The Loewner equation with branching and the continuum random tree

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics at Brown University

PROVIDENCE, RHODE ISLAND

May 2017

This dissertation by Vivian Olsiewski Healey is accepted in its present form
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Acknowledgements

First and foremost, I would like to thank my advisor, Govind Menon, for his invaluable help during my time at Brown. Govind, thank you for encouraging me to always follow my interests and never shy away from hard problems. Even more, thank you for your patience and for always believing in me.

I am deeply grateful to Steffen Rohde for the help and guidance he offered during my visit to the University of Washington and for the many productive conversations that helped get my project off the ground. Many thanks to Brent Werness for making the simulation shown in Figure 1.1 that gave the proof of concept for the tree embedding explored in this work. I also thank Richard Kenyon for discussions related to this work and for his service on my thesis committee.

Finally, on a personal note, I would like to thank my family, Paula, John, and Georgia, for their steadfast love and support and Susan for her endless hospitality and encouragement. Thank you to my friends from Brown, especially Liz, who brought joy to the hard times of graduate school. And most of all, thank you to my spouse and fellow mathematician, Wade, for accompanying me on this crazy journey.

Abstract of "The Loewner equation with branching and the continuum random tree" by Vivian Olsiewski Healey, Ph.D., Brown University, May 2017

Abstract

The present work brings together the fields of random maps and Loewner evolution by constructing explicit embeddings of critical Galton-Watson trees in the upper half-plane via the Loewner equation and considering the scaling limit of the associated time-dependent random driving measures as the finite trees converge to the continuum random tree. Chapter 2 addresses the (deterministic) conformal mapping problem of incorporating branching into the Loewner equation. We identify sufficient conditions on the driving measure for the Loewner equation to generate a union of two simple curves that meet at a fixed nontrivial angle on the real line, which is the fundamental step in generating graph embeddings of trees. Chapter 3 identifies a specific repulsive force (the deterministic part of Dyson Brownian motion) that, when used to describe the evolution of a random discrete measure whose atoms represent the particles of a Galton-Watson branching process, satisfies the conditions for tree embedding given in Chapter 2. Chapter 4 investigates the scaling limit of these time-dependent driving measures through the lens of superprocesses. In the setting when the critical Galton-Watson trees are conditioned to converge to the continuum random tree, the sequence of measure-valued processes is shown to be tight. In order to identify the limit, the question of convergence of the sequence of measures is reframed as a question concerning the associated sequence of Stieltjes transforms. For each measure-valued process in the sequence, the flow of the associated Stieltjes transform is shown to satisfy a particular SPDE that is related to the complex Burgers equation. Finally, in the unconditioned case, the density ρ of the limiting superprocess is conjectured to satisfy the equation $\partial_t \rho + \partial_x (\rho \cdot \mathcal{H}\rho) = \sigma\sqrt{\rho} \cdot \dot{W}$, where $\mathcal{H}$ is the Hilbert transform, $\dot{W}$ is space-time white noise, and σ is a positive constant.

Contents

Vitae **vii**

Acknowledgments **ix**

1 Introduction **1**

1.1 Overview . . . 2

1.2 The Continuum Random Tree . . . 8

1.3 The Loewner Equation . . . 13

2 Branching in the Loewner Equation **17**

2.1 Explicit conformal map calculation . . . 18

2.1.1 Fixing the preimage x of 0 in terms of a and b. . . . 23

2.2 A condition for branching . . . 26

2.2.1 Approach in (α, β)-direction . . . 26

2.2.2 A sufficient condition on the driving measure for (α, β)-approach . . 29

3 A Natural Tree Embedding **49**

3.1 Choosing the diffusion . . . 50

3.2 The tree embedding theorem . . . 51

4 The scaling limit of the driving measure **66**

4.1 Preliminaries: the driving measure as a superprocess . . . 70

4.2 Tightness of the sequence $\{\mu^k\}_{k\geq 1}$ . . . 74

4.3 Identifying the limit using the Stieltjes transform . . . 83

4.3.1 An SPDE for the flow of the Stieltjes transform . . . 83

4.3.2 A conjectural limiting equation . . . 91

4.3.3 The boundary SPDE . . . 95

A Additional estimates used in Chapter 2 **99**

Chapter One

Introduction

1.1 Overview

The continuum random tree, introduced by Aldous in [Ald91] and [Ald93], is a random metric space that arises as the scaling limit of many different finite tree processes, including the uniform distribution on rooted ordered trees (called plane trees) with n edges and critical discrete time Galton-Watson trees conditioned to have size n. Random plane trees are a specific instance of random planar maps (random graphs embedded in the sphere or plane up to orientation-preserving homeomorphisms) which provide models of two-dimensional random geometries and possess many links to random matrix theory, including those found in [BIZ80] and [Oko00]. Recently, it has been shown that there is a unique, universal scaling limit for large classes of random planar maps, which is called the Brownian map ([Mie13] and [LG13], or for an overview see [LGM12]). However, the continuum random tree and the Brownian map are not planar maps themselves, but rather metric spaces, so it is natural to ask how these may be embedded in the sphere or the plane. This embedding problem has been approached from a number of different directions, including the recent work of Miller and Sheffield uniting the theories of Liouville quantum gravity and the Brownian map ([MS15] [MS16a] [MS16b]) as well as from the point of view of conformally balanced trees as defined in [Bis14] whose scaling limits are investigated in [Bar14].

This work takes a different approach, constructing explicit embeddings of finite Galton-Watson trees in the half-plane using Loewner evolution with the goal of finding their geometric scaling limit. These embedded Galton-Watson trees are constructed as the hulls generated by the Loewner equation driven by discrete time-dependent driving measures that are indexed by Galton-Watson trees and have a specific power law repulsion between their point masses.

The (chordal) Loewner equation (1.1), originally proposed by Loewner in [Löw23], gives a bijection between certain families of hulls in the upper half-plane and certain families of real Borel measures, and it has been extensively studied in both deterministic and random settings. In particular, if $\{K_t\}_{t\geq 0}$ is an increasing family of hulls in the upper half-plane (subject to some minor assumptions), then there is a unique family of real Borel measures μ_t such that the unique (up to hydrodynamic normalization) conformal maps $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ satisfy the following initial value problem, called the (chordal) Loewner equation (see [Law05] and [Bau05]):

$$\dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_t(du)}{g_t(z) - u}, \quad g_0(z) = z. \tag{1.1}$$

Conversely, given an appropriate family of real Borel measures μ_t, equation (1.1) generates an increasing family of hulls K_t, which we call the hulls driven by μ_t.

In this work we restrict ourselves to the chordal version (1.1) of the Loewner equation, but it is important to note that there is also a radial version of the equation. The radial version describes conformal mappings on the unit disc instead of the upper half-plane, so that the driving measure is an evolving measure on the unit circle, and the normalization is chosen at 0 instead of ∞. Although the two settings are closely linked, there are subtle differences that arise from normalizing at an interior point rather than a boundary point. In the context of the radial Loewner equation, growth processes that exhibit branching behavior related to Diffusion Limited Aggregation and the Hastings-Levitov model have been studied in [CM01] and [JS09], respectively, by studying discontinuous driving functions.

The relationship between the geometry of the hulls generated by the chordal Loewner equation (1.1) and the associated driving measure is not fully characterized but has been studied in detail in a number of specific settings. When the time-dependent measure is a single point mass $\mu_t = \delta_{U(t)}$ for a continuous function U, the initial value problem reduces to a simpler form:

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U(t)}, \quad g_0(z) = z. \tag{1.2}$$

In this case, much is known about the relationship between the driving function $U(t)$ and the geometry of the hulls (see [Law05] for a summary). When $\dot{b}(t) \equiv 2$ and the driving function is $U(t) = \sqrt{\kappa}B_t$, for a linear Brownian motion B_t and a positive real constant κ, the generated hulls are random curves in the upper half-plane whose geometric properties are dependent on the value of κ (characterized in [RS05]), and this evolution is called Schramm-Loewner evolution (SLE$_\kappa$). Originally introduced by Schramm in [Sch00], SLE$_\kappa$ has been shown to be the scaling limit of many discrete growth processes that arise in statistical mechanics, including the loop-erased random walk ($\kappa = 2$) [LSW04]. Finally, when the driving measure is the discrete measure $\mu_t = \sum_{i=1}^N \omega_i(t)\delta_{U_i(t)}$ for nonintersecting continuous driving functions $U_i(t) \in \mathbb{R}$ and weights $\omega_i(t) \in \mathbb{R}^+$, the resulting equation is called the multi-slit Loewner equation, and it is studied extensively in [Sch12] and [Sch13].

In the spirit of understanding the relationship between deterministic driving functions and the geometry of the generated hulls, the first question we address is the following.

Question 1. *What hypotheses on μ_t guarantee that the hull generated by (1.1) is a graph embedding of a plane tree?*

In [Sch12], the author establishes a condition that guarantees that the multi-slit equation generates a disjoint union of simple curves. In order to embed trees as hulls generated by the Loewner equation, in the present work we build on this condition to understand the delicate situation when these curves meet, producing hulls in the upper half-plane with the tree property and nontrivial branching angles. Although at first glance it might appear that the results of [Sch12] could be applied directly to this situation, a fundamental difficulty lies in the fact that the geometric properties of Loewner hulls are not necessarily preserved under taking limits. In fact, one of the most important properties of the (single slit) Loewner equation is that the maps that produce curves are dense in schlicht mappings, so it should not be expected for any geometric property (e.g. the simple curve property) to persist in the limit. For this reason, the conditions that guarantee that a hull has the branching property are delicate, and much of the present work is devoted to the construction of the finite tree embeddings.

Chapter 2 begins with an explicit computation of a driving measure that generates a hull that is composed of two rays meeting on the real line at specified angles. This explicit driving measure is then used to establish a sufficient condition on discrete driving measures to guarantee that the generated hull has the desired tree property. The main work of the chapter consists in establishing a criterion that guarantees that the hull K_s approaches the real line in (α, β)-direction, which roughly means that for each $\epsilon > 0$ there is a small enough time s_ϵ such that the hull $K_{s_\epsilon} \subset K_s$ consists of two connected components, each of which lies in an ϵ-sector about angles α and $\pi - \beta$, respectively. (This definition and the corresponding sufficient condition are motivated by the idea of α-directional approach found in [Sch12], though the proof in our case requires lengthy explicit conformal radius estimates not present in [Sch12].) The sufficient condition for (α, β)-approach is given in the following theorem, for which the φ_1 and φ_2 will be made explicit upon the theorem's restatement in the body of the chapter.

Theorem 2.3. *For $t \in [0, T]$ and $c > 0$, let μ_t be the discrete measure given by*

$$\mu_t = c \sum_{i=1}^{N} \delta_{U_i(t)}, \tag{1.3}$$

where each $U_i : [0, T] \to \mathbb{R}$ is continuous, and $U_i(t) < U_{i+1}(t)$ for every $t \in [0, T]$ and every $1 \leq i < i + 1 \leq N$, except for a single index k for which $U_k(0) = U_{k+1}(0)$. Let $\{g_t\}$ be the unique family of conformal mappings with hydrodynamic normalization that satisfies the initial value problem (1.1), and let $\{K_t\}$ denote the corresponding hulls. There are algebraic functions $\varphi_1(\alpha, \beta)$ and $\varphi_2(\alpha, \beta)$ of α and β, which may be computed explicitly, such that if

$$\begin{aligned} \lim_{t \searrow 0} \frac{U_k(t) - U_k(0)}{\sqrt{t}} &= \varphi_1(\alpha, \beta) - \varphi_2(\alpha, \beta), \ \textit{and} \\ \lim_{t \searrow 0} \frac{U_{k+1}(t) - U_{k+1}(0)}{\sqrt{t}} &= \varphi_1(\alpha, \beta) + \varphi_2(\alpha, \beta), \end{aligned} \tag{1.4}$$

then the hulls K_t approach $\mathbb{R}$ at $U_k(0)$ in (α, β)-direction. In the case when $0 < \alpha = \beta < \frac{\pi}{2}$, condition (1.4) simplifies to

$$\begin{aligned} \lim_{t \searrow 0} \frac{U_k(t) - U_k(0)}{\sqrt{t}} &= -\sqrt{2c} \sqrt{\frac{\pi - 2\alpha}{\alpha}}, \ \textit{and} \\ \lim_{t \searrow 0} \frac{U_{k+1}(t) - U_{k+1}(0)}{\sqrt{t}} &= \sqrt{2c} \sqrt{\frac{\pi - 2\alpha}{\alpha}}. \end{aligned} \tag{1.5}$$

The theorem that is the goal of the chapter follows naturally from this result: if a driving measure satisfies the conditions of Theorem 2.3 and if, furthermore, for each $\epsilon > 0$ the hull generated on $[\epsilon, T]$ is a union of simple curves (this hull is simply $g_\epsilon(K_T)$), then the hull K_T is a union of simple curves with the branching property.

Chapter 3 is devoted to showing that a specific family of measures satisfies the hypotheses required for the results of Chapter 2 and that these measures embed finite Galton-Watson trees. The main result is the following.

Theorem 3.1. *Let $\mathcal{T}^* = \{(\nu, h_\nu)\}$ be a binary marked plane tree, with $h_\nu \neq h_\eta$ for all $\nu \neq \eta$. Let $p(\nu)$ denote the parent of ν, and let $\Delta_t\mathcal{T}^*$ denote the set of elements "alive" at time t:*

$$\Delta_t\mathcal{T}^* = \{\nu \in \mathcal{T}^* : h(p(\nu)) \leq t < h(\nu)\}.$$

For $c, c_1 > 0$, let

$$\mu_t = c \sum_{\nu \in \Delta_t\mathcal{T}^*} \delta_{U_\nu(t)},$$

where the U_ν evolve according to

$$\begin{aligned} \dot{U}_\nu(t) &= \sum_{\substack{\eta \in \Delta_t\mathcal{T} \\ \eta \neq \nu}} \frac{c_1}{U_\nu(t) - U_\eta(t)}, \\ U_\nu\left(h_{p(\nu)}\right) &= \lim_{t \nearrow h_{p(\nu)}} U_{p(\nu)}(t), \text{ and} \\ U_\emptyset(0) &= 0. \end{aligned}$$

Then for each $s \in [0, \max_{\nu \in \mathcal{T}^*} h_\nu]$, the hull K_s generated by the Loewner equation (1.1) with driving measure μ_t is a graph embedding in $\mathbb{H}$ of the (unmarked) plane tree

$$\mathcal{T}_s = \{\nu \in \mathcal{T}^* : h_{p(\nu)} < s\},$$

with the image of the root on $\mathbb{R}$.

Theorem 3.1 holds for arbitrary binary marked trees with distinct lifetimes, so in par-

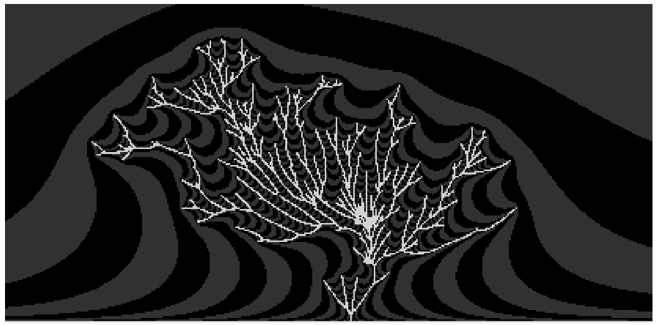

Figure 1.1: A sample of the random hull generated when $\mathcal{T}^*$ is a critical binary Galton-Watson tree with exponential lifetimes and the driving measure evolves according to (3.5). (Code for this image courtesy of Brent Werness.)

ticular it holds with probability one for critical binary (continuous time) Galton-Watson trees with exponential lifetimes of finite mean (an embedded sample of which is shown in Figure 1.1).

Finally, Chapter 4 investigates the limit of the driving measures μ_t^k from Chapter 3 through the lens of superprocesses with an eye toward determining the geometric scaling limit of the corresponding embedded trees. In particular, since the CRT is the scaling limit of the critical binary Galton-Watson trees with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$ discussed in Chapter 3, when these trees are conditioned to have k edges, the first step toward finding the geometric limit of the embedded trees is to understand the superprocess limit of the corresponding random driving measures. To this end, we show that the sequence of measure-valued processes $\{\mu^k\}$ is tight (so that at least one limit point exists), and in order to identify the limit, we reframe the problem in terms of the Stieltjes transform of the measures. In particular, we show that for each fixed k, the Stieltjes transform of these measures satisfies the differential equation (4.83), which is related to the complex Burgers equation. Using this equation, in the unconditioned case we conjecture that the limiting superprocess has density $\rho(x,t)$ that satisfies

$$\partial_t \rho + \partial_x \left(\rho \cdot \mathcal{H}\rho\right) = \sigma \sqrt{\rho}\dot{W}, \tag{1.6}$$

where $\dot{W}$ is space-time white noise, σ is a positive constant (see Conjecture 4.9 and equation

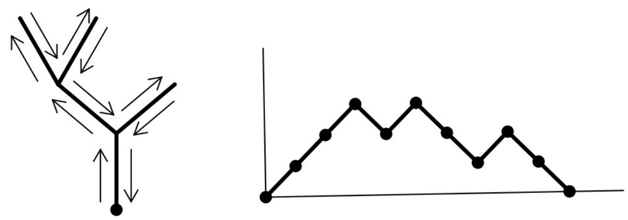

Figure 1.2: Left: tracing the tree. Right: its contour function.

(4.128)), and $\mathcal{H}$ is the Hilbert transform, defined by

$$\mathcal{H}\rho(x,t) = \frac{\text{p.v.}}{\pi} \int_{\mathbb{R}} \frac{1}{x-\xi}\rho(\xi,t)d\xi, \quad x \in \mathbb{R}. \tag{1.7}$$

Finding the limiting driving measure in the case when the trees are conditioned to be large and characterizing the geometry of the corresponding Loewner hull remain open problems.

We devote the rest of the introduction to background information. To motivate the work, we begin with a discussion of the continuum random tree. This is followed by a section on the Loewner equation, which details the requisite notation and foundational results.

1.2 The Continuum Random Tree

To motivate our investigation of embedded tress, we begin by giving an overview of the construction of the continuum random tree (CRT) as a limit of finite plane trees. A *plane tree* is a finite rooted tree $\mathcal{T}$, for which at each vertex the edges meeting there are endowed with a cyclic order. The cyclic order of the edges about each vertex guarantees that a plane tree is a unicellular planar map, i.e. an embedding of a graph in the sphere (or plane), up to orientation preserving homeomorphism, that has exactly one face. Given a plane tree $\mathcal{T}$ with k edges, there is an associated Dyck path on the interval $[0, 2k]$, called the contour function (or Harris path) of the tree, denoted by $C_{\mathcal{T}}$, obtained by tracing the tree in lexicographical order beginning at the root in the manner shown in Figure 1.2. In this construction, each step away from the root corresponds to an up step in the contour function (slope one), and every step towards the root corresponds to a down step in the contour function (slope negative one). In fact, this correspondence between plane trees with k edges and Dyck paths with $2k$ steps is a bijection. The graph distance d_{gr} between two vertices in the tree can be recovered from the contour function as follows. If v and v' are two vertices on the graph, and s and s' are (integer) times at which vertices v and v' are visited (according to the contour function construction), then

$$d_{gr}(v, v') = C_{\mathcal{T}}(s) + C_{\mathcal{T}}(s') - 2 \min_{t \in [s,s']} C_{\mathcal{T}}(t). \tag{1.8}$$

Extending slightly, a *marked plane tree* $\mathcal{T}^*$ is a finite plane tree $\mathcal{T}$ and a set of markings $\{h_\nu : \nu \in \mathcal{T}\}$ such that $h_\rho = 0$ (where ρ denotes the root of $\mathcal{T}$), and if η is an ancestor of ν, then $h_\eta < h_\nu$. These markings can be understood in terms of edge lengths on the graph: if $p(\nu)$ denotes the parent of ν, then

$$\text{length}(p(\nu), \nu) = h_\nu - h_{p(\nu)}. \tag{1.9}$$

Using this interpretation, we may construct a contour function just as before, except that now the length of each step up or down in the contour function is equal to the length of the corresponding edge. In this case, (1.8) again recovers the graph distance. Going forward, we will use a different, though equivalent, interpretation of the markings: we can consider a marked tree as the genealogical tree of a birth-death process, where for each element $\nu \in \mathcal{T}$, the marking h_ν denotes the time of death of ν, and the *lifetime* of ν is defined as the quantity in (1.9). With this interpretation, unmarked plane trees can be understood as encoding the genealogical structure of a birth-death process in which every individual has a lifetime of length one.

Galton-Watson trees are genealogical trees that correspond to a particular kind of random birth-death process. Specifically, a population begins with a single ancestor (the root), and at integer times each living element dies and independently gives rise to offspring according to a fixed offspring distribution ξ. For our purposes, will only be concerned with critical Galton-Watson trees, which are Galton-Watson trees for which ξ has expected value one and finite variance (but we exclude the trivial case when $\xi = \delta_1$, the Dirac mass at 1). With probability one, these trees die out after a finite number of generations. However, if these trees are conditioned to be large, the CRT (which we will define shortly) will give us a natural way to understand their infinite limit.

In order to make sense of what is meant by taking an infinite limit of finite trees, we will need a final definition, that of real trees.

Definition 1. A (compact, rooted) real tree is a compact metric space $(\mathcal{T}, d)$ where for every $a, b \in \mathcal{T}$ the following hold.

1. There is a unique isometric map $f_{a,b} : [0, d(a,b)] \hookrightarrow \mathcal{T}$ such that $f_{a,b}(0) = a$ and $f_{a,b}(d(a,b)) = b$.

2. For any injective map $f : [0,1] \hookrightarrow \mathcal{T}$ with $f(0) = a$ and $f(1) = b$, we have that $f([0,1]) = f_{a,b}([0, d(a,b)])$.

3. There is a unique distinguished point ρ, which is called the root.

The interpretation of the graph distance in terms of the contour function in (1.8) suggests a way to construct a real tree from an excursion. In particular, given a bounded continuous function $e : [0,T] \to \mathbb{R}_+$ such that $e(0) = e(T) = 0$, define a pseudometric $\tilde{d}_e$ on $[0,T]$ by

$$\tilde{d}_e(s,s') = e(s) + e(s') - 2 \inf_{t \in [s,s']} e(t). \tag{1.10}$$

Let $\sim_e$ denote the equivalence relation naturally induced by this pseudometric:

$$s \sim_e s' \iff d_e(s,s') = 0. \tag{1.11}$$

Then $\mathcal{T}_e = [0,T]/\sim_e$ is a real tree under the induced metric denoted by d_e given by

$$d_e([s],[s']) = \tilde{d}_e(s,s'), \quad s,s' \in [0,T], \tag{1.12}$$

whose distinguished point is $\rho := [0]$, where $[s]$ denotes the equivalence class of s. (See, for example, [LGM12] or [Pit06] for more details about this construction.) Since real trees are a subset of the space of pointed compact metric spaces, the usual Gromov-Hausdorff metric can be used to compute the distance between two real trees. Furthermore, the following theorem implies that convergence of a sequence of excursions in the sup norm is a sufficient condition for the convergence of the corresponding real trees in the Gromov-Hausdorff distance.

Theorem 1.1 ([LGM12] Corollary 3.5)**.** *If e and e' are two continuous functions from $[0,1]$ to $\mathbb{R}_+$ such that $e(0) = e(1) = e'(0) = e'(1) = 0$, then*

$$d_{GH}(\mathcal{T}_e, \mathcal{T}_{e'}) \leq 2 \sup_{t\in[0,1]} \left| e(t) - e'(t) \right|. \tag{1.13}$$

In the very same way that deterministic excursions code deterministic real trees, random real trees are coded by random excursions. The *continuum random tree* (CRT) is defined as the random real tree coded by the normalized Brownian excursion $\mathbb{e} : [0,1] \to \mathbb{R}_+$. The CRT can be obtained as a limit of the uniform distribution on finite plane trees as described in the following theorem.

Theorem 1.2. *[[LGM12] Theorem 3.6] Let θ_k be uniformly distributed over the set of plane trees with k edges, and equip θ_k with the graph distance d_{gr}. Then*

$$\left(\theta_k, \frac{1}{\sqrt{2k}} d_{gr}\right) \xrightarrow{(d)} (\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}), \tag{1.14}$$

as $k \to \infty$, in the sense of convergence in distribution of random variables with values in the metric space $\mathbb{K}$ of pointed compact metric spaces, where $\mathbb{K}$ is equipped with the Gromov-Hausdorff distance.

Using Theorem 1.1, the result of Theorem 1.2 follows from the fact that under proper rescaling, the uniform distribution on Dyck paths with $2k$ steps converges in distribution to the normalized Brownian excursion. Furthermore, Theorem 1.2 implies that the continuum random tree is a scaling limit of Galton-Watson trees that are conditioned to be large, since the uniform distribution on plane trees with k edges is the same as the distribution of (discrete time) Galton-Watson trees with offspring distribution

$$\xi(i) = \frac{1}{2^{i+1}}, \quad i = 0, 1, \ldots, \tag{1.15}$$

when these trees are conditioned to have k edges.

One important application of Theorem 1.2 comes from the close relationship between labeled plane trees and planar maps, which are connected by a number of bijections. The most famous of these is the Cori-Vauquelin-Schaeffer bijection ([CV81], [Sch98]), which provides a link between a particular class of labeled plane trees and planar quadrandulations. When the labeled trees are conditioned to converge to the CRT, the corresponding random planar quadrangulations converge to a limiting random surface called the Brownian map, which is universal in the sense that it is the scaling limit of planar p-angulations for $p = 3$ and all even $p \geq 4$ ([Mie13] and [LG13]) as well as other classes of random maps.

Although Theorem 1.2 gives a beautiful way to take a scaling limit of finite trees, it is important to notice that Gromov-Hausdorff convergence of real trees is merely a kind of convergence of metric spaces, so information is lost when we describe a limit of plane tress in this way. In particular, in addition to encoding the metric, the contour function also encodes the lexicographical order of the edges. This means that each excursion contains the information to construct a rooted planar unicellular map, and it endows such a tree with a root (first) edge and a metric. The limit in Theorem 1.2 retains only the metric information, ignoring the map structure that is also coded in the Dyck paths, except to the extent that it counts the multiplicity of each real tree according to the uniform distribution on rooted plane trees. This suggests the following question, which provides the motivation for this work.

Question 2. *Is there a way to take a geometric limit of embedded plane trees to obtain an embedding of the CRT?*

We approach this question in the present work by constructing tree embeddings via the Loewner equation. For technical reasons, it will be useful to consider trees for which there is only one branching event at a time (with probability one), so instead of working with discrete time Galton-Watson trees, we will work with continuous time Galton-Watson trees defined as follows. Each tree encodes the genealogy of a birth-death process starting from a single ancestor, where the lifetimes of the individuals are independent identically distributed exponential random variables (later we will fix these to have mean $\frac{1}{2\sqrt{k}}$), and upon the expiration of its lifetime each individual dies, leaving behind 0 or 2 offspring, each with probability one half. These trees will be referred to as critical binary Galton-Watson trees with exponential lifetimes, and it is well-known that these trees are almost surely finite. Furthermore, as we will see in Chapter 4, these Galton-Watson trees converge to the CRT when they are appropriately conditioned to be large.

