Abstract of "A Derivation and Analysis of the N-point Hierarchy For Conservation Laws And Multistable Dynamics with White Noise in the Allen-Cahn Equation" by Carey Caginalp, Ph.D., Brown University, May 2017.

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A Derivation and Analysis of the N-point Hierarchy For Conservation Laws And Multistable Dynamics with White Noise in the Allen-Cahn Equation

by

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A dissertation submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Division of Applied Mathematics at Brown University

Providence, Rhode Island

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Analytical and Numerical Results for Brownian Motion Through a Material, Workshop on Modeling of Materials with Fine Structure, Carnegie-Mellon University, May 2011. Abstract

Analytical and Numerical Results on Escape of Brownian particles - The 9th AIMS Conference on Dynamical Systems - Orlando, FL, July 2012. Abstract

Effects of White Noise in Multistable Dynamics in Differential Equations - AMS Sectional Meeting Special Session, Georgetown Univ, Washington, DC, March 7-8, 2015. Abstract

Propagation of Special Initial Data Through Burgers' Equation - Joint Dynamics and PDE Seminar – Brown University, April 25, 2016. Abstract

Effects of White Noise in Multistable Dynamics, Am. Inst. of Mathematics 11th Int. Conf on Dynamical Systems, Orlando, FL, July 1-5, 2016. Abstract

PUBLICATIONS

Analytical and numerical results for first escape time in 2D (with Xinfu Chen) - Comptes Rendus, C. R. Acad. Sci. Paris, Ser. I 349 (2011) 191– 194

Analytical and numerical results on escape (B. Phil. Thesis 2011) - University of Pittsburgh (2011)

Analytical and numerical results for an escape problem (with Xinfu Chen) - Archive for Rat. Mech. Analysis 203 (2012) 329– 342

Effects of white noise in multistable dynamics (with Xinfu Chen, Jianghao Hao and Yajing Zhang) - Discrete and Continuous Dynamical Systems B, 18 (2013)1805-1825

Preface

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Part I - A Derivation and Analysis of the N-point Hierarchy For Conservation Laws

Chapter 1

Introduction

1.1 Background on Conservation Laws

This dissertation consists of two parts. In the first part of this thesis, we develop kinetic equations for scalar conservation laws with a polygonal flux function. The classical prototype for nonlinear conservation laws is Burgers' equation, which has been studied in numerous works (e.g. [2], [3], [4], [5], [6], [14], [16], [20]). This is a simple model that produces shocks, given by:

$$\begin{aligned} u_t + uu_x &= 0 \\ u\left(x,0\right) &= g\left(x\right). \end{aligned} \tag{1.1}$$

This has also been studied extensively in Menon and Srivisan [31] involving random initial conditions, which is directly relevant to our current work. Under conditions of sufficient regularity, the solution of a generalized nonlinear conservation law

$$\begin{aligned} u_t + \left(f\left(u\right)\right)_x &= 0 \text{ on } \mathbb{R} \times (0, \infty) \\ u\left(x,0\right) &= g'\left(x\right) \text{ on } \mathbb{R} \times \{0\} \end{aligned} \tag{1.2}$$

can be related to that of the Hamilton-Jacobi problem

$$\begin{aligned} w_t + (f(w_x)) = 0 \text{ on } \mathbb{R}\times(0,\infty) \\ w = g \text{ on } \mathbb{R}\times\{0\} \end{aligned} \tag{1.3}$$

by the formula

$$w(x,t) = \int u(x,t)\,dx, \tag{1.4}$$

assuming $f \in C^2$. One can also consider an analogous problem for a discretized flux function, and we analyze this problem under a wide range of techniques, from Hopf-Lax analysis to front tracking, usage of the theorem of total probability, and finally the derivation of a formal hierarchy.

The classical Hopf-Lax formula is used to introduce the notion of an entropy solution to a scalar conservation law of the form (1.2). One method to construct the solution entails the use of curves (or as in this case, with no source term, straight lines) denoted *characteristics* in the space-time two-dimensional domain to simplify the PDE to a one-dimensional structure. Along these lines, the slope of which depends on the value at the initial condition, the solution is constant. If a point (x,t) can be traced back to a unique characteristic, the solution at this point is then prescribed by that value. However, it is often the case that multiple characteristics can be drawn back from a given point, leading to a multivalued solution if constructed solely from this method of tracing back characteristics. One method of well-posedness that was resolved by Hopf [22], and generalized by Lax [26], [27], involved the addition of a vanishing viscosity term εu_{xx} as a source to (1.1). Using various transformations and analytic techniques, they showed that there was indeed a closed form solution for the value of the solution. This solution identified the unique characteristic that would yield the solution with the desired properties. This solution formula is known as the Hopf-Lax formula and applies to convex, smooth flux functions f. However, this can also be used as a building block for some of our calculations with a piecewise linear flux, together with mollification and other arguments. In particular, if we denote the Legendre transform of the flux function f by

$$f^*(p) = \sup_{q\in\mathbb{R}} \{qp - f(q)\}, \tag{1.5}$$

the integrated initial condition as

$$G(p) = \int_0^p g(x)\,dx, \tag{1.6}$$

and the Hopf-Lax functional as

$$I(p,x,t) = G_0(p) + tf^*\left(\frac{x-s}{t}\right), \tag{1.7}$$

then the inverse Lagrangian point is expressed by the variational form

$$a(x,t) = \sup_{p\in\mathbb{R}}\left\{I(p,x,t) = \inf_{s\in\mathbb{R}} I(s,x,t)\right\}. \tag{1.8}$$

The solution is then expressed as

$$u(x,t) = \frac{x-a(x,t)}{t} \tag{1.9}$$

for Burgers' equation, or by

$$u(x,t) = (f')^{-1}\left(\frac{x-a(x,t)}{t}\right) \tag{1.10}$$

in the more general case. Of course, the further growth condition on the functional

$$\lim_{p\to\pm\infty} I(p,x,t) = \infty \tag{1.11}$$

is also needed in order to ensure $a(x,t)$ is finite and that the infimum is achieved. This is known as the Hopf-Lax formula for conservation laws. The notable case of Burgers' equation is achieved by setting $f(u) = \frac{1}{2}u^2$, upon which (1.8) may be rewritten as

$$a(x,t) = \arg^+\min_{p\in\mathbb{R}}\left\{G_0(p) + \frac{(x-p)^2}{2t}\right\},\ u(x,t) = \frac{x-a(x,t)}{t}. \tag{1.12}$$

The $^+$ means we pick the largest argument at which the minimum is obtained, should it occur at more than one point.

1.2 Evolution of Conservation Laws with Random Data

Burgers proposed his equation [4] as a model for turbulence of incompressible fluids. Even though it lacks some of the key features, it is a simple model that exhibits some key aspects of conservation laws. In particular it illustrates the development of shocks, even when the initial conditions are smooth. With the introduction of randomness in the initial conditions, a conservation law such as

Burgers' equation is useful to study to understand how the features of the randomness in the initial conditions evolve in time. In particular, questions that can be asked include how do the statistics of the solution evolve, whether Markov properties conserved, and whether there are universality classes for Burgers' equation, as in [30].

In Menon [29] and Menon and Srinivasan [31], further analysis on these equations was performed to gain deeper understanding of particle dynamics and the evolution of the system starting with random initial data. The analysis went beyond Burgers' equation to the more general case of a C^1, convex flux. Here, the initial conditions were restricted to processes that were *spectrally negative*, that is, stochastic processes with jumps but only in one direction. In particular, only downward jumps were permitted, as upward jumps lead to an immediate breakdown of the statistics. Furthermore, this initial stochastic process was also assumed to be Markov. Using the Levy-Khinchine representation for the Laplace exponent and other methods, formal calculations were used to establish a number of results. One remarkable assertion proven states that if one starts with strong Markov, spectrally negative initial data, then this Markov property persists in the entropy solution for any positive time $t > 0$. This closure result can also hold for a non-stationary process and can obtain an equation for a generator in the case of Feller processes. In general, Feller jump diffusions are described by a generator acting on C_c^∞ test functions, taking the form

$$\begin{aligned} A\varphi(u) &= a(u)\varphi''(u) + b(u)\varphi'(u) + c(u)\varphi(u) \\ &\quad + \int_{\mathbb{R}\setminus\{u\}} \left(\varphi(v) - \varphi(u) - \psi(u,v)\varphi'(u)\right) n(u,dv) \end{aligned} \tag{1.13}$$

for certain functions a, b, c, jump kernel $n(u, dv)$, and ψ describes certain types of interactions. For example, for a Feller process, the generator (1.13) becomes

$$A(t)\varphi(u) = b(u,t)\varphi'(u) + \int_{-\infty}^{u} \left(\varphi(v) - \varphi(u)\right) n(du, v, t). \tag{1.14}$$

Indeed, one main result of Menon and Srinivasan is that

$$\partial_t A = [A, B] \tag{1.15}$$

where brackets denote the Lie bracket, and the operator B is given by

$$B\varphi(u) = -f'(u)\, b(u,t)\, \varphi'(u) - \int_{-\infty}^{u} [f]_{u,v} (\varphi(v) - \varphi(u))\, n(u, dv, t) \tag{1.16}$$

where $[f]_{uv}$ is the slope given by the Rankine-Hugoniot condition

$$[f]_{uv} := \frac{f(u) - f(v)}{u - v}.$$

If the process is not stationary, (1.15) can be amended by adding in another term:

$$\partial_t A - \partial_x B = [A, B]. \tag{1.17}$$

Another surprising result along these lines is that these equations can admit exact solutions in certain special cases. The work of Groeneboom [20] calculates shock statistics for one such case with white noise initial data. A self-similar solution is obtained with explicit formulae for the jump density in terms of Laplace transforms of Airy functions.

Although a closure theorem is proved rigorously, many of the calculations in [31] such as the new kinetic equations and derivations of the Lax equation from four different approaches are formal. In [23], (1.15) is rigorously established. Specifically, they consider the Cauchy problem

$$\begin{cases} \rho_t = H(\rho)_x & \text{in } \mathbb{R} \times (0, \infty) \\ \rho = \xi & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{1.18}$$

for a stochastic process $\xi = \xi(x)$. Uniqueness of the solutions to marginal and kinetic equations was proven under the assumptions: (i) this process is a pure-jump process starting at $\xi(0) = 0$, evolving according to a specific rate kernel, (ii) that the Hamiltonian $H : [0, P] \to \mathbb{R}$ is smooth, convex, has downward jumps, nonnegative right derivative at 0, and noninfinite left-derivative at P. This is then used to prove the main result. In doing so, they consider a random particle system and write down equations for the infinite-dimensional system, much like in a statistical mechanics approach from physics.

1.3 Conservation Laws with Polygonal Flux

A polygonal flux function f consists of a finite number of piecewise linear segments. Denoting these slopes by $\{m_i\}_{i=1}^{n+1}$,we also require that they are in ascending order to force (non-strict) convexity of the flux function, and have break points at $\{c_i\}_{i=1}^{n}$. That is, we require $m_1 < m_2 < ... < m_n$. More precisely, define

$$f'(x) = m_i \text{ for } c_{i-1} < x < c_i,\ c_0 = -\infty,\ c_{n+1} = +\infty \tag{1.19}$$

As is readily calculated, the Legendre transform of such a function is another piecewise linear function on a bounded interval with the roles of the slopes and break points switched, and becomes infinite outside of that interval. Much of the original Hopf-Lax theory can still be implemented, as it involves minimization problems, which continue to make sense with the obvious correction of taking the minimum over the corresponding domain of the function argument rather than the entire real line. For the initial conditions, we take piecewise constant data whose range of values is specified by the slopes $\{m_i\}_{i=1}^{n+1}$ of the polygonal flux function. This sort of setup for the flux function was used in Dafermos [11] in an approximation argument, to build up to a smooth flux using increasingly fine polygonal approximations

Another related approach is that of a sticky particle system as considered by Brenier and Grenier [6]. As particles are discrete, this requires them to express the solution for the velocity field as a nonnegative measure rather than a function, and impose an entropy condition

$$\partial_t (\rho U(u)) + \partial_x (\rho u U(u)) \leq 0 \tag{1.20}$$

for all convex smooth functions U. They consider n particles on the real line, tracked by their weight (mass), position, and velocity. When two particles collide they "stick" together, and in this manner only a finite number of collisions (or shocks) can occur in this setup before a steady state is reached. A given pair of particles will either collide within a finite amount of time or never catch up due to their relative velocities. This is similar to the sort of behavior considered in this dissertation through the study of shocks both at the one and n-point levels.

1.4 Kinetic Equations Derived in This Dissertation

The main goal of this thesis is to derive kinetic equations that describe the evolution of the statistics of the solution to (1.2) when the flux function is polygonal, and the initial data is piecewise constant. This work improves on [31], as the results therein do not apply when the flux is not C^1. In this work, we pursue two different tracks for building a hierarchy of equations to describe the movement and interaction of shocks as they propagate through the system. In essence, the velocities at various points (and to a greater extent, shocks) can be treated as sticky particles. When an interaction occurs, they stick together and will not separate. Under the appropriate assumptions in regards to specific permitted initial data, there is no separation (i.e., rarefaction fanning), in such discrete cases. The virtue of this description of the problem and polygonal approximation scheme of Dafermos [11] lies in being able to enumerate possible interactions and illustrate results against a variety of examples. Further, in cases with such discrete states and corresponding initial data, the theorem in [31] does not hold, as the flux is no longer C^1.We derive results to fill in the gap and explain the results in our case of interest, with a polygonal flux. Similar work with a different approach is established in [23] with a finite boundary condition (that is, restriction of the problem to a box $0 \leq x \leq L$) and a representation of statistical solutions with bounded, monotone, and piecewise constant initial conditions.

In this dissertation, we consider a piecewise linear flux function as outlined in the proceeding subsections, using two different methods. In both approaches, a key object of importance to establish the hierarchy is the n-point function. This is the basic building block to track the important properties in a particle system in terms of probability distributions. For example, the one-point function of the form

$$p_1\left(x, t; u_i\right) \tag{1.21}$$

describes the probabilities (or distribution, in a continuous case) of a particle at location x with velocity state u_i, occurring at some time t in the system. Similarly, the two-point function may be described as

$$p_2\left(x_1, x_2, t; u_i, u_j\right) \tag{1.22}$$

and can be thought of in several ways. One interpretation is that we take a cross-section of the solution at time t and view it as a process, and (1.22) provides the information as to the probabilities (or distribution) of the solution with the values at the points x_1 and x_2 pinned to velocity states u_i and u_j, respectively. This definition can then easily be extended to any (finite) number of points, and in particular an n-point function of the form

$$p_n\left(x_1, ..., x_n, t; u_{i_1}, ..., u_{i_n}\right). \tag{1.23}$$

This simply corresponds to the probability that, at time t, the solution profile takes the value u_{i_1} at the point x_1, u_{i_2} at the point x_2, etc., up through u_{i_n} at x_n. The exact information specified can vary in different approaches that we consider, but the principle remains. At the n-th level, for a specific time t, there are n x-coordinates to which the n-point function ascribes information. Generally, the information at the nth level can be expressed in terms of various interactions at the (n+1) level. The goal is to ultimately provide a derivation of a set of equations known as a *hierarchy* and perform analysis and work various examples on it to check for consistency. It should be noted that later on in our work it becomes convenient to work with the tail cumulative distribution functions, that is

$$\begin{aligned} F_1\left(x, t; u_i\right) &= \sum_{l=i}^{n} p_1\left(x, t; u_l\right), \\ F_2\left(x_1, x_2, t; u_i, u_j\right) &= \sum_{l_1=i}^{n} \sum_{l_2=j}^{n} p_2\left(x_1, x_2, t; u_i, u_j\right), \end{aligned}$$

and so forth.

In the first of the approaches we consider, these particles are represented simply as the value of the velocity field at various points. When carried through, this method accurately describes the dynamics of the system and initial formation of shocks in a similar manner to the method of characteristics and front tracking. After the time at which shocks first encounter each other and combination of shocks occurs, it leads to a multivalued solution much like classical solutions. While this leads formally to a set of kinetic equations, they do not satisfy the entropy condition and do not ultimately give an adequate description for kinetic theory.

The latter, more cohesive and comprehensive approach comes from considering shocks rather than individual points in the velocity field as the basic building block of the particle system. In doing so, we change the definition and notation for the n-point function, writing it in the form

$$f_n\left(x_1, ..., x_n, t; u^1, ..., u^n, v^1, ..., v^n\right)$$

to represent the probability of the solution profile taking on values u^i at x_i- (the left-hand limit of the solution at x_i) and v^i at x_i. We assume right continuity, so one can regard it as the values at x_i- and x_i+, that is both limits, as one would consider it in a continuous case. In this approach it is necessary to make some reasonable assumptions on the initial data to avoid discretized rarefaction waves. In particular, with strictly ordered states $\{u_i\}$,the assumption that there are no *upward* jumps between non-neighboring states is required. With this particle structure, a complete (though not closed) hierarchy is obtained and tested rigorously against several examples. In particular, the n-point equation obtained by this method of focusing on shocks is given by

$$\begin{aligned}
&\partial_t f_n\left(x_1,...,x_n,t;u^1,...,u^n,v^1,...,v^n\right)+\sum_{i=1}^{n}c_{u_iv_i}\partial_{x_i}f_n\left(x_1,...,x_n,t;u^1,...,u^n,v^1,...,v^n\right)\\
&=\sum_{i=1}^{n}1_{\{v_i=u_i+1\}}\sum_{w}\left(c_{u_iw}-c_{wv_i}\right)\partial_i f_{n+1}\left(x_i,x_i,...,t;u^i,w,w,v^i....\right)\\
&-\sum_{i=1}^{n}\sum_{w\in W_2^i}\left(c_{u_iv}-c_{v_iw}\right)\partial_{i'}f_{n+1}\left(x_i,x_i,...,t;u^i,v^i,v^i,w,...\right)\\
&-\sum_{i=1}^{n}\sum_{w\in W_3^i}\left(c_{wu_i}-c_{u_iv_i}\right)\partial_i f_{n+1}\left(x_i,x_i,...,t;w,u^i,u^i,v^i,...\right). \qquad (1.24)
\end{aligned}$$

On the left-hand side of the equation we have a term representing the change in time of the n-point function, and a second *free-streaming* term keeping track of the relative motions of each pair $\left(u^i,v^i\right)$. The right-hand side contains three different types of collision terms. The first is a growth term, enumerating the possibilities by which a trajectory described by $\left\{u^i,v^i\right\}_{i=1}^n$ specified on the left-hand side can occur. Namely, this happens when a shock of the form $\left(u^i,w\right)$ collides with $\left(w,v^i\right)$ and forms $\left(u^i,v^i\right)$. The characteristic function in front represents the fact that this is only possible if the states u_i and v_i have an appropriate ordering, so that there is no upward jump between non-nearest neighbors formed, as it should be. The second and third terms refer to the possible ways in which this trajectory can be destroyed, and the sums enumerate the possibilities. Again depending on the ordering of u_i and v_i, a particular shock (u_i,v_i) can generally be destroyed by a shock from the right (second term) or shock from the left (third term). Only certain types of interactions are possible, as contained in the sets W_2^i and W_3^i for the indices. The intricacies and derivation of these exact lists of possibilities will be explained in greater detail later, but for completeness, we state the definitions here:

$$W_2^i = \left\{ w \mid \begin{array}{cc} w < u^i & \text{if } v^i = u^i + 1 \\ w \leq v^i + 1 & \text{if } v^i < u^i \end{array} \right.$$
$$W_3^i = \left\{ w \mid \begin{array}{cc} w > u^i + 1 & \text{if } v^i = u^i + 1 \\ w \geq u^i - 1 & \text{if } v^i < u^i \end{array} \right. \tag{1.25}$$

Unlike the first approach, this hierarchy of equations are consistent through collisions of shocks. When applied to examples with shock interactions, it is shown empirically that solving these equations yields a complete characterization of the solution profile. It persists even through shock interactions, an achievement that has eluded many classical methods without some sort of resetting or application of entropy conditions.

1.5 Multistable Dynamics in Allen-Cahn Equations

The second part of this thesis concerns a problem involving the introduction of white noise into Allen-Cahn equations. In basic ordinary differential equations, many become familiar with innocuous looking equations of the form

$$\frac{du}{dt} = 4\left(u - \frac{1}{2}\right)\left(1 - u^2\right). \tag{1.26}$$

It has a stable equilibrium points at ± 1 and an unstable equilibrium point at $\frac{1}{2}$, and analysis is clear-cut. It is somewhat less obvious, however, that under the addition of even an infinitesimal amount of randomness, the problem is transformed from simple and deterministic to probabilistic. Based on the relative strength of the associated potential calculated from the expression on the right-hand side, we delve into a rich collection of results regarding this and other stochastic ODEs. These results are interesting in many aspects, not the least of which that they initially counterintuitive to rudimentary ODE theory. We investigate the effect of replacing this bistable forcing term $4\left(u - \frac{1}{2}\right)\left(1 - u^2\right)$ with a generic drift $b\,(u)$, and consider the dynamics of the resulting stochastic differential equation

$$du_t = b(u_t)dt + \varepsilon\,\sigma(u_t)dW_t \tag{1.27}$$

where $\{W_t\}_{t\geqslant 0}$ is the Wiener process (Brownian motion), ε is a positive constant measuring the size of fluctuation, and $\sigma(\cdot)$ is a non-negative function modeling the dependence of fluctuation on system state. By standard Stochastic Differential Equation (SDE) theory ([38]), with certain conditions on σ and b, (1.27) admits a unique solution and the probability density of the random variable is the solution to the Fokker-Plank equation

$$\begin{cases} \partial_t \rho = \partial_u \left\{ \partial_u \left[\frac{1}{2}\, \varepsilon^2 \sigma^2 \rho \right] - b\rho \right\} & \text{on } \mathbb{R} \times (0,\infty), \\ \rho(\cdot,0) = \rho_0(\cdot) & \text{on } \mathbb{R} \times \{0\} \end{cases} \tag{1.28}$$

We expect the system to eventually achieve *stochastic equilibrium* in the sense that there is an invariant distribution ρ^* formed from the limit

$$\lim_{t\to\infty} \rho(\cdot,t) = \rho^*(\cdot). \tag{1.29}$$

In this second part we establish the existence of a unique invariant measure and analyze the asymptotic limit of vanishing noise, which in fact leads to Dirac measures. We also consider further results when the regularity and nondegeneracy conditions on the various parameters (σ, b) are relaxed. A more detailed introduction to this problem is presented in Part II.

Chapter 2

Background: A Discrete Example And Generalization for Piecewise Linear Flux

We start by providing a discrete pedagogical example with a piecewise linear flux, and analyze it using the Hopf-Lax formula. Our intention in doing so serves to illustrate the techniques for dealing with conservation laws with polygonal fluxes as in [11]. Although the assumptions of convexity and superlinearity are not met in a strict sense or do not hold, respectively, the solution formula can still be applied. The absence of the latter condition leads to infinite values for the Legendre transform of the flux function on all of $\mathbb{R}$ aside from a compact interval. However, this pathology can be mitigated by simply considering the minimization problem over this compact interval where the Legendre transform takes finite values.

As usual, any sort of example can be decomposed into two basic Riemann-style problems. The first is a downward jump in the initial conditions, which results in a single shock propitiating with a velocity determined by the Rankine-Hugoniot condition. We show that the Hamilton-Jacobi solution evaluated through the Legendre transform and minimization problem indeed matches the expected velocity of the shock. The second example is the reverse problem, with an upward jump in the initial solution profile. When passed through a smooth flux function, this sort of discontinuity results in a rarefaction fan. We argue that in the piecewise linear flux case, the solution profile will imitate this rarefaction structure to the best degree of fineness possible based on the available velocities contained in the flux function. In other words, a number of shocks appear between the two initial values, each between nearest neighbors, with velocities that match what would be given by the Rankine-Hugoniot condition. These two cases are illustrated below. Note that in our examples, the left and right slopes do not necessarily match with the slopes of the flux function. Thus, it does not truly correspond to the situation of kinetic theory, but it still provides some valuable insight into the behavior of solutions.

2.1 Downward Jump Case

We being by considering the Hamilton-Jacobi problem

$$u_t + H\left(u_x\right) = 0, \text{ in } \mathbb{R}\times \ (0,\infty) \tag{2.1}$$

$$u = g, \text{ on } \mathbb{R}\times \ \{0\}, \tag{2.2}$$

where H is a piecewise linear flux function, with slopes given by $\{m_i\}_{i=1}^{n+1}$ and break points $\{c_i\}_{i=1}^{n}$, with some initial condition that is piecewise linear with one break point at zero. In particular, choose the parameters $-c_7 = -2$, $-c_6 = -1.5$, $-c_5 = -.7$, $-c_4 = -.4$, $c_3 = -.2$, $-c_2 = .1$, $-c_1 = 2$, and set

$$g\left(y\right) = \begin{cases} .5y & y < 0 \\ -.3y & y > 0 \end{cases}. \tag{2.3}$$

The $\{m_i\}_{i=1}^{8}$ are arbitrary.

We want to minimize the expression $I\left(y;x,t\right) = tL\left(\frac{x-y}{t}\right) + g\left(y\right)$. The function I evaluated at the minimum is indeed the solution to the Hamilton-Jacobi equation. For the region $x < m_2 t$, the minimum will be attained at the value $y = x - m_5 t$, and we observe

$$u\left(x,t\right) = I\left(x - m_5 t; x, t\right) = tL\left(m_5\right) + .5\left(x - m_5 t\right) \tag{2.4}$$

and

$$\partial_x u\left(x,t\right) = .5. \tag{2.5}$$

Similarly, for the region $x > m_5 t$, the minimum is attained at $y = x - m_2 t$, and

$$u(x,t) = I(x - m_2 t; x, t) = tL(m_2) - .3(x - m_2 t) \tag{2.6}$$

so that

$$\partial_x u(x,t) = -.3. \tag{2.7}$$

The more interesting case occurs when $m_2 t \le x \le m_5 t$. In this case, we will have two local minima. They will be obtained at the y-values $y_1 = x - m_5 t$ and $y_2 = x - m_2 t$. We check when the values of I are equal at these two points:

$$\begin{aligned} I(x - m_5 t; x, t) = tL(m_5) + .5(x - m_5 t) &= tL(m_2) - .3(x - m_2 t) = I(x - m_2 t; x, t) \\ \Rightarrow x &= \frac{5}{4} t \{L(m_2) - L(m_5) + .5m_5 + .3m_2\} =: M^* t. \end{aligned} \tag{2.8}$$

Along $M^* t$, the value of y will not be unique. Both y_1 and y_2 give the value for the minimizer of $I(y; x, t)$. For $M^* t < x < m_5 t$, we have $I(...) = tL(m_5) + .5(x - m_5)t$, and for $m_2 t < x < M^* t$, we have $I(...) = tL(m_2) - .3(x - m_2 t)$. Hence the derivative is given by

$$\partial_x u(x,t) = \begin{cases} .5 & x > M^* t \\ -.3 & x < M^* t \end{cases}. \tag{2.9}$$

We want to compute precisely the slope of the shock formed in this case, and compare it with the result one would obtain by an application of the Rankine–Hugoniot condition. From the above calculation, we have

$$\begin{aligned} M^* &= \frac{5}{4} t \{L(m_2) - L(m_5) + .5m_5 + .3m_2\} \\ &= \frac{5}{4} \{c_4(m_5 - m_4) - c_3(m_4 - m_3) - c_2(m_3 - m_2) + .5m_5 + .3m_2\} \\ &= \frac{5}{4} \{-.4m_5 + .4m_4 - .2m_4 + .2m_3 + .1m_3 - .1m_2 + .5m_5 + .3m_2\} \\ &= \frac{5}{4} \{.1m_5 + .2m_4 + .3m_3 + .2m_2\} \\ &= \frac{1}{8} m_5 + \frac{1}{4} m_4 + \frac{3}{8} m_3 + \frac{1}{4} m_2. \end{aligned} \tag{2.10}$$

Now, from the Rankine-Hugoniot condition, one would expect to obtain

$$\sigma = \frac{[[H(u)]]}{[[u]]} = \frac{H(.5) - H(.3)}{.5 - (-.3)} = \frac{5}{4}(.2m_2 + .3m_3 + .2m_4 + .1m_5) \tag{2.11}$$
$$= \frac{1}{8}m_5 + \frac{1}{4}m_4 + \frac{3}{8}m_3 + \frac{1}{4}m_2,$$

and we see that they indeed agree.

2.2 Upward Jump (Rarefaction) Case

Now we observe we happens for an initial condition with an upward jump instead. Specifically, consider the initial condition defined by

$$g(y) = \begin{cases} -.5y & y < 0 \\ .3y & y > 0 \end{cases}. \tag{2.12}$$

The set of points which could potentially be minima for the expression $I(y; x, t) = tL\left(\frac{x-y}{t}\right) + g(y)$ for a given (x, t) is limited to a finite set, namely $y_i := x - m_i t$ and $\{0\}$. For $x - m_4 t < 0 < x - m_3 t$, i.e., $m_3 t < x < m_4 t$, the minimum is clearly attained at $y = 0$, so that

$$u(x,t) = I(0; x, t) = \min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} = tL\left(\frac{x}{t}\right) + g(0), \tag{2.13}$$

and differentiating yields

$$\partial_x u(x,t) = c_3. \tag{2.14}$$

For $x - m_3 t < 0 < x - m_2 t$, we again have

$$u(x,t) = I(0; x, t) = \min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} = tL\left(\frac{x}{t}\right) + g(0), \tag{2.15}$$

and differentiating yields

$$\partial_x u(x,t) = c_2. \tag{2.16}$$

Now, for $x - m_2 t < 0$, i.e. $x < m_2 t$, the slope of $I(y; x, t)$ is positive for $y > 0$, so the minimum must be attained for some $y < 0$, specifically at $y = x - m_2 t$. The minimum is then given by

$$I(x - m_2 t; x, t) = tL(m_2) + g(x - m_2 t) = tL(m_2) - .5(x - m_2 t), \tag{2.17}$$

and the derivative

$$\partial_x u(x,t) = -.5. \tag{2.18}$$

For $x - m_4 t > 0$, i.e. $x > m_4 t$, the slope of $I(y; x, t)$ is negative for $y < 0$, so the minimum must be achieved for some $y > 0$. Specifically, the minimum is then attained at $y = x - m_4 t$. The value at the minimum is then given by

$$I(x - m_4 t; x, t) = tL(m_2) + g(x - m_4 t) = tL(m_4) + .3(x - m_2 t), \tag{2.19}$$

and the derivative

$$\partial_x u(x,t) = .3. \tag{2.20}$$

Putting the value of $\partial_x u$ in all the different regions together, one has

$$\partial_x u(x,t) = \begin{cases} -.5 & x < m_2 t \\ c_2 & m_2 t < x < m_3 t \\ c_3 & m_3 t < x < m_4 t \\ +.3 & m_4 t < x \end{cases}. \tag{2.21}$$

From plotting this solution graphically, one can see that we obtain a rarefaction "fan" in a certain discretized sense. More precisely, note that from typical analysis using characteristics, one would expect the behavior in the regions $x < m_2 t$ and $x > m_4 t$. The region $m_2 t < x < m_4 t$ is left empty. For a smooth function, we would expect a rarefaction fan to open up in this wedge. Here, we observe that we still have a fan, in the sense of discretized wave speeds in accordance with the velocities of the flux function. In fact, it picks out the velocities c_i that are between the two slopes of the initial condition. In this case, this refers to $c_1 < -.5 < c_2 < c_3 < +.3 < c_4 < \ldots$ The solution profile and idea of this rarefaction in a discretized manner are illustrated in the figure below.

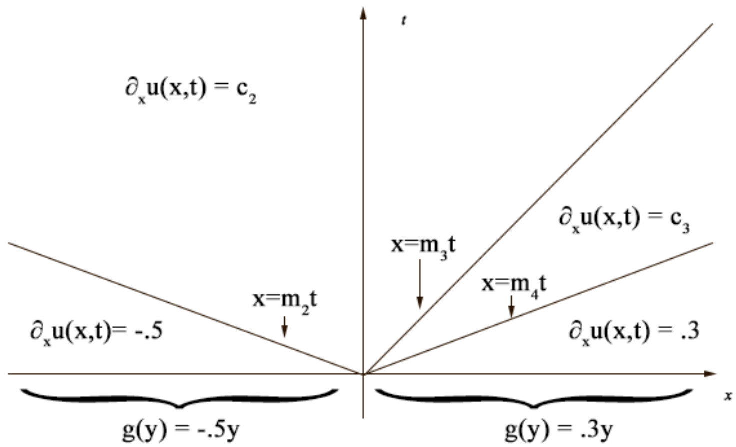

Figure 2.1: In the discrete case, an upward jump in the initial conditions results in the discretized version of a rarefaction wave. The region beteween splits into values between the two slopes, and the interface between regions is the line with slope prescribed by the Rankine-Hugoniot condition.

2.3 Generalization - Downward Jump

We now proceed to generalize the previous example in the form of a Proposition, from [11] as follows.

Proposition. *Let $H(\cdot)$ be a piecewise linear flux function with $n+1$ different slopes, denoted by $\{m_i\}_{i=1}^{n+1}$, $m_i < m_{i+1}$, and n corresponding break points $\{c_i\}_{i=1}^{n}$, $c_i < c_{i+1}$, and g be an initial condition of the form*

$$g(y) = \begin{cases} c_a y & y < 0 \\ c_b y & y > 0 \end{cases}. \tag{2.22}$$

where $c_b < c_a$. Consider the solution $u(x,t)$ to the corresponding Hamilton-Jacobi problem

$$u_t + H(u_x) = 0 \tag{2.23}$$

$$u = g.$$

Then we have

$$\partial_x u(x,t) = \begin{cases} c_a & x < m_{i+1}t \\ c_b & x > m_{i+1}t \end{cases}. \tag{2.24}$$

Proof. We consider two cases: (i) if there exists i such that $c_i < c_b < c_a < c_{i+1}$, that is, none of the c_i are between c_a and c_b, and (ii) otherwise, there exists some $j \geq 0$ such that $c_i < c_a < c_{i+1} <$

$... < c_{i+j} < c_b < c_{i+j+1}$.

To this end, consider an arbitrary initial condition of the form

$$g(y) = \begin{cases} c_a y & y < 0 \\ c_b y & y > 0 \end{cases} \tag{2.25}$$

with $c_a > c_b$ constant. We also take a flux function $H(\cdot)$ that has $n+1$ different slopes denoted by $\{m_i\}_{i=1}^{n+1}$, $m_i < m_{i+1}$, with n corresponding break points $\{c_i\}_{i=1}^{n}$, $c_i < c_{i+1}$. Take the Laplace transform in the usual way: $L(p) = \sup_{q\in\mathbb{R}} \{pq - H(q)\}$. Then $L(y)$ has slopes which will be given by $" -\infty"$, $\{c_i\}_{i=1}^{n}$, $" +\infty"$, $c_i < c_{i+1}$, and break points $\{m_i\}_{i=1}^{n+1}$. Now consider $f(y; x, t) := tL\left(\frac{x-y}{t}\right)$, as a function of y. The function will have slopes $-c_n, -c_{n-1}, ..., -c_1$, with break points at $y_{n+1}, y_n, ..., y_1$, where $y_i := y_i(x,t) = x - m_i t$, $1 \leq i \leq n+1$. Now, we find i, j such that $c_i < c_a < c_{i+j} \leq c_b < c_{i+j+1}$, noting that $j \geq 0$. If in fact $j = 0$, it means that there are no such i so that $c_a < c_i < c_b$, and if not, then we have c_i in this range from $i_* + 1$ to i^*. We split this into cases.

Case $j = 0$. In this case we have $c_i < c_b < c_a < c_{i+1}$, leading to the set of inequalities

$$\begin{aligned} &c_a - c_i > 0,\ c_a - c_{i+1} < 0, \\ &c_b - c_i > 0,\ c_b - c_{i+1} < 0. \end{aligned} \tag{2.26}$$

When $y_{i+1} > 0$, the slope of f near 0 is $-c_{i+1}$ on both sides. Since we have $c_a - c_{i+1} < 0$, $c_b - c_{i+1} < 0$, and $c_b - c_i > 0$, the minimum will be attained at y_{i+1}, and hence

$$\begin{aligned} u(x,t) &= tL\left(\frac{x - y_{i+1}t}{t}\right) + g(y_{i+1}) = tL(m_{i+1}) + g(x - m_{i+1}t) \\ \partial_x u(x,t) &= c_a. \end{aligned} \tag{2.27}$$

When $y_{i+1} < 0 < y_i$, the slope of f near 0 is $-c_i$ on both sides. Noting that $c_a - c_i > 0$, $c_a - c_{i+1} < 0$, and $c_b - c_i > 0$, we note that the minimum is attained at y_{i+1} (the vertex of the slopes $-c_{i+1}$ and $-c_i$ of f), and

$$\begin{aligned} u(x,t) &= tL\left(\frac{x - y_{i+1}t}{t}\right) + g(y_{i+1}) = tL(m_{i+1}) + g(x - m_{i+1}t) \\ \partial_x u(x,t) &= c_a\ (m_i t < x < m_{i+1} t) \end{aligned} \tag{2.28}$$

Finally, if $y_i < 0$, the slope of f near 0 is $-c_{i-1}$. Noting that $c_a - c_{i-1} > 0$, $c_b - c_{i-1} > 0$, $c_a - c_i > 0$, and $c_a - c_{i+1} < 0$, the minimum is attained at y_{i+1} (the vertex of the slopes $-c_{i+1}$ and $-c_i$), and

$$\begin{aligned} u(x,t) &= tL\left(\frac{x-y_{i+1}}{t}\right) + g(y_{i+1}) = tL(m_{i+1}) + g(x - m_{i+1}t) \\ \partial_x u(x,t) &= c_a \text{ (since } x - m_{i+1}t < x - m_i t < 0). \end{aligned} \tag{2.29}$$

Putting the pieces together, the solution is then given by

$$\partial_x u(x,t) = \begin{cases} c_a & x < m_{i+1}t \\ c_b & x > m_{i+1}t \end{cases}. \tag{2.30}$$

To check with the Rankine-Hugoniot condition, one observes that

$$\sigma = \frac{[[H(u)]]}{[[u]]} = \frac{H(c_b) - H(c_a)}{c_b - c_a} = \frac{(c_b - c_a)\, m_{i+1}}{c_b - c_a} = m_{i+1} \tag{2.31}$$

so it indeed matches.

