

Abstract of “Ultrasonic vibration potential imaging: theory and experiment” by Cuong Kieu Nguyen, Ph.D., Brown University, May 2009

This thesis aims to elaborate the theory of ultrasonic vibration potential imaging as well as to propose and implement new electro-acoustic imaging methodology for biomedical applications based on ultrasonic vibration potential.

Ultrasonic vibration potential refers to an electric signal developed across a colloidal or ionic solution with a presence of ultrasound. In colloidal suspensions, the vibration potential is generated as a consequence of the different motions of the colloidal particles and the solvent under the alternating acceleration in a sound wave. Upon passage of a sound wave, the normally spherical distribution of counter charge around those particles is periodically distorted resulting in the formation of dipoles at the sites of the colloidal particles. These dipoles add over a macroscopic distance to form a voltage. Thus, if electrodes are placed in the solution, an alternating voltage can be detected.

Since possessing electrolytes and containing red blood cells, whole blood is both colloidal and ionic. Experiments have shown that the whole canine blood produces vibration potentials on the order of about 500 times as large as those from various muscle tissues, which points to application of any biological or medical problem that requires blood detection. This thesis focuses on the detailed analytical explanation and theories of both the generation of current and the distribution of potential and electric field inside colloidal solutions. Experiments with different methodologies and samples were conducted to not only give results in an excellent agreement with the theory but also help proposing and implementing a new imaging method using ultrasonic vibration potential.

# Ultrasonic Vibration Potential Imaging: Theory and Experiment

by

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*Thesis*

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This dissertation by Cuong Kieu Nguyen  
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# Vita

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# Publications

**Cuong K. Nguyen** and Gerald Diebold. Ultrasonic Vibration Potential Imaging: 3D approach and experiments. (in preparation).

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**Cuong K. Nguyen**; Vitalyi Gusev and Gerald Diebold. Potential Distributions from Electroacoustic Polarization Sources. *Applied Physics Letters* **93** 184104 (2008).

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*December, 2008*

Cuong Kieu Nguyen

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*To my parents.*

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# Abstract

This thesis aims to elaborate the theory of ultrasonic vibration potential imaging as well as propose and implement new electro-acoustic imaging methodology for biomedical applications based on ultrasonic vibration potential.

Ultrasonic vibration potential refers to an electric signal developed across a colloidal or ionic solution with a presence of ultrasound. In colloidal suspensions, the vibration potential is generated as a consequence of the different motions of the colloidal particles and the solvent under the alternating acceleration in a sound wave. On passage of a sound wave, the normally spherical distribution of counter charge around those particles is periodically distorted resulting in the formation of dipoles at the sites of the colloidal particles. These dipoles add over a macroscopic distance to form a voltage. Thus, if electrodes are placed in the solution, an alternating voltage can be detected.

Since possessing electrolytes and containing red blood cells, whole blood is both colloidal and ionic. Experiments have shown that the whole canine blood produces vibration potentials on the order of about 500 times as large as those from various muscle tissues, which points to application of any biological or medical problem that requires blood detection. This thesis focuses on the detailed analytical explanation and theories of both the generation of current and the distribution of potential and electric field inside colloidal solutions. Experiments with different methodologies and samples were conducted to not only give results in an excellent agreement with the

theory but also help proposing and implementing a new imaging method using ultrasonic vibration potential.

# Chapter 1

## Introduction to imaging using ultrasonic vibration potential

### 1.1 Basic concepts and history

Ultrasonic vibration potentials refer to voltages that are developed across a colloidal or ionic solution as a result of the passage of ultrasound. These alternating potentials were first predicted for simple ionic solutions (hence called ionic vibration potentials) by Debye<sup>1</sup> in 1933 and were suggested by him as a means for studying ion-solvent interactions. Debye realized that, in the presence of a longitudinal sound wave, any differences in the effective mass or friction coefficient between anions and cations would result in different displacement amplitudes. In turn, this difference in displacement would create an alternating electric potential between points within the solution. Indeed, this phenomenon, usually referred to as an "Ion Vibration Potential", is measurable and can yield useful information about the properties of ions.

The first experimental observations of this ionic vibration potentials were reported by Yeager<sup>2</sup> in 1949 and then two years later by Derouet and Denizot<sup>3</sup> in 1951, some 16 years or more after Debye's original prediction. This delay reflects, in part, difficulties

associated with the detection and measurement of the effect principally because of its small magnitude (less than  $10 \mu\text{Vcm}^{-1}\text{sec}$ ). Substantial improvements in apparatus have made possible quantitative measurements of the ionic vibration potential with an accuracy of  $\pm 5\%$ . Data obtained with this apparatus have been used for the direct evaluation of the partial molal volumes for individual ions<sup>4;5</sup>. A thorough theoretical treatment of the ionic vibration potential was given in 1947 by Bugosh et al<sup>6</sup>. The results of two decade effort 1950's and 1960's of experiments on ionic vibration potentials was summarized by Zana and Yeager<sup>7</sup> in 1982. Sadly, this phenomenon has been largely forgotten, and virtually all of the interest in electroacoustic phenomena has shifted from pure electrolyte solutions to colloids.

AC potentials generated in colloidal suspensions, the so called colloid vibration potentials, were first predicted by Rutgers<sup>8</sup> in 1938 and have been treated theoretically by other workers<sup>9;10</sup>. Enderby and Booth<sup>11;12</sup> developed the first theory for the colloid vibration potential in the early 1950's. The first quantitative experiments were made in 1960 by Yeager's group<sup>7</sup>. The large magnitude of colloidal vibration potentials (up to  $10^{-2} \text{Vcm}^{-1}\text{sec}$ ) has facilitated the quantitative measurement of this effect<sup>13-15</sup>.

The further investigations of the ultrasonic vibration potentials, both theory and experiments, can be found in reviews by Babchin et al<sup>16</sup> in 1989 and Povey<sup>17</sup> in 1997 as well as in the work of O'Brien and coworkers<sup>18? -21</sup>, who have given the most sophisticated treatment of the vibration potential to date.

Imaging using ultrasonic vibration potentials is a new imaging method based on recording ultrasonic vibration potentials or currents, which aims to reconstruct images of colloidal objects. Several experimental papers have been published by Beveridge et al<sup>22;23</sup>. The theory of imaging based on the ultrasonic vibration potential has been given by Gusev and Diebold<sup>24</sup> as well as Beveridge et al<sup>25</sup>.

## 1.2 Production of electro-acoustic voltages

Let us take a close look at how an electric signal is produced across colloidal objects in the presence of ultrasound. Colloidal solutions are basically solutions containing suspensions of small particles typically in the range of nm to  $\mu\text{m}$ . Since the particles in a colloidal suspension have chemically attached charge, there is normally a spherically symmetric counter charge surrounding each particle, giving the suspension overall charge neutrality (see Figure 1.1).

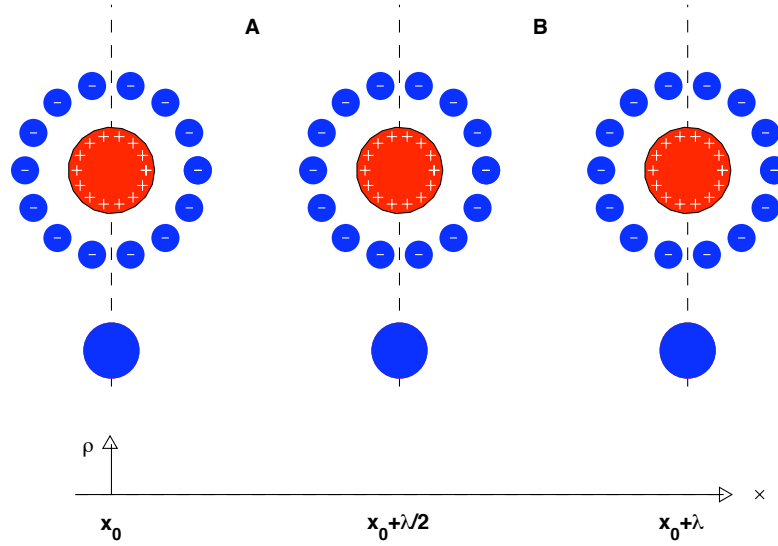


Figure 1.1: Schematic diagram of colloidal particles. The overall charge of colloidal suspension is normally neutral.

When an ultrasonic wave with wavelength  $\lambda$  traverses the solution, it produces periodic displacements for both the colloidal particles and the ionic layer surrounding those particles. Because of the differences in the apparent masses and frictional coefficients, those displacements are different: for example, light, negative ions in the

fluid would move much more than heavy, positive colloidal particles. The differences in motion of the particles and counter charges cause the distortion of the normally spherical charge distribution, resulting in the formation of local dipoles at the sites of particles in the suspension.

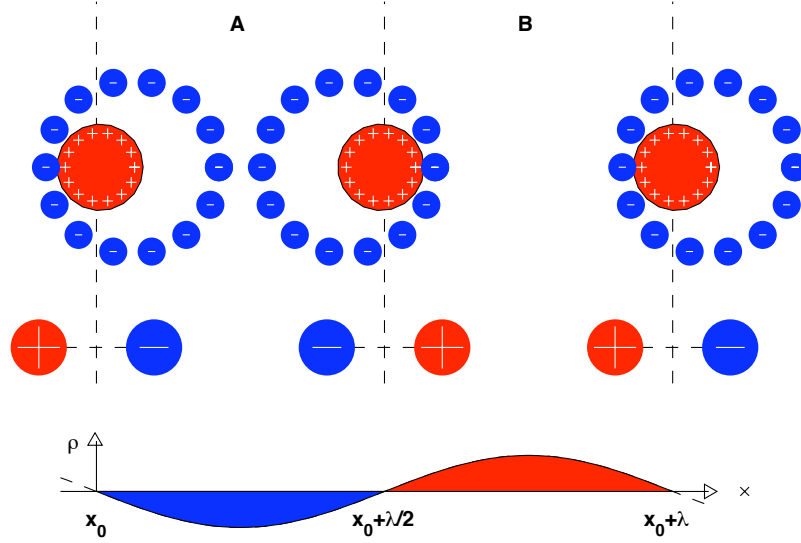


Figure 1.2: Diagram of colloidal particles and counter charge in the presence of ultrasound. At this point in the acoustic cycle, region A will be charged negatively relative to region B. The blue circles represent negative charges, and red circles are positive charges. The top big red circles are colloidal particles surrounded by small negative counter ions. The middle pair of circles represents dipoles formed when ultrasound propagated through the system.

Consider two colloidal particles at arbitrary points  $x_0$  and  $x_1$  which are separated by one half of a wavelength  $\lambda$  as shown in Figure 1.1. Thus,  $x_1 = x_0 + \lambda/2$ . The introduction of acoustic waves into the solution causes formation of two dipoles oscillating out of phase to each other. So, for the conditions represented in Figure 1.2, region A will be charged negatively relative to region B.



In the very next half period, those dipoles will move to be of opposite phase again, making region A positively charged relative to region B (see Figure 1.3). It is obvious that an alternating potential, referred to as the ultrasonic vibration potential, with the same frequency of ultrasound is formed between regions A and B, and that a pair of electrodes placed in those two regions can detect an alternating voltage.

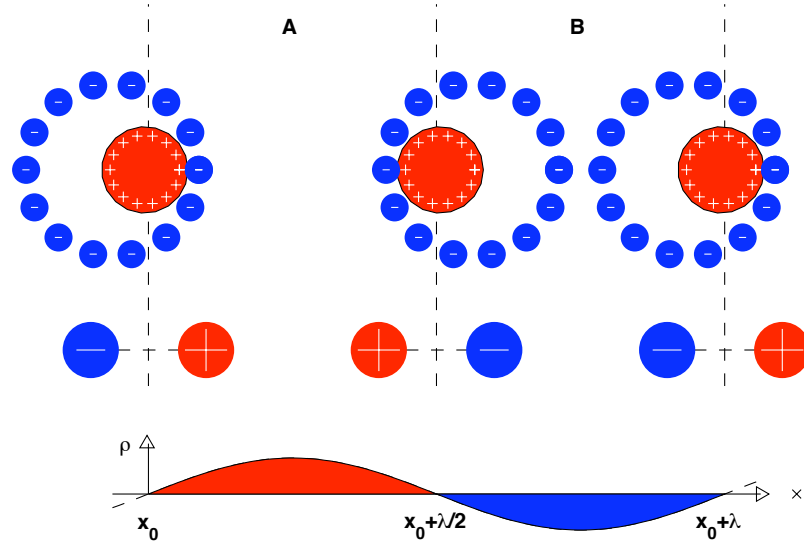


Figure 1.3: The same particles as shown in Figure 1.2 a half of period later. Region A is now charged positively relative to region B. The difference of charge concentration leads to an alternating voltage between regions A and B giving the so-called ultrasonic vibration potential.

### 1.3 Ultrasound imaging

Ultrasound refers to sound waves of high frequency that lies outside the human audible range. Music and common sounds that we consider pleasant are typically 12

kHz or less. Some humans can hear frequencies up to 20 kHz; thus ultrasonic waves are those with frequencies greater than 20 kHz. Ultrasonic waves are used in many applications including plastic welding, medicine, jewelry cleaning, and nondestructive testing. In the case of nondestructive testing, ultrasonic waves give us the ability to *see through* solid or opaque material and detect surface or internal flaws without affecting the material in an adverse manner.

Diagnostic sonography (or medical ultrasonography) is a diagnostic imaging technique based on ultrasound used to visualize muscles and internal organs to probe their size and structures as well. Typical diagnostic sonographic scanners operate in the frequency range of 2 to 18 MHz, hundreds of times greater than the limit of human hearing. The choice of frequency is a trade-off between spatial resolution of the image and imaging depth: lower frequencies produce less resolution but image deeper into the body.

### 1.3.1 Basic properties of ultrasonic waves

This section summarizes some basic properties of ultrasonic waves with respect to ultrasound and ultrasonic imaging applications. Being a harmonic wave, an ultrasound can be described by a sinusoidal function,

$$f(x, t) = A \sin(\omega t - kx) , \quad (1.1)$$

where  $A$  is the amplitude of the wave,  $\omega$  is an angular frequency which presents the frequency of the wave in terms of radians per second,  $k$  is a wavenumber,  $t$  is time and  $x$  is distance.

For an ultrasonic wave,  $f(x, t)$  can be a displacement of a component or particle in the medium, or a pressure in the direction of propagation of the wave. In the case of the former, amplitude  $A$  is a measure of the maximum displacement from the

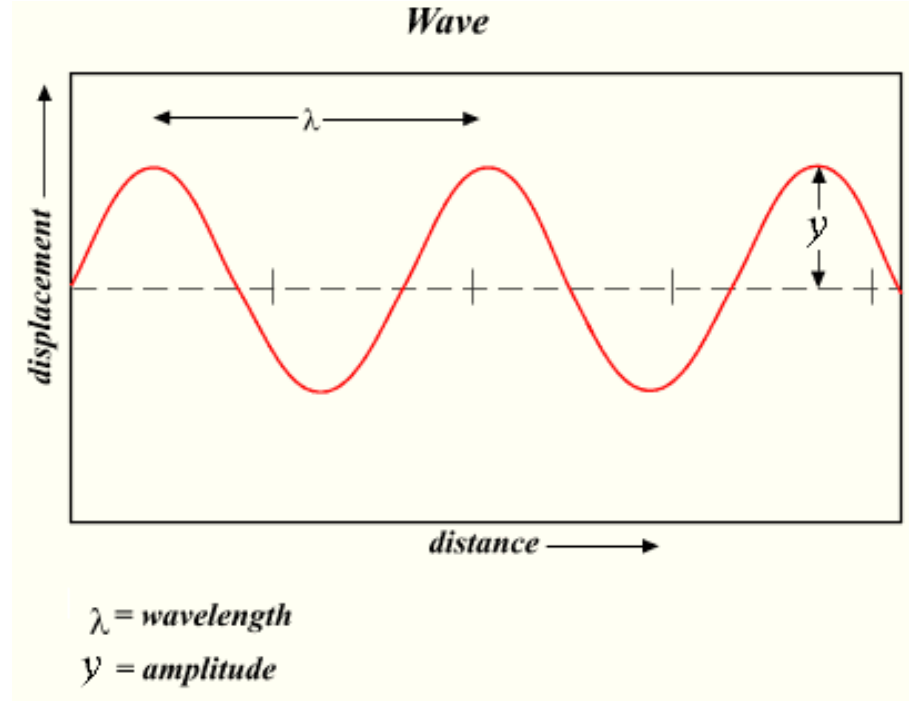


Figure 1.4: A mathematic diagram of an ultrasonic wave.

| Ratio $A_1/A_2$ | dB |
|-----------------|----|
| 1.4142          | 3  |
| 2               | 6  |
| 4               | 12 |
| 10              | 20 |
| 100             | 40 |
| 1000            | 60 |

Table 1.1: Conversion of dB and ratio of two amplitudes.

equilibrium position of a particle during one wave cycle.

In ultrasonic applications, the relative change in signal amplitude is commonly measured in decibels (dB). Decibel values are the logarithmic value of the ratio of two signal amplitudes. This can be calculated using the following equation:

$$\text{dB} = 20 \log_{10} \left( \frac{A_1}{A_2} \right), \quad (1.2)$$

where dB is decibels,  $A_1$  and  $A_2$  are amplitudes of signal 1 and 2 correspondingly. Some useful relationships between two signal amplitudes are displayed in Table (1.1).

Other characteristics of ultrasonic waves include the period  $T$ , the frequency  $f$  (also denoted as  $\nu$ ) and the wavelength  $\lambda$ . The period  $T$  is the time for one complete cycle for an oscillation of a wave. The frequency  $f$  is the number of periods per unit time, measured in hertz (Hz), and is related to the period and an angular frequency  $\omega$  by the following formula

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \quad (1.3)$$

The wavelength is the distance between two sequential crests and can be expressed in units of meters. The wavelength and wavenumber  $k$  are associated by the relation

$$k = \frac{2\pi}{\lambda}. \quad (1.4)$$

Sound speed  $c$  is the velocity that the wave propagates through the medium. The following obtains for linear acoustics

$$c = \frac{\omega}{k} = \lambda f. \quad (1.5)$$

Sound speed depends mainly on the properties and temperature of the medium in which the sound travels; it is generally given by

$$c = \sqrt{\frac{C}{\rho}}, \quad (1.6)$$

where  $C$  is a coefficient of stiffness and  $\rho$  is the density of the medium. Specifically for ideal gas then the sound speed is

$$c_{gas} = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\frac{\gamma R T}{\mu}} = \sqrt{\frac{\gamma k T}{m}}, \quad (1.7)$$

where  $\gamma$  is the heat capacity ratio for the medium,  $p$  is the pressure,  $R = 8.3145 \text{ J mol}^{-1} \text{ K}^{-1}$

is the molar gas constant,  $T$  is the absolute temperature in kelvins ( $^{\circ}\text{K}$ ),  $\mu$  is the molar mass in kilograms per mole,  $k = 1.380 \times 10^{-23} \text{ JK}^{-1}$  is the Boltzmann constant, and  $m$  is the mass of a single molecule in kilograms. If  $\tau$  is the temperature in degrees Celsius ( $^{\circ}\text{C}$ ) and is small compared to 273.15, then in dry air, we can easily find

$$c_{air} = 331.3 + 0.606 \times \tau, \quad (1.8)$$

where  $c_{air}$  is in m/s. For instance the sound speed in air at  $25^{\circ}\text{C}$  is 346.45 m/s.

In liquids, the only non-zero stiffness is the volumetric deformation, hence the speed is given by

$$c_{liquids} = \sqrt{\frac{K}{\rho}}, \quad (1.9)$$

where  $K$  is the bulk modulus of the liquid. In fresh water, sound travels at about 1497 m/s at  $25^{\circ}\text{C}$ .

In a solid, there is a non-zero stiffness both for volumetric and shear deformations. Hence in solid it is possible to generate sound wave with different velocities depending on the deformation mode. A sound wave generating volumetric deformations is called longitudinal wave. A transverse wave generates shear deformations. The velocities of these two different sound waves can be calculated in isotropic solids by

$$c_{long} = \sqrt{\frac{E}{\rho}}, \quad (1.10)$$

and

$$c_{trans} = \sqrt{\frac{G}{\rho}}, \quad (1.11)$$

where  $E$  is Young's modulus and  $G$  is the shear modulus. For example the longitudinal

sound speed in steel is about 5100 m/s.

Finally, the last but very important concept for ultrasonic wave propagation is the acoustic impedance  $Z$ , which is given by

$$Z = \rho c . \quad (1.12)$$

For reference, air has low acoustic impedance, water has higher impedance than air, and steel has higher impedance than water. Some acoustic properties of different materials are listed on Table (1.2)\*<sup>26</sup>.

When performing ultrasonic experiments, it is important to understand how effectively ultrasonic waves pass from one medium to another. Generally, when an ultrasonic wave is passed from one medium to another, some energy is reflected and the remaining energy is transmitted. The factor that describes this relationship is referred the acoustic impedance ratio. The acoustic impedance ratio is the impedance of the second material divided by the first. The higher the ratio, the more energy is reflected. For example, when ultrasonic waves are passed from water to steel, the acoustical impedance is approximately 20 to 1; whereas, when ultrasonic waves are passed from air to steel, the acoustical impedance is approximately 100,000 to 1. Almost 100% of the ultrasonic energy will be reflected when passing ultrasonic waves from air to a solid such as steel, making air a very poor ultrasonic couplant.

### 1.3.2 Fundamentals of conventional ultrasonic imaging

Ultrasonic waves, as well as electro-magnetic waves (visible light, x-rays etc), have many similar physical properties: they can be focused, reflected, and refracted. Because sound is transferred through media by the vibration of particles of that medium, ultrasonic waves can travel through gases, fluids, and solids such as steel. In a ho-

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\* Conversion factor:  $1 \text{ m/s} = 3.937 \times 10^{-5} \text{ in}/\mu\text{S}$

| Material                                 | Longitudinal velocity |        | Shear velocity       |        | Acoustic impedance<br>( $10^6 \text{ kg/m}^2\text{s}$ ) |
|------------------------------------------|-----------------------|--------|----------------------|--------|---------------------------------------------------------|
|                                          | (in/ $\mu\text{s}$ )  | (m/s)  | (in/ $\mu\text{s}$ ) | (m/s)  |                                                         |
| Acrylic resin (Perspex <sup>®</sup> )    | 0.107                 | 2,370  | 0.056                | 1,430  | 3.22                                                    |
| Aluminum                                 | 0.249                 | 6,320  | 0.123                | 3,130  | 17.06                                                   |
| Beryllium                                | 0.508                 | 12,900 | 0.350                | 8,880  | 23.5                                                    |
| Brass, naval                             | 0.174                 | 4,430  | 0.083                | 2,120  | 37.30                                                   |
| Cadmium                                  | 0.109                 | 2,780  | 0.059                | 1,500  | 24.02                                                   |
| Columbium                                | 0.194                 | 4,920  | 0.083                | 2,100  | 42.16                                                   |
| Copper                                   | 0.183                 | 4,660  | 0.089                | 2,260  | 41.61                                                   |
| Glycerine                                | 0.076                 | 1,920  | -                    | -      | 2.42                                                    |
| Gold                                     | 0.128                 | 3,240  | 0.047                | 1,200  | 62.60                                                   |
| Inconel <sup>®</sup>                     | 0.29                  | 5,820  | 0.119                | 3,020  | 49.47                                                   |
| Iron                                     | 0.232                 | 5,900  | 0.127                | 3,230  | 45.43                                                   |
| Iron, cast                               |                       |        |                      |        |                                                         |
| (slow)                                   | 0.138                 | 3,500  | 0.087                | 2,200  | 25.00                                                   |
| (fast)                                   | 0.220                 | 5,600  | 0.126                | 3,220  | 40.00                                                   |
| Lead                                     | 0.085                 | 2,160  | 0.028                | 700    | 24.49                                                   |
| Manganese                                | 0.183                 | 4,660  | 0.093                | 2,350  | 34.44                                                   |
| Mercury                                  | 0.057                 | 1,450  | -                    | -      | 19.66                                                   |
| Molybdenum                               | 0.246                 | 6,250  | 0.132                | 3,350  | 63.75                                                   |
| Motor Oil (SAE 20 or 30)                 | 0.069                 | 1,740  | -                    | -      | 1.51                                                    |
| Nickel, pure                             | 0.222                 | 5,630  | 0.117                | 2,960  | 49.99                                                   |
| Platinum                                 | 0.156                 | 3,960  | 0.066                | 71,670 | 84.74                                                   |
| Polyamide, (nylon, Perlon <sup>®</sup> ) |                       |        |                      |        |                                                         |
| (slow)                                   | 0.087                 | 2,200  | 0.043                | 1,100  | 0.40                                                    |
| (fast)                                   | 0.102                 | 2,600  | 0.047                | 1,200  | 3.10                                                    |
| Polystyrene                              | 0.092                 | 2,340  | -                    | -      | 2.47                                                    |
| Polyvinylchloride,PVC, hard              | 0.094                 | 2,395  | 0.042                | 1,060  | 3.35                                                    |
| Silver                                   | 0.142                 | 3,600  | 0.063                | 1,590  | 37.76                                                   |
| Steel, 302                               | 0.223                 | 5,660  | 0.123                | 3,120  | 45.45                                                   |
| austenitic stainless Steel, 347          | 0.226                 | 5,740  | 0.122                | 3,090  | 45.40                                                   |
| austenitic stainless Tin                 | 0.131                 | 3,320  | 0.066                | 1,670  | 24.20                                                   |
| Titanium, Ti 150A                        | 0.240                 | 6,100  | 0.123                | 3,120  | 27.69                                                   |
| Tungsten                                 | 0.204                 | 5,180  | 0.113                | 2,870  | 99.72                                                   |
| Uranium                                  | 0.133                 | 3,370  | 0.078                | 1,980  | 63.02                                                   |
| Water (20 °C)                            | 0.058                 | 1,480  | -                    | -      | 1.48                                                    |
| Zinc                                     | 0.164                 | 4,170  | 0.095                | 2,410  | 29.61                                                   |
| Zirconium                                | 0.183                 | 4,650  | 0.089                | 2,250  | 30.13                                                   |

Table 1.2: Acoustic properties of materials.

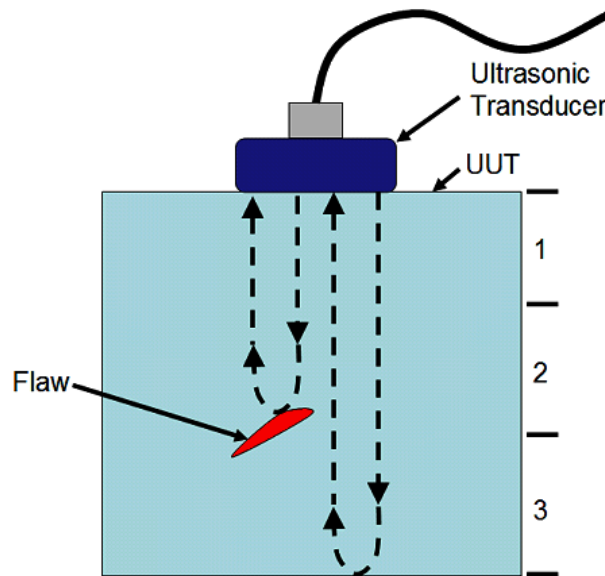


Figure 1.5: Diagram of apparatus for flaw detection using ultrasound.

in a homogeneous medium, ultrasonic waves travel just as light travels through a vacuum. But when traveling from one medium to another, on the boundary of the media, part of the ultrasonic wave is reflected, and the other part of the wave gets through the medium. The directed energy in an ultrasonic wave therefore is reflected by the boundaries between materials or media regardless of whether the material is gas, liquid, or solid. Ultrasonic waves are also reflected by any cracks or voids in solid materials. These reflected waves, which are caused by internal defects, can be compared to the reflected waves from the external surfaces, enabling the size and severity of internal defects to be identified.

Let us consider a basic ultrasonic test of mechanism used to detect internal defects or flaws inside the sample (we call it the unit under test, UUT) as shown in Figure 1.5. The ultrasonic transducer is used to generate and detect ultrasonic waves. In the ultrasonic transducer, piezoelectric ceramics are excited with electricity, typically between 50 and 1000 Volts - to produce the ultrasonic wave. This ultrasonic wave, which is carried from the transducer to the unit under test by a couplant - typically water, oil or gel, is reflected back to the transducer by both external surfaces and



internal defects (the red object in the figure).

In order to detect ultrasonic waves, the transducers have to be operated in pulse-echo mode and hence act as both emitters and receivers. The reflected ultrasonic waves vibrate the piezoelectric crystal within the transducer and generate voltages across the crystal that can be measured by a data acquisition systems. One can also detect reflected waves by operating in through-transmission mode, which gives transducers the function of an emitter only. In this case, at least two transducers are used, one to generate the wave and one to receive the reflection.

The most primitive method to analyze reflected ultrasonic signals is with a time-of-flight (TOF) display, or A-Scan. Discontinuities that are closer to the ultrasonic transducer are received sooner than those further away from the transducer. Figure 1.6 depicts the TOF display from a basic ultrasonic A mode scan.

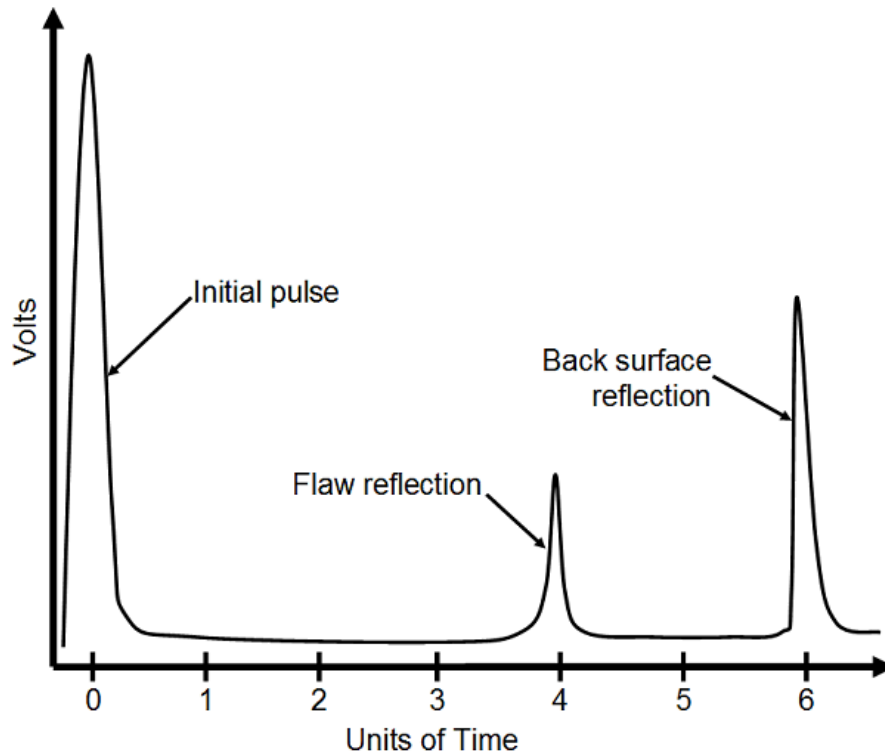


Figure 1.6: A-Scan method used to analyze reflected ultrasonic waves.

The x-axis on the A-scan is typically in units of time, and is usually converted

to distance using the sound speed. Although there are a few exceptions, the speed of sound through a material is governed largely by the density and elasticity of the material. For most materials, the speed of sound in homogenous materials is easy to research and find.

Most ultrasonic nondestructive testing applications range from 400 kHz to 25 MHz. The frequency of the ultrasonic sensor is chosen based on several factors including detectable flaw size, depth of penetration, and grain size of the material. Materials made up of fine grained material, such as metals, permit deep penetration by ultrasonic waves of all frequencies. However, coarse-grained material, including many plastics, scatter high frequency ultrasonic waves. The higher the frequency, the smaller the flaws that the system will detect, but the depth of penetration decreases.

## 1.4 Ultrasonic vibration potential imaging

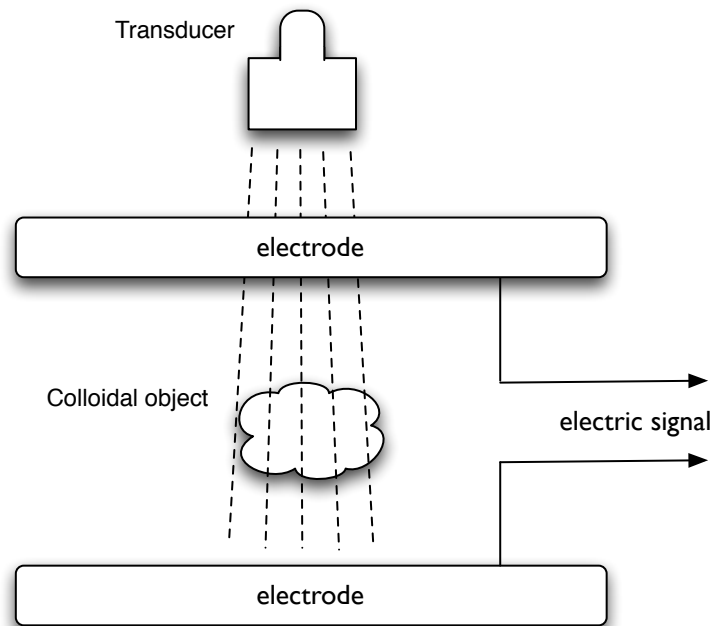


Figure 1.7: Schematic diagram of the electro-acoustic imaging method.

The ultrasonic vibration potential has the potential of a medical imaging method

as whole blood is both colloidal and ionic. Electro-acoustic imaging is obviously different from the conventional ultrasonic imaging in both nature and methodology.

Figure 1.7 shows the basic apparatus of a system for use of the ultrasonic vibration potential to detect a colloidal object. The transducer sends the sound throughout the whole system containing the colloidal object. The ultrasound causes an electric polarization to be developed across the object. With two electrodes placed to cover the colloidal object, we can easily capture an electric current or voltage produced by the colloidal object. The electric signal then can be analyzed based on a theory of ultrasonic vibration potential and with the help of computer programs, used to determine unknown characteristics of the object such as its dimensions, its depth in the system etc. In principle, it is also possible to use some techniques such as tomography to reconstruct the shape of the object.

## 1.5 Comparison between sonography and electro-acoustic imaging

As noted above, electro-acoustic imaging and conventional ultrasound imaging are much different from each other. Therefore both techniques will have certain advantages and disadvantages for tissue imaging that we will discuss as follows.

**Mechanism.** Conventional pulse-echo imaging method is a purely acoustic method, whereas the ultrasonic vibration potential is an electro-acoustic method. While the conventional ultrasound method uses a transducer to produce sound and then also uses that transducer to be a receiver to detect the echo, the electro-acoustic one uses the transducer to produce sound but then uses a pair of electrodes to detect the electric current developed in the body of interest. In conventional imaging method, mechanical energy, or sound energy is used and then mechanical energy is detected - there is no conversion of energy in the body. On the other hand, in electro-acoustic

imaging the mechanical energy in the ultrasound wave is converted to electrical energy directly by formation of a polarization.

As noted in section 1.1, the ultrasonic signal was so small that it took around 16 years or more after the prediction of Debye to first successfully detect the ultrasonic vibration potential in experiments. Depending on the colloids, the magnitude of the ultrasonic vibration potential signal ranges from  $\mu\text{V}$  to at most a few  $\text{mV}$ .

**Materials.** The nature of electro-acoustic imaging requires an object to be made of a colloidal or ionic material, whereas there is no actual requirement of materials for the conventional imaging method except that the object and body must have different acoustic impedance. This requirement obviously limits the applications where the ultrasonic vibration potential can be used.

**History.** Ultrasound was first discovered in 1974 by Lazzaro Spallanzani, an Italian biologist who demonstrated the ability of bats to navigate accurately in the dark through high frequency echo reflection from inaudible sound. The principle of piezoelectricity was discovered by Jacque Curie<sup>27</sup> which eventually led to the creation of the transducer. In the 1950s and 1960s Ian Donald, a Scottish doctor, developed practical ultrasonic technology for application in his research in obstetrics<sup>28–32</sup>.

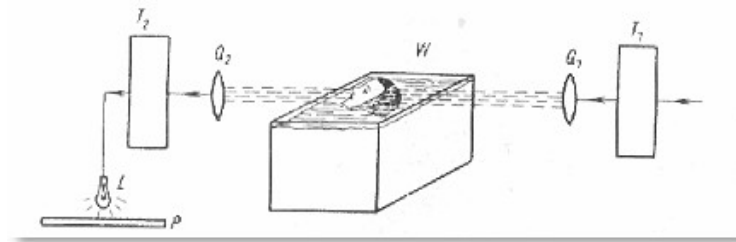


Figure 1.8: Diagram of the hyperphonograph of Dr. Dussik used to detect brain tumors. In the figure,  $T_1$  is an ultrasonic generator,  $T_2$  is a converter amplifier,  $Q_1$  is a transmitter,  $Q_2$  is a receiver,  $W$  is a water bath where the patient was placed,  $L$  is light and  $P$  is photographic-heat-sensitive paper.

The first use of diagnostic ultrasound was conducted in 1942 by Dr. Karl Dussik<sup>33</sup>,

a psychiatrist at the hospital in Bad Ischl, Austria. Dr. Dussik tried to locate brain tumors with a new imaging technique that he called *Hyperphonography*: a patient was placed between an ultrasound emitter and an ultrasound receiver as shown in Figure 1.8. After more than 60 years of research and development, sonography is now a mature research area and is widely used in hospitals and physical optics all over the world.

Although the effect of ultrasonic vibration potential was predicted more than 70 years ago, the ultrasonic vibration potential imaging is quite young compared with conventional imaging method. The idea of ultrasonic vibration potential imaging was first proposed by Diebold's group at Brown university around 2002. Since then several experiments have been conducted. First results of ultrasonic vibration potential imaging, both in experimental and theoretical, are reported in references 22–25.

# Chapter 2

## Instruments' overview

### 2.1 Transducers

A transducer is a device that converts one type of energy or physical attribute to another for various purposes including measurement or information transfer. However, most transducers convert electrical to other forms of energy. Transducers, depending on their usage, are also referred to as probes, detectors, pickup, sensors, gauges and electrodes.

Transducers that detect or transmit physical information include microphones, Geiger meters, potentiometers, pressure sensors, thermometers, and antennae. A microphone, for example, converts sound waves that strike its diaphragm into an electrical signal that can be transmitted over wires. A pressure sensor typically turns the physical force exerted on its sensing surface into an analog voltage that can be recorded with a volt-meter.

Most transducers have an inverse capability that allows for the converted energy to be returned to its original form. Audio cassettes, for example, operate by using a transducer to turn the electrical signal from a microphone into electrical energy which, in turn, is converted into magnetic fluctuations on the tape head which are

recorded in the tape as magnetization of ferromagnetic particles. These magnetic fluctuations can be read out and converted by another magnetic transducer into an electrical signal, which is then fed by wire to speakers, which act as yet another transducer to turn the electrical signal back into audio waves.

Other transducers convert one type of energy into another form, not for the purpose of measuring something in the external environment or to communicate information, but rather to make use of that energy in a more productive manner. A light bulb, for example, one of the many transducers around us in our day-to-day lives, converts electrical energy into visible light. Electric motors are electromechanical transducers that convert electrical energy into kinetic energy to perform mechanical tasks. The "inverse" of an electric motor is a generator, which is also a transducer, turning kinetic energy into electrical energy that can then be used by other devices.

As in all energy conversions, some energy is lost when the transducer operates. The efficiency of a transducer is found by comparing the total energy put into it to the transduced energy. Some transducers are very efficient, while others are extraordinarily inefficient. A radio antenna, for example, acts as a transducer to turn radio frequency waves into an oscillating current which can be upwards of 80 % efficient. Most electrical motors, by contrast, are well under 50 % efficient, and a common light bulb, because of the amount of energy lost as heat, is less than 10 % efficient.

### **2.1.1 Ultrasonic transducers**

The transducers used in all of the ultrasonic vibration potential experiments reported here are piezoelectric transducers.

In a piezoelectric transducer, the active element is basically a piece of polarized material (i.e. some parts of the molecule are positively charged, while other parts of the molecule are negatively charged) with electrodes attached to two of its opposite faces. When an electric field is applied across the material, the polarized molecules

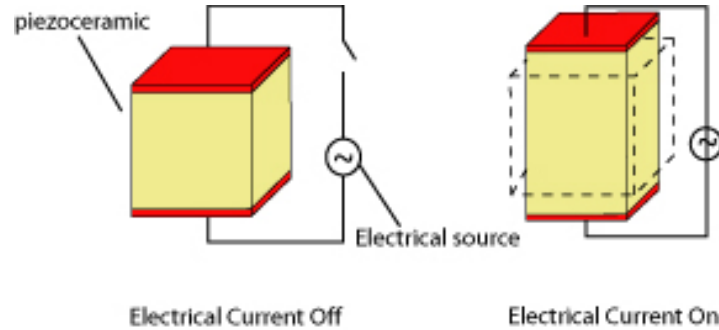


Figure 2.1: Illustration of the piezoelectric effect.

align themselves with the electric field, resulting in induced dipoles within the molecular or crystal structure of the material. This alignment of molecules causes the material to change dimensions. In addition, a permanently-polarized material such as quartz ( $\text{SiO}_2$ ) or barium titanate ( $\text{BaTiO}_3$ ) will produce an electric field when the material changes dimensions as a result of an imposed mechanical force. This phenomenon is known as the piezoelectric effect.



Figure 2.2: A side view of a piezoelectric transducer.

The active element of most acoustic transducers used today is a piezoelectric ceramic, which can be cut in various ways to produce different wave modes. In Figure 2.2, a large piezoelectric ceramic element can be seen in the image of a sectioned low frequency transducer. Preceding the advent of piezoelectric ceramics in the early 1950's, piezoelectric crystals made from quartz crystals and magnetostrictive materials were primarily used for production of ultrasound. When piezoelectric ceramics



were introduced, they soon became the dominant material for transducer construction due to their good piezoelectric properties and their ease of manufacture into a variety of shapes and sizes. They are also usable up to about 300 °C. The first piezoceramic in general use was barium titanate, which that was followed during the 1960's by lead zirconate titanate compositions, which are now the most commonly employed ceramic for making transducers. New materials such as piezo-polymers and composites are also being used in some applications.

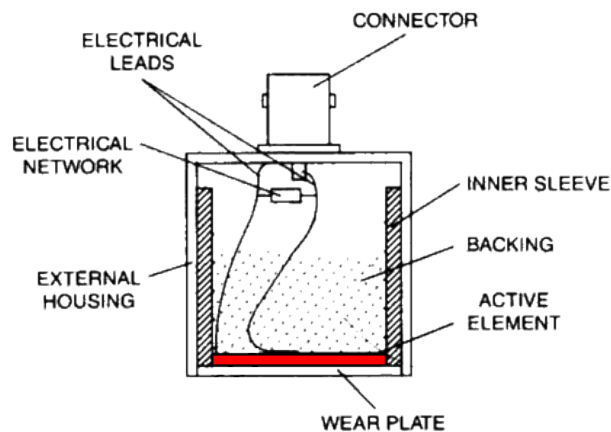


Figure 2.3: A diagram of a typical transducer.

