Small surface tension, infinite time, and light electrons: three results on the asymptotic analysis of PDEs

by

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The author received his BS at Oregon State University in 2018. Afterwards, he became a PhD student at Brown University. In the process, he received an MS at Brown University in 2020. He is advised by Benoît Pausader. His published works are [55–57]. His as-of-yet unpublished works include [53, 54].

Preface and Acknowledgments

I would like to thank my advisor, Benoît Pausader, for his strong guidance and encouragement. I only hope that some of his creativity, work ethic, and enthusiasm for science has rubbed off on me.

Thank you to Javier Gómez Serrano and Govind Menon, who have graciously agreed to serve on my committee. Beyond the defense, both have provided me invaluable support.

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CONTENTS

Chapter One

Introduction

1.1 Asymptotic behavior of PDEs

The study of fluids and plasmas is of central importance in modern science and engineering, due to their relevance to the realms of energy production, astrophysics, and climate change. However, the models which govern plasmas and fluids are often difficult to use in practice. This is for two reasons. First, the models are generally nonlinear, making it difficult to study exact solutions. Second, accurate numerical simulations come with an immense computational cost.

Nevertheless, all these models are expressed in terms of partial differential equations (PDEs), so we may subject them to mathematical analysis. The complexity of models in plasma and fluid physics necessitates the use of approximations to glean any useful information. As mathematicians, we can verify (or debunk) these approximations, and quantify their error. Typically, such approximations involve identifying the asymptotic behavior of solutions as time goes to infinity, or as a parameter goes to zero, so as to derive a simpler model.

This thesis is an anthology of four papers in this direction of research. Chapter 2 consists of the paper [57], joint work with Huy Nguyen, concerned with the vanishing surface tension limit for the Muskat equation. Chapter 3 consists of the paper [56], joint work with Zhimeng Ouyang, Benoît Pausader and Klaus Widmayer, concerned with the construction of the scattering map for the Vlasov-Poisson system. Chapter 4 consists of the work [54], joint work with Yan Guo, concerned with the massless electron limit for the Vlasov-Poisson-Landau system. Chapter 5 consists of the forthcoming work [53], which provides a local well-posedness theory for the Vlasov-Poisson-Landau system, and serves as a companion paper to the previous work. Summaries of these works, and potential future directions, are given below.

Each chapter is more or less self-contained, although there is overlap in notation, particularly between Chapter 4 and Chapter 5. Moreover, these two final chapters contain references to each other—in particular, Chapter 4 utilizes Lemma 4.3.2.

1.2 The Vanishing Surface Tension Limit of the Muskat problem

The Muskat problem is concerned with the interface dynamics of a fluid immersed in a porous medium. In the simplest setting, we can formulate the Muskat problem as follows. We write $(x, y) \in \mathbf{R}^{d-1} \times \mathbf{R}$, and let the fluid interface be given by the graph of $\eta(t, x)$, i.e. the fluid boundary is the set $\{(x, y) \ : \ y = \eta(t, x)\}$. Then, this function satisfies the equation

$$\partial_t \eta + G(\eta)(\eta + \sigma H(\eta)) = 0, \tag{1.2.1}$$

where $H(\eta)$ is the mean curvature of the interface, σ is the surface tension coefficient, and $G(\eta)$ is the *Dirichlet-to-Neumann* operator. To be more specific, if we let $\varphi(x, y)$ solve the Laplace equation in the fluid domain, and prescribe the Dirichlet boundary condition $\varphi(x, \eta(t, x)) = f$ on the interface, then $G(\eta)f$ gives the normal component of $\nabla_{x,y}\varphi$ along the fluid interface. If we linearize about a flat horizontal interface, then $G(\eta)$ behaves like the fractional Laplacian $(-\Delta_x)^{\frac{1}{2}}$, while $H(\eta)$ behaves like the full Laplacian $-\Delta_x$. As a PDE, the Muskat problem is quasilinear, nonlocal and parabolic in nature.

In many physically important situations, σ is very small, so one sets $\sigma = 0$. In Chapter 2, and also in [57], H. Nguyen and I show that this approximation can be made rigorous in ≥ 2 dimensions [57]. Specifically, we show that solutions to the Muskat problem with surface tension solve the model without surface tension in the limit as the surface tension coefficient tends to zero, with the optimal rate of convergence:

Theorem 1.2.1. *Let $d \geq 2$. There exist Hilbert spaces $X, Y \subset L^2(\mathbf{R}^{d-1})$ such that the following holds. Let $\eta_{in} \in X$. Then there exists constants C, T depending only on η_{in} such that for each $\sigma \in [0, 1]$, there exists a unique solution $\eta^{(\sigma)} \in C([0, T]; X)$ solving (1.2.1) with the given value of σ, and*

$$\|\eta^{(\sigma)} - \eta^{(0)}\|_{L^\infty([0,T];Y)} \leq C\sigma. \tag{1.2.2}$$

To accomplish this, we use the methodology of para-differential calculus, as had been applied in the previous result on local well-posedness of the Muskat problem [109], and the equation with surface tension [108]. This required a careful adaptation of the techniques in these previous works, including the paralinearization of the Dirichlet-to-Neumann operator and mean curvature, precise energy estimates, and contraction estimates.

1.3 The Scattering Problem for the Vlasov-Poisson System

Many important nonlinear PDEs, particularly wave type equations, are known to exhibit *scattering*: that is, as time goes to infinity, solutions asymptotically behave like a solution to the linearized problem. For certain equations, including the VP system, the limiting behavior is given by the linearization, plus some simple nonlinear correction. In this case, we say that the PDE exhibits *modified scattering*. After deriving this limiting behavior, one can also consider the inverse problem, where one constructs the *wave operator*: that is, the map which gives a solution to the PDE with prescribed asymptotic data. Then, one can construct the *scattering map*, which gives the asymptotic data of a solution as $t \to \infty$, with prescribed asymptotic data as $t \to -\infty$.

Next, we introduce the Vlasov-Poisson (VP) system, which is a kinetic model for distribution of matter $\mu(t, x, v)$[2] interacting through electrostatic force or classical gravitation, given as follows [1]

$$\partial_t \mu + v \cdot \nabla_x \mu - \lambda \nabla_x \phi \cdot \nabla_v \mu = 0, \quad -\Delta_x \phi = \int_{\mathbf{R}^3} \mu^2 dv. \tag{1.3.1}$$

Here, $\lambda = 1$ in the electrostatic case, and $\lambda = -1$ in the gravitational case. As a PDE, it is a transport equation whose characteristics are determined by an electric or gravitational field, which is in turn determined by Poisson's equation, resulting in a nonlinear term. While small solutions to (1.3.1) were known to have modified scattering (see [33], [90]), the question of the existence of the wave operators and scattering map had remained open. In Chapter 3, also in [56], Z. Ouyang, B. Pausader and K. Widmayer and I prove that these objects exist :

Theorem 1.3.1. *There exist $\varepsilon > 0$ and Hilbert spaces $X, Y, Z \subset L^2(\mathbf{R}^3 \times \mathbf{R}^3)$ such that the following hold*

(i) (Modified scattering) Suppose $\|\mu_0\|_X < \varepsilon$, and let $\mu(t,x,v)$ be the solution to (1.3.1) with $\mu(t=0) = \mu_0$. Then, this solution exhibits modified scattering, that is, there exists $\mu_\infty(x,v) \in Y$ and

$$E_\infty(v) = \nabla_v \Delta_v^{-1} \int_{\mathbf{R}^3} \mu_\infty(x,v)^2 dx \tag{1.3.2}$$

such that

$$\lim_{t\to\infty} \|\mu(t, x - tv - \lambda \ln(t) E_\infty(v), v) - \mu_\infty(x,v)\|_Y = 0. \tag{1.3.3}$$

(ii) (Existence of wave operators) Suppose $\|\mu_\infty\|_Z < \infty$, and define E_∞ as in (1.3.2). Then, there exists a classical solution $\mu(t,x,v)$ satisfying (1.3.1), with

[1] Usually, this equation is expressed in terms of $F = \mu^2$. It is convenient to express the VP system in terms of μ, as writing $F = \mu^2$ ensures positivity of F, and conservation of mass and energy are expressed in terms of Hilbertian norms of μ.

asymptotic behavior determined by (1.3.3).

(iii) (Existence of scattering map) There exists a map $\mathcal{S} : Z \to W$ *such that if we set* $\mathcal{S}\mu_{-\infty} = \mu_\infty$*, there exists a solution* μ *to* (1.3.1) *such that*

$$\lim_{t \to \pm\infty} \|\mu(t, x - tv - \lambda \ln(|t|) E_{\pm\infty}(v), v) - \mu_{\pm\infty}(x, v)\|_Y = 0 \tag{1.3.4}$$

Here, E_∞ *is defined as in* (1.3.2)*, and* $E_{-\infty}$ *is defined in the same way, except with* $\mu_{-\infty}$ *instead of* μ_∞*.*

Now, in (1.3.3), the transformation $(x, v) \mapsto (x - tv, v)$ accounts for the linear evolution, while the logarithmic term $\ln(t)E_\infty(v)$ accounts for long-range effects coming from the nonlinearity. As aforementioned, part (i) of this theorem had already been shown, although we prove modified scattering for more general initial data than prior works.

Drawing on [95], and inspiration from dispersive equations, (e.g. [129], [25]). One could use the *pseudo-conformal transformation* (PCT) to study the scattering problem for Vlasov-Poisson, where we define

$$s = \frac{1}{t}, \quad q = \frac{x}{t}, \quad p = x - tv \tag{1.3.5}$$

and set $\gamma(s, q, p) = \mu(t, x, v)$. Now, γ solves

$$\partial_s \gamma + p \cdot \nabla_q \gamma - \frac{\lambda}{s} \nabla_q \psi \cdot \nabla_p \gamma = 0, \quad -\Delta_x \psi = \int_{\mathbf{R}^3} \gamma^2 dp \tag{1.3.6}$$

Now, this transformation preserves the linearized VP system, while inverting time. The limit $t \to \infty$ becomes the limit as $s \downarrow 0$ in the new coordinates. The PCT nearly preserves the full VP system, except for the nonlinear term, which acquires a factor of $\frac{1}{s}$. This singularity is not integrable near $s = 0$, but nevertheless, the continuity of mass equation still holds in the new coordinates, i.e.

$$\partial_t \int_{\mathbf{R}^3} \gamma(s,q,p)^2 dp + \nabla_q \cdot \int_{\mathbf{R}^3} p\gamma(s,q,p)^2 dp = 0. \tag{1.3.7}$$

We then use this to show that $\lim_{s\downarrow 0} \nabla_q \psi(s,q) = E_\infty(q)$, provided γ does not blow-up too quickly (for instance, only logarithmically). To then construct γ, it is still necessary to remove the singularity. We accomplish this by using a canonical transformation, induced by a certain generating function. This allows us to reformulate the wave operator problem for the VP system as a standard initial value problem. When we express this transformation in terms of (x, v), we recover the asymptotic behavior appearing in (1.3.3). We show that this initial value problem is locally well-posed, via *a priori* estimates, and a fixed-point argument. This implies the existence of wave operators, and in turn existence of the scattering map.

1.4 The Massless Electron Limit for the Vlasov-Poisson-Landau-System

Physical plasmas, such as the sun, are primarily composed of ionized hydrogen and helium. The mass of electrons is roughly 2000 times smaller than the mass of a single proton, and 8000 times smaller than the mass of a helium nucleus. This motivates the massless electron approximation, where one sets the electron mass to zero. In this regime, one can approximate the electron distribution as a quasi-statically evolving equilibrium distribution, wholly determined by the dynamics of the ions. This gives rise to the Maxwell-Boltzmann relation, where the electron density is given by $e^{\beta\phi^0}$. Here, β is the inverse temperature of the electrons, and ϕ^0

is the limit of the electrostatic potential as $\varepsilon \to 0$.

To model such plasmas, we can use the Vlasov-Poisson-Landau (VPL) system. The VPL system is a kinetic model for the distributions of electrons F_- and positively charged ions F_+ in a plasma. Let ε^2 be the ratio of electron mass over ion mass. Now, in the regime where both species have roughly the same temperature (i.e., close to thermal equilibrium), the speed of a typical electron will be much larger than that of an ion, by a factor of roughly $1/\varepsilon$. Letting v be the velocity in the original frame, we use the re-scaled velocity $w = \frac{v}{\varepsilon}$ for the electron distribution $F_-(t,x,w)$. Under this re-scaling, the VPL equation becomes [2]

$$\begin{aligned}
&\{\partial_t + v\cdot\nabla_x - \nabla_x\phi^\varepsilon\cdot\nabla_v\}F_+^\varepsilon = Q(F_+^\varepsilon, F_+^\varepsilon) + Q_{-+}^\varepsilon(F_-^\varepsilon, F_+^\varepsilon),\\
&\{\varepsilon\partial_t + w\cdot\nabla_x + \nabla_x\phi^\varepsilon\cdot\nabla_w\}F_-^\varepsilon = Q(F_-^\varepsilon, F_-^\varepsilon) + Q_{+-}^\varepsilon(F_+^\varepsilon, F_-^\varepsilon),\\
&-\Delta_x\phi^\varepsilon = 4\pi\Big(\int_{\mathbf{R}^3} F_+^\varepsilon dv - \int_{\mathbf{R}^3} F_-^\varepsilon dw\Big).
\end{aligned} \tag{1.4.1}$$

Now, the VPL system is similar to the VP system, except that it contains the additional effects of collisions between particles, due to the collision operators Q, Q_{-+}^ε and Q_{+-}^ε. We will omit the exact form of these operators here. What is important to know is that while the VP system is reversible, the inclusion of the collision terms cause the VPL system to be parabolic in character. Moreover, these collisional effects force rapid decay to equilibrium solutions, as shown in the works [70], [71]. The equilibrium distributions in this case are the *Maxwellians*, that is, functions with Gaussian dependence on the velocity variable,

$$\mu_\gamma(t,v) = \left(\frac{\gamma}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{\gamma}{2}|v|^2}, \quad \gamma > 0. \tag{1.4.2}$$

Then, $F_+^\varepsilon = \mu_\gamma(v)$ and $F_-^\varepsilon = \mu_\gamma(w)$ give stationary solutions of (1.4.1) for all γ.

[2] We have also normalized some physical constants, such as the atomic mass of the ions.

Now, if we formally send $\varepsilon \to 0$, and assume $F_-^0 = \lim_{\varepsilon\to 0} F_-^\varepsilon$ and $F_+^0 = \lim_{\varepsilon\to 0} F_+^\varepsilon$ both exist, we have

$$\{w \cdot \nabla_x + \nabla_x \phi^0 \cdot \nabla_w\} F_-^0 = Q(F_-^0, F_-^0) + Q_{+-}^0(F_+^0, F_-^0), \tag{1.4.3}$$

We fix the spatial domain to be $x \in \mathbf{T}^3$, where $\mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$ is the 3D torus. We also assume $\int_{\mathbf{T}^3\times R^3} F_\pm^0 dxdw = 1$, so the only solutions to the above (which are sufficiently regular and decay rapidly) are $F_-^0(t,x,w) = \mu_{\beta(t)}(w)e^{\beta(t)\phi^0(t,x)}$. Here, $\beta(t)$ is inversely proportional to the total kinetic energy of F_-^0, and is in turn determined by conservation of energy. Moreover, the effects of ion-electron collisions vanish, i.e. $Q_{-+}^0(F_-^0, F_+^0) = 0$. This leads to the following system for (F_+^0, ϕ^0, β), which we call the VPL system with massless electrons (or VPLME),

$$\begin{aligned}
&\{\partial_t + v \cdot \nabla_x - \nabla_x \phi^0 \cdot \nabla_v\} F_+^0 = Q(F_+^0, F_+^0), \\
&\frac{d}{dt}\left(\frac{3}{2\beta} + \int_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2} F_+ dxdv + \int_{\mathbf{T}^3} |\nabla_x \phi|^2 dx\right) = 0. \\
&-\Delta_x \phi^0 = 4\pi\left(\int_{\mathbf{R}^3} F_+^0 dv - e^{\beta\phi^0}\right)
\end{aligned} \tag{1.4.4}$$

Now, the Maxwell-Boltzmann approximation is widely used in plasma physics, perhaps most notably to derive solitary wave solutions for the Euler-Poisson system [131], [16]. While the approximation has been derived in the case of fluid equations [64], no attempt to rigorously justify the limit for a kinetic equation had been made until the work [12], in the case of the Vlasov-Poisson-Boltzmann system. This result, however, is conditional, since it assumes *a priori* that the solution converges in the limit as the electron mass goes to zero, and then identifies the correct limit. This work also ignores the effects of inter-species collisions. In Chapter 4, and in [54], Yan Guo and I give a rigorous derivation of the massless electron limit for the VPL system on a periodic domain. This is stated in the following theorem:

Theorem 1.4.1. *There exist Hilbert spaces* $X, Y, Z, W \subset L^2(\mathbf{T}^3 \times \mathbf{R}^3)$ *such that the following holds. First, fix some solution* $(F_+^0, \phi^0, \beta) \in C([0,T]; X \times L^\infty(\mathbf{T}^3) \times (0,\infty))$ *to* (1.4.4) *with initial data* $(F_{+,in}^0, \phi_{in}^0, \beta_{in})$*. Then, for each* $\varepsilon \in (0,1]$*, take initial data* $F_{+,in}^\varepsilon, F_{-,in}^\varepsilon \in X$*. Take also* $F_{+,in} \in X$*,* $\beta_{in} \in (0,\infty)$*, and define* ϕ_{in}^0 *by the third line of* (1.4.4)*. Assume for all* $\varepsilon \in (0,1]$*,*

$$\|F_{+,in}^\varepsilon - F_{+,in}^0\|_Y + \|F_{-,in}^\varepsilon - \mu_{\beta_{in}} e^{\beta_{in}\phi_{in}^0}\|_Z \leq \varepsilon. \tag{1.4.5}$$

Then, for some $T > 0$*, there exist unique solutions* $(F_+^\varepsilon, F_-^\varepsilon) \in C([0,T]; X \times X)$ *to* (1.4.1) *with initial data* $(F_{+,in}^\varepsilon, F_{-,in}^\varepsilon)$ *for all* $\varepsilon \in (0,1]$*. These converge to the limiting system in the following sense*

$$\|F_+^\varepsilon - F_+^0\|_{L^\infty([0,T],Y)} + \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{L^2([0,T];W)} \leq C\varepsilon. \tag{1.4.6}$$

To prove the result above, we must verify a *singular limit*. Specifically, the time derivative of the electron distribution has a coefficient of ε, which could result in rapid growth or oscillation in the absence of any stabilization mechanism. In addition, since we do not rescale the ion velocity, we must account for two divergent velocity scales in the analysis of the ion-electron collision terms. This is reminiscent to the separate spatial scales that emerge in the asymptotic analysis of the Prandtl boundary layer problem [120]. Nevertheless, we show that the difference between the electron distribution and the limiting Maxwellian becomes arbitrarily small in a time proportional to ε, using the stabilization effect of the collisions. Moreover, we found that the effect of ion-electron collisions completely drop out of the equation for the ion distribution, and yet contribute to extra stabilization in the equation for the electron distribution.

To make this argument precise, we apply the framework of [71]. This requires some reformulation, as we need to handle variable temperature $\frac{1}{\beta(t)}$ of the underlying Maxwellian.

1.5 Local Well-Posedness of the Vlasov-Poisson-Landau system

The derivation of the massless limit given by Theorem 1.4.1 is conditioned on the existence of solutions to (1.4.4). In Chapter 5, also in the forthcoming paper [54], we state a local well-posedness Theorem for both (1.4.1) and (1.4.4). This chapter shares many of its methods of Chapter 4. An abridged version of this theorem for the system (1.4.4) is as follows:

Theorem 1.5.1. *There exists a Hilbert space $\tilde{X}$ such that the following holds. For all $(F_{+,in}, \beta) \in \tilde{X} \times \mathbf{R}_+$, such that $1/n_{+,in}$ is uniformly bounded, there exists $T > 0$ such that (1.4.4) has a unique weak solution (F_+, β, ϕ) to (1.4.4), with $F_+ \in C([0,T]; \tilde{X})$ and $\beta \in C([0,T]; \mathbf{R}_+)$.*

We emphasize that the space $\tilde{X}$ is different from the space X in Theorem 1.4.1, although only slightly. The $\tilde{X}$ norm is the same as the X norm, except that it also includes arbitrarily small control on v derivatives. The reason why this is necessary is rather technical. In the context of the massless limit, while it may seem that we are working with large initial data for F_+^{ε}, we rely on the fact that $F_+^{\varepsilon} - F_+^0$ is small to close our estimates. In the context of local well-posedness, we have no analogous smallness. To compensate for this, we utilize the regularity in v to interpolate problematic commutators.

In addition, Chapter 5 gives an account of a construction scheme for solutions to (1.4.1) and (1.4.4). While mostly standard, there are several difficulties coming from the degenerate, quasilinear nature of the dissipation. In the case of (1.4.4), constructing (β, ϕ^0) presents an additional challenge. For this, we adapt and extend some of the results in [12].

Chapter Two

The Vanishing Surface Tension Limit

Joint work with Huy Q. Nguyen.

2.1 Introduction

In the studies of fluid flows, interfacial dynamics is a broad and mathematically challenging topic. Some interfacial problems in fluid dynamics that have been rigorously studied include the water wave problem, the compressible free-boundary Euler equations, the Hele-Shaw problem, the Muskat problem and the Stefan problem. The dynamics of the interface between the fluids strongly depends on properties of the fluids and of the media through which they flow. However, a common feature in all the above problems is that the interface is driven by gravity and surface tension. Gravity is incorporated in the momentum equations as an external force. On the other hand, surface tension balances the pressure jump across the interface

(Young-Laplace equation):

$$[\![p]\!] = \mathfrak{s}H, \tag{2.1.1}$$

where $[\![p]\!]$ is the pressure jump, H is the mean-curvature of the interface, and $\mathfrak{s} > 0$ is the surface tension coefficient. Well-posedness in Sobolev spaces of high enough order always holds when surface tension is taken into account but only holds under the Rayleigh-Taylor stability condition on the initial data when surface tension is neglected. It is a natural problem to justify the models without surface tension as the limit of the corresponding full models as surface tension vanishes. This question was addressed in [5, 8, 9, 43, 78, 99, 121] for the problems listed above. The common theory is the following: if the initial data is stable, i.e. it satisfies the Rayleigh-Taylor stability condition, and is *sufficiently smooth*, then solutions to the problem with surface tension converge to the unique solution of the problem without surface tension locally in time. The general strategy of proof consists of two points.

(i) To leading order, the surface tension term $\mathfrak{s}H$ provides a regularizing effect. For sufficiently smooth solutions, the difference between $\mathfrak{s}H$ and its leading contribution can be controlled by the energy of the problem without surface tension. This yields a uniform time of existence T_* for the problem with surface tension $\mathfrak{s} \to 0$.

(ii) For sufficiently smooth solutions and for some $\theta \in [0, 1)$, the weighted mean curvature $\mathfrak{s}^\theta H$ is uniformly in $\mathfrak{s}$ bounded (in some appropriate Sobolev norm) by the energy of the problem without surface tension. It follows that $\mathfrak{s}H$, the difference between the two problems, vanishes as $\mathfrak{s}^{1-\theta}$ as $\mathfrak{s} \to 0$, establishing the convergence on the time interval $[0, T_*]$. Note that the optimal rate corresponds to $\theta = 0$.

Therefore, the vanishing surface tension limit becomes subtle if the initial data is sufficiently rough so that it can accommodate curvature singularities. As a matter of fact, in the aforementioned works, the initial curvature is at least bounded. In this paper, we prove that for the Muskat problem, the zero surface tension limit can be established for rough initial interfaces whose curvatures are *not bounded* or even *not locally* L^2. Regarding quantitive properties of the zero surface tension limit, the convergence rates in the aforementioned works are either unspecified or suboptimal. In this paper, we obtain the *optimal convergence rate* for the Muskat problem. The next subsections are devoted to a description of the Muskat problem and the statement of our main result.

2.1.1 The Muskat problem

The Muskat problem [107] concerns the interface evolution between two fluids of densities $\rho^\pm$ and viscosities $\mu^\pm$ governed by Darcy's law for flows through porous media. Specifically, the fluids occupy two domains $\Omega^\pm = \Omega^\pm_t \subset \mathbb{R}^{d+1}$ separated by an interface $\Sigma = \Sigma_t$, with Ω^+ confined below a rigid boundary Γ^+, and Ω^- likewise above Γ^-. We consider the case when the surfaces $\Gamma^\pm$ and Σ are given by the graphs of functions, that is, we designate $b^\pm : \mathbb{R}^d_x \to \mathbb{R}$ and $\eta : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}$ for which

$$\Sigma = \{(x, \eta(t,x)) \ : \ x \in \mathbb{R}^d\}, \tag{2.1.2}$$

$$\Gamma^\pm = \{(x, b^\pm(x)) \ : \ x \in \mathbb{R}^d\}, \tag{2.1.3}$$

$$\Omega^- = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} \ : \ b^-(x) < y < \eta(t,x)\}, \tag{2.1.4}$$

$$\Omega^+ = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} \ : \eta(t,x) < y < b^+(x)\}, \tag{2.1.5}$$

$$\Omega = \Omega^+ \cup \Omega^-. \tag{2.1.6}$$

We also consider the case where one or both of $\Gamma^\pm = \emptyset$. In each domain $\Omega^\pm$, the fluid velocity $u^\pm$ and pressure $p^\pm$ obey Darcy's law:

$$\mu^\pm u^\pm + \nabla_{x,y} p^\pm = -\rho^\pm g \vec{e}_{d+1}, \quad \mathrm{div}_{x,y} u^\pm = 0 \quad \text{ in } \Omega^\pm, \tag{2.1.7}$$

where g denotes the gravitational acceleration, and $\vec{e}_{d+1}$ is the upward unit vector in the vertical direction. For any two objects A^+ and A^- associated with the domains Ω^+ and Ω^- respectively, we denote the jump

$$[\![A]\!] = A^- - A^+$$

whenever this difference is well-defined. In particular, set

$$\mathfrak{g} = g[\![\rho]\!].$$

At the interface, there are three boundary conditions. First, the normal component of the fluid velocity is continuous across the interface

$$[\![u \cdot n]\!] = 0 \quad \text{on} \quad \Sigma, \tag{2.1.8}$$

where we fix n to be the upward normal of the interface, specifically $n = \langle \nabla \eta \rangle^{-1}(-\nabla \eta, 1)$ with

$$\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}.$$

Second, the interface is transported by the normal fluid velocity, leading to the kinematic boundary condition

$$\eta_t = \langle \nabla \eta \rangle u^- \cdot n|_{\Sigma_t}. \tag{2.1.9}$$

Third, according to the Young-Laplace equation, the pressure jump is proportional to the mean curvature through surface tension:

$$[\![p]\!] = \mathfrak{s}H(\eta) \equiv -\mathfrak{s}\mathrm{div}(\langle\nabla\eta\rangle^{-1}\nabla\eta) \text{ on } \Sigma_t \tag{2.1.10}$$

where $\mathfrak{s} \geq 0$ is the *surface tension coefficient* and

$$H(\eta) = -\mathrm{div}(\langle\nabla\eta\rangle^{-1}\nabla\eta) \tag{2.1.11}$$

is twice the mean curvature of Σ.

Finally, there is no transportation of fluid through the rigid boundaries:

$$u^{\pm} \cdot \nu^{\pm} = 0 \quad \text{on} \quad \Gamma^{\pm}, \tag{2.1.12}$$

where $\nu^{\pm} = \pm\langle\nabla b\rangle^{-1}(-\nabla b^{\pm}, 1)$ is the outward normal of $\Gamma^{\pm}$. If $\Gamma^{\pm} = \emptyset$, this condition is replaced by the decay condition

$$\lim_{y\to\pm\infty} u^{\pm}(x, y) = 0. \tag{2.1.13}$$

For the two-phase problem, we have $\rho^{\pm}$ and $\mu^{\pm}$ both as positive quantities. We will also consider the one-phase problem where the top fluid is treated as a vacuum by setting $\rho^{+} = \mu^{+} = 0$ and $\Gamma^{+} = \emptyset$.

In the absence of the boundaries $\Gamma^{\pm}$, both the Muskat problems with and without surface tension to leading order admit $\dot{H}^{1+\frac{d}{2}}(\mathbb{R}^d)$ as the scaling invariant Sobolev space in view of the scaling

$$\eta(x, t) \mapsto \lambda^{-1}\eta(\lambda x, \lambda^3 t) \quad \text{and} \quad \eta(x, t) \mapsto \lambda^{-1}\eta(\lambda x, \lambda t).$$

In either case, the problem is quasilinear. The literature on well-posedness for the Muskat problem is vast. Early results can be found in [6, 7, 31, 39, 52, 123, 132]. For more recent developments, we refer to [38, 40, 40, 41, 51, 63] for well-posedness, to [22,36–38,42,47,60,63] for global existence, and to [23,24] for singularity formation. Directly related to the problem addressed in the current paper is local well-posedness for low regularity large data. Consider first the problem without surface tension. In [32], the authors obtained well-posedness for $H^2(\mathbb{T})$ data for the one-phase problem, allowing for unbounded curvature. For the 2D Muskat problem without viscosity jump, i.e. $\mu^+ = \mu^-$, [38] proves well-posedness for data in all subcritical Sobolev spaces $W^{2,1+}(\mathbb{R})$. In L^2-based Sobolev spaces, [101] obtains well-posedness for data in all subcritical spaces $H^{\frac{3}{2}+}(\mathbb{R})$. We also refer to [2] for a generalization of this result to homogeneous Sobolev spaces $\dot{H}^1(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$, $s \in (\frac{3}{2}, 2)$, allowing non-L^2 data. In [109], local well-posedness for the Muskat problem in the general setting as described above was obtained for initial data in all subcritical Sobolev spaces $H^{1+\frac{d}{2}+}(\mathbb{R}^d)$, $d \geq 1$. The case of one fluid with infinite depth was independently obtained by [3]. Regarding the problem with surface tension, [101, 102] consider initial data in $H^{2+}(\mathbb{R})$. In the recent work [108], well-posedness for data in all subcritical Sobolev spaces $H^{1+\frac{d}{2}+}(\mathbb{R}^d)$, $d \geq 1$, was established.

2.1.2 Main Result

In order to state the Rayleigh-Taylor stability condition solely in terms of the interface, we define the operator

$$\mathrm{RT}(\eta) = 1 - [\![\mathfrak{B}(\eta)J(\eta)]\!]\eta \equiv 1 - \Big(\mathfrak{B}^-(\eta)J^-(\eta)\eta - \mathfrak{B}^+(\eta)J^+(\eta)\eta\Big), \tag{2.1.14}$$

where $J^{\pm}(\eta)$ and $\mathfrak{B}^{\pm}(\eta)$ are respectively defined by and (2.2.14) and (2.2.21) below. Our main result is the following.

Theorem 2.1.1. Consider either the one-phase Muskat problem or the two-phase Muskat problem in the stable regime $\rho^- > \rho^+$. The boundaries $\Gamma^{\pm}$ can be empty or graphs of Lipschitz functions $b^{\pm} \in \dot{W}^{1,\infty}(\mathbb{R}^d)$. For any $d \geq 1$, let $s > 1 + \frac{d}{2}$ be an arbitrary subcritical Sobolev index. Consider an initial datum $\eta_0 \in H^s(\mathbb{R}^d)$ satisfying

$$\inf_{x\in\mathbb{R}^d} \mathrm{RT}(\eta_0) \geq 2\mathfrak{a} > 0, \quad \mathrm{dist}(\eta_0, \Gamma^{\pm}) \geq 2h > 0. \tag{2.1.15}$$

Let $\mathfrak{s}_n$ be a sequence of surface tension coefficients converging to 0. Then, there exists $T_* > 0$ depending only on $\|\eta_0\|_{H^s}$ and $(\mathfrak{a}, h, s, \mu^{\pm}, \mathfrak{g})$ such that the following holds.

(i) The Muskat problems without surface tension and with surface tension $\mathfrak{s}_n$ have a unique solution on $[0, T_*]$, denoted respectively by η and η_n, that satisfy

$$\eta_n \in C([0,T_*]; H^s(\mathbb{R}^d)) \cap L^2([0,T_*]; H^{s+\frac{3}{2}}(\mathbb{R}^d)), \tag{2.1.16}$$

$$\eta \in C([0,T_*]; H^s(\mathbb{R}^d)) \cap L^2([0,T_*]; H^{s+\frac{1}{2}}(\mathbb{R}^d)) \cap C([0,T_*]; H^{s'}(\mathbb{R}^d)) \quad \forall s' < s, \tag{2.1.17}$$

$$\|(\eta_n,\eta)\|_{L^\infty([0,T_*];H^s)} + \|(\eta_n,\eta)\|_{L^2([0,T_*];H^{s+\frac{1}{2}})} + \sqrt{\mathfrak{s}}\|\eta_n\|_{L^2([0,T_*];H^{s+\frac{3}{2}})} \leq \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}), \tag{2.1.18}$$

$$\inf_{t\in[0,T_*]} \inf_{x\in\mathbb{R}^d} \mathrm{RT}(\eta_n(t)) > \mathfrak{a}, \qquad \inf_{t\in[0,T_*]} \mathrm{dist}(\eta_n(t), \Gamma^{\pm}) > h, \tag{2.1.19}$$

$$\inf_{t\in[0,T_*]} \inf_{x\in\mathbb{R}^d} \mathrm{RT}(\eta(t)) > \mathfrak{a}, \qquad \inf_{t\in[0,T_*]} \mathrm{dist}(\eta(t), \Gamma^{\pm}) > h, \tag{2.1.20}$$

where $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and depends only on $(h, s, \mu^{\pm}, \mathfrak{g})$.

(ii) As $n \to \infty$, η_n converges to η on $[0, T_*]$ with the rate $\sqrt{\mathfrak{s}_n}$:

$$\|\eta_n - \eta\|_{L^\infty([0,T_*];H^{s-1})} + \|\eta_n - \eta\|_{L^2([0,T_*];H^{s-\frac{1}{2}})} \le \sqrt{\mathfrak{s}_n}\mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}). \tag{2.1.21}$$

If in addition $s \ge 2$, then we have the convergence with optimal rate $\mathfrak{s}_n$:

$$\|\eta_n - \eta\|_{L^\infty([0,T_*];H^{s-2})} + \|\eta_n - \eta\|_{L^2([0,T_*];H^{s-\frac{3}{2}})} \le \mathfrak{s}_n\mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}). \tag{2.1.22}$$

The convergence (2.1.21) holds for initial data in *any subcritical Sobolev spaces* $H^{1+\frac{d}{2}+}(\mathbb{R}^d)$. In particular, this allows for initial interfaces whose curvatures are *unbounded* in all dimensions and *not locally square integrable* in one dimension. The former is because $H(\eta_0) \in H^{-1+\frac{d}{2}+}(\mathbb{R}^d) \not\subset L^\infty(\mathbb{R}^d)$ and latter is due to the fact that in one dimension we have $H(\eta_0) \in H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}) \not\subset L^2_{loc}(\mathbb{R})$. This appears to be the first result on vanishing surface tension that can accommodate curvature singularities of the initial interface. On the other hand, the convergence (2.1.22) has *optimal rate* $\mathfrak{s}_n$ and holds under the additional condition that $s \ge 2$. This is only a condition in one dimension since $s > 1 + \frac{d}{2} \ge 2$ for $d \ge 2$. Note also that $s \ge 2$ is the minimal regularity to ensure that the initial curvature is square integrable, yet it still allows for unbounded curvature. See the technical remark 2.1.3.

The proof of Theorem 2.1.1 exploits the Dirichlet-Neumann reformulation [108, 109] for the Muskat problem in a general setting. See also [3] for the one-fluid case. Part (i) of Theorem 2.1.1 is a uniform local well-posedness with respect to surface tension. The key tool in proving this is paralinearization results for the Dirichlet-Neumann operator taken from [1, 109]. The convergences (2.1.21) and (2.1.22) rely on contraction estimates for the Dirichlet-Neumann operator proved in [109] for a large Sobolev regularity range of the Dirichlet data. Together with [109] and

[108], Theorem 2.1.1 provides a rather complete local regularity theory for (large) subcritical data.

Remark 2.1.2. In [8] and [99], the first results on the zero surface tension limit for Muskat were obtained respectively in 2D and 3D for smooth initial data, i.e. $\Sigma_0 \in H^{s_0}$ for some sufficiently large s_0. The interface is not necessarily a graph but if it is then the convergence estimates therein translate into

$$\|\eta_n - \eta\|_{L^\infty([0,T_*];H^1(\mathbb{T}^d))} \le C\sqrt{\mathfrak{s}_n}, \quad d = 1, 2, \tag{2.1.23}$$

which has the same rate as (2.1.21).

Remark 2.1.3. The condition $s \ge 2$ in (2.1.22) is due to the control of low frequencies in the paralinearization and contraction estimates for the Dirichlet-Neumann operator $G(\eta)f$ (see (2.2.8) for its definition). Precisely, the best currently available results (see subsections 2.2.4 and 2.2.5 below) require $f \in H^\sigma(\mathbb{R}^d)$ with $\sigma \ge \frac{1}{2}$. The proof of the $L^\infty_t H^{s-2}_x$ convergence in (2.1.22) appeals to these results with $\sigma = s - \frac{3}{2}$.

Remark 2.1.4. It was proved in [109] that the Rayleigh-Taylor condition holds unconditionally in the following configurations:

- the one-phase problem without bottom or with Lipschitz bottoms;
- the two-phase problem with constant viscosity ($\mu^+ = \mu^-$).

When the Rayleigh-Taylor condition is violated, analytic solutions to the problem without surface tension exist [123]. The works [124,125] and [27,28] strongly indicate that these solutions are not limits of solutions to the problem with surface tension. We also refer to [72] for the instability of the trivial solution (with surface tension) and to [59] for the stability of bubbles (without surface tension).

Remark 2.1.5. In Theorem 2.1.1, the initial data is fixed for all surface tension coefficients $\mathfrak{s}_n$. In general, one can consider $\eta_n|_{t=0} = \eta_{n,0}$ uniformly bounded in $H^s(\mathbb{R}^d)$ such that the conditions in (2.1.15) hold uniformly in n. Then, for any $s > 1 + \frac{d}{2}$, we have

$$\|\eta_n - \eta\|_{L^\infty([0,T_*];H^{s-1})} + \|\eta_n - \eta\|_{L^2([0,T_*];H^{s-\frac{1}{2}})} \leq \big(\sqrt{\mathfrak{s}_n} + \|\eta_{n,0} - \eta_0\|_{H^{s-1}}\big)\mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}). \tag{2.1.24}$$

On the other hand, if $s > 1 + \frac{d}{2}$ and $s \geq 2$ then

$$\|\eta_n - \eta\|_{L^\infty([0,T_*];H^{s-2})} + \|\eta_n - \eta\|_{L^2([0,T_*];H^{s-\frac{3}{2}})} \leq \big(\mathfrak{s}_n + \|\eta_{n,0} - \eta_0\|_{H^{s-2}}\big)\mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}). \tag{2.1.25}$$

Remark 2.1.6. By interpolating the convergence estimate (2.1.21) and the uniform bounds in (2.1.18), we obtain the vanishing surface tension limit in $H^{s'}$ for all $s' \in [s-1, s)$:

$$\|\eta_n - \eta\|_{L^\infty([0,T_*];H^{s'})} + \|\eta_n - \eta\|_{L^2([0,T_*];H^{s'+\frac{1}{2}})} \leq \mathfrak{s}_n^{\frac{s-s'}{2}}\mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}). \tag{2.1.26}$$

Convergence in the highest regularity $L^\infty_t H^s_x$ is more subtle and can possibly be established using the Bona-Smith type argument [17]. This would imply in particular that η is continuous in time with values in H^s_x, $\eta \in C_t H^s_x$. In the context of vanishing viscosity limit, this question was addressed in [100], while convergence in lower Sobolev spaces (compared to initial data) was proved in [35]. For gravity water waves, the Bona-Smith type argument was applied in [110] to establish the continuity of the flow map in the highest regularity.

Remark 2.1.7. The proof of the local well-posedness in all subcritical Sobolev spaces in [109] uses a parabolic regularization. Theorem 2.1.1 provides an alternative proof via regularization by vanishing surface tension. We stress that the assertions about

η in Theorem 2.1.1 do not make use of the local well-posedness results in [109].

The rest of this paper is organized as follows. In section 2.2, we recall the reformulation of the Muskat problem in terms of the Dirichlet-Neumann operator together with results on the Dirichlet-Neumann operator established in [1, 109]. Section 2.3 is devoted to the proof of uniform-in-$\mathfrak{s}$ *a priori* estimates. In section 2.4, we prove contraction estimates for the operators $J^{\pm}$ which arise in the reformulation of the two-phase problem. The proof of Theorem 2.1.1 is given in section 2.5. Finally, in Appendix 2.6, we recall the symbolic paradifferential calculus and the Gårding inequality for paradifferential operators.

2.2 Preliminaries

2.2.1 Big-O notation

If X and Y are Banach spaces and $T : X \to Y$ is a bounded linear operator, $T \in \mathcal{L}(X, Y)$, with operator norm bounded by a constant multiple of A, we write

$$T = O_{X\to Y}(A).$$

We define the space of operators of order $m \in \mathbb{R}$ in the scale of Sobolev spaces $H^s(\mathbb{R}^d)$:

$$Op^m \equiv Op^m(\mathbb{R}^d) = \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^s(\mathbb{R}^d), H^{s-m}(\mathbb{R}^d)).$$

We shall write $T = O_{Op^m}(A)$ when $T \in Op^m$ and for all $s \in \mathbb{R}$, there exists $C = C(s)$ such that $T = O_{H^s\to H^{s-m}}(CA)$.

2.2.2 Function spaces

In the general setting, to reformulate the dynamics of the Muskat problem solely in terms of the interface, we require some function spaces.

We shall always assume that $\eta \in W^{1,\infty}(\mathbb{R}^d)$ and either $\Gamma^\pm = \emptyset$ or $b^\pm \in \dot{W}^{1,\infty}(\mathbb{R}^d)$ with $\mathrm{dist}(\eta, b^\pm) > 0$. Recall that the fluid domains $\Omega^\pm$ are given in (2.1.2). Define

$$\dot{H}^1(\Omega^\pm) = \{v \in L^1_{loc}(\Omega^\pm) : \nabla_{x,y} v \in L^2(\Omega^\pm)\}/\mathbb{R}, \quad \|v\|_{\dot{H}^1(\Omega^\pm)} = \|\nabla_{x,y} v\|_{L^2(\Omega^\pm)}. \tag{2.2.1}$$

For any $\sigma \in \mathbb{R}$, we define the 'slightly homogeneous' Sobolev space

$$H^{1,\sigma}(\mathbb{R}^d) = \left\{f \in L^1_{loc}(\mathbb{R}^d) : \nabla f \in H^{\sigma-1}(\mathbb{R}^d)\right\}/\mathbb{R}, \quad \|f\|_{H^{1,\sigma}(\mathbb{R}^d)} = \|\nabla f\|_{H^{\sigma-1}(\mathbb{R}^d)}. \tag{2.2.2}$$

When $b^\pm \in \dot{W}^{1,\infty}(\mathbb{R}^d)$, we fix an arbitrary number $a \in (0,1)$ and define the 'screened' fractional Sobolev spaces

$$\widetilde{H}^{\frac{1}{2}}_{\mp a(\eta - b^\pm)}(\mathbb{R}^d) = \left\{f \in L^1_{loc}(\mathbb{R}^d) : \int_{\mathbb{R}^d}\int_{|k| \le \mp a(\eta - b^\pm)} \frac{|f(x+k) - f(x)|^2}{|k|^{d+1}} dk dx < \infty\right\}/\mathbb{R}. \tag{2.2.3}$$

According to Proposition 3.2 [109], the spaces $\widetilde{H}^{\frac{1}{2}}_{\mp a(\eta - b^\pm)}(\mathbb{R}^d)$ are independent of η in $W^{1,\infty}(\mathbb{R}^d)$ satisfying $\mathrm{dist}(\eta, b^\pm) > h > 0$. Thus, we can set

$$\widetilde{H}^{\frac{1}{2}}_{\pm}(\mathbb{R}^d) = \begin{cases} \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) & \text{if } \Gamma^\pm = \emptyset, \\ \widetilde{H}^{\frac{1}{2}}_{\mp a(\eta - b^\pm)}(\mathbb{R}^d) & \text{if } b^\pm \in \dot{W}^{1,\infty}(\mathbb{R}^d). \end{cases} \tag{2.2.4}$$

It was proved in [97, 128] that there exist unique continuous trace operators

$$\mathrm{Tr}_{\Omega^\pm \to \Sigma} : \dot{H}^1(\Omega^\pm) \to \widetilde{H}^{\frac{1}{2}}_{\pm}(\Sigma) \equiv \widetilde{H}^{\frac{1}{2}}_{\pm}(\mathbb{R}^d) \tag{2.2.5}$$

with norm depending only on $\|\eta\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)}$ and $\|b^{\pm}\|_{\dot{W}^{1,\infty}(\mathbb{R}^d)}$.

Finally, for $\sigma > \frac{1}{2}$, we define

$$\widetilde{H}^{\sigma}_{\pm}(\mathbb{R}^d) = \widetilde{H}^{\frac{1}{2}}_{\pm}(\mathbb{R}^d) \cap H^{1,\sigma}(\mathbb{R}^d) \tag{2.2.6}$$

and equip it with the norm $\|\cdot\|_{\widetilde{H}^{\sigma}_{\pm}} = \|\cdot\|_{\widetilde{H}^{\frac{1}{2}}_{\pm}} + \|\cdot\|_{H^{1,\sigma}}$.

2.2.3 Dirichlet-Neumann formulation

Given a function $f \in \mathbb{R}^d$, let ϕ solve the Laplace equation

$$\begin{cases} \Delta_{x,y}\phi = 0 \text{ in } \Omega^-, \\ \phi = f \text{ on } \Sigma, \\ \frac{\partial \phi}{\partial \nu^-} = 0 \text{ on } \Gamma^-, \end{cases} \tag{2.2.7}$$

with the final condition replaced by decay at infinity of ϕ if $\Gamma^- = \emptyset$. Then, we define the (rescaled) Dirichlet-Neumann operator $G^- \equiv G^-(\eta)$ by

$$G^- f = \langle \nabla \eta \rangle \frac{\partial \phi^-}{\partial n}. \tag{2.2.8}$$

The operator $G^+(\eta)$ for the top fluid domain Ω^+ is defined similarly. The solvability of (2.2.7) is given in the next proposition.

Proposition 2.2.1 ([109] Propositions 3.4 and 3.6). Assume that either $\Gamma^- = \emptyset$ or $b^- \in \dot{W}^{1,\infty}(\mathbb{R}^d)$. If $\eta \in W^{1,\infty}(\mathbb{R}^d)$ and $\operatorname{dist}(\eta, \Gamma^-) > h > 0$, then for every $f \in \widetilde{H}^{\frac{1}{2}}_{-}(\mathbb{R}^d)$ there exists a unique variational solution $\phi^- \in \dot{H}^1(\Omega^-)$ to (2.2.7). Precisely,

ϕ satisfies $\mathrm{Tr}_{\Omega^-\to\Sigma}\phi = f$,

$$\int_{\Omega^-} \nabla_{x,y}\phi \cdot \nabla_{x,y}\varphi dxdy = 0 \quad \forall \varphi \in \big\{v \in \dot{H}^1(\Omega^-) :\ \mathrm{Tr}_{\Omega^-\to\Sigma} v = 0\big\}, \tag{2.2.9}$$

together with the estimate

$$\|\nabla_{x,y}\phi\|_{L^2(\Omega^-)} \le \mathcal{F}(\|\nabla\eta\|_{L^\infty})\|f\|_{\widetilde{H}^{\frac{1}{2}}_-(\mathbb{R}^d)} \tag{2.2.10}$$

for some $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on h and $\|\nabla_x b^-\|_{L^\infty(\mathbb{R}^d)}$.

As the functions $b^\pm$ are fixed in $\dot{W}^{1,\infty}(\mathbb{R}^d)$, we shall omit the dependence on $\|\nabla_x b^-\|_{L^\infty(\mathbb{R}^d)}$ in various estimates below.

The Muskat problem can be reformulated in terms of $G^\pm$ as follows.

Proposition 2.2.2 ([108] Proposition 1.1)*.* (i) If (u, p, η) solve the one-phase Muskat problem then $\eta : \mathbb{R}^d \to \mathbb{R}$ obeys the equation

$$\partial_t \eta = -\frac{1}{\mu^-} G^-(\eta)\big(g\rho^-\eta + \mathfrak{s}H(\eta)\big). \tag{2.2.11}$$

Conversely, if η is a solution of (2.2.11) then the one-phase Muskat problem has a solution which admits η as the free surface.

(ii) If $(u^\pm, p^\pm, \eta)$ is a solution of the two-phase Muskat problem then

$$\partial_t \eta = -\frac{1}{\mu^-} G^-(\eta) f^-, \tag{2.2.12}$$

where $f^{\pm} := p^{\pm}|_{\Sigma} + \rho^{\pm} g\eta$ satisfy

$$\begin{cases} f^- - f^+ = \mathfrak{g}\eta + \mathfrak{s}H(\eta), \\ \frac{1}{\mu^-}G^-(\eta)f^- - \frac{1}{\mu^+}G^+(\eta)f^+ = 0. \end{cases} \tag{2.2.13}$$

Conversely, if η is a solution of (2.2.12) where $f^{\pm}$ solve (2.2.13) then the two-phase Muskat problem has a solution which admits η as the free interface.

To make use of the results on the Dirichlet-Neumann operator established in [1, 108, 109], it is convenient to introduce the linear operators

$$J^{\pm} = J^{\pm}(\eta): \quad v \mapsto f^{\pm} \tag{2.2.14}$$

where $f^{\pm} : \mathbb{R}^d \to \mathbb{R}$ are solutions to the system

$$\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ \mu^+ G^-(\eta) & -\mu^- G^+(\eta) \end{pmatrix} \begin{pmatrix} f^- \\ f^+ \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{2.2.15}$$

Introduce

$$L = L(\eta) = \frac{\mu^+ + \mu^-}{\mu^-} G^- J^-. \tag{2.2.16}$$

For the two-phase case, L coincides with $\frac{\mu^+ + \mu^-}{\mu^+} G^+ J^+$ in view of (2.2.13). Thus, writing $L = (\frac{\mu^-}{\mu^+ + \mu^-} + \frac{\mu^+}{\mu^+ + \mu^-})L$ yields the symmetric formula

$$L = G^- J^- + G^+ J^+. \tag{2.2.17}$$

This formula holds for the one phase problem (2.2.11) as well. Indeed, when $\mu^+ = 0$ we have $J^+ = 0$ and $J^- = \mathrm{Id}$, and hence $L = G^-$. In view of (2.2.16), Proposition 2.2.2 implies the following.

Lemma 2.2.3. The Muskat problem (both one-phase and two-phase) is equivalent to

$$\partial_t \eta + \frac{1}{\mu^+ + \mu^-} L(\eta)(\mathfrak{g}\eta + \mathfrak{s}H(\eta)) = 0. \tag{2.2.18}$$

The next proposition gathers results on the existence and boundedness of the operators $J^\pm$, $G^\pm$, and L in Sobolev spaces.

Proposition 2.2.4. ([1] Theorem 3.12, and [109] Propositions 3.8, 4.8, 4.10 and Remark 4.9) Let $\mu^+ \geq 0$ and $\mu^- > 0$. Assume $\mathrm{dist}(\eta, \Gamma^\pm) > h > 0$.

(i) If $\eta \in W^{1,\infty}(\mathbb{R}^d)$ then

$$\|J^\pm\|_{\mathcal{L}(H^{\frac{1}{2}}, \widetilde{H}^{\frac{1}{2}}_{\pm})} + \|G^\pm\|_{\mathcal{L}(\widetilde{H}^{\frac{1}{2}}_{\pm}, H^{-\frac{1}{2}})} + \|L\|_{\mathcal{L}(H^{\frac{1}{2}}, H^{-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}), \tag{2.2.19}$$

where $\mathcal{F}$ is nondecreasing and depends only on $(h, \mu^\pm)$.

(ii) If $\eta \in H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2}$ then for any $\sigma \in [\frac{1}{2}, s]$, we have

$$\|J^\pm\|_{\mathcal{L}(H^\sigma, \widetilde{H}^\sigma_\pm)} + \|G^\pm\|_{\mathcal{L}(\widetilde{H}^\sigma_\pm, H^{\sigma-1})} + \|L\|_{\mathcal{L}(H^\sigma, H^{\sigma-1})} \leq \mathcal{F}(\|\eta\|_{H^s}), \tag{2.2.20}$$

where $\mathcal{F}$ is nondecreasing and depends only on $(h, \mu^\pm, s, \sigma)$.

2.2.4 Paralinearization

Given a function f, define the operators

$$\mathfrak{B}^\pm f \equiv \mathfrak{B}^\pm(\eta) f = \langle \nabla \eta \rangle^{-2} (\nabla \eta \cdot \nabla + G^\pm(\eta)) f, \tag{2.2.21}$$

$$\mathfrak{V}^\pm f \equiv \mathfrak{V}^\pm(\eta) f = \nabla f - \nabla \eta \mathfrak{B}^\pm f. \tag{2.2.22}$$

We note here that $\mathfrak{B}^{\pm} f = \partial_y \phi^{\pm}|_{\Sigma}$ and $\mathfrak{V}^{\pm} f = \nabla_x \phi^{\pm}|_{\Sigma}$, where ϕ solves (2.2.7). Moreover, as a consequence of (2.2.20) and product rules, we have

$$\|\mathfrak{B}^{\pm}(\eta)\|_{\widetilde{H}^{\sigma}_{\pm} \to H^{\sigma-1}} + \|\mathfrak{V}^{\pm}(\eta)\|_{\widetilde{H}^{\sigma}_{\pm} \to H^{\sigma-1}} \leq \mathcal{F}(\|\eta\|_{H^s}) \tag{2.2.23}$$

for all $s > 1 + \frac{d}{2}$ and $\sigma \in [\frac{1}{2}, s]$.

The principal symbol of $G^{-}(\eta)$ is

$$\lambda(x,\xi) = \sqrt{\langle \nabla \eta \rangle^2 |\xi|^2 - (\nabla \eta \cdot \xi)^2}, \tag{2.2.24}$$

while that of $G^{+}(\eta)$ is $-\lambda(x,\xi)$. Note that $\lambda(x,\xi) \geq |\xi|$ with equality when $d = 1$. Next we record results on paralinearization of $G^{\pm}(\eta)$.

Theorem 2.2.5 ([1] Propostion 3.13, [109] Theorem 3.18). Let $s > 1 + \frac{d}{2}$ and let $\delta \in (0, \frac{1}{2}]$ satisfy $\delta < s - 1 - \frac{d}{2}$. Assume that $\eta \in H^s$ and $\mathrm{dist}(\eta, \Gamma^{\pm}) > h > 0$.

(i) For any $\sigma \in [\frac{1}{2}, s - \delta]$, there exists a nondecreasing function $\mathcal{F}$ depending only on (h, s, σ, δ) such that

$$\mp G^{\pm}(\eta) = T_{\lambda} + O_{\widetilde{H}^{\sigma}_{\pm} \to H^{\sigma-1+\delta}}(\mathcal{F}(\|\eta\|_{H^s})), \tag{2.2.25}$$

$$L(\eta) = T_{\lambda} + O_{H^{\sigma} \to H^{\sigma-1+\delta}}(\mathcal{F}(\|\eta\|_{H^s})). \tag{2.2.26}$$

(ii) For any $\sigma \in [\frac{1}{2}, s]$, there exists a nondecreasing function $\mathcal{F}$ depending only on (h, s, σ, δ) such that for all $f \in \tilde{H}^\sigma_\pm(\mathbb{R}^d)$, we have

$$\mp G^\pm(\eta)f = T_\lambda(f - T_{\mathfrak{B}^\pm f}\eta) - T_{\mathfrak{V}^\pm f}\cdot\nabla\eta + O_{\widetilde{H}^\sigma_\pm \to H^{\sigma-\frac{1}{2}}}(\mathcal{F}(\|\eta\|_{H^s})(1+\|\eta\|_{H^{s+\frac{1}{2}-\delta}}))f, \tag{2.2.27}$$

$$L(\eta)f = T_\lambda(f - T_{[\![\mathfrak{B}J]\!]f}\eta) - T_{[\![\mathfrak{V}J]\!]f}\cdot\nabla\eta + O_{H^\sigma \to H^{\sigma-\frac{1}{2}}}(\mathcal{F}(\|\eta\|_{H^s})(1+\|\eta\|_{H^{s+\frac{1}{2}-\delta}}))f. \tag{2.2.28}$$

Proof. (2.2.25) was proved in Proposition 3.13 in [1] for $\sigma \in [\frac{1}{2}, s-\frac{1}{2}]$ but its proof allows for $\sigma \in [\frac{1}{2}, s-\delta]$. On the other hand, (2.2.27) was proved in Theorem 3.17 in [109]. Let us prove (2.2.26) and (2.2.28). We recall from (2.2.17) that $L = G^-J^- + G^+J^+$. Applying (2.2.25) we have

$$\mp G^\pm J^\pm f = T_\lambda J^\pm f + O_{\widetilde{H}^\sigma_\pm \to H^{\sigma-1+\delta}}(\mathcal{F}(\|\eta\|_{H^s}))J^\pm f.$$

By virtue of (2.2.20) we have $\|J^\pm\|_{\mathcal{L}(H^\sigma, \widetilde{H}^\sigma_\pm)} \le \mathcal{F}(\|\eta\|_{H^s})$, and thus

$$\mp G^\pm J^\pm f = T_\lambda J^\pm f + O_{H^\sigma \to H^{\sigma-1+\delta}}(\mathcal{F}(\|\eta\|_{H^s}))f.$$

It follows that

$$\begin{aligned} L(\eta)f &= T_\lambda [\![J]\!] f + O_{H^\sigma \to H^{\sigma-1+\delta}}(\mathcal{F}(\|\eta\|_{H^s}))f \\ &= T_\lambda f + O_{H^\sigma \to H^{\sigma-1+\delta}}(\mathcal{F}(\|\eta\|_{H^s}))f, \end{aligned}$$

where in the second equality we have used the fact that $[\![J]\!] = \mathrm{Id}$. This completes the proof of (2.2.26). Finally, (2.2.28) can be proved similarly upon using the paralinearizaion (2.2.27). $\square$

Finally, the mean curvature operator $H(\cdot)$, defined by (2.1.11), can be paralinearized as follows.

Proposition 2.2.6 ([108] Proposition 3.1). Let $s > 1 + \frac{d}{2}$ and let $\delta \in (0, \frac{1}{2}]$ satisfying $\delta < s - 1 - \frac{d}{2}$. Then there exists a nondecreasing function $\mathcal{F}$ depending only on s such that

$$H(\eta) = T_l\eta + O_{H^{s+\frac{3}{2}} \to H^{s-\frac{1}{2}+\delta}}(\mathcal{F}(\|\eta\|_{H^s}))\eta \tag{2.2.29}$$

where

$$l = \langle\nabla\eta\rangle^{-3}\lambda^2. \tag{2.2.30}$$

In addition, if $\sigma \geq -1$ then

$$\|H(\eta)\|_{H^\sigma} \leq \mathcal{F}(\|\eta\|_{H^s})\|\eta\|_{H^{\sigma+2}}. \tag{2.2.31}$$

2.2.5 Contraction estimates

Let $s > 1 + \frac{d}{2}$ and consider $\eta_j \in H^s(\mathbb{R}^d)$ satisfying $\mathrm{dist}(\eta_j, \Gamma^\pm) > h > 0$, $j = 1, 2$. We have the following contraction estimates for $G^\pm(\eta_1) - G^\pm(\eta_2)$.

Theorem 2.2.7 ([109] Corollary 3.25 and Proposition 3.31). For any $\sigma \in [\frac{1}{2}, s]$, there exists a nondecreasing function $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on (h, s, σ) such that

$$\|G^\pm(\eta_1) - G^\pm(\eta_2)\|_{\widetilde{H}^s_\pm \to H^{\sigma-1}} \leq \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{H^s}\big)\|\eta_1 - \eta_2\|_{H^\sigma} \tag{2.2.32}$$

and

$$\|G^\pm(\eta_1) - G^\pm(\eta_2)\|_{\widetilde{H}^\sigma_\pm \to H^{\sigma-1}} \leq \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{H^s}\big)\|\eta_1 - \eta_2\|_{H^s}. \tag{2.2.33}$$

Theorem 2.2.8 ([109] Theorem 3.24). Let $\delta \in (0, \frac{1}{2}]$ satisfy $\delta < s - 1 - \frac{d}{2}$. Let $\sigma \in [\frac{1}{2} + \delta, s]$. For any $f \in \widetilde{H}^s_\pm$, there exists a nondecreasing function $\mathcal{F}$ depending only on (h, s, σ) such that

$$\mp\big(G^{\pm}(\eta_1)f - G^{\pm}(\eta_2)f\big) = -T_{\lambda_1\mathfrak{B}^{\pm}(\eta_1)f}(\eta_1 - \eta_2) - T_{\mathfrak{V}^{\pm}(\eta_1)f} \cdot \nabla(\eta_1 - \eta_2)$$
$$+ O_{\widetilde{H}^s_{\pm} \to H^{\sigma-1}}\Big(\mathcal{F}\big(\|(\eta_1, \eta_2)\|_{H^s}\big)\|\eta_1 - \eta_2\|_{H^{\sigma-\delta}}\Big)f, \tag{2.2.34}$$

where λ_1 is defined by (2.2.24) with $\eta = \eta_1$.

2.3 Uniform *a priori* estimates

Conclusion (i) in Theorem 2.1.1 concerns the uniform local well-posedness of the Muskat problem with surface tension. Key to that is the following *a priori* estimates that are uniform in the vanishing surface tension limit $\mathfrak{s} \to 0$.

Proposition 2.3.1. Let $s > 1 + \frac{d}{2}$, $\mu^- > 0$, $\mu^+ \geq 0$, $\mathfrak{s} > 0$, and $h > 0$. Suppose

$$\eta \in C([0, T]; H^s(\mathbb{R}^d)) \cap L^2([0, T]; H^{s+\frac{3}{2}}(\mathbb{R}^d)) \tag{2.3.1}$$

is a solution to (2.2.18) with initial data $\eta_0 \in H^s(\mathbb{R}^d)$ such that

$$\inf_{t\in[0,T]} \inf_{x\in\mathbb{R}^d} \mathrm{RT}(\eta(t)) > \mathfrak{a} > 0, \tag{2.3.2}$$

$$\inf_{t\in[0,T]} \mathrm{dist}(\eta(t), \Gamma^{\pm}) > h. \tag{2.3.3}$$

Then, there exists a nondecreasing function $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ depending only on $(h, s, \mu^{\pm})$ such that

$$\|\eta\|_{L^\infty([0,T];H^s)} \leq \|\eta_0\|_{H^s} \exp\Big((\mathfrak{s} + \mathfrak{g})T\mathcal{F}\big(\|\eta\|_{L^\infty([0,T];H^s)}, \mathfrak{a}^{-1}\big)\Big) \tag{2.3.4}$$

and

$$\mathfrak{s}\|\eta\|^2_{L^2([0,T];H^{s+\frac{3}{2}})}+\mathfrak{g}\|\eta\|^2_{L^2([0,T];H^{s+\frac{1}{2}})} \leq \mathcal{F}_1\left(\|\eta_0\|_{H^s}\exp\Big((\mathfrak{s}+\mathfrak{g})T\mathcal{F}\big(\|\eta\|_{L^\infty([0,T];H^s)},\mathfrak{a}^{-1}\big)\Big),\mathfrak{a}^{-1}\right) \tag{2.3.5}$$

where $\mathcal{F}_1(m,n) = m^2\mathcal{F}(m,n)$.

Proof. Set $B = [\![\mathfrak{B}(\eta)J(\eta)]\!]\eta$ and $V = [\![\mathfrak{V}(\eta)J(\eta)]\!]\eta$. We shall write $\mathcal{Q} = \mathcal{Q}(t) = \mathcal{F}(\|\eta(t)\|_{H^s},\mathfrak{a}^{-1})$ when $\mathcal{F}(\cdot,\cdot)$ is nondecreasing and depends only on $(h,s,\mu^\pm)$. Note that $\mathcal{F}$ may change from line to line. From (2.2.18) we have

$$(\mu^+ + \mu^-)\partial_t\eta + \mathfrak{g}L(\eta)\eta + \mathfrak{s}L(\eta)H(\eta) = 0 \tag{2.3.6}$$

for both the one-phase and two-phase problems. Fix $\delta \in \big(0, \min(\frac{1}{2}, s-1-\frac{d}{2})\big)$. By virtue of the paralinearization (2.2.28) (with $\sigma = s$) we have

$$\mathfrak{g}L(\eta)\eta = \mathfrak{g}(T_\lambda(\eta - T_B\eta) - T_V\cdot\nabla\eta) + O_{H^s\to H^{s-\frac{1}{2}}}\big(\mathfrak{g}\mathcal{Q}(1+\|\eta\|_{H^{s+\frac{1}{2}-\delta}})\big)\eta.$$

On the other hand, (2.2.26) (with $\sigma = s-\frac{1}{2}$) together with (2.2.31) gives

$$\mathfrak{s}L(\eta)H(\eta) = \mathfrak{s}T_\lambda H(\eta) + O_{H^{s+\frac{3}{2}}\to H^{s-\frac{3}{2}+\delta}}(\mathfrak{s}\mathcal{F}(\|\eta\|_{H^s}))\eta$$

Combining this with the linearization (2.2.29) for $H(\eta)$ yields

$$\mathfrak{s}L(\eta)H(\eta) = \mathfrak{s}T_\lambda T_l\eta + O_{H^{s+\frac{3}{2}}\to H^{s-\frac{3}{2}+\delta}}(\mathfrak{s}\mathcal{F}(\|\eta\|_{H^s}))\eta,$$

where we have applied Theorem 2.6.2 to have $T_\lambda = O_{Op^1}(\mathcal{F}(\|\eta\|_{H^s}))$.

Then in view of (2.3.6) we obtain

$$(\mu^+ + \mu^-)\partial_t \eta + \mathfrak{s} T_\lambda T_l \eta + \mathfrak{g}(T_\lambda(\eta - T_B\eta) - T_V \cdot \nabla\eta)$$
$$= O_{H^s \to H^{s-\frac{1}{2}}}\Big(\mathfrak{g}\mathcal{F}(\|\eta\|_{H^s})(1 + \|\eta_s\|_{H^{\frac{1}{2}-\delta}})\Big)\eta + O_{H^{s+\frac{3}{2}} \to H^{s-\frac{3}{2}+\delta}}(\mathfrak{s}\mathcal{F}(\|\eta\|_{H^s}))\eta. \tag{2.3.7}$$

Note that $\lambda \in \Gamma^1_\delta$, $l \in \Gamma^2_\delta$ and $(B, V) \in W^{1+\delta,\infty} \subset \Gamma^0_\delta$, with seminorms bounded by $\mathcal{F}(\|\eta\|_{H^s})$. The symbolic calculus in Theorem 2.6.3 then gives

$$T_\lambda T_l \eta = T_{\lambda l}\eta + O_{Op^{3-\delta}}(\mathcal{F}(\|\eta\|_{H^s})),$$
$$T_\lambda(\mathrm{Id} - T_B) = T_{\lambda(1-B)} + O_{Op^{1-\delta}}(\mathcal{F}(\|\eta\|_{H^s})),$$
$$T_V \cdot \nabla = iT_{\xi\cdot V} = i\mathrm{Re}(T_{\xi\cdot V}) + O_{Op^{1-\delta}}(\mathcal{F}(\|\eta\|_{H^s})),$$

where $\mathrm{Re}(T_{\xi\cdot V}) = \frac{1}{2}(T_{\xi\cdot V} + T^*_{\xi\cdot V})$. It then follows from (2.3.7) that

$$(\mu^+ + \mu^-)\partial_t \eta + \mathfrak{s} T_{\lambda l}\eta + \mathfrak{g}\Big(T_{\lambda(1-B)}\eta - i\mathrm{Re}(T_{\xi\cdot V})\eta\Big)$$
$$= O_{H^s \to H^{s-\frac{1}{2}}}(\mathfrak{g}\mathcal{F}(\|\eta\|_{H^s})(1 + \|\eta_s\|_{H^{\frac{1}{2}-\delta}}))\eta + O_{H^{s+\frac{3}{2}} \to H^{s-\frac{3}{2}+\delta}}(\mathfrak{s}\mathcal{F}(\|\eta\|_{H^s}))\eta. \tag{2.3.8}$$

We set $\eta_s = \langle D\rangle^s \eta$. Appealing to Theorem 2.6.3 again we have

$$[\langle D\rangle^s, T_{\lambda l}] = O_{Op^{s+3-\delta}}(\mathcal{F}(\|\eta\|_{H^s})),$$
$$[\langle D\rangle^s, T_{\lambda(1-B)}] = O_{Op^{s+1-\delta}}(\mathcal{F}(\|\eta\|_{H^s}, \mathfrak{a}^{-1})),$$
$$[\langle D\rangle^s, \mathrm{Re}(T_{\xi\cdot V})] = O_{Op^{s+1-\delta}}(\mathcal{F}(\|\eta\|_{H^s})),$$

where $[A, B] = AB - BA$ and in the second line we have used the lower bound (2.3.2) for $(1 - B)$ together with the fact that $\lambda \geq |\xi|$. Note that we have adopted the convention that $\mathcal{F}(\|\eta\|_{H^s}) \equiv \mathcal{F}(\|\eta\|_{H^s}, 0)$. This implies

$$\begin{aligned}(\mu^+ + \mu^-)\partial_t \eta_s + \mathfrak{s} T_{\lambda l}\eta_s + \mathfrak{g}\Big(T_{\lambda(1-B)}\eta_s - \mathrm{Re}(T_{\xi\cdot V})\eta_s\Big) \\ = O_{L^2\to H^{-\frac{1}{2}}}(\mathfrak{g}\mathcal{Q}(1+\|\eta_s\|_{H^{\frac{1}{2}-\delta}}))\eta_s + O_{Op^{1-\delta}}(\mathfrak{g}\mathcal{Q})\eta_s + O_{H^{\frac{3}{2}}\to H^{-\frac{3}{2}+\delta}}(\mathfrak{s}\mathcal{Q})\eta_s.\end{aligned} \tag{2.3.9}$$

Since $i\mathrm{Re}(T_{\xi\cdot V})$ is skew-adjoint, by testing (2.3.9) against η_s, we obtain

$$\begin{aligned}&\frac{(\mu^+ + \mu^-)}{2}\frac{d}{dt}\|\eta_s\|_{L^2}^2 + \mathfrak{s}(T_{\lambda l}\eta_s, \eta_s)_{L^2} + \mathfrak{g}(T_{\lambda(1-B)}\eta_s, \eta_s)_{L^2} \\ &\quad\le \mathcal{Q}\Big\{\mathfrak{g}\Big[(1+\|\eta_s\|_{H^{\frac{1}{2}-\delta}})\|\eta_s\|_{L^2}\|\eta_s\|_{H^{\frac{1}{2}}} + \|\eta_s\|_{H^{\frac{1}{2}-\delta}}\|\eta_s\|_{H^{\frac{1}{2}}}\Big] + \mathfrak{s}\|\eta_s\|_{H^{\frac{3}{2}-\delta}}\|\eta_s\|_{H^{\frac{3}{2}}}\Big\},\end{aligned} \tag{2.3.10}$$

where $(\cdot,\cdot)_{L^2}$ denotes the L^2 pairing. The term involving $1+\|\eta_s\|_{H^{\frac{1}{2}-\delta}}$ is treated as follows:

$$\begin{aligned}(1+\|\eta_s\|_{H^{\frac{1}{2}-\delta}})\|\eta_s\|_{L^2}\|\eta_s\|_{H^{\frac{1}{2}}} &\le \|\eta_s\|_{L^2}\|\eta_s\|_{H^{\frac{1}{2}}} + \|\eta_s\|_{L^2}\|\eta_s\|_{H^{\frac{1}{2}-\delta}}\|\eta_s\|_{H^{\frac{1}{2}}} \\ &\le \mathcal{Q}\|\eta_s\|_{H^{\frac{1}{2}-\delta}}\|\eta_s\|_{H^{\frac{1}{2}}}.\end{aligned} \tag{2.3.11}$$

In view of (2.3.2) and the fact that $\lambda(x,\xi) \ge |\xi|$, we have the lower bounds

$$\lambda(1-B) \ge \mathfrak{a}|\xi|, \quad l\lambda = \langle\nabla\eta\rangle^{-3}\lambda^3 \ge \frac{1}{\langle\|\eta\|_{W^{1,\infty}}\rangle^3}|\xi|^3.$$

Moreover, $\lambda(1-B) \in \Gamma^1_\delta$ and $l\lambda \in \Gamma^3_\delta$ with seminorms bounded by $\mathcal{F}(\|\eta\|_{H^s})$. Then applying the Gårding's inequality (2.6.9) gives

$$\|\Psi(D)\eta_s\|_{H^{\frac{3}{2}}}^2 \le \mathcal{Q}\Big(\|\eta_s\|_{H^{\frac{3}{2}}}\|\eta_s\|_{H^{\frac{3}{2}-\delta}} + (T_{l\lambda}\eta_s, \eta_s)_{L^2}\Big), \tag{2.3.12}$$

$$\|\Psi(D)\eta_s\|_{H^{\frac{1}{2}}}^2 \le \mathcal{Q}\Big(\|\eta_s\|_{H^{\frac{1}{2}}}\|\eta_s\|_{H^{\frac{1}{2}-\delta}}^2 + (T_{\lambda(1-B)}\eta_s, \eta_s)_{L^2}\Big), \tag{2.3.13}$$

where $\Psi(D)$ denotes the Fourier multiplier with symbol Ψ defined by (2.6.3). In addition, we have

$$\|u\|_{H^r} \leq C(\|\Psi(D)u\|_{H^r} + \|u\|_{L^2}) \quad \forall r \in \mathbb{R}.$$

Thus, (2.3.10) amounts to

$$\begin{aligned}\frac{(\mu^+ + \mu^-)}{2}\frac{d}{dt}\|\eta_s\|_{L^2}^2 + \frac{1}{\mathcal{Q}}\Big(\mathfrak{s}\|\eta_s\|_{H^{\frac{3}{2}}}^2 + \mathfrak{g}\|\eta_s\|_{H^{\frac{1}{2}}}^2\Big) \\ \leq \mathcal{Q}(\mathfrak{g}\|\eta_s\|_{H^{\frac{1}{2}-\delta}}\|\eta_s\|_{H^{\frac{1}{2}}} + \mathfrak{s}\|\eta_s\|_{H^{\frac{3}{2}-\delta}}\|\eta_s\|_{H^{\frac{3}{2}}}).\end{aligned} \tag{2.3.14}$$

We use Young's inequality and interpolation as follows:

$$\|\eta_s\|_{H^{\frac{1}{2}-\delta}}\|\eta_s\|_{H^{\frac{1}{2}}} \leq \|\eta_s\|_{L^2}^{2\delta}\|\eta_s\|_{H^{\frac{1}{2}}}^{2(1-\delta)} \leq (10\mathcal{Q})^{\frac{2(1-\delta)}{\delta}}\|\eta_s\|_{L^2}^2 + \frac{1}{100\mathcal{Q}^2}\|\eta_s\|_{H^{\frac{1}{2}}}^2 \tag{2.3.15}$$

and similarly,

$$\|\eta_s\|_{H^{\frac{3}{2}-\delta}}\|\eta_s\|_{H^{\frac{3}{2}}} \leq \|\eta_s\|_{L^2}^{2\frac{\delta}{3}}\|\eta_s\|_{H^{\frac{3}{2}}}^{2(1-\frac{\delta}{3})} \leq (10\mathcal{Q})^{\frac{2(3-\delta)}{\delta}}\|\eta_s\|_{L^2}^2 + \frac{1}{100\mathcal{Q}^2}\|\eta_s\|_{H^{\frac{3}{2}}}^2. \tag{2.3.16}$$

Applying these inequalities to (2.3.14), and then subtracting terms involving $\|\eta_s\|_{H^{\frac{1}{2}}}$, we obtain for a larger $\mathcal{Q}$ if needed that

$$\frac{\mu^+ + \mu^-}{2}\frac{d}{dt}\|\eta_s\|_{L^2}^2 + \frac{1}{\mathcal{Q}}(\mathfrak{s}\|\eta_s\|_{H^{\frac{3}{2}}}^2 + \mathfrak{g}\|\eta_s\|_{H^{\frac{1}{2}}}^2) \leq (\mathfrak{g}+\mathfrak{s})\mathcal{Q}\|\eta_s\|_{L^2}^2. \tag{2.3.17}$$

A Grönwall's argument then leads to

$$\|\eta\|_{L^\infty([0,T];H^s)}^2 + \frac{1}{\mathcal{Q}_T}\Big(\mathfrak{s}\|\eta\|_{L^2([0,T];H^{s+\frac{3}{2}})}^2 + \mathfrak{g}\|\eta\|_{L^2([0,T];H^{s+\frac{1}{2}})}^2\Big) \leq \|\eta_0\|_{H^s}^2 \exp\Big((\mathfrak{s}+\mathfrak{g})T\mathcal{Q}_T\Big),$$

where $\mathcal{Q}_T = \mathcal{F}(\|\eta\|_{L^\infty([0,T];H^s)}, \mathfrak{a}^{-1})$ with $\mathcal{F}$ depending only on $(h, s, \mu^\pm)$. In particular, we have the H^s estimate (2.3.4). As for the dissipation estimate, we have

$$\mathfrak{s}\|\eta\|^2_{L^2([0,T];H^{s+\frac{3}{2}})} + \mathfrak{g}\|\eta\|^2_{L^2([0,T];H^{s+\frac{1}{2}})} \le \|\eta_0\|^2_{H^s} \exp\Big((\mathfrak{s}+\mathfrak{g})T\mathcal{F}\big(\|\eta\|_{L^\infty([0,T];H^s)}, \mathfrak{a}^{-1}\big)\Big)\mathcal{Q}_T.$$

On the other hand, plugging (2.3.4) into $\mathcal{Q}_T$ gives

$$\mathcal{Q}_T = \mathcal{F}(\|\eta\|_{L^\infty([0,T];H^s)}, \mathfrak{a}^{-1}) \le \mathcal{F}\Big(\|\eta_0\|_{H^s} \exp\Big((\mathfrak{s}+\mathfrak{g})T\mathcal{F}\big(\|\eta\|_{L^\infty([0,T];H^s)}, \mathfrak{a}^{-1}\big)\Big), \mathfrak{a}^{-1}\Big)$$

Therefore, upon setting $\mathcal{F}_1(m,n) = m^2\mathcal{F}(m,n)$ we obtain

$$\mathfrak{s}\|\eta\|^2_{L^2([0,T];H^{s+\frac{3}{2}})} + \mathfrak{g}\|\eta\|^2_{L^2([0,T];H^{s+\frac{1}{2}})} \le \mathcal{F}_1\Big(\|\eta_0\|_{H^s} \exp\Big((\mathfrak{s}+\mathfrak{g})T\mathcal{F}\big(\|\eta\|_{L^\infty([0,T];H^s)}, \mathfrak{a}^{-1}\big)\Big), \mathfrak{a}^{-1}\Big)$$

which finishes the proof of (2.3.5). □

2.4 Contraction estimates for $J^\pm$

Our goal in this section is to prove contraction estimates for $J^\pm(\eta)$ at two different surfaces η_1 and η_2. This is only a question for the two-phase problem since for the one-phase problem we have $J^- = \mathrm{Id}$ and $J^+ \equiv 0$. Given an object X depending on η, we shall denote $X_j = X|_{\eta=\eta_j}$ and the difference

$$X_\delta = X_1 - X_2.$$

Proposition 2.4.1. Let $s > 1 + \frac{d}{2}$ and consider $\eta_j \in H^s(\mathbb{R}^d)$ satisfying $\mathrm{dist}(\eta_j, \Gamma^\pm) > h > 0$, $j = 1, 2$. For any $\sigma \in [\frac{1}{2}, s]$, there exists $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on

$(h, s, \sigma, \mu^{\pm})$ such that

$$\|J_\delta^{\pm}\|_{H^s \to \widetilde{H}^{\sigma}_{\pm}} \leq \mathcal{F}(N_s)\|\eta_\delta\|_{H^\sigma}, \tag{2.4.1}$$

$$\|J_\delta^{\pm}\|_{H^\sigma \to \widetilde{H}^{\sigma}_{\pm}} \leq \mathcal{F}(N_s)\|\eta_\delta\|_{H^s}, \tag{2.4.2}$$

where we denoted

$$N_s = \|(\eta_1, \eta_2)\|_{H^s}. \tag{2.4.3}$$

We shall prove Proposition 2.4.1 for the most general case of two fluids and with bottoms, i.e. $\mu^+ > 0$ and $\Gamma^{\pm} \neq \emptyset$. Adaption to the other cases is straightforward.