Case $j > 0$. Then we have the string of inequalities $c_i < c_a < c_{i+1} < ... < c_{i+j} < c_b < c_{i+j+1}$. This leads to

$$\begin{aligned} &c_a - c_i > 0,\ c_a - c_{i+1} < 0, ...,\ c_a - c_{i+j} < 0,\ c_a - c_{i+j+1} < 0, \\ &c_b - c_i > 0,\ c_b - c_{i+1} > 0, ...,\ c_b - c_{i+j} > 0,\ c_b - c_{i+j+1} < 0. \end{aligned} \tag{2.32}$$

For $y_{i+j+1} > 0$, the slope of f near 0 is $-c_{i+j+1}$. Noting that $c_a - c_{i+j+1} < 0$ and $c_b - c_{i+j+1} > 0$, $c_b - c_{i+j} > 0$ we see that y_{i+j+1} is the minimum, hence

$$\begin{aligned} u(x,t) &= tL\left(\frac{x-y_{i+j+1}}{t}\right) + g(y_{i+j+1}) = tL(m_{i+j+1}) + g(x - m_{i+j+1}t) \\ \partial_x u(x,t) &= c_b. \end{aligned} \tag{2.33}$$

For $y_{k+1} < 0 < y_k$ for $i < k < i+j$, the slope of f near 0 is $-c_k$. Noting that $c_a - c_k < 0$ and

$c_b - c_k > 0$, we have that the minimum is at 0, hence

$$u(x,t) = tL\left(\frac{x}{t}\right) + g(0)$$
$$\partial_x u(x,t) = c_b \text{ (since } x > m_k t > m_i t\text{)}. \tag{2.34}$$

For $y_{i+1} < 0 < y_i$, the slope of f near 0 is $-c_i$. Since $c_a - c_i > 0$, $c_b - c_i > 0$, $c_a - c_{i+1} < 0$, the minimum is attained at y_{i+1} (at the vertex of the slopes $-c_i$ and $-c_{i+1}$ of f), so that

$$u(x,t) = tL\left(\frac{x - y_{i+1}}{t}\right) + g(y_{i+1}) = tL(m_{i+1}) + g(x - m_{i+1}t)$$
$$\partial_x u(x,t) = c_a \tag{2.35}$$

Finally, for $y_i < 0$, the slope of f near 0 is $-c_{i-1}$. We have $c_a - c_{i-1} > 0$, $c_b - c_{i-1} > 0$, $c_a - c_i > 0$, $c_a - c_{i+1}$, so that the minimum is attained at y_{i+1} (at the vertex of the slopes $-c_i$ and $-c_{i+1}$ of f), so that

$$u(x,t) = tL\left(\frac{x - y_{i+1}}{t}\right) + g(y_{i+1}) = tL(m_{i+1}) + g(x - m_{i+1}t)$$
$$\partial_x u(x,t) = c_a \text{ (since } x - m_{i+1}t < x - m_i t < 0\text{)}. \tag{2.36}$$

Hence, the solution in this case is still given by

$$\partial_x u(x,t) = \begin{cases} c_a & x < m_{i+1}t \\ c_b & x > m_{i+1}t \end{cases}. \tag{2.37}$$

although the computations are slightly more involved, as we have seen. Indeed, the Rankine-Hugoniot holds:

$$\sigma = \frac{[[H(u)]]}{[[u]]} \tag{2.38}$$
$$= \frac{m_{i+1}(c_{i+1} - c_a) + m_{i+2}(c_{i+2} - c_i) + ... + m_{i+j}(c_{i+j} - c_{i+j}) + m_{i+j+1}(c_b - c_a)}{c_b - c_a}$$

2.4 Generalization for Upward Jump (Rarefaction) Case

Now consider the case with the opposite initial condition, i.e.

$$g(y) = \begin{cases} c_a y & y < 0 \\ c_b y & y > 0 \end{cases}, \tag{2.39}$$

now with constants $c_a < c_b$. We argue that one will obtain a rarefaction fan of discretized slopes if there exist c_i in a suitable range, but not otherwise. To this end, we will consider cases.

Case (i). First suppose there does not exist i such that we have $c_a < c_i < c_b$. Then there exists i such that $c_i < c_a < c_b < c_{i+1}$. We then have the following inequalities:

$$\begin{aligned} &c_a - c_i > 0, \;\; c_a - c_{i+1} < 0 \\ &c_b - c_i > 0, \;\; c_b - c_{i+1} < 0. \end{aligned} \tag{2.40}$$

Consider first the range bounded by $y_{i+1} < 0 < y_i$. Under the above inequalities, we consider two subcases based on the vertex of the function $f(y; x, t) := tL\left(\frac{x-y}{t}\right)$ where the slopes $-c_{i+1}$ and $-c_i$ meet. More precisely, this occurs at the point $y_{i+1} = x - m_{i+1}t$.

First suppose $y_{i+1} < 0$. In this case the slope of f near zero is $-c_i$ on both sides. Since the sum of the slopes $c_b - c_i > 0$, the function $I(y; x, t)$ is increasing as one moves in the positive x direction. Further, in the other direction, we note that $c_b - c_i < 0$, so as one moves in the negative x direction, the function $I(y; x, t)$ is also increasing. This in conjunction with the convexity of the function outside the interval implies that the minimum indeed occurs at y_{i+1} in this case.

Now suppose that in fact $y_{i+1} > 0$. Then the slope of f near zero is $-c_{i+1}$ on both sides. Since the sum of the slopes $c_a - c_{i+1} > 0$, the function $I(y; x, t)$ is again increasing in the positive x direction. Furthermore, in the other direction, since $c_b - c_{i+1} < 0$, $I(y; x, t)$ is again increasing. Hence, by the same token, the minimum is attained at y_{i+1} again in this case.

Evaluating $I(y; x, t)$ at the minimum yields

$$u(x, t) = tL\left(\frac{x - y_{i+1}t}{t}\right) + g(y_{i+1}) = tL(m_{i+1}) + g(x - m_{i+1}t). \tag{2.41}$$

Differentiating yields

$$\partial_x u(x,t) = \begin{cases} c_a & m_i t < x < m_{i+1} t \\ c_b & x > m_{i+1} t \end{cases} \tag{2.42}$$

which in fact reduces to c_a inside the wedge $m_i t < x < m_{i+1} t$ we were considering. One can then show that $\partial_x u(x,t) = c_a$ for $x < m_i t$ and c_b for $x > m_{i+1} t$, so that for the final result we have

$$\partial_x u(x,t) = \begin{cases} c_a & x < m_{i+1} t \\ c_b & x > m_{i+1} t \end{cases}. \tag{2.43}$$

The boundary between c_a and c_b occurs at the line $x = m_{i+1} t$, which has slope m_{i+1}. From the Rankine-Hugoniot condition, we would expect the slope

$$\sigma = \frac{[[H(u)]]}{[[u]]} = \frac{(c_b - c_a)\, m_{i+1}}{c_b - c_a} = m_{i+1}, \tag{2.44}$$

which indeed agrees with what we have obtained above.

Case (ii). Now suppose there exists exactly one of the c_i that is between c_a and c_b, that is, $c_i < c_a < c_{i+1} < c_b < c_{i+2}$. This yields

$$\begin{aligned} &c_a - c_i > 0,\ c_a - c_{i+1} < 0,\ c_a - c_{i+2} < 0, \\ &c_b - c_i > 0,\ c_b - c_{i+1} > 0,\ c_b - c_{i+2} < 0. \end{aligned} \tag{2.45}$$

First suppose that $y_{i+2} > 0$. Then near zero, the slope of $f(y; x, t)$ is $-c_{i+2}$ on both sides. Since we have $c_a - c_{i+2} < 0$, $c_b - c_{i+2} < 0$, and $c_b - c_{i+1} > 0$, the minimum is attained at y_{i+2} (the vertex between the slopes $-c_{i+2}$ and $-c_{i+1}$). This means that

$$u(x,t) = tL\left(\frac{x - y_{i+2} t}{t}\right) + g(y_{i+2}) = tL(m_{i+2}) + g(x - m_{i+2} t). \tag{2.46}$$

Differentiating yields

$$\partial_x u(x,t) = c_b,\ x > m_{i+2} t. \tag{2.47}$$

Next consider the case $y_{i+2} < 0 < y_{i+1}$. Then near zero, the slope of $f(y; x, t)$ is $-c_{i+1}$ on both sides. Noting that $c_a - c_{i+1} < 0$ and $c_b - c_{i+1} > 0$, we observe that the minimum of $I(y; x, t)$ occurs at 0, hence

$$u(x,t) = tL\left(\frac{x}{t}\right) + g(0)$$
$$\partial_x u(x,t) = c_{i+1},\ \ m_{i+1}t < x < m_{i+2}t. \tag{2.48}$$

Now suppose $y_{i+1} < 0 < y_i$. Then near zero, the slope of $f(y;x,t)$ is $-c_i$ on both sides. Noting that $c_a - c_i > 0$, $c_b - c_i > 0$, we observe that the minimum of $I(y;x,t)$ occurs in this case at y_{i+1}, hence

$$u(x,t) = tL\left(\frac{x - y_{i+1}t}{t}\right) + g(y_{i+1}) = tL(m_{i+1}) + g(x - m_{i+1}t) \tag{2.49}$$

Differentiating then yields

$$\partial_x u(x,t) = c_a,\ \ m_i t < x < m_{i+1}t. \tag{2.50}$$

Finally, when $y_i < 0$, we have a minimum at y_i by a similar argument, so that

$$u(x,t) = tL\left(\frac{x - y_i t}{t}\right) + g(y_i) = tL(m_i) + g(x - m_i t)$$
$$\partial_x u(x,t) = c_a,\ \ x < m_i t. \tag{2.51}$$

Putting the different cases together, we have

$$\partial_x u(x,t) = \begin{cases} c_a & x < m_i t \\ c_i & m_i t < x < m_{i+1}t \\ c_b & m_{i+1}t < x \end{cases}. \tag{2.52}$$

This leaves us with two shocks, one on the line $x = m_i t$ and the other on $x = m_{i+1}t$. We verify that these shocks satisfy the R-H condition:

$$\sigma_i = \frac{[[H(u)]]}{[[u]]} = \frac{(c_i - c_a)\, m_i}{c_i - c_a} = m_i$$
$$\sigma_{i+1} = \frac{[[H(u)]]}{[[u]]} = \frac{(c_b - c_i)\, m_{i+1}}{c_b - c_i} = m_i. \tag{2.53}$$

Case (iii). Suppose there are multiple c_i between c_a and c_b, i.e., $c_i < c_a < c_{i+1} < c_{i+2} < ... <$

$c_{i+j} < c_b < c_{i+j+1}$. We then have the inequalities

$$\begin{aligned} &c_a - c_i > 0,\ c_a - c_{i+1} < 0, ...,\ c_a - c_{i+j} < 0,\ c_a - c_{i+j+1} < 0 \\ &c_b - c_i > 0,\ c_b - c_{i+1} > 0, ...,\ c_b - c_{i+j} > 0,\ c_b - c_{i+j+1} < 0. \end{aligned} \tag{2.54}$$

First suppose $y_{i+j+1} > 0$. Then near 0, the slope of $f(y; x, t)$ is $-c_{i+j+1}$ on both sides. Since $c_a - c_{i+j+1} < 0$, $c_b - c_{i+j+1} < 0$, and $c_b - c_{i+j} > 0$ we have that y_{i+j+1} (the vertex between slopes $-c_{i+j+1}$ and $-c_{i+j}$) is the minimum and

$$\begin{aligned} u(x,t) &= tL\left(\frac{x - y_{i+j+1}t}{t}\right) + g(y_{i+j+1}) = tL(m_{i+j+1}) + g(x - m_{i+j+1}) \\ \partial_x u(x,t) &= c_b. \end{aligned} \tag{2.55}$$

Now suppose that $y_{k+1} < 0 < y_k$ for $i < k < i + j$. Then near 0, the slope of $f(y; x, t)$ will be $-c_k$ on both sides. Since $c_a - c_k < 0$ and $c_b - c_k > 0$, clearly 0 will be the minimum, and

$$\begin{aligned} u(x,t) &= tL\left(\frac{x}{t}\right) + g(0) = tL\left(\frac{x}{t}\right) \\ \partial_x u(x,t) &= c_k. \end{aligned} \tag{2.56}$$

Then the value of $\partial_x u(x,t)$ takes on the different c_i bounded between c_a and c_b in the wedge $y_{i+j+1} > x > y_i$.

Finally we suppose $y_i < 0$. In this case, near 0, the slope of $f(y; x, t)$ is $-c_{i-1}$. We note that $c_a - c_{i-1} > 0$, $c_b - c_{i-1} > 0$, $c_a - c_i > 0$ and $c_a - c_{i+1} < 0$, so that the minimum is attained at y_{i+1} (at the vertex between the slopes $-c_{i+1}$ and $-c_i$), and we have

$$\begin{aligned} u(x,t) &= tL\left(\frac{x - y_{i+1}}{t}\right) + g(y_{i+1}) = tL(m_{i+1}) + g(x - m_{i+1}t) \\ \partial_x u(x,t) &= c_a. \end{aligned} \tag{2.57}$$

Now combining all of these cases, we have

$$\partial_x u(x,t) = \begin{cases} c_a & x < m_i t \\ c_i & m_i t < x < m_{i+1} t \\ \ldots & \ldots \\ c_{i+j} & m_{i+j} t < x < m_{i+j+1} t \\ c_b & m_{i+j+1} t < x \end{cases}. \tag{2.58}$$

The last step is to check the Rankine-Hugoniot conditions along the $j+2$ shocks. Indeed, we have

$$\begin{aligned} \sigma_0 &= \frac{H(c_i) - H(c_a)}{c_i - c_a} = \frac{m_i (c_i - c_a)}{c_i - c_a} = m_i, \\ \sigma_k &= \frac{H(c_{i+k+1}) - H(c_{i+k})}{c_{i+k+1} - c_{i+k}} = \frac{m_{i+k+1}(c_{i+k+1} - c_{i+k})}{c_{i+k+1} - c_{i+k}} = m_{i+k+1},\ 1 \le k \le j, \\ \sigma_j &= \frac{H(c_b) - H(c_{i+j})}{c_b - c_{i+j}} = \frac{m_{i+j+1}(c_b - c_{i+j})}{c_b - c_{i+j}} = m_{i+j+1}, \end{aligned} \tag{2.59}$$

which indeed matches the expression above.

Chapter 3

Derivation of n-point Distribution From Entropy-Entropy Flux Pair Form

In attempting to derive equations to represent the statistics of the system, a natural approach entails considering the velocities at certain points (x coordinates) as the basic building block. One might then start from an entropy-entropy flux pair form and derive algebraic identities to express the kinetic behavior at one level in terms of expressions at the next level. As outlined in the introduction, this behavior is expressed in terms of n-point functions. Recall that these expressions take the form, for example, of

$$
\begin{aligned}
&p_1\left(x, t; u_i\right) \\
&p_2\left(x, x+, t; u_i, u_j\right) \\
&p_2\left(x_1, x_2, t; u_i, u_j\right) \\
&p_n\left(x_1, ..., x_n, t; u_{i_1}, ..., u_{i_n}\right).
\end{aligned} \tag{3.1}
$$

The first equation in (3.1) represents the probability that the solution at the point x takes on the velocity u_i. The second expresses the probability that the solution takes the value u_i at x and the value u_j is the limiting value from the right - that is, there may be a jump at that point on the right.

That is, one hopes to obtain and solve an accurate collection of equations for the n-point distributions in terms of various other combinations of the n-point and n+1-point distribution functions.

The basis of our approach is the use of a piecewise linear, convex flux f and given test functions. The following sections highlight the steps in obtaining these equations, both for the simplest one-point distribution, the next level of the two-point distribution, and then finally for the generalized equation for the n-point distribution.

3.1 Derivation of equation for the 1-point distribution

We begin our study of the conservation law

$$u_t + (f(u))_x = 0, \tag{3.2}$$

where f is convex and polygonal (with finitely many break points) subject to piecewise constant initial conditions. By Corollary 2.8, p.67 of [21], there exists a unique solution that is a piecewise constant function of x for each t, and $u(x,t)$ takes values in the finite set comprising the break points of f and the values of the initial conditions. Also, there are only finitely many interactions between the fronts of u. The function u satisfies the Kruzkov entropy condition. For each fixed t as a function of x, the total variation of u is bounded by the initial total variation. In addition, the

$$\|u(\cdot,t) - u(\cdot,s)\|_{L^1} \leq \|f\|_{Lip} TV(u_0)\,|t-s|$$

Using this inequality in conjunction with basic BV results, one sees that $u(x,t)$ is of bounded variation in either x or t with the other variable fixed ([11], p.117, [40]). Thus, $u(x,t)$ is differentiable almost everywhere in x and t, and the derivatives are in L^1 ([35], p. 116) Thus, we can calculate

$$\partial_t\varphi(u) = \varphi'(u)\,\partial_t u = -\varphi'(u)\,(f(u))_x = -\varphi'(u)\,f'(u)\,\partial_x u = -\psi'(u)\,\partial_x u. \tag{3.3}$$

In other words, we have

$$\partial_t\varphi(u(x,t)) = -\partial_x\psi(u(x,t)). \tag{3.4}$$

almost everywhere. To be precise, this is a piecewise linear, cadlag, convex flux f defined by its values at points u_i together with piecewise linear interpolation; that is

$$f(u_i) = f_i,\ 1 \leq i \leq M$$

and slopes then given by

$$c_k = \frac{f_{k+1} - f_k}{u_{k+1} - u_k}. \tag{3.5}$$

We now take expectations of both sides of equation (3.4), assuming for the moment that we can pass the expectation on the left-hand side inside the time derivative, and have

$$\mathbb{E}\left\{\partial_t \varphi\left(u\left(x,t\right)\right)\right\} = \partial_t \mathbb{E}\left\{\varphi\left(u\left(x,t\right)\right)\right\} = -\mathbb{E}\left\{\partial_x \psi\left(u\left(x,t\right)\right)\right\} \tag{3.6}$$

We take a test function φ with discrete values on $x \in \{x_i\}_{i=1}^{M}$. Then the LHS of (3.6) becomes

$$\partial_t \sum_{l=1}^{M} \varphi\left(u_l\right) p_1\left(x; u_l; t\right), \tag{3.7}$$

and the RHS is

$$-\sum_{l=1}^{M} \sum_{m=1}^{M} \left(\psi\left(u_m\right) - \psi\left(u_l\right)\right) p_2\left(x, x+; u_l, u_m; t\right). \tag{3.8}$$

Therefore, in the general case (3.6) yields

$$\partial_t \sum_{l=1}^{M} \varphi\left(u_l\right) p_1\left(x; u_l; t\right) = -\sum_{l=1}^{M} \sum_{m=1}^{M} \left(\psi\left(u_m\right) - \psi\left(u_l\right)\right) p_2\left(x, x+; u_l, u_m; t\right). \tag{3.9}$$

Now, we wish to choose the derivative of φ as a discretized version of a δ-function at u_k, i.e. define

$$\varphi_k'\left(u\right) = \frac{1_{[u_k,,u_{k+1})}\left(u\right)}{u_{k+1} - u_k}. \tag{3.10}$$

Consequentially, one has (integrating φ up from $-\infty$ and holding the left limit at zero)

$$\varphi_k\left(u\right) = 1_{[u_{k+1},\infty)}\left(u\right),\ \psi_k'\left(u\right) = \frac{c_k 1_{[u_k,u_{k+1})}\left(u\right)}{u_{k+1} - u_k},\ \psi\left(u\right) = c_k 1_{[u_{k+1},\infty)}\left(u\right) \tag{3.11}$$

Substituting the expressions (3.11) into (3.9), we have

$$\begin{aligned} LHS &= \partial_t \sum_{l=1}^{M} 1_{[u_{k+1},\infty)}\left(u_l\right) p_1\left(x;u_l;t\right) = \partial_t \sum_{l=k}^{M} p_1\left(x;u_l;t\right) \\ RHS &= -\sum_{l=1}^{k}\sum_{m=k+1}^{M} c_k p_2\left(x,x+;u_l,u_m;t\right) + \sum_{l=k+1}^{M}\sum_{m=1}^{k} c_k p_2\left(x,x+;u_l,u_m;t\right) \end{aligned} \tag{3.12}$$

Hence, we have an equation for the 1-point equation in terms of the 2-point equation:

$$\sum_{l=k+1}^{M} \partial_t p_1\left(x;u_l;t\right) = -\sum_{l=1}^{k}\sum_{m=k+1}^{M} c_k p_2\left(x,x+;u_l,u_m;t\right) + \sum_{l=k+1}^{M}\sum_{m=1}^{k} c_k p_2\left(x,x+;u_l,u_m+;t\right) \tag{3.13}$$

The significance of this expression is that it presents an equation for the change in time of probabilities of a particle having certain velocities in terms of the behavior at the right-hand limit. In a way the terms on the right hand side of (3.13) can be interpreted as a discrete derivative or finite difference. We show in the following sections that this process can be generalized, for the 2-point and then n-point equations, establishing the pattern of deriving various combinations of the n-point distribution in terms of other expressions involving the n+1 point distributions. Thus, we will see that the change in time at one level of the system is related to changes at the limit points x_i+ at the next. We outline the procedure first for the 2-point distribution and then for the n-point to derive a complete hierarchy of the equations. Once the hierarchy is obtained, our aim is to use the theorem of total probability on the equations to show how they can be reduced to a set of transport equations in n dimensions. One can then solve these systems of equations and then analyze the resulting solutions, comparing them against classically obtained results in different examples.

3.2 Equation for the 2-point distribution

We now take φ_1, φ_2 that satisfy

$$\partial_t \varphi_i\left(u\left(x,t\right)\right) = -\partial_x \psi_i\left(u\left(x,t\right)\right), \ i = 1,2 \tag{3.14}$$

with $\psi_i' := f'\varphi_i'$ and note that then

$$\partial_t\left(\varphi_1\left(u\left(x_1,t\right)\right)\varphi_2\left(u\left(x_2,t\right)\right)\right) = -\partial_x\psi_1\left(u\left(x_1,t\right)\right)\varphi_2\left(u\left(x_2,t\right)\right) - \partial_x\psi_2\left(u\left(x_2,t\right)\right)\varphi_1\left(u\left(x_1,t\right)\right) \tag{3.15}$$

Taking expectations and passing to the limit as before, we have

$$\partial_t \mathbb{E}\left\{\varphi_1\left(u\left(x_1,t\right)\right)\varphi_2\left(u\left(x_2,t\right)\right)\right\} = -\mathbb{E}\left\{\partial_x \psi_2\left(u\left(x,t\right)\right)\varphi_1\left(u\left(x_2,t\right)\right)\right\}. \tag{3.16}$$

Now we consider test functions of the form

$$\varphi_i'\left(u\right) = 1_{[u_{k_i}, u_{k_i+1})}\left(u\right),\ \varphi_i\left(u\right) = 1_{[u_{k_i+1},\infty)}\left(u\right),\ i = 1,2$$

We then have

$$\psi_i\left(u\right) = \int_{-\infty}^{u} f'\left(s\right)\varphi_i'\left(s\right)ds = \begin{cases} 0 & u < u_{k_i+1} \\ c_{k_i} & u \geq u_{k_i+1} \end{cases},\ i = 1,2 \tag{3.17}$$

We now evaluate both sides of the equation (3.16) and have

$$\begin{aligned}
LHS &= \partial_t \sum_{l=1}^{M}\sum_{m=1}^{M}\varphi_1\left(u_l\right)\varphi_2\left(u_m\right)p_2\left(x_1,x_2;u_l,u_m;t\right) = \partial_t \sum_{l=k_1+1}^{M}\sum_{m=k_2+1}^{M} p_2\left(x_1,x_2;u_l,u_m;t\right) \\
RHS &= -\sum_{l=1}^{M}\sum_{m=1}^{M}\sum_{p=1}^{M}\left(\psi_1\left(u\left(x_m\right)\right)-\psi_1\left(u\left(x_l\right)\right)\right)\varphi_2\left(u\left(x_p\right)\right) \\
&\quad -\sum_{l=1}^{M}\sum_{m=1}^{M}\sum_{p=1}^{M}\left(\psi_2\left(u\left(x_m\right)\right)-\psi_2\left(u\left(x_l\right)\right)\right)\varphi_1\left(u\left(x_p\right)\right) \\
&= -\sum_{l=1}^{k_1}\sum_{m=k_1+1}^{M}\sum_{p=k_2+1}^{M} c_{k_1}p_3\left(x_1,x_1+,x_2;u_l,u_m,u_p;t\right) \\
&\quad +\sum_{l=k_1+1}^{M}\sum_{m=1}^{k_1}\sum_{p=k_2+1}^{M} c_{k_1}p_3\left(x_1,x_1+,x_2;u_l,u_m,u_p;t\right) \\
&\quad -\sum_{l=1}^{k_2}\sum_{m=k_2+1}^{M}\sum_{p=k_1+1}^{M} c_{k_2}p_3\left(x_2,x_2+,x_1;u_l,u_m,u_p;t\right) \\
&\quad +\sum_{l=k_2+1}^{M}\sum_{m=1}^{k_2}\sum_{p=k_1+1}^{M} c_{k_2}p_3\left(x_2,x_2+,x_1;u_l,u_m,u_p;t\right)
\end{aligned} \tag{3.18}$$

Hence, we have

$$\begin{aligned}
&\partial_t \sum_{l=k_1+1}^{M} \sum_{m=k_2+1}^{M} \partial_t p_2\left(x_1, x_2; u_l, u_m; t\right) \\
&= -\sum_{l=1}^{k_1} \sum_{m=k_1+1}^{M} \sum_{p=k_2+1}^{M} c_{k_1} p_3\left(x_1, x_1+, x_2; u_l, u_m, u_p; t\right) \\
&+ \sum_{l=k_1+1}^{M} \sum_{m=1}^{k_1} \sum_{p=k_2+1}^{M} c_{k_1} p_3\left(x_1, x_1+, x_2; u_l, u_m, u_p; t\right) \\
&- \sum_{l=1}^{k_2} \sum_{m=k_2+1}^{M} \sum_{p=k_1+1}^{M} c_{k_2} p_3\left(x_2, x_2+, x_1; u_l, u_m, u_p; t\right) \\
&+ \sum_{l=k_2+1}^{M} \sum_{m=1}^{k_2} \sum_{p=k_1+1}^{M} c_{k_2} p_3\left(x_2, x_2+, x_1; u_l, u_m, u_p; t\right)
\end{aligned} \tag{3.19}$$

3.3 Equation for the n-point distribution

In a similar fashion, we choose test functions φ_i with $\psi_i' = f_i^{'}\varphi_i'$such that

$$\partial_t \varphi_i\left(u\left(x,t\right)\right) = -\partial_x \psi_i\left(u\left(x,t\right)\right),\ 1 \le i \le n \tag{3.20}$$

and note that then

$$\mathbb{E}\left\{\partial_t \left(\prod_{i=1}^{n} \varphi_i\left(u\left(x_i,t\right)\right)\right)\right\} = -\mathbb{E}\left\{\sum_{i=1}^{n} \partial_x \psi_i\left(u\left(x_i,t\right)\right) \prod_{j\neq i} \varphi_j\left(u\left(x_j,t\right)\right)\right\} \tag{3.21}$$

Now specifically set

$$\varphi_i'\left(u\right) = 1_{\left[u_{k_i}, u_{k_i+1}\right)}\left(u\right),\ \varphi_i\left(u\right) = 1_{\left[u_{k_i}+1, \infty\right)}\left(u\right),\ 1 \le i \le n. \tag{3.22}$$

We then have

$$\psi_i\left(u\right) = \int_{-\infty}^{u} f'\left(s\right)\varphi_i'\left(s\right) ds = \begin{cases} 0 & u < u_{k_i+1} \\ c_{k_i} & u \ge u_{k_i+1} \end{cases},\ 1 \le i \le n. \tag{3.23}$$

We then evaluate both sides of equation (3.21) and have

$$
\begin{aligned}
LHS &= \partial_t \sum_{l_1=1}^{M} \dots \sum_{l_n=1}^{M} \prod_{i=1}^{n} \varphi_i\left(u_i\right) p_n\left(x_1,...,x_n;u_1,...,u_n;t\right) \\
&= \partial_t \sum_{l_1=k_1+1}^{M} \dots \sum_{l_n=k_n+1}^{M} p_n\left(x_1,...,x_n;u_{k_1},...,u_{k_n};t\right) \\
RHS &= -\mathbb{E} \sum_{l=1}^{M} \dots \sum_{l_n=1}^{M} \sum_{i=1}^{n} \partial_x \psi_i\left(u\left(x_{l_i}\right)\right) \prod_{j\neq i} \varphi_j\left(u\left(x_{l_j},t\right)\right) \\
&= -\mathbb{E}\left\{ \sum_{l_1=1}^{k_1} \sum_{l_2=k_1+1}^{M} \dots \sum_{l_{n+1}=1}^{M} \left(\psi_1\left(u\left(x_{l_2}\right)\right) - \psi_1\left(u\left(x_{l_1}\right)\right)\right) \prod_{j\neq i} \varphi_j\left(u\left(x_{l_j},t\right)\right) \right\} \\
&+ \mathbb{E}\left\{ \sum_{l_1=k_1+1}^{M} \sum_{l_2=1}^{M} \dots \sum_{l_{n+1}=1}^{M} \left(\psi_1\left(u\left(x_{l_2}\right)\right) - \psi_1\left(u\left(x_{l_1}\right)\right)\right) \prod_{j\neq i} \varphi_j\left(u\left(x_{l_j},t\right)\right) \right\} \\
&- \textit{other terms} \\
&= -\sum_{l_1=1}^{k_1} \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} \dots \sum_{l_n=k_n+1}^{M} c_{k_1} p_{n+1}\left(x_1,x_1+,x_2,...,x_n;u_{l_1},u_{l_1'},u_{l_2},...u_{l_n};t\right) \\
&+ \sum_{l_1=k_1+1}^{M} \sum_{l_1'=1}^{k_1} \sum_{l_2=k_2+1}^{M} \dots \sum_{l_n=k_n+1}^{M} c_{k_1} p_{n+1}\left(x_1,x_1+,x_2,...,x_n;u_{l_1},u_{l_1'},u_{l_2},...u_{l_n};t\right) \\
&- \textit{other terms} \\
&= -\sum_{i=1}^{n} \left\{ \begin{array}{c} \sum_{l_i=1}^{k_1} \sum_{l_i'=k_1+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_1,x_2,...,x_i,x_i+,...,x_n;u_{l_1},u_{l_2},...,u_{l_i},u_{l_i'},...u_{l_n};t\right) \\ -\sum_{l_i=k_i+1}^{M} \sum_{l_i'=1}^{k_1} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_1,x_2,...,x_i,x_i+,...,x_n;u_{l_1},u_{l_2},...,u_{l_i},u_{l_i'},...u_{l_n};t\right) \end{array} \right\}
\end{aligned} \tag{3.24}
$$

Hence,

$$
\begin{aligned}
&\sum_{l_1=k_1+1}^{M} \dots \sum_{l_n=k_n+1}^{M} \partial_t p_n\left(x_1,...,x_n;u_{l_1},...,u_{l_n};t\right) \\
&= -\sum_{i=1}^{n} \left\{ \begin{array}{c} \sum_{l_i=1}^{k_1} \sum_{l_i'=k_1+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_1,x_2,...,x_i,x_i+,...,x_n;u_{l_1},u_{l_2},...,u_{l_i},u_{l_i'},...u_{l_n};t\right) \\ -\sum_{l_i=k_i+1}^{M} \sum_{l_i'=1}^{k_1} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_1,x_2,...,x_i,x_i+,...,x_n;u_{l_1},u_{l_2},...,u_{l_i},u_{l_i'},...u_{l_n};t\right) \end{array} \right\}
\end{aligned} \tag{3.25}
$$

3.4 Reduction to Transport Type Equation and Properties

Under the assumption that the theorem of total probability holds without issue at a limit point $x+$, i.e. an equality of the form

$$\sum_{l_2} p_2\left(x, x+, t; u_{l_1}, u_{l_2}\right) = p_1\left(x, t; u_{l_1}\right), \tag{3.26}$$

we show that we can reduce and close the n-point hierarchy to a system of transport equations at each level. As expected, the complexity and number of equations increases with n, but the equations all express the time derivative of the n-point function in therms of the other spacial derivatives, as will be shown in the following section.

