bessel functions,

$$G(z, z', p^2) = (zz')^2 \int_0^\infty k dk \frac{J_{\Delta-2}(kz) J_{\Delta-2}(kz')}{k^2 - p^2} \quad (6.2.15)$$

Since there is no mass gap, $G(z, z', p^2)$ is analytic in p^2 , with a branch cut over $0 \leq p^2 < \infty$. Its discontinuity over the branch cut is

$$\text{Disc } G(z, z', p^2) = i\pi (zz')^2 J_{\Delta-2}(pz) J_{\Delta-2}(pz') \theta(p^2) \quad (6.2.16)$$

In the limit of $z, z' \rightarrow 0$, it approaches, up to a normalization constant, that for the Wightman function, (6.2.8), as it should. Indeed, this discontinuity formula, (6.2.16), together with (6.2.15), provides an alternative procedure for calculating energy distributions, $\gamma^* \rightarrow E(\hat{n}_1) + E(\hat{n}_2) + \cdots + X$, via Witten diagram in the AdS bulk.

Associated with each leg of the Green's function G_n is an off-shell wave-function, $e^{ip_\mu x^\mu}$ and

⁸It is meaningful to treat $\Delta = j + 2 + \gamma(j)$ as a function of j , or, equivalently, its inverse, $j(\Delta)$, as a continuous function of Δ . Conformal invariance leads to $j(\Delta)$ symmetric about $\Delta = 2$, with $j(\Delta)$ having a minimum at $\Delta = 2$. Indeed, the intercept of Pomeron [1], j_0 , can be found by $\Delta(j_0) = 2$. We shall return to this discussion in Sec. 6.3.

a bulk-to-boundary propagator, $G(x', z'; z, x)|_{z' \rightarrow 0}$. In order to define on-shell amplitudes, it is necessary to introduce confining deformation in the IR, leading to finite mass gaps. Under such a scenario, on-shell amplitudes are given by amputated Green's functions, with normal singularity structure as in standard flat space field theories. With confining deformation in IR, the spectrum also becomes discrete, and the propagator is now given by a discrete sum,

$$G(z, z', p^2) \rightarrow \sum_n \frac{\varphi_n(z) \varphi_n(z')}{m_n^2 - p^2} \quad (6.2.17)$$

where $\varphi_n(z)$ consists of a set of orthonormal wave functions, associated with an infinite set of scalar glueballs of increasing mass m_n . More importantly, the bulk-to-boundary propagator is also given by a discrete sum, $\sum_n \sum \frac{c_n \varphi_n(z)}{m_n^2 - p^2}$, with poles at $p^2 = m_n^2$.

It is now possible to define scattering amplitudes as amputated Green's functions by going on to the pole for each external state, leading to on-shell scattering amplitudes, T_n ,

$$T_n(p_1, p_2, \dots) = \int \dots \int \Pi_i \{d\mu(z_i) \varphi_i(z_i)\} \mathcal{T}_n(p_1, z_1, p_2, z_2, \dots) \quad (6.2.18)$$

That is, a bulk wave-function is associated with each external on-shell particle, $e^{-ipx} \varphi(z)$. This, in turn, allows us to extend our previous treatment for inclusive cross sections for on-shell scattering in terms of scattering in the AdS bulk. For instance, after applying (6.1.2) to a 4-point functions, one obtains, for $ab \rightarrow X$,

$$\sigma_{total} = (1/s) \int \{\Pi_{i=1-4} d\mu(z_i) \varphi_n(z_i)\} \text{Im } \mathcal{T}_4(p_1, z_1, \dots) \quad (6.2.19)$$

with $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2 = 0$, where we have switched from all-incoming to in- and out-convention where all energy components of 4-momenta are positive, $p_i^0 > 0$. For the single-particle production, applying (6.2.3), one has

$$\frac{d\sigma_{ab \rightarrow c+X}}{d^3\mathbf{p}_c/E_c} \simeq \frac{1}{2is} \int \{\Pi_{i=1-6} d\mu(z_i) \varphi_n(z_i)\} \text{Disc}_{M^2 > 0} \{\mathcal{T}_{abc' \rightarrow a'b'c}(p_i, z_i)\}. \quad (6.2.20)$$

with discontinuity taken in the forward limit. This key result can also be extended to multi-particle inclusive productions.

Although our discussion does rely on the IR confining deformation, the structure of Witten expansion, (6.2.18), is model independent. As stressed in [1, 26], it is possible to identify features which depend only on the conformal structure, e.g., large Q^2 behavior in DIS at small- x . We stress

here, in particular, the important fact that wave functions having universal behavior in the UV,

$$\varphi_n(z) \simeq z^\Delta, \quad (6.2.21)$$

i.e., as $z \rightarrow 0$. This behavior is independent of the confinement deformation and it depends only on the conformal properties. We shall make use of this fact when implementing the Polchinski-Strassler mechanism for large $p_\perp$ production ⁹.

6.3 String-Gauge Duality – Beyond $\lambda = \infty$ Limit:

In this section, we introduce and discuss stringy effects beyond the super-gravity limit of $\lambda \rightarrow \infty$. This is an important step both conceptually and phenomenologically. The standard Witten expansion involves only propagators and vertices of super-gravity fields in AdS_5 , e.g., dilaton φ , metric fluctuations $h_{\mu\nu}$, anti-symmetric tensor $B_{\mu\nu}$, etc. This dramatic reduction in the degrees of freedom can be understood by the rapidly increasing of anomalous dimensions for all unprotected gauge-invariant local operators in the large 't Hooft coupling limit, i.e., generically, their conformal dimensions grow as $\Delta(j) = j + 2 + \gamma_j = O(\sqrt{\lambda})$. As a consequence, their associated string duals decouples in AdS.

Incorporating stringy effects is also necessary phenomenologically. For instance, at high energy, as we will show shortly, the most important contribution to the Witten sum in the near-forward limit is due to a graviton exchange in the t-channel, which leads to too rapid an increase for the diffractive cross section. This effect is partially “tamed” due to stringy corrections. Starting with the $\lambda = \infty$ limit, we shall adopt the strategy of finding $1/\sqrt{\lambda}$ corrections in the context of $1/N_c$ expansion, i.e., we will treat stringy effects perturbatively in $1/\sqrt{\lambda}$ and in the leading $1/N_c$ limit.

We begin first by treating two special kinematic limits where consequence of stringy corrections can be formulated more easily. The first, fixed-angle limit, leads to “ultralocal” scattering in AdS, which in turn leads to the Polchinski-Strassler mechanism [12]. The second, eikonal limit, is appropriate for near-forward scattering at high energy. As shown in [1, 26], via AdS/CFT, this leads to reggeized AdS graviton, i.e., the BPST Pomeron. We next extend the BPST program to multi-particle near-forward scattering, which paves the way for treating central inclusive production below.

⁹Just for the record, in hard-wall deformation, $\Phi(z) \propto z^2 J_{\Delta-2}(m_n z)$. A similar explicit analytic expression can also be obtained for other deformations, e.g., that involving soft-wall deformation [1].

Ultralocal Scattering and Polchinski-Strassler Mechanism: It has been stressed in [12] that scattering amplitudes in gauge theories with good string dual description can often be simplified, where the underlying ten-dimensional, e.g., $AdS_5 \times S^5$, string scattering is effectively local. This simplification is particularly applicable in the limit of fixed angle-scattering when all four-dimensional Mandelstam invariants are large and *of the same order*, s . In this limit, gauge amplitudes can be expressed as a coherent sum of local scattering over the AdS bulk, (we ignore in this study fluctuations in S^5 throughout.) As an effective five-dimensional scattering process, momentum for external states, p_i^μ , seen by local observer in AdS bulk is red-shifted, with large components $\tilde{p}^\mu$ along p_i^μ where $\tilde{p}_i^\mu \simeq (z/R)p_i^\mu$. Here AdS radius R is large compared to string scale, $R^2 = \lambda^{1/2}\alpha'$, (in what follows, we shall set $R = 1$.)

In this limit, a 4-D scattering amplitude reduces to a coherent sum over local scattering over AdS bulk,

$$T_n(p_1, p_2, \dots) = \int d\mu(z) \mathcal{T}_n(\tilde{p}_1, \tilde{p}_2, \dots) \Pi_i\{\varphi_i(z)\} \quad (6.3.1)$$

where $\mathcal{T}_n$ corresponds to that for a flat-space bosonic string amplitudes. In terms of invariants, the arguments for $\mathcal{T}_n$ are red-shifted, $s_{ij} \rightarrow z^2 s_{ij}$. Let us recall that, a flat-space bosonic 4-point amplitude can be expressed via a Koba-Nelson representation involving an integral over a single modulus. In the limit of $-t \simeq (1 - \cos \theta_{cm})s \rightarrow \infty$, the integral is dominated by a saddle point. This leads to an exponential cutoff,

$$T_4(s, t) \sim e^{-f(\cos \theta_{cm})s}. \quad (6.3.2)$$

This exponential suppression is generic feature of flat-space string scattering, and holds also for multi-particle scattering in a similar generalized fixed-angle limit ¹⁰ As stressed in [12], in moving from $T_4(s, t)$ to $\mathcal{T}_4(\tilde{s}, \tilde{t})$, exponential suppression, (6.3.2), reduces the effective scattering region in (6.3.1), to $z = O(1/\sqrt{s})$. We shall refer this suppression as the Polchinski-Strassler mechanism. This leads to the dimensional counting rule, $\frac{d\sigma}{dt} \sim s^{-\tau_{total}}$ where the power is given by the sum of twists for each external lines, $\tau_i = \Delta_i - s_i$, with Δ_i the conformal dimension and s_i the spin,

$$\tau_{total} = \sum_i \tau_i, \quad (6.3.3)$$

consistent with the dimensional counting rule.