2.4.1 Flattening the domain

There exist $\eta_*^{\pm} \in C_b^{s+100}(\mathbb{R}^d)$ such that

$$b^-(x) + \frac{h}{2} \leq \eta_*^-(x) \leq \eta_j(x) - \frac{h}{2}, \quad \eta_j(x) \leq \eta_*^+(x) - \frac{h}{2} \leq b^+(x) - h \quad \forall x \in \mathbb{R}^d \tag{2.4.4}$$

and for some $C = C(h, s, d)$,

$$\|\eta_*^{\pm}\|_{C_b^{s+100}(\mathbb{R}^d)} \leq C(1 + \|\eta_1\|_{L^\infty} + \|\eta_2\|_{L^\infty}). \tag{2.4.5}$$

For $j = 1, 2$ we set $\Omega_* = \Omega_{j,*}^+ \cup \Omega_{j,*}^-$ where

$$\Omega_{*,j}^{\pm} = \{(x, y) : x \in \mathbb{R}^d, \pm\eta(x) \leq \pm y \leq \pm\eta_*^{\pm}(x)\}. \tag{2.4.6}$$

Note that $\Omega_* = \{(x,y) : x \in \mathbb{R}^d, \eta_*^-(x) \le y \le \eta_*^+(x)\}$ is independent of $j \in \{1,2\}$. For small $\tau > 0$ to be chosen, define $\rho_j(x,z) : \mathbb{R}^d \times [-1,1]$ by

$$\varrho_j(x,z) = (1-z^2)e^{-\tau|z|\langle D_x\rangle}\eta_j(x) - \frac{1}{2}z(1-z)\eta_*^-(x) + \frac{1}{2}z(1+z)\eta_*^+(x). \tag{2.4.7}$$

Lemma 2.4.2. There exists $K > 0$ depending only on (s,d) such that if $\tau K\|\eta_j\|_{H^s} \le \frac{h}{12}$ then

$$\partial_z\varrho_j(x,z) \ge \frac{h}{12} \quad a.e.\ (x,z) \in \mathbb{R}^d \times (-1,1),\ j = 1,2. \tag{2.4.8}$$

For $j = 1,2$, the mapping

$$\Phi_j : \mathbb{R}^d \times [-1,1] \ni (x,z) \mapsto (x, \varrho_j(x,z)) \in \Omega_*$$

is a Lipschitz diffeomorphism and respectively maps $\mathbb{R}^d \times [0,1]$ and $\mathbb{R}^d \times [-1,0]$ onto $\Omega_{*,j}^+$ and $\Omega_{*,j}^-$. Moreover, there exists $C = C(h,s,d)$ such that

$$\|\nabla\Phi_j\|_{L^\infty(\mathbb{R}^d\times(-1,1))} \le C(1+N_s), \tag{2.4.9}$$

$$\|\nabla(\Phi_j^{-1})\|_{L^\infty(\Omega_*)} \le C(1+N_s). \tag{2.4.10}$$

Proof. We first note that $\Phi_j(x,0) = \Sigma_j = \{(x,\eta_j(x)) : x \in \mathbb{R}^d\}$, $\Phi_j(x,1) = \{(x,\eta_*^+(x)) : x \in \mathbb{R}^d\}$ and $\Phi_j(x,-1) = \{(x,\eta_*^-(x)) : x \in \mathbb{R}^d\}$. Thus, in order to prove that Φ_j is one-to-one and onto, it suffices to prove that $\partial_z\varrho_j(x,z) \ge c > 0$ for a.e. $(x,z) \in \mathbb{R}^d \times (-1,1)$. For $z \in (-1,1)\setminus\{0\}$ we have

$$\begin{aligned}\partial_z\varrho_j(x,z) &= \frac{1}{2}(1-2z)(\eta_j(x) - \eta_*^-(x)) + \frac{1}{2}(1+2z)(\eta_*^+(x) - \eta_j(x)) \\ &\quad - 2z(e^{-\tau|z|\langle D_x\rangle} - 1)\eta_j(x) - \mathrm{sign}(z)\tau(1-z^2)e^{-\tau|z|\langle D_x\rangle}\langle D_x\rangle\eta_j(x).\end{aligned}$$

For $z \in [\frac{1}{3},1]$, $1-2z \in [-1,\frac{1}{3}]$ and $1+2z \in [\frac{5}{3},3]$. In addition, by (2.4.4) we have

$\mp(\eta_- \eta_*^\pm) \geq h/2$. Consequently,

$$\frac{1}{2}(1-2z)(\eta_j(x)-\eta_*^-(x))+\frac{1}{2}(1+2z)(\eta_*^+(x)-\eta_j(x)) \geq \frac{h}{6}. \tag{2.4.11}$$

Similarly we obtain (2.4.11) for $z \in (-1, -\frac{1}{3})$ and $z \in [-\frac{1}{3}, \frac{1}{3}] \setminus \{0\}$. Next writing

$$(e^{-\tau|z|\langle D_x\rangle}-1)\eta_j(x) = -\tau\int_0^{|z|} e^{-\tau z'\langle D_x\rangle}\langle D_x\rangle\eta_j(x)dz'$$

we obtain that

$$\partial_z\varrho_j(x,z) \geq \frac{h}{6} - \tau K\|\eta_j\|_{H^s} \quad \forall (x,z) \in \mathbb{R}^d \times (-1,1)$$

for some constant $K = K(s,d)$. Note that the condition $s > 1 + \frac{d}{2}$ has been used. Choosing $\tau > 0$ such that $\tau K\|\eta_j\|_{H^s} \leq h/12$ gives $\partial_z\varrho_j(x,z) \geq h/12$ for a.e. $(x,z) \in \mathbb{R}^d \times (-1,1)$.

Since

$$\begin{aligned}\partial_z\varrho_j(x,z) = &-2ze^{-\tau|z|\langle D_x\rangle}\eta_j(x) - \mathrm{sign}(z)\tau(1-z^2)e^{-\tau|z|\langle D_x\rangle}\langle D_x\rangle\eta_j(x)\\ &-\frac{1}{2}(1-2z)\eta_*^-(x)+\frac{1}{2}(1+2z)\eta_*^+(x),\end{aligned}$$

there exists $K' = K'(s,d)$ such that

$$\|\partial_z\varrho_j(x,z)\|_{W^{1,\infty}(\mathbb{R}^d\times(-1,1))} \leq K'\|\eta_j\|_{H^{s-1}} + K'\tau\|\eta_j\|_{H^s} + K'(1+N_{s-1}),$$

where we have used (2.4.5). Then in view of the fact that $\tau\|\eta_j\|_{H^s} \leq hK^{-1}$, we obtain

$$\|\varrho_j\|_{W^{1,\infty}(\mathbb{R}^d\times(-1,1))} \leq C(1+N_s), \quad C = C(h,s,d), \tag{2.4.12}$$

whence (2.4.9) follows. On the other hand, we have $\Phi_j^{-1}(x,y) = (x, \kappa_j(x,y))$ where

$$y = \varrho_j(x,z) \iff z = \kappa_j(x,y) \quad a.e.\ (x,z) \in \mathbb{R}^d \times (-1,1). \tag{2.4.13}$$

Then the relation $\kappa_j(x, \varrho_j(x,z)) = z$ yields

$$\partial_y \kappa_j(x, \varrho_j(x,z)) = \frac{1}{\partial_z \varrho_j(x,z))}, \quad \partial_x \kappa_j(x, \varrho_j(x,z)) = -\frac{\partial_x \varrho_j(x,z)}{\partial_z \varrho_j(x,z))}. \tag{2.4.14}$$

Thus, in view of (2.4.8) and (2.4.12), we obtain (2.4.10). □

Lemma 2.4.3. Set

$$\Upsilon(x,y) = \begin{cases} \Phi_1 \circ \Phi_2^{-1}, & (x,y) \in \Omega_*, \\ (x,y), & (x,y) \in \Omega \setminus \Omega_* \end{cases} \tag{2.4.15}$$

and

$$M = \frac{\nabla \Upsilon \nabla \Upsilon^t}{|\det \nabla \Upsilon|}. \tag{2.4.16}$$

Then, Υ is a Lipschitz diffeomorphism on Ω and

$$\frac{1}{C(1+N_s)} \le \det \nabla \Upsilon(x,y) \le C(1+N_s) \quad a.e.\ (x,y) \in \Omega, \tag{2.4.17}$$

$$\|\nabla \Upsilon\|_{L^\infty(\Omega)} + \|\nabla(\Upsilon^{-1})\|_{L^\infty(\Omega)} \le \mathcal{F}(N_s). \tag{2.4.18}$$

Moreover, M satisfies

$$\|M - \mathrm{Id}\|_{L^\infty(\Omega)} \le \mathcal{F}(N_s)\|\eta_\delta\|_{H^s}, \tag{2.4.19}$$

$$\|M - \mathrm{Id}\|_{L^2(\Omega)} \le \mathcal{F}(N_s)\|\eta_\delta\|_{H^{\frac{1}{2}}}. \tag{2.4.20}$$

Proof. According to Lemma 2.4.2, Υ is a Lipschitz diffeomorphism on Ω_*. For $(x,y) \in \Omega \setminus \Omega_*$ we have $\Upsilon = \mathrm{Id}$, and hence Υ is a Lipschitz diffeomorphism on

Ω and $M - \mathrm{Id} = 0$. It thus suffices to consider $(x, y) \in \Omega_*$. On Ω_* we have $\Upsilon(x, y) = (x, \varrho_1(x, \kappa_2(x, y)))$, and so

$$\nabla\Upsilon(x, y) = \begin{pmatrix} 1 & 0 \\ a(x, y) & b(x, y) \end{pmatrix}$$

where

$$a(x, y) = \partial_x\varrho_1(x, \kappa_2(x, y)) + \partial_z\varrho_1(x, \kappa_2(x, y))\partial_x\kappa_2(x, y), \quad b(x, y) = \partial_z\varrho_1(x, \kappa_2(x, y))\partial_y\kappa_2(x, y).$$

Using (2.4.14) (with $j = 2$) gives

$$\begin{aligned} a(x, y) &= \frac{\partial_x\varrho_1(x, \kappa_2(x, y))\partial_z\varrho_2(x, \kappa_2(x, y)) - \partial_x\varrho_2(x, \kappa_2(x, y))\partial_z\varrho_1(x, \kappa_2(x, y))}{\partial_z\varrho_2(x, \kappa_2(x, y))}, \\ b(x, y) &= \frac{\partial_z\varrho_1(x, \kappa_2(x, y))}{\partial_z\varrho_2(x, \kappa_2(x, y))}. \end{aligned}$$

In view of (2.4.8) and (2.4.12) we obtain

$$\frac{1}{C(1 + N_s)} \leq \det\nabla\Upsilon(x, y) = b(x, y) \leq C(1 + N_s) \quad a.e.\ (x, y) \in \Omega_*. \tag{2.4.21}$$

Next we compute

$$M - \mathrm{Id} = \frac{1}{b}\begin{pmatrix} 1 - b & a \\ a & a^2 + b(b - 1) \end{pmatrix}.$$

Using the above formulas for a and b together with (2.4.7) and (2.4.8) we deduce that

$$\begin{aligned} &\|(a, b - 1)\|_{L^\infty(\Omega_*)} \leq \mathcal{F}(N_s)\|\eta_\delta\|_{H^s}, \\ &\|(a, b - 1)\|_{L^2(\Omega_*)} \leq \mathcal{F}(N_s)\|\eta_\delta\|_{H^{\frac{1}{2}}}. \end{aligned}$$

This combined with (2.4.21) leads to (2.4.18), (2.4.19) and (2.4.20). $\square$

2.4.2 Proof of Proposition 2.4.1

The proof proceeds in three steps.

Step 1. We first recall from (2.2.15) that $J^{\pm}v = f^{\pm}$ where $f^{\pm}$ solve

$$\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ \mu^+ G^-(\eta) & -\mu^- G^+(\eta) \end{pmatrix} \begin{pmatrix} f^- \\ f^+ \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{2.4.22}$$

From the definition of the Dirichlet-Neumann operator we see that $f^{\pm} = q^{\pm}|_{\Sigma}$ where $q^{\pm}$ solve the two-phase elliptic problem

$$\begin{cases} \Delta q^{\pm} = 0 \quad \text{in } \Omega^{\pm}, \\ q^- - q^+ = v \quad \text{on } \Sigma, \\ \frac{\partial_n q^-}{\mu^-} - \frac{\partial_n q^+}{\mu^+} = 0 \quad \text{on } \Sigma, \\ \partial_{\nu^{\pm}} q^{\pm} = 0 \quad \text{on } \Gamma^{\pm}. \end{cases} \tag{2.4.23}$$

To remove the jump of q at Σ we take a function $\theta : \Omega \to \mathbb{R}$ satisfying

$$\theta(x, \eta(x)) = -\frac{1}{2}v(x), \quad \theta \equiv 0 \quad \text{near } \Gamma^{\pm}, \tag{2.4.24}$$

$$\|\theta\|_{\dot{H}^1(\Omega)} \le C(1 + \|\eta\|_{W^{1,\infty}})\|v\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}, \quad C = C(d). \tag{2.4.25}$$

Then, the solution of (2.4.23) can be taken to be $q^{\pm} := (r \pm \theta)|_{\Omega^{\pm}}$ where $r \in \dot{H}^1(\Omega)$ solves

$$\begin{cases} -\Delta r = \pm\Delta\theta \quad \text{in } \Omega^{\pm}, \\ \frac{\partial_n r}{\mu^+} - \frac{\partial_n r}{\mu^-} = -\partial_n \theta\left(\frac{1}{\mu^+} + \frac{1}{\mu^-}\right) \quad \text{on } \Sigma, \\ \partial_{\nu^{\pm}} r = 0 \quad \text{on } \Gamma^{\pm}. \end{cases} \tag{2.4.26}$$

A function θ satisfying (2.4.24) and (2.4.30) can be constructed as follows. Let $\varsigma(z) : \mathbb{R} \to \mathbb{R}^+$ be a cutoff function that is identically 1 for $|z| \leq \frac{1}{2}$ and vanishes for $|z| \geq 1$. Set

$$\underline{\theta}(x, z) = -\frac{1}{2}\varsigma(z)e^{-|z|\langle D_x\rangle}v(x), \quad \theta(x, y) = \theta\big(x, \frac{y-\eta(x)}{h}\big). \tag{2.4.27}$$

Then, $\theta(x, y) = 0$ for $|y - \eta(x)| \geq h$, and hence $\theta_1 \equiv 0$ near $\Gamma^\pm$ in view of the condition $\mathrm{dist}(\eta, \Gamma^\pm) > h$. Moreover, (2.4.30) is satisfied.

Integration by parts leads to the following variational form of (2.4.26):

$$\int_\Omega \Big(\frac{1_{\Omega^-}}{\mu^-} + \frac{1_{\Omega^+}}{\mu^+}\Big)\nabla r \cdot \nabla\phi \; dxdy = \int_\Omega \Big(\frac{1_{\Omega^-}}{\mu^-} - \frac{1_{\Omega^+}}{\mu^+}\Big)\nabla\theta \cdot \nabla\phi \; dxdy, \quad \forall\phi \in \dot{H}^1(\Omega). \tag{2.4.28}$$

For example, for $\varsigma(z) : \mathbb{R} \to \mathbb{R}^+$ a cutoff function that is identically 1 for $|z| \leq \frac{1}{2}$ and vanishes for $|z| \geq 1$, we set

$$\underline{\theta}(x, z) = -\frac{1}{2}\varsigma(z)e^{-|z|\langle D_x\rangle}v(x), \quad \theta(x, y) = \theta\big(x, \frac{y-\eta(x)}{h}\big). \tag{2.4.29}$$

Then, $\theta(x, y) = 0$ for $|y - \eta(x)| \geq h$, and hence $\theta_1 \equiv 0$ near $\Gamma^\pm$ in view of the condition $\mathrm{dist}(\eta, \Gamma^\pm) > h$. Moreover, we have

$$\|\theta\|_{\dot{H}^1(\Omega)} \leq C(1 + \|\eta\|_{W^{1,\infty}})\|v\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}, \quad C = C(d). \tag{2.4.30}$$

By virtue of the Lax-Milgram theorem, there exists a unique solution $r \in \dot{H}^1(\Omega)$ to (2.4.28) which obeys the bound

$$\|r\|_{\dot{H}^1(\Omega)} \leq C(\mu^\pm)\|\theta\|_{\dot{H}^1(\Omega)} \leq C(1 + \|\eta\|_{W^{1,\infty}})\|v\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}. \tag{2.4.31}$$

Consequently,

$$J^{\pm}v = f^{\pm} = \mathrm{Tr}_{\Omega^{\pm}\to\Sigma}(r \pm \theta), \tag{2.4.32}$$

and hence by the trace operation (2.2.5),

$$\|J^{\pm}v\|_{\widetilde{H}^{\frac{1}{2}}_{\pm}(\mathbb{R}^d)} \le \mathcal{F}(N_s)\|v\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}. \tag{2.4.33}$$

We note that θ defined by (2.4.27) depends on η, and so does r.

Step 2. In this step we prove contraction estimates for $J^{\pm}_{\delta} = J^{\pm}_1 - J^{\pm}_2$ in $\widetilde{H}^{\frac{1}{2}}_{\pm}$. Recall that Υ defined by (2.4.15) is a Lipschitz diffeomorphism on Ω. Let θ_1 be defined as in (2.4.27) with $\eta = \eta_1$, and let $\theta_2 = \theta_1 \circ \Upsilon$. Let us check that θ_2 obeys (2.4.24) and (2.4.30) for $\eta = \eta_2$. Indeed, using the fact that $\Upsilon : \Sigma_2 \to \Sigma_1$ we have

$$\theta_2(x, \eta_2(x)) = \theta_1(x, \Upsilon(x, \eta_2(x))) = \theta_1(x, \eta_1(x)) = -\frac{1}{2}v(x).$$

On the other hand, since $\Upsilon \equiv \mathrm{Id}$ near $\Gamma^{\pm}$ and $\theta_1 \equiv 0$ near $\Gamma^{\pm}$, we deduce that $\theta_1 \equiv 0$ near $\Gamma^{\pm}$. Finally, the bound (2.4.30) for θ_2 follows from (2.4.30) for θ_1 and the Lipschitz bound (2.4.18) for Υ.

According to Step 1, we have

$$J^{\pm}_j v = \mathrm{Tr}_{\Omega^{\pm}_j\to\Sigma_j}(r_j \pm \theta_j)$$

where $r_j \in \dot{H}^1(\Omega)$ satisfies

$$\int_{\Omega} \Big(\frac{1_{\Omega^-}}{\mu^-} + \frac{1_{\Omega^+}}{\mu^+}\Big)\nabla r_j \cdot \nabla\phi \, dxdy = \int_{\Omega} \Big(\frac{1_{\Omega^-}}{\mu^-} - \frac{1_{\Omega^+}}{\mu^+}\Big)\nabla\theta_j \cdot \nabla\phi \, dxdy \quad \forall \phi \in \dot{H}^1(\Omega). \tag{2.4.34}$$

Set $\widetilde{r}_2 = r_2 \circ \Upsilon^{-1}$ and recall that $\theta_1 = \theta_2 \circ \Upsilon^{-1}$. Combining (2.4.31) and (2.4.17) gives

$$\|\widetilde{r}_2\|_{\dot{H}^1(\Omega)} \le \mathcal{F}(N_s).$$

Since map $\phi \mapsto \phi \circ \Upsilon^{-1}$ is an isomorphism on $\dot{H}^1(\Omega)$, with $M = \nabla\Upsilon\nabla\Upsilon^t/|\det \nabla\Upsilon|$ we have for all $\phi \in \dot{H}^1(\Omega)$ that

$$\int_\Omega \Big(\frac{1_{\Omega_1^-}}{\mu^-} + \frac{1_{\Omega_1^+}}{\mu^+}\Big)\nabla\tilde{r}_2 M\nabla\phi^t \; dxdy = \int_\Omega \Big(\frac{1_{\Omega_1^-}}{\mu^-} - \frac{1_{\Omega_1^+}}{\mu^+}\Big)\nabla\theta_1 M\nabla\phi^t \; dxdy, \tag{2.4.35}$$

where gradients of scalar functions are understood as row vectors, and the rows of the Jacobian matrix $\nabla\Upsilon$ are the gradients of each component of Υ. Taking the difference between (2.4.35) with $j = 2$ and (2.4.28) with $j = 1$, we obtain

$$\begin{aligned}\int_\Omega \Big(\frac{1_{\Omega_1^-}}{\mu^-} + \frac{1_{\Omega_1^+}}{\mu^+}\Big)\nabla(\widetilde{r}_2 - r_1)\nabla\phi^t \; dxdy = &- \int_\Omega \Big(\frac{1_{\Omega_1^-}}{\mu^-} + \frac{1_{\Omega_1^+}}{\mu^+}\Big)\nabla\widetilde{r}_2(M - \mathrm{Id})\nabla\phi^t \; dxdy \\ &+ \int_\Omega \Big(\frac{1_{\Omega_1^-}}{\mu^-} - \frac{1_{\Omega_1^+}}{\mu^+}\Big)\nabla\theta_1(M - \mathrm{Id})\nabla\phi^t \; dxdy\end{aligned} \tag{2.4.36}$$

for all $\phi \in \dot{H}^1(\Omega)$. Setting $\phi = \widetilde{r}_2 - r_1$ and using the estimate (2.4.19) for $M - \mathrm{Id}$ in $L^\infty(\Omega)$ we obtain

$$\begin{aligned}\|\widetilde{r}_2 - r_1\|_{\dot{H}^1(\Omega)} &\le C(\mu^\pm)\|M - \mathrm{Id}\|_{L^\infty(\Omega)}(\|\widetilde{r}_2\|_{\dot{H}^1(\Omega)} + \|\theta_1\|_{\dot{H}^1(\Omega)}) \\ &\le \mathcal{F}(N_s)\|\eta_\delta\|_{H^s}\|v\|_{H^{\frac{1}{2}}}.\end{aligned} \tag{2.4.37}$$

On the other hand, using (2.4.20) for $M - \mathrm{Id}$ in $L^2(\Omega)$ instead gives

$$\begin{aligned}\|\widetilde{r}_2 - r_1\|_{\dot{H}^1(\Omega)} &\le C(\mu^\pm)\|M - \mathrm{Id}\|_{L^2(\Omega)}(\|\nabla\widetilde{r}_2\|_{L^\infty(\Omega)} + \|\nabla\theta_1\|_{L^\infty(\Omega)}) \\ &\le \mathcal{F}(N_s)\|\eta_\delta\|_{H^{\frac{1}{2}}}\|v\|_{H^s},\end{aligned} \tag{2.4.38}$$

where in the last inequality we have used the fact that

$$\|\widetilde{r}_2\|_{L^\infty(\Omega)} + \|\theta_1\|_{L^\infty(\Omega)} \le \mathcal{F}(N_s)\|v\|_{H^s}.$$

Since $r_2 = \widetilde{r}_2 \circ \Upsilon$, $\theta_2 = \theta \circ \Upsilon$, and $\Upsilon : \Omega_2^\pm \to \Omega_1^\pm,\ \Sigma_2 \to \Sigma_1$ we have

$$\mathrm{Tr}_{\Omega_2^\pm \to \Sigma_2}(r_2 \pm \theta_2) = \mathrm{Tr}_{\Omega_1^\pm \to \Sigma_1}(\widetilde{r}_2 \pm \theta_1),$$

and hence

$$\begin{aligned} J_\delta^\pm v &= \mathrm{Tr}_{\Omega_1^\pm \to \Sigma_1}(r_1 \pm \theta_1) - \mathrm{Tr}_{\Omega_2^\pm \to \Sigma_2}(r_2 \pm \theta_2) \\ &= \mathrm{Tr}_{\Omega_1^\pm \to \Sigma_1}(r_1 \pm \theta_1) - \mathrm{Tr}_{\Omega_1^\pm \to \Sigma_1}(\widetilde{r}_2 \pm \theta_1) \\ &= \mathrm{Tr}_{\Omega_1^\pm \to \Sigma_1}(r_1 - \widetilde{r}_2). \end{aligned}$$

In view of (2.4.37) and (2.4.38), the trace operation (2.2.5) yields

$$\|J_\delta^\pm v\|_{H^{\frac{1}{2}}_\pm} \le \mathcal{F}(N_s)\|\eta_\delta\|_{H^s}\|v\|_{H^{\frac{1}{2}}}, \tag{2.4.39}$$

$$\|J_\delta^\pm v\|_{H^{\frac{1}{2}}_\pm} \le \mathcal{F}(N_s)\|\eta_\delta\|_{H^{\frac{1}{2}}}\|v\|_{H^s}. \tag{2.4.40}$$

For the proof of (2.4.1), we shall only need (2.4.40).

Step 3. We have $J_j^\pm v \equiv J^\pm(\eta_j)v = f_j^\pm$ where

$$\begin{cases} f_j^- - f_j^+ = v, \\ \frac{1}{\mu^-}G_j^- f_j^- - \frac{1}{\mu^+}G_j^+ f_j^+ = 0, \end{cases} \qquad j = 1, 2. \tag{2.4.41}$$

By taking differences we obtain $f_\delta^- = f_\delta^+$ and

$$\frac{1}{\mu^-}G_1^- f_\delta^- - \frac{1}{\mu^+}G_1^+ f_\delta^+ = \frac{1}{\mu^+}G_\delta^+ f_2^+ - \frac{1}{\mu^-}G_\delta^- f_2^-, \tag{2.4.42}$$

where we recall the notation $G_\delta^\pm = G_1^\pm - G_2^\pm$. Combining the contraction estimate

(2.2.32) with the continuity (2.2.20) for $J^{\pm}$, we deduce that

$$\|G_\delta^+ f_2^+\|_{H^{\sigma-1}} + \|G_\delta^- f_2^-\|_{H^{\sigma-1}} \le \mathcal{F}(N_s)\|\eta_\delta\|_{H^\sigma}\|v\|_{H^s} \quad \forall\sigma \in [\frac{1}{2}, s]. \tag{2.4.43}$$

We take $\delta \in (0, \frac{1}{2}]$ satisfying $\sigma < s - 1 - \frac{d}{2}$. In light of the paralinearization (2.2.25) for $G_1^{\pm}$, we have

$$\begin{aligned}&\frac{1}{\mu^-}G_1^- f_\delta^- - \frac{1}{\mu^+}G_1^+ f_\delta^+ = T_{\lambda_1}\big(\frac{1}{\mu^-}f_\delta^- + \frac{1}{\mu^+}f_\delta^+\big) + R^1 = \frac{\mu^+ + \mu^-}{\mu^-\mu^+}T_{\lambda_1} f_\delta^- + R^1,\\ &\|R^1\|_{H^{\nu-1+\delta}} \le \mathcal{F}(N_s)\|f_\delta^-\|_{\widetilde{H}^\nu_-} \quad \forall\nu \in [\frac{1}{2}, s-\delta].\end{aligned} \tag{2.4.44}$$

It follows from (2.4.42),(2.4.43) and (2.4.44) that if

$$\nu \in [\frac{1}{2}, s-\delta], \quad \nu + \delta \le \sigma, \quad \sigma \in [\frac{1}{2}, s], \tag{2.4.45}$$

then

$$\|T_{\lambda_1} f_\delta^-\|_{H^{\nu-1+\delta}} \le \mathcal{F}(N_s)\|f_\delta^-\|_{\widetilde{H}^\nu_-} + \mathcal{F}(N_s)\|\eta_\delta\|_{H^\sigma}\|v\|_{H^s}. \tag{2.4.46}$$

Applying Lemma 2.6.6 we have

$$\|\Psi(D)f_\delta^-\|_{H^{\nu+\delta}} \le \mathcal{F}(\|\eta_1\|_{H^s})\|T_{\lambda_1} f_\delta^-\|_{H^{\nu+\delta-1}} + \mathcal{F}(\|\eta_1\|_{H^s})\|f_\delta^-\|_{H^{1,\nu}}.$$

But for $\tau > \frac{1}{2}$,

$$\|\cdot\|_{\widetilde{H}^\tau_\pm} \le C\|\Psi(D)\cdot\|_{H^\tau} + C\|\cdot\|_{\widetilde{H}^{\frac{1}{2}}_\pm},$$

whence (2.4.46) implies that

$$\|f_\delta^-\|_{\widetilde{H}^{\nu+\delta}_-} \le \mathcal{F}(N_s)\|f_\delta^-\|_{\widetilde{H}^\nu_-} + \mathcal{F}(N_s)\|\eta_\delta\|_{H^\sigma}\|v\|_{H^s} \tag{2.4.47}$$

provided that ν and σ satisfy (2.4.45).

We note that (2.4.40) implies that

$$\|f_\delta^-\|_{\widetilde{H}_-^{\frac{1}{2}}} \le \mathcal{F}(N_s)\|\eta_\delta\|_{H^{\frac{1}{2}}}\|v\|_{H^s}, \tag{2.4.48}$$

and hence (2.4.1) holds for $\sigma = \frac{1}{2}$. Now fix $\sigma \in (\frac{1}{2}, s]$. We use (2.4.47) to bootstrap the base estimate (2.4.48) to

$$\|f_\delta^-\|_{\widetilde{H}_-^{\sigma}} \le \mathcal{F}(N_s)\|\eta_\delta\|_{H^\sigma}\|v\|_{H^s}. \tag{2.4.49}$$

Indeed, we can always take δ smaller if necessary so that $\frac{1}{2}+\delta < \sigma$. Plugging (2.4.48) into (2.4.47) with $\nu = \frac{1}{2}$ yields (2.4.49) with $\frac{1}{2}+\delta$ in place of σ. Continuing this n steps, n being the greatest integer such that $\frac{1}{2}+n\delta \le \sigma$, we obtain (2.4.49) for $\frac{1}{2}+n\delta$ in place of σ. This is justified since $\nu = \frac{1}{2}+(n-1)\delta$ satisfies (2.4.45). Thus, for possibly one more step to gain $\sigma - (\frac{1}{2}+n\delta)$ derivative, we obtain (2.4.49). The proof of (2.4.1) is complete.

Finally, (2.4.2) can be proved similarly except that one uses the contraction estimate (2.2.33) to estimate $G_\delta^\pm f_2^\pm$ in (2.4.43).

2.5 Proof of Theorem 2.1.1

Let $s > 1+\frac{d}{2}$, $\mu^- > 0$, $\mu^+ \ge 0$, and $\mathfrak{s} \in (0,1]$. Consider an initial datum $\eta_0 \in H^s(\mathbb{R}^d)$ satisfying

$$\inf_{x\in\mathbb{R}^d} \mathrm{RT}(\eta_0) \ge 2\mathfrak{a} > 0, \tag{2.5.1}$$

$$\mathrm{dist}(\eta_0, \Gamma^\pm) \ge 2h > 0. \tag{2.5.2}$$

Theorem 2.1.1 will be proved in Propositions 2.5.1, 2.5.2 and 2.5.5 below. Precisely, Proposition 2.5.1 establishes the uniform local well-posedness for the Muskat problem with surface tension belonging to any bounded set, say $\mathfrak{s} \in (0,1]$. Then, in Propositions 2.5.2 and 2.5.5, we prove that in appropriate topologies, $\eta^{(\mathfrak{s})}$ converges to η with the rate $\sqrt{\mathfrak{s}}$ and $\mathfrak{s}$ respectively.

Proposition 2.5.1. There exists a time $T_* > 0$ depending only on $\|\eta_0\|_{H^s}$ and $(\mathfrak{a}, h, s, \mu^\pm, \mathfrak{g})$ such that the following holds. For each $\mathfrak{s} \in (0,1]$, there exists a unique solution

$$\eta^{(\mathfrak{s})} \in C([0,T_*]; H^s(\mathbb{R}^d)) \cap L^2([0,T_*]; H^{s+\frac{3}{2}}(\mathbb{R}^d)) \tag{2.5.3}$$

to the Muskat problem with surface tension $\mathfrak{s}$, $\eta^{(\mathfrak{s})}|_{t=0} = \eta_0$ and

$$\|\eta^{(\mathfrak{s})}\|^2_{L^\infty([0,T_*];H^s)} + \|\eta^{(\mathfrak{s})}\|^2_{L^2([0,T_*];H^{s+\frac{1}{2}})} + \mathfrak{s}\|\eta^{(\mathfrak{s})}\|^2_{L^2([0,T_*];H^{s+\frac{3}{2}})} \le \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}) \tag{2.5.4}$$

for some nondecreasing function $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ depending only on $(h, s, \mu^\pm, \mathfrak{g})$. Furthermore, for all $\mathfrak{s} \in (0,1]$ we have

$$\inf_{t\in[0,T_*]} \inf_{x\in\mathbb{R}^d} \mathrm{RT}(\eta^{(\mathfrak{s})}(t)) > \frac{3}{2}\mathfrak{a}, \tag{2.5.5}$$

$$\inf_{t\in[0,T_*]} \mathrm{dist}(\eta^{(\mathfrak{s})}(t), \Gamma^\pm) > \frac{3}{2}h. \tag{2.5.6}$$

Proof. According to Theorems 1.2 and 1.3 in [108], for each initial datum $\eta_0 \in H^s$ satisfying $\mathrm{dist}(\eta_0, \Gamma^\pm) \ge 2h > 0$ and for each $\mathfrak{s} > 0$, there exists $T_{\mathfrak{s}} > 0$ such that the Muskat problem has a unique solution

$$\eta^{(\mathfrak{s})} \in C([0,T_{\mathfrak{s}}]; H^s) \cap L^2([0,T_{\mathfrak{s}}]; H^{s+\frac{3}{2}})$$

satisfying $\inf_{t\in T_{\mathfrak{s}}} \mathrm{dist}(\eta^{(s)}(t), \Gamma^\pm) > \frac{3}{2}h$. We stress the continuity in time of the H^s norm of $\eta^{(\mathfrak{s})}$. Now we have in addition that η_0 satisfies the Rayleigh-Taylor condition

(2.5.1). Thus, we define

$$T^*_{\mathfrak{s}} = \sup\{T \in (0, T_{\mathfrak{s}}] : \inf_{t\in[0,T]} \inf_{x\in\mathbb{R}^d} \mathrm{RT}(\eta^{(\mathfrak{s})}(t)) > \frac{3}{2}\mathfrak{a}\}. \tag{2.5.7}$$

We shall prove that $T^*_{\mathfrak{s}} > 0$ for each $\mathfrak{s} \in (0, 1]$ and there exists $T_* > 0$ such that $T^*_{\mathfrak{s}} \geq T_*$ for all $s \in (0, 1]$.

Step 1. We claim that there exist $\theta > 0$ depending only on s, and $\mathcal{F}_0 : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on $(h, s, \mu^\pm, \mathfrak{g})$ such that

$$\Big\| [\![\mathfrak{B}(\eta^{(\mathfrak{s})}(t))J(\eta^{(\mathfrak{s})}(t))]\!]\eta^{(\mathfrak{s})}(t) - [\![\mathfrak{B}(\eta_0)J(\eta_0)]\!]\eta_0 \Big\|_{L^\infty(\mathbb{R}^d)} \leq (t^{\frac{\theta}{2}} + t^\theta)\mathcal{F}_0(E_{\mathfrak{s}}(t)) \tag{2.5.8}$$

for all $t \leq T_{\mathfrak{s}}$, where

$$E_{\mathfrak{s}}(t) = \|\eta^{(\mathfrak{s})}\|^2_{C([0,t];H^s)} + \mathfrak{s}\|\eta^{(\mathfrak{s})}\|^2_{L^2([0,t];H^{s+\frac{3}{2}})}. \tag{2.5.9}$$

Set

$$A(t) = [\![\mathfrak{B}(\eta^{(\mathfrak{s})}(t))J(\eta^{(\mathfrak{s})}(t))]\!]\eta^{(\mathfrak{s})}(t) - [\![\mathfrak{B}(\eta_0)J(\eta_0)]\!]\eta_0.$$

The continuity properties (2.2.20) and (2.2.23) of $J^\pm$ and $B^\pm$ imply that

$$\begin{aligned} \|A(t)\|_{H^{s-1}} &\leq \|[\![\mathfrak{B}(\eta^{(\mathfrak{s})}(t))J(\eta^{(\mathfrak{s})}(t))]\!]\eta^{(\mathfrak{s})}(t)\|_{H^{s-1}} + \|[\![\mathfrak{B}(\eta_0)J(\eta_0)]\!]\eta_0\|_{H^{s-1}} \\ &\leq \mathcal{F}\big(\|\eta^{(\mathfrak{s})}(t)\|_{H^s}\big) + \mathcal{F}\big(\|\eta^{(\mathfrak{s})}(0)\|_{H^s}\big) \\ &\leq \mathcal{F}(E_{\mathfrak{s}}(t)). \end{aligned} \tag{2.5.10}$$

On the other hand, denoting $\mathfrak{B}^{\pm}(\eta^{(\mathfrak{s})}(t)) = \mathfrak{B}^{\pm}_t$ and $J^{\pm}(\eta^{(\mathfrak{s})}(t)) = J^{\pm}_t$, we can write

$$\begin{aligned}A(t) &= \big(\mathfrak{B}^-_t J^-_t - \mathfrak{B}^+_t J^+_t\big)\eta^{(\mathfrak{s})}(t) - \big(\mathfrak{B}^-_0 J^-_0 - \mathfrak{B}^+_0 J^+_0\big)\eta_0 \\ &= (\mathfrak{B}^-_t - \mathfrak{B}^-_0)J^-_t\eta^{(\mathfrak{s})}(t) + \mathfrak{B}^-_0(J^-_t - J^-_0)\eta^{(\mathfrak{s})}(t) + \mathfrak{B}^-_0 J^-_0(\eta^{(\mathfrak{s})}(t) - \eta_0) \\ &\quad - (\mathfrak{B}^+_t - \mathfrak{B}^+_0)J^+_t\eta^{(\mathfrak{s})}(t) - \mathfrak{B}^+_0(J^+_t - J^+_0)\eta^{(\mathfrak{s})}(t) - \mathfrak{B}^+_0 J^+_0(\eta^{(\mathfrak{s})}(t) - \eta_0).\end{aligned}$$

We treat the first two terms since the other terms are either similar or easier. The contraction estimate (2.2.32) with $\sigma = s - \frac{1}{2}$ gives

$$\|G^{\pm}(\eta^{(\mathfrak{s})}(t)) - G^{\pm}(\eta_0)\|_{\widetilde{H}^s_{\pm} \to H^{s-\frac{3}{2}}} \le \mathcal{F}(E_{\mathfrak{s}}(t))\|\eta^{(\mathfrak{s})}(t) - \eta_0\|_{H^{s-\frac{1}{2}}}.$$

From this and the definition of $\mathfrak{B}^{\pm}$ it is easy to prove that

$$\|\mathfrak{B}^+_t - \mathfrak{B}^+_0\|_{\widetilde{H}^s_{\pm} \to H^{s-\frac{3}{2}}} \le \mathcal{F}(E_{\mathfrak{s}}(t))\|\eta^{(\mathfrak{s})}(t) - \eta_0\|_{H^{s-\frac{1}{2}}}.$$

Then recalling the continuity of $J^{\pm}$ from $H^{s-\frac{1}{2}} \to \widetilde{H}^{s-\frac{1}{2}}_{\pm}$ we obtain

$$\|(\mathfrak{B}^-_t - \mathfrak{B}^-_0)J^-_t\eta^{(\mathfrak{s})}(t)\|_{H^{s-\frac{3}{2}}} \le \mathcal{F}(E_{\mathfrak{s}}(t))\|\eta^{(\mathfrak{s})}(t) - \eta_0\|_{H^{s-\frac{1}{2}}}.$$

Next regarding $\mathfrak{B}^-_0(J^-_t - J^-_0)\eta^{(\mathfrak{s})}(t)$ we use the contraction estimate (2.4.1) with $\sigma = s - \frac{1}{2}$

$$\|J^-_t - J^-_0\|_{H^s \to \widetilde{H}^{s-\frac{1}{2}}_{\pm}} \le \mathcal{F}(E_{\mathfrak{s}}(t))\|\eta^{(\mathfrak{s})}(t) - \eta_0\|_{H^{s-\frac{1}{2}}}$$

together with the continuity (2.2.23) to have

$$\|\mathfrak{B}^-_0(J^-_t - J^-_0)\eta^{(\mathfrak{s})}(t)\|_{H^{s-\frac{3}{2}}} \le \mathcal{F}(E_{\mathfrak{s}}(t))\|\eta^{(\mathfrak{s})}(t) - \eta_0\|_{H^{s-\frac{1}{2}}}.$$

Therefore, we arrive at

$$\|A(t)\|_{H^{s-\frac{3}{2}}} \le \mathcal{F}(E_{\mathfrak{s}}(t))\|\eta^{(\mathfrak{s})}(t)-\eta_0\|_{H^{s-\frac{1}{2}}}. \tag{2.5.11}$$

To bound $\|\eta^{(\mathfrak{s})}(t)-\eta_0\|_{H^{s-\frac{1}{2}}}$ we first use the mean-value theorem and equation (2.2.18) to have

$$\begin{aligned}
\|\eta^{(\mathfrak{s})}(t)-\eta_0\|_{H^{-\frac{1}{2}}} &\le \int_0^t \|\partial_t\eta^{(\mathfrak{s})}(\tau)\|_{H^{-\frac{1}{2}}}d\tau\\
&\le \mathcal{F}(E_{\mathfrak{s}}(t))\int_0^t \mathfrak{g}\|\eta^{(\mathfrak{s})}(\tau)\|_{H^{\frac{1}{2}}} + \mathfrak{s}\|H\big(\eta^{(\mathfrak{s})}(\tau)\big)\|_{H^{\frac{1}{2}}}d\tau\\
&\le \mathcal{F}(E_{\mathfrak{s}}(t))\Big(t\mathfrak{g}\|\eta^{(\mathfrak{s})}\|_{C([0,t];H^s)} + t^{\frac{1}{2}}\mathfrak{s}\|\eta^{(\mathfrak{s})}\|_{L^2([0,t];H^{s+\frac{3}{2}})}\Big)\\
&\le (t^{\frac{1}{2}}+t)\mathcal{F}(E_{\mathfrak{s}}(t))\Big(\mathfrak{g}\|\eta^{(\mathfrak{s})}\|_{C([0,t];H^s)} + \sqrt{\mathfrak{s}}\|\eta^{(\mathfrak{s})}\|_{L^2([0,t];H^{s+\frac{3}{2}})}\Big)
\end{aligned}$$

for all $\mathfrak{s}\in(0,1]$ and $t\le T_{\mathfrak{s}}$. Interpolating this with the obvious bound $\|\eta^{(\mathfrak{s})}(t)-\eta_0\|_{H^s}\le \mathcal{F}(M_s(t))$ gives

$$\|\eta^{(\mathfrak{s})}(t)-\eta_0\|_{H^{s-\frac{1}{2}}} \le (t^{\frac{\theta_0}{2}}+t^{\theta_0})\mathcal{F}(E_{\mathfrak{s}}(t)) \tag{2.5.12}$$

for some $\theta_0\in(0,1)$ and $\mathcal{F}$ depending only on $(h,s,\mu^{\pm},\mathfrak{g})$. Then in view of (2.5.11), this implies

$$\|A(t)\|_{H^{s-\frac{3}{2}}} \le (t^{\frac{\theta_0}{2}}+t^{\theta_0})\mathcal{F}(E_{\mathfrak{s}}(t)).$$

Fixing $s'\in(\max\{1+\frac{d}{2},s-\frac{3}{2}\},s)$ and interpolating this with the H^s bound (2.5.10) we obtain

$$\|A(t)\|_{H^{s'-1}} \le (t^{\frac{\theta}{2}}+t^{\theta})\mathcal{F}(E_{\mathfrak{s}}(t))$$

for some $\theta\in(0,1)$. Using the embedding $H^{s'-1}\subset L^\infty(\mathbb{R}^d)$ we conclude the proof of (2.5.8).

Step 2. We note that (2.5.8) implies the continuity of

$$[0,T_{\mathfrak{s}}]\ni t\mapsto \inf_{x\in\mathbb{R}^d}\mathrm{RT}(\eta^{(\mathfrak{s})}(t)).$$

Thus, in view of the definition (2.5.7) and the initial condition (2.5.1), we have $T^*_{\mathfrak{s}}>0$ for all $\mathfrak{s}\in(0,1]$.

By the definition of $T^*_{\mathfrak{s}}$, conditions (2.3.1), (2.3.2) and (2.3.3) in Proposition 2.3.1 are satisfied for all $T\le T^*_{\mathfrak{s}}$. Thus, the estimates (2.3.4) and (2.3.5) imply the existence of a strictly increasing $\mathcal{F}_2:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ depending only on $(h,s,\mu^\pm,\mathfrak{g})$ such that

$$\mathcal{F}_2(m,0)>m^2\quad\forall m>0 \tag{2.5.13}$$

and

$$M_{\mathfrak{s}}(T)\le\mathcal{F}_2\Big(\|\eta_0\|_{H^s}+T\mathcal{F}_2\big(M_{\mathfrak{s}}(T),\mathfrak{a}^{-1}\big),\mathfrak{a}^{-1}\Big) \tag{2.5.14}$$

for all $\mathfrak{s}\in(0,1]$ and $T\le T^*_{\mathfrak{s}}$, where

$$M_{\mathfrak{s}}(T)=\|\eta^{(\mathfrak{s})}\|^2_{C([0,T];H^s)}+\|\eta^{(\mathfrak{s})}\|^2_{L^2([0,T];H^{s+\frac12})}+\mathfrak{s}\|\eta^{(\mathfrak{s})}\|^2_{L^2([0,T];H^{s+\frac32})}.$$

Set

$$T_2=\frac{\|\eta_0\|_{H^s}}{2\mathcal{F}_2\Big(\mathcal{F}_2\big(2\|\eta_0\|_{H^s},\mathfrak{a}^{-1}\big),\mathfrak{a}^{-1}\Big)} \tag{2.5.15}$$

independent of $\mathfrak{s}$. We claim that

$$M_{\mathfrak{s}}(T)\le K_0:=\mathcal{F}_2\big(2\|\eta_0\|_{H^s},\mathfrak{a}^{-1}\big)\quad\forall T\le\min\{T_2,T^*_{\mathfrak{s}}\},\quad\forall\mathfrak{s}\in(0,1]. \tag{2.5.16}$$

Assume not, then there exists $\mathfrak{s}_0\in(0,1]$ and $T_3\le\min\{T_2,T^*_{\mathfrak{s}_0}\}$ such that $M_{\mathfrak{s}_0}(T_3)>K_0$. Since $M_{\mathfrak{s}_0}(0)=\|\eta_0\|^2_{H^s}<K_0$, the continuity of $T\mapsto E_{\mathfrak{s}_0}(T)$ then yields the existence of $T_4\in(0,T_3)$ such that $M_{\mathfrak{s}_0}(T_4)=K_0$. Consequently, at $T=T_4$, (2.5.14)

gives

$$\begin{aligned}\mathcal{F}_2\big(2\|\eta_0\|_{H^s},\mathfrak{a}^{-1}\big) &\le \mathcal{F}_2\Big(\|\eta_0\|_{H^s}+T_4\mathcal{F}_2\Big(\mathcal{F}_2\big(2\|\eta_0\|_{H^s},\mathfrak{a}^{-1}\big),\mathfrak{a}^{-1}\Big),\mathfrak{a}^{-1}\Big)\\ &\le \mathcal{F}_2\Big(\|\eta_0\|_{H^s}+T_2\mathcal{F}_2\Big(\mathcal{F}_2\big(2\|\eta_0\|_{H^s},\mathfrak{a}^{-1}\big),\mathfrak{a}^{-1}\Big),\mathfrak{a}^{-1}\Big)\\ &\le \mathcal{F}_2\Big(\frac{3}{2}\|\eta_0\|_{H^s},\mathfrak{a}^{-1}\Big),\end{aligned}$$

where we have used the definition of T_2 in the last inequality. This contradicts the fact that $\mathcal{F}_2$ was chosen to be strictly increasing.

Now for all $T \le \min\{T_2, T_{\mathfrak{s}}^*\}$, we use (2.5.8) and the fact that $E_{\mathfrak{s}}(\cdot) \le M_{\mathfrak{s}}(\cdot)$ to have

$$\begin{aligned}\inf_{t\in[0,T]}\inf_{x\in\mathbb{R}^d}\mathrm{RT}(\eta^{(\mathfrak{s})}(t)) &\ge \inf_{x\in\mathbb{R}^d}\mathrm{RT}(\eta_0)-(T^{\frac{\theta}{2}}+T^{\theta})\mathcal{F}_0(M_{\mathfrak{s}}(T))\\ &\ge 2\mathfrak{a}-(T^{\frac{\theta}{2}}+T^{\theta})\mathcal{F}_0(K_0).\end{aligned}$$

Choosing $T_* \le T_2$ sufficiently small so that

$$(T_*^{\frac{\theta}{2}}+T_*^{\theta})\mathcal{F}_0(K_0) < \frac{1}{2}\mathfrak{a} \tag{2.5.17}$$

we obtain

$$\inf_{t\in[0,T]}\inf_{x\in\mathbb{R}^d}\mathrm{RT}(\eta^{(\mathfrak{s})}(t)) > \frac{3}{2}\mathfrak{a} \quad \forall T \le \min\{T_*, T_{\mathfrak{s}}^*\}. \tag{2.5.18}$$

Clearly, T_* is independent of $\mathfrak{s}$. Moreover, (2.5.18) and the definition of $T_{\mathfrak{s}}^*$ show that $T_* \le T_{\mathfrak{s}}^*$ for all $\mathfrak{s} \in (0,1]$. Finally, since $T_* \le \min\{T_2, T_{\mathfrak{s}}^*\}$, (2.5.16) and the definition of $T_{\mathfrak{s}}^*$ guarantee that the estimates (2.5.4), (2.5.5) and (2.5.6) hold true. $\square$

Proposition 2.5.2. There exists $\mathcal{F}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ depending only on $(h, s, \mu^{\pm}, \mathfrak{g})$

such that

$$\|\eta^{(\mathfrak{s}_1)} - \eta^{(\mathfrak{s}_2)}\|^2_{L^\infty([0,T_*];H^{s-1})} + \|\eta^{(\mathfrak{s}_1)} - \eta^{(\mathfrak{s}_2)}\|^2_{L^2([0,T_*];H^{s-\frac{1}{2}})} \le (\mathfrak{s}_1 + \mathfrak{s}_2)\mathcal{F}\big(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}\big) \tag{2.5.19}$$

for all $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in $(0, 1]$.

Proof. Denote $\eta_j = \eta^{(\mathfrak{s}_j)}$, $j = 1, 2$, and $\eta_\delta = \eta_1 - \eta_2$ which exists on $[0, T_*]$. We fix $\delta \in \big(0, \min(\frac{1}{2}, s - 1 - \frac{d}{2})\big)$. From (2.2.18) we have that η_δ evolves according to

$$(\mu^+ + \mu^-)\partial_t\eta_\delta = -\mathfrak{g}(L_\delta\eta_1 + L_2\eta_\delta) - \mathfrak{s}_1 L_1 H(\eta_1) + \mathfrak{s}_2 L_2 H(\eta_2). \tag{2.5.20}$$

By (2.2.20) and (2.2.31),

$$\|L_j H(\eta_j)\|_{H^{s-\frac{3}{2}}} \le \mathcal{F}(\|\eta_j\|_{H^s})\|H(\eta_j)\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}(\|\eta_j\|_{H^s})\|\eta_j\|_{H^{s+\frac{3}{2}}}. \tag{2.5.21}$$

We now paralinearize L_2 and L_δ. Applying (2.2.26) with $\sigma = s - \frac{1}{2} - \delta$ gives

$$L_2\eta_\delta = T_{\lambda_2}\eta_\delta + O_{H^{s-\frac{1}{2}-\delta} \to H^{s-\frac{3}{2}}}(\mathcal{F}(N_s))\eta_\delta. \tag{2.5.22}$$

As for L_δ we write

$$L_\delta = \sum_{\pm=+,-} G^\pm_\delta J^\pm_1 + G^\pm_2 J^\pm_\delta. \tag{2.5.23}$$

Using (2.2.25) at $\sigma = s - \frac{1}{2} - \delta$, we have

$$\sum_{\pm=+,-} G^\pm_2 J^\pm_\delta = T_{\lambda_2}(J^-_\delta - J^+_\delta) + \sum_{\pm=+,-} O_{\widetilde{H}^{s-\frac{1}{2}-\delta}_\pm \to H^{s-\frac{3}{2}}}(\mathcal{F}(N_s))J^\pm_\delta.$$

Recall that N_s is given by (2.4.3). However,

$$J_\delta^- - J_\delta^+ = (J_1^- - J_1^+) - (J_2^- - J_2^+) = \mathrm{Id} - \mathrm{Id} = 0$$

and by virtue of Proposition 2.4.1 (with $\sigma = s - \frac{1}{2} - \delta$),

$$J_\delta^\pm = O_{H^s \to \widetilde{H}_\pm^{s-\frac{1}{2}-\delta}}(\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}}).$$

We thus obtain

$$\sum_{\pm=+,-} G_2^\pm J_\delta^\pm = O_{H^s \to H^{s-\frac{3}{2}}}\big(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}}\big). \tag{2.5.24}$$

As for $G_\delta^\pm J_1^\pm$, we apply Theorem 2.2.8 with $\sigma = s - \frac{1}{2}$ and (2.2.20) to have

$$\mp G_\delta^\pm J_1^\pm f = -T_{\lambda_1 \mathfrak{B}_1^\pm J_1^\pm f}\eta_\delta - T_{\mathfrak{V}_1^\pm J_1^\pm f} \cdot \nabla \eta_\delta + O_{H^s \to H^{s-\frac{3}{2}}}\big(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}}\big) f,$$

and hence

$$\sum_{\pm=+,-} G_\delta^\pm J_1^\pm f = -T_{\lambda_1 [\![\mathfrak{B}_1 J_1]\!] f}\eta_\delta - T_{[\![\mathfrak{V}_1 J_1]\!] f} \cdot \nabla \eta_\delta + O_{H^s \to H^{s-\frac{3}{2}}}\big(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}}\big) f.$$

Combining this with (2.5.24) yields

$$L_\delta f = -T_{\lambda_1 [\![\mathfrak{B}_1 J_1]\!] f}\eta_\delta - T_{[\![\mathfrak{V}_1 J_1]\!] f} \cdot \nabla \eta_\delta + O_{H^s \to H^{s-\frac{3}{2}}}\big(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}}\big) f. \tag{2.5.25}$$

From (2.5.22) and (2.5.25) we have

$$\begin{aligned} L_2\eta_\delta + L_\delta\eta_1 &= T_{\lambda_2}\eta_\delta - T_{\lambda_1 [\![\mathfrak{B}_1 J_1]\!] \eta_1}\eta_\delta - T_{[\![\mathfrak{V}_1 J_1]\!] \eta_1} \cdot \nabla \eta_\delta \\ &\quad + O_{H^{s-\frac{1}{2}-\delta} \to H^{s-\frac{3}{2}}}(\mathcal{F}(N_s))\eta_\delta + O_{H^s \to H^{s-\frac{3}{2}}}\big(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}}\big)\eta_1 \end{aligned}$$

Interchanging η_1 and η_2 gives

$$L_1\eta_\delta + L_\delta\eta_2 = T_{\lambda_1}\eta_\delta - T_{\lambda_2[\![\mathfrak{B}_2 J_2]\!]\eta_2}\eta_\delta - T_{[\![\mathfrak{V}_2 J_2]\!]\eta_2}\cdot\nabla\eta_\delta$$
$$+ O_{H^{s-\frac{1}{2}-\delta}\to H^{s-\frac{3}{2}}}(\mathcal{F}(N_s))\eta_\delta + O_{H^s\to H^{s-\frac{3}{2}}}\big(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}}\big)\eta_2.$$

But $L_1\eta_\delta + L_\delta\eta_2 = L_2\eta_\delta + L_\delta\eta_1$, thus taking the average of the above identities yields

$$L_2\eta_\delta + L_\delta\eta_1 = T_{(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha}\eta_\delta - T_{([\![\mathfrak{V}J]\!]\eta)_\alpha}\cdot\nabla\eta_\delta$$
$$+ O_{H^{s-\frac{1}{2}-\delta}\to H^{s-\frac{3}{2}}}(\mathcal{F}(N_s))\eta_\delta + O_{H^s\to H^{s-\frac{3}{2}}}\big(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}}\big)(\eta_1+\eta_2),$$

where

$$\big(\lambda(1-[\![\mathfrak{B}J]\!]\eta)\big)_\alpha = \frac{1}{2}\Big(\lambda_1(1-[\![\mathfrak{B}_1 J_1]\!]\eta_1) + \lambda_2(1-[\![\mathfrak{B}_2 J_2]\!]\eta_2)\Big)$$

and similarly for $([\![\mathfrak{V}J]\!]\eta)_\alpha$. It then follows from (2.5.20) and (2.5.21) that

$$\begin{aligned}
&(\mu^+ + \mu^-)\partial_t\eta_\delta = -\mathfrak{g}T_{(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha}\eta_\delta + \mathfrak{g}T_{([\![\mathfrak{V}J]\!]\eta)_\alpha}\cdot\nabla\eta_\delta + \mathcal{R}_1,\\
&\|\mathcal{R}_1\|_{H^{s-\frac{3}{2}}} \le \mathfrak{g}\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{1}{2}-\delta}} + (\mathfrak{s}_1+\mathfrak{s}_2)\mathcal{F}(N_s)N_{s+\frac{3}{2}}.
\end{aligned} \tag{2.5.26}$$

Next we perform H^{s-1} energy estimate for (2.5.26). Introduce $\eta_{\delta,s-1} = \langle D\rangle^{s-1}\eta_\delta$. Upon commuting (2.5.26) with $\langle D\rangle^{s-1}$ and applying Theorem 2.6.3 we arrive at

$$\begin{aligned}
&(\mu^+ + \mu^-)\partial_t\eta_{\delta,s-1} = -\mathfrak{g}T_{(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha}\eta_{\delta,s-1} + \mathfrak{g}i\mathrm{Re}\big(T_{([\![\mathfrak{V}J]\!]\eta)_\alpha\cdot\xi}\big)\eta_{\delta,s-1} + \mathcal{R},\\
&\|\mathcal{R}\|_{H^{-\frac{1}{2}}} \le \mathfrak{g}\mathcal{F}(N_s)\|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}-\delta}} + \mathcal{F}(N_s)\sum_{j=1}^{2}\mathfrak{s}_j\|\eta_j\|_{H^{s+\frac{3}{2}}},
\end{aligned} \tag{2.5.27}$$

where $\mathcal{F}$ depends only on $(h, s, \mu^\pm)$. Moreover, the uniform estimate (2.5.4) implies that

$$N_s \le \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}). \tag{2.5.28}$$

Testing (2.5.27) against $\eta_{\delta,s-1}$ yields

$$\frac{(\mu^+ + \mu^-)}{2}\frac{d}{dt}\|\eta_{\delta,s-1}\|_{L^2}^2 = -\,\mathfrak{g}\big(T_{(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha}\eta_{\delta,s-1}, \eta_{\delta,s-1}\big)_{L^2} + (\mathcal{R}, \eta_{\delta,s-1})_{L^2},$$

where we have used the fact that $i\mathrm{Re}\big(T_{([\![\mathfrak{B}J]\!]\eta)_\alpha\cdot\xi}\big)$ is skew-adjoint.

From (2.5.5) we have that $(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha$ is an elliptic symbol in Γ^1_δ:

$$(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha \geq \mathfrak{a}|\xi|, \quad M^1_\delta\big((\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha\big) \leq \mathcal{F}(N_s, \mathfrak{a}^{-1}),$$

where $\mathcal{F}$ depends only on $(h, s, \mu^\pm, \mathfrak{g})$. Applying the Gårding inequality (2.6.9), we have

$$\|\eta_{\delta,s-1}\|^2_{H^{\frac{1}{2}}} \leq \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1})\Big(\big(T_{(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha}\eta_{\delta,s-1}, \eta_{\delta,s-1}\big)_{L^2} + \|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}}}\|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}-\delta}}\Big).$$

This combined with the estimate for $\mathcal{R}$ in (2.5.27) and (2.5.28) implies

$$\begin{aligned}&\frac{(\mu^+ + \mu^-)}{2}\frac{d}{dt}\|\eta_{\delta,s-1}\|^2_{L^2} + \frac{\mathfrak{g}}{\mathcal{Q}_{T_*}}\|\eta_{\delta,s-1}\|^2_{H^{\frac{1}{2}}}\\ &\quad\leq \mathfrak{g}\mathcal{Q}_{T_*}\|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}-\delta}}\|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}}} + \mathcal{Q}_{T_*}\|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}}}\sum_{j=1}^{2}\mathfrak{s}_j\|\eta_j\|_{H^{s+\frac{3}{2}}},\end{aligned} \tag{2.5.29}$$

where

$$\mathcal{Q}_{T_*} = \mathcal{F}\big(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}\big), \tag{2.5.30}$$

$\mathcal{F}$ depending only on $(h, s, \mu^\pm, \mathfrak{g})$. Using interpolation and Young's inequality, we have

$$\begin{aligned}&\|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}-\delta}}\|\eta_s\|_{H^{\frac{1}{2}}} \leq \|\eta_{\delta,s-1}\|^{2\delta}_{L^2}\|\eta_{\delta,s-1}\|^{2(1-\delta)}_{H^{\frac{1}{2}}} \leq (10\mathcal{Q}_{T_*})^{\frac{2(1-\delta)}{\delta}}\|\eta_{\delta,s-1}\|^2_{L^2} + \frac{1}{100\mathcal{Q}^2_{T_*}}\|\eta_{\delta,s-1}\|^2_{H^{\frac{1}{2}}},\\ &\mathfrak{s}_j\|\eta_j\|_{H^{s+\frac{3}{2}}}\|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}}} \leq \mathfrak{s}_j^2 100\mathcal{Q}^2_{T_*}\|\eta_j\|^2_{H^{s+\frac{3}{2}}} + \frac{1}{100\mathcal{Q}^2_{T_*}}\|\eta_{\delta,s-1}\|^2_{H^{\frac{1}{2}}}.\end{aligned} \tag{2.5.31}$$

Thus, for possibly a larger $\mathcal{F}$ in $\mathcal{Q}_{T_*}$, we obtain

$$\frac{(\mu^+ + \mu^-)}{2}\frac{d}{dt}\|\eta_{\delta,s-1}\|_{L^2}^2 + \frac{\mathfrak{g}}{\mathcal{Q}_{T_*}}\|\eta_{\delta,s-1}\|_{H^{\frac{1}{2}}}^2 \le \mathfrak{g}\mathcal{Q}_{T_1}\|\eta_{\delta,s-1}\|_{L^2}^2 + \mathcal{Q}_{T_*}\sum_{j=1}^{2}\mathfrak{s}_j^2\|\eta_j\|_{H^{s+\frac{3}{2}}}^2. \tag{2.5.32}$$

Finally, since $\eta_\delta|_{t=0} = 0$ and by (2.5.4)

$$\mathfrak{s}_j\int_0^{T_*}\|\eta_j\|_{H^{s+\frac{3}{2}}}^2 dt \le \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}),$$

an application of Grönwall's lemma leads to the estimate (2.5.19). □

Now let $\mathfrak{s}_n \to 0$ and rename $\eta_n = \eta^{(\mathfrak{s}_n)}$ solution to the Muskat problem with surface tension $\mathfrak{s}_n$ on $[0, T_*]$. The uniform estimates in (2.5.4) show that along a subsequence η_n converges weakly-* to

$$\eta \in L^\infty([0,T_*]; H^s) \cap L^2([0,T_*]; H^{s+\frac{1}{2}}) \tag{2.5.33}$$

together with the bounds

$$\|\eta\|_{L^\infty([0,T_*];H^s)} + \|\eta\|_{L^2([0,T_*];H^{s+\frac{1}{2}})} \le \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}). \tag{2.5.34}$$

The estimate (2.5.19) implies that $(\eta_n)_n$ is a Cauchy sequence in $C([0,T_*]; H^{s-1}) \cap L^2([0,T_*]; H^{s-\frac{1}{2}})$. Therefore,

$$\eta_n \to \eta \quad \text{in } C([0,T_*]; H^{s-1}) \cap L^2([0,T_*]; H^{s-\frac{1}{2}}); \tag{2.5.35}$$

in particular, $\eta|_{t=0} = \eta_0$. Moreover, by interpolating between $L_t^\infty H_x^s$ and $C_t H^{s-1}$, we deduce that $\eta \in C([0,T_*]; H^{s'})$ for all $s' < s$. Since $\eta_n \to \eta$ in $C_t H_x^{s-1} \subset C_t L_x^\infty$, (2.5.6) gives

$$\inf_{t\in[0,T_*]} \operatorname{dist}(\eta(t),\Gamma^\pm)\ge \frac{3}{2}h. \tag{2.5.36}$$

Lemma 2.5.3. η is a solution on $[0,T_*]$ of the Muskat problem without surface tension with initial data η_0.

Proof. For each n, we have from (2.2.18) that

$$\partial_t\eta_n+\frac{1}{\mu^++\mu^-}L(\eta_n)(\mathfrak{g}\eta_n+\mathfrak{s}H(\eta_n))=0 \tag{2.5.37}$$

For any compactly supported test function $\varphi\in C^\infty((0,T_*)\times\mathbb{R}^d)$, we have

$$\int_0^{T_*}\eta_n\partial_t\varphi dxdt=\frac{1}{\mu^++\mu^-}\int_0^{T_*}\int_{\mathbb{R}^d}\varphi L(\eta_n)(\mathfrak{g}\eta_n+\mathfrak{s}H(\eta_n))dxdt. \tag{2.5.38}$$

Clearly, (2.5.35) implies that

$$\int_0^{T_*}\eta_n\partial_t\varphi dxdt\to\int_0^{T_*}\eta\partial_t\varphi dxdt.$$

The continuity (2.2.20) of L combined with (2.2.31) and the uniform bound (2.5.4) yields

$$\begin{aligned}\mathfrak{s}_n\|L(\eta_n)H(\eta_n)\|_{L^2([0,T_*];H^{s-\frac32})}&\le\mathcal{F}(\|\eta_n\|_{L^\infty([0,T_*];H^s)})\mathfrak{s}_n\|\eta_n\|_{L^2([0,T_*];H^{s+\frac32})}\\&\lesssim\sqrt{\mathfrak{s}_n}\mathcal{F}(\|\eta_0\|_{H^s},\mathfrak{a}^{-1}).\end{aligned}$$

Since $s-\frac32>0$, this implies

$$\begin{aligned}\left|\int_0^{T_*}\int_{\mathbb{R}^d}\varphi L(\eta_n)\big(\mathfrak{s}_nH(\eta_n)\big)dxdt\right|&\le\|\varphi\|_{L^2_{x,t}}\|L(\eta_n)\big(\mathfrak{s}_nH(\eta_n)\big)\|_{L^2_{x,t}}\\&\lesssim\sqrt{\mathfrak{s}_n}\|\varphi\|_{L^2_{x,t}}\mathcal{F}(\|\eta_0\|_{H^s},\mathfrak{a}^{-1})\to 0.\end{aligned}$$

Next we write

$$
\begin{aligned}
L(\eta_n)\eta_n - L(\eta)\eta &= \big(L(\eta_n) - L(\eta)\big)\eta_n + L(\eta_n)(\eta_n - \eta)\\
&= \frac{\mu^+ + \mu^-}{\mu^-}\big(G^-(\eta_n)J^-(\eta_n) - G^-(\eta)J^-(\eta)\big)\eta_n + L(\eta_n)(\eta_n - \eta)\\
&= \frac{\mu^+ + \mu^-}{\mu^-}\Big\{\big(G^-(\eta_n) - G^-(\eta)\big)J^-(\eta_n)\eta_n - G^-(\eta)\big(J^-(\eta_n) - J^-(\eta)\big)\eta_n\Big\}\\
&\qquad + L(\eta_n)(\eta_n - \eta).
\end{aligned}
$$

Combining (2.2.32) and (2.2.20) we obtain

$$
\|\big(G^-(\eta_n) - G^-(\eta)\big)J^-(\eta_n)\eta_n\|_{L^2([0,T_*];H^{s-\frac{3}{2}})} \le \mathcal{F}(\|(\eta_n,\eta)\|_{L^\infty([0,T_*];H^s)})\|\eta_n - \eta\|_{L^2([0,T_*];H^{s-\frac{1}{2}})}.
$$

On the other hand, (2.2.20) and (2.4.1) yield

$$
\|G^-(\eta)\big(J^-(\eta_n) - J^-(\eta)\big)\eta_n\|_{L^2([0,T_*];H^{s-\frac{3}{2}})} \le \mathcal{F}(\|(\eta_n,\eta)\|_{L^\infty([0,T_*];H^s)})\|\eta_n - \eta\|_{L^2([0,T_*];H^{s-\frac{1}{2}})}.
$$

Finally, by (2.2.20) we have

$$
\|L(\eta_n)(\eta_n - \eta)\|_{L^2([0,T_*];H^{s-\frac{3}{2}})} \le \mathcal{F}(\|\eta_n\|_{L^\infty([0,T_*];H^s)})\|\eta_n - \eta\|_{L^2([0,T_*];H^{s-\frac{1}{2}})}.
$$

Putting together the above considerations, we obtain

$$
\left|\int_0^{T_*}\int_{\mathbb{R}^d}\varphi\big(L(\eta_n)\eta_n - L(\eta)\eta\big)\right| \le \|\varphi\|_{L^2_{x,t}}\mathcal{F}(\|(\eta_n,\eta)\|_{L^\infty([0,T_*];H^s)})\|\eta_n - \eta\|_{L^2([0,T_*];H^{s-\frac{1}{2}})} \to 0
$$

by virtue of the strong convergence (2.5.35) and the uniform H^s bound in (2.5.4). We have proved that

$$
\int_0^{T_*}\eta\partial_t\varphi dxdt = \frac{1}{\mu^+ + \mu^-}\int_0^{T_*}\int_{\mathbb{R}^d}\varphi L(\eta)(\mathfrak{g}\eta)dxdt
$$

for all compactly supported smooth test functions φ. Therefore, η is a solution on $[0, T_*]$ of the Muskat problem without surface tension. $\square$

Lemma 2.5.4. We have

$$\inf_{t\in[0,T_*]} \inf_{x\in\mathbb{R}^d} \mathrm{RT}(\eta(t)) \geq \frac{3}{2}\mathfrak{a}. \tag{2.5.39}$$

Proof. Set

$$K = [\![\mathfrak{B}(\eta_n)J(\eta_n)]\!]\eta_n - [\![\mathfrak{B}(\eta)J(\eta)]\!]\eta.$$

Arguing as in the proof of (2.5.11) we find that

$$\begin{aligned}\|K\|_{H^{s-2}} &\leq \mathcal{F}(\|(\eta_n, \eta)\|_{H^s})\|\eta_n - \eta\|_{H^{s-1}} \\ &\lesssim \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1})\|\eta_n - \eta\|_{H^{s-1}}.\end{aligned}$$

On the other hand, by estimating each term in K we have

$$\|K\|_{H^{s-1}} \leq \mathcal{F}(\|(\eta_n, \eta)\|_{H^s}) \lesssim \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}).$$

Choosing $s' \in (\max\{\frac{d}{2}, s-2\}, s-1)$, then interpolating the above estimates gives

$$\|K\|_{L^\infty([0,T_*];L^\infty(\mathbb{R}^d))} \leq \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1})\|\eta_n - \eta\|^{\theta}_{L^\infty([0,T];H^{s-1})}$$

for some $\theta \in (0,1)$. Then, (2.5.39) follows from this and (2.5.5).

$\square$

Now in view of the properties (2.5.34), (2.5.36) and (2.5.39) of η, we see that in the proof of (2.5.19), if we replace $\eta^{(\mathfrak{s}_1)}$ with η_n, $\eta^{(\mathfrak{s}_2)}$ with η, and $(\mathfrak{s}_1, \mathfrak{s}_2)$ with $(\mathfrak{s}_n, 0)$, then we obtain the convergence estimate

$$\|\eta_n-\eta\|_{L^\infty([0,T_*];H^{s-1})}+\|\eta_n-\eta\|_{L^2([0,T_*];H^{s-\frac12})}\le\sqrt{\mathfrak{s}_n}\mathcal{F}(\|\eta_0\|_{H^s},\mathfrak{a}^{-1}). \tag{2.5.40}$$

Furthermore, assume that η_1 and η_2 are two solutions on $[0,T_*]$ of the Muskat problem without surface tension with the same initial data η_0 and that both satisfy (2.5.34), (2.5.36) and (2.5.39). Then the proof of (2.5.19) with $\mathfrak{s}_1=\mathfrak{s}_2=0$ yields that $\eta_1\equiv\eta_2$ on $[0,T_*]$. This proves the uniqueness of η. In other words, we have obtained an alternative proof for the local well-posedness of the Muskat problem without surface tension for any subcritical data satisfying (2.5.1) and (2.5.2).

The next proposition improves the rate in (2.5.40) to the optimal rate.

Proposition 2.5.5. If in addition $s\ge 2$, then

$$\|\eta_n-\eta\|_{L^\infty([0,T_*];H^{s-2})}+\|\eta_n-\eta\|_{L^2([0,T_*];H^{s-\frac32})}\le\mathfrak{s}\mathcal{F}(\|\eta_0\|_{H^s},\mathfrak{a}^{-1}), \tag{2.5.41}$$

where $\mathcal{F}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ depends only on $(h,s,\mu^\pm,\mathfrak{g})$.

Proof. We follow the notation in the proof of Proposition 2.5.2 but set $\eta_1=\eta_n$ and $\eta_2=\eta$. Then, $\eta_\delta=\eta_n-\eta$ satisfies

$$(\mu^++\mu^-)\partial_t\eta_\delta=-\mathfrak{g}(L_\delta\eta_1+L_2\eta_\delta)-\mathfrak{s}_nL_1H(\eta_1). \tag{2.5.42}$$

Applying (2.2.20) and (2.2.31) with $\sigma=s-\frac32\ge\frac12$ yields

$$\|L_1H(\eta_1)\|_{H^{s-\frac52}}\le\mathcal{F}(\|\eta_1\|_{H^s})\|H(\eta_1)\|_{H^{s-\frac32}}\le\mathcal{F}(\|\eta_1\|_{H^s})\|\eta_1\|_{H^{s+\frac12}}. \tag{2.5.43}$$

Next we paralinearize L_2 and L_δ. For L_2 we apply (2.2.26) with $\sigma = s - \frac{3}{2} \geq \frac{1}{2}$

$$L_2\eta_\delta = T_{\lambda_2}\eta_\delta + O_{H^{s-\frac{3}{2}} \to H^{s-\frac{5}{2}+\delta}}(\mathcal{F}(N_s))\eta_\delta. \tag{2.5.44}$$

L_δ can be written as in (2.5.23). Using (2.2.25) with $\sigma = s - \frac{3}{2} \geq \frac{1}{2}$ together with the fact that $J_\delta^- - J_\delta^+ = 0$, we obtain

$$\sum_{\pm=+,-} G_2^\pm J_\delta^\pm = \sum_{\pm=+,-} O_{\widetilde{H}_\pm^{s-\frac{3}{2}} \to H^{s-\frac{5}{2}+\delta}}(\mathcal{F}(N_s))J_\delta^\pm.$$

Applying Proposition 2.4.1 with $\sigma = s - \frac{3}{2} \geq \frac{1}{2}$, we obain

$$J_\delta^\pm = O_{H^s \to \widetilde{H}_\pm^{s-\frac{3}{2}}}(\|\eta_\delta\|_{H^{s-\frac{3}{2}}}),$$

and hence

$$\sum_{\pm=+,-} G_2^\pm J_\delta^\pm = O_{H^s \to H^{s-\frac{5}{2}+\delta}}(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{3}{2}}}).$$

On the other hand, Theorem 2.2.8 can be applied with $\sigma = s - \frac{3}{2} \geq \frac{1}{2}$, implying

$$\sum_{\pm=+,-} G_\delta^\pm J_1^\pm f = -T_{\lambda_1[\![\mathfrak{B}_1 J_1]\!]f}\eta_\delta - T_{[\![\mathfrak{V}_1 J_1]\!]f} \cdot \nabla\eta_\delta + O_{H^s \to H^{s-\frac{5}{2}+\delta}}\big(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{3}{2}}}\big)f.$$

We thus obtain

$$L_\delta f = -T_{\lambda_1[\![\mathfrak{B}_1 J_1]\!]f}\eta_\delta - T_{[\![\mathfrak{V}_1 J_1]\!]f} \cdot \nabla\eta_\delta + O_{H^s \to H^{s-\frac{5}{2}+\delta}}(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{3}{2}}})f. \tag{2.5.45}$$

Applying this with $f = \eta_1$, then combining with (2.5.44) and symmetrizing we arrive at

$$L_2\eta_\delta + L_\delta\eta_1 = T_{(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha}\eta_\delta - T_{([\![\mathfrak{V}J]\!]\eta)_\alpha}\cdot\nabla\eta_\delta$$
$$+ O_{H^{s-\frac{3}{2}}\to H^{s-\frac{5}{2}+\delta}}(\mathcal{F}(N_s))\eta_\delta + O_{H^s\to H^{s-\frac{5}{2}+\delta}}(\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{3}{2}}})\eta_1.$$

Plugging this and (2.5.43) into (2.5.20) leads to

$$\begin{aligned}
&(\mu^+ + \mu^-)\partial_t\eta_\delta = -\mathfrak{g}T_{(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha}\eta_\delta + \mathfrak{g}T_{([\![\mathfrak{V}J]\!]\eta)_\alpha}\cdot\nabla\eta_\delta + \mathcal{R}_1' + \mathcal{R}_2',\\
&\|\mathcal{R}_1'\|_{H^{s-\frac{5}{2}+\delta}} \le \mathfrak{g}\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s-\frac{3}{2}}},\\
&\|\mathcal{R}_2'\|_{H^{s-\frac{5}{2}}} \le \mathfrak{s}_n\mathcal{F}(N_s)\|\eta_1\|_{H^{s+\frac{1}{2}}}.
\end{aligned} \tag{2.5.46}$$

Next we set $\eta_{\delta,s-2} = \langle D\rangle^{s-2}\eta_\delta$ and commute the first equation in (2.5.46) with $\langle D\rangle^{s-2}$ to obtain after applying Theorem 2.6.3 that

$$\begin{aligned}
&(\mu^+ + \mu^-)\partial_t\eta_{\delta,s-2} = -\mathfrak{g}T_{(\lambda(1-[\![\mathfrak{B}J]\!]\eta))_\alpha}\eta_{\delta,s-2} + \mathfrak{g}i\mathrm{Re}\big(T_{([\![\mathfrak{V}J]\!]\eta)_\alpha\cdot\xi}\big)\eta_{\delta,s-2} + \mathcal{R}_1 + \mathcal{R}_2,\\
&\|\mathcal{R}_1\|_{H^{-\frac{1}{2}+\delta}} \le \mathfrak{g}\mathcal{F}(N_s)\|\eta_{\delta,s-2}\|_{H^{\frac{1}{2}}},\\
&\|\mathcal{R}_2\|_{H^{-\frac{1}{2}}} \le \mathfrak{s}_n\mathcal{F}(N_s)\|\eta_1\|_{H^{s+\frac{1}{2}}},
\end{aligned} \tag{2.5.47}$$

where $\mathcal{F}$ depends only on $(h, s, \mu^\pm)$. An L^2 energy estimate as in (2.5.29) yields

$$\begin{aligned}
&\frac{(\mu^+ + \mu^-)}{2}\frac{d}{dt}\|\eta_{\delta,s-2}\|_{L^2}^2 + \frac{\mathfrak{g}}{\mathcal{Q}_{T_*}}\|\eta_{\delta,s-2}\|_{H^{\frac{1}{2}}}^2\\
&\qquad\le \mathfrak{g}\mathcal{Q}_{T_*}\|\eta_{\delta,s-2}\|_{H^{\frac{1}{2}-\delta}}\|\eta_{\delta,s-2}\|_{H^{\frac{1}{2}}} + \mathfrak{s}_n\mathcal{Q}_{T_*}\|\eta_1\|_{H^{s+\frac{1}{2}}}\|\eta_{\delta,s-2}\|_{H^{\frac{1}{2}}},
\end{aligned} \tag{2.5.48}$$

where $\mathcal{Q}_{T_*}$ is given by (2.5.30). Interpolating as in (2.5.31) we obtain

$$\frac{(\mu^+ + \mu^-)}{2}\frac{d}{dt}\|\eta_{\delta,s-2}\|_{L^2}^2 + \frac{\mathfrak{g}}{\mathcal{Q}_{T_*}}\|\eta_{\delta,s-2}\|_{H^{\frac{1}{2}}}^2 \le \mathfrak{g}\mathcal{Q}_{T_1}\|\eta_{\delta,s-2}\|_{L^2}^2 + \mathfrak{s}_n^2\mathcal{Q}_{T_*}\|\eta_1\|_{H^{s+\frac{1}{2}}}^2. \tag{2.5.49}$$

From the uniform estimate (2.5.4) we have

$$\int_0^{T_*} \|\eta_1\|^2_{H^{s+\frac{1}{2}}} dt \leq \mathcal{F}(\|\eta_0\|_{H^s}, \mathfrak{a}^{-1}).$$

Thus, applying Gönwall's lemma to (2.5.49) we arrive at (2.5.41). □

2.6 Paradifferential Calculus

In this appendix, we recall the symbolic calculus of Bony's paradifferential calculus. See [18, 103].

