Higher Dimensional Birational Geometry: Moduli and Arithmetic

by

Kenneth Ascher

B.A., Stony Brook University, Stony Brook, NY, 2012

M.Sc., Brown University, Providence, RI, 2014

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While the study of algebraic curves and their moduli has been a celebrated subject in algebraic and arithmetic geometry, generalizations of many results that hold in dimension 1 to higher dimensions has been a difficult task, and the subject of much active research. This thesis is devoted to studying topics in the moduli and arithmetic of certain classes of higher dimensional algebraic varieties, known as pairs of log general type. Chapters 3 and 4 of this thesis are concerned with using birational geometry of higher dimensional algebraic varieties to study the arithmetic of pairs of log general type. Building upon work of Caporaso-Harris-Mazur, Hassett, and Abramovich, Chapter 3 provides a careful analysis of the geometry of families of pairs of log general type, in an attempt to study the sparsity of integral points on such pairs under the assumption of the Lang-Vojta conjecture. Chapter 4 studies uniform boundedness statements for heights on hyperbolic varieties of general type that follow from Vojta's height conjecture. The final chapter (Chapter 5) follows a more geometric direction – we classify the log canonical models of pairs of elliptic fibrations with weighted marked fibers using techniques from the minimal model program. This chapter serves as the first step in constructing compactifications of the moduli space of elliptic fibration pairs which is pursued separately.

This dissertation by Kenneth Ascher is accepted in its present form
by the Department of Mathematics as satisfying the
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Date________________ ________________________________

Dan Abramovich, Ph.D., Advisor

Recommended to the Graduate Council

Date________________ ________________________________

Brendan Hassett, Ph.D., Reader

Date________________ ________________________________

Nathan Pflueger, Ph.D., Reader

Approved by the Graduate Council

Date________________ ________________________________

Andrew G. Campbell, Dean of the Graduate School

Vitae

Kenneth Ascher grew up on Long Island and graduated from Oceanside High School in 2008. He earned his B.S. in mathematics from Stony Brook University in 2012. In May 2017 he completed his PhD in mathematics at Brown University under the supervision of Dan Abramovich.

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CONTENTS

CHAPTER **One**

Introduction

1.1 Moduli of stable pairs

The moduli space of smooth curves of genus g and its Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ have been the subject of active algebraic geometry research for many decades. Not only is this space compact, but there is also a very explicit description of the objects which appear on the boundary – *stable curves.* These curves are characterized by their singularities (at worst nodal), and by their positivity (ample canonical bundle, equivalently finitely many automorphisms). The ubiquity of $\overline{\mathcal{M}}_g$ in algebraic geometry naturally led to a desire to obtain a similar story for higher dimensions. That is, to construct geometrically meaningful, or modular compactifications of moduli spaces of higher dimensional algebraic varieties. The standard constructions for moduli spaces of curves relied on geometric invariant theory (GIT), which became a standard technique for constructing compact moduli spaces. However, when Gieseker [Gie] used GIT to construct the moduli space of smooth surfaces of general type, he observed that his construction did *not* provide a modular compactification. Later, Wang and Xu [WX, Theorem 3] showed that GIT will not provide a compactification of moduli spaces of varieties of general type. The idea of a variety being of *general type* (see Definition 2.0.2) will be ubiquitous throughout this thesis. For now, we will say that these varieties are of interest from the viewpoint of arithmetic and moduli, as they have many remarkable properties: e.g. they have finitely many automorphisms, and the d-canonical map is generically injective.

As GIT was not going to be feasible tool in higher dimensions, it became clear that additional techniques would be necessary to classify algebraic varieties. The minimal model program (MMP) emerged as a tool to find the "simplest" birational representative of an algebraic variety. The ideas of the MMP led Kollár & Shepherd-Barron [KSB] to determine the correct class of objects, stable surfaces, that should appear on the boundary of the moduli space of smooth surfaces of general type. These objects generalize the notion of stable curves: they satisfy an analogous singularity condition (so called semi-log canonical singularities, see Definition 2.0.8), and have the same positivity requirement. However, even with this discovery of boundary objects, we still lacked many foundational aspects of the theory – e.g. existence and projectivity of a compact moduli space for surfaces as well as higher dimensions. The current status is much improved thanks to several people over the past 30 years (Abramovich, Alexeev, Fujino, Hacon, Hassett, Kollár, Kovács, McKernan, Patakfalvi, Shepherd-Barron, Viehweg, Xu, etc.).

More generally, we are often interested in moduli spaces of *stable pairs* (see Definition 2.0.10). Knudsen introduced the moduli space of n-pointed smooth curves of genus g, and a compactification thereof: $\overline{\mathcal{M}}_{g,n}$. The idea here, is that instead of considering just equivalence classes of stable curves, we can enlarge the moduli problem by studying equivalence classes of stable curves with n chosen points. That is, construct compactifications of the space of pairs $(C, P = p_1 + \ldots + p_n)$, where C is a smooth curve of genus g and P is a divisor on C. One beautiful example of this approach was carried about by Hassett, in his work on weighted stable pointed curves [Has3], which gave one of the first instances of the interplay of birational geometry and moduli theory. This work serves as the inspiration for Chapter 5.

A natural question arises: can we construct compactifications of moduli spaces of pairs in higher dimensions? While there is a great deal of work in this area, there are still quite a lot of foundational questions that still need to be answered.

The notion of what stable pairs should generalize stable pointed curves was identified first for surfaces [KSB], and later by Alexeev in all dimensions [Ale]. Moreover, projectivity of the moduli space of recently proven by Kovács-Patakfalvi [KP]. There are, however, still many issues – e.g. what is the "right" definition of the moduli functor? Some of the major issues stem from an observation of Hassett [KP, Section 1.2], showing that a smooth pair (X, D) where X is a surface, can deform in a way so that the limit of the divisor D is *no longer* a divisor. It is thus illuminating to find concrete examples of moduli spaces of stable pairs which we hope will elucidate some of the more mysterious phenomena.

The tools coming from birational geometry and the minimal model program are used in all three chapters of this thesis, and these tools will be further discussed below. Before doing so, we introduce some motivation for this thesis coming from number theory (which inspire Chapters 3 and 4). After which we give a brief introduction to the three subjects of this thesis, and discuss how they relate to the tools provided by the minimal model program.

1.2 Arithmetic of higher dimensional algebraic varieties

We remarked above that varieties of general type are interesting from the moduli viewpoint. It turns out that they are also interesting from an arithmetic perspective. For instance, Faltings' Theorem on the finiteness of k-rational points holds for curves of general type defined over k (e.g. smooth curves of genus $g \geq 2$). Conjecturally (due to Bombieri, Lang, Vojta), varieties of general type in higher dimensions satisfy similar properties – their k-rational points are believed to be Zariski sparse. Part of this thesis is devoted to understanding the geometry of varieties of general type, or pairs of *log general type* (see Definitions 2.0.5 and 2.0.6), with a view towards applications in arithmetic. These include understanding the sparsity of rational and integral points, as well as control on the arithmetic complexity (heights) of such points. The viewpoints we take here, are via birational geometry and the minimal model program, as well as studying the notion of hyperbolicity for families of Deligne-Mumford stacks representable by schemes.

We now give a brief introduction to the three subjects of this thesis and put them in context of the MMP.

1.3 Fibered Powers & Uniformity

Faltings' Theorem [Fal2], states that a smooth curve of genus $g \geq 2$ defined over a number field k has finitely many k points. Recall that a smooth curve of genus $g \geq 2$ is a curve of *general type*, a notion of positivity for algebraic varieties which will be the subject of much of this thesis. There are many natural questions that arise from this celebrated theorem – how does the number of points vary in a flat family? What can be said for higher dimensional algebraic varieties? The former question was investigated by Caporaso, Harris, and Mazur [CHM], and the latter is an open conjecture due to Lang.

Conjecture 1.3.1 (Lang)**.** *Let X be a variety of general type over a number field k. Then $X(k)$ is not Zariski dense.*

In 1997, Caporaso, Harris, and Mazur, show in their celebrated paper [CHM], that various versions of Lang's conjecture imply uniformed boundedness of rational points on curves of general type, answering the first question above. More precisely, they show that assuming Lang's conjecture, for every number field k and integer $g \geq 2$, there exists an integer $B(k, g)$ such that no smooth curve of genus $g \geq 2$ defined over k has more than $B(k, g)$ rational points. Similar statements were proven for the case of surfaces of general type by Hassett [Has1], and eventually all positive dimensional varieties of general type in a series of two papers by Abramovich and Abramovich-Voloch ([Abr1] and [AV]). The essence of all of these papers, is a purely algebro-geometric statement: the proof of a "fibered power theorem", which analyzes the behavior of families of varieties of general type.

The main idea is, given a family of varieties of general type $f : X \to B$, can one construct a variety of general type W and relate it back to the total space X of the starting family? If this is true, Lang's conjecture gives control over the rational points of W, and thus control over the points of X. As alluded to in the beginning of the introduction, it is often natural to ask how results for varieties extend to results for pairs of a variety and a divisor. In this setting, the guiding question is to understand how *integral points* behave on pairs of *log general type*. In a nutshell, we want to understand the behavior of points on the complement of a divisor inside a variety with some positivity. A conjecture due to Lang-Vojta predicts that the set of integral points on a pair of log general type is not Zariski dense on any model of the pair. We thus ask, assuming this conjecture, can we prove uniform boundedness of integral points on pairs of log general type, generalizing the result of Caporaso-Harris-Mazur (and their generalizations) to the setting of pairs?

The first step is to prove the aforementioned fibered power theorem, which we prove using the machinery of stable pairs, the minimal model program, and recent work of Kovács-Patakfalvi. The main theorem (joint with Amos Turchet), which appears in two slightly different versions, is the following.

Theorem 1.1. *[AT1, Theorem 1.1] Let $(X, D) \to B$ be a family of stable pairs with integral and log canonical general fiber over a smooth projective variety B. Then after a birational modification of the base $\widetilde{B} \to B$, there exists an integer $n > 0$, a positive dimensional pair $(\widetilde{W}, \widetilde{\Delta})$ of log general type, and a morphism $(\widetilde{X}^n_B, \widetilde{D}_n) \to (\widetilde{W}, \widetilde{\Delta})$.*

Theorem 1.2. *[AT1, Theorem 1.2] Let $(X, D) \to B$ be a family of stable pairs with integral and openly log canonical general fiber over a smooth projective variety B. Then there exists an integer $n > 0$, a positive dimensional pair (W, Δ) openly of log general type, and a morphism $(X^n_B, D_n) \to (W, \Delta)$.*

We note that the term *openly of log general type* is not quite standard, but will be introduced and motivated in the following chapter (see Definition 2.0.6 and 2.0.7).

This methods used in this chapter require a careful understanding of the geometry and moduli of stable pairs – both their positivity and singularities. In an upcoming paper [AT2], we apply this theorem to prove uniform boundedness results for integral points on pairs of log general type, assuming the conjecture of Lang & Vojta.

1.4 Height Uniformity

Yet another open conjecture in arithmetic geometry is Vojta's conjecture, about heights of points on algebraic varieties over number fields. The precise statement of the conjecture is as follows:

Conjecture 1.4.1 (Vojta). *[Voj, Conjecture 2.3] Let X be a nonsingular projective variety over a number field k. Let H be a big line bundle on X and fix $\delta > 0$. Then there exists a proper Zariski closed subset $Z \subset X$ such that, for all closed points $x \in X$ with $x \notin Z$,*

$$h_{K_X}(x) - \delta h_H(x) \leq d_k(k(x)) + O(1).$$

For a definition of the discriminant $d_k(k(x))$ see Section 4.2.4.

In fact, if X is a variety of general type, then Vojta's conjecture implies Lang's conjecture. As we have seen (e.g. [CHM]) that Lang's conjecture implies a uniform version of Lang's conjecture, it is natural to ask if Vojta's conjecture implies a similar uniformity statement for height bounds. Indeed, Su-Ion Ih has shown [Ih1] that Vojta's height conjecture implies that the height of a rational point on a smooth curve of general type is bounded uniformly in families. Ih later showed in [Ih2] that the same is true for integral points on elliptic curves. The goal of my work (joint with Ariyan Javanpeykar) is to generalize Ih's results in [Ih1] by investigating consequences of Vojta's height conjecture for families of (algebraically) hyperbolic varieties of general type – varieties where all integral subvarieties of X are of general type.

Our main result is as follows. As a note to the reader, to state our theorem, we will use heights on stacks as discussed in Section 4.2.4.

Theorem 1.3. *[AJ] Let k be a number field and let $f\colon X \to Y$ be a proper surjective morphism of proper Deligne-Mumford stacks over k which is representable by schemes. Let h be a height function on X and let h_Y be a height function on Y associated to an ample divisor with $h_Y \geq 1$. Assume Vojta's height conjecture (Conjecture 1.4.1). Let $U \subset Y$ be a constructible substack such that, for all $t \in U$, the variety X_t is smooth and hyperbolic. Then there is a real number $c > 0$ depending only on k, Y, X, and f such that, for all P in $X(k)$ with $f(P)$ in U, the following inequality holds*

$$h(P) \leq c \cdot \big(h_Y(f(P)) + d_k(\mathcal{T}_P)\big).$$

See Section 4.2.4 for a description of $d_k(\mathcal{T}_P)$. By *constructible substack*, we mean a substack of an algebraic stack that is a finite union of locally closed substacks.

Note that one cannot expect uniform height bounds in the naive sense. Indeed, for all $P \in \mathbb{P}^2(\mathbb{Q})$ and all $d \geq 4$, there is a smooth curve X of degree d in $\mathbb{P}^2_{\mathbb{Q}}$ with $P \in X(\mathbb{Q})$. Thus, for all $d \geq 4$, there is no real number $c > 0$ depending only on d such that for all smooth degree d hypersurfaces $X \subset \mathbb{P}^2_{\mathbb{Q}}$ and all $P \in X(\mathbb{Q})$ the inequality $h(P) \leq c$ holds. In particular, there is no real number $c > 0$ such that for all smooth quartic hypersurfaces $X \subset \mathbb{P}^2_{\mathbb{Q}}$ and all $P \in X(\mathbb{Q})$ the inequality $h(P) \leq c$ holds. On the other hand, Lang's conjecture on rational points of varieties of general type implies that there is a real number $c > 0$ such that the cardinality of $X(\mathbb{Q})$ is at most c [CHM].

Moreover, one also cannot expect a stronger uniformity type statement for heights on (not necessarily hyperbolic) varieties of general type. Indeed, if k is a number field and $f\colon X \to Y$ is a smooth proper morphism of k-schemes whose geometric fibres are varieties of general type and t is a point in Y such that X_t contains a copy of $\mathbb{P}^1_{k(t)}$, then there is no real number $c > 0$ such that for all $P \in X_t$, the inequality $h(P) \leq c \cdot h_Y(f(P))$ holds.

Our proof of Theorem 1.3 uses the recent [AMV], which shows that Vojta's conjecture actually implies a version of the conjecture for stacks.

We argue that it is more natural to work in the stacks setting, as this allows us to apply our results to moduli stacks of hyperbolic varieties, thus obtaining more concrete results. In fact, as a first corollary of Theorem 1.3 we obtain the following uniformity statement for curves.

Theorem 1.4. *[AJ] Assume Conjecture 1.4.1. Let $g \geq 2$ be an integer and let k be a number field. There is a real number c depending only on g and k satisfying the following. For all smooth projective curves X of genus g over k, and all P in $X(k)$, the following inequality holds*

$$h(P) \leq c(g,k) \cdot \big(h(X) + d_k(\mathcal{T}_X)\big).$$

Finally, we also obtain a uniformity statement for certain hyperbolic surfaces.

Theorem 1.5. *[AJ] Assume Conjecture 1.4.1. Fix an even integer a and a number field k. There is a real number c depending only on a and k satisfying the following. For all smooth hyperbolic surfaces S over k with $c_1^2(S) = a > c_2(S)$ and all P in $S(k)$, the following inequality holds*

$$h(P) \leq c \cdot \big(h(S) + d_k(\mathcal{T}_S)\big).$$

Theorem 1.3 applies to any family of varieties of general type for which the locus of hyperbolic varieties is constructible on the base. However, verifying the constructibility of the latter locus is not straightforward.

We note that a conjecture of Lang (see [Lan2]) asserts that our notion of hyperbolicity for X is equivalent to being *Brody hyperbolic*, i.e., that there are no non-constant holomorphic maps $f : \mathbb{C} \to X(\mathbb{C})$. In particular, as the property of being Brody hyperbolic is open in the analytic topology [Bro], Lang's conjecture implies that the property of being hyperbolic is open in the analytic topology. In particular, assuming Lang's conjecture, if the locus of smooth projective hyperbolic surfaces is constructible in the moduli stack of smooth canonically polarized surfaces, then [SGA, Exposé XII, Corollaire 2.3] implies that it is (Zariski) open.

1.5 Log canonical models of elliptic surfaces

As mentioned in the beginning of the introduction, one goal of the MMP is to find distinguished birational models for algebraic varieties – in dimension one these are the unique smooth projective models, but in higher dimensions we are led to minimal and canonical models which may have mild singularities. More generally, the log minimal model program takes as input a pair (X, D) consisting of a variety and a divisor with mild singularities and outputs a log minimal or log canonical model of the pair. Log canonical models and their non-normal analogues, semi-log canonical (slc) models, are the higher dimensional generalization of stable curves and lend themselves to admitting compact moduli spaces. Thus to compactify a moduli space using the MMP, we must first determine the log canonical models.

Inspired by La Nave's explicit stable reduction of elliptic surface pairs (X, S)

in [LN], we give a classification of the log canonical models of elliptic surface pairs $(f : X \to C, S + F_{\mathcal{A}})$, where $f : X \to C$ is an elliptic fibration, S is a chosen section, and $F_{\mathcal{A}}$ is a weighted sum of reduced marked fibers $F_{\mathcal{A}} = \sum a_i F_i$. We define an elliptic fibration as a surjective proper flat morphism $f : X \to C$ from an irreducible surface X to a proper smooth curve C with section S such that the generic fiber of f is a stable elliptic curve. The goal of a joint research program with Dori Bejleri, is to study the birational geometry of the moduli space of elliptic surfaces provided by the minimal model program and twisted stable maps (see also [AB2] and [AB3], which do not appear in this thesis).

Our first step, which is the subject of Chapter 5, is a complete description of the log canonical models building on the classification of singular fibers of minimal elliptic surfaces given by Kodaira and Nerón. The goal is to explicitly describe how the log canonical models of elliptic surface pairs depend on the choice of the weight vector $\mathcal{A}$. Drawing inspiration from the Hasset-Keel program for $\overline{\mathcal{M}}_{g,n}$, we will use these results to understand how the geometry of compactified moduli spaces of slc elliptic surface pairs vary as we change the weight vector $\mathcal{A}$ [AB3].

Our first main result is the following classification (see Figure 1.1). See Definition 5.4.9 for the definitions of twisted and intermediate fibers.

Theorem 1.6. *[AB1] Let $(f : X \to C, S + aF)$ be an elliptic surface pair over C the spectrum of a DVR with reduced special fiber F such that F is one of the Kodaira singular fiber types or f is isotrivial with constant j-invariant ∞.*

(i) If F is a type I_n fiber, the relative log canonical model is the Weierstrass model for all $0 \leq a \leq 1$.

(ii) For any other fiber type, there is an a_0 so the relative log canonical model is

(a) the Weierstrass model for any $0 \leq a \leq a_0$*,*

(b) a twisted fiber *consisting of a single non-reduced component when* $a = 1$*, or*

(c) an intermediate fiber *that interpolates between the above two models for any* $a_0 < a < 1$*.*

The constant $a_0 = 0$ *for fibers of type* I_n^*, II^*, III^* *and* IV^**, and* a_0 *is as follows for the other fiber types:*

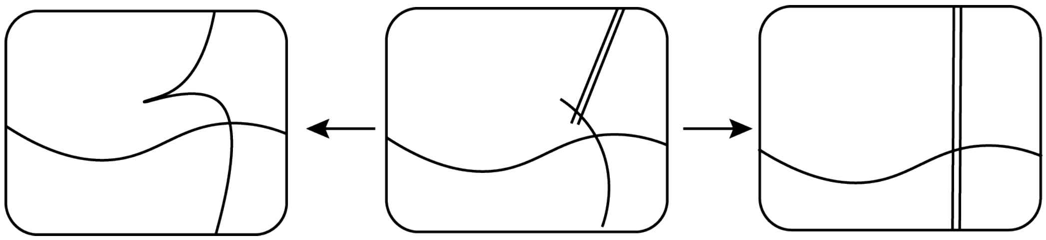
$$a_0 = \begin{cases} 5/6 & II \\ 3/4 & III \\ 2/3 & IV \end{cases}$$

We also describe the singularities of the relative log canonical models in each case.

Theorem 1.6 allows us to run the log minimal model program for $(X, S + F_{\mathcal{A}})$ relative to the map $f : X \to C$ to produce a relative log canonical (or *relatively stable*) model over the curve C. Indeed this question is local on the target so it reduces to the case $(X, S + aF)$ where $f : X \to C$ is an elliptic fibration over the spectrum of a DVR and F is the reduced special fiber. When the generic fiber of f is smooth, F is one of the singular fibers in Kodaira's classification. When the generic fiber of f is a nodal elliptic curve, the fibration f must be isotrivial with constant j-invariant ∞ and we classify the singular fibers by explicitly using their Weierstrass models.

In [LN], La Nave studied degenerations of *Weierstrass elliptic surfaces.* The approach was to replace any cuspidal fibers with twisted fibers, study degenerations using twisted stable maps of Abramovich-Vistoli, and then reinsert cuspidal fibers to obtain Weierstrass models. Theorem 1.6 can be seen as a generalization of La Nave's gluing procedure, which shows that instead the log minimal model program naturally interpolates between the Weierstrass and twisted fibers. Using the local computation

Figure 1.1: Transitions between (left to right): Weierstrass, intermediate, and twisted fibers.



of relative log canonical models we generalize the elliptic surface canonical bundle formula to elliptic surface pairs.

Theorem 1.7. *[AB1] Let $f : X \to C$ be an elliptic fibration with section S. Furthermore, let $F_{\mathcal{A}} = \sum a_i F_i$ be a sum of reduced marked fibers F_i with $0 \leq a_i \leq 1$. Suppose that $(X, S + F_{\mathcal{A}})$ is the relative log canonical model over C. Then*

$$\omega_X = f^*(\omega_C \otimes \mathbb{L}) \otimes \mathcal{O}_X(\Delta).$$

where $\mathbb{L}$ is the fundamental line bundle (see Definition 5.3.4) and Δ is an effective divisor supported on fibers of type II, III, and IV contained in $\mathrm{Supp}(F)$*. The contribution of a type II, III or IV fiber to Δ is given by αE where E supports the unique nonreduced component of the fiber and*

$$\alpha = \begin{cases} 4 & II \\ 2 & III \\ 1 & IV \end{cases}$$

In continuing the log minimal model program on the relatively stable pair $(X \to C, S + F_{\mathcal{A}})$, sometimes the section S is contracted. This was first observed by La

Nave when studying compactifications of the space of Weirstrass models (i.e. $a_i = 0$ for all i) [LN]. La Nave called the result of such a contraction a *pseudoelliptic surface*.

In Proposition 5.6.4 we compute the formula

$$(K_X + S + F_{\mathcal{A}}).S = 2g - 2 + \sum a_i$$

using [LN, Proposition 4.3.2] where $g = g(C)$ is the genus of the base curve. The section gets contracted by the log minimal model program precisely when $(K_X + S + F_{\mathcal{A}}).S \leq 0$.

It follows that the section does *not* get contracted if and only if the base curve is a Hassett *weighted stable pointed curve* [Has2] (Definition 5.6.5) with marked points $\sum a_i p_i$ where $p_i = f_* F_i$. In particular, the log minimal model program results in a pseudoelliptic only when $C \cong \mathbb{P}^1$ and $\sum a_i \leq 2$, or when C is an elliptic curve and $a_i = 0$ for all i.

Corollary 1.5.1. *Let $(f : X \to C, S + F_{\mathcal{A}})$ be an irreducible slc elliptic surface with section S and marked fibers $F_{\mathcal{A}}$. Suppose that $K_X + S + F_{\mathcal{A}}$ is big. Then the log canonical model of $(X, S + F_{\mathcal{A}})$ is either*

(i) the relative log canonical model as described in Theorem 1.6, or

(ii) a pseudoelliptic surface obtained by contracting the section of the relative log canonical model whenever $(C, f_* F_{\mathcal{A}})$ is not a weighted stable pointed curve (see Definition 5.6.5).