1.3 The Loewner Equation

We review the set-up for the chordal Loewner equation, primarily following [Law05]. A *compact* $\mathbb{H}$*-hull* is a bounded subset $K \subset \mathbb{H}$ such that $K = \overline{K} \cap \mathbb{H}$ and $\mathbb{H} \setminus K$ is simply connected. For brevity, we will refer to such sets simply as "hulls." By the Riemann mapping theorem, for each hull K there is a conformal map g_K such that $g_K(\mathbb{H} \setminus K) = \mathbb{H}$. Furthermore, since K is bounded, we can extend g_K by Schwartz reflection to a conformal mapping on $\hat{\mathbb{C}} \setminus \tilde{K}$, where $\tilde{K}$ is a bounded set containing $\overline{K}$ and its reflection about $\mathbb{R}$, so that it makes sense to take an expansion about ∞. Then the conformal map g_K is unique if we require that $\lim_{z\to\infty}(g_K(z) - z) = 0$. We refer to the latter condition by saying that

g_K has the *hydrodynamic normalization.* Under these conditions, g_K has the expansion

$$g_K(z) = z + \frac{b_K}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty, \tag{1.16}$$

where b_K is the *half-plane capacity* of K. A simple curve $\gamma : [0,T] \to \overline{\mathbb{H}}$ such that $\gamma(0) \in \mathbb{R}$ and $\gamma((0,T]) \subset \mathbb{H}$ is called a *slit.* Since each slit $\gamma((0,T])$ is a hull, we can consider the unique conformal map g_γ with hydrodynamic normalization such that $g_\gamma : \mathbb{H} \setminus \gamma((0,T]) \to \mathbb{H}$. In fact, we can consider the unique conformal map corresponding to each sub-slit of γ: for each t let $g_t := g_{\gamma((0,t])}$. It is a classical result that for each t there is a unique $U_t \in \mathbb{R}$ such that $\lim_{z \to \gamma(t)} g_t(z) = U_t$. Furthermore, $t \mapsto U_t$ is continuous, and if $b(t)$ is continuous, then g_t satisfies the initial value problem

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{1.17}$$

We call U the *driving function* for the slit γ.

In the opposite direction, one could start with a real-valued function U and study the geometry of the hulls generated by solving (1.17) with driving function U. If g_t is the family of conformal mappings that solves (1.17) for driving function U, the hulls K_t driven by U are defined by $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$. It is a classical question to ask under what circumstances the hulls K_t are simple curves. It is shown in [MR05] and [Lin05] that if U is Hölder continuous with exponent $\frac{1}{2}$ and $||U||_{\frac{1}{2}} < 4$, then each K_t is a simple curve. A related result concerning the multi-slit Loewner equation

$$\dot{g}_t(z) = \sum_{i=1}^{n} \frac{\dot{b}(t)}{g_t(z) - U_i(t)}, \quad g_0(z) = z, \tag{1.18}$$

is contained in [Sch12]. We recall this result here in its entirety, since we will refer to it in Chapter 3. Let $\text{Lip}(\frac{1}{2})$ denote the set of real functions that are Hölder continuous with exponent $\frac{1}{2}$.

Theorem 1.3. *[Thm 1.2 in [Sch12]] Let $U_1, \ldots, U_n \in Lip(\frac{1}{2})$ such that $U_i(t) < U_{i+1}(t)$ for each $i = 1, \ldots, n-1$ and all $t \in [0,T]$. Assume that for every $j \in \{1, \ldots, n\}$ and every*

$t \in [0,T]$ there exists an $\epsilon > 0$ such that

$$\sup_{\substack{r,s\in(0,T] \\ 0<|r-t|,|s-t|<\epsilon}} \frac{|U_j(r)-U_j(s)|}{\sqrt{|r-s|}} < 4\sqrt{c/2}. \tag{1.19}$$

Let $\{K_t\}_{t\in[0,T]}$ denote the hulls generated by solving Equation (1.18), where $\dot{b}(t) \equiv c > 0$, for $t \in [0,T]$. Then K_T consists of n disjoint connected components, and each component is a simple curve.

Equations (1.17) and (1.18) are special cases of equation (1.1), which is equivalent to the following inverse equation for $f_t := g^{-1}(t)$:

$$\dot{f}_t(z) = -f_t'(z)\int_{\mathbb{R}} \frac{\mu_t(dx)}{z-x}. \tag{1.20}$$

Endowing the space of real probability measures with the topology of weak convergence, it is shown in [Bau05] that for any measurable family of probability measures $\{\mu_t, t \in [0,\infty)\}$ (i.e. measurable with respect to the Borel σ-algebra for the topology of weak convergence on the space of probability measures) there is a unique family of conformal mappings f_t satisfying (1.20), whose images generate an increasing family of hulls in $\mathbb{H}$. Before moving on, we review a different version of this result, found in [Law05], which does not require the μ_t to be probability measures and includes an explicit interpretation of the total weight $\mu_t(\mathbb{R})$. The theorem shows that (1.1) relates real Borel measures to hulls in the same way that (1.17) relates driving measures to slits. In this case, instead of starting with the hull, we start with the measure.

Theorem 1.4 ([Law05], Thm 4.6)**.** *For $t \geq 0$, let μ_t be a one-parameter family of non-negative real Borel measures. Assume that $t \mapsto \mu_t$ is right continuos with left limits in the weak topology, and that for each t there is a constant $M_t < \infty$ such that* $\sup\{\mu_s(\mathbb{R}) : 0 \leq s \leq t\} < M_t$ *and supp $\mu_s \subset [-M_t, M_t]$ for all $s \leq t$. Let g_t be the solution of the initial value problem*

$$\dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_t(du)}{g_t(z)-u}, \quad g_0(z) = z. \tag{1.21}$$

Let $H_t = \{z \in \mathbb{H} : \text{ the solution } g_s(z) \text{ is well defined with } g_s(z) \in \mathbb{H} \text{ for } 0 \leq s \leq t\}$. *Then* g_t *is the unique conformal map from* H_t *to* $\mathbb{H}$ *with hydrodynamic normalization. Furthermore,* g_t *has the expansion*

$$g_t(z) = z + \frac{b(t)}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty,$$

where

$$b(t) = \int_0^t \mu_s(\mathbb{R})\, ds.$$

For each t let $K_t = \mathbb{H} \setminus H_t$. We call $\{K_t\}_{t\geq 0}$ the family of hulls driven by $\mu_t, t \geq 0$.

In this setting, our investigation begins in Chapter 2 by considering which families of measures $\{\mu_t\}$ generate hulls that are embeddings of trees.

Chapter Two

Branching in the Loewner Equation

In order to use the Loewner equation to embed marked plane trees, we will consider these trees as representing the genealogical structure of a birth-death process. The time parameter for the Loewner evolution will be the same as the time parameter in the birth-death process, which is given by the height of the contour function (see §1.2). If Γ is a hull that is a graph embedding of a combinatorial tree $\mathcal{T}$ such that the image of each edge is a simple curve in $\mathbb{H}$ and the image of the root lies on the real line, then Γ can be parametrized so that it is generated by equation (1.1), where the driving measure is of the form

$$\mu_t = c \sum_{\nu \in \mathcal{T}^*} \mathbb{1}_{[h_{p(\nu)}, h_\nu)} \delta_{U_\nu(t)}, \tag{2.1}$$

where $\mathcal{T}^* = (\mathcal{T}, \{h_\nu : \nu \in \mathcal{T}\})$ is a marked plane tree, each U_ν is a continuous function, and $U_\nu(h_{p(\nu)}) = \lim_{t \nearrow h_{p(\nu)}} U_{p(\nu)}(t)$. In this chapter, we consider the converse: for what measures are the hulls K_t graph embeddings of finite plane trees? As the simple curve question for the multislit equation is answered in [Sch12], this question centers on understanding the geometric properties of the hull when the driving measure splits (or, looking backward in time, when two driving functions collide). For this reason, we begin with an explicit calculation of the driving measure that generates a hull that is a union of two finite rays that meet on the real line. This calculation will be called upon in §2.2 in order to specify the angles of approach.

2.1 Explicit conformal map calculation

We explicitly compute the driving functions that generate a family of conformal maps with hydrodynamic normalization that take $\mathbb{H}$ to $\mathbb{H} \setminus \Gamma_t$, where Γ_t is the union of two finite rays, each starting at 0, forming angles $a\pi$ and $(1-b)\pi$ with the positive real line, and $\Gamma_s \subset \Gamma_t$ for all $s < t$. (This map is the inverse of the g_t from the Loewner equation.) An expert will quickly recognize that the basic Loewner scaling property, which we state later as Lemma 2.4, suggests that these driving functions should behave like $c_1\sqrt{t}$ and $c_2\sqrt{t}$ for some constants c_1 and c_2, so the main contribution of this section is the explicit computation of these constants.

To start, consider the map

$$f(z) = (z-1)^a z^{1-a-b}(z-x)^b. \tag{2.2}$$

If $x < 0$, this map takes $\mathbb{H}$ to $\mathbb{H} \setminus \Gamma$, where Γ is a union of two straight slits in $\mathbb{H}$, meeting the real line at 0 and forming angles $a\pi$ and $(1-b)\pi$ with the positive real line. (Notice that if $0 < x < 1$ or $x > 1$, then the angles are permuted, so the resulting hull has the correct shape, but the angles appear in the wrong order.) Although f generates the correct hull, it does not have the hydrodynamic normalization, so we will need to slightly modify it to get the map that we want. We will also have to introduce a parameter so that we have a family of maps that generates an increasing hull.

Let $\kappa_t : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function of t. (Eventually we will also want c to be increasing and $c(0) = 0$.) Let

$$f_t(z) = \big(z + (a+bx-1)\kappa_t\big)^a \big(z + (a+bx)\kappa_t\big)^{1-a-b} \big(z + (a+bx-x)\kappa_t\big)^b. \tag{2.3}$$

We see that

$$\lim_{z\to\infty} (f_t(z) - z) = 0, \tag{2.4}$$

so f_t has the hydrodynamic normalization for every $\kappa_t > 0$. Notice that f_t generates a hull of the type we want for every t. Next we will calculate the driving points U_k and weights λ_k so that f_t satisfies the inverse Loewner equation

$$-\frac{\dot{f}_t(z)}{f_t'(z)} = \sum_{k=1}^{n} \frac{\lambda_k(t)}{z - U_k(t)} \tag{2.5}$$

To simplify the calculation, let

$$w = z + (a+bx)\kappa_t, \tag{2.6}$$

so that

$$\frac{dw}{dz} = 1. \tag{2.7}$$

We compute $\frac{df_t(z)}{dz}$:

$$\begin{aligned} f_t'(z) = &[aw(w - x\kappa_t) + (1 - a - b)(w - \kappa_t)(w - x\kappa_t) + bw(w - \kappa_t)] \\ &\times \left[(w - \kappa_t)^{a-1} w^{-a-b} (w - x\kappa_t)^{b-1}\right] \\ = &\left[w^2 + w\kappa_t(a + bx - 1 - x) + \kappa_t^2 x(1 - a - b)\right] \\ &\times \left[(w - \kappa_t)^{a-1} w^{-a-b} (w - x\kappa_t)^{b-1}\right]. \end{aligned} \tag{2.8}$$

The poles of $f_t'(z)$ are $w = 0, \kappa_t, x\kappa_t$, and $f_t'(z)$ has zeros at

$$w = \frac{\kappa_t}{2}\left(1 + x - a - bx \pm \sqrt{(1 - a)^2 + 2x(a + b + ab - 1) + x^2(1 - b)^2}\right). \tag{2.9}$$

Undoing the original w substitution, the poles of $f_t'(z)$ are

$$z = \begin{cases} -(a + bx)\kappa_t, \\ -(a + bx - 1)\kappa_t \\ -(a + bx - x)\kappa_t, \end{cases} \tag{2.10}$$

and $f_t'(z)$ has zeros

$$z = \frac{\kappa_t}{2}\left(1 + x - 3a - 3bx \pm \sqrt{(1 - a)^2 + 2x(a + b + ab - 1) + x^2(1 - b)^2}\right). \tag{2.11}$$

On the other hand, we calculate $\frac{df_t(z)}{dt}$:

$$\begin{aligned} \dot{f}_t(z) = &\dot{\kappa}_t\big[a(a + bx - 1)w(w - x\kappa_t) + (1 - a - b)(a + bx)(w - x\kappa_t)(w - \kappa_t) + b(a + bx - x)w(w - \kappa_t)\big] \\ &\times \left[(w - \kappa_t)^{a-1} w^{-a-b} (w - x\kappa_t)^{b-1}\right] \\ = &\dot{\kappa}_t\left[\kappa_t w\Big(a(a - 1) + 2abx + b(b - 1)x^2\Big) + \kappa_t^2 x(a + bx)(1 - a - b)\right] \\ &\times \left[(w - \kappa_t)^{a-1} w^{-a-b} (w - x\kappa_t)^{b-1}\right]. \end{aligned} \tag{2.12}$$

So,

$$\frac{\dot{f}_t(z)}{f_t'(z)} = \dot{\kappa}_t\kappa_t \times \frac{w\Big(a(a-1)+2abx+b(b-1)x^2\Big)+x\kappa_t(a+bx)(1-a-b)}{w^2+w\kappa_t(a+bx-1-x)+\kappa_t^2x(1-a-b)}, \tag{2.13}$$

which by (2.11) has poles

$$\begin{aligned} U_1(t) &= \frac{\kappa_t}{2}\left(1+x-3a-3bx-\sqrt{(1-a)^2+2x(a+b+ab-1)+x^2(1-b)^2}\right) \\ U_2(t) &= \frac{\kappa_t}{2}\left(1+x-3a-3bx+\sqrt{(1-a)^2+2x(a+b+ab-1)+x^2(1-b)^2}\right). \end{aligned} \tag{2.14}$$

Since

$$\deg\left(-\frac{\dot{f}_t(z)}{f_t'(z)}\right) = -1, \tag{2.15}$$

there are constants $\lambda_1 = \lambda_1(t)$ and $\lambda_2 = \lambda_2(t)$ so that the the following partial fractions expansion holds:

$$-\frac{\dot{f}_t(z)}{f_t'(z)} = \frac{\lambda_1}{z-U_1(t)} + \frac{\lambda_2}{z-U_2(t)}. \tag{2.16}$$

To determine λ_1 and λ_2, we must solve

$$\frac{z\,(\lambda_1+\lambda_2)-(\lambda_1U_2+\lambda_2U_1)}{(z-U_1)\,(z-U_2)} = \frac{-\dot{\kappa}_t\kappa_t \times \Big(w\Big(a(a-1)+2abx+b(b-1)x^2\Big)+x\kappa_t(a+bx)(1-a-b)\Big)}{(z-U_1)\,(z-U_2)}, \tag{2.17}$$

where we have suppressed the dependence of U_1 and U_2 on t, and we are still employing the substitution (2.6). Since

$$\begin{aligned} -\dot{\kappa}_t\kappa_t\times & \Big(w\Big(a(a-1)+2abx+b(b-1)x^2\Big)+x\kappa_t(a+bx)(1-a-b)\Big) \\ &= -\dot{\kappa}_t\kappa_t w\Big(a(a-1)+2abx+b(b-1)x^2\Big) - \dot{\kappa}_t\kappa_t^2x(a+bx)(1-a-b) \\ &= -\dot{\kappa}_t\kappa_t z\Big(a(a-1)+2abx+b(b-1)x^2\Big) \\ &\quad - \dot{\kappa}_t\kappa_t^2\Big(a^2(a-1)+xa(1-a-2b+3ab)+x^2b(1-2a-b+3ab)+x^3b^2(b-1)\Big), \end{aligned} \tag{2.18}$$

(2.17) is equivalent to solving

$$\begin{cases} \lambda_1 + \lambda_2 = -\dot{\kappa}_t \kappa_t \Big(a(a-1) + 2abx + b(b-1)x^2 \Big) \\ \lambda_1 U_2 + \lambda_2 U_1 = \dot{\kappa}_t \kappa_t^2 \Big(a^2(a-1) + xa(1-a-2b+3ab) + x^2 b(1-2a-b+3ab) + x^3 b^2 (b-1) \Big). \end{cases} \tag{2.19}$$

For our purposes, we are interested in constant coefficients, so we let

$$\lambda_1(t) = \lambda_2(t) = c \tag{2.20}$$

so (2.19) becomes

$$\begin{cases} 2c = -\dot{\kappa}_t \kappa_t \Big(a(a-1) + 2abx + b(b-1)x^2 \Big) \\ c(U_1 + U_2) = \dot{\kappa}_t \kappa_t^2 \Big(a^2(a-1) + xa(1-a-2b+3ab) + x^2 b(1-2a-b+3ab) + x^3 b^2 (b-1) \Big). \end{cases} \tag{2.21}$$

Equation (2.11) implies that

$$U_1 + U_2 = \frac{\kappa_t}{2} \left(1 + x - 3a - 3bx \right), \tag{2.22}$$

so system (2.21) becomes

$$\begin{cases} 2c = -\dot{\kappa}_t \kappa_t \Big(a(a-1) + 2abx + b(b-1)x^2 \Big) \\ 1 + x - 3a - 3bx = \frac{2}{c} \dot{\kappa}_t \kappa_t \Big(a^2(a-1) + xa(1-a-2b+3ab) + x^2 b(1-2a-b+3ab) + x^3 b^2 (b-1) \Big). \end{cases} \tag{2.23}$$

If we require that $\kappa_0 = 0$, then the first equation in (2.23) has solution

$$\kappa_t = \sqrt{t} \frac{2\sqrt{c}}{\sqrt{a(1-a) - 2abx + b(1-b)x^2}}. \tag{2.24}$$

The second equation in (2.23) fixes x as a function of a and b. (More on this in the next section.) Our goal is to understand the behavior of the driving points that correspond to this chain. Substituting the value of κ_t given in (2.24) into (2.14) we see that the driving points are

$$\begin{aligned} U_i(t) =& \frac{\kappa_t}{2}\left((1+x-3a-3bx) \pm \sqrt{(1-a)^2+2x(a+b+ab-1)+x^2(1-b)^2}\right) \\ =& \sqrt{ct}\frac{(1+x-3a-3bx) \pm \sqrt{(1-a)^2+2x(a+b+ab-1)+x^2(1-b)^2}}{\sqrt{a(1-a)-2abx+b(1-b)x^2}}. \end{aligned} \tag{2.25}$$

If $a = b$, then $x = -1$, so

$$U_i(t) = \pm\sqrt{t}\sqrt{2c}\sqrt{\frac{1-2a}{a}}. \tag{2.26}$$

(While $x = -1$ is not the only solution to (2.23) when $a = b$ and κ_t is given by (2.24), it is the "correct" one, as will be explained in the next section.)

2.1.1 Fixing the preimage x of 0 in terms of a and b.

Equation (2.25) gives an expression for the location of the sought-after driving points. However, x is a function of a and b that is determined by the system (2.23), which has multiple solutions, so in order to make sense of (2.25), we must examine (2.23) more closely. We will show that system (2.23) has three real roots, explain the geometric significance of the three solutions, and give a justification for the choice of which root to use in equation (2.25).

Substituting the first equation of (2.23) into the second one, the second becomes

$$\begin{aligned} \left(\frac{4}{a(1-a)-2abx+b(1-b)x^2}\right)\left(a^2(a-1)+xa(1-a-2b+3ab)+x^2b(1-2a-b+3ab)+x^3b^2(b-1)\right) \\ = 1+x-3a-3bx. \end{aligned} \tag{2.27}$$

Let

$$Q(x) := -a+a^3+3ax-3a^2x-3abx+3a^2bx+3bx^2-3abx^2-3b^2x^2+3ab^2x^2-bx^3+b^3x^3. \tag{2.28}$$

Then (2.27) is equivalent to setting $Q(x) = 0$. To find the roots of Q, we can equivalently consider

$$0 = \frac{a(a^2-1)}{b(b^2-1)} + x\frac{3a(1+ab-a-b)}{b(b^2-1)} + x^2\frac{3(1+ab-a-b)}{(b^2-1)} + x^3. \tag{2.29}$$

Let

$$x = y - \frac{1+ab-a-b}{b^2-1}, \tag{2.30}$$

so that

$$Q(x) = P(y) = y^3 + py + q, \tag{2.31}$$

where

$$p = \frac{3(a-1)(a+b)}{b(b+1)^2}, \tag{2.32}$$

and

$$q = \frac{2(a-1)(a+b)(b+2a-1)}{b(b+1)^3(b-1)}. \tag{2.33}$$

Then the discriminant of $P(y)$ is

$$\begin{aligned} D &= -4p^3 - 27q^2 \\ &= -\frac{4 \cdot 27a(a-1)^2(a+b)(a+b-1)}{b^3(b+1)^4(b-1)^2}. \end{aligned} \tag{2.34}$$

Notice that the conditions $0 < a, b < 1$ and $0 < a + b < 1$ guarantee that $D > 0$, so $P(y) = Q(x)$ has exactly three distinct real roots.

In order to determine where these roots lie, first notice that since $0 < a < 1$ and $0 < a + b < 1$

$$Q(0) = a^3 - a < 0 \tag{2.35}$$

and

$$\begin{aligned} Q(1) &= 2a - 3a^2 + a^3 + 2b - 6ab + 3a^2b - 3b^2 + 3ab^2 + b^3 \\ &= (a+b)(a+b-1)(a+b-2) \\ &> 0. \end{aligned} \tag{2.36}$$

Also,

$$b^3 - b = b(b+1)(b-1) < 0, \tag{2.37}$$

so $Q(x)$ is positive for sufficiently large negative values of x and negative for sufficiently large positive values of x. We conclude that the three distinct real roots of $Q(x)$ satisfy

$$\begin{aligned} &\theta_1 \in (-\infty, 0) \\ &\theta_2 \in (0, 1) \\ &\theta_3 \in (1, \infty). \end{aligned} \tag{2.38}$$

For each real root,

$$f_t^i(z) = (z + (a + b\theta_i - 1)\kappa_t)^a (z + (a_b\theta_i)\kappa_t)^{1-a-b} (z + (a + b\theta_i - \theta_i)\kappa_t)^b \tag{2.39}$$

generates a hull that is a union of two rays, and

$$f_t^i(\kappa_t(1 - a - b\theta_i)) = f_t^i(\kappa_t(-a - b\theta_i)) = f_t^i(\kappa_t(\theta_i - a - b\theta_i)). \tag{2.40}$$

However, the angles between the real line and the first slit, the first slit and the second slit, and the second slit and the real line depend on the labeling of the three roots above. In order for the angles to appear in counter-clockwise order $a\pi$, $(1-a-b)\pi$, $b\pi$, it must be the case that

$$\theta_i < 0, \tag{2.41}$$

so that

$$\theta_i - a - b\theta_i < -a - b\theta_i < 1 - a - b\theta_i. \tag{2.42}$$

Thus, each root corresponds to a different permutation of the angles, and the unique negative root of $Q(x)$ is the unique value of x such that, for κ_t is given by (2.24), the one-parameter family of conformal maps

$$f_t(z) = (z + (a + bx - 1)\kappa_t)^a \, (z + (a + bx)\kappa_t)^{1-a-b} \, (z + (a + bx - x)\kappa_t)^b \tag{2.43}$$

generates the desired increasing family of $\mathbb{H}$-hulls.

2.2 A condition for branching

In this section we provide a condition on the driving measure that guarantees that the hull it generates is a union of simple curves, two of which begin at the same point on the real line. If $\{K_t\}_{t\geq 0}$ is a family of Loewner hulls, then for $s < t$, $g_s(K_t \setminus K_s)$ is homeomorphic to $K_t \setminus K_s$, which implies that a graph embedding of a tree can be built step-by-step from these simpler hulls.

If a driving measure generates disjoint simple curves on all intervals that do not contain branching times, then Theorem 2.1 states that (α, β)-directional approach at branching times turns out to be the additional condition we need. Theorem 2.3 is a sufficient condition on the behavior of the driving measure at time t to guarantee approach in (α, β)-direction at time t. This condition will be used in Chapter 3 to show that a specific family of driving measures generates tree embeddings. The definition of (α, β)-directional approach and the basic idea of the proof of the theorem mirror similar statements found in [Sch12] concerning the approach of a single slit in the context of the multislit Loewner equation. However, our proof is substantially more complicated, as extending these results to our setting requires explicit conformal radius estimates.

2.2.1 Approach in (α, β)-direction

We begin with the key definition.

Definition 2. Let $0 < \alpha < 1-\beta < \pi$. The hull K_t approaches $\mathbb{R}$ in (α, β)-direction at $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is an $s > 0$ such that there are exactly two connected components

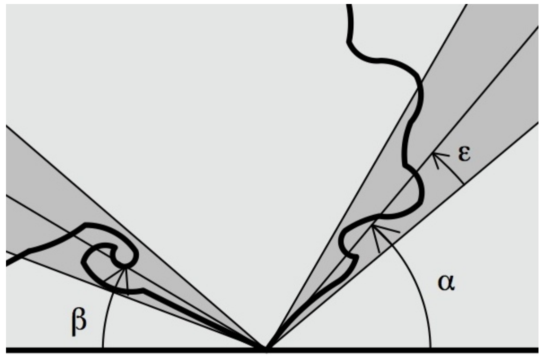

Figure 2.1: A hull approaching $\mathbb{R}$ in (α, β)-direction.

K_s^j and K_s^{j+1} of K_s that have $x = U_j(0) = U_{j+1}(0)$ as a boundary point and

$$K_s^j \subset \{z \in \mathbb{H} : \pi - \beta - \epsilon < \arg(z - U_j(0)) < \pi - \beta + \epsilon\} \tag{2.44}$$

and

$$K_s^{j+1} \subset \{z \in \mathbb{H} : \alpha - \epsilon < \arg(z - U_j(0)) < \alpha + \epsilon\}. \tag{2.45}$$

Figure 2.1 illustrates Definition 2. While the image depicts a hull that is a union of two simple curves, this need not be the case in general.

Theorem 2.1. *Let K_t be a family of Loewner hulls for $t \in [0, T]$ such that K_T approaches $x \in \mathbb{R}$ in (α, β)-direction for some $0 < \alpha < \pi - \beta < \pi$, and assume that $g_s(K_T)$ is a union of slits in $\mathbb{H}$ for all $s \in (0, T]$. Then there are exactly two connected components of K_T that have x as a boundary point, and each one is a slit.*

Proof. Fix ϵ. By the definition of approach in (α, β)-direction, there is s_ϵ such that for $t < s_\epsilon$ there are exactly two connected components of K_t that have x as a boundary point. Fix such a t, and denote these two connected components by K_t^k and K_t^{k+1}. Since $g_t(K_T)$ is a union of slits, this implies that K_T has exactly two connected components that have x as a boundary point, which we denote K_T^k and K_T^{k+1}, and $i = k, k+1$ these satisfy

1. $K_t^i \subseteq K_T^i$, and

2. $K_T^i \setminus K_t^i$ is a simple curve in $\mathbb{H}$.

Thus, the proof amounts to showing that K_t^i is a slit for $i = k, k+1$. We only give the proof for K_t^k since the proof for K_t^{k+1} is identical.

For every $0 < s < t$, by assumption, $g_s(K_t)$ is a union of slits, so in particular, $g_s(K_t^k)$ is a slit, which in turn implies that $K_t^k \setminus K_s^k$ is a simple curve in $\mathbb{H}$. Define the curve $\gamma : (0, t] \to \mathbb{H}$ by

$$\gamma(s) = K_s \setminus \bigcup_{u<s} K_u, \quad \text{for all } 0 < s \leq t, \tag{2.46}$$

so that $\gamma([s,t]) = K_t^k \setminus K_s^k$ is a simple curve for every $0 < s < t$. The proof will be complete if we can extend γ continuously to a well-defined $\gamma(0) \in \mathbb{R}$ and show that $\gamma([0,t])$ is simple. Let $\{t_i\}_{i\geq 1}$ be a sequence of times such that $t > t_1$ and $t_i \searrow 0$. Then $\{\gamma(t_i)\}$ is a sequence of points in K_t^k. Since $\overline{K_t^k}$ is compact, there is a subsequence $\{t_{i_j}\}$ such that

$$\gamma(t_{i_j}) \to x^* \in \overline{K_t^k}. \tag{2.47}$$

Notice that since K_t^k is a hull, $K_t^k = \overline{K_t^k} \cap \mathbb{H}$. Since K_T approaches x in (α, β)-direction, the only boundary point of K_t^k on $\mathbb{R}$ is x, so that $\overline{K_t^k} = K_t^k \cup \{x\}$. Therefore, either $x^* = x$ or $x^* \in K_t^k$. If $x^* \in K_t^k$, then there is an $s^* < t$ such that

$$\gamma(s^*) = x^*. \tag{2.48}$$

Now let $u < s^*$ and consider the sequence $\{\gamma(t_{i_j})\}$ for only $t_{i_j} < u$. This is still a sequence with limit point x^*, implying that $x^* \in K_u^k$, so that there is $u^* < u < s$ such that

$$\gamma(u^*) = x^*, \tag{2.49}$$

so $\gamma([u^*, s^*])$ is not simple, which is a contradiction. Therefore, $x^* = x$, so that every convergent subsequence $\gamma(t_{i_j})$ converges to x. This implies that $\{\gamma(t_i)\}$ converges to x, without passing to a subsequence. (If it did not converge to x, then there would be an infinite subsequence of points of bounded distance away from x. By the compactness of $\overline{K_t^k}$, this subsequence would have a further convergent subsequence that converges to a point other than x, which is a contradiction.) Therefore, $\gamma(0,t])$ can be continuously extended to $\gamma(0) = x$.