3.5 Two-point case

We begin our calculations with the long form of the two-point equation, which is given by

$$\begin{aligned}
&\partial_t \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} p_2\left(x_1, x_2, t; u_{l_1}, u_{l_2}\right) \\
&= -\sum_{l_1=1}^{k_1} \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3\left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \\
&+ \sum_{l_1=k_1+1}^{M} \sum_{l_1'=1}^{k_1} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3\left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \\
&- \sum_{l_2=1}^{k_2} \sum_{l_2'=k_2+1}^{M} \sum_{l_1=k_1+1}^{M} c_{k_2} p_3\left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right) \\
&+ \sum_{l_2=k_2+1}^{M} \sum_{l_2'=1}^{k_2} \sum_{l_1=k_1+1}^{M} c_{k_2} p_3\left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right)
\end{aligned} \tag{3.27}$$

We want to rewrite the terms and rearrange them using the theorem of total probability. We label the terms on the right-hand side as (1) through (4):

$$\begin{aligned}
(1) &:= -\sum_{l_1=1}^{k_1}\sum_{l_1'=k_1+1}^{M}\sum_{l_2=k_2+1}^{M} c_{k_1} p_3\left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right)\\
(2) &:= \sum_{l_1=k_1+1}^{M}\sum_{l_1'=1}^{k_1}\sum_{l_2=k_2+1}^{M} c_{k_1} p_3\left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right)\\
(3) &:= -\sum_{l_2=1}^{k_2}\sum_{l_2'=k_2+1}^{M}\sum_{l_1=k_1+1}^{M} c_{k_2} p_3\left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right)\\
(4) &:= \sum_{l_2=k_2+1}^{M}\sum_{l_2'=1}^{k_2}\sum_{l_1=k_1+1}^{M} c_{k_2} p_3\left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right)
\end{aligned} \tag{3.28}$$

and add or subtract terms to as follows:

$$\begin{aligned}
&-\sum_{l_1=k_1+1}^{M}\sum_{l_1'=k_1+1}^{M}\sum_{l_2=k_2+1}^{M} c_{k_1} p_3\left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \text{ to (1)},\\
&+\sum_{l_1=k_1+1}^{M}\sum_{l_1'=k_1+1}^{M}\sum_{l_2=k_2+1}^{M} c_{k_1} p_3\left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \text{ to (2)},\\
&-\sum_{l_1=k_1+1}^{M}\sum_{l_2=k_2+1}^{M}\sum_{l_2'=k_2+1}^{M} c_{k_2} p_3\left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right) \text{ to (3)},\\
&+\sum_{l_1=k_1+1}^{M}\sum_{l_2=k_2+1}^{M}\sum_{l_2'=k_2+1}^{M} c_{k_2} p_3\left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right) \text{ to (4)}.
\end{aligned} \tag{3.29}$$

Observe that the term we subtract from (1) is the same as the term we are adding to (2), and that the term we subtract from (3) is the same as the term we subtract add to (4). Therefore, defining the new quantities as (1'), (2'), (3'), and (4'), we have

$$\partial_t \sum_{l_1=k_1+1}^{M}\sum_{l_2=k_2+1}^{M} p_2\left(x_1, x_2, t; u_{l_1}, u_{l_2}\right) = (1') + (2') + (3') + (4') \tag{3.30}$$

where

$$
\begin{aligned}
(1') &= (1) - \sum_{l_1=k_1+1}^{M} \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3 \left(x_1, x_1+, x_2; u_{l_1}, u_{l_1'}, u_{l_2}; t\right) \\
(2') &= (2) + \sum_{l_1=k_1+1}^{M} \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3 \left(x_1, x_1+, x_2; u_{l_1}, u_{l_1'}, u_{l_2}; t\right) \\
(3') &= (3) - \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} \sum_{l_2'=k_2+1}^{M} c_{k_2} p_3 \left(x_2, x_2+, x_1; u_{l_2}, u_{l_2'}, u_{l_1}; t\right) \\
(4') &= (4) + \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} \sum_{l_2'=k_2+1}^{M} c_{k_2} p_3 \left(x_2, x_2+, x_1; u_{l_2}, u_{l_2'}, u_{l_1}; t\right)
\end{aligned}
\tag{3.31}
$$

Then we have

$$
\begin{aligned}
(1') = &- \sum_{l_1=1}^{k_1} \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3 \left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \\
&- \sum_{l_1=k_1+1}^{M} \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3 \left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \\
&= - \sum_{l_1=1}^{M} \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3 \left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \\
&= - \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_2 \left(x_1+, x_2, t; u_{l_1'}, u_{l_2}\right)
\end{aligned}
\tag{3.32}
$$

whereby in the last step we have used the theorem of total probability. For the other terms, one has

$$
\begin{aligned}
(2') = &\sum_{l_1=k_1+1}^{M} \sum_{l_1'=1}^{k_1} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3 \left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \\
&+ \sum_{l_1=k_1+1}^{M} \sum_{l_1'=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3 \left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \\
&= \sum_{l_1=k_1+1}^{M} \sum_{l_1'=1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_3 \left(x_1, x_1+, x_2, t; u_{l_1}, u_{l_1'}, u_{l_2}\right) \\
&= \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_2 \left(x_1, x_2, t; u_{l_1}, u_{l_2}\right)
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
(3') &= -\sum_{l_2=1}^{k_2} \sum_{l_2'=k_2+1}^{M} \sum_{l_1=k_1+1}^{M} c_{k_2} p_3 \left(x_2, x_2+, x_1, t; u_{l_2}, u'_{l_2}, u_{l_1}\right) \\
&\quad - \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} \sum_{l_2'=k_2+1}^{M} c_{k_2} p_3 \left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right) \\
&= -\sum_{l_1=k_1+1}^{M} \sum_{l_2=1}^{M} \sum_{l_2'=k_2+1}^{M} c_{k_2} p_3 \left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right) \\
&= -\sum_{l_1=k_1+1}^{M} \sum_{l_2'=k_2+1}^{M} c_{k_2} p_2 \left(x_2+, x_1, t; u_{l_2'}, u_{1_1}\right) \qquad (3.34)
\end{aligned}
$$

$$
\begin{aligned}
(4') &= \sum_{l_2=k_2+1}^{M} \sum_{l_2'=1}^{k_2} \sum_{l_1=k_1+1}^{M} c_{k_2} p_3 \left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right) \\
&\quad + \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} \sum_{l_2'=k_2+1}^{M} c_{k_2} p_3 \left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right) \\
&= \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} \sum_{l_2'=1}^{M} c_{k_2} p_3 \left(x_2, x_2+, x_1, t; u_{l_2}, u_{l_2'}, u_{l_1}\right) \\
&= \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_2} p_2 \left(x_1, x_2, t; u_{l_1}, u_{l_2}\right) \qquad (3.35)
\end{aligned}
$$

Relabelling dummy indices, we have

$$
\begin{aligned}
(1') &= -\sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_2 \left(x_1+, x_2, t; u_{l_1}, u_{l_2}\right) \\
(2') &= \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1} p_2 \left(x_1; x_2, t; u_{l_1}, u_{l_2}\right) \\
(3') &= -\sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_2} p_2 \left(x_1, x_2+, t; u_{l_1}, u_{l_2}\right) \\
(4') &= \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_2} p_2 \left(x_1, x_2, t; u_{l_1}, u_{l_2}\right). \qquad (3.36)
\end{aligned}
$$

Recalling equation (3.30), we can now put the right-hand side of the original equation back together, yielding

$$\begin{aligned}
&\partial_t \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} p_2\left(x_1, x_2, t; u_{l_1}, u_{l_2}\right) \\
&= -\sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} \left\{ \begin{array}{c} c_{k_1}\left(p_2\left(x_1+; x_2, t; u_{l_1}, u_{l_2}\right) - p_2\left(x_1, x_2, t; u_{l_1}, u_{l_2}\right)\right) \\ +c_{k_2}\left(p_2\left(x_1, x_2+, t; u_{l_1}, u_{l_2}\right) - p_2\left(x_1, x_2, t; u_{l_1}, u_{l_2}\right)\right) \end{array} \right\}.
\end{aligned} \tag{3.37}$$

Finally, we rewrite this as a derivative of its cumulative distribution function, i.e.

$$\begin{aligned}
&\partial_1 F_2\left(x_1, x_2, t; k_1, k_2\right) \\
&:= -\sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_1}\left(p_2\left(x_1+; x_2, t; u_{l_1}, u_{l_2}\right) - p_2\left(x_1; x_2, t; u_{l_1'}, u_{l_2}\right)\right) \\
&- \sum_{l_1=k_1+1}^{M} \sum_{l_2=k_2+1}^{M} c_{k_2}\left(p_2\left(x_1, x_2+, t; u_{l_1}, u_{l_2}\right) - p_2\left(x_1, x_2, t; u_{l_1}, u_{l_2}\right)\right)
\end{aligned} \tag{3.38}$$

The right-hand side can also be written in terms of a cdf, and so the two-point equation simplifies to

$$\partial_t F_2\left(x_1, x_2, t; k_1, k_2\right) + c \cdot \nabla F_2 = 0$$

where we define the constant vector c as

$$c = \left\{ \begin{array}{c} c_{k_1} \\ c_{k_2} \end{array} \right\}$$

At this point we have a set of transport equations for $F\left(x_1, x_2, \cdot, \cdot, t\right)$. We also want to have consistency conditions, i.e. that the functions F_2 satisfy

$$\begin{aligned}
0 &\leq F_2\left(x_1, x_2, t; \cdot, \cdot\right) \leq 1, \text{ all } t > 0 \\
1 &= F_2\left(x_1, x_2, t; 1, 1\right), \text{ all } t > 0.
\end{aligned} \tag{3.39}$$

and that they are increasing in the variables k_1 and k_2.

3.6 N-point case

In the previous section we reduced the two-point equation to a set of transport equations for the cumulative distribution function of the two-point equation. Now we will show that by using similar techniques, we can do the same for the n-point equation. Recall the equation for the n-point equation:

$$\sum_{l_1=k_1+1}^{M} ... \sum_{l_n=k_n+1}^{M} \partial_t p_n\left(x_1,...,x_n,t;u_{l_1},...,u_{l_n}\right)$$
$$= -\sum_{i=1}^{n} \left\{ \begin{array}{l} \sum_{l_i=1}^{k_i} \sum_{l_i'=k_i+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_1,x_2,...,x_i,x_i+,...,x_n,t;u_{l_1},u_{l_2},...,u_{l_i},u_{l_i'},...,u_{l_n}\right) \\ -\sum_{l_i=k_i+1}^{M} \sum_{l_i'=1}^{k_i} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_1,x_2,...,x_i,x_i+,...,x_n,t;u_{l_1},u_{l_2},...,u_{l_i},u_{l_i'},...,u_{l_n}\right) \end{array} \right\} \tag{3.40}$$

Indeed, for the left-hand side, by definition we have

$$\begin{aligned} LHS &= \sum_{l_1=k_1+1}^{M} ... \sum_{l_n=k_n+1}^{M} \partial_t p_n\left(x_1,...,x_n;u_{l_1},...,u_{l_n};t\right) \\ &=: \partial_t F_n\left(x_1,...,x_n,k_1,...,k_n;t\right). \end{aligned} \tag{3.41}$$

On the right-hand side, we once again add and subtract off terms and use the theorem of total probability on each term. For brevity we consider ith pair of terms, labelling them as A_i and B_i. These terms are given by

$$\begin{aligned} A_i &= \sum_{l_i=1}^{k_i} \sum_{l_i'=k_i+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_i,x_i+,...;u_{l_i},u_{l_i'},...;t\right) \\ B_i &= -\sum_{l_i=k_i+1}^{M} \sum_{l_i'=1}^{k_i} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_i,x_i+,...;u_{l_i},u_{l_i'},...;t\right) \end{aligned} \tag{3.42}$$

To the A_i term we add on a sum, defining

$$\tilde{A}_i = A_i + \sum_{l_i=k_i+1}^{M} \sum_{l_i'=k_i+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_i,x_i+,...;u_{l_i},u_{l_i'},...;t\right) \tag{3.43}$$

To B_i we subtract this same term, writing

$$\tilde{B}_i = B_i + \sum_{l_i=k_i+1}^{M} \sum_{l'_i=k_i+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1} \left(x_i, x_i+, ...; u_{l_i}, u_{l'_i}, ...; t\right) \tag{3.44}$$

These terms cancel, so we have

$$\begin{aligned} \sum_{l_1=k_1+1}^{M} ... \sum_{l_n=k_n+1}^{M} \partial_t p_n \left(x_1, ..., x_n; u_{l_1}, ..., u_{l_n}; t\right) &= -\sum_{i=1}^{n} \left\{A_i + B_i\right\} \\ &= -\sum_{i=1}^{n} \left\{\tilde{A}_i + \tilde{B}_i\right\} \end{aligned} \tag{3.45}$$

Further, we have

$$\begin{aligned} \tilde{A}_i = &\sum_{l_i=1}^{k_i} \sum_{l'_i=k_i+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1} \left(x_i, x_i+, ...; u_{l_i}, u_{l'_i}, ...; t\right) \\ &+ \sum_{l_i=k_i+1}^{M} \sum_{l'_i=k_i+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1} \left(x_i, x_i+, ...; u_{l_i}, u_{l'_i}, ...; t\right) \\ &= \sum_{l_i=1}^{M} \sum_{l'_i=k_i+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n+1} \left(x_i, x_i+, ...; u_{l_i}, u_{l'_i}, ...; t\right) \\ &= \sum_{l'_i=k_i+1}^{M} \sum_{\substack{l_j=k_j+1 \\ j\neq i}}^{M} c_{k_i} p_{n} \left(x_i+, ...; u_{l'_i}, ...; t\right) \end{aligned} \tag{3.46}$$

by the theorem of total probability on the index l_i. For the B_i term, we have

$$\begin{aligned}
\tilde{B}_i &= -\sum_{l_i=k_i+1}^{M}\sum_{l_i'=1}^{k_i}\sum_{\substack{l_j=k_j+1\\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_i, x_i+, ...; u_{l_i}, u_{l_i'}, ...; t\right) \\
&\quad -\sum_{l_i=k_i+1}^{M}\sum_{l_i'=k_i+1}^{M}\sum_{\substack{l_j=k_j+1\\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_i, x_i+, ...; u_{l_i}, u_{l_i'}, ...; t\right) \\
&= \sum_{l_i=k_i+1}^{M}\sum_{l_i'=1}^{M}\sum_{\substack{l_j=k_j+1\\ j\neq i}}^{M} c_{k_i} p_{n+1}\left(x_i, x_i+, ...; u_{l_i}, u_{l_i'}, ...; t\right) \\
&= \sum_{l_i=k_i+1}^{M}\sum_{\substack{l_j=k_j+1\\ j\neq i}}^{M} c_{k_i} p_n\left(x_i, ...; u_{l_i}, ...; t\right)
\end{aligned} \tag{3.47}$$

using the theorem of total probability on the index l_i'. We relabel the l_i' index in the ith term, so we have

$$\tilde{A}_i = \sum_{l_i=k_i+1}^{M}\sum_{\substack{l_j=k_j+1\\ j\neq i}}^{M} c_{k_i} p_n\left(x_i+, ...; u_{l_i}, ...; t\right) \tag{3.48}$$

and hence

$$\tilde{A}_i + \tilde{B}_i = \sum_{\substack{l_j=k_j+1\\ 1\leq j\leq n}}^{M} c_{k_i}\left\{\begin{array}{c} p_n\left(x_1, ..., x_i+, ..., x_n; u_{l_1}, ..., u_{l_i} ..., u_{l_n}; t\right) \\ -p_n\left(x_1, ...x_i, ..., x_n; u_{l_1}, ..., u_{l_i}, ..., u_{l_n}; t\right)\end{array}\right\} \tag{3.49}$$

Using (3.41), (3.45), and (3.49), we may write the n-point equation as

$$\begin{aligned}
\partial_t F_n\left(x_1, ..., x_n, k_1, ..., k_n; t\right) &= -\sum_{i=1}^{n} c_{k_i}\left(\tilde{A}_i + \tilde{B}_i\right) \\
&\quad -\sum_{i=1}^{n} c_{k_i}\left(\begin{array}{c} F_n\left(x_1, ...x_i+, ..., x_n; k_1, ..., k_n; t\right) \\ -F_n\left(x_1, ..., x_i, ..., x_n; k_1, ..., k_n; t\right)\end{array}\right)
\end{aligned} \tag{3.50}$$

or, more simply as

$$\partial_t F_n\left(x_1, ..., x_n, k_1, .., k_n; t\right) + c\cdot\nabla F_n = 0, \tag{3.51}$$

again a series of transport equations, where the vector c is given by

$$c = \left\{ \begin{array}{c} c_{k_1} \\ \ldots \\ c_{k_n} \end{array} \right\} \tag{3.52}$$

In this way, under the assumption that the theorem of total probability can be applied at the right-hand limit of some point, denoted by $x+$, the n-point equations can be reduced to a system of transport equations. As can be see in a number of examples following the solution of the equations, these algebraic identities yield a solution that persists until shock interactions form. Because the transport equation in both the two and higher dimensional cases is time-reversible, we know that the solution will break down once any of the initial shocks collide. This is explored further in several examples that follow. However, using a method similar to front tracking, one can use a bootstrapping method to continue the solution at each intersection of shocks. In essence, when shocks collide one state is eliminated and a recalibration of the set of discrete states is necessitated. The process continues until the next collision, etc.

Chapter 4

Solution and Illustration of Equations

4.1 Solving the Equations

The natural next step after working out the form for these equations is to solve them and apply the solutions to some simple cases to evaluate the results. For the one-point case, we drop the indices on x and k and have the set of one-dimensional transport equations given by

$$\begin{aligned} \partial_t F_1(x,k;t) + c_k \partial_1 F_1(x,k;t) &= 0 \\ F_1(x,k;0) &= g_k(x) \end{aligned} \tag{4.1}$$

where $g_k(x)$ are prescribed initial conditions satisfying the conditions

$$g_M(x) = 1,\ 0 \le g_k(x) \le g_l(x) \le 1,\ x \in \mathbb{R},\ l \le k \in \mathbb{N}. \tag{4.2}$$

The solution to the kth equation is given by

$$F_1(x,k;t) = g_k(x - c_k t) \tag{4.3}$$

In n dimensions, we have

$$\partial_t F_n\left(x_1, ..., x_n, k_1, .., k_n; t\right) + c \cdot \nabla F_n = 0$$
$$F_n\left(x_1, ..., x_n, k_1, ..., k_n; 0\right) = g_{k_1,...,k_n}\left(x\right). \tag{4.4}$$

with an associated initial and consistency conditions given by

$$g_{k_1,...,k_n}\left(x_1, ..., x_n\right)|_{k_i=M} = g_{k_1,...,k_{i-1},k_{i+1},...,k_n}\left(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n\right)$$
$$g_{M,...,M}\left(x_1, ..., x_n\right) = 1,\ 0 \le g_k\left(x\right) \le 1,\ x \in \mathbb{R} \tag{4.5}$$

The solution to this transport equation is given by

$$F_n\left(x_1, ..., x_n, k_1, .., k_n; t\right) = g_{k_1,...,k_n}\left(x - ct\right) \tag{4.6}$$

4.2 Examples: purely deterministic case with shocks

Now we want to consider some examples with purely deterministic cases. In these examples only the one-point functions are relevant, so we drop the subscript 1 on F. We consider several cases and observe behavior consistent with classical results.

Example. First consider the case of one shock with only two possible states. We have an initial condition of the form

$$g\left(x\right) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0 \end{cases} \tag{4.7}$$

which leads to the cdfs given by

$$g_1\left(x\right) = 1,\ g_2\left(x\right) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0 \end{cases} \tag{4.8}$$

Note that we are labelling the states in this and following examples in increasing order, i.e. 0 is the first and 1 is the second. Then the solution is given by

$$F(x,2;t) = g_2(x - c_2t) = \begin{cases} 1 & x \le c_1t \\ 0 & x > c_1t \end{cases}$$
$$F(x,1;t) = g_1(x - c_1t) = 1. \tag{4.9}$$

The values can then be calculated by subtraction:

$$p_1(x,2;t) = F(x,2;t) = \begin{cases} 1 & x \le c_1t \\ 0 & x > c_1t \end{cases}$$
$$p_1(x,1;t) = F(x,1;t) - F(x,2;t) = \begin{cases} 0 & x \le c_1t \\ 1 & x > c_1t \end{cases} \tag{4.10}$$

Hence, the solution is

$$u(x,t) = \begin{cases} 1 & x \le c_1t \\ 0 & x > c_1t \end{cases}. \tag{4.11}$$

Example. Now consider a rarefaction type case, where we have three possible states ($u = 0, \pm 1$) and an initial condition of the form

$$g(x) = \begin{cases} -1 & x \le 0 \\ 1 & x > 0 \end{cases}, \tag{4.12}$$

leading to the cdfs given by

$$g_1(x) = 1,\ g_2(x) = g_3(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}. \tag{4.13}$$

We then have

$$
\begin{aligned}
F(x,3;t) &= g_3(x-c_3t) = \begin{cases} 0 & x \le c_3t \\ 1 & x > c_3t \end{cases} \\
F(x,2;t) &= g_2(x-c_2t) = \begin{cases} 0 & x \le c_2t \\ 1 & x > c_2t \end{cases} \\
F(x,1;t) &= g_1(x-c_1t) = 1
\end{aligned}
\tag{4.14}
$$

and probabilities are computed by

$$
\begin{aligned}
p_1(x,3;t) &= F(x,3;t) = \begin{cases} 0 & x \le c_3t \\ 1 & x > c_3t \end{cases} \\
p_1(x,2;t) &= F(x,2;t) - F(x,3;t) = \begin{cases} 0 & x \le c_2t \\ 1 & c_2t < x \le c_3t \\ 0 & c_3t < x \end{cases} \\
p_1(x,1;t) &= F(x,1;t) - F(x,2;t) = \begin{cases} 1 & x \le c_2t \\ 0 & x > c_2t \end{cases}
\end{aligned}
\tag{4.15}
$$

Therefore, the solution in this case is then given by

$$
u(x,t) = \begin{cases} -1 & x \le c_1t \\ 0 & c_1t < x \le c_2t \\ 1 & c_2t < x \end{cases} .
\tag{4.16}
$$

Example. Now consider the evolution of the equations under a more complicated, 3-state case given by

$$
g(x) = \begin{cases} 3 & x \le 1 \\ 2 & 1 < x \le 2 \\ 1 & 2 < x \end{cases} , \; f_1 = 2, \; f_2 = 3, \; f_3 = 8.
\tag{4.17}
$$

First we need to rewrite this in terms of cumulative distribution functions for the different values of

k. We have

$$g_1(x) = 1, \text{ all } x$$
$$g_2(x) = \begin{cases} 1 & x \leq 2 \\ 0 & x > 2 \end{cases}$$
$$g_3(x) = \begin{cases} 1 & x \leq 1 \\ 0 & x > 1 \end{cases} \tag{4.18}$$

The resulting 1-point equation is then a series of simple transport equations that are given by

$$\partial_t F_1(x,k;t) + c_k \partial_1 F_1(x,k;t) = 0 \tag{4.19}$$
$$F_1(x,k;0) = g_k(x)$$

and the solution to each of these equations is

$$F_1(x,k;t) = g_k(x - c_k t) \tag{4.20}$$

The individual probabilities can be computed directly from the cumulative distribution functions, as before:

$$F(x,3;t) = g_3(x - c_3 t) = \begin{cases} 1 & x \leq 1 + c_3 t \\ 0 & x \leq 1 + c_3 t \end{cases}$$
$$F(x,2;t) = g_2(x - c_2 t) = \begin{cases} 1 & x > 2 + c_2 t \\ 0 & x \leq 2 + c_2 t \end{cases}$$
$$F(x,1;t) = g_1(x - c_1 t) = 1, \text{ all } x \tag{4.21}$$

so that

$$p_1(x,3;t) = F_1(x,3;t) = \begin{cases} 1 & x \leq 1 + c_3 t \\ 0 & x \leq 1 + c_3 t \end{cases}$$

$$p_1(x,2;t) = F_1(x,2;t) - F_1(x,3;t)$$

$$= \begin{cases} 1 & 2 + c_2 t < x \leq 1 + c_3 t \\ 0 & x \leq \min(1 + c_3 t, 2 + c_2 t) \text{ or } x > \max(1 + c_3 t, 2 + c_2 t) \\ -1 & 2 + c_2 t < x \leq 1 + c_3 t \end{cases}$$

$$p_1(x,1;t) = F_1(x,1;t) - F_1(x,2;t) = \begin{cases} 0 & x > 2 + c_2 t \\ 1 & x \leq 2 + c_2 t \end{cases} \tag{4.22}$$

This leads to the solution

$$f(x) = \begin{cases} 3 & x \leq 1 + 5t \\ 2 \quad = & 1 + 5t < x \leq 2 + t \\ 1 & 2 + t < x \end{cases} \tag{4.23}$$

up until the point $t = \frac{1}{4}$, $x = \frac{9}{4}$. At this point the solution becomes multivalued, as we have probability 1 for the states $u = 1$ and $u = 3$, that is, the two values seem to overlap. Thus at $\left(\frac{9}{4}, \frac{1}{4}\right)$, we need to reset the differential equation, eliminate any states in the middle (namely the value 2 in this case), and reapply the process with a new initial condition at $t = \frac{1}{4}$ and in this way can construct the solution for all time and space.

As is apparent from these examples, the application of the solution to the derived transport equations from the n-point equations aligns with the expected results from the method of characteristics. Similar to these classical results, solutions break down at some point, in particular when there is a collision of shocks. At time beyond that point, multiple, overlapping values for the solution are given, as would be obtained by continuing to draw characteristics from the conservation law. In order to obtain the full solution, one would reset the problem at each successive collision time t^*, perform the computations using the Rankine-Hugoniot condition to determine the new shock formed, and solve a new problem with initial condition $\tilde{u}(0,x) := u(x,t^*)$.

4.3 Selected examples with randomness introduced into the initial conditions

Another set of problems for consideration on this algebraic system of equations is formed by introducing some basic randomness into the initial conditions. We show that in several examples, the system of equations exhibits the same sort of qualitative behavior. For parts of the solution before potential shocks, the solutions to various examples will match up and be the same as drawing characteristics, and subsequent to collisions of shocks, different values will overlap. We being by considering the same set of transport equations for the tail end distributions as above:

$$\begin{aligned} \partial_t F(x,k;t) + c_k \partial_1 F(x,k;t) &= 0 \\ F(x,k;0) &= g_k(x) \end{aligned} \tag{4.24}$$

where the one-point function $F(x,k;t)$ and initial condition are defined as

$$\begin{aligned} F(x,k;t) &= \mathbb{P}\{u(x,t) \geq u_k\} \\ g_k(x) &= \mathbb{P}\{u(x,0) \geq u_k\} \end{aligned} \tag{4.25}$$

Recall that when attempting to use an initial condition of the form

$$g(x) = \begin{cases} 3 & x \leq 1 \\ 2 & 1 < x \leq 2 \\ 1 & 2 < x \end{cases}, \tag{4.26}$$

we obtained a region of overlap forming when the shocks collided. In this wedge, the probabilities (according to the solution of the transport equation) were given by

$$p(x,1;t) = p(x,3;t) = 1;\ p(x,2;t) = -1 \text{ for } 2+t \leq x < 1+5t,\ t > \frac{1}{4} \tag{4.27}$$

We now consider several examples where we induce randomness by prescribing it in the initial condition g for the location of the shocks to blur out the sharp discontinuity. We consider a uniform distribution, and an exponential distribution with and without an offset. As the uniform distribution is absolutely continuous with respect to the exponential, we expect qualitative similarities to arise in the results, as indeed becomes apparent through the calculations. More precisely, the prescription is to rewrite (4.26) as

$$\tilde{g}(x) = \begin{cases} 3 & x \leq 0 \\ 2 & 0 < x \leq x_* \\ 1 & x_* < x \end{cases}, \tag{4.28}$$

without loss of generality, where the value of x_* is prescribed not by a fixed value but by a given probability distribution $\rho_{x_*}(x)$, where

$$supp\,(\rho_{x_*}) \subseteq (0, \infty). \tag{4.29}$$

In one example below we shift the values slightly, but the principle remains the same.

4.4 Example 1: Uniform Distribution

To start off, we consider the simplest possible case, a uniform distribution for the location of the second jump (from the value 2 down to the value 1) at x_*. Fix the location of the discontinuity from 3 down to 2 at $x = 0$ without loss of generality, by translation. The location of the point $x_* \in [1, 2]$ is then ascribed randomly by the probability density

$$\rho_{x_*}(x) = \begin{cases} 1 & 1 < x \leq 2 \\ 0 & otherwise \end{cases} \tag{4.30}$$

As given by (4.28), our new initial condition is prescribed by

$$\tilde{g}(x) = \begin{cases} 3 & x \leq 0 \\ 2 & 0 < x \leq x_* \\ 1 & x_* < x \end{cases}. \tag{4.31}$$

Next, we need to rewrite this into the expressions g_1, g_2, g_3 (recall $g_i(x) = \mathbb{P}\{u(x,0) \geq u_i\}$; $u_1 = 1, u_2 = 2, u_3 = 3$). We have

$$
\begin{aligned}
g_1(x) &= 1, \text{ all } x \\
g_2(x) &= \begin{cases} 1 & x \leq 1 \\ 1 - x & 1 < x \leq 2 \\ 0 & 2 < x \end{cases} \\
g_3(x) &= \begin{cases} 1 & x \leq 0 \\ 0 & 0 < x \end{cases}
\end{aligned} \tag{4.32}
$$

The next step is to write out $F(x, \cdot; t)$; for this we have

$$
\begin{aligned}
F(x,1;t) &= g_1(x - c_1 t) = 1, \text{ all } x \\
F(x,2;t) &= g_2(x - c_2 t) = \begin{cases} 1 & x \leq 1 + c_2 t \\ 1 + c_2 t - x & 1 + c_2 t < x \leq 2 + c_2 t \\ 0 & 2 + c_2 t < x \end{cases} \\
F(x,3;t) &= g_3(x - c_3 t) = \begin{cases} 1 & x \leq c_3 t \\ 0 & c_3 t < x \end{cases}
\end{aligned} \tag{4.33}
$$

The expressions in (4.33) can now be used to compute the probability of each state at any point with a few simple computations. One has

$$
p_1(x,3;t) = F(x,3;t) = \begin{cases} 1 & x \leq c_3 t \\ 0 & c_3 t < x \end{cases}, \tag{4.34}
$$

$$\begin{aligned}
p_1(x,2;t) &= F(x,2;t) - F(x,3;t) \\
&= \begin{cases} 1 & x \le 1 + c_2 t \\ 1 + c_2 t - x & 1 + c_2 t < x \le 2 + c_2 t \\ 0 & 2 + c_2 t < x \end{cases} - \begin{cases} 1 & x \le c_3 t \\ 0 & c_3 t < x \end{cases} \\
&= \begin{cases} 1 & c_3 t < x \le 1 + c_2 t \\ 1 + c_2 t - x & \max\{1 + c_2 t, c_3 t\} < x \le 2 + c_2 t \\ 0 & \max\{2 + c_2 t, c_3 t\} < x \text{ or } x \le \min\{1 + c_2 t, c_3 t\} \\ c_2 t - x & 1 + c_2 t < x \le \min\{2 + c_2 t, c_3 t\} \\ -1 & 2 + c_2 t < x \le c_3 t \end{cases}, \qquad (4.35)
\end{aligned}$$

$$\begin{aligned}
p_1(x,1;t) &= F(x,1;t) - F(x,2;t) \\
&= 1 - \begin{cases} 1 & x \le 1 + c_2 t \\ 1 + c_2 t - x & 1 + c_2 t < x \le 2 + c_2 t \\ 0 & 2 + c_2 t < x \end{cases} \\
&= \begin{cases} 0 & x \le 1 + c_2 t \\ x - c_2 t & 1 + c_2 t < x \le 2 + c_2 t \\ 1 & 2 + c_2 t < x \end{cases} \qquad (4.36)
\end{aligned}$$

Making a sketch of the xt-plane, one has six separate regions bounded by the x-axis and the lines $x = 1 + c_2 t$, $x = 2 + c_2 t$, $x = c_3 t$. Labelling them counterclockwise and using the shorthand $p(i) := p(x; u_i; t)$, we can describe the probabilities of the solution to take on certain values and summarize the information obtained in equations (4.34)-(4.36). Specifically, let

$$\begin{aligned}
R_1 &:= \{(x,t) \mid x \le \min\{1 + c_2 t, c_3 t\}\} \\
R_2 &:= \{(x,t) \mid c_{3t} < x \le 1 + c_2 t\} \\
R_3 &:= \{(x,t) \mid \max\{1 + c_2 t, c_3 t\} < x \le 2 + c_2 t\} \\
R_4 &:= \{(x,t) \mid \max\{2 + c_2 t, c_3 t\} < x\} \\
R_5 &:= \{(x,t) \mid 2 + c_2 t < x \le c_3 t\} \\
R_6 &:= \{(x,t) \mid 1 + c_2 t < x < \min\{2 + c_2 t, c_3 t\}\}. \qquad (4.37)
\end{aligned}$$

For shorthand, denote $p(\cdot) = p_1(x, \cdot; t)$. Then for $u(x,t)$, we have

$$
\begin{aligned}
&p(1) = 0,\ p(2) = 0,\ p(3) = 1 \text{ in } R_1 \\
&p(1) = 0,\ p(2) = 1,\ p(3) = 0 \text{ in } R_2 \\
&p(1) = x - c_2 t,\ p(2) = 1 + c_2 t - x,\ p(3) = 0 \text{ in } R_3 \\
&p(1) = 1,\ p(2) = 0,\ p(3) = 0 \text{ in } R_4 \\
&p(1) = 1,\ p(2) = -1,\ p(3) = 1 \text{ in } R_5 \\
&p(1) = x - c_2 t,\ p(2) = c_2 t - x,\ p(3) = 1 \text{ in } R_6
\end{aligned}
\tag{4.38}
$$

In the regions R_1, R_2, and R_4, the solution proceeds as one might expect. However, in regions R_3, R_5, and R_6, we obtain probabilities outside the range $[0, 1]$. For example, in R_3 by definition $x - c_2 t > 1$, implying that $p(1) > 1$ and $p(2) < 0$, both of which are physical impossibilities.

4.5 Example 2: Exponential Distribution

Now consider a second example with a different kind of randomness. This time we choose an exponential distribution for the placement of the second shock. Recall the initial condition

$$
\tilde{g}(x) = \begin{cases} 3 & x \leq 0 \\ 2 & 0 < x \leq x_* \\ 1 & x_* < x \end{cases} .
\tag{4.39}
$$

This time, the density for the point x_* is prescribed by

$$
\rho_{x_*}(x) = \begin{cases} e^{-x} & 0 < x \\ 0 & otherwise \end{cases}
\tag{4.40}
$$

As before, we write out the expressions for $g_i\left(x\right)$ and have

$$
\begin{aligned}
g_1\left(x\right) &= 1, \text{ all } x \\
g_2\left(x\right) &= \begin{cases} 1 & x \leq 0 \\ e^{-x} & 0 < x \end{cases} \\
g_3\left(x\right) &= \begin{cases} 1 & x \leq 0 \\ 0 & 0 < x \end{cases}
\end{aligned}
\tag{4.41}
$$

and hence

$$
\begin{aligned}
F\left(x,1;t\right) &= g_1\left(x-c_1t\right) = 1, \text{ all } x \\
F\left(x,2;t\right) &= g_2\left(x-c_2t\right) = \begin{cases} 1 & x \leq c_2t \\ e^{c_2t-x} & c_2t < x \end{cases} \\
F\left(x,3;t\right) &= g_3\left(x-c_3t\right) = \begin{cases} 1 & x \leq c_3t \\ 0 & c_3t < x \end{cases}.
\end{aligned}
\tag{4.42}
$$

The probabilities of given states can then be calculated as follows:

$$
p_1\left(x,3;t\right) = F\left(x,3;t\right) = \begin{cases} 1 & x \leq c_3t \\ 0 & c_3t < x \end{cases}
\tag{4.43}
$$

$$
\begin{aligned}
p_1(x,2;t) &= F(x,2;t) - F(x,3;t) \\
&= \begin{cases} 1 & x \leq c_2 t \\ e^{c_2 t - x} & c_2 t < x \end{cases} - \begin{cases} 1 & x \leq c_3 t \\ 0 & c_3 t < x \end{cases} \\
&= \begin{cases} 1 & c_3 t < x \leq c_2 t \\ e^{c_2 t - x} - 1 & c_2 t < x \leq c_3 t \\ e^{c_2 t - x} & c_3 t < x \\ 0 & x \leq c_2 t \end{cases} \\
&= \begin{cases} 0 & x \leq c_2 t \\ e^{c_2 t - x} - 1 & c_2 t < x \leq c_3 t \\ e^{c_2 t - x} & c_3 t < x \end{cases}
\end{aligned} \tag{4.44}
$$

$$
\begin{aligned}
p_1(x,1;t) = F(x,1;t) - F(x,2;t) = 1 - &\begin{cases} 1 & x \leq c_2 t \\ e^{c_2 t - x} & c_2 t < x \end{cases} \\
= &\begin{cases} 0 & x \leq c_2 t \\ 1 - e^{c_2 t - x} & c_2 t < x \end{cases}
\end{aligned} \tag{4.45}
$$

Labelling regions from left to right as R_1, R_2, and R_3, where

$$
\begin{aligned}
R_1 &= \{(x,t) \mid x \leq c_2 t\} \\
R_2 &= \{(x,t) \mid c_2 t < x \leq c_3 t\} \\
R_3 &= \{(x,t) \mid c_3 t < x\}
\end{aligned} \tag{4.46}
$$

we then have

$$
\begin{aligned}
&p(1) = 0,\ p(2) = 0,\ p(3) = 1 \text{ in } R_1, \\
&p(1) = 1 - e^{c_2 t - x},\ p(2) = e^{c_2 t - x} - 1,\ p(3) = 1 \text{ in } R_2, \\
&p(1) = 1 - e^{c_2 t - x},\ p(2) = e^{c_2 t - x},\ p(3) = 0 \text{ in } R_3.
\end{aligned} \tag{4.47}
$$

Again, one observes that physical impossibilities arise in the cone between the two shocks, wherever they would collide (somewhere in the region R_2). For the other two regions, the computed probabilities make intuitive sense.

4.6 Example 3: Exponential Distribution (with offset)

We consider the exponential distribution again, but this time put an offset to introduce a set of fixed measure where the initial condition takes the intermediate value with probability 1. That is, modify the initial condition to the following:

$$\tilde{g}(x) = \begin{cases} 3 & x \leq -1 \\ 2 & -1 < x \leq x_* \\ 1 & x_* < x \end{cases} \tag{4.48}$$

and keep the probability distribution function for x_* identical to (4.40):

$$\rho_{x_*}(x) = \begin{cases} e^{-x} & 0 < x \\ 0 & otherwise \end{cases} \tag{4.49}$$

We then have

$$\begin{aligned} g_1(x) &= 1, \text{ all } x \\ g_2(x) &= \begin{cases} 1 & x \leq 0 \\ e^{-x} & 0 < x \end{cases} \\ g_3(x) &= \begin{cases} 1 & x \leq -1 \\ 0 & -1 < x \end{cases} \end{aligned} \tag{4.50}$$

and hence

$$\begin{aligned}
F\left(x,1;t\right) &= g_1\left(x-c_1t\right) = 1,\ \text{all } x \\
F\left(x,2;t\right) &= g_2\left(x-c_2t\right) = \begin{cases} 1 & x \le c_2t \\ e^{c_2t-x} & c_2t < x \end{cases} \\
F\left(x,3;t\right) &= g_3\left(x-c_3t\right) = \begin{cases} 1 & x \le c_3t-1 \\ 0 & c_3t-1 < x \end{cases}.
\end{aligned} \tag{4.51}$$

Again, we calculate the probabilities of the given states

$$p\left(x,3;t\right) = F\left(x,3;t\right) = \begin{cases} 1 & x \le c_3t-1 \\ 0 & c_3t-1 < x \end{cases} \tag{4.52}$$

$$\begin{aligned}
p\left(x,2;t\right) &= F\left(x,2;t\right) - F\left(x,3;t\right) \\
&= \begin{cases} 1 & x \le c_2t \\ e^{c_2t-x} & c_2t < x \end{cases} - \begin{cases} 1 & x \le c_3t-1 \\ 0 & c_3t-1 < x \end{cases} \\
&= \begin{cases} 1 & c_3t-1 < x \le c_2t \\ 1-e^{c_2t-x} & c_2t < x \le c_3t-1 \\ e^{c_2t-x} & \max\left\{c_2t, c_3t-1\right\} < x \\ 0 & x \le \min\left\{c_2t, c_3t-1\right\} \end{cases}
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
p_1\left(x,1;t\right) &= F\left(x,1;t\right) - F\left(x,2;t\right) = 1 - \begin{cases} 1 & x \le c_2t \\ e^{c_2t-x} & c_2t < x \end{cases} \\
&= \begin{cases} 0 & x \le c_2t \\ 1-e^{c_2t-x} & c_2t < x \end{cases}
\end{aligned} \tag{4.54}$$

The space is now separated into several regions bounded by c_2t and c_3t-1; label them going counterclockwise as follows:

$$\begin{aligned}
&R_1 = \{(x,t) \mid x \leq \min\{c_2t, c_3t-1\}\} \\
&R_2 = \{(x,t) \mid c_3t-1 < x \leq c_2t\} \\
&R_3 = \{(x,t) \mid \ \max\{c_2t, c_3t-1\}\} \\
&R_4 = \{(x,t) \mid c_2t < x \leq c_3t-1\}
\end{aligned} \tag{4.55}$$

The probabilities that the solution u will take certain values at points (x,t) is then given, by region, as

$$\begin{aligned}
&p(1) = 0,\ p(2) = 0,\ p(3) = 1 \text{ in } R_1, \\
&p(1) = 0,\ p(2) = 1,\ p(3) = 0 \text{ in } R_2, \\
&p(1) = 1 - e^{c_2t-x},\ p(2) = e^{c_2t-x} - 1,\ p(3) = 0 \text{ in } R_3 \\
&p(1) = 1 - e^{c_2t-x},\ p(2) = e^{c_2t-x},\ p(3) = 1 \text{ in } R_4
\end{aligned} \tag{4.56}$$

We once again see the pattern of an intuitively correct result before the first possible collision, i.e. in R_1, R_2, R_3, and afterwards results that seem counterintuitive. Subsequently, we have issues that are similar to what one obtains with classical front tracking without using the Rankine-Hugoniot condition.

