

Fast Pulses with Oscillatory Tails in the FitzHugh–Nagumo System

by

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A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics at Brown University

PROVIDENCE, RHODE ISLAND

May 2016

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Dedication

To the memory of Anthony Esposito, who nurtured my childhood curiosity in mathematics many years ago.

Acknowledgements

Above all, I want to thank my advisor Björn, whose advice, mentorship, and generosity have had a profoundly positive impact on my experience as a graduate student. Aside from being an inspiration as a researcher and teacher, he has been overwhelmingly supportive of me in all aspects of my own growth as a mathematician, from my research to my professional, academic, and personal development. I am honored to have him as a friend, mentor, and colleague.

I would also like to thank Justin Holmer and Walter Strauss for taking the time to serve on both my defense committee and my topics exam committee. Thanks to Martin Wechselberger and the group at the University of Sydney for hosting me, and for the helpful advice, discussions and feedback on my research, and to Björn de Rijk, for his friendship and the rewarding collaboration included in this thesis.

I would also like to thank the other members of the research group over my years at Brown: Al, Blake, Chao, John, Kelly, Laura, Liz, and Veronica, for the interesting group meetings, fruitful discussions, as well as friendly chats about just about everything, and my friends from the Brown math department: David, Kenny, Max, and Semin, for helping to make my experience navigating graduate school so enjoyable both in and outside of work.

Finally, I want to express special thanks to Julia, Deedee, my parents Mark and Cindy, and to all of my family and friends for their continued love and support.

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CHAPTER ONE

Introduction

1.1 The FitzHugh–Nagumo equation

The FitzHugh–Nagumo equation is a system of differential equations which arose as a simplification of the Hodgkin–Huxley model [28] for the propagation of nerve impulses in axons. The model FitzHugh [19] originally considered is given by

$$\begin{aligned}\frac{du}{dt} &= u(u - a)(1 - u) - w \\ \frac{dw}{dt} &= \delta(u - \gamma w),\end{aligned}\tag{1.1}$$

where $0 < a < \frac{1}{2}$, $0 < \delta \ll 1$, and $\gamma > 0$. Here u represents the electrical potential across the axonal membrane, and w is an aggregate recovery variable. This system was introduced as a means of capturing the essentials of excitability and generation of action potentials in a system more amenable to mathematical analysis than the more realistic, but complex Hodgkin–Huxley equations describing spatially homogeneous excitations in the case of a “space-clamped” axon. See [36] for an introduction to (1.1) and the relation to properties of the Hodgkin–Huxley equations.

Nagumo, Arimoto, and Yoshizawa [43] later introduced another version based on FitzHugh’s model, a reaction-diffusion partial differential equation (PDE) given by

$$\begin{aligned}u_t &= u_{xx} + f(u) - w, \\ w_t &= \delta(u - \gamma w),\end{aligned}\tag{1.2}$$

where $f(u) = u(u - a)(1 - u)$, $0 < a < \frac{1}{2}$, $0 < \delta \ll 1$, and $\gamma > 0$. With the diffusion term added, the system admits propagating solutions, dependent on time t and distance x along the axon.

The system (1.2) has since become a paradigm for singularly perturbed PDEs:

many of its features and solutions have been studied in great detail over the past decades (see [25] for an overview). Nerve impulses correspond to traveling waves that propagate with constant speed without changing their profile, and the FitzHugh-Nagumo system (1.2) indeed supports many different localized traveling waves, or pulses. It is this version of the FitzHugh-Nagumo equations that we consider in this work.

To find traveling waves, we search for solutions of the form $(u, w)(x, t) = (u, w)(x + ct)$ for wavespeed $c > 0$. Finding such solutions to (1.2) is equivalent to finding bounded solutions of the following system of ODEs

$$\begin{aligned}\frac{du}{d\xi} &= v \\ \frac{dv}{d\xi} &= cv - f(u) + w \\ \frac{dw}{d\xi} &= \varepsilon(u - \gamma w)\end{aligned}\tag{1.3}$$

where $\xi = x + ct$ is the traveling wave variable, and $0 < \varepsilon = \delta/c$. We assume $\varepsilon \ll 1$ so that we may view (1.3) as a singular perturbation problem in the parameter ε . In addition, we take $\gamma > 0$ sufficiently small so that $(u, v, w) = (0, 0, 0)$ is the only equilibrium of the system.

It is well known that for each $0 < a < 1/2$ and each sufficiently small $\varepsilon > 0$, (1.2) admits both slow and fast traveling pulse solutions. Equivalently, in (1.3) this corresponds to the existence of orbits homoclinic to the only equilibrium $(u, v, w) = (0, 0, 0)$ with constant wave speeds c . Slow pulses have wave speeds close to zero and arise as regular perturbations from the limit $\varepsilon \rightarrow 0$. Fast pulses, on the other hand, have speeds that are bounded away from zero as $\varepsilon \rightarrow 0$: their profiles do not arise as a regular perturbation from the $\varepsilon = 0$ limit. The existence result for fast pulses has been obtained using a number of different techniques: classical singular perturba-

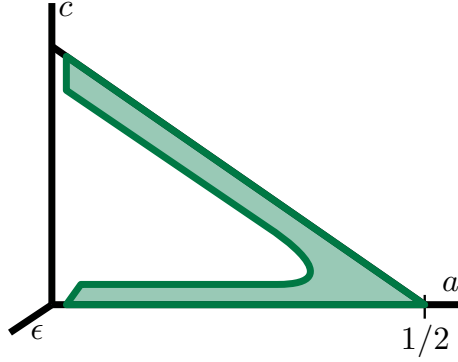


Figure 1.1: Shown is the bifurcation diagram indicating the known regions of existence for pulses in (1.2). Pulses on the upper branch are referred to as “fast” pulses, while those along the lower branch are called “slow” pulses. These two branches coalesce near the point $(c, a, \varepsilon) = (0, 1/2, 0)$.

tion theory [26], Conley index [6], and geometric singular perturbation theory [34]. This last viewpoint is the one we shall adopt. The main idea of geometric singular perturbation theory [18] is to use the small parameter ε to separate the analysis of the system (1.3) into slow and fast components. These components are studied separately and pieced together to construct solutions to the full system. An outline of this construction for fast pulses is revisited in greater detail in §2.

A schematic bifurcation diagram depicting the existence results for pulses is shown in Figure 1.1. The existence region is composed of two branches: the upper branch represents the fast pulses, and the lower branch represents the slow pulses. It has been shown [37] that near the point $(c, a, \varepsilon) = (0, 1/2, 0)$, these two branches coalesce and form a surface as shown.

While the slow pulses are known to be unstable in the PDE (1.2), it was proved independently by Jones [32] and Yanagida [54] that the fast pulses (with monotone tails) are stable for each fixed $0 < a < \frac{1}{2}$ provided $\varepsilon > 0$ is sufficiently small. The idea behind the stability proofs published in [32, 54] is as follows: first, (1.2) is linearized about a fast pulse, and the eigenvalue problem associated with the resulting linear operator is then analysed to see whether it has any eigenvalues with

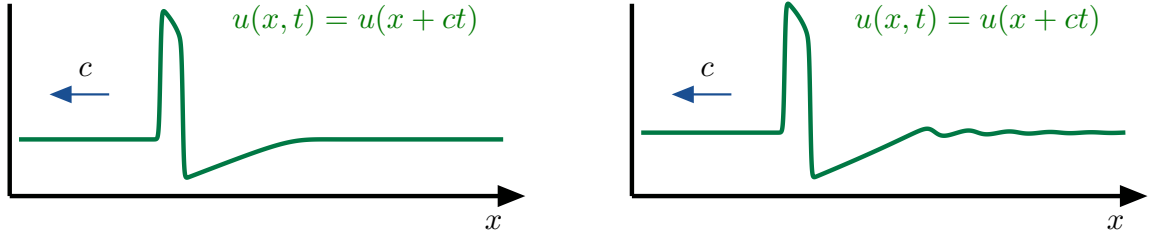


Figure 1.2: Shown are profiles of a fast pulse with monotone tail and fast pulse with oscillatory tail obtained numerically for the parameter values $(c, a, \varepsilon) = (0.593, 0.069, 0.0036)$ and $(c, a, \varepsilon) = (0.689, 0.002, 0.0036)$, respectively.

positive real part. Using an Evans-function analysis, it was shown in [32, 54] that there are at most two eigenvalues near or to the right of the imaginary axis: one of these eigenvalues stays at the origin due to translational invariance of the family of pulses (obtained by shifting the profile in space). The key was then to show that the second critical eigenvalue has a negative sign. In [32, 54], this was established using a parity argument by proving that the derivative of the Evans function at 0 is strictly positive, which, in turn, follows from geometric properties of the pulse profile in the limit $\varepsilon \rightarrow 0$. We mention that these results were extended in [14] to the long-wavelength spatially-periodic wave trains that accompany the fast pulses in the FitzHugh-Nagumo equation.

Both slow and fast pulses as described above have monotone tails as $x \rightarrow \pm\infty$. However, numerical simulations of (1.2) reveal that it also admits fast traveling pulses with small amplitude, exponentially decaying oscillatory tails: this observation is interesting as it opens up the possibility of constructing multi-pulses, which consist of several well-separated copies of the original pulses that are glued together and propagate without changes of speed and profile [30, §5.1.2]. The region in which the oscillatory tails is observed is in the upper left corner of the bifurcation diagram in Figure 1.1, near the point $(c, a, \varepsilon) = (1/\sqrt{2}, 0, 0)$. Figure 1.2 shows profiles a fast monotone and fast oscillatory pulse obtained numerically.

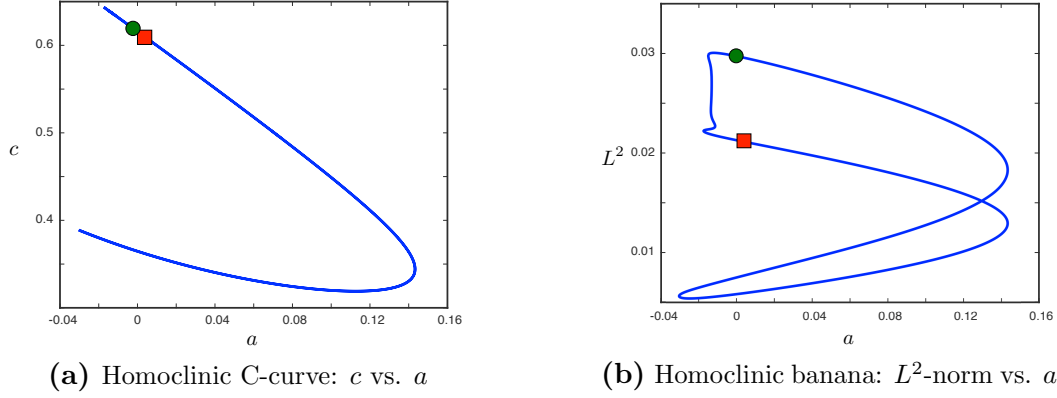
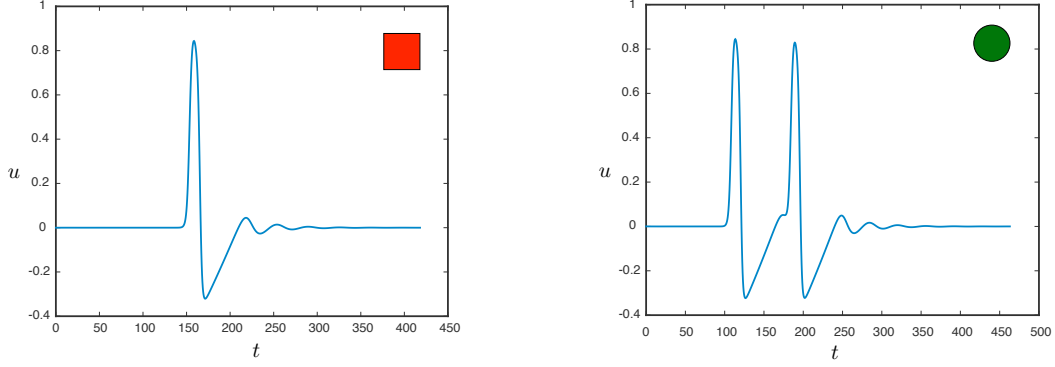


Figure 1.3: Plotted are the homoclinic C-curve and banana obtained by continuing the pulse solution in the parameters (a, c) for $\varepsilon = 0.021$. The red square and green circle refer to the locations of the oscillatory pulse and double pulse of Figure 1.4, respectively.

Further to this, when continuing a traveling pulse numerically in the parameters (c, a) for fixed ε , the continuation traces out a C-shaped (or rather, backwards C-shaped) curve. This is to be expected when considering an $\varepsilon = \text{const}$ slice of the bifurcation diagram in Figure 1.1. When approaching the upper left corner of this bifurcation diagram, the pulses develop oscillations in the tails as described above, but the curve does not terminate; rather the curve turns back sharply, and the oscillations in the tails of the pulses grow into a secondary pulse resembling the primary pulse. The curve then retraces itself, and the secondary pulse transitions back into a single pulse near the lower left corner of the bifurcation diagram. Plotting the parameter a versus the L^2 -norm of the solution shows that this C-curve is indeed composed of two curves forming a so-called homoclinic banana. This homoclinic C-curve and banana are shown in Figure 1.3, and a single and double pulse on either side of this sharp transition are shown in Figure 1.4.

The ultimate goal of this thesis is to explore these phenomena analytically. We first prove extensions of the above existence and stability results for fast pulses which also encompass the onset of oscillations in the tails of the pulses. We then consider the homoclinic banana, and analytically construct a one-parameter family



(a) Shown is a pulse with oscillatory tail for $(c, a, \varepsilon) = (0.005, 0.608, 0.021)$.

(b) Shown is a double pulse for $(c, a, \varepsilon) = (0.001, 0.612, 0.021)$.

Figure 1.4: Plotted are examples of an oscillatory pulse and a double pulse along the homoclinic C-curve. The colored shapes refer to their location along the homoclinic C-curve and banana of Figure 1.3.

of solutions describing the transition from the single to double pulse.

1.2 Outline and overview of results

We begin in §2 with an overview of the classical construction of fast pulses using geometric singular perturbation theory. We also outline how and where this approach breaks down when moving into the regime in which oscillatory tails are expected.

Existence. In §3, we prove the existence of traveling pulses with oscillatory tails. The general strategy behind the proof is similar to that of the classical existence result for fast pulses using geometric singular perturbation theory and the exchange lemma, albeit with a number of additional technical challenges due to the nature of the $(c, a, \varepsilon) \approx (1/\sqrt{2}, 0, 0)$ limit in which normal hyperbolicity is lost at two points on the critical manifold: these challenges will be described more precisely in §2. Related difficulties have also been encountered in other constructions of traveling wave solutions, e.g. in [5, 17], and we will discuss in §2.2 below how these results

differ from ours.

The results of §3 were published as joint work with B. Sandstede in [8].

Stability. In §4, we turn to investigating the stability of the traveling pulses with oscillatory tails, and we prove that the pulses with oscillatory tails are stable. In particular, we will show that, as in the monotone tail case, their stability is again determined by the location of two eigenvalues near the origin, and we will show that the nonzero critical eigenvalue has always negative real part. While the result is the expected one, the stability criterion that ensures negativity of the critical eigenvalue is actually very different from the criterion for monotone pulses. Furthermore, the nonzero eigenvalue scales differently in the monotone and oscillatory regimes: we show that the critical eigenvalue is of order ε for monotone pulses, while there are oscillatory pulses for which the eigenvalue scales with $\varepsilon^{2/3}$ as $\varepsilon \rightarrow 0$.

In contrast to [32, 54], our proof is not based on Evans functions but relies instead on Lin's method [31, 41, 48] to construct potential eigenfunctions of the linearization for each potential eigenvalue λ near and to the right of the imaginary axis. We show that we can construct a piecewise continuous eigenfunction with exactly two jumps for each choice of λ : finding proper eigenvalues then reduces to finding values of λ for which the two jumps vanish. While we restrict ourselves to the FitzHugh-Nagumo system, the approach applies more generally to stability problems of pulses in singularly perturbed reaction-diffusion systems.

We also comment on the presence of the second critical eigenvalue that determines stability. The fast traveling pulses are constructed by gluing pieces of the nullcline $w = u(u - a)(1 - u)$ together with traveling fronts and backs of the FitzHugh-

Nagumo system with $\varepsilon = 0$. These pulses will develop oscillatory tails when $a \approx 0$: this coincides with the region where the traveling fronts and backs jump off from the maxima and minima of the nullcline $w = u(u-a)(1-u)$ (we refer to Figure 2.7 below for an illustration). Depending on exactly how the back jumps off the maximum of the nullcline, the nontrivial second eigenvalue is either present or not: in previous work [5, 29] the stability of similar types of traveling pulses is considered, but the critical eigenvalue is not present and the pulses are therefore automatically stable. We comment in more detail in §6 on the differences between [5, 29] and the present work.

The results of §4 were submitted for publication as joint work with B. de Rijk and B. Sandstede in [7].

Transition. The geometric framework of the existence proof for pulses with oscillatory tails in §3 also provides insight into the mechanism responsible for the continuation of the branch of fast pulses with oscillatory tails via the homoclinic banana described above. In §5 we propose a geometric explanation and give an analytical construction describing the transition of a single fast pulse into a double pulse resembling two copies of the primary pulse.

At the present time, the results of §5 appear only in this thesis.

Finally, in §6, we give a brief discussion of the results and some directions for future work.

CHAPTER TWO

Background

2.1 Previously known existence results for pulses

It is known that for each $0 < a < 1/2$ and each sufficiently small $\varepsilon > 0$, there exists $c > 0$ such that the ODE

$$\begin{aligned}\frac{du}{d\xi} &= v \\ \frac{dv}{d\xi} &= cv - f(u) + w \\ \frac{dw}{d\xi} &= \varepsilon(u - \gamma w)\end{aligned}\tag{1.1}$$

admits an orbit homoclinic to $(u, v, w) = (0, 0, 0)$, the only equilibrium of the full system. In this section, we describe a proof of this result using geometric singular perturbation theory [18] and the exchange lemma [33], in the spirit of [34]. Many of the arguments carry over to the case of oscillatory tails, and we indicate where these arguments fail and more work is needed to establish this extension.

To keep similar notation to the relevant literature for geometric singular perturbation theory results, we abuse notation and denote the independent variable in (1.1) by t and write the system as

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= \varepsilon(u - \gamma w),\end{aligned}\tag{1.2}$$

where $\dot{} = \frac{d}{dt}$. We separately consider (1.2) above, which we call the fast system, and the system below obtained by rescaling time as $\tau = \varepsilon t$, which we call the slow

system:

$$\begin{aligned}\varepsilon u' &= v \\ \varepsilon v' &= cv - f(u) + w \\ w' &= (u - \gamma w),\end{aligned}\tag{1.3}$$

where $'$ denotes $\frac{d}{d\tau}$. The two systems (1.2) and (1.3) are equivalent for any $\varepsilon > 0$. The idea of geometric singular perturbation theory is to determine properties of the $\varepsilon > 0$ system by piecing together information from the simpler equations obtained by separately considering the fast and slow systems in the singular limit $\varepsilon = 0$.

We first set $\varepsilon = 0$ in (1.2), and we obtain the layer problem

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= 0,\end{aligned}\tag{1.4}$$

so that w becomes a parameter for the flow and $\mathcal{M}_0(c, a) = \{(u, v, w) : v = 0, w = f(u)\}$ is a set of equilibria (though the critical manifold does not depend on c , we keep track of this anyways for convenience later). Considering this system in the plane $w = 0$, we obtain the Nagumo system

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= cv - f(u).\end{aligned}\tag{1.5}$$

It can be shown that for each $0 \leq a \leq 1/2$, for $c = c^*(a) = \sqrt{2}(1/2 - a)$, this system possesses a heteroclinic connection φ_f (the Nagumo front) between the critical points $(u, v) = (0, 0)$ and $(u, v) = (1, 0)$. In (1.4), this manifests as a connection between the left and right branches of $\mathcal{M}_0(c, a)$ in the plane $w = 0$. By symmetry, there

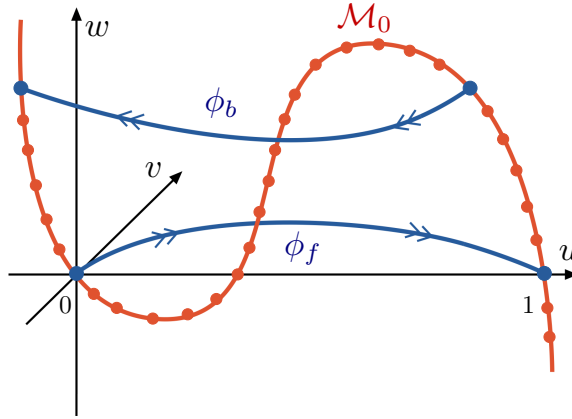


Figure 2.1: Shown is the fast subsystem for $\varepsilon = 0$ and $0 < a < 1/2$.

exists $w^*(a)$ such that there is a connection φ_b (which we call the Nagumo back) in the plane $w = w^*(a)$ between the right and left branches of $\mathcal{M}_0(c, a)$ traveling with the same speed $c = c^*(a)$. The layer problem is shown in Figure 2.1. We will use the notation $\mathcal{M}_0^r(c, a)$ and $\mathcal{M}_0^\ell(c, a)$ to denote the right and left branches of $\mathcal{M}_0(c, a)$, respectively.

Similarly, by setting $\varepsilon = 0$ in (1.3), we obtain the reduced problem

$$\begin{aligned} 0 &= v \\ 0 &= cv - f(u) + w \\ w' &= (u - \gamma w), \end{aligned} \tag{1.6}$$

where the flow is now restricted to the set $\mathcal{M}_0(c, a)$ with flow determined by the equation for w . This is shown in Figure 2.2.

Combining elements of both the fast and slow subsystems, we see that there is a singular $\varepsilon = 0$ “pulse” obtained by following φ_f , then up $\mathcal{M}_0^r(c, a)$, back across φ_b , then down $\mathcal{M}_0^\ell(c, a)$. This exists purely as a formal object as the two subsystems are not equivalent to (1.2) for $\varepsilon = 0$. This singular structure is shown in Figure 2.3.

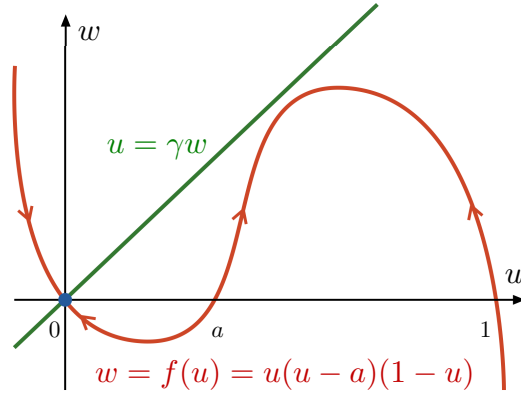


Figure 2.2: Shown is the slow subsystem for $\varepsilon = 0$ and $0 < a < 1/2$.

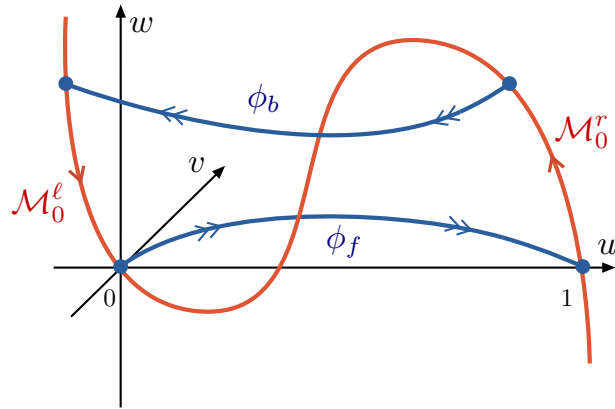


Figure 2.3: Shown is the singular pulse for $\varepsilon = 0$.

We now use Fenichel theory and the exchange lemma to construct a pulse for $\varepsilon > 0$ as a perturbation of this singular structure. The first thing to note is that for any $0 < a < 1/2$ the Nagumo front φ_f and Nagumo back φ_b leave and arrive at points on segments of $\mathcal{M}_0^r(c, a)$ and $\mathcal{M}_0^\ell(c, a)$ which are normally hyperbolic. Therefore such segments persist for $\varepsilon > 0$ as locally invariant manifolds $\mathcal{M}_\varepsilon^r(c, a)$ and $\mathcal{M}_\varepsilon^\ell(c, a)$. Also, the stable manifold $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$, consisting of the union of the stable fibers of the equilibria lying on $\mathcal{M}_0^\ell(c, a)$, also persists for $\varepsilon > 0$ as a two-dimensional manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. By Fenichel fibering, we in fact have that $\mathcal{W}_\varepsilon^{s,\ell}(c, a) = \mathcal{W}_\varepsilon^s(0; c, a)$, the stable manifold of the origin.

In addition, the origin has a one-dimensional unstable manifold $\mathcal{W}_0^u(0; c, a)$ which persists for $\varepsilon > 0$ as $\mathcal{W}_\varepsilon^u(0; c, a)$. The idea is to track $\mathcal{W}_\varepsilon^u(0; c, a)$ forwards and track $\mathcal{W}_\varepsilon^s(0; c, a)$ backwards and show that there is an intersection provided we adjust $c \approx c^*(a)$ appropriately. The difficulty in this procedure comes from trying to track these manifolds in a neighborhood of the right branch $\mathcal{M}_\varepsilon^r(c, a)$, where the flow spends time of order ε^{-1} . The exchange lemma is used to describe the flow in this region.

Since we are only concerned with a normally hyperbolic segment of $\mathcal{M}_0^r(c, a)$, as stated before it perturbs to a manifold $\mathcal{M}_\varepsilon^r(c, a)$. In addition its stable and unstable manifolds, $\mathcal{W}^s(\mathcal{M}_0^r(c, a))$ and $\mathcal{W}^u(\mathcal{M}_0^r(c, a))$ also perturb to locally invariant manifolds $\mathcal{W}_\varepsilon^{s,r}(c, a)$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$. Also, in a neighborhood of $\mathcal{M}_\varepsilon^r(c, a)$, there exists a smooth change of coordinates in which the flow takes a very simple form, the Fenichel

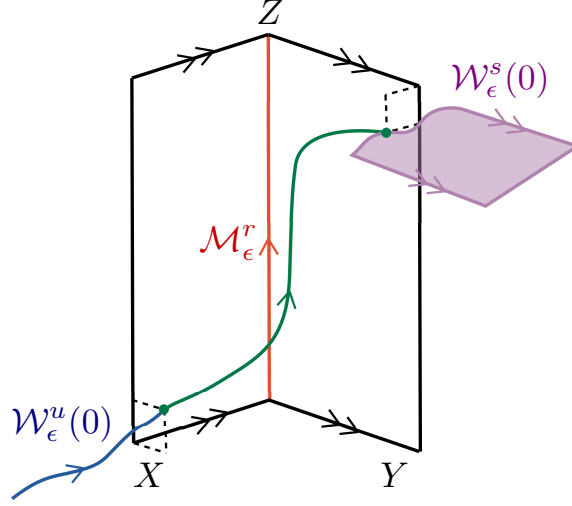


Figure 2.4: Shown is the setup for the exchange lemma.

normal form [18, 33]:

$$\begin{aligned}
 X' &= -A(X, Y, Z, c, a, \varepsilon)X \\
 Y' &= B(X, Y, Z, c, a, \varepsilon)Y \\
 Z' &= \varepsilon(1 + E(X, Y, Z, c, a, \varepsilon)XY),
 \end{aligned} \tag{1.7}$$

where $\mathcal{M}_\varepsilon^r(c, a)$ is given by $X = Y = 0$, and $\mathcal{W}_\varepsilon^{u,r}(c, a)$ and $\mathcal{W}_\varepsilon^{s,r}(c, a)$ are given by $X = 0$ and $Y = 0$, respectively, and the functions A and B are bounded below by some constant $\eta > 0$. The exchange lemma [33] then states that for sufficiently small $\Delta > 0$ and $\varepsilon > 0$, any sufficiently large T , and any Z_0 , there exists a solution to (1.7) satisfying $X(0) = \Delta$, $Z(0) = Z_0$, and $Y(T) = \Delta$ and the norms $|X(T)|$, $|Y(0)|$, and $|Z(T) - Z_0 - \varepsilon T|$ are of order $e^{-\eta T}$. The setup is shown in Figure 2.4.

The idea is now to follow $\mathcal{W}_\varepsilon^u(0; c, a)$ and $\mathcal{W}_\varepsilon^s(0; c, a)$ up to this neighborhood of $\mathcal{M}_\varepsilon^r(c, a)$ and determine how they behave at $X = \Delta$ and $Y = \Delta$. This gives a system of equations in c, T, ε which we can now solve to connect $\mathcal{W}_\varepsilon^u(0; c, a)$ and $\mathcal{W}_\varepsilon^s(0; c, a)$ using the solution given by the exchange lemma, completing the construction of the pulse which is shown in Figure 2.5.

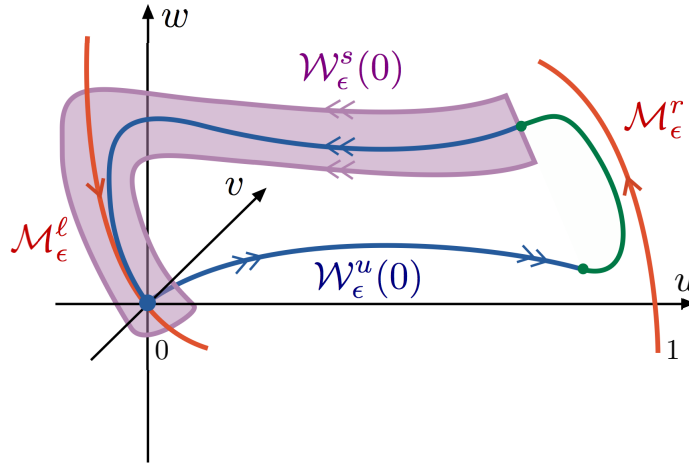


Figure 2.5: Shown is the construction of the pulse solution.

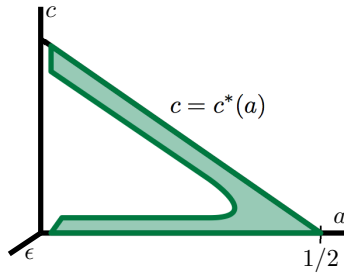


Figure 2.6: Shown is the bifurcation diagram indicating the known regions of existence for pulses in (1.2). Pulses on the upper branch are referred to as “fast” pulses, while those along the lower branch are called “slow” pulses. These two branches coalesce near the point $(c, a, \varepsilon) = (0, 1/2, 0)$.

The existence results for pulses in the FitzHugh–Nagumo system are collected in the bifurcation diagram in Figure 2.6 where the green surface denotes the existence region for pulses. The pulses constructed above for $c \approx c^*(a) > 0$ are called “fast” pulses and the region of existence is given by the upper branch. For each $0 < a < 1/2$, there are also “slow” pulses which bifurcate for small $c, \varepsilon > 0$, and the region of existence of such pulses is given by the lower branch. It is also known [37] that near the point $(c, a, \varepsilon) = (0, 1/2, 0)$, these two branches coalesce and form a surface as shown.

2.2 Motivation and complications for $a \approx 0$

Numerical evidence suggests that when one of the fast pulses constructed above is continued in (c, a) for fixed ε , the tail of the pulse becomes oscillatory as $a \rightarrow 0$, i.e. as one moves towards the upper left corner of the bifurcation diagram of Figure 2.6. Pulses with oscillatory tails correspond to homoclinic orbits of the travelling wave ODE (1.2) for which the origin is a saddle-focus with one strongly unstable eigenvalue and two weakly stable complex conjugate (non-real) eigenvalues: such homoclinic orbits are often referred to as Shilnikov saddle-focus homoclinic orbits. The numerical observation that pulses with oscillatory tails exist is of interest, because such pulses are typically accompanied by infinitely many distinct N -pulses for each given $N \geq 2$ [30, §5.1.2]: here, an N -pulse is a travelling pulse that resembles N well separated copies of the original pulse.

The goal of this current work is to prove the existence of pulses with oscillatory tails analytically by studying the branch of fast pulses in the regime near the singular point $(c, a, \varepsilon) = (1/\sqrt{2}, 0, 0)$ in the bifurcation diagram. We will accomplish this by looking for pulses which arise as perturbations from the singular $\varepsilon = 0$ structure for the case of $(c, a) = (1/\sqrt{2}, 0)$, which is shown in Figure 2.7. We note that the existence of pulses with oscillatory tails has been shown [27] previously for the FitzHugh–Nagumo system, but the manner of proof does not allow for the construction of multipulses due to the difficulty in obtaining a transversality condition with respect to the wave speed c . Our existence proof guarantees this transversality and also provides sufficient information to determine the stability of the pulses (see §4).

Proceeding as in the case of fast waves, we wish to find an intersection between the stable and unstable manifolds of the origin. Let $I_a = [-a_0, a_0]$ for some small

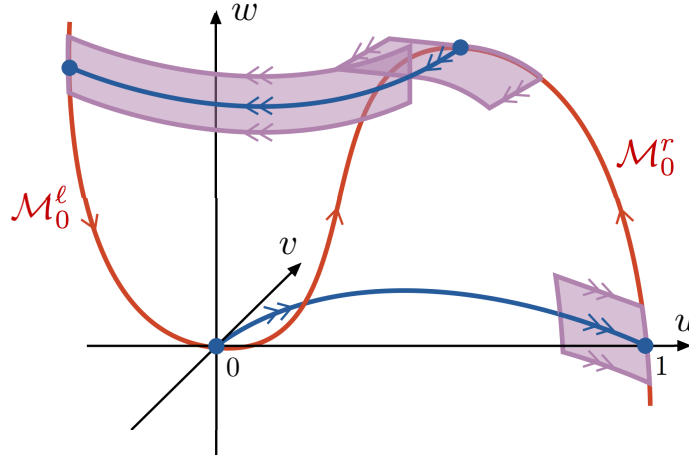


Figure 2.7: Shown is the singular pulse for $\varepsilon = 0$ in the case of $(c, a) = (1/\sqrt{2}, 0)$.

$a_0 > 0$. In the plane $w = 0$, the fast system for $a \in I_a$ reduces to

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - u(u - a)(1 - u). \end{aligned} \tag{2.1}$$

As stated previously, for $a > 0$ this system possesses Nagumo front type solutions connecting $u = 0$ to $u = 1$ for any $c = c^*(a)$. For $-a_0 < a < 0$ with a_0 sufficiently small, this system possesses front type solutions for any $c > 1/\sqrt{2}(1 + a)$ connecting $u = 0$ to $u = 1$. For the critical value $c = c^*(a) = \sqrt{2}(1/2 - a)$ the front leaves the origin along the strong unstable manifold of the origin, and for all other values of c , the front leaves the origin along a weak unstable direction. Our primary concern is the case of $a = 0$, in which (2.1) reduces to a Fisher-KPP type equation

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - u^2(1 - u). \end{aligned} \tag{2.2}$$

Again, it is known that this system possesses front type solutions connecting $u = 0$ to $u = 1$ for any $c \geq 1/\sqrt{2}$. For the critical value $c = 1/\sqrt{2}$ the front leaves the origin along the strong unstable manifold of the origin, and for $c > 1/\sqrt{2}$, the front leaves the origin along a center manifold. We are concerned with the case of

$(c, a) = (1/\sqrt{2}, 0)$ in which, as is the case with the Nagumo front, the singular fast front solution leaves the origin along the strong unstable manifold; here the solution is given explicitly by

$$\begin{aligned} u_f(t) &= \frac{1}{2} \left(\tanh \left(\frac{1}{2\sqrt{2}} t \right) + 1 \right) \\ v_f(t) &= \frac{1}{\sqrt{2}} u_f(t) (1 - u_f(t)). \end{aligned} \tag{2.3}$$

Note that by symmetry, for $(c, a) = (1/\sqrt{2}, 0)$, the fast singular back solution also leaves the upper right fold point along the strong unstable direction.

Thus from Fenichel theory, the origin has a strong unstable manifold $\mathcal{W}_0^u(0; c, a)$ for $c \in I_c$, $a \in I_a$, and $\varepsilon = 0$ which persists as an invariant manifold $\mathcal{W}_\varepsilon^u(0; c, a)$ for a, c in the same range and $\varepsilon \in [0, \varepsilon_0]$, some ε_0 . Here I_c is a fixed closed interval which contains the set $\{c^*(a) : a \in I_a\}$ in its interior. Recall $c^*(a)$ is the wavespeed for which the front solution in the strong unstable manifold exists for this choice of a , and $c^*(0) = 1/\sqrt{2}$. We note that for $-a_0 < a < 0$ with a_0 sufficiently small, though the origin sits on the unstable middle branch of the critical manifold, it still has a well defined strong unstable manifold.

Taking any piece of $\mathcal{M}_0^r(c, a)$ which is normally hyperbolic, i.e. away from the fold point, Fenichel theory again ensures that this persists a locally invariant manifold $\mathcal{M}_\varepsilon^r(c, a)$ for $\varepsilon \in (0, \varepsilon_0]$. Similarly outside of a small fixed neighborhood of the fold, $\mathcal{M}_0^r(c, a)$ has stable and unstable manifolds $\mathcal{W}^s(\mathcal{M}_0^r(c, a))$ and $\mathcal{W}^u(\mathcal{M}_0^r(c, a))$ which persist as locally invariant manifolds $\mathcal{W}_\varepsilon^{s,r}(c, a)$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$.

We follow $\mathcal{W}_\varepsilon^u(0; c, a)$ along the front into a neighborhood of the right branch $\mathcal{M}_\varepsilon^r(c, a)$, and using the exchange lemma, we can follow $\mathcal{W}_\varepsilon^u(0; c, a)$ along $\mathcal{M}_\varepsilon^r(c, a)$, but only up to a fixed neighborhood of the fold point. Here the exchange lemma

breaks down.

Another issue is that the origin does not have a well defined stable manifold as in the case of $0 < a < 1/2$. For $a = 0$, the origin sits on the fold of the critical manifold $\mathcal{M}_0(c, a)$ and thus does not lie in the region where the branch $\mathcal{M}_0^\ell(c, a)$ is normally hyperbolic. Therefore, we cannot use the results of Fenichel as before to deduce that any section of $\mathcal{M}_0^\ell(c, a)$ containing the origin persists as an invariant manifold for $\varepsilon > 0$. In the same vein, we cannot deduce that $\mathcal{W}_\varepsilon^{s,\ell}(c, a) = \mathcal{W}_\varepsilon^s(0; c, a)$.

However, outside any small fixed neighborhood of the origin, Fenichel theory applies, and we know that $\mathcal{M}_0^\ell(c, a)$ and its stable manifold $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ perturb to invariant manifolds $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ which enter this small fixed neighborhood of the origin. In addition, the origin remains an equilibrium for $\varepsilon > 0$, so it remains to find conditions which ensure that $\mathcal{M}_\varepsilon^\ell(c, a)$ and nearby trajectories on $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ in fact converge to zero. This is discussed in §3.5. It is important to note in this case that the manifolds $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ are not unique and are only defined up to errors exponentially small in $1/\varepsilon$. The forthcoming analysis is valid for any such choice of these manifolds and, in §3.5, we show that under certain conditions it is possible to choose $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ so that they in fact lie on $\mathcal{W}_\varepsilon^s(0; c, a)$.

We now follow the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ backwards along the back up to a small neighborhood of the fold point, where again the theory breaks down. Thus we may be able to find a connection between $\mathcal{W}_\varepsilon^u(0; c, a)$ and $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ up to understanding the flow near the fold point. The flow in this region and the interaction with the exchange lemma is discussed in §3.3 and §3.4.

Figure 2.8 summarizes what is given by the usual Fenichel theory arguments, which apply outside of small neighborhoods of the two fold points at which the

critical manifold is not normally hyperbolic.

We note that there have been other studies of constructing singular solutions passing near non-hyperbolic fold points. In [5], for instance, a pulse solution was constructed in a model of cardiac tissue: in this model, the fast ‘back’ portion of the pulse also originated from a non-hyperbolic fold point as in the case for FitzHugh–Nagumo above. Both models exhibit a Fisher–KPP type equation as described above when viewing the layer problem in the plane containing the singular fast ‘back’ solution. One difference between these two cases is that, in [5], only wavespeeds $c > 1/\sqrt{2}$ are considered, which means that the back solution leaves the fold point along the center manifold: in particular, the desired pulse solution can be constructed by following a continuation of the slow manifold in the center manifold of the fold point. A second difference is that the origin of the model considered in [5] is hyperbolic, instead of being a second fold point as in the situation discussed in this work. The setup discussed in [17] is similar to the one studied in [5] in that a condition is imposed on the wavespeed that ensures that the singular back solution leaves the fold along a center manifold rather than a strong unstable fiber.

In our case, we consider the critical wavespeed $c = 1/\sqrt{2}$ in which the back leaves along a strong unstable fiber. As in [5], we will use the blow up techniques of [38] to construct the desired pulse solution. However, a number of refinements of the results of [38] are needed to track the solution in a neighborhood of the fold point as the solution exits this neighborhood along a strong unstable fiber as opposed to remaining on the center manifold. This will be described in more detail in §3.3 and §3.4.

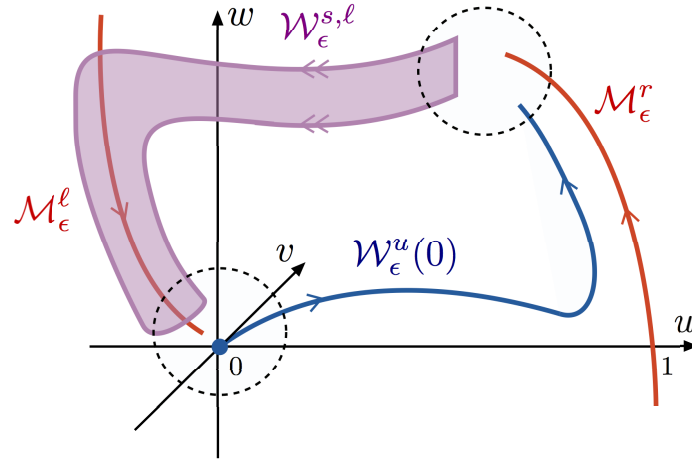


Figure 2.8: Shown are the regions of difficulty for the case of $a \approx 0$.

CHAPTER THREE

Existence of fast pulses with oscillatory tails

3.1 Introduction

In this chapter, we prove the existence of traveling pulse solutions to the PDE

$$\begin{aligned} u_t &= u_{xx} + f(u) - w, \\ w_t &= \delta(u - \gamma w), \end{aligned} \tag{1.1}$$

which exhibit oscillatory tails. Equivalently we search for homoclinic solutions of

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= \varepsilon(u - \gamma w), \end{aligned} \tag{1.2}$$

with wave speed $c > 0$. Recall that we take $f(u) = u(u - a)(1 - u)$, $0 < a < \frac{1}{2}$, $0 < \varepsilon = \delta/c \ll 1$, and $\gamma > 0$ sufficiently small so that $(u, v, w) = (0, 0, 0)$ is the only equilibrium of (1.2).

The main result of this chapter, Theorem 3.1, guarantees the existence of a surface of solutions near $(c, a, \varepsilon) = (1/\sqrt{2}, 0, 0)$ containing pulses with both monotone and oscillatory tails. The chapter is structured as follows. In §3.2, we state Theorem 3.1 and briefly outline its proof and the relation to oscillations in the tails of the pulses. The remainder of the chapter (§3.3-3.5) is then devoted to the proof of Theorem 3.1.

3.2 Statement of main result

We start by collecting a few results which follow from Fenichel theory. Define the closed intervals $I_a = [-a_0, a_0]$ for some small $a_0 > 0$ and $I_c = \{c^*(a) : a \in I_a\}$; recall

$c^*(a)$ is the wavespeed for which the Nagumo front exists for this choice of a . Then for sufficiently small ε_0 , standard geometric singular perturbation theory gives the following:

- (i) The origin has a strong unstable manifold $\mathcal{W}_0^u(0; c, a)$ for $c \in I_c$, $a \in I_a$, and $\varepsilon = 0$ which persists for a, c in the same range and $\varepsilon \in [0, \varepsilon_0]$.
- (ii) We consider the critical manifold defined by $\{(u, v, w) : v = 0, w = f(u)\}$. For each $a \in I_a$, we consider the right branch of the critical manifold $\mathcal{M}_0^r(c, a)$ up to a neighborhood of the knee for $\varepsilon = 0$. This manifold persists as a slow manifold $\mathcal{M}_\varepsilon^r(c, a)$ for $\varepsilon \in [0, \varepsilon_0]$. In addition, $\mathcal{M}_0^r(c, a)$ possesses stable and unstable manifolds $\mathcal{W}^s(\mathcal{M}_0^r(c, a))$ and $\mathcal{W}^u(\mathcal{M}_0^r(c, a))$ which also persist for $\varepsilon \in [0, \varepsilon_0]$ as invariant manifolds which we denote by $\mathcal{W}_\varepsilon^{s,r}(c, a)$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$.
- (iii) In addition, we consider the left branch of the critical manifold $\mathcal{M}_0^\ell(c, a)$ up to a neighborhood of the origin for $\varepsilon = 0$. This manifold persists as a slow manifold $\mathcal{M}_\varepsilon^\ell(c, a)$ for $\varepsilon \in [0, \varepsilon_0]$. In addition, $\mathcal{M}_0^\ell(c, a)$ possesses a stable manifold $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ which also persists for $\varepsilon \in [0, \varepsilon_0]$ as an invariant manifold which we denote by $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$.

The goal of this chapter is to prove the following theorem.

Theorem 3.1. *There exists $K^*, \mu > 0$ such that the following holds. For each $K > K^*$, there exists $a_0, \varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$, there exists $c = c(a, \varepsilon)$ given by*

$$c(a, \varepsilon) = \sqrt{2} \left(\frac{1}{2} - a \right) - \mu\varepsilon + \mathcal{O}(\varepsilon(|a| + \varepsilon)),$$

*such that (1.1) admits a traveling pulse solution. Furthermore, for $\varepsilon > K^*a^2$, the tail of the pulse is oscillatory.*

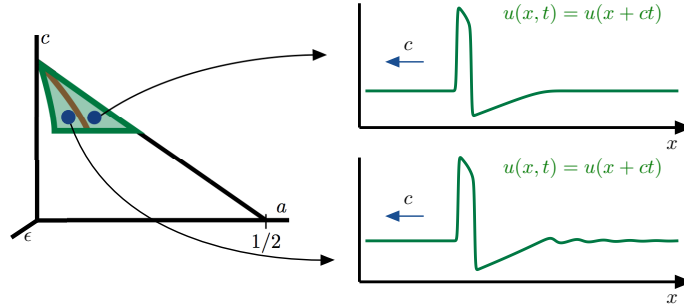


Figure 3.1: Shown is a schematic bifurcation diagram depicting the branch of pulses guaranteed by Theorem 3.1. The monotone pulse and oscillatory pulse shown were computed numerically for the parameter values $(c, a, \varepsilon) = (0.593, 0.069, 0.0036)$ and $(c, a, \varepsilon) = (0.689, 0.002, 0.0036)$, respectively.

We note here that this result extends the classical existence result by guaranteeing, at least near the point $(c, a, \varepsilon) \approx (1/\sqrt{2}, 0, 0)$, a surface of solutions which contains both pulses with monotone tails and pulses with oscillatory tails (see Figure 3.1).

In §3.4.5, the wave speed of the pulse is computed as

$$c(a, \varepsilon) = c^*(a) - \mu\varepsilon + \mathcal{O}(\varepsilon(|a| + \varepsilon)), \quad (2.1)$$

where $\mu > 0$. Figure 3.2 shows another schematic view of the surface of solutions guaranteed by the theorem in the bifurcation diagram for the parameters (c, a, ε) . We emphasize that this theorem does indeed guarantee the existence of the desired branch of pulses with oscillatory tails. The onset of the oscillations in the tail of the pulse is due to a transition occurring in the linearization of (1.2) about the origin in which the two stable real eigenvalues collide and emerge as a complex conjugate pair as a decreases for fixed ε . If a pulse/homoclinic orbit is present when eigenvalues changes in this fashion, then this situation is referred to as a Belyakov transition [30, §5.1.4]: all N -pulses that accompany a Shilnikov homoclinic orbit terminate near

the Belyakov transition point. The linearization of (1.2) about the origin is given by

$$J = \begin{pmatrix} 0 & 1 & 0 \\ a & c & 1 \\ \varepsilon & 0 & -\varepsilon\gamma \end{pmatrix}. \quad (2.2)$$

We can compute the location of the Belyakov transition for small (a, ε) by finding real eigenvalues which are double roots of the characteristic polynomial of J . Thus, we determine for which (ε, a) both the characteristic polynomial and its derivative vanish, and find that this holds when

$$\varepsilon = \frac{a^2}{4c} + \mathcal{O}(a^3). \quad (2.3)$$

This gives the location of the transition and allows us to choose the quantity $K^* > \frac{1}{4c^*(0)}$ for which the statement in Theorem 3.1 holds for all sufficiently small (a, ε) . Then by taking K sufficiently large in Theorem 3.1, we see that the surface of pulses in $ca\varepsilon$ -space which are given by the theorem encompasses both sides of this Belyakov transition and therefore captures both the monotone and oscillatory tails (see Figure 3.2).

The proof of Theorem 3.1 is presented in three parts:

- (i) In §3.3, we present an analysis of the flow in a small neighborhood of the upper right fold point.
- (ii) In §3.4, using the exchange lemma together with the analysis of §3.3, we show that for each $a \in I_a$ and $\varepsilon \in (0, \varepsilon_0)$, there exists $c = c(a, \varepsilon)$ such that $\mathcal{W}_\varepsilon^u(0; c, a)$ connects to $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ after passing near the upper right fold point.

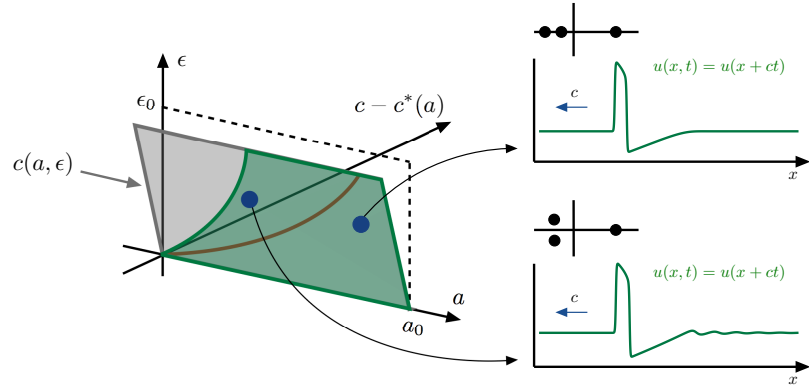


Figure 3.2: Schematic bifurcation diagram for the parameters (c, a, ε) . Here the green surface is the region of existence of pulses as in Theorem 3.1; in the grey region, Theorem 3.1 does not apply. The red curve denotes the location of the Belyakov transition which occurs at the origin.

- (iii) In §3.5, we show that for each (a, ε) satisfying the relation in the statement of Theorem 3.1, the manifold $\mathcal{M}_\varepsilon^\ell(c, a)$ and nearby solutions on $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ in fact converge to the equilibrium, completing the construction of the pulse.

3.3 Tracking around the fold

3.3.1 Preparation of equations

We append an equation for the parameter ε to (1.2) and arrive at the system

$$\begin{aligned}
 \dot{u} &= v \\
 \dot{v} &= cv - f(u) + w \\
 \dot{w} &= \varepsilon(u - \gamma w) \\
 \dot{\varepsilon} &= 0.
 \end{aligned} \tag{3.1}$$

For $(c, a) \in I_c \times I_a$, the fold point is given by the fixed point $(u, v, w, \varepsilon) = (u^*, 0, w^*, 0)$ of (3.1) where

$$u^* = \frac{1}{3} \left(a + 1 + \sqrt{a^2 - a + 1} \right), \quad (3.2)$$

and $w^* = f(u^*)$. The linearization of (3.1) about this point is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & u^* - \gamma w^* \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.3)$$

This matrix has one positive eigenvalue $\lambda = c$ with eigenvector $(1, c, 0, 0)$ as well as an eigenvalue $\lambda = 0$ with algebraic multiplicity three and geometric multiplicity one. The associated eigenvector is $(1, 0, 0, 0)$ and generalized eigenvectors are $(0, 1, -c, 0)$ and $(0, 0, u^* - \gamma w^*, -c)$. By making the coordinate transformation

$$\begin{aligned} z_1 &= u - u^* - \frac{v}{c} - \frac{w - w^*}{c^2} \\ z_2 &= -\frac{w - w^*}{c} \\ z_3 &= \frac{v}{c} + \frac{w - w^*}{c^2}, \end{aligned} \quad (3.4)$$

we arrive at the system

$$\begin{aligned}
\dot{z}_1 &= z_2 + \frac{-1}{c} \left(\sqrt{a^2 - a + 1} \right) (z_1 + z_3)^2 - \frac{1}{c} (z_1 + z_3)^3 \\
&\quad - \frac{\varepsilon}{c^2} (z_1 + z_3 + c\gamma z_2 + u^* - \gamma w^*) \\
\dot{z}_2 &= -\frac{\varepsilon}{c} (z_1 + z_3 + c\gamma z_2 + u^* - \gamma w^*) \\
\dot{z}_3 &= cz_3 + \frac{1}{c} \left(\sqrt{a^2 - a + 1} \right) (z_1 + z_3)^2 + \frac{1}{c} (z_1 + z_3)^3 \\
&\quad + \frac{\varepsilon}{c^2} (z_1 + z_3 + c\gamma z_2 + u^* - \gamma w^*) \\
\dot{\varepsilon} &= 0,
\end{aligned} \tag{3.5}$$

which, for $\varepsilon = 0$, is in Jordan normal form for the three dynamic variables (z_1, z_2, z_3) . To understand the dynamics near the fold point, we separate the nonhyperbolic dynamics which occur on a three-dimensional center manifold. In a small neighborhood of the fold point, this manifold can be represented as a graph

$$\begin{aligned}
z_3 &= F(z_1, z_2, \varepsilon) \\
&= \beta_0 z_1 + \beta_1 z_2 + \beta_2 z_1^2 + \mathcal{O}[\varepsilon, z_1 z_2, z_2^2, z_1^3].
\end{aligned} \tag{3.6}$$

We can directly compute the coefficients β_i , and we find that

$$\beta_0 = \beta_1 = 0, \quad \beta_2 = \frac{-1}{c^2} \left(\sqrt{a^2 - a + 1} \right). \tag{3.7}$$

We now make the following change of coordinates

$$\begin{aligned}
x &= \frac{-1}{c} \left(\sqrt{a^2 - a + 1} \right) z_1 \\
y &= \frac{-1}{c} \left(\sqrt{a^2 - a + 1} \right) z_2,
\end{aligned} \tag{3.8}$$

which gives the flow on the center manifold in the coordinates (x, y, ε) as

$$\begin{aligned} \dot{x} &= y + x^2 + \mathcal{O}(\varepsilon, xy, y^2, x^3) \\ \dot{y} &= \varepsilon \left[\frac{1}{c^2} \left(\sqrt{a^2 - a + 1} \right) (u^* - \gamma w^*) + \mathcal{O}(x, y, \varepsilon) \right] \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{3.9}$$

Making one further coordinate transformation in the variable z_3 to straighten out the unstable fibers and one further rescaling of (x, y, t) to rectify the flow in the y -direction, we arrive at the full system

$$\begin{aligned} \dot{x} &= y + x^2 + \mathcal{O}(\varepsilon, xy, y^2, x^3) \\ \dot{y} &= \varepsilon (1 + \mathcal{O}(x, y, \varepsilon)) \\ \dot{z} &= z (\theta_0 + \mathcal{O}(x, y, z, \varepsilon)) \\ \dot{\varepsilon} &= 0, \end{aligned} \tag{3.10}$$

where $\theta_0 > 0$ uniformly in $(c, a) \in I_c \times I_a$.

Let $V_f \subset \mathbb{R}^3$ be a small fixed neighborhood of $(x, y, z) = (0, 0, 0)$ where the above computations are valid. Define the neighborhood U_f by

$$U_f = \{(x, y, z, c, a) \in V_f \times I_c \times I_a\}, \tag{3.11}$$

and denote the change of coordinates from (x, y, z, c, a) to the original (u, v, w, c, a) coordinates by $\Phi_f : U_f \rightarrow O_f$ where O_f is the corresponding neighborhood of the fold in (u, v, w) -coordinates for $(c, a) \in I_c \times I_a$. We note that in the neighborhood U_f the equations for the variables (x, y) are in the canonical form for a fold point as

in [38], that is, we have

$$\begin{aligned}
\dot{x} &= y + x^2 + h(x, y, \varepsilon, c, a) \\
\dot{y} &= \varepsilon g(x, y, \varepsilon, c, a) \\
\dot{z} &= z (\theta_0 + \mathcal{O}(x, y, z, \varepsilon)) \\
\dot{\varepsilon} &= 0,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
h(x, y, \varepsilon, c, a) &= \mathcal{O}(\varepsilon, xy, y^2, x^3) \\
g(x, y, \varepsilon, c, a) &= 1 + \mathcal{O}(x, y, \varepsilon).
\end{aligned} \tag{3.13}$$

We assume that the neighborhood V_f has been chosen small enough so that the function $g(x, y, \varepsilon, c, a)$ is bounded away from zero, say $g_m < g(x, y, \varepsilon, c, a) < g_M$ with $g_m > 0$. We have thus factored out the one hyperbolic direction (given by z) and the flow consists of the flow on a three-dimensional center manifold, parametrized by (x, y, ε) and the one-dimensional flow along the fast unstable fibers (the z -direction).

3.3.2 Tracking solutions around the fold point: existing theory

Here we describe the existing theory for extending geometric singular perturbation theory to a fold point. Consider the two-dimensional system

$$\begin{aligned}
\dot{x} &= y + x^2 + h(x, y, \varepsilon, c, a) \\
\dot{y} &= \varepsilon g(x, y, \varepsilon, c, a),
\end{aligned} \tag{3.14}$$

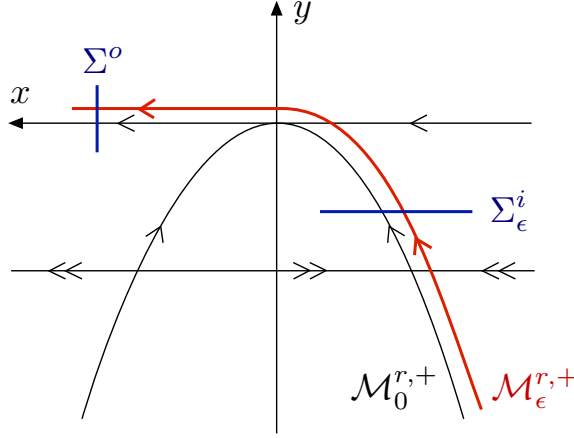


Figure 3.3: Shown is the setup of (3.14) for the passage near the fold point as in [38]: note that the positive x -axis points to the left, so that the attracting branch $S_0^+(c, a)$ corresponding to $x < 0$ is on the right.

with parameters (ε, c, a) . We collect a few relevant results from [38]. For $\varepsilon = 0$, this system possesses a critical manifold given by $\{(x, y) : y + x^2 + h(x, y, 0, c, a) = 0\}$, which in a sufficiently small neighborhood of the origin is shaped as a parabola opening downwards. The branch of this parabola corresponding to $x < 0$, which we denote by $S_0^+(c, a)$, is attracting and normally hyperbolic away from the fold point. Thus by Fenichel theory, this critical manifold persists as an attracting slow manifold $S_\varepsilon^+(c, a)$ for sufficiently small $\varepsilon > 0$ and consists of a single solution. This slow manifold is unique up to exponentially small errors. In [38], this slow manifold is tracked around the knee where normal hyperbolicity is lost. The set up is shown in Figure 3.3; note that the orientation is chosen so that the positive x -axis points to the left.

For sufficiently small $\rho > 0$ (to be chosen) and an appropriate interval J , define the following sections $\Delta^{in}(\rho) = \{(x, -\rho^2) : x \in J\}$ and $\Delta^{out}(\rho) = \{(\rho, y) : y \in \mathbb{R}\}$. Then we have the following

Theorem 3.2 ([38, Theorem 2.1]). *For each sufficiently small $\rho > 0$, there exists $\varepsilon_0 > 0$ such that for each $(c, a) \in I_c \times I_a$ and $\varepsilon \in (0, \varepsilon_0)$, the manifold $S_\varepsilon^+(c, a)$ passes through $\Delta^{out}(\rho)$ at a point $(\rho, \tilde{y}_\varepsilon(c, a))$ where $\tilde{y}_\varepsilon(c, a) = \mathcal{O}(\varepsilon^{2/3})$.*

This theorem describes how the slow manifold exits a neighborhood of the fold point but not the nature of the passage near the fold point. Since the solution we are trying to construct will leave the neighborhood U_f along a strong unstable fiber before reaching Δ^{out} , we need to extend the results of [38] to derive estimates which hold throughout this neighborhood, not just at the entry/exit sections.

3.3.3 Tracking solutions in a neighborhood of the fold point

For our purposes, we actually need to be able to say a bit more about the nature of $S_\varepsilon^+(c, a)$ as well as nearby solutions between the two sections $\Delta^{in}(\rho)$ and $\Delta^{out}(\rho)$. We can think of the slow manifold $S_\varepsilon^+(c, a)$ as being a one-dimensional slice of a two-dimensional critical manifold $M^+(c, a) = \cup_{\varepsilon < \varepsilon_0} S_\varepsilon^+(c, a)$ of the three-dimensional (x, y, ε) subsystem of (3.12). It will sometimes be useful to consider the manifold $M^+(c, a)$ instead as we utilize a number of different coordinate systems in the analysis below.

Let $\tilde{x}_\varepsilon(c, a)$ denote the x -value at which the manifold $S_\varepsilon^+(c, a)$ intersects the section $\Delta^{in}(\rho)$ and define the following set for small σ, ρ, δ to be chosen later:

$$\begin{aligned} \Sigma_i^+ = \{ & (\tilde{x}_\varepsilon(c, a) + x_0, -\rho^2, \varepsilon, c, a) : \\ & 0 \leq |x_0| < \sigma\rho\varepsilon, \varepsilon \in (0, \rho^3\delta), (c, a) \in I_c \times I_a \}. \end{aligned} \quad (3.15)$$

We also define the exit set

$$\Sigma_o^+ = \{(\rho, y, \varepsilon, c, a) : y \in \mathbb{R}, \varepsilon \in (0, \rho^3\delta), (c, a) \in I_c \times I_a\}. \quad (3.16)$$

Between the two sections Σ_i^+ and Σ_o^+ , the slow manifold $S_\varepsilon^+(c, a)$ consists of a single

solution $\gamma_\varepsilon(t; c, a)$ which can be written as

$$\gamma_\varepsilon(t; c, a) = (x_\varepsilon(t; c, a), y_\varepsilon(t; c, a), \varepsilon, c, a), \quad (3.17)$$

with $\gamma_\varepsilon(0; c, a) \in \Sigma_i^+$ and $\gamma_\varepsilon(\tau_\varepsilon; c, a) \in \Sigma_o^+$ for some time $\tau_\varepsilon = \tau_\varepsilon(c, a)$.

We define the $\mathcal{C}^1$ function $s_0(x; c, a)$ so that between $\Delta^{in}(\rho)$ and $\Delta^{out}(\rho)$, $y = s_0(x; c, a)$ is the graph of the singular solution obtained by following $S_0^+(c, a)$ to $(x, y) = (0, 0)$ then continuing on the fast fiber defined by $y = 0$.

The following Proposition 3.3.1 and Corollary 3.3.3, which will be proved in §3.3.7 and §3.3.8 below, are the main results of this section. Proposition 3.3.1 gives estimates on the flow of (3.12) in the center manifold $z = 0$ between the sections Σ_i^+ and Σ_o^+ . Corollary 3.3.3 then describes the implications for the full four dimensional flow of (3.12) where the dynamics of the basepoints of the unstable fibers are given by the flow on the center manifold.

Proposition 3.3.1. *Consider the flow of (3.12) in the three dimensional center manifold $z = 0$. There exists $\delta > 0$ such that for all sufficiently small choices of σ, ρ , all solutions starting in Σ_i^+ cross Σ_o^+ . Furthermore, there exists $\tilde{k} > 0$ such that the following holds. Given a solution $\gamma(t) = (x(t), y(t), \varepsilon, c, a)$ with $\gamma(0) \in \Sigma_i^+$, let τ denote the first time at which $\gamma(\tau) \in \Sigma_o^+$. Then*

$$(i) \quad \dot{x}(t) > \tilde{k}\varepsilon \text{ for } t \in [0, \tau]$$

In addition (see Remark 3.3.2 below), for each $(c, a) \in I_c \times I_a$, we can represent the manifold $S_\varepsilon^+(c, a)$ as a graph $(x, s_\varepsilon(x; c, a), \varepsilon)$ for $x \in [x_\varepsilon(0; c, a), \rho]$ where $s_\varepsilon(x; c, a)$ is an invertible function of x and

$$(ii) \quad |s_\varepsilon(x; c, a) - s_0(x; c, a)| = \mathcal{O}(\varepsilon^{2/3})$$

$$(iii) \quad \left| \frac{ds_\varepsilon}{dx}(x; c, a) - \frac{ds_0}{dx}(x; c, a) \right| = \mathcal{O}(\varepsilon^{1/3})$$

on the interval $[x_\varepsilon(0; c, a), \rho]$.

Remark 3.3.2. *The above result shows that there exists $\tilde{k} > 0$ such that for each $(c, a) \in I_c \times I_a$, we have $\dot{x}_\varepsilon(t; c, a) > \tilde{k}\varepsilon$ for $t \in [0, \tau_\varepsilon(c, a)]$. Note that due to the bounds on the function $g(x, y, \varepsilon, c, a)$ in system (3.12), there is a similar lower bound $\dot{y}_\varepsilon(t; c, a) \geq g_m\varepsilon$. Thus we can represent the manifold $S_\varepsilon^+(c, a)$ as a graph $(x, s_\varepsilon(x; c, a), \varepsilon)$ for $x \in [x_\varepsilon(0; c, a), \rho]$ where $s_\varepsilon(x; c, a)$ is an invertible function of x on the interval $[x_\varepsilon(0; c, a), \rho]$. Since this trajectory is contained in the neighborhood V_f , there exists an upper bound for the derivative*

$$\dot{x} = y + x^2 + h(x, y, \varepsilon, c, a) \leq \tilde{K}. \quad (3.18)$$

We therefore have the following bounds on the derivatives $\frac{ds_\varepsilon}{dx}(x; c, a)$ and $\frac{d(s_\varepsilon^{-1})}{dy}(y; c, a)$

$$\begin{aligned} \frac{g_m\varepsilon}{\tilde{K}} &\leq \frac{ds_\varepsilon}{dx}(x; c, a) \leq \frac{g_M}{\tilde{k}} \\ \frac{\tilde{k}}{g_M} &\leq \frac{d(s_\varepsilon^{-1})}{dy}(y; c, a) \leq \frac{\tilde{K}}{g_m\varepsilon}. \end{aligned} \quad (3.19)$$

We now fix ρ, σ small enough to satisfy Proposition 3.3.1. We have the following

Corollary 3.3.3. *There exists $\bar{x}_f, \varepsilon_0 > 0$ such that for each sufficiently small Δ_z , each*

$$(\varepsilon, c, a, x_f) \in (0, \varepsilon_0) \times I_c \times I_a \times [-\bar{x}_f, \bar{x}_f] \quad (3.20)$$

and each $0 \leq |x_i| < \sigma\rho\varepsilon$ there exists $z_i = z_i(\Delta_z, \varepsilon, x_i, x_f, c, a)$, $y_f = y_f(\varepsilon, x_i, x_f, c, a)$, time $T = T(\varepsilon, x_i, x_f, c, a)$, and a solution $\varphi(t; \varepsilon, x_i, x_f, c, a)$ to (3.12) satisfying

$$(i) \quad \varphi(0; \varepsilon, x_i, x_f, c, a) = (\tilde{x}_\varepsilon(c, a) + x_i, -\rho^2, z_i, \varepsilon)$$

$$(ii) \quad \varphi(T; \varepsilon, x_i, x_f, c, a) = (x_f, s_\varepsilon(x_f; c, a) - y_f, \Delta_z, \varepsilon)$$

where $|y_f| = \mathcal{O}(x_i)$, $|D_{\lambda_0} y_f| = \mathcal{O}(x_i/\varepsilon)$, $|D_{\lambda_1 \dots \lambda_n} T| = \mathcal{O}(\varepsilon^{-(n+1)})$ and $z_i = \mathcal{O}(e^{-\eta T})$, some $\eta > 0$, for $\lambda_j = \{x_i, x_f, c, a\}$, $j = 0, \dots, n$.

Remark 3.3.4. Corollary 3.3.3 solves a boundary value problem for (3.12) in the following sense. For each sufficiently small x_i, Δ_z, x_f , the result guarantees the existence of a solution to (3.12) whose basepoint in the center manifold is distance x_i in the x -direction from $S_\varepsilon^+(c, a)$ in Σ_i^+ and whose strong unstable z component reaches Δ_z at $x = x_f$. Also, the result gives estimates on the derivatives of the initial unstable component z_i in Σ_i^+ , the time T spent until $z = \Delta_z$, and the distance $y = y_f$ in the y -direction from $S_\varepsilon^+(c, a)$ when $(x, z) = (x_f, \Delta_z)$.

To prove these results we will use blow up techniques as in [38], and the proofs are given in §3.3.7 and §3.3.8, respectively. The blow up is essentially a rescaling which “blows up” the degenerate point $(x, y, \varepsilon) = (0, 0, 0)$ to a 2-sphere. The blow up transformation is given by

$$x = \bar{r}\bar{x}, \quad y = -\bar{r}^2\bar{y}, \quad \varepsilon = \bar{r}^3\bar{\varepsilon}. \quad (3.21)$$

Defining $B_f = S^2 \times [0, \bar{r}_0]$ for some sufficiently small $\bar{r}_0$, we consider the blow up as a mapping $B \rightarrow \mathbb{R}^3$ with $(\bar{x}, \bar{y}, \bar{\varepsilon}) \in S^2$ and $\bar{r} \in [0, \bar{r}_0]$. The point $(x, y, \varepsilon) = (0, 0, 0)$ is now represented as a copy of S^2 (i.e. $\bar{r} = 0$) in the blow up transformation. To study the flow on the manifold B_f and track solutions near $S_\varepsilon^+(c, a)$ around the fold, there are three relevant coordinate charts. Keeping the same notation as in [38], the

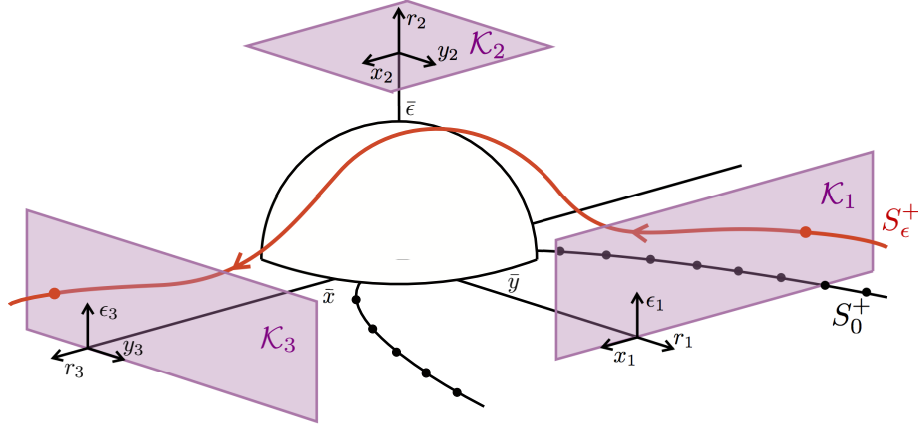


Figure 3.4: Shown is the set up for the coordinate charts $\mathcal{K}_i$, $i = 1, 2, 3$.

first is the chart $\mathcal{K}_1$ which uses the coordinates

$$x = r_1 x_1, \quad y = -r_1^2, \quad \varepsilon = r_1^3 \varepsilon_1, \quad (3.22)$$

the second chart $\mathcal{K}_2$ uses the coordinates

$$x = r_2 x_2, \quad y = -r_2^2 y_2, \quad \varepsilon = r_2^3, \quad (3.23)$$

and the third chart $\mathcal{K}_3$ uses the coordinates

$$x = r_3, \quad y = -r_3^2 y_3, \quad \varepsilon = r_3^3 \varepsilon_3. \quad (3.24)$$

The setup for the coordinate charts is shown in Figure 3.4. With these three sets of coordinates, a short calculation gives the following

Lemma 3.3.5. *The transition map $\kappa_{12} : K_1 \rightarrow K_2$ between the coordinates in $\mathcal{K}_1$ and $\mathcal{K}_2$ is given by*

$$x_2 = \frac{x_1}{\varepsilon_1^{1/3}}, \quad y_2 = \frac{1}{\varepsilon_1^{2/3}}, \quad r_2 = r_1 \varepsilon_1^{1/3}, \quad \text{for } \varepsilon_1 > 0, \quad (3.25)$$

and the transition map $\kappa_{23} : K_2 \rightarrow K_3$ between the coordinates in $\mathcal{K}_2$ and $\mathcal{K}_3$ is given

by

$$r_3 = r_2 x_2, \quad y_3 = \frac{y_2}{x_2^2}, \quad \varepsilon_3 = \frac{1}{x_2^3}, \quad \text{for } x_2 > 0. \quad (3.26)$$

3.3.4 Dynamics in $\mathcal{K}_1$

The desingularized equations in the new variables are given by

$$\begin{aligned} x_1' &= -1 + x_1^2 + \frac{1}{2}\varepsilon_1 x_1 + \mathcal{O}(r_1) \\ r_1' &= \frac{1}{2}r_1 \varepsilon_1 (-1 + \mathcal{O}(r_1)) \\ \varepsilon_1' &= \frac{3}{2}\varepsilon_1^2 (1 + \mathcal{O}(r_1)), \end{aligned} \quad (3.27)$$

where $' = \frac{d}{dt_1} = \frac{1}{r_1} \frac{d}{dt}$ denotes differentiation with respect to a rescaled time variable t_1 . Here we collect a few results from [38]. Firstly, there are two invariant subspaces for the dynamics of (3.27): the plane $r_1 = 0$ and the plane $\varepsilon_1 = 0$. Their intersection is the invariant line $l_1 = \{(x_1, 0, 0) : x_1 \in \mathbb{R}\}$, and the dynamics on l_1 evolve according to $x_1' = -1 + x_1^2$. There are two equilibria $p_a = (-1, 0, 0)$ and $p_r = (1, 0, 0)$. The equilibrium we are interested in, p_a has eigenvalue -2 for the flow along l_1 . In the plane $\varepsilon_1 = 0$, the dynamics are given by

$$\begin{aligned} x_1' &= -1 + x_1^2 + \mathcal{O}(r_1) \\ r_1' &= 0. \end{aligned} \quad (3.28)$$

This system has a normally hyperbolic curve of equilibria $S_{0,1}^+(c, a)$ emanating from p_a which exactly corresponds to the branch $S_0^+(c, a)$ of the critical manifold S in the original coordinates. Along $S_{0,1}^+(c, a)$ the linearization of (3.28) has one zero eigenvalue and one eigenvalue close to -2 for small r_1 .

In the invariant plane $r_1 = 0$, the dynamics are given by

$$\begin{aligned} x_1' &= -1 + x_1^2 + \frac{1}{2}\varepsilon_1 x_1 \\ \varepsilon_1' &= \frac{3}{2}\varepsilon_1^2. \end{aligned} \tag{3.29}$$

Here we still have the equilibrium p_a which now has an additional zero eigenvalue due to the second equation. The corresponding eigenvector is $(-1, 4)$ and hence there exists a one-dimensional center manifold $N_1^+(c, a)$ at p_a along which ε_1 increases. Note that the branch of $N_1^+(c, a)$ in the half space $\varepsilon_1 > 0$ is unique.

Restricting attention to the set

$$D_1 = \{(x_1, r_1, \varepsilon_1) : x_1 \in \mathbb{R}, 0 \leq r_1 \leq \rho, 0 \leq \varepsilon_1 \leq \delta\}, \tag{3.30}$$

we have the following result from [38]

Proposition 3.3.6 ([38, Proposition 2.6]). *For any $(c, a) \in I_c \times I_a$ and any sufficiently small $\rho, \delta > 0$, the following assertions hold for the dynamics of (3.27):*

- (i) *There exists an attracting center manifold $M_1^+(c, a)$ at p_a which contains the line of equilibria $S_{0,1}^+(c, a)$ and the center manifold $N_1^+(c, a)$. In D_1 , $M_1^+(c, a)$ is given as a graph $x_1 = h_+(r_1, \varepsilon_1, c, a) = -1 + \mathcal{O}(r_1, \varepsilon_1)$ with*

$$-3/2 < h_+(r_1, \varepsilon_1, c, a) < -1/2 \quad \text{on } D_1. \tag{3.31}$$

The branch of $N_1^+(c, a)$ in $r_1 = 0, \varepsilon > 0$ is unique. (Note that the manifold $M_1^+(c, a)$ is precisely the manifold $M^+(c, a)$ in the $\mathcal{K}_1$ coordinates.)

- (ii) *There exists a stable invariant foliation $\mathcal{F}^s(c, a)$ with base $M_1^+(c, a)$ and one-dimensional fibers. For any $\eta > -2$, for any sufficiently small ρ, δ , the con-*

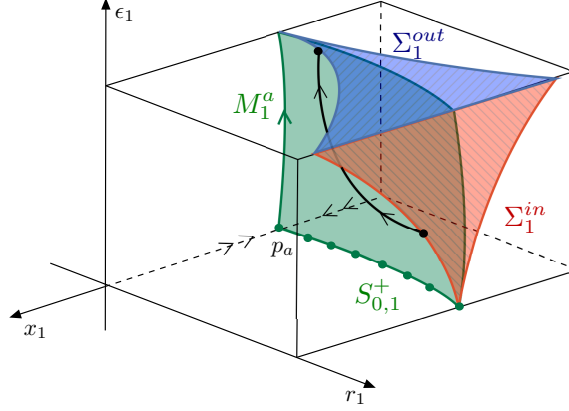


Figure 3.5: Shown is the set up in the chart $\mathcal{K}_1$.

traction along $\mathcal{F}^s(c, a)$ during a time interval $[0, T]$ is stronger than $e^{\eta T}$.

Making the change of variables $\tilde{x}_1 = x_1 - h_+(r_1, \varepsilon_1, c, a)$, we arrive at the system

$$\begin{aligned}\tilde{x}_1' &= \tilde{x}_1 (-2 + \tilde{x}_1 + \mathcal{O}(r_1, \varepsilon_1)) \\ r_1' &= \frac{1}{2} r_1 \varepsilon_1 (-1 + \mathcal{O}(r_1)) \\ \varepsilon_1' &= \frac{3}{2} \varepsilon_1^2 (1 + \mathcal{O}(r_1)).\end{aligned}\tag{3.32}$$

In the chart $\mathcal{K}_1$, the section Σ_i^+ is given by

$$\Sigma_1^{in} = \{(x_1, r_1, \varepsilon_1) : 0 < \varepsilon_1 < \delta, 0 \leq |\tilde{x}_1| < \sigma \rho^3 \varepsilon_1, r_1 = \rho\}.\tag{3.33}$$

We define the exit section

$$\Sigma_1^{out} = \{(x_1, r_1, \varepsilon_1) : \varepsilon_1 = \delta, 0 \leq |\tilde{x}_1| < \sigma r_1^3 \delta, 0 < r_1 \leq \rho\}.\tag{3.34}$$

The set up is shown in Figure 3.5. We have the following

Lemma 3.3.7. *Consider the system (3.32). There exists $k_1 > 0$ such that for all $(c, a) \in I_c \times I_a$ and all sufficiently small $\rho, \delta, \sigma > 0$, the following holds. Let $\gamma_1(t) = (x_1(t), r_1(t), \varepsilon_1(t))$ denote a solution with $\gamma_1(0) \in \Sigma_1^{in}$. Then γ_1 reaches Σ_1^{out} .*

In addition, letting τ_1 denote the first time at which $\gamma_1(\tau_1) \in \Sigma_1^{out}$, we have

$$\begin{aligned} \frac{dx}{dt_1} &= r_1 \frac{dx_1}{dt_1} + x_1 \frac{dr_1}{dt_1} \\ &= r_1 (\tilde{x}_1 + h_+(r_1, \varepsilon_1, c, a))' + r_1' (\tilde{x}_1 + h_+(r_1, \varepsilon_1, c, a)) \\ &> k_1 \rho r_1 \varepsilon_1, \quad \text{for } t \in [0, \tau_1]. \end{aligned} \tag{3.35}$$

Proof. Consider a solution $\gamma_1(t) = (x_1(t), r_1(t), \varepsilon_1(t))$ with $\gamma_1(0) \in \Sigma_1^{in}$. We note that for sufficiently small ρ, δ , since $r_1^3 \varepsilon_1$ is a constant of the motion, $|\tilde{x}_1|$ is decreasing, and γ_1 does indeed exit Σ_1^{out} .

To prove (3.35), we compute

$$\begin{aligned} r_1 (\tilde{x}_1 + h_+(r_1, \varepsilon_1, c, a))' + r_1' (\tilde{x}_1 + h_+(r_1, \varepsilon_1, c, a)) \\ = r_1 \tilde{x}_1' + \frac{1}{2} r_1 \varepsilon_1 (1 - \tilde{x}_1) + \mathcal{O}(r_1^2 \varepsilon_1, r_1 \varepsilon_1^2). \end{aligned} \tag{3.36}$$

Since $r_1^3 \varepsilon_1$ is a constant of the motion and $|\tilde{x}_1|$ is decreasing, $|\tilde{x}_1| < \sigma r_1^3 \varepsilon_1$. Also, from (3.32), we have $\tilde{x}_1' = \tilde{x}_1 (-2 + \tilde{x}_1 + \mathcal{O}(r_1, \varepsilon_1))$ so that

$$\begin{aligned} r_1 (\tilde{x}_1 + h_+(r_1, \varepsilon_1, c, a))' + r_1' (\tilde{x}_1 + h_+(r_1, \varepsilon_1, c, a)) \\ = r_1 \tilde{x}_1 (-2 + \tilde{x}_1) + \frac{1}{2} r_1 \varepsilon_1 (1 - \tilde{x}_1) + \mathcal{O}(r_1^2 \varepsilon_1, r_1 \varepsilon_1^2) \\ = \frac{1}{2} r_1 \varepsilon_1 (1 + \mathcal{O}(r_1, \varepsilon_1)). \end{aligned} \tag{3.37}$$

Thus there exists $k_1 > 0$ such that for all sufficiently small ρ, δ , the relation (3.35) holds. $\square$

3.3.5 Dynamics in $\mathcal{K}_3$

In the chart $\mathcal{K}_3$, the equations in the new variables are given by

$$\begin{aligned}\dot{r}_3 &= r_3^2 F(r_3, y_3, \varepsilon_3, c, a) \\ \dot{y}_3 &= r_3 [\varepsilon_3(-1 + \mathcal{O}(r_3)) - 2y_3 F(r_3, y_3, \varepsilon_3, c, a)] \\ \dot{\varepsilon}_3 &= -3r_3 \varepsilon_3 F(r_3, y_3, \varepsilon_3, c, a),\end{aligned}\tag{3.38}$$

where $F(r_3, y_3, \varepsilon_3, c, a) = 1 - y_3 + \mathcal{O}(r_3)$. For small $\beta > 0$, consider the set

$$\Sigma_3^{in} = \{(r_3, y_3, \varepsilon_3) : 0 < r_3 < \rho, y_3 \in [-\beta, \beta], \varepsilon_3 = \delta\},\tag{3.39}$$

The analysis in [38] shows that Σ_3^{in} is carried by the flow of (3.38) to the set

$$\Sigma_3^{out} = \{(r_3, y_3, \varepsilon_3) : r_3 = \rho, y_3 \in [-\beta, \beta], \varepsilon_3 \in (0, \delta)\}.\tag{3.40}$$

What we take from this is that for some fixed $k_3 \ll 1$, for all sufficiently small β, ρ, δ , between the sections Σ_3^{in} and Σ_3^{out} we have $F(r_3, y_3, \varepsilon_3) > k_3 \delta^{2/3}$ and thus $\dot{r}_3 > r_3^2 k_3 \delta^{2/3}$. So for a trajectory starting at $t = 0$ in Σ_3^{in} with initial $r_3(0) = r_0$, we have $\dot{r}_3 > r_0^2 k_3 \delta^{2/3}$. Since $\varepsilon_3 = \delta$ in Σ_3^{in} and $\varepsilon = r_3^3 \varepsilon_3$ is a constant of the flow, this implies $\dot{r}_3 > \varepsilon^{2/3} k_3$.

We can also compute an upper bound for the time spent between Σ_3^{in} and Σ_3^{out} . By integrating the estimate $\dot{r}_3 > r_3^2 k_3 \delta^{2/3}$ from 0 to t and using the relation $\varepsilon = r_0^3 \delta$, we obtain that $r_3(t) > \frac{\varepsilon^{1/3}}{\delta^{1/3} - \varepsilon^{1/3} \delta^{2/3} k_3 t}$. Thus any trajectory crossing Σ_3^{in} reaches Σ_3^{out} in time $t < \frac{1}{k_3} \left(\frac{1}{\varepsilon^{1/3} \delta^{1/3}} - \frac{1}{\rho} \right)$. We sum this up in the following

Lemma 3.3.8. *For any $(c, a) \in I_c \times I_a$ and all sufficiently small ρ, δ, β , any trajectory entering Σ_3^{in} exits Σ_3^{out} in time $t < \frac{1}{k_3} \left(\frac{1}{\varepsilon^{1/3} \delta^{1/3}} - \frac{1}{\rho} \right)$, and between these two sections,*

$$\dot{r}_3 > \varepsilon^{2/3} k_3.$$

We now fix β small enough so as to satisfy Lemma 3.3.8.

3.3.6 Dynamics in $\mathcal{K}_2$

In the chart $\mathcal{K}_2$, the desingularized equations in the new variables are given by

$$\begin{aligned} x'_2 &= -y_2 + x_2^2 + \mathcal{O}(r_2) \\ y'_2 &= -1 + \mathcal{O}(r_2) \\ r'_2 &= 0, \end{aligned} \tag{3.41}$$

where $' = \frac{d}{dt_2} = \frac{1}{r_2} \frac{d}{dt}$ denotes differentiation with respect to a rescaled time variable $t_1 = r_2 t$. For $r_2 = 0$, this reduces to the Riccati equation

$$\begin{aligned} x'_2 &= -y_2 + x_2^2 \\ y'_2 &= -1, \end{aligned} \tag{3.42}$$

whose solutions can be expressed in terms of special functions. We quote the relevant results:

Proposition 3.3.9 ([42, §II.9]). *The system (3.42) has the following properties:*

- (i) *There exists a special solution $\gamma_{2,0}(t) = (x_{2,0}(t), y_{2,0}(t))$ which can be represented as a graph $y_{2,0}(t) = s_{2,0}(x_{2,0}(t))$ for an invertible function $s_{2,0}$ which satisfies $s_{2,0}(x) > (x^2 + \frac{1}{2x})$ for $x < 0$ and $s_{2,0}(x) < x^2$ for all x . In addition,*

$$s_{2,0}(x) = -\Omega_0 + 1/x + \mathcal{O}(1/x^3) \quad \text{as } x \rightarrow \infty,$$

where Ω_0 is the smallest positive zero of

$$J_{-1/3}(2z^{3/2}/3) + J_{1/3}(2z^{3/2}/3),$$

where $J_{-1/3}, J_{1/3}$ are Bessel functions of the first kind.

- (ii) The special solution $\gamma_{2,0}(t) = (x_{2,0}(t), y_{2,0}(t))$ satisfies $x'_{2,0}(t), x''_{2,0}(t) > 0$ for all t and $x_{2,0}(t) \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$.

We now fix δ small enough to satisfy the results of §3.3.4 and §3.3.5 as well as taking $2\Omega_0\delta^{2/3} < \beta$, where β is the small constant fixed at the end of §3.3.5. The lemma below follows from a regular perturbation argument.

Lemma 3.3.10. *The special solution $\gamma_{2,0}$ has the following properties:*

- (i) Let τ_1, τ_2 be the times at which $y_{2,0}(\tau_1) = \delta^{-2/3}$ and $x_{2,0}(\tau_2) = \delta^{-1/3}$. Then there exists k_2 such that $x'_{2,0}(t) > 3k_2$ for $t \in [\tau_1, \tau_2]$.
- (ii) There exists $r_2^* > 0$ such that for any $(c, a) \in I_c \times I_a$ and any $0 < r_2 < r_2^*$, the special solution $\gamma_{2,0}$ persists as a solution

$$\gamma_{2,r_2}(t; c, a) = (x_{2,r_2}(t; c, a), y_{2,r_2}(t; c, a), r_2)$$

of (3.41), and solution similarly can be represented as a graph $y = s_{2,r_2}(x; c, a)$ for an invertible function $s_{2,r_2}(x; c, a)$ which is $\mathcal{C}^1$ - $\mathcal{O}(r_2)$ close to $s_{2,0}(x)$ on the interval $x \in [s_{2,r_2}^{-1}(\delta^{-2/3}; c, a), \delta^{-1/3}]$. Furthermore, we have $x'_{2,r_2}(t; c, a) > 2k_2$ for $x \in [s_{2,r_2}^{-1}(\delta^{-2/3}; c, a), \delta^{-1/3}]$.

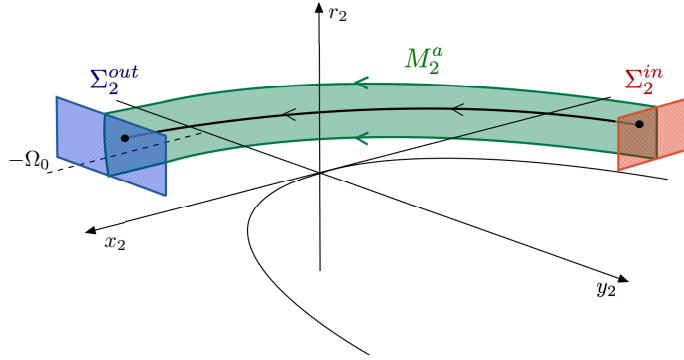


Figure 3.6: Shown is the set up in the chart $\mathcal{K}_2$.

Remark 3.3.11. *We note that the set*

$$M_2^+(c, a) := \{(x_{2,r_2}, s_{2,r_2}(x_{2,r_2}; c, a), r_2) : \quad (3.43)$$

$$x_{2,r_2} \in [s_{2,r_2}^{-1}(\delta^{-2/3}; c, a), \delta^{-1/3}], \quad 0 < r_2 < r_2^*\}$$

is in fact a piece of the manifold $M^+(c, a)$ in the $\mathcal{K}_2$ coordinates.

In the $\mathcal{K}_2$ coordinates, we have that $\kappa_{12}(\Sigma_1^{out})$ is contained in the set

$$\Sigma_2^{in} = \{(x_2, y_2, r_2) : \quad (3.44)$$

$$0 \leq |x_2 - s_{2,r_2}^{-1}(\delta^{-2/3}; c, a)| < \sigma \rho^3 \delta^{2/3}, y_2 = \delta^{-2/3}, 0 < r_2 \leq \rho \delta^{1/3}\}.$$

We also define the exit set

$$\Sigma_2^{out} = \{(x_2, y_2, r_2) : x_2 = \delta^{-1/3}, 0 < r_2 \leq \rho \delta^{1/3}\}. \quad (3.45)$$

The set up is shown in Figure 3.6. We have the following

Lemma 3.3.12. *For any $(c, a) \in I_c \times I_a$ and any sufficiently small σ, ρ , any solution $\gamma_2(t) = (x_2(t), y_2(t), r_2)$ satisfying $\gamma(0) \in \Sigma_2^{in}$ will reach Σ_2^{out} and between these two sections, this solution satisfies $x_2'(t) > k_2$ and $|y_2(t) - s_{2,r_2}(x_2(t); c, a)| \leq \Omega_0$.*

Proof. For sufficiently small $\rho < \delta^{-1/3}r_2^*$, we can appeal to Lemma 3.3.10 (ii), so that for any $r_2 < \rho\delta^{1/3}$, the special solution γ_{2,r_2} does in fact reach Σ_2^{out} with $x'_{2,r_2}(t) > 2k_2$ between Σ_2^{in} and Σ_2^{out} . We can also ensure that $y'_2 > -1/2$.

Now consider any solution $\gamma_2(t) = (x_2(t), y_2(t), r_2)$ with $\gamma(0) \in \Sigma_2^{in}$. By taking σ small, we can control how close γ_2 and γ_{2,r_2} are in Σ_2^{in} . Thus we can ensure that γ_2 reaches Σ_2^{out} and $x'_2(t) > k_2$ between Σ_2^{in} and Σ_2^{out} .

By shrinking σ if necessary, it is also possible to control the difference $|y_2(t) - s_{2,r_2}(x_2(t); c, a)|$. $\square$

3.3.7 Proof of Proposition 3.3.1

The following argument holds for any ρ, σ small enough to satisfy the analysis in §3.3.4, §3.3.5, and §3.3.6 (the parameters β and δ were already fixed in §3.3.5 and §3.3.6, respectively).

To prove (i), we follow the section Σ_i^+ , utilizing the results of the analysis in the previous sections. We consider a solution $\gamma(t) = (x(t), y(t), \varepsilon, c, a)$ which starts in Σ_i^+ . As outlined in §3.3.4, in the $\mathcal{K}_1$ coordinates, Σ_i^+ is given by the section Σ_1^{in} . The section Σ_1^{in} is carried to Σ_1^{out} by the flow and between these two sections, using (3.35)

we can also compute

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dr_1}{dt}x_1 + r_1 \frac{dx_1}{dt} \\
&= r_1 \left(\frac{dr_1}{dt_1}x_1 + r_1 \frac{dx_1}{dt_1} \right) \\
&> k_1 \rho r_1^2 \varepsilon_1 \\
&> k_1 \varepsilon.
\end{aligned} \tag{3.46}$$

As noted in §3.3.6, $\kappa_{12}(\Sigma_1^{out}) \subseteq \Sigma_2^{in}$. Between the two sections Σ_2^{in} and Σ_2^{out} , Lemma 3.3.12 gives

$$\begin{aligned}
\frac{dx}{dt} &= r_2 \frac{dx_2}{dt} \\
&= r_2^2 \frac{dx_2}{dt_2} \\
&> k_2 r_2^2 \\
&> k_2 \varepsilon^{2/3},
\end{aligned} \tag{3.47}$$

and in addition, by the choice of $2\Omega_0\delta^{2/3} < \beta$ in §3.3.6, we have that $\kappa_{23}(\Sigma_2^{out})$ is contained in the set Σ_3^{in} . In chart $\mathcal{K}_3$, Lemma 3.3.8 implies that $\dot{x}(t) > k_3 \varepsilon^{2/3}$ between Σ_3^{in} and Σ_3^{out} . Taking $\tilde{k} < \min\{k_i : i = 1, 2, 3\}$ gives $\dot{x}(t) > \tilde{k}\varepsilon$ between Σ^{in} and Σ^{out} , which completes the proof of (i).

It remains to prove the estimates (ii) and (iii) for the function $s_\varepsilon(x; c, a)$. In the chart $\mathcal{K}_1$, $S_0^+(c, a)$ is given by the graph $x_1 = h_+(r_1, 0, c, a) = -1 + \mathcal{O}(r_1)$, and for small positive ε , between the sections Σ_1^{in} and Σ_1^{out} , $S_\varepsilon^+(c, a)$ lies on the manifold defined by the graph

$$x_1 = h_+(r_1, \varepsilon_1, c, a) = h_+(r_1, 0, c, a) + \mathcal{O}(\varepsilon_1). \tag{3.48}$$

We now compute $\frac{ds_0}{dx}(x; c, a)$ and $\frac{ds_\varepsilon}{dx}(x; c, a)$ as

$$\begin{aligned}
\frac{ds_0}{dx}(x; c, a) &= \frac{dy/dr_1}{dx/dr_1} = \frac{2r_1}{h_+(r_1, 0, c, a) + r_1 \partial_{r_1} h_+(r_1, 0, c, a)} \\
&= r_1 (-2 + \mathcal{O}(r_1)) \\
\frac{ds_\varepsilon}{dx}(x; c, a) &= \frac{dy/dr_1}{dx/dr_1} = \frac{2r_1}{h_+(r_1, 0, c, a) + r_1 \partial_{r_1} h_+(r_1, 0, c, a) + \mathcal{O}(\varepsilon_1)} \\
&= \frac{ds_0}{dx}(x; c, a) + \mathcal{O}(r_1 \varepsilon_1).
\end{aligned} \tag{3.49}$$

Between Σ_1^{in} and Σ_1^{out} , we have that $r_1 \geq (\varepsilon/\delta)^{1/3}$. This implies that between Σ_1^{in} and Σ_1^{out} , we have

$$\frac{ds_\varepsilon}{dx}(x; c, a) - \frac{ds_0}{dx}(x; c, a) = \mathcal{O}(\varepsilon^{1/3}). \tag{3.50}$$

To estimate $|s_\varepsilon(x; c, a) - s_0(x; c, a)|$, we write

$$s_0(x; c, a) = s_\varepsilon(x; c, a) + \int_0^1 \frac{ds_0}{dx}(x + t(\bar{x} - x); c, a) \cdot (x - \bar{x}) dt, \tag{3.51}$$

where $\bar{x} = s_0^{-1}(s_\varepsilon(x; c, a); c, a)$. By (3.49), we have

$$\begin{aligned}
\frac{ds_0}{dx}(x; c, a) &= r_1(-2 + \mathcal{O}(r_1)) \\
&= \frac{r_1 x_1(-2 + \mathcal{O}(r_1))}{h_+(r_1, \varepsilon_1, c, a)} \\
&= \mathcal{O}(x),
\end{aligned} \tag{3.52}$$

by (3.31). Therefore

$$s_0(x; c, a) = s_\varepsilon(x; c, a) + \mathcal{O}(x(x - \bar{x}), (x - \bar{x})^2). \tag{3.53}$$

By (3.48), we have that $(x - \bar{x}) = \mathcal{O}(r_1 \varepsilon_1)$ which gives

$$\begin{aligned} |s_\varepsilon(x; c, a) - s_0(x; c, a)| &= \mathcal{O}(r_1^2 \varepsilon_1) \\ &= \mathcal{O}(\varepsilon^{2/3}), \end{aligned} \tag{3.54}$$

where again we used the fact that between Σ_1^{in} and Σ_1^{out} , we have that $r_1 \geq (\varepsilon/\delta)^{1/3}$.

In the chart $\mathcal{K}_2$, the function $s_\varepsilon(x; c, a)$ is given by $-r_2^2 s_{2,r_2}(x r_2^{-1}; c, a)$. By Lemma 3.3.10 (ii), we have that

$$s_\varepsilon(x; c, a) = -r_2^2 s_{2,r_2}(x r_2^{-1}; c, a) = \mathcal{O}(\varepsilon^{2/3}). \tag{3.55}$$

and

$$\frac{ds_\varepsilon}{dx}(x; c, a) = -r_2 \frac{ds_{2,r_2}}{dx}(x r_2^{-1}; c, a) = \mathcal{O}(\varepsilon^{1/3}), \tag{3.56}$$

between Σ_2^{in} and Σ_2^{out} . Since $s_0(x; c, a) = \mathcal{O}(x^2)$ near $x = 0$, we have that in this region in the chart $\mathcal{K}_2$, $s_0(x; c, a)$ satisfies $s_0(x; c, a) = \mathcal{O}(\varepsilon^{2/3})$ and $\frac{ds_0}{dx}(x; c, a) = \mathcal{O}(\varepsilon^{1/3})$.