The thickness of the active element determines the oscillation frequency of the transducer. A thin wafer element vibrates with a wavelength that is twice its thickness. Therefore, piezoelectric crystals are cut to a thickness that is  $1/2$  the desired radiated wavelength. The higher the frequency of the transducer, the thinner the active element. The primary reason that high frequency contact transducers are not produced is because the element must be very thin causing it to be fragile.

A diagram and a cut away of a typical contact transducer are shown in Figure (2.3) and Figure (2.4). It was noted above that the piezoelectric element is cut to  $1/2$  the desired wavelength. To get as much energy out of the transducer as possible, an impedance matching layer is placed between the active element and the face of the transducer. Optimal impedance matching is achieved by sizing the matching layer

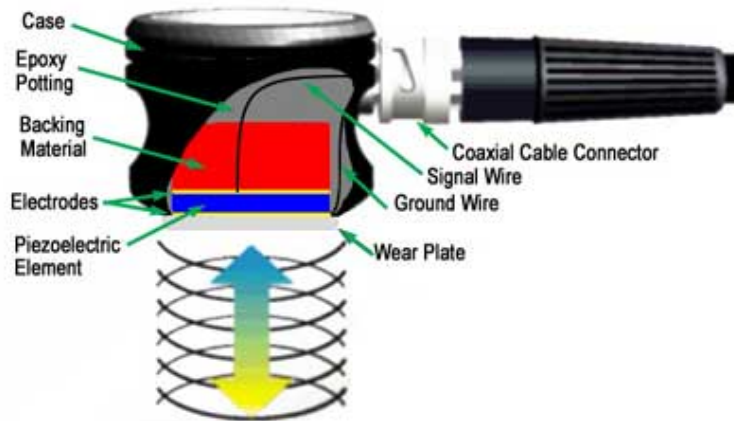


Figure 2.4: An internal view of a typical transducer.

so that its thickness is  $1/4$  of the desired wavelength. This keeps waves that were reflected within the matching layer in phase when they exit the layer as illustrated in Figure (2.5). For contact transducers, the matching layer is made from a material that has an acoustical impedance between the active element and irradiated medium. Immersion transducers have a matching layer with an acoustical impedance between the active element and water. Contact transducers also incorporate a wear plate to protect the matching layer and active element from scratching.

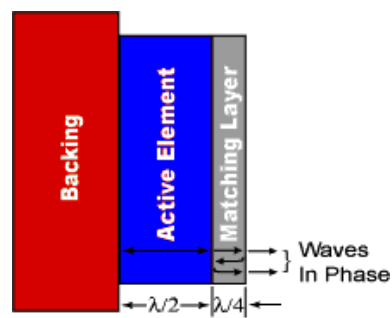


Figure 2.5: Optimal impedance matching layer.

The backing material supporting the crystal has a great influence on the damping characteristics of a transducer. Using a backing material with an impedance similar to that of the active element will produce the most effective damping. Such a transducer

will have a wider bandwidth than a single slab of ceramic.

### Waveform, spectrum and bandwidth

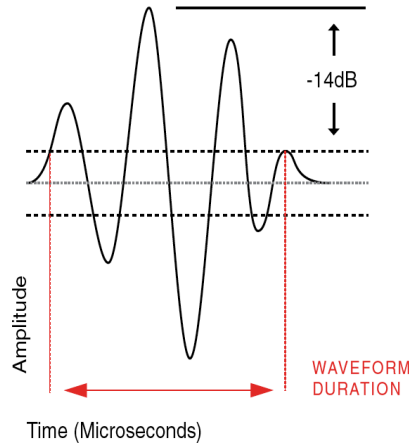


Figure 2.6: Waveform duration at the  $-14\text{ dB}$  level of a transducer.

Some important characteristics of a transducer are its impulse response waveform and frequency response spectrum. Typical units for those characteristics are MHz for frequency analysis,  $\mu\text{s}$  for waveform analysis and dB down from peak amplitude. Figure (2.6) illustrates the waveform duration at the  $-14\text{ dB}$  level or 20 % amplitude of peak. The  $-40\text{ dB}$  waveform duration corresponds to 1 % amplitude of peak.

It is very important to understand the concept of bandwidth, or range of frequencies, associated with a transducer. The frequency noted on a transducer is the central or center frequency which depends primarily on the sound speed and thickness of the transducer material. Highly damped transducers will respond to frequencies above and below the central frequency. For pulse-echo experiments, a broad frequency range provides a transducer with a high spatial resolving power. Less damped transducers will exhibit a narrower frequency range and poorer resolving power, but greater penetration. The central frequency will also define the capabilities of a transducer. Lower frequencies (0.5 – 2.25 MHz) provide greater energy and penetration in a ma-

terial, while high frequency crystals (15.0 – 25.0 MHz) have reduced penetration but greater sensitivity to small discontinuities. High frequency transducers, when used with the proper instrumentation, can improve flaw resolution and thickness measurement capabilities dramatically restrict to similar low frequency transducers. Broad-band transducers with frequencies up to 150 MHz are commercially available.

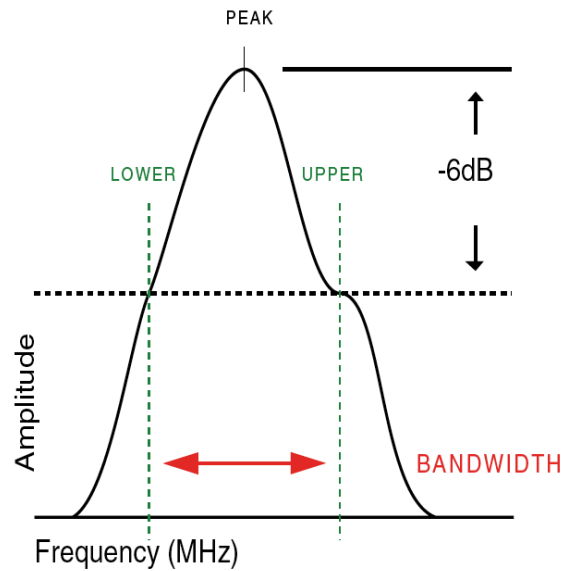


Figure 2.7: A spectrum of a typical transducer.

Figure (2.7) illustrates a spectrum of a typical transducer with peak frequency, upper and lower  $-6$  dB frequencies and MHz bandwidth measurements.

## Sound field

The sound field of a transducer can be divided into two zones: the near field and the far field. The near field is the region directly in front of the transducer where the acoustic amplitude goes through a series of maxima and minima and ends at the last maximum, at distance  $N$  from the transducer.

The location of the last maximum is known as the near field distance ( $N$  or  $Y_0^+$ ) and is the natural focus of the transducer. The far field is the area beyond  $N$  where

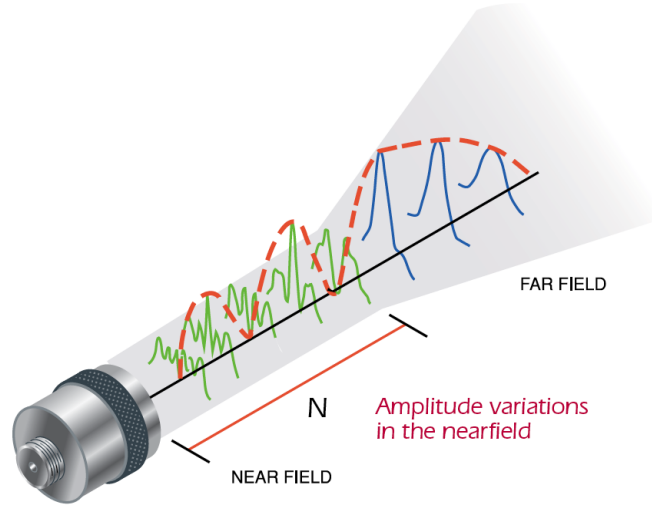


Figure 2.8: A typical sound field of a transducer.

the sound field pressure is uniform and gradually drops to zero at large distances. Because of the variations within the near field it can be difficult to accurately evaluate flaws using amplitude based techniques. The near field distance is a function of the transducer frequency  $f$ , element diameter  $D$ , wavelength  $\lambda$  and the sound velocity  $c$  of the material as shown by the following equation:

$$N = \frac{D^2 f}{4c} \left[ 1 - \left( \frac{\lambda}{D} \right)^2 \right] = \frac{D^2}{4\lambda} \left[ 1 - \left( \frac{\lambda}{D} \right)^2 \right]. \quad (2.1)$$

Table (2.1) lists the near field distances in water for several combinations of transducer frequency and element diameter.

Other sound field parameters including beam diameter  $BD$  and focal zone  $F_Z$  that are useful in describing the characteristics of a transducer are shown graphically on Figure (2.9).

**Beam diameter.** In a pulse-echo experiment, a transducer's sensitivity is affected by the beam diameter at the point of interest. The smaller the beam diameter, the greater the amount of energy is reflected by a flaw. The  $-6$  dB pulse-echo beam

| Frequency<br>( MHz) | Element<br>diameter ( in) | Near field<br>( in) | Focal length ( in) |        |
|---------------------|---------------------------|---------------------|--------------------|--------|
|                     |                           |                     | Min                | Max    |
| 0.5                 | 1.50                      | 4.757               | 2.15               | 3.80   |
|                     | 1.125                     | 2.661               | 1.50               | 2.10   |
|                     | 1.00                      | 2.095               | 1.25               | 1.65   |
|                     | 0.75                      | 1.164               | 0.78               | 0.93   |
| 1.0                 | 1.50                      | 9.559               | 2.50               | 7.65   |
|                     | 1.125                     | 5.366               | 1.90               | 4.30   |
|                     | 1.00                      | 4.235               | 1.625              | 3.38   |
|                     | 0.75                      | 2.372               | 1.00               | 1.90   |
|                     | 0.50                      | 1.043               | 0.60               | 0.80   |
| 2.25                | 1.50                      | 21.534              | 2.70               | 14.50  |
|                     | 1.125                     | 12.099              | 2.15               | 9.50   |
|                     | 1.00                      | 9.554               | 1.875              | 7.60   |
|                     | 0.75                      | 5.364               | 1.00               | 4.30   |
|                     | 0.50                      | 2.374               | 0.80               | 1.90   |
|                     | 0.375                     | 1.329               | 0.50               | 1.06   |
|                     | 0.25                      | 0.584               | 0.30               | 0.45   |
| 3.5                 | 1.00                      | 14.868              | 1.95               | 11.50  |
|                     | 0.75                      | 8.350               | 1.00               | 6.65   |
|                     | 0.50                      | 3.699               | 0.83               | 2.95   |
|                     | 0.375                     | 2.073               | 0.60               | 1.65   |
|                     | 0.25                      | 0.914               | 0.385              | 0.70   |
| 5.0                 | 1.00                      | 21.243              | 1.95               | 14.40  |
|                     | 0.75                      | 11.932              | 1.00               | 9.50   |
|                     | 0.50                      | 5.287               | 0.75               | 4.20   |
|                     | 0.375                     | 2.965               | 0.60               | 2.35   |
|                     | 0.25                      | 1.309               | 0.43               | 1.00   |
| 7.5                 | 0.75                      | 17.900              | 1.00               | 12.75  |
|                     | 0.50                      | 7.933               | 0.75               | 6.30   |
| 10                  | 1.00                      | 42.490              | 2.00               | 20.00  |
|                     | 0.75                      | 23.868              | 1.00               | 15.375 |
|                     | 0.50                      | 10.579              | 0.75               | 8.40   |
|                     | 0.375                     | 5.934               | 0.60               | 4.75   |
|                     | 0.25                      | 2.622               | 0.46               | 2.10   |
| 15                  | 0.50                      | 15.870              | 0.75               | 11.75  |
|                     | 0.375                     | 8.902               | 0.60               | 7.10   |
|                     | 0.25                      | 3.935               | 0.50               | 3.15   |
| 20                  | 0.25                      | 5.247               | 0.50               | 4.20   |
|                     | 0.125                     | 1.290               | 0.25               | 1.00   |
| 25                  | 0.25                      | 6.559               | 0.50               | 5.25   |

Table 2.1: Near field distance of flat transducers in water

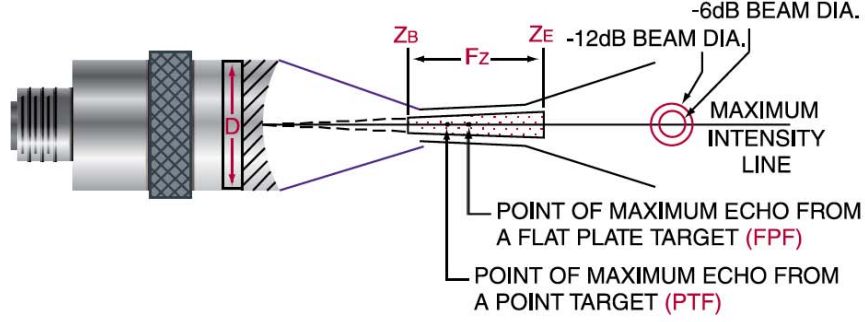


Figure 2.9: Graphical representation of sound beam parameters, where  $F_Z$  is the focal zone,  $Z_B$  is the beginning of the focal zone,  $Z_E$  is the end of the focal zone and  $D$  is the element diameter.

diameter at the focus can be calculated with the following equation:

$$BD(-6 \text{ dB}) = \frac{1.02Fc}{fD} = 0.2568DS_F, \quad (2.2)$$

where as  $BD$  is beam diameter,  $F$  is focal length,  $c$  is material sound velocity,  $D$  is element diameter and  $S_F$  is normalized focal length which is given by the following equation:

$$S_F = \frac{F}{N}. \quad (2.3)$$

For a flat transducer we can use Eq. (2.2) with  $S_F = 1$ .

**Focal zone.** The starting and ending points of the focal zone are located where the on-axis pulse-echo signal amplitude drops to  $-6 \text{ dB}$  of the amplitude at the focal point. The length of the focal zone is given by the following relation:

$$F_Z = NS_F^2 \left[ \frac{2}{1 + 0.5S_F} \right], \quad (2.4)$$

where  $F_Z$  is focal zone,  $N$  is near field and  $S_F$  is normalized focal length.

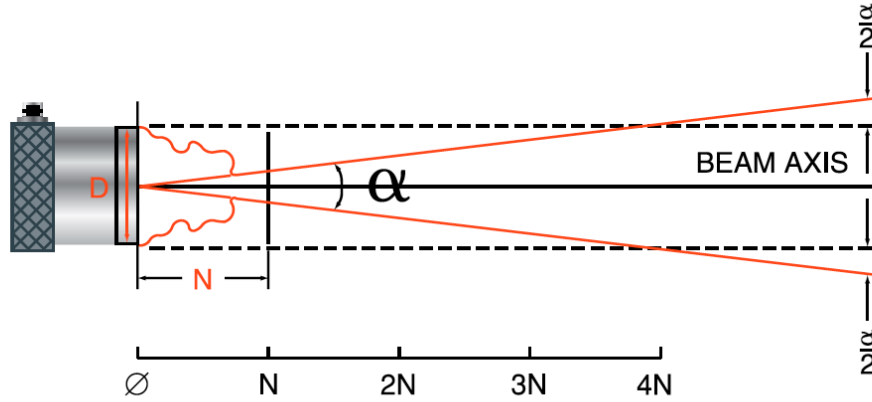


Figure 2.10: Simplified view of a sound beam for a flat transducer.

**Beam spread and half angle.** All ultrasonic beams diverge. In other words, all transducers have beam spread. Figure (2.10) gives a simplified view of a sound beam for a flat transducer. In the near field, the beam has a complex shape that narrows. In the far field, the beam diverges. For a flat transducer, the  $-6$  dB pulse-echo beam spread angle is given by

$$\sin(\alpha/2) = \frac{0.514c}{fD}, \quad (2.5)$$

where  $\alpha/2$  is half angle spread between  $-6$  dB points. It can be seen from Eq. (2.5) that beam spread from a transducer can be reduced by selecting a transducer with a higher frequency or a larger element diameter or both.

### 2.1.2 1 MHz flat transducer E1410

Flat transducer model E1410 is a customized immersed transducer manufactured by Valpey Fisher which is specially designed and built for high voltage (300 V to 500 V) usage. It has a wide spectral range (0.5 – 1.2 MHz) with a center frequency of 769 kHz. Detailed specification of transducer E1410 is listed in Table (2.2).



| Specification               | Value                 |
|-----------------------------|-----------------------|
| Serial number               | 07J281                |
| Part number                 | E1410                 |
| Nominal frequency           | 1.0 MHz               |
| Element dimension           | 1.0 in                |
| Focal length                | N/A                   |
| Focus type                  | FLAT                  |
| Connection type             | UHF                   |
| Peak frequency              | 0.757 MHz             |
| Center frequency            | 0.769 MHz             |
| −6 dB bandwidth             | 0.146 MHz             |
| −6 dB bandwidth             | 18.986 %              |
| −6 dB lower frequency       | 0.696 MHz             |
| −6 dB upper frequency       | 0.842 MHz             |
| −20 dB bandwidth            | 0.525 MHz             |
| Receive voltage             | 0.788 V <sub>pp</sub> |
| Loop gain                   | −51.609 dB            |
| Waveform duration at −20 dB | 11.360                |
| Near field in water         | 8.2 cm                |

Table 2.2: Specifications of flat transducer model E1410.

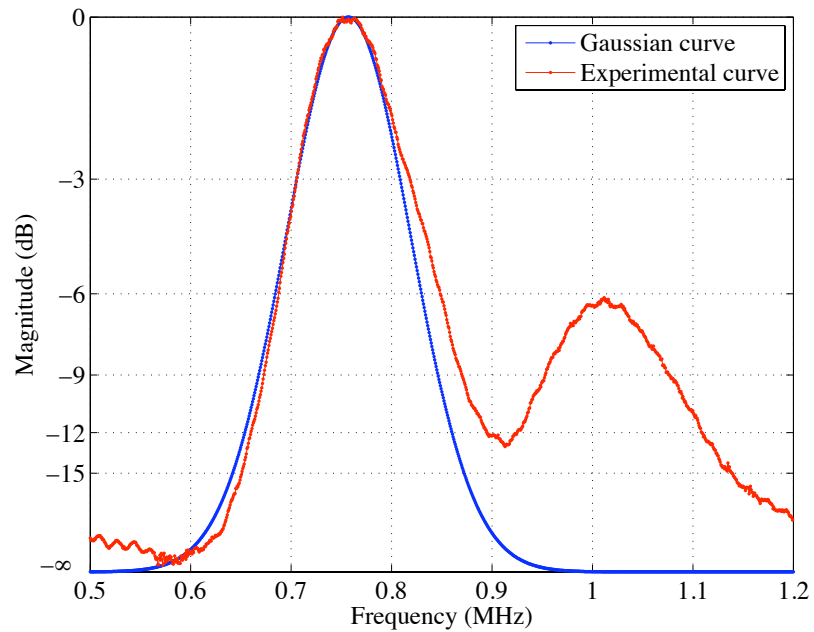


Figure 2.12: Gaussian spectrum (blue curve) and experimental spectrum (red curve) of transducer E1410.

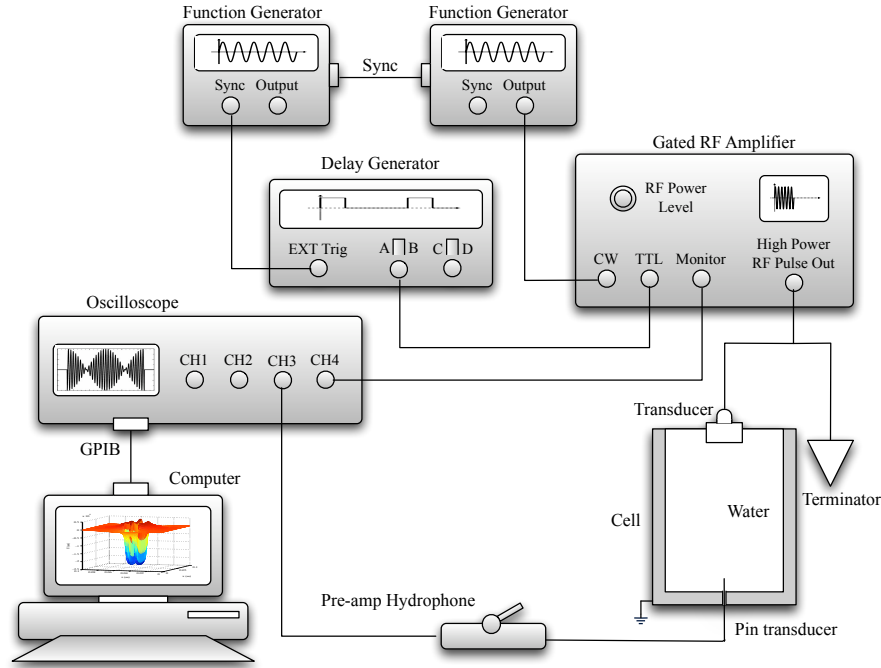


Figure 2.11: Diagram of the spectrum experiment setup.

In order to investigate the spectrum of transducer E1410, an experiment with a diagram as shown in Figure (2.11) was constructed. Transducer E1410 was driven by a high power Gated RF Amplifier (model GA 2500 from RITEC) through a  $50\ \Omega$  terminator (also from RITEC). To run gated amplifier, two function generators (model 33250A from Agilent) were used. One function generator is set to send out a continuous sinusoidal wave of frequency 769 kHz to the CW Input of the gated amplifier. The other function generator was set to send out a synchronized signal of frequency 50 Hz to an external trigger input (EXT TRIG) of a delay generator (model DG 535 from Stanford Research System, Inc.). The pulse output of the delay generator was sent to the TTL Gated Input of the gated amplifier.

Transducer E1410 was immersed in deionized water contained in a plastic cell which has a pin transducer (a small diameter PVDF thin film transducer) acting as a detector. Output of the pin transducer was sent to the oscilloscope (Infinium model 54845A from Agilent) after being amplified through a pre-amp (model AH-2020 from

ONDA). Finally, all the signals were sent to a computer through a GPIB port.

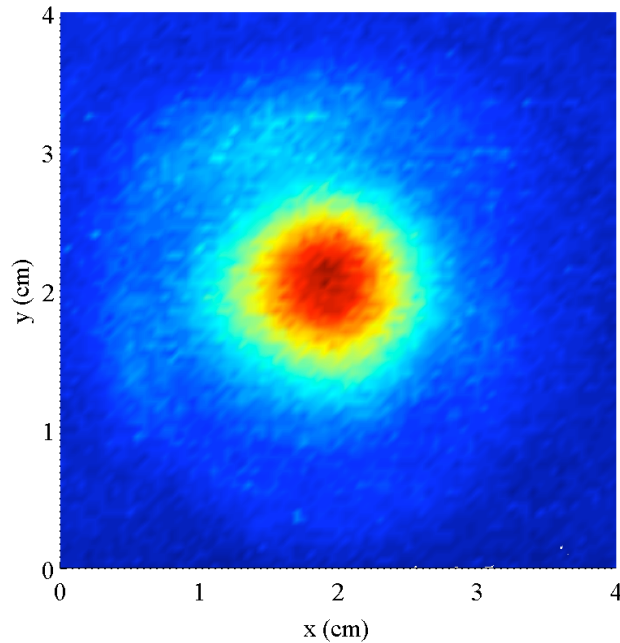


Figure 2.13: Sound intensity at the focal length of transducer E1410.

The spectrum of transducer E1410 is shown in Figure (2.12). The spacial distribution of the sound intensity of transducer E1410 at a distance of  $51\ \mu\text{s}$  from the surface of the transducer is shown in Figure (2.13).

### 2.1.3 Other transducers

Besides flat transducer E1410 which was used in the majority of the vibration potential experiments reported here, there are several other transducer, both focused and flat, which were occasionally used in some experiments, namely focused transducers for scanning experiments and transducers operating in the frequency range of  $1\ \text{MHz} - 5\ \text{MHz}$ . All of these transducers are listed in Table (2.3), including their characteristics.

| Model | Company          | Diameter (in) | Crystal            | Frequency (MHz) |
|-------|------------------|---------------|--------------------|-----------------|
| E1410 | Valpey Fisher    | 1.0           | LiNbO <sub>3</sub> | 0.76            |
| E1129 | Valpey Fisher    | 2.0           | LiNbO <sub>3</sub> | 1.0             |
| E1285 | Valpey Fisher    | 1.5           | LiNbO <sub>3</sub> | 1.0             |
| E1178 | Valpey Fisher    | 1.0           | LiNbO <sub>3</sub> | 1.0             |
| E1411 | Valpey Fisher    | 1.0           | LiNbO <sub>3</sub> | 3.5             |
| V301  | Panametrics Inc. | 1.0           | PZT                | 0.5             |
| V303  | Panametrics Inc. | 0.5           | PZT                | 1.0             |
| V382  | Panametrics Inc. | 0.5           | PZT                | 3.5             |
| V320  | Panametrics Inc. | 0.5           | PZT                | 7.5             |
| V309  | Panametrics Inc. | 0.5           | PZT                | 5.0             |
| V316  | Panametrics Inc. | 0.125         | PZT                | 20              |

Table 2.3: All transducers used in vibration potential experiments.

## 2.2 Gated RF pulse amplifier

### 2.2.1 Specifications

The gated RF pulse amplifier (model GA 2500 from RITEC) is a high power, limited duty cycle instrument designed to produce high amplitude RF tone bursts derived from either a continuous wave RF signal or from externally generated RF tone bursts. It's operating frequency range covers two decades of frequency selected by the user within the range of 250 kHz to 17.5 MHz. The specified output pulse power over the frequency range is  $1500 W_{\text{rms}}$  which is approximately 1260 V<sub>pp</sub> into  $50 \Omega$ ; however output pulse powers, up to 2.5 kW, are available at frequencies up to 2 MHz. Output level is controlled by a ten turn potentiometer on the front panel. Total gain available, when operating into a  $50 \Omega$  load, is approximately 70 dB. The specified maximum duty cycle for the instrument is 0.3%, but it will actually function up to approximately 0.4% under some conditions. The duty cycle should not be programmed higher than 0.2% when turning on the High Voltage. A large charging current is drawn from the high voltage supplies at turn on, and the built-in current limit in the supply will be activated if the duty cycle exceeds 0.3% along with the charge current. The instrument will not normally be damaged by operating into an

| Specification                | Value                                              |
|------------------------------|----------------------------------------------------|
| Model No                     | GA 2500                                            |
| Frequency range              | 250 kHz – 17.5 MHz                                 |
| RF Pulse power               | 1.5 kW                                             |
| Nominal output impedance     | 50 $\Omega$                                        |
| On/Off ratio                 | > 140 dB                                           |
| Output level control         | > 20 dB                                            |
| Maximum pulse width          | 200 $\mu$ s                                        |
| Maximum duty cycle           | 0.3 %                                              |
| External gate input          | +5 V level, 5 mA current draw                      |
| AC power supply requirements | 100, 120, 220 or 240 V <sub>rms</sub> , 50 – 60 Hz |
| Power requirements           | approx. 120 VA                                     |
| Dimensions                   | 17" $\times$ 5" $\times$ 9"                        |
| Weight                       | 18 pound                                           |

Table 2.4: Specifications of gated RF amplifier model GA 2500.

infinite VSWR, although such operation is, obviously, not recommended. The maximum pulse width is hardware limited to 200  $\mu$ s. Table (2.4) lists the full specifications of this gated amplifier.

The instrument requires an external positive TTL level gate coherent with either the tone burst or the CW signal at a 0.5 V<sub>pp</sub> level , and an appropriate AC line source (100, 120, 220 or 240 V<sub>rms</sub> at 50 – 60 Hz) to operate properly. A –40 dB monitor point for examining the output signal is provided at the front panel.

### 2.2.2 Operating instruction

If the RF pulse must be coherent with the gate, that function must be provided externally. There are two power switches: the one labeled Power controls the low voltage supplies; the one labeled High Voltage controls the +40 V and +200 V supplies. The unit is in a stand-by mode when the high voltage supplies are not energized. When first turning on the unit, the RF Power Level control should be set at its maximum counter clockwise position, and the device to be driven by the instrument should be connected to the Signal Out connector. Note that there is an approximate

10 s delay in the turn-on time of the +200 V supply. The output may be observed at the RF Pulse Monitor connector. This monitor point is at a  $-40$  dB level (when operating into a  $50\ \Omega$  load) with respect to the Signal Out connector. A positive TTL level gate must be connected to the TTL Gate In connector and either a CW signal or a tone burst (coincident with the gate) applied to the Signal In connector. At this point, the Power switch as well as the High Voltage switch may now be turned on. As the RF Power Level control is turned up (clockwise rotation of the control) a signal should be observed at the monitor point. The drive level may be increased until distortion occurs in the signal (distortion is typically seen as a squaring of the the normal sinusoidal output).

### 2.2.3 Spectral characteristics

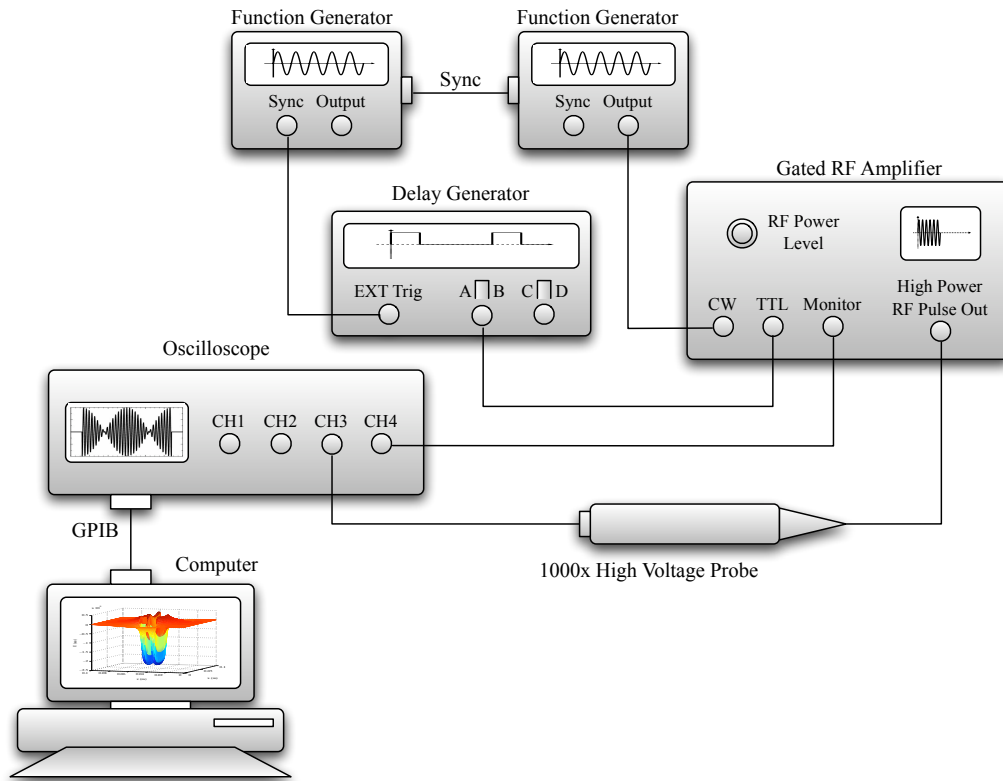


Figure 2.14: Diagram of the circuit to observe spectral characteristics of GA 2500.

A circuit as shown in Figure (2.14) was built to investigate both operational and spectral characteristics of the gated RF amplifier. Two function generators (model 33250A from Agilent) which were synchronized to each other through the 10 MHz trigger were used to supply inputs to the gated amplifier: one produces a continuous sinusoidal signal of 0.5 V<sub>pp</sub> with the output frequency automatically controllable by a LabView program on a computer fed to the CW In of the gated amplifier; the other function generator sent a sync squared signal at 50 Hz to a delay generator (model DG 535 from Stanford Research System, Inc.) to produce a fed pulse to the TTL Gate In input of the gated amplifier. The output of the gated amplifier was sent to an oscilloscope (Infinium model 54845A from Agilent) through a 1000× high voltage probe.

The voltage output versus gain factor setting of the gated amplifier is shown in Figure (2.15).

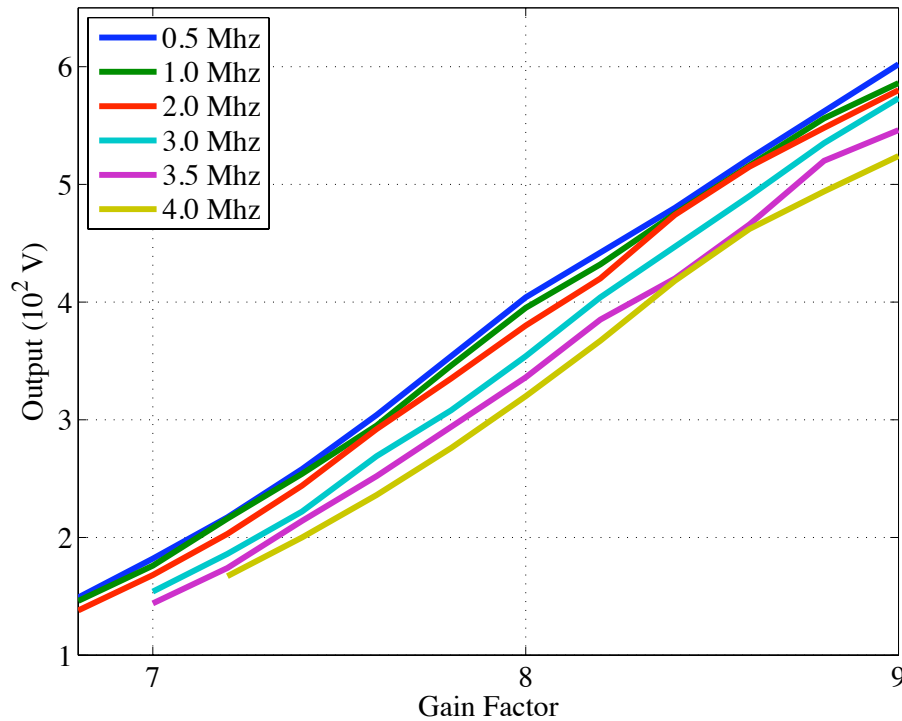


Figure 2.15: Spectral characteristics of gated amplifier model GA 2500.

## 2.3 Current amplifier

### 2.3.1 Overview and specifications

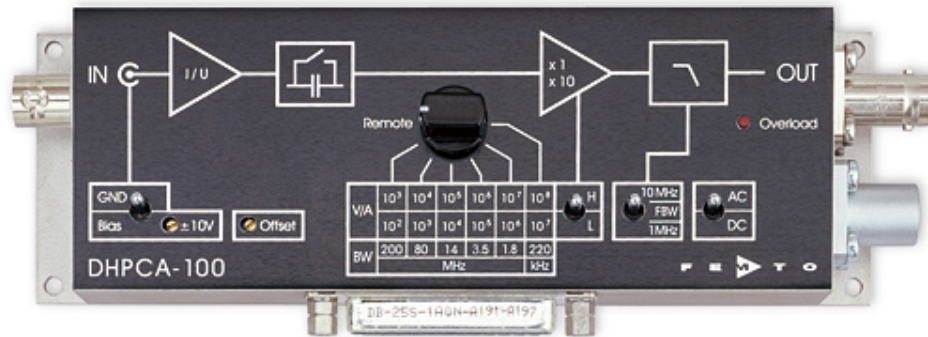


Figure 2.16: The variable gain high speed current amplifier model DHPCA-100.

The current amplifier (model DHPCA-100 from Femto, see Figure 2.16) is a variable gain, high speed transimpedance current amplifier that can amplify ultra-small electrical currents to usable voltages at a wide range of frequencies. Offering a maximum bandwidth of 200 MHz which equivalent to a signal rise time of 1.8 ns, it is ideally suited for DC and AC measurements up to 200 MHz, or for time resolved measurements for fast signals.

The gain of DHPCA-100 is adjustable from  $10^2$  to  $10^8$  V/A allowing the measurement of currents in the sub-nano to mili-Ampere range. An adjustable voltage supply is provided as bias for external photodiodes. An integrated offset control is useful for adjusting the baseline or subtracting photodiode dark currents.

The device offers many additional features like signal filters, switchable AC/DC coupling and an integrated digital interface for remote control of all main functions by a PC. The compact amplifier housing offers EMI shielding and allows use directly at the source before the signal gets degraded by long cables or interfering equipment.

The DHPCA-100 provides an optimum combination of gain and bandwidth. At



| Specification                               | Value                                         |        |        |                   |                   |                   |
|---------------------------------------------|-----------------------------------------------|--------|--------|-------------------|-------------------|-------------------|
| Model                                       | DHPCA-100                                     |        |        |                   |                   |                   |
| Performance range                           | <b>Low noise</b>                              |        |        |                   |                   |                   |
| Transimpedance ( V/A)                       | $10^2$                                        | $10^3$ | $10^4$ | $10^5$            | $10^6$            | $10^7$            |
| Bandwidth at $-3$ dB ( MHz)                 | 200                                           | 80     | 14     | 3.5               | 1.8               | 0.22              |
| Rise time (10 % – 90 %)                     | 1.8 ns                                        | 4.4 ns | 25 ns  | $0.1 \mu\text{s}$ | $0.2 \mu\text{s}$ | $1.6 \mu\text{s}$ |
| Equ. input noise ( pA/ $\sqrt{\text{Hz}}$ ) | 200                                           | 16     | 2.1    | 0.5               | 0.17              | 0.06              |
| Performance range                           | <b>High speed</b>                             |        |        |                   |                   |                   |
| Transimpedance ( V/A)                       | $10^3$                                        | $10^4$ | $10^5$ | $10^6$            | $10^7$            | $10^8$            |
| Bandwidth at $-3$ dB ( MHz)                 | 175                                           | 80     | 14     | 3.5               | 1.8               | 0.22              |
| Rise time (10 % – 90 %)                     | 2.0 ns                                        | 4.4 ns | 25 ns  | $0.1 \mu\text{s}$ | $0.2 \mu\text{s}$ | $1.6 \mu\text{s}$ |
| Equ. input noise ( pA/ $\sqrt{\text{Hz}}$ ) | 140                                           | 6.0    | 1.5    | 0.45              | 0.15              | 0.055             |
| Accuracy performance                        | Gain $\pm 1\%$                                |        |        |                   |                   |                   |
| Lowpass filter                              | Switchable to 1 MHz, 10 MHz or full bandwidth |        |        |                   |                   |                   |
| Output performance                          | $\pm 1$ V @ $50 \Omega$ load                  |        |        |                   |                   |                   |
| Bias voltage                                | $\pm 10$ V, max. 22 mA                        |        |        |                   |                   |                   |
| AC power supply                             | $\pm 15$ V, $+110$ mA / $-90$ mA              |        |        |                   |                   |                   |
| Control interface                           | Opto-Isolated Digital, TTL/CMOS Compatible    |        |        |                   |                   |                   |
| Dimensions (L $\times$ W $\times$ H)        | $170 \times 60 \times 45$ mm                  |        |        |                   |                   |                   |
| Weight                                      | 0.74 lb                                       |        |        |                   |                   |                   |

Table 2.5: Specifications of a model DHPCA-100 current amplifier.

the gain setting of  $10^3$  V/A it provides an extraordinary high 3 dB-bandwidth of 200 MHz; even at the highest gain setting of  $10^8$  V/A, the upper cut-off frequency is still an excellent 220 kHz. For those applications not requiring the full bandwidth, the unit has a switchable 10 MHz and 1 MHz lowpass filter to minimize broadband noise. Detailed specifications of DHPCA-100 are shown in Table (2.5).

The DHPCA-100 can amplify small currents generated by high impedance sources, such as photodiodes, PMTs or ionisation detectors. Typical applications include spectroscopy, laser physics, particle detection, and general lab use.

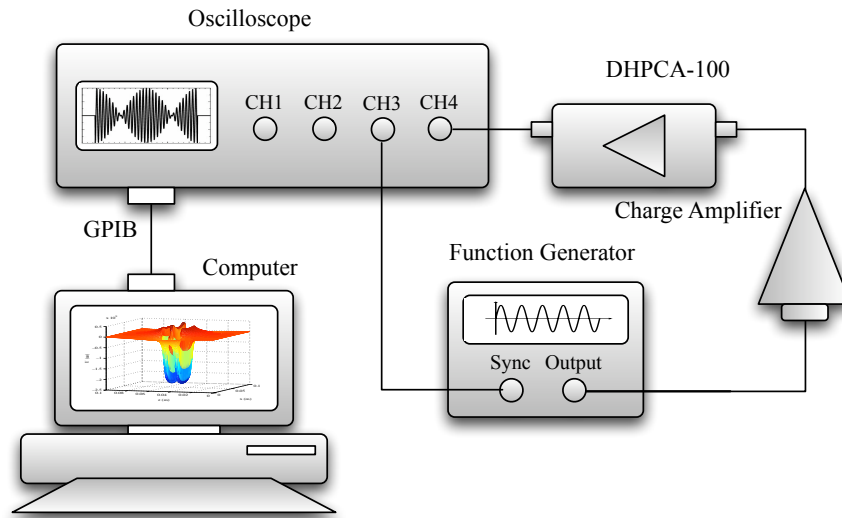


Figure 2.17: Investigation of spectral characteristics of DHPCA-100.

### 2.3.2 Spectral characteristics

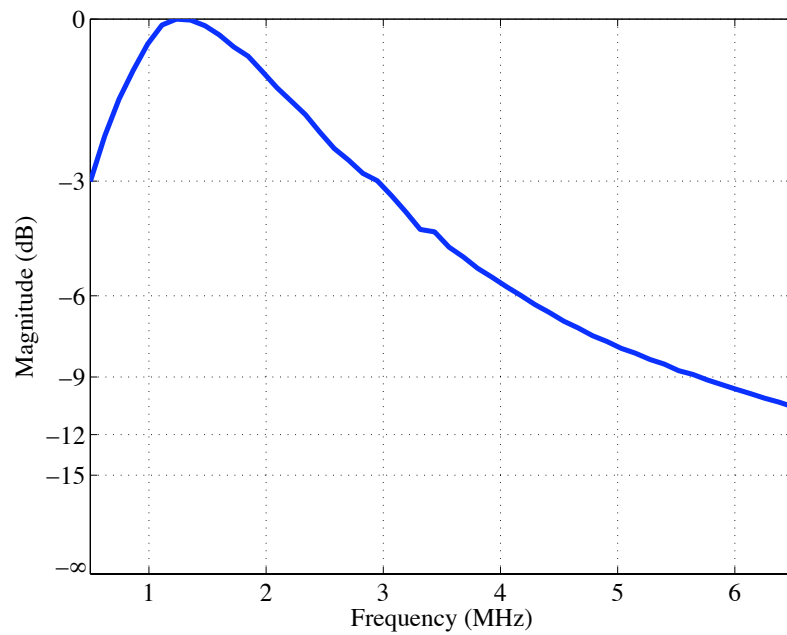


Figure 2.18: Spectral characteristics of the current amplifier model DHPCA-100 and the charge amplifier model 142A.