When $K_X + S + F_{\mathcal{A}}$ is *not* big, there is a log canonical contraction mapping the surface to a curve or point. We describe this with respect to a classification based on $\mathbb{L}$.

Furthermore, we present a wall-and-chamber decomposition of the space of weight vectors $\mathcal{A}$. The log canonical model of $(X \to C, S + F_{\mathcal{A}})$ remains the same within each chamber and we describe how it changes across each wall. Through an explicit example, we demonstrate a rational elliptic surface pair that exhibits each of these transitions.

CHAPTER **Two**

Background

2.0.1 Birational Geometry

Definition 2.0.1. *A line bundle $\mathcal{L}$ on a proper variety X is called **big** if the global sections of $\mathcal{L}^m$ define a birational map for $m > 0$. A Cartier divisor D is called big if $\mathcal{O}_X(D)$ is big.*

Definition 2.0.2. *A proper variety X is of **general type** if for any desingularization $\tilde{X} \to X$, the line bundle $\omega_{\tilde{X}}$ is big.*

From the point of view of birational geometry and the minimal model program, it has become convenient and standard to work with pairs. We define a pair (X, D) to be a variety X along with an effective $\mathbb{R}$-divisor $D = \sum d_i D_i$ which is a linear combination of distinct prime divisors.

Definition 2.0.3. *Let (X, D) be a pair where X is a normal variety and $K_X + D$*

is $\mathbb{Q}$-Cartier. Suppose that there is a log resolution $f : Y \to X$ such that

$$K_Y + \sum a_E E = f^*(K_X + D),$$

where the sum goes over all irreducible divisors on Y. We say that (X, D) is:

- ***canonical** if all $a_E \leq 0$,*
- ***log canonical** (lc) if all $a_E \leq 1$, and*
- ***Kawamata log terminal** (klt) if all $a_E < 1$.*

Remark 2.0.4. *In particular, for a klt pair, the coefficients d_i in the decomposition $D = \sum d_i D_i$ are all* strictly < 1. *Similarly, for a lc pair, the coefficients are $\leq$ 1.*

In what follows, we give the two definitions used for pairs (openly) of log general type.

Definition 2.0.5. *A pair (X, D) of a proper variety X and an effective $\mathbb{Q}$-divisor D is of **log general type** if:*

- *(X, D) has log canonical singularities and*
- *$\omega_X(D)$ is big.*

For applications to arithmetic (for instance in the upcoming [AT2]), it will be useful to consider the following.

Definition 2.0.6. *Let X be a quasi-projective variety and let $\widetilde{X} \to X$ be a desingularization. Let $\widetilde{X} \subset Y$ by a projective embedding and suppose $D = Y \setminus \widetilde{X}$ is a divisor of normal crossings. Then X is **openly of log general type** if $\omega_Y(D)$ is big.*

Note that this second definition is independent of both the choice of the desingularization as well as the embedding; it is also a birational invariant. Moreover, Definitions 2.0.5 and 2.0.6 are equivalent if the pair (X, D) has log canonical singularities and one considers the variety $X \setminus (D \cup \mathrm{Sing}(X))$.

Just to reiterate, we will refer to Definition 2.0.5 by saying (X, D) is a *pair of log general type*. We will refer to Definition 2.0.6 by stating that the pair is *openly of log general type*, as the definition is motivated by considering the complement $X \setminus D$. Throughout the course of this paper, we will take care to specify which definition we are using.

Definition 2.0.7. *By* openly canonical*, we mean that the variety $X \setminus D$ has canonical singularities.*

Definition 2.0.8. *A pair $(X, D = \sum d_i D_i)$ is **semi-log canonical** (slc) if X is reduced, $K_X + D$ is $\mathbb{Q}$-Cartier and:*

(i) The variety X satisfies Serre's condition S2,

(ii) X is Gorenstein in codimension 1, and

(iii) if $\nu : X^\nu \to X$ is the normalization, then the pair $(X^\nu, \sum d_i \nu^{-1}(D_i) + D^\nu)$ is log canonical, where D^ν denotes the preimage of the double locus on X^ν.

Remark 2.0.9. *Semi-log canonical singularities can be thought of as the extension of log canonical singularities to non-normal varieties. The only difference is that a log resolution is replaced by a* good semi-resolution *(see Definitions 5.2.2 and 5.2.3).*

Definition 2.0.10. *A pair (X, D) of a proper variety X and an effective $\mathbb{Q}$-divisor D, is a **stable pair** if:*

- *The $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\omega_X(D)$ is ample and*

- *The pair (X, D) has semi-log canonical singularities*

Definition 2.0.11. *An **slc family** is a flat morphism $f : (X, D) \to B$ such that for all $m \in \mathbb{Z}$:*

- *each fiber (X_b, D_b) is an slc pair,*
- *$\omega_f(D)^{[m]}$ is flat (see Definition 3.2.1), and*
- *(Kollár's Condition) for every base change $\tau : B' \to B$, given the induced morphism $\rho : (X_{B'}, D_{B'}) \to (X, D)$ we have that the natural homomorphism*

$$\rho^*(\omega_f(D)^{[m]}) \to \omega_{f'}(D)^{[m]}$$

is an isomorphism

*We say that $f : (X, D) \to B$ is a **stable family** if in addition to the above, each (X_b, D_b) is a stable pair. Equivalently, $K_{X_b} + D_b$ is ample for every $b \in B$.*

CHAPTER **Three**

Fibered Power Theorem for Pairs of Log General Type

3.1 Introduction

We work over an algebraically closed field of characteristic 0.

Recall from the introduction, that our main goal is to prove the following two theorems:

Theorem 3.1 (Theorem 1.1)**.** *Let $(X, D) \to B$ be a stable family with integral and log canonical general fiber over a smooth projective variety B. Then after a birational modification of the base $\widetilde{B} \to B$, there exists an integer $n > 0$, a positive dimensional pair $(\widetilde{W}, \widetilde{\Delta})$ of log general type, and a morphism $(\widetilde{X}^n_B, \widetilde{D}_n) \to (\widetilde{W}, \widetilde{\Delta})$.*

Theorem 3.2 (Theorem 1.2)**.** *Let $(X, D) \to B$ be a stable family with integral, openly canonical, and log canonical general fiber (see Definition 2.0.7) over a smooth projective variety B. Then there exists an integer $n > 0$, a positive dimensional pair*

(W, Δ) openly of log general type, and a morphism $(X^n_B, D_n) \to (W, \Delta)$.

To prove Theorem 1.1, we show that it suffices to prove the following:

Theorem 3.3 (See Theorem 3.11)**.** *Let $(X, D) \to B$ be a stable family with integral and log canonical general fiber over a smooth projective variety B. Suppose that the variation of the family f is maximal (see Definition 3.2.2). Let G be a finite group such that $(X, D) \to B$ is G-equivariant. Then there exists an integer $n > 0$ such that the quotient of the pair by a finite group of automorphisms, $(X^n_B/G, D_n/G)$ is of log general type.*

Similarly, to prove Theorem 1.2, it suffices to prove the following:

Theorem 3.4 (See Theorem 3.10)**.** *Let $f : (X, D) \to B$ be a stable family with integral, openly canonical, and log canonical general fiber over a smooth projective variety B. Suppose that the variation of the family f is maximal. Let G be a finite group such that $(X, D) \to B$ is G-equivariant. Then there exists an integer $n > 0$ such that the quotient $(X^n_B/G, D_n/G)$ is openly of log general type.*

We then obtain Theorem 1.2 by means of Theorem 1.1 and Theorem 3.11. More specifically, we show that there is a birational transformation from $(\widetilde{W}, \widetilde{\Delta}) \to (W, \Delta)$, such that $(\widetilde{W}, \widetilde{\Delta})$ manifests (W, Δ) as a pair openly of log general type.

The main tool of this paper is a recent result of Kovács-Patakfalvi which says that given a stable family with maximal variation $f : (X, D_\varepsilon) \to B$ where the general fiber is Kawamata log terminal (klt), then for large m the sheaf $f_*(\omega_f(D_\varepsilon))^m$ is *big* [KP, Theorem 7.1]. Here, the divisor D_ε denotes the divisor with lowered coefficients $(1-\varepsilon)D$ for a small rational number ε. Unfortunately the result of [KP] does *not* hold for log canonical pairs (see Example 7.5 of [KP]). As a result, since D is not assumed to be $\mathbb{Q}$-Cartier, one obstacle of this paper is showing that bigness of $\omega_{X_B^n}(D_{\varepsilon,n})$ for some n large enough allows you to conclude bigness of $\omega_{X_B^n}(D_n)$. To do so, one must first take a $\mathbb{Q}$-factorial dlt modification, followed by a relative log canonical model. The ideas here are present in Propositions 2.9 and 2.15 of [PX]. See Remark 3.3.4 below for a more detailed discussion.

Finally, we must guarantee that the fibered powers are not too singular. A priori, it is unclear if taking high fibered powers to ensure the positivity of $\omega_{X_B^n}(D_n)$ leads to a pair with good singularities. This is ensured by the following statement:

Proposition 3.1.1 (See Proposition 3.4.4)**.** *Let $f : (X, D) \to B$ be a stable family with integral and log canonical general fiber over a smooth projective variety B. Then for all $n > 0$ the fibered powers (X_B^n, D_n) have log canonical singularities.*

This statement also works after taking quotients by finite groups of automorphisms:

Proposition 3.1.2 (See Corollary 3.4.8)**.** *Let $f : (X, D) \to B$ be an slc family with integral and log canonical general fiber over a smooth projective variety B. Then for n large enough, the quotient pair $(X_B^n/G, D_n/G)$ also has log canonical singularities.*

In fact, although we do not use this result, we prove:

Proposition 3.1.3 (See Proposition 3.4.6)**.** *The total space of the fiber product of stable families over a stable base is stable.*

The main result we seek then follows via the above methods after applying standard tools from moduli theory.

Outline

We begin with some preliminary definitions and notation. In Section 3.3, we prove that $\omega_{X^n_B}(D_n)$ is big for a stable family of maximal variation with log canonical general fiber. We prove that the fibered power theorem holds for max variation families when the general fiber is both openly canonical and log canonical. Next in Section 3.4, we prove some results on singularities, namely we analyze the singularities of fibered powers and study the effect of group quotients. We prove the fibered power theorem for log canonical general fiber in the case of max variation. Finally, in Section 3.5 we prove the full fibered power theorems by reducing to families of maximal variation.

3.2 Preliminaries

We assume that all of our families (see Definition 2.0.11) of pairs satisfy *Kollár's condition*. Let X be a variety and $\mathcal{F}$ an $\mathcal{O}_X$-module. The dual of $\mathcal{F}$ is denoted $\mathcal{F}^\star := \mathcal{H}om_X(\mathcal{F}, \mathcal{O}_X)$.

Definition 3.2.1. *the m-th reflexive power of $\mathcal{F}$ to be the double dual (or reflexive hull) of the m-th tensor power of $\mathcal{F}$:*

$$\mathcal{F}^{[m]} := (\mathcal{F}^{\otimes m})^{\star\star}.$$

3.2.1 Moduli space of stable pairs

Constructing the moduli space of stable pairs, denoted below by $\overline{M}_h$, has been a difficult task. A discussion of the construction of the moduli space $\overline{M}_h$ is not necessary for this paper, but for sake of completeness we note that there exists a finite set of constants, which we denote by h, that allows for a compact moduli space. As long as the coefficients d_i appearing in the divisor decomposition are all $> \frac{1}{2}$, there are no issues and we do in fact have a well defined moduli space. There is no harm in assuming this outright.

We refer the reader to [Kol1] or to the introduction of [KP] for more details.

3.2.2 Variation of Moduli

Given a stable family $f : (X, D) \to B$, we obtain a *canonical morphism*:

$$\varphi : B \to \overline{M}_h$$

sending a point $b \in B$ to the point of the moduli space $\overline{M}_h$ of stable pairs, classifying the fiber (X_b, D_b). This motivates the following definition.

Definition 3.2.2. *A family has **maximal variation** of moduli if the corresponding canonical morphism is generically finite.*

Equivalently, the above definition means that the family is a truly varying family, diametrically opposed to one which is *isotrivial*, where the fibers do not vary at all.

3.2.3 Notation

Given a morphism of pairs $f : (X, D) \to B$, we denote by (X^n_B, D_n) the unique irreducible component of the nth fiber product of (X, D) over B dominating B. We define D_n to be the divisor $D_n := \sum_{i=1}^n \pi_i^*(D)$ where the maps $\pi_i : (X^n_B, D_n) \to B$ denote the projections onto the ith factors. We denote by f_n the maps $f_n : (X^n_B, D_n) \to B$. Finally, we denote by D_ε the divisor $(1-\varepsilon)D$ and by $D_{\varepsilon,n}$ the sum $D_{\varepsilon,n} := \sum_{i=1}^n \pi_i^*(D_\varepsilon)$.

3.3 Positivity of the relative anti-canonical sheaf

Recall that to prove that the pair (X^n_B, D_n) is a *pair of log general type*, we must show that

(a) $\omega_{X^n_B}(D_n)$ is big and

(b) The pair (X^n_B, D_n) has log canonical singularities.

We also remind the reader that we will demonstrate in Section 3.4 that Theorem 3.11 implies Theorem 1.1. Therefore, in this section we will assume that the variation of our family is maximal. More precisely, the goal of this section is to prove the following proposition, tackling (a) of the above definition:

Proposition 3.3.1. *Let $f : (X, D) \to B$ be a stable family with maximal variation over a smooth, projective variety B with integral and log canonical general fiber, then for n sufficiently large, the sheaf $\omega_{X^n_B}(D_n)$ is big.*

As mentioned in the introduction, we will prove the above proposition by means of the following slightly weaker statement:

Proposition 3.3.2. *Let $f : (X, D_\varepsilon) \to B$ be a stable family with maximal variation over a smooth, projective variety B with klt general fiber, then for n sufficiently large, the sheaf $\omega_{X^n_B}(D_{\varepsilon,n})$ is big.*

Our proof of Proposition 3.3.2 requires the recent Theorem of Kovács and Patakfalvi:

Theorem 3.5 ([KP, Theorem 7.1, Corollary 7.3])**.** *If $f : (X, D_\varepsilon) \to B$ is a stable family with maximal variation over a normal, projective variety B with klt general fiber, then $f_*(\omega_f(D_\varepsilon)^m)$ is big for m large enough. Moreover, $\omega_f(D_\varepsilon)$ is big.*

Let $S^{[n]}$ denote the reflexive hull of the nth symmetric power of a sheaf. Then the above theorem is equivalent to saying that, under the hypotheses, for any ample line bundle H on B there exists an integer n_0 such that

$$S^{[n_0]}(f_*(\omega_f(D_\varepsilon)^m) \otimes H^{-1} \tag{3.1}$$

is generically globally generated. We desire to show that this implies Proposition 3.3.2.

The proof of this statement essentially follows from Proposition 5.1 of [Has1], but we include the proof for completeness to show how it extends to the case of pairs. We begin with a lemma:

Lemma 3.3.3. *Let $f : (X, D) \to B$ be a stable family over a smooth projective variety B such that the general fiber has log canonical singularities. Then for all*

$n > 0$, the following formula holds:

$$\omega_{X_B^n}(D_n)^{[m]} = \pi_1^*\omega_f(D)^{[m]} \otimes \cdots \otimes \pi_n^*\omega_f(D)^{[m]} \otimes f_n^*\omega_B^m.$$

Proof. Recall that the relative dualizing sheaf satisfies the following equation:

$$\omega_{f_n}(D_n) = \pi_1^*\omega_f(D) \otimes \cdots \otimes \pi_n^*\omega_f(D)$$

where π_j denotes the projection $\pi_j : X^n \to X$ to the jth factor. Since B is smooth we obtain:

$$\omega_{X_B^n}(D_n)^{[m]} = \omega_{f_n}(D_n)^{[m]} \otimes f_n^*\omega_B^m.$$

Since $f : (X, D) \to B$ is a stable family, there exists an integer m such that for all $b \in B$, the sheaf $\omega_f(D)^{[m]}|_{X_b}$ is locally free. Moreover, since this sheaf is locally free on each fiber, $\omega_f(D)^{[m]}$ is also locally free for this m. We claim that the following holds:

$$\omega_{X_B^n}(D_n)^{[m]} = \pi_1^*\omega_f(D)^{[m]} \otimes \cdots \otimes \pi_n^*\omega_f(D)^{[m]} \otimes f_n^*\omega_B^m.$$

Both sides of the equation are reflexive – the left hand side by construction, and the right hand side because it is the tensor product of locally free sheaves. Therefore, to prove the equivalence, we must show the two sides agree on an open set whose complement has codimension at least two. Consider the locus consisting of both the general fibers, which are log canonical and hence $\mathbb{Q}$-Gorenstein, as well as the nonsingular parts of the special fibers. Note that the complement of this locus is of codimension at least two, because the singular parts of the special fiber are of codimension one, thus of at least codimension two in the total space. □

We will now give a proof of Proposition 3.3.2.

Proof of Proposition 3.3.2. Let $m \in \mathbb{Z}$ be such that both $\omega_f(D_\varepsilon)^{[m]}$ is locally free and $f_*(\omega_f(D_\varepsilon)^m)$ is big. First note that for n large enough, the sheaf

$$\big(f_*(\omega_f(D_\varepsilon)^m)\big)^{[n]} \otimes H^{-1}$$

is generically globally generated. This follows since by Proposition 5.2 of [Has1], for a r-dimensional vector space V, each irreducible component of the reflexive hull of the mth tensor power of V, is a quotient of a representation $S^{[q_1]}(V) \otimes \cdots \otimes S^{[q_k]}(V)$, where $k = r!$. Using this, we will prove that $\omega_{X^n_B}(D_{\varepsilon,n})$ is big for large n. To do so, it suffices to show that there are on the order of $m^{n \dim X_\eta + b}$ sections of $\omega_{X^n_B}(D_\varepsilon)^{[m]}$ where $b = \dim B$, and X_η denotes the general fiber.

By Lemma 3.3.3,

$$\omega_{X^n_B}(D_{\varepsilon,n})^{[m]} = \pi_1^*\omega_f(D_\varepsilon)^{[m]} \otimes \cdots \otimes \pi_n^*\omega_f(D_\varepsilon)^{[m]} \otimes f_n^*\omega_B^m.$$

The sheaf $\omega_f(D_\varepsilon)$ has good positivity properties – it is big by Corollary 7.3 of [KP], but the sheaf ω_B is somewhat arbitrary and could easily prevent $\omega_{X^n_B}(D_{\varepsilon,n})$ from being big. However, taking high enough powers of X allows the positivity of $\omega_f(D_\varepsilon)$ to overcome the possible negativity of ω_B.

Applying $(f_n)_*$ gives, via the projection formula:

$$(f_n)_*(\omega_{X^n_B}(D_{\varepsilon,n})^{[m]}) = \big(f_*(\omega_f(D_\varepsilon)^{[m]})\big)^n \otimes \omega_B^m$$

which is also a reflexive sheaf by Corollary 1.7 of [Har]. More specifically, it is the push forward of a reflexive sheaf under a proper dominant morphism. Then the inclusion map $\omega_f(D_\varepsilon)^m \to \omega_f(D_\varepsilon)^{[m]}$ induces a map of reflexive sheaves:

$$\left(f_*\omega_f(D_\varepsilon)^m\right)^{[n]} \to \left(f_*\omega_f(D_\varepsilon)^{[m]}\right)^n = (f_n)_*(\omega_{X_B^n}(D_{\varepsilon,n})^{[m]}) \otimes \omega_B^{-m}$$

which is an isomorphism at the generic point of B.

Let H be an invertible sheaf on B such that $H \otimes \omega_B$ is very ample. Then we can choose n so that $\left(f_*\omega_f(D_\varepsilon)^m\right)^{[n]} \otimes H^{-m}$ is generically globally generated for all admissible values of m. But then

$$\left(f_*\omega_f(D_\varepsilon)^m\right)^{[n]} \otimes H^{-m} = (f_n)_*(\omega_{X_B^n}(D_{\varepsilon,n})^{[m]}) \otimes (H \otimes \omega_B)^{-m}$$

is also generically globally generated for the same m.

This sheaf has rank on the order of $m^{n \dim X_\eta}$ so there are at least this many global sections. By our assumption on H, we have that $(H \otimes \omega_B)^m$ has on the order of m^b sections varying horizontally along the base B. By tensoring, we obtain that the sheaf

$$(f_n)_* \left(\omega_{X_B^n}(D_{\varepsilon,n})\right)^{[m]}$$

has on the order of $m^{n \dim X_\eta + b}$ global sections, and therefore $\omega_{X_B^n}(D_{\varepsilon,n})$ is big. □

Remark 3.3.4. *The above proposition assumed that the general fiber (X_b, D_b) had klt singularities, but to prove Theorem 1.1 as stated, we must allow the general fiber to have log canonical singularities. Unfortunately, we cannot just raise the coefficients of D so that the pair has log canonial singularities, via twisting by εD to conclude that $\omega_{X_B^n}(D_n)$ is also big. This is because we do* not *know that the divisor D is $\mathbb{Q}$-Cartier. We remedy this situation with a $\mathbb{Q}$-factorial divisorial log terminal (dlt)* *modification (see Section 1.4 of [Kol2] for an overview of dlt models), as explained below.*

First, the definition of a dlt pair.

Definition 3.3.5. *Let (X, D) be a log canonical pair such that X is normal an $D = \sum d_i D_i$ is the sum of distinct prime divisors. Then (X, D) is* ***divisorial log terminal*** *(dlt) if there exists a closed subset $Z \subset X$ such that:*

(i) $X \setminus Z$ is smooth and $D|_{X \setminus Z}$ is a snc divisor

(ii) If $f : Y \to X$ is birational and $E \subset Y$ is an irreducible divisor such that $center_X E \subset Z$ then the discrepancy $a(E, X, D) < 1$

See Definition 2.25 in [KM] for a definition of the discrepancy of a divisor E with respect to a pair (X, D).

Roughly speaking, a pair (X, D) is dlt if it is log canonical, and it is simple normal crossings at the places where it is not klt. The following theorem of Hacon guarantees the existence of dlt modifications.

Theorem 3.6. *[KK, Theorem 3.1] Let (X, D) be a pair of a projective variety and a divisor $D = \sum d_i D_i$ with coefficients $0 \leq d_i \leq 1$, such that $K_X + D$ is $\mathbb{Q}$-Cartier. Then (X, D) admits a $\mathbb{Q}$-factorial minimal dlt model $f^{min} : (X^{min}, D^{min}) \to (X, D)$.*

The upshot here is that, starting with a log canonical pair (X, D) we can obtain a model which is dlt and $\mathbb{Q}$-factorial.

The statement that we will actually apply follows from Proposition 2.9 of [PX].

Proposition 3.3.6. *[PX, Proposition 2.9] Let $f : (X, D) \to B$ be a stable family over a smooth variety B. Assume that the general fiber (X_b, D_b) has log canonical singularities and that the variation of the family is maximal. Then for each $0 < \varepsilon \ll 1$ there exists a pair (Z, Δ_ε), an effective divisor Δ on Z, and a morphism $p : Z \to X$ such that:*

(a) $K_Z + \Delta = p^*(K_X + D)$,

(b) (Z, Δ_ε) *is klt*

(c) $g : (Z, \Delta_\varepsilon) \to B$ *is a stable family*

(d) The variation of g *is maximal*

(e) $\Delta - \Delta_\varepsilon$ *is an effective divisor such that* $Supp(\varepsilon\Delta) \subset Ex(p) \cap Supp(p_*^{-1}\Delta)$

Sketch of proof. The rough idea is to take a $\mathbb{Q}$-factorial dlt modification of X, and then shrink the resulting divisor so that the new pair $(\widetilde{Z}, \widetilde{\Delta}_\varepsilon)$ is klt. Finally, taking the relative log canonical model of $(\widetilde{Z}, \widetilde{\Delta}_\varepsilon) \to X$ yields a stable family with klt general fiber and maximal variation. □

Using the above discussion, we are now in position to prove the main statement of this section, Proposition 3.3.1, whose proof is inspired by Proposition 2.15 of [PX].

Proof of Proposition 3.3.1. We begin with a stable family with maximal variation $f : (X, D) \to B$ such that the generic fiber is log canonical. The goal is to show that $\omega_{X_B^n}(D_n)$ is big for n sufficiently large.