Once the trajectory exits the chart $\mathcal{K}_2$ via Σ_2^{out} , we are in a region of positive x where $s_0(x; c, a) = \frac{ds_0}{dx}(x; c, a) = 0$, and we can determine the dynamics in the chart $\mathcal{K}_3$. From above, we know that in the chart $\mathcal{K}_3$, the y -coordinate changes by no more than $\mathcal{O}(\varepsilon^{2/3})$ so that $s_\varepsilon(x; c, a) = \mathcal{O}(\varepsilon^{2/3})$. Also, in the chart $\mathcal{K}_3$, we have that

$\dot{x} > k_3 \varepsilon^{2/3}$ which gives

$$\begin{aligned} \frac{ds_\varepsilon}{dx}(x; c, a) &= \frac{\dot{y}}{\dot{x}} \\ &\leq \frac{g_M \varepsilon}{k_3 \varepsilon^{2/3}} \\ &= \mathcal{O}(\varepsilon^{1/3}). \end{aligned} \tag{3.57}$$

This completes the proof of (ii) and (iii).

3.3.8 Proof of Corollary 3.3.3

Except for the estimates on T, y_f , the result follows directly from the statement of Proposition 3.3.1 and the implicit function theorem. To obtain the estimates on the time of flight T , we write

$$T(\varepsilon, x_i, x_f, c, a) = \int_{\tilde{x}_\varepsilon(c, a) + x_i}^{x_f} \frac{1}{\dot{x}} dx. \tag{3.58}$$

By Proposition 3.3.1 (i), we have that $T = \mathcal{O}(\varepsilon^{-1})$. Since the vector field is smooth, we can differentiate (3.58) and using Proposition 3.3.1 (i), we obtain the required bounds on the derivatives of T with respect to x_i, x_f, c, a .

To obtain $y_f = \mathcal{O}(x_i)$, we look at the evolution of $\tilde{y} = y - s_\varepsilon(x; c, a)$. We first note that since the graph $y = s_\varepsilon(x; c, a)$ defines a solution to (3.14), we can plug in this solution to get

$$\frac{d}{dt} s_\varepsilon(x; c, a) = \varepsilon g(x, s_\varepsilon(x; c, a), \varepsilon, c, a). \tag{3.59}$$

Plugging $y = \tilde{y} + s_\varepsilon(x; c, a)$ into (3.14) and using (3.59) gives

$$\begin{aligned}
\dot{\tilde{y}} &= \varepsilon (g(x, \tilde{y} + s_\varepsilon(x; c, a), \varepsilon, c, a) - g(x, s_\varepsilon(x; c, a), \varepsilon, c, a)) \\
&\quad - s'_\varepsilon(x; c, a) \tilde{y} (1 + \mathcal{O}(\varepsilon, x, s_\varepsilon(x; c, a), \tilde{y}))) \\
&= \int_0^1 \varepsilon g_y(x, s\tilde{y} + s_\varepsilon(x; c, a), \varepsilon, c, a) \tilde{y} ds \\
&\quad - s'_\varepsilon(x; c, a) \tilde{y} (1 + \mathcal{O}(\varepsilon, x, s_\varepsilon(x; c, a), \tilde{y}))),
\end{aligned} \tag{3.60}$$

and hence $\tilde{y}$ solves a differential equation of the form

$$\dot{\tilde{y}} = H(x, \tilde{y}, \varepsilon; c, a) \tilde{y}, \tag{3.61}$$

where

$$H(x, \tilde{y}, \varepsilon; c, a) \leq H_1 \varepsilon, \tag{3.62}$$

for some constant H_1 . Therefore $\tilde{y}$ can grow with rate at most $\mathcal{O}(\varepsilon)$, and we can deduce that

$$|y_f| = |\tilde{y}(T)| \leq |\tilde{y}(0)| e^{H_1 \varepsilon T}, \tag{3.63}$$

which, by using the bound on T above, we can reduce to

$$|y_f| \leq H_2 |\tilde{y}(0)|, \tag{3.64}$$

for some constant H_2 . To determine $\tilde{y}(0)$, we write

$$\begin{aligned} |\tilde{y}(0)| &= |s_\varepsilon(x_\varepsilon(c, a) + x_i; c, a) - s_\varepsilon(x_\varepsilon(c, a); c, a)| \\ &= \left| \int_0^1 \frac{ds_\varepsilon}{dx}(x_\varepsilon(c, a) + sx_i; c, a) \cdot x_i ds \right| \\ &\leq \frac{g_M}{\tilde{k}} |x_i|, \end{aligned} \quad (3.65)$$

where we used (3.19).

To obtain the bound on Dy_f , we integrate (3.60) to obtain

$$y_f = \tilde{y}(T) = \int_0^{T(\varepsilon, x_i, x_f, c, a)} \int_0^1 \varepsilon g_y(x, s\tilde{y} + s_\varepsilon(x; c, a), \varepsilon, c, a) \tilde{y}(t) ds dt. \quad (3.66)$$

Using the fact that the function g is smooth and the estimates on T and DT above, we obtain the desired estimate for the first derivative of y_f with respect to x_i, x_f, c, a .

The bound on z_i comes directly from the equations, but to ensure that z_i and its derivatives are exponentially small in $1/\varepsilon$, it is necessary to find a lower bound for the time of flight T . We now write

$$\begin{aligned} T(\varepsilon, x_i, x_f, c, a) &= \int_{-\rho^2}^{y_f} \frac{1}{\dot{y}} dy \\ &\geq \frac{1}{g_M \varepsilon} (y_f + \rho^2). \end{aligned} \quad (3.67)$$

We note that by Proposition 3.3.1 (ii) and the analysis above using the fact that $|x_i| \leq \sigma\varepsilon$, we have $y_f = s_0(x_f; c, a) + \mathcal{O}(\varepsilon^{2/3})$. Since $x_f \in [-\bar{x}_f, \bar{x}_f]$, we have $s_0(x_f; c, a) \geq s_0(-\bar{x}_f; c, a)$. So we can deduce the existence of τ_0 such that for all sufficiently small $\bar{x}_f$ and for all sufficiently small ε ,

$$T(\varepsilon, x_i, x_f, c, a) \geq \frac{\tau_0}{\varepsilon}. \quad (3.68)$$

3.4 Tying together the exchange lemma and fold analysis

3.4.1 Setup and transversality

To find connections between the strong unstable manifold $\mathcal{W}_\varepsilon^u(0; c, a)$ of the origin and the stable manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ of the left segment of the slow manifold, we will need two transversality results. The first describes transversality of the manifolds $\mathcal{W}_0^u(0; c, a)$ and $\mathcal{W}^s(\mathcal{M}_0^r(c, a))$ with respect to varying the wave speed parameter c .

Proposition 3.4.1. *There exists $\varepsilon_0 > 0$ and $\mu > 0$ such that for each $a \in I_a$ and $\varepsilon \in [0, \varepsilon_0]$, the manifold $\bigcup_{c \in I_c} \mathcal{W}_\varepsilon^u(0; c, a)$ intersects $\bigcup_{c \in I_c} \mathcal{W}_\varepsilon^{s,r}(c, a)$ transversely in uvw -space with the intersection occurring at $c = \tilde{c}(a, \varepsilon)$ for a smooth function $\tilde{c} : I_a \times [0, \varepsilon_0] \rightarrow I_c$ where $\tilde{c}(a, \varepsilon) = c^*(a) - \mu\varepsilon + \mathcal{O}(\varepsilon(|a| + \varepsilon))$.*

Proof. We aim to show that the manifold defined by $\bigcup_{c \in I_c} \mathcal{W}_0^u(0; c, a)$ intersects $\bigcup_{c \in I_c} \mathcal{W}^s(\mathcal{M}_0^r(c, a))$ transversely in uvw -space at $c = c^*(a)$ and that this transverse intersection persists for $\varepsilon \in [0, \varepsilon_0]$. To do this, we note that for each $a \in I_a$, there is an intersection of these manifolds occurring along the Nagumo front φ_f for $c = c^*(a)$, $\varepsilon = 0$ in the plane $w = 0$. It suffices to show that the intersection at $(c, a, \varepsilon) = (c^*(0), 0, 0)$ is transverse with respect to varying the wave speed c , so that for all sufficiently small $a \in I_a$ and $\varepsilon > 0$, we can solve for an intersection at $c = c(a, \varepsilon)$. This amounts to a Melnikov computation along the Nagumo front φ_f .

For each $a \in I_a$, we consider the planar system

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= cv - f(u),\end{aligned}\tag{4.1}$$

obtained by considering (1.2) with $w = \varepsilon = 0$. As stated above, for $(c, a) = (c^*(0), 0)$, this system possesses a heteroclinic connection $\varphi_f(t) = (u_f(t), v_f(t))$ (the Nagumo front) between the critical points $(u, v) = (0, 0) = p_0$ and $(u, v) = (1, 0) = p_1$ that lies in the intersection of $\mathcal{W}^u(p_0)$ and $\mathcal{W}^s(p_1)$. We now compute the distance between $\mathcal{W}^u(p_0)$ and $\mathcal{W}^s(p_1)$ to first order in $c - c^*(0)$. We consider the adjoint equation of the linearization of (4.1) about the Nagumo front φ_f given by

$$\dot{\psi} = \begin{pmatrix} 0 & \frac{df}{du}(u_f(t)) \\ -1 & -c^*(0) \end{pmatrix} \psi.\tag{4.2}$$

Let ψ_f be a nonzero bounded solution of (4.2), and let F_0 denote the right hand side of (4.1). Then

$$M_f^c = \int_{-\infty}^{\infty} D_c F_0(\varphi_f(t)) \cdot \psi_f(t) dt\tag{4.3}$$

measures the distance between $\mathcal{W}^u(p_0)$ and $\mathcal{W}^s(p_1)$ to first order in $c - c^*(0)$. Thus it remains to show that M_f^c is nonzero. Up to multiplication by a constant, we have that $\psi_f(t) = e^{-c^*(0)t}(-\dot{v}_f(t), \dot{u}_f(t)) = e^{-c^*(0)t}(-\dot{v}_f(t), v_f(t))$ which gives

$$M_f^c = \int_{-\infty}^{\infty} e^{-c^*(0)t} v_f(t)^2 dt > 0,\tag{4.4}$$

as required.

Similarly, we may also compute the distance between $\mathcal{W}^u(p_0)$ and $\mathcal{W}^s(p_1)$ to first

order in a as

$$\begin{aligned} M_f^a &= \int_{-\infty}^{\infty} D_a F_0(\varphi_f(t)) \cdot \psi_f(t) dt \\ &= \int_{-\infty}^{\infty} e^{-c^*(0)t} v_f(t) u_f(t) (1 - u_f(t)) dt. \end{aligned} \tag{4.5}$$

Using the explicit expressions

$$\begin{aligned} u_f(t) &= \frac{1}{2} \left(\tanh \left(\frac{1}{2\sqrt{2}} t \right) + 1 \right) \\ v_f(t) &= \frac{1}{\sqrt{2}} u_f(t) (1 - u_f(t)), \end{aligned}$$

for the Nagumo front for $a = 0$ from §2.2, we see that the Nagumo front satisfies the relation $v_f(t) = \frac{1}{\sqrt{2}} u_f(t) (1 - u_f(t))$. Hence

$$M_f^a = \sqrt{2} M_f^c. \tag{4.6}$$

To understand how the intersection of $\mathcal{W}_0^u(0; c^*(0), 0)$ and $\mathcal{W}^s(\mathcal{M}_0^r(c^*(0), 0))$ breaks as we vary ε , we now consider the full three-dimensional system (1.2)

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= \varepsilon(u - \gamma w). \end{aligned}$$

Note that the function $(u_f(t), v_f(t), 0)$ obtained by appending $w = 0$ to the Nagumo front $\varphi_f(t)$ is a solution to this system for $\varepsilon = a = 0$ and $c = c^*(0)$. We consider the

adjoint equation of the linearization of (1.2) about this solution given by

$$\dot{\Psi} = \begin{pmatrix} 0 & \frac{df}{du}(u_f(t)) & 0 \\ -1 & -c^*(0) & 0 \\ 0 & -1 & 0 \end{pmatrix} \Psi. \quad (4.7)$$

The space of solutions to (4.7) that grow at most algebraically is two-dimensional and spanned by

$$\Psi_1 = \left(-e^{-c^*(0)t} \dot{v}_f(t), e^{-c^*(0)t} \dot{u}_f(t), \int_0^t -e^{-c^*(0)s} v_f(s) ds \right) \quad (4.8)$$

and $\Psi_2 = (0, 0, 1)$. The function

$$\Psi = \left(-e^{-c^*(0)t} \dot{v}_f(t), e^{-c^*(0)t} \dot{u}_f(t), \int_t^\infty e^{-c^*(0)s} v_f(s) ds \right) \quad (4.9)$$

is the unique such solution to (4.7) (up to multiplication by a constant) satisfying $\Psi(t) \rightarrow 0$ as $t \rightarrow \infty$. Let F_1 denote the right hand side of (1.2); then by Melnikov theory, we can describe the distance between $\mathcal{W}_\varepsilon^u(c, a)$ and $\mathcal{W}_\varepsilon^{s,r}(c, a)$ to first order in ε by the integral:

$$\begin{aligned} M_f^\varepsilon &= \int_{-\infty}^\infty D_\varepsilon F_1(u_f(t), v_f(t), 0) \cdot \Psi(t) dt \\ &= \int_{-\infty}^\infty \left(\int_t^\infty e^{-c^*(0)s} v_f(s) ds \right) u_f(t) dt > 0. \end{aligned} \quad (4.10)$$

The distance function $d(c, a, \varepsilon)$ which defines the separation between $\mathcal{W}_\varepsilon^u(0; c, a)$ and $\mathcal{W}_\varepsilon^{s,r}(c, a)$ can now be expanded as

$$d(c, a, \varepsilon) = M_f^c(c - c^*(0) + \sqrt{2}a) + M_f^\varepsilon \varepsilon + \mathcal{O}(|c - c^*(0)| + a + \varepsilon)^2. \quad (4.11)$$

To find an intersection between $\mathcal{W}_\varepsilon^u(0; c, a)$ and $\mathcal{W}_\varepsilon^{s,r}(c, a)$, we now solve the equation

$d(c, a, \varepsilon) = 0$ for c and obtain $c = \tilde{c}(a, \varepsilon) = c^*(a) - \mu\varepsilon + \mathcal{O}(\varepsilon(|a| + \varepsilon))$ where $\mu := M_f^\varepsilon/M_f^c > 0$ due to (4.4) and (4.10), and we used the fact that $\sqrt{2}a = c^*(0) - c^*(a)$. The lack of $\mathcal{O}(a^2)$ terms in the expression for $\tilde{c}(a, \varepsilon)$ is due to the fact that for $\varepsilon = 0$ the intersection occurs at $c = c^*(a)$. $\square$

The second result needed is transversality of $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ and $\mathcal{W}^u(\mathcal{M}_0^r(c, a))$ along the back for $a = 0$. The problem here is that $\mathcal{M}_0^r(c, a)$ is not actually normally hyperbolic at the fold and therefore Fenichel theory does not ensure smooth persistence of the manifold $\mathcal{W}^u(\mathcal{M}_0^r(c, a))$ in this region for $\varepsilon > 0$: we will have to appeal to results from §3.3 to obtain the necessary transversality.

To start with, in the neighborhood O_f , in the center manifold near the fold point, we extend the right branch $\mathcal{M}_0^r(c, a)$ of the critical manifold by concatenating it with the fast unstable fiber leaving the fold point (see the description of the function $s_0(x; c, a)$ in §3.3.3) and call this new manifold $\mathcal{M}_0^{r,+}(c, a)$. It now makes sense to define $\mathcal{W}^u(\mathcal{M}_0^{r,+}(c, a))$ as the union of the strong unstable fibers of this singular trajectory. The advantage is now that Proposition 3.3.1 shows that $\mathcal{M}_0^{r,+}(c, a)$ persists as a trajectory $\mathcal{M}_\varepsilon^{r,+}(c, a)$ which is $\mathcal{C}^1$ - $\mathcal{O}(\varepsilon^{1/3})$ close to $\mathcal{M}_0^{r,+}(c, a)$. We can then define $\mathcal{W}_\varepsilon^{u,r}(c, a)$ to be the union of the strong unstable fibers of this perturbed solution. We are ready to state the following result.

Proposition 3.4.2. *For each $(c, a) \in I_c \times I_a$, the manifolds $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ and $\mathcal{W}^u(\mathcal{M}_0^{r,+}(c, a))$ intersect transversely in uvwc-space along the Nagumo back φ_b , and this transverse intersection persists for $\varepsilon \in [0, \varepsilon_0]$. Furthermore, for each $(c, a, \varepsilon) \in I_c \times I_a \times [0, \varepsilon_0]$, the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ intersects $\mathcal{W}_\varepsilon^{u,r}(c, a)$ transversely.*

Proof. We note that past the fold point, $\mathcal{M}_0^{r,+}(c, a)$ lies in a plane of constant w since in this region $\mathcal{M}_0^{r,+}(c, a)$ is described by the fast $\varepsilon = 0$ flow. Thus we proceed as in

the proof of Proposition 3.4.1, though now we show transversality of the manifolds $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ and $\mathcal{W}^u(\mathcal{M}_0^r(c, a))$ with respect to w , which is a parameter for the fast $\varepsilon = 0$ flow.

It suffices to prove transversality at $(\varepsilon, a) = (0, 0)$. By the $\mathcal{C}^1$ dependence of the manifolds with respect to a , this transversality persists for $a \in I_a$. The fact that this transversality persists for small $\varepsilon > 0$ follows from the $\mathcal{C}^1$ - $\mathcal{O}(\varepsilon^{1/3})$ closeness of $\mathcal{M}_\varepsilon^{r,+}(c, a)$ and $\mathcal{M}_0^{r,+}(c, a)$. This implies that $\mathcal{W}^u(\mathcal{M}_0^{r,+}(c, a))$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$ are also $\mathcal{C}^1$ - $\mathcal{O}(\varepsilon^{1/3})$ close.

To continue, we consider the planar system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - f(u) + w, \end{aligned} \tag{4.12}$$

obtained by considering (1.2) with $a = \varepsilon = 0$. For $c = c^*(0)$ and $w = w^*(0)$, this system possesses a heteroclinic connection $\varphi_b(t) = (u_b(t), v_b(t))$ (the Nagumo back) between the critical points $(u, v) = (u_1, 0) = q_1$ and $(u, v) = (u_0, 0) = q_0$ where u_0 and u_1 are the smallest and largest zeros of $w^*(0) - f(u)$, respectively. That is, there exists an intersection of $\mathcal{W}^u(q_1)$ and $\mathcal{W}^s(q_0)$ defined by the connection φ_b .

Thus the manifolds $\mathcal{W}^s(\mathcal{M}_0^\ell(c^*(0), 0))$ and $\mathcal{W}^u(\mathcal{M}_0^r(c^*(0), 0))$ intersect in the full system along the Nagumo back φ_b . Since $\mathcal{M}_0^{r,+}(c^*(0), 0)$ lies in the plane $w = w^*(0)$ past the fold point (and thus so do its fast fibers since the fast flow is confined to $w = \text{const}$ planes), we have that $\mathcal{W}^u(\mathcal{M}_0^r(c^*(0), 0))$ is tangent to the plane $w = w^*(0)$ along φ_b . In (4.12), from regular perturbation theory, the stable manifold of the leftmost equilibrium (given by the trajectory φ_b at $w = w^*(0)$) breaks smoothly in w and thus $\mathcal{W}^s(\mathcal{M}_0^\ell(c^*(0), 0))$ is transverse to planes $w = \text{const}$; in particular this gives the necessary transversality of $\mathcal{W}^s(\mathcal{M}_0^\ell(c^*(0), 0))$ and $\mathcal{W}^u(\mathcal{M}_0^r(c^*(0), 0))$.

We therefore obtain the desired transversality of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$ for all $(c, a, \varepsilon) \in I_c \times I_a \times [0, \varepsilon_0]$. $\square$

3.4.2 Exchange lemma

In this section we use the exchange lemma of [51] to track the manifold $\mathcal{W}_\varepsilon^u(0; c, a)$ near the right branch $\mathcal{M}_\varepsilon^r(c, a)$ of the slow manifold up to a fixed neighborhood of the fold point. The analysis of §3.3 defines a fixed neighborhood O_f of the fold point in uvw -coordinates for $(c, a) \in I_c \times I_a$ in which the flow is well understood. The neighborhood O_f corresponds to the neighborhood U_f in $xyzca$ -coordinates in which the section Σ_i^+ defines points along trajectories satisfying the desired estimates.

We may assume that the manifold $\mathcal{M}_\varepsilon^r(c, a)$ extends into this neighborhood past the section Σ_i^+ but ends before the fold (in U_f note that $\mathcal{M}_\varepsilon^r(c, a)$, where defined, coincides with $S_\varepsilon^+(c, a)$ up to errors exponentially small in $1/\varepsilon$ due to the non-uniqueness of the center manifold in §3.3). Here $\mathcal{M}_\varepsilon^r(c, a)$ is normally hyperbolic, and thus there exists a C^{r+1} Fenichel normal form for the equations in a neighborhood of $\mathcal{M}_\varepsilon^r(c, a)$:

$$\begin{aligned}
 X' &= -A(X, Y, Z, c, a, \varepsilon)X \\
 Y' &= B(X, Y, Z, c, a, \varepsilon)Y \\
 Z' &= \varepsilon(1 + E(X, Y, Z, c, a, \varepsilon)XY) \\
 c' &= 0 \\
 a' &= 0,
 \end{aligned} \tag{4.13}$$

where the functions A and B are positive and bounded away from 0 uniformly in all variables. These equations are valid in a neighborhood U_e of $\mathcal{M}_\varepsilon^r(c, a)$, $c \in I_c$, $a \in I_a$ which we assume to be given by $X, Y \in (-\Delta, \Delta)$ for some small $\Delta > 0$ and

$(Z, c, a) \in V = (-\Delta, Z_0 + \Delta) \times I_c \times I_a$ for appropriate $Z_0 > \Delta$. In U_e , for each c, a, ε , the manifold $\mathcal{M}_\varepsilon^r(c, a)$ is given by $X = Y = 0$. Similarly the manifolds $\mathcal{W}_\varepsilon^{u,r}(c, a)$ and $\mathcal{W}_\varepsilon^{s,r}(c, a)$ are given by $X = 0$ and $Y = 0$ respectively. We denote the change of coordinates from (X, Y, Z, c, a) to the (u, v, w, c, a) coordinates by $\Phi_e : U_e \rightarrow O_e$ where O_e is the corresponding neighborhood of $\mathcal{M}_\varepsilon^r(c, a)$ in (u, v, w) -coordinates for $(c, a) \in I_c \times I_a$. Since O_e is by construction a neighborhood of a normally hyperbolic segment of $\mathcal{M}_\varepsilon^r(c, a)$ which extends into O_f , there is an overlap of the neighborhoods O_e and O_f where the fold analysis is valid. We now comment on the constants Δ, Z_0 : since $\mathcal{M}_\varepsilon^r(c, a)$ extends past the section Σ_i^+ in the neighborhood U_f , for Δ sufficiently small, we can think of Z_0 as being the height in the U_e coordinates at which $\mathcal{M}_\varepsilon^r(c, a)$ hits Σ_i^+ for $(c, a, \varepsilon) = (c^*(0), 0, 0)$; see §3.4.3 for details.

We note that due to the non-uniqueness of the center manifold in §3.3, the coordinate descriptions of the manifolds $\mathcal{M}_\varepsilon^r(c, a)$, $\mathcal{W}_\varepsilon^{u,r}(c, a)$, and $\mathcal{W}_\varepsilon^{s,r}(c, a)$ in the two neighborhoods O_e and O_f are only equal up to errors exponentially small in $1/\varepsilon$. Since these errors are taken into account in the analysis below, for simplicity we will use the same notation for these manifolds in the different coordinate systems.

For each $\varepsilon \in [0, \varepsilon_0]$ we define the two-dimensional incoming manifold

$$N_\varepsilon^{in} = \left(\bigcup_{c \in I_c, a \in I_a} \mathcal{W}_\varepsilon^u(0; c, a) \right) \cap \{X = \Delta\}, \quad (4.14)$$

which, under the flow of (4.13) becomes a manifold N_ε^* of dimension three. Define

$$\mathcal{A} = \{(Y, Z, a) : Y \in (-\Delta, \Delta), Z \in (Z_0 - \Delta, Z_0 + \Delta), a \in I_a\}. \quad (4.15)$$

The necessary transversality of the incoming manifold N_ε^{in} with $\{Y = 0\}$ is given by Proposition 3.4.1. The generalized exchange lemma now gives the following

Theorem 3.3 ([51, Theorem 3.1]). *There exist functions $\tilde{X}, \tilde{W} : \mathcal{A} \times [0, \varepsilon_0] \rightarrow \mathbb{R}$ which satisfy*

(i) *For $\varepsilon > 0$, the set*

$$\{(X, Y, Z, a, c) : (Y, Z, a) \in \mathcal{A}, X = \tilde{X}(Y, Z, a, \varepsilon), c = \tilde{c}(a, \varepsilon) + \tilde{W}(Y, Z, a, \varepsilon)\}$$

is contained in N_ε^ .*

(ii) $\tilde{X}(Y, Z, a, 0) = 0, \tilde{W}(Y, Z, a, 0) = 0, \tilde{W}(0, Z, a, \varepsilon) = 0$

(iii) *There exists $q > 0$ such that $|D_j \tilde{X}|, |D_j \tilde{W}| = \mathcal{O}(e^{-q/\varepsilon})$ for any $0 \leq j \leq r$.*

We comment on the interpretation of Theorem 3.3. For each choice of a , height Z and unstable component Y lying in $\mathcal{A}$, provided the offset $c - \tilde{c}(a, \varepsilon)$ is adjusted by the quantity $\tilde{W}(Y, Z, a, \varepsilon)$, the theorem guarantees a solution which starts in N_ε^{in} which hits the point (X, Y, Z, a, c) where $X = \tilde{X}(Y, Z, a, c)$. In (ii), the property $\tilde{W}(0, Z, a, \varepsilon) = 0$ refers to the fact that for $c = \tilde{c}(a, \varepsilon)$, the manifold $\mathcal{W}_\varepsilon^u(0; c, a)$ in fact lies in the stable foliation $Y = 0$, which was proved in Proposition 3.4.1. The final properties $\tilde{X}(Y, Z, a, 0) = 0, \tilde{W}(Y, Z, a, 0) = 0$, and property (iii) state that the functions $\tilde{X}, \tilde{W} \rightarrow 0$ uniformly in the limit $\varepsilon \rightarrow 0$, and that this convergence is in fact exponential in derivatives up to order r .

3.4.3 Setup in U_e

We will use Theorem 3.3 to describe the flow up to a neighborhood of the fold point, then we will use the results of §3.3. We first place a section Σ^{in} in the neighborhood

U_f of the fold point which we define by

$$\begin{aligned} \Sigma^{in} = \{ & (x, y, z, c, a, \varepsilon) \in U_f : \\ & y = -\rho^2, |x - \tilde{x}_0(c^*(0), 0)| \leq \Delta', z \in [-\Delta', \Delta'], (c, a, \varepsilon) \in I \}, \end{aligned} \quad (4.16)$$

for some small choice of Δ' where $I = I_c \times I_a \times [0, \varepsilon_0]$. As described above, there is an overlap of the regions described by U_f and the neighborhood U_e where the Fenichel normal form is valid. We denote the change of coordinates between these neighborhoods by $\Phi_{\text{ef}} : \Phi_e^{-1}(O_e \cap O_f) \subseteq U_e \rightarrow U_f$ where $\Phi_{\text{ef}} = \Phi_f^{-1} \circ \Phi_e$. From the construction the section $\Phi_{\text{ef}}^{-1}(\Sigma^{in})$ will be given by a section in XYZ -space transverse to the sets $X = \text{const}$ and $Y = \text{const}$. We can therefore represent $\Phi_{\text{ef}}^{-1}(\Sigma^{in})$ in XYZ -space as $\Phi_{\text{ef}}^{-1}(\Sigma^{in}) = \{(X, Y, Z, c, a, \varepsilon) : Z = \psi(X, Y, c, a, \varepsilon)\}$ for some smooth function

$$\psi : [-\Delta, \Delta] \times [-\Delta, \Delta] \times I \rightarrow [-\Delta + Z_0, \Delta + Z_0], \quad (4.17)$$

where we assume that Z_0 has been chosen so that $\psi(0, 0, c^*(0), 0, 0) = Z_0$. It is important to note that since $\mathcal{M}_\varepsilon^r(c, a)$ and $S_\varepsilon^+(c, a)$ are equal up to exponentially small errors in the U_f coordinates, $\Phi_{\text{ef}}(0, 0, Z, c, a, \varepsilon)$ maps onto $S_\varepsilon^+(c, a)$ up to errors exponentially small in $1/\varepsilon$. Figure 3.7 shows the setup as well as the passage of a trajectory according to the exchange lemma. Figure 3.8 shows the continuation of this trajectory past the fold. The idea is to show that for each $a \in I_a$ and each $\varepsilon \in (0, \varepsilon_0)$, we can find c such that this solution connects $\mathcal{W}_\varepsilon^u(0; c, a)$ to $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ as shown.

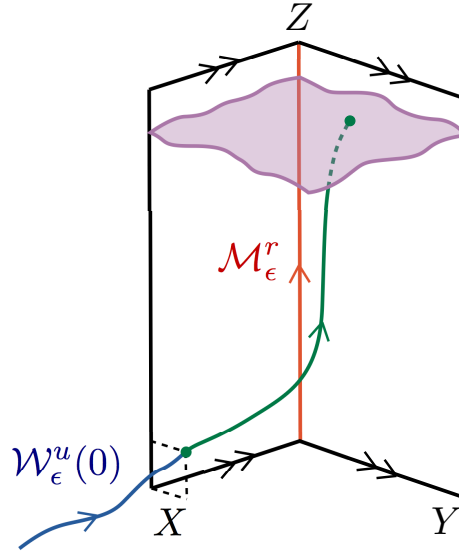


Figure 3.7: Shown is the set up of the exchange lemma (Theorem 3.3).

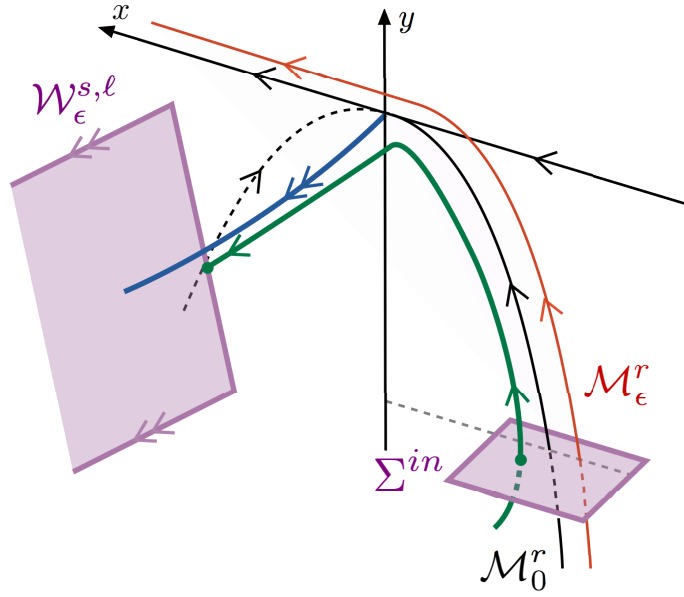


Figure 3.8: Shown is the flow near the fold point.

3.4.4 Entering U_f via the Exchange lemma

We now use the exchange lemma to solve for solutions which cross $\Phi_{\text{ef}}^{-1}(\Sigma^{\text{in}})$. For each $Y \in [-\Delta, \Delta]$, $a \in I_a$, and $\varepsilon \in (0, \varepsilon_0]$, we can find a solution which reaches the point

$$(X, Y, \psi(X, Y, c, a, \varepsilon), c, a) \in \Phi_{\text{ef}}^{-1}(\Sigma^{\text{in}}) \quad (4.18)$$

provided we can solve

$$\begin{aligned} X &= \tilde{X}(Y, \psi(X, Y, c, a, \varepsilon), a, \varepsilon) \\ c &= \tilde{c}(a, \varepsilon) + \tilde{W}(Y, \psi(X, Y, c, a, \varepsilon), a, \varepsilon) \end{aligned} \quad (4.19)$$

in terms of (Y, a, ε) where $\tilde{X}, \tilde{W}$ are the functions from Theorem 3.3. Using the fact that ψ is smooth and that $\tilde{X}, \tilde{W}$ and their derivatives are $\mathcal{O}(e^{-q/\varepsilon})$, we can solve by the implicit function theorem for $(X, c - \tilde{c}(a, \varepsilon))$ near $(0, 0)$ in terms of the variables (Y, a, ε) to obtain

$$\begin{aligned} X &= X^*(Y, a, \varepsilon) \\ c &= \tilde{c}(a, \varepsilon) + W^*(Y, a, \varepsilon), \end{aligned} \quad (4.20)$$

where the smooth functions X^*, W^* and their derivatives are $\mathcal{O}(e^{-q/\varepsilon})$, where we may need to take q smaller. To sum up, we have just shown the following:

Proposition 3.4.3. *For each $Y \in [-\Delta, \Delta]$, $a \in I_a$, and $\varepsilon \in (0, \varepsilon_0]$, we can find a solution which reaches the point*

$$(X, Y, \psi(X, Y, c, a, \varepsilon), c, a) \in \Phi_{\text{ef}}^{-1}(\Sigma^{\text{in}}), \quad (4.21)$$

where $X = X^*(Y, a, \varepsilon)$ and $c = \tilde{c}(a, \varepsilon) + W^*(Y, a, \varepsilon)$. The functions X^* and W^* and

their derivatives are $\mathcal{O}(e^{-q/\varepsilon})$.

3.4.5 Connecting to $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$: analysis in U_f

What we conclude from Proposition 3.4.3 is that for any sufficiently small choice of (Y, a, ε) we can find a solution which enters a neighborhood of the fold at a distance Y from $\mathcal{W}_\varepsilon^{s,r}(c, a)$ along the unstable fibers provided c is adjusted from $\tilde{c}(a, \varepsilon)$ by $\mathcal{O}(e^{-q/\varepsilon})$. In addition the distance from $\mathcal{W}_\varepsilon^{u,r}(c, a)$ is $\mathcal{O}(e^{-q/\varepsilon})$. By applying the smooth transition map Φ_{ef} , it is convenient to rewrite Proposition 3.4.3 in the U_f coordinates as shown in the following.

Proposition 3.4.4. *For each $z \in [-\Delta', \Delta']$, $a \in I_a$, and $\varepsilon \in (0, \varepsilon_0]$, we can find a solution which reaches the point*

$$(x, -\rho^2, z, c, a) \in \Sigma^{in}, \quad (4.22)$$

where $x = \tilde{x}_\varepsilon(c, a) + x^*(z, a, \varepsilon)$ and $c = \tilde{c}(a, \varepsilon) + w^*(z, a, \varepsilon)$. The functions x^* , w^* and their derivatives are $\mathcal{O}(e^{-q/\varepsilon})$.

Remark 3.4.5. *Though the manifolds $\mathcal{W}_\varepsilon^{s,r}(c, a)$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$ are not unique, the errors we incur by transforming to the U_f coordinates are exponentially small in $1/\varepsilon$ and can be absorbed in the functions x^*, w^* without changing the result.*

We will use this result along with the center manifold analysis of §3.3 to find such a solution for each (a, ε) which connects to $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. We first determine the location of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in the neighborhood U_f . From Proposition 3.4.2, we know that $\mathcal{W}^s(\mathcal{M}_0^\ell(c^*(0), 0))$ intersects $\mathcal{W}^u(\mathcal{M}_0^{r,+}(c^*(0), 0))$ transversely for $\varepsilon = 0$ along the Nagumo back φ_b , and this intersection persists for $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$.

This means that $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ will transversely intersect the manifold $\mathcal{W}_\varepsilon^{u,r}(c, a)$ which is composed of the union of the unstable fibers of the continuation of the slow manifold $\mathcal{M}_\varepsilon^{r,+}(c, a)$ found in §3.3. We therefore place an exit section Σ^{out} defined by

$$\Sigma^{out} = \{(x, y, z, c, a, \varepsilon) \in U_f : z = \Delta'\}. \quad (4.23)$$

For $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$, the intersection of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$ occurs at a point

$$(x, y, z, c, a, \varepsilon) = (x_\ell(c, a, \varepsilon), s_\varepsilon(x_\ell(c, a, \varepsilon); c, a), \Delta', c, a, \varepsilon) \in \Sigma^{out}, \quad (4.24)$$

and thus we may expand $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in Σ^{out} as

$$(x, y - s_\varepsilon(x; c, a)) = (x_\ell(c, a, \varepsilon) + \mathcal{O}(\tilde{y}, \varepsilon), \tilde{y}), \quad \tilde{y} \in [-\Delta_y, \Delta_y], \quad (4.25)$$

for some small Δ_y . Now using Corollary 3.3.3, we aim to find a solution to match with $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ at $z = \Delta'$. From Corollary 3.3.3 for each $(\varepsilon, c, a, x_i, x_f)$, we get a solution $\varphi(t; \varepsilon, x_i, x_f, c, a)$ and time of flight $T(\varepsilon, x_i, x_f, c, a)$ satisfying

$$\begin{aligned} \varphi(0; \varepsilon, x_i, x_f, c, a) &= (\tilde{x}_\varepsilon(c, a) + x_i, -\rho^2, z_i, \varepsilon) \\ \varphi(T; \varepsilon, x_i, x_f, c, a) &= (x_f, s_\varepsilon(x_f; c, a) - y_f, \Delta', \varepsilon). \end{aligned} \quad (4.26)$$

Thus finding a connection between $\mathcal{W}_\varepsilon^u(0; c, a)$ and $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ for a given (a, ε) amounts

to solving the following system of equations

$$\begin{aligned}
x_i &= x^*(z, a, \varepsilon) \\
c &= \tilde{c}(a, \varepsilon) + w^*(z, a, \varepsilon) \\
x_f &= x_\ell(c, a, \varepsilon) + \mathcal{O}(\tilde{y}, \varepsilon) \\
y_f(x_i, x_f, \varepsilon, a, c) &= \tilde{y} \\
z_i(\Delta', x_i, x_f, \varepsilon, a, c) &= z,
\end{aligned} \tag{4.27}$$

for all variables in terms of (a, ε) .

We start by substituting $x_i = x^*(z, a, \varepsilon)$ into the equation for $\tilde{y}$. Using the fact that $x^*(z, a, \varepsilon) = \mathcal{O}(e^{-q/\varepsilon})$ and the estimates $y_f = \mathcal{O}(x_i)$ and $Dy_f = \mathcal{O}(x_i/\varepsilon)$ from Corollary 3.3.3, we can solve for $\tilde{y} = \tilde{y}(z, a, \varepsilon)$ by the implicit function theorem where $\tilde{y}, D\tilde{y} = \mathcal{O}(e^{-q/\varepsilon})$, where q may need to be taken smaller. We now substitute everything into the equation for z . Using the estimates on z_i from Corollary 3.3.3 and the estimates on $\tilde{y}$ above, we can then solve for $z = z(a, \varepsilon)$ (and subsequently all other variables) by the implicit function theorem.

In particular, we note that the wave speed c is given by

$$\begin{aligned}
c(a, \varepsilon) &= \tilde{c}(a, \varepsilon) + \mathcal{O}(e^{-q/\varepsilon}) \\
&= c^*(a) - \mu\varepsilon + \mathcal{O}(\varepsilon(|a| + \varepsilon)),
\end{aligned} \tag{4.28}$$

where $\mu > 0$ is the constant from Proposition 3.4.1, and we have absorbed the exponentially small terms in the $\mathcal{O}(\varepsilon^2)$ term.

3.5 Convergence of $M_\varepsilon^\ell(c, a)$ to the equilibrium

The analysis of §3.3 and §3.4 shows that for each $a \in I_a$ and for any sufficiently small ε , there exists a wave speed c such that the manifold $\mathcal{W}_\varepsilon^u(0; c, a)$ intersects $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. Upon entering a neighborhood of the origin, this trajectory will be exponentially close to the perturbed slow manifold $\mathcal{M}_\varepsilon^\ell(c, a)$. It remains to show that $\mathcal{M}_\varepsilon^\ell(c, a)$ and as well as nearby trajectories on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in fact converge to the equilibrium at the origin. This result is not immediate, as for $a = 0$, the origin is on the lower left knee where $\mathcal{M}_0^\ell(c, a)$ is not normally hyperbolic. In this section, a center manifold analysis of the origin produces conditions on (a, ε) which ensure this result.

3.5.1 Preparation of equations

To study the stability properties of the equilibrium at $(u, v, w) = (0, 0, 0)$ of (1.2) for small ε, a , we append equations for a and ε to (1.2) and obtain

$$\begin{aligned}
 \dot{u} &= v \\
 \dot{v} &= cv - f(u) + w \\
 \dot{w} &= \varepsilon(u - \gamma w) \\
 \dot{a} &= 0 \\
 \dot{\varepsilon} &= 0.
 \end{aligned} \tag{5.1}$$

For $a = \varepsilon = 0$, the origin coincides with the lower left knee on the critical manifold. System (5.1) has the family of equilibria $(u, 0, f(u), a, 0)$ where u varies near 0. We are interested in the lower left knee of $w = f(u)$ as a function of a . For $(c, a) \in I_c \times I_a$,

the knee is given by the family $(u^\dagger(a), 0, w^\dagger(a), a, 0)$ where

$$\begin{aligned} u^\dagger(a) &= \frac{1}{3} \left(a + 1 - \sqrt{a^2 - a + 1} \right) \\ w^\dagger(a) &= f(u^\dagger(a)). \end{aligned} \tag{5.2}$$

The linearization of (5.1) about the knee at $(a, \varepsilon) = 0$ is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & c & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.3}$$

There is one positive eigenvalue $\lambda = c$ with eigenvector $(1, c, 0, 0, 0)$ and a quadruple zero eigenvalue. By making the coordinate transformation

$$\begin{aligned} z_1 &= u - u^\dagger - \frac{v}{c} - \frac{w - w^\dagger}{c^2} \\ z_2 &= -\frac{w - w^\dagger}{c} \\ z_3 &= \frac{v}{c} + \frac{w - w^\dagger}{c^2}, \end{aligned} \tag{5.4}$$

we arrive at the system

$$\begin{aligned}
\dot{z}_1 &= z_2 + \frac{1}{c} \left(\sqrt{a^2 - a + 1} \right) (z_1 + z_3)^2 - \frac{1}{c} (z_1 + z_3)^3 \\
&\quad - \frac{\varepsilon}{c^2} (z_1 + z_3 + c\gamma z_2 + u^\dagger - \gamma w^\dagger) \\
\dot{z}_2 &= -\frac{\varepsilon}{c} (z_1 + z_3 + c\gamma z_2 + u^\dagger - \gamma w^\dagger) \\
\dot{z}_3 &= cz_3 - \frac{1}{c} \left(\sqrt{a^2 - a + 1} \right) (z_1 + z_3)^2 + \frac{1}{c} (z_1 + z_3)^3 \\
&\quad + \frac{\varepsilon}{c^2} (z_1 + z_3 + c\gamma z_2 + u^\dagger - \gamma w^\dagger) \\
\dot{a} &= 0 \\
\dot{\varepsilon} &= 0,
\end{aligned} \tag{5.5}$$

which, for $\varepsilon = 0$, is in Jordan normal form for the three dynamic variables (z_1, z_2, z_3) . To understand the dynamics near the fold point, we separate the nonhyperbolic dynamics which occur on a four-dimensional center manifold. In a small neighborhood of the fold point, this manifold can be represented as a graph

$$\begin{aligned}
z_3 &= F(z_1, z_2, \varepsilon) \\
&= \beta_0 z_1 + \beta_1 z_2 + \beta_2 z_1^2 + \mathcal{O}[\varepsilon, z_1 z_2, z_2^2, z_1^3].
\end{aligned} \tag{5.6}$$

We can directly compute the coefficients β_i , and we find that

$$\beta_0 = \beta_1 = 0, \quad \beta_2 = \frac{1}{c^2} \left(\sqrt{a^2 - a + 1} \right) = \frac{1}{c^2} + \mathcal{O}(a). \tag{5.7}$$

We now make the following change of coordinates

$$\begin{aligned}
x &= \frac{1}{c^{1/2}} \left(\sqrt{a^2 - a + 1} \right) z_1 \\
y &= - \left(\sqrt{a^2 - a + 1} \right) z_2 \\
\alpha &= \frac{a}{2c^{1/2}},
\end{aligned} \tag{5.8}$$

and rescale time by $c^{-1/2}$ which gives the flow on the center manifold in the coordinates $(x, y, \alpha, \varepsilon)$ as

$$\begin{aligned}
 \dot{x} &= -y(1 + \mathcal{O}(x, y, \alpha, \varepsilon)) + x^2(1 + \mathcal{O}(x, y, \alpha, \varepsilon)) + \varepsilon \mathcal{O}(x, y, \alpha, \varepsilon) \\
 \dot{y} &= \varepsilon [x(1 + \mathcal{O}(x, y, \alpha, \varepsilon)) + \alpha(1 + \mathcal{O}(x, y, \alpha, \varepsilon)) + \mathcal{O}(y)] \\
 \dot{\alpha} &= 0 \\
 \dot{\varepsilon} &= 0.
 \end{aligned} \tag{5.9}$$

Making one further coordinate transformation in the variable z_3 to straighten out the unstable fibers, we arrive at the full system

$$\begin{aligned}
 \dot{x} &= -y + x^2 + \mathcal{O}(\varepsilon, xy, y^2, x^3) \\
 \dot{y} &= \varepsilon [x(1 + \mathcal{O}(x, y, \alpha, \varepsilon)) + \alpha(1 + \mathcal{O}(x, y, \alpha, \varepsilon)) + \mathcal{O}(y)] \\
 \dot{z} &= z(c^{3/2} + \mathcal{O}(x, y, z, \varepsilon)) \\
 \dot{\alpha} &= 0 \\
 \dot{\varepsilon} &= 0.
 \end{aligned} \tag{5.10}$$

We note that the (x, y) coordinates are in the canonical form for a canard point (compare [38]), that is,

$$\begin{aligned}
 \dot{x} &= -yh_1(x, y, \alpha, \varepsilon, c) + x^2h_2(x, y, \alpha, \varepsilon, c) + \varepsilon h_3(x, y, \alpha, \varepsilon, c) \\
 \dot{y} &= \varepsilon (xh_4(x, y, \alpha, \varepsilon, c) + \alpha h_5(x, y, \alpha, \varepsilon, c) + yh_6(x, y, \alpha, \varepsilon, c)) \\
 \dot{z} &= z(c^{3/2} + \mathcal{O}(x, y, z, \varepsilon)) \\
 \dot{\alpha} &= 0 \\
 \dot{\varepsilon} &= 0,
 \end{aligned} \tag{5.11}$$

where we have

$$\begin{aligned} h_3(x, y, \alpha, \varepsilon, c) &= \mathcal{O}(x, y, \alpha, \varepsilon) \\ h_j(x, y, \alpha, \varepsilon, c) &= 1 + \mathcal{O}(x, y, \alpha, \varepsilon), \quad j = 1, 2, 4, 5. \end{aligned} \tag{5.12}$$

We have now separated the hyperbolic dynamics (given by the z -coordinate) from the nonhyperbolic dynamics which are isolated on a four-dimensional center manifold parameterized by the variables $(x, y, \varepsilon, \alpha)$ on which the origin is a canard point in the sense of [38]. Geometrically, in a singularly perturbed system a canard point is characterized by a folded critical manifold with one attracting and one repelling branch and a singular “canard” trajectory traveling down the attracting branch and continuing up the repelling branch (see Figure 3.9). Such points are associated with “canard explosion” phenomena in which small scale oscillations near the equilibrium undergo a rapid transformation in an exponentially small region in parameter space and emerge as large relaxation cycles [39]. We note that Figure 5.4 in §5.2 provides a visualization of what such a canard explosion looks like, though in this case the solutions depicted are homoclinic pulse solutions rather than periodic orbits; we refer to §5.2 for a more detailed discussion.

3.5.2 Tracking $M_\varepsilon^l(c, a)$ close to the canard point - blowup and rescaling

From Fenichel theory, we know that away from the canard point, the left branch $M_0^l(c, a)$ of the critical manifold perturbs to a slow manifold $M_\varepsilon^l(c, a)$ for small $\varepsilon > 0$ (see Figure 3.9). This slow manifold is unique up to errors exponentially small in $1/\varepsilon$; as the preceding analysis is valid for any such choice of $M_\varepsilon^l(c, a)$, we may now fix a choice of $M_\varepsilon^l(c, a)$ which lies in the center manifold $z = 0$. In addition, there

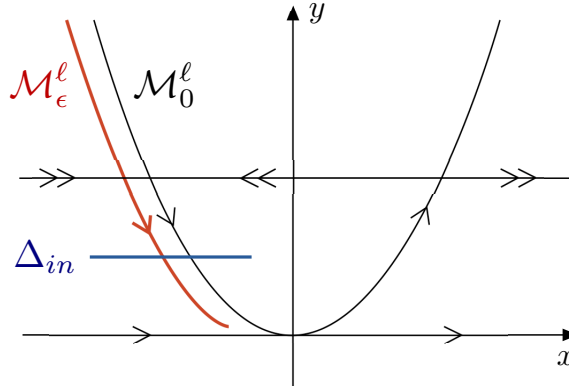


Figure 3.9: Shown is the flow near the canard point for $\varepsilon = \alpha = 0$. Away from the canard point, the left branch $M_0^l(c, a)$ of the critical manifold perturbs smoothly to a slow manifold $M_\varepsilon^l(c, a)$ for small $\varepsilon > 0$.

is a stable equilibrium $p_0 = p_0(\alpha)$ for the slow flow on $M_0^l(c, a)$ for $\alpha > 0$. The goal of this section is to show that under suitable conditions on ε and α , p_0 persists as a stable equilibrium $p_\varepsilon(\alpha)$, and the perturbed slow manifold $M_\varepsilon^l(c, a)$ and nearby trajectories converge to p_ε .

To do this, the idea will be to track a section $\Delta_{in}(\rho, \sigma) = \{(x, y) : |x + \rho| \leq \sigma\rho, y = \rho^2\}$ (see Figure 3.9) for some small $\rho, \sigma > 0$ and show that all trajectories crossing this section converge to the equilibrium. We have the following

Proposition 3.5.1. *Consider the section $\Delta_{in}(\rho, \sigma)$ for the system (5.11) in the center manifold $z = 0$. For each $K > 0$, there exists $\tilde{\rho}, \tilde{\sigma}, \tilde{\varepsilon}, \tilde{\alpha}$ such that for $(\rho, \sigma, \varepsilon, \alpha) \in D$ and $c \in I_c$, there is a stable equilibrium for (5.11), where D is given by*

$$D = \{(\rho, \sigma, \varepsilon, \alpha) : \rho \in (0, \tilde{\rho}), \sigma \in (0, \tilde{\sigma}), 0 < \varepsilon < \rho^2 \tilde{\varepsilon}, 0 < \alpha < \rho \tilde{\alpha}, 0 < \varepsilon < K \alpha^2\}. \quad (5.13)$$

Furthermore, under these conditions, all trajectories passing through $\Delta_{in}(\rho, \sigma)$ converge to the equilibrium.

From this we have the following

Corollary 3.5.2. *For each $K > 0$, there exists a choice of the parameters $\rho, \sigma, a_0, \varepsilon_0$ such that for all $(c, a, \varepsilon) \in I_c \times (0, a_0) \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$, the manifold $\mathcal{M}_\varepsilon^\ell(c, a)$ crosses the section $\Delta^{\text{in}}(\rho, \sigma)$ and thus converges to the equilibrium.*

Remark 3.5.3. *The aim of Corollary 3.5.2 is to prove convergence of the tails of the pulses constructed in §3.3 and §3.4. Thus far, we have found an intersection of $\mathcal{W}_\varepsilon^u(0; c, a)$ with $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$; this trajectory will therefore be exponentially attracted to the perturbed slow manifold $\mathcal{M}_\varepsilon^\ell(c, a)$ upon entering a neighborhood of the origin. The manifold $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$, however, is only unique up to errors exponentially small in $1/\varepsilon$, though the justification for the intersection holds for any such choice of $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$. Therefore, we may now fix $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ to be the manifold formed by evolving the section $\Delta^{\text{in}}(\rho, \sigma)$ in backwards time in the center manifold $z = 0$.*

We now fix an arbitrary $K > 0$. The section $\Delta_{\text{in}}(\rho, \sigma)$ will be tracked using blowup methods as in [38]. Restricting to the center manifold $z = 0$, we proceed as in §3.3, though now the blow up transformation is given by

$$x = \bar{r}\bar{x}, \quad y = \bar{r}^2\bar{y}, \quad \alpha = \bar{r}\bar{\alpha}, \quad \varepsilon = \bar{r}^2\bar{\varepsilon}, \quad (5.14)$$

defined on the manifold $B_c = S^2 \times [0, \bar{r}_0] \times [-\bar{\alpha}_0, \bar{\alpha}_0]$ for sufficiently small $\bar{r}_0, \bar{\alpha}_0$ with $(\bar{x}, \bar{y}, \bar{\varepsilon}) \in S^2$. There are three relevant coordinate charts which will be needed for the analysis of the flow on the manifold B_c . Keeping the same notation as in [38] and [39], the first is the chart $\mathcal{K}_1$ which uses the coordinates

$$x = r_1x_1, \quad y = r_1^2, \quad \alpha = r_1\alpha_1, \quad \varepsilon = r_1^2\varepsilon_1, \quad (5.15)$$

the second chart $\mathcal{K}_2$ uses the coordinates

$$x = r_2x_2, \quad y = r_2^2y_2, \quad \alpha = r_2\alpha_2, \quad \varepsilon = r_2^2, \quad (5.16)$$

and the third chart $\mathcal{K}_4$ uses the coordinates

$$x = r_4 x_4, \quad y = r_4^2 y_4, \quad \alpha = r_4, \quad \varepsilon = r_4^2 \varepsilon_4. \quad (5.17)$$

With these three sets of coordinates, a short calculation gives the following

Lemma 3.5.4. *The transition map $\kappa_{12} : K_1 \rightarrow K_2$ between the coordinates in $\mathcal{K}_1$ and $\mathcal{K}_2$ is given by*

$$x_2 = \frac{x_1}{\varepsilon_1^{1/2}}, \quad y_2 = \frac{1}{\varepsilon_1}, \quad \alpha_2 = \frac{\alpha_1}{\varepsilon_1^{1/2}}, \quad r_2 = r_1 \varepsilon_1^{1/2}, \quad \text{for } \varepsilon_1 > 0, \quad (5.18)$$

and the transition map $\kappa_{14} : K_1 \rightarrow K_4$ between the coordinates in $\mathcal{K}_1$ and $\mathcal{K}_4$ is given by

$$x_4 = \frac{x_1}{\alpha_1}, \quad y_4 = \frac{1}{\alpha_1^2}, \quad \varepsilon_4 = \frac{\varepsilon_1}{\alpha_1^2}, \quad r_4 = r_1 \alpha_1, \quad \text{for } \alpha_1 > 0. \quad (5.19)$$

3.5.3 Dynamics in $\mathcal{K}_1$

Here we outline the relevant dynamics in $\mathcal{K}_1$ as described in [38]. After desingularizing the equations in the new variables, we arrive at the following system

$$\begin{aligned} x_1' &= -1 + x_1^2 - \frac{1}{2} \varepsilon_1 x_1 F(x_1, r_1, \varepsilon_1, \alpha_1, c) + \mathcal{O}(r_1) \\ r_1' &= \frac{1}{2} r_1 \varepsilon_1 F(x_1, r_1, \varepsilon_1, \alpha_1, c) \\ \varepsilon_1' &= -\varepsilon_1^2 F(x_1, r_1, \varepsilon_1, \alpha_1, c) \\ \alpha_1' &= -\frac{1}{2} \alpha_1 \varepsilon_1 F(x_1, r_1, \varepsilon_1, \alpha_1, c), \end{aligned} \quad (5.20)$$

where

$$F(x_1, r_1, \varepsilon_1, \alpha_1, c) = x_1 + \alpha_1 + \mathcal{O}(r_1). \quad (5.21)$$

Here we collect a few results from [38]. The hyperplanes $r_1 = 0, \varepsilon_1 = 0, \alpha_1 = 0$ are all invariant. Their intersection is the invariant line $l_1 = \{(x_1, 0, 0, 0) : x_1 \in \mathbb{R}\}$, and the dynamics on l_1 evolve according to $x_1' = -1 + x_1^2$. There are two equilibria $p_a = (-1, 0, 0, 0)$ and $p_r = (1, 0, 0, 0)$. The equilibrium we are interested in, p_a has eigenvalue -2 for the flow along l_1 . There is a normally hyperbolic curve of equilibria $S_{0,1}^+(c)$ emanating from p_a which exactly corresponds to the manifold $\mathcal{M}_0^\ell(c, 0)$ in the original coordinates. Restricting attention to the set

$$D_1 = \{(x_1, r_1, \varepsilon_1) : -2 < x_1 < 2, 0 \leq r_1 \leq \rho, 0 \leq \varepsilon_1 \leq \tilde{\varepsilon}, -\tilde{\alpha} \leq \alpha_1 \leq \tilde{\alpha}\}, \quad (5.22)$$

we have the following result which will be useful in obtaining an expression for $\mathcal{M}_\varepsilon^\ell(c, a)$:

Proposition 3.5.5 ([38, Proposition 3.4]). *Consider the system (5.20). For any $c \in I_c$ and all sufficiently small $\rho, \tilde{\varepsilon}, \tilde{\alpha} > 0$, there exists a three-dimensional attracting center manifold $M_1^+(c)$ at p_a which contains the line of equilibria $S_{0,1}^+(c)$. In D_1 , $M_1^+(c)$ is given as a graph $x_1 = h_+(r_1, \varepsilon_1, \alpha_1, c) = -1 + \mathcal{O}(r_1, \varepsilon_1, \alpha_1)$.*

We now consider the following section

$$\Sigma_1^{\text{in}} := \{(x_1, r_1, \varepsilon_1, \alpha_1) : |1 + x_1| < \sigma, 0 < \varepsilon_1 \leq K\alpha_1^2, 0 < \alpha_1 \leq \tilde{\alpha}, r_1 = \rho\},$$

where $\sigma, \tilde{\alpha}, \rho$ will be chosen appropriately. It is clear that in the chart $\mathcal{K}_1$, $\Delta_{\text{in}}(\rho, \sigma)$ is contained in the section Σ_1^{in} for ε and α in the desired range, hence the goal will be to track the evolution of Σ_1^{in} . To accomplish this, we consider two subsets of Σ_1^{in}

defined by

$$\begin{aligned}\Sigma_{14}^{\text{in}} &:= \{(x_1, r_1, \varepsilon_1, \alpha_1) : |1 + x_1| < \sigma, \ 0 < \varepsilon_1 \leq 2\delta\alpha_1^2, \ 0 < \alpha_1 \leq \tilde{\alpha}, \ r_1 = \rho\} \\ \Sigma_{12}^{\text{in}} &:= \{(x_1, r_1, \varepsilon_1, \alpha_1) : |1 + x_1| < \sigma, \ \delta\alpha_1^2 < \varepsilon_1 \leq K\alpha_1^2, \ 0 < \varepsilon_1 \leq \tilde{\varepsilon}, \ r_1 = \rho\},\end{aligned}$$

where δ and $\tilde{\varepsilon} \ll \delta$ will be chosen appropriately small. The evolution of Σ_{14}^{in} and Σ_{12}^{in} will be governed by the dynamics of the charts $\mathcal{K}_4$ and $\mathcal{K}_2$, respectively.

We now consider the two exit sections defined by

$$\begin{aligned}\Sigma_{14}^{\text{out}} &:= \{(x_1, r_1, \varepsilon_1, \alpha_1) : |1 + x_1| < \sigma, \ 0 < \varepsilon_1 \leq 2\delta\tilde{\alpha}^2, \ 0 \leq r_1 \leq \rho, \ \alpha_1 = \tilde{\alpha}\} \\ \Sigma_{12}^{\text{out}} &:= \left\{ (x_1, r_1, \varepsilon_1, \alpha_1) : |1 + x_1| < \sigma, \ \frac{\tilde{\varepsilon}}{K} < \alpha_1^2 \leq \frac{\tilde{\varepsilon}}{\delta}, \ 0 \leq r_1 \leq \rho, \ \varepsilon_1 = \tilde{\varepsilon} \right\}.\end{aligned}$$

The following lemma describes the flow in the chart $\mathcal{K}_1$ for (5.20).

Lemma 3.5.6. *There exists $\tilde{\alpha} > 0$ and $k_1^* > 0$ such that the following hold for all $c \in I_c$:*

- (i) *For any $\rho, \sigma, \delta < k_1^*$, the flow maps Σ_{14}^{in} into Σ_{14}^{out} .*
- (ii) *Fix $\delta < k_1^*$. There exists $\tilde{\varepsilon}_1^* < \delta$ such that for any $\tilde{\varepsilon} < \tilde{\varepsilon}_1^*$ and any $\rho, \sigma < k_1^*$, the flow maps Σ_{12}^{in} into Σ_{12}^{out} .*

Thus once $\tilde{\alpha} > 0$ and $k_1^* > 0$ are fixed as in the above lemma, it is possible to define the transition map $\Pi_{14} : \Sigma_{14}^{\text{in}} \rightarrow \Sigma_{14}^{\text{out}}$ for any $\rho, \sigma, \delta < k_1^*$. Once $\delta < k_1^*$ is fixed, we may then also define the transition map $\Pi_{12} : \Sigma_{12}^{\text{in}} \rightarrow \Sigma_{12}^{\text{out}}$. Hence we first determine the evolution of $\Pi_{14}(\Sigma_{14}^{\text{in}}) \subseteq \Sigma_{14}^{\text{out}}$ in the chart $\mathcal{K}_4$ in order to choose δ appropriately. Then it will be possible to consider the evolution of $\Pi_{12}(\Sigma_{12}^{\text{in}}) \subseteq \Sigma_{12}^{\text{out}}$ in the chart $\mathcal{K}_2$.

3.5.4 Dynamics in $\mathcal{K}_4$

Fix $\tilde{\alpha}$ as in Lemma 3.5.6. We desingularize the equations in the new variables and arrive at the following system

$$\begin{aligned} x'_4 &= -y_4 + x_4^2 + \mathcal{O}(r_4) \\ y'_4 &= \varepsilon_4 (1 + x_4 + \mathcal{O}(r_4)) \\ r'_4 &= 0 \\ \varepsilon'_4 &= 0. \end{aligned} \tag{5.23}$$

We think of this system as a singularly perturbed system with two slow variables y_4 and r_4 , one fast variable x_4 , and singular perturbation parameter ε_4 .

It is possible to define a critical manifold $S_0(r_4)$ in each fixed r_4 slice for $r_4 \in [0, r_4^*]$ for some small r_4^* . At $r_4 = 0$ this critical manifold can be taken as any segment of the curve $y_4 = x_4^2$ for x_4 in any negative compact interval bounded away from 0, say for $x_4 \in [-x_4^\ell, -x_4^r]$ where we can take $x_4^\ell > 2/\tilde{\alpha}$ and $0 < x_4^r < 1/2$. For each fixed $r_4 \in [0, r_4^*]$, there is a similar critical manifold for the same range of x_4 . Define M_0 to be the union of the curves $S_0(r_4)$ over $r_4 \in [0, r_4^*]$. Then M_0 is a compact two-dimensional critical manifold for $\varepsilon_4 = 0$ for the full three-dimensional system. In addition, provided r_4^* is sufficiently small, for each fixed r_4 the slow flow on $S_0(r_4)$ has a stable equilibrium $p_0(r_4)$ with $p_0(0) = (-1, 1)$. Figure 3.10 shows the setup for $\varepsilon_4 = 0$.

In addition M_0 has a stable manifold $\mathcal{W}^s(M_0)$ consisting of the planes $y_4 = \text{const.}$

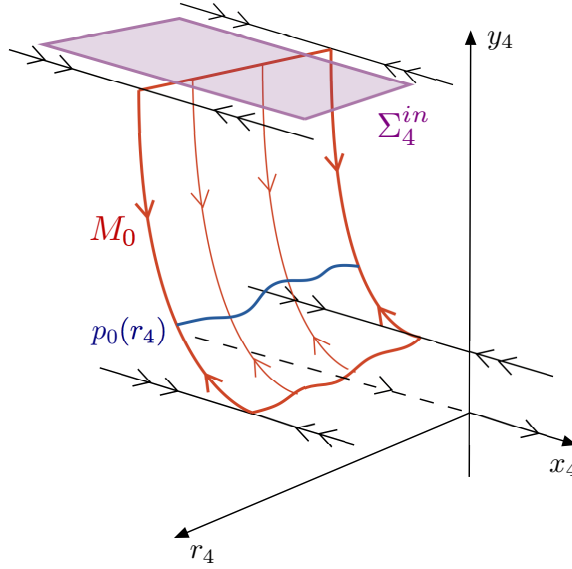


Figure 3.10: Shown is the critical manifold M_0 in chart $\mathcal{K}_4$ for $\varepsilon_4 = 0$. Also shown is the $\varepsilon_4 = 0$ curve of equilibria $p_0(r_4)$ for the slow flow on M_0 . The section Σ_4^{in} as in the proof of Proposition 3.5.1 is also shown.

In particular, we consider the subset of $\mathcal{W}^s(M_0)$ defined by

$$\begin{aligned} \mathcal{W}_0 = \{ (x_4, y_4, \varepsilon_4, r_4) : \\ x_4 \in [-x_4^\ell, -x_4^r], y_4 \in [(x_4^r)^2, (x_4^\ell)^2], \varepsilon_4 = 0, r_4 \in [0, r_4^*] \}. \end{aligned} \quad (5.24)$$

It follows from Fenichel theory that the critical manifold M_0 and its stable manifold $\mathcal{W}^s(M_0)$ perturb smoothly for small $\varepsilon_4 > 0$ to invariant manifolds M_{ε_4} and $\mathcal{W}^s(M_{\varepsilon_4})$. Further, the equilibria $p_0(r_4)$ persist as stable equilibria $p_{\varepsilon_4}(r_4)$, and in each fixed r_4 slice, all orbits lying on $\mathcal{W}^s(M_{\varepsilon_4})$ converge to $p_{\varepsilon_4}(r_4)$. The dynamics for $\varepsilon_4 > 0$ are shown in Figure 3.11. In particular, there exists $\tilde{\varepsilon}_4^* > 0$ such that for $0 < \varepsilon_4 < \tilde{\varepsilon}_4^*$, the set $\mathcal{W}_0$ perturbs to a set $\mathcal{W}_{\varepsilon_4}$, all points of which converge to $p_{\varepsilon_4}(r_4)$.

Using the transition map κ_{14} (5.19), we have in chart $\mathcal{K}_4$ that $\kappa_{14}(\Sigma_{14}^{\text{out}})$ is con-

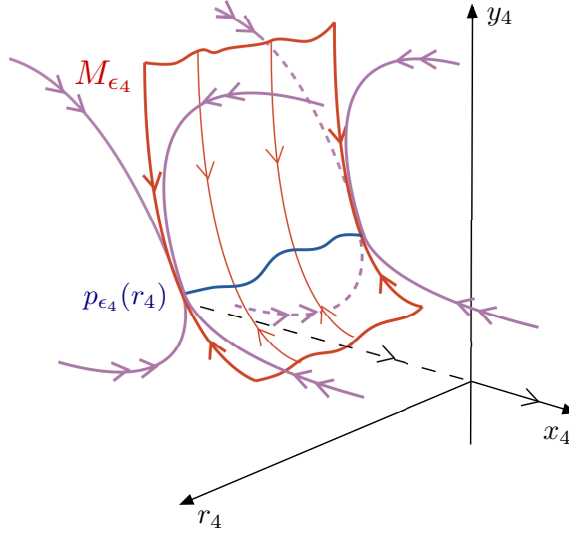


Figure 3.11: Shown is the perturbed slow manifold M_{ε_4} in chart $\mathcal{K}_4$ for $\varepsilon_4 > 0$. All trajectories on $\mathcal{W}^s(M_{\varepsilon_4})$ converge to $p_{\varepsilon_4}(r_4)$.

tained in the set

$$\Sigma_4^{\text{in}} = \left\{ (x_4, y_4, \varepsilon_4, r_4) : \left| \frac{1}{\tilde{\alpha}} + x_4 \right| < \frac{\sigma}{\tilde{\alpha}}, \right. \\ \left. 0 \leq \varepsilon_4 \leq 2\delta, \ 0 < r_4 \leq \rho\tilde{\alpha}, \ y_4 = \frac{1}{\tilde{\alpha}^2} \right\}, \quad (5.25)$$

which is shown in Figure 3.10 for $\varepsilon_4 = 0$.

We can now prove the following

Lemma 3.5.7. *There exists k_4^* such that for any $\rho, \delta < k_4^*$ and any $\sigma < 1/2$, there exists a curve of equilibria $p_{\varepsilon_4}(r_4)$ such that all trajectories crossing the section Σ_4^{in} converge to $p_{\varepsilon_4}(r_4)$.*

Proof. For any $\rho < r_4^*/\tilde{\alpha}$ and any $\sigma < 1/2$, the set $\Sigma_4^{\text{in}} \cap \{\varepsilon_4 = 0\}$ lies in $\mathcal{W}_0$. Thus by taking $\delta < \tilde{\varepsilon}_4^*/2$, we have that all trajectories passing through Σ_4^{in} converge to the unique equilibrium on the slow manifold $M_{\varepsilon_4}(r_4)$ for each $r_4 \in (0, \rho\tilde{\alpha})$. Thus taking $k_4^* < \min\left(\frac{r_4^*}{\tilde{\alpha}}, \frac{\tilde{\varepsilon}_4^*}{2}\right)$ proves the result. $\square$

3.5.5 Dynamics in $\mathcal{K}_2$

We now fix $\delta < \min(k_1^*, k_4^*)$ and desingularize the equations in the $\mathcal{K}_2$ coordinates to arrive at the following system

$$\begin{aligned} x_2' &= -y_2 + x_2^2 + \mathcal{O}(r_2) \\ y_2' &= x_2 + \alpha_2 + \mathcal{O}(r_2) \\ r_2' &= 0 \\ \alpha_2' &= 0. \end{aligned} \tag{5.26}$$

Making the change of variables $\tilde{x}_2 = x_2 + \alpha_2$ and $\tilde{y}_2 = y_2 - \alpha_2^2$, we arrive at the system

$$\begin{aligned} \tilde{x}_2' &= -\tilde{y}_2 + \tilde{x}_2^2 - 2\tilde{x}_2\alpha_2 + \mathcal{O}(r_2) \\ \tilde{y}_2' &= \tilde{x}_2 + \mathcal{O}(r_2) \\ r_2' &= 0 \\ \alpha_2' &= 0. \end{aligned} \tag{5.27}$$

For $r_2 = \alpha_2 = 0$, the system is integrable with constant of motion

$$H(\tilde{x}_2, \tilde{y}_2) = \frac{1}{2}e^{-2\tilde{y}_2} \left(\tilde{y}_2 - \tilde{x}_2^2 + \frac{1}{2} \right). \tag{5.28}$$

The function H has a continuous family of closed level curves

$$\Gamma^h = \{(\tilde{x}_2, \tilde{y}_2) : H(\tilde{x}_2, \tilde{y}_2) = h\}, \quad h \in (0, 1/4) \tag{5.29}$$

contained in the interior of the parabola $\tilde{y}_2 = \tilde{x}_2^2 - 1/2$, which is the level curve for $h = 0$ (see Figure 3.12).

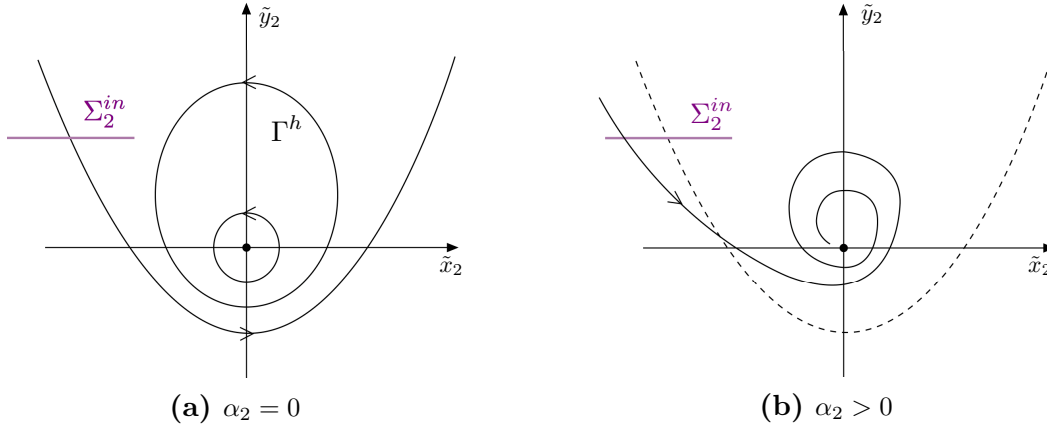


Figure 3.12: Shown are the dynamics in the chart $\mathcal{K}_2$ as well as the section Σ_2^{in} for the cases of $\alpha_2 = 0$ and $\alpha_2 > 0$. For $\alpha_2 = 0$, the orbits are given by the level curves Γ^h for the function $H(\tilde{x}_2, \tilde{y}_2)$.

For $r_2 = 0$, we have that

$$\frac{dH}{dt} = 2e^{-2\tilde{y}_2}\alpha_2\tilde{x}_2^2, \quad (5.30)$$

so that for positive α_2 , all trajectories in the interior of the parabola $\tilde{y}_2 = \tilde{x}_2^2 - 1/2$ converge to the unique equilibrium $(\tilde{x}_2, \tilde{y}_2) = (0, 0)$ corresponding to the maximum value $h = 1/4$. For sufficiently small r_2 , this equilibrium persists, and we denote it by $p_2(r_2)$ with $p_2(0) = (0, 0)$.

Using the transition map κ_{12} (5.18), we have in chart $\mathcal{K}_2$ that $\kappa_{12}(\Sigma_{12}^{\text{out}})$ is contained in the set

$$\Sigma_2^{\text{in}} = \left\{ (x_2, y_2, \alpha_2, r_2) : \left| \frac{1}{\tilde{\varepsilon}^{1/2}} + x_2 \right| < \frac{\sigma}{\tilde{\varepsilon}^{1/2}}, \right. \\ \left. \frac{1}{K^{1/2}} \leq \alpha_2 \leq \frac{1}{\delta^{1/2}}, \ 0 < r_2 \leq \rho^2 \tilde{\varepsilon}, \ y_2 = \frac{1}{\tilde{\varepsilon}} \right\}, \quad (5.31)$$

which in the coordinates $(\tilde{x}_2, \tilde{y}_2)$ is the set

$$\Sigma_2^{\text{in}} = \left\{ (\tilde{x}_2, \tilde{y}_2, \alpha_2, r_2) : \left| \frac{1}{\tilde{\varepsilon}^{1/2}} + \tilde{x}_2 - \alpha_2 \right| < \frac{\sigma}{\tilde{\varepsilon}^{1/2}} \right. \\ \left. \frac{1}{K^{1/2}} \leq \alpha_2 \leq \frac{1}{\delta^{1/2}}, \ 0 \leq r_2 \leq \rho^2 \tilde{\varepsilon}, \ \tilde{y}_2 = \frac{1}{\tilde{\varepsilon}} - \alpha_2^2 \right\}. \quad (5.32)$$

We assume that $\tilde{\varepsilon} < 1/K$ so that Σ_2^{in} lies in a region of positive $\tilde{y}_2$. We also define the set

$$\Sigma_{2,0}^{\text{in}} = \Sigma_2^{\text{in}} \cap \{r_2 = 0\}. \quad (5.33)$$

We can now prove the following

Lemma 3.5.8. *There exists $\tilde{\varepsilon}_2^* > 0$ such that the following holds. For each $\tilde{\varepsilon} < \tilde{\varepsilon}_2^*$, there exists $k_2^* > 0$ such that for $\rho < k_2^*$ and $\sigma < 1/2$, all trajectories crossing the section Σ_2^{in} converge to the equilibrium $p_2(r_2)$ of (5.26).*

Proof. It suffices to show that for σ small enough, all trajectories crossing $\Sigma_{2,0}^{\text{in}}$ eventually enter the interior of the parabola $\tilde{y}_2 = \tilde{x}_2^2 - 1/2$ when $r_2 = 0$ for any $\alpha_2 \in [1/K^{1/2}, 1/\delta^{1/2}]$. By a regular perturbation argument, this also holds for small $r_2 > 0$. Thus by taking ρ sufficiently small we can ensure all points in Σ_2^{in} converge to $p_2(r_2)$.

For $r_2 = 0$ and $\alpha_2 \in [1/K^{1/2}, 1/\delta^{1/2}]$, we consider the flow for points in the set $\Sigma_{2,0}^{\text{in}} \cap \{\tilde{x}_2^2 > \tilde{y}_2 + 1/2\}$ (as the other points already lie in the interior of the parabola $\tilde{y}_2 = \tilde{x}_2^2 - 1/2$). Note that all such points satisfy $\tilde{x}_2 < -1/\sqrt{2}$. In this region for any $\alpha_2 \in [1/K^{1/2}, 1/\delta^{1/2}]$, we have

$$\tilde{x}_2' > \frac{1}{2} + \left(\frac{1}{2K} \right)^{1/2}, \quad (5.34)$$

so that any orbit starting in Σ_2^{in} either reaches $\tilde{x}_2 = -1/\sqrt{2}$ in finite time or enters the interior of the parabola $\tilde{y}_2 = \tilde{x}_2^2 - 1/2$. The idea will be to show that all orbits starting in $\Sigma_{2,0}^{\text{in}}$ enter the interior of the parabola before reaching $x_0 = -1/\sqrt{2}$. For an orbit starting in $\Sigma_2^{\text{in}} \cap \{\tilde{x}_2^2 > \tilde{y}_2 + 1/2\}$ at $t = 0$ with $\tilde{x}_2(0) = x_0$, which reaches $\tilde{x}_2 = -1/\sqrt{2}$ at time $t = t_0$, the condition that this orbit has crossed $\tilde{y}_2 = \tilde{x}_2^2 - 1/2$ is satisfied if $\tilde{y}_2(t_0) > 0$. We have

$$\begin{aligned}
\tilde{y}_2(t_0) &= \tilde{y}_2(0) + \int_0^{t_0} \tilde{y}_2'(t) dt \\
&= \tilde{y}_2(0) + \int_0^{t_0} \tilde{x}_2(t) dt \\
&= \tilde{y}_2(0) + \int_0^{t_0} \tilde{x}_2(t) \frac{-\tilde{y}_2(t) + \tilde{x}_2^2(t) - 2\alpha_2\tilde{x}_2(t)}{-\tilde{y}_2(t) + \tilde{x}_2^2(t) - 2\alpha_2\tilde{x}_2(t)} dt \\
&> \tilde{y}_2(0) + \int_0^{t_0} -\frac{(-\tilde{y}_2(t) + \tilde{x}_2^2(t) - 2\alpha_2\tilde{x}_2(t))}{2\alpha_2} dt \\
&= \tilde{y}_2(0) + \int_{x_0}^{-1/\sqrt{2}} -\frac{1}{2\alpha_2} d\tilde{x}_2 \\
&= \frac{1}{\tilde{\varepsilon}} - \alpha_2^2 + \frac{1}{2\alpha_2} \left(x_0 + \frac{1}{\sqrt{2}} \right).
\end{aligned} \tag{5.35}$$

Thus the condition is satisfied if

$$\frac{1}{2\alpha_2} \left(x_0 + \frac{1}{\sqrt{2}} \right) + \frac{1}{\tilde{\varepsilon}} - \alpha_2^2 > 0 \tag{5.36}$$

for any initial condition x_0 of a trajectory in $\Sigma_2^{\text{in}} \cap \{\tilde{x}_2^2 > \tilde{y}_2 + 1/2\}$ and any $\alpha_2 \in [1/K^{1/2}, 1/\delta^{1/2}]$. In particular, this holds for any $\sigma < 1/2$ and any $\tilde{\varepsilon} < \min\left(\delta^3, \frac{1}{2\sqrt{K}}\right)$. Therefore we set $\tilde{\varepsilon}_2^* = \min\left(\delta^3, \frac{1}{2\sqrt{K}}\right)$.

Now fix any $\tilde{\varepsilon} < \tilde{\varepsilon}_2^*$. Then all points in $\Sigma_{2,0}^{\text{in}}$ converge to the equilibrium for $r_2 = 0$, and there exists r_2^* such that this continues to be true for $0 < r_2 < r_2^*$ and any $\sigma < 1/2$. Thus for any $\rho < (r_2^*/\tilde{\varepsilon})^{1/2}$, all points in Σ_2^{in} converge to the equilibrium. So we set $k_2^* = (r_2^*/\tilde{\varepsilon})^{1/2}$. $\square$

Proof of Proposition 3.5.1. To prove the main result, we just need to choose constants appropriately and identify $\Delta_{\text{in}}(\rho, \sigma)$ in the chart $\mathcal{K}_1$. We fix $\tilde{\alpha}$ and k_1^* as in Lemma 3.5.6. Then for $\rho, \sigma, \delta < k_1^*$ we have that $\Pi_{14}(\Sigma_{14}^{\text{in}}) \subseteq \Sigma_{14}^{\text{out}}$, and thus we may apply Lemma 3.5.7 from §3.5.4. Therefore for any $\rho, \delta < \min(k_1^*, k_4^*)$ and any $\sigma < \min(k_1^*, 1/2)$, all points in Σ_{14}^{in} converge to the equilibrium.

We now fix $\delta < \min(k_1^*, k_4^*)$. By Lemma 3.5.6 (ii) for any $\tilde{\varepsilon} < \tilde{\varepsilon}_1^*$ and any $\rho, \sigma < k_1^*$, we have that $\Pi_{12}(\Sigma_{12}^{\text{in}}) \subseteq \Sigma_{12}^{\text{out}}$ and we may apply Lemma 3.5.8 of §3.5.5. We fix $\tilde{\varepsilon} < \min(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*)$. Then Lemma 3.5.8 gives k_2^* such that for any $\rho < \min(k_1^*, k_2^*)$ and any $\sigma < \min(k_1^*, 1/2)$, all points in Σ_{12}^{in} converge to the equilibrium.

Taking $\tilde{\rho} < \min(k_1^*, k_2^*, k_4^*)$ and $\tilde{\sigma} < \min(k_1^*, 1/2)$, we have the following. For each $\rho < \tilde{\rho}$ and $\sigma < \tilde{\sigma}$, the union of $\Delta_{\text{in}}(\rho, \sigma)$ over $\alpha \in (0, \rho\tilde{\alpha})$, $\varepsilon \in (0, \rho^2\tilde{\varepsilon})$ and $0 < \varepsilon \leq K\alpha^2$ is contained in the union of the sections $\Sigma_{12}^{\text{in}} \cup \Sigma_{14}^{\text{in}}$, and we can apply Lemmas 3.5.7 and 3.5.8 as just described.

With these choices of $\tilde{\alpha}, \tilde{\varepsilon}, \tilde{\rho}, \tilde{\sigma}$, the result holds on all of D . $\square$

Proof of Corollary 3.5.2. Fix $K > 0$. Proposition 3.5.1 then gives $\tilde{\rho}, \tilde{\sigma}, \tilde{\varepsilon}, \tilde{\alpha}$ such that for all $(\rho, \sigma, \varepsilon, \alpha) \in D$, any trajectory crossing $\Delta^{\text{in}}(\rho, \sigma)$ converges to the equilibrium. We therefore need to show that the parameters can be chosen in such a way as to continue to satisfy Proposition 3.5.1 with $\mathcal{M}_\varepsilon^\ell(c, a)$ crossing $\Delta^{\text{in}}(\rho, \sigma)$. Using Proposition 3.5.5, we can obtain an expression for $\mathcal{M}_\varepsilon^\ell(c, a)$ at $y = r_1^2 = \rho^2$:

$$\begin{aligned} x = \rho x_1 &= -\rho + \mathcal{O}(\rho\alpha_1, \rho\varepsilon_1, \rho^2) \\ &= -\rho + \mathcal{O}\left(\alpha, \frac{\varepsilon}{\rho}, \rho^2\right). \end{aligned} \tag{5.37}$$

For $\mathcal{M}_\varepsilon^\ell(c, a)$ to hit $\Delta^{\text{in}}(\rho, \sigma)$, we need $|x + \rho| \leq \sigma\rho$. Provided $\alpha < \rho^2$ and $\varepsilon < \rho^3$,

we have that $\mathcal{M}_\varepsilon^\ell(c, a)$ reaches $y = \rho^2$ at

$$x = -\rho + \mathcal{O}(\rho^2). \quad (5.38)$$

So fix any $\sigma < \tilde{\sigma}$. Then for any sufficiently small ρ , we can ensure that $\mathcal{M}_\varepsilon^\ell(c, a)$ hits $\Delta^{in}(\rho, \sigma)$. Fix such a value of ρ . Now take $\varepsilon_0 = \min(\rho^3, \rho^2 \tilde{\varepsilon})$ and choose a_0 so that $a_0 c^{-1/2} < \min(\rho^2, \rho \tilde{\alpha})$ for all $c \in I_c$. Then the result follows from Proposition 3.5.1. □

CHAPTER FOUR

Stability of traveling pulses

4.1 Introduction

The goal of this chapter is to prove the stability of the pulses with oscillatory tails of Theorem 3.1 which were constructed in §3.

This chapter is organized as follows. The next section is devoted to an overview of our main results, including their precise statements which are contained in Theorems 4.2 and 4.4. In §4.3, we collect and prove pointwise estimates of the pulses in the limit $\varepsilon \rightarrow 0$ that will be crucial in our stability analysis, which will be carried out in §4.4 for the essential spectrum and in §4.5 for the point spectrum of the linearization about the pulses: these results are then collected in §4.6 to prove Theorems 4.2 and 4.4 and conclude stability. We illustrate our results with numerical simulations in §4.7.

4.2 Overview of main results

We consider the FitzHugh-Nagumo system

$$\begin{aligned} u_t &= u_{xx} + f(u) - w, \\ w_t &= \varepsilon(u - \gamma w), \end{aligned} \tag{2.1}$$

where $f(u) = f(u; a) = u(u - a)(1 - u)$, $0 < a < \frac{1}{2}$ and $0 < \varepsilon \ll 1$. Moreover, we take $0 < \gamma < 4$ such that (2.1) has a single equilibrium rest state $(u, w) = (0, 0)$. Using geometric singular perturbation theory [18] and the exchange lemma [33] one can construct traveling-pulse solutions to (2.1):

Theorem 4.1 ([8, 34]). *There exists $K^* > 0$ such that for each $\kappa > 0$ and $K > K^*$*

the following holds. There exists $\varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in [0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$ system (2.1) admits a traveling-pulse solution $\hat{\phi}_{a,\varepsilon}(x, t) := \tilde{\phi}_{a,\varepsilon}(x + \check{c}t)$ with wave speed $\check{c} = \check{c}(a, \varepsilon)$ approximated (a -uniformly) by

$$\check{c} = \sqrt{2} \left(\frac{1}{2} - a \right) + \mathcal{O}(\varepsilon).$$

Furthermore, if we have in addition $\varepsilon > K^*a^2$, then the tail of the pulse is oscillatory.

This theorem encompasses two different existence results: the well known classical existence result [34] in the region where $0 < \varepsilon \ll a < \frac{1}{2}$, and the extension [8] to the regime $0 < a, \varepsilon \ll 1$, where the onset of oscillations in the tails of the pulses is observed. In the following, we refer to these two regimes as the hyperbolic and nonhyperbolic regimes, respectively, due to the use of (non)-hyperbolic geometric singular perturbation theory in the respective existence proofs.

Remark 4.2.1. *The singular perturbation parameter ε from §3 has been rescaled by $\varepsilon \rightarrow \check{\varepsilon}$. For the purposes of the forthcoming stability analysis, it is more convenient to work with this rescaled parameter, though we emphasize that the results do not depend on this distinction due to the fact that $\check{c}$ can be bounded below by a positive constant uniformly in (a, ε) .*

In the co-moving frame $\xi = x + \check{c}t$, the solution $\tilde{\phi}_{a,\varepsilon}(\xi) = (u_{a,\varepsilon}(\xi), w_{a,\varepsilon}(\xi))$ is a stationary solution to

$$\begin{aligned} u_t &= u_{\xi\xi} - \check{c}u_\xi + f(u) - w, \\ w_t &= -\check{c}w_\xi + \varepsilon(u - \gamma w). \end{aligned} \tag{2.2}$$

We are interested in the stability of the traveling pulse $\hat{\phi}_{a,\varepsilon}(x, t)$ as solution to (2.1) or equivalently the stability of $\tilde{\phi}_{a,\varepsilon}(\xi)$ as solution to (2.2). Linearizing (2.2) about

$\tilde{\phi}_{a,\varepsilon}(\xi)$ yields a linear differential operator $\mathcal{L}_{a,\varepsilon}$ on $C_{ub}(\mathbb{R}, \mathbb{R}^2)$ given by

$$\mathcal{L}_{a,\varepsilon} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u_{\xi\xi} - \check{c}u_{\xi} + f'(u_{a,\varepsilon}(\xi))u - w \\ -\check{c}w_{\xi} + \varepsilon(u - \gamma w) \end{pmatrix}.$$

The stability of the pulse is determined by the spectrum of $\mathcal{L}_{a,\varepsilon}$, i.e. the values $\lambda \in \mathbb{C}$ for which the operator $\mathcal{L}_{a,\varepsilon} - \lambda$ is not invertible. The associated eigenvalue problem $\mathcal{L}_{a,\varepsilon}\psi = \lambda\psi$ can be written as the ODE

$$\begin{aligned} \psi_{\xi} &= A_0(\xi, \lambda)\psi, \\ A_0(\xi, \lambda) &= A_0(\xi, \lambda; a, \varepsilon) := \begin{pmatrix} 0 & 1 & 0 \\ \lambda - f'(u_{a,\varepsilon}(\xi)) & \check{c} & 1 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda + \varepsilon\gamma}{\check{c}} \end{pmatrix}. \end{aligned} \quad (2.3)$$

Invertibility of $\mathcal{L}_{a,\varepsilon} - \lambda$ can fail in two ways [49]: either the asymptotic matrix

$$\hat{A}_0(\lambda) = \hat{A}_0(\lambda; a, \varepsilon) := \begin{pmatrix} 0 & 1 & 0 \\ \lambda + a & \check{c} & 1 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda + \varepsilon\gamma}{\check{c}} \end{pmatrix},$$

of system (2.3) is nonhyperbolic (λ is in the essential spectrum), or there exists a nontrivial exponentially localized solution to (2.3) (λ is in the point spectrum). In the latter case we call λ an eigenvalue of $\mathcal{L}_{a,\varepsilon}$ or of (2.3). The spaces of exponentially localized solutions to $(\mathcal{L}_{a,\varepsilon} - \lambda)\psi = 0$ or to (2.3) are referred to as eigenspaces and its nontrivial elements are called eigenfunctions. This brings us to our main result.

Theorem 4.2. *There exists $b_0, \varepsilon_0 > 0$ such that the following holds. In the setting of Theorem 4.1, let $\tilde{\phi}_{a,\varepsilon}(\xi)$ denote a traveling-pulse solution to (2.2) for $0 < \varepsilon < \varepsilon_0$*

with associated linear operator $\mathcal{L}_{a,\varepsilon}$. The spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in

$$\{0\} \cup \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq -\varepsilon b_0\}.$$

More precisely, the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the half plane $\operatorname{Re}(\lambda) \leq -\varepsilon\gamma$. The point spectrum of $\mathcal{L}_{a,\varepsilon}$ to the right hand side of the essential spectrum consists of the simple translational eigenvalue $\lambda_0 = 0$ and at most one other real eigenvalue $\lambda_1 = \lambda_1(a, \varepsilon) < 0$.

Theorem 4.2 will be proved in §4.6. Combining Theorem 4.2 with [15] and [16, Theorem 2] yields nonlinear stability of the traveling pulse $\tilde{\phi}_{a,\varepsilon}(\xi)$.

Theorem 4.3. *In the setting of Theorem 4.2, the traveling pulse $\tilde{\phi}_{a,\varepsilon}(\xi)$ is nonlinearly stable in the following sense. There exists $d > 0$ such that, if $\phi(\xi, t)$ is a solution to (2.2) satisfying $\|\phi(\xi, 0) - \tilde{\phi}_{a,\varepsilon}(\xi)\| \leq d$, then there exists $\xi_0 \in \mathbb{R}$ such that $\|\phi(\xi + \xi_0, t) - \tilde{\phi}_{a,\varepsilon}(\xi)\| \rightarrow 0$ as $t \rightarrow \infty$.*

In specific cases we have more information about the critical eigenvalue λ_1 of $\mathcal{L}_{a,\varepsilon}$. In the hyperbolic regime, where a is bounded below by an ε -independent constant $a_0 > 0$, the nontrivial eigenvalue λ_1 can be approximated explicitly to leading order $\mathcal{O}(\varepsilon)$. In the nonhyperbolic regime we have $0 < a, \varepsilon \ll 1$; if we restrict ourselves to a wedge $K_0 a^3 < \varepsilon < K a^2$, then the second eigenvalue λ_1 can be approximated to leading order $\mathcal{O}(\varepsilon^{2/3})$ by an a -independent expression in terms of Bessel functions. Thus, regarding the potential other eigenvalue λ_1 we have the following result.

Theorem 4.4. *In the setting of Theorem 4.2, we have the following:*

- (i) (Hyperbolic regime) For each $a_0 > 0$ there exists $\varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in [a_0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ the potential eigenvalue $\lambda_1 < 0$ of $\mathcal{L}_{a,\varepsilon}$ is approximated

(a -uniformly) by

$$\lambda_1 = -M_1\varepsilon + \mathcal{O}(|\varepsilon \log \varepsilon|^2),$$

where $M_1 = M_1(a) > 0$ can be determined explicitly; see (6.1). If the condition $M_1 < \gamma + a^{-1}$ is satisfied, then λ_1 is contained in the point spectrum of $\mathcal{L}_{a,\varepsilon}$ and lies to the right hand side of the essential spectrum.

(ii) (Non-hyperbolic regime) There exists $\varepsilon_0 > 0$ and $K_0, k_0 > 1$ such that, if $(a, \varepsilon) \in (0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfies $K_0 a^3 < \varepsilon$, then the eigenvalue $\lambda_1 < 0$ of $\mathcal{L}_{a,\varepsilon}$ lies to the right hand side of the essential spectrum and satisfies

$$\varepsilon^{2/3}/k_0 < \lambda_1 < k_0 \varepsilon^{2/3}.$$

In particular, if $(a, \varepsilon) \in (0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfies $K_0 a^3 < \varepsilon^{1+\alpha}$ for some $\alpha > 0$, then λ_1 is approximated (a - and α -uniformly) by

$$\lambda_1 = -\frac{(18 - 4\gamma)^{2/3} \zeta_0}{3} \varepsilon^{2/3} + \mathcal{O}(\varepsilon^{(2+\alpha)/3}), \quad (2.4)$$

where $\zeta_0 \in \mathbb{R}$ is the smallest positive solution to the equation

$$J_{-2/3}\left(\frac{2}{3}\zeta^{3/2}\right) = J_{2/3}\left(\frac{2}{3}\zeta^{3/2}\right),$$

where J_r denote Bessel functions of the first kind.

The regions in (c, a, ε) -parameter space considered in Theorems 4.1 and 4.4 are shown in Figure 4.1. We emphasize that Theorem 4.4 (ii) covers the regime $\varepsilon > K^* a^2$ of oscillatory tails. Theorem 4.4 will be proved in §4.6.

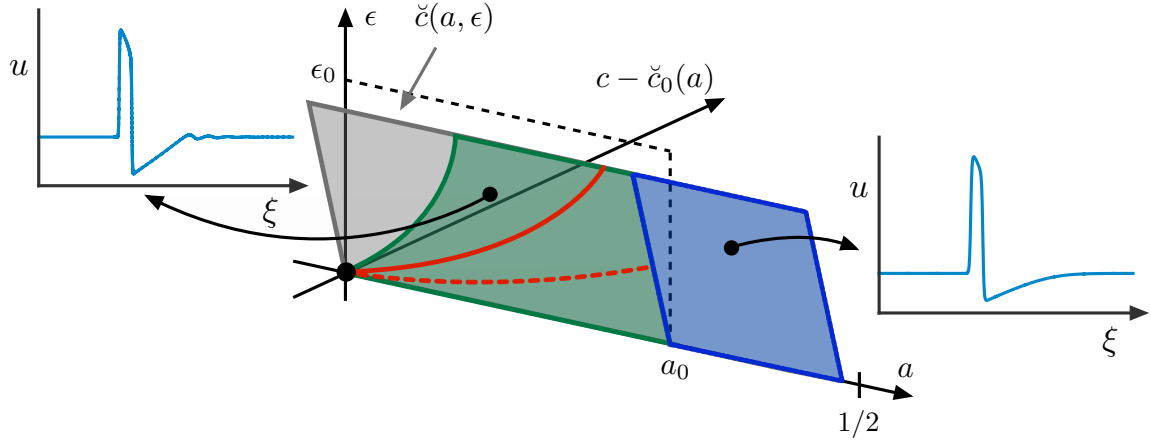


Figure 4.1: Shown is a schematic bifurcation diagram of the regions in (c, a, ε) -parameter space considered in Theorems 4.1 and 4.4. The green surface denotes the region of existence of pulses in the nonhyperbolic regime, and the blue surface represents the hyperbolic regime. The solid red curve $\varepsilon = K^* a^2$ represents the transition from monotone to oscillatory behavior in the tails of the pulses. The dashed red curve denotes $\varepsilon = K_0 a^3$; the region above this curve gives the parameter values for which the results of Theorem 4.4 (ii) are valid.

4.3 Pointwise approximation of pulse solutions

The traveling-pulse solutions in Theorem 4.1 arise from a concatenation of solutions to a series of reduced systems in the singular limit $\varepsilon \rightarrow 0$. It is essential for the forthcoming stability analysis to determine in what sense the pulse solutions are approximated by the singular limit structure. This can be understood best in the setting of the traveling-wave ODE

$$\begin{aligned} u_\xi &= v, \\ v_\xi &= cv - f(u) + w, \\ w_\xi &= \frac{\varepsilon}{c}(u - \gamma w), \end{aligned} \tag{3.1}$$

which is obtained from (2.1) by substituting the Ansatz $(u, w)(x, t) = (u, w)(x + ct)$ for wave speed $c > 0$ and putting $\xi = x + ct$. We consider a pulse solution $\tilde{\phi}_{a,\varepsilon}(\xi) = (u_{a,\varepsilon}(\xi), w_{a,\varepsilon}(\xi))$ as in Theorem 4.1. Equivalently, $\phi_{a,\varepsilon}(\xi) = (u_{a,\varepsilon}(\xi), u'_{a,\varepsilon}(\xi), w_{a,\varepsilon}(\xi))$ is a solution to (3.1) homoclinic to $(u, v, w) = (0, 0, 0)$ with wave speed $c = \check{c}(a, \varepsilon)$.