4.7 Analytic Solution

Now we try to solve the third example completely analytically, in the way the solutions would evolve according to the methods of characteristics, and doing computations over the appropriate probability measures. We begin with the initial condition from the third example

$$\tilde{g}(x) = \begin{cases} 3 & x \leq -1 \\ 2 & -1 < x \leq x_* \\ 1 & x_* < x \end{cases} \tag{4.57}$$

and aim to do computations over various values of x_*. We know that initially, we have two shocks, located along the lines $x = c_3 t - 1$ and $x = x_* + c_2 t$. This behavior will continue until the first possible point at which the shocks collide. In general, this will be at the value where

$$\begin{aligned} c_3 t - 1 &= x_* + c_2 t \\ t &= \frac{x_* + 1}{c_3 - c_2} \\ x &= \frac{x_* + 1}{c_3 - c_2} - 1 = \frac{x_* + c_3 - c_2 - 1}{c_3 - c_2} \\ (x_0, t_0) &= \left(\frac{x_* + 1}{c_3 - c_2}, \frac{x_* + c_3 - c_2 - 1}{c_3 - c_2} \right) \end{aligned} \tag{4.58}$$

The earliest possible time for this occurrence is when $x_* = 0$, yielding

$$(x_0, t_0) = \left(\frac{1}{c_3 - c_2}, \frac{c_3 - c_2 - 1}{c_3 - c_2} \right). \tag{4.59}$$

Subsequent to this point, we expect that the shocks collide, the state $u = 2$ disappears, and leaves us with a single shock moving with velocity

$$\tilde{c} = \frac{f(3) - f(1)}{3 - 1} = \frac{1}{2} \left\{ \frac{f(3) - f(2)}{3 - 2} + \frac{f(2) - f(1)}{2 - 1} \right\} = \frac{c_2 + c_3}{2}. \tag{4.60}$$

The complete description of the behavior of the solution can then be given by

$$\begin{aligned} u(x,t) &= \begin{cases} 3 & x \le c_3 t - 1 \\ 2 & c_3 t - 1 < x \le x_* + c_2 t \\ 1 & x_* + c_2 t < x \end{cases} \quad \text{for } t \le t_0 \\ &= \begin{cases} 3 & x \le x_0 + \tilde{c} t \\ 1 & x_0 + \tilde{c} t < x \end{cases} \quad \text{for } t_0 < t. \end{aligned} \tag{4.61}$$

Of course, we want the result in terms of probabilities rather than x_*, so we use the distribution to obtain this result. Recall the definition of the four regions used in the third example:

$$\begin{aligned}
R_1 &= \{(x,t) \mid x \le \min\{c_2 t, c_3 t - 1\}\} \\
R_2 &= \{(x,t) \mid c_3 t - 1 < x \le c_2 t\} \\
R_3 &= \{(x,t) \mid \ \max\{c_2 t, c_3 t - 1\}\} \\
R_4 &= \{(x,t) \mid c_2 t < x \le c_3 t - 1\}\,.
\end{aligned} \tag{4.62}$$

We then have

$$\begin{aligned}
&p(1) = 0,\ p(2) = 0,\ p(3) = 1 \text{ in } R_1 \\
&p(1) = 0,\ p(2) = 1,\ p(3) = 0 \text{ in } R_2, \\
&p(1) = 1 - e^{c_2 t - x},\ p(2) = e^{c_2 t - x} - 1,\ p(3) = 0 \text{ in } R_3
\end{aligned} \tag{4.63}$$

quite easily. For the fourth region, we first rewrite the solution out fully:

$$u(x,t) = \begin{cases} 3 & x \le \frac{x_*+1}{c_3-c_2} + \tilde{c}t \\ 1 & \frac{x_*+1}{c_3-c_2} + \tilde{c}t < x \end{cases}. \tag{4.64}$$

Hence,

$$\begin{aligned}
p(1) &= \mathbb{P}\left\{\frac{x_*+1}{c_3-c_2} + \tilde{c}t < x\right\} = \mathbb{P}\left\{x_* < (c_3-c_2)(x-\tilde{c}t) - 1\right\} \\
&= \int_0^{(c_3-c_2)(x-\tilde{c}t)-1} 1_{(0,\infty)}(y)\, e^{-y} dy = 1 - \exp\left(\max\{0, 1-(c_3-c_2)(x-\tilde{c}t)\}\right).
\end{aligned} \tag{4.65}$$

We can therefore represent the full solution probabilistically as

$$
\begin{aligned}
&p(1) = 0,\; p(2) = 0,\; p(3) = 1 \text{ in } R_1\\
&p(1) = 0,\; p(2) = 1,\; p(3) = 0 \text{ in } R_2,\\
&p(1) = 1 - e^{c_2 t - x},\; p(2) = e^{c_2 t - x} - 1,\; p(3) = 0 \text{ in } R_3\\
&\left\{\begin{array}{c} p(1) = 1 - \exp\left(\min\left\{0, 1 - (c_3 - c_2)(x - \tilde{c}t)\right\}\right)\\ p(2) = 0\\ p(3) = \exp\left(\min\left\{0, 1 - (c_3 - c_2)(x - \tilde{c}t)\right\}\right)\end{array}\right. \quad \text{in } R_4
\end{aligned}
\tag{4.66}
$$

It is not quite clear how to reconcile (4.56) with (4.66).

4.8 Calculating two-point functions for Example 3

To remedy the nonphysical probabilities we obtained in the examples, we now calculate out expressions for the two-point functions in the pursuit of a better understanding of the missing decay term on the right-hand side of (4.24). We perform our calculations in terms of x_* and the point of first collision, which we label as (x_0, t_0), and modify the notation to define several regions of xt-space:

$$
\begin{aligned}
R_1 &= \left\{(x,t) \mid x \le \min\left\{x_* + c_2 t, c_3 t - 1\right\}, t < t_0\right\}\\
R_2 &= \left\{(x,t) \mid c_3 t - 1 < x \le x_* + c_2 t\ \right\}\\
R_3 &= \left\{(x,t) \mid\ \max\left\{x_* + c_2 t, c_3 t - 1\right\}, t < t_0\right\}\\
R_4 &= \left\{(x,t) \mid x < x_0 + \tilde{c}t, t_0 \le t\right\}\\
R_5 &= \left\{(x,t) \mid x < x_0 + \tilde{c}(t - t_0), t_0 \le t\right\},
\end{aligned}
\tag{4.67}
$$

recalling that $\tilde{c} = \frac{c_1 + c_2}{2}$ Define in addition

$$
\begin{aligned}
l_1 &= \left\{(x,t) \mid x = c_3 t - 1,\; t < t_0\right\}\\
l_2 &= \left\{(x,t) \mid x = x_* + c_2 t,\; t < t_0\right\}\\
l_3 &= \left\{(x,t) \mid x = x_0 + \tilde{c}(t - t_0),\; t_0 \le t\right\}
\end{aligned}
\tag{4.68}
$$

We already know what the expected solution looks like for a given x_* and hence (x_0, t_0). In terms of these variables, the solution can be expressed as

$$u(x,t) = \begin{cases} 3 & \text{in } R_1 \\ 2 & \text{in } R_2 \\ 1 & \text{in } R_3 \\ 1 & \text{in } R_4 \\ 3 & \text{in } R_5 \end{cases} \tag{4.69}$$

For brevity, in this section we adopt the notation

$$F_2(k_1,k_2) := F_2(x_1,x_2;k_1,k_2;t) = \mathbb{P}\{u(x_1,t) \geq u_{k_1}, u(x_2,t) \geq u_{k_2}\} \tag{4.70}$$

and where further $u_k = k$, within the scope of this example. Clearly, with this setup in terms of x_*, one can write the two-point function in terms of the discretized heaviside function:

$$\begin{aligned} F_2(k_1,k_2) &= H_{u(x_1,t),k_1} H_{u(x_2,t),k_2}, \\ H_{i,j} &= \begin{cases} 1 & i \geq j \\ 0 & i < j \end{cases} \end{aligned} \tag{4.71}$$

i.e. the two point function vanishes if the value of the solution at either argument is greater than its k index, for either value. We assume that our solution is cadlag, i.e. define it to be left continuous and assume the right-hand limit exists. On a boundary of the shock, we then have

$$\begin{aligned} \lim_{x_2 \to x_1+} F_2(k_1,k_2) &= H_{2,k_2}, \ \ x_1 \in l_1 \\ \lim_{x_2 \to x_1+} F_2(k_1,k_2) &= \delta_{1,k_2} H_{2,k_1}, \ \ x_1 \in l_2 \\ \lim_{x_2 \to x_1+} F_2(k_1,k_2) &= \delta_{1,k_2}, \ \ x_1 \in l_3. \end{aligned} \tag{4.72}$$

We note that in the first equation, there is only one term since the solution takes the value of 3 on the left-hand side of l_1, so that whatever the value of k_1, one has $3 \geq k_1$. Define F_2^+, (relabelling the arguments x_1 as x, x_2 as x', and similarly for k_1, k_2) by

$$F_2^+(k,k') := F_2^+(x,x',k,k';t) := \lim_{x_2\to x_1} F_2(x_1,x_2,k_1,k_2;t) \tag{4.73}$$

which takes the value 1 with appropriate pairs of k_i on each shock, and is otherwise zero. We now modify (4.24) to the form

$$\begin{aligned}\partial_t F_1(x,k;t) + c_k\partial_1 F_1(x,k;t) &= \sum_{k'=1}^{3} C_{k,k'}F_2^+(k,k') \\ F_1(x,k;0) &= g_k(x),\end{aligned} \tag{4.74}$$

adding in any possible combination of F_2^+. The solution to (4.74), the inhomogeneous transport equation, is then given by

$$F_1(x,k;t) = g_k(x-c_kt) + \int_0^t \sum_{k'=1}^{3} C_{k,k'}F_2^+(x+(s-t)c_k,x',k,k';s)\,ds. \tag{4.75}$$

The entirety of the extra term will drop out for $x \notin \cup_{i=1}^3\{l_i\}$. Therefore, consider the behavior of the equations for x in one of these sets, starting with $x\in l_1$. Under this restriction, $F_2^+(k,k')\neq 0$ iff $k'\geq 2$. Thus, (4.75) becomes

$$\begin{aligned}F_1(x,k;t) &= g_k(x-c_kt) + \int_0^t \sum_{k'=2}^{3} C_{k,k'}F_2^+(x+(s-t)c_k,x',k,k';s)\,ds \\ &= g_k(x-c_kt) + t\sum_{k'=2}^{3} C_{k,k'}\end{aligned} \tag{4.76}$$

In order to match the solution (4.69), we must have $\sum_{k'=2}^{3} C_{k,k'} = C_{k,2}+C_{k,3}=0$, $1\leq k\leq 3$.

Similarly, choosing $x\in l_2$, we must have $k'=1$, $k\geq 2$ for $F_2^+(k,k')\neq 0$ and compute

$$\begin{aligned}F_1(x,k;t) &= g_k(x-c_kt) + \int_0^t C_{k,1}F_2^+(x+(s-t)c_k,x',k,1;s)\,ds \\ &= g_2(x-c_kt) + tC_{k,1},\ k\geq 2 \\ F_1(x,1;t) &= g_1(x-c_1t)\end{aligned} \tag{4.77}$$

requiring $C_{k,1}=0$ for $k\geq 2$ to reconcile (4.77) with (4.69).

We proceed to $x \in l_3$, and in this case $k' = 1$, k is free so that

$$\begin{aligned} F_1(x,k;t) &= g_3(x-tb) + \int_0^t C_{k,2} F_2^+ (x + (s-t) c_k, x', k, 1; s)\, ds \\ &= g_3(x-tb)\,. \end{aligned} \tag{4.78}$$

This second term in (4.78) is only nonzero when the first argument of F_2^+, that is the quantity $x + (s-t) c_k$, traces out the line $x = x_0 + \tilde{c}(t - t_0)$. However, as the slopes $\tilde{c} = \frac{c_2+c_3}{2}$ and $\{c_k\}$ will never match, this term is zero outside of a set of Lebesgue measure zero, and the second term drops out. Hence, we conclude that

$$\begin{aligned} &C_{1,1} = 0, C_{1,2} = -C_{1,3}, C_{1,3} \text{ free} \\ &C_{2,1} = 0, C_{2,2} = -C_{2,3}, C_{2,3} \text{ free} \\ &C_{3,1} = 0, C_{3,2} = -C_{3,3}, C_{3,3} \text{ free} \end{aligned} \tag{4.79}$$

Therefore, adding terms of the form F_2^+ would not seem to reconcile the differences between the solution computed from the transport equation and the expected solution from the method of characteristics combined with the Rankine-Hugoniot condition. Thus, a natural limitation of this approach appears to be the requirement of resetting the problem at the point when there is interaction between shocks. [It seems for this derivation that the method of solving the problem until there are interactions between shocks and then resetting the problem in the way described above is the furthest this approach can be carried out.]

Chapter 5

Utilizing Shocks as the Building Blocks

We now approach the problem in a distinctly different manner from the calculations above by making a fundamental change in how we view our system. In previous sections we had considered the discrete system as a set of particles and velocities at specific points and left-hand limits of these points. In this section, in contrast to this idea of using individual velocities as the main building block, we use the individual shocks as building blocks for the system. In this way we replace the one-point distribution with the density of a shock prescribed by its values at left and right states, the two-point distribution by a term tracking two shocks at these two points, and so forth, i.e.

$$\begin{aligned} p_1(x,t;u) &\to f_1(x_-,x,t;u,v) \\ p_2(x_1,x_2,t;u_1,u_2) &\to f_2(x_{1,-},x_1,x_{2,-},x_2,t;u_1,v_1,u_2,v_2)\,. \end{aligned} \tag{5.1}$$

We are also considering the left limit and value at the point and assuming right continuity for this chapter for consistency with the broader literature. We can think of shocks between different values as different species of particles, whose velocities are based only on the relative difference of these two states making up the species. The orientation will not matter in the sense that the shock speed will be the same whether we have a jump from u_1 to u_2 or from u_2 to u_1, but there will be changes that enter into interactions between the shock and other shocks. We assume without loss of generality that the original flux function is strictly increasing, so that all shocks are rightward moving. To simplify the notation, for the time being we will express this two-point distribution as

$$f_2\left(x, x', t; u, u', v, v'\right). \tag{5.2}$$

and the n-point distribution as

$$f_n\left(x_1, ..., x_n, t; u_1, ..., u_n; v_1, ..., v_n\right). \tag{5.3}$$

5.1 Compatibility Conditions for n-point Equations

Before starting to write out the equations for what is expected in the evolution of the n-point equations, starting with the 1-point equation, it is important to establish some compatibility conditions to ensure these objects make sense. For instance, the two-point distribution should reduce down to the one-point in several ways. Most simply, one would expect to find that

$$\sum_{u',v'} f_2\left(x, x', t; u, u', v, v'\right) = f_1\left(x, t; u, v\right) \text{ and } \sum_{u,v} f_2\left(x, x', t; u, u', v, v'\right) = f_1\left(x, t; u, v\right). \tag{5.4}$$

Equation (5.4) is the statement that the two-point distribution summed over one set of arguments reduces to the one-point distribution over the remaining set. Further, one expects convergence in some sense for the two-point distribution down to the one-point as the arguments converge, but only if the velocity pairs are identical, i.e.,

$$\lim_{x' \to x} f_2\left(x, x', t; u, u', v, v'\right) = f_1\left(x, t; u, v\right) \delta_{u,v} \delta_{u',v'}. \tag{5.5}$$

This is equivalent to statement that the probability in the limit is zero unless these shocks are indeed identical. In the n-point limit, equations (5.4) and (5.5) generalize to

$$\begin{aligned} &\sum_{u_i,v_i} f_n\left(x_1, ..., x_i, ..., x_n, t; u_1, ..., u_i, ..., u_n, v_1, ..., v_i, ..., v_n\right) \\ &= f_{n-1}\left(x_1, ..., x_{i-1}, x_{i+1}, t; u_1, ..., u_{i-1}, u_{i+1}, ..., u_n, v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\right) \end{aligned} \tag{5.6}$$

and

$$\lim_{x_j \to x_i} f_n\left(x_1, ..., x_j, ..., x_n, t; u_1, ..., u_j, ..., u_n, v_1, ..., v_j, ..., v_n\right)$$
$$= f_{n-1}\left(x_1, ..., x_{j-1}, x_{j+1}, t; u_1, ..., u_{j-1}, u_{j+1}, ..., u_n, v_1, ..., v_{j-1}, v_{j+1}, ..., v_n\right) \delta_{u_i,v_i} \delta_{u_j,v_j} \tag{5.7}$$

Further, we want to impose the restriction that any upward jumps in prescribed initial conditions can only be between nearest neighbor states when ordered as strictly increasing. This can be phrased in symbols as the following:

$$f_1\left(x, 0, u, v\right) = 0 \text{ for } v > u + 1, \tag{5.8}$$

assuming that the states are labelled by integers and in increasing order. Indeed, by simple convexity arguments, one can conclude further that no larger upward jumps (that is, an upward jump between two states that are not nearest neighbors) can ever form.

5.2 Derivation of 1-point Equation for Shocks

We want to write down an equation for the one-point density in (5.1) as a function of the two-point density. On the left-hand side we clearly have a term of the form

$$\partial_t f_1\left(x_-, x, t; u, v\right) \tag{5.9}$$

for the creation/destruction of the species. There will also be a free-streaming term, as the shocks will be moving with speed c_{uv} given from the Rankine-Hugoniot condition based on the values taken on by the solution on either side of the shock. This term will be given by

$$c_{uv} \partial_x f_1\left(x_-, x, t; u, v\right). \tag{5.10}$$

On the right-hand side of the equation, we must now account for creation or destruction of shocks of the flavor with u on the left side and v on the right side. Indeed, there is a birth term of the following sort:

$$\sum_w f_2\left(x, x + \Delta x, t; u, w, w, v\right). \tag{5.11}$$

That is, when we have one shock between u and an intermediate value w, and a second shock between that intermediate value w and the original terminal value v, these shocks will combine to produce a new member of the species uv. This can only occur when the shock between u and w can catch the shock between w and v, which is possible when

$$\frac{f(u)-f(w)}{u-w} = c_{uw} > c_{wv} = \frac{f(w)-f(v)}{w-v}. \tag{5.12}$$

We can also have a destruction of a shock with speed c_{uv}, upon collision with another shock. This leads to a term of the following

$$-\sum_{w} f_2(x, x+\Delta x, t; u, v, v, w). \tag{5.13}$$

There is a negative sign as this term leads to a destruction of the shock rather than creation as in (5.11). Again, this can only happen if the left shock can overtake the right one, i.e. if

$$\frac{f(u)-f(v)}{u-v} = c_{uv} > c_{vw} = \frac{f(v)-f(w)}{v-w} \tag{5.14}$$

Further, there can also be a loss if we have the shock between u and v collides with a different shock between w and u from the left, leading to a term of the form

$$-\sum_{w} f_2(x-\Delta x, x, t; w, u, u, v). \tag{5.15}$$

which can only occur if

$$\frac{f(w)-f(u)}{w-u} = c_{wu} > c_{uv} = \frac{f(u)-f(v)}{u-v} \tag{5.16}$$

We now want to express these terms in a Taylor expansion and rewrite the quantity Δx using the relative speeds of the collisions. Note that, starting with term under the summand in (5.11) we observe

$$f_2(x, x+\Delta x, t; u, w, w, v) \tilde{=} f_2(x, x, t; u, w, w, v) + \Delta x \partial_2 f_2(x, x, t; u, w, w, v) \tag{5.17}$$

We want to set

$$\Delta x = (c_{uw} - c_{wv}) \Delta t \tag{5.18}$$

yielding

$$f_2\left(x,x+\Delta x,t;u,w,w,v\right)=f_2\left(x,x,t;u,w,w,v\right)+\left(c_{wv}-c_{uw}\right)\Delta t\partial_2 f_2\left(x,x,t;u,w,w,v\right) \quad (5.19)$$

so the term we want is

$$\sum_w \left(c_{uw}-c_{wv}\right)\partial_2 f_2\left(x,x,u,w,w,v\right). \quad (5.20)$$

The first term vanishes since whenever u, v, w are not distinct, there are no shocks.

For the time being, we drop the first term in the right-hand side of (5.19) as it does not make a physical contribution. If u, v, w are not distinct, there is by definition no shock, and if they are equal, there are by definition no shocks. Similarly to this, we will obtain terms

$$-\sum_w \left(c_{uv}-c_{vw}\right)\partial_2 f_2\left(x,x,t;u,v,v,w\right) \quad (5.21)$$

and

$$-\sum_w \left(c_{wu}-c_{uv}\right)\partial_1 f_2\left(x,x,t;w,u,u,v\right) \quad (5.22)$$

There is an additional matter of which values the summand can take; that is, for which values of w do we indeed have growth and decay terms, and which simply give no contribution due to the left shock not being able to catch the right shock. These different cases can be separated into $u<v$ and $v<u$, and further bifurcated by the relative position of w with respect to the two values. Using the notion of the (piecewise linear) convexity of f, one can perform the bookkeeping to see which cases are possible and which are not. For example, for the term (5.20), we have shocks between the values (u,w) and (w,v). With $u<v$ we have

$$\begin{aligned}
w<u<v &\Rightarrow \frac{f\left(u\right)-f\left(w\right)}{u-w}=c_{uw}<c_{wv}=\frac{f\left(w\right)-f\left(v\right)}{w-v}\\
u<w<v &\Rightarrow \frac{f\left(u\right)-f\left(w\right)}{u-w}=c_{uw}<c_{wv}=\frac{f\left(w\right)-f\left(v\right)}{w-v}\\
u<v<w &\Rightarrow \frac{f\left(u\right)-f\left(w\right)}{u-w}=c_{uw}<c_{wv}=\frac{f\left(w\right)-f\left(v\right)}{w-v}.
\end{aligned} \quad (5.23)$$

Indeed, these calculations demonstrate that there cannot be a growth term for $u<v$, . On the other

hand, should one have $v < u$, one computes

$$\begin{aligned} w < v < u &\Rightarrow \frac{f(u) - f(w)}{u - w} = c_{uw} > c_{wv} = \frac{f(w) - f(v)}{w - v} \\ v < w < u &\Rightarrow \frac{f(u) - f(w)}{u - w} = c_{uw} > c_{wv} = \frac{f(w) - f(v)}{w - v} \\ v < u < w &\Rightarrow \frac{f(u) - f(w)}{u - w} = c_{uw} > c_{wv} = \frac{f(w) - f(v)}{w - v} \end{aligned} \tag{5.24}$$

meaning that the term can occur for any value of w as long as $v < u$. Next, for the decay term with collisions from the right (5.21), the relevant shocks are (u, v) and (v, w). We compute, for $u < v$:

$$\begin{aligned} w < u < v &\Rightarrow c_{uv} > c_{wv} \\ u < w < v &\Rightarrow c_{uv} < c_{wv} \\ u < v < w &\Rightarrow c_{uv} < c_{wv} \end{aligned} \tag{5.25}$$

and for $v < u$:

$$\begin{aligned} w < v < u &\Rightarrow c_{uv} > c_{wv} \\ v < w < u &\Rightarrow c_{uv} > c_{wv} \\ v < u < w &\Rightarrow c_{uv} < c_{wv} \end{aligned} \tag{5.26}$$

so this term will have a nonzero contribution if $u < v$ and $w < u$ or if $v < u$ and $w < u$. Finally, for the decay term with collisions from the left (5.22), the relevant shocks are (w, u) and (u, v). When $u < v$ we have

$$\begin{aligned} w < u < v &\Rightarrow c_{wu} < c_{uv} \\ u < w < v &\Rightarrow c_{wu} < c_{uv} \\ u < v < w &\Rightarrow c_{wu} > c_{uv} \end{aligned} \tag{5.27}$$

and for $v < u$ we have

$$
\begin{aligned}
w < v < u &\Rightarrow c_{wu} < c_{uv} \\
v < w < u &\Rightarrow c_{wu} > c_{uv} \\
v < u < w &\Rightarrow c_{wu} > c_{uv}
\end{aligned}
\tag{5.28}
$$

so that this term can occur for $u < v$ if $w > v$ or if $v < u$ and $w > v$. Putting together the terms (5.9), (5.10), and (5.20)-(5.22), we have

$$
\begin{aligned}
&\partial_t f_1 (x, t; u, v) + c_{uv} \partial_x f_1 (x, t; u, v) \\
&= - \sum_{w<u} (c_{uv} - c_{vw}) \, \partial_2 f_2 (x, x, t; u, v, v, w) \\
&- \sum_{v<w} (c_{wu} - c_{uv}) \, \partial_1 f_2 (x, x, t; w, u, u, v) \, .
\end{aligned}
\tag{5.29}
$$

for $u < v$ and

$$
\begin{aligned}
&\partial_t f_1 (x, t; u, v) + c_{uv} \partial_x f_1 (x, t; u, v) \\
&= \sum_{w} (c_{uw} - c_{wv}) \, \partial_2 f_2 (x, x, t; u, w, w, v) - \sum_{w<u} (c_{uv} - c_{vw}) \, \partial_2 f_2 (x, x, t; u, v, v, w) \\
&- \sum_{v<w} (c_{wu} - c_{uv}) \, \partial_1 f_2 (x, x, t; w, u, u, v) \, .
\end{aligned}
\tag{5.30}
$$

for $v < u$, where all summations are under the additional restriction that u, v, w are distinct. We need to incorporate the additional assumption that any upward jumps are only between nearest neighbors. This leaves us with the one-point equation written as

$$
\begin{aligned}
&\partial_t f_1 (x, t; u, u+1) + c_{u,u+1} \partial_x f_1 (x, t; u, u+1) \\
&= - \sum_{w<u} (c_{u,u+1} - c_{u+1,w}) \, \partial_2 f_2 (x, x, t; u, u+1, u+1, w) \\
&- \sum_{w>u+1} (c_{w,u} - c_{u,u+1}) \, \partial_1 f_2 (x, x, t; w, u, u, u+1)
\end{aligned}
\tag{5.31}
$$

for $u < v$, wherein the only meaningful case is when u and v are neighbors. We use the shorthand $v = u + 1$ to mean that for $u = u_i$, we set $v = u_{i+1}$, that is, the next state in increasing order. We use the same notation (where applicable) with the quantity $u - 1$ below in the equation for $v < u$

and have

$$\begin{aligned}
&\partial_t f_1(x,t;u,v) + c_{uv}\partial_x f_1(x,t;u,v) \\
&= \sum_{w\leq u+1} (c_{uw} - c_{wv})\,\partial_2 f_2(x,x,t;u,w,w,v) - \sum_{\substack{w<u\\ w\leq v+1}} (c_{uv} - c_{vw})\,\partial_2 f_2(x,x,t;u,v,v,w) \\
&- \sum_{\substack{v<w\\ w\geq u-1}} (c_{wu} - c_{uv})\,\partial_1 f_2(x,x,t;w,u,u,v)\,. \qquad (5.32)
\end{aligned}$$

In fact, the second term in equation (5.32) can have its summand rewritten as $w \leq v+1$ since $v < u$ implies $v \leq u+1$ and the last term can have its summand rewritten as $u-1 \leq w$ since $v < u$ implies $v \leq u-1$. Hence, one has

$$\begin{aligned}
&\partial_t f_1(x,t;u,v) + c_{uv}\partial_x f_1(x,t;u,v) \\
&= \sum_{w\leq u+1} (c_{uw} - c_{wv})\,\partial_2 f_2(x,x,t;u,w,w,v) - \sum_{w\leq v+1} (c_{uv} - c_{vw})\,\partial_2 f_2(x,x,t;u,v,v,w) \\
&- \sum_{w\geq u-1} (c_{wu} - c_{uv})\,\partial_1 f_2(x,x,t;w,u,u,v)\,. \qquad (5.33)
\end{aligned}$$

Equations (3.22) and (3.24) together describe the one-point equation in all cases. The further restriction that u, v, w are all distinct is not further written out in the equations for notational convenience, but always assumed in calculations following.

5.3 Example with two shocks

Consider an example with two downward shocks: one between u_3 and u_2, and a second between u_2 and u_1, where $u_3 > u_2 > u_1$. The various possible interactions are pictured in the figure on the next page. By convexity, one has

$$\frac{f(u_3) - f(u_2)}{u_3 - u_2} = c_{32} > c_{21} = \frac{f(u_2) - f(u_1)}{u_2 - u_1}, \qquad (5.34)$$

ensuring that the left shock will catch up to the right one in finite time. More precisely, write an initial condition of the form

$$g(x) = \begin{cases} u_3 & x < x_1 \\ u_2 & x_1 \leq x < x_2 \\ u_1 & x_2 \leq x \end{cases} \tag{5.35}$$

We will apply the one-point equation to all six species of shocks, denoting them by the notation (u, v), where u is the value of the "left" state and v the value of the "right" state. First we do this in the general framework of a three-state system, then apply it specifically to the initial condition in (5.35). We begin with the case $u = 3$, $v = 1$ (i.e., what is the equation describing the creation and destruction of shocks between the values $u = u_3$ and $v = u_1$), one has

$$\begin{aligned} \partial_t f_1(x; 3, 1) + c_{31}\partial_x f_1(x; 3, 1) &= \sum_{w \leq 4} (c_{3w} - c_{w1})\, \partial_2 f_2(x, x, t; 3, w, w, 1) \\ &\quad - \sum_{w \leq 2} (c_{31} - c_{1w})\, \partial_2 f_2(x, x, t; u, v, v, w) \\ &\quad - \sum_{w \geq 2} (c_{w3} - c_{31})\, \partial_1 f_2(x, x; w, 3, 3, 1)\,. \end{aligned} \tag{5.36}$$

Labelling the terms on the right-hand side of equation (4.3) as (I) and (II), and (III), we observe that each sum contributes only one term (using the restriction that u, v, w must all be distinct):

$$\begin{aligned} (I) &= (c_{32} - c_{21})\, \partial_2 f_2(x, x; 3, 2, 2, 1) \\ (II) &= -(c_{31} - c_{12})\, \partial_2 f_2(x, x; 3, 1, 1, 2) \\ (III) &= -(c_{23} - c_{31})\, \partial_1 f_2(x, x, 2, 3, 3, 1) \end{aligned} \tag{5.37}$$

Hence, (5.36) yields

$$\begin{aligned} \partial_t f_1(x; 3, 1) + c_{31}\partial_x f_1(x; 3, 1) &= (c_{32} - c_{21})\, \partial_2 f_2(x, x; 3, 2, 2, 1) \\ &\quad - (c_{31} - c_{12})\, \partial_2 f_2(x, x; 3, 1, 1, 2) \\ &\quad - (c_{23} - c_{31})\, \partial_1 f_2(x, x, 2, 3, 3, 1) \end{aligned} \tag{5.38}$$

which conforms to our expectations of the right equation; on the left hand side we have a creation and translation term between the states $u_3 = 3$ and $u_1 = 1$, and on the right we have a growth term

Production and destruction of shocks (left state, right state) in the three-state system. Pairs of shocks that will never collide due to relative velocities are marked with a (X), shocks that will interact are marked by a (✓), and interactions involving an upward jump not between nearest neighbors (here, from left state 1 to right state 3) are overlayed with a dashed (X).

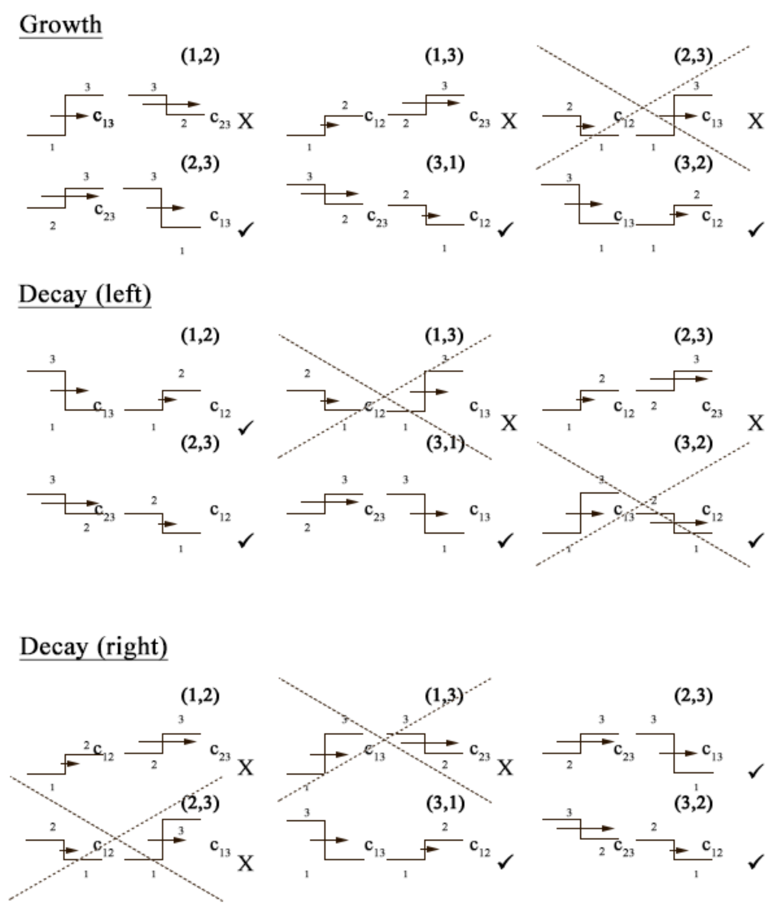

Figure 5.1: Interactions between possible shocks in the three-state system, categorized by growth and different kinds of decay. Depending on relative velocities of the shocks, one shock may catch up to another and interact to combine a new shock in the form of (left state, right state) above each figure.

when the pairs $(3,2)$, $(2,1)$ collide, and decay when $(3,1)$ hits $(1,2)$ or $(2,3)$ hits $(3,1)$.

In contrast, for the case $u=1$, $v=3$, we are considering a hypothetical upward jump between non-neighboring states, so we end up with a trivial equation of the form

$$\partial_t f_1 (x;1,3) + c_{13} \partial_x f_1 (x;1,3) = 0, \tag{5.39}$$

so that since the shock does not exist in the initial conditions, the ∂_x term is zero, hence the ∂_t term is also zero and such a shock can never be created, which is consistent with our assumptions and intuition.

If we try for example $u=1$, $v=2$, we have

$$\begin{aligned} &\partial_t f_1 (x;1,2) + c_{12} \partial_x f_1 (x;1,2) \\ &= -\sum_{w<1} (c_{12} - c_{2w}) \partial_2 f_2 (x,x,1,2,2,w) - \sum_{w>2} (c_{w1} - c_{12}) \partial_1 f_2 (x,x;w,1,1,2) \end{aligned} \tag{5.40}$$

and the right-hand side reduces to

$$(c_{31} - c_{12}) \partial_1 f_2 (x,x;3,1,1,2) \tag{5.41}$$

leading to the equation

$$\partial_t f_1 (x;1,2) + c_{12} \partial_x f_1 (x;1,2) = (c_{31} - c_{12}) \partial_1 f_2 (x,x;3,1,1,2). \tag{5.42}$$

That is, shocks between $u_1 = 1$ and $u_2 = 2$ can be destroyed by a collision between $(3,1)$ from the left with $(1,2)$, but not created.

For the case $u=2$, $v=3$, we have

$$\begin{aligned} &\partial_t f_1 (x;2,3) + c_{23} \partial_x f_1 (x;2,3) \\ &= -\sum_{w<2} (c_{23} - c_{3w}) \partial_2 f_2 (x,x,2,3,3,w) \\ &- \sum_{w>3} (c_{w2} - c_{23}) \partial_1 f_2 (x,x;w,2,2,3). \end{aligned} \tag{5.43}$$

or

$$\partial_t f_1(x; 2, 3,) + c_{23}\partial_x f_1(x; 2, 3) = -(c_{23} - c_{31})\partial_2 f_2(x, x, 2, 3, 3, 1) \tag{5.44}$$

so shocks of this form can only be destroyed by an interaction from the right between $(2, 3)$ and $(3, 1)$.