First take $\widetilde{p} : \widetilde{Z} \to X$ to be a $\mathbb{Q}$-factorial dlt modification of X, and let $\widetilde{\Delta}$ be a divisor on $\widetilde{Z}$ such that $\widetilde{p}^*(K_X + D) = K_{\widetilde{Z}} + \widetilde{\Delta}$. Since $\widetilde{Z}$ is $\mathbb{Q}$-factorial by construction, we can lower the coefficients of the divisor $\widetilde{\Delta}$ by $0 < \varepsilon \ll 1$, a rational number, to obtain a klt pair $(\widetilde{Z}, \widetilde{\Delta}_\varepsilon)$.

Define $p : Z \to X$ to be the relative log canonical model of $(\widetilde{Z}, \widetilde{\Delta}_\varepsilon) \to X$. Denoting by $q : \widetilde{Z} \dashrightarrow Z$ the induced morphism, we define Δ to be the pushforward $\Delta = q_*(\widetilde{\Delta})$. By Proposition 3.3.6, the new family $g : (Z, \Delta_\varepsilon) \to B$ is a stable family with maximal variation such that the generic fiber is klt. Thus, by Proposition 3.3.2, for n large enough, $\omega_{Z^n_B}(\Delta_{\varepsilon,n})$ is big.

Furthermore, since $(Z, \Delta_\varepsilon) \to X$ is the relative log canonical model of $(\widetilde{Z}, \widetilde{\Delta}_\varepsilon) \to X$, pluri-log canonical forms on $\widetilde{Z}$ are the pull back of pluri-log canonical forms on Z. From this we conclude that $\omega_{\widetilde{Z}^n_B}(\widetilde{\Delta}_{\varepsilon,n})$ is also big. Now since $\widetilde{Z}$ is $\mathbb{Q}$-factorial, we know that $\varepsilon\Delta$ is a $\mathbb{Q}$-Cartier divisor. This property allows us to enlarge the coefficients of $\widetilde{\Delta}$. Recall that $\varepsilon\Delta$ is effective by Proposition 3.3.6 (e), and thus $\omega_{\widetilde{Z}^n_B}(\widetilde{\Delta}_n)$ is big as well.

Since $\widetilde{p} : \widetilde{Z} \to X$ is a birational morphism and $\widetilde{p}^*(K_X + D) = K_{\widetilde{Z}} + \widetilde{\Delta}$, pulling back pluri-log canonical forms through $\widetilde{p}$ preserves the number of sections. Thus, we finally conclude that $\omega_{X^n_B}(D_n)$ is big. □

Finally, we prove the following theorem for pairs openly of log general type:

Theorem 3.7. *Let $f : (X, D) \to B$ be a stable family with integral, openly canonical, and log canonical general fiber over a smooth projective variety B. Suppose that the variation of the family f is maximal. Then $\omega_{X^n_{ss}}(D^{ss}_n)$ is big, where (X^n_{ss}, D^{ss}_n) denotes the nth fibered power of the weak semistable model of the pair (X, D).*

Proof. Consider the following diagram:

$$\begin{array}{ccccc} (X_{ss}, D^{ss}) & \xrightarrow{\varphi} & (\widetilde{X}, \widetilde{D}) = X \times_{B_1} B & \xrightarrow{\sigma} & (X, D) \\ \Big\downarrow{\scriptstyle \pi_{ss}} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ \Delta \subset B_1 & \longrightarrow & B_1 & \longrightarrow & B \end{array}$$

where $(X_{ss}, D^{ss}) \to B_1$ denotes the weak semistable model (see Definition 0.4 of [Kar]) of the family $(X, D) \to B$, and $\Delta \subset B_1$ denotes the discriminant divisor over which the exceptional lies. Such a model exists by [Kar]. Since taking the weak semistable model gives a pair which is at worst openly canonical and log canonical, we are not required to take a resolution of singularities. This is because, by definition of both openly canonical and log canonical singularities, sections of $\omega_{X_{ss}}(D^{ss})$ give regular sections of logarithmic pluricanonical sheaves of any desingularization.

More precisely, we have:

$$\varphi^*(\omega_g(\widetilde{D})) = \omega_{\pi_{\mathrm{ss}}}(D^{ss} + E) \subset \omega_{\pi_{\mathrm{ss}}}(D^{ss} + \pi^*_{\mathrm{ss}}(\Delta)).$$

Let $\pi^*_{\mathrm{ss}}\Delta = \Delta^{ss}$. Then since $\omega_g(\widetilde{D})$ is big by Theorem 3.5, so is $\omega_g(\widetilde{D} - \frac{1}{n}\Delta^{ss})$. Taking fibered powers as in Proposition 3.3.1, shows that $\omega_{\widetilde{X}^n_{B_1}}(\widetilde{D}_n(-\Delta^{ss}_n))$ is also big. Moreover,

$$\varphi_n^*(\omega_{\widetilde{X}^n_{B_1}}(\widetilde{D}_n(-\Delta^{ss}_n))) \subset \omega_{X^n_{ss}}(D^{ss}_n)(\Delta^{ss}_n - \Delta^{ss}_n) = \omega_{X^n_{ss}}(D^{ss}_n)$$

is big. □

The definition of openly of log general type then implies that we have actually shown the following:

Theorem 3.8. *Let $f : (X, D) \to B$ be a stable family with integral, openly canonical,* *and log canonical general fiber over a smooth projective variety B. Suppose that the variation of the family f is maximal. Then there exists an integer $n > 0$ such that (X^n_B, D_n) is openly of log general type.*

3.4 Singularities

The purpose of this section is to prove that, assuming we begin with a pair (X, D) with log canonical singularities, then fibered powers (X^n_B, D_n) also have log canonical singularities for all $n > 0$. As the following example shows, it *is* necessary to restrict the singularities, as there exist varieties Y such that ω_Y is a big, but Y is *not* of general type!

Example 3.4.1. *Let Y be the projective cone over a quintic plane curve C. Then ω_Y is big (even ample), but Y is birational to $\mathbb{P}^1 \times C$, which has Kodaira dimension $\kappa(\mathbb{P}^1 \times C) = -\infty$. So although ω_Y is big, Y is* not *openly of log general type.*

The following proposition is a version of log inversion of adjunction:

Proposition 3.4.2 (Lemma 2.12 [Pat1])**.** *The total space of an slc family over an slc base has slc singularities.*

This immediately implies:

Corollary 3.4.3. *The total space of the product of slc families over an slc base also has slc singularities.*

Proof. Let $f : (X_1, D_1) \to B$ and $g : (X_2, D_2) \to B$ be two slc families over an slc base B. Then the product family $g : (X, \Delta) \to B$ is the total space of an slc family over either of the factors. Therefore both the product family as well as its total space have slc singularities by Proposition 3.4.2. □

Inductively, this shows that the fibered powers (X^n_B, D_n) have semi-log canonical singularities. The statement that we will actually use to prove our result is the following:

Proposition 3.4.4. *Let $f : (X, D) \to B$ be a stable family with integral and log canonical general fiber over a smooth projective variety B. Then for all $n > 0$ the fibered powers (X^n_B, D_n) have log canonical singularities.*

Proof. By Proposition 3.4.2, the total space of the family (X, D) is slc. In fact, we will show that it is actually log canonical, which is equivalent to showing that (X, D) is normal. Recall to show that the pair (X, D) is normal, it suffices to show that it is regular in codimension one (R1) and satisfies Serre's condition S2. Since the general fiber has log canonical singularities, the fibers (X_b, D_b) are R1 over the general point of the base B. Over the special fibers, the singularities are of at least codimension one in the fiber, and are thus at least codimension two in the total space. Therefore, it follows that the total space (X, D) is R1. Finally, the pair (X, D) is S2 by definition, since it has semi-log canonical singularities.

Therefore, by Corollary 3.4.3, for all $n > 0$ the fibered powers (X^n_B, D_n) also have log canonical singularities. □

In fact, the following stronger statements are also true. Although we do not use them in this paper, we hope that they may be of interest to readers:

Proposition 3.4.5. *The fiber product of two stable families is a stable family.*

Proof. This result essentially follows from Proposition 2.12 in [BHPS]. We reproduce the argument for the convenience of the reader.

Let $f : (X, D) \to B$ and $g : (Y, E) \to B$ be two stable families, and denote the fiber product family by $h : (Z, F) \to B$. Since both families f and g are flat of finite type with S_2 fibers by assumption, and since we are assuming Kollár's condition, by Proposition 5.1.4 of [AH], we have that $\omega^{[k]}_{X/B}(D)$ is flat over B. Moreover, by Lemma 2.11 of [BHPS], we have that

$$p^*_X \omega^{[k]}_{X/B}(D) \otimes p^*_Y \omega^{[k]}_{Y/B}(E)$$

is a reflexive sheaf on the product. By Lemma 2.6 of [HK], the above sheaf is isomorphic to $\omega^{[k]}_{Z/B}(F)$ on an open subset whose complement has codimension at least two, and therefore we conclude that

$$\omega^{[k]}_{Z/B}(F) = p^*_X \omega^{[k]}_{X/B}(D) \otimes p^*_Y \omega^{[k]}_{Y/B}(E).$$

Moreover, Kollár's condition holds, as by assumption both components of this fiber product commute with arbitrary base change. Choosing a sufficient index k, namely the least common multiple of the index of the factors, we see that $\omega_{Z/B}(F)$ is a relatively ample $\mathbb{Q}$-line bundle and thus we conclude that $h : (Z, F) \to B$ is also a stable family. □

Proposition 3.4.6. *The total space of the fiber product of stable families over a stable base is stable.*

Proof. By Proposition 2.15 in [PX] (see also [Fuj2] Theorem 1.13), if $f : (X, D) \to (B, E)$ is a stable family whose variation is maximal over a normal base, then $\omega_f(D)$ is nef. First we note that it suffices to prove the statement over a normal base, since nef is a property which is decided on curves. Since normalization is a finite birational morphism, nonnegative intersection with a curve is preserved. Thus, we wish to show that this statement is true without the assumption that the variation of f is maximal. Let $B' \to B$ be a finite cover of the base so that the pullback family $f' : (X', D') \to B'$ maps to $g : (\mathcal{U}, \mathcal{D}) \to T$, a family of maximal variation. In this case $\omega_{f'}(D')$ is nef, as it is the pullback of $\omega_g(\mathcal{D})$ which is nef. Since $\omega_{f'}(D')$ is the pullback of $\omega_f(D)$ by the finite morphism $X' \to X$, the projection formula implies that $\omega_f(D)$ is nef as well. This shows that the sheaf $\omega_f(D)$ is nef, regardless of whether the variation of f is maximal or not. Then since $\omega_f(D)$ is nef and f-ample, and since the base is stable, $\omega_B(E)$ is ample. Therefore, we can conclude that $\omega_X(D + f^*E) = \omega_f(D) \otimes f^*\omega_B(E)$ is ample. □

The following theorem that we actually need follows from Proposition 3.3.1 and Proposition 3.4.4.

Theorem 3.9. *Let $(X, D) \to B$ be a stable family with integral and log canonical general fiber and maximal variation over a smooth projective variety B. Then there exists an integer $n > 0$ such that the pair (X^n_B, D_n) is of log general type.*

Proof. By Proposition 3.3.1, we have that $\omega_{X^n_B}(D_n)$ is big, and by Proposition 3.4.4, the fibered powers (X^n_B, D_n) have log canonical singularities. □

To prove the stronger Theorem 3.11, we must show that what we have proven also works after taking the quotient by a group of automorphisms. This is precisely the content of Proposition 3.4.7 and Corollary 3.4.9 below.

This claim essentially follows from the work of various authors in previous papers in the subject. The approach is present in, for example, Lemma 3.2.4 of [AM] as well as Lemma 2.4 of [Pac]. We reproduce the statement in our case below:

Proposition 3.4.7. *Let (X, D) be openly of log general type. There exists a positive integer n such that the pair $(X^n_B, D_n)/G$ is also openly of log general type.*

Proof. Let $H \subset X$ be the locus of fixed points of the action of $G \subseteq \mathrm{Aut}(X, D)$. Let $\mathcal{I}_H$ denote the corresponding sheaf of ideals. We have seen before that $\omega_f(D)$ is big. Then, for sufficiently large k, we have that the sheaf

$$\omega_f(D)^{\otimes k} \otimes f^*\omega_B^{\otimes k} \otimes \mathcal{I}_H^{|G|}$$

is big. If we pass to the kth fibered power, we have that

$$(\omega_{X^k_B}(D_k))^{\otimes k} \otimes f_k^*\omega_B^{\otimes k} \otimes \Pi_{i=1}^k \pi^{-1}\mathcal{I}_H^{|G|}$$

is also big.

The product $\Pi_{i=1}^k \pi_i^{-1}\mathcal{I}_H^{|G|} \subset \left(\sum_{i=1}^k \pi_i^{-1}\mathcal{I}_H^{|G|}\right)^k$, and the latter ideal vanishes to order at least $k|G|$ on the fixed points of the action of G. Moreover, we have that

$$(\omega_{f_k}(D_k))^{\otimes k} \otimes \pi_k^*\omega_B^{\otimes k} = (\omega_{X^k_B}(D_k))^{\otimes k}.$$

This allows us to conclude that for $n \gg 0$, there are enough invariant sections of $\omega_{X^k_B}(D_k)^{\otimes n}$ vanishing on the fixed point locus to order at least $n|G|$.

Now let

$$r : (\mathcal{X}, \mathcal{D}) \to (X^k_B, D_k)$$

be an equivariant good resolution of singularities so that $r^{-1}(D_k) = \mathcal{D}$. Note that such a resolution is guaranteed by Hironaka [Hir]. Since $X \setminus D$ does not necessarily have canonical singularities away from the general fiber, we have introduced exceptional divisors in the resolution that will alter sections of $\omega_X(D)$. To fix this, we simply apply the methods used in the proof of Theorem 3.7 – namely twist by some small negative multiple of the divisor Δ containing the exceptional.

To conclude the result, it suffices to show that invariant sections of $(\omega_{X^k_B}(D_k))^{\otimes n}$ vanishing on the fixed point locus to order at least $n \cdot |G|$, descend to sections of the pluri-log canonical divisors of a good resolution of the quotient pair $(X^n_B/G, D_n/G)$.

Denote by $q : (\mathcal{X}, \mathcal{D}) \to (\mathcal{X}/G, \mathcal{D}/G)$ the morphism to the quotient, and let

$$\varphi : (\widetilde{\mathcal{X}}/G, \widetilde{\mathcal{D}}/G) \to (\mathcal{X}/G, \mathcal{D}/G)$$

denote a good resolution. Then Lemma 4 from [Abr2] tells us that the invariant sections of $\omega_{\mathcal{X}}(\mathcal{D})^{\otimes n}$ vanishing on the fixed point locus to order $\geq n|G|$ come from sections of the pluri-log canonical divisors of a desingularization, i.e. sections of $\omega_{\widetilde{\mathcal{X}}}(\widetilde{\mathcal{D}})^{\otimes n}$. Therefore, for n sufficiently large, the quotient pair $(X^n_B, D_n)/G$ is openly of log general type. □

This also proves the following theorem:

Theorem 3.10. *Let $f : (X, D) \to B$ be a stable family with integral, openly canonical, and log canonical general fiber over a smooth projective variety B. Suppose that the variation of the family f is maximal. Let G be a finite group such that $(X, D) \to B$ is G-equivariant. Then there exists an integer $n > 0$ such that the quotient $(X^n_B/G, D_n/G)$ is openly of log general type.*

Furthermore, combining Proposition 3.4.7 with Proposition 3.4.4 yields:

Corollary 3.4.8. *Let $f : (X, D) \to B$ be an slc family with integral and log canonical general fiber over a smooth projective variety B. Then for n large enough, the quotient pair $(X_B^n/G, D_n/G)$ also has log canonical singularities.*

This then gives an analogue to Proposition 3.4.7 for pairs of log general type:

Corollary 3.4.9. *Let (X, D) be a pair of log general type. There exists a positive integer n such that the pair $(X_B^n, D_n)/G$ is also a pair of log general type.*

Thus we have completed the proof of the following Theorem 3.11.

Theorem 3.11. *Let $(X, D) \to B$ be a stable family with integral and log canonical general fiber over a smooth projective variety B. Suppose that the variation of the family f is maximal (see Definition 3.2.2). Let G be a finite group such that $(X, D) \to B$ is G-equivariant. Then there exists an integer $n > 0$ such that the quotient of the pair by a finite group of automorphisms, $(X_B^n/G, D_n/G)$ is of log general type.*

Proof. This follows from Theorem 3.9 and Corollary 3.4.9. □

The next and final section shows how to reduce the proof of the Theorem 1.1 to Theorem 3.11. Then, we show that Theorem 1.2 follows from Theorem 1.1.

3.5 Proof of Theorems 1.1 and 1.2 – Reduction to case of max variation

The final section of this chapter is devoted to reducing the proofs of our two main theorems to the case of maximal variation. We will use the existence of a tautological family over a finite cover of our moduli space to show that, after a birational modification of the base, the pullback of a stable family with integral and log canonical general fiber has a morphism to the quotient of a family of maximal variation by a finite group. Then using the fact that our result holds for families of maximal variation, we will conclude that, after a modification of the base, a high fibered power of the pullback of a stable family with integral and log canonical general fiber has a morphism to a pair of log general type.

Finally, we show that if we add the assumption that the general fiber of our family is openly canonical and log canonical, we can avoid taking a modification of the base to prove Theorem 1.2.

Remark 3.5.1. *As we will be using the moduli space of stable pairs $\overline{M}_h$, we remind the reader that we are in the situation where the coefficients of the divisor D are $> \frac{1}{2}$.*

Unfortunately the moduli space $\overline{M}_h$ that we are working with does *not* carry a universal family. The following lemma gives a *tautological family*, which can be thought of as an approximation of a universal family.

Lemma 3.5.2 ([KP, Corollary 5.19])**.** *There exists a tautological family $(\mathcal{T}, \mathcal{D})$ over a finite cover Ω of the moduli space $\overline{M}_h$ of stable log pairs. That is, there exists a variety Ω, a finite surjective map $\varphi : \Omega \to \overline{M}_h$ and a stable family $\mathcal{T} \to \Omega$ such that $\varphi(x) = [(\mathcal{T}_x, \mathcal{D}_x)]$.*

Proposition 3.5.3. *Let $f : (X, D) \to B$ be a stable family such that the general fiber is integral and has log canonical singularities. Then there exists a birational modification of the base $\widetilde{B} \to B$, and a morphism $(\widetilde{X}, \widetilde{D}) \to \widetilde{B}$ to $(\mathcal{T}_{\widetilde{\Sigma}}, \widetilde{\mathcal{D}})/G$, the quotient of a family of maximal variation by a finite group G*

Proof. Let $f : (X, D) \to B$ be a stable family such that the general fiber is integral and has log canonical singularities. In particular, we do *not* assume that the variation of f is maximal. There is a well defined canonical morphism $B \to \overline{M}_h$. Call the image of this morphism Σ. Over this Σ lies the universal family $(\mathcal{T}_\Sigma, \mathcal{D})$. Since $\overline{M}_h$ is a stack, the maps $(X, D) \to (\mathcal{T}_\Sigma, \mathcal{D})$ and $B \to \Sigma$ factor through the coarse spaces: $\underline{\Sigma}$ and $(\underline{\mathcal{T}}_\Sigma, \underline{\mathcal{D}})$. The general fiber of $(\underline{\mathcal{T}}_\Sigma, \underline{\mathcal{D}}) \to \underline{\Sigma}$ is simply $(S, D_S)/K$ where (S, D_S) is a pair of log general type and K is the finite automorphism group.

Unfortunately there is no control on the singularities of Σ – if the singularities are not too mild, the fibered powers $(\mathcal{T}_\Sigma^n, \mathcal{D}_n)$ have no chance of having log canonical singularities. To remedy this we take a resolution of singularities. Using Proposition 3.5.2, we take a Galois cover followed by an equivariant resolution of singularities to obtain $\widetilde{\Sigma} \to \underline{\Sigma}$. Call the Galois group of this cover H. Then over $\widetilde{\Sigma}$, we have a tautological family $(\mathcal{T}_{\widetilde{\Sigma}}, \widetilde{\mathcal{D}})$. Here the general fiber is simply (S, D_S), a pair of log general type.

Consider the quotient map $\widetilde{\Sigma} \to \widetilde{\Sigma}/H$. Taking the pullback of $(\underline{\mathcal{T}}_\Sigma, \underline{\mathcal{D}})$ through $\widetilde{\Sigma}/H$ yields $(\mathcal{T}_{\widetilde{\Sigma}/H}, \widetilde{D}')$. Letting G be the group $G = H \times K$, we can construct the following diagram:

$$\begin{array}{ccccccccc} (\mathcal{T}_{\widetilde{\Sigma}},\widetilde{D}) & \longrightarrow & (\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}})/G & \xrightarrow{\nu} & (\mathcal{T}_{\widetilde{\Sigma}/H},\widetilde{\mathcal{D}}') & \longrightarrow & (\underline{\mathcal{T}}_{\Sigma},\underline{\mathcal{D}}) & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widetilde{\Sigma} & \longrightarrow & \widetilde{\Sigma}/H & \longrightarrow & \widetilde{\Sigma}/H & \longrightarrow & \underline{\Sigma} & \longrightarrow & \overline{M}_h \end{array}$$

We claim that the map $\nu : (\mathcal{T}_{\widetilde{\Sigma}},\widetilde{D})/G \to (\mathcal{T}_{\widetilde{\Sigma}/H},\widetilde{\mathcal{D}}')$ is actually the normalization of $(\mathcal{T}_{\widetilde{\Sigma}/H},\widetilde{\mathcal{D}}')$. First note that $(\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}})/G$ is normal, and that the morphism ν is finite since the morphism $(\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}}) \to (\underline{\mathcal{T}}_{\Sigma},\underline{\mathcal{D}})$ is. Therefore, to prove ν is the normalization of $(\mathcal{T}_{\widetilde{\Sigma}/H},\widetilde{\mathcal{D}}')$, it suffices to prove that ν is birational. To do so, consider the following diagram:

$$\begin{array}{ccccc} (\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}}) & \longrightarrow & (\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}})/H & \longrightarrow & \Big((\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}})/H\Big)/K = (\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}})/G \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\Sigma} & \longrightarrow & \widetilde{\Sigma}/H & \longrightarrow & \widetilde{\Sigma}/H \end{array}$$

From this diagram it is clear that the general fiber of $(\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}})/G \to \widetilde{\Sigma}/H$ is precisely $(S, D_S)/K$ – the quotient by H identifies fibers and the quotient by K removes the automorphisms. Since the map ν is an isomorphism over the generic fibers, ν is a birational map and thus is the normalization of $(\mathcal{T}_{\widetilde{\Sigma}/H},\widetilde{\mathcal{D}}')$.

The pair (X, D) does *not* map to $(\mathcal{T}_{\widetilde{\Sigma}/H},\widetilde{\mathcal{D}}')$. Instead, consider a modification of the base $\widetilde{B} \to B$ where $\widetilde{B} = B \times_{\Sigma} \widetilde{\Sigma}/H$. Then the pullback $(\widetilde{X},\widetilde{D})$ maps to $(\mathcal{T}_{\widetilde{\Sigma}/H},\widetilde{\mathcal{D}}')$. Since the pair $(\widetilde{X},\widetilde{D})$ is normal and ν is the normalization, we see that $(\widetilde{X},\widetilde{D})$ also maps to $(\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}})/G$. Finally, because the family $(\mathcal{T}_{\widetilde{\Sigma}},\widetilde{\mathcal{D}})/G \to \widetilde{\Sigma}/H$ is the quotient of a family of maximal variation by a finite group, we have completed the proof of the proposition. $\square$

Proof of Theorem 1.1. Let $(\mathcal{T}_{\widetilde{\Sigma}}, \widetilde{\mathcal{D}})$ denote the tautological family of maximal variation obtained in the proof of Proposition 3.5.3. Passing to nth fibered powers, Theorem 3.11 guarantees that $(\mathcal{T}^n_{\widetilde{\Sigma}}, \widetilde{\mathcal{D}}_n)$ is of log general type. By Corollary 3.4.9, $(\mathcal{T}^n_{\widetilde{\Sigma}}, \widetilde{\mathcal{D}}_n)/G$ is also of log general type for n sufficiently large. Thus the proof of Theorem 1.1 follows from the above Proposition 3.5.3, as we have shown that after modifying the base, we obtain a morphism from a high fibered power of our family to a pair of log general type. □

Finally, we prove Theorem 1.2, the fibered power theorem for pairs openly of log general type.