The singular limit $\phi_{a,0}$ of $\phi_{a,\varepsilon}$ can be understood via the fast/slow decomposition of the traveling-wave ODE (3.1). Our main result of this section, Theorem 4.5, provides pointwise estimates describing the closeness of $\phi_{a,0}$ and $\phi_{a,\varepsilon}$ in $\mathbb{R}^3$. We begin by defining the singular limit $\phi_{a,0}$ and stating Theorem 4.5, followed by an overview of the existence analysis in both the hyperbolic and nonhyperbolic regimes, and finally the proof of Theorem 4.5.

4.3.1 Singular limit

We separately consider (3.1), which we call the fast system, and the system below obtained by rescaling $\hat{\xi} = \varepsilon\xi$, which we call the slow system

$$\begin{aligned}\varepsilon u_{\hat{\xi}} &= v, \\ \varepsilon v_{\hat{\xi}} &= cv - f(u) + w, \\ w_{\hat{\xi}} &= \frac{1}{c}(u - \gamma w).\end{aligned}\tag{3.2}$$

Note that (3.1) and (3.2) are equivalent for any $\varepsilon > 0$. Taking the singular limit $\varepsilon \rightarrow 0$ in each of (3.1) and (3.2) results in simpler lower dimensional systems from which enough information can be obtained to determine the behavior in the full system for $0 < \varepsilon \ll 1$. We first set $\varepsilon = 0$ in (3.1) and obtain the layer problem

$$\begin{aligned}u_{\xi} &= v, \\ v_{\xi} &= cv - f(u) + w, \\ w_{\xi} &= 0,\end{aligned}\tag{3.3}$$

so that w becomes a parameter for the flow, and the manifold

$$\mathcal{M}_0 := \{(u, v, w) \in \mathbb{R}^3 : v = 0, w = f(u)\},$$

defines a set of equilibria. Considering this layer problem in the plane $w = 0$ and for $c = \check{c}_0(a) = \sqrt{2}(\frac{1}{2} - a)$, we obtain the Nagumo system

$$\begin{aligned} u_\xi &= v, \\ v_\xi &= \check{c}_0 v - f(u). \end{aligned} \tag{3.4}$$

For each $0 \leq a \leq 1/2$, this system possesses a heteroclinic front solution $\phi_f(\xi) = (u_f(\xi), v_f(\xi))$ which connects the equilibria $p_f^0 = (0, 0)$ and $p_f^1 = (1, 0)$. In (3.3) this manifests as a connection in the plane $w = 0$ between the left and right branches of $\mathcal{M}_0$, when the wave speed c equals $\check{c}_0$. In addition, there exists a heteroclinic solution $\phi_b(\xi) = (u_b(\xi), v_b(\xi))$ (the Nagumo back) to the system

$$\begin{aligned} u_\xi &= v, \\ v_\xi &= \check{c}_0 v - f(u) + w_b^1, \end{aligned} \tag{3.5}$$

which connects the equilibria $p_b^1 = (u_b^1, 0)$ and $p_b^0 = (u_b^0, 0)$, where $u_b^0 = \frac{1}{3}(2a - 1)$ and $u_b^1 = \frac{2}{3}(1 + a)$ satisfy $f(u_b^0) = f(u_b^1) = w_b^1$. Thus, for the same wave speed $c = \check{c}_0$ there exists a connection between the left and right branches of $\mathcal{M}_0$ in system (3.3) in the plane $w = w_b^1$.

Remark 4.3.1. *The front $\phi_f(\xi)$ can be determined explicitly by substituting the Ansatz $v = bu(u - 1)$, $b \in \mathbb{R}$ in the Nagumo equations (3.4). Subsequently, the back $\phi_b(\xi)$ is established by using the symmetry of $f(u)$ about its inflection point.*

We obtain

$$\begin{aligned}\phi_f(\xi) &= \begin{pmatrix} u_\diamond(\xi + \xi_{f,0}) \\ u'_\diamond(\xi + \xi_{f,0}) \end{pmatrix}, \\ \phi_b(\xi) &= \begin{pmatrix} \frac{2}{3}(1+a) - u_\diamond(\xi + \xi_{b,0}) \\ -u'_\diamond(\xi + \xi_{b,0}) \end{pmatrix},\end{aligned}\tag{3.6}$$

with

$$u_\diamond(\xi) := \frac{1}{e^{-\frac{1}{2}\sqrt{2}\xi} + 1},\tag{3.7}$$

where $\xi_{b,0}, \xi_{f,0} \in \mathbb{R}$ depends on the initial translation. We emphasize that we do not use the explicit expressions in (3.6) to prove our main stability result Theorem 4.2. However, they are useful to evaluate the leading order expressions for the second eigenvalue close to 0; see Theorem 4.4. Here we make use of the explicit formulas above with $\xi_{b,0}, \xi_{f,0} = 0$, but we could have made any choice of initial translate.

We note that for any $0 < a < 1/2$ the heteroclitic orbits ϕ_f and ϕ_b connect equilibria which lie on normally hyperbolic segments of the right and left branches of $\mathcal{M}_0$ given by

$$\begin{aligned}\mathcal{M}_0^r &:= \{(u, 0, f(u)) : u \in [u_b^1, 1]\}, \\ \mathcal{M}_0^\ell &:= \{(u, 0, f(u)) : u \in [u_b^0, 0]\},\end{aligned}\tag{3.8}$$

respectively. However, for $a = 0$, ϕ_f and ϕ_b leave precisely at the fold points on the critical manifold where normal hyperbolicity is lost (see Figure 4.2). This determines the distinction in the singular structure between the hyperbolic and nonhyperbolic cases. Furthermore, we note that for $a = 1/2$, ϕ_f and ϕ_b form a heteroclinic loop, but we do not consider this case in this work; see [37].

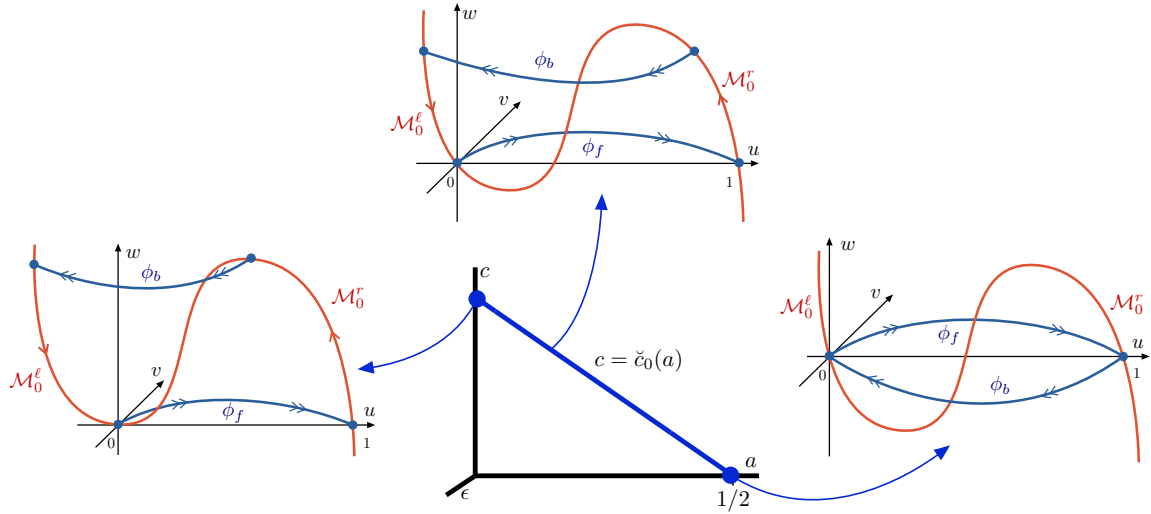


Figure 4.2: Shown is the singular pulse for $\varepsilon = 0$ in the nonhyperbolic regime (left), the hyperbolic regime (center), and the heteroclinic loop case [37] (right).

We now set $\varepsilon = 0$ in (3.2) and obtain the reduced problem

$$\begin{aligned} 0 &= v, \\ 0 &= cv - f(u) + w, \\ w_\xi &= \frac{1}{c}(u - \gamma w), \end{aligned}$$

where the flow is now restricted to the set $\mathcal{M}_0$ and the dynamics are determined by the equation for w . Putting together the information from the layer problem and reduced problem, there is for $c = \check{c}_0$ a singular homoclinic orbit $\phi_{a,0}$ obtained by following ϕ_f , then up $\mathcal{M}_0^r$, back across ϕ_b , then down $\mathcal{M}_0^\ell$; see Figure 4.2. Thus, we define $\phi_{a,0}$ as the singular concatenation

$$\phi_{a,0} := \{(\phi_f(\xi), 0) : \xi \in \mathbb{R}\} \cup \{(\phi_b(\xi), w_b^1) : \xi \in \mathbb{R}\} \cup \mathcal{M}_0^r \cup \mathcal{M}_0^\ell, \quad (3.9)$$

where $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$ are defined in (3.8). Note that $\phi_{a,0}$ exists purely as a formal object as the two subsystems are not equivalent to (3.1) for $\varepsilon = 0$.

4.3.2 Main approximation result

In the stability analysis we need to approximate the pulse $\phi_{a,\varepsilon}$ pointwise by its singular limit $\phi_{a,0}$. More specifically, we will cover the real line by four intervals J_f, J_r, J_b and J_ℓ . For ξ -values in J_r or J_ℓ the pulse $\phi_{a,\varepsilon}(\xi)$ is close to the right or left branches $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$ of the slow manifold $\mathcal{M}_0$, respectively. For ξ -values in J_f or J_b the pulse $\phi_{a,\varepsilon}(\xi)$ is approximated by (some translate of) the front $(\phi_f(\xi), 0)$ or back $(\phi_b(\xi), w_b^1)$, respectively.

To determine suitable endpoints of the intervals J_f and J_b we need to find $\xi \in \mathbb{R}$ such that $\phi_{a,\varepsilon}(\xi)$ can be approximated by one of the four non-smooth corners of the concatenation $\phi_{a,0}$; see Figure 4.3. By translational invariance, we can define the $\varepsilon \rightarrow 0$ limit of $\phi_{a,\varepsilon}(0)$ to be $(\phi_f(0), 0)$. Intuitively, one expects that, since the dynamics on the slow manifold is of the order $\mathcal{O}(\varepsilon)$, a point $\phi_{a,\varepsilon}(\Xi(\varepsilon))$ converges to the lower-right corner of $\phi_{a,0}$ as long as $\Xi(\varepsilon) \rightarrow \infty$ and $\varepsilon\Xi(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$; see also Theorem 4.8. This motivates to choose the upper endpoint of J_f to be an a - and ε -independent multiple of $-\log \varepsilon$. In a similar fashion one can determine endpoints for J_b .

We establish the following pointwise estimates for the traveling pulse $\phi_{a,\varepsilon}(\xi)$ along the front and back and along the right and left branches of the slow manifold.

Theorem 4.5. *For each sufficiently small $a_0, \sigma_0 > 0$ and each $\tau > 0$, there exists $\varepsilon_0 > 0$ and $C > 1$ such that the following holds. Let $\phi_{a,\varepsilon}(\xi)$ be a traveling-pulse solution as in Theorem 4.1 for $0 < \varepsilon < \varepsilon_0$, and define $\Xi_\tau(\varepsilon) := -\tau \log \varepsilon$. There exist $\xi_0, Z_{a,\varepsilon} > 0$ with ξ_0 independent of a and ε and $1/C \leq \varepsilon Z_{a,\varepsilon} \leq C$ such that:*

(i) For $\xi \in J_f := (-\infty, \Xi_\tau(\varepsilon)]$, $\phi_{a,\varepsilon}(\xi)$ is approximated by the front with

$$\left| \phi_{a,\varepsilon}(\xi) - \begin{pmatrix} \phi_f(\xi) \\ 0 \end{pmatrix} \right| \leq C\varepsilon\Xi_\tau(\varepsilon).$$

(ii) For $\xi \in J_b := [Z_{a,\varepsilon} - \Xi_\tau(\varepsilon), Z_{a,\varepsilon} + \Xi_\tau(\varepsilon)]$, $\phi_{a,\varepsilon}(\xi)$ is approximated by the back with

$$\left| \phi_{a,\varepsilon}(\xi) - \begin{pmatrix} \phi_b(\xi - Z_{a,\varepsilon}) \\ w_b^1 \end{pmatrix} \right| \leq C \begin{cases} \varepsilon\Xi_\tau(\varepsilon), & \text{if } a \geq a_0, \\ \varepsilon^{2/3}\Xi_\tau(\varepsilon), & \text{if } a < a_0. \end{cases}$$

(iii) For $\xi \in J_r := [\xi_0, Z_{a,\varepsilon} - \xi_0]$, $\phi_{a,\varepsilon}(\xi)$ is approximated by the right slow manifold $\mathcal{M}_0^r$ with

$$d(\phi_{a,\varepsilon}(\xi), \mathcal{M}_0^r) \leq \sigma_0.$$

(iv) For $\xi \in J_\ell := [Z_{a,\varepsilon} + \xi_0, \infty)$, $\phi_{a,\varepsilon}(\xi)$ is approximated by the left slow manifold $\mathcal{M}_0^\ell$ with

$$d(\phi_{a,\varepsilon}(\xi), \mathcal{M}_0^\ell) \leq \sigma_0.$$

As an immediate corollary, we obtain

Corollary 4.3.2. *For each sufficiently small $\sigma_0 > 0$, there exists $\varepsilon_0 > 0$ such that the following holds. Let $\phi_{a,\varepsilon}$ denote a pulse solution to (3.1) in the setting of Theorem 4.1 with $0 < \varepsilon < \varepsilon_0$. The Hausdorff distance between $\phi_{a,\varepsilon}$ and $\phi_{a,0}$ as geometric objects in $\mathbb{R}^3$ is smaller than σ_0 .*

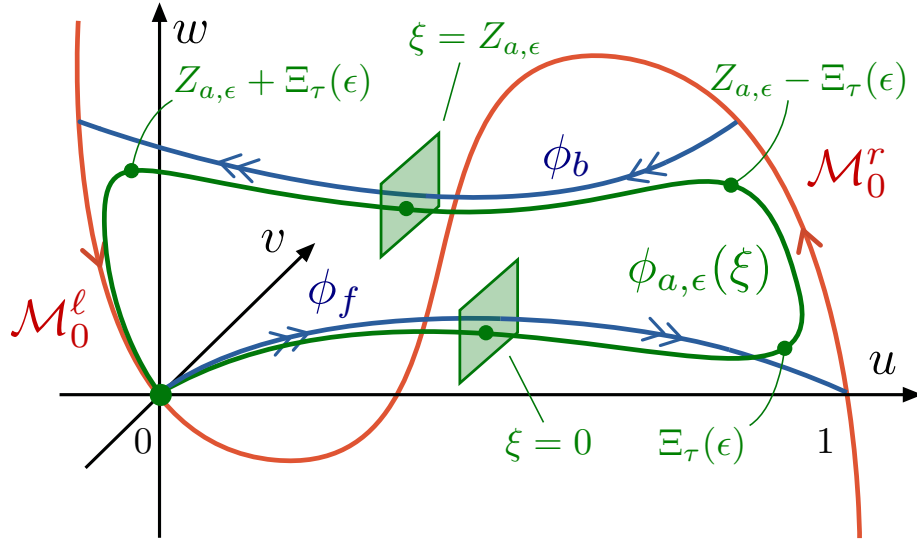


Figure 4.3: Shown is the pulse solution $\phi_{a,\epsilon}(\xi)$ along with the singular Nagumo front ϕ_f and back ϕ_b . The points $\phi_{a,\epsilon}(\Xi_\tau(\epsilon))$ and $\phi_{a,\epsilon}(Z_{a,\epsilon} \pm \Xi_\tau(\epsilon))$ approximate the intersection points of ϕ_f and ϕ_b with $\mathcal{M}_0^\ell$ and $\mathcal{M}_0^r$.

4.3.3 Overview of existence results

In this section, we give an overview of the existence results for the pulses considered in this chapter which are necessary in proving Theorem 4.5.

Theorem 4.1 combines the classical existence result for fast pulses as well as an extension to the regime of pulses with oscillatory tails proved in [8]. We begin by introducing the classical existence result and its proof in the context of geometric singular perturbation theory and then proceed by describing how to overcome the difficulties encountered in the case $0 < a, \epsilon \ll 1$. We refer to these cases as the hyperbolic and nonhyperbolic regimes, respectively.

Hyperbolic regime

The classical result is stated as follows

Theorem 4.6. *For each $0 < a < 1/2$, there exists $\varepsilon_0 = \varepsilon_0(a) > 0$ such that for $0 < \varepsilon < \varepsilon_0$ system (2.1) admits a traveling-pulse solution with wave speed $\check{c} = \check{c}(a, \varepsilon)$ satisfying*

$$\check{c}(a, \varepsilon) = \sqrt{2} \left(\frac{1}{2} - a \right) + \mathcal{O}(\varepsilon).$$

The above result is well known and has been obtained using a variety of methods including classical singular perturbation theory [26] and the Conley index [6]. We describe a proof of this result similar to that in [34], using geometric singular perturbation theory [18] and the exchange lemma [33].

It is possible to construct a pulse for $\varepsilon > 0$ as a perturbation of the singular structure $\phi_{a,0}$ given by (3.9) as follows. By Fenichel theory the segments $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$ persist for $\varepsilon > 0$ as locally invariant manifolds $\mathcal{M}_\varepsilon^r$ and $\mathcal{M}_\varepsilon^\ell$. In addition, the manifolds $\mathcal{W}^s(\mathcal{M}_0^r)$ and $\mathcal{W}^u(\mathcal{M}_0^r)$ defined as the union of the stable and unstable fibers, respectively, of $\mathcal{M}_0^r$ persist as locally invariant manifolds $\mathcal{W}_\varepsilon^{s,r}$ and $\mathcal{W}_\varepsilon^{u,r}$. Similarly the stable and unstable foliations of $\mathcal{M}_0^\ell$ persist as locally invariant manifolds $\mathcal{W}_\varepsilon^{s,\ell}$ and $\mathcal{W}_\varepsilon^{u,\ell}$. By Fenichel fibering the manifold $\mathcal{W}_\varepsilon^{s,\ell}$ coincides with $\mathcal{W}_\varepsilon^s(0)$, the stable manifold of the origin. The origin also has a one-dimensional unstable manifold $\mathcal{W}_0^u(0)$ which persists for $\varepsilon > 0$ as $\mathcal{W}_\varepsilon^u(0)$. By tracking $\mathcal{W}_\varepsilon^u(0)$ forwards and $\mathcal{W}_\varepsilon^s(0)$ backwards, it is possible to find an intersection provided that $c \approx \check{c}_0$ is chosen appropriately. The exchange lemma is needed to track these manifolds in a neighborhood of the right branch $\mathcal{M}_\varepsilon^r$, where the flow spends time of order ε^{-1} . There exists for any $r \in \mathbb{Z}_{>0}$ an ε -independent open neighborhood $\mathcal{U}_E$ of $\mathcal{M}_\varepsilon^r$ and a C^r -change of coordinates $\Psi_\varepsilon: \mathcal{U}_E \rightarrow \mathbb{R}^3$, depending C^r -smoothly on ε , in which the flow is given

by the Fenichel normal form [18, 33]

$$\begin{aligned} U' &= -\Lambda(U, V, W; c, a, \varepsilon)U, \\ V' &= \Gamma(U, V, W; c, a, \varepsilon)V, \\ W' &= \varepsilon(1 + H(U, V, W; c, a, \varepsilon)UV), \end{aligned} \tag{3.10}$$

where the functions Λ, Γ and H are C^r , and Λ and Γ are bounded below away from zero. In the local coordinates $\mathcal{M}_\varepsilon^r$ is given by $U = V = 0$, and $\mathcal{W}_\varepsilon^{u,r}$ and $\mathcal{W}_\varepsilon^{s,r}$ are given by $U = 0$ and $V = 0$, respectively. We assume that the Fenichel neighborhood contains a box

$$\Psi_\varepsilon(\mathcal{U}_E) \supseteq \{(U, V, W) : U, V \in [-\Delta, \Delta], W \in [-\Delta, W^* + \Delta]\}, \tag{3.11}$$

for $W^* > 0$ and some small $0 < \Delta \ll W^*$, both independent of ε . The exchange lemma [33] then states that for sufficiently small $\Delta > 0$ and $\varepsilon > 0$, any sufficiently large T , and any $|W_0| < \Delta$, there exists a solution $(U(\xi), V(\xi), W(\xi))$ to (3.10) that lies in $\Psi_\varepsilon(\mathcal{U}_E)$ for $\xi \in [0, T]$ and satisfies $U(0) = \Delta$, $W(0) = W_0$, and $V(T) = \Delta$ and the norms $|U(T)|$, $|V(0)|$, and $|W(T) - W_0 - \varepsilon W^*|$ are of order e^{-qT} for some $q > 0$, independent of ε .

We now track $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^s(0)$ up to the neighborhood $\mathcal{U}_E$ of $\mathcal{M}_\varepsilon^r$ and determine how they behave at $U = \Delta$ and $V = \Delta$. This gives a system of equations in c, T, ε which can be solved for $c = \check{c}(a, \varepsilon) = \check{c}_0(a) + \mathcal{O}(\varepsilon)$ to connect $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^s(0)$ via a solution given by the Exchange lemma, completing the construction of the pulse of Theorem 4.6. The full pulse solution $\phi_{a,\varepsilon}$ is shown in Figure 4.3.

Nonhyperbolic regime

We now move on to the case $0 < a, \varepsilon \ll 1$. For certain values of the parameters a, ε , the tails of the pulses develop small oscillations near the equilibrium. These oscillatory tails are due to a Belyakov transition occurring in the linearization of (3.1) about the origin where the two stable real eigenvalues collide and split as a complex conjugate pair. In [8], it was shown that for sufficiently small $a, \varepsilon > 0$ this transition occurs when

$$\varepsilon = \frac{a^2}{4} + \mathcal{O}(a^3), \quad (3.12)$$

and the following result capturing the existence of pulses on either side of this transition was proved.

Theorem 4.7. *[8, Theorem 1.1] There exists $K^*, \mu > 0$ such that the following holds. For each $K > K^*$, there exists $a_0, \varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$, system (2.1) admits a traveling-pulse solution with wave speed $\check{c} = \check{c}(a, \varepsilon)$ given by*

$$\check{c}(a, \varepsilon) = \sqrt{2} \left(\frac{1}{2} - a \right) - \mu\varepsilon + \mathcal{O}(\varepsilon(a + \varepsilon)).$$

*Furthermore, for $\varepsilon > K^*a^2$, the tail of the pulse is oscillatory.*

Remark 4.3.3. *In fact, by the identity (3.12), the constant $K^* > 0$ in Theorem 4.7 can be any value larger than $1/4$.*

The difficulties in the proof of Theorem 4.7 arise from the fact that the pulses are constructed as perturbations from the highly singular limit in which $a = \varepsilon = 0$ (see Figure 4.2). In this limit, the origin sits at the lower left fold on the critical

manifold $\mathcal{M}_0$, and the Nagumo front and back solutions $\phi_{f,b}$ leave $\mathcal{M}_0^\ell$ and $\mathcal{M}_0^r$ precisely at the folds where these manifolds are no longer normally hyperbolic. Near such points, standard Fenichel theory and the exchange lemma break down, and geometric blow-up techniques are used to track the flow in these regions.

However, away from the folds, standard geometric singular perturbation theory applies, and many of the arguments from the classical case carry over. Outside of neighborhoods of the two fold points, the manifolds $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$ persist for $\varepsilon > 0$ as locally invariant manifolds $\mathcal{M}_\varepsilon^r$ and $\mathcal{M}_\varepsilon^\ell$ as do their (un)stable foliations $\mathcal{W}_\varepsilon^{s,\ell}, \mathcal{W}_\varepsilon^{u,\ell}, \mathcal{W}_\varepsilon^{s,r}, \mathcal{W}_\varepsilon^{u,r}$. The origin has a strong unstable manifold $\mathcal{W}_\varepsilon^u(0)$ which persists for $\varepsilon > 0$ and can be tracked along $\mathcal{M}_\varepsilon^r$ through the neighborhood $\mathcal{U}_E$ given in (3.11) via the exchange lemma into a neighborhood $\mathcal{U}_F$ of the upper right fold point. The stable foliation $\mathcal{W}_\varepsilon^{s,\ell}$ of the left branch can be tracked backwards from a neighborhood of the equilibrium to a neighborhood of the upper right fold point. Constructing the pulse solution then amounts to the following two technical difficulties. First, one must find an intersection of $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^{s,\ell}$ near the upper right fold point. Second, since the exponentially attracting properties of the manifold $\mathcal{W}_\varepsilon^{s,\ell}$ are only defined along a normally hyperbolic segment of $\mathcal{M}_\varepsilon^\ell$, the flow can only be tracked up to a neighborhood of the equilibrium at the origin. Hence additional arguments are required to justify that the tails of the pulses in fact converge to the equilibrium upon entering this neighborhood. Overcoming these difficulties is therefore reduced to local analyses near the two fold points. We provide a few details regarding the flow in these regions which will be useful in the forthcoming stability analysis.

We begin with the upper right fold point; by the exchange lemma the manifold $\mathcal{W}_\varepsilon^u(0)$ is exponentially close to $\mathcal{M}_\varepsilon^r$ upon entering an a - and ε -independent neighborhood $\mathcal{U}_F$ of the fold point. The goal is therefore to track $\mathcal{M}_\varepsilon^r$ and nearby trajectories in this neighborhood. The fold point is given by the fixed point $(u^*, 0, w^*)$ of the

layer problem (3.3) where

$$u^* = \frac{1}{3} \left(a + 1 + \sqrt{a^2 - a + 1} \right),$$

and $w^* = f(u^*)$. The linearization of (3.3) about this fixed point has one positive real eigenvalue $c > 0$ and a double zero eigenvalue, since $f'(u^*) = 0$. As in [8] we can perform for any $r \in \mathbb{Z}_{>0}$ a C^r -change of coordinates $\Phi_\varepsilon: \mathcal{U}_F \rightarrow \mathbb{R}^3$ to (3.1), which is C^r -smooth in c, a and ε for (c, a, ε) -values restricted to the set $[\check{c}_0(a_0), \check{c}_0(-a_0)] \times [-a_0, a_0] \times [-\varepsilon_0, \varepsilon_0]$, where $a_0, \varepsilon_0 > 0$ are chosen sufficiently small and $\check{c}_0(a) = \sqrt{2}(\frac{1}{2} - a)$. Applying Φ_ε to the flow of (3.1) in the neighborhood $\mathcal{U}_F$ of the fold point yields

$$\begin{aligned} x' &= \theta_0 (y + x^2 + h(x, y, \varepsilon; c, a)), \\ y' &= \theta_0 \varepsilon g(x, y, \varepsilon; c, a), \\ z' &= z (c + \mathcal{O}(x, y, z, \varepsilon)), \end{aligned} \tag{3.13}$$

where

$$\theta_0 = \frac{1}{c} (a^2 - a + 1)^{1/6} (u^* - \gamma w^*)^{1/3} > 0, \tag{3.14}$$

uniformly in $|a| \leq a_0$ and $c \in [\check{c}_0(a_0), \check{c}_0(-a_0)]$, and h, g are C^r -functions satisfying

$$\begin{aligned} h(x, y, \varepsilon; c, a) &= \mathcal{O}(\varepsilon, xy, y^2, x^3), \\ g(x, y, \varepsilon; c, a) &= 1 + \mathcal{O}(x, y, \varepsilon), \end{aligned}$$

uniformly in $|a| \leq a_0$ and $c \in [\check{c}_0(a_0), \check{c}_0(-a_0)]$. The coordinate transform Φ_ε can be

decomposed in a linear and nonlinear part

$$\Phi_\varepsilon \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathcal{N} \left[\begin{pmatrix} u \\ v \\ w \end{pmatrix} - \begin{pmatrix} u^* \\ 0 \\ w^* \end{pmatrix} \right] + \tilde{\Phi}_\varepsilon \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where the nonlinearity $\tilde{\Phi}_\varepsilon$ satisfies $\tilde{\Phi}_\varepsilon(u^*, 0, w^*) = \partial \tilde{\Phi}_\varepsilon(u^*, 0, w^*) = 0$ and the linear part $\mathcal{N}$ is given by

$$\mathcal{N} = \partial \Phi_\varepsilon \begin{pmatrix} u^* \\ 0 \\ w^* \end{pmatrix} = \begin{pmatrix} -\beta_1 & \frac{\beta_1}{c} & \frac{\beta_1}{c^2} \\ 0 & 0 & \frac{\beta_2}{c} \\ 0 & \frac{1}{c} & \frac{1}{c^2} \end{pmatrix},$$

where

$$\begin{aligned} \beta_1 &= (a^2 - a + 1)^{1/3} (u^* - \gamma w^*)^{-1/3} > 0, \\ \beta_2 &= c (a^2 - a + 1)^{1/6} (u^* - \gamma w^*)^{-2/3} > 0, \end{aligned}$$

uniformly in $|a| < a_0$ and $c \in [\check{c}_0(a_0), \check{c}_0(-a_0)]$. Finally, there exists a neighborhood $\mathcal{U}'_F \subset \mathbb{R}^3$ of 0, which is independent of c, a and ε , such that $\mathcal{U}'_F \subset \Phi_\varepsilon(\mathcal{U}_F)$.

In the transformed system (3.13), the x, y -dynamics is decoupled from the dynamics in the z -direction along the straightened out strong unstable fibers. Thus, the flow is fully described by the dynamics on the two-dimensional invariant manifold $z = 0$ and by the one-dimensional dynamics along the fibers in the z -direction. On the invariant manifold $z = 0$, for $\varepsilon = 0$ we see that the critical manifold is given by $\{(x, y) : y + x^2 + h(x, y, 0; c, a) = 0\}$, which is approximately a downwards-opening parabola. The branch of this parabola for $x < 0$ is attracting and corresponds to the manifold $\mathcal{M}_0^*$. We define $\mathcal{M}_0^{r,+}$ to be the singular trajectory obtained by append-

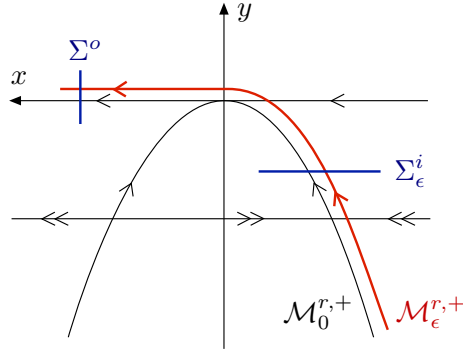


Figure 4.4: Shown is the flow on the invariant manifold $z = 0$ in the fold neighborhood $\mathcal{U}_F$. Note that x increases to the left.

ing the fast trajectory given by the line $\{(x, 0) : x > 0\}$ to the attracting branch $\mathcal{M}_0^r$ of the critical manifold. We note that $\mathcal{M}_0^{r,+}$ can be represented as a graph $y = s_0(x)$. In [8] it was shown that, for sufficiently small $\varepsilon > 0$, $\mathcal{M}_0^{r,+}$ perturbs to a trajectory $\mathcal{M}_\varepsilon^{r,+}$ on $z = 0$, represented as a graph $y = s_\varepsilon(x)$, which is a -uniformly $\mathcal{C}^0 - \mathcal{O}(\varepsilon^{2/3})$ -close to $\mathcal{M}_0^{r,+}$ (see Figure 4.4).

In addition, we have the following estimates on the flow in the invariant manifold $z = 0$. For each sufficiently small $\rho, \sigma > 0$, we define the following sections on $z = 0$. Let $\tilde{x}_\varepsilon(c, a)$ denote the x -value at which the manifold $\mathcal{M}_\varepsilon^{r,+}$ intersects $y = -\rho^2$, and define

$$\Sigma_\varepsilon^i = \Sigma_\varepsilon^i(\rho, \sigma) := \{(\tilde{x}_\varepsilon(c, a) + x_0, -\rho^2) : 0 \leq |x_0| < \sigma \rho \varepsilon\},$$

$$\Sigma^o = \Sigma^o(\rho) := \{(\rho, y) : y \in \mathbb{R}\}.$$

Proposition 4.3.4. *For each sufficiently small $\rho, \sigma > 0$, there exists $a_0, \varepsilon_0 > 0$ such that for $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ the following holds. The flow of (3.13) on the invariant manifold $z = 0$ maps $\Sigma_\varepsilon^i(\rho, \sigma)$ into $\Sigma^o(\rho)$. In addition, a trajectory Γ starting at $x = \tilde{x}_\varepsilon(c, a) + x_0$ in Σ_ε^i satisfies*

- (i) *Between Σ_ε^i and Σ^o we have that Γ is $\mathcal{O}(x_0)$ -close to the manifold $\mathcal{M}_\varepsilon^{r,+}$. In particular, we have, along Γ between Σ_ε^i and Σ^o , the bound $|y - s_\varepsilon(x)| < C|x_0|$*

for some constant $C > 0$ independent of a and ε .

- (ii) There exist constants $k, \tilde{k} > 0$, independent of ρ, σ, a and ε , such that, along Γ between Σ_ε^i and Σ° , we have $x' > (\tilde{k}/\rho)\varepsilon$. Furthermore, define the function $\Theta: (-\Omega_0, \infty) \rightarrow \mathbb{R}$ by

$$\Theta(\zeta) = \begin{cases} \sqrt{\zeta} \frac{I_{-2/3}\left(\frac{2}{3}\zeta^{3/2}\right) - I_{2/3}\left(\frac{2}{3}\zeta^{3/2}\right)}{I_{1/3}\left(\frac{2}{3}\zeta^{3/2}\right) - I_{-1/3}\left(\frac{2}{3}\zeta^{3/2}\right)}, & \text{if } \zeta > 0 \\ \sqrt{-\zeta} \frac{J_{2/3}\left(\frac{2}{3}(-\zeta)^{3/2}\right) - J_{-2/3}\left(\frac{2}{3}(-\zeta)^{3/2}\right)}{J_{1/3}\left(\frac{2}{3}(-\zeta)^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}(-\zeta)^{3/2}\right)}, & \text{if } \zeta \leq 0 \end{cases} \quad (3.15)$$

where J_r and I_r denote Bessel functions and modified Bessel functions of the first kind, respectively, and Ω_0 denotes the first positive zero of $J_{1/3}\left(\frac{2}{3}\zeta^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\zeta^{3/2}\right)$. Then, Θ is smooth, strictly decreasing and invertible and along Γ we approximate a -uniformly

$$x' = \theta_0 \left(x^2 - \Theta^{-1} \left(x\varepsilon^{-1/3} \right) \varepsilon^{2/3} \right) + \mathcal{O}(\varepsilon), \quad \text{for } 0 \leq |x| < k\varepsilon^{1/3},$$

where θ_0 is defined in (3.14).

Proof. In [8], using geometric blow-up techniques it was shown that between the sections Σ_ε^i and Σ° , the manifold $\mathcal{M}_\varepsilon^{r,+}$ is $\mathcal{O}(\varepsilon^{2/3})$ -close to $\mathcal{M}_0^{r,+}$ and can be represented as the graph of an invertible function $y = s_\varepsilon(x)$.

We consider the flow of (3.13) on the invariant manifold $z = 0$. We rescale $\bar{t} = \theta_0 \xi$

and append an equation for ε , arriving at the system

$$\begin{aligned}\frac{dx}{dt} &= y + x^2 + h(x, y, \varepsilon; c, a), \\ \frac{dy}{dt} &= \varepsilon g(x, y, \varepsilon; c, a), \\ \frac{d\varepsilon}{dt} &= 0.\end{aligned}\tag{3.16}$$

The blow up analysis in [8] makes use of three different rescalings in blow up charts $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ to track solutions between Σ_ε^i and Σ^o . The chart $\mathcal{K}_1$ is described by the coordinates

$$x = r_1 x_1, \quad y = -r_1^2, \quad \varepsilon = r_1^3 \varepsilon_1,\tag{3.17}$$

the second chart $\mathcal{K}_2$ uses the coordinates

$$x = r_2 x_2, \quad y = -r_2^2 y_2, \quad \varepsilon = r_2^3,\tag{3.18}$$

and the third chart $\mathcal{K}_3$ uses the coordinates

$$x = r_3, \quad y = -r_3^2 y_3, \quad \varepsilon = r_3^3 \varepsilon_3.\tag{3.19}$$

In each of the charts $\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{K}_3$, we define entry/exit sections

$$\Sigma_1^{in} := \{(x_1, r_1, \varepsilon_1) : 0 < \varepsilon_1 < \delta, 0 \leq |x_1 - \rho^{-1} s_\varepsilon^{-1}(-\rho^2)| < \sigma \rho^3 \varepsilon_1, r_1 = \rho\},$$

$$\Sigma_1^{out} := \{(x_1, r_1, \varepsilon_1) : \varepsilon_1 = \delta, 0 \leq |x_1 - r_1^{-1} s_\varepsilon^{-1}(-r_1^2)| < \sigma r_1^3 \delta, 0 < r_1 \leq \rho\},$$

$$\Sigma_2^{in} := \{(x_2, y_2, r_2) :$$

$$0 \leq |x_2 - r_2^{-1} s_\varepsilon^{-1}(-\delta^{-2/3} r_2^2)| < \sigma \rho^3 \delta^{2/3}, y_2 = \delta^{-2/3}, 0 < r_2 \leq \rho \delta^{1/3}\},$$

$$\Sigma_2^{out} := \{(x_2, y_2, r_2) : x_2 = \delta^{-1/3}, 0 < r_2 \leq \rho \delta^{1/3}\},$$

$$\Sigma_3^{in} := \{(r_3, y_3, \varepsilon_3) : 0 < r_3 < \rho, y_3 \in [-\beta, \beta], \varepsilon_3 = \delta\},$$

$$\Sigma_3^{out} := \{(r_3, y_3, \varepsilon_3) : r_3 = \rho, y_3 \in [-\beta, \beta], \varepsilon_3 \in (0, \delta)\},$$

for sufficiently small $\beta, \delta, \sigma, \rho > 0$ satisfying $2\Omega_0\delta^{2/3} < \beta$, where Ω_0 is the smallest positive zero of

$$J_{-1/3}\left(\frac{2}{3}z^{3/2}\right) + J_{1/3}\left(\frac{2}{3}z^{3/2}\right),$$

with J_r Bessel functions of the first kind. The set $\{(x, y, \varepsilon) \in \mathbb{R}^3 : (x, y) \in \Sigma_\varepsilon^i(\rho, \sigma), \varepsilon \in (0, \rho^3\delta)\}$ equals Σ_1^{in} in the $\mathcal{K}_1$ coordinates (3.17). Moreover, Σ_3^{out} is contained in the set $\{(x, y, \varepsilon) \in \mathbb{R}^3 : (x, y) \in \Sigma^o\}$, when converting to the $\mathcal{K}_3$ coordinates (3.19). In [8, §4], it was shown that the flow of (3.16) maps Σ_1^{in} into Σ_3^{out} via the sequence

$$\Sigma_1^{in} \longrightarrow \Sigma_1^{out} = \Sigma_2^{in} \longrightarrow \Sigma_2^{out} = \Sigma_3^{in} \longrightarrow \Sigma_3^{out},$$

taking into account the different coordinate systems to represent Σ_i^{in} and Σ_i^{out} for $i = 1, 2, 3$. The estimates on the flow between the various sections obtained in [8] enable us to prove (i) and (ii):

The proof of (i) follows from the proof of the estimates in [8, Corollary 4.1].

For (ii), we begin with the lower bound $x' > (\tilde{k}/\rho)\varepsilon$. Between the sections Σ_1^{in} and Σ_1^{out} , the existence of such a $\tilde{k} > 0$ follows from the proof of [8, Lemma 4.2]. In addition by [8, Lemmata 4.3, 4.4], by possibly taking $\tilde{k}$ smaller, the flow satisfies

$$x' = \theta_0 \frac{dx}{dt} > \tilde{k}\varepsilon^{2/3} > (\tilde{k}/\rho)\varepsilon,$$

between the sections Σ_2^{in} and Σ_3^{out} .

Finally, for any sufficiently small k , for $0 \leq |x| < k\varepsilon^{1/3}$, we are concerned with the flow in the chart $\mathcal{K}_2$ between the sections Σ_2^{in} and Σ_2^{out} . In the $\mathcal{K}_2$ coordinates (3.18),

the flow takes the form

$$\begin{aligned}\frac{dx_2}{dt_2} &= -y_2 + x_2^2 + \mathcal{O}(r_2), \\ \frac{dy_2}{dt_2} &= -1 + \mathcal{O}(r_2), \\ \frac{dr_2}{dt_2} &= 0,\end{aligned}\tag{3.20}$$

where $t_2 = r_2 \bar{t}$. We quote a few facts from [8, §4.6]. Between the sections Σ_2^{in} and Σ_2^{out} , the manifold $\mathcal{M}_\varepsilon^{r,+}$ can be represented as the graph $(x_2, s_2(x_2; r_2))$ of a smooth invertible function $y_2 = s_2(x_2; r_2)$ smoothly parameterized by $r_2 = \varepsilon^{1/3}$ with $s_2(x_2; r_2) = s_2(x_2; 0) + \mathcal{O}(r_2)$. Furthermore, using results from [42, § II.9], we have that $s_2(x_2; 0) = \Theta^{-1}(x_2)$, where the function Θ is defined in (3.15). The function Θ is smooth, strictly decreasing and maps $(-\Omega_0, \infty)$ bijectively onto $\mathbb{R}$. By part (i) above, we deduce that along Γ between Σ_2^{in} and Σ_2^{out} , we have $|y_2 - s_2(x_2; r_2)| = \mathcal{O}(r_2)$. Hence we compute

$$\begin{aligned}x' &= \theta \frac{dx}{d\bar{t}} \\ &= \theta_0 r_2^2 \frac{dx_2}{dt_2} \\ &= \theta_0 r_2^2 (x_2^2 - y_2) + \mathcal{O}(r_2^3) \\ &= \theta_0 r_2^2 (x_2^2 - \Theta^{-1}(x_2)) + \mathcal{O}(r_2^3) \\ &= \theta_0 (x^2 - \varepsilon^{2/3} \Theta^{-1}(x \varepsilon^{-1/3})) + \mathcal{O}(\varepsilon),\end{aligned}$$

which concludes the proof of assertion (ii). □

By tracking solutions close to $\mathcal{M}_\varepsilon^{r,+}$, it is possible to find a solution which connects

$\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^{s,\ell}$ near the fold. For small $z_0 > 0$, we define the sections

$$\begin{aligned}\Sigma_\varepsilon^{in} &= \Sigma_\varepsilon^{in}(\rho, \sigma, z_0) := \{(x, y, z) : (x, y) \in \Sigma_\varepsilon^i(\rho, \sigma), z \in [-z_0, z_0]\}, \\ \Sigma_\varepsilon^{out} &= \Sigma_\varepsilon^{out}(z_0) := \mathcal{U}'_F \cap \{z = z_0\}.\end{aligned}\tag{3.21}$$

We remark that for each sufficiently small $\rho, \sigma, z_0 > 0$, it is always possible to choose the fold neighborhood $\mathcal{U}_F$ and the Fenichel neighborhood $\mathcal{U}_E$ so that they intersect in a region containing the section Σ_ε^{in} . We have the following by [8, Proposition 4.1, Corollary 4.1 and §5.5].

Proposition 4.3.5. *There exists $\mu > 0$ such that for each sufficiently small $\sigma, \rho, z_0 > 0$ there exists $a_0, \varepsilon_0 > 0$ such that the following holds. For each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$, there exists $c = \check{c}(a, \varepsilon)$ satisfying*

$$\check{c}(a, \varepsilon) = \sqrt{2} \left(\frac{1}{2} - a \right) - \mu\varepsilon + \mathcal{O}(\varepsilon(a + \varepsilon)),$$

such that in system (3.1) the manifolds $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^{s,\ell}$ intersect. Denote by $\phi_{a,\varepsilon}(\xi)$ the solution to (3.1) lying in $\mathcal{W}_\varepsilon^u(0) \cap \mathcal{W}_\varepsilon^{s,\ell}$. The solution $\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi))$ to system (3.13) enters the fold neighborhood $\mathcal{U}'_F$ via the section $\Sigma_\varepsilon^{in}(\rho, \sigma, z_0)$ and exits via $\Sigma_\varepsilon^{out}(z_0)$. The intersection point of $\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi))$ with Σ_ε^{out} is a -uniformly $\mathcal{O}(\varepsilon^{2/3})$ -close to the intersection point between Σ_ε^{out} and the back solution $\Phi_0(\varphi_b(\xi), w_b^1)$ to system (3.13) at $\varepsilon = 0$.

We note that by taking $\rho, \sigma, z_0 > 0$ smaller, it is possible to ensure that the solutions considered in Proposition 4.3.5 pass as close to the fold as desired, at the expense of possibly taking a_0, ε_0 smaller.

After finding an intersection between $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^{s,\ell}$, it remains to show that solutions on the manifold $\mathcal{W}_\varepsilon^{s,\ell}$ converge to the equilibrium. As previously stated,

using standard geometric singular perturbation theory arguments, it is possible to track $\mathcal{W}_\varepsilon^{s,\ell}$ into a neighborhood of the origin, but more work is required to show that the tail of the pulse in fact converges to the equilibrium after entering this neighborhood. We have the following result which follows from the analysis in [8, §6].

Proposition 4.3.6. *For each $K > 0$ and each sufficiently small $\sigma_0 > 0$, there exists $a_0, \varepsilon_0, d_0 > 0$ such that the following holds. For each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$, the equilibrium $(u, v, w) = (0, 0, 0)$ in system (3.1) is stable with two-dimensional stable manifold $\mathcal{W}_\varepsilon^s(0)$. Furthermore, any solution on $\mathcal{W}_\varepsilon^{s,\ell}$ which enters the ball $B(0, \sigma_0)$ at a distance d_0 from $\mathcal{M}_\varepsilon^\ell$ lies in the stable manifold $\mathcal{W}_\varepsilon^s(0)$ and remains in $B(0, \sigma_0)$ until converging to the equilibrium.*

Theorem 4.7 then follows from Propositions 4.3.5 and 4.3.6.

Main existence result

Combining Theorems 4.6 and 4.7, we obtain Theorem 4.1, repeated here for convenience, which encompasses both the hyperbolic and nonhyperbolic regimes.

Theorem 4.1. *There exists $K^* > 0$ such that for each $\kappa > 0$ and $K > K^*$ the following holds. There exists $\varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in [0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$ system (2.1) admits a traveling-pulse solution $\hat{\phi}_{a,\varepsilon}(x, t) := \tilde{\phi}_{a,\varepsilon}(x + \check{c}t)$ with wave speed $\check{c} = \check{c}(a, \varepsilon)$ a -uniformly approximated by*

$$\check{c} = \sqrt{2} \left(\frac{1}{2} - a \right) + \mathcal{O}(\varepsilon).$$

*Furthermore, if we have in addition $\varepsilon > K^*a^2$, then the tail of the pulse is oscillatory.*

Proof. We take $K^* > \frac{1}{4}$ and fix K, κ satisfying $K > K^*$ and $\kappa > 0$. From Theorem 4.7 we obtain constants a_0, ε_0 and a traveling pulse for each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$, where the pulses for $K^*a^2 < \varepsilon < Ka^2$ have oscillatory tails. By shrinking $\varepsilon_0 > 0$ further if necessary, Theorem 4.6 yields the existence of pulse solutions for each $(a, \varepsilon) \in [a_0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$, where we use that $[a_0, \frac{1}{2} - \kappa]$ is compact to ensure $\varepsilon_0 > 0$ is independent of a . $\square$

4.3.4 Proof of Theorem 4.5

In this section, we provide a proof of Theorem 4.5; we take care to separate the cases corresponding to Theorem 4.6 and that of Theorem 4.7 in which the pulse passes by upper fold.

The estimates in Theorem 4.5 follow from standard Fenichel theory and the fold estimates along with the following argument from [14, 29]. Recall from §4.3.3 that in the ε -independent neighborhood $\mathcal{U}_E$ of $\mathcal{M}_\varepsilon^r$, there exists a C^r -change of coordinates $\Psi_\varepsilon: \mathcal{U}_E \rightarrow \mathbb{R}^3$ in which the flow is given by the Fenichel normal form (3.10). Here we have that $\mathcal{M}_\varepsilon^r$ is given by $U = V = 0$, $\mathcal{W}_\varepsilon^{u,r}$ and $\mathcal{W}_\varepsilon^{s,r}$ are given by $U = 0$ and $V = 0$, respectively, and the open Fenichel neighborhood $\Psi_\varepsilon(\mathcal{U}_E)$ contains a box $\{(U, V, W) : U, V \in [-\Delta, \Delta], W \in [-\Delta, W^* + \Delta]\}$ for $W^* > 0$ and some small $0 < \Delta \ll W^*$, both independent of ε . We define the following entry and exit manifolds

$$\begin{aligned} N_1 &:= \{(U, V, W) : U = \Delta, V \in [-\Delta, \Delta], W \in [-\Delta, \Delta]\}, \\ N_2 &:= \{(U, V, W) : U, V \in [-\Delta, \Delta], W = W_0\}, \end{aligned}$$

for the flow around the corner where $0 < W_0 < W^*$. We make use of the following

theorem, based on a result in [14].

Theorem 4.8 ([14, Theorem 4.1]). *Assume that $\Xi(\varepsilon)$ is a continuous function of ε into the reals satisfying*

$$\lim_{\varepsilon \rightarrow 0} \Xi(\varepsilon) = \infty, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \Xi(\varepsilon) = 0. \quad (3.22)$$

Moreover, assume that there is a one-parameter family of solutions $(U, V, W)(\xi, \cdot)$ to (3.10) with $(U, V, W)(\xi_1, \varepsilon) \in N_1$, $(U, V, W)(\xi_2(\varepsilon), \varepsilon) \in N_2$ and $\lim_{\varepsilon \rightarrow 0} W(\xi_1, \varepsilon) = 0$ for some $\xi_1, \xi_2(\varepsilon) \in \mathbb{R}$. Let $U_0(\xi)$ denote the solution to

$$U' = -\Lambda(U, 0, 0; c, a, 0)U, \quad (3.23)$$

satisfying $U_0(\xi_1) = \Delta + \tilde{U}_0$ where $|\tilde{U}_0| \ll \Delta$. Then, for $\varepsilon > 0$ sufficiently small, we have that

$$\|(U, V, W)(\xi, \varepsilon) - (U_0(\xi), 0, 0)\| \leq C \left(\varepsilon \Xi(\varepsilon) + |\tilde{U}_0| + |W(\xi_1, \varepsilon)| \right), \quad \text{for } \xi \in [\xi_1, \Xi(\varepsilon)],$$

where $C > 0$ is independent of a and ε .

Proof. This proof is based on an argument in [14]. In the box

$$\mathcal{U}'_E := \{(U, V, W) : U, V \in [-\Delta, \Delta], W \in [-\Delta, W^* + \Delta]\},$$

for sufficiently small $\varepsilon > 0$, there exist constants $\alpha_{\pm}^{u/s} > 0$ such that

$$0 < \alpha_-^s < \Lambda(U, V, W; c, a, \varepsilon) < \alpha_+^s,$$

$$0 < \alpha_-^u < \Gamma(U, V, W; c, a, \varepsilon) < \alpha_+^u,$$

We first consider the V -coordinate. For any $\xi > \xi_1$, we have

$$|V(\xi)| \geq |V(\xi_1)| e^{\alpha_-^u(\xi - \xi_1)}.$$

Since $V(\xi_2) \in N_2$, we also have

$$|V(\xi_1)| \leq \Delta e^{-\alpha_-^u(\xi_2 - \xi_1)}.$$

We note that since the solution enters $\mathcal{U}'_E$ via N_1 and reaches N_2 at $\xi_2(\varepsilon)$, using the equation for W in (3.10), we have that $\xi_2(\varepsilon)$ satisfies $\xi_2(\varepsilon) \geq (C\varepsilon)^{-1}$. Therefore, using the upper bound on Γ we have that

$$|V(\xi)| \leq \Delta e^{-\alpha_-^u \xi_2 + \alpha_+^u \xi - (\alpha_+^u - \alpha_-^u) \xi_1} \leq C e^{-\frac{1}{C\varepsilon}},$$

for $\xi \in [\xi_1, \Xi(\varepsilon)]$.

The solution in the slow W -component may be written as

$$W(\xi) = W(\xi_1, \varepsilon) + \int_{\xi_1}^{\xi} \varepsilon (1 + H(U(s), V(s), W(s), c, a, \varepsilon) U(s) V(s)) ds,$$

from which we infer that

$$|W(\xi) - W(\xi_1, \varepsilon)| \leq C\varepsilon(\xi - \xi_1) \leq C\varepsilon\Xi(\varepsilon), \quad \text{for } \xi \in [\xi_1, \Xi(\varepsilon)],$$

and hence

$$|W(\xi)| \leq C\varepsilon\Xi(\varepsilon) + |W(\xi_1, \varepsilon)|, \quad \text{for } \xi \in [\xi_1, \Xi(\varepsilon)].$$

Finally we consider the U -component. We have that the difference $(U(\xi) - U_0(\xi))$

satisfies

$$\begin{aligned} U' - U'_0 &= -(\Lambda(U, V, W, c, a, \varepsilon)U - \Lambda(U_0, 0, 0, c, a, 0)U_0) \\ &= -\Lambda(U_0, 0, 0, c, a, 0)(U - U_0) + \mathcal{O}(\varepsilon + |U - U_0| + |V| + |W|)U. \end{aligned}$$

with $U(\xi_1) - U_0(\xi_1) = \tilde{U}_0$ where $|\tilde{U}_0| \ll \Delta$. By possibly taking Δ smaller if necessary and using the fact that the rate of contraction in the U -component is stronger than α_-^s , we deduce that $(U(\xi) - U_0(\xi))$ satisfies a differential equation

$$X' = b_1(\xi)X + b_2(\xi), \quad X(\xi_1) = \tilde{U}_0,$$

where $b_1(\xi) < -\alpha_-^s/2 < 0$ and

$$|b_2(\xi)| \leq C(\varepsilon\Xi(\varepsilon) + |W(\xi_1, \varepsilon)|)e^{-\alpha_-^s\xi},$$

for $\xi \in [\xi_1, \Xi(\varepsilon)]$. Hence, it holds

$$|U(\xi) - U_0(\xi)| \leq C\left(\varepsilon\Xi(\varepsilon) + |\tilde{U}_0| + |W(\xi_1, \varepsilon)|\right),$$

for $\xi \in [\xi_1, \Xi(\varepsilon)]$, which completes the proof. $\square$

Remark 4.3.7. We note that Theorem 4.8 extends the result [14, Theorem 4.1] to account for the following minor technicalities. Firstly, the estimates obtained along the singular $\varepsilon = 0$ solution are shown to hold along the entire interval $[\xi_1, \Xi(\varepsilon)]$ rather than just at the endpoint $\xi = \Xi(\varepsilon)$. Second, we allow for an error $\tilde{U}_0$ in the case that the solution in question does not arrive in N_1 at the same time ξ_1 as the singular solution U_0 . Finally, no assumptions are made on the entry height $W(\xi_1, \varepsilon)$ other than continuity in ε with $\lim_{\varepsilon \rightarrow 0} W(\xi_1, \varepsilon) = 0$. This is necessary to deal with the $\mathcal{O}(\varepsilon^{2/3})$ estimates along the back arising from Proposition 4.3.5 in the nonhyperbolic

regime.

Proof of Theorem 4.5. We note that $\Xi_\tau(\varepsilon) := -\tau \log \varepsilon$ satisfies condition (3.22) in Theorem 4.8 for every $\tau > 0$.

We begin by showing (i). By standard geometric perturbation theory and the stable manifold theorem, the solution $\phi_{a,\varepsilon}(\xi)$ is a -uniformly $\mathcal{O}(\varepsilon)$ -close to $(\phi_f(\xi), 0)$ upon entry in N_1 at $\xi_f = \mathcal{O}(1)$. We apply the coordinate transform Ψ_ε in the neighborhood $\mathcal{U}_E$ of $\mathcal{M}_\varepsilon^r$, which brings system (3.1) into Fenichel normal form (3.10). For $\varepsilon = 0$, the orbit $(\phi_f(\xi), 0)$ converges exponentially to the equilibrium $(p_f^1, 0)$ and hence lies in $\mathcal{W}^s(\mathcal{M}_0^r)$. Therefore, we have that $\Psi_0(\phi_f(\xi), 0) = (U_0(\xi), 0, 0)$, where $U_0(\xi)$ solves (3.23). We denote $(U_{a,\varepsilon}(\xi), V_{a,\varepsilon}(\xi), W_{a,\varepsilon}(\xi)) = \Psi_\varepsilon(\phi_{a,\varepsilon}(\xi))$. By Theorem 4.8 we have $\|(U_{a,\varepsilon}(\xi), V_{a,\varepsilon}(\xi), W_{a,\varepsilon}(\xi)) - (U_0(\xi), 0, 0)\| \leq C\varepsilon\Xi_\tau(\varepsilon)$ for $\xi \in [\xi_f, \Xi_\tau(\varepsilon)]$. Since the transform Ψ_ε to the Fenichel normal form is C^r -smooth in ε , we incur at most $\mathcal{O}(\varepsilon)$ errors when transforming back to the (u, v, w) -coordinates. Therefore, $\phi_{a,\varepsilon}(\xi)$ is a -uniformly $\mathcal{O}(\varepsilon\Xi_\tau(\varepsilon))$ -close to $(\phi_f(\xi), 0)$ for $\xi \in [\xi_f, \Xi_\tau(\varepsilon)]$ and we obtain the estimate (i).

We now prove (ii). From Proposition 4.3.5, for each sufficiently small $a_0 > 0$ we have that for $0 < a < a_0$ the solution $\phi_{a,\varepsilon}$ leaves the neighborhood $\mathcal{U}_E$ of the slow manifold $\mathcal{M}_\varepsilon^r$ after passing the section Σ_ε^{in} , defined in (3.21), where the flow enters the neighborhood $\mathcal{U}_F$ governed by the fold dynamics. With appropriate choice of the neighborhood $\mathcal{U}_E$, the case $a \geq a_0$ bounded away from zero is covered by standard geometric singular perturbation theory and the exchange lemma. Hence the estimate (ii) is split into two cases.

We first consider the case $a \geq a_0$ in which the classical arguments apply. In this case, the pulse leaves $\mathcal{M}_\varepsilon^r$ via the Fenichel neighborhood $\mathcal{U}_E$, where the flow is

governed by the Fenichel normal form (3.10). By taking $Z_{a,\varepsilon} = \mathcal{O}_s(\varepsilon^{-1})$ to be at leading order the time at which the pulse solution exits the Fenichel neighborhood $\mathcal{U}_E$ of $\mathcal{M}_\varepsilon^r$ along the back and treating the flow in a neighborhood of the left slow manifold $\mathcal{M}_\varepsilon^\ell$ in a similar manner, the estimate (ii) follows from a similar argument as (i).

We now consider the case $a < a_0$ in which $\phi_{a,\varepsilon}$ leaves $\mathcal{U}_E$ via the fold neighborhood $\mathcal{U}_F$. We apply the coordinate transform $\Phi_\varepsilon: \mathcal{U}_F \rightarrow \mathbb{R}^3$ in the neighborhood $\mathcal{U}_F$ bringing system (3.1) into the canonical form (3.13); see §4.3.3. Take $Z_{a,\varepsilon} = \mathcal{O}_s(\varepsilon^{-1})$ to be at leading order the time at which the pulse solution exits the a - and ε -independent fold neighborhood $\mathcal{U}'_F \subset \Phi_\varepsilon(\mathcal{U}_F)$ via the section Σ^{out} , defined in (3.21); that is, we assume $\Phi_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_b)) \in \Sigma^{out}$, where $\xi_b = \mathcal{O}(1)$. We begin with establishing (ii) on the interval $J_{b,-} := [Z_{a,\varepsilon} - \Xi_\tau(\varepsilon), Z_{a,\varepsilon} - \xi_b]$.

The back solution $(\phi_b(\xi), w_b^1)$ to system (3.3) converges exponentially in backwards time to the equilibrium $(p_b^1, w_b^1) \in \mathcal{M}_0^r$ lying $\mathcal{O}(a)$ -close to the fold point $(u^*, 0, w^*)$. Therefore, the equilibrium (p_b^1, w_b^1) is contained in $\mathcal{U}_F$, for $a > 0$ sufficiently small. Thus, transforming to system (3.13) for $\varepsilon = 0$ yields $\Phi_0(p_b^1, w_b^1) = (x_b, y_b, 0)$, where $x_b < 0$ and the equilibrium (x_b, y_b) lies on the critical manifold $\mathcal{M}_0^r = \{(x, y) : x \leq 0, y + x^2 + h(x, y, 0, \check{c}, a) = 0\}$ of the invariant subspace $z = 0$. In addition, $\Phi_0(\phi_b(\xi), w_b^1)$ equals the solution $(x_b, y_b, z_b(\xi))$ to (3.13) for $\varepsilon = 0$, where we gauge $z_b(\xi)$ so that $(x_b, y_b, z_b(-\xi_b)) \in \Sigma^{out}$.

Recall that by Proposition 4.3.5 $\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi))$ enters the fold neighborhood $\mathcal{U}'_F$ via the section Σ_ε^{in} and leaves via the section Σ^{out} at $\xi = Z_{a,\varepsilon} - \xi_b$. Since the y -dynamics in (3.13) is $\mathcal{O}(\varepsilon)$, one readily observes that $\phi_{a,\varepsilon}(\xi)$ lies in $\mathcal{U}_F$ for $\xi \in J_{b,-} = [Z_{a,\varepsilon} - \Xi_\tau(\varepsilon), Z_{a,\varepsilon} - \xi_b]$. We claim that the pulse solution $\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi)) =$

$(x_{a,\varepsilon}(\xi), y_{a,\varepsilon}(\xi), z_{a,\varepsilon}(\xi))$ satisfies

$$\|\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi)) - \Phi_0(\phi_b(\xi - Z_{a,\varepsilon}), w_b^1)\| \leq C\varepsilon^{2/3}\Xi_\tau(\varepsilon), \quad \text{for } \xi \in J_{b,-}. \quad (3.24)$$

By Proposition 4.3.5, $\Phi_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_b)) \in \Sigma^{out}$ lies a -uniformly $\mathcal{O}(\varepsilon^{2/3})$ -close to $\Phi_0(\phi_b(-\xi_b), w_b^1) \in \Sigma^{out}$. Hence, it holds

$$\Phi_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_b)) = (x_b + \mathcal{O}(\varepsilon^{2/3}), y_b + \mathcal{O}(\varepsilon^{2/3}), z_0), \quad (3.25)$$

a -uniformly, for some $z_0 > 0$. First, since (x_b, y_b) lies on the critical manifold $\mathcal{M}_0^r$, we have $x_b \leq 0$. So, by (3.25) it holds $x_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_b) < C\varepsilon^{2/3}$. Second, Proposition 4.3.4 (ii) yields $x'_{a,\varepsilon}(\xi) > 0$ for $\xi \in J_{b,-}$. Combining these two observations, we establish $x_{a,\varepsilon}(\xi) < C\varepsilon^{2/3}$ for $\xi \in J_{b,-}$. Hence, by Proposition 4.3.4 (i) $(x_{a,\varepsilon}(\xi), y_{a,\varepsilon}(\xi))$ is $\mathcal{O}(\varepsilon^{2/3})$ -close to $\{(x, y) : y + x^2 + h(x, y, \varepsilon, \check{c}, a) = 0\}$ for $\xi \in J_{b,-}$. Thus, one observes directly from equation (3.13) that $|x'_{a,\varepsilon}(\xi)| < C\varepsilon^{2/3}$ and $|y'_{a,\varepsilon}(\xi)| < C\varepsilon$ for $\xi \in J_{b,-}$. Therefore, starting at $\xi = Z_{a,\varepsilon} - \xi_b$ and integrating backwards, we have

$$\begin{aligned} |x_{a,\varepsilon}(\xi) - x_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_b)| &\leq \int_\xi^{Z_{a,\varepsilon} - \xi_b} C\varepsilon^{2/3} dt \leq C\varepsilon^{2/3}\Xi_\tau(\varepsilon) \\ |y_{a,\varepsilon}(\xi) - y_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_b)| &\leq \int_\xi^{Z_{a,\varepsilon} - \xi_b} C\varepsilon dt \leq C\varepsilon\Xi_\tau(\varepsilon), \end{aligned} \quad (3.26)$$

for $\xi \in J_{b,-}$.

Define $\tilde{z}_b(\xi) := z_b(\xi - Z_{a,\varepsilon})$. In backwards time, trajectories in (3.13) are exponentially attracted to the invariant manifold $z = 0$ with rate greater than $\check{c}/2$ by taking $\mathcal{U}_F$ smaller if necessary. Note that $\check{c}(a, \varepsilon)$ is bounded from below away from 0 by an a -independent constant. Since $(x_b, y_b, z_b(\xi))$ solves (3.13) for $\varepsilon = 0$ the

difference $z_{a,\varepsilon}(\xi) - \tilde{z}_b(\xi)$ satisfies on $J_{b,-}$

$$\begin{aligned} z'_{a,\varepsilon} - \tilde{z}'_b &= (\check{c} + \mathcal{O}(x_{a,\varepsilon}, y_{a,\varepsilon}, z_{a,\varepsilon}, x_b, y_b, \tilde{z}_b, \varepsilon)) (z_{a,\varepsilon} - \tilde{z}_b) \\ &\quad + \mathcal{O}(|x_{a,\varepsilon} - x_b| + |y_{a,\varepsilon} - y_b| + \varepsilon) (|z_{a,\varepsilon}| + |\tilde{z}_b|), \end{aligned}$$

suppressing the ξ -dependence of terms. Hence, using (3.25), (3.26) and the fact that in backwards time $\tilde{z}_b(\xi)$ and $z_{a,\varepsilon}(\xi)$ are exponentially decaying with rate $\check{c}/2$, we deduce that $z_{a,\varepsilon} - \tilde{z}_b(\xi)$ satisfies a differential equation of the form

$$X' = b_1(\xi)X + b_2(\xi), \quad X(Z_{a,\varepsilon} - \xi_b) = 0,$$

where $b_1(\xi) > \check{c}/2 > 0$ and

$$|b_2(\xi)| \leq C\varepsilon^{2/3}\Xi_\tau(\varepsilon)e^{-\check{c}(Z_{a,\varepsilon}-\xi)/2}$$

for $\xi \in J_{b,-}$. Hence, we estimate

$$|z_{a,\varepsilon}(\xi) - \tilde{z}_b(\xi)| \leq C\varepsilon^{2/3}\Xi_\tau(\varepsilon).$$

for $\xi \in J_{b,-}$. Combining this with (3.25) and (3.26), we have that (3.24) holds. Hence, since the transform Φ_ε is C^r -smooth in a and ε , the pulse solution $\phi_{a,\varepsilon}(\xi)$ is a -uniformly $\mathcal{O}(\varepsilon^{2/3}\Xi_\tau(\varepsilon))$ -close to the back $(\phi_b(\xi), w_b^1)$ and the estimate (ii) holds for $\xi \in J_{b,-} = [Z_{a,\varepsilon} - \Xi_\tau(\varepsilon), Z_{a,\varepsilon} - \xi_b]$.

We now follow $\phi_{a,\varepsilon}$ along the back into a (Fenichel) neighborhood of $\mathcal{M}_\varepsilon^\ell$. Upon entry, $\phi_{a,\varepsilon}(\xi)$ is a -uniformly $\mathcal{O}(\varepsilon^{2/3})$ -close to $(\phi_b(\xi), w_b^1)$. Combining this with another application of Theorem 4.8, the estimate (ii) follows for $\xi \in J_{b,+} = [Z_{a,\varepsilon} - \xi_b, Z_{a,\varepsilon} + \Xi_\tau(\varepsilon)]$.

By taking the a - and ε -independent neighborhoods $\mathcal{U}_F$ and $\mathcal{U}_E$ smaller if necessary (and thus taking $a_0, \varepsilon_0 > 0$ smaller if necessary) and setting ξ_0 sufficiently large independent of a and ε , we have that $\phi_{a,\varepsilon}(\xi)$ lies in the union $\mathcal{U}_E \cup \mathcal{U}_F$ for $\xi \in [\xi_0, Z_{a,\varepsilon} - \xi_0]$. Hence we obtain (iii) along the right branch $\mathcal{M}_0^r$. Along the left branch $\mathcal{M}_0^\ell$, a similar argument combined with Proposition 4.3.6 gives the estimate (iv). $\square$

4.4 Essential spectrum

In this section we prove that the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the left half plane and that it is bounded away from the imaginary axis. Moreover, we compute the intersection points of the essential spectrum with the real axis. Explicit expressions of these points are useful to determine whether there is a second eigenvalue of $\mathcal{L}_{a,\varepsilon}$ to the right of the essential spectrum.

Proposition 4.4.1. *In the setting of Theorem 4.1, let $\tilde{\phi}_{a,\varepsilon}(\xi)$ denote a traveling-pulse solution to (2.2) with associated linear operator $\mathcal{L}_{a,\varepsilon}$. The essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the half plane $\operatorname{Re}(\lambda) \leq -\min\{\varepsilon\gamma, a\}$. Moreover, for all $\lambda \in \mathbb{C}$ to the right of the essential spectrum the asymptotic matrix $\hat{A}_0(\lambda) = \hat{A}_0(\lambda; a, \varepsilon)$ of system (2.3) has precisely one (spatial) eigenvalue of positive real part. Finally, the essential spectrum intersects with the real axis at points*

$$\lambda = \begin{cases} -\frac{1}{2}a - \frac{1}{2}\varepsilon\gamma \pm \frac{1}{2}\sqrt{(\varepsilon\gamma - a)^2 - 4\varepsilon}, & \text{for } a > \varepsilon\gamma + 2\sqrt{\varepsilon}, \\ -\varepsilon\gamma + \check{c}^2 - \frac{1}{2}\sqrt{(2\check{c}^2 - \varepsilon\gamma + a)^2 - (\varepsilon\gamma - a)^2 + 4\varepsilon}, & \text{for } a \leq \varepsilon\gamma + 2\sqrt{\varepsilon}. \end{cases} \quad (4.1)$$

Proof. The essential spectrum is given by the λ -values for which the asymptotic matrix $\hat{A}_0(\lambda)$ of system (2.3) is nonhyperbolic. Thus we are looking for solutions

$\lambda \in \mathbb{C}$ to

$$0 = \det(\hat{A}_0(\lambda) - i\tau) = \Delta \left(-i\tau - \frac{\lambda + \varepsilon\gamma}{\check{\varepsilon}} \right) + \frac{\varepsilon}{\check{\varepsilon}}, \quad (4.2)$$

with $\tau \in \mathbb{R}$ and $\Delta := -\tau^2 - \check{c}i\tau - a - \lambda$. For all $\tau \in \mathbb{R}$ and $\operatorname{Re}(\lambda) > -a$ we have that $\operatorname{Re}(\Delta) < 0$. For $\operatorname{Re}(\lambda) > -a$ we rewrite (4.2) as

$$\lambda = -\gamma\varepsilon + \varepsilon\Delta^{-1} - i\check{c}\tau.$$

Taking real parts in the latter equation yields $\operatorname{Re}(\lambda) < -\gamma\varepsilon$. This proves the first assertion.

One readily observes that for sufficiently large $\lambda > 0$, the asymptotic matrix $\hat{A}_0(\lambda)$ has precisely one unstable eigenvalue. By continuity this holds for all $\lambda \in \mathbb{C}$ to the right of the essential spectrum. This proves the second assertion.

For the third assertion we are interested in real solutions λ to the characteristic equation (4.2). Solving (4.2) yields

$$2\lambda = -\varepsilon\gamma - 2i\check{c}\tau - \tau^2 - a \pm \sqrt{(\varepsilon\gamma - a)^2 - 4\varepsilon + \tau^4 - 2(\varepsilon\gamma - a)\tau^2}. \quad (4.3)$$

Note that the square root in (4.3) is either real or purely imaginary. If the square root in (4.3) is real, it holds $0 = \operatorname{Im}(\lambda) = \check{c}\tau$ yielding $\tau = 0$. We obtain two real solutions given by (4.1) if and only if $(\varepsilon\gamma - a)^2 - 4\varepsilon > 0$. If the square root in (4.3) is purely imaginary it holds

$$\begin{aligned} 0 &= 2\operatorname{Im}(\lambda) = -2\check{c}\tau \pm \sqrt{-(\varepsilon\gamma - a)^2 + 4\varepsilon - \tau^4 + 2(\varepsilon\gamma - a)\tau^2}, \\ 2\lambda &= 2\operatorname{Re}(\lambda) = -\varepsilon\gamma - \tau^2 - a, \end{aligned}$$

yielding

$$\tau^2 = -2\check{c}^2 + \varepsilon\gamma - a \pm \sqrt{(2\check{c}^2 - \varepsilon\gamma + a)^2 - (\varepsilon\gamma - a)^2 + 4\varepsilon}.$$

Since we have $\tau^2 \geq 0$, we obtain one real solution given by (4.1) if and only if $(\varepsilon\gamma - a)^2 - 4\varepsilon \leq 0$. $\square$

4.5 Point spectrum

In order to prove Theorem 4.2, we need to show that the point spectrum of $\mathcal{L}_{a,\varepsilon}$ to the right of the essential spectrum consists at most of two eigenvalues. One of these eigenvalues is the simple translational eigenvalue $\lambda = 0$. The other eigenvalue is real and strictly negative. We will establish that this second eigenvalue is bounded away from the imaginary axis by εb_0 for some $b_0 > 0$. Moreover, we aim to provide a leading order expression of this eigenvalue in the hyperbolic and nonhyperbolic regimes to prove Theorem 4.4.

We cover the critical point spectrum by the following three regions (see Figure 4.5),

$$R_1 = R_1(\delta) := B(0, \delta),$$

$$R_2 = R_2(\delta, M) := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \geq -\delta, \delta \leq |\lambda| \leq M\},$$

$$R_3 = R_3(M) := \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq 2\pi/3, |\lambda| > M\},$$

where $\delta, M > 0$ are a - and ε -independent constants. Recall that the point spectrum of $\mathcal{L}_{a,\varepsilon}$ is given by the eigenvalues λ of the linear problem (2.3), i.e. the λ -values such that (2.3) has an exponentially localized solution.

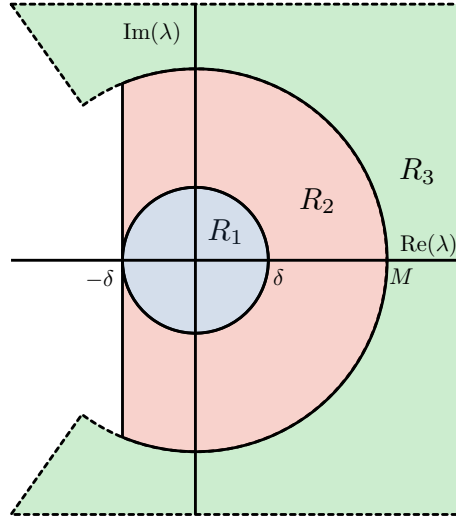


Figure 4.5: Shown are the regions $R_1(\delta)$, $R_2(\delta, M)$, $R_3(M)$ considered in the point spectrum analysis.

We start by showing that for $M > 0$ sufficiently large, the region $R_3(M)$ contains no point spectrum by rescaling the eigenvalue problem (2.3). The analysis in the regions R_1 and R_2 is more elaborate. The first step is to shift the essential spectrum away from the imaginary axis by introducing an exponential weight $\eta > 0$. The eigenvalues λ of system (2.3) and its shifted counterpart coincide to the right of the essential spectrum. Thus, it is sufficient to look at the eigenvalues λ of the shifted system to determine the critical point spectrum of $\mathcal{L}_{a,\varepsilon}$. We proceed by constructing a piecewise continuous eigenfunction for any prospective eigenvalue λ to the shifted problem. Finding eigenvalues then reduces to identifying the values of λ for which the discontinuous jumps vanish.

4.5.1 The region R_3

In this section we show that R_3 contains no point spectrum of $\mathcal{L}_{a,\varepsilon}$. Our approach is to prove that for $\lambda \in R_3(M)$, provided $M > 0$ is sufficiently large, a rescaled version of system (2.3) either has an exponential dichotomy on $\mathbb{R}$ or an exponential

trichotomy on $\mathbb{R}$ with one-dimensional center direction.

Definition 4.5.1. Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \text{Mat}_{n \times n}(\mathbb{C}))$. Denote by $T(x, y)$ the evolution operator of

$$\varphi_x = A(x)\varphi. \quad (5.1)$$

Equation (5.1) has an exponential dichotomy on J with constants $K, \mu > 0$ and projections $P^s(x), P^u(x): \mathbb{C}^n \rightarrow \mathbb{C}^n, x \in J$ if for all $x, y \in J$ it holds

- $P^u(x) + P^s(x) = 1;$
- $P^{u,s}(x)T(x, y) = T(x, y)P^{u,s}(y);$
- $\|T(x, y)P^s(y)\|, \|T(y, x)P^u(x)\| \leq Ke^{-\mu(x-y)}$ for $x \geq y$.

Equation (5.1) has an exponential trichotomy on J with constants $K, \mu, \nu > 0$ and projections $P^u(x), P^s(x), P^c(x): \mathbb{C}^n \rightarrow \mathbb{C}^n, x \in J$ if for all $x, y \in J$ it holds

- $P^u(x) + P^s(x) + P^c(x) = 1;$
- $P^{u,s,c}(x)T(x, y) = T(x, y)P^{u,s,c}(y);$
- $\|T(x, y)P^s(y)\|, \|T(y, x)P^u(x)\| \leq Ke^{-\mu(x-y)}$ for $x \geq y;$
- $\|T(x, y)P^c(y)\| \leq Ke^{\nu|x-y|}.$

Often we use the abbreviations $T^{u,s,c}(x, y) = T(x, y)P^{u,s,c}(y)$ leaving the associated projections of the dichotomy or trichotomy implicit. It is well-known that exponential separation is an important tool in studying spectral properties of traveling waves [49]. For an extensive introduction we refer to [11, 47].

We proceed by showing that in $R_3(M)$, the system (2.3) admits such an exponential separation and converges to the same asymptotic system as $\xi \rightarrow \infty$ and $\xi \rightarrow -\infty$, and thus can not have nontrivial exponentially localized solutions.

Proposition 4.5.2. *In the setting of Theorem 4.1, let $\hat{\phi}_{a,\varepsilon}(\xi)$ denote a traveling-pulse solution to (2.2) with associated linear operator $\mathcal{L}_{a,\varepsilon}$. There exists $M > 0$, independent of a and ε , such that the region $R_3(M)$ contains no point spectrum of $\mathcal{L}_{a,\varepsilon}$.*

Proof. Let $\lambda \in R_3$. We rescale system (2.3) by putting $\tilde{\xi} = \sqrt{|\lambda|}\xi$, $\tilde{u} = u$, $\sqrt{|\lambda|}\tilde{v} = v$ and $\tilde{w} = w$. The resulting system is of the form

$$\begin{aligned} \psi_\xi &= \check{A}(\xi, \lambda)\psi, \quad \check{A}(\xi, \lambda) = \check{A}(\xi, \lambda; a, \varepsilon) := \check{A}_1(\lambda) + \frac{1}{\sqrt{|\lambda|}}\check{A}_2(\xi, \lambda), \\ \check{A}_1(\lambda) &= \check{A}_1(\lambda; a, \varepsilon) := \begin{pmatrix} 0 & 1 & 0 \\ \frac{\lambda}{|\lambda|} & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{\check{c}\sqrt{|\lambda|}} \end{pmatrix}, \\ \check{A}_2(\xi, \lambda) &= \check{A}_2(\xi, \lambda; a, \varepsilon) := \begin{pmatrix} 0 & 0 & 0 \\ -\frac{f'(u)}{\sqrt{|\lambda|}} & \check{c} & \frac{1}{\sqrt{|\lambda|}} \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\varepsilon\gamma}{\check{c}} \end{pmatrix}, \end{aligned} \tag{5.2}$$

where we dropped the tildes. Note that $\check{A}_2$ is bounded on $\mathbb{R} \times R_3$ uniformly in $(a, \varepsilon) \in [0, \frac{1}{2} - \kappa] \times [0, \varepsilon_0]$. Our goal is to show that (5.2), and thus (2.3), admits no nontrivial exponentially localized solutions for $\lambda \in R_3$.

Since we have $|\arg(\lambda)| < 2\pi/3$ for all $\lambda \in R_3$, it holds $\operatorname{Re}(\sqrt{\lambda/|\lambda|}) > 1/2$. We distinguish between the cases $4|\operatorname{Re}(\lambda)| > \check{c}\sqrt{|\lambda|}$ and $4|\operatorname{Re}(\lambda)| \leq \check{c}\sqrt{|\lambda|}$. First, suppose $4|\operatorname{Re}(\lambda)| > \check{c}\sqrt{|\lambda|}$, then $\check{A}_1(\lambda)$ is hyperbolic with spectral gap larger than $1/4$. Thus, by roughness [11, p. 34] system (5.2) has an exponential dichotomy

on $\mathbb{R}$ for $M > 0$ sufficiently large (with lower bound independent of a , ε and λ). Hence, (5.2) admits no nontrivial exponentially localized solutions and λ is not in the point spectrum of $\mathcal{L}_{a,\varepsilon}$.

Second, suppose $4|\operatorname{Re}(\lambda)| \leq \check{c}\sqrt{|\lambda|}$, then $\check{A}_1(\lambda)$ has one (spatial) eigenvalue with absolute real part $\leq 1/4$ and two eigenvalues with absolute real part $\geq 1/2$. By roughness system (5.2) has an exponential trichotomy on $\mathbb{R}$ for $M > 0$ sufficiently large (with lower bound independent of a , ε and λ). Hence, all exponentially localized solution must be contained in the one-dimensional center subspace. Fix $0 < k < 1/8$. By continuity the eigenvalues of the asymptotic matrix $\check{A}_\infty(\lambda) := \lim_{\xi \rightarrow \pm\infty} \check{A}(\xi, \lambda)$ are separated in one eigenvalue v with absolute real part $\leq 1/4 + k$ and two eigenvalues with absolute real part $\geq 1/2 - k$ provided $M > 0$ is sufficiently large (with lower bound independent of a , ε and λ). Let β be the eigenvector associated with v . Using [40, Theorem 1] we conclude that any solution $\psi(\xi)$ in the center subspace of (5.2) satisfies $\lim_{\xi \rightarrow \pm\infty} \psi(\xi)e^{-v\xi} = b_\pm\beta$ for some $b_\pm \in \mathbb{C} \setminus \{0\}$ and is therefore only exponentially localized in case it is trivial. Therefore, λ is not in the point spectrum of $\mathcal{L}_{a,\varepsilon}$. $\square$

4.5.2 Setup for the regions R_1 and R_2

As described at the start of this section, we introduce a weight $\eta > 0$ and study the shifted system

$$\psi_\xi = A(\xi, \lambda)\psi, \quad A(\xi, \lambda) = A(\xi, \lambda; a, \varepsilon) := A_0(\xi, \lambda; a, \varepsilon) - \eta, \quad (5.3)$$

instead of the original eigenvalue problem (2.3) to determine the point spectrum of $\mathcal{L}_{a,\varepsilon}$ on the right hand side of the essential spectrum in the region $R_1 \cup R_2$. In

this section we describe the approach in more detail and fully formulate the shifted eigenvalue problem.

Approach

The structure (3.9) of the singular limit $\phi_{a,0}$ of the pulse $\phi_{a,\varepsilon}$ leads to our framework for the construction of exponentially localized solutions to (5.3) in the regions R_1 and R_2 . More specifically, depending on the value of $\xi \in \mathbb{R}$ the pulse $\phi_{a,\varepsilon}(\xi)$ is to leading order described by the front ϕ_f , the back ϕ_b or the left or right slow manifolds $\mathcal{M}_\varepsilon^\ell$ and $\mathcal{M}_\varepsilon^r$ (see Theorem 4.5). This leads to a partition of the real line in four intervals given by

$$\begin{aligned} I_f &= (-\infty, L_\varepsilon], & I_r &= [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon], \\ I_b &= [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon} + L_\varepsilon], & I_\ell &= [Z_{a,\varepsilon} + L_\varepsilon, \infty), \end{aligned}$$

where $Z_{a,\varepsilon} = \mathcal{O}_s(\varepsilon^{-1})$ is defined in Theorem 4.5 and stands for the time the traveling-pulse solution spends near the right slow manifold $\mathcal{M}_\varepsilon^r$, and L_ε is given by

$$L_\varepsilon := -\nu \log \varepsilon,$$

with $\nu > 0$ an a - and ε -independent constant. The endpoints of the above intervals correspond to the ξ -values for which $\phi_{a,\varepsilon}(\xi)$ converges to one of the four non-smooth corners of the singular concatenation $\phi_{a,0}$; see §4.3.4 and Figure 4.3. Recall from Theorem 4.5 that the pulse $\phi_{a,\varepsilon}(\xi)$ is for ξ in I_r or I_ℓ close to the right or left slow manifold, respectively. Moreover, for ξ in I_f or I_b the pulse $\phi_{a,\varepsilon}(\xi)$ is approximated by the front or the back, respectively.

When the weight $\eta > 0$ is chosen appropriately, the spectrum of the coefficient matrix $A(\xi, \lambda)$ of system (5.3) has for ξ -values in I_r and I_ℓ a consistent splitting into one unstable and two stable eigenvalues. This splitting along the slow manifolds guarantees the existence of exponential dichotomies on the intervals I_r and I_ℓ . Solutions to (5.3) can be decomposed in terms of these dichotomies. To obtain suitable expressions for the solutions in the other two intervals I_f and I_b we have to distinguish between the regions R_1 and R_2 .

We start with describing the set-up for the region R_1 . For $\xi \in I_f$ we establish a reduced eigenvalue problem by setting ε and λ to 0 in system (5.3), while approximating $\phi_{a,\varepsilon}(\xi)$ with the front $\phi_f(\xi)$. The reduced eigenvalue problem admits exponential dichotomies on both half-lines. The full eigenvalue problem (5.3) can be seen as a (λ, ε) -perturbation of the reduced eigenvalue problem. Hence, one can construct solutions to (5.3) using a variation of constants approach on intervals

$$I_{f,-} := (-\infty, 0], \quad I_{f,+} := [0, L_\varepsilon],$$

which partition I_f and correspond to the positive and negative half-lines in the singular limit. The perturbation term is kept under control by taking $\delta > 0$ and $\varepsilon > 0$ sufficiently small. Similarly, we establish a reduced eigenvalue problem along the back and one can construct solutions to (5.3) using a variation of constants approach on intervals

$$I_{b,-} := [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon}], \quad I_{b,+} := [Z_{a,\varepsilon}, Z_{a,\varepsilon} + L_\varepsilon].$$

In summary, we obtain variation of constants formulas for the solutions to (5.3) on the four intervals $I_{f,\pm}$ and $I_{b,\pm}$ and expressions for the solutions to (5.3) in terms of exponential dichotomies on the two intervals I_r and I_ℓ . Matching of these expres-

sions yields for any $\lambda \in R_1$ a piecewise continuous, exponentially localized solution to (5.3) which has jumps at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$. Finding eigenvalues then reduces to locating $\lambda \in R_1$ for which the two jumps vanish. Equating the jumps to zero leads to an analytic matching equation that is to leading order a quadratic in λ . The two solutions to this equation are the two eigenvalues of the shifted eigenvalue problem (5.3) in $R_1(\delta)$.

We know a priori that $\lambda = 0$ is a solution to the matching equation by translational invariance. The associated eigenfunction of (5.3) is the weighted derivative $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ of the pulse. This information can be used to simplify some of the expressions in the matching equation. In the hyperbolic regime, this leads to a leading order expression of the second nonzero eigenvalue. In the nonhyperbolic regime the expressions in the matching equations relate to the dynamics at the fold point. One needs detailed information about the dynamics in the blow-up coordinates to determine the sign and magnitude of these expressions, which eventually yield that the second eigenvalue is strictly negative and smaller than $b_0\varepsilon$ for some $b_0 > 0$ independent of a and ε . In the regime $K_0a^3 < \varepsilon$, a leading order expression for the second eigenvalue can be determined, which is of the order $\mathcal{O}(\varepsilon^{2/3})$.

Finally, we describe the set-up in the region R_2 . We establish reduced eigenvalue problems for $\lambda \in R_2$ by setting ε to 0 in (5.3), while approximating $\phi_{a,\varepsilon}(\xi)$ with (a translate of) the front $\phi_f(\xi)$ or the back $\phi_b(\xi)$. However, we do keep the λ -dependence in contrast to the reduction done in the region R_1 . In this case the reduced eigenvalue problems admit exponential dichotomies on the whole real line. By roughness these dichotomies transfer to exponential dichotomies of (5.3) on the two intervals I_f and I_b . Thus, the real line is partitioned in four intervals I_f, I_b, I_r and I_ℓ such that in each interval (5.3) admits an exponential dichotomy governing the solutions. By comparing the associated projections at the endpoints of these

intervals, we show that for $\lambda \in R_2(\delta, M)$ the shifted eigenvalue problem (5.3) can not have a nontrivial exponentially localized solution for any $M > 0$ and each $\delta > 0$ sufficiently small.

Formulation of the shifted eigenvalue problem

In this section we determine $\eta, \nu > 0$ such that the shifted system (5.3) admits exponential dichotomies on the intervals $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$ and $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$, where $L_\varepsilon = -\nu \log \varepsilon$ and $Z_{a,\varepsilon}$ is as in Theorem 4.5. Recall that for ξ -values in I_r and I_ℓ the pulse $\phi_{a,\varepsilon}(\xi)$ is close to the right and left slow manifold, respectively. The following technical result shows that for appropriate values of η the spectrum of the coefficient matrix $A(\xi, \lambda)$ of system (5.3) has for ξ -values in I_r and I_ℓ a consistent splitting into one unstable and two stable eigenvalues.

Lemma 4.5.3. *Let $\kappa, M > 0$ and define for $\sigma_0 > 0$*

$$\mathcal{U}(\sigma_0, \kappa) := \left\{ (a, u) \in \mathbb{R}^2 : a \in \left[0, \frac{1}{2} - \kappa\right], \right. \\ \left. u \in \left[\frac{1}{3}(2a - 1) - \sigma_0, \sigma_0\right] \cup \left[\frac{2}{3}(a + 1) - \sigma_0, 1 + \sigma_0\right] \right\}.$$

Take $\eta = \frac{1}{2}\sqrt{2}\kappa > 0$. For $\sigma_0, \delta > 0$ sufficiently small, there exists $\varepsilon_0 > 0$ and $0 < \mu \leq \eta$ such that the matrix

$$\hat{A} = \hat{A}(u, \lambda, a, \varepsilon) := \begin{pmatrix} -\eta & 1 & 0 \\ \lambda - f'(u) & \check{c} - \eta & 1 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda + \varepsilon\gamma}{\check{c}} - \eta \end{pmatrix},$$

has for $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$, $\lambda \in (R_1(\delta) \cup R_2(\delta, M))$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ a uniform spectral gap larger than $\mu > 0$ and precisely one eigenvalue of positive real part.

Proof. The matrix $\hat{A}(u, \lambda, a, \varepsilon)$ is nonhyperbolic if and only if

$$\begin{aligned} 0 &= \det(\hat{A}(u, \lambda, a, \varepsilon) - i\tau) \\ &= (\eta^2 - \tau^2 + 2i\tau\eta - \check{c}i\tau + f'(u) - \lambda - \check{c}\eta) \left(-i\tau - \frac{\lambda + \varepsilon\gamma}{\check{c}} - \eta\right) + \frac{\varepsilon}{\check{c}}, \end{aligned}$$

is satisfied for some $\tau \in \mathbb{R}$. Thus, all λ -values for which $\hat{A}(u, \lambda, a, 0)$ is nonhyperbolic are given by the union of a line and a parabola

$$\{-\check{c}_0\eta + i\check{c}_0\tau : \tau \in \mathbb{R}\} \cup \{\eta^2 - \tau^2 + 2i\tau\eta - \check{c}_0i\tau + f'(u) - \check{c}_0\eta : \tau \in \mathbb{R}\}. \quad (5.4)$$

Recall that $\check{c}_0 = \check{c}_0(a)$ is given by $\sqrt{2}(\frac{1}{2} - a)$. For any $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$, it holds $\check{c}_0 = \check{c}_0(a) \geq \sqrt{2}\kappa$ and $f'(u) = -3u^2 + 2(a+1)u - a \leq 3\sigma_0$. Hence, for $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$ the union (5.4) lies in the half plane

$$\operatorname{Re}(\lambda) \leq \max \left\{ -\check{c}_0\eta, \eta^2 - \sqrt{2}\kappa\eta + 3\sigma_0 \right\}.$$

Take $\eta = \frac{1}{2}\sqrt{2}\kappa$ and $3\sigma_0 < \frac{1}{4}\kappa^2$. We deduce that (5.4) is contained in $\operatorname{Re}(\lambda) \leq -\frac{1}{4}\kappa^2 < 0$ for any $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$. Hence, provided $\delta > 0$ is sufficiently small, the union (5.4) doesn't intersect the compact set $R_1(\delta) \cup R_2(\delta, M)$ for any (a, u) in the compact set $\mathcal{U}(\sigma_0, \kappa)$. By continuity we conclude that there exists $\varepsilon_0 > 0$ such that the matrix $\hat{A}(u, \lambda, a, \varepsilon)$ has for $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$, $\lambda \in (R_1(\delta) \cup R_2(\delta, M))$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ a uniform spectral gap larger than some $\mu > 0$. Note that $-\eta$ is in the spectrum of $\hat{A}(0, 0, a, 0)$. Therefore, we must have $\mu \leq \eta$.

In addition, one readily observes that for sufficiently large $\lambda > 0$ the matrix $\hat{A}(u, \lambda, a, 0)$ has precisely one eigenvalue of positive real part. On the other hand, the union (5.4) lies in the half plane $\operatorname{Re}(\lambda) \leq -\frac{1}{4}\kappa^2 < 0$ for $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$. So, by continuity $\hat{A}(u, \lambda, a, 0)$ has precisely one eigenvalue of positive real part for $\lambda \in \mathbb{C}$

lying to the right of (5.4). Taking $\delta, \varepsilon_0 > 0$ sufficiently small, we conclude that $\hat{A}(u, \lambda, a, \varepsilon)$ has precisely one eigenvalue of positive real part for $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$, $\lambda \in (R_1(\delta) \cup R_2(\delta, M))$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. $\square$

We are now able to state a suitable version of the shifted eigenvalue problem (5.3). Thus, we started with $\kappa > 0$ and $K > K^*$, where $K^* > 0$ is as in Theorem 4.1. Then, Theorem 4.1 provided us with an $\varepsilon_0 > 0$ such that for any $(a, \varepsilon) \in [0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$ there exists a traveling-pulse solution $\tilde{\phi}_{a,\varepsilon}(\xi)$ to (2.2). In Proposition 4.5.2 we obtained $M > 0$, independent of a and ε , such that the region $R_3(M)$ contains no point spectrum of the associated linear operator $\mathcal{L}_{a,\varepsilon}$. We fix

$$\eta := \frac{1}{2}\sqrt{2}\kappa > 0,$$

and take $\nu > 0$ an a - and ε -independent constant satisfying

$$\nu \geq \max \left\{ \frac{2}{\mu}, 2\sqrt{2} \right\} > 0, \quad (5.5)$$

where $\mu > 0$ is as in Lemma 4.5.3. The shifted eigenvalue problem is given by

$$\begin{aligned} \psi_\xi &= A(\xi, \lambda)\psi, \\ A(\xi, \lambda) &= A(\xi, \lambda; a, \varepsilon) := \begin{pmatrix} -\eta & 1 & 0 \\ \lambda - f'(u_{a,\varepsilon}(\xi)) & \check{c} - \eta & 1 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda + \varepsilon\gamma}{\check{c}} - \eta \end{pmatrix}, \\ (\lambda, a, \varepsilon) &\in (R_1(\delta) \cup R_2(\delta, M)) \times [0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0), \quad \varepsilon < Ka^2, \end{aligned} \quad (5.6)$$

where $u_{a,\varepsilon}(\xi)$ denotes the u -component of the pulse $\tilde{\phi}_{a,\varepsilon}(\xi)$ and $\delta > 0$ is as in Lemma 4.5.3. In the next section we will show that with the above choice of η, δ, M and ν system (5.6) admits for $\lambda \in R_1(\delta) \cup R_2(\delta, M)$ exponential dichotomies on the

intervals $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$ and $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$, where

$$L_\varepsilon = -\nu \log \varepsilon,$$

and $Z_{a,\varepsilon}$ is as in Theorem 4.5. However, before establishing these dichotomies, we prove that it is indeed sufficient to study the shifted eigenvalue problem (5.6) to determine the critical point spectrum of $\mathcal{L}_{a,\varepsilon}$ in $R_1 \cup R_2$.

Proposition 4.5.4. *In the setting of Theorem 4.1, let $\tilde{\phi}_{a,\varepsilon}(x, t)$ denote a traveling-pulse solution to (2.2) with associated linear operator $\mathcal{L}_{a,\varepsilon}$. A point $\lambda \in R_1 \cup R_2$ lying to the right of the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is in the point spectrum of $\mathcal{L}_{a,\varepsilon}$ if and only if it is an eigenvalue of the shifted eigenvalue problem (5.6).*

Proof. The spectra of the asymptotic matrices $\hat{A}_0(\lambda; a, \varepsilon)$ and $\hat{A}(0, \lambda, a, \varepsilon)$ of systems (2.3) and (5.6), respectively, are related via $\sigma(\hat{A}(0, \lambda, a, \varepsilon)) = \sigma(\hat{A}_0(\lambda; a, \varepsilon)) - \eta$. Moreover, both $\hat{A}(0, \lambda, a, \varepsilon)$ and $\hat{A}_0(\lambda; a, \varepsilon)$ have precisely one (spatial) eigenvalue of positive real part for $\lambda \in R_1 \cup R_2$ to the right of the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ by Proposition 4.4.1 and Lemma 4.5.3. Therefore, for $\lambda \in R_1 \cup R_2$ to the right of the essential spectrum of $\mathcal{L}_{a,\varepsilon}$, system (2.3) admits a nontrivial exponentially localized solution $\psi(\xi)$ if and only if system (5.6) admits one given by $e^{-\eta\xi}\psi(\xi)$. $\square$

Exponential dichotomies along the right and left slow manifolds

For ξ -values in I_ℓ or I_r the pulse $\phi_{a,\varepsilon}(\xi)$ is by Theorem 4.5 close to the right or left slow manifolds on which the dynamics is of the order $\mathcal{O}(\varepsilon)$. Hence, for $\xi \in I_\ell \cup I_r$ the coefficient matrix $A(\xi, \lambda)$ of the shifted eigenvalue problem (5.6) has slowly varying coefficients and is pointwise hyperbolic by Lemma 4.5.3. It is well-known that such systems admit exponential dichotomies; see [11, Proposition 6.1]. We will prove

below that the associated projections can be chosen to depend analytically on λ and are close to the spectral projections on the (un)stable eigenspaces of $A(\xi, \lambda)$. As described in §4.5.2 the exponential dichotomies provide the framework for the construction of solutions to (5.6) on I_r and I_ℓ . The approximations of the dichotomy projections by the spectral projections are needed to match solutions to (5.6) on I_r and I_ℓ to solutions on the other two intervals I_f and I_b .