Since $\gamma([s,t])$ is simple for every $0 < s < t$, to complete the proof we only need to show that there is no $s \in (0,t)$ such that $\gamma(s) = x$. Assume that such an s exists. Then $\gamma(s) = x \in \mathbb{R}$, so for $0 < u < s$, $g_u(\gamma(s)) \in \mathbb{R}$. But the existence of such a point contradicts the assumption that $g_u(K_t^k)$ is a slit. $\square$

2.2.2 A sufficient condition on the driving measure for (α, β)-approach

We will now turn out attention to finding an explicit condition on the driving measure itself that guarantees (α, β)-directional approach of the hulls.

If L is the union of two rays from 0 forming angles α and $\pi - \beta$ with the real axis, then it turns out that Hausdorff convergence of the sets $\rho K_{1/\rho^2}$ to L is a sufficient condition for (α, β)-approach, as we will prove in the next lemma.

Lemma 2.2. *Let L be the union of two rays in $\mathbb{H}$ emanating from 0 and forming angles α and $\pi - \beta$ with the real line. Fix $R > 0$. If*

$$D_R \cap \rho K_{1/\rho^2} \overset{Haus.}{\longrightarrow} D_R \cap L, \tag{2.50}$$

then K_t approaches 0 in (α, β)-direction.

Proof. Fix a small angle $\theta > 0$. The Hausdorff convergence of the hulls guarantees that there is $\hat{\rho}$ large enough such that for all $\rho \geq \hat{\rho}$, if $z \in \rho K_{1/\rho^2} \cap D_R$, then

$$d(z, L) < \frac{R}{2} \sin\theta. \tag{2.51}$$

Notice that z is in the set

$$C_{\alpha,\beta,\theta} = \{z \in \mathbb{H} : \pi - \beta - \theta < \arg(z) < \pi - \beta + \theta\} \cup \{z \in \mathbb{H} : \alpha - \theta < \arg(z) < \alpha + \theta\} \quad (2.52)$$

if and only if $az \in C_{\alpha,\beta,\theta}$ is for all $a > 0$. Notice also that $z \in C_{\alpha,\beta,\theta}$ if and only if

$$d(z, L) < |z| \sin\theta. \quad (2.53)$$

Let $w \in D_R \cap \rho K_{1/\rho^2}$ for $\rho \geq \hat{\rho}$. The lemma follows if we show that $\frac{w}{\rho} \in C_{\alpha,\beta,\theta}$. First, if $|w| > \frac{R}{2}$, then by assumption

$$d(w, L) < \frac{R}{2} \sin\theta < |w| \sin\theta, \quad (2.54)$$

so $w \in C_{\alpha,\beta,\theta}$, and therefore $\frac{w}{\rho} \in C_{\alpha,\beta,\theta}$, where $\frac{w}{\rho} \in K_{1/\rho^2}$.

Since the hypothesis holds for all $\rho \geq \hat{\rho}$, if $\frac{R}{4} < |w| \leq \frac{R}{2}$, then

$$d(2w, L) \leq \frac{R}{2} \sin\theta < |2w| \sin\theta, \quad (2.55)$$

so $2w \in C_{\alpha,\beta,\theta}$, and thus $\frac{w}{\rho} \in C_{\alpha,\beta,\theta}$. Similarly, if $\frac{R}{2^n} < |w| \leq \frac{R}{2^{n-1}}$, then

$$d(2^{n-1}w, L) < \frac{R}{2} \sin\theta \leq \left|2^{n-1}w\right| \sin\theta, \quad (2.56)$$

so $2^{n-1}w \in C_{\alpha,\beta,\theta}$, and again it follows that $\frac{w}{\rho} \in C_{\alpha,\beta,\theta}$. Since θ was arbitrary, this completes the proof of the lemma. $\square$

Theorem 2.3. *Consider the chordal multi-slit Loewner equation*

$$\dot{g}_t(z) = \sum_{i=1}^{n} \frac{c}{g_t(z) - U_i(t)}, \quad g_0(z) = z, \quad (2.57)$$

where each $U_i(t)$ is continuous from $[0,T]$ to $\mathbb{R}$. For each $t \in [0,T]$ let $\{K_t\}, t \in [0,T]$ denote the family of hulls generated by g_t, i.e. $g_t(\mathbb{H} \setminus K_t) = \mathbb{H}$. Let $U_i(t) < U_{i+1}(t)$ for all *i and all* $t \in [0,T]$ *except for*

$$U_k(0) = U_{k+1}(0). \tag{2.58}$$

Then the hulls K_t *approach* $\mathbb{R}$ *at* $U_k(0)$ *in* (α,β)*-direction if*

$$\begin{aligned} \lim_{t\searrow 0} \frac{U_k(t)-U_k(0)}{\sqrt{t}} &= \psi_1(\alpha,\beta) - \psi_2(\alpha,\beta) \\ \lim_{t\searrow 0} \frac{U_{k+1}(t)-U_{k+1}(0)}{\sqrt{t}} &= \psi_1(\alpha,\beta) + \psi_2(\alpha,\beta), \end{aligned} \tag{2.59}$$

where $\psi_1(\alpha,\beta)$ *and* $\psi_2(\alpha,\beta)$ *are given by*

$$\begin{aligned} \psi_1(\alpha,\beta) &= \sqrt{c}\frac{(1+x-3a-3bx)}{\sqrt{a(1-a)-2abx+b(1-b)x^2}} \\ \psi_2(\alpha,\beta) &= \sqrt{c}\sqrt{\frac{(1-a)^2+2x(a+b+ab-1)+x^2(1-b)^2}{a(1-a)-2abx+b(1-b)x^2}}, \end{aligned} \tag{2.60}$$

where $\alpha = a\pi$*,* $\beta = b\pi$*, and* $x = x(a,b)$ *is the unique negative root of (2.28).*

Remark 1. In the case when $0 < \alpha = \beta < \pi$, condition (2.59) simplifies to

$$\begin{aligned} \lim_{t\searrow 0} \frac{U_k(t)-U_k(0)}{\sqrt{t}} &= -\sqrt{2c}\sqrt{\frac{\pi-2\alpha}{\alpha}}, \text{ and} \\ \lim_{t\searrow 0} \frac{U_{k+1}(t)-U_{k+1}(0)}{\sqrt{t}} &= \sqrt{2c}\sqrt{\frac{\pi-2\alpha}{\alpha}}. \end{aligned} \tag{2.61}$$

Before proving Theorem 2.3, we recall the definition of conformal radius (Definition 3) and the Loewner scaling property (Lemma 2.4), both of which will be necessary for the proof.

Definition 3. For any compact $\mathbb{H}$-hull K and a point $w \in \mathbb{H} \setminus K$, the conformal radius is given by

$$\operatorname{rad}(w, \mathbb{H}\setminus K) = \frac{\Im(g_K(w))}{|g_K'(w)|}. \tag{2.62}$$

By the Kobe 1/4 theorem, the conformal radius satisfies

$$\frac{\operatorname{rad}(w,\mathbb{H}\setminus K)}{4} \le d\left(w, \partial\left(\mathbb{H}\setminus K\right)\right) \le \operatorname{rad}(w,\mathbb{H}\setminus K). \tag{2.63}$$

While Loewner scaling described in the next lemma is a basic and well-known property of the single-slit Loewner equation, we include the short proof here to make clear that it also holds more generally for measures μ_t.

Lemma 2.4. *If the hulls $\{K_t\}_{t\geq 0}$ are generated by the Loewner chain g_t with driving measure μ_t, then for each ρ, the chain g_t^ρ with driving measure $\rho\mu_{t/\rho^2}$ generates the family of hulls $\{\rho K_{t/\rho^2}\}_{t\geq 0}$.*

Proof. Let $g_t := g_{K_t}$ as has been our convention, and for each t let $g_{\rho K_t}$ be the unique conformal mapping with hydrodynamic normalization such that $g_{\rho K_t} : \mathbb{H} \setminus \rho K_t \to \mathbb{H}$, as in the introduction. Let $\hat{t} = \rho^2 t$, and let

$$g_{\hat{t}}^\rho(z) := g_{\rho K_t}(z), \tag{2.64}$$

so that the $g_{\hat{t}}^\rho$ generate the scaled hulls ρK_t. It is a basic property of conformal mappings that $g_{\rho K_t}(z) = \rho g_{K_t}(z/\rho)$, so that in fact $g_{\hat{t}}^\rho(z) = \rho g_{K_t}(z/\rho)$. We compute

$$\begin{aligned} \frac{d}{d\hat{t}} g_{\hat{t}}^\rho(z) &= \frac{1}{\rho^2} \cdot \frac{d}{dt} \rho g_{K_t}(z/\rho) \\ &= \frac{1}{\rho} \int_{\mathbb{R}} \frac{\mu_t(du)}{g_{K_t}(\frac{z}{\rho}) - u} \\ &= \int_{\mathbb{R}} \frac{\mu_{\hat{t}/\rho^2}(du)}{g_{\hat{t}}^\rho(z) - \rho u}, \end{aligned} \tag{2.65}$$

from which the conclusion of the lemma follows. $\square$

Proof of Theorem 2.3. So that the reader does not get bogged down in the details, we begin by outlining the arc of the proof. If L_t is the growing family of hulls parametrized by $2c$ times half-plane capacity such that each L_t is a union of two rays from the origin given by angles α and $\pi - \beta$, let g_t^∞ denote the corresponding Loewner chain, and let $V_1(t)$ and $V_2(t)$ denote the corresponding driving functions given by (2.25). Let g_t^ρ denote the Loewner chain with driving functions $\rho U_i(t/\rho^2)$, where $\{U_i\}_{i=1}^n$ are the driving functions of g_t. If $\{K_t\}$ is the family of hulls generated by g_t, then by Loewner scaling (Lemma 2.4)

g_t^ρ generates hulls $\{\rho K_{t/\rho^2}\}$. By Lemma 2.2, for $R > 0$, it is sufficient to show Hausdorff convergence inside the disk D_R of the rescaled hulls to L_1. Equation (2.63) allows us to use conformal radius to bound Hausdorff distance. Bounding the conformal radius requires that we show that for large enough ρ, $|\dot{g}_t^\infty - \dot{g}_t^\rho|$ and $\left|\frac{\partial}{\partial z} g_t^\infty - \frac{\partial}{\partial z} g_t^\rho\right|$ are uniformly bounded and arbitrarily small for $0 \leq t \leq 1$, and $z \in D_R$ such that the distance between z and the boundaries of the hulls is at least δ. These two estimates are assumed in the proof that follows and are proven separately in Lemmas 2.5 and 2.6. We will also make use of the elementary results A.1-A.3, the proofs of which are included in the appendix.

By the translation property of the Loewner equation, we can assume without loss of generality that

$$U_k(0) = U_{k+1}(0) = 0. \tag{2.66}$$

The results of §2.1 show that the following Loewner chain generates a union of two straight slits starting at 0 meeting the real line at angles α and $\pi - \beta$ for $0 < \alpha < \pi - \beta < \pi$:

$$\dot{g}_t^\infty(z) = \frac{c}{g_t^\infty(z) - V_1(t)} + \frac{c}{g_t^\infty(z) - V_2(t)}, \tag{2.67}$$

where

$$\begin{aligned} V_1(t) &= \sqrt{t}\left(\psi_1(\alpha,\beta) - \psi_2(\alpha,\beta)\right) \\ V_2(t) &= \sqrt{t}\left(\psi_1(\alpha,\beta) + \psi_2(\alpha,\beta)\right). \end{aligned} \tag{2.68}$$

Let L denote the hull generated by this evolution on the interval $t \in [0,1]$ (i.e. $L = L_1$ is the hull generated at time $t = 1$).

For each $\rho > 0$, let $g_t^{(\rho)}$ to be the family of conformal mappings that satisfy the Loewner equation with driving points $\rho U_i(t/\rho^2)$:

$$\begin{aligned} \dot{g}_t^\rho(z) &= \frac{c}{g_t^{(\rho)}(z) - \rho U_k(t/\rho^2)} + \frac{c}{g_t^{(\rho)}(z) - \rho U_{k+1}(t/\rho^2)} + \sum_{i \neq k, k+1} \frac{c}{g_t^{(\rho)}(z) - \rho U_i(t/\rho^2)}, \\ g_0^\rho(z) &= z. \end{aligned} \tag{2.69}$$

Let $K_t^{(\rho)}$ denote the hulls generated by g_t^ρ, and let K_t denote the hulls generated by g_t. Then by the scaling property of the Loewner equation (Lemma 2.4),

$$K_t^{(\rho)} = \rho K_{t/\rho^2}, \tag{2.70}$$

so in particular,

$$K_1^{(\rho)} = \rho K_{1/\rho^2}. \tag{2.71}$$

Let

$$D_R := \{z \in \mathbb{H} : |z| \leq R\}. \tag{2.72}$$

Lemma 3.8.3 in [Sch13] implies that for every $\epsilon > 0$, there is $\tau > 0$ such that $g_\epsilon(K_t)$ is a union of n connected components for $0 < t \leq \tau$. We can find a single $\tau > 0$ that holds for all small ϵ by the composition property of the Loewner equation. This implies that for $t \in (0, \tau]$, K_t is a union of at least $n-1$ connected components. (The k^{th} and $k+1^{th}$ connected components may coalesce, since $U_k(0) = U_{k+1}(0)$.) Since each component is bounded, and since 0 is only a boundary point of the k^{th} and $k+1^{th}$ components, for sufficiently large ρ, D_R contains only the connected components of $\rho K_{1/\rho^2}$ that correspond to U_k and U_{k+1} (which, again, may turn out to be a single connected component).

By Lemma 2.2, the hulls K_t approach $U_k(0)$ in (α, β)-direction if

$$\rho K_{1/\rho^2} \cap D_R \underset{\rho \to \infty}{\overset{\text{Haus.}}{\longrightarrow}} L \cap D_R. \tag{2.73}$$

We will show that

$$D_R \cap \partial \left(\mathbb{H} \setminus K_1^{(\rho)} \right) \overset{\text{Haus.}}{\longrightarrow} D_R \cap \partial \left(\mathbb{H} \setminus L \right). \tag{2.74}$$

Notice that there cannot be any components of $K_1^{(\rho)}$ that converge to subsets of $\mathbb{R}$ as ρ increases, because $K_1^{(\rho)} = \rho K_{1/\rho^2}$. For this reason (2.74) implies (2.73). In order to prove

(2.74), we will use the relationship between conformal radius and Hausdorff distance given in (2.63).

Fix $\epsilon > 0$. We prove the Hausdorff convergence (2.74) by verifying the following.

1. If ρ is sufficiently large, then every $v' \in D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right)$ is within distance 2ϵ of $\partial\left(\mathbb{H} \setminus L\right)$.
2. If ρ is sufficiently large, then every $u \in \partial\left(\mathbb{H} \setminus L\right)$ is within 2ϵ distance of $D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right)$.

We begin by proving 1. For $\delta > 0$ denote

$$K_{1,\delta}^{(\rho)} = \{z \in \mathbb{H} : d(z, K_1^{(\rho)}) \leq \delta\}. \tag{2.75}$$

We will first show that there is a small enough $\delta > 0$ and a large enough ρ^* such that for all $\rho > \rho^*$ and all $v \in D_R \cap \partial K_{1,\delta}^{(\rho)}$

$$d(v, L \cup \mathbb{R}) < \epsilon. \tag{2.76}$$

Once we have shown this, we will use it to show that all the points in the desired set $D_R \cap \partial K_1^{(\rho)}$ are close to $\partial\left(\mathbb{H} \setminus L\right)$, which takes a short argument, since it is not guaranteed that every point in the boundary of of $K_1^{(\rho)}$ is close to a point in the boundary of $K_{1,\delta}^{(\rho)}$.

Notice that the geometry of L implies that

$$\partial\left(\mathbb{H} \setminus L\right) = L \cup \mathbb{R}, \tag{2.77}$$

so we will use these expressions interchangeably.

Let $v \in D_R \cap \partial K_{1,\delta}^{(\rho)}$. If

$$d(v, L \cup \mathbb{R}) \leq \delta, \tag{2.78}$$

then there is nothing to prove, so in what follows we assume that

$$d(v, L \cup \mathbb{R}) > \delta. \tag{2.79}$$

In this case, $v \notin L$, so the conformal radius $\mathrm{rad}(v, \mathbb{H} \setminus L)$ is well defined, and

$$d(v, L \cup \mathbb{R}) \leq \mathrm{rad}(v, \mathbb{H} \setminus L), \tag{2.80}$$

implying that it will be sufficient to show that $\mathrm{rad}(v, \mathbb{H} \setminus L)$ is arbitrarily small for sufficiently large ρ.

The first two steps in this argument are to show that for large enough ρ and small enough $\delta > 0$, $|g_t^\rho(v) - g_t^\infty(v)|$ and $\left|\frac{d}{dz} g_t^\rho(v) - \frac{d}{dz} g_t^\infty(v)\right|$ are uniformly bounded and arbitrarily small for all

$$v \in D_R \cap \partial K_{1,\delta}^{(\rho)} \text{ such that } d(v, L \cup \mathbb{R}) > \delta. \tag{2.81}$$

These estimates are proven in Lemma 2.5 and Lemma 2.6, which follow this proof. Both arguments rely on a Grönwall inequality (as, for example, in the proof of Proposition 4.47 in [Law05]).

As a notational convention, we will let $'$ written directly above a function denote its partial derivative with respect to z, for example

$$\acute{g}_s^\rho(z) := \frac{\partial}{\partial z} g_s^\rho(z), \tag{2.82}$$

Assuming the results of Lemmas 2.5 and 2.6, we bound $\operatorname{rad}(v, \mathbb{H} \setminus L)$:

$$\begin{aligned}
\left|\operatorname{rad}(v,\mathbb{H}\setminus L) - \operatorname{rad}(v,\mathbb{H}\setminus K_1^{(\rho)})\right| &= \left|\frac{\Im g_1^\infty(v)}{|\acute{g}_1^\infty(v)|} - \frac{\Im g_t^\rho(v)}{|\acute{g}_1^\rho(v)|}\right| \\
&= \left|\frac{\Im g_1^\infty(v)\,|\acute{g}_1^\rho| - \Im g_1^\rho(v)\,|\acute{g}_1^\rho| + \Im g_1^\rho(v)\,|\acute{g}_1^\rho| - \Im g_1^\rho(v)\,|\acute{g}_1^\infty|}{|\acute{g}_1^\infty(v)|\,|\acute{g}_1^\rho(v)|}\right| \\
&\le \frac{|\Im g_1^\infty(v) - \Im g_1^\rho(v)|}{|\acute{g}_1^\infty(v)|} + \left|\frac{\Im g_1^\rho(v)\left(|\acute{g}_1^\rho(v)| - |\acute{g}_1^\infty(v)|\right)}{|\acute{g}_1^\infty(v)|\,|\acute{g}_1^\rho(v)|}\right| \\
&\le \frac{|g_1^\infty(v) - g_1^\rho(v)|}{|\acute{g}_1^\infty(v)|} + \operatorname{rad}(v,\mathbb{H}\setminus K_1^{(\rho)})\frac{|\acute{g}_1^\rho(v) - \acute{g}_1^\infty(v)|}{|\acute{g}_1^\infty(v)|} \\
&\le \frac{4R}{\Im g_1^\infty(v)}\left|g_1^\infty(v) - g_1^\rho(v)\right| + \operatorname{rad}(v,\mathbb{H}\setminus K_1^{(\rho)})\frac{4R}{\Im g_1^\infty(v)}\left|\acute{g}_1^\rho(v) - \acute{g}_1^\infty(v)\right|,
\end{aligned} \tag{2.83}$$

where the last inequality relied on the fact that

$$\frac{\Im g_1^\infty(v)}{|\acute{g}_1^\infty|} = \operatorname{rad}(v,\mathbb{H}\setminus L) \le 4d(v, L\cup\mathbb{R}) \le 4R. \tag{2.84}$$

Recall that by assumption $d(v, L\cup\mathbb{R}) \ge \delta$ (equation (2.79)), so in particular $\Im v \ge \delta$. Lemma A.1 guarantees that there is $\delta_2 > 0$ such that $\Im g_1^\infty(v) > \delta_2$. Since $|g_1^\rho(v) - g_1^\infty(v)|$ and $|\acute{g}_1^\rho(v) - \acute{g}_1^\infty(v)|$ are arbitrarily small for large ρ, we can choose ρ large enough that (2.83) is less than δ, so that

$$\begin{aligned}
\operatorname{rad}(v,\mathbb{H}\setminus L) &\le \operatorname{rad}\left(v,\mathbb{H}\setminus K_1^{(\rho)}\right) + \left|\operatorname{rad}(v,\mathbb{H}\setminus L) - \operatorname{rad}\left(v,\mathbb{H}\setminus K_1^{(\rho)}\right)\right| \\
&\le 5\delta.
\end{aligned} \tag{2.85}$$

This implies that

$$d\left(v, \partial\left(\mathbb{H}\setminus L\right)\right) \le 5\delta, \tag{2.86}$$

showing that every $v \in D_R \cap \partial K_{1,\delta}^{(\rho)}$ is close to $\partial\left(\mathbb{H}\setminus L\right)$. We note that the final steps above were necessary because v depends on ρ.

Next, we show that this implies that every point

$$v' \in D_R \cap K_1^{(\rho)} \tag{2.87}$$

is close to

$$D_R \cap \partial (\mathbb{H} \setminus L) = [-R, R] \cup (D_R \cap L) . \tag{2.88}$$

By the argument above, for large ρ and $0 < \delta < \frac{\epsilon}{5}$,

$$d (v, \partial (\mathbb{H} \setminus L)) < \epsilon, \tag{2.89}$$

for all

$$v \in D_R \cap \partial K_{1,\delta}^{(\rho)}. \tag{2.90}$$

But if $w_v \in \partial (\mathbb{H} \setminus L)$ is the point that minimizes the distance $d(v, w_v)$, then since $|v| \leq R$,

$$|w| \leq R + \epsilon. \tag{2.91}$$

The geometry of L implies that each point in

$$D_{R+\epsilon} \cap \partial (\mathbb{H} \setminus L) \tag{2.92}$$

is at most ϵ from a point in

$$D_R \cap \partial (\mathbb{H} \setminus L) . \tag{2.93}$$

This implies that

$$d (v, D_R \cap \partial (\mathbb{H} \setminus L)) < 2\epsilon. \tag{2.94}$$

Since $K_{1,\delta}^{(\rho)}$ is a hull (rather than an arbitrary set in $\mathbb{H}$), this implies that

$$D_R \cap \partial \left(\mathbb{H} \setminus K_{1,\delta}^{(\rho)}\right) \subset D_R \cap (L_{2\epsilon} \cup \{z \in \mathbb{H} : \Im z \leq 2\epsilon\}) . \tag{2.95}$$

Since the diameter of the set on the right-hand side is 2ϵ, and $K_1^{(\rho)} \subset K_{1,\delta}^{(\rho)}$, we conclude that

$$d \left(v', D_R \cap \partial (\mathbb{H} \setminus L)\right) \leq 2\epsilon. \tag{2.96}$$

for every

$$v' \in D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right), \tag{2.97}$$

proving 1.

Next, to prove 2, let $u \in D_R \cap L \subset D_R \cap \partial(\mathbb{H} \setminus L)$, and fix $0 < \delta < \frac{\epsilon}{5}$. We will show that

$$d\left(u, D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right)\right) < 2\epsilon. \tag{2.98}$$

Since L is the union of two rays meeting at 0, there is a $u' \in D_{R-\epsilon} \cap L$ such that

$$d(u, u') \leq \epsilon. \tag{2.99}$$

Again because of the specific geometry of L, we can find $w \in D_{R-\epsilon}$ such that $w \notin L$ and

$$d(u', w) = \delta. \tag{2.100}$$

We can assume that $w \notin K_1^{(\rho)}$, since otherwise trivially $d\left(u, \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right)\right) \leq \epsilon + \delta$, and we are done. We can further assume that

$$\begin{aligned} &\delta < d\left(w, \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right)\right), \quad \text{and} \\ &\delta < \Im w, \end{aligned} \tag{2.101}$$

since, again, otherwise there is nothing to prove. Let

$$\mathcal{T}_\rho := \{z \in \mathbb{H} : |z| \leq R, d(z, K_1^{(\rho)}) \geq \delta\}. \tag{2.102}$$

The method used in the proofs of Lemmas 2.5 and 2.6 can be used to show that there are large enough ρ and small enough δ such that for all $0 \leq t \leq 1$ and all $z \in \mathcal{T}_\rho$, $|\psi_t^\rho(z)|$ and $\left|\acute{\psi}_t^\rho(z)\right|$ are arbitrarily small. Since, in particular, $w \in \mathcal{T}_\rho$, we apply this result to conclude that if ρ and δ are chosen appropriately, then $|\psi_t^\rho(w)|$ and $\left|\acute{\psi}_t^\rho(w)\right|$ are arbitrarily small. Notice that w does not depend on ρ, so $|g_1^\infty(w)|$ and $|\acute{g}_1^\infty(w)|$ do not depend on ρ. If ρ is large enough that

$$|\acute{g}^{\rho}(w) - \acute{g}_1^{\infty}(w)| < |\acute{g}_1^{\infty}(w)|, \tag{2.103}$$

then

$$\begin{aligned}
\mathrm{rad}(w, \mathbb{H} \setminus K_1^{(\rho)}) &= \frac{\Im g_1^{\rho}(w)}{|\acute{g}_1^{\rho}(w)|} \\
&\leq \frac{|\Im g_1^{\rho}(w) - \Im g_1^{\infty}(w)| + \Im g_1^{\infty}(w)}{|\acute{g}_1^{\infty}(w)| - |\acute{g}^{\rho}(w) - \acute{g}_1^{\infty}(w)|} \\
&\leq \frac{|\Im g_1^{\rho}(w) - \Im g_1^{\infty}(w)|}{|\acute{g}_1^{\infty}(w)| - |\acute{g}^{\rho}(w) - \acute{g}_1^{\infty}(w)|} + \frac{\Im g_1^{\infty}(w)}{|\acute{g}_1^{\infty}(w)| - |\acute{g}^{\rho}(w) - \acute{g}_1^{\infty}(w)|} \\
&\leq \frac{|g_1^{\rho}(w) - g_1^{\infty}(w)|}{|\acute{g}_1^{\infty}(w)| - |\acute{g}^{\rho}(w) - \acute{g}_1^{\infty}(w)|} + \frac{\Im g_1^{\infty}(w)}{|\acute{g}_1^{\infty}(w)| - |\acute{g}^{\rho}(w) - \acute{g}_1^{\infty}(w)|} \\
&\leq \frac{|\psi_1^{\rho}(w)|}{|\acute{g}_1^{\infty}(w)| - \left|\acute{\psi}_1^{\rho}(w)\right|} + \frac{\Im g_1^{\infty}(w)}{|\acute{g}_1^{\infty}(w)| - \left|\acute{\psi}_1^{\rho}(w)\right|}.
\end{aligned} \tag{2.104}$$

This implies that for sufficiently large ρ,

$$\begin{aligned}
\mathrm{rad}(w, \mathbb{H} \setminus K_1^{(\rho)}) &\leq \mathrm{rad}(w, \mathbb{H} \setminus L) + \delta \\
&\leq 5\delta \\
&< \epsilon.
\end{aligned} \tag{2.105}$$

Since $|w| \leq R - \epsilon$, this implies that

$$d\left(w, D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right)\right) < \epsilon, \tag{2.106}$$

so that

$$d\left(u, D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right)\right) < 2\epsilon, \tag{2.107}$$

proving 2.

