Regularity Theory of Elliptic and Parabolic Equations and Systems

by

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Abstract of "Regularity Theory of Elliptic and Parabolic Equations and Systems", by Hong Zhang, Ph.D., Brown University, May 2017

In this dissertation, we study regularity theory of elliptic and parabolic equations and systems with irregular coefficients.

In the first part of the dissertation, we consider Schauder and L_p theories for general linear elliptic and parabolic systems with particular types of irregular coefficients. We also establish some new regularity results regarding a model from fiber reinforced composite material and expand the L_p and Schauder estimates of higher-order parabolic equations and systems to include a large class of irregular coefficients.

In the second part of the dissertation, we investigate the regularity problem of nonlinear nonlocal elliptic and parabolic equations. We prove the Schauder estimates for concave fully nonlinear nonlocal parabolic equations and then establish the Dini type estimates for such type of equations.

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Chapter 1

Introduction

The celebrated Hilbert's 19th problem was solved by De Giorgi [26] and Nash [91] in 1950s, who showed that the weak solution to the following divergence type uniformly elliptic equation with merely measurable coefficients

$$\sum_{i,j=1}^{d} D_i(a_{ij}(x)D_j u) = 0$$

is Hölder continuous. Later the corresponding result for non-divergence type elliptic equations with merely measurable coefficients

$$\sum_{i,j=1}^{d} a_{ij}(x)D_{ij}u = 0$$

was established by Krylov and Safonov [71, 70] in 1970s. Since then tremendous research efforts have been dedicated to the study of general linear elliptic and parabolic equations with particular type of irregular coefficients. The first part of the dissertation lies in that direction based on our previous work [42, 43, 52, 44], where we expanded the Schauder and L_p theory to include a large class of irregular coefficients.

Specifically in [42, 43], we considered Schauder estimates for higher-order parabolic systems under various boundary conditions with coefficients merely measurable in the time variable. In [43, 52], we studied L_p estimates of higher-order parabolic system under the conormal boundary condition and weighted L_p estimates of parabolic equation under the Neumann boundary condition, respectively. In both papers, the coefficients are in the space of vanishing mean oscillation (VMO), which is a large class of discontinuous functions introduced by Sarason [95]. In [44] we investigated a model from fiber-reinforced composite material and established a piecewise Schauder estimate for the following equation

$$Lu := \sum_{i=1}^{2} D_i(a(x)D_i u) = \sum_{i=1}^{2} D_i f_i \quad \text{in} \quad \mathcal{D},$$

where $a(x)$ and $f_i(x)$ are piecewise Hölder continuous in domain $\mathcal{D}$ containing two touching balls as subdomains. This completely addresses the open questions raised by Li and Vogalius [76] in dimension 2 and opens the door to fully understand the case of arbitrary shape of touching domains. Furthermore, our method has been applied to other problems such as conductivity problems on composite material [29].

The last decade has seen vigorous research activity in understanding nonlocal equations, which model systems involving long-range effects and have wide applications in biology, fluid mechanics, finance, geometry, image processing, etc. The most canonical example of nonlocal elliptic operator is the fractional Laplacian

$$(-\Delta)^{\sigma/2}u(x) = c_{d,\sigma}\int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+\sigma}}\,dy, \quad \sigma \in (0,2),$$

which is the infinitesimal generator of the radial symmetric and stable Lévy process of order σ. Notice that the canonical Laplacian is the asymptotic limit of the fractional Laplacian as $\sigma \to 2$. Apparently from the definition above, it is not hard to see the value of $(-\Delta)^{\sigma/2}u(x)$, unlike the classical differential operator, depends on values of u far away from x. Such nonlocal nature brings more difficulties into the analysis of nonlocal equations. In seminal works of Caffarelli and Silvestre [15, 17, 16], the classical regularity results of elliptic equation, such as Krylov-Safonov estimates and Evans-Krylov estimates [46, 61], were extented to nonlocal elliptic equations with symmetric and smooth kernels. In the second part of the thesis, based on our papers [31, 30, 32] we introduce our results on regularity theory of fully nonlinear nonlocal concave parabolic equations as follows

$$\partial_t u = \inf_{a\in\mathcal{A}}(L_a u + f_a),$$

where $\mathcal{A}$ is an index set and L_a is a nonlocal operator for each $a \in \mathcal{A}$. In [31], we established Schauder estimates for such equations with non-symmetric rough kernels and obtain an optimal estimate when the inhomogeneous term is merely bounded. Moreover, our estimates are uniform in the order of the operator. Namely, as $\sigma \to 2$ our results recover the classical result of second-order parabolic equations. We further studied Dini type estimates in [30, 32] for such equations by implementing a novel perturbation type arguments. We believe that such novel approach can be applied to other problems when dealing with nonlocal equations. In the rest of the chapter, we introduce these problems separately.

1.1 Regularity theory of classical elliptic and parabolic equations

1.1.1 On an elliptic equations with discontinuous coefficients

It is important to understand the regularity properties of elliptic equations with piecewise constant coefficients. In [44], we considered the following divergence type elliptic equation

$$Lu := \sum_{i=1}^{2} D_i(a(x)D_iu) = \sum_{i=1}^{2} D_if_i \quad \text{in} \quad \mathcal{D}, \tag{1.1.1}$$

where $\mathcal{D}$ is a bounded subset of $\mathbb{R}^2$,

$$a(x) = a_0\chi_{B_{r_1}(0,r_1)} + b_0\chi_{B_{r_2}(0,-r_2)} + \chi_{\mathbb{R}^2\setminus(B_{r_1}(0,r_1)\cup B_{r_2}(0,-r_2))}, \tag{1.1.2}$$

$a_0, b_0 > 0$ are constants, $r_1, r_2 \in (0,\infty)$, and χ is the indicator function. Notice that the two balls touch each other and therefore there exists a cusp like singularity at the origin. This problem was raised by Bonnetier and Vogelius [10], and can be considered as a simplified model for composite media with closely spaced interfacial boundaries. They proved that $u \in W^{1,\infty}(\mathcal{D})$ under the assumption that $a_0 = b_0$, $r_1 = r_2 = 1$, and $f_i \equiv 0$. Later, Li and Vogelius [76] improved this result by showing that under the extra condition that $\mathcal{D} = B_{R_0}$ for R_0 *sufficiently large*, u is piecewise smooth up to boundary. Here for piecewise smooth, we mean for any compact set $K \subset \mathcal{D}$,

$$u \in C^\infty(\overline{B_1(0,1)}), \quad u \in C^\infty(\overline{B_1(0,-1)}), \quad \text{and} \quad u \in C^\infty(K\setminus(B_1(0,1)\cup B_1(0,-1))).$$

In our paper [44], we first answered the question raised by Li and Vogelius [76] by relaxing the condition in their paper and showing that $R_0 > 2$ is sufficient to guarantee that u is piecewise smooth up to the boundary.

Theorem 1.1.1 ([44])**.** *Let $R_0 > 2$ and $g \in H^{1/2}(\partial B_{R_0})$. Suppose u is a weak solution of*

$$D_i(a(x)D_i u) = 0 \quad in \quad B_{R_0}, \quad u = g \quad on \quad \partial B_{R_0},$$

where a satisfies (1.1.2) with $a_0 = b_0$ and $r_1 = r_2 = 1$. Then u is piecewise smooth up to the boundary.

In order to fully understand the operator, we considered (1.1.1) with non-vanishing right-hand side and removed the geometric constraint of $\mathcal{D}$. We proved a piecewise interior Schauder type estimate.

Theorem 1.1.2 ([44])**.** *Let $\gamma \in (0, 1)$, and $n \geq 0$ be an integer. Assume that $\mathcal{D} \subset \mathbb{R}^2$ is a bounded open set containing 0. Suppose that for any $i = 1, 2$, f_i is piecewise $C^{n,\gamma}$, i.e.,*

$$f_i \in C^{n,\gamma}(\mathcal{D} \cap B_{r_1}(0, r_1)), \quad f_i \in C^{n,\gamma}(\mathcal{D} \cap B_{r_2}(0, -r_2)),$$

and

$$f_i \in C^{n,\gamma}(\mathcal{D} \setminus (B_{r_1}(0, r_1) \cup B_{r_2}(0, -r_2))).$$

Let u be a weak solution to (1.1.1). Define $\mathcal{D}_\varepsilon = \{x \in \mathcal{D}, dist(x, \partial\mathcal{D}) \geq \varepsilon\}$ for any $\varepsilon > 0$. Then u is piecewise $C^{n+1,\gamma}$ in D_ε.

In particular, when $f_i \equiv 0$, u is piecewise smooth in $\mathcal{D}_\varepsilon$ up to the boundary.

The main ingredient in our proof is the construction of the Green's function of the elliptic operator in (1.1.1). Then we are able to apply some known results of the

Poisson equation with piecewise Hölder continuous data. Our method gives a way to understand touching domains with arbitrary interfacial boundaries. It has been successfully implemented in the conductivity problems on composite material [29].

Moreover, by a perturbation argument, we are able to deal with the case that $a(x)$ is piecewise Hölder continuous. Specifically, if $a(x)$ is piecewise $C^{n,\gamma}$ and $\lambda \leq a \leq \Lambda$, we can prove the same result as in Theorem 1.1.2.

1.1.2 L_p and Schauder estimates for parabolic systems with rough coefficients

L_p and Schauder estimates are two major topics in elliptic and parabolic regularity theory. The elliptic and parabolic equations and systems with regular coefficients are well-understood, for example, see [1, 2]. For the parabolic equations, there is a less well-known result that the regularity assumptions in the t variable on the coefficients and data are sometimes not necessary, which was first considered by Knerr [56], and later by Lieberman [81]. In the same spirit, L_p estimates can be obtained under this relaxed condition, for instance see [66]. In several papers [42, 43, 52], we followed this line to consider L_p and Schauder estimates for higher-order parabolic systems with various boundary conditions and weighted L_p estimates for second-order parabolic equations with the Neumann boundary condition.

1.1.2.1 Schauder estimate for higher-order parabolic systems under the Dirichlet boundary condition

Divergence type system. In [42], we considered the divergence type higher-order parabolic systems,

$$u_t + (-1)^m \mathcal{L}u := u_t + (-1)^m \sum_{|\alpha|,|\beta| \le m} D^\alpha (A^{\alpha\beta} D^\beta u) = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad \text{in} \quad (0,T) \times \Omega, \tag{1.1.3}$$

where m is a positive integer, D is the differential operator in x variable, α, β are d-dimensional multi-indices, and, for each α and β, $A^{\alpha\beta} = [A^{\alpha\beta}_{ij}]^n_{i,j=1}$ is an $n \times n$ real matrix-valued function. The coefficients $A^{\alpha\beta}$ satisfy certain ellipticity condition. We obtain a Schauder type estimate under the Dirichlet boundary condition with coefficients and data irregular in the t variable. Specifically, we can show that

Theorem 1.1.3 ([42]). *Let u be a weak solution of* (1.1.3) *with the Cauchy-Dirichlet boundary condition. Assume that the coefficients $A^{\alpha\beta} \in C^a_x$. Then*

$$\|u\|_{\frac{a+m}{2m}, a+m; (0,T)\times\Omega} \le C(\|u\|_{L_2((0,T)\times\Omega)} + \sum_{|\alpha|=m} [f_\alpha]^x_{a;(0,T)\times\Omega} + \sum_{|\alpha|<m} \|f_\alpha\|_{L_\infty((0,T)\times\Omega)}).$$

Moreover, under the same condition, (1.1.3) *has a unique solution.*

Here, $\|\cdot\|_{a,b}$ is the Hölder norm, where a and b represent the regularity in t and x variables, respectively.

Non-Divergence type system. In [42], we also considered the non-divergence type systems

$$u_t + (-1)^m Lu := u_t + (-1)^m \sum_{|\alpha|,|\beta| \le m} A^{\alpha\beta} D^\alpha D^\beta u = f \quad \text{in} \quad (0,T) \times \Omega. \tag{1.1.4}$$

We studied the interior estimate and boundary estimates separately. For the interior estimate, we can control all the $2m$-th order spatial derivatives in both the t and x variables, and u_t in the x variable. Generally, one cannot expect that u_t is regular in the t variable under the reduced regularity assumption on the coefficients and data. For the boundary estimate, we can control all the $2m$-th order spatial derivatives with the exception of the highest order normal derivative. Even for heat equation in a half space, one cannot expect to bound $D_\nu^2 u$ if $f(t)$ has jump discontinuity, where ν is the unit normal vector. To summarize, we proved the following theorem.

Theorem 1.1.4 ([42])**.** *Let $a \in (0,1)$. Suppose that $A^{\alpha\beta} \in C_x^a$. Let u smooth be a solution to (1.1.4). For any $R \leq 1$,*

I. Interior case: if (1.1.4) holds in Q_{4R}, then

$$\|u_t\|^x_{a;Q_R} + \|D^{2m}u\|_{\frac{a}{2m},a;Q_R} \leq C(\|u\|_{L_2(Q_{4R})} + \|f\|^x_{a;Q_{4R}});$$

II. Boundary case: if (1.1.4) holds in Q^+_{4R} with the homogeneous Dirichlet boundary condition on $\{x_d = 0\} \cap Q_{4R}$, then

$$[D_{x'}D^{2m-1}u]_{\frac{a}{2m},a;Q^+_R} \leq C(\sum_{|\gamma|\leq 2m} \|D^\gamma u\|_{L_\infty(Q^+_{4R})} + [f]^x_{a;Q^+_{4R}}).$$

Here $Q_R := (-R^{2m}, 0) \times B_R$ is the parabolic cylinder, and $[\cdot]^x$ (or $\|\cdot\|^x$) represents the semi-norm (or norm) only in the x variable.

Our results extend the corresponding results found in Liberman [81] for second-order scalar equations. In the special case of second-order parabolic systems, compared to [96], our coefficients and data are more general.

1.1.2.2 L_p and Schauder estimates for higher-order parabolic systems under conormal boundary condition with coefficients irregular in the t variable

In [43], we considered the conormal problem for higher-order divergence type parabolic systems. We first obtained the L_p estimate, and then used the L_p estimate to prove the Schauder estimate.

For the L_p estimate, unlike the case of constant or continuous coefficients, which is based on the exact representation of solutions and the Calderón-Zygmund theorem, our proof is in the spirit of an approach introduced by Krylov [66, 67], to deal with elliptic and parabolic equations with VMO_x coefficients in the whole space. We considered the divergence type system with coefficients having vanishing mean oscillation in the x variable (VMO_x). For any $p \in (1, \infty)$, we proved the following:

Theorem 1.1.5 ([43]). *Under reasonable regularity assumptions on the boundary and ellipticity condition on the coefficients, there exist constants λ_0 and N such that, for any $u \in \mathcal{H}_p^m((-\infty, T) \times \Omega), T \in (-\infty, \infty]$ satisfying*

$$u_t + (-1)^m \mathcal{L}u + \lambda u = \sum_{|\alpha| \leq m} D^\alpha f_\alpha \quad \text{in} \quad (-\infty, T] \times \Omega \tag{1.1.5}$$

with the conormal derivative boundary conditions on $(-\infty, T) \times \partial\Omega$, we have

$$\sum_{k=0}^{m} \lambda^{1-\frac{k}{2m}} \|D^k u\|_{L_p((-\infty,T)\times\Omega)} \leq C \sum_{|\alpha| \leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p((-\infty,T)\times\Omega)} \tag{1.1.6}$$

provided that $\lambda \geq \lambda_0$.

Moreover, if $\lambda > \lambda_0$, there exists a unique solution $u \in \mathcal{H}_p^m((-\infty, T) \times \Omega)$ of (1.1.5) with the conormal derivative boundary conditions on $(-\infty, T) \times \partial\Omega$.

With the help of the L_p estimate, we applied Campanato's approach to obtain a Schauder type estimate. Similar to the non-divergence type system near the boundary, we can bound all the m-th order spatial derivative with the exception of the highest normal derivative, which is optimal in this case. Specifically, we considered the system (1.1.5) near a flat boundary. Under the assumption that the coefficients and data are only regular in the x variable and measurable in the t variable, we proved

Theorem 1.1.6 ([43]). *Assume that $a \in (0,1)$, u smooth in x satisfying*

$$u_t + (-1)^m \mathcal{L} u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad in \quad Q_{2R}^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2R}$, $A^{\alpha\beta} \in C_x^a(Q_{2R}^+)$. Then for any $R \le 1$, there exists a constant C such that

$$[D_{x'} D^{m-1} u]_{\frac{a}{2m}, a; Q_R^+} \le C\Big(\sum_{|\alpha| \le m} \|D^\alpha u\|_{L_\infty(Q_{2R}^+)} + \sum_{|\alpha| = m} [f_\alpha]^x_{a; Q_{2R}^+} + \sum_{|\alpha| < m} \|f_\alpha\|_{L_\infty(Q_{2R}^+)}\Big).$$

1.1.3 Weighted L_p estimate for non-divergence type parabolic equation with Neumann boundary condition

When the data may blow up near boundary, it is important to understand the behavior of the solutions to parabolic (or elliptic) equations near boundary. In [52], we studied a special case by considering non-divergence type second-order parabolic equations with the Neumann boundary condition in a half space $\{x_d > 0\}$. We established the L_p estimates with measure $\mu(dx\,dt) = x_d^\alpha\,dx\,dt$ in place of the Lebesgue measure, where $\alpha \in (-1, p-1)$. This is a continuation of [41], where Dong and Kim established the weighted L_p estimates for both divergence and non-divergence type parabolic equations under the Dirichlet boundary condition. We do not impose any regularity assumption in the t variable on the coefficients, and in the x variable we impose a small bounded mean oscillation (BMO_x) assumption, which extends the result in [59]. Specifically, we considered the following equation

$$-u_t + a_{ij}D_{ij}u + b_iD_iu + cu - \lambda u = f \tag{1.1.7}$$

in $(-\infty, T) \times \mathbb{R}^d_+$ with the Neumann boundary condition $D_du = 0$ on $\{x_d = 0\}$. We proved that

Theorem 1.1.7 ([52])**.** *Under the uniform ellipticity condition and BMO_x condition on the leading coefficients a_{ij}, there is a λ_0 such that for $\lambda > \lambda_0$, u_t, D^2u, and D_du/x_d are bounded by f in the weighted L_p spaces.*

Our weighted L_p estimates extend the result of [59] to a more general setting to include coefficients discontinuous in the x variable.

1.2 Regularity theory of nonlocal parabolic equations

In a series of works [31, 30, 32], we considered a class of fully nonlinear nonlocal concave elliptic and parabolic equations as follows

$$u_t - \inf_\beta (L_\beta u(t,x) + f_\beta(t,x)) = 0, \tag{1.2.1}$$

where L_β is a non-local operator defined by

$$L_\beta u(t,x) = \int_{\mathbb{R}^d} (u(t,x+y) - u(t,x))K_\beta(t,x,y)\,dy \quad \text{for } \sigma \in (0,1),$$
$$L_\beta u(t,x) = \int_{\mathbb{R}^d} (u(t,x+y) - u(t,x) - Du(t,x)y\chi_{B_1})K_\beta(t,x,y)\,dy \quad \text{for } \sigma = 1,$$
$$L_\beta u(t,x) = \int_{\mathbb{R}^d} (u(t,x+y) - u(t,x) - Du(t,x)y)K_\beta(t,x,y)\,dy \quad \text{for } \sigma \in (1,2),$$

and for some constants $0 < \lambda < \Lambda$

$$\frac{(2-\sigma)\lambda}{|y|^{d+\sigma}} \le K_\beta(t,x,y) \le \frac{(2-\sigma)\Lambda}{|y|^{d+\sigma}}.$$

In particular, when $\sigma = 1$ we assume

$$\int_{S_r} yK_\beta(t,x,y)\,dy = 0 \quad \text{for} \quad r \in (0,\infty). \tag{1.2.2}$$

This type of nonlocal operators was first considered by Komatsu [57], Mikulevičius and Pragarauskas [89, 88], and later by Dong and Kim [40, 39], and Schwab and Silvestre [97], to name a few.

1.2.1 Schauder estimates for fully nonlinear nonlocal equations

In [31], we first extended the Schauder estimate in [50, 98] to include concave nonlocal parabolic equations with non-symmetric rough kernels. Specifically, for any small α, if f_β and $K_\beta(t,x,y)$ are C^α in x and $C^{\alpha/\sigma}$ in t, then we have the following $C^{1+\alpha/\sigma,\sigma+\alpha}$ a priori estimate of any smooth solution to u of (1.2.1).

Theorem 1.2.1 ([31])**.** *There exists a constant $\hat{\alpha} \in (0,1)$ so that for any $\alpha \in (0,\hat{\alpha})$*

and $\sigma+\alpha$ not being an integer the following holds: Suppose that $u \in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_1)\cap C^{\alpha/\sigma,\alpha}((-1,0)\times\mathbb{R}^d)$ is a solution of (1.2.1) in Q_1. Then

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C(\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + \sup_\beta [f_\beta]_{\alpha/\sigma,\alpha;Q_1}),$$

where C remains bounded as $\sigma \to 2$.

As $\sigma \to 2$, our result recovers the classical Schauder estimates of concave fully nonlinear parabolic equations [104].

We also considered the end-point situation when $\alpha = 0$. When the kernels are independent of t and x for $\sigma \neq 1$. We prove a priori C^σ estimate in the x variable and Λ^1 estimate in the t variable when f_a is merely bounded and measurable, where Λ^1 is the Zygmund space. For $\sigma = 1$, we obtain a priori Λ^1 estimate in both t and x. Notice that such results do not hold for classical elliptic (or parabolic) equation. For example, even for the Poisson equation

$$\Delta u = f,$$

when f is merely bounded, u may fail to be $C^{1,1}$.

Theorem 1.2.2 ([31])**.** *Assume that u smooth satisfies (1.2.1) in $(-\infty,0)\times\mathbb{R}^d$ with K_β independent of t and x. Then there exists a constant C so that for $\sigma \in (1,2)$*

$$[u]^t_{\Lambda^1} + [u]^x_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}} \le C\sup_\beta \|f_\beta\|_{L_\infty};$$

for $\sigma < 1$,

$$[u]^t_{\Lambda^1} + [u]^x_\sigma \le C\sup_\beta \|f_\beta\|_{L_\infty};$$

for $\sigma = 1$,

$$[u]_{\Lambda^1} \leq C \sup_{\beta} \|f_\beta\|_{L_\infty}.$$

Moreover, we localized the result to prove an interior estimate corresponding to the previous theorem. Such result is new even for nonlocal elliptic equations with symmetric kernels and optimal even in the linear case.

1.2.2 Dini estimates for fully nonlinear nonlocal equations

As a continuation of our previous work, we considered the case that the kernel and data are Dini continuous and proved that the solution is C^σ in x and C^1 in t. In our proof, we implement a novel perturbation type argument which is well fitted to the nonlocal structure. We believe such novel technique is very useful in the study of nonlocal equations in the future. Specifically, we studied the following equations

$$\partial_t u = \inf_{\beta \in \mathcal{A}} (L_\beta u + b_\beta Du + f_\beta), \tag{1.2.3}$$

where, when $\sigma < 1$, $b_\beta(t,x) = 0$; when $\sigma = 1$, $b_\beta(t,x) = b(t,x)$. We assume that

$$K_\beta(t,x,y) = \frac{(2-\sigma)a_\beta(t,x,y)}{|y|^{d+\sigma}}, \quad \lambda \leq a_\beta \leq \Lambda,$$

and the moduli of continuity of $a_\beta(t,x,y)$ with respect to (t,x) is a Dini function. Furthermore, we require the moduli of continuity of f_β and b_β are Dini functions as well.

We obtain the following result.

Theorem 1.2.3. *Let* $\sigma \in (0,2)$, $0 < \lambda \leq \Lambda < \infty$, *and* $\mathcal{A}$ *be an index set. Assume for* *each $\beta \in \mathcal{A}$, K_β satisfies (1.2.2) when $\sigma = 1$. Suppose $u \in C^{1,\sigma^+}(B_1)$ is a solution of (1.2.3) in Q_1 and is Dini continuous in $(-1,0) \times \mathbb{R}^d$. Then under regularity assumptions of coefficients and data, we have the a priori estimate*

$$\|\partial_t u\|_{L_\infty(Q_{1/2})} + [u]^x_{\sigma;Q_{1/2}} \le C(\|u\|_{L_\infty}, \omega_u, \omega_f). \tag{1.2.4}$$

Moreover, when $\sigma \neq 1$, we have

$$\sup_{(t_0,x_0)\in Q_{1/2}} [u]^x_{\sigma;Q_r(t_0,x_0)} \to 0 \quad \text{as} \quad r \to 0.$$

When $\sigma = 1$, Du is uniformly continuous in $Q_{1/2}$.

By regarding the elliptic equations as steady-state of parabolic equation, we obtained the corresponding results of concave fully nonlinear nonlocal elliptic equations.

Part I

Regularity Theory for Linear Equations and Systems

Chapter 2

An elliptic equation arising from composite material

2.1 Introduction

In this section, we consider second-order divergence type elliptic equations with discontinuous coefficients and data

$$L_{r_1,r_2}u := D_i(a(x)D_iu) = D_if_i \quad \text{in } \mathcal{D}, \tag{2.1.1}$$

where $\mathcal{D}$ is a bounded subset of $\mathbb{R}^2$,

$$a(x) = a_0\chi_{B_{r_1}(0,r_1)\cup B_{r_2}(0,-r_2)} + \chi_{\mathbb{R}^2\setminus(B_{r_1}(0,r_1)\cup B_{r_2}(0,-r_2))},$$

$a_0 > 0$ is a constant, $r_1, r_2 \in (0,\infty)$, and χ is the indicator function. This problem was raised by Bonnetier and Vogelius [10], and can be considered as a simplified model for composite media with closely spaced interfacial boundaries. Here $\mathcal{D}$ models the cross-section of a fiber-reinforced composite and the balls $B_{r_1}(0, r_1)$ and $B_{r_2}(0, -r_2)$ represent the cross-sections of the fibers; the remaining subdomain represents the matrix surrounding the fibers. Moreover, $a(x)$ is the shear modulus, which is a constant on the fibers, and a different constant on the matrix surrounding the fibers. The function u stands for the out of plane elastic displacement.

Elliptic equations and systems arising from elasticity have been studied by many authors. See, for instance, [25, 10, 76, 4, 93, 33, 106, 3, 6, 74]. In [25], Chipot, Kinderlehrer, and Vergara-Caffarelli considered divergence type uniformly elliptic systems in a domain $\mathcal{D} \subset \mathbb{R}^d$ consisting of finite numbers of linearly elastic, homogeneous, parallel laminae, which models the equilibrium problem of linear laminates. In [76], Li and Vogelius studied divergence type elliptic equations in a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, where $\mathcal{D}$ can be divided into finite numbers of subdomains with $C^{1,\alpha}$ boundaries. The coefficients of the equations and data are Hölder continuous in each subdomain up to the boundary, but may have jump discontinuities across the boundaries of the subdomains. Under these conditions, they proved a global $W^{1,\infty}$ estimate and a piecewise $C^{1,\beta}$ estimate, for any $\beta \leq \frac{\alpha}{d(\alpha+1)}$. Notably, their estimate does not depend on the distance of discontinuous surfaces, which indicates that by an approximation argument, interfaces may touch each other, e.g., the geometry shown in Figure 2.1. Later, Li and Nirenberg [77] extended the result in [76] to elliptic systems under the same condition. They were able to improve the piecewise $C^{1,\beta}$ estimate in [76] to any $\beta \in (0, \frac{\alpha}{2(1+\alpha)}]$.

Regarding the operator in (2.1.1), Bonnetier and Vogelius [10] first considered the Dirichlet boundary value problem with $r_1 = r_2 = 1$ and $f_i \equiv 0$: $L_{1,1}u = 0$ in $\mathcal{D}$ and $u = \phi$ on $\partial\mathcal{D}$. They showed a global regularity result that the solution $u \in W^{1,\infty}(\mathcal{D})$. Later, Li and Vogelius [76] extended the result in [10] and proved

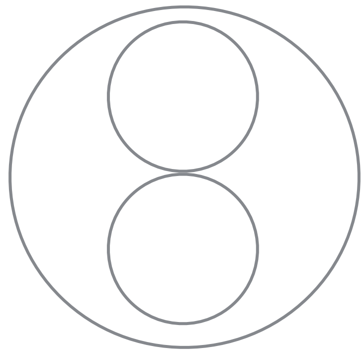

Figure 2.1: A disk contains two touching disks

that when $r_1 = r_2 = 1$, $f_i \equiv 0$, and $\mathcal{D} = B_{R_0}$ with R_0 sufficiently large, the weak solution u is piecewise smooth, i.e.,

$$u \in C^\infty(\overline{B_1(0,1)}),\ u \in C^\infty(\overline{B_1(0,-1)}),\ u \in C^\infty\big(K \setminus (B_1(0,1) \cup B_1(0,-1))\big),$$

where K is any compact subset of B_{R_0}. Then they asked the following natural question: can we drop the condition that R_0 being sufficiently large?

In our first result, we answer this question by proving that $R_0 > 2$ is sufficient to guarantee that u is piecewise smooth in the interior of B_{R_0}.

Theorem 2.1.1. *Let $R_0 > 2$ and $g \in H^{1/2}(\partial B_{R_0})$. Suppose u is a weak solution of*

$$D_i(a(x)D_i u) = 0 \quad in \quad B_{R_0}, \quad u = g \quad on \quad \partial B_{R_0},$$

where

$$a(x) = a_0 \chi_{B_1(0,1) \cup B_1(0,-1)} + \chi_{B_{R_0} \setminus (B_1(0,1) \cup B_1(0,-1))}. \tag{2.1.2}$$

Then

$$u \in C^{\infty}(\overline{B_1(0,1)}), \quad u \in C^{\infty}(\overline{B_1(0,-1)}), \quad u \in C^{\infty}\big(K \setminus (B_1(0,1) \cup B_1(0,-1))\big)$$

for any compact set $K \subset B_{R_0}$.

To prove Theorem 2.1.1, we borrow some ideas from [76]. In [76], Li and Vogelius constructed a sequence of piecewise smooth solutions $\{u_j\}$ to

$$D_i(a(x)D_i u) = 0 \quad \text{in } \mathbb{R}^2,$$

the linear combinations of which are dense in $H^s_{\text{sym}}(\partial B_{R_0})$ for R_0 sufficiently large, where $H^s_{\text{sym}}(\partial B_{R_0})$ denotes the space of functions even in x_1 with finite H^s norm for $s \geq 0$. The precise definition can be found in Section 2. Therefore, the solution u to the Dirichlet problem with the boundary condition $u = \phi \in H^s_{\text{sym}}(\partial B_{R_0})$ can be approximated by linear combinations of u_j's. Hence, by a classical elliptic regularity argument, one can show that $|D^k u| < \infty$ in each subdomain for any $k \geq 0$.

We carry out a more careful analysis on $\{u_j\}$ to show that $R_0 > 2$ is sufficient to guarantee that $\{u_j\}$ forms a Schauder basis for $H^s_{\text{sym}}(\partial B_{R_0})$. Precisely, it is obvious that

$$\{e_j, j \geq 0\} := \big\{(-1)^j \cos(2j\theta), (-1)^j \sin((2j+1)\theta), j \geq 0\big\}$$

is an orthogonal basis of $H^s_{\text{sym}}(\partial B_{R_0})$. Each u_j can be written as a linear combination of e_j's, i.e., $u_j = \sum_{k=0}^{\infty} M_{j,k} e_k$. We show that the infinite dimensional matrix $M := (M_{k,j})_{k,j=0}^{\infty}$ define a bounded and invertible operator on a Hilbert space l^s. For the definition of l^s, see Section 2. An important observation in our proof is that the submatrix $\{M_{k,j}\}_{k,j=1}^{\infty}$ is diagonally dominant by column. From this, we deduce that the map induced by M is invertible, which further implies that $\{u_j\}_{j \geq 0}$ forms a

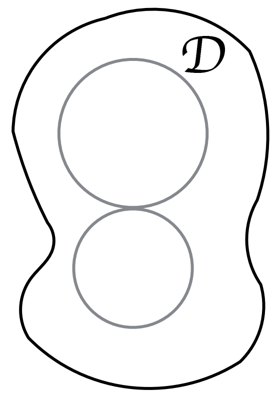

Figure 2.2: Arbitrary domain contains two touching disks

Schauder basis of $H^s_{\mathrm{sym}}(\{|x| = R_0\})$. The remaining proof then follows the lines in [76].

Another natural question to ask is if the geometry of the domain where the equation is satisfied affects the smoothness of the solution around the origin? In other words, if $Lu = 0$ in $\mathcal{D}$, is it necessary that $\mathcal{D}$ contains a ball with radius $R_0 > 2$ for u to be piecewise smooth around the origin? Or $\mathcal{D}$ can be any neighborhood of the origin? Our second result answers this question by proving an interior Schauder estimate for the non-homogeneous equation (2.1.1) in a general domain. Furthermore, we break the symmetry of the coefficients, meaning that $a(x)$ can be two different positive constants a_0 and b_0 in the two balls with different radii.

Theorem 2.1.2. *Let $r_1, r_2, a_0, b_0 \in (0, \infty)$, $\gamma \in (0,1)$, and $n \geq 0$ be an integer. Assume that $\mathcal{D} \subset \mathbb{R}^2$ is a bounded open set. Suppose that for any i, f_i is piecewise $C^{n,\gamma}$, i.e.,*

$$f_i \in C^{n,\gamma}(\mathcal{D} \cap B_{r_1}(0, r_1)), \quad f_i \in C^{n,\gamma}(\mathcal{D} \cap B_{r_2}(0, -r_2)),$$

and

$$f_i \in C^{n,\gamma}(\mathcal{D} \setminus (B_{r_1}(0, r_1) \cup B_{r_2}(0, -r_2))).$$

Let u be a weak solution to

$$D_i(a(x)D_iu(x)) = D_if_i \quad \text{in } \mathcal{D},$$

where

$$a(x) = a_0\chi_{B_{r_1}(0,r_1)} + b_0\chi_{B_{r_2}(0,-r_2)} + \chi_{\mathbb{R}^2\setminus(B_{r_1}(0,r_1)\cup B_{r_2}(0,-r_2))}.$$

Define $\mathcal{D}_\varepsilon = \{x \in \mathcal{D}, dist(x,\partial\mathcal{D}) \geq \varepsilon\}$ for any $\varepsilon \in (0,1)$. Then

$$u \in C^{n+1,\gamma}(\mathcal{D}_\varepsilon \cap B_{r_1}(0,r_1)), \quad u \in C^{n+1,\gamma}(\mathcal{D}_\varepsilon \cap B_{r_2}(0,-r_2)),$$

and

$$u \in C^{n+1,\gamma}(\mathcal{D}_\varepsilon \setminus (B_{r_1}(0,r_1) \cup B_{r_2}(0,-r_2))).$$

In particular, when $f_i \equiv 0$, u is piecewise smooth in $\mathcal{D}_\varepsilon$ up to the boundary.

For the proof, first we find a conformal mapping which maps two balls with different radii to two balls with the same radius, so it is sufficient to consider $r_1 = r_2$ and we denote the elliptic operator in (2.1.1) with $r_1 = r_2 = 1$ by L. Then the conformal mapping $\Gamma : z \to \mathrm{i}/z$ maps $\{|z - \mathrm{i}y| = 1\}$ and $\{|z + \mathrm{i}y| = 1\}$ to $\{\mathrm{Re}\, z = \frac{1}{2}\}$ and $\{\mathrm{Re}\, z = -\frac{1}{2}\}$, respectively, where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. We are able to construct Green's function $\tilde{G}(x,y)$ of the elliptic operator $\tilde{L}u = D_i(A(x)D_iu)$, where

$$A(x) = a_0\chi_{\{x_1 > \frac{1}{2}\}} + b_0\chi_{\{x_1 < -\frac{1}{2}\}} + \chi_{\{|x_1| < \frac{1}{2}\}}.$$

With the help of Γ and $\tilde{G}(x,y)$, we obtain Green's function $G(x,y)$ of the elliptic operator L in $\mathbb{R}^2$, which can be written as an infinite series of logarithmic function composed with smooth functions, for example, when $x \in \mathbb{R}^2 \setminus (B_1(0,1) \cup B_1(0,-1))$, and $y \in B_1(0,1)$,

$$G(x,y) = c_1 \sum_{k=0}^{\infty} (\alpha\beta)^k \log|X_{-2k}(x) - y| - c_2 \sum_{k=1}^{\infty} (\alpha\beta)^{k-1} \log|X_{2k-1}(x) - \overline{y}|,$$

where c_1, c_2, α, and β are constants with $|\alpha|, |\beta| < 1$, $\overline{y} = (y_1, -y_2)$, and $\{X_k\}$ are conformal maps and $X_0(x) = x$. Note that $\log|x-y|$ is Green's function of the Laplacian up to a factor. This observation allows us to implement some known results of the Laplace equation with piecewise Hölder continuous data on the right-hand side. More precisely, the original problem is decomposed to understand the regularity of solutions to the following equations

$$\begin{aligned}
&\Delta u = D_i(f_i \chi_{B_1(0,1)}),\\
&\Delta u = D_i(f_i \chi_{B_1(0,-1)}),\\
&\text{or } \Delta u = D_i(f_i \chi_{\mathbb{R}^2 \setminus (B_1(0,1) \cup B_1(0,-1))}),
\end{aligned}$$

where in each subdomain $f_i \in C^{n,\gamma}$. By locally flattening the boundary, the first two equations can be further reduced to the case that $f_i \in C^{n,\gamma}$ in two half spaces, i.e., $\{x_2 > 0\}$ and $\{x_2 < 0\}$. The detail can be found in the proof of Theorem 2.4.10 Case 1. The last equation needs an extension result to be reduced to the previous case. See Lemma 2.2.1. Combining with the smoothness of each X_k, we are able to estimate all the derivatives of the solution.

By a standard perturbation argument, we have the following corollary.

Corollary 2.1.3. *Let $0 < \lambda \le \Lambda < \infty$ and $n \ge 0$ be an integer. Assume $a(x)$ is piecewise $C^{n,\gamma}$, i.e.,*

$$a(x) \in C^{n,\gamma}(B_{r_1}(0,r_1)), \quad a(x) \in C^{n,\gamma}(B_{r_2}(0,-r_2)),$$

and

$$a(x) \in C^{n,\gamma}(\mathbb{R}^2 \setminus (B_{r_1}(0, r_1) \cup B_{r_2}(0, -r_2))),$$

and satisfies the ellipticity condition $\lambda \leq a \leq \Lambda$. *Suppose that for each* i, f_i *is piecewise* $C^{n,\gamma}$, *i.e.,*

$$f_i \in C^{n,\gamma}(\mathcal{D} \cap B_{r_1}(0, r_1)), \quad f_i \in C^{n,\gamma}(\mathcal{D} \cap B_{r_2}(0, -r_2)),$$

and

$$f_i \in C^{n,\gamma}(\mathcal{D} \setminus (B_{r_1}(0, r_1) \cup B_{r_2}(0, -r_2))).$$

Let u *be a weak solution to*

$$D_i(a(x)D_i u) = D_i f_i \quad in \quad \mathcal{D} \subset \mathbb{R}^2.$$

Denote $\mathcal{D}_\varepsilon = \{x \in \mathcal{D}, dist(x, \partial\mathcal{D}) \geq \varepsilon\}$ *for* $\varepsilon > 0$. *Then* u *is piecewise* $C^{n+1,\gamma}$ *in* $\mathcal{D}_\varepsilon$ *up to the boundary.*

The rest of this chapter is organized as follows. In the next section, we introduce some notation and preliminary results, which are needed in the proof of our main theorems. In Section 3, we prove Theorem 2.1.1. In Section 4, we make necessary preparations and prove Theorem 2.1.2 and Corollary 2.1.3.

2.2 Notation and preliminary results

In this section, we first introduce some notation used throughout this part of chapter. The Einstein summation convention is applied in this chapter. We use $B_R(x)$ to denote the Euclidean ball in $\mathbb{R}^2$ with radius R and center x. For simplicity, $B_1(0, 1)$

and $B_1(0,-1)$ are denoted by B_1 and B_2, respectively, and $\mathbb{R}^2 \setminus \overline{\mathrm{B}_1 \cup \mathrm{B}_2} = \mathrm{B}_0$. When there is no confusion, we use B_{R_0} to denote the ball with radius R_0 and center $(0,0)$. We use L to denote the operator L_{r_1,r_2} when $r_1 = r_2 = 1$.

Let $\mathcal{D}$ be a subset of $\mathbb{R}^2$ and $\beta \in (0,1]$. For any function f, we define

$$[f]_{\beta;\mathcal{D}} = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\beta} : x,y \in \mathcal{D}, x \neq y\right\}$$

and

$$\|f\|_{\beta;\mathcal{D}} = \|f\|_{L_\infty;\mathcal{D}} + [f]_{\beta;\mathcal{D}}.$$

We denote the space corresponding to $\|\cdot\|_{\beta;\mathcal{D}}$ by $C^\beta(\mathcal{D})$. For nonnegative integer m, we define

$$\|f\|_{m,\beta;\mathcal{D}} = \|f\|_{L_\infty;\mathcal{D}} + [D^m f]_{\beta;\mathcal{D}}.$$

The space corresponding to $\|\cdot\|_{m,\beta;\mathcal{D}}$ is denoted by $C^{m,\beta}(\mathcal{D})$.

Denote $L^2_{\mathrm{sym}}\big(\{x : |x| = R_0\}\big)$ to be the set of real-valued L^2 functions on the circle $\{|x| = R_0\}$ which are even with respect to x_1. We use a similar notation for the Sobolev spaces $H^s_{\mathrm{sym}}\big(\{x : |x| = R_0\}\big)$ for $s \geq 0$. Note that $L^2_{\mathrm{sym}}\big(\{x : |x| = R_0\}\big) = H^0_{\mathrm{sym}}\big(\{x : |x| = R_0\}\big)$.

Let V be a Banach space over $\mathbb{R}$. We say that a sequence $\{b_n\}$ in V is a Schauder basis of V if for every $v \in V$, there exists a unique sequence $\{a_n\}$ of scalars such that

$$v = \sum_{n=0}^{\infty} a_n b_n,$$

where the convergence is in the norm topology.

We first present an extension lemma, which is useful in our proofs.

Lemma 2.2.1. *Let $\gamma \in (0,1)$, $n \geq 0$, and $f \in C^{n,\gamma}(\mathbb{R}^2 \setminus (B_{r_1}(0,r_1) \cup B_{r_2}(0,-r_2)))$. Then there exists a function $F \in C^{n,\gamma}(\mathbb{R}^2)$ such that*

$$F|_{\mathbb{R}^2\setminus(B_{r_1}(0,r_1)\cup B_{r_2}(0,-r_2))} = f$$

and

$$\|F\|_{n,\gamma;\mathbb{R}^2} \leq C\|f\|_{n,\gamma;\mathbb{R}^2\setminus(B_{r_1}(0,r_1)\cup B_{r_2}(0,-r_2))},$$

where C is independent of f.

Proof. The proof can be found in [44]. □

Denote

$$\alpha = \frac{a_0 - 1}{a_0 + 1} \in (-1, 1).$$

Let $a(x)$ be defined as in (2.1.2). As mentioned in the introduction, Li and Vogelius [76] constructed a sequence solutions to

$$D_i(a(x)D_i u) = 0 \tag{2.2.1}$$

in $\mathbb{R}^2$, whose linear span is dense in $H^s_{\mathrm{sym}}(\{x : |x| = R_0\})$ for sufficiently large R_0. Following [76], we define Ψ_j as follows:

$$\Psi_j(z) = \frac{2}{a_0+1}\sum_{k=0}^{\infty}\alpha^k\frac{z^j}{(kz+\mathrm{i})^j} \quad \text{in } \{z : |z-\mathrm{i}| < 1\},$$

$$\Psi_j(z) = (-1)^{\frac{j+1}{2}}\mathrm{i}z^j + \sum_{k=1}^{\infty}\alpha^k\left[\frac{z^j}{(kz+\mathrm{i})^j} - \frac{z^j}{(kz-\mathrm{i})^j}\right]$$

$$\text{in } \{z : |z+\mathrm{i}| > 1 \text{ and } |z-\mathrm{i}| > 1\},$$

$$\Psi_j(z) = -\frac{2}{a_0+1}\sum_{k=0}^{\infty}\alpha^k\frac{z^j}{(kz-\mathrm{i})^j} \quad \text{in } \{z : |z+\mathrm{i}| < 1\}$$

for j odd, and

$$\Psi_j(z) = \frac{2}{a_0+1}\sum_{k=0}^{\infty}(-\alpha)^k \frac{z^j}{(kz+\mathrm{i})^j} \quad \text{in } \{z : |z-\mathrm{i}| < 1\},$$

$$\Psi_j(z) = (-1)^{j/2} z^j + \sum_{k=1}^{\infty}(-\alpha)^k \left[\frac{z^j}{(kz+\mathrm{i})^j} + \frac{z^j}{(kz-\mathrm{i})^j}\right]$$

$$\text{in } \{z : |z+\mathrm{i}| > 1 \,\text{and}\, |z-\mathrm{i}| > 1\},$$

$$\Psi_j(z) = \frac{2}{a_0+1}\sum_{k=0}^{\infty}(-\alpha)^k \frac{z^j}{(kz-\mathrm{i})^j} \quad \text{in} \quad \{z : |z+\mathrm{i}| < 1\}$$

for j even. Let $u_j(x_1, x_2) = R_0^{-j}\mathrm{Re}\,\Psi_j(z)$. It is shown in [76, Proposition 8.2] that $\{u_j\}$ are solutions to (2.2.1). In the lemma below, we first give an explicit representation of u_j's on $\{|x| = R_0\}$ in terms of trigonometric polynomials for $R_0 > 2$.

Lemma 2.2.2. *Let $R_0 > 2$ be a constant. For any $j \geq 0$, we have*

$$\begin{aligned} & u_{2j+1}(x_1, x_2)|_{|x|=R_0} \\ &= (-1)^j \sin(2j+1)\theta - 2\sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha^k \binom{2l+2j+1}{2j}(-1)^l \sin(2l+1)\theta}{(kR_0)^{2j+2l+2}}. \end{aligned} \tag{2.2.2}$$

For any $j \geq 1$, we have

$$\begin{aligned} & u_{2j}(x_1, x_2)|_{|x|=R_0} \\ &= (-1)^j \cos 2j\theta + 2\sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \frac{(-\alpha)^k \binom{2l+2j-1}{2j-1}(-1)^l \cos(2l\theta)}{(kR_0)^{2l+2j}}, \end{aligned} \tag{2.2.3}$$

and

$$u_0 = \frac{1-\alpha}{1+\alpha} = \frac{1}{a_0}. \tag{2.2.4}$$

Set

$$e_{2j-1} = (-1)^{j-1}\sin(2j-1)\theta, \quad e_{2j} = (-1)^j \cos 2j\theta$$

for $j \geq 1$, and $e_0 = u_0$. Clearly, for any $s \geq 0$, $\{e_j\}_{j=0}^{\infty}$ forms an orthogonal basis of $H^s_{\text{sym}}(\{x : |x| = R_0\})$. Define the Hilbert space

$$\mathrm{l}^s = \Big\{ a = (a_0, a_1, \ldots) : \sum_{j=0}^{\infty} |a_j|^2 (1+j)^{2s} < \infty \Big\}$$

with the norm $\|a\|_{\mathrm{l}^s} = \left(\sum_{j=0}^{\infty} |a_j|^2 (1+j)^{2s} \right)^{1/2}$. Then up to a constant factor, $H^s_{\text{sym}}(\{|x| = R_0\})$ is isometric to l^s. Denote $\vec{f} \in \mathrm{l}^s$ to be the infinite column vector $(f_j)_{j \geq 0}$. Let M be an infinite dimensional matrix such that its jth column is $\vec{u}_j$.

Since $u_0 = e_0$, we have $M_{0,0} = 1$ and $M_{j,0} = 0$ for $j \geq 1$. By Lemma 2.2.2, $M = id + B$, where id is the identity matrix, and B is defined as follows: for $l, j \geq 1$

$$B_{2l-1,2j-1} = -2 \sum_{k=1}^{\infty} \frac{\alpha^k \binom{2l+2j-3}{2j-2}}{(kR_0)^{2l+2j-2}}, \quad B_{2l,2j} = 2 \sum_{k=1}^{\infty} \frac{(-\alpha)^k \binom{2l+2j-1}{2j-1}}{(kR_0)^{2l+2j}}, \tag{2.2.5}$$

$$B_{0,2j} = 2 \sum_{k=1}^{\infty} \frac{(-\alpha)^k}{(kR_0)^{2j}}, \quad B_{2l-1,2j} = B_{2l,2j-1} = 0, \quad B_{l,0} = 0.$$

The following observation is crucial in our proof.

Lemma 2.2.3. *When $R_0 > 2$, the infinite dimensional matrix $\{M_{i,j}\}_{i,j=1}^{\infty}$ is diagonally dominant by column.*

Proof. Since $M = id + B$, it suffices to show that $\sum_{l=1}^{\infty} |B_{l,j}| < 1$ for $j \geq 1$. We first consider odd number columns. When $\alpha \in (0,1)$, obviously $B_{2l-1,2j-1} < 0$. On the other hand, when $\alpha \in (-1,0)$, by (2.2.5) we have $B_{2l-1,2j-1} > 0$. Thus, for $j \geq 1$ we have

$$\sum_{l=1}^{\infty} |B_{2l-1,2j-1}| = \sum_{l=1}^{\infty} 2 \Big| \sum_{k=1}^{\infty} \frac{\alpha^k \binom{2l+2j-3}{2j-2}}{(R_0 k)^{2l+2j-2}} \Big| = 2 \Big| \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \frac{\alpha^k \binom{2l+2j-3}{2j-2}}{(R_0 k)^{2l+2j-2}} \Big|. \tag{2.2.6}$$

Note that

$$
\begin{aligned}
2\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\frac{\alpha^k\binom{2l+2j-3}{2j-2}}{(R_0k)^{2l+2j-2}} &= 2\sum_{k=1}^{\infty}\frac{\alpha^k}{(kR_0)^{2j-1}}\sum_{l=1}^{\infty}\frac{\binom{2l+2j-3}{2j-2}}{(R_0k)^{2l-1}} \\
&= \sum_{k=1}^{\infty}\alpha^k\left(\Big(\frac{1}{kR_0-1}\Big)^{2j-1}-\Big(\frac{1}{kR_0+1}\Big)^{2j-1}\right). \qquad (2.2.7)
\end{aligned}
$$

Since $|\alpha|<1$ and $R_0>2$, we have

$$
\begin{aligned}
&\left|\sum_{k=1}^{\infty}\alpha^k\left(\Big(\frac{1}{kR_0-1}\Big)^{2j-1}-\Big(\frac{1}{kR_0+1}\Big)^{2j-1}\right)\right| \\
&<\sum_{k=1}^{\infty}\left(\Big(\frac{1}{2k-1}\Big)^{2j-1}-\Big(\frac{1}{2k+1}\Big)^{2j-1}\right)=1. \qquad (2.2.8)
\end{aligned}
$$

Then combining (2.2.6)-(2.2.8), we obtain

$$
\sum_{l=1}^{\infty}|B_{2l-1,2j-1}|<1.
$$

Similarly, for $j\geq 1$ we compute

$$
\begin{aligned}
\sum_{l=1}^{\infty}|B_{2l,2j}| &\leq \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}2\frac{|\alpha|^k\binom{2l+2j-1}{2j-1}}{(kR_0)^{2l+2j}} \qquad (2.2.9)\\
&= \sum_{k=1}^{\infty}|\alpha|^k\left(\Big(\frac{1}{kR_0-1}\Big)^{2j}+\Big(\frac{1}{kR_0+1}\Big)^{2j}-2\Big(\frac{1}{kR_0}\Big)^{2j}\right). \qquad (2.2.10)
\end{aligned}
$$

Note that for any $k\geq 1$, $R_0>2$, and $j\geq 1$, by convexity,

$$
\Big(\frac{1}{kR_0-1}\Big)^{2j}+\Big(\frac{1}{kR_0+1}\Big)^{2j}-2\Big(\frac{1}{kR_0}\Big)^{2j}>0.
$$

Therefore, the right-hand side of (2.2.10) is less than

$$
\sum_{k=1}^{\infty}\Big(\frac{1}{kR_0-1}\Big)^{2j}+\Big(\frac{1}{kR_0+1}\Big)^{2j}-2\Big(\frac{1}{kR_0}\Big)^{2j},
$$

which is decreasing with respect to R_0 because the right-hand side of (2.2.9) is decreasing. Thus, the left-hand side of (2.2.10) with $|\alpha| = 1$ is less than

$$\begin{aligned}
&\sum_{k=1}^{\infty} \Big(\frac{1}{2k-1}\Big)^{2j} + \Big(\frac{1}{2k+1}\Big)^{2j} - 2\Big(\frac{1}{2k}\Big)^{2j} \\
&= 1 + 2\sum_{k=1}^{\infty} \bigg(\Big(\frac{1}{2k+1}\Big)^{2j} - \Big(\frac{1}{2k}\Big)^{2j}\bigg) < 1.
\end{aligned}$$

Therefore,

$$\sum_{l=1}^{\infty} |B_{2l,2j}| < 1.$$

The lemma is proved. $\square$

In [76, Proposition 8.5], it is proved that for R_0 sufficiently large depending on s, $\mathrm{span}\{u_j|_{\{x:|x|=R_0\}}\}$ is dense in $H^s_{\mathrm{sym}}(\{|x| = R_0\})$. In the following proposition, by using Lemma 2.2.3 we prove that $R_0 > 2$ is sufficient to show that $\{u_j|_{\{x:|x|=R_0\}}\}$ forms a Schauder basis in $H^s_{\mathrm{sym}}(\{|x| = R_0\})$ for any $s \geq 0$.

Proposition 2.2.4. *For any $R_0 > 2$ and $s \geq 0$, $\{u_j|_{\{x:|x|=R_0\}}\}$ forms a Schauder basis in $H^s_{sym}(\{|x| = R_0\})$, i.e., for any $f \in H^s_{sym}(\{|x| = R_0\})$, there exists a unique sequence $\{a_j\}_{j=0}^{\infty}$ in $\mathbb{R}$ such that $f = \sum_{j=0}^{\infty} a_j u_j$. Moreover, we have*

$$\sum_{j=0}^{\infty} a_j^2 (1+j)^{2s} \leq C(\alpha, s, R_0)\|f\|^2_{H^s(\{|x|=R_0\})}. \tag{2.2.11}$$

Before proving Proposition 2.2.4, we first show that the matrix M defines a bounded and invertible operator on l^s in the following lemmas.

Lemma 2.2.5. *Let $R_0 > 0$ and $s \geq 0$. The matrix M defines a bounded operator on l^s. Consequently, $u_j \in H^s_{sym}\big(\{|x| = R_0\}\big)$ for any $j \geq 0$.*

Proof. For any nonnegative integer N, we first estimate the block $(B_{l,j})_{l\geq N, j\geq 0}$. Clearly,

$$\sum_{l\geq N}\sum_{j=0}^{\infty}|B_{l,j}| = \sum_{2l\geq N}\sum_{j=0}^{\infty}|B_{2l,2j}| + \sum_{2l-1\geq N}\sum_{j=1}^{\infty}|B_{2l-1,2j-1}|. \tag{2.2.12}$$

From (2.2.5) and the fact that $B_{l,0} = 0$ for $l \geq 0$, we have

$$\begin{aligned}&\sum_{2l\geq N}\sum_{j=0}^{\infty}|B_{2l,2j}| = \sum_{2l\geq N}\sum_{j=1}^{\infty} 2\left|\sum_{k=1}^{\infty}\frac{(-\alpha)^k\binom{2l+2j-1}{2l}}{(kR_0)^{2l+2j}}\right| \\ &\leq \sum_{2l\geq N}\frac{1}{R_0^{2l+1}}\sum_{k=1}^{\infty}|\alpha|^k\left(\left(\frac{1}{k-1/R_0}\right)^{2l+1} - \left(\frac{1}{k+1/R_0}\right)^{2l+1}\right).\end{aligned}$$

Since $R_0 > 2$,

$$\begin{aligned}&\sum_{k=1}^{\infty}|\alpha|^k\left(\left(\frac{1}{k-1/R_0}\right)^{2l+1} - \left(\frac{1}{k+1/R_0}\right)^{2l+1}\right) \\ &\leq 2\sum_{k=1}^{\infty}|\alpha|^k\left(\frac{1}{k-1/2}\right)^{2l+1} \leq C(\alpha),\end{aligned}$$

where $C(\alpha)$ only depends on α. Then we get

$$\sum_{2l\geq N}\sum_{j=0}^{\infty}|B_{2l,2j}| \leq \sum_{2l\geq N}\frac{C(\alpha)}{R_0^{2l+1}} \leq C(\alpha)R_0^{-N-1}. \tag{2.2.13}$$

Similarly by (2.2.5), we obtain

$$\begin{aligned}&\sum_{2l-1\geq N}\sum_{j=1}^{\infty}|B_{2l-1,2j-1}| \\ &\leq \sum_{2l-1\geq N}\sum_{k=1}^{\infty}|\alpha|^k\left(\left(\frac{1}{kR_0-1}\right)^{2l} + \left(\frac{1}{kR_0+1}\right)^{2l}\right) \leq CR_0^{-N-1}.\end{aligned} \tag{2.2.14}$$

Combining (2.2.12), (2.2.13), and (2.2.14), we have

$$\sum_{l\geq N}\sum_{j=0}^{\infty}|B_{l,j}|\leq CR_0^{-N-1}. \tag{2.2.15}$$

Now we are ready to show that M is a bounded operator on l^s. For any $f\in \mathrm{l}^s$ with

$$\|f\|_{\mathrm{l}^s}^2=\sum_{l=0}^{\infty}|f_l|^2(1+l)^{2s}=1,$$

we have

$$\|Mf\|_{\mathrm{l}^s}^2=\sum_{l=0}^{\infty}(1+l)^{2s}|(Mf)_l|^2=\sum_{l=0}^{\infty}(1+l)^{2s}\Big|\sum_{j=0}^{\infty}M_{l,j}f_j\Big|^2. \tag{2.2.16}$$

Since $M_{l,j}=\delta_{lj}+B_{l,j}$, by the Cauchy-Schwarz inequality

$$\Big|\sum_{j=0}^{\infty}M_{l,j}f_j\Big|^2=\Big|f_l+\sum_{j=0}^{\infty}B_{l,j}f_j\Big|^2\leq 2\Big(|f_l|^2+\Big|\sum_{j=0}^{\infty}B_{l,j}f_j\Big|^2\Big). \tag{2.2.17}$$

From (2.2.16) and (2.2.17), we get

$$\|Mf\|_{\mathrm{l}^s}^2\leq 2+2\sum_{l=0}^{\infty}(1+l)^{2s}\Big|\sum_{j=0}^{\infty}B_{l,j}f_j\Big|^2.$$

Notice that (2.2.15) implies for $l\geq 0$,

$$\sum_{j=0}^{\infty}|B_{l,j}|\leq CR_0^{-l-1}. \tag{2.2.18}$$

Combining with fact that $\|\{f_j\}\|_{l_\infty} \le 1$, we obtain

$$\begin{aligned}
&\sum_{l=0}^{\infty}(1+l)^{2s}\Big|\sum_{j=0}^{\infty}B_{l,j}f_j\Big|^2 \le \sum_{l=0}^{\infty}(1+l)^{2s}\Big(\sum_{j=0}^{\infty}|B_{l,j}|\Big)^2 \\
&\le C^2\sum_{l=0}^{\infty}(1+l)^{2s}R_0^{-2l-2} \le C(\alpha,s),
\end{aligned}$$

where $C(\alpha,s)$ depends on α and s. Therefore, the proof is completed. □

Lemma 2.2.6. *The operator on* l^s *defined by* M *is invertible.*

Proof. Let $N \ge 1$ be a large integer to be chosen later. First we estimate the block of $(B)_{l,j}$, where $l \in [1,N]$, $j > N$:

$$\sum_{l=1}^{N}\sum_{j>N}|B_{l,j}| \le \sum_{l=1}^{[N/2]+1}\sum_{2j>N}|B_{2l,2j}| + \sum_{l=1}^{[N/2]+1}\sum_{2j-1>N}|B_{2l-1,2j-1}|.$$

Using (2.2.10), the first summation on the right-hand side of the inequality above is bounded by

$$\begin{aligned}
&\sum_{2j>N}\sum_{k=1}^{\infty}\frac{2|\alpha|^k}{(kR_0)^{2j}}\sum_{l=1}^{[N/2]+1}\frac{\binom{2l+2j-1}{2j-1}}{(kR_0)^{2l}} \\
&< \sum_{2j>N}\sum_{k=1}^{\infty}\frac{2|\alpha|^k}{(kR_0)^{2j}}\left(\sum_{l=0}^{\infty}\frac{\binom{2l+2j-1}{2j-1}}{(kR_0)^{2l}}-1\right) \le CR_0^{-N-1},
\end{aligned} \tag{2.2.19}$$

where $C = C(\alpha)$. Similarly, by (2.2.6) and (2.2.7) we have

$$\sum_{l=1}^{[N/2]+1}\sum_{2j-1>N}|B_{2l-1,2j-1}| \le CR_0^{-N-1}. \tag{2.2.20}$$

Therefore, combining (2.2.19) and (2.2.20) we obtain

$$\sum_{l=1}^{N}\sum_{j>N}|B_{l,j}| \le CR_0^{-N-1}. \tag{2.2.21}$$

Next, we consider the N dimensional matrix $Q := \{M_{l,j}\}_{l,j=1}^N$. Since by Lemma 2.2.3 $\{M_{l,j}\}_{l,j=1}^\infty$ is diagonally dominant by column, Q is diagonally dominant by column as well, which implies that Q is invertible. We estimate Q^{-1} as follows. For odd number columns $2l-1 \in [1, N]$, by (2.2.7),

$$\begin{aligned}
&|Q_{2l-1,2l-1}| - \sum_{\{j\,:\,2j-1\in[1,N], j\neq l\}} |Q_{2j-1,2l-1}| \geq |Q_{2l-1,2l-1}| - \sum_{j=1, j\neq l}^{\infty} |\tilde{M}_{2j-1,2l-1}| \\
&\geq 1 - \sum_{k=1}^{\infty} \alpha^k \left(\left(\frac{1}{kR_0 - 1} \right)^{2l-1} - \left(\frac{1}{kR_0 + 1} \right)^{2l-1} \right) \geq C(\alpha, R_0) > 0,
\end{aligned}$$

where $C(\alpha, R_0)$ is a constant depending on α and R_0 but not on N. For even number columns, by (2.2.10) we obtain the same estimate,

$$|Q_{2l,2l}| - \sum_{\{j\,:\,2j\in[2,N], j\neq l\}} |Q_{2j,2l}| \geq C(\alpha, R_0) > 0.$$

Therefore, by [103, Corollary 1] we get

$$\|Q^{-1}\|_{l_1} \leq C(\alpha, R_0)^{-1},$$

which implies

$$\|Q^{-1}\|_{\mathrm{l}^s} \leq C(\alpha, R_0)^{-1} \sqrt{N} (1 + N)^s, \tag{2.2.22}$$

where $\|Q^{-1}\|_{\mathrm{l}^s}$ is the operator norm of Q^{-1} in a finite dimensional subspace of l^s. Indeed, for any $x = (x_1, x_2, \ldots, x_N)^T$, we compute

$$\|Q^{-1}x\|_{\mathrm{l}^s}^2 = \sum_{l=1}^N \left(\sum_{j=1}^N Q_{l,j}^{-1} x_j \right)^2 (1+l)^{2s} \leq (1+N)^{2s} \sum_{l=1}^N \left(\sum_{j=1}^N Q_{l,j}^{-1} x_j \right)^2. \tag{2.2.23}$$

Note that

$$\sum_{l=1}^N \left(\sum_{j=1}^N Q_{l,j}^{-1} x_j \right)^2 \leq \left(\sum_{l=1}^N \left| \sum_{j=1}^N Q_{l,j}^{-1} x_j \right| \right)^2 \leq \left(\|Q^{-1}\|_{l_1} \sum_{j=1}^N |x_j| \right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left(\|Q^{-1}\|_{l_1}\sum_{j=1}^{N}|x_j|\right)^2 &\leq \|Q^{-1}\|_{l_1}^2\left(\sum_{l=1}^{N}(1+l)^{-2s}\right)\|x\|_{\mathrm{l}^s}^2\\ &\leq N\|Q^{-1}\|_{l_1}^2\|x\|_{\mathrm{l}^s}^2. \qquad (2.2.24)\end{aligned}$$

Therefore, combining (2.2.23)-(2.2.24), (2.2.22) is proved.

Thanks to the estimates of the blocks, we are ready to prove the invertibility of M. For any fixed $y \in \mathrm{l}^s$, we need to find a unique $x \in \mathrm{l}^s$ such that $Mx = y$. Similar to l^s we define the space

$$\hat{\mathrm{l}}^s = \left\{a = (a_1, a_2, \cdots) : \sum_{j=1}^{\infty}|a_j|^2(1+j)^{2s} < \infty\right\}$$

with the norm $\|a\|_{\hat{\mathrm{l}}^s} = \left(\sum_{j=1}^{\infty}|a_j|^2(1+j)^{2s}\right)^{1/2}$. Notice that the summation index runs from 1 instead of 0 as in the definition of l^s.

By setting $x = (x_0, \hat{x})^T$ and $y = (y_0, \hat{y})^T$, where $x_0, y_0 \in \mathbb{R}$, $Mx = y$ is written as

$$\begin{pmatrix} 1 & (B_{0,j})_{j\geq 1} \\ 0 & (M_{l,j})_{l,j\geq 1}\end{pmatrix}\begin{pmatrix} x_0 \\ \hat{x}\end{pmatrix} = \begin{pmatrix} y_0 \\ \hat{y}\end{pmatrix}.$$

Clearly, it is sufficient to prove that for any $\hat{y} \in \hat{\mathrm{l}}^s$, there exists $\hat{x} \in \hat{\mathrm{l}}^s$ such that $(M_{l,j})_{l,j\geq 1}\hat{x} = \hat{y}$. In fact, if there exists $\hat{x} \in \hat{\mathrm{l}}^s$ so that $(M_{l,j})_{l,j\geq 1}\hat{x} = \hat{y}$, it remains to set $x_0 = y_0 - (B_{0,j})_{j\geq 1}\hat{x}$. From (2.2.18), x_0 is well defined. Obviously, $(M_{l,j})_{l,j\geq 1}\hat{x} = \hat{y}$ can be further written as

$$\begin{pmatrix} Q & B_1 \\ B_2 & id + \hat{B}\end{pmatrix}\begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2\end{pmatrix} = \begin{pmatrix} \mathrm{y}_1 \\ \mathrm{y}_2\end{pmatrix},$$

where $\hat{x} = (\mathrm{x}_1, \mathrm{x}_2)^T, \hat{y} = (\mathrm{y}_1, \mathrm{y}_2)^T$, $\mathrm{x}_1, \mathrm{y}_1$ are N dimensional vectors,

$$B_1 = (B_{i,j})_{i\in[1,N],j>N}, \quad B_2 = (B_{i,j})_{i>N,j\in[1,N]}, \quad \hat{B} = (B_{i,j})_{i,j>N}.$$

We rewrite the equation above as follows

$$Q\mathrm{x}_1 + B_1\mathrm{x}_2 = \mathrm{y}_1, \quad B_2\mathrm{x}_1 + \mathrm{x}_2 + \hat{B}\mathrm{x}_2 = \mathrm{y}_2.$$

We take Q^{-1} on the first equation to get

$$\mathrm{x}_1 = -Q^{-1}B_1\mathrm{x}_2 + Q^{-1}\mathrm{y}_1, \quad \mathrm{x}_2 = -B_2\mathrm{x}_1 - \hat{B}\mathrm{x}_2 + \mathrm{y}_2.$$

It is sufficient to find a unique fixed point for the map on the right-hand side of the system above. We claim when N is sufficiently large the map is a contraction, i.e., the operator norm of

$$\begin{pmatrix} 0 & -Q^{-1}B_1 \\ -B_2 & -\hat{B} \end{pmatrix}$$

is less than 1. Indeed, fix $\hat{x} = (\mathrm{x}_1, \mathrm{x}_2)^T \in \hat{\mathrm{l}}^s$ with $\|\hat{x}\|_{\hat{\mathrm{l}}^s} = 1$, and we compute

$$\begin{aligned}&\|Q^{-1}B_1\mathrm{x}_2\|^2_{\hat{\mathrm{l}}^s} + \|B_2\mathrm{x}_1 + \hat{B}\mathrm{x}_2\|^2_{\hat{\mathrm{l}}^s} \\ &= \sum_{k=1}^{N}(1+k)^{2s}\left(\sum_{j=1}^{N} Q^{-1}_{k,j}\Big(\sum_{l=N+1}^{\infty} B_{j,l}x_l\Big)\right)^2 + \sum_{k>N}(1+k)^{2s}\left(\sum_{j=1}^{\infty} B_{k,j}x_j\right)^2.\end{aligned}$$

By the definition of the l^s norm, we have

$$\begin{aligned}&\sum_{k=1}^{N}(1+k)^{2s}\left(\sum_{j=1}^{N} Q^{-1}_{k,j}\Big(\sum_{l=N+1}^{\infty} B_{j,l}x_l\Big)\right)^2 + \sum_{k>N}(1+k)^{2s}\Big(\sum_{j=1}^{\infty} B_{k,j}x_j\Big)^2 \\ &\le \|Q^{-1}\|^2_{\hat{\mathrm{l}}^s}\sum_{k=1}^{N}(1+k)^{2s}\sum_{l=N+1}^{\infty} B^2_{k,l} + \sum_{k>N}(1+k)^{2s}\Big(\sum_{j=1}^{\infty}|B_{k,j}|\Big)^2. \qquad (2.2.25)\end{aligned}$$

By (2.2.21), (2.2.18), and the fact that $|B_{k,l}| < 1$ for any $k, l \geq 1$, we know that

$$\sum_{k=1}^{N} \sum_{l=N+1}^{\infty} B_{k,l}^2 \leq \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} |B_{k,l}| \leq C R_0^{-N-1},$$

and

$$\sum_{j=1}^{\infty} |B_{k,j}| \leq C R_0^{-k-1}.$$

Therefore, combining the two inequalities above with (2.2.25) and (2.2.22), we have

$$\begin{aligned} &\|Q^{-1} B_1 \mathrm{x}_2\|_{\hat{\mathrm{l}}^s}^2 + \|B_2 \mathrm{x}_1 + \hat{B} x_2\|_{\hat{\mathrm{l}}^s}^2 \\ &\leq CN(1+N)^{4s} R_0^{-N-1} + C \sum_{k>N} (1+k)^{2s} R_0^{-2k-2} < 1, \end{aligned}$$

provided that N is sufficiently large only depending on α, R_0, and s. Hence $\tilde{M}$ and thus M are one-to-one and onto, and M is an invertible operator. Hence, we finish the proof. $\square$

Proof of Proposition 2.2.4. From Lemmas 2.2.5 and 2.2.6, M induces a bounded and invertible map T on $H^s_{\text{sym}}(\{|x| = R_0\})$. For any $f \in H^s_{\text{sym}}(\{|x| = R_0\})$, let $T^{-1}(f) = g$ and suppose that $g = \sum_{j=0}^{\infty} a_j e_j$. Since T is bounded, we have

$$f = T\Big(\sum_{j=0}^{\infty} a_j e_j\Big) = \sum_{j=0}^{\infty} a_j T(e_j). \tag{2.2.26}$$

Therefore, $\{u_j = T(e_j)\}$ is a Schauder basis in $H^s_{\text{sym}}(\{|x| = R_0\})$. By the Parseval identity,

$$\sum_{j=0}^{\infty} a_j^2 (1+j)^{2s} = C(s, R_0) \|g\|^2_{H^s(\{|x|=R_0\})} \leq C(\alpha, s, R_0) \|f\|^2_{H^s(\{|x|=R_0\})},$$

which gives (2.2.11). The proposition is proved. $\square$

Remark. In the previous proposition, we only consider functions even in x_1. For functions odd in x_1, the same result can be proved and we refer the reader to [44].

We finish this section by stating [76, Proposition 8.4].

Proposition 2.2.7. *Given any multi-index m there exists a constant C_m, independent of j and R_0, so that the functions u_j satisfy*

$$|D^m u_j(x_1, x_2)| \leq C_m R_0^{-j}(j + |m|)^{|m|}$$

in each of the three regions $\overline{B_1} \cap \mathrm{B}_1$, $\overline{B_1} \cap \mathrm{B}_2$, and $\overline{B_1} \cap \mathrm{B}_0$.

2.3 Proof of Theorem 2.1.1

Now we are ready to prove our first main theorem.

Proof of Theorem 2.1.1. As explained in Remark 2.2, we only need to consider g even in x_1. Without loss of generality, we may assume that g is smooth on $\partial B_{R_0} = \{|x| = R_0\}$. If not, we may simply choose $2 < R_0' < R_0$ such that $K \subset B_{R_0'}$. By the elliptic regularity, $u|_{\{|x|=R_0'\}}$ is smooth. Therefore, we can replace R_0 by R_0'. From Proposition 2.2.4 we have

$$g = \sum_{j=0}^{\infty} g_j u_j, \quad \sum_{j=0}^{\infty} |g_j|^2 (j+1)^{2s} < \infty \tag{2.3.1}$$

for some $s > 3/2$. It is easily seen that

$$\left| \frac{d}{dz} \Psi_j \right| \leq C(j+1)$$

in each of the three subdomains $\{z : |z + \mathrm{i}| < 1\}$, $\{z : |z - \mathrm{i}| < 1\}$, and $\{z : |z| \le R_0, |z + \mathrm{i}| > 1, |z - \mathrm{i}| > 1\}$, with constant C depending on R_0. Therefore,

$$\|u_j\|_{H^1(B_{R_0})} \le C(j + 1). \tag{2.3.2}$$

We consider $U_k = \sum_{j=0}^{k} g_j u_j$. Let h_k be the trace of U_k on $\{|x| = R_0\}$, i.e., $h_k = U_k|_{\{|x|=R_0\}}$. By Proposition 2.2.4, in particular (2.2.26), we have $h_k \to g$ in $H^{1/2}(\{|x| = R_0\})$. Moreover, by (2.3.2) and (2.3.1), we get

$$\begin{aligned}\sup_k \|U_k\|_{H^1(B_{R_0})} &\le C\sum_{j=0}^{\infty} |g_j|(j + 1)\\ &\le C\Big(\sum_{j=0}^{\infty} |g_j|^2(j + 1)^{2s}\Big)^{1/2}\Big(\sum_{j=0}^{\infty}(j + 1)^{2-2s}\Big)^{1/2} < \infty.\end{aligned}$$

From the construction of u_j, U_k is the solution of the following equation

$$D_j(aD_jU_k) = 0 \quad \text{in } B_{R_0}, \quad U_k = h_k \quad \text{on} \quad \partial B_{R_0}.$$

Since $h_k \to g$ in $H^{1/2}(\{|x| = R_0\})$, we have that $U_k \to u$ in $H^1(B_{R_0})$. Furthermore, for any multi-index m, by the interior elliptic estimates, we have the pointwise convergence

$$D^m U_k(x) \to D^m u(x) \tag{2.3.3}$$

with $x \in B_{R_0}$ but not on $\{|x - (0, 1)| = 1\}$ and $\{|x - (0, -1)| = 1\}$. By Proposition 2.2.7, and the Cauchy-Schwarz inequality,

$$\begin{aligned}|D^m U_k(x)| &\le \sum_{j=0}^{k} |g_j||D^m u_j(x)| \le C_m \sum_{j=0}^{k} |g_j| R_0^{-j}(j + |m|)^m\\ &\le C_m\Big(\sum_{j=0}^{\infty} |g_j|^2\Big)^{1/2}\Big(\sum_{j=0}^{\infty} R_0^{-2j}(j + |m|)^{2|m|}\Big)^{1/2} \le C_m\end{aligned} \tag{2.3.4}$$

for each multi-index m in each of the three regions $\overline{B_1} \cap \mathrm{B}_1$, $\overline{B_1} \cap \mathrm{B}_2$, and $\overline{B_1} \cap \mathrm{B}_0$. From (2.3.3) and (2.3.4), it follows immediately u has the desired smoothness in $\{|x| \leq 1\}$. In particular, $D^m u(x)$ has the same limit at the origin, whether we approach through the left cusp or through the right cusp. For $x \in K$ but outside $\{|x| \leq 1\}$, the piecewise smoothness of u follows from the classical elliptic regularity results; see, for instance, see [77, Proposition 1.4]. The theorem is proved. $\square$

2.4 Non-homogeneous equations with non-symmetric coefficients

In this section, we consider non-homogeneous equations with non-symmetric coefficients

$$D_i(a(x)D_i u) = D_i f_i, \tag{2.4.1}$$

where $a(x)$ is equal to a_0 in $B_{r_1}(0, r_1)$, b_0 in $B_{r_2}(0, -r_2)$, and 1 in $\mathbb{R}^2 \setminus (B_{r_1}(0, r_1) \cup B_{r_2}(0, -r_2))$, and $a_0, b_0 > 0$. The proof is divided into three steps. We shall first consider homogeneous equations with $r_1 = r_2 = 1$ in Section 2.4.1, and then non-homogeneous equations with $r_1 = r_2 = 1$ in Section 2.4.2, and finally the general case in Section 2.4.3.

2.4.1 Homogeneous equations

In this case, we basically adapt the proofs in Li and Vogelius [76], where they considered the special case $a_0 = b_0 > 0$.