Definition 2.6.1. 1. (Paradifferential symbols) Given $\rho \in [0, \infty)$ and $m \in \mathbb{R}$, $\Gamma^m_\rho(\mathbb{R}^d)$ denotes the space of locally bounded functions $a(x, \xi)$ on $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial^\alpha_\xi a(x, \xi)$ belongs to $W^{\rho,\infty}(\mathbb{R}^d)$ and there exists a constant C_α such that,

$$\forall |\xi| \geq \frac{1}{2}, \quad \|\partial^\alpha_\xi a(\cdot, \xi)\|_{W^{\rho,\infty}(\mathbb{R}^d)} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

Let $a \in \Gamma^m_\rho(\mathbb{R}^d)$, we define the semi-norm

$$M^m_\rho(a) = \sup_{|\alpha| \leq 2(d+2)+\rho} \sup_{|\xi| \geq \frac{1}{2}} \|(1 + |\xi|)^{|\alpha|-m} \partial^\alpha_\xi a(\cdot, \xi)\|_{W^{\rho,\infty}(\mathbb{R}^d)}. \tag{2.6.1}$$

2. (Paradifferential operators) Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \Psi(\eta) \widehat{u}(\eta)\, d\eta, \tag{2.6.2}$$

where $\widehat{a}(\theta, \xi) = \int e^{-ix\cdot\theta} a(x, \xi)\, dx$ is the Fourier transform of a with respect to the first variable; χ and Ψ are two fixed C^∞ functions such that:

$$\Psi(\eta) = 0 \quad \text{for } |\eta| \le \frac{1}{5}, \qquad \Psi(\eta) = 1 \quad \text{for } |\eta| \ge \frac{1}{4}, \tag{2.6.3}$$

and $\chi(\theta,\eta)$ satisfies, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta,\eta) = 1 \quad \text{if} \quad |\theta| \le \varepsilon_1 |\eta|, \qquad \chi(\theta,\eta) = 0 \quad \text{if} \quad |\theta| \ge \varepsilon_2 |\eta|,$$

and such that

$$\forall(\theta,\eta), \qquad |\partial_\theta^\alpha \partial_\eta^\beta \chi(\theta,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{-|\alpha|-|\beta|}.$$

Theorem 2.6.2. For all $m \in \mathbb{R}$, if $a \in \Gamma_0^m$ then

$$T_a = O_{Op^m}\big(M_0^m(a)\big). \tag{2.6.4}$$

Theorem 2.6.3 (Symbolic calculus). Let $a \in \Gamma_r^m, a' \in \Gamma_r^{m'}$ and set $\delta = \min\{1, r\}$. Then,

(i)

$$T_a T_{a'} = T_{aa'} + O_{Op^{m+m'-\delta}}\Big(M_r^m(a) M_0^{m'}(a') + M_0^{m_1}(a) M_r^{m'}(a')\Big); \tag{2.6.5}$$

(ii)

$$T_a^* = T_{\overline{a}} + O_{Op^{m-\delta}}\big(M_r^m(a)\big). \tag{2.6.6}$$

Remark 2.6.4. In the definition (2.6.2) of paradifferential operators, the cut-off Ψ removes the low frequency part of u. In particular, if $a \in \Gamma_0^m$ then

$$\|T_a u\|_{H^\sigma} \le C M_0^m(a) \|\nabla u\|_{H^{\sigma+m-1}} = C M_0^m(a) \|u\|_{H^{\sigma+m,1}},$$

and similarly for other estimates involving paradifferential operators.

Proposition 2.6.5 (Gårding's inequality). Assume $a \in \Gamma_r^m$ with $m \in \mathbb{R}$ and $r \in (0,1]$ such that for some $c > 0$

$$\inf_{(x,\xi)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})} \mathrm{Re}(a(x,\xi)) \geq c|\xi|^m. \tag{2.6.7}$$

Then, for all $\sigma \in \mathbb{R}$, there exists $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that

$$\|\Psi(D)u\|_{H^{\frac{m}{2}}}^2 \leq \mathcal{F}(M_r^m(a), c^{-1})\Big(\mathrm{Re}(T_a u, u)_{L^2} + \|u\|_{H^{1,\frac{m-r}{2}}}^2\Big) \tag{2.6.8}$$

and

$$\|\Psi(D)u\|_{H^{\frac{m}{2}}}^2 \leq \mathcal{F}(M_r^m(a), c^{-1})\Big(\mathrm{Re}(T_a u, u)_{L^2} + \|u\|_{H^{\frac{m}{2}}}\|u\|_{H^{1,\frac{m}{2}-r}}\Big) \tag{2.6.9}$$

provided that both sides are finite. Here, $\Psi(D)$ is the Fourier multiplier with symbol Ψ given by (2.6.3).

Proof. We have

$$\begin{aligned}\mathrm{Re}(T_a u, u)_{L^2} &= \frac{1}{2}\big((T_a u, u)_{L^2} + (T_a^* u, u)_{L^2}\big)\\ &= (T_{\mathrm{Re}(a)}u, u)_{L^2} + \frac{1}{2}((T_a^* - T_{\overline{a}})u, u)_{L^2}).\end{aligned}$$

According to Theorem 2.6.3 (ii), $T_a^* - T_{\overline{a}}$ is of order $m - r$ and

$$\|((T_a^* - T_{\overline{a}})u, u)_{L^2})\| \leq \|(T_a^* - T_{\overline{a}})u\|_{H^{\frac{m-r}{2}}}\|u\|_{H^{\frac{m-r}{2}}} \leq CM_r^m(a)\|u\|_{H^{\frac{m-r}{2}}}^2.$$

Set $b = (\mathrm{Re}(a))^{\frac{1}{2}}$. By virtue of (2.6.7) we have $b \in \Gamma_r^{\frac{m}{2}}$ and $M_r^{\frac{m}{2}}(b) \leq \mathcal{F}(M_r^m(a))$.

We write

$$
\begin{aligned}
(T_{\mathrm{Re}(a)}u, u)_{L^2} &= (T_b T_b u, u)_{L^2} + ((T_{b^2} - T_b T_b)u, u)_{L^2} \\
&= (T_b u, T_b^* u)_{L^2} + ((T_{b^2} - T_b T_b)u, u)_{L^2} \\
&= \|T_b u\|_{L^2}^2 + (T_b u, (T_b^* - T_b)u)_{L^2} + ((T_{b^2} - T_b T_b)u, u)_{L^2}.
\end{aligned}
$$

Applying Theorem 2.6.3 (ii) once again we deduce that $T_b^* - T_b$ is of order $\frac{m}{2} - r$ and

$$
\left|(T_b u, (T_b^* - T_b)u)_{L^2}\right| \le \|T_b u\|_{H^{-\frac{r}{2}}} \|(T_b^* - T_b)u)_{L^2}\|_{H^{\frac{r}{2}}} \le \mathcal{F}(M_r^m(a), c^{-1})\|u\|^2_{H^{1,\frac{m-r}{2}}},
$$

where we used Remark 2.6.4 in the last inequality. On the other hand, an application of Theorem 2.6.3 (i) yields

$$
\left|((T_{b^2} - T_b T_b)u, u)_{L^2}\right| \le \mathcal{F}(M_r^m(a), c^{-1})\|u\|^2_{H^{1,\frac{m-r}{2}}}.
$$

Thus, we obtain

$$
\|T_b u\|_{L^2}^2 \le (T_{\mathrm{Re}(a)}u, u)_{L^2} + \mathcal{F}(M_r^m(a), c^{-1})\|u\|^2_{H^{1,\frac{m-r}{2}}}. \tag{2.6.10}
$$

By shifting derivative differently in the above inner products, we have the variant

$$
\|T_b u\|_{L^2}^2 \le (T_{\mathrm{Re}(a)}u, u)_{L^2} + \mathcal{F}(M_r^m(a), c^{-1})\|u\|_{H^{1,\frac{m}{2}}}\|u\|_{H^{1,\frac{m}{2}-r}}. \tag{2.6.11}
$$

Next we note that $T_{b^{-1}}T_b - \Psi(D) = T_{b^{-1}}T_b - T_1$ is of order $-r$ and

$$
\begin{aligned}
\|\Psi(D)u\|_{H^{\frac{m}{2}}} &= \|T_{b^{-1}}T_b u\|_{H^{\frac{m}{2}}} + \|(T_{b^{-1}}T_b - \Psi(D))u\|_{H^{\frac{m}{2}}} \\
&\le \mathcal{F}(M_r^m(a), c^{-1})\|T_b u\|_{L^2} + \mathcal{F}(M_r^m(a), c^{-1})\|u\|_{H^{1,\frac{m}{2}-r}}.
\end{aligned} \tag{2.6.12}
$$

Finally, a combination of (2.6.10) and (2.6.12) leads to (2.6.8), and a combination of (2.6.11) and (2.6.12) leads to (2.6.9). □

The proof of (2.6.12) also proves the following lemma.

Lemma 2.6.6. Let $a \in \Gamma_r^m$, $r \in (0,1]$, be a real symbol satisfying $a(x,\xi) \geq c|\xi|^m$ for all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Then for all $s \in \mathbb{R}$ we have

$$\|\Psi(D)u\|_{H^s} \leq \mathcal{F}(M_r^m(a), c^{-1})\|T_a u\|_{H^{s-m}} + \mathcal{F}(M_r^m(a), c^{-1})\|u\|_{H^{1,s-r}}. \qquad (2.6.13)$$

Chapter Three

Scattering map for the Vlasov–Poisson system

Joint work with Zhimeng Ouyang, Benoît Pausader, and Klaus Widmayer.

3.1 Introduction

The three dimensional Vlasov-Poisson system describes the evolution of a particle distribution[1] $\mu(t,x,v):\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ satisfying

$$(\partial_t + v\cdot\nabla_x)\,\mu + \lambda\nabla_x\psi\cdot\nabla_v\mu = 0, \qquad \Delta_x\psi(t,x) = \rho(t,x), \qquad \rho(t,x) = \int_{\mathbb{R}^3}\mu^2(t,x,v)dv. \tag{3.1.1}$$

[1] To be precise, the physically relevant quantity $f(t,x,v) = \mu^2(t,x,v)$ is the square of our unknown μ – see also [91].

This is a model for a continuum limit of a classical many-body problem with Newtonian self-interactions through a force field $\nabla_x\psi$ that can be attractive ($\lambda = -1$) as in a galactic setting, or repulsive ($\lambda = 1$) as in a plasma or ion gas, and which is generated by the spatial density $\rho(t, x)$ of the particle distribution.

The mathematical theory for the initial value problem associated to (3.1.1) is classical and guarantees the global existence of unique solutions under suitable assumptions on the initial data [11,98,116,117]. In recent years there has been progress in understanding the long time asymptotic behavior: sharp decay rates of the density and force field are known in some settings [87, 89, 113, 115, 126, 130], and it has been shown that for sufficiently small initial data μ_0 the problem (3.1.1) exhibits a modified scattering dynamic [33, 91] defined in terms of a limit distribution μ_∞ and an asymptotic force field $E_\infty[\mu_\infty]$, defined by inverting the roles of x and v:

$$E_\infty[\mu](v) := \frac{1}{4\pi}\iint \frac{v-w}{|v-w|^3}\cdot\mu^2(y,w)\,dydw. \tag{3.1.2}$$

In this paper, using pseudo-conformal inversion, we prove the converse statement, namely that any solution of the asymptotic dynamic arises in a unique way as a limit of a solution to (3.1.1), i.e. we construct the wave operator $\mu_\infty \mapsto \mu_0$. Thus we obtain the existence of a scattering operator linking the asymptotic behavior in the past to the asymptotic behavior in the future ($\mu_{-\infty} \mapsto \mu_0 \mapsto \mu_{+\infty}$).

Our main results can be summarized as follows:

Theorem 3.1.1. *There exists $\varepsilon > 0$ such that:*

1. *(Global existence and modified scattering) Given* $\mu_1(x,v)$ *satisfying*

$$\|\mu_1\|_{L^2_{x,v}} + \left\|\langle x-v\rangle^2\mu_1\right\|_{L^\infty_{x,v}} + \|\nabla_{x,v}\mu_1\|_{L^\infty_{x,v}} \leq \varepsilon, \tag{3.1.3}$$

there exists a unique global strong solution μ *of the initial value problem for* (3.1.1) *with* $\mu(1,x,v)=\mu_1(x,v)$*. In addition, there exist* $\mu_\infty(x,v)$ *and* $E_\infty = E_\infty[\mu_\infty]$ *as in* (3.1.2) *such that, locally uniformly in* (x,v)*,*

$$\mu(t, x+tv-\lambda\ln(t)E_\infty(v), v) \to \mu_\infty(x,v), \qquad t\to+\infty. \tag{3.1.4}$$

2. *(Existence of modified wave operators) Given* $\mu_\infty \in W^{2,\infty}(\mathbb{R}^3_x\times\mathbb{R}^3_v)$ *and* $E_\infty = E_\infty[\mu_\infty]\in W^{3,\infty}(\mathbb{R}^3)$ *as in* (3.1.2) *satisfying*

$$\|\mu_\infty\|_{L^2_{x,v}} + \left\|\langle x\rangle^5\mu_\infty\right\|_{L^\infty_{x,v}} + \|\langle x\rangle\nabla_{x,v}\mu_\infty\|_{L^\infty_{x,v}} + \|\langle x\rangle^2\nabla^2_{x,v}\mu_\infty\|_{L^\infty_{x,v}} + \|E_\infty\|_{W^{3,\infty}} < \infty, \tag{3.1.5}$$

there exists a unique strong global solution μ *of* (3.1.1) *for which* (3.1.4) *holds.*

3. *(Scattering map) For any asymptotic state* $\mu_{-\infty}$ *with* $E_{-\infty} = E_\infty[\mu_{-\infty}] \in W^{3,\infty}(\mathbb{R}^3)$ *as in* (3.1.2)*,*

$$\|\mu_{-\infty}\|_{L^2_{x,v}} + \|\langle x,v\rangle^5\mu_{-\infty}\|_{L^\infty_{x,v}} + \|\langle x\rangle\nabla_{x,v}\mu_{-\infty}\|_{L^\infty_{x,v}} + \|\langle x\rangle^2\nabla^2_{x,v}\mu_{-\infty}\|_{L^\infty_{x,v}} \leq \varepsilon,$$

there exist a unique strong solution μ *of* (3.1.1)*,* $\mu_{+\infty}\in L^2_{x,v}\cap L^\infty_{x,v}$ *and* $E_{+\infty} = E_\infty[\mu_{+\infty}]$ *such that*

$$\mu(t, x+tv\mp\lambda\ln(\langle t\rangle)E_{\pm\infty}(v), v) \to \mu_{\pm\infty}(x,v), \qquad t\to\pm\infty. \tag{3.1.6}$$

We call the map defined in a neighborhood of the origin in the Schwartz space through 3 above,

$$\mathcal{S} : \mu_{-\infty} \mapsto \mu_{+\infty} \tag{3.1.7}$$

the *Scattering map*. We refer to Theorem 3.3.1 for a more precise statement of our results for 1 and to Theorem 3.4.1 for a more precise statement of 2. In particular we note that the force field has optimal decay $|\nabla\psi| \lesssim \langle t\rangle^{-2}$ in all cases.

Remark. We comment on some points:

1. The main novelty of this work is the construction of the wave operator 2. While the small data modified scattering dynamic (3.1.4) was already obtained in [91], the present result 1 is also of interest since it is stronger and the approach, while less generalizable, leads to a simple derivation of the asymptotic dynamic. We also refer to [114] for yet another point of view on the modified scattering as arising from mixing.

2. Our topology for small data/modified scattering in (3.1.5) is weaker than in all other works on asymptotic behavior that we are aware of [11,33,87,91,113,126, 130]. It is unclear what the optimal topology is, but to get almost Lipschitz bounds on the force field, by (3.1.9), one cannot work in a much weaker setting than ours.

3. We also obtain propagation of regularity: assuming more regularity on the initial data we obtain higher regularity on the final (scattering) data and vice-versa.

4. Our initial data for scattering may have infinite energy and momentum; in addition, a simple modification also allows for initial data of infinite mass. It is unclear which role (if any) the physical conservation laws play for the asymptotic behavior.

5. It is worth noting a curious fact: our proof can be adapted directly to the case of a plasma of two species (ions and electrons). In this case, using 2, one can construct solutions for which the asymptotic electric field profile $E_\infty \equiv 0$ vanishes and the solutions *scatter linearly*. In this case, the same equation allows two different asymptotic behaviors. It remains to be understood to which extent the linear scattering is nongeneric (say in case the total charge vanishes).

About the Proof of Theorem 3.1.1

In the spirit of the prior work [91] (see also [61, 62, 95, 96]), we build on parallels between kinetic and dispersive equations. In particular, the Hamiltonian structure of (3.1.1) guides our analysis.

The simplest case for asymptotic behavior of a nonlinear equation is *linear scattering* when the nonlinearity can simply be neglected to model asymptotic dynamics. For the Vlasov-Poisson system, this happens in the setting of Landau damping [14, 65, 106], the ion/screened problem [15, 82], and in higher dimensions [126], where solutions asymptotically satisfy $\mathcal{T}(\mu) = 0$ with $\mathcal{T}$ defined in (3.1.11). The asymptotic behavior of *modified scattering* as in (3.1.4) and (3.1.6) can be viewed as a manifestation of the unrelenting relevance of nonlinear interactions in (3.1.1) throughout time. In (3.1.1) the nonlinear, long-range interactions are governed by a force field which does not decay fast enough to produce only a finite correction as time tends to infinity and produces the logarithmic corrections identified in the above theorem – see also [33, 91, 114] for the Vlasov-Poisson setting, and [83, 84, 88, 93, 111] for related results on other equations.

In order to understand the asymptotic behavior, we need to (i) identify a mechanism for decay (here dispersion), (ii) prove global existence, (iii) isolate an asymptotic dynamic and (iv) prove convergence to it. We offload the dispersion to the pseudo-conformal transform $\mathcal{I}$ which compactifies time and reduces global existence to local existence for a singular equation in the transformed unknown

$$\gamma(s,q,p) := \mu(\frac{1}{s},\frac{q}{s},q-sp), \quad (s,q,p)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3, \tag{3.1.8}$$

see also [21,26,34,129] for similar ideas. At this point, the problem merely reduces to establishing convergence at the image of infinity, $s=0$, where, however, the equation has a violent singularity. We extend the force field $E=-\nabla\psi$ via a variant of the continuity equation:

$$\partial_s E + \nabla\Delta^{-1}\mathrm{div}(\mathbf{j}) = 0, \qquad \mathbf{j}(s,q) = \int p\gamma^2(s,q,p)dp, \tag{3.1.9}$$

which does not involve the (singular) acceleration and provides good control of E so long as we control some moments of γ. Once we obtain convergence of E to a fixed asymptotic field E_0, the equation becomes a simple perturbation of transport by a shear term:

$$\left(\partial_s + \lambda s^{-1}E_0(q)\cdot\nabla_p\right)\gamma = O(1),$$

which is easily integrated to recover the dynamic originally isolated in [91]. To make this rigorous, we need to propagate mild control on appropriate norms. This is done through a bootstrap that allows some deterioration over time in different ways depending on the scenarios: growth of nonconvergent norms in the case of modified scattering and loss of moment in the case of wave operators (where we start from the singular time $s=0$).

The proof of part 1 shows how natural the pseudo-conformal inversion $\mathcal{I}$ is to study asymptotics of (3.1.1): working with only moments that are conserved in the linear evolution of (3.1.1) one directly obtains global solutions in a bootstrap argument. Additional regularity as in (3.1.3) is easily propagated to yield unique strong solutions and to recover the asymptotic behavior (3.1.4) – see Section 3.3.

Part 2 is proved using a canonical change of variables in (3.1.14) to mitigate the strong singularity at $s = 0$ – see Section 3.4. The Cauchy problem for the resulting equations (3.4.7) can in fact be (locally) solved starting from $s = 0$ for a sufficiently large class of initial data as in (3.1.5). Again, moments are easily bootstrapped, while propagating derivatives requires us to identify a proper weighted norm which compensates for the *ill-conditioned* Hessian of the new Hamiltonian by allowing one loss of moment. Since via $\mathcal{I}$ this corresponds to a strong solution on $[T, \infty)$ for some $T > 0$, classical theory as in [98] then gives a global solution.

Finally 3 follows simply by combining 2 (backwards in time) to go from past-asymptotic data to initial data and 1 to go from initial data to future asymptotic data.

While it may be less intuitive, using the pseudo-conformal transformation simplifies the presentation over the physical space analysis as in [91], and quickly leads to the natural modified scattering behavior. It also sheds new light on some classical decay estimates like (3.1.15).

Open Questions

We list some open questions which remain outstanding:

- Is there a topology that makes the scattering operator in (3.1.7) an endomorphism?

- In the plasma case $\lambda = +1$, what is the asymptotic behavior for large data? Solutions are global, there are no nontrivial equilibriums and the wave operators are defined for large data, so it is tempting to believe that Theorem 3.1.1 may be extended to all solutions (see [89, 115, 118] and references therein for general results in this direction, and [113, 119] for the case of more symmetric data).

- In the gravitational case $\lambda = -1$, is there a "ground state", i.e. a smallest solution which does not scatter? Are there solutions which satisfy some form of modified scattering towards a nonzero stationary solution (of which there are many, see e.g. [75, 95, 105])? This appears very challenging, but we note [114] for an example of a stability result around a nonzero equilibrium in a related setting and [48, 77] for related works.

3.1.1 Pseudo-Conformal Inversion

We define the involution of $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ given by the *pseudo-conformal inversion* (see also [95])

$$\mathcal{I} : (t, x, v) \mapsto (1/t, x/t, x - tv). \tag{3.1.10}$$

This transformation interacts favorably with free streaming,

$$\mathcal{T} := \partial_t + v \cdot \nabla_x, \tag{3.1.11}$$

since heuristically it exchanges the role of v with that of $x - tv$, both of which are conserved along the evolution (i.e. commute with $\mathcal{T}$). Indeed, one can observe that if $(s, q, p) = \mathcal{I}(t, x, v)$,

$$\partial_s = -s^{-2}\left(\partial_t + q \cdot \nabla_x\right) - p \cdot \nabla_v, \qquad \nabla_q = s^{-1}\nabla_x + \nabla_v, \qquad \nabla_p = -s\nabla_v,$$

and

$$\mathcal{T}(f \circ \mathcal{I}) = -s^{-2}\mathcal{T}(f) \circ \mathcal{I}.$$

so that composition with $\mathcal{I}$ preserves the class of solutions of free streaming $\mathcal{T}f = 0$. The transformation $\mathcal{I}$ is almost symplectic in the sense that $dq \wedge dp = -dx \wedge dv$, and in particular the total charge is preserved:

$$\iint (f \circ \mathcal{I})^2 dqdp = \iint f^2 dxdv.$$

Recasting Vlasov-Poisson

Given a solution $\mu(t, x, v)$ of (3.1.1), we let $\gamma = \mu \circ \mathcal{I}$, so that

$$\gamma(s, q, p) := \mu(\frac{1}{s}, \frac{q}{s}, q - sp), \qquad \mu(t, x, v) = \gamma(\frac{1}{t}, \frac{x}{t}, x - tv). \tag{3.1.12}$$

The Vlasov-Poisson system involves a perturbation of free streaming (3.1.11) by a force field (in this paper, we stick to the plasma terminology and refer to it as the "Electric field"):

$$E[\mu](t, x) := \nabla_x \Delta_x^{-1} \int \mu^2(t, x, v)dv = \frac{1}{4\pi} \iint \frac{x - y}{|x - y|^3} \cdot \mu^2(t, y, v)dvdy, \tag{3.1.13}$$

which also transforms naturally:

$$E[\mu](t,tx) = \frac{1}{t^2}E[\gamma](\frac{1}{t},x),$$

and we see that μ solves (3.1.1) on $0 \leq T_* \leq t \leq T^*$ if and only if γ satisfies for $0 \leq (T^*)^{-1} \leq s \leq (T_*)^{-1}$

$$\left(\partial_s + p\cdot\nabla_q\right)\gamma + \lambda s^{-1}E[\gamma]\cdot\nabla_p\gamma = 0. \tag{3.1.14}$$

Remark. The natural energy estimate for (3.1.14) is

$$-s^2\frac{d}{ds}\left(\iint |p|^2\gamma^2(s,q,p)\,dqdp + \frac{\lambda}{s}\int |E[\gamma](q)|^2\,dq\right) = \lambda\int |E[\gamma](q)|^2\,dq \tag{3.1.15}$$

which, after rescaling, recovers one of the main integral estimates in [89, 115] and leads, for $\lambda > 0$, to the optimal control of $E[\gamma] \in L^\infty_s L^2_q$.[2]

3.2 The Force Field and the Continuity Equation

To prove both the modified scattering and wave operator theorems, we require general estimates on the electric field E defined in (3.1.13). In Lemma 3.2.1, we prove fix-time bounds on the operator $\gamma \mapsto E$. In Lemma 3.2.2 we obtain dynamic bounds for an electric field $E = E[\gamma]$ provided γ satisfies (3.2.8), a slight strengthening of the continuity equation.

Lemma 3.2.1. *Let $\gamma = \gamma(q,p)$ be such that $\gamma \in L^2_{q,p}$, $\langle p\rangle^2\gamma \in L^\infty_{q,p}$ and $\nabla_q\gamma \in L^\infty_{q,p}$*

[2]This in turn implies the optimal decay rate of $\|E[\mu](t)\|_{L^2_x} \lesssim \langle t\rangle^{-1/2}$ in the original variables.

and $E = E[\gamma]$ *defined by* (3.1.13)*. For all* $A > 0$ *and* $\kappa \in (0, 1/3)$ *we have*

$$\begin{aligned} \|E\|_{L^\infty_q} &\lesssim A\left[\|\gamma\|^2_{L^2_{q,p}} + \|\gamma\|^2_{L^\infty_{q,p}}\right] + A^{-1}\||p|^2\gamma\|^2_{L^\infty_{q,p}}, \\ \|\nabla_q E(s)\|_{L^\infty_q} &\lesssim A\|\gamma\|^2_{L^2_{q,p}} + A^{-\frac{\kappa}{3}}\||p|^2\gamma\|^2_{L^\infty_{q,p}} + A^{\kappa-\frac{1}{3}}\|\gamma\|_{L^\infty_{q,p}}\|\nabla_q\gamma\|_{L^\infty_{q,p}}. \end{aligned} \tag{3.2.1}$$

In fact, we will mostly make use of the second line of (3.2.1) corresponding to the choice $A = \langle \ln(s)\rangle^4$, $\kappa = \frac{1}{30}$, i.e. the bound

$$\|\nabla_q E(s)\|_{L^\infty_q} \lesssim \langle \ln(s)\rangle^4\|\gamma\|^2_{L^2_{q,p}} + \||p|^2\gamma\|^2_{L^\infty_{q,p}} + \langle \ln(s)\rangle^{-\frac{6}{5}}\|\gamma\|_{L^\infty_{q,p}}\|\nabla_q\gamma\|_{L^\infty_{q,p}}. \tag{3.2.2}$$

Remark. In the estimates of this section, up to minor modifications one may alternatively work with the $\langle p\rangle^{-1}L^4_{q,p}$ norm of γ, rather than its $L^2_{q,p}$ norm. This allows to consider initial data with infinite mass – see also Remark 3.1 (4).

Proof of Lemma 3.2.1. We decompose the electric field on different scales using a radially symmetric function $\chi \in C^\infty_c(\{1/2 \le |y| \le 2\})$ with $\int_{\mathbb{R}^3}\chi(y)dy = 1$, namely

$$E^j[\gamma](q) = c\int_{R=0}^\infty E^j_R(q)\frac{dR}{R^2}, \qquad E^j_R[\gamma](q) := \iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r))\cdot\gamma^2(r,u)\,drdu,$$

and we directly obtain the following elementary bounds

$$|E^j_R| \lesssim R^{-1}\|\gamma\|^2_{L^2_{q,p}}, \qquad |\partial_q E^j_R| \lesssim R^{-2}\|\gamma\|^2_{L^2_{q,p}}, \tag{3.2.3}$$

which is enough for large R. To go further, we introduce

$$E^j_{R,V}[\gamma](q) := \iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r))\cdot\chi(V^{-1}u)\cdot\gamma^2(r,u)\,drdu,$$

with $E^j[\gamma](q) = c\int_{R=0}^{\infty}\int_{V=0}^{\infty} E^j_{R,V}(q)\frac{dV}{V}\frac{dR}{R^2}$ and we estimate

$$\begin{aligned} |E^j_{R,V}| &\lesssim R^2\min\{V^3\|\gamma\|^2_{L^\infty_{q,p}}, V^{-1}\||p|^2\gamma\|^2_{L^\infty_{q,p}}\}, \\ |\partial_q E^j_{R,V}| &\lesssim R\min\{V^{-1}\||p|^2\gamma\|^2_{L^\infty_{q,p}}, RV^3\|\nabla_q\gamma\|_{L^\infty_{q,p}}\|\gamma\|_{L^\infty_{q,p}}\}. \end{aligned} \tag{3.2.4}$$

From this we deduce that

$$\begin{aligned} |E^j[\gamma]| &\lesssim \int_{R=A}^{\infty}|E^j_R|\frac{dR}{R^2} + \int_{R=0}^{A}\int_{V=0}^{B}|E^j_{R,V}|\frac{dR}{R^2}\frac{dV}{V} + \int_{R=0}^{A}\int_{V=B}^{\infty}|E^j_{R,V}|\frac{dR}{R^2}\frac{dV}{V} \\ &\lesssim A^{-2}\|\gamma\|^2_{L^2_{q,p}} + AB^3\|\gamma\|^2_{L^\infty_{q,p}} + AB^{-1}\||p|^2\gamma\|^2_{L^\infty_{q,p}} \end{aligned}$$

and choosing $A = B^{-1}$, we obtain the first line of (3.2.1). Similarly, we see that for $\kappa \in (0, 1/3)$

$$\begin{aligned} |\partial_q E^j| &\lesssim \int_{R=A}^{\infty}|\partial_q E^j_R|\frac{dR}{R^2} + \int_{R=0}^{A}\int_{V=0}^{R^{-\kappa}}|\partial_q E^j_{R,V}|\frac{dV}{V}\frac{dR}{R^2} + \int_{R=0}^{A}\int_{V=R^{-\kappa}}^{\infty}|\partial_q E^j_{R,V}|\frac{dV}{V}\frac{dR}{R^2} \\ &\lesssim A^{-3}\|\gamma\|^2_{L^2_{q,p}} + A^{\kappa}\||p|^2\gamma\|^2_{L^\infty_{q,p}} + A^{1-3\kappa}\|\gamma\|_{L^\infty_{q,p}}\|\nabla_q\gamma\|_{L^\infty_{q,p}}. \end{aligned}$$

After substituting A with $A^{-1/3}$, this gives the second line of (3.2.1). $\square$

Lemma 3.2.2. *Fix $0 < s_0 < s_1$ and let $\gamma \in L^\infty_s([s_0, s_1]; L^2_{q,p})$.*

1. *Assuming that $E = E[\gamma]$ satisfies (3.1.9), we see that*

$$\|E(s_1) - E(s_0)\|_{L^\infty_q} \lesssim \langle \ln(s_1 - s_0)\rangle (s_1 - s_0)\|\mathbf{j}\|_{L^\infty_{s,q}} + (s_1 - s_0)^2 \left[\|\langle p\rangle^2\gamma\|^2_{L^\infty_{s,q,p}} + \|\gamma\|^2_{L^\infty_s L^2_{q,p}}\right]. \tag{3.2.5}$$

We also have the corresponding estimate for $\nabla_q E = \nabla_q E[\gamma]$:

$$\|\nabla_q E(s_1) - \nabla_q E(s_0)\|_{L^\infty_q} \lesssim \langle \ln(s_1 - s_0)\rangle (s_1 - s_0)\|\nabla_q \mathbf{j}\|_{L^\infty_{s,q}} + (s_1 - s_0)^2 \left[\|\langle p\rangle^4\gamma\|^2_{L^\infty_{s,q,p}} + \|\nabla_q\gamma\|^2_{L^\infty_{s,q,p}} + \|\gamma\|^2_{L^\infty_s L^2_{q,p}}\right], \tag{3.2.6}$$

from which we deduce

$$\|\nabla_q E(s_1) - \nabla_q E(s_0)\|_{L^\infty_q} \lesssim \langle \ln(s_1 - s_0)\rangle (s_1 - s_0) \left[\|\langle p\rangle^5\gamma\|^2_{L^\infty_{s,q,p}} + \|\nabla_q\gamma\|^2_{L^\infty_{s,q,p}} + \|\gamma\|^2_{L^\infty_s L^2_{q,p}}\right]. \tag{3.2.7}$$

2. *If γ satisfies a slight strengthening of the continuity equation (3.1.9), namely*

$$\partial_s \left\{\gamma^2\right\} + div_q \left\{p\gamma^2\right\} + div_p\{F\gamma^2\} = 0 \tag{3.2.8}$$

for some force-field $F(s, q)$, then for $E = E[\gamma]$ there holds that

$$\|E(s_1) - E(s_0)\|_{L^\infty_q} \lesssim \langle \ln(s_1 - s_0)\rangle^2 (s_1 - s_0)\||p|^2\gamma\|^2_{L^\infty_{s,q,p}} + (s_1 - s_0)^2 \left[\|\gamma\|^2_{L^\infty_{s,q,p}} + \|\gamma\|^2_{L^\infty_s L^2_{q,p}}\right] + (s_1 - s_0)^3 \langle \ln(s_1/s_0)\rangle \|\langle p\rangle^2\gamma\|^2_{L^\infty_{s,q,p}}\|sF\|_{L^\infty_{s,q}}. \tag{3.2.9}$$

Proof. We start with *2*: using (3.2.3) and (3.2.4) we see that for $s \in \{s_0, s_1\}$,

$$\int_{R=A^{-1}}^{\infty} |E_R(s)| \frac{dR}{R^2} \lesssim A^2 \|\gamma(s)\|^2_{L^2_{q,p}}, \qquad \int_{R=0}^{A^2} |E_R(s)| \frac{dR}{R^2} \lesssim A^2 \|\langle p\rangle^2 \gamma(s)\|^2_{L^\infty_{q,p}}, \tag{3.2.10}$$

and

$$\int_{R=0}^{A^{-1}} \int_{V=0}^{B} |E_{R,V}(s)| \frac{dR}{R^2}\frac{dV}{V} \lesssim A^{-1}B^3 \|\gamma(s)\|^2_{L^\infty_{q,p}},$$
$$\int_{R=0}^{A^{-1}} \int_{V=B^{-3}}^{\infty} |E_{R,V}(s)| \frac{dR}{R^2}\frac{dV}{V} \lesssim A^{-1}B^3 \||p|^2\gamma(s)\|^2_{L^\infty_{q,p}},$$

and we conclude that

$$|E(s) - \int_{R=A^2}^{A^{-1}} E_{R,a}(s)\frac{dR}{R^2}| \leq A^{-1}B^3\|\langle p\rangle^2\gamma\|^2_{L^\infty_{q,p}} + A^2\left[\|\gamma\|^2_{L^2_{q,p}} + \|\langle p\rangle^2\gamma\|^2_{L^\infty_{q,p}}\right],$$
$$E_{R,a} := \iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r)) \cdot \chi_{\{B\leq\cdot\leq B^{-3}\}}(u) \cdot \gamma^2(r,u)\, drdu,$$

where

$$\chi_{\{B\leq\cdot\leq B^{-3}\}}(u) = \int_{\{B\leq V\leq B^{-3}\}} \chi(V^{-1}u)\frac{dV}{V}.$$

On the other hand, using the equation (3.2.8) we find that

$$\begin{aligned} 0 &= \int_{s=s_0}^{s_1} \iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r)) \cdot \chi_{\{B\leq\cdot\leq B^{-3}\}}(u) \cdot \left\{\partial_s\gamma^2 + \mathrm{div}_r(\gamma^2 u) + \mathrm{div}_u(F\gamma^2)\right\} drduds,\\ &= E_{R,a}(s_1) - E_{R,a}(s_0) + \int_{s=s_0}^{s_1} \iint R^{-2}u^k\{\partial_{q^j}\partial_{q^k}\chi\}(R^{-1}(q-r)) \cdot \chi_{\{B\leq\cdot\leq B^{-3}\}}(u) \cdot \gamma^2(s,r,u)\, drduds,\\ &\quad - \int_{s=s_0}^{s_1} \iint R^{-1}\partial_{q^j}\chi(R^{-1}(q-r)) \cdot \gamma^2(s,r,u) \cdot (F\cdot\nabla_u)\chi_{\{B\leq\cdot\leq B^{-3}\}}(u)\, drduds. \end{aligned} \tag{3.2.11}$$

Since

$$\left|\nabla_u \chi_{\{B\leq\cdot\leq B^{-3}\}}(u)\right| \lesssim B^{-1}\mathbf{1}_{\{|u|\leq 2B\}} + B^3\mathbf{1}_{\{|u|\geq B^{-3}/2\}},$$

we see that

$$\left|\iint R^{-1}\partial_{q^j}\chi(R^{-1}(q-r))\cdot\gamma^2(r,u)\cdot(F\cdot\nabla_u)\chi_{\{B\leq\cdot\leq B^{-3}\}}(u)drdu\right|$$
$$\lesssim\|sF\|_{L^\infty_q}\cdot s^{-1}R^2\cdot\left[B^2\|\gamma\|^2_{L^\infty_{q,p}}+B^6\||p|^2\gamma\|^2_{L^\infty_{q,p}}\right]$$

and using a crude bound for the second integral in (3.2.11), we find that

$$|\int_{R=A^2}^{A^{-1}}\{E_{R,a}(s_1)-E_{R,a}(s_0)\}\frac{dR}{R^2}|\lesssim(s_1-s_0)\cdot\||u|^2\gamma\|^2_{L^\infty_{s,r,u}}\cdot\int_{R=A^2}^{A^{-1}}\frac{dR}{R}\cdot\int_{V=B}^{B^{-3}}\frac{dV}{V}$$
$$+\langle\ln(s_1/s_0)\rangle\cdot\|sF\|_{L^\infty_{s,q}}\cdot A^{-1}B^2\|\langle p\rangle^2\gamma\|^2_{L^\infty_{s,q,p}}.$$

Letting $B=A^2=(s_1-s_0)^2$, we obtain the result. For the variant (3.2.5), we do not localize in u. In this case, we need only use (3.2.10) and the last term in (3.2.11) simplifies. We detail this in the similar analysis of $\nabla_q E$ in (i) below.

For *1* we use a similar analysis without localizing in u. Passing the derivative onto γ gives

$$\int_{R=A^{-\frac{2}{3}}}^{\infty}|\nabla_q E_R(s)|\frac{dR}{R^2}\lesssim A^2\|\gamma(s)\|^2_{L^2_{q,p}},\qquad\int_{R=0}^{A^2}|\nabla_q E_R(s)|\frac{dR}{R^2}\lesssim A^2\|\nabla_q\gamma(s)\|_{L^\infty_{q,p}}\cdot\|\langle p\rangle^4\gamma(s)\|_{L^\infty_{q,p}},$$

and the continuity equation (3.2.8) gives

$$0=\int_{s=s_0}^{s_1}\iint R^{-1}\{\partial_{q^j}\chi\}(R^{-1}(q-r))\cdot\partial_j\left\{\partial_s\gamma^2+\mathrm{div}_r(\gamma^2u)\right\}\,drduds$$
$$=\partial_j E_R(s_1)-\partial_j E_R(s_0)+2\int_{s=s_0}^{s_1}\iint R^{-2}u^k\{\partial_{q^j}\partial_{q^k}\chi\}(R^{-1}(q-r))\cdot\gamma\cdot\nabla_q\gamma(r,u)\,drduds,$$

from which we deduce that

$$\|\nabla_q E_R(s_1)-\nabla_q E_R(s_0)\|_{L^\infty_{q,p}}\lesssim(s_1-s_0)\cdot R\cdot\|\langle p\rangle^5\gamma\|_{L^\infty_{s,q,p}}\|\nabla_q\gamma\|_{L^\infty_{s,q,p}}$$

and integrating in $A^2\leq R\leq A^{-1}$, we obtain (3.2.7). □

Finally we collect the modifications of Lemma 3.2.1 and 3.2.2 above needed to consider smoother solutions. The proofs are similar (passing the derivative through the density) and are omitted.

Lemma 3.2.3. *There holds that for all* $\kappa \in (0, \frac{1}{3})$,

$$\|\nabla_q^2 E\|_{L_q^\infty} \lesssim A\|\gamma\|_{L_{q,p}^2}^2 + A^{-\frac{\kappa}{4}} \||p|^4\gamma\|_{L_{q,p}^\infty} \|\nabla_q \gamma\|_{L_{q,p}^\infty} + A^{\frac{3\kappa-1}{4}} \|\gamma\|_{L_{q,p}^\infty} \|\nabla_q^2 \gamma\|_{L_{q,p}^\infty}$$

and

$$\begin{aligned} \|\nabla_q^2 E(s_1) - \nabla_q^2 E(s_0)\|_{L_q^\infty} &\lesssim \langle \ln(s_1 - s_0)\rangle (s_1 - s_0) \|\nabla_q^2 \mathbf{j}\|_{L_{s,q}^\infty} \\ &+ (s_1 - s_0)^2 \left[\|\gamma\|_{L_s^\infty L_{q,p}^2}^2 + \|\langle p\rangle^5 \gamma\|_{L_{s,q,p}^\infty} \|\nabla_q^2 \gamma\|_{L_{s,q,p}^\infty} + \|\langle p\rangle^{2.1} \nabla_q \gamma\|_{L_{s,q,p}^\infty}^2 \right]. \end{aligned}$$

3.3 Modified Scattering

While we only need to study (3.1.14) on a compact time interval, this equation is now time dependent with a violent singularity at $s = 0$. This can be mitigated since the singular terms

$$\left(\partial_s + \lambda s^{-1} E(s,q) \cdot \nabla_p\right) \gamma = l.o.t.$$

can be integrated to main order:

$$\Gamma(s,q,p) = \gamma(s,q,p + \lambda \int_{s'=1}^{s} E(s',q) \frac{ds'}{s'}), \qquad \gamma(s,q,p) = \Gamma(s,q,p - \lambda \int_{s'=1}^{s} E(s',q) \frac{ds'}{s'}). \tag{3.3.1}$$

Since Γ satisfies an equivalent but more cumbersome equation, we prefer to work with (3.1.14) to bootstrap control of the norms, but a variant of (3.3.1) leads quickly to the modified dynamics (3.3.5) once E is shown to converge.

The main result of this section is the following statement about modified scattering:

Theorem 3.3.1. *There exists* $\varepsilon > 0$ *such that if* $\gamma_1(q,p)$ *satisfies*

$$\|\gamma_1\|_{L^2_{q,p}} + \|\langle p\rangle^2\gamma_1\|_{L^\infty_{q,p}} + \|\nabla_{p,q}\gamma_1\|_{L^\infty_{q,p}} \leq \varepsilon_0 \leq \varepsilon, \tag{3.3.2}$$

then there exists a unique solution γ *of* (3.1.14) *with "initial" data* $\gamma(s=1) = \gamma_1$ *for all times* $0 < s \leq 1$*, and* $\gamma \in L^\infty_s((0,1], L^\infty_{q,p} \cap L^2_{q,p})$ *satisfies*

$$\left\|\langle p\rangle^2\gamma(s)\right\|_{L^\infty_{q,p}} \lesssim \varepsilon_0\langle \ln(s)\rangle^2, \qquad \|\nabla_{p,q}\gamma(s)\|_{L^\infty_{q,p}} \lesssim \varepsilon_0\langle\ln(s)\rangle^5. \tag{3.3.3}$$

If in addition

$$\|\langle p\rangle\nabla_{p,q}\gamma_1\|_{L^\infty_{q,p}} \leq \varepsilon_0, \tag{3.3.4}$$

then $\|\langle p\rangle\nabla_{p,q}\gamma(s)\|_{L^\infty_{q,p}} \lesssim \varepsilon_0\langle\ln(s)\rangle^6$ *and there exist* $E_0 = E[\gamma_0] \in L^\infty_q$ *and* $\gamma_0 \in L^\infty_{q,p}$ *such that, uniformly in* q, p,

$$\gamma(s, q + ps + \lambda s\ln(s)E_0(q), p + \lambda\ln(s)E_0(q)) \to \gamma_0(q,p), \qquad s \to 0. \tag{3.3.5}$$

Remark. We comment on some points of interest:

1. In fact, as we will show below one can obtain global solutions in a bootstrap argument involving only the moments $\langle p\rangle^2\gamma$. The higher regularity of (3.3.2) is only used to make sense of the equations in a stronger sense.

2. The assumption (3.3.4) is used to guarantee the convergence (3.3.5). We note that this statement is slightly different from the one in Theorem 3.1.1, in that in (3.3.4) we start with *uniform* control of one additional moment in p on the gradients and obtain *uniform* (rather than local) convergence in (3.3.5). The

proofs are easily adapted to establish the corresponding local statement under local assumptions as in Theorem 3.1.1.

3. Our proof of Theorem 3.3.1 shows that control of higher moments (in both p and q) as well as higher regularity can be propagated. For higher moments in p this is explicitly done in Proposition 3.3.3, and from this the propagation of moments in q follows by the commutation relations (3.3.11). For higher regularity, by (3.3.11) one needs control of derivatives of the electric field; these in turn can be directly bounded by derivatives of γ via an adaptation of Lemmas 3.2.1 and 3.2.2 (see e.g. Lemma 3.2.3 for one additional derivative). As a consequence, given more regularity and/or moments on a solution, the convergence (3.3.5) can then be shown to hold in a correspondingly strengthened topology.

4. The convergence (3.3.5) implies the asymptotic dynamic (3.1.4) of Theorem 3.1.1: Letting

$$\mathcal{A} : (s, q, p) \mapsto (s, q + ps + \lambda s \ln(s) E_0(q), p + \lambda \ln(s) E_0(q)), \tag{3.3.6}$$

by $\mathcal{I}^2 = Id$ there holds that

$$\gamma \circ \mathcal{A}(s, q, p) = \mu \circ (\mathcal{I} \circ \mathcal{A})(s, q, p) = \mu \left(\frac{1}{s}, \frac{q}{s} + p + \lambda \ln(s) E_0(q), q \right), \tag{3.3.7}$$

which gives (3.1.4) with $\mu_\infty(x, v) = \gamma_0(v, x)$ by relabeling the arguments.

The proof of Theorem 3.3.1 makes frequent use of the fact that (3.1.14) is a transport equation and we can propagate uniform bounds using the maximum principle along the characteristics. In particular, writing

$$\mathcal{L} := \partial_s + p \cdot \nabla_q + \lambda s^{-1} E \cdot \nabla_p, \qquad \mathcal{L}[f] = \partial_s f + \mathrm{div}_{q,p} \left\{ (p, \lambda s^{-1} E(q)) \cdot f \right\} \tag{3.3.8}$$

we have that if h is a strong solution in a neighborhood of $s = 1$ to

$$\mathcal{L}[h] = F(s, q, p) \tag{3.3.9}$$

with $h(1) \in L^r_{q,p}$ for some $r \geq 1$, then since the transport field is divergence free there holds that

$$\|h(s)\|_{L^r_{q,p}} \leq \|h(1)\|_{L^r_{q,p}} + \int_s^1 \|F(s')\|_{L^r_{q,p}} \, ds' \tag{3.3.10}$$

for all $0 \leq s \leq 1$ in the interval of existence.

3.3.1 Commutation Relations

Now consider a solution γ to (3.1.14), i.e. $\mathcal{L}[\gamma] = 0$. In order to decide which equation we want to use, it will be convenient to compute some commutation relations: For any $m, n \in \{1, 2, 3\}$ we have

$$\begin{aligned} &\mathcal{L}[q^m\gamma] = \mathcal{L}[q^m]\gamma = p^m\gamma, \qquad \mathcal{L}[p^m\gamma] = \lambda s^{-1}E^m\gamma, \\ &\mathcal{L}[\partial_{q^m}\gamma] = \partial_{q^m}(\mathcal{L}[\gamma]) - (\partial_{q^m}\mathcal{L})[\gamma] = -\lambda s^{-1}\partial_{q^m}E^j\partial_{p^j}\gamma, \qquad \mathcal{L}[\partial_{p^m}\gamma] = -\partial_{q^m}\gamma, \end{aligned} \tag{3.3.11}$$

and we also remark that

$$\begin{aligned} \mathcal{L}[p^m\partial_{q^n}\gamma] &= -\lambda s^{-1}p^m\partial_{q^n}E^j\partial_{p^j}\gamma + \lambda s^{-1}E^m\partial_{q^n}\gamma, \\ \mathcal{L}[p^m\partial_{p^n}\gamma] &= -p^m\partial_{q^n}\gamma + \lambda s^{-1}E^m\partial_{p^n}\gamma. \end{aligned} \tag{3.3.12}$$

3.3.2 Bootstrap and Global Existence

As a first step we see that so long as the electric field remains bounded, we can propagate all the moments we want.

Lemma 3.3.2. *Let γ be a strong solution of (3.1.14) on $T^* \leq s \leq 1$ with "initial" data $\gamma(s=1) = \gamma_1$. Assume that γ_1 satisfies for some $a \in \mathbb{N}$, $r \in [2, \infty]$ that*

$$\|\langle p\rangle^a \gamma_1\|_{L^r_{q,p}} \leq \varepsilon_0, \tag{3.3.13}$$

and that

$$|E(s,q)| \leq D, \qquad T^* \leq s \leq 1. \tag{3.3.14}$$

Then there holds that

$$\begin{aligned} \|\gamma(s)\|_{L^r_{q,p}} &\leq \varepsilon_0, \\ \|\langle p\rangle^a \gamma(s)\|_{L^r_{q,p}} &\leq \varepsilon_0 + aD\varepsilon_0 \langle \ln(s)\rangle^a. \end{aligned} \tag{3.3.15}$$

Proof. The proof follows by applying (3.3.11) and (3.3.10) inductively to $p^\beta \gamma$, $\beta \in \mathbb{N}_0^3$, with $|\beta| \leq a$. □

Proposition 3.3.3. *Let $0 < \varepsilon_0 \leq \varepsilon_1 \ll 1$, and let γ be a solution of (3.1.14) on $T^* \leq s \leq 1$ with "initial" data $\gamma(s=1) = \gamma_1$ satisfying*

$$\|\gamma_1\|_{L^2_{q,p}} + \|\gamma_1\|_{L^\infty_{q,p}} \leq \varepsilon_0. \tag{3.3.16}$$

1. *(Moments and the electric field) If there holds that*

$$\|\langle p\rangle \gamma_1\|_{L^2_{q,p}} + \|\langle p\rangle^m \gamma_1\|_{L^\infty_{q,p}} \leq \varepsilon_1, \qquad m \geq 2 \tag{3.3.17}$$

then the electric field $E(s)$ remains bounded and the solution satisfies the bounds

$$\begin{aligned} \|\langle p\rangle\gamma(s)\|_{L^2_{q,p}} &\lesssim \varepsilon_1\langle\ln s\rangle, \\ \|\langle p\rangle^a\gamma(s)\|_{L^\infty_{q,p}} &\lesssim \varepsilon_1\langle\ln s\rangle^a, \qquad 0\le a\le m. \end{aligned} \tag{3.3.18}$$

Moreover, there exists $C>0$ (independent of T^*) such that for any $T^*\le s_1\le s_2\le 1$ there holds

$$|E(s_1,q)-E(s_2,q)|\le C\varepsilon_1^2\,\langle\ln(s_1)\rangle^4\langle\ln(s_2-s_1)\rangle^2(s_2-s_1). \tag{3.3.19}$$

2. *(Derivatives) Assume additionally that for some $b\in\{0,1\}$ there holds that*

$$\left\|\langle p\rangle^b\nabla_{p,q}\gamma_1\right\|_{L^\infty_{q,p}}\le\varepsilon_1. \tag{3.3.20}$$

Then we have the bounds

$$\begin{aligned} \|\langle p\rangle^a\nabla_p\gamma(s)\|_{L^\infty_{q,p}} &\lesssim \varepsilon_1\langle\ln s\rangle^a, \qquad 0\le a\le b, \\ \|\langle p\rangle^a\nabla_q\gamma(s)\|_{L^\infty_{q,p}} &\lesssim \varepsilon_1\langle\ln s\rangle^{5+a}, \qquad 0\le a\le b. \end{aligned} \tag{3.3.21}$$

Proof. We start by establishing claim (1). Let $C>0$ be a constant larger than all the implied constants appearing in Section 3.2 and let ε_1 be small enough so that

$$4C^2\varepsilon_1^2\le 1. \tag{3.3.22}$$

We make the following bootstrap assumption: Let $I\subset[T^*,1]$ be such that for $s\in I$ there holds

$$\|E(s)\|_{L^\infty_q}\le 2C^2\varepsilon_1^2. \tag{3.3.23}$$

By the first line of (3.2.1) (with $A=1$) and the assumptions (3.3.16), (3.3.17) we have that $1\in I\neq\emptyset$, and by continuity I is closed in $[T^*,1]$. To establish the claim it then suffices to prove that (3.3.23) holds with strictly smaller constants, implying that I is also open in $[T^*, 1]$.

To this end, note that by Lemma 3.3.2 we have that for $0 \le a \le m$,

$$\|\langle p\rangle^a \gamma(s)\|_{L^\infty_{q,p}} \le \varepsilon_1(1 + aC^2\varepsilon_1^2)\langle \ln(s)\rangle^a, \quad s \in I. \tag{3.3.24}$$

By Lemma 3.2.2 and (3.3.22), it then follows for $T^* \le s_1 \le s_2 \le 1$ that

$$\|E(s_1) - E(s_2)\|_{L^\infty_q} \le 4C\varepsilon_1^2 \left[\langle \ln(s_2 - s_1)\rangle^2 + \varepsilon_1^2\langle \ln(s_2/s_1)\rangle\right] \langle \ln(s_1)\rangle^4 (s_2 - s_1) \tag{3.3.25}$$

and (3.3.19) is proved. In particular, when $2^{-k} \le s_1 \le s_2 \le 2^{1-k}$, $k \ge 1$,

$$\|E(s_1) - E(s_2)\|_{L^\infty_q} \le 10C\varepsilon_1^2 k^6 2^{-k} \tag{3.3.26}$$

and since by (3.2.1) we have $\|E(1)\|_{L^\infty_q} \le 2C\varepsilon_1^2$, we see that for $s \in I$

$$\|E(s)\|_{L^\infty_q} \le \|E(1)\|_{L^\infty_q} + 10C\varepsilon_1^2 \sum_{k\ge 1} k^6 2^{-k} \ll C^2\varepsilon_1^2, \tag{3.3.27}$$

provided C is large enough.

To prove (2), we use a similar bootstrap argument based on the assumptions

$$\begin{aligned} \|\langle p\rangle^b \nabla_p \gamma(s)\|_{L^\infty_{q,p}} &\le 2C^4\varepsilon_1\langle \ln(s)\rangle^b, \\ \|\langle p\rangle^b \nabla_q \gamma(s)\|_{L^\infty_{q,p}} &\le 2C^2\varepsilon_1\langle \ln(s)\rangle^{5+b}. \end{aligned} \tag{3.3.28}$$

Using the commutation relations (3.3.11) and (3.3.10), we deduce from (3.3.20) and

(3.3.28) that

$$\left\|\nabla_p\gamma(s)\right\|_{L^\infty_{q,p}} \leq \left\|\nabla_p\gamma(1)\right\|_{L^\infty_{q,p}} + \int_s^1 \left\|\nabla_q\gamma(s')\right\|_{L^\infty_{q,p}} ds' \leq C^3 2\varepsilon_1, \tag{3.3.29}$$

provided $C > 0$ is large enough.

From the transport bounds and the commutation relations (3.3.11) we then deduce the estimate for $\nabla_q\gamma$: From (3.2.2) we have under our assumptions (3.3.23) and with (3.3.24) and (3.3.29) that

$$\begin{aligned}
\left\|\nabla_q\gamma(s)\right\|_{L^\infty_{q,p}} &\leq \left\|\nabla_q\gamma(1)\right\|_{L^\infty_{q,p}} + \int_s^1 \left\|\nabla_q E\right\|_{L^\infty_q} \left\|\nabla_p\gamma\right\|_{L^\infty_{q,p}} \frac{ds'}{s'} \\
&\leq \varepsilon_1 + \int_s^1 \left[\langle \ln s'\rangle^4 \left\|\gamma\right\|^2_{L^2_{q,p}} + \left\||p|^2\gamma\right\|^2_{L^\infty_{q,p}} + \langle \ln s'\rangle^{-\frac{6}{5}} \left\|\gamma\right\|_{L^\infty_{q,p}} \left\|\nabla_q\gamma\right\|_{L^\infty_{q,p}}\right] \left\|\nabla_p\gamma\right\|_{L^\infty_{q,p}} \frac{ds'}{s'} \\
&\leq \varepsilon_1 + \langle \ln s\rangle^5 \left(\varepsilon_0^2 + 4\varepsilon_1^2\right) \cdot 2C^4\varepsilon_1 + 2C^4\varepsilon_0\varepsilon_1 \int_s^1 \langle \ln s'\rangle^{-\frac{6}{5}} \left\|\nabla_q\gamma(s')\right\|_{L^\infty_{q,p}} \frac{ds'}{s'},
\end{aligned} \tag{3.3.30}$$

so that by Grönwall's lemma there holds that

$$\left\|\nabla_q\gamma(s)\right\|_{L^\infty_{q,p}} \leq 10\varepsilon_1 \langle \ln s\rangle^5, \tag{3.3.31}$$

provided ε_1 is small enough. A similar argument using (3.3.12), (3.3.23) and (3.3.28) shows that

$$\begin{aligned}
\left\|p^m\partial_{p^n}\gamma(s)\right\|_{L^\infty_{q,p}} &\leq \left\||p|\nabla_p\gamma(1)\right\|_{L^\infty_{q,p}} + \int_s^1 \left\||p|\nabla_q\gamma(s')\right\|_{L^\infty_{q,p}} ds' + \int_s^1 \left\|E\right\|_{L^\infty_q} \left\|\nabla_p\gamma(s')\right\|_{L^\infty_{q,p}} \frac{ds'}{s'} \\
&\leq C^3\varepsilon_1\langle \ln s\rangle.
\end{aligned} \tag{3.3.32}$$

For the last bound, we see from (3.3.12) that we need a bound on the derivative of the electric field. Using (3.2.2), (3.3.24) and (3.3.31), we find that

$$
\begin{aligned}
\|\nabla_q E(s)\|_{L^\infty_q} &\leq C\left[\langle \ln s\rangle^4 \|\gamma(s)\|^2_{L^2_{q,p}} + \left\|p^2\gamma(s)\right\|^2_{L^\infty_{q,p}} + \langle \ln s\rangle^{-\frac{6}{5}} \|\gamma(s)\|_{L^\infty_{q,p}} \|\nabla_q\gamma(s)\|_{L^\infty_{q,p}}\right] \\
&\leq C\left(\varepsilon_0^2 + 4\varepsilon_1^2\right)\langle \ln(s)\rangle^4 + 10C\varepsilon_0\varepsilon_1\langle \ln s\rangle^{5-\frac{6}{5}} \\
&\leq 10C\varepsilon_1^2\langle \ln s\rangle^4,
\end{aligned}
\tag{3.3.33}
$$

so that (3.3.12) with (3.3.10), (3.3.32), (3.3.23), (3.3.31) and (3.3.33) gives

$$
\begin{aligned}
\|p^m\nabla_q\gamma(s)\|_{L^\infty_{q,p}} &\leq \||p|\nabla_q\gamma(1)\|_{L^\infty_{q,p}} + \int_s^1 \left(\|\nabla_q E\|_{L^\infty_q} \||p|\nabla_p\gamma\|_{L^\infty_{q,p}} + \|E\|_{L^\infty_q}\|\nabla_q\gamma\|_{L^\infty_{q,p}}\right)\frac{ds'}{s'} \\
&\leq \varepsilon_1 + \varepsilon_1\int_s^1 \left(10C^4\varepsilon_1^3\langle \ln s'\rangle^5 + 20C^2\varepsilon_1^3\langle \ln s'\rangle^5\right)\frac{ds'}{s'} \\
&\leq \varepsilon_1\langle \ln s\rangle^6.
\end{aligned}
\tag{3.3.34}
$$

This closes the bootstrap (3.3.28). □

3.3.3 Asymptotic Behavior

From (3.3.19) we can deduce that the electric field has an asymptotic limit:

Corollary 3.3.4. *Let γ be a solution of (3.1.14) as in Proposition 3.3.3, which is moreover defined for $s \in (0,1]$. Then the limit*

$$E_0(q) := \lim_{s\to 0} E(s,q)$$

exists and is bounded

$$\|E_0\|_{L^\infty_q} \lesssim \varepsilon_1^2.$$

In addition we have the following convergence rate: if $0 \le s_1 \le s_2 \le 1$, *there holds*

$$\|E(s_1) - E(s_2)\|_{L^\infty_q} \lesssim \varepsilon^2 \langle \ln(s_2)\rangle^6 s_2. \tag{3.3.35}$$

The rate of convergence (3.3.35) is linked to the topology we choose through the continuity equation (3.1.9). Our assumptions scale like $\mathbf{j} \in L^\infty_q$ and we obtain almost Lipschitz bounds in time.

Proof. It follows from (3.3.26) in the proof above that $E(2^{-k})$ is Cauchy in L^∞_q. Summing again (3.3.26) gives (3.3.35). □

Now we are in the position to give the proof of the modified scattering result:

Proof of Theorem 3.3.1. From Proposition 3.3.3 we obtain a global solution γ on $(0,1]$, which satisfies (3.3.18), (3.3.21) and (3.3.35). Next we define

$$\nu(s,q,p) := \gamma(s, q + ps + \lambda s \ln(s) E_0(q), p + \lambda \ln(s) E_0(q)), \tag{3.3.36}$$

which satisfies

$$\begin{aligned}\partial_s \nu &= \partial_s \gamma + p \cdot \nabla_q \gamma + \lambda(1 + \ln(s)) E_0(q) \cdot \nabla_q \gamma + \lambda s^{-1} E_0(q) \cdot \nabla_p \gamma \\ &= \lambda E_0(q) \cdot \nabla_q \gamma + \lambda s^{-1} [E_0(q) - E(s, q + ps + \lambda s \ln(s) E_0(q)] \cdot \nabla_p \gamma,\end{aligned} \tag{3.3.37}$$

where

$$s^{-1} \left|E_0(q) - E(s, q + ps + \lambda s \ln(s) E_0(q)\right| \lesssim s^{-1} \left|E_0(q) - E(s,q)\right| + \left|p + \ln(s) E_0(q)\right| \|\nabla E(s)\|_{L^\infty_q}. \tag{3.3.38}$$

Hence by (3.2.2), Corollary 3.3.4 and (3.3.33) we have that

$$\|\partial_s \nu\|_{L^\infty_{q,p}} \lesssim \varepsilon_1^2 \|\nabla_q \gamma\|_{L^\infty_{q,p}} + \varepsilon_1^2 \langle \ln(s)\rangle^4 \|p\nabla_p \gamma\|_{L^\infty_{q,p}} + \varepsilon_1^2 \langle \ln(s)\rangle^6 \|\nabla_p \gamma\|_{L^\infty_{q,p}}, \quad (3.3.39)$$

which is integrable over $0 \leq s \leq 1$. $\square$

3.4 Wave Operators and Cauchy Problem at Infinity

Using the symplectic structure, the equations (3.1.14) can be written as

$$\gamma_s + \{\gamma, \mathcal{H}\} = 0, \qquad \{f, g\} := \nabla_q f \cdot \nabla_p g - \nabla_p f \cdot \nabla_q g \quad (3.4.1)$$

with the Hamiltonian

$$\mathcal{H}(s, q, p) := \frac{|p|^2}{2} - \lambda s^{-1} \phi(s, q), \quad (3.4.2)$$

where $\Delta\phi(s, q) = \int \gamma^2(s, q, p)dp$. We wish to find a new coordinate system (w, z) for which the Cauchy problem at $s = 0$ can be solved. For this, we introduce the type-3 generating function[3]

$$S(s, w, p) := w \cdot p + \frac{|p|^2}{2} s - \lambda \ln(s)\phi_0(w),$$

[3] see e.g. [104, Chapter 8]

where $\phi_0(q) = \phi(0, q)$. This gives rise to the canonical change of coordinates

$$z = \nabla_w S(w,p) = p - \lambda \ln(s) \nabla\phi_0(w),$$
$$q = \nabla_p S(w,p) = w + ps = w + zs + \lambda s \ln(s) \nabla\phi_0(w),$$

or

$$\begin{aligned} q = &\quad w + sz + \lambda s \ln(s) \nabla\phi_0(w), & w = &\quad q - sp, \\ p = &\quad z + \lambda \ln(s) \nabla\phi_0(w), & z = &\quad p - \lambda \ln(s) \nabla\phi_0(q - sp), \end{aligned} \tag{3.4.3}$$

with Jacobian matrix

$$\frac{\partial(w,z)}{\partial(q,p)} = \begin{pmatrix} Id & -sId \\ -\lambda \ln(s) \nabla E_0 & Id + \lambda s \ln(s) \nabla E_0 \end{pmatrix}, \tag{3.4.4}$$

with the usual notation $E = \nabla\phi$, $E_0 = \nabla\phi_0$. This corresponds to the new Hamiltonian

$$\mathcal{K}(s,w,z) := \mathcal{H}(s,q,p) - \partial_s S(s,w,p) = \lambda s^{-1} \big[\phi_0(w) - \phi(s,q)\big]$$

and vector field

$$\nabla_w \mathcal{K} = -\lambda s^{-1} \left\{ E(s,q) - E_0(w) \right\} - \lambda^2 \ln(s) E(s,q) \cdot \nabla E_0(w), \qquad \nabla_z \mathcal{K} = -\lambda E(q). \tag{3.4.5}$$

It follows that

$$\sigma(s,w,z) := \gamma(s,q,p) \tag{3.4.6}$$

solves

$$0 = \partial_s \sigma + \{\sigma, \mathcal{K}\} = \partial_s \sigma + \nabla_w \sigma \cdot \nabla_z \mathcal{K} - \nabla_z \sigma \cdot \nabla_w \mathcal{K}. \tag{3.4.7}$$

Remark. We note that the new variables (w,z) have a simple interpretation in terms of the original variables in (3.1.1): $w=v$, $z=x-tv-\lambda\ln(t)E_0(v)$, which are the variables in which the modified scattering of Theorem 3.1.1 and [91] is expressed.

The main result of this section then is the following:

Theorem 3.4.1. *Assume that initial data σ_0 and E_0*[4] *satisfy*

$$\|E_0\|_{W^{3,\infty}} \leq c_0^2, \tag{3.4.8}$$

and

$$\|\sigma_0\|_{L^2_{w,z}} + \|\langle z\rangle^5\sigma_0\|_{L^\infty_{w,z}} + \sum_{0\leq m+n\leq 2} \|\langle z\rangle^m \nabla_z^m \nabla_w^n \sigma_0\|_{L^\infty_{w,z}} \leq c_0. \tag{3.4.9}$$

Then there exists $T^*=T^*(c_0)>0$ and a unique solution $\sigma \in C^0_s([0,T^*) : L^2_{w,z})$ of (3.4.7) with "initial" data $\sigma(s=0)=\sigma_0$, and such that $s\partial_s\sigma$, $\nabla_{w,z}\sigma \in C^0_{s,w,z}$. Moreover, for $0\leq s<T^*$ we have that for any $\ell\in\mathbb{N}$

$$\|\sigma(s)\|_{L^2_{w,z}} + \left\|\langle z\rangle^5\sigma(s)\right\|_{L^\infty_{w,z}} + \|\nabla_{w,z}\sigma(s)\|_{L^\infty_{w,z}} \lesssim c_0, \qquad \|\langle w,z\rangle^\ell\sigma(s)\|_{L^\infty_{w,z}} \lesssim \|\langle w,z\rangle^\ell\sigma_0\|_{L^\infty_{w,z}}, \tag{3.4.10}$$

and if c_0 is sufficiently small we may take $T^*=1$.

The proof of Theorem 3.4.1 is given below in Section 3.4.3, after we have established some a priori estimates on the propagation of moments and derivatives for the system (3.4.7) in Sections 3.4.1 and 3.4.2.

[4]these are linked through (3.4.20)

3.4.1 Commutation Relations

Writing $\mathfrak{L} = \partial_s + \{\cdot, \mathcal{K}\}$, for moments in w, z we have the commutation relations

$$\mathfrak{L}[w_j\sigma] = -\lambda E_j(s,q)\sigma, \tag{3.4.11}$$

$$\mathfrak{L}[z_j\sigma] = \left(\lambda s^{-1}[E_j(s,q) - E_{0,j}(w)] + \lambda^2 \ln(s) E(s,q)\cdot\nabla_w E_{0,j}(w)\right)\sigma. \tag{3.4.12}$$

For the derivatives, we have

$$\mathfrak{L}(\partial_1\sigma) = \{\partial_1\mathcal{K}, \sigma\}, \qquad \mathfrak{L}(\partial_2\partial_1\sigma) = \{\partial_1\mathcal{K}, \partial_2\sigma\} + \{\partial_2\mathcal{K}, \partial_1\sigma\} + \{\partial_2\partial_1\mathcal{K}, \sigma\} \tag{3.4.13}$$

and this gives in block diagonal form

$$\mathfrak{L}\begin{pmatrix}\nabla_w\sigma\\ \nabla_z\sigma\end{pmatrix} = \begin{pmatrix}-\nabla_w\nabla_z\mathcal{K} & \nabla_w^2\mathcal{K}\\ -\nabla_z^2\mathcal{K} & \nabla_w\nabla_z\mathcal{K}\end{pmatrix}\begin{pmatrix}\nabla_w\sigma\\ \nabla_z\sigma\end{pmatrix}, \tag{3.4.14}$$

with

$$\begin{aligned}
\nabla^2_{w^jw^k}\mathcal{K} &= -\lambda s^{-1}\partial_j[E_k(s,q) - E_{0,k}(w)]\\
&\quad - \lambda^2\ln(s)\left\{\partial_j E(s,q)\cdot\partial_k E_0(w) + \partial_k E(s,q)\cdot\partial_j E_0(w) + \partial_j\partial_k E_0(w)\cdot E(s,q)\right\}\\
&\quad - \lambda^3 s\ln(s)^2\partial_k E_{0,a}(w)\partial_j E_{0,b}(w)\cdot\partial_a E_b(s,q),\\
\nabla_w\nabla_z\mathcal{K} &= -\lambda\nabla E(s,q) - \lambda^2 s\ln(s)(\nabla E(s,q)\cdot\nabla)E_0(w),\\
\nabla_z^2\mathcal{K} &= -\lambda s\nabla E(s,q).
\end{aligned} \tag{3.4.15}$$

We note that the matrix $\nabla^2_{w,z}\mathcal{K}$ is *ill-conditioned*, and to mitigate this effect, we introduce a weight on the gradient:

$$\theta(s,z) := \frac{\langle z\rangle}{1+s\langle z\rangle}, \qquad \frac{1}{2}\min\{\langle z\rangle, s^{-1}\} \leq \theta(\mathrm{s},z) \leq \min\{\langle z\rangle, s^{-1}\}, \tag{3.4.16}$$

which is linked to the vector field through (3.4.19) and satisfies nice differential equalities

$$\partial_s\theta = -\theta^2, \quad \nabla_z\theta = (z/\langle z\rangle^3)\cdot\theta^2.$$

3.4.2 A Priori Estimates

The goal of this section is to bootstrap the following assumptions: given c_0 as in (3.4.8), we assume that for $0 \leq s \leq T(c_0)$ there holds that

$$\begin{aligned}
\|\sigma(s)\|_{L^2_{w,z}} + \|\langle z\rangle^5\sigma(s)\|_{L^\infty_{w,z}} &\leq A \leq 4c_0,\\
\|\nabla_w\sigma(s)\|_{L^\infty_{w,z}} + \|\theta\nabla_z\sigma(s)\|_{L^\infty_{w,z}} &\leq B \leq 4c_0,\\
\|\nabla^2_{w,w}\sigma(s)\|_{L^\infty_{w,z}} + \|\theta\nabla^2_{w,z}\sigma(s)\|_{L^\infty_{w,z}} + \|\theta^2\nabla^2_{z,z}\sigma(s)\|_{L^\infty_{w,z}} &\leq C \leq 4c_0.
\end{aligned} \tag{3.4.17}$$

As we will show below in Section 3.4.2.1, this implies in particular that

$$\begin{aligned}
|\nabla_z\mathcal{K}(w,z)| &\leq 2c_0^2,\\
|\nabla_w\mathcal{K}(w,z)| &\leq c_0^2\left\{\min\{s^{-1},|z|\} + \langle\ln(s)\rangle^3\right\},
\end{aligned} \tag{3.4.18}$$

and that we have the derivative bounds

$$\|\nabla_w\nabla_z\mathcal{K}\|_{L^\infty_{w,z}} + \|\theta\nabla_z\nabla_z\mathcal{K}\|_{L^\infty_{w,z}} + \|\theta^{-1}\nabla_w\nabla_w\mathcal{K}\|_{L^\infty_{w,z}} \leq c_0^2\langle\ln(s)\rangle^4. \tag{3.4.19}$$

These in turn can then be used to close the bootstrap for (3.4.17), as in Section 3.4.2.2.

3.4.2.1 A Priori Control on the Electric Field

Here we consider a particle density $\sigma \in C^0([0,s] : L^2_{w,z})$ such that $\sigma(0) = \sigma_0$ and which satisfies the bounds (3.4.17). This creates an electric field $E(s)$ through the formula

$$E[\sigma](s,Q) = E(s,Q) = \iint \frac{Q - q(s,w,z)}{|Q - q(s,w,z)|^3}\sigma^2(s,w,z)dwdz = \iint \frac{Q-q}{|Q-q|^3}\gamma^2(s,q,p)dqdp \tag{3.4.20}$$

where σ and γ are related through (3.4.6). Simple bounds give uniformly in $R > 0$:

$$|E(s_2,Q) - E(s_1,Q)| \lesssim R\|\langle z\rangle^4\sigma\|^2_{L^\infty_{s,w,z}} + R^{-2}\|\sigma(s)\|_{L^\infty_s L^2_{w,z}}\|\sigma(s_2) - \sigma(s_1)\|_{L^2_{w,z}},$$

which ensures that E is continuous in time. In the next lemma we adapt the bounds from Section 3.2 to obtain stronger control as in (3.4.18), (3.4.19).

Lemma 3.4.2. *Let $\sigma \in C^0([0,T], L^2_{w,z})$ with $\sigma(0) = \sigma_0$ such that $E[\sigma]$ satisfies the continuity equation (3.1.9) and E_0 satisfies (3.4.8). Then there exists $T^*(c_0) \in (0,T]$ such that*

1. *Assuming the first line of (3.4.17) holds, we obtain (3.4.18) for $0 \le s \le T^*$.*
2. *Assuming the first two lines of (3.4.17) hold, we obtain (3.4.19) for $0 \le s \le T^*$.*

Proof. *1* In order to use Lemma 3.2.2, we observe that

$$|p - z| \leq c_0^2 \langle \ln(s) \rangle, \qquad |q - w| \leq s|z| + c_0^2 s \langle \ln(s) \rangle \tag{3.4.21}$$

and that the change of variable (3.4.3) preserves volume, so that

$$\begin{aligned}
&\|\gamma(s)\|_{L^r_{q,p}} = \|\sigma(s)\|_{L^r_{w,z}},\\
&\||p|^\alpha \gamma(s)\|_{L^r_{q,p}} \lesssim \||z|^\alpha \gamma(s)\|_{L^r_{q,p}} + c_0^{2\alpha} \langle \ln(s) \rangle^\alpha \|\gamma(s)\|_{L^r_{q,p}} \lesssim c_0^{2\alpha} \langle \ln(s) \rangle^\alpha \|\sigma(s)\|_{L^r_{w,z}} + \||z|^\alpha \sigma(s)\|_{L^r_{w,z}}.
\end{aligned}$$

In addition, since (see (3.4.4)) $\partial z / \partial p = Id + O(c_0 s \langle \ln(s) \rangle)$ has bounded Jacobian, we see that

$$\|\mathbf{j}(s)\|_{L^\infty_q} \leq \| \int \left[|z| + c_0^2 \langle \ln(s) \rangle \right] \sigma^2 dz \|_{L^\infty_w} \leq c_0^2 \langle \ln(s) \rangle \| \langle z \rangle^3 \sigma \|^2_{L^\infty_{w,z}}$$

and using (3.2.5), we obtain that for $2^{-k-1} \leq s_2 \leq s_1 \leq 2^{-k}$,

$$\|E(s_2, q) - E(s_1, q)\|_{L^\infty_q} \lesssim \langle c_0 \rangle^2 k^2 2^{-k} \| \langle z \rangle^3 \sigma \|^2_{L^\infty_{s,w,z}} + 2^{-2k} \|\sigma\|^2_{L^\infty_s L^2_{w,z}}$$

and summing we see that $E(2^{-k})$ is Cauchy in L^∞_q and that

$$\|E(s, q) - E_0(q)\|_{L^\infty_q} \lesssim \langle c_0 \rangle^2 s \langle \ln(s) \rangle^2 \| \langle z \rangle^3 \sigma \|^2_{L^\infty_{s,w,z}} + \langle c_0 \rangle^2 s^2 \|\sigma\|^2_{L^\infty_s L^2_{w,z}}. \tag{3.4.22}$$

Using the formulas in (3.4.5), the control on $\nabla_z \mathcal{K}$ follows directly, while we see that

$$\nabla_w \mathcal{K}(w, z) = -\lambda s^{-1} \left\{ E(s, q) - E_0(q) \right\} - \lambda s^{-1} \left\{ E_0(q) - E_0(w) \right\} + O(c_0^2 \langle \ln(s) \rangle)$$

and with (3.4.22), (3.4.17), (3.4.8) and (3.4.21), we obtain (3.4.18).

2 We want to use (3.2.6), which requires some additional control on the derivatives. From (3.4.4) we see that

$$\nabla_q \gamma = \nabla_w \sigma - \lambda \ln(s) \nabla E_0 \cdot \nabla_z \sigma$$

so that

$$\|\nabla_q \gamma\|_{L^r_{q,p}} \lesssim c_0^2 \langle \ln(s) \rangle \|\nabla_{w,z} \sigma\|_{L^r_{w,z}},$$

and

$$\begin{aligned}\|\nabla_q \mathbf{j}(s)\|_{L^\infty_q} &\lesssim c_0^2 \langle \ln(s) \rangle \| \int \left[|z| + c_0^2 \langle \ln(s) \rangle \right] |\sigma| \cdot |\nabla_{w,z} \sigma| dz \|_{L^\infty_w} \\ &\lesssim c_0^4 \langle \ln(s) \rangle^2 \left[\|\langle z \rangle^5 \sigma\|^2_{L^\infty_{w,z}} + \|\nabla_{w,z} \sigma\|^2_{L^\infty_{w,z}} \right].\end{aligned}$$

For $2^{-k-1} \leq s_2 \leq s_1 \leq 2^{-k}$ this gives by (3.2.6) that

$$\begin{aligned}\|\nabla_q E(s_2, q) - \nabla_q E(s_1, q)\|_{L^\infty_q} \lesssim \langle c_0 \rangle^4 k^3 2^{-k} \cdot \left(\|\langle z \rangle^5 \sigma\|^2_{L^\infty_{s,w,z)}} + \|\nabla_{w,z} \sigma\|^2_{L^\infty_{s,w,z}} \right) \\ + c_0^{10} 2^{-3k/2} \left[\|\langle z \rangle^5 \sigma\|^2_{L^\infty_{s,w,z}} + \|\sigma\|^2_{L^\infty_s L^2_{w,z}} \right],\end{aligned}$$

and applying similar arguments as before we obtain

$$\begin{aligned}\|\nabla_q E(s, q) - \nabla_q E_0(q)\|_{L^\infty_q} &\lesssim \langle c_0 \rangle^2 s \langle \ln(s) \rangle^3 \cdot \left(\|\langle z \rangle^5 \sigma\|^2_{L^\infty_{s,w,z}} + \|\nabla_{w,z} \sigma(s, w, z)\|^2_{L^\infty_{s,w,z}} \right) \\ &\quad + c_0^{10} s^{\frac{3}{2}} \left[\|\langle z \rangle^5 \sigma\|^2_{L^\infty_{s,w,z}} + \|\sigma\|^2_{L^\infty_s L^2_{w,z}} \right] \\ &\leq c_0^2 s \langle \ln(s) \rangle^4\end{aligned} \tag{3.4.23}$$

up to choosing $T(c_0) > 0$ small enough. Using the formulas in (3.4.15) we directly see that

$$\begin{aligned}\theta |\nabla^2_{z,z} \mathcal{K}| &\leq |\nabla E| \cdot s \min\{s^{-1}, |z|\} \leq c_0^2, \\ |\nabla^2_{w,z} \mathcal{K}| &\leq |\nabla E| (1 + s \langle \ln(s) \rangle |\nabla E_0|) \leq 2c_0^2.\end{aligned}$$

Moreover, using (3.4.8) and (3.4.23) we find that, up to choosing $T(c_0) > 0$ smaller,

$$
\begin{aligned}
|\nabla^2_{w,w}\mathcal{K}| &\leq s^{-1}|\nabla E_0(q) - \nabla E_0(w)| + s^{-1}|\nabla E(s,q) - \nabla E_0(w)| \\
&\qquad + \langle \ln(s)\rangle \left[2|\nabla E_0|^2|\nabla E| + |\nabla^2 E_0||E|\right] + s\langle \ln(s)\rangle^2|\nabla E_0|^2|\nabla E| \\
&\leq c_0^2 \min\{s^{-1}, |z|\} + c_0^2\langle \ln(s)\rangle^4,
\end{aligned}
$$

from which we deduce (3.4.19). □

3.4.2.2 A Priori Estimates on the Particle Density

Here we close the bootstrap of (3.4.17):

Lemma 3.4.3. *Assume that $\sigma \in C^0([0,T], L^2_{w,z})$ satisfies (3.4.7), for some Hamiltonian $\mathcal{K}$ (not necessarily related to σ) satisfying (3.4.18) and (3.4.19). If $\sigma_0 = \sigma(0)$ satisfies (3.4.9), there exists $T(c_0) > 0$ such that (3.4.17) holds for $A = B = C = 2c_0$.*

Proof. We first close the bootstrap for A, then for A, B. Finally we adapt the argument for A, B, C. The control follows from the commutation relations (compare with (3.4.14)):

$$
\begin{aligned}
\mathfrak{L}(\langle z\rangle^m\sigma) &= \sigma\{\langle z\rangle^m, \mathcal{K}\} = -m\sigma\langle z\rangle^{m-2}z\cdot\nabla_w\mathcal{K}, \\
\mathfrak{L}(\theta\nabla_z\sigma) &= (\mathfrak{L}\ln\theta)\cdot\theta\nabla_z\sigma + \theta\{\nabla_z\mathcal{K},\sigma\} = (\mathfrak{L}\ln\theta)\cdot\theta\nabla_z\sigma + \theta\nabla_z\sigma\cdot\nabla_w\nabla_z\mathcal{K} - \nabla_w\sigma\cdot\theta\nabla_z\nabla_z\mathcal{K}, \\
\mathfrak{L}(\nabla_w\sigma) &= \{\nabla_w\mathcal{K},\sigma\} = \theta\nabla_z\sigma\cdot\theta^{-1}\nabla_w\nabla_w\mathcal{K} - \nabla_w\sigma\cdot\nabla_z\nabla_w\mathcal{K}.
\end{aligned}
\tag{3.4.24}
$$

As in (3.3.10), we find that

$$
\|\langle z\rangle^m\sigma(s)\|_{L^r_{w,z}} \leq \|\langle z\rangle^m\sigma_0\|_{L^r_{w,z}} + m\int_0^s \|\langle z\rangle^{-1}\nabla_w\mathcal{K}(s')\|_{L^\infty_{w,z}}\|\langle z\rangle^m\sigma(s')\|_{L^r_{w,z}}ds',
$$

and we can easily propagate the first line of (3.4.17).