Since points in $[-R, R]$ are in both the boundary of $\mathbb{H} \setminus K_1^{(\rho)}$ and the boundary of $\mathbb{H} \setminus L$, the arguments above prove that if

$$v' \in D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right) \tag{2.108}$$

then

$$d\left(v', D_R \cap \partial\left(\mathbb{H} \setminus L\right)\right) < 2\epsilon, \tag{2.109}$$

and if

$$u \in D_R \cap \partial\left(\mathbb{H} \setminus L\right), \tag{2.110}$$

then

$$d\left(u, D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right)\right) < 2\epsilon. \tag{2.111}$$

Together, these imply that

$$D_R \cap \partial\left(\mathbb{H} \setminus K_1^{(\rho)}\right) \overset{\text{Haus.}}{\longrightarrow} D_R \cap \partial\left(\mathbb{H} \setminus L\right), \tag{2.112}$$

completing the proof, contingent upon assuming the results of Lemmas 2.5 and 2.6, which follow. □

Lemma 2.5. *Let $\delta^* > 0$. In the setting of Theorem 2.3 and the notation of its proof, let*

$$\mathcal{S} := \{z \in \mathbb{H} : |z| \leq R, d(z, L) \geq \delta\}. \tag{2.113}$$

Then there is a large enough ρ^* such that for all $\rho > \rho^*$, all $z \in \mathcal{S}$, and all $t \in [0,1]$

$$|\psi_s^\rho(z)| := |g_s^\rho(z) - g_s^\infty(z)| < \delta^*. \tag{2.114}$$

Proof. As before, let $V_1(t)$ and $V_2(t)$ denote the driving functions of g_t^∞. By Lemma A.2, there is $\delta_1 > 0$ such that for all $z \in \mathcal{S}$ and all $t \in [0,1]$,

$$\begin{aligned} &|g_t^\infty(z) - V_1(t)| \geq \delta_1, \quad \text{and} \\ &|g_t^\infty(z) - V_2(t)| \geq \delta_1. \end{aligned} \tag{2.115}$$

Then for all $z \in \mathcal{S}$ and $t \in [0,1]$,

$$\begin{aligned}
\left|\dot{\psi}_t^\rho(z)\right| =& \left|\frac{2}{g_t^\rho(z) - \rho U_k(t/\rho^2)} - \frac{2}{g_t^\infty(z) - V_1(t)} + \frac{2}{g_t^\rho(z) - \rho U_{k+1}(t\rho^2)} - \frac{2}{g_t^\infty(z) - V_2(t)}\right. \\
&\left. + \sum_{i \neq k,k+1} \frac{2}{g_t^\rho(z) - \rho U_i(t\rho^2)}\right| \\
\leq& \left|\frac{2}{g_t^\rho(z) - \rho U_k(t\rho^2)} - \frac{2}{g_t^\infty(z) - V_1(t)}\right| + \left|\frac{2}{g_t^\rho(z) - \rho U_{k+1}(t\rho^2)} - \frac{2}{g_t^\infty(z) - V_2(t)}\right| \\
&+ \left|\sum_{i \neq k,k+1} \frac{2}{g_t^\rho(z) - \rho U_i(t\rho^2)}\right| \\
=& 2\frac{|\psi_t^\rho(z)| + \left|\rho U_k(t/\rho^2) - V_1(t)\right|}{\left|\left(g_t^\rho(z) - \rho U_k(t/\rho^2)\right)\left(g_t^\infty(z) - V_1(t)\right)\right|} + 2\frac{|\psi_t^\rho(z)| + \left|\rho U_{k+1}(t/\rho^2) - V_2(t)\right|}{\left|\left(g_t^\rho(z) - \rho U_{k+1}(t/\rho^2)\right)\left(g_t^\infty(z) - V_2(t)\right)\right|} \\
&+ \left|\sum_{i \neq k,k+1} \frac{2}{g_t^\rho(z) - \rho U_i(t/\rho^2)}\right|
\end{aligned} \tag{2.116}$$

Let $\delta^* \in (0, \delta_1/4)$, and let

$$\sigma(\rho, z) = \min\{s : |g_t^\rho(z) - g_t^\infty(z)| \geq \delta^*\}. \tag{2.117}$$

We will show that $\sigma \geq 1$ for sufficiently large ρ and all $z \in \mathcal{S}$.

By Lemma A.3, $\rho U_k(t/\rho^2)$ and $\rho U_k(t/\rho^2)$ converge uniformly on $[0,1]$ to $V_1(t)$ and $V_2(t)$, respectively, so for any small $\kappa_1 > 0$ there is large enough ρ,

$$\begin{aligned}
\left|\rho U_k(t/\rho^2) - V_1(t)\right| &< \kappa_1 \\
\left|\rho U_{k+1}(t/\rho^2) - V_2(t)\right| &< \kappa_1.
\end{aligned} \tag{2.118}$$

Furthermore, we can bound the final term of (2.116) as follows. For each t, $\left|\rho U_i(t/\rho^2)\right| \underset{\rho\to\infty}{\longrightarrow} \infty$, and $g_t^\infty(z)$ is uniformly bounded for $z \in D_R$ and $0 \leq t \leq 1$, so if $0 \leq t \leq \sigma \wedge 1$, then

$$\begin{aligned}
\frac{2}{|g_t^\rho(z) - \rho U_i(t/\rho^2)|} &\leq \left|\frac{2}{|g_t^\infty(z) - \rho U_i(t/\rho^2)| - |\psi_t^\rho(z)|}\right| \\
&\leq \frac{2}{|g_t^\infty(z) - \rho U_i(t/\rho^2)| - \delta^*}.
\end{aligned} \tag{2.119}$$

This implies that for any small bound κ_2, there is sufficiently large ρ such that the final term in equation (2.116) is bounded by κ_2 for all $0 \leq t \leq 1$.

Next we use all of these estimates to bound the righthand side of (2.116). Of course

$$|g_t^\infty(z) - V_1(t)| \leq |g_t^\infty(z) - g_t^\rho(z)| + \left|g_t^\rho(z) - \rho U_k(t/\rho^2)\right| + \left|\rho U_k(t/\rho^2) - V_1(t)\right| \quad (2.120)$$

implies that

$$\left|g_t^\rho(z) - \rho U_k(t/\rho^2)\right| \geq |g_t^\infty(z) - V_1(t)| - |g_t^\infty(z) - g_t^\rho(z)| - \left|\rho U_k(t/\rho^2) - V_1(t)\right|, \quad (2.121)$$

so that if $0 \leq t \leq \sigma \wedge 1$, then

$$\left|g_t^\rho(z) - \rho U_k(t/\rho^2)\right| \geq \delta_1/2, \quad (2.122)$$

and similarly,

$$\left|g_t^\rho(z) - \rho U_{k+1}(t/\rho^2)\right| \geq \delta_1/2. \quad (2.123)$$

Then equation (2.116) becomes

$$\left|\dot{\psi}_t^\rho(z)\right| \leq (|\psi_t^\rho(z)| + \kappa_1)\left(\frac{8}{\delta_1^2}\right) + \kappa_2, \qquad 0 \leq t \leq \sigma \wedge 1. \quad (2.124)$$

In general, $\frac{d}{dt}|\psi_t^\rho(z)| \leq \left|\dot{\psi}_t^\rho(z)\right|$, so

$$\frac{d}{dt}|\psi_t^\rho(z)| \leq (|\psi_t^\rho(z)| + \kappa_1)\left(\frac{8}{\delta_1^2}\right) + \kappa_2, \qquad 0 \leq t \leq \sigma \wedge 1. \quad (2.125)$$

Solving this differential equation explicitly, we find that

$$|\psi_s^\rho(z)| \leq \left(\kappa_1 + \frac{\delta_1^2}{8}\kappa_2\right)\left(e^{8s/\delta_1^2} - 1\right), \qquad 0 \leq t \leq \sigma \wedge 1. \quad (2.126)$$

If κ_1 and κ_2 were initially chosen so that $\kappa_1 < \delta^*$,

$$\kappa_1(e^{8/\delta_1^2} - 1) < \delta^*/2, \quad (2.127)$$

and

$$\kappa_2 \delta_1^2 \left(e^{8/\delta_1^2} - 1 \right) < 4\delta^*, \tag{2.128}$$

then (2.126) implies that

$$|\psi_s^\rho(z)| < \delta^*, \qquad 0 \le t \le \sigma \wedge 1. \tag{2.129}$$

But σ was defined to be the first time s for which $|\psi_s^\rho(z)| \ge \delta^*$, so we conclude that $\sigma \ge 1$. In particular, $v \in \mathcal{S}$, so for sufficiently large ρ,

$$|\psi_s^\rho(v)| < \delta^*, \quad 0 \le t \le 1. \tag{2.130}$$

$\square$

Lemma 2.6. *Let $\delta^* > 0$. In the setting of Theorem 2.3 and the notation of its proof, let*

$$\mathcal{S} := \{z \in \mathbb{H} : |z| \le R, d(z, L) \ge \delta\}. \tag{2.131}$$

Then there is a large enough ρ^* such that for all $\rho > \rho^*$, all $z \in \mathcal{S}$, and all $t \in [0,1]$

$$\left| \frac{\partial}{\partial z} \psi_s^\rho(z) \right| := \left| \frac{\partial}{\partial z} g_s^\rho(z) - \frac{\partial}{\partial z} g_s^\infty(z) \right| < \delta^* \tag{2.132}$$

Proof. Recall our convention that (to avoid confusion with superscripts) the symbol ´ directly on top of a function denotes the partial derivative with respect to z. For example,

$$\acute{\dot{\psi}}_t^\rho(z) := \frac{\partial}{\partial z} \frac{\partial}{\partial t} \psi_t^\rho(z). \tag{2.133}$$

Fix δ_1 as in the proof of Lemma 2.5, and assume that $\delta^* \in (0, \delta_1/4)$.

For $z \in \mathcal{S}$ and $0 \le t \le 1$,

$$\begin{aligned}\left|\acute{\psi}_t^\rho(z)\right| =&\left|\frac{\acute{g}_t^\rho(z)}{(g_t^\rho(z)-\rho U_k(t/\rho^2))^2} - \frac{\acute{g}_t^\infty(z)}{(g_t^\infty(z)-V_1(t))^2} + \frac{\acute{g}_t^\rho(z)}{(g_t^\rho(z)-\rho U_{k+1}(t/\rho^2))^2} - \frac{\acute{g}_t^\infty(z)}{(g_t^\infty(z)-V_2(t))^2}\right.\\ &\left. + \sum_{i\neq k,k+1}\frac{\acute{g}_t^\rho(z)}{(g_t^\rho(z)-\rho U_i(t/\rho^2))^2}\right|\\ \le& \left|\frac{\acute{g}_t^\rho(z)}{(g_t^\rho(z)-\rho U_k(t/\rho^2))^2} - \frac{\acute{g}_t^\infty(z)}{(g_t^\infty(z)-V_1(t))^2}\right| + \left|\frac{\acute{g}_t^\rho(z)}{(g_t^\rho(z)-\rho U_{k+1}(t/\rho^2))^2} - \frac{\acute{g}_t^\infty(z)}{(g_t^\infty(z)-V_2(t))^2}\right|\\ &+\left|(\acute{g}_t^\rho(z)-\acute{g}_t^\infty(z)+\acute{g}_t^\infty(z))\sum_{i\neq k,k+1}\frac{1}{(g_t^\rho(z)-\rho U_i(t/\rho^2))^2}\right|\\ \le& \left|\frac{\acute{g}_t^\rho(z)}{(g_t^\rho(z)-\rho U_k(t/\rho^2))^2} - \frac{\acute{g}_t^\infty(z)}{(g_t^\infty(z)-V_1(t))^2}\right| + \left|\frac{\acute{g}_t^\rho(z)}{(g_t^\rho(z)-\rho U_{k+1}(t/\rho^2))^2} - \frac{\acute{g}_t^\infty(z)}{(g_t^\infty(z)-V_2(t))^2}\right|\\ &+|\acute{g}_t^\rho(z)-\acute{g}_t^\infty(z)|\left|\sum_{i\neq k,k+1}\frac{1}{(g_t^\rho(z)-\rho U_i(t/\rho^2))^2}\right| + |\acute{g}_t^\infty(z)|\left|\sum_{i\neq k,k+1}\frac{1}{(g_t^\rho(z)-\rho U_i(t/\rho^2))^2}\right|\end{aligned}\tag{2.134}$$

In the proof of Lemma 2.5, it was shown that the recurring sum term above is uniformly bounded by κ_2 satisfying (2.128), so

$$\begin{aligned}\left|\acute{\psi}_t^\rho(z)\right| \le& \left|\frac{\acute{g}_t^\rho(z)}{\left(g_t^\rho(z)-\rho U_k(t/\rho^2)\right)^2} - \frac{\acute{g}_t^\infty(z)}{\left(g_t^\infty(z)-V_1(t)\right)^2}\right| + \left|\frac{\acute{g}_t^\rho(z)}{\left(g_t^\rho(z)-\rho U_{k+1}(t/\rho^2)\right)^2} - \frac{\acute{g}_t^\infty(z)}{\left(g_t^\infty(z)-V_2(t)\right)^2}\right|\\ &+\left|\acute{\psi}_t^\rho(z)\right|\kappa_2 + |\acute{g}_t^\infty(z)|\,\kappa_2.\end{aligned}\tag{2.135}$$

The first two terms in (2.135) are nearly identical, so we consider only the first term. Then

$$\begin{aligned}
&\left|\frac{\acute{g}_t^\rho(z)}{(g_t^\rho(z)-\rho U_k(t/\rho^2))^2}-\frac{\acute{g}_t^\infty(z)}{(g_t^\infty(z)-V_1(t))^2}\right| \\
&\qquad\le \left|\frac{\acute{g}_t^\rho(z)(g_t^\infty(z)-V_1(t))^2-\acute{g}_t^\infty(z)\left(g_t^\rho(z)-\rho U_k(t/\rho^2)\right)^2}{(g_t^\rho(z)-\rho U_k(t/\rho^2))^2(g_t^\infty(z)-V_1(t))^2}\right| \\
&\qquad\le \left|\frac{\acute{g}_t^\rho(z)(g_t^\infty(z)-V_1(t))^2-\acute{g}_t^\infty(z)(g_t^\infty(z)-V_1(t))^2}{(g_t^\rho(z)-\rho U_k(t/\rho^2))^2(g_t^\infty(z)-V_1(t))^2}\right| \\
&\qquad\quad+\left|\frac{\acute{g}_t^\infty(z)(g_t^\infty(z)-V_1(t))^2-\acute{g}_t^\infty(z)\left(g_t^\rho(z)-\rho U_k(t/\rho^2)\right)^2}{(g_t^\rho(z)-\rho U_k(t/\rho^2))^2(g_t^\infty(z)-V_1(t))^2}\right| \\
&\qquad\le \frac{\left|\acute{\psi}_t^\rho(z)\right|}{|(g_t^\rho(z)-\rho U_k(t/\rho^2))^2|} \\
&\qquad\quad+|\acute{g}_t^\infty(z)|\left|\frac{(g_t^\infty(z)-V_1(t))^2-\left(g_t^\rho(z)-\rho U_k(t/\rho^2)\right)^2}{(g_t^\rho(z)-\rho U_k(t/\rho^2))^2(g_t^\infty(z)-V_1(t))^2}\right| \\
&\qquad= \frac{\left|\acute{\psi}_t^\rho(z)\right|}{|(g_t^\rho(z)-\rho U_k(t/\rho^2))^2|} \\
&\qquad\quad+|\acute{g}_t^\infty(z)|\,\frac{\left|g_t^\infty(z)-g_t^\rho(z)+\rho U_k(t/\rho^2)-V_1(t)\right|\left|g_t^\infty(z)-V_1(t)+g_t^\rho(z)-\rho U_k(t/\rho^2)\right|}{\left|g_t^\rho(z)-\rho U_k(t/\rho^2)\right|^2\left|g_t^\infty(z)-V_1(t)\right|^2} \\
&\qquad\le \frac{\left|\acute{\psi}_t^\rho(z)\right|}{|(g_t^\rho(z)-\rho U_k(t/\rho^2))^2|} \\
&\qquad\quad+|\acute{g}_t^\infty(z)|\,\frac{\left(|g_t^\infty(z)-g_t^\rho(z)|+\left|\rho U_k(t/\rho^2)-V_1(t)\right|\right)\left(|g_t^\infty(z)-V_1(t)|+\left|g_t^\rho(z)-\rho U_k(t/\rho^2)\right|\right)}{\left|g_t^\rho(z)-\rho U_k(t/\rho^2)\right|^2\left|g_t^\infty(z)-V_1(t)\right|^2} \\
&\qquad= \frac{\left|\acute{\psi}_t^\rho(z)\right|}{|(g_t^\rho(z)-\rho U_k(t/\rho^2))^2|}+|\acute{g}_t^\infty(z)|\left(|g_t^\infty(z)-g_t^\rho(z)|+\left|\rho U_k(t/\rho^2)-V_1(t)\right|\right) \\
&\qquad\qquad\qquad\cdot\left(\frac{1}{\left|g_t^\rho(z)-\rho U_k(t/\rho^2)\right|^2\left|g_t^\infty(z)-V_1(t)\right|}+\frac{1}{\left|g_t^\rho(z)-\rho U_k(t/\rho^2)\right|\left|g_t^\infty(z)-V_1(t)\right|^2}\right) \\
&\qquad\le \frac{\left|\acute{\psi}_t^\rho(z)\right|}{|(g_t^\rho(z)-\rho U_k(t/\rho^2))^2|}+|\acute{g}_t^\infty(z)|\left(|g_t^\infty(z)-g_t^\rho(z)|+\left|\rho U_k(t/\rho^2)-V_1(t)\right|\right)\left(\frac{6}{\delta_1^3}\right) \\
&\qquad\le \frac{\left|\acute{\psi}_t^\rho(z)\right|}{|(g_t^\rho(z)-\rho U_k(t/\rho^2))^2|}+|\acute{g}_t^\infty(z)|\left(|\psi_t^\rho(z)|+\kappa_1\right)\left(\frac{6}{\delta_1^3}\right) \\
&\qquad\le \frac{4}{\delta_1^2}\left|\acute{\psi}_t^\rho(z)\right|+\frac{6(|\psi_t^\rho(z)|+\kappa_1)}{\delta_1^3}\left|\acute{g}_t^\infty(z)\right| \\
&\qquad\le \frac{4}{\delta_1^2}\left|\acute{\psi}_t^\rho(z)\right|+\underbrace{\frac{6}{\delta_1^3}\left(\left(\kappa_1+\frac{\delta_1^2}{8}\kappa_2\right)\left(e^{8s/\delta_1^2}-1\right)+\kappa_1\right)}_{\kappa_3}\left|\acute{g}_t^\infty(z)\right|.
\end{aligned}\tag{2.136}$$

The calculation above made use of the following bounds from the proof of Lemma 2.5:

$$\begin{aligned}
\left|g_t^\rho(z) - \rho U_k(t/\rho^2)\right| \geq& \delta_1/2, \\
\left|g_t^\rho(z) - \rho U_{k+1}(t/\rho^2)\right| \geq& \delta_1/2, \\
\left|g_t^\infty(z) - V_1(t)\right| \geq& \delta_1, \\
\left|g_t^\infty(z) - V_2(t)\right| \geq& \delta_1, \\
\left|\rho U_k(t/\rho^2) - V_1(t)\right| <& \kappa_1 < \delta_1^4, \\
\left|\rho U_{k+1}(t/\rho^2) - V_2(t)\right| <& \kappa_1 < \delta_1^4, \quad \text{and} \\
|\psi_t^\rho(z)| = |g_t^\rho(z) - g_t^\infty(z)| \leq& \left(\kappa_1 + \frac{\delta_1^2}{8}\kappa_2\right)\left(e^{8s/\delta_1^2} - 1\right).
\end{aligned} \tag{2.137}$$

Then $\left|\acute{\psi}_t^\rho(z)\right|$ satisfies

$$\frac{d}{dt}\left|\acute{\psi}_t^\rho(z)\right| \leq \left|\acute{\psi}_t^\rho(z)\right|\left(\frac{8}{\delta_1^2} + \kappa_2\right) + (2\kappa_3 + \kappa_2)\left|\acute{g}_t^\infty(z)\right| \tag{2.138}$$

Since $|\acute{g}_t^\infty(z)| \leq \frac{R}{\delta}$ for all $z \in \mathcal{S}$ and all $0 \leq t \leq 1$,

$$\frac{d}{dt}\left|\acute{\psi}_t^\rho(z)\right| \leq \left|\acute{\psi}_t^\rho(z)\right|\left(\frac{8}{\delta_1^2} + \kappa_2\right) + \underbrace{\frac{R}{\delta}(2\kappa_3 + \kappa_2)}_{\kappa_4}. \tag{2.139}$$

Solving this differential equation explicitly, for sufficiently large ρ

$$\left|\acute{\psi}_s^\rho(z)\right| \leq \frac{\kappa_4(e^{s(8/\delta_1^2+\kappa_2)} - 1)}{8/\delta_1^2 + \kappa_2}, \qquad 0 \leq t \leq 1. \tag{2.140}$$

Since κ_2 and κ_1 can be chosen to be arbitrarily small, $\kappa_4 = \frac{R}{\delta}(2\kappa_3 + \kappa_2)$ is arbitrarily small for large ρ, so we can choose ρ large enough so that in particular

$$\left|\acute{\psi}_s^\rho(v)\right| \leq \delta^*, \qquad 0 \leq t \leq 1. \tag{2.141}$$

$\square$

Remark 2. Notice that for $0 < \alpha < \pi$, the range of $\sqrt{2c}\sqrt{\frac{\pi-2\alpha}{\alpha}}$ is $(0, \infty)$. This means that for any $0 < K < \infty$, if

$$\begin{aligned} \lim_{t \searrow 0} \frac{U_k(t) - U_k(0)}{\sqrt{t}} &= -K, \text{ and} \\ \lim_{t \searrow 0} \frac{U_{k+1}(t) - U_{k+1}(0)}{\sqrt{t}} &= K, \end{aligned} \tag{2.142}$$

then the hull meets $U_k(0)$ in (α, α)-direction, where

$$\alpha = \frac{\pi}{\frac{K^2}{4} + 2}. \tag{2.143}$$

Chapter Three

A Natural Tree Embedding

3.1 Choosing the diffusion

The results of Chapter 2 provide a sufficient condition for a driving measure to generate an embedding of a finite tree, and here we will apply those results to a specific driving measure. To specify the discrete driving measure μ_t on $0 \leq t \leq T$ that will embed a tree, there are two necessary pieces: first, a marked plane tree that will provide the underlying branching structure of the measure, and second, a specific time-evolution for the atoms of the measure during non-branching times. Both the tree and the evolution could be deterministic or random. We will show in Theorem 3.1 that the deterministic repulsion whereby each atom repels all the others according to the reciprocal of the distance between them satisfies the criteria laid out in Chapter 2, so that for any (deterministic or random) marked plane tree $\mathcal{T}^*$, the measure with the repulsion just described and branching structure given by $\mathcal{T}^*$ generates a graph embedding of $\mathcal{T}^*$ in $\mathbb{H}$.

Given the importance of the driving function $\sqrt{\kappa}B_t$ in the study of single slit Loewner equation (this is the driving function for which the evolution is SLE_κ), it is natural to consider the effect of using Dyson Brownian motion (n independent linear Brownian motions conditioned on non-intersection) as the driving measure for the multi-slit equation. Dyson Brownian motion is described by the stochastic differential equation

$$dx_i = \sum_{j \neq i} \frac{dt}{x_i - x_j} + dB^i, \tag{3.1}$$

where for each i, B^i is an independent linear Brownian motion. Intuitively, if this diffusion is used to prescribe the evolution of the atoms of the driving measure in between branching times, the result should be a kind of SLE tree. However, describing the geometric behavior of such a system would require advanced SLE techniques, and we do not attempt it here. Instead, we use only the deterministic part of Equation 3.1 in our construction, and to generalize slightly we scale by a constant $c_1 > 0$:

$$\frac{dx_i}{dt} = \sum_{j \neq i} \frac{c_1}{x_i - x_j}. \tag{3.2}$$

Theorem 3.1 below shows that the discrete driving measure with branching structure determined by a fixed marked plane tree $\mathcal{T}^*$ and evolution on non-branching intervals given by 3.2 generates an embedding of $\mathcal{T}^*$. Embedding continuous time Galton-Watson trees follows as a corollary.

The proof of Theorem 3.1 relies heavily on properties of the system of ordinary differential equations given by (3.2), so much of this section is devoted to ODE results.

3.2 The tree embedding theorem

We begin by stating the main result of this section.

Theorem 3.1 (Tree embedding theorem). *Let $\mathcal{T}^* = \{(\nu, h_\nu)\}$ be a binary marked plane tree, with $h_\nu \neq h_\eta$ for all $\nu \neq \eta$. Let $p(\nu)$ denote the parent of ν, and let $\Delta_t \mathcal{T}^*$ denote the set of elements "alive" at time t:*

$$\Delta_t \mathcal{T}^* = \{\nu \in \mathcal{T}^* : h(p(\nu)) \leq t < h(\nu)\}. \tag{3.3}$$

For $c, c_1 > 0$, let

$$\mu_t = c \sum_{\nu \in \Delta_t \mathcal{T}^*} \delta_{U_\nu(t)}, \tag{3.4}$$

where the U_ν evolve according to

$$\begin{aligned}
&\dot{U}_\nu(t) = \sum_{\substack{\eta \in \Delta_t \mathcal{T} \\ \eta \neq \nu}} \frac{c_1}{U_\nu(t) - U_\eta(t)}, \\
&U_\nu\left(h_{p(\nu)}\right) = \lim_{t \nearrow h_{p(\nu)}} U_{p(\nu)}(t), \ \ \textit{and} \\
&U_\emptyset(0) = 0.
\end{aligned} \tag{3.5}$$

Then for each $s \in [0, \max_{\nu \in \mathcal{T}^*} h_\nu]$, the hull K_s generated by the Loewner equation (1.1) with driving measure (3.4) is a graph embedding in $\mathbb{H}$ of the (unmarked) plane tree

$$\mathcal{T}_s = \{\nu \in \mathcal{T}^* : h_{p(\nu)} < s\}, \tag{3.6}$$

with the image of the root on $\mathbb{R}$.

The proof of this theorem relies on verifying a number of properties of system (3.5), which we will state as individual propositions in what follows.

We begin by considering the system on an interval where $\Delta_t \mathcal{T}^* = N$, and we change our notation to make clear to the reader that these results do not involve trees, but rather describe a particle system of N particles on the real line. For fixed N, let $\mathbb{R}^N_>$ denote the Weyl chamber:

$$\mathbb{R}^N_> = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < \dots < x_N\}. \tag{3.7}$$

Consider the initial value problem

$$\begin{aligned}
&\dot{X}(t) = \left(\sum_{j \neq 1} \frac{c_1}{x_1(t) - x_j(t)}, \dots, \sum_{j \neq N} \frac{c_1}{x_N(t) - x_j(t)} \right) \\
&X(t_1) = \left(x_1^0, \dots, x_N^0\right),
\end{aligned} \tag{3.8}$$

where x_i are the coordinates of $X = (x_1, \dots, x_N)$, and $(x_1^0, \dots, x_N^0) \in \mathbb{R}^N_>$ is a fixed initial condition at time $t_1 \in \mathbb{R}$.