Recall that we use B_1 and B_2 to denote $B_1(0,1)$ and $B_1(0,-1)$, respectively, and $\mathrm{B}_0 := \mathbb{R}^2 \setminus (\overline{\mathrm{B}_1 \cup \mathrm{B}_2})$. Let $\mathcal{D}$ be an open bounded subset of $\mathbb{R}^2$. The conformal mapping $z \to \mathrm{i}/z$ maps B_1 to $\{\mathrm{Re}\, z > \frac{1}{2}\}$, B_2 to $\{\mathrm{Re}\, z < -\frac{1}{2}\}$, and B_0 to $\{\mathrm{Re}\, z \in (-\frac{1}{2}, \frac{1}{2})\}$. This leads us to study the following homogeneous equation:

$$\tilde{L}u := D_i(A(x)D_i u) = 0, \tag{2.4.2}$$

where

$$A(x) = a_0 \chi_{\{x_1 > \frac{1}{2}\}} + \chi_{\{x_1 \in (-\frac{1}{2}, \frac{1}{2})\}} + b_0 \chi_{\{x_1 < -\frac{1}{2}\}}.$$

Let

$$\alpha = \frac{a_0 - 1}{a_0 + 1}, \quad \beta = \frac{b_0 - 1}{b_0 + 1}.$$

Choose a holomorphic function $\phi : \mathbb{C} \setminus (0,0) \to \mathbb{C}$ satisfying

$$\phi(\bar{z}) = \overline{\phi(z)} \tag{2.4.3}$$

and

$$|\phi(z)| \leq C\gamma^{|\mathrm{Re}\, z|}, \quad \text{when} \quad |\mathrm{Re}\, z| > \frac{1}{2}, \tag{2.4.4}$$

where $0 < \gamma < |\alpha\beta|^{-1}$ and C is a constant. We define Φ as follows:

$$\begin{aligned}
\Phi(z) &= (1-\alpha) \sum_{k=0}^{\infty} (\alpha\beta)^k \Big[\phi(z+2k) - \beta\phi(-z-(2k+1))\Big] \quad \text{in } \Big\{\mathrm{Re}\, z > \frac{1}{2}\Big\}, \\
\Phi(z) &= \phi(z) + \sum_{k=1}^{\infty} \Big[(\alpha\beta)^k \big(\phi(z+2k) + \phi(z-2k)\big) \\
&\quad - (\alpha\beta)^{k-1} \big(\alpha\phi(-z+2k-1) + \beta\phi(-z-(2k-1))\big)\Big] \quad \text{in } \Big\{\mathrm{Re}\, z \in (-\frac{1}{2}, \frac{1}{2})\Big\}, \\
\Phi(z) &= (1-\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k \Big[\phi(z-2k) - \alpha\phi(-z+2k+1)\Big] \quad \text{in } \Big\{\mathrm{Re}\, z < -\frac{1}{2}\Big\}.
\end{aligned}$$

Similar to [76, Proposition 8.2], we have the following proposition.

Proposition 2.4.1. *The function* $u(x_1, x_2) = Re\,\Phi(\mathrm{i}/z)$ *satisfies*

$$D_i(a(x)D_i u) = 0 \quad \mathit{in}\ \mathbb{R}^2. \tag{2.4.5}$$

Moreover, u *is even in* x_1.

When $\phi_j(z) = 1/z^j$ for $j \geq 0$, which is holomorphic in $\mathbb{C} \setminus (0,0)$ and satisfies (2.4.4), from Proposition 2.4.1, $u_j := R_0^{-j}\mathrm{Re}\,\Psi_j(z)$ is a solution to (2.4.5) for each j with

$$\begin{aligned}
\Psi_j(z) &= (1-\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\frac{z^j}{(\mathrm{i}+2kz)^j} + \beta\frac{z^j}{(\mathrm{i}+(2k+1)z)^j}\Big] \quad \text{in } \mathrm{B}_1,\\
\Psi_j(z) &= (-1)^{(j+1)/2}\mathrm{i}z^j + \sum_{k=1}^{\infty}(\alpha\beta)^k\Big[\frac{z^j}{(\mathrm{i}+2kz)^j} + \frac{z^j}{(\mathrm{i}-2kz)^j}\Big]\\
&\qquad - (\alpha\beta)^{k-1}\Big[\alpha\frac{z^j}{(-\mathrm{i}+(2k+1)z)^j} - \beta\frac{z^j}{(\mathrm{i}+(2k+1)z)^j}\Big] \quad \text{in } \mathrm{B}_0,\\
\Psi_j(z) &= (1-\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\frac{z^j}{(\mathrm{i}-2kz)^j} - \frac{z^j}{(-\mathrm{i}+(2k+1)z)^j}\Big] \quad \text{in } \mathrm{B}_2
\end{aligned}$$

for j odd, and

$$\begin{aligned}
\Psi_j(z) &= (1-\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\frac{z^j}{(\mathrm{i}+2kz)^j} - \beta\frac{z^j}{(\mathrm{i}+(2k+1)z)^j}\Big] \quad \text{in } \mathrm{B}_1,\\
\Psi_j(z) &= (-1)^{j/2}z^j + \sum_{k=1}^{\infty}(\alpha\beta)^k\Big[\frac{z^j}{(\mathrm{i}+2kz)^j} + \frac{z^j}{(\mathrm{i}-2kz)^j}\Big]\\
&\qquad - (\alpha\beta)^{k-1}\Big[\alpha\frac{z^j}{(-\mathrm{i}+(2k+1)z)^j} + \beta\frac{z^j}{(\mathrm{i}+(2k+1)z)^j}\Big] \quad \text{in } \mathrm{B}_0,\\
\Psi_j(z) &= (1-\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\frac{z^j}{(\mathrm{i}-2kz)^j} - \alpha\frac{z^j}{(-\mathrm{i}+(2k+1)z)^j}\Big] \quad \text{in } \mathrm{B}_2
\end{aligned}$$

for j even. By straightforward calculations, we have the following propositions similar to [76, Propositions 8.3 and 8.4], the proof of which can be found in [44].

Proposition 2.4.2. *Given any integer* m*, there exists a constant* C_m*, independent* *of j, so that the functions Ψ_j satisfy*

$$\Big|\Big(\frac{d}{dz}\Big)^m \Psi_j(z)\Big| \le C_m(j+m)^m$$

in each of the three regions

$$\Big\{z : |z| \le 1, |z+\mathrm{i}| > 1, |z-\mathrm{i}| > 1\Big\},$$

$$\Big\{z : |z| \le 1, |z-\mathrm{i}| < 1\Big\}, \quad \textit{and} \quad \Big\{z : |z| \le 1, |z+\mathrm{i}| < 1\Big\}.$$

Proposition 2.4.3. *Given any multi-index m, there exists a constant C_m independent of j and R_0, such that the functions u_j satisfy*

$$|D^m u_j(x_1,x_2)| \le C_m R_0^{-j}(j+|m|)^{|m|}$$

in each of the three regions: $\bar{B}_1 \cap \mathrm{B}_1$, $\bar{B}_1 \cap \mathrm{B}_2$, and $\bar{B}_1 \cap \mathrm{B}_0$.

Next, we investigate u_j restricted on $\{|x| = R_0\}$ with $R_0 > 2$. By setting $z = R_0 e^{\mathrm{i}\theta}$ for $j \ge 0$,

$$\begin{aligned}
u_{2j+1} = (-1)^j \sin(2j+1)\theta \\
&+ \mathrm{Re}\Big\{ \sum_{k=1}^{\infty} (\alpha\beta)^k \Big[\frac{e^{\mathrm{i}(2j+1)\theta}}{(\mathrm{i}+2kR_0e^{\mathrm{i}\theta})^{2j+1}} + \frac{e^{\mathrm{i}(2j+1)\theta}}{(\mathrm{i}-2kR_0e^{\mathrm{i}\theta})^{2j+1}}\Big] \\
&- (\alpha\beta)^{k-1}\Big[\alpha\frac{e^{\mathrm{i}(2j+1)\theta}}{(-\mathrm{i}+(2k+1)R_0e^{\mathrm{i}\theta})^{2j+1}} - \beta\frac{e^{\mathrm{i}(2j+1)\theta}}{(\mathrm{i}+(2k+1)R_0e^{\mathrm{i}\theta})^{2j+1}}\Big]\Big\}, \\
u_{2j} = (-1)^j \cos(2j\theta) + \mathrm{Re}\Big\{ \sum_{k=1}^{\infty} (\alpha\beta)^k \Big[\frac{e^{\mathrm{i}2j\theta}}{(\mathrm{i}+2kR_0e^{\mathrm{i}\theta})^{2j}} + \frac{e^{\mathrm{i}2j\theta}}{(\mathrm{i}-2kR_0e^{\mathrm{i}\theta})^{2j}}\Big] \\
&- (\alpha\beta)^{k-1}\Big[\alpha\frac{e^{\mathrm{i}2j\theta}}{(-\mathrm{i}+(2k+1)R_0e^{\mathrm{i}\theta})^{2j}} + \beta\frac{e^{\mathrm{i}2j\theta}}{(\mathrm{i}+(2k+1)R_0e^{\mathrm{i}\theta})^{2j}}\Big]\Big\}.
\end{aligned}$$

It is easy to see that

$$
\begin{aligned}
u_0 &= \frac{(\alpha-1)(\beta-1)}{1-\alpha\beta}, \\
u_{2j} &= (-1)^j \cos(2j\theta) + O(R_0^{-2j}) \quad \text{on} \quad \{|x| = R_0\},\ j \geq 1, \\
u_{2j+1} &= (-1)^j \sin(2j+1)\theta + O(R_0^{-2j-1}) \quad \text{on} \quad \{|x| = R_0\},\ \ j \geq 0.
\end{aligned}
$$

Therefore, similar to [76, Propositions 8.5 and 8.6], we obtain the following denseness result on $\{u_j\}$.

Proposition 2.4.4. *Given any $s \geq 0$, there exists a constant $C_s < \infty$, so that span$\{u_j|_{\{|x|=R_0\}}\}$ is dense in $H^s_{sym}(\{|x| = R_0\})$ provided that $R_0 > C_s$. Moreover, for any function $g \in H^s_{sym}(\{|x| = R_0\})$, we may approximate it by $g = \lim_k h_k$, with*

$$
h_k = \sum_{j=0}^{N_k} \gamma_j^{(k)} u_j|_{\{|x|=R_0\}} \in span\big\{u_j|_{\{|x|=R_0\}}\big\}
$$

and

$$
\Big(\sum_{j=0}^{N_k} |\gamma_j^{(k)}|^2 (j+1)^{2s}\Big)^{1/2} \leq C\|g\|_{H^s},
$$

where C depends on s and R_0, but independent of k and g, and $N_k \in \mathbb{N}$.

Remark. In Proposition 2.4.4 the solutions are even in x_1. For the case when solutions are odd in x_1, we define $\tilde{\Phi}$ as follows:

$$
\tilde{\Phi}(z) = (1-\alpha)\sum_{k=0}^{\infty} (\alpha\beta)^k \Big[\phi(z+2k) + \beta\phi(-z-(2k+1))\Big] \quad \text{in } \big\{\operatorname{Re} z > \frac{1}{2}\big\};
$$

$$
\begin{aligned}
\tilde{\Phi}(z) = \phi(z) + \sum_{k=1}^{\infty} \Big[&(\alpha\beta)^k(\phi(z+2k) + \phi(z-2k)) \\
&+ (\alpha\beta)^{k-1}\big(\alpha\phi(-z+2k+1) + \beta\phi(-z-(2k+1))\big)\Big] \quad \text{in } \big\{\operatorname{Re} z \in (-\frac{1}{2}, \frac{1}{2})\big\};
\end{aligned}
$$

$$\tilde{\Phi}(z) = (1-\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\phi(z-2k)+\alpha\phi(-z+2k+1)\Big] \quad \text{in } \big\{\operatorname{Re} z < -\frac{1}{2}\big\},$$

and $v(x_1, x_2) = \operatorname{Im}\tilde{\Phi}(\mathrm{i}/z)$. It is easy to see that v is odd in x_1 and following the proof of Proposition 2.4.1, we can show that v is a solution to (2.4.5). Further, we can prove similar result to Proposition 2.4.4.

Now we are in a good position to show the following theorem, the proof of which is similar to that of Theorem 2.1.1, and follows the lines in proving [76, Proposition 8.1].

Theorem 2.4.5. *Suppose $R_0 > C_0$, where C_0 is a constant depending on a_0 and b_0. Let g be in $H^{1/2}\big(\{|x| = R_0\}\big)$, and $u \in H^1(B_{R_0})$ denote the weak solution to*

$$D_i(a(x)D_i u) = 0 \quad \text{in } B_{R_0}, \quad u = g \quad \text{on} \quad \{|x| = R_0\}.$$

Then

$$u \in C^\infty(K \setminus (\mathrm{B}_1 \cup \mathrm{B}_2)), \quad u \in C^\infty(\overline{\mathrm{B}}_1), \quad \text{and} \quad u \in C^\infty(\overline{\mathrm{B}}_2)$$

for any compact set $K \subset B_{R_0}$.

2.4.2 Non-homogeneous equations

In this subsection, we consider the non-homogeneous equations by constructing Green's function of the operator in (2.4.1). By applying the conformal mapping $z \to \mathrm{i}/z$, we shall first construct Green's function of the operator $\tilde{L}$ defined in (2.4.2), i.e.,

$$D_i(A(x)D_i\tilde{G}(x,y)) = \delta(x-y),$$

where D_i is with respect to x_i.

It is well know that

$$-\frac{1}{2\pi}\Delta \log|x-y| = \delta(x-y).$$

In this section, for simplicity of exposition, we write $\Delta \log|x-y| = \delta(x-y)$. Denote $\mathbf{k} = (k,0)$, where $k \in \mathbb{Z}$. Let $\overline{y} = (y_1, -y_2)$. We define $\tilde{G}(x,y)$ as follows: <u>when $y_1 \in (-\frac{1}{2}, \frac{1}{2})$</u>,

$$\begin{aligned}
\tilde{G}(x,y) &= (1-\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\log|x+\mathbf{2k}-y| \\
&\qquad - \beta\log|x+\mathbf{2k}+\mathbf{1}+\overline{y}|\Big] \quad \text{in } \big\{x_1 > \frac{1}{2}\big\}, \\
\tilde{G}(x,y) &= \log|x-y| + \sum_{k=1}^{\infty}\Big[(\alpha\beta)^k\big(\log|x+\mathbf{2k}-y| + \log|x-\mathbf{2k}-y|\big) \\
&\qquad - (\alpha\beta)^{k-1}\big(\beta\log|x+\mathbf{2k}-\mathbf{1}+\overline{y}| + \alpha\log|x-(\mathbf{2k}-\mathbf{1})+\overline{y}|\big)\Big] \\
&\qquad \text{in } \big\{x_1 \in (-\frac{1}{2}, \frac{1}{2})\big\}, \\
\tilde{G}(x,y) &= (1-\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\log|x-\mathbf{2k}-y| \\
&\qquad - \alpha\log|x-(\mathbf{2k}+\mathbf{1})+\overline{y}|\Big] \quad \text{in } \big\{x_1 < -\frac{1}{2}\big\};
\end{aligned}$$

<u>when $y_1 > \frac{1}{2}$</u>,

$$\begin{aligned}
\tilde{G}(x,y) &= \log|x-y| + \alpha\log|x-\mathbf{1}+\overline{y}| \\
&\qquad - \frac{2\beta(1+\alpha)}{1+a_0}\sum_{k=0}^{\infty}(\alpha\beta)^k\log|x+\mathbf{2k}+\mathbf{1}+\overline{y}| \quad \text{in} \quad \big\{x_1 > \frac{1}{2}\big\}, \\
\tilde{G}(x,y) &= (1+\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\log|x-\mathbf{2k}-y| \\
&\qquad - \beta\log|x+\mathbf{2k}+\mathbf{1}+\overline{y}|\Big] \quad \text{in } \big\{x_1 \in (-\frac{1}{2}, \frac{1}{2})\big\}, \\
\tilde{G}(x,y) &= \frac{2(1+\alpha)}{1+b_0}\sum_{k=0}^{\infty}(\alpha\beta)^k\log|x-\mathbf{2k}-y| \quad \text{in } \big\{x_1 < -\frac{1}{2}\big\};
\end{aligned}$$

when $y_1 < -\frac{1}{2}$,

$$\tilde{G}(x,y) = \frac{2(1+\beta)}{1+a_0}\sum_{k=0}^{\infty}(\alpha\beta)^k \log|x+\mathbf{2k}-y| \quad \text{in } \{x_1 > \frac{1}{2}\},$$

$$\begin{aligned}\tilde{G}(x,y) = (1+\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[&\log|x+\mathbf{2k}-y| \\ &- \alpha\log|x-(\mathbf{2k}+\mathbf{1})+\overline{y}|\Big] \quad \text{in } \{x_1 \in (-\frac{1}{2},\frac{1}{2})\},\end{aligned}$$

$$\begin{aligned}\tilde{G}(x,y) = &\log|x-y| + \beta\log|x+\mathbf{1}+\overline{y}| \\ &- \frac{2\alpha(1+\beta)}{1+b_0}\sum_{k=0}^{\infty}(\alpha\beta)^k\log|x-(\mathbf{2k}+\mathbf{1})+\overline{y}| \quad \text{in } \{x_1 < -\frac{1}{2}\}.\end{aligned}$$

Proposition 2.4.6. *The function $\tilde{G}(x,y)$ defined above is Green's function of $\tilde{L}$, i.e.,*

$$D_i(A(x)D_i\tilde{G}(x,y)) = \delta(x-y).$$

Proof. To show $\tilde{G}(x,y)$ is Green's function, it is sufficient to prove that for $y \in \mathbb{R}^2$ a.e., $\Delta\tilde{G}(x,y) = \delta(x-y)$ for $x \notin \{x_1 = \frac{1}{2}\} \cup \{x_1 = -\frac{1}{2}\}$ and $\tilde{G}(x,y), A(x)D_1\tilde{G}(x,y)$ are continuous in x across the two lines $\{x_1 = \pm\frac{1}{2}\}$. The detail of the proof can be found in [44]. $\square$

Now, let us turn back to the original operator $Lu = D_i(a(x)D_iu(x))$. The conformal mapping $z \to \mathrm{i}/z$ can be written in real variables as $\Theta : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\Theta_1(x) = \frac{x_2}{x_1^2+x_2^2}, \quad \Theta_2(x) = \frac{x_1}{x_1^2+x_2^2}.$$

For any integer k, denote $X_k(x) = \Theta(\Theta(x)+k)$, which is a conformal map.

According to $\tilde{G}$, we define $\mathcal{G}(x,y)$ as follows: when $y \in \mathrm{B}_0$,

$$\mathcal{G}(x,y) = (1-\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k\big[\log|X_{2k}(x)-y| - \beta\log|X_{2k+1}(x)-\overline{y}|\big] \quad \text{for } x \in \mathrm{B}_1,$$

$$\begin{aligned}\mathcal{G}(x,y) = \log|x-y| + \sum_{k=1}^{\infty}\Big\{(\alpha\beta)^k\Big[\log|X_{2k}(x)-y| + \log|X_{-2k}(x)-y|\Big)\Big]\\ - (\alpha\beta)^{k-1}\Big[\beta\log|X_{2k-1}(x)-\overline{y}| + \alpha\log|X_{-(2k-1)}(x)-\overline{y}|\Big]\Big\} \quad \text{for } x \in \mathrm{B}_0,\end{aligned}$$

$$\begin{aligned}\mathcal{G}(x,y) = (1-\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k\big[\log|X_{-2k}(x)-y|\\ - \alpha\log|X_{-(2k+1)}(x)-\overline{y}|\big] \quad \text{for } x_1 \in \mathrm{B}_2;\end{aligned}$$

when $y \in \mathrm{B}_1$,

$$\begin{aligned}\mathcal{G}(x,y) = \log|x-y| + \alpha\log|X_{-1}(x)-\overline{y}|\\ - \frac{2\beta(1+\alpha)}{1+a_0}\sum_{k=0}^{\infty}(\alpha\beta)^k\log|X_{2k+1}(x)-\overline{y}| \quad \text{for} \quad x \in \mathrm{B}_1 \setminus \{(0,1)\},\end{aligned}$$

$$\mathcal{G}(x,y) = (1+\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k\big[\log|X_{-2k}(x)-y| - \beta\log|X_{2k+1}(x)-\overline{y}|\big] \quad \text{for } x \in \mathrm{B}_0,$$

$$\mathcal{G}(x,y) = \frac{2(1+\alpha)}{1+b_0}\sum_{k=0}^{\infty}(\alpha\beta)^k\log|X_{-2k}(x)-y| \quad \text{for } x \in \mathrm{B}_2;$$

when $y \in \mathrm{B}_2$,

$$\mathcal{G}(x,y) = \frac{2(1+\beta)}{1+a_0}\sum_{k=0}^{\infty}(\alpha\beta)^k\log|X_{2k}(x)-y| \quad \text{for } x_1 \in \mathrm{B}_1,$$

$$\begin{aligned}\mathcal{G}(x,y) = (1+\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\log|X_{2k}(x)-y|\\ - \alpha\log|X_{-(2k+1)}(x)-\overline{y}|\Big] \quad \text{for } x \in \mathrm{B}_0,\end{aligned}$$

$$\begin{aligned}\mathcal{G}(x,y) = \log|x-y| + \beta\log|X_1(x)-\overline{y}|\\ - \frac{2\alpha(1+\beta)}{1+b_0}\sum_{k=0}^{\infty}(\alpha\beta)^k\log|X_{-2k-1}(x)-\overline{y}| \quad \text{for } x \in \mathrm{B}_2 \setminus \{(0,-1)\}.\end{aligned}$$

Remark. Note that in the definition of $\mathcal{G}$, for the case $y \in \mathrm{B}_1$ (or $y_1 \in \mathrm{B}_2$), we excluded $(0,1)$ (or $(0,-1)$, respectively). This is because X_{-1} (or X_1) has a singularity at $(0,1)$ (or $(0,-1)$, respectively). All the other X_k for $k \in \mathbb{Z}$ appeared in the summation are regular.

Proposition 2.4.7. *Let $\mathcal{G}$ be defined above. Then we have*

$$\Delta_x \mathcal{G}(x,y) = \delta(x-y) \quad \textit{for} \quad y \in \mathrm{B}_0 \quad \textit{and} \quad x \notin \partial(\mathrm{B}_1 \cup \mathrm{B}_2);$$
$$\Delta_x \mathcal{G}(x,y) = \delta(x-y) \quad \textit{for} \quad y \in \mathrm{B}_1 \quad \textit{and} \quad x \notin \partial(\mathrm{B}_1 \cup \mathrm{B}_2) \cup \{(0,1)\};$$
$$\Delta_x \mathcal{G}(x,y) = \delta(x-y) \quad \textit{for} \quad y \in \mathrm{B}_2 \quad \textit{and} \quad x \notin \partial(\mathrm{B}_1 \cup \mathrm{B}_2) \cup \{(0,-1)\}.$$

Proof. We consider the case when $y \in \mathrm{B}_0$ as an example. When $x \in \mathrm{B}_1 \cup \mathrm{B}_2$, we show that $G(x,y)$ is harmonic. For instance, when $x \in \mathrm{B}_1$,

$$\Theta(x) + \mathbf{2k} \in \big\{x_1 > 1/2\big\} \quad \text{and} \quad \Theta(y) \in \big\{x_1 \in (-1/2, 1/2)\big\}.$$

Therefore, $\Theta(x)+\mathbf{2k} \neq \Theta(y)$, implying $X_{2k}(x) \neq y$. Similarly, we have $X_{2k-1}(x) \neq \overline{y}$. Combining with the facts that $\Delta_x \log|x-y| = 0$ when $x \neq y$ in $\mathbb{R}^2$ and that X_k is conformal, we obtain that $G(x,y)$ is harmonic in B_1. In the same way, we can show that $G(x,y)$ is harmonic in B_2 as well. When $x \in \mathrm{B}_0$, as we mentioned in the beginning of this section, we use the notation $\Delta_x \log|x-y| = \delta(x-y)$. Each term in the expression of $G(x,y)$, with the exception of $\log|x-y|$, is harmonic in B_0 by the same argument in proving $\Delta_x G(x,y) = 0$ in B_1. Hence, when $x \in \mathrm{B}_0$, $\Delta_x G(x,y) = \delta(x-y)$.

For the case when $y \in \mathrm{B}_1 \cup \mathrm{B}_2$, the same argument can be implemented to show that $\Delta_x G(x,y) = \delta(x-y)$ in the corresponding domains, and we omit the details. $\square$

Proposition 2.4.8. *The function $\mathcal{G}(x,y)$ defined above satisfies the compatibility* *condition, i.e.,*

$$\mathcal{G}(x,y) \quad \textit{and} \quad a(x)D_\nu \mathcal{G}(x,y)$$

are continuous across $\partial(\mathrm{B}_1 \cup \mathrm{B}_2)$*, where* ν *is the unite normal vector of* $\partial(\mathrm{B}_1 \cup \mathrm{B}_2)$*.*

Proof. Because Θ is a conformal map, the continuities of $\mathcal{G}(x,y)$ and $a(x)D_\nu\mathcal{G}(x,y)$ is equivalent to the continuities of $\mathcal{G}(\Theta(x),y)$ and $a(\Theta(x))D_1(\mathcal{G}(\Theta(x),y))$, respectively. Note that $a(\Theta(x)) = A(x)$ and

$$\frac{|x|}{|\Theta(y)|} = \frac{|y|}{|\Theta(x)|},$$

which by similarity of triangles implies that

$$\frac{|\Theta(x)-y|}{|x-\Theta(y)|} = \frac{|y|}{|x|}. \tag{2.4.6}$$

We take the case when $y \in \mathrm{B}_1$ as an example and the other cases can be verified in the same way. Since $\Theta(\Theta(x)) = x$, we have

$$\begin{aligned}
\mathcal{G}(\Theta(x),y) &= \log|\Theta(x)-y| + \alpha\log|\Theta(x-\mathbf{1})-\overline{y}| \\
&\quad - \frac{2\beta(1+\alpha)}{1+a_0}\sum_{k=0}^{\infty}(\alpha\beta)^k \log|\Theta(x+\mathbf{2k}+\mathbf{1})-\overline{y}| \quad \text{in} \quad \Big\{x_1 > \frac{1}{2}\Big\} \setminus \{(1,0)\}, \\
\mathcal{G}(\Theta(x),y) &= (1+\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big[\log|\Theta(x-\mathbf{2k})-y| \\
&\quad - \beta\log|\Theta(x+\mathbf{2k}+\mathbf{1})-\overline{y}|\Big] \quad \text{in } \Big\{x_1 \in (-\frac{1}{2},\frac{1}{2})\Big\}, \\
\mathcal{G}(\Theta(x),y) &= \frac{2(1+\alpha)}{1+b_0}\sum_{k=0}^{\infty}(\alpha\beta)^k\log|\Theta(x-\mathbf{2k})-y| \quad \text{in } \Big\{x_1 < -\frac{1}{2}\Big\}.
\end{aligned}$$

By taking (2.4.6) into account, $\mathcal{G}(\Theta(x), y)$ has the expression: in $\left\{x_1 > \frac{1}{2}\right\}$

$$\begin{aligned}
\mathcal{G}(\Theta(x), y) &= \log|x - \Theta(y)| + \alpha \log|x - \mathbf{1} + \overline{\Theta(y)}| \\
&- \frac{2\beta(1+\alpha)}{1+a_0} \sum_{k=0}^{\infty} (\alpha\beta)^k \log|x + \mathbf{2k} + \mathbf{1} + \overline{\Theta(y)}| - \log|x| - \alpha \log|x - \mathbf{1}| \\
&+ \frac{2\beta(1+\alpha)}{1+a_0} \sum_{k=0}^{\infty} (\alpha\beta)^k \log|x + \mathbf{2k} + \mathbf{1}| + \frac{(1-\beta)(1+\alpha)}{1-\alpha\beta} \log|y|;
\end{aligned}$$

in $\left\{x_1 \in (-\frac{1}{2}, \frac{1}{2})\right\}$

$$\begin{aligned}
\mathcal{G}(\Theta(x), y) = (1+\alpha) \sum_{k=0}^{\infty} (\alpha\beta)^k \Big[\log|x - \mathbf{2k} - \Theta(y)| - \beta \log|x + \mathbf{2k} + \mathbf{1} + \overline{\Theta(y)}| \Big] \\
- (1+\alpha) \sum_{k=0}^{\infty} (\alpha\beta)^k \Big[\log|x - \mathbf{2k}| - \beta \log|x + \mathbf{2k} + \mathbf{1}| \Big] + \frac{(1+\alpha)(1-\beta)}{1-\alpha\beta} \log|y|;
\end{aligned}$$

and in $\left\{x_1 < -\frac{1}{2}\right\}$

$$\begin{aligned}
\mathcal{G}(\Theta(x), y) &= \frac{2(1+\alpha)}{1+b_0} \sum_{k=0}^{\infty} (\alpha\beta)^k \log|x - \mathbf{2k} - \Theta(y)| \\
&- \frac{2(1+\alpha)}{1+b_0} \sum_{k=0}^{\infty} (\alpha\beta)^k \log|x - \mathbf{2k}| + \frac{(1-\beta)(1+\alpha)}{1-\alpha\beta} \log|y|.
\end{aligned}$$

Observe that

$$\mathcal{G}(\Theta(x), y) = \tilde{G}(x, \Theta(y)) - H(x) + \frac{(1+\alpha)(1-\beta)}{1-\alpha\beta} \log|y|,$$

where $\tilde{G}(x, y)$ is Green's function of $\tilde{L}$ and $H(x)$ is a function obtained by replacing y with 0 in the expression of $\tilde{G}(x, y)$. Since we verify the continuities of $\tilde{G}(x, y)$ and $A(x)D_1\tilde{G}(x, y)$ across the lines $\{x_1 = \pm\frac{1}{2}\}$ in Proposition 2.4.6, the same proof shows that $H(x)$ and $A(x)D_1H(x)$ are continuous across the lines $\{x_1 = \pm\frac{1}{2}\}$. Therefore, $\mathcal{G}(\Theta(x), y)$ and $A(x)D_1\mathcal{G}(\Theta(x), y)$ are continuous by linearity, and the proof is completed. $\square$

With Proposition 2.4.7 and 2.4.8, we are ready to construct Green's function $G(x,y)$ of operator L. Define

$$
\begin{aligned}
&G(x,y) = \mathcal{G}(x,y) \quad \text{when} \quad y \in \mathrm{B}_0, \\
&G(x,y) = \mathcal{G}(x,y) + \frac{\alpha}{1-\alpha}\mathcal{G}(x,(0,1)) \quad \text{when} \quad y \in \mathrm{B}_1, \\
&G(x,y) = \mathcal{G}(x,y) + \frac{\beta}{1-\beta}\mathcal{G}(x,(0,-1)) \quad \text{when} \quad y \in \mathrm{B}_2.
\end{aligned}
$$

When defining $\mathcal{G}$, we remove $(0,1)$ and $(0,-1)$, since X_{-1} and X_1 have singularity at these points. Nevertheless, in the following proposition we prove that for fixed y, $G(x,y)$ is the well-defined in $\mathbb{R}^2 \setminus \{y\}$. In particular,

$$
\lim_{x \to (0,1)(\text{ or }(0,-1))} G(x,y) \quad \text{exists.}
$$

Proposition 2.4.9. *$G(x,y)$ defined above is Green's function of L.*

Proof. Since from Proposition 2.4.8, $\mathcal{G}(x,y)$ satisfies the compatibility conditions, by linearity $G(x,y)$ satisfies the compatibility conditions as well. It remains to prove $\Delta G(x,y) = \delta(x-y)$ for $x \notin \partial(\mathrm{B}_1 \cup \mathrm{B}_2)$.

For $y \in \mathrm{B}_0$, this is proved in Proposition 2.4.7. It remains to consider $y \in \mathrm{B}_1 \cup \mathrm{B}_2$. Without loss of generality we assume $y \in \mathrm{B}_1$. As we mentioned in Remark 2.4.2, $X_{-1}(x)$ has singularity at $(0,1)$. By a simple computation,

$$
X_{-1}(x) = \left(\frac{x_1}{|x-(0,1)|^2}, \frac{x_2 - |x|^2}{|x-(0,1)|^2} \right).
$$

Hence for fixed $y \neq (0,1)$,

$$
\log|X_{-1}(x) - \bar{y}| \sim -\log|x-(0,1)| \quad \text{and} \quad \mathcal{G}(x,y) \sim -\alpha \log|x-(0,1)|
$$

when x is near $(0,1)$. As the singular part in $\mathcal{G}(x,(0,1))$ behaves like $(1-\alpha)\log|x-(0,1)|$, it is easy to see that $G(x,y)$ is bounded around $(0,1)$. This implies that $(0,1)$ is a removable singularity of $G(\cdot,y)$ for fixed $y \neq (0,1)$. Furthermore, y is the only singular point of $G(\cdot,y)$ for fixed $y \neq (0,1)$, i.e., $\Delta G(x,y) = \delta(x-y)$. For the case $y=(0,1)$, it suffices to consider $x \in \mathrm{B}_1$. By the definition of $G(x,y)$,

$$G(x,(0,1)) = \frac{1}{1-\alpha}\Big\{\log|x-(0,1)| + \alpha\log|X_{-1}(x)-(0,-1)| + H(x,(0,1))\Big\},$$

where $H(x,(0,1))$ is harmonic in B_1. Since

$$\log|X_{-1}(x)-(0,-1)| = \log\frac{1}{|x-(0,1)|},$$

we have

$$G(x,(0,1)) - \log|x-(0,1)| = \frac{1}{1-\alpha}H(x,(0,1)).$$

Since the right-hand side of the equality above is harmonic, when $y=(0,1)$, $G(x,y)$ is well defined as well and $\Delta G(x,(0,1)) = \delta(x-(0,1))$. Therefore, we finish the proof of the proposition. $\square$

With the help of Green's function constructed above, we are ready to consider the non-homogeneous equation

$$D_i(a(x)D_iu(x)) = D_if_i(x)$$

in general $\mathcal{D} \subset \mathbb{R}^2$. Now, we state our theorem in the case when $r_1 = r_2 = 1$.

Theorem 2.4.10. *Let $n \in \mathbb{N} \cup \{0\}$ and $\gamma \in (0,1)$. Assume that u is a weak solution to the equation*

$$D_i(a(x)D_iu(x)) = D_if_i \quad \textit{in } \mathcal{D},$$

where

$$a(x) = a_0 \quad in \ \mathrm{B}_1, \quad a(x) = b_0 \quad in \ \mathrm{B}_2, \quad a(x) = 1 \quad in \ \mathrm{B}_0,$$

and for each i, f_i is piecewise $C^{n,\gamma}$, i.e.,

$$f_i \in C^{n,\gamma}(\mathrm{B}_1), \quad f_i \in C^{n,\gamma}(\mathrm{B}_2), \quad f_i \in C^{n,\gamma}(\mathrm{B}_0).$$

Let $\mathcal{D}_\varepsilon = \{x \in \mathcal{D}, dist(x, \partial\mathcal{D}) \geq \varepsilon\}$ for $\varepsilon > 0$. Then u is piecewise $C^{n+1,\gamma}$ in $\mathcal{D}_\varepsilon$ up to the boundary, i.e.,

$$u \in C^{n+1,\gamma}(\mathcal{D}_\varepsilon \cap \mathrm{B}_1), \ u \in C^{n+1,\gamma}(\mathcal{D}_\varepsilon \cap \mathrm{B}_2), \quad u \in C^{n+1,\gamma}(\mathcal{D}_\varepsilon \cap \mathrm{B}_0).$$

Proof. We prove the theorem by considering two cases.

Case 1: $(0,0) \notin \mathcal{D}$. Define $\Omega_1 = \mathcal{D} \cap \mathrm{B}_1, \Omega_2 = \mathcal{D} \cap \mathrm{B}_2$, and $\Omega_0 = \mathcal{D} \cap \mathrm{B}_0$. Because $(0,0) \notin \mathcal{D}$, it is easy to see that any point in $\mathcal{D}$ belongs to at most two subdomains $\overline{\Omega_i}$, which is exactly the case in [33, Remark 3(ii)]. Therefore, we apply [33, Theorem 2 and Remark 3(ii)] to obtain that when $n = 0$, $u \in C^{1,\gamma}$ piecewise in $\mathcal{D}_\varepsilon$, i.e., for any $0 \leq i \leq 2$,

$$u \in C^{1,\gamma}(\mathcal{D}_\varepsilon \cap \Omega_i).$$

For $n > 0$, we use an induction argument. For a ball away from the circles $\{|x - (0, \pm 1)| = 1\}$, the conclusion follows from the classical Schauder estimate for Poisson's equation. We only need to consider a ball $B_l(x) \subset \mathcal{D}_\varepsilon$ and $x \in \{|x - (0, \pm 1)| = 1\}$. Notice that by locally flattening the boundary, it is sufficient to consider

$$D_i(a_{ij} D_j u) = D_i h_i,$$

where a_{ij} are piecewise smooth in $\mathbb{R}^2_+$ and $\mathbb{R}^2_-$ with bounded derivatives and $h_i \in C^{n,\gamma}(\mathbb{R}^2_+)$, $h_i \in C^{n,\gamma}(\mathbb{R}^2_-)$, where $\mathbb{R}^2_+$ (or $\mathbb{R}^2_-$) is the upper (lower) half plane. Here we only give a sketch of the proof. For $n = 1$, by taking derivative with respect to the tangential variable x_1, we have

$$D_i(a_{ij}D_jD_1u) = D_iD_1h_i - D_i(D_1(a_{ij})D_ju).$$

Thanks to the case $n = 0$, we have that $Du \in C^\gamma$ piecewise, which implies the right-hand side can be written as $D_i\mathrm{f}_i$ for some piecewise C^γ functions f_i. Therefore, we apply the result of the case $k = 0$ to obtain that $DD_1u \in C^\gamma$ piecewise. It remains to estimate D_2^2u, which is obtained from the formula

$$D_2^2u = a_{22}^{-1}(D_ih_i - D_1(a_{1j}D_ju) - D_2(a_{21}D_1u)).$$

Therefore, DD_2u is piecewise $C^{1,\gamma}$ as well. By induction, it is easy to prove that $u \in C^{n+1,\gamma}$ piecewise.

Case 2: $(0, 0) \in \mathcal{D}$. There exists $0 < l < 1$ such that $B_l \subset \mathcal{D}$. We define a cutoff function $\eta \in C_0^\infty(B_l)$, which equals 1 on $B_{l/2}$. Let $v = u\eta$, which satisfies

$$D_i(aD_iv(x)) = D_i\tilde{f}_i - f_iD_i\eta + aD_iuD_i\eta \quad \text{in } \mathbb{R}^2, \tag{2.4.7}$$

where

$$\tilde{f}_i = f_i\eta + auD_i\eta. \tag{2.4.8}$$

We define

$$
\begin{aligned}
&\tilde{u}(x) = \\
&\Big(-\int_{\mathrm{B}_0} D_{y_i} G(x,y)\tilde{f}_i(y)\,dy - \int_{\mathrm{B}_1} D_{y_i} G(x,y)\tilde{f}_i(y)\,dy - \int_{\mathrm{B}_2} D_{y_i} G(x,y)\tilde{f}_i(y)\,dy \Big) \\
&-\int_{\mathbb{R}^2} G(x,y)\big(f_i(y)D_i\eta(y) - aD_iuD_i\eta\big)\,dy \\
&:= u_1(x) + u_2(x), \qquad (2.4.9)
\end{aligned}
$$

Since G is Green's function of L, the function $\tilde{u}$ defined above is also a solution to (2.4.7). When u is restricted to $\{|x| > \frac{l}{2}\}$, the result follows from Case 1. Therefore, it remains to estimate u in $B_{l/2}$, where $v = u$. Because

$$D_i(a(x)D_i(\tilde{u} - v)) = 0$$

in $\mathbb{R}^2$, by Theorem 2.4.5 with a sufficiently large R_0, we know that $\tilde{u} - v$ is piecewise smooth. Hence, it suffices to estimate $\tilde{u}$ in $B_{l/2}$.

Note that $\mathrm{supp}(D_i\eta) \subset \{l/2 < |x| < l\}$, on which by Case 1 $u \in C^{n+1,\gamma}$ piecewise. Combining with the definition of $\tilde{f}_i$ in (2.4.8), we get that $\tilde{f}_i$ are piecewise $C^{n,\gamma}$. By (2.4.9), Since the estimates of u_1 and u_2 are quite similar, we only consider u_1 as follows:

$$
\begin{aligned}
&u_1(x) \\
&= -\int_{\mathrm{B}_1} D_{y_i} G(x,y)\tilde{f}_i(y)\,dy - \int_{\mathrm{B}_2} D_{y_i} G(x,y)\tilde{f}_i(y)\,dy - \int_{\mathrm{B}_0} D_{y_i} G(x,y)\tilde{f}_i(y)\,dy \\
&:= -w_1(x) - w_2(x) - w_3(x).
\end{aligned}
$$

We focus on the case $x \in \Omega_0 \cap B_1$ and the same argument can be applied to the other cases as well. By the definition of $G(x,y)$, we have

$$\begin{aligned} w_1(x) = (1+\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k \Big(\int_{\mathrm{B}_1} & D_{y_i}\log|X_{-2k}(x)-y|\tilde{f}_i(y)\,dy \\ & - \beta\int_{\mathrm{B}_1} D_{y_i}\log|X_{2k+1}(x)-\overline{y}|\tilde{f}_i(y)\,dy\Big) \\ = (1+\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k & \big[h(X_{-2k}(x)) - \beta\hat{h}(X_{2k+1}(x))\big], \end{aligned} \tag{2.4.10}$$

where

$$h(x) = \int_{\mathrm{B}_1} D_{y_i}\log|x-y|\tilde{f}_i(y)\,dy, \quad \hat{h}(x) = \int_{\mathrm{B}_1} D_{y_i}\log|x-\overline{y}|\tilde{f}_i(y)\,dy. \tag{2.4.11}$$

Since $\log|x-y|$ is the fundamental solution of the Laplace equation in $\mathbb{R}^2$, $h(x)$ satisfies

$$\Delta h = -D_i(\tilde{f}_i\chi_{\mathrm{B}_1}) \quad \text{in } \mathbb{R}^2.$$

Since $\tilde{f}$ is piecewise $C^{n,\gamma}$ and the interface $\partial\mathrm{B}_1$ is smooth by the same method in dealing with Case 1, we obtain that $h \in C^{n+1,\gamma}(B_1)$ piecewise.

For any $k \neq 0$, by the definition,

$$X_k(x_1,x_2) = \left(\operatorname{Re}\frac{\mathrm{i}}{\mathrm{i}/z+k}, \operatorname{Im}\frac{\mathrm{i}}{\mathrm{i}/z+k}\right) = \frac{1}{k}\left(\operatorname{Re}\frac{1}{kz+\mathrm{i}}, 1+\operatorname{Im}\frac{1}{kz+\mathrm{i}}\right),$$

where $z = x_1 + \mathrm{i}x_2$. Notice that for $x \in \mathrm{B}_0$, $|kz+\mathrm{i}| \geq 1$. By a straightforward calculation, it is easily seen that for any $j > 0$ and $x \in B_1 \cap \mathrm{B}_0$,

$$|D^j X_k(x)| \leq C|k|^{j+1}, \tag{2.4.12}$$

where C is independent of k. Moreover, since $x \in \Omega_0 \cap B_1$, $\Theta(x) \in \{|x|>1\} \cap \{x_1 \in$

$(-\frac{1}{2}, \frac{1}{2})\}$, which implies that $\Theta(x) - \mathbf{2k} \in \{x : x_1 < 1/2, |x| > 1\}$, and

$$X_{-\mathbf{2k}}(x) = \Theta(\Theta(x) - \mathbf{2k}) \in B_1 \setminus \overline{\mathrm{B}_1}. \tag{2.4.13}$$

Therefore, combining (2.4.12) and (2.4.13), with the chain rule, we have for any $k \geq 0$

$$\|h(X_{-2k}(x))\|_{n+1,\gamma;B_1\cap\Omega_0} \leq C(k+1)^{n+3}\Big(\|h\|_{n+1,\gamma;B_1\cap\mathrm{B}_2} + \|h\|_{n+1,\gamma;B_1\cap\mathrm{B}_0}\Big).$$

It remains to estimate $\hat{h}(X_{2k-1})$. Notice that

$$\Big|\int_{\mathrm{B}_1} D_{y_i} \log|x-\overline{y}|\tilde{f}_i(y)\,dy\Big| = \Big|\int_{\mathrm{B}_2} D_{y_i} \log|x-y|\tilde{f}_i(\overline{y})\,dy\Big|,$$

implying that

$$\Delta\hat{h}(x) = D_i\big(\tilde{f}_i(\overline{x})\chi_{\mathrm{B}_2}\big).$$

Note that for any $k \geq 1$ and $x \in B_1 \cap \mathrm{B}_0$,

$$X_{2k-1}(x) = \Theta(\Theta(x) + \mathbf{2k} - \mathbf{1}) \in B_1 \cap \mathrm{B}_1.$$

Furthermore, the regularity of $\tilde{f}_i(\overline{y})$ is the same as $\tilde{f}_i(y)$, which indicates that $\hat{h}(X_{2k-1})$ can be estimated in a similar way as $h(X_{-2k})$. Hence, combining the estimates of h and $\hat{h}$, we have

$$\begin{aligned}
&\|w_1\|_{n+1,\gamma;B_1\cap\mathrm{B}_0} \\
&\leq C(1+\alpha)\sum_{k=0}^{\infty}(\alpha\beta)^k\Big(\|h(X_{-2k})\|_{n+1,\gamma;B_1\cap\mathrm{B}_0} + \beta\|\hat{h}(X_{2k+1})\|_{n+1,\gamma;B_1\cap\mathrm{B}_0}\Big) \\
&\leq C\Big(\sum_{j=1}^{2}\|f_j\|_{n,\gamma;\mathrm{B}_1} + \|u\|_{n,\gamma;\mathrm{B}_1\cup(B_l\setminus B_{l/2})}\Big),
\end{aligned} \tag{2.4.14}$$

where the last term on the right-hand side is estimated in Case 1. Here in the last inequality above, we use the fact that

$$\begin{aligned}&\|h\|_{L_\infty(B_2)} + \|\hat{h}\|_{L_\infty(B_2)}\\&\le C\sum_{j=1}^{2}\|\tilde{f}_j\|_{L_\infty(\mathrm{B}_1)} \le C\Big(\sum_{j=1}^{2}\|f_j\|_{L_\infty(\mathrm{B}_1)} + \|u\|_{L_\infty(B_l\setminus B_{l/2})}\Big),\end{aligned}$$

which can be deduced directly from (2.4.11).

By symmetry, it is easy to see that the argument applied to w_1 can be implemented to w_2 as well. We omit the details. Now let us estimate w_3 with a little modification. Similar to the expression in (2.4.10), we have

$$\begin{aligned}w_3(x) = g(x) + \sum_{k=1}^{\infty}&(\alpha\beta)^k\Big[g(X_{2k}(x)) + g(X_{-2k}(x))\Big]\\&- (\alpha\beta)^{k-1}\Big[\beta\hat{g}(X_{2k+1}(x)) + \alpha\hat{g}(X_{-(2k+1)}(x))\Big],\end{aligned}$$

where

$$g(x) = \int_{\mathrm{B}_0} D_{y_i}\log|x-y|\tilde{f}_i(y)\,dy, \quad \hat{g}(x) = -\int_{\mathrm{B}_0} D_{y_i}\log|x-y|\tilde{f}_i(\overline{y})\,dy.$$

Since $\tilde{f}_i(\overline{y})$ has the same regularity as $\tilde{f}_i$, it is sufficient to consider g, which satisfies

$$\Delta g = -D_i(\tilde{f}_i\chi_{\mathrm{B}_0}).$$

Here, we cannot directly apply the result in Case 1 because of the singularity of the domain B_0. Nonetheless, by Lemma 2.2.1, there exists $F_i \in C^{n,\gamma}(\mathbb{R}^2)$ which is the extension of $\tilde{f}_i\chi_{\mathrm{B}_0}$ and satisfies

$$\|F_i\|_{n,\gamma;\mathbb{R}^2} \le C\big(\|f_i\|_{n,\gamma;B_l\cap\mathrm{B}_0} + \|u\|_{n,\gamma;(B_l\setminus B_{l/2})\cap\mathrm{B}_0}\big).$$

Therefore, define

$$\tilde{g} := \int_{\mathbb{R}^2} D_{y_i} \log|x-y| F_i(y)\, dy, \quad \mathrm{g}_1 := \int_{\mathrm{B}_1} D_{y_i} \log|x-y| F_i(y)\, dy,$$

and

$$\mathrm{g}_2 := \int_{\mathrm{B}_2} D_{y_i} \log|x-y| F_i(y)\, dy,$$

which satisfy

$$\Delta\tilde{g} = -D_i F_i, \quad \Delta\mathrm{g}_1 = -D_i(F_i \chi_{\mathrm{B}_1}), \quad \text{and} \quad \Delta\mathrm{g}_2 = -D_i(F_i \chi_{\mathrm{B}_2}).$$

From the classical Schauder estimate, we have $\tilde{g} \in C^{n+1,\gamma}(B_1)$. By the estimate of w_1 above, we have g_1, g_2 are all piecewise $C^{n+1,\gamma}$, which implies $g = \tilde{g} - \mathrm{g}_1 - \mathrm{g}_2$ is piecewise $C^{n+1,\gamma}$ as well. Then we can follow the same argument in the estimate of w_1 and obtain a similar estimate for w_3

$$\|w_3\|_{n+1,\gamma;B_1\cap\mathrm{B}_0} \leq C\big(\sum_{j=1}^{2} \|f_j\|_{n,\gamma;B_l\cap\mathrm{B}_0} + \|u\|_{n,\gamma;(B_l\setminus B_{l/2})\cap\mathrm{B}_0}\big).$$

Hence, we show that $\tilde{u}$ is piecewise $C^{n+1,\gamma}$ and the proof is completed. $\square$

2.4.3 Two balls with different radii

Next, we consider the general case that the two balls have different radii. Specifically,

$$a(x) = a_0\chi_{B_{r_1}(0,r_1)} + b_0\chi_{B_{r_2}(0,-r_2)} + \chi_{\mathcal{D}\setminus(B_{r_1}(0,r_1)\cup B_{r_2}(0,-r_2))}.$$

Denote $\partial B_{r_1}(0,r_1) = C_1$ and $\partial B_{r_2}(0,-r_2) = C_2$. By scaling and reflection, without loss of generality, we may assume that $r_2 > r_1 > 1/2$. Now, we consider a conformal map $\tilde{T} : \mathbb{C} \to \mathbb{C}, \tilde{T}(z) = \frac{1}{z-z_0}$, where $z_0 \in \mathbb{C}$ and $z_0 = z_1 + z_2\mathrm{i}$. It is well known that for $z_0 \notin C_1 \cup C_2$, $\tilde{T}$ maps C_1 and C_2 to two circles. We shall find a suitable z_0 such that the two circles have the same radius. Indeed, by a simple computation it is easy to see that $\tilde{T}(C_1)$ and $\tilde{T}(C_2)$ are circles with radii $|r_1/(z_1^2 + z_2^2 - 2r_1z_2)|$ and $|r_2/(z_1^2 + z_2^2 + 2r_2z_2)|$, respectively. Then, we only need to find (z_1, z_2) such that

$$\big|r_1/(z_1^2 + z_2^2 - 2r_1z_2)| = |r_2/(z_1^2 + z_2^2 + 2r_2z_2)\big|.$$

It is obvious that

$$\begin{aligned} r_1(z_1^2 + z_2^2 + 2r_2z_2) &= r_2(z_1^2 + z_2^2 - 2r_1z_2) \\ \Leftrightarrow (r_2 - r_1)(z_1^2 + z_2^2) &= 4z_1z_2r_1r_2. \end{aligned} \tag{2.4.15}$$

Note that (2.4.15) can be written as

$$\frac{z_1}{z_2} + \frac{z_2}{z_1} = \frac{4r_1r_2}{r_2 - r_1} > 2, \tag{2.4.16}$$

which implies the existence of (z_1, z_2). Moreover, if (z_1, z_2) is a solution of (2.4.16), for any $s > 0$ (sz_1, sz_2) solves (2.4.16) as well. Hence, we can pick z_0 outside $\mathcal{D}$, which indicates that $\tilde{T}$ is smooth in $\mathcal{D}$. After a translation, rotation, and scaling of the coordinates, we can assume that $\tilde{T}$ maps $B_{r_1}(0, r_1)$ and $B_{r_2}(0, -r_2)$ to B_1 and B_2, respectively. Therefore, we obtain the desired conformal map $\tilde{T}$, which is a diffeomorphism between $\mathcal{D}$ and $\tilde{T}(\mathcal{D})$.

Now we are ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. In order to consider the equation

$$D_i(a(x)D_iu(x)) = D_if_i(x),$$

we define $v(x) = u(\tilde{T}^{-1}x)$ in $\tilde{T}(\mathcal{D})$. By a straightforward calculation, we have

$$D_i(a(\tilde{T}^{-1}x)D_iv) = D_k(\eta_{ki}f_i(\tilde{T}^{-1}x))$$

in $\tilde{T}(\mathcal{D})$, where η_{ki} are some smooth functions in $\tilde{T}(\mathcal{D})$. The operator on the left-hand side of the equation above is the same as the operator in Theorem 2.4.10. Therefore, applying Theorem 2.4.10, we get $v \in C^{n+1,\gamma}$ piecewise in $\tilde{T}(\mathcal{D}) \cap B_l$, where B_l is a small ball around the origin. This, combined with the smoothness of $\tilde{T}$ and the chain rule, implies that $u \in C^{n+1,\gamma}$ piecewise in a small ball $B_{\hat{l}} \subset \tilde{T}^{-1}(B_l)$ around the origin. For the region outside $B_{\hat{l}}$, it follows from Case 1 in the proof of Theorem 2.4.10. Hence, the theorem is proved. $\square$

Chapter 3

L_p and Schauder estimates for higher-order parabolic systems

3.1 Introduction

This Chapter is devoted to the study of L_p and Schauder estimates for higher-order divergence type parabolic systems with various boundary conditions based on our papers [42, 43].

3.1.1 L_p estimates for higher-order parabolic systems

Many authors have studied the L_p estimates for parabolic equations and systems with discontinuous coefficients. It is of particular interest not only because of its various important applications in nonlinear equations and systems, but also due to its subtle link with the theory of stochastic processes. A good reference is [65].

In [43], we expand the L_p theory of higher-order parabolic systems to include a large class of discontinuous coefficients. For systems in the whole space and on a half space with the homogeneous Dirichlet boundary conditions, Dong and Kim [37] obtained the L_p estimates with leading coefficients having vanishing mean oscillations with respect to spatial variables (VMO_x) under the Legendre–Hadamard ellipticity condition (cf. (3.2.3)). Later in [38], they considered the conormal problem for higher-order elliptic systems with coefficients merely measurable in one direction and have small mean oscillation in orthogonal directions on each small ball, under the strong ellipticity condition (cf. (3.2.2)).

The L_p estimates of [43] can be viewed as a continuation of [37] and [38]. Compared to the strong ellipticity condition used in [38], the coefficients in the L_p estimates satisfy a Gårding type inequality (cf. (3.2.1)), which is weaker than the strong ellipticity condition. To present our results, we define the operator

$$\mathcal{L}u = \sum_{|\alpha|\le m, |\beta|\le m} D^\alpha (A^{\alpha\beta} D^\beta u),$$

where m is a positive integer, $A^{\alpha\beta}$ are $n \times n$ matrices, and $D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}$. The functions used throughout this chapter

$$u = (u^1, \ldots, u^n)^{\mathrm{tr}}, \quad f_\alpha = (f_\alpha^1, \ldots, f_\alpha^n)^{\mathrm{tr}}$$

are complex vector-valued functions. For the L_p estimates, the parabolic systems which we consider are of the form

$$u_t + (-1)^m \mathcal{L}u + \lambda u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in} \quad (-\infty, T) \times \Omega \tag{3.1.1}$$

with the conormal derivative boundary conditions on $(-\infty, T) \times \partial\Omega$ (cf. (3.2.5)), where $\lambda \geq 0$, $T \in (-\infty, \infty]$, and $\Omega \subset \mathbb{R}^d$ is either a half space or a $C^{m-1,1}$ domain.

We prove that if the leading coefficients $A^{\alpha\beta}(t,x)$ with $|\alpha| = |\beta| = m$ are VMO_x and satisfy a Gårding type inequality (3.2.1), then the solution u of (3.1.1) with the conormal derivative boundary conditions satisfies the estimate

$$\sum_{k=0}^{m} \lambda^{1-\frac{k}{2m}} \|D^k u\|_{L_p((-\infty,T)\times\Omega)} \leq N \sum_{|\alpha|\leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p((-\infty,T)\times\Omega)},$$

where N is a constant independent of u and λ is sufficiently large. To our best knowledge, this result is new in the higher-order case. We note that in the second-order case such result was proved in [36] by using the result in the whole space and the technique of odd/even extensions. However, such technique does not work for higher-order equations or systems.

For the proof, in many references including, for example, [1], the L_p estimates for second-order and higher-order systems with constant or continuous coefficients are obtained by relying on the exact representation of solutions and the Calderón–Zygmund theorem. Another approach for such L_p estimates is that of Campanato–Stampachia using Stampachia's interpolation theorem (see [48]). Our proof is in the spirit of an approach introduced by Krylov [66, 67] to deal with the second-order elliptic and parabolic equations with VMO_x coefficients in the whole space, which is well explained in his book [68]. Generally speaking, this approach consists of two steps. The first step is to establish mean oscillation estimates for systems with simple coefficients, i.e., the coefficients which depend only on t. The second step is to use a perturbation argument, which is well suited to the mean oscillation estimates, together with the Fefferman–Stein theorem on sharp functions and the Hardy–Littlewood theorem on maximal functions, in order to obtain the desired L_p estimates.

In our case, we estimate $D_d^m u$ and $D_{x'} D^{m-1} u$ separately. Here is an outline of our proof. We begin by considering special systems with simple coefficients as follows. Let u be a solution of

$$u_t + (-1)^m \tilde{\mathcal{L}}_0 u = 0 \quad \text{in} \quad \{x_d > 0\}$$

with the conormal derivative boundary conditions on $\{x_d = 0\}$, where

$$\tilde{\mathcal{L}}_0 u = \sum_{j=1}^{d-1} D_j^{2m} u + D^{\hat{\alpha}}(A^{\hat{\alpha}\hat{\alpha}}(t) D^{\hat{\alpha}} u)$$

and $\hat{\alpha} = (0, \ldots, 0, m)$. A key observation here is that the conormal derivative boundary conditions corresponding to the special systems are given by

$$D_d^m u = \cdots = D_d^{2m-1} u = 0 \quad \text{on} \quad \{x_d = 0\},$$

which allows us to use some estimates for non-divergence type systems with the Dirichlet boundary conditions obtained in [37] to get

$$\fint_{Q_r^+(X_0)} |D_d^m u - (D_d^m u)_{Q_r^+(X_0)}| \, dx \, dt \le C\kappa^{-1} \Big(\fint_{Q_{\kappa r}^+(X_0)} |D_d^m u|^2 \, dx \, dt \Big)^{1/2},$$

where $Q_r(X_0)$ is a parabolic cylinder with radius r and center $X_0 \in \{x_d = 0\}$,

$$Q_r^+(X_0) = Q_r(X_0) \cap \{x_d > 0\},$$

$(D_d^m u)_{Q_r^+(X_0)}$ is the average of $D_d^m u$ in $Q_r^+(X_0)$, $\kappa \ge 32$, and N is a constant independent of r, X_0, and u. The mean oscillation estimate above, together with the Fefferman–Stein theorem and the Hardy–Littlewood theorem, yields the L_p estimate of $D_d^m u$ on a half space for the special systems. For general systems on a half space with simple coefficients, which is

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha| \le m} D^\alpha f_\alpha, \tag{3.1.2}$$

with the conormal derivative boundary conditions on $\{x_d = 0\}$, where

$$\mathcal{L}_0 u = \sum_{|\alpha|=|\beta|=m} D^\alpha (A^{\alpha\beta}(t) D^\beta u),$$

$A^{\alpha\beta}(t)$ are functions of t, and $\lambda \ge 0$, we implement a scaling argument and the L_p estimates of $D_d^m u$ for the special systems to control the L_p norm of $D_d^m u$ by the L_p norms of f_α and $D_{x'} D^{m-1} u$. Then by the Sobolev embedding theorem and a bootstrap argument, we are able to obtain a local Hölder estimate of $D^{m-1} u$ for the homogeneous system

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = 0 \quad \text{in} \quad Q_2^+, \tag{3.1.3}$$

where $Q_2^+ = Q_2^+(0)$. Note that we can differentiate (3.1.3) with respect to x' without affecting the system and boundary conditions, which enables us to estimate the mean oscillation of $D_{x'} D^{m-1} u$ for (3.1.2). Finally, for coefficients depending on both x and t, we use the argument of freezing the coefficients to obtain the L_p estimate of $D_{x'} D^{m-1} u$, and apply the same method for the operator with simple coefficients to bound the L_p norm of $D_d^m u$ by those of f_α and $D_{x'} D^{m-1} u$.

Let us mention some other related results in the literature. In [23, 24], Chiarenza, Frasca, and Longo initiated the study of the W_p^2 estimates for second-order elliptic equations with VMO leading coefficients. Their proof is based on certain estimates of the Calderón–Zygmund theorem and the Coifman–Rochberg–Weiss commutator theorem. We refer the reader to Bramanti and Cerutti [12], Bramanti, Cerutti, and

Manfredini [11], Di Fazio [27], Maugeri, Palagachev, and Softova [86], Palagachev and Softova [92], Krylov [68], and the references therein.

3.1.2 Schauder estimates for higher-order parabolic systems

The second objective of this chapter is to obtain the Schauder estimates for solutions to the systems: 1. divergence type parabolic systems

$$u_t + (-1)^m \mathcal{L}u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in} \quad (0,T)\times\Omega \tag{3.1.4}$$

with the conormal derivative bondary conditions on $(0,T)\times\partial\Omega$ and the zero initial condition on $\{0\}\times\Omega$; 2. the divergence type and non-divergence type parabolic systems

$$u_t + (-1)^m \mathcal{L}u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in} \quad (0,T)\times\Omega,$$

$$u_t + (-1)^m Lu = f \quad \text{in} \quad (0,T)\times\Omega,$$

with the Dirichlet boundary conditions on $(0,T)\times\partial\Omega$ and zero initial condition on $\{0\}\times\Omega$, where

$$Lu = \sum_{|\alpha|\le m,\, |\beta|\le m} A^{\alpha\beta} D^\alpha D^\beta u.$$

It is well known that the Schauder estimates play an important role in the existence and regularity theory for elliptic and parabolic equations and systems. The classical approach is to first study the fundamental solutions for equations with constant coefficients, then use the argument of freezing the coefficients for general equations with regular coefficients; see, for instance, [1, 47, 63, 100]. For systems it has become customary to use Campanato's technique first introduced in [18], the application of which to second-order elliptic systems was explained comprehensively in Giaquinta [48]; see also [96, 34] for second-order parabolic systems. For higher-order systems, we also refer the reader to [85, Chap. 3] and the references therein.

The classical Schauder estimates were established under the assumption that the coefficients are regular in both space and time. In this chapter, we consider the coefficients which are regular only with respect to spatial variables. This type of coefficient has been studied by several authors mostly for second-order equations; see, for instance, [13, 56, 81, 84, 69]. In [81, 82] Lieberman studied interior and boundary Schauder estimates for second-order parabolic equations with time irregular coefficients using the maximum principle and a Campanato type approach. More recently, Boccia [9] considered higher-order non-divergence type parabolic systems in the whole space.

3.1.2.1 Dirichlet boundary conditions

For the homogeneous Dirichlet boundary condition, i.e., $|u| = |Du| = \ldots = |D^{m-1}u| = 0$ on $(0,T) \times \partial\Omega$, we assume the leading coefficients satisfy the so-called Legendre–Hadamard ellipticity condition (see (3.2.3)), which is more general than the usual strong ellipticity condition.

For the non-divergence form systems, we prove that $D^{2m}u$ is Hölder continuous with respect to both variables in the whole space, and in the half space all the $2m$th order derivatives of u are Hölder continuous with respect to both variables up to the boundary with the exception of $D_d^{2m}u$, where x_d is the normal direction of the boundary. For the divergence form systems, we prove that all the mth order derivatives are Hölder continuous with respect to both variables up to the boundary. We also prove an existence theorem for the divergence form systems in a cylindrical domain, provided that the boundary of the domain is sufficiently smooth. To our best knowledge, these results are new for higher-order systems and they extend the corresponding results found in Lieberman [81] for second-order scalar equations. In the special case of second-order parabolic systems, compared to [96] our conditions on the coefficients and data are more general. In particular, we do not require the data to vanish on the lateral boundary of the domain, i.e., the compatibility condition imposed in [96].

For the proof, we use some results in [37], in which the authors proved L_p estimates for divergence and non-divergence type higher-order parabolic systems with time-irregular coefficients. Let us give the outline of the proofs. In the divergence case, the classical L_2 estimates and an interior estimate obtained in [37] imply

$$\begin{aligned}&\int_{Q_r(X_0)} |D^m u - (D^m u)_{Q_r(X_0)}|^2 dx\,dt \\ &\le C(\frac{r}{R})^{2+d+2m} \int_{Q_R(X_0)} |D^m u - (D^m u)_{Q_R(X_0)}|^2 dx\,dt, \quad \forall\, r \le R,\end{aligned}$$

where $Q_r(X_0)$ (or $Q_R(X_0)$) is a parabolic cylinder with radius r (or R, respectively) and center X_0, $(D^m u)_{Q_r(X_0)}$ (or $(D^m u)_{Q_R(X_0)}$, respectively) is the average of $D^m u$ in $Q_r(X_0)$ (or $Q_R(X_0)$, respectively), C is a constant independent of r, R, X_0, and u, and u is a solution of

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \quad \text{in} \quad Q_{2R}.$$

Here

$$\mathcal{L}_0 = \sum_{|\alpha|=|\beta|=m} D^\alpha (A^{\alpha\beta}(t) D^\beta)$$

and $X_0 \in Q_R$. The coefficients $A^{\alpha\beta}$, which depend only on t, are called simple coefficients. For the boundary estimates, we combine the L_p estimates established in [37] and the Sobolev embedding theorem to prove the Hölder continuity and obtain, for instance, the following mean oscillation type estimate for systems with simple coefficients,

$$\begin{aligned}&\int_{Q_r^+(X_0)} |D_d^m u - (D_d^m u)_{Q_r^+(X_0)}|^2 \, dx \, dt \\ &\leq C(\frac{r}{R})^{2\gamma+d+2m} \int_{Q_R^+(X_0)} |D_d^m u - (D_d^m u)_{Q_R^+(X_0)}|^2 \, dx \, dt, \quad \forall \, r \leq R,\end{aligned}$$

where $X_0 \in \{x_d = 0\} \cap Q_R$, and $\gamma \in (0,1)$ are arbitrary, $Q_{r,R}^+(X_0) = Q_{r,R}(X_0) \cap \{x_d > 0\}$, C is a constant independent of r, R, X_0, and u, and u satisfies

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \quad \text{in} \quad Q_{2R}^+$$

with the Dirichlet boundary condition $|u| = |D_d u| = \ldots = |D_d^{m-1} u| = 0$ on $\{x_d = 0\}$. We then use a perturbation argument to treat the general systems.

Similar interior and boundary estimates for the non-divergence form systems can be established in the same fashion. Here the idea is that we can differentiate the system

$$\begin{cases} u_t + (-1)^m L_0 u = 0 \quad \text{in} \quad Q_{2R}^+ \\ u = 0, \; D_d u = 0, \ldots, D_d^{m-1} u = 0 \quad \text{on} \quad \{x_d = 0\} \cap Q_{2R} \end{cases}$$

with respect to tangential direction x'. This together with the $W_p^{1,2m}$ estimates implies that $D_{x'} D^{2m-1} u$ is in some Hölder space $C^{\frac{a}{2m}, a}(Q_R^+)$ with a arbitrarily close to 1 from below, which yields the mean oscillation type estimates.

3.1.2.2 Conormal boundary conditions

For the Schauder estimates under conormal boundary conditions, we assume the same ellipticity condition on the leading coefficients as in the L_p estimate for the conormal boundary condition, i.e., the Gårding type inequality (3.2.1). For systems on a half space, we estimate the Hölder norms of all the mth order spatial derivatives with the exception of $D_d^m u$, when the coefficients are only measurable in the t variable. Here we implement the L_p estimates with a bootstrap argument to obtain a local Hölder regularity for systems with simple coefficients, which yields the following mean oscillation estimate:

$$\begin{aligned}&\int_{Q_r^+} |D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q_r^+}|^2\,dx\,dt\\ &\leq C(\frac{r}{R})^{2\gamma+2m+d}\int_{Q_R^+} |D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q_R^+}|^2\,dx\,dt,\end{aligned}$$

where $\gamma \in (0,1)$, $0 < r < R < \infty$, and u is a solution of

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \quad \text{in } Q_R^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_R$. We then prove that if the coefficients $A^{\alpha\beta}$ and f_α in (3.1.4) are Hölder continuous in the spatial variables x, then $D_{x'}D^{m-1}u$ is Hölder continuous in both t and x.

In contrast to the Dirichlet boundary condition case in the previous subsection, more regularity assumptions on the coefficients and data are necessary. In fact, this is not surprising by considering the second-order equation

$$u_t - D_{11}u - D_2(a(t)D_2u) = D_2f(t) \quad \text{in} \quad (0,T)\times\mathbb{R}_+^2$$

with the conormal derivative boundary condition. It is easily seen that the corresponding boundary condition is given by

$$D_2 u = -f(t)/a(t) \quad \text{on} \quad \{x_2 = 0\},$$

which implies that $D_2 u$ is not necessarily continuous on $\{x_2 = 0\}$.

For linear systems, our Schauder estimate extends the results in Lieberman [79], in which the author considered second-order quasilinear parabolic equations with the conormal derivative boundary condition. For more results about the conormal derivative problems, we refer the reader to Lieberman [78, 80] and his new book [83].

The chapter is organized as follows. In the next section, we introduce some notation and state our main results. In Section 3, we present some technical lemmas. In Section 4, we treat the L_p estimates. In Section 5, we deal with the Schauder estimates under the Dirichlet boundary condition. In Section 6, we deal with the Schauder estimates under the conormal derivative boundary condition.

3.2 Main results

We first introduce some notation used throughout the chapter. A point in $\mathbb{R}^d$ is denoted by $x = (x_1, \ldots, x_d)$, also by $x = (x', x_d)$, where $x' \in \mathbb{R}^{d-1}$. A point in

$$\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$$

is denoted by $X=(t,x)$. For $T\in(-\infty,\infty]$, set

$$\mathcal{O}_T=(-\infty,T)\times\mathbb{R}^d,\quad \mathcal{O}_T^+=(-\infty,T)\times\mathbb{R}^d_+,$$

where $\mathbb{R}^d_+:=\{x=(x_1,\ldots,x_d)\in\mathbb{R}^d : x_d>0\}$. In particular, when $T=\infty$, we use $\mathcal{O}_\infty^+:=\mathbb{R}\times\mathbb{R}^d_+$. Denote

$$B_r(x_0)=\{y\in\mathbb{R}^d : |x_0-y|<r\},\quad Q_r(t_0,x_0)=(t_0-r^{2m},t_0)\times B_r(x_0),$$
$$B_r^+(x_0)=B_r(x_0)\cap\{x_d>0\},\quad Q_r^+(t_0,x_0)=Q_r(t_0,x_0)\cap\mathcal{O}_\infty^+.$$

We use the abbreviation, for instance, Q_r to denote the parabolic cylinder centered at $(0,0)$. We denote

$$\langle f,g\rangle_\Omega=\int_\Omega \overline{f}^{\mathrm{tr}}g=\sum_{j=1}^n\int_\Omega \overline{f}^j g^j.$$

The parabolic boundary of $Q_r(t_0,x_0)$ is defined to be

$$\partial_p Q_r(t_0,x_0)=[t_0,t_0-r^{2m})\times\partial B_r(x_0)\cup\{t=t_0-r^{2m}\}\times B_r(x_0).$$

The parabolic boundary of $Q_r^+(t_0,x_0)$ is

$$\partial_p Q_r^+(t_0,x_0)=(\partial_p Q_r(t_0,x_0)\cap\overline{\mathcal{O}_\infty^+})\cup(Q_r(t_0,x_0)\cap\{x_d=0\}).$$

For a function f on $\mathcal{D}\subset\mathbb{R}^{d+1}$, we set

$$[f]_{a,b,\mathcal{D}}:=\sup\left\{\frac{|f(t,x)-f(s,y)|}{|t-s|^a+|x-y|^b} : (t,x),(s,y)\in\mathcal{D},(t,x)\neq(s,y)\right\},$$

where $a,b\in(0,1]$. For $0<a\le 1$, let

$$\|f\|_{\frac{a}{2m},a,\mathcal{D}}=\|f\|_{L_\infty(\mathcal{D})}+[f]_{\frac{a}{2m},a,\mathcal{D}}.$$

The space corresponding to $\|\cdot\|_{\frac{a}{2m},a,\mathcal{D}}$ is denoted by $C^{\frac{a}{2m},a}(\mathcal{D})$. For $1<a<2m$ not an integer, we define

$$\|f\|_{\frac{a}{2m},a,\mathcal{D}} = \|f\|_{L_\infty(\mathcal{D})} + \sum_{|\alpha|<a} [D^\alpha f]_{\frac{a-|\alpha|}{2m},\{a\},\mathcal{D}},$$

where $\{a\} = a - [a]$.

The Hölder semi-norm with respect to t is denoted by

$$[f]^t_{a,\mathcal{D}} := \sup\left\{\frac{|f(t,x)-f(s,x)|}{|t-s|^a} : (t,x),(s,x)\in\mathcal{D}, t\neq s\right\},$$

where $a\in(0,1]$. Define $\|f\|^t_{a,\mathcal{D}} := \|f\|_{L_\infty(\mathcal{D})} + [f]^t_{a,\mathcal{D}}$ and denote the space corresponding to $\|\cdot\|^t_{a,\mathcal{D}}$ by $C^a_t(\mathcal{D})$.

We also define the Hölder semi-norm with respect to x

$$[f]^x_{a,\mathcal{D}} := \sup\left\{\frac{|f(t,x)-f(t,y)|}{|x-y|^a} : (t,x),(t,y)\in\mathcal{D}, x\neq y\right\},$$

and denote

$$\|f\|^x_{a,\mathcal{D}} = \|f\|_{L_\infty(\mathcal{D})} + [f]^x_{a,\mathcal{D}},$$

where $a\in(0,1]$. For a nonnegative integer m and $a\in(0,1]$, we define

$$\|f\|^x_{m+a,\mathcal{D}} = \|f\|_{L_\infty(\mathcal{D})} + [D^m f]^x_{a,\mathcal{D}}.$$

In particular,

$$\|f\|^x_{m,\mathcal{D}} = \|f\|_{L_\infty(\mathcal{D})} + \|D^m f\|_{L_\infty(\mathcal{D})}.$$

The space corresponding to $\|\cdot\|^x_{a,D}$ (or $\|\cdot\|^x_{m+a,D}$) is denoted by $C^a_x(\mathcal{D})$ (or $C^{(m+a)x}(\mathcal{D})$, respectively).

We denote the average of f in $\mathcal{D}$ to be

$$(f)_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} f(t,x)\,dx\,dt = \fint_{\mathcal{D}} f(t,x)\,dx\,dt.$$

Sometimes we take the average only with respect to x. For instance,

$$(f)_{B_R(x_0)}(t) = \fint_{B_R(x_0)} f(t,x)\,dx.$$

Throughout this chapter, we assume that all the coefficients are measurable and bounded:

$$|A^{\alpha\beta}| \le K.$$

In addition, for the case of conormal boundary condition, we introduce a Gårding type inequality: for fixed $X \in \mathbb{R}^{d+1}$ and any $u \in W_2^m(\mathbb{R}^d_+)$,

$$\operatorname{Re} \int_{\mathbb{R}^d_+} \sum_{|\alpha|=|\beta|=m} D^\alpha \bar{u}^{\mathrm{tr}}(y) A^{\alpha\beta}(X) D^\beta u(y)\,dy \ge \delta \int_{\mathbb{R}^d_+} \sum_{j=1}^{d} |D_j^m u(y)|^2\,dy, \tag{3.2.1}$$

where $\delta > 0$ is a constant and we use $\operatorname{Re} f$ to denote the real part of f. Clearly, this condition is weaker than the strong ellipticity condition used, for instance, in [38]: for any $\xi = (\xi_\alpha)_{|\alpha|=m}, \xi_\alpha \in \mathbb{C}^n$,

$$\sum_{|\alpha|=|\beta|=m} \operatorname{Re}\left(\bar{\xi}_\alpha^{\mathrm{tr}} A^{\alpha\beta} \xi_\beta\right) \ge \delta \sum_{|\alpha|=m} |\xi_\alpha|^2. \tag{3.2.2}$$

We will show in Lemma 3.4.1 that (3.2.1) is stronger than the Legendre–Hadamard ellipticity condition used, for instance, in [37] and our Schauder estimates for the Dirichlet boundary conditions: for any $\eta \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^n$,

$$\sum_{|\alpha|=|\beta|=m} \operatorname{Re}\left(A_{ij}^{\alpha\beta} \xi_i \bar{\xi}_j\ \eta^\alpha \eta^\beta\right) \ge \delta |\xi|^2 |\eta|^{2m}, \tag{3.2.3}$$

where $\alpha=(\alpha_1,\ldots,\alpha_d)$ and $\eta^\alpha=\eta_1^{\alpha_1}\eta_2^{\alpha_2}\cdots\eta_d^{\alpha_d}$. Here we call $A^{\alpha\beta}$ the leading coefficients if $|\alpha|=|\beta|=m$. All the other coefficients are called lower-order coefficients. When $A^{\alpha\beta}$ only depend on t, we assume that (3.2.1) holds with $A^{\alpha\beta}=A^{\alpha\beta}(t)$ for any $t\in\mathbb{R}$.