The next case is $u = 3$, $v = 2$, for which we have

$$\begin{aligned}
&\partial_t f_1(x; 3, 2) + c_{32}\partial_x f_1(x; 3, 2) \\
&= \sum_{w \le 4}(c_{3w} - c_{w2})\partial_2 f_2(x, x; 3, w, w, 2) - \sum_{w<3}(c_{32} - c_{2w})\partial_2 f_2(x, x; 3, 2, 2, w) \\
&- \sum_{w \ge 2}(c_{w3} - c_{32})\partial_1 f_2(x, x; w, 3, 3, 2).
\end{aligned} \tag{5.45}$$

or

$$\begin{aligned}
\partial_t f_1(x; 3, 2) + c_{32}\partial_x f_1(x; 3, 2) &= (c_{31} - c_{12})\partial_2 f_2(x, x; 3, 1, 1, 2) \\
&- (c_{32} - c_{21})\partial_2 f_2(x, x; 3, 2, 2, 1)
\end{aligned} \tag{5.46}$$

so this shock can be created by a collision between $(3, 1)$ and $(1, 2)$ and destroyed by a collision from the right with $(2, 1)$.

The final case is $u = 2$, $v = 1$, and we have

$$\begin{aligned}
&\partial_t f_1(x; 2, 1) + c_{21}\partial_x f_1(x; 2, 1) \\
&= \sum_{w \le 3}(c_{2w} - c_{w1})\partial_2 f_2(x, x; 2, w, w, 1) - \sum_{w \le 2}(c_{21} - c_{1w})\partial_2 f_2(x, x; 2, 1, 1, w) \\
&- \sum_{w \ge 1}(c_{w2} - c_{21})\partial_1 f_2(x, x; w, 2, 2, 1).
\end{aligned} \tag{5.47}$$

or

$$\begin{aligned}
\partial_t f_1(x; 2, 1) + c_{21}\partial_x f_1(x; 2, 1) &= (c_{23} - c_{31})\partial_2 f_2(x, x; 2, 3, 3, 1) \\
&- (c_{32} - c_{21})\partial_1 f_2(x, x, ; 3, 2, 2, 1)
\end{aligned} \tag{5.48}$$

so this shock can be created by a collision between $(2, 3)$ and $(3, 1)$ or decay via a collision from the left by $(3, 2)$.

In the case of (5.35), the densities of f_1 and f_2 for a specific time t are expressed as measures concentrated at discrete points, and can be written formally as δ-functions. It is known that the solution profile is given as

$$u(x,t) = \begin{cases} u_3 & x < \min\{x_1 + c_{23}t, x_* + c_{13}(t - t_*)\} \\ u_2 & x_1 + c_{23}t < x < x_2 + c_{12}t \\ u_1 & x > \max\{x_2 + c_{12}t, x_* + c_{13}(t - t_*)\} \end{cases} \tag{5.49}$$

where

$$\begin{aligned} t_* &= \frac{x_2 - x_1}{c_{23} - c_{12}} \\ x_* &= x_1 + c_{23}\frac{x_2 - x_1}{c_{23} - c_{12}} = \frac{(c_{23} - c_{12})x_1 + c_{23}x_2 - c_{23}x_1}{c_{23} - c_{12}} \\ &= \frac{c_{23}x_2 - c_{12}x_1}{c_{23} - c_{12}} \end{aligned} \tag{5.50}$$

Specifically, for the one- and two-point functions, one then has

$$\begin{aligned} f_1(x,t;3,1) &= \delta(x - x_* - c_{13}(t - t_*))\, 1_{\{t > t_*\}} \\ f_1(x,t;3,2) &= \delta(x - x_1 - c_{23}t)\, 1_{\{t \le t_*\}} \\ f_1(x,t;2,1) &= \delta(x - x_2 - c_{12}t)\, 1_{\{t \le t_*\}} \end{aligned} \tag{5.51}$$

and

$$f_2(x,y,t;3,2,2,1) = \delta(x - x_1 - c_{23}t)\,\delta(y - x_2 - c_{12}t)\, 1_{\{t \le t_*\}} \tag{5.52}$$

Evaluating these functions for any other combination of values in this problem yields identically zero. Testing this against (5.38), (5.46), and (5.48) yields

$$\begin{aligned} \partial_t f_1(x,t;3,1) + c_{13}\partial_x f_1(x.t;3,1) &= (c_{23} - c_{21})\,\partial_2 f_2(x,x,t;3,2,2,1) \\ \partial_t f_1(x,t;3,2) + c_{23}\partial_x f_1(x.t;3,2) &= -(c_{23} - c_{21})\,\partial_2 f_2(x,x,t;3,2,2,1) \\ \partial_t f_1(x,t;2,1) + c_{21}\partial_x f_1(x,t;2,1) &= -(c_{23} - c_{21})\,\partial_1 f_2(x,x,t;3,2,2,1) \end{aligned} \tag{5.53}$$

These equations formally give the correct conclusions. For $t > t_*$, the first equation describes the steady-state of the shock $(3,1)$ persisting indefinitely and moving to the left at speed c_{13} :

$$\partial_t f_1(x;3,1) + c_{13}\partial_x f_1(x;3,1) = 0 \tag{5.54}$$

and for $t < t_*$ yields identically zero. The second and third equations yield identically zero for $t > t_*$ and the steady-state of the shocks $(3,2)$ and $(2,1)$ persisting for $t < t_*$:

$$\begin{aligned}
&\partial_t f_1(x,t;3,2) + c_{23}\partial_x f_1(x,t;3,2) = 0\\
&\partial_t f_1(x,t;2,1) + c_{12}\partial_x f_1(x,t;2,1) = 0.
\end{aligned} \tag{5.55}$$

At the point (x_*, t_*) where the two shocks $(3,2)$ and $(2,1)$ collide and form $(3,1)$,

5.4 Derivation of n-point Equation for Shocks

In the same way as for the section on the derivation for the one-point equation, we now want to generalize the procedure and derive the equations for the n-point equation. For the left-hand side of the equation, the analog of the time-derivative term should clearly be represented by

$$\partial_t f_n\left(x_1, ...x_n, t; u^1, ..., u^n, v^1, ..., v^n\right). \tag{5.56}$$

We use superscripts to distinguish the different states $\{u_i\}$ from the $2n$ arguments in this term giving the left and right hand values of the solution on either side of the shocks located at the positions $\{x_i\}$. That is, $u^i, v^i \in \{u_1, u_2, ...\}$ for each $1 \leq i \leq n$. For the free-streaming term, as a replacement for the single term dealing with the interaction between two states in the one-point term, we now need to account for every possible combination of pairs of states, leading to the term $c_{u_i v_i}\partial_{x_i} f_n\left(x_1, ..., x_n, t; u^1, ..., u^n, v^1, ..., v^n\right)$ summed over all combinations:

$$\sum_{i=1}^{n} c_{u_i v_i}\partial_{x_i} f_n\left(x_1, ..., x_n, t; u^1, ..., u^n, v^1, ..., v^n\right). \tag{5.57}$$

We now want to consider the terms that will form on the right-hand side of the equation. In the one-point equation, these consisted of birth or decay terms for the two-point function arising from interactions of those shocks, modified by the appropriate rate constants c_{uv}. Our hypothesis here is that we have similar terms with a possible interaction involving any of the n shocks, with the same appropriate rate constants. For the birth term, we stipulate

$$\sum_{i=1}^{n}\sum_{w}\left(c_{u^iw}-c_{wv^i}\right)\partial_{i'}f_{n+1}\left(\begin{array}{c} x_i, x_ix_1, ..., x_{i-1}, x_{i+1}, ..., x_n, t; \\ u_i, w, w, v_i, u^1, ..., u^{i-1}, u^{i+1}, ...u^n, v^1, ..., v^{i-1}, v^{i+1}, ..., v^n \end{array}\right) \tag{5.58}$$

We have rearranged the order of arguments to put the ith component and intermediate shock first. The notation $\partial_{i'}$ refers to the derivative with respect to the index that appears second under f in (5.3), and ∂_i to the one appearing first (the ith component). For brevity, we will use abbreviated notation henceforth and suppress the remainder of the arguments, i.e. rewrite (5.58) as

$$\sum_{i=1}^{n}\sum_{w}\left(c_{u^iw}-c_{wv^i}\right)\partial_i f_{n+1}\left(x_i, x_i, ..., t; u^i, w, w, v^i....\right) \tag{5.59}$$

The range for w is precisely the same as for the one-point equation case, that is for $w \leq u^i + 1$ as long as $v^i < u^i$. We can also have a decay term formed by collisions from the right, given by

$$-\sum_{i=1}^{n}\sum_{w}\left(c_{u^iv}-c_{v^iw}\right)\partial_{i'} f_{n+1}\left(x_i, x_i, ..., t; u^i, v^i, v^i, w, ...\right) \tag{5.60}$$

and for collisions from the left, given by

$$-\sum_{i=1}^{n}\sum_{w}\left(c_{wu^i}-c_{u^iv^i}\right)\partial_i f_{n+1}\left(x_i, x_i, ..., t; w, u^i, u^i, v^i, ...\right) \tag{5.61}$$

The appropriate ranges for the terms (5.60) and (5.61) are given by $w < u^i$ and $w > u^i + 1$ for $u^i < v^i$ (i.e. when $v^i = u^i + 1$) and $w \leq v^i + 1$ and $w \geq u^i - 1$ for $v^i < u^i$. For ease of notation, define the sets

$$\begin{aligned} W_2^i &= \left\{ w \mid \begin{array}{cc} w < u^i & \text{if } v^i = u^i + 1 \\ w \leq v^i + 1 & \text{if } v^i < u^i \end{array} \right. \\ W_3^i &= \left\{ w \mid \begin{array}{cc} w > u^i + 1 & \text{if } v^i = u^i + 1 \\ w \geq u^i - 1 & \text{if } v^i < u^i \end{array} \right. \end{aligned} \tag{5.62}$$

Then the n-point equation is given by

$$\partial_t f_n\left(x_1,...,x_n,t;u^1,...,u^n,v^1,...,v^n\right)+\sum_{i=1}^{n}c_{u_iv_i}\partial_{x_i}f_n\left(x_1,...,x_n,t;u^1,...,u^n,v^1,...,v^n\right)$$
$$=\sum_{i=1}^{n}1_{\{v_i=u_i+1\}}\sum_{w}\left(c_{u_iw}-c_{wv_i}\right)\partial_i f_{n+1}\left(x_i,x_i,...,t;u^i,w,w,v^i....\right)$$
$$-\sum_{i=1}^{n}\sum_{w\in W_2^i}\left(c_{u_iv}-c_{v_iw}\right)\partial_{i'}f_{n+1}\left(x_i,x_i,...,t;u^i,v^i,v^i,w,...\right)$$
$$-\sum_{i=1}^{n}\sum_{w\in W_3^i}\left(c_{wu_i}-c_{u_iv_i}\right)\partial_i f_{n+1}\left(x_i,x_i,...,t;w,u^i,u^i,v^i,...\right). \tag{5.63}$$

In this fully written out n-point hierarchy, one can see all the expected interactions and dynamics between the modelled particle system of shocks. On the left-hand side we have two terms: the first of which the change in time of the distribution of a particular n-point state, a collection of n shocks between different state values at prescribed points. The other term is a free-streaming term representing the motion of each of these n shocks with relative velocities $\{c_{u_iv_i}\}$ along the x direction as time advances. On the right-hand side, we have three terms relating to the creation or destruction of this object by various types of interactions. The summand of the first term involves a growth by a shock of states $\left(u^i,w\right)$ combining with $\left(w,v^i\right)$, forming the shock $\left(u^i,v^i\right)$, hence the positive term. The latter two summands involve destruction of the shock $\left(u^i,v^i\right)$ from the right by a shock $\left(v^i,w\right)$ or from the left with the shock $\left(w,u^i\right)$, respectively. The sums then cover interactions with any of the n shocks involved. In this derivation, it should also be noted we have also implicitly ruled out the possibility of three or more shocks intersecting at the same point, which is a reasonable assumption for kinetic theory.

5.5 Reduction of n-point equation to 1-point

One important check that can be performed via a straightforward calculation is the reduction of the n-point equation down to the one-point. Substituting in $n=1$ into equation (5.63) yields for the left-hand side

$$\partial_t f_1\left(x_1,t;u^1,v^1\right)+c_{u_1v_1}f_1\left(x_1,t;u^1,v^1\right) \tag{5.64}$$

and for the right-hand side

$$
\begin{aligned}
&1_{\{u^1=v^1\}} \sum_{w} \left(c_{u^1 w} - c_{wv^1}\right) \partial_{1'} f_2 \left(x_1, x_1, t; u^1, w, w, v^1\right) \\
&- \sum_{w \in W_2^1} \left(c_{u^1 v^1} - c_{wv^1}\right) \partial_{1'} f_2 \left(x_1, x_1, t; u^1, v^1, w\right) \\
&- \sum_{w \in W_3^1} \left(c_{wu_1} - c_{u_1 v_1}\right) \partial_1 f_2 \left(x_1, x_1, t; w, u^1, u^1, v^1\right)
\end{aligned} \tag{5.65}
$$

Dropping the subscripts and breaking into cases, we have for $u < v$ $(v = u + 1)$ that

$$
\begin{aligned}
&\partial_t f_1 (x, t; u, v) + c_{uv} f (x, t; u, v) \\
&= \sum_{w} (c_{uw} - c_{wv})\, \partial_2 f_2 (x, x, t; u, w, w, v) - \sum_{w<u} (c_{uv} - c_{wv})\, \partial_2 f_2 (x, x, t; u, v, w) \\
&- \sum_{w>u+1} (c_{wv} - c_{uv})\, \partial_1 f_2 (x, x, t; w, u, u, v)
\end{aligned} \tag{5.66}
$$

and for $u > v$, we have

$$
\begin{aligned}
&\partial_t f_1 (x, t; u, v) + c_{uv} f (x, t; u, v) \\
&= - \sum_{w \leq v+1} (c_{uv} - c_{wv})\, \partial_2 f_2 (x, x, t; u, v, w) \\
&- \sum_{w \geq u-1} (c_{wv} - c_{uv})\, \partial_1 f_2 (x, x, t; w, u, u, v)\,,
\end{aligned} \tag{5.67}
$$

matching the one-point equation.

5.6 Flavor of a more Complex Example (5 shocks)

One can consider more complex examples and in theory, perform more computations on them to further verify the n-point hierarchy. As we will see in the following section, however, such examples become increasingly more complicated to work out. We limit ourselves to writing out the probability densities and showing how the equations match up intuitively with the expected terms. To this end we consider initial condition of the form

$$g(x) = \begin{cases} u_5 & x \leq x_1 \\ u_4 & x_1 < x \leq x_2 \\ u_1 & x_2 < x \leq x_3 \\ u_2 & x_3 < x \leq x_4 \\ u_3 & x_4 < x \end{cases} \tag{5.68}$$

Initially, there are four shocks with speeds c_{45}, c_{14}, c_{12}, and c_{23} (in order from left to right). The first interaction expected is c_{15} with c_{14}, forming a shock of speed c_{15}. This shock then collides subsequently with c_{12}, then c_{23}, to form a shock c_{35} which then persists indefinitely. Denoting the collision points by (x_i^*, t_i^*), the expected solution profile is given by the following:

$$u(x,t) = \begin{cases} u_5 & x \leq x_1 + c_{15}t \\ u_4 & x_1 + c_{15}t < x \leq x_2 + c_{14}t \\ u_1 & x_2 + c_{14}t < x \leq x_3 + c_{12}t \\ u_2 & x_3 + c_{12}t < x \leq x_4 + c_{23}t \\ u_3 & x_4 + c_{13}t < x \end{cases} \quad \text{for } t \leq t_1 \tag{5.69}$$

$$u(x,t) = \begin{cases} u_5 & x \leq x_1^* + (t - t_1^*)\, c_{15} \\ u_1 & x_1^* + (t - t_1^*)\, c_{15} < x \leq x_3 + c_{12}t \\ u_2 & x_3 + c_{12}t < x \leq x_4 + c_{23}t \\ u_3 & x_4 + c_{13}t < x \end{cases} \quad \text{for } t_1^* \leq t \leq t_2^* \tag{5.70}$$

$$u(x,t) = \begin{cases} u_5 & x \leq x_2^* + (t - t_2^*)\, c_{25} \\ u_2 & x_2^* + (t - t_2^*)\, c_{25} < x \leq x_4 + c_{23}t \\ u_3 & x_4 + c_{13}t < x \end{cases} \quad \text{for } t_2^* \leq t \leq t_3^* \tag{5.71}$$

$$u(x,t) = \begin{cases} u_5 & x \leq x_3^* + (t - t_3^*)\, c_{35} \\ u_3 & x_3^* + (t - t_3^*)\, c_{35} < x \end{cases} \quad \text{for } t_3^* \leq t \tag{5.72}$$

Given the solution profile (5.69)-(5.72), we can now write out the n-point probability distributions. Starting with the one-point distribution, we have

$$\begin{aligned}
f_1(x,t;5,4) &= \delta(x - x_1 - c_{15}t)\, 1_{\{t \le t_1^*\}} \\
f_1(x,t;4,1) &= \delta(x - x_2 - c_{14}t)\, 1_{\{t \le t_1^*\}} \\
f_1(x,t;1,2) &= \delta(x - x_3 - c_{12}t)\, 1_{\{t \le t_2^*\}} \\
f_1(x,t;2,3) &= \delta(x - x_4 - c_{23}t)\, 1_{\{t \le t_3^*\}} \\
f_1(x,t;5,1) &= \delta(x - x_1^* - (t - t_1^*)\, c_{15})\, 1_{\{t_1^* \le t \le t_2^*\}} \\
f_1(x,t;5,2) &= \delta(x - x_2^* - (t - t_2^*)\, c_{25})\, 1_{\{t_2^* \le t_3^*\}} \\
f_1(x,t;5,3) &= \delta(x - x_3^* - (t - t_3^*)\, c_{35})\, 1_{\{t_3^* \le t\}}
\end{aligned} \tag{5.73}$$

and zero for any other arguments. For the two-point distributions, we compute

$$\begin{aligned}
f_2(x,y,t;5,4,4,1) &= \delta(x - x_1 - c_{15}t)\, \delta(y - x_2 - c_{14}t)\, 1_{\{t \le t_1^*\}} \\
f_2(x,y,t;5,4,1,2) &= \delta(x - x_1 - c_{15}t)\, \delta(y - x_3 - c_{12}t)\, 1_{\{t \le t_2^*\}} \\
f_2(x,y,t;5,4,2,3) &= \delta(x - x_1 - c_{15}t)\, \delta(y - x_4 - c_{23}t)\, 1_{\{t \le t_2^*\}} \\
f_2(x,y,t;4,1,1,2) &= \delta(x - x_2 - c_{14}t)\, \delta(y - x_3 - c_{12}t)\, 1_{\{t \le t_2^*\}} \\
f_2(x,y,t;4,1,2,3) &= \delta(x - x_2 - c_{14}t)\, \delta(y - x_4 - c_{23}t)\, 1_{\{t \le t_2^*\}} \\
f_2(x,y,t;1,2,2,3) &= \delta(x - x_3 - c_{12}t)\, \delta(y - x_4 - c_{23}t)\, 1_{\langle t \le t_2^* \rangle} \\
f_2(x,y,t;5,1,1,2) &= \delta(x - x_1^* - (t - t_1^*)\, c_{15})\, \delta(y - x_3 - c_{12}t)\, 1_{\{t_1^* \le t \le t_2^*\}} \\
f_2(x,y,t;5,1,2,3) &= \delta(x - x_1^* - (t - t_1^*)\, c_{15})\, \delta(y - x_4 - c_{23}t)\, 1_{\{t_1^* \le t \le t_2^*\}} \\
f_2(x,y,t;5,2,2,3) &= \delta(x - x_2^* - (t - t_2^*)\, c_{25})\, \delta(y - x_4 - c_{23}t)\, 1_{\{t_2^* \le t \le t_3^*\}}
\end{aligned} \tag{5.74}$$

Aside from simple permutation of the arguments (i.e. $f_2(y,x,t;4,1,5,4) = f_2(x,y,t;5,4,4,1)$), the two-point distributions will be zero for other combinations of u_i in their arguments. Next, for the three-point distributions, we have

$$
\begin{aligned}
&f_3\left(x,y,z,t;5,4,4,1,1,2\right)=\delta\left(x-x_1-c_{15}t\right)\delta\left(y-x_2-c_{14}t\right)\delta\left(z-x_3-c_{12}t\right)1_{\left\{t\leq t_1^*\right\}}\\
&f_3\left(x,y,z,t;5,4,4,1,2,3\right)=\delta\left(x-x_1-c_{15}t\right)\delta\left(y-x_2-c_{14}t\right)\delta\left(z-x_4-c_{23}t\right)1_{\left\{t\leq t_1^*\right\}}\\
&f_3\left(x,y,z,t;5,4,1,2,2,3\right)=\delta\left(x-x_1-c_{15}t\right)\delta\left(y-x_3-c_{12}t\right)\delta\left(z-x_4-c_{23}t\right)1_{\left\{t\leq t_1^*\right\}}\\
&f_3\left(x,y,z,t;4,1,1,2,2,3\right)=\delta\left(x-x_2-c_{14}t\right)\delta\left(y-x_3-c_{12}t\right)\delta\left(z-x_4-c_{23}t\right)1_{\left\{t\leq t_1^*\right\}}\\
&f_3\left(x,y,z,t;5,1,1,2,2,3\right)=\delta\left(x-x_1^*-\left(t-t_1^*\right)c_{15}\right)\delta\left(y-x_3-c_{12}t\right)\delta\left(z-x_4-c_{23}t\right)1_{\left\{t_1^*\leq t\leq t_2^*\right\}}
\end{aligned}
\tag{5.75}
$$

and zero otherwise, except for permutations of the arguments. Finally, for the four-point distribution, we have simply

$$
\begin{aligned}
f_4\left(x,y,z,w,t;5,4,4,1,1,2,2,3\right)&=\delta\left(x-x_1-c_{15}t\right)\delta\left(y-x_2-c_{14}t\right)\\
&\quad\cdot\delta\left(z-x_3-c_{12}t\right)\delta\left(w-x_4-c_{23}t\right)1_{\left\{t\leq t_1^*\right\}}
\end{aligned}
\tag{5.76}
$$

Now we want to substitute the expressions (5.73)-(5.76) into the n-point equations. Recall the one-point equation is expressed by

$$
\begin{aligned}
&\partial_t f_1\left(x,t;u,u+1\right)+c_{u,u+1}\partial_x f_1\left(x,t;u,u+1\right)\\
&=-\sum_{w<u}\left(c_{u,u+1}-c_{u+1,w}\right)\partial_2 f_2\left(x,x,t;u,u+1,u+1,w\right)\\
&-\sum_{w>u+1}\left(c_{w,u}-c_{u,u+1}\right)\partial_1 f_2\left(x,x,t;w,u,u,u+1\right)
\end{aligned}
\tag{7.10}
$$

for $v=u+1$ and

$$
\begin{aligned}
&\partial_t f_1\left(x,t;u,v\right)+c_{uv}\partial_x f_1\left(x,t;u,v\right)\\
&=\sum_{w\leq u+1}\left(c_{uw}-c_{wv}\right)\partial_2 f_2\left(x,x,t;u,w,w,v\right)-\sum_{w\leq v+1}\left(c_{uv}-c_{vw}\right)\partial_2 f_2\left(x,x,t;u,v,v,w\right)\\
&-\sum_{w\geq u-1}\left(c_{wu}-c_{uv}\right)\partial_1 f_2\left(x,x,t;w,u,u,v\right).
\end{aligned}
\tag{5.77}
$$

for $v < u$. We start with equation (5.77) for $u = 1, 2, ...$, and have

$$\begin{aligned}
&\partial_t f_1(x,t;1,2) + c_{12}\partial_x f_1(x,t,1,2) \\
&= -\sum_{w<1}(c_{12} - c_{2w})\,\partial_2 f_2(x,x,t;1,2,2,w) + \sum_{w>2}(c_{w1} - c_{12})\,\partial_1 f_2(x,x,t;w,1,1,2) \\
&- (c_{31} - c_{12})\,f_2(x,x,t;3,1,1,2) - (c_{41} - c_{12})\,f_2(x,x,t;4,1,1,2) \\
&- (c_{51} - c_{12})\,f_2(x,x,t;5,1,1,2)
\end{aligned} \tag{5.78}$$

$$\begin{aligned}
&\partial_t f_1(x,t;2,3) + c_{23}\partial_x f_1(x,t,1,2) \\
&= -\sum_{w<2}(c_{23} - c_{3w})\,\partial_2 f_2(x,x,t;2,3,3,w) - \sum_{w>3}(c_{w2} - c_{23})\,\partial_1 f_2(x,x,t;w,2,2,3) \\
&= -(c_{23} - c_{31})\,\partial_2 f_2(x,x,t;2,3,3,1) - (c_{42} - c_{23})\,\partial_1 f_2(x,x,t;4,2,2,3) \\
&- (c_{52} - c_{23})\,\partial_1 f(x,x,t;5,2,2,3)
\end{aligned} \tag{5.79}$$

$$\begin{aligned}
&\partial_t f_1(x,t;3,4) + c_{34}\partial_x f_1(x,t,3,4) \\
&= -\sum_{w<3}(c_{34} - c_{4w})\,\partial_2 f_2(x,x,t;3,4,4,w) - \sum_{w>4}(c_{w3} - c_{34})\,\partial_2 f_2(x,x,t;3,4,4,w) \\
&- (c_{34} - c_{41})\,\partial_2(x,x,t;3,4,4,2) - (c_{34} - c_{42})\,\partial_2 f_2(x,x,t;3,4,4,2) \\
&- (c_{53} - c_{34})\,\partial_2 f_2(x,x,t;3,4,4,5)
\end{aligned} \tag{5.80}$$

$$\begin{aligned}
&\partial_t f_1(x,t;4,5) + c_{12}\partial_x f_1(x,t,4,5) \\
&= -\sum_{w<4}(c_{45} - c_{5w})\,\partial_2 f_2(x,x,t,;4,5,5,w) - \sum_{w>5}(c_{w4} - c_{45})\,\partial_1 f_2(x,x,t;w,4,4,5) \\
&= -(c_{45} - c_{51})\,\partial_2 f_2(x,x,t;4,5,5,1) - (c_{45} - c_{52})\,\partial_s f_2(x,x,t;4,5,5,2) \\
&- (c_{45} - c_{53})\,\partial_2 f_2(x,x,t;4,5,5,3)
\end{aligned} \tag{5.81}$$

Now we consider equation (5.77), the case $v < u$, first considering $v = 1$. This yields

$$\begin{aligned}
&\partial_t f_1(x,t;u,1) + c_{u1}\partial_x f_1(x,t;u,1) \\
&= \sum_{w\le u+1}(c_{uw}-c_{w1})\,\partial_2 f_2(x,x,t;u,w,w,1) - \sum_{w\le 2}(c_{u1}-c_{1w})\,\partial_2 f_2(x,x,t;u,1,1,w) \\
&- \sum_{w\ge u-1}(c_{wu}-c_{u1})\,\partial_1 f_2(x,x,t;w,u,u,1)\,. \qquad (5.82)
\end{aligned}$$

and substituting in for $u = 2, 3, ...$ will yield

$$\begin{aligned}
&\partial_t f_1(x,t;2,1) + c_{21}\partial_x f_1(x,t;2,1,) \\
&= \sum_{w\le 3}(c_{2w}-c_{w1})\,\partial_2 f_2(x,x,t;2,w,w,1) - \sum_{w\le 2}(c_{21}-c_{1w})\,\partial_2 f_2(x,x,t;2,1,1,w) \\
&- \sum_{w\ge 1}(c_{w2}-c_{21})\,\partial_1 f_2(x,x,t;w,2,2,1) \\
&= (c_{23}-c_{31})\,\partial_2 f_2(x,x,t;2,3,3,1) - (c_{32}-c_{21})\,\partial_1 f_2(x,x,t;3,2,2,1) \\
&- (c_{42}-c_{21})\,\partial_1 f_2(x,x,t;4,2,2,1) - (c_{52}-c_{21})\,\partial_1 f_2(x,x,t;5,2,2,1) \qquad (5.83)
\end{aligned}$$

$$\begin{aligned}
&\partial_t f_1(x,t;3,1) + c_{31}\partial_x f_1(x,t;3,1) \\
&= \sum_{w\le 4}(c_{4w}-c_{w1})\,\partial_2 f_2(x,x,t;3,w,w,1) - \sum_{w\le 2}(c_{31}-c_{1w})\,\partial_2 f_2(x,x,t;3,1,1,w) \\
&- \sum_{w\ge 2}(c_{w3}-c_{31})\,\partial_1 f_2(x,x,t;w,3,3,1) \\
&= (c_{42}-c_{21})\,\partial_2 f_2(x,x,t;3,2,2,1) + (c_{43}-c_{41})\,\partial_2 f_2(x,x,t;3,4,4,1) \\
&- (c_{31}-c_{12})\,\partial_2 f_2(x,x,t;3,1,1,2) - (c_{23}-c_{31})\,\partial_1 f_2(x,x,t;2,3,3,1) \\
&- (c_{43}-c_{31})\,\partial_1 f_2(x,x,t;4,3,3,1) - (c_{53}-c_{31})\,\partial_1 f_2(x,x,t;5,3,3,1) \qquad (5.84)
\end{aligned}$$

$$
\begin{aligned}
&\partial_t f_1 (x,t;4,1) + c_{41}\partial_x f_1 (x,t;4,1) \\
&= \sum_{w\le 5} (c_{4w} - c_{w1})\,\partial_2 f_2 (x,x,t;4,w,w,1) - \sum_{w\le 2} (c_{41} - c_{1w})\,\partial_2 f_2 (x,x,t;4,1,1,w) \\
&- \sum_{w\ge 3} (c_{w4} - c_{41})\,\partial_1 f_2 (x,x,t;w,4,4,1) \\
&= (c_{42} - c_{21})\,\partial_2 f_2 (x,x,t;4,2,2,1) + (c_{43} - c_{31})\,\partial_2 f_2 (x,x,t;4,3,3,1) \\
&+ (c_{45} - c_{51})\,\partial_2 f_2 (x,x,t;4,5,5,1) - (c_{41} - c_{12})\,\partial_2 f_2 (x,x,t;4,1,1,2) \\
&- (c_{34} - c_{41})\,\partial_1 f_2 (x,x,t;3,4,4,1) - (c_{54} - c_{41})\,\partial_1 f_2 (x,x,t;5,4,4,1)
\end{aligned}
\tag{5.85}
$$

$$
\begin{aligned}
&\partial_t f_1 (x,t,5,1) + c_{51}\partial_x (x,t;5,1) \\
&= -\sum_{w\le 6} (c_{5w} - c_{w1})\,\partial_2 f_2 (x,x,t;5,w,w,1) - \sum_{w\le 2} (c_{51} - c_{1w})\,\partial_2 f_2 (x,x,t;5,1,1,w) \\
&- \sum_{w\ge 4} (c_{w5} - c_{51})\,\partial_1 f_2 (x,x,t;w,5,5,1) \\
&= -(c_{54} - c_{41})\,\partial_2 f_2 (x,x,t;5,4,4,1) - (c_{53} - c_{31})\,\partial_2 f_2 (x,x,t;5,3,3,1) \\
&- (c_{52} - c_{21})\,\partial_2 f_2 (x,x,t;5,2,2,1) - (c_{51} - c_{12})\,\partial_2 f_2 (x,x,t;5,1,1,2) \\
&- (c_{45} - c_{51})\,\partial_1 f_2 (x,x,t;4,5,5,1)
\end{aligned}
\tag{5.86}
$$

These calculations help to give a flavor of future direction and serve to illustrate the various interactions by which perturbations at the one-point level can occur in a more complex system. Based on whether the velocity of one shock is sufficient to catch up to that of another, shocks of certain species can be created or destroyed, with rate constants specified by the relative velocities between the two. There is a free-streaming term on the right-hand side that tracks the movement of any existing shocks in that species. For future study, this lays the groundwork for either numerical computation or further analysis under special cases, such as more stringent initial conditions.

Chapter 6

Two interesting limits

In the previous section, we derived a set of exact equations for the n-point equations. Of particular interest is the application of this framework to two specific limits. The first such limit is the simplest nontrivial case, where we have three possible velocity states and hence, three different shock speeds in the system. The second limit of interest is obtained as the number of states tends to infinity, creating a finer approximation by piecewise linear functions to a smooth flux function, and passing to continuum limits in all the expressions, using theory of semigroups and generators. One would expect that in this latter limit and under appropriate assumptions, the discrete model in fact recovers the correct continuum dynamics of Menon and Srinivisan [31], etc.

6.1 Three-state limit

First consider the limit of the three velocity state system. Specifically, this means that the indices $\left\{u^i\right\}, \left\{v^i\right\}$ all take values from the set $\{u_1, u_2, u_3\}$. We may assume without loss of generality that

$$u_1 < u_2 < u_3. \tag{6.1}$$

The three possible species of shock are between u_1 and u_2, between u_2 and u_3, and finally between u_1 and u_3. We allow upward jumps only between nearest neighbors, so a shock between u_1 and u_3 is allowed only with u_3 as the left state and u_1 as the right state, and the other two are allowed with either orientation. The shocks then have speeds given by

$$\begin{aligned} c_{12} &= \frac{f(u_2) - f(u_1)}{u_2 - u_1} \\ c_{23} &= \frac{f(u_3) - f(u_2)}{u_3 - u_2} \\ c_{13} &= \frac{f(u_3) - f(u_1)}{u_3 - u_1} \end{aligned} \tag{6.2}$$

and by the usual convexity assumption on f, we have

$$c_{12} < c_{13} < c_{23}. \tag{6.3}$$

We start with the sets W_2^i, W_3^i defined in (5.62). Because of the restrictions on $\{u^i\}, \{v^i\}$, we may rewrite them based on cases. We have

$$\begin{aligned} &u^i = u_1, v^i = u_2 \Rightarrow W_2^i = \{w \mid w < u_1\} = \phi,\ W_3^i = \{w \mid w > u_2\} = \{u_3\} \\ &u^i = u_2, v^i = u_1 \Rightarrow W_2^i = \{w \mid w \leq u_2\} = \{u_1, u_2\},\ W_3^i = \{w \mid w \geq u_0\} = \{u_1, u_2, u_3\} \\ &u^i = u_2, v^i = u_3 \Rightarrow W_2^i = \{w \mid w < u_2\} = \{u_1\},\ W_3^i = \{w \mid w > u_3\} = \phi \\ &u^i = u_3, v^i = u_1 \Rightarrow W_2^i = \{w \mid w \leq u_2\} = \{u_1, u_2\},\ W_3^i = \{w \mid w \geq u_2\} = \{u_2, u_3\} \\ &u^i = u_3, v^i = u_2 \Rightarrow W_2^i = \{w \mid w \leq u_3\} = \{u_1, u_2, u_3\},\ W_3^i = \{w \mid w \geq u_2\} = \{u_2, u_3\} \end{aligned} \tag{6.4}$$

We can write out the exact form for the one-point equation, in the case of three states, as follows:

$$\begin{aligned} &\partial_t p_1\left(x_1, t; u^1, v^1\right) + c_{u^1 v^1} \partial_{x_1} f_1\left(x_1, t; u^1, v^1\right) \\ &= 1_{\langle v^1 - u^1 + 1\rangle} \sum_{w \leq u^1 + 1} \left(c_{u^1 w} - c_{w v^1}\right) \partial_2 f_2\left(x_1, x_1, t; u^1, w, w, v^1\right) \\ &- \sum_{w \in W_2^1} \left(c_{u^1 v^1} - c_{v^1 w}\right) \partial_2 f_2\left(x_1, x_1, t; u^1, v^1, v^1, w\right) \\ &- \sum_{w \in W_3^1} \left(c_{w u^1} - c_{u^1 v^1}\right) \partial_1 f_2\left(x_1, x_1, t; w, u^1, u^1, v^1\right) \end{aligned} \tag{6.5}$$

and the two-point equation as

$$\begin{aligned}
&\partial_t p_2\left(x_1, x_2, t; u^1, u^2, v^1, v_2\right) + \sum_{i=1}^{2} c_{u^i v^i} \partial_{x_i} f_2\left(x_1, x_2, t; u^1, u^2, v^1, v^2\right) \\
&= \sum_{i=1}^{2} 1_{\{v^i - u^i + 1\}} \sum_{w \leq u^i+1} \left(c_{u^i w} - c_{w v^i}\right) \partial_{i'} f_3\left(x_i, x_i, ..., t; u^i, w, w, v^i, ...\right) \\
&- \sum_{i=1}^{2} \sum_{w \in W_2^i} \left(c_{u^i v^i} - c_{v^i w}\right) \partial_{i'} f_3\left(x_i, x_i, ..., t; u^i, v^i, v^i, w, ...\right) \\
&- \sum_{i=1}^{2} \sum_{w \in W_3^i} \left(c_{w u^i} - c_{u^i v^i}\right) \partial_i f_3\left(x_i, x_i, ..., t; w, u^i, u^i, v^i, ...\right).
\end{aligned} \tag{6.6}$$

As one can see, in the simplest case the equations do simplify somewhat, as the range of indices in and number of W_j^i are reduced. However, in terms of notation, the two-point equation cannot be simplified much further, as there are certain combinations of possibilities allowed at the values of the given x-coordinates. For example, in the one-point equation with u^1 and v^1 taking on the values u_1 and u_2, respectively, one would have

$$\begin{aligned}
&\partial_t p_1\left(x_1, t; u_1, u_2\right) + c_{u_1 v_1} \partial_{x_1} f_1\left(x_1, t; u_1, v_1\right) \\
&= \sum_{w \leq 2} \left(c_{1w} - c_{w2}\right) \partial_2 f_2\left(x_1, x_1, t; u_1, w, w, u_2\right) \\
&- 0 - \sum_{w \in \{u_3\}} \left(c_{w1} - c_{12}\right) \partial_1 f_2\left(x_1, x_1, t; w, 1, 1, 2\right) \\
&= -\left(c_{31} - c_{12}\right) \partial_1 f_2\left(x_1, x_1, t; 3, 1, 1, 2\right).
\end{aligned} \tag{6.7}$$

In particular (6.7) indicates that, within the three-state system, the a shock between $(1, 2)$ can decay via an interaction with a shock of the species $(3, 1)$ from the left. It cannot decay within this limited state system from an interaction from the right or form from a collision of two other shocks.