For the derivatives, we also need to control θ. On the one hand, we can bound from above (note that $\mathfrak{L}(\ln\theta)$ can be very negative)

$$\mathfrak{L}(\ln\theta) = -(1+\frac{z}{\langle z\rangle^3}\nabla_w\mathcal{K})\theta \lesssim c_0^2 + c_0^2\langle \ln(s)\rangle^3$$

and we deduce from (3.4.24), (3.3.10) and (3.4.19) that

$$\begin{aligned}
\|\theta\nabla_z\sigma(s)\|_{L^r_{w,z}} &\leq \|\theta\nabla_z\sigma_0\|_{L^r_{w,z}} + c_0^2\int_0^s \langle \ln(s')\rangle^4 \left\{\|\theta\nabla_z\sigma(s')\|_{L^r_{w,z}} + \|\nabla_w\sigma(s')\|_{L^r_{w,z}}\right\} ds',\\
\|\nabla_w\sigma(s)\|_{L^r_{w,z}} &\leq \|\nabla_w\sigma_0\|_{L^r_{w,z}} + c_0^2\int_0^s \langle \ln(s')\rangle^4 \left\{\|\theta\nabla_z\sigma(s')\|_{L^r_{w,z}} ds + \|\nabla_w\sigma(s')\|_{L^r_{w,z}}\right\} ds',
\end{aligned}$$

and this allows us to propagate the second line of (3.4.17) for short time.

We now propagate higher order derivatives to bound the bootstrap for C. First by interpolation in (3.4.17), we observe that

$$\|\langle z\rangle^{2.1}\nabla_{w,z}\sigma\|_{L^\infty_{w,z}} \leq A + C.$$

We will use the weight θ to control the ∂_z derivatives. Using (3.4.13), we find that

$$\begin{aligned}
\mathfrak{L}(\partial_{w^j}\partial_{w^k}\sigma) &= \theta^{-1}\nabla_w\partial_{w^j}\mathcal{K}\cdot(\theta\nabla_z\partial_{w^k}\sigma) + \theta^{-1}\nabla_w\partial_{w^k}\mathcal{K}\cdot(\theta\nabla_z\partial_{w^j}\sigma) - \nabla_z\partial_{w^j}\mathcal{K}\cdot\nabla_w\partial_{w^k}\sigma \\
&\quad - \nabla_z\partial_{w^k}\mathcal{K}\cdot\nabla_w\partial_{w^j}\sigma + \theta^{-1}\nabla_w\partial_{w^j}\partial_{w^k}\mathcal{K}\cdot(\theta\nabla_z\sigma) - \nabla_z\partial_{w^j}\partial_{w^k}\mathcal{K}\cdot\nabla_w\sigma, \\
\mathfrak{L}(\theta\partial_{z^j}\partial_{w^k}\sigma) &= \mathfrak{L}(\ln\theta)\cdot\theta\partial_{z^j}\partial_{w^k}\sigma + \theta^{-1}\nabla_w\partial_{w^k}\mathcal{K}\cdot(\theta^2\nabla_z\partial_{z^j}\sigma) - \nabla_z\partial_{w^k}\mathcal{K}\cdot(\theta\nabla_w\partial_{z^j}\sigma) \\
&\quad + \nabla_w\partial_{z^j}\mathcal{K}\cdot(\theta\nabla_z\partial_{w^k}\sigma) - (\theta\nabla_z\partial_{z^j}\mathcal{K})\cdot\nabla_w\partial_{w^k}\sigma \\
&\quad + \nabla_w\partial_{z^j}\partial_{w^k}\mathcal{K}\cdot(\theta\nabla_z\sigma) - \theta\nabla_z\partial_{z^j}\partial_{w^k}\mathcal{K}\cdot\nabla_w\sigma, \\
\mathfrak{L}(\theta^2\partial_{z^j}\partial_{z^k}\sigma) &= 2\mathfrak{L}(\ln\theta)\cdot\theta^2\partial_{z^j}\partial_{z^k}\sigma + \nabla_w\partial_{z^k}\mathcal{K}\cdot(\theta^2\nabla_z\partial_{z^j}\sigma) - \theta\nabla_z\partial_{z^k}\mathcal{K}\cdot(\theta\nabla_w\partial_{z^j}\sigma) \\
&\quad + \nabla_w\partial_{z^j}\mathcal{K}\cdot(\theta^2\nabla_z\partial_{z^k}\sigma) - (\theta\nabla_z\partial_{z^j}\mathcal{K})\cdot(\theta\nabla_w\partial_{z^k}\sigma) \\
&\quad + \theta\nabla_w\partial_{z^j}\partial_{z^k}\mathcal{K}\cdot(\theta\nabla_z\sigma) - \theta^2\nabla_z\partial_{z^j}\partial_{z^k}\mathcal{K}\cdot\nabla_w\sigma,
\end{aligned}$$

and we can proceed as for the case of one derivative once we control the new terms

$$\|\theta^{-1}\nabla^3_{w,w,w}\mathcal{K}\|_{L^\infty_{w,z}} + \|\nabla^3_{w,w,z}\mathcal{K}\|_{L^\infty_{w,z}} + \|\theta\nabla^3_{w,z,z}\mathcal{K}\|_{L^\infty_{w,z}} + \|\theta^2\nabla^3_{z,z,z}\mathcal{K}\|_{L^\infty_{w,z}} \le c_0^2\langle\ln(s)\rangle^5. \tag{3.4.25}$$

It remains to prove (3.4.25). Starting from

$$\nabla_z\mathcal{K} = -\lambda E(q), \qquad \frac{\partial q^k}{\partial z^j} = s\delta^k_j, \qquad \frac{\partial q^k}{\partial w^j} = \delta^k_j + \lambda s\ln(s)\partial_j\partial_k\phi_0(w)$$

we deduce

$$\begin{aligned}
&\theta^2|\nabla^3_{z,z,z}\mathcal{K}| = (s\theta)^2|\nabla^2 E(q)|, \\
&\theta|\nabla^3_{w,z,z}\mathcal{K}| \le (s\theta)\cdot[1+s\langle\ln(s)\rangle|\nabla E_0|]\cdot|\nabla^2 E(q)|, \\
&|\nabla^3_{w,w,z}\mathcal{K}| \le [1+s\langle\ln(s)\rangle|\nabla E_0|]^2\cdot|\nabla^2 E(q)| + [1+s\langle\ln(s)\rangle|\nabla E_0|]\cdot\left[1+s\langle\ln(s)\rangle|\nabla^2 E_0|\right]\cdot|\nabla E(q)|,
\end{aligned}$$

and finally, from (3.4.15), we obtain that

$$\begin{aligned}\theta^{-1}|\nabla^3_{w,w,w}\mathcal{K}| \leq &\left[s^{-1} + \langle \ln(s)\rangle \cdot |\nabla E_0|\right] \cdot |\nabla^2 E(s,q) - \nabla^2 E_0(w)| \\ &+ \langle \ln(s)\rangle \cdot \left[|\nabla^2 E| \cdot |\nabla E_0| + |\nabla E| \cdot |\nabla^2 E_0| + |\nabla^3 E_0| \cdot |E|\right] \\ &+ s\langle \ln(s)\rangle^2 \cdot \left[|\nabla^2 E| \cdot |\nabla E_0|^2 + |\nabla E| \cdot |\nabla E_0| \cdot |\nabla^2 E_0|\right] \\ &+ s^2\langle \ln(s)\rangle^3 \cdot \left[|\nabla^2 E| \cdot |\nabla E_0|^3\right].\end{aligned}$$

Independently, we find that

$$\begin{aligned}\|\nabla^2 \mathbf{j}(s)\|_{L^\infty_q} &\lesssim c_0^2 \langle \ln(s)\rangle^2 \| \int \left[|z| + c_0^2\langle \ln(s)\rangle\right] \cdot \left[|\sigma| \cdot |\nabla^2_{w,z}\sigma| + |\nabla_{w,z}\sigma|^2\right] dz\|_{L^\infty_w} \\ &\lesssim c_0^4 \langle \ln(s)\rangle^3 \left[\|\langle z\rangle^5 \sigma\|_{L^\infty_{w,z}} \|\nabla^2_{w,z}\sigma\|_{L^\infty_{w,z}} + \|\langle z\rangle^{2.1}\nabla_{w,z}\sigma\|^2_{L^\infty_{w,z}}\right].\end{aligned}$$

Now using Lemma 3.2.3 and the bootstrap assumptions, we obtain that

$$\|\nabla^2 E(s,q) - \nabla^2 E_0(w)\|_{L^\infty_{w,z}} \leq c_0^2 \langle \ln(s)\rangle^5 + c_0^2 \min\{s^{-1}, |z|\}$$

which easily leads to (3.4.25). □

3.4.3 Local Solutions

We construct local solutions for the singular equation (3.4.7) via Picard iteration.

Proof of Theorem 3.4.1. We proceed in two steps.

3.4.3.0.1 Step 1: A priori estimates. We construct a sequence of approximate solutions on a time interval $[0, T]$ (with $T > 0$ to be chosen later) via Picard iteration:

We define $\sigma_{(0)}(s,w,z) := \sigma_0(w,z)$, and given $\sigma_{(n)} \in C^0_s([0,T], C^1_{w,z})$ satisfying (3.4.17) with $A = B = C = 4c_0$, we let $\sigma_{(n+1)} \in C^0_s([0,T], C^1_{w,z})$ be the solution of

$$\begin{aligned} &\partial_s \sigma_{(n+1)} + \{\sigma_{(n+1)}, \mathcal{K}_n\} = 0, \qquad \sigma_{(n+1)}(0) = \sigma_0,\\ &\mathcal{K}_n := \lambda s^{-1}(\phi_0(w) - \phi_n(s,q)), \qquad \Delta\phi_n = \int \gamma^2_{(n)}(s,q,p)dp, \end{aligned} \tag{3.4.26}$$

where $\gamma_{(n)}$ and $\sigma_{(n)}$ are related through (3.4.6). Using Lemma 3.4.2, we see that $\mathcal{K}_n$ satisfies (3.4.18) and (3.4.19). Using Lemma 3.4.3, we see that $\sigma_{(n+1)}$ satisfies (3.4.17) with $A = B = C = 2c_0$. We deduce that (3.4.17) holds uniformly in n with $A = B = C = 2c_0$ on a fixed time interval $0 \leq s \leq T(c_0)$.

In addition using the commutation relations (3.4.11), we easily propagate (3.4.10) uniformly in n.

3.4.3.0.2 Step 2: Contraction in $L^\infty_{s,w,z}$

Let

$$\delta_{(n)} := \sigma_{(n+1)} - \sigma_{(n)}, \qquad \delta\mathcal{K}_{(n)} := \mathcal{K}_n - \mathcal{K}_{n-1}, \qquad \mathfrak{L}_n := \partial_s + \{\cdot, \mathcal{K}_n\}, \qquad \delta\mathfrak{L}_n = \{\cdot, \delta\mathcal{K}_{(n)}\},$$

so that

$$\mathfrak{L}_n \delta_{(n)} = \delta\mathfrak{L}_n \sigma_{(n)}, \tag{3.4.27}$$

and we can express

$$\begin{aligned} \nabla_z \delta\mathcal{K}_{(n)} &= -\lambda(E_n(s,q) - E_{n-1}(s,q)),\\ \nabla_w \delta\mathcal{K}_{(n)} &= -\lambda s^{-1}(E_n(s,q) - E_{n-1}(s,q)) - \lambda^2 \ln(s)(E_n(s,q) - E_{n-1}(s,q)) \cdot \nabla E_0(q). \end{aligned} \tag{3.4.28}$$

Invoking the uniform bounds for $\sigma_{(n)}$ we will prove below that

$$\|\nabla_{w,z}\delta\mathcal{K}_{(n)}(s)\|_{L^\infty_{w,z}} \leq c_0\langle\ln(s)\rangle^6 \left\|\delta_{(n-1)}(s)\right\|_{L^\infty_{w,z}}. \tag{3.4.29}$$

In combination with (3.4.27), we find that

$$\|\delta_{(n)}(s)\|_{L^\infty_{w,z}} \lesssim \int_0^s \|\nabla_{w,z}\delta\mathcal{K}_{(n)}(s')\|_{L^\infty_{w,z}}\|\nabla_{w,z}\sigma_{(n)}(s')\|_{L^\infty_{w,z}}ds' \lesssim c_0^2\int_0^s \langle\ln(s')\rangle^6 \left\|\delta_{(n-1)}(s')\right\|_{L^\infty_{w,z}} ds',$$

from which we deduce that, possibly taking $T(c_0) > 0$ smaller, $\sigma_{(n)}$ form a Cauchy sequence in $L^\infty_{s,w,z}$, and thus $\sigma_{(n)} \to \sigma \in L^\infty_{s,w,z}$ as $n \to \infty$. Interpolation gives convergence in the other topologies. In particular

$$\left\|\nabla_{w,z}\delta_{(n)}\right\|_{L^\infty_{s,w,z}} \lesssim \left\|\delta_{(n)}\right\|^{1/2}_{L^\infty_{s,w,z}}\left[\left\|\nabla^2_{w,z}\sigma_{(n+1)}\right\|_{L^\infty_{s,w,z}} + \|\nabla^2_{w,z}\sigma_{(n)}\|_{L^\infty_{s,w,z}}\right]^{1/2} \tag{3.4.30}$$

so that $\sigma_{(n)}$ is Cauchy in $C^0_sC^1_{w,z}$ and the other bounds follow by Fatou's Lemma or by conservation. In particular, (3.4.10) follows by pointwise convergence. Finally we note that if c_0 is sufficiently small, the arguments give a contraction for any $T \leq 2$.

It remains to show (3.4.29). The main point is that E is quadratic in γ, so that in the estimates for $\delta\mathcal{K}_{(n)}$ we can always factor out the difference $\delta_{(n)}$ in $L^\infty_{w,z}$. The bound on $\nabla_z\delta\mathcal{K}_{(n)}$ follows from adaptation to Lemma 3.2.1 and this also allows to control all but the first term in $\nabla_w\delta\mathcal{K}_{(n)}$ as in (3.4.28). These then follow from (3.2.5) using the difference continuity equation

$$\partial_s(E_n - E_{n-1}) + \nabla\Delta^{-1}\mathrm{div}_q\left\{\delta\mathbf{j}_n\right\}, \qquad \delta\mathbf{j}_n = \int_{\mathbb{R}^3_p}(\gamma_n + \gamma_{n-1})(\gamma_n - \gamma_{n-1})\cdot p dp$$

with

$$\|\delta\mathbf{j}_n\|_{L^\infty_q} \lesssim \|\delta_{(n-1)}\|_{L^\infty_{w,z}} \cdot \left[\|\langle p\rangle^5\gamma_n\|_{L^\infty_{q,p}} + \|\langle p\rangle^5\gamma_{n-1}\|_{L^\infty_{q,p}}\right]$$

and simple adaptations of Lemma 3.2.2. □

Finally, we prove the main theorem.

Proof of Theorem 3.1.1. For 1, using (3.1.12), the assumptions (3.1.3) lead to (3.3.2) in Theorem 3.3.1 and the local convergence is easily adapted (see Remark 3.3). For 2, assumptions (3.1.5) lead to (3.4.8)-(3.4.9) and the conclusion follows from that of Theorem 3.4.1. Finally, for 3, we can apply Theorem 3.4.1 to $\mu_{-\infty}(x,-v)$ to get, using (3.4.6), (3.1.12) and (3.4.10) a solution for $-\infty < t \le -1$ such that

$$\|\mu(-1)\|_{L^2_{x,v}} + \|\langle x,v\rangle^5\mu(-1)\|_{L^\infty_{x,v}} + \|\nabla_{x,v}\mu(-1)\|_{L^\infty_{x,v}} \lesssim \varepsilon.$$

By local existence, we can extend these bounds for $-1 \le t \le 1$, at which point we can simply apply 1. □

Chapter Four

The Massless Electron limit of the Vlasov-Poisson-Landau system

Joint work with Yan Guo.

4.1 Introduction

A plasma is a collection of fast-moving, charged particles. It is believed that more than 95 percent of matter in the universe takes the form of a plasma. Besides this, a major impetus for the study of plasmas is to obtain nuclear fusion as a means of energy production. Due to its complex and rich nature, there are three main distinct families of models (kinetic, two-fluid and magnetohydrodynamic) for describing a plasma in different physical regimes. The two-fluid models (Euler-Poisson and Euler-Maxwell systems) describe dynamics of two *distinct* and *separate* compressible ion and electron fluids, interacting with their self-consistent electromagnetic field.

Such two-fluid models are important both from physical as well as mathematical standpoint, serving as an origin for most well-known dispersive PDEs, for instance KdV [16, 131], NLS [122], Zakharov [133], etc.

It is an important question to derive and justify two-fluid theory from more basic kinetic models for plasma. Consider a classical kinetic model for ions and electrons, the Vlasov-Poisson-Landau system, which models the distribution functions $F_+(t,x,v)$ and $F_-(t,x,v)$ for ions $(+)$ and electrons $(-)$ respectively:

$$\begin{aligned}
\{\partial_t + v\cdot\nabla_x + \frac{Z\mathsf{e}}{m_+}E\cdot\nabla_v\}F_+ &= 2\pi\mathsf{e}^4\ln(\Lambda)\{Z^4Q_{++}(F_+,F_+) + Z^2Q_{-+}(F_-,F_+)\},\\
\{\partial_t + v\cdot\nabla_x - \frac{\mathsf{e}}{m_-}E\cdot\nabla_v\}F_- &= 2\pi\mathsf{e}^4\ln(\Lambda)\{Q_{--}(F_-,F_-) + Z^2Q_{+-}(F_+,F_-)\},\\
-\Delta_x\phi &= 4\pi\mathsf{e}(Zn_+ - n_-).
\end{aligned} \tag{4.1.1}$$

In the above, e is the electron charge,[1] and Z is the atomic number of the ion species. The ion and electron masses are m_+ and m_- respectively. The precise values of these constants depend on the type of ion considered (hydrogen, helium, etc.), and the system of units used. In addition, we have the dimensionless parameter $\ln(\Lambda) > 0$ called the *Coulomb logarithm*, for which more is said in Section 4.2.1. Besides these parameters, we define the densities $n_\pm = \int_{\mathbf{R}^3} F_\pm dv$, the electric potential ϕ, the electric field $E := -\nabla_x\phi$. Finally, the Landau collision operators Q_{++}, Q_{-+}, Q_{--} and Q_{+-} are tabulated in Section 4.2.1 below. We refer the reader to [86], and Chapter 2 of [10] for the physical justification of this model.

[1] Not to be confused with the mathematical constant $e \approx 2.718$.

It is well known that any straightforward fluid limit must lead to

$$Z^4 Q_{++}(F_+, F_+) + Z^2 Q_{-+}(F_-, F_+) = 0, \tag{4.1.2}$$

$$Q_{+-}(F_+, F_+) + Z^2 Q_{--}(F_-, F_+) = 0. \tag{4.1.3}$$

However, due to the presence of ion and electron interactions Q_{+-} and Q_{-+}, the only possible solutions to the above are of the form[2]

$$F_+ = n_+ (\frac{m_+}{2\pi T})^{3/2} \exp\left(-\frac{m_+|v-u|^2}{2T} \right), \tag{4.1.4}$$

$$F_- = n_- (\frac{m_-}{2\pi T})^{3/2} \exp\left(-\frac{m_-|v-u|^2}{2T} \right). \tag{4.1.5}$$

Here, the electron and ion fluids exhibit the *same* velocity $u(t,x)$ and temperature $T(t,x)$, which *excludes* the possibility of justification of any two-fluid models from (4.1.1). It is well-known, however, that it is possible to derive two-fluid models from two-species kinetic equations *if ion-electron collisions* Q_{+-} *and* Q_{-+} *are disregarded*. This was done in the context of the Vlasov-Poisson-Boltzmann system [73, 74, 76], and more recently for single-species Landau-type equations [49, 112]. We highlight the work [13], which derives the fluid limit for the Vlasov-Poisson-Boltzmann system, after first deriving the massless electron limit. This work in pariticular served as a major inspiration for our work. Importantly, however, none of these works consider the inter-species collisions.[3]

In all physical situations, the electron mass m_- is much smaller than the ion mass m_+. For instance, $\frac{m_-}{m_+} \approx 0.005$ for a hydrogen plasma. This small parameter has been exploited in different contexts in plasma studies both from physical as well as mathematical standpoints. For instance, one can derive the Euler-Poisson system

[2] See equation (40) in [86].

[3] In the case of diffusive limits, the work [92] derives the single fluid Vlasov-Navier-Stokes-Fourier system from the two-species Vlasov-Maxwell-Boltzmann system.

for ions in this limit [64]. By sending the electron mass to zero, the density of the electrons (and, in turn, the electrostatic potential) becomes wholly determined by the density of the ions. This eliminates the need to solve a fluid or kinetic equation for the electrons separately from the ions, leading to a simpler model.

When the ions and electrons are near thermal equilibrium (say, at temperature 1), a typical ion speed will be comparable to $(\frac{1}{m_+})^{\frac{1}{2}}$, while a typical electron speed will be comparable to $(\frac{1}{m_-})^{\frac{1}{2}}$. Because of these divergent velocity scales, it is necessary to introduce the parameter $\varepsilon := (\frac{m_-}{m_+})^{\frac{1}{2}}$, and the rescaled electron velocity

$$\xi = \varepsilon v. \tag{4.1.6}$$

Applying this rescaling the system (4.1.1), we get the rescaled system (4.1.7) below. The ion-electron collisions then exhibit *two velocity scales*. Our main result is that by taking the limit $\varepsilon \to 0$ in the two-scale system, we are able to derive the massless electron system (4.1.18) below. A salient feature of this derived equation is that the *ion-electron collisions are no longer present*. In the future, we hope this work will clear the way for the derivation of the Euler-Poisson system for ions. This would be accomplished by first by taking $\varepsilon \to 0$, and then taking hydrodynamic limit $\kappa \to 0$, where $\kappa > 0$ is the analogue of the Knudsen number for our system.

Besides the relevance to the fluid limit, the massless electron limit in kinetic theory is of independent interest. There are a number of works that have handled some version of this limit. For instance, [45,46] give formal expansions for the kinetic equations with Boltzmann and (non-Coulombic) Landau-type collisions for systems of particles with disparate masses, including the cross collision terms. The works [20, 85], give derivations of the massless electron limit for Vlasov-Poisson system with linear Fokker-Planck collisions (and in the latter case, with a magnetic field).

The aforementioned work [13] derives the analogue of (4.1.18) for the Vlasov-Poisson-Boltzmann system. We note that they require a regularity assumption, and do not include ion-electron collisions.

There is also a growing literature on the Vlasov-Poisson system with massless electrons. We refer the reader to [19, 29, 58, 66–69, 79–81] and the references therein. In particular, the overview [67] has a formal derivation of the system with a Landau collision term. We also mention the recent work [30] for the Vlasov-Poisson-Landau system with weak collisions.

4.1.1 The Rescaled Vlasov-Poisson-Landau system

In Section 4.2.1, we apply a rescaling procedure to the system (4.1.1). In particular, we use the rescaled electron velocity $\xi = \varepsilon v$, as aforementioned. This system depends on three positive parameters: $\kappa > 0$ (defined in Section 4.2.1), the atomic number Z, and $\varepsilon = (\frac{m_-}{m_+})^{\frac{1}{2}}$. The rescaled versions of the distribution functions $F_+^\varepsilon(t, x, v)$ and $F_-^\varepsilon(t, x, \xi)$ then solve the system

$$\begin{aligned}
\{\partial_t + v \cdot \nabla_x + ZE^\varepsilon \cdot \nabla_v\}F_+^\varepsilon &= \kappa^{-1}\{Z^3 Q(F_+^\varepsilon, F_+^\varepsilon) + Z^2 Q_{-+}^\varepsilon(F_-^\varepsilon, F_+^\varepsilon)\}, \\
\{\varepsilon\partial_t + \xi \cdot \nabla_x - E^\varepsilon \cdot \nabla_\xi\}F_-^\varepsilon &= \kappa^{-1}\{Q(F_-^\varepsilon, F_-^\varepsilon) + ZQ_{+-}^\varepsilon(F_+^\varepsilon, F_-^\varepsilon)\}, \\
-\Delta_x \phi^\varepsilon &= 4\pi(n_+^\varepsilon - n_-^\varepsilon).
\end{aligned} \tag{4.1.7}$$

Above, ϕ^ε and $E^\varepsilon = -\nabla_x \phi^\varepsilon$ are the electric potential and field respectively. The functions $n_\pm(t, x)$ are the charge densities, defined by

$$n_+^\varepsilon = \int_{\mathbf{R}^3} F_+^\varepsilon dv, \quad n_-^\varepsilon = \int_{\mathbf{R}^3} F_-^\varepsilon d\xi. \tag{4.1.8}$$

The collision operators are given by

$$Q(G_1, G_2) = \nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v - v') \left\{ G_1(v') \nabla_v G_2(v) - G_2(v) \nabla_{v'} G_1(v') \right\} dv', \quad (4.1.9)$$

$$Q^\varepsilon_{-+}(G_1, G_2) = \nabla_v \cdot \int_{\mathbf{R}^3} \Phi(\varepsilon v - \xi') \left\{ \varepsilon G_1(\xi') \nabla_v G_2(v) - G_2(v) \nabla_{\xi'} G_1(\xi') \right\} d\xi', \quad (4.1.10)$$

$$Q^\varepsilon_{+-}(G_1, G_2) = \nabla_\xi \cdot \int_{\mathbf{R}^3} \Phi(\xi - \varepsilon v') \left\{ G_1(v') \nabla_\xi G_2(\xi) - \varepsilon G_2(\xi) \nabla_{v'} G_1(v') \right\} dv'. \quad (4.1.11)$$

Here, Φ gives the Landau collision kernel: for each $z \in \mathbf{R}^3$,

$$\Phi(z) := \frac{1}{|z|} \left(I - \frac{z \otimes z}{|z|^2} \right). \quad (4.1.12)$$

From here on in this paper, we only consider the case where $Z = \kappa = 1$, and fix the domain $(x, v) \in \mathbf{T}^3 \times \mathbf{R}^3$, where $\mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$ denotes the 3D torus.

We recall that the system (4.1.7) comes equipped with conservation of mass, momentum and energy:

$$\frac{d}{dt} \int_{\mathbf{T}^3 \times \mathbf{R}^3} F^\varepsilon_+ dx dv = \frac{d}{dt} \int_{\mathbf{T}^3 \times \mathbf{R}^3} F^\varepsilon_- dx d\xi = 0, \quad (4.1.13)$$

$$\frac{d}{dt} \left(\int_{\mathbf{T}^3 \times \mathbf{R}^3} v F^\varepsilon_+ dx dv + \varepsilon \int_{\mathbf{T}^3 \times \mathbf{R}^3} \xi F^\varepsilon_- dx d\xi \right) = 0, \quad (4.1.14)$$

$$\frac{d}{dt} \left(\int_{\mathbf{T}^3 \times \mathbf{R}^3} \frac{|v|^2}{2} F^\varepsilon_+ dx dv + \int_{\mathbf{T}^3 \times \mathbf{R}^3} \frac{|\xi|^2}{2} F^\varepsilon_- dx d\xi + \frac{1}{8\pi} \int_{\mathbf{T}^3} |E^\varepsilon|^2 dx \right) = 0. \quad (4.1.15)$$

From (4.1.7), it is clear that as $\varepsilon \to 0$, the the term involving ∂_t in the electron equation vanishes. This is a singular limit. Under appropriate circumstances, the electron distribution formally converges to an equilibrium solution at every time, evolving quasi-statically according to the ions. The ξ-dependence of these equilibrium solutions are Maxwellians (i.e. Gaussians). We denote the Maxwellians by

$$\mu_q(\xi) := \left(\frac{q}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{q}{2}|\xi|^2}, \quad q > 0. \tag{4.1.16}$$

On the other hand, the x dependence is determined by the electric field, specifically $\lim_{\varepsilon\to 0}\ln(n_-^\varepsilon)$ is proportional to $\phi^0 = \lim_{\varepsilon\to 0}\phi^\varepsilon$ (up to an additive constant). The main claim of this paper is that for some $\beta(t) > 0$, we have

$$\lim_{\varepsilon\to 0} F_-^\varepsilon(t,x,\xi) = \mu_{\beta(t)}(\xi) e^{\beta(t)\phi^0(t,x)}. \tag{4.1.17}$$

Here, β is the inverse temperature. This is known as the Maxwell-Boltzmann approximation. Here, we shift ϕ^0 by an additive constant to ensure $\int_{\mathbf{T}^3} e^{\beta(t)\phi^0(t,x)}dx = 1$.

The above claim is imprecise, because we have not yet made clear our assumptions, nor the sense in which this limit holds. Nevertheless, when the above holds, then the formal limit of (4.1.7), combined with conservation of energy, leads to the following equation, which we refer to as Vlasov-Poisson-Landau for ions:

$$\begin{aligned}
&\{\partial_t + v\cdot\nabla_x - E^0\cdot\nabla_v\}F_+^0 = Q(F_+^0, F_+^0)\\
&\frac{d}{dt}\left\{\frac{3}{2\beta} + \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2} F_+^0 dx dv + \frac{1}{8\pi}\int_{\mathbf{T}^3} |E^0|^2 dx\right\} = 0,\\
&-\Delta_x\phi^0 = 4\pi(n_+^0 - e^{\beta\phi^0}).
\end{aligned} \tag{4.1.18}$$

A formal derivation of this system is given in Section 4.2.2.

4.1.2 Main result

The goal of this paper is to show that for appropriate initial data, this formal limit holds on a small time interval. We first provide local well-posedness theorems for the systems (4.1.7) and (4.1.18) for general initial data. To do this, we must first define the function spaces which capture the dissipation rate from the collision kernels. We define the matrix $\sigma_{ij}(v) = \Phi_{ij} * \mu$, with $i, j \in \{1, 2, 3\}$. Given $u \in \mathbf{R}^3$, we have that

$$u^T \sigma(v) u \sim \frac{1}{\langle v \rangle^3} |P_v u|^2 + \frac{1}{\langle v \rangle} |P_{v^\perp} u|^2 \tag{4.1.19}$$

Here, $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$ is the Japanese bracket. The matrices P_v and $P_{v^\perp}$ denote the orthogonal projections onto span$\{v\}$ and $\{v\}^\perp$ respectively. Using this matrix, we define the following spaces which will capture the dissipation produced by the various collision operators. Let $\mathcal{H}_\sigma$ denote the Hilbert space with inner product (with $\psi = \psi(v), \zeta = \zeta(v)$)

$$\|\psi\|_{\mathcal{H}_\sigma}^2 = \langle \sigma_{ij} \partial_{v_i} \psi, \partial_{v_j} \psi \rangle_{L^2} + \langle \mathrm{tr}(\sigma) \psi, \psi \rangle_{L^2} \tag{4.1.20}$$

Here and throughout the paper, we use repeated index notation, i.e. $u_k u_k' = \sum_{k=1}^3 u_k u_k'$ given any two vectors $u, u' \in \mathbf{R}^3$. The above inner product captures the dissipation associated with self-collisions when linearized near Maxwellian [44,70,71]. We also define the semi-norms

$$\|\psi\|_{\dot{\mathcal{H}}_\sigma}^2 = \langle \sigma_{ij}(v) \partial_{v_i} \psi, \partial_{v_j} \psi \rangle_{L^2_v} \tag{4.1.21}$$

$$\|\psi\|_{\mathcal{H}^+_{\sigma;\varepsilon}}^2 = \varepsilon \langle \sigma_{ij}(\varepsilon v) \partial_{v_i} \psi, \partial_{v_j} \psi \rangle_{L^2_v} \tag{4.1.22}$$

$$\|\psi\|_{\mathcal{H}^-_{\sigma;\varepsilon}}^2 = \frac{1}{\varepsilon} \langle \sigma_{ij}(\frac{v}{\varepsilon}) \partial_{\xi_i} \psi, \partial_{\xi_j} \psi \rangle_{L^2_\xi}. \tag{4.1.23}$$

Using these norms, we now define the spaces $\mathfrak{E}$, $\mathfrak{D}$ and $\mathfrak{D}^\pm_\varepsilon$ by their (semi-)norms which we use to control $F_\pm$. Fix the parameters $m_k = 5(k+1)$ for $k \in \{0, 1, 2\}$, and

$s = 3$. [4] Given $u = u(x, v)$ (or $u = u(x, \xi)$), we define

$$\|u\|_{\mathfrak{E}}^2 := \|\langle v\rangle^{m_2} u\|_{L^2_{x,v}}^2 + \|\langle v\rangle^{m_1}\langle\nabla_x\rangle^s u\|_{L^2_{x,v}}^2, \tag{4.1.24}$$

$$\|u\|_{\mathfrak{D}}^2 := \|\langle v\rangle^{m_2} u\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}^2 + \|\langle v\rangle^{m_1}\langle\nabla_x\rangle^s u\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}^2, \tag{4.1.25}$$

$$\|u\|_{\mathfrak{D}_\varepsilon^\pm}^2 := \|\langle v\rangle^{m_2} u\|_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^\pm_{\sigma;\varepsilon})_v}^2 + \|\langle v\rangle^{m_1}\langle\nabla_x\rangle^s u\|_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^\pm_{\sigma;\varepsilon})_v}^2. \tag{4.1.26}$$

We note that as spaces, $\mathfrak{D}$ and $\mathfrak{D}_\varepsilon^\pm$ are the same. However, the norms are equivalent up to a constant which goes to infinity as $\varepsilon \to 0$.

To state the main theorem, we will also make use of the following spaces to measure the error in the ion distribution:

$$\|u\|_{\mathfrak{E}'}^2 := \|\langle v\rangle^{m_1} u\|_{L^2_{x,v}}^2 + \|\langle v\rangle^{m_0}\langle\nabla_x\rangle^s u\|_{L^2_{x,v}}^2, \tag{4.1.27}$$

$$\|u\|_{\mathfrak{D}'}^2 := \|\langle v\rangle^{m_1} u\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}^2 + \|\langle v\rangle^{m_0}\langle\nabla_x\rangle^s u\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}^2. \tag{4.1.28}$$

Finally, to measure the distance of F_-^ε from its limit, we let $\eta > 0$ be a constant to be determined later, and define $q_\alpha = e^{\alpha\eta}\beta_{in}$ for each $\alpha \in \{1, 2, 3\}$, where $\beta_{in} = \beta(0)$. Then, for $\alpha \in \{1, 2\}$, we define

$$\begin{aligned}\mathscr{E}^\varepsilon_{-,\alpha} &:= \|e^{q_\alpha|\xi|^2/4}\langle\nabla_x\rangle^s\{F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\}\|_{L^2_{x,\xi}}^2 \\ &\quad + \|e^{q_{\alpha+1}|\xi|^2/4}\{F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\}\|_{L^2_{x,\xi}}^2\end{aligned} \tag{4.1.29}$$

and

$$\begin{aligned}\mathscr{D}^\varepsilon_{-,\alpha} &:= \|e^{q_\alpha|\xi|^2/4}\langle\nabla_x\rangle^s\{F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\}\|_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}^2 \\ &\quad + \|e^{q_{\alpha+1}|\xi|^2/4}\{F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\}\|_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}^2.\end{aligned} \tag{4.1.30}$$

Also, throughout the rest of the paper, we will use the subscript "*in*" to denote the initial value of some quantity, for instance, $\phi^0_{in}(x) = \phi^0(0, x)$.

[4] Although these parameters may take on a range of values, it is simpler to fix their values to exact constants.

We can now state our main theorem:

Theorem 4.1.1. *Suppose for some $T_0 > 0$ and $M > 0$, there exists a weak solution (F_+^0, β, ϕ^0) to (4.1.7) with*

$$0 \le F_+^0 \in C([0,T_0];\mathfrak{E}') \cap L^\infty([0,T_0];\mathfrak{E}) \cap L^2([0,T_0];\mathfrak{D}), \tag{4.1.31}$$

as well as $\beta \in C([0,T_0])$ and $\phi^0 \in C([0,T]; H^{s+2})$, with

$$\sup_{t\in[0,T_0]} \{\|\frac{1}{n_+^0}\|_{L^\infty([0,T]\times\mathbf{T}^3)} + \|F_+^0\|_{\mathfrak{E}}\} + \|\ln(\beta(t))\|_{L^\infty([0,T_0])} \le M. \tag{4.1.32}$$

(i) There exist positive constants η, and $\overline{\varepsilon}$ depending only on M, as well as $\hat{T} \in [0, T_0]$ depending on (F_+^0, β, ϕ^0) such that the following holds. For each $\varepsilon \in (0, \overline{\varepsilon}]$, take $(F_{+,in}^\varepsilon, F_{-,in}^\varepsilon)$, with $0 \le F_{\pm,in}^\varepsilon \in \mathfrak{E}$, satisfying the estimates

$$\sup_{\varepsilon\in(0,\overline{\varepsilon}]} \|\frac{1}{n_{+,in}^\varepsilon}\|_{L_x^\infty} + \|F_{+,in}^\varepsilon\|_{\mathfrak{E}} \le M, \tag{4.1.33}$$

$$\|F_{+,in}^\varepsilon - F_{+,in}^0\|_{\mathfrak{E}'} + \mathscr{E}_{-,2,in}^\varepsilon \le M\varepsilon. \tag{4.1.34}$$

Then, for each $\varepsilon \in (0, \overline{\varepsilon}]$, there exists a solution $(F_+^\varepsilon, F_-^\varepsilon)$ to (4.1.7) with $0 \le F_\pm^\varepsilon \in C([0,\hat{T}];\mathfrak{E}) \cap L^2([0,\hat{T}];\mathfrak{D})$. This satisfies

$$\sup_{t\in[0,\hat{T}]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'} \lesssim_M \varepsilon \tag{4.1.35}$$

$$\sup_{t\in[0,\hat{T}]} \varepsilon\mathscr{E}_{-,2}^\varepsilon(t) + \int_0^{\hat{T}} \mathscr{D}_{-,2}^\varepsilon(t)dt \lesssim_M \varepsilon^2 \tag{4.1.36}$$

Moreover, for all $t \in [0, \hat{T}]$,

$$\mathscr{E}^{\varepsilon}_{-,1}(t) \lesssim_M \varepsilon^{\frac{1}{2}} e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}} + \varepsilon^{\frac{5}{3}} t^{\frac{1}{3}} + \varepsilon^2. \tag{4.1.37}$$

(ii) There exist positive constants η, δ *and* $\overline{\varepsilon}$ *depending on* M, *and* $\hat{T} \in (0, T_0]$ *depending on* (F^0_+, β, ϕ^0) *and* M *such that the following holds. For all* $\varepsilon \in (0, \overline{\varepsilon}]$, *we set* $F^{\varepsilon}_{+,in} = F^0_{+,in}$ *and* $F^{\varepsilon}_{-,in} = F_{-,in}$ *(where* $F^{\varepsilon}_{-,in}$ *is independent of* ε*) satisfying*

$$\mathscr{E}^{\varepsilon}_{-,2,in} \leq \delta. \tag{4.1.38}$$

We note the expression on the left is independent of ε. *We also impose the condition on the kinetic energy of* $F_{-,in}$*:*

$$\frac{1}{2} \iint_{\mathbf{T}^3 \times \mathbf{R}^3} |\xi|^2 F_{-,in} dx d\xi - \frac{3}{2\beta_{in}} + \frac{1}{8\pi} \int_{\mathbf{T}^3} |\nabla_x \phi_{in}|^2 - |\nabla_x \phi^0_{in}|^2 dx = 0. \tag{4.1.39}$$

Then, for each $\varepsilon \in (0, \overline{\varepsilon}]$, *there exists a solution* $(F^{\varepsilon}_+, F^{\varepsilon}_-)$ *to* (4.1.7) *with* $0 \leq F^{\varepsilon}_{\pm} \in C([0, \hat{T}]; \mathfrak{E}) \cap L^2([0, \hat{T}]; \mathfrak{D})$. *This satisfies*

$$\sup_{t \in [0, \hat{T}]} \|F^{\varepsilon}_+(t) - F^0_+(t)\|_{\mathfrak{E}'} \lesssim_M \sqrt{\varepsilon}\delta + \varepsilon \tag{4.1.40}$$

$$\sup_{t \in [0, \hat{T}]} \varepsilon \mathscr{E}^{\varepsilon}_{-,2}(t) + \int_0^{\hat{T}} \mathscr{D}^{\varepsilon}_{-,2}(t) dt \lesssim_M \varepsilon(\delta^2 + \varepsilon) \tag{4.1.41}$$

Moreover, for all $t \in [0, \hat{T}]$,

$$\mathscr{E}^{\varepsilon}_{-,1}(t) \lesssim_M \delta e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}} + \varepsilon^{\frac{5}{3}} t^{\frac{1}{3}} + \varepsilon\delta. \tag{4.1.42}$$

Remark. Similarly to [64], we give to separate derivations. Part (i) of the theorem above corresponds to initial data that converges to the limiting solution as $\varepsilon \downarrow 0$. This corresponds to the case of *well-prepared* initial data.. Part (ii) of the theorem shows that the derivation can also be performed with certain *ill-prepared* data, since $F^{\varepsilon}_{-,in} - F^{0}_{-,in}$ does not converge to 0 as $\varepsilon \downarrow 0$. The caveat is that one must assume the condition (4.1.39).

Remark. This theorem depends on the existence of solutions to the system (4.1.18). The local well-posedness theory for this system is given in Chapter 5.

4.1.3 Methodology and outline

Due to the nature of the rescaled velocity ξ in the two-scale system (4.1.7), the Landau kernel Q^{ε}_{+-} exhibits a severe singularity in ξ as $\varepsilon \to 0$. This is mirrored by the degeneracy of the Q^{ε}_{-+} in this limit. This makes it very difficult to propagate v derivatives of F^{ε}_{+} and ξ derivatives of F^{ε}_{-}. The main technical achievement of this paper is that we obtain uniform bound in ε in weighted Sobolev spaces, *without any* ξ *or* v *derivatives.* In other words, all regularity in v and ξ comes from the diffusive control from the top order part of the collision operators.

We highlight some of the techniques that go into this. In Lemma 4.3.2, we obtain a *non-perturbative* lower bound for $\Phi * G$ (where $G \geq 0$), which allows us to access diffusive control from collisions. One challenge we encounter is that we are unable to close a bootstrap estimate on F^{ε}_{+} solely in the $\mathfrak{E}$ norm. Proposition 4.4.1 states that if F^{ε}_{+} is of size M on $[0,T]$, and $\|F^{\varepsilon}_{+}\|_{L^2([0,T];\mathfrak{D}')}$ is small, then we can control the growth of $\|F^{\varepsilon}_{+}\|_{\mathfrak{E}}$ uniformly in ε. In order to assume $\|F^{\varepsilon}_{+}\|_{L^2([0,T];\mathfrak{D}')}$ is small uniformly ε, we T small enough that $\|F^{0}_{+}\|_{\mathfrak{D}'}$ is as small as desired, and then control the difference $\|F^{\varepsilon}_{+} - F^{0}_{+}\|_{\mathfrak{D}'}$. This control on the difference is in turn provided by Proposition 4.4.2.

The next challenge is to control the difference $F^{\varepsilon}_{-} - \mu_{\beta} e^{\beta\phi^{0}}$. This is roughly the content of Proposition 4.5.1, although we work with what we call the "intermediary quantities" $(\gamma^\varepsilon, \psi^\varepsilon)$ rather than (β, ϕ^0). This proposition utilizes the linear decay estimates of the linearized collision operator, via the framework developed in [70,71]. Because of the small parameter in front of the ∂_t term in the second line of (4.1.7), the question of convergence of F_-^ε to the Maxwellian $\mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}$ is in some sense equivalent to the question of asymptotic stability of the Maxwellian. However, there are a number of complications in our context. First, the underlying Maxwellian has a varying temperature. To deal with this, we take a window of time such that $\gamma^\varepsilon \approx \beta_{in}$, and choose $q_1 < q_2 < q_3$ (constant in time) close enough to β_{in}, and work with these constants instead. This results in a perturbed version of the linearized collision operator found in [70,71]. Showing that this operator retains the desired coercivity properties requires some care. There is also the ion-electron collision operator Q_{+-}^ε, which acquires a singularity at $\xi = 0$ as $\varepsilon \to 0$. To control this, we use Lemma 4.3.2 to extract a dissipative, albeit singular and degenerate, top order part. We then use the dissipation from the linearization of Q to control the singular lower order terms.

A key feature of the linearized collision operator is that it has a five dimensional kernel, corresponding to the macroscopic quantities of mass, momentum and energy density. Besides having to work with a perturbation of this operator, there are some additional challenges in adapting the macroscopic estimates from previous works. One issue is the effect of Q_{+-}^ε on the macroscopic quantities. Surprisingly, this term has a perturbative effect on the mass and energy density, and even contributes an extra dampening effect on the momentum density (see Lemma 4.5.6). A second challenge is how one controls the mean energy of the perturbation. The mean kinetic energy density of $F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}$ is not conserved, and corresponds to a neutral mode at the level of the linearization of the equation for $F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}$. In contrast, all other components of the macroscopic quantities are either conserved, or dissipated through hypo-coercive effects. In part (i) of Theorem 4.1.1, we can control the mean energy because it is initially small. In part (ii), we must impose (4.1.39) and use some nonlinear identities to control the average energy density, as in the first step of Proposition 4.5.7. This necessitates the use of the the intermediary quantities $(\gamma^{\varepsilon}, \psi^{\varepsilon})$ instead of (β, ϕ^0), since $\frac{3}{2\gamma^{\varepsilon}}$ serves as a better approximation of the mean kinetic energy of F_-^{ε} than $\frac{3}{2\beta}$.

The structure of the paper is as follows. Section 4.2 contains all of the formal analysis in this paper. In Section 4.2.1 we derive the rescaled system (4.1.7). In Section 4.2.2, we give a formal derivation of the system (4.1.18) from (4.1.7). In Section 4.2.2, we show how the analogous derivation does not work in the context of the Vlasov-Poisson-Boltzmann system, and we highlight the difficulties in defining the limiting system as $\varepsilon \to 0$.

Section 4.3 contains two preliminary results that will be used in the paper. The first is Lemma 4.3.1. This is a statement of local well-posedness that is compatible with the main theorem. We do not prove this result, as it follows from straightforward modifications of the *a priori* estimates shown in this work. The second result is Lemma 4.3.2, which gives upper and lower bounds on $\Phi * G$. This is particularly important for extracting diffusive control via the norms $\dot{\mathcal{H}}_{\sigma}$, and $\mathcal{H}^{\pm}_{\sigma;\varepsilon}$.

In Section 4.4, we prove *a priori* estimates on the ion distribution F_+^{ε}. First, we have Proposition 4.4.1, which gives control of $F_+^{\varepsilon} \in L^{\infty}(\mathfrak{E}) \cap L^2(\mathfrak{D})$ uniform in ε. Then, we have Proposition 4.4.2, which gives uniform control of $F_+^{\varepsilon} - F_+^0 \in L^{\infty}(\mathfrak{E}') \cap L^{\infty}(\mathfrak{D}')$. Finally, with Lemma 4.4.3, we construct the intermediary quantities $(\gamma^{\varepsilon}, \psi^{\varepsilon})$, which solve the system (4.4.140) below. These are defined in the same way as (β, ϕ^0), except using F_+^{ε} in place of F_+^0.

In Section 4.5, we prove Proposition 4.5.1, which gives the error estimate for the electrons under certain assumptions on the solutions. The proof of this is broken up into two components: energy estimates for $F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}$ (as in Proposition 4.5.4), and macroscopic estimates (as in Lemma 4.5.6 and Proposition 4.5.7).

Finally, in Section 4.6, we prove Theorem 4.1.1. For both parts (i) and (ii), we make five bootstrap assumptions, which allow us to satisfy the assumptions made by the various propositions throughout this paper. We then show that these assumptions can be propagated over a time interval independent of $\varepsilon > 0$. The error estimates then follow from Propositions 4.4.2 and 4.5.1.

4.2 Formal analysis

4.2.1 Non-dimensionalization of the Vlasov-Poisson-Landau System

Here, we show how to rescale the Vlasov-Poisson-Landau system (4.1.1) to the non-dimensionalized form (4.1.7). The collision operators in (4.1.1) are

$$Q_{++}(G_1, G_2)(v) = \frac{1}{m_+^2}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v-v')\left\{G_1(v')\nabla_v G_2(v) - G_2(v)\nabla_v G_1(v')\right\} dv', \tag{4.2.1}$$

$$Q_{-+}(G_1, G_2)(v) = \frac{1}{m_+}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v-v')\left\{\frac{1}{m_+}G_1(v')\nabla_v G_2(v) - \frac{1}{m_-}G_2(v)\nabla_v G_1(v')\right\} dv', \tag{4.2.2}$$

$$Q_{+-}(G_1, G_2)(v) = \frac{1}{m_-}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v-v')\left\{\frac{1}{m_-}G_1(v')\nabla_v G_2(v) - \frac{1}{m_+}G_2(v)\nabla_v G_1(v')\right\} dv', \tag{4.2.3}$$

$$Q_{--}(G_1, G_2)(v) = \frac{1}{m_-^2}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v-v')\left\{G_1(v')\nabla_v G_2(v) - G_2(v)\nabla_v G_1(v')\right\} dv', \tag{4.2.4}$$

In order to define the Coulomb logarithm, we assume F_+ and F_- have approximately a common temperature $\theta > 0$ (with the Boltzmann constant set to 1). To have a charge balance, the electrons will have on average N total number of particles per unit volume, and the ions will have total number N/Z per unit volume. For simplicity, one might take F_+ and F_- to be perturbations of such equilibrium, i.e.

$$F_+ \approx \frac{N}{Z}(\frac{m_+}{\theta})^{3/2} e^{-\frac{m_+|v|^2}{2\theta}}, \tag{4.2.5}$$

$$F_- \approx N(\frac{m_-}{\theta})^{3/2} e^{-\frac{m_-|v|^2}{2\theta}}. \tag{4.2.6}$$

When F_+ and F_- are equal to the Maxwellians, they solve (4.1.1) with $\phi = 0$.

Next, the Coulomb logarithm $\ln(\Lambda)$ arises as logarithmically divergent integral in the derivation of the Vlasov-Poisson-Landau system from the BBGKY hierarchy [10], which is then cutoff at length scales where the assumptions of the model break down. Several choices of cutoffs exist, the most common pair being the Debye length at large scales

$$\lambda_D = \left(\frac{\theta}{4\pi \mathsf{e}^2 N}\right)^{1/2} \tag{4.2.7}$$

and the distance of closest approach at small scales

$$b_0 = \frac{3\theta}{Z\mathsf{e}^2}. \tag{4.2.8}$$

This gives the expression

$$\ln(\Lambda) = \ln\left(\frac{\lambda_D}{b_0}\right). \tag{4.2.9}$$

Importantly, $\ln(\Lambda)$ does not depend on the m_- or m_+. Unfortunately, this choice fails in contexts where $\lambda_D \leq b_0$, for instance when N is large or θ is small, and another definition of $\ln(\Lambda)$ is needed. For our purposes, we shall simply take $\ln(\Lambda)$ to be some positive parameter independent of the masses. We note here that this logarithmic divergence can be resolved by using the Lenard-Balescu model for Coulombic collisions. Regarding this, see chapter 2 of [10], as well as the works [50, 127].

We define relevant length, velocity and time scales for our problem. This will allow us to redefine a non-dimensionalized equation depending on three parameters: the (square root of the) mass ratio, Z, and a dissipation rate. The relevant velocity scales for each species are

$$V_\pm = (\frac{\theta}{m_\pm})^{1/2}. \tag{4.2.10}$$

These are the the thermal speeds as in [4]. Next, we define the length scale

$$X = \left(\frac{\theta}{N\mathsf{e}^2}\right)^{1/2}, \tag{4.2.11}$$

which can be thought of the smallest length scale at which the electrostatic force is significant for a thermal particle. This gives us natural time scales for each species

$$T^{\pm} = \frac{X}{V^{\pm}} = \left(\frac{m_{\pm}}{N\mathsf{e}^2}\right)^{1/2} \tag{4.2.12}$$

While F_+ and F_- will be re-scaled in v differently, according to V_+ and V_- respectively, we will re-scale time by the ion-time scale for both species. The electrons then evolve according to a faster time-scale than the ions. The ratio of these time-scales is the same as the square-root of the mass ratio,

$$\varepsilon = \frac{T_-}{T_+} = \left(\frac{m_-}{m_+}\right)^{1/2} \tag{4.2.13}$$

The final non-dimensional parameter is the dissipation rate, which gives the size of the collisional effects:

$$\kappa^{-1} = 2\pi \ln(\Lambda)\frac{N^{1/2}\mathsf{e}^3}{\theta^{3/2}}. \tag{4.2.14}$$

We now perform the non-dimensionalization. Define the re-scaled coordinates

$$\bar{t} = \frac{t}{T_+}, \tag{4.2.15}$$

$$\overline{x} = \frac{x}{X}, \tag{4.2.16}$$

$$\overline{v}_{\pm} = \frac{v}{V_{\pm}}, \tag{4.2.17}$$

and define re-scaled versions of (F_-, F_+):

$$\tilde{F}_+(\bar{t}, \overline{x}, \overline{v}_+) = \frac{ZV_+^3}{N} F_+(t, x, v), \tag{4.2.18}$$

$$\tilde{F}_-(\bar{t}, \overline{x}, \overline{v}_-) = \frac{V_-^3}{N} F_-(t, x, v). \tag{4.2.19}$$

Moreover, define

$$\tilde{n}_\pm(\bar{t}, \overline{x}) = \int \tilde{F}_+(\bar{t}, \overline{x}, \overline{v}_\pm) d\overline{v}_\pm, \tag{4.2.20}$$

$$-\Delta_{\overline{x}} \tilde{\phi} = 4\pi(\tilde{n}_+(\bar{t}, \overline{x}) - \tilde{n}_-(\bar{t}, \overline{x})), \tag{4.2.21}$$

$$\tilde{E}(\bar{t}, \overline{x}) = -\nabla_{\overline{x}} \tilde{\phi}(\bar{t}, \overline{x}). \tag{4.2.22}$$

In particular,

$$\tilde{E}(\bar{t}, \overline{x}) = \frac{1}{NX\mathsf{e}} E(t, x) \tag{4.2.23}$$

The "Vlasov-Poisson" parts of the equations (4.1.1) transform like

$$\{\partial_t + v \cdot \nabla_x + \frac{Z\mathsf{e}}{m_+} E \cdot \nabla_v\} F_+ = \frac{N}{ZT_+V_+^3} \{\partial_{\bar{t}} + \overline{v}_+ \cdot \nabla_{\overline{x}} + Z\tilde{E} \cdot \nabla_{\overline{v}_+}\} \tilde{F}_+ \tag{4.2.24}$$

$$\{\partial_t + v \cdot \nabla_x - \frac{\mathsf{e}}{m_-} E \cdot \nabla_v\} F_- = \frac{N}{T_-V_-^3} \{\varepsilon \partial_{\bar{t}} + \overline{v}_- \cdot \nabla_{\overline{x}} - \tilde{E} \cdot \nabla_{\overline{v}_-}\} \tilde{F}_- \tag{4.2.25}$$

We now compute the transformations of the collision kernels:

$$Q_{++}(F_+,F_+)(v) \tag{4.2.26}$$
$$= \frac{N^2}{Z^2 m_+^2 V_+^6}\nabla_{\overline{v}_+}\cdot\int_{\mathbf{R}^3}\Phi(\overline{v}_+-\overline{v}'_+)\left\{\tilde{F}_+(\overline{v}'_+)\nabla_{\overline{v}_+}\tilde{F}_+(v_+)-\tilde{F}_+(\overline{v}_+)\nabla_{\overline{v}_+}\tilde{F}_+(\overline{v}'_+)\right\}d\overline{v}'_+, \tag{4.2.27}$$

$$Q_{-+}(F_-,F_+)(v) \tag{4.2.28}$$
$$= \frac{N^2}{Z m_+^2 V_+^6}\nabla_{\overline{v}_+}\cdot\int_{\mathbf{R}^3}\Phi(\varepsilon\overline{v}_+-\overline{v}'_-)\left\{\varepsilon\tilde{F}_-(\overline{v}'_-)\nabla_{\overline{v}_+}\tilde{F}_+(v_+)-\tilde{F}_+(\overline{v}_+)\nabla_{\overline{v}_-}\tilde{F}_-(\overline{v}'_-)\right\}d\overline{v}'_-, \tag{4.2.29}$$

$$Q_{--}(F_-,F_-)(v) \tag{4.2.30}$$
$$= \frac{N^2}{m_-^2 V_-^6}\nabla_{\overline{v}_-}\cdot\int_{\mathbf{R}^3}\Phi(\overline{v}_--\overline{v}'_-)\left\{\tilde{F}_-(\overline{v}'_-)\nabla_{\overline{v}_-}\tilde{F}_-(v_-)-\tilde{F}_-(\overline{v}_-)\nabla_{\overline{v}_-}\tilde{F}_-(\overline{v}'_-)\right\}d\overline{v}'_-, \tag{4.2.31}$$

$$Q_{+-}(F_+,F_-)(v) \tag{4.2.32}$$
$$= \frac{N^2}{Z m_-^2 V_-^6}\nabla_{\overline{v}_-}\cdot\int_{\mathbf{R}^3}\Phi(\overline{v}_--\varepsilon\overline{v}'_+)\left\{\tilde{F}_+(\overline{v}'_+)\nabla_{\overline{v}_-}\tilde{F}_-(\overline{v}_-)-\varepsilon\tilde{F}_-(\overline{v}_-)\nabla_{\overline{v}_+}\tilde{F}_+(\overline{v}'_+)\right\}d\overline{v}'_+ \tag{4.2.33}$$

Ignoring Z, the ratios of the pre-factors appearing in the expressions above for the collisions, over that of the transport terms for each species are

$$\frac{N^2}{m_\pm^2 V_\pm^6}\left(\frac{N}{T_\pm V_\pm^3}\right)^{-1} = \frac{NX}{m_\pm^2 V_\pm^4} = \frac{N^{1/2}}{\theta^{3/2}\mathsf{e}} \tag{4.2.34}$$

Multiplying the above by $2\pi\mathsf{e}^4\ln(\Lambda)$ gives κ^{-1}.

We now give the non-dimensionalized version of (4.1.1). By abuse of notation, we shall use the symbols (t,x,v) to denote the re-scaled coordinates for the ions; we will use (t,x,ξ) to denote the re-scaled coordinates of the electrons (i.e., we replace $\overline{t}\to t$, $\overline{x}\to x$, $\overline{v}_+\to v$, and $\overline{v}_-\to\xi$). Moreover, we write $F_\pm^\varepsilon=\tilde{F}_\pm$, and similarly for the potential and electric field. Then, these solve (4.1.7).

4.2.2 Formal derivation of the ion equation

We give a formal derivation of (4.1.18) from (4.1.7) by setting $\varepsilon = 0$. Then,

$$\{\partial_t + v \cdot \nabla_x + E^0 \cdot \nabla_v\} F_+^0 = Q(F_+^0, F_+^0) + Q_{-+}^0(F_-^0, F_+^0), \tag{4.2.35}$$

$$\{\xi \cdot \nabla_x - E^0 \cdot \nabla_\xi\} F_-^0 = Q(F_-^0, F_-^0) + Q_{+-}^0(F_+^0, F_-^0). \tag{4.2.36}$$

where

$$Q_{-+}^0(F_-^0, F_+^0)(v) = -\nabla_v \cdot \left(\int_{\mathbf{R}^3} \Phi(\xi') \nabla_\xi F_-^0(\xi') d\xi' F_+^0(v) \right), \tag{4.2.37}$$

$$Q_{+-}^0(F_+^0, F_-^0)(\xi) = \left(\int_{\mathbf{R}^3} F_+^0(v') dv' \right) \nabla_\xi \cdot (\Phi(\xi) \nabla_\xi F_-^0(\xi)). \tag{4.2.38}$$

We assume F_-^0 is positive at each (t, x, ξ), smooth in (x, ξ), and decays sufficiently fast in ξ. We can then use the entropy identity to deduce possible solutions. Multiplying (4.2.36) by $\ln(F_-^0)$, we have

$$0 = \iint_{\mathbf{T}^3 \times \mathbf{R}^3} (Q(F_-^0, F_-^0) + Q_{-+}^0(F_+^0, F_-^0)) \ln(F_-^0) dx dv \tag{4.2.39}$$

$$= -\frac{1}{2} \iiint_{\mathbf{T}^3 \times \mathbf{R}^3 \times \mathbf{R}^3} F_-^0(t, x, \xi) F_-^0(t, x, \xi') \tag{4.2.40}$$

$$\mathrm{tr}(\Phi(\xi - \xi') \{\nabla_\xi \ln(F_-^0(t, x, \xi)) - \nabla_{\xi'} \ln(F_-^0(t, x, \xi'))\}^{\otimes 2}) dx d\xi d\xi' \tag{4.2.41}$$

$$- n_+^0(t, x) \int_{\mathbf{T}^3 \times \mathbf{R}^3} F_-^0(t, x, \xi) \mathrm{tr}(\Phi(\xi) \{\nabla_\xi \ln(F_-^0(t, x, \xi))\}^{\otimes 2}) dx d\xi. \tag{4.2.42}$$

Note that both terms are nonpositive, and therefore zero. The null space of $\Phi(\xi)$ is $\mathrm{span}\{\xi\}$, so we deduce that $\nabla_\xi \ln(F_-^0(t, x)(t, x, \xi)) \in \mathrm{span}\{\xi\}$ for all (t, x, ξ). Thus $F_-(t, x, \xi)$ is radial in ξ for each (t, x). Taking $F_-^0(t, x, \xi) = \tilde{F}_-^0(t, x, |\xi|)$, we have that

$$0 = \frac{1}{2} \iiint_{\mathbf{T}^3 \times \mathbf{R}^3 \times \mathbf{R}^3} \tilde{F}^0_-(t,x,|\xi|)\tilde{F}^0_-(t,x,|\xi'|) \tag{4.2.43}$$

$$\mathrm{tr}(\Phi(\xi-\xi')\{\frac{\xi}{|\xi|}\partial_r \ln(\tilde{F}^0_-(t,x,r))|_{r=|\xi|} - \frac{\xi'}{|\xi'|}\partial_r \ln(\tilde{F}^0_-(t,x,r))|_{r=|\xi'|}\}^{\otimes 2})dxd\xi d\xi'. \tag{4.2.44}$$

Then, for every ξ, ξ' so that $\xi \neq 0$, $\xi' \neq 0$ and $\xi - \xi' \neq 0$, we have

$$\frac{\xi}{|\xi|}\partial_r \ln(\tilde{F}^0_-(t,x,r))|_{r=|\xi|} - \frac{\xi'}{|\xi'|}\partial_r \ln(\tilde{F}^0_-(t,x,r))|_{r=|\xi'|} \in \mathrm{span}\{\xi - \xi'\}. \tag{4.2.45}$$

The above implies that on the set where ξ and ξ' are linearly independent, we have

$$\frac{1}{|\xi|}\partial_r \ln(\tilde{F}^0_-(t,x,r))|_{r=|\xi|} = \frac{1}{|\xi'|}\partial_r \ln(\tilde{F}^0_-(t,x,r))|_{r=|\xi'|}. \tag{4.2.46}$$

For any given values of $|\xi|, |\xi'| > 0$, we can find ξ, ξ' which are linearly independent. Thus, for all $r, r' > 0$, we have

$$\frac{1}{r}\partial_r \ln(\tilde{F}^0_-(t,x,r)) = \frac{1}{r'}\partial_{r'} \ln(\tilde{F}^0_-(t,x,r')). \tag{4.2.47}$$

In particular, $\partial_r(\frac{1}{r}\partial_r \ln(\tilde{F}^0_-(t,x,r))) = 0$. Thus, there exist functions $\beta(t,x)$ and $\psi(t,x)$ such that

$$\ln(\tilde{F}^0_-(t,x,r)) = -\frac{\beta(t,x)}{2}r^2 + \psi(t,x) \tag{4.2.48}$$

In other words, $F^0_-(t,x,\xi)$ is a local Maxwellian centered at $\xi = 0$, i.e.

$$F^0_-(t,x,\xi) = e^{-\frac{\beta(t,x)|\xi|^2}{2} + \psi(t,x)} \tag{4.2.49}$$

Observe that $Q(F_-^0, F_-^0) = Q_{+-}^0(F_+^0, F_-^0) = 0$. Therefore,

$$\{\xi \cdot \nabla_x - E \cdot \nabla_\xi\} F_-^0 = 0. \tag{4.2.50}$$

Thus,

$$0 = \{\xi \cdot \nabla_x - E \cdot \nabla_\xi\} \ln(F_-^0) = \xi \cdot \nabla_x \psi + \xi \cdot \nabla_x \beta \frac{|\xi|^2}{2} + \xi \cdot E^0 \beta. \tag{4.2.51}$$

Taking $|\xi| \to \infty$, we deduce that $\nabla_x \beta = 0$, i.e. $\beta(t,x) = \beta(t)$. Finally, we conclude $\nabla_x \psi = -\beta E^0 = \nabla_x(\beta \phi^0)$. Thus, $\psi - \beta\phi^0$ is independent of x and ξ. We integrate in ξ, use conservation of mass (4.1.13) to determine this function, and add an appropriate constant to ϕ^0 to deduce

$$F_-^0(t,x,\xi) = \left(\frac{\beta(t)}{2\pi}\right)^{\frac{3}{2}} e^{-\beta(t)(\frac{|\xi|^2}{2} - \phi^0(t,x))} \tag{4.2.52}$$

This implies $n_-^0(t,x) = \frac{e^{\beta(t)\phi^0(t,x)}}{\int_{\mathbf{T}^3} e^{\beta(t)\phi^0(t,x')}dx'}$, giving the third line of (4.1.18). As for the first line, since $\nabla_\xi F_-^0 = -\beta w F_-^0$, we have the important vanishing property

$$Q_{-+}^0(F_-^0, F_+^0) = \beta \nabla_\xi \cdot \{\int_{\mathbf{R}^3} \Phi(\xi)\xi F_-^0(t,x,\xi) d\xi F_+^0(v)\} = 0, \tag{4.2.53}$$

as $\Phi(\xi)\xi = (0,0,0)^T$. Finally, for the second line, we invoke (4.1.15) and the identity

$$\iint_{\mathbf{T}^3 \times \mathbf{R}^3} \frac{|\xi|^2}{2} F_-^0 dx d\xi = \frac{3}{2\beta}. \tag{4.2.54}$$

4.2.3 The Boltzmann Case

We now apply the analogous re-scaling to the Vlasov-Poisson-Boltzmann system with hard spheres as was done in Section 4.2.1. Unlike the Vlasov-Poisson-Landau system, the ion-electron collisions do not seem to vanish in the limit $\varepsilon \downarrow 0$. Moreover, we show that the formal expansion in ε yields a system of equations that seems difficult, if not impossible, to solve. This is because the the $O(1)$ terms in the expansion depend on the $O(\varepsilon)$ terms in a nontrivial way, which in turn depend on the $O(\varepsilon^2)$ terms, and so on.

For simplicity, we let $Z = \mathsf{e} = 1$. Following [134], the Vlasov-Poisson-Boltzmann system reads

$$\begin{aligned} \{\partial_t + v\cdot\nabla_x + \frac{1}{m_-}E\cdot\nabla_v\}F_+ &= Q_{++}(F_+,F_+) + Q_{-+}(F_-,F_+)\}, \\ \{\partial_t + v\cdot\nabla_x - \frac{1}{m_+}E\cdot\nabla_v\}F_- &= Q_{--}(F_-,F_-) + Q_{+-}(F_+,F_-), \\ -\Delta_x\phi &= 4\pi(n_+ - n_-) \end{aligned} \tag{4.2.55}$$

Once again, $E = -\nabla_x\phi$. The Boltzmann collision operators are given as follows: with $\alpha,\beta \in \{-,+\}$, we have

$$Q_{\alpha\beta}(G_\alpha,G_\beta) = \frac{(\sigma_\alpha+\sigma_\beta)^2}{4}\iint_{\mathbf{R}^3\times\mathbf{S}^2}|(u-v)\cdot\omega|\{G_\alpha(u')G_\beta(v') - G_\alpha(u)G_\beta(v)\}dud\omega, \tag{4.2.56}$$

where $\sigma_\pm$ are the diameters of the particles, and

$$u' = u + \frac{2m_\beta}{m_\alpha+m_\beta}((v-u)\cdot\omega)\omega, \tag{4.2.57}$$

$$v' = v - \frac{2m_\alpha}{m_\alpha+m_\beta}((v-u)\cdot\omega)\omega. \tag{4.2.58}$$

For simplicity, we set $\sigma_\pm = 1$ as well. We now set $\xi = \varepsilon v$ (and $\zeta = \varepsilon u$), and

$\tilde{F}_-(\xi) = \varepsilon^{-3}F_-(\xi/\varepsilon)$. We now rewrite the first two lines of (4.2.55),

$$\{\partial_t + v\cdot\nabla_x + E\cdot\nabla_v\}F_+ = Q(F_+,F_+) + \tilde{Q}^\varepsilon_{-+}(\tilde{F}_-,F_+) \tag{4.2.59}$$

$$\{\varepsilon\partial_t + \xi\cdot\nabla_x - E\cdot\nabla_\xi\}\tilde{F}_- = Q(\tilde{F}_-,\tilde{F}_-) + \tilde{Q}^\varepsilon_{+-}(F_+,\tilde{F}_-). \tag{4.2.60}$$

Here, $Q = Q_{++} = Q_{--}$. Before we define the cross-collision terms, it is worthwhile to define the reflection matrix

$$R_\omega z = z - 2(z\cdot\omega)\omega, \tag{4.2.61}$$

where $\omega \in \mathbf{S}^2$. The ion-electron collisions have the following (singular!) effect on the ions:

$$\tilde{Q}^\varepsilon_{-+}(\tilde{F}_-,F_+) \tag{4.2.62}$$

$$= \frac{1}{\varepsilon}\iint_{\mathbf{R}^3\times\mathbf{S}^2} |(\zeta-\varepsilon v)\cdot\omega|\{\tilde{F}_-(\zeta')F_+(v') - \tilde{F}_-(\zeta)\tilde{F}_+(v)\}d\zeta d\omega, \tag{4.2.63}$$

where

$$\zeta' = \zeta + \frac{2}{1+\varepsilon^2}((\varepsilon v-\zeta)\cdot\omega)\omega = R_\omega\zeta + 2\varepsilon(v\cdot\omega)\omega + O(\varepsilon^2), \tag{4.2.64}$$

$$v' = v - \frac{2\varepsilon}{1+\varepsilon^2}((\varepsilon v-\zeta)\cdot\omega)\omega = v + 2\varepsilon(\zeta\cdot\omega)\omega + O(\varepsilon^2). \tag{4.2.65}$$

On the other hand, the effect of ion-electron collisions on the electrons are the following:

$$\tilde{Q}^\varepsilon_{+-}(F_+,\tilde{F}_-) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} |(\varepsilon u-\xi)\cdot\omega|\{F_+(u')\tilde{F}_-(\xi') - F_+(u)\tilde{F}_-(\xi)\}dud\omega, \tag{4.2.66}$$

where in the above,

$$u' = u + \frac{2\varepsilon}{1+\varepsilon^2}((\xi - \varepsilon u)\cdot\omega)\omega = u - 2\varepsilon(\xi\cdot\omega)\omega + O(\varepsilon^2), \tag{4.2.67}$$

$$\xi' = \xi - \frac{2}{1+\varepsilon^2}((\xi - \varepsilon u)\cdot\omega)\omega = R_\omega\xi + 2\varepsilon(u\cdot\omega)\omega + O(\varepsilon^2). \tag{4.2.68}$$

The ε^{-1} term in $\tilde{Q}^\varepsilon_{-+}$ is the source of the difficulty in describing the formal analysis as $\varepsilon \downarrow 0$. In particular, we have no reason to expect that the effect of $\tilde{Q}^\varepsilon_{-+}$ will vanish in this limit. To better understand this limit, we use a formal asymptotic expansion. First, we expand the collisons as follows:

$$\tilde{Q}^\varepsilon_{-+}(G_1, G_2) = \sum_{j=-1}^{\infty} \varepsilon^j q^j_{-+}(G_1, G_2), \tag{4.2.69}$$

$$\tilde{Q}^\varepsilon_{+-}(G_1, G_2) = \sum_{j=0}^{\infty} \varepsilon^j q^j_{+-}(G_1, G_2). \tag{4.2.70}$$

We now give the first two terms in these expansions. The first two terms in (4.2.69) are

$$q^{-1}_{-+}(G_1, G_2) = \left(\int_{\mathbf{R}^3\times\mathbf{S}^2} |\zeta\cdot\omega|\{G_1(R_\omega\zeta) - G_1(\zeta)\}d\zeta d\omega\right) G_2(v) \tag{4.2.71}$$

and

$$q^0_{-+}(G_1, G_2) = -\left(\int_{\mathbf{R}^3\times\mathbf{S}^2} \operatorname{sgn}(\zeta\cdot\omega)(v\cdot\omega)\{G_1(R_\omega\zeta) - G_1(\zeta)\}d\zeta d\omega\right) G_2(v) \tag{4.2.72}$$

$$+ 2\left(\int_{\mathbf{R}^3\times\mathbf{S}^2} |\zeta\cdot\omega|(v\cdot\omega)\omega\cdot\nabla_\zeta G_1(R_\omega\zeta)d\zeta d\omega\right) G_2(v) \tag{4.2.73}$$

$$- 2\left(\int_{\mathbf{R}^3\times\mathbf{S}^2} |\zeta\cdot\omega|(\zeta\cdot\omega)G_1(R_\omega\zeta)\omega_i d\zeta d\omega\right) \partial_{v_i} G_2(v). \tag{4.2.74}$$

On the other hand, the first two terms in (4.2.70) are

$$q^0_{+-}(G_1,G_2)=\left(\int_{\mathbf{R}^3}G_1(u)du\right)\int_{\mathbf{S}^2}|\xi\cdot\omega|\{G_1(R_\omega\xi)-G_1(\xi)\}d\omega. \tag{4.2.75}$$

and

$$q^1_{+-}(G_1,G_2)=-\left(\int_{\mathbf{R}^3}u_iG_1(u)du\right)\int_{\mathbf{S}^2}\operatorname{sgn}(\xi\cdot\omega)\omega_i\{G_2(R_\omega\xi)-G_2(\xi)\}d\omega \tag{4.2.76}$$
$$+2\left(\int_{\mathbf{R}^3}u_iG_1(u)du\right)\int_{\mathbf{S}^2}\operatorname{sgn}(\xi\cdot\omega)\omega_i\omega\cdot\nabla_\xi G_2(R_\omega\xi)d\omega. \tag{4.2.77}$$

In the above, there is another $O(\varepsilon)$ term arising from the expansion $G_1(u')=G_2(u)+2\varepsilon(\xi\cdot\omega)\omega\cdot\nabla_uG_1(u)+O(\varepsilon^2)$, but this integrates to zero.

Next, we expand F_+ and $\tilde F_-$ into an formal series in ε:

$$F_+\sim\sum_{j=0}^\infty\varepsilon^jF_{+,j},\quad \tilde F_-\sim\sum_{j=0}^\infty\varepsilon^j\tilde F_{-,j}, \tag{4.2.78}$$

where each $F_{\pm,j}$ is independent of ε. Similarly, we expand $\phi\sim\sum_{j=0}^\infty\varepsilon^j\phi_j$ and $E\sim\sum_{j=0}^\infty\varepsilon^jE_j$. Collecting the $O(1)$ terms of (4.2.60), we see that $F^0_{-,j}$ solves

$$\{\xi\cdot\nabla_x-E^0\cdot\nabla_\xi\}\tilde F_{-,0}=Q(\tilde F_{-,0},\tilde F_{-,0})+q^0_{-+}(F_{+,0},\tilde F_{-,0}). \tag{4.2.79}$$

It is straightforward to check that

$$\tilde F_{-,0}=\left(\frac{\beta(t)}{2\pi}\right)^{\frac32}e^{-\beta(t)(\frac{|v|^2}{2}-\phi^0)} \tag{4.2.80}$$

solves the above, with (β,ϕ^0) solving the same equation as in (4.1.7). Using a similar entropy identity as was used in Section 4.2.2, we expect that this is the unique such solution. With this ansatz, we have $q^j_{-,+}(\tilde F_{-,0},G)=0$ for both $j=-1$ and $j=0$, and any G, due to the radial symmetry of $\tilde F_{-,0}$. In particular, the $O(\frac{1}{\varepsilon})$ term in

(4.2.59) vanishes. Collecting all the $O(1)$ terms in (4.2.59), we then have

$$\partial_t F_{+,0} + \{v \cdot \nabla_x + E^0 \cdot \nabla_v\} F_{+,0} = Q(F_{+,0}, F_{+,0}) + q^{-1}_{-+}(\tilde{F}_{-,1}, F_{+,0}). \tag{4.2.81}$$

Thus, in order to solve for $F_{+,0}$, we must solve for $\tilde{F}_{-,1}$ as well. Then, collecting all the $O(\varepsilon)$ terms in (4.2.60), we have

$$\{\xi \cdot \nabla_x - E_0 \cdot \nabla_\xi\} \tilde{F}_{-,1} - E_1 \cdot \nabla_\xi F_{-,0} \tag{4.2.82}$$

$$- Q(\tilde{F}_{-,1}, \tilde{F}_{-,0}) - Q(\tilde{F}_{-,0}, \tilde{F}_{-,1}) - q^0_{+-}(F_{+,0}, \tilde{F}_{-,1}) \tag{4.2.83}$$

$$= \partial_t \tilde{F}_{-,0} + q^1_{-+}(F_{+,0}, \tilde{F}_{-,0}) \tag{4.2.84}$$

In the above, we used the fact that $q^0(F_{+,1}, \tilde{F}_{-,0}) = 0$. Now we arrive at the issue: in order to solve for $F_{+,0}$, we must solve for $\tilde{F}_{-,1}$ as well. However, the equation for $F_{-,1}$ depends on E_1, which in turn depends on $F_{+,1}$, which depends on $\tilde{F}_{-,2}$, and so on. In summary, it seems difficult to solve for the solve the terms in the expansion (4.2.78), in order to get a well-defined limiting equation.

4.3 Preliminaries

4.3.1 Well-posedness

In this section, we state a prerequisite local well-posedness result that guarantees solutions to (4.1.7) in our context.