Denote

$$f(t,X) = \left(\sum_{j\neq 1} \frac{c_1}{x_1 - x_j}, \ldots, \sum_{j\neq N} \frac{c_1}{x_N - x_j} \right), \tag{3.9}$$

Clearly,

$$|f(t,X)|^2 = \left(\sum_{j\neq 1} \frac{c_1}{x_1 - x_j} \right)^2 + \cdots + \left(\sum_{j\neq N} \frac{c_1}{x_N - x_j} \right)^2 \tag{3.10}$$

is continuous and finite in $\mathbb{R}^N_>$. Furthermore, for $X \in \mathbb{R}^N_>$, each partial derivative

$$\frac{\partial}{\partial x_i} f(t,X) \tag{3.11}$$

exists and is also continuous in $\mathbb{R}^N_>$, which is enough to guarantee that $f(t,X)$ is locally Lipschitz in the second variable. Since $f(t,X)$ doesn't depend on t at all, the Lipschitz constant does not depend on t. By the Picard-Lindelöf Theorem, there is an interval $(t_1 - \epsilon, t_1 + \epsilon)$ such that the solution exists and is unique. Let (t_0, T) denote the maximal interval where the unique solution exists. We seek to extend the solution to the left endpoint t_0 and show that uniqueness holds even if we take the initial condition to be defined at t_0.

Proposition 3.2. *If X is the unique solution to (3.8) on (t_0, T), then*

$$\min_{1\leq i<j\leq N} |x_i(t) - x_j(t)| \tag{3.12}$$

is a strictly increasing function of t.

Proof. For each i, let $D_i(t)$ denote the size of the i^{th} gap in the particle system:

$$D_i(t) = x_{i+1}(t) - x_i(t), \tag{3.13}$$

which is always nonnegative. Then

$$\begin{aligned}\dot{D}_i(t) &= \dot{x}_{i+1}(t) - \dot{x}_i(t) \\ &= \sum_{j\neq i+1} \frac{c_1}{x_{i+1}(t) - x_j(t)} - \sum_{j\neq i} \frac{c_1}{x_i(t) - x_j(t)} \\ &= \frac{2c_1}{x_{i+1}(t) - x_i(t)} + \sum_{j\neq i,i+1} \left(\frac{c_1}{x_{i+1}(t) - x_j(t)} - \frac{c_1}{x_i(t) - x_j(t)} \right).\end{aligned} \tag{3.14}$$

Fix s, and let k be an index (not necessarily unique) that minimizes $D_i(s)$:

$$D_k(s) \leq D_i(s) \quad \forall i \neq k. \tag{3.15}$$

Then

$$\dot{D}_k(s) = \frac{2c_1}{D_k(s)} - \sum_{i\neq k,k+1} \left(\frac{c_1 D_k(s)}{(x_{k+1}(s) - x_i(s))\,(x_k(s) - x_i(s))} \right). \tag{3.16}$$

Notice that since $D_k(s)$ is the minimum distance between any two x_j,

$$|x_{i+1}(s) - x_j(s)| \geq D_k(s) \cdot (i + 1 - j). \tag{3.17}$$

Also, for $j \neq i, i+1$, notice that the sign of $x_{i+1}(t) - x_j(t)$ is the same as that of $x_i(t) - x_j(t)$. Together, these facts imply that

$$\begin{aligned}\dot{D}_k(s) &\geq \frac{2c_1}{D_k(s)} - \sum_{j\neq k,k+1} \frac{c_1 D_k(s)}{D_k(s)^2 (k+1-j)(k-j)} \\ &= \frac{c_1}{D_k(s)} \left(2 - \sum_{l=1}^{k-1} \frac{1}{l(l+1)} - \sum_{l=1}^{N-k-1} \frac{1}{l(l+1)} \right).\end{aligned} \tag{3.18}$$

Noticing that

$$\frac{1}{l(l+1)} = \frac{1}{l} - \frac{1}{l+1}, \tag{3.19}$$

it is clear that for any $L < \infty$

$$\sum_{l=1}^{L} \frac{1}{l(l+1)} < 1, \tag{3.20}$$

so that the righthand side of (3.18) is always positive, so

$$\dot{D}_k(s) > 0. \tag{3.21}$$

Since the x_i are ordered on the real line according to their index,

$$\min_{i \neq j} |x_i(t) - x_j(t)| - \min_{i \neq j} |x_i(s) - x_j(s)| = \min_{1 \leq i \leq N-1} D_i(t) - \min_{1 \leq i \leq N-1} D_i(s). \tag{3.22}$$

Let $k_1, \ldots, k_l$ denote the indices that minimize $D_i(s)$:

$$D_{k_1}(s) = \cdots = D_{k_l}(s) < D_i(s), \quad \forall i \neq k_1, \ldots k_l. \tag{3.23}$$

Since l is finite and each $\dot{D}_{k_i}(s)$ is positive by (3.21), there is $\epsilon > 0$ such that

$$\dot{D}_{k_i}(t) > 0 \quad \forall t \in [s, s + \epsilon]. \tag{3.24}$$

Then for all $t \in (s, s + \epsilon]$,

$$\begin{aligned} \min_{i \neq j} |x_i(t) - x_j(t)| - \min_{i \neq j} |x_i(s) - x_j(s)| &= \min_{i = k_1, \ldots, k_l} D_i(t) - D_i(s) \\ &= \min_{i = k_1, \ldots, k_l} \int_s^t \dot{D}_i(\xi) d\xi \\ &> 0. \end{aligned} \tag{3.25}$$

□

Corollary 3.3. *The maximum interval where the solution to (3.8) is defined is* (t_0, ∞), *where either* $t_0 = -\infty$, *or there is a* k *such that*

$$\lim_{t \searrow t_0} |x_k(t) - x_{k+1}(t)| = 0. \tag{3.26}$$

Proof. It follows from Proposition 3.2 that as t decreases, the particles that are closest together continue to attract. □

Proposition 3.4. *Let* X *be the unique solution to (3.8), and let* (t_0, ∞) *be the maximum* *interval on which it is defined. Furthermore, assume that there is exactly one index k such that*

$$\lim_{t\searrow t_0} |x_{k+1}(t) - x_k(t)| = 0. \tag{3.27}$$

Then

$$\lim_{t\searrow t_0} X(t) \tag{3.28}$$

exists, so the solution X can be extended continuously at its left endpoint t_0. Furthermore, let $\hat{X}$ be the solution to the initial value problem

$$\begin{aligned} \frac{d}{dt}\widehat{X}(t) &= \left(\sum_{j\neq 1} \frac{c_1}{\hat{x}_1(t) - \hat{x}_j(t)}, \ldots, \sum_{j\neq N} \frac{c_1}{\hat{x}_N(t) - \hat{x}_j(t)}\right) \\ \widehat{X}(t_*) &= \left(\hat{x}_1^0, \ldots, \hat{x}_N^0\right), \end{aligned} \tag{3.29}$$

for some $t_* \in (t_0, \infty)$. If

$$\lim_{t\searrow t_0} X(t) = \lim_{t\searrow t_0} \widehat{X}(t), \tag{3.30}$$

then $X \equiv \widehat{X}$.

Proof. If $i \neq k, k+1$, then $\dot{x}_i(t)$ is bounded, so $\lim_{t\searrow t_0} x_i(t)$ exists, so we need only consider x_k and x_{k+1}.

To show that $\lim_{t\searrow t_0} x_k(t)$ and $\lim_{t\searrow t_0} x_{k+1}(t)$ exist, we will first solve a simplified version of the system (3.8) explicitly. For $t < \tau < T$, consider the related IVP in $\mathbb{R}^2_>$:

$$\begin{aligned} \dot{y}_k^{(\tau)}(t) &= \frac{c_1}{y_k^{(\tau)}(t) - y_{k+1}^{(\tau)}(t)} \\ \dot{y}_{k+1}^{(\tau)}(t) &= \frac{c_1}{y_{k+1}^{(\tau)}(t) - y_k^{(\tau)}(t)} \\ y_k^{(\tau)}(\tau) &= x_k(\tau) \\ y_{k+1}^{(\tau)}(\tau) &= x_{k+1}(\tau), \end{aligned} \tag{3.31}$$

where x_k and x_{k+1} are the k^{th} and $(k+1)^{th}$ coordinates of the unique solution to (3.8), and we have denoted the coordinates y_k, y_{k+1} to make the comparison with (3.8) more natural.

We can solve (3.31) explicitly for $y_k^{(\tau)}$ and $y_{k+1}^{(\tau)}$ by considering the equivalent integral equations:

$$\begin{aligned} y_k^{(\tau)}(t) &= x_k(\tau) - \int_t^\tau \frac{c_1}{y_k^{(\tau)}(s) - y_{k+1}^{(\tau)}(s)} ds, \quad \text{and} \\ y_{k+1}^{(\tau)}(t) &= x_{k+1}(\tau) - \int_t^\tau \frac{c_1}{y_{k+1}^{(\tau)}(s) - y_k^{(\tau)}(s)} ds. \end{aligned} \tag{3.32}$$

Notice that

$$y_{k+1}^{(\tau)} = -y_k^{(\tau)} + x_k(\tau) + x_{k+1}(\tau). \tag{3.33}$$

Substituting this identity into the differential equation for $y_k^{(\tau)}$, we have

$$\dot{y}_k^{(\tau)} = \frac{c_1}{2y_k^{(\tau)}(t) - x_k(\tau) - x_{k+1}(\tau)}. \tag{3.34}$$

Then

$$2y_k^{(\tau)}(t)\dot{y}_k^{(\tau)} - (x_k(\tau) + x_{k+1}(\tau))\dot{y}_k^{(\tau)} = c_1, \tag{3.35}$$

so that

$$\left(y_k^{(\tau)}(t)\right)^2 - (x_k(\tau) + x_{k+1}(\tau))y_k^{(\tau)} = c_1 t + C. \tag{3.36}$$

Solving for C by setting $t = \tau$, we see that

$$\left(y_k^{(\tau)}(t)\right)^2 - (x_k(\tau) + x_{k+1}(\tau))y_k^{(\tau)} + c_1(\tau - t) + x_k(\tau)x_{k+1}(\tau) = 0. \tag{3.37}$$

Solving this quadratic for $y_k^{(\tau)}$ yields exactly two solutions, which differ only in the order of the indices $y_k^{(\tau)}$ and $y_{k+1}^{(\tau)}$. Requiring that $y_k^{(\tau)}(t) \leq y_{k+1}^{(\tau)}(t)$ fixes a unique solution:

$$\begin{aligned} y_k^{(\tau)}(t) &= \frac{x_k(\tau) + x_{k+1}(\tau)}{2} - \frac{1}{2}\sqrt{(x_k(\tau) - x_{k+1}(\tau))^2 - 4c_1(\tau - t)}, \quad \text{and} \\ y_{k+1}^{(\tau)}(t) &= \frac{x_k(\tau) + x_{k+1}(\tau)}{2} + \frac{1}{2}\sqrt{(x_k(\tau) - x_{k+1}(\tau))^2 - 4c_1(\tau - t)}, \end{aligned} \tag{3.38}$$

which exists and is unique for $t \geq t_\tau$, where

$$t_\tau = \tau - \frac{1}{4c_1}\left(x_{k+1}(\tau) - x_k(\tau)\right)^2. \tag{3.39}$$

By assumption (3.27),

$$\begin{aligned}\lim_{\tau \searrow t_0} t_\tau &= \lim_{\tau \searrow t_0} \left(\tau - \frac{1}{4c_1} \left(x_{k+1}(\tau) - x_k(\tau) \right)^2 \right) \\ &= t_0.\end{aligned} \tag{3.40}$$

We want to show that for any sequence $\{s_n\}$ such that $s_n \searrow t_0$, the sequences $\{x_k(s_n)\}$ and $\{x_k(s_n)\}$ are Cauchy, and thus have limits. Notice that

$$\begin{aligned}|x_k(t) - x_k(s)| &= \left| y_k^{(t)}(t) - y_k^{(s)}(s) \right| \\ &\le \left| y_k^{(t)}(t) - y_k^{(t)}(s) \right| + \left| y_k^{(t)}(s) - y_k^{(s)}(s) \right|,\end{aligned} \tag{3.41}$$

and similarly,

$$|x_{k+1}(t) - x_{k+1}(s)| \le \left| y_{k+1}^{(t)}(t) - y_{k+1}^{(t)}(s) \right| + \left| y_{k+1}^{(t)}(s) - y_{k+1}^{(s)}(s) \right|, \tag{3.42}$$

so it will be sufficient to show that $\left| y_i^{(t)}(t) - y_i^{(t)}(s) \right|$ and $\left| y_i^{(t)}(s) - y_i^{(s)}(s) \right|$ are arbitrarily small when s and t are sufficiently close to t_0 for $i = k, k+1$. In what follows, we will formulate estimates only for $i = k+1$ since analogous estimates hold for $i = k$.

To bound $\left| y_{k+1}^{(t)}(t) - y_{k+1}^{(t)}(s) \right|$, notice that the general fact that $\left| \sqrt{A} - \sqrt{B} \right| \le \sqrt{|A - B|}$ implies that for $s, t \ge t_0$

$$\begin{aligned}\left| y_{k+1}^{(t)}(t) - y_{k+1}^{(t)}(s) \right| &= \frac{1}{2} \left| \sqrt{(x_{k+1}(t) - x_k(t))^2} - \sqrt{(x_{k+1}(t) - x_k(t))^2 - 4c_1(t-s)} \right| \\ &\le \sqrt{c_1 |t - s|}.\end{aligned} \tag{3.43}$$

Next, consider $\left| y_{k+1}^{(t)}(s) - y_{k+1}^{(s)}(s) \right|$. Let $s, t \ge t_0$. There exists M such that for all $t_0 \le \xi \le s \vee t$ and $\{i, j\} \ne \{k, k+1\}$,

$$\left| \frac{c_1}{x_i(\xi) - x_j(\xi)} \right| \le M. \tag{3.44}$$

Then

$$
\begin{aligned}
2\left|y_{k+1}^{(t)}(s)-y_{k+1}^{(s)}(s)\right| &= \left|x_{k+1}(t)+x_k(t)+\sqrt{(x_{k+1}(t)-x_k(t))^2-4c_1(t-s)}-2x_{k+1}(s)\right| \\
&= \Big|\,(x_{k+1}(t)+x_k(t)-x_{k+1}(s)-x_k(s)) \\
&\quad + \sqrt{(x_{k+1}(t)-x_k(t))^2-4c_1(t-s)}-(x_{k+1}(s)-x_k(s))\,\Big| \\
&= \left|\int_s^t (\dot{x}_{k+1}(\xi)+\dot{x}_k(\xi))\,d\xi \quad + \sqrt{(x_{k+1}(t)-x_k(t))^2-4c_1(t-s)}-(x_{k+1}(s)-x_k(s))\right| \\
&= \Big|\int_s^t \sum_{j\neq k,k+1}\left(\frac{c_1}{x_{k+1}(\xi)-x_j(\xi)}+\frac{c_1}{x_k(\xi)-x_j(\xi)}\right)d\xi \\
&\quad + \sqrt{(x_{k+1}(t)-x_k(t))^2-4c_1(t-s)}-(x_{k+1}(s)-x_k(s))\,\Big| \\
&\leq 2c_1M(N-2)\left|t-s\right| + \left|\sqrt{(x_{k+1}(t)-x_k(t))^2-4c_1(t-s)}\right| + \left|x_{k+1}(s)-x_k(s)\right|.
\end{aligned}
\tag{3.45}
$$

If $\epsilon > 0$, then there is $\delta > 0$ such that (3.43) and (3.45) are each less than $\epsilon/2$ whenever $|s-t_0| < \delta$ and $|t-t_0| < \delta$, so we conclude that if $\{s_n\}$ is a decreasing sequence with $s_n \searrow t_0$, then $\{x_{k+1}(s_n)\}$ is a Cauchy sequence, so its limit exists. The same argument holds for $\{x_k(s_n)\}$. Thus, we can continuously extend the solution X to the left endpoint t_0.

Uniqueness of the solution follows (analogously to the two-particle case) from the fact that solving the system near t_0 amounts to solving a quadratic equation, and only the solution that conforms with the labeling convention $x_k \leq x_{k+1}$ is kept. In particular, let

$$Z(t) = x_{k+1}(t-t_0) - x_k(t-t_0), \tag{3.46}$$

where x_k, x_{k+1} are the k^{th} and $k+1^{th}$ coordinate functions of the solution to system (3.8). Then $Z(0) = 0$ and Z satisfies

$$\frac{Z(t)}{2+\mathcal{C}_t Z^2(t)}\dot{Z}(t) = 1, \tag{3.47}$$

where

$$\mathcal{C}_t = \sum_{i\neq k,k+1} \frac{c_1}{(x_{k+1}(t) - x_i(t))\,(x_k(t) - x_i(t))}. \tag{3.48}$$

To see that (3.47) has exactly two solutions, notice that the function $\mathcal{C}_t$ is positive, continuous at t_0, and has bounded derivative on the interval $[t_0, t_0 + \delta)$ for $\delta > 0$ (in particular, it does not contain any terms of the form $x_{k+1}(t) - x_k(t)$). Considering the system on $[t_0, t_0 + \delta)$ for sufficiently small δ, the solution is closely approximated by solving (3.47) for $\mathcal{C}_t \equiv \mathcal{C}_{t_0}$, for which we have the explicit solution

$$Z^2(t) = \frac{2}{\mathcal{C}_{t_0}} \left(e^{2\mathcal{C}_{t_0}(t-t_0)} - 1 \right). \tag{3.49}$$

But only the positive square root of the righthand side above conforms to the labeling convention, so the solution is unique. $\square$

We now shift our attention to the rate at which the coordinates x_k and x_{k+1} separate from their initial position $x_k(t_0) = x_{k+1}(t_0)$. Eventually, this rate of separation will be used to determine the angles of approach of the Loewner hulls generated by μ_t defined above in (3.4).

Proposition 3.5. *Assume that there is a unique index k such that*

$$\lim_{t\searrow t_0} |x_{k+1}(t) - x_k(t)| = 0, \tag{3.50}$$

where $X(t) = (x_1(t), \ldots, x_N(t))$ is the unique solution to (3.8), which may be extended to the left endpoint t_0 by Proposition 3.4. Then

$$\begin{aligned} \lim_{t\searrow t_0} \frac{x_k(t) - x_k(t_0)}{\sqrt{t - t_0}} &= -\sqrt{c_1} \\ \lim_{t\searrow t_0} \frac{x_{k+1}(t) - x_{k+1}(t_0)}{\sqrt{t - t_0}} &= \sqrt{c_1}. \end{aligned} \tag{3.51}$$

Proof. We calculate

$$
\begin{aligned}
\lim_{t\searrow t_0} \frac{x_k(t)-x_k(t_0)}{\sqrt{t-t_0}} &= \lim_{t\searrow t_0}\lim_{\tau\searrow t_0}\left(\frac{x_k(t)-y_k^{(\tau)}(t)}{\sqrt{t-t_0}}+\frac{y_k^{(\tau)}(t)-x_k(t_0)}{\sqrt{t-t_0}}\right)\\
&= \underbrace{\lim_{t\searrow t_0}\lim_{\tau\searrow t_0}\frac{x_k(t)-y_k^{(\tau)}(t)}{\sqrt{t-t_0}}}_{A}+\underbrace{\lim_{t\searrow t_0}\lim_{\tau\searrow t_0}\frac{y_k^{(\tau)}(t)-x_k(t_0)}{\sqrt{t-t_0}}}_{B},
\end{aligned}
\tag{3.52}
$$

assuming that the limits A and B exist. We calculate A and B separately. First,

$$
\begin{aligned}
B &= \lim_{t\searrow t_0}\lim_{\tau\searrow t_0}\frac{y_k^{(\tau)}(t)-x_k(t_0)}{\sqrt{t-t_0}}\\
&= \lim_{t\searrow t_0}\lim_{\tau\searrow t_0}\frac{-x_k(t_0)+\frac{1}{2}(x_k(\tau)+x_{k+1}(\tau)-\sqrt{(x_{k+1}(\tau)-x_k(\tau))^2-4c_1(\tau-t)})}{\sqrt{t-t_0}}\\
&= \lim_{t\searrow t_0}\frac{-\sqrt{c_1}\sqrt{t-t_0}}{\sqrt{t-t_0}}\\
&= -\sqrt{c_1}.
\end{aligned}
\tag{3.53}
$$

Computing A using l'Hôpital's rule,

$$
\begin{aligned}
A &= \lim_{t\searrow t_0}\lim_{\tau\searrow t_0}\frac{x_k(t)-y_k^{(\tau)}(t)}{\sqrt{t-t_0}}\\
&= \lim_{t\searrow t_0}\lim_{\tau\searrow t_0} 2\sqrt{t-t_0}\left(\dot{x}_k(t)-\dot{y}_k^{(\tau)}(t)\right)\\
&= \lim_{t\searrow t_0}\lim_{\tau\searrow t_0} 2\sqrt{t-t_0}\left(\sum_{j\neq k,k+1}\frac{c_1}{x_k(t)-x_j(t)}+\frac{c_1}{x_k(t)-x_{k+1}(t)}-\frac{c_1}{y_k^{(\tau)}(t)-y_{k+1}^{(\tau)}(t)}\right)\\
&= \lim_{t\searrow t_0}\lim_{\tau\searrow t_0} 2\sqrt{t-t_0}\left(\frac{c_1}{x_k(t)-x_{k+1}(t)}-\frac{c_1}{y_k^{(\tau)}(t)-y_{k+1}^{(\tau)}(t)}\right)\\
&= \lim_{t\searrow t_0} 2\sqrt{t-t_0}\left(\frac{c_1}{x_k(t)-x_{k+1}(t)}-\frac{c_1}{-2\sqrt{c_1(t-t_0)}}\right)\\
&= \lim_{t\searrow t_0}\frac{2c_1\sqrt{t-t_0}}{x_k(t)-x_{k+1}(t)}+\sqrt{c_1},
\end{aligned}
\tag{3.54}
$$

where to obtain the fourth equality above we used the fact that for all $j \neq k, k+1$,

$\frac{c_1}{x_k(t)-x_j(t)} < M$, so that

$$\lim_{t\searrow t_0} 2\sqrt{t-t_0} \sum_{j\neq k,k+1} \frac{c_1}{x_k(t)-x_j(t)} = 0. \tag{3.55}$$

Adding A and B and substituting into (3.52),

$$\lim_{t\searrow t_0} \frac{x_k(t)-x_k(t_0)}{\sqrt{t-t_0}} = \lim_{t\searrow t_0} \frac{2c_1\sqrt{t-t_0}}{x_k(t)-x_{k+1}(t)}. \tag{3.56}$$

In order to simplify this expression, we notice that by a similar argument

$$\lim_{t\searrow t_0} \frac{x_{k+1}(t)-x_{k+1}(t_0)}{\sqrt{t-t_0}} = \lim_{t\searrow t_0} \frac{2c_1\sqrt{t-t_0}}{x_{k+1}(t)-x_k(t)}, \tag{3.57}$$

implying that

$$\lim_{t\searrow t_0} \frac{x_{k+1}(t)-x_{k+1}(t_0)}{\sqrt{t-t_0}} = -\lim_{t\searrow t_0} \frac{x_k(t)-x_k(t_0)}{\sqrt{t-t_0}}. \tag{3.58}$$

Using this equality and the fact that $x_k(t_0) = x_{k+1}(t_0)$,

$$\begin{aligned} \lim_{t\searrow t_0} \frac{x_k(t)-x_{k+1}(t)}{2\sqrt{t-t_0}} &= \lim_{t\searrow t_0} \frac{x_k(t)-x_k(t_0)+x_{k+1}(t_0)-x_{k+1}(t)}{2\sqrt{t-t_0}} \\ &= \lim_{t\searrow t_0} \frac{x_k(t)-x_k(t_0)}{\sqrt{t-t_0}}. \end{aligned} \tag{3.59}$$

Therefore, (3.56) becomes

$$\lim_{t\searrow t_0} \frac{x_k(t)-x_k(t_0)}{\sqrt{t-t_0}} = \lim_{t\searrow t_0} \frac{c_1\sqrt{t-t_0}}{x_k(t)-x_k(t_0)}, \tag{3.60}$$

so that

$$\lim_{t\searrow t_0} \frac{x_k(t)-x_k(t_0)}{\sqrt{t-t_0}} = \pm\sqrt{c_1}. \tag{3.61}$$

By the choice of labeling of x_k and x_{k+1}, we conclude that

$$\begin{aligned} &\lim_{t\searrow t_0} \frac{x_k(t)-x_k(t_0)}{\sqrt{t-t_0}} = -\sqrt{c_1}, \quad \text{and} \\ &\lim_{t\searrow t_0} \frac{x_{k+1}(t)-x_{k+1}(t_0)}{\sqrt{t-t_0}} = \sqrt{c_1}. \end{aligned} \tag{3.62}$$

□

Proposition 3.6. *On each interval $[s_i+\delta, s_{i+1})$, the measure μ_t defined by (3.4) and (3.5) for arbitrary $c, c_1 > 0$ generates simple curves.*

Proof. We show that the U_ν satisfy the simple curve condition given in [Sch12], Theorem 1.2, that is, we will show that U_ν is Hölder-1/2 and for any $t \in [s_i+\delta, s_{i+1})$ there is an ϵ such that

$$\frac{|U_\nu(r) - U_\nu(s)|}{\sqrt{r-s}} < 4\sqrt{c/2} \tag{3.63}$$

for all $s_i + \delta < s < r < s_{i+1}$ such that $|r-t| < \epsilon, |s-t| < \epsilon$.

For Hölder continuity, notice that Proposition 3.2 implies that $\dot{U}_\nu$ is bounded on $[s_i+\delta, s_{i+1})$, since the interval is bounded away from the birth time s_i. This implies that it is square integrable on the interval $[s_i+\delta, s_{i+1})$. Applying the Cauchy Schwartz inequality:

$$\begin{aligned} |U_\nu(r) - U_\nu(s)| &= \int_s^r \dot{U}_\nu(t)dt \\ &\leq \sqrt{\int_s^r dt}\sqrt{\int_s^r \dot{U}_\nu(t)^2 dt} \\ &= \sqrt{r-s}\sqrt{\int_s^r \dot{U}_\nu(t)^2 dt} \end{aligned} \tag{3.64}$$

so that for all $r, s \in [s_i+\delta, s_{i+1})$,

$$\frac{|U_\nu(r) - U_\nu(s)|}{\sqrt{r-s}} \leq \sqrt{\int_{s_i+\delta}^{s_{i+1}} \dot{U}_\nu(t)^2 dt} < \infty. \tag{3.65}$$

To show that U_ν satisfies the second criterion, fix $t \in (s_i+\delta, s_{i+1})$ and fix $\epsilon > 0$. Let $s_i + \delta < s < r < s_{i+1}$ such that

$$|r-t| < \epsilon, \quad |s-t| < \epsilon. \tag{3.66}$$

Then

$$\begin{aligned}
\frac{|U_\nu(r)-U_\nu(s)|}{\sqrt{r-s}} =& \frac{\left|\int_s^r \dot{U}_\nu(t)dt\right|}{\sqrt{r-s}} \\
\leq& \frac{\int_s^r \max_{t\in[r,s]}\left|\dot{U}_\nu(t)\right| dt}{\sqrt{r-s}} \\
=& \sqrt{r-s}\max_{t\in[r,s]}\left|\dot{U}_\nu(t)\right| \\
=& \sqrt{r-s}\max_{t\in[s_i+\delta,s_{i+1})}\left|\dot{U}_\nu(t)\right|.
\end{aligned} \tag{3.67}$$

This guarantees that (3.63) is satisfied, since $\max_{t\in[s_i+\delta,s_{i+1})}\left|\dot{U}_\nu(t)\right|$ is bounded. □

We now return to Theorem 3.1.