Part II

Part II - Multistable Dynamics in the Allen-Cahn Equation with White Noise

Chapter 7

Introduction

The Allen-Cahn equation

$$u_t = \Delta u + u - u^3$$

can also be studied without the presence of the diffusion term to see how solutions evolve toward the equilibrium points. In the theory of ordinary differential equations, the concept of stable equilibrium points plays a key role. The initial condition determines uniquely the evolution to the particular stable equilibrium point. An important question involves the introduction of white noise. In fact, as we will see, even a very small amount of noise will result in a small perturbation that influences which particular stable equilibrium point to which the trajectory evolves. As a practical example, multistable dynamics are well-known in the description of patterns in physics, chemistry, and biology; see for example a recent experimental work of Chang, Oh, Ingber, and Huang [10] analyzing a multistep dynamics of mammalian cell differentiation with multistability. Mathematically a prototype bistable dynamics is

$$\frac{du}{dt} = 4(u-\alpha)(1-u^2) \quad \text{or} \quad du = 4(u-\alpha)(1-u^2)dt \tag{7.1}$$

where $\alpha \in [0,1)$ is a constant and $u := 1$ and $u := -1$ are two stable equilibria. In the context of phase transition, the stability of an equilibrium is described by the smallness of the potential $P(u) = 4\int (u-\alpha)(u^2-1)du$. In particular, when $\alpha \in (0,1)$, the equilibrium $u \equiv -1$ is regarded as more stable than $u \equiv 1$ since $P(-1) < P(1)$.

It is commonplace that fluctuations gives rise to many interesting phenomena; see for example, molecular collisions in gasses and liquids [19] and electronic fluctuations in solids [25]. It has been long observed that external noise can induce a phase transition; see for example, Bulsara, Schieve, and Gragg [7], Porrá, Masoliver and Lindenberg [32, 33], Schimansky-Geier, Hesse, and Zülick [36, 37], Xiao, Yan, and Zhang [43], Zhang, Cao, and Wu [44], de Rueda, Izús and Borzi [39], Weidlich and Grabert [42], and references therein. It is well-known that white noise best models the fluctuations generated by microscopic effects in a homogeneous physical system. In the case of Allen–Cahn [1] bistable dynamics describing phase transition in binary alloys, interesting phenomena induced by white noises have been studied by Brassesco, De Masi, and Presutti [8], Erbar [13], Funika [15], Da Prato and J. Zabczyk [12], Fatkullin and Vanden-Eijnden [17], Katsoulakis, Kossioris, and Lakkis [24], Liu [28], Reznikoff & G. Vanden-Eijnden [34], Weber [41], etc.

In this part we investigate the effect of white noise on multistable dynamics. Replacing the bistable forcing $4(u-\alpha)(1-u^2)$ by a generic multistable forcing (or drift) $b(u)$, we consider the dynamics with white noise modeled by the stochastic differential equation (SDE)

$$du_t = b(u_t)dt + \varepsilon\,\sigma(u_t)dW_t \tag{7.2}$$

where $\{W_t\}_{t\geqslant 0}$ is the Wiener process (Brownian motion), ε is a positive constant measuring the size of fluctuation, and $\sigma(\cdot)$ is a non-negative function modeling the dependence of fluctuation on system state. By a standard theory of std (e.g. [38]), under certain technical conditions on σ and b, (7.2) subject to an initial condition admits a unique solution and the probability density $\rho = \rho(u,t)$ of the random variable u_t is the solution of the following Fokker-Plank or forward Kolmogorov equation

$$\begin{cases} \partial_t\rho = \partial_u\left\{\partial_u\left[\frac{1}{2}\,\varepsilon^2\sigma^2\rho\right] - b\rho\right\} & \text{on } \mathbb{R}\times(0,\infty), \\ \rho(\cdot,0) = \rho_0(\cdot) & \text{on } \mathbb{R}\times\{0\} \end{cases} \tag{7.3}$$

where ∂_t and ∂_u are partial derivatives and ρ_0 is the probability density function (pdf) of u_0. We expect that the system eventually achieve a stochastic equilibrium in the sense that

$$\lim_{t\to\infty}\rho(\cdot,t) = \rho^*(\cdot). \tag{7.4}$$

When ρ^* is non-trivial, it must be a pdf and is well-known as the **invariant measure** which describes the observed distribution density after a certain amount of initiation time that is used to get rid of the initial effect. If a non-trivial ρ^* exists then it must be a solution of the following linear second order ordinary differential equation (ode) subject to the pdf conditions:

$$\begin{cases} \left\{\left[\frac{1}{2}\,\varepsilon^2\sigma^2\rho^*\right]' - b\rho^*\right\}' = 0 \quad \text{on } \mathbb{R}, \\ \rho^* \geqslant 0 \text{ on } \mathbb{R}, \quad \int_{\mathbb{R}} \rho^*(u)du = 1. \end{cases} \tag{7.5}$$

Suppose there exists a unique solution ρ^* and that (7.4) holds. Then we conclude that fluctuation eventually drives a multistable system into an invariant state that does not depend on any initial setup. This is totally different from the ode dynamics (7.1) whose long time behavior of the solution crucially depends on the initial condition. This is indeed one of the main reason that multistable systems are studied with fluctuations.

In this part, we establish the existence of a unique invariant measure and analyze its asymptotic limit as the white noise vanishes (i.e. as $\varepsilon \searrow 0$). The limit is shown to be Dirac measures. When σ is a constant, the Dirac measure is concentrated on the most stable equilibria of the potential

$$P(u) := -\int b(u)du.$$

When σ is biased, the measure may support on those lesser stable equilibria but with smaller σ. The overall selection is the most stable equilibria of the effective potential Q defined by

$$Q(u) = -\int \frac{b(u)}{\sigma^2(u)}du.$$

Observe that Q shares the same set of equilibria and set of stable equilibria as P, but with different stability. The smaller the σ, the more stable the stable equilibrium of Q. We shall also address a few fundamental mathematical issues related to the degenerate case when σ is not everywhere positive and to the well-posedness of (7.3).

The rest of Part II is organized as follows. In the Chapter 8 we establish the existence of a unique solution of (7.5) under the following regularity, non-degeneracy, and stability conditions:

$$\sigma, b \in C(\mathbb{R}), \varepsilon \in \mathbb{R}; \quad \varepsilon > 0,\ \sigma(\cdot) > 0; \quad \overline{\lim_{u\to\infty}}\ b(u) < 0 < \underline{\lim_{u\to-\infty}}\ b(u). \tag{7.6}$$

In Chapter 9 we convert (7.3) to its classically equivalent version for the cumulative distribution function (cdf) to establish its well-posedness under the conditions only listed in (7.6). In Chapter 10 we show that every solution of (7.3) with ρ_0 being a pdf approaches ρ^* as $t \to \infty$. In Chapter 11 we study the asymptotic limit of the invariant measure as the size ε of the white noise approaches zero.

Finally, in Chapter 12 we consider the case when σ is not everywhere positive. We regard the invariant measure as the limit of invariant measures with σ replaced by an approximation sequence of positive functions. We show that if σ vanishes at a set $\mathcal{S}$ of (non-degenerate) stable equilibria, then for any probability measure μ supported on $\mathcal{S}$, there exists a sequence of positive functions approximating σ uniformly such that the corresponding invariant measures approach μ. On the other hand, if σ vanishes only outside of all stable equilibria, the limit, although highly non-trivial, does not depend the approximation in general, so an invariant measure can be uniquely defined.

Chapter 8

The Invariant Measure

In this section we prove the following:

Theorem 8.1. *Assume that the regularity, non-degeneracy, and stability conditions listed in (7.6) hold. Then (7.5) admits a unique solution and the solution is given by*

$$\rho^*(u) = \frac{1}{Z}\,\frac{e^{q(u)}}{\sigma^2(u)}, \qquad q(u) := \int_0^u \frac{2b(x)}{\varepsilon^2\sigma^2(x)}dx, \qquad Z := \int_{\mathbb{R}} \frac{e^{q(x)}}{\sigma^2(x)}dx. \tag{8.1}$$

Proof. Existence. We show that ρ^* defined in (8.1) is a solution. By the positivity assumption $\varepsilon, \sigma > 0$ and the regularity assumption $\sigma, b \in C(\mathbb{R})$, the function q is well-defined. To see that ρ^* is a pdf, we need only show that Z is finite. For this we use the stability assumption $\overline{\lim}_{u\to\infty} b(u) < 0$ and $\underline{\lim}_{u\to-\infty} b(u) > 0$. There exist positive constants η and M such that $b(u) \leqslant -\eta$ for every $u \geqslant M$ and $b(u) \geqslant \eta$ for every $u \leqslant -M$. It then follows that

$$\int_M^\infty \frac{e^{q(x)}}{\sigma^2(x)}dx \leqslant \int_M^\infty \frac{-2b(x)}{2\eta}\,\frac{e^{q(x)}}{\sigma^2(x)}dx = \int_M^\infty \frac{\varepsilon^2 q'(x)e^{q(x)}}{-2\eta}dx \leqslant \frac{\varepsilon^2 e^{q(M)}}{2\eta}. \tag{8.2}$$

Similarly, $\int_{-\infty}^{-M} \sigma^{-2}(x)e^{q(x)}dx$ is also finite. Thus, ρ^* defined in (8.1) is a pdf. By differentiation, one can verify that ρ^* satisfies (7.5) and therefore is a solution of (7.5).

Uniqueness. Let ρ^* be a generic solution of (7.5). The ode in (7.5) can be solved by an integration followed by another integration with integrating factor e^{-q}, giving the general solution

$$\rho^*(u) = \frac{e^{q(u)}}{\sigma^2(u)}\left(C_1\int_0^u e^{-q(x)}\,dx + C_2\right) \tag{8.3}$$

with integration constants C_1 and C_2. For such ρ^* to be non-negative, C_1 must be 0. Indeed,

$$\lim_{u \to \pm\infty} \int_0^u e^{-q(x)} dx = \int_0^{\pm M} e^{-q(x)} dx + \lim_{u \to \pm\infty} \int_{\pm M}^u e^{-q(x)} dx = \pm\infty \tag{8.4}$$

since $q(x) \leqslant q(M)$ for $x \geqslant M$ (as $b < 0$ in $[M, \infty)$) and $q(x) \leqslant q(-M)$ for $x \leqslant -M$. Thus, $C_1 = 0$. The condition $\int_{\mathbb{R}} \rho^*(x) dx = 1$ then gives $C_2 = 1/Z$. This completes the proof.

Chapter 9

Well-posedness of Kolmogorov Equation

Under the conditions listed in (7.6), the problem (7.3) may not admit a classical solution, i.e., a solution such that all derivatives appeared in (7.3) exist in the classical sense and the differential equation in (7.3) is satisfied. Hence, a certain notion of weak solutions is needed. From the probabilistic point of view and also to treat measure type initial values, it is natural to convert (7.3) to the cumulative distribution functions (cdf) defined by

$$f(u,t) := \int_{-\infty}^{u} \rho(x,t)dx, \quad f_0(u) := \mathbb{P}(\{u_0 < u\}), \quad f^*(u) := \int_{-\infty}^{u} \rho^*(x)dx. \tag{9.1}$$

Here $\mathbb{P}$ stands for the probability. Setting $a = \varepsilon^2\sigma^2/2$, one can derive formally that f satisfies

$$\begin{cases} \mathcal{L}f := \partial_t f - \partial_u(a\partial_u f) + b\,\partial_u f = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \lim_{t\searrow 0} f(u,t) = f_0(u) & \text{a.e. } u\in\mathbb{R}, \\ \lim_{u\to-\infty} f(u,t) = 0, \ \lim_{u\to\infty} f(u,t) = 1 & \text{locally uniformly in } t\in[0,\infty). \end{cases} \tag{9.2}$$

We define a solution of (9.2) as follows: f is called a classical solution of (9.2) if

$$f,\ \partial_u f,\ \partial_t f,\ \partial_u(a\partial_u f) \in C(\mathbb{R}\times(0,\infty)), \qquad \partial_u f \geqslant 0 \tag{9.3}$$

and each equation in (9.2) is satisfied. Also, when f_0 is continuous, we require $f \in C(\mathbb{R} \times [0, \infty))$. We say that ρ is a weak solution of (7.3) if $\rho = \partial_u f$ and f is a classical solution of (9.2).

Theorem 9.1. *Assume that ε, σ, and b satisfy (7.6) and $a = \varepsilon^2\sigma^2/2$. Then for each given cumulative distribution function f_0, (9.2) admits a unique classical solution.*

The classical theory of linear parabolic pde of divergence form (e.g. [18]) involves weak solutions in Sobolev spaces and does not seem to be enough to provide the stated assertion of the theorem. There are two concerns: (i) the regularity (9.3) is often lacking in high space dimensions, and (ii) the asymptotic behavior of the solution as $u \to \pm\infty$ depends on behavior of b; that is, the stability property of b is crucial in the analysis since without this property, drifted Brownian particles could escape from $\pm\infty$ in finite time, so the solution may not be a cdf.

Proof. We divide the proof into four steps. In the first step we establish the existence of a weak solution of $\mathcal{L}f = 0$ in $\mathbb{R} \times (0, \infty)$ with initial condition $f = f_0$ on $\mathbb{R} \times \{0\}$. In the second step we show that the weak solution satisfies the boundary condition at $u = \pm\infty$. In the third step we establish the regularity (9.3). Finally, we prove uniqueness.

1. The existence of a weak solution is standard so we keep it brief. Let $\{(a^n, b^n)\}_{n=1}^\infty$, where $a^n > 0$, be smooth functions that approach (a, b) locally uniformly in $\mathbb{R}$ as $n \to \infty$. Let $\{f_0^n\}_{n=1}^\infty$ be smooth increasing functions such that $f_0^n(-n) = 0$, $f_0^n(n) = 1$, and $f^n \to f_0$ a.e. in $\mathbb{R}$ as $n \to \infty$. Let f^n be the smooth solution of the initial boundary value problem

$$\begin{cases} \mathcal{L}^n f^n := \partial_t f^n - \partial_u(a^n \partial_u f^n) + b^n \partial_u f^n = 0 & \text{in } \ (-n, n) \times (0, \infty), \\ f^n(-n, t) = 0, \quad f^n(n, t) = 1 & \forall\, t > 0, \\ f^n(u, 0) = f_0^n(u) & \forall\, u \in [-n, n]. \end{cases} \tag{9.4}$$

The existence of a unique smooth solution f^n follows from the classical theory of parabolic pde [18]. By the maximum principle, $0 \leqslant f^n \leqslant 1$ on $[-n, n] \times [0, \infty)$, so $f^n(-n, t) = 0$ is the global minimum, $f^n(n, t) = 1$ is the global maximum, and $\partial_u f^n(\pm n, t) > 0$ for every $t > 0$. By the maximum principle again, $\partial_u f^n > 0$ in $[-n, n] \times (0, \infty)$. Standard local regularity estimates (c.f. Step **3** below) allows us to pass to a limit to obtain a weak (in the sense of distribution) solution of $\mathcal{L}f = 0$ in $\mathbb{R} \times (0, \infty)$

with initial condition $f(\cdot,0)=f_0$. The solution has the property

$$\partial_u f\in L^2_{\rm loc}(\mathbb{R}\times[0,\infty)),\qquad \partial_u f\geqslant 0,\qquad 0\leqslant f\leqslant 1. \tag{9.5}$$

2. We show that the solution obtained from Step **1** has the desired limit as $u\to\pm\infty$. It is in this step that the stability assumption on b is needed, by means of using the invariant measure.

Let η be any small positive constant. Since f_0 is a cdf, there exist $x_\eta<0$ and $y_\eta>0$ such that

$$f_0(u)\leqslant\eta\ \ \forall\, u\leqslant x_\eta,\qquad\qquad f_0(u)\geqslant 1-\eta\ \ \forall\, u\geqslant y_\eta. \tag{9.6}$$

Also, since $0\leqslant f_0\leqslant 1$, for every $u\in\mathbb{R}$ we have

$$\eta+\frac{f_n^*(u)}{f_n^*(x_\eta)}=:f_1^\eta(u)\geqslant f_0(u)\geqslant f_2^\eta(u):=1-\eta-\frac{1-f_n^*(u)}{1-f_n^*(y_\eta)} \tag{9.7}$$

where f_n^* is the analog of f^*, the cdf of the invariant measure ρ^* defined in the previous section with (a,b) replaced by (a^n,b^n). Note that $\mathcal{L}^nf_1^\eta=0$ and $\mathcal{L}^nf_2^\eta=0$ on $\mathbb{R}\times[0,\infty)$. Also, when $n\geqslant\max\{|x_\eta|,|y_\eta|\}$, $f_1^\eta\geqslant f^n\geqslant f_2^\eta$ on the parabolic boundary of $(-n,n)\times[0,\infty)$. Hence, applying the comparison principle on $[-n,n]\times[0,\infty)$ we derive that

$$f_2^\eta(u)\leqslant f^n(u,t)\leqslant f_1^\eta(u)\quad\forall\, u\in[-n,n],\ t>0. \tag{9.8}$$

Passing to the limit, we see that

$$1-\eta-\frac{1-f^*(u)}{1-f^*(y_\eta)}\leqslant f(u,t)\leqslant\eta+\frac{f^*(u)}{f^*(x_\eta)}\quad\forall\, u\in\mathbb{R},t\in[0,\infty). \tag{9.9}$$

Since $0\leqslant f\leqslant 1$, first sending $u\to\pm\infty$ and then $\eta\searrow 0$ we conclude that

$$\lim_{u\to-\infty}\sup_{t\in[0,\infty)}|f(u,t)|=0,\qquad \lim_{u\to\infty}\sup_{t\in[0,\infty)}|f(u,t)-1|=0. \tag{9.10}$$

3. Next we establish the needed regularity (9.3) for the solution obtained in Step **1.** We use energy estimates. For notational simplicity, we omit the superscript n so in the following energy estimates, (f,a,b) is indeed meant to be (f^n,a^n,b^n) with $n>R+2$.

Let $R \geqslant 1$ be any constant and let ζ be a cut-off function:

$$\zeta \in C^\infty(\mathbb{R}), \quad 0 \leqslant \zeta, |\zeta'| \leqslant 1 \text{ on } \mathbb{R}, \quad \zeta = 1 \text{ on } [-R,R], \quad \zeta(u) = 0 \text{ if } |u| > R+2. \tag{9.11}$$

For any non-negative integer k, integrating $\zeta^{4k+2}(u)[\partial_t^k f][\partial_t^k \mathcal{L}^n f] = 0$ over $\mathbb{R} \times \{t\}$ and using integration by parts we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \zeta^{4k+2}|\partial_t^k f|^2 + \int a\zeta^{4k+2}|\partial_t^k \partial_u f|^2 &= -\int \Big[(4k+2)a\zeta^{4k+1}\zeta' + b\zeta^{4k+2}\Big](\partial_t^k f)(\partial_t^k \partial_u f) \\ &\leqslant \frac{1}{2}\int a\zeta^{4k+2}|\partial_t^k \partial_u f|^2 + \int \Big[(4k+2)^2 a + \frac{b^2}{a}\Big]\zeta^{4k}|\partial_t^k f|^2 \end{aligned}\tag{9.12}$$

by Cauchy inequality. Multiplying the resulting inequality by $2t^{2k}$ we obtain

$$\frac{d}{dt}\int t^{2k}\zeta^{4k+2}|\partial_t^k f|^2 + \int at^{2k}\zeta^{4k+2}|\partial_t^k \partial_u f|^2 \leqslant \int \Big[2(4k+2)^2 at + \frac{2b^2 t}{a} + 2k\Big] t^{2k-1}\zeta^{4k}|\partial_t^k f|^2. \tag{9.13}$$

Similarly, integrating $2t^{2k+1}\zeta^{4k+4}(\partial_t^{k+1} f)(\partial_t^k \mathcal{L}^n f) = 0$ over $\mathbb{R} \times \{t\}$ we obtain

$$\int t^{2k+1}\zeta^{4k+4}|\partial_t^{k+1} f|^2 + \frac{d}{dt}\int at^{2k+1}\zeta^{4k+4}|\partial_t^k \partial_u f|^2 \leqslant \int \Big[2(4k+4)^2 at + \frac{2b^2 t}{a} + 2k+1\Big] at^{2k}\zeta^{4k+2}|\partial_t^k \partial_u f|^2. \tag{9.14}$$

Alternately integrating these two inequalities with $k = 0, 1$ over $[0,T]$ we obtain

$$\begin{aligned}\int \zeta^2 f^2(u,T)du + \int_0^T\int a\zeta^2|\partial_u f|^2 dudt &\leqslant \int_0^T\int_{-R-2}^{R+2}\Big(8a + \frac{2b^2}{a}\Big) f^2 dudt + \int \zeta^2 f_0^2 du, \\ T\int a\zeta^4|\partial_u f|^2 du + \int_0^T\int t\zeta^4|\partial_t f|^2 dudt &\leqslant \int_0^T\int\Big(32at + \frac{2b^2 t}{a} + 1\Big)a\zeta^2|\partial_u f|^2 dudt, \\ T^2\int \zeta^6|\partial_t f|^2 du + \int_0^T\int at^2\zeta^6|\partial_t\partial_u f|^2 dudt &\leqslant \int_0^T\int\Big(72at + \frac{2b^2 t}{a} + 2\Big)t\zeta^4|\partial_t f|^2 dudt, \\ T^3\int a\zeta^8|\partial_t\partial_u f|^2 du + \int_0^T\int t^3\zeta^8|\partial_t^2 f|^2 &\leqslant \int_0^T\int\Big(128a^2 t + \frac{2b^2 t}{a} + 3\Big)at^2\zeta^6|\partial_t\partial_u f|^2 dudt. \end{aligned}\tag{9.15}$$

These inequalities imply that

$$\int_0^T\int_{-R}^R t^3|\partial_t(\partial_t f)|^2 dudt + \sup_{t\in[0,T]}\int_{-R}^R t^3|\partial_u(\partial_t f(u,t))|^2 du \leqslant C(R,T) \tag{9.16}$$

where $C(R,T)$ depends only on T and on $\|a+b^2/a+1/a\|_{L^\infty([-R-2,R+2])}$. Thus, by Sobolev imbedding, changing notation f back to f^n which is what f meant to be here, we obtain

$$\|\partial_t f^n\|_{C^{1/4}([-R,R]\times[T,1/T])} \leqslant \hat{C}(R,T) \quad \forall\, R>1, T>1, n>R+2. \tag{9.17}$$

Passing this estimate to the limit, we see that $\partial_t f \in C^{1/4}(\mathbb{R}\times(0,\infty))$. Regarding $\partial_t f$ as a known function, writing $\mathcal{L}f = 0$ as $\partial_u(ae^{-q}\partial_u f) = e^{-q}\partial_t f$, and using $\partial_t(ae^{-q}\partial_u f) \in L^2_{\text{loc}}(\mathbb{R}\times(0,\infty))$ we derive that $ae^{-q}\partial_u f \in C^{1/4}(\mathbb{R}\times(0,\infty))$. Then, $\partial_u(a\partial_u f) = \partial_t f + b\partial_u f \in C(\mathbb{R}\times(0,\infty))$. Thus (9.2) admits a classical solution.

4. Finally we prove the uniqueness of the solution. The argument relies on the required behavior of the solution near $u = \pm\infty$. Suppose f_1 and f_2 are two solutions of (9.2). Let $T > 0$ be any positive constant. Fixing $\eta > 0$, we derive from the boundary condition $\lim_{u\to\pm\infty} f_i(u,t) = (1\pm 1)/2$ uniformly on $[0,T]$ that $|f_1(u,t) - f_2(u,t)| \leqslant \eta$ for all $|u| \geqslant y_\eta, t \in [0,T]$, for some large enough y_η. On the domain $[-y_\eta, y_\eta]\times[0,T]$ we can apply the maximum principle for weak solutions [18] to conclude that $|f_1 - f_2| \leqslant \eta$ in $[-y_\eta, y_\eta]\times[0,T]$. Sending $\eta \to 0$ we obtain $f_1 = f_2$ in $\mathbb{R}\times(0,T]$. As T is arbitrary, we see that $f_1 \equiv f_2$, and therefore obtain the uniqueness of the solution. This completes the proof of Theorem 9.

Remark. One can further show that $\partial_t(a\partial_u f) \in C(\mathbb{R}\times(0,\infty))$. Then $\rho_t = \partial_t\partial_u f = \partial_u j$ with $j = \partial_u[a\rho] - b\rho$ is also continuous, so ρ is indeed a classical solution too, though it may not be even differentiable in u. The delicacy here is that the combinations $a\rho$ and j are differentiable in u. These properties in general hold only in one space dimension.

Chapter 10

Long Time Behavior

In this section we prove (7.4).

Theorem 10.1. *Assume that ε, σ, b satisfy (7.6) and that ρ_0 is a probability density function. Let ρ^* be defined as in (8.1) and ρ be the weak solution of (7.3) defined in the previous section. Then*

$$\lim_{t\to\infty} \rho(\cdot, t) = \rho^*(\cdot) \quad \text{locally uniformly in } \mathbb{R} \text{ and in } L^1(\mathbb{R}). \tag{10.1}$$

Consequently, for any Borel set A in R, the solution $\{u_t\}_{t\geqslant 0}$ of the SDE (7.2) satisfies

$$\lim_{t\to\infty} \mathbb{P}(\{u_t \in A\}) = \lim_{t\to\infty} \int_A \rho(x,t)dx = \int_A \rho^*(x)dx. \tag{10.2}$$

Note that ρ^* defined in (8.1) depends only on ε, σ, and b; in particular it does not depend on the initial distribution of u_0. Theorem 10 proclaims that regardless of initial setup, any solution of the std (7.2) approaches the same statistical equilibrium distribution as ρ^* in the long run.

Proof. Let $f(\cdot, t)$ and $f^*(\cdot)$ be the cdfs of $\rho(\cdot, t)$ and $\rho^*(\cdot)$ respectively. Set $g = f - f^*$. Then

$$\mathcal{L}g = 0, \qquad g(\cdot, 0) = f_0(\cdot) - f^*(\cdot), \qquad \lim_{|u|\to\infty} \sup_{t\in[0,\infty)} |g(u,t)| = 0 \tag{10.3}$$

where the last equation follows from (9.10) (as the solution is unique).

Let $\eta > 0$ be any small positive constant. We shall show that $\|g(\cdot, t)\|_{L^\infty(\mathbb{R})} \leqslant 2\eta$ when t is large enough, by using comparison principle. The above asymptotic behavior of g as $|u| \to \infty$ implies that there exists $y_\eta > 0$ such that

$$|g(u,t)| \leqslant \eta \quad \forall\, u \in (-\infty, -y_\eta] \cup [y_\eta, \infty),\ t \in [0,\infty). \tag{10.4}$$

Next we focus the solution on the spatially bounded domain $[-y_\eta, y_\eta] \times [0,\infty)$. Let

$$\varphi(u,t) := e^{-\lambda t} \sin(\pi f^*(u)), \qquad \lambda := \min_{u\in[-y_\eta,y_\eta]} \Big\{\pi^2 a(u)\rho^*(u)^2\Big\}. \tag{10.5}$$

For $u \in [-y_\eta, y_\eta]$ and $t > 0$, direct computation gives

$$\mathcal{L}\varphi = [-\lambda + a\pi^2(\partial_u\ f^*)^2]\varphi + \pi e^{-\lambda t}\cos(\pi f^*)\mathcal{L}f^* = [-\lambda + \pi^2 a\rho^{*2}]\varphi \geqslant 0. \tag{10.6}$$

On the set $[-y_\eta, y_\eta] \times (0,\infty)$ let

$$\Psi(u,t) := \eta + \frac{e^{-\lambda t}\sin(\pi f^*(u))}{\min\{\sin(\pi f^*(-y_\eta)), \sin(\pi f^*(y_\eta))\}}. \tag{10.7}$$

It is easy to see that $\mathcal{L}\Psi \geqslant 0$ on $[-y_\eta, y_\eta] \times [0,\infty)$, $\Psi(u,0) > 1 \geqslant |g(u,0)|$ for all $u \in [-y_\eta, y_\eta]$ and $\Psi(\pm y_\eta, t) > \eta \geqslant |g(\pm y_\eta, t)|$ for all $t > 0$. Hence, comparing Ψ with $\pm g$ on $[-y_\eta, y_\eta] \times [0,\infty)$ we see that $\pm g < \Psi$ on $[-y_\eta, y_\eta] \times [0,\infty)$. Since this estimate is already known to be true when $u \notin [-y_\eta, y_\eta]$, we hence conclude that $|g(u,t)| \leqslant \Psi(u,t)$ for every $u \in \mathbb{R}, t \in [0,\infty)$. Therefore, there exists $T_\eta > 0$ such that $|g| \leqslant 2\eta$ on $\mathbb{R} \times [T_\eta, \infty)$. Since η is arbitrary, we conclude that

$$\lim_{t\to\infty} \|g(\cdot,t)\|_{L^\infty(\mathbb{R})} = 0. \tag{10.8}$$

Finally, we apply the energy estimates in Step **3** of the regularity proof in the previous section to the function $g(\cdot, h+\cdot)$ in the domain $\mathbb{R} \times [0,2]$ for any $h > 0$ to conclude that for every $R > 1$,

$$\lim_{t\to\infty} \|\rho(\cdot,t) - \rho^*(\cdot)\|_{L^\infty((-R,R))} = \lim_{t\to\infty} \|\partial_u g(\cdot,t)\|_{L^\infty((-R,R))} = 0. \tag{10.9}$$

This completes the proof.

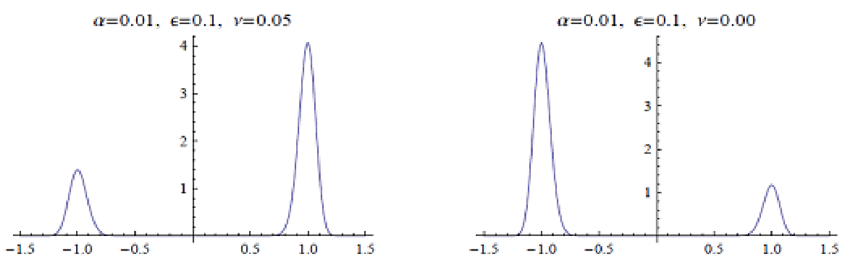

Figure 10.1: Graphs of $\rho = \rho^*(u)$ with $b(u) = (u - \alpha)(1 - u^2)$ and $\sigma(u) = 1 + \nu(u - 1)^2$. When $\nu = 0.05$, the less stable equilibrium $u \equiv 1$ is selected by the fluctuation.

Chapter 11

Limits of the Invariant Measure When White Noise Is Vanishing

In this section, we consider the asymptotic behavior of ρ^* as $\varepsilon \searrow 0$. We write it as

$$\rho^*(u) = \frac{\sigma^{-2}(u)\, e^{q(u)}}{\int \sigma^{-2}(y) e^{q(y)} dy}, \qquad q(u) := -\frac{2Q(u)}{\varepsilon^2}, \qquad Q(u) = -\int_0^u \frac{b(x)}{\sigma^2(x)} dx. \tag{11.1}$$

Figure 10.1 displays the probability density function ρ^* for the case $b(u) = (u - \alpha)(1 - u^2)$ and $\sigma(u) = 1 + \nu(u-1)^2$ with $(\alpha, \varepsilon, \nu) = (0.01, 0.1, 0.05)$ and $(0.01, 0.1, 0)$ respectively. Basic observation indicates that as $\varepsilon \searrow 0$, ρ^* approaches a Dirac type mass concentrated on the most stable equilibria of the effective potential Q. We confirm this observation into two Theorems, beginning with the model problem with constant σ.

Theorem 11.1. *Let $\sigma(u) \equiv 1$, $b(u) = (u - \alpha)(1 - u^2)$, $0 < \varepsilon \ll 1$, and $\{u_t\}_{t \geqslant 0}$ be any solution of (7.2).*

1. *If $\alpha = 0$, then $\int_{|u \pm 1| \leqslant \sqrt{\varepsilon}} \rho^*(u) du = \frac{1}{2} + O(e^{-1/\varepsilon})$. Consequently, for all t sufficiently large,*

$$\mathbb{P}(\{|u_t \pm 1| \leqslant \sqrt{\varepsilon}\}) = \tfrac{1}{2} + O(e^{-1/\varepsilon}).$$

2. If $\alpha \in (0,1)$, then $\int_{|u+1|>\sqrt{\varepsilon}} \rho^*(u)du = O(e^{-1/\varepsilon})$ and for each sufficiently large t,

$$\mathbb{P}(\{|u_t + 1| \geqslant \sqrt{\varepsilon}\}) = O(e^{-1/\varepsilon}).$$

The proof follows by a straightforward evaluation of the potential ρ^* and therefore is omitted.

Notice that without the white noise, i.e., for the ode solution of (7.1), we have

$$\lim_{t\to\infty} u(t) = \begin{cases} 1 & \text{if } \ u(0) > \alpha, \\ \alpha & \text{if } \ u(0) = \alpha, \\ -1 & \text{if } \ u(0) < \alpha. \end{cases}$$

Thus, adding an arbitrary small white noise totally eliminates the initial effect in the long run.

Statistically, it is important to find the mean time needed for the system to switch from one stable state to another by fluctuations; see [32, 33, 36, 37, 39, 43, 44] and references therein.

Next we consider the general case. We denote by $\delta(\cdot - x)$ the Dirac mass concentrated at x.

Theorem 11.2. *Let σ and b be in $C^1(R)$ and satisfy (7.6), $Q(u) = -\int_0^u b(x)\sigma^{-2}(x)dx$, and S be the set of global minimizers of Q. Assume that $b'(x) \neq 0$ for each $x \in S$. Then in the sense of measure*

$$\lim_{\varepsilon\searrow 0} \rho^*(\cdot) = \frac{1}{A}\sum_{x\in\mathcal{S}} \frac{1}{\sqrt{|b'(x)|}\ \sigma(x)}\ \delta(\cdot - x), \qquad \text{where } \ A = \sum_{x\in\mathcal{S}} \frac{1}{\sqrt{|b'(x)|}\ \sigma(x)}. \tag{11.2}$$

Remark. Note that $\mathcal{S}$ is a subset of the stable steady states $\{u \mid P'(u) = 0 \leqslant P''(u)\}$ since $Q' = -b/\sigma^2 = P'/\sigma^2$ and on $\mathcal{S}$, $Q'' = -b'/\sigma^2 = P''/\sigma^2$.

Proof. Assume, without loss of generality, that $0 \in \mathcal{S}$. Then $\min_{u\in\mathbb{R}} Q(u) = Q(0) = 0$.

Denote by M and η the positive constants such that $b(u) > \eta$ in $(-\infty, -M+\eta]$ and $b(u) < -\eta$ in $[M-\eta, \infty)$. Then $Q'(u) = -b(u)/\sigma^2(u) > 0$ in $[M-\eta, \infty)$ and $Q'(u) < 0$ in $(-\infty, -M+\eta]$, so that $\mathcal{S} \subset (-M+\eta, M-\eta)$. As $Q''(x) = -b'(x)/\sigma^2(x)$ is non-zero and therefore must be positive for every $x \in \mathcal{S}$, $\mathcal{S}$ is non-empty and has finitely many points. By taking smaller η if necessary we can assume that $Q''(y) > \frac{1}{2}Q''(x)$ if $|y-x| < \eta$ and $x \in \mathcal{S}$. We introduce

$$\mathrm{d}(y,\mathcal{S}) = \min_{x\in\mathcal{S}} |x-y|, \qquad c_1 = \min_{\mathrm{d}(y,\mathcal{S})\geqslant\eta} Q(y), \qquad c_2 = \min_{x\in\mathcal{S}} Q''(x). \tag{11.3}$$

We now calculate the contributions of the integrand towards the integral

$$Z := \frac{1}{\varepsilon}\int \frac{e^{q(y)}}{\sigma^2(y)}dy \qquad \text{with} \quad q(u) := -\frac{2Q(y)}{\varepsilon^2}. \tag{11.4}$$

We shall show that the contributions from the set $\{y \mid \mathrm{d}(y,\mathcal{S}) > 2\sqrt{\varepsilon}\}$ is negligible, whereas the contribution from each ball $\{y \mid \mathrm{d}(x,y) < 2\sqrt{\varepsilon}\}$ is proportional to $1/\sqrt{|b'(x)|\sigma^2(x)}$, if $x \in \mathcal{S}$.