Proposition 4.5.5. *For each sufficiently small $a_0 > 0$, there exists $\varepsilon_0 > 0$ such that system (5.6) admits for $0 < \varepsilon < \varepsilon_0$ exponential dichotomies on the intervals $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$ and $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$ with constants $C, \mu > 0$, where $\mu > 0$ is as in Lemma 4.5.3. The associated projections $\mathcal{Q}_{r,\ell}^{u,s}(\xi, \lambda) = \mathcal{Q}_{r,\ell}^{u,s}(\xi, \lambda; a, \varepsilon)$ are analytic in λ on $R_1 \cup R_2$ and are approximated at the endpoints $L_\varepsilon, Z_{a,\varepsilon} \pm L_\varepsilon$ by*

$$\begin{aligned} \|[\mathcal{Q}_r^s - \mathcal{P}](L_\varepsilon, \lambda)\| &\leq C\varepsilon|\log \varepsilon|, \\ \|[\mathcal{Q}_r^s - \mathcal{P}](Z_{a,\varepsilon} - L_\varepsilon, \lambda)\|, \|[\mathcal{Q}_\ell^s - \mathcal{P}](Z_{a,\varepsilon} + L_\varepsilon, \lambda)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|, \end{aligned}$$

where $\rho(a) = 1$ for $a \geq a_0$, $\rho(a) = \frac{2}{3}$ for $a < a_0$ and $\mathcal{P}(\xi, \lambda) = \mathcal{P}(\xi, \lambda; a, \varepsilon)$ are the spectral projections onto the stable eigenspace of the coefficient matrix $A(\xi, \lambda)$ of (5.6). In the above $C > 0$ is a constant independent of λ, a and ε .

Proof. We begin by proving the existence of the desired exponential dichotomy on the interval I_r . The construction on the interval I_ℓ is similar, and we outline the differences only. Denote $\hat{L}_\varepsilon := L_\varepsilon/2 = -\frac{\nu}{2} \log \varepsilon$. We introduce a smooth partition of

unity $\chi_i: \mathbb{R} \rightarrow [0, 1]$, $i = 1, 2, 3$, satisfying

$$\sum_{i=1}^3 \chi_i(\xi) = 1, \quad |\chi'_i(\xi)| \leq 2, \quad \xi \in \mathbb{R},$$

$$\text{supp}(\chi_1) \subset (-\infty, \hat{L}_\varepsilon),$$

$$\text{supp}(\chi_2) \subset (\hat{L}_\varepsilon - 1, Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1),$$

$$\text{supp}(\chi_3) \subset (Z_{a,\varepsilon} - \hat{L}_\varepsilon, \infty).$$

The equation

$$\psi_\xi = \mathcal{A}(\xi, \lambda)\psi, \tag{5.7}$$

with

$$\mathcal{A}(\xi, \lambda) = \mathcal{A}(\xi, \lambda; a, \varepsilon) := \chi_1(\xi)A(\hat{L}_\varepsilon, \lambda) + \chi_2(\xi)A(\xi, \lambda) + \chi_3(\xi)A(Z_{a,\varepsilon} - \hat{L}_\varepsilon, \lambda),$$

coincides with (5.6) on I_r . By Theorem 4.5 (iii) there exists, for any $\sigma_0 > 0$ sufficiently small, a constant $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ it holds

$$\|u'_{a,\varepsilon}(\xi)\| \leq \sigma_0, \quad u_{a,\varepsilon}(\xi) \in [u_b^1 - \sigma_0, 1 + \sigma_0] = \left[\frac{2}{3}(a+1) - \sigma_0, 1 + \sigma_0\right]. \tag{5.8}$$

for $\xi \in [\hat{L}_\varepsilon - 1, Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1]$. We calculate

$$\partial_\xi \mathcal{A}(\xi, \lambda) = \begin{cases} \chi_2(\xi) \partial_\xi A(\xi, \lambda), & \xi \in (\hat{L}_\varepsilon, Z_{a,\varepsilon} - \hat{L}_\varepsilon), \\ \chi_2'(\xi)(A(\xi, \lambda) - A(\hat{L}_\varepsilon, \lambda)) \\ \quad + \chi_2(\xi) \partial_\xi A(\xi, \lambda), & \xi \in [\hat{L}_\varepsilon - 1, \hat{L}_\varepsilon], \\ \chi_2'(\xi)(A(\xi, \lambda) - A(Z_{a,\varepsilon} - \hat{L}_\varepsilon, \lambda)) \\ \quad + \chi_2(\xi) \partial_\xi A(\xi, \lambda), & \xi \in [Z_{a,\varepsilon} - \hat{L}_\varepsilon, Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1], \\ 0, & \text{otherwise.} \end{cases} \quad (5.9)$$

First, we have that $\|\partial_\xi \mathcal{A}(\xi, \lambda)\| \leq C\sigma_0$ on $\mathbb{R} \times (R_1 \cup R_2)$ by the mean value theorem and identities (5.8) and (5.9). Second, by Lemma 4.5.3 and (5.8) the matrix $\mathcal{A}(\xi, \lambda)$ is hyperbolic on $\mathbb{R} \times (R_1 \cup R_2)$ with a - and ε -uniform spectral gap larger than $\mu > 0$. Third, $\mathcal{A}(\xi, \lambda)$ can be bounded on $\mathbb{R} \times (R_1 \cup R_2)$ uniformly in a and ε . Combining these three items with [11, Proposition 6.1] gives that system (5.7) has, provided $\sigma_0 > 0$ is sufficiently small, an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu > 0$, independent of λ, a and ε , and projections $\mathcal{Q}_r^{u,s}(\xi, \lambda) = \mathcal{Q}_r^{u,s}(\xi, \lambda; a, \varepsilon)$. Since (5.7) coincides with (5.6) on $[\hat{L}_\varepsilon, Z_{a,\varepsilon} - \hat{L}_\varepsilon]$, we have established the desired exponential dichotomy of (5.6) on I_r with constants $C, \mu > 0$ and projections $\mathcal{Q}_r^{u,s}(\xi, \lambda)$.

The next step is to prove that the projections $\mathcal{Q}_r^{u,s}(\xi, \lambda)$ are analytic in λ on $R_1 \cup R_2$. Any solution to the constant coefficient system $\psi_\xi = A(\hat{L}_\varepsilon, \lambda)\psi$ that converges to 0 as $\xi \rightarrow -\infty$ must be in the kernel of the spectral projection $\mathcal{P}(\hat{L}_\varepsilon, \lambda)$ on the stable eigenspace of $A(\hat{L}_\varepsilon, \lambda)$. Hence, it holds $R(1 - \mathcal{P}(\hat{L}_\varepsilon, \lambda)) = R(\mathcal{Q}_r^u(\hat{L}_\varepsilon - 1, \lambda))$ by construction of (5.7). Moreover, the spectral projection $\mathcal{P}(\hat{L}_\varepsilon, \lambda)$ is analytic in λ , since $A(\hat{L}_\varepsilon, \lambda)$ is analytic in λ . Thus, $R(\mathcal{Q}_r^u(\hat{L}_\varepsilon - 1, \lambda))$ and similarly $R(\mathcal{Q}_r^s(Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1, \lambda))$ must be analytic subspaces in λ . Denote by $\mathcal{T}(\xi, \hat{\xi}, \lambda) = \mathcal{T}(\xi, \hat{\xi}, \lambda; a, \varepsilon)$ the

evolution of (5.7), which is analytic in λ . We conclude that both $\ker(\mathcal{Q}_r^s(\hat{L}_\varepsilon - 1, \lambda))$ and

$$R(\mathcal{Q}_r^s(\hat{L}_\varepsilon - 1, \lambda)) = R(\mathcal{T}(\hat{L}_\varepsilon - 1, Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1, \lambda)\mathcal{Q}_r^s(Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1, \lambda)),$$

are analytic subspaces. Therefore, the projection $\mathcal{Q}_r^s(\hat{L}_\varepsilon - 1, \lambda)$ (and thus any projection $\mathcal{Q}_r^{u,s}(\xi, \lambda)$, $\xi \in \mathbb{R}$) is analytic in λ on $R_1 \cup R_2$.

Finally, we shall prove that the projections $\mathcal{Q}_r^s(\xi, \lambda)$ are close to the spectral projections $\mathcal{P}(\xi, \lambda)$ on the stable eigenspace of $A(\xi, \lambda)$ at the points $\xi = L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon$. First, observe that we have,

$$\begin{aligned} |u'_{a,\varepsilon}(\xi)| &\leq C\varepsilon|\log \varepsilon|, \quad \xi \in [\hat{L}_\varepsilon, 3\hat{L}_\varepsilon], \\ |u'_{a,\varepsilon}(\xi)| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|, \quad \xi \in [Z_{a,\varepsilon} - 3\hat{L}_\varepsilon, Z_{a,\varepsilon} - \hat{L}_\varepsilon], \end{aligned} \tag{5.10}$$

by Theorem 4.5 (i)-(ii). Consider the family of constant coefficient systems

$$\psi_\xi = \hat{A}(u, \lambda)\psi, \tag{5.11}$$

parameterized over $u \in \mathbb{R}$, where $\hat{A}(u, \lambda) = \hat{A}(u, \lambda; a, \varepsilon)$ is defined in Lemma 4.5.3. Denote by $\hat{\mathcal{P}}(u, \lambda) = \hat{\mathcal{P}}(u, \lambda; a, \varepsilon)$ the spectral projection on the stable eigenspace of $\hat{A}(u, \lambda)$ and by $\hat{\mathcal{T}}(\xi, \hat{\xi}, u, \lambda) = \hat{\mathcal{T}}(\xi, \hat{\xi}, u, \lambda; a, \varepsilon)$ the evolution operator of (5.11). Thus, we have $\hat{A}(u_{a,\varepsilon}(\xi), \lambda) = A(\xi, \lambda)$ and $\hat{\mathcal{P}}(u_{a,\varepsilon}(\xi), \lambda) = \mathcal{P}(\xi, \lambda)$ for $\xi \in \mathbb{R}$. Let $b_1 \in R(\mathcal{P}(L_\varepsilon, \lambda))$. Observe that

$$\hat{\psi}(\xi) := \mathcal{P}(\xi, \lambda)\hat{\mathcal{T}}(\xi, L_\varepsilon, u_{a,\varepsilon}(\xi), \lambda)b_1,$$

satisfies the inhomogeneous equation

$$\psi_\xi = A(\xi, \lambda)\psi + \hat{g}(\xi), \quad \hat{g}(\xi) := \partial_u \hat{\mathcal{P}}(u, \lambda) \hat{\mathcal{T}}(\xi, L_\varepsilon, u, \lambda) \Big|_{u=u_{a,\varepsilon}(\xi)} u'_{a,\varepsilon}(\xi) b_1.$$

By the variation of constants formula there exists $b_2 \in \mathbb{C}^3$ such that

$$\begin{aligned} \hat{\psi}(\xi) &= \mathcal{T}(\xi, L_\varepsilon + \hat{L}_\varepsilon, \lambda) b_2 + \int_{L_\varepsilon}^{\xi} \mathcal{Q}_r^s(\xi, \lambda) \mathcal{T}(\xi, \hat{\xi}, \lambda) \hat{g}(\hat{\xi}) d\hat{\xi} \\ &\quad + \int_{L_\varepsilon + \hat{L}_\varepsilon}^{\xi} \mathcal{Q}_r^u(\xi, \lambda) \mathcal{T}(\xi, \hat{\xi}, \lambda) \hat{g}(\hat{\xi}) d\hat{\xi}, \end{aligned} \quad (5.12)$$

for $\xi \in [L_\varepsilon, L_\varepsilon + \hat{L}_\varepsilon]$. By [47, Lemma 1.1] and (5.10) we have

$$\|\hat{\psi}(\xi)\| \leq C e^{-\mu(\xi - L_\varepsilon)} \|b_1\|, \quad \|\hat{g}(\xi)\| \leq C \varepsilon |\log \varepsilon| e^{-\mu(\xi - L_\varepsilon)} \|b_1\|, \quad (5.13)$$

for $\xi \in [L_\varepsilon, L_\varepsilon + \hat{L}_\varepsilon]$. Evaluating (5.12) at $L_\varepsilon + \hat{L}_\varepsilon$ while using (5.13), we derive $\|b_2\| \leq C \varepsilon |\log \varepsilon| \|b_1\|$, since $\nu \geq \mu/2$ by (5.5). Thus, applying $\mathcal{Q}_r^u(L_\varepsilon, \lambda)$ to (5.12) at L_ε yields the bound $\|\mathcal{Q}_r^u(L_\varepsilon, \lambda) b_1\| \leq C \varepsilon |\log \varepsilon| \|b_1\|$ for every $b_1 \in R(\mathcal{P}(L_\varepsilon, \lambda))$ by (5.13). Similarly, one shows that for every $b_1 \in \ker(\mathcal{P}(L_\varepsilon, \lambda))$ we have $\|\mathcal{Q}_r^s(L_\varepsilon, \lambda) b_1\| \leq C \varepsilon |\log \varepsilon| \|b_1\|$. Thus, we obtain

$$\|[\mathcal{Q}_r^s - \mathcal{P}](L_\varepsilon, \lambda)\| \leq \|[\mathcal{Q}_r^u \mathcal{P}](L_\varepsilon, \lambda)\| + \|[\mathcal{Q}_r^s(1 - \mathcal{P})](L_\varepsilon, \lambda)\| \leq C \varepsilon |\log \varepsilon|.$$

The bound at $Z_{a,\varepsilon} - L_\varepsilon$ is obtain analogously.

In a similar way one obtains for $\lambda \in R_1 \cup R_2$ the desired exponential dichotomy for (5.6) on I_ℓ with constants $C, \mu > 0$ and projections $\mathcal{Q}_\ell^{u,s}(\xi, \lambda)$. The only fundamental difference in the analysis is that the analyticity of the range of $\mathcal{Q}_\ell^s(\xi, \lambda)$ is immediate, since the asymptotic system $\lim_{\xi \rightarrow \infty} A(\xi, \lambda)$ is analytic in λ , see [50, Theorem 1]. $\square$

4.5.3 The region $R_1(\delta)$

A reduced eigenvalue problem

As described in §4.5.2 we establish for ξ in I_f or I_b a reduced eigenvalue problem by setting ε and λ to 0 in system (5.6), while approximating $\phi_{a,\varepsilon}(\xi)$ with (a translate of) the front $\phi_f(\xi)$ or the back $\phi_b(\xi)$, respectively. Thus, the reduced eigenvalue problem reads

$$\begin{aligned} \psi_\xi &= A_j(\xi)\psi, \\ A_j(\xi) &= A_j(\xi; a) := \begin{pmatrix} -\eta & 1 & 0 \\ -f'(u_j(\xi)) & \check{c}_0 - \eta & 1 \\ 0 & 0 & -\eta \end{pmatrix}, \quad j = f, b, \end{aligned} \quad (5.14)$$

where $u_j(\xi)$ denotes the u -component of $\phi_j(\xi)$ and a is in $[0, \frac{1}{2} - \kappa]$. Now, for ξ -values in $I_f = (-\infty, L_\varepsilon]$, problem (5.6) can be written as the perturbation

$$\begin{aligned} \psi_\xi &= (A_f(\xi) + B_f(\xi, \lambda))\psi, \\ B_f(\xi, \lambda) &= B_f(\xi, \lambda; a, \varepsilon) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda - [f'(u_{a,\varepsilon}(\xi)) - f'(u_f(\xi))] & \check{c} - \check{c}_0 & 0 \\ \frac{\lambda\varepsilon}{\check{c}} & 0 & -\frac{\lambda+\varepsilon\gamma}{\check{c}} \end{pmatrix}. \end{aligned} \quad (5.15)$$

To define (5.6) as a proper perturbation of (5.14) along the back, we introduce the translated version of (5.6)

$$\psi_\xi = A(\xi + Z_{a,\varepsilon}, \lambda)\psi. \quad (5.16)$$

For ξ -values in $[-L_\varepsilon, L_\varepsilon]$ problem (5.16) can be written as the perturbation

$$\begin{aligned} \psi_\xi &= (A_b(\xi) + B_b(\xi, \lambda)) \psi, \\ B_b(\xi, \lambda) &= B_b(\xi, \lambda; a, \varepsilon) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda - [f'(u_{a,\varepsilon}(\xi + Z_{a,\varepsilon})) - f'(u_b(\xi))] & \check{c} - \check{c}_0 & 0 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda + \varepsilon \gamma}{\check{c}} \end{pmatrix}. \end{aligned} \quad (5.17)$$

The reduced eigenvalue problem (5.14) has an upper triangular block structure. Consequently, system (5.14) leaves the subspace $\mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$ invariant and the dynamics of (5.14) on that space is given by

$$\begin{aligned} \varphi_\xi &= C_j(\xi) \varphi, \\ C_j(\xi) &= C_j(\xi; a) := \begin{pmatrix} -\eta & 1 \\ -f'(u_j(\xi)) & \check{c}_0 - \eta \end{pmatrix}, \quad j = \text{f, b}. \end{aligned} \quad (5.18)$$

Before studying the full reduced eigenvalue problem (5.14) we study the dynamics on the invariant subspace. We observe that system (5.18) has a one-dimensional space of bounded solution spanned by

$$\varphi_j(\xi) = \varphi_j(\xi; a) := e^{-\eta \xi} \phi'_j(\xi), \quad j = \text{f, b}.$$

Therefore, the adjoint system

$$\varphi_\xi = -C_j(\xi)^* \varphi, \quad j = \text{f, b}, \quad (5.19)$$

also has a one-dimensional space of bounded solution spanned by

$$\varphi_{j,\text{ad}}(\xi) = \varphi_{j,\text{ad}}(\xi; a) := \begin{pmatrix} v'_j(\xi) \\ -u'_j(\xi) \end{pmatrix} e^{(\eta - \tilde{c}_0)\xi}, \quad j = \text{f, b.} \quad (5.20)$$

We emphasize that φ_j and $\varphi_{j,\text{ad}}$ can be determined explicitly using the expressions in (3.6) for ϕ_j , $j = \text{f, b.}$ We establish exponential dichotomies for subsystem (5.18) on both half-lines.

Proposition 4.5.6. *Let $\kappa > 0$. For each $a \in [0, \frac{1}{2} - \kappa]$, system (5.18) admits exponential dichotomies on both half-lines $\mathbb{R}_\pm$ with a -independent constants $C, \mu > 0$ and projections $\Pi_{j,\pm}^{u,s}(\xi) = \Pi_{j,\pm}^{u,s}(\xi; a)$, $j = \text{f, b.}$ Here, $\mu > 0$ is as in Lemma 4.5.3 and the projections can be chosen in such a way that*

$$\begin{aligned} R(\Pi_{j,+}^s(0)) &= \text{Span}(\varphi_j(0)) = R(\Pi_{j,-}^u(0)), \\ R(\Pi_{j,+}^u(0)) &= \text{Span}(\varphi_{j,\text{ad}}(0)) = R(\Pi_{j,-}^s(0)), \quad j = \text{f, b.} \end{aligned} \quad (5.21)$$

Proof. Define the asymptotic matrices $C_{j,\pm\infty} = C_{j,\pm\infty}(a) := \lim_{\xi \rightarrow \pm\infty} C_j(\xi)$ of (5.18) for $j = \text{f, b.}$ Consider the matrix $\hat{A}(u, \lambda, a, \varepsilon)$ from Lemma 4.5.3. The spectra of $C_{\text{f},-\infty}$ and $C_{\text{f},\infty}$ are contained in the spectra of $\hat{A}(0, 0, a, 0)$ and $\hat{A}(1, 0, a, 0)$, respectively. Similarly, we have the spectral inclusions $\sigma(C_{\text{b},-\infty}) \subset \sigma(\hat{A}(u_{\text{b}}^1, 0, a, 0))$ and $\sigma(C_{\text{b},\infty}) \subset \sigma(\hat{A}(u_{\text{b}}^0, 0, a, 0))$. By Lemma 4.5.3 the matrices $\hat{A}(u, 0, a, 0)$ have for $u = 0, 1, u_{\text{b}}^0, u_{\text{b}}^1$ and $a \in [0, \frac{1}{2} - \kappa]$ a uniform spectral gap larger than $\mu > 0$. Thus, the same holds for the asymptotic matrices $C_{j,\pm\infty}$, $j = \text{f, b.}$ Hence, it follows from [47, Lemmata 1.1 and 1.2] that system (5.18) admits exponential dichotomies on both half-lines with constants $C, \mu > 0$ and projections as in (5.21). By compactness of $[0, \frac{1}{2} - \kappa]$ the constant $C > 0$ can be chosen independent of a . $\square$

We shift our focus to the full reduced eigenvalue problem (5.14). One readily

observes that

$$\omega_j(\xi) = \omega_j(\xi; a) := \begin{pmatrix} \varphi_j(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-\eta\xi} \phi'_j(\xi) \\ 0 \end{pmatrix}, \quad j = f, b, \quad (5.22)$$

is a bounded solution to (5.14). Moreover, using variation of constants formulas the exponential dichotomies of the subsystem (5.18) can be transferred to the full system (5.14).

Corollary 4.5.7. *Let $\kappa > 0$. For each $a \in [0, \frac{1}{2} - \kappa]$ system (5.14) admits exponential dichotomies on both half-lines $\mathbb{R}_\pm$ with a -independent constants $C, \mu > 0$ and projections $Q_{j,\pm}^{u,s}(\xi) = Q_{j,\pm}^{u,s}(\xi; a)$, $j = f, b$, given by*

$$\begin{aligned} Q_{j,+}^s(\xi) &= \begin{pmatrix} \Pi_{j,+}^s(\xi) & \int_\infty^\xi e^{\eta(\xi-\hat{\xi})} \Phi_{j,+}^u(\xi, \hat{\xi}) F d\hat{\xi} \\ 0 & 1 \end{pmatrix} = 1 - Q_{j,+}^u(\xi), \quad \xi \geq 0, \\ Q_{j,-}^s(\xi) &= \begin{pmatrix} \Pi_{j,-}^s(\xi) & \int_0^\xi e^{\eta(\xi-\hat{\xi})} \Phi_{j,-}^u(\xi, \hat{\xi}) F d\hat{\xi} \\ 0 & 1 \end{pmatrix} = 1 - Q_{j,-}^u(\xi), \quad \xi \leq 0, \end{aligned} \quad (5.23)$$

where F is the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\Phi_{j,\pm}^{u,s}(\xi, \hat{\xi}) = \Phi_{j,\pm}^{u,s}(\xi, \hat{\xi}; a)$ denotes the (un)stable evolution of system (5.18) under the exponential dichotomies established in Proposition 4.5.6. Here, $\mu > 0$ is as in Lemma 4.5.3 and the projections satisfy

$$\begin{aligned} R(Q_{j,+}^u(0)) &= \text{Span}(\Psi_{1,j}), & R(Q_{j,+}^s(0)) &= \text{Span}(\omega_j(0), \Psi_2), \\ R(Q_{j,-}^u(0)) &= \text{Span}(\omega_j(0)), & R(Q_{j,-}^s(0)) &= \text{Span}(\Psi_{1,j}, \Psi_2), \end{aligned} \quad (5.24)$$

where ω_j is defined in (5.22) and

$$\Psi_{1,j} = \Psi_{1,j}(a) := \begin{pmatrix} \varphi_{j,\text{ad}}(0) \\ 0 \end{pmatrix}, \quad \Psi_2 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad j = f, b, \quad (5.25)$$

with $\varphi_{j,\text{ad}}(\xi)$ defined in (5.20).

Proof. By variation of constants, the evolution $T_j(\xi, \hat{\xi}) = T_j(\xi, \hat{\xi}; a)$ of the triangular block system (5.14) is given by

$$T_j(\xi, \hat{\xi}) = \begin{pmatrix} \Phi_j(\xi, \hat{\xi}) & \int_{\hat{\xi}}^{\xi} \Phi_j(\xi, z) F e^{-\eta(z-\hat{\xi})} dz \\ 0 & e^{-\eta(\xi-\hat{\xi})} \end{pmatrix}, \quad j = \text{f, b.}$$

Hence, using Proposition 4.5.6, one readily observes that the projections defined in (5.23) yield exponential dichotomies on both half-lines for (5.14) with constants $C, \min\{\mu, \eta\} > 0$, where $C > 0$ is independent of a . The result follows, since $\mu \leq \eta$ by Lemma 4.5.3. $\square$

Along the front

In the previous section we showed that the eigenvalue problem (5.6) can be written as a (λ, ε) -perturbation (5.15) of the reduced eigenvalue problem (5.14). Moreover, we established an exponential dichotomy of (5.14) on $(-\infty, 0]$ in Corollary 4.5.7. Hence, solutions to (5.6) can be expressed by a variation of constant formula on $(-\infty, 0]$. This leads to an exit condition at $\xi = 0$ for exponentially decaying solutions to (5.6) in backward time.

Eventually, our plan is to also obtain entry and exit conditions for solutions to (5.6) on $[0, Z_{a,\varepsilon}]$ and for exponentially decaying solutions to (5.6) in forward time on $[Z_{a,\varepsilon}, \infty)$. As outlined in §4.5.2 equating these exit and entry conditions at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$ leads to a system of equations that can be reduced to a single analytic matching equation, whose solutions are λ -values for which (5.6) admits an exponentially localized solution.

Simultaneously, we evaluate the obtained exit condition at $\lambda = 0$ using that we know a priori that the weighted derivative $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ of the pulse is the eigenfunction of (5.6) at $\lambda = 0$. As described in §4.5.2 this leads to extra information needed to simplify the expressions in the final matching equation.

Proposition 4.5.8. *Let B_f be as in (5.15) and ω_f as in (5.22). Denote by $T_{f,-}^{u,s}(\xi, \hat{\xi}) = T_{f,-}^{u,s}(\xi, \hat{\xi}; a)$ the (un)stable evolution of system (5.14) under the exponential dichotomy on $I_{f,-} = (-\infty, 0]$ established in Corollary 4.5.7 and by $Q_{f,-}^{u,s}(\xi) = Q_{f,-}^{u,s}(\xi; a)$ the associated projections.*

(i) *There exists $\delta, \varepsilon_0 > 0$ such that for $\lambda \in R_1(\delta)$ and $\varepsilon \in (0, \varepsilon_0)$ any solution $\psi_{f,-}(\xi, \lambda)$ to (5.6) decaying exponentially in backward time satisfies*

$$\begin{aligned} \psi_{f,-}(0, \lambda) &= \beta_{f,-}\omega_f(0) + \beta_{f,-} \int_{-\infty}^0 T_{f,-}^s(0, \hat{\xi}) B_f(\hat{\xi}, \lambda) \omega_f(\hat{\xi}) d\hat{\xi} + \mathcal{H}_{f,-}(\beta_{f,-}), \\ Q_{f,-}^u(0) \psi_{f,-}(0, \lambda) &= \beta_{f,-}\omega_f(0), \end{aligned} \tag{5.26}$$

for some $\beta_{f,-} \in \mathbb{C}$, where $\mathcal{H}_{f,-}$ is a linear map satisfying the bound

$$\|\mathcal{H}_{f,-}(\beta_{f,-})\| \leq C(\varepsilon|\log \varepsilon| + |\lambda|)^2 |\beta_{f,-}|,$$

with $C > 0$ independent of λ, a and ε . Moreover, $\psi_{f,-}(\xi, \lambda)$ is analytic in λ .

(ii) *The derivative $\phi'_{a,\varepsilon}$ of the pulse solution satisfies*

$$Q_{f,-}^s(0) \phi'_{a,\varepsilon}(0) = \int_{-\infty}^0 T_{f,-}^s(0, \hat{\xi}) B_f(\hat{\xi}, 0) e^{-\eta\hat{\xi}} \phi'_{a,\varepsilon}(\hat{\xi}) d\hat{\xi}. \tag{5.27}$$

Proof. We begin with (i). Take $0 < \hat{\mu} < \mu$ with $\mu > 0$ as in Lemma 4.5.3. Denote by $C_{\hat{\mu}}(I_{f,-}, \mathbb{C}^3)$ the space of $\hat{\mu}$ -exponentially decaying, continuous functions $I_{f,-} \rightarrow \mathbb{C}^3$

endowed with the norm $\|\psi\|_{\hat{\mu}} = \sup_{\xi \leq 0} \|\psi(\xi)\|e^{\hat{\mu}|\xi|}$. By Theorem 4.5 (i) we bound the perturbation matrix B_f by

$$\|B_f(\xi, \lambda; a, \varepsilon)\| \leq C(\varepsilon|\log \varepsilon| + |\lambda|), \quad (5.28)$$

for $\xi \in I_{f,-}$. Let $\beta \in \mathbb{C}$ and $\lambda \in R_1(\delta)$. Combining (5.28) with Corollary 4.5.7 the function $\mathcal{G}_{\beta,\lambda}: C_{\hat{\mu}}(I_{f,-}, \mathbb{C}^3) \rightarrow C_{\hat{\mu}}(I_{f,-}, \mathbb{C}^3)$ given by

$$\begin{aligned} \mathcal{G}_{\beta,\lambda}(\psi)(\xi) &= \beta\omega_f(\xi) + \int_0^\xi T_{f,-}^u(\xi, \hat{\xi})B_f(\hat{\xi}, \lambda)\psi(\hat{\xi})d\hat{\xi} \\ &\quad + \int_{-\infty}^\xi T_{f,-}^s(\xi, \hat{\xi})B_f(\hat{\xi}, \lambda)\psi(\hat{\xi})d\hat{\xi}, \end{aligned}$$

is a well-defined contraction mapping for each $\delta, \varepsilon > 0$ sufficiently small (with upper bound independent of β and a). By the Banach Contraction Theorem there exists a unique fixed point $\psi_{f,-} \in C_{\hat{\mu}}(I_{f,-}, \mathbb{C}^3)$ satisfying

$$\psi_{f,-} = \mathcal{G}_{\beta,\lambda}(\psi_{f,-}), \quad \xi \in I_{f,-}. \quad (5.29)$$

Observe that $\psi_{f,-}(\xi, \lambda)$ is analytic in λ , because the perturbation matrix $B_f(\xi, \lambda)$ is analytic in λ . Moreover, $\psi_{f,-}$ is linear in β by construction. Hence, using estimate (5.28) we derive the bound

$$\|\psi_{f,-}(\xi, \lambda) - \beta\omega_f(\xi)\| \leq C|\beta|(\varepsilon|\log \varepsilon| + |\lambda|), \quad (5.30)$$

for $\xi \in I_{f,-}$.

The solutions to the family of fixed point equations (5.29) parameterized over $\beta \in \mathbb{C}$ form a one-dimensional space of exponentially decaying solutions as $\xi \rightarrow -\infty$ to (5.6). By Lemma 4.5.3 the asymptotic matrix $\hat{A}(0, \lambda, a, \varepsilon)$ of system (5.6)

has precisely one eigenvalue of positive real part. Therefore, the space of decaying solutions in backward time to (5.6) is one-dimensional. This proves that any solution $\psi_{f,-}(\xi, \lambda)$ to (5.6) that converges to 0 as $\xi \rightarrow -\infty$, satisfies (5.29) for some $\beta \in \mathbb{C}$. Evaluating (5.29) at $\xi = 0$ and using estimates (5.28) and (5.30) yields (5.26).

For (ii), note that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an eigenfunction of (5.6) at $\lambda = 0$. Therefore, $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ satisfies the fixed point identity (5.29) at $\lambda = 0$ for some $\beta \in \mathbb{C}$ and identity (5.27) follows. $\square$

Passage near the right slow manifold

Using the exponential dichotomies of system (5.14) established in Corollary 4.5.7 one can construct expressions for solutions to (5.6) via a variation of constants approach on the intervals $I_{f,+} = [0, L_\varepsilon]$ and $I_{b,-} = [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon}]$. Moreover, the exponential dichotomies established in Proposition 4.5.5 govern the solutions to (5.6) on $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$. Matching the solutions on these three intervals we obtain the following entry and exit conditions at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$.

Proposition 4.5.9. *Let B_j be as in (5.15) and (5.17), Ψ_2 as in (5.25) and ω_j as in (5.22) for $j = f, b$. Denote by $T_{j,\pm}^{u,s}(\xi, \hat{\xi}) = T_{j,\pm}^{u,s}(\xi, \hat{\xi}; a)$ the (un)stable evolution of system (5.14) under the exponential dichotomies established in Corollary 4.5.7 and by $Q_{j,\pm}^{u,s}(\xi) = Q_{j,\pm}^{u,s}(\xi; a)$ the associated projections for $j = f, b$.*

(i) *For each sufficiently small $a_0 > 0$, there exists $\delta, \varepsilon_0 > 0$ such that for $\lambda \in R_1(\delta)$*

and $\varepsilon \in (0, \varepsilon_0)$ any solution $\psi^{\text{sl}}(\xi, \lambda)$ to (5.6) satisfies

$$\begin{aligned} \psi^{\text{sl}}(0, \lambda) &= \beta_{\text{f}} \omega_{\text{f}}(0) + \zeta_{\text{f}} Q_{\text{f},+}^s(0) \Psi_2 \\ &\quad + \beta_{\text{f}} \int_{L_\varepsilon}^0 T_{\text{f},+}^u(0, \hat{\xi}) B_{\text{f}}(\hat{\xi}, \lambda) \omega_{\text{f}}(\hat{\xi}) d\hat{\xi} + \mathcal{H}_{\text{f}}(\beta_{\text{f}}, \zeta_{\text{f}}, \beta_{\text{b}}), \end{aligned} \quad (5.31)$$

$$Q_{\text{f},-}^u(0) \psi^{\text{sl}}(0, \lambda) = \beta_{\text{f}} \omega_{\text{f}}(0),$$

and

$$\begin{aligned} \psi^{\text{sl}}(Z_{a,\varepsilon}, \lambda) &= \beta_{\text{b}} \omega_{\text{b}}(0) + \beta_{\text{b}} \int_{-L_\varepsilon}^0 T_{\text{b},-}^s(0, \hat{\xi}) B_{\text{b}}(\hat{\xi}, \lambda) \omega_{\text{b}}(\hat{\xi}) d\hat{\xi} \\ &\quad + \mathcal{H}_{\text{b}}(\beta_{\text{f}}, \zeta_{\text{f}}, \beta_{\text{b}}), \end{aligned} \quad (5.32)$$

$$Q_{\text{b},-}^u(0) \psi^{\text{sl}}(Z_{a,\varepsilon}, \lambda) = \beta_{\text{b}} \omega_{\text{b}}(0),$$

for some $\beta_{\text{f}}, \beta_{\text{b}}, \zeta_{\text{f}} \in \mathbb{C}$, where $\mathcal{H}_{\text{f}}$ and $\mathcal{H}_{\text{b}}$ are linear maps satisfying the bounds

$$\begin{aligned} \|\mathcal{H}_{\text{f}}(\beta_{\text{f}}, \zeta_{\text{f}}, \beta_{\text{b}})\| &\leq C \left((\varepsilon |\log \varepsilon| + |\lambda|) |\zeta_{\text{f}}| + (\varepsilon |\log \varepsilon| + |\lambda|)^2 |\beta_{\text{f}}| + e^{-q/\varepsilon} |\beta_{\text{b}}| \right), \\ \|\mathcal{H}_{\text{b}}(\beta_{\text{f}}, \zeta_{\text{f}}, \beta_{\text{b}})\| &\leq C \left((\varepsilon^{\rho(a)} |\log \varepsilon| + |\lambda|)^2 |\beta_{\text{b}}| + e^{-q/\varepsilon} (|\beta_{\text{f}}| + |\zeta_{\text{f}}|) \right), \end{aligned}$$

where $\rho(a) = \frac{2}{3}$ for $a < a_0$ and $\rho(a) = 1$ for $a \geq a_0$ and $q, C > 0$ independent of λ, a and ε . Moreover, $\psi^{\text{sl}}(\xi, \lambda)$ is analytic in λ .

(ii) The derivative $\phi'_{a,\varepsilon}$ of the pulse solution satisfies

$$\begin{aligned} Q_{\text{f},+}^u(0) \phi'_{a,\varepsilon}(0) &= T_{\text{f},+}^u(0, L_\varepsilon) e^{-\eta L_\varepsilon} \phi'_{a,\varepsilon}(L_\varepsilon) \\ &\quad + \int_{L_\varepsilon}^0 T_{\text{f},+}^u(0, \hat{\xi}) B_{\text{f}}(\hat{\xi}, 0) e^{-\eta \hat{\xi}} \phi'_{a,\varepsilon}(\hat{\xi}) d\hat{\xi}, \\ Q_{\text{b},-}^s(0) \phi'_{a,\varepsilon}(Z_{a,\varepsilon}) &= T_{\text{b},-}^s(0, -L_\varepsilon) e^{\eta L_\varepsilon} \phi'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ &\quad + \int_{-L_\varepsilon}^0 T_{\text{b},-}^s(0, \hat{\xi}) B_{\text{f}}(\hat{\xi}, 0) e^{-\eta \hat{\xi}} \phi'_{a,\varepsilon}(Z_{a,\varepsilon} + \hat{\xi}) d\hat{\xi}. \end{aligned} \quad (5.33)$$

Proof. We begin with (i). For the matching procedure, we need to compare projec-

tions $Q_{f,+}^{u,s}(\xi)$ of the exponential dichotomies of (5.14) established in Corollary 4.5.7 with the projections $\mathcal{Q}_r^{u,s}(\xi, \lambda)$ of the dichotomy of (5.6) on I_r established in Proposition 4.5.5. First, recall that the front $\phi_f(\xi)$ is a heteroclinic to the fixed point $(1, 0)$ of (3.4). By looking at the linearization of (3.4) about $(1, 0)$ we deduce that $\phi_f(\xi)$, and thus the coefficient matrix $A_f(\xi)$ of (5.14), converges at an exponential rate $\frac{1}{2}\sqrt{2}$ to some asymptotic matrix $A_{f,\infty}$ as $\xi \rightarrow \infty$. Hence, by [46, Lemma 3.4] and its proof the projections $Q_{f,+}^{u,s}$ associated with the exponential dichotomy of system (5.14) satisfy for $\xi \geq 0$

$$\|Q_{f,+}^{u,s}(\xi) - P_f^{u,s}\| \leq C \left(e^{-\frac{1}{2}\sqrt{2}\xi} + e^{-\mu\xi} \right), \quad (5.34)$$

where $P_f^{u,s} = P_f^{u,s}(a)$ denotes the spectral projection on the (un)stable eigenspace of the asymptotic matrix $A_{f,\infty}$. Moreover, the coefficient matrix $A(\xi, \lambda)$ of (5.6) is approximated at $L_\varepsilon = -\nu \log \varepsilon$ by

$$\|A(L_\varepsilon, \lambda) - A_{f,\infty}\| \leq C(\varepsilon|\log \varepsilon| + |\lambda|),$$

by Theorem 4.5 (i) and the fact that $A_f(\xi)$ converges to $A_{f,\infty}$ at an exponential rate $\frac{1}{2}\sqrt{2}$ as $\xi \rightarrow \infty$, using that ν is chosen larger than $2\sqrt{2}$ in (5.5). By continuity the same bound holds for the spectral projections associated with the matrices $A(L_\varepsilon, \lambda)$ and $A_{f,\infty}$. Combining the latter facts with (5.34) and the bounds in Proposition 4.5.5 we obtain

$$\|\mathcal{Q}_r^{u,s}(L_\varepsilon, \lambda) - Q_{f,+}^{u,s}(L_\varepsilon)\| \leq C(\varepsilon|\log \varepsilon| + |\lambda|), \quad (5.35)$$

using $\nu \geq \max\{\frac{2}{\mu}, 2\sqrt{2}\}$. In a similar way we obtain an estimate at $Z_{a,\varepsilon} - L_\varepsilon$

$$\|\mathcal{Q}_r^{u,s}(Z_{a,\varepsilon} - L_\varepsilon, \lambda) - Q_{b,-}^{u,s}(-L_\varepsilon)\| \leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|). \quad (5.36)$$

using Theorem 4.5 (ii).

By the variation of constants formula, any solution $\psi_f^{\text{sl}}(\xi, \lambda)$ to (5.6) must satisfy on $I_{f,+}$

$$\begin{aligned} \psi_f^{\text{sl}}(\xi, \lambda) &= T_{f,+}^u(\xi, L_\varepsilon)\alpha_f + \beta_f\omega_f(\xi) + \zeta_f T_{f,+}^s(\xi, 0)\Psi_2 \\ &\quad + \int_0^\xi T_{f,+}^s(\xi, \hat{\xi})B_f(\hat{\xi}, \lambda)\psi_f^{\text{sl}}(\hat{\xi}, \lambda)d\hat{\xi} + \int_{L_\varepsilon}^\xi T_{f,+}^u(\xi, \hat{\xi})B_f(\hat{\xi}, \lambda)\psi_f^{\text{sl}}(\hat{\xi}, \lambda)d\hat{\xi}, \end{aligned} \quad (5.37)$$

for some $\beta_f, \zeta_f \in \mathbb{C}$ and $\alpha_f \in R(Q_{f,+}^u(L_\varepsilon))$. By Theorem 4.5 (i) we bound the perturbation matrix B_f as

$$\|B_f(\xi, \lambda; a, \varepsilon)\| \leq C(\varepsilon|\log \varepsilon| + |\lambda|), \quad (5.38)$$

for $\xi \in I_{f,+}$. Hence, for all sufficiently small $|\lambda|, \varepsilon > 0$, there exists a unique solution ψ_f^{sl} to (5.37) by the contraction mapping principle. Note that ψ_f^{sl} is linear in $(\alpha_f, \beta_f, \zeta_f)$ and satisfies the bound

$$\sup_{\xi \in [0, L_\varepsilon]} \|\psi_f^{\text{sl}}(\xi, \lambda)\| \leq C(|\alpha_f| + |\beta_f| + |\zeta_f|), \quad (5.39)$$

by estimate (5.38), taking $\delta, \varepsilon_0 > 0$ smaller if necessary.

Denote by $\mathcal{T}_r^{u,s}(\xi, \hat{\xi}, \lambda) = \mathcal{T}_r^{u,s}(\xi, \hat{\xi}, \lambda; a, \varepsilon)$ the (un)stable evolution of system (5.6) under the exponential dichotomy on I_r established in Proposition 4.5.5. Any solution ψ_r to (5.6) on I_r is of the form

$$\psi_r(\xi, \lambda) = \mathcal{T}_r^u(\xi, Z_{a,\varepsilon} - L_\varepsilon, \lambda)\alpha_r + \mathcal{T}_r^s(\xi, L_\varepsilon, \lambda)\beta_r, \quad (5.40)$$

for some $\alpha_r \in R(\mathcal{Q}_r^u(Z_{a,\varepsilon} - L_\varepsilon, \lambda))$ and $\beta_r \in R(\mathcal{Q}_r^s(L_\varepsilon, \lambda))$. Applying the projection

$\mathcal{Q}_r^u(L_\varepsilon, \lambda)$ to the difference $\psi_r(L_\varepsilon, \lambda) - \psi_f^{\text{sl}}(L_\varepsilon, \lambda)$ yields the matching condition

$$\alpha_f = \mathcal{H}_1(\alpha_f, \beta_f, \alpha_r), \quad (5.41)$$

$$\|\mathcal{H}_1(\alpha_f, \beta_f, \alpha_r)\| \leq C((\varepsilon|\log \varepsilon| + |\lambda|)(\|\alpha_f\| + |\beta_f| + |\zeta_f|) + e^{-q/\varepsilon}\|\alpha_r\|),$$

where we use (5.35), (5.38), (5.39) and the fact that $Z_{a,\varepsilon} = \mathcal{O}_s(\varepsilon^{-1})$ (see Theorem 4.5) to obtain the bound on the linear map $\mathcal{H}_1$. Similarly, applying the projection $\mathcal{Q}_r^s(L_\varepsilon, \lambda)$ to the difference $\psi_r(L_\varepsilon, \lambda) - \psi_f^{\text{sl}}(L_\varepsilon, \lambda)$ yields the matching condition

$$\beta_r = \mathcal{H}_2(\alpha_f, \beta_f, \zeta_f), \quad (5.42)$$

$$\|\mathcal{H}_2(\alpha_f, \beta_f, \zeta_f)\| \leq C(\varepsilon|\log \varepsilon| + |\lambda|)(\|\alpha_f\| + |\beta_f| + |\zeta_f|),$$

where we use (5.35), (5.38), (5.39) and $\nu \geq 2/\mu$ to obtain the bound on the linear map $\mathcal{H}_2$.

Consider the translated version (5.16) of system (5.6). By the variation of constants formula, any solution $\psi_b^{\text{sl}}(\xi, \lambda)$ to (5.16) on $[-L_\varepsilon, 0]$ must satisfy

$$\begin{aligned} \psi_b^{\text{sl}}(\xi, \lambda) &= T_{b,-}^s(\xi, -L_\varepsilon)\alpha_b + \beta_b\omega_b(\xi) + \int_0^\xi T_{b,-}^u(\xi, \hat{\xi})B_b(\hat{\xi}, \lambda)\psi_b^{\text{sl}}(\hat{\xi}, \lambda)d\hat{\xi} \\ &\quad + \int_{-L_\varepsilon}^\xi T_{b,-}^s(\xi, \hat{\xi})B_b(\hat{\xi}, \lambda)\psi_b^{\text{sl}}(\hat{\xi}, \lambda)d\hat{\xi}, \end{aligned} \quad (5.43)$$

for some $\beta_b \in \mathbb{C}$ and $\alpha_b \in R(Q_{b,-}^s(-L_\varepsilon))$. By Theorem 4.5 (ii) we estimate

$$\|B_b(\xi, \lambda; a, \varepsilon)\| \leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|), \quad (5.44)$$

for $\xi \in [-L_\varepsilon, 0]$. For all sufficiently small $|\lambda|, \varepsilon > 0$, there exists a unique solution ψ_b^{sl}

of (5.43). Note that ψ_b^{sl} is linear in (α_b, β_b) and using (5.44) we obtain the bound

$$\sup_{\xi \in [-L_\varepsilon, 0]} \|\psi_b^{\text{sl}}(\xi, \lambda)\| \leq C(\|\alpha_b\| + |\beta_b|), \quad (5.45)$$

taking $\delta, \varepsilon_0 > 0$ smaller if necessary. The matching of $\psi_b^{\text{sl}}(-L_\varepsilon, \lambda)$ with $\psi_r(Z_{a,\varepsilon} - L_\varepsilon, \lambda)$ is completely similar to the matching of $\psi_f^{\text{sl}}(L_\varepsilon, \lambda)$ with $\psi_r(L_\varepsilon, \lambda)$ in the previous paragraph using (5.45) instead of (5.39) and (5.36) instead of (5.35). Hence we give only the resulting matching conditions

$$\alpha_r = \mathcal{H}_3(\alpha_b, \beta_b), \quad (5.46)$$

$$\|\mathcal{H}_3(\alpha_b, \beta_b)\| \leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|)(\|\alpha_b\| + |\beta_b|),$$

$$\alpha_b = \mathcal{H}_4(\alpha_b, \beta_b, \beta_r), \quad (5.47)$$

$$\|\mathcal{H}_4(\alpha_b, \beta_b, \beta_r)\| \leq C((\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|)(\|\alpha_b\| + |\beta_b|) + e^{-q/\varepsilon}\|\beta_r\|),$$

where $\mathcal{H}_3$ and $\mathcal{H}_4$ are again linear maps.

We now combine the above results regarding the solution on $[0, Z_{a,\varepsilon}]$ to obtain the relevant conditions satisfied at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$. Combining equations (5.42) and (5.47), we obtain a linear map $\mathcal{H}_5$ satisfying

$$\alpha_b = \mathcal{H}_5(\alpha_b, \beta_b, \alpha_f, \beta_f, \zeta_f),$$

$$\|\mathcal{H}_5(\alpha_b, \beta_b, \beta_r, c_r)\| \leq C((\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|)(\|\alpha_b\| + |\beta_b|) + e^{-q/\varepsilon}(\|\alpha_f\| + |\beta_f| + |\zeta_f|)). \quad (5.48)$$

Thus, solving (5.48) for α_b , we obtain for all sufficiently small $|\lambda|, \varepsilon > 0$

$$\alpha_b = \alpha_b(\alpha_f, \beta_b, \beta_f, \zeta_f), \quad (5.49)$$

$$\|\alpha_b(\alpha_f, \beta_b, \beta_f, \zeta_f)\| \leq C((\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|)|\beta_b| + e^{-q/\varepsilon}(\|\alpha_f\| + |\beta_f| + |\zeta_f|)).$$

From (5.41), (5.46) and (5.49) we obtain a linear map $\mathcal{H}_6$ satisfying

$$\alpha_f = \mathcal{H}_6(\alpha_f, \beta_f, \zeta_f, \beta_b), \quad (5.50)$$

$$\|\mathcal{H}_6(\alpha_f, \beta_b, \beta_f, \zeta_f)\| \leq C \left((\varepsilon |\log \varepsilon| + |\lambda|)(\|\alpha_f\| + |\beta_f| + |\zeta_f|) + e^{-q/\varepsilon} |\beta_b| \right),$$

We solve (5.50) for α_f for each sufficiently small $|\lambda|, \varepsilon > 0$ and obtain

$$\alpha_f = \alpha_f(\beta_b, \beta_f, \zeta_f), \quad (5.51)$$

$$\|\alpha_f(\beta_b, \beta_f, \zeta_f)\| \leq C \left((\varepsilon |\log \varepsilon| + |\lambda|)(|\beta_f| + |\zeta_f|) + e^{-q/\varepsilon} |\beta_b| \right).$$

Substituting (5.51) into (5.37) at $\xi = 0$ we deduce, using $\nu \geq \mu/2$ and identities (5.24), (5.38) and (5.39), that any solution $\psi^{\text{sl}}(\xi, \lambda)$ to (5.6) satisfies the entry condition (5.31). Similarly, we substitute (5.51) into (5.49) and substitute the resulting expression for α_b into (5.43) at $\xi = 0$. Using estimates (5.44) and (5.45) and we obtain the exit condition (5.32). Since the perturbation matrices $B_j(\xi, \lambda), j = f, b$, the evolution $\mathcal{T}(\xi, \hat{\xi}, \lambda)$ of system (5.6) and the projections $\mathcal{Q}_r^{u,s}(\xi, \lambda)$ associated with the exponential dichotomy of (5.6) are analytic in λ , all quantities occurring in this proof depend analytically on λ . Thus, $\psi^{\text{sl}}(\xi, \lambda)$ is analytic in λ .

For (ii), we note that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an eigenfunction of (5.6) at $\lambda = 0$. Therefore, there exists $\beta_{f,0}, \zeta_{f,0} \in \mathbb{C}$ and $\alpha_{f,0} \in R(Q_{f,+}^u(L_\varepsilon))$ such that (5.37) is satisfied at $\lambda = 0$ with $\psi_f^{\text{sl}}(\xi, 0) = e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ and $(\alpha_f, \beta_f, \zeta_f) = (\alpha_{f,0}, \beta_{f,0}, \zeta_{f,0})$. We derive $\alpha_{f,0} = Q_{f,+}^u(L_\varepsilon)e^{-\eta L_\varepsilon}\phi'_{a,\varepsilon}(L_\varepsilon)$ by applying $Q_{f,+}^u(L_\varepsilon)$ to (5.37) at $\xi = L_\varepsilon$. Therefore, the first identity in (5.33) follows by applying $\mathcal{Q}_{f,+}^u(0)$ to (5.37) at $\xi = 0$. The second identity in (5.33) follows in a similar fashion using that there exists $\beta_{b,0} \in \mathbb{C}$ and $\alpha_{b,0} \in R(Q_{b,-}^s(-L_\varepsilon))$ such that (5.43) is satisfied at $\lambda = 0$ with $\psi_b^{\text{sl}}(\xi, 0) = e^{-\eta(\xi+Z_{a,\varepsilon})}\phi'_{a,\varepsilon}(Z_{a,\varepsilon} + \xi)$ and $(\alpha_b, \beta_b) = (\alpha_{b,0}, \beta_{b,0})$. $\square$

Along the back

Finally, we establish an entry condition for exponentially decaying solution to (5.6) on the interval $[Z_{a,\varepsilon}, \infty)$.

Proposition 4.5.10. *Let B_b be as in (5.17), Ψ_2 as in (5.25) and ω_b as in (5.22). Denote by $T_{b,\pm}^{u,s}(\xi, \hat{\xi}) = T_{b,\pm}^{u,s}(\xi, \hat{\xi}; a)$ the (un)stable evolution of system (5.14) under the exponential dichotomies established in Corollary 4.5.7 and by $Q_{b,\pm}^{u,s}(\xi) = Q_{b,\pm}^{u,s}(\xi; a)$ the associated projections.*

- (i) *For each sufficiently small $a_0 > 0$, there exists $\delta, \varepsilon_0 > 0$ such that for $\lambda \in R_1(\delta)$ and $\varepsilon \in (0, \varepsilon_0)$ any solution $\psi_{b,+}(\xi, \lambda)$ to (5.6), which is exponentially decaying in forward time, satisfies*

$$\begin{aligned} \psi_{b,+}(Z_{a,\varepsilon}, \lambda) &= \beta_{b,+}\omega_b(0) + \zeta_{b,+}Q_{b,+}^s(0)\Psi_2 \\ &\quad + \beta_{b,+} \int_{L_\varepsilon}^0 T_{b,+}^u(0, \hat{\xi})B_b(\hat{\xi}, \lambda)\omega_b(\hat{\xi})d\hat{\xi} + \mathcal{H}_{b,+}(\beta_{b,+}, \zeta_{b,+}), \\ Q_{b,-}^u(0)\psi_{b,+}(Z_{a,\varepsilon}, \lambda) &= \beta_{b,+}\omega_b(0), \end{aligned} \tag{5.52}$$

for some $\beta_{b,+}, \zeta_{b,+} \in \mathbb{C}$, where $\mathcal{H}_{b,+}$ is a linear map satisfying the bound

$$\|\mathcal{H}_{b,+}(\beta_{b,+}, \zeta_{b,+})\| \leq C \left((\varepsilon^{\rho(a)} |\log \varepsilon| + |\lambda|) |\zeta_{b,+}| + (\varepsilon^{\rho(a)} |\log \varepsilon| + |\lambda|)^2 |\beta_{b,+}| \right),$$

with $\rho(a) = \frac{2}{3}$ for $a < a_0$ and $\rho(a) = 1$ for $a \geq a_0$ and $C > 0$ independent of λ, a and ε . Moreover, $\psi_{b,+}(\xi, \lambda)$ is analytic in λ .

(ii) The derivative $\phi'_{a,\varepsilon}$ of the pulse solution satisfies

$$\begin{aligned} Q_{b,+}^u(0)\phi'_{a,\varepsilon}(Z_{a,\varepsilon}) &= T_{b,+}^u(0, L_\varepsilon)e^{-\eta L_\varepsilon}\phi'_{a,\varepsilon}(Z_{a,\varepsilon} + L_\varepsilon) \\ &\quad + \int_{L_\varepsilon}^0 T_{b,+}^u(0, \hat{\xi})B_b(\hat{\xi}, 0)e^{-\eta\hat{\xi}}\phi'_{a,\varepsilon}(Z_{a,\varepsilon} + \hat{\xi})d\hat{\xi}. \end{aligned} \quad (5.53)$$

Proof. We begin with (i). Consider the translated version (5.16) of system (5.6). By the variation of constants formula, any solution $\hat{\psi}_{b,+}(\xi, \lambda)$ to (5.16) on $[0, L_\varepsilon]$ must satisfy

$$\begin{aligned} \hat{\psi}_{b,+}(\xi, \lambda) &= T_{b,+}^u(\xi, L_\varepsilon)\alpha_{b,+} + \beta_{b,+}\omega_b(\xi) + \zeta_{b,+}T_{b,+}^s(\xi, 0)\Psi_2 \\ &\quad + \int_0^\xi T_{b,+}^s(\xi, \hat{\xi})B_b(\hat{\xi}, \lambda)\hat{\psi}_{b,+}(\hat{\xi}, \lambda)d\hat{\xi} + \int_{L_\varepsilon}^\xi T_{b,+}^u(\xi, \hat{\xi})B_b(\hat{\xi}, \lambda)\hat{\psi}_{b,+}(\hat{\xi}, \lambda)d\hat{\xi}, \end{aligned} \quad (5.54)$$

for some $\beta_{b,+}, \zeta_{b,+} \in \mathbb{C}$ and $\alpha_{b,+} \in R(Q_{b,+}^u(L_\varepsilon))$. By Theorem 4.5 (ii) we estimate

$$\|B_b(\xi, \lambda; a, \varepsilon)\| \leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|), \quad (5.55)$$

for $\xi \in [0, L_\varepsilon]$. For all sufficiently small $|\lambda|, \varepsilon > 0$, there exists a unique solution $\hat{\psi}_{b,+}$ of (5.54). Note that $\hat{\psi}_{b,+}$ is linear in $(\alpha_{b,+}, \beta_{b,+}, \zeta_{b,+})$ and using (5.55) we obtain the bound,

$$\sup_{\xi \in [0, L_\varepsilon]} \|\hat{\psi}_{b,+}(\xi, \lambda)\| \leq C(\|\alpha_{b,+}\| + |\beta_{b,+}| + |\zeta_{b,+}|), \quad (5.56)$$

taking $\delta, \varepsilon_0 > 0$ smaller if necessary.

Consider the exponential dichotomies of (5.6) on $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$ established in Proposition 4.5.5 with associated projections $\mathcal{Q}_\ell^{u,s}(\xi, \lambda)$. Completely analogous to

the derivation of (5.36) in the proof of Proposition 4.5.9 we establish

$$\|\mathcal{Q}_\ell^{u,s}(Z_{a,\varepsilon} + L_\varepsilon, \lambda) - \mathcal{Q}_{b,+}^{u,s}(L_\varepsilon)\| \leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|). \quad (5.57)$$

The image of any exponentially decaying solution to (5.6) at $Z_{a,\varepsilon} + L_\varepsilon$ under $\mathcal{Q}_\ell^u(Z_{a,\varepsilon} + L_\varepsilon, \lambda)$ must be 0, i.e. any solution $\psi_\ell(\xi, \lambda)$ to (5.6) decaying in forward time can be written as

$$\psi_\ell(\xi, \lambda) = \mathcal{T}_\ell^s(\xi, Z_{a,\varepsilon} + L_\varepsilon, \lambda)\beta_\ell, \quad (5.58)$$

for some $\beta_\ell \in R(\mathcal{Q}_\ell^s(Z_{a,\varepsilon} + L_\varepsilon, \lambda))$, where $\mathcal{T}_\ell^s(\xi, \hat{\xi}, \lambda)$ denotes the stable evolution of system (5.6). Thus, by applying $\mathcal{Q}_\ell^u(Z_{a,\varepsilon} + L_\varepsilon, \lambda)$ to $\hat{\psi}_{b,+}(L_\varepsilon, \lambda)$ we obtain a linear map $\mathcal{H}_1$ satisfying

$$\begin{aligned} \alpha_{b,+} &= \mathcal{H}_1(\alpha_{b,+}, \beta_{b,+}, \zeta_{b,+}), \\ \|\mathcal{H}_1(\alpha_{b,+}, \beta_{b,+}, \zeta_{b,+}, \beta_r)\| &\leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|)(\|\alpha_{b,+}\| + |\beta_{b,+}| + |\zeta_{b,+}|), \end{aligned} \quad (5.59)$$

where we have used (5.55), (5.56) and (5.57). So, for sufficiently small $|\lambda|, \varepsilon > 0$, solving (5.59) for $\alpha_{b,+}$ yields

$$\begin{aligned} \alpha_{b,+} &= \alpha_{b,+}(\beta_{b,+}, \zeta_{b,+}) \\ \|\alpha_{b,+}(\beta_{b,+}, \zeta_{b,+})\| &\leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|)(|\beta_{b,+}| + |\zeta_{b,+}|). \end{aligned} \quad (5.60)$$

Substituting (5.60) into (5.54) we deduce with the aid of (5.24), (5.55) and (5.56) that any exponentially decaying solution $\psi_{b,+}(\xi, \lambda) = \hat{\psi}_{b,+}(\xi - Z_{a,\varepsilon}, \lambda)$ to (5.6) satisfies the entry condition (5.52) at $\xi = Z_{a,\varepsilon}$. Moreover, analyticity of $\psi_{b,+}(\xi, \lambda)$ in λ follows from the analyticity of $B_b(\xi, \lambda)$, of the evolution $\mathcal{T}(\xi, \hat{\xi}, \lambda)$ and of the projections $\mathcal{Q}_\ell^{u,s}(\xi, \lambda)$.

We now prove (ii). Identity (5.53) follows in a similar fashion as (5.27) in the proof of Proposition 4.5.9 using that there exists $\beta_{b,+}, \zeta_{b,+} \in \mathbb{C}$ and $\alpha_{b,+} \in R(Q_{b,+}^u(L_\varepsilon))$ such that (5.43) is satisfied at $\lambda = 0$ with $\hat{\psi}_{b,+}(\xi, 0) = e^{-\eta(\xi + Z_{a,\varepsilon})} \phi'_{a,\varepsilon}(Z_{a,\varepsilon} + \xi)$. $\square$

The matching procedure

In the previous sections we constructed a piecewise continuous, exponentially localized solution to the shifted eigenvalue problem (5.6) for any $\lambda \in R_1(\delta)$. At the two discontinuous jumps at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$ we obtained expressions for the left and right limits of the solution; these are the so-called exit and entry conditions. Finding eigenvalues now reduces to locating $\lambda \in R_1$ for which the exit and entry conditions match up. Equating the exit and entry conditions leads, after reduction, to a single analytic matching equation in λ .

During the matching process we simplify terms in the following way. Recall that we evaluated the obtained exit and entry conditions at $\lambda = 0$ using that the weighted derivative $e^{-\eta\xi} \phi'_{a,\varepsilon}(\xi)$ of the pulse is an eigenfunction of (5.6) at $\lambda = 0$. This leads to identities that can be substituted in the matching equations; see Remark 4.5.11.

Since the final analytic matching equation is to leading order a quadratic in λ , it has precisely two solutions in R_1 . These solutions are the eigenvalues of $\mathcal{L}_{a,\varepsilon}$ in R_1 . A priori we know that $\lambda_0 = 0$ must be one of these two eigenvalues by translational invariance. In the next section 4.5.4 we show that λ_0 is in fact a simple eigenvalue of $\mathcal{L}_{a,\varepsilon}$. The other eigenvalue λ_1 can be determined to leading order. Section 4.5.5 is devoted to the calculation of this second eigenvalue, which differs between the hyperbolic and nonhyperbolic regime.

Thus, our aim is to prove the following result.

Theorem 4.9. *For each sufficiently small $a_0 > 0$, there exists $\delta, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ system (5.6) has precisely two different eigenvalues $\lambda_0, \lambda_1 \in R_1(\delta)$. The eigenvalue λ_0 equals 0 and the corresponding eigenspace is spanned by the solution $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ to (5.6). The other eigenvalue λ_1 is a -uniformly approximated by*

$$\lambda_1 = -\frac{M_{b,2}}{M_{b,1}} + \mathcal{O}\left(|\varepsilon^{\rho(a)}\log \varepsilon|^2\right),$$

with

$$M_{b,1} := \int_{-\infty}^{\infty} (u'_b(\xi))^2 e^{-\check{c}_0\xi} d\xi,$$

$$M_{b,2} := \langle \Psi_*, \phi'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \rangle, \quad \Psi_* := \begin{pmatrix} e^{\check{c}_0 L_\varepsilon} v'_b(-L_\varepsilon) \\ -e^{\check{c}_0 L_\varepsilon} u'_b(-L_\varepsilon) \\ \int_{-\infty}^{-L_\varepsilon} e^{-\check{c}_0 \hat{\xi}} u'_b(\hat{\xi}) d\hat{\xi} \end{pmatrix}, \quad (5.61)$$

where $(u_b(\xi), v_b(\xi)) = \phi_b(\xi)$ denotes the heteroclinic back solution to the Nagumo system (3.5) and the exponent $\rho(a)$ equals $\frac{2}{3}$ for $a < a_0$ and 1 for $a \geq a_0$. The corresponding eigenspace is spanned by a solution $\psi_1(\xi)$ to (5.6) satisfying

$$\begin{aligned} \|\psi_1(\xi + Z_{a,\varepsilon}) - \omega_b(\xi)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|, \quad \xi \in [-L_\varepsilon, L_\varepsilon], \\ \|\psi_1(\xi + Z_{a,\varepsilon})\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|, \quad \xi \in \mathbb{R} \setminus [-L_\varepsilon, L_\varepsilon], \end{aligned} \quad (5.62)$$

where ω_b is as in (5.22) and $C > 1$ is independent of a and ε . Finally, the quantities $M_{b,1}$ and $M_{b,2}$ satisfy the bounds

$$1/C \leq M_{b,1} \leq C, \quad |M_{b,2}| \leq C\varepsilon^{\rho(a)}|\log \varepsilon|.$$

Proof. We start the proof with some estimates from the existence problem. By

Theorem 4.5 (i)-(ii) we have the bounds

$$\begin{aligned} \|B_f(\xi, \lambda; a, \varepsilon)\| &\leq C(\varepsilon|\log \varepsilon| + |\lambda|), \quad \xi \in (-\infty, L_\varepsilon], \\ \|B_b(\xi, \lambda; a, \varepsilon)\| &\leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|), \quad \xi \in [-L_\varepsilon, L_\varepsilon]. \end{aligned} \quad (5.63)$$

where B_f and B_b are as in (5.15) and (5.17). Moreover, we use the equations (3.4) and (3.5) for ϕ_f and ϕ_b and the equation (3.1) for $\phi_{a,\varepsilon}$ in combination with Theorem 4.5 (i)-(ii) to estimate the difference between the derivatives

$$\begin{aligned} \left\| \begin{pmatrix} \phi'_f(\xi) \\ 0 \end{pmatrix} - \phi'_{a,\varepsilon}(\xi) \right\| &\leq C\varepsilon|\log \varepsilon|, \quad \xi \in (-\infty, L_\varepsilon], \\ \left\| \begin{pmatrix} \phi'_b(\xi) \\ 0 \end{pmatrix} - \phi'_{a,\varepsilon}(Z_{a,\varepsilon} + \xi) \right\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|, \quad \xi \in [-L_\varepsilon, L_\varepsilon]. \end{aligned} \quad (5.64)$$

We outline the matching procedure that yields the two λ -values for which (5.6) admits nontrivial exponentially localized solutions. By Proposition 4.5.8 any solution $\psi_{f,-}(\xi, \lambda)$ to (5.6) decaying exponentially in backward time satisfies (5.26) at $\xi = 0$ for some constant $\beta_{f,-} \in \mathbb{C}$. Moreover, by Proposition 4.5.9 any solution $\psi^{\text{sl}}(\xi, \lambda)$ to (5.6) satisfies (5.31) at $\xi = 0$ for some $\beta_f, \zeta_f \in \mathbb{C}$ and (5.32) at $\xi = Z_{a,\varepsilon}$ for some $\beta_b \in \mathbb{C}$. Finally, by Proposition 4.5.10 any solution $\psi_{b,+}(\xi, \lambda)$ to (5.6) decaying exponentially in forward time satisfies (5.52) at $\xi = Z_{a,\varepsilon}$ for some $\beta_{b,+}, \zeta_{b,+} \in \mathbb{C}$. To obtain an exponentially localized solution to (5.6) we match the solutions $\psi_{f,-}, \psi^{\text{sl}}$ and $\psi_{b,+}$ at $\xi = 0$ and at $\xi = Z_{a,\varepsilon}$. It suffices to require that the differences $\psi_{f,-}(0, \lambda) - \psi^{\text{sl}}(0, \lambda)$ and $\psi^{\text{sl}}(Z_{a,\varepsilon}, \lambda) - \psi_{b,+}(Z_{a,\varepsilon}, \lambda)$ vanish under the projections $Q_{f,-}^{u,s}(0)$ and $Q_{b,-}^{u,s}(0)$ associated with the exponential dichotomy of (5.14) established in Corollary 4.5.7.

We first apply the projections $Q_{j,-}^u(0)$, $j = f, b$ to the differences $\psi_{f,-}(0, \lambda) - \psi^{\text{sl}}(0, \lambda)$ and $\psi^{\text{sl}}(Z_{a,\varepsilon}, \lambda) - \psi_{b,+}(Z_{a,\varepsilon}, \lambda)$ and immediately obtain $\beta_f = \beta_{f,-}$ and $\beta_b =$

$\beta_{b,+}$ using (5.26), (5.31), (5.32) and (5.52). For the remaining matching conditions, consider the vectors $\Psi_{1,j}$ and Ψ_2 defined in (5.25) and the bounded solution $\varphi_{j,\text{ad}}$, given by (5.20), to the adjoint equation (5.19) of the reduced eigenvalue problem (5.14). By (5.24) the vectors Ψ_2 and

$$\Psi_{j,\perp} := \Psi_{1,j} - \int_{\infty}^0 e^{-\eta\xi} \langle \varphi_{j,\text{ad}}(\xi), F \rangle d\xi \Psi_2, \quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad j = \text{f}, \text{b},$$

span $R(Q_{j,-}^s(0))$ and $\Psi_{j,\perp}$ is contained in $\ker(Q_{j,+}^s(0)^*) = R(Q_{j,+}^u(0)^*) \subset R(Q_{j,-}^s(0)^*)$ for $j = \text{f}, \text{b}$. Thus, we obtain four other matching conditions by requiring that the inner products of the differences $\psi_{\text{f},-}(0, \lambda) - \psi^{\text{sl}}(0, \lambda)$ and $\psi^{\text{sl}}(Z_{a,\varepsilon}, \lambda) - \psi_{\text{b},+}(Z_{a,\varepsilon}, \lambda)$ with Ψ_2 and $\Psi_{j,\perp}$ vanish for $j = \text{f}, \text{b}$. With the aid of the identities (5.26), (5.31), (5.32) and (5.52) we obtain the first two matching conditions by pairing with Ψ_2

$$\begin{aligned} 0 &= \langle \Psi_2, \psi_{\text{f},-}(0, \lambda) - \psi^{\text{sl}}(0, \lambda) \rangle = -\zeta_{\text{f}} + \mathcal{H}_1(\beta_{\text{b}}, \beta_{\text{f}}, \zeta_{\text{f}}), \\ 0 &= \langle \Psi_2, \psi^{\text{sl}}(Z_{a,\varepsilon}, \lambda) - \psi_{\text{b},+}(Z_{a,\varepsilon}, \lambda) \rangle = -\zeta_{\text{b},+} + \mathcal{H}_2(\beta_{\text{b}}, \zeta_{\text{b},+}, \beta_{\text{f}}, \zeta_{\text{f}}), \end{aligned} \tag{5.65}$$

where the linear maps $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy by (5.63) the bounds

$$\begin{aligned} |\mathcal{H}_1(\beta_{\text{b}}, \beta_{\text{f}}, \zeta_{\text{f}})| &\leq C \left((\varepsilon |\log \varepsilon| + |\lambda|) (|\beta_{\text{f}}| + |\zeta_{\text{f}}|) + e^{-q/\varepsilon} |\beta_{\text{b}}| \right), \\ |\mathcal{H}_2(\beta_{\text{b}}, \zeta_{\text{b},+}, \beta_{\text{f}}, \zeta_{\text{f}})| &\leq C \left((\varepsilon^{\rho(a)} |\log \varepsilon| + |\lambda|) (|\beta_{\text{b}}| + |\zeta_{\text{b},+}|) + e^{-q/\varepsilon} (|\beta_{\text{f}}| + |\zeta_{\text{f}}|) \right), \end{aligned}$$

with $q > 0$ independent of λ, a and ε . Hence, we can solve system (5.65) for ζ_{f} and $\zeta_{\text{b},+}$, provided $|\lambda|, \varepsilon > 0$ are sufficiently small, and obtain

$$\begin{aligned} \zeta_{\text{f}} &= \zeta_{\text{f}}(\beta_{\text{b}}, \beta_{\text{f}}), \quad |\zeta_{\text{f}}(\beta_{\text{b}}, \beta_{\text{f}})| \leq C \left((\varepsilon |\log \varepsilon| + |\lambda|) |\beta_{\text{f}}| + e^{-q/\varepsilon} |\beta_{\text{b}}| \right), \\ \zeta_{\text{b},+} &= \zeta_{\text{b},+}(\beta_{\text{b}}, \beta_{\text{f}}), \quad |\zeta_{\text{b},+}(\beta_{\text{b}}, \beta_{\text{f}})| \leq C \left((\varepsilon^{\rho(a)} |\log \varepsilon| + |\lambda|) |\beta_{\text{b}}| + e^{-q/\varepsilon} |\beta_{\text{f}}| \right). \end{aligned} \tag{5.66}$$

For the last two matching conditions we substitute (5.66) into the identities (5.26), (5.31), (5.32) and (5.52). Moreover, we estimate the tail of the integral in (5.26), i.e.

the part from $-\infty$ to $-L_\varepsilon$, using that the exponential dichotomy of (5.14) on $\mathbb{R}_-$ has exponent μ by Corollary 4.5.7 and it holds $\nu \geq \mu/2$. Thus, we obtain the last two matching conditions by pairing with $\Psi_{f,\perp} \in \ker(Q_{f,+}^s(0)^*)$ and $\Psi_{b,\perp} \in \ker(Q_{b,+}^s(0)^*)$

$$\begin{aligned} 0 &= \langle \Psi_{f,\perp}, \psi_{f,-}(0, \lambda) - \psi^{\text{sl}}(0, \lambda) \rangle \\ &= \beta_f \int_{-L_\varepsilon}^{L_\varepsilon} \langle T_f(0, \xi)^* \Psi_{f,\perp}, B_f(\xi, \lambda) \omega_f(\xi) \rangle d\xi + \mathcal{H}_3(\beta_b, \beta_f), \end{aligned} \quad (5.67)$$

$$\begin{aligned} 0 &= \langle \Psi_{b,\perp}, \psi^{\text{sl}}(Z_{a,\varepsilon}, \lambda) - \psi_{b,+}(Z_{a,\varepsilon}, \lambda) \rangle \\ &= \beta_b \int_{-L_\varepsilon}^{L_\varepsilon} \langle T_b(0, \xi)^* \Psi_{b,\perp}, B_b(\xi, \lambda) \omega_b(\xi) \rangle d\xi + \mathcal{H}_4(\beta_b, \beta_f), \end{aligned} \quad (5.68)$$

where the linear maps $\mathcal{H}_3$ and $\mathcal{H}_4$ satisfy the bounds

$$\begin{aligned} |\mathcal{H}_3(\beta_b, \beta_f)| &\leq C \left((\varepsilon |\log \varepsilon| + |\lambda|)^2 |\beta_f| + e^{-q/\varepsilon} |\beta_b| \right), \\ |\mathcal{H}_4(\beta_b, \beta_f)| &\leq C \left((\varepsilon^{\rho(a)} |\log \varepsilon| + |\lambda|)^2 |\beta_b| + e^{-q/\varepsilon} |\beta_f| \right). \end{aligned}$$

The same procedure can be done using the expressions (5.27), (5.33) and (5.53) instead. We approximate a -uniformly

$$\begin{aligned} 0 &= \langle \Psi_{f,\perp}, \phi'_{a,\varepsilon}(0) - \phi'_{a,\varepsilon}(0) \rangle = \langle \Psi_{f,\perp}, Q_{f,-}^s(0) \phi'_{a,\varepsilon}(0) - Q_{f,+}^u(0) \phi'_{a,\varepsilon}(0) \rangle \\ &= \int_{-L_\varepsilon}^{L_\varepsilon} \langle e^{-\xi\eta} T_f(0, \xi)^* \Psi_{f,\perp}, B_f(\xi, 0) \phi'_{a,\varepsilon}(\xi) \rangle d\xi + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (5.69)$$

$$\begin{aligned} 0 &= \langle \Psi_{b,\perp}, \phi'_{a,\varepsilon}(0) - \phi'_{a,\varepsilon}(0) \rangle = \langle \Psi_{b,\perp}, Q_{b,-}^s(0) \phi'_{a,\varepsilon}(0) - Q_{b,+}^u(0) \phi'_{a,\varepsilon}(0) \rangle \\ &= \int_{-L_\varepsilon}^{L_\varepsilon} \langle e^{-\xi\eta} T_b(0, \xi)^* \Psi_{b,\perp}, B_b(\xi, 0) \phi'_{a,\varepsilon}(Z_{a,\varepsilon} + \xi) \rangle d\xi \\ &\quad + \langle e^{\eta L_\varepsilon} T_b(0, -L_\varepsilon)^* \Psi_{b,\perp}, \phi'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \rangle + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (5.70)$$

using $\nu \geq \mu/2 \geq \eta/2$ (see (5.5) and Lemma 4.5.3).

Our plan is to use the identities (5.69) and (5.70) to simplify the expressions

in (5.67) and (5.68). First, we calculate

$$\begin{aligned}
e^{-\eta\xi}T_j(0,\xi)^*\Psi_{j,\perp} &= \begin{pmatrix} e^{-\eta\xi}\varphi_{j,\text{ad}}(\xi) \\ -\int_{\infty}^{\xi} e^{-\eta\hat{\xi}} \left\langle \varphi_{j,\text{ad}}(\hat{\xi}), F \right\rangle d\hat{\xi} \end{pmatrix} \\
&= \begin{pmatrix} e^{-\check{c}_0\xi}v'_j(\xi) \\ -e^{-\check{c}_0\xi}u'_j(\xi) \\ \int_{\infty}^{\xi} e^{-\check{c}_0\hat{\xi}}u'_j(\hat{\xi})d\hat{\xi} \end{pmatrix}, \quad \xi \in \mathbb{R}, \quad j = \text{f}, \text{b},
\end{aligned} \tag{5.71}$$

where $(u_j(\xi), v_j(\xi)) = \phi_j(\xi)$. Recall that the front ϕ_{f} is a heteroclinic connection between the fixed points $(0, 0)$ and $(1, 0)$ of the Nagumo system (3.4). By looking at the linearization of (3.4) about $(0, 0)$ and $(1, 0)$ we deduce that $\phi'_{\text{f}}(\xi)$ converges to 0 at an exponential rate $\frac{1}{2}\sqrt{2}$ as $\xi \rightarrow \pm\infty$. The same holds for $\phi'_{\text{b}}(\xi)$ by symmetry. Recall that $\check{c}_0$ is given by $\sqrt{2}(\frac{1}{2} - a)$. So, for all $a \geq 0$, the upper two entries of (5.71) are bounded on $\mathbb{R}$ by some constant $C > 0$, independent of a , whereas the last entry is bounded by $C|\log \varepsilon|$ on $[-L_{\varepsilon}, L_{\varepsilon}]$. Further, by (5.63) the upper two rows of $B_{\text{f}}(\xi, 0)$ are bounded by $C\varepsilon|\log \varepsilon|$ on $[-L_{\varepsilon}, L_{\varepsilon}]$, whereas the last row is bounded by $C\varepsilon$ as can be observed from (5.15). Combining these bounds with $\nu \geq 2\sqrt{2}$, (5.64) and (5.69) we approximate a -uniformly

$$\begin{aligned}
\int_{-L_{\varepsilon}}^{L_{\varepsilon}} \langle T_{\text{f}}(0, \xi)^*\Psi_{\text{f},\perp}, B_{\text{f}}(\xi, \lambda)\omega_{\text{f}}(\xi) \rangle d\xi &= \int_{-L_{\varepsilon}}^{L_{\varepsilon}} \langle e^{-\xi\eta}T_{\text{f}}(0, \xi)^*\Psi_{\text{f},\perp}, B_{\text{f}}(\xi, 0)\phi'_{a,\varepsilon}(\xi) \rangle d\xi \\
&\quad - \lambda \int_{-L_{\varepsilon}}^{L_{\varepsilon}} e^{-\check{c}_0\xi} (u'_{\text{f}}(\xi))^2 d\xi + \mathcal{O}(|\varepsilon \log \varepsilon|^2) \\
&= -\lambda \int_{-\infty}^{\infty} e^{-\check{c}_0\xi} (u'_{\text{f}}(\xi))^2 d\xi + \mathcal{O}(|\varepsilon \log \varepsilon|^2).
\end{aligned} \tag{5.72}$$

Similarly, we estimate a -uniformly

$$\begin{aligned} \int_{-L_\varepsilon}^{L_\varepsilon} \langle T_b(0, \xi)^* \Psi_{b,\perp}, B_b(\xi, \lambda) \omega_b(\xi) \rangle d\xi &= - \langle e^{\eta L_\varepsilon} T_b(0, -L_\varepsilon)^* \Psi_{b,\perp}, \phi'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \rangle \\ &\quad - \lambda \int_{-\infty}^{\infty} e^{-\check{c}_0 \xi} (u'_b(\xi))^2 d\xi + \mathcal{O}(|\varepsilon^{\rho(a)} \log \varepsilon|^2), \end{aligned} \quad (5.73)$$

using (5.70) instead of (5.69). Substituting identities (5.72) and (5.73) into the remaining matching conditions (5.67) and (5.68) we arrive at the linear system

$$\begin{pmatrix} -\lambda \mathcal{M}_f + \mathcal{O}((\varepsilon |\log \varepsilon| + |\lambda|)^2) & \mathcal{O}(e^{-q/\varepsilon}) \\ \mathcal{O}(e^{-q/\varepsilon}) & -\lambda M_{b,1} - M_{b,2} + \mathcal{O}((\varepsilon^{\rho(a)} |\log \varepsilon| + |\lambda|)^2) \end{pmatrix} \times \begin{pmatrix} \beta_f \\ \beta_b \end{pmatrix} = 0, \quad (5.74)$$

where the approximations are a -uniformly and with $M_{b,1}$ and $M_{b,2}$ as in (5.61) and

$$M_f := \int_{-\infty}^{\infty} (u'_f(\xi))^2 e^{-\check{c}_0 \xi} d\xi > 0. \quad (5.75)$$

Thus, any nontrivial solution (β_b, β_f) to (5.74) corresponds to an eigenfunction of (5.6).

Since the perturbation matrices $B_j(\xi, \lambda), j = f, b$, the evolution $\mathcal{T}(\xi, \hat{\xi}, \lambda)$ of system (5.6) and the projections $\mathcal{Q}_{r,\ell}^{u,s}(\xi, \lambda)$ associated with the exponential dichotomy of (5.6) established in Proposition 4.5.5 are analytic in λ , all quantities occurring in this section are analytic in λ . Thus, the matrix in (5.74) and its determinant $D(\lambda) = D(\lambda; a, \varepsilon)$ are analytic in λ .

Observe that the ε -independent quantities M_f and $M_{b,1}$ are to leading order bounded away from 0, i.e. it holds $1/C \leq M_f, M_{b,1} \leq C$, since $u'_j(\xi)$ converges to 0 as $\xi \rightarrow \pm\infty$ at an exponential rate $\frac{1}{2}\sqrt{2}$; see also (3.6). Second, we estimate a -uniformly $M_{b,2} = \mathcal{O}(\varepsilon^{\rho(a)}|\log \varepsilon|)$ by combining (5.63) and (5.70). Hence, provided $\delta, \varepsilon > 0$ are sufficiently small, we have for $\lambda \in \partial R_1(\delta) = \{\lambda \in \mathbb{C} : |\lambda| = \delta\}$

$$|D(\lambda) - \lambda M_f(\lambda M_{b,1} + M_{b,2})| < |\lambda M_f(\lambda M_{b,1} + M_{b,2})|.$$

By Rouché's Theorem $D(\lambda)$ has in $R_1(\delta)$ precisely two roots λ_0, λ_1 that are a -uniformly $\mathcal{O}(|\varepsilon^{\rho(a)} \log \varepsilon|^2)$ -close to the roots of the quadratic $\lambda M_f(\lambda M_{b,1} + M_{b,2})$ given by 0 and $-M_{b,2}M_{b,1}^{-1}$. We conclude that (5.6) has two eigenvalues λ_0, λ_1 in the region R_1 .

We are interested in an eigenfunction $\psi_1(\xi)$ of (5.6) corresponding to the eigenvalue λ_1 that is a -uniformly $\mathcal{O}(|\varepsilon^{\rho(a)} \log \varepsilon|^2)$ -close to $-M_{b,2}M_{b,1}^{-1}$. The associated solution to (5.74) is given by the eigenvector $(\beta_f, \beta_b) = (\mathcal{O}(e^{-q/\varepsilon}), 1)$. In the proofs of Propositions 4.5.8, 4.5.9 and 4.5.10 we established a piecewise continuous eigenfunction to (5.6) for any prospective eigenvalue $\lambda \in R_1$. Thus, the eigenfunction $\psi_1(\xi)$ to (5.6), corresponding to the eigenvalue λ_1 , satisfies (5.29) on $I_{f,-}$, (5.37) on $I_{f,+}$, (5.40) on I_r , (5.43) on $I_{b,-}$, (5.54) on $I_{b,+}$ and (5.58) on I_ℓ . The variables occurring in these six expressions can all be expressed in $\beta_f = \mathcal{O}(e^{-q/\varepsilon})$ and $\beta_b = 1$. This leads to the approximation (5.62) of $\psi_1(\xi)$.

By translational invariance we know a priori that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an eigenfunction of (5.6) at $\lambda = 0$. Therefore, $\lambda = 0$ is one of the two eigenvalues $\lambda_0, \lambda_1 \in R_1$ of (5.6). With the aid of the bounds (5.62) one observes that the eigenfunction $\psi_1(\xi)$ is not a multiple of $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$. On the other hand, the space of exponentially decaying solutions in backward time to (5.6) is one-dimensional, because the asymptotic ma-

trix $\hat{A}(0, \lambda, a, \varepsilon)$ of system (5.6) has precisely one eigenvalue of positive real part by Lemma 4.5.3. Hence, the eigenfunctions $\psi_1(\xi)$ and $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ must correspond to different eigenvalues. We conclude $\lambda_0 = 0$ and $\lambda_1 \neq \lambda_0$. $\square$

Remark 4.5.11. *In the proof of Theorem 4.9 we simplified the final matching equation by using that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an exponentially localized solution to (5.6) at $\lambda = 0$. More precisely, during the matching procedure we substituted the expressions*

$$\int_{-L_\varepsilon}^{L_\varepsilon} \langle T_j(0, \xi)^* \Psi_{j,\perp}, B_j(\xi, 0) \omega_j(\xi) \rangle d\xi, \quad j = f, b, \quad (5.76)$$

by (5.72) and (5.73). Alternatively, one could try to calculate (5.76) directly using (5.71). The most problematic term is the difference $f'(u_{a,\varepsilon}(\xi)) - f'(u_j(\xi))$ in $B_j(\xi, 0)$. This difference can be calculated using an identity of the form

$$(\partial_\xi - C_j(\xi)) \begin{bmatrix} e^{-\eta\xi} \begin{pmatrix} u'_{a,\varepsilon}(\xi) \\ v'_{a,\varepsilon}(\xi) \end{pmatrix} \end{bmatrix} = e^{-\eta\xi} \begin{pmatrix} 0 \\ (c(\varepsilon) - c(0))v'_{a,\varepsilon}(\xi) - (f'(u_{a,\varepsilon}(\xi)) - f'(u_j(\xi)))u'_{a,\varepsilon}(\xi) + w'_{a,\varepsilon}(\xi) \end{pmatrix},$$

$j = f, b$, where C_j is the coefficient matrix of (5.18). The equivalent of the latter is done in [31] in the context of the lattice Fitzhugh-Nagumo equations.

Remark 4.5.12. *The proof of Theorem 4.9 shows that any eigenfunction of problem (5.6) corresponds to an eigenvector (β_f, β_b) of (5.74). Such an eigenfunction is obtained by pasting together the eigenfunctions $\omega_f(\xi)$ and $\omega_b(\xi)$ to the reduced eigenvalue problems (5.14) with amplitudes β_f and β_b , respectively.*

The eigenvector $(\beta_f, \beta_b) = (1, \mathcal{O}(e^{-q/\varepsilon}))$ of (5.74) corresponds to the eigenfunction $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ of (5.6) at $\lambda = 0$. Indeed, this eigenfunction is centered at the front and close to $\omega_f(\xi)$. Switching back to the unshifted eigenvalue problem (2.3), we ob-

serve that the corresponding eigenfunction $\phi'_{a,\varepsilon}(\xi)$ to (2.3) is close to a concatenation of $\omega_f(\xi)$ and $\omega_b(\xi)$; see also Theorem 4.5.

The other eigenvector $(\beta_f, \beta_b) = (\mathcal{O}(e^{-q/\varepsilon}), 1)$ of (5.74) corresponds to the eigenfunction $\psi_1(\xi)$ of (5.6) at $\lambda = \lambda_1$. The eigenfunction $\psi_1(\xi)$ is centered at the back and close to $\omega_b(\xi)$; see also estimate (5.62). When λ_1 lies to the right of the essential spectrum of $\mathcal{L}_{a,\varepsilon}$, it is also an eigenvalue of the unshifted eigenvalue problem (2.3) by Proposition 4.5.4. An eigenfunction of (2.3) corresponding to this potential second eigenvalue λ_1 is given by $\tilde{\psi}_1(\xi) := e^{\eta(\xi - Z_{a,\varepsilon})}\psi_1(\xi)$. Using the estimate (5.62) we conclude that $\tilde{\psi}_1(\xi)$ is centered at the back and the left slow manifold and close to $\omega_b(\xi)$ along the back, i.e. it holds

$$\begin{aligned} \|\tilde{\psi}_1(\xi)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|e^{-\eta(Z_{a,\varepsilon}-\xi)}, \quad \xi \in (-\infty, Z_{a,\varepsilon} - L_\varepsilon], \\ \|\tilde{\psi}_1(\xi + Z_{a,\varepsilon}) - \omega_b(\xi)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|e^{\eta\xi}, \quad \xi \in [-L_\varepsilon, L_\varepsilon]. \end{aligned}$$

We emphasize that in contrast to the shifted eigenvalue problem, we do not obtain that the eigenfunction $\tilde{\psi}_1(\xi)$ is small along the left slow manifold, i.e. for $\xi \in I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$. This observation agrees with the numerics done in §4.7; compare Figures 4.6a and 4.7.

4.5.4 The translational eigenvalue is simple

In this section we prove that $\lambda_0 = 0$ is a simple eigenvalue of $\mathcal{L}_{a,\varepsilon}$. This is an essential ingredient to establish nonlinear stability of the traveling pulse $\tilde{\phi}_{a,\varepsilon}(\xi)$; see [15, 16] and Theorem 4.3. By Theorem 4.9 λ_0 has geometric multiplicity one. To prove that λ_0 also has algebraic multiplicity one we consider the associated shifted generalized eigenvalue problem at $\lambda = \lambda_0$. Particular solutions to this inhomogeneous problem

are given by the λ -derivatives of solutions $\psi(\xi, \lambda)$ to the shifted eigenvalue problem (5.6). By differentiating the exit and entry conditions at $\xi = 0$ and at $\xi = Z_{a,\varepsilon}$ established in Propositions 4.5.8, 4.5.9 and 4.5.10 we obtain exit and entry conditions for exponentially localized solutions to the generalized eigenvalue problem. Matching of these expression leads to a contradiction showing that λ_0 also has algebraic multiplicity one.

Proposition 4.5.13. *In the setting of Theorem 4.1, let $\tilde{\phi}_{a,\varepsilon}(\xi)$ denote a traveling-pulse solution to (2.2) with associated linear operator $\mathcal{L}_{a,\varepsilon}$. The translational eigenvalue $\lambda_0 = 0$ of $\mathcal{L}_{a,\varepsilon}$ is simple.*

Proof. By Theorem 4.9 the eigenspace of the shifted eigenvalue problem (5.6) at $\lambda = \lambda_0$ is spanned by the weighted derivative $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$. Translating back to the original system (2.3) we deduce $\ker(\mathcal{L}_{a,\varepsilon})$ is one-dimensional and spanned by $\tilde{\phi}'_{a,\varepsilon}(\xi)$. So the geometric multiplicity of λ_0 equals one. Regarding the algebraic multiplicity of the eigenvalue λ_0 we are interested in exponentially localized solutions $\tilde{\psi}$ to the generalized eigenvalue problem $\mathcal{L}_{a,\varepsilon}\tilde{\psi} = \tilde{\phi}'_{a,\varepsilon}(\xi)$. This problem can be represented by the inhomogeneous ODE

$$\tilde{\psi}_\xi = A_0(\xi, 0)\tilde{\psi} + [\partial_\lambda A_0](\xi, 0)\phi'_{a,\varepsilon}(\xi), \quad (5.77)$$

where $A_0(\xi, \lambda)$ is the coefficient matrix of (2.3). The asymptotic matrices of (2.3) and the shifted version (5.6) have precisely one eigenvalue of positive real part at $\lambda = 0$ by Proposition 4.4.1 and Lemma 4.5.3. Moreover, the weighted derivative $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is exponentially localized. Therefore, $\tilde{\psi}(\xi)$ is an exponentially localized solution to (5.77) if and only if $\psi(\xi) = e^{-\eta\xi}\tilde{\psi}(\xi)$ is an exponentially localized solution

to

$$\psi_\xi = A(\xi, 0)\psi + e^{-\eta\xi} [\partial_\lambda A](\xi, 0)\phi'_{a,\varepsilon}(\xi), \quad (5.78)$$

where $A(\xi, \lambda)$ is the coefficient matrix of the shifted eigenvalue problem (5.6).

Since $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an exponentially localized solution to (5.6) at $\lambda = 0$, there exists by Propositions 4.5.8, 4.5.9 and 4.5.10 solutions $\psi_{f,-}(\xi, \lambda)$, $\psi^{\text{sl}}(\xi, \lambda)$ and $\psi_{b,+}(\xi, \lambda)$ to (5.6), which are analytic in λ and satisfy (5.26), (5.31), (5.32) and (5.52) for some $\beta_{f,-}, \beta_f, \zeta_f, \beta_b, \beta_{b,+}, \zeta_{b,+} \in \mathbb{C}$, such that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ equals $\psi_{f,-}(\xi, 0)$ on $(-\infty, 0]$, $\psi^{\text{sl}}(\xi, 0)$ on $[0, Z_{a,\varepsilon}]$ and $\psi_{b,+}(\xi, 0)$ on $[Z_{a,\varepsilon}, \infty)$. As in the proof of Theorem 4.9 we match $\psi_{f,-}(0, 0)$ to $\psi^{\text{sl}}(0, 0)$ and $\psi^{\text{sl}}(Z_{a,\varepsilon}, 0)$ to $\psi_{b,-}(Z_{a,\varepsilon}, 0)$. Applying the projections $Q_{j,-}^u(0), j = f, b$ to the differences $\psi_{f,-}(0, 0) - \psi^{\text{sl}}(0, 0)$ and $\psi^{\text{sl}}(Z_{a,\varepsilon}, 0) - \psi_{b,-}(Z_{a,\varepsilon}, 0)$ yields $\beta_{f,-} = \beta_f$ and $\beta_b = \beta_{b,+}$. Taking the inner products $0 = \langle \Psi_2, \psi_{f,-}(0, 0) - \psi^{\text{sl}}(0, 0) \rangle$ and $0 = \langle \Psi_2, \psi^{\text{sl}}(Z_{a,\varepsilon}, 0) - \psi_{b,-}(Z_{a,\varepsilon}, 0) \rangle$ we obtain that ζ_f and $\zeta_{b,+}$ can be expressed in β_b and β_f as

$$\begin{aligned} \zeta_f &= \zeta_f(\beta_b, \beta_f), \quad |\zeta_f(\beta_b, \beta_f)| \leq C (\varepsilon |\log \varepsilon| |\beta_f| + e^{-q/\varepsilon} |\beta_b|) \\ \zeta_{b,+} &= \zeta_{b,+}(\beta_b, \beta_f), \quad |\zeta_{b,+}(\beta_b, \beta_f)| \leq C (\varepsilon^{2/3} |\log \varepsilon| |\beta_b| + e^{-q/\varepsilon} |\beta_f|), \end{aligned} \quad (5.79)$$

where $C > 0$ is independent of a and ε .

Observe that the derivatives $[\partial_\lambda \psi_{f,-}](\xi, 0)$, $[\partial_\lambda \psi^{\text{sl}}](\xi, 0)$ and $[\partial_\lambda \psi_{b,+}](\xi, 0)$ are particular solutions to the equation (5.78) on $(-\infty, 0]$, $[0, Z_{a,\varepsilon}]$ and $[Z_{a,\varepsilon}, \infty)$, respectively. Moreover, $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ spans the space of exponentially localized solutions to the homogeneous problem (5.6) associated to (5.78). Now suppose that $\psi(\xi)$ is an

exponentially localized solution to (5.78). By the previous two observations it holds

$$\begin{aligned}
\psi(\xi) &= [\partial_\lambda \psi_{f,-}](\xi, 0) + \alpha_1 e^{-\eta\xi} \phi'_{a,\varepsilon}(\xi), \quad \xi \in (-\infty, 0], \\
\psi(\xi) &= [\partial_\lambda \psi^{\text{sl}}](\xi, 0) + \alpha_2 e^{-\eta\xi} \phi'_{a,\varepsilon}(\xi), \quad \xi \in [0, Z_{a,\varepsilon}], \\
\psi(\xi) &= [\partial_\lambda \psi_{b,+}](\xi, 0) + \alpha_3 e^{-\eta\xi} \phi'_{a,\varepsilon}(\xi), \quad \xi \in [Z_{a,\varepsilon}, \infty),
\end{aligned} \tag{5.80}$$

for some $\alpha_{1,2,3} \in \mathbb{C}$. We differentiate the analytic expressions (5.26) and (5.31) with respect to λ and obtain by the Cauchy estimates and (5.79)

$$\begin{aligned}
[\partial_\lambda \psi_{f,-}](\xi, 0) &= \beta_f \int_{-\infty}^0 T_{f,-}^s(0, \hat{\xi}) \tilde{B} \omega_f(\hat{\xi}) d\hat{\xi} + \mathcal{H}_1(\beta_f), \\
\|\mathcal{H}_1(\beta_f)\| &\leq C\varepsilon |\log \varepsilon| |\beta_{f,-}|, \\
[\partial_\lambda \psi^{\text{sl}}](\xi, 0) &= \beta_f \int_{L_\varepsilon}^0 T_{f,+}^u(0, \hat{\xi}) \tilde{B} \omega_f(\hat{\xi}) d\hat{\xi} + \mathcal{H}_2(\beta_f, \beta_b), \\
\|\mathcal{H}_2(\beta_f, \beta_b)\| &\leq C (\varepsilon |\log \varepsilon| |\beta_f| + e^{-q/\varepsilon} |\beta_b|),
\end{aligned} \tag{5.81}$$

where ω_f is as in (5.22), $\mathcal{H}_{1,2}$ are linear maps and $\tilde{B}$ denotes the derivative of the perturbation matrix

$$\tilde{B} = \tilde{B}(a, \varepsilon) := [\partial_\lambda] B_f(\xi, \lambda) := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\varepsilon} \end{pmatrix}.$$

On the other hand, we estimate using Theorem 4.5 (i)

$$\|\Psi_{f,a,\varepsilon} - \Psi_{1,f}\| \leq C\varepsilon |\log \varepsilon|, \quad \text{where } \Psi_{f,a,\varepsilon} := \begin{pmatrix} v'_{a,\varepsilon}(0) \\ -u'_{a,\varepsilon}(0) \\ 0 \end{pmatrix}, \tag{5.82}$$

and $\Psi_{1,f}$ is defined in (5.25). Note that $\Psi_{f,a,\varepsilon}$ is perpendicular to the derivative $\phi'_{a,\varepsilon}(0)$. As in the proof of Theorem 4.9 note that the front $\phi'_f(\xi) = (u'_f(\xi), v'_f(\xi))$

decays to 0 as $\xi \rightarrow \pm\infty$ with an exponential rate $\frac{1}{2}\sqrt{2}$. Thus, we calculate using $\nu \geq 2\sqrt{2}$, (5.80), (5.81) and (5.82)

$$\begin{aligned}
0 &= \langle \Psi_{f,a,\varepsilon}, [\partial_\lambda \psi_f, -](0, 0) - [\partial_\lambda \psi^{sl}](0, 0) + (\alpha_1 - \alpha_2) \phi'_{a,\varepsilon}(0) \rangle \\
&= \beta_f \left(\int_{-\infty}^{L_\varepsilon} \left\langle T_f(0, \xi)^* \Psi_{1,f}, \tilde{B} \omega_f(\xi) \right\rangle d\xi + \mathcal{O}(\varepsilon |\log \varepsilon|) \right) + \beta_b \mathcal{O}(e^{-q/\varepsilon}) \\
&= \beta_f (-M_f + \mathcal{O}(\varepsilon |\log \varepsilon|)) + \beta_b \mathcal{O}(e^{-q/\varepsilon}),
\end{aligned} \tag{5.83}$$

a -uniformly, where M_f is defined in (5.75). Let $\Psi_{b,a,\varepsilon} = (v'_{a,\varepsilon}(Z_{a,\varepsilon}), -u'_{a,\varepsilon}(Z_{a,\varepsilon}), 0)$. A similar calculation shows

$$\begin{aligned}
0 &= \langle \Psi_{b,a,\varepsilon}, [\partial_\lambda \psi^{sl}](Z_{a,\varepsilon}, 0) - [\partial_\lambda \psi_{b,+}](Z_{a,\varepsilon}, 0) + (\alpha_2 - \alpha_3) e^{-\eta Z_{a,\varepsilon}} \phi'_{a,\varepsilon}(Z_{a,\varepsilon}) \rangle \\
&= \beta_b (-M_{b,1} + \mathcal{O}(\varepsilon^{2/3} |\log \varepsilon|)) + \beta_f \mathcal{O}(e^{-q/\varepsilon}),
\end{aligned} \tag{5.84}$$

a -uniformly, where $M_{b,1}$ is defined in (5.61). The conditions (5.83) and (5.84) form a system of linear equations in β_f and β_b . The only solution to this system is $\beta_f = \beta_b = 0$, because $M_f, M_{b,1} > 0$ are independent of ε and bounded below away from 0 uniformly in a . This is a contradiction with the fact that $e^{-\eta \xi} \phi'_{a,\varepsilon}(\xi)$ is not the zero solution to (5.6). We conclude that (5.78) has no exponentially localized solution and that also the algebraic multiplicity of the eigenvalue $\lambda = 0$ of $\mathcal{L}_{a,\varepsilon}$ equals one. $\square$

4.5.5 Calculation of second eigenvalue

By Theorem 4.9 the second eigenvalue $\lambda_1 \in R_1$ of (5.6) is a -uniformly $\mathcal{O}(|\varepsilon^{\rho(a)} \log \varepsilon|^2)$ -close to the quotient $-M_{b,2} M_{b,1}^{-1}$. Thus, to prove our main stability results in Theorem 4.2, we need to show $-M_{b,2} M_{b,1}^{-1} \leq -\varepsilon b_0$, where b_0 is independent of a and ε . Since $M_{b,1} > 0$ is independent of ε and bounded by an a -independent constant, the

problem amounts to proving that $M_{b,2}$ is bounded below by $\varepsilon \tilde{b}_0$ for some $\tilde{b}_0 > 0$. We distinguish between the hyperbolic and nonhyperbolic regime.