¹⁰Indeed, as mentioned in the chapter introduction, this feature represents a serious failure for earlier attempt in formulating gauge theories as strings.

Eikonal Limit and Pomeron: We turn next to 2-to-2 near-forward scattering in the high energy, corresponding to the limit $s \rightarrow \infty$ and t fixed. In terms of the scattering angle θ in the CM, it corresponds to the limit $\theta \sim 1/\sqrt{s}$, which will be referred to as the eikonal limit. Under plausible assumption, it can be shown that flat-space scattering in this limit is determined by an eikonal phase over the two-dimensional impact space, $\vec{b}$, with the amplitude given by a 2-D Fourier transform,

$$T_{1+2 \rightarrow 3+4}(s, t) \simeq (-2is) \int d\vec{b} e^{-i\vec{q}_\perp \cdot \vec{b}} \left\{ e^{i\chi(s, \vec{b})} - 1 \right\}. \quad (6.3.4)$$

Here $t = -q_\perp^2$, and can also be considered as the Laplacian in $\vec{b}$. Expanding in χ and applying (6.1.2), one sees that exchanging eikonals directly contributes to the total cross section, $\sigma_{total}(s) \simeq 2 \int d\vec{b} \chi(s, \vec{b}) + O(\chi^2)$. Consider the case where χ increases with energy squared, $\chi \sim s$, which corresponds to that expected in flat-space closed string theory due to a spin-2 exchange¹¹. For string-like theories, with $t \simeq \nabla_b^2$, this can be considered as a diffusion process [1, 129] in $\vec{b}$, with $\ln s$ serving as the diffusion time. The eikonal function for gravitational scattering can also be understood from the perspective of shock-wave due to exchanging of a graviton.

Correspondingly, with AdS/CFT, near-forward scattering can again take on an eikonal form, with

$$\tilde{t} \rightarrow \nabla_{AdS_\perp}^2, \quad (6.3.5)$$

i.e., it corresponds to diffusion in 3-dimensional transverse space, between $(x_\perp, z)$ and $x'_\perp, z'$, with $\vec{b} = x'_\perp - x_\perp$. In the eikonal limit, one finds

$$T_{1+2 \rightarrow 3+4}(s, t) \simeq (-2is) \int d\vec{b} e^{-i\vec{q}_\perp \cdot \vec{b}} \int dz dz' P_{(13)}(z) P_{(24)}(z') \left\{ e^{i\chi(s, \vec{b}, z, z')} - 1 \right\} \quad (6.3.6)$$

Expanding to first order in χ , it is possible to attribute the eikonal with leading t-channel exchange, as is done in [26], i.e., the Pomeron kernel, i.e., $\mathcal{K}_P(\vec{s}, \vec{t}, z, z') \simeq (2s) \int d\vec{b} e^{i\vec{q}_\perp \cdot \vec{b}} \chi(s, \vec{b}, z, z')$.

For elastic scattering, the contribution from a single Pomeron exchange can be represented schematically in a factorized form,

$$T_{2 \rightarrow 2} = \Phi_{13} * \tilde{\mathcal{K}}_P * \Phi_{24}, \quad (6.3.7)$$

where Φ_{13} and Φ_{24} represented two elastic vertex couplings, and $\tilde{\mathcal{K}}_P$ is an universal Pomeron kernel,

¹¹To be more precise, in the planar limit, one has $T(s, t) \sim s^{2+\alpha' t/4}$ for t fixed and s large. At $t = 0$, it corresponds to spin-2 exchange. One also note, $\chi = O(1/N_c)$. Therefore, keeping higher order terms in an eikonal expansion goes beyond keeping only the leading $1/N_c$ term. This is necessary for maintaining unitarity.

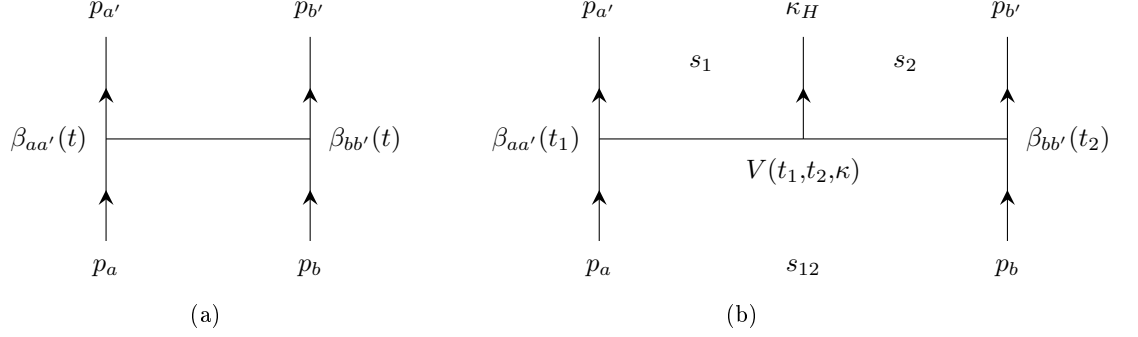


Figure 6.3.1: Kinematics for (a) the 2-to-2 amplitude in the Regge limit and (b) the 2-to-3 amplitude in the double Regge limit.

as indicated by Fig. 6.3.1a.

The convolution, denoted by the $*$ -operation, involves an integration over the AdS location in the bulk. For many purposes, e.g., for DIS at HERA, keeping this single term is often sufficient, although this eikonal summation will be ultimately be important in moving into higher energies. For now, we will focus in what follows on the kernel first.

As pointed earlier, through AdS/CFT, one can identify the Pomeron with Reggeized graviton in AdS. This kernel can be introduced by perturbing about the super-gravity limit, through a world-sheet OPE, as is done in [1]. In the conformal limit, this can be most effectively accomplished through the use of Pomeron vertex operator, $\mathcal{V}_P$. An equivalent but more explicit treatment is through the use complex j -plane representation. The Pomeron kernel exhibits a characteristic power behavior at large $s \gg |t|$, $\tilde{\mathcal{K}}_P \sim s^{j_0}$, The ‘‘Pomeron intercept’’, j_0 , lies in the range $1 < j_0 < 2$ and is a function of the ’t Hooft coupling, $\lambda = g^2 N_c$.

Alternatively, in the conformal limit, this can also be achieved from the perspective of t -channel OPE, with the Pomeron kernel representing a coherent sum of leading twist two conformal primaries, with dimensions interpolated by $\Delta(j)$, with $\Delta(2) = 4$ due to conserved stress-energy tensor. Conformal invariance also dictates that the spectral curve is symmetric under $\Delta(j) \leftrightarrow 4 - \Delta(j)$, with its inverse, $j(\Delta)$, taking on a minimum at $\Delta = 2$. As explained in [45], the Pomeron intercept is given by $j_0 = j(2)$, which admits an asymptotic expansion, $j_0 = 2 - 2/\sqrt{\lambda} + 0(1/\lambda)$.

BPST Program and 3-to-3 Amplitude: Before addressing inclusive production, it is useful to first provide a brief comment here on the BPST program for exclusive production. This formulation of AdS/CFT for high energy near-forward scattering has already generated a rather extensive literature [25–27, 113]. ‘‘Factorization in AdS’’ has emerged as a *universal* feature, applicable to scat-

tering involving both particles and currents. For elastic scattering, the amplitude can be represented schematically in a factorized form, Eq. (6.3.7). Similarly, because of factorization, it is also possible to extend this treatment to 2-to-3 amplitudes [130–132], thus providing a holographic description for an exclusive single-particle central production. In this treatment, one generalizes Eq. (6.3.7) to a 2-to-3 amplitude, with amplitude schematically represented by

$$T_{2 \rightarrow 3} = \Phi_{13} * \tilde{\mathcal{K}}_P * V_c * \tilde{\mathcal{K}}_P * \Phi_{24}, \quad (6.3.8)$$

which involves a new 3-point central production vertex, V_c , as shown in Fig. 6.3.1b. Similarly, it is possible to generalize above treatment to n -to- n scattering, $n \geq 3$, in the near forward limit. This can be carried out, following the same steps in moving from $T_{2 \rightarrow 2}$ to $T_{2 \rightarrow 3}$. For inclusive single-particle production in the central region, will be concerned with the limit where the 3-to-3 amplitude takes on the factorized form schematically given by

$$T_{abc' \rightarrow a'b'c} = \Phi_{13} * \tilde{\mathcal{K}}_P * V_{c\bar{c}} * \tilde{\mathcal{K}}_P * \Phi_{24} \quad (6.3.9)$$

which involves a new central vertex $V_{c\bar{c}}$, as illustrated by Fig. 6.3.2 below.

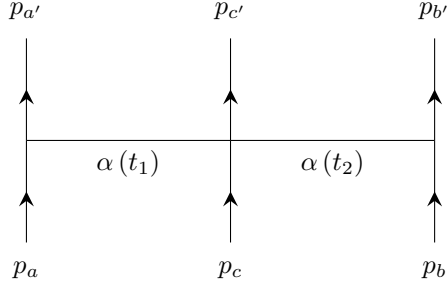


Figure 6.3.2: Schematic representation of 3→3 scattering of strings.