Proof of Theorem 3.1. Proposition 3.4 shows that the solution to the system is well-defined and unique on each interval $[s_i, s_{i+1})$. Proposition 3.5 guarantees that at each birth time, the two connected components that share a boundary point approach $\mathbb{R}$ in (α,α)-direction, for

$$\alpha = \frac{\pi}{2+\frac{c_1}{2c}}, \tag{3.68}$$

which we obtain by setting $\kappa = 2\frac{c_1}{c_2}$ in Equation (2.143) at the end of Theorem 2.3. Combining Propositions 3.5 and 3.6, we conclude that on each interval $[s_i, s_{i+1})$ the generated hull is a union of simple curves. Piecing together the solutions, we conclude that at each $0 \leq s \leq T$, the hull K_s is a graph embedding of the subtree

$$\mathcal{T}_s = \{\nu \in T^* : h_{p(\nu)} < s\}. \tag{3.69}$$

□

Remark 3. In particular, if $c = c_1$ in Theorem 3.1, then the curves approach in $(\frac{4}{9}\pi, \frac{4}{9}\pi)$-direction. In Chapter 4, we will consider sequences of measures that fall under this framework, where for each k, μ_t^k depends on constants $c^{(k)}$ and $c_1^{(k)}$. Requiring that $c^{(k)} = c_1^{(k)}$ guarantees that the angles do not depend on k.

Corollary 3.7. *If* θ_k *is distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean* $\frac{1}{2\sqrt{k}}$*, conditioned to have* k *edges, then with probability one Theorem 3.1 holds for* $\mathcal{T}^* = \theta_k$*.*

Proof. The only hypothesis that needs to be checked is that $h_\nu \neq h_\eta$ for all $\nu \neq \eta \in \theta_k$, but this holds with probability one. □

Remark 4. While Theorem 3.1 guarantees a graph embedding of the marked tree $\mathcal{T}^*$, there is no known sense in which it provides an isometric embedding. It is true that the distance between two points on $\mathcal{T}^*$ can be easily recovered from the family of hulls $\{K_s\}_{0 \leq s \leq T}$, since the instant each point $\nu \in \mathcal{T}^*$ is embedded is given by the distance on $\mathcal{T}^*$ from ν to the root of $\mathcal{T}^*$. However, there is not a strictly geometric interpretation of this quantity given only the hull K_T. The total weight of the driving measure is the derivative of the half-plane capacity of the growing hull, but there is no known geometric interpretation of the growth rate of the individual curves.

Chapter Four

The scaling limit of the driving measure

Theorem 3.1 provides a way to embed finite trees in the upper half-plane using the Loewner equation, and Corollary 3.7 specifies that this embedding holds for a specific class of critical binary Galton-Watson trees. We now turn our attention to the question of whether these finite tree embeddings have a well-defined scaling limit. We begin with a theorem that clarifies exactly how continuous time Galton-Watson trees converge to the CRT.

Theorem 4.1. *Let θ_k be distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned on the event that θ_k has exactly k edges. Then*

$$(\theta_k, d_{gr}) \xrightarrow{(d)} (\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}), \tag{4.1}$$

as $k \to \infty$ in the same sense as in Theorem 1.2.

Proof. Let $T_k = \text{Shape}(\theta_k)$, i.e. the plane tree that encodes the genealogical structure of θ_k but for which each edge has length one. (See [Pit06] for more detail on this construction. Elsewhere, Shape is referred to as the "skeleton".) Then T_k is a discrete time Galton-Watson tree with offspring distribution with generating function

$$\Phi(s) = \frac{1}{2} + \frac{1}{2}s^2. \tag{4.2}$$

Aldous proved that if $\mathfrak{T}_k$ is distributed as a Galton-Watson tree conditioned to have exactly k edges whose offspring distribution is of mean 1 and variance $\sigma^2 < \infty$ (as usual, excluding the trivial case where the offspring distribution is the Dirac mass at 1), then

$$\left(\frac{1}{\sqrt{2k}} C_{\mathfrak{T}_k}(2kt)\right)_{0\le t\le 1} \xrightarrow{\text{(d)}} \left(\frac{\sqrt{2}}{\sigma}\mathbb{e}_t\right)_{0\le t\le 1}. \tag{4.3}$$

In our case, the variance is one, so we conclude that

$$\left(\frac{1}{2\sqrt{k}} C_{T_k}(2kt)\right)_{0\le t\le 1} \xrightarrow{\text{(d)}} (\mathbb{e}_t)_{0\le t\le 1}. \tag{4.4}$$

Recall that there is a natural pairing of the segments of C_{T_k}, since for each edge of T_k there is an upward segment and a downward segment corresponding to this edge in the path

C_{T_k}. For each such pair, rescale their slope by an independent random variable distributed according to exp(1). Let $C_{\tilde{\theta}_k}$ denote the resulting excursion, and $\tilde{\theta}_k$ the associated real tree. (Only the heights are rescaled when passing from C_{θ_k} to $C_{\tilde{\theta}_k}$, not the width.) Appealing to a conditional version of Donsker's theorem, we conclude that

$$\left(\frac{1}{2\sqrt{k}} C_{\tilde{\theta}_k}(2kt)\right)_{0\le t\le 1} \xrightarrow{(d)} (\mathbb{e}_t)_{0\le t\le 1}\,. \tag{4.5}$$

Noticing that the real tree coded by $\frac{1}{2\sqrt{k}} C_{\tilde{\theta}_k}$ is $\frac{1}{2\sqrt{k}}\tilde{\theta}_k$, Theorem 1.1 implies that

$$\left(\frac{1}{2\sqrt{k}}\tilde{\theta}_k, d_{gr}\right) \xrightarrow{(d)} (\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})\,. \tag{4.6}$$

But $\frac{1}{2\sqrt{k}}\tilde{\theta}_k$ has the same distribution as θ_k, so (4.1) follows. $\square$

Since the CRT can be constructed as a limit of these continuous time Galton-Watson trees, it is natural to consider whether the corresponding random hulls generated in Theorem 3.1 have a nontrivial scaling limit. In particular, our investigation focuses on the following question.

Question 3. *For $k \ge 1$, let θ_k be distributed as a random critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned to have k edges. With probability one, Theorem 3.1 gives an embedding of θ_k in $\mathbb{H}$, so the distribution of θ_k induces a probability distribution on hulls in the upper half-plane. As $k \to \infty$, does this law converge to a nontrivial scaling limit?*

To gain intuition about the precise rescaling of the measure that will be needed to give rise to a geometric scaling limit of the hulls, we prove a bound on the Euclidean size of the generated hull for the scaled multi-slit equation, which depends on the scaling parameter c, the number of driving points n, and the time t. This result is a minor modification of [Law05] Lemma 4.13, but we include the proof for completeness, since it illuminates the role of each parameter.

Lemma 4.2. *For $t \in [0,T]$, let $g_t(z)$ be the solution to the initial value problem*

$$\dot{g}_t(z) = \sum_{i=1}^{n} \frac{c}{g_t(z) - U_i(t)}, \quad g_0(z) = z, \tag{4.7}$$

for continuous driving functions $U_1, \ldots, U_n$, and define K_t by $g_t(\mathbb{H} \setminus K_t) = \mathbb{H}$. Let

$$R_t = \max\left\{\sqrt{\frac{cnt}{2}}, \max_i \sup_{0 \le t \le T} |U_i(t)|\right\}. \tag{4.8}$$

If $|z| > 4R_t$, then $|g_s(z) - z| \le R_t$, for $0 \le s \le t$. As a consequence,

$$rad(K_t) \le 4R_t. \tag{4.9}$$

Proof of Lemma 4.2. Following the proof of [Law05] Lemma 4.13, let $|z| > 4R_t$, and let

$$\sigma = \sigma(z,t) := \min\{s : \max_i |g_s(z) - z| \ge R_t\}. \tag{4.10}$$

If $0 \le s \le t \wedge \sigma$, then

$$\begin{aligned} |\dot{g}_s(z)| &= \left|\sum_{i=1}^{n} \frac{c}{g_s(z) - U_i(s)}\right| \\ &\le \sum_{i=1}^{n} \frac{c}{|g_s(z) - U_i(s)|} \\ &\le \frac{cn}{2R_t}, \end{aligned} \tag{4.11}$$

since for each i,

$$\begin{aligned} |g_s(z) - U_i(s)| &\ge |z - U_i(s)| - |z - g_s(z)| \\ &\ge |z| - |U_i(s)| - |z - g_s(z)| \\ &> 4R_t - R_t - R_t \\ &= 2R_t. \end{aligned} \tag{4.12}$$

In this case,

$$\begin{aligned} \frac{d}{ds}|g_s(z) - z| &\le \left|\frac{d}{ds}(g_s(z) - z)\right| \\ &\le \frac{cn}{2R_t}, \end{aligned} \tag{4.13}$$

which implies that

$$|g_s(z) - z| \leq \frac{cns}{2R_t}. \tag{4.14}$$

By the definition of σ, either $\sigma > t$ or $R_t \leq \frac{cn\sigma}{2R_t}$. But since $\sqrt{\frac{cnt}{2}} \leq R_t$, this case is ruled out, implying that $\sigma \geq t$.

The fact that $\text{rad}(K_t) \leq 4R_t$ is seen as follows. Assume that there is $z \in K_t$ such that $|z| > 4R_t$. Since $z \in K_t$, there exists $\hat{s} \in (0, t]$ and $i \in \{1, \ldots, n\}$ such that $g_{\hat{s}}(z) = U_i(\hat{s})$. Then

$$\begin{aligned} |g_{\hat{s}}(z) - z| &= |U_i(\hat{s}) - z| \\ &\geq |z| - |U_i(\hat{s})| \\ &> 3R_t. \end{aligned} \tag{4.15}$$

But by the first part of the lemma, $|z| > R_t$ implies that $|g_{\hat{s}}(z) - z| < R_t$ for all $0 \leq \hat{s} \leq t$, so no such z exists. □

If N_t^k is the conditioned Galton-Watson process corresponding to θ_k, then we will see shortly that $\frac{1}{\sqrt{k}} N_t^k$ has a nontrivial scaling limit. Plugging in N_t^k for n in the lemma above suggests that as long as the support of the measure does not spread out too quickly, choosing $c = \frac{1}{\sqrt{k}}$ is the rescaling of the total mass of the driving measure that we expect will yield a nontrivial geometric limit.

4.1 Preliminaries: the driving measure as a superprocess

As a first step toward answering Question 3, we consider whether the corresponding random driving measures μ_t^k have a scaling limit. The language of spatial branching processes and superprocesses gives the correct framework for considering this question, so we begin by stating some definitions and classical results. (See [Eth00] for an overview of the subject.)

A *spatial branching process* is a discrete branching process (in continuous time) where each individual is endowed with a spatial motion in a Polish space E (in our case $E = \mathbb{R}$) and each offspring begins its spatial motion at the final location of its parent. It is common to study such processes in the case where the spatial motion of each particle is independent. (For example, in population genetics such spatial branching processes are used to model genetic variation, so that location in the Polish space E is interpreted as genetic type.) When the spatial motion of each particle is given by an independent linear Brownian motion, the resulting spatial branching process is called *branching Brownian motion.* In order for a spatial branching process to be a Markov process, the lifetimes of the particles must be independent and exponentially distributed. We will let V denote the parameter for these exponential lifetimes. Finally, to specify a spatial branching process we need specify an offspring distribution, which we record by its probability generating function:

$$\Phi(s) = \sum_{i=0}^{\infty} p_i s^i, \tag{4.16}$$

where p_i is the probability that an individual has i offspring.

Let $\mathcal{M}_F(\mathbb{R})$ denote the space of finite Borel measures on $\mathbb{R}$, equipped with the topology of weak convergence. If a branching process has genealogical tree $\mathcal{T}^*$ and particle locations denoted by $\{U_\nu(t) : \nu \in \Delta_t \mathcal{T}^*\}$, then we can equivalently consider the $\mathcal{M}_F(\mathbb{R})$-valued process that represents the entire state of the population, which is given by

$$\tilde{\mu}_t = \sum_{\nu \in \Delta_t \mathcal{T}^*} \delta_{U_\nu(t)}, \tag{4.17}$$

where δ_x denotes a Dirac mass at x.

Given a sequence $\{\mu^k\}_{k \geq 1}$ of $\mathcal{M}_F(\mathbb{R})$-valued processes, we want to determine whether the sequence converges to a limiting process. (This will require a rescaling of the total mass of the system, which we will specify shortly.) To find a limit, it is not sufficient to merely have convergence of the measure at each fixed time t, but instead we need to consider convergence of the whole processes. To do this, we work in the Skorokhod space $D_{\mathcal{M}_F(\mathbb{R})}[0, \infty)$ of càdlàg (i.e. right continuous with left limits) paths in $\mathcal{M}_F(\mathbb{R})$, which is endowed with the usual topology induced by the Skorokhod metric (see Definition 1.12 in [Eth00] for an equivalent and more transparent definition of this topology). Considering spatial branching processes as elements of $D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$ clarifies that we are aiming for a result that proves weak convergence of random variables in $D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$. Consequently, we aim to first show tightness of the sequence $\{\mu^k\}$ in $D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$, which guarantees that the sequence has at least one limit point, and then prove that in fact the limit point is unique.

To understand convergence of spatial branching processes as superprocesses, we first take a step back and examine the convergence of the Galton-Watson processes that describe the size of the total population. A continuous-state branching process is a Markov process $(Y_t, t \geq 0)$ in $\mathbb{R}_+$ that has càdlàg sample paths and whose transition kernels $P_t(x, dy)$ satisfy

$$P_t(x + x', \cdot) = P_t(x, \cdot) * P_t(x', \cdot). \tag{4.18}$$

If a collection of rescaled discrete branching processes $\tilde{N}_t^k$ (which represent the total population at time t) converge, then they converge to a continuous state branching process. An important example of this is the Galton-Watson approximation of the Feller diffusion. In particular, if $\{\tilde{N}_t^k\}_{k\geq 1}$ is a sequence of critical (discrete time) Galton-Watson processes, and if the initial population $x_k := \tilde{N}_0^k$ satisfies

$$\frac{x_k}{k} \to x_0, \tag{4.19}$$

for some $x_0 \geq 0$, then

$$\frac{1}{k}\tilde{N}^k_{\lfloor kt \rfloor} \tag{4.20}$$

converges weakly to the Feller diffusion, which is the solution to the stochastic differential equation

$$dY_t = \sqrt{\sigma Y} dB_t, \tag{4.21}$$

where σ is a diffusion parameter determined by the variance of the offspring distribution.

We can use this result to find the limit of a sequence of continuous time Galton-Watson processes $\{N_t^k\}_{k\geq 1}$ by approximating each N_t^k by a discrete time Galton-Watson process $\tilde{N}_t^k$ obtained by evaluating N_t^k at integer multiples of the mean lifetime. In our case, we will fix the mean lifetime to be $\frac{1}{V\sqrt{k}}$ so we define

$$\tilde{N}_t^k = N^k_{\lfloor V\sqrt{k}t\rfloor/V\sqrt{k}}. \tag{4.22}$$

If the lifetimes are exponential with mean α, then N_t^k is described by the generating function given in [NP89]:

$$\frac{1}{1-F_t(s)} = \frac{1}{1-s} + \frac{t}{2\alpha}, \quad t \in \mathbb{R}_+, 0 \leq s \leq 1. \tag{4.23}$$

Evaluating this for $t = \alpha$ (in our case, $\alpha = \frac{1}{V\sqrt{k}}$) we see that

$$F(s) = \frac{1}{3} + 4\sum_{i=1}^{\infty} \frac{s^i}{3^{i+1}}. \tag{4.24}$$

Since continuous time Galton-Watson processes are Markov processes, we can use this as the offspring distribution for the discrete time approximation $\tilde{N}_t^k$, which is a Markov chain. Since (4.24) has mean 1, it is again critical, so rescaling $\tilde{N}_t^k$ by the length of the intervals gives the Feller diffusion:

$$\frac{1}{V\sqrt{k}}\tilde{N}_t^k \to Y_t, \tag{4.25}$$

where Y_t is given by (4.21). Since $\tilde{N}_t^k$ closely approximates N_t^k, this same scaling holds for the continuous time process:

$$\frac{1}{V\sqrt{k}}N_t^k \to Y_t. \tag{4.26}$$

The discussion above clarifies why the same rescaling is used in both the discrete time and continuous time settings. Returning to the setting of superprocesses, this rescaling by mean lifetime can be used to obtain the Dawson-Watanabe superprocess (which is called "superbrownian motion" in [LG99] and elsewhere) as a limit of branching Brownian motions. In particular, for each k, let $\xi^k \in D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$ be a branching Brownian motion, i.e.

a spatial branching process where the particles move independently according to Brownian motion, and assume that the lifetimes of the particles of ξ^k are exponential with parameter $V\sqrt{k}$ and that at every stage the branching mechanism is the same and is critical (mean one). If the rescaled initial states $\frac{\xi_0^k}{\sqrt{k}}$ form a tight sequence of measures, then the sequence $\{X^k\} \subset D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$ defined by

$$X_t^k = \frac{1}{\sqrt{k}}\xi_{kt}^k \tag{4.27}$$

converges to a unique limit, which is called the Dawson-Watanabe superprocess.

We would like to produce a similar result for the sequence of measures $\{\mu^k\}$ defined by (3.4) and (3.5) when the branching structure is determined by a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$ and the scaling constants are

$$c^{(k)} = c_1^{(k)} = \frac{1}{\sqrt{k}} \tag{4.28}$$

(so that, as described in Remark 3, the corresponding Loewner hulls are tree embeddings whose approach angles do not depend on k). However, the standard proof of the convergence to the Dawson-Watanabe superprocess relies on the independent motions of the particles. Since in our setting the atoms comprising μ_t^k evolve dependently, our approach to finding a limiting system is different. In §4.2 below, we will show tightness of the sequence $\{\mu^k\}$, and in §4.3 we will make progress toward identifying the limiting process by reframing the problem in terms of the Stieltjes transform of the measures.

4.2 Tightness of the sequence $\{\mu^k\}_{k\geq 1}$

In order to prove tightness of $\{\mu^k\}_{k\geq 1}$, we first define the metric on $\mathcal{M}_F(\mathbb{R})$. Let

$$Lip^1(\mathbb{R}) := \{\varphi : \mathbb{R} \to \mathbb{R} \text{ such that } \|\varphi\|_\infty \leq 1 \text{ and } |\varphi(x) - \varphi(y)| \leq |x-y|\,, \forall x, y \in \mathbb{R}\}. \tag{4.29}$$

Definition 4. The Wasserstein metric on $\mathcal{M}_F(\mathbb{R})$ is defined by

$$\rho(\mu,\nu) = \sup\{|\langle\varphi,\mu\rangle| - |\langle\varphi,\nu\rangle|\} : \varphi \in Lip^1(\mathbb{R})\}. \tag{4.30}$$

In fact, the Wasserstein metric induces the topology of weak convergence on $\mathcal{M}_F(\mathbb{R})$, so it is well suited for our purposes. It plays a key role in Theorem 4.3 below, which we will use to prove that $\{\mu^k\}_{k\geq 1}$ is tight. We define one more piece of notation before stating the theorem. Let

$$w'(x,\delta,T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} \rho(x(s),x(t)), \tag{4.31}$$

where $\{t_i\}$ ranges over all partitions

$$0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n \tag{4.32}$$

with $n \geq 1$ and

$$\min_i |t_i - t_{i-1}| > \delta. \tag{4.33}$$

Theorem 4.3. *[[EK], Corollary 7.4 of chapter 3] Let $\{X_k\}_{k\geq 1}$ be a family of random variables with sample paths in $D_{\mathcal{M}_F(\mathbb{R})[0,\infty)}$. If the following two conditions hold, then $\{X_k\}_{k\geq 1}$ is tight.*

1. *For every $\epsilon > 0$ and rational $t \geq 0$, there exists a compact set $\Gamma_{\epsilon,t} \subset \mathcal{M}_f(\mathbb{R})$ such that*

$$\liminf_{k\to\infty} P\left[X_k(t) \in \Gamma_{\epsilon,t}\right] \geq 1-\epsilon. \tag{4.34}$$

2. *For every $\epsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that*

$$\limsup_{k\to\infty} P\left[w'(X_k,\delta,T) \geq \epsilon\right] \leq \epsilon. \tag{4.35}$$

Theorem 4.4. *Let $\{\mu^k\}_{k\geq 1} \subset D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$ be the sequence of measure-valued processes, where for each k, μ^k_t is the discrete measure defined by (3.4) and (3.5) for $c^{(k)} = c_1^{(k)} = \frac{1}{\sqrt{k}}$, with initial value $\mu^k_0 = \frac{1}{\sqrt{k}}\delta_0$ whose branching structure is given by a random tree θ_k*

distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$*, conditioned to have* k *edges. Then* $\{\mu^k\}_{k\geq 1}$ *is tight in* $D_{\mathcal{M}_F(\hat{\mathbb{R}})[0,\infty)}$.

Proof. We show that $\{\mu^k\}_{k\geq 1}$ satisfies the conditions of Theorem 4.3.

1. Following the discussion in [Eth00] §1.4, let $\hat{\mathbb{R}}$ denote the compactification of $\mathbb{R}$. We will show that the compact containment condition holds in $\mathcal{M}_F(\hat{\mathbb{R}})$. Because of this extension, later we will need to show that in the limit no mass "escapes to infinity" so, in fact, the limit is truly a process in $D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$.

In fact, it is known that the rescaled total mass process converges to the total local time at level t of the normalized Brownian excursion, which we denote by $L^t_{\mathbb{e}} := L^t_{\mathbb{e}}(1)$ [Pit06]:

$$\langle 1, \mu^k_t \rangle = \frac{1}{\sqrt{k}} N^k_t \xrightarrow{(\mathrm{d})} L^t_{\mathbb{e}}. \tag{4.36}$$

(This is analogous to convergence of the unconditioned process to the Feller diffusion.) To compute probabilities, we use the following comparison between the supremum of the local time and the supremum of the reflected Brownian bridge (see [Pit99], equation (35)):

$$\sup_{t\geq 0} L^t_{\mathbb{e}} \overset{(\mathrm{d})}{=} 4 \sup_{0\leq t\leq 1} B^{|\mathrm{br}|,1}_t, \tag{4.37}$$

where $B^{|\mathrm{br}|,1}$ is the reflected Brownian bridge of length 1. The righthand side of (4.37) is given by the well-known Kolmogorov-Smirnov formula:

$$P\left(\sup_{0\leq t\leq 1} B^{|\mathrm{br}|,1}_t \leq x\right) = 1 + 2\sum_{i=-\infty}^{\infty} (-1)^i e^{-2i^2x^2}, \tag{4.38}$$

implying that for each $\epsilon > 0$ there exists a K_ϵ such that

$$\mathbb{P}\left(\sup_{t\geq 0} \langle 1, \mu^k_t \rangle \geq K_\epsilon\right) < \epsilon. \tag{4.39}$$

2. In order to show that condition 2 holds for $\{\mu^k\}$, we first notice that for fixed k and fixed times t and $t+s$, the Wasserstein distance between μ_t^k and μ_{t+s}^k can be decomposed into a pure drift part and a pure jump part as follows. By construction, for each fixed k, the number of jumps on an interval $[t, t+s)$ is fewer than k (since θ_k is conditioned to have k total edges). Let $t \leq \tau_1 < \tau_2 < \cdots < \tau_{J-1} < t+s \leq \tau_J$ denote the jump times for μ^k on the interval $[t, t+s)$. Setting $\tau_0 = t$ (even if t is not a jump time), we can bound the Wasserstein distance by decomposing into a jump term and a drift term as follows:

$$\rho(\mu_t^k, \mu_{t+s}^k) \leq \underbrace{\left|\left|\mu_{t+s}^k\right| - \left|\mu_t^k\right|\right|}_{\text{jumps}} + \underbrace{\sum_{i=1}^{J} \rho(\mu_{\tau_i^-}^k, \mu_{\tau_{i-1}}^k)}_{\text{drift}}, \tag{4.40}$$

where by convention $|\mu| = \int_{\mathbb{R}} \mu(dx)$. We consider the jump and drift terms separately in parts A and B below.

A. Contribution of the jump term. We first focus our attention on the first term above, which represents the change in total mass over the interval. Notice that if the lifetime of x_l expires at time τ_i, then

$$\left|\int_{\mathbb{R}} \varphi d\mu_{\tau_i}^k - \int_{\mathbb{R}} \varphi d\mu_{\tau_i^-}^k\right| = \frac{1}{\sqrt{k}} \left|\varphi(x_l(\tau_i^-))\right| \leq \frac{1}{\sqrt{k}}, \tag{4.41}$$

for any $\varphi \in Lip^1(\mathbb{R})$, regardless of whether the number of offspring of x_l is 0 or 2.

Following the method of [Eth00] §5.3, we fix t and s and let $N_b^k\left([t, t+s)\right)$ denote the (unweighted) number of births in the interval $[t, t+s)$ and $N_d^k\left([t, t+s)\right)$ denote the number of deaths in the same interval. Then clearly

$$\begin{aligned} \left|\left|\mu_{t+s}^k\right| - \left|\mu_t^k\right|\right| &= \left|\frac{1}{\sqrt{k}}\left(N_{t+s}^k - N_t^k\right)\right| \\ &= \left|\frac{1}{\sqrt{k}}\left(N_b^k\left([t, t+s)\right) - N_d^k\left([t, t+s)\right)\right)\right|. \end{aligned} \tag{4.42}$$

Furthermore, births and deaths happen according to independent Poisson processes with parameter equal to $2\sqrt{k}N_t^k$, so

$$N_b^k\left([t,t+s)\right) = V_1\left(\frac{k}{2}\int_t^{t+s}\frac{N_\xi^k}{\sqrt{k}}d\xi\right)$$
$$N_d^k\left([t,t+s)\right) = V_2\left(\frac{k}{2}\int_t^{t+s}\frac{N_\xi^k}{\sqrt{k}}d\xi\right), \tag{4.43}$$

where V_1, V_2 are independent Poisson processes of rate 1. For $i = 1, 2$, let $\tilde{V}_i(u) = V_i(u) - u$ denote the centered Poisson process. Then

$$\begin{aligned} \left|\left|\mu_{t+s}^k\right| - \left|\mu_t^k\right|\right| &= \left|\frac{1}{\sqrt{k}}V_1\left(\frac{k}{2}\int_t^{t+s}\frac{N_\xi^k}{\sqrt{k}}d\xi\right) - \frac{1}{\sqrt{k}}V_2\left(\frac{k}{2}\int_t^{t+s}\frac{N_\xi^k}{\sqrt{k}}d\xi\right)\right| \\ &= \left|\frac{1}{\sqrt{k}}\tilde{V}_1\left(\frac{k}{2}\int_t^{t+s}\frac{N_\xi^k}{\sqrt{k}}d\xi\right) - \frac{1}{\sqrt{k}}\tilde{V}_2\left(\frac{k}{2}\int_t^{t+s}\frac{N_\xi^k}{\sqrt{k}}d\xi\right)\right| \\ &\xrightarrow[k\to\infty]{(d)} \left|B_1\left(\int_t^{t+s}\frac{1}{2}L_{\mathrm{e}}^\xi d\xi\right) - B_2\left(\int_t^{t+s}\frac{1}{2}L_{\mathrm{e}}^\xi d\xi\right)\right|, \end{aligned} \tag{4.44}$$

where B_1 and B_2 are independent Brownian motions, $L_{\mathrm{e}}^\xi = L_{\mathrm{e}}^\xi(1)$ is the total local time at level ξ of the normalized Brownian excursion, and the convergence is in the sense of convergence in distribution of random variables. (Recall that t and $t+s$ are fixed, so the total weight increment $\left|\left|\mu_{t+s}^k\right| - \left|\mu_t^k\right|\right|$ on the lefthand side of (4.44) is a real-valued random variable; (4.44) does not address convergence of the total weight as a process.) This implies that for $\epsilon > 0$,

$$\begin{aligned} \lim_{k\to\infty} P\left[\left|\mu_{t+s}^k\right| - \left|\mu_t^k\right| < \epsilon\right] &\geq \left(P\left[|B|\left(\int_t^{t+s}\frac{1}{2}L_{\mathrm{e}}^\xi d\xi\right) < \frac{\epsilon}{2}\right]\right)^2 \\ &\geq \left(P\left[|B|\left(\frac{s}{2}\sup_{\xi\geq 0}L_{\mathrm{e}}^\xi\right) < \frac{\epsilon}{2}\right]\right)^2 \end{aligned} \tag{4.45}$$

where $|B|$ denotes reflected Brownian motion. Appealing to the fact that $P\left[|B|(u) < \frac{\epsilon}{2}\right]$ is decreasing in u, and recalling that the distribution of $\sup_{\xi\geq 0} L_{\mathrm{e}}^\xi$ is tight, we conclude that s can be chosen small enough that the righthand side above is greater than $1-\epsilon$.