In order to state and prove our results in Sobolev spaces, in addition to the well-known L_p and W_p^k spaces, we introduce the following function spaces. If $\Omega=\mathbb{R}^d$, let

$$\mathcal{H}_p^m((S,T)\times\mathbb{R}^d)=(1-\Delta)^{\frac{m}{2}}W_p^{1,2m}((S,T)\times\mathbb{R}^d)$$

equipped with the norm

$$\|u\|_{\mathcal{H}_p^m((S,T)\times\mathbb{R}^d)}=\|(1-\Delta)^{-\frac{m}{2}}u\|_{W_p^{1,2m}((S,T)\times\mathbb{R}^d)}.$$

Here

$$W_p^{1,2m}((S,T)\times\mathbb{R}^d)=\{u:u_t,D^\alpha u\in L_p((S,T)\times\mathbb{R}^d),0\le|\alpha|\le 2m\}$$

equipped with its natural norm. Notice that if we set

$$\mathbb{H}_p^{-m}((S,T)\times\mathbb{R}^d)=(1-\Delta)^{\frac{m}{2}}L_p((S,T)\times\mathbb{R}^d),$$
$$\|f\|_{\mathbb{H}_p^{-m}((S,T)\times\mathbb{R}^d)}=\|(1-\Delta)^{-\frac{m}{2}}f\|_{L_p((S,T)\times\mathbb{R}^d)},$$

then

$$\|u\|_{\mathcal{H}_p^m((S,T)\times\mathbb{R}^d)}\cong\|u_t\|_{\mathbb{H}_p^{-m}((S,T)\times\mathbb{R}^d)}+\sum_{|\alpha|\le m}\|D^\alpha u\|_{L_p((S,T)\times\mathbb{R}^d)}.$$

For a general domain $\Omega \subset \mathbb{R}^d$, we set

$$\mathbb{H}_p^{-m}((S,T)\times\Omega) = \Big\{ f : f = \sum_{|\alpha|\le m} D^\alpha f_\alpha,\ f_\alpha \in L_p((S,T)\times\Omega) \Big\},$$
$$\|f\|_{\mathbb{H}_p^{-m}((S,T)\times\Omega)} = \inf\Big\{ \sum_{|\alpha|\le m} \|f_\alpha\|_{L_p((S,T)\times\Omega)} : f = \sum_{|\alpha|\le m} D^\alpha f_\alpha \Big\},$$

and

$$\begin{aligned}&\mathcal{H}_p^m((S,T)\times\Omega)\\ &= \{u : u_t \in \mathbb{H}_p^{-m}((S,T)\times\Omega), D^\alpha u \in L_p((S,T)\times\Omega), 0 \le |\alpha| \le m\},\\ &\|u\|_{\mathcal{H}_p^m((S,T)\times\Omega)} = \|u_t\|_{\mathbb{H}_p^{-m}((S,T)\times\Omega)} + \sum_{|\alpha|\le m} \|D^\alpha u\|_{L_p((S,T)\times\Omega)}.\end{aligned}$$

Let $T \in (-\infty, \infty]$ and $\mathcal{Q} = \{Q_r(t,x) \cap \mathcal{O}_T^+ : (t,x) \in \overline{\mathcal{O}_T^+}, r \in (0,\infty)\}$. For a function g defined on $\mathcal{O}_T^+$, we denote its (parabolic) maximal and sharp functions, respectively, by

$$Mg(t,x) = \sup_{Q\in\mathcal{Q}:(t,x)\in Q} \fint_Q |g(s,y)|\,dy\,ds,$$
$$g^{\#}(t,x) = \sup_{Q\in\mathcal{Q}:(t,x)\in Q} \fint_Q |g(s,y) - (g)_Q|\,dy\,ds.$$

Then

$$\|g\|_{L_p(\mathcal{O}_T^+)} \le N\|g^{\#}\|_{L_p(\mathcal{O}_T^+)}, \quad \|Mg\|_{L_p(\mathcal{O}_T^+)} \le N\|g\|_{L_p(\mathcal{O}_T^+)},$$

if $g \in L_p$, where $1 < p < \infty$ and $N = N(d,p)$. As is well known, the first inequality above is due to the Fefferman–Stein theorem on sharp functions and the second one is the Hardy–Littlewood maximal function theorem.

Now we state our regularity assumption on the leading coefficients for the L_p

estimates. Let

$$\operatorname{osc}_x(A^{\alpha\beta}, Q_r^+(t,x)) = \fint_{t-r^{2m}}^{t} \fint_{B_r^+(x)} |A^{\alpha\beta}(s,y) - \fint_{B_r^+(x)} A^{\alpha\beta}(s,z)\,dz|\,dy\,ds,$$

which is the mean oscillation in the spatial variables. Then we set

$$A_R^{\#} = \sup_{(t,x)\in\mathcal{O}_\infty^+} \sup_{r\le R} \sup_{|\alpha|=|\beta|=m} \operatorname{osc}_x(A^{\alpha\beta}, Q_r^+(t,x)).$$

We impose on the leading coefficients a small mean oscillation condition with a parameter $\rho > 0$, which is specified later.

Assumption 3.2.1. (ρ). There is a constant $R_0 \in (0,1]$ such that $A_{R_0}^{\#} \le \rho$.

We are now ready to present our main results.

L_p **estimates with conormal boundary conditions.** For any $-\infty \le S < T \le \infty$ and $\Omega \subset \mathbb{R}^d$, we say that $u \in \mathcal{H}_p^m((S,T)\times\Omega)$ is a solution to

$$u_t + (-1)^m \mathcal{L}u + \lambda u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in} \quad (S,T)\times\Omega \tag{3.2.4}$$

with the conormal derivative boundary conditions on $(S,T)\times\partial\Omega$ if the equality

$$\begin{aligned} &-\langle u, \phi_t\rangle_{(S,T)\times\Omega} + \sum_{|\alpha|\le m, |\beta|\le m} (-1)^{m+|\alpha|} \langle A^{\alpha\beta} D^\beta u, D^\alpha \phi\rangle_{(S,T)\times\Omega} \\ &+ \lambda\langle u, \phi\rangle_{(S,T)\times\Omega} = \sum_{|\alpha|\le m} (-1)^{|\alpha|} \langle f_\alpha, D^\alpha\phi\rangle_{(S,T)\times\Omega} \end{aligned} \tag{3.2.5}$$

holds for any $\phi \in C_0^\infty((S,T)\times\overline{\Omega})$.

Theorem 3.2.2. *Let $p \in (1,\infty)$ be a constant, $\partial\Omega \in C^{m-1,1}$, $T \in (-\infty,\infty]$, and $f_\alpha \in L_p((-\infty,T)\times\Omega)$ for $|\alpha| \le m$. The operator L satisfies* (3.2.1) *for any* $X \in (-\infty, T] \times \overline{\Omega}$. *Then there exist constants* $\rho = \rho(d, n, m, \delta, p, K)$, *and* $\lambda_0 = \lambda_0(d, n, m, \delta, p, K, R_0)$ *such that, under Assumption 3.2.1* (ρ), *for any* $u \in \mathcal{H}_p^m((-\infty, T) \times \Omega)$ *satisfying* (3.2.4) *with the conormal derivative boundary conditions on* $(-\infty, T) \times \partial\Omega$, *we have*

$$\sum_{k=0}^{m} \lambda^{1-\frac{k}{2m}} \|D^k u\|_{L_p((-\infty,T)\times\Omega)} \le C \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_p((-\infty,T)\times\Omega)} \tag{3.2.6}$$

provided that $\lambda \ge \lambda_0$, *where* C *depends only on* d, n, m, δ, p, K, *and* Ω.

Moreover, if $\lambda > \lambda_0$, *there exists a unique solution* $u \in \mathcal{H}_p^m((-\infty, T) \times \Omega)$ *of* (3.2.4) *with the conormal derivative boundary conditions on* $(-\infty, T) \times \partial\Omega$.

Schauder estimates under the Dirichlet boundary conditions The following result is about the solvability of divergence type higher-order parabolic systems with the Dirichlet boundary condition.

Theorem 3.2.3. *Let* $a \in (0, 1)$ *and* $T \in (0, \infty]$. *Assume* $f_\alpha \in C_x^a$ *for* $|\alpha| = m$ *and* $f_\alpha \in L_\infty$ *for* $|\alpha| < m$. *Suppose that the operator* $\mathcal{L}$ *satisfies the Legendre–Hadamard condition* (3.2.3), *and* $A^{\alpha\beta} \in C_x^a$. *Let* g *be a smooth function in* $\mathbb{R}^{d+1}$ *and* $\Omega \in C^{m,a}$. *Then*

$$\begin{cases} u_t + (-1)^m \mathcal{L} u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in} \quad (0,T)\times\Omega, \\ u = g,\ Du = Dg,\ \ldots,\ D^{m-1}u = D^{m-1}g \quad \text{on} \quad [0,T)\times\partial\Omega, \\ u = g \quad \text{on} \quad \{0\}\times\Omega \end{cases}$$

has a unique solution u *such that* $u \in C^{\frac{a+m}{2m}, a+m}([0, \min(T,k)) \times \bar{\Omega})$ *for any* $k > 0$, *and it satisfies*

$$\|u\|_{\frac{a+m}{2m}, a+m, (0,T)\times\Omega} \le C\big(\|u\|_{L_2((0,T)\times\Omega)} + F + G\big), \tag{3.2.7}$$

where

$$F = \sum_{|\alpha|=m} [f_\alpha]^x_{a,(0,T)\times\Omega} + \sum_{|\alpha|<m} \|f_\alpha\|_{L_\infty((0,T)\times\Omega)},$$
$$G = \sum_{|\alpha|=m} \|D^\alpha g\|^x_{a,(0,T)\times\Omega} + \sum_{|\alpha|<m} \|D^\alpha g\|_{L_\infty((0,T)\times\Omega)} + \|g_t\|_{L_\infty((0,T)\times\Omega)},$$

and $C > 0$ *is a constant depending only on* d, n, m, λ, K, $\|A^{\alpha\beta}\|^x_a$, Ω, *and* a. *Moreover, for any constant* $k > 0$, *we have*

$$\|u\|_{\frac{a+m}{2m},a+m,(0,\min(T,k))\times\Omega} \le Ce^{C_1 k}\big(F_k + G_k\big), \tag{3.2.8}$$

where

$$F_k = \sum_{|\alpha|=m} [f_\alpha]^x_{a,(0,\min(T,k))\times\Omega} + \sum_{|\alpha|<m} \|f_\alpha\|_{L_\infty((0,\min(T,k))\times\Omega)},$$
$$G_k = \sum_{|\alpha|=m} \|D^\alpha g\|^x_{a,(0,\min(T,k))\times\Omega} + \sum_{|\alpha|<m} \|D^\alpha g\|_{L_\infty((0,\min(T,k))\times\Omega)}$$
$$+ \|g_t\|_{L_\infty((0,\min(T,k))\times\Omega)},$$

$C > 0$ *is a constant depending only on* d, n, m, λ, K, $\|A^{\alpha\beta}\|^x_a$, Ω, *and* a, *and* $C_1 > 0$ *is a constant depending only on* d, n, m, λ, *and* K.

The next theorem is regarding the a priori interior and boundary Schauder estimates for the non-divergence type systems under *the Dirichlet boundary conditions*.

Theorem 3.2.4. *Let* $a \in (0,1)$. *Suppose that* $A^{\alpha\beta} \in C^a_x$ *and* $f \in C^a_x$. *Let* $u \in C^\infty_{loc}(\mathbb{R}^{d+1})$ *be a solution to*

$$u_t + (-1)^m Lu = f, \tag{3.2.9}$$

where L *satisfies* (3.2.3). *For any* $R \le 1$, *I. Interior case: if* (3.2.9) *holds in Q_{4R}, then there exists a constant C depending only on d, n, m, λ, K, $\|A^{\alpha\beta}\|_a^x$, R, and a such that*

$$\|u_t\|_{a,Q_R}^x + \|D^{2m}u\|_{\frac{a}{2m},a,Q_R} \le C\big(\|f\|_{a,Q_{4R}}^x + \|u\|_{L_2(Q_{4R})}\big);$$

II. Boundary case: if (3.2.9) *holds in Q_{4R}^+ with the homogeneous Dirichlet boundary condition on $\{x_d = 0\} \cap Q_{4R}$, then there exists a constant C depending only on d, n, m, λ, K, $\|A^{\alpha\beta}\|_a^x$, R, and a such that*

$$[D_{x'}D^{2m-1}u]_{\frac{a}{2m},a,Q_R^+} \le C[f]_{a,Q_{4R}^+}^x + C\sum_{|\gamma|\le 2m}\|D^\gamma u\|_{L_\infty(Q_{4R}^+)}.$$

We note that the boundary estimate in Theorem 3.2.4 is optimal even for the heat equation, in the sense that near the boundary $D_d^2 u$ might be discontinuous with respect to x if f has jump discontinuities in t. See, for instance, [101, §3] and [81, §16]. In order to estimate the Hölder semi-norm of $D_d^{2m}u$, we need to impose more regularity conditions on $A^{\alpha\beta}$ and f. See Remark 3.5.2.2 below for a discussion.

Schauder estimates under conormal derivative boundary conditions.

We say $u \in \mathcal{H}_2^m((0,T)\times\Omega)$ is a weak solution to

$$u_t + (-1)^m \mathcal{L}u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in} \quad (0,T)\times\Omega$$

with the conormal derivative boundary conditions on $(0,T)\times\partial\Omega$ and the zero initial condition on $\{0\}\times\Omega$ if (3.2.5) with $S = \lambda = 0$ holds for any $\phi \in C^\infty([0,T)\times\overline{\Omega})$.

The following result is regarding the Schauder estimates near a flat boundary with coefficients and data only measurable in the t variable under *the conormal boundary condidtions.*

Theorem 3.2.5. *Assume that $a \in (0,1)$, $u \in C^m_{x,loc}(\overline{\mathcal{O}_0^+})$ satisfying*

$$u_t + (-1)^m \mathcal{L}u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad in \quad Q_{2R}^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2R}$, where $f_\alpha \in C^a_x(Q^+_{2R})$ if $|\alpha| = m$, $f_\alpha \in L_\infty(Q^+_{2R})$ if $|\alpha| < m$, and $A^{\alpha\beta} \in C^a_x(Q^+_{2R})$. Suppose that the operator $\mathcal{L}$ satisfies (3.2.1) for any $X \in \overline{Q^+_{2R}}$. Then for any $R \le 1$, there exists a constant N depending only on d, n, m, δ, K, a, $\|A^{\alpha\beta}\|^x_{a,Q^+_{2R}}$, and R such that

$$\begin{aligned}
&[D_{x'} D^{m-1} u]_{\frac{a}{2m}, a, Q_R^+} \\
&\le N \Big(\sum_{|\alpha| \le m} \|D^\alpha u\|_{L_\infty(Q^+_{2R})} + \sum_{|\alpha| = m} [f_\alpha]^x_{a,Q^+_{2R}} + \sum_{|\alpha| < m} \|f_\alpha\|_{L_\infty(Q^+_{2R})} \Big).
\end{aligned}$$

Our last result is regarding the Schauder estimates in cylindrical domains with more regularity assumptions on the coefficients and data under *the conormal boundary conditions.*

Theorem 3.2.6. *Assume that $a \in (0,1)$, $\partial\Omega \in C^{m,a}, T \in (0,\infty)$, $f_\alpha \in C^a_x([0,T) \times \Omega) \cap C_t^{\frac{a}{2m}}([0,T) \times \partial\Omega)$, $f_\alpha = 0$ on $\{0\} \times \partial\Omega$ if $|\alpha| = m$, and $f_\alpha \in L_\infty((0,T) \times \Omega)$ if $|\alpha| < m$. The operator $\mathcal{L}$ satisfies (3.2.1) for any $X \in [0,T] \times \overline{\Omega}$, and $A^{\alpha\beta} \in C^a_x([0,T) \times \Omega) \cap C_t^{\frac{a}{2m}}([0,T) \times \partial\Omega)$. Then the equation*

$$u_t + (-1)^m \mathcal{L}u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad in \quad (0,T) \times \Omega$$

with the conormal derivative boundary conditions on $(0,T) \times \partial\Omega$ and the zero initial condition on $\{0\} \times \Omega$ has a unique solution $u \in C^{\frac{a+m}{2m}, a+m}([0,T) \times \overline{\Omega})$. Moreover, there *exists a constant N depending only on d, n, m, δ, K, $\|A^{\alpha\beta}\|^x_{a,[0,T)\times\Omega}$, $[A^{\alpha\beta}]^t_{\frac{a}{2m},[0,T)\times\partial\Omega}$, Ω, and a such that,*

$$\|u\|_{\frac{a+m}{2m},a+m,(0,T)\times\Omega} \le N\big(\|u\|_{L_2((0,T)\times\Omega)} + G\big), \tag{3.2.10}$$

where

$$G := \sum_{|\alpha|=m}([f_\alpha]^x_{a,[0,T)\times\Omega} + [f_\alpha]^t_{\frac{a}{2m},[0,T)\times\partial\Omega}) + \sum_{|\alpha|\le m}\|f_\alpha\|_{L_\infty((0,T)\times\Omega)}.$$

It is worth noting that in view of the example given at the end of the introduction, without the compatibility condition that $f_\alpha = 0$ on $\{0\}\times\partial\Omega$ for $|\alpha| = m$, in general $D^m u$ is not Hölder continuous near $\{t=0\}\times\partial\Omega$. This is in contrast to the Dirichlet case, where such condition is not needed.

3.3 Technical preparation

We first present some technical lemmas which are useful in our proofs. The following one is well known and its proof can be found in [48].

Lemma 3.3.1. *Let Φ be a nonnegative, nondecreasing function on $(0, r_0]$ such that*

$$\Phi(\rho) \le A(\frac{\rho}{r})^a\Phi(r) + Br^b$$

for $0 < \rho < r \le r_0$, where $0 < b < a$ are fixed constants. Then

$$\Phi(r) \le Nr^b(r_0^{-b}\Phi(r_0) + B) \quad \forall\, r \in (0, r_0)$$

with a constant $N = N(A, a, b)$.

The following version of Campanato's theorem can be found in [48] and [96].

Lemma 3.3.2. *(i) Let $f \in L_2(Q_2)$ and $a \in (0, 1]$. Assume that*

$$\fint_{Q_r(t_0,x_0)} |f - (f)_{Q_r(t_0,x_0)}|^2 \, dx \, dt \le A^2 r^{2a}, \quad \forall \, (t_0, x_0) \in Q_1, \tag{3.3.1}$$

and any $0 < r \le 1$. Then we have $f \in C^{\frac{a}{2m}, a}(Q_1)$ and

$$[f]_{\frac{a}{2m}, a, Q_1} \le CA$$

with $C = C(d, m, a)$.

(ii) Let $f \in L_2(Q_2^+)$ and $a \in (0, 1]$. Assume that (3.3.1) holds for any $r \in (0, x_{0d})$. Moreover,

$$\fint_{Q_r^+(t_1,x_1)} |f - (f)_{Q_r^+(t_1,x_1)}|^2 \, dx \, dt \le A^2 r^{2a}, \quad \forall \, (t_1, x_1) \in \{x_d = 0\} \cap Q_1,$$

and any $0 < r \le 1$. Then we have $f \in C^{\frac{a}{2m}, a}(Q_1^+)$ and

$$[f]_{\frac{a}{2m}, a, Q_1^+} \le CA$$

with $C = C(d, m, a)$.

Here is another technical lemma.

Lemma 3.3.3. *Assume that Ω is an open bounded domain in $\mathbb{R}^d$. Then there exists a function $\xi \in C_0^\infty(\Omega)$ with a unit integral such that for any $0 < |\alpha| \le m$*

$$\int_\Omega \xi(y) y^\alpha dy = 0.$$

Next, we state a parabolic type Sobolev embedding theorem (cf. [7, pp. 187] and [8, pp. 72]).

Lemma 3.3.4. *Let $r \in (0, \infty)$ and $1 \le q \le p < \infty$. Assume that*

$$\frac{1}{q} - \frac{1}{p} \le \frac{1}{d+2m}.$$

Let $\zeta \in C_0^\infty$ be such that $\zeta = 1$ in Q_r^+. Then for any function u such that $u\zeta \in \mathcal{H}_q^m(\mathcal{O}_0^+)$, we have $u, D^{m-1}u \in L_p(Q_r^+)$ and

$$\|u\|_{L_p(Q_r^+)} + \|D^{m-1}u\|_{L_p(Q_r^+)} \le C\|u\zeta\|_{\mathcal{H}_q^m(\mathcal{O}_0^+)},$$

where $C = C(d, m, r, p, q)$. Moreover, for any $q > d + 2m$, we have

$$\|u\|_{\frac{\gamma}{2m},\gamma,Q_r^+} \le C\|u\|_{\mathcal{H}_q^m(Q_r^+)},$$

where $\gamma = 1 - (d + 2m)/q$.

3.4 L_p estimates for higher-order parabolic systems

3.4.1 Some auxiliary result under conormal boundary conditions

In this section, we consider operators without lower-order terms. Denote

$$\mathcal{L}_0 u = \sum_{|\alpha|=|\beta|=m} D^\alpha (A^{\alpha\beta} D^\beta u),$$

where $A^{\alpha\beta} = A^{\alpha\beta}(t)$. Let $C_0^\infty(\overline{\mathcal{O}_T^+})$ be the collection of infinitely differentiable functions defined on $\overline{\mathcal{O}_T^+}$ vanishing for large $|(t,x)|$. We prove (3.2.1) implies (3.2.3) in the following lemma. Recall that when the coefficients only depends on t, we assume that (3.2.1) holds with $A^{\alpha\beta} = A^{\alpha\beta}(t)$ for any $t \in \mathbb{R}$.

Lemma 3.4.1. *Under the condition (3.2.1), (3.2.3) holds with a possibly different $\delta > 0$.*

Proof. We refer the interested reader to [43]. □

We state the following L_2 estimate for parabolic operators in the divergence form with measurable coefficients satisfying (3.2.1).

Theorem 3.4.2. *There exists a constant $C = C(d, m, n, \delta)$ such that under the condition (3.2.1), for any $\lambda \geq 0$ and $T \in (-\infty, +\infty]$,*

$$\sum_{|\alpha|\leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha u\|_{L_2(\mathcal{O}_T^+)} \leq C \sum_{|\alpha|\leq m} \lambda^{\frac{|\alpha|}{2m}} \|f_\alpha\|_{L_2(\mathcal{O}_T^+)} \tag{3.4.1}$$

provided that $u \in \mathcal{H}_2^m(\mathcal{O}_T^+)$ *satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \tag{3.4.2}$$

in $\mathcal{O}_T^+$ *with the conormal derivative boundary conditions on* $\{x_d = 0\}$*, and* $f_\alpha \in L_2(\mathcal{O}_T^+)$*,* $|\alpha| \le m$*. Furthermore, for any* $\lambda > 0$ *and* $f_\alpha \in L_2(\mathcal{O}_T^+)$*, there exists a unique solution* $u \in \mathcal{H}_2^m(\mathcal{O}_T^+)$ *to the system* (3.4.2) *with the conormal derivative boundary conditions on* $\{x_d = 0\}$*.*

By Theorem 3.4.2 and adapting the proofs of [35, Lemmas 3.2 and 7.2] to the conormal case, we have the following local L_2 estimates.

Lemma 3.4.3. *Let* $0 < r < R < \infty$*. Assume that* $u \in C_{loc}^\infty(\overline{\mathcal{O}_0^+})$ *satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \quad in \quad Q_R^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_R$*. Then under the condition* (3.2.1)*, there exists a constant* $C = C(d, m, n, \delta, K)$ *such that for* $j = 1, \ldots, m$*,*

$$\|D^j u\|_{L_2(Q_r^+)} \le C(R-r)^{-j} \|u\|_{L_2(Q_R^+)}.$$

The technical lemma below is useful in our proof. Note that we do not require any ellipticity condition for this lemma.

Lemma 3.4.4. *Let* $R \in (0, \infty)$ *and* $f_\alpha \in L_\infty(Q_{2R}^+)$*. Assume that* $u \in C_{loc}^\infty(\overline{\mathcal{O}_0^+})$ *satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad in \quad Q_{2R}^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2R}$*. Let* $P(x)$ *be a* *vector-valued polynomial of order* $m-1$ *and satisfy*

$$(D^\alpha P(x))_{Q_R^+} = (D^\alpha u)_{Q_R^+} \tag{3.4.3}$$

for $|\alpha| < m$. *Let* $v = u - P(x)$. *Then there exists a constant* $C = C(d,n,m,K)$ *such that*

$$\begin{aligned} &\|D^\alpha v\|_{L_2(Q_R^+)} \\ &\le CR^{m-|\alpha|}\|D^m u\|_{L_2(Q_R^+)} + C\sum_{|\beta|\le m} R^{3m+d/2-|\alpha|-|\beta|}\|f_\beta\|_{L_\infty(Q_R^+)}, \end{aligned}$$

where $|\alpha| < m$. *If* $f_\alpha = 0$ *for any* α, *then it holds that*

$$\|D^\alpha v\|_{L_2(Q_R^+)} \le CR^{|\beta|-|\alpha|}\|D^\beta v\|_{L_2(Q_R^+)},$$

where $|\alpha| < |\beta| \le m$.

3.4.2 L_p estimate of $D_d^m u$ for systems with special coefficients

In this section, we consider the special operator:

$$\tilde{\mathcal{L}}_0 = \mathcal{D}_{d-1}^m I_{n\times n} + D^{\hat\alpha}(A^{\hat\alpha\hat\alpha}(t)D^{\hat\alpha}),$$

where

$$\hat\alpha = (0,\dots,0,m), \quad \mathcal{D}_{d-1}^m = \sum_{j=1}^{d-1} D_j^{2m},$$

and $A^{\hat\alpha\hat\alpha}(t)$ is only measurable in t and satisfies $A^{\hat\alpha\hat\alpha} \ge \delta I_{n\times n}$. Note that $\tilde{\mathcal{L}}_0$ satisfies (3.2.1) so that Theorem 3.4.2 is applicable.

We have the following observation.

Lemma 3.4.5. *Assume that* $u \in C^\infty_{loc}(\overline{\mathcal{O}_0^+})$ *satisfies*

$$u_t + (-1)^m \tilde{\mathcal{L}}_0 u = 0 \quad in \quad Q_4^+ \tag{3.4.4}$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_4$. *Then the boundary conditions are given by*

$$D_d^m u = \cdots = D_d^{2m-1} u = 0 \quad on \quad \{x_d = 0\} \cap Q_4.$$

Proof. By the weak formulation of the system,

$$-\langle u, \phi_t \rangle_{Q_4^+} + \sum_{j=1}^{d-1} \langle D_j^m u, D_j^m \phi \rangle_{Q_4^+} + \langle A^{\hat{\alpha}\hat{\alpha}}(t) D_d^m u, D_d^m \phi \rangle_{Q_4^+} = 0 \tag{3.4.5}$$

for any $\phi \in C_0^\infty(Q_4)$. Since only the boundary conditions on $\{x_d = 0\}$ are considered, we integrate by parts and boundary terms only appear in the last term of the equation above. Let us denote $\Sigma := \{x_d = 0\} \cap Q_4$. First, we integrate by parts once and get

$$\begin{aligned} &\langle A^{\hat{\alpha}\hat{\alpha}}(t) D_d^m u, D_d^m \phi \rangle_{Q_4^+} \\ &= -\int_\Sigma A^{\hat{\alpha}\hat{\alpha}}(t) D_d^m u D_d^{m-1} \phi \, dx' \, dt - \langle A^{\hat{\alpha}\hat{\alpha}}(t) D_d^{m+1} u, D_d^{m-1} \phi \rangle_{Q_4^+}. \end{aligned}$$

We continue this process by integrating by parts the second term on the right-hand side of the equality above. By induction, it is easily seen that

$$\begin{aligned} &\langle A^{\hat{\alpha}\hat{\alpha}}(t) D_d^m u, D_d^m \phi \rangle_{Q_4^+} \\ &= \sum_{j=1}^m \int_\Sigma (-1)^j A^{\hat{\alpha}\hat{\alpha}} D_d^{m+j-1} u \, D_d^{m-j} \phi \, dx' \, dt + (-1)^m \langle A^{\hat{\alpha}\hat{\alpha}}(t) D_d^{2m} u, \phi \rangle_{Q_4^+}. \end{aligned}$$

Plug the equality above into (3.4.5). Since ϕ is an arbitrary smooth function, we get

$$\sum_{j=1}^{m} \int_{\Sigma} (-1)^j A^{\hat{\alpha}\hat{\alpha}} D_d^{m+j-1} u \, D_d^{m-j} \phi \, dx' \, dt = 0,$$

and thus

$$A^{\hat{\alpha}\hat{\alpha}} D_d^{m+j-1} u = 0 \quad \text{on} \quad \Sigma \quad \text{for} \quad j = 1, 2, \ldots, m.$$

Since $A^{\hat{\alpha}\hat{\alpha}}$ is positive definite, we obtain

$$D_d^m u = \ldots = D_d^{2m-1} u = 0 \quad \text{on} \quad \Sigma.$$

The lemma is proved. $\square$

We state a conclusion in [37, Remark 6] and notice that by Lemma 3.4.1 $\tilde{\mathcal{L}}_0$ satisfies the Legendre–Hadamard condition (3.2.3).

Lemma 3.4.6. *Let $R \in (0, \infty)$ be a constant. Assume that $u \in C^{\infty}_{loc}(\overline{\mathcal{O}^+_{\infty}})$ satisfies*

$$\begin{aligned} &u_t + (-1)^m \tilde{\mathcal{L}}_0 u = 0 \quad \textit{in} \quad Q^+_{2R}, \\ &u = \cdots = D_d^{m-1} u = 0 \quad \textit{on} \quad Q_{2R} \cap \{x_d = 0\}. \end{aligned}$$

Then for any $r \in (0, R)$, there exists a constant C depending only on d, n, m, δ, K, r, R, and a such that

$$\|u\|_{\frac{a}{2m}, a, Q^+_r} \le C \|u\|_{L_2(Q^+_R)},$$

where $0 < a < 2m$.

As a consequence of Lemma 3.4.6, we get the following Hölder estimate of $D_d^m u$.

Lemma 3.4.7. *Let $\lambda \geq 0$ be a constant. Assume that $u \in C^\infty_{loc}(\overline{\mathcal{O}^+_\infty})$ satisfies*

$$u_t + (-1)^m \tilde{\mathcal{L}}_0 u + \lambda u = 0 \tag{3.4.6}$$

in Q^+_2 with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_2$. Then there exists $C = C(d, m, n, \delta, K)$ such that

$$[D^m_d u]_{\frac{1}{2m},1,Q^+_1} \leq N \|D^m_d u\|_{L_2(Q^+_2)}.$$

Proof. For the case when $\lambda = 0$, as noted in Lemma 3.4.5, the conormal derivative boundary conditions for (3.4.4) are given by

$$D^m_d u = \cdots = D^{2m-1}_d u = 0 \quad \text{on} \quad \{x_d = 0\}.$$

We differentiate (3.4.4) m times with respect to x_d and let $v = D^m_d u$. Then we arrive at

$$\begin{aligned} &v_t + (-1)^m \tilde{\mathcal{L}}_0 v = 0 \quad \text{in } Q^+_2, \\ &v = D_d v = \cdots = D^{m-1}_d v = 0 \quad \text{on } \{x_d = 0\} \cap Q_2. \end{aligned}$$

By Lemma 3.4.6 with $a = 1$,

$$[v]_{\frac{1}{2m},1,Q^+_1} \leq C \|v\|_{L_2(Q^+_2)}.$$

Since $v = D^m_d u$, we prove the case when $\lambda = 0$. For the case when $\lambda > 0$, we apply an idea of S. Agmon, the details of which can be found in Corollary 3.4.14. □

Lemma 3.4.8. *Let $r \in (0, \infty)$, $\kappa \in [32, \infty)$, $\lambda > 0$, and $X_0 = (t_0, x_0) \in \overline{\mathcal{O}^+_\infty}$. Assume that $u \in C^\infty_{loc}(\overline{\mathcal{O}^+_\infty})$ satisfies (3.4.6) in $Q^+_{\kappa r}(X_0)$ with the conormal derivative* *boundary conditions on* $\{x_d = 0\} \cap Q_{\kappa r}(X_0)$*. Then we have*

$$(|D_d^m u - (D_d^m u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} \le C\kappa^{-1}(|D_d^m u|^2)^{\frac{1}{2}}_{Q_{\kappa r}^+(X_0)},$$

where $C = C(d, n, m, \delta, K)$*.*

Proof. By scaling, it suffices to prove the inequality for $r = 8/\kappa \in (0, 1/4]$. We consider the following two cases.

Case 1: the last coordinate of $x_0 \in [0, 1)$. In this case, by denoting $Y_0 = (t_0, x_0', 0)$, we have

$$Q_r(X_0) \subset Q_2(Y_0) \subset Q_4(Y_0) \subset Q_{\kappa r}(X_0).$$

After applying Lemma 3.4.7 to u with a scaling and translation of the coordinates, we obtain

$$\begin{aligned}(|D_d^m u - (D_d^m u)_{Q_r^+(X_0)}|)_{Q_r^+(X_0)} &\le Cr[D_d^m u]_{\frac{1}{2m},1,Q_2^+(Y_0)} \\ &\le C\kappa^{-1}(|D_d^m u|^2)^{\frac{1}{2}}_{Q_4^+(Y_0)} \le C\kappa^{-1}(|D_d^m u|^2)^{\frac{1}{2}}_{Q_{\kappa r}^+(X_0)}.\end{aligned}$$

Case 2: the last coordinate of $x_0 \ge 1$. This case is indeed an interior case. From Lemmas 2 and 3 in [37], we have

$$[v]_{\frac{1}{2m},1,Q_1} \le C\|v\|_{L_2(Q_4)}, \tag{3.4.7}$$

where v smooth is a solution of

$$v_t + (-1)^m \tilde{\mathcal{L}}_0 v + \lambda v = 0 \quad \text{in} \quad Q_4.$$

Taking $v = D_d^m u$, we get

$$
\begin{aligned}
(|D_d^m u - (D_d^m u)_{Q_r(X_0)}|)_{Q_r(X_0)} &\le Cr[D_d^m u]_{\frac{1}{2m},1,Q_{1/4}(X_0)} \\
&\le Cr\|D_d^m u\|_{L_2(Q_1(X_0))} \le C\kappa^{-1}(|D_d^m u|^2)^{\frac{1}{2}}_{Q^+_{\kappa r}(X_0)},
\end{aligned}
$$

where we use (3.4.7) with a scaling and translation of the coordinates in the second inequality. Hence we prove the lemma. □

Now we are ready to establish a mean oscillation estimate of $D_d^m u$ for systems with special coefficients on a half space.

Theorem 3.4.9. *Let $r \in (0, \infty)$, $T \in (-\infty, \infty]$, $\kappa \in [64, \infty)$, $\lambda > 0$, $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_T^+}$, and $f_\alpha \in L_{2,loc}(\overline{\mathcal{O}_T^+})$, where $|\alpha| \le m$. Assume that $u \in \mathcal{H}^m_{2,loc}(\overline{\mathcal{O}_T^+})$ satisfies*

$$
u_t + (-1)^m \tilde{\mathcal{L}}_0 u + \lambda u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \tag{3.4.8}
$$

in $Q^+_{2\kappa r}(X_0)$ with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2\kappa r}(X_0)$. Then we have

$$
\begin{aligned}
&\left(|D_d^m u - (D_d^m u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} \\
&\le C\kappa^{-1}(|D_d^m u|^2)^{\frac{1}{2}}_{Q^+_{\kappa r}(X_0)} + C\kappa^{m+\frac{d}{2}} \sum_{|\alpha| \le m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|f_\alpha|^2)^{\frac{1}{2}}_{Q^+_{\kappa r}(X_0)},
\end{aligned} \tag{3.4.9}
$$

where $C = C(d, m, n, \delta, K)$.

Proof. Choose two smooth functions ζ and ζ_1 defined on $\mathbb{R}^{d+1}$ such that

$$
\begin{aligned}
&\zeta = 1 \quad \text{on} \quad Q_{\kappa r/2}(X_0), \\
&\zeta = 0 \quad \text{outside} \quad (t_0 - (\kappa r)^{2m}, t_0 + (\kappa r)^{2m}) \times B_{\kappa r}(x_0).
\end{aligned}
$$

and

$$\zeta_1 = 1 \quad \text{on} \quad Q_{\kappa r}(X_0),$$

$$\zeta_1 = 0 \quad \text{outside} \quad (t_0 - (2\kappa r)^{2m}, t_0 + (2\kappa r)^{2m}) \times B_{2\kappa r}(x_0).$$

Define $\tilde{u} = \zeta_1 u$. We claim that $\tilde{u}$ satisfies the following equation in $\mathcal{O}_{t_0}^+$

$$\tilde{u}_t + (-1)^m \tilde{\mathcal{L}}_0 \tilde{u} + \lambda \tilde{u} = \sum_{|\alpha| \le m} D^\alpha F_\alpha \tag{3.4.10}$$

with the conormal derivative boundary conditions on $\{x_d = 0\}$, where F_α is the linear combination of terms like

$$f_\alpha \zeta_1, \quad u\zeta_{1t}, \quad D^k \zeta_1 f_\alpha, \quad A^{\hat{\alpha}\hat{\alpha}} D^m u D^k \zeta_1, \quad D^m u D^k \zeta_1,$$
$$A^{\hat{\alpha}\hat{\alpha}} D^{m-k} u D^k \zeta_1, \quad D^{m-k} u D^k \zeta_1, \quad k \ge 1.$$

In order to show (3.4.10), we compute

$$-\langle \tilde{u}, \phi_t \rangle_{\mathcal{O}_{t_0}^+} + \sum_{j=1}^{m-1} \langle D_j^m \tilde{u}, D_j^m \phi \rangle_{\mathcal{O}_{t_0}^+} + \langle A^{\hat{\alpha}\hat{\alpha}} D_d^m \tilde{u}, D_d^m \phi \rangle_{\mathcal{O}_{t_0}^+} \tag{3.4.11}$$

for any $\phi \in C_0^\infty((-\infty, t_0) \times \overline{\mathbb{R}^d_+})$. Notice that

$$D^m(\tilde{u}) = D^m(u\zeta_1) = D^m u \zeta_1 + \sum_{j=0}^{m-1} C(d,m,j) D^j u D^{m-j} \zeta_1,$$

$$\zeta_1 D^m \phi = D^m(\zeta_1 \phi) - \sum_{j=1}^{m} C'(d,m,j) D^j \zeta_1 D^{m-j} \phi,$$

and $\zeta_1 \phi$ vanishes outside $Q_{2\kappa r}(X_0)$. Upon plugging the two equalities above into (3.4.11) and using the fact that u satisfies (3.4.8) in $Q_{2\kappa r}^+(X_0)$ with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2\kappa r}(X_0)$, where we take $\zeta_1 \phi$ as a test function, we obtain (3.4.10). On the other hand, by our assumption that $u \in \mathcal{H}^m_{2,\text{loc}}(\overline{\mathcal{O}^+_T})$ and $f_\alpha \in L_{2,\text{loc}}(\overline{\mathcal{O}^+_T})$, it is easy to see that $F_\alpha \in L_2(\mathcal{O}^+_T)$. Now we consider the values of u and f_α in $Q_{\kappa r}(X_0)$. Because $\tilde{u} = u$ and $f_\alpha = F_\alpha$ in $Q_{\kappa r}(X_0)$, after extending f_α to be zero when $t > T$, without loss of generality we can assume that $f_\alpha \in L_2(\mathcal{O}^+_\infty)$.

By a standard approximation argument, we can assume $A^{\hat{\alpha}\hat{\alpha}}$ and f_α are smooth in $\overline{\mathcal{O}^+_\infty}$ and (3.4.8) holds in $\mathcal{O}^+_\infty$. Indeed, let $A^{\hat{\alpha}\hat{\alpha}}_\varepsilon$ be the mollification of $A^{\hat{\alpha}\hat{\alpha}}$ in the t variable and $f_{\alpha\varepsilon}$ be the mollification of f_α (we may further extend f_α to be 0 when $x_d < 0$). Therefore, $f_{\alpha\varepsilon} \to f_\alpha$ in $L_2(\mathcal{O}^+_\infty)$ and $A^{\alpha\beta}_\varepsilon \to A^{\alpha\beta}$ (a.e) as $\varepsilon \to 0$. Let u^ε be the unique weak solution to the system

$$u^\varepsilon_t + (-1)^m \tilde{\mathcal{L}}_{0\varepsilon} u^\varepsilon + \lambda v^\varepsilon = \sum_{|\alpha| \le m} D^\alpha f_{\alpha\varepsilon}$$

in $\mathcal{O}^+_\infty$ with the conormal derivative boundary conditions on $\{x_d = 0\}$. By Theorem 3.4.2 and the dominated convergence theorem, u^ε converges to u in $\mathcal{H}^m_2(\mathcal{O}^+_{t_0})$ as $\varepsilon \to 0$. Obviously, if (3.4.9) holds for u^ε, then it holds for u as well. Thus, it is sufficient to assume that f_α and $A^{\hat{\alpha}\hat{\alpha}}$ are smooth and u satisfies (3.4.8) in $\mathcal{O}^+_\infty$, which implies $u \in C^\infty_{\text{loc}}(\overline{\mathcal{O}^+_\infty})$ (see, for instance, [1]).

By Theorem 3.4.2, for any $\lambda > 0$, there exists a unique solution $w \in \mathcal{H}^m_2(\mathcal{O}^+_\infty)$, which is also smooth, to the equation

$$w_t + (-1)^m \tilde{\mathcal{L}}_0 w + \lambda w = \sum_{|\alpha| \le m} D^\alpha (\zeta f_\alpha)$$

in $\mathcal{O}^+_\infty$ with the conormal derivative boundary conditions on $\{x_d = 0\}$. Let $v :=$

$u - w \in C^\infty_{\mathrm{loc}}(\overline{\mathcal{O}^+_\infty})$. Then v satisfies

$$v_t + (-1)^m \tilde{\mathcal{L}}_0 v + \lambda v = 0 \quad \text{in} \quad Q^+_{\kappa r/2}(X_0)$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{\kappa r/2}(X_0)$. By applying Lemma 3.4.8 to v (note that $\kappa/2 \geq 32$), we have

$$\big(|D^m_d v - (D^m_d v)_{Q^+_r(X_0)}|\big)_{Q^+_r(X_0)} \leq C\kappa^{-1}(|D^m_d v|^2)^{\frac{1}{2}}_{Q^+_{\kappa r/2}(X_0)}. \tag{3.4.12}$$

By Theorem 3.4.2 with $T = t_0$, we get

$$\sum_{|\alpha|\leq m} \lambda^{1-\frac{|\alpha|}{2m}} \|D^\alpha w\|_{L_2(\mathcal{O}^+_{t_0})} \leq C \sum_{|\alpha|\leq m} \lambda^{\frac{|\alpha|}{2m}} \|\zeta f_\alpha\|_{L_2(\mathcal{O}^+_{t_0})}. \tag{3.4.13}$$

In particular,

$$(|D^m w|^2)^{\frac{1}{2}}_{Q^+_r(X_0)} \leq C\kappa^{m+d/2} \sum_{|\alpha|\leq m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} (|f_\alpha|^2)^{\frac{1}{2}}_{Q^+_{\kappa r}(X_0)}. \tag{3.4.14}$$

Now let us prove (3.4.9). By the triangle inequality,

$$\big(|D^m_d u - (D^m_d u)_{Q^+_r(X_0)}|\big)_{Q^+_r(X_0)} \leq 2(|D^m_d u - c|)_{Q^+_r(X_0)}$$

for any constant c. By taking $c = (D^m_d v)_{Q^+_r(X_0)}$, we have

$$\big(|D^m_d u - (D^m_d u)_{Q^+_r(X_0)}|\big)_{Q^+_r(X_0)} \leq 2(|D^m_d u - (D^m_d v)_{Q^+_r(X_0)}|)_{Q^+_r(X_0)}.$$

By using the triangle inequality and the Cauchy–Schwarz inequality, the right-hand side of the inequality above can be bounded by

$$C\big(|D^m_d v - (D^m_d v)_{Q^+_r(X_0)}|\big)_{Q^+_r(X_0)} + C(|D^m w|^2)^{\frac{1}{2}}_{Q^+_r(X_0)}.$$

By using (3.5.35) and (3.4.14), the quantity above is less than

$$C\kappa^{-1}(|D_d^m v|^2)^{\frac{1}{2}}_{Q^+_{\kappa r/2}(X_0)} + C\kappa^{m+d/2}\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}(|f_\alpha|^2)^{\frac{1}{2}}_{Q^+_{\kappa r}(X_0)}. \tag{3.4.15}$$

Finally, from (3.4.13) we know that

$$(|D_d^m w|^2)^{\frac{1}{2}}_{Q^+_{\kappa r/2}(X_0)} \le C\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}(|f_\alpha|^2)^{\frac{1}{2}}_{Q^+_{\kappa r}(X_0)}.$$

Combining the inequality above and the fact that $v = u - w$, we see that the first term of (3.4.15) is less than the right-hand side of (3.4.9). The theorem is proved. □

Now we are ready to prove an L_p estimate of $D_d^m u$ for the system with special coefficients.

Theorem 3.4.10. *Let $p \in [2,\infty)$, $\lambda \ge 0$, $T \in (-\infty,\infty]$, and $f_\alpha \in L_p(\mathcal{O}_T^+)$ for $|\alpha| \le m$. Then for any $u \in \mathcal{H}_p^m(\mathcal{O}_T^+)$ satisfying*

$$u_t + (-1)^m\tilde{\mathcal{L}}_0 u + \lambda u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in } \mathcal{O}_T^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\}$, we have

$$\lambda^{\frac{1}{2}}\|D_d^m u\|_{L_p(\mathcal{O}_T^+)} \le C\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}}\|f_\alpha\|_{L_p(\mathcal{O}_T^+)},$$

where $C = C(d,m,n,p,\delta,K)$.

Proof. First we suppose that $p \in (2,\infty)$. From Theorem 3.4.9, we deduce that

$$
\begin{aligned}
&(D_d^m u)^{\#}(X_0)\\
&\le C\kappa^{-1}(M(D_d^m u)^2(X_0))^{\frac{1}{2}} + C\kappa^{m+\frac{d}{2}} \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} (M(f_\alpha)^2(X_0))^{\frac{1}{2}}
\end{aligned}
$$

for any $\kappa \ge 64$ and $X_0 \in \overline{\mathcal{O}_T^+}$. This, together with the Fefferman–Stein theorem and the Hardy–Littlewood maximal function theorem, yields

$$
\begin{aligned}
&\|D_d^m u\|_{L_p(\mathcal{O}_T^+)} \le C\|(D_d^m u)^{\#}\|_{L_p(\mathcal{O}_T^+)}\\
&\le C\kappa^{-1}\|D_d^m u\|_{L_p(\mathcal{O}_T^+)} + C\kappa^{m+\frac{d}{2}} \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \|f_\alpha\|_{L_p(\mathcal{O}_T^+)}.
\end{aligned}
$$

Now we choose κ sufficiently large such that the first term on the right-hand side of the inequality above is absorbed in the left-hand side. Then we obtain the desired estimate. The case when $p = 2$ follows from Theorem 3.4.2. The theorem is proved. □

3.4.3 Mean oscillation estimate for $D_{x'}D^{m-1}u$

In this section, we prove a mean oscillation estimate for $D_{x'}D^{m-1}u$. The following lemma shows that $\|D_d^m u\|_{L_p}$ can be bounded by $\|D_{x'}D^{m-1}u\|_{L_p}$ and $\|f_\alpha\|_{L_p}$ for systems with simple coefficients.

Lemma 3.4.11. *Let $T \in (-\infty,\infty]$ and $p \in [2,\infty)$. Assume that $u \in \mathcal{H}_p^m(\mathcal{O}_T^+)$ and*

$$
u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha|\le m} D^\alpha f_\alpha
$$

with the conormal derivative boundary conditions on $\{x_d = 0\}$, where $\lambda \ge 0$ and

$f_\alpha \in L_p(\mathcal{O}_T^+)$. Then under the condition $A^{\hat\alpha\hat\alpha} \geq \delta I_{n\times n}$, there exists a constant N, depending only on d, m, n, δ, K, and p, such that

$$\lambda^{\frac{1}{2}}\|D^m u\|_{L_p(\mathcal{O}_T^+)} \leq C\sum_{|\alpha|\leq m}\lambda^{\frac{|\alpha|}{2m}}\|f_\alpha\|_{L_p(\mathcal{O}_T^+)} + C\lambda^{\frac{1}{2}}\|D_{x'}D^{m-1}u\|_{L_p(\mathcal{O}_T^+)}.$$

Especially, if $\lambda = 0$ and $f_\alpha = 0$ when $|\alpha| < m$, then

$$\|D^m u\|_{L_p(\mathcal{O}_T^+)} \leq C\|D_{x'}D^{m-1}u\|_{L_p(\mathcal{O}_T^+)} + C\sum_{|\alpha|=m}\|f_\alpha\|_{L_p(\mathcal{O}_T^+)}.$$

Proof. The case when $\lambda = 0$ follows by letting $\lambda \searrow 0$ after the estimate with $\lambda > 0$ is proved.

We use a scaling argument. Let $v(t,x',x_d) = u(\mu^{-2m}t, x', \mu^{-1}x_d)$ with a sufficiently large constant μ to be chosen later. Then v satisfies in $\mathcal{O}_{\mu^{2m}T}^+$,

$$\begin{aligned} & v_t + (-1)^m \sum_{|\alpha|=|\beta|=m}\mu^{\alpha_d+\beta_d-2m}D^\alpha(\tilde A^{\alpha\beta}D^\beta v) + \mu^{-2m}\lambda v \\ & = \sum_{|\alpha|\leq m}\mu^{\alpha_d-2m}D^\alpha \tilde f_\alpha \end{aligned}$$

with the conormal derivative boundary conditions on $\{x_d = 0\}$, where

$$\tilde A^{\alpha\beta}(t) = A^{\alpha\beta}(\mu^{-2m}t), \quad \tilde f_\alpha(t,x',x_d) = f_\alpha(\mu^{-2m}t, x', \mu^{-1}x_d).$$

We leave the term $(-1)^m D^{\hat\alpha}(\tilde A^{\hat\alpha\hat\alpha}D^{\hat\alpha}v)$ on the left-hand side and move all the other spatial derivatives to the right-hand side and add $(-1)^m\mathcal{D}_{d-1}^m v$ to both sides of the system,

$$v_t + (-1)^m\big(D^{\hat\alpha}(A^{\hat\alpha\hat\alpha}D^{\hat\alpha}v) + \mathcal{D}_{d-1}^m v\big) + \mu^{-2m}\lambda v = \sum_{|\alpha|\leq m}D^\alpha \hat f_\alpha + (-1)^m\mathcal{D}_{d-1}^m v,$$

where $\hat{f}_\alpha = \mu^{\alpha_d - 2m}\tilde{f}_\alpha$ for $|\alpha| < m$,

$$\hat{f}_\alpha = \mu^{\alpha_d-2m}\tilde{f}_\alpha + \sum_{|\beta|=m}(-1)^{m+1}\mu^{\alpha_d+\beta_d-2m}\tilde{A}^{\alpha\beta}(t)D^\beta v$$

for $|\alpha| = m$ but $\alpha \neq \hat{\alpha}$, and

$$\hat{f}_{\hat{\alpha}} = \mu^{-m}\tilde{f}_{\hat{\alpha}} + \sum_{|\beta|=m,\beta\neq\hat{\alpha}}(-1)^{m+1}\mu^{\beta_d-m}\tilde{A}^{\hat{\alpha}\beta}(t)D^\beta v.$$

Then we implement Theorem 3.4.10 to get

$$\begin{aligned}
&\mu^{-m}\lambda^{\frac{1}{2}}\|D^m v\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})}\\
&\leq C\sum_{|\alpha|\leq m}(\mu^{-2m}\lambda)^{\frac{|\alpha|}{2m}}\|\hat{f}_\alpha\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})} + C\mu^{-m}\lambda^{\frac{1}{2}}\|D_{x'}D^{m-1}v\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})}\\
&\leq C\sum_{|\alpha|\leq m}(\mu^{-2m}\lambda)^{\frac{|\alpha|}{2m}}\mu^{\alpha_d-2m}\|\tilde{f}_\alpha\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})}\\
&\quad + C\mu^{-m}\lambda^{\frac{1}{2}}\Big(\sum_{|\alpha|=m,\alpha\neq\hat{\alpha}}\sum_{|\beta|=m}\mu^{\alpha_d+\beta_d-2m}\|D^\beta v\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})}\\
&\quad + \sum_{|\beta|=m,\beta\neq\hat{\alpha}}\mu^{\beta_d-m}\|D^\beta v\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})}\Big) + C\mu^{-m}\lambda^{\frac{1}{2}}\|D_{x'}D^{m-1}v\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})}.
\end{aligned}$$

Let μ be sufficiently large such that

$$C\Big(\sum_{|\alpha|=m,\alpha\neq\hat{\alpha}}\sum_{|\beta|=m}\mu^{\alpha_d+\beta_d-2m} + \sum_{|\beta|=m,\beta\neq\hat{\alpha}}\mu^{\beta_d-m}\Big) \leq 1/2.$$

Then we fix this μ and obtain

$$\begin{aligned}
\mu^{-m}\lambda^{\frac{1}{2}}\|D^m v\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})} \leq & C\sum_{|\alpha|\leq m}(\mu^{-2m}\lambda)^{\frac{|\alpha|}{2m}}\mu^{\alpha_d-2m}\|\tilde{f}_\alpha\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})}\\
& + C\mu^{-m}\lambda^{\frac{1}{2}}\|D_{x'}D^{m-1}v\|_{L_p(\mathcal{O}^+_{\mu^{2m}T})}.
\end{aligned}$$

After returning to u and f_α, we complete the proof of the lemma. $\square$

We localize Lemma 3.4.11 to get Lemma 3.4.12 following the proof of [37, Lemma 1].

Lemma 3.4.12. *Let $0 < r < R < \infty$ and $p \in [2, \infty)$. Assume $u \in \mathcal{H}_p^m(Q_R^+)$ and*

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \quad in \quad Q_R^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_R$. Then under the condition $A^{\hat{\alpha}\hat{\alpha}} \geq \delta I_{n\times n}$, there exists a constant C depending only on d, m, n, δ, r, R, p, and K such that

$$\|D^m u\|_{L_p(Q_r^+)} \leq C(\|D_{x'} D^{m-1} u\|_{L_p(Q_R^+)} + \|u\|_{L_p(Q_R^+)}). \tag{3.4.16}$$

In the next lemma, we obtain a Hölder estimate of $D^{m-1}u$.

Lemma 3.4.13. *Assume that $u \in C^\infty_{loc}(\overline{\mathcal{O}_0^+})$ satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \quad in \quad Q_2^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_2$. Then under the condition (3.2.1), for any $\gamma \in (0, 1)$ there exists a constant $C = C(d, m, n, \delta, \gamma, K)$ such that

$$\|D^{m-1} u\|_{\frac{\gamma}{2m}, \gamma, Q_1^+} \leq C \|u\|_{L_2(Q_2^+)}.$$

Proof. Due to Lemma 3.4.3 and the definition of $\|\cdot\|_{\mathcal{H}_p^m}$, we have

$$\|u\|_{\mathcal{H}_2^m(Q_{r_1}^+)} \leq C(d, r_1) \|u\|_{L_2(Q_2^+)},$$

where $1 < r_1 < 2$. From Lemma 3.3.4, we know that there is a p_1 satisfying

$\frac{1}{2} - \frac{1}{p_1} \leq \frac{1}{d+2m}$ such that

$$\|u\|_{L_{p_1}(Q^+_{r_1'})} + \|D^{m-1}u\|_{L_{p_1}(Q^+_{r_1'})}$$
$$\leq C(d, r_1', r_1, p_1)\|u\|_{\mathcal{H}^m_2(Q^+_{r_1})} \leq C(d, r_1', r_1, p_1)\|u\|_{L_2(Q^+_2)}, \tag{3.4.17}$$

where $1 < r_1' < r_1$. Since $D_{x'}u$ satisfies the same system and boundary conditions as u, with slight modifications of the argument above and Lemma 3.4.3 we get

$$\|D_{x'}D^{m-1}u\|_{L_{p_1}(Q^+_{r_1'})} \leq C(d, r_1', r_1, p_1)\|u\|_{L_2(Q^+_2)}.$$

Choose $r_2 \in (1, r_1')$. From Lemma 3.4.1, (3.4.16), and (3.4.17), we have

$$\|D^m u\|_{L_{p_1}(Q^+_{r_2})} \leq C\big(\|D_{x'}D^{m-1}u\|_{L_{p_1}(Q^+_{r_1'})} + \|u\|_{L_{p_1}(Q^+_{r_1'})}\big) \leq C\|u\|_{L_2(Q^+_2)},$$

which, combining with the fact that u satisfies the system, implies

$$\|u\|_{\mathcal{H}^m_{p_1}(Q^+_{r_2})} \leq C\|u\|_{L_2(Q^+_2)}.$$

By induction, we can choose an increasing sequence p_1, $p_2, \ldots$, such that

$$\frac{1}{p_i} - \frac{1}{p_{i+1}} \leq \frac{1}{d+2m},$$

and a sequence of decreasing domains $Q^+_{r_1} \supset Q^+_{r_2} \supset \cdots$, such that

$$\|u\|_{\mathcal{H}^m_{p_i}(Q^+_{r_{2i}})} \leq C\|u\|_{L_2(Q^+_2)}.$$

It is obvious that for any $\gamma \in (0,1)$, in finite steps we can always take $p_n = (d + 2m)/(1-\gamma)$ and $Q^+_{r_{2n}} \supset Q^+_1$. Finally applying the last assertion of Lemma 3.3.4 for $p > 2m + d$, we prove the lemma. $\square$

Corollary 3.4.14. *Let* $\lambda \geq 0$ *and* $\gamma \in (0,1)$*. Assume that* $u \in C^\infty_{loc}(\overline{\mathcal{O}_0^+})$ *satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = 0$$

in Q_2^+ *with the conormal derivative boundary conditions on* $\{x_d = 0\} \cap Q_2$*. Then under the condition* (3.2.1)*, there exists a constant* C *depending only on* d*,* m*,* n*,* δ*,* γ*, and* K *such that*

$$[D_{x'} D^{m-1} u]_{\frac{\gamma}{2m},\gamma,Q_1^+} + \lambda^{\frac{1}{2}} [u]_{\frac{\gamma}{2m},\gamma,Q_1^+} \leq C \sum_{k=0}^{m} \lambda^{\frac{1}{2} - \frac{k}{2m}} \|D^k u\|_{L_2(Q_2^+)}. \tag{3.4.18}$$

Proof. We refer the interested reader to [43]. □

In the next lemma, we obtain a mean oscillation estimate of $D_{x'} D^{m-1} u$ for homogeneous systems. The proof is similar to Lemma 3.4.8.

Lemma 3.4.15. *Let* $r \in (0,\infty)$*,* $\kappa \in [32,\infty)$*,* $\gamma \in (0,1)$*, and* $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_0^+}$*. Assume that* $u \in C^\infty_{loc}(\overline{\mathcal{O}_0^+})$ *satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = 0 \quad in \quad Q_{\kappa r}^+(X_0)$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{\kappa r}(X_0)$*. Then under the condition* (3.2.1)*, we have*

$$\begin{aligned}&\left(|D_{x'} D^{m-1} u - (D_{x'} D^{m-1} u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} + \lambda^{\frac{1}{2}} \left(|u - (u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} \\ &\leq C \kappa^{-\gamma} \sum_{k=0}^{m} \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k u|^2)^{\frac{1}{2}}_{Q_{\kappa r}^+(X_0)},\end{aligned}$$

where $C = C(d, n, m, \delta, \gamma, K)$*.*

In the following proposition, we obtain a mean oscillation estimate of $D_{x'} D^{m-1} u$

for systems with simple coefficients, the proof of which is similar to that of Theorem 3.4.9.

Proposition 3.4.16. *Let $r \in (0,\infty)$, $T \in (-\infty,\infty]$, $\kappa \in [64,\infty)$, $\lambda > 0$, $\gamma \in (0,1)$, and $X_0 = (t_0, x_0) \in \overline{\mathcal{O}_T^+}$. Assume that $u \in \mathcal{H}^m_{2,loc}(\overline{\mathcal{O}_T^+})$ satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u + \lambda u = \sum_{|\alpha| \le m} D^\alpha f_\alpha$$

in $Q^+_{2\kappa r}(X_0)$ with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2\kappa r}(X_0)$, where $f_\alpha \in L_{2,loc}(\overline{\mathcal{O}_T^+})$, $|\alpha| \le m$. Then under the condition (3.2.1), we have

$$\begin{aligned}
&\left(|D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} + \lambda^{\frac{1}{2}}\left(|u - (u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} \\
&\le C\kappa^{-\gamma} \sum_{k=0}^m \lambda^{\frac{1}{2} - \frac{k}{2m}} (|D^k u|^2)^{\frac{1}{2}}_{Q^+_{\kappa r}(X_0)} + C\kappa^{m+\frac{d}{2}} \sum_{|\alpha| \le m} \lambda^{\frac{|\alpha|}{2m} - \frac{1}{2}} (|f_\alpha|^2)^{\frac{1}{2}}_{Q^+_{\kappa r}(X_0)},
\end{aligned}$$

where $C = C(d, m, n, \delta, \gamma, K)$.

Next, we consider the case that $A^{\alpha\beta}$ are functions of both x and t and use the argument of freezing the coefficients to obtain:

Lemma 3.4.17. *Let $\mathcal{L}$ be the operator in Theorem 3.2.2 and $T \in (-\infty, \infty]$. Suppose the lower-order coefficients of $\mathcal{L}$ are all zero. Let $\xi, \nu \in (1,\infty)$ satisfying $1/\xi + 1/\nu = 1$, $\gamma \in (0,1)$, and $\lambda, R \in (0,\infty)$. Assume that $u \in \mathcal{H}^m_{2,loc}(\overline{\mathcal{O}_T^+})$ vanishes outside $Q^+_R(X_1)$, where $X_1 \in \overline{\mathcal{O}_\infty^+}$, and satisfies*

$$u_t + (-1)^m \mathcal{L} u + \lambda u = \sum_{|\alpha| \le m} D^\alpha f_\alpha$$

in $\mathcal{O}_T^+$ with the conormal derivative boundary conditions on $\{x_d = 0\}$, where $f_\alpha \in L_{2,loc}(\overline{\mathcal{O}_T^+})$. Suppose that (3.2.1) holds for any $X \in \overline{\mathcal{O}_T^+}$. Then there exists a constant

$C = C(d, m, n, \delta, \xi, K, \gamma)$ such that for any $r \in (0, \infty), \kappa \geq 64$, and $X_0 \in \overline{\mathcal{O}_T^+}$, we have

$$
\begin{aligned}
&\left(|D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} + \lambda^{\frac{1}{2}}\left(|u - (u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} \\
&\leq C\kappa^{m+\frac{d}{2}}\Big(\sum_{|\alpha|\leq m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}(|f_\alpha|^2)^{\frac{1}{2}}_{Q_{\kappa r}^+(X_0)} + (A_R^{\#})^{\frac{1}{2\nu}}(|D^m u|^{2\xi})^{\frac{1}{2\xi}}_{Q_{\kappa r}^+(X_0)}\Big) \\
&\quad + C\kappa^{-\gamma}\sum_{k=0}^{m} \lambda^{\frac{1}{2}-\frac{k}{2m}}(|D^k u|^2)^{\frac{1}{2}}_{Q_{\kappa r}^+(X_0)}.
\end{aligned}
$$

Proof. Throughout the proof, we assume that $Q_R^+(X_1) \cap Q_{\kappa r}^+(X_0) \neq \emptyset$. Otherwise, the conclusion holds trivially. Fix a $y \in \overline{\mathbb{R}_+^d}$ and set

$$\mathcal{L}_y u = \sum_{|\alpha|=|\beta|=m} D^\alpha(A^{\alpha\beta}(t, y)D^\beta u(t, x)).$$

Then we have

$$u_t + (-1)^m \mathcal{L}_y u + \lambda u = \sum_{|\alpha|\leq m} D^\alpha \tilde{f}_\alpha,$$

where

$$
\begin{aligned}
&\tilde{f}_\alpha = f_\alpha + (-1)^m \sum_{|\beta|=m}(A^{\alpha\beta}(t, y) - A^{\alpha\beta}(t, x))D^\beta u \quad \text{when} \quad |\alpha| = m, \\
&\tilde{f}_\alpha = f_\alpha \quad \text{otherwise.}
\end{aligned}
$$

It follows from Proposition 3.4.16 that

$$
\begin{aligned}
&\left(|D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} + \lambda^{\frac{1}{2}}\left(|u - (u)_{Q_r^+(X_0)}|\right)_{Q_r^+(X_0)} \\
&\leq C\kappa^{-\gamma}\sum_{k=0}^{m} \lambda^{\frac{1}{2}-\frac{k}{2m}}(|D^k u|^2)^{\frac{1}{2}}_{Q_{\kappa r}^+(X_0)} + C\kappa^{m+\frac{d}{2}}\sum_{|\alpha|\leq m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}(|\tilde{f}_\alpha|^2)^{\frac{1}{2}}_{Q_{\kappa r}^+(X_0)}.
\end{aligned}
\tag{3.4.19}
$$

Note that

$$\int_{Q^+_{\kappa r}(X_0)} |\tilde{f}_\alpha|^2 \, dx \, dt \le C \int_{Q^+_{\kappa r}(X_0)} |f_\alpha|^2 \, dxdt + CI_y, \tag{3.4.20}$$

where, for $|\alpha| = m$,

$$I_y = \int_{Q^+_{\kappa r}(X_0)} |(A^{\alpha\beta}(t,y) - A^{\alpha\beta}(t,x))D^\beta u|^2 \, dx \, dt.$$

Denote B^+ to be $B^+_{\kappa r}(x_0)$ if $\kappa r < R$, or to be $B^+_R(x_1)$ otherwise; denote Q^+ in the same fashion. Now we take the average of I_y with respect to y in B^+. Since u vanishes outside $Q^+_R(X_1)$, by Hölder's inequality we get

$$\begin{aligned} \fint_{B^+} I_y \, dy \le \fint_{B^+} \Big(\int_{Q^+} |A^{\alpha\beta}(t,y) - A^{\alpha\beta}(t,x)|^{2\nu} \, dx \, dt \Big)^{\frac{1}{\nu}} \, dy \\ \cdot \Big(\int_{Q^+_{\kappa r}(X_0) \cap Q^+_R(X_1)} |D^m u|^{2\xi} \, dx \, dt \Big)^{\frac{1}{\xi}}, \end{aligned}$$

where, by the boundedness of $A^{\alpha\beta}$, Hölder's inequality as well as the definition of osc_x and $A^{\#}_R$, the integral over B^+ in the last term above is estimated as follows

$$\begin{aligned} &\fint_{B^+} \Big(\int_{Q^+} |A^{\alpha\beta}(t,y) - A^{\alpha\beta}(t,x)|^{2\nu} \, dx \, dt \Big)^{\frac{1}{\nu}} \, dy \\ &\le C(|Q^+| \operatorname{osc}_x(A^{\alpha\beta}, Q^+))^{\frac{1}{\nu}} \le C((\kappa r)^{2m+d} A^{\#}_R)^{\frac{1}{\nu}}. \end{aligned}$$

This together with (3.4.19) and (3.4.20) completes the proof of the lemma. □

3.4.4 Proof of Theorem 3.2.2

Proof of Theorem 3.2.2. Due to the method of continuity, it suffices to prove the a priori estimate. First we note that the interior estimates are obtained in [37, Theorem 1]. With the standard arguments of partition of unity and flattening the boundary, it suffices to consider the case when $\Omega = \mathbb{R}^d_+$.

We may assume that all the lower-order coefficients are zero. Indeed, if we get the a priori estimate without the lower-order terms, for general systems, we can move the lower-order terms to the right-hand side and apply the interpolation inequality. Taking λ large enough, we then obtain the estimate for general systems.

Case 1: $p \in (2, \infty)$. First we suppose $u \in \mathcal{H}^m_p(\overline{\mathcal{O}^+_T})$ and vanishes on $\mathcal{O}^+_T \setminus \tilde{Q}^+_{R_0}(X_1)$ for some $X_1 = (t_1, x_1) \in \overline{\mathcal{O}^+_\infty}$, where

$$\tilde{Q}^+_{R_0}(X_1) = \{(\mu^{-2m}t, x', \mu^{-1}x_d) : (t, x) \in Q^+_{R_0}(X_1)\},$$

$\mu \geq 1$ is a parameter which will be determined later, and R_0 is the constant in Assumption 3.2.1 (ρ). Then it follows $\tilde{Q}^+_{R_0}(X_1) \subset Q^+_{R_0}(\tilde{X}_1)$, where

$$\tilde{X}_1 = (\mu^{-2m}t_1, x'_1, \mu^{-1}x_{1d}).$$

Choose ξ and $\nu \in (1, \infty)$ such that $2\xi < p$ and $1/\nu + 1/\xi = 1$, and fix $\gamma \in (0, 1)$. Under these assumptions, from Lemma 3.4.17 we easily deduce

$$\begin{aligned}(D_{x'}D^{m-1}u)^\#(X_0) + \lambda^{\frac{1}{2}}u^\#(X_0) &\leq C\kappa^{-\gamma}\sum_{k=0}^{m}\lambda^{\frac{1}{2}-\frac{k}{2m}}(M(D^k u)^2(X_0))^{\frac{1}{2}} \\ &+C\kappa^{m+\frac{d}{2}}\left(\sum_{|\alpha|\leq m}\lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}(M(f_\alpha)^2(X_0))^{\frac{1}{2}} + \rho^{\frac{1}{2\nu}}(M(D^m u)^{2\xi}(X_0))^{\frac{1}{2\xi}}\right)\end{aligned}$$

for any $\kappa \geq 64$ and $X_0 \in \overline{\mathcal{O}^+_T}$. This, together with the Fefferman–Stein theorem and the Hardy–Littlewood maximal function theorem, yields

$$\|D_{x'}D^{m-1}u\|_{L_p} + \lambda^{\frac{1}{2}}\|u\|_{L_p} \le C(\|(D_{x'}D^{m-1}u)^{\#}\|_{L_p} + \lambda^{\frac{1}{2}}\|u^{\#}\|_{L_p})$$
$$\le C\kappa^{m+\frac{d}{2}} \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}\|f_\alpha\|_{L_p} + C(\kappa^{-\gamma} + \kappa^{m+\frac{d}{2}}\rho^{\frac{1}{2\nu}}) \sum_{k=0}^{m} \lambda^{\frac{1}{2}-\frac{k}{2m}}\|D^k u\|_{L_p} \tag{3.4.21}$$

for any $\kappa \ge 64$, where $L_p = L_p(\mathcal{O}_T^+)$.

Now we use the arguments of freezing the coefficients and scaling to get the estimate of $D_d^m u$. As in Lemma 3.4.11, let

$$v(t, x', x_d) = u(\mu^{-2m}t, x', \mu^{-1}x_d),$$

$$\tilde{A}^{\alpha\beta}(t, x', x_d) = A^{\alpha\beta}(\mu^{-2m}t, x', \mu^{-1}x_d),$$

$$\tilde{f}_\alpha(t, x', x_d) = f_\alpha(\mu^{-2m}t, x', \mu^{-1}x_d).$$

Then v satisfies

$$v_t + (-1)^m D_d^m(\tilde{A}^{\hat{\alpha}\hat{\alpha}} D_d^m v) + (-1)^m \sum_{\substack{|\alpha|=|\beta|=m \\ (\alpha,\beta)\neq(\hat{\alpha},\hat{\alpha})}} \mu^{\alpha_d+\beta_d-2m} D^\alpha(\tilde{A}^{\alpha\beta}D^\beta v)$$
$$+ \lambda\mu^{-2m}v = \sum_{|\alpha|\le m} \mu^{\alpha_d-2m} D^\alpha f_\alpha$$

in $\mathcal{O}^+_{\mu^{2m}T}$ with the conormal derivative boundary conditions on $\{x_d = 0\}$. We fix $(t, y) \in \overline{\mathcal{O}^+_\infty}$ and move all the spacial derivatives to the right-hand side of the equation, then add $(-1)^m(\mathcal{D}^m_{d-1}v + D_d^m(\tilde{A}^{\hat{\alpha}\hat{\alpha}}(t, y)D_d^m v))$ to both sides of the equation so that

$$v_t + (-1)^m(D_d^m(\tilde{A}^{\hat{\alpha}\hat{\alpha}}(t, y)D_d^m v) + \mathcal{D}^m_{d-1}v) + \lambda\mu^{-2m}v$$
$$= D^\alpha \hat{f}_\alpha + (-1)^m D_d^m\big((\tilde{A}^{\hat{\alpha}\hat{\alpha}}(t, y) - \tilde{A}^{\hat{\alpha}\hat{\alpha}}(t, x))D_d^m v\big) + (-1)^m\mathcal{D}^m_{d-1}v,$$

where $\hat{f}_\alpha = \mu^{\alpha_d - 2m}\tilde{f}_\alpha$ for $|\alpha| < m$,

$$\hat{f}_\alpha = \mu^{\alpha_d - 2m}\tilde{f}_\alpha + \sum_{|\beta|=m} (-1)^{m+1}\mu^{\alpha_d+\beta_d-2m}\tilde{A}^{\alpha\beta}(t,x)D^\beta v$$

for $|\alpha| = m$ but $\alpha \neq \hat{\alpha}$, and

$$\hat{f}_{\hat{\alpha}} = \mu^{-m}\tilde{f}_{\hat{\alpha}} + \sum_{\beta\neq\hat{\alpha},|\beta|=m} (-1)^{m+1}\mu^{\beta_d-m}\tilde{A}^{\hat{\alpha}\beta}(t,x)D^\beta v.$$

We follow the proof of Lemma 3.4.17. From (3.4.9), we know that

$$\begin{aligned}
&\big(|D_d^m v - (D_d^m v)_{Q_r^+(X_0)}|\big)_{Q_r^+(X_0)} \\
&\le C\kappa_1^{-1}(|D_d^m v|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)} + C\kappa_1^{m+\frac{d}{2}}\Big(\sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}(|\hat{f}_\alpha|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)} \\
&\quad + \big(|(\tilde{A}^{\hat{\alpha}\hat{\alpha}}(t,y) - \tilde{A}^{\hat{\alpha}\hat{\alpha}}(t,x))D_d^m v|^2\big)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)} + (|D_{x'}^m v|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)}\Big)
\end{aligned}$$

for any $X_0 \in \overline{\mathcal{O}^+_{\mu^{2m}T}}$ and $\kappa_1 \ge 64$. It is easy to check that $\tilde{A}^{\alpha\beta}$ satisfies the Assumption 3.2.1($2\mu^{2m+1}\rho$) with the same R_0 as $A^{\alpha\beta}$ and the support of v is contained in $Q^+_{R_0}(X_1)$. Therefore, applying the same argument as in Lemma 3.4.17, we obtain for $\xi, \nu \in (1,\infty)$ satisfying $1/\xi + 1/\nu = 1$,

$$\begin{aligned}
&\big(|D_d^m v - (D_d^m v)_{Q_r^+(X_0)}|\big)_{Q_r^+(X_0)} \\
&\le C\kappa_1^{-1}(|D_d^m v|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)} + C\kappa_1^{m+\frac{d}{2}}\sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}(|\hat{f}_\alpha|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)} \\
&+ C\kappa_1^{m+\frac{d}{2}}(2\mu^{2m+1}\rho)^{\frac{1}{2\nu}}(|D_d^m v|^{2\xi})^{\frac{1}{2\xi}}_{Q^+_{\kappa_1 r}(X_0)} + C\kappa_1^{m+\frac{d}{2}}(|D_{x'}^m v|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)}.
\end{aligned}$$

By the definition of $\hat{f}_\alpha$, the right-hand side of the inequality above can be bounded from above by

$$
\begin{aligned}
&C\kappa_1^{-1}(|D_d^m v|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)} + C\kappa_1^{m+\frac{d}{2}}(2\mu^{2m+1}\rho)^{\frac{1}{2\nu}}(|D_d^m v|^{2\xi})^{\frac{1}{2\xi}}_{Q^+_{\kappa_1 r}(X_0)} \\
&+C\kappa_1^{m+\frac{d}{2}}\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}\mu^{\alpha_d-2m}(|\tilde f_\alpha|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)} + C\kappa_1^{m+\frac{d}{2}}\mu^{-1}(|D_d^m v|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)} \\
&+ C\kappa_1^{m+\frac{d}{2}}(|D_{x'}D^{m-1}v|^2)^{\frac{1}{2}}_{Q^+_{\kappa_1 r}(X_0)}
\end{aligned}
$$

provided that $\mu \ge 1$. Therefore, we obtain that, for any $X_0 \in \overline{\mathcal{O}^+_{\mu^{2m}T}}$,

$$
\begin{aligned}
&(D_d^m v)^{\#}(X_0) \\
&\le C\kappa_1^{-1}(M(D_d^m v)^2(X_0))^{\frac{1}{2}} + C\kappa_1^{m+\frac{d}{2}}(2\mu^{2m+1}\rho)^{\frac{1}{2\nu}}(M(D_d^m v)^{2\xi}(X_0))^{\frac{1}{2\xi}} \\
&\quad + C\kappa_1^{m+\frac{d}{2}}\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}\mu^{\alpha_d-2m}(M(\tilde f_\alpha)^2(X_0))^{\frac{1}{2}} \\
&\quad + C\kappa_1^{m+\frac{d}{2}}\mu^{-1}\big(M(D_d^m v)^2(X_0)\big)^{\frac{1}{2}} + C\kappa_1^{m+\frac{d}{2}}\big(M(D_{x'}D^{m-1}v)^2(X_0)\big)^{\frac{1}{2}}.
\end{aligned}
$$

By the Fefferman–Stein theorem, the Hardy–Littlewood maximal function theorem, and choosing $\xi > 1$ satisfying $2\xi < p$, we get

$$
\begin{aligned}
&\|D_d^m v\|_{L_p} \le C\|(D_d^m v)^{\#}\|_{L_p} \\
&\le C\kappa_1^{-1}\|D_d^m v\|_{L_p} + C\kappa_1^{m+\frac{d}{2}}(2\mu^{2m+1}\rho)^{\frac{1}{2\nu}}\|D_d^m v\|_{L_p} \\
&\quad + C\kappa_1^{m+\frac{d}{2}}\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}}\mu^{\alpha_d-2m}\|\tilde f_\alpha\|_{L_p} + C\kappa_1^{m+\frac{d}{2}}\mu^{-1}\|D_d^m v\|_{L_p} \\
&\quad + C\kappa_1^{m+\frac{d}{2}}\|D_{x'}D^{m-1}v\|_{L_p},
\end{aligned}
$$

where $L_p = L_p(\mathcal{O}^+_{\mu^{2m}T})$. We first let κ_1 be sufficiently large, then μ be sufficiently large, and finally ρ be sufficiently small so that

$$
C\kappa_1^{-1} + C\kappa_1^{m+\frac{d}{2}}(2\mu^{2m+1}\rho)^{\frac{1}{2\nu}} + C\kappa_1^{m+\frac{d}{2}}\mu^{-1} \le 1/2.
$$

We then obtain

$$\|D_d^m v\|_{L_p} \le C \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \|\tilde{f}_\alpha\|_{L_p} + C\|D_{x'}D^{m-1}v\|_{L_p},$$

where $L_p = L_p(\mathcal{O}^+_{\mu^{2m}T})$. After changing back to u and f_α, we get

$$\|D_d^m u\|_{L_p(\mathcal{O}_T^+)} \le C \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \|f_\alpha\|_{L_p(\mathcal{O}_T^+)} + C\|D_{x'}D^{m-1}u\|_{L_p(\mathcal{O}_T^+)}.$$

Combining the inequality above, the interpolation inequalities, and (3.4.21), we know that

$$\begin{aligned}
&\sum_{k=0}^m \lambda^{\frac{1}{2}-\frac{k}{2m}} \|D^k u\|_{L_p(\mathcal{O}_T^+)} \le C\|D^m u\|_{L_p(\mathcal{O}_T^+)} + C\lambda^{\frac{1}{2}}\|u\|_{L_p(\mathcal{O}_T^+)} \\
&\le C\|D_{x'}D^{m-1}u\|_{L_p(\mathcal{O}_T^+)} + C\lambda^{\frac{1}{2}}\|u\|_{L_p(\mathcal{O}_T^+)} + C\|D_d^m u\|_{L_p(\mathcal{O}_T^+)} \\
&\le C\kappa^{m+\frac{d}{2}} \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}-\frac{1}{2}} \|f_\alpha\|_{L_p(\mathcal{O}_T^+)} + C(\kappa^{-\gamma} + \kappa^{m+\frac{d}{2}}\rho^{\frac{1}{2\nu}}) \sum_{k=0}^m \lambda^{\frac{1}{2}-\frac{k}{2m}} \|D^k u\|_{L_p(\mathcal{O}_T^+)}.
\end{aligned}$$

We take κ sufficiently large and ρ sufficiently small so that the terms involving u on the right-hand side are absorbed in the left-hand side. The desired estimate then follows.