Turning next to inclusive production. Consider first the case of DIS(DIS), one first replaces Φ_{13} in (6.3.7) by the appropriate product of propagators for external currents [46, 47], and next perform the crucial step of taking discontinuity,

$$\sigma_{\gamma^* p}^{total} \simeq \frac{1}{s} \Phi_{13} * [\text{Im } \tilde{\mathcal{K}}_P] * \Phi_{24} . \quad (6.3.10)$$

This has been carried out previously. Moving on to the next simplest single-particle production in the “central region”, where the relevant amplitude is given by (6.3.9), the appropriate discontinuity

can again be taken [121–123], leading to

$$(1/2i)\text{Disc}_{M^2}T_{abc'\rightarrow a'b'c} = \Phi_{13} * [\text{Im } \tilde{\mathcal{K}}_P] * [\text{Im } V_{c\bar{c}}] * [\text{Im } \tilde{\mathcal{K}}_P] * \Phi_{24}. \quad (6.3.11)$$

At this point, it is useful to be more explicit. For (6.3.12), as explained in greater detail in chapter 3, one has

$$T_{2\rightarrow 2} = \int d\mu(z) \int d\mu(z') \Phi_{13}(z) \tilde{\mathcal{K}}_P(\tilde{s}, \tilde{t}, z, z') \Phi_{24}(z'), \quad (6.3.12)$$

More formally, as well as for establishing with the string connection, one can introduce a Pomeron vertex operator in AdS, so that

$$T_{2\rightarrow 2} = \langle \varphi_1 \varphi_3 | \mathcal{V}_P \rangle \Pi(\alpha_P) s^{\alpha_P} \langle \mathcal{V}_P | \varphi_2 \varphi_4 \rangle \quad (6.3.13)$$

where $\Pi(\alpha_P)$ carries the phase information for taking the discontinuity. For (6.3.8), the central 3-point central production vertex, V_c , in terms of Pomeron vertex operator, can be expressed as $V_c = \langle \mathcal{V}_P | \varphi_c | \mathcal{V}_P \rangle$. Its structure is similar to that from flat string theory.e.g, for open string, one has

$$V_c(t_1, t_2, \kappa) = \int_0^1 dy_1 \int_0^1 dy_2 y_1^{-\alpha_1-1} y_2^{-\alpha_2-1} e^{-y_1-y_2+\frac{y_1 y_2}{\kappa}} \quad (6.3.14)$$

where

$$\kappa = \frac{s_1 s_2}{s} \quad (6.3.15)$$

A similar expression also holds for closed-strings. In moving to AdS/CFT, one finds that the central vertex takes on the form $V_c(\tilde{t}_1, \tilde{t}_2, \tilde{\kappa})$ with all invariants red-shifted [130–132], as discussed in Sec. 6.3. For the purpose of interest to us, where t_1 and t_2 are kept fixed, it is sufficient to replace in the 5 point central vertex with invariants which are “red-shifted”, i.e., $t_1 \rightarrow \tilde{t}_1$, $t_2 \rightarrow \tilde{t}_2$, and $\kappa \rightarrow \tilde{\kappa}$.

Turning next to the new central vertex $V_{c\bar{c}}$ in (eq:ads3to3PomeronScheme), it can be expressed in terms of Pomeron vertex operator

$$V_{c\bar{c}} = \langle \mathcal{V}_P | \varphi_c \varphi_{\bar{c}} | \mathcal{V}_P \rangle \quad (6.3.16)$$

This new vertex, $V_{c\bar{c}}(\tilde{t}_1, \tilde{t}_2, \tilde{\kappa}, z)$, is also structurally similar to that from flat string theory, with with

red-shifted invariants. For the flat-space, one has

$$V_{c\bar{c}}(t_1, t_2, \kappa) \simeq \int_0^1 dy y^{-\alpha_{a\bar{a}c}-1} (1-y)^{-\alpha_{a\bar{a}\bar{c}}-1} V_c(t_1, t_2, \frac{\kappa}{y(1-y)}) \quad (6.3.17)$$

Here,

$$\kappa = \frac{s_{\bar{a}c}s_{\bar{b}c}}{M^2} \quad (6.3.18)$$

6.4 Inclusive Single-Particle Production in the Central Region:

Event Kinematics To discuss single particle production we first need to identify a set of kinematic variables that will make our description simple. As described in the introduction to this chapter, for $ab \rightarrow c + X$, treating it as a 2-to-2 process, with a variable mass, M^2 , we have three independent invariants, (s, t, M^2) . Equally well one can use the traditional

$$s = (p_a + p_b)^2, t = (p_a - p_c)^2, u = (p_b - p_c)^2$$

with M^2 as a dependent variable where

$$M^2 = s + t + u - m_a^2 - m_b^2 - m_c^2$$

In a typical hadronic collision in LHC, we are essentially in the CM frame of the colliding particles a and b, each with large and opposite momentum defining a longitudinal direction. It is convenient to specify produced particles with transverse momenta $\vec{p}_\perp$ and their longitudinal momenta $p_\parallel$ by a rapidity $p_\parallel = \sqrt{m^2 + p_\perp^2} \sinh y$. Phenomenologically, instead of s, t, u , one often prefers $(s, y, p_\perp^2)$. In a typically collider event, we are interested in events where y in the CM is small or held fixed. Such events can then be described by two variables s and $p_\perp^2$, such that $s \simeq M^2 \gg p_\perp^2, m_a^2, \text{etc.}$. One also finds that the center of mass energy and rapidity can be expressed as

$$s \simeq M^2 \simeq \frac{(-t)(-u)}{m_T^2} \quad (6.4.1)$$

$$e^{2y} \simeq \frac{t}{u} \quad (6.4.2)$$

Finally, it is important to visualize that, for central productions, produced particles can be

grouped as $a + b \rightarrow X_1 + c + X_2$, where X_1 is a left-moving group and X_2 is the right-moving group, relatively to the centrally produced particle c . If we denote the invariant mass squared for the left-moving group, X_1 -c, by s_2 , one has $s_2 \simeq -u = -(p_c - p_b)^2$, and, similarly, for the right-moving group, $s_1 \simeq -t = -(p_c - p_a)^2$. In what follows, we shall use s_1 , s_2 , and $\kappa \equiv p_\perp^2 + m_c^2$ as three independent variables, where

$$s \simeq M^2 \simeq \frac{s_1 s_2}{\kappa} \quad (6.4.3)$$

These are more convenient for our use for taking discontinuity for the 3-to-3 amplitude in M^2 .

6.4.1 The Six-Point Amplitude in AdS/CFT

We are now ready to discuss the scattering of six strings of the form $ab\bar{c} \rightarrow \bar{a}\bar{b}c$. To introduce a mass scale we will work with the hard wall model. A Witten diagram for this scattering is shown in Figure 6.4.1.

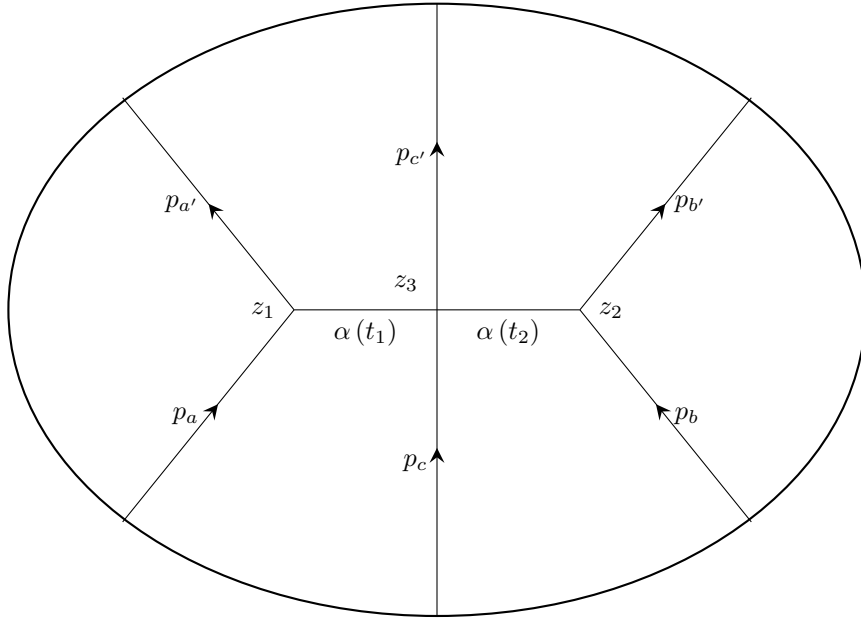


Figure 6.4.1: Witten diagram for $3 \rightarrow 3$ scattering in curved space.