B. Contribution of the drift term.

Let $\varphi \in Lip^1(\mathbb{R})$. Notice that N_ξ is constant for $\xi \in [\tau_i, \tau_{i+1})$, and denote this value by N_i. Denote the particles comprising μ^k on the interval by $x_1, \ldots, x_{N_t}$ so that in this notation

$$\mu^k(\xi) = \frac{1}{\sqrt{k}} \sum_{j=1}^{N_t} \delta_{x_j(\xi)}. \tag{4.46}$$

Assume that x_l and x_{l+1} are the two offspring born at time τ_i. Then

$$\begin{aligned}
\left| \int_{\mathbb{R}} \varphi(x)\, \mu^k_{\tau_{i+1}^-}(dx) - \int_{\mathbb{R}} \varphi(x)\, \mu^k_{\tau_i}(dx) \right| =& \frac{1}{\sqrt{k}} \left| \sum_{j=1}^{N_i} \varphi\left(x_j(\tau_{i+1})\right) - \sum_{i=1}^{N_i} \varphi\left(x_j(\tau_i)\right) \right| \\
\leq& \frac{1}{\sqrt{k}} \sum_{j=1}^{N_i} \left| \varphi\left(x_j(\tau_{i+1})\right) - \varphi\left(x_j(\tau_i)\right) \right| \\
\leq& \frac{1}{\sqrt{k}} \sum_{j=1}^{N_i} \left| x_j(\tau_{i+1}) - x_j(\tau_i) \right| \\
\leq& \frac{1}{\sqrt{k}} \sum_{j \neq l, l+1} \left| x_j(\tau_{i+1}) - x_j(\tau_i) \right| \\
&+ \frac{1}{\sqrt{k}} \left| x_l(\tau_{i+1}) - x_l(\tau_i) \right| + \frac{1}{\sqrt{k}} \left| x_{l+1}(\tau_{i+1}) - x_{l+1}(\tau_i) \right|
\end{aligned} \tag{4.47}$$

For the second and third terms, (3.51) implies that for $j = l, l+1$:

$$\begin{aligned}
\frac{1}{\sqrt{k}} \left| \varphi\left(x_j(\tau_{i+1})\right) - \varphi\left(x_j(\tau_i)\right) \right| &\leq \frac{1}{\sqrt{k}} \left| x_j(\tau_i + 1) - x_j(\tau_i) \right| \\
&= \frac{1}{\sqrt{k}} \left(\sqrt{\frac{\tau_{i+1} - \tau_i}{\sqrt{k}}} + o\left(\sqrt{\tau_{i+1} - \tau_i} \right) \right).
\end{aligned} \tag{4.48}$$

To address the first term in (4.47), we let M_t denote the minimum gap in the particle system, excluding the gap between x_l and x_{l+1} at time t:

$$M_t = \min_{i \neq l} \left| x_{i+1}(t) - x_i(t) \right|. \tag{4.49}$$

By assumption, $M_{\tau_i} > 0$. Let

$$M_i := M_{\tau_i}. \tag{4.50}$$

Then $M_t \geq \frac{M_i}{2}$ for all $t \in [\tau_i, \tau_{i+1})$. For each $j \neq l, l+1$, we will bound $|\dot{x}_j(\tau_i)|$ by considering the corresponding quantity for a modified particle system. To motivate the construction, first notice that the repulsive force on $\tilde{x}_m$ generated by having one particle located at distance $\frac{M_i}{2}$ and one at distance M_i is greater than the force generated when there are two particles at distance M_i. We will use this observation to bound the force generated by the newly doubled atom at the birth time τ_i. Also notice that the particles on opposite sides of x_j push x_j in opposite directions.

Using these observations, we construct a new particle system as follows with the aim of bounding $|x_j(t)|$. Assume for now that $j > \frac{N_i}{2}$. Then there at least as many particles to the left of x_j as to the right. Consider the particle system consisting of $\{\tilde{x}_1, \ldots, \tilde{x}_j\}$ with initial position at time τ_i given by

$$\tilde{x}_m(\tau_i) = x_j(\tau_i) - (j-m)\frac{M_i}{2}, \tag{4.51}$$

and then evolving on (τ_i, τ_{i+1}) according to the usual repulsion

$$\frac{d}{dt}\tilde{x}_m(\xi) = \frac{1}{\sqrt{k}} \sum_{n \neq m} \frac{1}{\tilde{x}_m(\xi) - \tilde{x}_n(\xi)}. \tag{4.52}$$

Then

$$|\dot{x}_j(t)| \le \left| \frac{d}{dt}\tilde{x}_j(t) \right|, \qquad \text{for all } t \in [\tau_i, \tau_{i+1}). \tag{4.53}$$

If $j \le \frac{N_i}{2}$, we could construct an analogous particle system $\{\tilde{x}_j, \ldots, \tilde{x}_{N_i}\}$ and draw the same conclusion. This insight shows that

$$|\dot{x}_j(t)| \le \frac{2}{\sqrt{k}M_i} \sum_{m=1}^{(j-1)\vee(N_i-j)} \frac{1}{m}, \qquad \text{for all } t \in [\tau_i, \tau_{i+1}), \tag{4.54}$$

which we can use to bound $\frac{1}{\sqrt{k}}$ times the total displacement of x_j over the interval $[\tau_i, \tau_{i+1})$:

$$\begin{aligned}
\rho(\mu^k_{\tau^-_{i+1}}, \mu^k_{\tau_i}) &\leq \frac{1}{\sqrt{k}} \sum_{j \neq l, l+1} |x_j(\tau_{i+1}) - x_j(\tau_i)| \\
&\leq \frac{2(\tau_{i+1} - \tau_i)}{kM_i} \sum_{j=1}^{N_i} \sum_{m=1}^{(j-1)\vee(N_i - j)} \frac{1}{m} \\
&< \frac{2(\tau_{i+1} - \tau_i)}{kM_i} \sum_{j=1}^{N_i} 2\log\big((j-1) \vee (N_i - j)\big) \\
&< \frac{2(\tau_{i+1} - \tau_i)}{kM_i} \big(2N_i \log(N_i - 1)\big) \\
&\leq \frac{4(\tau_{i+1} - \tau_i)}{M_i} \frac{N_i}{\sqrt{k}} \frac{\log(N_i - 1)}{\sqrt{k}}.
\end{aligned} \tag{4.55}$$

Then the full drift term is

$$\sum_{i=0}^{J-1} \rho(\mu^k_{\tau^-_{i+1}}, \mu^k_{\tau_i}) \leq \sum_{i=0}^{J-1} \frac{4(\tau_{i+1} - \tau_i)}{M_i} \frac{N_i}{\sqrt{k}} \frac{\log(N_i - 1)}{\sqrt{k}}, \tag{4.56}$$

so for fixed $\epsilon > 0$ we need to show that there is a small enough s and large enough k_0 such that

$$\mathbb{P}\left[\sum_{i=0}^{J-1} \frac{4(\tau_{i+1} - \tau_i)}{M_i} \frac{N_i}{\sqrt{k}} \frac{\log(N_i - 1)}{\sqrt{k}} > \epsilon\right] < \epsilon, \quad \forall k > k_0. \tag{4.57}$$

However, $\frac{N_i}{\sqrt{k}} \to L^{\tau_i}_{\mathrm{e}}$, and $\frac{\log(N_i - 1)}{\sqrt{k}}$ becomes arbitrarily small as k gets large, implying that the event inside the probability above only happens if M_i is extremely small. In this case, M_i is close to $k^{-1/4}\sqrt{\tau_i - \tau_{i-1}}$. Recall that $\tau_i - \tau_{i-1}$ is distributed as an exponential random variable of mean $(2\sqrt{k}N_i)^{-1}$, so

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=0}^{J-1} \frac{4(\tau_{i+1} - \tau_i)}{k^{-1/4}\sqrt{\tau_i - \tau_{i-1}}} \frac{N_i}{\sqrt{k}} \frac{\log(N_i - 1)}{\sqrt{k}}\right] &= \mathbb{E}\left[\sum_{i=0}^{J-1} \frac{4\sqrt{2}\sqrt{k}\sqrt{N_{i-1}}\phi_1}{2\sqrt{k}N_i\sqrt{\phi_2}} \frac{N_i}{\sqrt{k}} \frac{\log(N_i - 1)}{\sqrt{k}}\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{J-1} \frac{2\sqrt{2}}{\sqrt{k}} \frac{\phi_1}{\sqrt{\phi_2}} \frac{\sqrt{N_{i-1}}\log(N_i - 1)}{\sqrt{k}}\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{J-1} \frac{2\sqrt{2}}{\sqrt{k}} \frac{\phi_1}{\sqrt{\phi_2}} \frac{1}{k^{1/8}} \frac{N_{i-1}^{3/4}}{k^{3/8}} \frac{\log(N_i - 1)}{N_{i-1}^{1/4}}\right] \\
&\leq \mathbb{E}\left[\frac{J}{\sqrt{k}} \max_i \left(\frac{2\sqrt{2}}{k^{1/8}} \frac{\phi_1}{\sqrt{\phi_2}} \frac{N_{i-1}^{3/4}}{k^{3/8}} \frac{\log(N_i - 1)}{N_{i-1}^{1/4}}\right)\right],
\end{aligned} \tag{4.58}$$

where ϕ_1 and ϕ_2 are independent exp(1) random variables. Notice that $\frac{\log(N_i-1)}{N_i^{1/4}}$ is bounded independently of ϵ, s, and k, and $\mathbb{E}\left[\frac{\phi_1}{\sqrt{\phi_2}}\right]$ may be factored out of the expectation since ϕ_1 and ϕ_2 are independent of all other terms. Also, $\frac{N_{i-1}^{3/4}}{k^{3/8}} \xrightarrow{(d)} (L_{\mathbb{e}}^{\tau_i})^{3/4}$, which has a tight probability distribution. Recall that $J-1$ is the random number of jumps in interval $[t, t+s)$ (where t and s are fixed). Changing the minus sign to a plus in (4.44) shows that

$$\frac{J}{\sqrt{k}} \underset{k\to\infty}{\xrightarrow{(d)}} B_1\left(\int_t^{t+s} \frac{1}{2} L_{\mathbb{e}}^{\xi} d\xi\right) + B_2\left(\int_t^{t+s} \frac{1}{2} L_{\mathbb{e}}^{\xi} d\xi\right), \tag{4.59}$$

where B_1 and B_2 are two independent Brownian motions, and the convergence is in the sense of convergence of distribution of random variables. Combining these observations implies that the limit of (4.58) is 0 as $k \to \infty$, which implies (4.57), completing the proof.

□

An analogous result holds for the unconditioned case, which is of independent interest.

Theorem 4.5. *Let $\{\mu^k\}_{k\geq 1} \subset D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$ be the sequence of measure-valued processes such that for each k, μ_t^k is the discrete measure defined by (3.4) and (3.5) for $c^{(k)} = c_1^{(k)} = \frac{1}{\sqrt{k}}$, with the modification that the initial measures are only required to be a tight sequence $\{\mu_0^k\}_{k\geq 1}$ (rather than requiring $\mu_0^k = \frac{1}{\sqrt{k}}\delta_0$). Furthermore, assume that for each k, the branching structure θ_k of μ^k is distributed according to the genealogy of a critical binary Galton-Watson process with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$ (unconditioned). Then $\{\mu^k\}_{k\geq 1}$ is tight in $D_{\mathcal{M}_F(\hat{\mathbb{R}})[0,\infty)}$.*

Proof. The proof is analogous to that of Theorem 4.2, but in this case the convergence of the total population process to the local time of the normalized Brownian excursion is replaced by

$$\frac{N_t^k}{\sqrt{k}} \xrightarrow{(d)} Y_t, \tag{4.60}$$

where Y_t is the Feller diffusion. □

4.3 Identifying the limit using the Stieltjes transform

In §4.2, we saw that $\{\mu^k\}$ is a tight sequence in $D_{\mathcal{M}_F(\hat{\mathbb{R}})}[0,\infty)$, which implies that it has at least one limit point. We seek to show that there is exactly one limit point, and that it lies in $D_{\mathcal{M}_F(\mathbb{R})}[0,\infty)$. To identify the limit, we convert the problem to a limit problem for the Stieltjes transforms of the measures. (Elsewhere these are called Cauchy transforms). This is a natural method because Stieltjes transforms are a representing class for finite measures on $\mathbb{R}$ and also because conditions are known that guarantee that the limit of a sequence of Stieltjes transforms is itself the Stieltjes transform of the weak limit of the corresponding measures. (See [GH03] for the case of probability measures.)

4.3.1 An SPDE for the flow of the Stieltjes transform

With the goal of understanding the scaling limit of the measure-valued process, we describe the time evolution of μ_t^k via its Stieltjes transform.

For now, we work in a slightly generalized setting as the following results are not specific to the particular choice of the distribution of the trees. Let $\mathcal{T}^* = \{(\nu, h_\nu) : \nu \in \mathcal{T}\}$ be a (deterministic or random) marked plane forest with distinct lifetimes ($h_\nu \neq h_\eta$ for all $\eta \neq \nu$) and total population process N_t:

$$N_t = |\Delta_t \mathcal{T}^*| \,. \tag{4.61}$$

(We generalize to let $\mathcal{T}^*$ be a plane forest instead of a plane tree so that the initial number of individuals need not be constrained to equal one.) Since it will be convenient to refer to time intervals on which there are no birth or death events, let $\{s_i\}$ be a relabeling of the h_ν so that $s_i < s_{i+1}$. Recalling the setup, we consider the following time-dependent measure given in (3.4) and (3.5), where for the moment we preserve full generality and do not fix the c, c_1 or the initial condition. In particular, for $t \in [0, \max_{\nu \in \mathcal{T}^*} h_\nu]$, let

$$\mu_t = c \sum_{\nu \in \Delta_t \mathcal{T}^*} \delta_{U_\nu(t)}, \tag{4.62}$$

where the U_ν evolve according to

$$\begin{aligned} \dot{U}_\nu(t) &= \sum_{\substack{\eta \in \Delta_t \mathcal{T}^* \\ \eta \neq \nu}} \frac{c_1}{U_\nu(t) - U_\eta(t)}, \text{ and} \\ U_\nu(h_{p(\nu)}) &= \lim_{t \nearrow h_{p(\nu)}} U_{p(\nu)}(t). \end{aligned} \tag{4.63}$$

For now, the initial measure μ_0 is not fixed.

In what follows, let $f = f(z,t)$ be the Stieltjes transform of μ_t:

$$f(z,t) = \int_{\mathbb{R}} \frac{1}{z - x} \mu_t(dx). \tag{4.64}$$

When $\mathcal{T}^*$ is deterministic, the flow of f satisfies a particular integral equation, which we describe in the following proposition.

Proposition 4.6. *Assume that $\mathcal{T}^*$ is a deterministic marked plane forest with binary branching and distinct lifetimes. Then f satisfies the integral equation*

$$f(s) - f(0) = -\int_0^s \frac{c_1}{c} f \partial_z dt f - \int_0^s \frac{c_1}{2} \partial_z^2 f dt + \sum_{\nu \in \mathcal{T} : h_\nu \leq s} (\mathfrak{c}(\nu) - 1) \frac{c}{z - U_\nu(h_\nu)} \tag{4.65}$$

on the interval $[0, \max_{\nu \in \mathcal{T}^*} h_\nu]$, where for each $\nu \in \mathcal{T}^*$, $\mathfrak{c}(\nu)$ denotes the number of offspring of ν.

Proof. The proof is a direct computation. To begin, assume that (s_i, s_{i+1}) is an interval containing none of the h_ν. Once we have computed the equation describing the flow of f on this interval, we will conclude the proof by treating the birth and death events.

Let φ be a test function, and denote $\varphi'(x) := \frac{d}{dx}\varphi(x)$. Then

$$\begin{aligned}
\frac{d}{dt}\langle \mu_t, \varphi\rangle &= \int_{\mathbb{R}} \dot{\varphi}(x)\mu_t(dx) \\
&= c \sum_{\nu \in \Delta_t \mathcal{T}^*} \varphi'(U_\nu(t))\dot{U}_\nu(t) \\
&= c \sum_{\nu \in \Delta_t \mathcal{T}^*} \left(\varphi'(U_\nu(t)) \cdot c_1 \sum_{\eta \neq \nu} \frac{1}{U_\nu - U_\eta} \right) \\
&= \frac{c_1}{c} \int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{\varphi'(x)}{x-y} \mu_t(dx)\mu_t(dy).
\end{aligned} \tag{4.66}$$

But

$$\begin{aligned}
\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{\varphi'(x)}{x-y}\mu_t(dx)\mu_t(dy) &= \int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \left(\frac{\varphi'(x)-\varphi'(y)}{x-y} + \frac{\varphi'(y)}{x-y} \right) \mu_t(dx)\mu_t(dy) \\
&= \int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \left(\frac{\varphi'(x)-\varphi'(y)}{x-y} - \frac{\varphi'(y)}{y-x} \right) \mu_t(dx)\mu_t(dy) \\
&= \int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{\varphi'(x)-\varphi'(y)}{x-y} \mu_t(dx)\mu_t(dy) \\
&\quad - \int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{\varphi'(y)}{y-x} \mu_t(dx)\mu_t(dy),
\end{aligned} \tag{4.67}$$

so

$$\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{\varphi'(x)}{x-y}\mu_t(dx)\mu_t(dy) = \frac{1}{2}\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{\varphi'(x)-\varphi'(y)}{x-y}\mu_t(dx)\mu_t(dy). \tag{4.68}$$

Substituting this into (4.66), we conclude that

$$\frac{d}{dt}\langle \mu_t, \varphi\rangle = \frac{c_1}{2c}\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{\varphi'(x)-\varphi'(y)}{x-y}\mu_t(dx)\mu_t(dy). \tag{4.69}$$

For fixed $z \in \mathbb{H}$, set the test function to be

$$\varphi(x) = \frac{1}{z-x}, \tag{4.70}$$

so that

$$\frac{d}{dt}\langle \mu_t, \varphi\rangle = \partial_t f. \tag{4.71}$$

Substituting back into (4.69), we have

$$\begin{aligned}\partial_t f &= \frac{c_1}{2c}\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}}\left(\frac{1}{(z-x)^2}-\frac{1}{(z-y)^2}\right)\frac{1}{x-y}\mu_t(dx)\mu_t(dy)\\ &= \frac{c_1}{2c}\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}}\frac{2z-y-x}{(z-x)^2(z-y)^2}\mu_t(dx)\mu_t(dy)\\ &= \frac{c_1}{2c}\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}}\left(\frac{2z}{(z-x)^2(z-y)^2}-\frac{y}{(z-x)^2(z-y)^2}-\frac{x}{(z-x)^2(z-y)^2}\right)\mu_t(dx)\mu_t(dy).\end{aligned} \tag{4.72}$$

We compute this integral one term at a time. The first term is

$$\begin{aligned}&\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}}\frac{2z}{(z-x)^2(z-y)^2}\mu_t(dx)\mu_t(dy) =\\ &\qquad = 2z\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}}\frac{1}{(z-x)^2(z-y)^2}\mu_t(dx)\mu_t(dy)\\ &\qquad = 2z\int\int_{\mathbb{R}\times\mathbb{R}}\frac{1}{(z-x)^2(z-y)^2}\mu_t(dx)\mu_t(dy) - 2z\int\int_{\mathbb{R}\times\mathbb{R}:x=y}\frac{1}{(z-x)^2(z-y)^2}\mu_t(dx)\mu_t(dy)\\ &\qquad = 2z\int_{\mathbb{R}}\frac{1}{(z-x)^2}\mu_t(dx)\int_{\mathbb{R}}\frac{1}{(z-y)^2}\mu_t(dy) - 2cz\int_{\mathbb{R}}\frac{1}{(z-x)^4}\mu_t(dx).\end{aligned} \tag{4.73}$$

The factor of c in the last term appears because (4.62) implies that

$$\begin{aligned}\mu_t(x)\mu_t(y) &= \left(c\sum_{\nu\in\Delta_t\mathcal{T}^*}\delta_{U_\nu(t)}(x)\right)\left(c\sum_{\eta\in\Delta_t\mathcal{T}^*}\delta_{U_\eta(t)}(y)\right)\\ &= c^2\sum_{\eta,\nu\in\Delta_t\mathcal{T}^*}\delta_{U_\nu(t)}(x)\delta_{U_\eta(t)}(y),\end{aligned} \tag{4.74}$$

so that if we fix $x=y$, then

$$\begin{aligned}\mu_t(x)\mu_t(x) &= c^2\sum_{\nu\in\Delta_t\mathcal{T}^*}\left(\delta_{U_\nu(t)}(x)\right)^2\\ &= c\mu_t(x).\end{aligned} \tag{4.75}$$

Before simplifying (4.73), we record the first three z derivatives of f for the reader's ease:

$$\begin{aligned}
\partial_z f &= -\int_{\mathbb{R}} \frac{1}{(z-x)^2}\mu_t(dx), \\
\partial_z^2 f &= 2\int_{\mathbb{R}} \frac{1}{(z-x)^3}\mu_t(dx), \text{ and} \\
\partial_z^3 f &= -6\int_{\mathbb{R}} \frac{1}{(z-x)^4}\mu_t(dx).
\end{aligned} \tag{4.76}$$

Then (4.73) is equal to

$$\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{2z}{(z-x)^2(z-y)^2}\mu_t(dx)\mu_t(dy) = 2z\left(\partial_z f\right)^2 + \frac{cz}{3}\partial_z^3 f. \tag{4.77}$$

Next we compute the second term of (4.72):

$$\begin{aligned}
&\int\int_{\mathbb{R}\times\mathbb{R}\setminus\{x=y\}} \frac{y}{(z-x)^2(z-y)^2}\mu_t(dx)\mu_t(dy) = \\
&\quad= \int\int_{\mathbb{R}\times\mathbb{R}} \frac{y}{(z-x)^2(z-y)^2}\mu_t(dx)\mu_t(dy) - c\int_{\mathbb{R}} \frac{x}{(z-x)^4}\mu_t(dx) \\
&\quad= \int_{\mathbb{R}} \frac{1}{(z-x)^2}\mu_t(dx)\int_{\mathbb{R}} \frac{y}{(z-y)^2}\mu_t(dy) - c\int_{\mathbb{R}} \frac{x}{(z-x)^4}\mu_t(dx) \\
&\quad= -\partial_z f\int_{\mathbb{R}} \frac{(y-z)+z}{(z-y)^2}\mu_t(dy) - c\int_{\mathbb{R}} \frac{x}{(z-x)^4}\mu_t(dx) \\
&\quad= \partial_z f\int_{\mathbb{R}} \left(\frac{1}{z-y} - \frac{z}{(z-y)^2}\right)\mu_t(dy) - c\int_{\mathbb{R}} \frac{x}{(z-x)^4}\mu_t(dx) \\
&\quad= \partial_z f\left(\int_{\mathbb{R}} \frac{1}{z-y}\mu_t(dy) - \int_{\mathbb{R}} \frac{z}{(z-y)^2}\mu_t(dy)\right) - c\int_{\mathbb{R}} \frac{x}{(z-x)^4}\mu_t(dx) \\
&\quad= \partial_z f\left(f + z\partial_z f\right) - \int_{\mathbb{R}} \frac{x}{(z-x)^4}\mu_t(dx) \\
&\quad= f\partial_z f + z(\partial_z f)^2 - c\int_{\mathbb{R}} \left(\frac{(x-z)}{(z-x)^4} + \frac{z}{(z-x)^4}\right)\mu_t(dx) \\
&\quad= f\partial_z f + z(\partial_z f)^2 - c\int_{\mathbb{R}} \frac{(x-z)}{(z-x)^4}\mu_t(dx) - c\int_{\mathbb{R}} \frac{z}{(z-x)^4}\mu_t(dx) \\
&\quad= f\partial_z f + z(\partial_z f)^2 + c\int_{\mathbb{R}} \frac{1}{(z-x)^3}\mu_t(dx) - cz\int_{\mathbb{R}} \frac{1}{(z-x)^4}\mu_t(dx) \\
&\quad= f\partial_z f + z(\partial_z f)^2 + \frac{c}{2}\partial_z^2 f + \frac{cz}{6}\partial_z^3 f.
\end{aligned} \tag{4.78}$$

Subtracting twice (4.78) from (4.77), and multiplying by the factor of $\frac{c_1}{2c}$, we see that f satisfies the equation

$$\partial_t f = -\frac{c_1}{c} f\partial_z f - \frac{c_1}{2}\partial_z^2 f \tag{4.79}$$

on the interval (s_i, s_{i+1}).

In the computation above, we assumed that $\#\operatorname{supp}\mu_t$ was constant, i.e. that we were working in an interval where μ_t had no birth or death events. In order to modify equation (4.79) so that it holds on intervals containing birth and death events, we notice that if h_ν is the time of death of ν, and if ν has no offspring, then at time h_ν the measure jumps $\mu_t \mapsto \mu_t - c\delta_{U_\nu(h_\nu)}$, so its Stieltjes transform jumps as well:

$$f(t^+) = f(t^-) - \frac{c}{z - U_\nu(h_\nu)}. \tag{4.80}$$

Similarly, if ν has two offspring, then

$$f(t^+) = f(t^-) + \frac{c}{z - U_\nu(h_\nu)}. \tag{4.81}$$

By assumption, $\mathcal{T}^*$ is binary, so for each ν, the number of offspring $\mathfrak{c}(\nu)$ will always be 0 or 2, so these are the only cases we need. Then for each h_ν equations (4.80) and (4.81) can be combined:

$$f(h_\nu) = \lim_{t \nearrow h_\nu} f(t) + (\mathfrak{c}(\nu) - 1)\frac{c}{z - U_\nu(h_\nu)}. \tag{4.82}$$

We conclude that the Stieltjes transform of μ_t satisfies (4.65). □

The utility of equation (4.65) is hampered by the fact that it contains explicit references to the U_ν. We would prefer to have a self-contained differential equation, which is what we will obtain in the following proposition, when we consider the stochastic version of (4.65).

Proposition 4.7. *Let θ be distributed as the genealogy of a critical binary Galton-Watson process $N(t)$ with exponential lifetimes of fixed finite mean, and let $0 < s_1 < \cdots < s_n$ denote the times of discontinuity of N. Let μ_t be defined as in (4.62) and (4.63). Then with probability one, the Stieltjes transform f of μ_t has the same distribution as the solution to*

$$f(s) - f(0) = -\frac{c_1}{c}\int_0^s f\partial_z f dt - \frac{c_1}{2}\int_0^s \partial_z^2 f dt + \int_0^s \frac{c}{z - Y(t)}\partial_t N(t) dt, \tag{4.83}$$

where for each t, $Y(t)$ is a random variable distributed as $\frac{\mu_{t-}}{|\mu_{t-}|}$.