Lemma 4.3.1. *Fix $M, T > 0$, and suppose (F^0_+, β, ϕ^0) is a solution to (4.1.18) satisfying the hypothesis of Theorem 4.1.1. Fix $\varepsilon > 0$. Then there exists $T_\varepsilon \in (0, T_0]$ (depending on (F^0_+, β, ϕ^0), $\varepsilon > 0$ and M), and $\varrho = \varrho(M)$ (depending only on M)*

such that the following holds. If we take $(F^{\varepsilon}_{+,in}, F^{\varepsilon}_{+,in})$ *satisfying*

$$\|\frac{1}{n^{\varepsilon}_{+,in}}\|_{L^{\infty}_x} + \|F^{\varepsilon}_{+,in}\|_{\mathfrak{E}} \le M, \tag{4.3.1}$$

$$\|F^{\varepsilon}_{+,in} - F^{0}_{+,in}\|_{\mathfrak{E}'} + \sqrt{\mathscr{E}^{\varepsilon}_{-,2,in}} \le \varrho. \tag{4.3.2}$$

then there exists a unique weak solution $(F^{\varepsilon}_{+}, F^{\varepsilon}_{-})$ *with* $0 \le F^{\varepsilon}_{\pm} \in C([0,T_{\varepsilon}];\mathfrak{E}') \cap L^{\infty}([0,T_{\varepsilon}];\mathfrak{E}) \cap L^{2}([0,T_{\varepsilon}];\mathfrak{D})$ *to* (4.1.7) *with the initial data, satisfying the estimates*

$$\sup_{t\in[0,T_{\varepsilon}]}\left(\|\frac{1}{n^{\varepsilon}_{+}(t)}\|_{L^{\infty}_x} + \|F^{\varepsilon}_{+}(t)\|_{\mathfrak{E}}\right) \le 2M, \tag{4.3.3}$$

$$\sup_{t\in[0,T_{\varepsilon}]}\left(\|F^{\varepsilon}_{+}(t) - F^{0}_{+}(t)\|_{\mathfrak{E}'} + \sqrt{\mathscr{E}^{\varepsilon}_{-,2}(t)}\right) \lesssim_M \|F^{\varepsilon}_{+,in} - F^{0}_{+,in}\|_{\mathfrak{E}'} + \sqrt{\mathscr{E}^{\varepsilon}_{-,2,in}}. \tag{4.3.4}$$

We only sketch the proof. First, construct smooth approximate solutions via the iteration method laid out in Section 5.3 in the following chapter. The estimates shown in this paper can then be applied to these iterations, at least for short time in ε, if one replaces the intermediate system $(\gamma^{\varepsilon}, \psi^{\varepsilon})$ with (β, ϕ^{0}). This gives uniform boundedness of the iterates on some ε dependent interval. We can then show that these iterates converge in $\mathfrak{E}'$, uniformly in time. This gives us a local solution. We emphasize that unlike in Chapter 5, the uniform boundedness of the iterates does not require propagation of regularity in v, allowing us to get local solutions in a lower regularity class.

4.3.2 Lower and upper bounds on the diffusion matrix

In order to prove local well-posedness for the system (4.1.7) and (4.1.18), we first prove the following lemma, which gives upper and lower bounds on the diffusion matrix appearing in the collision kernels:

Lemma 4.3.2. *Let* $G(v) = G : \mathbf{R}^3 \to \mathbf{R}$, $\nu \in \mathbf{S}^2$. *Then for all* $v \in \mathbf{R}^3$,

$$|\Phi_{ij} * G(v)\nu_i\nu_j| \lesssim \|\langle v\rangle^5 G\|_{L^2}\sigma_{ij}(v)\nu_i\nu_j. \tag{4.3.5}$$

Assuming $G \geq 0$, *we have the lower bound*

$$\frac{\|G\|_{L^1}}{\langle \|\langle v\rangle^2 G\|_{L^2}/\|G\|_{L^1}\rangle^{17}}\sigma_{ij}(v)\nu_i\nu_j \lesssim \Phi_{ij} * G(v)\nu_i\nu_j. \tag{4.3.6}$$

Proof. We first prove (4.3.5). Observe that $|\Phi_{ij} * G(v)\nu_i\nu_j| \leq \Phi_{ij} * |G|(v)\nu_i\nu_j$. Thus, it suffices to consider the case when $G \geq 0$. First we have uniform boundedness,

$$\|\Phi_{ij} * G(v)\nu_i\nu_j\|_{L^\infty} \lesssim \|(-\Delta_v)^{-1}G\|_{L^\infty} \lesssim \|\langle v\rangle^2 G\|_{L^2} \tag{4.3.7}$$

Alternatively,

$$\Phi_{ij} * G(v)\nu_i\nu_j \leq \int_{2|v'|<|v|} \frac{|v-v'|^2 - \langle v-v',\nu\rangle^2}{|v-v'|^3}G(v')dv' \tag{4.3.8}$$

$$+ \int_{2|v'|\geq|v|} \frac{|v-v'|^2 - \langle v-v',\nu\rangle^2}{|v-v'|^3}G(v')dv'. \tag{4.3.9}$$

Now,

$$\sqrt{|v-v'|^2 - \langle v-v',\nu\rangle^2} = |P_{\nu^\perp}(v-v')| \leq |P_{\nu^\perp}v| + |P_{\nu^\perp}v'|, \tag{4.3.10}$$

so

$$(4.3.8) \lesssim \int_{2|v'|<|v|} \frac{|P_{\nu^\perp} v|^2 + |P_\nu v|^2}{|v-v'|^3} G(v') dv' \tag{4.3.11}$$
$$\lesssim \frac{|P_{\nu^\perp} v|^2}{|v|^3} \|G\|_{L^1} + \frac{1}{|v|^3} \||v'|^2 G\|_{L^1} \tag{4.3.12}$$
$$\lesssim (\frac{|P_{v^\perp} \nu|^2}{|v|} + \frac{1}{|v|^3}) \|\langle v'\rangle^5 G(v')\|_{L^2_{v'}} \tag{4.3.13}$$

On the other hand, we have

$$(4.3.9) \le \int_{2|v'|\ge|v|} \frac{1}{|v-v'|} G(v') dv' \tag{4.3.14}$$
$$\le \frac{1}{|v|^3} \int_{2|v'|\ge|v|} \frac{1}{|v-v'|} |v'|^3 G(v') dv' \tag{4.3.15}$$
$$\le \frac{1}{|v|^3} \|(-\Delta_{v'})^{-1} (\langle v'\rangle^3 G(v'))\|_{L^\infty_{v'}} \tag{4.3.16}$$
$$\le \frac{1}{|v|^3} \|\langle v'\rangle^5 G(v')\|_{L^2_{v'}}. \tag{4.3.17}$$

Thus,

$$\Phi_{ij} * G(v) \nu_i \nu_j \lesssim \min\{1, \frac{|P_{v^\perp}\nu|^2}{|v|} + \frac{1}{|v|^3}\} \|\langle v'\rangle^5 G(v')\|_{L^2_{v'}}, \tag{4.3.18}$$

which implies (4.3.5).

We now prove (4.3.6). By re-scaling $G \mapsto G/\|G\|_{L^1}$, it suffices to show (4.3.6) in the case $\|G\|_{L^1} = 1$. For convenience, we define

$$k_G = \langle \|\langle v\rangle^2 G\|_{L^2_v} \rangle. \tag{4.3.19}$$

Let $\nu \in \mathbf{S}^2$,

$$\Phi_{ij} * G(v) \nu_i \nu_j = \int_{\mathbf{R}^3} \frac{\sin^2\theta}{|v'|} G(v - v') dv' \tag{4.3.20}$$

where $\theta = \theta(v')$ is the angle between v' and ν. Taking $\theta_0 > 0$ to be a constant to be chosen later, we bound the above from below by

$$\Phi_{ij} * G(v)\nu_i\nu_j \geq \sin^2\theta_0 \left(\int_{\mathbf{R}^3} \frac{1}{|v'|} G(v-v')dv' - \int_{\sin^2\theta < \sin^2\theta_0} \frac{1}{|v'|} G(v-v')dv' \right) \tag{4.3.21}$$

Now,

$$\int_{\mathbf{R}^3} \frac{1}{|v'|} G(v-v')dv' = \int_{|v|\geq 10|v'|} \frac{1}{|v-v'|} G(v')dv' \tag{4.3.22}$$

$$\geq \frac{9}{10|v|} \int_{|v|\geq 10|v'|} G(v')dv' \tag{4.3.23}$$

$$\geq \frac{9}{10|v|}(1 - \int_{|v|\leq 10|v'|} G(v')dv') \tag{4.3.24}$$

$$\geq \frac{9}{10|v|}(1 - \||v'|^{-2} 1_{|v|\leq 10|v'|}\|_{L^2_{v'}} \||v'|^2 G(v')\|_{L^2_{v'}}) \tag{4.3.25}$$

$$\geq \frac{9}{10|v|}(1 - \frac{k_G}{|v|}) \tag{4.3.26}$$

On the other hand, for any $\lambda > 0$,

$$\int_{\mathbf{R}^3} \frac{1}{|v'|} G(v-v')dv' \geq \frac{1}{\lambda}\left(1 - \int_{|v'|>\lambda} G(v-v')dv'\right) \tag{4.3.27}$$

$$\geq \frac{1}{\lambda}\left(1 - \||v'|^{-2} 1_{|v'|>\lambda}\|_{L^2_{v'}} \||v-v'|^2 G(v')\|_{L^2_{v'}}\right) \tag{4.3.28}$$

$$\geq \frac{1}{\lambda}(1 - \frac{k_G \langle v\rangle^2}{\lambda^{\frac{1}{2}}}) \tag{4.3.29}$$

Taking $\lambda = 4k_G^2\langle v\rangle^4$, and combining with (4.3.26),

$$\int_{\mathbf{R}^3} \frac{1}{|v'|} G(v-v')dv' \gtrsim \min\{\frac{1}{|v|}(1 - \frac{k_G}{|v|}), \frac{1}{k_G^2\langle v\rangle^4}\} \gtrsim \frac{1}{k_G^5\langle v\rangle} \tag{4.3.30}$$

We now consider the second term in (4.3.21). First,

$$\int_{\sin^2\theta<\sin^2\theta_0}\frac{1}{|v'|}G(v-v')dv' \lesssim k_G\left(\int_{\sin^2\theta<\sin^2\theta_0}\frac{1}{|v'|^2\langle v-v'\rangle^4}dv'\right)^{\frac{1}{2}} \tag{4.3.31}$$

$$\lesssim k_G\left(\int_{\sin^2\theta<\sin^2\theta_0}\frac{1}{|v'|^2(1+|v|^2+2\langle v,v'\rangle+|v'|^2)^2}dv'\right)^{\frac{1}{2}}. \tag{4.3.32}$$

We then use spherical coordinates to evaluate the integral above, setting the span of ν to be the z-axis, and the span of $P_{\nu^\perp}v$ to be the x axis. Letting $v_\| = \langle v,\nu\rangle$, and $|P_{\nu^\perp}v| = v_\perp$, we get

$$\int_{\sin^2\theta<\sin^2\theta_0}\frac{1}{|v'|^2(1+|v|^2+2\langle v,v'\rangle+|v'|^2)^2}dv' \tag{4.3.33}$$

$$=\int_0^\infty\int_0^{2\pi}\int_{[0,\theta_0]\cup[\pi-\theta_0,\pi]}\frac{\sin\theta}{(1+v_\|^2+v_\perp^2+2r(v_\|\cos\theta+v_\perp\cos\alpha\sin\theta)+r^2)^2}d\theta d\alpha dr \tag{4.3.34}$$

We now rewrite

$$v_\|^2+v_\perp^2+2r(v_\|\cos\theta+v_\perp\cos\alpha\sin\theta)+r^2 \tag{4.3.35}$$

$$=v_\|^2\sin^2\theta+v_\perp^2(\cos^2\theta+\sin^2\alpha\sin^2\theta)+(r+v_\|\cos\theta+v_\perp\cos\alpha\sin\theta)^2 \tag{4.3.36}$$

$$\geq v_\perp^2\cos^2\theta+(r+v_\|\cos\theta+v_\perp\cos\alpha\sin\theta)^2 \tag{4.3.37}$$

Extending the domain of integration to be $r\in(-\infty,\infty)$, and using shift invariance in r, and then the symmetries of $\sin\theta$, we bound the integral by

$$(4.3.33)\lesssim\int_{\mathbf{R}}\int_0^{\theta_0}\frac{\sin\theta}{(1+v_\perp^2\cos^2\theta+r^2)^2}d\theta dr \tag{4.3.38}$$

Taking $\theta_0 \leq \frac{\pi}{4}$, we have $\cos^2\theta \geq \frac{1}{2}$. This ensures that

$$(4.3.33) \lesssim \int_{\mathbf{R}} \int_0^{\theta_0} \frac{\sin\theta}{(1+v_\perp+|r|)^4} d\theta dr \tag{4.3.39}$$

$$\lesssim \frac{1}{\langle v_\perp \rangle^3} \int_0^{\theta_0} \sin\theta d\theta \tag{4.3.40}$$

$$\lesssim \frac{\theta_0^2}{\langle v_\perp \rangle^3}. \tag{4.3.41}$$

Combining with (4.3.21), we have

$$\Phi_{ij} * G(v)\nu_i\nu_j \gtrsim \theta_0^2 \left(\frac{1}{k_G^4 \langle v \rangle} - k_G \frac{\theta_0}{\langle v_\perp \rangle^{\frac{3}{2}}} \right) \tag{4.3.42}$$

At this point, we take $\theta_0 = \lambda \frac{\langle v_\perp \rangle}{\langle v \rangle}$, where $\lambda \in (0, \frac{\pi}{4})$ is to be chosen later (not necessarily the same λ from before). In particular, $v_\parallel \theta_0 \lesssim \lambda \langle v_\perp \rangle$, so

$$\Phi_{ij} * G(v)\nu_i\nu_j \gtrsim \frac{\lambda^2 \langle v_\perp \rangle^2}{\langle v \rangle^2} \left(\frac{1}{k_G^5 \langle v \rangle} - k_G \frac{\lambda}{\langle v_\perp \rangle^{\frac{1}{2}} \langle v \rangle} \right). \tag{4.3.43}$$

Taking $\lambda = \frac{1}{10 k_G^6}$, we deduce

$$\Phi_{ij} * G(v)\nu_i\nu_j \gtrsim \frac{\langle v_\perp \rangle^2}{k_G^{17} \langle v \rangle^3} = \frac{1}{k_G^{17}} \left(\frac{|P_v \nu|^2}{\langle v \rangle^3} + \frac{|P_{v^\perp} \nu|^2}{\langle v \rangle} \right). \tag{4.3.44}$$

We deduce (4.3.6). □

4.4 Estimates on the ion distribution

4.4.1 Boundedness of the ion distribution

We now prove a priori estimates for the ion distribution.

Proposition 4.4.1. *Let $M, T, \varepsilon > 0$. Suppose $(F^\varepsilon_-, F^\varepsilon_+)$ is a weak solution to (4.1.7) with $0 \leq F^\varepsilon_\pm \in C([0,T];\mathfrak{E}) \cap L^2([0,T];\mathfrak{D})$. Assume the following bootstrap assumptions:*

$$\sup_{\pm\in\{-,+\}, 0\leq t\leq T} (\|F^\varepsilon_\pm(t)\|^2_{\mathfrak{E}} + \|\frac{1}{n^\varepsilon_\pm(t)}\|_{L^\infty_x}) \leq M \tag{4.4.1}$$

In particular, there exists $\mathscr{X}^\varepsilon_+(t) \sim_M \|F^\varepsilon_+\|^2_{\mathfrak{E}}$ (depending on M) such that

$$\frac{d}{dt}\mathscr{X}^\varepsilon_+ + \|F^\varepsilon_+(t)\|^2_{\mathfrak{D}^+_\varepsilon} \leq C_M \langle \|F^\varepsilon_-\|_{\mathfrak{D}}\rangle^{\frac{3}{2}}. \tag{4.4.2}$$

Alternatively,

$$\frac{d}{dt}\|F^\varepsilon_+\|^2_{\mathfrak{E}} + \frac{1}{C_M}\|F^\varepsilon_+(t)\|^2_{\mathfrak{D}^+_\varepsilon} \leq C_M(\langle \|F^\varepsilon_-\|_{\mathfrak{D}}\rangle^{\frac{3}{2}} + \|\langle v\rangle^{m_1} F^\varepsilon_+(t)\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^+_{\sigma;\varepsilon})_v}). \tag{4.4.3}$$

Proof. It is convenient to ignore dependence on M: we write $C = C_M$, and "$\lesssim$" instead of "$\lesssim_M$." We also drop the dependence on ε, for instance, $F_+ = F^\varepsilon_+$. Given $m', s' \in \mathbf{R}$, we define

$$F_+^{(m',s')} := \langle v\rangle^{m'}\langle\nabla_x\rangle^{s'} F_+, \tag{4.4.4}$$

$$F_-^{(m',s')} := \langle \xi\rangle^{m'}\langle\nabla_x\rangle^{s'} F_-. \tag{4.4.5}$$

It is also convenient to split the collision operators into their "diffusion" (second order) and "transport" (first order) parts in divergence form:

$$Q(G_1, G_2) = Q_D(G_1, G_2) + Q_T(G_1, G_2), \tag{4.4.6}$$

$$\tag{4.4.7}$$

where

$$Q_D(G_1, G_2) := \partial_{v_j}((\Phi_{ij} *_v G_1)\partial_{v_i} G_2), \tag{4.4.8}$$

$$Q_T(G_1, G_2) := \partial_{v_j}(\partial_{v_i}(\Phi_{ij} *_v G_1) G_2). \tag{4.4.9}$$

We split $Q^\varepsilon_{\pm\mp} = Q^\varepsilon_{\pm\mp,D} + Q^\varepsilon_{\pm\mp,T}$ similarly. We now proceed with the proof, broken into several steps.

Step 1: We have the following estimates:

$$\frac{d}{dt}\|F_+^{(m_2,0)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|F_+^{(m_2,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} \leq C(1 + \|F_-\|^{\frac{3}{2}}_{\mathfrak{D}}), \tag{4.4.10}$$

and for all $\kappa > 0$,

$$\begin{aligned} &\frac{d}{dt}\|F_+^{(m_1,s)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|F_+^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} \\ &\quad \leq C(1 + \|F_+^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} + \|F_-\|^{\frac{3}{2}}_{\mathfrak{D}}). \end{aligned} \tag{4.4.11}$$

The proof of (4.4.10) follows by a similar method as (4.4.11), so we only show the latter. [5] To prove (4.4.11), observe that $F_+^{(m_1,s)}$ satisfies

$$(\partial_t + v \cdot \nabla_x + E \cdot \nabla_v) F_+^{(m_1,s)} + [\langle v\rangle^{m_1}\langle\nabla_x\rangle^s, E \cdot \nabla_v] F_+ \tag{4.4.12}$$

$$= \langle v\rangle^{m_1}\langle\nabla_x\rangle^s\{Q(F_+, F_+) + Q^\varepsilon_{-+}(F_-, F_+)\}. \tag{4.4.13}$$

[5] We note that the RHS of (4.4.11) contains the term $\|F_+^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v}$, while (4.4.10) does not. This terms arises from the commutator of $\langle\nabla_x\rangle^s$ with collisional terms, for which there is no analogue in the proof of the estimate (4.4.10).

Multiplying the above by by $F_+^{(m_1,s)}$ integrating, we have

$$\frac{1}{2}\frac{d}{dt}\|F_+^{(m_1,s)}\|^2_{L^2_{x,v}} \tag{4.4.14}$$

$$= \langle [E\cdot\nabla_v, \langle v\rangle^m \langle \nabla_x\rangle^s]F_+, F_+^{(m_1,s)}\rangle_{L^2_{x,\xi}} \tag{4.4.15}$$

$$+ \langle \langle v\rangle^{m_1}\langle \nabla_x\rangle^s Q(F_+,F_+), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.4.16}$$

$$+ \langle \langle v\rangle^{m_1}\langle \nabla_x\rangle^s Q^\varepsilon_{-+}(F_-,F_+), F_+^{(m_1,s)}\rangle_{L^2_{x,v}}. \tag{4.4.17}$$

What follows are estimates for each term on the right-hand side.

Step 1.1 (electric field commutator): We now bound (4.4.15):

$$(4.4.15) \lesssim 1 + \|F_+^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_v}. \tag{4.4.18}$$

Observe first that

$$[E\cdot\nabla_v, \langle v\rangle^m\langle\nabla_x\rangle^s]F_+ = E\cdot\nabla_v(\langle v\rangle^{m_1}\langle\nabla_x\rangle^s F_+) - \langle v\rangle^{m_1}E\cdot\nabla_v\langle\nabla_x\rangle^s F_+ \tag{4.4.19}$$

$$+ \langle v\rangle^{m_1}E\cdot\nabla_v\langle\nabla_x\rangle^s F_+ - \langle v\rangle^{m_1}\langle\nabla_x\rangle^s(E\cdot\nabla_v F_+) \tag{4.4.20}$$

$$= m_1 v\cdot EF_+^{(m_1-2,s)} \tag{4.4.21}$$

$$+ \langle v\rangle^{m_1}\nabla_v\cdot(E\langle\nabla_x\rangle^s F_+ - \langle\nabla_x\rangle^s(EF_+)). \tag{4.4.22}$$

Thus,

$$
\begin{aligned}
(4.4.15) &= m_1\langle v\cdot EF_+^{(m-2,s)}, F_+^{(m_1,s)}\rangle_{L^2_{x,v}} && (4.4.23)\\
&\quad - \langle E\langle\nabla_x\rangle^s F_+ - \langle\nabla_x\rangle^s(EF_+), \nabla_v(\langle v\rangle^{m_1}F_+^{(m_1,s)})\rangle_{L^2_{x,v}} && (4.4.24)\\
&\lesssim \|E\|_{L^\infty_x}\|F_+^{(m_1,s)}\|^2_{L^2_{x,v}} && (4.4.25)\\
&\quad + (\|\nabla_x E\|_{L^\infty_x}\|F_+^{(m_1+\frac{3}{2},s-1)}\|_{L^2_{x,v}} + \|E\|_{H^s_x}\|F_+^{(m_1+\frac{3}{2},0)}\|_{L^2_{x,v}})\|F_+^{(m_1-\frac{3}{2},s)}\|_{L^2_xH^1_v} && (4.4.26)\\
&\lesssim (\|F_+^{(m_1+\frac{3}{2},s-1)}\|_{L^2_{x,v}} + \|F_+^{(m_2,0)}\|_{L^2_{x,v}})(1+\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}). && (4.4.27)
\end{aligned}
$$

Next, note that we have the interpolation inequality

$$
\|F^{(m_1+\frac{3}{2},s-1)}\|_{L^2_{x,v}} \lesssim \|F_+^{(m_1,s)}\|^{\frac{s-1}{s}}_{L^2_{x,v}}\|F^{(m_1+\frac{3s}{2},0)}\|^{\frac{1}{s}}_{L^2_{x,v}} \lesssim \|F_+\|_{\mathfrak{E}} \lesssim 1, \qquad (4.4.28)
$$

since $m_1+\frac{3s}{2}\le m_2$. We conclude with (4.4.18).

Step 1.2 (self-collisions): Next, we have the following bound on (4.4.16):

$$
(4.4.16) \le -\frac{1}{C}\|F_+^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + C(1+\|F_+^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}). \qquad (4.4.29)
$$

For this, we separate

$$
\begin{aligned}
(4.4.16) &= \langle Q_D(F_+, F_+^{(m_1,s)}), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} && (4.4.30)\\
&\quad + \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q_D(F_+,F_+) - Q_D(F_+,F_+^{(m_1,s)}), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} && (4.4.31)\\
&\quad + \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q_T(F_+,F_+), F_+^{(m_1,s)}\rangle_{L^2_{x,v}}. && (4.4.32)
\end{aligned}
$$

Using (4.3.6), we have the bound

$$(4.4.30) = -\langle (\Phi_{ij} *_v F_+)\partial_{v_i} F_+^{(m_1,s)}, \partial_{v_j} F_+^{(m_1,s)} \rangle_{L^2_{x,v}} \tag{4.4.33}$$

$$\leq -\frac{1}{C_{k_{F_+}}} \| F_+^{(m_1,s)} \|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.4.34}$$

where

$$k_{F_+} = \max\{1, \|\langle v \rangle^5 F_+\|_{L^\infty_x L^2_v}, \|n_+^{-1}\|_{L^\infty_x}\}. \tag{4.4.35}$$

Now,

$$\|\langle v \rangle^5 F_+\|_{L^\infty_x L^2_v} \lesssim \|F_+^{(5,2)}\|_{L^2_{x,v}} \lesssim 1. \tag{4.4.36}$$

so $k_{F_+} \lesssim 1$. Using these estimates, we get that

$$(4.4.30) \leq -\frac{1}{C} \|F_+\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.4.37}$$

Next, we have the commutator term (4.4.31). Let us first write

$$\langle v \rangle^{m_1} \langle \nabla_x \rangle^s Q_D(F_+, F_+) - Q_D(F_+, F_+^{(m_1,s)}) \tag{4.4.38}$$

$$= \langle \nabla_x \rangle^s Q_D(F_+, F_+^{(m_1,0)}) - Q_D(F_+, F_+^{(m_1,s)}) \tag{4.4.39}$$

$$+ \langle \nabla_x \rangle^s \{ \langle v \rangle^m Q_D(F_+, F_+) - Q_D(F_+, F_+^{(m_1,0)}) \}. \tag{4.4.40}$$

Now, let

$$\mathcal{C}_j := \langle \nabla_x \rangle^s ((\Phi_{ij} *_v F_+)\partial_{v_i} F_+^{(m_1,0)}) - (\Phi_{ij} *_v F_+)\partial_{v_i} F_+^{(m_1,s)} \tag{4.4.41}$$

Then

$$\langle (4.4.39), F_+^{(m_1,s)} \rangle_{L^2_{x,v}} = \langle \mathcal{C}_i, \partial_{v_i} F_+^{(m_1,s)} \rangle_{L^2_{x,v}} \tag{4.4.42}$$

$$\leq \langle (\sigma^{-1})_{ij} \mathcal{C}_i, \mathcal{C}_j \rangle^{\frac{1}{2}}_{L^2_{x,v}} \langle \sigma_{ij} \partial_{v_i} F_+^{(m_1,s)}, \partial_{v_j} F_+^{(m_1,s)} \rangle^{\frac{1}{2}}_{L^2_{x,v}} \tag{4.4.43}$$

Now, the latter factor is $\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}$. On the other hand, writing

$$A_G = \sigma^{-\frac{1}{2}}\Phi *_v G\sigma^{-\frac{1}{2}}. \tag{4.4.44}$$

Note that by (4.3.5), $\|A_G\|_{L^\infty_v} \lesssim \|\langle v\rangle^5 G\|_{L^2_v}$ given a function $G = G(v)$. Therefore, we have

$$\begin{aligned}
\|\sigma^{-\frac{1}{2}}\mathcal{C}\|_{L^2_{x,v}} &= \|\langle\nabla_x\rangle^s(A_{F_+}\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1,0)}) - A_{F_+}\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1,s)}\|_{L^2_{x,v}} && (4.4.45)\\
&\lesssim \|A_{F_+^{(0,s)}}\|_{L^2_x L^\infty_v}\|\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1,s-1)}\|_{L^2_{x,v}} && (4.4.46)\\
&\lesssim \|F_+^{(m_1,s-1)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} && (4.4.47)\\
&\lesssim \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F_+^{(m_1,s)}\|^{\frac{s-1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. && (4.4.48)
\end{aligned}$$

Thus,

$$\langle(4.4.39), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \lesssim \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F_+^{(m_1,s)}\|^{\frac{2s-1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.4.49}$$

Next, we have

$$(4.4.40) = -m_1\langle\nabla_x\rangle^s\{v_j(\Phi_{ij} *_v F_+)\partial_{v_i}F_+^{(m_1-2,0)} + \partial_{v_j}(v_i(\Phi_{ij} *_v F_+)F_+^{(m_1-2,0)}) \tag{4.4.50}$$

$$- (m_1-2)v_i v_j(\Phi_{ij} *_v F_+)F_+^{(m_1-4,0)}\}. \tag{4.4.51}$$

Thus,

$$\langle (4.4.40), F_+^{(m_1,s)} \rangle_{L^2_{x,v}} = -m_1 \langle \langle \nabla_x \rangle^s \{ v_j (\Phi_{ij} *_v F_+) \partial_{v_i} F_+^{(m_1-2,0)} \}, F_+^{(m_1,s)} \rangle_{L^2_{x,v}} \quad (4.4.52)$$

$$+ m_1 \langle \langle \nabla_x \rangle^s \{ \langle v \rangle^{m_1-2} v_i (\Phi_{ij} *_v F_+) F_+^{(m_1-2,0)} \}, \partial_{v_j} F_+^{(m_1,s)} \rangle_{L^2_{x,v}} \quad (4.4.53)$$

$$+ m_1(m_1-2) \langle \langle \nabla_x \rangle^s \{ v_i v_j (\Phi_{ij} *_v F_+) F_+^{(m_1-4,0)} \}, F_+^{(m_1,s)} \rangle_{L^2_{x,v}} \quad (4.4.54)$$

$$\lesssim \Big\| |\sigma^{\frac{1}{2}} v| \|A_{F_+^{(0,s)}}\|_{L^2_x} \|\sigma^{\frac{1}{2}} \nabla_v F_+^{(m_1-2,s)}\|_{H^s_x} \Big\|_{L^2_v} \|F_+^{(m_1,s)}\|_{L^2_{x,v}} \quad (4.4.55)$$

$$+ \Big\| |\sigma^{\frac{1}{2}} v| \|A_{F_+^{(0,s)}}\|_{L^2_x} \|F_+^{(m_1-2,s)}\|_{H^s_x} \Big\|_{L^2_v} \|\sigma^{\frac{1}{2}} \nabla_v F_+^{(m_1,s)}\|_{L^2_{x,v}} \quad (4.4.56)$$

$$+ \Big\| (v \cdot (\sigma^{\frac{1}{2}} v)) \|A_{F_+^{(0,s)}}\|_{L^2_x} \|F_+^{(m_1-4,s)}\|_{H^s_x} \Big\|_{L^2_v} \|F_+^{(m_1,s)}\|_{L^2_{x,v}} \quad (4.4.57)$$

$$\lesssim \|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \quad (4.4.58)$$

Therefore,

$$(4.4.31) \lesssim \|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \|F_+^{(m_1,s)}\|^{\frac{2s-1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \quad (4.4.59)$$

Next we estimate (4.4.32). We bound this as follows:

$$(4.4.32) = \langle \langle \nabla_x \rangle^s \{ \partial_{v_i} (\Phi_{ij} *_v F_+) F_+^{(m_1,0)} \}, \partial_{v_j} F^{(m_1,s)} \rangle_{L^2_{x,v}} \tag{4.4.60}$$

$$+ \langle \langle \nabla_x \rangle^s \{ v_j \partial_{v_i} (\Phi_{ij} *_v F_+) F_+^{(m_1-2,0)} \}, F^{(m_1,s)} \rangle_{L^2_{x,v}} \tag{4.4.61}$$

$$\lesssim \| \langle \nabla_x \rangle^s \{ \sigma^{-\frac{1}{2}} (\nabla_v \cdot \Phi *_v F_+) F_+ \} \|_{L^2_{x,v}} \| \sigma^{\frac{1}{2}} \nabla_v F_+^{(m_1,s)} \|_{L^2_{x,v}} \tag{4.4.62}$$

$$+ \| \langle \nabla_x \rangle^s \{ v_j \partial_{v_i} (\Phi_{ij} *_v F_+) F_+^{(m_1-2,0)} \} \|_{L^2_{x,v}} \| F_+^{(m_1,s)} \|_{L^2_{x,v}} \tag{4.4.63}$$

$$\lesssim \| \sigma^{-\frac{1}{2}} \nabla_v (-\Delta_v)^{-1} F_+^{(0,s)} \|_{L^2_x L^\infty_v} \| F_+^{(m_1,s)} \|_{L^2_{x,v}} \| F_+^{(m_1,s)} \|_{L^2_x (\dot{\mathcal{H}}_\sigma)_v} \tag{4.4.64}$$

$$+ \| \nabla_v (-\Delta_v)^{-1} F_+^{(0,s)} \|_{L^2_x L^\infty_v} \| F_+^{(m_1,s)} \|_{L^2_{x,v}} \| F_+^{(m_1,s)} \|_{L^2_{x,v}} \tag{4.4.65}$$

$$\lesssim \| \langle v \rangle^{\frac{3}{2}} \nabla_v (-\Delta_v)^{-1} F_+^{(0,s)} \|_{L^2_x L^\infty_v} (1 + \| F_+^{(m_1,s)} \|_{L^2_x (\dot{\mathcal{H}}_\sigma)_v}). \tag{4.4.66}$$

Now, by

$$\langle v \rangle^{\frac{3}{2}} | \nabla_v (-\Delta_v)^{-1} F_+^{(0,s)} | \lesssim \langle v \rangle^{\frac{3}{2}} \int_{\langle v \rangle > 2|v'|} \frac{|F_+^{(0,s)}(v')|}{|v-v'|^2} dv' + \langle v \rangle^{\frac{3}{2}} \int_{\langle v \rangle \leq 2|v'|} \frac{|F_+^{(0,s)}(v')|}{|v-v'|^2} dv' \tag{4.4.67}$$

$$\lesssim \langle v \rangle^{-\frac{1}{2}} \| F_+^{(0,s)} \|_{L^1_v} + \int_{\mathbf{R}^3} \frac{|F_+^{(\frac{3}{2},s)}(v')|}{|v-v'|^2} dv' \tag{4.4.68}$$

Thus,

$$\| \langle v \rangle^{\frac{3}{2}} \nabla_v (-\Delta_v)^{-1} F_+^{(0,s)} \|_{L^2_x L^\infty_v} \lesssim \| F_+^{(0,s)} \|_{L^2_x L^1_v} + \| |\nabla_v|^{-1} |F_+^{(\frac{3}{2},s)}| \|_{L^2_x L^\infty_v} \tag{4.4.69}$$

$$\lesssim \| F_+^{(2,s)} \|_{L^2_{x,v}} + \| \langle \nabla_v \rangle^{\frac{3}{4}} |F_+^{(\frac{3}{2},s)}| \|_{L^2_{x,v}} \tag{4.4.70}$$

$$\lesssim 1 + \| |F_+^{(\frac{3}{2},s)}| \|^{\frac{3}{4}}_{L^2_x H^1_v}. \tag{4.4.71}$$

$$\lesssim 1 + \| F_+^{(m_1,s)} \|^{\frac{3}{4}}_{L^2_x (\dot{\mathcal{H}}_\sigma)_v} \tag{4.4.72}$$

In the above, we use the interpolation inequality $\|u\|_{H^{3/4}} \lesssim \|u\|^{\frac{1}{4}}_{L^2} \|u\|^{\frac{3}{4}}_{H^1}$, followed by the fact that $\big|\nabla_v|u|\big| = |\nabla_v u|$ a.e., for any function $u \in H^1(\mathbf{R}^3)$. Thus,

$$(4.4.32) \lesssim 1 + \|F_+^{(m_1,s)}\|^{\frac{7}{4}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.4.73}$$

Combining the estimates (4.4.30), (4.4.31) and (4.4.32), we conclude

$$\begin{aligned}(4.4.16) \leq & -\frac{1}{C}\|F_+^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} && (4.4.74)\\ & + C(1 + \|F_+^{(m_1,s)}\|^{\frac{7}{4}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F_+^{(m_1,s)}\|^{\frac{2s-1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}). && (4.4.75)\end{aligned}$$

We then use Hölder's inequality to get (4.4.29).

Step 1.3 (control of ion-electron collisions): Now, we turn to (4.4.17), for which we have the bound

$$\begin{aligned}(4.4.17) \leq & -\frac{1}{C}\{\|F_+^{(m_1,s)}\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} - \lambda\|F_+^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\}\\ & + C\|F_+^{(m_1,0)}\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}\\ & + C_\lambda(1 + \|F_-^{(m_1,s)}\|^{\frac{3}{2}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_\xi}).\end{aligned} \tag{4.4.76}$$

for all $\lambda > 0$. We decompose this term in the same way as (4.4.16),

$$\begin{aligned}(4.4.17) = & \langle Q^\varepsilon_{-+,D}(F_-, F_+^{(m_1,s)}), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} && (4.4.77)\\ & + \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q^\varepsilon_{-+,D}(F_-, F_+) - Q^\varepsilon_{-+,D}(F_-, F_+^{(m_1,s)}), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} && (4.4.78)\\ & + \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q^\varepsilon_{-+,T}(F_-, F_+), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} && (4.4.79)\end{aligned}$$

The terms (4.4.77) and (4.4.78) using the same approach as (4.4.30) and (4.4.31),

$$(4.4.77) \le -\frac{1}{C}\|F_+^{(m_1,s)}\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}, \tag{4.4.80}$$

$$(4.4.78) \lesssim \|F_+^{(m_1,s)}\|_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} + \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}\|F_+^{(m_1,s)}\|^{\frac{2s-1}{s}}_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}. \tag{4.4.81}$$

The final term (4.4.79) requires a different approach: we separate it into a main term and commutators,

$$(4.4.79) = \langle Q^\varepsilon_{-+,T}(F_-, F_+^{(m_1,s)}), F^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.4.82}$$

$$+ \langle \langle \nabla_x\rangle^s Q^\varepsilon_{-+,T}(F_-, F_+^{(m_1,0)}) - Q^\varepsilon_{-+,T}(F_-, F_+^{(m_1,s)}), F^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.4.83}$$

$$+ \langle \langle \nabla_x\rangle^s \{\langle v\rangle^{m_1} Q^\varepsilon_{-+,T}(F_-, F_+) - Q^\varepsilon_{-+,T}(F_-, F_+^{(m_1,0)})\}, F^{(m_1,s)}\rangle_{L^2_{x,v}}. \tag{4.4.84}$$

For the first term, we note that $\partial_{v_i}\partial_{v_j}\Phi_{ij}(v) = -8\pi\delta(v)$ in the sense of distributions, where δ is the Dirac mass. Hence,

$$(4.4.82) = 4\pi\varepsilon\langle F_-|_{\xi=\varepsilon v}F_+^{(m_1,s)}, F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.4.85}$$

$$\lesssim \varepsilon\|\langle\varepsilon v\rangle^{\frac{3}{4}}F_-|_{\xi=\varepsilon v}\|_{L^\infty_x L^6_v}\|\langle\varepsilon v\rangle^{-\frac{3}{4}}F_+^{(m_1,s)}\|_{L^2_x L^3_v}\|F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.4.86}$$

$$\lesssim \varepsilon^{\frac{1}{2}}\|F_-^{(\frac{3}{4},0)}\|_{L^\infty_x L^6_\xi}\|\langle\varepsilon v\rangle^{-\frac{3}{4}}F_+^{(m_1,s)}\|_{L^2_x L^3_v} \tag{4.4.87}$$

Now, using the generalized Minkowski inequality and Sobolev embedding,

$$\|F_-^{(\frac{3}{4},0)}\|_{L^\infty_x L^6_\xi} \lesssim \|\nabla_v F_-^{(\frac{3}{4},0)}\|_{L^\infty_x L^2_\xi} \lesssim \|\nabla_v F_-^{(\frac{3}{4},s)}\|_{L^2_{x,\xi}} \lesssim 1 + \|F_-^{(\frac{9}{4},s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.4.88}$$

On the other hand,

$$\|\langle\varepsilon v\rangle^{-\frac{3}{4}}F_+^{(m_1,s)}\|_{L^2_x L^3_v} \lesssim \|F_+^{(m_1,s)}\|^{\frac{1}{2}}_{L^2_{x,v}}\|\langle\varepsilon v\rangle^{\frac{3}{2}}F_+^{(m_1,s)}\|^{\frac{1}{2}}_{L^2_x H^1_v} \tag{4.4.89}$$

$$\lesssim \varepsilon^{-\frac{1}{4}}\|F_+^{(m_1,s)}\|^{\frac{1}{2}}_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} \tag{4.4.90}$$

We conclude that

$$(4.4.82) \leq C\varepsilon^{\frac{1}{4}}\|F_+^{(m_1,s)}\|_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}^{\frac{1}{2}}\|F_-^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.4.91}$$

As for (4.4.83), we have

$$(4.4.83) = 8\pi\langle\langle\nabla_x\rangle^s\{\nabla_\xi(-\Delta_\xi)^{-1}F_-|_{\xi=\varepsilon v}F_+^{(m_1,0)}\} - \nabla_\xi(-\Delta_\xi)^{-1}F_-|_{\xi=\varepsilon v}F_+^{(m_1,s)}, \nabla_v F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.4.92}$$

$$\lesssim \left\|\langle v\rangle^{\frac{3}{2}}\||\nabla_\xi|^{-1}|F_-^{(0,s)}|_{\xi=\varepsilon v}|\|_{L^2_x}\|F_+^{(m_1,s-1)}\|_{L^2_x}\right\|_{L^2_v}\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.4.93}$$

$$\lesssim \||\nabla_\xi|^{-1}|F_-^{(0,s)}|\|_{L^2_xL^\infty_\xi}\|F_+^{(m+\frac{3}{2},s-1)}\|_{L^2_{x,v}}\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.4.94}$$

$$\lesssim \||F_-^{(0,s)}|\|_{L^2_xH^{\frac{3}{4}}_\xi}\|F_+^{(m+\frac{3s}{2},0)}\|_{L^2_{x,v}}^{\frac{1}{s}}\|F_+^{(m_1,s)}\|_{L^2_{x,v}}^{\frac{s-1}{s}}\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.4.95}$$

$$\lesssim \|F_-^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}^{\frac{3}{4}}\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.4.96}$$

The second commutator (4.4.83) is bounded as follows:

$$(4.4.83) = 8\pi\langle\langle\nabla_x\rangle^s\{v\cdot\nabla_\xi(-\Delta_\xi)^{-1}F_-|_{\xi=\varepsilon v}F_+^{(m_1-2,0)}\}, F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.4.97}$$

$$\lesssim \||\nabla_v|^{-1}F_-^{(0,s)}\|_{L^2_xL^\infty_\xi}\|F_+^{(m_1,s)}\|_{L^2_{x,v}}^2 \tag{4.4.98}$$

$$\lesssim \|F_-^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.4.99}$$

Thus,

$$(4.4.79) \lesssim (\|F_-^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} + \varepsilon^{\frac{1}{4}}\|F_+^{(m_1,s)}\|_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}^{\frac{1}{2}}\|F_-^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.4.100}$$

$$+ \|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F_-^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}^{\frac{3}{4}}). \tag{4.4.101}$$

Now, combining the bounds on (4.4.77), (4.4.78), (4.4.79), and Young's inequality, we conclude (4.4.76).

To conclude step 1, we combine (4.4.18), (4.4.29) and (4.4.76), taking λ sufficiently small, to get (4.4.11).

Step 2 (combining the estimates): Take $0 < \kappa_1 < \kappa_2$, and set $\mathscr{X}_+ := \kappa_1 \|F_+^{(m_1,s)}\|^2_{L^2_{x,v}} + \kappa_2 \|F_+^{(m_2,0)}\|^2_{L^2_{x,v}}$. Then by (4.4.10) and (4.4.11), we have

$$\frac{d}{dt}\mathscr{X}_+ + \frac{\kappa_1}{C}\|F^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} + \left(\frac{\kappa_2}{C} - C\kappa_1\right)\|F^{(m_2,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} \tag{4.4.102}$$

$$\leq C(\kappa_1 + \kappa_2)\langle \|F_-\|_{\mathfrak{D}}\rangle^{\frac{3}{2}} \tag{4.4.103}$$

Taking κ_2 sufficiently large and κ_1 sufficiently small depending on M, we get (4.4.2).

Alternatively, by simply adding (4.4.10) and (4.4.11), we have (4.4.3). □

4.4.2 Error estimate for the ion distribution

Proposition 4.4.2. *Let $M, T > 0$, and let $(F_+^\varepsilon, F_-^\varepsilon)$ be a solution to (4.1.7) satisfying the same hypotheses as in 4.4.1, and (F_+^0, β, ϕ^0) be a solution to (4.1.18), satisfying the hypothesis of Theorem 4.1.1. Then, there is a function $\mathscr{Y}_+^\varepsilon : [0, T^*) \to \mathbf{R}_+$ such that $\mathscr{Y}_+^\varepsilon \sim_M \|F_+^\varepsilon - F_+^0\|_{\mathfrak{E}'}$ and*

$$\begin{aligned}&\frac{d}{dt}\mathscr{Y}_+^\varepsilon + \|F_+^\varepsilon - F_+^0\|^2_{\mathfrak{D}'}\\ &\quad\lesssim_M \langle \|(F_+^0, F_-^\varepsilon)\|_{\mathfrak{D}}\rangle^2(\varepsilon^2 + \mathscr{Y}_+^\varepsilon) + \mathscr{D}^\varepsilon_{-,2}.\end{aligned} \tag{4.4.104}$$

Proof. Let $G = F_+^\varepsilon - F_+^0$. Then,

$$\partial_t G + \{v \cdot \nabla_x + E^\varepsilon \cdot \nabla_v\}G - (E^\varepsilon - E^0) \cdot \nabla_v F_+^0 - Q(F_+^\varepsilon, G) - Q(G, F_+^0) \tag{4.4.105}$$

$$= Q_{-+}(F_-^\varepsilon, F_+^\varepsilon) \tag{4.4.106}$$

Similarly to the proof of the previous proposition, given $m', s' \in \mathbf{R}$, we denote $G^{(m',s')} = \langle v\rangle^{m'}\langle\nabla_x\rangle^s G$, $(F_\pm^\varepsilon)^{(m',s')} = \langle v\rangle^{m'}\langle\nabla_x\rangle^s F_\pm^\varepsilon$, etc. The above gives

$$\frac{1}{2}\frac{d}{dt}\|G^{(m_0,s)}\|_{L^2_{x,v}}^2 = \langle[\langle v\rangle^{m_0}\langle\nabla_x\rangle^s, E^\varepsilon \cdot \nabla_v]G, G^{(m_0,s)}\rangle_{L^2_x} \tag{4.4.107}$$

$$+ \langle\langle v\rangle^{m_0}\langle\nabla_x\rangle^s\{(E^0 - E^\varepsilon) \cdot \nabla_v F_+^0\}, G^{(m_0,s)}\rangle_{L^2_{x,\xi}} \tag{4.4.108}$$

$$+ \langle\langle v\rangle^{m_0}\langle\nabla_x\rangle^s Q(F_+^\varepsilon, G), G^{(m_0,s)}\rangle_{L^2_{x,v}} \tag{4.4.109}$$

$$+ \langle\langle v\rangle^{m_0}\langle\nabla_x\rangle^s Q(G, F_+^0), G^{(m_0,s)}\rangle_{L^2_{x,v}} \tag{4.4.110}$$

$$+ \langle\langle v\rangle^{m_0}\langle\nabla_x\rangle^s Q_{-+}^\varepsilon(F_-^\varepsilon, F_+^\varepsilon), G^{(m_0,s)}\rangle_{L^2_{x,v}}. \tag{4.4.111}$$

We modify the estimates from Proposition 4.4.1 to get

$$|(4.4.107)| \lesssim \|G\|_{\mathfrak{E}'}(\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}), \tag{4.4.112}$$

$$|(4.4.109)| \le -\frac{1}{C}\|G^{(m_0,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + C(\langle\|F_+^\varepsilon\|_{\mathfrak{D}}\rangle^2\|G\|^2_{\mathfrak{E}'} + \|G^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}), \tag{4.4.113}$$

$$|(4.4.110)| \lesssim \|G\|^2_{\mathfrak{E}'} + \|G\|^{\frac{1}{4}}_{\mathfrak{E}'}\|G^{(m_0,s)}\|^{\frac{7}{4}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \langle\|F_+^0\|_{\mathfrak{D}}\rangle\|G\|_{\mathfrak{E}'}\|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.4.114}$$

What's left is to bound (4.4.108) and (4.4.111). For the former, we have

$$|(4.4.108)| \lesssim \|E^0 - E^\varepsilon\|_{H^s_x}(\|(F^0_+)^{(m_0,s)}\|_{L^2_{x,\xi}} + \|(F^0_+)^{(m_0+\frac{3}{2},s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v})\|G^{(m_0,s)}\|_{L^2_{x,v}} \tag{4.4.115}$$

$$\lesssim (\|G\|_{L^1_v H^{s-1}_x} + \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|_{L^1_\xi H^{s-1}_x})\langle \|F^0_+\|_{\mathfrak{D}}\rangle \tag{4.4.116}$$

$$\lesssim (\|G\|_{\mathfrak{E}'} + \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|_{\mathfrak{E}'})\langle \|F^0_+\|_{\mathfrak{D}}\rangle \tag{4.4.117}$$

In the above, we used the fact that $m_0 + \frac{3}{2} \leq m_1$, and

$$-\Delta_x(\phi^\varepsilon - \phi^0) = \int_{\mathbf{R}^3} G dv - \int_{\mathbf{R}^3} F^\varepsilon_- - \mu_\beta e^{\beta\phi^0} d\xi. \tag{4.4.118}$$

As for (4.4.111), we split the term as follows:

$$(4.4.111) = \langle \langle v\rangle^{m_0} \langle \nabla_x\rangle^s Q^\varepsilon_{-+}(F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}, F^\varepsilon_+), G^{(m_0,s)}\rangle_{L^2_{x,v}} \tag{4.4.119}$$

$$+ \langle \langle v\rangle^{m_0} \langle \nabla_x\rangle^s Q^\varepsilon_{-+}(\mu_\beta e^{\beta\phi^0}, F^\varepsilon_+), G^{(m_0,s)}\rangle_{L^2_{x,v}} \tag{4.4.120}$$

Now,

$$|(4.4.119)| \lesssim \varepsilon \|\Delta^{-1}_\xi (F^\varepsilon_- - \mu_\beta e^{\beta\phi^0})\|_{L^\infty_\xi H^s_x} \|(F^\varepsilon_+)^{(m_0+\frac{3}{2},s)}\|_{L^2_x H^1_v} \|G^{(m_0-\frac{3}{2},s)}\|_{L^2_x H^1_v} \tag{4.4.121}$$

$$+ \|\nabla_\xi \Delta^{-1}_\xi (F^\varepsilon_- - \mu_\beta e^{\beta\phi^0})\|_{L^\infty_\xi H^s_x} \|(F^\varepsilon_+)^{(m_0+\frac{3}{2},s)}\|_{L^2_{x,v}} \|G^{(m_0-\frac{3}{2},s)}\|_{L^2_x H^1_v} \tag{4.4.122}$$

$$\lesssim \varepsilon \langle \|F^\varepsilon_+\|_{\mathfrak{D}}\rangle (\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}) \tag{4.4.123}$$

$$+ \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|^{\frac{1}{4}}_{\mathfrak{E}'} \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|^{\frac{3}{4}}_{\mathfrak{D}'} (\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}) \tag{4.4.124}$$

In the above, we use $m_0 + 3 \leq m_1$. Next, in order to bound (4.4.120), we compute directly from (4.1.10) that

$$
\begin{aligned}
&Q^{\varepsilon}_{-+}(\mu_\beta e^{\beta\phi^0}, F^{\varepsilon}_{+}) && (4.4.125)\\
&= e^{\beta\phi^0}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(\xi - \varepsilon v)\{\varepsilon\mu_\beta(\xi)\nabla_v F^{\varepsilon}_{+}(v) + \beta\xi\mu_\beta(\xi)F^{\varepsilon}_{+}(v)\}d\xi && (4.4.126)\\
&= \varepsilon e^{\beta\phi^0}\nabla_v \cdot \{\Phi * \mu_\beta(\varepsilon v)(\beta v + \nabla_v)F^{\varepsilon}_{+}(v)\}. && (4.4.127)
\end{aligned}
$$

In the above, we use $\Phi(\xi - \varepsilon v)\xi = \varepsilon\Phi(\xi - \varepsilon v)v$. Now, $\|\Phi_{ij} * \mu_\beta(v)\|_{L^\infty} \lesssim 1$, so

$$
|(4.4.120)| \lesssim \varepsilon(\|(F^{\varepsilon}_{+})^{(m_0+\frac{5}{2},s)}\|_{L^2_{x,v}}\|G^{(m_0-\frac{3}{2},s)}\|_{L^2_x H^1_v} + \|(F^{\varepsilon}_{+})^{(m_0+\frac{3}{2},s)}\|_{L^2_x H^1_v}\|G^{(m_0-\frac{3}{2},s)}\|_{L^2_x H^1_v}) \quad (4.4.128)
$$

$$
\lesssim \varepsilon\langle\|F^{\varepsilon}_{+}\|_{\mathfrak{D}}\rangle(\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\mathcal{H}_\sigma)_v}). \quad (4.4.129)
$$

Thus,

$$
\begin{aligned}
|(4.4.111)| &\lesssim (\varepsilon\langle\|F^{\varepsilon}_{+}\|_{\mathfrak{D}}\rangle + \|F^{\varepsilon}_{-} - \mu_\beta e^{\beta\phi^0}\|^{\frac{1}{4}}_{\mathfrak{E}'}\|F^{\varepsilon}_{-} - \mu_\beta e^{\beta\phi^0}\|^{\frac{3}{4}}_{\mathfrak{D}'}) && (4.4.130)\\
&\quad \cdot (\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}). && (4.4.131)
\end{aligned}
$$

Combining the bounds on (4.4.107) through (4.4.111), we have

$$
\begin{aligned}
&\frac{d}{dt}\|G^{(m_0,s)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|G^{(m_0,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} && (4.4.132)\\
&\quad \le C\{\langle\|(F^{\varepsilon}_{+}, F^0_{+})\|_{\mathfrak{D}}\rangle^2(\varepsilon^2 + \|G\|^2_{\mathfrak{E}'}) + \|G^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} && (4.4.133)\\
&\quad\quad + \|F^{\varepsilon}_{-} - \mu_\beta e^{\beta\phi^0}\|^{\frac{1}{2}}_{\mathfrak{E}'}\|F^{\varepsilon}_{-} - \mu_\beta e^{\beta\phi^0}\|^{\frac{3}{2}}_{\mathfrak{D}'}\} && (4.4.134)
\end{aligned}
$$

Following a similar argument, we can show the following upper bound on $G^{(m_1,0)}$:

$$
\begin{aligned}
&\frac{d}{dt}\|G^{(m_1,0)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|G^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} && (4.4.135)\\
&\le C(\langle\|(F^{\varepsilon}_{+}, F^0_{+})\|_{\mathfrak{D}}\rangle^2(\varepsilon^2 + \|G\|^2_{\mathfrak{E}'}) + \|F^{\varepsilon}_{-} - \mu_\beta e^{\beta\phi^0}\|^{\frac{1}{2}}_{\mathfrak{E}'}\|F^{\varepsilon}_{-} - \mu_\beta e^{\beta\phi^0}\|^{\frac{3}{2}}_{\mathfrak{D}'}). && (4.4.136)
\end{aligned}
$$

Now, choosing $0 < \kappa_1 < \kappa_2$ appropriately, we have that

$$\mathscr{Y}_+^\varepsilon = \kappa_2 \|G^{(m_1,0)}(t)\|_{L^2_{x,v}}^2 + \kappa_1 \|G^{(m_0,s)}(t)\|_{L^2_{x,v}}^2 \tag{4.4.137}$$

saqtisfies

$$\frac{d}{dt}\mathscr{Y}_+^\varepsilon + \|G\|_{\mathfrak{D}'}^2 \tag{4.4.138}$$

$$\lesssim \langle \|(F_+^\varepsilon, F_+^0)\|_{\mathfrak{D}}\rangle^2 (\varepsilon^2 + \|G\|_{\mathfrak{E}'}^2) + \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{\mathfrak{E}'}^{\frac{1}{2}} \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{\mathfrak{D}'}^{\frac{3}{2}}. \tag{4.4.139}$$

From this, we deduce (4.4.104). $\square$

4.4.3 The intermediary quantities

In preparation for Section 4.5, we introduce the what we call the intermediary potential ψ^ε and intermediary inverse temperature γ^ε. Fixing an initial inverse temperature β_{in}, the pair $(\gamma^\varepsilon, \psi^\varepsilon)$ solve the Poincaré-Poisson system for each $\varepsilon > 0$:

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{3}{2\gamma^\varepsilon} + \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2} F_+^\varepsilon dxdv + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \psi^\varepsilon|^2 dx\right\} = 0, \\ &\gamma^\varepsilon(0) = \beta_{in} \\ &-\Delta_x \psi^\varepsilon = 4\pi(n_+^\varepsilon - e^{\gamma^\varepsilon \psi^\varepsilon}) \end{aligned} \tag{4.4.140}$$

We call ψ^ε the intermediary potential because like ϕ^0, it solves a Poincaré-Poisson system; however, unlike ϕ^0, we use F_+^ε instead of F_+^0. Thus ψ serves as a better approximation to ϕ^ε than ϕ^0. Similarly, $\frac{3}{2\gamma^\varepsilon}$ serves as a better approximation of $\iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|\xi|^2}{2} F_-^\varepsilon dxdv$ than $\frac{3}{2\beta}$. The lemma below gives existence and uniqueness for the pair $(\gamma^\varepsilon, \psi^\varepsilon)$, for a given F_+^ε, along with bounds that will be used in the coming section.

Lemma 4.4.3. *Let M, T, $\eta \in (0,1]$ and $\beta_{in} \in (e^{-M}, e^{M})$. Assume also that $(F_+^\varepsilon, F_-^\varepsilon)$ is a weak solution to (4.1.7) with $0 \le F_\pm^\varepsilon \in C([0,T];\mathfrak{E}) \cap L^2([0,T];\mathfrak{D})$ satisfying*

$$\sup_{\pm \in \{-,+\}, t \in [0,T]} \| \frac{1}{n_\pm^\varepsilon(t)} \|_{L_x^\infty} + \|F_\pm^\varepsilon(t)\|_{\mathfrak{e}} \le M. \tag{4.4.141}$$

Then there exists $0 < T^*$ such that there exists a unique solution $(\gamma^\varepsilon, \psi^\varepsilon) \in C([0,T] \cap [0,T^*); \mathbf{R}_+ \times H^{s+2})$ to (4.4.140). Moreover, if $T^* \le T$, then $\beta(t) \uparrow +\infty$ as $t \uparrow T^*$.

Moreover, assume that for some $T' \in (0,T]$, that

$$\sup_{t \in [0,T']} |\ln(\frac{\gamma^\varepsilon(t)}{\beta_{in}})| \le \eta, . \tag{4.4.142}$$

Note that $T < T^*$ necessarily. Then, we have the estimates

$$\sup_{t \in [0,T']} \|\psi^\varepsilon\|_{H_x^{s+2}} \lesssim_M 1, \tag{4.4.143}$$

$$\sup_{t \in [0,T']} \|\partial_t \psi^\varepsilon\|_{H_x^{s+1}} + |\dot{\gamma}^\varepsilon| \lesssim_M \langle \| \langle \xi \rangle^5 F_-^\varepsilon \|_{L_x^2(\dot{\mathcal{H}}_\sigma)_\xi} \rangle, \tag{4.4.144}$$

Next, assume (F_+^0, β, ϕ^0) is a weak solution to (4.1.18) and satisfies the hypotheses of Theorem 4.1.1 (with $T_0 = T$), with the given β_{in}. Then, we also control the difference of $(\gamma^\varepsilon, \psi^\varepsilon)$ and (β, ϕ^0):

$$\sup_{t \in [0,T']} \{ |\gamma^\varepsilon(t) - \beta(t)| + \|\psi^\varepsilon(t) - \phi^0(t)\|_{H_x^{s+1}} \} \lesssim \sup_{t \in [0,T]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{e}'}. \tag{4.4.145}$$

Proof. We break the proof up into 3 parts: first, the bounds (4.4.143), and (4.4.144); second, the bound (4.4.145); and third, we sketch how to construct the solutions. Once again, all bounds involved may depend on M.

Step 1: We now show (4.4.143), and (4.4.144). For this step, it is convenient to drop the superscripts of ε, i.e. $F_+^\varepsilon = F_+$, $F_-^\varepsilon = F_-$, etc. We now prove (4.4.143), from the third line of (4.4.140), and the maximum principle, we have the estimate

$$\|\psi\|_{L_x^\infty} \lesssim \frac{1}{\gamma}\|\ln(n_+)\|_{L_x^\infty} \lesssim 1. \tag{4.4.146}$$

Next, applying $|\nabla_x|^s$ to both sides of the third line of (4.4.140),

$$\frac{1}{4\pi}|\nabla_x|^{s'+2}\psi = |\nabla_x|^{s'}\{n_+ - e^{\gamma\psi}\}. \tag{4.4.147}$$

Using Theorem 5.2.6 in [103],

$$\|e^{\gamma\psi}\|_{H_x^s} \leq C_{\|\psi\|_{L_x^\infty}}(\|\psi\|_{H_x^s} + 1) \lesssim \|\psi\|_{H_x^s} + 1. \tag{4.4.148}$$

Hence,

$$\|\psi\|_{H_x^{s+2}} \lesssim 1 + \|n_+\|_{H_x^s} + \|\psi\|_{H_x^s} \lesssim 1 + \|\psi\|_{H_x^{s+2}}^{\frac{s}{s+2}}\|\psi\|_{L_x^2}^{\frac{2}{s+2}}. \tag{4.4.149}$$

Now, using $\|\psi\|_{L_x^2} \lesssim \|\psi\|_{L_x^\infty} \lesssim 1$ and Young's inequality, we deduce $\|\psi\|_{H_x^{s+2}} \lesssim 1$.

Next, we have

$$(\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)\partial_t\psi = \partial_t n_+ - \dot{\gamma}e^{\gamma\psi}\psi. \tag{4.4.150}$$

The maximum principle implies

$$e^{-C\|\psi\|_{L_x^\infty}}\|\partial_t\psi\|_{L_x^\infty} \leq C\|\partial_t n_+\|_{L_x^\infty} + |\dot{\gamma}|e^{C\|\psi\|_{L_x^\infty}}\|\psi\|_{L_x^\infty}, \tag{4.4.151}$$

From the continuity equation, we know $\|\partial_t n_+\|_{L_x^\infty} \lesssim \|\partial_t n_+\|_{H_x^{s-1}} \lesssim 1$; in combination with the estimates on ψ, this implies

$$\|\partial_t\psi\|_{L^\infty_x} \lesssim 1+|\dot{\gamma}|. \tag{4.4.152}$$

Similarly as was done to get the bound on $\|\psi\|_{H^{s+2}_x}$, we deduce

$$\|\partial_t\psi\|_{H^{s+1}_x} \lesssim 1+|\dot{\gamma}|. \tag{4.4.153}$$

We now estimate $\dot{\gamma}$: first,

$$\frac{3\dot{\gamma}}{2\gamma^2} = \frac{1}{2}\iint_{\mathbf{T}^3\times\mathbf{R}^3}|v|^2\partial_t F_+ dxdv + \frac{1}{4\pi}\int_{\mathbf{T}^3}\nabla_x\psi\cdot\partial_t\nabla_x\psi dx. \tag{4.4.154}$$

We first analyze the last term. From (4.4.150),

$$\begin{aligned}\frac{1}{4\pi}\int_{\mathbf{T}^3}\nabla_x\psi\cdot\partial_t\nabla_x\psi dx &= -\frac{1}{4\pi}\langle\Delta_x\psi,\partial_t\psi\rangle_{L^2_x} &(4.4.155)\\ &= -\frac{1}{4\pi}\langle\Delta_x\psi,(\gamma e^{\gamma\psi}-\frac{1}{4\pi}\Delta_x)^{-1}\partial_t n_+\rangle_{L^2_x} &(4.4.156)\\ &\quad+\frac{\dot{\gamma}}{4\pi}\langle\Delta_x\psi,(\gamma e^{\gamma\psi}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} &(4.4.157)\end{aligned}$$

Looking more closely at the latter term, we observe

$$\begin{aligned}&\frac{1}{4\pi}\langle\Delta_x\psi,(\gamma e^{\gamma\psi}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} &(4.4.158)\\ &= -\langle(\gamma e^{\gamma\psi}-\frac{1}{4\pi}\Delta_x)\psi,(\gamma e^{\gamma\psi}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} &(4.4.159)\\ &\quad+\gamma\langle e^{\gamma\psi}\psi,(\gamma e^{\gamma\psi}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} &(4.4.160)\\ &= -\langle\psi,e^{\gamma\psi}\psi\rangle_{L^2_x}-\gamma\langle e^{\gamma\psi}\psi,(\gamma e^{\gamma\psi}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x}\le 0. &(4.4.161)\end{aligned}$$

Thus, we have that

$$\dot{\gamma} = \frac{\frac{1}{2}\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2\partial_t F_+ dxdv - \frac{1}{4\pi}\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}\partial_t n_+\rangle_{L^2_x}}{\frac{3}{2\gamma^2} - \frac{1}{4\pi}\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x}}, \tag{4.4.162}$$

with the denominator being strictly bounded from below by $\frac{3}{2\gamma^2}$. On the other hand, by the continuity equation $\partial_t n_+ + \int_{\mathbf{R}^3} v\cdot\nabla_x F_+ dxdv = 0$, we have

$$|\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}\partial_t n_+\rangle_{L^2_x}| \lesssim 1. \tag{4.4.163}$$

Thus,

$$|\dot{\gamma}| \lesssim 1 + |\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2\partial_t F_+ dxdv|. \tag{4.4.164}$$

Now,

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2\partial_t F_+ dxdv = 2\iint_{\mathbf{T}^3\times\mathbf{R}^3} v\cdot EF_+ dxdv + \iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 Q^{\varepsilon}_{-+}(F_-, F_+)dxdv \tag{4.4.165}$$

The first term has the bound

$$\begin{aligned} 2\iint_{\mathbf{T}^3\times\mathbf{R}^3} v\cdot EF_+ dxdv &\lesssim \|E\|_{L^2_x}\||v|F_+\|_{L^2_xL^1_v} & (4.4.166)\\ &\lesssim (\|\langle\xi\rangle^2 F_-\|_{L^2_{x,v}} + \|\langle v\rangle^2 F_+\|_{L^2_{x,v}})\|\langle v\rangle^3 F_+\|_{L^2_{x,v}} & (4.4.167)\\ &\lesssim 1. & (4.4.168)\end{aligned}$$

Next,

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 Q^{\varepsilon}_{-+}(F_-,F_+)dxdv \tag{4.4.169}$$
$$= -2\iint_{\mathbf{T}^3\times\mathbf{R}^3} v_j\{\varepsilon\Phi_{ij} *_\xi F_-|_{\xi=\varepsilon v}\partial_{v_i}F_+ - \partial_{\xi_i}\Phi_{ij} *_\xi F_-|_{\xi=\varepsilon v}F_+\}dxdv \tag{4.4.170}$$
$$= 2\iint_{\mathbf{T}^3\times\mathbf{R}^3} \{\varepsilon\ \mathrm{tr}(\Phi *_\xi F_-|_{\xi=\varepsilon v}) + (1+\varepsilon^2)v_j\partial_{\xi_i}\Phi_{ij} *_\xi F_-|_{\xi=\varepsilon v}\}F_+dxdv \tag{4.4.171}$$

Therefore,

$$|\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 Q^{\varepsilon}_{-+}(F_-,F_+)dxdv| \tag{4.4.172}$$
$$\lesssim (\|(-\Delta_\xi)^{-1}F_-\|_{L^2_xL^\infty_\xi} + \|\nabla_\xi(-\Delta_\xi)^{-1}F_-\|_{L^2_xL^\infty_\xi})\|vF_+\|_{L^2_xL^1_v} \tag{4.4.173}$$
$$\lesssim \|\langle\xi\rangle^2\langle\nabla_v\rangle F_-\|_{L^2_{x,v}}\|\langle v\rangle^3F_+\|_{L^2_{x,v}} \tag{4.4.174}$$
$$\lesssim \langle\|\langle\xi\rangle^5F_-\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_\xi}\rangle\|\langle v\rangle^3F_+\|_{L^2_{x,v}} \tag{4.4.175}$$
$$\lesssim \langle\|\langle\xi\rangle^5F_-\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_\xi}\rangle. \tag{4.4.176}$$

From this, we conclude (4.4.144).

Part 2: We now control the difference (4.4.145). reintroduce the superscripts of ε, so as to distinguish F^ε_+ from F^0_+, etc, although we still ignore the dependence of constants on M. We first bound the difference $\psi^\varepsilon_{in} - \phi^0_{in}$. It suffices to show this in the case

$$\sup_{t\in[0,T]} \|F^\varepsilon_+(t) - F^0_+(t)\|_{\mathfrak{E}'} \le \alpha, \tag{4.4.177}$$

where $\alpha = \alpha(M)$ is sufficiently small. If this bound does not hold, then we use the bound

$$\sup_{t\in[0,T]} |\gamma^{\varepsilon}(t)-\beta(t)| + \|\psi^{\varepsilon}(t)-\phi^0(t)\|_{H_x^{s+1}} \leq C_M \tag{4.4.178}$$

$$\leq \frac{C_M}{\alpha} \sup_{t\in[0,T]} \|F_+^{\varepsilon}(t)-F_+^0(t)\|_{\mathfrak{E}'}. \tag{4.4.179}$$

We first control the difference at time zero:

$$-\frac{1}{4\pi}\Delta_x(\psi_{in}^{\varepsilon}-\phi_{in}^0) = n_{+,in}^{\varepsilon} - n_{+,in}^0 + e^{\beta_{in}\phi_{in}^0} - e^{\beta_{in}\psi_{in}^{\varepsilon}}. \tag{4.4.180}$$

Using the maximum principle, we have

$$\|e^{\beta_{in}\psi_{in}^{\varepsilon}} - e^{\beta_{in}\phi_{in}^0}\|_{L_x^\infty} \lesssim \|n_{+,in}^{\varepsilon}-n_{+,in}^0\|_{L_x^\infty} \lesssim \sup_{t\in[0,T]} \|F_+^{\varepsilon}(t)-F_+^0(t)\|_{\mathfrak{E}'}. \tag{4.4.181}$$

By the mean-value theorem, this implies $\|\psi_{in}^{\varepsilon}-\phi_{in}^0\|_{L_x^\infty} \lesssim \sup_{t\in[0,T]} \|F_+^{\varepsilon}(t)-F_+^0(t)\|_{\mathfrak{E}'}$. In fact, by writing

$$e^{\beta_{in}\phi^0} - e^{\beta_{in}\psi_{in}^{\varepsilon}} = \beta_{in}\int_0^1 e^{\lambda\phi^0+(1-\lambda)\psi^{\varepsilon}}(\phi^0-\psi^{\varepsilon})d\lambda, \tag{4.4.182}$$

it is easy to show that after applying $\langle\nabla_x\rangle^s$ to (4.4.180), we get

$$\|\psi_{in}^{\varepsilon}-\phi_{in}^0\|_{H_x^{s+2}} \lesssim \|n_{+,in}^{\varepsilon}-n_{+,in}^0\|_{H_x^s} + \|\psi_{in}^{\varepsilon}-\phi_{in}^0\|_{H_x^s}. \tag{4.4.183}$$

Thus, by interpolation, we get

$$\|\psi_{in}^{\varepsilon}-\phi_{in}^0\|_{H_x^{s+2}} \lesssim \sup_{t\in[0,T]} \|F_+^{\varepsilon}(t)-F_+^0(t)\|_{\mathfrak{E}'}. \tag{4.4.184}$$

Now, we show how to bound the differences for positive times. Take $\kappa \in (0,1]$ to be a constant to be determined later, and let $\tilde{T}$ be the longest time such that $\tilde{T} \leq T$ and

$$\sup_{t\in[0,\tilde{T}]} |\beta - \gamma^\varepsilon| + \|\psi^\varepsilon - \phi^0\|_{H_x^{s+2}} \leq \kappa. \tag{4.4.185}$$

The equation we have for the difference $\psi^\varepsilon - \phi^0$ is

$$-\Delta_x(\psi^\varepsilon - \phi^0) = 4\pi(n_+^\varepsilon - n_+^0 + e^{\beta\phi^0} - e^{\gamma^\varepsilon\psi^\varepsilon}). \tag{4.4.186}$$

Rearranging, we have

$$(\gamma^\varepsilon e^{\gamma^\varepsilon\psi^\varepsilon} - \frac{1}{4\pi}\Delta_x)(\psi^\varepsilon - \phi^0) \tag{4.4.187}$$

$$= n_+^\varepsilon - n_+^0 + e^{\gamma^\varepsilon\psi^\varepsilon}(e^{\beta\phi^0-\gamma^\varepsilon\psi^\varepsilon} - 1 - (\beta\phi^0 - \gamma\psi^\varepsilon) + (\beta-\gamma^\varepsilon)(\phi^0 - \psi^\varepsilon)) \tag{4.4.188}$$

$$+ (\beta - \gamma^\varepsilon)e^{\gamma^\varepsilon\psi^\varepsilon}\psi^\varepsilon. \tag{4.4.189}$$

Then, on the interval $t \in [0, \tilde{T}_\varepsilon]$, we have

$$\begin{aligned} &\|\psi^\varepsilon - \phi^0 - (\beta - \gamma^\varepsilon)(\gamma^\varepsilon e^{\gamma^\varepsilon\psi^\varepsilon} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^\varepsilon\psi^\varepsilon}\psi^\varepsilon)\|_{H_x^{s+2}} \\ &\quad\lesssim \|F_+^\varepsilon - F_+^0\|_{\mathfrak{E}'} + \kappa(|\gamma^\varepsilon - \beta| + \|\psi^\varepsilon - \phi^0\|_{H_x^{s+2}}). \end{aligned} \tag{4.4.190}$$

In particular, taking κ small enough,

$$\|\psi^\varepsilon - \phi^0\|_{H_x^{s+2}} \lesssim \sup_{t\in[0,T]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'} + |\gamma^\varepsilon - \beta|. \tag{4.4.191}$$

Now,

$$\Big|\frac{1}{8\pi}\int_{\mathbf{T}^3}\nabla_x(\psi^\varepsilon-\phi^0)\cdot\nabla_x(\psi^\varepsilon+\phi^0)dx \tag{4.4.192}$$

$$+\frac{1}{4\pi}\int_{\mathbf{T}^3}(\beta-\gamma^\varepsilon)(\gamma^\varepsilon e^{\gamma^\varepsilon\psi^\varepsilon}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^\varepsilon\psi^\varepsilon}\psi^\varepsilon)\Delta_x\psi^\varepsilon dx\Big| \tag{4.4.193}$$

$$\lesssim\left|\int_{\mathbf{T}^3}\{(\psi^\varepsilon-\phi^0)-(\beta-\gamma^\varepsilon)(\gamma^\varepsilon e^{\gamma^\varepsilon\psi^\varepsilon}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^\varepsilon\psi^\varepsilon}\psi^\varepsilon)\}\Delta_x\psi^\varepsilon dx\right| \tag{4.4.194}$$

$$+\int_{\mathbf{T}^3}|\nabla_x(\psi^\varepsilon-\phi^0)|^2dx \tag{4.4.195}$$

$$\lesssim\sup_{t\in[0,T]}\|F_+^\varepsilon(t)-F_+^0(t)\|_{\mathfrak{E}'}+|\gamma^\varepsilon-\beta| \tag{4.4.196}$$

Next,

$$\frac{d}{dt}\left(\frac{3}{2\beta}-\frac{3}{2\gamma^\varepsilon}\right) \tag{4.4.197}$$

$$=\frac{d}{dt}\left(\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_+^\varepsilon-F_+^0)dxdv+\frac{1}{8\pi}\int_{\mathbf{T}^3}\nabla_x(\psi^\varepsilon-\phi^0)\cdot\nabla_x(\psi^\varepsilon+\phi^0)dx\right) \tag{4.4.198}$$

Then, using $\beta_{in}=\gamma(t=0)^\varepsilon$ and integrating the above, we have

$$\frac{3(\gamma^\varepsilon-\beta)}{2\beta\gamma^\varepsilon}=\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_+^\varepsilon-F_+^0)dxdv \tag{4.4.199}$$

$$+\frac{1}{8\pi}\int_{\mathbf{T}^3}\nabla_x(\psi^\varepsilon-\phi^0)\cdot\nabla_x(\psi^\varepsilon+\phi^0)dx \tag{4.4.200}$$

$$-\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_{+,in}^\varepsilon-F_{+,in}^0)dxdv \tag{4.4.201}$$

$$-\frac{1}{8\pi}\int_{\mathbf{T}^3}\nabla_x(\psi_{in}^\varepsilon-\phi_{in}^0)\cdot\nabla_x(\psi_{in}^\varepsilon+\phi_{in}^0)dx. \tag{4.4.202}$$

Using the previous bounds, plus (4.4.184), we have

$$|\gamma^\varepsilon-\beta|\left|\frac{3}{2\beta\gamma^\varepsilon}-\frac{1}{4\pi}\int_{\mathbf{T}^3}(\gamma^\varepsilon e^{\gamma^\varepsilon\psi^\varepsilon}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^\varepsilon\psi^\varepsilon}\psi^\varepsilon)\Delta_x\psi^\varepsilon dx\right| \tag{4.4.203}$$

$$\lesssim|\gamma^\varepsilon-\beta|^2+\sup_{t\in[0,T]}\|F_+^\varepsilon(t)-F_+^0(t)\|_{\mathfrak{E}'}. \tag{4.4.204}$$

By a similar argument as was used to bound $\dot{\gamma}$, we have that

$$\int_{\mathbf{T}^3} (\gamma^\varepsilon e^{\gamma^\varepsilon \psi^\varepsilon} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^\varepsilon \psi^\varepsilon}\psi^\varepsilon)\Delta_x \psi^\varepsilon dx \leq 0. \tag{4.4.205}$$

We conclude that for all $t \in [0, \tilde{T}]$,

$$|\gamma^\varepsilon(t) - \beta(t)| \lesssim \kappa |\gamma^\varepsilon(t) - \beta(t)| + \sup_{t\in[0,T]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'} \tag{4.4.206}$$

Taking κ sufficiently small, we have $|\gamma^\varepsilon(t) - \beta(t)| \leq C_0\alpha$ for some C_0. So, by taking $\alpha = \frac{\kappa}{2C_0}$, we have $\tilde{T} = T$.

Step 3: Finally, we sketch how to construct solutions $(\gamma^\varepsilon, \psi^\varepsilon)$ satisfying (4.4.140). We refer the reader (ii) of Theorem 1.4 in [13] and its proof. Using standard elliptic theory, there exists a unique $\psi_{in}^\varepsilon \in H^{s+2}$ solving the system

$$-\Delta_x \psi_{in}^\varepsilon = 4\pi(n_{+,in}^\varepsilon - e^{\beta_{in}\psi_{in}^\varepsilon}). \tag{4.4.207}$$

Now, for $t > 0$, define

$$E(t) := \frac{3}{2\beta_{in}} + \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2}(F_{+,in}^\varepsilon - F_+^\varepsilon(t))dxdv + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \psi_{in}^\varepsilon|^2 dx. \tag{4.4.208}$$

Then, for all $t > 0$ such that $E(t) > 0$, there exists $(\gamma(t), \psi(t)) \in \mathbf{R}_+ \times H^{s+2}$ solving third line of (4.4.140), and the first line integrated on $[0, t]$. Since $t \mapsto \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2} F_+^\varepsilon(t) dxdv$ is continuous, and the above condition holds at $t = 0$, we take $T^* > 0$ to be the first time less that T such that the above condition does not hold, or take $T^* = T$ if no such time exists. Then there exists a unique (γ, ψ) satisfying (4.4.140), with $\ln(\gamma) \in L_{loc}^\infty([0, T^*))$ and $\psi \in L_{loc}^\infty([0, T^*); H^{s+2})$. Moreover, $\frac{3}{2\gamma^\varepsilon(t)} \leq E(t)$, so $\gamma^\varepsilon(t) \to \infty$ as $t \to T^*$ whenever $T^* \leq T$.

Next, we note that γ is continuous in time. This follows by the identity (4.4.162), and the fact that

$$\|\partial_t n_+^\varepsilon(t)\|_{L_x^2}, \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2}\partial_t F_+^\varepsilon(t,x,v)dxdv \in L^1([0,T]), \tag{4.4.209}$$

also shown above. It is straightforward to show $\psi \in C([0,T^*); H^{s+2})$ from this, and (4.4.150).

$\square$

4.5 Estimates on the electron distribution

4.5.1 Stability of the Maxwellian

The main result of this section is Proposition 4.5.1, which, roughly speaking, shows that the Maxwellian $\mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}$ is stable; that is, if $F_{-,in}^\varepsilon$ is close to $\mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}$, then it will remain close. We also utilize the stretched exponential decay to acquire some form of asymptotic stability.