In the hyperbolic regime, it is possible to determine the quantity $M_{b,2}$ to leading order. This relies on the fact that the solution $\varphi_{b,\text{ad}}(\xi)$, defined in (5.20), to the adjoint system (5.19) converges exponentially to 0 as $\xi \rightarrow -\infty$ with rate $\sqrt{2}a$. Since a is bounded below in the hyperbolic regime, the first two coordinates of Ψ_* , defined in (5.61), are of higher order by choosing ν sufficiently large.

Therefore, the calculation for $M_{b,2}$ reduces to approximating the product

$$w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \int_{-\infty}^{L_\varepsilon} u'_b(\xi) e^{-\check{c}_0 \xi} d\xi.$$

This leads to the following result.

Proposition 4.5.14. *For each $a_0 > 0$ there exists $\varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in [a_0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ the quantity $M_{b,2}$ in Theorem 4.9 is approximated (a -uniformly) by*

$$M_{b,2} = \frac{\varepsilon}{\check{c}_0} (\gamma w_b^1 - u_b^1) \int_{-\infty}^{\infty} u'_b(\xi) e^{-\check{c}_0 \xi} d\xi + \mathcal{O}(\varepsilon^2 |\log \varepsilon|), \quad (5.85)$$

In particular, we have $M_{b,2} > \varepsilon/k_0$ for some $k_0 > 1$, independent of a and ε .

Proof. The Nagumo back solution $\phi_b(\xi)$ to system (3.5) converges to the fixed point $p_b^1 = (u_b^1, 0)$ as $\xi \rightarrow -\infty$. By looking at the linearization of (3.5) about p_b^1 we deduce that the convergence of $\phi_b(\xi)$ to p_b^1 is exponential at a rate $\frac{1}{2}\sqrt{2}$. Combining this

with Theorem 4.5 (ii), $\nu \geq 2\sqrt{2}$ and $\check{c} - \check{c}_0 = \mathcal{O}(\varepsilon)$ we estimate

$$\begin{aligned} w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) &= \frac{\varepsilon}{\check{c}} (u_{a,\varepsilon}(-L_\varepsilon) - \gamma w_{a,\varepsilon}(-L_\varepsilon)) \\ &= \frac{\varepsilon}{\check{c}_0} (u_b^1 - \gamma w_b^1) + \mathcal{O}(\varepsilon^2 |\log \varepsilon|). \end{aligned}$$

In addition, the derivative $\phi'_b(\xi)$ converges exponentially to 0 at a rate $\frac{1}{2}\sqrt{2}$ as $\xi \rightarrow -\infty$. Finally, recall that $\check{c}_0(a) = \sqrt{2}(\frac{1}{2} - a)$. Using all the previous observations, we estimate

$$\begin{aligned} M_{b,2} &= \left\langle \begin{pmatrix} e^{\check{c}_0 L_\varepsilon} v'_b(-L_\varepsilon) \\ -e^{\check{c}_0 L_\varepsilon} u'_b(-L_\varepsilon) \\ \int_{-\infty}^{-L_\varepsilon} e^{-\check{c}_0 \hat{\xi}} u'_b(\hat{\xi}) d\hat{\xi} \end{pmatrix}, \begin{pmatrix} u'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ v'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle \\ &= -\frac{\varepsilon}{\check{c}_0} (u_b^1 - \gamma w_b^1) \int_{-\infty}^{\infty} u'_b(\xi) e^{-\check{c}_0 \xi} d\xi + \mathcal{O}(\varepsilon^2 |\log \varepsilon|, \varepsilon^{\sqrt{2}a_0\nu}). \end{aligned}$$

Without loss of generality we may assume $\nu \geq \sqrt{2}/a_0$. Thus, we take

$$\nu \geq \max\{2\sqrt{2}, \sqrt{2}/a_0, 2/\mu\} > 0$$

(see (5.5)). With this choice of ν the approximation result follows. Since we have $0 < \gamma < 4$, the line $w = \gamma^{-1}u$ intersects the cubic $w = f(u)$ only at $u = 0$. So, it holds $u_b^1 - \gamma w_b^1 > 0$. Moreover, we have $u'_b(\xi) = v_b(\xi) < 0$ for all $x \in \mathbb{R}$. Combing these two items, it follows $M_{b,2} > \varepsilon/k_0$. $\square$

Recall that the solution $\varphi_{b,\text{ad}}(\xi)$, defined in (5.20), to the adjoint system (5.19) converges exponentially to 0 as $\xi \rightarrow -\infty$ with rate $\sqrt{2}a$. Thus, in the nonhyperbolic regime $0 < a \ll 1$, the first two coordinates of Ψ_* , defined in (5.61), are no longer of higher-order, as was the case in the hyperbolic regime. Therefore, in addition to the

product $w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \int_{-\infty}^{L_\varepsilon} u'_b(\xi) e^{-\check{c}_0 \xi} d\xi$, we also have to bound the inner product

$$\left\langle \begin{pmatrix} \varphi_{b,\text{ad}}(-L_\varepsilon) \\ 0 \end{pmatrix}, \phi'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \right\rangle, \quad (5.86)$$

from below away from 0. Recall from §4.3.3 that the pulse solution $\phi_{a,\varepsilon}(\xi)$ is at $\xi = Z_{a,\varepsilon} - L_\varepsilon$ in the neighborhood $\mathcal{U}_F$ of the fold point $(u^*, 0, w^*)$, where $u^* = \frac{1}{3}(a + 1 + \sqrt{a^2 - a + 1})$ and $w^* = f(u^*)$. In $\mathcal{U}_F$ there exists a coordinate transform $\Phi_\varepsilon: \mathcal{U}_F \rightarrow \mathbb{R}^3$ bringing system (3.1) into the canonical form (3.13). In system (3.13) the dynamics on the two-dimensional invariant manifold $z = 0$ is decoupled from the dynamics along the straightened out strong unstable fibers in the z -direction. The flow on the invariant manifold $z = 0$ can be estimated; see Propositions 4.3.4 and 4.3.5. Therefore, our approach is to transfer to local coordinates by applying Φ_ε to the inner product (5.86). The estimates on the dynamics of (3.13) leads to bounds on $\phi'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)$ in the local coordinates. In addition, the other term $(\phi'_{b,\text{ad}}(-L_\varepsilon), 0)$ in the inner product (5.86) can be determined to leading order in the local coordinates, since the linear action of Φ_ε is explicit. Furthermore, if we have $\varepsilon > K_0 a^3$, then the leading order of $\phi'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)$ can also be determined in local coordinates using the estimates on the x -derivative given in Proposition 4.3.4 (ii). The procedure described above leads to the following result.

Proposition 4.5.15. *For each sufficiently small $a_0 > 0$, there exists $\varepsilon_0 > 0$ and $K_0, k_0 > 1$, such that for each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ the quantity $M_{b,2}$ in Theorem 4.9 satisfies $M_{b,2} > \varepsilon/k_0$. If we have in addition $\varepsilon > K_0 a^3$, then $M_{b,2}$ is bounded as $\varepsilon^{2/3}/k_0 < M_{b,2} < \varepsilon^{2/3} k_0$ and can be approximated a -uniformly by*

$$M_{b,2} = \frac{a^2}{4\sqrt{2}} - \frac{(18 - 4\gamma)^{2/3}}{9\sqrt{2}} \Theta^{-1} \left(\frac{-3a}{2(18 - 4\gamma)^{1/3} \varepsilon^{1/3}} \right) \varepsilon^{2/3} + \mathcal{O}(\varepsilon |\log \varepsilon|),$$

where Θ is defined in (3.15).

Proof. We start by estimating the lower term in the inner product $M_{b,2}$. Similarly as in the proof of Proposition 4.5.14, we estimate a -uniformly

$$w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) = \frac{\varepsilon}{\check{c}_0} (u_b^1 - \gamma w_b^1) + \mathcal{O}(\varepsilon^{5/3} |\log \varepsilon|),$$

using Theorem 4.5 (ii). The ε -independent quantity $u_b^1 - \gamma w_b^1 > 0$ is approximated by $\frac{2}{3} - \frac{4}{27}\gamma + \mathcal{O}(a)$ and is bounded away from 0, since $u_b^1 = \frac{2}{3}(1+a)$, $w_b^1 = f(u_b^1)$ and $0 < \gamma < 4$. In addition, $u'_b(\xi)$ is strictly negative, independent of ε and a and converges to 0 at an exponential rate $\frac{1}{2}\sqrt{2}$ as $\xi \rightarrow \pm\infty$; see (3.6). Therefore, we estimate

$$\tilde{k}_0 \varepsilon < \left\langle \int_{-\infty}^{-L_\varepsilon} e^{-\check{c}_0 \hat{\xi}} u'_b(\hat{\xi}) d\hat{\xi}, w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \right\rangle < \varepsilon |\log \varepsilon| / \tilde{k}_0 \quad (5.87)$$

for some $\tilde{k}_0 > 0$ independent of a and ε .

We continue by estimating the upper terms in the inner product $M_{b,2}$. The linearization about the fixed point $(u_b^1, 0)$ of (3.5) has eigenvalues $\frac{1}{2}\sqrt{2}$ and $-\sqrt{2}a$ and corresponding eigenvectors $v_+ = (1, \frac{1}{2}\sqrt{2})$ and $v_- = (1, -\sqrt{2}a)$, respectively. By [40, Theorem 1] $\phi'_b(\xi)e^{-\xi/\sqrt{2}}$ converges at an exponential rate $\frac{1}{2}\sqrt{2}$ to an eigenvector $\alpha_+ v_+$ as $\xi \rightarrow -\infty$ for some $\alpha_+ \in \mathbb{R} \setminus \{0\}$. Using the explicit formula (3.6) for $\phi_b(\xi)$, we deduce $\alpha_+ = -\frac{1}{2}\sqrt{2}e^{\xi_{b,0}/\sqrt{2}}$, where $\xi_{b,0} \in \mathbb{R}$ denotes the initial translation. Without loss of generality we take $\xi_{b,0} = 0$ so that $\alpha_+ = -\frac{1}{2}\sqrt{2}$; see Remark 4.3.1. Thus, we approximate a -uniformly

$$e^{\check{c}_0 L_\varepsilon} \begin{pmatrix} v'_b(-L_\varepsilon) \\ -u'_b(-L_\varepsilon) \end{pmatrix} = \frac{1}{2} e^{-\sqrt{2}a L_\varepsilon} \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} + \mathcal{O}(\varepsilon^2), \quad (5.88)$$

using $\nu \geq 2\sqrt{2}$. For the remaining computations, we transform into local coordinates in the neighborhood $\mathcal{U}_F$ of the fold point $(u^*, 0, w^*)$; see §4.3.3. Recall from the

proof of Theorem 4.5 that $\phi_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)$ is contained in the fold neighborhood $\mathcal{U}_F$ for $a_0, \varepsilon_0 > 0$ sufficiently small. We apply the coordinate transform $\Phi_\varepsilon: \mathcal{U}_F \rightarrow \mathbb{R}^3$ bringing system (3.1) into the canonical form (3.13). Recall from §4.3.3 that Φ_ε is C^r -smooth in a and ε in a neighborhood of $(a, \varepsilon) = 0$. Moreover, Φ_ε can be decomposed about $(u^*, 0, w^*)$ into a linear and a nonlinear part

$$\begin{aligned} \Phi_\varepsilon \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= \mathcal{N} \left[\begin{pmatrix} u \\ v \\ w \end{pmatrix} - \begin{pmatrix} u^* \\ 0 \\ w^* \end{pmatrix} \right] + \tilde{\Phi}_\varepsilon \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \\ \mathcal{N} = \partial \Phi_\varepsilon \begin{pmatrix} u^* \\ 0 \\ w^* \end{pmatrix} &= \begin{pmatrix} -\beta_1 & \frac{\beta_1}{\varepsilon} & \frac{\beta_1}{\varepsilon^2} \\ 0 & 0 & \frac{\beta_2}{\varepsilon} \\ 0 & \frac{1}{\varepsilon} & \frac{1}{\varepsilon^2} \end{pmatrix}, \end{aligned} \quad (5.89)$$

where

$$\begin{aligned} \beta_1 &= (a^2 - a + 1)^{1/3} (u^* - \gamma w^*)^{-1/3} > 0, \\ \beta_2 &= \check{c} (a^2 - a + 1)^{1/6} (u^* - \gamma w^*)^{-2/3} > 0, \end{aligned}$$

uniformly in a and ε . The nonlinearity $\tilde{\Phi}_\varepsilon$ satisfies $\tilde{\Phi}_\varepsilon(u^*, 0, w^*) = \partial \tilde{\Phi}_\varepsilon(u^*, 0, w^*) = 0$ and $\partial \tilde{\Phi}_\varepsilon$ is bounded a - and ε -uniformly. Differentiating $(x_{a,\varepsilon}(\xi), y_{a,\varepsilon}(\xi), z_{a,\varepsilon}(\xi)) = \Phi_\varepsilon(\phi_{a,\varepsilon}(\xi))$ yields

$$\begin{pmatrix} x'_{a,\varepsilon}(\xi) \\ y'_{a,\varepsilon}(\xi) \\ z'_{a,\varepsilon}(\xi) \end{pmatrix} = [\mathcal{N} + \partial \tilde{\Phi}(\phi_{a,\varepsilon}(\xi))] \begin{pmatrix} u'_{a,\varepsilon}(\xi) \\ v'_{a,\varepsilon}(\xi) \\ w'_{a,\varepsilon}(\xi) \end{pmatrix}.$$

Recall that $(\phi_b(\xi), w_b^1)$ converges at an exponential rate $\frac{1}{2}\sqrt{2}$ to $(u_b^1, 0, w_b^1)$. Thus,

by Theorem 4.5 (ii) and $\nu \geq 2\sqrt{2}$ we have

$$\|\phi_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) - (u_b^1, 0, w_b^1)\| \leq C\varepsilon^{2/3}|\log \varepsilon|, \quad (5.90)$$

where $C > 0$ denotes a constant independent of a and ε . Recall that $u_b^1 = \frac{2}{3}(1+a)$, $u^* = \frac{1}{3}(a+1+\sqrt{a^2-a+1})$, $w_b^1 = f(u_b^1)$, $w^* = f(u^*)$ and $f'(u^*) = 0$. Therefore, we estimate

$$|u^* - \frac{2}{3}|, |w^* - \frac{4}{27}| \leq Ca \quad |u_b^1 - u^* - \frac{1}{2}a|, |w_b^1 - w^*| \leq Ca^2. \quad (5.91)$$

Combining estimates (5.90) and (5.91) with $\partial\tilde{\Phi}_\varepsilon(u^*, 0, w^*) = 0$, we estimate

$$\|\partial\tilde{\Phi}_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon))\| \leq C(\varepsilon^{2/3}|\log \varepsilon| + a). \quad (5.92)$$

Using (5.88) and

$$(\mathcal{N}^{-1})^* = \begin{pmatrix} -\frac{1}{\beta_1} & 0 & 0 \\ 0 & -\frac{1}{\beta_2} & \frac{\check{c}}{\beta_2} \\ 1 & \check{c} & 0 \end{pmatrix},$$

we approximate a -uniformly

$$\begin{aligned}
& e^{\check{c}_0 L_\varepsilon} \left\langle \begin{pmatrix} v'_b(-L_\varepsilon) \\ -u'_b(-L_\varepsilon) \end{pmatrix}, \begin{pmatrix} u'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ v'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle \\
&= \left\langle \frac{1}{2} e^{-\sqrt{2}aL_\varepsilon} \begin{pmatrix} -1 \\ \sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} u'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ v'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle + \mathcal{O}(\varepsilon^2) \\
&= \left\langle \frac{1}{2} e^{-\sqrt{2}aL_\varepsilon} (\mathcal{N}^{-1})^* \begin{pmatrix} -1 \\ \sqrt{2} \\ 0 \end{pmatrix}, \mathcal{N} \begin{pmatrix} u'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ v'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle + \mathcal{O}(\varepsilon^2) \tag{5.93} \\
&= \left\langle \frac{1}{2} e^{-\sqrt{2}aL_\varepsilon} \begin{pmatrix} \frac{1}{\beta_1} \\ -\frac{\sqrt{2}}{\beta_2} \\ \sqrt{2}\check{c} - 1 \end{pmatrix}, (I + \Delta) \begin{pmatrix} x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ y'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ z'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where $\Delta := -\partial\tilde{\Phi}_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)) \left(\mathcal{N} + \partial\tilde{\Phi}_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)) \right)^{-1}$. First, by (5.92) it holds $\|\Delta\| \leq C(\varepsilon^{2/3}|\log \varepsilon| + a)$. Second, from the equations (3.13) one observes that $|y'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)| < C\varepsilon$. Third, by Theorem 4.5 the pulse $\phi_{a,\varepsilon}(\xi)$ exits the fold neighborhood at $\xi = Z_{a,\varepsilon} - \xi_b$, where $\xi_b = \mathcal{O}(1)$. The dynamics in the z -component in (3.13) decays exponentially in backward time with rate greater than $\check{c}/2$ by taking the neighborhood $\mathcal{U}_F$ smaller if necessary. Note that $\check{c}$ is bounded from below away from 0 by an a -independent constant. Thus, we may assume that the a -independent constant ν satisfies $\nu \geq 2\check{c}^{-1}$, i.e. we take $\nu \geq \max\{2\sqrt{2}, 2\check{c}^{-1}, 2/\mu\} > 0$ (see (5.5)). With this choice of ν , we estimate $|z_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)| \leq C\varepsilon$. So, using the equation for z' in (3.13), one observes that $|z'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)| \leq C\varepsilon$. Combining the previous three

observations with (5.93), we approximate a -uniformly

$$\begin{aligned}
& e^{(\sqrt{2}a+\check{c}_0)L_\varepsilon} \left\langle \begin{pmatrix} v'_b(-L_\varepsilon) \\ -u'_b(-L_\varepsilon) \end{pmatrix}, \begin{pmatrix} u'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ v'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle \\
&= \frac{1}{2} x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \left\langle \begin{pmatrix} \frac{1}{\beta_1} \\ -\frac{\sqrt{2}}{\beta_2} \\ \sqrt{2}\check{c} - 1 \end{pmatrix}, (I + \Delta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle + \mathcal{O}(\varepsilon), \\
&= \frac{1}{2\beta_1} x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) (1 + \mathcal{O}(\varepsilon^{2/3}|\log \varepsilon| + a)) + \mathcal{O}(\varepsilon).
\end{aligned} \tag{5.94}$$

From Propositions 4.3.4 and 4.3.5 it follows that for any $k^\dagger > 0$ there exists $\varepsilon_0, a_0 > 0$ such that for $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ it holds $x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) > k^\dagger \varepsilon$. Moreover, $\beta_1 > 0$ is bounded by an a -independent constant. Thus, by taking $k^\dagger > 0$ sufficiently large, we estimate

$$M_{b,2} > e^{-\sqrt{2}aL_\varepsilon} \frac{k^\dagger \varepsilon}{4\beta_1} + \tilde{k}_0 \varepsilon, \tag{5.95}$$

using (5.87) and (5.94). This proves the first assertion.

Suppose we are in the regime $\varepsilon > K_0 a^3$ for some $K_0 > 0$, so that $a = \mathcal{O}(\varepsilon^{1/3})$. On the one hand, using (5.89) and (5.91) we approximate the x -coordinate x_b of $\Phi_\varepsilon(u_b^1, 0, w_b^1)$ by

$$x_b = -\beta_1 (u_b^1 - u^*) + \frac{\beta_1}{\check{c}^2} (w_b^1 - w^*) + \mathcal{O}(a^2) = -\frac{\beta_1 a}{2} + \mathcal{O}(a^2).$$

On the other hand, since $\partial \Phi_\varepsilon$ is bounded a - and ε -uniformly, we have by (5.90) that $|x_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) - x_b| \leq C\varepsilon^{2/3}|\log \varepsilon|$. Hence, using $K_0 a^3 < \varepsilon$, we estimate

$$|x_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) + \frac{1}{2}\beta_1 a| \leq C(\varepsilon^{2/3}|\log \varepsilon| + a^2) \leq C\varepsilon^{2/3}|\log \varepsilon|. \tag{5.96}$$

Therefore, Propositions 4.3.4 and 4.3.5 yield, provided $K_0 > 0$ is chosen sufficiently large (with lower bound independent of a and ε),

$$x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) = \theta_0 (x_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)^2 - \Theta^{-1}(x_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)\varepsilon^{-1/3})\varepsilon^{2/3}) + \mathcal{O}(\varepsilon). \quad (5.97)$$

First, by (5.91) it holds

$$\begin{aligned} \theta_0 &= \frac{1}{\check{c}}(a^2 - a + 1)^{1/6} (u^* - \gamma w^*)^{1/3} = \frac{\sqrt{2}}{3} (18 - 4\gamma)^{1/3} + \mathcal{O}(a), \\ \beta_1 &= (a^2 - a + 1)^{1/3} (u^* - \gamma w^*)^{-1/3} = 3(18 - 4\gamma)^{-1/3} + \mathcal{O}(a). \end{aligned}$$

Second, in the regime $K_0 a^3 < \varepsilon$ we have

$$\left| e^{-\sqrt{2}aL_\varepsilon} - 1 \right| \leq C\varepsilon^{1/3} |\log \varepsilon|.$$

Third, by combining (5.96) and (5.97), we observe $x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) = \mathcal{O}(\varepsilon^{2/3})$. We substitute (5.96) and (5.97) into (5.94) and approximate $M_{b,2}$ with the aid of the previous three observations and identity (5.87) by

$$\begin{aligned} M_{b,2} &= \frac{1}{2\beta_1} x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) + \mathcal{O}(\varepsilon |\log \varepsilon|) \\ &= \frac{\theta_0}{2\beta_1} (x_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)^2 - \Theta^{-1}(x_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)\varepsilon^{-1/3})\varepsilon^{2/3}) + \mathcal{O}(\varepsilon |\log \varepsilon|) \\ &= \frac{a^2}{4\sqrt{2}} - \frac{(18 - 4\gamma)^{2/3}}{9\sqrt{2}} \Theta^{-1} \left(\frac{-3a}{2(18 - 4\gamma)^{1/3} \varepsilon^{1/3}} \right) \varepsilon^{2/3} + \mathcal{O}(\varepsilon |\log \varepsilon|). \end{aligned}$$

This is the desired leading order approximation of $M_{b,2}$. In the regime $K_0 a^3 < \varepsilon$, for $K_0 > 1$ sufficiently large, the bound $\varepsilon^{2/3}/k_0 < M_{b,2} < \varepsilon^{2/3}k_0$ follows from this approximation, using that Θ^{-1} is smooth and $\Theta^{-1}(0) < 0$. $\square$

Remark 4.5.16. By Theorem 4.9 the second eigenvalue λ_1 of (5.6) is to leading order approximated by the quotient $M_{b,2}M_{b,1}^{-1}$. We give a geometric interpretation of

the quantities $M_{b,1}$ and $M_{b,2}$ in both the hyperbolic and nonhyperbolic regimes.

For the interpretation of the quantity $M_{b,1}$ we append the Nagumo eigenvalue problem to the Nagumo existence problem (3.5) along the back

$$\begin{aligned}
 u_\xi &= v, \\
 v_\xi &= \check{c}_0 v - f(u) + w_b^1, \\
 \tilde{u}_\xi &= \tilde{v}, \\
 \tilde{v}_\xi &= \check{c}_0 \tilde{v} - f'(u) \tilde{u} + \lambda \tilde{u}.
 \end{aligned} \tag{5.98}$$

Note that $(\phi_b(\xi), \phi'_b(\xi))$ is a heteroclinic solution to (5.98) for $\lambda = 0$ connecting the equilibria $(p_b^1, 0)$ and $(p_b^0, 0)$. The space of bounded solutions to the adjoint equation of the linearization of (5.98) at $\lambda = 0$ about $(\phi_b(\xi), \phi'_b(\xi))$ is spanned by $(\psi_{ad,1}(\xi), 0)$ and $(\psi_{ad,2}(\xi), \psi_{ad,1}(\xi))$, where $\psi_{ad,1}(\xi) = (v'_b(\xi), -u'_b(\xi))e^{-\check{c}_0\xi}$. The Melnikov integral

$$M_{b,1} = \int_{-\infty}^{\infty} (u'_b(\xi))^2 e^{-\check{c}_0\xi} d\xi,$$

measures how the intersection between the stable manifold $\mathcal{W}^s(p_b^0, 0)$ and unstable manifold $\mathcal{W}^u(p_b^1, 0)$ breaks at $(\phi_b(0), \phi'_b(0))$ in the direction of $(\psi_{ad,2}(0), \psi_{ad,1}(0))$ as we vary λ . Note that the quantity M_f , defined in (5.75), has a similar interpretation.

In the hyperbolic regime $M_{b,2}$ is to leading order given by (5.85). The positive sign of the quantity $u_b^1 - \gamma w_b^1$ in (5.85) corresponds to the fact that solutions on the right slow manifold move in the direction of positive w . For the geometric interpretation of the integral

$$\int_{-\infty}^{\infty} u'_b(\xi) e^{-\check{c}_0\xi} d\xi, \tag{5.99}$$

in (5.85) we observe that the dynamics in the layers of the fast problem (3.3) are

given by the Nagumo systems

$$\begin{aligned} u_\xi &= v, \\ v_\xi &= \check{c}_0 v - f(u) + w. \end{aligned} \tag{5.100}$$

For $w = w_b^1$ system (5.100) admits the heteroclinic solution $\phi_b(\xi)$ connecting the equilibria p_b^1 and p_b^0 . The space of bounded solutions to the adjoint problem of the linearization of (5.100) at $w = w_b^1$ about $\phi_b(\xi)$ is spanned by $\psi_{ad,1}(\xi)$. One readily observes that (5.99) is a Melnikov integral measuring how the intersection between the stable manifold $\mathcal{W}^s(p_b^0)$ and unstable manifold $\mathcal{W}^u(p_b^1)$ breaks at $\phi_b(0)$ in the direction of $\psi_{ad,1}(0)$ as we vary w in (5.100), i.e. as we move through the fast fibers in the layer problem (3.3).

In the nonhyperbolic regime $M_{b,2}$ is estimated by (5.95). As can be observed from the proof of Proposition 4.5.15, the sign of $M_{b,2}$ is dominated by the inner product

$$\left\langle \begin{pmatrix} v'_b(-L_\varepsilon) \\ -u'_b(-L_\varepsilon) \end{pmatrix}, \begin{pmatrix} u'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \\ v'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle$$

of the adjoint of the singular back solution and the derivative of the pulse solution near the fold point. This inner product determines the orientation of the pulse solution as it passes over the fold before jumping off in the strong unstable direction along the singular back solution. In essence, upon passing up and over the fold, the solution jumps off along a strong unstable fiber to the left. In the fold coordinates, the sign of this inner product amounts to the sign of the derivative $x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)$ of the x -coordinate of the pulse solution in the local coordinates around the fold (3.13). The sign of this derivative is determined by the direction of the Riccati flow in the blow up charts near the fold; see system (3.20).

4.5.6 The region R_2

The goal of the section is to prove that the region $R_2(\delta, M)$ contains no eigenvalues of (5.6) for any $M > 0$ and each $\delta > 0$ sufficiently small. As described in §4.5.2 our approach is to show that problem (5.6) admits exponential dichotomies on each of the intervals I_f, I_r, I_b and I_ℓ , which together form a partition of the whole real line $\mathbb{R}$. The exponential dichotomies on I_r and I_ℓ are yet established in Proposition 4.5.5. The exponential dichotomies on I_f and I_b are generated from exponential dichotomies of a reduced eigenvalue problem via roughness results. Our plan is to compare the projections of the aforementioned exponential dichotomies at the endpoints of the intervals. The obtained estimates yield that any exponentially localized solution to (5.6) must be trivial for $\lambda \in R_2$.

A reduced eigenvalue problem

We establish for ξ in I_f or I_b a reduced eigenvalue problem by setting ε to 0 in system (5.6), while approximating $\phi_{a,\varepsilon}(\xi)$ with (a translate of) the front $\phi_f(\xi)$ or the back $\phi_b(\xi)$, respectively. However, we do keep the λ -dependence in contrast to the reduction done in the region R_1 . Thus, the reduced eigenvalue problem reads

$$\begin{aligned} \psi_\xi &= A_j(\xi, \lambda)\psi, \\ A_j(\xi, \lambda) &= A_j(\xi, \lambda; a) := \begin{pmatrix} -\eta & 1 & 0 \\ \lambda - f'(u_j(\xi)) & \check{c}_0 - \eta & 1 \\ 0 & 0 & -\frac{\lambda}{\check{c}_0} - \eta \end{pmatrix}, \quad j = f, b, \end{aligned} \quad (5.101)$$

where $u_j(\xi)$ denotes the u -component of $\phi_j(\xi)$, λ is in R_2 and a is in $[0, \frac{1}{2} - \kappa]$. By its triangular structure, system (5.101) leaves the subspace $\mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$ invariant.

The dynamics of (5.101) on that space is given by

$$\begin{aligned} \varphi_\xi &= C_j(\xi, \lambda)\varphi, \\ C_j(\xi, \lambda) &= C_j(\xi, \lambda; a) := \begin{pmatrix} -\eta & 1 \\ \lambda - f'(u_j(\xi)) & \check{c}_0 - \eta \end{pmatrix}, \quad j = \text{f, b}. \end{aligned} \quad (5.102)$$

We remark that problem (5.102) corresponds to the weighted eigenvalue problem of the Nagumo systems $u_t = u_{xx} + f(u)$ and $u_t = u_{xx} + f(u) - w_b^1$ about the traveling-wave solutions $u_f(x + \check{c}_0 t)$ and $u_b(x + \check{c}_0 t)$, respectively.

We show that systems (5.101) and (5.102) admit exponential dichotomies on both half-lines. The translated derivative $e^{-\eta\xi}\phi'_j(\xi)$ is an exponentially localized solution to (5.102) at $\lambda = 0$, which admits no zeros. Therefore, by Sturm-Liouville theory, $\lambda = 0$ is the eigenvalue of largest real part of (5.102). So, problems (5.102) admit no exponentially localized solutions for $\lambda \in R_2(\delta, M)$ by taking $\delta > 0$ sufficiently small. This fact allows us to paste the exponential dichotomies on both half-lines of systems (5.102) and (5.101) to a single exponential dichotomy on $\mathbb{R}$. This is the content of the following result.

Proposition 4.5.17. *Let $\kappa, M > 0$. For each $\delta > 0$ sufficiently small, $a \in [0, \frac{1}{2} - \kappa]$ and $\lambda \in R_2(\delta, M)$ system (5.101) admit exponential dichotomies on $\mathbb{R}$ with λ - and a -independent constants $C, \frac{\mu}{2} > 0$, where $\mu > 0$ is as in Lemma 4.5.3.*

Proof. By Lemma 4.5.3, provided $\delta > 0$ is sufficiently small, the asymptotic matrices $C_{j,\pm\infty}(\lambda) = C_{j,\pm\infty}(\lambda; a) := \lim_{\xi \rightarrow \pm\infty} C_j(\xi, \lambda)$ of (5.102) have for $a \in [0, 1/2 - \kappa]$ and $\lambda \in R_2(\delta, M)$ a uniform spectral gap larger than $\mu > 0$. Hence, it follows from [47, Lemmata 1.1 and 1.2] that system (5.102) admits for $(\lambda, a) \in R_2 \times [0, 1/2 - \kappa]$ exponential dichotomies on both half-lines with constants $C, \mu > 0$ and projections $\Pi_{j,\pm}^{u,s}(\xi, \lambda) = \Pi_{j,\pm}^{u,s}(\xi, \lambda; a)$, $j = \text{f, b}$. We emphasize that the constant $C > 0$ is inde-

pendent of λ and a , because $R_2 \times [0, 1/2 - \kappa]$ is compact.

By Sturm-Liouville theory (see e.g. [35, Theorem 2.3.3]) system (5.102) has precisely one eigenvalue $\lambda = 0$ on $\text{Re}(\lambda) \geq -\delta$ (taking $\delta > 0$ smaller if necessary). Therefore, system (5.102) admits no bounded solutions for $\lambda \in R_2$. Hence, we can paste the exponential dichotomies as in [11, p. 16-19] by defining $\Pi_j^s(0, \lambda)$ to be the projection onto $R(\Pi_{j,+}^s(0, \lambda))$ along $R(\Pi_{j,-}^u(0; \lambda))$. Thus, system (5.102) admits for $(\lambda, a) \in R_2 \times [0, 1/2 - \kappa]$ an exponential dichotomy on $\mathbb{R}$ with λ - and a -independent constants $C, \mu > 0$ and projections $\Pi_j^{u,s}(\xi, \lambda) = \Pi_j^{u,s}(\xi, \lambda; a), j = f, b$.

By the triangular structure of system (5.101) the exponential dichotomy on $\mathbb{R}$ of the subsystem (5.102) can be transferred to the full system (5.101) using a variation of constants formula; see also the proof of Corollary 4.5.7. The exponential dichotomy on $\mathbb{R}$ of system (5.101) has constants $C, \min\{\mu, \eta - \frac{\delta}{\varepsilon_0}\} > 0$, where $C > 0$ is independent of a and λ . The result follows by taking $\delta > 0$ sufficiently small using that $\mu \leq \eta$ by Lemma 4.5.3. $\square$

Absence of point spectrum in R_2

With the aid of the following lemma we show that the region R_2 contains no eigenvalues of (5.6).

Lemma 4.5.18 ([29, Lemma 6.10]). *Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ with $a < b$ and $A \in C([a, b], \text{Mat}_{n \times n}(\mathbb{C}))$. Suppose the equation*

$$\varphi_x = A(x)\varphi, \tag{5.103}$$

has an exponential dichotomy on $[a, b]$ with constants $C, m > 0$ and projections

$P_1^{u,s}(x)$. Denote by $T(x, y)$ the evolution of (5.103). Let P_2 be a projection such that $\|P_1^s(b) - P_2\| \leq \delta_0$ for some $\delta_0 > 0$ and let $v \in \mathbb{C}^n$ a vector such that $\|P_1^s(a)v\| \leq k\|P_1^u(a)v\|$ for some $k \geq 0$. If we have $\delta_0(1 + kC^2e^{-2m(b-a)}) < 1$, then it holds

$$\|P_2T(b, a)v\| \leq \frac{\delta_0 + kC^2e^{-2m(b-a)}(1 + \delta_0)}{1 - \delta_0(1 + kC^2e^{-2m(b-a)})} \|(1 - P_2)T(b, a)v\|.$$

Proposition 4.5.19. *Let $M > 0$ be as in Proposition 4.5.2. There exists $\delta, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ system (5.6) admits no nontrivial exponentially localized solution for $\lambda \in R_2(\delta, M)$.*

Proof. We start by establishing exponential dichotomies of system (5.6) on the intervals $I_f = (-\infty, L_\varepsilon]$ and $I_b = [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon} + L_\varepsilon]$. Let $\lambda \in R_2(\delta, M)$. We regard the eigenvalue problem (5.6) as an ε -perturbation of system (5.101). Indeed, by Theorem 4.5 (i)-(ii), for each sufficiently small $a_0 > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ we estimate the difference between the coefficient matrices of both systems along the front and the back by

$$\begin{aligned} \|A(\xi, \lambda) - A_f(\xi, \lambda)\| &\leq C\varepsilon|\log \varepsilon|, \quad \xi \in (-\infty, L_\varepsilon], \\ \|A(Z_{a,\varepsilon} + \xi, \lambda) - A_b(\xi, \lambda)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|, \quad \xi \in [-L_\varepsilon, L_\varepsilon], \end{aligned} \tag{5.104}$$

where $\rho(a) = \frac{2}{3}$ for $a < a_0$ and $\rho(a) = 1$ for $a \geq a_0$ and C is independent of λ, a and ε . By Proposition 4.5.17 system (5.101) has an exponential dichotomy on $\mathbb{R}$ with λ - and a -independent constants $C, \frac{\mu}{2} > 0$ and projections $Q_j^{u,s}(\xi, \lambda) = Q_j^{u,s}(\xi, \lambda; a)$ for $j = f, b$. Denote by $P_j^{u,s}(\lambda) = P_j^{u,s}(\lambda; a)$ the spectral projection onto the (un)stable eigenspace of the asymptotic matrices $A_{j,\pm\infty}(\lambda) = A_{j,\pm\infty}(\lambda; a)$ of system (5.101). As in the proof of Proposition 4.5.9 we obtain the estimate

$$\|Q_j^{u,s}(\pm\xi, \lambda) - P_{j,\pm}^{u,s}(\lambda)\| \leq C \left(e^{-\frac{1}{2}\sqrt{2}\xi} + e^{-\frac{\mu}{2}\xi} \right), \quad j = f, b, \tag{5.105}$$

for $\xi \geq 0$. By estimate (5.104) roughness [10, Theorem 2] yields exponential dichotomies on $I_f = (-\infty, L_\varepsilon]$ and $I_b = [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon} + L_\varepsilon]$ for system (5.6) with λ - and a -independent constants $C, \frac{\mu}{2} > 0$ and projections $\mathcal{Q}_j^{u,s}(\xi, \lambda) = \mathcal{Q}_j^{u,s}(\xi, \lambda; a, \varepsilon)$, which satisfy

$$\begin{aligned} \|\mathcal{Q}_f^{u,s}(\xi, \lambda) - \mathcal{Q}_f^{u,s}(\xi, \lambda)\| &\leq C\varepsilon|\log \varepsilon|, \\ \|\mathcal{Q}_b^{u,s}(Z_{a,\varepsilon} + \xi, \lambda) - \mathcal{Q}_b^{u,s}(\xi, \lambda)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|, \end{aligned} \quad (5.106)$$

for $|\xi| \leq L_\varepsilon$.

On the other hand, system (5.6) admits by Proposition 4.5.5 exponential dichotomies on $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$ and on $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$ with constants $C, \mu > 0$ and projections $\mathcal{Q}_{r,\ell}^{u,s}(\xi, \lambda) = \mathcal{Q}_{r,\ell}^{u,s}(\xi, \lambda; a, \varepsilon)$. The projections satisfy at the endpoints

$$\begin{aligned} \|[\mathcal{Q}_r^s - \mathcal{P}](L_\varepsilon, \lambda)\| &\leq C\varepsilon|\log \varepsilon|, \\ \|[\mathcal{Q}_r^s - \mathcal{P}](Z_{a,\varepsilon} - L_\varepsilon, \lambda)\|, \|[\mathcal{Q}_\ell^s - \mathcal{P}](Z_{a,\varepsilon} + L_\varepsilon, \lambda)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|, \end{aligned} \quad (5.107)$$

where $\mathcal{P}(\xi, \lambda) = \mathcal{P}(\xi, \lambda; a, \varepsilon)$ denote the spectral projections onto the stable eigenspace of $A(\xi, \lambda)$.

Having established exponential dichotomies for (5.6) on the intervals I_f, I_r, I_b and I_ℓ , our next step is to compare the associated projections at the endpoints of the intervals. Recall that $A_j(\xi, \lambda)$ converges at an exponential rate $\frac{1}{2}\sqrt{2}$ to the asymptotic matrix $A_{j,\pm\infty}(\lambda)$ as $\xi \rightarrow \pm\infty$ for $j = f, b$. Combining this with (5.104) and $\nu \geq 2\sqrt{2}$ we estimate

$$\begin{aligned} \|A(L_\varepsilon, \lambda) - A_{f,\infty}(\lambda)\| &\leq C\varepsilon|\log \varepsilon|, \\ \|A(Z_{a,\varepsilon} \pm L_\varepsilon, \lambda) - A_{b,\pm\infty}(\lambda)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|. \end{aligned}$$

By continuity the same bound holds for the spectral projections associated with these matrices. Combining this fact with $\nu \geq \max\{2\sqrt{2}, 2/\mu\}$, (5.105), (5.106) and (5.107) we obtain

$$\begin{aligned} \|[\mathcal{Q}_r^{u,s} - \mathcal{Q}_f^{u,s}](L_\varepsilon, \lambda)\| &\leq C\varepsilon|\log \varepsilon|, \\ \|[\mathcal{Q}_\ell^{u,s} - \mathcal{Q}_b^{u,s}](Z_{a,\varepsilon} + L_\varepsilon, \lambda)\|, \|[\mathcal{Q}_r^{u,s} - \mathcal{Q}_b^{u,s}](Z_{a,\varepsilon} - L_\varepsilon, \lambda)\| &\leq C\varepsilon^{\rho(a)}|\log \varepsilon|. \end{aligned} \quad (5.108)$$

The last step is an application of Lemma 4.5.18. Let $\psi(\xi)$ be an exponentially localized solution to (5.6) at some $\lambda \in R_2$. This implies $\mathcal{Q}_f^s(0, \lambda)\psi(0) = 0$. An application of Lemma 4.5.18 yields

$$\|\mathcal{Q}_r^s(L_\varepsilon, \lambda)\psi(L_\varepsilon)\| \leq C\varepsilon|\log \varepsilon|\|\mathcal{Q}_r^u(L_\varepsilon, \lambda)\psi(L_\varepsilon)\|, \quad (5.109)$$

using (5.108) and $\nu \geq 2/\mu$. We proceed in a similar fashion by applying Lemma 4.5.18 to the inequality (5.109) and using (5.108) to obtain a similar inequality at the endpoint $Z_{a,\varepsilon} - L_\varepsilon$. Applying the Lemma once again, we eventually obtain

$$\|\mathcal{Q}_\ell^s(Z_{a,\varepsilon} + L_\varepsilon, \lambda)\psi(Z_{a,\varepsilon} + L_\varepsilon)\| \leq C\varepsilon^{\rho(a)}|\log \varepsilon|\|\mathcal{Q}_\ell^u(Z_{a,\varepsilon} + L_\varepsilon, \lambda)\psi(Z_{a,\varepsilon} + L_\varepsilon)\| = 0,$$

where the latter equality is due to the fact that $\psi(\xi)$ is exponentially localized. Thus, ψ is the trivial solution to (5.6). $\square$

4.6 Proofs of main stability results

We studied the essential spectrum in §4.4 and the point spectrum in §4.5 of the linearization $\mathcal{L}_{a,\varepsilon}$. In this section we complete the proofs of the main stability results:

Theorem 4.2 and Theorem 4.4.

Proof of Theorem 4.2. In the regime $\varepsilon < Ka^2$, the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the half-plane $\operatorname{Re}(\lambda) \leq -\min\{\varepsilon\gamma, a\} = -\varepsilon\gamma$ by Theorem 4.4.1. Consider the regions R_1, R_2 and R_3 defined in §4.5.2. By Propositions 4.5.2, 4.5.4 and 4.5.19 there is no point spectrum of $\mathcal{L}_{a,\varepsilon}$ in the regions R_2 and R_3 to the right hand side of the essential spectrum. By Proposition 4.5.4, Theorem 4.9 and Proposition 4.5.13 the point spectrum in R_1 to the right hand side of the essential spectrum consists of the simple translational eigenvalue $\lambda_0 = 0$ and at most one other real eigenvalue λ_1 approximated by $-M_{b,2}M_{b,1}^{-1}$, where $M_{b,1} > 0$ is independent of ε and bounded by an a -independent constant. Subsequently, we use Propositions 4.5.14 and 4.5.15 to estimate $M_{b,2}$. We conclude that there exists a constant $b_0 > 0$ such that $\lambda_1 < -\varepsilon b_0$. $\square$

Proof of Theorem 4.4. It follows by Proposition 4.5.14 that the potential eigenvalue $\lambda_1 < 0$ of $\mathcal{L}_{a,\varepsilon}$ is approximated (a -uniformly) by $\lambda_1 = -M_1\varepsilon + \mathcal{O}(|\varepsilon \log \varepsilon|^2)$ in the hyperbolic regime, where M_1 is given by

$$\begin{aligned} M_1 = M_1(a) &:= \frac{(\gamma w_b^1 - u_b^1) \int_{-\infty}^{\infty} u_b'(\xi) e^{-\check{c}_0 \xi} d\xi}{\check{c}_0 \int_{-\infty}^{\infty} (u_b'(\xi))^2 e^{-\check{c}_0 \xi} d\xi} \\ &= \frac{18(a+1) - \gamma(4a^3 - 6a^2 - 6a + 4)}{9a(1-a)(1-2a)} \\ &> 0, \end{aligned} \tag{6.1}$$

where we used the explicit expressions for the front and the back given in (3.6) and substituted $u_b^1 = \frac{2}{3}(1+a)$, $w_b^1 = f(u_b^1)$ and $\check{c}_0 = \sqrt{2}(\frac{1}{2} - a)$.

By Proposition 4.4.1 the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ intersects the real axis only at points $\lambda \leq -\varepsilon(\gamma + a^{-1})$ in the hyperbolic regime. So, if $M_1 < \gamma + a^{-1}$ is satisfied, then λ_1 lies to the right hand side of the essential spectrum. In that case, λ_1 is

by Proposition 4.5.4 contained in the point spectrum of $\mathcal{L}_{a,\varepsilon}$. This proves the first assertion.

By Theorem 4.9 and Proposition 4.5.15, there exists $K_0, k_0 > 1$, independent of a and ε , such that, if $\varepsilon > K_0 a^3$, then

$$\lambda_1 = -\frac{M_{b,2}}{M_{b,1}} + \mathcal{O}\left(|\varepsilon^{2/3} \log \varepsilon|^2\right),$$

satisfies $1/k_0 \varepsilon^{2/3} < \lambda_1 < k_0 \varepsilon^{2/3}$. By Theorem 4.4.1 the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ intersects the real axis only at points $\lambda \leq -\min\{\varepsilon(\gamma + a^{-1}), \frac{1}{2}a + \frac{1}{2}\varepsilon\gamma\}$. Thus, in the regime $K_0 a^3 < \varepsilon < K a^2$ the essential spectrum intersects the real axis at points $\lambda < -K_0^{1/3} \varepsilon^{2/3}$. Taking $K_0 > 1$ larger if necessary, it follows that λ_1 lies to the right hand side of the essential spectrum and λ_1 is by Proposition 4.5.4 an eigenvalue of $\mathcal{L}_{a,\varepsilon}$.

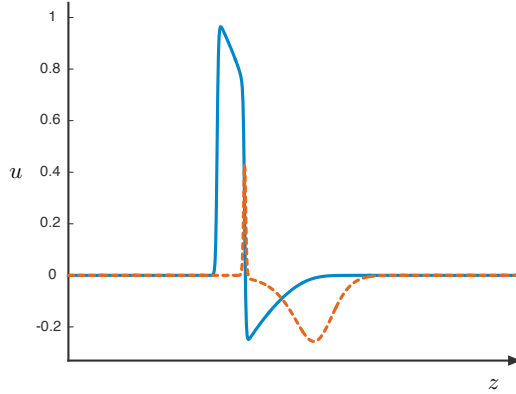
With the aid of (3.6) we calculate

$$M_{b,1} = \int_{-\infty}^{\infty} (u'_b(\xi))^2 e^{-\tilde{c}_0 \xi} d\xi = \frac{1}{3\sqrt{2}} + \mathcal{O}(a),$$

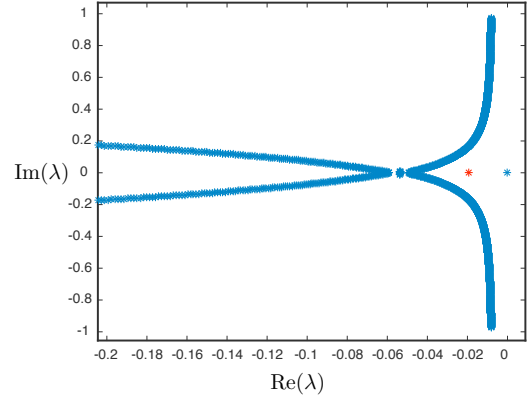
taking the initial translation $\xi_{b,0}$ of $u_b(\xi)$ equal to 0; see Remark 4.3.1. Moreover, if $K_0 a^3 < \varepsilon^{1+\alpha}$ for some $\alpha > 0$, then we compute with the aid of Proposition 4.5.15

$$M_{b,2} = -\frac{(18 - 4\gamma)^{2/3}}{9\sqrt{2}} \Theta^{-1}(0) \varepsilon^{2/3} + \mathcal{O}\left(\varepsilon^{(2+\alpha)/3}, \varepsilon |\log \varepsilon|\right),$$

uniformly in a and α , where Θ is defined in (3.15). With these leading order computations of $M_{b,1}$ and $M_{b,2}$ the approximation (2.4) of λ_1 follows in the regime $K_0 a^3 < \varepsilon^{1+\alpha}$, $\varepsilon < K a^2$. $\square$



(a) Shown is the u -component of the pulse solution (blue) obtained numerically for $(a, \varepsilon, \gamma) = (0.0997, 0.0021, 3.5)$. Also plotted is the u -component of the eigenfunction (dashed red) corresponding to the eigenvalue $\lambda_1 = -0.0194$.



(b) Shown is the spectrum of the operator $\mathcal{L}_{a,\varepsilon}$ associated with the pulse in Figure 4.6a. Note that the eigenvalue $\lambda_1 = -0.0194$ (shown in red) lies to the right of the essential spectrum.

Figure 4.6: Shown is a sample pulse solution and its spectrum. Also shown is the eigenfunction corresponding to the critical eigenvalue $\lambda_1 = -0.0194$.

4.7 Numerics

In this section, we discuss numerical results pertaining to Theorem 4.4; in particular, we focus on the location of the potential second eigenvalue λ_1 of $\mathcal{L}_{a,\varepsilon}$ with respect to the essential spectrum and its asymptotic behavior as $\varepsilon \rightarrow 0$.

4.7.1 Position of λ_1 with respect to the essential spectrum

In the nonhyperbolic regime $K_0 a^3 < \varepsilon$ it is always the case that λ_1 lies to the right of the essential spectrum and is in fact an eigenvalue of $\mathcal{L}_{a,\varepsilon}$ by Theorem 4.4 (ii). In the hyperbolic regime there is a condition in Theorem 4.4 (i) which ensures that λ_1 lies to the right of the essential spectrum and is an eigenvalue of $\mathcal{L}_{a,\varepsilon}$. We comment on this condition. Note that for parameter values $(a, \gamma) = (0.0997, 3.5)$ the condition

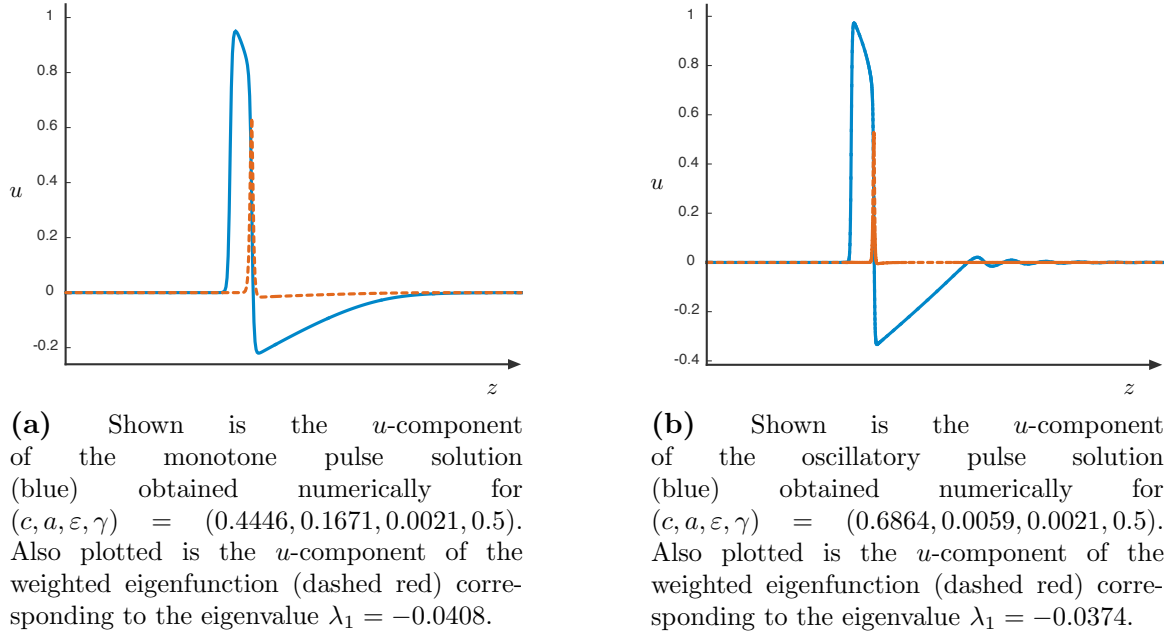


Figure 4.7: Sample monotone and oscillatory pulses and the weighed eigenfunctions corresponding to the critical eigenvalue λ_1 .

is satisfied

$$M_1 = 12.498 < 13.530 = \gamma + a^{-1}.$$

Here M_1 is calculated with the aid of formula (6.1). In Matlab, we solve for stationary solutions of (2.2) numerically for the parameter values $(a, \varepsilon, \gamma) = (0.0997, 0.0021, 3.5)$ where we obtain the monotone pulse solution shown in Figure 4.6a; we also solve the eigenvalue problem (2.3) and obtain a solution with eigenvalue $\lambda_1 = -0.0194$; the corresponding eigenfunction of $\mathcal{L}_{a,\varepsilon}$ is plotted along with the pulse in Figure 4.6a. The spectrum associated with the pulse is plotted in Figure 4.6b. Note that the eigenvalue $\lambda_1 = -0.0194$ appears indeed to the right of the essential spectrum.

4.7.2 Asymptotics of λ_1 as $\varepsilon \rightarrow 0$

We now turn to the asymptotics of the eigenvalue λ_1 of $\mathcal{L}_{a,\varepsilon}$ as $\varepsilon \rightarrow 0$. To study this, we continue traveling-pulse solutions to (2.1) numerically along different curves in the parameters c, a and ε in order to illustrate the behavior of the eigenvalue λ_1 in the hyperbolic and nonhyperbolic regimes treated in Theorem 4.4. In order to ensure that we obtain the correct value for λ_1 , we use a small exponential weight $\eta > 0$ to shift the essential spectrum away from the imaginary axis, i.e. we look for solutions to the eigenvalue problem (2.3) bounded in the weighted norm $\|\psi\|_{-\eta} = \sup_{\xi \in \mathbb{R}} \|\psi(\xi)e^{-\eta\xi}\|$. This amounts to replacing (2.3) with the shifted version

$$\psi_\xi = (A_0(\xi, \lambda) - \eta) \psi. \quad (7.1)$$

This procedure is justified and explained in detail in §4.5.2. In short, if $[0, 1/2 - \kappa]$ is the allowed range for a in the existence result Theorem 4.1, then for the choice $\eta = \frac{1}{2}\sqrt{2}\kappa$, λ_1 lies to the right of the shifted essential spectrum and is always an eigenvalue of the shifted problem (7.1). In the following, we fix $\eta = 0.1$. Thus, we restrict to a -values in $[0, 0.3586]$.

Hyperbolic regime

We first consider the hyperbolic regime: according to Theorem 4.4 (i), for sufficiently small $\varepsilon > 0$, the eigenvalue λ_1 of (7.1) is approximated by

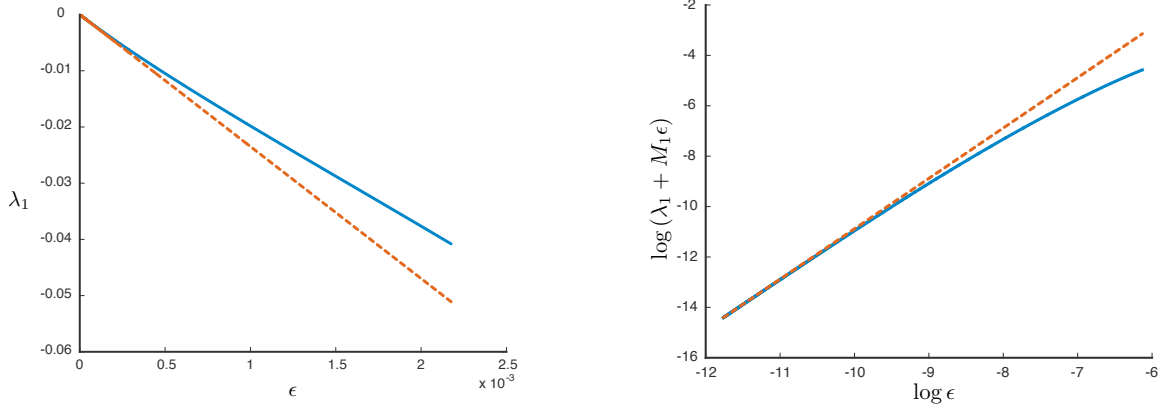
$$\lambda_1 = -M_1\varepsilon + \mathcal{O}(|\varepsilon \log \varepsilon|^2), \quad (7.2)$$

where $M_1 > 0$ is given by (6.1). If $(u(x - ct), w(x - ct))$ is a traveling-wave solution to (2.1) with wave speed c , then $(u(\xi), u'(\xi), w(\xi))$ satisfies the ODE

$$\begin{aligned} u_\xi &= v, \\ v_\xi &= cv - f(u) + w, \\ w_\xi &= \frac{\varepsilon}{c}(u - \gamma w). \end{aligned} \tag{7.3}$$

Using Matlab, we solve (7.3) numerically for the parameter values $(c, a, \varepsilon, \gamma) = (0.4446, 0.1671, 0.0021, 0.5)$ where we obtain the monotone pulse solution shown in Figure 4.7a. In addition, we solve the eigenvalue problem (7.1) and obtain a solution with eigenvalue $\lambda_1 = -0.0408$; the corresponding weighted eigenfunction of (7.1) is plotted along with the pulse in Figure 4.7a. To see whether (7.2) gives a good prediction for the location of the eigenvalue λ_1 in the hyperbolic regime, we fix the parameter a and using the continuation software package AUTO, we append the weighted eigenvalue problem (7.1) to the existence problem (7.3) and continue in the parameters (c, ε) letting $\varepsilon \rightarrow 0$ to determine the asymptotics of the eigenvalue λ_1 . We regard c here as a free parameter, because the value of $c = \check{c}(a, \varepsilon)$ for which (2.1) admits a traveling-pulse solution depends on a and ε by Theorem 4.1. Thus, instead of prescribing $c = \check{c}(a, \varepsilon)$ we require AUTO to continue along a 1-dimensional curve in the (c, ε) -plane of homoclinic solutions to 0 of (7.3).

The results of the continuation process are plotted in Figure 4.8. In Figure 4.8a, the continuation of the eigenvalue λ_1 is plotted against ε along with the first order approximation $\lambda_1 \approx -M_1\varepsilon$ for the eigenvalue λ_1 from Theorem 4.4 (i). There is good agreement as $\varepsilon \rightarrow 0$. In addition, in Figure 4.8b, a log-log plot of the difference of the two curves in Figure 4.8a is plotted along with a straight line of slope 2. Asymptotically, there is good agreement between these two curves, which suggests that the difference between the numerically computed values for λ_1 and the approximation



(a) Plotted is the curve (blue) obtained for the continuation of the eigenvalue λ_1 as $\epsilon \rightarrow 0$ in the monotone pulse case. Here we have fixed $a = 0.1671$ and the wave speed c varies along the continuation. For comparison, we also plot the first order approximation (dashed red) $\lambda \approx -M_1\epsilon$ for the eigenvalue λ_1 from Theorem 4.4 (i).

(b) Shown is a log-log plot of the differences (blue) of the two curves in Figure 4.8a, that is, we plot $\log(\lambda_1 + M_1\epsilon)$ vs. $\log \epsilon$ where the values for λ_1 were obtained using the numerical continuation. Also plotted (dashed red) is a straight line of slope 2.

Figure 4.8: Asymptotics of the critical eigenvalue λ_1 in the hyperbolic regime.

$\lambda_1 \approx -M_1\epsilon$ is indeed higher order.

Nonhyperbolic regime

We next consider the nonhyperbolic regime. Take $K^* > 1/4$. By Theorem 4.1 and Remark 4.3.3, provided $a, \epsilon > 0$ are sufficiently small with $K^*a^2 < \epsilon$, the tail of the pulse solution is oscillatory. Hence for sufficiently small $\epsilon > 0$, in the region of oscillatory pulses, one expects by Theorem 4.4 (ii) that the eigenvalue λ_1 of (7.1) becomes asymptotically $\mathcal{O}(\epsilon^{2/3})$. Using Matlab, we solve (7.3) numerically for the parameter values $(c, a, \epsilon, \gamma) = (0.6864, 0.0059, 0.0021, 0.5)$ and obtain the oscillatory pulse solution shown in Figure 4.7b. We also solve the eigenvalue problem (7.1) and obtain a solution with eigenvalue $\lambda_1 = -0.0374$ and corresponding weighted eigenfunction which is plotted along with the pulse in Figure 4.7b. To determine the asymptotics of the eigenvalue λ_1 in the oscillatory regime, we now continue this

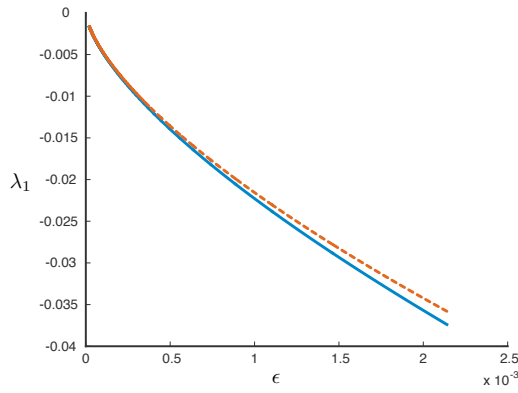
solution letting $\varepsilon \rightarrow 0$ along the curve $\varepsilon = 61.9026a^2$ so that it holds $\varepsilon > K^*a^2$ along this curve. Note that we regard c again as a free parameter for the same reasons as in §4.7.2.

We compare the results of the continuation process with the results of Theorem 4.4. Along the curve $\varepsilon = 61.9026a^2$, for sufficiently small $a, \varepsilon > 0$, by Theorem 4.4 (ii) the eigenvalue is given by

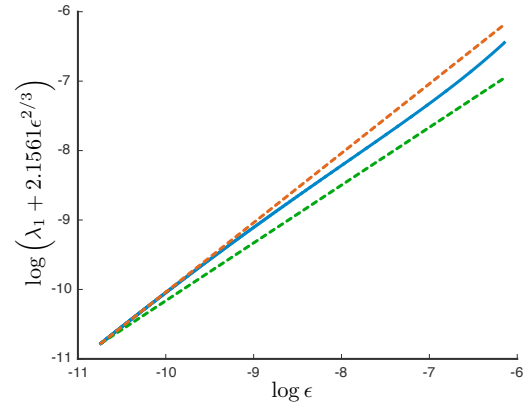
$$\lambda_1 = -\frac{1}{3}(18 - 4\gamma)^{2/3} \zeta_0 \varepsilon^{2/3} + \mathcal{O}(\varepsilon^{5/6}) \approx -2.1561 \varepsilon^{2/3}, \quad (7.4)$$

where we used that $\zeta_0 \approx 1.0187$.

The results of the continuation process are shown in Figure 4.9; in Figure 4.9a, the continuation of the eigenvalue λ_1 is plotted against ε in blue along with the first order approximation (7.4) in red. In Figure 4.9b, a log-log plot of the difference of the two curves in Figure 4.9 is plotted along with straight lines of slope 1 and $5/6$. Asymptotically, the log of the difference lies between these two lines, which suggests that the difference between the numerically computed values for λ_1 and the approximation is indeed higher order.



(a) Plotted is the curve obtained for the continuation of the eigenvalue λ_1 as $\varepsilon \rightarrow 0$ in the oscillatory pulse case. Here we continue along the curve $\varepsilon = 61.9026a^2$, and the wave speed $c = \check{c}(a, \varepsilon)$ varies along the continuation. For comparison, we also plot the first order approximation (dashed red) $\lambda \approx -2.1561\varepsilon^{2/3}$ for the eigenvalue λ_1 from Theorem 4.4 (ii).



(b) Shown is a log-log plot of the differences (blue) of the two curves in Figure 4.9a, that is, we plot $\log(\lambda_1 + 2.1561\varepsilon^{2/3})$ vs. $\log \varepsilon$ where the values for λ_1 were obtained using the numerical continuation. Also plotted are a straight line (dashed red) of slope 1 and a straight line (dashed green) of slope $5/6$.

Figure 4.9: Asymptotics of the critical eigenvalue λ_1 in the nonhyperbolic regime.

CHAPTER FIVE

Unpeeling a homoclinic banana

5.1 Introduction

We return to the FitzHugh-Nagumo traveling wave ODE:

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= \varepsilon(u - \gamma w) ,\end{aligned}\tag{1.1}$$

where $\dot{} = \frac{d}{dt}$, and where the nonlinearity $f(u) = u(u - a)(1 - u)$, $-1/2 < a < 1/2$, $\gamma > 0$, and $0 < \varepsilon \ll 1$. In this chapter, we discuss a phenomenon, which has previously been observed numerically, in which the oscillation in the tails of the pulses constructed in [8, Theorem 1.1] grow into a secondary excursion upon continuation in the parameters (c, a) . This second excursion grows into a second copy of the primary pulse via a mechanism resembling a canard explosion. The goal of this chapter is to construct this transition analytically using geometric singular perturbation theory and blow up techniques as in the construction of the oscillatory pulses in [8, Theorem 1.1]

We begin in §5.2 by describing the transition mechanism in more detail with the aid of numerical computations. In §5.3, we outline the setup and give a statement of the main result. The pulse solutions are constructed in §5.4 save for a few technical results which are proved in §5.5 and §5.6.

5.2 Homoclinic C-curve and single-to-double pulse transition

The following computations were performed using the continuation software AUTO. Throughout, we fix $\gamma = 1/2$.

5.2.1 The homoclinic banana

Starting with a monotone 1-pulse, we begin by fixing $\varepsilon = 0.021$ and continuing in the parameters (c, a) and obtain the homoclinic “C-curve” (Figure 5.2a) which connects the branch of fast monotone pulses with the branch of slow pulses (see the schematic diagram in Figure 5.1). When continuing along the upper branch towards the left corner of the diagram, due to the Belyakov transition occurring at the origin, the tail of the pulse solution changes from monotone to oscillatory. The branch eventually turns around sharply as the pulse undergoes a transition from a single to a double pulse. The continuation then follows the C-curve in reverse, and similar sharp turn occurs when the lower branch appears to terminate in the lower left corner of the diagram, during which the double pulse transitions back to a single pulse. To better visualize this curve of solutions, in Figure 5.2b, the L^2 -norm of the solutions is plotted against the parameter a . This gives the homoclinic “banana” as described in [9]. Figure 5.3 shows an example oscillatory pulse solution and a nearby double pulse solution after the transition along the banana.

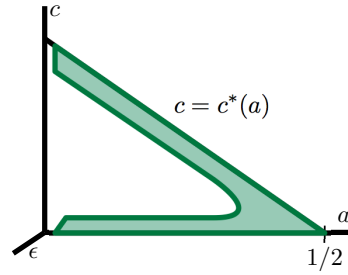
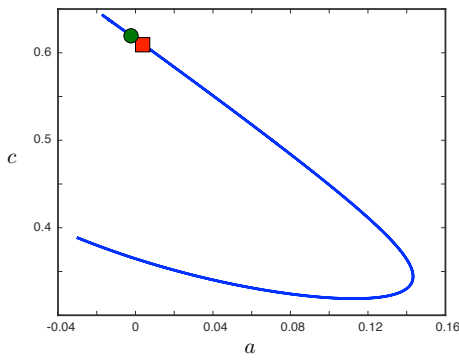
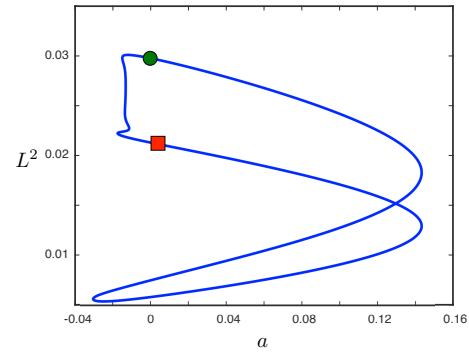


Figure 5.1: Shown is the bifurcation diagram indicating the known regions of existence for pulses in (1.1). Pulses on the upper branch are referred to as “fast” pulses, while those along the lower branch are called “slow” pulses. These two branches coalesce near the point $(c, a, \varepsilon) = (0, 1/2, 0)$.

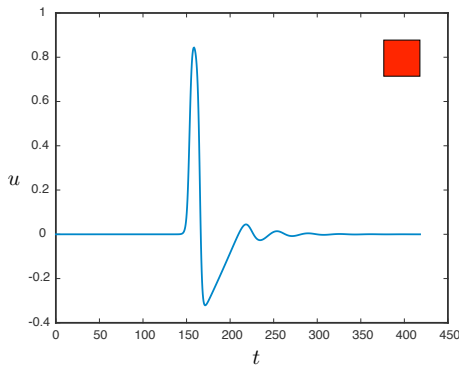


(a) Homoclinic C-curve: c vs. a

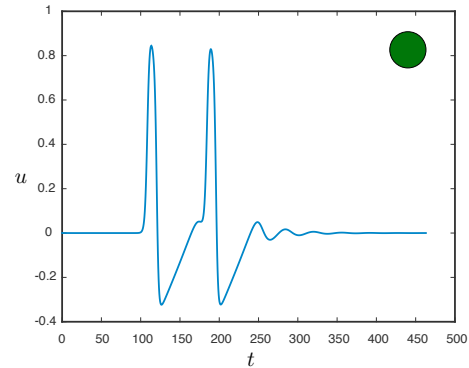


(b) Homoclinic banana : L^2 -norm vs. a

Figure 5.2: Plotted are the homoclinic C-curve and banana obtained by continuing the pulse solution in the parameters (a, c) for $\varepsilon = 0.021$. The red square and green circle refer to the locations of the oscillatory pulse and double pulse of Figure 5.3, respectively.



(a) Shown is a pulse with oscillatory tail for $(c, a, \varepsilon) = (0.005, 0.608, 0.021)$.



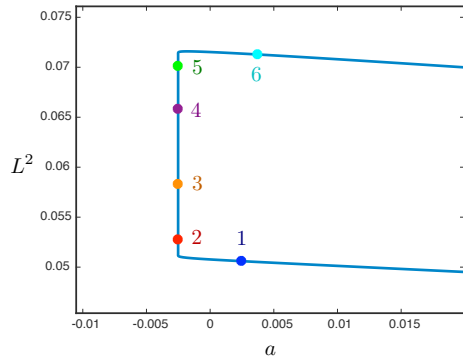
(b) Shown is a double pulse for $(c, a, \varepsilon) = (0.001, 0.612, 0.021)$.

Figure 5.3: Plotted are examples of an oscillatory pulse and a double pulse along the homoclinic C-curve. The colored shapes refer to their location along the homoclinic C-curve and banana of Figure 5.2.

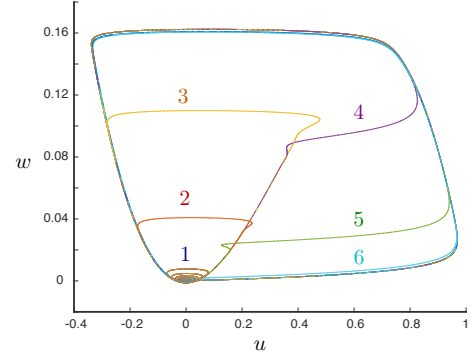
5.2.2 Transition mechanism

Numerical explorations of the FitzHugh–Nagumo system have resulted in possible explanations for the termination of the branch of pulses in the upper left corner of the C-curve and the structure of the homoclinic banana [9, 22, 23]. The major contributing factor to this behavior is the singular Hopf bifurcation occurring at the origin. As the Hopf bifurcation is subcritical in the region in question, the onset of small unstable periodic solutions nearby block the convergence of the homoclinic to the equilibrium. However, the exact nature of the sharp turn in the C-curve, and in particular the relation to the transition between the single and double pulse, is not well understood. Guided by the analysis to construct the pulse in the previous sections and the investigation of the canard point at the origin, we propose a geometric mechanism for the transition from the single to double pulse, and we use the numerical continuation to visualize this transition. Figure 5.4 shows a zoom of the upper left part of the banana for a lower value of ε as well as six different pulses along the curve plotted together.

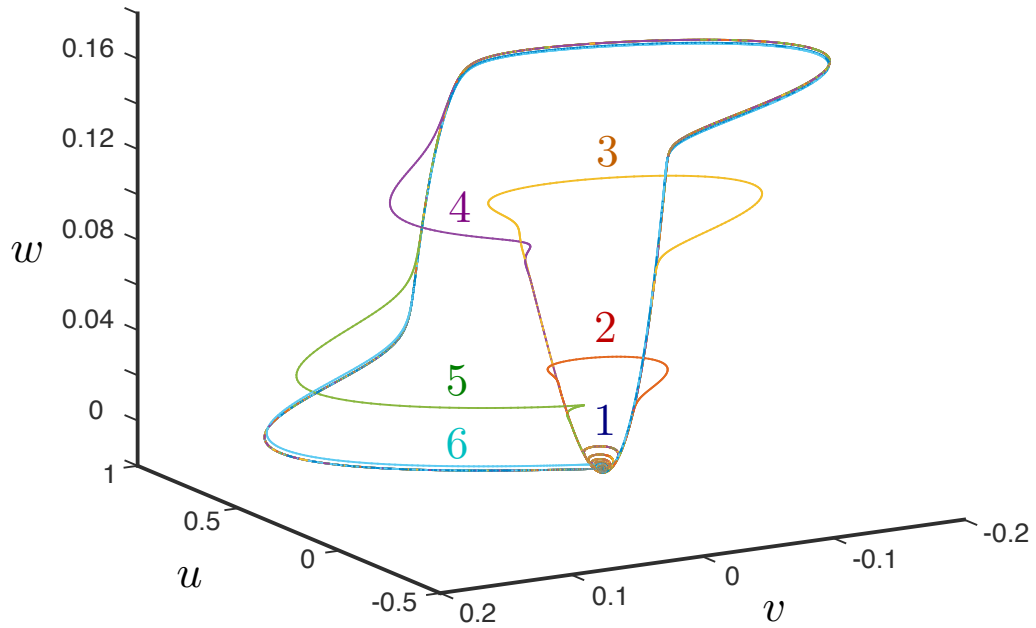
When viewing the progression in Figure 5.4, it becomes clear how the second pulse is added. Starting from the oscillatory pulse, after passing near the equilibrium, the tail follows the completely unstable middle branch of the slow manifold for some amount of time before jumping off and returning to $\mathcal{M}_\varepsilon^\ell(c, a)$ and then converging to the equilibrium. Eventually the pulse follows the entire middle branch up to the fold point before jumping back to $\mathcal{M}_\varepsilon^\ell(c, a)$. In Figure 5.5a, we see this progression fills out a surface when many such pulses are plotted together. As the transition continues, the pulse instead jumps from the middle branch to $\mathcal{M}_\varepsilon^r(c, a)$ which it follows until reaching the fold point, then jumps back to $\mathcal{M}_\varepsilon^\ell(c, a)$, culminating in a double pulse. Figure 5.5b shows a surface filled out by this part of the sequence with many pulses



(a) Zoom of homoclinic banana: L^2 -norm vs. a

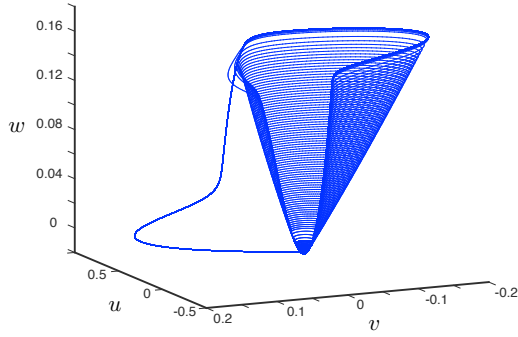


(b) Six pulses along the banana projected onto the uw -plane showing the transition from a single to a double pulse.

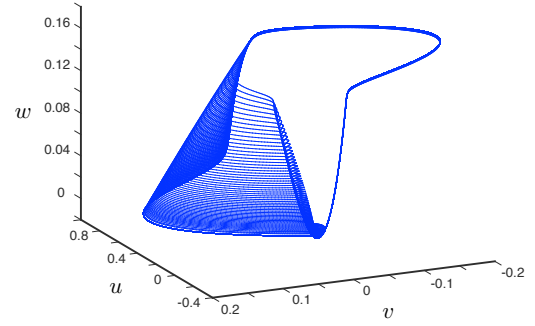


(c) Six pulses along the banana in the full uvw -space showing the transition from a single to a double pulse.

Figure 5.4: Transition from single to double pulse in the top left of the homoclinic banana for $\varepsilon = 0.0036$. The solutions labelled 1, 2, 3 are left pulses, and those labelled 4, 5, 6 are right pulses.



(a) Plotted are left pulses along the transition from single to double pulse.



(b) Plotted are right pulses along the transition from single to double pulse.

Figure 5.5: Transition from single to double pulse in the top left of the homoclinic banana for $\varepsilon = 0.0036$.

plotted together. The entire progression of the tail from small oscillations to a full additional pulse resembles a classical canard explosion (see Figure 5.4b).

In the following, we will construct this entire sequence analytically using the same geometric framework used to construct the pulse with oscillatory tail. In Figure 5.6, we show the two new types of singular pulses from which the transitional pulses will be constructed; these will be defined in more detail in §5.3. As with the proof of [8, Theorem 1.1], up to some technical difficulties, each pulse along the transition can be constructed using classical geometric singular perturbation theory and the exchange lemma. The main challenges involve the flow near the fold points, for which we use blow up analysis, as well as constructing the tails of the pulses. The structure of the middle branch turns out to be very important in this regard. While the entire branch is completely unstable, there is a point along the middle branch (in fact two due to symmetry) in which the flow transitions from node to focus behavior, which is crucial to understanding the nature of the tails. In what follows, this point will be referred to as the Airy point, and we will use another blow up to treat this region.

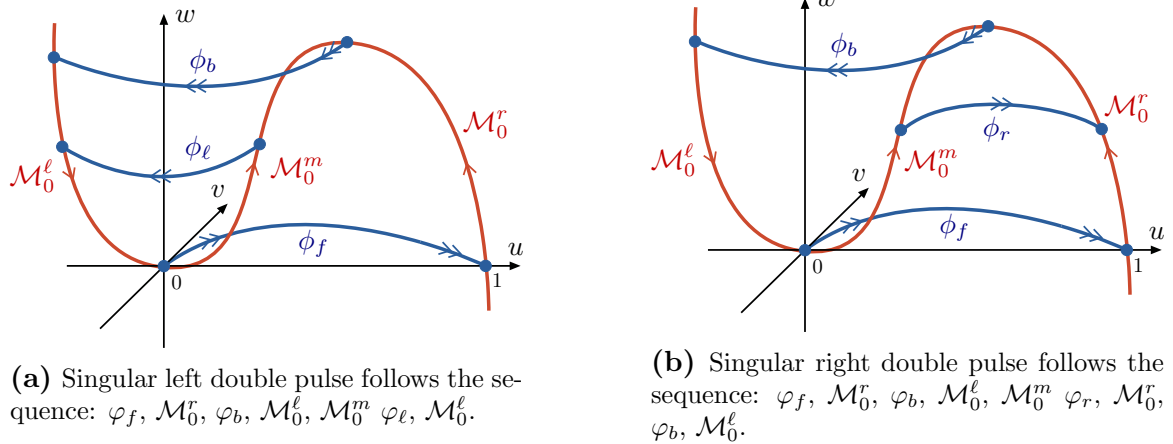


Figure 5.6: Singular $\varepsilon = 0$ double pulses for $(c, a) = (1/\sqrt{2}, 0)$.

5.3 Setup

We start by collecting a few results from [8]. Define the closed intervals $I_a = [-a_0, a_0]$ for sufficiently small $a_0 > 0$ and $I_c = \{c^*(a) : a \in I_a\}$; here $c^*(a) = 1/\sqrt{2}(1 - 2a)$ is the wavespeed for which the Nagumo front exists for this choice of a . Then for sufficiently small ε_0, a_0 , we have the following:

- (i) The origin has a strong unstable manifold $\mathcal{W}_0^u(0; c, a)$ for $c \in I_c$, $a \in I_a$, and $\varepsilon = 0$ which persists for a, c in the same range and $\varepsilon \in [0, \varepsilon_0]$.
- (ii) We consider the critical manifold defined by $\{(u, v, w) : v = 0, w = f(u)\}$. For each $a \in I_a$, we consider the right branch of the critical manifold $\mathcal{M}_0^r(c, a)$ up to a neighborhood of the knee for $\varepsilon = 0$. This manifold persists as a slow manifold $\mathcal{M}_\varepsilon^r(c, a)$ for $\varepsilon \in [0, \varepsilon_0]$. In addition, $\mathcal{M}_0^r(c, a)$ possesses stable and unstable manifolds $\mathcal{W}^s(\mathcal{M}_0^r(c, a))$ and $\mathcal{W}^u(\mathcal{M}_0^r(c, a))$ which also persist for $\varepsilon \in [0, \varepsilon_0]$ as invariant manifolds which we denote by $\mathcal{W}_\varepsilon^{s,r}(c, a)$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$.
- (iii) In addition, we consider the left branch of the critical manifold $\mathcal{M}_0^\ell(c, a)$ up to a neighborhood of the origin for $\varepsilon = 0$. This manifold persists as a slow manifold

$\mathcal{M}_\varepsilon^\ell(c, a)$ for $\varepsilon \in [0, \varepsilon_0]$. In addition, $\mathcal{M}_0^\ell(c, a)$ possesses a stable manifold $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ which also persists for $\varepsilon \in [0, \varepsilon_0]$ as an invariant manifold which we denote by $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. In §5.3.2, we show that there is a way to extend $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in such a manner that it also encompasses a center manifold near the origin which will be useful in the existence proof.

- (iv) Finally, we consider the middle branch of the critical manifold $\mathcal{M}_0^m(c, a)$ away from neighborhoods of the origin and the upper right fold point for $\varepsilon = 0$. This manifold persists as a slow manifold $\mathcal{M}_\varepsilon^m(c, a)$ for $\varepsilon \in [0, \varepsilon_0]$. In addition, $\mathcal{M}_0^m(c, a)$ possesses a three-dimensional unstable manifold $\mathcal{W}^u(\mathcal{M}_0^m(c, a))$ which also persists for $\varepsilon \in [0, \varepsilon_0]$ as an invariant manifold which we denote by $\mathcal{W}_\varepsilon^{u,m}(c, a)$. The stable manifolds $\mathcal{W}^s(\mathcal{M}_0^r(c, a))$ and $\mathcal{W}^s(\mathcal{M}_0^\ell(c, a))$ form part of $\mathcal{W}^u(\mathcal{M}_0^m(c, a))$ for $\varepsilon = 0$ and hence for sufficiently small $\varepsilon > 0$, we have that the foliations $\mathcal{W}_\varepsilon^{s,r}(c, a)$ and $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ are contained in $\mathcal{W}_\varepsilon^{u,m}(c, a)$.

We also have the following proposition, which follows from the analysis in [8, §5].

Proposition 5.3.1. *There exists $\varepsilon_0 > 0$ and $\mu > 0$ such that for each $a \in I_a$ and $\varepsilon \in (0, \varepsilon_0)$, the manifold $\bigcup_{c \in I_c} \mathcal{W}_\varepsilon^u(0; c, a)$ intersects $\bigcup_{c \in I_c} \mathcal{W}_\varepsilon^{s,\ell}(c, a)$ near the upper right fold point transversely in uvw -space with the intersection occurring at $c = \check{c}(a, \varepsilon)$ for a smooth function $\check{c} : I_a \times (0, \varepsilon_0) \rightarrow I_c$ where $\check{c}(a, \varepsilon) = c^*(a) - \mu\varepsilon + \mathcal{O}(\varepsilon(|a| + \varepsilon))$.*

5.3.1 Layer analysis

The construction of transitional pulses involves concatenating pieces of slow manifolds with fast jumps along fronts/backwards at different heights along the slow manifolds.

In this section we outline the structure of the layer problem

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= 0,\end{aligned}\tag{3.1}$$

at $(c, a) = (1/\sqrt{2}, 0)$, where w now acts as a parameter (see Figure 5.7). Hence we study the two-dimensional system

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= cv - u^2(1 - u) + w,\end{aligned}\tag{3.2}$$

for values of $w \in [0, w^\dagger]$, where $(u^\dagger, 0, w^\dagger) = (2/3, 0, 8/27)$ denotes the location of the upper right fold point for $(c, a) = (1/\sqrt{2}, 0)$. For the values $w = 0$, (3.2) has two equilibria at $(u, v) = (0, 0)$ and $(u, v) = (1, 0)$, and when $c = 1/\sqrt{2}$, there is a front φ_f connecting these two equilibria. By symmetry, when $w = w^\dagger$ there is a ‘back’ φ_b connecting the two equilibria at $(u, v) = (2/3, 0)$ and $(u, v) = (-1/3, 0)$.

For values of $w \in (0, w^\dagger)$, (3.2) has three equilibria $p_i = (u_i(w), 0)$, where $u_i(w), i = 1, 2, 3$ are the three solutions of $f(u) = w$ in increasing order. The outer equilibria are saddles, while the middle equilibrium p_2 is completely unstable. To compute the type, we note that the eigenvalues at p_2 are given by

$$\lambda = \frac{c \pm \sqrt{c^2 - 4f'(u_2(w))}}{2}.\tag{3.3}$$

We define w_A to be the lesser of the two solutions of $c^2 - 4f'(u_2(w))$, and we refer to the point $(u_2(w_A), 0, w_A)$ as the Airy point. Hence for $(c, a) = (1/\sqrt{2}, 0)$, the equilibrium p_2 is an unstable node for $w \in (0, w_A) \cup (w^\dagger - w_A, w^\dagger)$, a degenerate node at $w = w_A, w^\dagger - w_A$, and an unstable spiral for $w \in (w_A, w^\dagger - w_A)$.

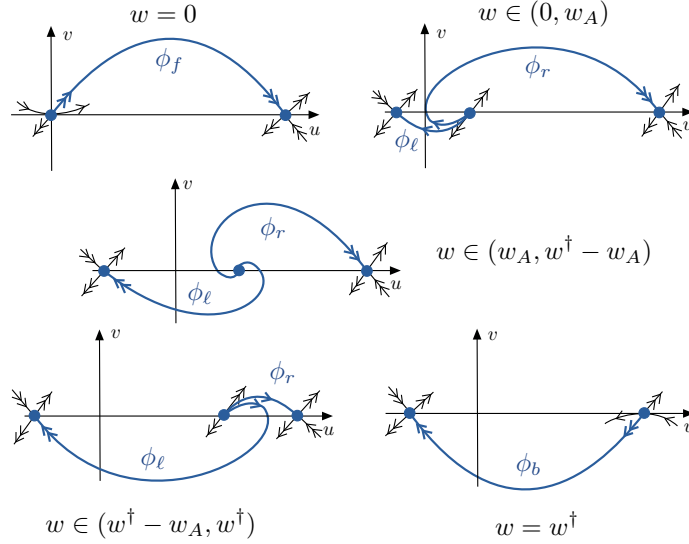


Figure 5.7: Shown is the structure of the fronts $\varphi_\ell, \varphi_r, \varphi_f, \varphi_b$ given by Proposition 5.3.2 for values of $w \in [0, w^\dagger]$.

We have the following proposition regarding heteroclinic connections between the equilibria p_i for values of $w \in [0, w^\dagger]$. The results of Proposition 5.3.2 are shown in Figure 5.7.

Proposition 5.3.2. *Consider the system (3.2) for $(c, a) = (1/\sqrt{2}, 0)$. For each $w \in (0, w^\dagger)$, there exists a front $\varphi_\ell(w)$ connecting the equilibria p_2 and p_1 , and a front $\varphi_r(w)$, connecting the equilibria p_2 and p_3 . Furthermore,*

- (i) *For $w \in (0, w_A)$, the front φ_ℓ leaves p_2 along a weak unstable direction and remains in $\{(u, v) : u_1(w) < u < u_2(w), v < 0\}$. The front φ_r leaves p_2 along φ_ℓ , then crosses into the half space $v > 0$, where it remains until arriving at p_3 .*
- (ii) *When $w = w_A$, the fronts φ_ℓ, φ_r leave p_2 along the line $v = \frac{u - u_2(w)}{2\sqrt{2}}$ in the half space $v < 0$. There exist A_ℓ, A_r and $B_\ell, B_r > 0$ such that φ_ℓ, φ_r satisfy*

$$\begin{aligned} u(t) &= u_2(w) + (A_j + B_j t)e^{\frac{t}{2\sqrt{2}}} + \mathcal{O}(t^2 e^{\frac{t}{\sqrt{2}}}) \\ v(t) &= \frac{1}{2\sqrt{2}}(A_j + B_j t)e^{\frac{t}{2\sqrt{2}}} + B_j e^{\frac{t}{2\sqrt{2}}} + \mathcal{O}(t^2 e^{\frac{t}{\sqrt{2}}}), \end{aligned} \tag{3.4}$$

$j = \ell, r$, asymptotically as $t \rightarrow -\infty$. There exists $\Delta > 0$ such that these solutions can be written as graphs $v = v_j(u)$, $j = \ell, r$, for $u \in [u_2(w) - \Delta, u_2(w)]$ with $v_r(u) > v_\ell(u)$ for all $u \in [u_2(w) - \Delta, u_2(w)]$.

(iii) When $w = w^\dagger - w_A$, the fronts φ_ℓ, φ_r leave p_2 along the line $v = \frac{u - u_2(w)}{2\sqrt{2}}$ in the half space $v > 0$.

(iv) For $w \in (w^\dagger - w_A, w^\dagger)$, the front φ_r leaves p_2 along a weak unstable direction and remains in $\{(u, v) : u_2(w) < u < u_3(w), v > 0\}$. The front φ_ℓ leaves p_2 along φ_r , then crosses into the half space $v < 0$, where it remains until arriving at p_1 .

Proof. We prove (i) and (ii); the remaining two assertions follow from the symmetry of the cubic nonlinearity. The claims regarding the front φ_ℓ follow from analysis of traveling fronts [2, 20, 24].

It remains to show the properties of the front φ_r . We first consider the case of small w . When $w = 0$, the equilibria p_1 and p_2 collide, and p_1 and p_3 are connected by the Nagumo front φ_f . Hence for small $w > 0$ property (i) follows from the fact that φ_f breaks regularly as w increases; this can be shown in a manner similar to the proof of [8, Proposition 5.2]. Hence the result holds for $w \in (0, \Delta_w)$ sufficiently small.

We next examine the linearization of (3.2) at the equilibria p_2, p_3 . At p_i , the linearization of (3.2) is given by

$$J_2 = \begin{pmatrix} 0 & 1 \\ -f'(u_i(w)) & c \end{pmatrix}, \quad (3.5)$$

which has eigenvalues

$$\lambda_i^\pm = \frac{c \pm \sqrt{c^2 - 4f'(u_i(w))}}{2}. \quad (3.6)$$

For all $w \in (0, w_A)$ and all $c \geq 1/\sqrt{2}$, p_2 is an unstable node (which is degenerate in the critical case of $w = w_A, c = 1/\sqrt{2}$) with corresponding eigenvectors

$$e_2^\pm = \begin{pmatrix} 1 \\ \frac{c \pm \sqrt{c^2 - 4f'(u_2(w))}}{2} \end{pmatrix}. \quad (3.7)$$

For $w \in (0, w_A)$, the equilibrium p_2 has a well defined strong unstable eigenspace with nonzero (u, v) -components. Hence the front φ_r leaves the equilibrium along a trajectory tangent to this subspace with u initially either increasing or decreasing. Proving (i) amounts to showing that the former is always the case.

For all $w \in (0, w_A)$ and all $c \geq 1/\sqrt{2}$, p_3 is a saddle with corresponding eigenvectors

$$e_2^\pm = \begin{pmatrix} 1 \\ \frac{c \pm \sqrt{c^2 - 4f'(u_3(w))}}{2} \end{pmatrix}. \quad (3.8)$$

Hence for each $w \in (\Delta_w, w_A)$ and each $c \geq 1/\sqrt{2}$, the equilibrium p_2 has a well defined strong unstable manifold $\mathcal{W}^{uu}(p_2)$, and the equilibrium p_3 has a well defined stable manifold $\mathcal{W}^s(p_3)$. If a front were to exist as an intersection of $\mathcal{W}^{uu}(p_2)$ and $\mathcal{W}^s(p_3)$ lying in the half space $v > 0$ for some $c_w \geq \sqrt{2}$, then by monotonicity of the flow with respect to c , this connection will break upon varying c : for $c < c_w$, we

must have that $\mathcal{W}^{uu}(p_2)$ lies below $\mathcal{W}^s(p_3)$ and vice-versa for $c > c_w$, and hence this value of c_w for which a connection exists is unique among $c \geq 1/\sqrt{2}$. We show that for each $w \in (0, w_A)$, such a value $c_w > 1/\sqrt{2}$ exists by explicitly constructing the associated front.

Using the ansatz $v = b(u - u_2(w))(u - u_3(w))$, we deduce that there is a front connecting p_2 and p_3 given by

$$\begin{aligned} u(t) &= \frac{u_3(w) + u_2(w)}{2} + \frac{u_3(w) - u_2(w)}{2} \tanh \left(\frac{u_3(w) - u_2(w)}{2\sqrt{2}} t \right) \\ v(t) &= \frac{(u_3(w) - u_2(w))^2}{4\sqrt{2}} \operatorname{sech} \left(\frac{u_3(w) - u_2(w)}{2\sqrt{2}} t \right), \end{aligned} \quad (3.9)$$

with wave speed

$$\begin{aligned} c &= \frac{1}{\sqrt{2}} (u_2(w) + u_3(w) - 2u_1(w)) \\ &= \frac{1}{\sqrt{2}} (u_1(w) + u_2(w) + u_3(w) - 3u_1(w)) \\ &= \frac{1}{\sqrt{2}} (1 - 3u_1(w)) \\ &> \frac{1}{\sqrt{2}}, \end{aligned} \quad (3.10)$$

for all $w \in (\Delta_w, w_A)$. Hence for each $w \in (\Delta_w, w_A)$, for $c = 1/\sqrt{2}$, we must have that $\mathcal{W}^{uu}(p_2)$ lies below $\mathcal{W}^s(p_3)$.

Finally, we can apply the same argument as above to the case of $w = w_A$. For $c = 1/\sqrt{2}$, there is a unique trajectory decaying exponentially in backwards time

along the eigenvector

$$e_2^A = \begin{pmatrix} 1 \\ \frac{1}{2\sqrt{2}} \end{pmatrix}, \quad (3.11)$$

with exponential rate $e^{\frac{t}{2\sqrt{2}}}$, whereas all other trajectories decay with algebraic exponential rate $te^{\frac{t}{2\sqrt{2}}}$. We abuse notation and refer to this trajectory as $\mathcal{W}^{uu}(p_2)$. For $c > 1/\sqrt{2}$, p_2 is an unstable node, and as above we can find a front solution connecting $\mathcal{W}^{uu}(p_2)$ and $\mathcal{W}^s(p_3)$ at with wave speed

$$\begin{aligned} c &= \frac{1}{\sqrt{2}} (1 - 3u_1(w_A)) \\ &> \frac{1}{\sqrt{2}}, \end{aligned} \quad (3.12)$$

and hence, by the above monotonicity argument, we deduce that $\mathcal{W}^{uu}(p_2)$ lies below $\mathcal{W}^s(p_3)$ for $c = 1/\sqrt{2}$, $w = w_A$, which completes the proof of (ii). $\square$

5.3.2 Existence of the center-stable manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$

There are a number of invariant manifolds near the origin which will be involved in the construction of the transitional pulses outlined above. In particular there is the local center manifold near the origin considered in [8, §6] and the stable foliation $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ of the left slow manifold $\mathcal{M}_\varepsilon^\ell(c, a)$. These two manifolds are only unique up to exponentially small errors and were chosen in such a manner that they overlap to form one larger extended center-type manifold.

In the following, as certain parts of our analysis is sensitive to exponentially small errors, it will be convenient to consider an even larger center manifold in this

region which contains all of the essential dynamics to reduce dependence on matching conditions containing exponentially small errors.

The goal is to show that for any small Δ_w , for any sufficiently small ε_0, a_0 , there exists a center type manifold at the origin (as in [8, §6]) which we can extend up to $w = w_A - \Delta_w$, where w_A is the height of the Airy point, due to the consistent exponential separation away from the Airy point. We will be able to choose this manifold in such a way that it contains any part of $\mathcal{M}_\varepsilon^m(c, a)$ lying below $w = w_A - \Delta_w$ and the entirety of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. Hence we abuse notation and refer to this new (larger) manifold as $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$.

To see this we look at the linearization of (1.1) at $(c, a, \varepsilon) = (1/\sqrt{2}, 0, 0)$ given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -f'(u) & c & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.13)$$

There are three eigenvalues $\lambda_0 = 0, \lambda_\pm = \frac{c \pm \sqrt{c^2 - 4f'(u)}}{2}$. A quick computation shows that $\text{Re}(\lambda_+) > \text{Re}(\lambda_-)$ provided $c^2 > 4f'(u)$. In particular for $(c, a) = (1/\sqrt{2}, 0)$, this holds for any $u < u_A = \frac{1}{3} \left(1 - \sqrt{\frac{5}{8}}\right)$.