Proof of Theorem 1.2. This proof essentially follows from the proof of Proposition 3.5.3. Assuming that the general fiber is openly canonical and log canonical, Theorem 3.10 shows that, for n sufficiently large, the pair $(\mathcal{T}^n_{\widetilde{\Sigma}}, \widetilde{\mathcal{D}}_n)/G$ is openly of log general type. Since there is a birational morphism $(\mathcal{T}^n_{\widetilde{\Sigma}}, \widetilde{\mathcal{D}}_n)/G \to (\underline{\mathcal{T}}^n_{\Sigma}, \underline{\mathcal{D}}_n)$, it follows that $(\underline{\mathcal{T}}^n_{\Sigma}, \underline{\mathcal{D}}_n)$ is also openly of log general type. Therefore, we have constructed a morphism from a high fibered power of our family to a pair openly of log general type, and have thus completed the proof of the theorem. The upshot here, is that we do not have to modify the base of our starting family. □

CHAPTER **Four**

Bounding Heights Uniformly in Families of Hyperbolic Varieties

4.1 Introduction

In the statement of our main result we consider morphisms of algebraic stacks $f\colon X \to Y$ which are representable by schemes, i.e., for all schemes S and all morphisms $S \to Y$, the algebraic stack $X \times_Y S$ is (representable by) a scheme. Furthermore, a substack of an algebraic stack is constructible if it is a finite union of locally closed substacks. Moreover, we will use the relative discriminant $d_k(\mathcal{T}_P)$ of a point on an algebraic stack over a number field k; we refer the reader to Section 4.2.4 for a precise definition of $\mathcal{T}_P$ as well as the relative discriminant $d_k(\mathcal{T}_P)$. Also, to state our theorem, we will use heights on stacks as discussed in Section 4.2.4.

We state our main theorem.

Theorem 4.1 (Theorem 1.3)**.** *Let k be a number field and let $f\colon X \to Y$ be a proper surjective morphism of proper Deligne-Mumford stacks over k which is representable by schemes. Let h be a height function on X and let h_Y be a height function on Y associated to an ample divisor with $h_Y \geq 1$. Assume Vojta's height conjecture (Conjecture 1.4.1). Let $U \subset Y$ be a constructible substack such that, for all $t \in U$, the variety X_t is smooth and hyperbolic. Then there is a real number $c > 0$ depending only on k, Y, X, and f such that, for all P in $X(k)$ with $f(P)$ in U, the following inequality holds*

$$h(P) \leq c \cdot \big(h_Y(f(P)) + d_k(\mathcal{T}_P)\big).$$

Our proof of Theorem 1.3 uses the recent [AMV], which shows that Vojta's conjecture actually implies a version of the conjecture for stacks. Moreover, to prove Theorem 1.3 we follow the strategy of Ih. Indeed, we combine an induction argument with an application of Vojta's conjecture to a desingularization of X (Proposition 4.3.1). This line of reasoning was also used in Ih's work [Ih1, Ih2].

Theorem 4.2 (Theorem 1.4)**.** *Assume Conjecture 1.4.1. Let $g \geq 2$ be an integer and let k be a number field. There is a real number c depending only on g and k satisfying the following. For all smooth projective curves X of genus g over k, and all P in $X(k)$, the following inequality holds*

$$h(P) \leq c(g,k) \cdot \big(h(X) + d_k(\mathcal{T}_X)\big).$$

Finally, we also obtain a uniformity statement for certain hyperbolic surfaces.

Theorem 4.3 (Theorem 1.5)**.** *Assume Conjecture 1.4.1. Fix an even integer a and a number field k. There is a real number c depending only on a and k satisfying the* *following. For all smooth hyperbolic surfaces S over k with $c_1^2(S) = a > c_2(S)$ and all P in $S(k)$, the following inequality holds*

$$h(P) \leq c \cdot \big(h(S) + d_k(\mathcal{T}_S)\big).$$

We refer the reader to Section 4.5 for precise definitions of the height functions appearing in Theorems 1.4 and 1.5. We prove Theorems 1.4 and 1.5 by applying Theorem 1.3 to the universal family of the moduli space of curves and the moduli space of surfaces of general type, respectively. The technical difficulty in applying Theorem 1.3 is to prove the constructibility of the locus of points corresponding to hyperbolic varieties. In the setting of curves (Theorem 1.4) this is simple, whereas the case of surfaces (Theorem 1.5) requires deep results of Bogomolov and Miyaoka [Bog, Miy].

4.1.1 Hyperbolicity

In this subsection the base field k is a field of arbitrary characteristic, and all divisors D are assumed to be Cartier.

Definition 4.1.1. *Let X be a proper Deligne-Mumford stack of dimension n over k. A divisor D on X is* ***big*** *if $h^0(X, \mathcal{O}_X(mD)) > c \cdot m^n$ for some $c > 0$ and $m \gg 1$.*

Recall that a projective geometrically irreducible variety X over k is of general type if for a desingularization $\widetilde{X} \to X_{\mathrm{red}}$ of the reduced scheme X_{red}, the sheaf $\omega_{\widetilde{X}}$ is big. Note that, if X is of general type and $\widetilde{X} \to X_{\mathrm{red}}$ is any desingularization, then $\omega_{\widetilde{X}}$ is big.

Definition 4.1.2. *A projective scheme X over k is **hyperbolic** (over k) if for all its closed subschemes Z, any irreducible component of $Z_{\overline{k}}$ is of general type.*

Note that, if X is a hyperbolic projective scheme over k, then X and all of its closed subvarieties are of general type. Moreover, if L/k is a field extension, then X is hyperbolic over k if and only if X_L is hyperbolic over L.

For example, a smooth proper geometrically connected curve X over k is hyperbolic if and only if the genus of X is at least two. For another example, let X be a smooth projective scheme over $\mathbb{C}$ and suppose that there exists a smooth proper morphism $Y \to X$ whose fibres have ample canonical bundle such that, for all a in $X(\mathbb{C})$, the set of b in $X(\mathbb{C})$ with $X_a \cong X_b$ is finite. Then X is hyperbolic. This is a consequence of Viehweg's conjecture for "compact" base varieties [Pat2].

4.1.2 Kodaira's criterion for bigness

We assume in this section that k is of characteristic zero. Recall that for a big divisor D on a projective variety, there exists a positive integer n such that $nD \sim_{\mathbb{Q}} A + E$, where A is ample and E is effective [KM, Lemma 2.60]. We state a generalization of this statement (see Lemma 4.1.4) which is presumably known; we include a proof for lack of reference.

Lemma 4.1.3. *Let $\pi\colon X \to Y$ be a quasi-finite morphism of proper Deligne-Mumford stacks over k. Let D be a divisor on Y. The divisor D is big on Y if and only if π^*D is big on X.*

Proof. This follows from the definition of bigness, and the fact that $\pi_*\pi^*D$ is linearly equivalent to mD, where $m \geq 1$ is some integer. □

If D is a divisor on a finite type separated Deligne-Mumford stack $\mathcal{X}$ over k with coarse space $\mathcal{X} \to \mathcal{X}^c$, then D is *ample* (resp. *effective*) on $\mathcal{X}$ if there exists a positive integer n such that nD is the pull-back of an ample (resp. *effective*) divisor on $\mathcal{X}^c$. Note that, if $\mathcal{X}$ has an ample divisor, then $\mathcal{X}^c$ is a quasi-projective scheme over k.

Lemma 4.1.4. *Let $\mathcal{X}$ be a proper Deligne-Mumford stack over k with projective coarse moduli space $\mathcal{X}^c$. If D is a big divisor on $\mathcal{X}$, then there exists a positive integer n such that $nD \sim_{\mathbb{Q}} \mathcal{A} + \mathcal{E}$, where $\mathcal{A}$ is ample and $\mathcal{E}$ is effective.*

Proof. Let $\pi\colon \mathcal{X} \to \mathcal{X}^c$ denote the morphism from $\mathcal{X}$ to its coarse moduli space $\mathcal{X}^c$. It follows from [Ols, Proposition 6.1] that there exists a positive integer m such that mD is $\mathbb{Q}$-linearly equivalent to the pullback of a divisor D_0 on $\mathcal{X}^c$. As mD is a big divisor on $\mathcal{X}$, the divisor D_0 is big on $\mathcal{X}^c$ (Lemma 4.1.3). By Kodaira's criterion for bigness, there exists a positive integer m_2 such that $m_2 D_0$ is $\mathbb{Q}$-linearly equivalent to $A + E$, where A is an ample divisor on $\mathcal{X}^c$ and E is an effective divisor on $\mathcal{X}^c$. Write $n = m \cdot m_2$. We now see that $nD = m \cdot m_2 \cdot D \sim_{\mathbb{Q}} \pi^* m_2 D_0 \sim_{\mathbb{Q}} \pi^*(A + E)$. Since $\mathcal{A} := \pi^* A$ is ample, and $\mathcal{E} := \pi^* E$ is effective, this concludes the proof of the lemma. □

4.2 Vojta's conjecture for varieties and stacks

In this section, we let k be a number field. We begin by recalling Vojta's conjecture for heights of points on schemes, using [AMV] and [Voj]. Our statement of the conjecture is perhaps not the most standard, but is more natural for our setting as we will need the extension of the conjecture to algebraic stacks.

4.2.1 Discriminants of fields

We first recall discriminants of fields following Section 2 of [AMV].

Definition 4.2.1. *Given a finite extension E/k, define the* ***relative logarithmic discriminant*** *to be:*

$$d_k(E) = \frac{1}{[E:k]} \log|\mathrm{Disc}(\mathcal{O}_E)| - \log|\mathrm{Disc}(\mathcal{O}_k)| = \frac{1}{[E:k]} \deg(\Omega_{\mathcal{O}_E/\mathcal{O}_k}), \qquad (2.1)$$

where the second equality follows from the equality of ideals $(\mathrm{Disc}(\mathcal{O}_k)) = N_{k/\mathbb{Q}} \det \Omega_{\mathcal{O}_k/\mathbb{Z}}$.

4.2.2 Heights

In this paper we will use *logarithmic* (Weil) heights. For more details, we refer the reader to [BG, HS].

Definition 4.2.2. *Let d be $[k:\mathbb{Q}]$, and let M_k be a complete set of normalized absolute values on k. The* ***(logarithmic) height*** *of a point $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$ is defined to be:*

$$h_k(P) = \frac{1}{d} \sum_{v \in M_k} \log(\max_{0 \le i \le n} \{\|x_i\|_v\}).$$

If X is a projective variety with a projective embedding $\varphi : X \hookrightarrow \mathbb{P}^n$, we can define a height function $h_\varphi \colon X \to \mathbb{R}$ given by

$$h_\varphi(P) = h(\varphi(P)).$$

More generally, given a very ample divisor D on X, we define $h_D(P) = h(\varphi_D(P))$, where φ_D is the natural embedding of X in $\mathbb{P}^n$ given by D. (We stress that h_D is well-defined, up to a bounded function.)

Proposition 4.2.3. *The Weil height machine satisfies the following properties.*

(i) *If* $f\colon X \to Y$ *is a morphism, then* $h_{X,f^*D} = h_{Y,D} + O(1)$.

(ii) *If* D *and* E *are both divisors, then* $h_{D+E} = h_D + h_E + O(1)$.

(iii) *If* D *is effective,* $h_D \geq O(1)$ *for all points not in the base locus of* D.

Proof. See [HS, Theorems B.3.2.b, B.3.2.c, and B.3.2.e]. □

4.2.3 Vojta's conjecture

We again recall from the introduction Vojta's conjecture for schemes (Conjecture 1.4.1).

Conjecture 4.2.4 (Vojta)**.** *[Voj, Conjecture 2.3] Let* X *be a nonsingular projective scheme over* k*. Let* H *be a big line bundle on* X *and fix* $\delta > 0$*. Then there exists a proper Zariski closed subset* $Z \subset X$ *such that, for all closed points* $x \in X$ *with* $x \notin Z$*,*

$$h_{K_X}(x) - \delta h_H(x) \leq d_k(k(x)) + O(1).$$

Note that the discriminant term $d_k(k(x))$ equals zero when x is a rational point of X.

4.2.4 Vojta's conjecture for stacks

Before stating the version of Vojta's conjecture for Deligne-Mumford stacks, we introduce some preliminaries, following Section 3 of [AMV]. If S is a finite set of finite places of k, we let $\mathcal{O}_{k,S}$ be the ring of S-integers in k.

The stacky discriminant

Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_{k,S})$ be a finite type separated Deligne-Mumford stack. Given a point $x \in \mathcal{X}(\bar{k}) = X(\bar{k})$, we define $\mathcal{T}_x \to \mathcal{X}$ to be the normalization of the closure of x in $\mathcal{X}$. Note that $\mathcal{T}_x$ is a normal proper Deligne-Mumford stack over $\mathcal{O}_{k,S}$ whose coarse moduli scheme is $\mathrm{Spec}(\mathcal{O}_{k(x),S_{k(x)}})$. Here $S_{k(x)}$ is the set of finite places of $k(x)$ lying over S.

Relative discriminants for stacks

Definition 4.2.5. *Let E be a finite field extension of k, and let $\mathcal{T}$ be a normal separated Deligne-Mumford stack over $\mathcal{O}_E$ whose coarse moduli scheme is* $\mathrm{Spec}\mathcal{O}_E$. *We define the **relative discriminant** of $\mathcal{T}$ over $\mathcal{O}_k$ as follows:*

$$d_k(\mathcal{T}) = \frac{1}{\deg(\mathcal{T}/\mathcal{O}_k)} \deg(\Omega_{\mathcal{T}/\mathrm{Spec}(\mathcal{O}_k)}). \tag{2.2}$$

Note that $d_k(\mathcal{T})$ is a well-defined real number.

Heights on stacks

Let X be a finite type Deligne-Mumford stack over k with finite inertia whose coarse space X^c is a quasi-projective scheme over k. Fix a finite set of finite places S of K and a finite type separated Deligne-Mumford stack $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_{K,S})$ such that $\mathcal{X}_K \cong X$. Let H be a divisor on X. Let $n \geq 1$ be an integer such that nH is the pull-back of a divisor H^c on X^c. Fix a height function h_{H^c} for H^c on X^c.

Definition 4.2.6. *We define the* ***height function*** h_H *on* $X(k)$ *with respect to* H *to be*

$$h_H(x) := h_{H^c}(\pi(x)).$$

Note that h_H is a well-defined function on $X(k)$.

We now give another way to compute the height function, under suitable assumptions on X. By [KV, Theorem 2.1], a finite type separated Deligne-Mumford stack over k which is a quotient stack and has a quasi-projective coarse moduli space admits a finite flat surjective morphism $f\colon Y \to \mathcal{X}$, where Y is a quasi-projective scheme. Fix a height function h_{f^*H} on Y. We define the height $h_H(x)$ of $x \in \mathcal{X}(\overline{k})$ as follows. If $x \in \mathcal{X}(\overline{k})$, then we choose $y \in Y(\overline{k})$ to be a point over x, and we define

$$h_H(x) := h_{f^*(H)}(y).$$

It follows from the projection formula (which holds for Deligne-Mumford stacks, in particular see the introduction of [Vis]) that h_H is a well-defined function on $\mathcal{X}(\overline{k})$. Moreover, if H is ample, for all $d \geq 1$ and $C \in \mathbb{R}$, the set of isomorphism classes of $\overline{k}$-points x of $\mathcal{X}$ such that $h_H(x) \leq C$ and $[k(x) : k] \leq d$ is finite. The analogous finiteness statement for k-isomorphism classes can fail. However, the set of k-isomorphism classes of k-points x of $\mathcal{X}$ such that $h_H(x) + d_k(\mathcal{T}_x) \leq C$ and $[k(x) : k] \leq d$ is finite. In particular, as $h_H(x) + d_k(\mathcal{T}_x)$ has the Northcott property, the expression $h_H(x) + d_k(\mathcal{T}_x)$ can be considered as "the" height of x [AMV].

Proposition 4.2.7 (Vojta's Conjecture for stacks)**.** *Assume Conjecture 1.4.1 holds and fix $\delta > 0$. Let S be a finite set of finite places of k. Let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack over $\mathcal{O}_{k,S}$ whose generic fibre $X = \mathcal{X}_k$ is geometrically irreducible over k. There is a proper Zariski closed substack $Z \subset X$ such that, for all $x \in X(k) \setminus Z$ the following inequality holds*

$$h_{K_X}(x) - \delta h_H(x) \leq d_k(\mathcal{T}_x) + O(1).$$

Proof. This is [AMV, Proposition 3.2]. □

4.3 Applying the stacky Vojta conjecture

We prove a generalization of [Ih2, Proposition 2.5.1] to morphisms of proper Deligne-Mumford stacks, under suitable assumptions. We stress that our reasoning follows Ih's arguments in *loc. cit.* in several parts of the proof.

Let k be a number field, and let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of proper integral Deligne-Mumford stacks over $B = \mathrm{Spec}\mathcal{O}_{k,S}$, where $\mathcal{X}$ is a smooth algebraic stack with a projective coarse moduli space. Let h be a height function on $\mathcal{X}$ and let $h_{\mathcal{Y}}$ be a height function on $\mathcal{Y}$ associated to an ample divisor such that $h_{\mathcal{Y}} \geq 1$. Let η be the generic point of $\mathcal{Y}$, let $\mathcal{X}_\eta$ be the generic fibre of $f\colon \mathcal{X} \to \mathcal{Y}$, and let $\mathcal{X}_k$ be the generic fibre of $\mathcal{X} \to B$.

Proposition 4.3.1. *Assume Conjecture 1.4.1. Suppose that the morphism f is* *representable by schemes, and that $\mathcal{X}_\eta$ is smooth and of general type. Then there exists a real number $c(k, S, \mathcal{Y}, f)$ and a proper Zariski closed substack $\mathcal{Z} \subset \mathcal{X}$ such that, for all P in $\mathcal{X}(B) \setminus \mathcal{Z}$, the following inequality holds:*

$$h(P) \leq c(k, \mathcal{Y}, f) \cdot \big(d_k(\mathcal{T}_P) + h_{\mathcal{Y}}(f(P))\big).$$

Proof. Let Δ be an ample divisor on $\mathcal{X}$ such that the associated height h_Δ on $\mathcal{X}$ satisfies $h_\Delta \geq 1$. Since $\mathcal{X}_\eta$ is smooth and of general type, by the Kodaira criterion for bigness (Lemma 4.1.4), there exists an ample divisor A on $\mathcal{X}_\eta$, an effective divisor E on $\mathcal{X}_\eta$, and a positive integer n such that

$$n(K_{\mathcal{X}_\eta}) \sim_{\mathbb{Q}} A + E.$$

For a small enough $\varepsilon \in \mathbb{Q}_{>0}$, we can rewrite

$$\begin{aligned}(K_{\mathcal{X}} - \varepsilon\Delta)|_\eta = K_{\mathcal{X}_\eta} - \varepsilon\Delta|_\eta &\sim_{\mathbb{Q}} \left(\frac{1}{n}A + \frac{1}{n}E\right) - \varepsilon\Delta|_\eta \\ &= \left(\frac{1}{n}A - \varepsilon\Delta|_\eta\right) + \frac{1}{n}E.\end{aligned}$$

Thus, there exists an effective divisor E' on $\mathcal{X}_\eta$ and a positive integer m such that

$$m\left(\left(\frac{1}{n}A - \varepsilon\Delta|_\eta\right) + \frac{1}{n}E\right) \sim_{\mathbb{Q}} E'.$$

Taking Zariski closures of these divisors in $\mathcal{X}$, it follows that there exists a vertical $\mathbb{Q}$-divisor $\mathcal{F}$ on $\mathcal{X}$ and an effective divisor $\mathcal{E}$ on $\mathcal{X}$ such that

$$K_{\mathcal{X}} - \varepsilon\Delta + \mathcal{F} \sim_{\mathbb{Q}} \frac{1}{m}\mathcal{E}.$$

Recall that $\mathcal{X}_k$ denotes the generic fibre of $\mathcal{X} \to B$. Note that $\mathcal{X}_k$ is a smooth proper Deligne-Mumford stack over k with a projective coarse space. Moreover, Vojta's conjecture (Conjecture 1.4.1) implies Vojta's conjecture for stacks (Proposition 4.2.7). Therefore, by Vojta's conjecture for stacks (Proposition 4.2.7) applied to $\mathcal{X}_k$, there exists a proper Zariski closed substack $Z \subset \mathcal{X}_k$ such that, for all $P \in \mathcal{X}_k(k) \setminus Z$, the following inequality

$$h_{K_{\mathcal{X}_k}}(P) - \frac{1}{2}\varepsilon h_{\Delta}(P) \leq d_k(\mathcal{T}_P) + O(1)$$

holds, where we compute all invariants with respect to the model $\mathcal{X}$ for $\mathcal{X}_k$ over B. In particular, there exists a proper closed substack $\mathcal{Z}$ of $\mathcal{X}$ (namely, the closure of Z in $\mathcal{X}$) such that, for all P in $\mathcal{X}(B)$ not in $\mathcal{Z}$, the following inequality holds

$$h_{K_{\mathcal{X}}}(P) - \frac{1}{2}\varepsilon h_{\Delta}(P) \leq d_k(\mathcal{T}_P) + O(1). \tag{3.1}$$

Since $\mathcal{F}$ is a vertical divisor on $\mathcal{X}$, there is an effective divisor $\mathcal{G}$ on $\mathcal{Y}$ such that $\mathcal{F} \leq f^*\mathcal{G}$. Therefore, by Proposition 4.2.3, the inequality $h_{\mathcal{F}} \leq h_{f^*\mathcal{G}} + O(1)$ holds, outside of Supp $(f^*\mathcal{G})$, and $h_{f^*\mathcal{G}} = (h_{\mathcal{G}} \circ f) + O(1)$. In particular, since $h_{\mathcal{Y}}$ is a height associated to an ample divisor, we see that $h_{\mathcal{G}} \leq O(h_{\mathcal{Y}})$ by [Lan1, Proposition 5.4]. Therefore, for all points t in $\mathcal{Y}(k)$ and all $P \in \mathcal{X}_t(B) \setminus \mathrm{Supp}(f^*G)$, the inequality

$$h_{\mathcal{F}}(P) \;\; \leq \;\; h_{f^*\mathcal{G}}(P) + O(1) = h_{\mathcal{G}}(f(P)) + O(1) \leq O\big(h_{\mathcal{Y}}(f(P))\big) + O(1)$$

holds, outside of Supp $(f^*\mathcal{G})$. In particular, replacing $\mathcal{Z}$ by the union of $\mathcal{Z}$ with $\mathrm{Supp}(f^*\mathcal{G})$, it follows that

$$h_{\mathcal{F}} \leq O(h_{\mathcal{Y}} \circ f) + O(1) \tag{3.2}$$

outside $\mathcal{Z}$. Since $K_{\mathcal{X}} - \varepsilon\Delta + F$ is effective, it follows that, replacing $\mathcal{Z}$ by a larger proper closed substack of $\mathcal{X}$ if necessary, the inequality

$$h_{K_{\mathcal{X}}-\varepsilon\Delta+F} \geq O(1) \tag{3.3}$$

holds outside $\mathcal{Z}$ by Proposition 4.2.3 (3).

Let $d_k(\mathcal{T})$ be the function that assigns to a point P in $\mathcal{X}(\overline{k})$ the real number $d_k(\mathcal{T}_P)$. In particular, we obtain that

$$\begin{aligned} O(1) \leq h_{K_{\mathcal{X}}-\varepsilon\Delta+F} &\leq (h_{K_{\mathcal{X}}} - \frac{1}{2}\varepsilon h_\Delta) - \frac{1}{2}\varepsilon h_\Delta + h_F + O(1) \\ &\leq (h_{K_{\mathcal{X}}} - \frac{1}{2}\varepsilon h_\Delta) - \frac{1}{2}\varepsilon h_\Delta + O(h_{\mathcal{Y}} \circ f) + O(1) \\ &\leq d_k(\mathcal{T}) - \frac{1}{2}\varepsilon h_\Delta + O(h_{\mathcal{Y}} \circ f) + O(1), \end{aligned}$$

where the inequalities follow from Equation (3.3), Proposition 4.2.3.(2), Equation (3.2), and Vojta's conjecture (3.1) respectively.