To state the result, let $\eta > 0$, and recall $q_\alpha = e^{\alpha\eta}\beta_{in}$ for each $\alpha \in \{1,2,3\}$. Then, similarly to $\mathscr{E}_{-,\alpha}$ and $\mathscr{D}_{-,\alpha}$, we define for each $\alpha \in \{1,2\}$,

$$\begin{aligned} \widetilde{\mathscr{E}}_{-,\alpha}^\varepsilon &:= \|e^{q_\alpha|\xi|^2/4}\langle\nabla_x\rangle^s\{F_-^\varepsilon - \mu_{\gamma^\varepsilon\psi^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon}\}\|_{L^2_{x,\xi}}^2 \\ &\quad + \|e^{q_{\alpha+1}|\xi|^2/4}\{F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon}\}\|_{L^2_{x,\xi}}^2 \end{aligned} \tag{4.5.1}$$

and

$$\widetilde{\mathscr{D}}^{\varepsilon}_{-,\alpha} := \|e^{q_\alpha|\xi|^2/4}\langle\nabla_x\rangle^s\{F^\varepsilon_- - \mu_{\gamma^\varepsilon\psi}e^{\gamma^\varepsilon\psi^\varepsilon}\}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \|e^{q_{\alpha+1}|\xi|^2/4}\{F^\varepsilon_- - \mu_{\gamma^\varepsilon}e^{\gamma^\varepsilon\psi^\varepsilon}\}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}. \tag{4.5.2}$$

Proposition 4.5.1. *Fix $M > 0$. Then there exists positive constants η, ς and $\bar{\varepsilon}$ such that the following holds. Assume $0 < \varepsilon \le \bar{\varepsilon}$. For each such ε, let $(F^\varepsilon_+, F^\varepsilon_-)$ be solution to the system (4.1.7) with $0 \le F^\varepsilon_\pm \in C([0,T];\mathfrak{E}) \cap L^2([0,T];\mathfrak{D})$. Fixing $\beta_{in} \in (e^{-M}, e^M)$, let $(\psi^\varepsilon, \gamma^\varepsilon)$ be the corresponding solution to the system (4.4.140), assuming that*

$$\sup_{t\in[0,T]}(\|\frac{1}{n^\varepsilon_+(t)}\|_{L^\infty_x} + \|F^\varepsilon_+(t)\|_{\mathfrak{E}}) \le M, \tag{4.5.3}$$

$$\sup_{t\in[0,T]}\sqrt{\widetilde{\mathscr{E}}^{\varepsilon}_{-,2}(t)} \le \varsigma, \tag{4.5.4}$$

$$\sup_{t\in[0,T]}|\ln(\frac{\gamma^\varepsilon(t)}{\beta_{in}})| \le \eta, \tag{4.5.5}$$

$$\int_0^T \|\partial_t(n^\varepsilon_- - e^{\gamma^\varepsilon\psi^\varepsilon})\|_{L^2_x}dt \le 1. \tag{4.5.6}$$

We also impose the following assumption on the initial data:

$$\left|\int_{\mathbf{R}^3\times\mathbf{T}^3}\frac{|\xi|^2}{2}(F^\varepsilon_{-,in} - \mu_{\beta_{in}}e^{\beta_{in}\psi^\varepsilon_{in}})dxd\xi + \frac{1}{8\pi}\int_{\mathbf{T}^3}|\nabla_x\phi^\varepsilon_{in}|^2 - |\nabla_x\psi^\varepsilon_{in}|^2dx\right| \le M\varepsilon. \tag{4.5.7}$$

Then, we have

$$\varepsilon\sup_{t\in[0,T]}\widetilde{\mathscr{E}}^{\varepsilon}_{-,2}(t) + \int_0^T\widetilde{\mathscr{D}}^{\varepsilon}_{-,2}(t)dt \le C_M(\varepsilon\widetilde{\mathscr{E}}^{\varepsilon}_{-,2,in} + \varepsilon^2) \tag{4.5.8}$$

and for all $t \in [0,T]$, we have

$$\mathscr{E}^{\varepsilon}_{-,1}(t) \le C_M(e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\sup_{t'\in[0,T]}\mathscr{E}^{\varepsilon}_{-,2}(t') + \varepsilon^{\frac{5}{3}}t^{\frac{1}{3}}) \tag{4.5.9}$$

Throughout the rest of this section, we will not make reference to (F_+^0, β, ϕ^0) and its derived quantities, and instead only work with $(F_-^\varepsilon, F_+^\varepsilon)$. With this understood, throughout this section, we will write $F_+^\varepsilon = F_+$, $F_-^\varepsilon = F_-$, and so on, with the dependence on ε being implicit.

4.5.2 Setup

We introduce $a = n_- - e^{\gamma\psi}$ and note that

$$4\pi a = -\Delta_x(\phi - \psi), \tag{4.5.10}$$

Using (4.5.5), we note

$$\gamma(t) \leq q_1 < q_2 < q_3. \tag{4.5.11}$$

Next, we separate

$$F_- - \mu_\gamma e^{\gamma\psi} = \sqrt{\mu_\gamma} f = \sqrt{\mu_{q_\alpha} e^{q_\alpha \phi}} g_\alpha \tag{4.5.12}$$

where $\alpha \in \{1, 2, 3\}$.

The equation for f reads as follows

$$\begin{aligned} &\varepsilon\mu_\gamma^{-\frac{1}{2}}\partial_t\{\mu_\gamma^{\frac{1}{2}} f\} + \{\xi \cdot \nabla_x + E \cdot (\frac{\xi}{2} - \nabla_\xi)\} f + e^{\gamma\psi}\mathcal{L}_\gamma f + \mathcal{M}_{\gamma,F_+} f \\ &\quad - 4\pi\gamma\xi \cdot \nabla_x \Delta_x^{-1} a \mu_\gamma^{\frac{1}{2}} e^{\gamma\psi} \\ &= -\varepsilon\mu_\gamma^{\frac{1}{2}}\partial_t\{\mu_\gamma e^{\gamma\psi}\} - e^{\gamma\psi}\mathcal{M}_{\gamma,F_+}\mu_\gamma^{\frac{1}{2}} + \Gamma_\gamma(f, f) \end{aligned} \tag{4.5.13}$$

where

$$\mathcal{L}_\gamma h = \mu_\gamma^{-\frac{1}{2}}\{Q(\mu_\gamma^{\frac{1}{2}}h, \mu_\gamma) + Q(\mu_\gamma, \mu_\gamma^{\frac{1}{2}}h)\}, \tag{4.5.14}$$

$$\mathcal{M}_{\gamma,F_+}h = \mu_\gamma^{-\frac{1}{2}}Q^\varepsilon_{+-}(F_+, \mu_\gamma^{\frac{1}{2}}h), \tag{4.5.15}$$

$$\Gamma_\gamma(h_1, h_2) = -\mu_\gamma^{-\frac{1}{2}}Q(\mu_\gamma^{\frac{1}{2}}h_1, \mu_\gamma^{\frac{1}{2}}h_2). \tag{4.5.16}$$

On the other hand, g_α, $\alpha \in \{1,2,3\}$ solve the equation

$$\begin{aligned}
&\varepsilon\{\partial_t + q\partial_t\phi\}g_\alpha + \{v\cdot\nabla_x - E\cdot\nabla_\xi\}g_\alpha + e^{\gamma\psi}\widetilde{\mathcal{L}}_{\gamma,q}g_\alpha + \mathcal{M}_{q_\alpha,F_+}g_\alpha \\
&\quad - 4\pi\gamma\xi\cdot\nabla_x\Delta_x^{-1}a\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}e^{\gamma\psi-\frac{q_\alpha}{2}\phi} \\
&= -\varepsilon\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\psi}}} - e^{\gamma\psi-\frac{q_\alpha}{2}\phi}\mathcal{M}_{q_\alpha,F_+}(\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma) + e^{\frac{q_\alpha}{2}\phi}\Gamma_{q_\alpha}(g_\alpha, g_\alpha)
\end{aligned} \tag{4.5.17}$$

where $\mathcal{M}_{q_\alpha,F_+}$ and Γ_{q_α} are defined the same way as $\mathcal{M}_{\gamma,F_+}$ and Γ_γ respectively, with γ substituted with q_α. The operator $\widetilde{\mathcal{L}}_{\gamma,q}$ is defined as follows.

$$\widetilde{\mathcal{L}}_{\gamma,q}\psi = \mu_q^{-\frac{1}{2}}\{Q(\mu_q^{\frac{1}{2}}\psi, \mu_\gamma) + Q(\mu_\gamma, \mu_q^{\frac{1}{2}}\psi)\} \tag{4.5.18}$$

Now, in addition, we define the operator $\mathcal{P}_\gamma$ to be the L^2-projection (in the ξ variable) to the kernel of $\mathcal{L}_\gamma$, and we take $\mathcal{P}_\gamma^\perp = \mathrm{Id}_{L^2(\mathbf{R}^3)} - \mathcal{P}_\gamma$. From [70], we have that this kernel is

$$N(\mathcal{L}_\gamma) = \mathrm{span}\{\mu_\gamma^{\frac{1}{2}}(\xi), w_1\mu_\gamma^{\frac{1}{2}}, \xi_2\mu_\gamma^{\frac{1}{2}}, \xi_3\mu_\gamma^{\frac{1}{2}}, |\xi|^2\mu_\gamma^{\frac{1}{2}}\}. \tag{4.5.19}$$

In addition to the density of the perturbation a already defined, we define the macroscopic variables $c : \mathbf{T}^3 \to \mathbf{R}$ and $b : \mathbf{T}^3 \to \mathbf{R}^3$ as the coefficients of this projection on

f:

$$\mathcal{P}_\gamma f = a\mu_\gamma^{\frac{1}{2}} + \gamma^{\frac{1}{2}} b \cdot w \mu_\gamma^{\frac{1}{2}} + c\frac{\gamma|\xi|^2 - 3}{\sqrt{6}}\mu_\gamma^{\frac{1}{2}} \tag{4.5.20}$$

We note that the representation above is in terms of an orthonormal basis for $N(\mathcal{L}_\gamma)$. Moreover,

$$a = \int_{\mathbf{R}^3} \{F_- - \mu_\gamma e^{\gamma\psi}\} d\xi = n_- - e^{\gamma\psi} \tag{4.5.21}$$

$$b = \int_{\mathbf{R}^3} \gamma^{\frac{1}{2}} \xi \{F_- - \mu_\gamma e^{\gamma\psi}\} d\xi = \gamma^{\frac{1}{2}} \int_{\mathbf{R}^3} \xi F_- d\xi \tag{4.5.22}$$

$$c = \int_{\mathbf{R}^3} \frac{(\gamma|\xi|^2 - 3)}{\sqrt{6}} \{F_- - \mu_\gamma e^{\gamma\psi}\} d\xi = \frac{1}{\sqrt{6}} \left(\gamma \int_{\mathbf{R}^3} |\xi|^2 F_- d\xi - 3n_- \right). \tag{4.5.23}$$

Thus, (a, b, c) form a linear transformation of the physical macroscopic variables of density, current and local kinetic energy of the electrons. Given $r \geq 0$, we shall denote $a^{(r)} = \langle \nabla_x \rangle^r a$, $g_\alpha^{(r)} = \langle \nabla_x \rangle^r g_\alpha$, and so forth.

Finally, we close this section by mentioning some commonly occurring estimates that will occur throughout this section. We note that under the bootstrap assumptions, $\|\phi\|_{H_x^{s+2}} + \|\psi\|_{H_x^{s+2}} \lesssim_M 1$. Therefore,

$$\|f\|_{L^2_{x,\xi}} \lesssim_M \|g_1\|_{L^2_{x,\xi}} \lesssim_M \|g_2\|_{L^2_{x,\xi}} \lesssim_M \|g_3\|_{L^2_{x,\xi}} \tag{4.5.24}$$

$$\|f^{(s)}\|_{L^2_{x,\xi}} \lesssim_M \|g_1^{(s)}\|_{L^2_{x,\xi}} \lesssim_M \|g_2^{(s)}\|_{L^2_{x,\xi}} \lesssim_M \|g_3^{(s)}\|_{L^2_{x,\xi}} \tag{4.5.25}$$

The same inequalities hold over the spaces $L^2_x(\mathcal{H}_\sigma)_\xi$ and $L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi$. Moreover,

$$\widetilde{\mathscr{E}}_{-,\alpha} \sim_M \|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} + \|g_{\alpha+1}\|^2_{L^2_{x,\xi}} \tag{4.5.26}$$

$$\widetilde{\mathscr{D}}_{-,\alpha} \sim_M \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma \cap \mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma \cap \mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{4.5.27}$$

4.5.3 Preliminary estimates

Next, we have upper and lower bounds on the linearized collision operators $\widetilde{\mathcal{L}}_{q,\gamma}$.

Lemma 4.5.2. *Assume (4.5.5). Let $h_1, h_2, h_3 : \mathbf{R}^3 \to \mathbf{R}$. We have the following:*

(i) Then

$$\|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma}^2 \lesssim \langle \mathcal{L}_\gamma h_1, h_1\rangle_{L^2} \lesssim \|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma}^2. \tag{4.5.28}$$

(ii) Taking $q \in \{q_1, q_2, q_3\}$, and $h_1, h_2 : \mathbf{R}^3 \to \mathbf{R}$,

$$\langle \widetilde{\mathcal{L}}_{\gamma,q} h_1, h_2\rangle_{L^2} \lesssim \|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma}\|\mathcal{P}_\gamma^\perp h_2\|_{\mathcal{H}_\sigma} + \eta\|h_1\|_{\mathcal{H}_\sigma}\|h_2\|_{\mathcal{H}_\sigma}, \tag{4.5.29}$$

$$\|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma}^2 \lesssim \langle \widetilde{\mathcal{L}}_{\gamma,q} h_1, h_1\rangle_{L^2} + \eta^2\|\mathcal{P}_\gamma h_1\|_{L^2}^2. \tag{4.5.30}$$

(iii) Take $\eta > 0$ sufficiently small. Suppose $h_1 = \sqrt{\frac{\mu_q}{\mu_\gamma}} h_2$ where $q \in \{q_1, q_2, q_3\}$. Then,

$$\|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma}^2 \lesssim \eta\|\mathcal{P}_\gamma h_1\|_{L^2}^2 + \|\mathcal{P}_\gamma^\perp h_2\|_{\mathcal{H}_\sigma}^2. \tag{4.5.31}$$

Also,

$$\|h_1\|_{\mathcal{H}_\sigma} \lesssim \|\mathcal{P}_\gamma h_1\|_{L^2} + \|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma}. \tag{4.5.32}$$

(iv) If $q \in \{q_1, q_2\}$, then, taking $q' \in [0, q)$, we have

$$\langle \Gamma_q(h_1, h_2), h_3\rangle_{L^2} \lesssim_{q'} (\|e^{\frac{q'|\xi|^2}{4}} h_1\|_{L^2}\|h_2\|_{\mathcal{H}_\sigma} + \|e^{\frac{q'|\xi|^2}{4}} h_1\|_{\mathcal{H}_\sigma}\|h_2\|_{L^2})\|h_3\|_{\mathcal{H}_\sigma}. \tag{4.5.33}$$

Proof. For (i), the equivalence (4.5.28) in the case $\gamma = 2$ can be found in [70]. The case of general γ follows by re-scaling in ξ.

For (ii), we compute

$$\langle \widetilde{\mathcal{L}}_{\gamma,q} h_1, h_2\rangle_{L^2} = \langle \mathcal{L}_\gamma h_1, h_2\rangle_{L^2} + \frac{\gamma - q}{2}\{\langle \sigma^\gamma_{ij}\xi_i h_1, \partial_{\xi_j} h_2\rangle_{L^2} - \langle \sigma^\gamma_{ij}\xi_j \partial_{\xi_i} h_1, h_2\rangle_{L^2}\} \tag{4.5.34}$$

$$- \frac{(\gamma - q)^2}{2}\langle \sigma^\gamma_{ij}\xi_i\xi_j h_1, h_2\rangle_{L^2} \tag{4.5.35}$$

It is clear from the above that we have (4.5.29). In the case $h_2 = h_1$, we have by (4.5.28) and the computation above that

$$\langle \widetilde{\mathcal{L}}_{\gamma,q} h_1, h_1\rangle_{L^2} \geq \langle \mathcal{L}_\gamma h_1, h_1\rangle_{L^2_\xi} - C\eta^2 \|h_1\|^2_{\mathcal{H}_\sigma} \tag{4.5.36}$$

$$\geq (\frac{1}{C} - C\eta^2)\|\mathcal{P}^\perp_\gamma h_1\|^2_{\mathcal{H}_\sigma} - C\eta^2|\mathcal{P}_\gamma h_1|^2_{\mathcal{H}_\sigma}. \tag{4.5.37}$$

Taking η sufficiently small, we get (4.5.30).

We now prove (iii). We write $\zeta = \sqrt{\frac{\mu_q}{\mu_\gamma}}$. Now,

$$\|\mathcal{P}^\perp_\gamma h_1\|_{\mathcal{H}_\sigma} = \|\mathcal{P}^\perp_\gamma(\zeta h_2)\|_{\mathcal{H}_\sigma} \leq \|\mathcal{P}^\perp_\gamma(\zeta\mathcal{P}_\gamma h_2)\|_{\mathcal{H}_\sigma} + \|\mathcal{P}^\perp_\gamma(\zeta\mathcal{P}^\perp_\gamma h_2)\|_{\mathcal{H}_\sigma} \tag{4.5.38}$$

$$\lesssim \|\mathcal{P}^\perp_\gamma((\zeta - 1)\mathcal{P}_\gamma h_2)\|_{\mathcal{H}_\sigma} + \|\mathcal{P}^\perp_\gamma h_2\|_{\mathcal{H}_\sigma} \tag{4.5.39}$$

$$\lesssim \|(\zeta - 1)\mathcal{P}_\gamma h_2\|_{\mathcal{H}_\sigma} + \|\mathcal{P}^\perp_\gamma h_2\|_{\mathcal{H}_\sigma} \tag{4.5.40}$$

Now, $|\zeta - 1| \lesssim \eta|\xi|^2$, and $|\mathcal{P}_\gamma h_2(\xi)| \lesssim \langle\xi\rangle^2 e^{-\frac{\gamma}{2}|\xi|^2}\|\mathcal{P}_\gamma h_2\|_{L^2}$, $\|(\zeta - 1)\mathcal{P}_\gamma h_2\|_{\mathcal{H}_\sigma} \lesssim \eta\|\mathcal{P}_\gamma h_2\|_{L^2}$. Now,

$$\|\mathcal{P}_\gamma h_2\|_{L^2} \leq \|\mathcal{P}_\gamma(\zeta^{-1}\mathcal{P}_\gamma h_1)\|_{L^2} + \|\mathcal{P}_\gamma((\zeta^{-1} - 1)\mathcal{P}^\perp_\gamma h_1)\|_{L^2} \tag{4.5.41}$$

$$\lesssim \|\mathcal{P}_\gamma h_1\|_{L^2} + \eta\|\mathcal{P}^\perp_\gamma h_1\|_{L^2}. \tag{4.5.42}$$

Combining these bounds, we deduce (4.5.31) after taking η sufficiently small. From the latter estimate, we deduce (4.5.32).

Finally, we prove (iv). The bound (4.5.33) in the case $q = 2$ follows from the proof of Theorem 3 in [70]. The case of general q follows by re-scaling in ξ. $\square$

Lemma 4.5.3. *Suppose $q \in \{q_1, q_2, q_3\}$ and assume (4.5.5) with η taken sufficiently small. Let $G, h_1, h_2 : \mathbf{R}^3 \to \mathbf{R}$, $G = G(v)$, $h_1 = h_1(\xi)$, $h_2 = h_2(\xi)$.*

(i) We have the upper bound

$$\begin{aligned}\langle \mathcal{M}_{q,G} h_1, h_2\rangle_{L^2} \lesssim \|\langle v\rangle^3 G\|_{L^2}(\|h_1\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}} h_1\|_{L^2})(\|h_2\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}} h_2\|_{L^2})\\ + \varepsilon^{\frac{1}{2}}\|\langle v\rangle^{\frac{7}{2}} G\|_{\mathcal{H}_\sigma}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}} h_1\|_{L^2}\|h_2\|_{\mathcal{H}^-_{\sigma;\varepsilon}}.\end{aligned} \tag{4.5.43}$$

(ii) Assume $G \geq 0$, $\|\langle v\rangle^3 G\|_{L^2}, \|\langle v\rangle^{\frac{7}{2}} G\|_{\mathcal{H}_\sigma} < \infty$ and $\|G\|_{L^1_v} > 0$. Let

$$k_G := \sup\{1, \|\langle v\rangle^3 G\|_{L^2_v}, \frac{1}{\|G\|_{L^1_v}}\}, \tag{4.5.44}$$

Then,

$$\langle \mathcal{M}_{q,G} h_1, h_1\rangle_{L^2_\xi} \geq \frac{1}{C_{k_G}}\|h_1\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} - C_{k_G}\varepsilon\langle\|\langle v\rangle^{\frac{7}{2}} G\|_{\mathcal{H}_\sigma}\rangle^2\|h_1\|^2_{L^2}. \tag{4.5.45}$$

Proof. Recall that $\mathcal{M}_{q,G} h = \frac{1}{\sqrt{\mu_q}} Q^\varepsilon_{+-}(\mu_q, \sqrt{\mu_q} h)$, with Q^ε_{+-} defined in (4.1.11). Then,

$$\begin{aligned}&\langle \mathcal{M}_{q,G} h_1, h_2\rangle_{L^2_\xi} &(4.5.46)\\ &= \frac{1}{\varepsilon}\langle (\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}}(\partial_{\xi_i} - \frac{q\xi_i}{2})h_1, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_1\rangle_{L^2_\xi} &(4.5.47)\\ &- \langle (\partial_{v_i}\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}} h_1, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_1\rangle_{L^2_\xi}. &(4.5.48)\end{aligned}$$

For the first term, we have

$$(4.5.47) \lesssim \frac{1}{\varepsilon}\langle (\Phi_{ij} * |G|)|_{v=\frac{\xi}{\varepsilon}}(\partial_{\xi_i} - \frac{q\xi_i}{2})h_1, (\partial_{\xi_j} - \frac{q\xi_i}{2})h_1\rangle^{\frac{1}{2}}_{L^2_\xi} \tag{4.5.49}$$

$$\cdot \langle (\Phi_{ij} * |G|)|_{v=\frac{\xi}{\varepsilon}}(\partial_{\xi_i} + \frac{q\xi_i}{2})h_2, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_2\rangle^{\frac{1}{2}}_{L^2_\xi} \tag{4.5.50}$$

$$\lesssim \|\langle v\rangle^3 G\|_{L^2_v}(\|h_1\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} + \frac{1}{\varepsilon}\langle \sigma_{ij}|_{v=\frac{\xi}{\varepsilon}}\xi_i\xi_j h_1, h_1\rangle_{L^2_\xi})^{\frac{1}{2}} \tag{4.5.51}$$

$$\cdot (\|h_2\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} + \frac{1}{\varepsilon}\langle \sigma_{ij}|_{v=\frac{\xi}{\varepsilon}}\xi_i\xi_j h_2, h_2\rangle_{L^2_\xi})^{\frac{1}{2}}. \tag{4.5.52}$$

Above we used the upper bound (4.3.5). Now, recall that by (4.1.19), $\sigma_{ij}(z)z_iz_j \sim \frac{|z|^2}{\langle z\rangle^3}$. Hence,

$$\frac{1}{\varepsilon}\sigma_{ij}(\frac{\xi}{\varepsilon})\xi_i\xi_j \sim \frac{|\xi|^2}{\varepsilon\langle \xi/\varepsilon\rangle^3} = \frac{\varepsilon|\xi/\varepsilon|^2}{\langle \xi/\varepsilon\rangle^3} \leq \frac{\varepsilon}{\langle \xi/\varepsilon\rangle} \leq \min\{\varepsilon, \frac{1}{|\xi|}\}. \tag{4.5.53}$$

Thus,

$$(4.5.47) \lesssim \|\langle v\rangle^3 G\|_{L^2_v}(\|h_1\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|h_1\|_{L^2_\xi})(\|h_2\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|h_2\|_{L^2_\xi}). \tag{4.5.54}$$

Next, we consider the second term: using (4.1.19) again, and recalling the definition of $\mathcal{H}^\varepsilon_{\sigma,-}$ in (4.1.23), we have

$$|\langle (\partial_{v_i}\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}}h_1, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_2\rangle_{L^2_\xi}| \tag{4.5.55}$$

$$\lesssim \left(\int_{\mathbf{R}^3}(\sigma^{-1})_{ij}(\frac{\xi}{\varepsilon})\partial_{v_k}\Phi_{ki} * G|_{v=\frac{\xi}{\varepsilon}}\partial_{v_l}\Phi_{lj} * G|_{v=\frac{\xi}{\varepsilon}}h_1^2(\xi)d\xi\right)^{\frac{1}{2}} \tag{4.5.56}$$

$$\cdot \langle \sigma_{ij}(\frac{\xi}{\varepsilon})(\partial_{\xi_i} + \frac{q\xi_i}{2})h_2, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_2\rangle^{\frac{1}{2}}_{L^2_\xi} \tag{4.5.57}$$

$$\lesssim \left(\int_{\mathbf{R}^3}\langle \frac{\xi}{\varepsilon}\rangle^3|(\nabla_v(-\Delta_v)^{-1}G)(\frac{\xi}{\varepsilon})|^2 h_1^2(\xi)d\xi\right)^{\frac{1}{2}} \tag{4.5.58}$$

$$\lesssim \varepsilon^{\frac{1}{2}}\sup_{v\in\mathbf{R}^3, i\in\{1,2,3\}}\{\langle v\rangle^2|\nabla_v(-\Delta_v)^{-1}G(v)|\}\|\langle \frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_1\|_{L^2_\xi}(\|h_2\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|\langle \frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_2\|_{L^2_\xi}) \tag{4.5.59}$$

Now, we have

$$\sup_{v\in\mathbf{R}^3}\|\langle v\rangle^2|\partial_{v_j}\Phi_{ij}*G(v)\|\lesssim\|\langle v\rangle^2 G\|_{H^1_v}\lesssim\|\langle v\rangle^{\frac{7}{2}}G\|_{\mathcal{H}_\sigma}. \tag{4.5.60}$$

Proof of (ii): We write

$$\langle\mathcal{M}_{q,G}h_1,h_1\rangle_{L^2_\xi}=\frac{1}{\varepsilon}\langle(\Phi_{ij}*G)|_{v=\frac{\xi}{\varepsilon}}(\partial_{\xi_i}-\frac{q\xi_i}{2})h_1,(\partial_{\xi_j}+\frac{q\xi_i}{2})h_1\rangle_{L^2_\xi} \tag{4.5.61}$$

$$-\langle(\partial_{v_i}\Phi_{ij}*G)|_{v=\frac{\xi}{\varepsilon}}h_1,(\partial_{\xi_j}+\frac{q\xi_i}{2})h_1\rangle_{L^2_\xi} \tag{4.5.62}$$

$$=\frac{1}{\varepsilon}\langle(\Phi_{ij}*G)|_{v=\frac{\xi}{\varepsilon}}\partial_{\xi_i}h_1,\partial_{\xi_j}h_1\rangle_{L^2_\xi} \tag{4.5.63}$$

$$-\frac{1}{\varepsilon}\langle(\Phi_{ij}*G)|_{v=\frac{\xi}{\varepsilon}}\xi_i\xi_j h_1,h_1\rangle_{L^2_\xi} \tag{4.5.64}$$

$$-\langle(\partial_{v_i}\Phi_{ij}*G)|_{v=\frac{\xi}{\varepsilon}}h_1,(\partial_{\xi_j}+\frac{q\xi_i}{2})h_1\rangle_{L^2_\xi} \tag{4.5.65}$$

Now, applying (4.3.6), we have

$$(4.5.63)\geq\frac{1}{C_{k_G}\varepsilon}\langle\sigma_{ij}|_{v=\frac{\xi}{\varepsilon}}\partial_{\xi_i}h_1,\partial_{\xi_j}h_1\rangle_{L^2_\xi}. \tag{4.5.66}$$

Once again, we apply (4.3.5) to get

$$\frac{1}{\varepsilon}\sup_{\xi\in\mathbf{R}^3}\left|(\Phi_{ij}*G)(\frac{\xi}{\varepsilon})\xi_i\xi_j\right|\lesssim\frac{k_G}{\varepsilon}\sigma_{ij}(\frac{\xi}{\varepsilon})\xi_i\xi_j\lesssim\frac{k_G|\xi|^2}{\varepsilon\langle\xi/\varepsilon\rangle^3}\leq\frac{k_G\varepsilon}{\langle\xi/\varepsilon\rangle}. \tag{4.5.67}$$

Hence,

$$|(4.5.64)|\leq C_{k_G}\varepsilon\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_1\|^2_{L^2} \tag{4.5.68}$$

Finally, we reuse the bound from part (i) to get for all $\lambda > 0$,

$$|(4.5.65)| \lesssim \varepsilon^{\frac{1}{2}} \|\langle v\rangle^{\frac{7}{2}} G\|_{\mathcal{H}_\sigma} \|\langle \frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}} h_1\|_{L^2_\xi} \|h_1\|_{\mathcal{H}^-_{\sigma;\varepsilon}} \tag{4.5.69}$$

$$\leq \frac{\lambda}{C_{k_G}} \|h_1\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} + \frac{C_{k_G}\varepsilon}{\lambda} \|\langle v\rangle^{\frac{7}{2}} G\|^2_{\mathcal{H}_\sigma} \|\langle \frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}} h_1\|^2_{L^2}. \tag{4.5.70}$$

Taking λ sufficiently small so that the first term in the above is dominated by (4.5.63), we deduce (4.5.45). $\square$

4.5.4 Energy estimates

Proposition 4.5.4. *We assume that (4.5.3), (4.5.4) and (4.5.5) hold. Then, for all $\kappa > 0$, $t \in [0, T]$:*

(i) for $\alpha \in \{1, 2, 3\}$, we have

$$\begin{aligned}
&\varepsilon \frac{d}{dt}\{\|g_\alpha\|^2_{L^2_{x,\xi}} + 4\pi\sqrt{\gamma}\||\Delta_x|^{-1} a\|^2_{L^2_x}\} + \frac{1}{C}\{\|\mathcal{P}^\perp_\gamma g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\} \\
&\quad \leq C(\varepsilon + \varsigma + \eta + \kappa)\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \\
&\qquad + C\varepsilon\langle \|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle \|g_\alpha\|^2_{L^2_{x,\xi}} \\
&\qquad + C_\kappa \varepsilon^2 \langle \|F_+\|_{\mathfrak{D}}\rangle^2
\end{aligned} \tag{4.5.71}$$

(ii) For $\alpha \in \{1, 2\}$,

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}} + \frac{1}{C}\{\|\mathcal{P}^\perp_\gamma g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\} \\
&\quad \leq C(\varepsilon + \varsigma + \eta + \kappa)\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + C_{\eta,\kappa}\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \\
&\qquad + C\varepsilon\langle \|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle \|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}} \\
&\qquad + C_\kappa \varepsilon^2 \langle \|F_+\|_{\mathfrak{D}}\rangle^2
\end{aligned} \tag{4.5.72}$$

Proof. We separate the proof into each part:

Proof of (i): We multiply (4.5.17) by g_α and integrate:

$$\frac{\varepsilon}{2}\frac{d}{dt}\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{4.5.73}$$
$$+\frac{q_\alpha\varepsilon}{2}\langle\partial_t\phi g_\alpha, g_\alpha\rangle_{L^2_{x,\xi}} \tag{4.5.74}$$
$$+\langle e^{\gamma\psi}\widetilde{\mathcal{L}}_{\gamma,q_\alpha}g_\alpha, g_\alpha\rangle_{L^2_{x,\xi}} \tag{4.5.75}$$
$$+\langle\mathcal{M}_{q_\alpha,F_+}g_\alpha, g_\alpha\rangle_{L^2_{x,\xi}} \tag{4.5.76}$$
$$-4\pi\gamma\langle\xi\cdot\nabla_x\Delta_x^{-1}a\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}e^{\gamma\psi-\frac{q_\alpha}{2}\phi}, g_\alpha\rangle_{L^2_{x,\xi}} \tag{4.5.77}$$
$$=-\varepsilon\langle\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}}, g_\alpha\rangle_{L^2_{x,\xi}} \tag{4.5.78}$$
$$-\langle e^{\gamma\psi-\frac{q_\alpha}{2}\phi}\mathcal{M}_{q_\alpha,F_+}(\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}), g_\alpha\rangle_{L^2_{x,\xi}} \tag{4.5.79}$$
$$+\langle e^{\frac{q_\alpha}{2}\phi}\Gamma_{q_\alpha}(g_\alpha,g_\alpha), g_\alpha\rangle_{L^2_{x,\xi}}. \tag{4.5.80}$$

By (4.4.144),

$$|(4.5.74)|\lesssim\varepsilon(\|\partial_t\psi\|_{L^\infty_x}+\|\Delta_x^{-1}\partial_t a\|_{L^\infty_x})\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{4.5.81}$$
$$\lesssim\varepsilon\langle\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi}+\|\partial_t a\|_{L^2_x}\rangle\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{4.5.82}$$
$$\lesssim\varepsilon^2+\varepsilon\langle\|\partial_t a\|_{L^2_x}\rangle\|g_\alpha\|^2_{L^2_{x,\xi}}+\varsigma\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.83}$$

Next, we apply (4.5.30) to get

$$(4.5.75)\geq\frac{1}{C}e^{-C\|\psi\|_{L^\infty_x}}\|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_{x,\xi}}-Ce^{C\|\psi\|_{L^\infty_x}}\eta^2\|\mathcal{P}_\gamma g_\alpha\|^2_{L^2_{x,\xi}} \tag{4.5.84}$$
$$\geq\frac{1}{C}\|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_{x,\xi}}-C\eta^2\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.85}$$

In the second line, we use $\|\phi\|_{L^\infty_x} \lesssim M + \varsigma \lesssim 1$. Next, we apply (4.5.45) to get

$$(4.5.76) \geq \frac{1}{C_{k_{F_+}}} \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C_{K_{F_+}} \varepsilon \langle \|\langle v\rangle^{\frac{7}{2}} F_+\|_{L^\infty_x(\mathcal{H}_\sigma)_\xi}\rangle^2 \|g_\alpha\|^2_{L^2_{x,\xi}} \tag{4.5.86}$$

where

$$k_{F_+} = \max\{1, \|\langle v\rangle^3 F_+\|_{L^\infty_x L^2_v}, \|n_+^{-1}\|_{L^\infty_x}\} \lesssim 1. \tag{4.5.87}$$

Using this, we have

$$(4.5.76) \geq \frac{1}{C} \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C\varepsilon \langle \|F_+\|_{\mathfrak{D}}\rangle^2 \|g_\alpha\|^2_{L^2_{x,\xi}} \tag{4.5.88}$$

For the next term, we write

$$\frac{e^{\frac{q_\alpha}{2}\phi} g_\alpha}{\sqrt{\mu_{q_\alpha}}} = \frac{\sqrt{\mu_\gamma} f}{\mu_{q_\alpha}} = \frac{f}{\sqrt{\mu_\gamma}} + \left(\frac{1}{\sqrt{\mu_\gamma}} - \frac{\sqrt{\mu_\gamma}}{\mu_{q_\alpha}}\right) f = \frac{f}{\sqrt{\mu_\gamma}} + \left(\frac{\mu_{q_\alpha}}{\mu_\gamma} - 1\right) \frac{e^{\frac{q_\alpha}{2}\phi} g_\alpha}{\sqrt{\mu_{q_\alpha}}} \tag{4.5.89}$$

Then,

$$(4.5.77) = -4\pi\gamma \langle \xi \cdot \nabla_x \Delta_x^{-1} a \sqrt{\mu_\gamma} e^{\gamma\psi - q_\alpha\phi}, f\rangle_{L^2_{x,\xi}} \tag{4.5.90}$$

$$- 4\pi\gamma \langle \xi \cdot \nabla_x \Delta_x^{-1} a \frac{\mu_{q_\alpha} - \mu_\gamma}{\sqrt{\mu_{q_\alpha}}} e^{\gamma\psi - \frac{q_\alpha}{2}\phi}, g_\alpha\rangle_{L^2_{x,\xi}} \tag{4.5.91}$$

$$= -4\pi\gamma \langle \nabla_x \Delta_x^{-1} a, b\rangle_{L^2_{x,\xi}} \tag{4.5.92}$$

$$- 4\pi\gamma \langle (e^{\gamma\psi - q_\alpha\phi} - 1) \nabla_x \Delta_x^{-1} a, b\rangle_{L^2_{x,\xi}} \tag{4.5.93}$$

$$- 4\pi\gamma \langle \xi \cdot \nabla_x \Delta_x^{-1} a \frac{\mu_{q_\alpha} - \mu_\gamma}{\sqrt{\mu_{q_\alpha}}} e^{\gamma\psi - \frac{q_\alpha}{2}\phi}, g_\alpha\rangle_{L^2_{x,\xi}} \tag{4.5.94}$$

For term (4.5.92), we use the continuity equation (see (4.5.201) below)

$$\varepsilon\partial_t a + \sqrt{\gamma}\nabla_x \cdot b = -\varepsilon\partial_t(e^{\gamma\psi}), \tag{4.5.95}$$

which gives

$$\begin{aligned}
(4.5.92) &= 2\pi\sqrt{\gamma}\varepsilon\frac{d}{dt}\||\nabla_x|^{-1}a\|_{L_x^2}^2 - 4\pi\sqrt{\gamma}\varepsilon\langle\partial_t(e^{\gamma\psi}), a\rangle_{L_x^2} &(4.5.96)\\
&= 2\pi\varepsilon\frac{d}{dt}\{\sqrt{\gamma}\||\nabla_x|^{-1}a\|_{L_x^2}^2\} &(4.5.97)\\
&\quad - \frac{\varepsilon\pi\dot{\gamma}}{\sqrt{\gamma}}\||\nabla_x|^{-1}a\|_{L_x^2}^2 - 4\pi\sqrt{\gamma}\varepsilon\langle\partial_t(e^{\gamma\psi}), a\rangle_{L_x^2} &(4.5.98)
\end{aligned}$$

Now, by (4.4.144),

$$\begin{aligned}
|\frac{\varepsilon\pi\dot{\gamma}}{\sqrt{\gamma}}\||\nabla_x|^{-1}a\|_{L_x^2}^2 + 4\pi\sqrt{\gamma}\varepsilon\langle\partial_t(e^{\gamma\psi}), a\rangle_{L_x^2}| &\lesssim \varepsilon(1+|\dot{\gamma}|+\|\partial_t\psi\|_{L_x^\infty})\|a\|_{L_x^2} &(4.5.99)\\
&\lesssim \varepsilon(1+\|g_\alpha\|_{L_x^2(\mathcal{H}_\sigma)_\xi})\|g_\alpha\|_{L_x^2(\mathcal{H}_\sigma)_\xi} &(4.5.100)\\
&\lesssim \varepsilon^2 + \varepsilon\|g_\alpha\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2. &(4.5.101)
\end{aligned}$$

Hence,

$$|(4.5.92) - 2\pi\varepsilon\frac{d}{dt}\{\sqrt{\gamma}\||\nabla_x|^{-1}a\|_{L_x^2}^2\}| \lesssim \varepsilon^2 + \varepsilon\|g_\alpha\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2. \tag{4.5.102}$$

For (4.5.93), we have

$$\|e^{\gamma\psi - q_\alpha\phi} - 1\|_{L_x^\infty} = \|e^{(\gamma-q_\alpha)\psi - 4\pi q_\alpha\Delta_x^{-1}a} - 1\|_{L_x^\infty} \lesssim \varsigma + \eta. \tag{4.5.103}$$

Thus,

$$|(4.5.93)| \lesssim (\varsigma+\eta)\|a\|_{L^2_x}\|b\|_{L^2_x} \tag{4.5.104}$$

$$\lesssim (\varsigma+\eta)\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.105}$$

Third, we have (4.5.94). For this, we note that when η is taken sufficiently small, we can guarantee that

$$\left|\frac{\mu_{q_\alpha}-\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}\right| \lesssim \eta e^{-\frac{\beta_{in}}{8}|\xi|^2} \tag{4.5.106}$$

Hence,

$$|(4.5.94)| \lesssim \eta\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.107}$$

Combining these bounds, we find

$$|(4.5.77) - 2\pi\varepsilon\frac{d}{dt}\{\sqrt{\gamma}\||\nabla_x|^{-1}a\|_{L^2_x}\}| \lesssim \varepsilon^2 + (\varepsilon+\varsigma+\eta)\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.108}$$

We move on to our next term:

$$(4.5.78) \lesssim \varepsilon\|\langle\xi\rangle^{\frac{1}{2}}\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}}\|_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha\|_{L^2_{x,\xi}} \tag{4.5.109}$$

$$\lesssim \varepsilon\|\langle\xi\rangle^{\frac{1}{2}}\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}}\|_{L^2_{x,\xi}}\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.110}$$

Now,

$$\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}} = (\frac{\dot{\gamma}}{\gamma}(\frac{3}{2}-\gamma(|\xi|^2-\psi))+\gamma\partial_t\psi)\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}e^{\gamma\psi-\frac{q_\alpha}{2}\phi}. \tag{4.5.111}$$

Taking η sufficiently small, we ensure that $\langle\xi\rangle^{\frac{1}{2}}\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma \lesssim e^{-\frac{\beta_{in}|\xi|^2}{8}}$. Thus, combining this with (4.4.144),

$$(4.5.78) \lesssim \varepsilon(|\dot{\gamma}| + \|\partial_t \psi\|_{L^\infty_x})\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.112}$$

$$\leq C_\kappa \varepsilon^2 + (\varepsilon+\kappa)\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.113}$$

Next, we bound (4.5.79) using (4.5.43):

$$(4.5.79) \lesssim (\|\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{L^2_\xi})(\|g_\alpha\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}g_\alpha\|_{L^2_{x,\xi}}) \tag{4.5.114}$$

$$+\,\varepsilon^{\frac{1}{2}}\|F_+\|_{\mathfrak{D}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{L^2_\xi}\|g_\alpha\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{4.5.115}$$

Now, by (4.1.19) again, we have

$$\|\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{\mathcal{H}^-_{\sigma;\varepsilon}} \lesssim \frac{1}{\varepsilon}\int_{\mathbf{R}^3}\sigma_{ij}(\frac{\xi}{\varepsilon})\xi_i\xi_j e^{-\frac{\beta_{in}}{10}|\xi|^2}d\xi \tag{4.5.116}$$

$$\lesssim \frac{1}{\varepsilon}\int_{\mathbf{R}^3}\frac{|\xi|^2}{\langle\xi/\varepsilon\rangle^3}e^{-\frac{\beta_{in}}{10}|\xi|^2}d\xi \tag{4.5.117}$$

$$\leq \varepsilon^2\int_{\mathbf{R}^3}\frac{1}{|\xi|}e^{-\frac{\beta_{in}}{10}|\xi|^2}d\xi \tag{4.5.118}$$

$$\lesssim \varepsilon^2. \tag{4.5.119}$$

since $\frac{1}{|\xi|}$ is locally integrable. Similarly,

$$\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{L^2_{x,\xi}} \lesssim \varepsilon. \tag{4.5.120}$$

Moreover, by Sobolev embedding

$$\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}g_\alpha\|_{L^2_{x,\xi}} \lesssim \varepsilon^{\frac{1}{2}}\|1_{B_1}(\xi)|\xi|^{-\frac{1}{2}}g_\alpha\|_{L^2_{x,\xi}} + \|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha\|_{L^2_{x,\xi}} \lesssim \|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.121}$$

Hence,

$$|(4.5.79)| \lesssim \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{4.5.122}$$

Thus, taking $\lambda > 0$ to be chosen later,

$$|(4.5.79)| \leq \kappa\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + C_\kappa\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2. \tag{4.5.123}$$

Finally, applying (4.5.33), we have

$$\begin{aligned}
(4.5.80) &\lesssim \|g_1\|_{L^\infty_x L^2_\xi}\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} && (4.5.124)\\
&\lesssim \|g_1^{(s)}\|_{L^2_{x,\xi}}\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} && (4.5.125)\\
&\lesssim \varsigma\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. && (4.5.126)
\end{aligned}$$

Combining the upper bounds for (4.5.74) through (4.5.80), we conclude

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\{\|g_\alpha\|^2_{L^2_{x,\xi}} + 4\pi\sqrt{\gamma}\||\Delta_x|^{-1}a\|^2_{L^2_x}\} + \frac{1}{C}\{\|\mathcal{P}^\perp_\gamma g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\} && (4.5.127)\\
&\quad \leq C(\varepsilon+\varsigma+\eta+\kappa)\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} && (4.5.128)\\
&\qquad + C\varepsilon\langle\|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle\|g_\alpha\|^2_{L^2_{x,\xi}} && (4.5.129)\\
&\qquad + C_\kappa\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2. && (4.5.130)
\end{aligned}$$

Using (4.5.32) and taking $\varepsilon, \varsigma, \eta$ and κ sufficiently small, conclude (4.5.71).

We now prove (ii). Let $g_j^{(s)} := \langle\nabla_x\rangle^s g_\alpha$. Then,

$$\frac{\varepsilon}{2}\frac{d}{dt}\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{4.5.131}$$
$$+ q_\alpha\varepsilon\langle\langle\nabla_x\rangle^s\{\partial_t\phi g_\alpha\}, g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.132}$$
$$+ \langle\langle\nabla_x\rangle^s\{Eg_\alpha\} - Eg_\alpha^{(s)}, \nabla_v g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.133}$$
$$+ \langle\langle\nabla_x\rangle^s(e^{\gamma\psi}\widetilde{\mathcal{L}}_{\gamma,q_\alpha}g_\alpha), g_j^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.134}$$
$$- 4\pi\gamma\langle\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}\langle\nabla_x\rangle^s\{\xi\cdot\nabla_x\Delta_x^{-1}ae^{\gamma\psi-\frac{q_\alpha}{2}\phi}\}, g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.135}$$
$$+ \langle\langle\nabla_x\rangle^s(\mathcal{M}_{q_\alpha,F_+}g_\alpha), g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.136}$$
$$= -\varepsilon\langle\langle\nabla_x\rangle^s\left(\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}}\right), g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.137}$$
$$- \langle\langle\nabla_x\rangle^s\{e^{\gamma\psi-\frac{q_\alpha}{2}\phi}\mathcal{M}_{q_\alpha,F_+}(\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}})\}, g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.138}$$
$$+ \langle\langle\nabla_x\rangle^s\{e^{\frac{q_\alpha}{2}\phi}\Gamma_{q_\alpha}(g_\alpha,g_\alpha)\}, g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.139}$$

First,

$$|(4.5.132)| = q_\alpha\varepsilon|\langle\langle\nabla_x\rangle^s\{(\partial_t\psi + 4\pi\Delta_x^{-1}\partial_t a)g_\alpha\}, g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{4.5.140}$$
$$\lesssim \varepsilon\int_{\mathbf{R}^3}\big(\|\partial_t\psi\|_{H^s_x}\|g_\alpha^{(s)}\|^2_{L^2_x} + \|\partial_t a\|_{H^{s-2}_x}\|g_\alpha\|_{L^\infty_x}\|g_\alpha^{(s)}\|_{L^2_x} \tag{4.5.141}$$
$$+ \|\Delta_x^{-1}\partial_t a\|_{L^\infty_x}\|g_\alpha^{(s)}\|_{L^2_x}\big)d\xi \tag{4.5.142}$$

Applying (4.5.95) to the second term, we get

$$\varepsilon\int_{\mathbf{R}^3}\|\partial_t a\|_{H^{s-2}_x}\|g_\alpha\|_{L^\infty_x}\|g_\alpha^{(s)}\|_{L^2_x}d\xi \tag{4.5.143}$$
$$\lesssim \int_{\mathbf{R}^3}\|b\|_{H^{s-1}_x}\|g_\alpha^{(s-1)}\|_{L^2_x}\|g_\alpha^{(s)}\|_{L^2_x}d\xi + \varepsilon(1+|\dot\gamma| + \|\partial_t\psi\|_{H^s_x})\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{4.5.144}$$
$$\lesssim \|b\|_{H^{s-1}_x}\|\langle\xi\rangle^{\frac12}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac12}g_\alpha^{(s)}\|_{L^2_{x,\xi}} + \varepsilon(1+|\dot\gamma| + \|\partial_t\psi\|_{H^s_x})\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{4.5.145}$$

Collecting terms,

$$|(4.5.132)| \lesssim \|b\|_{H_x^{s-1}}\|\langle\xi\rangle^{\frac{1}{2}}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}} \tag{4.5.146}$$

$$+\varepsilon(1+\|\partial_t a\|_{L^\infty_x}+|\dot{\gamma}|+\|\partial_t\psi\|_{H^s_x})\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{4.5.147}$$

Applying (4.4.144), we get

$$\varepsilon(|\dot{\gamma}|+\|\partial_t\psi\|_{H^s_x})\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \lesssim \varepsilon^2+\varepsilon\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}}+\varsigma\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.148}$$

On the other hand, using interpolation, we have

$$\|\langle\xi\rangle^{\frac{1}{2}}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha^{(s)}\|_{L^2_{x,\xi}} \leq (C_\eta\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}). \tag{4.5.149}$$

Thus,

$$|(4.5.132)| \lesssim \varepsilon\langle\|\partial_t a\|_{L^2_x}\rangle\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}}+C\varsigma\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+C_\eta\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.150}$$

Next,

$$|(4.5.133)| \lesssim \|\langle\xi\rangle^{\frac{1}{2}}(\langle\nabla_x\rangle^s\{Eg_\alpha\}-Eg_\alpha^{(s)})\|_{L^2_{x,\xi}}\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.151}$$

$$\lesssim (\|E\|_{H^s_x}\|\langle\xi\rangle^{\frac{1}{2}}g_\alpha\|_{L^\infty_x L^2_\xi}+\|\nabla_x E\|_{L^\infty_x}\|\langle\xi\rangle^{\frac{1}{2}}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}})\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.152}$$

Above we use the usual commutator estimate for $[\langle\nabla_x\rangle^s, E_j]$. Next we interpolate

$$\|\langle\xi\rangle^{\frac{1}{2}}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}} \lesssim \|\langle\xi\rangle^{s-\frac{1}{2}}g_\alpha\|^{\frac{1}{s}}_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha^{(s)}\|^{\frac{s-1}{s}}_{L^2_{x,\xi}} \tag{4.5.153}$$

$$\lesssim C_\eta\|\langle\xi\rangle^{-\frac{1}{2}}g_{\alpha+1}\|^{\frac{1}{s}}_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha^{(s)}\|^{\frac{s-1}{s}}_{L^2_{x,\xi}} \tag{4.5.154}$$

$$\lesssim C_\eta\|\langle\xi\rangle^{-\frac{1}{2}}g_{\alpha+1}\|^{\frac{1}{s}}_{L^2_x(\mathcal{H}_\sigma)_\xi}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha^{(s)}\|^{\frac{s-1}{s}}_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.155}$$

This implies

$$|(4.5.133)| \leq \kappa \|g_\alpha^{(s)}\|_{L_x^2}^2 + C_{\eta,\kappa} \|\langle \xi \rangle^{-\frac{1}{2}} g_{\alpha+1}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2. \tag{4.5.156}$$

Next, applying (4.5.29) and (4.5.30), we have

$$\begin{aligned}
(4.5.134) = & \langle e^{\gamma\psi} \widetilde{\mathcal{L}}_{\gamma,q} g_\alpha^{(s)}, g_\alpha^{(s)} \rangle_{L_{x,\xi}^2} && (4.5.157)\\
& + \langle \widetilde{\mathcal{L}}_{\gamma,q} \{ \langle \nabla_x \rangle^s (e^{\gamma\psi} g_\alpha) - e^{\gamma\psi} g_\alpha^{(s)} \}, g_1^{(s)} \rangle_{L_{x,\xi}^2} && (4.5.158)\\
\geq & \frac{1}{C} \|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 - C\varsigma^2 \|g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 && (4.5.159)\\
& - C \|\langle \nabla_x \rangle^s (e^{\gamma\psi} g_\alpha) - e^{\gamma\psi} g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \|g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} && (4.5.160)
\end{aligned}$$

Now,

$$\begin{aligned}
\|\langle \nabla_x \rangle^s (e^{\gamma\psi} g_\alpha) - e^{\gamma\psi} g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} & \lesssim \|\langle \nabla_x \rangle^s e^{\gamma\psi}\|_{L_x^2} \|g_\alpha^{(s-1)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} && (4.5.161)\\
& \lesssim \|g_\alpha^{(s-1)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}. && (4.5.162)
\end{aligned}$$

Hence, we can interpolate to get

$$\begin{aligned}
(4.5.134) \geq & \frac{1}{C} \|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 - \kappa \|g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 && (4.5.163)\\
& - C_\kappa \|g_{\alpha+1}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 && (4.5.164)
\end{aligned}$$

For the next term, we have

$$\begin{aligned}
|(4.5.135)| & \lesssim \|e^{\gamma\psi - \frac{q_\alpha}{2}\phi} \nabla_x \Delta_x^{-1} a\|_{H^s} \|g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} && (4.5.165)\\
& \lesssim \|g_\alpha^{(s-1)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \|g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} && (4.5.166)\\
& \leq \kappa \|g_\alpha^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 + C_\kappa \|g_{\alpha+1}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2. && (4.5.167)
\end{aligned}$$

For our next term, we have

$$
\begin{aligned}
(4.5.136) = {} & \langle \mathcal{M}_{q_1,F_+} g_\alpha^{(s)}, g_\alpha^{(s)} \rangle_{L^2_{x,\xi}} && (4.5.168)\\
& + \langle \langle \nabla_x \rangle^s (\mathcal{M}_{q_1,F_+} g_\alpha) - \mathcal{M}_{q_1,F_+} g_\alpha^{(s)}, g_\alpha^{(s)} \rangle_{L^2_{x,\xi}} && (4.5.169)
\end{aligned}
$$

For the first term, similarly to (4.5.134), we have

$$
(4.5.168) \geq \frac{1}{C} \| g_\alpha^{(s)} \|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C\varepsilon \langle \| F_+^{(s,m)} \|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \rangle^2 \| g_\alpha^{(s)} \|^2_{L^2_{x,\xi}} \qquad (4.5.170)
$$

Next, combining the commutator estimate with the proof of (4.5.43), we have

$$
\begin{aligned}
|(4.5.169)| \lesssim {} & \| F_+^{(m_1,s)} \|_{L^2_{x,v}} (\| g_1^{(s-1)} \|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon^{\frac{1}{2}} \| \langle \frac{\xi}{\varepsilon} \rangle^{-\frac{1}{2}} g_1^{(s-1)} \|_{L^2_{x,\xi}}) && (4.5.171)\\
& \cdot (\| g_1^{(s)} \|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon^{\frac{1}{2}} \| \langle \frac{\xi}{\varepsilon} \rangle^{-\frac{1}{2}} g_1^{(s)} \|_{L^2_{x,\xi}}) && (4.5.172)\\
& + \varepsilon^{\frac{1}{2}} \| F_+ \|_{\mathfrak{D}} \| g_1^{(s-1)} \|_{L^2_{x,\xi}} \| g_1^{(s)} \|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} && (4.5.173)\\
\lesssim {} & (\| g_\alpha \|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon \| g_\alpha \|_{L^2_x(\mathcal{H}_\sigma)_\xi})^{\frac{1}{s}} (\| g_\alpha \|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon \| g_\alpha \|_{L^2_x(\mathcal{H}_\sigma)_\xi})^{\frac{2s-1}{s}} && (4.5.174)\\
& + \varepsilon^{\frac{1}{2}} \| F_+ \|_{\mathfrak{D}} \| g_1^{(s)} \|_{L^2_{x,\xi}} \| g_1^{(s)} \|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}. && (4.5.175)
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
(4.5.136) \geq {} & \frac{1}{C} \| g_\alpha^{(s)} \|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C \{ \| g_{\alpha+1} \|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \| g_{\alpha+1} \|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \} && (4.5.176)\\
& - C\varepsilon \langle \| F_+ \|_{\mathfrak{D}} \rangle^2 \| g_\alpha^{(s)} \|^2_{L^2_{x,\xi}} && (4.5.177)
\end{aligned}
$$

Next, similarly to (4.5.78),

$$
(4.5.137) \leq C_\kappa \varepsilon + \kappa \| g_\alpha^{(s)} \|_{L^2_x(\mathcal{H}_\sigma)_\xi} \qquad (4.5.178)
$$

and, as with (4.5.79), we use (4.5.43) to get

$$(4.5.138) \leq \kappa \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + C_\kappa \varepsilon^2 \langle \|F_+\|_{\mathfrak{D}} \rangle^2. \tag{4.5.179}$$

Finally, using (4.5.33),

$$(4.5.139) \lesssim \|g_\alpha^{(s)}\|_{L^2_{x,\xi}} \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.180}$$

$$\lesssim \varsigma \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.181}$$

We now combine the estimates for the terms (4.5.132) through (4.5.139):

$$\frac{\varepsilon}{2}\frac{d}{dt}\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} + \frac{1}{C}\{\|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\} \tag{4.5.182}$$

$$\leq C(\varepsilon + \varsigma + \eta + \kappa)\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + C_{\eta,\kappa}\{\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}\} \tag{4.5.183}$$

$$+ C\varepsilon\langle \|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle \|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{4.5.184}$$

$$+ C_\kappa \varepsilon^2 \langle \|F_+\|_{\mathfrak{D}}\rangle^2. \tag{4.5.185}$$

Using (4.5.32) again, we conclude with (4.5.72). □

4.5.5 Macroscopic estimates

What remains is to get bounds on $\mathcal{P}_\gamma f$. This requires analysis of the local conservation laws. In order to derive these equations efficiently, and, in particular, see how the transport part of (4.5.13) couples (a, b, c), we introduce the ladder operators: the lowering and raising operators are respectively

$$\mathcal{A}_j = \gamma^{\frac{1}{2}}\frac{\xi_j}{2} + \gamma^{-\frac{1}{2}}\partial_{\xi_j}, \quad \mathcal{A}_j^* = \gamma^{\frac{1}{2}}\frac{\xi_j}{2} - \gamma^{-\frac{1}{2}}\partial_{\xi_j}, \quad j \in \{1, 2, 3\}. \tag{4.5.186}$$

We recall the identities

$$\mathcal{A}_j\mu^{\frac{1}{2}} = 0, \quad [\mathcal{A}_j, \mathcal{A}_k^*] = \varsigma_{jk}, \quad j,k \in \{1,2,3\}. \tag{4.5.187}$$

Recall also that the family of Hermite functions

$$\{\frac{1}{\sqrt{n_1!n_2!n_3!}}(\mathcal{A}_1^*)^{n_1}(\mathcal{A}_2^*)^{n_2}(\mathcal{A}_3^*)^{n_3}\mu_\gamma^{\frac{1}{2}}\}_{n_1,n_2,n_3\in\mathbf{N}_0^3} \tag{4.5.188}$$

gives a complete orthonormal basis for $L^2_\xi(\mathbf{R}^3)$. We shall denote the (unnormalized) Hermite functions

$$\mathfrak{h} = \mu_\gamma^{\frac{1}{2}}, \quad \mathfrak{h}_{j_1\cdots j_m} = \mathcal{A}_{j_1}\cdots\mathcal{A}_{j_n}\mu_\gamma^{\frac{1}{2}}, \quad j_1,\ldots,j_m \in \{1,2,3\}. \tag{4.5.189}$$

We can represent the kernel of $\mathcal{L}_\gamma$ using the hermite functions as follows:

$$N(\mathcal{L}) = \mathrm{span}\{\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \frac{1}{\sqrt{6}}\mathfrak{h}_{jj}\}. \tag{4.5.190}$$

We can thus rewrite (4.5.20) as

$$\mathcal{P}_\gamma f = a\mathfrak{h} + b_j\mathfrak{h}_j + c\frac{1}{\sqrt{6}}\mathfrak{h}_{jj} \tag{4.5.191}$$

It is also convenient to define the following projection of h, involving third-order hermite functions:

$$d_j = \frac{1}{\sqrt{10}}\langle\mathfrak{h}_{jkk}, f\rangle_{L^2_\xi}, \quad j \in \{1,2,3\} \tag{4.5.192}$$

We now rewrite the transport part of (4.5.13) in terms of $\mathcal{A}, \mathcal{A}^*$

$$\xi\cdot\nabla_x + E\cdot(\nabla_\xi - \frac{\xi}{2}) = (\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_j)\mathcal{A}_j^* + \gamma^{-\frac{1}{2}}\partial_{x_j}\mathcal{A}_j. \tag{4.5.193}$$

We also will need to evaluate projections of $\mu^{-\frac{1}{2}}\partial_t(\mu^{\frac{1}{2}}f)$. Let $\zeta(\xi)$ be some linear combination of the hermite functions in $\gamma^{\frac{1}{2}}\xi$, i.e. there exists a polynomial $p:\mathbf{R}^3\to\mathbf{R}$ (with coefficients independent of γ) such that

$$\zeta(\xi)=p(\gamma^{\frac{1}{2}}\xi)\mu(\xi)^{\frac{1}{2}}. \tag{4.5.194}$$

Then,

$$\langle\zeta,\mu^{-\frac{1}{2}}\partial_t(\mu^{\frac{1}{2}}f)\rangle_{L^2_\xi}=\partial_t\langle\zeta,f\rangle_{L^2_\xi}-\langle\partial_t(\zeta\mu^{-\frac{1}{2}})\mu^{\frac{1}{2}},f\rangle_{L^2_\xi} \tag{4.5.195}$$

Now,

$$\partial_t\{\zeta\mu^{-\frac{1}{2}}\}=\partial_t\{p(\gamma^{\frac{1}{2}}\xi)\}=\frac{\dot\gamma}{2\gamma}\xi\cdot\nabla_\xi\{p(\gamma^{\frac{1}{2}}\xi)\}=\frac{\dot\gamma}{2\gamma}\xi\cdot\nabla_\xi\{\zeta\mu^{-\frac{1}{2}}\}. \tag{4.5.196}$$

Therefore,

$$\begin{aligned}\partial_t\{\zeta\mu^{-\frac{1}{2}}\}\mu^{\frac{1}{2}}&=\frac{\dot\gamma}{2\gamma}(\xi\cdot\nabla_\xi+\frac{\gamma|\xi|^2}{2})\zeta && (4.5.197)\\ &=\frac{\dot\gamma}{2\gamma}\xi\cdot(\nabla_\xi+\frac{\gamma\xi}{2})\zeta && (4.5.198)\\ &=\frac{\dot\gamma}{2\gamma}(\mathcal{A}_j+\mathcal{A}_j^*)\mathcal{A}_j\zeta && (4.5.199)\end{aligned}$$

In summary,

$$\langle\zeta,\mu^{-\frac{1}{2}}\partial_t(\mu^{\frac{1}{2}}f)\rangle_{L^2_\xi}=\partial_t\langle\zeta,f\rangle_{L^2_\xi}-\frac{\dot\gamma}{2\gamma}\langle(\mathcal{A}_j+\mathcal{A}_j^*)\mathcal{A}_j\zeta,f\rangle_{L^2_\xi}. \tag{4.5.200}$$

The following lemma contains formulas for relevant projections of the equation (4.5.13).

Lemma 4.5.5. *The following formulas hold:*

$$\varepsilon\partial_t a+\gamma^{-\frac{1}{2}}\partial_{x_j}b_j=-\varepsilon\partial_t(e^{\gamma\psi}), \tag{4.5.201}$$

$$\varepsilon(\partial_t - \frac{\dot{\gamma}}{2\gamma})b_j + (\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_j)a - 4\pi\gamma^{\frac{1}{2}}e^{\gamma\psi}\partial_{x_j}\Delta_x^{-1}a + \sqrt{\frac{2}{3}}\partial_{x_j}c \\ + \partial_{x_k}\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L^2_\xi} + \langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi} = -e^{\gamma\psi}\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L^2_\xi}, \tag{4.5.202}$$

$$\varepsilon(\partial_t c - \frac{\dot{\gamma}}{\gamma}(c + \sqrt{\frac{3}{2}}a)) + \sqrt{\frac{2}{3}}(\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_j)b_j + \sqrt{\frac{5}{3}}\gamma^{-\frac{1}{2}}\partial_{x_j}d_j \\ + \frac{1}{\sqrt{6}}\langle\mathfrak{h}_{jj}, \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi} = \varepsilon\sqrt{\frac{3}{2}}\frac{\dot{\gamma}}{\gamma}e^{\gamma\psi} + \frac{e^{\gamma\psi}}{\sqrt{6}}\langle\mathfrak{h}_{jj}, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L^2_\xi}, \tag{4.5.203}$$

$$\varepsilon(\partial_t d_j - \frac{\dot{\gamma}}{2\gamma}(3d_j + \frac{3\sqrt{2}}{\sqrt{5}}b_j)) + \frac{3\sqrt{3}}{\sqrt{5}}(\gamma^{-\frac{1}{2}}\partial_{x^j} + \gamma^{\frac{1}{2}}E_j)c \\ + \sqrt{\frac{2}{5}}(\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k)\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L^2_\xi} + \gamma^{-\frac{1}{2}}\frac{1}{\sqrt{10}}\partial_{x_k}\langle\mathfrak{h}_{jkll}, f\rangle_{L^2_\xi} \\ + \frac{1}{\sqrt{10}}\langle\mathfrak{h}_{jll}, \mathcal{L}_\gamma f + \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi} = \langle\mathfrak{h}_{jll}, -e^{\gamma\psi}\mathcal{M}_{\gamma,F_+}\mathfrak{h} + \Gamma_\gamma(f,f)\rangle. \tag{4.5.204}$$

Proof. These formulas follow from direct computation of the ξ integral of (4.5.13) multiplied by $\mathfrak{h}, \mathfrak{h}_j, \frac{1}{\sqrt{6}}\mathfrak{h}_{jj}$ and $\frac{1}{\sqrt{10}}\mathfrak{h}_{jll}$. We only give the details for (4.5.202). The proof of the other formulas are similar. Using (4.5.193) and (4.5.200), and the fact that $\mathcal{L}_\gamma f$, $\mu^{-\frac{1}{2}}\partial_t\nu_\gamma$ and $\Gamma_\gamma(f,f)$ are orthogonal to $\mathfrak{h}_j$, we have

$$\varepsilon(\partial_t b_j - \frac{\dot{\gamma}}{2\gamma}\langle(\mathcal{A}_k\mathcal{A}_k + \mathcal{A}_k^*\mathcal{A}_k)\mathfrak{h}_j, f\rangle_{L^2_\xi}). \tag{4.5.205}$$

$$+ \langle\mathfrak{h}_j, \{(\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k)\mathcal{A}_k^* + \gamma^{-\frac{1}{2}}\partial_{x_k}\mathcal{A}_k\}f\rangle_{L^2_\xi} \tag{4.5.206}$$

$$+ \langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi} \tag{4.5.207}$$

$$= -\langle\mathfrak{h}_j, e^{\gamma\psi}\mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L^2_\xi}. \tag{4.5.208}$$

It remains to evaluate terms involving the ladder operators. Using (4.5.187), one computes

$$(\mathcal{A}_k + \mathcal{A}_k^*)\mathcal{A}_k \mathfrak{h}_j = (\mathcal{A}_k^* + \mathcal{A}_k)\mathcal{A}_k \mathcal{A}_j^* \mathfrak{h} \tag{4.5.209}$$

$$= (\mathcal{A}_j^* + \mathcal{A}_j)\mathfrak{h} \tag{4.5.210}$$

$$= \mathfrak{h}_j \tag{4.5.211}$$

and

$$\langle \mathfrak{h}_j, \{(\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k)\mathcal{A}_k^* + \gamma^{-\frac{1}{2}}\partial_{x_k}\mathcal{A}_k\} f\rangle_{L^2_\xi} \tag{4.5.212}$$

$$= \langle \mathcal{A}_k \mathcal{A}_j^* \mathfrak{h}, (\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k) f\rangle_{L^2_\xi} \tag{4.5.213}$$

$$+ \gamma^{-\frac{1}{2}}\langle \mathcal{A}_k^* \mathcal{A}_j^* \mathfrak{h}, \partial_{x_k} f\rangle_{L^2_\xi} \tag{4.5.214}$$

$$= (\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_k)a \tag{4.5.215}$$

$$+ \gamma^{-\frac{1}{2}}\partial_{x_k}\langle \mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L^2_\xi} \tag{4.5.216}$$

$$+ \frac{1}{\sqrt{6}}\gamma^{-\frac{1}{2}}\langle \mathfrak{h}_{jk}, \mathfrak{h}_{ll}\rangle_{L^2_\xi}\partial_{x_k} c. \tag{4.5.217}$$

Finally, noting that in the last term,

$$\langle \mathfrak{h}_{jk}, \mathfrak{h}_{ll}\rangle_{L^2_\xi} = \sqrt{2}\delta_{jk}, \tag{4.5.218}$$

we conclude with (4.5.202). $\square$

Lemma 4.5.6. *Assume the hypotheses of Proposition 4.5.1. Then, for all $r \geq 0$,*

$$\|b\|_{H^r_x} \lesssim_M \|\mathcal{P}_\gamma^\perp f^{(r)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f^{(r)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon^2\|(a,c)\|^2_{H^r_x}. \tag{4.5.219}$$

Proof. It suffices to show the case when $r = 0$. Because of radial symmetry of $\mathfrak{h}$ and

$\mathfrak{h}_{jj}$, we have

$$\|(\mathfrak{h}, \mathfrak{h}_{jj})\|_{\mathcal{H}^-_{\sigma;\varepsilon}} \lesssim \varepsilon. \tag{4.5.220}$$

Hence,

$$\|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \geq \frac{1}{2}\|\mathcal{P}^\perp_\gamma f + b_j\mathfrak{h}_j\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C\varepsilon^2\|(a,c)\|^2_{L^2_x} \tag{4.5.221}$$

Now, writing $u = \mathcal{P}^\perp_\gamma f + b_j\mathfrak{h}_j$, we have

$$\|u\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} = \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}} \geq \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}}. \tag{4.5.222}$$

On the other hand, for all $\nu \in \mathbf{S}^2$, and $\xi \in \mathbf{R}^3 \setminus B_1$, we have

$$\sigma_{ij}(\frac{\xi}{\varepsilon})\nu_i\nu_j \lesssim \sigma_{ij}(\xi)\nu_i\nu_j \tag{4.5.223}$$

Now, we expand u as follows:

$$\frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}} \tag{4.5.224}$$

$$= \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}(b_k\mathfrak{h}_k), \partial_{\xi_j}(b_l\mathfrak{h}_l)\rangle_{L^2_{x,\xi}} \tag{4.5.225}$$

$$+ \frac{2}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}(b_k\mathfrak{h}_k), \partial_{\xi_j}\mathcal{P}^\perp_\gamma f\rangle_{L^2_{x,\xi}} \tag{4.5.226}$$

$$+ \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}\mathcal{P}^\perp_\gamma f, \partial_{\xi_j}\mathcal{P}^\perp_\gamma f\rangle_{L^2_{x,\xi}}. \tag{4.5.227}$$

Now $\frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}\cdot, \partial_{\xi_j}\cdot\rangle_{L^2_{x,\xi}}$ defines a semi-inner product. So, we apply Cauchy-

Schwarz and Young's inequality to the cross term to get

$$\frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i} u, \partial_{\xi_j} u\rangle_{L^2_{x,\xi}} \tag{4.5.228}$$

$$\geq \frac{1}{2\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i} (b_k\mathfrak{h}_k), \partial_{\xi_j} (b_l\mathfrak{h}_l)\rangle_{L^2_{x,\xi}} - \frac{2}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i} \mathcal{P}_\gamma^\perp f, \partial_{\xi_j} \mathcal{P}_\gamma^\perp f\rangle_{L^2_{x,\xi}} \tag{4.5.229}$$

$$\geq \frac{1}{C} W_{kl}\langle b_l, b_k\rangle_{L^2_x} - C\|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}, \tag{4.5.230}$$

where

$$W_{kl} := \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}\mathfrak{h}_k, \partial_{\xi_j}\mathfrak{h}_l\rangle_{L^2_\xi}. \tag{4.5.231}$$

We now show that

$$W_{kl}\nu_k\nu_l \geq \frac{1}{C} \tag{4.5.232}$$

for all $\nu \in \mathbf{S}^2$. Now,

$$\partial_{\xi_i}\mathfrak{h}_k = \frac{\gamma^{\frac{1}{2}}}{2}(\delta_{ik} - \frac{\gamma}{2}\xi_i\xi_k)\mu_\gamma^{\frac{1}{2}} \tag{4.5.233}$$

Then, by the reverse triangle inequality,

$$\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\partial_{\xi_i}\mathfrak{h}_k, \partial_{\xi_j}\mathfrak{h}_l = \frac{\gamma}{4\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}(\nu_i - \frac{\gamma}{2}(\nu\cdot\xi)\xi_i)(\nu_j - \frac{\gamma}{2}(\nu\cdot\xi)\xi_j)\mu_\gamma \tag{4.5.234}$$

$$\geq \{\frac{\gamma}{8\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\nu_i\nu_j - C\frac{(\nu\cdot\xi)^2}{\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\xi_i\xi_j\}\mu_\gamma \tag{4.5.235}$$

Now, for $|\xi| \geq 1$, we have

$$\frac{1}{\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\xi_i\xi_j \sim \frac{|\xi|^2}{\langle \xi/\varepsilon\rangle^3} \sim \varepsilon^2\frac{1}{|\xi|}. \tag{4.5.236}$$

On the other hand,

$$\frac{1}{\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\nu_i\nu_j \gtrsim \frac{1}{\varepsilon}\frac{|P_{\xi^\perp}\nu|^2}{\langle \xi/\varepsilon\rangle} \gtrsim \frac{|P_{\xi^\perp}\nu|^2}{|\xi|}. \tag{4.5.237}$$

Hence,

$$W_{kl}\nu_k\nu_l \geq \frac{1}{C}\int_{\mathbf{R}^3\setminus B_1}\frac{|P_{\xi^\perp}\nu|^2}{|\xi|}\mu_\gamma d\xi - \varepsilon^2\int_{\mathbf{R}^3\setminus B_1}|\xi|\mu_\gamma d\xi \tag{4.5.238}$$

$$\geq \frac{1}{C} - C\varepsilon^2. \tag{4.5.239}$$

Taking ε small enough, we deduce (4.5.232), and thus

$$\frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}} \geq \frac{1}{C}\|b\|^2_{L^2_x} - C\|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.240}$$

From this, we conclude (4.5.219) in the case $r = 0$. □

The next lemma allows us to control (a, b, c).

Proposition 4.5.7. *Let $M \geq 1$, and let η, ς and ε be sufficiently small depending on M. Assume the bootstrap assumptions (4.5.3) through (4.5.6). Then, exists two real valued functions $\mathscr{G} = \mathscr{G}^{(M)}$ and $\mathscr{G}_{reg} = \mathscr{G}_{reg}^{(M)}$ on $[0, T]$ satisfying, for all $t \in [0, T]$*

$$|\mathscr{G}(t)| \lesssim_M \|f(t)\|^2_{L^2_{x,\xi}}, \quad |\mathscr{G}_{reg}(t)| \lesssim_M \|f^{(s)}(t)\|^2_{L^2_{x,\xi}} \tag{4.5.241}$$

such that

$$\begin{aligned}&\varepsilon\frac{d}{dt}\mathscr{G}(t) + \|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}\\ &\quad\lesssim_M \varepsilon^2\langle \|F_+\|_{\mathfrak{D}}\rangle^2 + \min_{\alpha\in\{1,2,3\}}\{\|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\},\end{aligned} \tag{4.5.242}$$

and

$$\begin{aligned}&\varepsilon\frac{d}{dt}\mathscr{G}_{reg}(t)+\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}\\&\quad\lesssim_M \varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2+\min_{\alpha\in\{1,2\}}\{\|\mathcal{P}^\perp_\gamma g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+\|g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\}.\end{aligned}\tag{4.5.243}$$

Proof. We break the proof into a number of steps. Given $u:\mathbf{T}^3\to\mathbf{R}$, we split $u=u^\times+\overline{u}$, where $\overline{u}(t)=\int_{\mathbf{T}^3}u(t,x)dx$. The first step concerns the bounds on $(\overline{a},\overline{c})$. In step 2, we bound $\|a\|_{L^2_x}$. In step 3, we bound $(e^{\gamma\psi}c)^\times$. In step 4, we bound $\|(a,c)\|_{H^s_x}$. In step 5, we synthesize these bounds.

Step 1 (estimate on $\overline{a}$ and $\overline{c}$): Regarding $\overline{a}$, from (4.5.21), we simply have

$$\int_{\mathbf{T}^3}a(t,x)dx\equiv 0\tag{4.5.244}$$

for all times. In particular, $\Delta_x^{-1}a$ is well defined.

We now discuss the zero mode of c. Subtracting the first line of (4.4.140) from (4.1.15), we have

$$\sqrt{\frac{3}{2}}\frac{d}{dt}\left(\frac{\overline{c}(t)}{\gamma}\right)=\frac{d}{dt}\int_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|\xi|^2}{2}(F_--\mu_\gamma e^{\gamma\psi})dxd\xi\tag{4.5.245}$$

Integrating in time, we have

$$\sqrt{\frac{3}{2}}\frac{\overline{c}(t)}{\gamma(t)}=\sqrt{\frac{3}{2}}\frac{\overline{c}_{in}}{\beta_{in}}+\frac{1}{8\pi}\int_{\mathbf{T}^3}|\nabla_x\phi_{in}(x)|^2-|\nabla_x\psi_{in}(x)|^2dx\tag{4.5.246}$$

$$+\frac{1}{8\pi}\int_{\mathbf{T}^3}|\nabla_x\psi(t,x)|^2-|\nabla_x\phi(t,x)|^2dx\tag{4.5.247}$$

On the other hand, from (4.5.10),

$$\int_{\mathbf{T}^3} |\nabla_x\psi(t,x)|^2 - |\nabla_x\phi(t,x)|^2 dx = -2\langle \nabla_x\psi, \nabla_x\Delta_x^{-1}a\rangle_{L^2_x} - \|\nabla_x\Delta_x^{-1}a\|_{L^2_x} \quad (4.5.248)$$

$$= 2\langle\psi, a\rangle_{L^2_x} - \|\Delta_x^{-\frac{1}{2}}a\|^2_{L^2_x}. \quad (4.5.249)$$

Combining these with (4.5.7), we get

$$\left|\sqrt{\frac{3}{2}}\frac{\overline{c}(t)}{\gamma(t)} - \frac{1}{4\pi}\langle\psi, a\rangle_{L^2_x}\right| \lesssim \varepsilon + \|a\|^2_{L^2_x}. \quad (4.5.250)$$

In what follows, it will become clear that we cannot control $c^\times$ directly. Instead, we can only get a bound on $(ce^{\gamma\psi})^\times$. It is then necessary to compute $c^\times$ in terms of $(ce^{\gamma\psi})^\times$ and $\overline{c}$. Observe that

$$e^{\gamma\psi}(e^{-\gamma\psi}c)^\times + e^{\gamma\psi}\int_{\mathbf{R}^3} e^{-\gamma\psi}c dx' = c^\times + \overline{c}. \quad (4.5.251)$$

Since $\int_{\mathbf{T}^3} e^{\gamma\psi}dx = 1$, the above integrated in x yields

$$\int_{\mathbf{R}^3} e^{-\gamma\psi}cdx = \overline{c} - \int_{\mathbf{T}^3} e^{\gamma\psi}(e^{-\gamma\psi}c)^\times dx. \quad (4.5.252)$$

Thus, we have the identity

$$c^\times = (e^{\gamma\psi}-1)\overline{c} + e^{\gamma\psi}(e^{-\gamma\psi}c)^\times - e^{\gamma\phi_0}\int_{\mathbf{T}^3} e^{\gamma\psi}(e^{-\gamma\psi}c)^\times dx \quad (4.5.253)$$

In particular, $\|c\|_{L^2_x} \lesssim |\overline{c}| + \|(e^{-\gamma\psi}c)^\times\|_{L^2_x}$.