Due to the argument of partition of unity, we can remove the assumption that u vanishes on $\mathcal{O}_T^+ \backslash \tilde{Q}^+_{R_0}(X_1)$ for some $X_1 \in \overline{\mathcal{O}^+_\infty}$ by choosing λ_0 large enough.

Case 2: $p \in (1, 2)$. Since the system is in the divergence form, this case follows from the previous case by using the duality argument.

Case 3: $p = 2$. Denote λ_1 and λ_2 to be the constant λ_0 in Theorem 3.2.2 corresponding to $p = 3/2$ and $p = 3$, respectively. Let $\lambda_3 = \max\{\lambda_1, \lambda_2\}$. When $p = 3/2, 3$, (3.2.6) is proved for any $\lambda \ge \lambda_3$. Applying the Riesz–Thorin theorem, we then obtain (3.2.6) for $p = 2$ and $\lambda \ge \lambda_3$. Therefore, the proof is completed. $\square$

3.5 Schauder estimates under the Dirichlet boundary condition

3.5.1 Estimates for divergence type systems

This section is devoted to the interior and boundary a priori estimates for the divergence type higher-order systems under under the *Dirichlet boundary condition*. We assume the leading coefficients satisfy the Legendre-Hadamard ellipticity condition.

3.5.1.1 Interior estimates

The main result of this subsection is the following proposition.

Proposition 3.5.1. *Let $a \in (0,1)$. Assume that $R \le 1$ and $u \in C^\infty_{loc}(\mathbb{R}^{d+1})$ satisfying*

$$u_t + (-1)^m \mathcal{L}u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad in \quad Q_{2R}. \tag{3.5.1}$$

Suppose that $A^{\alpha\beta} \in C^a_x$, $f_\alpha \in C^a_x$ if $|\alpha| = m$, and $f_\alpha \in L_\infty$ if $|\alpha| < m$. Then there exists a constant C depending only on d, n, m, λ, K, $\|A^{\alpha\beta}\|^x_a$, R, and a such that

$$[D^m u]_{\frac{a}{2m}, a, Q_R} \le C\Big(\sum_{|\beta| \le m} \|D^\beta u\|_{L_\infty(Q_{2R})} + F \Big), \tag{3.5.2}$$

$$[u]^t_{\frac{1}{2} + \frac{a}{2m}, Q_R} \le C\Big(\sum_{|\beta| \le m} \|D^\beta u\|_{L_\infty(Q_{2R})} + F \Big), \tag{3.5.3}$$

where

$$F = \sum_{|\alpha|=m} [f_\alpha]^x_{a,Q_{2R}} + \sum_{|\alpha|<m} \|f_\alpha\|_{L_\infty(Q_{2R})}.$$

For the proof of the proposition, we first consider homogeneous systems with the simple coefficients.

Lemma 3.5.2. *Assume that $u \in C^\infty_{loc}(\mathbb{R}^{d+1})$ and satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \quad \text{in} \quad Q_{2R}, \tag{3.5.4}$$

where $R \in (0, \infty)$. Then for any $X_0 \in Q_R$ and $r < R$, There is a constant C depending only on d, n, m, and λ such that

$$\begin{aligned}&\int_{Q_r(X_0)} |u - (u)_{Q_r(X_0)}|^2 \, dx\, dt \\ &\le C(\frac{r}{R})^{2+2m+d} \int_{Q_R(X_0)} |u - (u)_{Q_R(X_0)}|^2 \, dx\, dt.\end{aligned} \tag{3.5.5}$$

Proof. By scaling and translation of the coordinates, without loss of generality, we can assume that $R = 1$ and $X_0 = (0, 0)$. First we consider the case $r \le 1/4$. Since

$$\int_{Q_r} |u - (u)_{Q_r}|^2 \, dx\, dt \le Cr^{2+2m+d} \sup_{Q_r} |Du|^2 + Cr^{4m+2m+d} \sup_{Q_r} |u_t|^2. \tag{3.5.6}$$

By Lemma 2 in [37],

$$\sup_{Q_{1/4}} (|Du| + |u_t|) \le C\|u\|_{L_2(Q_1)}.$$

Hence, by scaling and (3.5.6),

$$\int_{Q_r} |u - (u)_{Q_r}|^2 \, dx\, dt \le Cr^{2+2m+d} \|u\|^2_{L_2(Q_1)}.$$

Clearly, the inequality above holds true in the case when $r \in [1/4, 1)$ as well. Since

$u - (u)_{Q_1}$ also satisfies (3.5.4), upon substituting u by $u - (u)_{Q_1}$, the lemma is proved. $\square$

Lemma 3.5.3. *Let $a \in (0,1)$. Assume that $r < R \le 1$ and $u \in C^\infty_{loc}(\mathbb{R}^{d+1})$ satisfying*

$$u_t + (-1)^m \mathcal{L}_0 u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad in \quad Q_{2R},$$

$f_\alpha \in C^a_x$ if $|\alpha| = m$, and $f_\alpha \in L_\infty$ if $|\alpha| < m$. Then for any $X_0 \in Q_R$, there exists a constant C depending only on n, m, d, and λ such that

$$\begin{aligned}
&\int_{Q_r(X_0)} |D^m u - (D^m u)_{Q_r(X_0)}|^2 \, dx \, dt \\
&\le C(\frac{r}{R})^{2+2m+d} \int_{Q_R(X_0)} |D^m u - (D^m u)_{Q_R(X_0)}|^2 \, dx \, dt \\
&+ C \sum_{|\alpha| = m} [f_\alpha]^{x2}_{a,Q_{2R}} R^{2a+2m+d} + C \sum_{|\alpha| < m} \|f_\alpha\|^2_{L_\infty(Q_{2R})} R^{2+2m+d}.
\end{aligned}$$

Applying Lemmas 3.3.1 and 3.5.3, we get the Campanato type estimate

$$\begin{aligned}
&\int_{Q_r(X_0)} |D^m u - (D^m u)_{Q_r(X_0)}|^2 \, dx \, dt \\
&\le C(\frac{r}{R})^{2a+d+2m} \int_{Q_R(X_0)} |D^m u - (D^m u)_{Q_R(X_0)}|^2 \, dx \, dt \\
&+ C\Big(\sum_{|\alpha| = m} [f_\alpha]^{x2}_{a,Q_{2R}} + \sum_{|\alpha| < m} \|f_\alpha\|^2_{L_\infty(Q_{2R})} \Big) r^{2a+d+2m}
\end{aligned}$$

for any $r < R \le 1$ and $X_0 \in Q_R$. Then,

$$\begin{aligned}
&\frac{1}{r^{2a+d+2m}} \int_{Q_r(X_0)} |D^m u - (D^m u)_{Q_r(X_0)}|^2 \, dx \, dt \\
&\le \frac{C}{R^{2a+d+2m}} \int_{Q_R(X_0)} |D^m u - (D^m u)_{Q_R(X_0)}|^2 \, dx \, dt + CF^2 \\
&\le \frac{C}{R^{2a+2m+d}} \int_{Q_{2R}} |D^m u|^2 \, dx \, dt + CF^2.
\end{aligned}$$

Thanks to Lemma 3.3.2, we get

$$[D^m u]_{\frac{a}{2m},a,Q_R} \le C\Big(R^{-(2a+d+2m)}\int_{Q_{2R}}|D^m u|^2\,dx\,dt + F^2\Big)^{\frac{1}{2}}. \tag{3.5.7}$$

As for the coefficients which also depend on x, the estimates follow from the standard argument of freezing the coefficients. Specifically, we first consider the operator $\mathcal{L}$ that only has the highest-order terms. We fix a $y \in B_R$, and define

$$\mathcal{L}_{0y} = \sum_{|\alpha|=|\beta|=m} D^\alpha(A^{\alpha\beta}(t,y)D^\beta).$$

Then

$$u_t + (-1)^m\mathcal{L}_{0y}u = \sum_{|\alpha|\le m} D^\alpha f_\alpha - (-1)^m(\mathcal{L}-\mathcal{L}_{0y})u = \sum_{|\alpha|\le m} D^\alpha \tilde{f}_\alpha,$$

where

$$\tilde{f}_\alpha = f_\alpha + (-1)^m\Big(\sum_{|\beta|=m}(A^{\alpha\beta}(t,y)-A^{\alpha\beta}(t,x))D^\beta u\Big) \qquad \text{if} \quad |\alpha| = m,$$

$$\tilde{f}_\alpha = f_\alpha \quad \text{if} \quad |\alpha| < m.$$

Applying the method in proving Lemma 3.5.3, we obtain

$$\|D^m w\|^2_{L_2(Q_{2R})} \le C\Big(\sum_{|\alpha|=|\beta|=m}[A^{\alpha\beta}]^{x2}_a\|D^m u\|^2_{L_\infty(Q_{2R})}R^{2a+2m+d} + \sum_{|\alpha|<m}\|f_\alpha\|^2_{L_\infty(Q_{2R})}R^{2+2m+d} + \sum_{|\alpha|=m}[f_\alpha]^{x2}_{a,Q_{2R}}R^{2a+2m+d}\Big).$$

Following the proof of (3.5.7), we reach

$$[D^m u]_{\frac{a}{2m},a,Q_R} \le C\Big(R^{-(2a+d+2m)}\int_{Q_{2R}} |D^m u|^2\,dx\,dt \\ + \sum_{|\alpha|=|\beta|=m} [A^{\alpha\beta}]_a^{x2}\|D^m u\|^2_{L_\infty(Q_{2R})} + F^2\Big)^{\frac{1}{2}}. \quad (3.5.8)$$

For systems with lower-order terms, we move all the lower-order terms to the right-hand side to get

$$u_t + (-1)^m \mathcal{L}_h u = \sum_{|\alpha|=m} D^\alpha\Big(f_\alpha + (-1)^{m+1}\sum_{|\beta|<m} A^{\alpha\beta}D^\beta u\Big) \\ + \sum_{|\alpha|<m} D^\alpha\Big(f_\alpha + (-1)^{m+1}\sum_{|\beta|\le m} A^{\alpha\beta}D^\beta u\Big),$$

where $\mathcal{L}_h u$ denotes the sum of the highest-order terms. It suffices to substitute $\hat{f}_\alpha = f_\alpha + (-1)^{m+1}\sum_\beta A^{\alpha\beta}D^\beta u$ into the estimate (3.5.8) and notice that when $|\alpha| = m$

$$\sum_{|\beta|<m} [A^{\alpha\beta}D^\beta u]^x_{a,Q_{2R}} \\ \le C\sum_{|\beta|<m}\Big(\|A^{\alpha\beta}\|_{L_\infty}[D^\beta u]^x_{a,Q_{2R}} + [A^{\alpha\beta}]^x_a\|D^\beta u\|_{L_\infty(Q_{2R})}\Big).$$

An easy calculation then completes the proof of (3.5.2).

It remains to prove (3.5.3). We estimate $|u(t,x_0) - u(s,x_0)|$, where (t,x_0), $(s,x_0) \in Q_R$. Let $\rho = |t-s|^{\frac{1}{2m}}$ and $\eta(x) = \frac{1}{\rho^d}\xi(\frac{x}{\rho})$, where $\xi(x)$ is the function in Lemma 3.3.3 with $\Omega = B_1$. We define

$$\overline{u}(t,y) = u(t,y) - T_{m,x_0}u(t,y) + u(t,x_0),$$

where $T_{m,x_0}u(t,y)$ is the Taylor expansion of $u(t,y)$ in y at x_0 up to mth order. By the triangle inequality,

$$
\begin{aligned}
\Big|u(t,x_0)-u(s,x_0)\Big| &\le \Big|u(t,x_0)-\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(t,y)\,dy\Big| \\
&+\Big|u(s,x_0)-\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(s,y)\,dy\Big| \\
&+\Big|\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(s,y)\,dy-\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(t,y)\,dy\Big|. \qquad (3.5.9)
\end{aligned}
$$

The first two terms on the right-hand side of (3.5.9) can be estimated in a similar fashion: noting that $\int_{B_\rho}\eta(x)\,dx=1$,

$$
\begin{aligned}
&\Big|u(t,x_0)-\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(t,y)\,dy\Big| \\
&=\Big|\int_{B_\rho(x_0)}\eta(y-x_0)(u(t,x_0)-\overline{u}(t,y))\,dy\Big|.
\end{aligned}
$$

Since

$$
|\overline{u}(t,y)-u(t,x_0)|=|u(t,y)-T_{m,x_0}u(t,y)|\le C[D^m u]^x_{a,Q_{2R}}|y-x_0|^{m+a},
$$

we get

$$
\Big|u(t,x_0)-\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(t,y)\,dy\Big|\le C[D^m u]^x_{a,Q_{2R}}\rho^{m+a}. \qquad (3.5.10)
$$

For the last term on the right-hand side of (3.5.9), by the definition of η,

$$
\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(s,y)\,dy=\int_{B_\rho(x_0)}\eta(y-x_0)u(s,y)\,dy.
$$

Therefore,

$$
\begin{aligned}
&\Big|\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(t,y)\,dy-\int_{B_\rho(x_0)}\eta(y-x_0)\overline{u}(s,y)\,dy\Big|\\
&=\Big|\int_{B_\rho(x_0)}\eta(y-x_0)u(t,y)\,dy-\int_{B_\rho(x_0)}\eta(y-x_0)u(s,y)\,dy\Big|\\
&=\Big|\int_s^t\int_{B_\rho(x_0)}\eta(y-x_0)u_t(\tau,y)\,dy\,d\tau\Big|.
\end{aligned}
$$

Because u satisfies (3.5.1),

$$
\begin{aligned}
&\Big|\int_s^t\int_{B_\rho(x_0)}\eta(y-x_0)u_t(\tau,y)\,dy\,d\tau\Big|\\
&=\Big|\int_s^t\int_{B_\rho(x_0)}\eta(y-x_0)\big((-1)^{m+1}\sum_{|\alpha|\le m,|\beta|\le m}D^\alpha(A^{\alpha\beta}D^\beta u(\tau,y))\\
&\quad+\sum_{|\alpha|\le m}D^\alpha f_\alpha\big)\,dy\,d\tau\Big|.
\end{aligned}
\tag{3.5.11}
$$

Since $\eta(y-x_0)$ has compact support in $B_\rho(x_0)$, we use $f_\alpha-(f_\alpha)_{B_\rho(x_0)}$ to substitute f_α when $|\alpha|=m$. Moreover, for $|\alpha|=m$,

$$
\begin{aligned}
&D^\alpha(A^{\alpha\beta}D^\beta u)\\
&=D^\alpha\big((A^{\alpha\beta}-(A^{\alpha\beta})_{B_\rho(x_0)})D^\beta u\big)+D^\alpha\big((A^{\alpha\beta})_{B_\rho(x_0)}(D^\beta u-(D^\beta u)_{B_\rho(x_0)})\big).
\end{aligned}
$$

We plug these into (3.5.11) and integrate by parts. It follows easily that

$$
\begin{aligned}
&\Big|\int_s^t\int_{B_\rho(x_0)}\eta(y-x_0)u_t(\tau,y)\,dy\,d\tau\Big|\\
&\le C\sum_{|\beta|\le m}\|D^\beta u\|^x_{a,Q_{2R}}\rho^{m+a}+C\sum_{|\alpha|=m}[f_\alpha]^x_{a,Q_{2R}}\rho^{m+a}\\
&\quad+C\sum_{|\alpha|<m}\Big(\|f_\alpha\|_{L_\infty(Q_{2R})}+\sum_{|\beta|\le m}\|A^{\alpha\beta}\|_{L_\infty}\|D^\beta u\|_{L_\infty(Q_{2R})}\Big)\rho^{m+1},
\end{aligned}
$$

where we use the fact that $\|D^\alpha\eta\|_{L_\infty}\le C\rho^{-d-|\alpha|}$ for any $|\alpha|\le m$. Hence, combining with (3.5.10), we reach

$$|u(t,x_0)-u(s,x_0)| \le C\rho^{m+a}\Big(\sum_{|\beta|\le m}\|D^\beta u\|^x_{a,Q_{2R}}+F\Big),$$

which, together with (3.5.2) and the interpolation inequality, implies (3.5.3). This completes the proof of the proposition.

3.5.1.2 Boundary estimates

As in the previous subsection, we first consider systems with the simple coefficients without lower-order terms:

$$u_t+(-1)^m\mathcal{L}_0 u=0 \quad \text{in} \quad Q^+_{2R}, \tag{3.5.12}$$

$$u=0,\ D_d u=0,\ \ldots,\ D_d^{m-1}u=0 \quad \text{on} \quad \{x_d=0\}\cap Q_{2R}. \tag{3.5.13}$$

Thanks to the L_p estimates in a half space obtained in [37], we are able to derive a local $W^{1,2m}_p$ estimate for solution of (3.5.12)–(3.5.13) by the method in Lemma 1 in [37]. Namely,

$$\|u\|_{W^{1,2m}_p(Q^+_R)} \le C(R,p)\|u\|_{L_p(Q^+_{2R})}, \tag{3.5.14}$$

where $C(R,p)$ is a constant. Now we are ready to prove the following lemma.

Lemma 3.5.4. *Assume that $0<r\le R<\infty$, $\gamma\in(0,1)$, and $u\in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ satisfying (3.5.12)–(3.5.13). Then there exists a constant C depending only on d, n,*

m, λ, K, and γ, such that

$$\int_{Q_r^+(X_0)} |u|^2\,dx\,dt \le C(\frac{r}{R})^{d+4m} \int_{Q_R^+(X_0)} |u|^2\,dx\,dt, \tag{3.5.15}$$

$$\int_{Q_r^+(X_0)} |D_d^l u|^2\,dx\,dt \le C(\frac{r}{R})^{2(m-l)+d+2m} \int_{Q_R^+(X_0)} |D_d^l u|^2\,dx\,dt, \tag{3.5.16}$$

$$\begin{aligned} &\int_{Q_r^+(X_0)} |D_d^m u - (D_d^m u)_{Q_r^+(X_0)}|^2\,dx\,dt \\ &\le C(\frac{r}{R})^{2\gamma+d+2m} \int_{Q_R^+(X_0)} |D_d^m u - (D_d^m u)_{Q_R^+(X_0)}|^2\,dx\,dt, \end{aligned} \tag{3.5.17}$$

where $X_0 \in \{x_d = 0\} \cap Q_R$ and $0 < l < m$.

Proof. By scaling and translation of the coordinates, without loss of generality, we can assume $R = 1$ and $X_0 = (0,0)$. From (3.5.14), for any $r_1 \in [1,2)$, there exists a constant $C = C(r_1)$ so that

$$\|u\|_{W_2^{1,2m}(Q_{r_1}^+)} \le C\|u\|_{L_2(Q_2^+)}.$$

By the Sobolev embedding theorem,

$$\|u\|_{L_p(Q_{r_1}^+)} \le C\|u\|_{W_2^{1,2m}(Q_{r_1}^+)},$$

where $1/p > 1/2 - 1/(d+1)$. Using (3.5.14) again, we get

$$\|u\|_{W_p^{1,2m}(Q_{r_2}^+)} \le C\|u\|_{L_p(Q_{r_1}^+)} \le C\|u\|_{L_2(Q_2^+)},$$

where $r_2 < r_1$ and $C = C(r_1, r_2)$. Using a standard argument of bootstrap with a sequence of shrinking cylinders Q_{r_l}, for any $p > 2$ there is a C depending on p such that

$$\|u\|_{W_p^{1,2m}(Q_1^+)} \le C\|u\|_{L_2(Q_2^+)}. \tag{3.5.18}$$

By the Sobolev embedding theorem, we have

$$\begin{aligned} &W_p^{1,2m}(Q_1^+) \hookrightarrow C^{\frac{a}{2m},a}(Q_1^+),\\ &\|u\|_{\frac{a}{2m},a,Q_1^+} \le C\|u\|_{W_p^{1,2m}(Q_1^+)}, \end{aligned} \tag{3.5.19}$$

where $a = 2m-(d+2m)/p$. Since $p<\infty$ can be arbitrarily large, a can be arbitrarily close to $2m$ from below.

Let us first prove (3.5.15). We only need to consider $r \in (0, 1/2]$, otherwise it is obvious. By the Poincaré inequality and (3.5.19) with some $a > m$,

$$\begin{aligned} \int_{Q_r^+} |u|^2\,dx\,dt &\le Cr^{2m}\|D_d^m u\|^2_{L_\infty(Q_r^+)}|Q_r^+| \le Cr^{d+4m}\|D_d^m u\|^2_{L_\infty(Q_{1/2}^+)}\\ &\le Cr^{d+4m}\|u\|^2_{W_p^{1,2m}(Q_{1/2}^+)} \le Cr^{d+4m}\int_{Q_1^+}|u|^2\,dx\,dt. \end{aligned}$$

The last inequality is due to the L_p estimate in (3.5.18) with $1/2$ and 1 in place of 1 and 2, respectively. Next, we prove (3.5.16) and only consider the case $r \in (0, 1/2]$ as in the proof of (3.5.15). By the Poincaré inequality and (3.5.19) with some $a > m$, we have

$$\begin{aligned} &\int_{Q_r^+} |D_d^l u|^2\,dx\,dt \le r^{2(m-l)}\int_{Q_r^+}|D_d^m u|^2\,dx\,dt\\ &\le Cr^{2(m-l)+d+2m}\sup_{Q_{1/2}^+}|D_d^m u|^2 \le Cr^{2(m-l)+d+2m}\|u\|^2_{W_p^{1,2m}(Q_{1/2}^+)}\\ &\le Cr^{2(m-l)+d+2m}\int_{Q_1^+}|u|^2\,dx\,dt \le Cr^{2(m-l)+d+2m}\int_{Q_1^+}|D_d^l u|^2\,dx\,dt. \end{aligned}$$

Finally for (3.5.17), let $v = u - \frac{x_d^m}{m!}(D_d^m u)_{Q_1^+}$ so that v also satisfies the same system as u. Moreover, $D_d^m v = D_d^m u - (D_d^m u)_{Q_1^+}$. Then, by (3.5.19), (3.5.18), and the

Poincaré inequality, for $a = m + \gamma \in (m, m+1)$ and $r \in (0, 1/2]$,

$$\begin{aligned} \int_{Q_r^+} |D_d^m v - (D_d^m v)_{Q_r^+}|^2 \, dx \, dt &\le C r^{2(a-m)+d+2m} \|v\|^2_{\frac{a}{2m}, a, Q_r^+} \\ &\le C r^{2\gamma+d+2m} \int_{Q_1^+} |v|^2 \, dx \, dt \le C r^{2\gamma+d+2m} \int_{Q_1^+} |D_d^m v|^2 \, dx \, dt. \end{aligned}$$

The lemma is proved. □

Similar to Lemma 3.5.3, we consider

$$u_t + (-1)^m \mathcal{L}_0 u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad \text{in} \quad Q_{2R}^+ \tag{3.5.20}$$

and the boundary condition (3.5.13).

Lemma 3.5.5. *Let $a \in (0, 1)$. Assume that $u \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ satisfying (3.5.20) and (3.5.13), $f_\alpha \in C^{ax}$ for $|\alpha| = m$, and $f_\alpha \in L_\infty$ for $|\alpha| < m$. Then for any $0 < r < R \le 1, \gamma \in (0, 1)$, and $X_0 \in \{x_d = 0\} \cap Q_R$, there exists a constant C depending only on d, m, n, λ, K, and γ such that*

$$\begin{aligned} &\int_{Q_r^+(X_0)} |D_{x'}^m u|^2 \, dx \, dt \\ &\le C(\frac{r}{R})^{4m+d} \int_{Q_R^+(X_0)} |D_{x'}^m u|^2 \, dx \, dt + C \sum_{|\alpha|=m} [f_\alpha]^{x2}_{a, Q_{2R}^+} R^{2a+2m+d} \\ &+ C \sum_{|\alpha|<m} \|f_\alpha\|^2_{L_\infty(Q_{2R}^+)} R^{2+2m+d}, \end{aligned} \tag{3.5.21}$$

$$\begin{aligned} &\int_{Q_r^+(X_0)} |D_d^m u - (D_d^m u)_{Q_r^+(X_0)}|^2 \, dx \, dt \\ &\le C(\frac{r}{R})^{2\gamma+d+2m} \int_{Q_R^+(X_0)} |D_d^m u - (D_d^m u)_{Q_R^+(X_0)}|^2 \, dx \, dt \\ &+ C \sum_{|\alpha|=m} [f_\alpha]^{x2}_{a, Q_{2R}^+} R^{2a+2m+d} + C \sum_{|\alpha|<m} \|f_\alpha\|^2_{L_\infty(Q_{2R}^+)} R^{2+2m+d}, \end{aligned} \tag{3.5.22}$$

and

$$\int_{Q_r^+(X_0)} |D_d^l D_{x'}^{m-l} u|^2 \, dx \, dt$$
$$\leq C(\frac{r}{R})^{2(m-l)+d+2m} \int_{Q_R^+(X_0)} |D_d^l D_{x'}^{m-l} u|^2 \, dx \, dt$$
$$+ C \sum_{|\alpha|=m} [f_\alpha]^{x2}_{a,Q_{2R}^+} R^{2a+2m+d} + C \sum_{|\alpha|<m} \|f_\alpha\|^2_{L_\infty(Q_{2R}^+)} R^{2+2m+d}, \tag{3.5.23}$$

where $0 < l < m$.

Now we are ready to prove the boundary Hölder estimates similar to the interior case. From (3.5.21), (3.5.22), and (3.5.23), taking $\gamma > a$ and using Lemma 3.3.1, we get that for any $X_0 \in \{x_d = 0\} \cap Q_R$ and $r \in (0, R)$,

$$\int_{Q_r^+(X_0)} |D_{x'}^m u|^2 + |D_d^m u - (D_d^m u)_{Q_r^+(X_0)}|^2 + \sum_{l=1}^{m-1} |D_d^l D_{x'}^{m-l} u|^2 \, dx \, dt \leq I,$$

where

$$I := C\Big(R^{-(2a+2m+d)} \int_{Q_{2R}^+} |D^m u|^2 \, dx \, dt + F^2\Big) r^{2a+2m+d},$$
$$F = \sum_{|\alpha|=m} [f_\alpha]^x_{a,Q_{2R}^+} + \sum_{|\alpha|<m} \|f_\alpha\|_{L_\infty(Q_{2R}^+)}.$$

This estimate, together with the interior estimates and Lemma 3.3.2, implies that

$$[D^m u]_{\frac{a}{2m},a,Q_R^+} \leq C\Big(R^{-(2a+2m+d)} \int_{Q_{2R}^+} |D^m u|^2 \, dx \, dt + F^2\Big)^{\frac{1}{2}}.$$

Similar to the proof of (3.5.2), we can apply the argument of freezing the coefficients to deal with the general operator $\mathcal{L}$ with lower-order terms. Here we just state the conclusion as the following proposition.

Proposition 3.5.6. *Assume that $R \le 1$ and $u \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ satisfying*

$$\begin{cases} u_t + (-1)^m \mathcal{L} u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad \text{in} \quad Q^+_{2R}, \\ u = 0,\ D_d u = 0,\ \ldots,\ D_d^{m-1} u = 0 \quad \text{on} \quad \{x_d = 0\} \cap Q_{2R}. \end{cases}$$

Suppose that $\mathcal{L}$ and f_α satisfy the conditions in Theorem 3.2.3. Then there exists a constant C depending only on d, n, m, λ, K, $\|A^{\alpha\beta}\|^x_a$, R, and a such that

$$[u]^t_{\frac{1}{2}+\frac{a}{2m}, Q^+_R} + [D^m u]_{\frac{a}{2m}, a, Q^+_R} \le C\Big(\sum_{|\beta| \le m} \|D^\beta u\|_{L_\infty(Q^+_{2R})} + F \Big).$$

3.5.1.3 Proof of Theorem 3.2.3

Before proving the main theorem, let us state a technical lemma.

Lemma 3.5.7. *Assume that $a \in (0,1)$, $u \in C^\infty_{loc}(\mathbb{R}^{d+1})$, Ω is an open set in $\mathbb{R}^d$, $\partial\Omega \in C^2$, and $T \in (0, \infty]$. Then for any $\varepsilon > 0$ sufficiently small, we have*

$$\|u\|_{L_\infty((0,T)\times\Omega)} \le \varepsilon [u]_{\frac{a}{2m}, a, (0,T)\times\Omega} + C(\varepsilon) \|u\|_{L_2((0,T)\times\Omega)},$$

where $C(\varepsilon)$ is a constant depending on ε.

Next, we show a global a priori estimate.

Proposition 3.5.8. *Let $\mathcal{L}$, Ω, and f_α satisfy the conditions in Theorem 3.2.3. As-*

sume that $u \in C^\infty((0,T)\times\Omega)$ *and satisfies the following system*

$$\begin{cases} u_t + (-1)^m \mathcal{L} u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad in \quad (0,T)\times\Omega, \\ |u| = 0,\ |Du| = 0,\ \ldots,\ |D^{m-1}u| = 0 \quad on \quad [0,T)\times\partial\Omega, \\ u = 0 \quad on \quad \{0\}\times\Omega. \end{cases} \tag{3.5.24}$$

Then (3.2.10) *holds with* $G = 0$.

Proof. Using the standard arguments of partition of the unity and flattening the boundary, we combine the interior and boundary estimates to get

$$[u]^t_{\frac{a+m}{2m},(0,T)\times\Omega} + [D^m u]_{\frac{a}{2m},a,(0,T)\times\Omega} \le C\Big(\sum_{|\beta|\le m} \|D^\beta u\|_{L_\infty((0,T)\times\Omega)} + F\Big).$$

By the interpolation inequalities in Hölder spaces, for instance, see [63, Section 8.8],

$$[u]^t_{\frac{a+m}{2m},(0,T)\times\Omega} + [D^m u]_{\frac{a}{2m},a,(0,T)\times\Omega} \le C\big(\|u\|_{L_\infty((0,T)\times\Omega)} + F\big).$$

Applying Lemma 3.5.7 and the interpolation inequalities again, we get

$$\begin{aligned} &\|u\|_{L_\infty((0,T)\times\Omega)} \\ &\le C(\varepsilon)\|u\|_{L_2((0,T)\times\Omega)} + \varepsilon[D^m u]_{\frac{a}{2m},a,(0,T)\times\Omega} + \varepsilon[u]^t_{\frac{a+m}{2m},(0,T)\times\Omega}. \end{aligned}$$

Upon taking ε sufficiently small, we arrive at (3.2.10). The proposition is proved. □

In order to implement the method of continuity, we need the right-hand side of (3.2.10) to be independent of u and this leads us to consider the following system for $T < \infty$:

$$\begin{cases} u_t + (-1)^m \mathcal{L} u + \kappa u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad \text{in} \quad (0,T) \times \Omega, \\ |u| = 0, \ |Du| = 0, \ldots, |D^{m-1} u| = 0 \quad \text{on} \quad [0,T) \times \partial\Omega, \\ u = 0 \quad \text{on} \quad \{0\} \times \Omega, \end{cases} \tag{3.5.25}$$

where κ is a large constant to be specified later. We rewrite the system as

$$u_t + (-1)^m \mathcal{L}_h u + \kappa u = \sum_{|\alpha| \le m} D^\alpha f_\alpha + (-1)^m (\mathcal{L}_h - \mathcal{L}) u, \tag{3.5.26}$$

where $\mathcal{L}_h u$ is the sum of the highest-order terms. Then multiply u to both sides of (3.5.26) and integrate over $(0,T) \times \Omega$. Thus

$$\begin{aligned} &\lambda \|D^m u\|^2_{L_2((0,T)\times\Omega)} + \kappa \|u\|^2_{L_2((0,T)\times\Omega)} \\ &\le \sum_{|\alpha| \le m} (-1)^{|\alpha|} \int_{(0,T)\times\Omega} f_\alpha D^\alpha u \, dx \, dt \\ &\quad + \sum_{|\alpha| < m, |\beta| \le m} (-1)^{m+|\alpha|+1} \int_{(0,T)\times\Omega} A^{\alpha\beta} D^\beta u D^\alpha u \, dx \, dt \\ &\quad + \sum_{|\alpha| = m, |\beta| < m} (-1)^{m+|\alpha|+1} \int_{(0,T)\times\Omega} A^{\alpha\beta} D^\beta u D^\alpha u \, dx \, dt. \end{aligned} \tag{3.5.27}$$

Take a point $x_0 \in \Omega$. Due to the homogeneous Dirichlet boundary condition, if $|\alpha| = m$, the factor f_α in the first integral on the right-hand side above can be replaced by $f_\alpha(t,x) - f_\alpha(t,x_0)$. We use the Cauchy–Schwarz inequality, Young's inequality, and the interpolation inequality to bound the right-hand side by

$$\begin{aligned} C(n,m,d,\lambda,K,\varepsilon) \Big(&\sum_{|\alpha| \le m} \int_{(0,T)\times\Omega} |f_\alpha - f_\alpha(t,x_0) 1_{|\alpha|=m}|^2 \, dx \, dt \\ &+ \int_{(0,T)\times\Omega} |u|^2 \, dx \, dt \Big) + \varepsilon \|D^m u\|^2_{L_2((0,T)\times\Omega)}. \end{aligned}$$

After taking ε sufficiently small to absorb the term $\varepsilon\|D^m u\|^2_{L_2((0,T)\times\Omega)}$ to the left-hand side of (3.5.27) and choosing κ sufficiently large depending on n, m, d, λ, K, and ε, we reach

$$\|u\|_{L_2((0,T)\times\Omega)} \le C\sqrt{T}F, \tag{3.5.28}$$

where C depends only on d, n, m, λ, K, and Ω.

Now we are ready to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. By considering $u-g$ instead of u, without loss of generality, we can assume $g=0$. We only consider the case $T<\infty$, as the case when $T=\infty$ then follows by considering $\min(T,k)$ in place of T. It remains to prove the solvability. Let κ be the constant in the derivation of (3.6.21). Since u satisfies (3.5.24), the function $v := e^{-\kappa t}u$ satisfies

$$v_t + (-1)^m\mathcal{L}v + \kappa v = e^{-\kappa t}\sum_{|\alpha|\le m} D^\alpha f_\alpha.$$

By (3.6.21), we have

$$\|v\|_{L_2((0,T)\times\Omega)} \le C\sqrt{T}F,$$

which implies that

$$\|u\|_{L_2((0,T)\times\Omega)} \le C\sqrt{T}e^{\kappa T}F \le Ce^{2\kappa T}F.$$

The inequality above combined with (3.2.10) yields

$$\|u\|_{\frac{a+m}{2m},a+m,(0,T)\times\Omega} \le Ce^{2\kappa T}F, \tag{3.5.29}$$

which gives (3.2.8). Next we consider the following equation

$$u_t + (-1)^m\big(s\mathcal{L}u + (1-s)\Delta^m I_{n\times n}u\big) = \sum_{|\alpha|\le m} D^\alpha f_\alpha$$

with the same initial and boundary conditions, where the parameter $s \in [0,1]$. It is well known that when $s = 0$ there is a unique solution in $C^{\frac{a+m}{2m},a+m}([0,T)\times\Omega)$. Then by the method of continuity and the a priori estimate (3.5.29), we find a solution when $s = 1$. The theorem is proved. $\square$

Remark. Actually the condition $f_\alpha \in L_\infty$ for $|\alpha| < m$ can be relaxed. With slight modification in our proof, f_α can be in some Morrey spaces.

3.5.2 Estimate for non-divergence type systems

In this subsection, we deal with the non-divergence type systems under the *Dirichlet boundary conditions*.

3.5.2.1 Interior estimates

First, we consider the interior estimates for the non-divergence form system

$$u_t + (-1)^m L_0 u = f \quad \text{in} \quad Q_{2R}.$$

For this system, the interior estimates are simple consequences of the corresponding estimates for the divergence form systems. Indeed, we differentiate the system m

times with respect to x to get

$$D^m u_t + (-1)^m L_0 D^m u = D^m f.$$

Thanks to the estimates for the divergence type systems, by (3.5.7),

$$[D^{2m}u]_{\frac{a}{2m},a,Q_R} \le C\big(\|D^{2m}u\|_{L_\infty(Q_{2R})} + [f]^x_{a,Q_{2R}}\big).$$

Next we deal with the general non-divergence form systems with lower-order terms and coefficients depending on both t and x. The coefficients are all in C^a_x. Following exactly the same idea as in handling the divergence form systems, we first consider $L = \sum_{|\alpha|=|\beta|=m} A^{\alpha\beta} D^\alpha D^\beta$. Fix $y \in B_R^+$ and let

$$L_{0y} = \sum_{|\alpha|=|\beta|=m} A^{\alpha\beta}(t,y) D^\alpha D^\beta.$$

We rewrite the systems as follows

$$u_t + (-1)^m L_{0y} u = f + (-1)^m (L_{0y} - L)u.$$

Set ζ to be an infinitely differentiable function in $\mathbb{R}^{d+1}$ such that

$$\zeta = 1 \quad \text{in} \quad Q_{2R}, \quad \zeta = 0 \quad \text{outside} \quad (-(4R)^{2m}, (4R)^{2m}) \times B_{4R}. \tag{3.5.30}$$

From Theorem 2 and Remark 1 of [37], for $T = (4R)^{2m}$, there is a unique solution $w \in W_2^{1,2m}$ to the following system

$$w_t + (-1)^m L_{0y} w = \zeta\big(f - (f)_{B_{4R}(t)}\big) + \zeta(-1)^m (L_{0y} - L)u$$

in $(-T,0)\times\mathbb{R}^d$ with the zero initial condition at $t=-T$, and w satisfies

$$\begin{aligned}&\|D^{2m}w\|_{L_2(Q_{4R})}^2\le C\|f-(f)_{B_{4R}}(t)\|_{L_2(Q_{4R})}^2+C\|(L_{0y}-L)u\|_{L_2(Q_{4R})}^2\\&\le CR^{2a+2m+d}[f]_{a,Q_{4R}}^{x2}+CR^{2a+2m+d}\sum_{|\alpha|=|\beta|=m}[A^{\alpha\beta}]_a^{x2}\|D^{2m}u\|_{L_\infty(Q_{4R})}^2.\end{aligned}$$

We apply the mollification argument so that w is smooth. Let $v=u-w$ which is also a smooth function. Moreover, since $\zeta=1$ in Q_{2R}, $D^{2m}v$ satisfies (3.5.4). Therefore, (3.5.5) holds for $D^{2m}v$. Combining the estimate of w and $D^{2m}v$, similar to the proof of Proposition 3.5.1, we get

$$[D^{2m}u]_{\frac{a}{2m},a,Q_R}\le C(\|D^{2m}u\|_{L_\infty(Q_{4R})}+[f]_{a,Q_{4R}}^x). \tag{3.5.31}$$

For the operator L with the lower-order terms, we rewrite the systems as follows

$$u_t+(-1)^mL_hu=f+(-1)^{m+1}\sum_{|\alpha|+|\beta|<2m}A^{\alpha\beta}D^\alpha D^\beta u,$$

where $L_h=\sum_{|\alpha|=|\beta|=m}A^{\alpha\beta}D^\alpha D^\beta$. Let

$$\tilde f=f+(-1)^{m+1}\sum_{|\alpha|+|\beta|<2m}A^{\alpha\beta}D^\alpha D^\beta u.$$

An easy calculation shows that

$$\begin{aligned}[\tilde f]_{a,Q_{4R}}^x\le C\Big([f]_{a,Q_{4R}}^x+\sum_{|\alpha|+|\beta|<2m}(\|A^{\alpha\beta}\|_{L_\infty}[D^\alpha D^\beta u]_{a,Q_{4R}}^x\\+[A^{\alpha\beta}]_a^x\|D^\alpha D^\beta u\|_{L_\infty(Q_{4R})})\Big).\end{aligned}$$

From (3.5.31) with $\tilde{f}$ in place of f, the following estimate holds

$$[D^{2m}u]_{\frac{a}{2m},a,Q_R} \le C\Big(\sum_{|\gamma|\le 2m}\|D^\gamma u\|_{L_\infty(Q_{4R})} + [f]^x_{a,Q_{4R}}\Big),$$

where C depends only on d, m, n, λ, K, $\|A^{\alpha\beta}\|^x_a$, R, and a. Because

$$u_t = (-1)^{m+1}Lu + f,$$

it follows immediately that

$$\|u_t\|^x_{a,Q_R} + [D^{2m}u]_{\frac{a}{2m},a,Q_R} \le C\Big(\sum_{|\gamma|\le 2m}\|D^\gamma u\|_{L_\infty(Q_{4R})} + \|f\|^x_{a,Q_{4R}}\Big).$$

Implementing a standard interpolation argument, for instance, see [63, Section 8.8] and Lemma 5.1 of [48], we are able to prove the first part of Theorem 3.2.4.

3.5.2.2 Boundary estimates

In the non-divergence case, in order to prove the boundary Schauder estimates, better estimates than these in Lemma 3.5.4 are necessary. For (3.5.17), we actually can estimate more normal derivatives up to $2m-1$-th order. In order to show this, let us present the following lemma. A similar result can be found in Lemma 3.3 of [35].

Lemma 3.5.9. *Assume that $u \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ satisfying (3.5.12)–(3.5.13). Let $Q(x)$ be a vector-valued polynomial of order $m-1$ and $P(x) = x_d^m Q(x)$. Suppose that*

$$(D^k D_d^m P(x))_{Q_R^+} = (D^k D_d^m u(t,x))_{Q_R^+},$$

where $0 \leq k \leq m-1$*. Let* $v = u - P(x)$*, then there exists a constant* C *depending only on* d*,* m*,* n*, and* K *such that*

$$\|D^k D_d^m v\|_{L_2(Q_R^+)} \leq C R^{m-1-k} \|D^{m-1} D_d^m v\|_{L_2(Q_R^+)}$$

for any $0 \leq k \leq m-1$*.*

Next, we prove an estimate for $D^{2m-1}u$.

Lemma 3.5.10. *Assume that* $u \in C_{loc}^{\infty}(\overline{\mathbb{R}_+^{d+1}})$ *and satisfies* (3.5.12)–(3.5.13)*. Then for any* $0 < r \leq R < \infty$ *and* $\gamma \in (0,1)$*, there exists a constant* C *depending only on* d*,* n*,* m*,* λ*, and* γ *such that for any* $X_0 \in \{x_d = 0\} \cap Q_R$

$$\begin{aligned}
&\int_{Q_r^+(X_0)} |D^{2m-1}u - (D^{2m-1}u)_{Q_r^+(X_0)}|^2 \, dx \, dt \\
&\leq C(\frac{r}{R})^{2\gamma+d+2m} \int_{Q_R^+(X_0)} |D^{2m-1}u - (D^{2m-1}u)_{Q_R^+(X_0)}|^2 \, dx \, dt.
\end{aligned}$$

Proof. By scaling and translation of the coordinates, without loss of generality, we can assume $R = 1$ and $X_0 = (0,0)$. Moreover, we only need to consider the case $r \in (0, 1/2]$ for the same reason as in the proof of Lemmas 3.5.2 and 3.5.4. Recall that in Lemma 3.5.4 we have

$$\|u\|_{\frac{a}{2m}, a, Q_1^+} \leq C\|u\|_{W_p^{1,2m}(Q_1^+)} \leq C\|u\|_{L_2(Q_2^+)},$$

where $a < 2m$ can be arbitrarily close to $2m$. Set $a = \gamma + 2m - 1 \in (2m-1, 2m)$. Applying the inequality above with $1/2$ and 1 in place of 1 and 2, and the Poincaré inequality, we have

$$\begin{aligned}
&\int_{Q_r^+} |D^{2m-1}u - (D^{2m-1}u)_{Q_r^+}|^2 \, dx \, dt \\
&\le C r^{2\gamma+d+2m} \|u\|^2_{\frac{a}{2m}, a, Q_r^+} \le C r^{2\gamma+d+2m} \|u\|^2_{W_p^{1,2m}(Q_{1/2}^+)} \\
&\le C r^{2\gamma+d+2m} \|u\|^2_{L_2(Q_1^+)} \le C r^{2\gamma+d+2m} \|D_d^m u\|^2_{L_2(Q_1^+)}.
\end{aligned} \tag{3.5.32}$$

Let P and v be the functions in Lemma 3.5.9. It is easily seen that for a given function u, such P and thus v exist and are unique. Note that v also satisfies (3.5.12)–(3.5.13). Then the inequality (3.5.32) holds with v in place of u. Since $P(x)$ is a polynomial of degree $2m-1$, using Lemma 3.5.9, we have

$$\begin{aligned}
&\int_{Q_r^+} |D^{2m-1}u - (D^{2m-1}u)_{Q_r^+}|^2 \, dx \, dt \\
&= \int_{Q_r^+} |D^{2m-1}v - (D^{2m-1}v)_{Q_r^+}|^2 \, dx \, dt \\
&\le C r^{2\gamma+d+2m} \int_{Q_1^+} |D_d^m v|^2 \, dx \, dt \le C r^{2\gamma+d+2m} \int_{Q_1^+} |D^{m-1} D_d^m v|^2 \, dx \, dt \\
&= C r^{2\gamma+d+2m} \int_{Q_1^+} |D^{m-1} D_d^m u - (D^{m-1} D_d^m u)_{Q_1^+}|^2 \, dx \, dt \\
&\le C r^{2\gamma+d+2m} \int_{Q_1^+} |D^{2m-1}u - (D^{2m-1}u)_{Q_1^+}|^2 \, dx \, dt.
\end{aligned}$$

The lemma is proved. $\square$

Let us turn to the non-homogeneous systems. Consider a smooth solution $u \in C_{loc}^\infty(\overline{\mathbb{R}_+^{d+1}})$ of

$$\begin{cases}
u_t + (-1)^m L_0 u = f \quad \text{in} \quad Q_{4R}^+, \\
u = 0, \; D_d u = 0, \dots, D_d^{m-1} u = 0 \quad \text{on} \quad Q_{4R} \cap \{x_d = 0\}.
\end{cases}$$

We shall show the estimates of all derivatives up to order $2m$ with the exception of

$D_d^{2m}u$. Let ζ be the function in (3.5.30). For $T = (4R)^{2m}$, consider the following system

$$\begin{cases} w_t + (-1)^m L_0 w = \zeta\Big(f - (f)_{B^+_{4R}}(t)\Big) \quad \text{in} \quad (-T, 0) \times \mathbb{R}^d_+, \\ w = 0,\ D_d w = 0, \ldots,\ D_d^{m-1} w = 0 \quad \text{on} \quad (-T, 0) \times \{x_d = 0\}, \\ w = 0 \quad \text{on} \quad \{t = -T\}. \end{cases}$$

We know from Theorem 4 and Remark 1 of [37] that the system above has a unique $W_2^{1,2m}$-solution which satisfies the following estimate

$$\|D^{2m} w\|^2_{L_2(Q^+_{4R})} \leq C\|f - (f)_{B^+_{4R}}(t)\|^2_{L_2(Q^+_{4R})} \leq CR^{2a+2m+d}[f]^{x2}_{a,Q^+_{4R}}. \tag{3.5.33}$$

We again apply the mollification argument so that w is smooth. Observe that $v := u - w$ satisfies

$$\begin{cases} v_t + (-1)^m L_0 v = (1-\zeta)f + \zeta(f)_{B^+_{4R}}(t) \quad \text{in} \quad Q^+_{2R}, \\ v = 0,\ D_d v = 0, \ldots, D_d^{m-1} v = 0 \quad \text{on} \quad \{x_d = 0\} \cap Q_{2R}. \end{cases}$$

Since $\zeta = 1$ in Q^+_{2R}, $\tilde{v} := D_{x'} v$ satisfies

$$\begin{cases} \tilde{v}_t + (-1)^m L_0 \tilde{v} = 0 \quad \text{in} \quad Q^+_{2R}, \\ \tilde{v} = 0,\ D_d \tilde{v} = 0, \ldots, D_d^{m-1} \tilde{v} = 0 \quad \text{on} \quad \{x_d = 0\} \cap Q_{2R}. \end{cases} \tag{3.5.34}$$

Thanks to Lemma 3.5.10, for any $X_0 \in \{x_d = 0\} \cap Q_R$ and $r \in (0, R)$, we get

$$\begin{aligned} &\int_{Q^+_r(X_0)} |D_{x'} D^{2m-1} v - (D_{x'} D^{2m-1} v)_{Q^+_r(X_0)}|^2 \, dx \, dt \\ &\leq C(\frac{r}{R})^{2\gamma+d+2m} \int_{Q^+_R(X_0)} |D_{x'} D^{2m-1} v - (D_{x'} D^{2m-1} v)_{Q^+_R(X_0)}|^2 \, dx \, dt. \end{aligned} \tag{3.5.35}$$

Therefore, by the triangle inequality,

$$\int_{Q_r^+(X_0)} |D_{x'}D^{2m-1}u - (D_{x'}D^{2m-1}u)_{Q_r^+(X_0)}|^2\,dx\,dt$$
$$\leq C(\frac{r}{R})^{2\gamma+d+2m}\int_{Q_R^+(X_0)} |D_{x'}D^{2m-1}u - (D_{x'}D^{2m-1}u)_{Q_R^+(X_0)}|^2\,dx\,dt$$
$$+C\int_{Q_{2R}^+} |D^{2m}w|^2\,dx\,dt.$$

Combining with (3.5.33), we obtain

$$\int_{Q_r^+(X_0)} |D_{x'}D^{2m-1}u - (D_{x'}D^{2m-1}u)_{Q_r^+(X_0)}|^2\,dx\,dt$$
$$\leq C(\frac{r}{R})^{2\gamma+d+2m}\int_{Q_R^+(X_0)} |D_{x'}D^{2m-1}u - (D_{x'}D^{2m-1}u)_{Q_R^+(X_0)}|^2\,dx\,dt$$
$$+CR^{2a+d+2m}[f]^{x2}_{a,Q_{4R}^+}.$$

Taking $\gamma > a$, from Lemmas 3.3.1 and 3.3.2, and the corresponding interior estimates, we obtain the estimate for the Hölder semi-norm,

$$[D_{x'}D^{2m-1}u]_{\frac{a}{2m},a,Q_R^+} \leq C\Big(\|D^{2m}u\|_{L_2(Q_{4R}^+)} + [f]^x_{a,Q_{4R}^+}\Big).$$

Let us turn to the case that the coefficients are functions of both t and x. The method of freezing the coefficients is implemented in this case. We first consider the case when $L = \sum_{|\alpha|=|\beta|=m} A^{\alpha\beta}D^\alpha D^\beta$. Fix $y \in B_R^+$ and rewrite the system as follows

$$u_t + (-1)^m L_{0y}u = f + (-1)^m(L_{0y} - L)u.$$

We use the same ζ as in (3.5.30) and let $T = (4R)^{2m}$. By Theorem 4 and Remark 1 of [37], there is a unique solution $w \in W_2^{1,2m}$ to the system

$$w_t + (-1)^m L_{0y} w = \zeta\big(f - (f)_{B_{4R}^+}(t)\big) + \zeta(-1)^m (L_{0y} - L)u$$

in $(-T,0)\times\mathbb{R}^d_+$ with the Dirichlet boundary conditions $w = 0,\ D_d w = 0, \ldots,\ D_d^{m-1} w = 0$ on $(-T,0)\times\{x_d = 0\}$ and the zero initial condition at $t = -T$. Moreover,

$$\begin{aligned}
&\|D^{2m} w\|^2_{L_2(Q^+_{4R})} \le C\|f - (f)_{B^+_{4R}}(t)\|^2_{L_2(Q^+_{4R})} + C\|(L_{0y} - L)u\|^2_{L_2(Q^+_{4R})}\\
&\le C R^{2a+2m+d}[f]^{x2}_{a,Q^+_{4R}} + C R^{2a+2m+d} \sum_{|\alpha|=|\beta|=m} [A^{\alpha\beta}]^{x2}_a \|D^{2m} u\|^2_{L_\infty(Q^+_{4R})}.
\end{aligned}$$

Let $v = u - w$. Then $\tilde{v} := D_{x'} v$ satisfies (3.5.34), which implies (3.5.35) holds for v. Consequently, taking $\gamma > a$, we get the following inequality

$$[D_{x'} D^{2m-1} u]_{\frac{a}{2m},a,Q^+_R} \le C\Big([f]^x_{a,Q^+_{4R}} + \|D^{2m} u\|_{L_\infty(Q^+_{4R})}\Big). \tag{3.5.36}$$

When the operator L has lower-order terms, we can move all the lower-order terms to the right-hand side regarded as a part of f. Hence, we proved the second part of Theorem 3.2.4, i.e.,

$$[D_{x'} D^{2m-1} u]_{\frac{a}{2m},a,Q^+_R} \le C\Big([f]^x_{a,Q^+_{4R}} + \sum_{|\gamma|\le 2m} \|D^\gamma u\|_{L_\infty(Q^+_{4R})}\Big).$$

The proof of Theorem 3.2.4 is completed.

Remark. We need more regularity assumptions to get the estimate of $[D_d^{2m} u]_{\frac{a}{2m},a,Q^+_R}$. For example, if assuming $A^{\alpha\beta}, f \in C^{\frac{a}{2m},a}$, then for $R \le 1$, by using a similar method one can show that

$$[D_d^{2m} u]_{\frac{a}{2m},a,Q^+_R} + [u_t]_{\frac{a}{2m},a,Q^+_R} \le C\Big(\|f\|_{\frac{a}{2m},a,Q^+_{4R}} + \|u\|_{L_2(Q^+_{4R})}\Big). \tag{3.5.37}$$

3.6 Schauder estimates under the conormal boundary conditions

3.6.1 Schauder estimates for systems on a half space under the conormal boundary condition

In this subsection, we prove the Schauder estimates for (3.1.4) on a half space under the conormal derivative boundary conditions.

3.6.1.1 The estimate of $D_{x'}D^{m-1}u$

Systems with coefficients depending only on t. First we consider

$$u_t + (-1)^m \mathcal{L}_0 u = 0 \tag{3.6.1}$$

in Q_{2R}^+ with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2R}$. We have the following mean oscillation estimate, the proof of which can be found in [43].

Lemma 3.6.1. *Let $R \in (0,\infty)$ be a constant. Assume that $u \in C_{loc}^{\infty}(\overline{\mathcal{O}_0^+})$ satisfies (3.6.1) in Q_R^+ with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_R$. Then under the condition (3.2.1), for any $r \in (0,R)$ and $\gamma \in (0,1)$, there exists a constant $C = C(d,m,n,\delta,K,\gamma)$ such that*

$$\begin{aligned}&\int_{Q_r^+} |D^{m-1}u - (D^{m-1}u)_{Q_r^+}|^2 \, dx\, dt \\ &\le C(\frac{r}{R})^{2\gamma+2m+d} \int_{Q_R^+} |D^{m-1}u - (D^{m-1}u)_{Q_R^+}|^2 \, dx\, dt.\end{aligned} \tag{3.6.2}$$

Lemma 3.6.2. *Let $R \in (0,1]$ and $a \in (0,1)$ be constants. Assume that $u \in C^{(m)x}_{loc}(\overline{\mathcal{O}^+_0})$ satisfies*

$$u_t + (-1)^m \mathcal{L}_0 u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad in \quad Q^+_{2R} \tag{3.6.3}$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2R}$ and $f_\alpha \in C^a_x(Q^+_{2R})$ if $|\alpha| = m$, $f_\alpha \in L_\infty(Q^+_{2R})$ if $|\alpha| < m$. Then under the condition (3.2.1), for any $r \in (0,R)$, $\gamma \in (0,1)$, there exists a constant $C = C(n,m,d,\delta,\gamma,K)$ such that

$$\begin{aligned}
&\int_{Q^+_r} |D_{x'} D^{m-1} u - (D_{x'} D^{m-1} u)_{Q^+_r}|^2 \, dx \, dt \\
&\le C(\frac{r}{R})^{2\gamma+2m+d} \int_{Q^+_R} |D_{x'} D^{m-1} u - (D_{x'} D^{m-1} u)_{Q^+_R}|^2 \, dx \, dt \\
&\quad + C \sum_{|\alpha|=m} [f_\alpha]^{x2}_{a,Q^+_{2R}} R^{2a+2m+d} + C \sum_{|\alpha|<m} \|f_\alpha\|^2_{L_\infty(Q^+_{2R})} R^{2+2m+d}.
\end{aligned}$$

Proof. By using the approximation argument as in Theorem 3.4.9, without loss of generality we may assume that $A^{\alpha\beta}$, f_α, and u are smooth functions. Let $\zeta \in C^\infty_0(\mathbb{R}^{d+1})$ and

$$\zeta = 1 \quad \text{in} \quad Q_R, \quad \zeta = 0 \quad \text{outside} \quad (-(2R)^{2m}, (2R)^{2m}) \times B_{2R}.$$

For $T = (2R)^{2m}$, we consider the system

$$w_t + (-1)^m \mathcal{L}_0 w = \sum_{|\alpha| \le m} D^\alpha (\zeta \tilde{f}_\alpha)$$

in $(-T,0) \times \mathbb{R}^d_+$ with the conormal derivative boundary conditions on $\{x_d = 0\}$ and the zero initial condition on $\{-T\} \times \mathbb{R}^d_+$, where

$$\tilde{f}_\alpha(t,x) = f_\alpha(t,x) - f_\alpha(t,0)$$

if $|\alpha| = m$, and $\tilde{f}_\alpha = f_\alpha$ otherwise. From Theorem 3.4.2, the above system has a unique solution $w \in \mathcal{H}^m_2((-T,0) \times \mathbb{R}^d_+)$. Indeed, we can consider the system which $\tilde{w} = e^{-\lambda t} w$ satisfies, where $\lambda > 0$. It is easy to see that from Theorem 3.4.2 and [37, Remark 1], we can solve for $\tilde{w}$, and then obtain w. By the classical theory, w is smooth. Let $v = u - w$, which satisfies

$$v_t + (-1)^m \mathcal{L}_0 v = \sum_{|\alpha|=m} D^\alpha f_\alpha(t,0) \quad \text{in} \quad Q^+_R$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_R$.

We differentiate the above system with respect to x' and let $\hat{v} = D_{x'} v$. Then $\hat{v}$ satisfies all the conditions in Lemma 3.6.1 so that (3.6.2) holds for $\hat{v}$. Therefore, we obtain

$$\begin{aligned}
&\int_{Q^+_r} |D_{x'} D^{m-1} v - (D_{x'} D^{m-1} v)_{Q^+_r}|^2 \, dx \, dt \\
&\leq C(\frac{r}{R})^{2\gamma+2m+d} \int_{Q^+_R} |D_{x'} D^{m-1} v - (D_{x'} D^{m-1} v)_{Q^+_R}|^2 \, dx \, dt.
\end{aligned} \tag{3.6.4}$$

From the proof of Theorem 3.4.2,

$$\begin{aligned}
&\delta \int_{Q^+_{2R}} |D^m w|^2 \, dx \, dt \leq \varepsilon \|D^m w\|^2_{L_2(Q^+_{2R})} + C(\varepsilon) \sum_{|\alpha|=m} \|f_\alpha(t,x) - f_\alpha(t,0)\|^2_{L_2(Q^+_{2R})} \\
&\quad + \sum_{|\alpha|<m} |\langle f_\alpha, D^\alpha w \rangle_{Q^+_{2R}}|.
\end{aligned} \tag{3.6.5}$$

We take ε sufficiently small so that the first term on the right-hand side is absorbed in the left-hand side. The second term is less than

$$\sum_{|\alpha|=m} C[f_\alpha]^{x2}_{a,Q^+_{2R}} R^{2a+d+2m}.$$

The last term can be estimated as follows. We choose a vector-valued polynomial $P(x)$ as in Lemma 3.4.4 with respect to w. Let $h = w - P(x)$. Because $P(x)$ is of degree $m-1$, h satisfies the same system and boundary conditions as w. Moreover $D^m h{=}D^m w$. Therefore, (3.6.5) holds with h in place of w in the last term of the right-hand side. By Lemma 3.4.4 with $2R$ in place of R and the Cauchy–Schwarz inequality,

$$\begin{aligned}
&|\langle f_\alpha, D^\alpha h\rangle_{Q^+_{2R}}| \le C\|f_\alpha\|_{L_\infty(Q^+_{2R})}|Q^+_{2R}|^{\frac{1}{2}}\|D^\alpha h\|_{L_2(Q^+_{2R})}\\
&\le C\|f_\alpha\|_{L_\infty(Q^+_{2R})}|Q^+_{2R}|^{\frac{1}{2}}\big(R^{m-|\alpha|}\|D^m w\|_{L_2(Q^+_{2R})}\\
&\quad + \sum_{|\beta|\le m} R^{3m+d/2-|\alpha|-|\beta|}\|\tilde f_\beta\|_{L_\infty(Q^+_{2R})}\big).
\end{aligned} \tag{3.6.6}$$

Since $R \le 1$ and for $|\beta| = m$,

$$\|\tilde f_\beta\|_{L_\infty(Q^+_{2R})} \le C[f_\beta]^x_{a,Q^+_{2R}} R^a,$$

by Young's inequality and the Cauchy–Schwarz inequality, the right-hand side of (3.6.6) is bounded by

$$\varepsilon\|D^m w\|^2_{L_2(Q^+_{2R})} + C(\varepsilon)\Big(\sum_{|\alpha|<m}\|f_\alpha\|^2_{L_\infty(Q^+_{2R})}R^{2+2m+d} + \sum_{|\alpha|=m}[f_\alpha]^{x2}_{a,Q^+_{2R}}R^{2a+2m+d}\Big). \tag{3.6.7}$$

Finally, choosing ε sufficiently small and combining (3.6.4)-(3.6.7), by the triangle inequality we immediately prove the lemma. □

Thanks to Lemma 3.3.1, by using Lemma 3.6.2 with $\gamma > a$, a translation of the coordinates, and scaling, we have

$$\begin{aligned}&\int_{Q_r^+(X_0)} |D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q_r^+(X_0)}|^2\,dx\,dt\\&\le C(\frac{r}{R})^{2a+2m+d}\int_{Q_R^+(X_0)} |D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q_R^+(X_0)}|^2\,dx\,dt\\&\quad + C\sum_{|\alpha|=m}[f_\alpha]^{x2}_{a,Q_{2R}^+}r^{2a+2m+d} + C\sum_{|\alpha|<m}\|f_\alpha\|^2_{L_\infty(Q_{2R}^+)}r^{2a+2m+d}\end{aligned}$$

for any $r \in (0, R/2)$, where $R \le 1$ and $X_0 \in \{x_d = 0\} \cap Q_R$. By Lemma 3.3.2 together with the corresponding interior estimate (cf. [42, Proposition 4.1]), we obtain

$$[D_{x'}D^{m-1}u]_{\frac{a}{2m},a,Q_R^+} \le C\big(R^{-(2a+d+2m)}\int_{Q_{2R}^+}|D^m u|^2\,dx\,dt + F^2\big)^{\frac12},$$

where

$$F = \sum_{|\alpha|<m}\|f_\alpha\|_{L_\infty(Q_{2R}^+)} + \sum_{|\alpha|=m}[f]^x_{a,Q_{2R}^+}. \tag{3.6.8}$$

Variable coefficients depending on both x and t. In this part, we use the argument of freezing the coefficients to deal with the case when $A^{\alpha\beta}$ depend on both x and t. First, let us consider systems which only consist of highest order terms:

$$u_t + (-1)^m\mathcal{L}_h u := u_t + (-1)^m\sum_{|\alpha|=|\beta|=m}D^\alpha(A^{\alpha\beta}D^\beta u) = \sum_{|\alpha|\le m}D^\alpha f_\alpha \tag{3.6.9}$$

in Q_{2R}^+ and follow the steps in the simple coefficient case. For any $\hat{x} \in \overline{B_R^+}$, the equation above can be rewritten as

$$u_t + (-1)^m\mathcal{L}_{0\hat{x}}u = \sum_{|\alpha|\le m}D^\alpha\tilde{f}_\alpha,$$

where

$$\mathcal{L}_{0\hat{x}} = \sum_{|\alpha|=|\beta|=m} D^\alpha(A^{\alpha\beta}(t,\hat{x})D^\beta),$$

and

$$\tilde{f}_\alpha = f_\alpha + (-1)^m \sum_{|\beta|=m} \big(A^{\alpha\beta}(t,\hat{x}) - A^{\alpha\beta}(t,x)\big) D^\beta u$$

when $|\alpha| = m$, and $\tilde{f}_\alpha = f_\alpha$ otherwise. Using exactly the same argument as in Lemma 3.6.2 with $\tilde{f}_\alpha$ in place of f_α, we can prove the following lemma corresponding to Lemma 3.6.2.

Lemma 3.6.3. *Let $R \in (0,1]$ and $a \in (0,1)$ be constants. Assume that $u \in C^m_{x,loc}(\overline{\mathcal{O}^+_0})$ satisfies (3.6.9) in Q^+_{2R} with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2R}$, $f_\alpha \in C^a_x(Q^+_{2R})$ if $|\alpha| = m$, $f_\alpha \in L_\infty(Q^+_{2R})$ if $|\alpha| < m$, and $A^{\alpha\beta} \in C^a_x(Q^+_{2R})$. Suppose that the condition (3.2.1) holds for any $X \in \overline{Q^+_{2R}}$. Then for any $r \in (0,R)$ and $\gamma \in (0,1)$, there exists a constant N depending only on d, n, m, δ, K, γ, and $\|A^{\alpha\beta}\|^x_{a,Q^+_{2R}}$ such that*

$$\begin{aligned}
&\int_{Q^+_r} |D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q^+_r}|^2\,dx\,dt \\
&\leq C(\frac{r}{R})^{2\gamma+2m+d} \int_{Q^+_R} |D_{x'}D^{m-1}u - (D_{x'}D^{m-1}u)_{Q^+_R}|^2\,dx\,dt \\
&\quad + C(\sum_{|\alpha|=m} [f_\alpha]^{x2}_{a,Q^+_{2R}} + \|D^m u\|^2_{L_\infty(Q^+_{2R})})R^{2a+2m+d} \\
&\quad + C\sum_{|\alpha|<m} \|f_\alpha\|^2_{L_\infty(Q^+_{2R})} R^{2+2m+d}.
\end{aligned}$$

By Lemmas 3.3.1, 3.3.2, a translation of the coordinates and scaling, we derive the following corollary by taking $\gamma > a$ in Lemma 3.6.3 and the corresponding interior estimate.

Corollary 3.6.4. *Under the conditions in Lemma 3.6.3, we have*

$$[D_{x'}D^{m-1}u]_{\frac{a}{2m},a,Q_R^+} \le C(\|D^m u\|_{L_\infty(Q_{2R}^+)} + F),$$

where C depends only on d, m, n, δ, $\|A^{\alpha\beta}\|^x_{a,Q_{2R}^+}$, K, R, and a, and F is defined in (3.6.8).

Proof of Theorem 3.2.5. Now we are ready to handle the general system

$$u_t + (-1)^m \mathcal{L}u = \sum_{|\alpha|\le m} D^\alpha f_\alpha,$$

which can be written as

$$u_t + (-1)^m \mathcal{L}_h u = \sum_{|\alpha|\le m} D^\alpha \tilde{f}_\alpha,$$

where

$$\tilde{f}_\alpha = f_\alpha + (-1)^{m+1} \sum_{|\beta|<m} A^{\alpha\beta} D^\beta u \quad \text{if} \quad |\alpha| = m,$$

$$\tilde{f}_\alpha = f_\alpha + (-1)^{m+1} \sum_{|\beta|\le m} A^{\alpha\beta} D^\beta u \quad \text{otherwise.}$$

Note that

$$[A^{\alpha\beta} D^\beta u]^x_{a,Q_{2R}^+} \le C\|A^{\alpha\beta}\|_{L_\infty(Q_{2R}^+)} [D^\beta u]^x_{a,Q_{2R}^+} + C[A^{\alpha\beta}]^x_{a,Q_{2R}^+} \|D^\beta u\|_{L_\infty(Q_{2R}^+)}.$$

We substitute f_α with $\tilde{f}_\alpha$ in the estimate of Corollary 3.6.4 to get

$$[D_{x'}D^{m-1}u]_{\frac{a}{2m},a,Q_R^+} \le C\big(\|D^m u\|_{L_\infty(Q_{2R}^+)} + \sum_{|\beta|<m} \|D^\beta u\|^x_{a,Q_{2R}^+} + F\big).$$

Since $R \le 1$, it is easily seen that

$$\sum_{|\beta|<m} \|D^\beta u\|^x_{a,Q^+_{2R}} \le \sum_{|\beta|\le m} \|D^\beta u\|_{L_\infty(Q^+_{2R})}.$$

Therefore, Theorem 3.2.5 is proved. □

3.6.1.2 The estimate of $D_d^m u$

In this subsection, we prove the Schauder estimate for $D_d^m u$ near a flat boundary. In this case, in view of the example in the introduction it is not sufficient that the coefficients are merely measurable in t. We require the coefficients be Hölder continuous in t as well but only on the boundary where the conormal derivative boundary conditions are given.

Special systems with coefficients depending only on t. In this part, we study the special systems which we introduced in Lemma 3.4.5.

Lemma 3.6.5. *Let $R \in (0,1]$ be a constant. Assume that $u \in C^\infty_{loc}(\overline{\mathcal{O}^+_0})$ satisfies*

$$u_t + (-1)^m \tilde{\mathcal{L}}_0 u = \sum_{|\alpha|=m} D^\alpha f_\alpha \quad in \quad Q^+_R$$

with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_R$. Moreover, $A^{\hat\alpha\hat\alpha}$ and $f_{\hat\alpha}$ are constants, and $f_\alpha = f_\alpha(t)$ for $|\alpha| = m, \alpha \ne \hat\alpha$. Then under the condition $A^{\hat\alpha\hat\alpha} \ge \delta I_{n\times n}$, for any $r \in (0,R)$,

$$\int_{Q^+_r} |D_d^m u - c|^2 \, dx\, dt \le C(\frac{r}{R})^{4m+d} \int_{Q^+_R} |D_d^m u - c|^2 \, dx\, dt,$$

where $C = C(d,n,m,\delta,K)$ and $c = (-1)^m (A^{\hat\alpha\hat\alpha})^{-1} f_{\hat\alpha}$ is a constant vector.

Proof. Using the method in the proof of Lemma 3.4.5, one can show that the system and boundary conditions can be written as

$$
\begin{aligned}
&u_t + (-1)^m \tilde{\mathcal{L}}_0 u = 0 \quad \text{in} \quad Q_R^+,\\
&D_d^m u = (-1)^m (A^{\hat\alpha\hat\alpha})^{-1} f_{\hat\alpha}, \quad D_d^{m+1}u = \cdots = D_d^{2m-1}u = 0 \text{ on } \{x_d = 0\} \cap Q_R.
\end{aligned}
$$

Indeed, by the definition of the conormal derivative boundary conditions,

$$
\begin{aligned}
&-\langle u, \phi_t\rangle_{Q_R^+} + \sum_{j=1}^{m-1} \langle D_j^m u, D_j^m \phi\rangle_{Q_R^+} + \langle A^{\hat\alpha\hat\alpha} D_d^m u, D_d^m \phi\rangle_{Q_R^+}\\
&= (-1)^m \sum_{|\alpha|=m} \langle f_\alpha, D^\alpha \phi\rangle_{Q_R^+}
\end{aligned}
$$

for any $\phi \in C_0^\infty(Q_R)$. We integrate by parts as in the proof of Lemma 3.4.5 and the left-hand side becomes

$$
\begin{aligned}
&\langle u_t, \phi\rangle_{Q_R^+} + (-1)^m \sum_{j=1}^{m-1} \langle D_j^{2m} u, \phi\rangle_{Q_R^+} + (-1)^m \langle A^{\hat\alpha\hat\alpha} D_d^{2m} u, \phi\rangle_{Q_R^+}\\
&+ \int_\Sigma \sum_{j=1}^{m} (-1)^j A^{\hat\alpha\hat\alpha} D_d^{m+j-1} u \, D_d^{m-j} \phi \, dx' \, dt,
\end{aligned}
$$

where $\Sigma = \{x_d = 0\} \cap Q_R$. For the right-hand side, since $f_\alpha = f_\alpha(t)$ when $\alpha \neq \hat\alpha$, after integrated by parts the terms vanish when $\alpha \neq \hat\alpha$. Moreover, because $f_{\hat\alpha}$ is constant, the only boundary term is

$$
(-1)^{m+1} \int_\Sigma f_{\hat\alpha} D_d^{m-1} \phi \, dx' \, dt.
$$

By the arbitrariness of ϕ, it follows that

$$
\langle u_t, \phi\rangle_{Q_R^+} + (-1)^m \sum_{j=1}^{m-1} \langle D_j^{2m} u, \phi\rangle_{Q_R^+} + (-1)^m \langle A^{\hat\alpha\hat\alpha} D_d^{2m} u, \phi\rangle_{Q_R^+} = 0
$$

and

$$\int_{\Sigma}\sum_{j=1}^{m}(-1)^j A^{\hat{\alpha}\hat{\alpha}} D_d^{m+j-1}u\, D_d^{m-j}\phi\, dx'\, dt = (-1)^{m+1}\int_{\Sigma} f_{\hat{\alpha}} D_d^{m-1}\phi\, dx'\, dt,$$

which yield

$$u_t + (-1)^m \tilde{\mathcal{L}}_0 u = 0$$

and

$$D_d^m u = (-1)^m (A^{\hat{\alpha}\hat{\alpha}})^{-1} f_{\hat{\alpha}}, \quad D_d^{m+1}u = \cdots = D_d^{2m-1}u \text{ on } \{x_d = 0\} \cap Q_R.$$

Recall that $A^{\hat{\alpha}\hat{\alpha}}$ and $f_{\hat{\alpha}}$ are constants. Hence $v := D_d^m u - (-1)^m (A^{\hat{\alpha}\hat{\alpha}})^{-1} f_{\hat{\alpha}}$ satisfies

$$\begin{aligned} &v_t + (-1)^m \tilde{\mathcal{L}}_0 v = 0 \quad \text{in} \quad Q_R^+, \\ &v = \cdots = D_d^{m-1} v = 0 \quad \text{on} \quad \{x_d = 0\} \cap Q_R. \end{aligned}$$

From [42, Lemma 4.4],

$$\int_{Q_r^+} |v|^2\, dx\, dt \le C(\frac{r}{R})^{4m+d} \int_{Q_R^+} |v|^2\, dx\, dt.$$

We note that although in [42, Lemma 4.4] we assume $C^\infty_{\text{loc}}(\overline{\mathcal{O}^+_\infty})$, from the proof it is easy to see that we only need $u \in C^\infty_{\text{loc}}(\overline{\mathcal{O}^+_0})$. Then the lemma is proved. □

We now use a scaling argument to handle $D_d^m u$ for systems with simple coefficients. Assume that u satisfies (3.6.3) and as in the proof of Theorem 3.2.2, we define v, $\tilde{f}_\alpha$, and $\tilde{A}^{\alpha\beta}$. Then v satisfies

$$v_t + (-1)^m \sum_{|\alpha|=|\beta|=m} \mu^{\alpha_d+\beta_d-2m} D^\alpha(\tilde{A}^{\alpha\beta}(t) D^\beta v) = \sum_{|\alpha|\le m} \mu^{\alpha_d-2m} D^\alpha \tilde{f}_\alpha \tag{3.6.10}$$

in $T_\mu(Q_{2R}^+)$ with the conormal derivative boundary conditions on $\{x_d=0\}\cap T_\mu(Q_{2R})$, where

$$T_\mu(Q_{2R}^+)=\{(\mu^{2m}t,x',\mu x_d):(t,x',x_d)\in Q_{2R}^+\}$$

and $T_\mu(Q_{2R})$ is defined in a similar way. From now on, we consider the system of v with $\mu\geq 1$ and it is easy to see that $Q_{2R}^+\subset T_\mu(Q_{2R}^+)$. Note that the regularity assumptions on $A^{\alpha\beta}$ and f_α are naturally inherited by $\tilde{A}^{\alpha\beta}$ and $\tilde{f}_\alpha$. For instance, if $f_\alpha\in C_x^a(Q_{2R}^+)$, then $\tilde{f}_\alpha\in C_x^a(T_\mu(Q_{2R}^+))$.