In order to investigate the spectral characteristics of DHP-100, an experiment was conducted as shown in Figure (2.17). A function generator (model 33250A from Agilent) sent a continuous sinusoidal signal of amplitude 10 mV and controllable frequency to the DHP-100 through a charge sensitive amplifier (model 142A from EG&G Ortec). The lowpass filter of DHP-100 was set at 10 MHz, and the gain was set at  $10^2$ . Output of DHP-100 was connected to the oscilloscope (Infinium model 54845A from Agilent). Results were recorded by the computer connected to the oscilloscope through GPIB port. The investigated spectrum of DHP-100 and the charge amplifier 142A is shown in Figure (2.18).

## 2.4 Experiment cells

### 2.4.1 Aluminum cell

An older version of the working cell for vibration potential experiments is a cylindrical aluminum cell with outer diameter of 4.5 in, inner diameter of 3 in and a height of 4.85 in. The aluminum housing is not only an excellent electric ground but also a Faraday shield (Faraday cage) to prevent noise and extra RF signals from the outside environment. Near the bottom of the cell, there is a hole of diameter 0.25 in for an SMA feedthrough with a test spring probe (model 50F5570 from Newark) which is an electric connector to pickup vibration potential signals from the electrode inside the cell.

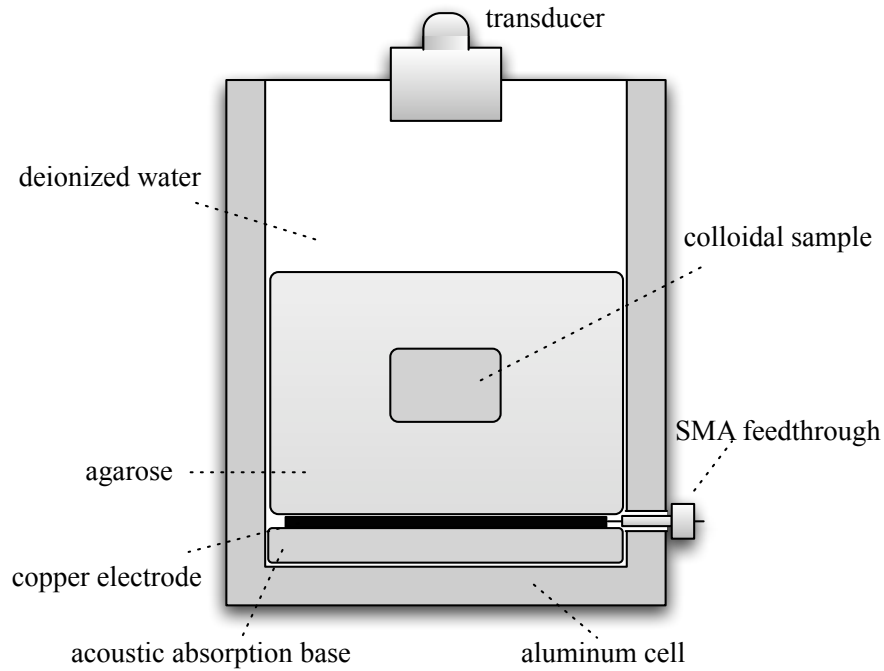


Figure 2.19: An inside view of an aluminum cell.

To eliminate the reflection when the sound meets the cell bottom, an acoustic absorption material (Aptflex F28 from Precision Acoustic Ltd.) is used to act as a base which carries a thin copper plate on its surface to act as an electrode. Agarose (product number A9539 from Sigma-Aldrich) solution is used as the acoustic media. Containing 99 % water, the acoustic impedance of agarose to the sample is in excellent match to that of the deionized water so that the sound can travel without significant reflection. Agarose solution, after being boiled, becomes a flexible gel on cooling down so that it is rigid enough to hold any sample placed inside. The detailed inside view of the aluminum cell is shown in Figure (2.19).

#### 2.4.2 Acrylic cell

A newer version of a working cell was designed for 3D scanning experiments where the transducer needs more space to scan the whole investigated area, or if the trans-

ducer has a long near field distance. Made of clear acrylic, the cell has an inner diameter of 6 in, an outer diameter of 6.5 in and a height of 12 in. Around one end of the cell is an acrylic ring of thickness 0.75 in and outer diameter of 8.75 in. The ring contains ten screw holes used to attach it to the bottom, and one hole for the SMA feedthrough.

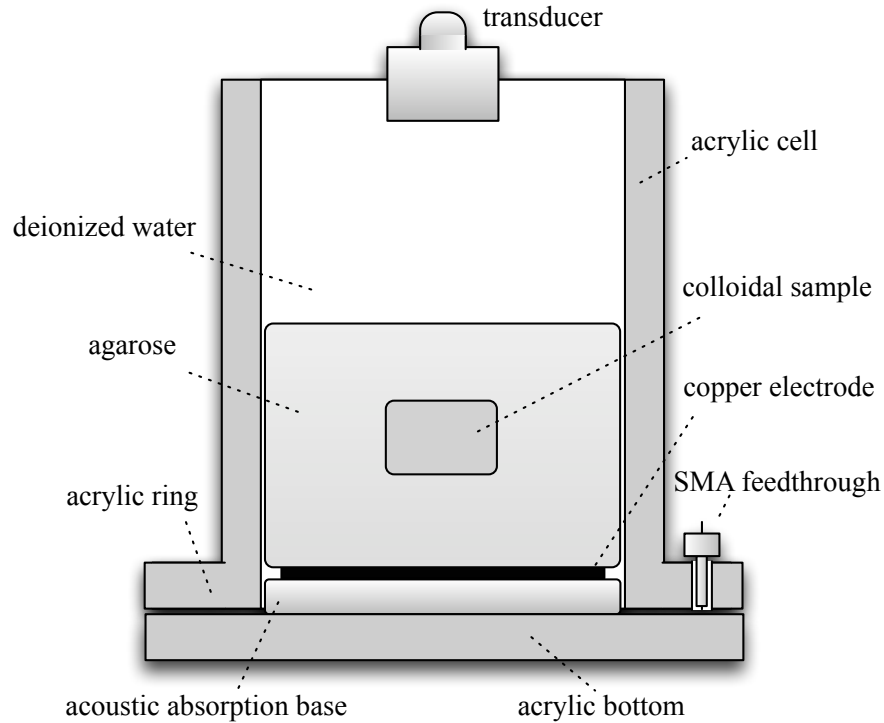


Figure 2.20: An inside view of an acrylic cell.

The cell bottom, also made of acrylic, is separate from the cell so that it can be detached to replace the agarose or colloidal samples. Having a thickness of 0.45 in and a diameter of 12 in, the bottom also has ten screw holes and carries an acoustic absorber (Aptflex F28 from Precision Acoustic Ltd.) of diameter 5.9 in on its surface to help in reducing sound reflection. The electrode, made of a copper plate, has a width of 1 in and is glued on top of the absorber. The plate has a "tail" glued on its surface to connect to the SMA feedthrough. Similar to the aluminum cell, after the bottom is screwed with the cell, agarose and sample are placed inside the cell in a

desired distance. A detailed inside view of the acrylic cell is shown in Figure (2.20).

## 2.5 Agarose

### 2.5.1 Overview and specifications

Agarose is a polymeric cross-linked, high molecular weight polysaccharide extracted from the seaweed agar. Agarose molecules have molecular weights of about 120,000, The gel network of agarose contains double helices formed from left-handed threefold helices. These double helices are stabilized by the presence of water molecules bound inside the double helical cavity. Exterior hydroxyl groups allow aggregation of up to 10,000 of these helices to form suprafibers.

Having low melting temperature, agarose is often used to make gels to act as a solidifying agent in microbial cultures and in gel electrophoresis. It is also used as a separation medium in biochromatography.

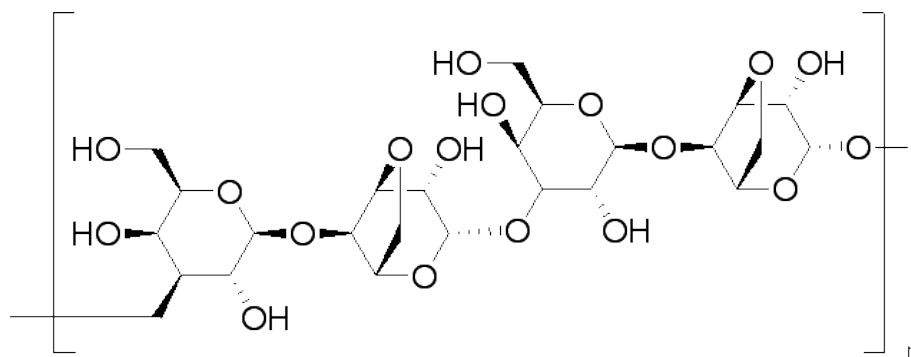


Figure 2.21: Molecular formula of agarose A9539.

Agarose used in vibration potential experiments is agarose gel (product number A9539) from Sigma-Aldrich Corporation. Detailed specifications of agarose are listed in Table (2.6). Its molecular formula is shown in Figure (2.21).

| Specification           | Value                                                   |
|-------------------------|---------------------------------------------------------|
| Product number          | A9539                                                   |
| CAS number              | 9012-36-6                                               |
| MDL number              | MFCD00081294                                            |
| PubChem Substance ID    | 24891469                                                |
| EG/EC Number            | 232-731-8                                               |
| Total impurities        | $\leq 1.0\%$ Ash, $\leq 10\%$ water                     |
| Electroendosmosis (EEO) | 0.09 – 0.13                                             |
| Transition temperature  | gel point $36^{\circ}\text{C}(\pm 1.5^{\circ}\text{C})$ |
| Foreign activity        | DNase, RNase, NICKase                                   |

Table 2.6: Specifications of agarose used in vibration potential experiments.

### 2.5.2 Procedure of making gel

Agarose solution used in vibration potential experiments has 1 % concentration. It is made by dissolving 2 g agarose powder into a pyrex bottle containing 200 ml deionized water. The bottle should be shaken several times to make sure that all agarose powder is dissolved completely. Agarose solution then is heated until boiled by using a microwave oven for about 3.5 minutes. To make sure that agarose is completely dissolved and that the agarose solution is homogeneous, the bottle can be shaken several times in the middle of the heating.

Boiled agarose solution then can be cast to make any desired shape, by letting it cool down to room temperature. In the aluminum cell, which has a relatively high thermal conductivity, a layer of thickness 1 in usually takes two hours to cool down and solidify as a gel. In an acrylic cell, it usually takes longer times, about three to four hours to form a gel.

## 2.6 Colloidal silica

### 2.6.1 Overview

Colloidal silicas are suspensions of fine sized amorphous, nonporous, and typically spherical silica particles in a liquid phase. Sometimes referred to as a sol or silica sol, colloidal silica consists of an electrostatically stable dispersion silica particles. To achieve this, the silica particles must be small enough such that they are largely unaffected by gravity. Therefore, silica particle sizes are usually of the order of less than 100 nm. Colloidal silicas exhibit particle densities in the range of 2.1 to 2.3 g/cc. The molecular formula of colloidal silica is  $\text{SiO}_2$ .

Most colloidal silicas are prepared as monodisperse suspensions with particle sizes ranging from approximately 5 to 100 nm in diameter. Polydisperse suspensions can also be synthesized and have roughly the same limits in particle size. Smaller particles are difficult to stabilize while particles much greater than 150 nm are subject to sedimentation.

Colloidal silicas are most often prepared in a multi-step process where an alkali-silicate solution is partially neutralized, leading to the formation of silica nuclei. The resulting suspension is then concentrated and stabilized. The initial neutralization step can be accomplished by acidification or electro-dialysis, but it most often achieved by contacting the silicate with a hydrogen form ion exchange resin. The maximum concentration at which this neutralization can be carried out is in the range of 10 to 15 %. Higher concentrations result in gelation and plugging of the resin pores.

The colloidal suspension is stabilized by pH adjustment and then concentrated, usually by evaporation. The maximum concentration obtainable depends on the on particle size. For example, 50 nm particles can be concentrated to greater than 50 wt% solids while 10 nm particles can only be concentrated to approximately 30 wt% solids before the suspension becomes too unstable.



### 2.6.2 SNOWTEX specifications

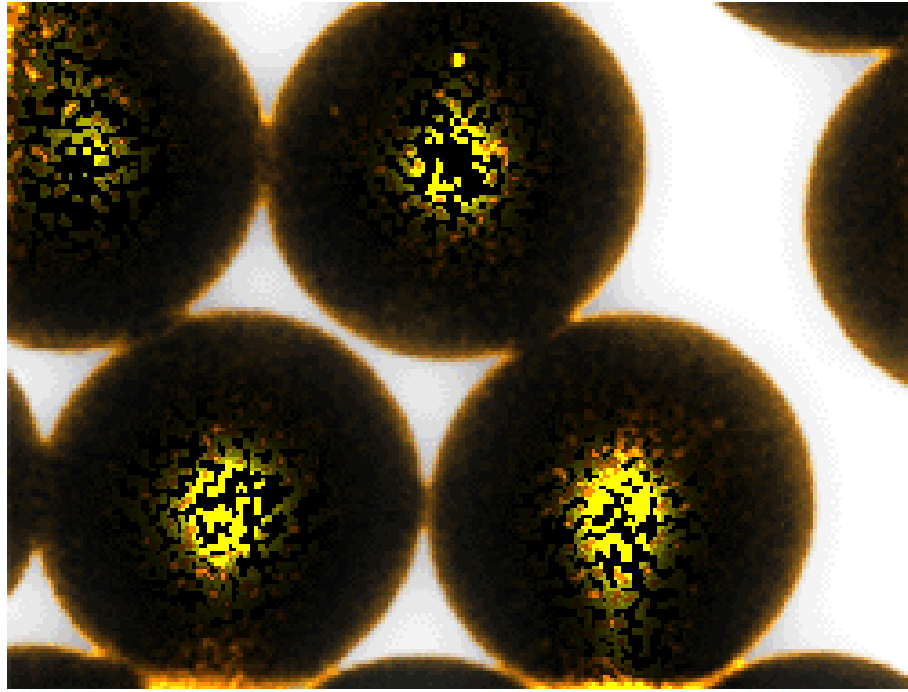


Figure 2.22: Electron micrograph of SNOWTEX from Nissan Chemical.

Colloidal silica used in vibration potential experiments is SNOWTEX<sup>®</sup> colloidal silica from Nissan Chemical America Corporation. SNOWTEX is made by growing mono-dispersed, negatively charged, amorphous silica particles in water.  $\text{OH}^-$  ions exist at the surface of the particles with an electric double layer formed by alkali ions. Stabilization is achieved by the repulsion between the same negatively charged particles. Disturbance of the charge balance will cause the colloidal silica to aggregate, resulting in high viscosity or gelation of the solution. Detailed specifications of different SNOWTEX products are listed in Table (2.7).

| <b>SNOWTEX<sup>®</sup></b>  | <b>ST-20L</b> | <b>ST-OL</b> | <b>ST-ZL</b> |
|-----------------------------|---------------|--------------|--------------|
| $SiO_2$ ( wt%)              | 20-21         | 20-21        | 40-41        |
| $Na_2O$ ( wt%)              | < 0.3         | < 0.05       | < 0.07       |
| pH                          | 9.5-11.0      | 2-4          | 9.0-10.0     |
| Particle shape              | spherical     | spherical    | spherical    |
| Particle size ( nm)         | 40-50         | 40-50        | 70-100       |
| Viscosity ( mPa.s at 25 °C) | < 3           | < 3          | < 5          |
| Specific gravity ( °C)      | 1.12-1.14     | 1.12-1.14    | 1.29-1.32    |
| Appearance                  | opalescent    | opalescent   | opalescent   |

Table 2.7: Specifications of agarose used in vibration potential experiments.

# Chapter 3

## Ultrasonic vibration potential theory

### 3.1 Fundamental approach

The basics of ultrasonic vibration potential theory is based on the four Maxwell's equations which, in SI units<sup>34</sup>, are

$$\nabla \cdot \mathbf{D} = \rho, \quad (3.1)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (3.2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3.4)$$

where  $\rho$  and  $\mathbf{J}$  are the charge and current densities of the considered object respectively;  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic fields respectively;  $\mathbf{D}$  is the electric displacement and  $\mathbf{B}$  is the magnetic-flux density (or magnetic induction). Among  $\mathbf{E}$ ,

$\mathbf{H}$ ,  $\mathbf{D}$  and  $\mathbf{B}$  we have relations

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} , \quad (3.5)$$

$$\mathbf{B} = \mu \mu_0 \mathbf{H} , \quad (3.6)$$

where  $\mathbf{P}$  is the electric polarization;  $\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$  and  $\mu_0 = 1.256 \times 10^{-6} \text{ H/m}$ . We also need here Ohm's law

$$\mathbf{J} = \sigma \mathbf{E} , \quad (3.7)$$

and the continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} . \quad (3.8)$$

With the help of equations (3.1) and (3.7), we can rewrite equation (3.8) as follows

$$\nabla \cdot (\sigma \mathbf{E}) = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{D}) = -\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} , \quad (3.9)$$

or, in a more convenient form,

$$\nabla \cdot \left( \sigma \mathbf{E} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 . \quad (3.10)$$

Suppose that the source of  $\mathbf{P}$  is  $\mathbf{P}_a$  - an acoustic source, and notice that  $(\epsilon - \epsilon_0)\mathbf{E}$  is the electrical polarizability, then we have

$$\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E} + \mathbf{P}_a , \quad (3.11)$$

and from equation (3.5) then

$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P}_a . \quad (3.12)$$

Substitution of equation (3.10) into equation (3.12) gives

$$\nabla \cdot \left[ \sigma \mathbf{E} + \frac{\partial}{\partial t} (\epsilon \mathbf{E} + \mathbf{P}_a) \right] = 0. \quad (3.13)$$

For the quasi-electrostatic approximation, when an averaging is made of the homogeneous equation, we have  $\nabla \times \mathbf{E}_{micro} = 0$ , then the same equation, namely

$$\nabla \times \mathbf{E} = 0, \quad (3.14)$$

holds for the averaged, that is, the macroscopic, electric field  $\mathbf{E}$ . This means that the electric field is still derivable from a potential  $\varphi(\mathbf{x})$ , or

$$\mathbf{E} = -\nabla \varphi. \quad (3.15)$$

Now from equations (3.13) and (3.15) we have

$$\nabla \cdot \left[ \sigma (-\nabla \varphi) + \frac{\partial}{\partial t} \left( \epsilon (-\nabla \varphi) + \mathbf{P}_a \right) \right] = 0, \quad (3.16)$$

or

$$\nabla \cdot \left[ \sigma \nabla \varphi + \frac{\partial}{\partial t} (\epsilon \nabla \varphi) \right] = \frac{\partial}{\partial t} (\nabla \cdot \mathbf{P}_a). \quad (3.17)$$

For simplicity, suppose that the considered object has  $\sigma = \text{const}$  and  $\epsilon = \text{const}$  then

$$\Delta \left[ \varphi + \frac{\epsilon}{\sigma} \frac{\partial \varphi}{\partial t} \right] = \frac{1}{\sigma} \nabla \cdot \frac{\partial \mathbf{P}_a}{\partial t}. \quad (3.18)$$

Now suppose that:

$$\varphi = \varphi \cos \omega t = \text{Re } \varphi e^{i\omega t}, \quad (3.19)$$

$$\mathbf{P}_a = \mathbf{P}_a \cos \omega t = \text{Re } \mathbf{P}_a e^{i\omega t}, \quad (3.20)$$

then equation (3.18) leads to

$$\Delta \left[ \varphi + i\omega \frac{\epsilon}{\sigma} \varphi \right] = i\omega \frac{1}{\sigma} \nabla \cdot \mathbf{P}_a, \quad (3.21)$$

or

$$(\sigma + i\omega\epsilon) \Delta \varphi = i\omega \nabla \cdot \mathbf{P}_a. \quad (3.22)$$

Then, we have

$$\Delta \varphi = \frac{i\omega}{\sigma + i\omega\epsilon} \nabla \cdot \mathbf{P}_a = \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \nabla \cdot \mathbf{P}_a. \quad (3.23)$$

We denote the acoustically induced polarization as  $\mathbf{P}_a$ , which is given by  $\mathbf{P}_a = \alpha(\omega) \nabla p$ , where  $\nabla p$  is the acoustic pressure gradient and<sup>19</sup>  $\alpha(\omega)$  is given by the following formula

$$\alpha(\omega) = \epsilon f \frac{\Delta \rho_m}{\rho_m} \frac{\mu_E}{\sigma^*} \quad (3.24)$$

In the above equation,  $\sigma^*$  is the complex conductivity, the coefficient  $\mu_E$  which relates particle velocity to field strength is the electrophoretic mobility,  $f$  is the particle volume fraction,  $\rho_m$  is the suspension density and  $\rho_m + \Delta \rho_m$  is the particle density. Equation (3.23) then leads to

$$\Delta \varphi = \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \nabla \cdot (\alpha(\omega) \nabla p) = \nabla \cdot \left( \frac{\alpha(\omega)}{\epsilon - i\frac{\sigma}{\omega}} \nabla p \right) = \nabla (\alpha^\dagger \nabla p), \quad (3.25)$$

where

$$\alpha^\dagger(\omega) = \frac{\alpha(\omega)}{\epsilon - i\frac{\sigma}{\omega}}. \quad (3.26)$$

Equation (3.25) has the form of Poisson's equation

$$\Delta \varphi = \nabla (\alpha^\dagger \nabla p). \quad (3.27)$$

Notice that  $\varphi$  has to satisfy the boundary conditions. For the sake of simplicity, we

consider the potential  $\varphi$  between two plates, then there are two known possibilities. In the case of Neumann's boundary condition<sup>34</sup>, the two plates are disconnected and they have infinite conductivity, which causes an infinite fast relaxation of charges on the plates meaning  $E_n = 0$ . In the case of Dirichlet's boundary condition, the plates are connected by an infinitely small resistance, meaning  $\varphi|_{\partial V} = 0$ . So the conditions are

$$\text{Neumann's condition:} \quad \left. \frac{\partial \varphi}{\partial n} \right|_{\partial V} = 0, \quad (3.28)$$

or

$$\text{Dirichlet's condition:} \quad \varphi|_{\partial V} = 0, \quad (3.29)$$

where  $\partial V$  is the boundary surface of the volume  $V$  to be considered.

In order to solve equation (3.27), we use the method of Green's functions and construct the Green's function  $G = G(\mathbf{r}; \mathbf{r}')$  so that

$$\Delta G = \delta(\mathbf{r} - \mathbf{r}'). \quad (3.30)$$

$G$  must also satisfy the boundary condition

$$\text{Neumann's condition:} \quad \left. \frac{\partial G}{\partial n} \right|_{\partial V} = 0, \quad (3.31)$$

or

$$\text{Dirichlet's condition:} \quad G|_{\partial V} = 0, \quad (3.32)$$

Using the Green's second identity for the two functions  $\varphi(\mathbf{r})$  and  $G(\mathbf{r}; \mathbf{r}')$

$$\int_V [\varphi \Delta' G - G \Delta' \varphi] dV' = \oint_{\partial V} \left[ \varphi \frac{\partial G}{\partial n'} - G \frac{\partial \varphi}{\partial n'} \right] dS', \quad (3.33)$$

which, in more detail, is

$$\int_V [\varphi(\mathbf{r}') \Delta' G(\mathbf{r}; \mathbf{r}') - G(\mathbf{r}; \mathbf{r}') \Delta' \varphi(\mathbf{r}')] dV' = \oint_{\partial V} \left[ \varphi(\mathbf{r}') \frac{\partial G}{\partial n'} \Big|_{\partial V} - G \frac{\partial \varphi(\mathbf{r}')}{\partial n'} \Big|_{\partial V} \right] dS', \quad (3.34)$$

where the prime sign denotes the operators are functions of the primed coordinates.

Using equation (3.30) we can write

$$\int_V \varphi(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') dV' = \int_V G(\mathbf{r}; \mathbf{r}') \Delta' \varphi(\mathbf{r}') dV' + \oint_{\partial V} \left[ \varphi(\mathbf{r}') \frac{\partial G}{\partial n'} \Big|_{\partial V} - G \frac{\partial \varphi(\mathbf{r}')}{\partial n'} \Big|_{\partial V} \right] dS'. \quad (3.35)$$

Notice that  $\varphi(\mathbf{r}) = \int_V \varphi(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') dV'$  and with the help of the Neumann's conditions (equations (3.28) and (3.31)) or the Dirichlet's conditions (equations (3.29) and (3.32)) of  $\varphi$  and  $G$  we always have

$$\varphi(\mathbf{r}) = \int_V G(\mathbf{r}; \mathbf{r}') \Delta' \varphi(\mathbf{r}') dV', \quad (3.36)$$

or when using equation (3.27)<sup>24</sup>

$$\varphi(\mathbf{r}) = \int_V G(\mathbf{r}; \mathbf{r}') \nabla' (\alpha^\dagger \nabla' p) dV'. \quad (3.37)$$

From equation (3.26) then we have

$$\varphi(\mathbf{r}) = \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \int_V G(\mathbf{r}; \mathbf{r}') \nabla' [\alpha(\mathbf{r}', \omega) \nabla' p] dV'. \quad (3.38)$$

Equation (3.38) is the basic equation for the ultrasonic vibration potential theory.



## 3.2 Green's functions

### 3.2.1 Green's function in Cartesian coordinates

In this section, the Dirichlet Green's function in the simplest case of two parallel electrodes separated a distance of  $h$  will be considered. Suppose that two electrodes are in the  $x$ - $y$  plane and perpendicular to the  $z$ -axis. The method of solution to be used is Fourier transformation. Our task is to find the Green's function that obeys

$$\nabla^2 G(\mathbf{r}; \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \quad (3.39)$$

and obeys the condition

$$G|_{\partial D} = G|_{z=0, z=h} = 0 \quad (3.40)$$

Consider the Fourier transformation of the spatial coordinate in equation (3.39) through the transform pair

$$\tilde{f}(\kappa_x, \kappa_y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(\kappa_x x + \kappa_y y)} dx dy \quad (3.41a)$$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(\kappa_x, \kappa_y) e^{i(\kappa_x x + \kappa_y y)} d\kappa_x d\kappa_y \quad (3.41b)$$

which gives

$$\tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, z; \mathbf{r}') e^{-i(\kappa_x x + \kappa_y y)} dx dy \quad (3.42a)$$

$$G(x, y, z; \mathbf{r}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') e^{i(\kappa_x x + \kappa_y y)} d\kappa_x d\kappa_y \quad (3.42b)$$

For the Dirac delta function we have

$$\tilde{\delta}(\kappa_x, \kappa_y, z; \mathbf{r}') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\mathbf{r} - \mathbf{r}') e^{-i(\kappa_x x + \kappa_y y)} dx dy, \quad (3.43a)$$

$$\delta(\mathbf{r} - \mathbf{r}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\delta}(\kappa_x, \kappa_y, z; \mathbf{r}') e^{i(\kappa_x x + \kappa_y y)} d\kappa_x d\kappa_y. \quad (3.43b)$$

Note that

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\mathbf{r} - \mathbf{r}') e^{-i(\kappa_x x + \kappa_y y)} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - x') \delta(y - y') \delta(z - z') e^{-i(\kappa_x x + \kappa_y y)} dx dy \\ &= \delta(z - z') \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - x') \delta(y - y') e^{-i(\kappa_x x + \kappa_y y)} dx dy \\ &= \delta(z - z') e^{-i(\kappa_x x' + \kappa_y y')} , \end{aligned} \quad (3.44)$$

so equation (3.43a) gives

$$\tilde{\delta}(\kappa_x, \kappa_y, z; \mathbf{r}') = \frac{1}{(2\pi)^2} \delta(z - z') e^{-i(\kappa_x x' + \kappa_y y')} . \quad (3.45)$$

Now from equation (3.39) we have

$$\begin{aligned} \nabla^2 G &= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) G(x, y, z; \mathbf{r}') \\ &= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') e^{i(\kappa_x x + \kappa_y y)} d\kappa_x d\kappa_y \\ &= \left( -\kappa_x^2 - \kappa_y^2 + \frac{\partial^2}{\partial z^2} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') e^{i(\kappa_x x + \kappa_y y)} d\kappa_x d\kappa_y \\ &= \delta(\mathbf{r} - \mathbf{r}') \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\delta}(\kappa_x, \kappa_y, z; \mathbf{r}') e^{i(\kappa_x x + \kappa_y y)} d\kappa_x d\kappa_y , \end{aligned} \quad (3.46)$$

or

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \left( -\kappa_x^2 - \kappa_y^2 + \frac{\partial^2}{\partial z^2} \right) \tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') - \tilde{\delta}(\kappa_x, \kappa_y, z; \mathbf{r}') \right] e^{i(\kappa_x x + \kappa_y y)} d\kappa_x d\kappa_y = 0 . \quad (3.47)$$

This gives us

$$\left( -\kappa_x^2 - \kappa_y^2 + \frac{\partial^2}{\partial z^2} \right) \tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') = \tilde{\delta}(\kappa_x, \kappa_y, z; \mathbf{r}') . \quad (3.48)$$

From equations (3.45) and (3.48) we have

$$\left( -\kappa_x^2 - \kappa_y^2 + \frac{\partial^2}{\partial z^2} \right) \tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') = \frac{1}{(2\pi)^2} \delta(z - z') e^{-i(\kappa_x x' + \kappa_y y')} . \quad (3.49)$$

In order to be terminated at  $z = 0$  and  $z = h$ ,  $\tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}')$  should have the form of  $\sum_{m=1}^{\infty} A_m \sin(d_m z)$  where  $d_m = \frac{m\pi}{h}$  and  $A_m$  are constants independent of  $z$  when  $m$  runs from 1 to  $\infty$ . We can find  $A_m$  as follow: substitute  $\tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}')$  into Eq. (3.49) giving

$$\begin{aligned} \frac{1}{(2\pi)^2} \delta(z - z') e^{-i(\kappa_x x' + \kappa_y y')} &= \left( -\kappa_x^2 - \kappa_y^2 + \frac{\partial^2}{\partial z^2} \right) \tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') , \\ &= \left( -\kappa_x^2 - \kappa_y^2 + \frac{\partial^2}{\partial z^2} \right) \sum_{m=1}^{\infty} A_m \sin(d_m z) , \\ &= \sum_{m=1}^{\infty} A_m (-\kappa_x^2 - \kappa_y^2 - d_m^2) \sin(d_m z) . \end{aligned} \quad (3.50)$$

Multiplying two sides of Eq. (3.50) with  $\sin d_{m'} z$  and integrating the resulting expression from 0 to  $h$  we have

$$\begin{aligned} \int_0^h \frac{1}{(2\pi)^2} \delta(z - z') \sin d_{m'} z e^{-i(\kappa_x x' + \kappa_y y')} dz \\ = \int_0^h \sum_{m=1}^{\infty} A_m (-\kappa_x^2 - \kappa_y^2 - d_m^2) \sin(d_m z) \sin d_{m'} z dz . \end{aligned} \quad (3.51)$$

Note that

$$\begin{aligned}
\int_0^h \sin(d_m z) \sin d_{m'} z dz &= \int_0^h \frac{\cos(d_m - d_{m'})z - \cos(d_m + d_{m'})z}{2} dz , \\
&= \frac{1}{2} \int_0^h \left( \cos\left(\frac{(m-m')\pi z}{h}\right) - \cos\left(\frac{(m+m')\pi z}{h}\right) \right) dz , \\
&= \begin{cases} \frac{1}{2} \int_0^h (1 - \cos \frac{2m'\pi z}{h}) dz & \text{if } m = m' , \\ 0 & \text{if } m \neq m' , \end{cases} \\
&= \begin{cases} \frac{h}{2} & \text{if } m = m' , \\ 0 & \text{if } m \neq m' , \end{cases} \\
&= \delta_{mm'} \frac{h}{2} , \tag{3.52}
\end{aligned}$$

and that

$$\begin{aligned}
\int_0^h \sin(d_{m'} z) \delta(z - z') dz &= \int_{-\infty}^{\infty} \sin(d_{m'} z) \delta(z - z') dz , \\
&= \sin(d_{m'} z') , \tag{3.53}
\end{aligned}$$

so from equation (3.51) we have

$$\frac{1}{(2\pi)^2} \sin(d_{m'} z') e^{-i(\kappa_x x' + \kappa_y y')} = -A_{m'} \left[ \kappa_x^2 + \kappa_y^2 + \left(\frac{m'\pi}{h}\right)^2 \right] \frac{h}{2} , \tag{3.54}$$

or (dropping the prime sign for more convenience),

$$A_m = -\frac{1}{2\pi^2 h} \frac{\sin(d_m z')}{\left[ \kappa_x^2 + \kappa_y^2 + \left(\frac{m\pi}{h}\right)^2 \right]} e^{-i(\kappa_x x' + \kappa_y y')} , \tag{3.55}$$

Because  $\tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') = \sum_{m=1}^{\infty} A_m \sin(d_m z)$  then we have

$$\tilde{G}(\kappa_x, \kappa_y, z; \mathbf{r}') = -\frac{1}{2\pi^2 h} \sum_{m=1}^{\infty} \frac{\sin\left(\frac{m\pi}{h} z'\right) \sin\left(\frac{m\pi}{h} z\right)}{\left[\kappa_x^2 + \kappa_y^2 + \left(\frac{m\pi}{h}\right)^2\right]} e^{-i(\kappa_x x' + \kappa_y y')} . \quad (3.56)$$

Finally, from equations (3.42b) and (3.56) we have a Dirichlet Green's function in the case of two parallel electrodes

$$G(\mathbf{r}; \mathbf{r}') = -\frac{1}{2\pi^2 h} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{h} z\right) \sin\left(\frac{m\pi}{h} z'\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} d\kappa_x d\kappa_y}{\left[\kappa_x^2 + \kappa_y^2 + \left(\frac{m\pi}{h}\right)^2\right]} . \quad (3.57)$$

### 3.2.2 Green's function in cylindrical coordinates

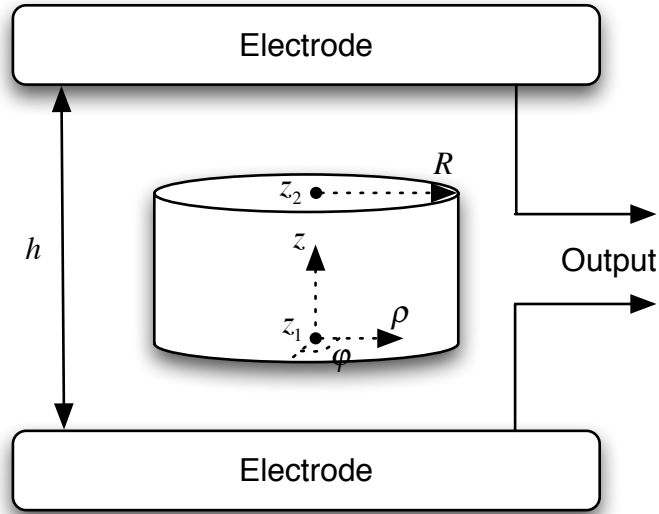


Figure 3.1: Cylinder of radius  $R$  lying along  $z$  axis from  $z_1$  to  $z_2$ .

In order to find the potential distribution in this case of cylinder of radius  $R$  lying along  $z$  axis from  $z_1$  to  $z_2$  as shown in Figure (3.1), we will need the solution of Green function in cylindrical coordinates. The Poisson's equation for Green function

in cylindrical coordinates  $(\rho, \varphi, z)$  is written as

$$\Delta G(\mathbf{r}, \mathbf{r}') = \delta(\varphi - \varphi')\delta(z - z')\frac{\delta(\rho - \rho')}{\rho}, \quad (3.58)$$

where

$$\Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, \quad (3.59)$$

and  $G(\mathbf{r}, \mathbf{r}')$  must satisfy the Dirichlet's condition  $G|_{z=0} = G|_{z=h} = 0$ .

We will find the solution of Green function by using the method of separation of variable

$$G(\mathbf{r}, \mathbf{r}') = \Gamma(\rho)Z(z)\Phi(\varphi), \quad (3.60)$$

with the obvious boundary conditions

$$\begin{cases} G|_{z=0} = Z(0) = 0, \\ G|_{z=h} = Z(h) = 0. \end{cases} \quad (3.61)$$

Using the Laplace operator on both sides of (3.60) and substituting the result into equation (3.58) we have

$$\begin{aligned} Z(z)\Phi(\varphi)\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Gamma(\rho)}{\partial \rho}\right) + \Gamma(\rho)Z(z)\frac{1}{\rho^2}\frac{\partial^2 \Phi(\varphi)}{\partial \varphi^2} + \Gamma(\rho)\Phi(\varphi)\frac{\partial^2 Z(z)}{\partial z^2} \\ = \delta(\varphi - \varphi')\delta(z - z')\frac{\delta(\rho - \rho')}{\rho}. \end{aligned} \quad (3.62)$$

Dividing both side of Eq. (3.62) by  $\Gamma(\rho)Z(z)\Phi(\varphi)$  ( with the condition  $z \neq z'$ ) we have

$$\frac{1}{\Gamma(\rho)}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Gamma(\rho)}{\partial \rho}\right) + \frac{1}{\Phi(\varphi)}\frac{1}{\rho^2}\frac{\partial^2 \Phi(\varphi)}{\partial \varphi^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = 0. \quad (3.63)$$

By introducing the separation constants  $m$  and  $k_n$  we get

$$\ddot{\Phi}(\varphi) + m^2\Phi(\varphi) = 0, \quad (3.64)$$

$$\ddot{Z}(z) + k_n^2 Z(z) = 0, \quad (3.65)$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma(\rho)}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma(\rho) = 0. \quad (3.66)$$

We can easily see the solution of equation (3.64) is

$$\Phi(\varphi) = A_m \cos(m\varphi), \quad (3.67)$$

and, it is evident that solution of equation (3.65), together with the boundary condition (3.61), is

$$Z(z) = B_n \sin(k_n z), \quad (3.68)$$

where  $k_n = \frac{n\pi}{h}$ . The remaining equation (3.66), is just the equation for the modified Bessel functions,  $I_m(k_n \rho)$  and  $K_m(k_n \rho)$  which gives when  $\rho \neq \rho'$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{mn}}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma_{mn} = \frac{\delta(\rho - \rho')}{\rho}. \quad (3.69)$$

Suppose that  $\psi_1(k_n \rho)$  is some linear combination of  $I_m$  and  $K_m$  which satisfies the correct boundary conditions for  $\rho < \rho'$ , and that  $\psi_2(k_n \rho)$  is a linearly independent combination that satisfies the proper boundary conditions for  $\rho > \rho'$ . Then the symmetry of the Green function in  $\rho$  and  $\rho'$  requires that

$$\Gamma_{mn} = \psi_1(k_n \rho_{<}) \psi_1(k_n \rho_{>}) \quad (3.70)$$

where  $\rho_{<} = \min(\rho, \rho')$  and  $\rho_{>} = \max(\rho, \rho')$ . Multiplication both sides of equation (3.69) by  $\rho$  and then integration over  $\rho$  from  $\rho' - \epsilon$  to  $\rho' + \epsilon$  with any very small

positive  $\epsilon$ , we have

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{mn}}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma_{mn} \right] \rho \, d\rho = \int_{\rho'-\epsilon}^{\rho'+\epsilon} \frac{\delta(\rho - \rho')}{\rho} \rho \, d\rho, \quad (3.71)$$

or

$$\rho \frac{\partial \Gamma_{mn}}{\partial \rho} \Big|_{\rho'-\epsilon}^{\rho'+\epsilon} = -1. \quad (3.72)$$

Substitution (3.70) into the left hand side of (3.72) leads to

$$\rho k_n (\psi_1 \psi_2' - \psi_2 \psi_1') = 1, \quad (3.73)$$

or

$$k_n \rho W[\psi_1, \psi_2] = 1, \quad (3.74)$$

where  $W[\psi_1, \psi_2]$  is the Wronskian of  $\psi_1$  and  $\psi_2$ .