We fix Δ_w sufficiently small and consider the union of the fronts φ_ℓ for $w \in (0, w_A - \Delta_w)$ for $(c, a, \varepsilon) = (1/\sqrt{2}, 0, 0)$ and refer to this invariant manifold as $\mathcal{W}_0^{s,\ell}(c, a)$. This manifold is normally hyperbolic with the rate of expansion in the normal direction stronger than the expansion rates on $\mathcal{W}_0^{s,\ell}(c, a)$. Therefore this manifold persists [3, 4] as a normally repelling locally invariant manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ containing a neighborhood of the slow manifolds $\mathcal{M}_\varepsilon^\ell(c, a), \mathcal{M}_\varepsilon^m(c, a)$ and the local

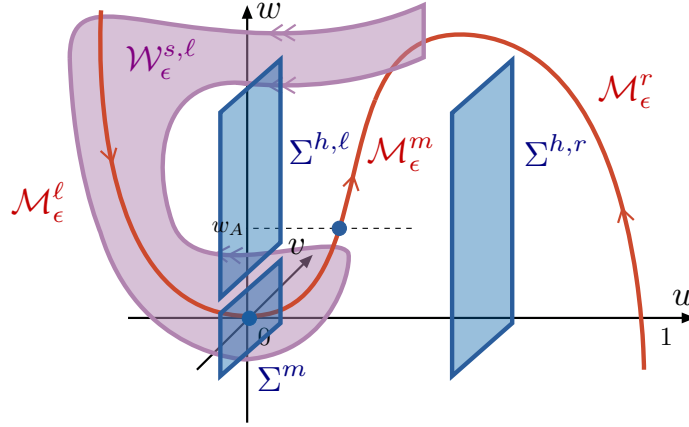


Figure 5.8: The extended center-stable manifold $\mathcal{W}_\varepsilon^{s,\ell}(c,a)$ and the sections $\Sigma^m, \Sigma^{h,\ell}, \Sigma^{h,r}$.

center manifold near the origin. Taking the intersection of this manifold in a plane of fixed $w < w_A - \Delta_w$ which intersects the manifold $\mathcal{M}_\varepsilon^\ell(c,a)$ and evolving backwards in time determines a choice of $\mathcal{W}_\varepsilon^{s,\ell}(c,a)$ which also contains the strong stable fibers of the manifold $\mathcal{M}_\varepsilon^\ell(c,a)$ for $w > w_A - \Delta_w$. This extended center-stable manifold $\mathcal{W}_\varepsilon^{s,\ell}(c,a)$ is shown in Figure 5.8.

5.3.3 Existence of maximal canards

In the two-dimensional manifold $\mathcal{W}_\varepsilon^{s,\ell}(c,a)$, for certain parameter values, there exist canard solutions near the attracting slow manifold $\mathcal{M}_\varepsilon^\ell(c,a)$ which pass near the origin and then follow the repelling slow manifold $\mathcal{M}_\varepsilon^m(c,a)$ for some time. Using the results in [38], there is a maximal canard solution which occurs when $\mathcal{M}_\varepsilon^\ell(c,a)$ and $\mathcal{M}_\varepsilon^m(c,a)$ coincide. We have the following.

Theorem 5.1. *There exists $\varepsilon_0 > 0$ and a smooth function $a^c(\sqrt{\varepsilon}, c) : (0, \varepsilon_0) \times I_c \rightarrow I_a$ such that there is a maximal canard solution connecting the manifolds $\mathcal{M}_\varepsilon^\ell(c,a)$ and $\mathcal{M}_\varepsilon^m(c,a)$ when $a = a^c(c, \sqrt{\varepsilon})$. We have that*

$$a^c(\sqrt{\varepsilon}, c) = -m\varepsilon + \mathcal{O}(\varepsilon^{3/2}) \quad (3.14)$$

where $m = m(c)$ is positively bounded away from zero uniformly in $c \in I_c$.

5.3.4 Singular transitional pulses

In this section we construct singular transitional pulses for $(c, a, \varepsilon) = (1/\sqrt{2}, 0, 0)$ using the layer analysis above. We define

$$\mathcal{M}(u_1, u_2) := \{(u, 0, f(u)) : u \in [u_1, u_2]\} \quad (3.15)$$

All singular pulses consist of a single pulse

$$\Gamma_0^1 = \varphi_f \cup \mathcal{M}(u^\dagger, 1) \cup \varphi_b \cup \mathcal{M}(u^\dagger - 1, 0), \quad (3.16)$$

followed by a secondary pulse. The secondary pulse follows a canard-like explosion which we parametrize by $s \in (0, 2w^\dagger)$. We define the singular secondary pulses

$$\Gamma_0^2(s) := \begin{cases} \mathcal{M}(0, u_2(s)) \cup \varphi_\ell(s) \cup \mathcal{M}(u_1(s), 0), & s \in (0, w^\dagger) \\ \mathcal{M}(0, u^\dagger) \cup \varphi_b \cup \mathcal{M}(u^\dagger - 1, 0), & s = w^\dagger \\ \mathcal{M}(0, u_2(2w^\dagger - s)) \cup \varphi_r(2w^\dagger - s) \\ \cup \mathcal{M}(u^\dagger, u_3(2w^\dagger - s)) \cup \varphi_b \cup \mathcal{M}(u^\dagger - 1, 0), & s \in (w^\dagger, 2w^\dagger) \end{cases}. \quad (3.17)$$

We refer to singular transitional pulses $\Gamma_0(s) = \Gamma_0^1 \cup \Gamma_0^2(s)$ as “left” transitional pulses for $s \in (0, w^\dagger)$ and “right” double pulses for $s \in (w^\dagger, 2w^\dagger)$. The left/right descriptor refers to whether the double pulse involves a jump from $\mathcal{M}_0^m$ to the left branch $\mathcal{M}_0^\ell$ or a jump to the right branch $\mathcal{M}_0^r$. These two types of singular pulses are shown in Figure 5.6.

We also define the two dimensional singular tail manifolds $\mathcal{T}_0(\bar{w})$ for $\bar{w} \in (0, w_A)$ by

$$\mathcal{T}_0(\bar{w}) = \bigcup_{w \in (0, \bar{w})} \varphi_\ell(w),$$

where the fronts $\varphi_\ell(w)$ are defined as in Proposition 5.3.2.

5.3.5 Statement of the main result

The goal of this chapter is to prove the following existence theorem for a one-parameter family of homoclinic solutions to (1.1) which encompasses the transition from single pulses with oscillatory tails from [8, Theorem 1.1] to double pulse solutions comprised of a single pulse followed by a secondary excursion which is close to the original pulse.

Theorem 5.2. *For each sufficiently small $\Delta_w > 0$, there exists $\varepsilon_0, q, C > 0$ such that the following holds. For each $0 < \varepsilon < \varepsilon_0$, there exists a one-parameter family of traveling pulse solutions to (1.1)*

$$s \rightarrow (c(s, \sqrt{\varepsilon}), a(s, \sqrt{\varepsilon}), \Gamma(s, \sqrt{\varepsilon})), \quad s \in (0, 2w^\dagger - \Delta_w) \quad (3.18)$$

which is C^1 in $(s, \sqrt{\varepsilon})$. Furthermore

(i) *The family $\Gamma(s, \sqrt{\varepsilon})$ is approximated by the singular transitional pulses $\Gamma_0(s) = \Gamma_0^1 \cup \Gamma_0^2(s)$ in the following sense:*

For $s \in (0, w_A)$, the pulse $\Gamma(s, \sqrt{\varepsilon})$ consists of a primary excursion which lies within $\mathcal{O}(\sqrt{\varepsilon})$ of Γ_0^1 , followed by an oscillatory tail which remains within an $\mathcal{O}(\sqrt{\varepsilon})$ neighborhood of $\mathcal{T}_0(\max\{s, \Delta_w\})$.

For $s \in (w_A, 2w^\dagger - \Delta_w)$, the pulse $\Gamma(s, \sqrt{\varepsilon})$ consists of a primary excursion which lies within $\mathcal{O}(\sqrt{\varepsilon})$ of Γ_0^1 , followed by a secondary excursion which lies within $\mathcal{O}(\sqrt{\varepsilon})$ of $\Gamma_0^2(s)$, followed by an oscillatory tail which remains within an $\mathcal{O}(\sqrt{\varepsilon})$ neighborhood of $\mathcal{T}_0(w_A)$.

(ii) For all sufficiently small $s > 0$, $a(s, \sqrt{\varepsilon})$ is monotone decreasing in s and the pulses $\Gamma(s, \sqrt{\varepsilon})$ correspond to the pulses constructed in [8, Theorem 1.1] with

$$c(s, \sqrt{\varepsilon}) = \check{c}(a(s, \sqrt{\varepsilon}), \varepsilon). \quad (3.19)$$

(iii) For all $s \in (0, 2w^\dagger - \Delta_w)$, we have that

$$|c(s, \sqrt{\varepsilon}) - \check{c}(a(s, \sqrt{\varepsilon}), \varepsilon)| \leq Ce^{-q/\varepsilon}. \quad (3.20)$$

(iv) For all $s \in (\Delta_w, 2w^\dagger - \Delta_w)$, we have that

$$|a(s, \sqrt{\varepsilon}) - a^c(\sqrt{\varepsilon}, c(s, \sqrt{\varepsilon}))| \leq Ce^{-q/\varepsilon}. \quad (3.21)$$

5.4 Constructing transitional pulses

In this section, we construct transitional pulses in pieces and obtain matching conditions near the canard point at the origin which are solved using an implicit function theorem.

The general construction for pulses of all types involves three pieces: the primary pulse, a secondary excursion, and a tail manifold which is formed by an appropriately defined subset of the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. Hence the procedure involves obtaining two

conditions: one which matches the primary pulse to the secondary excursion, and one matching the secondary excursion with the tail manifold. Once these matching conditions are obtained, it remains to show that solutions on the tail manifold in fact converge to the equilibrium (and are not blocked by periodic orbits, etc.). This is treated in §5.5, where we show that the tail manifold forms part of the stable manifold of the origin, which completes the construction.

Due to various interactions of the pulses with the fold points and Airy point, the construction of the transitional pulses breaks down into six types, five of which we are able to construct. All pulse types have the same primary excursion, and hence the pulse types are determined by properties of the secondary excursion.

- Type 1: $\{\Gamma(s) : s \in (0, w_A + \Delta_w)\}$

Type 1 pulses are left pulses with a secondary excursion of height $w \in (0, w_A + \Delta_w)$, where w_A represents the height of the Airy point, and Δ_w is sufficiently small. For these pulses, we show that this secondary excursion already lies in the tail manifold, and no further matching is required.

- Type 2: $\{\Gamma(s) : s \in (w_A + \Delta_w, w^\dagger - \Delta_w)\}$

Type 2 pulses are left pulses with a secondary excursion of height $w \in (w_A + \Delta_w, w^\dagger - \Delta_w)$, where w_A represents the height of the ‘Airy’ point, and $w^\dagger$ is the height of the upper right fold point. For these pulses, we show that a second matching condition is necessary to ensure that the secondary excursion can be matched with the tail manifold.

- Type 3: $\{\Gamma(s) : s \in (w^\dagger - \Delta_w, w^\dagger + \Delta_w)\}$

Type 3 pulses pass near the upper right fold point and encompass the transition between left and right pulses. These are constructed in much the same

way as type 2 pulses, but there are additional difficulties encountered in parameterizing these pulses (w is not a natural parameter in this regime), and in verifying that the interaction with the upper right fold does not break down the argument.

- Type 4: $\{\Gamma(s) : s \in (w^\dagger + \Delta_w, 2w^\dagger - w_A - \Delta_w)\}$

Type 4 pulses are right pulses with secondary excursion of height $w \in (w_A + \Delta_w, w^\dagger - \Delta_w)$. There is a technical difficulty involved in obtaining the first matching condition which involves balance of exponential contraction/expansion along the slow manifolds $\mathcal{M}_\varepsilon^r(c, a), \mathcal{M}_\varepsilon^\ell(c, a)$.

- Type 5: $\{\Gamma(s) : s \in (2w^\dagger - w_A - \Delta_w, 2w^\dagger - \Delta_w)\}$

Type 5 pulses are right pulses with a secondary excursion of height $w \in (0, w_A + \Delta_w)$. For these pulses, the secondary excursions have a more delicate interaction with the Airy point and therefore also introduce complications when trying to determine the final matching condition with the tail manifold.

- Type 6: $\{\Gamma(s) : s \in (2w^\dagger - \Delta_w, 2w^\dagger)\}$

Type 6 pulses are essentially two copies of the primary pulse. Our results do not cover this regime, though it is an area for further study. When trying to construct these pulses, we approach the Belyakov transition where we expect the branch of pulses will terminate for ε sufficiently small [9], so we do not expect that it is possible to construct pulses for s arbitrarily close to $2w^\dagger$.

The different pulse types are shown in Figure 5.4 with one caveat: the type 3 pulses as defined above should actually appear somewhere in between the pulses with labels 3 and 4 in Figure 5.4.

We begin with setting up the blown-up coordinate system near the canard point

at the origin in which the matching will occur, followed by constructing pulses of type 1,2. We then outline the difficulties/differences in constructing pulses of type 3,4,5 and how to overcome these. The construction is then complete up to two technical results: first, the convergence of the tails, proved in §5.5, and second, a transversality condition which arises due to interaction with the Airy point, proved in §5.6.

5.4.1 Flow near the canard point

We collect some results from [8, 38] which will be useful in the forthcoming analysis for obtaining matching conditions for the transitional pulses near the equilibrium. In [8], it was shown that in a neighborhood of the origin, after a change of coordinates, we obtain the system

$$\begin{aligned}
 \dot{x} &= -y + x^2 + \mathcal{O}(\varepsilon, xy, y^2, x^3) \\
 \dot{y} &= \varepsilon [x(1 + \mathcal{O}(x, y, \alpha, \varepsilon)) + \alpha(1 + \mathcal{O}(x, y, \alpha, \varepsilon)) + \mathcal{O}(y)] \\
 \dot{z} &= z(c^{3/2} + \mathcal{O}(x, y, z, \varepsilon)) \\
 \dot{\alpha} &= 0 \\
 \dot{\varepsilon} &= 0,
 \end{aligned} \tag{4.1}$$

where $\alpha = \frac{a}{2c^{1/2}}$. The manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ is given by $z = 0$, where the strong unstable fibers have been straightened. We note that the (x, y) coordinates are in

the canonical form for a canard point (compare [38]), that is,

$$\begin{aligned}
\dot{x} &= -yh_1(x, y, \alpha, \varepsilon, c) + x^2h_2(x, y, \alpha, \varepsilon, c) + \varepsilon h_3(x, y, \alpha, \varepsilon, c) \\
\dot{y} &= \varepsilon (xh_4(x, y, \alpha, \varepsilon, c) + \alpha h_5(x, y, \alpha, \varepsilon, c) + yh_6(x, y, \alpha, \varepsilon, c)) \\
\dot{z} &= z (c^{3/2} + \mathcal{O}(x, y, z, \varepsilon)) \\
\dot{\alpha} &= 0 \\
\dot{\varepsilon} &= 0 ,
\end{aligned} \tag{4.2}$$

where we have

$$\begin{aligned}
h_3(x, y, \alpha, \varepsilon, c) &= \mathcal{O}(x, y, \alpha, \varepsilon) \\
h_j(x, y, \alpha, \varepsilon, c) &= 1 + \mathcal{O}(x, y, \alpha, \varepsilon), \quad j = 1, 2, 4, 5 .
\end{aligned} \tag{4.3}$$

We have now separated the hyperbolic dynamics (given by the z -coordinate) from the nonhyperbolic dynamics which are isolated on a four-dimensional center manifold parameterized by the variables $(x, y, \varepsilon, \alpha)$ on which the origin is a canard point in the sense of [38]. Such points are characterized by “canard” trajectories which follow a strongly attracting manifold (in this case $\mathcal{M}_\varepsilon^\ell(c, a)$), pass near the equilibrium and continue along a strongly repelling manifold (in this case $\mathcal{M}_\varepsilon^m(c, a)$) for some time. To understand the flow near this point, we use blowup methods as in [38]. Restricting to the center manifold $z = 0$, the blow up transformation is given by

$$x = \bar{r}\bar{x}, \quad y = \bar{r}^2\bar{y}, \quad \alpha = \bar{r}\bar{\alpha}, \quad \varepsilon = \bar{r}^2\bar{\varepsilon}, \tag{4.4}$$

defined on the manifold $B_c = S^2 \times [0, \bar{r}_0] \times [-\bar{\alpha}_0, \bar{\alpha}_0]$ for sufficiently small $\bar{r}_0, \bar{\alpha}_0$ with $(\bar{x}, \bar{y}, \bar{\varepsilon}) \in S^2$. There is one relevant coordinate chart which will be needed for the matching analysis. Keeping the same notation as in [38] and [39], the chart $\mathcal{K}_2$ uses

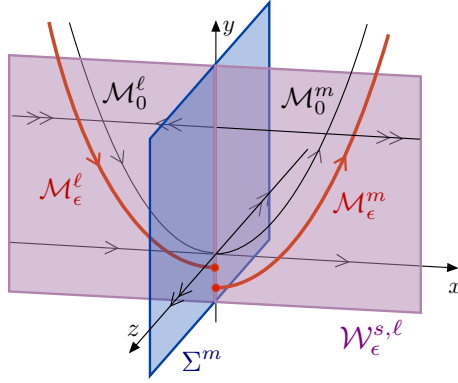


Figure 5.9: The local coordinates near the canard point and the section Σ^m . The manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ coincides with the subspace $z = 0$.

the coordinates

$$x = r_2 x_2, \quad y = r_2^2 y_2, \quad \alpha = r_2 \alpha_2, \quad \varepsilon = r_2^2. \quad (4.5)$$

Using these blow-up charts, in [38], the authors studied the behavior of the manifolds $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{M}_\varepsilon^m(c, a)$ near the equilibrium, and in particular determined conditions under which these manifolds coincide along a canard trajectory. We place a section $\Sigma^m = \{x = 0, |y| < \Delta_y, |z| \leq \Delta_z\}$ for small fixed Δ_z and $\Delta_y = 2\Delta_w$ in which most of our computations will take place (see Figure 5.9).

In the chart $\mathcal{K}_2$, the section Σ^m is given by $\Sigma_2^m = \left\{x_2 = 0, |y_2| < \frac{\Delta_y}{r_2^2}, |z| \leq \Delta_z\right\}$. It was shown in [38] that for all sufficiently small r_2, α_2 , the manifolds $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{M}_\varepsilon^m(c, a)$ reach Σ_2^m at $y = y_2^{\mathcal{M},\ell}(c, a)$ and $y = y_2^{\mathcal{M},m}(c, a)$, respectively. Furthermore, we have the following result which will be useful in the coming analysis.

Proposition 5.4.1. [38, Proposition 3.5] *The distance between the slow manifolds $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{M}_\varepsilon^m(c, a)$ in Σ^m is given by*

$$y_2^{\mathcal{M},\ell} - y_2^{\mathcal{M},m} = \mathcal{D}_0(\alpha_2, r_2; c) = d_{\alpha_2} \alpha_2 + d_{r_2} r_2 + \mathcal{O}(r_2^2 + \alpha_2^2), \quad (4.6)$$

where the coefficients d_{α_2}, d_{r_2} are positive constants. Hence we can solve for when

this distance vanishes which occurs when

$$\alpha_2 = \alpha_2^c = -\frac{d_{r_2}}{d_{\alpha_2}} r_2 + \mathcal{O}(r_2^2). \quad (4.7)$$

Remark 5.4.2. *In the following, many computations will be performed in the $\mathcal{K}_2$ coordinates before transforming back into the original coordinates/parameters as the results of Theorem 5.2 are stated in terms of the original parameters (c, a, ε) , rather than (c, α_2, r_2) . To obtain (a, ε) from (α_2, r_2) , we have*

$$\begin{aligned} a &= 2c^{1/2} \alpha_2 r_2 \\ \varepsilon &= r_2^2, \end{aligned} \quad (4.8)$$

which are smooth functions of (c, α_2, r_2) for (c, α_2, r_2) near $(1/\sqrt{2}, 0, 0)$.

We remark that results involving transversality with respect to parameter variations due to the exchange lemma [51] which are obtained for the original system (1.1) can likewise be shown to hold in the $\mathcal{K}_2$ coordinates by instead considering the system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= r_2^2(u - \gamma w), \end{aligned} \quad (4.9)$$

where $f(u) = u(u - 2c^{1/2} \alpha_2 r_2)(1 - u)$ for (c, α_2, r_2) near $(1/\sqrt{2}, 0, 0)$.

5.4.2 Type 1 pulses

Type 1 pulses are the simplest of the transitional pulses and are really just single pulses with oscillatory tails. In this section, we deduce the existence of transitional

left pulses with secondary excursion of height $w \leq w_A + \Delta_w$ and show that these pulses are in fact a continuation of the family of pulses with oscillatory tails constructed in [8]. To construct a type 1 pulse, we need a single matching condition which matches the primary pulse with a tail of height $w \leq w_A + \Delta_w$.

We break this into two parts. We first construct pulses of height $w \in (\Delta_w, w_A + \Delta_w)$ and then move onto pulses with ‘small’ oscillatory tails, that is, pulses with tails of height $w \leq \Delta_w$.

Matching condition for pulses $\Gamma(s, \sqrt{\varepsilon})$, $s \in (\Delta_w, w_A + \Delta_w)$

We match the various components of the solution in the section Σ^m in the $\mathcal{K}_2$ coordinates. First, we have the following lemma, which will be useful in solving the matching conditions.

Lemma 5.4.3. *For each sufficiently small $\Delta_z > 0$, there exists $C, q, \varepsilon_0 > 0$ and $q_1 > q_2 > 0$ such that the following holds. For each $0 < \varepsilon < \varepsilon_0$ and each $|z| < \Delta_z$, there exists c with $|c - \check{c}(a, \varepsilon)| = \mathcal{O}(e^{-q/\varepsilon})$ such that $\mathcal{W}_\varepsilon^u(0; c, a)$ intersects Σ^m at the point $(y_2^u(z; c, a), z)$ where*

$$\begin{aligned} e^{-q_1/\varepsilon}/C &\leq y_2^u(0; c, a) - y_2^{\mathcal{M}, \ell} \leq Ce^{-q_2/\varepsilon} \\ |y_2^u(z; c, a) - y_2^u(0; c, a)| &= \mathcal{O}(ze^{-q/\varepsilon}). \end{aligned} \tag{4.10}$$

Proof. We have that $\mathcal{W}_\varepsilon^u(0; \check{c}(a, \varepsilon), a)$ is $\mathcal{O}(e^{-q/\varepsilon})$ -close to $\mathcal{M}_\varepsilon^\ell(c, a)$ in Σ^m . Since by Proposition 5.3.1, $\mathcal{W}_\varepsilon^u(0; c, a)$ transversely intersects $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ upon varying $c \approx \check{c}(a, \varepsilon)$, the result follows from the exchange lemma. $\square$

Consider the solution $\gamma^f(s; c, a)$ on the stable foliation $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ which intersects

the section $\Sigma^{h,\ell} := \{u = 0, \Delta_w < w < w^\dagger - \Delta_w\}$ at height $w = s \in (\Delta_w, w_A + \Delta_w)$. This intersection occurs at a point $(u, v, w) = (0, v^f(s; c, a), s)$. Evolving $\gamma^f(s; c, a)$ backwards, we have that $\gamma^f(s; c, a)$ is exponentially close to $\mathcal{M}_\varepsilon^m(c, a)$ in Σ^m . Thus we have that $\gamma^f(s; c, a)$ intersects Σ^m at a point $(y_2, z) = (y_2^b, z^b)(s; c, a)$ which satisfies

$$\begin{aligned} |y_2^b(s; c, a) - y_2^{\mathcal{M},m}| &= \mathcal{O}(e^{-q/\varepsilon}) \\ |z^b(s; c, a)| &= \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \tag{4.11}$$

uniformly in (c, a) . Thus by Proposition 5.4.1 we can match $\mathcal{W}_\varepsilon^u(0; c, a)$ with the trajectory $\gamma^f(s; c, a)$ by solving

$$\begin{aligned} \mathcal{D}_0(\alpha_2, r_2; c) &= \left(y_2^{\mathcal{M},\ell} - y_2^u(z; c, a) \right) + \left(y_2^b(s; c, a) - y_2^{\mathcal{M},m} \right) = \mathcal{O}(e^{-q/\varepsilon}) \\ z &= z^b(s; c, a), \end{aligned} \tag{4.12}$$

We obtain a solution by solving

$$\begin{aligned} a &= 2c^{1/2}\varepsilon^{1/2}(\alpha_2^c + \mathcal{O}(e^{-q/\varepsilon})) \\ c &= \check{c}(a, \varepsilon) + \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \tag{4.13}$$

by the implicit function theorem to find $(c, a) = (c, a)(s, \sqrt{\varepsilon})$.

Connection to pulses with small oscillatory tails

We now consider the case of pulses with tails of height $w \leq \Delta_w$. In [8], it was shown that there exists K^* , such that for each K and each sufficiently small (a, ε) satisfying $\varepsilon < Ka^2$, there exists a pulse solution with wave speed $c = \check{c}(a, \varepsilon)$. For $\varepsilon > K^*a^2$, the tail of the pulse decays exponentially to zero in an oscillatory fashion.

We deduce that such pulses exist for (α_2, r_2) for any $r_2 > 0$ sufficiently small and

$\alpha_2 > \frac{1}{2c\sqrt{K}} > \frac{1}{2\sqrt{K}}$, since

$$c = 1/\sqrt{2} + \mathcal{O}(\alpha_2 r_2, r_2^2) < 1 \quad (4.14)$$

for α_2 bounded and $r_2 > 0$ sufficiently small. In this section we show that these pulses overlap with the type 1 pulses constructed above, forming a one-parameter family. It turns out that this family is naturally parameterized by α_2 .

To see this, we proceed as follows. Setting $s = \Delta_w$, for sufficiently small $\varepsilon > 0$, we can follow the procedure above in constructing a type 1 pulse with a tail of height $s = \Delta_w$. We note that in this case, the backwards evolution of the trajectory $\gamma^f(\Delta_w; c, a)$ remains in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ until reaching the section Σ^m and therefore intersects this section at a point $(y_2, z) = (y_2^b, z^b)(\Delta_w; c, a)$ which satisfies

$$\begin{aligned} e^{-q_1/\varepsilon}/C &\leq y_2^b(\Delta_w; c, a) - y_2^{\mathcal{M},m} \leq C e^{-q_2/\varepsilon} \\ z^b(\Delta_w, c, a) &= 0, \end{aligned} \quad (4.15)$$

for some $q_1 > q_2 > 0$. Thus we can match $\mathcal{W}_\varepsilon^u(0; c, a)$ with $\gamma^f(\Delta_w; c, a)$ by solving

$$\mathcal{D}_0(\alpha_2, r_2; c) = \left(y_2^{\mathcal{M},\ell} - y_2^u(0; c, a) \right) + \left(y_2^b(\Delta_w; c, a) - y_2^{\mathcal{M},m} \right) \quad (4.16)$$

We obtain a solution by solving

$$\begin{aligned} \alpha_2 &= \alpha_2^c + \mathcal{O}(e^{-q/\varepsilon}) \\ c &= \check{c} \left(2c^{1/2} r_2 \alpha_2, r_2^2 \right), \end{aligned} \quad (4.17)$$

by the implicit function theorem to find a solution at $(c, \alpha_2) = (c^u, \alpha_2^u)(r_2)$. We now

consider the function $\bar{\mathcal{D}}(\alpha_2, r_2, c)$ defined to be the difference

$$\bar{\mathcal{D}}(\alpha_2, r_2, c) = y_2^u(0; c, a) - y_2^b(\Delta_w; c, a) \quad (4.18)$$

in Σ^m . From the construction above for the pulse with tail of height $\bar{w} = \Delta_w$, we have that

$$\bar{\mathcal{D}}(\alpha_2^u, r_2, c^u) = 0 \quad (4.19)$$

and

$$\begin{aligned} \bar{\mathcal{D}}(\alpha_2, r_2, c) &= y_2^u(0; c, a) - y_2^b(\Delta_w; c, a) \\ &= y_2^{\mathcal{M}, \ell} - y_2^{\mathcal{M}, m} + \mathcal{O}(e^{-q/\varepsilon}) \\ &= \mathcal{D}_0(\alpha_2, r_2; c) + \mathcal{O}(e^{-q/\varepsilon}) \\ &= d_{\alpha_2} \alpha_2 + d_{r_2} r_2 + \mathcal{O}(\alpha_2^2, \alpha_2 r_2, r_2^2). \end{aligned} \quad (4.20)$$

Hence we have that

$$\frac{\partial}{\partial \alpha_2} \bar{\mathcal{D}}(\alpha_2, r_2, c) = d_{\alpha_2} + \mathcal{O}(\alpha_2, r_2) > 0 \quad (4.21)$$

for any sufficiently small $r_2 > 0$ and $|\alpha_2| \leq \kappa$, uniformly in $c \approx 1/\sqrt{2}$.

Hence for sufficiently small $r_2 > 0$, for $\alpha_2^u < \alpha_2 < \kappa$, we can ensure that $\mathcal{W}_\varepsilon^u(0; c, a)$ lands in $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ by solving

$$c = \check{c}(2c^{1/2}r_2\alpha_2, r_2^2), \quad (4.22)$$

for $c = c(\alpha_2, r_2)$ by the implicit function theorem. Furthermore, we have that the distance $\bar{\mathcal{D}}(\alpha_2, r_2, c)$ is positive, and hence we obtain a pulse whose tail reaches a

height lower than Δ_w , but remains in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ and converges to the equilibrium. For fixed r_2 , such pulses are therefore parameterized by $\alpha_2^u < \alpha_2 < \kappa$.

By taking $K > \frac{1}{4\kappa^2}$ in [8, Theorem 1.1], we deduce that these pulses form a continuous family with the pulses constructed in [8] for $\alpha_2 > \frac{1}{2\sqrt{K}}$.

5.4.3 The tail manifold

We will match the various components of the solution in the section $\Sigma^m = \{x = 0, |y| < \Delta_y, |z| \leq \Delta_z\}$. These matching conditions will be determined by intersections of various invariant manifolds evolved forwards/backwards under the flow between the sections $\Sigma^{h,\ell}, \Sigma^m$. To avoid confusion, when referring to an invariant manifold evolved in *backwards* time from $\Sigma^{h,\ell}$ to Σ^m , we use the notation ‘ $\widehat{}$ ’, e.g. when referring to the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$, we mean the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ under the backwards evolution of (1.1).

We now identify the ‘tail’ manifold in which the desired pulse solutions will be trapped. In §5.5, we will show that this manifold indeed forms part of the stable manifold $\mathcal{W}_\varepsilon^s(0; c, a)$ of the equilibrium. We consider the backwards evolution of $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ from $\Sigma^{h,\ell}$ to Σ^m . The manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ intersects $\Sigma^{h,\ell}$ in a curve $(u, v, w) = (0, v^f(w; c, a), w)$. We have the following proposition, which states that the backwards evolution of $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ intersects Σ^m in a curve transverse to the strong unstable fibers $y_2 = \text{const}$. Except for an exponentially thin region around $\mathcal{M}_\varepsilon^m(c, a)$, this curve coincides with $z = 0$. The intersection of $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ with the section Σ^m is shown in Figure 5.10.

Proposition 5.4.4. *For each sufficiently small $\Delta_w > 0$, there exists $C, \kappa, \varepsilon_0, q > 0$ and sufficiently small choice of the intervals I_c, I_a such that for each $(c, a, \varepsilon) \in I_c \times$*

$I_a \times (0, \varepsilon_0)$, there exists $w_\varepsilon^\Delta(c, a) \in [w_A - 3\Delta_w, w_A - \Delta_w]$ and $w_\varepsilon^\natural(c, a) > w_A + \kappa\varepsilon^{2/3}$ such that the following holds. Let $\widehat{\mathcal{W}}_\varepsilon^{s, \ell, *}(c, a)$ denote the backwards evolution of the curve $\{(u, v, w) = (0, v^f(w; c, a), w); w_\varepsilon^\Delta < w < w_\varepsilon^\natural\}$. Then $\widehat{\mathcal{W}}_\varepsilon^{s, \ell, *}(c, a)$ intersects Σ^m in a curve $z = z^{\ell, *}(y_2; c, a)$ for $y_{2,0}^{\ell, *}(c, a) \leq y_2 \leq \hat{y}_{2,0}^\ell(c, a)$ where

(i) The interval $[y_{2,0}^{\ell, *}(c, a), \hat{y}_{2,0}^\ell(c, a)]$ satisfies $0 < \hat{y}_{2,0}^\ell(c, a) - y_{2,0}^{\ell, *}(c, a) < Ce^{-q/\varepsilon}$ uniformly in $(c, a) \in I_c \times I_a$.

(ii) There exists $\tilde{y}_{2,0}^\ell(c, a) \in [y_{2,0}^{\ell, *}(c, a), \hat{y}_{2,0}^\ell(c, a)]$ such that $z^{\ell, *}(y_2; c, a) \equiv 0$ for $y_2 \in [\tilde{y}_{2,0}^\ell(c, a), \hat{y}_{2,0}^\ell(c, a)]$.

(iii) The function $z^{\ell, *}$ and its derivatives are $\mathcal{O}(e^{-q/\varepsilon})$ uniformly in $y_2 \in [y_{2,0}^{\ell, *}(c, a), \hat{y}_{2,0}^\ell(c, a)]$ and $(c, a) \in I_c \times I_a$.

Hence we have that $\widehat{\mathcal{W}}_\varepsilon^{s, \ell, *}(c, a)$ intersects Σ^m in a curve which can be represented as a graph $z = z^{\ell, *}(y_2; c, a)$ for $y_2 \geq y_{2,0}^{\ell, *}(c, a)$ where the function $z^{\ell, *}$ and its derivatives are $\mathcal{O}(e^{-q/\varepsilon})$. The proof of this proposition will be given in §5.6.

We can now define the tail manifold in the section Σ^m . We define $\mathcal{T}_\varepsilon(c, a)$ to be the forward evolution of trajectories which intersect the section Σ^m in the graph of a smooth function given by $z = z_\varepsilon^{\mathcal{T}}(y_2; c, a)$ for $y_2 \geq y_{2,0}^{\ell, *}(c, a)$ where

$$z_\varepsilon^{\mathcal{T}}(y_2; c, a) = \begin{cases} 0 & y_2 \geq \hat{y}_{2,0}^\ell(c, a) \\ z^{\ell, *}(y_2; c, a) & y_{2,0}^{\ell, *}(c, a) \leq y_2 \leq \hat{y}_{2,0}^\ell(c, a) \end{cases}. \quad (4.23)$$

The intersection of the tail manifold with the section Σ^m is shown in Figure 5.10. In §5.5, it is shown that all trajectories which hit $\mathcal{T}_\varepsilon(c, a)$ in fact lie in the stable manifold $\mathcal{W}_\varepsilon^s(0; c, a)$.

Figure 5.10: Shown is the setup for the matching conditions in the section Σ^m .

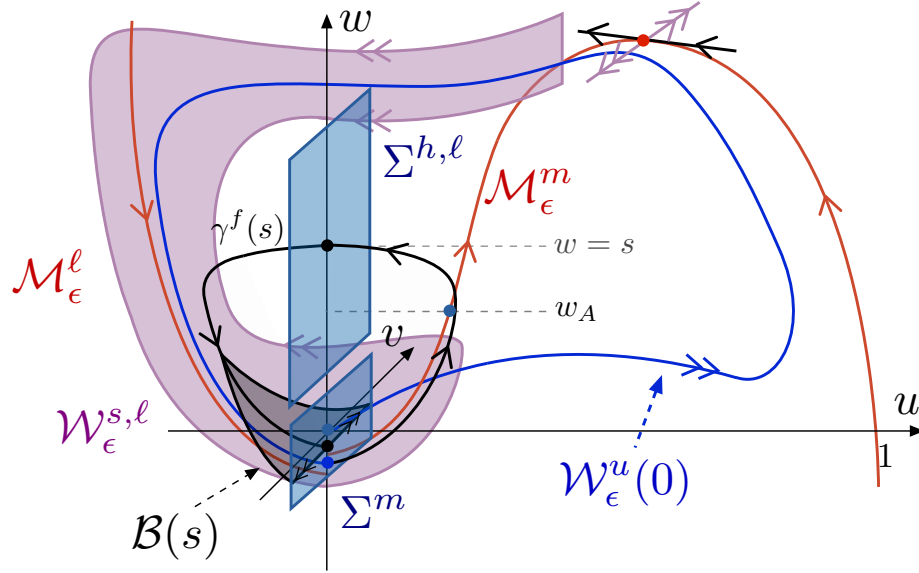


Figure 5.11: Shown is the geometry for constructing a type 2 left transitional pulse.

5.4.4 Type 2 pulses

To construct a left transitional pulse with secondary excursion of height $s \in (w_A + \Delta_w, w^\dagger - \Delta_w)$, we consider a two dimensional manifold $\mathcal{B}(s; c, a)$ (to be chosen below) which intersects the manifold $\mathcal{W}_\epsilon^{s,\ell}(c, a)$ in the section $\Sigma^{h,\ell} := \{u = 0, \Delta_w < w < w^\dagger - \Delta_w\}$ at $w = s$. Evolving $\widehat{\mathcal{B}}(s; c, a)$ backwards to the section Σ^m , we show that each trajectory passing through $\mathcal{B}(s; c, a)$ can be matched with $\mathcal{W}_\epsilon^u(0; c, a)$ by choosing (c, a) appropriately. By evolving $\mathcal{B}(s; c, a)$ forwards, we show that precisely one of these choices results in $\mathcal{W}_\epsilon^u(0; c, a)$ becoming trapped in the tail manifold $\mathcal{T}_\epsilon(c, a)$ as $t \rightarrow \infty$. The setup is shown in Figure 5.11.

Therefore, to construct a transitional pulse we need two matching conditions near the equilibrium: the first matches $\mathcal{W}_\epsilon^u(0; c, a)$ with $\widehat{\mathcal{B}}(s; c, a)$ to guarantee height s for the second excursion, and the second matches $\mathcal{B}(s; c, a)$ with the tail manifold $\mathcal{T}_\epsilon(c, a)$, which we will show in §5.5 forms part of the two-dimensional stable manifold of the equilibrium. The local geometry for the matching conditions is shown in Figure 5.12. The setup for the matching conditions in the section Σ^m is shown in

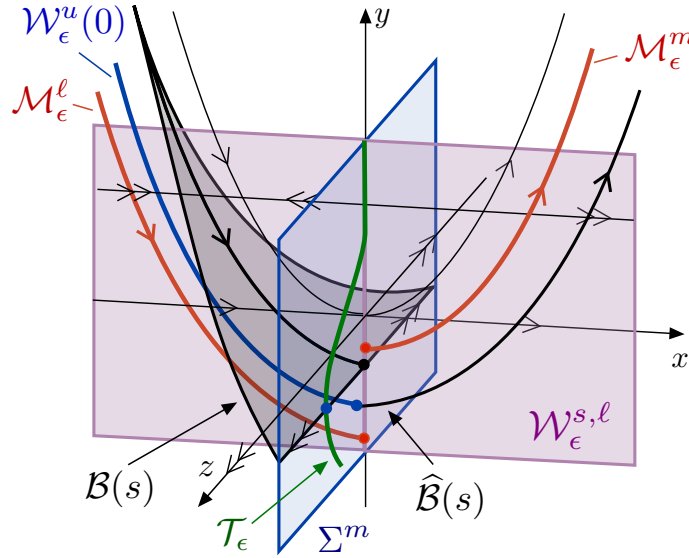


Figure 5.12: Shown is the setup for the matching conditions in the section Σ^m . There are two matching conditions: (i) match $\mathcal{W}_\epsilon^u(0; c, a)$ with $\hat{\mathcal{B}}(s; c, a)$ (ii) match $\mathcal{B}(s; c, a)$ with the tail manifold $\mathcal{T}_\epsilon(c, a)$.

Figure 5.10.

Matching conditions for pulses $\Gamma(s, \sqrt{\epsilon})$, $s \in (w_A + \Delta_w, w^\dagger - \Delta_w)$

We match the various components of the solution in the section Σ^m in the $\mathcal{K}_2$ coordinates. From Lemma 5.4.3, for each $|z| < \Delta_z$, there exists c with $|c - \check{c}(a, \epsilon)| = \mathcal{O}(e^{-q/\epsilon})$ such that $\mathcal{W}_\epsilon^u(0; c, a)$ intersects Σ^m at the point $(y_2^u(z; c, a), z)$ where

$$\begin{aligned} e^{-q_1/\epsilon}/C &\leq y_2^u(0; c, a) - y_2^{\mathcal{M}, \ell} \leq C e^{-q_2/\epsilon} \\ |y_2^u(z; c, a) - y_2^u(0; c, a)| &= \mathcal{O}(z e^{-q/\epsilon}), \end{aligned} \tag{4.24}$$

for some $q_1 > q_2 > 0$.

Consider the solution $\gamma^f(s; c, a)$ on the stable foliation $\mathcal{W}_\epsilon^{s, \ell}(c, a)$ which intersects the section $\Sigma^{h, \ell}$ at height $w = s$. This intersection occurs at a point $(u, v, w) = (0, v^f(s; c, a), s)$. This solution is exponentially attracted in forward time to $\mathcal{M}_\epsilon^\ell(c, a)$

and hence intersects Σ^m at the point $(y_2^f(s; c, a), 0)$ where

$$e^{-q_1/\varepsilon}/C \leq y_2^f(s; c, a) - y_2^u(0; c, a) \leq Ce^{-q_2/\varepsilon}. \quad (4.25)$$

The geometry of the setup for type 2 pulses and the solution $\gamma^f(s; c, a)$ is shown in Figure 5.11.

Define the manifold $\mathcal{B}(s; c, a)$ to be the backwards evolution of the fiber

$$\{(0, y_2^f(s; c, a), z) : |z| \leq \Delta_z\} \subseteq \Sigma^m. \quad (4.26)$$

We parameterize $\mathcal{B}(s; c, a)$ by $|z_B| \leq \Delta_z$, where z_B denotes the height along the fiber. In backwards time, this fiber is exponentially contracted to the solution $\gamma^f(s; c, a)$ and hence intersects $\Sigma^{h,\ell}$ in a one-dimensional curve which is $\mathcal{O}(e^{-q/\varepsilon})$ -close to $(u, v, w) = (0, v^f(s; c, a), s)$, uniformly in (c, a) . A schematic of the manifold $\mathcal{B}(s; c, a)$ and its relation to $\gamma^f(s; c, a)$ is shown in Figure 5.11.

Evolving $\widehat{\mathcal{B}}(s; c, a)$ backwards, we have that $\widehat{\mathcal{B}}(s; c, a)$ is exponentially close to $\mathcal{M}_\varepsilon^m(c, a)$ in Σ^m . Thus we have that in Σ^m , $\widehat{\mathcal{B}}(s; c, a)$ is given by a curve $(y_2, z) = (y_2^b, z^b)(z_B, s; c, a)$ which satisfies

$$\begin{aligned} |y_2^b(z_B, s; c, a) - y_2^{\mathcal{M},m}| &= \mathcal{O}(e^{-q/\varepsilon}) \\ |z^b(z_B, s; c, a)| &= \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \quad (4.27)$$

uniformly in $|z_B| \leq \Delta_z$ and $(c, a) \in I_c \times I_a$. The derivatives of the above expressions with respect to (c, a, z_B) are also $\mathcal{O}(e^{-q/\varepsilon})$, by taking q a bit smaller if necessary.

Recall from §5.4.3 that in the section Σ^m , the tail manifold $\mathcal{T}_\varepsilon(c, a)$ is defined by

the graph of a smooth function given by $z = z_\varepsilon^T(y_2; c, a)$ for $y_2 \geq y_{2,0}^{\ell,*}(c, a)$ where

$$z_\varepsilon^T(y_2; c, a) = \begin{cases} 0 & y_2 \geq \hat{y}_{2,0}^\ell(c, a) \\ z^{\ell,*}(y_2; c, a) & y_{2,0}^{\ell,*}(c, a) \leq y_2 \leq \hat{y}_{2,0}^\ell(c, a) \end{cases} \quad (4.28)$$

and the function $z^{\ell,*}$ and its derivatives are $\mathcal{O}(e^{-q/\varepsilon})$. We have the following.

Lemma 5.4.5. *For each $s \in (w_A + \Delta_w, w^\dagger - \Delta_w)$ and each sufficiently small $\varepsilon > 0$, the backwards evolution of the manifold $\widehat{\mathcal{B}}(s; c, a)$ intersects Σ^m in a curve $(y_2, z) = (y_2^b, z^b)(z_B, s; c, a)$ which satisfies*

$$\begin{aligned} |y_2^b(z_B, s; c, a) - y_2^{\mathcal{M},m}| &= \mathcal{O}(e^{-q/\varepsilon}) \\ |z^b(z_B, s; c, a)| &= \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \quad (4.29)$$

uniformly in $|z_B| \leq \Delta_z$ and $(c, a) \in I_c \times I_a$. The derivatives of the above expressions with respect to (c, a, z_B) are also $\mathcal{O}(e^{-q/\varepsilon})$. Furthermore

$$y_{2,0}^{\ell,*}(c, a) < \inf_{|z_B| \leq \Delta_z} y_2^b(z_B, s; c, a), \quad (4.30)$$

for all $(c, a) \in I_c \times I_a$.

The first assertion of Lemma 5.4.5 follows from the analysis above; the proof of the second assertion will be given in §5.6.

The final matching conditions are obtained as follows. Starting in $\Sigma^{h,\ell}$, we evolve $\widehat{\mathcal{B}}(s; c, a)$ back to the section Σ^m and show that for each $|z_B| \leq \Delta_z$, $\mathcal{W}_\varepsilon^u(0; c, a)$ can be matched with the corresponding solution on $\widehat{\mathcal{B}}(s; c, a)$ by adjusting (c, a) . We then evolve $\mathcal{B}(s; c, a)$ forwards from $\Sigma^{h,\ell}$ to Σ^m and show that $\mathcal{B}(s; c, a)$ transversely intersects $\mathcal{T}_\varepsilon(c, a)$ for each such (c, a) as z_B varies. This implies the existence of parameter values (c, a) for which $\mathcal{W}_\varepsilon^u(0; c, a)$ completes one full pulse and a secondary

pulse of height s before landing in the tail manifold $\mathcal{T}_\varepsilon(c, a)$. Convergence of the tails is proved in §5.5. The setup for the matching conditions is shown in Figures 5.10 and 5.12.

From Proposition 5.4.1, we have that the distance between the manifolds $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{M}_\varepsilon^m(c, a)$ in Σ^m is given by

$$y_2^{\mathcal{M},\ell} - y_2^{\mathcal{M},m} = \mathcal{D}_0(\alpha_2, r_2; c) = d_{\alpha_2}\alpha_2 + d_{r_2}r_2 + \mathcal{O}(2) . \quad (4.31)$$

Using Lemma 5.4.3 and Proposition 5.4.1, by varying c, α_2 we can match $\mathcal{W}_\varepsilon^u(0; c, a)$ with any solution in $\widehat{\mathcal{B}}(s; c, a)$ by solving

$$\begin{aligned} \mathcal{D}_0(\alpha_2, r_2; c) &= \left(y_2^{\mathcal{M},\ell} - y_2^u(z; c, a) \right) + \left(y_2^b(z_B, s; c, a) - y_2^{\mathcal{M},m} \right) = \mathcal{O}(e^{-q/\varepsilon}) \\ z &= z^b(z_B, s; c, a), \end{aligned} \quad (4.32)$$

for each $|z_B| \leq \Delta_z$. For each such z_B , we obtain a solution by solving

$$\begin{aligned} a &= 2c^{1/2}\varepsilon^{1/2}(\alpha_2^c + \mathcal{O}(e^{-q/\varepsilon})) \\ c &= \check{c}(a, \varepsilon) + \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \quad (4.33)$$

by the implicit function theorem to find $(a, c) = (a^u, c^u)(z_B, s, \sqrt{\varepsilon})$. We now evolve $\mathcal{W}_\varepsilon^u(0; c, a)$ forwards; for each z_B we can hit the corresponding point on $\mathcal{B}(s; c, a)$, hence $\mathcal{W}_\varepsilon^u(0; c, a)$ intersects Σ^m at the point $(y_2^f(s; c, a), z_B)$ when $(c, a) = (c^u, a^u)(z_B)$ defined above. We now match with the tail manifold $\mathcal{T}_\varepsilon(c, a)$ by solving

$$z_B = z_\varepsilon^T(y_2^f(s; c, a); c^u(z_B), a^u(z_B)), \quad (4.34)$$

which, using Lemma 5.4.5, we can solve by the implicit function theorem when

$z_B = z_B^s = \mathcal{O}(e^{-q/\varepsilon})$ to find the desired pulse solution when

$$\begin{aligned} a &= a(s, \sqrt{\varepsilon}) := a^u(z_B^s; s, \sqrt{\varepsilon}) \\ c &= c(s, \sqrt{\varepsilon}) := a^u(z_B^s; s, \sqrt{\varepsilon}). \end{aligned} \tag{4.35}$$

5.4.5 Type 4 & 5 pulses

Type 4/5 pulses correspond to $\Gamma(s), s \in (w^\dagger + \Delta_w, 2w^\dagger - \Delta_w)$. Type 4 pulses are right transitional pulses with a secondary pulse of heights $w \in (w_A + \Delta_w, w^\dagger - \Delta_w)$, and type 5 pulses are right transitional pulses with a secondary pulse of height $w \in (\Delta_w, w_A + \Delta_w)$. For type 4/5 pulses, the secondary pulses pass close to the upper right fold point. These pulses are constructed in much the same way as type 2 pulses, except with a different definition of the solution $\gamma^f(s; c, a)$ and the associated manifold $\mathcal{B}(s; c, a)$. In terms of the actual construction of the pulses, there is no distinction between pulses of type 4 and 5. We distinguish these pulses, however, due to the technical difficulties associated with proving Lemma 5.4.9 for the case of type 5 pulses, which is crucial to solving the final matching conditions.

To construct a right transitional pulse with secondary height $w = 2w^\dagger - s$, we first consider the plane $w = 2w^\dagger - s$ which intersects the section $\Sigma^{h,r} := \{u = 2/3, \Delta_w < w < w^\dagger - \Delta_w\}$ in a line $\{u = 2/3, w = 2w^\dagger - s\}$. This line transversely intersects the manifold $\mathcal{W}_\varepsilon^{s,r}(c, a)$ for all $(c, a) \in I_c \times I_a$. Using arguments similar to those in [8, §5] in the proof of Proposition 5.3.1, it follows that the forward evolution of this line transversely intersects $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ for each $(c, a) \in I_c \times I_a$ and each sufficiently small $\varepsilon > 0$ along a trajectory $\gamma^f(s; c, a)$. Furthermore, the solution $\gamma^f(s; c, a)$ is exponentially close to $\mathcal{W}_\varepsilon^{s,r}(c, a)$ in $\Sigma^{h,r}$ and passes $\mathcal{O}(\varepsilon^{2/3} + |a|)$ close to the fold before intersecting $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$.

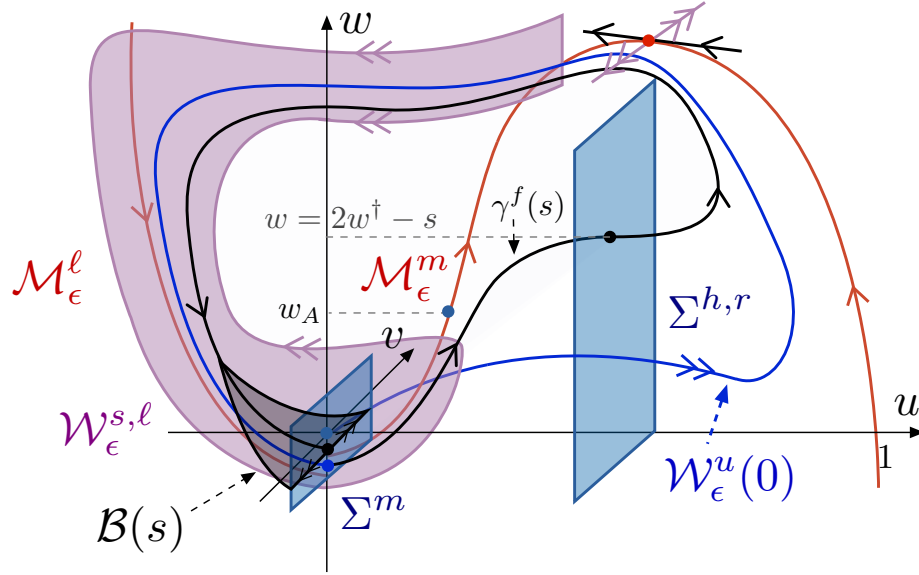


Figure 5.13: Shown is the geometry for constructing a type 4 right transitional pulse.

The geometry of the setup for type 4 pulses and the solution $\gamma^f(s; c, a)$ is shown in Figure 5.13.

Proceeding as with Type 2 pulses, we follow $\gamma^f(s; c, a)$ along $\mathcal{W}_\epsilon^{s,\ell}(c, a)$ where it is exponentially contracted to $\mathcal{M}_\epsilon^r(c, a)$ and intersects the section Σ^m at a point $(y_2^f(s; c, a), 0)$. We again define $\mathcal{B}(s; c, a)$ to be the backwards evolution of the fiber $\{(0, y_2^f(s; c, a), z) : |z| \leq \Delta_z\}$. We parametrize the manifold $\mathcal{B}(s; c, a)$ by $\{z_B, |z_B| \leq \Delta_z\}$ corresponding to the initial height along the fiber in Σ^m . Assuming this manifold is well defined and exponentially close to $\gamma^f(s; c, a)$ in $\Sigma^{h,r}$ (and the derivatives of the transition maps with respect to (c, a, z_B) are also exponentially small), the remainder of the construction follows similarly to the case of type 2 pulses. A schematic of the manifold $\mathcal{B}(s; c, a)$ and its relation to $\gamma^f(s; c, a)$ is shown in Figure 5.13.

Contraction/expansion rates along $\mathcal{M}_\varepsilon^\ell(c, a)$, $\mathcal{M}_\varepsilon^r(c, a)$

To construct pulses of type 4, 5, we need more explicit bounds on the rates of contraction and expansion along solutions near the slow manifolds $\mathcal{M}_\varepsilon^\ell(c, a)$, $\mathcal{M}_\varepsilon^r(c, a)$. We consider the flow in neighborhoods of each of these slow manifolds in which they are normally hyperbolic, and we make coordinate transformations to put the equations in a preliminary Fenichel normal form which identifies the stable/unstable subspaces and corresponding contraction/expansion rates.

The ultimate goal is to show that the manifold $\mathcal{B}(s; c, a)$ is well defined and exponentially close to $\gamma^f(s; c, a)$ in $\Sigma^{h,r}$ for each $(c, a) \in I_c \times I_a$. The existence of the solution $\gamma^f(s; c, a)$ for $(c, a) \in I_c \times I_a$ is clear; however, it is not immediately obvious that the fiber of this solution in the section Σ^m is exponentially contracted to $\gamma^f(s; c, a)$ in backwards time to $\Sigma^{h,r}$. Along the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$, this is clear as this fiber is defined by the fact that it contracts exponentially to $\gamma^f(s; c, a)$ in backwards time. However, after passing near the fold, in backwards time, $\gamma^f(s; c, a)$ is near the slow manifold $\mathcal{M}_\varepsilon^r(c, a)$ and solutions near $\gamma^f(s; c, a)$ undergo exponential expansion. We claim that the contraction along $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ compensates for this expansion.

We proceed by determining the balance of contraction/expansion along the slow manifolds $\mathcal{M}_\varepsilon^{r,\ell}(c, a)$ in backwards time from Σ^m to $\Sigma^{h,r}$. We break this into three pieces: first the transition from Σ^m to Σ^{out} , where $\gamma^f(s; c, a)$ exits a neighborhood $\mathcal{U}_F$ of the upper right fold point along the fast jump φ_b , second the transition from Σ^{out} to $\Sigma^{i,-}$ encompassing the passage near the fold point, and finally the transition from $\Sigma^{i,-}$ to $\Sigma^{h,r}$ describing the passage near the right slow manifold $\mathcal{M}_\varepsilon^r(c, a)$.

We first follow $\gamma^f(s; c, a)$ backwards from Σ^m into a neighborhood of $\mathcal{M}_\varepsilon^\ell(c, a)$

at a height $w = \Delta_w$, so that we are away from the lower fold point. By construction $\gamma^f(s; c, a)$ lies in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ and remains in this neighborhood of $\mathcal{M}_\varepsilon^\ell(c, a)$ until some height $w = w^\dagger + \mathcal{O}(\varepsilon^{2/3}, a)$ corresponding to the fast jump to Σ^{out} in the neighborhood $\mathcal{U}_F$ of the upper right fold point. During this entire passage, solutions corresponding to the fiber $\{(0, y_2^f(s; c, a), z) : |z| \leq \Delta_z\}$ in the section Σ^m are contracted exponentially to $\gamma^f(s; c, a)$ in backwards time, and hence we have the following.

Lemma 5.4.6. *For each sufficiently small Δ_w , there exists $\Delta > 0, \varepsilon_0 > 0$ and sufficiently small choice of the intervals I_c, I_a , such that for each $0 < \varepsilon < \varepsilon_0$, each $(c, a) \in I_c \times I_a$, and each $s \in (w^\dagger + \Delta_w, 2w^\dagger - \Delta_w)$, the following holds. The backwards evolution $\mathcal{B}(s; c, a)$ of the fiber $\{(0, y_2^f(s; c, a), z) : |z| \leq \Delta_z\}$ in Σ^m reaches the section Σ^{out} near the upper right fold point in a curve which is $\mathcal{O}\left(e^{\Lambda^\ell(\Delta_w, w^\dagger - \Delta_w)}\right)$ close to $\gamma^f(s; c, a)$ uniformly in $(c, a, z_B) \in I_c \times I_a \times [-\Delta_z, \Delta_z]$ where*

$$\begin{aligned} \Lambda^\ell(\Delta_w, w^\dagger - \Delta_w) &= \int_{u_1(\Delta_w)}^{u_1(w^\dagger - \Delta_w)} \frac{c + \sqrt{c^2 - 4f'(u)}}{2\varepsilon(u - \gamma f(u))} f'(u) du \\ &< 0. \end{aligned} \tag{4.36}$$

Furthermore the derivatives of the transition map from Σ^m to Σ^{out} for solutions on $\mathcal{B}(s; c, a)$ with respect to (c, a, z_B) are also $\mathcal{O}\left(e^{\Lambda^\ell(\Delta_w, w^\dagger - \Delta_w)}\right)$.

Proof. To see this, we consider the flow in a neighborhood of $\mathcal{M}_\varepsilon^\ell(c, a)$; essentially we perform coordinate transformations to explicitly determine the expansion along $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ away from the fold at the origin. Away from the origin, we can parametrize $\mathcal{M}_\varepsilon^\ell(c, a)$ by w , that is, the slow manifold $\mathcal{M}_\varepsilon^\ell(c, a)$ is given as a graph

$$\begin{aligned} u &= H(w, \varepsilon) = f^{-1}(w) + \varepsilon h(w, \varepsilon) \\ v &= G(w, \varepsilon) = \varepsilon g(w, \varepsilon), \end{aligned} \tag{4.37}$$

where we take the negative root $u_1(w)$ for $f^{-1}(w)$, and the functions H, G satisfy

$$\begin{aligned}\varepsilon D_w H(w, \varepsilon)(H(w, \varepsilon) - \gamma w) &= G(w, \varepsilon) \\ \varepsilon D_w G(w, \varepsilon)(H(w, \varepsilon) - \gamma w) &= cG(w, \varepsilon) - f(H(w, \varepsilon)) + w,\end{aligned}\tag{4.38}$$

and the flow on $\mathcal{M}_\varepsilon^\ell(c, a)$ is given by

$$\dot{w} = \varepsilon(H(w, \varepsilon) - \gamma w).\tag{4.39}$$

We now write

$$\begin{aligned}u &= \tilde{u} + H(w, \varepsilon) \\ v &= \tilde{v} + G(w, \varepsilon),\end{aligned}\tag{4.40}$$

and compute the flow nearby for small $\tilde{u}, \tilde{v}$ as

$$\begin{aligned}\dot{\tilde{u}} &= \tilde{v} - \varepsilon \tilde{u} D_w H(w, \varepsilon) \\ \dot{\tilde{v}} &= c\tilde{v} - \tilde{u} f'(H(w, \varepsilon)) - \varepsilon \tilde{u} D_w G(w, \varepsilon) + \mathcal{O}(\tilde{u}^2) \\ \dot{w} &= \varepsilon(\tilde{u} + H(w, \varepsilon) - \gamma w).\end{aligned}\tag{4.41}$$

We consider the linearization of the two dimensional $(\tilde{u}, \tilde{v})$ system about $(\tilde{u}, \tilde{v}, \varepsilon) = (0, 0, 0)$ for each w . There is one stable and one unstable eigenvalue

$$\lambda^\pm = \frac{c \pm \sqrt{c^2 - 4f'(f^{-1}(w))}}{2},\tag{4.42}$$

with corresponding eigenvectors

$$e^\pm = \begin{pmatrix} 1 \\ \lambda^\pm \end{pmatrix}.\tag{4.43}$$

We now introduce the coordinates

$$\begin{aligned} U &= \tilde{v} - \lambda^+ \tilde{u} \\ V &= \tilde{v} - \lambda^- \tilde{u}, \end{aligned} \tag{4.44}$$

which, using the identities

$$\begin{aligned} \lambda^+ \lambda^- &= f'(f^{-1}(w)) \\ \lambda^\pm &= c - \lambda^\mp, \end{aligned} \tag{4.45}$$

results in the system

$$\begin{aligned} \dot{U} &= \lambda^- U + F^-(U, V, w, \varepsilon) \\ \dot{V} &= \lambda^+ V + F^+(U, V, w, \varepsilon) \\ \dot{w} &= \varepsilon(f^{-1}(w) - \gamma w + F^s(U, V, w, \varepsilon)), \end{aligned} \tag{4.46}$$

where

$$\begin{aligned} F^\pm(U, V, w, \varepsilon) &= \mathcal{O}(\varepsilon U, \varepsilon V, U^2, UV, V^2) \\ F^s(U, V, w, \varepsilon) &= \mathcal{O}(U, V, \varepsilon). \end{aligned} \tag{4.47}$$

We now identify the part of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ which intersects this neighborhood as a graph $V = V^*(U, w, \varepsilon)$. This manifold is foliated by strong unstable fibers tangent to lines $(U, w) = \text{const}$ for $\varepsilon = 0$. Setting $\tilde{V} = V - V^*(U, w, \varepsilon)$ and performing a coordinate change

$$\begin{pmatrix} U \\ w \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{U} \\ \tilde{W} \end{pmatrix} = \begin{pmatrix} U \\ w \end{pmatrix} + \mathcal{O}(\tilde{V}), \tag{4.48}$$

to straighten out the unstable fibers, we arrive at the system

$$\begin{aligned}
\dot{\tilde{U}} &= \lambda^- \tilde{U} + \tilde{F}^-(\tilde{U}, \tilde{W}, \varepsilon) \\
\dot{\tilde{V}} &= \lambda^+ \tilde{V} + \tilde{F}^+(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) \tilde{V} \\
\dot{\tilde{W}} &= \varepsilon(f^{-1}(\tilde{W}) - \gamma \tilde{W} + \tilde{F}^s(\tilde{U}, \tilde{W}, \varepsilon)),
\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}
\tilde{F}^-(\tilde{U}, \tilde{W}, \varepsilon) &= \mathcal{O}(\tilde{U}^2, \varepsilon \tilde{U}) \\
\tilde{F}^+(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) &= \mathcal{O}(\tilde{U}, \tilde{V}, \varepsilon) \\
\tilde{F}^s(\tilde{U}, \tilde{W}, \varepsilon) &= \mathcal{O}(\tilde{U}, \varepsilon).
\end{aligned} \tag{4.50}$$

We can now estimate the contraction rate $\Lambda^\ell(\tilde{W}_1, \tilde{W}_2)$ in backwards time along the fiber of a given trajectory lying on $\mathcal{W}_\varepsilon^{\ell}(c, a)$ between heights $\tilde{W}_1$ and $\tilde{W}_2$, under the assumption that this trajectory remains in a small neighborhood of $\mathcal{M}_\varepsilon^\ell(c, a)$, say $|\tilde{U}|, |\tilde{V}| \leq \Delta \ll 1$, for $\tilde{W} \in [\tilde{W}_1, \tilde{W}_2]$. We compute

$$\begin{aligned}
\Lambda^\ell(\tilde{W}_1, \tilde{W}_2) &= \int_{\tilde{W}_1}^{\tilde{W}_2} \frac{\lambda^+ + \tilde{F}^+(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon)}{\varepsilon(f^{-1}(\tilde{W}) - \gamma \tilde{W} + \tilde{F}^s(\tilde{U}, \tilde{W}, \varepsilon))} d\tilde{w} \\
&= \int_{\tilde{W}_1}^{\tilde{W}_2} \frac{\lambda^+}{\varepsilon(f^{-1}(\tilde{W}) - \gamma \tilde{W})} (1 + \mathcal{O}(\varepsilon, \Delta)) d\tilde{W} \\
&= \int_{\tilde{W}_1}^{\tilde{W}_2} \frac{c + \sqrt{c^2 - 4f'(f^{-1}(\tilde{W}))}}{2\varepsilon(f^{-1}(\tilde{W}) - \gamma \tilde{W})} (1 + \mathcal{O}(\varepsilon, \Delta)) d\tilde{W} \\
&= \int_{u_1(\tilde{W}_1)}^{u_1(\tilde{W}_2)} \frac{c + \sqrt{c^2 - 4f'(u)}}{2\varepsilon(u - \gamma f(u))} f'(u) (1 + \mathcal{O}(\varepsilon, \Delta)) du.
\end{aligned} \tag{4.51}$$

Hence, by fixing $\Delta_w > 0$ small, and taking $\Delta, \varepsilon > 0$ sufficiently small, we obtain the result. $\square$

We proceed by considering the flow near the upper right fold point. Using the analysis in [8, 38], it is clear that the transition in backwards time from Σ^{out} to $\Sigma^{i,-}$

in the neighborhood $\mathcal{U}_F$ of the upper right fold point can be bounded by $e^{\eta/\varepsilon}$ for each $\eta > 0$ by taking the neighborhood $\mathcal{U}_F$ sufficiently small, that is, by shrinking Δ_w . The derivatives of the transition map also satisfy the same bounds.

Finally, we consider the transition from $\Sigma^{i,-}$ to $\Sigma^{h,r}$. We first prove the following technical lemma.

Lemma 5.4.7. *For each sufficiently small Δ_w and $(c, a) \in I_c \times I_a$, we have that*

$$\int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c + \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du + \int_{u_3(-\Delta_w)}^{u_3(w^\dagger)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du > 0. \quad (4.52)$$

Proof. We first write

$$\begin{aligned} \int_{u_3(-\Delta_w)}^{u_3(w^\dagger)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du \\ = \int_{u_3(\Delta_w)}^{u_3(w^\dagger - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du + \mathcal{O}(\Delta_w). \end{aligned} \quad (4.53)$$

Hence it suffices to show that there exists $C > 0$ such that

$$\int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c + \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du + \int_{u_3(\Delta_w)}^{u_3(w^\dagger - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du > C, \quad (4.54)$$

for $(c, a) = (1/\sqrt{2}, 0)$ uniformly in $\Delta_w > 0$ sufficiently small; the result then follows by continuity provided Δ_w and the intervals I_c, I_a are sufficiently small. For $(c, a) = (1/\sqrt{2}, 0)$, we have that $w^\dagger = \frac{4}{27}$, and the following identities hold for each $w \in (0, w^\dagger)$

and $u < 0$.

$$\begin{aligned}
u_3(w) &= \frac{2}{3} - u_1(w^\dagger - w) \\
f(u) &= \frac{4}{27} - f\left(\frac{2}{3} - u\right) \\
f'(u) &= f'\left(\frac{2}{3} - u\right).
\end{aligned} \tag{4.55}$$

Hence

$$\begin{aligned}
\int_{u_3(\Delta_w)}^{u_3(w^\dagger - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du &= \int_{\frac{2}{3} - u_1(w^\dagger - \Delta_w)}^{\frac{2}{3} - u_1(\Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du \\
&= - \int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c - \sqrt{c^2 - 4f'\left(\frac{2}{3} - u\right)}}{\frac{2}{3} - u - \gamma f\left(\frac{2}{3} - u\right)} f'\left(\frac{2}{3} - u\right) du \\
&= \int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u) + \frac{4\gamma}{27} - \frac{2}{3}} f'(u) du \\
&> \int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du,
\end{aligned} \tag{4.56}$$

since $0 < \gamma < 4$. We therefore have that

$$\begin{aligned}
&\int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c + \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du + \int_{u_3(\Delta_w)}^{u_3(w^\dagger - \Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du \\
&> \int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c + \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du + \int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c - \sqrt{c^2 - 4f'(u)}}{u - \gamma f(u)} f'(u) du \\
&= \int_{u_1(w^\dagger - \Delta_w)}^{u_1(\Delta_w)} \frac{c}{u - \gamma f(u)} f'(u) du \\
&> C,
\end{aligned} \tag{4.57}$$

uniformly in $\Delta_w > 0$ sufficiently small, which completes the proof. $\square$

In combination with the above results, we now have the following.

Lemma 5.4.8. *For each sufficiently small Δ_w , there exists $\varepsilon_0, q > 0$ and sufficiently small choice of the intervals I_c, I_a , such that for each $0 < \varepsilon < \varepsilon_0$, each $(c, a) \in I_c \times I_a$, and each $s \in (w^\dagger + \Delta_w, 2w^\dagger - \Delta_w)$, the following holds. In $\Sigma^{h,r}$, $\mathcal{B}(s; c, a)$ is given by a set of points (u, v, w) in $\Sigma^{h,r}$ satisfying*

$$(u, v, w) = (2/3, v^f(s; c, a), 2w^\dagger - s) + (0, v_B(z_B, s; c, a), w_B(z_B, s; c, a)), \quad (4.58)$$

where

$$|v_B(z_B, s; c, a)|, |w_B(z_B, s; c, a)| = \mathcal{O}(e^{-q/\varepsilon}), \quad (4.59)$$

along with their derivatives with respect to (z_B, c, a) uniformly in $|z_B| \leq \Delta_z$ and $(c, a) \in I_c \times I_a$.

Proof. In a neighborhood of $\mathcal{M}_\varepsilon^r(c, a)$, we can put the flow into the Fenichel normal form

$$\begin{aligned} \dot{U} &= -\lambda^- U + F^-(U, V, \tilde{w}, \varepsilon)U \\ \dot{V} &= -\lambda^+ V + F^+(U, V, \tilde{w}, \varepsilon)V \\ \dot{\tilde{w}} &= \varepsilon(-f^{-1}(\tilde{w}) + \gamma\tilde{w} + F^{sl}(U, V, \tilde{w}, \varepsilon)), \end{aligned} \quad (4.60)$$

where

$$\begin{aligned} F^-(U, V, \tilde{w}, \varepsilon) &= \mathcal{O}(U, V, \varepsilon) \\ F^+(U, V, \tilde{w}, \varepsilon) &= \mathcal{O}(U, V, \varepsilon) \\ F^{sl}(U, V, \tilde{w}, \varepsilon) &= \mathcal{O}(UV, \varepsilon), \end{aligned} \quad (4.61)$$

$$\lambda^\pm = \frac{c \pm \sqrt{c^2 - 4f'(f^{-1}(\tilde{w}))}}{2}, \quad (4.62)$$

and $f^{-1}(\tilde{w})$ refers to the largest root $u_3(\tilde{w})$ of $f(u) = \tilde{w}$. We note that the flow is now in *backwards* time. By construction, up to a reparameterization of $\gamma^f(s; c, a)$ according to the smooth coordinate transformation $(u, v, w) \rightarrow (U, V, \tilde{w})$, in backwards time $\gamma^f(s; c, a)$ exits at height $\tilde{w} = 2w^\dagger - s$. Between $\tilde{w} = 2w^\dagger - s$ and $\tilde{w} = w^\dagger - \Delta_w$, $\gamma^f(s; c, a)$ is given as a solution

$$\begin{aligned} U &= U^f(t; c, a, \varepsilon) \\ V &= V^f(t; c, a, \varepsilon) \\ \tilde{w} &= \tilde{w}^f(t; c, a, \varepsilon) \end{aligned} \quad (4.63)$$

where $|U^f|, |V^f| \leq \Delta$ for $\tilde{w} \in (2w^\dagger - s, w^\dagger - \Delta_w)$. We now obtain estimates on this solution and its derivatives. We first recall/comment on how the solution $\gamma^f(s; c, a)$ is constructed.

For a given value of (s, c, a) , $\gamma^f(s; c, a)$ is defined as the unique transverse intersection of the forward evolution of the line $\{u = 2/3, w = 2w^\dagger - s\}$ with the manifold $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$. Equivalently, for the same effect we could have worked in this Fenichel neighborhood of $\mathcal{M}_\varepsilon^r(c, a)$ and considered constructing $\gamma^f(s; c, a)$ as the unique transverse intersection of the forward evolution of the line $\{U = \Delta, |V| \leq \Delta, \tilde{w} = 2w^\dagger - s\}$ with the manifold $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$. Using arguments similar to those in [8, §5] in the proof of Proposition 5.3.1, we obtain the solution $\gamma^f(s; c, a) = (U^f, V^f, \tilde{w}^f)$ which satisfies $(U^f, V^f) = (\mathcal{O}(e^{-q/\varepsilon}), \mathcal{O}(e^{-\eta/\varepsilon}))$ at $\tilde{w} = w^\dagger - \Delta_w$ and $(U^f, V^f) = (\Delta, \mathcal{O}(e^{-q/\varepsilon}))$ at $\tilde{w} = 2w^\dagger - s$, where $q > \eta > 0$. Furthermore, the derivatives with respect to (c, a) of these boundary values satisfy similar bounds, where q, η may need to be taken slightly smaller.

We now obtain more precise bounds for this solution and its derivatives. We write $\tilde{w} = w^* + W$ where $w^*(t)$ is the solution to

$$\dot{\tilde{w}} = \varepsilon(-f^{-1}(\tilde{w}) + \gamma\tilde{w} + F^{sl}(0, 0, \tilde{w}, \varepsilon)), \quad (4.64)$$

satisfying $\tilde{w}(0) = w^\dagger - \Delta_w$, $\tilde{w}(T) = 2w^\dagger - s$, where we note that $\varepsilon/C < T < C/\varepsilon$ for some $C > 0$. This results in the equations

$$\begin{aligned} \dot{U} &= \Lambda^u(t)U + G^-(U, V, W, \varepsilon)U \\ \dot{V} &= -\Lambda^s(t)V + G^+(U, V, W, \varepsilon)V \\ \dot{W} &= \varepsilon(-(f^{-1})'(w^*)W + \gamma W + F_{\tilde{w}}^{sl}(0, w^*, \varepsilon)W + G^{sl}(U, V, W, \varepsilon)), \end{aligned} \quad (4.65)$$

where

$$\begin{aligned} \Lambda^u(t) &= -\lambda^-(w^*(t)) + \mathcal{O}(\varepsilon) \\ \Lambda^s(t) &= \lambda^+(w^*(t)) + \mathcal{O}(\varepsilon) \\ G^-(U, W, \varepsilon) &= \mathcal{O}(U, V, W) \\ G^+(U, V, W, \varepsilon) &= \mathcal{O}(U, V, W) \\ G^{sl}(U, V, W, \varepsilon) &= \mathcal{O}(UV, W^2). \end{aligned} \quad (4.66)$$

We now define for each sufficiently small $\delta > 0$ the functions

$$\begin{aligned} \beta_\delta^-(t, s) &= \int_s^t \Lambda^u(\tau) - \delta d\tau \\ \beta_\delta^+(t, s) &= \int_s^t -\Lambda^s(\tau) + \delta d\tau \\ \beta^{sl}(t, s) &= \varepsilon \int_s^t -(f^{-1})'(w^*(\tau)) + \gamma + F_{\tilde{w}}^{sl}(0, w^*(\tau), \varepsilon) d\tau. \end{aligned} \quad (4.67)$$

Hence the solution $\gamma^f(s; c, a)$ given by (U^f, V^f, W^f) , $W^f = w^f - w^*$, solves

$$\begin{aligned}
U(t) &= e^{\beta_0^-(t,T)} \Delta + \int_T^t e^{\beta_0^-(t,s)} G^-(U(s), V(s), W(s), \varepsilon) U(s) ds \\
&:= \mathcal{F}^-(U, V, W, \Delta, V_0; c, a)(t) \\
V(t) &= e^{\beta_0^+(t,0)} V_0 + \int_0^t e^{\beta_0^+(t,s)} G^+(U(s), V(s), W(s), \varepsilon) V(s) ds \\
&:= \mathcal{F}^+(U, V, W, \Delta, V_0; c, a)(t) \\
W(t) &= \int_T^t \varepsilon e^{\beta^{sl}(t,s)} G^{sl}(U(s), V(s), W(s), \varepsilon) ds \\
&:= \mathcal{F}^{sl}(U, V, W, \Delta, V_0; c, a)(t).
\end{aligned} \tag{4.68}$$

We define the spaces

$$\begin{aligned}
V_\delta^- &= \left\{ U : [0, T] \rightarrow \mathbb{R}^2 : \|U\|_\delta^- = \sup_{t \in [0, T]} e^{\beta_\delta^-(T,t)} |U(t)| < \infty \right\} \\
V_\delta^+ &= \left\{ V : [0, T] \rightarrow \mathbb{R} : \|V\|_\delta^+ = \sup_{t \in [0, T]} e^{\beta_\delta^+(0,t)} |V(t)| < \infty \right\} \\
V^{sl} &= \left\{ W : [0, T] \rightarrow \mathbb{R} : \|W\|^{sl} = \sup_{t \in [0, T]} |W(t)| < \infty \right\},
\end{aligned} \tag{4.69}$$

and for each fixed small $\delta > 0$ we have that

$$\begin{aligned}
\|\mathcal{F}^-(U, V, W, \Delta, V_0; c, a)\|_\delta^- &= \Delta + \mathcal{O}(\|U\|_\delta^- (\|U\|_\delta^- + \|V\|_\delta^+ + \|W\|^{sl})) \\
\|\mathcal{F}^+(U, V, W, \Delta, V_0; c, a)\|_\delta^+ &= V_T + \mathcal{O}(\|V\|_\delta^+ (\|U\|_\delta^- + \|V\|_\delta^+ + \|W\|^{sl})) \\
\|\mathcal{F}^{sl}(U, V, W, \Delta, V_0; c, a)\|^{sl} &= \mathcal{O}((\|W\|^{sl})^2, \|U\|_\delta^- \|V\|_\delta^+),
\end{aligned} \tag{4.70}$$

and hence $\gamma^f(s; c, a)$ satisfies

$$\begin{aligned}
\|U^f\|_\delta^- &= \mathcal{O}(\Delta) \\
\|V^f\|_\delta^+ &= \mathcal{O}(V_0) = \mathcal{O}(e^{-\eta/\varepsilon}) \\
\|W^f\|^{sl} &= \mathcal{O}(e^{-\eta/\varepsilon}).
\end{aligned} \tag{4.71}$$

Taking derivatives of (4.68) with respect to the parameters (c, a) and taking δ slightly larger and η slightly smaller if necessary, we can bound the derivatives

$$\begin{aligned}\|D_\nu U^f\|_\delta^- &= \mathcal{O}(\Delta) \\ \|D_\nu V^f\|_\delta^+ &= \mathcal{O}(e^{-\eta/\varepsilon}) \\ \|D_\nu W^f\|^{sl} &= \mathcal{O}(e^{-\eta/\varepsilon}),\end{aligned}\tag{4.72}$$

for $\nu = (c, a)$.

To determine the contraction/expansion of solutions along $\gamma^f(s; c, a)$, we write

$$\begin{aligned}U &= U^f(t; c, a, \varepsilon) + \tilde{U} \\ V &= V^f(t; c, a, \varepsilon) + \tilde{V} \\ W &= W^f(t; c, a, \varepsilon) + \tilde{W}\end{aligned}\tag{4.73}$$

and obtain the equations

$$\begin{aligned}\dot{\tilde{U}} &= \Lambda^u \tilde{U} + \tilde{G}_1^-(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) \tilde{U} + \tilde{G}_2^-(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) U^f \\ \dot{\tilde{V}} &= -\Lambda^s \tilde{V} + \tilde{G}_1^+(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) \tilde{V} + \tilde{G}_2^+(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) V^f \\ \dot{\tilde{W}} &= \varepsilon(-(f^{-1})'(w^*) \tilde{W} + \gamma \tilde{W} + F_w^{sl}(0, w^*, \varepsilon) \tilde{W} + \tilde{G}^{sl}(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon)),\end{aligned}\tag{4.74}$$

where

$$\begin{aligned}\tilde{G}_1^-(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) &= \mathcal{O}(U^f, V^f, W^f, \tilde{U}, \tilde{V}, \tilde{W}) \\ \tilde{G}_2^-(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) &= \mathcal{O}(\tilde{U}, \tilde{V}, \tilde{W}) \\ \tilde{G}_1^+(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) &= \mathcal{O}(U^f, V^f, W^f, \tilde{U}, \tilde{V}, \tilde{W}) \\ \tilde{G}_2^+(\tilde{U}, \tilde{V}, \tilde{W}, \varepsilon) &= \mathcal{O}(\tilde{U}, \tilde{V}, \tilde{W}) \\ \tilde{G}^{sl}(\tilde{U}, \tilde{W}, \varepsilon) &= \mathcal{O}(\tilde{U} V^f, U^f \tilde{V}, \tilde{U} \tilde{V}, \tilde{W} W^f, \tilde{W}^2).\end{aligned}\tag{4.75}$$

We can write this as the integral equation

$$\begin{aligned}
\tilde{U}(t) &= e^{\beta_0^-(t,T)} \tilde{U}_T \\
&\quad + \int_T^t e^{\beta_0^-(t,s)} \tilde{G}_1^-(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), \varepsilon) \tilde{U}(s) + \tilde{G}_2^-(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), \varepsilon) U^f(s) ds \\
\tilde{V}(t) &= e^{\beta_0^+(t,0)} \tilde{V}_0 \\
&\quad + \int_0^t e^{\beta_0^+(t,s)} \tilde{G}_1^+(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), \varepsilon) \tilde{V}(s) + \tilde{G}_2^+(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), \varepsilon) V^f(s) ds \\
\tilde{W}(t) &= e^{\beta^{sl}(t,0)} \tilde{W}_0 + \int_0^t \varepsilon e^{\beta^{sl}(t,s)} \tilde{G}^{sl}(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), \varepsilon) ds.
\end{aligned} \tag{4.76}$$

Provided $|\tilde{U}_T|$, $|\tilde{V}_0|$, and $|\tilde{W}_0|$ are sufficiently small, we can solve this by the implicit function theorem and obtain a solution satisfying

$$\begin{aligned}
\|\tilde{U}\|_\delta^- &= \mathcal{O}\left(|\tilde{U}_T| + \Delta(|\tilde{V}_0| + |\tilde{W}_0|)\right) \\
\|\tilde{V}\|_\delta^+ &= \mathcal{O}\left(|\tilde{V}_0| + e^{-\eta/\varepsilon}(|\tilde{U}_T| + |\tilde{W}_0|)\right) \\
\|\tilde{W}\|^{sl} &= \mathcal{O}\left(|\tilde{W}_0| + e^{-\eta/\varepsilon}|\tilde{U}_T| + \Delta|\tilde{V}_0|\right).
\end{aligned} \tag{4.77}$$

Taking derivatives of (4.76) with respect to the parameters (c, a) and taking δ slightly larger if necessary, we can bound the derivatives

$$\begin{aligned}
\|D_\nu \tilde{U}\|_\delta^- &= \mathcal{O}\left(|\tilde{U}_T| + |D_\nu \tilde{U}_T| + \Delta(|\tilde{V}_0| + |D_\nu \tilde{V}_0| + |\tilde{W}_0| + |D_\nu \tilde{W}_0|)\right) \\
\|D_\nu \tilde{V}\|_\delta^+ &= \mathcal{O}(|\tilde{V}_0| + |D_\nu \tilde{V}_0| + e^{-\eta/\varepsilon}(|\tilde{U}_T| + |D_\nu \tilde{U}_T| + |\tilde{W}_0| + |D_\nu \tilde{W}_0|)) \\
\|D_\nu \tilde{W}\|^{sl} &= \mathcal{O}\left(|\tilde{W}_0| + |D_\nu \tilde{W}_0| + e^{-\eta/\varepsilon}(|\tilde{U}_T| + |D_\nu \tilde{U}_T|) + \Delta(|\tilde{V}_0| + |D_\nu \tilde{V}_0|)\right),
\end{aligned} \tag{4.78}$$

for $\nu = (z_B, c, a)$.

By Lemma 5.4.6, at $t = 0$ the manifold $\mathcal{B}(s; c, a)$ is $\mathcal{O}(e^{\Lambda^\ell(\Delta_w, w^\dagger - \Delta_w)})$ -close to $\gamma^f(s; c, a)$. That is, the solutions on $\mathcal{B}(s; c, a)$ can be represented in the above coor-

dinates by solutions $(\tilde{U}, \tilde{V}, \tilde{W})$ satisfying

$$(\tilde{U}, \tilde{V}, \tilde{W})(0) = (\tilde{U}_0^B, \tilde{V}_0^B, \tilde{W}_0^B) = \mathcal{O}(e^{\Lambda^\ell(\Delta_w, w^\dagger - \Delta_w)})$$

uniformly along with their derivatives with respect to (z_B, c, a) . We now solve for the solution to (4.76) which satisfies $(\tilde{V}_0, \tilde{W}_0) = (\tilde{V}_0^B, \tilde{W}_0^B)$ and

$$\begin{aligned} \tilde{U}_0^B &= e^{\beta_0^-(0,T)} \tilde{U}_T \\ &+ \int_T^0 e^{\beta_0^-(0,s)} \tilde{G}_1^-(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), \varepsilon) \tilde{U}(s) + \tilde{G}_2^-(\tilde{U}(s), \tilde{V}(s), \tilde{W}(s), \varepsilon) U^f(s) ds. \end{aligned} \quad (4.79)$$

Provided $e^{\beta_0^-(T,0)} \tilde{U}_0^B$ is sufficiently small, we can find a solution $\tilde{U}_T$ satisfying (4.79). Performing a similar computation as in the proof of Lemma 5.4.6 shows that the expansion $\beta_0^-(T, 0)$ in backwards time from $w = w^\dagger - \Delta_w$ to $w = 2w^\dagger - s$ can be estimated by

$$\Lambda^r(w^\dagger - \Delta_w, 2w^\dagger - s) = \int_{u_3(w^\dagger - \Delta_w)}^{u_3(2w^\dagger - s)} \frac{c - \sqrt{c^2 - 4f'(u)}}{2\varepsilon(u - \gamma f(u))} (1 + \mathcal{O}(\varepsilon, \Delta)) f'(u) du. \quad (4.80)$$

Using this in combination with the $\mathcal{O}(e^{\Lambda^\ell(\Delta_w, w^\dagger - \Delta_w)})$ bounds on $\tilde{U}_0^B, \tilde{V}_0^B, \tilde{W}_0^B$ for the contraction/expansion from Σ^m to Σ^{out} and Σ^{out} to $\Sigma^{i,-}$ and Lemma 5.4.7, we deduce that $\mathcal{B}(s; c, a)$ is $\mathcal{O}(e^{-q/\varepsilon})$ -close to $\gamma^f(s; c, a)$ in $\Sigma^{h,r}$. $\square$

Matching conditions for pulses $\Gamma(s, \sqrt{\varepsilon})$, $s \in (w^\dagger + \Delta_w, 2w^\dagger - \Delta_w)$

Evolving $\mathcal{B}(s; c, a)$ backwards and using Lemma 5.4.8, we have that $\hat{\mathcal{B}}(s; c, a)$ is exponentially close to $\mathcal{M}_\varepsilon^m(c, a)$ in Σ^m . Thus we have that in Σ^m , $\hat{\mathcal{B}}(s; c, a)$ is given

by a curve $(y_2, z) = (y_2^b, z^b)(z_B, s; c, a)$ which satisfies

$$\begin{aligned} |y_2^b(z_B, s; c, a) - y_2^{\mathcal{M}, m}| &= \mathcal{O}(e^{-q/\varepsilon}) \\ |z^b(z_B, s; c, a)| &= \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \quad (4.81)$$

uniformly in $|z_B| \leq \Delta_z$.

Recall from §5.4.3 that in the section Σ^m , the tail manifold $\mathcal{T}_\varepsilon(c, a)$ is defined by the graph of a smooth function given by $z = z_\varepsilon^T(y_2; c, a)$ for $y_2 \geq y_{2,0}^{\ell,*}(c, a)$ where

$$z_\varepsilon^T(y_2; c, a) = \begin{cases} 0 & y_2 \geq \hat{y}_{2,0}^\ell(c, a) \\ z^{\ell,*}(y_2; c, a) & y_{2,0}^{\ell,*}(c, a) \leq y_2 \leq \hat{y}_{2,0}^\ell(c, a) \end{cases} \quad (4.82)$$

and the function $z^{\ell,*}$ and its derivatives are $\mathcal{O}(e^{-q/\varepsilon})$. We have the following analogue of Lemma 5.4.5 which will be proved in §5.6.

Lemma 5.4.9. *For each sufficiently small $\Delta_w > 0$, there exists $\varepsilon_0, \Delta_z > 0$ and sufficiently small choice of the intervals I_c, I_a such that for each $s \in (w^\dagger + \Delta_w, 2w^\dagger - \Delta_w)$ and each $0 < \varepsilon < \varepsilon_0$, the backwards evolution of the manifold $\widehat{\mathcal{B}}(s; c, a)$ intersects Σ^m in a curve $(y_2, z) = (y_2^b, z^b)(z_B, s; c, a)$ which satisfies*

$$\begin{aligned} |y_2^b(z_B, s; c, a) - y_2^{\mathcal{M}, m}| &= \mathcal{O}(e^{-q/\varepsilon}) \\ |z^b(z_B, s; c, a)| &= \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \quad (4.83)$$

uniformly in $|z_B| \leq \Delta_z$ and $(c, a) \in I_c \times I_a$. The derivatives of the above expressions with respect to (c, a, z_B) are also $\mathcal{O}(e^{-q/\varepsilon})$. Furthermore

$$y_{2,0}^{\ell,*}(c, a) < \inf_{|z_B| \leq \Delta_z} y_2^b(z_B, s; c, a), \quad (4.84)$$

for all $(c, a) \in I_c \times I_a$.

The first assertion of Lemma 5.4.9 follows from the analysis above; the proof of the second assertion will be given in §5.6.

The final matching conditions can then be obtained analogously as in the case of type 2 pulses. Starting in $\Sigma^{h,r}$, we evolve $\widehat{\mathcal{B}}(s; c, a)$ back to the section Σ^m and show that for each $|z_B| \leq \Delta_z$, $\mathcal{W}_\varepsilon^u(0; c, a)$ can be matched with the corresponding solution on $\widehat{\mathcal{B}}(s; c, a)$ by adjusting (c, a) using Lemma 5.4.9. We then evolve $\mathcal{B}(s; c, a)$ forwards from $\Sigma^{h,r}$ to Σ^m and show that $\mathcal{B}(s; c, a)$ transversely intersects $\mathcal{T}_\varepsilon(c, a)$ for each such (c, a) as z_B varies. Convergence of the tails is proved in §5.5. The setup for the matching conditions in the section Σ^m is shown in Figures 5.10 and 5.12.

From Proposition 5.4.1, we have that the distance between the manifolds $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{M}_\varepsilon^m(c, a)$ in Σ^m is given by

$$y_2^{\mathcal{M},\ell} - y_2^{\mathcal{M},m} = \mathcal{D}_0(\alpha_2, r_2; c) = d_{\alpha_2}\alpha_2 + d_{r_2}r_2 + \mathcal{O}(2) . \quad (4.85)$$

Thus we can match $\mathcal{W}_\varepsilon^u(0; c, a)$ with any solution in $\widehat{\mathcal{B}}(s; c, a)$ by solving

$$\begin{aligned} \mathcal{D}_0(\alpha_2, r_2; c) &= y_2^u(z; c, a) - y_2^{\mathcal{M},\ell} - \left(y_2^b(z_B, s; c, a) - y_2^{\mathcal{M},m} \right) \\ z &= z^b(z_B, s; c, a), \end{aligned} \quad (4.86)$$

for each $|z_B| \leq \Delta_z$. For each such z_B , we obtain a solution by solving

$$\begin{aligned} a &= 2c^{1/2}\varepsilon^{1/2}(\alpha_2^c + \mathcal{O}(e^{-q/\varepsilon})) \\ c &= \check{c}(a, \varepsilon) + \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \quad (4.87)$$

by the implicit function theorem to find $(a, c) = (a^u, c^u)(z_B, s, \sqrt{\varepsilon})$. We now evolve $\mathcal{W}_\varepsilon^u(0; c, a)$ forwards; for each z_B we can hit the corresponding point on $\mathcal{B}(s; c, a)$,

hence $\mathcal{W}_\varepsilon^u(0; c, a)$ intersects Σ^m at the point $(y_2^f(s; c, a), z_B)$ when $(c, a) = (c^u, a^u)(z_B)$ defined above. We now match with the tail manifold $\mathcal{T}_\varepsilon(c, a)$ by solving

$$z_B = z_\varepsilon^T(y_2^f(s; c, a); c^u(z_B), a^u(z_B)), \quad (4.88)$$

which, using Lemma 5.4.9, we can solve by the implicit function theorem when $z_B = z_B^s = \mathcal{O}(e^{-q/\varepsilon})$ to find the desired pulse solution when

$$\begin{aligned} a &= a(s, \sqrt{\varepsilon}) := a^u(z_B^s; s, \sqrt{\varepsilon}) \\ c &= c(s, \sqrt{\varepsilon}) := a^u(z_B^s; s, \sqrt{\varepsilon}). \end{aligned} \quad (4.89)$$

5.4.6 Type 3 pulses

Finally, we consider type 3 pulses. Type 3 pulses are those with secondary heights which are close to the upper right fold point; these pulses encompass the transition from left pulses to right pulses and hence form a bridge between type 2 pulses and type 4 pulses. We construct these in a manner similar to type 2 pulses, but they are not parametrized naturally by the height of the secondary pulse. To set up a parametrization of these pulses, we change to local coordinates in a neighborhood of the upper right fold point [8, §4].

The fold point is given by the fixed point $(u^*, 0, w^*)$ of the layer problem (3.1) where

$$u^* = \frac{1}{3} \left(a + 1 + \sqrt{a^2 - a + 1} \right),$$

and $w^* = f(u^*)$. The linearization of (3.1) about this fixed point has one positive real eigenvalue $c > 0$ and a double zero eigenvalue, since $f'(u^*) = 0$. As in [8] we can

perform for any $r \in \mathbb{Z}_{>0}$ a C^r -change of coordinates $\Phi_\varepsilon: \mathcal{U}_F \rightarrow \mathbb{R}^3$ to (1.1), which is C^r -smooth in c, a and ε for (c, a, ε) -values restricted to the set $I_c \times I_a \times [0, \varepsilon_0]$, where $\varepsilon_0 > 0$ is chosen sufficiently small. We apply Φ_ε in the neighborhood $\mathcal{U}_F$ of the fold point and rescale time by a positive constant θ_0 given by

$$\theta_0 = \frac{1}{c^{2/3}} (a^2 - a + 1)^{1/6} (u^* - \gamma w^*)^{1/3} > 0, \quad (4.90)$$

uniformly in $(c, a) \in I_c \times I_a$, so that (1.1) becomes

$$\begin{aligned} x' &= (y + x^2 + h(x, y, \varepsilon; c, a)), \\ y' &= \varepsilon g(x, y, \varepsilon; c, a), \\ z' &= z \left(\frac{c}{\theta_0} + \mathcal{O}(x, y, z, \varepsilon) \right), \end{aligned} \quad (4.91)$$

where h, g are C^r -functions satisfying

$$\begin{aligned} h(x, y, \varepsilon; c, a) &= \mathcal{O}(\varepsilon, xy, y^2, x^3), \\ g(x, y, \varepsilon; c, a) &= 1 + \mathcal{O}(x, y, \varepsilon), \end{aligned}$$

uniformly in $(c, a) \in I_c \times I_a$. In the transformed system (4.91), the x, y -dynamics is decoupled from the dynamics in the z -direction along the straightened out strong unstable fibers. Thus, the flow is fully described by the dynamics on the two-dimensional invariant manifold $z = 0$ and by the one-dimensional dynamics along the fibers in the z -direction.

We consider the flow of (4.91) on the invariant manifold $z = 0$. We append an

equation for ε , arriving at the system

$$\begin{aligned}x' &= y + x^2 + h(x, y, \varepsilon; c, a), \\y' &= \varepsilon g(x, y, \varepsilon; c, a), \\ \varepsilon' &= 0.\end{aligned}\tag{4.92}$$

For $\varepsilon = 0$, this system possesses a critical manifold given by $\{(x, y) : y + x^2 + h(x, y, 0, c, a) = 0\}$, which in a sufficiently small neighborhood of the origin is shaped as a parabola opening downwards. The branch of this parabola for $x < 0$ is attracting and corresponds to the manifold $\mathcal{M}_0^r$. We define $\mathcal{M}_0^{r,+}$ to be the singular trajectory obtained by appending the fast trajectory given by the line $\{(x, 0) : x > 0\}$ to the attracting branch $\mathcal{M}_0^r$ of the critical manifold. In [8] it was shown that, for sufficiently small $\varepsilon > 0$, $\mathcal{M}_0^{r,+}$ perturbs to a trajectory $\mathcal{M}_\varepsilon^{r,+}$ on $z = 0$, represented as a graph $y = s_\varepsilon(x; c, a)$, which is a -uniformly $\mathcal{C}^0 - \mathcal{O}(\varepsilon^{2/3})$ -close to $\mathcal{M}_0^{r,+}$ (see Figure 5.14 – note that in this figure, x increases to the left). The branch of the critical manifold corresponding to $x > 0$, which we denote by $S_0^-(c, a)$, is repelling and corresponds to the manifold $\mathcal{M}_0^m$ and is normally hyperbolic away from the fold point. Thus by Fenichel theory, this critical manifold persists as an attracting slow manifold $S_\varepsilon^-(c, a)$ for sufficiently small $\varepsilon > 0$ and consists of a single solution. This slow manifold is unique up to exponentially small errors. We will be concerned with trajectories which are exponentially contracted to $S_\varepsilon^-(c, a)$ in backwards time (see Figure 5.14).