We conclude that, for all t in $\mathcal{Y}(B)$ and all P in $\mathcal{X}_t(B) \setminus \mathcal{Z}$ the inequality

$$\frac{1}{2}\varepsilon h_\Delta(P) \leq d_k(\mathcal{T}_P) + O(h_{\mathcal{Y}}(t)) + O(1)$$

holds. Therefore, there is a real number $c > 0$ such that, for all t in $\mathcal{Y}(t)$ and all P in $\mathcal{X}_t$ not in $\mathcal{Z}$, the inequality

$$h_\Delta(P) \leq c \cdot \Big(d_k(\mathcal{T}_P) + O\big(h_{\mathcal{Y}}(t)\big)\Big) + O(1)$$

holds. In particular, replacing c by a larger real number if necessary, we conclude that

$$h_\Delta(P) \le c \cdot \Big(d_k(\mathcal{T}_P) + h_{\mathcal{Y}}(t)\Big) + O(1).$$

As Δ is ample and $h_\Delta \ge 1$, we conclude that, using [Lan1, Proposition 5.4] and replacing c by a larger real number if necessary, for all t in $\mathcal{Y}(t)$ and all P in $\mathcal{X}_t$ not in $\mathcal{Z}$, the inequality

$$h(P) \le O\big(h_\Delta(P)\big) \le c \cdot \big(d_k(\mathcal{T}_p) + h_{\mathcal{Y}}(f(P))\big) + O(1)$$

holds. In particular, replacing c by a larger real number $c(k, \mathcal{Y}, f)$ if necessary, we conclude that the following inequality

$$h(P) \le c(k, \mathcal{Y}, f) \cdot (d_k(\mathcal{T}_P) + h_{\mathcal{Y}}(f(P)))$$

holds. $\square$

4.4 Uniformity results

In this section we prove Theorem 1.3.

Lemma 4.4.1. *Let $f\colon X \to Y$ be a proper surjective morphism of proper Deligne-Mumford stacks over k which is representable by schemes. Let h be a height function on X and let h_Y be a height function on Y associated to an ample divisor with $h_Y \ge 1$. Assume Conjecture 1.4.1. Suppose that the generic fibre X_η of $f\colon X \to Y$ is smooth and of general type. There exists a proper Zariski closed substack $Z \subset X$ and a real number c depending only on k, X, Y, and f, such that, for all P in*

$X(k) \setminus Z$, the following inequality holds

$$h(P) \leq c \cdot \big(h_Y(f(P)) + d_k(\mathcal{T}_P)\big).$$

Proof. We may and do assume that X and Y are geometrically integral over k.

Let $\mu\colon \widetilde{X} \to X$ be a desingularization of X; see [Tem, Theorem 5.3.2]. Note that $\widetilde{f}\colon \widetilde{X} \to Y$ is a proper surjective morphism of proper Deligne-Mumford stacks whose generic fibre is of general type. Define $X_{exc} \subset X$ to be the exceptional locus of $\mu\colon \widetilde{X} \to X$, so that μ induces an isomorphism of stacks from $\widetilde{X} \setminus \mu^{-1}(X_{exc})$ to $X \setminus X_{exc}$. Note that X_{exc} is a proper closed substack of X, as X is reduced.

Let $\widetilde{h}$ be the height function on $\widetilde{X}$ associated to h, so that, for all $\widetilde{P}$ in $\widetilde{X}$, we have $\widetilde{h}(\widetilde{P}) = h(P)$. As we are assuming Conjecture 1.4.1, it follows from Proposition 4.3.1 that there exists a proper Zariski closed substack $\widetilde{Z} \subset \widetilde{X}$ such that, for all $\widetilde{P}$ in $\widetilde{X}(k) \setminus \widetilde{Z}$, the following inequality

$$\widetilde{h}(P) \leq c \cdot \big(h_Y(\widetilde{f}(P)) + d_k(\mathcal{T}_P)\big)$$

holds, where c is a real number depending only on k, Y, X, and f. (Here we use that $\widetilde{X} \to X$ only depends on X.)

Define Z to be the closed substack $\mu(\widetilde{Z}) \cup X_{exc}$ in X. Note that μ induces an isomorphism from $\widetilde{X} \setminus \mu^{-1}(Z)$ to $X \setminus Z$. Therefore, we conclude that, for all P in $X(k) \setminus Z$, the inequality

$$h(P) = \widetilde{h}(\widetilde{P}) \leq c \cdot \big(h_Y(f(P)) + d_k(\mathcal{T}_P)\big)$$

holds, where $\widetilde{P}$ is the unique point in $\widetilde{X}$ mapping to X. □

Proof of Theorem 1.3. Since U is constructible, we have that $U = \cup_{i=1}^n U_i$ is a finite union of locally closed substacks $U_i \subset Y$. Let Y_i be the closure of U_i in Y, let $X_i = X \times_Y Y_i$, and let $f_i : X_i \to Y_i$ be the associated morphism. Note that U_i is open in Y_i. In particular, to prove the theorem, replacing X by X_i, Y by Y_i, U by U_i, and $f : X \to Y$ by $f_i : X_i \to Y_i$ if necessary, we may and do assume that U is open in Y.

We now argue by induction on $\dim X$. If $\dim X = 0$, then the statement is clear.

As we are assuming Conjecture 1.4.1, it follows from Lemma 4.4.1 that there exists a proper Zariski closed substack $Z \subset X$ and a real number $c_0 > 0$ depending only on k, X, Y, and f such that, for all P in $X(k) \setminus Z$, the inequality

$$h(P) \quad \leq \quad c_0 \cdot \big(h_Y(f(P)) + d_k(\mathcal{T}_P)\big) \tag{4.1}$$

holds.

Let $X_1, \ldots, X_s \subset Z$ be the irreducible components of Z. For $i \in \{1, \ldots, s\}$, let $Y_i = f(X_i)$ be the image of Y_i in Y. Note that $f_i := f|_{X_i} \colon X_i \to Y_i$ is a proper morphism of proper integral Deligne-Mumford stacks which is representable by schemes. Moreover, for t in the open subscheme $Y_i \cap U$ of Y_i, the proper variety $X_{i,t}$ is hyperbolic, as $X_{i,t}$ is a closed subvariety of the hyperbolic variety X_t. Let h_i be the restriction of h to X_i, and let h_{Y_i} be the restriction of h_Y to Y_i.

Since X_i is a proper Zariski closed substack of X, it follows that $\dim X_i < \dim X$. Therefore, by the induction hypothesis, we conclude that there is a real number $c_i > 0$ depending only on k, X_i, Y_i, and f_i such that, for all P in $X_i(k)$, the following inequality

$$h(P) = h_i(P) \;\; \leq \;\; c_i \cdot \big(h_{Y_i}(f_i(P)) + d_k(\mathcal{T}_P)\big) = c_i \cdot \big(h_Y(f(P)) + d_k(\mathcal{T}_P)\big). \quad (4.2)$$

holds. Let $c' := \max(c_1, \ldots, c_s)$. By (4.2), we conclude that, for all P in $Z(k)$, the inequality

$$h(P) \;\; \leq \;\; c' \cdot \big(h_Y(f(P)) + d_k(\mathcal{T}_P)\big) \quad (4.3)$$

holds.

Combining (4.1) and (4.3), we conclude the proof of the theorem with $c := \max(c_0, c')$. □

4.5 Applications

In this section we apply our main result (Theorem 1.3) to some explicit families of hyperbolic varieties, and prove Theorems 1.4 and 1.5.

4.5.1 Application to curves

For $g \geq 2$ an integer, let $\mathcal{M}_g$ be the stack over $\mathbb{Z}$ of smooth proper genus g curves. Let $\overline{\mathcal{M}}_g$ be the stack of stable genus g curves. Note that $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are smooth finite type separated Deligne-Mumford stacks. Moreover, $\mathcal{M}_g \to \overline{\mathcal{M}}_g$ is an open immersion, and $\overline{\mathcal{M}}_g$ is proper over $\mathbb{Z}$. These properties of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are proven in [DM]. We fix an ample divisor H on $\overline{\mathcal{M}}_g$.

If X is a smooth projective curve of genus at least two over a number field k, we let $h\colon X(\overline{k}) \to \mathbb{R}$ be the height with respect to the canonical embedding $X \to \mathbb{P}^{5g-6}_k$. Moreover, we define the height of X to be the height of the corresponding k-rational point of $\overline{\mathcal{M}}_g$ with respect to the fixed ample divisor H on $\overline{\mathcal{M}}_g$ (following Section 4.2.4).

If X is a smooth projective curve of genus at least two over k, and P is the corresponding rational point of $\mathcal{M}_g$, we let $d_k(\mathcal{T}_X)$ denote $d_k(\mathcal{T}_P)$, as defined in Section 4.2.4.

Proof of Theorem 1.4. Since $U := \mathcal{M}_g$ is open in $Y := \overline{\mathcal{M}_g}$, we can apply Theorem 1.3 to the universal family of stable genus g curves $f\colon X \to Y$. □

Remark 4.5.1. *In Theorem 1.4, one can also use the (stable) Faltings height* $h_{\mathrm{Fal}}(X)$ *of* X *(instead of the height* h *introduced above). Indeed, it follows from [Fal1, Paz] that the Faltings height* $h_{\mathrm{Fal}}(X)$ *is bounded by* $h(X) + c$*, where* c *is a real number depending only on the genus of* X*.*

4.5.2 Hyperbolic surfaces

Recall that, if S is a smooth projective surface, then $c_1^2(S) = K_S^2$ and $c_2(S) = e(S)$ is the topological Euler characteristic. Moreover, by Noether's lemma, they are related by the following equality:

$$\chi(S, \mathcal{O}_S) = \frac{c_1(S)^2 + c_2(S)^2}{12}.$$

In particular, the information of K_S^2 and $\chi(S)$ is equivalent to the data of $c_1(S)$ and $c_2(S)$. Finally, we note that $c_2(S) \geq 1$ for any surface of general type S [Bea, X.1 and X.4].

A smooth proper morphism $f : X \to Y$ of schemes is a canonically polarized smooth surface over Y if, for all y in Y, the scheme X_y is connected and $\omega_{X_y/k(y)}$ is ample. If a and b are integers, we let $\mathcal{M}_{a,b}$ over $\mathbb{Z}$ be the stack of smooth canonically polarized surfaces S with $c_1(S)^2 = a$ and $c_2(S) = b$. Note that $\mathcal{M}_{a,b}$ is a finite type algebraic stack over $\mathbb{Z}$ with finite diagonal (cf. [MM, Tan]).

We start by proving the following lemma.

Lemma 4.5.2. *If S is a smooth hyperbolic surface over a field k, then S is canonically polarized.*

Proof. If S is a (smooth) minimal surface of general type, then the canonical model S^c is obtained by contracting all rational curves with self intersection -2 [Liu, Chapter 9]. Consequently, the singularities on a singular surface in $\mathcal{M}_{a,b}(k)$ are rational double points arising from the contraction of these -2 curves. As having a -2 rational curve would contradict S being hyperbolic, we see that S^c must be smooth, and thus equal to S. As the canonical bundle on S^c is ample, we conclude that S is canonically polarized. □

Let $\mathcal{M}^h_{a,b} \subset \mathcal{M}_{a,b}$ be the substack of hyperbolic surfaces, i.e., for a scheme S, the objects $f : X \to S$ of the full subcategory $\mathcal{M}^h_{a,b}(S)$ of $\mathcal{M}_{a,b}(S)$ satisfy the property that, for all s in S, the surface X_s is hyperbolic (Definition 4.1.2). We do not know of any result on the algebraicity of $\mathcal{M}^h_{a,b}$ (nor the algebraicity of $\mathcal{M}^h_{a,b} \times_{\mathbb{Z}} \mathrm{Spec}\mathbb{C}$). However, if S is a minimal projective surface of general type over $\mathbb{C}$ and $c_1^2(S) > c_2(S)$, then Bogomolov proved [Bog] that S contains only a finite number of curves of bounded genus, and thus S contains only finitely many rational and elliptic curves.

Yoichi Miyaoka [Miy, Theorem 1.1] proved a more effective version of Bogomolov's result, showing that in fact the canonical degree of such curves is bounded in terms of c_1^2 and c_2. Using these results we are able to prove the following.

Lemma 4.5.3. *If $a > b$, then $\mathcal{M}^h_{a,b} \times_{\mathbb{Z}} \mathrm{Spec}\mathbb{C}$ is a constructible substack of $\mathcal{M}_{a,b} \times_{\mathbb{Z}}$ $\mathrm{Spec}\mathbb{C}$.*

Proof. Let a and b be integers such that $a > b$. Let N be an integer such that, for all S in $\mathcal{M}_{a,b}(\mathbb{C})$, the ample line bundle $\omega_{S/\mathbb{C}}^{\otimes N}$ is very ample. In particular, S is embedded in $\mathbb{P}^n \cong \mathbb{P}(\mathrm{H}^0(S, \omega_{S/\mathbb{C}}^{\otimes N}))$. Let $\mathrm{Hilb}_{a,b}$ be the Hilbert scheme of N-canonically embedded smooth surfaces, and note that $\mathcal{M}_{a,b} = [\mathrm{Hilb}_{a,b}/\mathrm{PGL}_{n+1}]$.

Let H_d be the Hilbert scheme of (possibly singular) curves of canonical degree d in $\mathbb{P}^n$. Let $\mathrm{H}_d^{\mathrm{int}}$ be the subfunctor of geometrically integral curves. Since the universal family over H_d is flat and proper, the subfunctor $\mathrm{H}_d^{\mathrm{int}}$ is an open subscheme of H_d; see [GW, Appendix E.1.(12)].

Let $\mathcal{W}_{a,b,d} \subset \mathrm{H}_d^{\mathrm{int}} \times \mathrm{Hilb}_{a,b}$ be the incidence correspondence subscheme parametrizing parametrizing pairs (C, S) where the curve C is inside the surface S. (Note that $\mathcal{W}_{a,b,d}$ is a closed subscheme of $\mathrm{H}_d^{\mathrm{int}} \times \mathrm{Hilb}_{a,b}$.)

By Miyaoka's theorem [Miy, Theorem 1.1], there exist integers $d_1, \ldots, d_m$ which depend only on a and b with the following property. A surface $S \in \mathcal{M}_{a,b}(\mathbb{C})$ is hyperbolic if and only if, for all $i = 1, \ldots, m$, it does not contain an integral curve of degree d_i.

Note that, by Chevalley's theorem, for all $d \in \mathbb{Z}$, the image of the composed morphism

$$\mathcal{W}_{a,b,d} \subset \mathrm{H}_d \times \mathrm{Hilb}_{a,b} \to \mathrm{Hilb}_{a,b} \to \mathcal{M}_{a,b}$$

is constructible. Let $\mathcal{M}_{a,b,d_i}$ be the stack-theoretic image of $\mathcal{W}_{a,b,d_i}$ in $\mathcal{M}_{a,b}$. Since a finite union of constructible substacks is constructible, the union $\bigcup_{i=1}^m \mathcal{M}_{a,b,d_i}$ is a constructible substack of $\mathcal{M}_{a,b}$.

Finally, by construction, a surface S in $\mathcal{M}_{a,b}(\mathbb{C})$ is hyperbolic if and only if it is not (isomorphic to an object) in the constructible substack $\bigcup_{i=1}^m \mathcal{M}_{a,b,d_i}$. As the complement of a constructible substack is constructible, we conclude that $\mathcal{M}_{a,b} \times_{\mathbb{Z}}$ Spec$\mathbb{C}$ is a constructible substack of $\mathcal{M}_{a,b} \times_{\mathbb{Z}} \mathbb{C}$. □

We let $\overline{\mathcal{M}_{a,b,\mathbb{Q}}}$ be a compactification of $\mathcal{M}_{a,b,\mathbb{Q}}$ with a projective coarse moduli space; see [Hac, Section 2.5] for an explicit construction of such a compactification. (As the stack of smooth canonically polarized surfaces is open in the stack of canonical models, it suffices to compactify the latter, as is achieved in *loc. cit.* for all a and b.) We now choose $\overline{\mathcal{M}}_{a,b}$ to be a compactification of $\mathcal{M}_{a,b}$ over $\mathbb{Z}$ whose generic fibre $\overline{\mathcal{M}}_{a,b} \times_{\mathbb{Z}}$ Spec$\mathbb{Q}$ is isomorphic to $\overline{\mathcal{M}_{a,b,\mathbb{Q}}}$. If S is a smooth projective canonically polarized hyperbolic surface over a number field k, we let $h : S(\overline{k}) \to \mathbb{R}$ be the height with respect to the very ample divisor $\omega_{S/k}^{\otimes 34}$ (see [Tan]). Moreover, we define the height of S in $\mathcal{M}_{a,b,\mathbb{Q}}(k)$ to be the height of the corresponding k-rational point of $\overline{\mathcal{M}}_{a,b}$ with respect to some fixed ample divisor H on $\overline{\mathcal{M}}_{a,b,\mathbb{Q}}$ (following Section 4.2.4). Also, if P is the rational point of $\mathcal{M}_{a,b}^h$ corresponding to S, then we let $d_k(\mathcal{T}_S)$ denote $d_k(\mathcal{T}_P)$ as defined in Section 4.2.4. (Here we compute $d_k(\mathcal{T}_P)$ with respect to the fixed $\mathbb{Z}$-model $\overline{\mathcal{M}}_{a,b}$ of $\overline{\mathcal{M}}_{a,b,\mathbb{Q}}$.)

Proof of Theorem 1.5. By Lemma 4.5.3 and standard descent arguments, we conclude that $\mathcal{M}_{a,b}^h \times_{\mathbb{Z}}$ Spec$\mathbb{Q}$ is a constructible substack of $\mathcal{M}_{a,b} \times_{\mathbb{Z}}$ Spec$\mathbb{Q}$. Also, a smooth hyperbolic surface is canonically polarized by Lemma 4.5.2. Therefore, the result follows from an application of Theorem 1.3 to the universal family over

$Y := \overline{\mathcal{M}}_{a,b,\mathbb{Q}}$ and the constructible substack $U := \mathcal{M}^h_{a,b}$ in Y. $\square$

Remark 4.5.4. *There are many examples of surfaces of general type with* $c_1^2 > c_2$*. Some of the simplest examples are surfaces* S *with ample canonical bundle such that there exist a smooth proper curve* C *and a smooth proper morphism* $S \to C$ *(see for instance [Kod]).*

CHAPTER **Five**

Log Canonical Models of Elliptic Surfaces

5.1 Introduction

We begin by recalling the main result (see Figure 1.1):

Theorem 5.1 (Theorem 1.6). *Let $(f : X \to C, S + aF)$ be an elliptic surface pair over C the spectrum of a DVR with reduced special fiber F such that F is one of the Kodaira singular fiber types (see Table 1), or f is isotrivial with constant j-invariant ∞.*

(i) If F is a type I_n or N_0 fiber (see Definition 5.5.3), the relative log canonical model is the Weierstrass model (see Definition 5.3.3) for all $0 \leq a \leq 1$.

(ii) For any other fiber type, there is an a_0 such that the relative log canonical model is

(a) the Weierstrass model for any $0 \leq a \leq a_0$*,*

(b) a twisted fiber *(see Defintion 5.4.9) consisting of a single non-reduced component when* $a = 1$*, or*

(c) an intermediate fiber *(Definition 5.4.9) that interpolates between the above two models for any* $a_0 < a < 1$*.*

The constant $a_0 = 0$ *for fibers of type* I_n^*, II^*, III^* *and* IV^**, and* a_0 *is as follows for the other fiber types:*

$$a_0 = \begin{cases} 5/6 & II \\ 3/4 & III \\ 2/3 & IV \\ 1/2 & N_1 \end{cases}$$

We also describe the singularities of the relative log canonical models in each case.

Theorem 5.2 (see Theorem 1.7)**.** *Let* $f : X \to C$ *be an elliptic fibration with section* S*. Furthermore, let* $F_{\mathcal{A}} = \sum a_i F_i$ *be a sum of reduced marked fibers* F_i *with* $0 \leq a_i \leq 1$*. Suppose that* $(X, S + F_{\mathcal{A}})$ *is the relative log canonical model over* C*. Then*

$$\omega_X = f^*(\omega_C \otimes \mathbb{L}) \otimes \mathcal{O}_X(\Delta).$$

where $\mathbb{L}$ *is the fundamental line bundle (see Definition 5.3.4) and* Δ *is an effective divisor supported on fibers of type* II, III*, and* IV *contained in* $\mathrm{Supp}(F)$*. The contribution of a type* II*,* III *or* IV *fiber to* Δ *is given by* αE *where* E *supports the* *unique nonreduced component of the fiber and*

$$\alpha = \begin{cases} 4 & II \\ 2 & III \\ 1 & IV \end{cases}$$

Corollary 5.1.1. *Let* $(f : X \to C, S + F_{\mathcal{A}})$ *be an irreducible slc elliptic surface with section* S *and marked fibers* $F_{\mathcal{A}}$. *Suppose that* $K_X + S + F_{\mathcal{A}}$ *is big. Then the log canonical model of* $(X, S + F_{\mathcal{A}})$ *is either*

(i) the relative log canonical model as described in Theorem 1.6, or

(ii) a pseudoelliptic surface obtained by contracting the section of the relative log canonical model whenever $(C, f_*F_{\mathcal{A}})$ *is not a weighted stable pointed curve (see Definition 5.6.5).*

We work over an algebraically closed field k of characteristic 0.

5.2 The log minimal model program in dimension two

First we recall some facts about the log minimal model program that we will use later. The standard reference is [KM] (for generalizations to log canonical surface pairs see [Fuj1]).

Throughout this section, X will denote a connected surface, $D = \sum a_i D_i$ will be a $\mathcal{Q}$-divisor with $0 \leq a_i \leq 1$, and (X, D) will be referred to as a surface pair.

We let $NS(X)$ denote the $\mathcal{Q}$-vector space of $\mathcal{Q}$-divisors modulo numerical equivalence. If $f : X \to S$ is a projective morphism, denote by $N_1(X/S)$ the $\mathcal{Q}$-vector space generated by irreducible curves $C \subset X$ contracted by f modulo numerical equivalence.

Let $\overline{NE}(X/S) \subset N_1(X/S)$ be the closure of the cone generated by effective curve classes. For any divisor $D \in NS(X)$, we let

$$\overline{NE}(X/S)_{D\geq 0} = \overline{NE}(X/S) \cap \{C : D.C \geq 0\}.$$

The first step of the log minimal model program is to understand these cones:

Theorem 5.3. *(Cone and contraction theorems for log canonical surfaces) Let (X, D) be a log canonical surface pair and $f : X \to S$ a projective morphism.*

(i) There exist countably many rational curves $C_j \subset X$ contracted by f and

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{(K_X+D)\geq 0} + \sum_j \mathbb{R}_{\geq 0}[C_j]$$

such that $R_j := \mathbb{R}_{\geq 0}[C_j]$ is an extremal ray for each j. That is, R_j satisfies $x, y \in R$ whenever $x + y \in R$ for any curve classes x, y.

(ii) For each extremal ray R as above, there exists a unique morphism $\varphi_R : X \to Y$ such that $(\varphi_R)_*\mathcal{O}_X = \mathcal{O}_Y$ and $\varphi_R(C) = 0$ for an integral curve C if and only if $[C] \in R$. In particular, $f : X \to S$ factors as $g \circ \varphi_R$ for a unique $g : Y \to S$. The pair $(Y, (\varphi_R)_*D)$ is a log canonical surface pair.

The morphism φ_R is called an **extremal contraction**. The log minimal model program takes as input a pair (X, D) and applies the above theorem repeatedly to construct a sequence of extremal contractions in hopes of reaching a birational model $f_0 : (X_0, D_0) \to S$ so that $K_{X_0} + D_0$ is f_0-nef. That is, with $\overline{NE}(X_0/S) = \overline{NE}(X_0/S)_{(K_{X_0}+D_0)\geq 0}$. If it exists, the pair (X_0, D_0) is a log minimal model of (X, D) over S. Note that this pair is not unique and depends on the sequence of extremal contractions. For surfaces, it can be shown (e.g. using the Picard rank) that a sequence of extremal contractions leads to a log minimal model after finitely many steps. Once we reach a log minimal model, the next step is provided by the abundance theorem:

Proposition 5.2.1. *[AFKM, Kaw](Abundance theorem for log canonical surfaces) Let (X, D) be a log canonical surface pair and $f : X \to S$ a projective morphism. If $K_X + D$ is f-nef, then it is f-semiample.*

A divisor B is *f-semiample* if the linear series $|mB|$ induces a morphism $\varphi_{|mB|} : X \to \mathbb{P}^N_S$ over S for $m \gg 0$. In this case, $\varphi_{|mB|}$ is a projective morphism with connected fibers satisfying $\varphi^*_{|mB|}H = mB$ where H is the hyperplane class on $\mathbb{P}^N_S$. In particular, $\varphi_{|mB|}(C) = 0$ if and only if $B.C = 0$ so $\varphi_{|mB|}$ is determined by numerical data of the divisor B.