Since there are no boundary surfaces,  $\Gamma_{mn}$  must be finite at  $\rho = 0$  and vanish at  $\rho \rightarrow \infty$ . Consequently  $\psi_1(k_n \rho) = A I_m(k_n \rho)$  and  $\psi_2(k_n \rho) = K_m(k_n \rho)$  where  $A$  is a constant. Substitution of  $\psi_1$  and  $\psi_2$  into the Wronskian condition (3.73) leads to

$$A(k_n \rho) W[I_m(k_n \rho), K_m(k_n \rho)] = 1, \quad (3.75)$$

and because<sup>34</sup>

$$W[I_m(x), K_m(x)] = -\frac{1}{x}, \quad (3.76)$$



then  $A = -1$  and the solution for  $\Gamma_{mn}$  is

$$\Gamma_{mn}(\rho) = -I_m(k_n \rho_<) K_m(k_n \rho_>) . \quad (3.77)$$

Results (3.67,3.68) and (3.77) leads us to the Green's function of the following form

$$G(\mathbf{r}, \mathbf{r}') = \sum_{m,n} A_{mn} \sin(k_n z) \cos(m\varphi) \Gamma_{mn}(\rho) . \quad (3.78)$$

Operating with the Laplace operator on both sides of equation (3.78) and substituting the result into equation (3.58) we have

$$\begin{aligned} \sum_{m,n} A_{mn} \sin(k_n z) \cos(m\varphi) \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{mn}}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma_{mn} \right] \\ = \delta(\varphi - \varphi') \delta(z - z') \frac{\delta(\rho - \rho')}{\rho} . \end{aligned} \quad (3.79)$$

Notice that for any integers  $m, m' \geq 0$

$$\int_0^{2\pi} \cos(m\varphi) \cos(m'\varphi) d\varphi = \frac{2\pi \delta_{mm'}}{\varepsilon_m} , \quad (3.80)$$

where  $\varepsilon_m = 1$  if  $m = 0$  and  $\varepsilon_m = 2$  if  $m > 0$ . Also for any integer  $n \geq 1$  we have

$$\int_0^h \sin(k_n z) \sin(k'_n z) dz = \frac{h \delta_{nn'}}{2} . \quad (3.81)$$

Now we multiple two sides of equation (3.79) with  $\cos(m'\varphi)$  and integrate from 0 to  $2\pi$  to give

$$\begin{aligned} \sum_{m,n} A_{mn} \sin(k_n z) \int_0^{2\pi} \cos(m\varphi) \cos(m'\varphi) d\varphi \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{mn}}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma_{mn} \right] \\ = \delta(z - z') \frac{\delta(\rho - \rho')}{\rho} \int_0^{2\pi} \delta(\varphi - \varphi') \cos(m'\varphi) d\varphi , \end{aligned} \quad (3.82)$$

or, by using (3.80)

$$\begin{aligned} \sum_{m,n} A_{mn} \sin(k_n z) \frac{2\pi \delta_{mm'}}{\varepsilon_m} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{mn}}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma_{mn} \right] \\ = \delta(z - z') \frac{\delta(\rho - \rho')}{\rho} \cos(m' \varphi'), \end{aligned} \quad (3.83)$$

and hence

$$\begin{aligned} \sum_n A_{m'n} \sin(k_n z) \frac{2\pi}{\varepsilon_{m'}} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{m'n}}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma_{m'n} \right] \\ = \delta(z - z') \frac{\delta(\rho - \rho')}{\rho} \cos(m' \varphi'). \end{aligned} \quad (3.84)$$

Multiplication the two sides of (3.83) by  $\sin(k_{n'} z)$  and then integration over  $z$  from 0 to  $h$  gives

$$\begin{aligned} \sum_n A_{m'n} \int_0^h \sin(k_n z) \sin(k_{n'} z) dz \frac{2\pi}{\varepsilon_{m'}} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{m'n}}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma_{m'n} \right] \\ = \frac{\delta(\rho - \rho')}{\rho} \cos(m' \varphi') \int_0^h \delta(z - z') \sin(k_{n'} z) dz, \end{aligned} \quad (3.85)$$

or through use of (3.81)

$$\begin{aligned} \sum_n A_{m'n} \frac{h \delta_{nn'}}{2} \frac{2\pi}{\varepsilon_{m'}} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{m'n}}{\partial \rho} \right) - \left( k_n^2 + \frac{m^2}{\rho^2} \right) \Gamma_{m'n} \right] \\ = \frac{\delta(\rho - \rho')}{\rho} \cos(m' \varphi') \sin(k_{n'} z'), \end{aligned} \quad (3.86)$$

and hence

$$A_{m'n'} \frac{h\pi}{\varepsilon_{m'}} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Gamma_{m'n'}}{\partial \rho} \right) - \left( k_{n'}^2 + \frac{m'^2}{\rho^2} \right) \Gamma_{m'n'} \right] = \frac{\delta(\rho - \rho')}{\rho} \cos(m' \varphi') \sin(k_{n'} z'), \quad (3.87)$$

Because the function  $\Gamma_{mn}$  satisfies equation (3.69) then we have

$$A_{m'n'} \frac{h\pi}{\varepsilon_{m'}} = \cos(m'\varphi') \sin(k_n z'), \quad (3.88)$$

or

$$A_{m'n'} = \frac{\varepsilon_{m'} \cos(m'\varphi') \sin(k_n z')}{\pi h}. \quad (3.89)$$

Substitution (3.77,3.89) into (3.78) gives the final solution of Green's function

$$G(\mathbf{r}, \mathbf{r}') = - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \varepsilon_m \frac{\cos(m\varphi) \cos(m\varphi') \sin(k_n z) \sin(k_n z')}{\pi h} I_m(k_n \rho_{<}) K_m(k_n \rho_{>}). \quad (3.90)$$

### 3.3 The acoustic wave

In general, a Gaussian sound wave with a pulse width  $\tau$  moving in the  $z$ -direction and having an arbitrary center point  $(x_0, y_0)$  has the following form in Cartesian coordinates:

$$p(x, y, z, t) = p_0 e^{-i\kappa z} e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{a^2}} e^{-\left(\frac{t-\frac{z}{c}}{\tau}\right)^2}, \quad (3.91)$$

where  $a$  is a parameter describing the diameter of the burst and  $p_0$  is a constant. Some limiting cases in different coordinates are discussed as below

#### 3.3.1 Cartesian coordinates

##### Large beam - Plane wave

In this case,  $a$  and  $\tau$  are large. Equation (3.91) then reduces to

$$p(x, y, z, t) = p_0 e^{-i\kappa z}. \quad (3.92)$$

### Narrow beam

We will use here the Delta Dirac function; first, we use one of its representations and two properties

$$\delta(x) = \lim_{a \rightarrow 0} \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}}, \quad (3.93)$$

$$\delta(ax) = \frac{\delta(x)}{|a|}, \quad (3.94)$$

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \quad (3.95)$$

where in equation (3.95)  $x_i$  satisfies  $g(x_i) = 0$  and the sum is taken in all the roots of equation  $g(x) = 0$ . The Gaussian wave in equation (3.91) can be rewritten as

$$\begin{aligned} p(x, y, z, t) &= p_0 e^{-i\kappa z} e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{a^2}} e^{-\left(\frac{t-\frac{z}{c}}{2}\right)^2}, \\ &= p_0 e^{-i\kappa z} a\sqrt{\pi} \frac{1}{a\sqrt{\pi}} e^{-\frac{(x-x_0)^2}{a^2}} a\sqrt{\pi} \frac{1}{a\sqrt{\pi}} e^{-\frac{(y-y_0)^2}{a^2}} e^{-\left(\frac{t-\frac{z}{c}}{2}\right)^2}, \\ &= \pi a^2 p_0 \delta(x - x_0) \delta(y - y_0) e^{-i\kappa z} e^{-\left(\frac{t-\frac{z}{c}}{2}\right)^2}. \end{aligned} \quad (3.96)$$

### 3.3.2 Spherical coordinates

The spherical coordinates are related to the Cartesian ones as follow

$$\begin{cases} x = r \sin \varphi \cos \theta, \\ y = r \sin \varphi \sin \theta, \\ z = r \cos \varphi, \end{cases} \quad (3.97)$$

We have

$$\begin{aligned}
& (x - x_0)^2 + (y - y_0)^2 \\
&= (r \sin \varphi \cos \theta - r_0 \sin \varphi_0 \cos \theta_0)^2 + (r \sin \varphi \sin \theta - r_0 \sin \varphi_0 \sin \theta_0)^2, \\
&= r^2 \sin^2 \varphi \cos^2 \theta - 2rr_0 \sin \varphi \cos \theta \sin \varphi_0 \cos \theta_0 + r_0^2 \sin^2 \varphi_0 \cos^2 \theta_0 + \\
&\quad + r^2 \sin^2 \varphi \sin^2 \theta - 2rr_0 \sin \varphi \sin \theta \sin \varphi_0 \sin \theta_0 + r_0^2 \sin^2 \varphi_0 \sin^2 \theta_0, \\
&= r^2 \sin^2 \varphi - 2rr_0 \sin \varphi \sin \varphi_0 \cos(\theta - \theta_0) + r_0^2 \sin^2 \varphi_0, \tag{3.98}
\end{aligned}$$

so the pressure in equation (3.91) can be written in spherical coordinates as follow

$$\begin{aligned}
p &= p_0 e^{-i\kappa r \cos \varphi} e^{-\frac{r^2 \sin^2 \varphi - 2rr_0 \sin \varphi \sin \varphi_0 \cos(\theta - \theta_0) + r_0^2 \sin^2 \varphi_0}{a^2}} e^{-\frac{4(ct - r \cos \varphi)^2}{(c\tau)^2}}, \\
&= p_0 e^{-i\kappa r \cos \varphi} e^{-\frac{(r \sin \varphi - r_0 \sin \varphi_0 \cos(\theta - \theta_0))^2 + r_0^2 \sin^2 \varphi_0 \sin^2(\theta - \theta_0)}{a^2}} e^{-\frac{4(ct - r \cos \varphi)^2}{(c\tau)^2}}. \tag{3.99}
\end{aligned}$$

### Large beam - Plane wave

In this case,  $a$  and  $\tau$  are large. Equation (3.91) is reduced to

$$p = p_0 e^{-i\kappa r \cos \varphi}. \tag{3.100}$$

### Narrow beam

For simplicity, I will take  $\varphi_0 = \pi/2$ ; then, equation (3.99) can be rewritten as

$$p = p_0 e^{-i\kappa r \cos \varphi} e^{-\frac{(r \sin \varphi - r_0 \cos(\theta - \theta_0))^2 + r_0^2 \sin^2(\theta - \theta_0)}{a^2}} e^{-\frac{4(ct - r \cos \varphi)^2}{(c\tau)^2}}. \tag{3.101}$$

Using equations (3.93 - 3.95), we have

$$\begin{aligned}
 e^{-\frac{(r_0 \sin(\theta - \theta_0))^2}{a^2}} &= a\sqrt{\pi} \frac{1}{a\sqrt{\pi}} e^{-\frac{(r_0 \sin(\theta - \theta_0))^2}{a^2}}, \\
 &= a\sqrt{\pi} \delta[r_0 \sin(\theta - \theta_0)], \\
 &= \frac{a\sqrt{\pi}}{r_0} \delta[\sin(\theta - \theta_0)], \\
 &= \frac{a\sqrt{\pi}}{r_0} \delta(\theta - \theta_0),
 \end{aligned} \tag{3.102}$$

and that

$$\begin{aligned}
 e^{-\frac{(r \sin \varphi - r_0 \cos(\theta - \theta_0))^2}{a^2}} &= a\sqrt{\pi} \frac{1}{a\sqrt{\pi}} e^{-\frac{(r \sin \varphi - r_0 \cos(\theta - \theta_0))^2}{a^2}}, \\
 &= a\sqrt{\pi} \delta[r \sin \varphi - r_0 \cos(\theta - \theta_0)], \\
 &= \frac{a\sqrt{\pi}}{r} \delta\left(\sin \varphi - \frac{r_0}{r} \cos(\theta - \theta_0)\right).
 \end{aligned} \tag{3.103}$$

Combining equations (3.101, 3.102) and (3.103) we have

$$p = \frac{\pi a^2 p_0}{r r_0} \delta(\theta - \theta_0) \delta\left(\sin \varphi - \frac{r_0}{r} \cos(\theta - \theta_0)\right) e^{-i\kappa r \cos \varphi - \frac{4(ct - r \cos \varphi)^2}{(c\tau)^2}}. \tag{3.104}$$

### 3.3.3 Horizontally cylindrical coordinates

The horizontally cylindrical coordinates relate to the Cartesian ones as follows

$$\begin{cases} x = x, \\ y = r \sin \varphi, \\ z = r \cos \varphi, \end{cases} \tag{3.105}$$

so the pressure in equation (3.91) can be written in horizontally cylindrical coordinates as follows

$$p = p_0 e^{-i\kappa r \cos \varphi} e^{-\frac{(x-x_0)^2 + (r \sin \varphi - r_0 \sin \varphi_0)^2}{a^2}} e^{-\frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}}. \quad (3.106)$$

### Large beam - Plane wave

In this case,  $a$  and  $\tau$  are large. Equation (3.91) is then reduced to

$$p = p_0 e^{-i\kappa r \cos \varphi}. \quad (3.107)$$

### Narrow beam

We can easily see

$$\begin{aligned} p &= p_0 e^{-i\kappa r \cos \varphi} e^{-\frac{(x-x_0)^2 + (r \sin \varphi - r_0 \sin \varphi_0)^2}{a^2}} e^{-\frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}}, \\ &= p_0 e^{-i\kappa r \cos \varphi} e^{-\frac{(x-x_0)^2}{a^2}} e^{-\frac{(r \sin \varphi - r_0 \sin \varphi_0)^2}{a^2}} e^{-\frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}}, \\ &= \pi a^2 p_0 e^{-i\kappa r \cos \varphi} \delta(x - x_0) \delta(r \sin \varphi - r_0 \sin \varphi_0) e^{-\frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}}. \end{aligned} \quad (3.108)$$

## 3.4 The generation of current

This section describes how the current in the simple case of two parallel electrodes is calculated with Dirichlet's conditions. As discussed above, our task is to solve the following Dirichlet's problem

$$\text{Dirichlet's problem} \quad \begin{cases} \varphi(\mathbf{r}) &= \int_V G(\mathbf{r}; \mathbf{r}') \nabla(\alpha' \nabla p) dV', \\ \Delta G &= \delta(\mathbf{r} - \mathbf{r}'), \\ G|_{\partial V} &= 0. \end{cases} \quad (3.109)$$

The current flowing between the electrodes is generated by two mechanisms, induced displacement and the flow of free charge. In the first mechanism, supposing that the region between the two plates is nonconducting dielectric, the time-varying polarization vector  $\mathbf{D}$  induced by the ultrasound source in colloidal or ionic object will generate the current  $\mathbf{J}_P$  as follow

$$\mathbf{J}_P = \frac{\partial \mathbf{D}}{\partial t}, \quad (3.110)$$

where the displacement vector can be found from the potential through equation (3.15) and the constitutive relation (3.5). Thus  $\mathbf{J}_P$  is found from the potential as

$$\mathbf{J}_P = -\epsilon \frac{\partial}{\partial t} \nabla \varphi(\mathbf{r}), \quad (3.111)$$

In the second mechanism, the time-varying polarization produces an electric field  $\mathbf{E}$  in a weak conductor, and from the relation (3.7) the free charge current can be expressed in terms of the potential as

$$\mathbf{J}_F(\mathbf{r}) = -\sigma \nabla \varphi(\mathbf{r}). \quad (3.112)$$

The total current density then is

$$\mathbf{J} = \mathbf{J}_P + \mathbf{J}_F = -\epsilon \frac{\partial}{\partial t} \nabla \varphi(\mathbf{r}) - \sigma \nabla \varphi(\mathbf{r}), \quad (3.113)$$

Substituting (3.19) into (3.113) gives

$$\mathbf{J} = -\epsilon i \omega \nabla \varphi(\mathbf{r}) - \sigma \nabla \varphi(\mathbf{r}) = -(\sigma + i \omega \epsilon) \nabla \varphi(\mathbf{r}). \quad (3.114)$$

The current flow between the electrodes then is found by integrating the normal component of the electric field, that is, the component perpendicular to one of the



surfaces, over the surface of one of the electrodes  $S$ ; hence, the expression for the frequency domain current  $I$  in the external circuit is given by

$$I = \int_S J_n dS_n = -(\sigma + i\omega\epsilon) \int_S \nabla_n \varphi(\mathbf{r}) dS_n . \quad (3.115)$$

In this case, the electrodes are in parallel with x-y plane, hence the normal component of the electric field is in parallel with z-axis, namely  $\nabla_n \varphi(\mathbf{r}) = \nabla_z \varphi(\mathbf{r}) = \frac{\partial}{\partial z} \varphi(\mathbf{r})$  and  $dS_n = dxdy$ . Using Green's function found in section 3.2, ie. equation (3.57), and the potential  $\varphi(\mathbf{r})$  in equation (3.38) we have

$$\begin{aligned} \varphi(\mathbf{r}) = & \frac{-1}{2\pi^2 h (\epsilon - i\frac{\sigma}{\omega})} \sum_{m=1}^{\infty} \int_V \sin\left(\frac{m\pi}{h} z\right) \sin\left(\frac{m\pi}{h} z'\right) \\ & \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{[\kappa_x^2 + \kappa_y^2 + (\frac{m\pi}{h})^2]} d\kappa_x d\kappa_y \nabla'(\alpha \nabla' p) dV' . \end{aligned} \quad (3.116)$$

Notice that the current should be conserved, that is, it is unchanged when equation (3.115) is calculated on any surface  $S$  perpendicular to the flow. For simplicity, we will calculate the current at the surface  $z = h$

$$I = -(\sigma + i\omega\epsilon) \int_S \left. \frac{\partial \varphi(\mathbf{r})}{\partial z} \right|_{z=h} dxdy . \quad (3.117)$$

The gradient of  $\varphi$  in the  $z$  direction is

$$\begin{aligned} \frac{\partial \varphi(\mathbf{r})}{\partial z} = & \frac{-1}{2\pi^2 h (\epsilon - i\frac{\sigma}{\omega})} \sum_{m=1}^{\infty} \int_V \frac{m\pi}{h} \cos\left(\frac{m\pi}{h} z\right) \sin\left(\frac{m\pi}{h} z'\right) \\ & \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{[\kappa_x^2 + \kappa_y^2 + (\frac{m\pi}{h})^2]} d\kappa_x d\kappa_y \nabla'(\alpha \nabla' p) dV' . \end{aligned} \quad (3.118)$$

Given the relation

$$\begin{aligned} \cos\left(\frac{m\pi}{h}z\right)\Big|_{z=h} &= \cos(m\pi) , \\ &= (-1)^m , \end{aligned} \quad (3.119)$$

it follows that

$$\begin{aligned} \frac{\partial\varphi(\mathbf{r})}{\partial z}\Big|_{z=h} &= \frac{-i\omega}{2\pi^2h(i\omega\epsilon + \sigma)} \sum_{m=1}^{\infty} \int_V (-1)^m \frac{m\pi}{h} \sin\left(\frac{m\pi}{h}z'\right) \\ &\quad \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x')+\kappa_y(y-y')]} }{[\kappa_x^2 + \kappa_y^2 + (\frac{m\pi}{h})^2]} d\kappa_x d\kappa_y \nabla'(\alpha \nabla' p) dV' . \end{aligned} \quad (3.120)$$

From equations (3.117) and (3.120) we have

$$\begin{aligned} I &= \frac{i\omega}{2\pi^2h} \int_S dx dy \sum_{m=1}^{\infty} \int_V (-1)^m \frac{m\pi}{h} \sin\left(\frac{m\pi}{h}z'\right) \\ &\quad \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x')+\kappa_y(y-y')]} }{[\kappa_x^2 + \kappa_y^2 + (\frac{m\pi}{h})^2]} d\kappa_x d\kappa_y \nabla'(\alpha \nabla' p) dV' , \end{aligned} \quad (3.121)$$

or

$$\begin{aligned} I &= \frac{i\omega}{2\pi^2h} \int_V \sum_{m=1}^{\infty} (-1)^m \frac{m\pi}{h} \sin\left(\frac{m\pi}{h}z'\right) \\ &\quad \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\int_S e^{i[\kappa_x(x-x')+\kappa_y(y-y')]} dx dy}{[\kappa_x^2 + \kappa_y^2 + (\frac{m\pi}{h})^2]} d\kappa_x d\kappa_y \nabla'(\alpha \nabla' p) dV' . \end{aligned} \quad (3.122)$$

Note that

$$\int_{-\infty}^{\infty} e^{i\kappa_t t} dt = 2\pi\delta(\kappa_t) , \quad (3.123)$$

it follows that

$$\begin{aligned}
\int_S e^{i[\kappa_x(x-x')+\kappa_y(y-y')]} dx dy &= e^{-i(\kappa_x x' + \kappa_y y')} \int_{-\infty}^{\infty} e^{i\kappa_x x} dx \int_{-\infty}^{\infty} e^{i\kappa_y y} dy , \\
&= e^{-i(\kappa_x x' + \kappa_y y')} \cdot 2\pi\delta(\kappa_x) \cdot 2\pi\delta(\kappa_y) , \\
&= (2\pi)^2 e^{-i(\kappa_x x' + \kappa_y y')} \delta(\kappa_x) \delta(\kappa_y) , \tag{3.124}
\end{aligned}$$

so from equation (3.122) we have

$$\begin{aligned}
I &= \frac{(2\pi)^2 i\omega}{2\pi^2 h} \int_V \sum_{m=1}^{\infty} (-1)^m \frac{m\pi}{h} \sin\left(\frac{m\pi}{h} z'\right) \\
&\quad \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i(\kappa_x x' + \kappa_y y')} \delta(\kappa_x) \delta(\kappa_y)}{[\kappa_x^2 + \kappa_y^2 + (\frac{m\pi}{h})^2]} d\kappa_x d\kappa_y \nabla'(\alpha \nabla' p) dV' . \tag{3.125}
\end{aligned}$$

Note that for any function  $f(t)$ ,

$$\int_{-\infty}^{\infty} f(t) \delta(t) dt = f(0) , \tag{3.126}$$

so

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i(\kappa_x x' + \kappa_y y')} \delta(\kappa_x) \delta(\kappa_y)}{[\kappa_x^2 + \kappa_y^2 + (\frac{m\pi}{h})^2]} d\kappa_x d\kappa_y &= \int_{-\infty}^{\infty} \frac{e^{-i\kappa_y y'} \delta(\kappa_y)}{[\kappa_y^2 + (\frac{m\pi}{h})^2]} d\kappa_y , \\
&= \frac{1}{(\frac{m\pi}{h})^2} , \tag{3.127}
\end{aligned}$$

it follows that

$$\begin{aligned}
I &= \frac{2i\omega}{h} \int_V \sum_{m=1}^{\infty} (-1)^m \frac{m\pi}{h} \sin\left(\frac{m\pi}{h} z'\right) \frac{1}{(\frac{m\pi}{h})^2} \nabla'(\alpha \nabla' p) dV' , \\
&= \frac{2i\omega}{h} \int_V \sum_{m=1}^{\infty} \frac{(-1)^m \sin(\frac{m\pi}{h} z')}{\frac{m\pi}{h}} \nabla'(\alpha \nabla' p) dV' . \tag{3.128}
\end{aligned}$$

Using the following identities for any  $0 \leq t \leq \pi$

$$\sum_{m=1}^{\infty} (-1)^m \frac{\sin mt}{m} = -\frac{t}{2} , \quad (3.129)$$

$$\sum_{m=1}^{\infty} \frac{\sin mt}{m} = \frac{\pi - t}{2} , \quad (3.130)$$

equation (3.128) leads to

$$I = -\frac{i\omega}{h} \int_V z' \nabla (\alpha \nabla' p) dV' . \quad (3.131)$$

Dropping the prime sign for convenience, we can rewrite equation (3.131) as

$$\begin{aligned} I &= -\frac{i\omega}{h} \int_V z \nabla (\alpha \nabla p) dV , \\ &= -\frac{i\omega}{h} \int_V z \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \left( \alpha \frac{\partial p}{\partial x} \mathbf{i} + \alpha \frac{\partial p}{\partial y} \mathbf{j} + \alpha \frac{\partial p}{\partial z} \mathbf{k} \right) dxdydz , \\ &= -\frac{i\omega}{h} \int_V z \left[ \frac{\partial}{\partial x} \left( \alpha \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \alpha \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial z} \left( \alpha \frac{\partial p}{\partial z} \right) \right] dxdydz . \end{aligned} \quad (3.132)$$

The above integral can be simplified with the assumption of the limited dimensions of the object, as in an experiment. In this case, the value of  $\alpha$  at  $x = \pm\infty, y = \pm\infty$  and  $z = 0$  or  $z = h$  can be taken as zero. Consider each of the terms in equation (3.132)

$$\begin{aligned} \int_V z \frac{\partial}{\partial x} \left( \alpha \frac{\partial p}{\partial x} \right) dxdydz &= \int_0^h zdz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left( \alpha \frac{\partial p}{\partial x} \right) dx , \\ &= \int_0^h zdz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d \left( \alpha \frac{\partial p}{\partial x} \right) , \\ &= \int_0^h zdz \int_{-\infty}^{\infty} dy \left( \alpha \frac{\partial p}{\partial x} \right) \Big|_{x=-\infty}^{x=\infty} , \\ &= 0 , \end{aligned} \quad (3.133)$$

$$\begin{aligned}
\int_V z \frac{\partial}{\partial y} \left( \alpha \frac{\partial p}{\partial y} \right) dx dy dz &= \int_0^h z dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \left( \alpha \frac{\partial p}{\partial y} \right) dy , \\
&= \int_0^h z dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d \left( \alpha \frac{\partial p}{\partial y} \right) , \\
&= \int_0^h z dz \int_{-\infty}^{\infty} dx \left( \alpha \frac{\partial p}{\partial y} \right) \Big|_{y=-\infty}^{y=\infty} , \\
&= 0 ,
\end{aligned} \tag{3.134}$$

$$\begin{aligned}
\int_V z \frac{\partial}{\partial z} \left( \alpha \frac{\partial p}{\partial z} \right) dx dy dz &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_0^h z \frac{\partial}{\partial z} \left( \alpha \frac{\partial p}{\partial z} \right) dz , \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_0^h z d \left( \alpha \frac{\partial p}{\partial z} \right) , \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[ z \alpha \frac{\partial p}{\partial z} \Big|_0^h - \int_0^h \alpha \frac{\partial p}{\partial z} dz \right] , \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[ 0 - \int_0^h \alpha \frac{\partial p}{\partial z} dz \right] , \\
&= - \int_V \alpha \frac{\partial p}{\partial z} dx dy dz .
\end{aligned} \tag{3.135}$$

The current in (3.132) then can be rewritten as:

$$I = \frac{i\omega}{h} \int_V \alpha(\mathbf{r}, \omega) \frac{\partial p(\mathbf{r})}{\partial z} dx dy dz . \tag{3.136}$$

Integrating by parts, equation (3.136) leads to

$$\begin{aligned}
I &= \frac{i\omega}{h} \int_V \alpha(\mathbf{r}, \omega) \frac{\partial p(\mathbf{r})}{\partial z} dx dy dz , \\
&= \frac{i\omega}{h} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \int_0^h \alpha(\mathbf{r}, \omega) \frac{\partial p(\mathbf{r})}{\partial z} dz , \\
&= \frac{i\omega}{h} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \int_{z=0}^{z=h} \alpha(\mathbf{r}, \omega) dp(\mathbf{r}) , \\
&= \frac{i\omega}{h} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \left[ \alpha(\mathbf{r}, \omega) p(\mathbf{r}) \Big|_{z=0}^{z=h} - \int_0^h p(\mathbf{r}) \frac{\partial}{\partial z} \alpha(\mathbf{r}) dz \right] , \\
&= - \frac{i\omega}{h} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \int_0^h p(\mathbf{r}) \frac{\partial}{\partial z} \alpha(\mathbf{r}) dz ,
\end{aligned} \tag{3.137}$$

or

$$I = -\frac{i\omega}{h} \int_V [\nabla_z \alpha(\mathbf{r})] p(\mathbf{r}) dx dy dz . \quad (3.138)$$

Using the linearized Navier-Stokes law for an inviscid fluid

$$\nabla p(\mathbf{r}) = -\rho \ddot{\mathbf{u}}, \quad (3.139)$$

where  $\mathbf{u}(\mathbf{r})$  is the displacement of the medium in the acoustic field which is related to the acceleration  $\ddot{\mathbf{u}}(\mathbf{r})$  in a plane wave as  $\ddot{\mathbf{u}} = -\omega^2 \mathbf{u}$ , hence the expression for the current in terms of the displacement in the plane wave can be written as

$$I = \frac{i\omega^3 \rho}{h} \int_V \alpha(\mathbf{r}, \omega) u_z(\mathbf{r}) dx dy dz . \quad (3.140)$$

### 3.4.1 Infinite layer of the thickness of $\Delta$

In the case of a plane wave that propagates in the z-direction through an infinite layer of thickness  $\Delta$ , we can take the pressure  $p$  in Eq. (3.92) and  $\alpha$  as

$$\alpha(x, y, z) = \alpha_0 [H(z - z_0) - H(z - z_1)] , \quad (3.141)$$

where  $z_1 - z_0 = \Delta$ ,  $\alpha_0$  and  $p_0$  are constant and  $H(t)$  is the Heaviside function

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 , \\ 0 & \text{if } t < 0 . \end{cases} \quad (3.142)$$

Using (3.138) and noticing that  $\frac{\partial}{\partial z}H(z - z_0) = \delta(z - z_0)$  we have

$$\begin{aligned}
I &= -\frac{i\omega}{h} \int_V \nabla_z \alpha(\mathbf{r}) p(\mathbf{r}) dx dy dz , \\
&= -\frac{i\omega}{h} \int_V \frac{\partial}{\partial z} \left( \alpha_0 [H(z - z_0) - H(z - z_1)] \right) p_0 e^{-i\kappa z} dx dy dz , \\
&= -\frac{i\omega\alpha_0 p_0}{h} \int_V \frac{\partial}{\partial z} \left( [H(z - z_0) - H(z - z_1)] \right) e^{-i\kappa z} dx dy dz , \\
&= -\frac{i\omega\alpha_0 p_0}{h} \int_V [\delta(z - z_0) - \delta(z - z_1)] e^{-i\kappa z} dx dy dz , \\
&= \int_S \left( -\frac{i\omega\alpha_0 p_0}{h} \int_0^h [\delta(z - z_0) - \delta(z - z_1)] e^{-i\kappa z} dz \right) dx dy , \\
&= \int_S \left( -\frac{i\omega\alpha_0 p_0}{h} [e^{-i\kappa z_0} - e^{-i\kappa z_1}] \right) dx dy . \tag{3.143}
\end{aligned}$$

Equation (3.143) gives us the current density  $J$  flowing through the unit area of these electrodes

$$J = -\frac{i\omega\alpha_0 p_0}{h} [e^{-i\kappa z_0} - e^{-i\kappa z_1}] . \tag{3.144}$$

With the notation  $I_0 = \frac{\omega\alpha_0 p_0}{h}$ , equation (3.144) reads

$$J = -iI_0 [e^{-i\kappa z_0} - e^{-i\kappa z_1}] . \tag{3.145}$$

### 3.4.2 Infinite thin layer

Applying equation (3.144) for the case of thin layer, meaning the thickness of the thick layer is very small  $\Delta \ll z_0, z_1, h$ , and using the following Taylor series for any real function  $f(t)$  to be expanded about a point  $t = t_0$

$$f(t) = f(t_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(t_0)}{n!} (t - t_0)^n , \tag{3.146}$$

where  $f^{(n)}(t_0)$  is the n-th order derivative of the function  $f(t)$  at a point  $t = t_0$ . In the case of  $t - t_0 = \delta$  and  $\delta$  is very small, we have

$$\begin{aligned} f(t) &= f(t_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(t_0)}{n!} (t - t_0)^n, \\ &= f(t_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(t_0)}{n!} \delta^n, \\ &\approx f(t_0) + \delta f'(t_0). \end{aligned} \tag{3.147}$$

In our case

$$\begin{aligned} e^{-i\kappa z_1} &= e^{-i\kappa(z_0 + \Delta)}, \\ &\approx e^{-i\kappa z_0} + \Delta \left. \frac{de^{-i\kappa z}}{dz} \right|_{z=z_0}, \\ &\approx e^{-i\kappa z_0} - i\kappa \Delta e^{-i\kappa z_0}, \end{aligned} \tag{3.148}$$

so we have

$$\begin{aligned} J &= -\frac{i\omega\alpha_0 p_0}{h} [e^{-i\kappa z_0} - e^{-i\kappa z_1}], \\ &= -\frac{i\omega\alpha_0 p_0}{h} [e^{-i\kappa z_0} - (e^{-i\kappa z_0} - i\kappa \Delta e^{-i\kappa z_0})], \\ &= \frac{\kappa\omega\alpha_0 p_0 \Delta}{h} e^{-i\kappa z_0}, \end{aligned} \tag{3.149}$$

or

$$J = I_0(\kappa \Delta) e^{-i\kappa z_0}, \tag{3.150}$$

or because  $\omega = \kappa c$  where  $c$  is the velocity of sound moving inside the object, we have

$$J = \frac{\omega^2 \alpha_0 p_0 \Delta}{hc} e^{-\frac{i\omega z_0}{c}}. \tag{3.151}$$



### 3.4.3 The spherical object of radius $R$

Consider a spherical colloidal object with a radius  $R$  in the spherical coordinate  $(r, \theta, \varphi)$ . The expression for  $\alpha$  should read

$$\alpha(\mathbf{r}) = \alpha_0 H(R - r) , \quad (3.152)$$

and the pressure should have the form as in equation (3.101). The Cartesian partial derivative of  $z$  in spherical coordinates is

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial r}{\partial z} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial \varphi} , \\ &= \cos \varphi \frac{\partial}{\partial r} - \frac{\sin \varphi}{r} \frac{\partial}{\partial \varphi} , \end{aligned} \quad (3.153)$$

hence it follows that

$$\begin{aligned} \frac{\partial}{\partial z} H(R - r) &= \left( \cos \varphi \frac{\partial}{\partial r} - \frac{\sin \varphi}{r} \frac{\partial}{\partial \varphi} \right) H(R - r) , \\ &= \cos \varphi \frac{\partial}{\partial r} H(R - r) , \\ &= -\cos \varphi \delta(R - r) . \end{aligned} \quad (3.154)$$

In the case that we choose  $(x_0, y_0, z_0) = (0, 0, r_0)$  then  $(r_0, \theta_0, \varphi_0) = (r_0, \theta_0, 0)$  so

$$p = p_0 e^{-i\kappa r \cos \varphi} e^{-\frac{r^2 \sin^2 \varphi}{a^2}} e^{-\frac{4(ct - r \cos \varphi)^2}{(c\tau)^2}} , \quad (3.155)$$

and hence the current in (3.138) now is

$$\begin{aligned}
I &= -\frac{i\omega}{h} \int_V \nabla_z \alpha(\mathbf{r}) p(\mathbf{r}) dV , \\
&= -\frac{i\omega}{h} \int_V \frac{\partial}{\partial z} \left( \alpha_0 H(R-r) \right) p_0 e^{-i\kappa r \cos \varphi} e^{-\frac{r^2 \sin^2 \varphi}{a^2}} e^{-\frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}} dV , \\
&= \frac{i\omega \alpha_0 p_0}{h} \int_V \cos \varphi \delta(R-r) e^{-i\kappa r \cos \varphi} e^{-\frac{r^2 \sin^2 \varphi}{a^2}} e^{-\frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}} dV , \\
&= \frac{i\omega \alpha_0 p_0}{h} \int_0^\pi \sin \varphi \cos \varphi d\varphi \int_0^{2\pi} d\theta \int_0^\infty \delta(R-r) e^{-i\kappa r \cos \varphi} e^{-\frac{r^2 \sin^2 \varphi}{a^2}} e^{-\frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}} r^2 dr , \\
&= \frac{2\pi i\omega \alpha_0 p_0 R^2}{h} \int_0^\pi \sin \varphi \cos \varphi e^{-i\kappa R \cos \varphi} e^{-\frac{R^2 \sin^2 \varphi}{a^2}} e^{-\frac{4(ct-R \cos \varphi)^2}{(c\tau)^2}} d\varphi , \tag{3.156}
\end{aligned}$$

or when denoting  $\xi = \cos \varphi$ , because  $\varphi \in (0, \pi)$  so  $\xi \in (1, -1)$ ,  $\sin \varphi d\varphi = -d \cos \varphi = -d\xi$ ,  $\int_1^{-1} d\xi = -\int_{-1}^1 d\xi$ , we have

$$I = \frac{2\pi i\omega \alpha_0 p_0 R^2}{h} \int_{-1}^1 \xi e^{-i\kappa R \xi} e^{-\frac{R^2(1-\xi^2)}{a^2}} e^{-\frac{4(ct-R\xi)^2}{(c\tau)^2}} d\xi . \tag{3.157}$$

### Large beam continuous wave experiment

In this case, set  $a \rightarrow \infty$  and  $\tau \rightarrow \infty$  then from equation (3.157) we have

$$\begin{aligned}
I &= \frac{2\pi i\omega \alpha_0 p_0 R^2}{h} \int_{-1}^1 \xi e^{-i\kappa R \xi} d\xi , \\
&= \frac{2\pi i\omega \alpha_0 p_0 R^2}{h} \frac{1}{-i\kappa R} \int_{-1}^1 \xi d e^{-i\kappa R \xi} , \\
&= -\frac{2\pi \omega \alpha_0 p_0 R}{\kappa h} \left( \xi e^{-i\kappa R \xi} \Big|_{-1}^1 - \int_{-1}^1 e^{-i\kappa R \xi} d\xi \right) , \\
&= -\frac{2\pi \omega \alpha_0 p_0 R}{\kappa h} \left( \xi e^{-i\kappa R \xi} \Big|_{-1}^1 - \frac{e^{-i\kappa R \xi}}{-i\kappa R} \Big|_{-1}^1 \right) , \\
&= -\frac{2\pi \omega \alpha_0 p_0 R}{\kappa h} \left( e^{-i\kappa R} + e^{i\kappa R} - \frac{e^{i\kappa R} - e^{-i\kappa R}}{i\kappa R} \right) . \tag{3.158}
\end{aligned}$$

Using Euler formula

$$e^{i\varphi} = \cos \varphi + i \sin \varphi , \tag{3.159}$$

we have

$$\begin{cases} \cos \varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \\ \sin \varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}, \end{cases} \quad (3.160)$$

giving

$$I = -\frac{4\pi\omega\alpha_0 p_0 R}{\kappa h} \left( \cos \kappa R - \frac{\sin \kappa R}{\kappa R} \right). \quad (3.161)$$

or

$$I = -4\pi R^2 I_0 \frac{1}{\kappa R} \left( \cos \kappa R - \frac{\sin \kappa R}{\kappa R} \right). \quad (3.162)$$

### Narrow beam continuous wave experiment

In this case, the cross section  $a$  of the beam is very small,  $a \rightarrow 0$ , thus we can use the pressure as in equation (3.104). The current in the case of narrow beam should read

$$\begin{aligned} I &= -\frac{i\omega}{h} \int_V \nabla_z \alpha(\mathbf{r}) p(\mathbf{r}) dV, \\ &= -\frac{i\omega}{h} \int_V \frac{\partial}{\partial z} \left( \alpha_0 H(R-r) \right) \frac{\pi a^2 p_0}{r r_0} \delta(\theta - \theta_0) \\ &\quad \times \delta \left( \sin \varphi - \frac{r_0}{r} \cos(\theta - \theta_0) \right) e^{-i\kappa r \cos \varphi - \frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}} dV, \\ &= \frac{i\omega \pi a^2 \alpha_0 p_0}{h r_0} \int_0^\pi \sin \varphi d\varphi \int_0^{2\pi} d\theta \int_0^\infty \frac{1}{r} \cos \varphi \delta(R-r) \delta(\theta - \theta_0) \\ &\quad \times \delta \left( \sin \varphi - \frac{r_0}{r} \cos(\theta - \theta_0) \right) e^{-i\kappa r \cos \varphi - \frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}} r^2 dr, \\ &= \frac{i\omega \pi a^2 \alpha_0 p_0}{h r_0} \int_0^\pi \sin \varphi d\varphi \int_0^\infty \cos \varphi \delta(R-r) \delta \left( \sin \varphi - \frac{r_0}{r} \right) e^{-i\kappa r \cos \varphi - \frac{4(ct-r \cos \varphi)^2}{(c\tau)^2}} r dr, \\ &= \frac{i\omega \pi a^2 \alpha_0 p_0}{h r_0} \int_0^\pi \sin \varphi \cos \varphi \delta \left( \sin \varphi - \frac{r_0}{R} \right) e^{-i\kappa R \cos \varphi - \frac{4(ct-R \cos \varphi)^2}{(c\tau)^2}} R d\varphi. \end{aligned} \quad (3.163)$$

Let us denote  $\cos \varphi = \xi$  because  $\varphi \in (0, \pi)$  so  $\xi \in (1, -1)$ ,  $\sin \varphi d\varphi = -d \cos \varphi = -d\xi$ ,  $\int_1^{-1} d\xi = -\int_{-1}^1 d\xi$  and obviously  $\sin \varphi \geq 0$  hence  $\sin \varphi = \sqrt{1 - \xi^2}$ , equation

(3.163) can be rewritten as

$$I = \frac{i\omega\pi a^2 \alpha_0 p_0 R}{hr_0} \int_{-1}^1 \xi \delta\left(\sqrt{1-\xi^2} - \frac{r_0}{R}\right) e^{-i\kappa R\xi - \frac{4(ct-R\xi)^2}{(c\tau)^2}} d\xi . \quad (3.164)$$

In order to calculate (3.164), we use the property of delta function (3.95) with  $g(\xi) = \sqrt{1-\xi^2} - \frac{r_0}{R}$ . We know that equation  $g(\xi) = 0$  has two roots  $\xi_1 = \sqrt{1 - \frac{r_0^2}{R^2}}$  and  $\xi_2 = -\sqrt{1 - \frac{r_0^2}{R^2}}$ , and because  $g'(\xi) = -\frac{\xi}{\sqrt{1-\xi^2}}$  then  $g'(\xi_1) = -\frac{\sqrt{R^2-r_0^2}}{r_0}$  and  $g'(\xi_2) = \frac{\sqrt{R^2-r_0^2}}{r_0}$ . So we have

$$\begin{aligned} \delta\left(\sqrt{1-\xi^2} - \frac{r_0}{R}\right) &= \delta(g(\xi)), \\ &= \frac{\delta(\xi - \xi_1)}{|g'(\xi_1)|} + \frac{\delta(\xi - \xi_2)}{|g'(\xi_2)|}, \\ &= -\frac{r_0 \delta(\xi - \xi_1)}{\sqrt{R^2 - r_0^2}} + \frac{r_0 \delta(\xi - \xi_2)}{\sqrt{R^2 - r_0^2}}, \end{aligned} \quad (3.165)$$

and equation (3.164) now reads

$$\begin{aligned} I &= \frac{i\omega\pi a^2 \alpha_0 p_0 R}{hr_0} \int_{-1}^1 \xi \left[ -\frac{r_0 \delta(\xi - \xi_1)}{\sqrt{R^2 - r_0^2}} + \frac{r_0 \delta(\xi - \xi_2)}{\sqrt{R^2 - r_0^2}} \right] e^{-i\kappa R\xi - \frac{4(ct-R\xi)^2}{(c\tau)^2}} d\xi , \\ &= \frac{i\omega\pi a^2 \alpha_0 p_0 R}{h\sqrt{R^2 - r_0^2}} \int_{-1}^1 \xi \left[ -\delta(\xi - \xi_1) + \delta(\xi - \xi_2) \right] e^{-i\kappa R\xi - \frac{4(ct-R\xi)^2}{(c\tau)^2}} d\xi , \\ &= -\frac{i\omega\pi a^2 \alpha_0 p_0}{h} \left[ e^{-i\kappa\sqrt{R^2-r_0^2} - \frac{4\left(ct-\sqrt{R^2-r_0^2}\right)^2}{(c\tau)^2}} + e^{i\kappa\sqrt{R^2-r_0^2} - \frac{4\left(ct+\sqrt{R^2-r_0^2}\right)^2}{(c\tau)^2}} \right] , \end{aligned} \quad (3.166)$$

or

$$I = -i\pi a^2 I_0 \left[ e^{-i\kappa\sqrt{R^2-r_0^2} - \frac{4\left(ct-\sqrt{R^2-r_0^2}\right)^2}{(c\tau)^2}} + e^{i\kappa\sqrt{R^2-r_0^2} - \frac{4\left(ct+\sqrt{R^2-r_0^2}\right)^2}{(c\tau)^2}} \right] . \quad (3.167)$$

### 3.4.4 Spherical layer with outer radius $R_o$ and inner radius

$$R_i$$

In the case of a thick layer hollow sphere with the outer radius  $R_o$  and the inner radius  $R_i$ , the colloidal distribution can be taken as  $\alpha(\mathbf{r}) = \alpha_0[H(R_o - r) - H(R_i - r)]$ . The calculation in this section will be as the same as subsection (3.4.3), and the result can be found as

$$I = I(R_o) - I(R_i), \quad (3.168)$$

where  $I(R_o)$  and  $I(R_i)$  are the above results applied for  $R_o$  and  $R_i$  correspondingly.