The external lines are scalar glueballs with conformal dimension $\Delta = 4$. We are interested in a high energy regge limit where the internal bulk-to-bulk lines are Pomeron exchanges. From this

Witten diagram we can read off that the full six-point amplitude

$$\begin{aligned}
T_6(\kappa, s_1, s_2, t_1, t_2) = & \frac{g_0^2}{R^4} \int_0^{z_{max}} dz_1 \sqrt{-g(z_1)} \int_0^{z_{max}} dz_2 \sqrt{-g(z_2)} \int_0^{z_{max}} dz_3 \sqrt{-g(z_3)} \\
& \times e^{-2A(z_1)} \varphi_a(z_1) \varphi_{\bar{a}}(z_1) K_P(\tilde{s}_1, \tilde{t}_1, z_1, z_3) z_3^2 \varphi_c(z_3) V_6(\tilde{\kappa}, \tilde{t}_1, \tilde{t}_2, z_3) \\
& \times z_3^2 \varphi_{\bar{c}}(z_3) K_P(\tilde{s}_2, \tilde{t}_2, z_2, z_3) e^{-2A(z_2)} \varphi_b(z_2) \varphi_{\bar{b}}(z_2). \tag{6.4.4}
\end{aligned}$$

where the φ_i are the wavefunctions of the i -th glueball, V_6 is the vertex operator defined in equation 6.3.16, and K_P are the Pomeron kernels. In the Regge limit, the large invariants are

$$\tilde{s} = (\tilde{k}_a + \tilde{k}_b)^2 \sim \frac{z_1 z_2}{R^2} s \tag{6.4.5a}$$

$$\tilde{s}_1 = (k_a + k_{\bar{c}})^2 \sim \frac{z_1 z_3}{R^2} s_1 \tag{6.4.5b}$$

$$\tilde{s}_2 = (k_b + k_{\bar{c}})^2 \sim \frac{z_2 z_3}{R^2} s_2 \tag{6.4.5c}$$

$$\tilde{\kappa} = \frac{\tilde{s}_1 \tilde{s}_2}{\tilde{s}} \sim \frac{z_3^2}{R^2} \kappa. \tag{6.4.5d}$$

Again, we should note that we assume the vertex operator V_6 , taken to be the same in form as in flat space, is now a function of $\tilde{\kappa} \equiv \frac{z_3^2}{R^2} \kappa$. We must now compute the complete discontinuity in this amplitude across the missing mass M^2 . This comes from two sources: The vertex operator V_6 has a discontinuity across a $\kappa=0$ branch cut, and the two Pomeron propagators K_P both contain discontinuities in the partial energies s_i . By inspection of the imaginary part of the kernel in chapter 4 we see

$$\text{Disc}_{s_i} K_G(s_i, t_i, z_i, z_3) \propto s_i^{\alpha_0}. \tag{6.4.6}$$

The total 6 point discontinuity can be written

$$\begin{aligned}
\text{Disc}_{M^2} T_6 = & \frac{g_0^2}{R^4} \int_0^{z_{max}} dz_1 \sqrt{-g(z_1)} \int_0^{z_{max}} dz_2 \sqrt{-g(z_2)} \int_0^{z_{max}} dz_3 \sqrt{-g(z_3)} \\
& \times e^{-2A(z_1)} \varphi_a(z_1) \varphi_{\bar{a}}(z_1) \text{Disc} [K_G(\tilde{s}_1, \tilde{t}_1, z_1, z_3)] z_3^2 \varphi_c(z_3) \\
& \times \text{Disc} [V_6(\tilde{\kappa}, \tilde{t}_1, \tilde{t}_2, z_3)] z_3^2 \varphi_{\bar{c}}(z_3) \text{Disc} [K_G(\tilde{s}_2, \tilde{t}_2, z_2, z_3)] e^{-2A(z_2)} \varphi_b(z_2) \varphi_{\bar{b}}(z_2). \tag{6.4.7}
\end{aligned}$$

Which becomes

$$\begin{aligned} \text{Disc}_{M^2} T_6 = & \frac{g_0^2}{R^4} \int_0^{z_{max}} dz_1 \sqrt{-g(z_1)} \int_0^{z_{max}} dz_2 \sqrt{-g(z_2)} \int_0^{z_{max}} dz_3 \sqrt{-g(z_3)} \\ & \times e^{-2A(z_1)} \varphi_a(z_1) \varphi_{\bar{a}}(z_1) \tilde{s}_1^{\alpha_0} z_3^2 \varphi_c(z_3) e^{-2\alpha' \tilde{\kappa} z_3^2} \varphi_{\bar{c}}(z_3) \tilde{s}_2^{\alpha_0} e^{-2A(z_2)} \varphi_b(z_2) \varphi_{\bar{b}}(z_2). \end{aligned} \quad (6.4.8)$$

Crucially, the discontinuity in V_6 induces an exponential cutoff in $\tilde{\kappa}$, which itself exponentially suppresses the z_3 integrand:

$$e^{-2\alpha' \tilde{\kappa}} = e^{-2\alpha' \kappa z^2 / R^2}. \quad (6.4.9)$$

Thus, for large values of z_3 the contribution is small, and there exists some scattering length $z_s < z_{max}$ past which the z_3 integral becomes negligible; hence, we are free to integrate only over z_3 in the domain $[0, z_s]$. From Eq. 6.4.9, we read off that

$$z_s \sim \frac{R}{\sqrt{2\alpha' \kappa}}. \quad (6.4.10)$$

Because we work in a limit where $\kappa \rightarrow \infty$, all z in this interval are small, and we can approximate $e^{-2\alpha' \tilde{\kappa}} \sim 1$ and neglect the exponential term in the integral. Now, writing

$$\tilde{s}_1^{\alpha_0} \tilde{s}_2^{\alpha_0} = \left(s_1 \frac{z_1}{R} \frac{z_3}{R} \right)^{\alpha_0} \left(s_2 \frac{z_2}{R} \frac{z_3}{R} \right)^{\alpha_0} = (s_1 s_2)^{\alpha_0} \frac{z_1^{\alpha_0}}{R^{\alpha_0}} \frac{z_2^{\alpha_0}}{R^{\alpha_0}} \frac{z_3^{2\alpha_0}}{R^{2\alpha_0}} = s^{\alpha_0} \kappa^{\alpha_0} \frac{z_1^{\alpha_0}}{R^{\alpha_0}} \frac{z_2^{\alpha_0}}{R^{\alpha_0}} \frac{z_3^{2\alpha_0}}{R^{2\alpha_0}}, \quad (6.4.11)$$

we have

$$\begin{aligned} \text{Disc}_{M^2} T_6 = & \frac{g_0^2 s^{\alpha_0} \kappa^{\alpha_0}}{R^{4+4\alpha_0}} \int_0^{z_{max}} dz_1 \sqrt{-g(z_1)} \int_0^{z_{max}} dz_2 \sqrt{-g(z_2)} \int_0^{z_s} dz_3 \sqrt{-g(z_3)} \\ & \times e^{-2A(z_1)} \varphi_a(z_1) \varphi_{\bar{a}}(z_1) z_3^2 \varphi_c(z_3) z_3^2 \varphi_{\bar{c}}(z_3) e^{-2A(z_2)} \varphi_b(z_2) \varphi_{\bar{b}}(z_2) [z_1^{\alpha_0} z_2^{\alpha_0} z_3^{2\alpha_0}]. \end{aligned} \quad (6.4.12)$$

In what remains, the z_3 terms are completely independent of the z_1 and z_2 terms, and moreover all residual κ dependence stems from the κ dependence of z_s . Thus the z_1 and z_2 integrals contribute only an overall numerical prefactor, which we write as β . Then

$$\text{Disc}_{M^2} T_6 = \frac{g_0^2 s^{\alpha_0} \kappa^{\alpha_0} \beta}{R^{4+4\alpha_0}} \int_0^{z_s} dz_3 \sqrt{-g(z_3)} z_3^{4+2\alpha_0} \varphi_c(z_3) \varphi_{\bar{c}}(z_3). \quad (6.4.13)$$

We will now analyze the components of this integrand to extract the final κ dependence of the discontinuity. The open-string wavefunctions near $z = 0$ are

$$\varphi(z) = \frac{z^2}{R^2} J_{\Delta-2}(m_0 z) \quad (6.4.14)$$

To leading order in small z this becomes

$$\varphi(z) \approx R^{-2} z^\Delta \quad (6.4.15)$$

Inserting these results into Eq. 6.4.13, we find that

$$\text{Disc}_{M^2} T_6 = \frac{g_0^2 s^{\alpha_0} \kappa^{\alpha_0} \beta}{R^{3+4\alpha_0}} \int_0^{z_s} dz_3 z_3^{-5} z_3^{4+2\alpha_0} z_3^{\Delta_c} z_3^{\Delta_{\bar{c}}} \quad (6.4.16)$$

$$\begin{aligned} &= \frac{g_0^2 s^{\alpha_0} \kappa^{\alpha_0} \beta}{R^{3+4\alpha_0}} \int_0^{z_s} \frac{dz_3}{z_3} z_3^{\Delta_c + \Delta_{\bar{c}} + 2\alpha_0} \\ &= \frac{g_0^2 s^{\alpha_0} \kappa^{\alpha_0} \beta}{R^{3+4\alpha_0} (\Delta_c + \Delta_{\bar{c}} + 2\alpha_0)} z_3^{\Delta_c + \Delta_{\bar{c}} + 2\alpha_0} \Big|_0^{z_s} \\ &= \frac{g_0^2 s^{\alpha_0} \kappa^{\alpha_0} \beta}{R^{3+4\alpha_0} (\Delta_c + \Delta_{\bar{c}} + 2\alpha_0)} z_s^{\Delta_c + \Delta_{\bar{c}} + 2\alpha_0}. \end{aligned} \quad (6.4.17)$$

Finally, we insert Eq. 6.4.10 into this result to find that

$$\begin{aligned} \text{Disc}_{M^2} T_6 &= \frac{g_0^2 s^{\alpha_0} \kappa^{\alpha_0} \beta}{R^{3+4\alpha_0} (\Delta_c + \Delta_{\bar{c}} + 2\alpha_0)} \left(\frac{R}{\sqrt{2\alpha' \kappa}} \right)^{\Delta_c + \Delta_{\bar{c}} + 2\alpha_0} \\ &= \frac{R^{\Delta_c + \Delta_{\bar{c}} - 2\alpha_0 - 3} g_0^2 \beta}{(2\alpha')^{(\Delta_c + \Delta_{\bar{c}} + 2\alpha_0)/2} (\Delta_c + \Delta_{\bar{c}} + 2\alpha_0)} s^{\alpha_0} \kappa^{-(\Delta_c + \Delta_{\bar{c}})/2}. \end{aligned} \quad (6.4.18)$$