Lemma 3.6.6. *Assume that $a\in(0,1)$, $R\in(0,1]$, $v\in C_{loc}^{\frac{a+m}{2m},a+m}(\overline{\mathcal{O}_0^+})$ satisfies (3.6.10) in Q_{2R}^+ with the conormal derivative boundary conditions on $\{x_d=0\}\cap Q_{2R}$, $\tilde{f}_\alpha\in C_x^a(Q_{2R}^+)$ if $|\alpha|=m$, $\tilde{f}_\alpha\in L_\infty(Q_{2R}^+)$ if $|\alpha|<m$, $f_{\hat{\alpha}}\in C_t^a(\{x_d=0\}\cap Q_{2R})$, and $\tilde{A}^{\hat{\alpha}\beta}(t)\in C^{\frac{a}{2m}}(\mathbb{R})$. Then under the condition $\tilde{A}^{\hat{\alpha}\hat{\alpha}}(X_0)\geq\delta I_{n\times n}$ for any $X_0\in\overline{Q_R^+}$, there exist two constants $C_1=C_1(d,n,m,\delta,K,[\tilde{A}^{\hat{\alpha}\beta}]^t_{\frac{a}{2m},Q_{2R}^+},R,a)$ and $C_2=C_2(d,n,m,\delta,K,R,a)$ such that*

$$\begin{aligned}[D_d^m v]_{\frac{a}{2m},a,Q_R^+}\leq& C_1\|D^m v\|_{L_\infty(Q_{2R}^+)}+C_2([D_{x'}D^{m-1}v]_{\frac{a}{2m},a,Q_{2R}^+}\\&+\mu^{-1}[D_d^m v]_{\frac{a}{2m},a,Q_{2R}^+}+\tilde{G}),\end{aligned}$$

where

$$\tilde{G}:=\sum_{|\alpha|=m}[\tilde{f}_\alpha]^x_{a,Q_{2R}^+}+[\tilde{f}_{\hat{\alpha}}]^t_{\frac{a}{2m},\{x_d=0\}\cap Q_{2R}}+\sum_{|\alpha|\leq m}\|\tilde{f}_\alpha\|_{L_\infty(Q_{2R}^+)}. \tag{3.6.11}$$

Proof. As before, we may assume that $\tilde{A}^{\alpha\beta}$, $\tilde{f}_\alpha$, and v are smooth functions. We move all the spatial derivatives to the right-hand side and then add

$$(-1)^m\mathcal{D}_{d-1}^m v+(-1)^m D^{\hat{\alpha}}(\tilde{A}^{\hat{\alpha}\hat{\alpha}}(0)D^{\hat{\alpha}}v)$$

to both sides of the system so that

$$\begin{aligned}
& v_t + (-1)^m \big(D^{\hat\alpha}(\tilde A^{\hat\alpha\hat\alpha}(0) D^{\hat\alpha} v) + \mathcal{D}^m_{d-1} v \big) \\
& = \sum_{|\alpha|\le m} D^\alpha \hat f_\alpha + (-1)^m \mathcal{D}^m_{d-1} v + (-1)^{m+1} D^{\hat\alpha}\big((\tilde A^{\hat\alpha\hat\alpha}(t) - \tilde A^{\hat\alpha\hat\alpha}(0)) D^{\hat\alpha} v \big),
\end{aligned} \tag{3.6.12}$$

where $\hat f_\alpha = \mu^{\alpha_d - 2m} \tilde f_\alpha$ for $|\alpha| < m$,

$$\hat f_\alpha = \mu^{\alpha_d - 2m} \tilde f_\alpha + (-1)^{m+1} \sum_{|\beta| = m} \mu^{\alpha_d + \beta_d - 2m} \tilde A^{\alpha\beta}(t) D^\beta v$$

for $|\alpha| = m$ but $\alpha \neq \hat\alpha$, and

$$\hat f_{\hat\alpha} = \mu^{-m} \tilde f_{\hat\alpha} + (-1)^{m+1} \sum_{|\beta| = m, \beta \neq \hat\alpha} \mu^{\beta_d - m} \tilde A^{\hat\alpha\beta}(t) D^\beta v.$$

Notice that the left-hand side of (3.6.12) satisfies the conditions in Lemma 3.6.5, and similar to $\tilde{\mathcal{L}}_0$ we denote this operator by $\hat{\mathcal{L}}_0$. Note that in the definition of $\tilde{\mathcal{L}}_0$ in Section 4, $A^{\hat\alpha\hat\alpha}$ is not required to be constant, while for $\hat{\mathcal{L}}_0$ it is assumed to be constant.

Let r and τ be constants such that $0 < r < \tau \le R/2$. We use basically the same argument as in Lemma 3.6.2 with slight modifications. Let $w \in \mathcal{H}^m_2((-(2\tau)^{2m}, 0) \times \mathbb{R}^d_+)$ be a smooth solution of the system

$$\begin{aligned}
& w_t + (-1)^m \hat{\mathcal{L}}_0 w = \sum_{|\alpha|\le m} D^\alpha(\zeta \overline{f}_\alpha) + (-1)^m \sum_{j=1}^{d-1} D^m_j \big(\zeta (D^m_j v - D^m_j v(t,0)) \big) \\
& + (-1)^{m+1} D^{\hat\alpha}\big(\zeta (\tilde A^{\hat\alpha\hat\alpha}(t) - \tilde A^{\hat\alpha\hat\alpha}(0)) D^{\hat\alpha} v \big) \quad \text{in} \quad (-(2\tau)^{2m}, 0) \times \mathbb{R}^d_+
\end{aligned} \tag{3.6.13}$$

with the conormal derivative boundary conditions on $\{x_d = 0\}$ and the zero initial condition on $\{-(2\tau)^{2m}\} \times \mathbb{R}^d_+$, where ζ is the same smooth function as in Lemma

3.6.2 with τ in place of R, $\overline{f}_{\hat\alpha} = \hat f_{\hat\alpha}(t,x) - \hat f_{\hat\alpha}(0,0)$, $\overline{f}_\alpha = \hat f_\alpha(t,x) - \hat f_\alpha(t,0)$ if $|\alpha| = m$ and $\alpha \neq \hat\alpha$, and $\overline{f}_\alpha = \hat f_\alpha$ otherwise. Then $\mathrm{v} := v - w$ is also smooth and satisfies

$$\mathrm{v}_t + (-1)^m \hat{\mathcal{L}}_0 \mathrm{v} = \sum_{|\alpha|=m, \alpha\neq\hat\alpha} D^\alpha \hat f_\alpha(t,0) + D^{\hat\alpha} \hat f_{\hat\alpha}(0,0) + (-1)^m \sum_{j=1}^{d-1} D_j^{2m} v(t,0)$$

in $Q_{2\tau}^+$ with the conormal derivative boundary conditions on $\{x_d = 0\} \cap Q_{2\tau}$. Due to Lemma 3.6.5, we have

$$\int_{Q_r^+} |D_d^m \mathrm{v} - c|^2 \, dx \, dt \leq C(\frac{r}{\tau})^{4m+d} \int_{Q_\tau^+} |D_d^m \mathrm{v} - c|^2 \, dx \, dt, \tag{3.6.14}$$

where $c = (-1)^m (\tilde A^{\hat\alpha\hat\alpha}(0,0))^{-1} \hat f_{\hat\alpha}(0,0)$ is a constant vector. Next we estimate w by applying the same idea as in Lemma 3.6.2,

$$\begin{aligned}
\delta \int_{Q_{2\tau}^+} |D^m w|^2 \, dx \, dt &\leq \sum_{|\alpha|<m} |\langle \zeta \mu^{\alpha_d - 2m} \tilde f_\alpha, D^\alpha w \rangle_{Q_{2\tau}^+}| \\
&+ |\langle \zeta (\tilde A^{\hat\alpha\hat\alpha} - \tilde A^{\hat\alpha\hat\alpha}(0)) D^{\hat\alpha} v, D^{\hat\alpha} w \rangle_{Q_{2\tau}^+}| + |\langle \zeta (\hat f_{\hat\alpha} - \hat f_{\hat\alpha}(0,0)), D^{\hat\alpha} w \rangle_{Q_{2\tau}^+}| \\
&+ C \sum_{|\alpha|=m, \alpha\neq\hat\alpha} |\langle \zeta (\hat f_\alpha - \hat f_\alpha(t,0)), D^\alpha w \rangle_{Q_{2\tau}^+}| \\
&+ \sum_{j=1}^{d-1} |\langle \zeta (D_j^m v - D_j^m v(t,0)), D_j^m w \rangle_{Q_{2\tau}^+}|.
\end{aligned} \tag{3.6.15}$$

The first term on the right-hand side of (3.6.15) can be dealt with in the same way as (3.6.6) in Lemma 3.6.2. Indeed, we use the same h as in the proof of Lemma 3.6.2, i.e., $h = w - P(x)$ where $P(x)$ is the vector-valued polynomial as in Lemma 3.4.4. Therefore, the inequality above holds with h in place of w in the first term on the right-hand side of the inequality above and this term can be estimated as follows:

$$\begin{aligned}
&|\langle \zeta \mu^{\alpha_d - 2m} \tilde f_\alpha, D^\alpha h \rangle_{Q_{2\tau}^+}| \\
&\leq C \mu^{\alpha_d - 2m} \|\tilde f_\alpha\|_{L_\infty(Q_{2\tau}^+)} |Q_{2\tau}^+|^{\frac{1}{2}} \|D^\alpha h\|_{L_2(Q_{2\tau}^+)}.
\end{aligned}$$

Then we apply Lemma 3.4.4 to h, which satisfies (3.6.13), and note that $\tilde{A}^{\hat{\alpha}\hat{\alpha}} \in C^{\frac{a}{2m}}(\mathbb{R})$ to bound the first term from above by

$$
\begin{aligned}
&C\mu^{\alpha_d-2m}\|\tilde{f}_\alpha\|_{L_\infty(Q^+_{2\tau})}|Q^+_{2\tau}|^{\frac{1}{2}}\bigg(\tau^{m-|\alpha|}\|D^m w\|_{L_2(Q^+_{2\tau})}\\
&+\sum_{|\beta|\le m}\tau^{3m+d/2-|\alpha|-|\beta|}\|\overline{f}_\beta\|_{L_\infty(Q^+_{2\tau})}\\
&+\tau^{2m+d/2-|\alpha|+a}\big([D^m_{x'}v]^x_{a,Q^+_{2\tau}}+[\tilde{A}^{\hat{\alpha}\hat{\alpha}}]^t_{\frac{a}{2m},Q^+_{2\tau}}\|D^m_d v\|_{L_\infty(Q^+_{2\tau})}\big)\bigg).
\end{aligned}
$$

Using the definition of $\overline{f}_\alpha$, we get

$$
\begin{aligned}
&\|\overline{f}_\alpha\|_{L_\infty(Q^+_{2\tau})}\\
&\le C_2\tau^a[\tilde{f}_\alpha]^x_{a,Q^+_{2\tau}}+C_2\tau^a\big([D_{x'}D^{m-1}v]^x_{a,Q^+_{2\tau}}+\mu^{-1}[D^m_d v]^x_{a,Q^+_{2\tau}}\big)
\end{aligned}
$$

for $|\alpha|=m$ but $\alpha\neq\hat{\alpha}$. For $\alpha=\hat{\alpha}$, noticing that for any smooth function f,

$$
\begin{aligned}
|f(t,x)-f(0,0)|&=|f(t,x)-f(t,0)+f(t,0)-f(0,0)|\\
&\le [f]^x_a|x|^a+[f]^t_{\frac{a}{2m}}|t|^{\frac{a}{2m}},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\|\overline{f}_{\hat{\alpha}}\|_{L_\infty(Q^+_{2\tau})}\le C_2\tau^a\big([\tilde{f}_{\hat{\alpha}}]^x_{a,Q^+_{2\tau}}+[\tilde{f}_{\hat{\alpha}}]^t_{\frac{a}{2m},\{x_d=0\}\cap Q_{2\tau}}\big)\\
&+C_2\tau^a\big([\tilde{A}^{\hat{\alpha}\beta}]^t_{\frac{a}{2m},Q^+_{2\tau}}\|D^m v\|_{L_\infty(Q^+_{2\tau})}+C_2[D_{x'}D^{m-1}v]_{\frac{a}{2m},a,Q^+_{2\tau}}\big).
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
&\sum_{|\alpha|<m} |\langle \zeta(x)\mu^{\alpha_d-2m}\tilde{f}_\alpha, D^\alpha w\rangle_{Q_{2\tau}^+}| \\
&\leq \varepsilon\|D^m w\|^2_{L_2(Q_{2\tau}^+)} + C_2(\varepsilon)\sum_{|\alpha|<m}\|\tilde{f}_\alpha\|^2_{L_\infty(Q_{2\tau}^+)}\tau^{2+2m+d} \\
&\quad + C_2(\varepsilon)\Big(\sum_{|\alpha|=m}[\tilde{f}_\alpha]^{x2}_{a,Q_{2\tau}^+} + [\tilde{f}_{\hat{\alpha}}]^{t2}_{\frac{a}{2m},\{x_d=0\}\cap Q_{2\tau}} + [D_{x'}D^{m-1}v]^2_{\frac{a}{2m},a,Q_{2\tau}^+} \\
&\quad + \mu^{-1}[D_d^m v]^2_{\frac{a}{2m},a,Q_{2\tau}^+}\Big)\tau^{2a+d+2m} + C_1\|D^m v\|^2_{L_\infty(Q_{2\tau}^+)}\tau^{2a+d+2m}
\end{aligned}
$$

provided that $\mu \geq 1$.

For the other terms on the right-hand side of (3.6.15), following the proof of Lemma 3.6.2, we apply Young's inequality so that, for any $\varepsilon > 0$, the right-hand side of (3.6.15) is bounded by

$$
\begin{aligned}
&\varepsilon\|D^m w\|^2_{L_2(Q_{2\tau}^+)} + C_2(\varepsilon)\big(\sum_{|\alpha|=m}[\tilde{f}_\alpha]^{x2}_{a,Q_{2\tau}^+} + [\tilde{f}_{\hat{\alpha}}]^{t2}_{\frac{a}{2m},\{x_d=0\}\cap Q_{2\tau}} \\
&+ [D_{x'}D^{m-1}v]^2_{\frac{a}{2m},a,Q_{2\tau}^+} + \mu^{-1}[D_d^m v]^2_{\frac{a}{2m},a,Q_{2\tau}^+}\big)\tau^{2a+2m+d} \\
&+ C_1\|D^m v\|^2_{L_\infty(Q_{2\tau}^+)}\tau^{2a+2m+d} + C_2\sum_{|\alpha|<m}\|\tilde{f}_\alpha\|^2_{L_\infty(Q_{2\tau}^+)}\tau^{2+2m+d}.
\end{aligned}
\tag{3.6.16}
$$

After choosing ε sufficiently small, by the triangle inequality, (3.6.14), and (3.6.16) we obtain

$$
\begin{aligned}
&\int_{Q_r^+} |D_d^m v - c|^2 \, dx \, dt \\
&\leq C_2\Big(\frac{r}{\tau}\Big)^{4m+d}\int_{Q_\tau^+}|D_d^m v - c|^2 \, dx \, dt + I\tau^{2a+2m+d},
\end{aligned}
$$

where

$$
\begin{aligned}
I :=& C_2\Big(\sum_{|\alpha|=m} [\tilde{f}_\alpha]^{x2}_{a,Q^+_{2\tau}} + C_1\|D^m v\|^2_{L_\infty(Q^+_{2\tau})} + \sum_{|\alpha|<m} \|\tilde{f}_\alpha\|^2_{L_\infty(Q^+_{2\tau})} \\
&+ [\tilde{f}_{\hat{\alpha}}]^{t2}_{\frac{a}{2m},\{x_d=0\}\cap Q_{2\tau}} + [D_{x'}D^{m-1}v]^2_{\frac{a}{2m},a,Q^+_{2\tau}} + \mu^{-1}[D_d^m v]^2_{\frac{a}{2m},a,Q^+_{2\tau}}\Big).
\end{aligned}
$$

It then follows from Lemma 3.3.1 that

$$
\begin{aligned}
&\int_{Q^+_r} |D_d^m v - c|^2 \, dx \, dt \\
&\le C_2(\frac{r}{R})^{2a+2m+d} \int_{Q^+_{R/2}} |D_d^m v - c|^2 \, dx \, dt + I r^{2a+2m+d},
\end{aligned}
$$

which together with a translation of the coordinates, a corresponding interior estimate, and Lemma 3.3.2 yields

$$
\begin{aligned}
&[D_d^m v]^2_{\frac{a}{2m},a,Q^+_R} \le C_2(R^{-(2a+d+2m)}\|D^m v - c\|^2_{L_2(Q^+_{2R})} + I) \\
&\le C_2([D_{x'}D^{m-1}v]^2_{\frac{a}{2m},a,Q^+_{2R}} + \mu^{-1}[D_d^m v]^2_{\frac{a}{2m},a,Q^+_{2R}} + \tilde{G}^2) + C_1\|D^m v\|^2_{L_\infty(Q^+_{2R})}.
\end{aligned}
$$

Therefore, the lemma is proved. □

General systems with coefficients depending on both x and t. Similar to Lemma 3.6.6, we can estimate the highest normal derivative in the case of variable coefficients depending on both x and t. As before, we need more regularity assumptions on $\tilde{A}^{\alpha\beta}$ and $\tilde{f}_\alpha$. Similar to Lemma 3.6.3 and Corollary 3.6.4, following the proof of Lemma 3.6.6, we can prove the lemma below.

Lemma 3.6.7. *Assume that $a \in (0,1)$, $v \in C_{loc}^{\frac{a+m}{2m},a+m}(\overline{\mathcal{O}_0^+})$ satisfies*

$$
v_t + (-1)^m \sum_{|\alpha|=|\beta|=m} \mu^{\alpha_d+\beta_d-2m} D^\alpha(\tilde{A}^{\alpha\beta}D^\beta v) = \sum_{|\alpha|\le m} \mu^{\alpha_d-2m} D^\alpha \tilde{f}_\alpha
$$

in Q_{2R}^+ *with the conormal derivative boundary conditions on* $\{x_d = 0\} \cap Q_{2R}$*, where* $\tilde{f}_\alpha \in C_x^a(Q_{2R}^+)$ *if* $|\alpha| = m$*,* $\tilde{f}_\alpha \in L_\infty(Q_{2R}^+)$ *if* $|\alpha| < m$*,* $\tilde{A}^{\alpha\beta} \in C_x^a(Q_{2R}^+)$ *for* $|\alpha| = m$*, and* $f_{\hat{\alpha}}, \tilde{A}^{\hat{\alpha}\beta} \in C_t^{\frac{a}{2m}}(\{x_d = 0\} \cap Q_{2R})$*. Then under the condition* $\tilde{A}^{\hat{\alpha}\hat{\alpha}}(X_0) \geq \delta I_{n\times n}$ *for any* $X_0 \in \overline{Q_R^+}$*, there exist two constants*

$$C_1 = C_1\big(d, n, m, \delta, K, [\tilde{A}^{\hat{\alpha}\beta}]^t_{\frac{a}{2m},\{x_d=0\}\cap Q_{2R}}, \|\tilde{A}^{\alpha\beta}\|^x_{a,Q_{2R}^+}, R, a\big)$$

and $C_2 = C_2(d, n, m, \delta, K, R, a)$ *such that*

$$\begin{aligned}[D_d^m v]_{\frac{a}{2m},a,Q_R^+} \leq C_1 \|D^m v\|_{L_\infty(Q_{2R}^+)} + C_2\big([D_{x'} D^{m-1} v]_{\frac{a}{2m},a,Q_{2R}^+} \\ + \mu^{-1}[D_d^m v]_{\frac{a}{2m},a,Q_{2R}^+} + \tilde{G}\big),\end{aligned}$$

where $\tilde{G}$ *is defined in* (3.6.11)*.*

Furthermore, from Lemma 3.6.7 we follow the proof of the Theorem 3.2.5 moving the lower-order terms to the right-hand side regarded as part of $\tilde{f}_\alpha$ and noticing that,

$$\begin{aligned}\sum_{|\beta|<m} \langle \tilde{A}^{\hat{\alpha}\beta} D^\beta v\rangle_{\frac{a}{2m},\{x_d=0\}\cap Q_{2R}} \leq C \sum_{|\beta|<m} \big(\|\tilde{A}^{\hat{\alpha}\beta}\|_{L_\infty(Q_{2R}^+)} [D^\beta v]_{\frac{a}{2m},a,Q_{2R}^+} \\ + [\tilde{A}^{\hat{\alpha}\beta}]^t_{\frac{a}{2m},\{x_d=0\}\cap Q_{2R}} \|D^\beta v\|_{L_\infty(Q_{2R}^+)}\big)\end{aligned}$$

to get the following lemma.

Lemma 3.6.8. *Assume that* $a \in (0,1)$*,* $v \in C_{loc}^{\frac{a+m}{2m}, a+m}(\overline{\mathcal{O}_0^+})$ *satisfies*

$$v_t + (-1)^m \sum_{|\alpha|\leq m, |\beta|\leq m} \mu^{\alpha_d+\beta_d-2m} D^\alpha(\tilde{A}^{\alpha\beta} D^\beta v) = \sum_{|\alpha|\leq m} \mu^{\alpha_d - 2m} D^\alpha \tilde{f}_\alpha \tag{3.6.17}$$

in Q_{2R}^+ *with the conormal derivative boundary conditions on* $\{x_d = 0\} \cap Q_{2R}$ *and* $\tilde{f}_\alpha \in C_x^a(Q_{2R}^+)$ *if* $|\alpha| = m$*,* $\tilde{f}_\alpha \in L_\infty(Q_{2R}^+)$ *if* $|\alpha| < m$*,* $\tilde{A}^{\alpha\beta} \in C_x^a(Q_{2R}^+)$ *for* $|\alpha| = m$*,* *and* $\tilde{f}_{\hat{\alpha}}, \tilde{A}^{\hat{\alpha}\beta} \in C_t^{\frac{a}{2m}}(\{x_d = 0\} \cap Q_{2R})$*. Then under the condition* $\tilde{A}^{\hat{\alpha}\hat{\alpha}}(X_0) \geq \delta I_{n\times n}$ *for any* $X_0 \in \overline{Q_R^+}$*, there exist two constants*

$$C_1 = C_1(d, n, m, \delta, K, [\tilde{A}^{\hat{\alpha}\beta}]^t_{\frac{a}{2m}, \{x_d=0\}\cap Q_{2R}}, \|\tilde{A}^{\alpha\beta}\|^x_{a, Q^+_{2R}}, R, a)$$

and $C_2 = C_2(d, n, m, \delta, K, R, a)$ *such that*

$$\begin{aligned}[D_d^m v]_{\frac{a}{2m}, a, Q_R^+} \leq C_1 \sum_{|\beta|\leq m} \|D^\beta v\|_{L_\infty(Q^+_{2R})} &+ C_2([D_{x'} D^{m-1} v]_{\frac{a}{2m}, a, Q^+_{2R}} \\ &+ \mu^{-1}[D_d^m v]_{\frac{a}{2m}, a, Q^+_{2R}} + \sum_{|\beta|<m} [D^\beta v]_{\frac{a}{2m}, a, Q^+_{2R}} + \tilde{G}),\end{aligned}$$

where $\tilde{G}$ *is defined in* (3.6.11)*.*

3.6.1.3 Proof of Theorem 3.2.6

The interior Schauder estimates of divergence type higher-order parabolic systems were established in Proposition 3.5.1. We note that although we assumed $u \in C^\infty_{\mathrm{loc}}(\mathbb{R}^{d+1})$ in Proposition 3.5.1, by using the approximation argument as in Theorem 3.4.9, the result still holds for any $u \in C^{\frac{a+m}{2m}, a+m}_{\mathrm{loc}}(\mathbb{R}^{d+1}_0)$.

We have the following global estimate in $(0, T) \times \mathbb{R}^d_+$. For simplicity, we use C^a_x (or $C_t^{\frac{a}{2m}}$) to denote $C^a_x((0, T) \times \mathbb{R}^d_+)$ (or $C_t^{\frac{a}{2m}}((0, T) \times \{x : x_d = 0\})$, respectively).

Proposition 3.6.9. *Let* $a \in (0, 1)$*,* $u \in C^{\frac{a+m}{2m}, m+a}_{loc}(\overline{(0, T) \times \mathbb{R}^d_+})$ *satisfy*

$$u_t + (-1)^m \mathcal{L} u = \sum_{|\alpha|\leq m} D^\alpha f_\alpha \quad \text{in} \quad (0, T) \times \mathbb{R}^d_+$$

with the conormal derivative boundary conditions on $(0, T) \times \{x : x_d = 0\}$*, the zero initial condition on* $\{0\} \times \mathbb{R}^d_+$*, and* $f_\alpha \in C^a_x$ *if* $|\alpha| = m$*,* $f_\alpha \in L_\infty$ *if* $|\alpha| < m$*,*

$A^{\alpha\beta} \in C^{a}_{x}$ for $|\alpha| = m$, $f_{\hat{\alpha}}, A^{\hat{\alpha}\beta} \in C^{\frac{a}{2m}}_{t}$, and $f_{\hat{\alpha}} = 0$ on $\{0\} \times \{x : x_d = 0\}$. Then under condition (3.2.1) for any $X \in [0,T] \times \overline{\mathbb{R}^d_+}$, there exists a constant C depending only on d, n, m, δ, K, $[A^{\hat{\alpha}\beta}]^{t}_{\frac{a}{2m}}$, $\|A^{\alpha\beta}\|^{x}_{a}$, and a such that

$$
\begin{aligned}
&[D^m u]_{\frac{a}{2m},a,(0,T)\times\mathbb{R}^d_+} \\
&\leq C\big(\|D^m u\|_{L_\infty((0,T)\times\mathbb{R}^d_+)} + \sum_{|\beta|<m} \|D^\beta u\|_{\frac{a}{2m},a,(0,T)\times\mathbb{R}^d_+} + G\big),
\end{aligned}
$$

where

$$
G := [f_{\hat{\alpha}}]^{t}_{\frac{a}{2m},(0,T)\times\{x:\,x_d=0\}} + \sum_{|\alpha|=m} [f_\alpha]^{x}_{a,(0,T)\times\mathbb{R}^d_+} + \sum_{|\alpha|\leq m} \|f_\alpha\|_{L_\infty((0,T)\times\mathbb{R}^d_+)}.
$$

Proof. We extend u and f_α to be zero when $t < 0$. By checking the definition (3.2.5), we see that u satisfies the system in $(-\infty, T) \times \mathbb{R}^d_+$ with the conormal derivative boundary conditions on $\{x_d = 0\}$. Moreover, by our compatibility condition $f_{\hat{\alpha}}$ is Hölder continuous in t on $(-\infty, T] \times \{x : x_d = 0\}$. We then use a scaling argument and consider the equation of v in the corresponding domain $(-\infty, T_1) \times \mathbb{R}^d_+$, where $T_1 = \mu^{2m} T$. If we can prove the inequality above for v, then after changing back to u, we prove the lemma. By translation of the coordinates and applying Theorem 3.2.5 to (3.6.17), we get

$$
\begin{aligned}
[D_{x'} D^{m-1} v]_{\frac{a}{2m},a,Q^+_R(X_0)} &\leq C(\mu)\big(\sum_{|\alpha|\leq m} \|D^\alpha v\|_{L_\infty(Q^+_{2R}(X_0))} \\
&+ \sum_{|\alpha|=m} [\tilde{f}_\alpha]^{x}_{a,Q^+_{2R}(X_0)} + \sum_{|\alpha|<m} \|\tilde{f}_\alpha\|_{L_\infty(Q^+_{2R}(X_0))}\big),
\end{aligned}
\tag{3.6.18}
$$

for any $X_0 \in (-\infty, T_1] \times \partial\mathbb{R}^d_+$. Similarly, translating of the coordinates and applying

Lemma 3.4.1, Proposition 3.5.1 to (3.6.17), we get

$$\begin{aligned}[][D^m v]_{\frac{a}{2m},a,Q_R(X_0)} \le C(\mu)\Big(&\sum_{|\beta|\le m}\|D^\beta v\|_{L_\infty(Q_{2R}(X_0))}\\ &+\sum_{|\alpha|=m}[\tilde f_\alpha]^x_{a,Q_{2R}(X_0)}+\sum_{|\alpha|<m}\|\tilde f_\alpha\|_{L_\infty(Q_{2R}(X_0))}\Big),\end{aligned} \tag{3.6.19}$$

for any $X_0 \in (-\infty, T_1] \times \mathbb{R}^d_+$ satisfying $Q_{2R}(X_0) \subset (-\infty, T_1) \times \mathbb{R}^d_+$. Using the arguments of partition of unity and translation of the coordinates, we deduce from Lemma 3.6.8, (3.6.18), and (3.6.19) that

$$\begin{aligned}&[D^m v]_{\frac{a}{2m},a,(0,T_1)\times\mathbb{R}^d_+}\\ &\le C_1(\mu)\big(\|D^m v\|_{L_\infty((0,T_1)\times\mathbb{R}^d_+)}+\sum_{|\beta|<m}\|D^\beta v\|_{\frac{a}{2m},a,(0,T_1)\times\mathbb{R}^d_+}+\tilde G\big)\\ &\quad + C_2\mu^{-1}[D^m_d v]_{\frac{a}{2m},a,(0,T_1)\times\mathbb{R}^d_+},\end{aligned}$$

where $\tilde G$ is defined in (3.6.11) with $(0,T_1)\times\mathbb{R}^d_+$ in place of Q^+_{2R}, $C_1(\mu)$ depends on μ, but C_2 does not. Let μ be sufficiently large such that $C_2\mu^{-1}[D^m_d v]_{\frac{a}{2m},a,(0,T)\times\mathbb{R}^d_+}$ is absorbed in the left-hand side. Then we fix this μ and obtain that

$$\begin{aligned}&[D^m v]_{\frac{a}{2m},a,(0,T_1)\times\mathbb{R}^d_+}\\ &\le C\big(\|D^m v\|_{L_\infty((0,T_1)\times\mathbb{R}^d_+)}+\sum_{|\beta|<m}\|D^\beta v\|_{\frac{a}{2m},a,(0,T_1)\times\mathbb{R}^d_+}+\tilde G\big).\end{aligned}$$

Therefore, the proposition is proved. □

Next, we obtain the estimate in the t variable by applying the method in Proposition 3.5.1.

Proposition 3.6.10. *Let $a \in (0,1)$ and $R \in (0,1]$ be constants. Suppose that*

$u \in C_{loc}^{\frac{a+m}{2m}, m+a}(\overline{\mathcal{O}_0^+})$ *satisfies*

$$u_t + (-1)^m \mathcal{L} u = \sum_{|\alpha| \le m} D^\alpha f_\alpha \quad in \quad Q_{2R}^+$$

with the conormal derivative boundary conditions on $\{x_d = 0\}$. *The coefficients of* $\mathcal{L}$ *and* f_α *satisfy the same condition as in Theorem 3.2.6. Then there exists a constant* C *depending only on* d, m, n, δ, K, $\|A^{\alpha\beta}\|^x_{a,Q_{2R}^+}$, R, *and* a *such that*

$$[u]^t_{\frac{1}{2}+\frac{a}{2m},Q_R^+} \le C(\|D^m u\|^x_{a,Q_{2R}^+} + \sum_{|\beta|<m} \|D^\beta u\|_{L_\infty(Q_{2R}^+)} + F),$$

where F *is defined in* (3.6.8).

Due to Propositions 3.6.9, 3.6.10, and 3.5.1, we use the standard arguments of partition of unity and flattening the boundary to obtain

$$\begin{aligned} &[u]^t_{\frac{a+m}{2m},(0,T)\times\Omega} + [D^m u]_{\frac{a}{2m},a,(0,T)\times\Omega} \\ &\le C\big(\|D^m u\|_{L_\infty((0,T)\times\Omega)} + \sum_{|\beta|<m} \|D^\beta u\|_{\frac{a}{2m},a,(0,T)\times\Omega} + G\big). \end{aligned}$$

By the interpolation inequalities in Hölder spaces (see, for instance, [63, §8.8]),

$$[u]^t_{\frac{a+m}{2m},(0,T)\times\Omega} + [D^m u]_{\frac{a}{2m},a,(0,T)\times\Omega} \le C(\|u\|_{L_\infty((0,T)\times\Omega)} + G).$$

Applying the interpolation inequalities again, we get

$$\begin{aligned} &\|u\|_{L_\infty((0,T)\times\Omega)} \\ &\le C(\varepsilon)\|u\|_{L_2((0,T)\times\Omega)} + \varepsilon\big([D^m u]_{\frac{a}{2m},a,(0,T)\times\Omega} + [u]^t_{\frac{a+m}{2m},(0,T)\times\Omega}\big). \end{aligned}$$

Upon taking ε sufficiently small, we arrive at (3.2.10). In order to implement the method of continuity to prove the existence of solutions, we need the right-hand side of (3.2.10) to be independent of u and this leads us to consider the system:

$$u_t + (-1)^m \mathcal{L}u + \kappa u = \sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in} \quad (0,T)\times\Omega \tag{3.6.20}$$

with the conormal derivative boundary conditions on $(0,T)\times\partial\Omega$ and the zero initial condition on $\{0\}\times\Omega$. We claim that for κ sufficiently large,

$$\|u\|_{L_2((0,T)\times\Omega)} \le C\sqrt{T}G, \tag{3.6.21}$$

where $C = C(d,n,m,K,\|A^{\alpha\beta}\|_a^x,\Omega)$. We extend $u = 0$ and $f_\alpha = 0$ for any α when $t < 0$. It is easy to check that (3.6.20) holds in $(-\infty,T)\times\Omega$ with the conormal derivative boundary conditions on $(-\infty,T)\times\partial\Omega$. Furthermore, since $A^{\alpha\beta}$ are Hölder continuous in x, which implies $A^{\alpha\beta}$ are VMO_x, we can apply Theorem 3.2.2 and set $\kappa = \lambda_0 + 1$ to obtain

$$\|u\|_{L_2((0,T)\times\Omega)} \le C\sum_{|\alpha|\le m} \kappa^{\frac{|\alpha|}{2m}-1}\|f_\alpha\|_{L_2((0,T)\times\Omega)} \le C\sum_{|\alpha|\le m}\|f_\alpha\|_{L_2((0,T)\times\Omega)}.$$

Therefore, we reach (3.6.21).

Finally, we follows the proof of Theorem 3.2.3 to show the solvability in Theorem 3.2.6 and omit the proof.

Chapter 4

Neumann problem for non-divergence elliptic and parabolic equations with BMO$_x$ coefficients in weighted sobolev spaces

4.1 Introduction

In this chapter, we study L_p estimates for elliptic and parabolic equations in non-divergence form:

$$a_{ij}D_{ij}u + b_iD_iu + cu - \lambda u = f \quad \text{in } \mathbb{R}^d_+,$$

$$-u_t + a_{ij}D_{ij}u + b_iD_iu + cu - \lambda u = f \quad \text{in } (-\infty, T) \times \mathbb{R}^d_+,$$

with the homogeneous Neumann boundary condition, where λ is a nonnegative constant and $\mathbb{R}^d_+$ is a half space defined by

$$\mathbb{R}^d_+ = \{x = (x_1, \cdots, x_d) = (x', x_d) : x_d > 0\}.$$

We consider the equations in weighted Sobolev spaces with measures

$$\mu_d(dx) = x_d^{\theta-d}\, dx \quad \text{and} \quad \mu(dx\, dt) = x_d^{\theta-d}\, dx\, dt$$

in the elliptic and parabolic cases, respectively, for some $\theta \in (d-1, d-1+p)$.

Krylov [64] first studied Laplace's equation and the heat equation in weighted Sobolev spaces $H^\gamma_{p,\theta}$ and $\mathbb{H}^\gamma_{p,\theta}$; see Section 2 for precise definitions. After [64], there has been quite a few work on the solvability theory for elliptic and parabolic equations in weighted Sobolev spaces, for instance, see [54, 60, 55, 53]. In particular, the authors of [55, 53] studied second-order parabolic equations with the Dirichlet boundary condition in weighted Sobolev spaces with leading coefficients having small mean oscillations. The motivation of such theory came from stochastic partial differential equations (SPDEs) and is well explained in [62].

Recently, Dong and Kim [41] studied both divergence and non-divergence type elliptic and parabolic equations on a half space in weighted Sobolev spaces with the Dirichlet boundary condition. The coefficients in [41] are contained in a larger class than those in [55, 53]. Namely, the leading coefficients are assumed to be only measurable in t and x_d except a_{dd}, which is measurable in either t or x_d, where x_d is the normal direction. Kozlov and Nazarov [59] considered an oblique derivative problem for non-divergence type parabolic equations on a half space with coefficients discontinuous in t (and continuous in x) in a weighted Sobolev space. Their proof is based on a careful investigation of Green's functions. In this chapter, we extend the result in [59] to a more general setting. Namely, the coefficients considered in this chapter are measurable in the time variable and have small mean oscillations with respect to a weighted measure in the spatial variables. We call this class of coefficients BMO_x. The weight, for instance, for the parabolic case is $\mu(dx\,dt) = x_d^{\theta-d}\,dx\,dt$, where $\theta \in (d-1, d-1+p)$. The condition $\theta \in (d-1, d-1+p)$ is sharp even for the heat equation; see [64]. We note that the coefficients a_{ij} in [41] also have small mean oscillations with respect to a weighted measure as functions of $x' \in \mathbb{R}^{d-1}$ (whereas, in this chapter as functions of $x \in \mathbb{R}^d$), but the size of the modulus of regularity of a_{ij} is proportional to the distance to the boundary. See Assumption 4.2.1 and Remark 4.2.

Our proof is in the spirit of the approach by Krylov [63] and one of the main steps is to get the mean oscillation estimates of D^2u. For a simple equation

$$-u_t + a_{ij}(t)D_{ij}u = f$$

in $\mathbb{R} \times \mathbb{R}^d_+$ with the Neumann boundary condition $D_d u = 0$ on $\{x_d = 0\}$, we treat $DD_d u$ and $D^2_{x'}u$ separately. We estimate $DD_d u$ as follows. Differentiating the equation above with respect to x_d, it is easily seen that $D_d u$ satisfies the divergence type parabolic equation

$$-(D_d u)_t + D_i(a_{ij}D_j(D_d u)) = D_d f$$

in $\mathbb{R} \times \mathbb{R}^d_+$ with $D_d u = 0$ on $\{x_d = 0\}$. Therefore, we can apply a result in [41] to obtain the mean oscillation estimates of $DD_d u$. On the other hand, the estimates of $D^2_{x'}u$ are much involved. We treat the mean oscillations of $D^2_{x'}u$ in the x_d variable and x' variables differently. By integrating by parts and the Poincaré inequality in weighted spaces, we manage to bound the mean oscillations of $D^2_{x'}u$ in the x_d variable by the maximal functions of $DD_d u$. For the mean oscillations in x' variables, we write the equation in the following form

$$-u_t + \sum_{i,j<d} a_{ij} D_{ij} u = f - \sum_{j=1}^{d-1} (a_{jd} + a_{dj}) D_{dj} u - a_{dd} D_{dd} u,$$

which can be regarded as a non-divergence type parabolic equation in $\mathbb{R} \times \mathbb{R}^{d-1}$. Then by applying an interior estimate result without weights for $D^2 u$, where u is, as a function of $x' \in \mathbb{R}^{d-1}$, a solution of a non-divergence type equation in the whole space $\mathbb{R}^{d-1}$ (see, for instance, [63]), we bound the mean oscillations of $D_{x'}^2 u$ in the x' variables by the maximal functions of f, $DD_d u$ and $D_{x'}^2 u$.

This chapter is organized as follows. In the next section, we introduce some notation and state our main results. In Section 3, we obtain the mean oscillation estimates for $D_{x'}^2 u$ and $DD_d u$ separately for a parabolic equation with simple coefficients. In Section 4, we prove our main theorem (Theorem 4.2.2).

4.2 Preliminaries and main results

In addition to the notation of the previous chapter, we set $\mathbb{R}_+^{d+1} = \{(t,x) : x_d > 0\}$. For $r > 0$, let $B'_r(x')$ be the open ball in $\mathbb{R}^{d-1}$ of radius r with center x'. Note that the definition of the ball in $\mathbb{R}^d$ is different

$$B_r(x) = B_r(x', x_d) = B'_r(x') \times (x_d - r, x_d + r),$$

$$Q_r(t,x) = (t - r^2, t) \times B_r(x),$$

$$B_r^+(x) = B_r(x) \cap \mathbb{R}_+^d, \quad Q_r^+(t,x) = Q_r(t,x) \cap \mathbb{R}_+^{d+1}.$$

For $a \in \mathbb{R}$, we use $Q_r(a)$ to denote

$$Q_r(0,0,a) = (-r^2,0) \times (a-r,a+r) \times B'_r(0),$$

and $Q_r = Q_r(0)$. Similarly, we define $Q_r^+(a)$ and Q_r^+. By a^+ we mean $\max\{a,0\}$.

Throughout the chapter, we assume that the leading coefficients a_{ij} are bounded, measurable, and satisfy the ellipticity condition:

$$a_{ij}\eta_i\eta_j \geq \delta|\eta|^2, \quad |a_{ij}| \leq 1/\delta$$

for any $\eta \in \mathbb{R}^d$, where $\delta > 0$ is a constant.

To introduce the function spaces used in this chapter, we first recall the weighted Sobolev spaces $H^\gamma_{p,\theta}$ introduced in [64]. If γ is a non-negative integer

$$H^\gamma_{p,\theta} = H^\gamma_{p,\theta}(\mathbb{R}^d_+) = \{u : x_d^{|\alpha|}D^\alpha u \in L_{p,\theta}(\mathbb{R}^d_+) \quad \forall\, \alpha : 0 \leq |\alpha| \leq \gamma\},$$

where $L_{p,\theta}(\mathbb{R}^d_+)(= L_{p,\theta})$ is a Lebesgue space with the measure $\mu_d(dx) = x_d^{\theta-d}\,dx$. For a general real number γ, $H^\gamma_{p,\theta}$ is defined as follows. Take and fix a nonnegative function $\zeta \in C_0^\infty(0,\infty)$ such that

$$\sum_{n=-\infty}^{\infty} \zeta^p(e^{x_d-n}) \geq 1$$

for all $x_d \in \mathbb{R}$. For any $\gamma, \theta \in \mathbb{R}$ and $p \in (1,\infty)$, let $H^\gamma_{p,\theta}$ be the set of all functions u on $\mathbb{R}^d_+$ such that

$$\|u\|^p_{\gamma,p,\theta} = \sum_{n=-\infty}^{\infty} e^{n\theta}\|u(e^n\cdot)\zeta(x_d)\|^p_{\gamma,p} < \infty,$$

where $\|\cdot\|_{\gamma,p}$ is the norm in the Bessel potential space $H^\gamma_p(\mathbb{R}^d)$. For any $a \in \mathbb{R}$, let

M^α be the operator of multiplication by $(x_d)^\alpha$ and $M := M^1$. We write $u \in M^\alpha H^\gamma_{p.\theta}$ if $M^{-\alpha}u \in H^\gamma_{p,\theta}$. We set

$$\mathbb{H}^\gamma_{p,\theta}(S,T) = L_p((S,T), H^\gamma_{p,\theta}), \quad \mathbb{L}_{p,\theta}(S,T) = L_p((S,T), L_{p,\theta}),$$

where $-\infty \le S < T \le \infty$.

Our solution spaces are defined as follows. For the elliptic case, we set

$$W^2_{p,\theta}(\mathbb{R}^d_+) = \{u : u, Du, D^2u \in L_{p,\theta}(\mathbb{R}^d_+)\}.$$

For the parabolic case,

$$W^{1,2}_{p,\theta}(S,T) = \{u : u, Du, D^2u, u_t \in \mathbb{L}_{p,\theta}(S,T)\},$$

where $-\infty \le S < T \le \infty$.

Throughout the chapter, we use the weighted measures:

$$\mu_d(dx) := (x_d)^{\theta-d}\,dx, \quad \mu(dx\,dt) := (x_d)^{\theta-d}\,dx\,dt,$$

where $\theta \in (d-1, d-1+p)$ and $p \in (1,\infty)$.

Now we state our regularity assumption on the leading coefficients. For a function g on $\mathbb{R}^{d+1}_+$, denote

$$[g(t,\cdot)]_{B_r^+(x)} = \fint_{B_r^+(x)} |g(t,y) - \fint_{B_r^+(x)} g(t,z)\,\mu_d(dz)|\,\mu_d(dy).$$

Then we define the mean oscillation of g in $Q_r^+(s,y)$ with respect to x as

$$\mathrm{osc}_x(g, Q_r^+(s,y)) = \fint_{s-r^2}^{s} [g(\tau,\cdot)]_{B_r^+(y)}\, d\tau,$$

and denote

$$g_R^{\#} = \sup_{(s,y)\in\mathbb{R}_+^{d+1}} \sup_{r\le R} \mathrm{osc}_x(g, Q_r^+(s,y)). \tag{4.2.1}$$

Using the notation above with a_{ij} in place of g, we state the following regularity assumption on a_{ij} with a sufficiently small parameter $\rho > 0$ to be specified later.

Assumption 4.2.1 (ρ)**.** There exists a positive constant R_0 such that

$$A_{R_0}^{\#} := \sup_{i,j} (a_{ij})_{R_0}^{\#} \le \rho.$$

Note that under this assumption, the coefficients a_{ij} may not have any regularity with respect to t.

Remark. While we have a fixed size of the modulus of regularity R_0 above, in [41] the size of the modulus is proportional to the distance from the boundary to the location where the mean oscillations of a_{ij} are measured. To express this, one can replace R in (4.2.1) by $y_d R$. This means, in particular, that the coefficients a_{ij} in [41] are allowed to be much rougher *near the boundary* than those in this chapter.

For lower-order terms, we assume that the coefficients b_i and c are only measurable (without any regularity assumptions) and bounded so that

$$|b_i|,\ |c| \le K$$

for some constant $K > 0$.

The following theorems are our main results, the first of which is the unique solvability of parabolic equations.

Theorem 4.2.2. *Let $T \in (-\infty, \infty]$, $1 < p < \infty$, and $\theta \in (d-1, d-1+p)$ be constants. Then there exist constants $\rho = \rho(d, \delta, p, \theta) > 0$ and $\lambda_0 = \lambda_0(d, \delta, p, \theta, K, R_0) \geq 0$ such that under Assumption 4.2.1 (ρ) the following assertions hold.*

(i) Suppose that $u \in W^{1,2}_{p,\theta}(-\infty, T)$ satisfies

$$-u_t + a_{ij} D_{ij} u + b_i D_i u + cu - \lambda u = f \tag{4.2.2}$$

in $(-\infty, T) \times \mathbb{R}^d_+$ with the Neumann boundary condition $D_d u = 0$ on $\{x_d = 0\}$, where $f \in \mathbb{L}_{p,\theta}(-\infty, T)$. Then we have

$$\|u_t\|_{p,\theta} + \|M^{-1} D_d u\|_{p,\theta} + \|D^2 u\|_{p,\theta} + \sqrt{\lambda}\|Du\|_{p,\theta} + \lambda\|u\|_{p,\theta} \leq C_0 \|f\|_{p,\theta} \tag{4.2.3}$$

provided that $\lambda \geq \lambda_0$, where $\|\cdot\|_{p,\theta} = \|\cdot\|_{\mathbb{L}_{p,\theta}(-\infty,T)}$ and $C_0 = C_0(d, \delta, p, \theta)$. In particular, when $b_i = c = 0$ and $a_{ij} = a_{ij}(t)$, we can take $\lambda_0 = 0$.

(ii) For any $\lambda > \lambda_0$ and $f \in \mathbb{L}_{p,\theta}(-\infty, T)$, there is a unique solution $u \in W^{1,2}_{p,\theta}(-\infty, T)$ to the equation (4.2.2).

We now present our results for elliptic equations, where the coefficients are independent of t. Since Assumption 4.2.1 (ρ) does not concern the regularity of coefficients in t, we still require the coefficients to satisfy Assumption 4.2.1 (ρ).

By adapting, for example, the proof of [63, Theorem 2.6] to the results above for parabolic equations, i.e., by regarding elliptic equations as steady state parabolic equations, we obtain the following theorem for elliptic equations.

Theorem 4.2.3. *Let $1 < p < \infty$ and $\theta \in (d-1, d-1+p)$ be constants. Then there exist constants $\rho = \rho(d, \delta, p, \theta) > 0$ and $\lambda_0 = \lambda_0(d, \delta, p, \theta, K, R_0) \geq 0$ such that under Assumption 4.2.1 (ρ) the following assertions hold.*

(i) Suppose that $u \in W^2_{p,\theta}(\mathbb{R}^d_+)$ satisfies

$$a_{ij}D_{ij}u + b_iD_iu + cu - \lambda u = f \tag{4.2.4}$$

in $\mathbb{R}^d_+$ with the Neumann boundary condition $D_du = 0$ on $\{x_d = 0\}$, where $f \in L_{p,\theta}(\mathbb{R}^d_+)$. Then we have

$$\|M^{-1}D_du\|_{p,\theta} + \|D^2u\|_{p,\theta} + \sqrt{\lambda}\|Du\|_{p,\theta} + \lambda\|u\|_{p,\theta} \leq C_0\|f\|_{p,\theta}$$

provided that $\lambda \geq \lambda_0$, where $\|\cdot\|_{p,\theta} = \|\cdot\|_{L_{p,\theta}(\mathbb{R}^d_+)}$ and $C_0 = C_0(d, \delta, p, \theta)$. In particular, when $b_i = c = 0$ and a_{ij} are constant, we can take $\lambda_0 = 0$.

(ii) For any $\lambda > \lambda_0$ and $f \in L_{p,\theta}(\mathbb{R}^d_+)$, there is a unique solution $u \in W^2_{p,\theta}(\mathbb{R}^d_+)$ to the equation (4.2.4).

4.3 Equations with coefficients independent of x

In this section, we deal with equations in the form

$$u_t - a_{ij}(t)D_{ij}u = f \quad \text{in} \quad \mathbb{R}^{d+1}_+, \tag{4.3.1}$$

$$D_du = 0 \quad \text{on} \quad \{x_d = 0\}. \tag{4.3.2}$$

Note that now the coefficients a_{ij} depend only on t. Let us state several technical lemmas. The first one is Hardy's inequality, which can be found in [72].

Lemma 4.3.1. *Let $p \in (1,\infty)$, $\theta \in (d-1, d-1+p)$, and $v \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ such that $v = 0$ on $\{x_d = 0\}$. Then*

$$\int_{B_r^+} |v|^p y_d^{\theta-d-p}\, dy \le C \int_{B_r^+} |D_d v|^p y_d^{\theta-d}\, dy$$

for any $r \in (0,\infty]$, where $C = C(d,p,\theta)$.

We summarize Lemmas 4.2 and 4.3 in [41] as the following results, which were proved by localizing the results in [51, 28] and using the Sobolev embedding theorem.

Lemma 4.3.2. *Let $p \in (1,\infty)$ and $\alpha \in (0,1)$ be constants. Assume that $u \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ satisfies (4.3.1) in Q_2^+ with $f = 0$, and $u = 0$ on $\{x_d = 0\}$. Then we have*

$$\|Du\|_{C^{\alpha/2,\alpha}(Q_1^+)} \le C\|u\|_{L_p(Q_2^+)},$$

where $C = C(d,\delta,p,\alpha)$. If Q_1 and Q_2 replace Q_1^+ and Q_2^+, respectively, then

$$\|Du\|_{C^{\alpha/2,\alpha}(Q_1)} \le C\|Du\|_{L_p(Q_2)}.$$

For a domain $\mathcal{D} \subset \mathbb{R}^{d+1}_+$, we denote $(u)_{\mathcal{D}}$ to be the average of u in $\mathcal{D}$ with respect to the measure $\mu(dx\,dt) = x_d^{\theta-d}\,dx\,dt$. Precisely,

$$(u)_{\mathcal{D}} = \frac{1}{\mu(\mathcal{D})}\int_{\mathcal{D}} u(t,x)\mu(dx\,dt) = \fint_{\mathcal{D}} u(t,x)\mu(dx\,dt),$$

where

$$\mu(\mathcal{D}) = \int_{\mathcal{D}} \mu(dx\,dt).$$

Using Lemma 4.3.2, we obtain the following mean oscillation type estimate.

Corollary 4.3.3. *Let $p \in (1,\infty)$, $\alpha \in (0,1)$, $y_d \in [0,1]$, and $\theta \in (d-1, d-1+p)$ be constants. Assume that $u \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ satisfies*

$$u_t - a_{ij}(t) D_{ij} u = 0 \quad \text{in} \quad Q_4^+$$

with the Dirichlet boundary condition $u = 0$ on $\{x_d = 0\}$. Then there exists a constant $C = C(d, \delta, p, \theta, \alpha)$ such that for any $r < 1$,

$$(|Du - (Du)_{Q_r^+(y_d)}|^p)_{Q_r^+(y_d)} \le C r^{\alpha p} \int_{Q_4^+} |D_d u|^p \, \mu(dx\, dt).$$

Before we state the next theorem, we introduce a function space. For $\lambda \ge 0$, we denote $u \in \mathcal{H}^{1,\lambda}_{p,\theta}(-\infty, T)$ if

$$\sqrt{\lambda} u,\ M^{-1} u,\ Du \in \mathbb{L}_{p,\theta}(-\infty, T),$$

and $u_t \in M^{-1}\mathbb{H}^{-1}_{p,\theta}(-\infty,T) + \sqrt{\lambda}\mathbb{L}_{p,\theta}(-\infty,T)$. Let us write $\mathcal{H}^1_{p,\theta}(-\infty,T)$ if $\lambda = 0$. By [64, Remark 5.3], one can find $f, g = (g_1, \ldots, g_d) \in \mathbb{L}_{p,\theta}(-\infty, T)$ such that

$$u_t = D_i g_i + \sqrt{\lambda} f$$

in $(-\infty, T) \times \mathbb{R}^d_+$. We set

$$\begin{aligned} \|u\|_{\mathcal{H}^{1,\lambda}_{p,\theta}(-\infty,T)} = \inf \Big\{ & \sqrt{\lambda}\|u\|_{p,\theta} + \|M^{-1}u\|_{p,\theta} + \|Du\|_{p,\theta} \\ & + \|g\|_{p,\theta} + \|f\|_{p,\theta} : u_t = D_i g_i + \sqrt{\lambda} f \Big\}. \end{aligned}$$

Now we state a special case ($a_{ij} = a_{ij}(t)$) of [41, Theorem 3.9], where a_{ij} are allowed to be merely measurable in (t, x_d) except that $a_{dd} = a_{dd}(t)$ or $a_{dd} = a_{dd}(x_d)$. Note that in the theorem below there is no specification of the boundary condition, but functions in the solution space $\mathcal{H}^{1,\lambda}_{p,\theta}(-\infty, T)$ necessarily satisfies $u = 0$ on the boundary. Hence the theorem is about the Dirichlet boundary value problem for divergence type equations in weighted Sobolev spaces.

Theorem 4.3.4. *Let $T \in (-\infty, \infty]$, $\lambda \geq 0$, $p \in (1, \infty)$, $\theta \in (d-1, d-1+p)$, and $u \in \mathcal{H}^{1,\lambda}_{p,\theta}(-\infty, T)$ satisfy*

$$u_t - D_i(a_{ij} D_j u) + \lambda u = D_i g_i + f \tag{4.3.3}$$

in $(-\infty, T) \times \mathbb{R}^d_+$, where $g = (g_1, g_2, \ldots, g_d)$, $g, f \in \mathbb{L}_{p,\theta}(-\infty, T)$, and $f \equiv 0$ if $\lambda = 0$. Then we have

$$\sqrt{\lambda}\|u\|_{p,\theta} + \|M^{-1}u\|_{p,\theta} + \|Du\|_{p,\theta} \leq C\|g\|_{p,\theta} + \frac{C}{\sqrt{\lambda}}\|f\|_{p,\theta},$$

where $C = C(d, \delta, p, \theta)$. Moreover, for any $g, f \in \mathbb{L}_{p,\theta}(-\infty, T)$ such that $f \equiv 0$ if $\lambda = 0$, there exists a unique solution $u \in \mathcal{H}^{1,\lambda}_{p,\theta}(-\infty, T)$ to the equation (4.3.3).

Next we consider (4.3.1)-(4.3.2) with non-vanishing right-hand side.

Lemma 4.3.5. *Let $p \in (1, \infty)$, $\theta \in (d-1, d-1+p)$, $\alpha \in (0, 1)$, $r > 0, \kappa \geq 32$, and $y_d \geq 0$. Assume that $f \in \mathbb{L}_{p,\theta}(Q^+_{\kappa r}(y_d))$ and $u \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ is a solution of*

$$u_t - a_{ij} D_{ij} u = f$$

in $Q^+_{\kappa r}(y_d)$ with the Neumann boundary condition $D_d u = 0$ on $\{x_d = 0\}$. Then we *have*

$$
\begin{aligned}
&(|DD_du - (DD_du)_{Q_r^+(y_d)}|^p)_{Q_r^+(y_d)} \\
&\leq C\kappa^{-\alpha p}(|DD_du|^p)_{Q_{\kappa r}^+(y_d)} + C\kappa^{d+\theta+2}(|f|^p)_{Q_{\kappa r}^+(y_d)},
\end{aligned}
$$

where $C = C(d, \delta, p, \theta, \alpha)$.

Proof. Denote $v = D_du$. Then v satisfies

$$
v_t - D_i(a_{ij}D_jv) = D_df
$$

in $Q_{\kappa r}^+(y_d)$ with the Dirichlet boundary condition $v = 0$ on $\{x_d = 0\}$. By a scaling argument, it is sufficient to set $\kappa r = 8$. We consider two cases.

Case 1: $y_d \in [0, 1]$. Since $r = 8/\kappa \leq 1/4$, we have

$$
Q_r^+(y_d) \subset Q_2^+ \subset Q_4^+ \subset Q_{\kappa r}^+(y_d).
$$

Let η be a smooth function with support in $(-(\kappa r)^2, (\kappa r)^2) \times B_{\kappa r}(0, y_d)$ and $\eta = 1$ in Q_4. By Theorem 4.3.4, there exists a unique solution $w \in \mathcal{H}_{p,\theta}^{1,0}(-\infty, 0) = \mathcal{H}_{p,\theta}^1(-\infty, 0)$ to the equation

$$
w_t - D_i(a_{ij}D_jw) = D_d(f\eta)
$$

in $(-\infty, 0) \times \mathbb{R}_+^d$, satisfying

$$
\|Dw\|_{p,\theta} \leq C\|f\eta\|_{p,\theta}.
$$

Due to the definition of η, this implies

$$\|Dw\|_{p,\theta}^p \le C\mu(Q_{\kappa r}^+(y_d))\, (|f|^p)_{Q_{\kappa r}^+(y_d)}\,. \tag{4.3.4}$$

By a standard mollification argument (see, for instance, [43, Theorem 4.7]), we may assume that w is smooth. Let $\hat{u} = v - w$, which is also smooth, and satisfies $\hat{u} = 0$ on $\{x_d = 0\}$ and

$$\hat{u}_t - D_i(a_{ij} D_j \hat{u}) = 0$$

in Q_4^+. By Corollary 4.3.3, we have

$$(|D\hat{u} - (D\hat{u})_{Q_r^+(y_d)}|^p)_{Q_r^+(y_d)} \le Cr^{\alpha p} \int_{Q_4^+} |D_d \hat{u}|^p \, \mu(dx\, dt). \tag{4.3.5}$$

Combining (4.3.4), (4.3.5), and the triangle inequality, we reach

$$\begin{aligned}
&(|DD_d u - (DD_d u)_{Q_r^+(y_d)}|^p)_{Q_r^+(y_d)} \\
&\le C(|D\hat{u} - (D\hat{u})_{Q_r^+(y_d)}|^p)_{Q_r^+(y_d)} + C(|Dw|^p)_{Q_r^+(y_d)} \\
&\le Cr^{\alpha p}(|DD_d u|^p)_{Q_4^+} + C\mu(Q_r^+(y_d))^{-1}\mu(Q_{\kappa r}^+(y_d))(|f|^p)_{Q_{\kappa r}^+(y_d)}.
\end{aligned}$$

Bearing in mind that $\mu(Q_r^+(y_d)) \ge Cr^{\sigma+2}$, where $\sigma = \max\{d, \theta\}$ and $r = 8/\kappa$, we prove the lemma for Case 1.

Case 2: $y_d > 1$. This is essentially an interior case. Since $r = 8/\kappa \le 1/4$, we have

$$Q_r^+(y_d) = Q_r(y_d) \subset Q_{1/4}(y_d) \subset Q_{1/2}(y_d) \subset Q_{\kappa r}^+(y_d).$$

As in Case 1, we take a smooth function η with support in $(-(\kappa r)^2, (\kappa r)^2) \times B_{\kappa r}(0, y_d)$ and $\eta = 1$ on $Q_{1/2}(y_d)$. Then by Theorem 4.3.4, there exists a unique solution

$w \in \mathcal{H}^1_{p,\theta}(-\infty, 0)$ of the equation

$$w_t - D_i(a_{ij} D_j w) = D_d(f\eta)$$

in $(-\infty, 0) \times \mathbb{R}^d_+$, satisfying

$$\|Dw\|_{p,\theta} \le C\|f\eta\|_{p,\theta}.$$

Then we get

$$\|Dw\|^p_{p,\theta} \le C\mu(Q^+_{\kappa r}(y_d)) \left(|f|^p\right)_{Q^+_{\kappa r}(y_d)}. \tag{4.3.6}$$

For the same reason as before, we may assume that w is smooth and so is $\hat{u} = D_d u - w$. It is easily seen that

$$\hat{u}_t - a_{ij} D_{ij} \hat{u} = 0 \quad \text{in} \quad Q_{1/2}(y_d).$$

By Lemma 4.3.2, we have

$$\|D\hat{u}\|_{C^{\alpha/2,\alpha}(Q_{1/4}(y_d))} \le C\|D\hat{u}\|_{L_p(Q_{1/2}(y_d))}.$$

From this and (4.3.6), we apply the triangle inequality

$$\begin{aligned}
&\left(|DD_d u - (DD_d u)_{Q_r(y_d)}|^p\right)_{Q_r(y_d)} \\
&\le C\left(|D\hat{u} - (D\hat{u})_{Q_r(y_d)}|^p\right)_{Q_r(y_d)} + C\left(|Dw|^p\right)_{Q_r(y_d)} \\
&\le Cr^{\alpha p}\|D\hat{u}\|^p_{L_p(Q_{1/2}(y_d))} + C\mu(Q_r(y_d))^{-1}\mu(Q^+_{\kappa r}(y_d))(|f|^p)_{Q^+_{\kappa r}(y_d)}.
\end{aligned} \tag{4.3.7}$$

Since for any $x_d \in (y_d - 1/2, y_d + 1/2)$,

$$C_1 \le \frac{x_d^{\theta-d}}{\mu(Q_{1/2}(y_d))} \le C_2,$$

where $C_{1,2} = C_{1,2}(d,\theta)$, by (4.3.7), (4.3.6), and the triangle inequality, it holds that

$$\begin{aligned}
&(|DD_d u - (DD_d u)_{Q_r(y_d)}|^p)_{Q_r(y_d)} \\
&\le Cr^{\alpha p}(|DD_d u|^p)_{Q_{1/2}(y_d)} + C\mu(Q_r(y_d))^{-1}\mu(Q^+_{\kappa r}(y_d))(|f|^p)_{Q^+_{\kappa r}(y_d)}.
\end{aligned}$$

Taking into account that $\mu(Q_r(y_d)) \ge Nr^{d+2}\mu(Q_8^+(y_d))$, we prove the lemma for the second case. $\square$

It remains to estimate $D_{ij}u$ with $i,j > 1$. Let us first state a Poincaré inequality in weighted L_p spaces.

Lemma 4.3.6. *Let $a \in \overline{\mathbb{R}^+}$, $\alpha \in (-1,\infty)$, $r \in (0,\infty)$, $p \in [1,\infty)$, $\nu(dx) = (x_d)^\alpha\, dx$, $x \in \mathbb{R}^d_+$, and $u \in C^\infty_{loc}(\overline{\mathbb{R}^d_+})$. Then*

$$\begin{aligned}
&\int_{B_r^+(a)}\int_{B_r^+(a)} |u(x)-u(y)|^p\,\nu(dx)\,\nu(dy) \\
&\le Cr^p\nu(B_r^+(a))\int_{B_r^+(a)} |Du|^p\,\nu(dx),
\end{aligned} \tag{4.3.8}$$

where $C = C(d,p,\alpha)$.

Proof. When $\alpha \in [0,\infty)$, the inequality is proved in [55, Lemma 4.1] with a missing constant depending on d. For the sake of completeness, we here present a proof when $\alpha \in (-1,\infty)$. Since the weight is with respect to x_d, we only prove (4.3.8) for $d = 1$. In fact, to prove (4.3.8) for $d > 1$ we just need to combine the case when $d = 1$ with the unweighted Poincaré inequality.

Due to scaling, it suffices to prove (4.3.8) with $r = 1$. We further assume $a \in (0, 2)$. Indeed, if $a \geq 2$, the inequality (4.3.8) is equivalent to the usual Poincaré inequality without weights. For each $x, y \in ((a-1)^+, a+1)$, by Hölder's inequality,

$$
\begin{aligned}
&|u(x) - u(y)|^p = \Big| \int_{\min\{x,y\}}^{\max\{x,y\}} u'(z)\, dz \Big|^p \\
&\leq C \left(y^{1-\alpha q/p} + x^{1-\alpha q/p} \right)^{p/q} \int_{\min\{x,y\}}^{\max\{x,y\}} |u'(z)|^p z^\alpha \, dz \\
&\leq C \left(y^{1-\alpha q/p} + x^{1-\alpha q/p} \right)^{p/q} \int_{(a-1)^+}^{a+1} |u'(z)|^p z^\alpha \, dz,
\end{aligned}
$$

where $1/p + 1/q = 1$ and $C = C(p, \alpha)$. Then, to conclude (4.3.8), we integrate the above inequalities with respect to $\nu(dx)$ and $\nu(dy)$, and use the fact that $a \in (0, 2)$. $\square$

In the sequel, we denote the standard parabolic cylinder in $\mathbb{R}^{d+1}$ as

$$\hat{Q}_r^{d+1}(X) = (t - r^2, t) \times \hat{B}_r(x),$$

where $\hat{B}_r(x)$ is the Euclidean ball in $\mathbb{R}^d$ with radius r and center x. For a function f on $\mathbb{R}^{d+1}$, we define the average of f in $\hat{Q}_r^{d+1}(X)$ without weight as

$$\langle f \rangle_{\hat{Q}_r^{d+1}(X)} = \frac{1}{|\hat{Q}_r^{d+1}(X)|} \int_{\hat{Q}_r^{d+1}(X)} f(t, x)\, dx\, dt.$$

Theorem 4.3.7. *Let $p \in (1, \infty)$ and $u \in C_{loc}^\infty(\mathbb{R}^{d+1})$ satisfy*

$$-u_t + a_{ij}(t) D_{ij} u = f \quad \textit{in} \quad \mathbb{R}^{d+1}.$$

Then there exists a constant $C = C(d, \delta, p)$ such that for any $\kappa \geq 4, r > 0$, we have

$$\Big\langle |D^2 u - \langle D^2 u\rangle_{\hat{Q}_r^{d+1}}|^p \Big\rangle_{\hat{Q}_r^{d+1}} \leq C\kappa^{-p} \langle |D^2 u|^p\rangle_{\hat{Q}_{\kappa r}^{d+1}} + C\kappa^{d+2}\langle |f|^p\rangle_{\hat{Q}_{\kappa r}^{d+1}}.$$

Proof. See [67, Theorem 5.1]. □

To estimate $D_{x'}^2 u$, we introduce the following notation. For $\mathrm{y}_d \geq 0$, set $B_r^{1+}(\mathrm{y}_d) = ((\mathrm{y}_d - r)^+, \mathrm{y}_d + r) \subset \mathbb{R}_+$, $\mu_1(dx_d) = x_d^{\theta - d}\, dx_d$, and

$$\mu_1(B_r^{1+}(\mathrm{y}_d)) = \int_{(\mathrm{y}_d - r)^+}^{\mathrm{y}_d + r} x_d^{\theta - d}\, dx_d.$$

Lemma 4.3.8. *Let $p \in (1, \infty), \theta \in (d-1, d-1+p), r > 0, \kappa \geq 32$, and $\mathrm{y}_d \geq 0$. Assume that $f \in \mathbb{L}_{p,\theta}(Q_{\kappa r}^+(\mathrm{y}_d))$ and $u \in C_{loc}^\infty(\overline{\mathbb{R}_+^{d+1}})$ is a solution of*

$$-u_t + a_{ij} D_{ij} u = f \tag{4.3.9}$$

in $Q_{\kappa r}^+(\mathrm{y}_d)$ with $D_d u = 0$ on $\{x_d = 0\}$. Then there exist constants $C = C(d, \delta, p, \theta)$ and $\alpha = \alpha(d, p, \theta) > 0$ such that

$$\begin{aligned} &(|D_{x'}^2 u - (D_{x'}^2 u)_{Q_r^+(\mathrm{y}_d)}|^p)_{Q_r^+(\mathrm{y}_d)} \\ &\leq C\kappa^{-\alpha}(|D_{x'}^2 u|^p)_{Q_{\kappa r}^+(\mathrm{y}_d)} + C\kappa^{d+\theta+2}\big((|f|^p)_{Q_{\kappa r}^+(\mathrm{y}_d)} + (|DD_d u|^p)_{Q_{\kappa r}^+(\mathrm{y}_d)}\big). \end{aligned}$$

Proof. By scaling, we may assume that $\kappa r = 8$. In this case $r \leq 1/4$ because $\kappa \geq 32$. Let $\eta = \eta(t, x') \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{d-1})$ with a unit integral such that $\mathrm{supp}(\eta) \subset Q_r^d$ and

$$|D_{x'}\eta| \leq C(d)\, r^{-d-2} \quad \text{on } Q_r^d, \tag{4.3.10}$$

where

$$Q_r^d := (-r^2, 0) \times B_r^{d-1} := (-r^2, 0) \times \{|x'| \le r\}.$$

We consider two cases.

Case (i): $\mathrm{y}_d \in [0, 1]$. By Hölder's inequality and the triangle inequality, it is easily seen that

$$\begin{aligned}
&\fint_{Q_r^+(\mathrm{y}_d)} |D_{x'}^2 u - (D_{x'}^2 u)_{Q_r^+(\mathrm{y}_d)}|^p \, \mu(dx\, dt) \\
&\le C \fint_{Q_r^+(\mathrm{y}_d)} \fint_{Q_r^+(\mathrm{y}_d)} |D_{x'}^2 u(t,x) - D_{x'}^2 u(s,y)|^p \, \mu(dx\, dt)\, \mu(dy\, ds). \qquad (4.3.11)
\end{aligned}$$

Since, by the triangle inequality,

$$\begin{aligned}
&|D_{x'}^2 u(t,x) - D_{x'}^2 u(s,y)|^p \\
&\le C \Big| D_{x'}^2 u(t,x) - \int_{Q_r^d} \eta(\sigma, z') D_{x'}^2 u(\sigma, x_d, z')\, dz'\, d\sigma \Big|^p \\
&\quad + C \Big| D_{x'}^2 u(s,y) - \int_{Q_r^d} \eta(\sigma, z') D_{x'}^2 u(\sigma, y_d, z')\, dz'\, d\sigma \Big|^p \\
&\quad + C \Big| \int_{Q_r^d} \eta(\sigma, z') (D_{x'}^2 u(\sigma, x_d, z') - D_{x'}^2 u(\sigma, y_d, z'))\, dz'\, d\sigma \Big|^p,
\end{aligned}$$

the right-hand side of (4.3.11) is bounded by $C(I + II)$, where

$$\begin{aligned}
I :&= \fint_{Q_r^+(\mathrm{y}_d)} \Big| D_{x'}^2 u(t, x_d, x') - \int_{Q_r^d} \eta(s, y') D_{x'}^2 u(s, x_d, y')\, dy'\, ds \Big|^p \mu(dx\, dt), \\
II :&= \fint_{B_r^{1+}(\mathrm{y}_d)} \fint_{B_r^{1+}(\mathrm{y}_d)} \Big| \int_{Q_r^d} (D_{x'}^2 u(t, x_d, z') - D_{x'}^2 u(t, y_d, z')) \eta(t, z')\, dz'\, dt \Big|^p \\
&\quad \cdot \mu_1(dx_d)\, \mu_1(dy_d).
\end{aligned}$$

Let us now estimate I and II separately and first consider II. By integrating by parts and Hölder's inequality,

$$\begin{aligned}
&\Big|\int_{Q_r^d}(D_{x'}^2u(t,x_d,z')-D_{x'}^2u(t,y_d,z'))\eta(t,z')dz'\,dt\Big|^p\\
&=\Big|\int_{Q_r^d}(D_{x'}u(t,x_d,z')-D_{x'}u(t,y_d,z'))D_{x'}\eta(t,z')dz'\,dt\Big|^p\\
&\le\int_{Q_r^d}|D_{x'}u(t,x_d,z')-D_{x'}u(t,y_d,z')|^pdz'\,dt\cdot(\int_{Q_r^d}|D_{x'}\eta|^qdz'\,dt)^{\frac{p}{q}},
\end{aligned}$$

where $1/p+1/q=1$. From (4.3.10), we have

$$\Big(\int_{Q_r^d}|D_{x'}\eta|^qd\sigma\,dz'\Big)^{\frac{p}{q}}\le Cr^{-(d+1+p)}.$$

We plug the two inequalities above into II to achieve

$$\begin{aligned}
&II\le\\
&Cr^{-(d+1+p)}\fint_{B_r^{1+}(\mathrm{y}_d)}\fint_{B_r^{1+}(\mathrm{y}_d)}\int_{Q_r^d}|D_{x'}u(t,x_d,z')-D_{x'}u(t,y_d,z')|^pdz'\,dt\\
&\cdot\mu_1(dx_d)\,\mu_1(dy_d).
\end{aligned}$$

Applying Lemma 4.3.6 with $d=1$, we get

$$II\le C\fint_{Q_r^+(\mathrm{y}_d)}|D_dD_{x'}u|^p\,\mu(dx\,dt)\le C\kappa^{d+\theta+2}(|DD_du|^p)_{Q_{\kappa r}^+(\mathrm{y}_d)}$$

because $|Q_r^+(\mathrm{y}_d)|\ge Cr^{d+\theta+2}$.

Next, we estimate I, which can be written as

$$I=\fint_{Q_r^+(\mathrm{y}_d)}\Big|\int_{Q_r^d}\eta(s,y')(D_{x'}^2u(t,x_d,x')-D_{x'}^2u(s,x_d,y'))\,dy'\,ds\Big|^p\,\mu(dx\,dt).$$

By Hölder's inequality,

$$I \le \fint_{Q_r^+(\mathrm{y}_d)} \int_{Q_r^d} |D_{x'}^2 u(t,x_d,x') - D_{x'}^2 u(s,x_d,y')|^p \, dy' \, ds \, \mu(dx\,dt) \cdot \left(\int_{Q_r^d} |\eta|^q \, dy' \, ds \right)^{\frac{p}{q}}.$$

Since

$$\left(\int_{Q_r^d} |\eta|^q \, dy' \, ds \right)^{\frac{p}{q}} \le C r^{-(d+1)},$$

we have

$$\begin{aligned} I \le & C \fint_{Q_r^+(\mathrm{y}_d)} \fint_{Q_r^d} |D_{x'}^2 u(t,x_d,x') - D_{x'}^2 u(s,x_d,y')|^p \, dy' \, ds \, \mu(dx\,dt) \\ = & \, C \fint_{B_r^{1+}(\mathrm{y}_d)} I(x_d) \, x_d^{\theta-d} \, dx_d, \end{aligned} \tag{4.3.12}$$

where

$$I(x_d) = \fint_{Q_r^d} \fint_{Q_r^d} |D_{x'}^2 u(t,x_d,x') - D_{x'}^2 u(s,x_d,y')|^p \, dy' \, ds \, dx' \, dt.$$

We now estimate $I(x_d)$ by writing the equation (4.3.9) as

$$-u_t + \sum_{i,j<d} a_{ij} D_{ij} u = -a_{dd} D_{dd} u - \sum_{j=1}^{d-1} (a_{dj} + a_{jd}) D_{dj} u + f.$$

Here, for each fixed $x_d \in (0,\infty)$, we regard $u(t,x',x_d)$ as a solution to the above equation defined in $\mathbb{R} \times \mathbb{R}^{d-1}$. Then thanks to Theorem 4.3.7 with d in place of $d+1$ and the triangle inequality, we get

$$\begin{aligned} I(x_d) &\le C \fint_{Q_r^d} |D_{x'}^2 u(t,x',x_d) - (D_{x'}^2 u)_{Q_r^d}(x_d)|^p \, dx' \, dt \\ &\le C\kappa^{-p} \langle |D_{x'}^2 u|^p \rangle_{Q_{\kappa r}^d}(x_d) + C\kappa^{d+1} \langle |f|^p \rangle_{Q_{\kappa r}^d}(x_d) + C\kappa^{d+1} \langle |DD_d u|^p \rangle_{Q_{\kappa r}^d}(x_d), \end{aligned}$$

where, for fixed x_d, $\langle |g|^p \rangle_{Q_{\kappa r}^d}(x_d)$ is the unweighted average of g with respect to (t,x')

in the d dimensional parabolic cylinder with radius κr. We plug the inequality above into (4.3.12) to obtain

$$I \leq C \fint_{B_r^{1+}(\mathrm{y}_d)} \Big(\kappa^{-p} \langle |D_{x'}^2 u|^p \rangle_{Q_{\kappa r}^d}(x_d) + \kappa^{d+1} \langle |f|^p \rangle_{Q_{\kappa r}^d}(x_d) + \kappa^{d+1} \langle |DD_d u|^p \rangle_{Q_{\kappa r}^d}(x_d) \Big) x_d^{\theta - d}\, dx_d. \tag{4.3.13}$$

Bearing in mind that $\mu_1(B_r^{1+}(\mathrm{y}_d)) \geq C r^{\zeta+1-d}$, where $\zeta = \max\{d, \theta\}$, and $r = 8/\kappa$, we obtain from (4.3.13) that

$$I \leq C \kappa^{-(p+d-\zeta-1)} (|D_{x'}^2 u|^p)_{Q_{\kappa r}^+(\mathrm{y}_d)} + C\kappa^{\zeta+2} \left((|f|^p)_{Q_{\kappa r}^+(\mathrm{y}_d)} + (|DD_d u|^p)_{Q_{\kappa r}^+(\mathrm{y}_d)} \right).$$

By the definition of ζ, $p + d - \zeta - 1 > 0$. Combining the estimates of I and II, we complete the proof of Case (i) with $\alpha = p + d - \zeta - 1$.

Case (ii): $\mathrm{y}_d > 1$. Set $v = D_k u$, $k = 1, \ldots, d$, and note that v satisfies the divergence type equation

$$v_t - D_i(a_{ij} D_j v) = D_k f$$

in $B_{\kappa r}(\mathrm{y}_d)$. Since this is an interior estimate, we do not care about the boundary value of v on $\{x_d = 0\}$. Then we repeat the second part of the proof of Lemma 4.3.5 with D_k in place of D_d. The lemma is proved. □

4.4 Proof of theorem 4.2.2

In this section, we deal with operators with coefficients depending on both x and t. We denote $L = a_{ij} D_{ij}$ and assume $p > 1$, $\lambda \geq 0$, and $\theta \in (d-1, d-1+p)$.