In the setting of the abundance theorem, the map $\varphi_{|m(K_X+D)|}$ is the log canonical contraction and the image $(Y, (\varphi_{|m(K_X+D)|})_*D)$ is the log canonical model of the pair (X, D) over S. Thus

$$Y = \mathrm{Proj}_S \left(\bigoplus_{n\geq 0} H^0(X, n(K_X + D)) \right).$$

More generally, if (X, D) is any pair projective over S, the log canonical model of (X, D) over S is the pair $(Y, \varphi_* D)$ where

$$\varphi : X \dashrightarrow Y = \mathrm{Proj}_S \left(\bigoplus_{n \geq 0} H^0(X, n(K_X + D)) \right)$$

is the rational map induced by the linear series $|m(K_X + D)|$ for $m \gg 0$. Note that the log canonical model is unique and $K_Y + \varphi_* D$ is relatively ample over S.

The log minimal model program then gives us a method to compute the unique log canonical model of an lc surface pair (X, D) projective over a base S. First use the cone and contraction theorems to perform finitely many extremal contractions of $K_X + D$-negative curves to obtain a log minimal model (X_0, D_0) over S. Then $K_{X_0} + D_0$ will be relatively semiample by the abundance theorem so that the log canonical model is the image of the log canonical contraction $\varphi_0 : X_0 \to Y$ that contracts precisely the curves C such that $(K_{X_0} + D_0).C = 0$.

5.2.1 SLC pairs and stable pairs

To obtain compact moduli spaces, one is naturally forced to allow non-normal singularities as in the case of stable curves. The non-normal analogue of log canonical pairs are semi-log canonical pairs.

Definition 5.2.2. *A surface is* ***semi-smooth*** *if it only has the following singularities:*

(i) 2-fold normal crossings (locally $x^2 = y^2$), or

(ii) pinch points (locally $x^2 = zy^2$).

This naturally leads to the definition of a *semi-resolution.*

Definition 5.2.3. *A* ***semi-resolution*** *of a surface X is a proper map $g : Y \to X$ such that Y is semi-smooth and g is an isomorphism over the semi-smooth locus of X. A semi-log resolution of a pair (X, D) is a semi-resolution $g : Y \to X$ such that $g_*^{-1}D + \mathrm{Exc}(g)$ is normal crossings.*

Remark 5.2.4. *As in the case of a log canonical pair, the definition of a semi-log canonical pair can be rephrased in terms of a semi-log resolution (see Lemma 5.6.2).*

Note in particular that log canonical models are stable pairs and often we will use the words log canonical model and stable model interchangeably.

The cone, contraction, and abundance theorems for log surfaces hold as stated above for slc surfaces (see [Fuj3]). Thus one hopes that one can run the log minimal model program as described above to obtain the stable model of an slc pair (X, D). However, this is not always the case. An extremal contraction of an slc pair might result in a pair (X, D) where $K_X + D$ is *not* $\mathcal{Q}$-Cartier! This becomes an important issue in moduli theory. However, this issue does not occur for the specific non-normal elliptic surfaces we consider in this paper.

5.3 Preliminaries on elliptic surfaces

In this section we summarize the basic theory of elliptic surfaces.

5.3.1 Standard elliptic surfaces

We point the reader to [Mir] for a detailed exposition.

Definition 5.3.1. *An irreducible* ***elliptic surface with section*** $(f : X \to C, S)$ *is an irreducible surface* X *together with a surjective proper flat morphism* $f : X \to C$ *to a smooth curve and a section* S *such that:*

(i) the generic fiber of f *is a stable elliptic curve, and*

(ii) the generic point of the section is contained in the smooth locus of f*.*

We call $(f : X \to C, S)$ ***standard*** *if* S *is contained in the smooth locus of* f*.*

This differs from the usual definition in that we only require the generic fiber to be a *stable* elliptic curve rather than a smooth one.

Definition 5.3.2. *A surface is called* ***relatively minimal*** *if it semi-smooth and there are no* (-1)*-curves in any fiber.*

Note that a relatively minimal elliptic surface with section must be standard. If $(f : X \to C, S)$ is relatively minimal, then there are finitely many fiber components not intersecting the section. Contracting these, we obtain an elliptic surface with all fibers reduced and irreducible:

Definition 5.3.3. *A* ***minimal Weierstrass fibration*** *is an elliptic surface obtained from a relatively minimal elliptic surface* $(f : X \to C, S)$ *by contracting all fiber components not meeting* S*. We refer to this as the* ***minimal Weierstrass model*** *of* $(f : X \to C, S)$*.*

Definition 5.3.4. *The **fundamental line bundle** of a standard elliptic surface* $(f : X \to C, S)$ *is* $\mathbb{L} := (f_* N_{S/X})^{-1}$ *where* $N_{S/X}$ *denotes the normal bundle of* S *in* X*. For* $(f : X \to C, S)$ *an arbitrary elliptic surface, we define* $\mathbb{L} := (f'_* N_{S'/X'})^{-1}$ *where* $(f' : X' \to C, S')$ *is a semi-resolution.*

Since $N_{S/X}$ only depends on a neighborhood of S in X, the line bundle $\mathbb{L}$ is invariant under taking a semi-resolution or the Weierstrass model of a standard elliptic surface. Therefore $\mathbb{L}$ is well defined and equal to $(f'_* N_{S'/X'})^{-1}$ for $(f' : X' \to C, S')$ the unique minimal Weierstrass model of $(f : X \to C, S)$.

The fundamental line bundle greatly influences the geometry of a minimal Weierstrass fibration. The line bundle $\mathbb{L}$ has non-negative degree on C and is independent of choice of section S [Mir]. Furthermore, $\mathbb{L}$ determines the canonical bundle of X:

Proposition 5.3.5. *[Mir, Proposition III.1.1] Let* $(f : X \to C, S)$ *be either*

- *a Weierstrass fibration, or*
- *a relatively minimal smooth elliptic surface*

. Then $\omega_X = f^*(\omega_C \otimes \mathbb{L})$.

We prove a more general canonical bundle formula in Theorem 1.7.

Definition 5.3.6. *We say that* $f : X \to C$ *is **properly elliptic** if* $\deg(\omega_C \otimes \mathbb{L}) > 0$.

final section of this chapter

It is clear that X is properly elliptic if and only if the Kodaira dimension $\kappa(X) = 1$.

When $(f : X \to C, S)$ is a smooth relatively minimal elliptic surface, then f has finitely many singular fibers. These are unions of rational curves with possibly non-reduced components whose dual graphs are ADE Dynkin diagrams. The possible singular fibers were classified independently by Kodaira and Néron. Table 5.1 gives the full classification in Kodaira's notation for the fiber. Fiber types I_n for $n \geq 1$ are reduced and normal crossings, fibers of type I_n^*, II^*, III^*, and IV^* are normal crossings but nonreduced, and fibers of type II, III and IV are reduced but not normal crossings. With the exception of type I_0, I_1 and II, all irreducible components of the fibers are -2 curves.

Definition 5.3.7. *We will use **reduced fiber** to mean the reduced divisor $F = (f^*(p))^{red}$ corresponding to a (possibly nonreduced) fiber $f^*(p)$ for $p \in C$.*

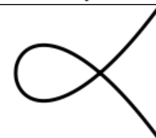
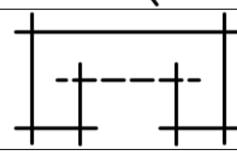
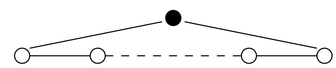
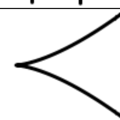
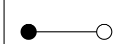
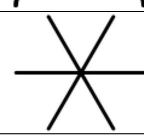
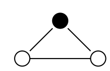
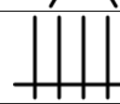
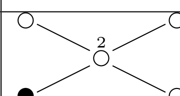
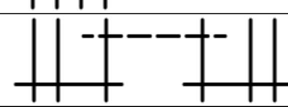
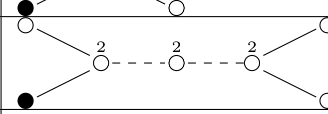
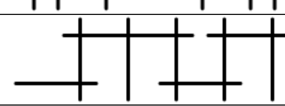
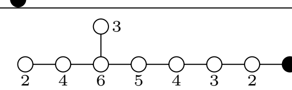
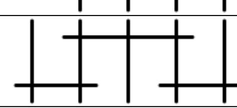
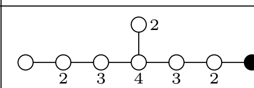
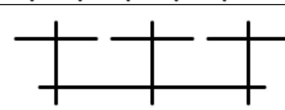
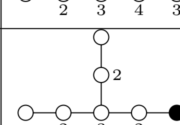
5.3.2 Weighted stable elliptic surfaces

Following the log minimal model program, in [AB3] we will study compactifications of the moduli space of irreducible elliptic surfaces with section and marked fibers obtained by allowing our surface pairs to degenerate to semi-log canonical (slc) stable pairs.

Let $\mathcal{A} = (a_1, \ldots, a_n) \in \mathcal{Q}^n$ such that $0 < a_i \leq 1$ for all i be a *weight vector*. We consider elliptic surfaces marked by an $\mathcal{A}$-weighted sum $F_{\mathcal{A}} = \sum_{i=1}^n a_i F_i$ with F_i reduced fibers.

Definition 5.3.8. *An **$\mathcal{A}$-weighted slc elliptic surface** is a tuple $(f : X \to C, S + F_{\mathcal{A}})$ where $(f : X \to C, S)$ is an elliptic surface with section and $(X, S + F_{\mathcal{A}})$ is an slc pair. An **$\mathcal{A}$-weighted stable elliptic surface** is an $\mathcal{A}$-weighted slc elliptic surface such that $(X, S + F_{\mathcal{A}})$ is a stable pair.*

Table 5.1: Kodaira's classification of fibers. The numbers in the Dynkin diagrams represent multiplicities of nonreduced components. Pictures shamelessly taken from Wikipedia.

Kodaira Type	# of components	Fiber	Dynkin Diagrams
I_0	1		
I_1	1 (double pt)		
I_n, $n \geq 2$	n (n intersection pts)		
II	1 (cusp)		
III	2 (meet at double pt)		
IV	3 (meet at 1 pt)		
I_0^*	5		
I_n^*, $n \geq 1$	$5 + n$		
II^*	9		
III^*	8		
IV^*	7		

In this paper we consider only irreducible elliptic surfaces. As observed by La Nave [LN], sometimes the log minimal model program contracts the section of an slc elliptic surface.

Definition 5.3.9. *A* ***pseudoelliptic surface*** *is a surface pair (Z, F) obtained by contracting the section of an irreducible slc elliptic surface pair $(f : X \to C, S + \bar{F})$. By abuse of terminology, we call F the marked fibers of Z.*

5.4 Relative log canonical models I: smooth generic fiber

We begin by computing the relative log canonical models of an $\mathcal{A}$-weighted elliptic surface $(f : X \to C, S + F_{\mathcal{A}})$ with smooth generic fiber.

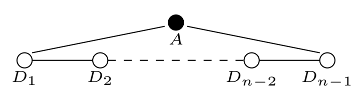
The question of whether $K_X + S + F_{\mathcal{A}}$ is f-ample is local on the base. Therefore we may assume that $C = \mathrm{Spec} R$ is a DVR with closed point s and generic point η. We then have the surface pair $(f : X \to C, S + aF)$ where $F = f^{-1}(s)_{red}$, and $0 \leq a \leq 1$, with generic fiber $X_\eta = f^{-1}(\eta)$ a smooth elliptic curve. In particular, we can make use of the classification of central fibers F in Table 1.

Since X is a normal 2-dimensional scheme and the pair is log canonical, we may replace X with a minimal log resolution before running the log minimal model program over C. For most central fiber types, this is the unique relatively minimal smooth model. However, for the special fiber types II, III, IV the special fiber of the relatively minimal model is not normal crossings so we must further resolve to obtain a log resolution (see Remark 5.4.4).

The section S passes through the smooth locus of f. In particular, S meets the special fiber in the smooth locus of a uniquely determined reduced fiber component. We fix notation and denote this component A (denoted by a black node in the dual graph; see Table 1).

Lemma 5.4.1. *Suppose F is a fiber of type I_n for $n \geq 0$. Then for any $0 \leq a \leq 1$, the stable model of $(X, S+aF)$ is the surface pair obtained by contracting all components of F not meeting the section so that only A remains. In particular, the stable model is the Weierstrass model of (X, S) with a single A_{n-1} singularity.*

Proof. Denote the components of F not meeting the section by D_i for $i = 1, \ldots, n-1$.



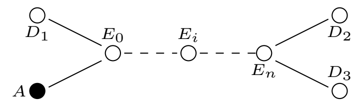
Note that the surface is relatively minimal so that K_X is pulled back from C by Proposition 5.3.5. This allows us to conclude that $K_X.D_i = K_X.A = 0$ for any fiber component. Furthermore, since F is reduced and normal crossings, the pair $(X, S + aF)$ is log canonical. We compute

$$(K_X + S + aF).D_i = 0$$
$$(K_X + S + aF).A = 1.$$

As a result, we must contract all components D_i independent of the coefficient a. □

Next we consider a type I_n^* fiber. The support of this fiber consists of $n + 5$ rational (-2)-curves and has dual graph affine D_{n+4}, where the root corresponds to the component A. There is a chain of nonreduced multiplicity 2 components

$E_0, \dots, E_n$ as well as 3 reduced components D_1, D_2 and D_3, corresponding to the remaining vertices of valence 1.



Lemma 5.4.2. *Suppose F supports an I_n^* fiber. Let $\varphi : (X, S + aF) \to (Y, \varphi_*(S + aF))$ be the stable model of $(X, S + F)$ over C. Then we have the following:*

(i) If $a = 0$ then φ contracts all components except A so that Y is the Weierstrass model with a single D_{n+4} singularity at the cusp of $\varphi_* A$;

(ii) If $0 < a < 1$ then φ contracts all components except E_0 and A. For $n = 0$ there are three A_1 singularities along $\varphi_* E_0$. For $n = 1$, there is an A_1 singularity and an A_3 singularity along $\varphi_* E_0$. For $n \geq 2$, there is an A_1 and a D_{n+2} singularity along $\varphi_* E_0$.

(iii) If $a = 1$ then φ contracts every component except E_0. When $n = 0$, there are four A_1 singularities along $\varphi_* E_0$. When $n = 1$ there are two A_1 singularities and one A_3 singularity along $\varphi_* E_0$. When $n \geq 2$, there are two A_1 singularities and a D_{n+2} singularity along $\varphi_* E_0$.

Proof. Again since X is relatively minimal over C, the canonical divisor $K_X \sim_{\mathcal{Q},f} 0$ so that $K_X.D = 0$ for any fiber component D. Furthermore, the pair $(X, S + aF)$ is log canonical since X is smooth and $S + aF$ is a normal crossings divisor. We compute

$$(K_X + S + aF).D_i = -a$$

for each leaf D_i, since D_i is disjoint from all components except either E_0 or E_n.

First suppose that $a > 0$. Then the log MMP contracts each leaf D_i to obtain a log terminal model with one A_1 singularity along E_0, and two A_1 singularities along E_n. Calling this map $\mu : X \to X'$, we see that $\varphi : X \to Y$ factors as $\varphi' \circ \mu$, where $\varphi' : X' \to Y$ is the log canonical contraction of $(X', S' + aF')$. Here, for any divisor D on X, we denote by $D' := \mu_* D$.

Now we compute

$$(K_{X'} + S' + aF').A' = S'.A' + a(A')^2 + aA'.E_0' = 1 - a.$$

$$(K_{X'} + S' + aF').E_i' = a(E_i')^2 + E_i'.(aE_{i-1}' + aE_{i+1}') = 0$$

for $i = 1, \ldots, n-1$. We have used that μ is an isomorphism in a neighborhood of A and of E_i for $1 \leq i \leq n-1$.

Next, we compute $\mu^*(E_0') = E_0 + 1/2D_1$ and $\mu^*(E_n') = E_n + 1/2D_2 + 1/2D_3$ so that

$$(E_0')^2 = (E_0 + 1/2D_1)^2 = -3/2$$
$$(E_n')^2 = (E_n + 1/2D_2 + 1/2D_3)^2 = -1.$$

It follows that

$$(K_{X'} + S' + aF').E_0' = 1/2a$$
$$(K_{X'} + S' + aF').E_n' = 0.$$

Therefore, when $0 < a < 1$, the morphism φ' contracts E_i' for $i = 1, \ldots, n$ leaving $\varphi_*' A'$ and $\varphi_*' E_0'$. When $a = 1$, the morphism φ' also contracts A' leaving just $\varphi_*' E_0'$.

Finally, if $a = 0$, the intersection $(K_X + S).A = 1$, and $(K_X + S).D = 0$ for any other fiber component D. Therefore, the morphism φ contracts all components except A, and $(Y, \varphi_*(S))$ is the Weierstrass model. The resulting singularities of $(Y, \varphi_*(S + aF))$ are deduced from the dual graphs of the trees of contracted components. □

Next we consider the Kodaira fibers of type II^*, III^* and IV^*, which have dual graph affine E_8, E_7 and E_6 respectively. There is a unique component E of valence 3, two leaves D_1 and D_2, and several valence 2 components B_j.

Proposition 5.4.3. *Suppose that F supports a fiber of type II^*, III^*, or IV^*. Let $\varphi : X \to Y$ be the stable model of $(X, S + aF)$ over C. Then we have the following:*

(i) If $a = 0$ then φ contracts all components except A so that Y is the Weierstrass model,

(ii) if $0 < a < 1$ then φ contracts all components except E and A, and

(iii) if $a = 1$ then φ contracts all components except E.

The singularities in each case are summarized in the table below:

	$a = 0$	$0 < a < 1$	$a = 1$
II^*	E_8	A_1, A_2, A_4	A_1, A_2, A_5
III^*	E_7	A_1, A_2, A_3	A_1, $2A_3$
IV^*	E_6	A_1, $2A_2$	$3A_2$

Proof of Proposition 5.4.3. As before, $(X, S + aF)$ is log canonical and $K_X \sim_{\mathbb{Q},f} 0$

so that $K_X.D = 0$ for any fiber component D. We compute:

$$(K_X + S + aF).A = 1 - a$$

$$(K_X + S + aF).E = a$$

$$(K_X + S + aF).D_i = -a$$

$$(K_X + S + aF).B_j = 0.$$

If $a = 0$ then $K_X + S$ is nef and induces the morphism φ. In this case φ contracts E, D_i and B_j, and gives the Weierstrass model.

If $a > 0$, then $K_X + S + aF$ is no longer nef. Therefore, by the log MMP there is an extremal contraction $\mu_1 : X \to X'$ contracting each leaf D_i. As before, for any divisor D on X, we denote by $D' := \mu_{1*}D$. Suppose one such D_i meets the valence 3 component E. Then

$$\mu_1^* E' = E + 1/2 D_i$$

so that by the projection formula

$$(E')^2 = (E + 1/2 D_i).E = -3/2.$$

This allows us to compute

$$(K_{X'} + S' + aF').E' = 1/2a.$$

If E does not meet any D_i, then since μ_1 is an isomorphism in a neighborhood of E:

$$(K_{X'} + S' + aF').E' = a.$$

Let B be a valence 2 component of F on X that meets one of D_i. Then B' is a valence one component passing through an A_1 singularity, and therefore

$$\mu_1^*(B') = B + 1/2D_i$$

for some i, from which it follows that

$$(B')^2 = (B + 1/2D_i)^2 = -3/2.$$

Now we can compute

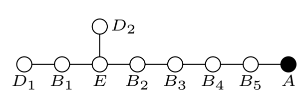
$$(K_{X'} + S' + aF').B' = -1/2a.$$

On the other hand, if B is a valence 2 component of F not meeting either D_1 or D_2, then μ is an isomorphism in a neighborhood of B so that

$$(K_{X'} + S' + aF').B' = 0.$$

Thus we must perform another extremal contraction $\mu_2 : X' \to X''$ that contracts exactly the valence 2 components that meet either D_1 or D_2, producing A_2 singularities.

For type II^* fiber there is exactly one of these, denoted B_1, meeting D_1, and then there are 4 valence 2 components $(B_2, \ldots, B_5)$ not meeting D_1 or D_2 (see figure below).



As before we use the notation $\mu_{2*}D' = D''$, and then

$$(\mu_2 \circ \mu_1)^* E'' = E + 1/3D_1 + 1/2D_2 + 2/3B_1$$

so that by the projection formula,

$$(E'')^2 = (E + 1/3D_1 + 1/2D_2 + 2/3B_1).E = -5/6.$$

The μ_i are isomorphisms in a neighborhood of $A, B_2, \ldots, B_5$. This lets us compute:

$$(K_{X''} + S'' + aF'').A'' = 1 - a$$

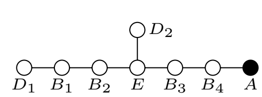
$$(K_{X''} + S'' + aF'').B_i'' = 0$$

$$(K_{X''} + S'' + aF'').E'' = 1/6a$$

so that $(K_{X''} + S'' + aF'')$ is f-nef and thus f-semiample by abundance (Proposition 5.2.1).

We take the log canonical contraction $\varphi'' : X'' \to Y$ over C to obtain the stable model with $\varphi = \varphi'' \circ \mu_2 \circ \mu_1$. If $0 < a < 1$ then φ'' contracts B_i'' and if $a = 1$ φ'' contracts B_i'' and A'' completing the claim.

In a type III^* fiber there is exactly one valence 2 component B_1 meeting D_1, and valence 2 components B_2, B_3, B_4 meeting neither D_1 nor D_2 as below:



In this case we have

$$(\mu_2 \circ \mu_1)^* B_2'' = B_2 + 2/3B_1 + 1/3D_1,$$

and μ_2 is an isomorphism away from B_2''. By the projection formula,

$$(B_2'')^2 = (B_2 + 2/3B_1 + 1/3D_1).B_2 = -4/3.$$

Then we can compute:

$$(K_{X''} + S'' + aF'').A'' = 1 - a$$
$$(K_{X''} + S'' + aF'').B_2'' = -1/3a$$
$$(K_{X''} + S'' + aF'').B_3'' = 0$$
$$(K_{X''} + S'' + aF'').B_4'' = 0$$
$$(K_{X''} + S'' + aF'').E'' = 1/2a$$

Therefore we must perform a third extremal contraction $\mu_3 : X'' \to X'''$, that contracts B_2'' and results in an A_3 singularity along E'''. We have

$$(\mu_3 \circ \mu_2 \circ \mu_1)^* E''' = E + 1/2D_2 + 1/4D_1 + 1/2B_1 + 3/4B_2$$

and so by the projection formula:

$$(E''')^2 = (E + 1/2D_2 + 1/4D_1 + 1/2B_1 + 3/4B_2).E = -3/4$$

Therefore

$$(K_{X'''} + S''' + aF''').A''' = 1 - a$$

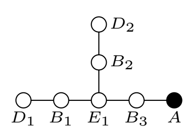
$$(K_{X'''} + S''' + aF''').B_3''' = 0$$

$$(K_{X'''} + S''' + aF''').B_4''' = 0$$

$$(K_{X'''} + S''' + aF''').E''' = 1/4a.$$

It follows that $K_{X'''} + S''' + aF'''$ is nef and thus f-semiample by abundance (Proposition 5.2.1), so that there is a log canonical contraction $\varphi''' : X''' \to Y$ to the stable model over C so that $\varphi = \mu_3 \circ \mu_2 \circ \mu_1 \circ \varphi'''$. If $0 < a < 1$, then φ''' contracts B_3''' and B_4''' and if $a = 1$, then φ''' contracts B_i''' and A''' as claimed.

Finally, in a type IV^* fiber there are two valence 2 components B_1 and B_2 meeting D_1 and D_2 respectively, and a valence 2 component B_3 meeting neither component:



We have two A_2 singularities along E''. Therefore

$$(\mu_2 \circ \mu_1)^* E'' = E + 1/3D_1 + 1/3D_2 + 2/3B_1 + 2/3B_2$$

and by the projection formula,

$$(E'')^2 = (E + 1/3D_1 + 1/3D_2 + 2/3B_1 + 2/3B_2).E = -2/3.$$

Furthermore, μ_i are isomorphisms in a neighborhood of B_3 and A. This lets us compute:

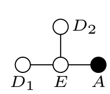
$$(K_{X''} + S'' + aF'').A'' = 1 - a$$
$$(K_{X''} + S'' + aF'').B_3'' = 0$$
$$(K_{X''} + S'' + aF'').E'' = 1/3a.$$

Thus $K_{X''} + S'' + aF''$ is f-nef and thus f-semiample by Proposition 5.2.1, so there is a log canonical contraction $\varphi'' : X'' \to Y$ to the stable model over C so that $\varphi = \mu_2 \circ \mu_1 \circ \varphi''$. If $0 < a < 1$, then φ'' is the contraction of B_3'' and if $a = 1$, then φ'' contracts both B_3'' and A''. □

Finally, we are left with type II, III, IV fibers in Kodaira's classification.