Step 2 (estimate for a): Now, we turn to bounding a. We claim that the following estimate holds:

$$
\begin{aligned}
&-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, b\rangle_{L_x^2} + \frac{1}{C}\|a\|_{L_x^2}^2 \\
&\quad\le C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + \|b\|_{L_x^2}^2 + \|(e^{-\gamma\psi}c)^\times\|_{L_x^2}^2 + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 + \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}^2\}.
\end{aligned}
\tag{4.5.254}
$$

To prove this, first observe

$$
(\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E)a = \gamma^{-\frac{1}{2}}e^{\gamma\psi}\nabla_x(e^{-\gamma\psi}a) - 4\pi a\nabla_x\Delta_x^{-1}a, \tag{4.5.255}
$$

Therefore,

$$
-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, b\rangle_{L_x^2} + \gamma^{-\frac{1}{2}}\|e^{-\frac{\gamma\psi}{2}}a\|_{L_x^2}^2 + 4\pi\|e^{\frac{\gamma\psi}{2}}\nabla_x\Delta_x^{-1}a\|_{L_x^2}^2 + 4\pi\overline{c}^2 \tag{4.5.256}
$$

$$
= -\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, \varepsilon\partial_t b + (\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E - 4\pi\gamma^{\frac{1}{2}}e^{\gamma\psi}\nabla_x\Delta_x^{-1})a + \sqrt{\frac{2}{3}}\nabla_x c\rangle_{L_x^2} \tag{4.5.257}
$$

$$
-\varepsilon\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}\partial_t a, b\rangle_{L_x^2} \tag{4.5.258}
$$

$$
+\varepsilon\langle\partial_t(\gamma\psi)e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, b\rangle_{L_x^2} \tag{4.5.259}
$$

$$
+4\pi\varepsilon\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, a\nabla_x\Delta_x^{-1}a\rangle_{L_x^2} \tag{4.5.260}
$$

$$
+4\pi\overline{c}^2 + \sqrt{\frac{2}{3}}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, \nabla_x c\rangle_{L_x^2}. \tag{4.5.261}
$$

From (4.5.202), we have

$$\|\varepsilon\partial_t b_j + (\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_j)a - 4\pi\gamma^{\frac{1}{2}}e^{\gamma\psi}\nabla_x\Delta_x^{-1}a\|_{H_x^{-1}} \tag{4.5.262}$$

$$= \Big\|\varepsilon\frac{\dot{\gamma}}{2\gamma}b_j + \sqrt{\frac{2}{3}}\partial_{x_j}c + \partial_{x_k}\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L_\xi^2} + \langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L_\xi^2} \tag{4.5.263}$$

$$+ e^{\gamma\psi}\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L_\xi^2}\Big\|_{H_x^{-1}} \tag{4.5.264}$$

$$\lesssim \varepsilon|\dot{\gamma}|\|b\|_{L_x^2} + \|c\|_{L_x^2} + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{4.5.265}$$

$$+ \|\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L_\xi^2}\|_{H_x^{-1}} \tag{4.5.266}$$

$$+ \|e^{\gamma\psi}\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L_\xi^2}\|_{H_x^{-1}} \tag{4.5.267}$$

Applying (4.5.43) to the latter two terms, we have

$$(4.5.266) \lesssim \Big\|\|F_+^{(3,0)}\|_{L_v^2}(\|f\|_{(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi} + \varepsilon\|f\|_{(\mathcal{H}_\sigma)_\xi})\Big\|_{H_x^{-1}} \tag{4.5.268}$$

$$+ \varepsilon\Big\|\|F_+^{(\frac{7}{2},0)}\|_{(\dot{\mathcal{H}}_\sigma)_v}\|f\|_{(\mathcal{H}_\sigma)_\xi}\Big\|_{H_x^{-1}} \tag{4.5.269}$$

$$\lesssim \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi} + \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}. \tag{4.5.270}$$

and

$$(4.5.267) \lesssim \varepsilon\|F_+^{(3,0)}\|_{H_x^{-1}L_v^2} + \varepsilon\|F_+^{(\frac{7}{2},0)}\|_{H_x^{-1}(\dot{\mathcal{H}}_\sigma)_\xi} \tag{4.5.271}$$

$$\lesssim \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle. \tag{4.5.272}$$

Then,

$$|(4.5.257)| \lesssim \|e^{\gamma\psi}\nabla_x(-\Delta_x)^{-1}a\|_{H_x^1} \tag{4.5.273}$$

$$\cdot \|\varepsilon\partial_t b + (\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E)a - 4\pi\gamma^{\frac{1}{2}}e^{\gamma\psi}\nabla_x\Delta_x^{-1}a\|_{H_x^{-1}} \tag{4.5.274}$$

$$\lesssim \|a\|_{L_x^2}\{\varepsilon\|b\|_{L_x^2} + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{4.5.275}$$

$$+ \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi} + \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}\rangle\}. \tag{4.5.276}$$

Now, note that

$$\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \lesssim \|a\|_{L^2_x} + \|b\|_{L^2_x} + |\bar{c}| + \|(e^{\gamma\psi}c)^\times\|_{L^2_x} + \|\mathcal{P}_\gamma^\perp f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.277}$$

Then, taking $\lambda > 0$, and using Young's inequality, and the bootstrap assumptions, we have

$$|(4.5.257)| \leq C(\varepsilon + \varsigma + \lambda)(\|a\|_{L^2_x}^2 + \bar{c}^2) \tag{4.5.278}$$

$$+ C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + \|(b,(e^{\gamma\psi}c)^\times)\|_{L^2_x}^2 + \|\mathcal{P}_\gamma^\perp f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}^2 + \|f\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}^2\} \tag{4.5.279}$$

We now turn to bounding (4.5.258) and (4.5.259). Using (4.4.144) and (4.5.201), we have

$$\varepsilon\|\partial_t a\|_{H^{-1}_x} \lesssim \|b\|_{L^2_x} + \varepsilon\|\partial_t e^{\gamma\psi}\|_{H^{-1}_x} \lesssim \|b\|_{L^2_x} + \varepsilon\langle\|f\|_{L^2(\mathcal{H}_\sigma)_\xi}\rangle. \tag{4.5.280}$$

Hence,

$$|(4.5.258)| \lesssim (\varepsilon\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle + \|b\|_{L^2_x})\|b\|_{L^2_x} \tag{4.5.281}$$

$$\lesssim \varepsilon^2\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle^2 + \|b\|_{L^2_x}^2 \tag{4.5.282}$$

$$\lesssim \varepsilon^2(\|a\|_{L^2_x} + |\bar{c}|^2) + \|(b,(e^{\gamma\psi}c)^\times)\|_{L^2_x}^2 + \|\mathcal{P}_\gamma^\perp f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}^2 \tag{4.5.283}$$

Next,

$$|(4.5.259)| \lesssim \varepsilon(\|\partial_t\psi\|_{L^\infty_x} + |\dot{\gamma}|)\|a\|_{L^2_x}\|b\|_{L^2_x}. \tag{4.5.284}$$

Applying (4.4.144) again, the above reduces to

$$
\begin{aligned}
|(4.5.259)| &\lesssim \varepsilon\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle\|a\|_{L^2_x}\|b\|_{L^2_x} && (4.5.285)\\
&\lesssim \varepsilon^2(\|a\|_{L^2_x}+|\overline{c}|^2)+\|(b,(e^{\gamma\psi}c)^\times)\|^2_{L^2_x}+\|\mathcal{P}^\perp_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. && (4.5.286)
\end{aligned}
$$

Thus,

$$
|(4.5.260)| \lesssim \varepsilon\|a\|_{L^\infty}\|a\|^2_{H^{-1}} \lesssim \varepsilon\|a\|^2_{L^2_x}. \tag{4.5.287}
$$

Now to bound (4.5.261), we use (4.5.253) to get

$$
\|\nabla_x c-\gamma\nabla_x\psi e^{\gamma\psi}\overline{c}\|_{H^{-1}_x} \lesssim \|(e^{-\gamma\psi}c)^\times\|_{L^2_x}. \tag{4.5.288}
$$

Hence,

$$
\left|\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a,\nabla_x c\rangle_{L^2_x}-\gamma\langle\nabla_x\Delta_x^{-1}a,\nabla_x\phi_0\rangle_{L^2_x}\overline{c}\right| \lesssim \|a\|_{L^2_x}\|(e^{-\gamma\psi}c)^\times\|_{L^2_x}. \tag{4.5.289}
$$

Combining the above with (4.5.250), we have

$$
\begin{aligned}
(4.5.261) &\le \sqrt{\frac{2}{3}}\left|\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a,\nabla_x c\rangle_{L^2_x}-\gamma\langle\nabla_x\Delta_x^{-1}a,\nabla_x\phi_0\rangle_{L^2_x}\overline{c}\right| && (4.5.290)\\
&\quad+\left|4\pi\overline{c}-\sqrt{\frac{2}{3}}\gamma\langle\phi_0,a\rangle_{L^2_x}\right||\overline{c}| && (4.5.291)\\
&\lesssim \varsigma\|a\|^2_{L^2_x}+\|a\|_{L^2_x}\|(e^{-\gamma\psi}c)^\times\|_{L^2_x}+\varepsilon|\overline{c}| && (4.5.292)
\end{aligned}
$$

Now, combining the bounds on (4.5.257) through (4.5.261), we now have

$$
\begin{aligned}
& -\varepsilon \frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, b\rangle_{L^2_x} + \gamma^{-\frac{1}{2}}\|e^{-\frac{\gamma\psi}{2}}a\|^2_{L^2_x} + 4\pi\bar{c}^2 && (4.5.293)\\
& \le C(\varepsilon+\varsigma+\lambda)(\|a\|^2_{L^2_x} + \bar{c}^2) && (4.5.294)\\
& \quad + C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + \|b\|^2_{L^2_x} + \|(e^{-\gamma\psi}c)^\times\|^2_{L^2_x} + \|P_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\}. && (4.5.295)
\end{aligned}
$$

Thus, taking ε, ς and λ sufficiently small, we have (4.5.254).

Step 3 (estimate for $a^{(s)}$): Next, we have the following estimate for $a^{(s)}$:

$$
\begin{aligned}
& -\varepsilon \frac{d}{dt}\langle \nabla_x\Delta_x^{-1}a^{(s)}, b^{(s)}\rangle_{L^2_x} + \frac{1}{C}\|a\|^2_{H^s} && (4.5.296)\\
& \lesssim \varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + \|a\|^2_{L^2_x} + \|(b,c)\|^2_{H^s_x} + \|\mathcal{P}_\gamma^\perp f^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} && (4.5.297)
\end{aligned}
$$

To show this, we have

$$
\begin{aligned}
& -\varepsilon \frac{d}{dt}\langle \nabla_x\Delta_x^{-1}a^{(s)}, b^{(s)}\rangle_{L^2_x} + \gamma^{-\frac{1}{2}}\|a^{(s)}\|^2_{L^2_x} && (4.5.298)\\
& = -\langle \nabla_x\Delta_x^{-1}a^{(s)}, \varepsilon\partial_t b^{(s)} + \gamma^{-\frac{1}{2}}\nabla_x a^{(s)}\rangle_{L^2_x} && (4.5.299)\\
& \quad - \varepsilon\langle \nabla_x\Delta_x^{-1}\partial_t a^{(s)}, b^{(s)}\rangle_{L^2_x}. && (4.5.300)
\end{aligned}
$$

From (4.5.202), we have

$$\|\varepsilon\partial_t b_j^{(s)} + \gamma^{-\frac{1}{2}}\partial_{x_j} a^{(s)}\|_{H_x^{-1}} \tag{4.5.301}$$

$$= \Big\|\varepsilon\frac{\dot\gamma}{2\gamma}b_j + \gamma^{\frac{1}{2}}E_j a - 4\pi\gamma^{\frac{1}{2}}e^{\gamma\psi}\partial_{x_j}\Delta_x^{-1}a + \sqrt{\frac{2}{3}}\partial_{x_j}c + \partial_{x_k}\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L_\xi^2} \tag{4.5.302}$$

$$+ \langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L_\xi^2} + e^{\gamma\psi}\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L_\xi^2}\Big\|_{H_x^{s-1}} \tag{4.5.303}$$

$$\lesssim \varepsilon|\dot\gamma|\|b\|_{H_x^{s-1}} + \langle\|E\|_{H_x^{s-1}}\rangle\|a\|_{H_x^{s-1}} + \|c\|_{H^s} + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{4.5.304}$$

$$+ \|\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L_\xi^2}\|_{H_x^{s-1}} + \|e^{\gamma\psi}\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L_\xi^2}\|_{H_x^{s-1}} \tag{4.5.305}$$

$$\lesssim \varepsilon\langle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}\rangle\|b\|_{H_x^{s-1}} + \|a\|_{H_x^{s-1}} + \|c\|_{H_x^s} + \|\mathcal{P}_\gamma^\perp f^{(s-1)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{4.5.306}$$

$$+ \|f^{(s-1)}\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi} + \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f^{(s-1)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}\rangle \tag{4.5.307}$$

In the final line, we use the bootstrap assumptions, plus (4.4.144) and (4.5.43) as before, combined with algebra estimates for H^{s-1}. Hence, with the bootstrap assumptions, Young's inequality and interpolation, for any $\lambda > 0$ we have

$$|(4.5.299)| \le C(\varepsilon + \varsigma + \lambda)\|a^{(s)}\|_{L_x^2}^2 \tag{4.5.308}$$

$$+ C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle + \|a\|_{L_x^2}^2 + \|(b,c)\|_{H_x^s} + \|P_\gamma^\perp f^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi} + \|f^{(s)}\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}\} \tag{4.5.309}$$

Next, from (4.4.144) and (4.5.201),

$$|(4.5.300)| \lesssim \|b^{(s)}\|_{L_x^2}^2 + \varepsilon\|\partial_t(e^{\gamma\psi})\|_{H_x^{s-1}}\|b^{(s)}\|_{L_x^2} \tag{4.5.310}$$

$$\lesssim (\varepsilon\langle\|f^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}\rangle + \|b^{(s)}\|_{L_x^2})\|b^{(s)}\|_{L_x^2} \tag{4.5.311}$$

$$\lesssim \varepsilon^2\|a\|_{H_x^2} + \|(b,c)\|_{H_x^s}^2 + \|P_\gamma^\perp f^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}. \tag{4.5.312}$$

Taking ε, ς and λ sufficiently small, we conclude (4.5.296).

Step 4 (estimate on $(e^{\gamma\psi}c)^{\times}$): We now show the following bound,

$$
\begin{aligned}
&-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^{\times}, d\rangle_{L_x^2} + \frac{1}{C}\|(e^{-\gamma\psi}c)^{\times}\|_{L_x^2}^2 && (4.5.313)\\
&\le C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon+\varsigma)(\|a\|_{L_x^2}^2 + |\overline{c}|^2) && (4.5.314)\\
&\quad + \|b\|_{L_x^2}^2 + \|\mathcal{P}_\gamma^{\perp} f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 + \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}^2\}. && (4.5.315)
\end{aligned}
$$

By writing

$$
(\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E)c = \gamma^{-\frac{1}{2}}e^{\gamma\psi}\nabla_x(e^{-\gamma\psi}c) - 4\pi\gamma^{\frac{1}{2}}c\nabla_x\Delta_x^{-1}a, \tag{4.5.316}
$$

we have

$$
\begin{aligned}
&-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^{\times}, d\rangle_{L_x^2} + \frac{3\sqrt{3}}{5\sqrt{\gamma}}\|(e^{-\gamma\psi}c)^{\times}\|_{L_x^2}^2 && (4.5.317)\\
&= -\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^{\times}, \varepsilon\partial_t d + \frac{3\sqrt{3}}{5}(\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E)c\rangle_{L_x^2} && (4.5.318)\\
&\quad - \varepsilon\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}\partial_t c)^{\times}, d\rangle_{L_x^2} && (4.5.319)\\
&\quad - \varepsilon\langle \partial_t(e^{-\gamma\psi})\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^{\times}, d\rangle_{L_x^2} && (4.5.320)\\
&\quad - \varepsilon\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(\partial_t(e^{-\gamma\psi})c)^{\times}, d\rangle_{L_x^2} && (4.5.321)\\
&\quad - 4\pi\frac{3\sqrt{3}}{5}\gamma^{\frac{1}{2}}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^{\times}, c\nabla_x\Delta_x^{-1}a\rangle_{L_x^2}. && (4.5.322)
\end{aligned}
$$

First, from (4.5.204), we have

$$\|\varepsilon\partial_t d + \frac{3\sqrt{3}}{5}(\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E)c\|_{H_x^{-1}} \tag{4.5.323}$$

$$\lesssim \varepsilon|\dot{\gamma}|(\|b\|_{H_x^{-1}} + \|d\|_{H_x^{-1}}) \tag{4.5.324}$$

$$+ \sup_{j,k\in\{1,2,3\}}\{\|(\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k)\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L_\xi^2}\|_{H_x^{-1}} \tag{4.5.325}$$

$$+ \|\partial_{x_k}\langle\mathfrak{h}_{jkll}, f\rangle_{L_\xi^2}\|_{H_x^{-1}} \tag{4.5.326}$$

$$+ \|\langle\mathcal{L}_\gamma\mathfrak{h}_{jll}, f\rangle_{L_\xi^2}\|_{H_x^{-1}} \tag{4.5.327}$$

$$+ \|\langle\mathfrak{h}_{jll}, \mathcal{M}_{\gamma,F_+}f\rangle_{L_\xi^2}\|_{H_x^{-1}} \tag{4.5.328}$$

$$+ \|\langle\mathfrak{h}_{jll}, e^{\gamma\psi}\mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle\|_{H_x^{-1}} \tag{4.5.329}$$

$$+ \|\langle\mathfrak{h}_{jll}, \Gamma_\gamma(f,f)\rangle\|_{H_x^{-1}}\} \tag{4.5.330}$$

Going line by line, we see that

$$(4.5.324) \lesssim \varepsilon(\|b\|_{L_x^2} + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}) \tag{4.5.331}$$

Next up, we have

$$(4.5.325) + (4.5.326) + (4.5.327) \leq \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}. \tag{4.5.332}$$

Next, by (4.5.43), we have

$$(4.5.328) \lesssim \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi} + \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{4.5.333}$$

$$(4.5.329) \lesssim \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle \tag{4.5.334}$$

Finally, from (4.5.33), we have

$$(4.5.330) \lesssim \varsigma\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{4.5.335}$$

Hence,

$$|(4.5.318)| \lesssim \|(ce^{\gamma\psi})^{\times}\|_{L^2_x}\{\|b\|_{L^2_x} + \|\mathcal{P}_\gamma^{\perp} f\|_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{4.5.336}$$

$$+ \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle + \varsigma\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\}. \tag{4.5.337}$$

Using (4.5.277), we deduce that for any $\lambda > 0$,

$$|(4.5.318)| \le C(\varepsilon + \varsigma + \lambda)\|(e^{\gamma\psi}c)^{\times}\|^2_{L^2_x} \tag{4.5.338}$$

$$+ C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon + \varsigma)(\|a\|^2_{L^2_x} + |\overline{c}|^2) \tag{4.5.339}$$

$$+ \|b\|^2_{L^2_x} + \|\mathcal{P}_\gamma^{\perp} f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\} \tag{4.5.340}$$

Next, we have the term (4.5.319). From (4.5.203), and (4.4.144),

$$\varepsilon\|\partial_t c\|_{H^{-1}_x} \lesssim \varepsilon|\dot{\gamma}|\langle\|(a,c)\|_{L^2_x}\rangle + \|(b,d)\|_{L^2_x} + \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle \tag{4.5.341}$$

$$\lesssim \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle + \varepsilon\|(a,c)\|_{L^2_x} + \|b\|_{L^2_x} + \|\mathcal{P}_\gamma^{\perp} f\|_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{4.5.342}$$

Thus

$$|(4.5.319)| \lesssim \varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + \varepsilon\|(a,c)\|^2_{L^2_x} + \|b\|^2_{L^2_x} + \|\mathcal{P}_\gamma^{\perp} f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{4.5.343}$$

Next,

$$|(4.5.320)| + |(4.5.321)| \lesssim \varepsilon\|c\|_{L^2_x}\|d\|_{L^2_x} \lesssim \varepsilon^2 + \|\mathcal{P}_\gamma^{\perp} f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.344}$$

Finally,

$$|(4.5.322)| \lesssim \|(e^{-\gamma\psi}c)^{\times}\|_{L_x^2}(\|c\nabla_x\Delta_x^{-1}a\|_{L_x^2} + \|c\nabla_x\Delta_x^{-1}(n_+^{\varepsilon} - n_+^0)\|_{L_x^2}) \quad (4.5.345)$$

$$\lesssim \|a\|_{H_x^s}\|(e^{-\gamma\psi}c)^{\times}\|_{L_x^2}(\|c\|_{L_x^2} + \|F_+^{\varepsilon} - F_+^0\|_{\mathfrak{E}'}) \quad (4.5.346)$$

$$\lesssim \varsigma\|(e^{-\gamma\psi}c)^{\times}\|_{L_x^2}(\|c\|_{L_x^2} + \|F_+^{\varepsilon} - F_+^0\|_{\mathfrak{E}'}) \quad (4.5.347)$$

Combining these bounds for (4.5.318) through (4.5.322), we conclude that

$$-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^{\times}, d\rangle_{L_x^2} + \frac{3\sqrt{3}}{5\sqrt{\gamma}}\|(e^{-\gamma\psi}c)^{\times}\|_{L_x^2}^2 \quad (4.5.348)$$

$$\leq C(\varepsilon + \varsigma + \lambda)\|(e^{\gamma\psi}c)^{\times}\|_{L_x^2}^2 \quad (4.5.349)$$

$$+ C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon + \varsigma)(\|a\|_{L_x^2}^2 + |\overline{c}|^2) \quad (4.5.350)$$

$$+ \|b\|_{L_x^2}^2 + \|\mathcal{P}_\gamma^{\perp}f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 + \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}^2\} \quad (4.5.351)$$

By taking ε, ς and λ sufficiently small, we conclude (4.5.313).

Step 5: Estimate on $c^{(s)}$: Following the same method as previous bounds, we have the estimate

$$-\varepsilon\frac{d}{dt}\langle\nabla_x\Delta_x^{-1}c^{(s)}, d^{(s)}\rangle_{L_x^2} + \frac{1}{C}\|c\|_{H_x^s}^2 \quad (4.5.352)$$

$$\leq C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon + \varsigma)\|a\|_{L_x^2}^2 \quad (4.5.353)$$

$$+ \|c\|_{L_x^2}^2 + \|b\|_{H_x^s}^2 + \|\mathcal{P}_\gamma^{\perp}f^{(s)}\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 + \|f^{(s)}\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}^2\} \quad (4.5.354)$$

The proof is similar to that of (4.5.296).

Step 6: Combining the bounds: Combining the upper bounds on (4.5.254), (4.5.296), (4.5.219), (4.5.313), (4.5.352) there is a choice of constant $\kappa > 0$ taken sufficiently small, such that the functional

$$\mathscr{G}_{reg}(t) := \kappa\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, b\rangle_{L^2_x} - \kappa^3\langle\nabla_x\Delta_x^{-1}a^{(s)}, b^{(s)}\rangle_{L^2_x} \tag{4.5.355}$$

$$- \langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^{\times}, d\rangle_{L^2_x} - \kappa^2\langle\nabla_x\Delta_x^{-1}c^{(s)}, d^{(s)}\rangle_{L^2_x} \tag{4.5.356}$$

satisfies

$$\varepsilon\frac{d}{dt}\mathscr{G}_{reg}(t) + \frac{1}{C}\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.357}$$

$$\leq C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon+\varsigma)\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_{x,\xi}} \tag{4.5.358}$$

$$+ \|\mathcal{P}_\gamma^\perp f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \|F_+^\varepsilon - F_+^0\|^2_{\mathfrak{E}'}\} \tag{4.5.359}$$

On the other hand by (4.5.31), for each $\alpha \in \{1, 2\}$,

$$\|\mathcal{P}_\gamma^\perp f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \lesssim \|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{4.5.360}$$

$$+ \eta\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.5.361}$$

Thus, by taking ε, ς and η sufficiently small, and re-scaling $\mathscr{G}$ if necessary, we conclude with (4.5.243). To get (4.5.242), we take $\kappa > 0$ and

$$\mathscr{G}(t) := -\kappa\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, b\rangle_{L^2_x} - \langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^{\times}, d\rangle_{L^2_x} \tag{4.5.362}$$

so that

$$\varepsilon\frac{d}{dt}\mathscr{G}(t) + \frac{1}{C}\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \leq C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon+\varsigma)\|\mathcal{P}_\gamma f\|^2_{L^2_{x,\xi}} \tag{4.5.363}$$

$$+ \|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\}. \tag{4.5.364}$$

And, once again, by (4.5.31), for each $\alpha \in \{1,2,3\}$,

$$\|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \lesssim \|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \eta\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.365}$$

Again, taking ε, ς and η sufficiently small, and re-scaling $\mathscr{G}$ if necessary, we deduce (4.5.242). □

We conclude this section with a corollary of Propositions 4.5.4 and 4.5.7:

Corollary 4.5.8. *Let $M \geq 1$, and let η be sufficiently small depending on M. We let ς and ε be sufficiently small depending on M and η. Assume the bootstrap assumptions (4.5.3) through (4.5.6). Then, for each $M, \eta > 0$, and $\alpha \in \{1,2,3\}$ there exists a function $\mathscr{Y}_{-,\alpha} = \mathscr{Y}^{(M,\eta)}_{-,\alpha} : [0,T] \to \mathbf{R}_+$ such that*

$$\mathscr{Y}_{-,\alpha} \sim_{M,\eta} \widetilde{\mathscr{E}}_{-,\alpha} \tag{4.5.366}$$

and

$$\varepsilon \frac{d}{dt}\mathscr{Y}_{-,\alpha} + \widetilde{\mathscr{D}}_{-,\alpha} \lesssim_{M,\eta} \varepsilon^2. \tag{4.5.367}$$

Proof. We take $\kappa_0, \kappa_1, \kappa_2$ and $\kappa_3 \in (0,1)$ be constants to be determined. We combine

(4.5.71) in the case $\alpha = 2$ and (4.5.242) as follows:

$$\varepsilon\frac{d}{dt}\{\|g_\alpha\|^2_{L^2_{x,\xi}} + 4\pi\sqrt{\gamma}\||\Delta_x|^{-1}a\|^2_{L^2_x} + \kappa_1\mathscr{G}\} \tag{4.5.368}$$

$$+ (\frac{1}{C} - C\kappa_1)\{\|\mathcal{P}^\perp_\gamma g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\} + \kappa_0\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.369}$$

$$\leq C(\varepsilon + \varsigma + \eta + \kappa_0)\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.5.370}$$

$$+ C\varepsilon\langle\|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{4.5.371}$$

$$+ C_{\kappa_0}\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2. \tag{4.5.372}$$

Using (4.5.32), we first fix κ_1 sufficiently small such that the "$\frac{1}{C} - C\kappa_1$" in the above is strictly positive, and

$$\mathscr{X}_{-,\alpha} := \|g_\alpha\|^2_{L^2_{x,\xi}} + 4\pi\sqrt{\gamma}\||\Delta_x|^{-1}a\|^2_{L^2_x} + \kappa_1\mathscr{G} \tag{4.5.373}$$

satisfies

$$\mathscr{X}_{-,\alpha} \sim \|g_\alpha\|^2_{L^2_{x,\xi}}. \tag{4.5.374}$$

Then, we fix κ_0 to be a sufficiently small constant, and also require that ε, ς and η are sufficiently small so that

$$\varepsilon\frac{d}{dt}\mathscr{X}_{-,\alpha} + \frac{1}{C}\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{4.5.375}$$

$$\leq C\{\varepsilon\langle\|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle\mathscr{X}_{-,\alpha} + \varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2\}. \tag{4.5.376}$$

Next, we combine this bound with (4.5.72),

$$\varepsilon\frac{d}{dt}(\|g_\alpha^{(s)}\|_{L^2_{x,\xi}}^2+\kappa_3\mathscr{G}_{reg}) \tag{4.5.377}$$

$$+(\frac{1}{C}-C\kappa_3)\{\|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}^2+\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}^2\}+\kappa_3\|\mathcal{P}_\gamma f^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}^2 \tag{4.5.378}$$

$$\leq C(\varepsilon+\varsigma+\eta+\kappa_2)\|\mathcal{P}_\gamma f^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}^2+C_{\eta,\kappa_2}\|g_{\alpha+1}\|_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}^2 \tag{4.5.379}$$

$$+C\varepsilon\langle\|\partial_t a\|_{H_x^{s-\frac{1}{4}}}+\|F_+\|_{\mathfrak{D}}^2\rangle\|g_\alpha^{(s)}\|_{L^2_{x,\xi}}^2 \tag{4.5.380}$$

$$+C_{\kappa_2}\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 \tag{4.5.381}$$

We now fix κ_3 small enough so that the "$\frac{1}{C}-C\kappa_3$" in the above is positive, and

$$\|g_\alpha^{(s)}\|_{L^2_{x,\xi}}^2+\kappa_3\mathscr{G}_{reg}\sim\|g_\alpha^{(s)}\|_{L^2_{x,\xi}}^2, \tag{4.5.382}$$

and fix κ_2, and take ε,ς and η small enough that

$$\varepsilon\frac{d}{dt}(\|g_\alpha^{(s)}\|_{L^2_{x,\xi}}^2+\kappa_3\mathscr{G}_{reg})+\frac{1}{C}\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}^2 \tag{4.5.383}$$

$$\leq C_\eta\|g_{\alpha+1}\|_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}^2 \tag{4.5.384}$$

$$+C\{\varepsilon\langle\|\partial_t a\|_{L^2_x}+\|F_+\|_{\mathfrak{D}}^2\rangle\|g_\alpha^{(s)}\|_{L^2_{x,\xi}}^2+\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2\}. \tag{4.5.385}$$

FInally, we take $\kappa_4^{(\eta)}$ sufficiently small depending on η so that

$$\mathscr{X}_{-,\alpha,reg}^{(\eta)}:=\kappa_4^{(\eta)}(\|g_\alpha^{(s)}\|_{L^2_{x,\xi}}^2+\kappa_3\mathscr{G}_{reg})+\mathscr{X}_{-,\alpha+1} \tag{4.5.386}$$

satisfies

$$\frac{d}{dt}\mathscr{X}_{-,reg,\alpha}^{(\eta)}+\frac{1}{C_\eta}\widetilde{\mathscr{D}}_{-,\alpha} \tag{4.5.387}$$

$$\leq C_\eta\{\varepsilon\langle\|\partial_t a\|_{L^2_x}+\|F_+\|_{\mathfrak{D}}^2\rangle\mathscr{X}_{-,reg,\alpha}^{(\eta)}+\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2\}. \tag{4.5.388}$$

By (4.4.2), we have

$$\frac{1}{C}\langle\|F_+\|_{\mathfrak{D}}\rangle^2 \leq -\frac{d}{dt}\mathscr{X}_+ + C(1+\widetilde{\mathscr{D}}_{-,\alpha}). \tag{4.5.389}$$

Taking ε and ς sufficiently small depending on η now, we can hide the $\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}$ term and get for some collection of constants $C_1^{(\eta)}, C_2^{(\eta)}$ and $C_3^{(\eta)}$,

$$\varepsilon(\frac{d}{dt} + C_1^{(\eta)}\frac{d}{dt}\mathscr{X}_+ - C_2^{(\eta)}\langle\|\partial_t a\|_{L^2_x}\rangle)\mathscr{X}^{(\eta)}_{-,reg,\alpha} + \frac{1}{C_3^{(\eta)}}\widetilde{\mathscr{D}}_{-,\alpha} \tag{4.5.390}$$

$$\leq C_3^{(\eta)}\varepsilon^2. \tag{4.5.391}$$

We now define

$$\Omega^{(\eta)}(t) := C_1^{(\eta)}\mathscr{X}_+(t) - C_2^{(\eta)}\int_0^t \langle\|\partial_t a(t')\|_{L^2_x}\rangle dt' \tag{4.5.392}$$

By the (4.5.3) and (4.5.6), we have that $|\Omega^{(\eta)}|$ is less than or equal to some constant depending on η, say $C_4^{(\eta)} > 0$. Thus, if we set $\mathscr{Y}_j^{(\eta)} = C_3^{(\eta)} e^{\Omega + C_4^{(\eta)}}\mathscr{X}^{(\eta)}_{-,reg,\alpha}$, we have a function which satisfies (4.5.366) and (4.5.367). □

4.5.6 Proof of Proposition 4.5.1

We now conclude this section with a proof of Proposition 4.5.1.

Proof of Proposition 4.5.1: Throughout this proof, we take η sufficiently small so as to satisfy the hypothesis of Corollary 4.5.8. Any of the constants appearing in the estimates will implicitly depend on the constants M, $\eta > 0$ i.e. $C = C_{M,\eta}$.

For each $\alpha \in \{1, 2\}$, we integrate the bound in Corollary 4.5.8 on $t \in [0, T]$ and use (4.5.366) to get

$$\varepsilon \sup_{t\in[0,\hat{T}]} \widetilde{\mathscr{E}}_{-,\alpha}(t) + \int_0^{\hat{T}} \widetilde{\mathscr{D}}_{-,\alpha}(t)dt \tag{4.5.393}$$

$$\leq C(\varepsilon^2 + \varepsilon\widetilde{\mathscr{E}}_{-,\alpha}(0)). \tag{4.5.394}$$

Taking $j = 2$, this implies (4.5.8).

We now show (4.5.9). For this, we introduce $\theta > 0$, to be chosen latter. Now, for any $h = h(\xi)$, observe that

$$\|h\|^2_{\mathcal{H}_\sigma} \gtrsim \int_{\mathbf{R}^3} \frac{1}{\langle\xi\rangle} h(\xi)^2 d\xi \tag{4.5.395}$$

$$\geq \frac{1}{\theta t^{\frac{1}{3}}} \int_{\langle\xi\rangle\leq\theta t^{\frac{1}{3}}} h(\xi)^2 d\xi \tag{4.5.396}$$

$$\geq \frac{1}{\theta t^{\frac{1}{3}}} \left(\|h\|^2_{L^2} - \int_{\langle\xi\rangle\geq\theta t^{\frac{1}{3}}} h(\xi)^2 d\xi \right) \tag{4.5.397}$$

$$\geq \frac{1}{\theta t^{\frac{1}{3}}} \left(\|h\|^2_{L^2} - \int_{\langle\xi\rangle\geq\theta t^{\frac{1}{3}}} e^{\frac{1}{4}(e^\eta-1)\beta_{in}(|\xi|^2+1-\theta^2 t^{\frac{2}{3}})} h(\xi)^2 d\xi \right) \tag{4.5.398}$$

$$\geq \frac{1}{\theta t^{\frac{1}{3}}} \left(\|h\|^2_{L^2} - e^{-\frac{1}{4}(e^\eta-1)\beta_{in}\theta^2 t^{\frac{2}{3}})} \|e^{\frac{1}{4}(e^\eta-1)\beta_{in}(|\xi|^2+1)} h\|^2_{L^2} \right) \tag{4.5.399}$$

Applying this to $h = g_1^{(s)}$ and g_2 and integrating in x, and re-scaling θ, we have

$$\widetilde{\mathscr{D}}_{-,1} \gtrsim \frac{1}{\theta t^{\frac{1}{3}}} \left(\mathscr{Y}_{-,1} - e^{-\theta^2 t^{\frac{2}{3}}} \mathscr{Y}_{-,2} \right) \tag{4.5.400}$$

Thus, using (4.5.4) with (4.5.367), we have

$$\varepsilon \frac{d}{dt} \mathscr{Y}_{-,1} + \frac{2}{3C\theta t^{\frac{1}{3}}} \mathscr{Y}_{-,1} \leq C(\varepsilon^2 + \frac{2}{3\theta t^{\frac{1}{3}}} e^{-\theta^2 t^{\frac{2}{3}}} \sup_{t'\in[0,T]} \widetilde{\mathscr{E}}_{-,2}(t'). \tag{4.5.401}$$

and so

$$\frac{d}{dt}(e^{\frac{1}{C\theta\varepsilon}t^{\frac{2}{3}}}\mathscr{Y}_{-,1}) \leq C(\varepsilon e^{\frac{1}{C\theta\varepsilon}t^{\frac{2}{3}}} + \frac{1}{\varepsilon\theta t^{\frac{1}{3}}}e^{(\frac{1}{C\theta\varepsilon}-\theta^2)t^{\frac{2}{3}}} \sup_{t'\in[0,T]} \widetilde{\mathscr{E}}_{-,2}(t'). \tag{4.5.402}$$

Hence, by taking $\theta = \left(\frac{2}{C\varepsilon}\right)^{\frac{1}{3}}$ with C as in the above, we get some for some C_0, C_1, C_2,

$$\frac{d}{dt}(e^{\frac{1}{C_0}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\mathscr{Y}_{-,1}) \leq C_2(\varepsilon e^{\frac{1}{C_0}(\frac{t}{\varepsilon})^{\frac{2}{3}}} + \frac{d}{dt}e^{-\frac{1}{C_1}(\frac{t}{\varepsilon})^{\frac{2}{3}}} \sup_{t'\in[0,T]} \widetilde{\mathscr{E}}_{-,2}(t') \tag{4.5.403}$$

Now,

$$\int_0^t e^{\frac{1}{C_0}(\frac{t'}{\varepsilon})^{\frac{2}{3}}}dt' \leq \varepsilon^{\frac{2}{3}}t^{\frac{1}{3}}\int_0^t \frac{1}{\varepsilon^{\frac{2}{3}}(t')^{\frac{1}{3}}}e^{\frac{1}{C_0}(\frac{t'}{\varepsilon})^{\frac{2}{3}}}dt' \lesssim \varepsilon^{\frac{2}{3}}t^{\frac{1}{3}}e^{\frac{1}{C_0}(\frac{t}{\varepsilon})^{\frac{2}{3}}}. \tag{4.5.404}$$

Thus, by integrating (4.5.403), we get

$$\mathscr{Y}_{-,1}(t) \lesssim e^{-\frac{1}{C}(\frac{t}{\varepsilon})^{\frac{2}{3}}} \sup_{t'\in[0,T]} \widetilde{\mathscr{E}}_{-,2}(t') + \varepsilon^{\frac{5}{3}}t^{\frac{1}{3}}. \tag{4.5.405}$$

for all $t \in [0,T]$. Using (4.5.366), we conclude (4.5.9). □

4.6 Proof of Theorem 4.1.1

In this section, we conclude with the proof of the main theorem.

Proof of Theorem 4.1.1: We break the proof of the Theorem into the proofs of parts (i) and (ii).

Proof of part (i): We break the proof of (i) into a number of steps: the setup of the bootstrap argument, combining the estimates of the preceding results, and closing the bootstrap argument.

Step 1 of part (i) (setup & bootstrap assumptions): We take $\kappa = \kappa(M) > 0$ to be a constant to be determined later. We take (F_+^0, β, ϕ^0) to be a weak solution to (4.1.7) on $[0, T_0]$ satisfying the hypotheses of the theorem. In addition, we may as well take $T_0 > 0$ small enough so that

$$\int_0^{T_0} \|F_+^0(t)\|_{\mathfrak{D}}^2 dt \leq \kappa^2. \tag{4.6.1}$$

Now, for all ε sufficiently small, there exists a unique weak solution $(F_+^\varepsilon, F_-^\varepsilon)$ to (4.1.7) on some interval containing zero, in the sense described in Lemma 4.3.1. Moreover, we let $(\psi^\varepsilon, \gamma^\varepsilon)$ be the solution to (4.4.140) in the sense of Lemma 4.4.3, defined on some time interval containing 0. We then take η, $\bar{\epsilon}$, and ς as in Proposition 4.5.1, and take $\varepsilon \in (0, \bar{\epsilon}]$.

By continuity in time of the quantities considered, there exists $\hat{T}_\varepsilon > 0$ such that the following conditions hold:

$$\sup_{t \in [0, \hat{T}_\varepsilon]} (\|\frac{1}{n_+^\varepsilon(t)}\|_{L_x^\infty} + \|F_+^\varepsilon(t)\|_{\mathfrak{E}}) \leq 2M, \tag{4.6.2}$$

$$\sup_{t \in [0, \hat{T}_\varepsilon]} \|F_+^\varepsilon(t) - F_-^0(t)\|_{\mathfrak{E}'} \leq \kappa \tag{4.6.3}$$

$$\sup_{t \in [0, \hat{T}_\varepsilon]} \sqrt{\mathscr{E}_{-,2}^\varepsilon(t)} \leq \kappa, \tag{4.6.4}$$

$$\sup_{t \in [0, \hat{T}_\varepsilon]} |\ln(\frac{\gamma^\varepsilon(t)}{\beta_{in}})| \leq \eta, \tag{4.6.5}$$

$$\int_0^{\hat{T}_\varepsilon} \|\partial_t (n_-^\varepsilon - e^{\gamma^\varepsilon \psi^\varepsilon})\|_{L_x^2} dt \leq 1. \tag{4.6.6}$$

By Lemmas 4.3.1 (taking $\kappa < \frac{\varrho(2M)}{2}$) and 4.4.3 (using condition (4.6.5)), we may take

κ small enough so that the interval of existence of $(F_+^\varepsilon, F_-^\varepsilon)$ and $(\gamma^\varepsilon, \psi^\varepsilon)$ is strictly larger than $[0, \hat{T}_\varepsilon]$. Hence, we may take $\hat{T}_\varepsilon > 0$ to be the largest such time, so that either at least one of the above conditions holds with equality, or $\hat{T}_\varepsilon = T_0$.

Next, we note that by condition (4.6.4) and (4.6.5), we have

$$\sup_{t\in[0,T]} \{ \| \frac{1}{n_-^\varepsilon(t)} \|_{L^\infty_x} + \| F_-^\varepsilon(t) \|_{\mathfrak{E}} \} \leq C_M, \tag{4.6.7}$$

so the consequences of Propositions 4.4.1 and 4.4.2 both hold, up to replacing M with some C_M in their hypotheses.

Step 2 of part (i) (combining estimates): We now combine the estimates of the preceding propositions and lemmas. We first show that the hypotheses of Proposition 4.5.1 are satisfied. By (4.4.145), for each $\alpha \in \{1, 2\}$, and $T \in [0, \hat{T}_\varepsilon]$, we have

$$\begin{aligned} &\sup_{t\in[0,T]} (\| e^{\frac{q_\alpha |\xi|^2}{2}} \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle (\mu_\beta e^{\beta \phi^0} - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}) \|_{L^2_{x,\xi}} & (4.6.8) \\ &\qquad + \| e^{\frac{q_{\alpha+1} |\xi|^2}{2}} \langle \nabla_\xi \rangle (\mu_\beta e^{\beta \phi^0} - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}) \|_{L^2_{x,\xi}}) & (4.6.9) \\ &\lesssim_M \sup_{t\in[0,T]} (|\beta - \gamma^\varepsilon| + \| \phi^0 - \psi^\varepsilon \|_{H^s_x}) & (4.6.10) \\ &\lesssim_M \sup_{t\in[0,T]} \| F_+^\varepsilon(t) - F_+^0(t) \|_{\mathfrak{E}'}. & (4.6.11) \end{aligned}$$

Hence, for all such T, and for each $\alpha \in \{1, 2\}$, we have

$$\sup_{t\in[0,T]} |\mathscr{E}_{-,\alpha}^\varepsilon(t) - \widetilde{\mathscr{E}}_{-,\alpha}^\varepsilon(t)| \leq C_M \sup_{t\in[0,T]} \| F_+^\varepsilon(t) - F_+^0(t) \|_{\mathfrak{E}'}^2, , \tag{4.6.12}$$

$$\sup_{t\in[0,T]} |\mathscr{D}_{-,\alpha}^\varepsilon(t) - \widetilde{\mathscr{D}}_{-,\alpha}^\varepsilon(t)| \leq C_M \sup_{t\in[0,T]} \| F_+^\varepsilon(t) - F_+^0(t) \|_{\mathfrak{E}'}^2.. \tag{4.6.13}$$

Using (4.6.12) with $T=0$, we have

$$\widetilde{\mathscr{E}}^{\varepsilon}_{-,\alpha,in} \lesssim_M \mathscr{E}^{\varepsilon}_{-,\alpha,in} + C_M\varepsilon^2 \lesssim \varepsilon^2. \tag{4.6.14}$$

By the condition $\mathscr{E}^{\varepsilon}_{-,2,in} \leq M\varepsilon$, we have

$$\left|\int_{\mathbf{R}^3\times\mathbf{T}^3} \frac{|\xi|^2}{2}(F^{\varepsilon}_{-,in} - \mu_{\gamma^{\varepsilon}_{in}} e^{\beta_{in}\psi^{\varepsilon}_{in}})dxd\xi + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x\phi^{\varepsilon}_{in}|^2 - |\nabla_x\psi^{\varepsilon}_{in}|^2 dx\right| \tag{4.6.15}$$

$$\lesssim \sqrt{\widetilde{\mathscr{E}}_{-,2,in}} + \widetilde{\mathscr{E}}_{-,2,in} \tag{4.6.16}$$

$$\lesssim C_M(\varepsilon+\varepsilon^2). \tag{4.6.17}$$

Thus, we have that (4.5.7) holds in Proposition 4.5.1. On the other hand, by (4.6.3), (4.6.4) and (4.6.12), we have $\sup_{t\in[0,T]}\sqrt{\widetilde{\mathscr{E}}^{\varepsilon}_{-,\alpha}(t)} \lesssim_M \kappa$. Taking $\kappa \leq \frac{\varsigma}{C_M}$, we have that the conclusions of Proposition 4.5.1 are hold on $[0,\hat{T}_\varepsilon]$.

From (4.5.8), we have

$$\varepsilon \sup_{t\in[0,\hat{T}_\varepsilon]} \mathscr{E}^{\varepsilon}_{-,2}(t) + \int_0^{\hat{T}_\varepsilon} \mathscr{D}^{\varepsilon}_{-,2}(t)dt \leq C_M(\varepsilon\mathscr{E}^{\varepsilon}_{-,2,in} + \varepsilon^2). \tag{4.6.18}$$

Combining the above with (4.5.9), we have and for all $t\in[0,\hat{T}_\varepsilon]$, we have

$$\mathscr{E}^{\varepsilon}_{-,1}(t) \leq C_M(e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\varepsilon + \varepsilon^{\frac{5}{3}}t^{\frac{1}{3}} + \varepsilon^2). \tag{4.6.19}$$

We now prove the estimate

$$\sup_{t\in[0,\hat{T}_\varepsilon]} \|F^{\varepsilon}_{+}(t) - F^{0}_{+}(t)\|^2_{\mathfrak{E}'} + \int_0^{\hat{T}_\varepsilon} \|F^{\varepsilon}_{+}(t) - F^{0}_{+}(t)\|^2_{\mathfrak{D}'}dt \lesssim_M \varepsilon^2. \tag{4.6.20}$$

Using the above and (4.5.8), then for all $T\leq\hat{T}_\varepsilon$, we integrate (4.4.104) on $t\in[0,T]$

to get

$$\sup_{t\in[0,T]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'}^2 + \int_0^T \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{D}'}^2 dt \tag{4.6.21}$$

$$\lesssim_M \|F_+^\varepsilon(0) - F_+^0(0)\|_{\mathfrak{E}'}^2 \tag{4.6.22}$$

$$+ \int_0^T \langle \|(F_+^\varepsilon, F_+^0)\|_{\mathfrak{D}}\rangle^2 (\varepsilon^2 + \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'}) + \mathscr{D}_{-,2}(t) dt \tag{4.6.23}$$

$$\lesssim_M \int_0^T \langle \|(F_+^\varepsilon, F_+^0)\|_{\mathfrak{D}}\rangle^2 (\varepsilon^2 + \sup_{t'\in[0,t]} \|F_+^\varepsilon(t') - F_+^0(t')\|_{\mathfrak{E}'}^2) dt + \varepsilon^2. \tag{4.6.24}$$

In the final line, we used (4.6.18). Using the time integrated form of Grönwall's inequality, we deduce (4.6.20), provided the follow bound holds true:

$$\int_0^{\hat{T}_\varepsilon} \langle \|(F_+^0(t), F_+^\varepsilon(t))\|_{\mathfrak{D}}\rangle^2 dt \lesssim_M 1. \tag{4.6.25}$$

Indeed, by integrating (4.4.2) in Proposition 4.4.1, and using (4.6.18) to control the right-hand side, we have

$$\int_0^{\hat{T}_\varepsilon} \|F_+^\varepsilon(t)\|_{\mathfrak{D}}^2 dt \lesssim_M 1. \tag{4.6.26}$$

Combined with (4.6.1), we have (4.6.25).

Step 3 of part (i) (closing the bootstrap): We now show that $\hat{T} := \inf_{\varepsilon\in(0,\bar{\varepsilon}]} \hat{T}_\varepsilon \geq \kappa$. To show this, we suppose $\hat{T}_\varepsilon < \kappa$ to yield a contradiction. Now, one of (4.6.2) through (4.6.6) holds with equality. We now show that in each of these five cases, the condition $\hat{T}_\varepsilon < \kappa$ cannot hold.

First, suppose (4.6.2) holds with equality. By integrating (4.4.3), and using

(4.6.18) to control the right-hand side, we have for all $t \in [0, \hat{T}_\varepsilon]$,

$$\|F_+^\varepsilon(t)\|_{\mathfrak{E}}^2 \leq \|F_{+,in}^\varepsilon\|_{\mathfrak{E}}^2 + C_M(t + \|\langle v\rangle^{m_1} F_+^\varepsilon\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} dt') \tag{4.6.27}$$

We claim that

$$\int_0^t \|\langle v\rangle^{m_1} F_+^\varepsilon(t')\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} dt' \lesssim_M \kappa + \varepsilon. \tag{4.6.28}$$

Indeed,

$$\int_0^t \|\langle v\rangle^{m_1} F_+^\varepsilon(t')\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} dt' \leq \int_0^t \|F_+^\varepsilon(t')\|_{\mathfrak{D}'} dt' + \int_0^t \|\langle v\rangle^{m_1} F_+^\varepsilon(t')\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} dt' \tag{4.6.29}$$

Now, by (4.6.1) and (4.6.20),

$$\int_0^t \|F_+^\varepsilon(t')\|^2_{\mathfrak{D}'} dt' \lesssim \int_0^t \|F_+^\varepsilon\|^2_{\mathfrak{D}'} dt' + \int_0^t \|F_+^\varepsilon(t') - F_+^0(t')\|^2_{\mathfrak{D}'} dt' \lesssim_M \varepsilon^2 + \kappa^2. \tag{4.6.30}$$

On the other hand, recalling (4.1.22), we note that for all $\nu \in \mathbf{S}^2$,

$$\varepsilon \sigma_{ij}(\varepsilon v)\nu_i\nu_j \lesssim \varepsilon \lesssim \varepsilon \langle v\rangle^3 \sigma_{ij}(v)\nu_i\nu_j. \tag{4.6.31}$$

Hence, using (4.6.25),

$$\begin{aligned}
\int_0^t \|\langle v\rangle^{m_1} F_+^\varepsilon\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} dt' &\lesssim t + \int_0^t \|\langle v\rangle^{m_2} F_+^\varepsilon\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} dt' &\quad (4.6.32)\\
&\lesssim_M t + \varepsilon \int_0^t \|F_+^\varepsilon\|^2_{\mathfrak{D}} dt' &\quad (4.6.33)\\
&\lesssim_M t + \varepsilon. &\quad (4.6.34)
\end{aligned}$$

Using $t \leq \hat{T}_\varepsilon \leq \kappa$, we conclude (4.6.28) holds true. Taking the supremum of (4.6.27) over all $t \in [0, \hat{T}_\varepsilon]$, we deduce

$$\sup_{t\in[0,\hat{T}\varepsilon]} \|F^\varepsilon_+(t)\|_{\mathfrak{e}} \leq \|F^\varepsilon_{+,in}\|_{\mathfrak{e}} + C_M(\kappa+\varepsilon). \tag{4.6.35}$$

Next, by the continuity equation, we have for all $t \in [0, \hat{T}_\varepsilon]$,

$$\frac{1}{n^\varepsilon_+(t,x)} = \frac{1}{n^\varepsilon_{+,in}(x) - \int_0^t \int_{\mathbf{R}^3} v \cdot \nabla_x F_+(t,x,v) dv dt} \tag{4.6.36}$$

Taking κ smaller if necessary, we get

$$\sup_{t\in[0,\hat{T}_\varepsilon]} \|\frac{1}{n^\varepsilon_+(t)}\|_{L^\infty_x} \leq \|\frac{1}{n^\varepsilon_{+,in}}\|_{L^\infty_x} \frac{1}{1 - C\|\frac{1}{n^\varepsilon_{+,in}}\|_{L^\infty_x} \|F^\varepsilon_+\|_{\mathfrak{e}}\kappa} \leq \frac{3}{2}\|\frac{1}{n^\varepsilon_{+,in}}\|_{L^\infty_x}. \tag{4.6.37}$$

Now, taking ε and κ sufficiently small, we have

$$2M = \sup_{t\in[0,\hat{T}_\varepsilon]} (\|\frac{1}{n^\varepsilon_+(t)}\|_{L^\infty_x} + \|F^\varepsilon_+(t)\|_{\mathfrak{e}}) \leq \frac{3}{2}\|\frac{1}{n^\varepsilon_{+,in}}\|_{L^\infty_x} + \|F^\varepsilon_{+,in}\|_{\mathfrak{e}} + C_M(\kappa+\varepsilon) \tag{4.6.38}$$

$$= \frac{3}{2}M. \tag{4.6.39}$$

Next, assume (4.6.3) holds with equality. However, by (4.6.20), we have $\sup_{t\in[0,\hat{T}_\varepsilon]} \|F^\varepsilon_+(t) - F^0_+(t)\|_{\mathfrak{e}'} \lesssim_M \varepsilon$, so we may take ε sufficiently small to reach a contradiction.

Next, assume (4.6.4) holds with equality. However by (4.6.18), $\sqrt{\mathscr{E}^\varepsilon{}_{-,2}} \lesssim_M \sqrt{\varepsilon}$. Thus, by taking ε small enough, we ensure

$$\kappa = \sqrt{\mathscr{E}^\varepsilon{}_{-,2}(\hat{T}_\varepsilon)} \leq \frac{\kappa}{2}, \tag{4.6.40}$$

a contradiction.

Next, assume (4.6.5) holds with equality. By (4.4.144) and (4.6.18),

$$\eta = |\ln(\frac{\gamma^{\varepsilon}(\hat{T}_{\varepsilon})}{\beta_{in}})| \tag{4.6.41}$$

$$\leq \int_0^{\hat{T}_{\varepsilon}} |\frac{\dot{\gamma}^{\varepsilon}(t)}{\gamma^{\varepsilon}(t)}|dt \tag{4.6.42}$$

$$\lesssim \int_0^{\hat{T}_{\varepsilon}} 1 + \sqrt{\mathscr{D}^{\varepsilon}_{-,2}}dt \tag{4.6.43}$$

$$\leq \hat{T}_{\varepsilon} + \hat{T}_{\varepsilon}^{\frac{1}{2}} \left(\int_0^{T_{\varepsilon}} \mathscr{D}^{\varepsilon}_{-,2} dt \right)^{\frac{1}{2}}. \tag{4.6.44}$$

Combining with (4.6.18), we conclude $\eta \lesssim \hat{T}_{\varepsilon}$. We conclude $\eta = \eta(M) \lesssim_M \hat{T}_{\varepsilon} \leq \kappa$. Taking κ sufficiently small, we reach a contradiction.

Now, assume (4.6.6) holds with equality. By (4.5.201), (4.4.144), (4.5.8), we have

$$1 = \int_0^{\hat{T}_{\varepsilon}} \| \int_{\mathbf{R}^3} (F^{\varepsilon}_{-} - \mu_{\gamma^{\varepsilon}} e^{\gamma^{\varepsilon}\psi^{\varepsilon}}) d\xi \|_{L^2_x} dt \tag{4.6.45}$$

$$\lesssim_M \int_0^{\hat{T}_{\varepsilon}} \frac{1}{\varepsilon} \| \int_{\mathbf{R}^3} v \cdot \nabla_x (F^{\varepsilon}_{-} - \mu_{\gamma^{\varepsilon}} e^{\gamma^{\varepsilon}\psi^{\varepsilon}}) d\xi \|_{L^2_x} + \| \partial_t (\gamma^{\varepsilon}\psi^{\varepsilon}) e^{\gamma^{\varepsilon}\psi^{\varepsilon}} \|_{L^2_x} dt \tag{4.6.46}$$

$$\lesssim_M \int_0^{\hat{T}_{\varepsilon}} 1 + \frac{1}{\varepsilon} \sqrt{\widetilde{\mathscr{D}}^{\varepsilon}_{-,2}} dt \tag{4.6.47}$$

$$\lesssim_M \hat{T}_{\varepsilon} + \hat{T}_{\varepsilon}^{\frac{1}{2}} \tag{4.6.48}$$

This implies $1 \lesssim_M \hat{T}_{\varepsilon} \leq \kappa$. Taking κ small enough, we reach a contradiction.

In conclusion, $\hat{T}_{\varepsilon} \geq \kappa$ for all ε small enough, so $\hat{T} \geq \kappa$. The bounds (4.6.18) and (4.6.19) imply (4.1.35) and (4.1.36) respectively.

Proof of part (ii): The proof is similar to part (i), so we omit some details. In the same way as in part (i), we take solutions (F_+, β, ϕ^0) to (4.1.18) (satisfying (4.6.1)). We also take the solution $(F^{\varepsilon}_+, F^{\varepsilon}_-)$ to (4.1.7) and $(\gamma^{\varepsilon}, \psi^{\varepsilon})$ satisfying the conditions

(4.6.2) through (4.6.6) for some $\hat{T}_\varepsilon > 0$. We note that the conclusions of Propositions 4.4.1 and 4.4.2 both hold.

By (4.1.39), and the fact that $\psi^\varepsilon_{in} = \phi^0_{in}$, we have that the expression in (4.5.7) is exactly zero. The bounds (4.6.12) and (4.6.13) both hold in this context as well, so we can guarantee $\sup_{t\in[0,\hat{T}_\varepsilon]} \sqrt{\mathscr{E}^\varepsilon_{-,2}} < \varsigma$ by taking κ small enough. Thus all the hypotheses of Proposition 4.5.1 are satisfied up to $T = \hat{T}_\varepsilon$. Next, we have that $\mathscr{E}^\varepsilon_{-,\alpha,in} = \widetilde{\mathscr{E}}^\varepsilon_{-,\alpha,in} \lesssim_M \delta^2$ (and this expression is independent of ε). Therefore, by (4.5.8), we have

$$\varepsilon \sup_{t\in[0,\hat{T}_\varepsilon]} \widetilde{\mathscr{E}}^\varepsilon_{-,2}(t) + \int_0^{\hat{T}_\varepsilon} \widetilde{\mathscr{D}}^\varepsilon_{-,2}(t)dt \le C_M(\varepsilon\delta^2 + \varepsilon^2). \tag{4.6.49}$$

On the other hand, the above and (4.5.9) give the bound

$$\widetilde{\mathscr{E}}^\varepsilon_{-,1}(t) \lesssim_M e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\delta^2 + \varepsilon^{\frac{5}{3}}t^{\frac{1}{3}} + \varepsilon^2. \tag{4.6.50}$$

Next, similarly to (4.6.20) in part (i), we have

$$\sup_{t\in[0,\hat{T}_\varepsilon]} \|F^\varepsilon_+(t) - F^0_+(t)\|^2_{\mathfrak{E}'} + \int_0^{\hat{T}_\varepsilon} \|F^\varepsilon_+(t) - F^0_+(t)\|^2_{\mathfrak{D}'}dt \lesssim \varepsilon(\varepsilon+\delta^2). \tag{4.6.51}$$

Now, combining (4.6.51) with (4.6.12) , (4.6.13), and (4.6.49) and (4.6.50), we have

$$\varepsilon \sup_{t\in[0,\hat{T}_\varepsilon]} \mathscr{E}^\varepsilon_{-,2}(t) + \int_0^{\hat{T}_\varepsilon} \mathscr{D}^\varepsilon_{-,2}(t)dt \le C_M(\varepsilon\delta^2 + \varepsilon^2) \tag{4.6.52}$$

and for all $t \in [0, \hat{T}_\varepsilon]$, we have

$$\mathscr{E}^\varepsilon_{-,1}(t) \lesssim_M e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\delta^2 + \varepsilon^{\frac{5}{3}}t^{\frac{1}{3}} + \varepsilon\delta^2. \tag{4.6.53}$$

We now show that $\hat{T} = \inf_{\varepsilon\in(0,\bar{\varepsilon}]} \hat{T}_\varepsilon$ is positive. As before, we take $\hat{T}_\varepsilon < \kappa$, and show that each of (4.6.2) through (4.6.6) holding with equality yields a contradiction, provided κ is taken small enough. In the cases of (4.6.2), (4.6.3), (4.6.4) and (4.6.5), the proof is essentially same as in the case of part (i), up to taking δ sufficiently small in addition to κ and ε.

We now address the case of when (4.6.6) holds with equality at $T = \hat{T}_\varepsilon$. Next, take $0 < t_0 < \hat{T}_\varepsilon$ to be chosen later. Through a trivial modification of the proof of Proposition 4.5.1, we have that the estimate (4.5.8) holds with $\alpha = 1$ instead of $\alpha = 2$, and $[t_0, T]$ instead of $[0, T]$. This gives

$$\varepsilon \sup_{t\in[t_0,T]} \widetilde{\mathscr{E}}^\varepsilon_{-,1}(t) + \int_{t_0}^{\hat{T}_\varepsilon} \mathscr{D}_{-,1}(t)dt \lesssim_M \mathscr{E}^\varepsilon_{-,1}(t_0) + \varepsilon^2 \tag{4.6.54}$$

$$\lesssim_M \varepsilon(e^{-\frac{1}{C_M}(\frac{t_0}{\varepsilon})^{\frac{2}{3}}}\delta^2 + \varepsilon^{\frac{5}{3}}t_0^{\frac{1}{3}}) + \varepsilon^2 \tag{4.6.55}$$

Taking $t_0 = \varepsilon(C_M|\ln(\varepsilon)|)^{\frac{3}{2}}$, we can bound last expression by by ε^2. On the other hand, integrating (4.6.50) on $[0, t_0]$, we have

$$\int_0^{t_0} \sqrt{\widetilde{\mathscr{E}}^\varepsilon_{-,1}(t)}dt \lesssim_M \varepsilon\delta + \varepsilon^{\frac{5}{4}}t_0^{\frac{7}{6}} + \varepsilon t_0 \lesssim_M \varepsilon\delta + \varepsilon^{\frac{3}{2}} \tag{4.6.56}$$

Therefore, combining these bounds, we deduce

$$\int_0^{\hat{T}_\varepsilon} \| \int_{\mathbf{R}^3} \xi \cdot \nabla_x (F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}) d\xi \|_{L_x^2}^2 dt \tag{4.6.57}$$

$$\lesssim_M \int_0^{t_0} \sqrt{\widetilde{\mathscr{E}}_{-,1}^\varepsilon(t)} dt + \left(\int_0^{t_0} \widetilde{\mathscr{D}}_{-,1}^\varepsilon(t) dt \right)^{\frac{1}{2}} \lesssim_M \varepsilon. \tag{4.6.58}$$

Now, combining the above with (4.5.201), (4.4.144), and (4.6.52),

$$1 = \int_0^t \| \int_{\mathbf{R}^3} (F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}) d\xi \|_{L_x^2} dt \tag{4.6.59}$$

$$\lesssim_M \int_0^{\hat{T}_\varepsilon} \frac{1}{\varepsilon} \| \int_{\mathbf{R}^3} \xi \cdot \nabla_x (F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}) d\xi \|_{L_x^2} + \| \partial_t (\gamma^\varepsilon \psi^\varepsilon) e^{\gamma^\varepsilon \psi^\varepsilon} \|_{L_x^2} dt \tag{4.6.60}$$

$$\lesssim_M \varepsilon\delta + \varepsilon^{\frac{3}{2}} + \hat{T}_\varepsilon + \hat{T}_\varepsilon^{\frac{1}{2}} \tag{4.6.61}$$

$$\leq \varepsilon\delta + \varepsilon^{\frac{3}{2}} + \kappa^{\frac{1}{2}}. \tag{4.6.62}$$

Taking ε, δ and κ sufficiently small, we have a contradiction.

In conclusion, we have $\hat{T} \geq \kappa > 0$. Moreover, (4.1.40), (4.1.41) and (4.1.42) follow from (4.6.51), (4.6.52), and (4.6.53) respectively. □

Chapter Five

Local Well-Posedness for the VPL System

5.1 Introduction

Consider a plasma whose ion and electron distributions are given by $F_+(t,x,v)$, $F_-(t,x,v)$, respectively. These functions represent the amount of particles at a given time t, position x and velocity v. These unknowns satisfy the two-species Vlasov-Poisson-Landau system,[1]

$$\begin{aligned} \{\partial_t + v\cdot\nabla_x + E\cdot\nabla_v\}F_+ &= Q(F_+ + F_-, F_+), \\ \{\partial_t + v\cdot\nabla_x - E\cdot\nabla_w\}F_- &= Q(F_+ + F_-, F_-), \\ -\Delta_x\phi &= 4\pi(n_+ - n_-). \end{aligned} \tag{5.1.1}$$

[1] We have set all physically relevant parameters to 1.

Above, ϕ and $E = -\nabla_x \phi$ are the electric potential and field respectively. The functions $n_\pm(t,x)$ are the charge densities, defined by

$$n_+ = \int_{\mathbf{R}^3} F_+ dv, \quad n_- = \int_{\mathbf{R}^3} F_- dv. \tag{5.1.2}$$

The collision kernels are given by

$$Q(G_1, G_2) = \nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v - v') \left\{ G_1(v') \nabla_v G_2(v) - G_2(v) \nabla_{v'} G_1(v') \right\} dv'. \tag{5.1.3}$$

Here, Φ gives the Landau collision kernel: for each $z \in \mathbf{R}^3$,

$$\Phi(z) := \frac{1}{|z|} \left(I - \frac{z \otimes z}{|z|^2} \right). \tag{5.1.4}$$

Finally, we chose the domain to be the 3D torus, i.e. $x \in \mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$.

In Chapter 4, we gave a derivation of the Vlasov-Poisson-Landau system with massless electrons. However, this theorem was conditioned on the assumption that one could construct solutions to such a system. In such a limit, the electron distribution F_- is wholly determined by the ion distribution F_+, the electrostatic potential ϕ, and a time dependent number $\beta(t)$ representing the inverse temperature of the electrons. The system for (F_+, β, ϕ) reads

$$\begin{aligned}
&\{\partial_t + v \cdot \nabla_x - E \cdot \nabla_v\} F_+ = Q(F_+, F_+) \\
&\frac{d}{dt} \left\{ \frac{3}{2\beta} + \iint_{\mathbf{T}^3 \times \mathbf{R}^3} \frac{|v|^2}{2} F_+ dx dv + \frac{1}{8\pi} \int_{\mathbf{T}^3} |E|^2 dx \right\} = 0, \\
&- \Delta_x \phi = 4\pi (n_+ - e^{\beta \phi^0}).
\end{aligned} \tag{5.1.5}$$

5.1.1 Main Result

The goal of this paper is to show that for appropriate initial data, this formal limit holds on a small time interval. We first provide local well-posedness theorems for the systems (5.1.1) and (5.1.5) for general initial data. To do this, we must first define the function spaces which capture the dissipation rate from the collision kernels. We define the matrix $\sigma_{ij}(v) = \Phi_{ij} * \mu$, with $i, j \in \{1, 2, 3\}$. Given $u \in \mathbf{R}^3$, we have that

$$u^T \sigma(v) u \sim \frac{1}{\langle v \rangle^3} |P_v u|^2 + \frac{1}{\langle v \rangle} |P_{v^\perp} u|^2 \tag{5.1.6}$$

Here, $P_v + P_{v^\perp} = I$ denote the orthogonal projections onto $\text{span}\{v\}$ and $\{v\}^\perp$ respectively. Using this matrix, we define the following spaces which will capture the dissipation produced by the various collision operators. Let $\dot{\mathcal{H}}_\sigma$ denote the Hilbert space with inner product (with $\psi = \psi(v), \zeta = \zeta(v)$)

$$\|\psi\|^2_{\dot{\mathcal{H}}_\sigma} = \langle \sigma_{ij} \partial_{v_i} \psi, \partial_{v_j} \psi \rangle_{L^2_v} \tag{5.1.7}$$

Fix parameters $s > (\frac{5}{2}, 3]$, $r \in (0, 1]$, and $m_j = 5(j+1)$ with $j \in \{0, 1, 2\}$. Given $u = u(x, v)$, we define

$$\|u\|^2_{\mathfrak{E}} := \|\langle v \rangle^{m_2} u\|^2_{L^2_{x,v}} + \|\langle v \rangle^{m_1} \langle \nabla_x \rangle^s u\|^2_{L^2_{x,v}} + \|\langle \nabla_v \rangle^r u\|^2_{L^2_{x,v}}, \tag{5.1.8}$$

$$\|u\|^2_{\mathfrak{D}} := \|\langle v \rangle^{m_2} u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|\langle v \rangle^{m_1} \langle \nabla_x \rangle^s u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|\langle \nabla_v \rangle^r u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}, \tag{5.1.9}$$

$$\|u\|^2_{\mathfrak{E}'} := \|\langle v \rangle^{m_1} u\|^2_{L^2_{x,v}} + \|\langle v \rangle^{m_0} \langle \nabla_x \rangle^s u\|^2_{L^2_{x,v}}, \tag{5.1.10}$$

$$\|u\|^2_{\mathfrak{D}'} := \|\langle v \rangle^{m_1} u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|\langle v \rangle^{m_0} \langle \nabla_x \rangle^s u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{5.1.11}$$

We also define $\mathfrak{E}_T = L^\infty([0, T]; \mathfrak{E})$, $\mathfrak{D}_T = L^2([0, T]; \mathfrak{D})$ and similarly for the primed spaces.

We now state the theorem on local well-posedness.

Theorem 5.1.1. *Let $F_{+,in}(x,v), F_{-,in}(x,w)$ be nonnegative functions in $L^1(\mathbf{T}^3\times\mathbf{R}^3)$ with unit L^1 norm.*

(i) Assume

$$\|(F_{+,in},F_{-,in})\|_{\mathfrak{E}}+\|(\frac{1}{n_{+,in}},\frac{1}{n_{-,in}})\|_{L^\infty}<\infty \tag{5.1.12}$$

Then, there exists $T^*>0$, such that there exists a unique weak solution (F_+,F_-) to (5.1.1) with initial data $(F_+,F_-)|_{t=0}=(F_{+,in},F_{-,in})$ on $t\in[0,T^*)$ in the space

$$F_\pm\in C([0,T^*);\mathfrak{E}')\cap L^\infty_{loc}([0,T^*);\mathfrak{E})\cap L^2_{loc}([0,T^*);\mathfrak{D}), \tag{5.1.13}$$

Moreover, either $T^*=\infty$ or

$$\lim_{T\uparrow T^*}\|(F_+,F_-)\|_{\mathfrak{E}_T}+\|(\frac{1}{n_+},\frac{1}{n_-})\|_{L^\infty([0,T]\times\mathbf{T}^3)}=\infty. \tag{5.1.14}$$

(ii) Assume $\beta_{in}>0$ and

$$\|F_{+,in}\|_{\mathfrak{E}}+\|\frac{1}{n_{+,in}}\|_{L^\infty}<\infty \tag{5.1.15}$$

Then, there exists $T^*>0$, such that there exists a unique weak solution (F_+,β,ϕ) to (5.1.5) with initial data $(F_+,\beta)|_{t=0}=(F_{+,in},\beta_{in})$ on $t\in[0,T^*)$

$$F_+\in C([0,T^*);\mathfrak{E}')\cap L^\infty_{loc}([0,T^*);\mathfrak{E})\cap L^2_{loc}([0,T^*);\mathfrak{D}), \tag{5.1.16}$$

$$\beta\in C([0,T^*);\mathbf{R}_+), \tag{5.1.17}$$

$$\phi\in C([0,T^*);H^{s+2}). \tag{5.1.18}$$

Moreover, either $T^* = \infty$ *or*

$$\lim_{T \uparrow T^*} \|F_+\|_{\mathfrak{E}_T} + \|\frac{1}{n_+}\|_{L^\infty([0,T]\times \mathbf{T}^3)} = \infty. \tag{5.1.19}$$

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Remark. The spaces $\mathfrak{E}$ and $\mathfrak{D}$ differ from the spaces considered in Chapter 4 through the $\langle \nabla_v \rangle^r$ term. Since the norms here are stronger, we have that the local well-posedness theorem proved here implies that one can construct a solution to (5.1.5) compatible with the assumption of Theorem 4.1.1. As noted in Chapter 5, we are able to avoid propagation of ∇_v derivatives since the setting of the massless electron limit is perturbative. Indeed, it appears that if we consider perturbations of smooth solutions to (5.1.1) or (5.1.5), then we can prove a local well-posedness result for the perturbations without propagating any regularity in v.

Remark. Using this method, it is straightforward (in fact, easier) to prove a local well-posedness theorem for the inhomogeneous Landau equation,

$$(\partial_t + v \cdot \nabla_x)F = Q(F, F), \tag{5.1.20}$$

in the same topology as in Theorem 5.1.1.

5.1.2 Method of the proof

The proof method involves fairly straightforward propagation of the norm $\mathfrak{E}$ via energy estimates, and commutator estimates. We also reuse Lemma 4.3.2 from the previous chapter to control collisions.

Building on the methodology of Chapter 4, we encountered one main difficulty, foreshadowed in the remark above. While in the previous chapter we were able to close estimates without any v derivatives (and indeed, it was necessary), this framework was insufficient for general large data. This is because we were unable to close standard bootstrap assumptions without propagating v derivatives, due to the commutators of x derivatives with the diffusion matrix.

5.2 *A priori* estimates

5.2.1 Energy estimates

It will be convenient to split the collision operators into their "diffusion" (second order) and "transport" (first order) parts in divergence form:

$$Q(G_1, G_2) = Q_D(G_1, G_2) + Q_T(G_1, G_2) \tag{5.2.1}$$

$$Q_D(G_1, G_2) := \partial_{v_j}((\Phi_{ij} *_v G_1)\partial_{v_i} G_2) \tag{5.2.2}$$

$$Q_T(G_1, G_2) := -\partial_{v_j}(\partial_{v_i}(\Phi_{ij} *_v G_1)G_2) \tag{5.2.3}$$

We now prove a priori estimates for the following system:

$$\partial_t F + \{v \cdot \nabla_x - \nabla_x \psi \cdot \nabla_v\} F = Q(G, F). \tag{5.2.4}$$

Proposition 5.2.1. *Let $M > 0$. Suppose F is a weak solution to (5.1.1) on the interval $t \in [0, T]$. Assume the following on $G = G(t, x, v) \geq 0$:*

$$\sup_{t \in [0,T]} \left(\| \frac{1}{\int_{\mathbf{R}^3} G dv} \|_{L^\infty_x} + \|G\|_{\mathfrak{E}} + \|\psi\|_{H^{s+1}_x} \right) \leq M < \infty \tag{5.2.5}$$

Then,

$$\|F\|_{\mathfrak{E}_T\cap\mathfrak{D}_T} \le e^{C_M(T+\sqrt{T}\|G\|^{\frac{3}{2}}_{\mathfrak{D}_T})}\|F_{in}\|_{\mathfrak{E}}. \tag{5.2.6}$$

Proof. For every m', s', r', we denote $F^{(m',r',s')} = \langle v\rangle^{m'}\langle\nabla_x\rangle^{s'}\langle\nabla_v\rangle^{r'}F$, and similarly for $G^{(m',s',r')}$.

Step 1: We prove

$$\frac{d}{dt}\|F^{(m_2,0,0)}\|^2_{L^2x,v} + \frac{1}{C}\|F^{(m_2,0,0)}\|^2_{L^2x(\mathcal{H}_\sigma)_v} \le C_\kappa\langle\|G\|_{\mathfrak{D}}\rangle^{\frac{3}{2}}\|F\|^2_{\mathfrak{E}} + \kappa\|F\|^2_{\mathfrak{D}}. \tag{5.2.7}$$

Observe that $F^{(m_2,0,0)}$ this satisfies

$$(\partial_t + v\cdot\nabla_x - \nabla_x\psi\cdot\nabla_v)F^{(m_2,0,0)} - [\langle v\rangle^{m_2}, \nabla_x\psi\cdot\nabla_v]F = \langle v\rangle^{m_2}Q(G,F). \tag{5.2.8}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|F^{(m_2,0,0)}\|^2_{L^2x,v} = \langle[\langle v\rangle^{m_2}, \nabla_x\psi\cdot\nabla_v]F, F^{(m_2,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.9}$$

$$+ \langle\langle v\rangle^{m_2}Q_D(G,F), F^{(m_2,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.10}$$

$$+ \langle\langle v\rangle^{m_2}Q_T(G,F), F^{(m_2,0,0)}\rangle_{L^2_{x,v}}. \tag{5.2.11}$$

First, we see that

$$(5.2.9) = -m_2\langle \nabla_x\psi \frac{v}{\langle v\rangle^2} F^{(m_2,0,0)}, F^{(m_2,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.12}$$

$$\lesssim \|\nabla_x \psi\|_{L^\infty_x} \|F^{(m_2,0,0)}\|^2_{L^2_{x,v}} \tag{5.2.13}$$

$$\lesssim \|\psi\|_{H^s_x} \|F^{(m_2,0,0)}\|^2_{L^2_{x,v}} \tag{5.2.14}$$

$$\lesssim \|F^{(m_2,0,0)}\|^2_{L^2_{x,v}}. \tag{5.2.15}$$

Next,

$$\langle v\rangle^{m_2} \partial_{v_j}(\Phi_{ij} * G\partial_{v_i} F) = \partial_{v_j}(\Phi_{ij} * G\partial_{v_i} F^{(m_2,0,0)} \tag{5.2.16}$$

$$- m_2 \frac{v_j}{\langle v\rangle^2} \Phi_{ij} * G\partial_{v_i} F^{(m_2,0,0)} \tag{5.2.17}$$

$$- m_2 \partial_{v_j}(\frac{v_i}{\langle v\rangle^2} \Phi_{ij} * G F^{(m_2,0,0)}) \tag{5.2.18}$$

$$+ m_2(m_2-2)\frac{v_i v_j}{\langle v\rangle^4} \Phi_{ij} * GF. \tag{5.2.19}$$

Thus,

$$(5.2.10) \le -\langle \Phi_{ij} * G\partial_{v_i} F^{(m_2,0,0)}, \partial_{v_j} F^{(m_2,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.20}$$

$$+ C\|\langle v\rangle^{\frac{1}{2}} \Phi_{ij} * G\|_{L^\infty_{x,v}} \|F^{(m_2,0,0)}\|_{L^2_{x,v}} \|F^{(m_2,0,0)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{5.2.21}$$

$$+ C\|\Phi_{ij} * G\|_{L^\infty_{x,v}} \|F^{(m_2,0,0)}\|^2_{L^2_{x,v}} \tag{5.2.22}$$

Using Lemma 4.3.2,

$$\langle \Phi_{ij} * G\partial_{v_i} F^{(m_2,0,0)}, \partial_{v_j} F^{(m_2,0,0)}\rangle_{L^2_{x,v}} \gtrsim \|F^{(m_2,0,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}, \tag{5.2.23}$$

and

$$\|\langle v\rangle^{\frac{1}{2}} \Phi_{ij} * G\|_{L^\infty_{x,v}} \lesssim \|\langle v\rangle^5 G\|_{L^\infty_x L^2_v} \lesssim \|G^{(m_1,s,0)}\|_{L^2_{x,v}}. \tag{5.2.24}$$

Hence,

$$(5.2.10) \leq -\frac{1}{C}\|F^{(m_2,0,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + C\|F^{(m_2,0,0)}\|^2_{L^2_{x,v}}. \tag{5.2.25}$$

Next, we have

$$(5.2.11) = \langle \partial_{v_i}\Phi_{ij} * GF^{(m_2,0,0)}, \partial_{v_j}F^{(m_2,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.26}$$

$$+ m\langle \frac{v}{\langle v\rangle^2}\partial_{v_i}\Phi_{ij} * GF^{(m_2,0,0)}, F^{(m_2,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.27}$$

$$= \|\langle v\rangle^{\frac{3}{2}}\nabla_v\Phi * G\|_{L^\infty_{x,v}}\|F^{(m_2,0,0)}\|_{L^2_x(L^2\cap\dot{\mathcal{H}}_\sigma)_v}\|F^{(m_2,0,0)}\|_{L^2_{x,v}}. \tag{5.2.28}$$

Now, we claim that

$$\|\langle v\rangle^2\nabla_v\Phi * G\|_{L^\infty_{x,v}} \lesssim \|G\|_{\mathfrak{E}} + \|G\|^{\frac{1}{4}}_{\mathfrak{E}}\|G\|^{\frac{3}{4}}_{\mathfrak{D}}. \tag{5.2.29}$$

To see this, first observe that $|\nabla_v\Phi| \lesssim \frac{1}{|v|^2}$. Thus, we have for all (t,x,v),

$$|\langle v\rangle^2\partial_{v_i}\Phi_{ij} * G(t,x,v)| \leq \langle v\rangle^2\int_{|v|>2\langle v'\rangle}\frac{1}{|v-v'|^2}|G(t,x,v')|dv' \tag{5.2.30}$$

$$+ \langle v\rangle^2\int_{|v|\leq 2\langle v'\rangle}\frac{1}{|v-v'|^2}|G(t,x,v')|dv' \tag{5.2.31}$$

$$\lesssim \int_{\mathbf{R}^3} G(t,x,v')dv' + |\nabla_v|^{-1}\{\langle v\rangle^2|G|\}(t,x,v) \tag{5.2.32}$$

Hence,

$$\|\langle v\rangle^2\partial_{v_i}\Phi_{ij} * G\|_{L^\infty_{x,v}} \lesssim \|G\|_{L^\infty_x L^1_v} + \||\nabla_v|^{-1}\{\langle v\rangle^2|G|\}\|_{L^\infty_{x,v}} \tag{5.2.33}$$

$$\lesssim \|G^{(2,s,0)}\|_{L^2_{x,v}} + \|\ |G^{(2,0,0)}|\ \|_{L^\infty_x H^{3/4}_v} \tag{5.2.34}$$

$$\lesssim \|G^{(m_1,s,0)}\|_{L^2_{x,v}} + \|G^{(2,0,0)}\|^{\frac{1}{4}}_{L^\infty_x L^2_v}\|G^{(2,0,0)}\|^{\frac{3}{4}}_{L^\infty_x H^1_v} \tag{5.2.35}$$

$$\lesssim \|G\|_{\mathfrak{E}} + \|G\|^{\frac{1}{4}}_{\mathfrak{E}}\|G\|^{\frac{3}{4}}_{\mathfrak{D}}. \tag{5.2.36}$$

In the above, we used the fact that $\| \, |u| \, \|_{H^1} = \|u\|_{H^1}$. Combining these estimates on (5.2.9), (5.2.10), and (5.2.11), we get (5.2.7).