#### Large beam

Using equation (3.168) and result (3.162), we find

$$I = -4\pi I_0 \left[ \frac{R_o^2}{\kappa R_o} \left( \cos \kappa R_o - \frac{\sin \kappa R_o}{\kappa R_o} \right) - \frac{R_i^2}{\kappa R_i} \left( \cos \kappa R_i - \frac{\sin \kappa R_i}{\kappa R_i} \right) \right]. \quad (3.169)$$

#### Narrow beam

Using equation (3.168) and (3.167), the result should read

$$I = -i\pi a^2 I_0 \left[ e^{-i\kappa\sqrt{R_o^2 - r_0^2} - \frac{4\left(ct - \sqrt{R_o^2 - r_0^2}\right)^2}{(c\tau)^2}} + e^{i\kappa\sqrt{R_o^2 - r_0^2} - \frac{4\left(ct + \sqrt{R_o^2 - r_0^2}\right)^2}{(c\tau)^2}} - e^{-i\kappa\sqrt{R_i^2 - r_0^2} - \frac{4\left(ct - \sqrt{R_i^2 - r_0^2}\right)^2}{(c\tau)^2}} - e^{i\kappa\sqrt{R_i^2 - r_0^2} - \frac{4\left(ct + \sqrt{R_i^2 - r_0^2}\right)^2}{(c\tau)^2}} \right]. \quad (3.170)$$

### 3.4.5 Spherical sheet thickness $\Delta$ and radius $R$

In the case of spherical sheet, the results in subsection (3.4.4) can be applied with  $R_o - R_i = \Delta$  is taken to be very small, so we can use the Taylor expansion given by (3.147).

### Large beam

Notice result (3.162), denote that

$$f(r) = r \cos \kappa r - \frac{\sin \kappa r}{\kappa}, \quad (3.171)$$

so the result should have the form

$$I = -4\pi I_0 \frac{\Delta f'(R)}{\kappa}. \quad (3.172)$$

We have

$$\begin{aligned} f'(R) &= \frac{\partial}{\partial r} \left( r \cos \kappa r - \frac{\sin \kappa r}{\kappa} \right) \Big|_{r=R}, \\ &= (\cos \kappa r - \kappa r \sin \kappa r - \cos \kappa r) \Big|_{r=R}, \\ &= -\kappa R \sin \kappa R, \end{aligned} \quad (3.173)$$

The combination of equations (3.172) and (3.173) gives

$$I = 4\pi I_0 (\Delta R) \sin \kappa R. \quad (3.174)$$

### Narrow beam

From the result given by equation (3.167), the function  $f$  becomes

$$f(r) = e^{-i\kappa\sqrt{r^2-r_0^2} - \frac{4\left(ct - \sqrt{r^2-r_0^2}\right)^2}{(c\tau)^2}} + e^{i\kappa\sqrt{r^2-r_0^2} - \frac{4\left(ct + \sqrt{r^2-r_0^2}\right)^2}{(c\tau)^2}}, \quad (3.175)$$

so the result for the current should have the form

$$I = -i\pi a^2 I_0 \Delta f'(R). \quad (3.176)$$

We have

$$\begin{aligned}
f'(R) &= \frac{\partial}{\partial r} \left( e^{-i\kappa\sqrt{r^2-r_0^2}-\frac{4\left(ct-\sqrt{r^2-r_0^2}\right)^2}{(c\tau)^2}} + e^{i\kappa\sqrt{r^2-r_0^2}-\frac{4\left(ct+\sqrt{r^2-r_0^2}\right)^2}{(c\tau)^2}} \right) \Big|_{r=R}, \\
&= \left( \frac{r}{\sqrt{r^2-r_0^2}} \left[ \frac{8(ct-\sqrt{r^2-r_0^2})}{(c\tau)^2} + i\kappa \right] e^{-i\kappa\sqrt{r^2-r_0^2}-\frac{4\left(ct-\sqrt{r^2-r_0^2}\right)^2}{(c\tau)^2}} - \right. \\
&\quad \left. - \frac{r}{\sqrt{r^2-r_0^2}} \left[ \frac{8(ct+\sqrt{r^2-r_0^2})}{(c\tau)^2} - i\kappa \right] e^{i\kappa\sqrt{r^2-r_0^2}-\frac{4\left(ct+\sqrt{r^2-r_0^2}\right)^2}{(c\tau)^2}} \right) \Big|_{r=R}, \\
&= \frac{R}{\sqrt{R^2-r_0^2}} \left[ \left( \frac{8(ct-\sqrt{R^2-r_0^2})}{(c\tau)^2} + i\kappa \right) e^{-i\kappa\sqrt{R^2-r_0^2}-\frac{4\left(ct-\sqrt{R^2-r_0^2}\right)^2}{(c\tau)^2}} - \right. \\
&\quad \left. - \left( \frac{8(ct+\sqrt{R^2-r_0^2})}{(c\tau)^2} - i\kappa \right) e^{i\kappa\sqrt{R^2-r_0^2}-\frac{4\left(ct+\sqrt{R^2-r_0^2}\right)^2}{(c\tau)^2}} \right]. \tag{3.177}
\end{aligned}$$

The combination of (3.176) and (3.177) gives

$$\begin{aligned}
I &= -i\pi a^2 I_0 \frac{R\Delta}{\sqrt{R^2-r_0^2}} \left[ \left( \frac{8(ct-\sqrt{R^2-r_0^2})}{(c\tau)^2} + i\kappa \right) e^{-i\kappa\sqrt{R^2-r_0^2}-\frac{4\left(ct-\sqrt{R^2-r_0^2}\right)^2}{(c\tau)^2}} - \right. \\
&\quad \left. - \left( \frac{8(ct+\sqrt{R^2-r_0^2})}{(c\tau)^2} - i\kappa \right) e^{i\kappa\sqrt{R^2-r_0^2}-\frac{4\left(ct+\sqrt{R^2-r_0^2}\right)^2}{(c\tau)^2}} \right]. \tag{3.178}
\end{aligned}$$

### 3.4.6 Cylinder of radius R - Large beam

Suppose that an infinite colloidal cylinder of radius R lies along x-axis. In the case of large beam, we can use the following expansion for  $\alpha$

$$\alpha(\mathbf{r}) = \alpha_0 H(R-r), \tag{3.179}$$

and take  $p$  as in equation (3.107). I will solve the problem in the cylindrical coordinates  $(x, r, \varphi)$  which are related to the Cartesian coordinates  $(x, y, z)$  as follows

$$\begin{cases} x = x, \\ y = r \cos \varphi, \\ z = r \sin \varphi, \varphi \in (0, 2\pi), \end{cases} . \quad (3.180)$$

Because  $\alpha(\mathbf{r})$  depends only on  $r$ , we have

$$\begin{aligned} \frac{\partial}{\partial z} \alpha(\mathbf{r}) &= \left( \frac{\partial x}{\partial z} \frac{\partial}{\partial x} + \frac{\partial r}{\partial z} \frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial \varphi} \right) \alpha_0 H(R - r), \\ &= \alpha_0 \frac{\partial}{\partial r} H(R - r) \frac{\partial \sqrt{y^2 + z^2}}{\partial z}, \\ &= -\alpha_0 \delta(R - r) \frac{z}{\sqrt{y^2 + z^2}}, \\ &= -\alpha_0 \sin \varphi \delta(R - r) . \end{aligned} \quad (3.181)$$

Now, using (3.138) and (3.181) we have

$$\begin{aligned} I &= -\frac{i\omega}{h} \int_V \frac{\partial \alpha(\mathbf{r})}{\partial z} p(\mathbf{r}) dV , \\ &= -\frac{i\omega}{h} \int_V -\alpha_0 \sin \varphi \delta(R - r) p_0 e^{-i\kappa r \sin \varphi} dV , \\ &= \frac{i\omega \alpha_0 p_0}{h} \int_V \sin \varphi \delta(R - r) e^{-i\kappa r \sin \varphi} dV , \\ &= iI_0 \int dx \int_0^{2\pi} d\varphi \int_0^\infty \sin \varphi \delta(R - r) e^{-i\kappa r \sin \varphi} r dr , \\ &= iI_0 \int dx \int_0^{2\pi} \sin \varphi e^{-i\kappa R \sin \varphi} R d\varphi . \end{aligned} \quad (3.182)$$

Given the integral expression for the Bessel's function  $J_n(x)$

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin \varphi - in\varphi} d\varphi , \quad (3.183)$$



and the identity

$$\frac{\partial}{\partial x} J_0(x) = -J_1(x), \text{ for } x > 0, \quad (3.184)$$

we have

$$\begin{aligned} \int_0^{2\pi} \sin \varphi e^{-ix \sin \varphi} d\varphi &= \int_0^{2\pi} \sin(-\varphi) e^{ix \sin(-\varphi)} d(-\varphi), \\ &= \int_0^{-2\pi} \sin \varphi' e^{ix \sin \varphi'} d\varphi', \quad (\varphi' = -\varphi), \\ &= - \int_0^{2\pi} \sin \varphi e^{ix \sin \varphi} d\varphi, \quad (\varphi = \varphi' + 2\pi), \\ &= i \int_0^{2\pi} i \sin \varphi e^{ix \sin \varphi} d\varphi, \\ &= i \frac{\partial}{\partial x} \left( \int_0^{2\pi} e^{ix \sin \varphi} d\varphi \right), \\ &= i \frac{\partial}{\partial x} 2\pi J_0(x), \\ &= -2\pi i J_1(x). \end{aligned} \quad (3.185)$$

Combining (3.182) and (3.185) we have

$$\begin{aligned} I &= iI_0 \int dx \int_0^{2\pi} \sin \varphi e^{-i\kappa R \sin \varphi} R d\varphi, \\ &= iI_0 R \int dx [-2\pi i J_1(\kappa R)], \\ &= 2\pi I_0 R J_1(\kappa R) \int dx. \end{aligned} \quad (3.186)$$

Equation (3.186) means that the current per unit length of the cylinder is

$$J = 2\pi I_0 R J_1(\kappa R). \quad (3.187)$$

### 3.4.7 Cylindrical layer with outer radius $R_o$ and inner radius $R_i$ - Large beam

The calculation for this section will be nearly identical to that given in section (3.4.6) with the colloidal distribution  $\alpha(r)$  given as  $\alpha(\mathbf{r}) = \alpha_0[H(R_o - r) - H(R_i - r)]$ . Obviously the result is

$$J = 2\pi I_0 [R_o J_1(\kappa R_o) - R_i J_1(\kappa R_i)] . \quad (3.188)$$

### 3.4.8 Cylindrical sheet with thickness $\Delta$ and radius $R$ - Large beam

The result for a cylindrical sheet with a thickness  $\Delta \ll R$  must be of the form

$$J = 2\pi I_0 \Delta f'(R) , \quad (3.189)$$

where  $f(r) = r J_1(\kappa r)$ . Because  $(x J_1(x))' = x J_0(x)$  then we have

$$\begin{aligned} f'(R) &= \left. \frac{\partial}{\partial r} r J_1(\kappa r) \right|_{r=R} , \\ &= \left. \frac{\partial}{\partial(\kappa r)} \kappa r J_1(\kappa r) \right|_{\kappa r = \kappa R} , \\ &= \kappa r I_0(\kappa r) \Big|_{r=R} , \\ &= \kappa R I_0(\kappa R) . \end{aligned} \quad (3.190)$$

The final result is then

$$J = 2\pi I_0 \Delta \kappa R J_0(\kappa R) . \quad (3.191)$$

## 3.5 The potential distribution

### 3.5.1 Infinite rectangular layer

The potential  $\phi(\mathbf{r})$  in a medium with conductivity  $\sigma$  and dielectric constant  $\epsilon$  containing a colloidal object whose spatial dependence is given by the function  $\alpha(\mathbf{r}, \omega)$ <sup>19</sup> can be written as a volume integral over a Dirichlet Green function  $G_D(\mathbf{r}, \mathbf{r}')$

$$\phi(\mathbf{r}) = \frac{i\omega}{\sigma + i\omega\epsilon} \int_V G_D(\mathbf{r}; \mathbf{r}') \nabla' \cdot [\alpha(\mathbf{r}', \omega) \nabla' p(\mathbf{r}')] dV'. \quad (3.192)$$

In the case of a plane wave traveling along  $z$  axis, the pressure is written as

$$p(\mathbf{r}) = p_0 e^{-i\kappa z}. \quad (3.193)$$

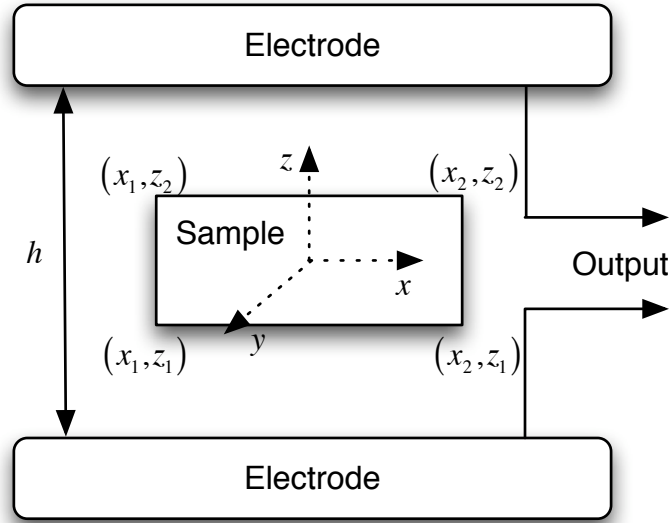


Figure 3.2: Rectangular layer with infinite length lying along  $y$  axis

If a colloidal object is a rectangle placed at  $x_1, x_2, y_1, y_2, z_1, z_2$  as shown in Figure

(3.2) then the colloidal distribution in space can be written as

$$\alpha(\mathbf{r}) = \alpha_0 \Theta_{x_1 x_2}(x) \Theta_{y_1 y_2}(y) \Theta_{z_1 z_2}(z), \quad (3.194)$$

where  $\Theta_{t_1 t_2}(t)$  is 1 if  $t_1 \leq t \leq t_2$  and 0 if otherwise with  $t = \{x, y, z\}$ .

The Dirichlet Green function in Cartesian coordinates, as discussed in section (3.2.1), is

$$G_D(\mathbf{r}; \mathbf{r}') = -\frac{1}{2\pi^2 h} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{h} z\right) \sin\left(\frac{m\pi}{h} z'\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} d\kappa_x d\kappa_y}{\left[\kappa_x^2 + \kappa_y^2 + \left(\frac{m\pi}{h}\right)^2\right]} \quad (3.195)$$

Now the gradient of pressure gives

$$\begin{aligned} \nabla p(\mathbf{r}) &= \nabla p_0 e^{-ikz}, \\ &= p_0 \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) e^{-ikz}, \\ &= -ik p_0 e^{-ikz} \hat{\mathbf{z}}, \end{aligned} \quad (3.196)$$

and hence

$$\begin{aligned} \nabla(\alpha \nabla p) &= \nabla[\alpha_0 \Theta_{x_1 x_2}(x) \Theta_{y_1 y_2}(y) \Theta_{z_1 z_2}(z) (-ik p_0 e^{-ikz} \hat{\mathbf{z}})], \\ &= \left( \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) [\alpha_0 \Theta_{x_1 x_2}(x) \Theta_{y_1 y_2}(y) \Theta_{z_1 z_2}(z) (-ik p_0 e^{-ikz} \hat{\mathbf{z}})], \\ &= -ik p_0 \alpha_0 \Theta_{x_1 x_2}(x) \Theta_{y_1 y_2}(y) \frac{\partial}{\partial z} [\Theta_{z_1 z_2}(z) e^{-ikz}]. \end{aligned} \quad (3.197)$$

On substitution of (3.197) and (3.195) into (3.192) we have

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \int_V \left[ \frac{-1}{2\pi^2 h} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \sin\left(\frac{n\pi}{h} z'\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} d\kappa_x d\kappa_y}{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} \right] \\ &\times \left[ -ik \alpha_0 p_0 \Theta_{x_1 x_2}(x') \Theta_{y_1 y_2}(y') \frac{\partial}{\partial z'} (\Theta_{z_1 z_2}(z') e^{-ikz'}) \right] dx' dy' dz', \end{aligned}$$

$$\begin{aligned}
&= \frac{ik\alpha_0 p_0}{2\pi^2 h \left(\epsilon - i\frac{\sigma}{\omega}\right)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \int_0^h \sin\left(\frac{n\pi}{h} z'\right) \frac{\partial}{\partial z'} \left( \Theta_{z_1 z_2}(z') e^{-ikz'} \right) dz' \\
&\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Theta_{x_1 x_2}(x') \Theta_{y_1 y_2}(y') \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y \right) dx' dy', \\
&= \frac{ik\alpha_0 p_0}{2\pi^2 h \left(\epsilon - i\frac{\sigma}{\omega}\right)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \int_0^h \sin\left(\frac{n\pi}{h} z'\right) \frac{\partial}{\partial z'} \left( \Theta_{z_1 z_2}(z') e^{-ikz'} \right) dz' \\
&\times \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y \right) dx' dy', \\
&= \frac{ik\alpha_0 p_0}{2\pi^2 h \left(\epsilon - i\frac{\sigma}{\omega}\right)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \cdot A_{z_1 z_2} \cdot B(x, y), \tag{3.198}
\end{aligned}$$

where

$$A_{z_1 z_2} = \int_0^h \sin\left(\frac{n\pi}{h} z'\right) \frac{\partial}{\partial z'} \left( \Theta_{z_1 z_2}(z') e^{-ikz'} \right) dz', \tag{3.199}$$

and

$$B(x, y) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y \right) dx' dy'. \tag{3.200}$$

### Calculation of $A_{z_1 z_2}$

We find factor  $A_{z_1 z_2}$  by using the method of integration by parts

$$\begin{aligned}
A_{z_1 z_2} &= \int_0^h \sin\left(\frac{n\pi}{h} z'\right) \frac{\partial}{\partial z'} \left( \Theta_{z_1 z_2}(z') e^{-ikz'} \right) dz', \\
&= \sin\left(\frac{n\pi}{h} z'\right) \Theta_{z_1 z_2}(z') e^{-ikz'} \Big|_0^h - \int_0^h \Theta_{z_1 z_2}(z') e^{-ikz'} d \sin\left(\frac{n\pi}{h} z'\right), \\
&= -\frac{n\pi}{h} \int_0^h \Theta_{z_1 z_2}(z') e^{-ikz'} \cos\left(\frac{n\pi}{h} z'\right) dz', \\
&= -\frac{n\pi}{h} \int_{z_1}^{z_2} e^{-ikz'} \cos\left(\frac{n\pi}{h} z'\right) dz'. \tag{3.201}
\end{aligned}$$

For real numbers  $a, b$  where  $a \neq b$  the integral,  $I_{ab}$  in (3.201) can be written

$$I_{ab} = \int e^{-iat} \cos(bt) dt,$$

$$\begin{aligned}
&= \frac{1}{b} \int e^{-iat} d \sin(bt), \\
&= \frac{1}{b} e^{-iat} \sin(bt) - \frac{1}{b} \int \sin(bt) (-ia) e^{-iat} dt, \\
&= \frac{1}{b} e^{-iat} \sin(bt) + \frac{ia}{b} \int e^{-iat} \left(\frac{-1}{b}\right) d \cos bt, \\
&= \frac{1}{b} e^{-iat} \sin(bt) - \frac{ia}{b^2} \int e^{-iat} d \cos(bt), \\
&= \frac{1}{b} e^{-iat} \sin(bt) - \frac{ia}{b^2} \left( e^{-iat} \cos(bt) - \int (-ia) e^{-iat} \cos(bt) dt \right), \\
&= \frac{be^{-iat} \sin(bt) - ia e^{-iat} \cos(bt)}{b^2} + \frac{a^2}{b^2} \int \cos(bt) e^{-iat} dt, \\
&= \frac{be^{-iat} \sin(bt) - ia e^{-iat} \cos(bt)}{b^2} + \frac{a^2}{b^2} I_{ab}, \tag{3.202}
\end{aligned}$$

or

$$I_{ab}(b^2 - a^2) = e^{-iat} [b \sin(bt) - ia \cos(bt)]. \tag{3.203}$$

Finally we have

$$I_{ab} = \frac{e^{-iat} [ia \cos(bt) - b \sin(bt)]}{a^2 - b^2}. \tag{3.204}$$

In the case of  $a = b$  we have

$$\begin{aligned}
I_{ab} &= \int e^{-iat} \cos(at) dt, \\
&= \int e^{-iat} \frac{e^{iat} + e^{-iat}}{2} dt, \\
&= \frac{1}{2} \int (1 + e^{-2iat}) dt, \\
&= \frac{1}{2} \left( t + \frac{e^{-2iat}}{-2ia} \right), \\
&= \frac{2iat + ie^{-2iat}}{4a}. \tag{3.205}
\end{aligned}$$

The above can be written

$$I_{ab} = \begin{cases} \frac{e^{-iat} [ia \cos(bt) - b \sin(bt)]}{a^2 - b^2}, & \text{if } a \neq b \\ \frac{2at + ie^{-2iat}}{4}, & \text{if } a = b \end{cases} \quad (3.206)$$

Consider the case  $k \neq \frac{n\pi}{h}$ . Using (3.204) with  $a = -ik$ ,  $b = \frac{n\pi}{h}$  and  $k \neq \frac{n\pi}{h}$  we have

$$A_{z_1 z_2} = \frac{n\pi}{h} \frac{e^{-ikz'} \left[ \frac{n\pi}{h} \sin\left(\frac{n\pi}{h} z'\right) - ik \cos\left(\frac{n\pi}{h} z'\right) \right]}{\left(\frac{n\pi}{h}\right)^2 - k^2} \Big|_{z_1}^{z_2}. \quad (3.207)$$

For the case  $k = \frac{n\pi}{h}$ , using Euler's formula we have

$$\begin{aligned} A_{z_1 z_2} &= -k \int_{z_1}^{z_2} e^{-ikz'} \cos(kz') dz', \\ &= -\frac{2kz' + ie^{-2ikz'}}{4} \Big|_{z_1}^{z_2}. \end{aligned} \quad (3.208)$$

### Calculation $B(x, y)$

We consider the case of an infinite rod ( $x_1, x_2$  is finite but  $y_1, y_2 \rightarrow \infty$ ). Noting that  $\int_{-\infty}^{\infty} e^{-i\kappa t} dt = 2\pi\delta(\kappa)$ , we have

$$\begin{aligned} B(x, y) &= \int_{x_1}^{x_2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y \right) dx' dy', \\ &= \int_{x_1}^{x_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y y]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y dx' \int_{-\infty}^{\infty} e^{-i\kappa_y y'} dy', \\ &= 2\pi \int_{x_1}^{x_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y y]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} \delta(\kappa_y) d\kappa_x d\kappa_y dx', \\ &= 2\pi \int_{x_1}^{x_2} \int_{-\infty}^{\infty} \frac{e^{i\kappa_x x - i\kappa_x x'}}{\kappa_x^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x dx'. \end{aligned} \quad (3.209)$$

We know that for any  $b > 0$ ,  $\int_{-\infty}^{\infty} \frac{e^{i\kappa t}}{t^2 + b^2} dt = \frac{\pi e^{-b|\kappa|}}{b}$  thus

$$B(x, y) = \frac{2\pi h}{n} \int_{x_1}^{x_2} e^{-\frac{n\pi}{h}|x-x'|} dx'. \quad (3.210)$$

Consider the following three cases:

(a)  $x < x_1 < x' < x_2$

In this case,  $|x' - x| = x' - x$  then we have

$$\begin{aligned}
 B(x, y) &= \frac{2\pi h}{n} \int_{x_1}^{x_2} e^{-\frac{n\pi(x'-x)}{h}} dx', \\
 &= \frac{2\pi h}{n} \frac{e^{-\frac{n\pi(x'-x)}{h}}}{-\frac{n\pi}{h}} \Big|_{x_1}^{x_2}, \\
 &= -\frac{2h^2}{n^2} \left[ e^{-\frac{n\pi}{h}(x_2-x)} - e^{-\frac{n\pi}{h}(x_1-x)} \right]. \tag{3.211}
 \end{aligned}$$

(b)  $x_1 < x' < x_2 < x$

In this case,  $|x' - x| = x - x'$  then we have

$$\begin{aligned}
 B(x, y) &= \frac{2\pi h}{n} \int_{x_1}^{x_2} e^{\frac{n\pi(x'-x)}{h}} dx', \\
 &= \frac{2\pi h}{n} \frac{e^{\frac{n\pi(x'-x)}{h}}}{\frac{n\pi}{h}} \Big|_{x_1}^{x_2}, \\
 &= \frac{2h^2}{n^2} \left[ e^{\frac{n\pi}{h}(x_2-x)} - e^{\frac{n\pi}{h}(x_1-x)} \right]. \tag{3.212}
 \end{aligned}$$

(c)  $x_1 < x < x_2$

In this case, divide region  $[x_1, x_2]$  into two sub-regions  $[x_1, x]$  and  $[x, x_2]$  we have

$$\begin{aligned}
 B(x, y) &= \frac{2\pi h}{n} \left( \int_{x_1}^x e^{\frac{n\pi(x'-x)}{h}} dx' + \int_x^{x_2} e^{-\frac{n\pi(x'-x)}{h}} dx' \right), \\
 &= \frac{2\pi h}{n} \left( \frac{e^{\frac{n\pi(x'-x)}{h}}}{\frac{n\pi}{h}} + \frac{e^{-\frac{n\pi(x'-x)}{h}}}{-\frac{n\pi}{h}} \right) \Big|_{x_1}^{x_2}, \\
 &= -\frac{2h^2}{n^2} \left[ e^{-\frac{n\pi}{h}(x_2-x)} + e^{\frac{n\pi}{h}(x_1-x)} - 2 \right]. \tag{3.213}
 \end{aligned}$$



The final result for the potential distribution for an infinite rod is

$$\phi(\mathbf{r}) = \frac{ik\alpha_0 p_0}{2\pi^2 h (\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \cdot A_{z_1 z_2} \cdot B(x, y), \quad (3.214)$$

where

$$A_{z_1 z_2} = \begin{cases} \frac{n\pi}{h} \frac{e^{-ikz'} \left[ \frac{n\pi}{h} \sin\left(\frac{n\pi}{h} z'\right) - ik \cos\left(\frac{n\pi}{h} z'\right) \right]}{\left(\frac{n\pi}{h}\right)^2 - k^2} \Bigg|_{z_1}^{z_2}, & \text{if } k \neq \frac{n\pi}{h} \\ -\frac{2kz' + ie^{-2ikz'}}{4} \Bigg|_{z_1}^{z_2}, & \text{if } k = \frac{n\pi}{h} \end{cases} \quad (3.215)$$

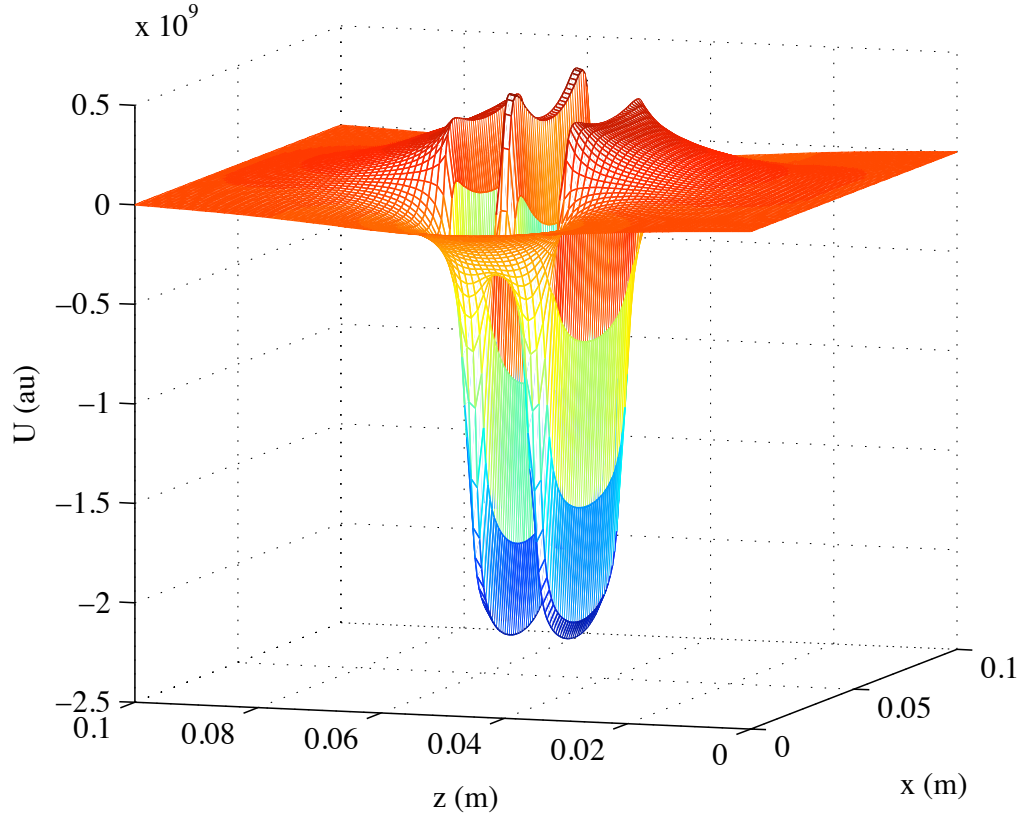


Figure 3.3: Potential distribution for a colloidal silica rectangular layer of infinite length lying along  $y$  axis. The layer has a width of 4 cm and height of 2 cm.

and

$$B(x, y) = \begin{cases} -\frac{2h^2}{n^2} \left( e^{-\frac{n\pi}{h}(x_2-x)} - e^{-\frac{n\pi}{h}(x_1-x)} \right) & \text{for } x < x_1 \\ -\frac{2h^2}{n^2} \left( e^{-\frac{n\pi}{h}(x_2-x)} + e^{\frac{n\pi}{h}(x_1-x)} - 2 \right) & \text{for } x_1 < x < x_2 \\ -\frac{2h^2}{n^2} \left( e^{\frac{n\pi}{h}(x_1-x)} - e^{\frac{n\pi}{h}(x_2-x)} \right) & \text{for } x_2 < x. \end{cases} \quad (3.216)$$

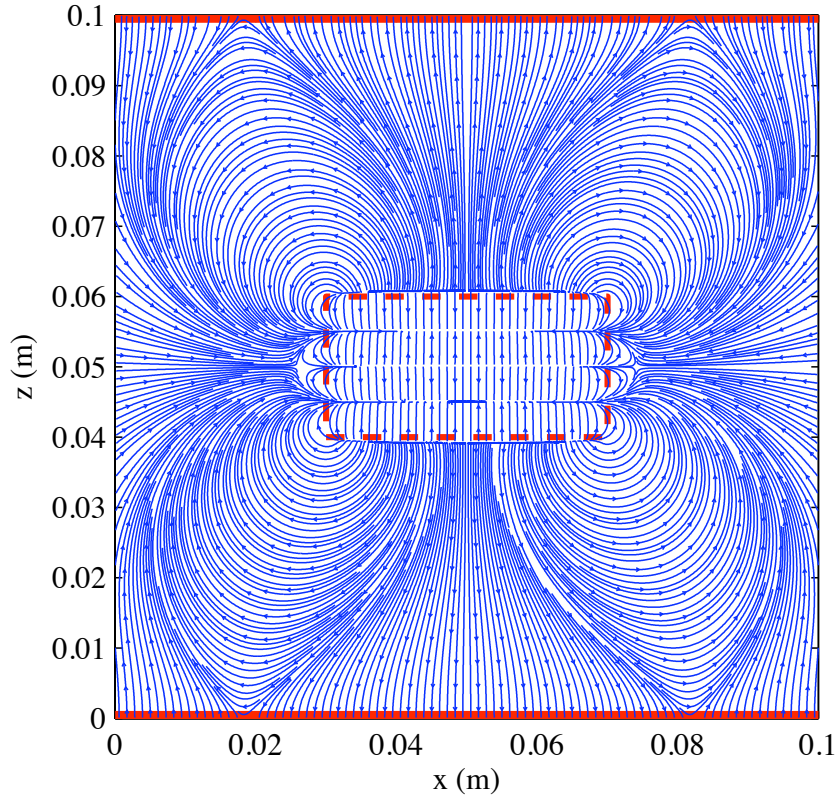


Figure 3.4: Electric lines of force in the case of a colloidal silica rectangular layer of infinite length lying along  $y$  axis of width 4 cm and height 2 cm in the view along  $y$  axis. The top and the bottom red lines are the electrodes and the red box in the center is where the layer is located.

The potential distribution for a rectangular layer of silica colloidal of infinite length and width 4 cm and height 2 cm with  $\sigma = 6.8 \times 10^{-6}$  S/m and  $\epsilon/\epsilon_0 = 84$ <sup>35</sup> embedded in a body of agarose traversed by a sound wave of frequency  $f = 150$  kHz and sound speed  $c = 1500$  m/s is plotted on Fig. (3.3). The electric lines of force for the same

sample is found by taking derivatives of the potential [plotted in Fig. (3.4)].

### 3.5.2 Triangular wedge with infinite length along y axis

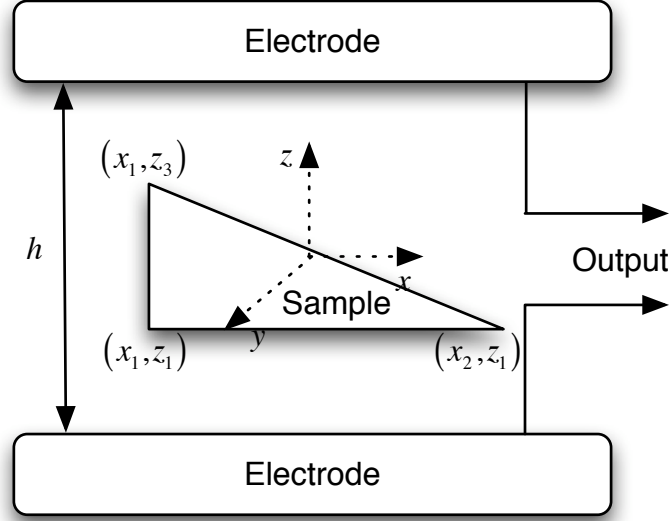


Figure 3.5: Infinite triangular wedge lying along the  $y$  axis

In the case of triangle wedge of infinite length lying along  $y$  axis, the colloidal distribution in space can be written as

$$\alpha(\mathbf{r}) = \alpha_0 \Theta_w(x, z), \quad (3.217)$$

where  $\Theta_w(x, z) = 1$  if  $x_1 \leq x \leq x_2, z_1 \leq z \leq z_x$  and  $\Theta_w(x, z) = 0$  otherwise. Here  $z_x = z_1 + \frac{z_3 - z_1}{x_2 - x_1}(x_2 - x)$ . The gradient in equation (3.197) now becomes

$$\nabla(\alpha \nabla p) = -ikp_0 \alpha_0 \frac{\partial}{\partial z} [\Theta_w(x, z) e^{-ikz}]. \quad (3.218)$$

In this case the potential distribution is

$$\phi(\mathbf{r}) = \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \int_V \left[ \frac{-1}{2\pi^2 h} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \sin\left(\frac{n\pi}{h} z'\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} d\kappa_x d\kappa_y}{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} \right]$$

$$\begin{aligned}
& \times \left[ -ik\alpha_0 p_0 \frac{\partial}{\partial z'} \left( \Theta_w(x', z') e^{-ikz'} \right) \right] dx' dy' dz' , \\
& = \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_0^h \int_{-\infty}^{\infty} \sin \left( \frac{n\pi}{h} z' \right) \frac{\partial}{\partial z'} \left( \Theta_w(x', z') e^{-ikz'} \right) \\
& \times \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left( \frac{n\pi}{h} \right)^2} d\kappa_x d\kappa_y dy' \right) dx' dz' . \tag{3.219}
\end{aligned}$$

Note that from the calculation of  $B(x, y)$  in the previous section, equation (3.200), we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left( \frac{n\pi}{h} \right)^2} d\kappa_x d\kappa_y dy' = \frac{2\pi h}{n} e^{-\frac{n\pi}{h} |x'-x|} , \tag{3.220}$$

so the potential distribution now is calculated as

$$\begin{aligned}
\phi(\mathbf{r}) &= \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_0^h \int_{-\infty}^{\infty} \sin \left( \frac{n\pi}{h} z' \right) \frac{\partial}{\partial z'} \left( \Theta_w(x', z') e^{-ikz'} \right) \\
& \times \left( \frac{2\pi h}{n} e^{-\frac{n\pi}{h} |x'-x|} \right) dx' dz' , \\
& = \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_{-\infty}^{\infty} \frac{2\pi h}{n} e^{-\frac{n\pi}{h} |x'-x|} \\
& \times \left( \int_0^h \sin \left( \frac{n\pi}{h} z' \right) \frac{\partial}{\partial z'} \left( \Theta_w(x', z') e^{-ikz'} \right) dz' \right) dx' , \\
& = \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_{-\infty}^{\infty} \frac{2\pi h}{n} e^{-\frac{n\pi}{h} |x'-x|} \\
& \times \left( \Theta_w(x', z') e^{-ikz'} \sin \left( \frac{n\pi}{h} z' \right) \Big|_0^h - \int_0^h \Theta_w(x', z') e^{-ikz'} d \sin \left( \frac{n\pi}{h} z' \right) \right) dx' , \\
& = \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_{-\infty}^{\infty} \frac{2\pi h}{n} e^{-\frac{n\pi}{h} |x'-x|} \\
& \times \left( -\Theta_w(x', z') e^{-ikz'} \frac{n\pi}{h} \cos \left( \frac{n\pi}{h} z' \right) dz' \right) dx' , \\
& = \frac{-ik\alpha_0 p_0}{h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_0^h \int_{-\infty}^{\infty} e^{-\frac{n\pi}{h} |x'-x|} \Theta_w(x', z') e^{-ikz'} \cos \left( \frac{n\pi}{h} z' \right) dx' dz' , \\
& = \frac{-ik\alpha_0 p_0}{h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_{x_1}^{x_2} e^{-\frac{n\pi}{h} |x'-x|} \left( \int_{z_1}^{z_{x'}} e^{-ikz'} \cos \left( \frac{n\pi}{h} z' \right) dz' \right) dx' ,
\end{aligned}$$

$$= \frac{-ik\alpha_0 p_0}{h(\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h}z\right) \int_{x_1}^{x_2} e^{-\frac{n\pi}{h}|x'-x|} f(x') dx', \quad (3.221)$$

where  $f(x)$  is found by applying (3.206)

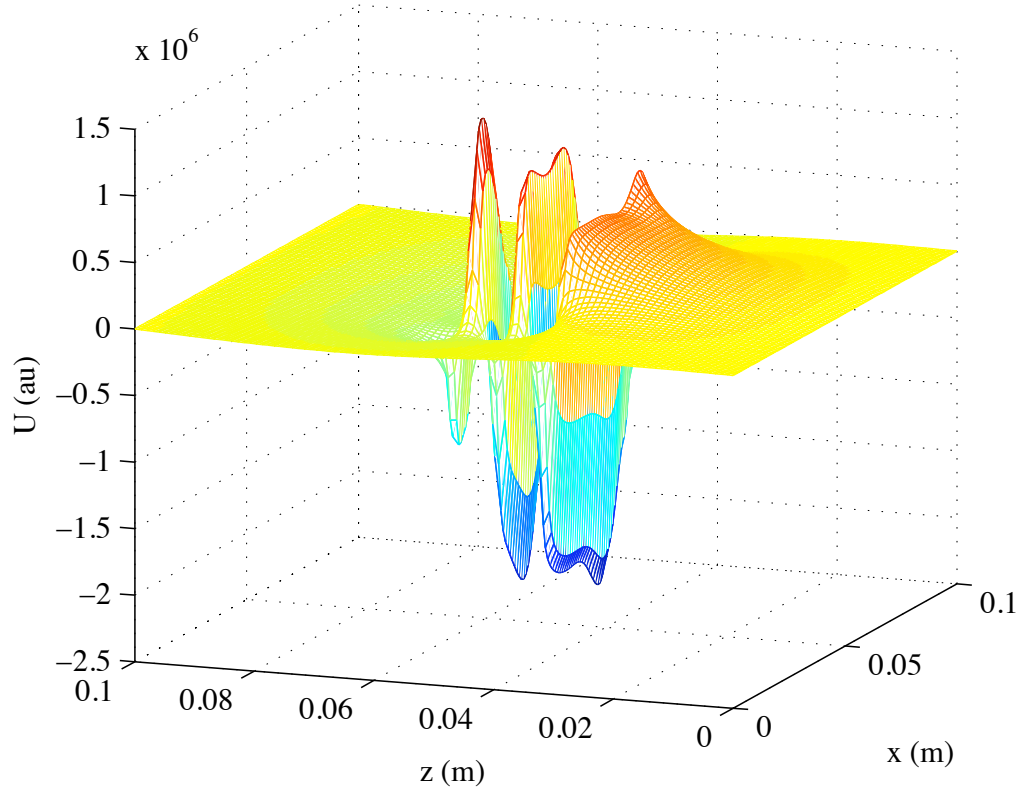


Figure 3.6: Potential distribution in the case of a colloidal silica triangular wedge of infinite length lying along  $y$  axis.

$$f(x) = \int_{z_1}^{z_x} e^{-ikz'} \cos\left(\frac{n\pi}{h}z'\right) dz',$$

$$= \begin{cases} \frac{n\pi}{h} \frac{e^{-ikz'} \left[ \frac{n\pi}{h} \sin\left(\frac{n\pi}{h}z'\right) - ik \cos\left(\frac{n\pi}{h}z'\right) \right]}{\left(\frac{n\pi}{h}\right)^2 - k^2} \Bigg|_{z_1}^{z_x}, & \text{if } k \neq \frac{n\pi}{h} \\ -\frac{2kz' + ie^{-2ikz'}}{4} \Bigg|_{z_1}^{z_x}, & \text{if } k = \frac{n\pi}{h} \end{cases} \quad (3.222)$$

with  $z_x = z_1 + \frac{z_3 - z_1}{x_2 - x_1}(x_2 - x)$ .