With this result, we directly calculate the inclusive cross section

$$\frac{d^3 \sigma_{ab \rightarrow X}}{d\mathbf{p}_c^3} = \frac{1}{s} \text{Disc}_{M^2} T_6 = \frac{R^{\Delta_c + \Delta_{\bar{c}} - 2\alpha_0 - 3} g_0^2 \beta}{(2\alpha')^{(\Delta_c + \Delta_{\bar{c}} + 2\alpha_0)/2} (\Delta_c + \Delta_{\bar{c}} + 2\alpha_0)} s^{\alpha_0 - 1} \kappa^{-(\Delta_c + \Delta_{\bar{c}})/2}. \quad (6.4.19)$$

For the calculation we have performed, the external particles c and $\bar{c}$ are scalar glueballs, so

$\Delta_c = \Delta_\varepsilon = 4$ [133]. Then our cross section is given essentially by

$$\frac{d^3\sigma_{ab\rightarrow X}}{d\mathbf{p}_c^3} \sim s^{\alpha_0-1}\kappa^{-4}. \quad (6.4.20)$$

6.4.2 Application to LHC Data

We will now attempt to demonstrate the validity of our model by fitting to published experimental data. We will focus on differential cross section measurements at high $\sqrt{s}$. Many recent measurements are in the form of a double differential cross section measurement, in which particle production is binned both in the transverse momentum p_T and the pseudorapidity η (or equivalently the rapidity y); symbolically, these studies measure the cross section $\frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T d\eta}$. In principle, this is not precisely the quantity we have computed above, as our calculation has no dependence on the pseudorapidity. However, these two cross sections encode essentially the same information, so we expect essentially the same dependence on the kinematic variables.

Our goal is to fit our predictions to differential cross sections $\frac{d\sigma}{dp_T}$. However, the result in 6.4.20 is only expected to hold at high p_T ; at low p_T , much more complicated behavior can occur [134,135]. To obtain a model that we expect to correspond to the full experimental spectrum, we modify this prediction in two ways. First, we include an exponential term in the model to explicitly account for low- p_T production unexplained by our model:

$$\left. \frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T d\eta} \right|_{\text{exponential}} = ae^{-bp_t+c}. \quad (6.4.21)$$

Alternate parameterizations of this low- p_T background exist for comparison. We also slightly modify the form of the power law fit:

$$\left. \frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T d\eta} \right|_{\text{power}} = A(p_T + C)^{-B}. \quad (6.4.22)$$

In the limit $p_T \rightarrow \infty$, i.e. in the high- κ regime where our results are most accurate, the added p_T offset C vanishes; physically, this offset corresponds to the emergence of Regge behavior only after some characteristic energy scale, such as the average momentum $\langle p_T \rangle$. Such phenomena are known to be poorly described by Regge kinematics, so our hope is to allow for them in the experimental fit. Thus, the total model which we will fit to data is

$$\frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T d\eta} = A(p_T + C)^{-B} + ae^{-bp_t+c}. \quad (6.4.23)$$

Dataset	a	b	c	A/1000	B	C
ALICE, 900 GeV	3.55 ± 4.29	2.60 ± 0.0229	4.49 ± 1.22	0.152 ± 0.176	6.11 ± 0.309	0.117 ± 0.668
ALICE, 2.76 TeV	2.96 ± 3.60	2.74 ± 0.0234	4.42 ± 1.22	0.189 ± 0.366	6.62 ± 0.544	0.352 ± 0.947
ALICE, 7 TeV	3.74 ± 4.05	2.95 ± 0.0230	3.93 ± 1.09	0.642 ± 3.32	7.65 ± 1.56	0.891 ± 1.97
CMS, 900 GeV	2.53 ± 6.02	4.07 ± 0.114	1.33 ± 2.38	0.00572 ± 0.0294	7.44 ± 2.28	0.524 ± 1.07
CMS, 2360 GeV	1.73 ± 3.26	3.62 ± 0.0766	1.772 ± 1.88	0.00101 ± 0.00187	5.76 ± 0.935	0.345 ± 0.420

Table 6.1: Fitted values of parameters in Eq. 6.4.23 for five datasets from p - p collisions. Both central values and statistical errors are quoted.

This model is in good agreement with the model suggested in [136, 137].

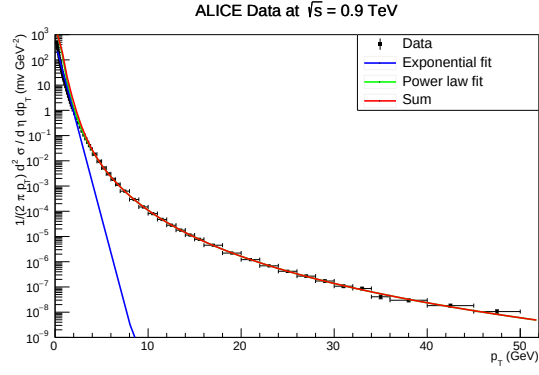
Explicitly, we are treating the exponential low- p_T production given in Eq. 6.4.21 as a background in our study; we fit this exponential term only up to some maximum p_T ; past this cutoff, we fit instead the power-law model given in Eq. 6.4.22. In effect, this transition is a selection cut. The stability of fit parameters as this cutoff has been analyzed, but is omitted for clarity. Following these studies, we take the cutoff momentum to be $p_T = 2.5$ GeV, except for a few datasets with reduced statistics for which we take the cutoff to be 2 GeV.

Theoretically, our results are most strongly suited to describe glueballs; because glueballs are not experimentally observable, we will instead focus on the production of other QCD bound states, namely hadrons. We will in particular study hadron production at the LHC in both p - p collisions.

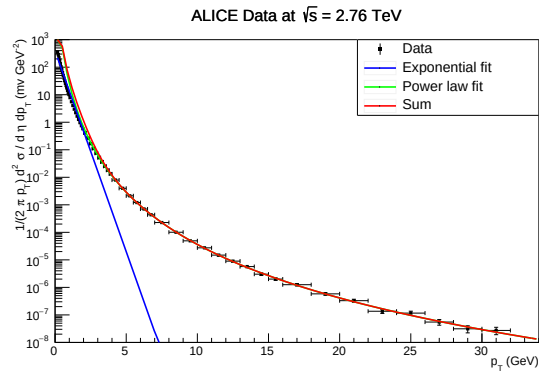
6.4.3 Hadron Production in p - p Collisions

Hadron production from p - p at a variety of center of mass energies has been studied by both the ALICE and CMS experiments. We fit the models described above to five different datasets; three are from ALICE, corresponding to $\sqrt{s} = 900$ GeV, 2.76 TeV, and 7 TeV [2] and $|\eta| < 0.8$, and two are from CMS, at $\sqrt{s} = 900$ GeV and 2.36 TeV and $|\eta| < 2.4$. First we will show plots of the fit and data in Figures 6.4.2-6.4.3, and then in Table 6.1 we will give numerical information on the fit parameters.

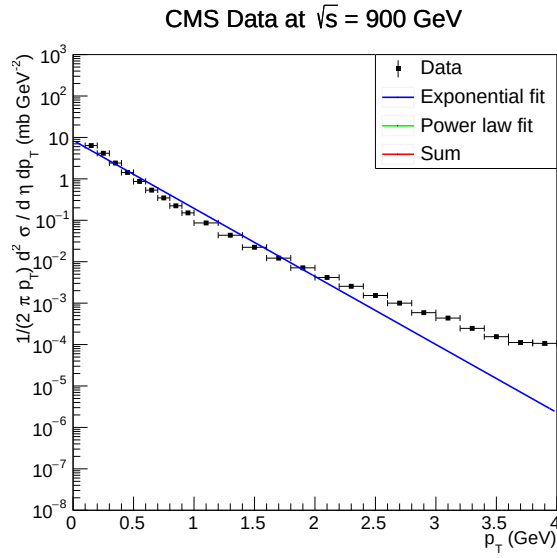
Results are found to be consistent throughout. The most important parameter for our analysis is the exponent B ; these are found to be in relative agreement with our prediction of $B = 8$, corresponding to a $\Delta = 4$ state in the supergravity theory. It is tempting to interpret the apparent increase of B with $\sqrt{s}$ as a transition between a $\Delta = 3$ process at low energy to a $\Delta = 4$ process at high energy, corresponding to direct production and glueball-mediated production, respectively. This hypothesis is, however, made difficult to take seriously by the large uncertainty in the exponent B ; within 1 standard deviation of the central value the $B = 6$ hypothesis is still consistent with the $\sqrt{s} = 7$ TeV results. Indeed, the CMS datasets are even worse for this hypothesis, in that the central



(a)



(b)



(c)

Figure 6.4.2: Charged hadron production cross section measured in p - p collisions recorded by ALICE at $\sqrt{s} = .9$ TeV (a), $\sqrt{s} = 2.79$ TeV (b), $\sqrt{s} = 7$ TeV (c). A fit to the model given in Eq. 6.4.23 is superimposed. Data from [2]. In color online.