Remark 5.4.10. *We use the notation $S_\varepsilon^-(c, a)$ rather than $\mathcal{M}_\varepsilon^m(c, a)$ as in general these manifolds do not coincide. This is due to the fact that the choice of $\mathcal{M}_\varepsilon^m(c, a)$ was made so that $\mathcal{M}_\varepsilon^m(c, a)$ would lie in the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in a neighborhood of the canard point at the origin.*

We determine the location of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in the neighborhood $\mathcal{U}_F$. From [8, §5],

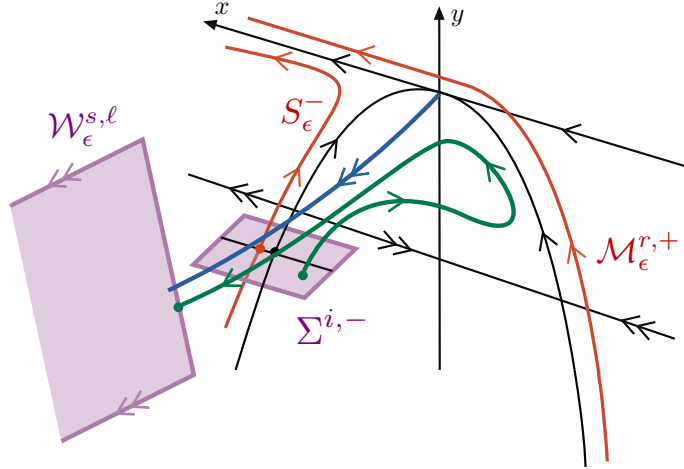


Figure 5.14: Shown is the setup near the upper right fold point. Note that x increases to the left.

we know that $\mathcal{W}^s(\mathcal{M}_0^\ell(c^*(0), 0))$ intersects $\mathcal{W}^u(\mathcal{M}_0^{r,+}(c^*(0), 0))$ transversely for $\varepsilon = 0$ along the Nagumo back φ_b , and this intersection persists for $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$. This means that $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ will transversely intersect the manifold $\mathcal{W}_\varepsilon^{u,r}(c, a)$ which is composed of the union of the unstable fibers of the continuation of the slow manifold $\mathcal{M}_\varepsilon^{r,+}(c, a)$ found in [8, §4]. We recall the exit section Σ^{out} defined by

$$\Sigma^{out} = \{z = \Delta'\}. \quad (4.93)$$

For $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$, the intersection of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ and $\mathcal{W}_\varepsilon^{u,r}(c, a)$ occurs at a point

$$(x, y, z) = (x_\ell(c, a, \varepsilon), s_\varepsilon(x_\ell(c, a, \varepsilon); c, a), \Delta') \in \Sigma^{out}, \quad (4.94)$$

and thus we may expand $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in Σ^{out} as

$$(x, y) = (x_\ell(c, a, \varepsilon) + \mathcal{O}(y - s_\varepsilon(x_\ell(c, a, \varepsilon); c, a), \varepsilon), y), \quad y \in [-\Delta_y, \Delta_y], \quad (4.95)$$

for some small $\Delta_y > 0$. The goal is now to parametrize the construction of type 3 pulses by the coordinate y , which parametrizes trajectories on the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$.

By taking Δ_w sufficiently small, it is clear that there is overlap with the construction of type 2 pulses. We will argue that there is also overlap with the construction of type 4 pulses by considering an appropriate range of y -values.

We now place a section $\Sigma^{i,-}$ defined by

$$\Sigma^{i,-} = \{(x, y, z) : 0 \leq x \leq 2\rho, y = -\rho^2, |z| \leq \Delta'\}, \quad (4.96)$$

and we note that the manifold $S_\varepsilon^-(c, a)$ intersects $\Sigma^{i,-}$ for all sufficiently small $\varepsilon > 0$. We define two other sections

$$\Sigma^{i,+} = \{(x, y, z) : -2\rho \leq x \leq 0, y = -\rho^2, |z| \leq \Delta'\} \quad (4.97)$$

$$\Sigma^{i,v} = \{(x, y, z) : x = -2\rho, |y| \leq \rho^2, |z| \leq \Delta'\}. \quad (4.98)$$

We now note that, provided Δ_y is sufficiently small, any trajectory $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in Σ^{out} leaves a neighborhood of the fold point through one of the sections $\Sigma^{i,-}, \Sigma^{i,+}, \Sigma^{i,v}$ in backwards time. We construct type 3 pulses by considering only heights y which correspond to trajectories on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ which exit via $\Sigma^{i,-}$ in backwards time. The setup is shown in Figure 5.14.

Firstly, all solutions in $\Sigma^{i,-}$ are exponentially contracted in backwards time to $\mathcal{M}_\varepsilon^m(c, a)$ upon arrival in the section Σ_m near the lower left fold point. We first consider the trajectory $S_\varepsilon^-(c, a)$. Tracking backwards down the middle slow manifold, we have that $S_\varepsilon^-(c, a)$ is uniformly $\mathcal{O}(e^{-q/\varepsilon})$ -close to $\mathcal{M}_\varepsilon^m(c, a)$ in Σ^m for $(c, a) \in I_c \times I_a$.

Using similar arguments as in the construction of type 2 pulses, for each sufficiently small $\varepsilon > 0$ we can find values of $(c, a) = (c^-, a^-)(\sqrt{\varepsilon})$ such that the

backwards evolution of $S_\varepsilon^-(c, a)$ can be matched with $\mathcal{W}_\varepsilon^u(0; c, a)$ in the section Σ^m . We now consider trajectories on $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ in Σ^{out} which pass through $\Sigma^{i, -}$ in backwards time. As stated above, trajectories on $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ in Σ^{out} are parametrized by $y \in [-\Delta_y, \Delta_y]$. For $(c, a) = (c^-, a^-)$, for each $\bar{y}$ sufficiently small, (for instance, $-\Delta_y < \bar{y} < -\Delta_y/2$), for all $\varepsilon \in (0, \varepsilon_0)$, the trajectory which intersects $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$ in Σ^{out} at $\bar{y}$ contracts exponentially in backwards time to $S_\varepsilon^-(c, a)$ and therefore passes through $\Sigma^{i, -}$. We can therefore define the supremum of all such values of $\bar{y}$ to be y_ε^- . That is, y_ε^- is defined as the supremum of $\bar{y}$ -values such that the trajectory which intersects $\mathcal{W}_\varepsilon^{s, \ell}(c^-, a^-)$ in Σ^{out} at height $\bar{y}$ passes through $\Sigma^{i, -}$ in backwards time.

We now show that for each sufficiently small $\varepsilon > 0$ and each $\bar{y} \in [-\Delta_y, y_\varepsilon^-]$, we can construct a transitional pulse with secondary excursion which passes near the upper right fold and intersects the section Σ^{out} passing exponentially close to the point $(x, y) = (x_\ell(c, a, \varepsilon) + \mathcal{O}(\bar{y} - s_\varepsilon(x; c, a), \varepsilon), \bar{y})$ on the manifold $\mathcal{W}_\varepsilon^{s, \ell}(c, a)$. In this sense, $\bar{y}$ will serve as a parameterization of such transitional pulses passing near the fold. We then argue below that there is overlap between these and the construction of pulses of type 2 and 4.

To proceed, we show that for each such $\bar{y}$, a transitional pulse can be constructed following the same procedure as with type 2 pulses, though extra care is needed to make sure that the passage near the fold does not cause the argument to break down. We define the solution $\gamma^f(\bar{y}; c^-, a^-)$ on the stable foliation $\mathcal{W}_\varepsilon^{s, \ell}(c^-, a^-)$ which intersects the section Σ^{out} at height $y = \bar{y}$. This solution is exponentially attracted in forward time to $\mathcal{M}_\varepsilon^\ell(c^-, a^-)$ and hence intersects Σ^m at the point $(y_2^f(c^-, a^-), 0)$ where

$$e^{-q_1/\varepsilon}/C \leq y_2^f(c^-, a^-) - y_2^u(0; c^-, a^-) \leq Ce^{-q_2/\varepsilon}. \quad (4.99)$$

Define the manifold $\mathcal{B}(\bar{y}; c^-, a^-)$ to be the backwards evolution of the fiber

$$\{(0, y_2^f(c^-, a^-), z) : |z| \leq \Delta_z\}. \quad (4.100)$$

In backwards time, this fiber is exponentially contracted to the solution $\gamma^f(\bar{y}; c^-, a^-)$ and hence intersects Σ^{out} in a one-dimensional curve which is $\mathcal{O}(e^{-q/\varepsilon})$ -close to $\gamma^f(\bar{y}; c^-, a^-)$. Because $\bar{y} < y_\varepsilon^-$, in backwards time $\gamma^f(\bar{y}; c^-, a^-)$ hits the section $\Sigma^{i,-}$. The passage from Σ^{out} to $\Sigma^{i,-}$ defines a map which is at worst expands exponentially at a rate $e^{\eta/\varepsilon}$, where η can be made arbitrarily small by shrinking the fold neighborhood. In particular, we can ensure that $\eta \ll q$. Hence we can ensure that this potential expansion is always compensated by the contraction occurring along the fibers of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in the passage in backwards time from Σ^m to Σ^{out} . Hence the backwards evolution of $\widehat{\mathcal{B}}(\bar{y}; c^-, a^-)$ also defines a one-dimensional manifold in $\Sigma^{i,-}$ which is $\mathcal{O}(e^{-q/\varepsilon})$ -close to $\gamma^f(\bar{y}; c^-, a^-)$, where q may have to be slightly decreased.

We will now show that the results above hold for an interval of parameters (c, a) exponentially close to (c^-, a^-) , that is, we write $(c, a) = (c^-, a^-) + (\tilde{c}, \tilde{a})$ and consider values $|\tilde{c}|, |\tilde{a}| \leq Ce^{-2\eta/\varepsilon}$. For all sufficiently small $\varepsilon > 0$, we claim that the above assertions continue to hold uniformly for all such $(\tilde{c}, \tilde{a})$. We define the solution $\gamma^f(\bar{y}; c, a)$ on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ which intersects the section Σ^{out} at height $y = \bar{y}$. This solution intersects Σ^m at the point $(y_2^f(\bar{y}; c, a), 0)$ where

$$e^{-q_1/\varepsilon}/C \leq y_2^f(\bar{y}; c, a) - y_2^u(0; c, a) \leq Ce^{-q_2/\varepsilon}. \quad (4.101)$$

uniformly in (c, a) . Again we define the manifold $\mathcal{B}(\bar{y}; c, a)$ to be the intersection of the backwards evolution of the fiber $\{(0, y_2^f(\bar{y}; c, a), z) : |z| \leq \Delta_z\}$ with the section Σ^{out} . In backwards time, this fiber is exponentially contracted to the solution $\gamma^f(\bar{y}; c, a)$ and hence intersects Σ^{out} in a one-dimensional curve which is $\mathcal{O}(e^{-q/\varepsilon})$ -

close to $\gamma^f(\bar{y}; c, a)$. In backwards time, for any $|\tilde{c}|, |\tilde{a}| \leq Ce^{-2\eta/\varepsilon}$, $\gamma^f(\bar{y}; c, a)$ hits the section $\Sigma^{i,-}$, and the backwards evolution of $\widehat{\mathcal{B}}(\bar{y}; c, a)$ also defines a one-dimensional manifold in $\Sigma^{i,-}$ which is $\mathcal{O}(e^{-q/\varepsilon})$ -close to $\gamma^f(\bar{y}; c, a)$ uniformly in $|\tilde{c}|, |\tilde{a}| \leq Ce^{-2\eta/\varepsilon}$, where q may have to be slightly decreased. Similar estimates also hold for the derivatives of the transition maps from Σ^m to $\Sigma^{i,-}$.

Since $\mathcal{B}(\bar{y}; c, a)$ defines a vertical fiber $\{(0, y_2^f(\bar{y}; c, a), z) : |z| \leq \Delta_z\}$ in Σ^m , we parameterize $\mathcal{B}(\bar{y}; c, a)$ by $|z_B| \leq \Delta_z$, where z_B denotes the height along the fiber. Evolving $\widehat{\mathcal{B}}(\bar{y}; c, a)$ backwards, for any $|\tilde{c}|, |\tilde{a}| \leq Ce^{-2\eta/\varepsilon}$, we have that $\widehat{\mathcal{B}}(\bar{y}; c, a)$ is $\mathcal{O}(e^{-q/\varepsilon})$ -close to $\mathcal{M}_\varepsilon^m(c, a)$ in Σ^m . Thus we have that in Σ^m , $\widehat{\mathcal{B}}(\bar{w}; c, a)$ is given by a curve $(y_2, z) = (y_2^b, z^b)(z_B, \bar{y}; c, a)$ which satisfies

$$\begin{aligned} |y_2^b(z_B, \bar{y}; c, a) - y_2^{\mathcal{M}, m}| &= \mathcal{O}(e^{-q/\varepsilon}) \\ |z^b(z_B, \bar{y}; c, a)| &= \mathcal{O}(e^{-q/\varepsilon}), \end{aligned} \tag{4.102}$$

uniformly in $|z_B| \leq \Delta_z$ and $|\tilde{c}|, |\tilde{a}| \leq Ce^{-2\eta/\varepsilon}$.

We can now repeat argument in the construction of type 2 pulses, given the uniformity of the above estimates in $|\tilde{c}|, |\tilde{a}| \leq Ce^{-2\eta/\varepsilon}$ and the fact that we only need freedom in the variation in the bifurcation parameters (c, a) of $\mathcal{O}(e^{-q/\varepsilon})$ for perhaps a slightly smaller value of q to solve the corresponding matching conditions.

Overlap with pulses of type 2, 4

The pulses now form a one-parameter family parametrized by the height $\bar{y}$. By taking Δ_w sufficiently small with respect to Δ_y , it is clear that there is overlap in the construction of type 3 pulses above and the construction of type 2 pulses. That is, type 3 pulses for $\bar{y}$ near Δ_y are constructed in much the same manner as

type 2 pulses for s near $w^\dagger - \Delta_w$. Furthermore, again with Δ_w sufficiently small with respect to Δ_y , there is also overlap in the construction of type 4 pulses. Type 4 pulses constructed for s near $w^\dagger + \Delta_w$ will pass through the section $\Sigma^{i,-}$ before passing $\mathcal{O}(\varepsilon^{2/3} + |a|)$ -close to the fold and leaving the section Σ^{out} exponentially close to $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. These pulses therefore overlap with the construction of type 3 pulses for $\bar{y}$ near y_ε^- .

As all pulses of types 2, 3, 4 were constructed using the implicit function theorem, by local uniqueness the one-parameter families of pulses of types 2, 3, 4 in fact form one continuous family. Hence we reparameterize the family of type 3 pulses by $s \in (w^\dagger - \Delta_w, w^\dagger + \Delta_w)$ rather than the height $\bar{y}$.

5.5 Analysis of tails

In this section, we show that any transitional pulse landing in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in the section $\Sigma^{h,\ell} := \{u = 0, \Delta_w < w < w^\dagger - \Delta_w\}$ at a height no higher than $w \leq w_A + \Delta_w$ in fact lies in the stable manifold $\mathcal{W}_\varepsilon^s(0; c, a)$ of the equilibrium $(u, v, w) = (0, 0, 0)$. We break down the argument into the following two steps:

- (i) By possibly shrinking Δ_w if necessary, we show that for sufficiently small $\varepsilon > 0$, any trajectory on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ starting in $\Sigma^{h,\ell}$ at a height $w \leq w_A + \Delta_w$ returns at a height $w \leq w_A - \Delta_w$ and remains in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$.
- (ii) Next we show any such trajectory remains in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ with the height on each return to $\Sigma^{h,\ell}$ monotonically decreasing until entering an arbitrarily small $\mathcal{O}(1)$ neighborhood of the equilibrium, and any such trajectory which reaches this neighborhood in fact converges to the equilibrium. This amounts to showing

that any periodic orbits lying in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ are repelling.

5.5.1 Way-in-way-out function

We define a way-in-way-out function as in [39] which essentially determines the difference in contraction/expansion rates along the critical manifold $\mathcal{M}_0$. We consider the slow flow restricted to the critical manifold $\{v = 0, w = f(u)\}$ for $(a, \varepsilon) = (0, 0)$; the flow satisfies

$$u' = \frac{u - \gamma f(u)}{f'(u)}, \quad (5.1)$$

where $f(u) = u^2(1 - u)$. For $w \in (0, w^\dagger)$, the equation $f(u) - w = 0$ has three solutions $u_i(w)$, $i = 1, 2, 3$ where the zeros are indexed in increasing order. For $w^+, w^- \in (0, w_A)$, we now define the way-in-way-out function

$$R(w^+, w^-) := \int_{u_1(w^+)}^{u_2(w^-)} \left(c - \sqrt{c^2 - 4f'(u)} \right) \frac{f'(u)}{u - \gamma f(u)} du, \quad (5.2)$$

and for short, we denote $R(w) := R(w, w)$. We note that $-u_1(w) < u_2(w) < \frac{2}{3}$ for all $w \in (0, w^\dagger)$; hence for $\gamma > 0$ and $w \in (0, w_A)$, we have that

$$\begin{aligned} R(w) &> \frac{1}{2} \int_{u_1(w)}^{-u_1(w)} \left(c - \sqrt{c^2 - 4f'(u)} \right) (2 - 3u) du \\ &> -2cu_1(w) - \frac{16}{3} (-u_1(w))^{3/2} \\ &> 0. \end{aligned} \quad (5.3)$$

5.5.2 Initial exit point

In this section, we primarily refer to the results of [12] in which entry-exit or way-in-way-out functions are described for systems with a critical manifold containing a turning point. We consider the flow on a piece of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ which contains everything below $w = w_A - \Delta_w$ as well as trajectories to the left of $\Sigma^{h,\ell}$ for $w \in [w_A - \Delta_w, w_A + \Delta_w]$. Within this manifold, away from the fold point, define $\mathcal{S}_{R/L}^\ell, \mathcal{S}_{R/L}^m$ to be sections transverse to the strong stable (resp. unstable) fibers of $\mathcal{M}_\varepsilon^\ell(c, a)$ (resp. $\mathcal{M}_\varepsilon^m(c, a)$), where the R, L notation refers to whether the sections sit to the left or right of the corresponding slow manifold. That is, $\mathcal{S}_R^\ell$ sits to the right of $\mathcal{M}_\varepsilon^\ell(c, a)$ and $\mathcal{S}_L^m$ sits to the left of $\mathcal{M}_\varepsilon^m(c, a)$.

We define boundary entry/exit points within $\mathcal{S}_R^\ell, \mathcal{S}_L^m$ as follows. Let $w^+ \in [\Delta_w, w_A + \Delta_w]$ and $w^- \in [\Delta_w, w_A - \Delta_w]$, and let $b_\varepsilon^\ell(c, a), b_\varepsilon^r(c, a)$ define points on $\mathcal{S}_R^\ell, \mathcal{S}_L^m$ depending continuously on (c, a, ε) satisfying $\lim_{\varepsilon \rightarrow 0} b_\varepsilon^\ell(c, a) = b_0^\ell(c, a) \in \mathcal{S}_R^\ell \cap \{w = w^+\}$ and $\lim_{\varepsilon \rightarrow 0} b_\varepsilon^m(c, a) = b_0^m(c, a) \in \mathcal{S}_L^m \cap \{w = w^-\}$ so that the limits b_0^ℓ, b_0^m lie on the stable (resp. unstable) foliation of $\mathcal{M}_0^\ell$ (resp. $\mathcal{M}_0^m$).

The next result follows from [12, §9].

Proposition 5.5.1. *Suppose that for each sufficiently small $\varepsilon > 0$ there exists (c, a) such that there is a canard trajectory $\gamma_\varepsilon^C(c, a)$ which meets the sections $\mathcal{S}_R^\ell, \mathcal{S}_L^m$ at the entry/exit points $b_\varepsilon^\ell(c, a), b_\varepsilon^r(c, a)$. Suppose $R(w^-, w^+) > 0$. Then there exists $w^e < w^-$ such that the following holds. Take any other entry point $b_\varepsilon^+ \in \mathcal{S}_{R/L}^\ell$ with $\lim_{\varepsilon \rightarrow 0} b_\varepsilon^+ = b_0^+ \in \mathcal{S}_{R/L}^\ell \cap \{w = w_C\}$ where $w_C > w^+$. Then there is a canard solution connecting b_ε^+ with a corresponding exit point $b_\varepsilon^- \in \mathcal{S}_R^m$ satisfying $\lim_{\varepsilon \rightarrow 0} b_\varepsilon^- = b_0^- \in \mathcal{S}_R^m \cap \{w = w^e\}$.*

We claim that the trajectory on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ starting in $\Sigma^{h,\ell}$ at height $w = w_A + \Delta_w$ returns to $\Sigma^{h,\ell}$ at a height $w \leq w_A - \Delta_w$. Suppose for contradiction that the backwards evolution of the trajectory starting in $\Sigma^{h,\ell}$ at height $w = w_A - \Delta_w$ ends up in $\Sigma^{h,\ell}$ at a height $w = w_C < w_A + \Delta_w$. Proposition 5.5.1 implies the existence of $w^e < w_A - \Delta_w$ such that all trajectories entering $\mathcal{S}_R^\ell$ above $w = w_A + \Delta_w$ exit $\mathcal{S}_R^m$ at height w^e . However we know the basepoint of the second excursion of our pulse solution (which is above $w = w + \Delta_w$) does not exit near this buffer point but rather continues up to have a tail at some $w_A - \Delta_w \leq w \leq \widehat{w}_\varepsilon$; this is a contradiction.

Hence the backwards evolution of the trajectory starting in $\Sigma^{h,\ell}$ at height $w = w_A - \Delta_w$ ends up in $\Sigma^{h,\ell}$ at a height $w = w_C \geq w_A + \Delta_w$. In other words, this means that the trajectory on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ starting in $\Sigma^{h,\ell}$ at height $w = w_A + \Delta_w$ returns to $\Sigma^{h,\ell}$ at a height $w \leq w_A - \Delta_w$. We therefore have the following.

Proposition 5.5.2. *For each sufficiently small $\Delta_w > 0$, for sufficiently small $\varepsilon > 0$, let $(c, a) \in I_c \times I_a$ be such that there exists a transitional pulse with a tail in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ starting in $\Sigma^{h,\ell}$ at a height $w \leq w_A + \Delta_w$. Then this tail trajectory remains in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$, returning to $\Sigma^{h,\ell}$ at a height $w \leq w_A - \Delta_w$.*

5.5.3 Periodic orbits

Once a tail trajectory ends up below $w = w_A - \Delta_w$, it is stuck in the two-dimensional manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ and its height is monotonically decreasing on each return to $\Sigma^{h,\ell}$. Hence such a trajectory is approaching the equilibrium; the only way it can fail to lie in the stable manifold $\mathcal{W}_\varepsilon^s(0; c, a)$ is if it is blocked by a periodic orbit (in this case $\mathcal{W}_\varepsilon^s(0; c, a)$ would topologically take the form of a disc). The aim of this section is to show that any periodic orbit lying in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ must be repelling.

In [39], the authors constructed periodic canard orbits in a class of planar systems. Although the entire canard explosion is not possible to construct in the same manner in our case (there is no such two-dimensional invariant manifold which contains the entire S-shaped critical manifold), the construction procedure is valid in the two-dimensional manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ for canard orbits up to height $w = w_A - \Delta_w$. We collect the following results from [39] regarding such periodic solutions.

Proposition 5.5.3. *For each $c \in I_c$ and $\varepsilon > 0$ sufficiently small, there exists a family of periodic orbits*

$$(w, \varepsilon) \rightarrow (a(w, \varepsilon), \Gamma(w, \varepsilon)) \quad (5.4)$$

parameterized by the height $w \in (0, w_A - \Delta_w)$ such that $\Gamma(w, \varepsilon) \subset \mathcal{W}_\varepsilon^{s,\ell}(c, a(w, \varepsilon))$.

(i) *Any periodic orbit passing near the critical manifold $\mathcal{M}_0$ which is entirely contained in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ for $-\Delta_w \leq w \leq w_A - \Delta_w$ is part of this family.*

(ii) *For $\Delta_w < w < w_A - \Delta_w$, the Floquet exponent $P(w, \varepsilon)$ satisfies*

$$P(w, \varepsilon) = \frac{1}{\varepsilon} (R(w) + \theta(w, \varepsilon)), \quad (5.5)$$

where θ and $\frac{\partial \theta}{\partial w} \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$.

(iii) *For $0 < w < \Delta_w$ all of the $\Gamma(w, \varepsilon)$ are repelling.*

In particular, the above implies that for any sufficiently small $\varepsilon > 0$, there are no nonrepelling periodic orbits in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ between $-\Delta_w \leq w \leq w_A - \Delta_w$.

5.5.4 Convergence of tails

We now combine Propositions 5.5.2 and 5.5.3 to obtain the following.

Proposition 5.5.4. *Fix Δ_w sufficiently small. For each sufficiently small $\varepsilon > 0$, consider a transitional pulse with tail landing in the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in $\Sigma^{h,\ell}$ at a height $w \leq w_A + \Delta_w$. Then the tail of this pulse in fact lies in the stable manifold $\mathcal{W}_\varepsilon^s(0; c, a)$ of the equilibrium $(u, v, w) = (0, 0, 0)$. In particular, every solution on the tail manifold $\mathcal{T}_\varepsilon(c, a)$ lies on $\mathcal{W}_\varepsilon^s(0; c, a)$.*

5.6 Flow near the Airy point

The goal of this section is to prove Proposition 5.4.4, regarding properties of the backwards evolution of certain trajectories on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ between the sections $\Sigma^{h,\ell}$ and Σ^m . We need to track the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in backwards time into a neighborhood of $\mathcal{M}_\varepsilon^m(c, a)$ and determine its behavior near the canard point, in particular its transversality with respect to the strong unstable fibers in the section Σ^m . For trajectories on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ for $w < w_A - \Delta_w$, this behavior is clear: these trajectories are attracted exponentially close to $\mathcal{M}_\varepsilon^m(c, a)$ and remain in the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ upon entering a neighborhood of the canard point. This is due to the fact that the backwards evolution of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ from $\Sigma^{h,\ell}$ to Σ^m and forwards evolution from Σ^m to $\Sigma^{h,\ell}$ in fact coincide for $w < w_A - \Delta_w$. Hence transversality with respect to the strong unstable fibers in Σ^m is clear.

However, for $w \geq w_A - \Delta_w$, the behavior is not so clear since near $\mathcal{M}_\varepsilon^m(c, a)$, the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ is not defined for $w \geq w_A - \Delta_w$ due to the lack of spectral gap in the linearization of the vector field. Such trajectories on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ are still

exponentially attracted to $\mathcal{M}_\varepsilon^m(c, a)$ in backwards time, but in general do not coincide with $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ upon reaching $w = w_A - \Delta$ due to the interaction with the focus-like properties of the manifold $\mathcal{M}_\varepsilon^m(c, a)$ (see Figure 5.15 – note that the flow direction is reversed in this figure). To understand the transition from node to focus, we must understand the flow near the Airy point. The goal is to show that even though the backwards and forward evolution of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ between $\Sigma^{h,\ell}$ and Σ^m do not coincide, we retain the desired transversality properties for trajectories on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ a bit above the Airy point, specifically for $w < w_A + C\varepsilon^{2/3}$ for some $C > 0$.

The Airy point (u_A, w_A) is defined by the conditions $c^2 = 4f'(u_A)$ and $w_A = f(u_A)$. In a neighborhood of this point, the manifold $\mathcal{M}_\varepsilon^m(c, a)$ can be written as a graph $(u, v) = (u_A + h(w, \varepsilon), g(w, \varepsilon))$ where

$$\begin{aligned} h(w, \varepsilon) &= \frac{1}{f'(u_A)}(w - w_A) + \mathcal{O}(\varepsilon, (w - w_A)^2) \\ g(w, \varepsilon) &= \mathcal{O}(\varepsilon, (w - w_A)). \end{aligned} \tag{6.1}$$

We make the coordinate transformation

$$\begin{aligned} x &= \frac{2}{c}(v - g(w, \varepsilon)) - (u - u_A - h(w, \varepsilon)) \\ y &= \frac{4f''(u_A)}{c^2 f'(u_A)}(w - w_A) \\ z &= u - u_A - h(w, \varepsilon), \end{aligned} \tag{6.2}$$

and rescale time by $-(c/2)$ to arrive at the system

$$\begin{aligned} \dot{x} &= -x + yz + \mathcal{O}(z^2, \varepsilon z, y^2 z) \\ \dot{z} &= -x - z \\ \dot{y} &= \varepsilon(-k + \mathcal{O}(y, z, \varepsilon)), \end{aligned} \tag{6.3}$$

where

$$k = \frac{4f''(u_A)}{c^2 f'(u_A)} > 0. \quad (6.4)$$

Note that the flow direction has been reversed. We make a final coordinate change $y \rightarrow y + \mathcal{O}(y^2, \varepsilon)$ to simplify the equation for $\dot{x}$ and arrive at the system

$$\begin{aligned} \dot{x} &= -x + yz + \mathcal{O}(z^2) \\ \dot{z} &= -x - z \\ \dot{y} &= \varepsilon (-k + \mathcal{O}(x, y, z, \varepsilon)), \end{aligned} \quad (6.5)$$

We consider solutions entering via the section

$$\Sigma_A^{in} = \{(x, y, z, \varepsilon) : x = \rho^4, |y| \leq \rho^2, |z| \leq \rho^3 \mu, 0 < \varepsilon \leq \rho^3 \delta\}. \quad (6.6)$$

Such solutions exit via the section

$$\Sigma_A^{out} = \{(x, y, z, \varepsilon) : |x| \leq \rho^4, y = -\rho^2, |z| \leq \rho^3 \mu, 0 < \varepsilon \leq \rho^3 \delta\}. \quad (6.7)$$

The goal of this section is to prove Proposition 5.4.4, that is, we track $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ near the Airy point until it exits via Σ_A^{out} , where we then use an exchange lemma type argument to track the rest of the way to Σ^m . To start, we have the following regarding the entry of the manifolds $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a), \widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ in Σ_A^{in} .

Lemma 5.6.1. *For each sufficiently small $\Delta_y > 0$ there exists $\varepsilon_0 > 0$ and sufficiently small choice of the intervals $I_c \times I_a$, such that for $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$, the manifolds $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a), \widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ intersect Σ_A^{in} in smooth curves $z = z_\varepsilon^{s,\ell}(y; c, a), z_\varepsilon^{s,r}(y; c, a)$*

for $|y| \leq \Delta_y$. Furthermore, there exists a constant $\kappa = \kappa(\rho) > 0$ such that

$$z_\varepsilon^{s,r}(y; c, a) - z_\varepsilon^{s,\ell}(y; c, a) > \rho^3 \kappa(\rho) \quad (6.8)$$

uniformly in $|y| \leq \Delta_y$.

Proof. Using Proposition 5.3.2, we find that for $(c, a, \varepsilon) = (1/\sqrt{2}, 0, 0)$, the front φ_ℓ is asymptotic (in forward time according to 6.5) to the Airy point $(x, y, z) = (0, 0, 0)$ and satisfies

$$\begin{aligned} x(s) &= 2\sqrt{2}B_\ell e^{\frac{-s}{2\sqrt{2}}} + \mathcal{O}(s^2 e^{\frac{-s}{\sqrt{2}}}) \\ z(s) &= (A_\ell - B_\ell s)e^{\frac{-s}{2\sqrt{2}}} + \mathcal{O}(s^2 e^{\frac{-s}{\sqrt{2}}}). \end{aligned} \quad (6.9)$$

Therefore, φ_ℓ intersects Σ_A^{in} at the point $(x, y, z) = (\rho^4, 0, z_0^\ell)$ where

$$\begin{aligned} z_0^\ell &= \rho^4 \left(\frac{A_\ell}{2\sqrt{2}B_\ell} + \log \left(\frac{\rho^4}{2\sqrt{2}B_\ell} \right) \right) + o(\rho^4) \\ &= 4\rho^4 \log \left(\frac{\rho}{2\sqrt{2}B_\ell} \right) + \mathcal{O}(\rho^4). \end{aligned} \quad (6.10)$$

Therefore, by a regular perturbation argument, we have for sufficiently small Δ_y and any $(c, a) \in I_c \times I_a$, for $|y| \leq \Delta_y$, the manifold $\mathcal{W}_0^{s,\ell}(c, a)$ intersects Σ_A^{in} in a curve $z = z_0^{s,\ell}(y; c, a)$ given by

$$z_0^{s,\ell}(y; c, a) = z_0^\ell + \mathcal{O}(y, (c - c^*), a). \quad (6.11)$$

By using standard geometric singular perturbation theory, for sufficiently small $\varepsilon > 0$, this manifold perturbs to a locally invariant manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ which intersects Σ_A^{in} in a smooth curve $z = z_\varepsilon^{s,\ell}(y; c, a)$ given by

$$z_\varepsilon^{s,\ell}(y; c, a) = z_0^\ell + \mathcal{O}(y, (c - c^*), a, \varepsilon), \quad (6.12)$$

for $|y| \leq \Delta_y$. We similarly obtain that $\mathcal{W}_\varepsilon^{s,r}(c, a)$ intersects Σ_A^{in} in a smooth curve $z = z_\varepsilon^{s,r}(y; c, a)$ given by

$$z_\varepsilon^{s,r}(y; c, a) = z_0^r + \mathcal{O}(y, (c - c^*), a, \varepsilon), \quad (6.13)$$

for $|y| \leq \Delta_y$. Using Proposition 5.3.2, and taking $\Delta_y \ll \rho^4$ sufficiently small, we deduce that there exists $\kappa = \kappa(\rho) > 0$ such that

$$z_\varepsilon^{s,r}(y; c, a) - z_\varepsilon^{s,\ell}(y; c, a) > \rho^3 \kappa(\rho) \quad (6.14)$$

uniformly in $|y| \leq \Delta_y$. □

By taking $\Delta_y := 2k\Delta_w$ sufficiently small, we reduce the study of Proposition 5.4.4 to just understanding the passage of trajectories on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ which enter a neighborhood of the Airy point in backwards time in a manner governed by Lemma 5.6.1; these solutions interact with the flow near the Airy point in a nontrivial manner (see Figure 5.15). All solutions on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ entering a neighborhood of $\mathcal{M}_\varepsilon^m(c, a)$ in backwards time at heights lower than this remain in $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ until arriving at the section Σ^m due to the nature of the construction of this manifold in §5.3.2.

To accomplish this, we need to understand detailed properties of the flow of (6.5). Ultimately, we will show that the flow of (6.5) is qualitatively similar to the flow of the simpler system

$$\begin{aligned} \dot{x} &= -x + yz \\ \dot{z} &= -x - z \\ \dot{y} &= -\varepsilon, \end{aligned} \quad (6.15)$$

which are essentially the Airy equations on a slow timescale coupled with exponential

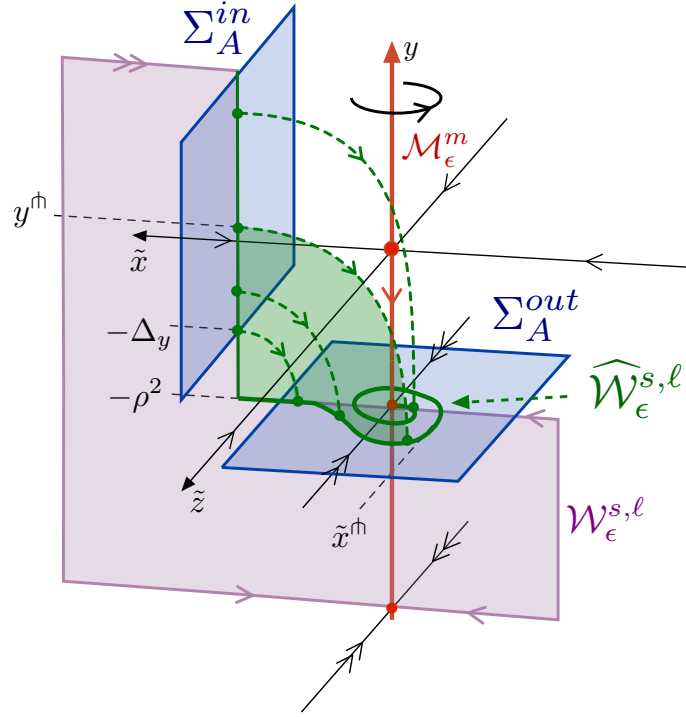


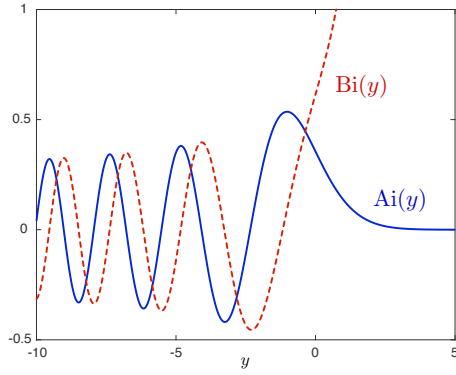
Figure 5.15: Shown is the schematic of the flow near the Airy point. Note that the flow direction corresponds to that of (6.5), which is the reverse of (1.1).

decay. The solutions of this system are given in terms of the Airy functions Ai , Bi , and their derivatives, which are shown in Figure 5.16.

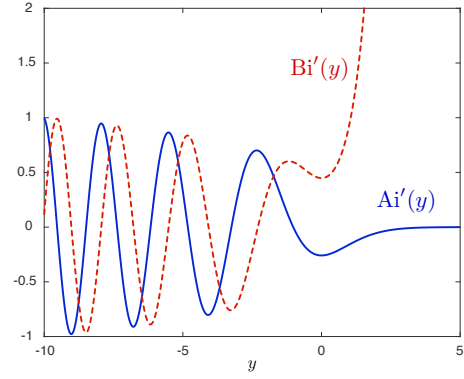
We begin by solving the simpler system (6.15) to demonstrate why it is reasonable to expect that the transversality properties of Proposition 5.4.4 should indeed hold. Then we will use blow up techniques to study (6.5) directly to show that Proposition 5.4.4 continues to be valid when including the higher order terms.

5.6.1 A simpler system

In this section, we consider the simpler system (6.15), which are essentially the Airy equations on a slow timescale coupled with exponential decay. To see this, we rescale



(a) The Airy functions $\text{Ai}(y)$ (blue) and $\text{Bi}(y)$ (dashed red).



(b) The derivatives $\text{Ai}'(y)$ (blue) and $\text{Bi}'(y)$ (dashed red) of the Airy functions $\text{Ai}(y), \text{Bi}(y)$.

Figure 5.16: Shown are graphs of the Airy functions Ai, Bi and their derivatives.

$(x, z) = (e^{-t}\bar{x}, e^{-t}\bar{z})$ and obtain the equations

$$\begin{aligned}\dot{\bar{x}} &= y\bar{z} \\ \dot{\bar{z}} &= -\bar{x} \\ \dot{y} &= -\varepsilon.\end{aligned}\tag{6.16}$$

The solutions of this system can be given explicitly in terms of Airy functions Ai, Bi (see Figure 5.16)

$$\begin{aligned}\bar{x}(t) &= \pi \left[\left(\text{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} + \varepsilon^{1/3}t \right) - \text{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} + \varepsilon^{1/3}t \right) \right) x_0 \right. \\ &\quad \left. + \varepsilon^{1/3} \left(\text{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} + \varepsilon^{1/3}t \right) - \text{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} + \varepsilon^{1/3}t \right) \right) z_0 \right] \\ \bar{z}(t) &= \frac{\pi}{\varepsilon^{1/3}} \left[\left(\text{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} + \varepsilon^{1/3}t \right) - \text{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} + \varepsilon^{1/3}t \right) \right) x_0 \right. \\ &\quad \left. + \varepsilon^{1/3} \left(\text{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} + \varepsilon^{1/3}t \right) - \text{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} + \varepsilon^{1/3}t \right) \right) z_0 \right] \\ y(t) &= y_0 - \varepsilon t,\end{aligned}\tag{6.17}$$

where $y_0 = y(0)$, $x_0 = \bar{x}(0) = x(0)$, and $z_0 = \bar{z}(0) = z(0)$. This solution reaches $y = -\rho^2$ at time $T = \frac{y_0 + \rho^2}{\varepsilon}$ with

$$\begin{aligned}
x(T) &= \pi e^{-\frac{y_0 + \rho^2}{\varepsilon}} \left[\left(\text{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Bi}' \left(\frac{\rho^2}{\varepsilon^{2/3}} \right) - \text{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Ai}' \left(\frac{\rho^2}{\varepsilon^{2/3}} \right) \right) x_0 \right. \\
&\quad \left. + \varepsilon^{1/3} \left(\text{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Bi}' \left(\frac{\rho^2}{\varepsilon^{2/3}} \right) - \text{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Ai}' \left(\frac{\rho^2}{\varepsilon^{2/3}} \right) \right) z_0 \right] \\
z(T) &= \frac{\pi}{\varepsilon^{1/3}} e^{-\frac{y_0 + \rho^2}{\varepsilon}} \left[\left(\text{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Ai} \left(\frac{\rho^2}{\varepsilon^{2/3}} \right) - \text{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Bi} \left(\frac{\rho^2}{\varepsilon^{2/3}} \right) \right) x_0 \right. \\
&\quad \left. + \varepsilon^{1/3} \left(\text{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Ai} \left(\frac{\rho^2}{\varepsilon^{2/3}} \right) - \text{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \text{Bi} \left(\frac{\rho^2}{\varepsilon^{2/3}} \right) \right) z_0 \right] \\
y(T) &= -\rho^2.
\end{aligned} \tag{6.18}$$

Using asymptotic properties of Airy functions [1, §10.4], we have the following

Lemma 5.6.2. *The Airy functions $\text{Ai}(y)$, $\text{Bi}(y)$ have the following asymptotics for all sufficiently large $y \gg 1$*

$$\begin{aligned}
\text{Ai}(y) &= \frac{e^{-\frac{2}{3}y^{3/2}}}{2\sqrt{\pi}y^{1/4}} \left(1 - \frac{15}{144y^{3/2}} + \mathcal{O}(y^{-3}) \right), \\
\text{Ai}'(y) &= \frac{-y^{1/4}e^{-\frac{2}{3}y^{3/2}}}{2\sqrt{\pi}} \left(1 + \frac{21}{144y^{3/2}} + \mathcal{O}(y^{-3}) \right), \\
\text{Bi}(y) &= \frac{e^{\frac{2}{3}y^{3/2}}}{\sqrt{\pi}y^{1/4}} \left(1 + \frac{15}{144y^{3/2}} + \mathcal{O}(y^{-3}) \right), \\
\text{Bi}'(y) &= \frac{y^{1/4}e^{\frac{2}{3}y^{3/2}}}{\sqrt{\pi}} \left(1 + \frac{21}{144y^{3/2}} + \mathcal{O}(y^{-3}) \right).
\end{aligned} \tag{6.19}$$

Considering the linearization of (6.15) for $\varepsilon = 0$ in the plane $y = -\rho^2$, we see that there are two eigenvalues $\lambda = -1 \pm \rho$ with corresponding eigenvectors $(\mp\rho, 1)$. We now change coordinates $\tilde{x} = x - \rho z$, $\tilde{z} = x + \rho z$ and using Lemma 5.6.2 under the assumption that $0 < \varepsilon^{2/3} \ll \Delta_y \ll \rho^2 \ll 1$, we can expand the terms dependent

on the fixed argument $\frac{\rho^2}{\varepsilon^{2/3}}$ to obtain

$$\begin{aligned}
\tilde{x}(T) &= \frac{\rho^{1/2}\sqrt{\pi}}{\varepsilon^{1/6}} e^{-\frac{y_0+\rho^2}{\varepsilon}} \left[\left(x_0 \operatorname{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) + \varepsilon^{1/3} z_0 \operatorname{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \right) e^{\frac{2\rho^3}{3\varepsilon}} (2 + \mathcal{O}(\varepsilon)) \right. \\
&\quad \left. + \left(x_0 \operatorname{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) + \varepsilon^{1/3} z_0 \operatorname{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \right) e^{-\frac{2\rho^3}{3\varepsilon}} \left(\frac{\varepsilon}{8\rho^3} + \mathcal{O}(\varepsilon^2) \right) \right] \\
\tilde{z}(T) &= \frac{\rho^{1/2}\sqrt{\pi}}{\varepsilon^{1/6}} e^{-\frac{y_0+\rho^2}{\varepsilon}} \left[\left(x_0 \operatorname{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) + \varepsilon^{1/3} z_0 \operatorname{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \right) e^{\frac{2\rho^3}{3\varepsilon}} \left(\frac{\varepsilon}{24\rho^3} + \mathcal{O}(\varepsilon^2) \right) \right. \\
&\quad \left. + \left(x_0 \operatorname{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) + \varepsilon^{1/3} z_0 \operatorname{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \right) e^{-\frac{2\rho^3}{3\varepsilon}} (1 + \mathcal{O}(\varepsilon)) \right] \\
y(T) &= -\rho^2.
\end{aligned} \tag{6.20}$$

We now consider solutions on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ which enter via Σ_A^{in} , with $(x, y, z)(0) = (x_0, y_0, z_0) = (\rho^4, y_0, z_\varepsilon^{s,\ell}(y_0; c, a))$, where $z_\varepsilon^{s,\ell}(y_0; c, a) < 0$, so that $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ is parameterized in Σ_A^{in} by $|y_0| \leq \Delta_y$. Using the above analysis, $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ exits via Σ_A^{out} in a curve $(\tilde{x}, \tilde{z}) = (\tilde{x}^\ell, \tilde{z}^\ell)(y_0)$ given by

$$\begin{aligned}
\tilde{x}^\ell(y_0) &= \frac{\rho^{1/2}\sqrt{\pi}}{\varepsilon^{1/6}} e^{-\frac{y_0+\rho^2}{\varepsilon}} \tilde{X}^\ell(y_0) \\
\tilde{z}^\ell(y_0) &= \frac{\rho^{1/2}\sqrt{\pi}}{\varepsilon^{1/6}} e^{-\frac{y_0+\rho^2}{\varepsilon}} \tilde{Z}^\ell(y_0)
\end{aligned} \tag{6.21}$$

where

$$\begin{aligned}
\tilde{X}^\ell(y_0) &= \left(\rho^4 \operatorname{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) + \varepsilon^{1/3} z_\varepsilon^{s,\ell}(y_0) \operatorname{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \right) e^{\frac{2\rho^3}{3\varepsilon}} (2 + \mathcal{O}(\varepsilon)) \\
&\quad + \left(\rho^4 \operatorname{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) + \varepsilon^{1/3} z_\varepsilon^{s,\ell}(y_0) \operatorname{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \right) e^{-\frac{2\rho^3}{3\varepsilon}} \left(\frac{\varepsilon}{8\rho^3} + \mathcal{O}(\varepsilon^2) \right) \\
\tilde{Z}^\ell(y_0) &= \left(\rho^4 \operatorname{Ai} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) + \varepsilon^{1/3} z_\varepsilon^{s,\ell}(y_0) \operatorname{Ai}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \right) e^{\frac{2\rho^3}{3\varepsilon}} \left(\frac{\varepsilon}{24\rho^3} + \mathcal{O}(\varepsilon^2) \right) \\
&\quad + \left(\rho^4 \operatorname{Bi} \left(-\frac{y_0}{\varepsilon^{2/3}} \right) + \varepsilon^{1/3} z_\varepsilon^{s,\ell}(y_0) \operatorname{Bi}' \left(-\frac{y_0}{\varepsilon^{2/3}} \right) \right) e^{-\frac{2\rho^3}{3\varepsilon}} (1 + \mathcal{O}(\varepsilon)).
\end{aligned} \tag{6.22}$$

We now want to understand the transversality of this curve with respect to the fibers

of the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in the section Σ_A^{out} . Then, using the exchange lemma, we deduce that the transversality holds in the section Σ^m as well. We note that for trajectories entering Σ_A^{in} for $y < -\Delta_y$, this transversality is clear as due to the construction of the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in §5.3.2, the forward/backward evolution of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ coincide in this region.

Hence we are primarily concerned with the trajectories above, which conveniently we have explicitly computed for $y_0 > -\Delta_y$. This is precisely the regime in which the manifolds $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ and $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ begin to deviate. Under the transformation to the $\sim$ coordinates corresponding to the strong/weak eigenspaces of the linearization of (6.15), the manifold $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ will manifest as a curve in Σ_A^{out} aligned approximately with the subspace $\tilde{z} = 0$ and its fibers will manifest as curves aligned approximately with $\tilde{x} \approx \text{const}$. It is clear from the expressions above that the same does not hold for $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ when y_0 gets too large, as the Airy functions transition to oscillatory behavior.

We compute the derivatives

$$\begin{aligned} (\tilde{x}^\ell)'(y_0) &= -\frac{\rho^{1/2}\sqrt{\pi}}{\varepsilon^{7/6}}e^{-\frac{y_0+\rho^2}{\varepsilon}}\left(\tilde{X}^\ell(y_0) + \varepsilon(\tilde{X}^\ell)'(y_0)\right) \\ (\tilde{z}^\ell)'(y_0) &= -\frac{\rho^{1/2}\sqrt{\pi}}{\varepsilon^{7/6}}e^{-\frac{y_0+\rho^2}{\varepsilon}}\left(\tilde{Z}^\ell(y_0) + \varepsilon(\tilde{Z}^\ell)'(y_0)\right), \end{aligned} \quad (6.23)$$

and hence $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ can be written as a graph $\tilde{z} = \tilde{z}(\tilde{x})$ with

$$\frac{d\tilde{z}}{d\tilde{x}} = \frac{(\tilde{z}^\ell)'(y_0)}{(\tilde{x}^\ell)'(y_0)} = \frac{\tilde{Z}^\ell(y_0) + \varepsilon(\tilde{Z}^\ell)'(y_0)}{\tilde{X}^\ell(y_0) + \varepsilon(\tilde{X}^\ell)'(y_0)}, \quad (6.24)$$

provided that the denominator does not vanish. Points at which the denominator vanishes are essentially those at which this curve becomes tangent to the fibers $\tilde{x} \approx \text{const}$ of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. Hence we reduce our study to finding zeros of this expression.

This will be carried out in detail in the following sections, but we note that they occur approximately at the zeros of $\tilde{X}^\ell(y_0)$, which are approximately the zeros of $\text{Ai}(-y_0/\varepsilon^{2/3})$ for all sufficiently small $\varepsilon > 0$. Hence we are primarily concerned with studying the Airy function Ai . We have the following [1, §10.4]

Lemma 5.6.3. *There exists $y^* < 0$ such that the Airy function Ai satisfies the following*

- (i) $\text{Ai}(y^*) = 0$
- (ii) $\text{Ai}'(y^*) > 0$
- (iii) $\text{Ai}(y) > 0$ for all $y > y^*$.

We can therefore find the first zero of the denominator or equivalently, the first turning point of $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$, which occurs when $y_0 = y_0^\natural \approx -y^*\varepsilon^{2/3} > 0$. Therefore $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ is transverse to the fibers of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in Σ_A^{out} up to the fiber passing through the point $(\tilde{x}^\natural, \tilde{z}(\tilde{x}^\natural)) = (\tilde{x}^\ell(y_0^\natural), \tilde{z}^\ell(y_0^\natural))$. A schematic of this result is depicting in Figure 5.15.

Using the exchange lemma, we continue to track $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ backwards from Σ_A^{out} to the section Σ^m and deduce that this transversality holds there also.

In the coming sections, we consider the full system (6.5), and we make the above computations precise in this context.

5.6.2 Blow up transformation

To study the flow of the full equations (6.5)

$$\begin{aligned}\dot{x} &= -x + yz + \mathcal{O}(z^2) \\ \dot{z} &= -x - z \\ \dot{y} &= \varepsilon(-k + \mathcal{O}(x, y, z, \varepsilon)),\end{aligned}$$

we will use blow up techniques. The blow up is a rescaling which blows up the degenerate point $(x, y, z, \varepsilon) = (0, 0, 0, 0)$ to a 3-sphere. The blow up transformation is given by

$$x = \bar{r}^4 \bar{x}, \quad y = \bar{r}^2 \bar{y}, \quad z = \bar{r}^3 \bar{z}, \quad \varepsilon = \bar{r}^3 \bar{\varepsilon}. \quad (6.25)$$

Defining $B_A = S^3 \times [0, \bar{r}_0]$ for some sufficiently small $\bar{r}_0$, we consider the blow up as a mapping $B_A \rightarrow \mathbb{R}^4$ with $(\bar{x}, \bar{y}, \bar{z}, \bar{\varepsilon}) \in S^3$ and $\bar{r} \in [0, \bar{r}_0]$. The point $(x, y, z, \varepsilon) = (0, 0, 0, 0)$ is now represented as a copy of S^3 (i.e. $\bar{r} = 0$) in the blow up transformation. To study the flow on the manifold B_A , there are three relevant coordinate charts. The first is the chart $\mathcal{K}_1$ which uses the coordinates

$$x = r_1^4, \quad y = r_1^2 y_1, \quad z = r_1^3 z_1, \quad \varepsilon = r_1^3 \varepsilon_1, \quad (6.26)$$

the second chart $\mathcal{K}_2$ uses the coordinates

$$x = r_2^4 x_2, \quad y = r_2^2 y_2, \quad z = r_2^3 z_2, \quad \varepsilon = r_2^3, \quad (6.27)$$

and the third chart $\mathcal{K}_3$ uses the coordinates

$$x = r_3^4 x_3, \quad y = -r_3^2, \quad z = r_3^3 z_3, \quad \varepsilon = r_3^3 \varepsilon_3, \quad (6.28)$$

With these three sets of coordinates, a short calculation gives the following.

Lemma 5.6.4. *The transition map $\kappa_{12} : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ between the coordinates in $\mathcal{K}_1$ and $\mathcal{K}_2$ is given by*

$$x_2 = \frac{1}{\varepsilon_1^{4/3}}, \quad y_2 = \frac{y_1}{\varepsilon_1^{2/3}}, \quad z_2 = \frac{z_1}{\varepsilon_1}, \quad r_2 = r_1 \varepsilon_1^{1/3}, \quad \text{for } \varepsilon_1 > 0, \quad (6.29)$$

the transition map $\kappa_{13} : \mathcal{K}_1 \rightarrow \mathcal{K}_3$ between the coordinates in $\mathcal{K}_1$ and $\mathcal{K}_3$ is given by

$$x_3 = \frac{1}{y_1^2}, \quad r_3 = r_1(-y_1)^{1/2}, \quad z_3 = \frac{z_1}{(-y_1)^{3/2}}, \quad \varepsilon_3 = \frac{\varepsilon_1}{(-y_1)^{3/2}}, \quad \text{for } y_1 < 0, \quad (6.30)$$

and the transition map $\kappa_{23} : \mathcal{K}_2 \rightarrow \mathcal{K}_3$ between the coordinates in $\mathcal{K}_2$ and $\mathcal{K}_3$ is given by

$$x_3 = \frac{x_2}{y_2^2}, \quad r_3 = r_2(-y_2)^{1/2}, \quad z_3 = \frac{z_2}{(-y_2)^{3/2}}, \quad \varepsilon_3 = \frac{1}{(-y_2)^{3/2}}, \quad \text{for } y_2 < 0. \quad (6.31)$$

Solutions on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ will enter via the section Σ_A^{in} and exit via Σ_A^{out} . During this passage, it will be necessary to track different parts of the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ in the different charts $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$. A diagram of the sequence through which the solutions will be tracked is shown in Figure 5.17. We begin in §5.6.3 with a study of the chart $\mathcal{K}_1$, where all solutions enter via the section Σ_A^{in} .

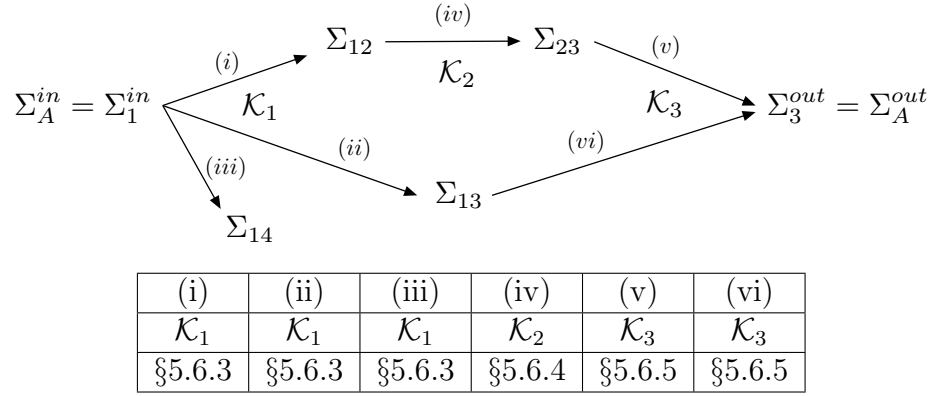


Figure 5.17: Shown is the sequence of sections through which the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c,a)$ will be tracked. The table displays the charts and sections in the text in which the various transitions will be studied.

5.6.3 Dynamics in $\mathcal{K}_1$

In the $\mathcal{K}_1$ coordinates, the equations are given by

$$\begin{aligned}
 \dot{r}_1 &= -\frac{1}{4}r_1 + \frac{1}{4}r_1^2 y_1 z_1 + \mathcal{O}(r_1^3 z_1^2) \\
 \dot{z}_1 &= -\frac{1}{4}z_1 - r_1 - \frac{3}{4}r_1 y_1 z_1^2 + \mathcal{O}(r_1^2 z_1^3) \\
 \dot{y}_1 &= \frac{1}{2}y_1 - kr_1 \varepsilon_1 - \frac{1}{2}r_1 y_1^2 z_1 + \mathcal{O}(r_1^2 y_1 z_1^2, r_1^5 \varepsilon_1, r_1^3 \varepsilon_1 y_1, r_1^4 \varepsilon_1 z_1, r_1^4 \varepsilon_1^2) \\
 \dot{\varepsilon}_1 &= \frac{3}{4}\varepsilon_1 - \frac{3}{4}r_1 y_1 z_1 \varepsilon_1 + \mathcal{O}(r_1^2 z_1^2 \varepsilon_1).
 \end{aligned} \tag{6.32}$$

In these coordinates, the section Σ_A^{in} is given by

$$\Sigma_1^{in} = \{(r_1, y_1, z_1, \varepsilon_1) : r_1 = \rho, |y_1| \leq 1, |z_1| \leq \mu, 0 < \varepsilon_1 \leq \delta\}. \tag{6.33}$$

Define the set

$$D_1 = \{(r_1, y_1, z_1, \varepsilon_1) : 0 \leq r_1 \leq \rho, |y_1| \leq 1, |z_1| \leq \mu, 0 \leq \varepsilon_1 \leq \delta\}. \tag{6.34}$$

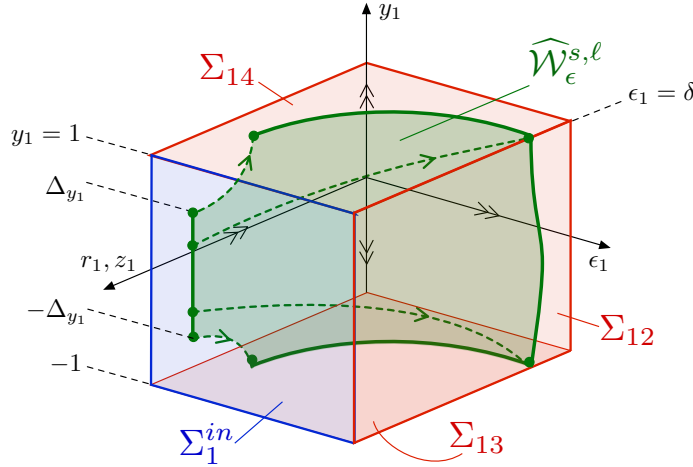


Figure 5.18: Shown is the setup in the chart $\mathcal{K}_1$.

Under the flow of (6.32), any solution starting in Σ_1^{in} exits D_1 via one of the sections

$$\Sigma_{12} = \{(r_1, y_1, z_1, \varepsilon_1) : r_1 \leq \rho, |y_1| \leq 1, |z_1| \leq \mu, \varepsilon_1 = \delta\} \quad (6.35)$$

$$\Sigma_{13} = \{(r_1, y_1, z_1, \varepsilon_1) : r_1 \leq \rho, y_1 = -1, |z_1| \leq \mu, 0 < \varepsilon_1 \leq \delta\} \quad (6.36)$$

$$\Sigma_{14} = \{(r_1, y_1, z_1, \varepsilon_1) : r_1 \leq \rho, y_1 = 1, |z_1| \leq \mu, 0 < \varepsilon_1 \leq \delta\}. \quad (6.37)$$

The setup in the chart $\mathcal{K}_1$ is shown in Figure 5.18. It turns out that we only need to consider those solutions exiting via Σ_{12} and Σ_{13} , which will be tracked in the charts $\mathcal{K}_2$ and $\mathcal{K}_3$, respectively (see Figure 5.17). Solutions exiting via Σ_{14} will not be analyzed.

The following result gives estimates for solutions on the manifolds $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ and $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ which exit via the sections Σ_{12} and Σ_{13} .

Proposition 5.6.5. *For each sufficiently small $\rho, \delta > 0$, there exists $\varepsilon_0 > 0$ and sufficiently small choice of the intervals I_c, I_a such that the following holds. For each sufficiently small $\Delta_{y_1} > 0$ and each $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$, the manifolds $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ and $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ intersect Σ_1^{in} in smooth curves $z_1 = z_{1,0}^\ell(y_1; c, a, \varepsilon)$ and $z_1 = z_{1,0}^r(y_1; c, a, \varepsilon)$ for $|y_1| \leq \Delta_{y_1}$. Furthermore, there exists $C > 0$ independent of*

c, a, ε and $0 < \kappa(\rho) \leq C_0 \rho |\log \rho|$ where C_0 is independent of $c, a, \varepsilon, \rho, \delta$ such that for any $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$, the following hold

- (i) The parts of the manifolds $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a), \widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ which exit via Σ_{13} intersect Σ_{13} in curves $z_1 = z_1^{\ell,r}(r_1)$ which satisfy $\left| \frac{dz_1^{\ell,r}}{dr_1} \right| \leq C |\log \varepsilon|$ uniformly in y_1 .
- (ii) The parts of the manifolds $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a), \widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ which exit via Σ_{12} intersect Σ_{12} in curves $z_1 = z_1^{\ell,r}(y_1)$ which satisfy

$$\begin{aligned} \left| z_1^{\ell,r} \right| &\leq C \varepsilon^{1/3} |\log \varepsilon| \\ \left| \frac{dz_1^{\ell,r}}{dy_1} \right| &\leq C \varepsilon |\log \varepsilon| \end{aligned} \tag{6.38}$$

and

$$0 < \kappa(\rho) \varepsilon^{1/3} < z_1^r(y_1) - z_1^\ell(y_1) < C \varepsilon^{1/3} |\log \varepsilon|. \tag{6.39}$$

uniformly in y_1 .

Proof. We focus on the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$; the computations for $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ are similar.

First we consider the function $z_{1,0}^\ell(y_1; c, a, \varepsilon)$. By taking $\Delta_{y_1} \ll \rho^2$, for any sufficiently small ρ we have that

$$\begin{aligned} \sup_{|y_1| \leq \Delta_{y_1}} |z_{1,0}^\ell(y_1; c, a, \varepsilon)| &\leq C_0 \rho |\log \rho| \\ \sup_{|y_1| \leq \Delta_{y_1}} \left| \frac{dz_{1,0}^\ell(y_1; c, a, \varepsilon)}{dy_1} \right| &\leq \frac{C_0}{\rho}, \end{aligned}$$

for some C_0 independent of $(c, a, \varepsilon, \rho, \delta)$, provided ε and the intervals I_c, I_a are suffi-

ciently small. This follows from Lemma 5.6.1 by taking $\rho^2 \Delta_{y_1} = \Delta_y \ll \rho^4$.

To prove (i), for each sufficiently small $|y_{1,0}| \leq \Delta_{y_1}$, we consider solutions starting in Σ_A^{in} with $(r_1, z_1, y_1, \varepsilon_1)(0) = (\rho, z_{1,0}^\ell(y_{1,0}), y_{1,0}, \varepsilon/\rho^3)$ which exit via Σ_{13} at time $T_1^*(y_{1,0}; c, a, \varepsilon)$. As the solution exits via Σ_{13} , we must have $y_1(T_1^*) = -1$ and $\varepsilon/\rho^3 < \varepsilon_1(T_1^*) = \varepsilon_1^* \leq \delta$.

We define $\Phi_1(t, s)$ to be the linear evolution of the constant coefficient system

$$\begin{pmatrix} \dot{r}_1 \\ \dot{z}_1 \end{pmatrix} = \begin{pmatrix} -1/4 & 0 \\ -1 & -1/4 \end{pmatrix} \begin{pmatrix} r_1 \\ z_1 \end{pmatrix}. \quad (6.40)$$

We set

$$U_1 = \begin{pmatrix} r_1 \\ z_1 \end{pmatrix} \quad (6.41)$$

$$U_{1,0} = \begin{pmatrix} r_{1,0} \\ z_{1,0}^\ell(y_{1,0}) \end{pmatrix}, \quad (6.42)$$

and we rewrite (6.32) as the integral equation

$$\begin{aligned} U_1(t) &= \Phi_1(t, 0)U_{1,0} + \int_0^t \Phi_1(t, s)g_{U_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))ds \\ &=: F_{U_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a) \\ y_1(t) &= -e^{\frac{1}{2}(t-T_1^*)} + \int_{T_1^*}^t e^{\frac{1}{2}(t-s)} g_{y_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))ds \\ &=: F_{y_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a) \\ \varepsilon_1(t) &= \frac{\varepsilon}{\rho^3} e^{\frac{3}{4}t} + \int_0^t e^{\frac{3}{4}(t-s)} g_{\varepsilon_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))ds \\ &=: F_{\varepsilon_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a), \end{aligned} \quad (6.43)$$

where

$$\begin{aligned}
g_{U_1}(r_1, z_1, y_1, \varepsilon_1) &= \begin{pmatrix} \frac{1}{4}r_1^2 y_1 z_1 + \mathcal{O}(r_1^3 z_1^2) \\ -\frac{3}{4}r_1 y_1 z_1^2 + \mathcal{O}(r_1^2 z_1^3) \end{pmatrix} \\
&= \mathcal{O}(|U_1|^3) \\
g_{y_1}(r_1, z_1, y_1, \varepsilon_1) &= -kr_1 \varepsilon_1 - \frac{1}{2}r_1 y_1^2 z_1 + \mathcal{O}(r_1^2 y_1 z_1^2, r_1^5 \varepsilon_1, r_1^3 \varepsilon_1 y_1, r_1^4 \varepsilon_1 z_1, r_1^4 \varepsilon_1^2) \quad (6.44) \\
&= \mathcal{O}(|U_1||\varepsilon_1| + |U_1|^2|y_1|) \\
g_{\varepsilon_1}(r_1, z_1, y_1, \varepsilon_1) &= -\frac{3}{4}r_1 y_1 z_1 \varepsilon_1 + \mathcal{O}(r_1^2 z_1^2 \varepsilon_1) \\
&= \mathcal{O}(|U_1|^2|\varepsilon_1|),
\end{aligned}$$

and we assume $T_1^* \geq 0$ is such that $\left| \frac{\varepsilon}{\rho^3} e^{\frac{3}{4}T_1^*} \right| \leq 2\delta$. We define the spaces

$$\begin{aligned}
V_{\frac{1}{4}}^- &= \left\{ U_1 : [0, T_1^*] \rightarrow \mathbb{R}^2 : \|U_1\|_{\frac{1}{4}}^- = \sup_{t \in [0, T_1^*]} \frac{e^{\frac{1}{4}t}}{1+|t|} |U_1(t)| < \infty \right\} \\
V_{\frac{1}{2}}^+ &= \left\{ y_1 : [0, T_1^*] \rightarrow \mathbb{R} : \|y_1\|_{\frac{1}{2}}^+ = \sup_{t \in [0, T_1^*]} e^{\frac{1}{2}(T_1^*-t)} |y_1(t)| < \infty \right\} \quad (6.45) \\
V_{\frac{3}{4}}^+ &= \left\{ \varepsilon_1 : [0, T_1^*] \rightarrow \mathbb{R} : \|\varepsilon_1\|_{\frac{3}{4}}^+ = \sup_{t \in [0, T_1^*]} e^{\frac{3}{4}(T_1^*-t)} |\varepsilon_1(t)| < \infty \right\},
\end{aligned}$$

and search for solutions $(U_1, y_1, \varepsilon_1) \in V_{\frac{1}{4}}^- \times V_{\frac{1}{2}}^+ \times V_{\frac{3}{4}}^+$ to (6.43). We note that

$$\|U_1\|_{\infty} \leq \|U_1\|_{\frac{1}{4}}^-, \quad \|y_1\|_{\infty} \leq \|y_1\|_{\frac{1}{2}}^+, \quad \|\varepsilon_1\|_{\infty} \leq \|\varepsilon_1\|_{\frac{3}{4}}^+, \quad (6.46)$$

where $\|X\|_{\infty} = \sup_{t \in [0, T_1^*]} |X(t)|$ denotes the C^0 -norm.

First we show that for each fixed $(U_{1,0}, T_1^*)$, the mapping

$$(U_1, y_1, \varepsilon_1) \rightarrow F_1(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a), \quad (6.47)$$

defined by

$$F_1(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a) = \begin{pmatrix} F_{U_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a) \\ F_{y_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a) \\ F_{\varepsilon_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a) \end{pmatrix} \quad (6.48)$$

maps the space $V_{\frac{1}{4}}^- \times V_{\frac{1}{2}}^+ \times V_{\frac{3}{4}}^+$ into itself. We compute

$$\begin{aligned} & \|F_{U_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a)\|_{\frac{1}{4}}^- \\ &= \sup_{t \in [0, T_1^*]} \frac{e^{\frac{1}{4}t}}{1 + |t|} \left(\Phi_1(t, 0)U_{1,0} + \int_0^t \Phi_1(t, s)g_{U_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))ds \right) \\ &\leq C|U_{1,0}| + C \left(\|U_1\|_{\frac{1}{4}}^- \right)^3, \end{aligned} \quad (6.49)$$

where we used (6.46) and the fact that $|\Phi_1(t, s)| \leq |t - s|e^{-\frac{1}{4}(t-s)}$.

Similarly, we compute

$$\begin{aligned} & \|F_{y_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a)\|_{\frac{1}{2}}^+ \\ &= \sup_{t \in [0, T_1^*]} e^{\frac{1}{2}(T_1^* - t)} \left(e^{\frac{1}{2}(t - T_1^*)} + \int_{T_1^*}^t e^{\frac{1}{2}(t-s)} g_{y_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))ds \right) \\ &\leq 1 + Ce^{\frac{1}{2}T_1^*} \int_0^{T_1^*} e^{-\frac{1}{2}s} (|U_1(s)||\varepsilon_1(s)| + |U_1(s)|^2|y_1(s)|) ds \\ &\leq 1 + C \left(\|U_1\|_{\frac{1}{4}}^- \|\varepsilon_1\|_{\frac{3}{4}}^+ + \left(\|U_1\|_{\frac{1}{4}}^- \right)^2 \|y_1\|_{\frac{1}{2}}^+ \right), \end{aligned} \quad (6.50)$$

and

$$\begin{aligned}
& \|F_{\varepsilon_1}(U_1, y_1, \varepsilon_1, U_{1,0}, T_1^*; c, a)\|_{\frac{3}{4}}^+ \\
&= \sup_{t \in [0, T_1^*]} e^{\frac{3}{4}(T_1^* - t)} \left(\frac{\varepsilon}{\rho^3} e^{\frac{3}{4}t} + \int_0^t e^{\frac{3}{4}(t-s)} g_{\varepsilon_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s)) ds \right) \\
&\leq \left| \frac{\varepsilon}{\rho^3} e^{\frac{3}{4}T_1^*} \right| + C e^{\frac{3}{4}T_1^*} \int_0^{T_1^*} e^{-\frac{3}{4}s} (|U_1(s)|^2 |\varepsilon_1(s)|) ds \\
&\leq \left| \frac{\varepsilon}{\rho^3} e^{\frac{3}{4}T_1^*} \right| + C \left(\|U_1\|_{\frac{1}{4}}^- \right)^2 \|\varepsilon_1\|_{\frac{3}{4}}^+.
\end{aligned} \tag{6.51}$$

Provided ρ, δ are sufficiently small, for each sufficiently small $U_{1,0}$ and for $\left| \frac{\varepsilon}{\rho^3} e^{\frac{3}{4}T_1^*} \right| \leq 2\delta$ sufficiently small, that is T_1^* is not too large, we can solve (6.43) to find a unique solution satisfying

$$\begin{aligned}
\|U_1\|_{\frac{1}{4}}^- &= \mathcal{O}(|U_{1,0}|) \\
\|y_1\|_{\frac{1}{2}}^+ &= 1 + \mathcal{O}(|U_{1,0}|(\delta + |U_{1,0}|)) \\
\|\varepsilon_1\|_{\frac{3}{4}}^+ &= \mathcal{O}(\delta)
\end{aligned} \tag{6.52}$$

By our assumption that we consider only solutions exiting via Σ_{13} , and so $\varepsilon_1 \leq \delta$, the time T_1^* satisfies $0 \leq T_1^* \leq C(\rho, \delta) |\log \varepsilon|$ for all sufficiently small $\varepsilon > 0$.

To obtain estimates on the derivatives of the solutions with respect to $U_{1,0}, c, a$, we consider the variational equation

$$\begin{aligned}
dU_1 &= \begin{pmatrix} -1/4 & 0 \\ -1 & -1/4 \end{pmatrix} dU_1 + dg_{U_1}(r_1, z_1, y_1, \varepsilon_1) \\
dy_1 &= \frac{1}{2} dy_1 + dg_{y_1}(r_1, z_1, y_1, \varepsilon_1) \\
d\varepsilon_1 &= \frac{3}{4} d\varepsilon_1 + dg_{\varepsilon_1}(r_1, z_1, y_1, \varepsilon_1),
\end{aligned} \tag{6.53}$$

where

$$\begin{aligned}
dg_{U_1}(r_1, z_1, y_1, \varepsilon_1) &= \mathcal{O}(|U_1|^2 dU_1, |U_1|^3(|dy_1| + |d\varepsilon_1|), |U_1|^3) \\
dg_{y_1}(r_1, z_1, y_1, \varepsilon_1) &= \mathcal{O}\left(|U_1|d\varepsilon_1, (|\varepsilon_1| + |U_1|)dU_1, \right. \\
&\quad \left. (|U_1|^2 + |U_1||\varepsilon_1|)|dy_1|, |U_1|^2|y_1|, |U_1||\varepsilon_1| \right) \\
dg_{\varepsilon_1}(r_1, z_1, y_1, \varepsilon_1) &= \mathcal{O}(|U_1||\varepsilon_1|dU_1, |U_1|^2d\varepsilon_1, |U_1|^2|\varepsilon_1|dy_1, |U_1|^2|\varepsilon_1|).
\end{aligned} \tag{6.54}$$

Proceeding as above, we can rewrite this as an integral equation; using the estimates obtained for the solutions $(U_1, y_1, \varepsilon_1)$ and noting that the derivatives of k with respect to (c, a) are uniformly bounded, we can solve for the derivatives of the solutions on the same spaces and obtain

$$\|D_\nu U_1\|_{\frac{1}{4}}^-, \|D_\nu y_1\|_{\frac{1}{2}}^+, \|D_\nu \varepsilon_1\|_{\frac{3}{4}}^+ \leq C, \tag{6.55}$$

$\nu = U_{1,0}, c, a$, uniformly in $(U_{1,0}, T_1^*, c, a, \varepsilon)$ for all sufficiently small ρ, δ .

We also need estimates on the derivatives with respect to T_1^* . First, we show that these derivatives exist; then we show that they are in fact bounded uniformly in T_1^* . To compute the derivative with respect to T_1^* at some $T_1^* = T_0$, we rescale time by $t = (1 + \omega)\tau$, which results in the differential equation

$$\dot{\hat{X}} = (1 + \omega)F(\hat{X}), \tag{6.56}$$

where $X = (r_1, z_1, y_1, \varepsilon_1)$ and $F(X)$ denotes the RHS of (6.32). Proceeding as above, we can now find solutions to this new system, keeping T_0 fixed and allowing ω to vary as a small parameter, with $|\omega| \leq \omega_0$, where ω_0 is sufficiently small. We obtain

a new integral equation

$$\begin{aligned}
U_1(t) &= \Phi_{1,\omega}(t, 0)U_{1,0} + \int_0^t \Phi_{1,\omega}(t, s)g_{U_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))ds \\
y_1(t) &= -e^{\frac{1}{2}(1+\omega)(t-T_0)} + \int_{T_0}^t e^{\frac{1}{2}(1+\omega)(t-s)}g_{y_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))ds \\
\varepsilon_1(t) &= \frac{\varepsilon}{\rho^3}e^{\frac{3}{4}(1+\omega)t} + \int_0^t e^{\frac{3}{4}(1+\omega)(t-s)}g_{\varepsilon_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))ds,
\end{aligned} \tag{6.57}$$

where the functions $g_{U_1}, g_{y_1}, g_{\varepsilon_1}$ are defined as in (6.44), and $\Phi_{1,\omega}$ denotes the evolution of the constant coefficient system

$$\begin{pmatrix} \dot{r}_1 \\ \dot{z}_1 \end{pmatrix} = \begin{pmatrix} -(1+\omega)/4 & 0 \\ -(1+\omega) & -(1+\omega)/4 \end{pmatrix} \begin{pmatrix} r_1 \\ z_1 \end{pmatrix}. \tag{6.58}$$

We now slightly decrease the exponential weights and solve (6.57) for $(U_1, y_1, \varepsilon_1) \in V_{\frac{1}{4}(1-\omega_0)}^- \times V_{\frac{1}{2}(1-\omega_0)}^+ \times V_{\frac{3}{4}(1-\omega_0)}^+$, where the spaces $V_\eta^\pm$ are defined analogously to (6.45). Further, as above we can use the corresponding variational equation to estimate the derivatives of the solution with respect to the parameters, including ω , noting that they are bounded uniformly in T_0 .

Let $\hat{X}(\tau; T_0, \omega, U_{1,0}, c, a) = (U_1, y_1, \varepsilon_1)(\tau; T_0, \omega, U_{1,0}, c, a)$ denote a solution to (6.57), and let $X(t; T_1^*, U_{1,0}, c, a) = (U_1, y_1, \varepsilon_1)(t; T_1^*, U_{1,0}, c, a)$ denote a solution to the original equation (6.43). By uniqueness, we have that $\hat{X}(T_0, \omega, U_{1,0}, c, a) = X((1+\omega)T_0, U_{1,0}, c, a)$. We now differentiate

$$\begin{aligned}
D_\omega \hat{X}(\tau; T_0, \omega, U_{1,0}, c, a) &= \tau \dot{X}((1+\omega)\tau; (1+\omega)T_0, U_{1,0}, c, a) \\
&\quad + T_0 D_{T_1^*} X((1+\omega)\tau; (1+\omega)T_0, U_{1,0}, c, a),
\end{aligned} \tag{6.59}$$

from which we deduce that the derivative $D_{T_1^*}X$ exists and is bounded in the norms

$$\|D_{T_1^*}U_1\|_{\frac{1}{4}(1-\omega_0)}^-, \|D_{T_1^*}y_1\|_{\frac{1}{2}(1-\omega_0)}^+, \|D_{T_1^*}\varepsilon_1\|_{\frac{3}{4}(1-\omega_0)}^+ \leq C \quad (6.60)$$

uniformly in $(U_{1,0}, T_1^*, c, a, \varepsilon)$ for all sufficiently small ρ, δ .

We can now write the unique solution of (6.43) satisfying

$$\begin{aligned} r_1(0) &= \rho \\ z_1(0) &= z_{1,0}^\ell(y_{1,0}), \end{aligned} \quad (6.61)$$

for sufficiently small $0 > y_{1,0} > -\Delta_{y_1}$ so that $z_1^\ell(y_{1,0}) = \mathcal{O}(\rho \log \rho)$. Recalling $U_1 = (r_1, z_1)$, we have that this solution is given by

$$\begin{aligned} \begin{pmatrix} r_1(t) \\ z_1(t) \end{pmatrix} &= \begin{pmatrix} \rho e^{-\frac{1}{4}t} \\ z_{1,0}^\ell(y_{1,0})e^{-\frac{1}{4}t} + \rho t e^{-\frac{1}{4}t} \end{pmatrix} \\ &\quad + \int_0^t \Phi_1(t, s) g_{U_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s)) ds, \end{aligned} \quad (6.62)$$

where

$$y_{1,0} = -e^{-\frac{1}{2}T_1^*} + \int_{T_1^*}^0 e^{-\frac{1}{2}s} g_{y_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s)) ds. \quad (6.63)$$

We consider only T_1^* large enough so that $y_{1,0} \geq -\Delta_{y_1}$, and we recall that $\Delta_{y_1} < \rho^2$.

This gives

$$\begin{aligned} -\rho^2 &\leq -e^{-\frac{1}{2}T_1^*} + \int_{T_1^*}^0 e^{-\frac{1}{2}s} g_{y_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s)) ds \\ &= -e^{-\frac{1}{2}T_1^*} + \mathcal{O}\left(e^{-\frac{1}{2}T_1^*} \|U_1\|_{1/4}^- \left(\|\varepsilon_1\|_{3/4}^+ + \|U_1\|_{1/4}^-\right)\right) \\ &= -e^{-\frac{1}{2}T_1^*} (1 + \mathcal{O}(\rho^2, \rho\delta)), \end{aligned} \quad (6.64)$$

so that

$$e^{-\frac{1}{2}T_1^*} \leq \rho^2 (1 + \mathcal{O}(\rho^2, \rho\delta)), \quad (6.65)$$

and

$$\begin{aligned} \frac{dy_{1,0}}{dT_1^*} &= \frac{1}{2}e^{-\frac{1}{2}T_1^*} + e^{-\frac{1}{2}T_1^*} g_{y_1}(r_1(T_1^*), z_1(T_1^*), y_1(T_1^*), \varepsilon_1(T_1^*)) \\ &\quad + \int_{T_1^*}^0 e^{-\frac{1}{2}s} \frac{d}{dT_1^*} [g_{y_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))] ds \\ &= e^{-\frac{1}{2}T_1^*} \left(\frac{1}{2} + \mathcal{O}(\rho \log \rho, \delta) \right) \\ &= \rho^2 (1 + \mathcal{O}(\rho \log \rho, \delta)) \end{aligned} \quad (6.66)$$

where we used (6.44), (6.60), and (6.65).

We have

$$\begin{aligned} \begin{pmatrix} r_1(T_1^*) \\ z_1(T_1^*) \end{pmatrix} &= \begin{pmatrix} \rho e^{-\frac{1}{4}T_1^*} \\ z_{1,0}^\ell(y_{1,0}) e^{-\frac{1}{4}T_1^*} + \rho T_1^* e^{-\frac{1}{4}T_1^*} \end{pmatrix} \\ &\quad + \int_0^{T_1^*} \Phi_1(T_1^*, s) g_{U_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s)) ds, \end{aligned} \quad (6.67)$$

and we now compute

$$\begin{aligned} \frac{d}{dT_1^*} \begin{pmatrix} r_1(T_1^*) \\ z_1(T_1^*) \end{pmatrix} &= \begin{pmatrix} -\frac{\rho}{4} e^{-\frac{1}{4}T_1^*} \\ \left(-\frac{z_{1,0}^\ell(y_{1,0})}{4} + (z_{1,0}^\ell)'(y_{1,0}) \frac{dy_{1,0}}{dT_1^*} + \rho - \frac{\rho}{4} T_1^* \right) e^{-\frac{1}{4}T_1^*} \end{pmatrix} \\ &\quad + g_{U_1}(r_1(T_1^*), z_1(T_1^*), y_1(T_1^*), \varepsilon_1(T_1^*)) \\ &\quad + \int_0^{T_1^*} \frac{d}{dT_1^*} [\Phi_1(T_1^*, s) g_{U_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))] ds, \end{aligned} \quad (6.68)$$

where

$$\begin{aligned} z_{1,0}^\ell(y_{1,0}) &= \mathcal{O}(\rho \log \rho) \\ (z_{1,0}^\ell)'(y_{1,0}) \frac{dy_{1,0}}{dT_1^*} &= \mathcal{O}\left(\frac{1}{\rho} e^{-\frac{1}{2}T_1^*} (1 + \mathcal{O}(\rho \log \rho, \delta))\right) = \mathcal{O}(\rho), \end{aligned}$$

by (6.66), and

$$\begin{aligned} g_{U_1}(r_1(T_1^*), z_1(T_1^*), y_1(T_1^*), \varepsilon_1(T_1^*)) &= \mathcal{O}\left((\rho \log \rho)^3 e^{-\frac{3}{4}T_1^*}\right) \\ \int_0^{T_1^*} \frac{d}{dT_1^*} [\Phi_1(T_1^*, s) g_{U_1}(r_1(s), z_1(s), y_1(s), \varepsilon_1(s))] ds &= \begin{pmatrix} \mathcal{O}\left((\rho \log \rho)^2 e^{-\frac{1}{4}T_1^*}\right) \\ \mathcal{O}\left(T_1^* (\rho \log \rho)^2 e^{-\frac{1}{4}T_1^*}\right) \end{pmatrix}, \end{aligned}$$

by (6.44), (6.52), and (6.60).

Therefore we have that

$$\begin{aligned} \frac{dr_1(T_1^*)}{dT_1^*} &= \left(-\frac{\rho}{4} + \mathcal{O}((\rho \log \rho)^2)\right) e^{-\frac{1}{4}T_1^*} \\ \frac{dz_1(T_1^*)}{dT_1^*} &= (\mathcal{O}(\rho \log \rho, (1 + T_1^*)\rho)) e^{-\frac{1}{4}T_1^*} \end{aligned} \tag{6.69}$$

so that in Σ_{13} , for each fixed ρ, δ sufficiently small, we obtain a curve $z_1 = z_1(r_1)$ satisfying

$$\left| \frac{dz_1}{dr_1} \right| \leq C(\rho, \delta)(1 + T_1^*), \tag{6.70}$$

uniformly. Using the fact that $|T_1^*| \leq C(\rho, \delta)|\log \varepsilon|$, we therefore obtain

$$\left| \frac{dz_1}{dr_1} \right| \leq C(\rho, \delta)|\log \varepsilon|, \tag{6.71}$$

uniformly in (c, a, ε) , which completes the proof of (i).

The proof of (ii) is similar and we omit the details. $\square$

5.6.4 Dynamics in $\mathcal{K}_2$

In the $\mathcal{K}_2$ coordinates, the equations are given by

$$\begin{aligned}\dot{x}_2 &= -x_2 + r_2 y_2 z_2 + \mathcal{O}(r_2^2 z_2^2) \\ \dot{z}_2 &= -z_2 - r_2 x_2 \\ \dot{y}_2 &= -k r_2 + \mathcal{O}(r_2^3 y_2, r_2^4) \\ \dot{r}_2 &= 0.\end{aligned}\tag{6.72}$$

Solutions enter via Σ_{12} which is given in the $\mathcal{K}_2$ coordinates by

$$\Sigma_{12} = \left\{ (x_2, y_2, z_2, r_2) : x_2 = \frac{1}{\delta^{4/3}}, |y_2| \leq \frac{1}{\delta^{2/3}}, |z_2| \leq \frac{\mu}{\delta}, 0 < r_2 \leq \rho \delta^{1/3} \right\}, \tag{6.73}$$

and exit via

$$\Sigma_{23} = \left\{ (x_2, y_2, z_2, r_2) : |x_2| \leq \frac{1}{\delta^{4/3}}, y_2 = -\frac{1}{\delta^{2/3}}, |z_2| \leq \frac{\mu}{\delta}, 0 < r_2 \leq \rho \delta^{1/3} \right\}. \tag{6.74}$$

The setup in the chart $\mathcal{K}_2$ is shown in Figure 5.19.

In this chart we can determine formulae for the solutions as follows. First, we consider solutions starting in Σ_{12} as time $t = 0$. We set $x_2 = e^{-t} \tilde{x}_2$, $z_2 = e^{-t} \tilde{z}_2$ and

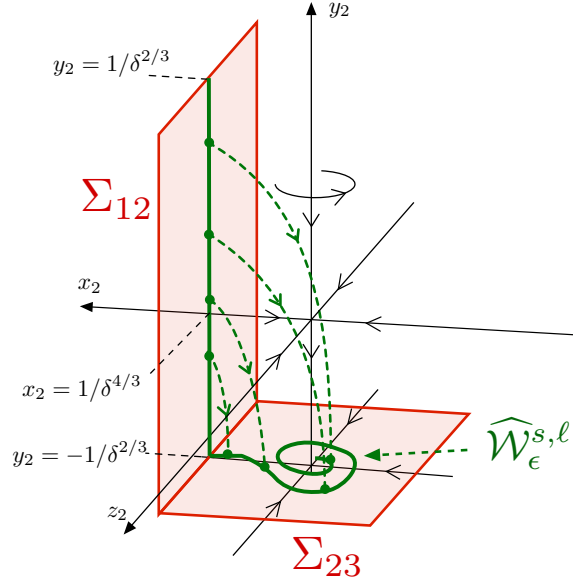


Figure 5.19: Shown is the setup in the chart $\mathcal{K}_2$.

obtain the system

$$\begin{aligned}
 \dot{\tilde{x}}_2 &= r_2 y_2 \tilde{z}_2 + \mathcal{O}(e^{-t} r_2^2 \tilde{z}_2^2) \\
 \dot{\tilde{z}}_2 &= -r_2 \tilde{x}_2 \\
 \dot{y}_2 &= -k r_2 + \mathcal{O}(r_2^3) \\
 \dot{r}_2 &= 0.
 \end{aligned} \tag{6.75}$$

We now rescale time by $t = t_2/r_2$ to desingularize the system

$$\begin{aligned}
 \tilde{x}'_2 &= y_2 \tilde{z}_2 + \mathcal{O}(e^{-t_2/r_2} r_2) \\
 \tilde{z}'_2 &= -\tilde{x}_2 \\
 y'_2 &= -k + \mathcal{O}(r_2^2) \\
 r'_2 &= 0,
 \end{aligned} \tag{6.76}$$

where $\prime$ denotes $\frac{d}{dt_2}$. Setting $r_2 = 0$ we obtain the Airy equations

$$\begin{aligned}\tilde{x}'_2 &= y_2 \tilde{z}_2 \\ \tilde{z}'_2 &= -\tilde{x}_2 \\ y'_2 &= -k,\end{aligned}\tag{6.77}$$

which have the following explicit solutions in terms of Airy functions Ai, Bi

$$\begin{aligned}\tilde{x}_2 &= \pi \left[\left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} + k^{1/3} t_2 \right) \right. \right. \\ &\quad \left. \left. - \text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} + k^{1/3} t_2 \right) \right) \frac{1}{\delta^{4/3}} \right. \\ &\quad \left. + k^{1/3} \left(\text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} + k^{1/3} t_2 \right) \right. \right. \\ &\quad \left. \left. - \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} + k^{1/3} t_2 \right) \right) z_{2,0} \right] \\ \tilde{z}_2 &= \frac{\pi}{k^{1/3}} \left[\left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} + k^{1/3} t_2 \right) \right. \right. \\ &\quad \left. \left. - \text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} + k^{1/3} t_2 \right) \right) \frac{1}{\delta^{4/3}} \right. \\ &\quad \left. + k^{1/3} \left(\text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} + k^{1/3} t_2 \right) \right. \right. \\ &\quad \left. \left. - \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} + k^{1/3} t_2 \right) \right) z_{2,0} \right] \\ y_2 &= y_{2,0} - k t_2,\end{aligned}\tag{6.78}$$

where $y_{2,0} = y_2(0)$ and $z_{2,0} = \tilde{z}_2(0) = z_2(0)$.

Lemma 5.6.6. *For each fixed $\delta, \mu > 0$, there exists $r_{2,0} > 0$ such that for any $0 < r_2 < r_{2,0}$, any solution of (6.75) with initial condition in Σ_{12} given by $(x_2, y_2, z_2)(0) =$*

$(1/\delta^{4/3}, y_{2,0}, z_{2,0})$ reaches Σ_{23} with

$$\begin{aligned}
x_2 &= \pi e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{kr_2}} \left[\left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi}' \left(\frac{1}{k^{2/3} \delta^{2/3}} \right) - \text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai}' \left(\frac{1}{k^{2/3} \delta^{2/3}} \right) \right) \frac{1}{\delta^{4/3}} \right. \\
&\quad \left. + k^{1/3} \left(\text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi}' \left(\frac{1}{k^{2/3} \delta^{2/3}} \right) - \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai}' \left(\frac{1}{k^{2/3} \delta^{2/3}} \right) \right) z_{2,0} + \mathcal{O}(r_2^2) \right] \\
z_2 &= \frac{\pi e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{kr_2}}}{k^{1/3}} \left[\left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai} \left(\frac{1}{k^{2/3} \delta^{2/3}} \right) - \text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi} \left(\frac{1}{k^{2/3} \delta^{2/3}} \right) \right) \frac{1}{\delta^{4/3}} \right. \\
&\quad \left. + k^{1/3} \left(\text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai} \left(\frac{1}{k^{2/3} \delta^{2/3}} \right) - \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi} \left(\frac{1}{k^{2/3} \delta^{2/3}} \right) \right) z_{2,0} + \mathcal{O}(r_2^2) \right] \\
y_2 &= -\frac{1}{\delta^{2/3}}.
\end{aligned} \tag{6.79}$$

Proof. Considering the equations (6.77), solutions given by (6.78) with initial conditions $(x_2, y_2, z_2)(0) = (1/\delta^{4/3}, y_{2,0}, z_{2,0})$ in Σ_{12} exit Σ_{23} in time

$$T_2 = \frac{1}{k} \left(y_{2,0} + \frac{1}{\delta^{2/3}} \right). \tag{6.80}$$

For each fixed $\delta > 0$, T_2 is bounded uniformly in $|y_{2,0}| \leq 1/\delta^{2/3}$. Hence by a regular perturbation argument, and returning to the original coordinates x_2, z_2 , we obtain the result. $\square$

5.6.5 Dynamics in $\mathcal{K}_3$

In the $\mathcal{K}_3$ coordinates, the equations are given by

$$\begin{aligned}\dot{x}_3 &= -x_3 - r_3 z_3 - 2kr_3 x_3 \varepsilon_3 + \mathcal{O}(r_3^2 z_3^2, r_3^3 \varepsilon_3 x_3) \\ \dot{z}_3 &= -z_3 - r_3 x_3 - \frac{3}{2}kr_3 \varepsilon_3 z_3 + \mathcal{O}(r_3^3 z_3 \varepsilon_3) \\ \dot{r}_3 &= \frac{1}{2}r_3^2 \varepsilon_3 (k + \mathcal{O}(r_3^2)) \\ \dot{\varepsilon}_3 &= -\frac{3}{2}r_3 \varepsilon_3^2 (k + \mathcal{O}(r_3^2))\end{aligned}\tag{6.81}$$

Solutions enter via Σ_{13} or Σ_{23} which are given in the $\mathcal{K}_3$ coordinates by

$$\Sigma_{13} = \{(x_3, z_3, r_3, \varepsilon_3) : x_3 = 1, |z_3| \leq \mu, 0 < r_3 \leq \rho, 0 < \varepsilon_3 \leq \delta\} \tag{6.82}$$

$$\Sigma_{23} = \{(x_3, z_3, r_3, \varepsilon_3) : |x_3| \leq 1, |z_3| \leq \mu, 0 < r_3 \leq \rho, \varepsilon_3 = \delta\}, \tag{6.83}$$

respectively, and exit via

$$\Sigma_3^{out} = \{(x_3, z_3, r_3, \varepsilon_3) : |x_3| \leq 1, |z_3| \leq \mu, r_3 = \rho, 0 < \varepsilon_3 \leq \delta\}. \tag{6.84}$$

We need to determine the behavior of solutions which enter via Σ_{13} or Σ_{23} upon exit in Σ_3^{out} . The setup is shown in Figure 5.20. Between these sections, due to the relation $r_3^3 \varepsilon_3 = \varepsilon$, such solutions are restricted to the region $(\varepsilon/\delta)^{1/3} \leq r_3 \leq \rho$ in which r_3 is strictly increasing. The linearization of (6.81) in the (x_3, z_3) -plane has approximate eigenvalues $(-1 \pm r_3)$. Hence, informally one expects that the flow should separate into strong and weak stable directions with an exponential separation that is initially $\mathcal{O}(\varepsilon^{1/3})$ and grows to $\mathcal{O}(1)$ at Σ_3^{out} . We begin by deriving a change of coordinates $(x_3, z_3) \rightarrow (\tilde{x}_3, \tilde{z}_3)$ which more clearly separates these strong/weak directions.

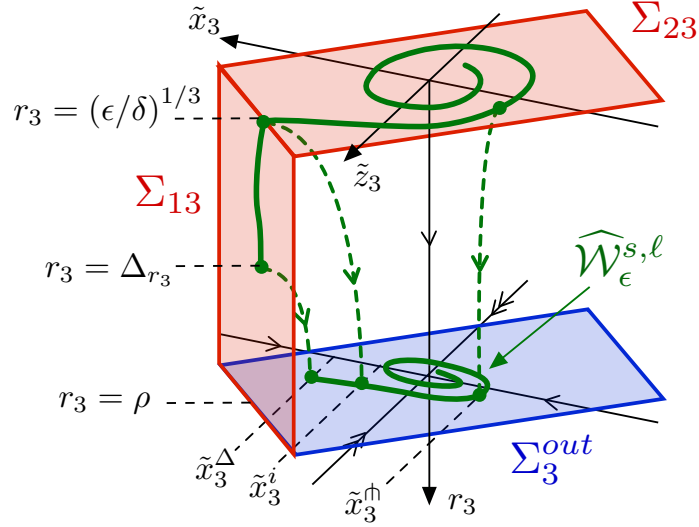


Figure 5.20: Shown is the setup in the chart $\mathcal{K}_3$, including a schematic of the results of Lemma 5.6.10, Lemma 5.6.11, and Lemma 5.6.12.

To see this, we add an equation for the ratio $\theta_3 := z_3/x_3$

$$\begin{aligned}
 \dot{\theta}_3 &= \frac{\dot{z}_3}{x_3} - \theta_3 \frac{\dot{x}_3}{x_3} \\
 &= -r_3 + r_3 \theta_3^2 - \frac{3}{2} k r_3 \varepsilon_3 \theta_3 + 2 k r_3 \varepsilon_3 \theta_3 + \mathcal{O}(r_3^3 \theta_3 \varepsilon_3, r_3^2 \theta_3^2) \\
 &= r_3 \left(\theta_3^2 - 1 + \frac{1}{2} k \varepsilon_3 \theta_3 \right) + \mathcal{O}(r_3^2),
 \end{aligned} \tag{6.85}$$

and we consider the extended system

$$\begin{aligned}
 \dot{x}_3 &= -x_3 - r_3 z_3 - 2 k r_3 x_3 \varepsilon_3 + \mathcal{O}(r_3^2(|x_3| + |z_3|)) \\
 \dot{z}_3 &= -z_3 - r_3 x_3 - \frac{3}{2} k r_3 \varepsilon_3 z_3 + \mathcal{O}(r_3^3 z_3 \varepsilon_3) \\
 \dot{\theta}_3 &= r_3 \left(\theta_3^2 - 1 + \frac{1}{2} k \varepsilon_3 \theta_3 \right) + \mathcal{O}(r_3^2) \\
 \dot{r}_3 &= \frac{1}{2} r_3^2 \varepsilon_3 (k + \mathcal{O}(r_3^2)) \\
 \dot{\varepsilon}_3 &= -\frac{3}{2} r_3 \varepsilon_3^2 (k + \mathcal{O}(r_3^2)).
 \end{aligned} \tag{6.86}$$

Solutions are exponentially attracted to the subspace $x_3 = z_3 = 0$ on which the

flow is given by

$$\begin{aligned}\dot{\theta}_3 &= r_3 \left(\theta_3^2 - 1 + \frac{1}{2} k \varepsilon_3 \theta_3 \right) + \mathcal{O}(r_3^2) \\ \dot{r}_3 &= \frac{1}{2} r_3^2 \varepsilon_3 (k + \mathcal{O}(r_3^2)) \\ \dot{\varepsilon}_3 &= -\frac{3}{2} r_3 \varepsilon_3^2 (k + \mathcal{O}(r_3^2)).\end{aligned}\tag{6.87}$$

Rescaling time by $t_3 = r_3 t$, we obtain

$$\begin{aligned}\theta'_3 &= \theta_3^2 - 1 + \frac{1}{2} k \varepsilon_3 \theta_3 + \mathcal{O}(r_3) \\ r'_3 &= \frac{1}{2} r_3 \varepsilon_3 (k + \mathcal{O}(r_3^2)) \\ \varepsilon'_3 &= -\frac{3}{2} \varepsilon_3^2 (k + \mathcal{O}(r_3^2)).\end{aligned}\tag{6.88}$$

Firstly, there are two invariant subspaces for the dynamics of (6.88): the plane $r_3 = 0$ and the plane $\varepsilon_3 = 0$. Their intersection is the invariant line $l_3 = \{(\theta_3, 0, 0) : \theta_3 \in \mathbb{R}\}$, and the dynamics on l_3 evolve according to $\theta'_3 = -1 + \theta_3^2$. There are two equilibria $p^- = (-1, 0, 0)$ and $p^+ = (1, 0, 0)$, with eigenvalues -2 and 2 , respectively, for the flow along l_3 . In the plane $\varepsilon_3 = 0$, the dynamics are given by

$$\begin{aligned}\theta'_3 &= \theta_3^2 - 1 + \mathcal{O}(r_3) \\ r'_3 &= 0.\end{aligned}\tag{6.89}$$

This system has normally hyperbolic curves $S_{0,3}^\pm(c, a)$ of equilibria emanating from $p^\pm$ (see Figure 5.21). Along $S_{0,3}^\pm(c, a)$ the linearization has one zero eigenvalue and one eigenvalue close to ± 2 for small r_3 .