The potential distribution for a triangular wedge of blood of infinite length and

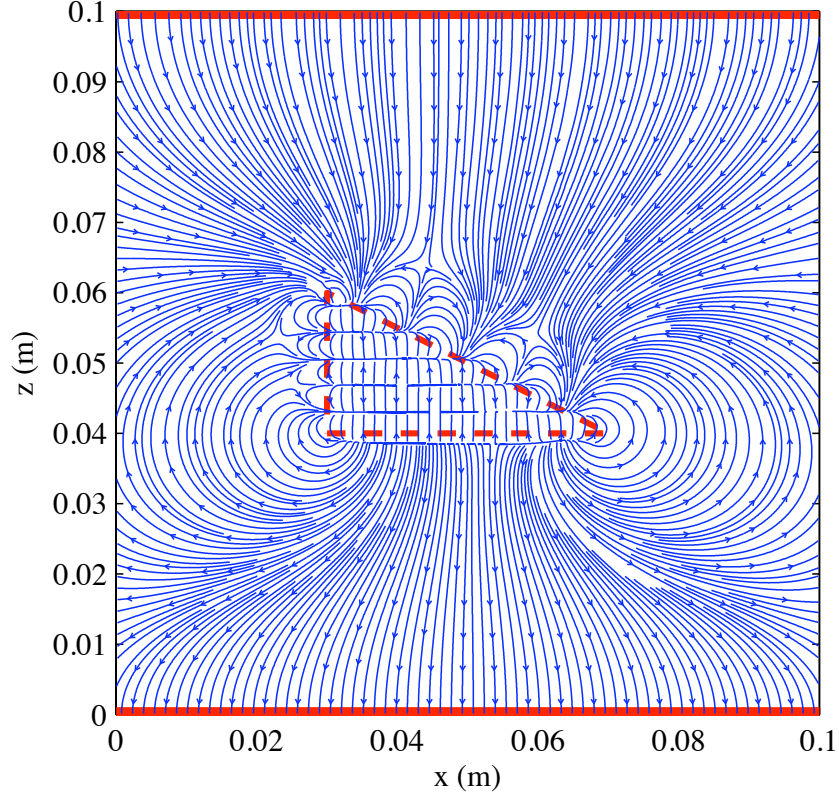


Figure 3.7: Electric lines of force in the case of a colloidal silica triangular wedge of infinite length lying along the  $y$  axis in a view along  $y$  axis. The top and the bottom red lines are the electrodes and the red triangle in the center locates the wedge.

width 4 cm and height 2 cm with  $\sigma = 0.822$  S/m and  $\epsilon/\epsilon_0 = 3026$ <sup>35</sup> embedded in a body traversed by a sound wave of frequency  $f = 200$  kHz and sound speed  $c = 1500$  m/s is plotted on Figure (3.6). The electric lines of force plotted in Figure (3.7) for the same sample are found by taking derivatives of the potential found above.

### 3.5.3 Cylinder of infinite length lying along $y$ axis

Consider a cylinder of radius  $R$  and infinite length and center at  $(x_0, z_0)$  lying along the  $y$  axis as shown in Figure (3.8). The colloidal distribution in this case is

$$\alpha(\mathbf{r}) = \alpha_0 \Theta_R(x, z), \quad (3.223)$$

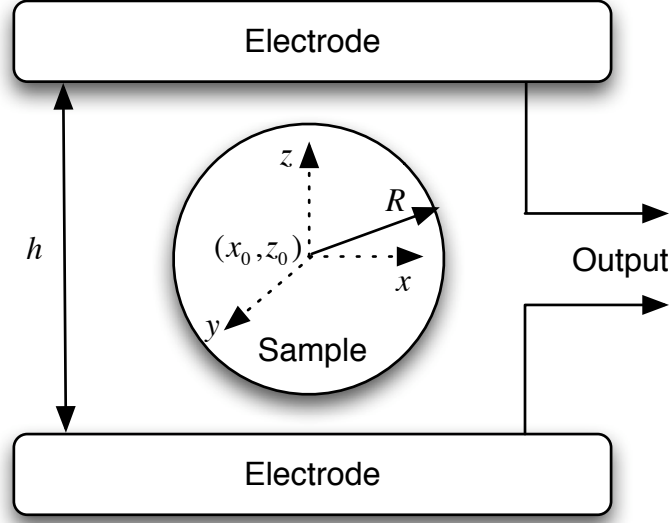


Figure 3.8: Cylinder of radius  $R$  with infinite length lying along  $y$  axis

where  $\Theta_R(x, z) = 1$  if  $(x, z)$  lies inside the circle of radius  $R$  and  $\Theta_R(x, z) = 0$  otherwise. The gradient as in Eq. (3.197) now becomes

$$\nabla(\alpha \nabla p) = -ikp_0\alpha_0 \frac{\partial}{\partial z} [\Theta_R(x, z)e^{-ikz}] . \quad (3.224)$$

A similar calculation as given in section (3.5.2) yields

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \int_V \left[ \frac{-1}{2\pi^2 h} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \sin\left(\frac{n\pi}{h} z'\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y \right] \\ &\times \left[ -ik\alpha_0 p_0 \frac{\partial}{\partial z'} \left( \Theta_R(x', z') e^{-ikz'} \right) \right] dx' dy' dz' , \\ &= \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \int_0^h \int_{-\infty}^{\infty} \sin\left(\frac{n\pi}{h} z'\right) \frac{\partial}{\partial z'} \left( \Theta_R(x', z') e^{-ikz'} \right) \\ &\times \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y dy' \right) dx' dz' , \\ &= \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \int_{-\infty}^{\infty} \frac{2\pi h}{n} e^{-\frac{n\pi}{h} |x'-x|} \end{aligned}$$

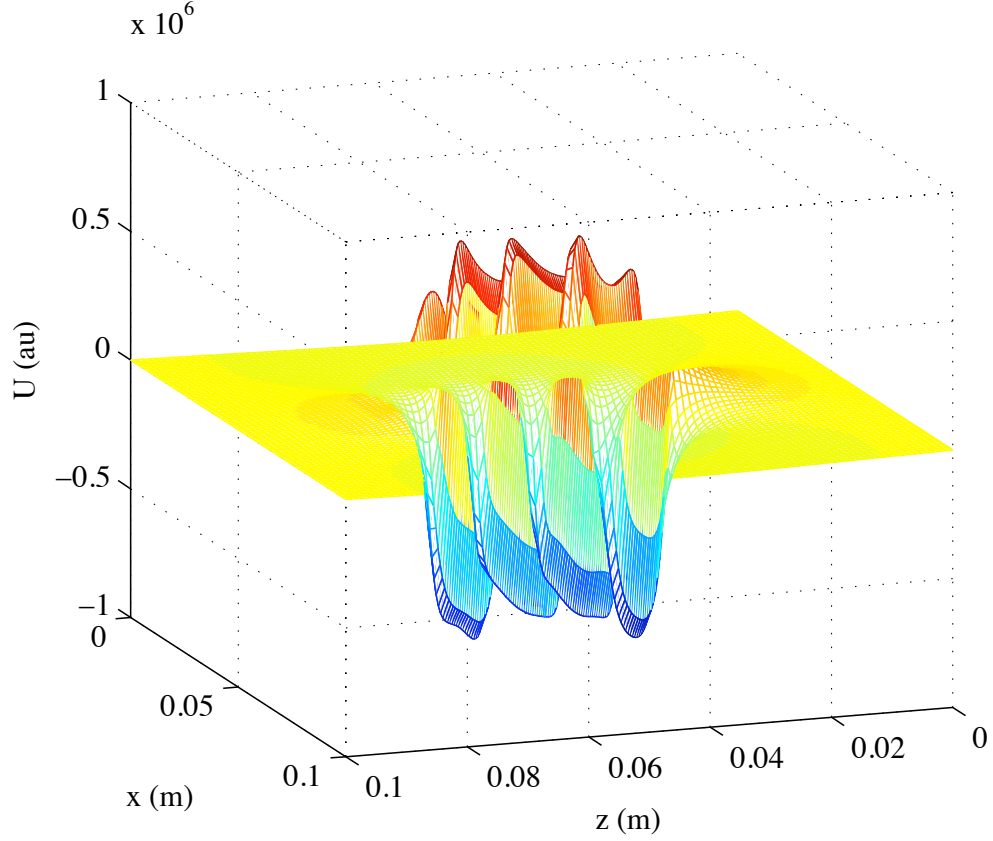


Figure 3.9: Potential distribution in the case of a cylinder of body fluid of radius  $R$  and infinite length lying along  $y$  axis.

$$\begin{aligned}
 & \times \left( \int_0^h \sin \left( \frac{n\pi}{h} z' \right) \frac{\partial}{\partial z'} \left( \Theta_R(x', z') e^{-ikz'} \right) dz' \right) dx' \\
 &= \frac{-ik\alpha_0 p_0}{h \left( \epsilon - i \frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_0^h \int_{-\infty}^{\infty} e^{-\frac{n\pi}{h} |x'-x|} \Theta_R(x', z') e^{-ikz'} \cos \left( \frac{n\pi}{h} z' \right) dx' dz' , \\
 &= \frac{-ik\alpha_0 p_0}{h \left( \epsilon - i \frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_{x_0-R}^{x_0+R} e^{-\frac{n\pi}{h} |x'-x|} \left( \int_{z'_1}^{z'_2} e^{-ikz'} \cos \left( \frac{n\pi}{h} z' \right) dz' \right) dx' , \\
 &= \frac{-ik\alpha_0 p_0}{h \left( \epsilon - i \frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{h} z \right) \int_{x_0-R}^{x_0+R} e^{-\frac{n\pi}{h} |x'-x|} f(x') dx' , \tag{3.225}
 \end{aligned}$$

where  $f(x)$  is found by applying (3.206)

$$f(x) = \int_{z'_1}^{z'_2} e^{-ikz'} \cos \left( \frac{n\pi}{h} z' \right) dz' ,$$



$$= \begin{cases} \frac{n\pi}{h} \frac{e^{-ikz'} \left[ \frac{n\pi}{h} \sin\left(\frac{n\pi}{h} z'\right) - ik \cos\left(\frac{n\pi}{h} z'\right) \right]}{\left(\frac{n\pi}{h}\right)^2 - k^2} \Bigg|_{z'_1}^{z'_2}, & \text{if } k \neq \frac{n\pi}{h} \\ -\frac{2kz' + ie^{-2ikz'}}{4k} \Bigg|_{z'_1}^{z'_2}, & \text{if } k = \frac{n\pi}{h} \end{cases} \quad (3.226)$$

with  $z'_1 = z_0 - \sqrt{R^2 - (x' - x_0)^2}$  and  $z'_2 = z_0 + \sqrt{R^2 - (x' - x_0)^2}$ .

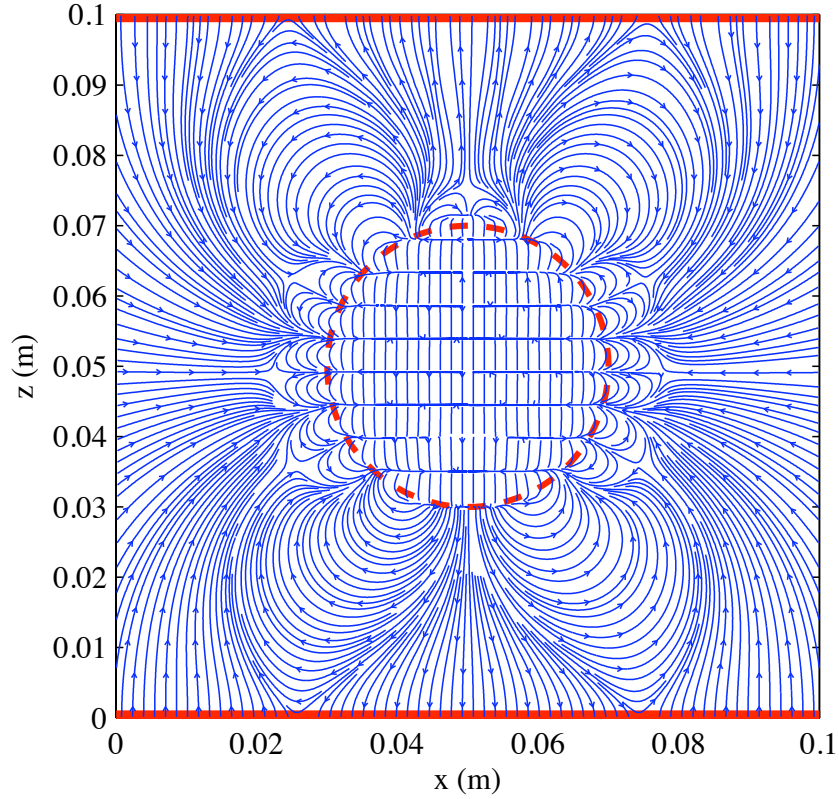


Figure 3.10: Electric lines of force in the case of a cylinder of body fluid of radius  $R$  and infinite length lying along the  $y$  axis. The top and the bottom red lines are the electrodes and the red circle in the center is a cross section of the cylinder in the  $(x, z)$  plane.

The potential distribution for a cylinder of body fluid of radius  $R = 2$  cm and infinite length with  $\sigma = 1.5$  S/m and  $\epsilon/\epsilon_0 = 84^{35}$  embedded in a body with a sound speed  $c = 1500$  m/s traversed by a sound wave of frequency  $f = 160$  kHz is plotted on Figure (3.9). The electric lines of force of the same sample are found by taking derivatives of the potential found above as plotted in Figure (3.10).

### 3.5.4 Ellipse of infinite length lying along y axis

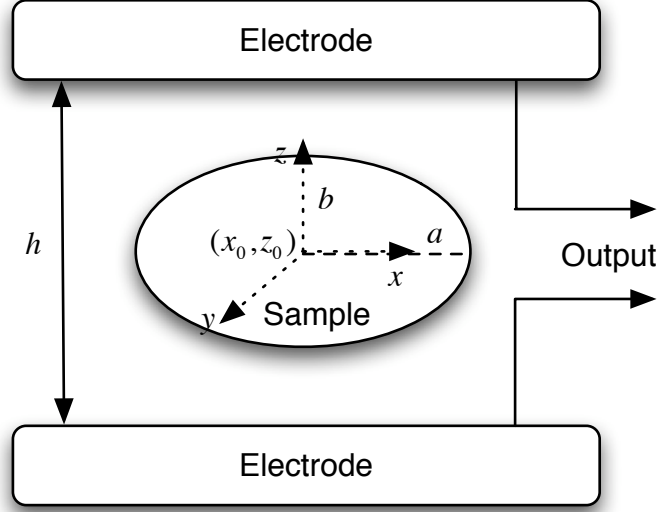


Figure 3.11: Ellipse with semimajor axis  $a$  and semiminor axis  $b$  with infinite length, lying along the  $y$  axis.

In the case of a colloidal ellipse with semimajor axis  $a$  and semiminor axis  $b$  with infinite length lying along the  $y$  axis as shown in Figure (3.11), the colloidal distribution can be written as

$$\alpha(\mathbf{r}) = \alpha_0 \Theta_E(x, z), \quad (3.227)$$

where  $\Theta_E(x, z) = 1$  if  $(x, z)$  lies inside the ellipse and  $\Theta_R(x, z) = 0$  otherwise. The gradient as in Eq. (3.197) now becomes

$$\nabla(\alpha \nabla p) = -ikp_0\alpha_0 \frac{\partial}{\partial z} [\Theta_E(x, z)e^{-ikz}]. \quad (3.228)$$

Calculation similar to the one in Section (3.5.2) gives

$$\phi(\mathbf{r}) = \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \int_V \left[ \frac{-1}{2\pi^2 h} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \sin\left(\frac{n\pi}{h} z'\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y \right]$$

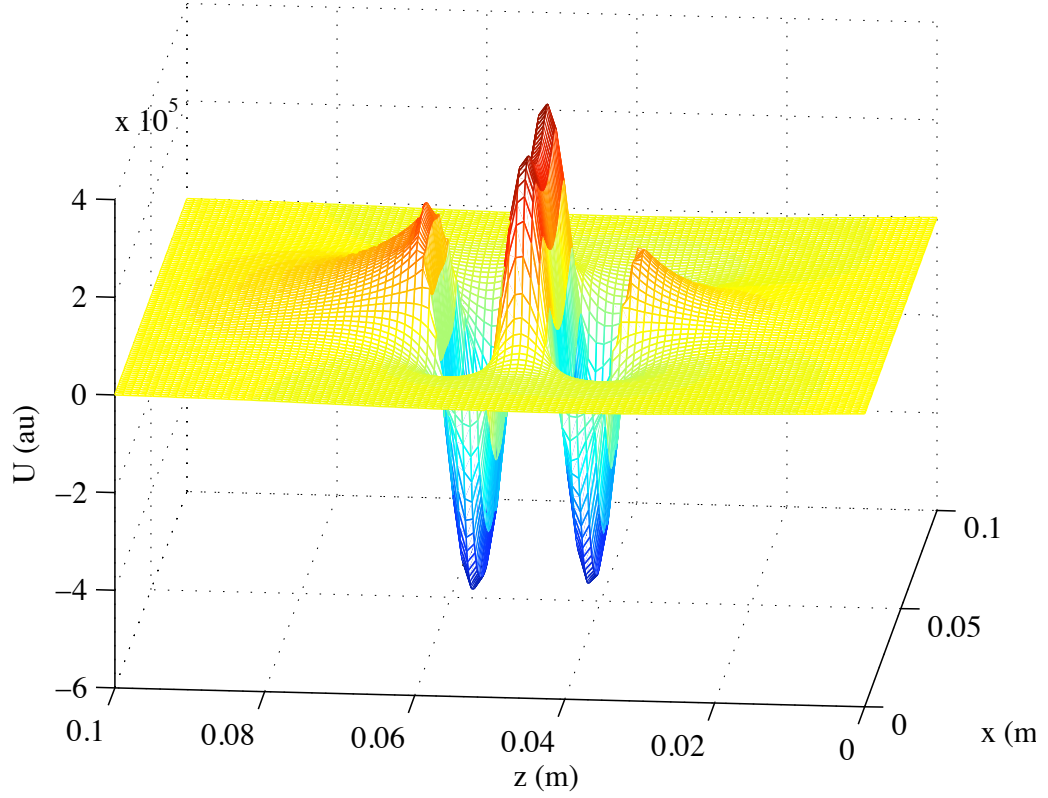


Figure 3.12: Potential distribution in the case of an ellipse of body fluid of semimajor axis  $a = 2$  cm and semiminor axis  $b = 1.5$  cm, infinite in length, lying along the  $y$  axis.

$$\begin{aligned}
& \times \left[ -ik\alpha_0 p_0 \frac{\partial}{\partial z'} \left( \Theta_E(x', z') e^{-ikz'} \right) \right] dx' dy' dz', \\
& = \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \int_0^h \int_{-\infty}^{\infty} \sin\left(\frac{n\pi}{h} z'\right) \frac{\partial}{\partial z'} \left( \Theta_E(x', z') e^{-ikz'} \right) \\
& \times \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i[\kappa_x(x-x') + \kappa_y(y-y')]} }{\kappa_x^2 + \kappa_y^2 + \left(\frac{n\pi}{h}\right)^2} d\kappa_x d\kappa_y dy' \right) dx' dz', \\
& = \frac{ik\alpha_0 p_0}{2\pi^2 h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \int_{-\infty}^{\infty} \frac{2\pi h}{n} e^{-\frac{n\pi}{h} |x'-x|} \\
& \times \left( \int_0^h \sin\left(\frac{n\pi}{h} z'\right) \frac{\partial}{\partial z'} \left( \Theta_E(x', z') e^{-ikz'} \right) dz' \right) dx' \\
& = \frac{-ik\alpha_0 p_0}{h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \int_0^h \int_{-\infty}^{\infty} e^{-\frac{n\pi}{h} |x'-x|} \Theta_E(x', z') e^{-ikz'} \cos\left(\frac{n\pi}{h} z'\right) dx' dz',
\end{aligned}$$

$$\begin{aligned}
&= \frac{-ik\alpha_0 p_0}{h(\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h}z\right) \int_{x_0-a}^{x_0+a} e^{-\frac{n\pi}{h}|x'-x|} \left( \int_{z'_1}^{z'_2} e^{-ikz'} \cos\left(\frac{n\pi}{h}z'\right) dz' \right) dx', \\
&= \frac{-ik\alpha_0 p_0}{h(\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h}z\right) \int_{x_0-a}^{x_0+a} e^{-\frac{n\pi}{h}|x'-x|} f(x') dx', \tag{3.229}
\end{aligned}$$

where  $f(x)$  is found by applying equation (3.206) to give

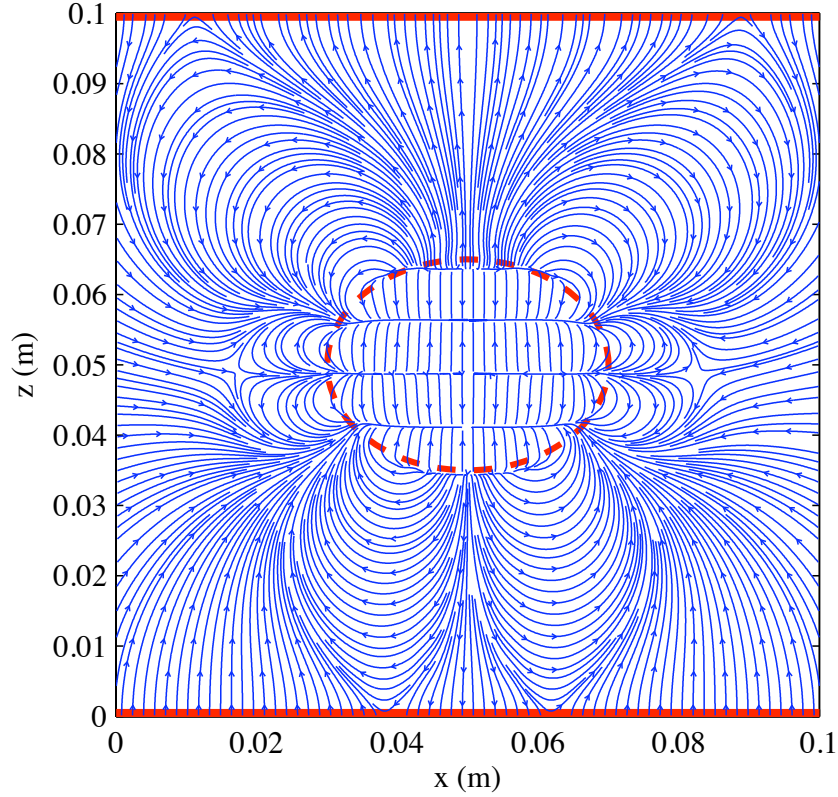


Figure 3.13: Electric lines of force in the case of an elliptical rod of body fluid of semimajor axis  $a = 2$  cm and semiminor axis  $b = 1.5$  cm and infinite length lying along the  $y$  axis. The top and the bottom red lines are the electrodes and the red ellipse in the center is a cross section of the ellipse in the  $(x, z)$  plane.

$$f(x) = \int_{z'_1}^{z'_2} e^{-ikz'} \cos\left(\frac{n\pi}{h}z'\right) dz',$$

$$= \begin{cases} \frac{n\pi}{h} \frac{e^{-ikz'} \left[ \frac{n\pi}{h} \sin\left(\frac{n\pi}{h} z'\right) - ik \cos\left(\frac{n\pi}{h} z'\right) \right]}{\left(\frac{n\pi}{h}\right)^2 - k^2} \Bigg|_{z'_1}^{z'_2}, & \text{if } k \neq \frac{n\pi}{h} \\ -\frac{2kz' + ie^{-2ikz'}}{4} \Bigg|_{z'_1}^{z'_2}, & \text{if } k = \frac{n\pi}{h} \end{cases} \quad (3.230)$$

with  $z'_1 = z_0 - b\sqrt{1 - \frac{(x' - x_0)^2}{a^2}}$  and  $z'_2 = z_0 + b\sqrt{1 - \frac{(x' - x_0)^2}{a^2}}$ .

The potential distribution for an ellipse of body fluid of semimajor axis  $a = 2$  cm and semiminor axis  $b = 1.5$  cm and infinite length with  $\sigma = 1.5$  S/m and  $\epsilon/\epsilon_0 = 84$ <sup>35</sup> embedded in a body with a sound speed  $c = 1500$  m/s traversed by a sound wave of frequency  $f = 160$  kHz and is plotted on Figure (3.12). The electric lines of force of the same sample are found by taking derivatives of the potential found above as plotted in Figure (3.13).

### 3.5.5 Cylinder of radius $R$ lying along the $z$ axis

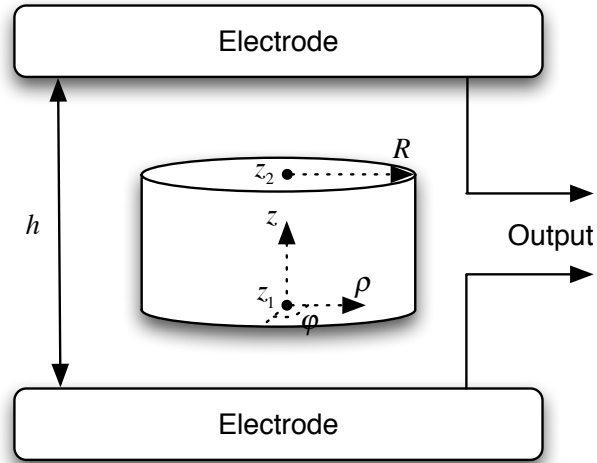


Figure 3.14: Cylinder of radius  $R$  lying along  $z$  axis from  $z_1$  to  $z_2$ .

The colloidal distribution in the case of cylinder sample of radius  $R$  lying along  $z$

axis from  $z_1$  to  $z_2$  can be written as

$$\alpha(\mathbf{r}) = \alpha_0 \Theta_R(\rho) \Theta_{z_1 z_2}(z), \quad (3.231)$$

where  $\Theta_R(\rho)$  is 1 if  $\rho \leq R$  and 0 otherwise, and  $\Theta_{z_1 z_2}(z)$  is 1 if  $z_1 \leq z \leq z_2$  and 0 otherwise. The gradient as in Eq. (3.197) now becomes

$$\begin{aligned} \nabla(\alpha \nabla p) &= \left( \frac{\partial}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \varphi} \hat{\varphi} + \frac{\partial}{\partial z} \hat{z} \right) (-ik\alpha_0 p_0 \Theta_R(\rho) \Theta_{z_1 z_2}(z) e^{-ikz} \hat{z}), \\ &= -ik\alpha_0 p_0 \Theta_R(\rho) \frac{\partial}{\partial z} (\Theta_{z_1 z_2}(z) e^{-ikz}). \end{aligned} \quad (3.232)$$

Substitution of (3.90) and (3.232) into (3.192) gives

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \int_V \left[ - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \varepsilon_m \frac{\cos(m\varphi) \cos(m\varphi') \sin(k_n z) \sin(k_n z')}{\pi h} I_m(k_n \rho_{<}) K_m(k_n \rho_{>}) \right] \\ &\times \left[ -ik\alpha_0 p_0 \Theta_R(\rho) \frac{\partial}{\partial z} (\Theta_{z_1 z_2}(z) e^{-ikz}) \right] d\varphi' \rho' d\rho' dz', \\ &= \frac{ik\alpha_0 p_0}{h\pi (\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin(k_n z) \sum_{m=0}^{\infty} \varepsilon_m \cos(m\varphi) \int_0^{\infty} \Theta_R(\rho') I_m(k_n \rho_{<}) K_m(k_n \rho_{>}) \rho' d\rho' \\ &\times \int_0^{2\pi} \cos(m\varphi') d\varphi' \times \int_0^h \sin(k_n z') \frac{\partial}{\partial z'} (\Theta_{z_1 z_2}(z') e^{-ikz'}) dz'. \end{aligned} \quad (3.233)$$

It is evident that

$$\begin{aligned} \int_0^{2\pi} \cos(mt) dt &= \begin{cases} \int_0^{2\pi} dt, & \text{if } m = 0, \\ \int_0^{2\pi} \cos(mt) dt, & \text{if } m \neq 0, \end{cases} \\ &= \begin{cases} 2\pi, & \text{if } m = 0, \\ \left. \frac{\sin(mt)}{m} \right|_0^{2\pi}, & \text{if } m \neq 0, \end{cases} \\ &= \begin{cases} 2\pi, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0, \end{cases} \end{aligned}$$

$$= 2\pi\delta(m) , \quad (3.234)$$

so (3.233) becomes

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{ik\alpha_0 p_0}{h\pi(\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin(k_n z) \sum_{m=0}^{\infty} 2\pi\delta(m)\varepsilon_m \cos(m\varphi) \\ &\times \int_0^{\infty} \Theta_R(\rho') I_m(k_n \rho_{<}) K_m(k_n \rho_{>}) \rho' d\rho' \int_0^h \sin(k_n z') \frac{\partial}{\partial z'} \left( \Theta_{z_1 z_2}(z') e^{-ikz'} \right) dz' , \\ &= \frac{2ik\alpha_0 p_0}{h(\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin(k_n z) \int_0^R I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \rho' d\rho' \\ &\times \int_0^h \sin(k_n z') \frac{\partial}{\partial z'} \left( \Theta_{z_1 z_2}(z') e^{-ikz'} \right) dz' , \\ &= \frac{2ik\alpha_0 p_0}{h(\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin(k_n z) \cdot F(\rho) \cdot A_{z_1 z_2} , \end{aligned} \quad (3.235)$$

where

$$\begin{aligned} A_{z_1 z_2} &= \int_0^h \sin(k_n z') \frac{\partial}{\partial z'} \left( \Theta_{z_1 z_2}(z') e^{-ikz'} \right) dz' , \\ &= \sin(k_n z') \Theta_{z_1 z_2}(z') e^{-ikz'} \Big|_0^h - \int_0^h \Theta_{z_1 z_2}(z') e^{-ikz'} d \sin(k_n z') , \\ &= -k_n \int_{z_1}^{z_2} e^{-ikz'} \cos(k_n z') dz' , \\ &= \begin{cases} -\frac{n\pi}{h} \frac{e^{-ikz'} \left[ \frac{n\pi}{h} \sin\left(\frac{n\pi}{h} z'\right) - ik \cos\left(\frac{n\pi}{h} z'\right) \right]}{\left(\frac{n\pi}{h}\right)^2 - k^2} \Big|_{z'_1}^{z'_2} , & \text{if } k \neq \frac{n\pi}{h} \\ -\frac{2kz' + ie^{-2ikz'}}{4} \Big|_{z'_1}^{z'_2} , & \text{if } k = \frac{n\pi}{h} \end{cases} \end{aligned} \quad (3.236)$$

and  $F(\rho) = \int_0^R I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \rho' d\rho'$ . In order to calculate  $F(\rho)$ , we consider two cases as follows

(a)  $\rho > R$

If  $\rho > R$  then obviously  $\rho > \rho'$  so we have

$$F(\rho) = \int_0^R I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \rho' d\rho' ,$$

$$\begin{aligned}
&= \int_0^R I_0(k_n \rho') K_0(k_n \rho) \rho' d\rho' , \\
&= K_0(k_n \rho) \int_0^R I_0(k_n \rho') \rho' d\rho' , \\
&= K_0(k_n \rho) \frac{RI_1(k_n R)}{k_n} , \\
&= \frac{R}{k_n} K_0(k_n \rho) I_1(k_n R) .
\end{aligned} \tag{3.237}$$

(b)  $\rho \leq R$

If  $\rho \leq R$ , notice the useful property of modified Bessel functions  $K_0(x)I_1(x) + K_1(x)I_0(x) = \frac{1}{x}$ , we have

$$\begin{aligned}
F(\rho) &= \int_0^R I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \rho' d\rho' , \\
&= \int_0^\rho I_0(k_n \rho') K_0(k_n \rho) \rho' d\rho' + \int_\rho^R I_0(k_n \rho) K_0(k_n \rho') \rho' d\rho' , \\
&= \frac{\rho}{k_n} K_0(k_n \rho) I_1(k_n \rho) + I_0(k_n \rho) \int_\rho^R K_0(k_n \rho') \rho' d\rho' , \\
&= \frac{\rho}{k_n} K_0(k_n \rho) I_1(k_n \rho) + I_0(k_n \rho) \frac{\rho K_1(k_n \rho) - R K_1(k_n R)}{k_n} , \\
&= \frac{1 - (k_n R) K_1(k_n R) I_0(k_n \rho)}{k_n^2} .
\end{aligned} \tag{3.238}$$

The final solution of the potential in the case of cylinder lying along z axis is

$$\phi(\mathbf{r}) = \frac{2ik\alpha_0 p_0}{h(\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{h} z\right) \cdot F(\rho) \cdot A_{z_1 z_2} , \tag{3.239}$$

where

$$A_{z_1 z_2} = \begin{cases} -\frac{n\pi}{h} \frac{e^{-ikz'} \left[ \frac{n\pi}{h} \sin\left(\frac{n\pi}{h} z'\right) - ik \cos\left(\frac{n\pi}{h} z'\right) \right]}{\left(\frac{n\pi}{h}\right)^2 - k^2} \Bigg|_{z'_1}^{z'_2} , & \text{if } k \neq \frac{n\pi}{h} \\ -\frac{2kz' + ie^{-2ikz'}}{4} \Bigg|_{z'_1}^{z'_2} , & \text{if } k = \frac{n\pi}{h} \end{cases} \tag{3.240}$$



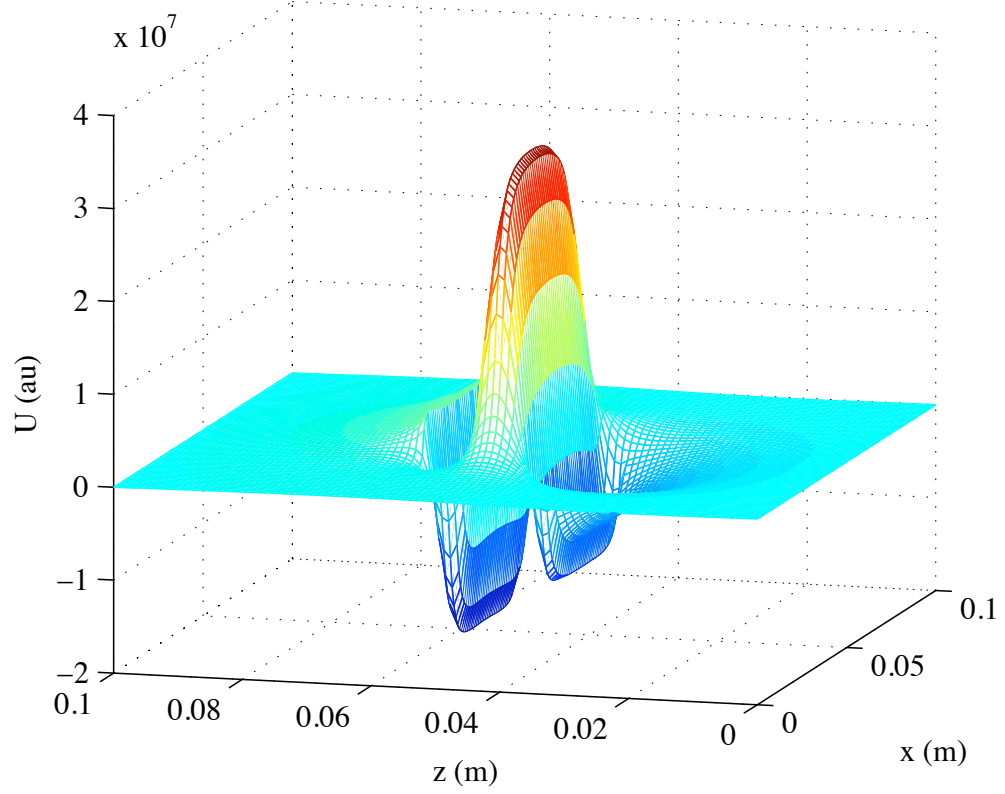


Figure 3.15: Potential distribution in the case of a cylinder of breast fat of radius  $R = 2$  cm and height 2 cm lying along  $z$  axis.

and

$$F(\rho) = \begin{cases} \frac{Rh}{n\pi} K_0\left(\frac{n\pi}{h}\rho\right) I_1\left(\frac{n\pi}{h}R\right) & , \quad \text{if } \rho > R \\ \frac{1 - \left(\frac{n\pi}{h}R\right) K_1\left(\frac{n\pi}{h}R\right) I_0\left(\frac{n\pi}{h}\rho\right)}{\left(\frac{n\pi}{h}\right)^2} & , \quad \text{if } \rho \leq R \end{cases} \quad (3.241)$$

The potential distribution for a cylinder of breast fat of radius  $R = 2$  cm and height 2 cm with  $\sigma = 0.02576$  S/m and  $\epsilon/\epsilon_0 = 23.66$ <sup>35</sup> embedded in a body with sound speed  $c = 1500$  m/s traversed by a sound wave of frequency  $f = 100$  kHz is plotted in Figure (3.15). The electric lines of force of the same sample plotted in Figure (3.16) are found by taking derivatives of the potential found above.

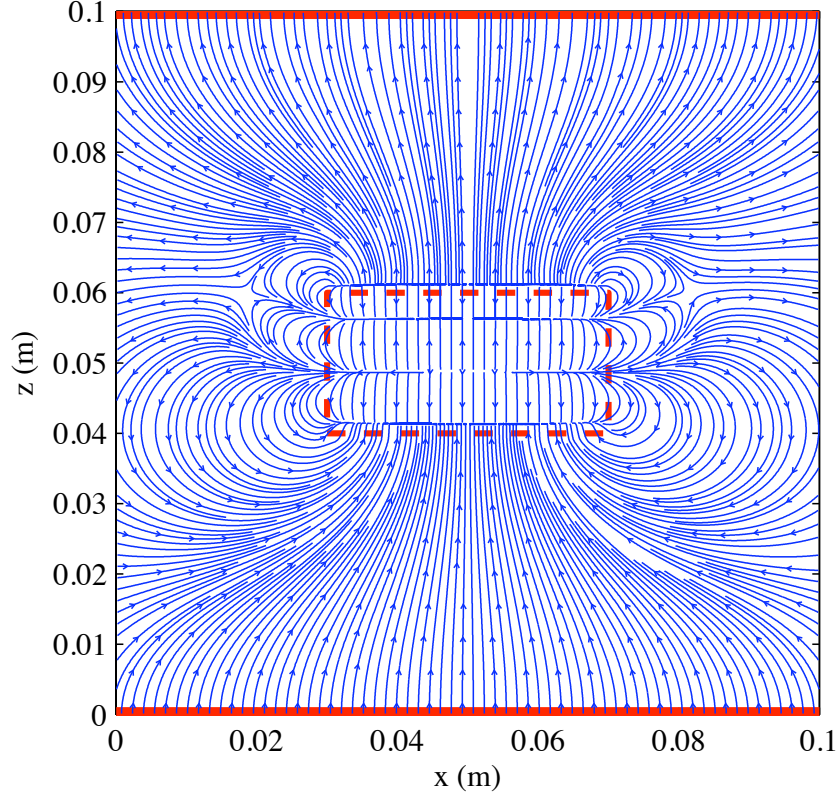


Figure 3.16: Electric lines of force in the case of a cylinder of breast fat of radius  $R = 2$  cm and height 2 cm lying along the  $z$  axis. The top and bottom red lines are the electrodes and the red rectangle in the center is a cross section of the cylinder along the cylinder's axis.

### 3.5.6 Sphere of radius $R$ centered at $(0, 0, z_0)$

In the case of a colloidal sphere of radius  $R$  centered at  $(0, 0, z_0)$  as shown in Figure (3.17), the colloidal distribution in space can be written as

$$\alpha(\mathbf{r}) = \alpha_0 \Theta_R \left( R - \sqrt{\rho^2 + (z - z_0)^2} \right) , \quad (3.242)$$

where  $\Theta_R(R - \sqrt{\rho^2 + (z - z_0)^2})$  equals 1 if  $0 \leq \rho \leq \sqrt{R^2 - (z - z_0)^2}$  and 0 if  $\sqrt{R^2 - (z - z_0)^2} < \rho$ . The gradient as in (3.197) becomes

$$\nabla(\alpha \nabla p) = \left( \frac{\partial}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial}{\partial \varphi} \hat{\boldsymbol{\varphi}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \left( -ik\alpha_0 p_0 \Theta_R \left( R - \sqrt{\rho^2 + (z - z_0)^2} \right) e^{-ikz} \hat{\mathbf{z}} \right) ,$$

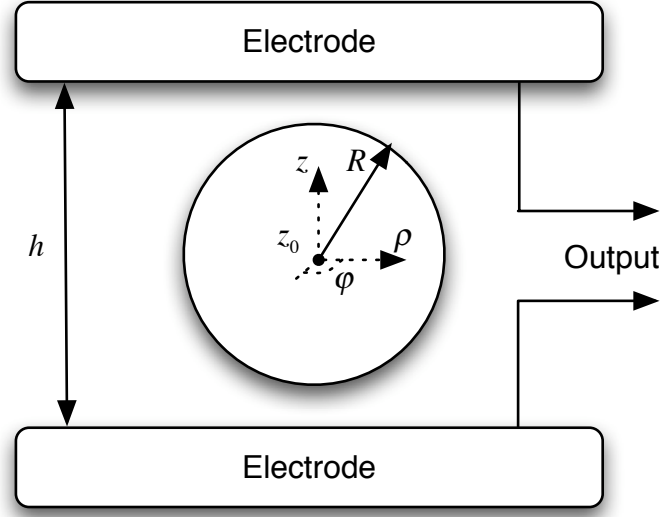


Figure 3.17: Sphere of radius  $R$  centering at  $(0, 0, z_0)$ .

$$= -ik\alpha_0 p_0 \frac{\partial}{\partial z} \left[ \Theta_R \left( R - \sqrt{\rho^2 + (z - z_0)^2} \right) e^{-ikz} \right]. \quad (3.243)$$

Using the Green's function in cylindrical coordinate (3.90) together with the coloidal distribution (3.242), Eq. (3.192) becomes

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{\epsilon - i\frac{\sigma}{\omega}} \int_V \left[ - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \varepsilon_m \frac{\cos(m\varphi) \cos(m\varphi') \sin(k_n z) \sin(k_n z')}{\pi h} I_m(k_n \rho_{<}) K_m(k_n \rho_{>}) \right] \\ &\times \left[ -ik\alpha_0 p_0 \frac{\partial}{\partial z} \left[ \Theta_R \left( R - \sqrt{\rho^2 + (z' - z_0)^2} \right) e^{-ikz} \right] d\varphi' \rho' d\rho' dz' \right], \\ &= \frac{ik\alpha_0 p_0}{h\pi \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin(k_n z) \sum_{m=0}^{\infty} \varepsilon_m \cos(m\varphi) \int_0^{\infty} \left[ I_m(k_n \rho_{<}) K_m(k_n \rho_{>}) \right. \\ &\times \left. \int_0^{2\pi} \cos(m\varphi') d\varphi' \int_0^h \sin(k_n z') \frac{\partial}{\partial z'} \left[ \Theta_R \left( R - \sqrt{\rho^2 + (z' - z_0)^2} \right) e^{-ikz'} \right] dz' \right] \rho' d\rho', \\ &= \frac{ik\alpha_0 p_0}{h\pi \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin(k_n z) \sum_{m=0}^{\infty} \varepsilon_m \cos(m\varphi) \int_0^{\infty} \left[ I_m(k_n \rho_{<}) K_m(k_n \rho_{>}) 2\pi \delta(m) \right. \\ &\times \left. \int_0^h \sin(k_n z') \frac{\partial}{\partial z'} \left[ \Theta_R \left( R - \sqrt{\rho^2 + (z' - z_0)^2} \right) e^{-ikz'} \right] dz' \right] \rho' d\rho', \\ &= \frac{2ik\alpha_0 p_0}{h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin(k_n z) \int_0^{\infty} \left[ I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \right. \end{aligned}$$

$$\begin{aligned}
& \times \int_0^h \sin(k_n z') \frac{\partial}{\partial z'} \left[ \Theta_R \left( R - \sqrt{\rho^2 + (z' - z_0)^2} \right) e^{-ik_n z'} \right] dz' \rho' d\rho', \\
& = \frac{-2ik\alpha_0 p_0}{h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} \sin(k_n z) \int_0^{\infty} \left[ I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \right. \\
& \times \int_0^h k_n \cos(k_n z') \Theta_R \left( R - \sqrt{\rho^2 + (z' - z_0)^2} \right) e^{-ik_n z'} dz' \left. \right] \rho' d\rho', \\
& = \frac{-2ik\alpha_0 p_0}{h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} k_n \sin(k_n z) \int_{z_0-R}^{z_0+R} \cos(k_n z') e^{-ik_n z'} \\
& \times \left( \int_0^{\rho'_z} I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \rho' d\rho' \right) dz', \\
& = \frac{-2ik\alpha_0 p_0}{h \left( \epsilon - i\frac{\sigma}{\omega} \right)} \sum_{n=1}^{\infty} k_n \sin(k_n z) \int_{z_0-R}^{z_0+R} \cos(k_n z') e^{-ik_n z'} f(z') dz', \tag{3.244}
\end{aligned}$$

where  $f(z') = \int_0^{\rho'_z} I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \rho' d\rho'$  and  $\rho'_z = \sqrt{R^2 - (z' - z_0)^2}$ .