First we need the following lemma.

Lemma 4.4.1. *Let* $0 < r \le R < \infty$ *and* $X = (t,x)$, $Y = (s,y) \in \overline{\mathbb{R}^{d+1}_+}$. *Then*

$$\mu\left(Q_r^+(X)\right) \le C(d,\theta)\,\mu\left(Q_R^+(Y)\right) \tag{4.4.1}$$

provided that $Q_r^+(X) \cap Q_R^+(Y) \neq \emptyset$.

The following two lemmas are mean oscillation estimates for the operator $a_{ij}(t,x)D_{ij}$. We prove them by using the mean oscillation estimates for $a_{ij}(t)D_{ij}$ proved in Section 4.3 combined with a perturbation argument.

Lemma 4.4.2. *Let* $R > 0$, $\kappa \ge 32$, *and* $\beta, \beta' \in (1,\infty)$ *satisfying* $1/\beta + 1/\beta' = 1$. *Suppose that* $u \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ *is compactly supported on* $\overline{Q_R^+(x_d)}$, *where* $x_d = (\hat{t},\hat{x}) \in \overline{\mathbb{R}^{d+1}_+}$. *Moreover,* $D_d u = 0$ *on* $\{x_d = 0\}$ *and* $f := -u_t + Lu$. *Then for any* $r > 0$ *and* $Y = (s,y) \in \overline{\mathbb{R}^{d+1}_+}$, *we have*

$$\begin{aligned}
&(|DD_d u - (DD_d u)_{Q_r^+(Y)}|^p)_{Q_r^+(Y)} \le C_0 \kappa^{-p/2} (|DD_d u|^p)_{Q^+_{\kappa r}(Y)} \\
&+ C_0 \kappa^{d+\theta+2} (|f|^p)_{Q^+_{\kappa r}(Y)} + C_1 \kappa^{d+\theta+2} (A_R^{\#})^{1/\beta'} (|D^2 u|^{\beta p})^{1/\beta}_{Q^+_{\kappa r}(Y)},
\end{aligned} \tag{4.4.2}$$

where $C_0 = C_0(d,\delta,p,\theta)$ *and* $C_1 = C_1(d,\delta,p,\theta,\beta)$.

Then we obtain the following Lemma.

Lemma 4.4.3. *Let* $R > 0, \kappa \ge 32$, *and* $\beta, \beta' \in (1,\infty)$ *satisfying* $1/\beta + 1/\beta' = 1$. *Let* $u \in C^\infty_{loc}(\overline{\mathbb{R}^{d+1}_+})$ *be compactly supported on* $\overline{Q_R^+(x_d)}$, *where* $x_d \in \overline{\mathbb{R}^{d+1}_+}$. *Moreover,* $D_d u = 0$ *on* $\{x_d = 0\}$ *and* $f := -u_t + Lu$. *Then for any* $r > 0$ *and* $Y = (s,y) \in \overline{\mathbb{R}^{d+1}_+}$, *we have*

$$
\begin{aligned}
&(|D^2_{x'}u-(D^2_{x'}u)_{Q^+_r(Y)}|^p)_{Q^+_r(Y)}\\
&\le C_0\kappa^{-\alpha}(|D^2_{x'}u|^p)_{Q^+_{\kappa r}(Y)}+C_0\kappa^{d+\theta+2}(|f|^p+|DD_du|^p)_{Q^+_{\kappa r}(Y)}\\
&\quad +C_1\kappa^{d+\theta+2}(A^\#_R)^{1/\beta'}(|D^2u|^{\beta p})^{1/\beta}_{Q^+_{\kappa r}(Y)},
\end{aligned}
$$

where $C_0=C_0(d,\delta,p,\theta)$, $C_1=C_1(d,\delta,p,\theta,\beta)$, *and* $\alpha=\alpha(d,\theta,p)>0$.

Next we recall the Hardy–Littlewood maximal function theorem and the Fefferman–Stein theorem on sharp functions. Let

$$
\mathcal{Q}=\{Q^+_r(z):z=(t,x)\in\overline{\mathbb{R}^{d+1}_+},r>0\}.
$$

For a function g defined on $\mathbb{R}^{d+1}_+$, the weighted (parabolic) maximal and sharp functions of g are given by

$$
\begin{aligned}
\mathcal{M}g(t,x)&=\sup_{Q\in\mathcal{Q},(t,x)\in Q}\fint_Q|g(s,y)|\,\mu(dy\,ds),\\
g^\#(t,x)&=\sup_{Q\in\mathcal{Q},(t,x)\in Q}\fint_Q|g(s,y)-(g)_Q|\,\mu(dy\,ds).
\end{aligned}
$$

For any $\theta>d-1$ and $g\in\mathbb{L}_{p,\theta}(\mathbb{R}^{d+1}_+)$, we have

$$
\|g\|_{\mathbb{L}_{p,\theta}(\mathbb{R}^{d+1}_+)}\le C\|g^\#\|_{\mathbb{L}_{p,\theta}(\mathbb{R}^{d+1}_+)},\quad \|\mathcal{M}g\|_{\mathbb{L}_{p,\theta}(\mathbb{R}^{d+1}_+)}\le C\|g\|_{\mathbb{L}_{p,\theta}(\mathbb{R}^{d+1}_+)},
$$

where $p\in(1,\infty)$ and $C=C(d,p,\theta)$. The first inequality above is known as the Fefferman–Stein theorem on sharp functions and the second one is the Hardy–Littlewood maximal function theorem, for instance see [68, Chapter 3].

Now we are ready to prove our main theorem.

Proof of Theorem 4.2.2. By the method of continuity, it is enough to prove the a priori estimate (4.2.3). Moreover, since the set of functions in $C^\infty(\overline{(-\infty,T)\times\mathbb{R}^d_+})$ vanishing for large (t,x) is dense in $W^{1,2}_{p,\theta}(-\infty,T)$, we only need to prove (4.2.3) for infinitely differentiable functions with compact support. In this case, the proof of (4.2.3) can be divided into several steps.

Step 1: We consider $\lambda=0$, $b_i=c=0$, $T=\infty$, and $u\in C^\infty(\overline{\mathbb{R}^{d+1}_+})$ vanishing outside $\overline{Q^+_{R_0}(x_d)}$ for some $x_d\in\overline{\mathbb{R}^{d+1}}$, where R_0 is from Assumption 4.2.1. Let $\kappa\ge 32$ be a constant to be determined later. We fix $q\in(1,p)$ and $\beta\in(1,\infty)$, depending only on p and θ, such that $\beta q<p$ and $\theta<d-1+q$. Let β' be such that $1/\beta+1/\beta'=1$. By applying Lemma 4.4.2 with q in place of p and using Assumption 4.2.1, we obtain

$$\begin{aligned}(DD_du)^{\#}(Y)\le& C_0\kappa^{-1/2}\mathcal{M}^{1/q}(|DD_du|^q)(Y)+C_0\kappa^{(d+\theta+2)/q}\mathcal{M}^{1/q}(|f|^q)(Y)\\&+C_0\kappa^{(d+\theta+2)/q}\rho^{1/(\beta'q)}\mathcal{M}^{1/(\beta q)}(|D^2u|^{\beta q})(Y)\end{aligned}$$

for any $Y\in\overline{\mathbb{R}^{d+1}_+}$, where $C_0=C_0(d,\delta,p,\theta)$. This estimate, together with Fefferman–Stein theorem on sharp functions and the Hardy–Littlewood theorem on maximal functions, gives

$$\begin{aligned}&\|DD_du\|_{p,\theta}\le C\|(DD_du)^{\#}\|_{p,\theta}\\&\le C_0\kappa^{-1/2}\|\mathcal{M}^{1/q}(|DD_du|^q)\|_{p,\theta}+C_0\kappa^{(d+\theta+2)/q}\|\mathcal{M}^{1/q}(|f|^q)\|_{p,\theta}\\&\quad+C_0\kappa^{(d+\theta+2)/q}\rho^{1/(\beta'q)}\|\mathcal{M}^{1/(\beta q)}(|D^2u|^{\beta q})\|_{p,\theta}\\&\le C_0\kappa^{-1/2}\|DD_du\|_{p,\theta}+C_0\kappa^{(d+\theta+2)/q}\|f\|_{p,\theta}+C_0\kappa^{(d+\theta+2)/q}\rho^{1/(\beta'q)}\|D^2u\|_{p,\theta}.\end{aligned}\tag{4.4.3}$$

In the same way, we apply Lemma 4.4.3 instead of Lemma 4.4.2 to obtain the estimate of $D_{x'}^2 u$:

$$\|D_{x'}^2 u\|_{p,\theta} \leq C_0 \kappa^{-\alpha/q} \|D_{x'}^2 u\|_{p,\theta} + C_0 \kappa^{(d+\theta+2)/q} (\|f\|_{p,\theta} + \|DD_d u\|_{p,\theta}) \\ + C_0 \kappa^{(d+\theta+2)/q} \rho^{1/(\beta' q)} \|D^2 u\|_{p,\theta}. \tag{4.4.4}$$

By choosing κ sufficiently large, from (4.4.3) and (4.4.4) we obtain

$$\|DD_d u\|_{p,\theta} \leq C_0 \|f\|_{p,\theta} + C_0 \rho^{1/(\beta' q)} \|D^2 u\|_{p,\theta}, \tag{4.4.5}$$

$$\|D_{x'}^2 u\|_{p,\theta} \leq C_0 (\|f\|_{p,\theta} + \|DD_d u\|_{p,\theta}) + C_0 \rho^{1/(\beta' q)} \|D^2 u\|_{p,\theta}. \tag{4.4.6}$$

We combine (4.4.5) and (4.4.6) together to get

$$\|D^2 u\|_{p,\theta} \leq C_0 \|f\|_{p,\theta} + C_0 \rho^{1/(\beta' q)} \|D^2 u\|_{p,\theta}.$$

Taking ρ sufficiently small depending only on d, δ, p, θ, we arrive at

$$\|D^2 u\|_{p,\theta} \leq C_0 \|f\|_{p,\theta}. \tag{4.4.7}$$

On the other hand, since

$$u_t = a_{ij} D_{ij} u - f,$$

we have

$$\|u_t\|_{p,\theta} \leq C_0 \|f\|_{p,\theta} + C_0 \|D^2 u\|_{p,\theta}. \tag{4.4.8}$$

By Hardy's inequality (Lemma 4.3.1) with $r = \infty$,

$$\|M^{-1} D_d u\|_{p,\theta} \leq C_0 \|D_{11} u\|_{p,\theta}. \tag{4.4.9}$$

Combining (4.4.7), (4.4.8), and (4.4.9), we obtain (4.2.3) for $\lambda = 0$.

Step 2: We remove the assumption that u is compactly supported in $Q^+_{R_0}(x_d)$. By a standard partition of the unity argument with (4.4.7)–(4.4.9) (cf. [68, Theorem 1.6.4]) we see that

$$\|u_t\|_{p,\theta} + \|M^{-1}D_d u\|_{p,\theta} + \|D^2 u\|_{p,\theta} \le C_0\|f\|_{p,\theta} + C_1\|u\|_{p,\theta}, \tag{4.4.10}$$

where $C_0 = C_0(d,\delta,p,\theta)$ and $C_1 = C_1(d,\delta,p,\theta,R_0)$.

Step 3: We still assume that $b_i = c = 0$ and $T = \infty$, but λ is not necessarily zero. In this case, we follow an idea of S. Agmon. Since

$$-u_t + a_{ij}D_{ij}u = f - \lambda u,$$

by (4.4.10) we have

$$\begin{aligned}\|u_t\|_{p,\theta} + \|M^{-1}D_d u\|_{p,\theta} + \|D^2 u\|_{p,\theta} &\le C_0\|f - \lambda u\|_{p,\theta} + C_1\|u\|_{p,\theta} \\ &\le C_0\|f\|_{p,\theta} + (C_0\lambda + C_1)\|u\|_{p,\theta}.\end{aligned}$$

Hence it is sufficient to show that for large λ

$$\lambda\|u\|_{p,\theta} \le C_0\|f\|_{p,\theta}.$$

We pick a function $\zeta(y) \in C_0^\infty(\mathbb{R}), \zeta \not\equiv 0$ and introduce the following notation

$$z = (x,y), \quad \hat{u}(t,z) = u(t,x)\zeta(y)\cos(\sqrt{\lambda}y), \quad \hat{L}u = L(t,x)u(t,z) + u_{yy}(t,z).$$

Finally, set $\mathbb{R}^{d+2}_+ = \{(t,z) : x_d > 0\}$ and

$$\mathbb{B}_r(z) = (x_d - r, x_d + r) \times B'_r(x') \times (y - r, y + r), \quad \mathbb{Q}_r(t,z) = (t - r^2, t) \times \mathbb{B}_r(z),$$

$$\mathbb{B}^+_r(z) = \mathbb{B}_r(z) \cap \{x_d > 0\}, \quad \mathbb{Q}^+_r(t,z) = (t - r^2, t) \times \mathbb{B}^+_r(z).$$

For any $r > 0$, $Z = (t,z) \in \overline{\mathbb{R}^{d+2}_+}$, and $\hat{a}(t)$, we have

$$\fint_{\mathbb{B}^+_r(z)} |a_{ij}(t,x) - \hat{a}(t)| \, \mu_d(dx) \, dy = \fint_{B^+_r(x)} |a_{ij}(t,x) - \hat{a}(t)| \, \mu_d(dx),$$

where $C = C(d)$. In particular, by the definition of the osc_x and setting

$$\hat{a}(t) = \fint_{B^+_r(x)} a_{ij}(t,x) \, \mu_d(dx),$$

we have $\mathrm{osc}_z(a_{ij}, \mathbb{Q}^+_r(t,z)) = \mathrm{osc}_x(a_{ij}, Q^+_r(t,x))$. By a simple calculation,

$$\hat{L}\hat{u} = f \cos(\sqrt{\lambda} y) \zeta(y) + u \zeta'' \cos(\sqrt{\lambda} y) - 2\sqrt{\lambda} u \zeta' \sin(\sqrt{\lambda} y) := \hat{f}.$$

By (4.4.10) with $\hat{u}$ instead of u in dimension $d+2$,

$$|||D^2 \hat{u}|||_{p,\theta} \le C_0 |||\hat{f}|||_{p,\theta} + C_1 |||\hat{u}|||_{p,\theta}, \tag{4.4.11}$$

where $|||\cdot|||_{p,\theta}$ is the weighted L_p norm in $\mathbb{R}^{d+2}_+$ with respect to $\mu(dx\,dt)\,dy$. Note that for $\lambda > 1$,

$$c_1 \le \int_{\mathbb{R}} |\zeta(y) \cos(\sqrt{\lambda} y)|^p \, dy := A \le c_2, \tag{4.4.12}$$

where c_1 and c_2 are positive constants independent of λ. Moreover, A is bounded from above with ζ replaced by any derivatives of ζ and $\cos(\sqrt{\lambda} y)$ replaced by $\sin(\sqrt{\lambda} y)$.

Then from (4.4.11), we have

$$|||D_{yy}\hat{u}|||_{p,\theta} \le |||D^2\hat{u}|||_{p,\theta} \le C_0\|f\|_{p,\theta} + C_1(1+\sqrt{\lambda})\|u\|_{p,\theta}. \tag{4.4.13}$$

Since

$$\begin{aligned} D_{yy}\hat{u} = u(t,x)\zeta''(y)\cos(\sqrt{\lambda}y) - 2\sqrt{\lambda}u(t,x)\zeta'(y)\sin(\sqrt{\lambda}y) \\ - \lambda u(t,x)\zeta(y)\cos(\sqrt{\lambda}y), \end{aligned} \tag{4.4.14}$$

Combining (4.4.12), (4.4.13), and (4.4.14), we have

$$\lambda\|u\|_{p,\theta} \le C_0\|f\|_{p,\theta} + C_1(1+\sqrt{\lambda})\|u\|_{p,\theta}. \tag{4.4.15}$$

After choosing λ sufficiently large depending on $d, \delta, p, \theta, R_0$ to absorb the term of u on the right-hand side of (4.4.15) to the left-hand side, we have

$$\lambda\|u\|_{p,\theta} \le C_0\|f\|_{p,\theta},$$

where $C_0 = C_0(d, \delta, p, \theta)$. By the interpolation inequality

$$\sqrt{\lambda}\|Du\|_{p,\theta} \le C_0\|D^2u\|_{p,\theta} + C_0\lambda\|u\|_{p,\theta},$$

we finish the proof of Step 3.

Step 4: We remove the assumption that $b_i = c = 0$ by moving the terms of b_i and c to the right-hand side

$$-u_t + a_{ij}D_{ij}u - \lambda u = f - b_iD_iu - cu.$$

By the conclusion in Step 3 with $b_i = c = 0$, there exists $\lambda_0 = \lambda_0(d, \delta, p, \theta, R_0)$ such that for any $\lambda \geq \lambda_0$,

$$\begin{aligned}
&\|u_t\|_{p,\theta} + \lambda\|u\|_{p,\theta} + \sqrt{\lambda}\|Du\|_{p,\theta} + \|M^{-1}D_d u\|_{p,\theta} + \|D^2 u\|_{p,\theta} \\
&\leq C_0\|f - b_i D_i u - cu\|_{p,\theta} \\
&\leq C_0\|f\|_{p,\theta} + C_0 K\|Du\|_{p,\theta} + C_0 K\|u\|_{p,\theta}.
\end{aligned}$$

By taking λ sufficiently large depending on d, δ, p, θ, R_0, and K, we get

$$\|u_t\|_{p,\theta} + \lambda\|u\|_{p,\theta} + \sqrt{\lambda}\|Du\|_{p,\theta} + \|M^{-1}u\|_{p,\theta} + \|D^2 u\|_{p,\theta} \leq C_0\|f\|_{p,\theta}.$$

Step 5: To remove the assumption that $T = \infty$, we simply follow the standard step in [41, Theorem 2.1] or [68, Theorem 6.4.1]. Therefore, the estimate (4.2.3) is proved.

Finally, in the case when $b_i = c = 0$ and $a_{ij} = a_{ij}(t)$, by using a scaling argument we can take $R_0 = 0$. The theorem is proved. □

Part II

Regularity Theory for Nonlocal Equations

Chapter 5

Schauder estimates for a class of nonlocal fully nonlinear parabolic equations

5.1 Introduction

This chapter is based on [31] about the study of Schauder estimates for a class of concave fully nonlinear nonlocal parabolic equations. An example of nonlocal operators, which is associated with pure jump processes (see, for instance, [89]), is the following

$$
\begin{aligned}
L_a u &= \int_{\mathbb{R}^d} \big(u(t,x+y) - u(t,x) - y^T Du(t,x)\big) K_a(t,x,y)\, dy \quad \text{for } \sigma \in (1,2), \\
L_a u &= \int_{\mathbb{R}^d} \big(u(t,x+y) - u(t,x) - y^T Du(t,x)\chi_{B_1}\big) K_a(t,x,y)\, dy \quad \text{for } \sigma = 1 \\
&\quad \text{with} \quad \int_{S_r} y K_a(t,x,y)\, ds = 0 \quad \forall\, r > 0, \qquad (5.1.1) \\
L_a u &= \int_{\mathbb{R}^d} \big(u(t,x+y) - u(t,x)\big) K_a(t,x,y)\, dy \quad \text{for } \sigma \in (0,1),
\end{aligned}
$$

where

$$
K_a \in \mathcal{L}_0 := \Big\{ K : \frac{\lambda(2-\sigma)}{|y|^{d+\sigma}} \le K(t,x,y) \le \frac{\Lambda(2-\sigma)}{|y|^{d+\sigma}} \Big\}
$$

for some ellipticity constants $0 < \lambda \le \Lambda$, with no regularity assumption imposed with respect to the y variable. This type of nonlocal operators was first considered by Komatsu [57], Mikulevičius and Pragarauskas [89, 88], and later by Dong and Kim [40, 39], and Schwab and Silvestre [97], to name a few.

The fully nonlinear nonlocal parabolic equation that we are interested in is of the form

$$
u_t = \inf_{a \in \mathcal{A}} (L_a u + f_a), \tag{5.1.2}
$$

where $K_a \in \mathcal{L}_0$ for $a \in \mathcal{A}$ and $\mathcal{A}$ is an index set. For fully nonlinear second-order equations with $f_a \equiv 0$, the celebrated $C^{2,\alpha}$ estimate was established independently by Evans [46] and Krylov [61] in early nineteen-eighties. Nonhomogeneous second-order equations were considered a bit later by Safonov [94]. Recently, Caffarelli and Silvestre [16] investigated the nonlocal version of Evans-Krylov theorem with translation invariant and symmetric kernels, i.e., $K_a(x,y) = K_a(y) = K_a(-y)$, satisfying additional regularity assumptions

$$
[K_a]_{C^2(\mathbb{R}^d \setminus B_\rho)} \le \Lambda(2-\sigma)\rho^{-d-\sigma-2}. \tag{5.1.3}
$$

More recently, their result was extended to nonhomogeneous fully nonlinear elliptic equations by Jin and Xiong [50] by using a recursive Evans-Krylov theorem. At almost the same time, Serra [98] removed the regularity assumption (5.1.3) and proved the Evans-Krylov theorem and Schauder estimates with symmetric kernels. His proof relies on a Liouville type theorem and a blow-up analysis. In this chapter, we do not assume that the kernels are symmetric, which is certainly more general than the kernels considered in [16, 50, 98]. Specifically, when the kernels are symmetric, (5.1.1) is satisfied automatically, and

$$L_a u = \frac{1}{2}\int_{\mathbb{R}^d} \big(u(x+y) + u(x-y) - 2u(x)\big) K_a(y)\,dy,$$

which is the form of the operators considered in [16, 50, 98].

For equations with non-symmetric kernels, Dong and Kim [39, 40] proved L_p and Schauder estimates for linear elliptic equations. Chang-Lara and Dávila [73, 20] considered nonlocal parabolic equations with non-symmetric kernels and critical drift, and proved the corresponding C^α and $C^{1,\alpha}$ estimate. Recently in [21], they proved a version of the Evans-Krylov theorem for concave nonlocal parabolic equations with critical drift, where they assumed the kernels to be non-symmetric but translation invariant and smooth (5.1.3). We also mention that Schauder estimates for linear nonlocal parabolic equations were studied in [49, 88].

First we extend the previous results in [98, 21, 50, 49] to include concave nonlocal parabolic equations with non-symmetric rough kernels. More specifically, for any small α, if f_a and $K_a(t,x,y)$ are C^α in x and $C^{\alpha/\sigma}$ in t, then we have the following $C^{1+\alpha/\sigma,\sigma+\alpha}$ a priori estimate of any smooth solution u to (5.1.2) in $(-1,0)\times B_1$.

Theorem 5.1.1. *Let $\sigma \in (0,2)$, $0 < \lambda \le \Lambda < \infty$, and $\mathcal{A}$ be an index set. There is a constant $\hat{\alpha} \in (0,1)$ depending on $d, \sigma, \lambda,$ and Λ so that the following holds. Let*

$\alpha \in (0, \hat{\alpha})$ such that $\sigma + \alpha$ is not an integer. Assume $K_a \in \mathcal{L}_0$ and satisfies (5.1.1) when $\sigma = 1$, and

$$\left|K_a(t,x,y) - K_a(t',x',y)\right| \le A\big(|x-x'|^\alpha + |t-t'|^{\alpha/\sigma}\big)\frac{\Lambda(2-\sigma)}{|y|^{d+\sigma}}, \tag{5.1.4}$$

where $A \ge 0$ is a constant. Suppose $u \in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_1) \cap C^{\alpha/\sigma,\alpha}((-1,0)\times\mathbb{R}^d)$ is a solution of

$$u_t = \inf_{a\in\mathcal{A}}(L_a u + f_a) \quad \text{in } Q_1, \tag{5.1.5}$$

where $f_a \in C^{\alpha/\sigma,\alpha}(Q_1)$ satisfying

$$C_0 := \sup_{a\in\mathcal{A}}[f_a]_{\alpha/\sigma,\alpha;Q_1} < \infty, \qquad \sup_{(t,x)\in Q_1}\left|\inf_{a\in\mathcal{A}} f_a(t,x)\right| < \infty.$$

Then,

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + CC_0, \tag{5.1.6}$$

where $C > 0$ is a constant depending only on $d, \lambda, \Lambda, \alpha, A$, and σ, and is uniformly bounded as $\sigma \to 2$.

In the theorem above, $\|\cdot\|_{\alpha/\sigma,\alpha;\Omega}$ is the Hölder norm of order α/σ in t and α in x with underlying domain Ω. We used Q_r to denote the parabolic cylinder with radius r centered at the origin. For precise definitions, see Section 2. As pointed out in [98], the $C^{\alpha/\sigma,\alpha}$ Hölder norm of u on the right-hand side of (5.1.6) is necessary and cannot be replaced by the L_∞ norm or any lower-order Hölder norm of u. We also note that by keeping track of the constants in the proofs below, in the symmetric case, if $\sigma \in [\sigma_0, 2)$ for some $\sigma_0 \in (0,1)$, then the constant C in the estimate (5.1.6) depends on σ_0, not σ. In the non-symmetric case, if $0 < \sigma_0 \le \sigma \le \sigma_1 < 1$ (or $1 < \sigma_2 \le \sigma < 2$), then the constant C depends on σ_0 and σ_1 (or σ_2), not σ. In particular, C does not blow up as σ approaches 2.

Roughly speaking, the proof of Theorem 5.1.1 can be divided into three steps. First we prove a Liouville type theorem for solutions in $(-\infty, 0) \times \mathbb{R}^d$. For the classical PDEs, we generally apply interpolation and iteration to obtain $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates. One notable feature of nonlocal operators is that the boundary data is prescribed on the complement of the domain where the equation is satisfied, which makes it difficult to implement these techniques. However, if we assume that (5.1.2) is satisfied in $(-\infty, 0) \times \mathbb{R}^d$, then we do not need to handle boundary data any more, which is the advantage of considering an equation satisfied in the whole space. Second, we prove the a priori estimate for equations with translation invariant kernels by combining the Liouville theorem and a blow-up analysis. Particularly in this step, the extension from symmetric kernels to non-symmetric kernels is non-trivial. A key idea in the classical Evans-Krylov theorem for $F(D^2u) = 0$ is that, since the function F is concave, any second directional derivative $D^2_{ee}u$ is a subsolution. It is relatively easy to adapt this idea to the nonlocal equations with symmetric kernels due to the appearance of centered second order difference in the definition of the operator. For nonsymmetric kernels, some new ideas are required to obtain a similar subsolution as in the symmetric case. Moreover, the dependence of the t variable also makes the proof more involved. Finally, we implement a more or less standard perturbation argument to treat the general case.

The second objective of this chapter is to consider the end-point situation when $\alpha = 0$. For second-order elliptic equations, even the Poisson equation $\Delta u = f$, when f is merely bounded, it is well known that u may fail to be $C^{1,1}$. However, this is not the case for nonlocal equations. When $\sigma \neq 1$ and the kernels are independent of t and x, we prove a priori C^σ estimate in the x variable and Λ^1 estimate in the t variable when f_a is merely bounded and measurable, where Λ^1 is the Zygmund space. When $\sigma = 1$, we obtain a priori Λ^1 estimate in both t and x. We assume that the solution is smooth because the spaces in which estimates are obtained are not fine enough for the nonlocal operators to be defined pointwise.

Theorem 5.1.2. *(i) Let $\sigma \neq 1$. Assume that u is a smooth solution to (5.1.2) in $(-\infty, 0) \times \mathbb{R}^d$ with K_a independent of t and x. When $\sigma \in (1,2)$, we also assume that Du is $C^{(\sigma-1)/\sigma}$ in t. Then there exists a constant C depending on d, λ, Λ, and σ such that for $\sigma > 1$,*

$$[u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}} \leq C \sup_a \|f_a\|_{L_\infty};$$

and for $\sigma < 1$,

$$[u]^t_{\Lambda^1} + [u]^*_\sigma \leq C \sup_a \|f_a\|_{L_\infty}.$$

(ii) Let $\sigma = 1$. Assume that u is a smooth solution to (5.1.2) in $(-\infty, 0) \times \mathbb{R}^d$ with K_a independent of t and x. Then there exists a constant C depending on d, λ, Λ, and σ such that

$$[u]_{\Lambda^1} \leq C \sup_a \|f_a\|_{L_\infty}.$$

Here all the norms are taken in $\mathbb{R}^{d+1}_0 := (-\infty, 0) \times \mathbb{R}^d$.

In the theorem above, $[\,\cdot\,]^*$, $[\,\cdot\,]^t_\alpha$, $[\,\cdot\,]^t_{\Lambda^1}$, and $[\,\cdot\,]_{\Lambda^1}$ are the Hölder semi-norms in x, t, the Zygmund semi-norm in t, and the Zygmund semi-norm with respect to (t,x), respectively. See the precise definitions in Section 2.

We localize Theorem 5.1.2 to obtain the following corollary.

Corollary 5.1.3. *(i) Let $\sigma \neq 1$. Assume that u is a smooth solution to (5.1.2) with K_a independent of t and x. Then for $\sigma > 1$,*

$$[u]^*_{\sigma;Q_{1/2}} + [u]^t_{\Lambda^1(Q_{1/2})} + [Du]^t_{\frac{\sigma-1}{\sigma};Q_{1/2}} \leq C\Big(\sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty(Q_1)} + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}\Big);$$

and for $\sigma < 1$,

$$[u]^*_{\sigma;Q_{1/2}} + [u]^t_{\Lambda^1(Q_{1/2})} \le C\Big(\sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty(Q_1)} + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}\Big).$$

(ii) Let $\sigma = 1$. *Assume that* u *is a smooth solution to* (5.1.5) *with* K_a *independent of* t *and* x. *Then we have*

$$[u]_{\Lambda^1;Q_{1/2}} \le C\Big(\sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty(Q_1)} + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}\Big).$$

To our best knowledge, such result is new even for nonlocal elliptic equations with symmetric kernels. A similar result was obtained very recently by Mou [90] for elliptic equations with symmetric, smooth kernels and Dini continuous data. With merely bounded and measurable data, the best estimates known in the literature are the C^β regularity of u in t for $\beta < 1$ and the C^γ regularity of u in x for $\gamma < \min\{\sigma, 1+\alpha\}$, where $\alpha > 0$ is a small constant. See [99, 19]. Because $\Lambda^1 \subsetneq C^\beta$ for any $\beta < 1$, Theorem 5.1.2 improves these results and is optimal even in the linear case. See Remark 5.1 below.

The proof of Theorem 5.1.2 is based on a perturbation type argument using Campanato's approach. We first refine the estimate in Theorem 5.1.1 when the operator is translation invariant. In particular, we replace $\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}$ on the right-hand side of (5.1.6) by

$$[u]_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + \sum_{j=2}^{\infty} 2^{-j\sigma}[u]_{\alpha/\sigma,\alpha;(-1,0)\times(B_{2^j}\setminus B_{2^{j-1}})}. \tag{5.1.7}$$

The advantage of the replacement will be explained below. Another important ingredient in the proof is the fact that, for example, when $\sigma > 1$,

$$[u]^t_{\Lambda^1(\mathbb{R}^{d+1}_0)} + [u]^*_{\sigma;\mathbb{R}^{d+1}_0} + [Du]^t_{\frac{\sigma-1}{\sigma};\mathbb{R}^{d+1}_0} \le C \sup_{r>0} \sup_{(t,x)\in\mathbb{R}^{d+1}_0} E[u; Q_r(t,x)], \tag{5.1.8}$$

where

$$E[u; Q_r(t,x)] := \inf_{p\in\mathcal{P}_1} r^{-\sigma}\|u-p\|_{L_\infty(Q_r(t,x))},$$

$\mathcal{P}_1$ is the set of linear functions in (t,x), and $Q_r(t,x)$ is the parabolic cylinder with center (t,x); see (5.2.1). Therefore, instead of directly estimating

$$[u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}},$$

we estimate $E[u; Q_r(t,x)]$ for any fixed r and (t,x). It is worth noting that in view of the proofs of Lemmas 5.5.2, 5.5.4, and 5.5.5, the two quantities on the left and right hand sides of (5.1.8) are actually equivalent.

More specifically, without loss of generality, we set $(t,x) = (0,0)$ and let v_K solve the homogeneous equation

$$\begin{cases} \partial_t v_K = \inf_{a\in\mathcal{A}} L_a v_K & \text{in } Q_{2R} \\ v_K = g_K := \max\{-K, \min\{u-p, K\}\} & \text{in } (-(2R)^\sigma, 0)\times B^c_{2R} \end{cases},$$

where K is a large constant, p is a carefully chosen linear function, and $R > 2r$ is a constant to be determined. Now we apply Theorem 5.1.1 to v_K and control $[v_K]_{1+\alpha/\sigma,\alpha+\sigma;Q_{R/2}}$ by using scaling argument and replacing $\|v_K\|_{\alpha/\sigma,\alpha}$ by (5.1.7). It is easily seen that in each cylindrical domain $(-R^\sigma, 0)\times(B_{2^jR}\setminus B_{2^{j-1}R})$, the Hölder norm of g_K is bounded and independent of K, but globally it depends on K and goes to infinity as $K\to\infty$. This is also the advantage of decomposing the domain into annuli. We then set q_K to be the first-order Taylor expansion of v_K and estimate

$$\|u - p - q_K\|_{L_\infty(Q_r)} \le \|u - p - v_K\|_{L_\infty(Q_r)} + \|v_K - q_K\|_{L_\infty(Q_r)},$$

where the first term is bounded by CR^σ due to the Aleksandrov-Bakelman-Pucci estimate and second term is controlled by $[v_K]_{1+\alpha/\sigma,\alpha+\sigma;Q_r}$. Finally, we are able to obtain

$$r^{-\sigma}\|u - p - q_K\|_{L_\infty(Q_r)} \le C(r/R)^\alpha\big([u]^*_\sigma + [u]^t_{\Lambda^1} + [Du]^t_{\frac{\sigma-1}{\sigma}}\big) + C(R/r)^\sigma\|f\|_{L_\infty}.$$

By setting $R = Mr$, using (5.1.8), and taking M sufficiently large, the terms involving u on the right-hand side above are absorbed in the left-hand side.

Remark. We give two simple examples which indicate that the estimates in Theorem 5.1.2 are optimal even in the linear case. Set $d = 1$. Let $f = f(t,x) = \chi_{\{t<-M|x|^\sigma\}}$ for some constant $M > 0$ and u be a solution to the equation

$$u_t + (-\Delta)^{\sigma/2}u = f \quad \text{in } (-\infty, 0) \times \mathbb{R}^d.$$

Then by the explicit representation of solutions, it is easily seen that for sufficiently large M,

$$\lim_{t\nearrow 0} u_t(t,0) = -\infty.$$

Thus u cannot be Lipschitz in t in this case. Next we set $\sigma = 1$, $g = g(t,x) = \chi_{\{t<0,x>t\}}$, and v be a solution to

$$v_t + (-\Delta)^{1/2}u = g \quad \text{in } (-\infty, 0) \times \mathbb{R}^d.$$

Again by the explicit representation of solutions,

$$\lim_{x\to 0} v_x(0,x) = \infty.$$

Therefore, v cannot be Lipschitz in x in this case.

By viewing solutions to elliptic equations as steady state solutions to parabolic equations, from Theorems 5.1.1, 5.1.2, and Corollary 5.1.3, we obtain the corresponding results for nonlocal elliptic equations with nonsymmetric and rough kernels.

The organization of this chapter is as follows. In the next section, we introduce some notation and preliminary results that are necessary in the proof of our main results. We prove the Liouville theorem in Section 3 and Theorem 5.1.1 in Section 4. In Section 5, we apply Theorem 5.1.1 to prove Theorem 5.1.2.

5.2 Notation and preliminary results

In this section, we introduce some notation which will be used throughout this chapter and some preliminary results which are useful in our proof. We use $B_r(x)$ to denote the Euclidean ball in $\mathbb{R}^d$ with center x and radius r. The parabolic cylinder $Q_r(t,x)$ is defined as follows

$$Q_r(t,x) = (t - r^\sigma, t) \times B_r(x). \tag{5.2.1}$$

We simply use Q_r to denote $Q_r(0,0)$ and $\mathbb{R}^{d+1}_0 := (-\infty, 0) \times \mathbb{R}^d$. For $\alpha \in (0,2)$, we define the Lipschitz-Zygmund semi-norm and norm by

$$
\begin{aligned}
[u]_{\Lambda^\alpha} &= \sup_{|h|>0} |h|^{-\alpha} \|u(\cdot + h) + u(\cdot - h) - 2u(\cdot)\|_{L_\infty}, \\
\|u\|_{\Lambda^\alpha} &= \|u\|_{L_\infty} + [u]_{\Lambda^\alpha}.
\end{aligned}
$$

We say $u \in \Lambda^\alpha$ if $\|u\|_{\Lambda^\alpha} < \infty$.

For simplicity of notation, we denote

$$
\delta u(t,x,y) = \begin{cases} u(t,x+y) - u(t,x) - y^T Du(t,x) & \text{for } \sigma \in (1,2), \\ u(t,x+y) - u(t,x) - y^T Du(t,x)\chi_{B_1} & \text{for } \sigma = 1, \\ u(t,x+y) - u(t,x) & \text{for } \sigma \in (0,1). \end{cases} \tag{5.2.2}
$$

The Pucci extremal operator is defined as follows: for $\sigma \neq 1$

$$
\begin{aligned}
\mathcal{M}^+ u(t,x) &= \int_{\mathbb{R}^d} \left(\Lambda \delta u(t,x,y)^+ - \lambda \delta u(t,x,y)^-\right) \frac{2-\sigma}{|y|^{d+\sigma}}\, dy, \\
\mathcal{M}^- u(t,x) &= \int_{\mathbb{R}^d} \left(\lambda \delta u(t,x,y)^+ - \Lambda \delta u(t,x,y)^-\right) \frac{2-\sigma}{|y|^{d+\sigma}}\, dy.
\end{aligned}
$$

When $\sigma = 1$, the extremal operator cannot be written out explicitly, due to the condition (5.1.1). Nevertheless, we do not use exact representation directly and define the extremal operator by

$$
\mathcal{M}^+ u = \sup_a L_a u \qquad \text{and} \quad \mathcal{M}^- u = \inf_a L_a u,
$$

where the infimum (or supremum) is taken with respect to all L_a's with kernels K_a satisfying (5.1.1).

We recall the weak Harnack inequality of [97, Theorem 6.1].

Proposition 5.2.1. *Assume that $0 < \sigma_0 \leq \sigma < 2$ and $C > 0$ is a constant. Let u be a function such that*

$$u_t - \mathcal{M}^- u \geq -C \quad in \quad Q_1, \quad u \geq 0 \quad in \quad (-1,0) \times \mathbb{R}^d.$$

Then there are constants $C_1 > 0$ and $\varepsilon_1 \in (0,1)$ depending only on $\sigma_0, \lambda, \Lambda,$ and d, such that

$$\Big(\int_{(-1,-2^{-\sigma}) \times B_{1/4}} u^{\varepsilon_1} \, dx \, dt \Big)^{1/\varepsilon_1} \leq C_1 \Big(\inf_{Q_{1/4}} u + C \Big).$$

From Proposition 5.2.1, we obtain the following corollary for $\sigma \in (1,2)$, the proof of which is provided in the appendix.

Corollary 5.2.2. *Let $\sigma_2 \in (1,2)$, $\sigma \in (\sigma_2, 2)$, $C > 0$ be constants, and u satisfy*

$$u_t - \mathcal{M}^- u \geq -C \quad in \quad Q_{2r}, \quad u \geq 0 \quad in \quad (-(2r)^\sigma, 0) \times \mathbb{R}^d.$$

Let ε_1 be the constant in Proposition 5.2.1. For any $r, \delta \in (0,1)$, denote $\tilde{Q}_{\delta r} = (-r^\sigma, -(\delta r)^\sigma) \times B_r$. Then we have

$$r^{-(d+\sigma)/\varepsilon_1} \Big(\int_{\tilde{Q}_{\delta r}} u^{\varepsilon_1} \, dx \, dt \Big)^{1/\varepsilon_1} \leq C_2 \Big(\inf_{Q_{\delta r/2}} u + C r^\sigma \Big),$$

where $C_2 > 0$ is a constant depending only on δ, σ_2, λ, Λ, and d.

We state the following local boundedness estimate from [22, Corollary 6.2].

Proposition 5.2.3. *Let $\Omega \subset \mathbb{R}^d$, $t_1 < t_2$, and u satisfy*

$$u_t - \mathcal{M}^+ u \leq 0 \quad in \quad (t_1, t_2] \times \Omega.$$

Then for any $(t_1', t_2] \times \Omega' \subset\subset (t_1, t_2] \times \Omega$,

$$\sup_{\Omega' \times (t_1', t_2]} u^+ \le C(2-\sigma) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{u^+}{1+|x|^{d+\sigma}}\, dx\, dt,$$

where C *depends on* Ω, Ω', t_1, t_2, *and* t_1'.

The next proposition is [97, Theorem 7.1].

Proposition 5.2.4. *Let* $0 < \sigma_0 \le \sigma < 2$ *and* u *satisfy in* Q_1

$$u_t - \mathcal{M}^+ u \le C_0 \quad \text{and} \quad u_t - \mathcal{M}^- u \ge -C_0.$$

Then there are constants $\gamma \in (0,1)$ *and* $C > 0$ *only depending on* d, σ_0, λ, *and* Λ *such that*

$$[u]_{\gamma/\sigma, \gamma; Q_{1/2}} \le C \|u\|_{L_\infty((-1,0);L_1(\omega_\sigma))} + CC_0.$$

Here

$$\|u\|_{L_\infty((-1,0);L_1(\omega_\sigma))} = (2-\sigma) \sup_{t \in (0,1)} \int_{\mathbb{R}^d} \frac{|u(t,x)|}{1+|x|^{d+\sigma}}\, dx.$$

Note that we replaced $\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}$ by $\|u\|_{L_\infty((-1,0);L_1(\omega_\sigma))}$, which follows from a simple localization argument. See, for instance, [22, Corollary 7.1]. In the sequel, we always assume $\gamma < \sigma$.

We finish this section by proving the following global Hölder estimate.

Lemma 5.2.5. *Let* u *satisfy in* Q_1

$$u_t - \mathcal{M}^+ u \le C_0 \quad \text{and} \quad u_t - \mathcal{M}^- u \ge -C_0,$$

where C_0 *is a constant and* $u \equiv 0$ *in* $\mathbb{R}^{d+1}_0 \setminus Q_1$. *Then there exists a constant* $\alpha \in (0,1)$

depending on $d, \lambda, \Lambda,$ and σ (uniformly as $\sigma \to 2$), so that

$$[u]_{\alpha/\sigma,\alpha;Q_1} \leq CC_0,$$

where C depends on $d, \lambda, \Lambda,$ and σ, which is uniformly bounded as $\sigma \to 2$.

Proof. Thanks to the interior Hölder estimate Proposition 5.2.4, it suffices to prove the estimate near the parabolic boundary of Q_1. We consider the lateral boundary and bottom separately. Define $\phi : \mathbb{R}^d \to \mathbb{R}^+$ as

$$\phi(x) = \begin{cases} x_d^\beta & \text{for } x_d > 0 \\ 0 & \text{for } x_d \leq 0 \end{cases},$$

where $\beta \in (0, 1)$. We claim that for sufficiently small $\beta \in (0, \sigma_0)$ depending on d, λ, Λ, and σ_0, we have

$$\mathcal{M}^+\phi(x) < -\hat{C}x_d^{\beta-\sigma} \quad \text{in} \quad \{x_d > 0\},$$

where $\hat{C}$ depends on $d, \lambda, \Lambda,$ and σ_0. By scaling, it is obvious that

$$\mathcal{M}^+\phi(x) = x_d^{\beta-\sigma}\mathcal{M}^+\phi(e),$$

where $e = (0, 0, \ldots, 0, 1)$. Therefore, we only need to estimate $\mathcal{M}^+\phi(e)$.

Case 1: $\sigma > 1$. By definition,

$$\begin{aligned} &\mathcal{M}^+\phi(e) \\ &= (2-\sigma)x_d^{\beta-\sigma}\left(\int_{\{y_d>-1\}} + \int_{\{y_d<-1\}}\right)\left(\Lambda(\delta\phi(e,y))^+ - \lambda(\delta\phi(e,y))^-\right)\frac{1}{|y|^{d+\sigma}}\,dy \\ &=: (2-\sigma)x_d^{\beta-\sigma}(\mathrm{I}_1 + \mathrm{I}_2). \end{aligned}$$

When $y_d > -1$, by concavity, it follows that

$$\phi(e+y) = (1+y_d)^\beta < 1+\beta y_d \quad \text{and} \quad \delta\phi(e,y) \le 0.$$

Therefore,

$$\begin{aligned} \mathrm{I}_1 &\le \int_{\{|y_d|<1\}} \lambda \delta\phi(e,y) \frac{1}{|y|^{d+\sigma}}\, dy \\ &= \lambda \int_{y_d \in (0,1)} \big((1+y_d)^\beta + (1-y_d)^\beta - 2\big) \frac{1}{|y|^{d+\sigma}}\, dy. \end{aligned}$$

Notice that for any $s \in (-1,1)$,

$$(1+s)^\beta < 1 + \beta s + \frac{\beta(\beta-1)}{2} s^2 + \frac{\beta(\beta-1)(\beta-2)}{6} s^3,$$

which implies that for $y_d \in (0,1)$

$$(1+y_d)^\beta + (1-y_d)^\beta - 2 < \beta(\beta-1) y_d^2.$$

Therefore,

$$\mathrm{I}_1 < \lambda\beta(\beta-1) \int_{y_d\in(0,1)} \frac{y_d^2}{|y|^{d+\sigma}}\, dy = C_1 \frac{\beta(\beta-1)}{2-\sigma},$$

where C_1 depends on d and λ.

Now we turn to I_2. Since $\phi(e+y) = 0$ when $y_d < -1$, we have

$$\begin{aligned} \mathrm{I}_2 &= \int_{\{y_d<-1\}} \frac{\Lambda(-\beta y_d - 1)^+ - \lambda(-\beta y_d - 1)^-}{|y|^{d+\sigma}}\, dy \\ &\le \int_{\{y_d<-1/\beta\}} \frac{\Lambda(-\beta y_d - 1)}{|y|^{d+\sigma}}\, dy = C_2 \beta^\sigma, \end{aligned}$$

where C_2 depends on Λ, d, and σ, and is uniformly bounded as $\sigma \to 2$. Thanks to the estimates of I_1 and I_2 above, it follows that

$$\mathcal{M}^+\phi(e) \le C_1\beta(\beta-1) + C_2\beta^\sigma.$$

By choosing β sufficiently small depending on λ, Λ, d, and σ (but uniformly as $\sigma \to 2$) so that

$$C_1(\beta-1) + C_2\beta^{\sigma-1} \le -C_1/2,$$

the claim is proved.

Case 2: $\sigma < 1$. Let I_1 and I_2 be defined as before. Since $\phi(x) = 0$ for $x_d < 0$, we get

$$\mathrm{I}_2 = -(2-\sigma)\int_{\{y_d<-1\}} \frac{\lambda}{|y|^{d+\sigma}}\,dy = -C_3, \tag{5.2.3}$$

where $C_3 > 0$ depends on σ, λ, and d. For I_1, we have

$$\begin{aligned}\mathrm{I}_1 &= (2-\sigma)\int_{\{y_d>0\}} \frac{\Lambda((1+y_d)^\beta - 1)}{|y|^{d+\sigma}}\,dy - (2-\sigma)\int_{y_d\in(-1,0)} \frac{\lambda(1-(1+y_d)^\beta)}{|y|^{d+\sigma}}\,dy \\ &\le (2-\sigma)\Lambda\int_{\{y_d>0\}} \frac{(1+y_d)^\beta - 1}{|y|^{d+\sigma}}\,dy \to 0 \quad \text{as} \quad \beta \to 0\end{aligned}$$

by the monotone convergence theorem. Therefore, we can choose β small depending on Λ, λ, d, and σ so that

$$\mathcal{M}^+\phi(e) \le -C_3/2.$$

The claim is proved.

Case 3: $\sigma = 1$. In this case, we still have (5.2.3). For I_1, we notice that integrand in the region $\{-1 < y_d < 0\} \cup \{|y| < 1\}$ is negative, and

$$\int_{\{y_d>0\}\cap\{|y|>1\}} \frac{(1+y_d)^\beta - 1}{|y|^{d+\sigma}}\, dy \to 0 \quad \text{as} \quad \beta \to 0$$

by the monotone convergence theorem. Thus the claim follows as well.

Now we are ready to consider u near the lateral boundary. By a translation and rotation of the coordinates, we replace the ball B_1 by $B_1(e)$ and estimate u near the origin. Define the barrier function $\psi(t,x) = \frac{C_0}{2^{\beta-\sigma}\hat{C}}\phi(x)$. Obviously,

$$\partial_t\psi(t,x) - \mathcal{M}^+\psi(t,x) = -\frac{C_0}{2^{\beta-\sigma}\hat{C}}\mathcal{M}^+\phi(x) > \frac{C_0}{2^{\beta-\sigma}}x_d^{\beta-\sigma} > C_0$$

when $x \in B_1(e)$. On the other hand, $\psi \geq 0$ in $\mathbb{R}\times\mathbb{R}^d$. Since

$$u_t - \mathcal{M}^+u \leq C_0$$

and $u \equiv 0$ outside Q_1, by the comparison principle,

$$u(t,x) \leq \psi(t,x) = \frac{C_0}{2^{\beta-\sigma}\hat{C}}x_d^\beta.$$

By considering $-u$ instead of u, we have $u \geq -\frac{C_0}{2^{\beta-\sigma}\hat{C}}x_d^\beta$. Hence, around the origin $|u| \leq C|x|^\beta$. By rotation of the coordinate, we obtain the estimate near the lateral boundary.

For the bottom, let $\tilde{\phi} = C_0(t+1)$ so that $\tilde{\phi}(-1) = 0$ and $\tilde{\phi}'(t) = C_0$. This yields that

$$\partial_t\tilde{\phi} - \mathcal{M}^+\phi = C_0.$$

Moreover, $\tilde{\phi} \geq 0$ in $(-1,0)\times\mathbb{R}^d$. By the comparison principle again, $u \leq \phi$ in Q_1. In particular, near the bottom $u \leq C_0(t+1)$, which further implies $|u| \leq C_0(t+1)$

by symmetry.

Combining the estimates of lateral boundary and bottom with the interior Hölder estimate, we prove the lemma. □

5.3 A Liouville theorem

The aim of this section is to prove the following Liouville theorem for the fully nonlinear parabolic nonlocal equation with non-symmetric kernels. The elliptic version for symmetric kernels was established in [98].

Theorem 5.3.1. *Let $\sigma \in (0,2)$. There is a constant $\hat{\alpha} \in (0,1/2)$ depending on d, λ, Λ, and σ (but is uniform as $\sigma \to 2$) such that the following statement holds. Let $\alpha \in (0,\hat{\alpha})$ be such that $[\sigma+\hat{\alpha}] < \sigma+\alpha$ and suppose that $u \in C_{loc}^{1+\frac{\alpha}{\sigma},\sigma+\alpha}(\mathbb{R}_0^{d+1})$ satisfies the following properties:*

(i) For any $\beta \in [0,\sigma+\alpha]$ and $R \geq 1$, we have

$$[u]_{\beta/\sigma,\beta;Q_R} \leq N_0 R^{\sigma+\alpha-\beta}; \tag{5.3.1}$$

(ii) For any $(s,h) \in \mathbb{R}_0^{d+1}$, we have

$$\partial_t\big(u(\cdot+s,\cdot+h)-u\big) - \mathcal{M}^-\big(u(\cdot+s,\cdot+h)-u\big) \geq 0, \tag{5.3.2}$$

$$\partial_t\big(u(\cdot+s,\cdot+h)-u\big) - \mathcal{M}^+\big(u(\cdot+s,\cdot+h)-u\big) \leq 0; \tag{5.3.3}$$

(iii) If $\sigma > 1$, for any nonnegative measure μ in $\mathbb{R}^d$ with compact support,

$$\partial_t u_\mu - \mathcal{M}^+ u_\mu \le 0, \quad \textit{where} \quad u_\mu(t,x) = \int_{\mathbb{R}^d} \delta u(t,x,h)\, d\mu(h).$$

Then u is a polynomial of degree ν in x and 1 in t, where ν is the integer part of $\sigma + \alpha$.

Remark. As in [98], it is possible to relax Condition (i) in Theorem 5.3.1 by assuming that (5.3.1) is satisfied for any $\beta \in [0, \sigma + \alpha']$ and $R \ge 1$, where $\alpha' \in (0,\alpha)$ is a constant satisfying $\sigma + \alpha' > \nu$. A simple computation reveals that in the case we also require that $\sigma > 1 + \alpha - \alpha'$ when $\sigma > 1$ and $\nu = 1$; and $\sigma > \alpha - \alpha'$ when $\sigma < 1$ and $\nu = 0$.

To prove Theorem 5.3.1, we first present a few lemmas. Define

$$\begin{aligned} P(t,x) &= \int_{\mathbb{R}^d} \big(\delta u(t,x,y) - \delta u(0,0,y)\big)^+ \frac{2-\sigma}{|y|^{d+\sigma}}\, dy, \\ N(t,x) &= \int_{\mathbb{R}^d} \big(\delta u(t,x,y) - \delta u(0,0,y)\big)^- \frac{2-\sigma}{|y|^{d+\sigma}}\, dy. \end{aligned}$$

Lemma 5.3.2. *Let $\hat\alpha \in (0, 1/2)$ be a constant satisfying $\hat\alpha < \sigma/2$. Under the conditions (i) and (ii) of Theorem 5.3.1, for any $\kappa \ge 2$ and $l \in \mathbb{N} \cup \{0\}$, we have*

$$\sup_{Q_{\kappa^l}} \big(P + N + |u_t - u_t(0,0)|\big) \le C N_0 \kappa^{\alpha l} \le C N_0 \kappa^{\hat\alpha l} \tag{5.3.4}$$

and

$$[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}} \le C N_0 \kappa^{\hat\alpha}, \tag{5.3.5}$$

where C depends only on d, λ, Λ, and σ, and is uniformly bounded as $\sigma \to 2$.

Proof. We first estimate P and N assuming that $\nu = 2$. Fix $(t,x), (t',x') \in Q_1$ and set $l=|x-x'|+|t-t'|^{1/\sigma}$. By Condition (i), when $|y|<l$,

$$
\begin{aligned}
&|\delta u(t,x,y)-\delta u(t',x',y)|\\
&=\Big|\int_0^1 y\Big[Du(t,x+sy)-Du(t,x)-\big(Du(t',x'+sy)-Du(t',x')\big)\Big]\,ds\Big|\\
&\le C|y|^2 l^{\sigma+\alpha-2}[u]_{1+\alpha/\sigma,\sigma+\alpha;Q_2}\le CN_0|y|^2 l^{\sigma+\alpha-2}. \qquad (5.3.6)
\end{aligned}
$$

Similarly, when $|y|\ge l$,

$$
|\delta u(t,x,y)-\delta u(t',x',y)|\le C|y|^{\sigma+\alpha-1}l[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1+|y|}}\le CN_0 l|y|^{\sigma+\alpha-1}. \qquad (5.3.7)
$$

Combining (5.3.6) and (5.3.7), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d}|\delta u(t,x,y)-\delta u(t',x',y)|\frac{2-\sigma}{|y|^{d+\sigma}}\,dy\\
&\le CN_0 l^{\sigma+\alpha-2}\int_{B_l}\frac{(2-\sigma)|y|^2}{|y|^{d+\sigma}}\,dy+CN_0 l\int_{\mathbb{R}^d\setminus B_l}\frac{(2-\sigma)|y|^{\sigma+\alpha-1}}{|y|^{d+\sigma}}\,dy\\
&\le CN_0 l^{\alpha}. \qquad (5.3.8)
\end{aligned}
$$

Hence $P,N\in C^{\alpha,\alpha/\sigma}(Q_1)$. Because $P(0,0)=N(0,0)=0$, we have

$$
P(t,x)+N(t,x)\le CN_0 \quad \text{in } Q_1.
$$

By modifying the estimate above, we can prove the same estimate for P when $\nu=0$ or 1.

We then use a scaling argument. Define $\hat u(t,x)=\eta^{-\alpha-\sigma}u(\eta^\sigma t,\eta x)$ for any $\eta>1$. It is easily seen that $\hat u$ satisfies all the conditions in this lemma. Hence, we know that

$$
\int_{\mathbb{R}^d}\frac{(2-\sigma)\big|\delta\hat u(t,x,y)-\delta\hat u(0,0,y)\big|}{|y|^{d+\sigma}}\,dy\le CN_0 \quad \text{in } Q_1.
$$

Therefore,

$$P(\eta^\sigma t, \eta x) + N(\eta^\sigma t, \eta x) \le CN_0\eta^\alpha \quad \text{in } Q_1,$$

which together with (5.3.2) and (5.3.3) implies (5.3.4).

To prove (5.3.5), we take $h = 0$ in (5.3.2) and (5.3.3), and then multiply them by $1/s$. By letting $s \to 0$, we know that u_t as well as $u_t - u_t(0,0)$ are sub and super-solutions at the same time. By Proposition 5.2.4, we obtain that $u_t \in C^{\gamma/\sigma,\gamma}(Q_{1/2})$ for some $\gamma > 0$ depending on d, λ, Λ, and σ_0, and

$$[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}} = [u_t - u_t(0,0)]_{\gamma/\sigma,\gamma;Q_{1/2}} \le C \sup_{t\in(-1,0)} \int_{\mathbb{R}^d} \frac{|u_t - u_t(0,0)|}{1+|x|^{\sigma+d}}\,dx.$$

Using (5.3.4), for any $t \in (-1,0)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|u_t - u_t(0,0)|}{1+|x|^{d+\sigma}}\,dx\,dt \\
&\le \int_{B_1} \frac{|u_t - u_t(0,0)|}{1+|x|^{d+\sigma}}\,dx\,dt + \sum_{i=0}^{\infty} \int_{B_{\kappa^{i+1}}\setminus B_{\kappa^i}} \frac{|u_t - u_t(0,0)|}{1+|x|^{d+\sigma}}\,dx\,dt \\
&\le CN_0 \int_{B_1} \frac{1}{1+|x|^{d+\sigma}}\,dx\,dt + CN_0 \sum_{i=0}^{\infty} \int_{B_{\kappa^{i+1}}\setminus B_{\kappa^i}} \frac{\kappa^{\hat\alpha(i+1)}}{1+|x|^{d+\sigma}}\,dx\,dt \le CN_0\kappa^{\hat\alpha},
\end{aligned}$$

where C depends only on d, λ, Λ, and σ_0. Here we used the fact $\hat\alpha < \sigma/2$ and $\kappa \ge 2$ in the last inequality. Therefore, the lemma is proved. □

Dividing u by a CN_0, where C is the constant in (5.3.4), and using Lemma 5.3.2, we have that for any $\kappa \ge 2$ and $l \in \mathbb{N} \cup \{0\}$,

$$\sup_{Q_{\kappa^l}} P \le \kappa^{\hat\alpha l}, \quad \sup_{Q_{\kappa^l}} N \le \kappa^{\hat\alpha l}. \tag{5.3.9}$$

We are going to prove inductively that there exists a sufficiently large $\kappa \ge 2$ and sufficiently small $\hat{\alpha} \in (0, 1/2)$ such that

$$\sup_{Q_{\kappa^{-l}}} P \leq \kappa^{-\hat{\alpha} l}, \quad \sup_{Q_{\kappa^{-l}}} N \leq \kappa^{-\hat{\alpha} l} \quad \text{for any } l \in \mathbb{N}.$$

For a fixed $r \in (0, 1)$, assume that P attains its maximum in $\overline{Q_r}$ at (t_0, x_0). Denote

$$\mathbb{A} := \{y : \delta u(t_0, x_0, y) - \delta u(0, 0, y) > 0\}.$$

Then

$$\begin{aligned} P(t_0, x_0) &= \int_{\mathbb{A}} \big(\delta u(t_0, x_0, y) - \delta u(0, 0, y)\big) \frac{2 - \sigma}{|y|^{d+\sigma}}\, dy, \\ N(t_0, x_0) &= \int_{\mathbb{R}^d \setminus \mathbb{A}} \big(\delta u(t_0, x_0, y) - \delta u(0, 0, y)\big) \frac{2 - \sigma}{|y|^{d+\sigma}}\, dy. \end{aligned}$$

We define

$$v(t, x) = \int_{\mathbb{A}} \big(\delta u(t, x, y) - \delta u(0, 0, y)\big) \frac{2 - \sigma}{|y|^{d+\sigma}}\, dy.$$

Notice that $v \leq P$, and in particular $v \leq 1$ in Q_1. Moreover, $P(t_0, x_0) = v(t_0, x_0)$. We denote $\mathrm{v} = (1 - v)^+$.

Lemma 5.3.3. *Suppose that $\hat{\alpha} \in (0, 1/2)$ satisfying $\hat{\alpha} < \sigma/2$ and $\kappa \geq 2$. Then we have*

$$\mathrm{v}_t - \mathcal{M}^- \mathrm{v} \geq -C(\kappa^{\hat{\alpha}} - 1) \quad \text{in } Q_{3/4}, \tag{5.3.10}$$

where C is a positive constant depending only on d, λ, Λ, and σ, and is uniformly bounded as $\sigma \to 2$.

Proof. Since $v \le 1$ in Q_1, for any $(t,x) \in Q_1$ we have $\mathrm{v}(t,x) = 1 - v(t,x)$, thus

$$\begin{aligned}\delta\mathrm{v}(t,x,y) &= \mathrm{v}(t,x+y) - \mathrm{v}(t,x) - D\mathrm{v}(t,x)y \\ &= (1-v)^+(t,x+y) - (1-v)(t,x) + Dv(t,x)y \\ &= (v-1)^+(t,x+y) - \delta v(t,x,y).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\big(\delta\mathrm{v}(t,x,y)\big)^- &\ge \big(\delta v(t,x,y)\big)^+ - (v-1)^+(t,x+y), \\ \big(\delta\mathrm{v}(t,x,y)\big)^+ &\le \big(\delta v(t,x,y)\big)^- + (v-1)^+(t,x+y).\end{aligned}$$

These imply

$$\begin{aligned}\mathrm{v}_t - \mathcal{M}^-\mathrm{v} &= -v_t - (2-\sigma)\int_{\mathbb{R}^d} \frac{\lambda(\delta\mathrm{v}(t,x,y))^+ - \Lambda(\delta\mathrm{v}(t,x,y))^-}{|y|^{d+\sigma}}\,dy \\ &\ge -v_t + \mathcal{M}^+v - (2-\sigma)(\Lambda+\lambda)\int_{\mathbb{R}^d} \frac{(v-1)^+(t,x+y)}{|y|^{d+\sigma}}\,dy. \qquad (5.3.11)\end{aligned}$$

From Condition (iii) and an approximation (see, for instance, the proof of Lemma 5.4.4 below), v satisfies

$$v_t - \mathcal{M}^+v \le 0. \qquad (5.3.12)$$

On the other hand, we have

$$\int_{\mathbb{R}^d} \frac{(v-1)^+(t,x+y)}{|y|^{d+\sigma}}\,dy \le \int_{\mathbb{R}^d} \frac{(P-1)^+(t,x+y)}{|y|^{d+\sigma}}\,dy.$$

Since P satisfies (5.3.9), for $(t,x) \in Q_{3/4}$, we have $P(t,x+y) \le 1$ when $y \in B_{1/4}$, and thus the right-hand side above is equal to

$$\sum_{i=0}^{\infty} \int_{B_{\kappa^{i+1}-3/4} \setminus B_{\kappa^i-3/4}} \frac{(P-1)^+(t,x+y)}{|y|^{d+\sigma}}\,dy,$$

which by (5.3.9) is bounded by

$$\sum_{i=0}^{\infty}\int_{B_{\kappa^{i+1}-3/4}\setminus B_{\kappa^{i}-3/4}}\frac{\kappa^{(i+1)\hat{\alpha}}-1}{|y|^{d+\sigma}}\,dy$$
$$\le C\sum_{i=0}^{\infty}(\kappa^{(i+1)\hat{\alpha}}-1)\kappa^{-i\sigma}=C\Big(\frac{\kappa^{\hat{\alpha}}}{1-\kappa^{\hat{\alpha}-\sigma}}-\frac{1}{1-\kappa^{-\sigma}}\Big)\le C\big(\kappa^{\hat{\alpha}}-1\big), \tag{5.3.13}$$

where C only depends on $d,\lambda,\Lambda,$ and σ_0. Here we used the fact $\hat{\alpha}<\sigma/2$ and $\kappa\ge 2$ in the last inequality. Combining (5.3.12)-(5.3.13) with (5.3.11), we prove the lemma. □

Let $\hat{\theta}=\lambda/(4\Lambda)$ and γ be the constant in Proposition 5.2.4. For any $r_1>0$, define the set

$$\mathcal{D}_{r_1}=\{(t,x)\in Q_{r_1}:v\ge 1-\hat{\theta}\}.$$

Lemma 5.3.4. *Suppose that $\hat{\alpha}\in(0,1/2)$ satisfying $\hat{\alpha}<\sigma/2$ and $\kappa\ge 2$. There exist some $\eta\in(0,1)$ sufficiently close to 1 and $c\in(0,1)$ sufficiently small, both depending only on d, λ, Λ, and σ (and is uniform as $\sigma\to 2$), such that for $r_1=c\kappa^{-\hat{\alpha}/\gamma}$,*

$$|\mathcal{D}_{r_1}|\le\eta|Q_{r_1}|. \tag{5.3.14}$$

Proof. By contradiction we assume that $|\mathcal{D}_{r_1}|>\eta|Q_{r_1}|$, and consider

$$w:=\int_{\mathbb{R}^d\setminus\mathbb{A}}\big(\delta u(t,x,y)-\delta u(0,0,y)\big)\frac{2-\sigma}{|y|^{d+\sigma}}\,dy.$$

By Condition (iii) and an approximation argument, w is a subsolution, i.e.,

$$w_t-\mathcal{M}^+w\le 0\quad\text{in}\quad\mathbb{R}^{d+1}_0. \tag{5.3.15}$$

From (5.3.2) and (5.3.3), we know that in $\mathbb{R}^{d+1}_0$,

$$\frac{\lambda}{\Lambda}P - \frac{u_t - u_t(0,0)}{\Lambda} \le N \le \frac{\Lambda}{\lambda}P - \frac{u_t - u_t(0,0)}{\lambda}, \tag{5.3.16}$$

implying that in $Q_{1/2}$

$$\frac{\lambda}{\Lambda}P - \frac{[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}}}{\Lambda}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \le N \le \frac{\Lambda}{\lambda}P + \frac{[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}}}{\lambda}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big).$$

From (5.3.5), (5.3.9), and the above inequality, we obtain

$$\begin{aligned} w = P - v - N &\le \big(1 - \lambda/\Lambda\big)P - v + C\kappa^{\hat\alpha}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \\ &\le 1 - \lambda/\Lambda - (1-\hat\theta) + C\kappa^{\hat\alpha}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \\ &\le -\lambda/\Lambda + \hat\theta + C\kappa^{\hat\alpha}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \quad \text{in } \mathcal{D}_{r_1}, \end{aligned}$$

where C only depends on d, λ, Λ, and σ, and is uniformly bounded as $\sigma \to 2$. Now we choose c sufficiently small depending only on d, λ, Λ, and σ (uniformly as $\sigma \to 2$), such that

$$\begin{aligned} &-\lambda/\Lambda + \hat\theta + C\kappa^{\hat\alpha}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \\ &\le -3\hat\theta + C\kappa^{\hat\alpha}\big(c\kappa^{-\hat\alpha/\gamma}\big)^\gamma \le -3\hat\theta + Cc^\gamma \le -\hat\theta \quad \text{in } Q_{r_1}, \end{aligned} \tag{5.3.17}$$

which implies $w \le -\hat\theta$ in $\mathcal{D}_{r_1}$. Since w is a subsolution (5.3.15), it follows immediately that for any $\varepsilon \in (0, r_1)$, $\bar w_\varepsilon(t,x) := (w + \hat\theta)^+(\varepsilon^\sigma t, \varepsilon x)$ is a subsolution as well. Moreover,

$$\Big|\{\bar w_\varepsilon \le 0\} \cap Q_{r_1/\varepsilon}\Big| > \eta\Big|Q_{r_1/\varepsilon}\Big|. \tag{5.3.18}$$

We estimate $\bar{w}_\varepsilon$ by applying Proposition 5.2.3 with $t_1 = -1$, $t_2 = 0$, and $\Omega = \mathbb{R}^d$

$$\begin{aligned}\bar{w}_\varepsilon(0,0) &\le C\int_{-1}^{0}\int_{\mathbb{R}^d}\frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt\\ &= C\int_{-1}^{0}\int_{B_{r_1/\varepsilon}}\frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt + C\int_{-1}^{0}\int_{B^c_{r_1/\varepsilon}}\frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt. \end{aligned} \tag{5.3.19}$$

We first consider the second term on the right-hand side of the inequality above

$$\begin{aligned}&\int_{-1}^{0}\int_{B^c_{r_1/\varepsilon}}\frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt\\ &\le \int_{-1}^{0}\int_{B^c_{r_1/\varepsilon}}\frac{\hat{\theta}}{1+|x|^{d+\sigma}}\,dx\,dt + \int_{-1}^{0}\int_{B^c_{r_1/\varepsilon}}\frac{|w|(\varepsilon^\sigma t,\varepsilon x)}{1+|x|^{d+\sigma}}\,dx\,dt\\ &\le C\hat{\theta}(\varepsilon/r_1)^\sigma + \int_{-\varepsilon^\sigma}^{0}\int_{B^c_{r_1}}\frac{|w|(t,x)}{\varepsilon^{d+\sigma}+|x|^{d+\sigma}}\,dx\,dt. \end{aligned} \tag{5.3.20}$$

Since $|w| \le \max\{P,N\}$, from (5.3.9), for any $l \ge 0$,

$$\sup_{Q_{\kappa^l}}|w| \le \kappa^{\hat{\alpha}l}.$$

Therefore,

$$\begin{aligned}&\int_{-\varepsilon^\sigma}^{0}\int_{B^c_{r_1}}\frac{|w|(t,x)}{\varepsilon^{d+\sigma}+|x|^{d+\sigma}}\,dx\,dt\\ &\le \int_{-\varepsilon^\sigma}^{0}\int_{B_1\setminus B_{r_1}}\frac{|w|}{\varepsilon^{d+\sigma}+|x|^{d+\sigma}}\,dx\,dt + \sum_{i=0}^{\infty}\int_{-\varepsilon^\sigma}^{0}\int_{B_{\kappa^{i+1}}\setminus B_{\kappa^i}}\frac{|w|}{|x|^{d+\sigma}}\,dx\,dt\\ &\le C\big((\varepsilon/r_1)^\sigma + \varepsilon^\sigma\kappa^{\hat{\alpha}}\big), \end{aligned} \tag{5.3.21}$$

where C only depends on d, λ, Λ, and σ, and is uniformly bounded as $\sigma \to 2$. We combine (5.3.20) and (5.3.21) to obtain that

$$C\int_{-1}^{0}\int_{B^c_{r_1/\varepsilon}}\frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt \le C\big((\varepsilon/r_1)^\sigma + \varepsilon^\sigma\kappa^{\hat{\alpha}}\big).$$

Recall that $r_1 = c\kappa^{-\hat{\alpha}/\gamma}$. For the right-hand side of the inequality above, we want to choose ε sufficiently small such that $C\varepsilon^\sigma(r_1^{-\sigma} + \kappa^{\hat{\alpha}}) \leq \hat{\theta}/4$. Indeed, since $c \in (0,1)$ and $\sigma > \gamma$, we have $r_1^{-\sigma} \geq \kappa^{\hat{\alpha}}$. It is sufficient to fix ε such that $2C\varepsilon^\sigma r_1^{-\sigma} = \hat{\theta}/4$, i.e.,

$$\varepsilon/r_1 = \left(\hat{\theta}/(8C)\right)^{1/\sigma} := c_1,$$

where c_1 only depends on d, λ, Λ, and σ, and is uniformly bounded as $\sigma \to 2$. In other words, by taking $\varepsilon = c_1 r_1$ we have

$$C\int_{-1}^{0}\int_{B^c_{r_1/\varepsilon}} \frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt \leq \hat{\theta}/4. \tag{5.3.22}$$

Next we estimate the first term on the right-hand side of (5.3.19) using (5.3.18):

$$\begin{aligned}
&C\int_{-1}^{0}\int_{B_{1/c_1}} \frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt \\
&= C\int_{\left((-1,0)\times B_{1/c_1}\right)\cap\{\bar{w}_\varepsilon>0\}} \frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt \\
&\leq C(1-\eta)|Q_{1/c_1}| \sup_{(-1,0)\times B_{r_1}} (w+\hat{\theta})^+ \leq \hat{\theta}/4
\end{aligned} \tag{5.3.23}$$

upon taking η sufficiently close to 1 depending only on d, λ, Λ, and σ (uniformly as $\sigma \to 2$).

Combining (5.3.23) and (5.3.22) with (5.3.19), we have $\bar{w}_\varepsilon(0,0) \leq \hat{\theta}/2$ indicating that $w(0,0) \leq -\hat{\theta}/2$, which contradicts with $w(0,0) = 0$ by the definition of w. Therefore, the lemma is proved. □

Now we are ready to prove Theorem 5.3.1.

Proof of Theorem 5.3.1. Below we first elaborate on the case when $\sigma > 1$. At the end, we briefly discuss the case when $\sigma = 1$. The proof of the case when $\sigma < 1$ is omitted due to similarity to the first case.

Let η and c be the constants in Lemma 5.3.4 and $r_1 = c\kappa^{-\hat{\alpha}/\gamma}$. From (5.3.14),

$$\big|\{1 - v \geq \hat{\theta}\} \cap Q_{r_1}\big| \geq (1-\eta)\big|Q_{r_1}\big|.$$

Recall that $\tilde{Q}_{\delta r_1} = (-{r_1}^{\sigma}, -(\delta r_1)^{\sigma}) \times B_{r_1}$. For δ sufficiently small depending on η, we have

$$\big|\{1 - v \geq \hat{\theta}\} \cap \tilde{Q}_{\delta r_1}\big| \geq \frac{1-\eta}{2}\big|\tilde{Q}_{\delta r_1}\big|.$$

We set $r = \delta r_1/2$ and apply the weak Harnack inequality Corollary 5.2.2 to $(1-v)^+$ in $Q_{3/4}$ with (5.3.10) to obtain

$$\begin{aligned}
&\inf_{Q_r}(1-v) + C(\kappa^{\hat{\alpha}} - 1)r_1^{\sigma} \\
&\geq C(\delta)\|(1-v)^+\|_{L_{\varepsilon_1}(\tilde{Q}_{\delta r_1})}{r_1}^{-(d+\sigma)/\varepsilon_1} \\
&\geq C(\delta)\hat{\theta}\big((1-\eta)|\tilde{Q}_{\delta r_1}|/2\big)^{1/\varepsilon_1} r_1^{-(d+\sigma)/\varepsilon_1} \\
&= C(\delta)\hat{\theta}\big((1-\eta)(1-\delta^{\sigma})/2\big)^{1/\varepsilon_1} =: 2\theta,
\end{aligned} \tag{5.3.24}$$

where θ is a small constant depending only on d, λ, Λ, and σ, and is uniformly bounded as $\sigma \to 2$. We fix $\kappa = 4/(c\delta)^2$, where c is chosen according to (5.3.17), which guarantees that

$$\kappa^{-1} \leq c\delta/2 \cdot \kappa^{-\hat{\alpha}/\gamma}$$

for any $\hat{\alpha} \in (0, \gamma/2)$. Since $r = \delta r_1/2$ and $r_1 = c\kappa^{-\hat{\alpha}/\gamma}$, the inequality above can be written as $\kappa^{-1} < r$.