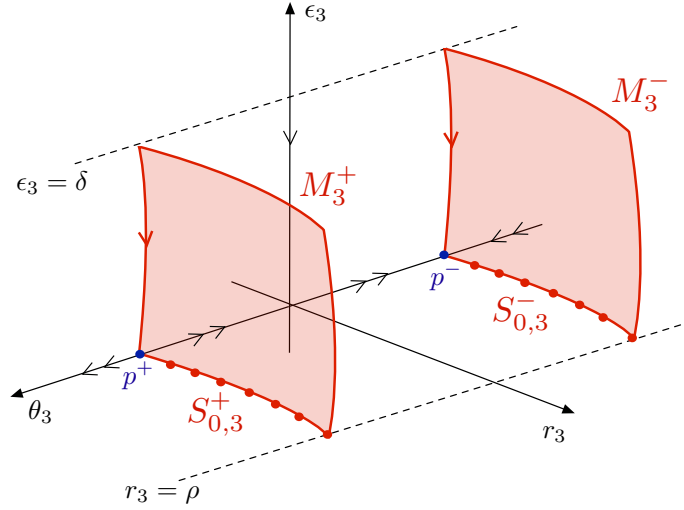


Figure 5.21: Shown are the invariant manifolds $M_3^\pm(c, a)$ corresponding to the dynamics of (6.88).

In the invariant plane $r_3 = 0$, the dynamics are given by

$$\begin{aligned}\theta'_3 &= \theta_3^2 - 1 + \frac{1}{2}k\varepsilon_3\theta_3 \\ \varepsilon'_3 &= -\frac{3}{2}\varepsilon_3^2.\end{aligned}\tag{6.90}$$

Here we still have the equilibria $p^\pm$ which now have an additional zero eigenvalue due to the second equation. The corresponding eigenvector is $(-k, 4)$ and hence there exist one-dimensional center manifolds $N_3^\pm(c, a)$ at $p^\pm$ along which ε_3 decreases. Note that the branch of $N_3^+(c, a)$ in the half space $\varepsilon_1 > 0$ is unique.

Restricting attention to the set

$$D_3 = \{(\theta_3, r_3, \varepsilon_3) : \theta_3 \in \mathbb{R}, 0 \leq r_3 \leq \rho, 0 \leq \varepsilon_1 \leq \delta\},\tag{6.91}$$

we have the following (see Figure 5.21).

Proposition 5.6.7. *For any $(c, a) \in I_c \times I_a$ and any sufficiently small $\rho, \delta > 0$, the following assertions hold for the dynamics of (6.88):*

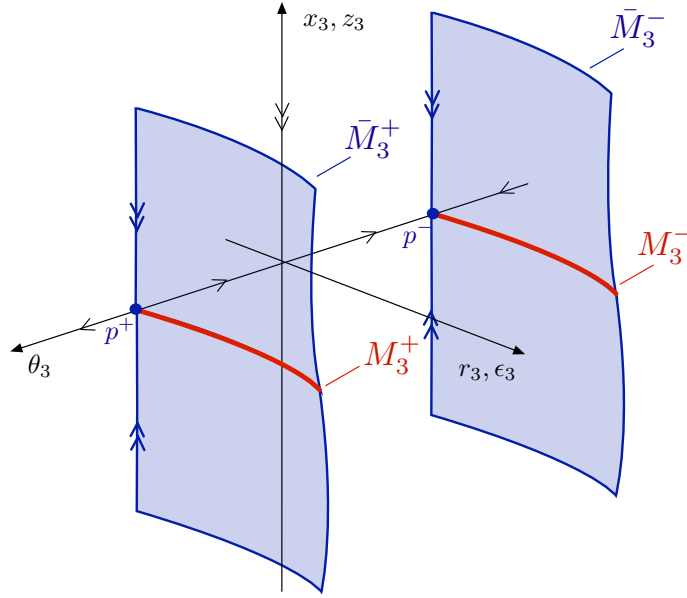


Figure 5.22: Shown are the invariant manifolds $\bar{M}_3^\pm(c, a)$ corresponding to the dynamics of (6.86).

- (i) *There exists a repelling two-dimensional center manifold $M_3^+(c, a)$ at p^+ which contains the line of equilibria $S_{0,3}^+(c, a)$ and the center manifold $N_3^+(c, a)$. In D_3 , $M_3^+(c, a)$ is given as a graph $\theta_3 = h^+(r_3, \varepsilon_3, c, a) = 1 + \mathcal{O}(r_3, \varepsilon_3)$. The branch of $N_3^+(c, a)$ in $r_3 = 0, \varepsilon_3 > 0$ is unique.*
- (ii) *There exists an attracting two-dimensional center manifold $M_3^-(c, a)$ at p^- which contains the line of equilibria $S_{0,3}^-(c, a)$ and the center manifold $N_3^-(c, a)$. In D_3 , $M_3^-(c, a)$ is given as a graph $\theta_3 = h^-(r_3, \varepsilon_3, c, a) = -1 + \mathcal{O}(r_3, \varepsilon_3)$.*

We now return to the full system (6.86), in which the flow on the subspace $x_3 = z_3 = 0$ is foliated by flow along strong stable fibers. Hence in the full five-dimensional space, there exist four-dimensional invariant manifolds $\bar{M}_3^\pm(c, a)$ (see Figure 5.22) given by the strong stable foliations of the two-dimensional manifolds $M_3^\pm(c, a)$. The manifolds $\bar{M}_3^\pm(c, a)$ can be written as graphs $\theta_3 = H^\pm(x_3, z_3, r_3, \varepsilon_3, c, a) = \pm 1 + \mathcal{O}(r_3, \varepsilon_3)$.

Now using the relation $\theta_3 = z_3/x_3$, we see that the dynamics are in fact restricted

to three-dimensional invariant submanifolds $\tilde{M}_3^\pm(c, a)$ of $\bar{M}_3^\pm(c, a)$. The manifolds $\tilde{M}_3^\pm(c, a)$ are given by $z_3 = x_3 H^\pm(x_3, z_3, r_3, \varepsilon_3, c, a)$. By the implicit function theorem, for any sufficiently small $\rho, \delta > 0$, we can now solve to find $\tilde{M}_3^\pm(c, a)$ as graphs

$$\begin{aligned} z_3 &= F^-(x_3, r_3, \varepsilon_3, c, a) = x_3(-1 + \mathcal{O}(r_3, \varepsilon_3)) \\ x_3 &= F^+(z_3, r_3, \varepsilon_3, c, a) = z_3(1 + \mathcal{O}(r_3, \varepsilon_3)). \end{aligned} \quad (6.92)$$

We now change coordinates by

$$\begin{aligned} \tilde{z}_3 &= z_3 - F^-(x_3, r_3, \varepsilon_3, c, a) = z_3 + x_3(1 + \mathcal{O}(r_3, \varepsilon_3)) \\ \tilde{x}_3 &= x_3 - F^+(z_3, r_3, \varepsilon_3, c, a) = x_3 - z_3(1 + \mathcal{O}(r_3, \varepsilon_3)) \end{aligned} \quad (6.93)$$

In these coordinates, (6.81) becomes

$$\begin{aligned} \dot{\tilde{x}}_3 &= (-1 + r_3 + r_3 h_+(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3)) \tilde{x}_3 \\ \dot{\tilde{z}}_3 &= (-1 - r_3 + r_3 h_-(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3)) \tilde{z}_3 \\ \dot{r}_3 &= \frac{1}{2} r_3^2 \varepsilon_3 (k + g_1(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3)) \\ \dot{\varepsilon}_3 &= -\frac{3}{2} r_3 \varepsilon_3^2 (k + g_2(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3)), \end{aligned} \quad (6.94)$$

where

$$\begin{aligned} h_+(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3) &= \mathcal{O}(r_3, \varepsilon_3) \\ h_-(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3) &= \mathcal{O}(r_3, \varepsilon_3) \\ g_1(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3) &= \mathcal{O}(r_3^2) \\ g_2(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3) &= \mathcal{O}(r_3^2). \end{aligned} \quad (6.95)$$

In (6.94), it is clear that the strong attraction in the variables (x_3, z_3) splits into strong/weak directions where the exponential splitting increases as r_3 increases. By changing coordinates to $(\tilde{x}_3, \tilde{z}_3)$, we straighten out the invariant manifolds $\tilde{M}_3^\pm(c, a)$

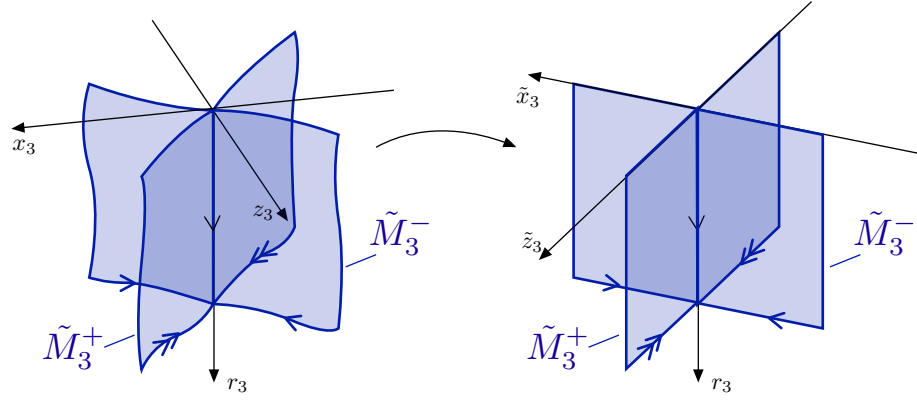


Figure 5.23: Shown are the invariant manifolds $\tilde{M}_3^\pm(c, a)$ transformed to the $(\tilde{x}_3, \tilde{z}_3)$ -coordinates.

(see Figure 5.23).

Solutions with initial conditions in Σ_{23}

We first consider solutions entering $\mathcal{K}_3$ via Σ_{23} . Using $r_3^3 \varepsilon_3 = \varepsilon$, we have that such solutions satisfy $\varepsilon_3 = \delta, r_3 = (\varepsilon/\delta)^{1/3}$ in Σ_{23} . We have the following.

Lemma 5.6.8. *For all sufficiently small $\rho, \delta > 0$, any solution to (6.94) with initial condition $(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3)(0) = (\tilde{x}_{3,0}, \tilde{z}_{3,0}, (\varepsilon/\delta)^{1/3}, \delta) \in \Sigma_{23}$ which reaches the section Σ_3^{out} at time $t = T^* = T^*(\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \varepsilon)$ satisfies*

$$\begin{aligned}
 \tilde{x}_3(T^*) &= \tilde{x}_{3,0} \exp \left(\beta_+^2(\rho, \delta, \varepsilon) + \eta_+^2(\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \varepsilon) \right) \\
 \tilde{z}_3(T^*) &= \tilde{z}_{3,0} \exp \left(\beta_-^2(\rho, \delta, \varepsilon) + \eta_-^2(\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \varepsilon) \right) \\
 r_3(T^*) &= \rho \\
 \varepsilon_3(T^*) &= \frac{\varepsilon}{\rho^3},
 \end{aligned} \tag{6.96}$$

where

$$\begin{aligned}
\beta_-^2(\rho, \delta, \varepsilon) &= \frac{\rho^2}{\varepsilon} \left(-1 - \frac{2\rho}{3} + \mathcal{O}(\rho^2, \rho\delta) \right) \\
\beta_+^2(\rho, \delta, \varepsilon) &= \frac{\rho^2}{\varepsilon} \left(-1 + \frac{2\rho}{3} + \mathcal{O}(\rho^2, \rho\delta) \right) \\
\eta_\pm^2(\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \varepsilon) &= \mathcal{O} \left(\left(\frac{\varepsilon}{\delta} \right)^{1/3} (|\tilde{x}_{3,0}| + |\tilde{z}_{3,0}|) \right).
\end{aligned} \tag{6.97}$$

Proof. It is clear from (6.94) that the $(\tilde{x}_3, \tilde{z}_3)$ -coordinates decay exponentially for all sufficiently small $\rho, \delta > 0$. By directly integrating (6.94), we obtain the following expressions

$$\begin{aligned}
\tilde{x}_3(T^*) &= \tilde{x}_{3,0} \exp \left(-T^* + \int_0^{T^*} r_3(t) (1 + h_+(\tilde{x}_3(t), \tilde{z}_3(t), r_3(t), \varepsilon_3(t))) dt \right) \\
\tilde{z}_3(T^*) &= \tilde{z}_{3,0} \exp \left(-T^* - \int_0^{T^*} r_3(t) (1 + h_-(\tilde{x}_3(t), \tilde{z}_3(t), r_3(t), \varepsilon_3(t))) dt \right) \\
r_3(T^*) &= \rho \\
\varepsilon_3(T^*) &= \frac{\varepsilon}{\rho^3}.
\end{aligned} \tag{6.98}$$

We determine the functions β_+^2, η_+^2 . The computation of β_-^2, η_-^2 is similar. We now write

$$\begin{aligned}
T^* &= \int_{(\varepsilon/\delta)^{1/3}}^{\rho} \frac{1}{\dot{r}_3} dr_3 \\
&= \frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} r_3 (1 + \mathcal{O}(r_3^2)) dr_3,
\end{aligned} \tag{6.99}$$

using $r_3^3 \varepsilon_3 = \varepsilon$. We also have

$$\int_0^{T^*} r_3(t) (1 + h_+(\tilde{x}_3(t), \tilde{z}_3(t), r_3(t), \varepsilon_3(t))) dt = \frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} r_3^2 (1 + \mathcal{O}(r_3, \varepsilon_3)) dr_3, \tag{6.100}$$

and hence

$$\begin{aligned}
-T^* + \int_0^{T^*} r_3(t) (1 + \mathcal{O}(r_3(t), \varepsilon_3(t))) dt \\
= -\frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} r_3 - r_3^2 + \mathcal{O}(r_3^3, r_3^2 \varepsilon_3) dr_3 \\
= -\frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} r_3 - r_3^2 + h_1(r_3, \varepsilon_3) + h_2(r_3, \varepsilon_3, \tilde{x}_3, \tilde{z}_3) dr_3,
\end{aligned} \tag{6.101}$$

where we have separated out the $\tilde{x}_3, \tilde{z}_3$ dependence through the functions h_1, h_2 .

That is, we have $\partial_{\tilde{x}_3} h_1 = \partial_{\tilde{z}_3} h_1 = 0$ and

$$\begin{aligned}
h_1(r_3, \varepsilon_3) &= \mathcal{O}(r_3^3, r_3^2 \varepsilon_3) \\
h_2(r_3, \varepsilon_3, \tilde{x}_3, \tilde{z}_3) &= \mathcal{O}(r_3^2(|r_3| + |\varepsilon_3|)(|\tilde{x}_3| + |\tilde{z}_3|)).
\end{aligned} \tag{6.102}$$

We now define

$$\begin{aligned}
\beta_+^2(\rho, \delta, \varepsilon) &= -\frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} r_3 - r_3^2 + h_1(r_3, \varepsilon_3) dr_3 \\
&= -\frac{2}{\varepsilon} \left(\frac{\rho^2}{2} - \frac{\rho^3}{3} + \mathcal{O}(\rho^4, \rho^3 \delta) \right) \\
\eta_+^2(\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \varepsilon) &= -\frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} h_2(r_3, \varepsilon_3, \tilde{x}_3, \tilde{z}_3) dr_3.
\end{aligned} \tag{6.103}$$

To estimate η_+^2 , we first note that for any sufficiently small ρ, δ , we can bound

$$\begin{aligned}
|\tilde{x}_3(t)| &\leq \tilde{x}_{3,0} \exp(-t/2) \\
|\tilde{z}_3(t)| &\leq \tilde{z}_{3,0} \exp(-t/2),
\end{aligned} \tag{6.104}$$

for any $0 \leq t \leq T^*$. Furthermore, we have

$$\begin{aligned}
t &= \int_{(\varepsilon/\delta)^{1/3}}^{r_3(t)} \frac{1}{\dot{r}_3} dr_3 \\
&= \frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{r_3(t)} r_3 (1 + \mathcal{O}(r_3^2)) dr_3 \\
&> \frac{1}{2\varepsilon} (r_3(t)^2 - (\varepsilon/\delta)^{2/3}),
\end{aligned} \tag{6.105}$$

for each sufficiently small fixed $\rho, \delta > 0$. Hence we have

$$\begin{aligned}
&\eta_+^2(\rho, \delta, \tilde{x}_{3,0}, \tilde{z}_{3,0}, \varepsilon) \\
&= -\frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} h_2(r_3, \varepsilon_3, \tilde{x}_3, \tilde{z}_3) dr_3 \\
&= -\frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} \mathcal{O}(r_3^2(|r_3| + |\varepsilon_3|)(|\tilde{x}_3| + |\tilde{z}_3|)) dr_3 \\
&= -\frac{2}{\varepsilon} \int_{(\varepsilon/\delta)^{1/3}}^{\rho} \mathcal{O}\left(r_3^2(|r_3| + |\varepsilon_3|)(|\tilde{x}_{3,0}| + |\tilde{z}_{3,0}|) \exp\left(-\frac{1}{4\varepsilon}(r_3^2 - (\varepsilon/\delta)^{2/3})\right)\right) dr_3 \\
&= \mathcal{O}\left(\left(\frac{\varepsilon}{\delta}\right)^{1/3} (|\tilde{x}_{3,0}| + |\tilde{z}_{3,0}|)\right)
\end{aligned} \tag{6.106}$$

that is, the dependence on the initial $(\tilde{x}_{3,0}, \tilde{z}_{3,0})$ of the exponential contraction between Σ_{23} and Σ_3^{out} is very small. $\square$

We now consider solutions on $\widehat{\mathcal{W}}_\varepsilon^{s,j}(c, a)$, $j = \ell, r$ passing through $\Sigma_A^{in} = \Sigma_1^{in} \rightarrow \Sigma_{12} \rightarrow \Sigma_{23}$. We obtain estimates for these solutions upon entry in the chart $\mathcal{K}_3$ in Σ_{23} and exit via Σ_3^{out} .

Lemma 5.6.9. *Solutions on the manifolds $\widehat{\mathcal{W}}_\varepsilon^{s,j}(c, a)$, $j = \ell, r$, which have initial*

conditions in Σ_{23} define curves in Σ_{23} parametrized by $|y_{2,0}| \leq 1/\delta^{2/3}$ given by

$$\begin{aligned}\tilde{x}_{3,0}^j(y_{2,0}) &= \frac{\sqrt{\pi} e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{k\varepsilon^{1/3}}}}{k^{1/6} \delta^{1/6}} \tilde{X}_3^j(y_{2,0}) \\ \tilde{z}_{3,0}^j(y_{2,0}) &= \frac{\sqrt{\pi} e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{k\varepsilon^{1/3}}}}{k^{1/6} \delta^{1/6}} \tilde{Z}_3^j(y_{2,0}),\end{aligned}\tag{6.107}$$

where

$$\begin{aligned}\tilde{X}_3^j(y_{2,0}) &= \left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{\frac{2}{3} \frac{1}{k\delta}} (2 + \mathcal{O}(\delta)) \\ &\quad + \mathcal{O}(\delta) \left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{-\frac{2}{3} \frac{1}{k\delta}} + \mathcal{O}(\varepsilon^{2/3}) \\ \tilde{Z}_3^j(y_{2,0}) &= \left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{-\frac{2}{3} \frac{1}{k\delta}} (1 + \mathcal{O}(\delta)) \\ &\quad + \mathcal{O}(\delta) \left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{\frac{2}{3} \frac{1}{k\delta}} + \mathcal{O}(\varepsilon^{2/3}),\end{aligned}\tag{6.108}$$

for $j = \ell, r$, where

$$\begin{aligned}|z_{2,0}^j(y_{2,0})| &\leq C\varepsilon^{1/3} |\log \varepsilon| \\ \left| \frac{dz_{2,0}^j}{dy_{2,0}} \right| &\leq C\varepsilon^{1/3} |\log \varepsilon|\end{aligned}\tag{6.109}$$

and

$$\kappa \varepsilon^{1/3} < z_{2,0}^r(y_{2,0}) - z_{2,0}^\ell(y_{2,0}) < C\varepsilon^{1/3} |\log \varepsilon|.\tag{6.110}$$

uniformly in $y_{2,0}$ for some $C, \kappa > 0$ independent of c, a, ε .

Proof. Using the analysis in §5.6.4, Lemma 5.6.6 and the estimates in Proposition 5.6.5 (ii), we deduce that solutions on the manifolds $\widehat{\mathcal{W}}_\varepsilon^{s,j}(c, a)$ define curves

in Σ_{23} parametrized by $|y_{2,0}| \leq 1/\delta^{2/3}$ as

$$\begin{aligned}
x_3 &= x_{3,0}^j(y_{2,0}) \\
&= \pi e^{-\frac{\frac{1}{\delta^{2/3}} + y_{2,0}}{kr_2}} \left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi}' \left(\frac{1}{k^{2/3}\delta^{2/3}} \right) - \text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai}' \left(\frac{1}{k^{2/3}\delta^{2/3}} \right) \right. \\
&\quad \left. + k^{1/3}\delta^{4/3} \left(\text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi}' \left(\frac{1}{k^{2/3}\delta^{2/3}} \right) - \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai}' \left(\frac{1}{k^{2/3}\delta^{2/3}} \right) \right) z_{2,0}^j(y_{2,0}) \right. \\
&\quad \left. + \mathcal{O}(r_2^2) \right) \\
z_3 &= z_{3,0}^j(y_{2,0}) \\
&= \frac{\pi e^{-\frac{\frac{1}{\delta^{2/3}} + y_{2,0}}{kr_2}}}{k^{1/3}\delta^{1/3}} \left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai} \left(\frac{1}{k^{2/3}\delta^{2/3}} \right) - \text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi} \left(\frac{1}{k^{2/3}\delta^{2/3}} \right) \right. \\
&\quad \left. + k^{1/3}\delta^{4/3} \left(\text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Ai} \left(\frac{1}{k^{2/3}\delta^{2/3}} \right) - \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \text{Bi} \left(\frac{1}{k^{2/3}\delta^{2/3}} \right) \right) z_{2,0}^j(y_{2,0}) \right. \\
&\quad \left. + \mathcal{O}(r_2^2) \right) \\
\varepsilon_3 &= \delta,
\end{aligned} \tag{6.111}$$

for $j = \ell, r$, where

$$\begin{aligned}
|z_{2,0}^j(y_{2,0})| &\leq C\varepsilon^{1/3} |\log \varepsilon| \\
\left| \frac{dz_{2,0}^j}{dy_{2,0}} \right| &\leq C\varepsilon |\log \varepsilon|
\end{aligned} \tag{6.112}$$

and

$$\kappa\varepsilon^{1/3} < z_{2,0}^r(y_{2,0}) - z_{2,0}^\ell(y_{2,0}) < C\varepsilon^{1/3} |\log \varepsilon|. \tag{6.113}$$

uniformly in $y_{2,0}$ for some $C, \kappa > 0$ independent of c, a, ε . Using asymptotic properties

of Airy functions (6.19), we have

$$\begin{aligned} x_{3,0}^j(y_{2,0}) &= \frac{\sqrt{\pi} e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{kr_2}}}{k^{1/6} \delta^{1/6}} X_3^j(y_{2,0}) \\ z_{3,0}^j(y_{2,0}) &= \frac{\sqrt{\pi} e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{kr_2}}}{k^{1/6} \delta^{1/6}} Z_3^j(y_{2,0}), \end{aligned} \quad (6.114)$$

where

$$\begin{aligned} X_3^j(y_{2,0}) &= \left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{\frac{2}{3} \frac{1}{k\delta}} (1 + \mathcal{O}(\delta)) \\ &\quad + \left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) \frac{e^{-\frac{2}{3} \frac{1}{k\delta}}}{2} (1 + \mathcal{O}(\delta)) \\ &\quad + \mathcal{O}(r_2^2) \\ Z_3^j(y_{2,0}) &= - \left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) e^{\frac{2}{3} \frac{1}{k\delta}} (1 + \mathcal{O}(\delta)) \\ &\quad + \left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^j(y_{2,0}) \right) \frac{e^{-\frac{2}{3} \frac{1}{k\delta}}}{2} (1 + \mathcal{O}(\delta)) \\ &\quad + \mathcal{O}(r_2^2). \end{aligned} \quad (6.115)$$

Using (6.93), in the ‘ $\sim$ ’ coordinates we have

$$\begin{aligned} \tilde{z}_{3,0}^j(y_{2,0}) &= z_{3,0}^j(y_{2,0}) - F^- \left(x_{3,0}^j(y_{2,0}), (\varepsilon/\delta)^{1/3}, \delta, c, a \right) \\ \tilde{x}_{3,0}^j(y_{2,0}) &= x_{3,0}^j(y_{2,0}) - F^+ \left(z_{3,0}^j(y_{2,0}), (\varepsilon/\delta)^{1/3}, \delta, c, a \right), \end{aligned} \quad (6.116)$$

from which the result follows, noting $r_2 = \varepsilon^{1/3}$. □

We now obtain estimates for solutions on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ with initial conditions in Σ_{23} upon exit in Σ_3^{out} . We have the following lemma regarding $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ (an analogous result holds for $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$).

Lemma 5.6.10. *Consider solutions on the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$, with initial conditions as in Lemma 5.6.9 parameterized by $|y_{2,0}| \leq 1/\delta^{2/3}$. Such solutions exit Σ_3^{out}*

in time $T^* = T^*(y_{2,0})$ in a curve $(\tilde{x}_3^\ell(T^*(y_{2,0})), \tilde{z}_3^\ell(T^*(y_{2,0})))$. For each sufficiently small $\delta, \rho > 0$, there exists $C > 0$ independent of (c, a, ε) and $y_{2,0}^\dagger > 0$ such that the following holds. Let $\tilde{x}_3^i = \tilde{x}_3^\ell(T^*(-1/\delta^{2/3}))$, and let $\tilde{x}_3^\dagger = \tilde{x}_3^\ell(T^*(y_{2,0}^\dagger))$. Then

$$\tilde{x}_3^\dagger \leq -\frac{\varepsilon^{1/3}}{C} \exp\left(\beta_+^2(\rho, \delta, \varepsilon) - \frac{C}{\varepsilon^{1/3}}\right), \quad (6.117)$$

and for $y_{2,0} \in (-1/\delta^{2/3}, y_{2,0}^\dagger)$, the curve $(\tilde{x}_3^\ell(T^*(y_{2,0})), \tilde{z}_3^\ell(T^*(y_{2,0})))$ can be expressed as a graph $\tilde{z}_3 = \widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)$ for $\tilde{x}_3 \in (\tilde{x}_3^\dagger, \tilde{x}_3^i)$ which satisfies

$$\begin{aligned} |\widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)| &\leq C \exp\left(-\frac{\rho^2}{\varepsilon} \left(1 + \frac{2\rho}{3} + \mathcal{O}(\rho^2, \rho\delta)\right)\right) \\ \frac{d\widehat{z}_3}{d\tilde{x}_3}(\tilde{x}_3; c, a, \varepsilon) &\leq \frac{C}{\varepsilon^{2/3}} \exp\left(-\frac{4\rho^3}{3\varepsilon} (1 + \mathcal{O}(\rho, \delta))\right) \end{aligned} \quad (6.118)$$

uniformly in $\tilde{x}_3 \in (\tilde{x}_3^\dagger(c, a, \varepsilon), \tilde{x}_3^i(c, a, \varepsilon))$ and (c, a, ε) .

Proof. Using Lemma 5.6.8, we have that solutions with initial conditions given by Lemma 5.6.9 for $|y_{2,0}| \leq 1/\delta^{2/3}$ reach Σ_3^{out} at time $T^* = T^*(y_{2,0})$ in curves

$$\begin{aligned} \tilde{x}_3^\ell(T^*(y_{2,0})) &= \tilde{x}_{3,0}^\ell(y_{2,0}) \exp\left(\beta_+^2(\rho, \delta, \varepsilon) + \eta_+^2(\rho, \delta, \tilde{x}_{3,0}^\ell(y_{2,0}), \tilde{z}_{3,0}^\ell(y_{2,0}), \varepsilon)\right) \\ \tilde{z}_3^\ell(T^*(y_{2,0})) &= \tilde{z}_{3,0}^\ell(y_{2,0}) \exp\left(\beta_-^2(\rho, \delta, \varepsilon) + \eta_-^2(\rho, \delta, \tilde{x}_{3,0}^\ell(y_{2,0}), \tilde{z}_{3,0}^\ell(y_{2,0}), \varepsilon)\right), \end{aligned} \quad (6.119)$$

where β_+^2, η_+^2 are given by the integrals (6.103) (and analogously for β_-^2, η_-^2). It remains to estimate the derivatives $\frac{d\tilde{x}_3^\ell(T^*)}{dy_{2,0}}, \frac{d\tilde{z}_3^\ell(T^*)}{dy_{2,0}}$.

To obtain estimates on the derivatives of the solutions with respect to $y_{2,0}, c, a$, we consider the variational equation of (6.94). Using the estimates (6.119), for $K = 1/2$ and each small $\kappa > 0$, there exists C such that for all sufficiently small ρ, δ , we can

estimate

$$\begin{aligned}
\frac{d\tilde{x}_3^\ell(t)}{dy_{2,0}} &\leq C \left(|(\tilde{x}_{3,0}^\ell)'(y_{2,0})| + |(\tilde{z}_{3,0}^\ell)'(y_{2,0})| \right) e^{-Kt} \\
\frac{d\tilde{z}_3^\ell(t)}{dy_{2,0}} &\leq C \left(|(\tilde{x}_{3,0}^\ell)'(y_{2,0})| + |(\tilde{z}_{3,0}^\ell)'(y_{2,0})| \right) e^{-Kt} \\
\frac{dr_3^\ell(t)}{dy_{2,0}} &\leq C \left(|(\tilde{x}_{3,0}^\ell)'(y_{2,0})| + |(\tilde{z}_{3,0}^\ell)'(y_{2,0})| \right) e^{\kappa t} \\
\frac{d\varepsilon_3^\ell(t)}{dy_{2,0}} &\leq C \left(|(\tilde{x}_{3,0}^\ell)'(y_{2,0})| + |(\tilde{z}_{3,0}^\ell)'(y_{2,0})| \right) e^{\kappa t}
\end{aligned} \tag{6.120}$$

for solutions on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ with initial conditions

$$(\tilde{x}_3, \tilde{z}_3, r_3, \varepsilon_3)(0) = (\tilde{x}_{3,0}^\ell(y_{2,0}), \tilde{z}_{3,0}^\ell(y_{2,0}), (\varepsilon/\delta)^{1/3}, \delta) \in \Sigma_{23}. \tag{6.121}$$

Differentiating (6.119), we have that

$$\begin{aligned}
&\frac{d\tilde{x}_3^\ell(T^*)}{dy_{2,0}} \\
&= \left((\tilde{x}_{3,0}^\ell)'(y_{2,0}) + \tilde{x}_{3,0}^\ell \frac{d}{dy_{2,0}} \left(\frac{2}{\varepsilon} \int_{r_{3,0}}^\rho \mathcal{O}(r_3^3(|\tilde{x}_3^\ell| + |\tilde{z}_3^\ell|), r_3^2 \varepsilon_3(|\tilde{x}_3^\ell| + |\tilde{z}_3^\ell|)) dr_3 \right) \right) \\
&\quad \times \exp \left(\beta_+^2(\rho, \delta, \varepsilon) + \eta_+^2(\rho, \delta, \tilde{x}_{3,0}^\ell, \tilde{z}_{3,0}^\ell, \varepsilon) \right),
\end{aligned} \tag{6.122}$$

where $r_{3,0} = (\varepsilon/\delta)^{1/3}$. It remains to prove the following estimate for the second term

$$\begin{aligned}
&\left| \frac{d}{dy_{2,0}} \left(\frac{2}{\varepsilon} \int_{r_{3,0}}^\rho \mathcal{O}(r_3^3(|\tilde{x}_3^\ell| + |\tilde{z}_3^\ell|), r_3^2 \varepsilon_3(|\tilde{x}_3^\ell| + |\tilde{z}_3^\ell|)) dr_3 \right) \right| \\
&\leq C\delta \left(|(\tilde{x}_{3,0}^\ell)'(y_{2,0})| + |(\tilde{z}_{3,0}^\ell)'(y_{2,0})| \right),
\end{aligned} \tag{6.123}$$

where $C > 0$ is independent of $\delta, \rho, \varepsilon$. We begin with estimating a term of the form

$$\frac{d}{dy_{2,0}} \left(\frac{2}{\varepsilon} \int_{r_{3,0}}^\rho r_3^3 \tilde{x}_3^\ell dr_3 \right), \tag{6.124}$$

as the others will be similar. As the endpoints of the integral are fixed, we obtain

$$\frac{d}{dy_{2,0}} \left(\frac{2}{\varepsilon} \int_{r_{3,0}}^{\rho} r_3^3 \tilde{x}_3^{\ell} dr_3 \right) = \frac{2}{\varepsilon} \int_{r_{3,0}}^{\rho} r_3^3 \frac{d\tilde{x}_3^{\ell}}{dy_{2,0}} dr_3 + \frac{2}{\varepsilon} \int_{r_{3,0}}^{\rho} 3r_3^2 \frac{dr_3}{dy_{2,0}} \tilde{x}_3^{\ell} dr_3. \quad (6.125)$$

Using the estimates (6.120) and noting that $t > \frac{1}{2\varepsilon}(r_3(t)^2 - r_{3,0}^2)$ (as in the proof of Lemma 5.6.8), we see that we can bound the above integrals by an integral of the form

$$\begin{aligned} \frac{2}{\varepsilon} (|(\tilde{x}_{3,0}^{\ell})'(y_{2,0})| + |(\tilde{z}_{3,0}^{\ell})'(y_{2,0})|) \left| \int_{r_{3,0}}^{\rho} r_3^2 \exp \left(-\frac{1}{4\varepsilon} (r_3^2 - r_{3,0}^2) \right) dr_3 \right| \\ \leq C (|(\tilde{x}_{3,0}^{\ell})'(y_{2,0})| + |(\tilde{z}_{3,0}^{\ell})'(y_{2,0})|) (r_{3,0} + \mathcal{O}(r_{3,0}^2)), \end{aligned} \quad (6.126)$$

where C is independent of $\rho, \delta, \varepsilon$. Proceeding similarly, we estimate

$$\frac{d}{dy_{2,0}} \left(\frac{2}{\varepsilon} \int_{r_{3,0}}^{\rho} r_3 \varepsilon_3 \tilde{x}_3^{\ell} dr_3 \right) \leq C \delta (|(\tilde{x}_{3,0}^{\ell})'(y_{2,0})| + |(\tilde{z}_{3,0}^{\ell})'(y_{2,0})|) (1 + \mathcal{O}(r_{3,0})), \quad (6.127)$$

where C is independent of $\rho, \delta, \varepsilon$. Using the fact that $r_{3,0} = \left(\frac{\varepsilon}{\delta}\right)^{1/3}$, we obtain

$$\begin{aligned} \frac{d\tilde{x}_3^{\ell}(T^*)}{dy_{2,0}} &= ((\tilde{x}_{3,0}^{\ell})'(y_{2,0}) + \mathcal{O}(\delta \tilde{x}_{3,0}^{\ell}(y_{2,0}) (|(\tilde{x}_{3,0}^{\ell})'(y_{2,0})| + |(\tilde{z}_{3,0}^{\ell})'(y_{2,0})|))) \\ &\quad \times \exp(\beta_+^2(\rho, \delta, \varepsilon) + \eta_+^2(\rho, \delta, \tilde{x}_{3,0}^{\ell}(y_{2,0}), \tilde{z}_{3,0}^{\ell}(y_{2,0}), \varepsilon)) \\ \frac{d\tilde{z}_3^{\ell}(T^*)}{dy_{2,0}} &= ((\tilde{z}_{3,0}^{\ell})'(y_{2,0}) + \mathcal{O}(\delta \tilde{z}_{3,0}^{\ell}(y_{2,0}) (|(\tilde{x}_{3,0}^{\ell})'(y_{2,0})| + |(\tilde{z}_{3,0}^{\ell})'(y_{2,0})|))) \\ &\quad \times \exp(\beta_-^2(\rho, \delta, \varepsilon) + \eta_-^2(\rho, \delta, \tilde{x}_{3,0}^{\ell}(y_{2,0}), \tilde{z}_{3,0}^{\ell}(y_{2,0}), \varepsilon)), \end{aligned} \quad (6.128)$$

for $j = \ell, r$, where $\beta_{\pm}^2, \eta_{\pm}^2$ are given by Lemma 5.6.8.

We now can compute the slope of $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ in Σ_3^{out}

$$\begin{aligned} \frac{d\tilde{z}_3^\ell}{d\tilde{x}_3^\ell}(y_{2,0}) &= \frac{((\tilde{z}_{3,0}^\ell)'(y_{2,0}) + \mathcal{O}(\delta\tilde{z}_{3,0}^\ell(|(\tilde{x}_{3,0}^\ell)'(y_{2,0})| + |(\tilde{z}_{3,0}^\ell)'(y_{2,0})|)))}{((\tilde{x}_{3,0}^\ell)'(y_{2,0}) + \mathcal{O}(\delta\tilde{x}_{3,0}^\ell(|(\tilde{x}_{3,0}^\ell)'(y_{2,0})| + |(\tilde{z}_{3,0}^\ell)'(y_{2,0})|)))} e^{\left(-\frac{4\rho^3}{3\varepsilon}(1+\mathcal{O}(\rho,\delta))\right)} \\ &= \frac{\left(\tilde{Z}_3^\ell - k\varepsilon^{1/3}(\tilde{Z}_3^\ell)' + \mathcal{O}\left(\delta\tilde{z}_{3,0}^\ell\left(|\tilde{X}_3^\ell| + |\tilde{Z}_3^\ell| + \mathcal{O}(\varepsilon^{1/3})\right)\right)\right)}{\left(\tilde{X}_3^\ell - k\varepsilon^{1/3}(\tilde{X}_3^\ell)' + \mathcal{O}\left(\delta\tilde{x}_{3,0}^\ell\left(|\tilde{X}_3^\ell| + |\tilde{Z}_3^\ell| + \mathcal{O}(\varepsilon^{1/3})\right)\right)\right)} e^{\left(-\frac{4\rho^3}{3\varepsilon}(1+\mathcal{O}(\rho,\delta))\right)} \end{aligned} \quad (6.129)$$

where we used that $r_2 = \varepsilon^{1/3}$. For each fixed small $\delta, \rho > 0$, the numerator in the above fraction is a bounded function for sufficiently small $\varepsilon > 0$. Hence it is clear that the tangent space to $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ has exponentially small slope except near points where the denominator is also exponentially small. Hence we proceed by obtaining a lower bound for the denominator for an appropriate range of $y_{2,0}$.

From properties of Airy functions in Lemma 5.6.3 and the bounds in Lemma 5.6.9, we know that the function $\tilde{X}_3^\ell(y_{2,0})$ is smooth and is positive for $y_{2,0} \leq 0$. For $y_{2,0}$, $\tilde{X}_3^\ell(y_{2,0})$ transitions to oscillatory behavior: the first zero occurs at $y_{2,0} = y_{2,0}^\ell > 0$ and here $(\tilde{X}_3^\ell)'(y_{2,0}^\ell) < 0$. Hence by the implicit function theorem we can solve for the first zero of the denominator

$$\left(\tilde{X}_3^\ell - k\varepsilon^{1/3}(\tilde{X}_3^\ell)' + \mathcal{O}\left(\delta\tilde{x}_{3,0}^\ell\left(|\tilde{X}_3^\ell| + |\tilde{Z}_3^\ell| + \mathcal{O}(\varepsilon^{1/3})\right)\right)\right) = 0. \quad (6.130)$$

We first argue that the $\mathcal{O}$ -term does not contribute to finding zeros in this expression.

To see this, we note that for δ fixed sufficiently small, we can bound

$$|\tilde{X}_3^\ell| + |\tilde{Z}_3^\ell| + \mathcal{O}(\varepsilon^{1/3}) \leq 4|\tilde{X}_3^\ell| \quad (6.131)$$

uniformly in $y_{2,0} \in (-1/\delta^{2/3}, -1/\delta^{2/3} + \delta)$, provided ε is taken sufficiently small.

Hence there are no zeros of (6.130) for $y_{2,0} \in (-1/\delta^{2/3}, -1/\delta^{2/3} + \delta)$ and ε sufficiently

small. For $y_{2,0} > -1/\delta^{2/3} + \delta$, we have that

$$\tilde{x}_{3,0}^\ell = \frac{\sqrt{\pi} e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{k\varepsilon^{1/3}}}}{k^{1/6} \delta^{1/6}} \tilde{X}_3^\ell(y_{2,0}), \quad (6.132)$$

where we used the fact that $r_2 = \varepsilon^{1/3}$. Hence by taking ε sufficiently small, we can bound $\tilde{x}_{3,0}^\ell = \mathcal{O}(\varepsilon^{2/3})$. Hence the first zero of (6.130) occurs when

$$y_{2,0} = y_{2,0}^\ell + k\varepsilon^{1/3} + \mathcal{O}(\varepsilon^{2/3}). \quad (6.133)$$

Hence there exists C such that for all

$$y_{2,0} \leq y_{2,0}^\dagger := y_{2,0}^\ell + k\varepsilon^{1/3} - C\varepsilon^{2/3}, \quad (6.134)$$

the slope $\frac{d\tilde{z}}{d\tilde{x}}(y_{2,0})$ of the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ in Σ_3^{out} is exponentially small. We now define

$$\begin{aligned} \tilde{x}_3^\dagger(c, a, \varepsilon) &= \tilde{x}_{3,0}(y_{2,0}^\dagger) \exp\left(\beta_+^2(\rho, \delta, \varepsilon) + \eta_+^2(\rho, \delta, \tilde{x}_{3,0}(y_{2,0}^\dagger), \tilde{z}_{3,0}(y_{2,0}^\dagger), \varepsilon)\right) \\ &\leq -\frac{\varepsilon^{1/3}}{C} \exp\left(\beta_+^2(\rho, \delta, \varepsilon) - \frac{C}{\varepsilon^{1/3}}\right) \end{aligned} \quad (6.135)$$

for some $C > 0$ independent of (c, a, ε) . The result follows. $\square$

Solutions with initial conditions in Σ_{13}

The above takes care of the solutions on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ entering via Σ_{23} , but we still have those which enter via Σ_{13} directly from chart $\mathcal{K}_1$. We have the following.

Lemma 5.6.11. *For each sufficiently small $\rho, \delta > 0$, there exists $C, \eta, \varepsilon_0 > 0$ and sufficiently small choice of the intervals I_a, I_c such that for each $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$, there exists $x_3^\Delta(c, a, \varepsilon) > 0$ such that the following holds. Consider*

solutions on the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$, with initial conditions in Σ_{13} . All such solutions exit Σ_3^{out} in a curve which can be represented as a graph $\tilde{z}_3 = \widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)$ for $\tilde{x}_3 \in (\tilde{x}_3^i(c, a, \varepsilon), \tilde{x}_3^\Delta(c, a, \varepsilon))$ which satisfies

$$\begin{aligned} |\widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)| &\leq C e^{-\eta/\varepsilon} \\ \frac{d\widehat{z}_3}{d\tilde{x}_3}(\tilde{x}_3; c, a, \varepsilon) &\leq C e^{-\eta/\varepsilon}, \end{aligned} \tag{6.136}$$

uniformly in $\tilde{x}_3 \in (\tilde{x}_3^i(c, a, \varepsilon), \tilde{x}_3^\Delta(c, a, \varepsilon))$ and $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$.

Proof. From Proposition 5.6.5 (i), we have that such solutions enter Σ_{13} in a curve $z_3 = z_{3,0}(r_{3,0})$, for $r_{3,0} \in ((\varepsilon/\delta)^{1/3}, \Delta_{r_3})$ which satisfies $|z'_{3,0}(r_{3,0})| \leq C|\log \varepsilon|$ uniformly in $r_{3,0}$, where we may assume $\Delta_{r_3} \ll \rho$ by taking Δ_{y_1} smaller in Proposition 5.6.5 if necessary. In the $\sim$ coordinates, we have that this curve can be parameterized by $r_{3,0}$ as $(\tilde{x}_3, \tilde{z}_3) = (\tilde{x}_{3,0}(r_{3,0}), \tilde{z}_{3,0}(r_{3,0}))$ where

$$\begin{aligned} \tilde{x}_{3,0}(r_{3,0}) &= 1 - F^+(z_{3,0}(r_{3,0}), r_{3,0}, \varepsilon_{3,0}(r_{3,0}), c, a) \\ \tilde{z}_{3,0}(r_{3,0}) &= z_{3,0}(r_{3,0}) - F^-(1, r_{3,0}, \varepsilon_{3,0}(r_{3,0}), c, a) \\ \varepsilon_{3,0}(r_{3,0}) &= \frac{\varepsilon}{r_{3,0}^3}. \end{aligned} \tag{6.137}$$

Similarly to the proof of Lemma 5.6.8 we track these solutions through to Σ_3^{out} , where they intersect in a curve

$$\begin{aligned} \tilde{x}_3(r_{3,0}) &= \tilde{x}_{3,0}(r_{3,0}) \exp(\beta_+^1(\rho, \delta, r_{3,0}, \varepsilon)) \\ \tilde{z}_3(r_{3,0}) &= \tilde{z}_{3,0}(r_{3,0}) \exp(\beta_-^1(\rho, \delta, r_{3,0}, \varepsilon)), \end{aligned} \tag{6.138}$$

where

$$\begin{aligned}
\beta_-^1(\rho, \delta, r_{3,0}, \varepsilon) &= \frac{2}{\varepsilon} \int_{r_{3,0}}^{\rho} (-r_3 - r_3^2 + \mathcal{O}(r_3^3, r_3^2 \varepsilon_3)) dr_3 \\
&= \frac{1}{\varepsilon} \left(-\rho^2 + r_{3,0}^2 - \frac{2}{3}(\rho^3 - r_{3,0}^3) + \mathcal{O}(\rho^4, \rho^3 \delta, r_{3,0}^4) \right) \\
\beta_+^1(\rho, \delta, r_{3,0}, \varepsilon) &= \frac{2}{\varepsilon} \int_{r_{3,0}}^{\rho} (-r_3 + r_3^2 + \mathcal{O}(r_3^3, r_3^2 \varepsilon_3)) dr_3 \\
&= \frac{1}{\varepsilon} \left(-\rho^2 + r_{3,0}^2 + \frac{2}{3}(\rho^3 - r_{3,0}^3) + \mathcal{O}(\rho^4, \rho^3 \delta, r_{3,0}^4) \right).
\end{aligned} \tag{6.139}$$

Using similar arguments as in the proof of Lemma 5.6.10, we estimate

$$\begin{aligned}
\frac{d\tilde{x}_3}{dr_{3,0}} &= \left(\frac{d\tilde{x}_{3,0}}{dr_{3,0}} + \frac{2}{\varepsilon} \tilde{x}_{3,0} (r_{3,0} + \mathcal{O}(r_{3,0}^2)) + \mathcal{O} \left(\tilde{x}_{3,0} \left(\varepsilon^{-1/3} + \left| \frac{d\tilde{x}_{3,0}}{dr_{3,0}} \right| + \left| \frac{d\tilde{z}_{3,0}}{dr_{3,0}} \right| \right) \right) \right) \\
&\quad \times \exp(\beta_+^1(\rho, \delta, r_{3,0}, \varepsilon)) \\
\frac{d\tilde{z}_3}{dr_{3,0}} &= \left(\frac{d\tilde{z}_{3,0}}{dr_{3,0}} + \frac{2}{\varepsilon} \tilde{z}_{3,0} (r_{3,0} + \mathcal{O}(r_{3,0}^2)) + \mathcal{O} \left(\tilde{z}_{3,0} \left(\varepsilon^{-1/3} + \left| \frac{d\tilde{x}_{3,0}}{dr_{3,0}} \right| + \left| \frac{d\tilde{z}_{3,0}}{dr_{3,0}} \right| \right) \right) \right) \\
&\quad \times \exp(\beta_-^1(\rho, \delta, r_{3,0}, \varepsilon)),
\end{aligned} \tag{6.140}$$

and we define

$$\begin{aligned}
\tilde{x}_3^i(c, a, \varepsilon) &:= \tilde{x}_{3,0}((\varepsilon/\delta)^{1/3}) \exp \left(\beta_+^1(\rho, \delta, (\varepsilon/\delta)^{1/3}, \varepsilon) \right) \\
\tilde{x}_3^\Delta(c, a, \varepsilon) &:= \tilde{x}_{3,0}(\Delta_{r_3}) \exp \left(\beta_+^1(\rho, \delta, \Delta_{r_3}, \varepsilon) \right),
\end{aligned} \tag{6.141}$$

noting that the definition of $\tilde{x}_3^i(c, a, \varepsilon)$ coincides with that in Lemma 5.6.10. We therefore can estimate

$$\frac{d\tilde{z}_3}{d\tilde{x}_3}(r_{3,0}) = \frac{\tilde{z}_{3,0} + \mathcal{O}(\varepsilon^{1/3} \log \varepsilon)}{\tilde{x}_{3,0} + \mathcal{O}(\varepsilon^{1/3} \log \varepsilon)} \exp \left(-\frac{4}{3\varepsilon} (\rho^3 - r_{3,0}^3) + \mathcal{O}(\rho^4, \rho^3 \delta, r_{3,0}^4) \right), \tag{6.142}$$

from which we obtain the required estimates, recalling the choice of $0 < \Delta_{r_3} \ll \rho$. $\square$

Estimates for $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ in Σ_3^{out}

We now fix ρ, δ sufficiently small and combine the results of Lemma 5.6.9 and Lemma 5.6.11 into the following.

Lemma 5.6.12. *For each sufficiently small $\Delta_y > 0$, there exists $C, \eta, \varepsilon_0 > 0$ and sufficiently small choice of the intervals I_a, I_c such that for each $(c, a, \varepsilon) \in I_c \times I_a \times (0, \varepsilon_0)$, there exists $y_\varepsilon^*(c, a) > \varepsilon^{2/3}/C$ such that the following holds. Define $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ to be the backwards evolution of initial conditions $\{(\rho^4, y, z, \varepsilon) : z = z_\varepsilon^{s,\ell}(y; c, a), -\Delta_y \leq y \leq y_\varepsilon^*(c, a)\}$ on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ in Σ_A^{out} . The intersection of $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ with $\Sigma_A^{out} = \Sigma_3^{out}$ is given by a curve $\tilde{z}_3 = \widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)$ in the $\mathcal{K}_3$ coordinates for $\tilde{x}_3 \in (\tilde{x}_3^\nabla(c, a, \varepsilon), \tilde{x}_3^\Delta(c, a, \varepsilon))$ which satisfies*

$$\begin{aligned} |\widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)| &\leq C e^{-\eta/\varepsilon} \\ \frac{d\widehat{z}_3}{d\tilde{x}_3}(\tilde{x}_3; c, a, \varepsilon) &\leq C e^{-\eta/\varepsilon}, \end{aligned} \tag{6.143}$$

uniformly in $\tilde{x}_3 \in (\tilde{x}_3^\nabla(c, a, \varepsilon), \tilde{x}_3^\Delta(c, a, \varepsilon))$ and (c, a, ε) . Here $\tilde{x}_3^\nabla(c, a, \varepsilon), \tilde{x}_3^\Delta(c, a, \varepsilon)$ are defined as in Lemma 5.6.9 and Lemma 5.6.11 and

$$\tilde{x}_3^\nabla(c, a, \varepsilon) \leq -\frac{\varepsilon^{1/3}}{C} \exp\left(\beta_+^2(\rho, \delta, \varepsilon) - \frac{C}{\varepsilon^{1/3}}\right). \tag{6.144}$$

5.6.6 Proofs of transversality results

To measure the transversality properties of $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ with respect to the strong unstable fibers of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in the section Σ^m , we use the transversality results for the backwards evolution of fibers on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ with height $y > -\Delta_y$ obtained above in the section Σ_3^{out} combined with the fact that the backwards/forwards evolution of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ between the sections $\Sigma^m, \Sigma^{h,\ell}$ coincide for $y < -\Delta_w$.

Proof of Proposition 5.4.4. We note that the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ is defined in terms of the (u, v, w) -coordinates in $\Sigma^{h,\ell}$ in Proposition 5.4.4, but defined in terms of the (x, y, z) -coordinates in Σ_A^{in} in Lemma 5.6.12. In the analysis below, it is more natural to determine transversality properties in the section Σ_3^{out} , and hence also more natural to represent this manifold in the (x, y, z) -coordinates near the Airy point. To obtain the corresponding definition in $\Sigma^{h,\ell}$, we evolve $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ *forwards* from Σ_A^{in} to $\Sigma^{h,\ell}$ to obtain the w -coordinate endpoints w_ε^Δ and w_ε^∇ corresponding to the solutions on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ with initial conditions in Σ_A^{in} at $y = -\Delta_y$ and $y = y_\varepsilon^*(c, a)$, respectively.

Using Lemma 5.6.12, we are able to track trajectories on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ with initial conditions in Σ_A^{in} with $y \in (-\Delta_y, y_\varepsilon^*)$ down to the section Σ_3^{out} . In the chart $\mathcal{K}_3$, the intersection of this manifold with Σ_3^{out} is given by a curve $\tilde{z}_3 = \widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)$ for $\tilde{x}_3 \in (\tilde{x}_3^\nabla(c, a, \varepsilon), \tilde{x}_3^\Delta(c, a, \varepsilon))$ which satisfies

$$|\widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)| \leq Ce^{-\eta/\varepsilon}, \quad \frac{d\tilde{z}_3}{d\tilde{x}_3}(\tilde{x}_3; c, a, \varepsilon) \leq Ce^{-\eta/\varepsilon}, \quad (6.145)$$

uniformly in $\tilde{x}_3 \in (\tilde{x}_3^\nabla(c, a, \varepsilon), \tilde{x}_3^\Delta(c, a, \varepsilon))$ and (c, a, ε) .

We now investigate the intersection of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ (integrated forwards from Σ^m up to the section Σ_3^{out}). This manifold will intersect Σ_3^{out} in a curve $\tilde{z}_3 = \tilde{z}_3^s(\tilde{x}_3; c, a, \varepsilon)$ which satisfies

$$\tilde{z}_3^s(\tilde{x}_3; c, a, \varepsilon) = \tilde{z}_3^s(0; c, a, \varepsilon) + \frac{d\tilde{z}_3^s}{d\tilde{x}_3}(0; c, a, \varepsilon)\tilde{x}_3 + o(\tilde{x}_3) \quad (6.146)$$

where

$$\tilde{z}_3^s(0; c, a, \varepsilon) = \mathcal{O}(e^{-q/\varepsilon}), \quad \frac{d\tilde{z}_3^s}{d\tilde{x}}(0; c, a, \varepsilon) = \mathcal{O}(e^{-q/\varepsilon}), \quad (6.147)$$

uniformly in (c, a, ε) . This follows from the fact that $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ contains a (non-unique) choice of the slow manifold $\mathcal{M}_\varepsilon^m(c, a)$ which will be exponentially close to the point $(\tilde{x}, \tilde{z}) = (0, 0)$. Furthermore, at this point, $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ will be (up to exponentially small errors) tangent to the weak unstable subspace $\tilde{z} = 0$, and the strong unstable fiber at this point will be (up to exponentially small errors) tangent to the strong unstable subspace $\tilde{x} = 0$. Therefore, the strong unstable fiber of a basepoint $(\tilde{x}_3, \tilde{z}_3^s(\tilde{x}_3))$ on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ is given by a graph

$$\tilde{x}_3^s(\tilde{z}_3; \tilde{x}_3, c, a, \varepsilon) = \tilde{x}_3 + \frac{d\tilde{x}_3^s}{d\tilde{z}_3}(\tilde{z}_3^s(\tilde{x}_3); \tilde{x}_3, c, a, \varepsilon)(\tilde{z}_3 - \tilde{z}_3^s) + \mathcal{O}((\tilde{z}_3 - \tilde{z}_3^s)^2) \quad (6.148)$$

where

$$\frac{d\tilde{x}_3^s}{d\tilde{z}_3}(\tilde{z}_3^s; \tilde{x}_3, c, a, \varepsilon) = \mathcal{O}(\tilde{x}_3, \tilde{z}_3^s(\tilde{x}_3), e^{-q/\varepsilon}). \quad (6.149)$$

Finally, since the forward/backward evolution of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ coincide for $y < -\Delta_y$, we have that $\tilde{z}_3^s(\tilde{x}_3^\Delta; c, a, \varepsilon) = \hat{z}_3(\tilde{x}_3^\Delta; c, a, \varepsilon)$ and trajectories on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ evolved backwards from Σ_A^{in} for $y < -\Delta_y$ in fact again *land in* $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$. Since $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ is certainly transverse to its own strong unstable fibers, we are only concerned for values $\tilde{x}_3 \in (\tilde{x}_3^\dagger(c, a, \varepsilon), \tilde{x}_3^\Delta(c, a, \varepsilon))$ as here the backwards/forwards evolution of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ separates in the section Σ_3^{out} into curves given by the two functions described above.

From the estimates above, we can deduce that the backwards evolution $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ given by the curve $\tilde{z}_3 = \hat{z}_3(\tilde{x}_3; c, a, \varepsilon)$ in the section Σ_3^{out} remains transverse to the strong unstable fibers of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ at least up to the fiber which passes through the point $(\tilde{x}_3^\dagger, \hat{z}_3(\tilde{x}_3^\dagger))$.

We now evolve the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ backwards from Σ_3^{out} to Σ^m . Using the exchange lemma, we deduce that the above transversality also holds in the section Σ^m and all trajectories are exponentially contracted to $\mathcal{M}_\varepsilon^m(c, a)$. We denote by $\tilde{y}_{2,0}^\ell(c, a)$ the y_2 coordinate in Σ^m of the backwards evolution of the solution $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ passing through $(\tilde{x}_3^\Delta(c, a, \varepsilon), \tilde{z}_3(\tilde{x}_3^\Delta(c, a, \varepsilon)))$ in Σ_3^{out} , and we denote by $y_{2,0}^{\ell,*}(c, a)$ the y_2 coordinate in Σ^m of the backwards evolution of the basepoint on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ corresponding to the fiber containing the solution on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ passing through $(\tilde{x}_3^\dagger, \tilde{z}_3(\tilde{x}_3^\dagger))$ in Σ_3^{out} . With these definitions, we see that the assertions of the proposition hold in the section Σ^m . $\square$

Proof of Lemma 5.4.5. The estimates (4.29) are shown in §5.4.4. Hence it remains to show that the transversality of $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$ with respect to the fibers of $\mathcal{W}_\varepsilon^{s,\ell,*}(c, a)$ in Σ^m includes the fibers through all points on the backwards evolution of $\mathcal{B}(s; c, a)$. By Proposition 5.4.4, this amounts to proving (4.30). As in the proof of Proposition 5.4.4, we work in a neighborhood of the Airy point and determine transversality conditions in the section Σ_3^{out} and use this information to deduce what happens in Σ^m .

Here we consider pulses of Type 2,3 so $s \in (w_A + \Delta_w, w^\dagger + w_A)$. Evolving $\mathcal{B}(s; c, a)$ backwards from $\Sigma^{h,\ell}$, these solutions are already exponentially contracted to $\mathcal{M}_\varepsilon^m(c, a)$ above the Airy point, and we see that they eventually enter the chart $\mathcal{K}_3$ via the section Σ_{23} where their $(\tilde{x}, \tilde{z})$ -coordinates are *already* $\mathcal{O}(e^{-q/\varepsilon})$ uniformly in (c, a) .

Suppose we take any such solution which enters Σ_{23} at a point $(\tilde{x}_{3,0}^b, \tilde{z}_{3,0}^b) = \mathcal{O}(e^{-q/\varepsilon})$ where we drop the dependence on (c, a) . This solution reaches Σ_3^{out} at

$(\tilde{x}_3, \tilde{z}_3) = (\tilde{x}_{3,1}^b, \tilde{z}_{3,1}^b)$ where

$$\begin{aligned}\tilde{x}_{3,1}^b &= \tilde{x}_{3,0}^b \exp \left(\beta_+^2(\rho, \delta, \varepsilon) + \eta_+^2(\rho, \delta, \tilde{x}_{3,0}^b, \tilde{z}_{3,0}^b, \varepsilon) \right) \\ \tilde{z}_{3,1}^b &= \tilde{z}_{3,0}^b \exp \left(\beta_-^2(\rho, \delta, \varepsilon) + \eta_-^2(\rho, \delta, \tilde{x}_{3,0}^b, \tilde{z}_{3,0}^b, \varepsilon) \right),\end{aligned}\tag{6.150}$$

We then need to show that $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ is transverse to the fiber in Σ_3^{out} passing through the point $(\tilde{x}_{3,1}^b, \tilde{z}_{3,1}^b)$. One way to do this is to find the intersection of this fiber with $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ and show that it occurs for some $\tilde{x}_3 > \tilde{x}_3^\natural$, where we know this transversality holds.

The fiber through $(\tilde{x}_3, \tilde{z}_3) = (\tilde{x}_{3,1}^b, \tilde{z}_{3,1}^b)$ is given by the set of $(\tilde{x}_3, \tilde{z}_3)$ satisfying

$$\tilde{x}_3 = \tilde{x}_{3,1}^b + \mathcal{O} \left((|\tilde{x}_{3,1}^b| + |\tilde{z}_{3,1}^b| + |e^{-q/\varepsilon}|) (\tilde{z}_3 - \tilde{z}_{3,1}^b), (\tilde{z}_3 - \tilde{z}_{3,1}^b)^2 \right). \tag{6.151}$$

We can solve for when this intersects $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ by substituting the expressions $(\tilde{x}_3, \tilde{z}_3) = (\tilde{x}_3, \widehat{z}_3(\tilde{x}_3; c, a, \varepsilon))$ to obtain

$$\tilde{x}_3 = \tilde{x}_{3,1}^b + \mathcal{O} \left((|\tilde{x}_{3,1}^b| + |\tilde{z}_{3,1}^b| + |e^{-q/\varepsilon}|) (\widehat{z}_3(\tilde{x}_3) - \tilde{z}_{3,1}^b), (\widehat{z}_3(\tilde{x}_3) - \tilde{z}_{3,1}^b)^2 \right), \tag{6.152}$$

which we can solve by the implicit function theorem to find an intersection at

$$\tilde{x}_3^* = \mathcal{O} \left(\exp \left(-\frac{q}{\varepsilon} + \beta_+^2(\rho, \delta, \varepsilon) \right) \right), \tag{6.153}$$

which indeed satisfies $\tilde{x}_3^* > \tilde{x}_3^\natural$. As the chosen solution on $\mathcal{B}(s; c, a)$ was arbitrary, this shows that $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ is transverse to the fibers passing through each solution on $\mathcal{B}(s; c, a)$ in the section Σ_3^{out} for all $(c, a) \in I_c \times I_a$.

As in the proof of Proposition 5.4.4, we track these solutions in backwards time

from Σ_3^{out} to Σ^m to deduce that the transversality holds there also. We recall that $y_{2,0}^{\ell,*}(c, a)$ denotes the y_2 coordinate in Σ^m of the backwards evolution of the basepoint on $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ in Σ_3^{out} corresponding to the fiber containing the solution on $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ passing through $(\tilde{x}_3^\dagger, \widehat{z}_3(\tilde{x}_3^\dagger))$. Hence following the solutions on $\mathcal{B}(s; c, a)$ from Σ_3^{out} to Σ^m in backwards time gives the result (4.30). $\square$

Proof of Lemma 5.4.9. For the case of type 4 pulses, the argument proceeds as in the proof of Lemma 5.4.5. To treat the case of Type 5 pulses, more care is needed. Using Lemma 5.3.2 and the fact that the forward/backward evolution of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$ coincide for $w \leq w_A - \Delta_w$, the transversality result (4.84) hold easily for type 5 pulses with $s \in (2w^\dagger - w_A + \Delta_w, 2w^\dagger - \Delta_w)$, that is, with secondary right pulses of height $w \in (\Delta_w, w_A - \Delta_w)$. For type 5 pulses with $s \in (2w^\dagger - w_A - \Delta_w, 2w^\dagger - w_A + \Delta_w)$, that is, with secondary height $w \in (w_A - \Delta_w, w_A + \Delta_w)$ the backwards evolution of $\mathcal{B}(s; c, a)$ interacts with the Airy point before reaching the section Σ^m , and hence the result (4.84) is not clear.

For type 5 pulses with secondary height $w \in (w_A - \Delta_w, w_A + \Delta_w)$, the manifolds $\mathcal{B}(s; c, a)$ in fact approach the Airy point exponentially close to $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ in backwards time. Hence these trajectories reach Σ_3^{out} after passing through different charts, as with $\widehat{\mathcal{W}}_\varepsilon^{s,\ell}(c, a)$. We need to ensure that $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ is transverse to the fibers in Σ_3^{out} passing through each point on the intersection of $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ with Σ_3^{out} . Similar to the above analysis for tracking $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$, the manifold $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ intersects Σ_3^{out} curve defined in terms of Airy functions which winds around the origin in an exponentially decaying manner.

We focus on the part of $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ which reaches Σ_3^{out} after passing through the charts $\mathcal{K}_1 \rightarrow \mathcal{K}_2 \rightarrow \mathcal{K}_3$ (see §5.6.5) as solutions entering $\mathcal{K}_3$ via different charts do not cause issues. From Lemma 5.6.9, we have that $\widehat{\mathcal{W}}_\varepsilon^{s,r}(c, a)$ intersects Σ_3^{out} in a

curve parameterized by $y_{2,0}$ as

$$\begin{aligned}\tilde{x}_{3,1}^r(y_{2,0}) &= \tilde{x}_{3,0}^r(y_{2,0}) \exp \left(\beta_+^2(\rho, \delta, \varepsilon) + \eta_+^2(\rho, \delta, \tilde{x}_{3,0}^r, \tilde{z}_{3,0}^r, \varepsilon) \right) \\ \tilde{z}_{3,1}^r(y_{2,0}) &= \tilde{z}_{3,0}^r(y_{2,0}) \exp \left(\beta_-^2(\rho, \delta, \varepsilon) + \eta_-^2(\rho, \delta, \tilde{x}_{3,0}^r, \tilde{z}_{3,0}^r, \varepsilon) \right),\end{aligned}\quad (6.154)$$

where

$$\tilde{x}_{3,0}^r(y_{2,0}) = \frac{\sqrt{\pi} e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{k\varepsilon^{1/3}}}}{k^{1/6} \delta^{1/6}} \tilde{X}_3^r(y_{2,0}), \quad \tilde{z}_{3,0}^r(y_{2,0}) = \frac{\sqrt{\pi} e^{-\frac{1}{\delta^{2/3} + y_{2,0}} \frac{1}{k\varepsilon^{1/3}}}}{k^{1/6} \delta^{1/6}} \tilde{Z}_3^r(y_{2,0}), \quad (6.155)$$

and

$$\begin{aligned}\tilde{X}_3^r(y_{2,0}) &= \left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^r(y_{2,0}) \right) e^{\frac{2}{3} \frac{1}{k\delta}} (2 + \mathcal{O}(\delta)) \\ &\quad + \mathcal{O}(\delta) \left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^r(y_{2,0}) \right) e^{-\frac{2}{3} \frac{1}{k\delta}} + \mathcal{O}(\varepsilon^{2/3}) \\ \tilde{Z}_3^r(y_{2,0}) &= \left(\text{Bi} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^r(y_{2,0}) \right) e^{-\frac{2}{3} \frac{1}{k\delta}} (1 + \mathcal{O}(\delta)) \\ &\quad + \mathcal{O}(\delta) \left(\text{Ai} \left(-\frac{y_{2,0}}{k^{2/3}} \right) + k^{1/3} \delta^{4/3} \text{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) z_{2,0}^r(y_{2,0}) \right) e^{\frac{2}{3} \frac{1}{k\delta}} + \mathcal{O}(\varepsilon^{2/3}).\end{aligned}\quad (6.156)$$

The fiber through $(\tilde{x}_3, \tilde{z}_3) = (\tilde{x}_{3,1}^r(y_{2,0}), \tilde{z}_{3,1}^r(y_{2,0}))$ is given by $(\tilde{x}_3, \tilde{z}_3)$ satisfying

$$\begin{aligned}\tilde{x}_3 &= \tilde{x}_{3,1}^r(y_{2,0}) \\ &\quad + \mathcal{O} \left((|\tilde{x}_{3,1}^r(y_{2,0})| + |\tilde{z}_{3,1}^r(y_{2,0})| + |e^{-q/\varepsilon}|) (\tilde{z}_3 - \tilde{z}_{3,1}^r(y_{2,0})), (\tilde{z}_3 - \tilde{z}_{3,1}^r(y_{2,0}))^2 \right).\end{aligned}\quad (6.157)$$

We can solve for when this intersects $\widehat{\mathcal{W}}_\varepsilon^{s,\ell,*}(c, a)$ by plugging in $(\tilde{x}_3, \tilde{z}_3) = (\tilde{x}_3, \widehat{z}_3(\tilde{x}_3))$ to obtain

$$\begin{aligned}\tilde{x}_3 &= \tilde{x}_{3,1}^r(y_{2,0}) \\ &\quad + \mathcal{O} \left((|\tilde{x}_{3,1}^r(y_{2,0})| + |\tilde{z}_{3,1}^r(y_{2,0})| + |e^{-q/\varepsilon}|) (\widehat{z}_3(\tilde{x}_3) - \tilde{z}_{3,1}^r(y_{2,0})), (\widehat{z}_3(\tilde{x}_3) - \tilde{z}_{3,1}^r(y_{2,0}))^2 \right),\end{aligned}\quad (6.158)$$

which we can solve by the implicit function theorem to find an intersection at

$$\tilde{x}_3^*(y_{2,0}) = \tilde{x}_{3,1}^r(y_{2,0}) + \mathcal{O}\left(e^{2\beta_+^2(\rho,\delta,\varepsilon)}\right), \quad (6.159)$$

provided $\tilde{x}_3^* > \tilde{x}_3^{\text{th}}$ (i.e. we need to be careful not to leave the domain on which $\widehat{z}_3(\tilde{x}_3; c, a, \varepsilon)$ is both well-defined and transverse to the fibers of $\mathcal{W}_\varepsilon^{s,\ell}(c, a)$). To determine this, we note that the minimum possible $\tilde{x}_3^*(y_{2,0})$ -value achieved is at a value of $y_{2,0}$ which is exponentially close to that which gives the minimum value of $\tilde{x}_{3,1}^r(y_{2,0})$. We hence proceed as above by computing the first ‘turning point’ on this curve, that is, the minimum (or largest negative) $\tilde{x}_3$ -value achieved by $\tilde{x}_{3,1}^r(y_{2,0})$.

Similar to the proof of Lemma 5.6.10, we search for the first zero of $(\tilde{x}_{3,1}^r)'(y_{2,0})$, which amounts to solving for the first zero of

$$\left(\tilde{X}_3^r(y_{2,0}) - k\varepsilon^{1/3}(\tilde{X}_3^r)'(y_{2,0}) + \mathcal{O}\left(\delta\tilde{x}_{3,0}^r(y_{2,0})\left(|\tilde{X}_3^r| + |\tilde{Z}_3^r| + \mathcal{O}(\varepsilon^{1/3})\right)\right)\right) = 0, \quad (6.160)$$

which occurs when

$$y_{2,0} = y_{2,0}^r + k\varepsilon^{1/3} + \mathcal{O}(\varepsilon^{2/3}), \quad (6.161)$$

where $y_{2,0}^r$ is the first zero of $\tilde{X}_3^r(y_{2,0})$. Hence the minimum of $\tilde{x}_3^*(y_{2,0})$ occurs at some

$$y_{2,0}^{*,r} = y_{2,0}^r + k\varepsilon^{1/3} + \mathcal{O}(\varepsilon^{2/3}). \quad (6.162)$$

We now note that for $y_{2,0}$ near $y_{2,0}^r$, for all sufficiently small ε , we have that

$$\begin{aligned}\tilde{X}_3^r(y_{2,0}) &= \tilde{X}_3^\ell(y_{2,0}) + \left(\left(k^{1/3} \delta^{4/3} \operatorname{Ai}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \right) e^{\frac{2}{3} \frac{1}{k\delta}} (2 + \mathcal{O}(\delta)) \right. \\ &\quad \left. + \mathcal{O} \left(\delta e^{-\frac{2}{3} \frac{1}{k\delta}} \left(k^{1/3} \delta^{4/3} \operatorname{Bi}' \left(-\frac{y_{2,0}}{k^{2/3}} \right) \right) \right) \right) (z_{2,0}^r(y_{2,0}) - z_{2,0}^\ell(y_{2,0})) + \mathcal{O}(\varepsilon^{2/3}),\end{aligned}\tag{6.163}$$

and hence

$$\begin{aligned}0 &= \tilde{X}_3^r(y_{2,0}^r) = \tilde{X}_3^\ell(y_{2,0}^\ell)(y_{2,0}^r - y_{2,0}^\ell) + \mathcal{O}((y_{2,0}^r - y_{2,0}^\ell)^2) \\ &\quad + \left(\left(k^{1/3} \delta^{4/3} \operatorname{Ai}' \left(-\frac{y_{2,0}^r}{k^{2/3}} \right) \right) e^{\frac{2}{3} \frac{1}{k\delta}} (2 + \mathcal{O}(\delta)) \right. \\ &\quad \left. + \mathcal{O} \left(\delta e^{-\frac{2}{3} \frac{1}{k\delta}} \left(k^{1/3} \delta^{4/3} \operatorname{Bi}' \left(-\frac{y_{2,0}^r}{k^{2/3}} \right) \right) \right) \right) (z_{2,0}^r(y_{2,0}^r) - z_{2,0}^\ell(y_{2,0}^r)) + \mathcal{O}(\varepsilon^{2/3}),\end{aligned}\tag{6.164}$$

from which we deduce that

$$y_{2,0}^r - y_{2,0}^\ell = \mu (z_{2,0}^r(y_{2,0}^r) - z_{2,0}^\ell(y_{2,0}^r)) + \mathcal{O}((z_{2,0}^r(y_{2,0}^r) - z_{2,0}^\ell(y_{2,0}^r))^2, \varepsilon^{2/3}), \tag{6.165}$$

for some constant $\mu > 0$ bounded away from zero uniformly in ε . Hence we have

$$y_{2,0}^{*,r} - y_{2,0}^\natural = \mu (z_{2,0}^r(y_{2,0}^r) - z_{2,0}^\ell(y_{2,0}^r)) + \mathcal{O}((z_{2,0}^r(y_{2,0}^r) - z_{2,0}^\ell(y_{2,0}^r))^2, \varepsilon^{2/3}). \tag{6.166}$$

Finally, using (6.130), (6.160), (6.166), Lemma 5.6.9, and the definitions of $y_{2,0}^{*,r}, y_{2,0}^\natural$, we have that

$$\begin{aligned}\tilde{X}_3^r(y_{2,0}^{*,r}) - \tilde{X}_3^\ell(y_{2,0}^\natural) &= k\varepsilon^{1/3} \left((\tilde{X}_3^r)'(y_{2,0}^{*,r}) - (\tilde{X}_3^\ell)'(y_{2,0}^\natural) \right) + \mathcal{O}(\varepsilon^{2/3}) \\ &= \mathcal{O}(\varepsilon^{1/3} (y_{2,0}^{*,r} - y_{2,0}^\natural), \varepsilon^{2/3}).\end{aligned}\tag{6.167}$$

We now estimate

$$\begin{aligned}
\tilde{x}_3^* - \tilde{x}_3^\dagger &= \tilde{x}_{3,1}^r(y_{2,0}^{*,r}) - \tilde{x}_3^\dagger + \mathcal{O}\left(e^{\beta_+^2(\rho,\delta,\varepsilon)-q/\varepsilon}\right) \\
&= \tilde{x}_{3,0}^r(y_{2,0}^{*,r}) \exp\left(\beta_+^2(\rho,\delta,\varepsilon) + \eta_+^2(\rho,\delta, \tilde{x}_{3,0}^r(y_{2,0}^{*,r}), \tilde{z}_{3,0}^r(y_{2,0}^{*,r}), \varepsilon)\right) \\
&\quad - \tilde{x}_{3,0}^\dagger(y_{2,0}^\dagger) \exp\left(\beta_+^2(\rho,\delta,\varepsilon) + \eta_+^2(\rho,\delta, \tilde{x}_{3,0}^\dagger(y_{2,0}^\dagger), \tilde{z}_{3,0}^\dagger(y_{2,0}^\dagger), \varepsilon)\right) \\
&\quad + \mathcal{O}\left(e^{\beta_+^2(\rho,\delta,\varepsilon)-q/\varepsilon}\right) \\
&= \left(\tilde{x}_{3,0}^r(y_{2,0}^{*,r})(1 + \mathcal{O}(\varepsilon^{2/3})) - \tilde{x}_{3,0}^\dagger(y_{2,0}^\dagger)(1 + \mathcal{O}(\varepsilon^{2/3})) + \mathcal{O}(e^{-q/\varepsilon})\right) e^{\beta_+^2(\rho,\delta,\varepsilon)} \\
&= \left(\frac{\sqrt{\pi}e^{-\frac{1}{\delta^{2/3}}+y_{2,0}^{*,r}}}{k^{1/6}\delta^{1/6}} \tilde{X}_3^r(y_{2,0}^{*,r})(1 + \mathcal{O}(\varepsilon^{2/3})) \right. \\
&\quad \left. - \frac{\sqrt{\pi}e^{-\frac{1}{\delta^{2/3}}+y_{2,0}^\dagger}}{k^{1/6}\delta^{1/6}} \tilde{X}_3^\ell(y_{2,0}^\dagger)(1 + \mathcal{O}(\varepsilon^{2/3})) + \mathcal{O}(e^{-q/\varepsilon})\right) e^{\beta_+^2(\rho,\delta,\varepsilon)} \\
&= \left(\tilde{X}_3^r(y_{2,0}^{*,r})(1 + \mathcal{O}(\varepsilon^{2/3})) - e^{\frac{y_{2,0}^{*,r}-y_{2,0}^\dagger}{k\varepsilon^{1/3}}} \tilde{X}_3^\ell(y_{2,0}^\dagger)(1 + \mathcal{O}(\varepsilon^{2/3})) + \mathcal{O}(e^{-q/\varepsilon})\right) \\
&\quad \times \frac{\sqrt{\pi}e^{-\frac{1}{\delta^{2/3}}+y_{2,0}^{*,r}}}{k^{1/6}\delta^{1/6}} e^{\beta_+^2(\rho,\delta,\varepsilon)} \\
&= \left(\left(\tilde{X}_3^\ell(y_{2,0}^\dagger) + \mathcal{O}(\varepsilon^{1/3}(y_{2,0}^{*,r} - y_{2,0}^\dagger), \varepsilon^{2/3})\right)(1 + \mathcal{O}(\varepsilon^{2/3})) \right. \\
&\quad \left. - e^{\frac{y_{2,0}^{*,r}-y_{2,0}^\dagger}{k\varepsilon^{1/3}}} \tilde{X}_3^\ell(y_{2,0}^\dagger)(1 + \mathcal{O}(\varepsilon^{2/3}))\right) \frac{\sqrt{\pi}e^{-\frac{1}{\delta^{2/3}}+y_{2,0}^{*,r}}}{k^{1/6}\delta^{1/6}} e^{\beta_+^2(\rho,\delta,\varepsilon)} \\
&= \left(\tilde{X}_3^\ell(y_{2,0}^\dagger) \left(1 - e^{\frac{y_{2,0}^{*,r}-y_{2,0}^\dagger}{k\varepsilon^{1/3}}}(1 + \mathcal{O}(\varepsilon^{2/3}))\right) + \mathcal{O}(\varepsilon^{1/3}(y_{2,0}^{*,r} - y_{2,0}^\dagger), \varepsilon^{2/3})\right) \\
&\quad \times \frac{\sqrt{\pi}e^{-\frac{1}{\delta^{2/3}}+y_{2,0}^{*,r}}}{k^{1/6}\delta^{1/6}} e^{\beta_+^2(\rho,\delta,\varepsilon)} \\
&> \left(\varepsilon^{1/3}(\tilde{X}_3^\ell)'(y_{2,0}^\dagger) \left(-\frac{\mu\kappa(\rho)}{\delta} + \mathcal{O}\left(\left(\frac{\kappa(\rho)}{\delta}\right)^2\right)\right) + \mathcal{O}(\varepsilon^{2/3}\log\varepsilon)\right) \\
&\quad \times \frac{\sqrt{\pi}e^{-\frac{1}{\delta^{2/3}}+y_{2,0}^{*,r}}}{k^{1/6}\delta^{1/6}} e^{\beta_+^2(\rho,\delta,\varepsilon)} \\
&> 0,
\end{aligned}$$

for all sufficiently small $\varepsilon > 0$. From this we deduce that $\tilde{x}_3^* > \tilde{x}_3^\natural$ as required. The remainder of the proof follows as in the proof of Lemma 5.4.5.

□

CHAPTER SIX

Conclusion

In the preceding we showed that the FitzHugh–Nagumo system

$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \varepsilon(u - \gamma w), \end{aligned} \tag{0.1}$$

admits a traveling pulse solution $(u, w)(x, t) = (u, w)(x + ct)$ with wave speed $c = c(a, \varepsilon)$ for $0 < a < 1/2$ and sufficiently small $\varepsilon > 0$. We showed that the region of existence near $a = 0$ encompasses a Belyakov transition occurring at the equilibrium $(u, v, w) = (0, 0, 0)$ where two real stable eigenvalues split as a complex conjugate pair and thus describes the onset of small scale oscillations in the tails of the pulses. This result extends the classical existence result for traveling pulses in FitzHugh–Nagumo.

We employed many of the same techniques used in the classical existence proof in the context of geometric singular perturbation theory. Fenichel’s theorems and the exchange lemma were used to construct the pulse up to understanding the flow near two non-hyperbolic fold points of the critical manifold. To understand the flow near the folds, we used blow up techniques to extend results from [38] and obtain estimates on the flow in small neighborhoods of these points.

We next proved the spectral and nonlinear stability of fast pulses with oscillatory tails in the regime where $0 < a, \varepsilon \ll 1$. We showed that the linearization of this PDE about a fast pulse has precisely two eigenvalues near the origin when considered in an appropriate weighted function space. One of these eigenvalues λ_0 is situated at the origin due to translational invariance, and we proved that the second nontrivial eigenvalue λ_1 is real and strictly negative, thus yielding stability. Our proof also recovers the known result that fast pulses with monotone tails, which exist for fixed $0 < a < \frac{1}{2}$, are stable. Comparing the case of monotone versus oscillatory tails, there

are some challenges present in the oscillatory case due to the nonhyperbolicity of the slow manifolds at the two fold points where the Nagumo front and back jump off to the other branches of the slow manifold. Our results show that these challenges are not just technical but rather result in qualitatively different behaviors. First, the fold at the equilibrium rest state facilitates the onset of the oscillations in the tails of the pulses. Second, the symmetry present due to the cubic nonlinearity means that the back has to jump off the other fold point. Due to the interaction of the back with this second fold point, the scaling of the critical eigenvalue λ_1 in the oscillatory case is given by $\varepsilon^{2/3}$, in contrast to the monotone case where it scales with ε . Moreover, the criterion that needs to be checked to ascertain the sign of λ_1 is different in these two cases.

Our proof of spectral stability is based on Lyapunov-Schmidt reduction, and, more specifically, on the approach taken in [31] to prove the stability of fast pulses with monotone tails for the discrete FitzHugh-Nagumo system. We begin with the linearization of the FitzHugh-Nagumo equation about the fast pulse and write the associated eigenvalue problem as

$$\psi_\xi = A(\xi, \lambda)\psi, \tag{0.2}$$

where $A(\xi, \lambda) \rightarrow \hat{A}(\lambda)$ as $|\xi| \rightarrow \infty$. The ξ -dependence in the matrix $A(\xi, \lambda)$ reflects the passage of the fast pulse along the front, through the right branch of the slow manifold, the jump-off at the upper-right knee along the back, and down the left branch of the slow manifold. Key to our approach is the fact that the spectrum of the matrix $A(\xi, \lambda)$ near the slow manifolds has a consistent splitting into one unstable and two center-stable eigenvalues, and that an exponential weight moves the center eigenvalue into the left half-plane. Eigenfunctions therefore correspond to solutions that decay exponentially as $\xi \rightarrow -\infty$, while they may grow algebraically or

even with a small exponential rate (corresponding to the center-stable matrix eigenvalues) as $\xi \rightarrow \infty$. The splitting along the slow manifolds guarantees the existence of exponential dichotomies along the slow manifolds and shows that they cannot contribute point eigenvalues. The splitting allows us also to decide whether the front and the back will contribute eigenvalues. For the FitzHugh-Nagumo system, both will contribute because their derivatives decay exponentially as $\xi \rightarrow -\infty$ so that they emerge along the unstable direction. In contrast, for the cases studied in [5, 29], the back decays algebraically as $\xi \rightarrow -\infty$ and therefore emerges from the center-stable direction instead of the unstable direction as required for eigenfunctions: hence, the back does not contribute an eigenvalue. Thus, for FitzHugh-Nagumo, both front and back will contribute an eigenvalue, and our approach consists of constructing, for each prospective eigenvalue λ in the complex plane, a piecewise continuous eigenfunction of the linearization, that is a piecewise continuous solution to (0.2), where we allow for precisely two jumps that occur in the middle of the front and the back. Finding eigenvalues then reduces to identifying values of λ for which these jumps vanish. Melnikov theory allows us to find expressions for these jumps that can then be solved.

We emphasize that this approach applies to the more general situation of a pulse that is constructed by concatenating several fronts and backs with parts of the slow manifolds: as long as there is a consistent splitting of eigenvalues, we can decide which fronts and backs contribute an eigenvalue, and then construct prospective eigenfunctions with as many jumps as expected eigenvalues, where the jumps occur near the fronts and backs that contribute. Equation (0.2) will have exponential dichotomies along the slow manifolds and along the fronts and backs that do not contribute eigenvalues, which allows for a reduction to a finite set of jumps with expansions that can be calculated using Melnikov theory.

Our method provides a piecewise continuous eigenfunction for any prospective eigenvalue λ . Thus, by finding the eigenvalues λ for which the finite set of jumps vanishes, we have therefore determined the corresponding eigenfunctions. In our analysis, this amounts to the observation that eigenfunctions are found by piecing together multiples of the derivatives of the Nagumo front $\beta_f \phi'_f$ and back $\beta_b \phi'_b$, where the ratio of the amplitudes (β_f, β_b) is determined by the corresponding eigenvalue (see Remark 4.5.12). As expected, the eigenfunction corresponding to the translational eigenvalue $\lambda_0 = 0$ is represented by $(\beta_f, \beta_b) = (1, 1)$. Moreover, assuming the second eigenvalue $\lambda_1 < 0$ lies to the right of the essential spectrum, the corresponding eigenfunction is centered at the back as we have $(\beta_f, \beta_b) = (0, 1)$. The implications for the dynamics of the pulse profile under small perturbations are as follows. If a perturbation is localized near the back of the pulse, then it excites only the eigenfunction corresponding to λ_1 , and the back will move with exponential rate back to its original position relative to the front without interacting with the front. On the other hand, perturbations that affect also the front will cause a shift of the full profile. These two mechanisms provide a detailed description of the way in which solutions near the traveling pulse converge over time to an appropriate translate of the pulse.

Finally, we described a phenomenon, previously observed numerically [9, 22, 23] in which the branch of fast pulse solutions to (0.1) described in [8] turns sharply when continued numerically in the parameters (c, a) . In addition, we contextualized the above existence result in the study of FitzHugh–Nagumo as a “CU-system”, and we described a geometric mechanism for how the onset of oscillations leads to the addition of a full second pulse and the appearance of a homoclinic banana bifurcation diagram. This transition also explains the termination of the branch of pulses constructed above as we approach the region in parameter space containing

canard solutions arising from the singular Hopf bifurcation occurring at the origin.

Using geometric singular perturbation theory and blow-up techniques, we constructed this transition analytically. Our results guarantee, for sufficiently small $\varepsilon > 0$, the existence of a one-parameter family of traveling pulse solutions, which encompasses the onset of oscillations in the tails and all intermediate pulses between the single and double pulse. The procedure works for constructing pulses along the transition which are arbitrarily close to the double pulse, but not all double pulses along the homoclinic banana. This seems in line with the observation that for sufficiently small $\varepsilon > 0$, the homoclinic banana cannot be continued beyond the Belyakov transition [9]. Further study is required to determine exactly how this branch terminates.

The analysis in the construction may also shed light on the phenomenon of spike adding, which has been studied and observed in a variety of contexts [13, 21, 44, 45, 52, 53]. A common theme in such studies is the connection with a canard explosion type mechanism, and previous studies have been primarily numerical. A first question is whether there is a common geometric set-up which generates this type of behavior. The single-to-double pulse transition described above in relation to the FitzHugh-Nagumo system is a starting point for understanding the geometry and analysis required in constructing such a transition.

Bibliography

- [1] Abramowitz, M., Stegun, I.A., editors. Handbook of mathematical functions: with formulas, graphs, and mathematical tables. Dover, 1964.
- [2] Aronson, D.G., Weinberger, H.F.. Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics* 1978;30(1):33–76.
- [3] Bates, P.W., Lu, K., Zeng, C.. Existence and persistence of invariant manifolds for semiflows in Banach space. volume 645. American Mathematical Soc., 1998.
- [4] Bates, P.W., Lu, K., Zeng, C.. Invariant foliations near normally hyperbolic invariant manifolds for semiflows. *Transactions of the American Mathematical Society* 2000;352(10):4641–4676.
- [5] Beck, M., Jones, C., Schaeffer, D., Wechselberger, M.. Electrical waves in a one-dimensional model of cardiac tissue. *SIAM J Applied Dynamical Systems* 2008;7(4):1558–1581.
- [6] Carpenter, G.. A geometric approach to singular perturbation problems with applications to nerve impulse equations. *J Differential Equations* 1977;23(3):335–367.
- [7] Carter, P., de Rijk, B., Sandstede, B.. Stability of traveling pulses with oscillatory tails in the FitzHugh–Nagumo system; 2015. Submitted for publication.
- [8] Carter, P., Sandstede, B.. Fast pulses with oscillatory tails in the FitzHugh–Nagumo system. *SIAM J Math Anal* 2015;47(5):3393–3441.
- [9] Champneys, A., Kirk, V., Knobloch, E., Oldeman, B., Sneyd, J.. When Shil’nikov meets Hopf in excitable systems. *SIAM J Applied Dynamical Systems* 2007;6(4):663–693.
- [10] Coppel, W.A.. Dichotomies and reducibility. *J Differential Equations* 1967;3:500–521.
- [11] Coppel, W.A.. Dichotomies in stability theory. volume 629 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1978.

- [12] De Maesschalck, P., Dumortier, F.. Time analysis and entry–exit relation near planar turning points. *Journal of Differential Equations* 2005;215(2):225–267.
- [13] Desroches, M., Kaper, T., Krupa, M.. Mixed-mode bursting oscillations: Dynamics created by a slow passage through spike-adding canard explosion in a square-wave burster. *Chaos: An Interdisciplinary Journal of Nonlinear Science* 2013;23(4):046106.
- [14] Eszter, E.G.. An Evans function analysis of the stability of periodic travelling wave solutions of the FitzHugh-Nagumo system. Ph.D. thesis; University of Massachusetts; 1999.
- [15] Evans, J.W.. Nerve axon equations. I. Linear approximations. *Indiana Univ Math J* 1971/72;21:877–885.
- [16] Evans, J.W.. Nerve axon equations. III. Stability of the nerve impulse. *Indiana Univ Math J* 1972/73;22:577–593.
- [17] Faye, G.. Existence and stability of traveling pulses in a neural field equation with synaptic depression. *SIAM J Applied Dynamical Systems* 2013;12(4):2032–2067.
- [18] Fenichel, N.. Geometric singular perturbation theory for ordinary differential equations. *J Differential Equations* 1979;31(1):53–98.
- [19] FitzHugh, R.. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal* 1961;1(6):445–466.
- [20] Gallay, T.. Local stability of critical fronts in nonlinear parabolic partial differential equations. *Nonlinearity* 1994;7(3):741.
- [21] Guckenheimer, J., Kuehn, C.. Computing slow manifolds of saddle type. *SIAM Journal on Applied Dynamical Systems* 2009;8(3):854–879.
- [22] Guckenheimer, J., Kuehn, C.. Homoclinic orbits of the FitzHugh-Nagumo equation: The singular-limit. *Discrete and Continuous Dynamical Systems Series S* 2009;2(4):851.
- [23] Guckenheimer, J., Kuehn, C.. Homoclinic orbits of the FitzHugh-Nagumo equation: Bifurcations in the full system. *SIAM J Applied Dynamical Systems* 2010;9(1):138–153.
- [24] Haaf, H.. Existence of periodic travelling waves to reaction-diffusion equations with excitable-oscillatory kinetics. Ph.D. thesis; University of Warwick; 1993.
- [25] Hastings, S.P.. Some mathematical problems from neurobiology. *American Mathematical Monthly* 1975;;881–895.
- [26] Hastings, S.P.. On the existence of homoclinic and periodic orbits for the FitzHugh-Nagumo equations. *Quarterly J Mathematics* 1976;27(1):123–134.
- [27] Hastings, S.P.. Single and multiple pulse waves for the FitzHugh–Nagumo equations. *SIAM Journal on Applied Mathematics* 1982;42(2):247–260.

- [28] Hodgkin, A.L., Huxley, A.F.. A quantitative description of membrane current and its application to conduction and excitation in nerve. *The Journal of physiology* 1952;117(4):500.
- [29] Holzer, M., Doelman, A., Kaper, T.J.. Existence and stability of traveling pulses in a reaction–diffusion-mechanics system. *J Nonlinear Sci* 2013;23(1):129–177.
- [30] Homburg, A., Sandstede, B.. Homoclinic and heteroclinic bifurcations in vector fields. *Handbook of Dynamical Systems* 2010;3:379–524.
- [31] Hupkes, H., Sandstede, B.. Stability of pulse solutions for the discrete FitzHugh-Nagumo system. *Trans American Mathematical Society* 2013;365(1):251–301.
- [32] Jones, C.. Stability of the travelling wave solution of the FitzHugh-Nagumo system. *Trans American Mathematical Society* 1984;286(2):431–469.
- [33] Jones, C., Kaper, T., Kopell, N.. Tracking invariant manifolds up to exponentially small errors. *SIAM J Mathematical Analysis* 1996;27(2):558–577.
- [34] Jones, C., Kopell, N., Langer, R.. Construction of the FitzHugh-Nagumo pulse using differential forms. In: *Patterns and dynamics in reactive media*. Springer; 1991. p. 101–115.
- [35] Kapitula, T., Promislow, K.. Spectral and dynamical stability of nonlinear waves. volume 185 of *Applied Mathematical Sciences*. Springer, New York, 2013. With a foreword by C. K. R. T. Jones.
- [36] Keener, J.P., Sneyd, J.. *Mathematical physiology*. volume 1. Springer, 1998.
- [37] Krupa, M., Sandstede, B., Szmolyan, P.. Fast and slow waves in the FitzHugh-Nagumo equation. *J Differential Equations* 1997;133(1):49–97.
- [38] Krupa, M., Szmolyan, P.. Extending geometric singular perturbation theory to nonhyperbolic points—fold and canard points in two dimensions. *SIAM J Mathematical Analysis* 2001;33(2):286–314.
- [39] Krupa, M., Szmolyan, P.. Relaxation oscillation and canard explosion. *J Differential Equations* 2001;174(2):312–368.
- [40] Levinson, N.. The asymptotic nature of solutions of linear systems of differential equations. *Duke Math J* 1948;15:111–126.
- [41] Lin, X.B.. Using Melnikov’s method to solve Shilnikov’s problems. *Proc Roy Soc Edinburgh Sect A* 1990;116(3-4):295–325.
- [42] Mishchenko, E., Rozov, N.. *Differential equations with small parameters and relaxation oscillations*. Mathematical Concepts and Methods in Science and Engineering. Boston: Springer, 1980.
- [43] Nagumo, J., Arimoto, S., Yoshizawa, S.. An active pulse transmission line simulating nerve axon. *Proceedings of the IRE* 1962;50(10):2061–2070.

- [44] Nowacki, J., Osinga, H., Tsaneva-Atanasova, K.. Dynamical systems analysis of spike-adding mechanisms in transient bursts. *The Journal of Mathematical Neuroscience (JMN)* 2012;2(1):1–28.
- [45] Osinga, H., Tsaneva-Atanasova, K.. Dynamics of plateau bursting depending on the location of its equilibrium. *Journal of Neuroendocrinology* 2010;22(12):1301–1314.
- [46] Palmer, K.J.. Exponential dichotomies and transversal homoclinic points. *J Differential Equations* 1984;55(2):225–256.
- [47] Sandstede, B.. Verzweigungstheorie homokliner Verdopplungen. Ph.D. thesis; University of Stuttgart; 1993.
- [48] Sandstede, B.. Stability of multiple-pulse solutions. *Trans Amer Math Soc* 1998;350(2):429–472.
- [49] Sandstede, B.. Stability of travelling waves. In: *Handbook of dynamical systems*. North-Holland, Amsterdam; volume 2; 2002. p. 983–1055.
- [50] Sandstede, B., Scheel, A.. Absolute and convective instabilities of waves on unbounded and large bounded domains. *Phys D* 2000;145(3-4):233–277.
- [51] Schechter, S.. Exchange lemmas 2: General exchange lemma. *J Differential Equations* 2008;245(2):411–441.
- [52] Terman, D.. Chaotic spikes arising from a model of bursting in excitable membranes. *SIAM Journal on Applied Mathematics* 1991;51(5):1418–1450.
- [53] Tsaneva-Atanasova, K., Osinga, H., Rieß, T., Sherman, A.. Full system bifurcation analysis of endocrine bursting models. *Journal of Theoretical Biology* 2010;264(4):1133–1146.
- [54] Yanagida, E.. Stability of fast travelling pulse solutions of the FitzHugh-Nagumo equations. *J Mathematical Biology* 1985;22(1):81–104.