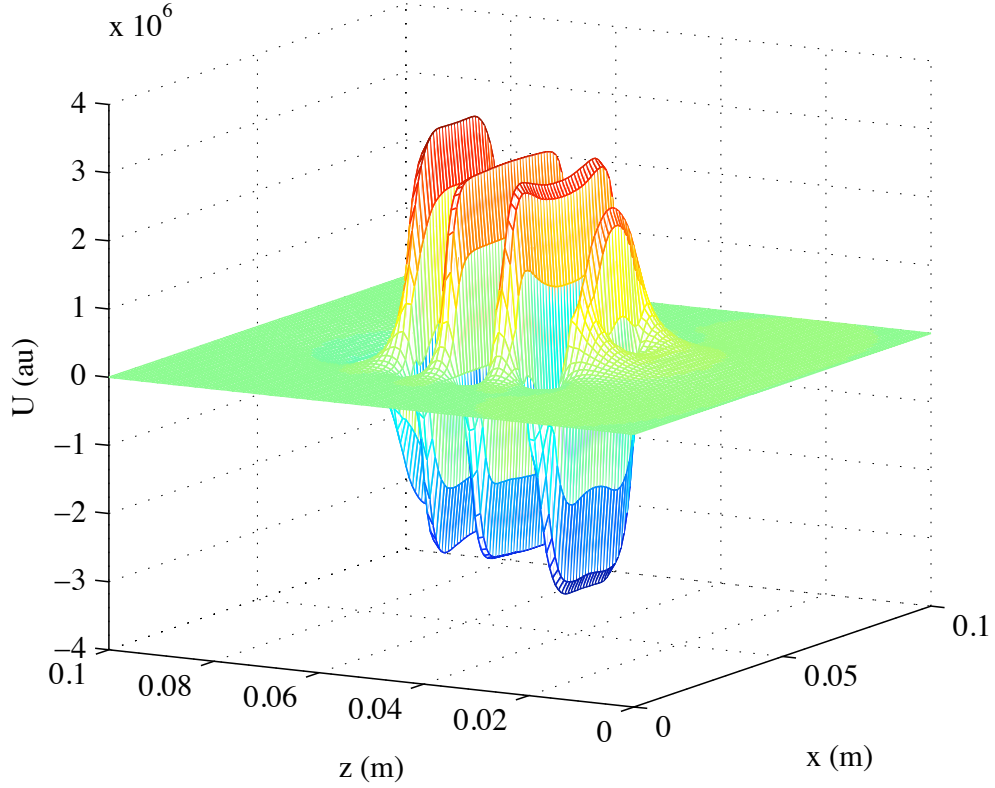


Figure 3.18: Potential distribution in the case of a sphere of breast fat of radius  $R = 2$  cm centered at  $z_0 = 5$  cm.

If we denote  $\zeta = \frac{z' - z_0}{R}$ , then when  $z'$  runs from  $z_0 - R$  to  $z_0 + R$ ,  $\zeta$  runs from  $-1$  to  $1$ , hence we can rewrite the potential distribution as

$$\phi(\mathbf{r}) = \frac{-2ik\alpha_0 p_0}{h(\epsilon - i\frac{\sigma}{\omega})} \sum_{n=1}^{\infty} (k_n R) \sin(k_n z) \int_{-1}^1 \cos[k_n(R\zeta + z_0)] e^{-ik(R\zeta + z_0)} f(\zeta) d\zeta, \quad (3.245)$$

where  $f(\zeta) = \int_0^{R\sqrt{1-\zeta^2}} I_0(k_n \rho_{<}) K_0(k_n \rho_{>}) \rho' d\rho'$  or, in a manner similar to calculation

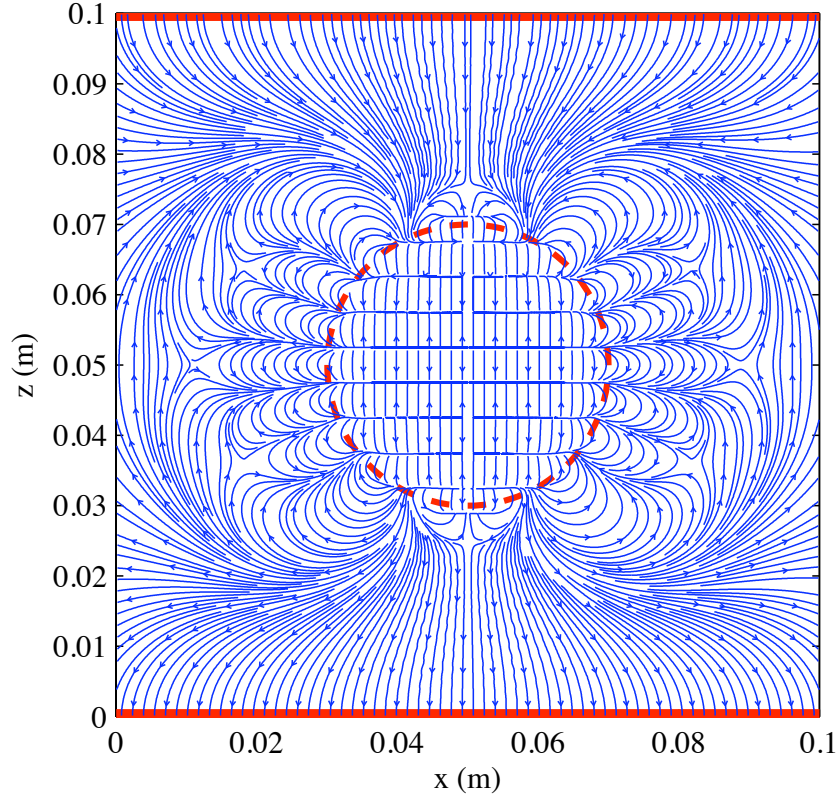


Figure 3.19: Electric lines of force in the case of a sphere of blood of radius  $R = 2$  cm lying on the symmetric axis of the system at  $z_0 = 5$  cm. The top and the bottom red lines are the electrodes and the red circle in the center is a cross section of the sphere.

given by (3.241), we have

$$f(\zeta) = \begin{cases} \frac{R\sqrt{1-\zeta^2}}{k_n} K_0(k_n \rho) I_1(k_n R) , & \text{if } \rho > R \\ \frac{1 - (k_n R \sqrt{1-\zeta^2}) K_1(k_n R \sqrt{1-\zeta^2}) I_0(k_n \rho)}{(k_n^2)} , & \text{if } \rho \leq R \end{cases} \quad (3.246)$$

with  $k_n = \frac{n\pi}{h}$ .

The potential distribution for a sphere of blood of radius  $R = 2$  cm lying on the symmetry axis of the system at  $z_0 = 5$  cm with  $\sigma = 0.3267$  S/m and  $\epsilon/\epsilon_0 = 218^{35}$  embedded in a body with sound speed  $c = 1500$  m/s traversed by a sound wave of frequency  $f = 150$  kHz is plotted on Fig. (3.18). The electric lines of force for the same sample are found by taking derivatives of the above potential as can be seen in Fig. (3.19).

# Chapter 4

## Characteristics of vibration potential signals

As discussed earlier in section (3.4), there are two mechanisms for current generation in the external circuit that follow from the creation of a polarization within the region between the two electrodes: the electric current and the displacement current. First, in the presence of a sound wave, the ultrasonically generated polarization  $\mathbf{P}_a$  produces an electric field within the inert medium. If the conductivity of the medium is  $\sigma$  then from Ohm's law, an electric current density  $\mathbf{J}_F$  for the flow of electric free charge is related to the electric field through the following equation

$$\mathbf{J}_F = \sigma \mathbf{E}, \quad (4.1)$$

or if using relation (3.15) between the electric field  $\mathbf{E}(\mathbf{x})$  and the potential  $\varphi(\mathbf{x})$

$$\mathbf{J}_F(\mathbf{x}) = -\sigma \nabla \varphi(\mathbf{x}). \quad (4.2)$$

The second mechanism for current generation arises from the time dependence of the ultrasonically induced polarization  $\mathbf{P}_a = \mathbf{P}_a(t)$ , which also leads to the time-

dependence of the electrical displacement  $\mathbf{D} = \mathbf{D}(\mathbf{t})$  through the constitution relation (3.12). From Maxwell's theory, the time derivative of the electrical displacement  $\frac{\partial \mathbf{D}}{\partial t}$  is a displacement current. The polarization current density  $\mathbf{J}_P$  then is found by the following equation

$$\mathbf{J}_P = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} = -\varepsilon \frac{\partial}{\partial t} \nabla \varphi. \quad (4.3)$$

The current in the circuit is found by adding the electric current and displacement current densities then integrating over the area of an electrode, as in equation (3.115)

$$I = \int_S (\mathbf{J}_P + \mathbf{J}_F) \cdot d\mathbf{S} = -(\sigma + i\omega\varepsilon) \int_S \nabla_n \varphi(\mathbf{r}) dS_n. \quad (4.4)$$

The ratio between magnitudes of the electric component  $I_F = -\sigma \int_S \nabla_n \varphi(\mathbf{r}) dS_n$  and the displacement component  $I_P = -i\omega\varepsilon \int_S \nabla_n \varphi(\mathbf{r}) dS_n$  is

$$\frac{I_F}{I_P} = \frac{\sigma}{\omega\varepsilon}. \quad (4.5)$$

It is obvious that if  $\sigma/\omega\varepsilon$  is very small compared to one, then the electric current from the flow of free charge can be neglected. On the contrary, if the ratio is big compared to one, then the dominant mechanism is the polarization current. Careful measurements of conductivity  $\sigma$  of deionized water and agarose, which are used mainly in vibration potential experiments, show that this ratio at the frequency of 1 MHz, as show in Table (4), is exceedingly small. The same statement is not true for a number of other body tissues<sup>35</sup>.

Two important questions arising in the study of vibration potential imaging when developing imaging modality in biomedical applications, specially in imaging of blood within soft tissues:

- (1) Is it the conduction of free charges or the polarization current that is the dom-



| Sample                          | $\sigma$ (S/m)        | $\hat{\epsilon}$ | $\sigma/\omega\epsilon$ |
|---------------------------------|-----------------------|------------------|-------------------------|
| Deionized water                 | $3.6 \times 10^{-6}$  | 84               | $7.71 \times 10^{-4}$   |
| Ethanol                         | $1.35 \times 10^{-7}$ | 24.5             | $9.88 \times 10^{-5}$   |
| Agarose                         | $6.8 \times 10^{-6}$  | 84               | $1.46 \times 10^{-3}$   |
| Blood <sup>35</sup>             | 0.822                 | 3026             | 4.89                    |
| Body fluid <sup>35</sup>        | 1.501                 | 84               | 321                     |
| Breast fat <sup>35</sup>        | 0.0258                | 23.7             | 19.6                    |
| Fat <sup>35</sup>               | 0.0251                | 27.2             | 16.6                    |
| Brain grey matter <sup>35</sup> | 0.163                 | 860              | 3.41                    |
| Liver <sup>35</sup>             | 0.187                 | 1536             | 2.19                    |

Table 4.1: Conductivity, relative dielectric constant  $\hat{\epsilon}$  and ratio  $\sigma/\omega\epsilon$  at 1 MHz.

inant mechanism for signal generation?

- (2) Does the current become independent of the frequency altogether at high frequencies?

We address these two questions with a series of experiments, the first making use of purely dielectric substances placed across the area of one electrode, and the second through measurement of the frequency dependency of the current.

## 4.1 Mechanism of voltage production in soft tissues

In mechanism experiments to determine, the aluminum cell was prepared as shown in Figure (4.1). The colloidal sample inside the cell was placed inside a cylindrical acrylic container with a thin bottom. The bottom of the acrylic container rested atop an arbitrary thickness acrylic block located at the bottom of the cell. The copper electrode was placed at the bottom of the acrylic block and was connected with a short length of insulated wire to an electric SMA feedthrough which led to the input of the preamplifier. The ground electrode was provided by a layer of degassed, deionized water placed above the colloidal sample. The deionized water acts as an acoustic

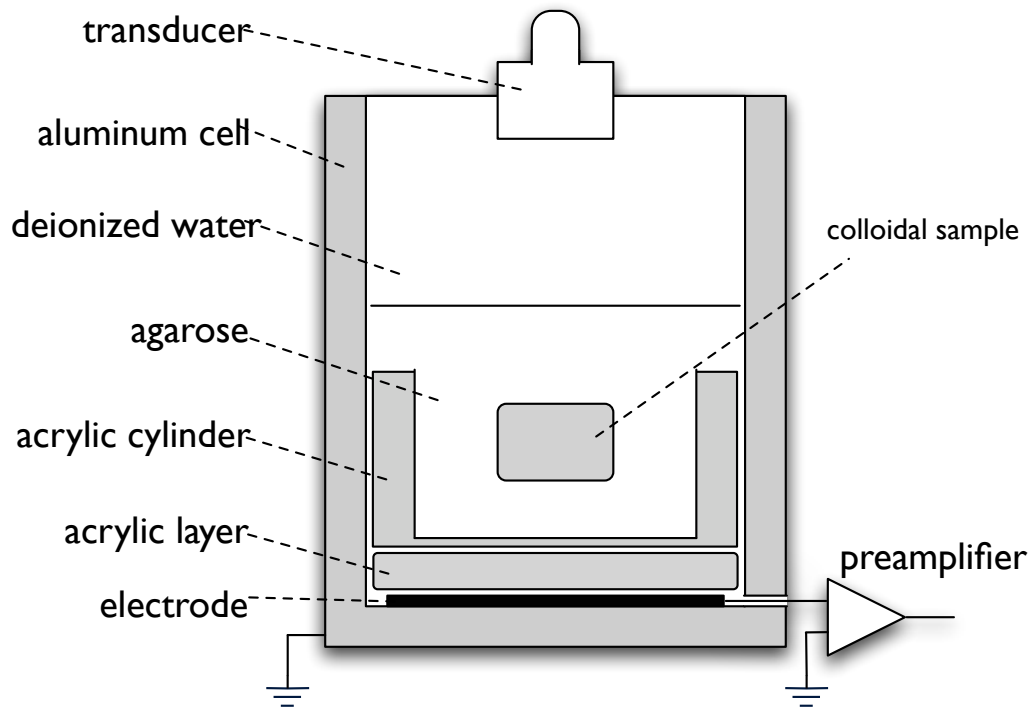


Figure 4.1: Diagram of the cell used for determination of the mechanism of current generation.

delay line, and, as a result of its small, but finite conductivity, serves as a ground since it contacts the inner surface of the aluminum cylinder as well as the immersed transducer.

A continuous sinusoidal voltage from a function generator (model 33250A from Agilent) was fed to a gated RF power amplifier (model GA 2500 from Ritec, Inc.) that drove an unfocused 25.4 mm diameter  $\text{LiNbO}_3$  transducer (model E1178 from Valpey Fisher Co.) terminated by a water cooled  $50\ \Omega$  terminator. The signal preamplifier connected to the electrodes was a current amplifier (model DHPA-100 from Femto Co.) and had a voltage amplification of 100. Its output was fed to a gated amplifier whose output, in turn, was fed to an RF lock-in amplifier (model SP 844 from Stanford Research Instruments) whose reference frequency was supplied by the function generator. The lock-in amplifier, which was connected to a computer through a GPIB port, can be controlled by a LabView program on a computer. The signal amplitude

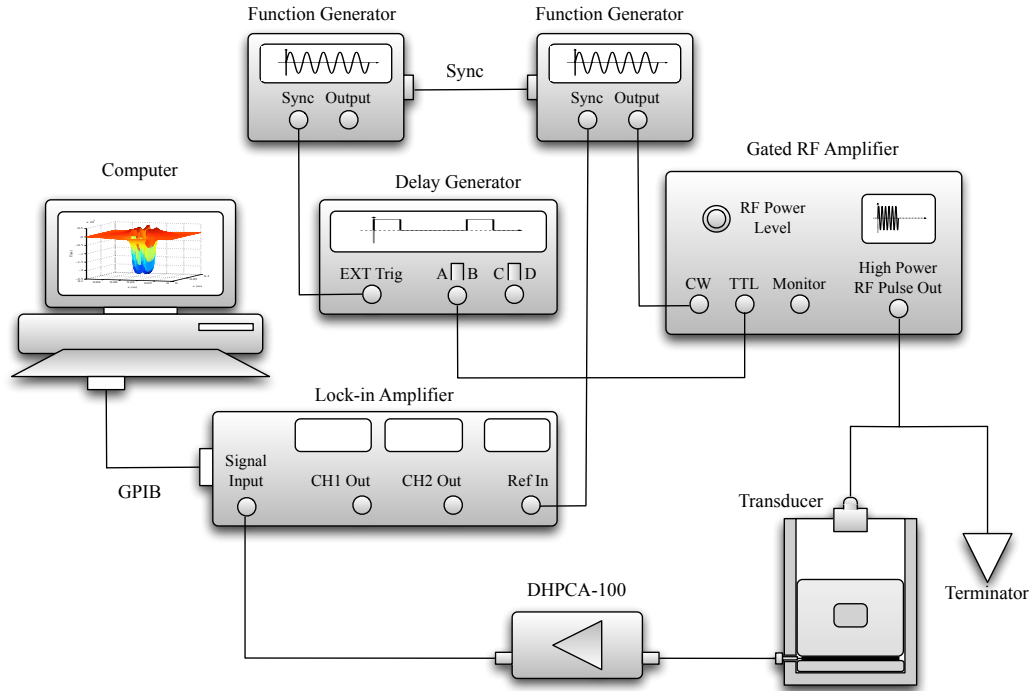


Figure 4.2: Experimental apparatus for determination of the mechanism of current generation.

and phase with respect to the input voltage from the function generator were then recorded from the lock-in amplifier. A detailed circuit is shown on Figure (4.2).

To make sure that the gated RF amplifier works properly and that the working high voltage burst would not exceed the duty cycle of 0.3%, the continuous voltage was an RF burst of 20 cycles of amplitude 0.5 V<sub>pp</sub> and frequency 1 MHz. The experiments were carried out with different blocks of acrylic with various thicknesses placed between the acrylic container and the copper electrode. The plot in Figure (4.3) shows the effect of increasing the layer thickness. Since the acrylic can be approximated as having essentially no electrical conductivity, the role of ionic conductivity as the primary mechanism of current generation can be seen to be negligible: a complete loss of signal would be expected for insertion of even the thinnest of the layers used.

A quantitative description of the curve for signal versus insulating layer thickness shown in Figure (4.3) requires a knowledge of the effect of addition of layers of differ-

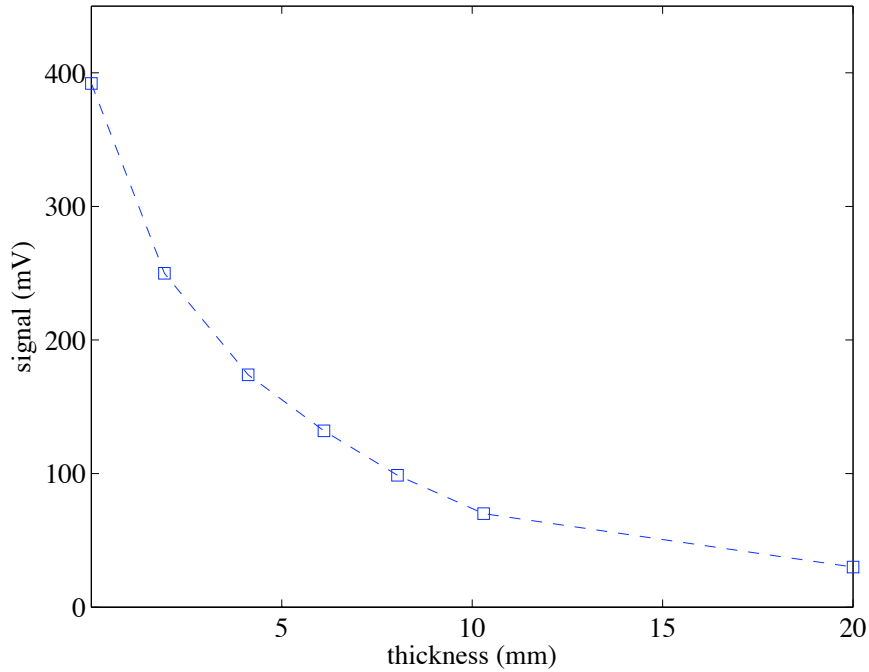


Figure 4.3: Vibration potential signal versus layer thickness for several acrylic layers. Data were obtained at 1 MHz.

ent conductivity and dielectric constant on the current from the cell which requires substantial modification of the Green's function that is used in the formulation of the theory that gives the integral formula in Eq. (4.4). In the absence of such theory, however, it can be concluded that the current generation mechanism relies on the production of a time dependent polarization and is wholly independent of the conductivity for the suspensions studied here.

## 4.2 Frequency dependence of the vibration potential

Figure (4.4) shows the preparation of the cell for frequency response experiments. In these setups, colloidal solution inside the aluminum cell was placed inside a cylindrical acrylic container equipped with two plastic films at the top and bottom that

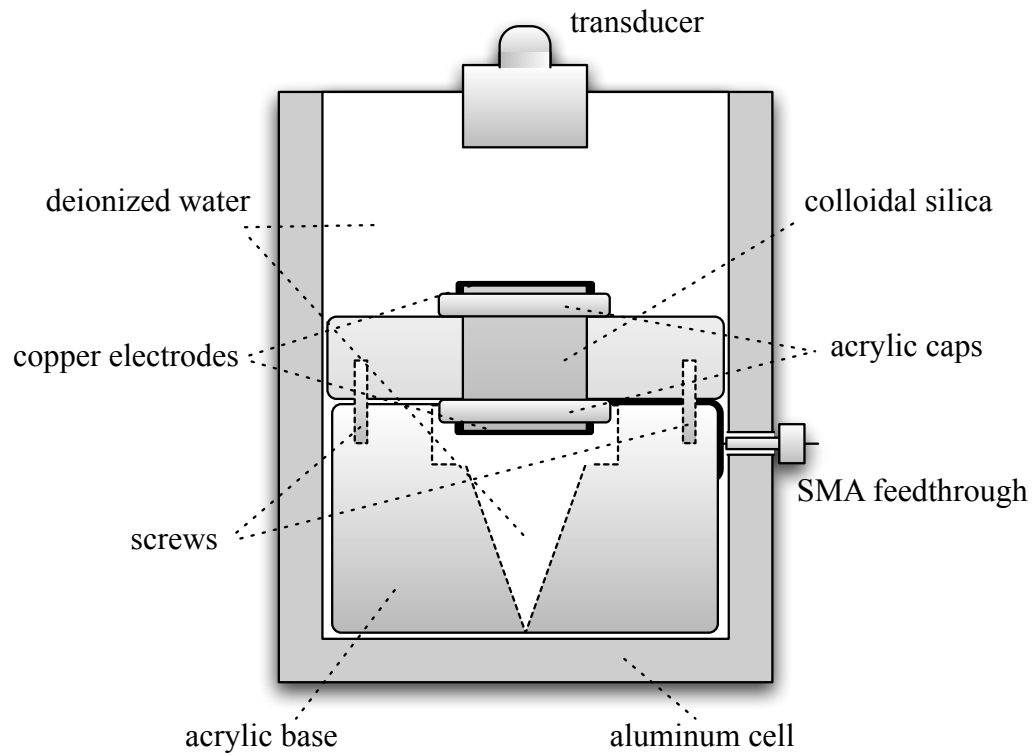


Figure 4.4: Diagram of the cell used for frequency response experiments.

permitted entry and exit of the ultrasound, but which confined the colloidal solution. Two thin copper tapes were attached to the inner surfaces of those plastic films to act as electrodes and then were kept on top and bottom of the acrylic container by two acrylic caps. The bottom electrode was connected with a short length of insulated wire to an electrical SMA feedthrough which led to the input of the preamplifier. The acrylic container was placed on top of an acrylic base with a conic hole which both confined deionized water and helped terminating the ultrasound reflection. On top of the acrylic container was deionized water which acts as an acoustic delay line, and, as a result of its small, but finite conductivity, serves as a ground since it contacts the inner surface of the aluminum cylinder and the housing of the transducer.

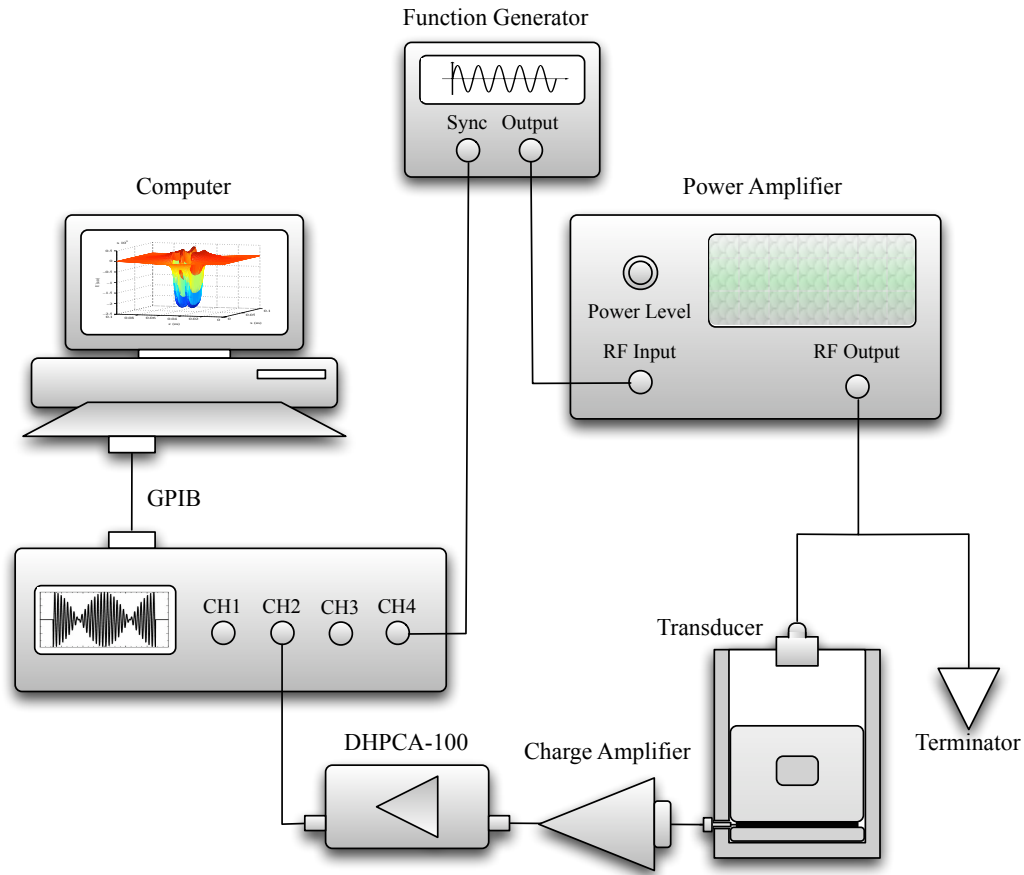


Figure 4.5: Diagram of the apparatus for the frequency response experiments.

The transducer used in this series of experiments is a flat PZT transducer with the center frequency of 3.5 MHz (model V382 from Panametrics, Inc.). It was fed by a high voltage signal from a power amplifier (model GC S0C2CRR from Empower, Inc.) through a high power  $50\ \Omega$  terminator which was cooled by circulated tape water. A function generator (model 33250A from Agilent) was used to supply sinusoidal bursts of amplitude 0 dB at a computationally controlled frequency to the power amplifier. The ultrasonic vibration signal from the electrodes was fed to a charge sensitive amplifier (model 142A from EG&G Ortec) whose output, in turn, was passed to an RF current amplifier (model DHPCA 100 from Femto) operated with an amplification of 100. The magnitude of the amplified signal was then recorded on an oscilloscope

(Infinium model 54845A from Agilent) by a computer connected with the oscilloscope by a GPIB port. The circuit diagram for frequency response setups is shown in Figure (4.5).

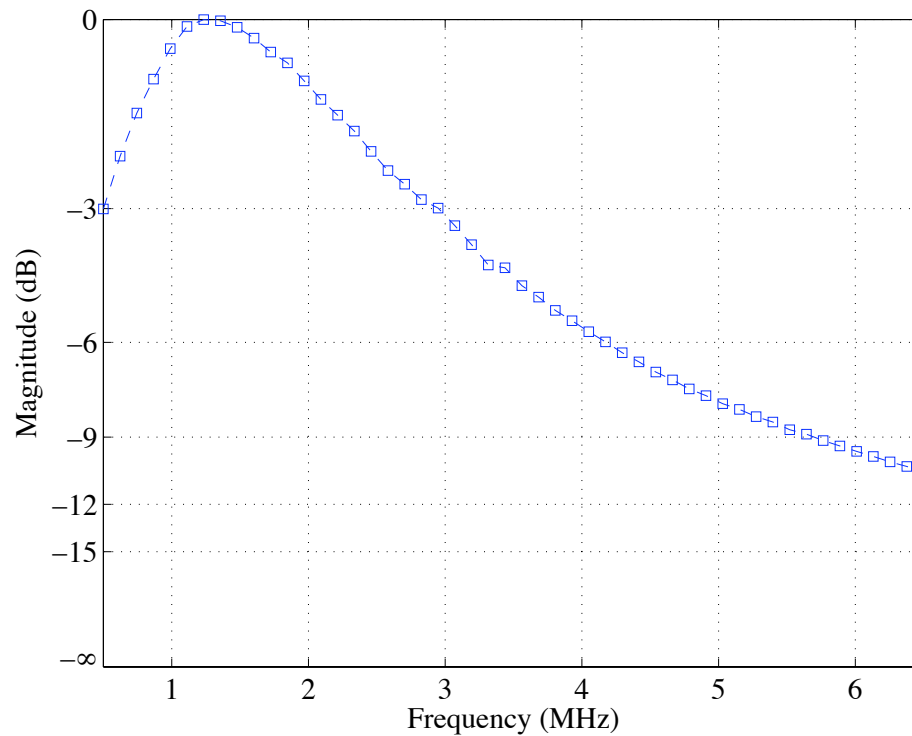


Figure 4.6: Output signal versus frequency for the combination of the Ortec and Femto amplifiers.

Experiments to determine the frequency dependence of colloidal suspensions required careful calibration of the combined power amplifier and transducer, as well as the signal amplifiers including charge sensitive and current amplifiers. Calibration was carried out using a function generator which produced a sinusoidal wave with a constant amplitude over the frequency range of the experiment. The results of the calibration, as shown in Figure (4.6), show a falloff in gain with frequency as expected for a charge sensitive amplifier. The decrease in gain seen below about 1.2 MHz is attributed to the integration time constant of the charge amplifier. The distance

between the PVDF transducer used for calibration and the colloidal sample used in the vibration current experiments were made the same so that viscous attenuation of the ultrasound is taken into account in the calibration.

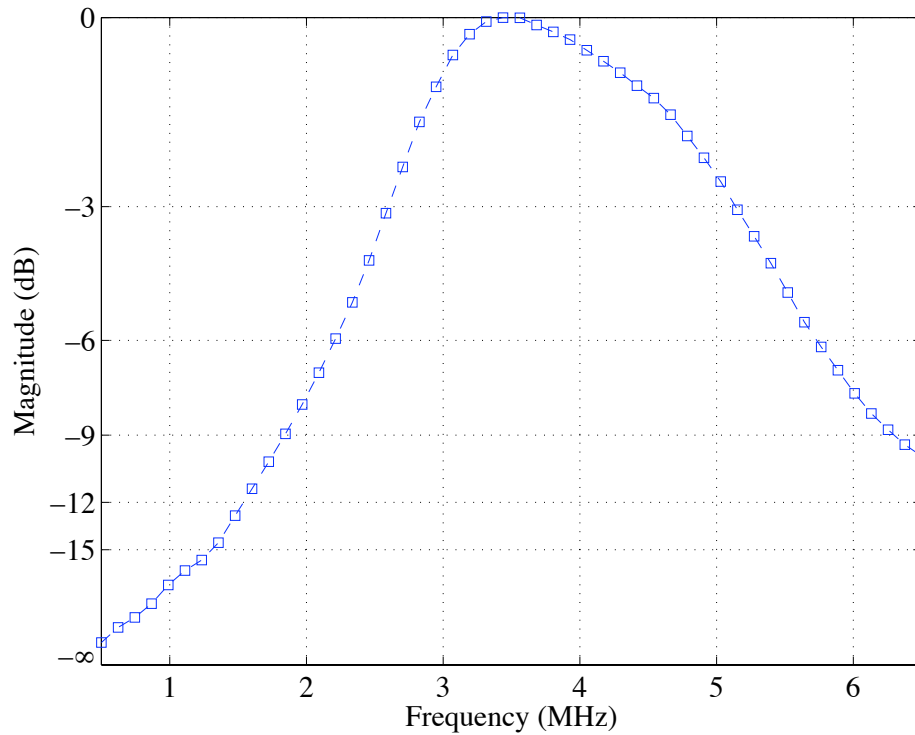


Figure 4.7: Frequency response of the flat 3.5 MHz transducer.

The active element in the ultrasonic transducer is a resonant PZT piezoelectric disk attached to a backing material; hence its frequency response is expected to be peaked around its 3.5 MHz resonance frequency. The data shown in Figure (4.7) were taken using a PVDF transducer with a flat frequency response as the function generator that drove the power amplifier was slowly swept over the range from 0.5 to 6.5 MHz.



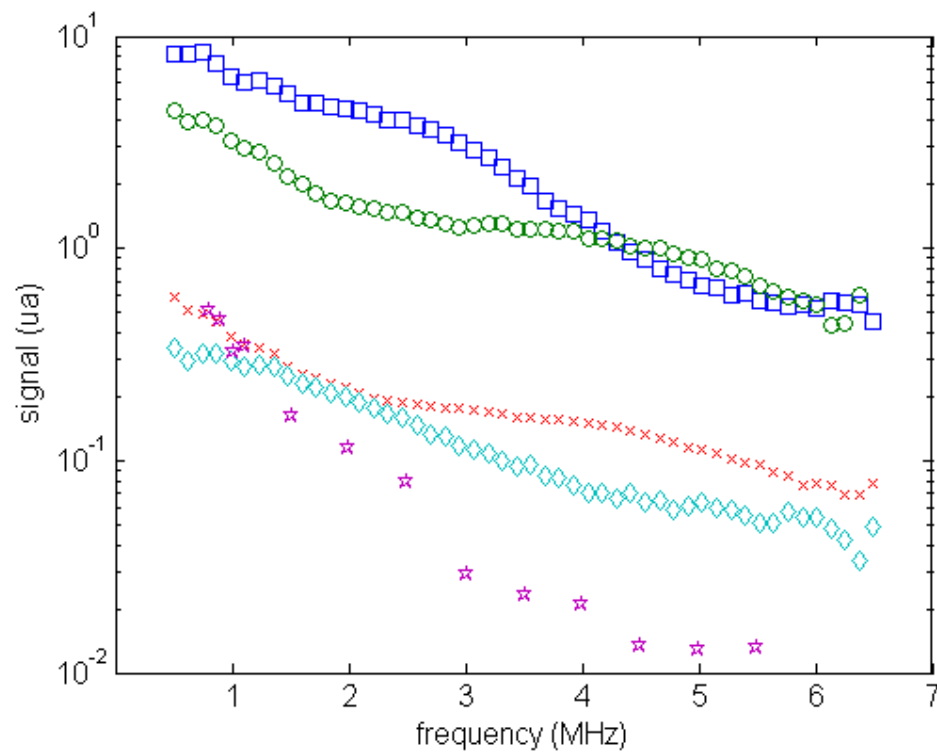


Figure 4.8: Current in arbitrary units versus frequency for blue, undiluted; green, 50 %; red, 20 %; cyan, 10 % dilutions of colloidal silica in deionized water; and pink, whole canine blood.

All data for the frequency response of the vibration current from the colloidal suspensions were corrected for the frequency response of both the power amplifier and transducer combination, as well as that of the amplifiers. The data shown in Figure (4.8) give the amplitude of the vibration current versus frequency for four different dilutions of the silica colloid in water, and that of whole canine blood preserved with heparin. For frequencies below 5 MHz; the amplitude of the signal produced by whole blood can be seen to be nearly inversely proportional to frequency.

### 4.3 Temperature dependence of the vibration potential

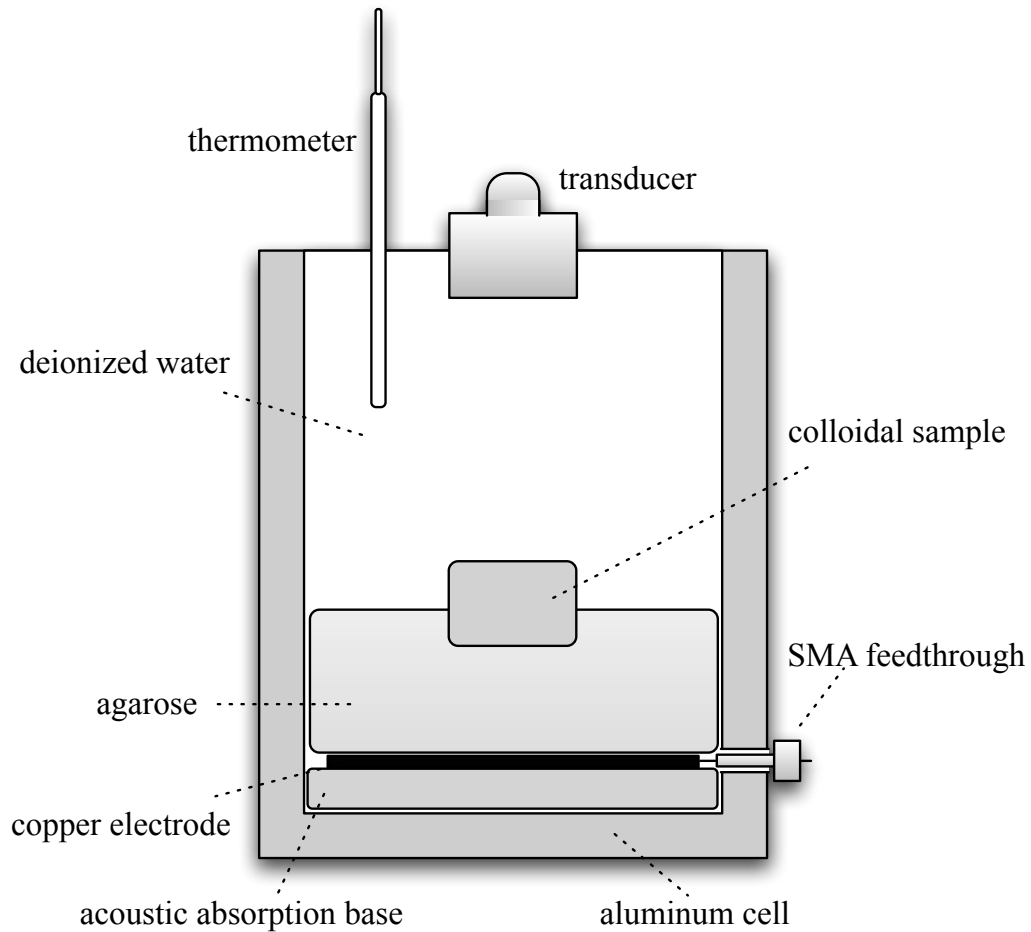


Figure 4.9: Diagram of the cell used for temperature dependence experiments.

Experiments investigating the temperature dependence of the ultrasonic vibration signal were conducted with the cell prepared as shown in Figure (4.9). A colloidal silica layer, which was prepared by mixing colloidal silica solution and boiled agarose in deionized water, is held inside the aluminum cell and on top of an agarose block, which, in turn, rested on top of the copper electrode. On the bottom of the system, an acoustic absorption base (Aptflex F28 from Precision Acoustic Ltd.) was used to terminate the sound reflection. The cell then was filled with hot deionized

water (around 35 °C). As a result, the colloidal sample was quickly heated by the heat conducting from the water. The temperature of the system was measured by a thermometer placed inside the aluminum cell.

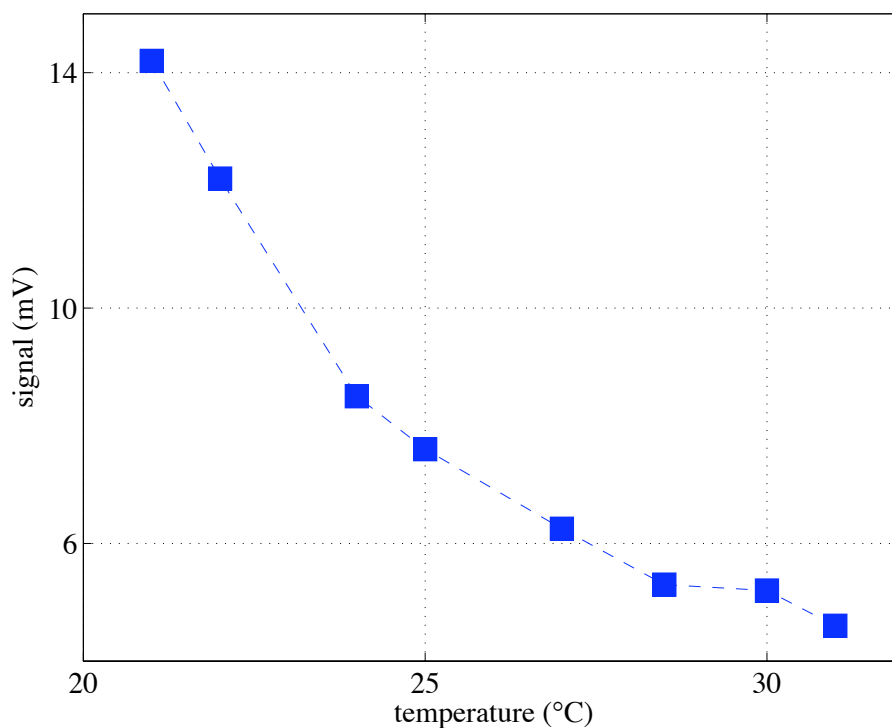


Figure 4.10: Vibration signal versus temperature of a colloidal layer.