Next, we choose $\hat{\alpha}_1 = \log(1+\theta/C)/\log\kappa$ such that for any $\hat{\alpha} \in (0, \hat{\alpha}_1)$,

$$C(\kappa^{\hat{\alpha}} - 1) r_1^{\sigma} < C(\kappa^{\hat{\alpha}} - 1) < \theta.$$

Therefore, from (5.3.24) and the inequality above we obtain that $\sup_{Q_r} v \leq 1 - \theta$. Since $\kappa^{-1} < r$,

$$\sup_{Q_{\kappa^{-1}}} P \leq \sup_{Q_r} P = P(t_0, x_0) = v(t_0, x_0) = \sup_{Q_r} v \leq 1 - \theta. \tag{5.3.25}$$

Similarly,

$$\sup_{Q_{\kappa^{-1}}} N \leq 1 - \theta. \tag{5.3.26}$$

Set

$$\hat{\alpha} = \min\big\{ -\log(1-\theta)/\log\kappa, \hat{\alpha}_1, \gamma/2 \big\}.$$

Then (5.3.25) and (5.3.26) imply

$$\sup_{Q_{\kappa^{-1}}} \kappa^{\hat{\alpha}} P \leq 1, \quad \sup_{Q_{\kappa^{-1}}} \kappa^{\hat{\alpha}} N \leq 1. \tag{5.3.27}$$

Let

$$\tilde{P}(t, x) = \kappa^{\hat{\alpha}} P(\kappa^{-1} x, \kappa^{-\sigma} t), \quad \tilde{N}(t, x) = \kappa^{\hat{\alpha}} N(\kappa^{-1} x, \kappa^{-\sigma} t),$$

and

$$\tilde{u}(t, x) = \kappa^{\sigma + \hat{\alpha}} u(\kappa^{-1} x, \kappa^{-\sigma} t).$$

From (5.3.16), we have

$$\frac{\lambda}{\Lambda}\tilde{P} - \frac{\tilde{u}_t - \tilde{u}_t(0,0))}{\Lambda} \leq \tilde{N} \leq \frac{\Lambda}{\lambda}\tilde{P} - \frac{\tilde{u}_t - \tilde{u}_t(0,0)}{\lambda}.$$

Since $\kappa \geq 2$ and $\hat{\alpha} \leq \gamma$, we get

$$[\tilde{u}_t]_{\gamma/\sigma,\gamma;Q_{1/2}} \leq \kappa^{\hat{\alpha}-\gamma}[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}} \leq [u_t]_{\gamma/\sigma,\gamma;Q_{1/2}}.$$

On the other hand, for any $l \geq 0$,

$$\sup_{Q_{\kappa^l}} \tilde{P} = \kappa^{\hat{\alpha}} \sup_{Q_{\kappa^{l-1}}} P \leq \kappa^{\hat{\alpha}}\kappa^{(l-1)\hat{\alpha}} = \kappa^{l\hat{\alpha}}, \quad \sup_{Q_{\kappa^l}} \tilde{N} \leq \kappa^{l\hat{\alpha}}.$$

Therefore, $\tilde{P}$ and $\tilde{N}$ satisfy all the conditions that P and N satisfied. Applying (5.3.27) to $\tilde{P}$ and $\tilde{N}$, we have

$$\sup_{Q_{\kappa^{-1}}} \tilde{P} \leq \kappa^{-\hat{\alpha}}, \quad \sup_{Q_{\kappa^{-1}}} \tilde{N} \leq \kappa^{-\hat{\alpha}},$$

which further implies that

$$\sup_{Q_{\kappa^{-2}}} P \leq \kappa^{-2\hat{\alpha}}, \quad \sup_{Q_{\kappa^{-2}}} N \leq \kappa^{-2\hat{\alpha}}.$$

By induction, for any $l \in \mathbb{N}$,

$$\sup_{Q_{\kappa^{-l}}} P \leq \kappa^{-l\hat{\alpha}}, \quad \sup_{Q_{\kappa^{-l}}} N \leq \kappa^{-l\hat{\alpha}}.$$

Therefore, we have in Q_1

$$P(t,x) \leq C\big(|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}\big), \quad N(t,x) \leq C\big(|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}\big).$$

Since for any $\eta \geq 1$, $\hat{u}(t,x) = \eta^{-\sigma-\alpha}u(\eta^\sigma t, \eta x)$ satisfies the same condition as u, replacing u by $\hat{u}$ in the definition of P and denoting it as $P_{\hat{u}}$, we obtain

$$P_{\hat{u}}(t,x) \leq C\big(|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}\big) \quad \text{in } Q_1.$$

Returning to P, we have

$$\eta^{-\alpha}P(\eta^{\sigma}t,\eta x)\le C\big(|x|^{\hat\alpha}+|t|^{\hat\alpha/\sigma}\big)$$

in Q_1, which further implies that

$$\sup_{Q_\eta}\frac{P(t,x)}{|x|^{\hat\alpha}+|t|^{\hat\alpha/\sigma}}\le C\eta^{\alpha-\hat\alpha}.$$

Let $\eta\to\infty$ yields

$$\sup_{(t,x)\in\mathbb{R}^{d+1}_0}\frac{P(t,x)}{|x|^{\hat\alpha}+|t|^{\hat\alpha/\sigma}}=0,$$

which gives $P=0$. Similarly, $N=0$.

From the definition of P and N, we have

$$u(t,x+y)-u(t,x)-Du(t,x)y=u(0,y)-u(0,0)-Du(0,0)y.$$

Taking derivative in y, we have, for any $t\in(-\infty,0)$ and $x,y\in\mathbb{R}^d$

$$Du(t,x+y)-Du(t,x)=Du(0,y)-Du(0,0),$$

which implies for fixed t, u is a polynomial in x of order at most two. Using Condition (i) with $\beta=0$, we infer that this order is at most ν. Condition (ii), together with $P=N=0$, yields $u_t=c$ for some constant c. The proof is completed for $\sigma>1$.

Finally, we sketch the proof for $\sigma=1$. From Condition (ii), we know that $u(\cdot+s,\cdot+h)-u$, and thus Du and u_t are both sub and supersolutions, and are in

$C^{\gamma/\sigma,\gamma}(Q_{1/2})$. By Proposition 5.2.4, we have

$$[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}} + [Du]_{\gamma/\sigma,\gamma;Q_{1/2}} \leq C \sup_{t\in[0,1]} \int_{\mathbb{R}^d} \frac{|Du| + |u_t|}{1 + |x|^{d+1}}\, dx.$$

From Condition (i), the right-hand side of the inequality above is less than

$$\int_{B_1} \frac{1}{1 + |x|^{1+d}}\, dx + \sum_{j=1}^{\infty} \int_{B_{2^j}\setminus B_{2^{j-1}}} \frac{C2^{j\alpha}}{1 + |x|^{1+d}}\, dx \leq \frac{C}{1 - 2^{\alpha-1}}$$

for any $\alpha \in (0,1)$. By taking $\hat{\alpha} \leq \gamma$ and scaling as before, we can prove that

$$[u]_{1+\gamma,1+\gamma;Q_R} \leq CR^{\alpha-\gamma},$$

i.e., u must be a linear function by sending $R \to \infty$. The theorem is proved. $\square$

5.4 Schauder estimate for nonlocal parabolic equations

In this section, we prove Theorem 5.1.1 by applying the Liouville theorem, a blow-up analysis, and a localization procedure. In the rest of the chapter, we do not specify the domain associated with the norm when it is $\mathbb{R}^{d+1}_0 = (-\infty, 0) \times \mathbb{R}^d$.

5.4.1 Equations with translation invariant kernels

In this subsection, we consider equations with translation invariant kernels, i.e., $K = K(y)$. The main result of the subsection is the following theorem.

Theorem 5.4.1. *Let $\sigma \in (0,2)$ be a constant and $\mathcal{A}$ be an index set. There exists a constant $\hat{\alpha} > 0$ depending on d, λ, Λ, and σ (uniformly as $\sigma \to 2$), such that given $0 < \alpha' < \alpha < \hat{\alpha}$ satisfying $[\sigma + \alpha] < \sigma + \alpha' < \sigma + \alpha$ the following holds. Let $u \in C^{1+\alpha'/\sigma,\alpha'+\sigma}\big((-1,0)\times\mathbb{R}^d\big) \cap C^{1+\alpha/\sigma,\alpha+\sigma}(Q_1)$ satisfy*

$$u_t = \inf_{a\in\mathcal{A}} \big(L_a u + f_a\big) \quad \text{in } Q_1,$$

where $L_a \in \mathcal{L}_0(\sigma,\lambda,\Lambda)$ with $K_a = K_a(y)$ for any $a \in \mathcal{A}$. Assume that

$$\sup_{(t,x)\in Q_1} \Big|\inf_{a\in\mathcal{A}} f_a(t,x)\Big| < \infty.$$

Then

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C[u]_{1+\alpha'/\sigma,\alpha'+\sigma;(-1,0)\times\mathbb{R}^d} + C\sup_{a\in\mathcal{A}}[f_a]_{\alpha/\sigma,\alpha;Q_1},$$

where C only depends on $d, \lambda, \Lambda, \sigma, \alpha$, and α', and is uniformly bounded as $\sigma \to 2$.

We denote

$$Q^k = \big(-1 + 2^{-(k+1)\sigma}/(1-2^{-\sigma}), 0\big) \times B_{1-2^{-k}}(0) \tag{5.4.1}$$

for all sufficiently large integers k such that $2^{-(k+1)\sigma} < 1 - 2^{-\sigma}$. We shall prove a stronger result:

$$\begin{aligned}&\sup_k 2^{-k(\alpha-\alpha')}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k}\\ &\le C[u]_{1+\alpha'/\sigma,\alpha'+\sigma;(-1,0)\times\mathbb{R}^d} + C\sup_{a\in\mathcal{A}}[f_a]_{\alpha/\sigma,\alpha;Q_1}.\end{aligned} \tag{5.4.2}$$

The conclusion of the theorem is a particular case for k large only depending on σ (uniformly as $\sigma \to 2$) so that $Q_{1/2} \subset Q^k$. Since we assume that $u \in C^{1+\alpha/\sigma,\alpha+\sigma}(Q_1)$, there exists an integer k such that

$$2^{-k(\alpha-\alpha')}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k} = \sup_l 2^{-l(\alpha-\alpha')}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^l}.$$

Next, we prove (5.4.2) by contradiction. Assume that we can find solutions u_j and index sets $\mathcal{A}_j$ such that

$$\begin{aligned}
&\partial_t u_j = \inf_{a\in\mathcal{A}_j}(L_a u_j + f_a) \quad \text{in} \quad Q_1, \qquad \sup_{(t,x)\in Q_1}\left|\inf_{a\in\mathcal{A}_j} f_a(t,x)\right| < \infty, \\
&[u_j]_{1+\alpha'/\sigma,\sigma+\alpha';(-1,0)\times\mathbb{R}^d} + \sup_{a\in\mathcal{A}_j}[f_a]_{\alpha/\sigma,\alpha;Q_1} \le 1, \\
&\text{and} \quad \sup_k 2^{-k(\alpha-\alpha')}[u_j]_{1+\alpha/\sigma,\sigma+\alpha;Q^k} \ge j,
\end{aligned} \tag{5.4.3}$$

where for any $a \in \mathcal{A}_k$, $L_a \in \mathcal{L}_0$ with $K_a = K_a(y)$. As explained above, for each j there exists an integer k_j so that

$$2^{-k_j(\alpha-\alpha')}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} = \sup_k 2^{-k(\alpha-\alpha')}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^k}.$$

Lemma 5.4.2. *For any $j \ge 1$, we have*

$$\begin{aligned}
[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} &\le \sup_{r>0}\sup_{(t,x)\in Q^{k_j}} r^{-(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_r(t,x)\cap\{t>-1\}} \\
&\le 4^{\alpha-\alpha'}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}.
\end{aligned} \tag{5.4.4}$$

Moreover, we can find $(t_j,x_j) \in Q^{k_j}$ and r_j such that

$$\frac{1}{2}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \le r_j^{-(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{r_j}(t_j,x_j)\cap\{t>-1\}} \tag{5.4.5}$$

and

$$2^{k_j} r_j \to 0 \quad \text{as } j \to \infty. \tag{5.4.6}$$

Proof. The first inequality in (5.4.4) follows from the fact that for any $(t,x),(s,y) \in Q^{k_j}$ with $t \geq s$, we have $(s,y) \in Q_r(t,x) \cap \{t > -1\}$, where $r = \max(|x-y|, |t-s|^{1/\sigma})$. See, for instance, Claim 3.2 of [98]. For the second inequality, if $r \leq 2^{-(k_j+1)}$, for any $(t,x) \in Q^{k_j}$, we have $Q_r(t,x) \subset Q^{k_j+1}$ and

$$\begin{aligned}
&r^{-(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_r(t,x)\cap\{t>-1\}} \\
&\leq 2^{\alpha-\alpha'}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j+1}} \leq 4^{\alpha-\alpha'}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}},
\end{aligned}$$

where the last inequality is due to the choice of k_j. On the other hand, if $r > 2^{-(k_j+1)}$, for any $(t,x) \in Q^{k_j}$,

$$\begin{aligned}
&r^{-(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_r(t,x)\cap\{t>-1\}} \\
&\leq 2^{(k_j+1)(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;(-1,0)\times\mathbb{R}^d} \leq 2^{\alpha-\alpha'}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}},
\end{aligned}$$

where the last inequality follows from (5.4.3). Thus, we obtain the second inequality in (5.4.4).

Due to (5.4.4), we can find $(t_j, x_j) \in Q^{k_j}$ and r_j such that (5.4.5) is satisfied and thus by (5.4.3),

$$(2^{k_j} r_j)^{\alpha-\alpha'} \leq \frac{2[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{r_j}(t_j,x_j)\cap\{t>-1\}}}{2^{-k_j(\alpha-\alpha')}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \to 0 \quad \text{as } j \to \infty,$$

which further implies (5.4.6). The lemma is proved. □

Let T_j be the Taylor expansion of u_j at $X_j = (t_j, x_j)$ of order $\nu = [\sigma+\alpha]$ in x and 1 in t. Now we consider the blow-up sequence

$$v_j(t,x) = \frac{u_j(t_j + r_j^\sigma t, x_j + r_j x) - T_j(t_j + r_j^\sigma t, x_j + r_j x)}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}.$$

Here (t_j, x_j) and r_j are from Lemma 5.4.2. Note that v_j is well defined on $(-R_j^\sigma, 0) \times \mathbb{R}^d$, where by Lemma 5.4.2,

$$R_j := 2^{-(k_j+1)} r_j^{-1} \to \infty \quad \text{as} \quad j \to \infty.$$

Observe that from (5.4.5) and (5.4.6), for sufficiently large j such that $r_j \le 2^{-(k_j+1)}$,

$$[v_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_1} = \frac{r_j^{\sigma+\alpha'}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{r_j}(t_j,x_j)}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \ge 1/2. \tag{5.4.7}$$

Lemma 5.4.3. *For any $R > 0$ and $\beta \in [0, \sigma + \alpha']$, we have*

$$[v_j]_{\beta/\sigma,\beta;Q_R \cap \{t > -R_j^\sigma\}} \le C R^{\sigma+\alpha-\beta}, \tag{5.4.8}$$

where C depends only on α and α'. Moreover, for any $0 < R < R_j$ and $\beta \in [0, \sigma+\alpha]$, we have

$$[v_j]_{\beta/\sigma,\beta;Q_R} \le C R^{\sigma+\alpha-\beta}, \tag{5.4.9}$$

where C depends only on α and α'. Thus, we can find $v \in C^{1+\alpha/\sigma,\sigma+\alpha}(\mathbb{R}_0^{d+1})$ such that v satisfies (5.4.9) for any $R > 0$ and $\beta \in [0, \sigma + \alpha]$, and along a subsequence $v_j \to v$ in $C^{\beta/\sigma,\beta}$ locally uniformly for any $\beta \in [0, \sigma + \alpha)$.

We remark that (5.4.8) will be used below to prove that v satisfies Condition (iii) in Theorem 5.3.1, and (5.4.9) will be used to show that v satisfies Condition (i).

Proof of Lemma 5.4.3. For any $R > 0$ and $\beta \in [0, \sigma + \alpha']$,

$$
\begin{aligned}
&[v_j]_{\beta/\sigma,\beta;Q_R\cap\{t>-R_j^\sigma\}} \\
&= \frac{[u_j(t_j + r_j^\sigma\cdot, x_j + r_j\cdot) - T_j(t_j + r_j^\sigma\cdot, x_j + r_j\cdot)]_{\beta/\sigma,\beta;Q_R\cap\{t>-R_j^\sigma\}}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \\
&\le \frac{r_j^\beta (Rr_j)^{\sigma+\alpha'-\beta}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{Rr_j}(t_j,x_j)\cap\{t>-1\}}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \le CR^{\sigma+\alpha-\beta},
\end{aligned}
$$

where we used (5.4.4) in the last inequality.

For any $R < R_j$, by the choice of k_j we have

$$
\begin{aligned}
[v_j]_{1+\alpha/\sigma,\alpha+\sigma;Q_R} &= \frac{[u_j(t_j + r_j^\sigma\cdot, x_j + r_j\cdot)]_{1+\alpha/\sigma,\alpha+\sigma;Q_R}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \\
&= \frac{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q_{Rr_j}(t_j,x_j)}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \le \frac{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j+1}}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \le 2^{\alpha-\alpha'}.
\end{aligned}
$$

Using the interpolation inequality, we reach (5.4.9). The last statement of the lemma follows from the Arzela-Ascoli theorem and the Cauchy diagonal method. □

Lemma 5.4.4. *The function v in Lemma 5.4.3 satisfies the conditions in Theorem 5.3.1.*

Proof. By Lemma 5.4.3, Condition (i) is satisfied. Next we verify Condition (iii) for $\sigma \in (1, 2)$. For any measure μ with compact support and $\delta \in (0, 1)$, we define

$$
V_j(t, x) = \int_{\mathbb{R}^d} v_j(t, x + h) - v_j(t, x) - \frac{v_j(t, x) - v_j(t, x - \delta h)}{\delta}\, d\mu(h).
$$

Since T_j is linear in t, from the definition of v_j, we have

$$\begin{aligned}\partial_t V_j(t,x) &= \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}\\ &\cdot \int_{\mathbb{R}^d}\Big[\partial_t u_j(t_j+r_j^\sigma t, x_j+r_j(x+h)) - \partial_t u_j(t_j+r_j^\sigma t, x_j+r_jx)\\ &\quad - \frac{\partial_t u_j(t_j+r_j^\sigma t, x_j+r_jx) - \partial_t u_j(t_j+r_j^\sigma t, x_j+r_j(x-\delta h))}{\delta}\Big]\,d\mu(h),\end{aligned}$$

which is equal to

$$\begin{aligned}&\frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}\Big[\int_{\mathbb{R}^d}\partial_t u_j(t_j+r_j^\sigma t, x_j+r_j(x+h))\,d\mu(h)\\ &\quad + \int_{\mathbb{R}^d}\frac{\partial_t u_j(t_j+r_j^\sigma t, x_j+r_j(x-\delta h))}{\delta}\,d\mu(h)\\ &\quad - (1+1/\delta)\|\mu\|_{L_1}\partial_t u_j(t_j+r_j^\sigma t, x_j+r_jx)\Big].\end{aligned} \tag{5.4.10}$$

For any $a\in\mathcal{A}_j$, define $\hat{K}_a(y)=r_j^{d+\sigma}K_a(r_jy)$, which satisfies

$$\frac{\lambda(2-\sigma)}{|y|^{d+\sigma}}\le \hat{K}_a(y)\le \frac{\Lambda(2-\sigma)}{|y|^{d+\sigma}},$$

and $\hat{L}_a$ be the corresponding operator with kernel $\hat{K}_a$.

Clearly,

$$\begin{aligned}\hat{L}_aV_j &= \int_{\mathbb{R}^d}\Big[\hat{L}_a(v_j(t,x+h)-v_j(t,x)) - \frac{\hat{L}_a(v_j(t,x)-v_j(t,x-\delta h))}{\delta}\Big]d\mu(h)\\ &= \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}\int_{\mathbb{R}^d}\Big\{(L_au_j)(t_j+r_j^\sigma t, x_j+r_jx+r_jh)\\ &\quad - (L_au_j)(t_j+r_j^\sigma t, x_j+r_jx)\\ &\quad - \frac{(L_au_j)(t_j+r_j^\sigma t, x_j+r_jx) - (L_au_j)(t_j+r_j^\sigma t, x_j+r_j(x-\delta h))}{\delta}\Big\}\,d\mu(h),\end{aligned}$$

where in the second equality, we used the definitions of $\hat{L}_a$, v_j and the fact that T_j

is at most second-order in x variable, so that for $\sigma > 1$ and any $y \in \mathbb{R}^d$

$$\delta T_j(t, x+h, y) - \delta T_j(t, x, y) = 0.$$

Therefore, for any $(t, x) \in (-R_j^\sigma, 0) \times \mathbb{R}^d$,

$$\sup_{a \in \mathcal{A}_j} \hat{L}_a(V_j) = \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma, \alpha+\sigma; Q^{k_j}}} \sup_{a \in \mathcal{A}_j} \Big\{ \int_{\mathbb{R}^d} \Big((L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x + r_j h) \\ + \frac{(L_a u_j)(t_j + r_j^\sigma t, x_j + r_j(x - \delta h))}{\delta} \Big) d\mu(h) - (1 + \frac{1}{\delta})(L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x) \Big\}$$

$$= \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma, \alpha+\sigma; Q^{k_j}}} \sup_{a \in \mathcal{A}_j} \Big\{ \int_{\mathbb{R}^d} (L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x + r_j h) \\ + f_a(t_j + r_j^\sigma r, x_j + r_j x + r_j h) + \frac{1}{\delta}\big((L_a u_j)(t_j + r_j^\sigma t, x_j + r_j(x - \delta h)) \\ + f_a(t_j + r_j^\sigma t, x_j + r_j(x - \delta h)) \big) \, d\mu(h) \\ - (1 + \frac{1}{\delta})((L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x) + f_a(t_j + r_j^\sigma t, x_j + r_j x)) \cdot \|\mu\|_{L_1} \\ - \int_{\mathbb{R}^d} \Big(f_a(t_j + r_j^\sigma t, x_j + r_j x + r_j h) - f_a(t_j + r_j^\sigma t, x_j + r_j x) \\ + \frac{1}{\delta}(f_a(t_j + r_j^\sigma t, x_j + r_j(x - \delta h)) - f_a(t_j + r_j^\sigma t, x_j + r_j x) \Big) \, d\mu(h) \Big\}.$$

Note that for sufficiently large j such that $\max((-t)^{1/\sigma}, |x| + |h|) \leq R_j$ whenever $h \in \operatorname{supp}\mu$, we have

$$|f_a(t_j + r_j^\sigma t, x_j + r_j x + r_j h) - f_a(t_j + r_j^\sigma t, x_j + r_j x)| \leq [f_a]_{\alpha/\sigma, \alpha; Q_1} |r_j h|^\alpha,$$

$$|f_a(t_j + r_j^\sigma t, x_j + r_j(x - \delta h)) - f_a(t_j + r_j^\sigma t, x_j + r_j x)| \leq [f_a]_{\alpha/\sigma, \alpha; Q_1} |\delta r_j h|^\alpha.$$

Therefore, by the inequality

$$\sup\{f + g - h\} \geq \inf f + \inf g - \inf h,$$

we have that for $(t,x) \in \mathbb{R}^{d+1}_0$ and $h \in \operatorname{supp}\mu$ so that $\max((-t)^{1/\sigma}, |x|+|h|) \le R_j$,

$$\begin{aligned}
&\sup_{a\in\mathcal{A}_j} \hat{L}_a(V_j)(t,x)\\
&\ge \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}\Big[\inf_{a\in\mathcal{A}_j}\int_{\mathbb{R}^d}(L_a u_j)(t_j+r_j^\sigma t, x_j+r_jx+r_jh)\\
&\quad + f_a(t_j+r_j^\sigma r, x_j+r_jx+r_jh)\,d\mu(h)\\
&\quad + \inf_{a\in\mathcal{A}_j}\int_{\mathbb{R}^d}\frac{1}{\delta}\Big((L_a u_j)(t_j+r_j^\sigma t, x_j+r_j(x-\delta h))\\
&\quad + f_a(t_j+r_j^\sigma t, x_j+r_j(x-\delta h))\Big)\,d\mu(h)\\
&\quad - \frac{\delta+1}{\delta}\|\mu\|_{L_1}\inf_a[(L_a u_j)(t_j+r_j^\sigma t, x_j+r_jx)+f_a(t_j+r_j^\sigma t, x_j+r_jx)]\\
&\quad - (1+\delta^{\alpha-1})r_j^\alpha \sup_{a\in\mathcal{A}_j}[f_a]_{\alpha/\sigma,\alpha;Q_1}\int_{\mathbb{R}^d}|h|^\alpha\,d\mu(h)\Big].
\end{aligned} \tag{5.4.11}$$

Since each u_j satisfies

$$\partial_t u_j = \inf_{a\in\mathcal{A}_j}(L_a u_j + f_a) \quad \text{in } Q_1, \tag{5.4.12}$$

it follows from (5.4.10) and (5.4.11) that in any bounded subset of $\mathbb{R}^{d+1}_0$, for sufficiently large j,

$$\partial_t V_j - \sup_{a\in\mathcal{A}_j}\hat{L}_a V_j \le \frac{1+\delta^{\alpha-1}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}\sup_{a\in\mathcal{A}_j}[f_a]_{\alpha/\sigma,\alpha}\int_{\mathbb{R}^d}|h|^\alpha\,d\mu(h). \tag{5.4.13}$$

We denote

$$V(t,x) := \int_{\mathbb{R}^d}\Big[v(t,x+h)-v(t,x)-\frac{v(t,x)-v(t,x-\delta h)}{\delta}\Big]\,d\mu(h).$$

For fixed $(t,x)\in\mathbb{R}^{d+1}_0$, by (5.4.8) in Lemma 5.4.3 and using the fact that μ has compact support, we have

$$\lim_j \partial_t V_j(t,x) = \partial_t \lim_j V_j(t,x) = \partial_t V(t,x), \tag{5.4.14}$$

for $|y| \leq 1$,

$$|\delta V_j(t,x,y)| \leq C|y|^{\sigma+\alpha'}, \quad |\delta V(t,x,y)| \leq C|y|^{\sigma+\alpha'}, \tag{5.4.15}$$

and for $|y| > 1$,

$$|V_j(t,y)|, |V(t,y)| \leq C|y|^{\alpha}, \tag{5.4.16}$$

where C depends on μ. Clearly,

$$\sup_{a\in\mathcal{A}_j} \left|L_a(V_j - V)(t,x)\right| \leq C \int_{\mathbb{R}^d} \left|\delta(V_j - V)(t,x,y)\right| |y|^{-d-\sigma}\, dy.$$

It follows from Lemma 5.4.3 that $\delta(V_j - V) \to 0$ locally uniformly. Therefore, by (5.4.15), (5.4.16), and the dominated convergence theorem, we have

$$\lim_j \sup_{a\in\mathcal{A}_j} \left|(L_a(V_j - V)(t,x))\right| = 0,$$

i.e.,

$$\lim_j \sup_{a\in\mathcal{A}_k} L_a V_j(t,x) = \sup_{a\in\mathcal{A}_k} L_a V(t,x). \tag{5.4.17}$$

Since μ has compact support, by Lemma 5.4.2 and (5.4.3), we have $R_j \to \infty$ and

$$[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \geq 2^{-k_j(\alpha-\alpha')}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \to \infty.$$

For fixed $\delta \in (0,1)$, we send j to infinity to get from (5.4.13), (5.4.14), and (5.4.17) that

$$\partial_t V - \mathcal{M}^+ V \leq 0 \quad \text{in } \mathbb{R}^{d+1}_0.$$

By sending δ to 0 and using the dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^d} \big(v(t,x+h) - v(t,x) - h^T Dv(t,x)\big)\, d\mu(h)$$

is a subsolution as well. Therefore, for $\sigma > 1$, v satisfies Condition (iii).

It remains to verify that v satisfies Condition (ii). Clearly, for fixed $(t,x),(s,h)\in\mathbb{R}^{d+1}_0$, when j is sufficiently large,

$$\begin{aligned}&\partial_t\big(v_j(t+s,x+h)-v_j(t,x)\big)=r_j^{-\alpha}[u_j]^{-1}_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}\\&\quad\cdot\big(\partial_t u_j(t_j+r_j^\sigma(t+s),x_j+r_j(x+h))-\partial_t u_j(t_j+r_j^\sigma t,x_j+r_jx)\big).\end{aligned}\tag{5.4.18}$$

On the other hand,

$$\begin{aligned}&\mathcal{M}^-\big(v_j(t+s,x+h)-v_j(t,x)\big)=r_j^{-\alpha}[u_j]^{-1}_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}\\&\quad\cdot\mathcal{M}^-\big(u_j(t_j+r_j^\sigma(t+s),x_j+r_j(x+h))-u_j(t_j+r_j^\sigma t,x_j+r_jx)\big).\end{aligned}\tag{5.4.19}$$

Combining (5.4.12), (5.4.18), and (5.4.19), we obtain that for j sufficiently large,

$$\begin{aligned}&\partial_t\big(v_j(t+s,x+h)-v_j(t,x)\big)-\mathcal{M}^-\big(v_j(t+s,x+h)-v_j(t,x)\big)\\&\geq-[u_j]^{-1}_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}\sup_a[f_a]_{\alpha/\sigma,\alpha;Q_1}.\end{aligned}$$

By sending j to infinity, we get for any $(t,x)\in\mathbb{R}^{d+1}_0$,

$$\partial_t\big(v(t+s,x+h)-v(t,x)\big)-\mathcal{M}^-\big(v(t+s,x+h)-v(t,x)\big)\geq 0.$$

Similarly,

$$\partial_t\big(v(t+s,x+h)-v(t,x)\big)-\mathcal{M}^+\big(v(t+s,x+h)-v(t,x)\big)\leq 0.$$

The lemma is proved. $\square$

Now we are ready to finish

Proof of Theorem 5.4.1. By Lemma 5.4.4 and Theorem 5.3.1, v is a polynomial of order ν in x and 1 in t. Since at the origin v_j along with its first derivative in t and up to ν-th order derivatives in x are 0, by Lemma 5.4.3 the same is true for v. Therefore, $v \equiv 0$. This gives us a contradiction with (5.4.7) and Lemma 5.4.3. The proof is completed. □

5.4.2 Equations with (t, x)-dependent kernels

In this subsection, we consider the case that kernels also depend on (t, x) and Hölder continuous in (t, x), i.e., there exists $A > 0$ such that for any $a \in \mathcal{A}$, (5.1.4) is satisfied. We only prove Theorem 5.1.1 in the case when $\sigma + \alpha > 2$ and the proof of the cases $\sigma + \alpha < 2$ is similar and actually simpler. Below we divide the proof into several steps.

Let η be a nonnegative smooth cutoff function with $\eta \equiv 1$ in Q_1 and vanishes outside $(-(5/4)^\sigma, (5/4)^\sigma) \times B_{5/4}$. Set $v := \eta u$ and note that in Q_1,

$$\begin{aligned} v_t = \eta u_t + \eta_t u &= \inf_{a \in \mathcal{A}} (\eta L_a u + \eta f_a + \eta_t u) \\ &= \inf_{a \in \mathcal{A}} (L_a^0 v + (L_a - L_a^0) v + \eta L_a u - L_a v + \eta f_a + \eta_t u) \end{aligned}$$

where

$$L_a^0 v = \int_{\mathbb{R}^d} \delta v(t, x, y) K_a(0, 0, y)\, dy.$$

We further define

$$\begin{aligned} h_a &:= \eta L_a u - L_a v \\ &= \int_{\mathbb{R}^d} \big((\eta(t,x) - \eta(t,x+y))u(t,x+y) + y^T D\eta(t,x)u(t,x)\big) K_a(t,x,y)\, dy \end{aligned}$$

and

$$g_a := (L_a - L_a^0)v = \int_{\mathbb{R}^d} \delta v(t,x,y)\big(K_a(t,x,y) - K_a(0,0,y)\big)\, dy.$$

Here in order to apply the argument of freezing the coefficients, we subtracted and added $K_a(0,0,y)$ in the formula above.

Lemma 5.4.5. *Assume that* $u \in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_{11/8}) \cap C^{\alpha/\sigma,\alpha}((-(11/8)^\sigma, 0) \times \mathbb{R}^d)$. *Let* h_a *and* g_a *be functions defined above. Then for any* $\alpha \in \mathcal{A}$, *we have*

$$[g_a]_{\alpha/\sigma,\alpha;Q_1} \le CA\big([v]_{1+\alpha/\sigma,\alpha+\sigma} + [v]_{\alpha/\sigma,\alpha}\big), \tag{5.4.20}$$

$$[h_a]_{\alpha/\sigma,\alpha;Q_1} \le C(A+1)\big(\|u\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d} + \|D^2 u\|_{L_\infty(Q_{11/8})}\big). \tag{5.4.21}$$

Moreover,

$$\begin{aligned} [u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} &\le C\Big([u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{5/4}} + C_0 + A[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5/4}} \\ &+ (A+1)(\|u\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d} + \|D^2 u\|_{L_\infty(Q_{11/8})})\Big). \end{aligned} \tag{5.4.22}$$

Here the constant C *depends only on* d, λ, Λ, α, *and* σ, *and is uniformly bounded as* $\sigma \to 2$.

Proof. For $(t,x),(t',x') \in Q_1$, set $l = \max(|x-x'|, |t-t'|^{1/\sigma})$. Without loss of generality, we may assume that $l \le 1/4$.

Estimates of g_a: From the definition and the triangle inequality,

$$
\begin{aligned}
&\big|g_a(t,x) - g_a(t',x')\big| \\
&\le \Big| \int_{\mathbb{R}^d} \big(\delta v(t,x,y) - \delta v(t',x',y)\big)\big(K_a(t,x,y) - K_a(0,0,y)\big)\,dy \Big| \\
&\quad + \Big| \int_{\mathbb{R}^d} \delta v(t',x',y)\big(K_a(t,x,y) - K_a(t',x',y)\big)\,dy \Big| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

Then we estimate I and II separately. First, similar to (5.3.8), I is less than

$$
\begin{aligned}
&\int_{B_l} \Big| \big(\delta v(t,x,y) - \delta v(t',x',y)\big)\big(K_a(t,x,y) - K_a(0,0,y)\big) \Big|\,dy \\
&\quad + \int_{\mathbb{R}^d \setminus B_l} \Big| \big(\delta v(t,x,y) - \delta v(t',x',y)\big)\big(K_a(t,x,y) - K_a(0,0,y)\big) \Big|\,dy := \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}
$$

Applying (5.3.6), we have

$$
\begin{aligned}
\mathrm{I}_1 &\le C[v]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5/4}} \int_{B_l} l^{\alpha+\sigma-2}|y|^2\big(K_a(t,x,y) - K_a(0,0,y)\big)\,dy \\
&\le CA(2-\sigma)[v]_{1+\alpha/\sigma,\alpha+\sigma} l^{\alpha+\sigma-2}\big(|x|^\alpha + |t|^{\alpha/\sigma}\big) \int_{B_l} |y|^2|y|^{-d-\sigma}\,dy \\
&= CAl^\alpha[v]_{1+\alpha/\sigma,\alpha+\sigma}.
\end{aligned}
$$

For I_2, we have

$$
\begin{aligned}
\mathrm{I}_2 &\le C[v]_{1+\alpha/\sigma,\alpha+\sigma} l \int_{\mathbb{R}^d \setminus B_l} |y|^{\sigma+\alpha-1}\big|K_a(t,x,y) - K_a(0,0,y)\big|\,dy \\
&\le CA(2-\sigma)[v]_{1+\alpha/\sigma,\alpha+\sigma} l^\alpha\big(|x|^\alpha + |t|^{\alpha/\sigma}\big) \le CAl^\alpha[v]_{1+\alpha/\sigma,\alpha+\sigma}.
\end{aligned}
$$

Next, we bound

$$\begin{aligned}\mathrm{II} &\le \int_{\mathbb{R}^d\setminus B_1} \big([v]_{\alpha/\sigma,\alpha}|y|^\alpha + \|Dv\|_{L_\infty}|y|\big)\big|K_a(t,x,y)-K_a(t',x',y)\big|\,dy\\
&\quad + \int_{B_1} \|D^2v\|_{L_\infty}|y|^2\big|K_a(t,x,y)-K_a(t',x',y)\big|\,dy\\
&\le CAl^\alpha\big([v]_{\alpha/\sigma,\alpha} + \|Dv\|_\infty + \|D^2v\|_{L_\infty}\big).\end{aligned}$$

Combining the estimates of I, II, and the interpolation inequality, we get

$$|g_a(t,x)-g_a(t',x')| \le CAl^\alpha\big([v]_{1+\alpha/\sigma,\alpha+\sigma} + [v]_{\alpha/\sigma,\alpha}\big),$$

which implies (5.4.20).

Estimates of h_a: For simplicity of notation, we denote

$$\xi(t,x,y) = \big(\eta(t,x)-\eta(t,x+y)\big)u(t,x+y) + y^T D\eta(t,x)u(t,x). \tag{5.4.23}$$

By the Leibniz rule, we have

$$\begin{aligned}\xi(t,x,y) &= y^T\int_0^1 \big(D\eta(t,x)u(t,x) - D\eta(t,x+sy)u(t,x+y)\big)\,ds\\
&= -\int_0^1\int_0^1 \big(u(t,x)sy^T D^2\eta(t,x+s'sy)y\\
&\qquad + y^T D\eta(t,x+sy)y^T Du(t,x+s'y)\big)\,ds'\,ds,\end{aligned} \tag{5.4.24}$$

which implies that when $|y|\le 1/8$,

$$|\xi(t,x,y)| \le C|y|^2\big(\|u\|_{L_\infty(Q_{11/8})} + \|Du\|_{L_\infty(Q_{11/8})}\big). \tag{5.4.25}$$

On the other hand, clearly when $|y| \geq 1/8$,

$$|\xi(t,x,y)| \leq C\big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + |y|\|u\|_{L_\infty(Q_1)}\big). \tag{5.4.26}$$

Note that

$$\begin{aligned}\big|h_a(t,x) - h_a(t',x')\big| &\leq \int_{\mathbb{R}^d} \big|\xi(t,x,y) - \xi(t',x',y)\big| K_a(t,x,y)\,dy \\ &+ \int_{\mathbb{R}^d} |\xi(t',x',y)|\big|K_a(t,x,y) - K_a(t',x',y)\big|\,dy =: \mathrm{III} + \mathrm{IV}.\end{aligned} \tag{5.4.27}$$

Estimate of III: By (5.4.24) when $|y| \leq 1/8$, we have

$$\begin{aligned}&\big|\xi(t,x,y) - \xi(t',x',y)\big| \\ &= \Big|\int_0^1\int_0^1 s\big(u(t',x')y^T D^2\eta(t',x'+ss'y)y - u(t,x)y^T D^2\eta(t,x+ss'y)y\big)\,dx\,ds' \\ &\quad + \int_0^1\int_0^1 \Big[y^T D\eta(t',x'+sy)y^T Du(t',x'+s'y) \\ &\qquad\qquad - y^T D\eta(t,x+sy)y^T Du(t,x+s'y)\Big]\,ds\,ds'\Big| \\ &\leq C|y|^2 l^\alpha\big(\|u\|_{L_\infty(Q_{11/8})} + \|D^2u\|_{L_\infty(Q_{11/8})}\big),\end{aligned}$$

where we used the interpolation inequalities in the last inequality. On the other hand, when $|y| > 1/8$,

$$\big|y^T D\eta(t,x)u(t,x) - y^T D\eta(t',x')u(t',x')\big| \leq |y| l^\alpha \|u\|_{\alpha/\sigma,\alpha;Q_1}$$

and

$$
\begin{aligned}
&\big|(\eta(t,x)-\eta(t,x+y))u(t,x+y)-(\eta(t',x')-\eta(t',x'+y))u(t',x'+y)\big| \\
&= \big|u(t,x+y)\big(\eta(t,x)-\eta(t',x')-\eta(t,x+y)+\eta(t',x'+y)\big) \\
&\quad + \big(\eta(t',x')-\eta(t',x'+y)\big)\big(u(t,x+y)-u(t',x'+y)\big)\big| \\
&\le C(l\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}+l^\alpha\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}),
\end{aligned}
$$

which imply that when $|y|>1/8$,

$$
\begin{aligned}
&\big|\xi(t,x,y)-\xi(t',x',y)\big| \\
&\le C\big(l\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}+l^\alpha\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}\big)+C|y|l^\alpha\|u\|_{\alpha/\sigma,\alpha;Q_1}.
\end{aligned}
$$

Now with the above estimates, we obtain

$$
\begin{aligned}
\mathrm{III} \le &\int_{B_{1/8}} C|y|^2 l^\alpha\big(\|u\|_{L_\infty(Q_{11/8})}+\|D^2u\|_{L_\infty(Q_{11/8})}\big)K_a(t,x,y)\,dy \\
&+ Cl^\alpha\|u\|_{\alpha/\sigma,\alpha;Q_1}\int_{B^c_{1/8}}|y|K_a(t,x,y)\,dy \\
&+ C\big(l\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}+l^\alpha\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}\big)\int_{B^c_{1/8}}K_a(t,x,y)\,dy \\
\le &\, Cl^\alpha\big(\|D^2u\|_{L_\infty(Q_{11/8})}+\|u\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d}\big).
\end{aligned}
$$

<u>Estimate of IV:</u> By (5.4.25) and (5.4.26), we have

$$
\mathrm{IV} \le Cl^\alpha A\big(\|u\|_{L_\infty((-(11/8)^\sigma,0)\times\mathbb{R}^d)}+\|Du\|_{L_\infty(Q_{11/8})}\big).
$$

The estimates of III and IV with the interpolation inequalities give (5.4.21).

Now we apply Theorem 5.4.1 to v with the estimates of g_a and h_a in Lemma 5.4.5 to obtain

$$[v]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C\Big([v]_{1+\alpha'/\sigma,\alpha'+\sigma} + A[v]_{1+\alpha/\sigma,\alpha+\sigma} + A[v]_{\alpha/\sigma,\alpha} + (A+1)(\|u\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d} + \|D^2u\|_{L_\infty(Q_{11/8})}) + \sup_a[\eta f_a]_{\alpha/\sigma,\alpha;Q_1}\Big).$$

Since $\eta \equiv 1$ in Q_1 and has compact support in $(-(5/4)^\sigma,(5/4)^\sigma)\times B_{5/4}$, we get (5.4.21). The lemma is proved. □

Proof of Theorem 6.1.1. We first use a scaling argument. For any $\varepsilon > 0$, set $\hat{u}(t,x) := \varepsilon^{-\sigma}u(\varepsilon^\sigma t,\varepsilon x)$. Since u satisfies (5.1.5), we have

$$\hat{u}_t(t,x) = \inf_a \Big\{ \int_{\mathbb{R}^d} \delta\hat{u}(t,x,y)K_a^\varepsilon(t,x,y)\,dy + f_a(\varepsilon^\sigma t,\varepsilon x)\Big\} \quad \text{in } Q_{1/\varepsilon},$$

where

$$K_a^\varepsilon(t,x,y) = \varepsilon^{d+\sigma}K_a(\varepsilon^\sigma t,\varepsilon x,\varepsilon y).$$

Clearly,

$$\big|K_a^\varepsilon(t,x,y) - K_a^\varepsilon(t',x',y)\big| \le A(2-\sigma)\varepsilon^\alpha\big(|x-x'|^\alpha + |t-t'|^{\alpha/\sigma}\big)\frac{\Lambda}{|y|^{d+\sigma}}.$$

Then we apply (5.4.22) to $\hat{u}$ and get

$$[\hat{u}]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C\Big([\hat{u}]_{1+\alpha'/\sigma,\alpha+\sigma;Q_{5/4}} + C_0\varepsilon^\alpha + A\varepsilon^\alpha[\hat{u}]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5/4}} + (A\varepsilon^\alpha+1)\big(\|\hat{u}\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d} + \|D^2\hat{u}\|_{L_\infty(Q_{11/8})}\big)\Big).$$

Returning back to u, we have

$$[u]_{1+\alpha/\sigma,\sigma+\alpha;Q_{\varepsilon/2}} \leq C\Big(\varepsilon^{\alpha'-\alpha}[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{5\varepsilon/4}} + C_0 + A\varepsilon^{\alpha}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5\varepsilon/4}}$$
$$+ (A\varepsilon^{\alpha}+1)\big(\varepsilon^{-\sigma-\alpha}\|u\|_{\alpha/\sigma,\alpha;(-(11\varepsilon/8)^{\sigma},0)\times\mathbb{R}^d} + \varepsilon^{2-\sigma-\alpha}\|D^2u\|_{L_\infty(Q_{11\varepsilon/8})}\big)\Big).$$

By a translation of the coordinates, the inequality above holds for any $(t,x) \in Q_1$ for sufficiently small $\varepsilon > 0$

$$\begin{aligned}
&[u]_{1+\alpha/\sigma,\sigma+\alpha;Q_{\varepsilon/2}(t,x)} \\
&\leq C\Big(\varepsilon^{\alpha'-\alpha}[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{5\varepsilon/4}(t,x)} + C_0 + A\varepsilon^{\alpha}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5\varepsilon/4}(t,x)} \\
&\quad + (A\varepsilon^{\alpha}+1)\big(\varepsilon^{-\sigma-\alpha}\|u\|_{\alpha/\sigma,\alpha;(t-(11\varepsilon/8)^{\sigma},t)\times\mathbb{R}^d} + \varepsilon^{2-\sigma-\alpha}\|D^2u\|_{L_\infty(Q_{11\varepsilon/8}(t,x))}\big)\Big).
\end{aligned} \tag{5.4.28}$$

Let Q^k be defined in (5.4.1). It is obvious that Q^k monotonically increases to Q_1. Then for any $(t,x),(s,y) \in Q^k$ such that $t \geq s$, we set $l := \max(|t-s|^{1/\sigma}, |x-y|)$. When $l \geq \varepsilon/2$,

$$\begin{aligned}
&\frac{|D^2u(t,x) - D^2u(s,y)|}{l^{\sigma+\alpha-2}} + \frac{|u_t(t,x) - u_t(s,y)|}{l^{\sigma+\alpha-2}} \\
&\leq 2^{\sigma+\alpha-1}\varepsilon^{2-\sigma-\alpha}\big(\|u_t\|_{L_\infty(Q^k)} + \|D^2u\|_{L_\infty(Q^k)}\big);
\end{aligned}$$

when $l < \varepsilon/2$,

$$\frac{|D^2u(t,x) - D^2u(s,y)|}{l^{\sigma+\alpha-2}} + \frac{|u_t(t,x) - u_t(s,y)|}{l^{\sigma+\alpha-2}} \leq 2[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{\varepsilon/2}(t,x)}.$$

Now we choose $\varepsilon = 2^{-k-2}$ so that for any $(t,x) \in Q^k$, $Q_{11\varepsilon/8}(t,x) \subset Q^{k+1}$ and $(t-(11\varepsilon/8)^{\sigma}, t) \subset (-1,0)$. Combining the two inequalities above with (5.4.28), we obtain

$$
\begin{aligned}
[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k} &\le 2^{(k+3)(\sigma+\alpha-2)+1}(\|u_t\|_{L_\infty(Q^k)} + \|D^2u\|_{L_\infty(Q^k)})\\
&+ C\Big(2^{(k+2)(\alpha-\alpha')}[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q^{k+1}} + C_0 + 2^{-(k+2)\alpha}A[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}}\\
&+ (2^{-(k+2)\alpha}A+1)\big(2^{(k+2)(\sigma+\alpha)}\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}\\
&+ 2^{(k+2)(\sigma+\alpha-2)}\|D^2u\|_{L_\infty(Q^{k+1})}\big)\Big).
\end{aligned} \tag{5.4.29}
$$

By the interpolation inequalities

$$
\begin{aligned}
&[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q^{k+1}} \le 2^{-2(k+2)(\alpha-\alpha')}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}} + C2^{2(k+2)(\sigma+\alpha')}\|u\|_{L_\infty},\\
&\|D^2u\|_{L_\infty(Q^{k+1})} + \|u_t\|_{L_\infty(Q^{k+1})}\\
&\le 2^{-2(k+1)(\sigma+\alpha-2)}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}} + C2^{4(k+1)}\|u\|_{L_\infty(Q^{k+1})},
\end{aligned}
$$

we reorganize the right-hand side of (5.4.29) to get

$$
\begin{aligned}
[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k} &\le C\Big(\big(2^{-(k+1)(\sigma+\alpha-2)} + 2^{-(k+2)(\alpha-\alpha')}\big)[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}}\\
&+ 2^{5k(\sigma+\alpha)}\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + C_0\Big),
\end{aligned}
$$

where C depends on A. Obviously, $\sigma+\alpha<3$ and there exists a constant k_0 depends on d, σ_0, α, λ, Λ, and A such that $Q_{1/2}\subset Q^{k_0}$ and for any $k\ge k_0$,

$$
C\big(2^{-(k+1)(\sigma+\alpha-2)} + 2^{-2(k+1)(\alpha-\alpha')}\big) < 2^{-16}.
$$

Therefore, we have for any $k\ge k_0$,

$$
[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k} \le 2^{-16}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}} + C2^{15k}\|u\|_{\alpha/\sigma,\alpha} + CC_0.
$$

We multiply both sides above by $2^{-16(k-k_0)}$ and then sum from $k=k_0$ to infinity and obtain that

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_0}} \le C2^{15k_0}\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + CC_0.$$

In particular,

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C(\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + C_0).$$

The proof is completed. □

5.4.3 An improved estimate

By a more careful analysis, we obtain the following corollary when the kernels depend only on y.

Corollary 5.4.6. *Let $\sigma \in (0,2)$ and $0 < \lambda \le \Lambda$. Assume that for any $a \in \mathcal{A}$, K_a only depends on y. There is a constant $\hat{\alpha} \in (0,1)$ depending on d, σ, λ, and Λ (uniformly as $\sigma \to 2$) so that the following holds. Let $\alpha \in (0,\hat{\alpha})$. Suppose $u \in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_1) \cap C^{\alpha/\sigma,\alpha}_{loc}([-1,0]\times\mathbb{R}^d)$ is a solution of*

$$u_t = \inf_{a\in\mathcal{A}}(L_a u + f_a) \quad \text{in } Q_1.$$

Then,

$$\begin{aligned}&[u]_{1+\alpha/\sigma,\alpha+\sigma,Q_{1/2}}\\ &\le C[u]_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + C\sum_{j=2}^{\infty} 2^{-j\sigma}[u]_{\alpha/\sigma,\alpha;(-1,0)\times(B_{2^j}\setminus B_{2^{j-1}})} + CC_0,\end{aligned} \tag{5.4.30}$$

where $C_0 = \sup_a [f_a]_{\alpha/\sigma,\alpha;Q_1}$ and $C > 0$ depends only on d, λ, Λ, α, and σ, and is uniformly bounded as $\sigma \to 2$.

Proof. Since the proof is quite similar to the proof of Theorem 5.1.1, we only provide a sketch here. By a standard scaling and covering argument, we may assume that $u \in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_2) \cap C^{\alpha/\sigma,\alpha}_{\mathrm{loc}}([-2,0] \times \mathbb{R}^d)$ and the equation is satisfied in Q_2. Let η be a cutoff function such that $\eta \in C_0^\infty((-2^\sigma, 2^\sigma) \times B_2)$ and $\eta \equiv 1$ in $Q_{5/4}$. Let $v = \eta u$, which satisfies

$$v_t = \inf_a (L_a v + h_a + \eta f_a + \eta_t u),$$

where

$$h_a = \int_{\mathbb{R}^d} \xi(t,x,y) K_a(y)\, dy$$

and ξ is defined in (5.4.23). It is sufficient to estimate $[h_a]_{\alpha/\sigma,\alpha;Q_1}$. Since K_a only depends on y, it follows that

$$|h_a(t,x) - h_a(t',x')| = \mathrm{III},$$

where III is defined in (5.4.27). The estimate is similar to the one in the proof of Lemma 5.4.5. For any $(t,x),(t',x') \in Q_1$, since $\eta \equiv 1$ in $Q_{5/4}$, $D\eta(t,x) = D\eta(t',x') = 0$. When $|y| \le 1/4$, $\xi(t,x,y) = 0$; When $|y| > 1/4$, we have

$$\begin{aligned}
&\big|\xi(t,x,y) - \xi(t',x',y)\big| \\
&\le \big|\eta(t,x+y)u(t,x+y) - \eta(t',x'+y)u(t',x'+y)\big| + \big|u(t,x+y) - u(t',x'+y)\big|.
\end{aligned}$$

Recall that $l = \max\{|x-x'|, |t-t'|^{1/\sigma}\}$. Combining the estimate above, we obtain

$$\begin{aligned}
\mathrm{III} &= \int_{B^c_{1/4}} \big|\xi(t,x,y) - \xi(t',x',y)\big| K_a(y)\,dy \\
&\le Cl^\alpha \|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + \sum_{j=-1}^{\infty} \int_{B_{2^j}\setminus B_{2^{j-1}}} |u(t,x+y) - u(t',x'+y)| K_a(y)\,dy \\
&\le Cl^\alpha \Big(\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + \sum_{j=1}^{\infty} 2^{-j\sigma} [u]_{\alpha/\sigma,\alpha;(-1,0)\times(B_{2^j}\setminus B_{2^{j-1}})} \Big),
\end{aligned}$$

which implies that

$$[h_a]_{\alpha/\sigma,\alpha;Q_1} \le C\Big(\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + \sum_{j=1}^{\infty} 2^{-j\sigma} [u]_{\alpha/\sigma,\alpha;(-1,0)\times(B_{2^j}\setminus B_{2^{j-1}})} \Big).$$

Then we apply Theorem 5.1.1 to v and obtain

$$\begin{aligned}
[v]_{1+\alpha/\sigma,\sigma+\alpha;Q_{1/2}} \le C\Big(&\|v\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + \|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} \\
&+ \sum_{j=1}^{\infty} 2^{-j\sigma} [u]_{\alpha/\sigma,\alpha;(-1,0)\times(B_{2^j}\setminus B_{2^{j-1}})} + C_0 \Big).
\end{aligned}$$

Combining the fact that $\eta \equiv 1$ in $Q_{5/4}$ and replacing u by $u - u(0,0)$, we reach (5.4.30). Therefore, the proof is completed. $\square$

5.5 Equations with bounded inhomogeneous terms

In this section, we present an application of Corollary 5.4.6 to nonlocal parabolic equations with merely bounded nonhomogeneous terms:

$$u_t = \inf_{a\in\mathcal{A}} (L_a u + f_a), \tag{5.5.1}$$

where $\sup_a \|f_a\|_{L_\infty} < \infty$ and

$$L_a u(x) = \int_{\mathbb{R}^d} \delta u(t,x,y) K_a(y)\, dy.$$

Before proving Theorem 5.1.2, we first present an interpolation inequality involving the Zygmund semi-norm. The proof can be found in the appendix of [31].

Lemma 5.5.1. *Let $\alpha \in (0,1)$ and $f \in \Lambda^1((-1,0)) \cap L_\infty((-1,0))$. Then we have $f \in C^\alpha((-1,0))$ and*

$$[f]_{\alpha;(-1,0)} \le C\|f\|_{L_\infty((-1,0))} + C[f]_{\Lambda^1((-1,0))}, \tag{5.5.2}$$

where C depends only on α.

In the sequel, we set

$$[u]^t_{\Lambda^1} := [u]^t_{\Lambda^1(\mathbb{R}^{d+1}_0)}, \quad [u]^*_\sigma := [u]^*_{\sigma;\mathbb{R}^{d+1}_0}, \quad \text{and} \quad [Du]^t_{\frac{\sigma-1}{\sigma}} =: [Du]^t_{\frac{\sigma-1}{\sigma};\mathbb{R}^{d+1}_0}.$$

Let η be a smooth even nonnegative function in $\mathbb{R}$ with unit integral and vanishing outside $(-1,1)$. For $R > 0$, we define the mollification of u with respect to t as

$$u^{(R)}(t,x) = \int_{\mathbb{R}} \big(2u(t - R^\sigma s, x) - u(t - 2R^\sigma s, x)\big)\eta(s-1)\, ds.$$

The following lemmas will also be used in our proof. We present their proofs in the appendix of [31].

Lemma 5.5.2. *Let $\beta \in (0,1]$ and $R > 0$. Then we have*

$$[\partial_t u^{(R)}]^t_{\beta;\mathbb{R}^{d+1}_0} \le C(\beta) R^{-\beta\sigma}[u]^t_{\Lambda^1}. \tag{5.5.3}$$

Lemma 5.5.3. *Let $\sigma \in (1,2)$, $\alpha \in (0,1)$, and $R > 0$ be constants. Assume that u defined on $\mathbb{R}^{d+1}_0$ is C^σ in x, Λ^1 in t, and Du is $C^{(\sigma-1)/\sigma}$ in t. Let $p = p(t,x)$ be the first-order Taylor expansion of $u^{(R)}$ at the origin. Then for any integer $j \geq 0$, we have*

$$
\begin{aligned}
&[u-p]^*_{\alpha;(-R^\sigma,0)\times B_{2^jR}} \\
&\leq C(2^jR)^{\sigma-\alpha}[u]^*_\sigma + C2^{(1-\alpha)j}R^{\sigma-\alpha}[Du]^t_{\frac{\sigma-1}{\sigma}} + C2^{-j\alpha}R^{\sigma-\alpha}[u]^t_{\Lambda^1}
\end{aligned}
\tag{5.5.4}
$$

and

$$
\begin{aligned}
[u-p]^t_{\alpha/\sigma;(-R^\sigma,0)\times B_{2^jR}} \leq{}& C2^{j(\sigma-\alpha/2)}R^{\sigma-\alpha}[u]^*_\sigma + C2^{j(\sigma+1-\alpha)/2}R^{\sigma-\alpha}[Du]^t_{\frac{\sigma-1}{\sigma}} \\
&+ C2^{j(\sigma-\alpha/2)}R^{\sigma-\alpha}[u]^t_{\Lambda^1},
\end{aligned}
\tag{5.5.5}
$$

where $C > 0$ is a constant depending only on d, σ, and α.

In the case when $\sigma = 1$, we define $u^{(R)}$ differently. Let $\zeta \in C_0^\infty(B_1)$ be a radial nonnegative function with unit integral. For $R > 0$, we define

$$
u^{(R)}(t,x) = \int_{\mathbb{R}^{d+1}} \big(2u(t-Rs, x-Ry) - u(t-2Rs, x-Ry)\big)\eta(s-1)\zeta(y)\,dy\,ds.
$$

Lemma 5.5.4. *Let $\alpha \in (0,1)$, and $R > 0$ be constants. Assume that u defined on $\mathbb{R}^{d+1}_0$ is Λ^1 in (t,x). Let $p = p(t,x)$ be the first-order Taylor expansion of $u^{(R)}$ at the origin. Then for any integer $j \geq 0$, we have*

$$
[u-p]_{\alpha,\alpha;(-R,0)\times B_{2^jR}} \leq C2^{j(1-\alpha/2)}R^{1-\alpha}[u]_{\Lambda^1}, \tag{5.5.6}
$$

where $C > 0$ is a constant depending only on d and α.

Define $\mathcal{P}_0$ to be the set of first-order polynomials of t, and $\mathcal{P}_1$ to be the set of first-order polynomials of (t,x).

Lemma 5.5.5. *(i) When $\sigma \in (0,1)$, we have*

$$[u]^t_{\Lambda^1} + [u]^*_\sigma \le C \sup_{r>0} \sup_{(t,x)\in\mathbb{R}^{d+1}_0} r^{-\sigma} \inf_{p\in\mathcal{P}_0} \|u-p\|_{L_\infty(Q_r(t,x))},$$

where $C>0$ is a constant depending only on d and σ.

(ii) When $\sigma \in (1,2)$, we have

$$[u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}} \le C \sup_{r>0} \sup_{(t,x)\in\mathbb{R}^{d+1}_0} r^{-\sigma} \inf_{p\in\mathcal{P}_1} \|u-p\|_{L_\infty(Q_r(t,x))},$$

where $C>0$ is a constant depending only on d and σ.

(iii) We have

$$[u]_{\Lambda^1} \le C \sup_{r>0} \sup_{(t,x)\in\mathbb{R}^{d+1}_0} r^{-1} \inf_{p\in\mathcal{P}_1} \|u-p\|_{L_\infty(Q_r(t,x))},$$

where $C>0$ is a constant depending only on d.

Proof. The estimates of $[u]^*_\sigma$ and $[Du]^t_{\frac{\sigma-1}{\sigma}}$ are standard. See, for instance, [63, Section 3.3]. We only consider $[u]^t_{\Lambda^1}$. For any polynomial p which is linear in t, by the triangle inequality,

$$\begin{aligned}
&\big|u(t+s,x)+u(t-s,x)-2u(t,x)\big| \\
&= \big|u(t+s,x)-p(t+s,x)+u(t-s,x)-p(t-s,x)-2(u(t,x)-p(t,x))\big| \\
&\le 4\|u-p\|_{L_\infty(Q_r(t+s,x))},
\end{aligned}$$

where $r^\sigma = 2s$. Since p is arbitrary, the inequality above implies that

$$\big|u(t+s,x)+u(t-s,x)-2u(t,x)\big| \le 8sr^{-\sigma}\inf_p \|u-p\|_{L_\infty(Q_r(t+s,x))}.$$

Similarly, we can prove Assertion (iii). The lemma is proved. □

Proof of Theorem 5.1.2. We only treat the case when $\sigma \ge 1$. For the case when $\sigma < 1$, the proof is almost the same with minor modifications.

First we assume $\sigma > 1$. Let $\hat\alpha$ be the constant in Corollary 5.4.6 and $\alpha \in (0,\hat\alpha)$ be such that $\alpha+\sigma \le 2$. Let $R > 0$ be a constant, p be defined as in Lemma 5.5.3, and $c_0 = \partial_t p$. Let $K \ge 2\|u-p\|_{L_\infty(Q_{2R})}$ be a constant to be specified later and denote

$$g_K = \max\big(\min(u-p,K),-K\big).$$

Clearly, $g_K \in C^{\alpha/\sigma,\alpha}$ and any $C^{\alpha/\sigma,\alpha}$ norm (or semi-norm) of g_K is less than or equal to that of $u-p$.

Let v_K be the solution to

$$\begin{cases} \partial_t v_K = \inf_a (L_a v_K - c_0) & \text{in } Q_{2R}, \\ v_K = g_K & \text{in } \mathbb{R}^{d+1}_0 \setminus Q_{2R}. \end{cases} \tag{5.5.7}$$

Noting that g_k is Hölder continuous in both t and x, the solvability follows from Theorem 5.1.1 and a regularization argument; see [16, 98]. We apply Corollary 5.4.6

to v_K with a scaling to get

$$
\begin{aligned}
&[v_K]_{1+\alpha/\sigma,\alpha+\sigma;Q_{R/2}}\\
&\le C\Big(R^{-\sigma}[v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2R}}+\sum_{j=2}^{\infty}2^{-j\sigma}R^{-\sigma}[u-p]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times(B_{2^jR}\setminus B_{2^{j-1}R})}\Big),
\end{aligned}
\tag{5.5.8}
$$

where in the last equality we used the fact that $v_K=g_K$ in $(-(2R)^\sigma,0)\times B_{2R}^c$ and the Hölder norm of g_K is less than the Hölder norm of $u-p$. By Lemma 5.5.3,

$$
\begin{aligned}
[u-p]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2^jR}} &\le C2^{j(\sigma-\alpha/2)}R^{\sigma-\alpha}[u]^t_{\Lambda^1}\\
&\quad+C2^{j(\sigma-\alpha/2)}R^{\sigma-\alpha}[u]^*_\sigma+C2^{j(\sigma+1-\alpha)/2}R^{\sigma-\alpha}[Du]^t_{\frac{\sigma-1}{\sigma}},
\end{aligned}
\tag{5.5.9}
$$

which together with (5.5.8) gives

$$
\begin{aligned}
&[v_K]_{1+\alpha/\sigma,\alpha+\sigma;Q_{R/2}}\\
&\le C\big(R^{-\sigma}[v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2R}}+R^{-\alpha}[u]^t_{\Lambda^1}+R^{-\alpha}[u]^*_\sigma+R^{-\alpha}[Du]^t_{\frac{\sigma-1}{\sigma}}\big).
\end{aligned}
\tag{5.5.10}
$$

Next we estimate $w_K:=g_K-v_K$, which is equal to $u-p-v_K$ in Q_{2R} by the choice of K. By (5.5.1) and (5.5.7), w_K satisfies

$$
\begin{cases}
\partial_t w_K\le\mathcal{M}^+w_K+h_K+\sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty} & \text{in } Q_{2R},\\
\partial_t w_K\ge\mathcal{M}^-w_K+\hat{h}_K-\sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty} & \text{in } Q_{2R},\\
w_K=0 & \text{in } \mathbb{R}^{d+1}_0\setminus Q_{2R},
\end{cases}
$$

where

$$
h_K:=\mathcal{M}^+(u-p-g_K),\quad \hat{h}_K:=\mathcal{M}^-(u-p-g_K).
$$

By the dominated convergence theorem, it is not hard to see that

$$\|h_K\|_{L_\infty(Q_{2R})},\ \|\hat{h}_K\|_{L_\infty(Q_{2R})} \to 0 \quad \text{as} \quad K \to \infty.$$

We then fix K large enough so that

$$\|h_K\|_{L_\infty(Q_{2R})} + \|\hat{h}_K\|_{L_\infty(Q_{2R})} \le \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty}.$$

From Lemma 5.2.5, we have

$$\|w_K\|_{L_\infty(Q_{2R})} \le CR^\sigma \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty}, \quad [w_K]_{\alpha/\sigma,\alpha;Q_{2R}} \le CR^{\sigma-\alpha} \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty}, \tag{5.5.11}$$

where C depends on d, σ, λ, and Λ.

Now let q_K be the first-order Taylor expansion of v_K at the origin. Then by (5.5.10), for any $r \in (0, R/2)$,

$$\begin{aligned}
&\|u - p - q_K\|_{L_\infty(Q_r)} \le \|u - p - v_K\|_{L_\infty(Q_r)} + \|v_K - q_K\|_{L_\infty(Q_r)} \\
&\le \|u - p - v_K\|_{L_\infty(Q_r)} + Cr^{\sigma+\alpha} R^{-\sigma} [v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2R}} \\
&\quad + Cr^{\sigma+\alpha} R^{-\alpha} \big([u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}}\big).
\end{aligned} \tag{5.5.12}$$

Since $w_K = u - p - v_K$ in Q_{2R}, we plug (5.5.9) with $j = 0$, and (5.5.11) to (5.5.12) and obtain

$$\|u - p - q_K\|_{L_\infty(Q_r)} \le CR^\sigma \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty} + Cr^{\sigma+\alpha} R^{-\alpha} \big([u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}}\big).$$

Dividing both sides of the inequality above by r^σ, we have

$$\begin{aligned}
&r^{-\sigma}\|u-p-q_K\|_{L_\infty(Q_r)}\\
&\leq C(R/r)^\sigma \sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty} + C(r/R)^\alpha\big([u]^t_{\Lambda^1}+[u]^*_\sigma+[Du]^t_{\frac{\sigma-1}{\sigma}}\big).
\end{aligned}$$

Set $r = R/M$, where $M \geq 2$ is a constant to be determined. Note that the center of the cylinder can be replaced by any point (t,x) in $\mathbb{R}^{d+1}_0$, i.e.,

$$\begin{aligned}
&r^{-\sigma}\|u-p-q_K\|_{L_\infty(Q_r(t,x))}\\
&\leq CM^\sigma \sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty} + CM^{-\alpha}\big([u]^t_{\Lambda^1}+[u]^*_\sigma+[Du]^t_{\frac{\sigma-1}{\sigma}}\big),
\end{aligned} \tag{5.5.13}$$

which together with Lemma 5.5.5 implies

$$\begin{aligned}
&[u]^t_{\Lambda^1}+[u]^*_\sigma+[Du]^t_{\frac{\sigma-1}{\sigma}}\\
&\leq C\sup_{r>0}\sup_{(t,x)\in\mathbb{R}^{d+1}_0} r^{-\sigma}\inf_{p\in\mathcal{P}_1}\|u-p\|_{L_\infty(Q_r(t,x))}\\
&\leq CM^\sigma \sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty} + CM^{-\alpha}\big([u]^t_{\Lambda^1}+[u]^*_\sigma+[Du]^t_{\frac{\sigma-1}{\sigma}}\big).
\end{aligned} \tag{5.5.14}$$

By taking M sufficiently large in (5.5.14) so that $CM^{-\alpha} < 1/2$, we obtain

$$[u]^t_{\Lambda^1}+[u]^*_\sigma+[Du]^t_{\frac{\sigma-1}{\sigma}} \leq C\sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty}.$$

In the case when $\sigma = 1$, by Lemma 5.5.4 and (5.5.8),

$$[v_K]_{1+\alpha,1+\alpha;Q_{R/2}} \leq C\big(R^{-1}[v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2R}} + R^{-\alpha}[u]_{\Lambda^1}\big).$$

Then by the same proof, similar to (5.5.13), we get

$$r^{-1}\|u-p-q_K\|_{L_\infty(Q_r(t,x))} \le CM\sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty} + CM^{-\alpha}[u]_{\Lambda^1},$$

which together with Lemma 5.5.5 gives

$$[u]_{\Lambda^1} \le C\sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty}$$

by taking M sufficiently large. The theorem is proved. $\square$

Chapter 6

Dini estimates for nonlocal fully nonlinear parabolic equations

6.1 Introduction and main results

This chapter is a continuation of the work in the previous chapter. We consider concave fully nonlinear nonlocal parabolic equations with Dini continuous coefficients and nonhomogeneous terms, and establish a $C^{1,\sigma}$ estimate under these assumptions.

The study of classical elliptic/parabolic equations with Dini continuous coefficients and data has a long history. Burch [14] first considered divergence type linear elliptic equations with Dini continuous coefficients and data, and estimated the modulus of continuity of the derivatives of solutions. The corresponding result for concave fully nonlinear elliptic equations was obtained by Kovats [58], which generalized a previous result by Safonov [94]. Wang [105] studied linear non-divergence type elliptic and parabolic equations with Dini continuous coefficients and data, and gave a simple proof to estimate the modulus of continuity of the second-order derivatives of solutions. See, also [79, 102, 5, 45, 87, 75], and the references therein.

Recently, there is extensive work on the regularity theory for nonlocal elliptic and parabolic equations. For example, C^α estimates, $C^{1+\alpha}$ estimates, Evans-Krylov type theorem, and Schauder estimates were established in the past decade. In particular, Mou [90] investigated a class of concave fully nonlinear nonlocal elliptic equations with smooth symmetric kernels, and obtained the C^σ estimate under a slightly stronger assumption than the usual Dini continuity on the coefficients and data. The author implemented a recursive Evans-Krylov theorem, which was first studied by Jin and Xiong [50], as well as a perturbation type argument. In this chapter, by using a novel perturbation type argument, we relax the regularity assumption to simply Dini continuity and also remove the symmetry and smoothness assumptions on the kernels.

To be more specific, we are interested in fully nonlinear nonlocal parabolic equations in the form

$$u_t = \inf_{\beta \in \mathcal{A}} (L_\beta u + b_\beta Du + f_\beta), \tag{6.1.1}$$

where $\mathcal{A}$ is an index set and for each $\beta \in \mathcal{A}$,

$$L_\beta u = \int_{\mathbb{R}^d} \delta u(t,x,y) K_\beta(t,x,y)\, dy,$$

$\delta u(t,x,y)$ is defined (5.2.2) and

$$K_\beta(t,x,y) = (2-\sigma) a_\beta(t,x,y) |y|^{-d-\sigma}.$$

We assume that $a(\cdot,\cdot,\cdot) \in [\lambda, \Lambda]$ for some ellipticity constants $0 < \lambda \leq \Lambda$, and is merely measurable with respect to the y variable. When $\sigma = 1$, we additionally assume that

$$\int_{S_r} yK_\beta(t,x,y)\,ds_y = 0, \tag{6.1.2}$$

for any $r > 0$, where S_r is the sphere of radius r centered at the origin. We say that a function f is Dini continuous if its modulus of continuity ω_f is a Dini function, i.e.,

$$\int_0^1 \omega_f(r)/r\,dr < \infty.$$

The following theorem is our main result.

Theorem 6.1.1. *Let $\sigma \in (0,2)$, $0 < \lambda \leq \Lambda < \infty$, and $\mathcal{A}$ be an index set. Assume for each $\beta \in \mathcal{A}$, K_β satisfies (6.1.2) when $\sigma = 1$, and in $(-1,0)\times\mathbb{R}^d$*

$$\begin{aligned}
&\left|a_\beta(t,x,y) - a_\beta(t',x',y)\right| \leq \Lambda\omega_a(\max\{|x-x'|,|t-t'|^{1/\sigma}\}),\\
&|f_\beta(t,x) - f_\beta(t',x')| \leq \omega_f(\max\{|x-x'|,|t-t'|^{1/\sigma}\}), \quad \sup_{\beta\in\mathcal{A}} \|f_\beta\|_{L_\infty(Q_1)} < \infty,\\
&\sup_\beta \|b_\beta\|_{L_\infty(Q_1)} \leq N_0, \quad |b_\beta(t,x) - b_\beta(t',x')| \leq \omega_b(\max\{|x-x'|,|t-t'|^{1/\sigma}\}),
\end{aligned}$$

where ω_a, ω_b, and ω_f are Dini functions. Suppose $u \in C^{1,\sigma^+}(Q_1)$ is a solution of (6.1.1) in Q_1 and is Dini continuous in $(-1,0)\times\mathbb{R}^d$. Then we have that $\partial_t u$ is uniform continuous and the a priori estimate: for $\sigma \in (0,1)$

$$\begin{aligned}
&\|\partial_t u\|_{L_\infty(Q_{1/2})} + [u]^x_{\sigma;Q_{1/2}}\\
&\leq C\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + C\sup_\beta \|f_\beta\|_{L_\infty(Q_1)} + C\sum_{j=1}^\infty \left(\omega_u(2^{-j}) + \omega_f(2^{-j})\right);
\end{aligned} \tag{6.1.3}$$

for $\sigma = 1$

$$\|\partial_t u\|_{L_\infty(Q_{1/2})} + \|Du\|_{L_\infty(Q_{1/2})}$$
$$\leq C\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d))} + C\sup_\beta \|f_\beta\|_{L_\infty(Q_1)} + C\sum_{j=1}^{\infty}\big(\omega_u(2^{-j}) + \omega_f(2^{-j})\big); \tag{6.1.4}$$

for $\sigma \in (1,2)$

$$\|\partial_t u\|_{L_\infty(Q_{1/2})} + [Du]^t_{\frac{\sigma-1}{\sigma};Q_{1/2}} + [u]^x_{\sigma;Q_{1/2}}$$
$$\leq C\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d))} + C\sup_\beta \|f_\beta\|_{L_\infty(Q_1)} + C\sum_{j=1}^{\infty}\big(\omega_u(2^{-j}) + \omega_f(2^{-j})\big), \tag{6.1.5}$$

where $C > 0$ *is a constant depending only on* d, σ, λ, Λ, N_0, ω_b, *and* ω_a. *Moreover, when* $\sigma \neq 1$, *we have*

$$\sup_{(t_0,x_0)\in Q_{1/2}} [u]^x_{\sigma;Q_r(t_0,x_0)} \to 0 \quad as \quad r \to 0$$

with a decay rate depending only on d, σ, λ, Λ, ω_a, ω_f, ω_u, N_0, ω_b, *and* $\sup_{\beta\in\mathcal{A}}\|f_\beta\|_{L_\infty(Q_1)}$. *When* $\sigma = 1$, Du *is uniformly continuous in* $Q_{1/2}$ *with a modulus of continuity controlled by the quantities before.*

Here for simplicity we assume $u \in C^{1,\sigma^+}(Q_1)$, which means that $u \in C^{1,\sigma+\varepsilon}(Q_1)$ for some arbitrary $\varepsilon > 0$. This condition is only needed for $L_\beta u$ to be well defined, and it may be replaced by other weaker conditions.

Remark. By a careful inspection of the proofs below, one can see that the estimates above in fact only depend on d, σ, λ, Λ, $\sup_{\beta\in\mathcal{A}}\|f_\beta\|_{L_\infty(Q_1)}$, the modulus continuity ω_f of f_β in B_1, $\omega_a(r)$, $\omega_u(r)$ for $r \in (0,1)$, and $\|u\|_{L_{1,w}}$, where the weight $w = w(x)$ is equal to $(1+|x|)^{-d-\sigma}$. In particular, u does not need to be globally bounded in $\mathbb{R}^d$.

By regarding elliptic equations as steady-state parabolic equations, we obtain the following corollary.

Corollary 6.1.2. *Let $\sigma \in (0,2)$, $0<\lambda\le\Lambda<\infty$, and $\mathcal{A}$ be an index set. Assume for each $\beta\in\mathcal{A}$, K_β satisfies (6.1.2) when $\sigma=1$, and in $\mathbb{R}^d$*

$$
\begin{aligned}
&\left|a_\beta(x,y)-a_\beta(x',y)\right| \le \Lambda\omega_a(|x-x'|),\\
&|f_\beta(x)-f_\beta(x')| \le \omega_f(|x-x'|), \quad \sup_{\beta\in\mathcal{A}}\|f_\beta\|_{L_\infty(B_1)}<\infty,\\
&\sup_\beta \|b_\beta\|_{L_\infty(B_1)} \le N_0, \quad |b_\beta(x)-b_\beta(x')|\le\omega_b(|x-x'|),
\end{aligned}
$$

where ω_a, ω_b, and ω_f are Dini functions. Suppose $u\in C^{\sigma^+}(B_1)$ is a solution

$$
\inf_\beta(L_\beta u + b_\beta Du + f_\beta) = 0,
$$

B_1 and is Dini continuous in $\mathbb{R}^d$. Then we have for $\sigma\ne1$

$$
\begin{aligned}
&[u]^x_{\sigma;B_{1/2}}\\
&\le C\|u\|_{L_\infty(\mathbb{R}^d)} + C\sup_\beta\|f_\beta\|_{L_\infty(B_1)} + C\sum_{j=1}^\infty\big(\omega_u(2^{-j})+\omega_f(2^{-j})\big);
\end{aligned}
\tag{6.1.6}
$$

for $\sigma=1$

$$
\begin{aligned}
&\|Du\|_{L_\infty(B_{1/2})}\\
&\le C\|u\|_{L_\infty(\mathbb{R}^d)} + C\sup_\beta\|f_\beta\|_{L_\infty(B_1)} + C\sum_{j=1}^\infty\big(\omega_u(2^{-j})+\omega_f(2^{-j})\big);
\end{aligned}
$$

where $C>0$ is a constant depending only on d, σ, λ, Λ, N_0, ω_b, and ω_a.

Roughly speaking, the proof can be divided into two steps: We first show that Theorem 6.1.1 holds when the equation is satisfied in the whole space; Then we implement a localization argument to treat the general case. In Step one, our proof is based on a refined $C^{1+\alpha/\sigma,\sigma+\alpha}$ estimate in our previous paper [31] and a new perturbation type argument, as the standard perturbation techniques do not seem to work here. The novelty of this method is that instead of estimating $C^{1,\sigma}$ semi-norm of the solution, we construct and bound certain semi-norms of the solution, see Lemma 6.2.1. When $\sigma < 1$, such semi-norm is defined as a series of lower-order Hölder semi-norms of u. This is in the spirit of Campanato's approach first developed in [18]. Heuristically, in order for the nonlocal operator to be well defined, the solution needs to be smoother than $C^{1,\sigma}$. To resolve this problem, we divide the integral domain into annuli, which allows us to use a lower-order semi-norm to estimate the integral in each annulus. The series of lower-order semi-norms, which turns out to be slightly stronger than the $C^{1,\sigma}$ semi-norm, further implies that

$$[u]^x_{\sigma;Q_r(x_0)} \to 0 \quad \text{as} \quad r \to 0$$

uniformly in x_0. In particular, when $\sigma = 1$ we are able to estimate the modulus of continuity of the gradient of solutions.

The organization of this chapter is as follows. In the next section, we introduce some notation and preliminary results that are necessary in the proof of our main theorem. Some of these results might be of independent interest. In section 3, we first prove a global version of Theorem 6.1.1 and then localize the result to obtain Theorem 6.1.1.