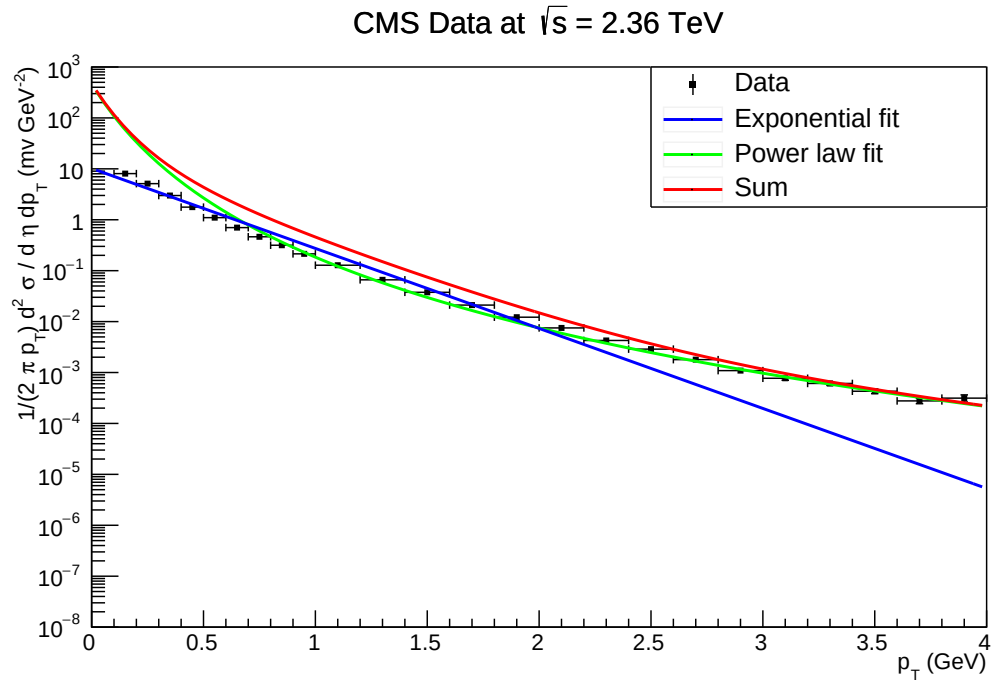
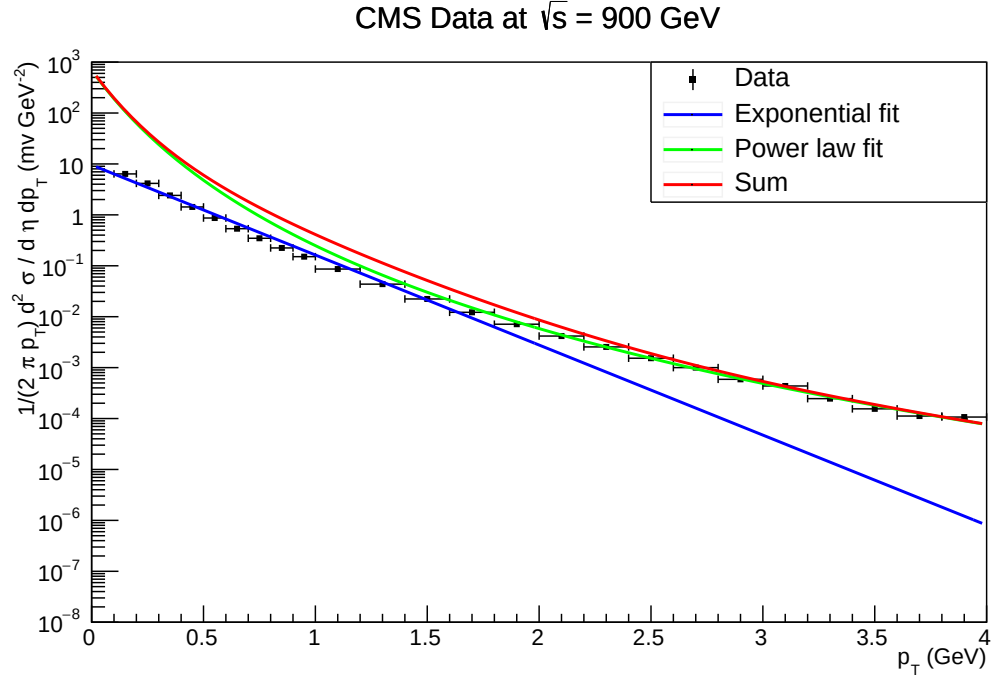


Figure 6.4.3: Charged hadron production cross section measured in p - p collisions recorded by CMS at $\sqrt{s} = .9$ TeV (a) and $\sqrt{s} = 2.36$ TeV (b). A fit to the model given in Eq. 6.4.23 is superimposed. Data from [2].

value decreases with $\sqrt{s}$; however, these datasets are statistically limited, and should be given less weight than the other datasets. As this analysis is refined we will comment on the stability of the fit parameters and repeat the fit for lead collisions coming from the ALICE experiment.

Appendix A

Regge Limits

Here we review some of the details of the 4 and 5 point string amplitudes and Regge limit(s). (The 6 point is more thoroughly discussed in the text) These amplitudes will generally be functions of external 4 momenta, p_i . It is convenient however to define *mandelstam variables* which are squares of linear combinations of momenta.

$$s_{ij\dots k} \equiv \pm(p_i + p_j \dots - p_k)^2 \quad (\text{A.0.1})$$

where the signs inside the square are determined by whether the particle is incoming or outgoing and the overall sign is fixed via the momentum conserving constraint. For n-particle scattering we can construct $n(n-3)/2$ independent mandelstam variables to categorize momentum in different channels.¹

A.1 n=4

Let's begin with the 4-point amplitude 2.1.1 and it's regge limit A.1.1. We choose a "scattering" orientation convention (in-in-out-out) for our momenta so that we have the standard

$$s_{12} = s = -(p_1 + p_2)^2 \quad s_{14} = t = -(p_1 - p_4)^2 \quad s_{13} = -(p_1 - p_3)^2 \quad (\text{A.1.1})$$

where $s + t + u = \sum_1^4 m_i^2$. For simplicity, let's examine the classic closed string tachyon amplitude in critical string theory. Let's drop overall constants and numerical factors as we are really interested

¹In the context of Mellin amplitudes these satisfy the Mellin variable constraints

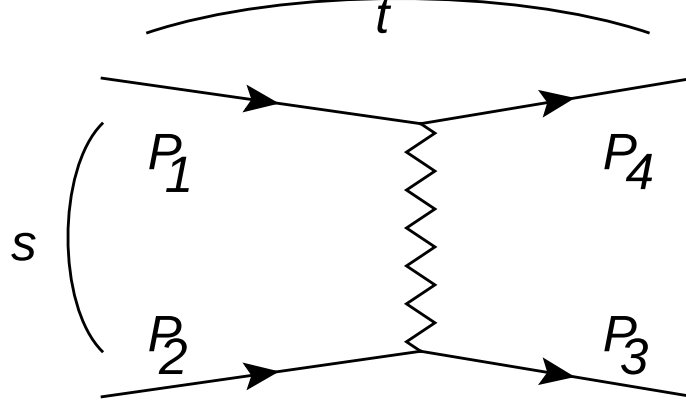


Figure A.1.1: Pomeron exchange in 2-to-2 scattering

in extracting the Regge limit behavior.

$$S_4^{tree} \sim \langle \mathcal{V}_1^T \mathcal{V}_2^T \mathcal{V}_3^T \mathcal{V}_4^T \rangle \sim \langle \prod_1^3 : c_i \bar{c}_i e^{ip_i \cdot X} : \int d^2 z_4 : e^{ip_4 \cdot X} : \rangle \quad (\text{A.1.2})$$

$$\sim |z_1 2|^2 |z_1 3|^2 |z_2 3|^2 \int d^2 z_4 \prod_{i < j=1}^4 |z_{ij}|^{\alpha' p_i \cdot p_j} \quad (\text{A.1.3})$$

The $PSL(2, \mathbb{C})$ symmetry allows one to fix $z_1 = 0$, $z_2 = 1$, and $z_3 \rightarrow \infty$ so that the result becomes

$$S_4^{tree} \sim \int d^2 z |z|^{\alpha' t/2} |1-z|^{\alpha' u/2} \sim \frac{\Gamma(-1 - \alpha' \frac{s}{4}) \Gamma(-1 - \alpha' \frac{t}{4}) \Gamma(-1 - \alpha' \frac{u}{4})}{\Gamma(2 + \alpha' \frac{s}{4}) \Gamma(2 + \alpha' \frac{t}{4}) \Gamma(2 + \alpha' \frac{u}{4})} \quad (\text{A.1.4})$$

which is the celebrated Virasoro-Shapiro amplitude. If we expand in the Regge limit (s large, t fixed)

$$S_4^{tree} \xrightarrow{\text{Regge}} s^{2 + \alpha' t/2} \frac{\Gamma(-1 - \alpha' t/4)}{\Gamma(2 + \alpha' t/4)} \quad (\text{A.1.5})$$

where we see Regge behavior. In fact, if we further take the limit where t and u are large as well we will see the exponentially suppressed behavior that plagued flat space string theory as a theory of hadrons. The extension to AdS space is discussed in great detail in [32] and reviewed above in chapter 3.