(i) For the set $(-\infty,-M]\cup[M,\infty)$, we use the estimate in the proof of Theorem 8 to derive

$$\frac{1}{\varepsilon}\int_{|y|>M} \frac{e^{q(y)}}{\sigma^2(y)}dy \leqslant \frac{\varepsilon[e^{q(-M)}+e^{q(M)}]}{2\eta} \leqslant \frac{\varepsilon e^{-2c_1/\varepsilon^2}}{2\eta}. \tag{11.5}$$

(ii) In the bounded set $\{y \in [-M,M] \mid \mathrm{d}(y,\mathcal{S}) \geqslant \eta\}$ we have $Q \geqslant c_1$ so that

$$\frac{1}{\varepsilon}\int_{|y|<M,\mathrm{d}(y,\mathcal{S})>\eta} \frac{e^{q(y)}}{\sigma^2(y)}dy \leqslant \frac{2Me^{-2c_1/\varepsilon^2}}{\varepsilon\ \min\{\sigma^2(u) \mid u\in[-M,M]\}}. \tag{11.6}$$

(iii) Suppose $2\sqrt{\varepsilon} < |y-x| < \eta$ for some $x \in \mathcal{S}$. Then $Q(x) = 0$ and $Q'(x) = 0 < Q''(x)$. Using $Q''(\tilde{y}) \geqslant \frac{1}{2}Q''(x)$ for every $\tilde{y} \in [x-\eta, x+\eta]$ and Taylor expansion, we derive that

$$\begin{aligned} &Q(y) = \tfrac{1}{2}Q''(\tilde{y})|y-x|^2 \geqslant Q''(x)\varepsilon \geqslant c_2\varepsilon, \\ &\frac{1}{\varepsilon}\int_{2\sqrt{\varepsilon}<\mathrm{d}(y,\mathcal{S})\leqslant\eta} \frac{e^{q(y)}}{\sigma^2(y)}dy \leqslant \frac{2Me^{-2c_2/\varepsilon}}{\varepsilon\min\{\sigma^2(u) \mid u\in[-M,m]\}}. \end{aligned} \tag{11.7}$$

(iv) Now for each $x \in \mathcal{S}$ and $|y-x| \leqslant 2\sqrt{\varepsilon}$, we have

$$q(y) = -\frac{2Q(y)}{\varepsilon^2} = -\Big[Q''(x)+o(1)\Big]\frac{(x-y)^2}{\varepsilon^2}, \quad \sigma^2(y) = \sigma^2(x)+o(1) \tag{11.8}$$

where $o(1) \searrow 0$ as $\varepsilon \searrow 0$. The change of variable $y = x + \varepsilon z$ gives

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{|y-x| \leqslant 2\sqrt{\varepsilon}} \frac{e^{q(y)}}{\sigma^2(y)} dy &= \lim_{\varepsilon \searrow 0} \int_{-2/\sqrt{\varepsilon}}^{2/\sqrt{\varepsilon}} \frac{e^{-[Q''(x)+o(1)]z^2}}{\sigma^2(x) + o(1)} \, dz \\ &= \int \frac{e^{-Q''(x) z^2}}{\sigma^2(x)} dz = \frac{\sqrt{\pi}}{\sqrt{Q''(x)} \sigma^2(x)} = \frac{\sqrt{\pi}}{\sqrt{|b'(x)|}\, \sigma(x)}. \end{aligned} \tag{11.9}$$

The above estimates can be summarized as follows:

$$\begin{aligned} \lim_{\varepsilon \searrow 0} Z &= \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int \frac{e^{q(y)}}{\sigma^2(y)} dx = \sum_{x \in \mathcal{S}} \frac{\sqrt{\pi}}{\sqrt{|b'(x)|}\ \sigma(x)} = \sqrt{\pi} A, \\ \lim_{\varepsilon \searrow 0} \int_{|y-x| \leqslant 2\sqrt{\varepsilon}} \rho^*(y) dy &= \lim_{\varepsilon \searrow 0} \frac{1}{Z\ \varepsilon} \int_{|y-x| < 2\sqrt{\varepsilon}} \frac{e^{q(y)}}{\sigma^2(y)} dy = \frac{1}{A \sqrt{|b'(x)|} \sigma(x)} \quad \forall\, x \in \mathcal{S}, \\ \lim_{\varepsilon \searrow 0} \int_{\mathrm{d}(y, \mathcal{S}) > 2\sqrt{\varepsilon}} \rho^*(y) dy &= \lim_{\varepsilon \searrow 0} \frac{1}{Z\ \varepsilon} \int_{\mathrm{d}(y, \mathcal{S}) > 2\sqrt{\varepsilon}} \frac{e^{q(y)}}{\sigma^2(y)} dy = 0. \end{aligned} \tag{11.10}$$

The assertion of Theorem 11.2 thus follows.

Chapter 12

The Degenerate Case Where σ^2 Is Not Everywhere Positive

Here we consider the degenerate case where σ is not everywhere positive. Our prototype is

$$\begin{aligned} b(u) &= (u-\alpha)(1-u^2) && (\ \alpha \in [0,1)\ ), \\ \sigma(u) &= (\max\{\ell^2 - u^2, 0\})^\nu && (\ \nu > 0, \ell > 0\ ). \end{aligned} \tag{12.1}$$

For the invariant measure ρ^* in (8.1) to be well-defined, we go through a regularization process defining ρ^* as the $\eta \searrow 0$ limit of those regular ρ_η obtained by replacing σ by positive σ_η:

$$\rho_\eta(x) := \frac{1}{Z_\eta} \frac{e^{q_\eta(x)}}{\sigma_\eta^2(x)}, \quad q_\eta(x) := \int_0^x \frac{b(s)}{\sigma_\eta^2(s)} ds, \quad Z_\eta := \int \frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)} du. \tag{12.2}$$

Here by absorbing ε into σ, we assume for simplicity that $\varepsilon = \sqrt{2}$. It turns out that when $\ell \in (0,1]$, the limit depends sensitively on the regularization process, whereas when $\ell > 1$, the limit does not depend on the regularization process and therefore ρ^* is well-defined.

We shall consider a special regularization for the case $\ell = 1$ in the first subsection. Then consider a general regularization covering the case $\ell \in (0,1]$ in second subsection, showing that

$$\lim_{\eta \searrow 0} \rho_\eta(\cdot) \ = \ \theta\, \delta(\cdot + 1) + [1-\theta]\, \delta(\cdot - 1) \tag{12.3}$$

where $\theta \in [0,1]$ is a constant that depends on the regularization. Indeed, for any constant $\theta \in [0,1]$, there is a regularization such that (12.3) is true. The conclusion is then generalized to a much broader situation where σ vanishes at certain stable equilibria of a quite general potential. In the last subsection we consider a general regularization covering the case $\ell > 1$, proving that

$$\lim_{\eta \searrow 0} \rho_\eta(\cdot) = \frac{m_1\, \delta(\cdot + \ell) \ + \ m_2\, \delta(\cdot - \ell) \ + \ \hat{\rho}(\cdot)\, \chi_{(-\ell,\ell)}(\cdot)}{m_1 \ + \ m_2 \ + \ m} \tag{12.4}$$

where χ_A is the characteristic function of the set A, $\hat{\rho} = e^q/\sigma^2$, m is a positive constant, and m_1, m_2 are non-negative constants depending only on α, ℓ, and ν; $m_1 = m_2 = 0$ if and only if $\nu \geqslant 1/2$.

12.1 The Model Case With Special Regularization

We begin with a special regularization for a special case where σ vanishes at the stable equilibria.

Theorem 12.1. *Let $\alpha \in [0,1)$ and $\nu > 0$ be constants and $b(u) = (u-\alpha)(1-u^2)$ and $\sigma = (\max\{1-u^2, 0\})^\nu$ be functions of $u \in \mathbb{R}$. For $\eta \in (0,1]$ define ρ_η by (12.2) with $\sigma_\eta = \max\{\eta, \sigma\}$. Then (12.3) holds in the sense of measure with*

$$\theta = \frac{1}{1+m}, \qquad m = \begin{cases} 1 & \text{if } \ \alpha = 0, \\ \sqrt{\frac{b'(1)}{b'(-1)}} \exp\left(\int_{-1}^{1} \frac{b(u)}{\sigma^2(u)} ds\right) & \text{if } \ \alpha \in (0,1), \nu \in (0,1), \\ 0 & \text{if } \ \alpha \in (0,1), \nu \geqslant 1. \end{cases}$$

To prove the theorem, we first prove a lemma. For this we introduce

$$\rho_\eta^+(x) := \frac{e^{q_\eta(x)}}{\sigma_\eta^2(x)}\, \eta e^{-q_\eta(1)} \chi_{[\alpha,\infty)}(x), \qquad \rho_\eta^-(x) := \frac{e^{q_\eta(x)}}{\sigma_\eta^2(x)}\, \eta e^{-q_\eta(-1)} \chi_{(-\infty,\alpha)}(x). \tag{12.5}$$

Lemma 12.1. *As $\eta \searrow 0$, $\rho_\eta^\pm(\cdot) \longrightarrow M^\pm\, \delta(\cdot \mp 1)$ where, with $k = |b'(\pm 1)|/2$,*

$$M^\pm = \begin{cases} \int_{-\infty}^{\infty} e^{-kz^2}\, dz & \text{if } \ \nu > 1, \\ \int_{-1/2}^{\infty} e^{-kz^2}\, dz + \int_{-\infty}^{-1/2} \frac{e^{-k/4}\, dz}{|2z|^{k/2+2}} & \text{if } \ \nu = 1, \\ \int_{0}^{\infty} e^{-kz^2}\, dz & \text{if } \ \nu \in (0,1). \end{cases}$$

Proof of Theorem 12.1. Setting $M_\eta^\pm = \int \rho_\eta^\pm(x)dx$ and $m_\eta = M_\eta^+ e^{q_\eta(1)}/[M_\eta^- e^{q_\eta(-1)}]$. Then

$$\rho_\eta(x) = \frac{m_\eta}{1+m_\eta}\,\frac{\rho_\eta^+(x)}{M_\eta^+} + \frac{1}{1+m_\eta}\,\frac{\rho_\eta^-(x)}{M_\eta^-}. \tag{12.6}$$

The assertion of Theorem 12.1 thus follows from Lemma 12.1 and the computation

$$\lim_{\eta\searrow 0} m_\eta = \lim_{\eta\searrow 0}\frac{M_\eta^+}{M_\eta^-}e^{q_\eta(1)-q_\eta(-1)} = \frac{M^+}{M^-}\lim_{\eta\searrow 0}\exp\Big(\int_{-1}^{1}\frac{b(u)}{\sigma_\eta^2(u)}du\Big) = m. \tag{12.7}$$

Proof of Lemma 12.1. First consider ρ_η^+. We shall show that the major contribution toward the total mass M_η^+ of ρ_η^+ comes from the interval $(1-\eta|\ln\eta|, 1+\eta|\ln\eta|)$ for the case $\nu \geqslant 1$, and from the interval $[1, 1+\eta|\ln\eta|)$ for the case $\nu \in (0,1)$.

Set $\eta_\nu = 1-\sqrt{1-\eta^{1/\nu}}$. Then $\sigma_\eta(u) = \eta$ when $|u| \geqslant 1-\eta_\nu$ and $\sigma_\eta(u) = (1-u^2)^\nu$ when $|u| \leqslant 1-\eta_\nu$. Denote $k = -b'(1)/2$. Then for $x \geqslant 1-\eta_\nu$,

$$q_\eta(x) - q_\eta(1) = \int_1^x \frac{b(u)}{\eta^2}du = -k\Big(\frac{x-1}{\eta}\Big)^2\Big(1+\frac{b''(\hat{x})}{3b'(1)}(x-1)\Big) \tag{12.8}$$

by Taylor expansion, where $\hat{x}$ lies between 1 and x. Since $b'(1)<0$ and $b''<0$ in $[1,\infty)$,

$$\overline{\lim_{\eta\searrow 0}}\int_{1+\eta|\ln\eta|}^{\infty}\rho_\eta^+(x)dx = \overline{\lim_{\eta\searrow 0}}\int_{1+\eta|\ln\eta|}^{\infty}\frac{e^{q_\eta(x)-q_\eta(1)}}{\eta}dx \leqslant \lim_{\eta\searrow 0}\int_{|\ln\eta|}^{\infty}e^{-kz^2}dz = 0 \tag{12.9}$$

by the change of variable $x = 1+\eta z$. Also, by Lebesgue's Dominated Convergence Theorem,

$$\lim_{\eta\searrow 0}\int_{1-\eta_\nu}^{1+\eta|\ln\eta|}\rho_\eta^+(x)dx = \lim_{\eta\searrow 0}\int_{-\eta_\nu/\eta}^{|\ln\eta|}e^{-kz^2(1+O(z\eta))}dz = \begin{cases}\int_{-\infty}^{\infty}e^{-kz^2}\,dz & \text{if } \nu>1,\\[4pt] \int_{-1/2}^{\infty}e^{-kz^2}\,dz & \text{if } \nu=1,\\[4pt] \int_0^{\infty}e^{-kz^2}\,dz & \text{if } \nu\in(0,1).\end{cases} \tag{12.10}$$

Finally consider the integral on $(\alpha, 1-\eta_\nu]$. We shall use $\eta_\nu = \frac{1}{2}\eta^{\frac{1}{\nu}}[1+O(\eta^{\frac{1}{\nu}})]$ and

$$q_\eta(1-\eta_\nu) - q_\eta(1) = -k\Big(\frac{\eta_\nu}{\eta}\Big)^2\Big(1-\frac{b''(\hat{x})}{3b'(1)}\eta_\nu\Big) = -\eta^{2/\nu-2}\Big[\frac{k}{4}+O(\eta_\nu)\Big]. \tag{12.11}$$

(a) When $\nu > 1$, using $q_\eta(x) \leqslant q_\eta(1-\eta_\nu)$ and $\sigma_\eta(x) \geqslant (1+\alpha)^\nu(1-x)^\nu$ for $x \in (\alpha, 1-\eta_\nu]$, we obtain

$$\int_\alpha^{1-\eta_\nu} \rho_\eta^+(x)dx \leqslant \int_\alpha^{1-\eta_\nu} \frac{\eta e^{q_\eta(1-\eta_\nu)-q_\eta(1)}}{(1+\alpha)^{2\nu}(1-x)^{2\nu}}dx \leqslant \frac{\eta e^{-\eta^{2/\nu-2}[k/4+O(\eta_\nu)]}}{(2\nu-1)(1+\alpha)^{2\nu}\eta_\nu^{2\nu-1}} \longrightarrow 0 \quad \text{as } \eta \searrow 0. \tag{12.12}$$

(b) When $\nu \in (0,1)$, considering separately cases $\nu \in (0, 1/2), \nu = 1/2$, and $\nu \in (1/2, 1)$ we derive that

$$\int_\alpha^{1-\eta_\nu} \rho_\eta^+(x)dx \leqslant \int_\alpha^{1-\eta_\nu} \frac{\eta\, dx}{(1-x^2)^{2\nu}} \leqslant O(1)\eta[1+|\ln \eta_\nu| + \eta_\nu^{1-2\nu}] \longrightarrow 0 \quad \text{as } \eta \searrow 0. \tag{12.13}$$

(c) When $\nu = 1$, we have $\eta_\nu = \frac{\eta}{2} + O(\eta^2)$ and for $x \in (\alpha, 1-\eta_\nu]$,

$$\rho_\eta^+(x) = \frac{\eta e^{q_\eta(1-\eta_\nu)-q_\eta(1)} e^{\int_{1-\eta_\nu}^x \frac{(u-\alpha)}{(1-u^2)}du}}{(1-x^2)^2} = \frac{e^{k/4+O(\eta)}}{\eta}\Big(\frac{\eta_\nu}{1-x}\Big)^{\frac{k}{2}+2}\Big(\frac{2-\eta_\nu}{1+x}\Big)^{\frac{1+\alpha}{2}+2}. \tag{12.14}$$

Using the change of variable $x = 1+\eta z$ we obtain

$$\begin{aligned} &\lim_{\eta\searrow 0}\int_{1-\eta|\ln\eta|}^{1-\eta_\nu} \rho_\eta^+(x)dx = \lim_{\eta\searrow 0}\int_{-|\ln\eta|}^{-\eta_\nu/\eta}\Big(\frac{\eta_\nu}{-\eta z}\Big)^{\frac{k}{2}+2}\Big[e^{-k/4}+O(\eta\ln\eta)\Big]dz = \int_{-\infty}^{-1/2}\frac{e^{-k/4}\,dz}{|2z|^{k/2+2}},\\ &\overline{\lim_{\eta\searrow 0}}\int_\alpha^{1-\eta|\ln\eta|}\rho_\eta^+(x)dx \leqslant \overline{\lim_{\eta\searrow 0}}\int_{-\infty}^{-|\ln\eta|}\frac{O(1)}{|z|^{k/2+2}}dz = 0. \end{aligned} \tag{12.15}$$

In conclusion, defining $f_\eta(x) = \int_{-\infty}^x \rho_\eta^+(u)du$, we find that $\lim_{\eta\searrow 0} f_\eta(x) = 0$ if $x < 1$ and $= M^+$ if $x > 1$. Thus, $\rho_\eta^+(\cdot) \to M^+\delta(\cdot - 1)$. Similarly, by considering $\tilde{\rho}_\eta(x) = \rho_\eta(-x)$ with $k = -b'(-1)/2$, we can show that $\rho_\eta^-(\cdot) \to M^-\delta(\cdot+1)$. This completes the proof of Lemma 12.1.

Remark. From the proof, one sees that $\rho^\pm$ has the following limiting profile:

$$\lim_{\eta\searrow 0} \eta\rho_\eta^\pm(\pm 1 \pm z\eta) = \begin{cases} e^{-kz^2} & \text{if } \nu > 1, \\ e^{-kz^2}\chi_{[0,\infty)} & \text{if } \nu \in (0,1), \\ e^{-kz^2}\chi_{[-\frac{1}{2},\infty)} + \frac{e^{-k/4}}{|2z|^{k/2+2}}\chi_{(-\infty,-\frac{1}{2})} & \text{if } \nu = 1. \end{cases} \tag{12.16}$$

12.2 General Regularization When σ Vanishes at Stable States

Here we extend Theorem 12.1 to a general regularization for the general case when σ vanishes at stable equilibria. We assume that σ vanishes near ± 1 where -1 is the smallest and 1 is the largest zeros of b. Also σ is positive in $[x_1, x_2] \subset (-1, 1)$ where (x_1, x_2) is an interval contains all zeros of b except ± 1.

Theorem 12.2. *Let b be a continuous function that satisfies, for some $x_2 \in (0, 1)$ and $x_1 \in (-1, 0)$,*

$$b < 0 \text{ in } (-1, x_1] \cup (1, \infty), \quad b(u) > 0 \text{ in } (-\infty, -1) \cup [x_2, 1), \quad \varliminf_{|u| \to \infty} |b(u)| > 0. \tag{12.17}$$

Let σ be a non-negative continuous function such that, for some $\epsilon > 0$,

$$\sigma(u) > 0 \text{ in } \ [x_1, x_2], \qquad \sigma(u) = 0 \text{ in } \ [-1-\epsilon, -1] \cup [1, 1+\epsilon]. \tag{12.18}$$

For σ_η to be specified below, let ρ_η be defined as in (12.2). The following holds:

1. *Suppose $\{\sigma_\eta\}_{\eta \in (0,1]}$ is a family of continuous positive functions satisfying $\lim_{\eta \searrow 0} \sigma_\eta(\cdot) = \sigma(\cdot)$ uniformly on $[-1-\epsilon, 1+\epsilon]$. Then for some $\theta \in [0, 1]$, (12.3) holds along a sequence $\eta \searrow 0$;*
2. *For every $\theta \in [0, 1]$, there exists a family $\{\sigma_\eta\}_{\eta \in (0,1]}$ of continuous positive functions satisfying $\lim_{\eta \searrow 0} \sigma_\eta = \sigma$ uniformly on $\mathbb{R}$ such that (12.3) holds.*

Proof. Similar to the proof of Theorem 12.1, we introduce

$$\rho_\eta^\pm(x) = \frac{e^{q_\eta(x) - q_\eta(\pm 1)}}{\sigma_\eta^2(x)} \chi_{[0,\infty)}(\pm x), \qquad M_\eta^\pm = \int \rho_\eta^\pm(x) dx. \tag{12.19}$$

1. We first show that $\lim_{\eta \searrow 0} M_\eta^+ = \infty$. For each $x > 1$ setting $b^*(x) = \max_{u \in [1,x]} |b(u)|$ and using $q'_\eta = b/\sigma_\eta^2$, $b<0$ in $(1,x)$, $\lim_{\eta\searrow 0}\sigma_\eta = 0$ on $[1,1+\epsilon]$, and $b(1)=0$, we derive that

$$\begin{aligned} M_\eta^+ &\geqslant \int_1^x \frac{|b(u)|}{b^*(x)}\rho_\eta^+(u)du = \frac{1-\exp(\int_1^x \frac{b(u)}{\sigma_\eta^2(u)}du)}{b^*(x)}, \\ \varliminf_{\eta\searrow 0} M_\eta^+ &\geqslant \varliminf_{x\searrow 1}\varliminf_{\eta\searrow 0}\frac{1-\exp(\int_1^x \frac{b(u)}{\sigma_\eta^2(u)}du)}{b^*(x)} = \lim_{x\searrow 1}\frac{1}{b^*(x)} = \infty. \end{aligned} \tag{12.20}$$

2. Next we estimate the mass on $[x,\infty)$ for each $x>1$. Setting $b_*(x) = \inf_{u\geqslant x}|b(u)|$ we have

$$\begin{aligned} \varlimsup_{\eta\searrow 0}\int_x^\infty \rho_\eta^+(u)du &\leqslant \varlimsup_{\eta\searrow 0}\int_x^\infty \frac{|b(u)|}{b_*(x)}\rho^+(u)du \\ &\leqslant \varlimsup_{\eta\searrow 0}\frac{e^{q_\eta(x)-q_\eta(1)}}{b_*(x)} = \lim_{\eta\searrow 0}\frac{1}{b_*(x)}\exp\Big(\int_1^x \frac{b(u)}{\sigma_\eta^2}du\Big) = 0. \end{aligned} \tag{12.21}$$

3. For any $x\in(x_2,1)$, setting $B_*(x) = \min_{u\in[x^+,x]} b(u)$ we derive that

$$\begin{aligned} \int_{x^+}^x \rho_\eta^+(u)du &\leqslant \int_{x^+}^x \frac{b(u)}{B_*(x)}\rho_\eta^+(u)du = \frac{e^{-q_\eta(1)}}{B_*(x)}\int_{x^+}^x q'(u)e^{q_\eta(u)}du \leqslant \frac{1}{B_*(x)}, \\ \varlimsup_{\eta\searrow 0}\frac{1}{M_\eta^+}\int_{-\infty}^x \rho_\eta^+(u)du &\leqslant \varlimsup_{\eta\searrow 0}\frac{1}{M_\eta^+}\Big(\int_0^{x_2}\frac{e^{q_\eta(u)-q_\eta(x_2)}}{\sigma_\eta^2(u)}du + \frac{1}{B^*(x)}\Big) = 0. \end{aligned} \tag{12.22}$$

4. The above three estimates shows that as $\eta\searrow 0$, $\rho_\eta^+(x)/M_\eta^+ \longrightarrow \delta(x-1)$. Similarly, as $\eta\searrow 0$, $\rho_\eta^-(x)/M_\eta^- \longrightarrow \delta(x+1)$. Finally, note that

$$\rho_\eta(x) = \theta_\eta\,\frac{\rho_\eta^-(x)}{M_\eta^-} + [1-\theta_\eta]\frac{\rho_\eta^+(x)}{M^+(\eta)}, \qquad \theta_\eta := \frac{M_\eta^- e^{q_\eta(-1)}}{M_\eta^+ e^{q_\eta(1)} + M_\eta^- e^{q_\eta(-1)}}. \tag{12.23}$$

As $\{\theta_\eta\}_{\eta\in(0,1]}$ is a family in $(0,1)$, there exists a sequence $\{\eta_i\}_{i=1}^\infty$ in $(0,1)$ such that as $i\to\infty$, $\eta_i\searrow 0$ and $\theta_{\eta_i}\to\theta$ for some $\theta\in[0,1]$. This implies that $\rho_{\eta_i}(\cdot)\to\theta\delta(\cdot+1)+[1-\theta]\delta(\cdot-1)$ as $i\to\infty$. The first assertion of the theorem thus follows. As the second assertion is a special case of the theorem to be presented next, this completes the proof.

Remark. Following the same proof, one sees that the first assertion of Theorem **12.2** remains true if condition (12.17) is replaced by the following more applicable conditions:

$$\int_{x_1}^{x_2}\frac{du}{\sigma^2(u)} \leqslant \varlimsup_{\eta\searrow 0}\int_{x_1}^{x_2}\frac{du}{\sigma_\eta^2(u)} < \infty = \int_{-1-\varepsilon}^{-1}\frac{du}{\sigma^2(u)} = \int_1^{1+\varepsilon}\frac{du}{\sigma^2(u)}du \quad \forall\,\varepsilon>0. \tag{12.24}$$

For example, $\sigma(u) = \sqrt{|u|}\,|\ln u^2|$.

Since the effective potential is $Q(u) = \int b(u)/\sigma^2(u)du$, the second assertion of Theorem 12.2 can be extended to a much more general case where σ vanishes at multiple stable equilibria.

Theorem 12.3. *Let σ be a non-negative continuous function and b be a $C^1(R)$ function satisfying*

$$\overline{\lim_{u\to\infty}}\, b(u) < 0 < \underline{\lim_{u\to-\infty}}\, b(u). \tag{12.25}$$

Assume that the set $S := \{u \in R \mid b(u) = 0, b'(0) < 0, \sigma(u) = 0\}$ is non-empty. Then for every probability measure μ supported on S, these exists a family of continuous positive functions $\{\sigma_\eta\}_{0<\eta<1}$ such that $\lim_{\eta\searrow 0}\sigma_\eta = \sigma$ uniformly on R and the function ρ_η defined in (12.2) satisfies

$$\lim_{\eta\searrow 0}\rho_\eta = \mu \qquad \text{in the sense of measure.} \tag{12.26}$$

Proof. 1. Let $\eta \in (0, 1]$ be any constant.

Observe that $\mathcal{S}$ is bounded and has only finitely many points. As $\sigma = 0$ and $b' < 0$ on $\mathcal{S}$, there exists $\epsilon > 0$ such that $\sigma(y) \leqslant \eta$ and $|b'(y)/b'(x) - 1| \leqslant 1/2$ when $y \in [x-\epsilon, x+\epsilon]$ and $x \in \mathcal{S}$. The distance between different points in $\mathcal{S}$ exceeds 2ϵ since $b(x-\epsilon) > 0 > b(x+\epsilon)$ for each $x \in \mathcal{S}$. Denote

$$\begin{aligned} &A = \cup_{x\in\mathcal{S}}[x-\epsilon, x+\epsilon], \qquad A^c := \mathbb{R}\setminus A, \qquad \hat\sigma(u) = \max\{\sigma(u), \eta\}, \\ &\hat q(u) = \int_0^u \frac{b(y)}{\hat\sigma^2(y)}dy, \qquad \hat Z = \int_{\mathbb{R}} \frac{e^{\hat q(y)}}{\hat\sigma^2(y)}dy, \qquad \hat Z_c = \int_{A^c} \frac{e^{\hat q(y)}}{\hat\sigma^2(y)}dy. \end{aligned} \tag{12.27}$$

Here without loss of generality, we assume that $0 \in A^c$. Recall that the stability assumption on b implies that $\hat Z$ is finite. For non-negative functions $s(x)$ and $L(x, u)$ to be chosen later, we define

$$\sigma_\eta(u) = \hat\sigma(u)\chi_{A^c}(u) + \sum_{x\in\mathcal{S}} \frac{\hat\sigma(u)}{\sqrt{1+\hat\sigma^2(u)s(x)L(x,u)}}\chi_{[x-\varepsilon,x+\varepsilon]}(u) \tag{12.28}$$

and the corresponding $\rho_\eta, q_\eta, Z_\eta$ as in (12.2). The central idea here is to choose an appropriate L such that $q_\eta = \hat q$ on A^c. Then the integral of ρ_η on A^c is still $\hat Z_c$, whereas in each interval $[x-\epsilon, x+\epsilon]$ with $x \in \mathcal{S}$, we still have the freedom to choose $s(x)$. For this we notice the following

$$\frac{1}{\sigma_\eta^2(u)} = \frac{1}{\hat\sigma^2(u)} + \sum_{x\in\mathcal{S}} s(x)L(x,u)\chi_{[x-\epsilon,x+\epsilon]}(u), \tag{12.29}$$

$$q_\eta(u) = \hat q(u) + \sum_{x\in\mathcal{S}} s(x)\int_0^u b(y)L(x,y)\chi_{[x-\epsilon,x+\epsilon]}(y)dy. \tag{12.30}$$

Hence, for $q_\eta \equiv \hat{q}$ on A^c, we need only $\int_{x-\epsilon}^{x+\epsilon} b(y)L(x,y)dy = 0$ for each $x \in \mathcal{S}$. More details will be given later. Here we point out that if $L(x,u)$ is continuous in u and $L(x, x\pm\varepsilon) = 0$ for each $x \in \mathcal{S}$, then σ_η is positive and continuous. In addition, since $\sigma \leqslant \eta$ on A,

$$\|\sigma - \sigma_\eta\|_{L^\infty(\mathbb{R})} \leqslant \eta. \tag{12.31}$$

2. Now we define $L(x,\cdot)$ for each $x \in \mathcal{S}$ as follows: $L(x,\cdot)$ is zero in $(-\infty, x-\epsilon/2]$, linear with slope 2 in $[x-\epsilon/2, x]$, linear with slop $-k$ in $[x, x+\epsilon/k]$ and zero in $[x+\epsilon/k, \infty)$; more precisely,

$$L(x,u) = [2(u-x)+\epsilon]\, \chi_{[x-\epsilon/2,x]}(u) + [\epsilon + (x-u)k]\chi_{(x,x+\epsilon/k]}(u). \tag{12.32}$$

Here $k = k(x) \geqslant 1$ is the root of the equation $j(k) = 0$ where

$$\begin{aligned} j(k) := \int L(x,u)b(u)du &= \int_{x-\epsilon/2}^{x} [2(u-x)+\epsilon]b(u)du + \epsilon \int_x^{x+\epsilon/k} b(u)[\epsilon + (x-u)k]du \\ &= \epsilon^2 \int_0^1 (1-s)\Big[\frac{b(x-\epsilon s/2)}{2} + \frac{b(x+\epsilon s/k)}{k}\Big]ds \\ &= \epsilon^3 \int_0^1 (1-s)s \int_0^1 \Big[\frac{b'(x+\epsilon\theta s/k)}{k^2} - \frac{b'(x-\epsilon\theta s/2)}{4}\Big]d\theta ds. \end{aligned} \tag{12.33}$$

There exists a (unique) $k \in [1,4]$ such that $j(k) = 0$ since $|b'(y)/b'(x) - 1| \leqslant 1/2$ for $y \in [x-\epsilon, x+\epsilon]$.

Since $0 \notin A$, we see from (12.29) that $q_\eta = \hat{q}$ on A^c. In addition, since $\int_{x-\varepsilon}^{u} b(y)L(x,y)dy$ is increasing in $(-\infty, x]$ and decreasing in $[x, \infty)$, it is non-negative in $\mathbb{R}$.

3. Now we define $s(x) = s$ for $x \in \mathcal{S}$ as the root of

$$\int_{x-\epsilon}^{x+\epsilon} \Big(\frac{1}{\hat{\sigma}^2(u)} + sL(x,u)\Big) \exp\Big(\hat{q}(y) + s\int_{x-\epsilon}^{u} b(y)L(x,y)dy\Big)du = \frac{\hat{Z}\,[\mu(\{x\}) + \eta]}{\eta}. \tag{12.34}$$

There exists a unique solution $s > 0$ since the left-hand side is an increasing function of $s \in [0,\infty)$ (as $\int_{x-\varepsilon}^{u} b(y)L(x,y) \geqslant 0$ for each $u \in [x-\epsilon, x+\epsilon]$), approaches ∞ as $s \to \infty$, and is less than the right-hand side when $s = 0$.

4. Now we are ready to complete the proof. Since $q_\eta = \hat{q}$ on A^c, we have

$$\int_{A^c} \frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)}du = \int_{A^c} \frac{e^{\hat{q}(u)}}{\hat{\sigma}^2(u)}du = \hat{Z}_c \leqslant \hat{Z}. \tag{12.35}$$

Also, for each $x \in \mathcal{S}$ using (12.28), (12.29) and the definition of $s = s(x)$ in (12.34) we find that

$$\int_{x-\epsilon}^{x+\epsilon} \frac{e^{q_\eta(y)}}{\sigma_\eta^2(x)} dx = \frac{\hat{Z}\,[\mu(\{x\}) + \eta]}{\eta}. \tag{12.36}$$

Consequently,

$$Z_\eta = \int_{A^c} \frac{e^{q_\eta(y)}}{\sigma_\eta^2(y)} dy + \sum_{x\in\mathcal{S}} \int_{x-\epsilon}^{x+\epsilon} \frac{e^{q_\eta(y)}}{\sigma_\eta^2(y)} dy = \hat{Z}_c + \sum_{x\in\mathcal{S}} \frac{\hat{Z}\,[\mu(\{x\}) + \eta]}{\eta} = \hat{Z}_c + |\mathcal{S}|\hat{Z} + \frac{\hat{Z}}{\eta}$$

since $\sum_{x\in\mathcal{S}} \mu(\{x\}) = \mu(\mathbb{R}) = 1$, where $|\mathcal{S}|$ is number of points of $\mathcal{S}$. Hence,

$$\begin{aligned} \int_{A^c} \rho_\eta(y)dy &= \frac{1}{Z_\eta}\int_{A^c} \frac{e^{q_\eta}}{\sigma_\eta^2} dy = \frac{\eta\hat{Z}_c}{(1+\eta|\mathcal{S}|)\hat{Z} + \eta\hat{Z}_c}, \\ \int_{x-\epsilon}^{x+\epsilon} \rho_\eta(y)dy &= \frac{1}{Z_\eta}\int_{x-2\epsilon}^{x+2\epsilon} \frac{e^{q_\eta}}{\sigma_\eta^2} dy = \frac{\mu\{x\} + \eta}{1 + \eta|\mathcal{S}| + \eta\hat{Z}_c/\hat{Z}} \quad \forall\, x \in \mathcal{S}. \end{aligned} \tag{12.37}$$

Sending $\eta \searrow 0$ we then conclude that $\rho_\eta \longrightarrow \mu$. This completes the proof.

12.3 The Case When σ Vanishes Only Beyond All Critical Points

Here we consider a case based on the prototype $b(u) = (u-\alpha)(1-u^2)$ and $\sigma(u) = (\max\{\ell^2 - u^2, 0\})^\nu$ with $\ell > 1$. Let $[-1, 1]$ be an interval that contains all critical points of the potential, i.e., the set $\{x \in \mathbb{R} \mid b(x) = 0\}$. We consider a degenerate case where σ is positive in an open interval (x_1, x_2) that contains $[-1, 1]$ and vanishes outside (x_1, x_2). It turns out that the limit of the regularized invariant measure does not depend on the modification. Therefore, the limit can be defined as the invariant measure.

Theorem 12.4. *Let b be a continuous function on R that satisfies*

$$b > 0 \text{ in } (-\infty, -1), \quad b < 0 \text{ in } (1, \infty), \quad \varliminf_{|u|\to\infty} |b(u)| > 0. \tag{12.38}$$

Let σ be a continuous function on R such that, for some $x_1 < -1$, $x_2 > 1$ and $\epsilon > 0$,

$$\sigma > 0 \text{ in } (x_1, x_2), \qquad \sigma = 0 \text{ in } [x_1 - \epsilon, x_1] \cup [x_2, x_2 + \epsilon]. \tag{12.39}$$

Let $\rho_\eta(\cdot)$ be defined as in (12.2) where $\{\sigma_\eta\}_{\eta\in(0,1]}$ is any family of continuous positive functions in R that satisfies $\lim_{\eta\searrow 0}\sigma_\eta(\cdot)=\sigma(\cdot)$ uniformly on $[x_1-\epsilon, x_2+\epsilon]$. Then in measure

$$\lim_{\eta\searrow 0}\rho_\eta(x)=\frac{1}{m_1+m_2+m}\left(m_1\delta(x-x_1)+m_2\delta(x-x_2)+\frac{e^{q(x)}}{\sigma^2(x)}\,\chi_{(x_1,x_2)}(x)\right) \tag{12.40}$$

where

$$q(x):=\int_0^x\frac{b(s)}{\sigma^2(x)}\,ds\ \forall\, x\in[x_1,x_2],\quad m_i:=\frac{e^{q(x_i)}}{|b(x_i)|},\quad m:=\int_{x_1}^{x_2}\frac{e^{q(x)}}{\sigma^2(x)}dx. \tag{12.41}$$

Before the proof, we remark on the sizes of m, m_1 and m_2.