Remark 5.4.4. *These fibers F are such that $S+F$ is* not *a normal crossings divisor. As such, $(X, S + aF)$ does not have log canonical singularities for $a > 0$, and so we must first take a log resolution $q : Z \to X$ before we can run the log MMP with the pair $(Z, S + a\widetilde{F} + \mathrm{Exc}(q))$ to obtain the stable model. Here $\widetilde{F}$ is the strict transform of F, and we note that $(Z, S + a\widetilde{F} + \mathrm{Exc}(q))$ is log canonical.*

The dual graph of the special fiber of the log resolution looks as follows in each case:



However, the multiplicities and self intersections of the components vary depending on the type. Furthermore, since Z is *not* minimal over C, then $K_Z \not\sim_{\mathbb{Q},f} 0$.

Rather, the canonical class depends on the number of blowups performed to obtain the log resolution.

Proposition 5.4.5. *Suppose F supports a fiber of type II and let $q : Z \to X$ as above. Let $\varphi : Z \to Y$ be the stable model of $(Z, S + a\widetilde{F} + \mathrm{Exc}(q))$ over C. Then:*

(i) if $0 \leq a \leq 5/6$ then φ contracts all components except A so that Y is the Weierstrass model of (X, S),

(ii) if $5/6 < a < 1$ then φ contracts D_1 and D_2, and

(iii) if $a = 1$ then φ contracts all components except E.

Proof. The minimal log resolution q is obtained by three successive blowups so that D_1, D_2 and E are exceptional divisors with self intersection -3, -2, and -1 respectively. Furthermore, one can compute that A is a -6 curve. Here $\widetilde{F} = A$ and $\mathrm{Exc}(q) = D_1 + D_2 + E$. The canonical bundle is given by $K_Z = q^*(K_X) + D_1 + 2D_2 + 4E$. We compute:

$$(K_Z + S + aA + D_1 + D_2 + E).A = 6 - 6a$$

$$(K_Z + S + aA + D_1 + D_2 + E).D_1 = -1$$

$$(K_Z + S + aA + D_1 + D_2 + E).D_2 = -1$$

$$(K_Z + S + aA + D_1 + D_2 + E).E = a$$

Therefore, there is an extremal contraction $\mu : Z \to Z'$ contracting D_1 and D_2. Denote by $\Delta' := \mu_*\Delta$ for any divisor Δ on Z. Then $\mu^*E = E + 1/3D_1 + 1/2D_2$ and

$\mu^*(K_{Z'}) = K_Z + 1/3D_1$. Using the projection formula, we calculate

$$(E')^2 = (E + 1/3D_1 + 1/2D_2).E = -1/6$$
$$K_{Z'}.E' = (K_Z + 1/3D_1).E = -2/3,$$

which gives

$$(K_{Z'} + S' + aA' + E').E' = a - 5/6$$
$$(K_{Z'} + S' + aA' + E').A' = 6 - 6a.$$

When $0 \leq a < 5/6$, there is another extremal contraction $\varphi : Z' \to Y$ contracting E' resulting in a stable model Y that is the Weierstrass model. When $a = 5/6$, we see that $K_{Z'} + S' + aF'$ is f-nef and thus f-semiample by abundance (Proposition 5.2.1). The log canonical contraction contracts E' resulting in the Weierstrass model $\varphi : Z' \to Y$. In either case the stable model is the same, but $(Y, \varphi_*(S' + aA'))$ has log terminal (resp. log canonical) singularities for $a < 5/6$ (resp $a = 5/6$).

When $5/6 < a < 1$, then $K_{Z'} + S' + aF' + E'$ is ample so that $(Z', S' + aF' + E')$ is the stable model over C. This leaves the case $a = 1$: here $K_{Z'} + S' + aF' + E'$ is f-nef and thus f-semiample by (abundance) Proposition 5.2.1, and the log canonical contraction $\varphi : Z' \to Y$ contracts A' leaving just $\varphi_* E$. □

Proposition 5.4.6. *Suppose F supports a fiber of type III and let $q : Z \to X$ as above. Let $\varphi : Z \to Y$ be the stable model of $(Z, S + a\widetilde{F} + \mathrm{Exc}(q))$ over C. Then:.*

(i) If $0 \leq a \leq 3/4$ then φ contracts all components except for A resulting in the Weierstrass model,

(ii) if $3/4 < a < 1$ then φ contracts D_1 and D_2, and

(iii) if $a = 1$ then φ contracts all components except E.

Proof. The minimal log resolution $q : Z \to X$ is obtained by two successive blowups at the point of tangency. We have exceptional divisors $\mathrm{Exc}(q) = D_2 + E$ and $\widetilde{F} = A + D_1$ with self intersections $A^2 = D_1^2 = -4$, $D_2^2 = -2$ and $E^2 = -1$. Furthermore $K_Z = q^*K_X + D_2 + 2E$. Then:

$$(K_Z + S + a\widetilde{F} + \mathrm{Exc}(q)).A = 4 - 4a$$
$$(K_Z + S + a\widetilde{F} + \mathrm{Exc}(q)).D_1 = 3 - 4a$$
$$(K_Z + S + a\widetilde{F} + \mathrm{Exc}(q)).D_2 = -1$$
$$(K_Z + S + a\widetilde{F} + \mathrm{Exc}(q)).E = 2a - 1$$

Suppose that $0 \leq a < 1/2$. In this case there is an extremal contraction blowing back down E and D_2 – this is precisely the blow down $q : Z \to X$. Denoting by $\Delta' := q_*\Delta$ for any divisor Δ on Z, we have

$$(K_X + S' + a(A' + D_1')).A' = 1$$
$$(K_X + S' + a(A' + D_1')).D_1' = 0$$

so that $K_X + S' + a(A' + D_1')$ is f-nef and thus f-semiample by abundance (Proposition 5.2.1). The log canonical contraction $\varphi' : X \to Y$ contracts D_1' resulting in the Weierstrass model.

Now let $1/2 \leq a < 3/4$, in which case the first extremal contraction $\mu_1 : Z \to Z'$ contracts D_2. Note that this is an isomorphism away from E, and we calculate that $\mu_1^*(E') = E + 1/2D_2$, giving $(E')^2 = -1/2$. Since $\mu_1^*K_{Z'} = K_Z$, we also have that

$K_{Z'}.E' = K_Z.E = -1$. Thus:

$$(K_{Z'} + S' + a(A' + D_1') + E').E' = 2a - 3/2.$$

Since $a < 3/4$, there is a second extremal contraction $\mu_2 : Z' \to X$ contracting E' and resulting in the minimal model X again. As before, the $K_X + S'' + a(A'' + D_1'')$ is f-semiample and the log canonical contraction blows down D_1'' to obtain the Weierstrass model.

When $a = 3/4$, we perform the first extremal contraction of D_2 via $\mu_1 : Z \to Z'$ as above. Then $K_{Z'} + S' + a(A' + D_1') + E'$ is f-nef and the log canonical contraction will contract E' and D_1' resulting in the Weierstrass model.

Now suppose $3/4 < a < 1$; the first extremal contraction $\mu : Z \to Z'$ contracts D_1 and D_2 (since $3 - 4a < 0$). We compute that $\mu^*(E') = E + 1/2D_2 + 1/4D_1$ and $\mu^* K_Z' = K_Z + 1/2D_1$, which gives

$$E'^2 = (E + 1/2D_2 + 1/4D_1).E = -1 + 1/2 + 1/4 = -1/4$$
$$K_Z'.E' = (K_Z + 1/2B).E = -1/2.$$

This allows us to recompute

$$(K_{Z'} + S' + aA' + E').E' = a - 3/4 > 0.$$

Therefore $K_{Z'} + S' + aA' + E'$ is f-ample and Z' is the stable model over C.

Finally, suppose that $a = 1$. As above, there is an extremal contraction $\mu : Z \to Z'$ contracting D_1 and D_2 but now $(K_{Z'} + S' + A' + E').A = 0$ so that the log canonical contraction $\varphi : Z' \to Y$ contracts A. □

Proposition 5.4.7. *Suppose F supports a fiber of type IV and let $q : Z \to X$ be the minimal log resolution. Let $\varphi : Z \to Y$ be the stable model of $(Z, S + a\widetilde{F} + \mathrm{Exc}(q))$ over C. Then:*

(i) if $0 \le a \le 2/3$ then φ contracts all components except A resulting in the Weierstrass model,

(ii) if $2/3 < a < 1$ then φ contracts the leaves D_i, and

(iii) if $a = 1$, φ contracts all components except E.

Proof. The minimal log resolution $q : Z \to X$ is the blowup of the triple point. The exceptional divisor $\mathrm{Exc}(q) = E$ and $\widetilde{F} = A + D_1 + D_2$. The fiber components in X are -2 curves therefore their strict transforms A and D_i are -3 curves. Furthermore $K_Z = q^*(K_X) + E$. Therefore $K_Z.A = K_Z.D_i = 1$ and $K_Z.E = -1$. It follows that

$$(K_Z + S + aF + E).A = 3 - 3a$$

$$(K_Z + S + aF + E).D_i = 2 - 3a$$

$$(K_Z + S + aF + E).E = 3a - 2.$$

We begin with the case $0 \le a < 2/3$. Here, we see that $K_Z + S + aF + E$ is not nef and there exists an extremal contraction of E. This is precisely the blowup $q : Z \to X$. Denoting $\Delta' = q_*\Delta$ for any divisor Δ on Z, we have $(A')^2 = (D_i')^2 = -2$. Therefore

$$(K_{Z'} + S' + aF').A' = 1$$

$$(K_{Z'} + S' + aF').D_i' = 0.$$

Here we used that $K_X.\Delta = 0$ for any fiber component Δ as X is relatively minimal over C.

Now, we see that $K_X + S' + aF'$ is f-semiample and the morphism $\varphi' : X \to Y$ contracts D_i' resulting in the Weierstrass model.

When $a = 2/3$ we have a similar outcome. In this case $K_Z + S + aF + E$ is already and the morphism $\varphi : Z \to Y$ contracts D_i and E so again we obtain the Weierstrass model. The difference is that for $a = 2/3$, the singularities of the stable model are strictly log canonical while they are log terminal for $a < 2/3$.

Next suppose that $2/3 < a < 1$. In this case there is an extremal contraction $\mu : Z \to Z'$ that contracts the D_i. Note that $\mu^*(E') = E + 1/3D_1 + 1/3D_2$, thus by the projection formula $(E')^2 = -1/3$. Furthermore, $\mu^* K_{Z'} = K_Z + 1/3D_1 + 1/3D_2$. Since $K_Z \sim_{\mathcal{Q},f} E$ then by pushing forward we see that $K_{Z'} \sim_{\mathcal{Q},f'} E'$ where $f' : Z' \to C$ is the elliptic fibration; that is E' and $K_{Z'}$ are rationally equivalent over C. It follows that $K_{Z'}.A' = 1$ and $K_{Z'}.E = -1/3$.

This allows us to compute:

$$(K_{Z'} + S' + aF' + E').A' = 3 - 3a$$

$$(K_{Z'} + S' + aF' + E').E' = a - 2/3$$

Therefore, $K_{Z'} + S' + aF' + E'$ is f'-ample $\mu : Z \to Z'$ is the stable model over C.

Finally, suppose that $a = 1$. As above, there is an extremal contraction $\mu : Z \to Z'$ contracting the D_i. Based on the above calculation, $K_{Z'} + S' + aF' + E'$ is f'-nef and f'-semiample and the morphism $\varphi' : Z' \to Y$ contracts A'. This leaves just $\varphi'_* E'$. $\square$

In each of fibers type II, III and IV, the stable model over C is the Weierstrass model for $a \le a_0$ where $a_0 = 5/6, 3/4, 2/3$ respectively. We summarize the singularities obtained in the log canonical model for these fibers:

	$0 \le a \le a_0$	$a_0 < a < 1$	$a = 1$
II	A_0	A_1^*, A_2^*	A_1^*, A_2^*, A_5^*
III	A_1	A_1^*, A_3^*	A_1^*, $2A_3^*$
IV	A_2	$2A_2^*$	$3A_2^*$

Here A_0 denotes a smooth point at the cusp of the central fiber and A_{n-1}^* denotes the singularity obtained by contracting a rational $-n$ curve on a smooth surface. We use this dual notation suggestively – this is further discussed in [AB2].

Remark 5.4.8. *The numbers a_0 above can easily be seen to be the log canonical thresholds of the Weierstrass model with respect to the the corresponding central fiber.*

Definition 5.4.9. *Given a relative log canonical model of an elliptic surface with section $f : X \to C$, we say that a fiber of f is a* ***twisted fiber*** *if it is irreducible but non-reduced. We say that a fiber is an* ***intermediate fiber*** *if it is a nodal union of a reduced component A and a non-reduced component E such that the section meets the fiber along the smooth locus of A.*

Remark 5.4.10. *By the computations of this section, we see that the following are equivalent:*

- *log canonical models at $a = 1$ of fibers that are not of type I_n;*
- *twisted fibers.*

We can summarize the results above as stating that as we vary the coefficient a of the central fiber, the log canonical model interpolates between a twisted fiber at $a = 1$

and the Weierstrass fiber at $a = 0$. Below, we will see the same behavior in the non-normal case.

5.5 Relative log canonical models II: nodal generic fiber

We now turn to elliptic surfaces with nodal generic fiber. These must necessarily have constant j-invariant ∞. Such an elliptic surface is non-normal with normalization a birationally ruled surface over the same base curve.

As above we let $X \to \mathrm{Spec} R = C$ be a flat elliptic fibration with section S over the spectrum of a complete DVR. As usual F denotes the reduced divisor corresponding to the central fiber. We begin with the Weierstrass fibrations (see also [LN, Lemma 3.2.2]):

Lemma 5.5.1. *A Weierstrass fibration $f : X \to \mathrm{Spec} R$ with nodal generic fiber has equations $y^2 = x^2(x - \lambda t^k)$ where t is the uniformizer in R and λ is a unit. Furthermore, (X, S) is an slc pair if and only if $k \leq 2$.*

Proof. The form of the Weierstrass equation is given in [LN, Lemma 3.2.2]. The section S is a smooth divisor passing through the smooth locus of f so (X, S) is slc if and only if $(X, 0)$ is slc. Let $\nu : X^\nu \to X$ be the normalization. Then X^ν is defined by $w^2 = x - \lambda t^k$ and ν is induced by the homomorphism

$$\frac{k[x,y,t]}{(y^2 - x^2(x - \lambda t^k))} \to \frac{k[w,x,t]}{(w^2 - (x - \lambda t^k))}$$

$$y \mapsto wx$$

One can check that X^ν is a smooth surface. The double locus $x = y = 0$ in X pulls back to the locus $x = 0$ in X^ν. This is the divisor $D^\nu = \{w^2 = -\lambda t^k\}$. The pair (X^ν, D^ν) is slc if and only if D^ν is an at worst nodal curve and this occurs exactly when $k \leq 2$. □

After reparametrizing, we may suppose $\lambda = 1$. Next we compute the fiber types that appear in minimal log semi-resolutions of these slc Weierstrass models analogous to Kodaira's classification:

Lemma 5.5.2. *Consider the equation $y^2 = x^2(x - t^k)$ as above.*

(i) $k = 0$: $y^2 = x^2(x-1)$ is a semi-smooth surface and the elliptic fibration is a trivial family with all fibers nodal cubics;

(ii) $k = 1$: the minimal log semi-resolution of $y^2 = x^2(x - t)$ is an elliptic surface $f : Y \to \mathrm{Spec}R$ where the reduced central fiber is a nodal chain of rational curves A, B, E and E supports a multiplicity two fiber component intersecting both A and B;

(iii) $k = 2$: the minimal log semi-resolution of $y^2 = x^2(x - t^2)$ is an elliptic surface with $f : Y \to \mathrm{Spec}R$ with central fiber a nodal union of E and A where A is a reduced rational curve and E is a nodal cubic.

Proof. (i) is clear.

For (ii), we blow up once at $(0, 0, 0)$ to obtain a surface with two central fiber components: a nonreduced component of multiplicity 2 supported on the exceptional divisor of the blowup, and a reduced rational component given by the strict transform of the central fiber of the Weierstrass model. In local coordinates, two of the charts are smooth and the relevant chart is $u^2 = v^2 t(u-1)$ which has an A_1 singularity at $(u,v,t) = (1,0,0)$ but is semi-smooth elsewhere. Blowing this up yields the semi-resolution as described.

For (iii) we take the normalization of $y^2 = x^2(x-t^2)$ as in the proof of the above lemma. This is the smooth surface $w^2 = x - t^2$ with $wx = y$ and double locus $w^2 = -t^2$ the union of two components D_i intersecting at $(x,w,t) = (0,0,0)$. The central fiber of the normalization is the rational curve $w^2 = x$. Then blow up $(0,0,0)$ to obtain a rational surface X' and let E' be the exceptional divisor, A' the strict transform of the fiber and D_i' the strict transforms of the double locus. We may glue back together the D_i' to obtain a map $\mu : X' \to X$. Then X is a semi-smooth elliptic surface resolving our Weierstrass fibration and the central fiber consists of $E = \mu_*(E')$ and $A = \mu_*(A')$ as described. □

Definition 5.5.3. *The fibers N_k are the slc fiber types with Weierstrass equation $y^2 = x^2(x-t^k)$ for $k = 0,1,2$.*

The N_0 case is clear:

Lemma 5.5.4. *Let $f : X \to C$ with central fiber F of type N_0. Then $(X, S + aF)$ is relatively stable over C for all $0 \leq a \leq 1$.*

Proposition 5.5.5. *Suppose $f : X \to C$ is a type N_1 fiber. Let $q : Z \to X$ be the minimal log resolution. Let $\varphi : Z \to Y$ be the stable model of $(Z, S + a\widetilde{F} + \mathrm{Exc}(q))$ over C. Then:*

(i) when $0 \leq a \leq 1/2$, φ contracts all components except A giving the Weierstrass model;

(ii) when $1/2 < a < 1$, then φ contracts all components except for A and E; and

(iii) if $a = 1$, then φ contracts all components except E.

Proof. In the notation of Lemma 5.5.2, $\widetilde{F} = A$ and $\mathrm{Exc}(q) = B + E$. The central fiber is $A + B + 2E$. One can check that $A^2 = B^2 = -2$ and $E^2 = -1$ and that $K_Z.A = K_Z.B = 0$, $K_Z.E = 0$ by adunction ([KSB, Proposition 4.6]). We compute:

$$(K_Z + S + aA + B + E).A = 2 - 2a$$
$$(K_Z + S + aA + B + E).B = -1$$
$$(K_Z + S + aA + B + E).E = a.$$

There is an extremal contraction $\mu : Z \to Z'$ contracting the (-2) curve B. Letting $D' = \mu_* D$ for any divisor D on Z, then $(E')^2 = -1/2$ and the other intersection numbers remain unchanged since μ is crepant. Thus

$$(K_{Z'} + S' + aA' + E').A' = 2 - 2a$$
$$(K_{Z'} + S' + aA' + E').E' = a - 1/2.$$

Therefore when $a = 1$, there is a semi-log canonical contraction $\varphi : Z' \to Y$ contracting A'. When $1/2 < a < 1$ then Z' is the stable model over C and $\varphi = \mu$. When $a = 1/2$ there is a semi-log canonical contraction $\mu : Z' \to Y$ that contracts E' obtaining the Weierstrass model. Finally when $a < 1/2$, there is still an extremal contraction $\mu : Z' \to Y$ contracting E' yielding again the Weierstrass model. □

Proposition 5.5.6. *Suppose $f : X \to C$ is a type N_2 Weierstrass model, and let $q : Z \to X$ be the minimal semi-log resolution as in Lemma 5.5.2 with reduced central fiber F. Then the stable model $\varphi : Z \to Y$ of $(Z, S + aF)$ over C is a type N_0 Weierstrass model.*

Proof. The central fiber of $Z \to C$ is reduced with $F = A+E$ where A is rational and E is a nodal cubic such that $A.E = 1$. By adjunction ([KSB, Proposition 4.6]) we deduce that $K_Z.E = 1$ and $K_Z.A = -1$. Furthermore, we must have $A^2 = E^2 = -1$ so we see that

$$(K_Z + S + a(A+E)).A = -1 + 1 - a + a = 0$$
$$(K_Z + S + a(A+E)).E = 1.$$

Therefore, $K_Z + S + aF$ is relatively semiample over C and the semi-log canonical contraction $\varphi : Z \to Y$ contracts A yielding a Weierstrass model of type N_0. □

5.6 Canonical Bundle Formula

Using the above results, we can now compute the canonical bundle of a relatively stable elliptic surface pair.

Theorem 5.4. *Let $f : X \to C$ be a fibration where X is an irreducible elliptic surface with section S. Furthermore, let $F_{\mathcal{A}} = \sum a_i F_i$ be a sum of reduced marked fibers F_i with $0 \leq a_i \leq 1$. Suppose that $(X, S + F_{\mathcal{A}})$ is the relative log canonical model over C. Then*

$$\omega_X = f^*(\omega_C \otimes \mathbb{L}) \otimes \mathcal{O}_X(\Delta).$$

where Δ is effective and supported on fibers of type II, III, and IV contained in Supp(F)*. The contribution of a type II, III or IV fiber to Δ is given by αE where*

E supports the unique nonreduced component of the fiber and

$$\alpha = \begin{cases} 4 & II \\ 2 & III \\ 1 & IV \end{cases}$$

It is important to emphasize here that only type II, III or IV fibers that are *not* in Weierstrass form affect the canonical bundle. If all of the type II, III, and IV fibers of $f : X \to C$ are Weierstrass, then the usual canonical bundle formula $\omega_X = f^*(\omega_C \otimes \mathbb{L})$ holds.

Before proceeding with the proof, we will need the following two lemmas:

Lemma 5.6.1. *Let X be seminormal and $\mu : Y \to X$ a projective morphism with connected fibers. Then for any coherent sheaf $\mathcal{F}$ on X, we have that $\mu_*\mu^*\mathcal{F} = \mathcal{F}$.*

Proof. Note that $\mu_*\mathcal{O}_Y = \mathcal{O}_X$ by the defining property of being seminormal. The result then follows by the projection formula. □

Lemma 5.6.2. *Let (X, Δ) be an slc pair and $\mu : Y \to X$ a partial semi-resolution. Write*

$$K_Y + \mu_*^{-1}\Delta + \Gamma = \mu^*(K_X + \Delta) + B$$

where $\Gamma = \sum_i E_i$ is the exceptional divisor of μ and B is effective and exceptional. Then

$$\mu_*\mathcal{O}_Y(m(K_Y + \mu_*^{-1}\Delta + \Gamma)) \cong \mathcal{O}_X(m(K_X + \Delta)).$$

Proof. There is an exact sequence

$$0 \to \mu^*\mathcal{O}_X(m(K_X + \Delta)) \to \mathcal{O}_Y(m(K_Y + \mu_*^{-1}\Delta + \Gamma)) \to \mathcal{O}_{mB}(mB|_{mB}) \to 0.$$

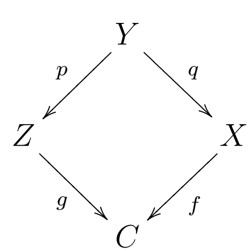
If $B = 0$ then $\mu^*\mathcal{O}_X(m(K_X + \Delta)) \cong \mathcal{O}_Y(m(K_Y + \mu_*^{-1}\Delta + \Gamma))$. Otherwise $B^2 < 0$, since $B \geq 0$ is exceptional and the intersection form on exceptional curves is negative definite [Kol3, Theorem 10.1]. Therefore $\mathcal{O}_{mB}(mB|_{mB})$ has no sections and so $\mu_*\mathcal{O}_{mB}(mB|_{mB}) = 0$. In either case,

$$\mu_*\mu^*\mathcal{O}_X(m(K_X + \Delta)) \cong \mu_*\mathcal{O}_Y(m(K_Y + \mu_*^{-1}\Delta + \Gamma)).$$

On the other hand, $\mu_*\mu^*\mathcal{O}_X(m(K_X + \Delta)) = \mathcal{O}_X(m(K_X + \Delta))$ by Lemma 5.6.1. □

Proof of Theorem 1.7. The formula is true for Weierstrass fibrations by [Mir, Proposition III.1.1]. These include fiber types N_0, N_2 and I_n for any coefficients as well as the relative log canonical models of any fiber with coefficient $a = 0$. For marked fibers of type I_n, I_n^*, II^*, III^*, and IV^* the minimal semi-resolution of the relative log canonical model is crepant. It follows that $f^*(\omega_C \otimes \mathbb{L}) \cong \omega_X$ away from type II, III, IV, and N_1 fibers contained in Supp(F).