A.2 n=5

Let's begin by examining the flat space 5-point closed string amplitude and then evaluate its Regge limit ala the diffractive Higgs analysis in [131]. The tree level amplitude is

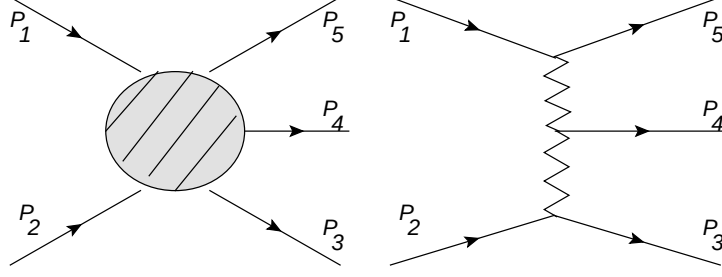


Figure A.2.1: (a) General 5 point diagram (b) double Regge limit for the 5 point amplitude

$$S_5^{tree} \sim \langle \mathcal{V}_1^T \mathcal{V}_2^T \mathcal{V}_3^T \mathcal{V}_4^T \mathcal{V}_5^T \rangle \sim \langle \prod_{\{1,2,5\}} : c_i \bar{c}_i e^{ip_i \cdot X} : \int d^2 z_3 d^2 z_4 : e^{ip_3 \cdot X} :: e^{ip_4 \cdot X} : \rangle \quad (\text{A.2.1})$$

Again, using the $PSL(2, \mathbb{C})$ symmetry to fix $z_1 = 0$, $z_2 = 1$, $z_5 \rightarrow \infty$ we can re-write this amplitude as

$$S_5^{tree} \sim \int d^2 z_3 d^2 z_4 |z_3|^{\alpha' p_1 \cdot p_3} |z_4|^{\alpha' p_1 \cdot p_4} |1 - z_3|^{\alpha' p_2 \cdot p_3} |1 - z_4|^{\alpha' p_2 \cdot p_4} |z_3 - z_4|^{\alpha' p_3 \cdot p_4} \quad (\text{A.2.2})$$

$$\sim \int d^2 u d^2 v |u|^{\frac{-\alpha' t_1}{2} + 3} |v|^{\frac{-\alpha' t_2}{2} + \frac{5}{2}} |1 - u|^{\frac{-\alpha' s_1}{2} - 1} |1 - v|^{\frac{-\alpha' s_2}{2} - 1} |1 - uv|^{\frac{1}{2}(s + s_1 + s_2)} \quad (\text{A.2.3})$$

$$(\text{A.2.4})$$

where we have performed conformal transformations $z_4 \rightarrow z_3/v$ and $z_3 \rightarrow 1/u$. At this point one can take the Regge limit via a saddle point approximation, but we turn to AdS space before going forward.

AdS Double Regge Limit The AdS Regge analysis for 2-to-3 amplitude follows the same path as for the elastic amplitude but is considerably more subtle. Let's focus on the double diffractive Higgs production [131] amplitude, $A(s, s_1, s_2, t_1, t_2)$,

$$a(k_1) + b(k_2) \rightarrow a(k_3) + c(q) + b(k_4) \quad (\text{A.2.5})$$

has 5 Mandelstam invariants. Again we can express the invariants, $s = (k_1 + k_2)^2$, $s_1 = (k_3 + q)^2$, $t_1 = (k_1 - k_3)^2$, $s_2 = (k_4 - q)^2$, $t_2 = (k_2 - k_4)^2$ in terms of light cone coordinates by choosing an appropriate

frame:

$$\begin{aligned} k_1 &= (m_1 e^{y/2}, m_1 e^{-y/2}, 0_\perp) \quad , & k_2 &= (m_2 e^{-y/2}, m_2 e^{y/2}, 0_\perp) \\ k_3 &= (m_{3\perp} e^{y_3}, m_{3\perp} e^{-y_3}, -q_{1\perp}) \quad , & k_4 &= (m_{4\perp} e^{-y_4}, m_{4\perp} e^{y_4}, -q_{2\perp}) \end{aligned} \quad (\text{A.2.6})$$

where the transverse mass is $m_\perp = m^2 + k_\perp^2$ for any on shell state $k_i^2 = m^2$. The momentum for the central particle, $q = k_1 + k_2 - k_3 - k_4$ be parametrized by $q = (m_{H\perp} e^{y_H}, m_{H\perp} e^{-y_H}, q_{H\perp})$ with transverse mass $m_{H\perp}$,

$$m_{H\perp}^2 = m_H^2 + q_\perp^2 . \quad (\text{A.2.7})$$

The double Regge limit is $|y_3 - y_H| \sim \log(s_1) \rightarrow \infty$, $|y_4 + y_H| \sim \log(s_2) \rightarrow \infty$ at fixed momentum transfers $t_i \simeq -k_{i\perp}^2$ and $\mathbf{q}_\perp^2$. In this limit, one also has $y_3 \simeq -y_4 \simeq y/2$. The Higgs momentum components are fixed by energy-momentum conservation, e.g.,

$$y_H = \frac{1}{2} [\log(s_1/s_2) - \log(m_{3\perp}/m_{4\perp})] . \quad (\text{A.2.8})$$

The transverse mass is specified by the relation,

$$\kappa = \frac{s_1 s_2}{s} \simeq m_H^2 + q_\perp^2 = m_{H\perp}^2 , \quad (\text{A.2.9})$$

referred to as the “kappa” variable for the Regge-Regge particle vertex, $V(t_t, t_2, \kappa)$. It is natural to introduce the two outgoing rapidity gaps separating the Higgs as

$$\Delta y_1 = y_3 - y_H \quad , \quad \Delta y_2 = y_4 + y_H . \quad (\text{A.2.10})$$

The light-cone parametrization provides exact change of coordinates: $s, s_i, t_i \rightarrow \Delta y_i, q_{i\perp}^2$. The double diffractive production limit is characterized by small transverse momenta, $t_i \simeq -q_{i\perp}^2$ and large rapidity gaps, $\Delta y_1 \simeq \log(s_1/(m_{3\perp} m_{H\perp}))$ and $\Delta y_2 \simeq \log(s_2/(m_{4\perp} m_{H\perp}))$.

In the double Regge limit the only “ κ ” dependence occurs in the Regge-Regge Higgs vertex, $V(t_t, t_2, \kappa)$. Note also since $\kappa = m_H^2 + q_{1\perp}^2 + q_{2\perp}^2 + 2q_{1\perp} \cdot q_{2\perp}$, κ is solely responsible for the so-called Toller angular dependence in $q_{1\perp} \cdot q_{2\perp}$ between the two protons scattering planes (see Fig. A.2.1). The key to a double-Regge expansion is factorization in the J -plane. It is possible to obtain the full amplitude via a double-inverse-Mellin transform. We refer to the literature for further discussion of this formalism for some of the subtleties involved [138]. Most of these can be neglected due to

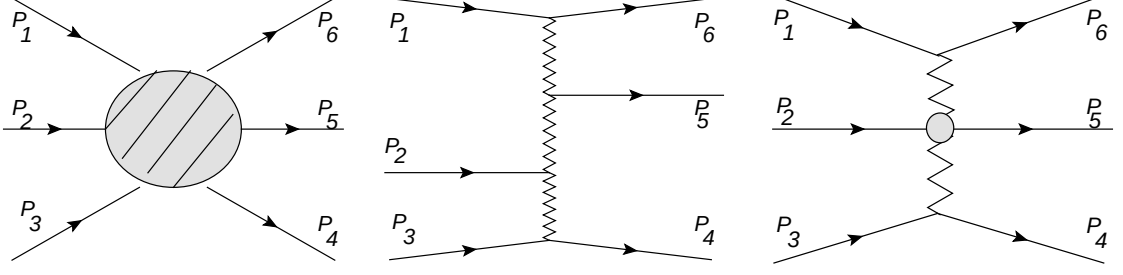


Figure A.2.2: (a) General 6 point diagram (b) triple Regge limit for the 6 point amplitude (c) double Regge limit for the 6 point amplitude

the high mass of the Higgs at the central vertex. Consequently, with this *proviso*, we arrive at the amplitude for Higgs production expressed in terms of two Pomeron kernels, $\tilde{\mathcal{K}}_P(s_1, t_1, z_1, z'_1)$ and $\tilde{\mathcal{K}}_P(s_2, t_2, z'_2, z_2)$, together with a Higgs vertex, $V_H(t_1, t_2, \kappa, z'_1, z'_2)$. Finally we are able to give an explicit form to the symbolic expression for the double-Pomeron amplitude in Eq. 6.3.8 of the Introduction:

$$A(s, s_1, s_2, t_1, t_2) \simeq \int dz_1 dz'_1 dz'_2 dz_2 \sqrt{-g(z_1)} \sqrt{-g(z'_1)} \sqrt{-g(z'_2)} \sqrt{-g(z_2)} \quad (\text{A.2.11})$$

$$\times \Phi_{13}(t_1, z_1) \tilde{\mathcal{K}}_P(s_1, t_1, z_1, z'_1) V_H(t_1, t_2, \kappa, z'_1, z'_2) \tilde{\mathcal{K}}_P(s_2, t_2, z'_2, z_2) \Phi_{24}(t_2, z_2) .$$

In the above expression, impact factors, $\Phi_{13}(t_1, z_1)$ and $\Phi_{24}(t_2, z_2)$ are identical to those entering the elastic amplitude in chapter 3. This represents a generalization of the elastic scattering to central diffractive Higgs production. The only new component is the Pomeron-Pomeron-Higgs vertex, $V_H(t_1, t_2, \kappa, z'_1, z'_2)$, which in general allows non-local interaction in AdS_3 , e.g., its dependence on z'_1 and z'_2 .