Remark. Clearly, m_1, m_2, and m are finite positive constants if $\int_{x_1}^{x_2}\sigma^{-2}(x)dx<\infty$.

Suppose $\int_0^{x_2}\sigma^{-2}(x)dx=\infty$. Then $q(x_2)=-\infty$ since $b(x_2)<0$. This implies that $m_2=0$. In addition, setting

$$\epsilon_1:=\frac{1}{2}\min\{x_2-1,-1-x_1\},\qquad b_*:=\inf_{x\in(-\infty,x_1+\epsilon_1]\cup[x_2-\epsilon_1,\infty)}|b(x)|,$$

we have $b_*>0$ and

$$\int_{x_2-\epsilon_1}^{x_2}\frac{e^{q(x)}}{\sigma^2(x)}dx\leqslant-\frac{1}{b_*}\int_{x_2-\varepsilon}^{y}q'(x)e^{q(x)}dx=\frac{e^{q(x_2-\epsilon_1)}}{b_*}<\infty.$$

Similarly, if $\int_{x_1}^{0}\sigma^{-2}(x)dx=\infty$, then $q(x_1)=-\infty, m_1=0$, and $\int_{x_1}^{x_1+\varepsilon_1}\sigma^{-2}e^{q(x)}dx<\infty$. Thus, *$m$ is a finite positive constant whereas m_1 and m_2 are non-negative finite constants.*

Remark. The limit does not depend on regularization, so the right-hand side of (12.40) can well be defined as the invariant measure ρ^*. Here the contribution of the point masses at x_1 and x_2 is highly non-trivial and has to be obtained through a regularization process.

Proof of Theorem 12.4. **1.** Introduce

$$F_\eta(x):=\int_0^x\frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)}du\quad\forall\, x\in\mathbb{R},\qquad F(x):=\int_0^x\frac{e^{q(u)}}{\sigma^2(u)}du\quad\forall\, x\in[x_1,x_2]. \tag{12.42}$$

By Remark 12.3, F is continuous on $[x_1, x_2]$. Since $\sigma > 0$ in (x_1, x_2) and $\lim_{\eta\searrow 0} \sigma_\eta = \sigma$ uniformly,

$$\lim_{\eta\searrow 0} q_\eta(x) = q(x), \quad \lim_{\eta\to 0} F_\eta(x) = F(x) \quad \forall\, x \in (x_1, x_2). \tag{12.43}$$

2. Next we consider $F_\eta(\infty) - F_\eta(x)$ for $x \in (x_2, \infty)$. Let b_* be as above. For $\beta > 0$,

$$\int_{x_2+\beta}^{\infty} \frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)} du \leqslant \int_{x_2+\beta}^{\infty} \frac{|b(u)|}{b_*\sigma_\eta^2(u)} e^{q_\eta(x)} du = -\frac{1}{b^*} \int_{x_2+\beta}^{\infty} q_\eta'(u) e^{q_\eta(u)} du \leqslant \frac{e^{q_\eta(x_2+\beta)}}{b^*}. \tag{12.44}$$

In addition, since $\lim_{\eta\searrow 0} \sigma_\eta = 0$ in $[x_2, x_2 + \epsilon]$ and $b < 0$ in $(1, \infty)$,

$$\lim_{\eta\searrow 0} q_\eta(x_2 + \beta) = q(1) + \lim_{\eta\searrow 0} \int_1^x \frac{b(s)}{\sigma_\eta^2(s)} ds = -\infty. \tag{12.45}$$

It then follows that

$$\lim_{\eta\searrow 0} \int_{x_2+\beta}^{\infty} \frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)} du = 0 \quad \forall\, \beta > 0. \tag{12.46}$$

3. Now we consider the jump of F_η across x_2. For every $\beta \in (0, \varepsilon_1)$, by the mean value theorem, there exists $x_{\eta,\beta} \in [x_2 - \beta, x_2 + \beta]$ such that

$$\begin{aligned} e^{q_\eta(x_2-\beta)} - e^{q_\eta(x_2+\beta)} &= -\int_{x_2-\beta}^{x_2+\beta} q_\eta'(x) e^{q_\eta(x)} dx = -\int_{x_2-\beta}^{x_1+\beta} \frac{b(x)}{\sigma_\eta^2(x)} e^{q_\eta(x)} dx \\ &= -b(x_{\eta,\beta}) \int_{x_2-\beta}^{x_2+\beta} \frac{e^{q_\eta(x)}}{\sigma_\eta^2(x)} dx = |b(x_{\eta,\beta})| [F_\eta(x_2 + \beta) - F_\eta(x_2 - \beta)]. \end{aligned} \tag{12.47}$$

Thus, for any fixed $x > x_2$, in view of (12.46), (12.45) and (12.43), we have

$$\begin{aligned} \overline{\lim_{\eta\searrow 0}}\, F_\eta(x) &= \lim_{\beta\searrow 0} \overline{\lim_{\eta\searrow 0}}\, F_\eta(x_2 + \beta) \\ &= \lim_{\beta\searrow 0} \overline{\lim_{\eta\searrow 0}} \left(F_\eta(x_2 - \beta) + \frac{e^{q_\eta(x_2-\beta)} - e^{q_\eta(x_2+\beta)}}{|b(x_{\eta,\beta})|} \right) \\ &= \lim_{\beta\searrow 0} \left(F(x_2 - \beta) + \frac{e^{q(x_2-\beta)}}{\underline{\lim}_{\eta\searrow 0} |b(x_{\eta,\beta})|} \right) = F(x_2) + \frac{e^{q(x_2)}}{|b(x_2)|} = F(x_2) + m_2. \end{aligned} \tag{12.48}$$

Similarly, we derive $\underline{\lim}_{\eta\searrow 0} F_\eta(x) = F(x_2) + m_2$. Hence, $\lim_{\eta\searrow 0} F_\eta(x) = F(x_2) + m_2$ for each $x > x_2$.

In summary, we have

$$\lim_{\eta \searrow 0} F_\eta(x) = \begin{cases} F(x) & \text{if } \ x \in (x_1, x_2), \\ m_2 + F(x_2) & \text{if } \ x > x_2. \end{cases} \tag{12.49}$$

After a similar analysis for $F_\eta(x)$ for $x \leqslant x_1$, we then obtain the assertion of the Theorem.

Remark. Following the same proof, one sees that Theorem **12.3** remains true if the assumptions are replaced by the following more applicable conditions:

$$\lim_{\eta \searrow 0} \int_{x_1+\epsilon}^{x_2-\epsilon} \frac{du}{\sigma_\eta^2(u)} = \int_{x_1+\epsilon}^{x_2-\epsilon} \frac{du}{\sigma^2(u)} < \infty = \int_{x_1-\epsilon}^{x_1} \frac{du}{\sigma^2(u)} = \int_{x_2}^{x_2+\varepsilon} \frac{du}{\sigma^2(u)} du \quad \forall \, \epsilon \in (0, \tfrac{x_2-x_1}{2}). \tag{12.50}$$

For example, $\sigma(u) = \sqrt{|(u-x_1)(x_2-u)|}\, |\sin(2\pi u)|^{1/4}$ with $x_1 < -1$ and $x_2 > 1$.

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Appendix A

Hopf-Lax Style Existence and Uniqueness of Solution

By relating the conservation law to the corresponding Hamilton-Jacobi problem, it is possible to prove both uniqueness and existence of such a solution. The argument involves using mollification techniques and analysis, as well as a solution candidate from the conservation law. We have the following.

Theorem. Let H be a piecewise linear function with a finite number of break points, and g' be a step function with a finite number of steps, and fix $T > 0$. Consider the PDE given by

$$\begin{aligned} w_t + \left(H\left(w\right)\right)_x &= 0 \text{ on } \mathbb{R}\times\left(0, \infty\right) \\ w &= g' \text{ on } \mathbb{R}\times\left\{0\right\}. \end{aligned}$$

Then there exists an integral solution $w \in L^{\infty}\left(\mathbb{R}\times[0, T)\right)$ which satisfies

$$\int_0^{\infty}\int_{-\infty}^{\infty} w\psi_t + H\left(w\right)\psi_x dxdt + \int_{-\infty}^{\infty} w\psi dx|_{t=0} = 0$$

for each test function ψ where $\psi : \mathbb{R}\times[0, T) \to \mathbb{R}$ is smooth with compact support. Furthermore, this solution is unique.

A.1 Proving existence of a solution

I. Clarification of sufficient conditions for existence in Evans text

We go over Lemma 1 in Section 3.3 without the assumption of superlinearity on the flux function on H. This means that some quantities in L will be infinite, but this will not matter since we are then taking the minimum which is not affected by the infinite values.

Also, the Lemma 1 in Evans depends on part of Lemma 2, and we write this in a more coherent manner using Lemma A, B etc.

The first thing to do is to make sure that we have the key theorem that H and L are Legendre transforms of one another. We do not need to use any of the theorems in Evans that rely on superlinearity. We only assume that H is Lipschitz continuous, which is obvious from the definition. We also assume that g (which will be the initial condition for the Hamilton-Jacobi equation) is Lipschitz on $\mathbb{R}$.

In all of the theorems we start with the assumption that $H(q)$ is polygonal convex with the last segments having slopes m_1 at the left and m_{N+1} at the right, with $c_1 < c_2 < ... < c_N$ at the break points with $c_1 < 0 < c_N$. Assume $m_1 < 0 < m_{N+1}$.

Definition. We define the usual Legendre transform, and let $L(p)$ denote it:

$$L(p) := H^*(q) := \sup_{q\in\mathbb{R}} \{pq - H(q)\} \tag{A.1}$$

A computation shows that this is a convex polygonal shape such that $L(p) < \infty$ iff $p \in (m_1, m_N + 1)$. It has break points at $m_1 < m_2 < ... < m_{N+1}$ and slopes $c_1, c_2, .., c_N$. The last break point is at m_{N+1} where the slope and $L(m_{N+1})$ become infinite. Note that L is Lipschitz on (m_1, m_{N+1}).

Prop. P1. Let $L(p)$ be as defined above (with $L(p) < \infty$ iff $p \in (m_1, m_{N+1})$). Then the Legendre transform (defined above) of $L(p)$ is

$$L^*(q) := \sup_{p\in\mathbb{R}} \{pq - L(q)\} = H(q). \tag{A.2}$$

Remark. In other words, if we define $L(p)$ as polygonal, convex between $p \in (m_1, m_{N+1})$ as in the figure, with $L(p) = \infty$ for $p \notin (m_1, m_{N+1})$ then the operation $\sup_{p\in\mathbb{R}} \{pq - L(q)\}$ yields the function H defined above. This is all that one needs in order to prove (below and in Evans, Theorem 5 or 6) that if u is defined by the Hopf-Lax formula $u(x,t) := \min_{y\in\mathbb{R}} \left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}$, then it satisfies the Hamilton-Jacobi equation, $u_t + H(Du) = 0$ at points where u is differentiable.

Lemma A. Suppose L and g are Lipschitz continuous and $L(\cdot) < \infty$ only on (m_1, m_{N+1}). Then u defined by the Hopf-Lax formula is Lipschitz continuous in x.

Proof. Fix $t > 0$, $x, \hat{x} \in \mathbb{R}$. Choose $y \in \mathbb{R}$ (depending on x, t) such that

$$u(x,t) = \min_z \left\{ tL\left(\frac{x-z}{t}\right) + g(z) \right\} = tL\left(\frac{x-y}{t}\right) + g(y). \tag{A.3}$$

The minimum is attained since both L and g are continuous. Note that while there may be values of $(x-z)/t$ such that $L((x-z)/t) = \infty$, these are irrelevant, as there are some finite values, and $L((x-y)/t)$ will be one of those. Now use (1) to write

$$u(\hat{x},t) - u(x,t) = \inf_{\tilde{z}} \left\{ tL\left(\frac{\hat{x}-\tilde{z}}{t}\right) + g(\tilde{z}) \right\} - tL\left(\frac{x-y}{t}\right) - g(y) \ . \tag{A.4}$$

We define $z := \hat{x} - x + y$ such that

$$\frac{x-y}{t} = \frac{\hat{x}-z}{t} \tag{A.5}$$

and substitute this z in place of $\tilde{z}$ in (A.4) which can only increase the RHS. This yields the inequality

$$\begin{aligned} u(\hat{x},t) - u(x,t) &\leq tL\left(\frac{\hat{x}-z}{t}\right) + g(z) - tL\left(\frac{x-y}{t}\right) - g(y) \\ &= tL\left(\frac{x-y}{t}\right) + g(\hat{x}-x+y) \ - tL\left(\frac{x-y}{t}\right) - g(y) \\ &= g(\hat{x}-x+y) - g(y). \end{aligned}$$

Using the assumption that g is Lipschitz, one has

$$u(\hat{x},t) - u(x,t) \leq Lip(g) \cdot |\hat{x} - x| \ . \tag{A.6}$$

Note that in obtaining this inequality, x and $\hat{x}$ were arbitrary (without any assumption on order). Hence, we can interchange them. I.e., we start by defining y such that, instead of (1), it satisfies

$$u(\hat{x},t) = tL\left(\frac{\hat{x}-y}{t}\right) + g(y).$$

Thus, we obtain the same inequality as (A.6) with the x and $\hat{x}$ interchanged, yielding

$$|u(\hat{x},t) - u(x,t)| \leq Lip(g) \cdot |\hat{x} - x| \ . \tag{5}$$

Lemma 1. Suppose L and g are Lipschitz continuous and $L(\cdot) < \infty$ only on (m_1, m_N). For each $x \in \mathbb{R}$ and $0 \leq s < t$ we have

$$u(x,t) = \min_{y \in \mathbb{R}} \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}.$$

Proof of Lemma 1. Part 1. Fix $y \in \mathbb{R}$, $0 < s < t$. Since u and L are continuous, the minimum in the definition of $u(x,t)$ is attained on the interval (m_1, m_{N+1}) where L is finite. Thus we can find $z \in \mathbb{R}$ such that

$$u(y,s) = sL\left(\frac{y-z}{s}\right) + g(z). \tag{A.7}$$

Note that since $(y-z)/s$ is the minimizer, we know that $L((y-z)/s)$ is finite.

By convexity of L we can write (as in Evans) using

$$\frac{x-z}{t} = \left(1 - \frac{s}{t}\right)\left(\frac{x-y}{t-s}\right) + \frac{s}{t}\left(\frac{y-z}{s}\right) \tag{A.8}$$

$$L\left(\frac{x-z}{t}\right) \leq \left(1 - \frac{s}{t}\right) L\left(\frac{x-y}{t-s}\right) + \frac{s}{t} L\left(\frac{y-z}{s}\right). \tag{A.9}$$

Note that $L((y-z)/s)$ is finite, and, from the geometry of convexity it is clear that for any fixed $y \in \mathbb{R}$, $0 < s < t$ we have

$$L\left(\frac{x-z}{t}\right) = \infty \ \ iff \ \ L\left(\frac{x-y}{t-s}\right) = \infty. \tag{A.10}$$

Next, we have from our basic assumption that $u(x,t)$ is defined by the Hopf-Lax formula, the identity

$$u(x,t) = \min_{\hat{z}} \left\{ tL\left(\frac{x-\hat{z}}{t}\right) + g(\hat{z}) \right\} \tag{A.11}$$

so substituting the z defined above in (A.7) yields the inequality

$$u(x,t) \leq tL\left(\frac{x-z}{t}\right) + g(z) \tag{A.12}$$

and now using (A.9) yields

$$u(x,t) \le (t-s)\, L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-z}{s}\right) + g(z)\,. \tag{A.13}$$

Now note that the last two terms, by (A.7) are $u(y,s)$. This yields

$$u(x,t) \le (t-s)\, L\left(\frac{x-y}{t-s}\right) + u(y,s)\,. \tag{A.14}$$

Now we take the minimum over all $y \in \mathbb{R}$. Note that y has been arbitrary. As we take the minimum over y we note that there are values of y for which **??** is infinite, but there are also values for which it is finite. Thus in taking the minimum, the values for which it is infinite are irrelevant. Hence, we can write

$$u(x,t) \le \min_{y\in\mathbb{R}} \left\{ (t-s)\, L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}. \tag{A.15}$$

Part 2. Next, we know again that there exists $w \in \mathbb{R}$ (depending on x and t that we regard as fixed) such that

$$u(x,t) = tL\left(\frac{x-w}{t}\right) + g(w)\,. \tag{A.16}$$

We choose $y := \frac{s}{t}x + \left(1-\frac{s}{t}\right)w$, which implies

$$\frac{x-y}{t-s} = \frac{x-w}{t} = \frac{y-w}{s}. \tag{A.17}$$

We know that w is the minimizer (and of course, g is finite in the domain (m_1, m_N)) so that $L((x-w)/t)$ is finite. Thus, by the identity (A.17) above, so are $L((x-w)/t)$ and $L((y-w)/s)$. Thus we can write, using the basic definition of $u(y,s)$, the equality

$$\begin{aligned} (t-s)\, L\left(\frac{x-y}{t-s}\right) + u(y,s) &= (t-s)\, L\left(\frac{x-y}{t-s}\right) + \min_{\hat{z}} \left\{ sL\left(\frac{y-\hat{z}}{x}\right) + g(\hat{z}) \right\} \\ &\le (t-s)\, L\left(\frac{x-y}{t-s}\right) + \left\{ sL\left(\frac{y-w}{x}\right) + g(w) \right\} \end{aligned} \tag{A.18}$$

where the inequality is obtained simply by substituting a particular value for $\hat{z}$, namely the w that we defined above in (A.16).

We can use (A.17) in order to re-write the arguments of L in equivalent forms. By the equality and the fact that $L((x-w)/t)$ is finite, so are so are $L((x-w)/t)$ and $L((y-w)/s)$. Hence, replacing the two expressions involving L on the RHS of (A.18) yields

$$\begin{aligned}(t-s)\,L\left(\frac{x-y}{t-s}\right)+u\,(y,s) &\leq (t-s)\,L\left(\frac{x-w}{t}\right)+\left\{sL\left(\frac{x-w}{t}\right)+g\,(w)\right\} \\ &= tL\left(\frac{x-w}{t}\right)+g\,(w)=u\,(x,t) \qquad (8)\end{aligned}$$

where the last identity follows from the expression (A.16) that defines w. Thus (A.18) gives us an identity for a particular y that we defined, namely

$$(t-s)\,L\left(\frac{x-y}{t-s}\right)+u\,(y,s)\leq u\,(x,t)\,. \qquad \text{(A.19)}$$

If we replace y be a minimum over all $\hat{y}$ we obtain the inequality

$$\min_{\hat{y}}\left\{(t-s)\,L\left(\frac{x-y}{t-s}\right)+u\,(y,s)\right\}\leq u\,(x,t)\,. \qquad \text{(A.20)}$$

Combining (A.15) and (A.20) proves Lemma 1. ///

Lemma B (This is the analog of Lemma 2 - proof in part 2 of Evans. Part 2 is essentially the same; one needs only pay attention to finiteness of the terms)

Under the same conditions (L and g Lipschitz continuous), one has $u\,(x,0)=g\,(x)\,.$

Proof. Since $0\in(m_1,m_{N+1})$, the interval on which L is finite, one has

$$u\,(x,t)=\min_{y}\left\{tL\left(\frac{x-y}{t}\right)+g\,(y)\right\}\leq tL\,(0)+g\,(x)\,, \qquad \text{(A.21)}$$

upon choosing $y=x$. Also,

$$\begin{aligned}u\,(x,t) &= \min_{y}\left\{tL\left(\frac{x-y}{t}\right)+g\,(y)\right\} \\ &\geq \min_{y}\left\{tL\left(\frac{x-y}{t}\right)+g\,(x)-Lip\,(g)\cdot|x-y|\right\} \\ &= g\,(x)+\min_{y}\left\{tL\left(\frac{x-y}{t}\right)-Lip\,(g)\cdot|x-y|\right\} \\ &= g\,(x)-t\max_{y}\left\{Lip\,(g)\cdot\frac{|x-y|}{t}-L\left(\frac{x-y}{t}\right)\right\} \\ &= g\,(x)-t\max_{z}\left\{|z|\,Lip\,(g)-L\,(z)\right\}. \qquad \text{(A.22)}\end{aligned}$$

Note that $\max_z \{|z| \, Lip\,(g) - L\,(z)\}$ is a finite number since $-L\,(z)$ is bounded above and is $-\infty$ outside of the range (m_1, m_N). Thus we define

$$C := \max\left\{|L\,(0)|\,, \max_z \{|z| \, Lip\,(g) - L\,(z)\}\right\} \tag{A.23}$$

and combine (A.21) and (A.22) to write

$$|u\,(x,t) - g\,(x)| \le Ct \;\text{ for all } x \in \mathbb{R} \text{ and } t > 0. \;\; /// \tag{A.24}$$

Lemma C (This is the analog of Lemma 3– proof in part 3 of Evans. This is just sketched out, even in the case he considers). If L and g are Lipschitz, one has for any $x \in \mathbb{R}$ and $0 < \hat{t} < t$, the inequalities,

$$\left|u\,(x,t) - u\left(x,\hat{t}\right)\right| \le C\left|t - \hat{t}\right|$$

$$C := \max\left\{|L\,(0)|\,, \quad \max_z \{|z| \, Lip\,(g) - L\,(z)\}\right\}.$$

Proof. Let $x \in \mathbb{R}$ and $0 < \hat{t} < t$. By Lemma A one has

$$|u\,(x,t) - u\,(\hat{x},t)| \le Lip\,(g)\,|x - \hat{x}|$$

i.e., $Lip\,(u\,(\cdot,t)) = Lip\,(g)$. From Lemma 1, we have for $0 \le s = \hat{t} < t$, the inequality

$$\begin{aligned}
u\,(x,t) &= \min_y \left\{(t-s)\,L\left(\frac{x-y}{t-s}\right) + u\,(y,s)\right\} \\
&\ge \min_y \left\{(t-s)\,L\left(\frac{x-y}{t-s}\right) + u\,(x,s) - Lip\,(u)\cdot|x-y|\right\} \\
&= u\,(x,s) + \min_y \left\{(t-s)\,L\left(\frac{x-y}{t-s}\right) - Lip\,(g)\cdot|x-y|\right\} \\
&= u\,(x,s) + (t-s)\min_y \left\{L\left(\frac{x-y}{t-s}\right) - Lip\,(g)\cdot\frac{|x-y|}{t-s}\right\} \\
&= u\,(x,s) + (t-s)\min_z \{L\,(z) - Lip\,(g)\cdot|z|\}
\end{aligned} \tag{A.25}$$

where we have just defined $z := |x-y|\,/\,(t-s)$. We can also write this as

$$u\,(x,t) \ge u\,(x,s) - (t-s)\max_z \{Lip\,(g)\cdot|z| - L\,(z)\}\,. \tag{A.26}$$

Using $C_1 := \max_z \{Lip(g) \cdot |z| - L(z)\}$ which, as discussed above, is clearly finite, one has then

$$u(x,t) - u(x,s) \geq -C_1 (t-s). \tag{A.27}$$

The other direction in the inequality is given by using Lemma 1 directly. Substituting x in place of the y in the minimizer, we have

$$\begin{aligned} u(x,t) &= \min_y \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\} \\ &\leq (t-s) L\left(\frac{x-x}{t-s}\right) + u(x,s) \\ &= (t-s) L(0) + u(x,s) \end{aligned} \tag{A.28}$$

yielding the inequality

$$u(x,t) - u(x,s) \leq (t-s) L(0). \tag{A.29}$$

Combining (A.28) and (A.29) yields the Lipschitz inequality in t, namely,

$$|u(x,t) - u(x,s)| \leq C |t-s| \ . \tag{A.30}$$

Lemma D. Under the conditions of Lemmas A and C one has

$$\left|u\left(\hat{x},\hat{t}\right) - u(x,t)\right| \leq C \left\| \left(\hat{x},\hat{t}\right) - (x,t) \right\|_2 \tag{A.31}$$

where $\|\cdot\|_2$ is the usual Euclidean norm.

Proof. We use the triangle inequality and Lemmas A and C, one has

$$\begin{aligned} \left|u\left(\hat{x},\hat{t}\right) - u(x,t)\right| &\leq \left|u\left(\hat{x},\hat{t}\right) - u\left(x,\hat{t}\right)\right| + \left|u\left(x,\hat{t}\right) - u(x,t)\right| \\ &\leq C|x-\hat{x}| + C\left|t-\hat{t}\right| \\ &\leq 2C\left\{|x-\hat{x}|^2 + \left|t-\hat{t}\right|^2\right\} = 2C\left\|\left(\hat{x},\hat{t}\right) - (x,t)\right\|_2 \ . \end{aligned} \tag{A.32}$$

Lemma E. If L and g are Lipschitz continuous then $u : \mathbb{R}^2 \to \mathbb{R}$ is differentiable almost everywhere on $\mathbb{R}\times(0,\infty)$.

Proof. Follows from Rademacher's theorem and Lemma D.

Theorem 5 (Modified Evans' Sec. 3.3) Let $x \in \mathbb{R}$ and $t > 0$. Let u be defined by the Hopf-Lax formula and differentiable at $(x,t) \in \mathbb{R}\times(0,\infty)$. Then

$$u_t(x,t) + H(Du(x,t)) = 0. \tag{A.33}$$

Proof. Part 1. Fix $q \in (m_1, m_{N+1})$ and $h > 0$. By Lemma 1, we have

$$u(x+hq, t+h) = \min_{y\in\mathbb{R}} \left\{ hL\left(\frac{x+hq-y}{h}\right) + u(y,t) \right\}. \tag{A.34}$$

(Once again since there are some finite values over which we are taking the minimum, the expression is well-defined). Upon setting y as x, we can only obtain a larger quantity on the RHS, yielding

$$u(x+hq, t+h) \leq hL(q) + u(x,t). \tag{A.35}$$

Hence, we have that for $q \in (m_1, m_{N+1})$ the inequality

$$\frac{u(x+hq,t+h) - u(x,t)}{h} \leq L(q). \tag{A.36}$$

Since we are assuming that u is differentiable at (x,t) we have the existence of the limit of the LHS of (A.36) thereby yielding

$$\partial_t u(x,t) + qDu(x,t) \leq L(q), \ \textit{i.e.},$$

$$\partial_t u(x,t) + qDu(x,t) - L(q) \leq 0. \tag{A.37}$$

We now use the definition, $\sup_{q\in\mathbb{R}}\{qw - L(q)\} =: H(w)$ that we made earlier (note that we are taking the sup over q and not p now), and write

$$H(Du(x,t)) = \sup_{q\in\mathbb{R}}\{qDu - L(q)\}. \tag{A.38}$$

Note that the values of q for which $L(q) = \infty$ are clearly not candidates for the $\sup_p$ since $-L(q) = -\infty$. Hence, we can take the sup over all q in (2), (which is equivalent to taking the sup over $q \in (m_1, m_{N+1})$) to obtain

$$\partial_t u(x,t) + \sup_{q \in \mathbb{R}} \{\partial_t u(x,t) + qDu(x,t) - L(q)\} \leq 0, \quad or,$$
$$\partial_t u(x,t) + H(u(x,t)) \leq 0. \tag{A.39}$$

Part 2. Now use the definition

$$u(x,t) = \min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

Since L and g are continuous, the minimizer exists and for some $z \in \mathbb{R}$ depending on (x,t) we have

$$u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z). \tag{A.40}$$

Define $s := t-h$, $y := \frac{s}{t}x + \left(1 - \frac{s}{t}\right)z$ so $\frac{x-z}{t} = \frac{y-z}{s}$. Then we can write, using the definition of $u(y,s)$,

$$u(x,t) - u(y,s) = tL\left(\frac{x-z}{t}\right) + g(z) - \min_{\hat{y}} \left\{ sL\left(\frac{y-\hat{y}}{s}\right) + g(\hat{y}) \right\}. \tag{A.41}$$

By substituting z (defined by (A.40)) in place of $\hat{y}$ in this expression, we subtract out at least as much and obtain the inequality

$$\begin{aligned} u(x,t) - u(y,s) &\geq tL\left(\frac{x-z}{t}\right) + g(z) - \left\{ sL\left(\frac{y-z}{s}\right) + g(z) \right\} \\ &= (t-s)L\left(\frac{x-z}{t}\right) \end{aligned} \tag{A.42}$$

by virtue of the equality $\frac{x-z}{t} = \frac{y-z}{s}$. Note that by definition, $L\left(\frac{x-z}{t}\right) = L\left(\frac{y-z}{s}\right) < \infty$, so there is no divergence problem there. Now replace y with its definition above, and use $t-s=h$ to write (5) as

$$\frac{u(x,t) - u\left(x + \frac{h}{t}(z-x), t-h\right)}{h} \geq L\left(\frac{x-z}{t}\right). \tag{A.43}$$

Since we are assuming that the derivative exists at (x,t) we can take the limit as $h \to 0+$ and obtain

$$\frac{z-x}{t} Du(x,t) + \partial_t u(x,t) \geq L\left(\frac{x-z}{t}\right) \tag{A.44}$$

We use the definition of H again, and write

$$\begin{aligned} u_t(x,t) + H(Du(x,t)) &= u_t(x,t) + \max_{q\in\mathbb{R}} \{qDu - L(q)\} \\ &\geq u_t(x,t) + \frac{z-x}{t} Du(x,t) - L\left(\frac{x-z}{t}\right) \end{aligned} \tag{A.45}$$

where we have chosen one value of q, namely $(x-z)/t$ to obtain the inequality. Note, again, that from the original definition in (4), $L\left(\frac{x-z}{t}\right)$ must be finite. Hence, the RHS of (8) is well-defined. Combining (7) and (8) yields the inequality

$$\partial_t u(x,t) + H(Du(x,t)) \geq 0. \tag{A.46}$$

Combining (A.46) with the result from Part 1 of the proof, namely (A.40), we obtain the result that u satisfies the Hamilton-Jacobi equation at (x,t). ///

Thus, Theorem 5 in Evans (Section 3.3) is fine as stated for the polynomial H.

Theorem 6. The function $u(x,t)$ defined by the Hopf-Lax formula is Lipschitz continuous, differentiable a.e. in $\mathbb{R}\times(0,\infty)$ and solves the initial value problem

$$\begin{aligned} u_t + H(Du) = 0 \quad &a.e.\ in\ \mathbb{R}\times(0,\infty) \\ u(x,0) = g(x) \quad &for\ all\ x\in\mathbb{R}\ . \end{aligned} \tag{A.47}$$

Definition. We say that $w \in L^\infty(\mathbb{R}\times(0,\infty))$ is an **integral solution** of

$$\begin{aligned} w_t + H(w)_x = 0\ in\ \mathbb{R}\times(0,\infty) \\ w(x,0) = h(x)\ \ for\ all\ x\in\mathbb{R} \end{aligned} \tag{A.48}$$

if for all test functions $\phi : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ (i.e., ϕ that are smooth and have compact support) one has the identity

$$\int_0^\infty \int_{-\infty}^\infty w\phi_t dxdt + \int_{-\infty}^\infty w\phi dx|_{t=0} + \int_0^\infty \int_{-\infty}^\infty H(w)\phi_x dxdt = 0. \tag{A.49}$$

Theorem 2 (Section 3.4 of Evans, but statement of the theorem is somewhat different) Under the assumptions that g is Lipschitz and that H is polygonal and convex (as described above), the function $w(x,t) := \partial_x u(x,t)$ where u is the Hopf-Lax function is an integral solution of the initial value problem for the conservation law above.

Proof. From Theorem 6 above we know that u is Lipschitz continuous, differentiable a.e. in $\mathbb{R}\times(0,\infty)$ and solves the Hamilton-Jacobi initial value problem subject to initial condition $h(x)$ where $g(x) = \int_0^x h(z)\,dz$. We multiply $u_t + H(u_x) = 0$ with a test function ϕ_x and integrate over $\int_0^\infty \int_{-\infty}^\infty ...dxdt$. Upon integrating by parts one obtains the relation above in the definition of integral solution. The integration by parts operations are justified by the fact that $u(x,t)$ is Lipschitz in both x and t. Also,

$$w(x,0) = \partial_x u(x,0) = g'(x) = h(x). \tag{A.50}$$

A.2 Uniqueness

Suppose that we have two integral solutions, w and $\tilde{w}$, to the conservation law IVP:

$$w_t + H(w)_x = 0, \quad w(x,0) = g'(x). \tag{A.51}$$

I.e., these satisfy the integral identity that we obtain by multiplying by the derivative of a test function, ϕ_x.

We also assume that they satisfy the condition

$$w(x+z,t) - w(x,t) \le C\left(1 + \frac{1}{t}\right) z. \tag{A.52}$$

We define $u(x,t) := \int_0^x w(\tilde{x},t)\,d\tilde{x}$ and $\tilde{u}(x,t) := \int_0^x \tilde{w}(\tilde{x},t)\,d\tilde{x}$. We have seen that u and $\tilde{u}$ will then be solutions to the Hamilton-Jacobi equations a.e. in (x,t).

We then take the equations and mollify them in $\mathbb{R}^2$, i.e.,

$$(u_t)_\varepsilon + (H(u_x))_\varepsilon = 0, \quad (\tilde{u}_t)_\varepsilon + (H(\tilde{u}_x))_\varepsilon = 0. \tag{A.53}$$

Note that $(u_t)_\varepsilon = (u_\varepsilon)_t$ and similarly for x.

Let $v_\varepsilon(x,t) := \tilde{u}_\varepsilon(x,t) - u_\varepsilon(x,t)$ so that

$$\partial_t v_\varepsilon = (H(u_x))_\varepsilon - (H(\tilde{u}_x))_\varepsilon\ ,\ v_\varepsilon(x,0) = 0. \tag{A.54}$$

We add and subtract terms to write

$$\begin{aligned} \partial_t v_\varepsilon &= H^\varepsilon(u_{\varepsilon,x}) + \{(H(u_x))_\varepsilon - H^\varepsilon(u_{\varepsilon,x})\} && \text{(A.55)} \\ &\quad - H^\varepsilon(\tilde{u}_{\varepsilon,x}) - \{(H(\tilde{u}_x))_\varepsilon - H^\varepsilon(\tilde{u}_{\varepsilon,x})\} \\ &=: H^\varepsilon(u_{\varepsilon,x}) - H^\varepsilon(\tilde{u}_{\varepsilon,x}) + R_1(\varepsilon) + R_2(\varepsilon) && \text{(A.56)} \end{aligned}$$

By the usual theorems on mollification, we note that R_1 and R_2 converge (uniformly on compact sets in (x,t)) to zero as $\varepsilon \to 0$.

We define $Z_\varepsilon(x,t)$ by leaving out the R_1 and R_2 terms:

$$\partial_t Z_\varepsilon = H^\varepsilon(u_{\varepsilon,x}) - H^\varepsilon(\tilde{u}_{\varepsilon,x}), \quad Z_\varepsilon(x,0) = 0. \tag{A.57}$$

We then use the arguments of Evans (Section 3.3 Theorem 7). Note that these use convexity, but not uniform convexity or superlinearity. Also, (1) implies, for small $z > 0$,

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \le C\left(1 + \frac{1}{t}\right) z^2, \tag{A.58}$$

which is the key condition needed for uniqueness. The inequality is inherited by the mollified functions u_ε and $\tilde{u}_\varepsilon$. Note that all we need for the proof is for (A.58) to hold for small, positive z, since we use it to bound smoothed out derivatives. To establish that (A.52) implies (A.58) for small, positive z, note the following. By definition of w one has from (A.52),

$$\partial_x u(x+z) - \partial_x u(x) \le C\left(1 + \frac{1}{t}\right) z. \tag{A.59}$$

For sufficiently small $h > 0$, one must have for some C'

$$\frac{u(x+z) - u(x+z-h)}{h} - \frac{u(x) - u(x-h)}{h} \le C'\left(1 + \frac{1}{t}\right) z. \tag{A.60}$$

Setting z at h, and dividing by h, one has

$$\frac{u\left(x+h\right)-2u\left(x\right)+u\left(x-h\right)}{h^{2}}C'\left(1+\frac{1}{t}\right)h. \tag{A.61}$$

Hence, when (A.52) applies, we can use (A.58) at least for small $h>0$.

Thus, we can use the proof of Evans identically on (A.57), noting that H is convex and u_ε and $\tilde{u}_\varepsilon$ satisfy (5) . This allows us to conclude that $Z_\varepsilon\left(x,t\right)=0$ and so $\partial_t Z_\varepsilon\left(x,t\right)=0$ both on $\mathbb{R}\times[0,\infty)$. From (4) this means that $H^\varepsilon\left(u_{\varepsilon,x}\right)-H^\varepsilon\left(\tilde{u}_{\varepsilon,x}\right)=0$. We can then use this in (A.56) to obtain

$$v_t\left(x,t\right)=R_1\left(\varepsilon\right)+R_2\left(\varepsilon\right), \quad v\left(x,0\right)=0. \tag{A.62}$$

From the uniform convergence properties of R_i we conclude that

$$\left|v\left(x,t\right)\right|\leq CT\varepsilon, \quad on \;\; \mathbb{R}\times\left[0,T\right]. \tag{A.63}$$

Since this difference can be made arbitrarily small, we have uniqueness. Another approach to proving these results is to follow [21].

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