We can compute the contributions of these fiber types explicitly. In the minimal log resolution Y, the fibers consist of components E, A, and D_i, where A is a reduced fiber intersecting the section S, and the components D_i and S are disjoint, each intersecting E transversely. Note that E and D_i may support nonreduced fiber components. We have a diagram



where X is the log canonical model over C, Z is the Weierstrass model, and Y is the minimal log resolution obtained by finitely many blowups. In each case we have $p^*\omega_X = \omega_Y \otimes \mathcal{O}_Y(B)$ where B is effective and p-exceptional.

Since the formula holds for Weierstrass models, we need to consider the other fibers of type II, III, IV, and N_1 appearing in X. These are obtained either by contracting D_i in Y, or by contracting D_i and A. First consider when q contracts D_i. Since the D_i are rational curves with negative self intesection on a smooth surface Y, the singularities of $(X, 0)$ are log canonical. In particular, by Lemma 5.6.2 we have

$$q_*(\omega_Y(\sum D_i)) = \omega_X.$$

On the other hand,

$$\omega_Y = p^*g^*(\omega_C \otimes \mathbb{L}) \otimes \mathcal{O}_Y(B) = q^*f^*(\omega_C \otimes \mathbb{L}) \otimes \mathcal{O}_Y(B).$$

Therefore, by the projection formula:

$$\omega_X = q_*(q^*f^*(\omega_C \otimes \mathbb{L}) \otimes \mathcal{O}_Y(B + \sum D_i)) = f^*(\omega_C \otimes \mathbb{L}) \otimes q_*\mathcal{O}_Y(B + \sum D_i).$$

Now $\mathcal{O}_Y(B + \sum D_i)$ is effective and isomorphic to $\mathcal{O}_Y$ away from $E \cup \big(\cup D_i\big)$. Since $q_*\mathcal{O}_Y = \mathcal{O}_X$, it follows that $q_*\mathcal{O}_Y(B + \sum D_i) = \mathcal{O}_X(\Delta)$ where Δ is effective and supported on $q(E \cup \big(\cup D_i\big))$. The same argument works when q contracts the D_i and A.

Now we compute the contribution to Δ from each type of fiber. This is a local question in the neighborhood of such a fiber. Let $\varphi : Y \to Y'$ be the contraction of the component A meeting the section induced by the transition from intermediate to twisted fiber in the relative log canonical model. Let E denote the divisor supporting the nonreduced component of the intermediate fiber of Y and denote $\varphi_* D := D'$ for any divisor D on Y.

As above, $A^2 = -n$ for $n = 6, 4, 3, 2$ for fibers of type II, III, IV, or N_1 respectively. Then

$$\varphi^* K_{Y'} = K_Y + \frac{n-2}{n} A.$$

Furthermore, by the above, we know that

$$K_{Y'} = (f')^*(K_C + \mathbb{L}) + \alpha E'$$

for some α. Here $f' : Y' \to C$ and $f : Y \to C$ are the corresponding elliptic fibrations. Then

$$\varphi^*((f')^*(K_C + \mathbb{L})) + \alpha\varphi^*(E') = f^*(K_C + \mathbb{L}) + \alpha E + \frac{\alpha}{n} A$$

Since $\varphi_* K_Y = K_{Y'}$ as divisors and $K_Y - f^*(K_C + \mathbb{L})$ is supported on E, we see that

$$K_Y = f^*(K_C + \mathbb{L}) + \alpha E$$

and by equating the two expressions for $\varphi^*(K_{Y'})$ we get

$$\frac{\alpha}{n} = \frac{n-2}{n}$$

so $\alpha = n - 2$. □

Remark 5.6.3. *For a type N_1 fiber, $\alpha = 0$ so N_1 fibers don't contribute to ω_X* a posteriori.

Next we describe how the log canonical divisor intersects the section:

Proposition 5.6.4. *Let $(f : X \to C, S + F_{\mathcal{A}})$ be an $\mathcal{A}$-weighted slc elliptic surface that is stable over C. Then*

$$(K_X + S + F_{\mathcal{A}}).S = 2g - 2 + \sum_i a_i.$$

Proof. Let $I \subset \{1, \ldots, n\}$ be the indices such that $a_i = 1$, and let J be the complement of I. The section passses through the smooth locus of the surface in a neighborhood of any fiber that is not marked with coefficient $a_i = 1$. This includes F_j for $j \in J$. Therefore this formula follows from the adjunction formula away from F_i for $i \in I$. On the other hand, for the twisted fibers F_i, this is the content of Proposition 4.3.2 of [LN]. □

Definition 5.6.5. *[Has3] Let $g \in \mathbb{Z}_{\geq 0}$ and $\mathcal{A} = (a_1, \cdots, a_n) \in \mathcal{Q}^n$ be such that $0 < a_i \leq 1$ and $2g - 2 + \sum a_i > 0$. An $\mathcal{A}$-weighted stable pointed curve is a pair $(C, D_{\mathcal{A}} = \sum a_i p_i)$ such that C is a nodal curve of genus g, the p_i are in the smooth locus of C, and $\omega_C(D_{\mathcal{A}})$ is ample.*

Corollary 5.6.6. *If $(f : X \to C, S + F_{\mathcal{A}})$ is an $\mathcal{A}$-weighted stable elliptic surface, then $(C, \sum a_i p_i)$ is an $\mathcal{A}$-weighted stable pointed curve where $p_i = f_* F_i$.*

Let $(f : X \to C, S + F_{\mathcal{A}})$ be an slc elliptic surface such that $(C, \sum a_i p_i)$ is a weighted stable pointed curve where $p_i = f_* F_i$. In light of the above, we have that the log canonical model of $(X, S + F_{\mathcal{A}})$ is the same as the log canonical model of $(X, S + F_{\mathcal{A}})$ relative to the base curve C:

Corollary 5.6.7. *Let $(f : X \to C, S + F_{\mathcal{A}})$ be a relatively stable elliptic surface such that $(C, \sum a_i p_i)$ is a weighted stable curve. Then $(X, S + F_{\mathcal{A}})$ is stable. In particular, $(X, S + F_{\mathcal{A}})$ is of log general type and its log canonical model is an elliptic surface.*

We are left to consider the following:

Corollary 5.6.8. *The log minimal model program contracts the section of an $\mathcal{A}$-weighted slc elliptic surface if and only if either*

(i) $C \cong \mathbb{P}^1$ *and* $\sum a_i \leq 2$, *or*

(ii) C *is a genus* 1 *curve and* $a_i = 0$ *for all* i.

In either of the two cases above, if $X = E \times C$ is a product then the contraction of the section S is the projection $X \to E$ resulting in an elliptic curve as the log canonical model. Otherwise, the contraction of the section is birational and we obtain a pseudoelliptic.

In case (a), if the pair is of log general type then the resulting pseudoelliptic is the log canonical model. However, it is possible that that the pair is not of log general type in which case the log minimal model program will continue with either an extremal or log canonical contraction to produce a curve or point. In the next section, we describe how to determine the coefficients for which this happens. This is also discussed in greater detail in [AB3].

In case (b), the contraction of the section is necessarily the log canonical contraction and the resulting pseudoelliptic is the log canonical model.

5.7 Base curve of genus 0

In the last section we arrived at the log canonical model of an $\mathcal{A}$-weighted elliptic surface whenever the base curve has genus $g \geq 1$. We are left to analyze genus 0 base curve case.

Proposition 5.7.1. *Let $f : X \to C$ be a properly elliptic surface with section S. Then $K_X + S$ is big. In particular, any $\mathcal{A}$-weighted slc properly elliptic surface is of log general type.*

Proof. By assumption, $K_X = G + E$ where G is an effective sum of fibers and E is an effective divisor supported on fibral components. Then $(K_X + S).G > 1$ so $K_X + S$ is f-big and for a generic horizontal divisor D, the intersection $(K_X + S).D > 0$. It follows that $K_X + S + F_\mathcal{A}$ is big for any $F_\mathcal{A}$. □

Corollary 5.7.2. *Let $(f : X \to C, S)$ be a properly elliptic surface over $\mathbb{P}^1$. Then the log canonical model of $(f : X \to C, S + F_\mathcal{A})$ for any choice of marked fibers $F_\mathcal{A}$ is either*

(i) the relative log canonical model over C, or

(ii) the pseudoelliptic formed by contracting the section of the relative log canonical model.

This leaves $\deg \mathbb{L} = 1, 2$. Note that if the generic fiber of $f : X \to \mathbb{P}^1$ is smooth, then $\deg \mathbb{L} = 1, 2$ are exactly the cases corresponding to X being rational ($\deg \mathbb{L} = 1$) or birational to a K3 surface ($\deg \mathbb{L} = 2$).

Proposition 5.7.3. *Let $(f : X \to \mathbb{P}^1, S + F_\mathcal{A})$ be an $\mathcal{A}$-weighted slc elliptic surface with section and marked fibers and suppose* $\deg \mathbb{L} = 2$.

(i) If $\mathcal{A} > 0$, then $K_X + S + F_\mathcal{A}$ is big and the log canonical model is the pseudoelliptic obtained by contracting the section of the relative log canonical model;

(ii) If $\mathcal{A} = 0$, then the minimal model program results in a pseudoelliptic surface and the log canonical contraction contracts this surface to a point.

Proof. (i) As a big divisor plus an effective divisor is big, it suffices to prove the result for $\mathcal{A} = (\varepsilon, \ldots, \varepsilon)$ for some $0 < \varepsilon \ll 1$. In this case, each type II, III and IV fiber in the relative log canonical model $(g : Y \to \mathbb{P}^1, S + F_{\mathcal{A}})$ is a Weierstrass model. Then $\omega_Y = g^*(\omega_{\mathbb{P}^1} \otimes \mathbb{L})$ by the canonical bundle formula, but $\omega_{\mathbb{P}^1} \otimes \mathbb{L} = \mathcal{O}_{\mathbb{P}^1}$ since $\mathbb{L}$ is degree 2. Therefore

$$K_Y + S + F_{\mathcal{A}} = S + \varepsilon \left(\sum F_i \right)$$

and the result follows as in Proposition 5.7.1.

(ii) If $F_{\mathcal{A}} = 0$ then the relative log canonical model is the Weierstrass model $(g : Y \to \mathbb{P}^1, S)$ and $K_Y = 0$ as in part (a) so $K_Y + S = S$. We have $S^2 = -2$ by the adjunction formula so there is an extremal contraction of S to obtain a pseudoelliptic $\mu : Y \to Y_0$ and $\mu_*(K_Y + S) = K_{Y_0} \sim_{\mathcal{Q}} 0$. Therefore $|mK_{Y_0}|$ is basepoint free and induces a log canonical contraction to a point.

□

Proposition 5.7.4. *Let $(X, F_{\mathcal{A}})$ be an $\mathcal{A}$-weighted slc pseudoelliptic surface with marked fibers $F_{\mathcal{A}}$. Denote by Y the corresponding elliptic surface and $\mu : Y \to X$ the contraction of the section. Suppose* $\deg \mathbb{L} = 1$ *and* $0 < \mathcal{A} \leq 1$ *such that $K_X + F_{\mathcal{A}}$ is a nef and $\mathcal{Q}$-Cartier $\mathcal{Q}$-divisor. Then either*

(i) $K_X + F_{\mathcal{A}}$ is big and the log canonical model is an elliptic or pseudoelliptic surface;

(ii) $K_X + F_{\mathcal{A}} \sim_{\mathcal{Q}} \mu_*\Sigma$ where Σ is a multisection of Y and the log canonical map contracts X onto a rational curve;

(iii) $K_X + F_{\mathcal{A}} \sim_{\mathcal{Q}} 0$ and the log canonical map contracts X onto a point.

The cases above correspond to $K_X + F_{\mathcal{A}}$ *having Iitaka dimension* $2, 1$ *and* 0 *respectively.*

Proof. By the Abundance Conjecture in dimension two (Proposition 5.2.1), we know that $K_X + F_{\mathcal{A}}$ is semiample. Let $\varphi : X \to Z$ be the Iitaka fibration. If φ is birational, we are in situation (i) and $\kappa(X, K_X + F_{\mathcal{A}}) = 2$. Thus suppose φ is not birational.

Let $f : Y \to C$ be the elliptic fibration whose section S is contracted to obtain X and let $\mu : Y \to X$ be this contraction. Consider $g = \varphi \circ \mu : Y \to Z$. Let G be a generic fiber of f. Then $G^2 = G.B = 0$ for any fiber component B of the elliptic fibration. Writing

$$\mu^*(K_X + F_{\mathcal{A}}) = K_Y + tS + \widetilde{F}_{\mathcal{A}}$$

where $\widetilde{F}_{\mathcal{A}}$ is the strict transform of $F_{\mathcal{A}}$, we have that

$$\mu^*(K_X + F_{\mathcal{A}}).G = t.$$

On the other hand,

$$\mu^*(K_X + F_{\mathcal{A}}).G = (K_X + F_{\mathcal{A}}).\mu_* G \geq 0$$

by the projection formula and the assumption that $K_X + F_{\mathcal{A}}$ is nef.

Suppose $t = 0$ so that $\mu^*(K_X + F_{\mathcal{A}}).G = 0$ for a general fiber G. It follows that $(K_Y + tS + \widetilde{F}_{\mathcal{A}}).B = 0$ for *all* fiber components B. Indeed in the case of a Weierstrass of twisted fiber there is a unique fiber component B, and $dB \sim_{\mathbb{Q}} G$ for some $d \geq 1$. For an intermediate fiber consisting of a reduced component A and a component E

supporting a nonreduced component, we have that $A + dE \sim_{\mathbb{Q}} G$ for some $d \geq 2$ so

$$(A + dE).\mu^*(K_X + F_{\mathcal{A}}) = 0$$

but $K_X + F_{\mathcal{A}}$ is nef so $A.\mu^*(K_X + F_{\mathcal{A}}) = E.\mu^*(K_X + F_{\mathcal{A}}) = 0$.

Therefore $\mu^*(K_X + F_{\mathcal{A}}) = K_Y + \widetilde{F}_{\mathcal{A}}$ is trivial on both the fibers and the section and so must be numerically trivial. By abundance, it must be rationally equivalent to 0. Therefore, $K_X + F_{\mathcal{A}} \sim_{\mathbb{Q}} 0$ so we are in case (iii) and $\varphi : X \to Z$ is the contraction to a point. On the other hand, if $\varphi : X \to Z$ is the contraction to a point, then it is immediate that $K_X + F_{\mathcal{A}} \sim_{\mathbb{Q}} 0$ so that we are in case (iii) if and only if $t = 0$.

This leaves only the case where $t > 0$ and $\varphi : X \to Z$ is a contraction onto a curve. Note that Z is necessarily rational since the normalization of X is a rational surface. Now $\mu^*(K_X + F_{\mathcal{A}})$ is ample on the generic fiber of $f : Y \to C$ and $K_X + F_{\mathcal{A}}$ is base point free so it is linearly equivalent to an effective nonzero divisor D that avoids the point $\mu(S)$. Therefore $\mu^*(K_X + F_{\mathcal{A}})$ is linearly equivalent to an effective horizontal divisor. That is, $\mu^*(K_X + F_{\mathcal{A}}) \sim_{\mathbb{Q}} \Sigma$ where Σ is an effective multisection and $K_X + F_{\mathcal{A}} \sim_{\mathbb{Q}} \mu_*\Sigma$ since Σ is contained in the locus where μ is an isomorphism. □

Remark 5.7.5. *The proposition above then gives us a method for determining which situation of* $(i), (ii)$, *and* (iii) *we are in. Indeed since* $K_X + F_{\mathcal{A}}$ *is nef, it is big if and only if* $(K_X + F_{\mathcal{A}})^2 > 0$. *Furthermore,* $K_X + F_{\mathcal{A}} \sim_{\mathbb{Q}} 0$ *if and only if* $t = 0$. *Thus when* $K_X + F_{\mathcal{A}}$ *is not big, it suffices to compute whether* $t > 0$ *or not to decide whether the log canonical contraction morphism contracts the pseudoelliptic to a curve or a point.*

5.8 Wall and chamber structure

In this section we briefly discuss the wall and chamber in the domain of weights $\mathcal{A}$ for an $\mathcal{A}$-weighted slc elliptic surface $(f : X \to C, S + F_{\mathcal{A}})$. By the results in the rest of the paper the log canonical model remains the same within each chamber and changes across each wall. We use these walls in [AB3] to describe how compactifications of the moduli space of $\mathcal{A}$-weighted stable elliptic surfaces vary as the weight vector $\mathcal{A}$ varies.

Finally we end with a detailed example of a rational elliptic surface to demonstrate the various transitions the log canonical model undergoes across each type of wall.

5.8.1 Transitions from twisted to Weierstrass form

First we note the weights for which the relative log canonical models change as the weight decreases from 1 to 0:

- There is a wall at $a_i = 1$ where a non-stable fiber transitions between twisted and intermediate inside the chamber $a_i = 1 - \varepsilon$ for $0 < \varepsilon \ll 1$.
- There is a wall at $a_i = 5/6$ where a type II fiber transitions between intermediate and Weierstrass.
- There is a wall at $a_i = 3/4$ where a type III fiber transitions between intermediate and Weierstrass.
- There is a wall at $a_i = 2/3$ where a type IV fiber transitions between interme-

diate and Weierstrass.

- There is a wall at $a_i = 1/2$ where a type N_1 transitions between intermediate and Weierstrass.
- There is a wall at $a_i = 0$ where a non-stable fiber that is not of the above form transitions from intermediate to Weierstrass.

Across each of these walls, the relative log canonical model exhibits a birational transformation.

5.8.2 Contraction of the section

By Proposition 5.6.4, there is a wall at $2g(C) - 2 + \sum a_i = 0$ where the section is contracted by the log canonical contraction. In the chambers below the wall the section is contracted by an extremal contraction. This contraction is birational except in the case when X is birational to a product $E \times C$ in which case it is the projection to E. Note that this wall only exists when $g = 0$ or when both $g = 1$ and $\mathcal{A} = 0$.

5.8.3 Pseudoelliptic Contractions

These transitions occur when a pseudoelliptic surface X is contracted to a rational curve or a point. Let $(f : Y \to \mathbb{P}^1, S + F_{\mathcal{A}})$ be the corresponding elliptic surface with $\mu : Y \to X$ the contraction of the section. By Proposition 5.7.1, these walls do *not* occur for $\deg \mathbb{L} \geq 3$. When $\deg \mathbb{L} = 2$, there is a single such wall at $\mathcal{A} = 0$ when the log canonical contraction contracts X to a point by Proposition 5.7.3.

For $\deg \mathbb{L} = 1$, Proposition 5.7.4 guarantees that there are possibly two such walls. The first is when $(K_X + \mu_* F_{\mathcal{A}})^2 = 0$, so that $(X, F_{\mathcal{A}})$ is not of log general type. If $K_X + \mu_* F_{\mathcal{A}} \sim_{\mathbb{Q}} 0$ then the log canonical contraction maps to a point. If $K_X + \mu_* F_{\mathcal{A}} \not\sim_{\mathbb{Q}} 0$, the log canonical contraction maps to a rational curve and there is a further wall when $t = (K_X + \mu_* F_{\mathcal{A}}).\mu_* G = 0$, where G is a general fiber of f. At this wall the log canonical contraction maps to a point.

These walls are less explicit in that they depend on the particular configurations of singular fibers that are marked. However, since there are only finitely many combinations of singular fibers on Y with $\deg \mathbb{L} = 1$, one may compute these walls explicitly in any particular case as is illustrated by the following lemma and the example in the next subsection.

Lemma 5.8.1. *In the situation above, suppose $a_i < 1$ for all i. Then there is a wall at $\sum a_i = 1$ where the log canonical contraction maps to a point.*

Proof. Without loss of generality we take $f : Y \to \mathbb{P}^1$ to be the relative log canonical model. Since $a_i < 1$ for all i, the surface Y has no twisted fibers and so S passes through the smooth locus of f and $S^2 = -1$. Therefore $\mu : Y \to X$ is the contraction of a (-1) curve and we can compute explicitly

$$\mu^*(K_X + \mu_* F_{\mathcal{A}}) = K_Y + \left(\sum a_i - 1\right) S + F_{\mathcal{A}}.$$

Therefore the coefficient in front of S becomes 0 precisely when $\sum a_i = 1$ and the result follows by Proposition 5.7.4. □

5.8.4 A rational example

Let $X \to \mathbb{P}^1$ be a rational elliptic surface that contains exactly two singular fibers of type I_0^* whose existence follows from Persson's classification of rational elliptic surfaces [Per]. Denote the reduced singular fibers by F_0 and F_1. All other fibers are smooth and we denote the class of a general fiber by G. We fix F_1 to have coefficient 1 and give F_0 and G the same coefficient α. Then $\mathcal{A} = (\alpha, \alpha, 1)$ and we have the pair

$$(X \to C, S + \alpha(G + F_0) + F_1)$$

Since F_1 is kept with coefficient 1 it is a twisted model for all α. Thus $F_1 \sim_{\mathcal{Q}} 1/2G$. Furthermore, by the canonical bundle formula, $K_X \sim_{\mathcal{Q}} -G$ since $\deg \mathbb{L} = 1$ and there are no fibers of type II, III, or IV. Putting this together (with $F_{\mathcal{A}} = \alpha(G+F_0)+F_1$), we have

$$K_X + S + F_{\mathcal{A}} = S + \alpha F_0 + (\alpha - 1/2)G.$$

When $\alpha = 1$, the log canonical model is $f : X_1 \to \mathbb{P}^1$ with two twisted I_0^* fibers.

For $1/2 < \alpha < 1$, F_0 becomes an intermediate fiber with components A and E in the relative log canonical model. The log canonical model is an elliptic surface $X_{1-\varepsilon}$ with a map $X_{1-\varepsilon} \to X_1$ contracting A. Next we check

$$(K_X + S + \alpha(G + F_0) + F_1).S = 2\alpha - 1.$$

At $\alpha = 1/2$ the section of X_1 is contracted to a pseudoelliptic by the log canonical contraction $\mu : X_{1-\varepsilon} \to X_{1/2}$.

For $\alpha < 1/2$ the map $\mu : X_{1-\varepsilon} \to X_{1/2}$ is an extremal contraction. Writing

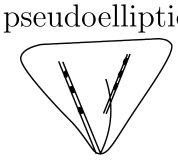
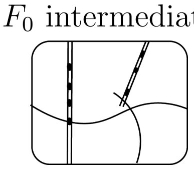
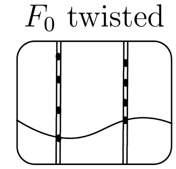
$$\mu^*(K_{X_{1/2}} + \mu_*(F_{\mathcal{A}})) = K_{X_{1-\varepsilon}} + tS + F_A$$

we compute $t = 4\alpha - 1$ by intersecting both sides with S and using that $S^2 = -1/2$ since S passes through an A_1 singularity along the twisted I_0^* fiber. Furthermore, using $F_0^2 = -1/2$ for an intermediate I_0^* fiber,

$$\begin{aligned}(K_{X_{1-\varepsilon}} + tS + F_A)^2 &= ((4\alpha - 1)S + \alpha F_0 + (\alpha - 1/2)G)^2 \\ &= (4\alpha - 1)^2(-1/2) + 2(4\alpha - 1)(2\alpha - 1/2) - \alpha^2/2 \\ &= \frac{1}{2}[(4\alpha - 1)^2 - \alpha^2] = \frac{1}{2}(3\alpha - 1)(5\alpha - 1).\end{aligned}$$

Therefore there is a pseudoelliptic contraction at $\alpha = 1/3$ where $K_{X_{1/2}} + \mu_*(F_{\mathcal{A}})$ is no longer big. Since $t > 0$ for $1/4 < \alpha \leq 1/3$, the log canonical class is a multisection and the log canonical contraction maps onto a rational curve. Finally at $\alpha = 1/4$, $t = 0$ so the log canonical class is trivial and the log canonical contraction maps to a point.

Table 5.2: We show the transformation of the elliptic surface $X \to \mathbb{P}^1$ as we lower the weight α on F_0 and G. We always keep F_1 with a fixed weight 1.

$0 \leq \alpha \leq 1/4$	$1/4 < \alpha \leq 1/3$	$1/3 < \alpha \leq 1/2$	$1/2 < \alpha < 1$, elliptic	$\alpha = 1$, elliptic
pt	curve	pseudoelliptic	F_0 intermediate	F_0 twisted
				

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