6.2 Preliminary

We will frequently use the following identities:

$$
\begin{aligned}
&2^j(u(t,x+2^{-j}l)-u(x))-(u(t,x+l)-u(t,x))\\
&=\sum_{k=1}^{j}2^{k-1}(2u(t,x+2^{-k}l)-u(t,x+2^{-k+1}l)-u(t,x))
\end{aligned}
\tag{6.2.1}
$$

and

$$
\begin{aligned}
&2^j(u(t-2^{-j},x)-u(t,x))-(u(t-1,x)-u(t,x))\\
&=\sum_{k=1}^{j}2^{k-1}(2u(t-2^{-k},x)-u(t-2^{-k+1},x)-u(t,x)),
\end{aligned}
\tag{6.2.2}
$$

which hold for any unit vector $l\in\mathbb{R}^d$ and $j\in\mathbb{N}$. Denote $\mathcal{P}_t$ (or $\mathcal{P}_x$) to be the set of first order polynomial in t (or x), respectively. Let $\mathcal{P}_1$ be the set of first order polynomial in both t and x.

Lemma 6.2.1. *Let $\alpha\in(0,\sigma)$ be a constant.*

(i) When $\sigma\in(0,1)$, we have

$$
\begin{aligned}
&[u]^x_\sigma+\|\partial_t u\|_{L_\infty}\\
&\le C\sum_{k=1}^{\infty}2^{k(\sigma-\alpha)}\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0}\inf_{p\in\mathcal{P}_t}[u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)}+C\|u\|_{L_\infty},
\end{aligned}
\tag{6.2.3}
$$

where C is a constant depending only on d, σ, and α. Moreover, the modulus of continuity of $\partial_t u$ is bounded by the tail of the summation on the right-hand side of (6.2.3).

(ii) When $\sigma \in (1,2)$, we have

$$
\begin{aligned}
&[u]^x_\sigma + \|\partial_t u\|_{L_\infty} + [Du]^t_{\frac{\sigma-1}{\sigma}} \\
&\le C \sum_{k=1}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)} + C\|u\|_{L_\infty},
\end{aligned}
\tag{6.2.4}
$$

where C is a constant depending on d, α, and σ. The modulus of continuity of $\partial_t u$ is bounded by the tail of the summation above.

(iii) When $\sigma = 1$, we have

$$
\begin{aligned}
\|Du\|_{L_\infty} + \|\partial_t u\|_{L_\infty} \le{}& C \sum_{k=1}^{\infty} 2^{k(1-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-k}}(t_0,x_0)} \\
&+ C \sup_{\substack{(t,x),(t',x')\in\mathbb{R}^{d+1}_0 \\ \max\{|t-t'|,|x-x'|\}=1}} |u(t,x)-u(t',x')|,
\end{aligned}
\tag{6.2.5}
$$

where C is a constant depending on d, α, and σ. The modulus of continuity of $\partial_t u$ and Du are bounded by the tail of the summation above.

Proof. We first prove the estimate of $\partial_t u$ for $\sigma \in (0,2)$ by showing that

$$
\begin{aligned}
\|\partial_t u\|_{L_\infty} \le{}& C \sum_{k=1}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]^t_{\alpha/\sigma;Q_{2^{-k}}(t_0,x_0)} \\
&+ 2 \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} |u(t_0-1,x_0)-u(t_0,x_0)|.
\end{aligned}
\tag{6.2.6}
$$

Indeed, from (6.2.2)

$$
\begin{aligned}
&2^j|u(t-2^{-j},x)-u(t,x)| \\
&\le 2|u(t-1,x)-u(t,x)| + \sum_{k=1}^{\infty} 2^{k-1}|2u(t-2^{-k},x)-u(t-2^{-k+1},x)-u(t,x)| \\
&\le 2|u(t-1,x)-u(t,x)| + C\sum_{k=1}^{\infty} 2^{k(1-\alpha/\sigma)}[u]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-k^*}}(t,x))},
\end{aligned}
\tag{6.2.7}
$$

where C only depends on σ and $k^* = [\frac{k}{\sigma}]$, i.e., the largest integer which is smaller than k/σ. Obviously, $k^* \ge k/2$. Therefore, the right-hand side of the inequality above is less than

$$
\begin{aligned}
&2|u(t-1,x)-u(t,x)| + C\sum_{k=1}^{\infty} 2^{(k^*\sigma+\sigma)(1-\alpha/\sigma)}[u]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-k^*}}(t,x))} \\
&\le 2|u(t-1,x)-u(t,x)| + C\sum_{k=1}^{\infty} 2^{k^*(\sigma-\alpha)} \inf_{p\in\mathcal{P}_t}[u-p]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-k^*}}(t,x))}.
\end{aligned}
$$

By using the definition of k^*, it is easy to see the second term on the right-hand side of the inequality above is bounded by

$$
\begin{aligned}
&2C\sum_{k=1}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t}[u-p]^t_{\alpha/\sigma;Q_{2^{-k}}(t_0,x_0)} \\
&\quad + 2\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0}|u(t_0-1,x_0)-u(t_0,x_0)|.
\end{aligned}
$$

Therefore, by sending $j\to\infty$ in (6.2.7), we prove that $\|\partial_t u\|_{L_\infty}$ is bounded by the right-hand side of (6.2.6). Notice that

$$
\inf_{p\in\mathcal{P}_t}[u-p]^t_{\alpha/\sigma;Q_{2^{-k}}(t_0,x_0)} \le \inf_{p\in\mathcal{P}_t}[u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)},
$$

and

$$\inf_{p\in\mathcal{P}_t}[u-p]^t_{\alpha/\sigma;Q_{2^{-k}}(t_0,x_0)} \le \inf_{p\in\mathcal{P}_1}[u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)}.$$

Then the right-hand side of (6.2.6) is bounded by that of (6.2.3), (6.2.4), and (6.2.5). We obtain the bound of $\|\partial_t u\|_{L_\infty}$.

Next, we bound the modulus of continuity of $\partial_t u$. Assume that

$$|t-t'|+|x-x'|^\sigma \in [2^{-(i+1)}, 2^{-i}).$$

From (6.2.2), for any $j\ge i+1$,

$$\begin{aligned}&2^j(u(t-2^{-j},x)-u(t,x))-2^i(u(t-2^{-i},x)-u(t,x))\\&=\sum_{k=i+1}^{j}2^{k-1}(2u(t-2^{-k},x)-u(t-2^{-k+1},x)-u(t,x)),\end{aligned}$$

and the same identity holds with (t',x') in place of (t,x). Then we have

$$\begin{aligned}&|\partial_t u(t,x)-\partial_t u(t',x')|\\&=\lim_{j\to\infty}|2^j(u(t-2^{-j},x)-u(t,x))-2^j(u(t'-2^{-j},x')-u(t',x'))|\\&\le|2^i(u(t-2^{-i},x)-u(t,x))-2^i(u(t'-2^{-i},x')-u(t',x'))|\\&\quad+C\sum_{k=i+1}^{\infty}\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0}2^{k(1-\alpha/\sigma)}[u]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-k^*}}(t_0,x_0))},\end{aligned}$$

where k^* is defined above. By the triangle inequality, the first term on the right-hand side is bounded by

$$2^i|u(t-2^{-i},x)+u(t',x')-2u(\bar t,\bar x)|+2^i|u(t'-2^{-i},x')-2u(\bar t,\bar x)+u(t,x)|,$$

where $\bar{t} = (t + t' - 2^{-i})/2$ and $\bar{x} = (x + x')/2$, which is further bounded by

$$2^{i(1-\alpha/\sigma)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \big([u]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-i^*}}(t_0,x_0))} + [u]^x_{\alpha;Q_{2^{-i^*}}(t_0,x_0)}\big),$$

where $i^* = [\frac{i}{\sigma}]$. Therefore,

$$\begin{aligned}
&|\partial_t u(t,x) - \partial_t u(t',x')| \\
&\le C \sum_{k=i}^{\infty} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} 2^{k(1-\alpha/\sigma)} \big([u]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-i^*}}(t_0,x_0))} + [u]^x_{\alpha;Q_{2^{-i^*}}(t_0,x_0)}\big) \\
&\le C \sum_{k=i}^{\infty} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} 2^{k(1-\alpha/\sigma)} \inf_{p\in\mathcal{P}_t} [u-p]^t_{\alpha/\sigma,\alpha;Q_{2^{-k^*}}(t_0,x_0)},
\end{aligned}$$

which, from the definition of k^*, converges to 0 as $i \to \infty$.

In the rest of the proof, we consider the three cases separately.

Case 1. $\sigma \in (0,1)$. The estimates of $[u]^x_\sigma$ is the same as [30, Lemma 2.1] and we only provide a sketch here. Let $x, x' \in \mathbb{R}^d$ be two different points. Denote $h = |x - x'|$. Since

$$\begin{aligned}
&u(t,x') - u(t,x) \\
&= \frac{1}{2}\big(u(t,2x'-x) - u(t,x)\big) - \frac{1}{2}\big(u(t,2x'-x) - 2u(t,x') + u(t,x)\big),
\end{aligned}$$

we get

$$\begin{aligned}
&h^{-\sigma}|u(t,x') - u(t,x)| \\
&\le 2^{\sigma-1}(2h)^{-\sigma}\big(u(t,2x'-x) - u(t,x)\big) + h^{-\sigma}|u(t,2x'-x) - 2u(t,x') + u(t,x)| \\
&\le 2^{\sigma-1}(2h)^{-\sigma}\big(u(t,2x'-x) - u(t,x)\big) + \sup_{x\in\mathbb{R}^d} h^{\alpha-\sigma}[u(t,\cdot)]_{\Lambda^\alpha(B_h(x))}.
\end{aligned}$$

Taking the supremum with respect to t, x, and x' on both sides, we get

$$[u]^x_\sigma \le 2^{\sigma-1}[u]^x_\sigma + \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sup_{h>0} h^{\alpha-\sigma}[u]^x_{\alpha;Q_h(t_0,x_0)},$$

which further implies

$$\begin{aligned}[u]^x_\sigma &\le C \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sup_{h>0} h^{\alpha-\sigma}[u]^x_{\alpha;Q_h(t_0,x_0)} \\ &\le C\sum_{k=1}^\infty 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} [u]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} + C\|u\|_{L_\infty},\end{aligned}$$

where we used the obvious inequality

$$h^{\alpha-\sigma}[u]^x_{\alpha;Q_h(t_0,x_0)} \le C\|u\|_{L_\infty}$$

when $h > 1/2$. Notice that

$$[u]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} = \inf_{p\in\mathcal{P}_t}[u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le \inf_{p\in\mathcal{P}_t}[u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)}.$$

The proof of Case 1 is completed.

Case 2. $\sigma \in (1,2)$. Similar to the previous case, we only provide the sketch of the proof following that of [30, Lemma 2.1]. Let $\ell \in \mathbb{R}^d$ be a unit vector and $\varepsilon \in (0, 1/16)$ be a small constant to be specified later. For any two distinct points $x, x' \in \mathbb{R}^d$, we denote $h = |x - x'|$. By the triangle inequality,

$$h^{1-\sigma}|D_\ell u(t,x) - D_\ell u(t,x')| \le I_1 + I_2 + I_3, \tag{6.2.8}$$

where

$$I_1 := h^{1-\sigma}|D_\ell u(t,x) - (\varepsilon h)^{-1}(u(t,x+\varepsilon h\ell) - u(t,x))|,$$
$$I_2 := h^{1-\sigma}|D_\ell u(t,x') - (\varepsilon h)^{-1}(u(t,x'+\varepsilon h\ell) - u(t,x'))|,$$
$$I_3 := h^{1-\sigma}(\varepsilon h)^{-1}|(u(t,x+\varepsilon h\ell) - u(t,x)) - (u(t,x'+\varepsilon h\ell) - u(t,x'))|.$$

By the mean value theorem,

$$I_1 + I_2 \le 2\varepsilon^{\sigma-1}[Du]^x_\sigma. \tag{6.2.9}$$

Now we choose and fix a ε sufficiently small depending only on σ such that $2\varepsilon^{\sigma-1} \le 1/2$. Using the triangle inequality, we have

$$\begin{aligned} I_3 \le Ch^{-\sigma}\big(&|u(t,x+\varepsilon h\ell) + u(t,x') - 2u(t,\bar{x})| \\ &+ |u(t,x'+\varepsilon h\ell) + u(t,x) - 2u(t,\bar{x})|\big), \end{aligned}$$

where $\bar{x} = (x+\varepsilon h\ell + x')/2$. Thus,

$$I_3 \le Ch^{\alpha-\sigma}[u(t,\cdot)]^x_{\Lambda^\alpha(B_h(\bar{x}))}. \tag{6.2.10}$$

Combining (6.2.8), (6.2.9), and (6.2.10), we get

$$[u]^x_\sigma \le C\sum_{k=1}^\infty 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} + C\|u\|_{L_\infty}.$$

By noting that

$$\inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)},$$

we bound $[u]^x_\sigma$ by the right-hand side of (6.2.4).

The estimate of $[Du]^t_{\frac{\sigma-1}{\sigma}}$ follows from [63, Section 3.3]. Therefore, (6.2.4) is proved.

Case 3. $\sigma = 1$. We give the estimate of $\|Du\|_{L_\infty}$. It follows from (6.2.1) that

$$\begin{aligned}
&2^j\big|u(t,x+2^{-j}\ell)-u(t,x)\big| \\
&\le 2|u(t,x+\ell)-u(t,x)| + \sum_{k=1}^{j} 2^{-k(\alpha-1)}[u(t,\cdot)]^x_{\Lambda^\alpha(B_{2^{-k}}(x+2^{-k}\ell))}.
\end{aligned}$$

Taking $j\to\infty$, we obtain that

$$\begin{aligned}
\|Du\|_{L_\infty} \le C\sum_{k=1}^{\infty} 2^{k(1-\alpha)} &\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0}\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \\
&+ 2\sup_{\substack{(t,x),(t,x')\in\mathbb{R}^{d+1}_0\\ |x-x'|=1}} |u(t,x)-u(t,x')|.
\end{aligned}$$

The estimate of the continuity of Du is the same as $\partial_t u$, and thus omitted. $\square$

Let η be a smooth nonnegative function in $\mathbb{R}$ with unit integral and vanishing outside $(0,1)$. For $R>0$ and $\sigma\in(0,1)$, we define the mollification of u with respect to t as

$$u^{(R)}(t,x)=\int_{\mathbb{R}} u(t-R^\sigma s,x)\eta(s)\,ds.$$

For the case $\sigma\in[1,2)$, we define $u^{(R)}$ differently by mollifying the x variable as well. Let $\zeta\in C_0^\infty(B_1)$ be a radial nonnegative function with unit integral. For $R>0$, we define

$$u^{(R)}(t,x)=\int_{\mathbb{R}^{d+1}} u(t-Rs,x-Ry)\eta(s)\zeta(y)\,dy\,ds.$$

The following lemma is for the case $\sigma\in(0,1)$.

Lemma 6.2.2. *Let $\alpha \in (0,\sigma)$ and $R > 0$ be constants. Let $p_0 = p_0(t)$ be the first-order Taylor expansion of $u^{(R)}$ at the origin in t and $\tilde{u} = u - p_0$. Then for any integer $j \geq 0$, we have*

$$[\tilde{u}]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2^jR}} \leq C \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2^jR}}, \tag{6.2.11}$$

where C is a constant only depending on d and α.

Proof. It is easily seen that $\tilde{u}$ is invariant up to a constant if we replace u by $u - p$ for any $p \in \mathcal{P}_t$. Thus to prove the lemma, we only need to bound the left-hand side of (6.2.11) by

$$C[u]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2^jR}}.$$

Since $\tilde{u} = u - p(t)$, it suffices to observe that

$$[p]^t_{\alpha/\sigma;(-R^\sigma,0)} = R^{\sigma-\alpha}|\partial_t u^{(R)}(0,0)| \leq C[u]^t_{\alpha/\sigma;Q_R}.$$

The lemma is proved. □

The following lemma is useful in dealing with the case $\sigma \in (1,2)$.

Lemma 6.2.3. *Let $\alpha \in (0,1)$ and $\sigma \in (1,2)$ be constant. Then for any $u \in C^1$, we have*

$$\begin{aligned}
&\sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} [u-p_0]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \\
&\leq C \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)},
\end{aligned} \tag{6.2.12}$$

where p_0 is the first-order Taylor's expansion of u in the x variable at (t_0, x_0), and $C > 0$ is a constant depending only on d, α, and σ.

Proof. Denote

$$b_k := 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}_0^{d+1}} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}.$$

Then for any $(t_0, x_0) \in \mathbb{R}_0^{d+1}$ and each $k = 0, 1, \ldots$, there exists $\mathrm{p}_k \in \mathcal{P}_x$ such that

$$[u-\mathrm{p}_k]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le 2b_k 2^{-k(\sigma-\alpha)}.$$

By the triangle inequality, for $k \ge 1$ we have

$$[\mathrm{p}_{k-1}-\mathrm{p}_k]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le 2b_k 2^{-k(\sigma-\alpha)} + 2b_{k-1}2^{-(k-1)(\sigma-\alpha)}. \tag{6.2.13}$$

It is easily seen that

$$[\mathrm{p}_{k-1}-\mathrm{p}_k]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} = |\nabla \mathrm{p}_{k-1} - \nabla \mathrm{p}_k| 2^{-(k-1)(1-\alpha)},$$

which together with (6.2.13) implies that

$$|\nabla \mathrm{p}_{k-1} - \nabla \mathrm{p}_k| \le C 2^{-k(\sigma-1)}(b_{k-1}+b_k). \tag{6.2.14}$$

Since $\sum_k b_k < \infty$, from (6.2.14) we see that $\{\nabla \mathrm{p}_k\}$ is a Cauchy sequence in $\mathbb{R}^d$. Let $q = q(t_0, x_0) \in \mathbb{R}^d$ be the limit, which clearly satisfies for each $k \ge 0$,

$$|q - \nabla \mathrm{p}_k| \le C \sum_{j=k}^{\infty} 2^{-j(\sigma-1)} b_j.$$

By the triangle inequality, we get

$$
\begin{aligned}
&[u - q\cdot x]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le [u - \mathrm{p}_k]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} + [\mathrm{p}_k - q\cdot x]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}\\
&\le C2^{-k(1-\alpha)}\sum_{j=k}^{\infty} 2^{-j(\sigma-1)} b_j \le C2^{-k(\sigma-\alpha)}, \qquad (6.2.15)
\end{aligned}
$$

which implies that

$$
\|u(t_0,\cdot) - u(t_0,x_0) - q\cdot(x-x_0)\|_{L_\infty(B_{2^{-k}}(x_0))} \le C2^{-k\sigma},
$$

and thus $q = \nabla u(t_0,x_0)$. It then follows from (6.2.15) that

$$
\begin{aligned}
&\sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} [u - p_0]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}\\
&\le C\sum_{k=0}^{\infty} 2^{k(\sigma-1)} \sum_{j=k}^{\infty} 2^{-j(\sigma-1)} b_j\\
&= C\sum_{j=0}^{\infty} 2^{-j(\sigma-1)} b_j \sum_{k=0}^{j} 2^{k(\sigma-1)} \le C\sum_{j=0}^{\infty} b_j.
\end{aligned}
$$

This completes the proof of (6.2.12). □

The last lemma in this section is for the case when $\sigma \in [1,2)$.

Lemma 6.2.4. *Let $\alpha \in (0,1)$, $\sigma \in [1,2)$, and $R > 0$ be constants. Let $p_0 = p_0(t,x)$ be the first order Taylor's expansion of $u^{(R)}$ at the origin and $\tilde{u} = u - p_0$. Then for any integer $j \ge 0$, we have*

$$
\begin{aligned}
&\sup_{\substack{(t,x),(t',x')\in(-R^\sigma,0)\times B_{2^j R}\\ (t,x)\neq(t',x'),\,0\le|x-x'|<2R}} \frac{|\tilde{u}(t,x)-\tilde{u}(t',x')|}{|x-x'|^\alpha + |t-t'|^{\alpha/\sigma}}\\
&\le C\inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2^j R}}, \qquad (6.2.16)
\end{aligned}
$$

where $C > 0$ is a constant depending only on d, α, and σ.

Proof. It is easily seen that $\tilde{u}$ is invariant up to a constant if we replace u by $u - p$ for any $p \in \mathcal{P}_1$. Thus to show (6.2.16), we only bound the left-hand side of (6.2.16) by

$$C[u]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2^j R}}.$$

Since $\tilde{u} = u - p(t,x)$, it suffices to observe that for any two distinct $(t,x),(t',x') \in (-R^\sigma,0)\times B_{2^j R}$ such that $0 \le |x - x'| < 2R$,

$$\begin{aligned}
&|p(t,x) - p(t',x')| \\
&\le |x - x'||Du^{(R)}(0,0)| + |t - t'||\partial_t u^{(R)}(0,0)| \\
&\le C|x - x'|R^{\alpha-1}[u]^x_{\alpha;Q_R} + C|t - t'|R^{\sigma(\alpha/\sigma-1)}[u]^t_{\alpha/\sigma;Q_R} \\
&\le C\big(|x - x'|^\alpha + |t - t'|^{\alpha/\sigma}\big)[u]_{\alpha/\sigma,\alpha;Q_R}.
\end{aligned}$$

The lemma is proved. □

6.3 Proofs of Theorem 1.2.3

The following proposition is a further refinement of [31, Corollary 4.6].

Proposition 6.3.1. *Let $\sigma \in (0,2)$ and $0 < \lambda \le \Lambda$. Assume that for any $\beta \in \mathcal{A}$, K_β only depends on y. There is a constant $\hat{\alpha}$ depending on d, σ, λ, and Λ so that the following holds. Let $\alpha \in (0,\hat{\alpha})$ such that $\sigma + \alpha$ is not an integer. Suppose $u \in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_1) \cap C^{\alpha/\sigma,\alpha}((-1,0)\times\mathbb{R}^d)$ is a solution of*

$$\partial_t u = \inf_{\beta\in\mathcal{A}}(L_\beta u + f_\beta) \quad \textit{in} \quad Q_1.$$

Then,

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C\sum_{j=1}^{\infty} 2^{-j\sigma} M_j + C\sup_{\beta}[f_\beta]_{\alpha/\sigma,\alpha;Q_1},$$

where

$$M_j = \sup_{\substack{(t,x),(t',x')\in(-1,0)\times B_{2^j},\\ (t,x)\neq(t',x'),\, 0\le|x-x'|<2}} \frac{|u(t,x)-u(t',x')|}{|x-x'|^\alpha + |t-t'|^{\alpha/\sigma}}.$$

Proof. This follows from the proof of [31, Corollary 4.6] by observing that in the estimate of $[h_\beta]_{\alpha/\sigma,\alpha;Q_1}$, the term $[u]_{\alpha/\sigma,\alpha;(-1,0)\times B_{2^j}}$ can be replaced by M_j. Moreover, by replacing u by $u-u(0,0)$, we see that

$$\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} \le C[u]_{\alpha/\sigma,\alpha;(-1,0)\times B_2}.$$

The lemma is proved. □

In the rest of this section, we consider three cases separately.

6.3.1 The case $\sigma \in (0,1)$

Proposition 6.3.2. *Suppose that* (6.1.1) *is satisfied in* $\mathbb{R}^{d+1}_0$*. Then under the condition of Theorem 1.2.3, we have*

$$[u]^x_\sigma + \|\partial_t u\|_{L_\infty} \le C\|u\|_{L_\infty} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}), \tag{6.3.1}$$

where $C>0$ *is a constant depending only on* d, λ, Λ, ω_a, *and* σ.

Proof. For $k \in \mathbb{N}$, let v be the solution of

$$\begin{cases} \partial_t v = \inf_{\beta\in\mathcal{A}}(L_\beta(0,0)v + f_\beta(0,0) - \partial_t p_0) & \text{in } Q_{2^{-k}}, \\ v = u - p_0(t) & \text{in } (-2^{-k\sigma}, 0) \times B^c_{2^{-k}}, \end{cases}$$

where $L_\beta(0,0)$ is the operator with kernel $K_\beta(0,0,y)$, and $p_0(t)$ is the Taylor's expansion of $u^{(2^{-k})}$ in t at the origin. Then by Proposition 6.3.1 with scaling, we have

$$[v]_{1+\alpha/\sigma,\alpha+\sigma;Q_{2^{-k-1}}} \le C\sum_{j=1}^{\infty} 2^{(k-j)\sigma} M_j + C2^{k\sigma}[v]_{\alpha/\sigma,\alpha;Q_{2^{-k}}}, \tag{6.3.2}$$

where $\alpha \in (0, \hat{\alpha})$ satisfying $\sigma + \alpha < 1$ and

$$M_j = \sup_{\substack{(t,x),(t',x')\in(-2^{-k\sigma},0)\times B_{2^{j-k}},\\ (t,x)\neq(t',x'),\, 0\le|x-x'|<2^{-k+1}}} \frac{|\tilde{u}(t,x) - \tilde{u}(t',x')|}{|x-x'|^\alpha + |t-t'|^{\alpha/\sigma}}.$$

Let $k_0, k_1 \ge 1$ be integers to be specified and $p_1 = p_1(t)$ be the Taylor's expansion of v in t at the origin. By the mean value formula,

$$\|v - p_1\|_{L_\infty(Q_{2^{-k-k_0}})} \le 2^{-(k+k_0)(\sigma+\alpha)}[v]_{1+\alpha/\sigma,\sigma+\alpha;Q_{2^{-k-k_0}}},$$

and the interpolation inequality

$$\begin{aligned} &[v - p_1]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}} \\ &\le C\big(2^{(k+k_0)\alpha}\|v - p_1\|_{L_\infty(Q_{2^{-k-k_0}})} + 2^{-(k+k_0)\sigma}[v - p_1]_{1+\alpha/\sigma,\alpha+\sigma;Q_{2^{-k-k_0}}}\big), \end{aligned}$$

we obtain

$$[v-p_1]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}} \le C2^{-(k+k_0)\sigma}[v]_{1+\alpha/\sigma,\alpha+\sigma;Q_{2^{-k-k_0}}}.$$

This and (6.3.2) give

$$\begin{aligned}
[v-p_1]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}} &\le C2^{-(k+k_0)\sigma}\sum_{j=1}^{k}2^{(k-j)\sigma}M_j \\
&\quad + C2^{-(k+k_0)\sigma}[u-p_0]_{\alpha/\sigma,\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d} + C2^{-k_0\sigma}[v]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} \\
&\le C2^{-(k+k_0)\sigma}\sum_{j=1}^{k}2^{(k-j)\sigma}M_j + C2^{-(k+k_0)\sigma}\big([u]_{\alpha/\sigma,\alpha}+\|\partial_t u\|_{L_\infty}\big) \\
&\quad + C2^{-k_0\sigma}[v]_{\alpha/\sigma,\alpha;Q_{2^{-k}}}.
\end{aligned} \tag{6.3.3}$$

Next, $w := u - p_0 - v$ satisfies

$$\begin{cases}
w_t - M^+ w \le C_k & \text{in } Q_{2^{-k}}, \\
w_t - M^- w \ge -C_k & \text{in } Q_{2^{-k}}, \\
w = 0 & \text{in } (-2^{-k\sigma},0)\times B^c_{2^{-k}},
\end{cases} \tag{6.3.4}$$

where

$$C_k = \sup_{\beta\in\mathcal{A}} \|f_\beta - f_\beta(0,0) + (L_\beta - L_\beta(0,0))u\|_{L_\infty(Q_{2^{-k}})}.$$

It is easily seen that

$$\begin{aligned}
C_k &\le \omega_f(2^{-k}) + C\omega_a(2^{-k})\int_{\mathbb{R}^d}|u(t,x+y)-u(t,x)||y|^{-d-\sigma}\,dy \\
&\le \omega_f(2^{-k}) + C\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}}\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big).
\end{aligned}$$

Then by the Hölder estimate [31, Lemma 2.5], we have

$$
\begin{aligned}
&[w]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} \le C2^{-k(\sigma-\alpha)}C_k \le C2^{-k(\sigma-\alpha)} \\
&\cdot \Big[\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big) + \omega_f(2^{-k})\Big].
\end{aligned} \tag{6.3.5}
$$

From Lemma 6.2.2, we have

$$
M_j \le C \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(-2^{-k\sigma},0)\times B_{2^{j-k}}}. \tag{6.3.6}
$$

Notice that from the triangle inequality and Lemma 6.2.2 with $j=0$

$$
\begin{aligned}
[v]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} &\le [w]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} + [u-p_0]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} \\
&\le [w]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} + C \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(-2^{-k\sigma},0)\times B_{2^{-k}}}.
\end{aligned}
$$

Combining (6.3.3), with (6.3.5) and (6.3.6), yields

$$
\begin{aligned}
&2^{(k+k_0)(\sigma-\alpha)}[u-p_0-p_1]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}} \\
&\le C2^{-(k+k_0)\alpha} \sum_{j=1}^{k} 2^{(k-j)\sigma} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(-2^{-k\sigma},0)\times B_{2^{j-k}}} \\
&\quad + C2^{-(k+k_0)\alpha}([u]^x_\alpha + \|\partial_t u\|_{L_\infty}) + C2^{-k_0\alpha+k(\sigma-\alpha)} \\
&\quad\ \cdot \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(-2^{-k\sigma},0)\times B_{2^{-k}}} + C2^{k_0(\sigma-\alpha)}\omega_f(2^{-k}) \\
&\quad + C2^{k_0(\sigma-\alpha)}\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}(t_0,x_0)}} + \|u\|_{L_\infty}\Big).
\end{aligned} \tag{6.3.7}
$$

By translation of the coordinates, from (6.3.7) we have

$$\begin{aligned}
&2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} [u-p_0-p_1]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\le C2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{k} 2^{(k-j)\sigma} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\
&\quad + C2^{-(k+k_0)\alpha}\big([u]^x_\alpha + \|\partial_t u\|_{L_\infty}\big) + C2^{k_0(\sigma-\alpha)}\Big[\omega_f(2^{-k}) + \omega_a(2^{-k}) \\
&\quad \cdot\Big(\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big)\Big]. \qquad (6.3.8)
\end{aligned}$$

Then we take the sum (6.3.8) in $k = k_1, k_1+1, \ldots$ to obtain

$$\begin{aligned}
&\sum_{k=k_1}^{\infty} 2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\le C\sum_{k=k_1}^{\infty} 2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{k} 2^{(k-j)\sigma} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\
&\quad + C2^{-(k_1+k_0)\alpha}\big([u]^x_\alpha + \|\partial_t u\|_{L_\infty}\big) + C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \omega_f(2^{-k}) + C2^{k_0(\sigma-\alpha)} \\
&\quad \cdot \sum_{k=k_1}^{\infty} \omega_a(2^{-k}) \Big(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big).
\end{aligned}$$

By switching the order of summations and then replacing k by $k+j$, the first term on the right-hand side is bounded by

$$\begin{aligned}
&C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{-j\sigma} \sum_{k=j}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\
&\le C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{-j\alpha} \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{-k}}(x_0)} \\
&\le C2^{-k_0\alpha} \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)}.
\end{aligned}$$

With the inequality above, we have

$$
\begin{aligned}
&\sum_{k=k_1}^{\infty} 2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\le C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\quad + C2^{-(k_1+k_0)\alpha}\big([u]^x_\alpha + \|\partial_t u\|_{L_\infty}\big) + C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \omega_f(2^{-k}) \\
&\quad + C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \omega_a(2^{-k}) \cdot \Big(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\quad + \|u\|_{L_\infty}\Big).
\end{aligned}
$$

The bound above together with the obvious inequality

$$
\sum_{j=0}^{k_1+k_0-1} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma.\alpha;Q_{2^{-j}}(t_0,x_0)} \le C2^{(k_1+k_0)(\sigma-\alpha)}[u]_{\alpha/\sigma,\alpha},
$$

implies that

$$
\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\le C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&+ C2^{(k_1+k_0)(\sigma-\alpha)}[u]_{\alpha/\sigma,\alpha} + C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \omega_f(2^{-k}) \\
&+ C2^{-(k_1+k_0)\alpha}([u]^x_\alpha + \|\partial_t u\|_{L_\infty}) \\
&+ C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \omega_a(2^{-k}) \cdot \Big(\sum_{j=0}^{\infty} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big).
\end{aligned}
$$

By first choosing k_0 sufficiently large and then k_1 sufficiently large and using (6.2.3), we get

$$\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}_0^{d+1}} \inf_{p\in\mathcal{P}_t} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)}$$
$$\leq C\|u\|_{L_\infty} + C[u]_{\alpha/\sigma,\alpha} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}),$$

which together with Lemma 6.2.1 (i) and the interpolation inequality gives (6.3.1) and the continuity of $\partial_t u$. □

Theorem 1.2.3 follows from Proposition 6.3.2 by the argument of freezing the coefficients. Since the proof of it is almost the same and even simpler than the case $\sigma \in (1,2)$, we only present the detailed proof of Theorem 1.2.3 for $\sigma \in (1,2)$ in the next section.

6.3.2 The case when $\sigma \in (1,2)$

Proposition 6.3.3. *Suppose that* (6.1.1) *is satisfied in* $\mathbb{R}_0^{d+1}$. *Then under the condition of Theorem 1.2.3, we have for* $\sigma \in (1,2)$

$$[u]_\sigma^x + [Du]_{\frac{\sigma-1}{\sigma}}^t + \|\partial_t u\|_{L_\infty} \leq C\|u\|_{L_\infty} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}), \tag{6.3.9}$$

where $C > 0$ *is a constant depending only on* d, λ, Λ, ω_a, ω_b, N_0, *and* σ.

Proof. For $k \in \mathbb{N}$, let v_M be the solution of

$$\begin{cases} \partial_t v_M = \inf_{\beta\in\mathcal{A}}\big(L_\beta(0,0)v_M + f_\beta(0,0) + b_\beta(0,0)Du(0,0) - \partial_t p_0\big) & \text{in } Q_{2^{-k}} \\ v_M = g_M & \text{in } \big(-2^{-k\sigma},0\big)\times B_{2^{-k}}^c \end{cases},$$

where $M \geq 2\|u - p_0\|_{L_\infty(Q_{2^{-k}})}$ is a constant to be specified later,

$$g_M = \max(\min(u - p_0, M), -M),$$

and $p_0 = p_0(t, x)$ is the first-order Taylor's expansion of $u^{(2^{-k})}$ at the origin. By Proposition 6.3.1, we have

$$\begin{aligned}
[v_M]_{1+\alpha/\sigma,\alpha+\sigma;Q_{2^{-k-1}}} &\leq C\sum_{j=1}^{\infty} 2^{(k-j)\sigma} M_j + C2^{k\sigma}[v_M]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} \\
&\leq C\sum_{j=1}^{k} 2^{(k-j)\sigma} M_j + C\big(\|Du\|_{L_\infty} + \|\partial_t u\|_{L_\infty}\big) + C2^{k\sigma}[v_M]_{\alpha/\sigma,\alpha;Q_{2^{-k}}},
\end{aligned} \tag{6.3.10}$$

where $\alpha \in (0, \min\{\hat{\alpha}, (\sigma - 1)/2\})$ and

$$M_j = \sup_{\substack{(t,x),(t',x') \in (-2^{-k\sigma},0)\times B_{2^{j-k}} \\ (t,x)\neq(t',x'),\, 0\leq|x-x'|<2^{-k+1}}} \frac{|u(t,x) - p_0(t,x) - u(t',x') + p_0(t',x')|}{|t-t'|^{\alpha/\sigma} + |x-x'|^{\alpha}}.$$

From Lemma 6.2.4 with $\sigma \in (1, 2)$, it follows

$$M_j \leq C \inf_{p\in\mathcal{P}_1} [u - p]_{\alpha/\sigma,\alpha;(-2^{-k\sigma},0)\times B_{2^{j-k}}}. \tag{6.3.11}$$

From (6.3.10), and the mean value formula,

$$\begin{aligned}
\|v_M - p_1\|_{L_\infty(Q_{2^{-k-k_0}})} &\leq C2^{-(k+k_0)(\sigma+\alpha)} \sum_{j=1}^{k} 2^{(k-j)\sigma} M_j \\
&\quad + C2^{-(k+k_0)(\sigma+\alpha)}\big(\|Du\|_{L_\infty} + \|\partial_t u\|_{L_\infty}\big) + C2^{-k\alpha-k_0(\sigma+\alpha)}[v_M]_{\alpha/\sigma,\alpha;Q_{2^{-k}}},
\end{aligned}$$

where p_1 is the first-order Taylor's expansion of v_M at the origin. The above inequality, (6.3.10), and the interpolation inequality imply

$$[v_M - p_1]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}} \le C2^{-(k+k_0)\sigma}\sum_{j=1}^{k} 2^{(k-j)\sigma}M_j$$
$$+ C2^{-(k+k_0)\sigma}\big(\|Du\|_{L_\infty} + \|\partial_t u\|_{L_\infty}\big) + C2^{-k_0\sigma}[v_M]_{\alpha/\sigma,\alpha;Q_{2^{-k}}}. \tag{6.3.12}$$

Next $w_M := g_M - v_M$ satisfies

$$\begin{cases} \partial_t w_M \le \mathcal{M}^+ w_M + h_M + C_k & \text{in } Q_{2^{-k}} \\ \partial_t w_M \ge \mathcal{M}^- w_M + \hat{h}_M - C_k & \text{in } Q_{2^{-k}} \\ w_M = 0 & \text{in } (-2^{-k\sigma},0)\times B^c_{2^{-k}} \end{cases},$$

where

$$h_M := \mathcal{M}^-\big(g_M - (u-p_0)\big), \quad \hat{h}_M := \mathcal{M}^+\big(g_M - (u-p_0)\big).$$

Here

$$C_k = \sup_{\beta\in\mathcal{A}}\big\|f_\beta - f_\beta(0,0) + b_\beta Du - b_\beta(0,0)Du(0,0) + (L_\beta - L_\beta(0,0))u\big\|_{L_\infty(Q_{2^{-k}})}.$$

It follows easily that

$$C_k \le \omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty} + \sup_\beta \|b_\beta\|_{L_\infty} 2^{-k\alpha}[Du]_{\alpha/\sigma,\alpha;Q_{2^{-k}}}$$
$$+ C\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}}\sum_{j=0}^\infty 2^{j(\sigma-\alpha)}[u-p_0]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|Du\|_{L_\infty}\Big).$$

From Lemma 6.2.3, we obtain

$$C_k \le \omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty} + \sup_\beta \|b_\beta\|_{L_\infty} 2^{-k\alpha}[Du]_{\alpha/\sigma,\alpha;Q_{2^{-k}}}$$
$$+ C\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}}\sum_{j=0}^\infty 2^{j(\sigma-\alpha)}\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|Du\|_{L_\infty}\Big).$$

By the dominated convergence theorem, it is easy to see that

$$\|h_M\|_{L_\infty(Q_{2^{-k}})},\ \|\hat{h}_M\|_{L_\infty(Q_{2^{-k}})} \to 0 \quad \text{as} \quad M \to \infty.$$

Thus similar to (6.3.5), choosing M sufficiently large so that

$$\|h_M\|_{L_\infty(Q_{2^{-k}})},\ \|\hat{h}_M\|_{L_\infty(Q_{2^{-k}})} \le C_k/2,$$

we have

$$\begin{aligned}
&[w_M]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} \\
&\le C2^{-k(\sigma-\alpha)}\Big(\omega_f(2^{-k}) + \big(\omega_b(2^{-k}) + \omega_a(2^{-k})\big)\|Du\|_{L_\infty} + 2^{-k\alpha}[Du]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} \\
&\quad + \omega_a(2^{-k}) \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)}\Big).
\end{aligned} \tag{6.3.13}$$

Clearly,

$$\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} = \inf_{p\in\mathcal{P}_1}[u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)}. \tag{6.3.14}$$

From the triangle inequality and Lemma 6.2.4 with $j=0$,

$$\begin{aligned}
[v_M]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} &\le [w_M]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} + [u-p_0]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} \\
&\le [w_M]_{\alpha/\sigma,\alpha;Q_{2^{-k}}} + C\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}}.
\end{aligned}$$

Combining (6.3.12), (6.3.13) with (6.3.14), and (6.3.11), similar to (6.3.8), we get

$$
\begin{aligned}
&2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-(k_0+k)}}(t_0,x_0)} \\
&\le C2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=1}^{k} 2^{(k-j)\sigma} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\
&\quad + C2^{-(k+k_0)\alpha}(\|Du\|_{L_\infty} + \|\partial_t u\|_{L_\infty}) + \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} 2^{-k\alpha}[Du]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)} \\
&\quad + C2^{-k_0\alpha}2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)} \\
&\quad + C2^{k_0(\sigma-\alpha)}\Big[\omega_f(2^{-k}) + \big(\omega_b(2^{-k}) + \omega_a(2^{-k})\big)\|Du\|_{L_\infty} \\
&\quad + \omega_a(2^{-k}) \sup_{(t,x)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)}\Big].
\end{aligned}
$$

Summing the above inequality in $k = k_1, k_1+1, \ldots$ as before, we obtain

$$
\begin{aligned}
&\sum_{k=k_1}^{\infty} 2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\le C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t,x)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&+ C2^{-(k_0+k_1)\alpha}(\|u\|_1 + \|\partial_t u\|_{L_\infty}) \\
&+ C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} 2^{-k\alpha} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} [Du]_{\alpha/\sigma,\alpha;Q_{2^{-k}}(t_0,x_0)} \\
&+ C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \Big(\omega_f(2^{-k}) + \big(\omega_b(2^{-k}) + \omega_a(2^{-k})\big)\|Du\|_{L_\infty}\Big) \\
&+ C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \omega_a(2^{-k}) \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \inf_{p\in\mathcal{P}_1} [u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)},
\end{aligned}
\tag{6.3.15}
$$

and

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)}\\
&\le C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)}\\
&\quad + 2^{(k_1+k_0)(\sigma-\alpha)}[u]_{\alpha/\sigma,\alpha} + C2^{-(k_1+k_0)\alpha}\big(\|u\|_1 + \|\partial_t u\|_{L_\infty}\big)\\
&\quad + C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \Big(\omega_f(2^{-k}) + (\omega_b(2^{-k}) + \omega_a(2^{-k}))\|Du\|_{L_\infty}\Big)\\
&\quad + C2^{k_0(\sigma-\alpha)-k_1\alpha}[Du]_{\alpha/\sigma,\alpha}\\
&\quad + C2^{k_0(\sigma-\alpha)} \sum_{k=k_1}^{\infty} \omega_a(2^{-k}) \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)}.
\end{aligned}$$

By choosing k_0 and k_1 sufficiently large, we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)}\\
&\le C\big(\|Du\|_{L_\infty} + \|\partial_t u\|_{L_\infty} + [Du]_{\alpha/\sigma,\alpha}\big) + C\sum_{k=1}^{\infty} \omega_f(2^{-k}).
\end{aligned} \tag{6.3.16}$$

Finally, by Lemma 6.2.1 (ii), recalling that $\alpha < (\sigma-1)/2$, and the interpolation inequality, we obtain (6.3.9). □

Then we use the localization argument to prove Theorem 1.2.3 for $\sigma \in (1,2)$.

Proof of Theorem 1.2.3 when $\sigma \in (1,2)$. We divide the proof into three steps.

Step 1. For $k = 1, 2, \ldots$, denote $Q^k := Q_{1-2^{-k}}$. Let $\eta_k \in C_0^\infty(Q^{k+1})$ be a sequence of nonnegative smooth cutoff functions satisfying $\eta \equiv 1$ in Q^k, $|\eta| \le 1$ in Q^{k+1}, $\|\partial_t^j D^i \eta_k\|_{L_\infty} \le C2^{k(i+j)}$ for each $i, j \ge 0$. Set $v_k := u\eta_k \in C^{1,\sigma+}$ and notice that in

$\mathbb{R}^{d+1}_0$

$$\partial_t v_k = \eta_k \partial_t u + \partial_t \eta_k u = \inf_{\beta\in\mathcal{A}} (\eta_k L_\beta u + \eta_k b_\beta D u + \eta_k f_\beta + \partial_t \eta_k u)$$
$$= \inf_{\beta\in\mathcal{A}} (L_\beta v_k + b_\beta D v_k - b_\beta u D\eta_k + h_{k\beta} + \eta_k f_\beta + \partial_t \eta_k u),$$

where

$$h_{k\beta} = \eta_k L_\beta u - L_\beta v_k = \int_{\mathbb{R}^d} \frac{\xi_k(t,x,y) a_\beta(t,x,y)}{|y|^{d+\sigma}}\, dy,$$

and

$$\xi_k(t,x,y) = u(t,x+y)(\eta_k(t,x+y) - \eta_k(t,x)) - y\cdot D\eta_k(t,x)u(t,x).$$

Obviously, $\eta_k f_\beta$ is a Dini continuous function in $\mathbb{R}^{d+1}_0$ and

$$\begin{aligned}
&|\eta_k(t,x) f_\beta(t,x) - \eta_k(t',x') f_\beta(t',x')| \\
&\le \|\eta_k\|_{L_\infty} \omega_f(\max\{|x-x'|, |t-t'|^{1/\sigma}\}) + \|f_\beta\|_{L_\infty(Q_1)}\big(\|D\eta_k\|_{L_\infty}|x-x'| \\
&\quad + \|\partial_t \eta_k\|_{L_\infty}|t-t'|\big) \\
&\le C\omega_f(\max\{|x-x'|, |t-t'|^{1/\sigma}\}) + C2^k \|f_\beta\|_{L_\infty(Q_1)}(|x-x'| + |t-t'|),
\end{aligned}$$

where C only depends on d and σ. Similarly, we consider $\partial_t\eta_k u$ and $b_\beta u D\eta_k$ by estimating their moduli of continuity and L_∞ norm. It is easy to see that

$$\|\partial_t \eta_k u\|_{L_\infty} + \|b_\beta u D\eta_k\|_{L_\infty} \le C 2^k \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}.$$

Furthermore, we have

$$\begin{aligned}
&|\partial_t\eta_k(t,x)u(t,x) - \partial_t\eta_k(t',x')u(t',x')| \\
&\le C2^{2k}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}(|x-x'| + |t-t'|) + C2^k\omega_u(\max\{|x-x'|, |t-t'|^{1/\sigma}\})
\end{aligned}$$

and

$$
\begin{aligned}
&|b_\beta u D\eta_k(t,x) - b_\beta u D\eta_k(t',x')| \\
&\le C2^k \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} \omega_b(\max\{|x-x'|, |t-t'|^{1/\sigma}\}) \\
&\quad + C2^k \omega_u(\max\{|x-x'|, |t-t'|^{1/\sigma}\}) + C2^{2k}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}(|x-x'|+|t-t'|).
\end{aligned}
$$

Step 2. We first estimate the L_∞ norm of $h_{k\beta}$. By the fundamental theorem of calculus,

$$
\xi_k(t,x,y) = y \cdot \int_0^1 u(t,x+y)D\eta_k(t,x+sy) - u(t,x)D\eta_k(t,x)\,ds.
$$

For $|y| \ge 2^{-k-3}$,

$$
|\xi_k(t,x,y)| \le C2^k |y| \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}.
$$

For $|y| < 2^{-k-3}$, we can further write

$$
\begin{aligned}
\xi_k(t,x,y) = y \cdot \int_0^1 &(u(t,x+y) - u(t,x))D\eta_k(t,x+sy) \\
&+ u(t,x)(D\eta_k(t,x+sy) - D\eta_k(t,x))\,ds,
\end{aligned}
$$

where the second term on the right-hand side is bounded by $C2^{2k}|y|^2|u(t,x)|$. To estimate the first term, we consider two cases: when $|x| \ge 1-2^{-k-2}$, because $|y| < 2^{-k-3}$, $\xi_k(t,x,y) \equiv 0$; when $|x| < 1-2^{-k-2}$, we have

$$
\left| y \cdot \int_0^1 (u(t,x+y) - u(t,x))D\eta_k(t,x+sy)\,ds \right| \le C2^k |y|^2 \|Du\|_{L_\infty(Q^{k+3})}.
$$

Hence for $|y| < 2^{-k-3}$,

$$
|\xi_k(t,x,y)| \le C|y|^2\big(2^{2k}|u(t,x)| + 2^k\|Du\|_{L_\infty(Q^{k+3})}\big).
$$

Combining with the case when $|y| > 2^{-k-3}$, we see that

$$\|h_{k\beta}\|_{L_\infty} \le C2^{\sigma k}\big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + \|Du\|_{L_\infty(Q^{k+3})}\big). \tag{6.3.17}$$

Next we estimate the modulus of continuity of $h_{k\beta}$. By the triangle inequality,

$$\begin{aligned} &|h_{k\beta}(t,x) - h_{k\beta}(t',x')| \\ &\le \int_{\mathbb{R}^d} \frac{|(\xi_k(t,x,y) - \xi_k(t',x',y))a_\beta(t,x,y)|}{|y|^{d+\sigma}} \\ &\quad + \frac{|\xi_k(t',x',y)(a_\beta(t,x,y) - a_\beta(t',x',y))|}{|y|^{d+\sigma}}\,dy := \mathrm{I} + \mathrm{II}. \end{aligned} \tag{6.3.18}$$

Similar to (6.3.17), by the estimates of $|\xi_k(t,x,y)|$ above, we have

$$\mathrm{II} \le C2^{\sigma k}\big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + \|Du\|_{L_\infty(Q^{k+3})}\big)\omega_a(\max\{|x-x'|, |t-t'|^{1/\sigma}\}), \tag{6.3.19}$$

where C depends on d, σ, and Λ. For I, by the fundamental theorem of calculus,

$$\begin{aligned} \xi_k(t,x,y) - \xi_k(t',x',y) = y \cdot \int_0^1 \Big(&u(t,x+y)D\eta_k(t,x+sy) - u(t,x)D\eta_k(t,x) \\ &- u(t',x'+y)D\eta_k(t',x'+sy) + u(t',x')D\eta_k(t',x')\Big)\,ds. \end{aligned}$$

When $|y| \ge 2^{-k-3}$, similar to the estimate of $\xi_k(t,x,y)$, it follows that

$$\begin{aligned} |\xi_k(t,x,y) - \xi_k(t',x',y)| \le C|y|\big(&2^k\omega_u(\max\{|x-x'|, |t-t'|^{1/\sigma}\}) \\ &+ 2^{2k}\|u\|_{L_\infty}(|x-x'| + |t-t'|)\big). \end{aligned} \tag{6.3.20}$$

The case when $|y| < 2^{-k-3}$ is more delicate. First, by the fundamental theorem of calculus,

$$\begin{aligned}
&|\xi_k(t,x,y)-\xi_k(t',x',y)|\\
&\le |y|\int_0^1 |(u(t,x+y)-u(t,x))D\eta_k(t,x+sy)\\
&\quad -(u(t',x'+y)-u(t',x'))D\eta_k(t',x'+sy)|\,ds\\
&\quad +|y|^2\int_0^1\int_0^1 |u(t,x)D^2\eta_k(t,x+s'sy)-u(t',x')D^2\eta_k(t',x'+s'sy)|\,ds\,ds'\\
&:= \mathrm{III}+\mathrm{IV}.
\end{aligned}$$

It is easily seen that

$$\mathrm{IV}\le C|y|^2\Big(2^{2k}\omega_u(\max\{|x-x'|,|t-t'|^{1/\sigma}\})+2^{3k}\|u\|_{L^\infty}(|x-x'|+|t-t'|)\Big).$$

Next we bound III by considering four cases. When $(t,x),(t',x')\in(Q^{k+2})^c$, we have $\mathrm{III}\equiv 0$. When $(t,x),(t',x')\in Q^{k+2}$, III is bounded by

$$\begin{aligned}
&|y|^2\int_0^1\int_0^1 |Du(t,x+sy)D\eta_k(t,x+s'y)-Du(t',x'+s'y)D\eta_k(t',x'+sy)|\,ds\,ds'\\
&\le C|y|^2\Big[2^k\big([Du]^x_{\alpha;Q^{k+3}}|x-x'|^\alpha+[Du]^t_{\alpha/\sigma;Q^{k+3}}|t-t'|^{\alpha/\sigma}\big)\\
&\quad +2^{2k}\|Du\|_{L_\infty(Q^{k+3})}(|x-x'|+|t-t'|)\Big],
\end{aligned}$$

where we choose $\alpha=\frac{\sigma-1}{2}$. When $(t,x)\in Q^{k+2}$ and $(t',x')\in(Q^{k+2})^c$,

$$\begin{aligned}
\mathrm{III}&=|y|\int_0^1 |(u(t,x+y)-u(t,x))D\eta_k(t,x+sy)|\,ds\\
&\le |y|^2\int_0^1\int_0^1 |Du(t,x+s'y)(D\eta_k(t,x+sy)-D\eta_k(t',x'+sy))|\,ds\,ds'\\
&\le C|y|^2 2^{2k}\|Du\|_{L_\infty(Q^{k+3})}(|x-x'|+|t-t'|).
\end{aligned}$$

The last case is similar. In conclusion, we obtain

$$\begin{aligned}\mathrm{III} \le C|y|^2 \Big[& 2^k \big([Du]^x_{\alpha;Q^{k+3}} |x-x'|^\alpha + [Du]^t_{\alpha/\sigma;Q^{k+3}} |t-t'|^{\alpha/\sigma} \big) \\ & + 2^{2k} \|Du\|_{L_\infty(Q^{k+3})} (|x-x'| + |t-t'|) \Big].\end{aligned}$$

From the estimates of $\mathrm{III}, \mathrm{IV}$, and (6.3.20), we obtain

$$\begin{aligned}\mathrm{I} \le C 2^{k(\sigma+1)} \Big[& \omega_u(\max\{|x-x'|, |t-t'|^{1/\sigma}\}) \\ & + [Du]^x_{\alpha;Q^{k+3}} |x-x'|^\alpha + [Du]^t_{\alpha/\sigma;Q^{k+3}} |t-t'|^{\alpha/\sigma} \\ & + \big(\|Du\|_{L_\infty(Q^{k+3})} + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} \big) (|x-x'| + |t-t'|) \Big].\end{aligned} \tag{6.3.21}$$

By combining (6.3.18), (6.3.19), and (6.3.21), we obtain

$$|h_{k\beta}(t,x) - h_{k\beta}(t',x')| \le \omega_h(\max\{|x-x'|, |t-t'|^{1/\sigma}\}),$$

where

$$\begin{aligned}\omega_h(r) := {} & C 2^{\sigma k} \big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + \|Du\|_{L_\infty(Q^{k+3})} \big) \omega_a(r) \\ & + C 2^{k(\sigma+1)} \Big[\omega_u(r) + \big([Du]^x_{\alpha;Q^{k+3}} + [Du]^t_{\alpha/\sigma;Q^{k+3}} \big) r^\alpha \\ & + \big(\|Du\|_{L_\infty(Q^{k+3})} + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} \big) (r + r^\sigma) \Big]\end{aligned} \tag{6.3.22}$$

is a Dini function.

Step 3. We apply Proposition 6.3.3 to v_k to obtain

$$
\begin{aligned}
&\|\partial_t v_k\|_{L_\infty} + [v_k]^x_\sigma + [v_k]^t_{\frac{\sigma-1}{\sigma}} \\
&\le C\|v_k\|_{L_\infty} + C\sum_{j=1}^\infty \big(\omega_h(2^{-j}) + \omega_f(2^{-j})\big) + C2^k \sup_\beta \|f_\beta\|_{L_\infty(Q_1)} \\
&\quad + C2^{2k}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + C2^k\sum_{j=1}^\infty \omega_u(2^{-j}) + C2^k\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}\sum_{j=1}^\infty \omega_b(2^{-j}) \\
&\le C\|v_k\|_{L_\infty} + C2^{k(\sigma+1)}\big([Du]^x_{\alpha;Q^{k+3}} + [Du]^t_{\alpha/\sigma;Q^{k+3}} + \|Du\|_{L_\infty(Q^{k+3})}\big) \\
&\quad + C\sum_{j=1}^\infty \big(2^{k(\sigma+1)}\omega_u(2^{-j}) + \omega_f(2^{-j})\big) + C2^k \sup_\beta \|f_\beta\|_{L_\infty(Q_1)} \\
&\quad + C2^{k(\sigma+1)}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)},
\end{aligned}
$$

where C depends on d, λ, Λ, σ, N_0, ω_b, and ω_a, but independent of k. Since $\eta_k \equiv 1$ in Q^k, it follows that

$$
\begin{aligned}
&\|\partial_t u\|_{L_\infty(Q^k)} + [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k} \le C2^{k(\sigma+1)}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} \\
&\quad + C_0 2^{k(\sigma+1)}\big([Du]^x_{\alpha;Q^{k+3}} + [Du]^t_{\alpha/\sigma,Q^{k+3}} + \|Du\|_{L_\infty(Q^{k+3})}\big) \\
&\quad + C\sum_{j=1}^\infty \big(2^{k(\sigma+1)}\omega_u(2^{-j}) + \omega_f(2^{-j})\big) + C2^k \sup_\beta \|f_\beta\|_{L_\infty(Q_1)}.
\end{aligned} \tag{6.3.23}
$$

By the interpolation inequality, for any $\varepsilon \in (0,1)$

$$
\begin{aligned}
&[Du]^x_{\alpha;Q^{k+3}} + [Du]^t_{\alpha/\sigma;Q^{k+3}} + \|Du\|_{L_\infty(Q^{k+3})} \\
&\le \varepsilon(\|\partial_t u\|_{L_\infty(Q^{k+3})} + [u]^x_{\sigma;Q^{k+3}} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^{k+3}}) + C\varepsilon^{-\frac{1+\alpha}{\sigma-(1+\alpha)}}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}.
\end{aligned} \tag{6.3.24}
$$

Recall that $\alpha = \frac{\sigma-1}{2}$ and denote

$$
N := \frac{1+\alpha}{\sigma-(1+\alpha)} = \frac{\sigma+1}{\sigma-1} (> 3).
$$

Combining (6.3.23) and (6.3.24) with $\varepsilon = C_0^{-1}2^{-3k-12N-1}$, we obtain

$$
\begin{aligned}
&\|\partial_t u\|_{L_\infty(Q^k)} + [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k} \le C2^{3k+(3k+12N)N}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} \\
&\quad + 2^{-12N-1}([u]^x_{\sigma;Q^{k+3}} + \|\partial_t u\|_{L_\infty(Q^{k+3})} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^{k+3}}) \\
&\quad + C2^k \sup_\beta \|f_\beta\|_{L_\infty(Q_1)} + C\sum_{j=1}^\infty \big(2^{3k}\omega_u(2^{-j}) + \omega_f(2^{-j})\big).
\end{aligned}
$$

Then we multiply 2^{-4kN} to both sides of the inequality above and get

$$
\begin{aligned}
&2^{-4kN}(\|\partial_t u\|_{L_\infty(Q^k)} + [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k}) \\
&\le C2^{3k-kN}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + 2^{-4N(k+3)-1}(\|\partial_t u\|_{L_\infty(Q^k)} + [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k}) \\
&\quad + C2^{-4kN+k}\sup_\beta \|f_\beta\|_{L_\infty(Q_1)} + C2^{-kN}\sum_{j=1}^\infty \big(\omega_u(2^{-j}) + \omega_f(2^{-j})\big).
\end{aligned}
$$

We sum up the both sides of the inequality above and obtain

$$
\begin{aligned}
&\sum_{k=1}^\infty 2^{-4kN}(\|\partial_t u\|_{L_\infty(Q^k)} + [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k}) \\
&\le C\sum_{k=1}^\infty 2^{3k-kN}\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} \\
&\quad + \frac{1}{2}\sum_{k=4}^\infty 2^{-4kN}(\|\partial_t u\|_{L_\infty(Q^k)} + [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k}) \\
&\quad + C\sum_{k=1}^\infty 2^{-4kN+k}\sup_\beta \|f_\beta\|_{L_\infty(Q_1)} + C\sum_{j=1}^\infty \big(\omega_u(2^{-j}) + \omega_f(2^{-j})\big),
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
&\sum_{k=1}^\infty 2^{-4kN}(\|\partial_t u\|_{L_\infty(Q^k)} + [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k}) \\
&\le C\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + C\sup_\beta \|f_\beta\|_{L_\infty(Q_1)} + C\sum_{j=1}^\infty \big(\omega_u(2^{-j}) + \omega_f(2^{-j})\big),
\end{aligned}
$$

where C depends on d, λ, Λ, σ, ω_b, N_0, and ω_a. In particular, when $k=4$, we deduce

$$\begin{aligned}&\|\partial_t u\|_{L_\infty(Q^4)}+[u]^x_{\sigma;Q^4}+[Du]^t_{\frac{\sigma-1}{\sigma};Q^4}\\&\le C\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}+C\sup_\beta\|f_\beta\|_{L_\infty(Q_1)}+C\sum_{j=1}^\infty\big(\omega_u(2^{-j})+\omega_f(2^{-j})\big),\end{aligned}\tag{6.3.25}$$

which apparently implies (6.1.6). Finally, since $[v_1]_{\alpha/\sigma,\alpha}, \|Dv_1\|_{L_\infty}, \|\partial_t v_1\|_{L_\infty}$ are bounded by the right-hand side of (6.3.25), from (6.3.16), we see that

$$\sum_{j=0}^\infty 2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0}\inf_{p\in\mathcal{P}_1}[v_1-p]_{\alpha/\sigma,\alpha;Q_{2^{-j}}(t_0,x_0)}\le C.$$

This and (6.3.15) with u replaced by v_1 and f_β replaced by $h_{1\beta}+\eta_1 f_\beta+\partial_t\eta_1 u-b_\beta u D\eta_1$ give

$$\begin{aligned}&\sum_{j=k_1}^\infty 2^{(j+k_0)(\sigma-\alpha)}\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0}\inf_{p\in\mathcal{P}_1}[v_1-p]_{\alpha/\sigma,\alpha;Q_{2^{-j-k_0}}(t_0,x_0)}\\&\le C2^{-k_0\alpha}+C2^{k_0(\sigma-\alpha)}\sum_{j=k_1}^\infty\big(\omega_f(2^{-j})+\omega_a(2^{-j})+\omega_u(2^{-j})+\omega_b(2^{-j})+2^{-j\alpha}\big).\end{aligned}$$

Here we also used (6.3.22) with $k=1$. Therefore, for any small $\varepsilon>0$, we can find k_0 sufficiently large then k_1 sufficiently large, depending only on C, σ, N_0, α, ω_f, ω_a, ω_f, ω_b, and ω_u, such that

$$\sum_{j=k_1}^\infty 2^{(j+k_0)(\sigma-\alpha)}\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0}\inf_{p\in\mathcal{P}_1}[v_1-p]_{\alpha/\sigma,\alpha;Q_{2^{-j-k_0}}(t_0,x_0)}<\varepsilon,$$

which, together with the fact that $v_1=u$ in $Q_{1/2}$ and the proof of Lemma 6.2.1 (ii), indicates that

$$\sup_{(t_0,x_0)\in Q_{1/2}}\Big([u]^x_{\sigma;Q_r(t_0,x_0)}+[Du]^t_{\frac{\sigma-1}{\sigma};Q_r(t_0,x_0)}\Big)\to 0\quad\text{as}\quad r\to 0$$

with a decay rate depending only on d, λ, N_0, Λ, ω_a, ω_f, ω_b, ω_u, $\sup_{\beta\in\mathcal{A}}\|f_\beta\|_{L_\infty(Q_1)}$, and σ. Hence, the proof of the case when $\sigma\in(1,2)$ is completed. $\square$

6.3.3 The case when $\sigma = 1$

Proposition 6.3.4. *Suppose that* (6.1.1) *is satisfied in* $\mathbb{R}^{d+1}_0$. *Then under the condition of Theorem 1.2.3,*

$$\|Du\|_{L_\infty}+\|\partial_t u\|_{L_\infty}\le C\|u\|_{L_\infty}+C\sum_{k=1}^{\infty}\omega_f(2^{-k}),\tag{6.3.26}$$

where $C>0$ is a constant depending only on d, λ, Λ, N_0, ω_a, and ω_b.

Proof. Set $b_0=b(0,0)$ and we define

$$\hat{u}(t,x)=u(t,x-b_0t),\qquad \hat{f}_\beta(t,x)=f_\beta(t,x-b_0t),\quad\text{and}\quad \hat{b}(t,x)=b(t,x-b_0t).$$

It is easy to see that in $\mathbb{R}^{d+1}_0$

$$\partial_t\hat{u}(t,x)=\partial_t u(t,x-b_0t)-b_0\nabla u(t,x-b_0t),$$

and for $(t,x)\in Q_{2^{-k}}$

$$\begin{aligned}|\hat{f}_\beta(t,x)-\hat{f}_\beta(0,0)|&\le\omega_f((1+N_0)2^{-k}),\\ |b-b_0|&\le\omega_b((1+N_0)2^{-k}).\end{aligned}$$

It follows immediately that

$$\hat{u}_t=\inf_\beta(L_\beta\hat{u}+\hat{f}_\beta+(\hat{b}-b_0)\nabla\hat{u}).\tag{6.3.27}$$

Furthermore,

$$\|Du\|_{L_\infty} + \|\partial_t u\|_{L_\infty} \le (1+N_0)\big(\|D\hat{u}\|_{L_\infty} + \|\partial_t \hat{u}\|_{L_\infty}\big),$$

so it is sufficient to bound $\hat{u}$. In the rest of the proof, we estimate the solution to (6.3.27) and abuse the notation to use u instead of $\hat{u}$ for simplicity. We are going to choose ε sufficiently small so that

The proof is similar to the case $\sigma \in (1,2)$ and we indeed proceed as in the previous case. Take p_0 to be the first-order Taylor's expansion of $u^{(2^{-k})}$ at the origin. We also assume that the solution v to the following equation

$$\begin{cases} \partial_t v = \inf_{\beta\in\mathcal{A}}(L_\beta(0,0)v + f_\beta(0,0) - \partial_t p_0) & \text{in } Q_{2^{-k}} \\ v = u - p_0 & \text{in } (-2^{-k\sigma},0)\times B^c_{2^{-k}} \end{cases},$$

exists without carrying out another approximation argument. By Proposition 6.3.1 and Lemma 6.2.4 with $\sigma = 1$,

$$\begin{aligned}
&[v]_{1+\alpha,1+\alpha;Q_{2^{-k-1}}} \le C\sum_{j=1}^{\infty} 2^{k-j} M_j + C2^k[v]_{\alpha,\alpha;Q_{2^{-k}}} \\
&\le C\sum_{j=1}^{\infty} 2^{k-j} \inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;(-2^{-k},0)\times B_{2^{j-k}}} + C2^k[v]_{\alpha,\alpha;Q_{2^{-k}}} \\
&\le C\sum_{j=1}^{k} 2^{k-j} \inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;(-2^{-k},0)\times B_{2^{j-k}}} + C[u]_{\alpha,\alpha} + C2^k[v]_{\alpha,\alpha;Q_{2^{-k}}}. \qquad (6.3.28)
\end{aligned}$$

From (6.3.28) and the interpolation inequality, we obtain

$$
\begin{aligned}
&[v-p_1]_{\alpha,\alpha;Q_{2^{-k-k_0}}} \\
&\le C2^{-(k+k_0)}\sum_{j=1}^{k} 2^{k-j}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;(-2^{-k},0)\times B_{2^{j-k}}} \\
&\quad + C2^{-k_0}[v]_{\alpha,\alpha;Q_{2^{-k}}} + C2^{-(k+k_0)}[u]_{\alpha,\alpha},
\end{aligned}
\tag{6.3.29}
$$

where p_1 is the first-order Taylor's expansion of v at the origin. Next $w := u-p_0-v$ satisfies (6.3.4), where by the cancellation property,

$$
\begin{aligned}
C_k \le{}& \omega_f((1+N_0)2^{-k}) + \omega_b((1+N_0)2^{-k})\|Du\|_{L_\infty} + C\omega_a((1+N_0)2^{-k}) \\
&\cdot\Big(\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0}\sum_{j=0}^{\infty} 2^{j(1-\alpha)}\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big).
\end{aligned}
$$

Clearly, for any $r\ge 0$,

$$
\omega_\bullet((1+N_0)r)\le(2+N_0)\omega_\bullet(r).
$$

Therefore, similar to (6.3.5), we have

$$
\begin{aligned}
&[w]_{\alpha,\alpha;Q_{2^{-k}}} \le C2^{-k(1-\alpha)}\Big(\omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty} \\
&+ \omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}}\sum_{j=0}^{\infty} 2^{j(1-\alpha)}\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big).
\end{aligned}
\tag{6.3.30}
$$

From (6.2.16) and the triangle inequality,

$$
\begin{aligned}
[v]_{\alpha,\alpha;Q_{2^{-k}}} &\le [w]_{\alpha,\alpha;Q_{2^{-k}}} + [u-p_0]_{\alpha,\alpha;Q_{2^{-k}}} \\
&\le [w]_{\alpha,\alpha;Q_{2^{-k}}} + C\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;Q_{2^{-k}}}.
\end{aligned}
$$

Similar to (6.3.8), by combining (6.3.29) and (6.3.30), shifting the coordinate, and using the inequality above, we obtain

$$
\begin{aligned}
&2^{(k+k_0)(1-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-k-k_0}(t_0,x_0)}} \\
&\le C2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{k} 2^{k-j} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;(-2^{-k},0)\times B_{2^{j-k}}} \\
&\quad + C2^{k_0(1-\alpha)}\Big[\omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty} \\
&\quad + \omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha,Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big)\Big] \\
&\quad + C2^{-(k+k_0)\alpha}[u]_{\alpha,\alpha},
\end{aligned}
$$

which by summing in $k = k_1, k_1+1, \ldots$, implies that

$$
\begin{aligned}
&\sum_{k=k_1}^{\infty} 2^{(k+k_0)(1-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\le C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\quad + C[u]_{\alpha,\alpha} + C2^{k_0(1-\alpha)} \sum_{k=k_1}^{\infty} \omega_f(2^{-k}) \\
&\quad + C2^{k_0(1-\alpha)} \sum_{k=k_1}^{\infty} \Big[\omega_b(2^{-k})\|Du\|_{L_\infty} + \omega_a(2^{-k}) \\
&\quad \cdot \Big(\sup_{(t_0,x_0)\in\mathbb{R}^{d+1}_0} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big)\Big],
\end{aligned}
$$

where for the first term on the right-hand side, we replaced j by $k-j$ and switched the order of the summation to get

$$\sum_{k=k_1}^{\infty} 2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in\mathbb{R}_0^{d+1}} \sum_{j=0}^{k} 2^{k-j} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;(t_0-2^{-k},t_0)\times B_{2^{j-k}}(x_0)}$$
$$\leq \sum_{k=0}^{\infty} 2^{-(k+k_0)\alpha} \sum_{j=0}^{k} 2^{j} \sup_{(t_0,x_0)\in\mathbb{R}_0^{d+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} + C[u]_{\alpha,\alpha}$$
$$= 2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j} \sup_{(t_0,x_0)\in\mathbb{R}_0^{d+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \sum_{k=j}^{\infty} 2^{-k\alpha} + C[u]_{\alpha,\alpha}$$
$$\leq C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}_0^{d+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} + C[u]_{\alpha,\alpha}.$$

Therefore,

$$\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}_0^{d+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)}$$
$$\leq C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}_0^{d+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)}$$
$$+ C2^{(k_1+k_0)(1-\alpha)}[u]_{\alpha,\alpha} + C2^{k_0(1-\alpha)} \sum_{k=k_1}^{\infty} \omega_f(2^{-k})$$
$$+ C2^{k_0(1-\alpha)} \sum_{k=k_1}^{\infty} \omega_b(2^{-k})\|Du\|_{L_\infty} + C2^{k_0(1-\alpha)} \sum_{k=k_1}^{\infty} \omega_a(2^{-k})$$
$$\cdot \Big(\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty} \Big).$$

Then we choose k_0 and k_1 sufficiently large, and apply Lemma 6.2.1 (iii) and the interpolation inequality to get (6.3.26). The proposition is proved. □

Next we employ a localization argument as before.

Proof of Theorem 1.2.3 with $\sigma = 1$*.* The proof is very similar to the case when $\sigma \in (1,2)$ and we only provide a sketch here. We use the same notation as in the previous case

$$h_{k\beta}(t,x) = \int_{\mathbb{R}^d} \frac{\xi_k(t,x,y)a_\beta(t,x,y)}{|y|^{d+1}}\,dy,$$

where

$$\xi_k(t,x,y) := u(t,x+y)(\eta_k(t,x+y)-\eta_k(t,x)) - u(t,x)y\cdot D\eta_k(t,x)\chi_{B_1}.$$

It is easy to see that when $|y| \geq 2^{-k-3}$,

$$|\xi_k(t,x,y)| \leq C2^k|y|\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}.$$

On the other hand, when $|y| < 2^{-k-3}$,

$$\begin{aligned}|\xi_k(t,x,y)| &\leq |y|\int_0^1 |u(t,x+y)D\eta_k(t,x+sy) - u(t,x)D\eta_k(t,x)|\,ds\\ &\leq C2^k|y|w_u(|y|) + C2^{2k}|y|^2|u(t,x)|.\end{aligned}$$

Therefore,

$$\|h_{k\beta}\|_{L_\infty} \leq C2^k\Big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + \int_0^1 \frac{w_u(r)}{r}\,dr\Big).$$

Next we estimate the modulus of continuity of $h_{k\beta}$ and proceed as in the case when $\sigma \in (1,2)$. Indeed, recall the definitions of and I and II in (6.3.18). It is easily seen that

$$\mathrm{II} \leq C2^k\Big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + \int_0^1 \frac{\omega_u(r)}{r}\,dr\Big)\,\omega_a(\max\{|x-x'|,|t-t'|\}).$$

To estimate I, we write

$$\begin{aligned}
&\xi_k(t,x,y) - \xi_k(t',x',y) \\
&= u(t,x+y)(\eta_k(t,x+y) - \eta_k(t,x)) - u(t,x)y\cdot D\eta_k(t,x)\chi_{B_1} \\
&\quad - u(t',x'+y)(\eta_k(t',x'+y) - \eta_k(t',x')) + u(t',x')y\cdot D\eta_k(t',x')\chi_{B_1}.
\end{aligned}$$

Obviously, when $|y| \geq 2^{-k-3}$

$$\begin{aligned}
&|\xi_k(t,x,y) - \xi_k(t',x',y)| \\
&\leq C2^{2k}|y|\big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}(|x-x'| + |t-t'|) + \omega_u(\max\{|x-x'|,|t-t'|\})\big).
\end{aligned} \tag{6.3.31}$$

When $|y| < 2^{-k-3}$, we have $\chi_{B_1}(y) = 1$. Thus similar to the previous case,

$$\begin{aligned}
&|\xi_k(t,x,y) - \xi_k(t',x',y)| \\
&\leq |y|\int_0^1 |(u(t,x+y) - u(t,x))D\eta_k(t,x+sy) \\
&\quad - (u(t',x'+y) - u(t',x'))D\eta_k(t',x'+sy)|\,ds \\
&\quad + |y|^2\int_0^1\int_0^1 |u(t,x)D^2\eta_k(t,x+s'sy) - u(t',x')D^2\eta_k(t',x'+s'sy)|\,ds'\,ds \\
&:= \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Clearly,

$$\mathrm{IV} \leq C2^{3k}|y|^2(\omega_u(\max\{|x-x'|,|t-t'|\}) + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}(|x-x'| + |t-t'|)).$$

When $(t,x),(t',x') \in (Q^{k+2})^c$, we have $\mathrm{III} \equiv 0$. When $(t,x),(t',x') \in Q^{k+2}$, by the triangle inequality, III is bounded by

$$
\begin{aligned}
&|y|\int_0^1 |u(t,x+y)-u(t,x)-(u(t',x'+y)-u(t',x'))||D\eta_k(t,x+sy)|\,ds \\
&\quad+|y|\int_0^1 |u(t',x'+y)-u(t',x')||D\eta_k(t,x+sy)-D\eta_k(t',x'+sy)|\,ds \\
&\le C2^k|y|^{1+\gamma}(|x-x'|^\zeta+|t-t'|^\zeta)[u]_{\zeta+\gamma,\zeta+\gamma;Q^{k+3}} \\
&\quad+C2^{2k}|y|\omega_u(|y|)(|x-x'|+|t-t'|),
\end{aligned}
$$

where C depends on d, and $\zeta+\gamma<1$. Here we used the inequality

$$
\begin{aligned}
&|u(t,x+y)-u(t,x)-(u(t',x'+y)-u(t',x'))| \\
&\le C[u]_{\gamma+\zeta,\gamma+\zeta}(|x-x'|^\zeta+|t-t'|^\zeta)|y|^\gamma.
\end{aligned}
$$

Set $\gamma=\zeta=1/4$. When $(t,x)\in Q^{k+2}$ and $(t',x')\in(Q^{k+2})^c$,

$$
\begin{aligned}
\mathrm{III}&=|y|\int_0^1 |(u(t,x+y)-u(t,x))D\eta_k(x+sy)|\,ds \\
&=|y|\int_0^1 |(u(t,x+y)-u(t,x))(D\eta_k(t,x+sy)-D\eta_k(t',x'+sy))|\,ds \\
&\le C2^{2k}|y|\omega_u(|y|)(|x-x'|+|t-t'|).
\end{aligned}
$$

The case when $(t',x')\in Q^{k+2}$ and $(t,x)\in(Q^{k+2})^c$ is similar. Then with the estimates of III and IV above, we obtain that when $|y|<2^{-k-3}$,

$$
\begin{aligned}
&|\xi_k(t,x,y)-\xi_k(t',x',y)| \\
&\le C2^{3k}|y|^2\big(\omega_u(\max\{|x-x'|+|t-t'|\})+\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}(|x-x'|+|t-t'|)\big) \\
&\quad+C2^k|y|^{5/4}(|x-x'|^{1/4}+|t-t'|^{1/4})[u]_{1/2,1/2;Q^{k+3}} \\
&\quad+C2^{2k}|y|\omega_u(|y|)(|x-x'|+|t-t'|),
\end{aligned}
$$

which, combined with (6.3.31) for the case when $|y| \geq 2^{-k-3}$, further implies that

$$
\begin{aligned}
\mathrm{I} \leq C2^{2k}\Big(&\omega_u(\max\{|x-x'|, |t-t'|\}) + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}(|x-x'| + |t-t'|)\\
&+ [u]_{1/2,1/2;Q^{k+3}}(|x-x'|^{1/4} + |t-t'|^{1/4}) + (|x-x'| + |t-t'|)\int_0^1 \frac{w_u(r)}{r}\,dr\Big),
\end{aligned}
$$

where C depends on d and Λ. Hence, we obtain the estimate of the modulus of continuity of $h_{k\beta}(x)$:

$$
\begin{aligned}
\omega_h(r) = C2^{2k}\Big(&\omega_u(r) + [u]_{1/2,1/2;Q^{k+3}} r^{1/4} + \big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}\\
&+ \int_0^1 \frac{\omega_u(r)}{r}\,dr\big)\big(r + \omega_a(r)\big)\Big).
\end{aligned}
$$

Applying Proposition 6.3.4, the rest of the proof is the same as the previous case. $\square$

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