Step 2: We now prove the bound

$$\frac{d}{dt}\|F^{(0,0,r)}\|^2_{L^2x,v} + \frac{1}{C}\|F^{(0,0,r)}\|^2_{L^2x(\mathcal{H}_\sigma)_v} \leq C_\kappa \langle \|G\|_{\mathfrak{D}}\rangle^{\frac{3}{2}}\|F\|^2_{\mathfrak{E}} + \kappa\|F\|^2_{\mathfrak{D}}. \tag{5.2.37}$$

First, we compute

$$\frac{1}{2}\frac{d}{dt}\|F^{(0,0,r)}\|^2_{L^2x,v} = \langle [\langle \nabla_v\rangle^r, v\cdot\nabla_x]F, F^{(0,0,r)}\rangle_{L^2_{x,v}} \tag{5.2.38}$$

$$+ \langle \langle \nabla_v\rangle^r Q_D(G,F), F^{(0,0,r)}\rangle_{L^2_{x,v}} \tag{5.2.39}$$

$$+ \langle \langle \nabla_v\rangle^r Q_T(G,F), F^{(0,0,r)}\rangle_{L^2_{x,v}}.. \tag{5.2.40}$$

We now bound (5.2.38).

$$[\langle \nabla_v\rangle^r, v\cdot\nabla_x]F = -r\langle\nabla_v\rangle^{r-2}\nabla_v\cdot\nabla_x F. \tag{5.2.41}$$

Hence,

$$(5.2.38) \lesssim \|\langle\nabla_v\rangle^{r-1}\nabla_x F\|_{L^2_{x,v}}\|F^{(0,0,r)}\|_{L^2_{x,v}} \lesssim \|F\|^2_{\mathfrak{E}}. \tag{5.2.42}$$

To bound the remaining terms, we need the following weighted commutator estimate: Given appropriate $u_1(v), u_2(v)$, and $m \geq 0$, we have

$$\|\langle v\rangle^m[\langle\nabla_v\rangle^r, u_1]u_2\|_{L^2_v} \lesssim \|\langle\nabla_v\rangle^r u_1\|_{(\mathcal{F}L^1)_v}\|\langle v\rangle^m u_2\|_{L^2_v} \tag{5.2.43}$$

$$\lesssim \|\langle\nabla_v\rangle^{r+\frac{3}{2}}u_1\|_{L^2_v}\|u_2\|_{L^2_v}, \tag{5.2.44}$$

where $\mathcal{F}L^1$ is the space of functions whose Fourier transform is in L^1. By interpolation, it suffices to show this when $m \in 2\mathbf{N}_0$. We take the Fourier transform,

$$\mathcal{F}\{\langle v\rangle^m[\langle\nabla_v\rangle^r, u_1]u_2\}(\xi) = (1-\Delta_\xi)^{\frac{m}{2}}\int(\langle\xi\rangle^r - \langle\xi-\eta\rangle^r)\hat{u}_1(\eta)\hat{u}_2(\xi-\eta)d\eta \quad (5.2.45)$$

$$= \int(\langle\xi\rangle^r - \langle\xi-\eta\rangle^r)\hat{u}_1(\eta)(1-\Delta_\xi)^{\frac{m}{2}}\hat{u}_2(\xi-\eta)d\eta \quad (5.2.46)$$

$$+ \sum_{|\alpha+\alpha'|=\frac{m}{2},\alpha\neq 0} c_{\alpha,\alpha'}\int\partial_\xi^\alpha(\langle\xi\rangle^r - \langle\xi-\eta\rangle^r)\hat{u}_1(\eta)\partial_\xi^{\alpha'}\hat{u}_2(\xi-\eta)d\eta \quad (5.2.47)$$

where $c_{\alpha,\alpha'}$ are some collection of constants. Next, $|\langle\xi\rangle - \langle\xi-\eta\rangle^r| \leq \langle\eta\rangle^r$ (recall $r \in (0,1]$). Moreover, $|\partial_\xi^\alpha(\langle\xi\rangle^r - \langle\xi-\eta\rangle^r)| \lesssim 1$ for all multi-indices $\alpha \neq 0$. Thus, pointwise in ξ, we have

$$|\mathcal{F}\{\langle v\rangle^m[\langle\nabla_v\rangle^r, u_1]u_2\}(\xi)| \lesssim \langle\eta\rangle^r|\hat{u}_1(\eta)(1-\Delta_\xi)^{\frac{m}{2}}\hat{u}_2(\xi-\eta)|d\eta \quad (5.2.48)$$

$$+ \sum_{|\alpha+\alpha'|=\frac{m}{2},\alpha\neq 0}\int|\hat{u}_1(\eta)\partial_\xi^{\alpha'}\hat{u}_2(\xi-\eta)|d\eta \quad (5.2.49)$$

The estimate (5.2.43) directly follows from Young's inequality, and Plancherel's identity.

Next, we split

$$\langle\nabla_v\rangle^r Q_D(G,F) = Q_D(G, F^{(0,0,r)}) + \nabla_v\cdot([\langle\nabla_v\rangle^r, \Phi * G]\nabla_v F) \quad (5.2.50)$$

$$(5.2.51)$$

Therefore, we have

$$(5.2.39) \leq -\frac{1}{C}\|F^{(0,0,r)}\|^2_{L^2_x(\mathcal{H}_\sigma)_v} + \langle[\langle\nabla_v\rangle^r, \Phi * G]\nabla_v F, \nabla_v F^{(0,0,r)}\rangle_{L^2_{x,v}} \quad (5.2.52)$$

The latter term is bounded as follows,

$$\langle [\langle \nabla_v \rangle^r, \Phi * G] \nabla_v F, \nabla_v F^{(0,0,r)} \rangle_{L^2_{x,v}} \tag{5.2.53}$$

$$\lesssim \| \langle v \rangle^{\frac{3}{2}} [\langle \nabla_v \rangle^r, \Phi * G] \nabla_v F \|_{L^2_{x,v}} \| \langle v \rangle^{-\frac{3}{2}} \nabla_v F^{(0,0,r)} \|_{L^2_{x,v}} \tag{5.2.54}$$

$$\lesssim \| \Phi * G \|_{L^2_x \mathcal{F} L^1} \| \langle v \rangle^{\frac{3}{2}} \nabla_v F \|_{L^2_{x,v}} (\| F^{(0,0,r)} \|_{L^2_x (\mathcal{H}_\sigma)_v} + \| F \|_{\mathfrak{E}}), \tag{5.2.55}$$

having applied (5.2.43) in the last line. Now, using $\hat{G}(t,x,\xi)$ to denote the Fourier transform of G in v, we have

$$\| \Phi * G \|_{L^2_x (\mathcal{F} L^1)_v} \lesssim \| \int \frac{1}{|\xi|^2} |\hat{G}(t,x,\xi)| d\xi \|_{L^2_x} \tag{5.2.56}$$

$$\le \| \int_{B(0,1)} \frac{1}{|\xi|^2} |\hat{G}(t,x,\xi)| d\xi \|_{L^2_x} + \| \int_{B(0,1)^c} \frac{1}{|\xi|^2} |\hat{G}(t,x,\xi)| d\xi \|_{L^2_x} \tag{5.2.57}$$

$$\le \| \hat{G} \|_{L^2_x L^\infty_\xi} + \| \hat{G} \|_{L^2_{x,v}} \| \frac{1_{B(0,1)^c}}{|\xi|^2} \|_{L^2_\xi} \tag{5.2.58}$$

$$\lesssim \| \langle v \rangle^4 G \|_{L^2_{x,v}} \tag{5.2.59}$$

$$\lesssim 1 \tag{5.2.60}$$

In the penultimate line, we used Sobolev embedding $L^\infty \subset H^4$. Combining all the bounds, we get

$$(5.2.39) \lesssim -\frac{1}{C} \| F^{(0,0,r)} \|^2_{L^2_x (\mathcal{H}_\sigma)_v} + \| \langle v \rangle^{\frac{3}{2}} \nabla_v F \|_{L^2_{x,v}} (\| F^{(0,0,r)} \|_{L^2_x (\mathcal{H}_\sigma)_v} + \| F \|_{\mathfrak{E}}). \tag{5.2.61}$$

Next, we verify the following claim: for all $\kappa > 0$,

$$\| \langle v \rangle^{\frac{3}{2}} \nabla_v F^{(m_0,0,0)} \|_{L^2_{x,v}} \le C_\kappa \| F \|_{\mathfrak{E}} + \kappa \| F \|_{\mathfrak{D}}. \tag{5.2.62}$$

First, we apply the interpolation inequality: since $m_2 - \frac{3}{2} > \frac{3}{2}$,

$$\|\langle v\rangle^{\frac{3}{2}}\nabla_v F^{(m_0,0,0)}\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^{m_2-\frac{3}{2}}\nabla_v F^{(m_0,0,0)}\|^{1-\theta}_{L^2_{x,v}}\|\langle v\rangle^{-\frac{3}{2}}\nabla_v F^{(m_0,0,0)}\|^{\theta}_{L^2_{x,v}} \tag{5.2.63}$$

for some $\theta \in (0,1)$. The first factor is controlled by $\|F^{(m_2,0,0)}\|_{L^2_x(L^2\cap\dot{\mathcal{H}}_\sigma)_v} \lesssim \|F\|_{\mathfrak{E}\cap\mathfrak{D}}$.

We further interpolate the second factor as follows:

$$\|\langle v\rangle^{-\frac{3}{2}}\nabla_v F\|_{L^2_{x,v}} \lesssim \|\langle\nabla_v\rangle^{-1}(\langle v\rangle^{-\frac{3}{2}}\nabla_v F\|^{\frac{r}{1+r}}_{L^2_{x,v}} \tag{5.2.64}$$

$$\cdot\, \|\langle\nabla_v\rangle^{r}(\langle v\rangle^{-\frac{3}{2}}\nabla_v F^{(m_0,0,0)})\|^{\frac{1}{1+r}}_{L^2_{x,v}} \tag{5.2.65}$$

Writing $\langle v\rangle^{-\frac{3}{2}}\nabla_v F = \nabla_v(\langle v\rangle^{-\frac{3}{2}}F) - \nabla_v\langle v\rangle^{-\frac{3}{2}}F$, we see that the first factor in the above is bounded by $\|F\|_{L^2_{x,v}}$.

As for the second, we use (5.2.43),

$$\|\langle\nabla_v\rangle^{r}(\langle v\rangle^{-\frac{3}{2}}\nabla_v F)\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^{-\frac{3}{2}}\nabla_v\langle\nabla_v\rangle^{r}F\|_{L^2_{x,v}} \tag{5.2.66}$$

$$+\, \|[\langle\nabla_v\rangle^{r}, \langle v\rangle^{-\frac{3}{2}}]\nabla_v F\|_{L^2_{x,v}} \tag{5.2.67}$$

$$\lesssim \|F^{(0,0,r)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|\langle\nabla_v\rangle^{r}\langle v\rangle^{-\frac{3}{2}}\|_{\mathcal{F}L_1}\|\nabla_v F\|_{L^2_{x,v}} \tag{5.2.68}$$

Now, $\mathcal{F}\{\langle v\rangle^{-\frac{3}{2}}\}(\xi)$ behaves like $|\xi|^{-\frac{3}{2}}$ near the origin, and decays rapidly as $|\xi| \to \infty$. Thus,

$$\|\langle\nabla_v\rangle^{r}(\langle v\rangle^{-\frac{3}{2}}\nabla_v F)\|_{L^2_{x,v}} \lesssim \|F^{(0,0,r)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|\nabla_v F\|_{L^2_{x,v}}. \tag{5.2.69}$$

Combining with (5.2.63) and (5.2.64), we get

$$\|\langle v\rangle^{\frac{3}{2}}\nabla_v F\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^{m_2-\frac{3}{2}}\nabla_v F\|^{1-\theta}_{L^2_{x,v}}\|F\|^{\frac{\theta r}{1+r}}_{L^2_{x,v}}(\|F^{(0,0,r)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}+\|\nabla_v F\|_{L^2_{x,v}})^{\frac{\theta}{1+r}}. \tag{5.2.70}$$

Hence, for all κ, we can bound

$$\|\langle v\rangle^{\frac{3}{2}}\nabla_v F\|_{L^2_{x,v}} \le C_\kappa\|F\|_{\mathfrak{E}}+\kappa\|F\|_{\mathfrak{D}}. \tag{5.2.71}$$

We conclude that

$$(5.2.39) \le -\frac{1}{C}\|F^{(0,0,r)}\|^2_{L^2_x(\mathcal{H}_\sigma)_v}+C_\kappa\|F\|^2_{\mathfrak{E}}+\kappa\|F\|^2_{\mathfrak{D}}. \tag{5.2.72}$$

Finally, we bound (5.2.40). We write

$$(5.2.40) = \langle\partial_{v_i}\Phi_{ij}*GF^{(0,0,r)},\partial_{v_j}F^{(0,0,r)}\rangle_{L^2_{x,v}}+\langle[\langle\nabla_v\rangle^r,\partial_{v_i}\Phi_{ij}*G]F,\partial_{v_j}F^{(0,0,r)}\rangle_{L^2_{x,v}} \tag{5.2.73}$$

Similarly to bounding (5.2.10),

$$\langle\partial_{v_i}\Phi_{ij}*GF^{(0,0,r)},\partial_{v_j}F^{(0,0,r)}\rangle_{L^2_{x,v}} \lesssim \langle\|G\|_{\mathfrak{D}}\rangle^{\frac{3}{4}}\|F\|_{\mathfrak{E}}\|F\|_{\mathfrak{E}\cap\mathfrak{D}}. \tag{5.2.74}$$

On the other hand, using (5.2.43), we have

$$\langle[\langle\nabla_v\rangle^r,\partial_{v_i}\Phi_{ij}*G]F,\partial_{v_j}F^{(0,0,r)}\rangle_{L^2_{x,v}} \lesssim \|\langle v\rangle^2[\langle\nabla_v\rangle^r,\partial_{v_i}\Phi_{ij}*G]F\|_{L^2_{x,v}}\|F\|_{\mathfrak{E}\cap\mathfrak{D}} \tag{5.2.75}$$

$$\lesssim \||\nabla_v|^{-1}\langle\nabla_v\rangle^r G\|_{L^\infty_x(\mathcal{F}L^1)_v}\|F\|_{\mathfrak{E}}\|F\|_{\mathfrak{E}\cap\mathfrak{D}}. \tag{5.2.76}$$

Similarly to the bound on $\Phi * G \in \mathcal{F}L^1$, we have

$$\||\nabla_v|^{-1}\langle\nabla_v\rangle^r G\|_{L^\infty_x(\mathcal{F}L^1)_v} \lesssim \langle\|G\|_{\mathfrak{D}}\rangle^{\frac{3}{4}}. \tag{5.2.77}$$

Thus, altogether,

$$(5.2.40) \lesssim C_\kappa\langle\|G\|\rangle^{\frac{3}{2}}\|F\|^2_{\mathfrak{E}} + \kappa\|F\|^2_{\mathfrak{D}}. \tag{5.2.78}$$

Step 3: We now prove that for all $\kappa > 0$,

$$\frac{d}{dt}\|F^{(m_1,s,0)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|F^{(m_1,s,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \leq C_\kappa\langle\|G\|\rangle^{\frac{3}{2}}\|F\|^2_{\mathfrak{E}} + \kappa\|F\|^2_{\mathfrak{D}}. \tag{5.2.79}$$

Observe that $F^{(m_1,s,0)}$ satisfies

$$(\partial_t + v\cdot\nabla_x - \nabla_x\psi\cdot\nabla_v)F^{(m_1,s,0)} - [\langle v\rangle^{m_1}\langle\nabla_x\rangle^s, \nabla_x\psi\cdot\nabla_v]F \tag{5.2.80}$$

$$= \langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q(G,F). \tag{5.2.81}$$

Then,

$$\frac{1}{2}\frac{d}{dt}\|F^{(m_1,s,0)}\|^2_{L^2x,v} = \langle[\langle v\rangle^{m_1}\langle\nabla_x\rangle^s, \nabla_x\psi\cdot\nabla_v]F, F^{(m_1,s,0)}\rangle_{L^2_{x,v}} \tag{5.2.82}$$

$$+ \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q_D(G,F), F^{(m_1,s,0)}\rangle_{L^2_{x,v}} \tag{5.2.83}$$

$$+ \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q_T(G,F), F^{(m_1,s,0)}\rangle_{L^2_{x,v}}. \tag{5.2.84}$$

Now,

$$(5.2.84) = \langle \langle v\rangle^{m_1}[\langle \nabla_x\rangle^s, \nabla_x\psi]\cdot\nabla_v F, F^{(m_1,r)}\rangle_{L^2_{x,v}} \tag{5.2.85}$$

$$+ \langle [\langle v\rangle^{m_1}, \nabla_x\psi\cdot\nabla_v]F^{(0,s)}, F^{(m_1,r)}\rangle_{L^2_{x,v}} \tag{5.2.86}$$

$$\lesssim \|\nabla_x\psi\|_{H^s_x}(\|\langle v\rangle^{m_1}\nabla_v F^{(0,s-1)}\|_{L^2_{x,v}} + \|F^{(m_1,s,0)}\|_{L^2_{x,v}})\|F^{(m_1,s,0)}\|_{L^2_{x,v}}. \tag{5.2.87}$$

Now,

$$\|\langle v\rangle^{m_1}\nabla_v F^{(0,s-1)}\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^{m_1+\frac{3}{2}(s-1)}\nabla_v F\|^{\frac{1}{s}}_{L^2_{x,v}}\|\langle v\rangle^{m_1-\frac{3}{2}}\nabla_v F^{(0,s,0)}\|^{\frac{s-1}{s}}_{L^2_{x,v}} \tag{5.2.88}$$

$$\lesssim \|F^{(m_1,s,0)}\|_{L^2_x(L^2\cap\dot{\mathcal{H}}_\sigma)_v}. \tag{5.2.89}$$

Thus,

$$(5.2.82) \lesssim \|F\|_{\mathfrak{E}\cap\mathfrak{D}}\|F\|_{\mathfrak{E}}. \tag{5.2.90}$$

Next, we have

$$(5.2.83) = -\langle \Phi_{ij} * G\partial_{v_i}F^{(m_1,s,0)}, \partial_{v_j}F^{(m_1,s,0)}\rangle_{L^2_{x,v}} \tag{5.2.91}$$

$$- \langle \langle\nabla_x\rangle^s\{\Phi_{ij} * G\partial_{v_i}F^{(m_1,0,0)}\} - \Phi_{ij} * G\partial_{v_i}F^{(m_1,s,0)}, \partial_{v_j}F^{(m_1,s,0)}\rangle_{L^2_{x,v}} \tag{5.2.92}$$

$$+ m_1\langle \langle\nabla_x\rangle^s\{\Phi_{ij} * G\frac{v_i}{\langle v\rangle^2}F^{(m_1,0,0)}\}, \partial_{v_j}F^{(m_1,s,0)}\rangle_{L^2_{x,v}} \tag{5.2.93}$$

$$- m_1\langle v_j\langle v\rangle^{m_1-2}\langle\nabla_x\rangle^s\{\Phi_{ij} * G\partial_{v_i}F\}, F^{(m_1,s,0)}\rangle_{L^2_{x,v}}. \tag{5.2.94}$$

For (5.2.91), we use (4.3.6) to get

$$(5.2.91) \leq -\frac{1}{C}\|F^{(m_1,s,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{5.2.95}$$

Using (4.3.5) we have

$$(5.2.92) \lesssim \|\sigma^{-\frac{1}{2}}\Phi * G^{(0,s)}\sigma^{-\frac{1}{2}}\|_{L^\infty_{x,v}}\|F^{(m_1,s-1,0)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F^{(m_1,s,0)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{5.2.96}$$

$$\lesssim \|G\|_{\mathfrak{e}}\|F^{(m_1,0,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F^{(m_1,s,0)}\|^{2-\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{5.2.97}$$

Similarly as with (5.2.71), we can bound

$$\|F^{(m_1,0,0)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \lesssim \|F^{(m_1,0,0)}\|_{L^2_{x,v}} + \|\langle v\rangle^{m_1-\frac{1}{2}}\nabla_v F\|_{L^2_{x,v}} \tag{5.2.98}$$

$$\leq C_\kappa\|F\|_{\mathfrak{e}} + \kappa\|F\|_{\mathfrak{D}}. \tag{5.2.99}$$

for all $\kappa > 0$. Thus,

$$(5.2.92) \leq C_\kappa\|F\|^2_{\mathfrak{e}} + \kappa\|F\|^2_{\mathfrak{D}}. \tag{5.2.100}$$

Finally, we have

$$(5.2.93) + (5.2.94) \leq C\|G\|_{\mathfrak{e}}\|F^{(m_1,s,0)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F^{(m_1,s,0)}\|_{L^2_{x,v}} \tag{5.2.101}$$

$$\leq C_\kappa\|F\|^2_{\mathfrak{e}} + \kappa\|F\|^2_{\mathfrak{D}}. \tag{5.2.102}$$

Thus, altogether,

$$(5.2.83) \leq -\frac{1}{C}\|F^{(m_1,s,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + C_\kappa\|F\|^2_{\mathfrak{e}} + \kappa\|F\|^2_{\mathfrak{D}}. \tag{5.2.103}$$

Finally,

$$
\begin{aligned}
(5.2.84) &= \langle \langle \nabla_x \rangle^s \{ \partial_{v_i} \Phi_{ij} * G F^{(m_1,0,0)} \}, \partial_{v_j} F^{(m_1,s,0)} \rangle_{L^2_{x,v}} && (5.2.104)\\
&\quad + m_1 \langle \frac{v_j}{\langle v \rangle^2} \langle \nabla_x \rangle^s \{ \partial_{v_i} \Phi_{ij} * G F^{(m_1,0,0)} \}, F^{(m_1,s,0)} \rangle_{L^2_{x,v}} && (5.2.105)\\
&\lesssim \| \langle v \rangle^2 \nabla_v \Phi * G^{(0,s)} \|_{L^2_x L^\infty_v} \|F\|_{\mathfrak{E}\cap\mathfrak{D}} \|F\|_{\mathfrak{E}} && (5.2.106)\\
&\lesssim \|G\|_{\mathfrak{E}}^{\frac{1}{4}} \|G\|_{\mathfrak{E}\cap\mathfrak{D}}^{\frac{3}{4}} \|F\|_{\mathfrak{E}\cap\mathfrak{D}} \|F\|_{\mathfrak{E}} && (5.2.107)\\
&\lesssim C_\kappa \langle \|G\|_{\mathfrak{E}\cap\mathfrak{D}} \rangle^{\frac{3}{2}} \|F\|_{\mathfrak{E}}^2 + \kappa \|F\|_{\mathfrak{D}}^2. && (5.2.108)
\end{aligned}
$$

In the above, we used a similar estimate as in (5.2.29). Combining the bounds for (5.2.82),(5.2.83) and (5.2.84), we have proved (5.2.79). Combining with the previous steps, we conclude with (5.2.6).

Step 4: Adding (5.2.7), (5.2.37) and (5.2.79), and taking κ sufficiently small, we have

$$
\frac{d}{dt}\|F\|_{\mathfrak{E}}^2 + \frac{1}{C}\|F\|_{\mathfrak{D}}^2 \lesssim \langle \|G\|_{\mathfrak{D}} \rangle^{\frac{3}{2}} \|F\|_{\mathfrak{E}}^2. \tag{5.2.109}
$$

Then, (5.2.6) follows from Grönwall's inequality.

□

5.2.2 Contraction estimates

Proposition 5.2.2. *Let $M > 0$, and $G_\alpha(t,x,v) \geq 0$ and $\psi_\alpha(t,x)$ with $\alpha \in \{1,2\}$ satisfy*

$$\sup_{t\in[0,T],\alpha\in\{1,2\}} \left(\| \frac{1}{\int_{\mathbf{R}^3} G_\alpha dv} \|_{L^\infty_x} + \|G_\alpha\|_{\mathfrak{E}} + \|\psi_\alpha\|_{H^{s+1}_x} \right) \leq M \tag{5.2.110}$$

Suppose F_α is a solution to (5.2.4) with $\psi = \psi_\alpha$ and $G = G_\alpha$ and initial data $F_{\alpha,in}$. Let $\delta F = F_2 - F_1$, $\delta G = G_2 - G_1$ and $\delta\psi = \psi_2 - \psi_1$. Then,

$$\begin{aligned} &\|\delta F\|^2_{\mathfrak{E}'_T} + \|\delta F\|^2_{\mathfrak{D}'_T} \\ &\quad \lesssim_M e^{C_M(T+\sqrt{T}\|G_2\|^{\frac{3}{2}}_{\mathfrak{D}_T})} \{ \|\delta F_{in}\|^2_{\mathfrak{E}'} \\ &\qquad + \int_0^T (\|\delta\psi(t)\|^2_{H^{s+1}} + \|\delta G(t)\|^2_{\mathfrak{E}'}) \|F_1(t)\|^2_{\mathfrak{E}\cap\mathfrak{D}'} dt \\ &\qquad + \int_0^T \|\delta G(t)\|^{\frac{1}{2}}_{\mathfrak{E}'} \|\delta G(t)\|^{\frac{3}{2}}_{\mathfrak{D}'} \|F_1(t)\|^2_{\mathfrak{E}} dt \}. \end{aligned} \tag{5.2.111}$$

Proof. Similarly as in Proposition 5.2.1, we denote $F^{(m',r',s')} = \langle v\rangle^{m'} \langle \nabla_x\rangle^{s'} \langle \nabla_v\rangle^{r'} F$, and similarly for G, δG and δF. Then, δF solves

$$\partial_t \delta F + \{v\cdot\nabla_x - \nabla_x\psi_2 \cdot \nabla_v\}\delta F - Q(G_2,\delta F) = Q(\delta G, F_1) + \nabla_x\delta\psi \cdot \nabla_v F_1. \tag{5.2.112}$$

Observe that δF satisfies the same type of equation as (5.2.4), but with the forcing term $Q(\delta G, F_1) + \nabla_x \delta\psi \cdot \nabla_v F_1$. We note here that the proof of Proposition 5.2.1 requires that $5 \leq m_1$ and $m_1 + \frac{3}{2}s \leq m_2$. Thus, modifying the proof of (5.2.7), we have

$$\frac{d}{dt}\|\delta F^{(m_1,0,0)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|\delta F^{(m_1,0,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{5.2.113}$$

$$\lesssim (1+\|G_2\|^{\frac{3}{2}}_{\mathfrak{D}})\|\delta F\|^2_{\mathfrak{E}'} \tag{5.2.114}$$

$$+ \langle \langle v\rangle^{m_1} Q(\delta G, F_1), \delta F^{(m_1,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.115}$$

$$+ \langle \langle v\rangle^{m_1} \nabla_x \delta\psi \cdot \nabla_v F_1, \delta F^{(m_1,0,0)}\rangle_{L^2_{x,v}}. \tag{5.2.116}$$

Now, again modifying the proof of (5.2.7), we have

$$\langle \langle v\rangle^{m_1} Q(\delta G, F_1), \delta F^{(m_1,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.117}$$

$$\lesssim \|\delta G\|_{\mathfrak{E}'}\|F_1^{(m_1,0,0)}\|_{L^2_x(L^2\cap\dot{\mathcal{H}}_\sigma)_v}\|\delta F^{(m_1,0,0)}\|_{L^2_x(L^2\cap\dot{\mathcal{H}}_\sigma)_v} \tag{5.2.118}$$

$$+ \|\delta G\|^{\frac{1}{4}}_{\mathfrak{E}'}\|\delta G\|^{\frac{3}{4}}_{\mathfrak{E}'\cap\mathfrak{D}'}\|F_1^{(m_1,0,0)}\|_{L^2_{x,v}}\|\delta F^{(m_1,0,0)}\|_{L^2_x(L^2\cap\dot{\mathcal{H}}_\sigma)_v} \tag{5.2.119}$$

Also,

$$\langle \langle v\rangle^{m_1} \nabla_x \delta\psi \cdot \nabla_v F_1, \delta F^{(m_1,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.120}$$

$$= -\langle \nabla_x \delta\psi F_1^{(m_1+\frac{3}{2},0)}, \nabla_v \delta F^{(m_1-\frac{3}{2},0)}\rangle_{L^2_{x,v}} \tag{5.2.121}$$

$$-(m_1+\frac{3}{2})\langle \frac{v}{\langle v\rangle^2}\cdot\nabla_x\delta\psi F_1^{(m_1,0,0)}, \delta F^{(m_1,0,0)}\rangle_{L^2_{x,v}} \tag{5.2.122}$$

$$\lesssim \|\delta\psi\|_{H^{s+1}_x}\|F_1\|_{\mathfrak{E}}\|(\delta F^{(m_1,0,0)}, \nabla_v \delta F^{(m_1-\frac{3}{2},0,0)})\|_{L^2_{x,v}} \tag{5.2.123}$$

$$\lesssim \|\delta\psi\|_{H^{s+1}_x}\|F_1\|_{\mathfrak{E}}\|\delta F^{(m_1,0,0)}\|_{L^2_x(L^2\cap\dot{\mathcal{H}}_\sigma)_v}. \tag{5.2.124}$$

Altogether,

$$\frac{d}{dt}\|\delta F^{(m_1,0,0)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|\delta F^{(m_1,0,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{5.2.125}$$

$$\leq C\{(1+\|G_2\|^{\frac{3}{2}}_{\mathfrak{D}})\|\delta F\|^2_{\mathfrak{E}'} + (\|\delta\psi\|^2_{H^{s+1}_x} + \|\delta G\|^2_{\mathfrak{E}'})\|F_1\|^2_{\mathfrak{E}\cap\mathfrak{D}'} \tag{5.2.126}$$

$$+ \|\delta G\|^{\frac{1}{2}}_{\mathfrak{E}'}\|\delta G\|^{\frac{3}{2}}_{\mathfrak{D}'}\|F_1\|^2_{\mathfrak{E}}\}. \tag{5.2.127}$$

Similarly, we also have

$$\frac{d}{dt}\|\delta F^{(m_0,s)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|\delta F^{(m_0,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{5.2.128}$$

$$\leq C\{\langle\|G_2\|_{\mathfrak{D}}\rangle^{\frac{3}{2}}\|\delta F\|^2_{\mathfrak{E}} + (\|\delta\psi\|_{H^{s+1}_x} + \|\delta G\|^2_{\mathfrak{E}'})\|F_1\|^2_{\mathfrak{E}\cap\mathfrak{D}'} \tag{5.2.129}$$

$$+ \|\delta G\|^{\frac{1}{2}}_{\mathfrak{E}'}\|\delta G\|^{\frac{3}{2}}_{\mathfrak{D}'}\|F_1\|^2_{\mathfrak{E}} + \|\delta F^{(m_1,0,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\}. \tag{5.2.130}$$

Combining the bounds on $F^{(m_1,0,0)}$ and $F^{(m_0,s,0)}$, we have the following: for some choice of $\lambda > 0$ sufficiently small depending on $M > 0$, we have

$$\frac{d}{dt}\left(\lambda\|\delta F^{(m_0,s)}\|^2_{L^2_{x,v}} + \|\delta F^{(m_1,0,0)}\|^2_{L^2_{x,v}}\right) + \frac{1}{C}\|\delta F\|^2_{\mathfrak{D}'} \tag{5.2.131}$$

$$\leq C\{\langle\|G_2\|_{\mathfrak{D}}\rangle^{\frac{3}{2}}\|\delta F\|^2_{\mathfrak{E}'} + (\|\delta\psi\|^2_{H^{s+1}_x} + \|\delta G\|^2_{\mathfrak{E}'})\|F_1\|^2_{\mathfrak{E}\cap\mathfrak{D}'} \tag{5.2.132}$$

$$+ \|\delta G\|^{\frac{1}{2}}_{\mathfrak{E}'}\|\delta G\|^{\frac{3}{2}}_{\mathfrak{D}'}\|F_1\|^2_{\mathfrak{E}}\}. \tag{5.2.133}$$

Now, $\lambda\|\delta F^{(m_0,s,0)}\|^2_{L^2_{x,v}} + \|\delta F^{(m_1,0,0)}\|^2_{L^2_{x,v}} \sim \|\delta F\|^2_{\mathfrak{E}'}$. Hence, by Grönwall's inequality, we deduce (5.2.111). □

5.3 Construction of Solutions

5.3.1 Solutions to the linear system

In this section, we use the *a priori* estimates to give a construction scheme for the systems (5.1.1) and (5.1.5) using an iteration scheme. Before we can do this, however, we need the following Lemma which guarantees solutions to the linear system (under very strong hypotheses):

Proposition 5.3.1. *Suppose $F_{in} = F_{in}(x, v)$ satisfies*

$$F_{in} \geq 0 \tag{5.3.1}$$

$$e^{p|v|^2} \nabla^r_{x,v} F_{in} \in L^\infty(\mathbf{T}^3 \times \mathbf{R}^3), \qquad \text{for all } p \geq 0, r \in \mathbf{N}_0. \tag{5.3.2}$$

Next, let $T > 0$ and suppose $G = G(t, x, v)$ satisfies

$$G \geq 0 \tag{5.3.3}$$

$$e^{p|v|^2} \nabla^r_{t,x,v} G \in L^\infty([0, T] \times \mathbf{T}^3 \times \mathbf{R}^3), \qquad \text{for all } p \geq 0, r \in \mathbf{N}_0. \tag{5.3.4}$$

and $\psi = \psi(t, x) \in C^\infty([0, T] \times \mathbf{T}^3)$ satisfies

$$\psi \in C^\infty(\mathbf{T}^3 \times \mathbf{R}^3) \tag{5.3.5}$$

Then, there exists a unique $F = F(t, x, v)$ satisfying the same hypotheses as G that (5.2.4) with initial data $F(t = 0) = F_{in}$.

Proof. Fix $p \geq 0$. Observe that if such a solution to (5.2.4) exists, then $U_p = e^{p(|v|^2+2\psi)} F$ solves

$$(\partial_t - \partial_t \psi) U_p + \{v \cdot \nabla_x - \nabla_x \psi \cdot \nabla_v\} U_p = e^{p|v|^2} Q(G, e^{-p|v|^2} U_p). \tag{5.3.6}$$

We now construct solutions to the above. Take $\lambda \in (0, 1]$. Consider the following initial value problem:

$$\begin{aligned} &(\partial_t - \partial_t \psi) U_p^{(\lambda)} + \{\frac{v}{\langle \lambda v \rangle} \cdot \nabla_x - \nabla_x \psi \cdot \nabla_v\} U_p^{(\lambda)} \\ &\qquad = e^{p|v|^2} Q(G, e^{-p|v|^2} U_p) + \lambda \Delta_{x,v} U_p^{(\lambda)}, \\ &U_p^{(\lambda)}(t = 0) = e^{p(|w|^2+\psi)} F_{in}. \end{aligned} \tag{5.3.7}$$

Observe that

$$e^{p|v|^2}Q(G, e^{-p|v|^2}U_p^{(\lambda)}) = \partial_{v_j}(\Phi_{ij} * G\partial_{v_i}U_p^{(\lambda)} - \partial_{v_i}\Phi_{ij} * GU_p^{(\lambda)}) \tag{5.3.8}$$

$$+ p(pv_iv_j - \delta_{ij})\Phi_{ij} * GU_p^{(\lambda)} - pv_i\Phi_{ij} * G\partial_{v_j}U_p^{(\lambda)}. \tag{5.3.9}$$

By Lemma 4.3.2, we see that $\partial_{v_i}\Phi_{ij} * G$, $\Phi_{ij} * Gv_i$ and $p(pv_iv_j - \delta_{ij})\Phi_{ij} * G$ are all bounded. Hence, this a uniformly parabolic PDE with C^∞ coefficients, and thus there exists a unique strong solution $U_p^{(\lambda)}$ to the above in $\bigcap_{r\geq 0} H^r([0,T] \times \mathbf{T}^3 \times \mathbf{R}^3)$ for all $r > 0$. This fact follows from Chapter 2 of [94], with small modifications needed for the domain $\mathbf{T}^3 \times \mathbf{R}^3$. By applying $\nabla_{x,v}^r$ to both sides, and integrating against $\nabla_{x,v}^r U_p^{(\lambda)}$, we have for all $m \geq 0$, and $r \in \mathbf{N}_0$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla_{x,v}^r U_p^{(\lambda)}\|_{L^2_{x,v}}^2 + \frac{\lambda}{2}\|\nabla_{x,v}^{r+1} U_p^{(\lambda)}\|_{L^2_{x,v}}^2 \tag{5.3.10}$$

$$\leq \langle \nabla_{x,v}^r U_p^{(\lambda)}, \nabla_{x,v}^r \{e^{-p|w|^2} Q(G, e^{-p|w|^2}U_p^{(\lambda)})\}\rangle_{L^2_{x,v}} \tag{5.3.11}$$

$$+ C\sum_{k=1}^r \||\nabla_v^k \left(\frac{v}{\langle \lambda v\rangle}\right)||\nabla_{x,v}^{r-k+1}U_p^{(\lambda)}|\|_{L^2_{x,v}}\|\nabla_{x,v}^r U_p^{(\lambda)}\|_{L^2_{x,v}} \tag{5.3.12}$$

$$+ C\sum_{k=1}^r \||\nabla_x^{k+1}\psi||\nabla_{x,v}^{r-k+1}U_p^{(\lambda)}|\|_{L^2_{x,v}}\|\nabla_{x,v}^r U_p^{(\lambda)}\|_{L^2_{x,v}} \tag{5.3.13}$$

$$\leq \langle \nabla_{x,v}^r U_p^{(\lambda)}, \nabla_{x,v}^r \{e^{-p|w|^2} Q(G, e^{p|w|^2}U_p^{(\lambda)})\}\rangle_{L^2_{x,v}} + C_{\psi,r}\|U_p^{(\lambda)}\|_{H^k_{x,v}}. \tag{5.3.14}$$

In the above, we use $\nabla_v^k\left(\frac{v}{\langle\lambda v\rangle}\right)\lesssim_k 1$. Now, using (5.3.8),

$$\langle\nabla_{x,v}^r U_p^{(\lambda)},\nabla_{x,v}^r\{e^{-p|w|^2}Q(G,e^{p|w|^2}U_p^{(\lambda)})\}\rangle_{L^2_{x,v}} \tag{5.3.15}$$

$$= -\langle\nabla_{x,v}^r\{\Phi_{ij}*G\partial_{v_i}U_p^{(\lambda)}\},\partial_{v_j}\nabla_{x,v}^r U_p^{(\lambda)}\rangle_{L^2_{x,v}} \tag{5.3.16}$$

$$+\langle\nabla_{x,v}^r\{\partial_{v_i}\Phi_{ij}*GU_p^{(\lambda)}\},\partial_{v_j}\nabla_{x,v}^r U_p^{(\lambda)}\rangle_{L^2_{x,v}} \tag{5.3.17}$$

$$+p\langle\nabla_{x,v}^r\{(pv_iv_j-\delta_{ij})\Phi_{ij}*GU_p^{(\lambda)}\},\nabla_{x,v}U_p^{(\lambda)}\rangle_{L^2_{x,v}} \tag{5.3.18}$$

$$-p\langle\nabla_{x,v}^r\{v_i\Phi_{ij}*G\partial_{v_j}U_p^{(\lambda)}\},\nabla_{x,v}^r U_p^{(\lambda)}\rangle_{L^2_{x,v}} \tag{5.3.19}$$

$$\le -\langle\Phi_{ij}*G\partial_{v_i}\nabla_{x,v}^r U_p^{(\lambda)},\partial_{v_j}\nabla_{x,v}^r U_p^{(\lambda)}\rangle_{L^2_{x,v}} \tag{5.3.20}$$

$$+C_{G,p,r}\|U_p^{(\lambda)}\|^2_{H^r_{x,v}} \tag{5.3.21}$$

Therefore,

$$\frac{d}{dt}\|\nabla_{x,v}^r U_p^{(\lambda)}\|^2_{L^2_{x,v}}\le C_{G,\psi,p,r}\|\nabla_{x,v}^r U_p^{(\lambda)}\|^2_{L^2_{x,v}}. \tag{5.3.22}$$

Thus, for all $p\ge 0$, $r\in\mathbf{N}_0$,

$$\|\nabla_{x,v}^r U_p^{(\lambda)}\|_{L_t^\infty([0,T];L^2_{x,v})}\lesssim_{F_{in},G,\psi,p,r}1. \tag{5.3.23}$$

Next, for two different λ_1 and λ_2, we have that $U_p^{(\lambda_2)}-U_p^{(\lambda_1)}$ solves

$$\partial_t(U_p^{(\lambda_2)}-U_p^{(\lambda_1)})+\{\frac{v}{\langle\lambda_1 v\rangle}\cdot\nabla_x+\nabla_x\psi\cdot\nabla_v\}(U_p^{(\lambda_2)}-U_p^{(\lambda_1)}) \tag{5.3.24}$$

$$=e^{p|v|^2}Q(G,e^{-p|v|^2}U)+\lambda_1\Delta_{x,v}(U_p^{(\lambda_2)}-U_p^{(\lambda_1)}) \tag{5.3.25}$$

$$+\left(\frac{v}{\langle\lambda_1 v\rangle}-\frac{v}{\langle\lambda_2 v\rangle}\right)\cdot\nabla_x U_p^{(\lambda_2)}+(\lambda_2-\lambda_1)\Delta_{x,v}U_p^{(\lambda_2)}. \tag{5.3.26}$$

Now, using (5.3.23), and following similar *a priori* estimates, we have

$$\|U_p^{(\lambda_2)}-U_p^{(\lambda_1)}\|_{L_t^\infty([0,T];L^2_{x,v})}\lesssim|\lambda_1-\lambda_2|. \tag{5.3.27}$$

Thus, there exists $U_p \in L^\infty_t([0,T];L^2_{x,v})$ such that $\lim_{\lambda\downarrow 0}\|U_p^{(\lambda)} - U_p\|_{L^\infty_t([0,T];L^2_{x,v})} = 0$. Moreover, we have that U_p also satisfies (5.3.23). In particular, U_p is a classical solution to (5.3.6). Next, by a similar estimate, we have that $U_p = e^{p(|v|^2+2\psi)}U_0$ for all $p \geq 0$. Setting $F = U_0$, we have that F is a classical solution to (5.2.4). Then, for all p, r, we have

$$\|e^{p|w|^2}F\|_{L^\infty_t([0,T];H^r_{x,v})} \lesssim_{\psi,p,r} \|U_{2p}\|_{L^\infty_t([0,T];H^r_{x,v})} \lesssim_{F_{in},G,\psi,p,r} 1. \tag{5.3.28}$$

Combining the above estimate with (5.2.4), we can show that $\partial_t^k F$ satisfies the same estimate as the above, allowing us to conclude that (5.3.4) holds for F.

It remains to check that $F \geq 0$. Now, recall that $\nabla_{t,x,v}(\mathbf{1}_{\{F<0\}}F) = \mathbf{1}_{\{F<0\}}\nabla_{t,x,v}F$ a.e. Thus, we have

$$\frac{d}{dt}\|\mathbf{1}_{\{F<0\}}F\|^2_{L^2_{x,v}} = \langle \partial_t F, \mathbf{1}_{\{F<0\}}F\rangle_{L^2_{x,v}}, \tag{5.3.29}$$

$$\langle \{v\cdot\nabla_x - \nabla_x\psi\}F, \mathbf{1}_{\{F<0\}}F\rangle_{L^2_{x,v}} = 0, \tag{5.3.30}$$

$$\langle Q(G,F), \mathbf{1}_{\{F<0\}}F\rangle_{L^2_{x,v}} = -\langle \Phi_{ij} * G\mathbf{1}_{\{F<0\}}\partial_{v_i}F, \partial_{v_j}F\rangle_{L^2_{x,v}} \tag{5.3.31}$$

$$+ \langle \partial_{v_i}\partial_{v_j}\Phi_{ij} * G\mathbf{1}_{\{F<0\}}F, F\rangle_{L^2_{x,v}} \tag{5.3.32}$$

Hence,

$$\frac{d}{dt}\|\mathbf{1}_{F<0}\nabla_{t,x,v}F\|^2_{L^2_{x,v}} \leq C_G\|\mathbf{1}_{F<0}\nabla_{t,x,v}F\|^2_{L^2_{x,v}}, \tag{5.3.33}$$

So, by Grönwall's inequality,

$$0 \leq \|\mathbf{1}_{F<0}\nabla_{t,x,v}F\|^2_{L^2_{x,v}} \leq C_G\|\mathbf{1}_{F<0}\nabla_{t,x,v}F_{in}\|^2_{L^2_{x,v}} = 0. \tag{5.3.34}$$

We conclude $F \geq 0$. $\square$

5.3.2 Construction of solutions to the VPL system

We now have everything we need to prove the first part of the main theorem.

Proof of Theorem 5.1.1-(i): We break the proof into a number of steps—first, construction of solutions for smooth, compactly supported initial data; second, general data in $\mathfrak{E}$; third, we show continuity in time in $\mathfrak{E}$; and finally, we show the solutions constructed here are unique.

Step 1 (iteration scheme, boundedness): For all $N \geq 1$, fix nonnegative $F^N_{+,in}, F^N_{-,in} \in C^\infty_c(\mathbf{T}^3 \times \mathbf{R}^3)$, and $0 < M < \infty$ sufficiently large such that

$$\|(\frac{1}{n^N_{+,in}}, \frac{1}{n^N_{-,in}})\|_{L^\infty} + \|(F^N_{+,in}, F^N_{-,in})\|_{\mathfrak{E}} \leq \frac{M}{2} \tag{5.3.35}$$

Moreover, we assume $\|F^N_{\pm,in} - F_{\pm,in}\|_{\mathfrak{E}} \to 0$ as $N \to \infty$. In fact, up to taking to a subsequence, we may as well take $\sum_{N\geq 1} \|F^N_{\pm,in} - F^{N-1}_{\pm,in}\|_{\mathfrak{E}} < \infty$.

The solution will be given by the following iteration scheme. Let $(F^0_+(t,x,v), F^0_-(t,x,v)) = (\eta(v), \eta(v))$ where $\eta(v)$ is any nonnegative standard C^∞, compactly supported cutoff. We also set $\phi^0(t,x) = 0$.

Up to taking M larger, we can guarantee that (F^0_+, F^0_-) satisfy the same condition as (5.3.35). Then, for each $N \in \mathbf{N}_0$, define $(F^{N+1}_+, F^{N+1}_-) \in L^2([0,1] \times \mathbf{T}^3 \times \mathbf{R}^3)$ to be the solution to

$$\{\partial_t + v \cdot \nabla_x - \nabla_x \phi^N \cdot \nabla_v\} F^{N+1}_+ = Q(F^N_+ + F^N_-, F^{N+1}_+), \tag{5.3.36}$$

$$\{\partial_t + v \cdot \nabla_x + \nabla_x \phi^N \cdot \nabla_v\} F^{N+1}_- = Q(F^N_+ + F^N_-, F^{N+1}_-), \tag{5.3.37}$$

where $\phi^N = -4\pi\Delta_x^{-1}\int_{\mathbf{R}^3} F_+^N - F_-^N dv$ with initial data $(F_{+,in}, F_{-,in})$. The existence of the iterates is guaranteed by Proposition 5.3.1. In particular, we have $e^{p|w|^2}\nabla_{t,x,v}^r F_\pm^N \in L^\infty([0,1]\times\mathbf{T}^3\times\mathbf{R}^3)$ for all $r, N \in \mathbf{N}_0$, $p \geq 0$. We will also denote $\delta F_\pm^N = F_\pm^N - F_\pm^{N-1}$, and similarly for ϕ^N, etc. Moreover, for each N, we have $F_\pm^N \geq 0$.

Now, for each $N \geq 0$, let $[0, T_N] \subset [0,1]$ be the largest such interval such that (5.2.5) holds with $G \in \{F_-^N, F_+^N\}$ with $T = T_N$, in addition to

$$\|(\frac{1}{n_+^N}, \frac{1}{n_-^N})\|_{L^\infty([0,T_N]\times\mathbf{T}^3)} + \|(F_+^N, F_-^N)\|_{\mathfrak{E}_{T_N}} + \|(F_+^N, F_-^N)\|_{\mathfrak{D}_{T_N}} \leq M. \tag{5.3.38}$$

Let $\tau \in (0,1]$ also be a small constant to be chosen later. We shall show that $\tau \leq T_N$ for all $N \geq 1$.

We prove this by induction. In the base case $N = 0$, this holds trivially, taking M larger if necessary. Now, assume $\tau \leq T_N$ for some $N \geq 1$. Note that $\|\phi^n\| \leq C_M$ for all $t \in [0, T_N]$. Now, since $\tau < T_{N-1}$, we have by Proposition 5.2.1,

$$\|F_\pm^N\|_{\mathfrak{E}_\tau} + \frac{1}{C_M}\|F_\pm^N\|_{\mathfrak{D}_\tau} \leq e^{C_M\sqrt{\tau}}\|F_{\pm,in}^N\|_{\mathfrak{E}} \tag{5.3.39}$$

Since $\|F_{\pm,in}\|_{\mathfrak{E}} \leq \frac{M}{2} \leq \|F_\pm^N\|_{\mathfrak{E}_\tau}$, we may choose $\sqrt{\tau}$ sufficiently small so that

$$\|F_\pm^N\|_{\mathfrak{E}_\tau} + \|F_\pm^N\|_{\mathfrak{D}_\tau} \leq (1 + C_M\sqrt{\tau})\frac{M}{2}. \tag{5.3.40}$$

Moreover, we have the continuity equation

$$\partial_t n_\pm^N(t,x) + \nabla_x \cdot \int_{\mathbf{R}^3} F_\pm^N(t,x,v)dv \tag{5.3.41}$$

which implies

$$\|\partial_t n_\pm^N\|_{L^\infty([0,\tau];H^{s-1})} \lesssim \|F_\pm^N\|_{\mathfrak{E}}. \tag{5.3.42}$$

so $\|n_\pm^N(t,x) - n_{\pm,in}(x)\|_{L^\infty([0,\tau]\times\mathbf{T}^3)} \lesssim_M \tau$. In particular, taking τ small enough,

$$\|(\frac{1}{n_+^N}, \frac{1}{n_-^N})\|_{L^\infty([0,T_N]\times\mathbf{T}^3)} \leq (1+C_M\tau)\|(\frac{1}{n_{+,in}}, \frac{1}{n_{-,in}})\|_{L^\infty([0,T_N]\times\mathbf{T}^3)}. \tag{5.3.43}$$

Combining this with (5.3.40), we see that τ can be taken small enough, depending only on M, such that $\tau < T_N$. This completes the inductive step. Hence, there exists some $\tau = \tau(M)$ such that

$$\|(\frac{1}{n_+^N}, \frac{1}{n_-^N})\|_{L^\infty([0,\tau]\times\mathbf{T}^3)} + \|(F_+^N, F_-^N)\|_{\mathfrak{E}_\tau} + \|(F_+^N, F_-^N)\|_{\mathfrak{D}_\tau} \leq M, \tag{5.3.44}$$

for all $N \geq 1$.

Step 2 (convergence): For all $N \geq 0$, let $\delta F_\pm^N = F_\pm^N - F^{N-1}$, etc. Then, by Proposition 5.2.2, and the boundedness proved in previous step, we have for all $N \geq 2$

$$\|(\delta F_+^N, \delta F_-^N)\|_{\mathfrak{E}'_\tau\cap\mathfrak{D}'_\tau} \lesssim_M \|(\delta F_{+,in}^N, \delta F_{-,in}^N)\|_{\mathfrak{E}'}^2 \tag{5.3.45}$$

$$+ \int_0^\tau \|(\delta F_+^{N-1}, \delta F_-^{N-1})\|_{\mathfrak{E}'}^{\frac{1}{2}} \|(\delta F_+^{N-1}, \delta F_-^{N-1})\|_{\mathfrak{E}'\cap\mathfrak{D}'}^{\frac{3}{2}} dt \tag{5.3.46}$$

$$\lesssim_M \|(\delta F_{+,in}^N, \delta F_{-,in}^N)\|_{\mathfrak{E}'}^2 + \sqrt{\tau}\|(\delta F_+^{N-1}, \delta F_-^{N-1})\|_{\mathfrak{E}'_\tau\cap\mathfrak{D}'_\tau}^2. \tag{5.3.47}$$

Taking $N' \geq 2$, and summing the above over all $2 \leq N \leq N'$, we have

$$\sum_{1\leq N\leq N'} \|(\delta F^N_+, \delta F^N_-)\|_{\mathfrak{E}'_\tau \cap \mathfrak{D}'_\tau} \lesssim_M \sum_{1\leq N\leq N'} \|(\delta F^N_{+,in}, \delta F^N_{-,in})\|_{\mathfrak{E}'_\tau \cap \mathfrak{D}'_\tau} \tag{5.3.48}$$

$$\|(\delta F^1_+, \delta F^1_-)\|_{\mathfrak{E}'_\tau \cap \mathfrak{D}'_\tau} + \tau^{\frac{1}{4}} \sum_{1\leq N\leq N'} \|(\delta F^N_+, \delta F^N_-)\|_{\mathfrak{E}'_\tau \cap \mathfrak{D}'_\tau} \tag{5.3.49}$$

Now, by step 1, we know $\|(\delta F^1_+, \delta F^1_-)\|_{\mathfrak{E}'_\tau \cap \mathfrak{D}'_\tau} \lesssim_M 1$ and $\sum_{1\leq N\leq N'} \|(\delta F^N_{+,in}, \delta F^N_{-,in})\|_{\mathfrak{E}'_\tau \cap \mathfrak{D}'_\tau} < \infty$. Hence, by Taking τ sufficiently small depending on M, we have that $\|(\delta F^N_+, \delta F^N_-)\|_{\mathfrak{E}'_\tau \cap \mathfrak{D}'_\tau} \in l^1$. In particular, (F^N_+, F^N_-) are Cauchy in $\mathfrak{E}'_\tau \cap \mathfrak{D}'_\tau$. Letting (F_+, F_-) be the limit, we see that $(F_+, F_-) \in C([0,\tau]; \mathfrak{E}') \cap \mathfrak{E}_\tau \cap \mathfrak{D}_\tau$, and n_+, n_- enjoy uniform bounds from below. It is easy to check that (F_+, F_-) satisfy (5.1.1) in a weak sense.

Step 3 (uniqueness): The solution constructed by the previous steps is unique as a direct consequence of Proposition 5.2.2.

Step 4 (blow-up criterion): Since τ depends only on M, we can continue the solution uniquely up until either $\|1/n_\pm(t,\cdot)\|_{L^\infty}$ or $\|F_+(t,\cdot)\|_{\mathfrak{E}}$ blow-up. □

5.3.3 Extension to the VPL-Ion system and Landau equation

The extension to the Landau equation is a trivial modification (and simplification) of the proof given above. On the other hand, the extension to the (5.1.5), we must show that solutions Poincaré-Poisson system (5.3.50) below have good boundedness and continuity properties: fixing $G(t,x,v)$, we consider solutions to the system

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{3}{2\beta(t)}+\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}G(t,x,v)dxdv+\frac{1}{8\pi}\int_{\mathbf{T}^3}|\nabla_x\phi(t,x)|^2dx\right\}=0,\\
&\beta(0)=\beta_{in},\\
&-\Delta_x\phi(t,x)=4\pi(\int_{\mathbf{T}^3}G(t,x,v)dv-e^{\beta(t)\phi(t,x)}).
\end{aligned}\tag{5.3.50}$$

Lemma 5.3.2. *The following statements hold:*

(i) Suppose $G\in C([0,T];\mathfrak{E}')$, $\inf_x\int_{\mathbf{R}^3}Gdv>0$, and $\beta_{in}>0$. Then for some $T^*>0$, then there exists a unique solution $(\beta,\phi)\in C([0,T^*];\mathbf{R}_+\times H^{s+1}(\mathbf{T}^3))$ to (5.3.50). More precisely, let $M>0$ be large enough that

$$|\ln(\beta_{in})|+\|\frac{1}{\int Gdv}\|_{L^\infty([0,T]\times\mathbf{T}^3)}+\|G\|_{\mathfrak{E}_T}\leq M \tag{5.3.51}$$

Moreover, we allow the kinetic energy to increase by only a very small amount:

$$\sup_{t\in[0,T]}\int_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(G(t,\cdot)-G_{in})dxdv\leq e^{-M}. \tag{5.3.52}$$

Then, (β,ϕ) exists for all $t\in[0,T]$ and

$$\|\ln(\beta)\|_{L^\infty([0,T]])}+\|\phi\|_{L^\infty([0,T];H^{s+1})}\lesssim_M 1. \tag{5.3.53}$$

(ii) For each $\alpha\in\{1,2\}$ suppose $G^\alpha\in C([0,T];\mathfrak{E})$, $\beta_{in}^\alpha\in\{1,2\}$, and let $(\beta^\alpha,\phi^\alpha)$ be the corresponding solution to (5.3.50) by setting $G=G^\alpha$ and $\beta_{in}=\beta_{in}^\alpha$. Assume also that each (β^α,G^α) satisfy the estimate (5.3.51). Letting $\delta G=G^2-G^1$, $\delta\phi=\phi^2-\phi^1$

and so on, we have the estimates

$$\|\delta\beta\|_{L^\infty([0,T])} + \|\delta\phi\|_{L^\infty([0,T];H^{s+1})} \lesssim_M \|\delta G\|_{\mathfrak{E}'} + |\delta\beta_{in}|. \tag{5.3.54}$$

(iii) In addition to the assumptions in part (i), also assume G satisfies (5.3.4) *in Proposition 5.3.1. Then $\beta \in C^\infty([0,T])$ and $\phi \in C^\infty([0,T] \times \mathbf{T}^3)$.*

Proof. **Step 1 (proof of (i), sans continuity in time):** We first construct solutions to (5.3.50) obeying the estimates as in part (i). We shall show $(\beta, \phi) \in L^\infty([0,T]; \mathbf{R}_+ \times H^{s+2}(\mathbf{T}^3))$ for now, and prove continuity in a later step.

For existence and uniqueness of $(\beta, \phi) \in L^\infty([0,T]; \mathbf{R}_+ \times H^2(\mathbf{T}^3))$ solving (5.3.50), we state (a special case of) part (ii) of Theorem 1.14 in [13]: for any $0 \le f(x) \in L^2$ and $E > 0$, there exists a unique solution $(\gamma, \psi) \in \mathbf{R}_+ \times H^2(\mathbf{T}^3)$ to the system

$$\frac{3}{2\gamma} + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \psi|^2 dx = E, \tag{5.3.55}$$

$$-\Delta_x \psi = 4\pi(f(x) - e^{\gamma\psi}). \tag{5.3.56}$$

Moreover, for all $\gamma > 0$, we can find a unique $\psi \in H^2$ solving only the second line. Now, to construct (β, ϕ), we first take $\phi_{in} = \psi$ to solve (5.3.56) with $\gamma = \beta_{in}$. Now, ϕ_{in} exists uniquely in H^2.

In order to solve for (β, ϕ) for $t > 0$, we must ensure that the first line of (5.3.50), when integrated on $[0, t]$, gives an admissible condition on (β, ϕ). Specifically, we take $(\beta, \phi) = (\gamma, \psi)$ as in (5.3.55) and (5.3.56), with $f(t, x) = n_+(t, x)$. The quantity E, in accordance with (5.3.50), is given by

$$E(t) := \int_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2}(G_{in} - G(t,\cdot))dxdv + \frac{3}{2\beta_{in}} + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \phi_{in}|^2 dx. \tag{5.3.57}$$

Such a solution (β, ϕ) with $E = \frac{3}{2\beta} + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \phi|^2 dx$ exists as long as $E(t) > 0$. Now, by (5.3.51) and (5.3.52),

$$E(t) \geq -e^{-M} + \frac{3}{2}e^{-M} \geq \frac{1}{2}e^{-M}, \tag{5.3.58}$$

guaranteeing the desired condition.

Next, we prove that the assumptions (5.3.51) and (5.3.52) imply (5.3.53). First, as we have seen,

$$\frac{3}{2\beta} + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \phi|^2 dx = E(t) \geq \frac{1}{2}e^{-M} \tag{5.3.59}$$

Alternatively, $E(t) \lesssim_M 1$, so $|\ln(\beta)| \lesssim_M 1$.

Next, by the maximum principle,

$$\|\phi\|_{L^\infty_x} \leq \frac{1}{\beta}\|\ln(\int_{\mathbf{R}^3} Gdv)\|_{L^\infty_x} \lesssim_M 1. \tag{5.3.60}$$

Next,

$$\frac{1}{4\pi}|\nabla_x|^s \phi = |\nabla_x|^s\{\int_{\mathbf{R}^3} Gdv - e^{\beta\phi}\}. \tag{5.3.61}$$

Using Theorem 5.2.6 in [103],

$$\|e^{\beta\phi}\|_{H^s_x} \leq C_{\beta,\|\phi\|_{L^\infty_x}}(\|\phi\|_{H^s_x} + 1) \lesssim_M \|\phi\|_{H^s_x} + 1. \tag{5.3.62}$$

Hence,

$$\|\phi\|_{H_x^{s+2}} \lesssim_M 1 + \| \int_{\mathbf{R}^3} G dv \|_{H_x^s} + \|\phi\|_{H_x^s} \lesssim_M 1 + \|\phi\|_{H_x^{s+2}}^{\frac{s}{s+2}} \|\phi\|_{L_x^2}^{\frac{2}{s+2}}. \tag{5.3.63}$$

Now, using $\|\phi\|_{L_x^2} \lesssim \|\phi\|_{L_x^\infty} \lesssim_M 1$ and Young's inequality, we deduce $\|\phi\|_{H_x^{s+2}} \lesssim_M 1$.

Step 2 (control on differences, linearization): Before proving (ii) and (iii), we first demonstrate how to control differences in the system (5.3.55) and (5.3.56): first, define the map $\mathcal{F} : \mathbf{R}_+ \times H^{s+2}(\mathbf{T}^3) \to \mathbf{R}_+ \times H^s(\mathbf{T}^3)$, given by

$$\mathcal{F}(\gamma, \psi) = \left(\frac{3}{2\gamma} + \frac{1}{8\pi} \int_{\mathbf{T}^3} |\nabla_x \psi|^2 dx, \quad -\Delta_x \psi + 4\pi e^{\gamma\psi} \right) \tag{5.3.64}$$

Now, let $d\mathcal{F}_{(\gamma,\psi)}$ denote the Gateaux derivative of $\mathcal{F}$ at (γ, ψ), i.e. given $(\dot{\gamma}, \dot{\psi}) \in \mathbf{R} \times H^{s+2}(\mathbf{T}^3)$, define

$$d\mathcal{F}_{(\gamma,\psi)}(\dot{\gamma}, \dot{\psi}) = \left(-\frac{3\dot{\gamma}}{2\gamma^2} + \frac{1}{4\pi} \int_{\mathbf{T}^3} \nabla_x \psi \cdot \nabla_x \dot{\psi} dx, \quad -\Delta_x \dot{\psi} + 4\pi e^{\gamma\psi} (\dot{\gamma}\psi + \gamma\dot{\psi}) \right) \tag{5.3.65}$$

Clearly, for $(\gamma, \psi) \in \mathbf{R}_+ \times H^{s+2}$, this is a bounded linear operator in the sense $dF_{\gamma,\psi} \in \mathcal{L}(\mathbf{R} \times H^{s+2}, \mathbf{R} \times H^s)$. We can also show a lower bound on this operator,

$$\|(\dot{\gamma}, \dot{\psi})\|_{\mathbf{R} \times H^{s+2}} \leq C_{\gamma,\psi} \|d\mathcal{F}_{(\gamma,\psi)}(\dot{\gamma}, \dot{\psi})\|_{\mathbf{R} \times H^s} \tag{5.3.66}$$

More precisely, $C_{\gamma,\psi}$ is an increasing function of $|\ln(\gamma)|$ and $\|\psi\|_{H^{s+2}}$. To see this, first set $d\mathcal{F}_{(\gamma,\psi)}(\dot{\gamma}, \dot{\psi}) = (\dot{E}, \dot{f})$. We first solve for $\dot{\psi}$,

$$\dot{\psi} = (\gamma e^{\gamma\psi} - \frac{1}{4\pi} \Delta_x)^{-1} (\dot{f} - e^{\gamma\psi} \psi \dot{\gamma}). \tag{5.3.67}$$

Hence,

$$\dot{E} = -\frac{3\dot{\gamma}}{2\gamma^2} - \frac{1}{4\pi}\int_{\mathbf{T}^3} \Delta_x\psi(\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(\dot{f} - e^{\gamma\psi}\psi\dot{\gamma})dx \tag{5.3.68}$$

This allows us to solve for $\dot{\gamma}$:

$$\dot{\gamma} = -\frac{\dot{E} + \frac{1}{4\pi}\langle\Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}\dot{f}\rangle_{L^2}}{\frac{3}{2\gamma^2} - \frac{1}{4\pi}\langle\Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2}} \tag{5.3.69}$$

To then bound $|\dot{\gamma}| \leq C_{\gamma,\psi}\|(\dot{E}, \dot{f})\|_{\mathbf{R}\times H^s}$, it suffices to show the denominator is bounded from below by some $1/C_\gamma$. This follows by showing the second term in the denominator is nonnegative:

$$-\frac{1}{4\pi}\langle\Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2} \geq 0. \tag{5.3.70}$$

Indeed,

$$\begin{aligned}
& -\frac{1}{4\pi}\langle\Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} && (5.3.71)\\
&= \langle(\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} && (5.3.72)\\
&\quad - \gamma\langle e^{\gamma\psi}\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} && (5.3.73)\\
&= \langle\psi, e^{\gamma\psi}\psi\rangle_{L^2_x} + \gamma\langle e^{\gamma\psi}\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} \geq 0. && (5.3.74)
\end{aligned}$$

Now that we have bounded $\dot{\gamma}$, we rely on (5.3.67) to get $\|\dot{\psi}\|_{H^{s+2}} \leq C_{\gamma,\psi}\|(\dot{E}, \dot{f})\|_{\mathbf{R}\times H^s}$. We conclude (5.3.66) holds true.

We can use this bound to control finite differences of $\mathcal{F}$, as well. Consider two (γ_1, ψ_1) and (γ_2, ψ_2). For each $\lambda \in [1, 2]$, define $(\gamma_\lambda, \psi_\lambda) = (\lambda - 1)(\gamma_2, \psi_2) + (2 -$

$\lambda)(\gamma_1, \psi_1)$. Then, by the mean value theorem, there exists some $\rho \in [1, 2]$ such that

$$\mathcal{F}(\gamma_2, \psi_2) - \mathcal{F}(\gamma_1, \psi_1) = \int_1^2 \frac{d}{d\lambda}\mathcal{F}(\gamma_\lambda, \psi_\lambda) d\lambda = d\mathcal{F}_{(\gamma_\rho, \psi_\rho)}(\gamma_2 - \gamma_1, \psi_2 - \psi_1). \quad (5.3.75)$$

Now $|\ln(\gamma_\rho)| \lesssim \sup_{j\in\{1,2\}} |\ln(\gamma_j)|$ and $\|\psi_\rho\|_{H^{s+2}} \leq \sup_{j\in\{1,2\}} \|\psi_j\|_{H^{s+2}}$. Therefore,

$$\|(\gamma_2 - \gamma_1, \psi_2 - \psi_1)\|_{\mathbf{R}\times H^{s+2}} \leq (\sup_{j\in\{1,2\}} C_{\gamma_j,\psi_j})\|\mathcal{F}(\gamma_2, \psi_2) - \mathcal{F}(\gamma_1, \psi_1)\|_{\mathbf{R}\times H^s}. \quad (5.3.76)$$

This estimate will prove useful in the subsequent steps.

Step 3 (time continuity): Using the previous step, we complete the proof of (ii). For any two $t_1, t_2 \in [0, T]$, we have by (5.3.76)

$$|(\beta(t_2) - \beta(t_1), \phi(t_2, \cdot) - \phi(t_1, \cdot))\|_{\mathbf{R}\times H^{s+2}} \quad (5.3.77)$$

$$\lesssim_M \|(E(t_2) - E(t_1), n_+(t_2, \cdot) - n_+(t_1, \cdot))\|_{\mathbf{R}\times H^s}. \quad (5.3.78)$$

Recall (5.3.57) for the definition of E. Hence,

$$\|(\beta(t_2) - \beta(t_1), \phi(t_2, \cdot) - \phi(t_1, \cdot))\|_{\mathbf{R}\times H^{s+2}} \lesssim_M \|G(t_2, \cdot) - G(t_1, \cdot)\|_{\mathfrak{E}}. \quad (5.3.79)$$

Now since $G \in C([0, T]; \mathfrak{E})$, we conclude that $(\beta, \phi) \in C([0, T]; \mathbf{R}_+ \times H^{s+2})$. This concludes

Step 4 (proof of (ii)): Similarly as in the previous step, we apply (5.3.76). For each $t \in [0, T]$,

$$\|(\delta\beta(t), \delta\phi(t))\|_{\mathbf{R}\times H^{s+2}} \lesssim_M \|(\delta E(t), \delta n_+(t, \cdot))\|_{\mathbf{R}\times H^s} \quad (5.3.80)$$

Here,

$$\delta E(t) = E^2(t) - E^1(t), \tag{5.3.81}$$

$$E^j(t) = \int_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2}(G^j_{in} - G^j(t,\cdot))dxdv + \frac{3}{2\beta^j_{in}} + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \phi^j_{in}|^2 dx. \tag{5.3.82}$$

It is then straightforward to get

$$\|(\delta\beta(t), \delta\phi(t))\|_{\mathbf{R}\times H^{s+2}} \lesssim \|\delta G\|_{\mathfrak{E}'} + \|(\delta\beta_{in}, \delta\phi_{in})\|_{\mathbf{R}\times H^{s+2}} \tag{5.3.83}$$

Modifying the proof of (5.3.76) slightly, we can show that $\|\delta\phi_{in}\|_{H^{s+2}} \lesssim_M \|\delta G\|_{\mathfrak{E}'} + |\delta\beta|$, proving (5.3.54).

Step 5 (proof of (iii)): Since $n_+ \in C^\infty([0,T]\times\mathbf{T}^3)$, it follows that $\phi \in L^\infty([0,T]; C^\infty(\mathbf{T}^3))$ easily from the estimates in step 1.

It suffices to then show that all time derivatives of (β,ϕ) are bounded. Let $k \geq 0$. Then

$$d\mathcal{F}_{(\beta,\phi)}(\partial_t^k\beta, \partial_t^k\phi) = (\partial_t^k E, \partial_t^k n_+) + R_k. \tag{5.3.84}$$

where in R_k, we have terms involving the time derivatives of E, n_+, β and ϕ, *up to order* $k-1$. In particular, $\|R\|_{L^\infty}$ is controlled by some constant depending only on the C^{k-1} norms of E, n_+, β and ϕ. Note also that all derivatives of E and n_+ are both C^∞ on their respective domains.

Modifying step 2 slightly, it is straightforward to show that

$$\|(\partial_t^k\beta,\partial_t^k\phi)\|_{L^\infty([0,T])\times L^\infty([0,T]\times\mathbf{T}^3)} \leq C_M\|d\mathcal{F}_{(\beta,\phi)}(\partial_t^k\beta,\partial_t^k\phi)\|_{L^\infty([0,T])\times L^\infty([0,T]\times\mathbf{T}^3)}. \tag{5.3.85}$$

Hence, if $\partial_t^{k-1}\beta$ and $\partial_t^{k-1}\phi$ are both bounded on their respective domains, then so are $\partial_t^k\beta$ and $\partial_t^k\phi$. This allows us to induct on k, and conclude β and ϕ are both C^∞ on their respective domains. □

With Lemma 5.3.2 in hand, we are ready to prove part (ii) of Theorem 5.1.1.

Proof of Theorem 1.1-(ii): Some details shall be omitted, as the proof is similar to part (i) in certain aspects.

Once again, we let $F_{+,in}^N \in C_c^\infty(\mathbf{T}^3\times\mathbf{R}^3)$ be a nonnegative approximating sequence for $F_{+,in}$ in $\mathfrak{E}$. We also let $F_+^0(t,x,v)=\eta(v)$, where $\eta(v)$ is any nonnegative standard C^∞, compactly supported cutoff, $\beta^0=1$ and $\phi^0=0$. We also fix $0<M<\infty$ large enough so that

$$|\ln(\beta_{in})| + \|\frac{1}{n_{+,in}^N}\|_{L^\infty(\mathbf{T}^3)} + \|F_{+,in}^N\|_{\mathfrak{E}} \leq \frac{M}{2}. \tag{5.3.86}$$

We now describe the iteration scheme. Suppose we are given (F_+^N,β^N,ϕ^N) with $N\in\mathbf{N}_0$, with (F_+^N,ϕ^N) satisfying the same hypotheses as G and ϕ in Proposition 5.3.1, with $t\in[0,1]$, and $\beta\in C^\infty([0,1])$. We then construct $(F^{N+1},\beta^{N+1},\phi^{N+1})$. First, F_+^{N+1} is the solution to

$$(\partial_t + v\cdot\nabla_x - \nabla_x\phi^N)F_+^{N+1} = Q(F_+^N,F_+^{N+1}), \tag{5.3.87}$$

guaranteed by Proposition 5.3.1. As for $(\beta^{N+1}, \phi^{N+1})$, we would like this pair to solve (5.3.50) with $G = F_+^N$. Unfortunately Lemma 5.3.2 only guarantees existence for some small time.

To get around this, we let $T_N \in [0,1]$ be the largest time such that

$$\|\ln(\beta^N)\|_{L^\infty([0,T_N])} + \|\frac{1}{n_+}\|_{L^\infty([0,T_N]\times\mathbf{T}^3)} + \|F_+^N\|_{\mathfrak{E}_{T_N}\cap\mathfrak{D}_{T_N}} \le M, \tag{5.3.88}$$

$$\sup_{t\in[0,T]}\int_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_+^N(t,\cdot)-F_{+,in}^N)dxdv \le e^{-M}. \tag{5.3.89}$$

Clearly, these conditions imply that (5.3.51) and (5.3.52) both hold with $G = F_+^N$. Then, we let $(\underline{\beta}^{N+1}, \underline{\phi}^{N+1})$ be the unique smooth solution to (5.3.50) on the time interval $[0, T_N]$. We then take $0 \le \chi(t)$ to be a smooth cutoff function on $[0,\infty)$, which is identically 1 on $[0, \frac{1}{2}]$, and identically zero on $[1,\infty)$. We then define

$$\beta^{N+1}(t) = \chi(\frac{t}{T_N})\underline{\beta}^N(t) + (1-\chi(\frac{t}{T_N}))\beta_{in}, \tag{5.3.90}$$

$$\phi^{N+1}(t) = \chi(\frac{t}{T_N})\underline{\phi}^{N+1}(t). \tag{5.3.91}$$

so that $(\beta^{N+1}, \phi^{N+1})$ are smooth functions defined for all $t \in [0,1]$.

Next, we show boundedness of the iterates on some small time interval $[0,\tau]$ depending only on M. We let $T_N \in [0,1]$ be the largest time such that

$$\|\ln(\beta^N)\|_{L^\infty([0,T_N])} + \|\frac{1}{n_+}\|_{L^\infty([0,T_N]\times\mathbf{T}^3)} + \|F_+^N\|_{\mathfrak{E}_{T_N}\cap\mathfrak{D}_{T_N}} \le M. \tag{5.3.92}$$

We argue by induction that $\tau < T_N$. As before, the case $N = 0$ is trivial (taking M larger if necessary). Now, assume that for some N, we have $\tau < T_{N'}$ for all

$0 \leq N' \leq N$. As in the proof of part (i), we have

$$\|\frac{1}{n_+^N}\|_{L^\infty([0,\tau]\times\mathbf{T}^3)} + \|F_+^N\|_{\mathfrak{E}_\tau\cap\mathfrak{D}_\tau} \leq \frac{3}{2}\left(\|\frac{1}{n_{+,in}}\|_{L^\infty([0,\tau]\times\mathbf{T}^3)} + \|F_{+,in}^N\|_{\mathfrak{E}}\right). \tag{5.3.93}$$

On the other hand, repeating a computation in step 2 of the proof of Lemma 5.3.2, in particular the identity (5.3.69), we have for all $t \in [0, T_N]$ that

$$\dot{\underline{\beta}}^{N+1} = \frac{\partial_t \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2} F_+^N dxdv - \frac{1}{4\pi}\langle \Delta_x \underline{\phi}^{N+1}, (\underline{\beta}^{N+1} e^{\underline{\beta}^{N+1}\underline{\phi}^{N+1}} - \frac{1}{4\pi}\Delta_x)^{-1}\partial_t n_+^N\rangle_{L^2}}{\frac{3}{2(\underline{\beta}^{N+1})^2} - \frac{1}{4\pi}\langle \Delta_x \underline{\phi}^{N+1}, (\underline{\beta}^{N+1} e^{\underline{\beta}^{N+1}\underline{\phi}^{N+1}} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\underline{\beta}^{N+1}\underline{\phi}^{N+1}}\underline{\phi}^{N+1})\rangle_{L^2}} \tag{5.3.94}$$

With the estimates stated Lemma 5.3.2 in hand, and the positivity of the denominator (also shown in step 2 of the proof of the Lemma), we see

$$|\dot{\underline{\beta}}^{N+1}(t)| \lesssim_M |\iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2}\partial_t F_+^N(t,\cdot)dxdv| + \|\partial_t n_+^N(t,\cdot)\|_{L^\infty(\mathbf{T}^3)} \tag{5.3.95}$$

As shown in the proof of part (i), we can control $\|\partial_t n_+^N(t,\cdot)\|_{L^\infty(\mathbf{T}^3)} \lesssim_M 1$. As for the kinetic energy, we have either we are in the case $N = 0$, where $\partial_t F_+^N = 0$, or

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2}\partial_t F_+^N(t,\cdot)dxdv = \iint_{\mathbf{T}^3\times\mathbf{R}^3} v\cdot\nabla_x\phi^{N-1}F_+^N + \frac{|v|^2}{2}Q(F_+^{N-1}, F_+^N)dxdv. \tag{5.3.96}$$

The first term is easily bounded in terms of some C_M. As for the second, we have

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2}Q(F_+^{N-1}, F_+^N)dxdv = \iint_{\mathbf{T}^3\times\mathbf{R}^3} (\mathrm{tr}(\Phi * F^{N-1}) + 2v_i\partial_{v_j}\Phi * F^{N-1})F^N dxdv. \tag{5.3.97}$$

It's then clear that we can bound

$$|\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}\partial_t F_+^N(t,\cdot)dxdv|\lesssim_M 1. \tag{5.3.98}$$

We conclude that $\|\dot{\underline{\beta}}^{N+1}\|_{L^\infty([0,T_N])}\lesssim_M 1$. Thus,

$$\|\beta^{N+1}(t)-\beta_{in}\|_{L^\infty([0,\tau])}=\|\chi(\frac{t}{T_N})(\underline{\beta}^{N+1}(t)-\beta_{in})\|_{L^\infty([0,\tau])}\lesssim_M \tau. \tag{5.3.99}$$

In particular, we can take τ small enough depending on M so that

$$\|\ln(\beta^{N+1}(t))\|_{L^\infty([0,\tau])}\leq\frac{3}{2}|\ln(\beta_{in})|. \tag{5.3.100}$$

We thus have that

$$\|\ln(\beta^{N+1}(t))\|_{L^\infty([0,\tau])}+\|\frac{1}{n_+^N}\|_{L^\infty([0,\tau]\times\mathbf{T}^3)}+\|F_+^N\|_{\mathfrak{E}_\tau\cap\mathfrak{D}_\tau}\leq\frac{3}{2}M. \tag{5.3.101}$$

On the other hand, by (5.3.98), we can take τ small enough that

$$\sup_{t\in[0,\tau]}\int_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_+^N(t,\cdot)-F_{+,in}^N)dxdv\leq C_M\tau\leq\frac{e^{-M}}{2}. \tag{5.3.102}$$

We have thus shown that $\tau<T_{N+1}$, completing the inductive step. Hence, we have uniform boundedness of the iterates (F_+^N,β^N,ϕ^N) on $[0,\tau]$ in the sense of (5.3.88).

In a similar fashion as part (i), we can argue that these iterates converge to a weak solution to (5.1.5). Up to dividing τ by 2, we have that $(\beta^N,\phi^N)=(\underline{\beta}^N,\phi^N)$ on $t\in[0,\tau]$. Using Proposition 5.2.2 and part (ii) of Lemma 5.3.2, we can show that the increments of (F_+^N,β^N,ϕ^N) are summable in $\mathfrak{E}_\tau'\times L^\infty([0,\tau])\times L^\infty([0,\tau]\times H^{s+2})$, and thus Cauchy. Letting (F_+,β,ϕ) be the limit of this sequence, it is straightforward to show that it defines a weak solution. By Proposition 5.2.2 and Lemma 5.3.2, this solution is unique as well.

As τ depends only on M, we can continue the solution (F_+, β, ϕ) for some maximal interval $[0, T^*)$. As $T \uparrow T^*$, we have either $\|F_+\|_{\mathfrak{E}_T} \to \infty$, $\|1/n_+\|_{L^\infty([0,T]\times\mathbf{T}^3)} \to \infty$, or $\|\ln(\beta)\|_{L^\infty([0,T])} \to \infty$. In the third case, Clearly β is bounded uniformly from below, due to conserved energy involving $\frac{3}{2\beta}$. On the other hand, the conclusion of Corollary 4.4 in holds in [13] the case of our system as well, so we have β is uniformly bounded from above as well. Thus β cannot blow-up. $\square$

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