Proof. With probability one, the h_ν are distinct, so they can be relabeled as $0 = s_0 < s_1 < s_2, \cdots < s_n$, and these times partition the interval where the Galton-Watson process N is defined. On each interval (s_i, s_{i+1}), N is constant, and at each s_i, N jumps up or down by 1. The evolution of the U_ν is deterministic on intervals (s_i, s_{i+1}), so just as in the proof of Proposition 4.6, on each of these intervals f satisfies

$$f(s) - f(s_i) = -\frac{c_1}{c}\int_{s_i}^{s} f\partial_z f dt - \frac{c_1}{2}\int_{s_i}^{s} \partial_z^2 f dt. \tag{4.84}$$

Since the lifetimes of the elements of θ are independent and exponential, they possess the Markov property, and each element is equally likely to be the next to die. This implies that

$$f(s_i) - \lim_{t \nearrow s_i} f(t) \overset{(\mathrm{d})}{=} (N(s_i) - N(s_{i-1}))\frac{c}{z - Y(s_i)}, \tag{4.85}$$

where $Y(s_i)$ is a random variable distributed as $\frac{\mu_{s_i^-}}{|\mu_{s_i^-}|}$. This implies that the distribution of f is equal to the solution to the equation

$$f(s)-f(0) = -\frac{c_1}{c}\int_0^s f\partial_z f dt - \frac{c_1}{2}\int_0^s \partial_z^2 f dt + \sum_{i:0<s_i\leq s}\Big(N(s_i)-N(s_{i-1})\Big)\frac{c}{z - Y(s_i)}, \tag{4.86}$$

which holds on the whole interval $[0, s_n]$. Equation (4.86) can be rewritten

$$f(s)-f(0) = -\frac{c_1}{c}\int_0^s f\partial_z f dt - \frac{c_1}{2}\int_0^s \partial_z^2 f dt + \int_0^s \Big(N(s_{i+1})-N(s_i)\Big)\frac{c}{z - Y(s_i)}\delta_{s_i} dt. \tag{4.87}$$

However,

$$\partial_t N(t) = \begin{cases} (N(s_i) - N(s_{i-1}))\,\delta_{s_i} & t = s_i \\ 0 & t \notin \ \{s_1, \ldots, s_N\}, \end{cases} \tag{4.88}$$

so (4.87) is equivalent to (4.83). □

Remark 5. In order for a continuous time branching process to have the Markov property, which we needed in the calculation above, its lifetimes must be i.i.d. exponential, as in our case. Since in our case the deterministic movement of the particles also possesses the

Markov property, each μ^k is itself a Markov process.

We conclude this section by remarking that (4.84) is simply the complex viscous Burgers equation, which can be solved using the Cole-Hopf transformation. In particular, if $|\mu_t|$ is constant and $c = c_1$, then

$$f(z,t) = c\sum_{i=1}^{n} \frac{1}{z - U_i(t)}, \tag{4.89}$$

which has no zeros in the upper half-plane. Define

$$g(z,t) = c\sum_{i=1}^{n} \log\left(z - U_i(t)\right), \tag{4.90}$$

so that

$$\partial_z g(z,t) = f(z,t). \tag{4.91}$$

Since $f(z,t)$ satisfies (4.84), $g(z,t)$ satisfies

$$g_t + \frac{1}{2}g_z^2 = -\frac{c_1}{2}g_{zz}. \tag{4.92}$$

Define $w(z,t)$ by

$$w(z,t) = e^{\frac{1}{c_1}g(z,t)} = \prod_{i=1}^{n}\left(z - U_i(t)\right). \tag{4.93}$$

Clearly,

$$\begin{aligned} w_t &= \frac{1}{c_1}g_t e^{g/c_1} \\ w_z &= \frac{1}{c_1}g_z e^{g/c_1} \\ w_{zz} &= \frac{1}{c_1}g_{zz}e^{g/c_1} + \frac{1}{c_1^2}g_z^2 e^{g/c_1}. \end{aligned} \tag{4.94}$$

Rearranging these to solve for g_t, g_z, and g_{zz}, respectively, and substituting the resulting values into (4.92) shows that $w(z,t)$ satisfies

$$w_t = -\frac{c_1}{2}w_{zz}. \tag{4.95}$$

This is simply the linear heat equation, which is exactly solvable.

4.3.2 A conjectural limiting equation

In pursuit of an answer to answer Question 3, here we describe progress toward find the scaling limit of the integral equation (4.83), the solution to which will identify the limit of the Stieltjes transforms $f_k(z,t) = \int \frac{1}{z-x}\mu_t^k(dx)$. In particular, we assume that $\{\theta_k\}_{k\geq 1}$ is a sequence of random trees distributed as binary Galton-Watson trees with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned to have k edges, and for each k, N^k is the corresponding Galton-Watson process. For each θ_k, let μ_t^k be the time-dependent measure defined as in (4.62) and (4.63) for the random tree θ_k with scaling constants $c_1^{(k)}$ and $c^{(k)}$ and initial measures $\mu_0^k = c^k\delta_0$. For each k, let f_k be the Stieltjes transform of μ_t^k:

$$f_k(z,t) = \int \frac{1}{z-x}\mu_t^k(dx). \tag{4.96}$$

Then for each k, with probability one, the f_k satisfy (4.83), which we rewrite here in the current notation:

$$f_k(s) - f_k(0) = -\frac{c_1^{(k)}}{c^{(k)}}\int_0^s f_k\partial_z f_k dt - \frac{c_1^{(k)}}{2}\int_0^s \partial_z^2 f_k dt + \int_0^s \frac{c^{(k)}}{z-Y_k(t)}\partial_t N^k(t)dt, \tag{4.97}$$

where for each t, $Y_k(t)$ is a random variable distributed as $\frac{\mu_{t-}^k}{\left|\mu_{t-}^k\right|}$.

As in §4.2, we set

$$c_1^{(k)} = c^{(k)} = \frac{1}{\sqrt{k}}, \tag{4.98}$$

since $c^{(k)} = \frac{1}{\sqrt{k}}$ is the total mass rescaling for which the Galton-Watson process has a nontrivial scaling limit, and $c^{(k)} = c_1^{(k)}$ guarantees consistent branching angles independent of k for the tree embeddings.

For this choice of constants, we expect the second term on the righhand side of (4.97) to go to 0 as $k \to \infty$, and clearly $\frac{c_1}{c} = 1$, so we expect that if

$$\tilde{f}(z,t) = \lim_{k\to\infty} f_k(z,t), \tag{4.99}$$

then $\tilde{f} = \tilde{f}(\xi, z)$ satisfies

$$\tilde{f}(t+s,z) - \tilde{f}(z,t) = -\int_t^{t+s} \tilde{f}\partial_z \tilde{f} d\xi + \lim_{k\to\infty} \int_t^{t+s} \frac{1/\sqrt{k}}{z - Y_k(\xi)} \partial_\xi N^k(\xi) d\xi \tag{4.100}$$

We focus on finding the scaling limit of the final term, which has two equivalent forms:

$$\int_t^{t+s} \frac{1/\sqrt{k}}{z - Y_k(\xi)} \partial_\xi N^k(\xi) d\xi = \sum_{i: t < \tau_i \leq t+s} \frac{N^k(\tau_i) - N^k(\tau_{i-1})}{\sqrt{k}} \frac{1}{z - Y(s_i)}, \tag{4.101}$$

where τ_i are the jump times in $[t, t+s)$.

If the sequence of measures $\{\mu_t^k\}_{k\geq 1}$ is defined as above, but for an unconditioned Galton-Watson process $N^k(t)$, then we notice that for large k on a small interval, $N^k(t)$ is close to a simple random walk. Furthermore, as $k \to \infty$ the jumps in the Galton-Watson process N^k occur very rapidly compared to the diffusion of the measure. With this in mind, we calculate the a scaling limit of (4.101) when the measure is fixed and $N^k(t)$ is replaced by a simple random walk.

Proposition 4.8. *For real numbers $U_1, \ldots, U_N$, let μ to be the real empirical measure*

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{U_i}, \tag{4.102}$$

Furthermore, for $i = 1, \ldots, n$, let X_i be independent identically distributed random variables taking values ± 1, each with probability $\frac{1}{2}$, and let Y_i be independent and identically distributed according to μ. Then

$$\sum_{i=1}^{n} \frac{X_i}{z - Y_i} \xrightarrow[n\to\infty]{(d)} h(z), \tag{4.103}$$

where $h(z)$ is the Gaussian analytic function with covariance kernel

$$\mathbb{E}\left(h(z)h(\overline{w})\right) = \int_{\mathbb{R}} \frac{1}{z-x} \frac{1}{\overline{w}-x} \mu(dx). \tag{4.104}$$

Proof. The quantity
$$\sum_{i=1}^{n} \frac{X_i}{z - Y_i} \tag{4.105}$$
is a sum of random Herglotz functions, so the limit is also a random Herglotz function. This sum can be partitioned based on the value of each $Y_i \in \{U_1, \dots, U_N\}$:

$$\sum_{i=1}^{n} \frac{X_i}{z - Y_i} = \frac{1}{z - U_1} S_{n,1} + \cdots + \frac{1}{z - U_N} S_{n,N}, \tag{4.106}$$

where
$$S_{n,j} = \sum_{i=1}^{n} X_i \mathbb{1}_{Y_i = U_j}. \tag{4.107}$$

We know that
$$\mathbb{P}(Y_i = U_j) = \frac{1}{N}, \tag{4.108}$$

and
$$\frac{1}{\sqrt{n}} S_n \to \mathcal{N}_1, \tag{4.109}$$

where $\mathcal{N}_1$ is a standard normal. Since as n gets large the number of elements in the sum $S_{n,i}$ is close to its expected value $\frac{n}{N}$,

$$\sqrt{\frac{N}{n}} S_{n,i} \to \mathcal{N}_1, \tag{4.110}$$

so that
$$\frac{1}{\sqrt{n}} S_{n,i} = \frac{1}{\sqrt{N}} \sqrt{\frac{N}{n}} S_{n,i} \to \frac{1}{\sqrt{N}} \mathcal{N}_1. \tag{4.111}$$

We conclude that as $n \to \infty$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i}{z - Y_i} \xrightarrow{(\mathrm{d})} \sum_{i=1}^{N} \frac{1}{\sqrt{N}} \frac{1}{z - U_i} \mathcal{N}_i, \tag{4.112}$$

where N_i are iid standard normal. In order to understand this random Herglotz function, we recall that a Gaussian analytic function is determined by its covariance kernel. Let

$$h(z) = \sum_{i=1}^{N} \frac{1}{\sqrt{N}} \frac{1}{z - U_i} \mathcal{N}_i. \tag{4.113}$$

Then

$$
\begin{aligned}
C(z,w) &= \mathbb{E}\left(h(z)h(\bar{w})\right) \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{N}\frac{1}{\sqrt{N}}\frac{1}{z-U_i}\mathcal{N}_i\right)\left(\sum_{j=1}^{N}\frac{1}{\sqrt{N}}\frac{1}{\bar{w}-U_j}\mathcal{N}_j\right)\right] \\
&= \sum_{i=1}^{N}\frac{1}{N}\frac{1}{z-U_i}\frac{1}{\bar{w}-U_i}\mathbb{E}\left((\mathcal{N}_i)^2\right) \\
&= \sum_{i=1}^{N}\frac{1}{N}\frac{1}{z-U_i}\frac{1}{\bar{w}-U_i} \\
&= \int_{\mathbb{R}}\frac{1}{z-y}\frac{1}{\bar{w}-y}\mu(dy).
\end{aligned}
\tag{4.114}
$$

□

This result motivates the following conjecture.

Conjecture 4.9. *For $k = 1, 2, \ldots$, let θ_k be distributed as the genealogy of a critical binary Galton-Watson process with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$. For each k, let μ_t^k be defined as in (4.62) and (4.63), and let $c_1^{(k)} = c^{(k)} = \frac{1}{\sqrt{k}}$. Furthermore, assume that $\{\mu_0^k\}_{k\geq 1}$ is a tight sequence of measures, where $\sqrt{k}\left|\mu_0^k\right|$ is the initial population of θ_k. Then the limit $f = \lim_{k\to\infty} f_k$ exists, and there is a real constant σ such that*

$$f = \int_{\mathbb{R}} \frac{1}{x-z}\mu_t^{\infty}(dx) \tag{4.115}$$

is the solution to the equation

$$\partial_t f = -f\partial_z f + \sigma h(z,t), \tag{4.116}$$

where $h(z,t)$ is the Gaussian analytic function with covariance kernel

$$\mathbb{E}\left(h(z,t)h(\overline{w},t')\right) = \delta(t-t')\int_{\mathbb{R}}\frac{1}{z-x}\frac{1}{\overline{w}-x}\mu_t^{\infty}(dx). \tag{4.117}$$

It is important to notice that the forests θ_k in Conjecture 4.9 are not conditioned to converge to the continuum random tree, so the conjecture does not address the limiting measure corresponding to the continuum random tree. Identifying the limiting equation in the conditioned case remains an open question.

4.3.3 The boundary SPDE

We conclude this work by identifying a conjectural SPDE for the density of the limiting superprocess of Conjecture 4.9 that is equivalent to (4.116) on the boundary of $\mathbb{H}$. (This equation is (4.128) below.) The line of inquiry that leads to this equation is distinct from that in §4.3.2, providing further evidence for Conjecture 4.9.

In what follows, we change notation slightly to let X_t and X_t^k denote superprocesses (instead of using μ_t and μ_t^k). If each X_t^k is a superprocess with branching structure given by the genealogy of a critical binary Galton-Watson process and spatial motion of each particle given by an independent Markov process with generator A, then the limiting superprocess X_t satisfies the differential equation ([Daw75], equation (6.2)):

$$dX_t = AX_t + \sigma(X_t)^{\frac{1}{2}} dB_t. \tag{4.118}$$

Let $\rho(x,t)$ denote the density of X_t. If the spatial motion is standard linear Brownian motion, then (4.118) is equivalent to

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{1}{2}\partial_x^2\rho(x,t) + \sqrt{\sigma^2\rho(x,t)} \cdot \dot{W}, \tag{4.119}$$

where $\dot{W}$ denotes space-time white noise, and $\rho(x,t)$ denotes the density of X_t ([LG99], [KS88]).

Recall that if $\tilde{\rho}(x,t)$ is the density of a standard linear Brownian motion, then $\tilde{\rho}(x,t)$ satisfies the heat equation

$$\partial_t\tilde{\rho}(x,t) = \frac{1}{2}\partial_x^2\tilde{\rho}(x,t). \tag{4.120}$$

This suggests that the first term on the righthand side of (4.119) is the time derivative of the density of the superprocess in the case when there is no branching, and the second term describes the branching structure of the process. Thus, we expect that if $\rho(x,t)$ is the density of the limiting superprocess from Conjecture 4.9 (the scaling limit of the superprocesses with critical binary Galton-Watson branching structure and the fixed deterministic repulsion) then the evolution of $\rho(x,t)$ will be given by an equation of the form

$$\frac{\partial}{\partial t}\rho(x,t) = (\text{motion term}) + \sqrt{2\sigma^2\rho(x,t)}\cdot \dot{W}. \tag{4.121}$$

In particular, the "motion term" above will be the time derivative of the density $\rho(x,t)$ in the setting when the spatial motion is the same but there is no branching. To determine $\partial_t\rho(x,t)$ in this setting, we first notice that if there were no branching, the limiting differential equation for the Stieltjes transform would be

$$\partial_t f + f\partial_z f = 0, \tag{4.122}$$

where

$$f(z,t) = \int_{\mathbb{R}} \frac{1}{z-\xi}\rho(\xi,t)d\xi. \tag{4.123}$$

This equation is obtained by setting $c^k = c_1^k = \frac{1}{\sqrt{k}}$ in (4.79) and taking the limit as $k \to \infty$. Equation (4.122) is the complex Burgers equation (also known as the Hopf equation), which can be thought of as the analogue of the heat equation for free Brownian motion.

Letting $z \to x \in \mathbb{R}$, we obtain the boundary value of the Stieltjes transform:

$$f(x,t) = \pi\mathcal{H}\rho(x,t) + i\pi\rho(x,t), \quad x \in \mathbb{R}, \tag{4.124}$$

where $\mathcal{H}\rho(x,t)$ is the Hilbert transform of ρ defined by

$$\mathcal{H}\rho(x,t) = \frac{\text{p.v.}}{\pi}\int_{\mathbb{R}} \frac{1}{x-\xi}\rho(\xi,t)d\xi, \quad x \in \mathbb{R}. \tag{4.125}$$

(Many references define the Hilbert transform as the negative of the definition above.)

Then for $x \in \mathbb{R}$, (4.122) is equivalent to the system of equations

$$\begin{cases} \partial_t \rho + \partial_x (\rho \cdot \mathcal{H}\rho) = 0 \\ \partial_t (\mathcal{H}\rho) + (\mathcal{H}\rho) \cdot \partial_x (\mathcal{H}\rho) - \rho \partial_x \rho = 0. \end{cases} \tag{4.126}$$

The first equation can be trivially rewritten as

$$\partial_t \rho(x,t) = -\partial_x (\rho \cdot \mathcal{H}\rho), \tag{4.127}$$

which makes it clear that $-\partial_x(\rho \cdot \mathcal{H}\rho)$ plays the role in this setting that is played by $\frac{1}{2}\partial_x^2 \rho$ for the Dawson-Watanabe superprocess. We conclude that the density $\rho(x,t)$ of the limiting superprocess of Conjecture 4.9 satisfies the equation

$$\partial_t \rho + \partial_x \left(\rho \cdot \mathcal{H}\rho\right) = \sigma \sqrt{\rho} \cdot \dot{W}. \tag{4.128}$$

In fact, equation (4.128) is the boundary limit of (4.116) as $z \to x \in \mathbb{R}$, which provides further evidence for Conjecture 4.9. To see this, we first integrate the lefthand side of (4.128):

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{z-x} \left(\partial_t \rho + \partial_x \left(\rho \cdot \mathcal{H}\rho\right)\right) dx &= \int_{\mathbb{R}} \frac{1}{z-x} \partial_t \rho(x,t) dx - \int_{\mathbb{R}} \frac{1}{(z-x)^2} \rho(x,t) \left(\int_{\mathbb{R}} \frac{1}{z-\xi} \rho(\xi,t) \right) dx \\ &= \partial_t f + f \partial_z f. \end{aligned} \tag{4.129}$$

On the other hand, recall that space-time white noise $\dot{W}(x,t)$ is the stationary Gaussian process such that

$$\mathbb{E}\left(\dot{W}(x,t)\dot{W}(y,t')\right) = \delta(x-y)\delta(t-t'), \tag{4.130}$$

so for each t,

$$\sigma h(z,t) := \int_{\mathbb{R}} \frac{1}{z-x} \sigma \sqrt{\rho} \dot{W}(x,t) dx, \tag{4.131}$$

where $h(z,t)$ is a Gaussian analytic function. To specify a Gaussian analytic function, it is sufficient to calculate the covariance kernel:

$$
\begin{aligned}
\mathbb{E}\left(h(z,t)h(\bar{w},t')\right) &= \mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sqrt{\rho(x,t)}\sqrt{\rho(y,t')}}{(z-x)(\bar{w}-y)}\dot{W}(x,t)\dot{W}(y,t')dxdy\right) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sqrt{\rho(x,t)}\sqrt{\rho(y,t')}}{(z-x)(\bar{w}-y)}\mathbb{E}\left(\dot{W}(x,t)\dot{W}(y,t')\right)dxdy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sqrt{\rho(x,t)}\sqrt{\rho(y,t')}}{(z-x)(\bar{w}-y)}\delta(x-y)\delta(t-t')dxdy \\
&= \delta(t-t')\int_{\mathbb{R}} \frac{\rho(x,t)}{(z-x)(\bar{w}-x)}dx,
\end{aligned}
\tag{4.132}
$$

which is exactly the covariance kernel (4.117) in Conjecture 4.9.

While there is still work to be done to make this argument rigorous, this line of reasoning shows why Conjecture 4.9 is natural from the point of view of superprocesses. We expect that in the future a similar analysis will provide a natural candidate for the limiting superprocess in the conditioned case, which would be a further step towards the desired embedding of the continuum random tree.

Appendix A

Additional estimates used in Chapter 2

Here we collect the proofs of three elementary results that are used in the proofs of Theorem 2.3, Lemma 2.5, and Lemma 2.6.

We begin with a lemma that gives a uniform lower bound for $|\Im g_t^\infty(z)|$ when $z \in D_R$ is bounded away from L and $\mathbb{R}$. This is a crucial step for bounding the conformal radius in the proof of Theorem 2.3.

Lemma A.1. *Fix $R > 0$ and $\delta_0 > 0$. Let*

$$\mathcal{T} = \{z \in \mathbb{H} : |z| \leq R, d(z, L \cup \mathbb{R}) \geq \delta_0\}. \tag{A.1}$$

Then there is a $\delta_1 > 0$ such that for all $z \in \mathcal{T}$ and all $0 \leq t \leq 1$,

$$\Im g_t(z) > \delta_1. \tag{A.2}$$

Proof. For each $z \in \mathcal{T}$, there is a δ_z for which $\Im g_t(z) > \delta_z$ for all $0 \leq t \leq 1$, but the point of the lemma is that there is a single δ_1 that works for all $z \in \mathcal{T}$.

We proceed by contradiction. Assume that $\{z_n\}$ is a sequence in $\mathcal{T}$ and $\{t_n\}$ a sequence in $[0, 1]$ such that for every $\epsilon > 0$ there is a N_ϵ such that for all $N > N_\epsilon$

$$\Im g_{t_N}(z_N) \leq \epsilon. \tag{A.3}$$

Since $\overline{\mathcal{T}}$ is compact, $\{z_n\}$ has an infinite convergent subsequence, which we will call $\{z_{n_k}\}$, so that

$$z_{n_k} \to z^* \in \overline{\mathcal{T}}. \tag{A.4}$$

Now consider the sequence $\{t_{n_k}\}$. Since $[0, 1]$ is compact, it has a convergent subsequence $\{t_{\tilde{n}}\}$ such that

$$t_{\tilde{n}} \to t^* \in [0, 1]. \tag{A.5}$$

We consider the sequence $z_{\tilde{n}}$:

$$|\Im g_{t^*}(z^*)| \leq |\Im g_{t^*}(z^*) - \Im g_{t^*}(z_{\tilde{n}})| + |\Im g_{t^*}(z_{\tilde{n}}) - \Im g_{t_{\tilde{n}}}(z_{\tilde{n}})| + |\Im g_{t_{\tilde{n}}}(z_{\tilde{n}})| \,. \tag{A.6}$$

Each term on the right-hand side can be made arbitrarily small for sufficiently large $\tilde{n}$ by the continuity of $g_t(z)$ in both t and z and the assumption. Therefore,

$$|\Im g_{t^*}(z^*)| = 0, \tag{A.7}$$

so $z^* \in L \cup \mathbb{R}$. This is a contradiction, since $z_{\tilde{n}} \to z^*$ and $d(z_{\tilde{n}}, L \cup \mathbb{R}) \geq \delta_0$. □

The following lemma is the full justification of Equation (2.115) in the proof of Theorem 2.3.

Lemma A.2. *Fix $R > 0$ and $\delta_0 > 0$, and let*

$$\mathcal{S} = \{z \in \mathbb{H} : |z| \leq R, d(z, L) \geq \delta_0\}, \tag{A.8}$$

where L is as in Chapter 2, and V_1 and V_2 are its driving functions. Then there is a δ_1 such that $\mathcal{S} \subset \mathcal{V}_{\delta_1}$, where

$$\mathcal{V}_{\delta_1} = \{z \in \mathbb{H} : \min\{|g_t(z) - V_1(t)| \,, |g_t(z) - V_2(t)|\} \geq \delta_1, \ \textit{for } 0 \leq t \leq 1\} \tag{A.9}$$

Proof. For each z, there is a δ_z for which $z \in \mathcal{V}_{\delta_z}$, but the claim of the lemma is that there is a single δ_1 that works for all points in $\mathcal{S}$.

We proceed by contradiction. Assume that $\{z_n\}$ is a sequence in $\mathcal{S}$ and $\{t_n\}$ a sequence in $[0, 1]$ such that for every $\epsilon > 0$ there is a N_ϵ such that for all $N > N_\epsilon$

$$\left|g^{\infty}_{t_N}(z_N) - V_1(t_N)\right| \leq \epsilon. \tag{A.10}$$

Since $\overline{\mathcal{S}}$ is compact, $\{z_n\}$ has an infinite convergent subsequence, which we will call $\{z_{n_k}\}$, so that

$$z_{n_k} \to z^* \in \overline{\mathcal{S}}. \tag{A.11}$$

Now consider the sequence $\{t_{n_k}\}$. Since $[0,1]$ is compact, it has a convergent subsequence $\{t_{\tilde{n}}\}$ such that

$$t_{\tilde{n}} \to t^* \in [0,1]. \tag{A.12}$$

We consider the sequence $z_{\tilde{n}}$:

$$|g_{t^*}^{\infty}(z^*) - V_1(t^*)| \leq |g_{t^*}^{\infty}(z^*) - g_{t^*}^{\infty}(z_{\tilde{n}})| + \left|g_{t^*}^{\infty}(z_{\tilde{n}}) - g_{t_{\tilde{n}}}^{\infty}(z_{\tilde{n}})\right| + \left|g_{t_{\tilde{n}}}^{\infty}(z_{\tilde{n}}) - V_1(t_{\tilde{n}})\right| + |V_1(t_{\tilde{n}}) - V_1(t^*)|\,. \tag{A.13}$$

Each term on the right-hand side can be made arbitrarily small for sufficiently large $\tilde{n}$ by the continuity of $g_t^{\infty}(z)$ in both t and z, the assumption, and the continuity of V_1. Therefore,

$$|g_{t^*}^{\infty}(z^*) - V_1(t^*)| = 0, \tag{A.14}$$

so $z^* \in L$. This is a contradiction, since $z_{\tilde{n}} \to z^*$ and $d(z_{\tilde{n}}, L) \geq \delta_0$. The same is true when V_1 is replaced by V_2 in the argument. □

Lemma A.3. *In the notation of Theorem 2.3 and Lemma 2.5, $\rho U_k(t/\rho^2)$ converges to $V_1(t)$ uniformly on $[0,1]$, and $\rho U_{k+1}(t/\rho^2)$ converges to $V_2(t)$ uniformly on $[0,1]$.*

Proof. Assumption (2.59) implies that

$$U_{k+1}(t) = V_2(t) + \varphi(t), \tag{A.15}$$

where

$$\lim_{t \to 0} \frac{\varphi(t)}{\sqrt{t}} = 0. \tag{A.16}$$

Substituting t/ρ^2 for t, this becomes

$$\lim_{\rho \to \infty} \frac{\varphi(t/\rho^2)}{\sqrt{t/\rho^2}} \to 0, \tag{A.17}$$

so

$$\lim_{\rho\to\infty} \rho \frac{\varphi(t/\rho^2)}{\sqrt{t}} \to 0. \tag{A.18}$$

Let $\epsilon > 0$. There exists $\rho_\epsilon > 0$ such that if $\rho > \rho_\epsilon$, then

$$\rho\varphi(t/\rho^2) < \epsilon\sqrt{t} \leq \epsilon, \tag{A.19}$$

since $t \in [0,1]$. But V_2 satisfies

$$\rho V_2(t/\rho^2) = V_2(t), \tag{A.20}$$

so

$$\rho U_{k+1}(t/\rho^2) = V_2(t) + \rho\varphi(t/\rho^2), \tag{A.21}$$

implying that

$$\left|\rho U_{k+1}(t/\rho^2) - V_2(t)\right| < \epsilon \tag{A.22}$$

for all $\rho > \rho_\epsilon$ and all $t \in [0,1]$. The proof for the uniform convergence of $\rho U_k(t/\rho^2)$ to $V_1(t)$ is similar. $\square$

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