Just as the case of 2-to-2 scattering, it is often simpler to think of scattering in terms of transverse coordinates $\mathbf{b} = (x_\perp, z)$. Instead of the vertex $V_H(t_1, t_2, \kappa, z'_1, z'_2)$ in momentum space, one can work directly with $V_H(x'_{1\perp} - x_{H\perp}, z'_1; x'_{2\perp} - x_{H\perp}, z'_2)$. The κ -dependence enters through the possible dependence on the angular correlation between the two proton's scattering planes, $x'_{1\perp} - x_{H\perp}$ and $x'_{2\perp} - x_{H\perp}$ or the scalar product: $(x'_{1\perp} - x_{H\perp}) \cdot (x'_{2\perp} - x_{H\perp})$. As we will see, for Higgs production, due to its large mass, this dependence is suppressed and we will choose to drop it. Indeed, (A.2.12) would take on a more complicated form if non-trivial $O(1/m_H^2)$ dependence on κ turns out to be important. We will not address this issue here.

Appendix B

Mellin Transformation

When written in terms of analytic momenta, scattering amplitudes can very naturally be expressed in terms of a Mellin transformation. An example is our pomeron kernel whos Mellin representation is given in chapter 3. This formulation is historically common in terms of analytic S-matrix theory and Regge theory. Recently Mellin transformations have again become vogue as they were shown by Mack to give a natural formulation of CFT correlators. [30,139] This is briefly touched upon in chapter 5. In turn, Mack's results have been extended to show that CFT correlators with an AdS dual [140–144] and superstring amplitudes [145] have a very convenient Mellin representations.

Since we aim here for pedagogy, and since Mellin transformations aren't standard in textbook physics, we review here some properties of Mellin transformations. See [146] for more details and more rigorous exploration. We will see that many functions appearing in scattering amplitudes have very simple Mellin transformations.

The Mellin transformation is defined as

$$\mathcal{M}[f, s] \equiv F[J] = \int_0^\infty f(s) s^{J-1} ds \quad (\text{B.0.1})$$

Where the variables have been suggestively choosen to look like center of mass energy $\sqrt{s}$ and angular momentum J . The above Mellin transformation is defined for real s such that $0 < s < \infty$. The transformed function $F[J]$ is defined for complex $J = a + ib$ in some strip of the complex plane $a_1 < a < a_2$. (Often one takes a_1, a_2 , or both to infinity) However, there are straightforward procedures to continue the Mellin transform out of these regions.

Examples: Γ & β functions String amplitudes are often written in terms of Γ & β functions. These have very simple Mellin transformations. The Γ function

$$F_1[J] = \int_0^\infty e^{-\rho s} s^{J-1} ds = \rho^{-J} \Gamma[J] \quad (\text{B.0.2})$$

for $\rho > 0$ and $0 < a < \infty$ and the β function can be written

$$\begin{aligned} F_2[J] &= \int_0^\infty \frac{1}{1+s} s^{J-1} ds \quad \text{for } x = \frac{s}{s+1} \\ &\rightarrow \int_0^\infty x^{J-1} (1-x)^{-J} dx = \beta(J, 1-J) = \Gamma(J) \Gamma(1-J) = \frac{\pi}{\sin(\pi J)} \end{aligned} \quad (\text{B.0.3})$$

Relation to Other Transforms We can shine a brighter light on the Mellin transform by changing coordinates $s = \exp(-x)$. The integral transform becomes

$$F[J] = \int_{-\infty}^\infty g(x) e^{-Jx} dx \quad (\text{B.0.4})$$

which can be identified as a two-sided Laplace transformation. The Mellin strip of analyticity comes from the fact that the right Laplace transform is defined for $\text{Re}(J) < a_2$ and the left Laplace transform is defined for $\text{Re}(J) > a_1$. If we now write out J in components, $J = a + ib$, we can see the Mellin transformation in terms of a Fourier transformation

$$F[b] = \int_{-\infty}^\infty h(x) e^{-2\pi i b x} dx \quad (\text{B.0.5})$$

Inverse Transformation The amplitudes described in chapter 4 are derived in a J -plane representation. An *inverse* Mellin transformation is needed to look at the amplitude in terms of $\sqrt{s}$. The inverse Mellin transform can be understood by inverting the Fourier transform above and working backwards. Again, letting $s = e^{-x}$ we have

$$h(x) = \int_{-\infty}^\infty \tilde{F}[b] e^{2\pi i b x} dx \quad (\text{B.0.6})$$

$$\Rightarrow f(e^{-x}) e^{-ax} = \int_{-\infty}^\infty \tilde{F}[b] e^{2\pi i b x} dx \quad (\text{B.0.7})$$

$$\Rightarrow f(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F[J] s^{-J} dJ \quad (\text{B.0.8})$$

where now $a \in (a_1, a_2)$ is our strip of analyticity. We would like to continue out of this region. We can try to close the inverse contour by extending a closed contour to positive or negative infinity. Clearly, we should pick up contributions to our inverse transformation from any poles, cuts, or essential singularities. We see that by including an explicit factor of $1/\sin(\pi J)$ we are forced to pick up contributions from all J even as expected from the Pomeron trajectory.

Appendix C

Confluent Hypergeometric Functions

Here we detail some of the important features of the confluent hypergeometric functions used in chapter (4). We will also perform some basic checks and calculations.

The solutions to Whittaker's equation can be written as in sec (4.2.3)

$$f_1(y) = M_{\kappa,\mu}(y) = e^{-y/2} y^{\mu+1/2} M(\mu + \frac{1}{2} - \kappa, 2\mu + 1, y) \quad (\text{C.0.1})$$

$$f_2(y) = W_{\kappa,\mu}(y) = e^{-y/2} y^{\mu+1/2} U(\mu + \frac{1}{2} - \kappa, 2\mu + 1, y) \quad (\text{C.0.2})$$

$$(\text{C.0.3})$$

M and U are solutions to Kummer's equation and, like the solution to many differential equations in physics, can be written in terms of generalized hypergeometric functions. Specifically,

$$M(\mu + \frac{1}{2} - \kappa, 2\mu + 1, y) = {}_1F_1(\mu + \frac{1}{2} - \kappa, 2\mu + 1; y) \quad (\text{C.0.4})$$

Near the IR/UV boundaries of the gravity theory one of the solutions becomes normalizable as seen by

$$y \rightarrow 0 \quad f_1(y) \rightarrow y^{\mu+1/2} \quad (\text{C.0.5})$$

$$y \rightarrow \infty \quad f_2(y) \rightarrow y^\kappa e^{-y/2} \quad (\text{C.0.6})$$

Although the functions are linearly independent (see below) they can be written in terms of each

other, specifically we can write U as a linear combination of M with different arguments. This will simplify many calculations. In terms of this dependence the second solutions becomes

$$f_2(y) = \left(\frac{\pi e^{-y/2} y^{1/2}}{\sin(2\pi\mu)} \right) \left(y^{-\mu} \frac{M(\mu + 1/2 - \kappa, -2\mu + 1, y)}{\Gamma(\mu + 1/2 - \kappa)\Gamma(-2\mu + 1)} - y^\mu \frac{M(\mu + 1/2 - \kappa, 2\mu + 1, y)}{\Gamma(-\mu + 1/2 - \kappa)\Gamma(2\mu + 1)} \right) \quad (\text{C.0.7})$$

We can check linear independence by showing the wronskian, $W(f, g) = f g' - f' g$, does not vanish. At arbitrary y one can show it is non-zero, but by looking near $0, \infty$ we can find it's explicit form. Near $y \sim 0$ we have

$$M(\mu + 1/2 - \kappa, 2\mu + 1, y) \approx \left(1 + \frac{\mu + 1/2 - \kappa}{2\mu + 1} y + \dots \right) \quad (\text{C.0.8})$$

substituting this into the wronskian we find

$$W(f_1, f_2) \approx \frac{\pi}{\sin(2\pi\mu)} \frac{1}{\Gamma(\mu + 1/2 - \kappa)\Gamma(-2\mu + 1)} \quad (\text{C.0.9})$$

$$\begin{aligned} & \times \left(((\mu + 1/2)y^{\mu-1/2} + \dots)(y^{-\mu+1/2} + \dots) - (y^{\mu+1/2} + \dots)((-\mu + 1/2)y^{-\mu-1/2} + \dots) \right) \\ & \approx \frac{\Gamma(2\mu + 1)}{\Gamma(\mu + 1/2 - \kappa)} \end{aligned} \quad (\text{C.0.10})$$

Near $y \rightarrow \infty$ we have

$$M(\mu + 1/2 - \kappa, 2\mu + 1, y) \approx \left(\frac{\Gamma(2\mu + 1)}{\Gamma(\mu + 1/2 - \kappa)} e^y y^{-\mu-1/2-\kappa} \right) (1 + O(1/y)) \quad (\text{C.0.11})$$

Plugging this in we find for the Wronskian

$$W(f_1, f_2) \approx \frac{\Gamma(2\mu + 1)}{\Gamma(\mu + 1/2 - \kappa)} \quad (\text{C.0.12})$$

$$\begin{aligned} & \times \left(((1/2 - \kappa/y)e^{y/2} y^{-\kappa})(e^{-y/2} y^\kappa) - (e^{y/2} y^{-\kappa})(-1/2 + \kappa/y + \dots)e^{-y/2} y^\kappa \right) \\ & \approx \frac{\Gamma(2\mu + 1)}{\Gamma(\mu + 1/2 - \kappa)} \end{aligned} \quad (\text{C.0.13})$$

in agreement with the previous calculation.

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