The circuit diagram for temperature dependence setup is identical to that for mechanism determination, which is shown in Figure (4.2). A 1 MHz flat transducer was used to produce ultrasound. The vibration signal was recorded by a LabView program on a computer connected to the lock-in amplifier via a GPIB port. In the experiments, the system was allowed to cool down slowly to the room temperature while the current and the temperature were recorded at the same time. The result of these experiments, which is shown in the plot in Figure (4.10), shows that the vibration signal decreases rapidly with temperature.

Although it is known that the sound speed of the colloid depends on temperature,

as discussed earlier in section (1.3.1), a full quantitative explanation of the curve for signal versus temperature requires a careful theoretical investigation of the temperature dependence of the electrophoretic mobility  $\mu_E$  of the colloid, which was not studied in this thesis.

# Chapter 5

## 3-D ultrasonic vibration imaging

As discussed in Chapter (3), the ultrasonic vibration current produced from acoustical polarizations in colloidal objects depends on both the frequency (or more precisely the wave vector  $\kappa$ ) of the incident ultrasound and the dimensions of the objects such as thickness of layer, radius of the sphere or a cylindrical object, etc. For example, for a case of spherical colloidal object radius  $R$ , the vibration current, as calculated in section (3.4.3), is

$$I = -4\pi R^2 I_0 \frac{1}{\kappa R} \left( \cos \kappa R - \frac{\sin \kappa R}{\kappa R} \right) , \quad (5.1)$$

or because  $I_0 = \frac{2\pi f \alpha_0 p_0}{h}$  and  $\kappa = \frac{2\pi f}{c}$  where  $f$  is the frequency of the incident ultrasound and  $c$  is the sound speed

$$I = -\frac{4\pi R \alpha_0 p_0 c}{h} \left[ \cos \left( \frac{2\pi f R}{c} \right) - \frac{c \sin \left( \frac{2\pi f R}{c} \right)}{2\pi f R} \right] . \quad (5.2)$$

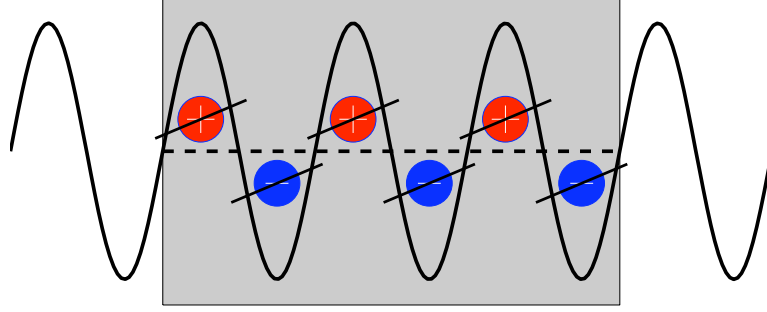


Figure 5.1: Diagram of a colloidal object and ultrasonic wave. When there are an even number of half wave cycles inside the colloidal objects, the positive and negative polarizations cancel out each other which leads to a zero net total polarization charge at the boundaries of the object.

Formula (5.2) shows that there are some certain frequencies where the current magnitude is zero; at other frequencies the current magnitude takes on its maximum value. The physical meaning of these extreme values is that, at some frequencies, there is an odd or even number of half wave cycles residing inside the colloidal object. In the case of an even number of half wave cycles as shown in Figure (5.1), the number of positive acoustic polarization dipoles produced inside the colloidal layer equals the number of negative acoustic polarization dipoles, resulting in a cancellation. The total dipole moment of the sample then is zero which leads to zero current in the external circuit. If  $L$  is the length of the colloidal layer and  $\lambda$  is the wavelength of the ultrasound, then

$$L = n\lambda, \quad (5.3)$$

where  $n$  is any positive integer number.

In the case of an odd number of half wave cycles as shown in Figure (5.2), the

positive and negative acoustic polarization dipoles do not cancel each other, hence the colloidal sample has a net dipole moment which leads to an alternating current flowing through the external circuit. The relation between  $L$  and  $\lambda$  in this case is

$$L = \left(n - \frac{1}{2}\right) \lambda. \quad (5.4)$$

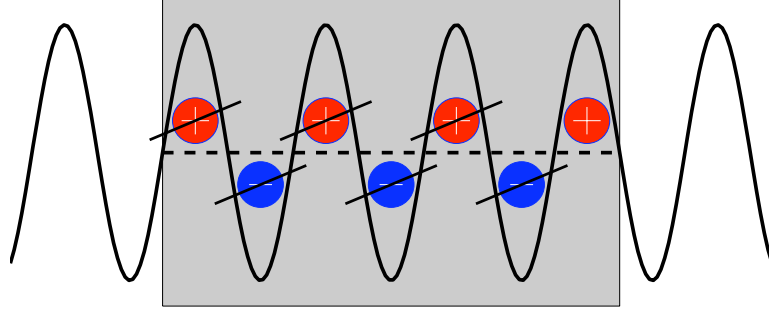


Figure 5.2: When there are an odd number of half wave cycles inside the colloidal objects, the positive and negative polarizations do not cancel out each other leading to a current in the external circuit.

Experiments can be conducted to measure the magnitude of the ultrasonic vibration current  $I$  for a colloidal sphere versus the frequency  $f$  of the ultrasound. The experimental data then can be used to calculate the dimensions of the colloidal object, such as diameter  $R$  of the sphere based on formula (5.2).

## 5.1 Experiment setup

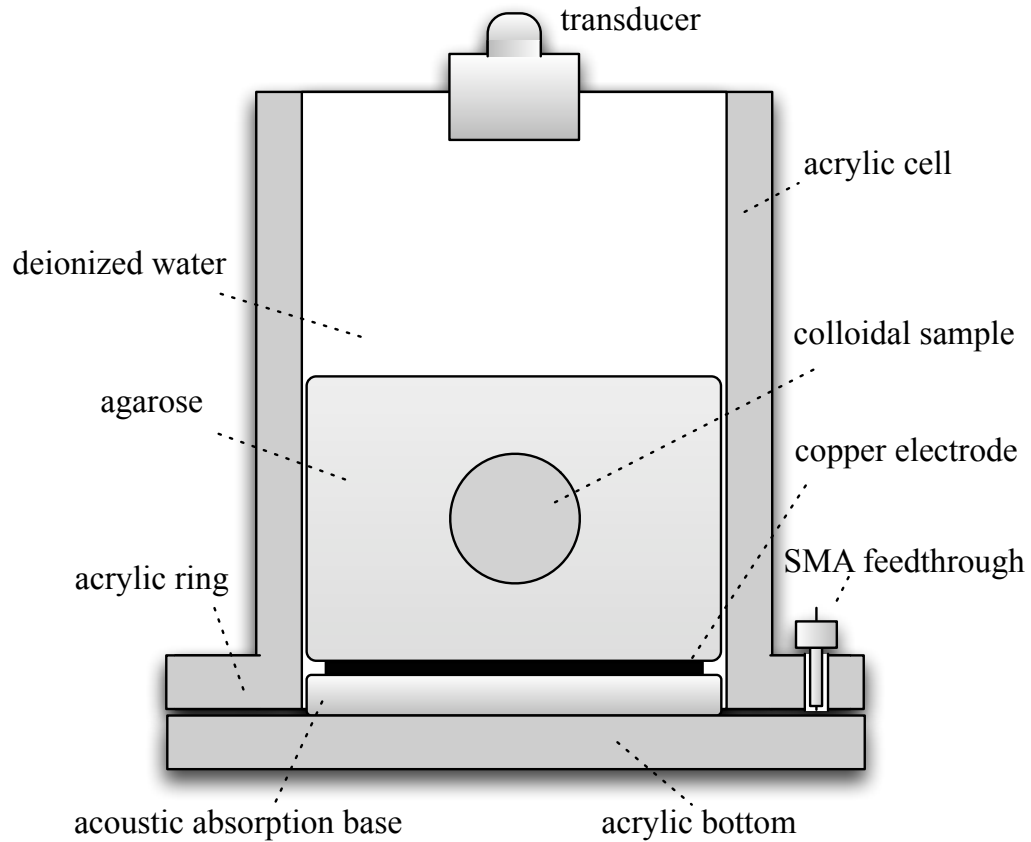


Figure 5.3: Cell geometry for 3D imaging experiments.

Preparation of the cell used in 3D imaging experiments is shown in Figure (5.3). Colloidal samples (spheres or cylinders) were held inside the acrylic cell by an agarose block located on top of a copper electrode connected to the SMA feedthrough output by a thin copper tape. An acoustic absorption base made of Aptflex F28 from Precision Acoustic Ltd. was placed on the bottom of the cell to help reduce ultrasound reflection. The cell was then filled with deionized water to act as an acoustic delay line as well as a ground since it contacts the housing of the transducer, which in turn, was grounded by a copper wire.



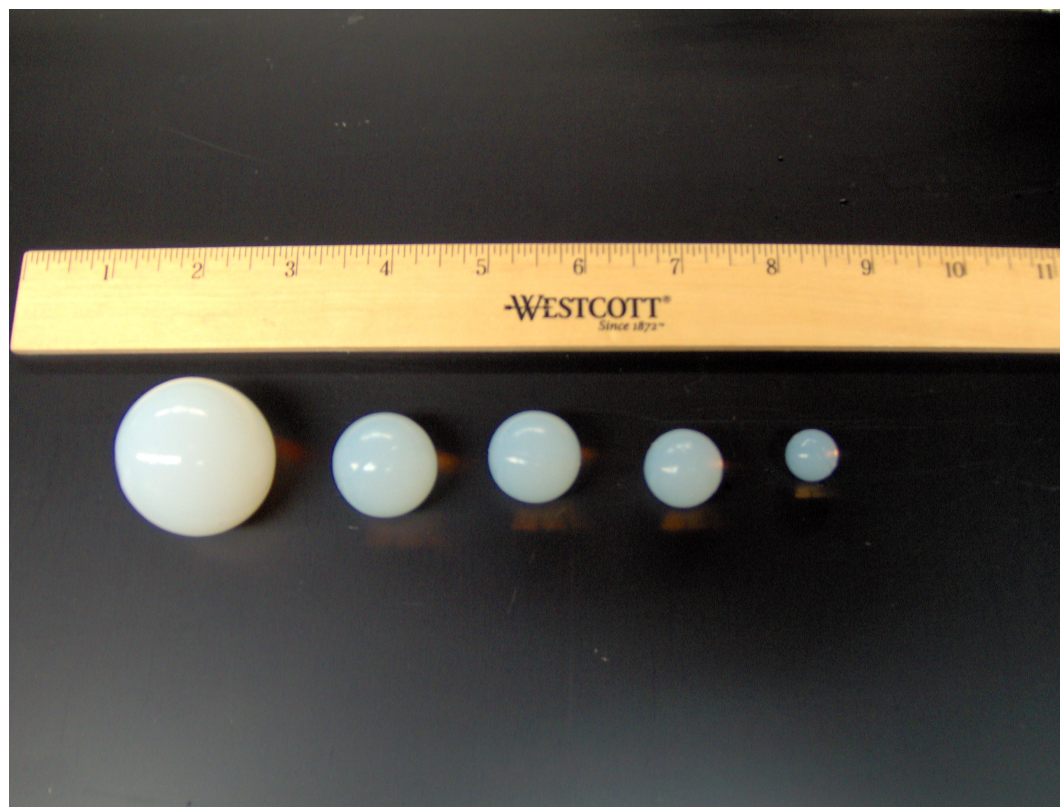


Figure 5.4: Photograph of spheres with different diameters made of colloidal silica.

Colloidal samples were prepared by mixing boiled agarose solution with silica colloid which was allowed to solidify in a mold. Figure (5.4) and (5.5) are pictures of silica spheres and other samples made of colloidal silica.



Figure 5.5: Photograph of different samples made of colloidal silica.

In order to act as a support which holds the colloidal sample inside the cell, boiled

agarose was first filled inside the acrylic cell to make a delay line of about 20 to 50  $\mu\text{s}$ , which corresponds to an agarose layer with thickness of 4.5 to 7.5 cm. The cell then was allowed to cool down in about 2 to 3 hours so that the agarose layer became a gel block. The colloidal sample then was placed gently on top of the agarose block. Boiled agarose solution then was added in the cell so that the second layer completely covered the sample, and then again, was allowed to cool down in about 2 to 3 hours to become a solid block. Finally, when the second agarose block was solid enough, deionized water was filled in the cell up to the desired depth to make sure the distance between the transducer and the colloidal sample is appropriate.

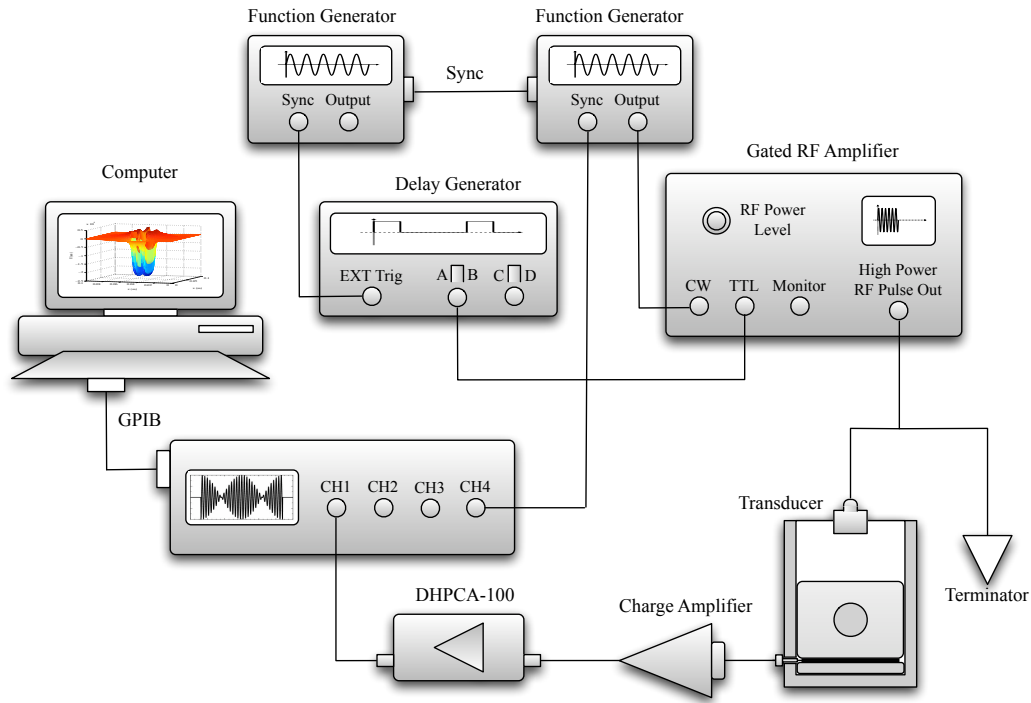


Figure 5.6: Circuit diagram of the apparatus used for 3D imaging experiments.

The cell then was placed in the circuit diagram of which is shown in Figure (5.6). The transducer used in these experiments is a customized 1 in diameter flat  $\text{LiNbO}_3$  transducer (model E1410 from Valpey Fisher) with a center frequency of 0.769 MHz. This transducer was fed by the high voltage output from the gated RF amplifier (GA

2500 from Ritec) through a high power  $50\ \Omega$  terminator. Two function generators (model 33250A from Agilent) were used to supply inputs for the gated amplifier: one sends a continuous sinusoidal signal to the CW Input of the power amplifier and one send a 50 Hz and 4 Vpp trigger to a delay generator (model DG 535 from Stanford Research System, Inc.) whose AB output, in turn, was fed to the TTL input of the power amplifier. The output signal from the SMA feedthrough of the cell was sent to a charge amplifier (model 142A from EG&G Ortec) to act as a preamplifier. The output of the charge amplifier then was sent to a current amplifier and then to an oscilloscope (model 54845A from Agilent). The signal detected and averaged on the oscilloscope was then recorded by a LabView program on a computer connected to the oscilloscope by a GPIB port.

The LabView program on the computer can simultaneously control both the function generator and the oscilloscope via GPIB interfaces connected from those devices to the computer. In the appointed range of frequency from 0.5 to 1.2 MHz, the program, in each cycle of runtime, set a certain frequency which was increased a constant increment each runtime which controlled the function generator that sent out a continuous sinusoidal signal with that frequency and an amplitude of 0.5 Vpp. This sinusoidal signal, when fed to the CW input of the gated amplifier whose RF power level set to an arbitrary instrumental gain setting of "7.5", was mixed with the 50 Hz trigger from the other function generator and amplified to produce a high voltage sinusoidal burst. This high voltage burst, at an RF power level of 7.5, according to the frequency response curve of the GA 2500 in Figure (2.15), has an amplitude around 270 to 290 V which is an appropriate value to drive the transducer.

The ultrasonic vibration signal coming out of the SMA feedthrough, after being amplifier by the charge and current amplifiers, was then sent to the oscilloscope by a BNC cable. The signal values then were recorded in the memory of the oscilloscope. The LabView program then read those values, calculated the amplitudes of the signal

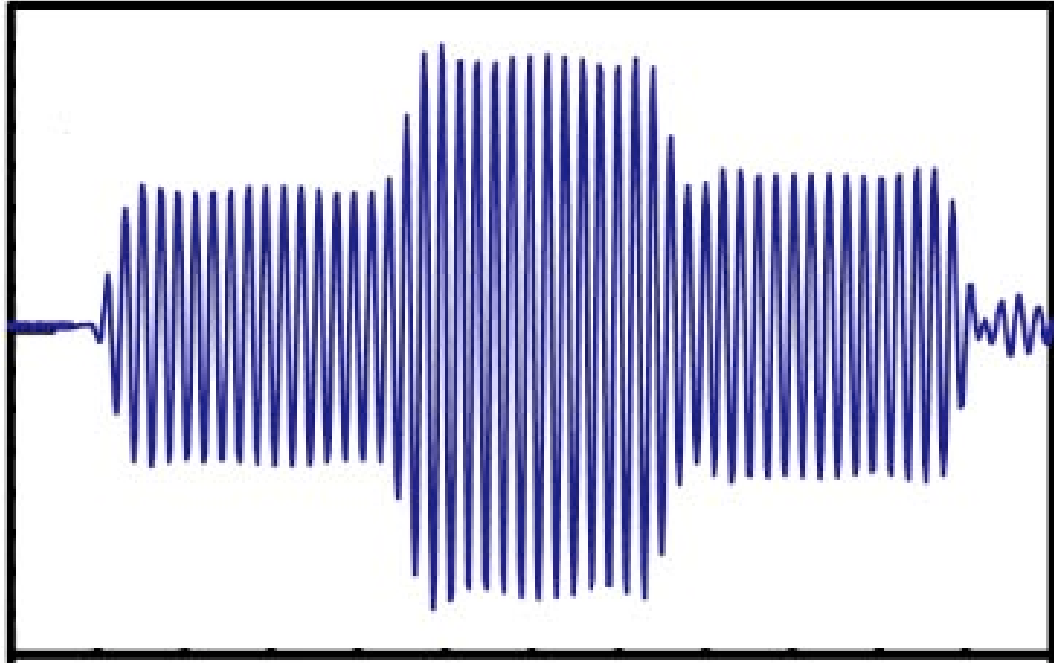


Figure 5.7: Typical vibration signal captured on the screen of the oscilloscope.

and then wrote these values together with the corresponding frequencies to a text file, which could be analyzed later.

## 5.2 Data treatment and calculations

There are many factors that affect on the experiment data that should be considered. Each electronic device such as the gated amplifier, the charge and current amplifier has its own operational frequency dependence. One should also take into account the frequency dependence of the pressure produced by the transducer as well as the frequency response of colloidal silica. Experiment data stored in text files recorded in the computer are just raw data which must be corrected for the factors noted above. These data must be corrected by all above factors before they can be compared to the theory. Figure (5.8) and (5.9), for example, show the raw signal versus frequency curves for the sphere of diameter 22.2 mm and cylinder of diameter 19 mm correspondingly.

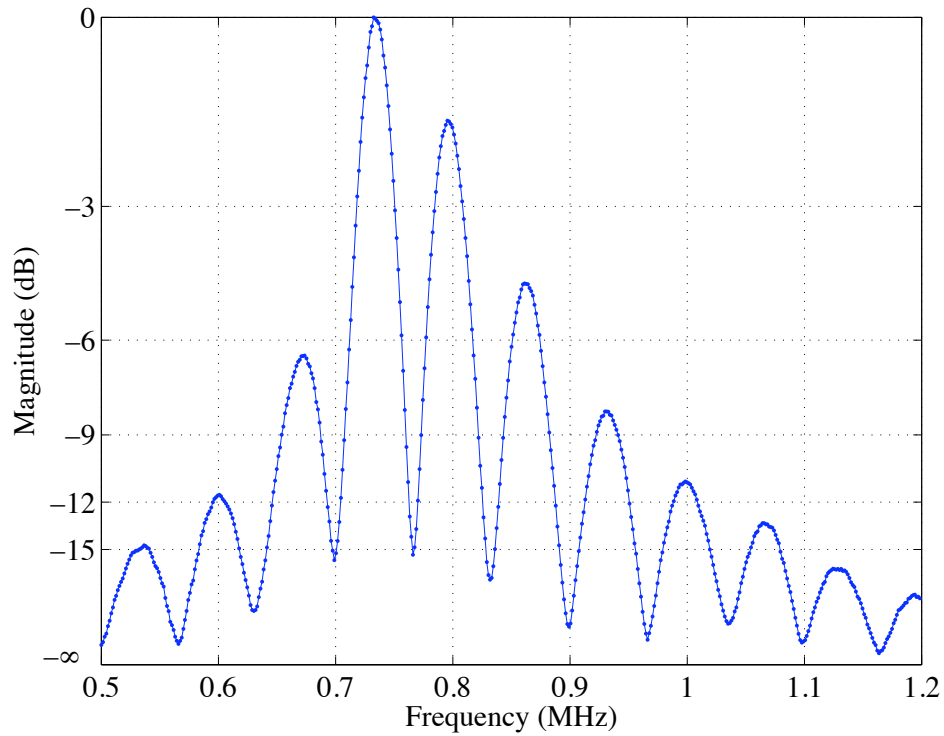


Figure 5.8: Raw vibration signal versus frequency data for a colloidal silica sphere of diameter 22.2 mm.

In order to account on all external factors noted above, two series of measurements were taken in every experiment for each colloidal sample. The first series measured the amplitude of the center burst of the signal shown in Figure (5.7), which is the portion of the waveform where the entire object is exposed to ultrasound. The other series measured the amplitude of the incoming burst (the left burst in Figure (5.7)) which represents all other factors, including the frequency dependence of colloidal silica, transducer and other electronic devices such as gated amplifier, current amplifier etc. The corrected data then were found by dividing data from the first experiment by data from the second series of measurements.

After correction, the data then were used to find characteristics of the colloidal objects by using a nonlinear least squares fitting algorithm. For a sphere of radius  $R$ ,

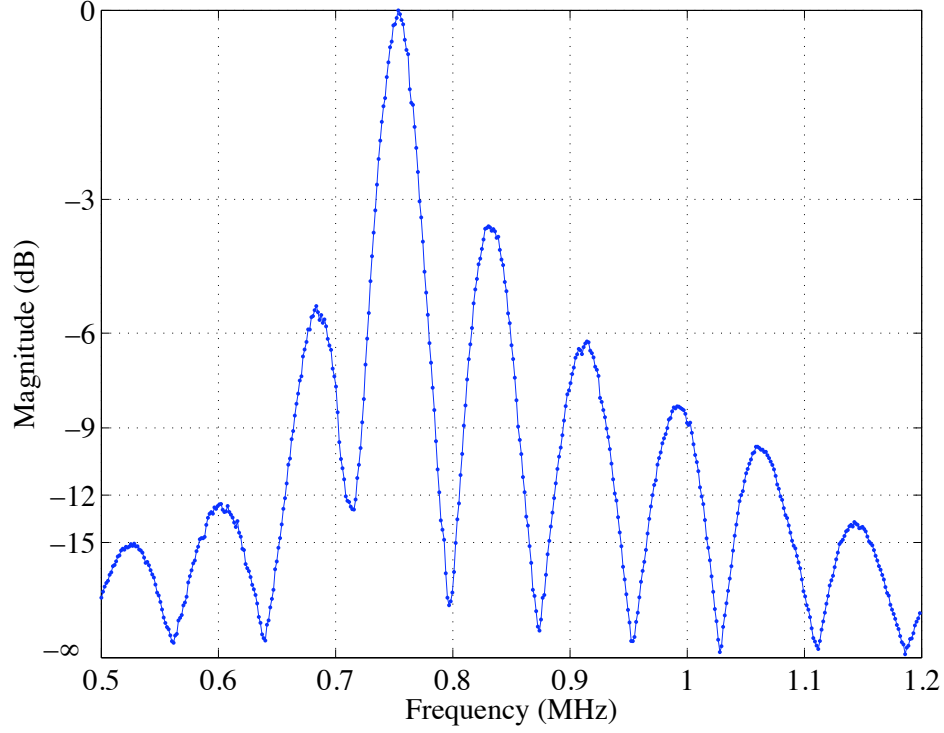


Figure 5.9: Raw vibration signal versus frequency data for a colloidal silica cylinder of diameter 19 mm.

amplitude of the current, from equation (5.2), can be written as

$$I(f) = I_0 R \left| \cos\left(\frac{2\pi f R}{c}\right) - \frac{c \sin\left(\frac{2\pi f R}{c}\right)}{2\pi f R} \right|. \quad (5.5)$$

Using the sound speed  $c = 1480$  m/s ( $\pi = 3.1416$ ) we have

$$I(f) = I_0 \left| R \cos(4.245 \times 10^{-3} f R) - 2.355 \times 10^2 \frac{\sin(4.245 \times 10^{-3} f R)}{f} \right|. \quad (5.6)$$

Equation (5.6) represents a nonlinear dependence of the current amplitude  $I$  versus frequency  $f$  where  $I_0$  and  $R$  are two unknown constants. For each set of data  $(f_i, I_i)$  where index  $i$  represents the  $i$ 'th frequency, current data point, a Matlab program (see appendix (B.3)) using the least squares fitting procedure gave the experimental values for  $I_0$  and  $R$ .

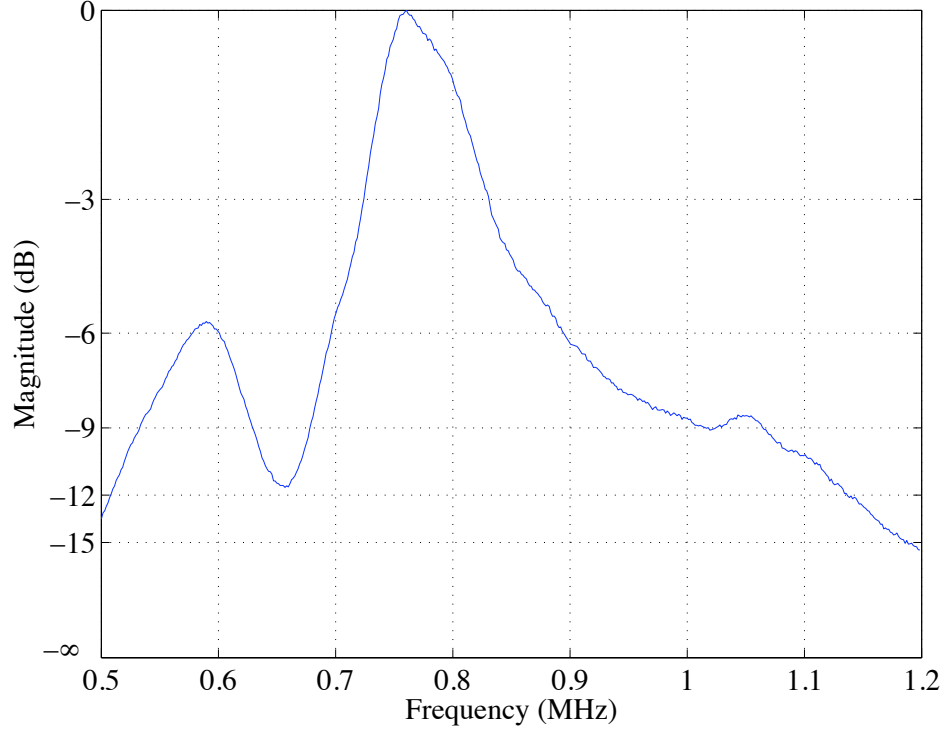


Figure 5.10: Data of the second series of measurements of the incoming burst which represents the frequency dependence of all external factors.

For cylinder of radius  $R$ , according to equation (3.187), the magnitude of the current per unit length of the cylinder is

$$I(f) = \frac{4\pi^2 R f \alpha_0 p_0}{ch} \left| J_1 \left( \frac{2\pi f}{c} R \right) \right| , \quad (5.7)$$

which, on substitution of  $c = 1480$  m/s gives

$$I(f) = I_0 f R \left| J_1 \left( 4.245 \times 10^{-3} f R \right) \right| , \quad (5.8)$$

where  $J_1(x)$  is Bessel function of the first order of  $x$  and  $I_0$  and  $R$  are unknown constants.

For convenience, all the signal values, obtained both from experiments and calculations (which are in unit of mV) were compared to the maximum value and then

were converted to dB.

### 5.3 Experiment results

Experiments conducted with six different colloidal spheres with diameters 12.79, 18.55, 22.20, 25.19, 31.30 and 38.50 mm, and two colloidal cylinders with diameters 19.08 and 24.88 mm were carried out. The results of these experiments are shown in Figures (5.11) to (5.18).

In the calculations, the theoretical curves are equations (5.6) and (5.8) multiplied by the correction curves for the frequency response of the transducers, amplifiers and colloidal silica. The measured values of diameter of the samples were obtained by using the digital caliper which has the instrumental error of 0.01 mm. Calculation error of diameter was obtained by calculating the 95 % confidence interval of that diameter from the nonlinear least squares algorithm.

Table (5.1) summaries the experimental results of the data fits and their corresponding errors when comparing the calculated values of diameter using nonlinear least squares fitting procedure and the measured values.



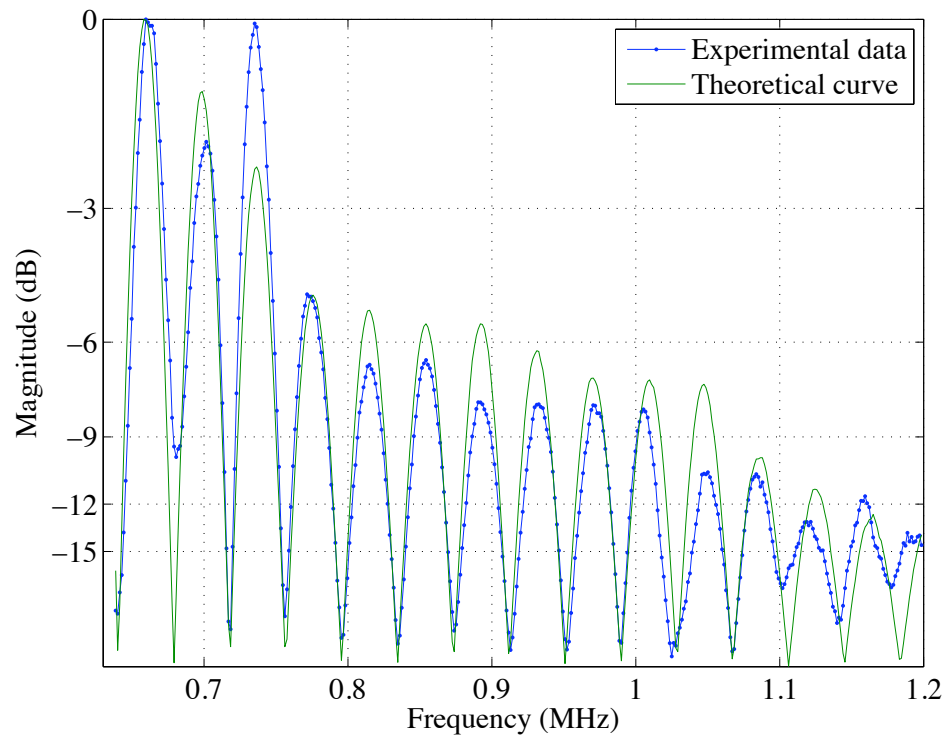


Figure 5.11: Experimental data (blue) and theoretical curve (green) for a colloidal sphere of diameter  $38.05 \pm 0.01$  mm. The nonlinear least squares fit to the data gave the diameter of the sphere to be  $38.12 \pm 0.03$  mm.

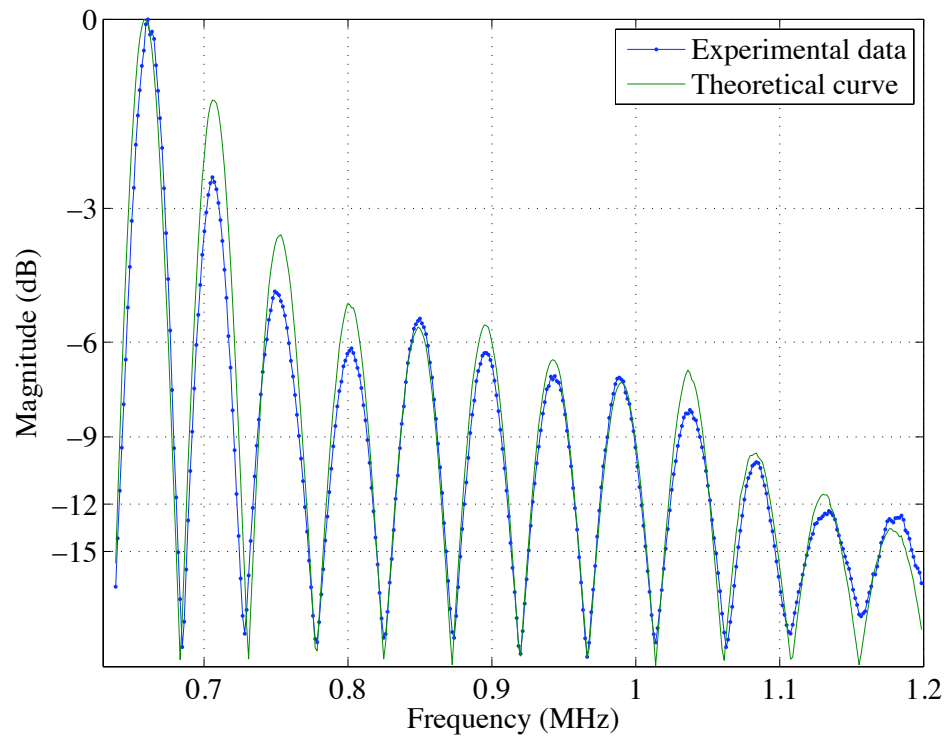


Figure 5.12: Experimental data (blue) and theoretical curve (green) for a colloidal sphere of diameter  $31.30 \pm 0.01$  mm. The nonlinear least squares fit to the data gave the diameter of the sphere to be  $31.38 \pm 0.02$  mm

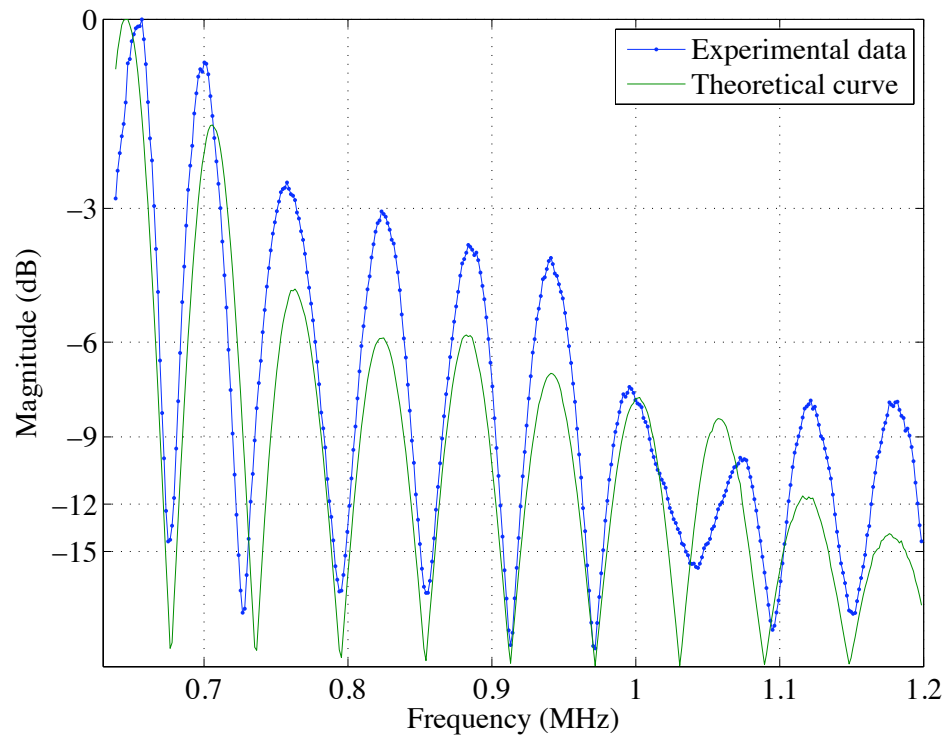


Figure 5.13: Experimental data (blue) and theoretical curve (green) for a colloidal sphere of diameter  $25.19 \pm 0.01$  mm. The nonlinear least squares fit to the data gave the diameter of the sphere to be  $25.12 \pm 0.02$  mm

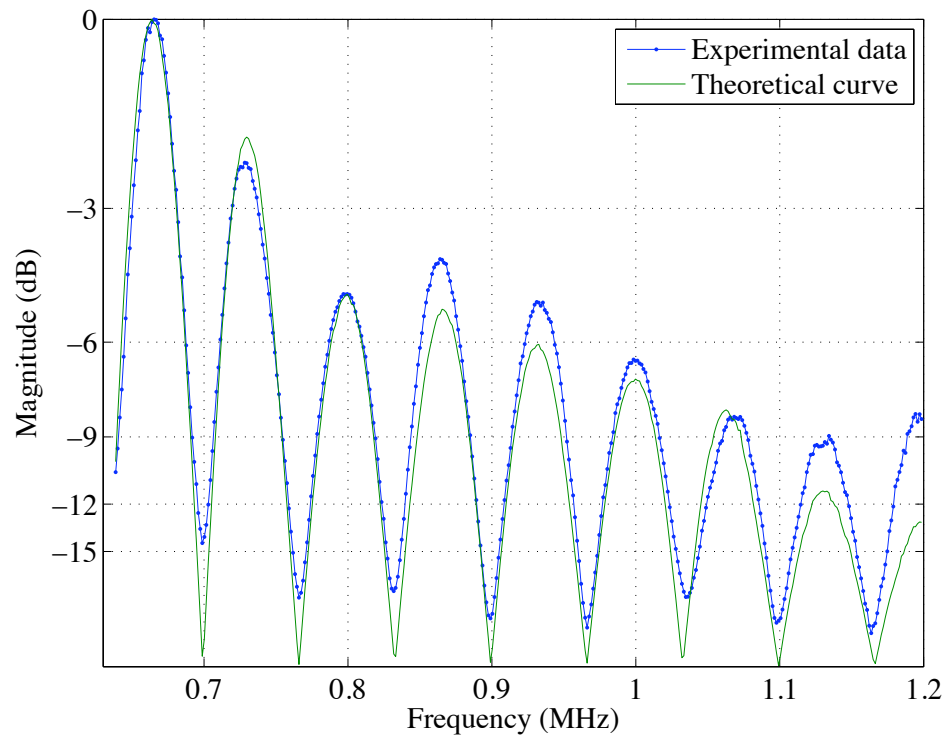


Figure 5.14: Experimental data (blue) and theoretical curve (green) for a colloidal sphere of diameter  $22.20 \pm 0.01$  mm. The nonlinear least squares fit to the data gave the diameter of the sphere to be  $22.21 \pm 0.02$  mm

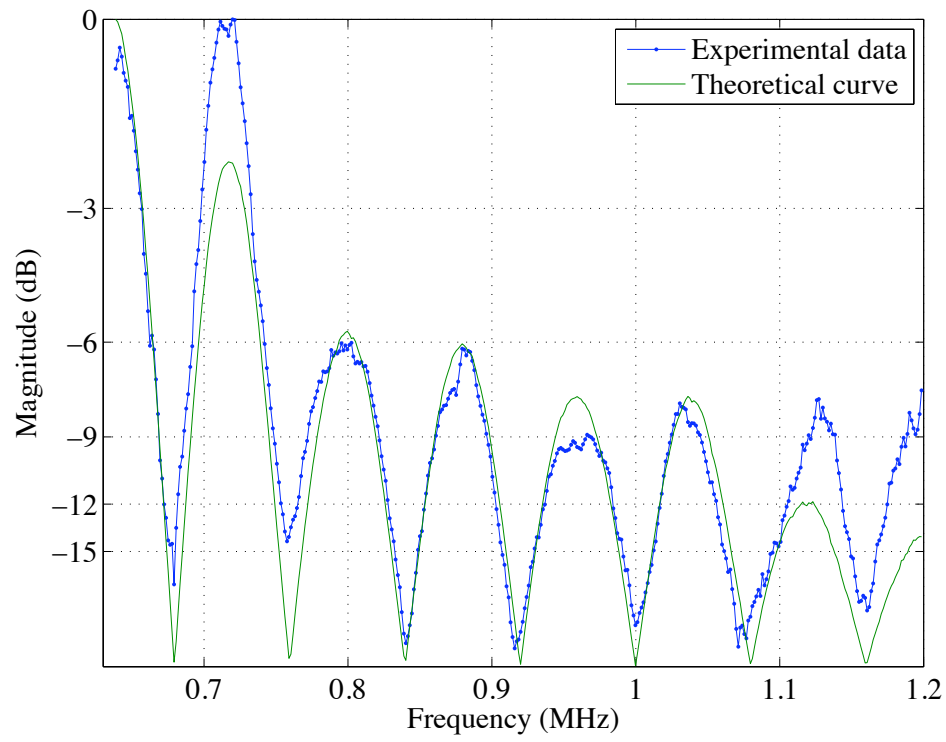


Figure 5.15: Experimental data (blue) and theoretical curve (green) for a colloidal sphere of diameter  $18.55 \pm 0.01$  mm. The nonlinear least squares fit to the data gave the diameter of the sphere to be  $18.49 \pm 0.02$  mm

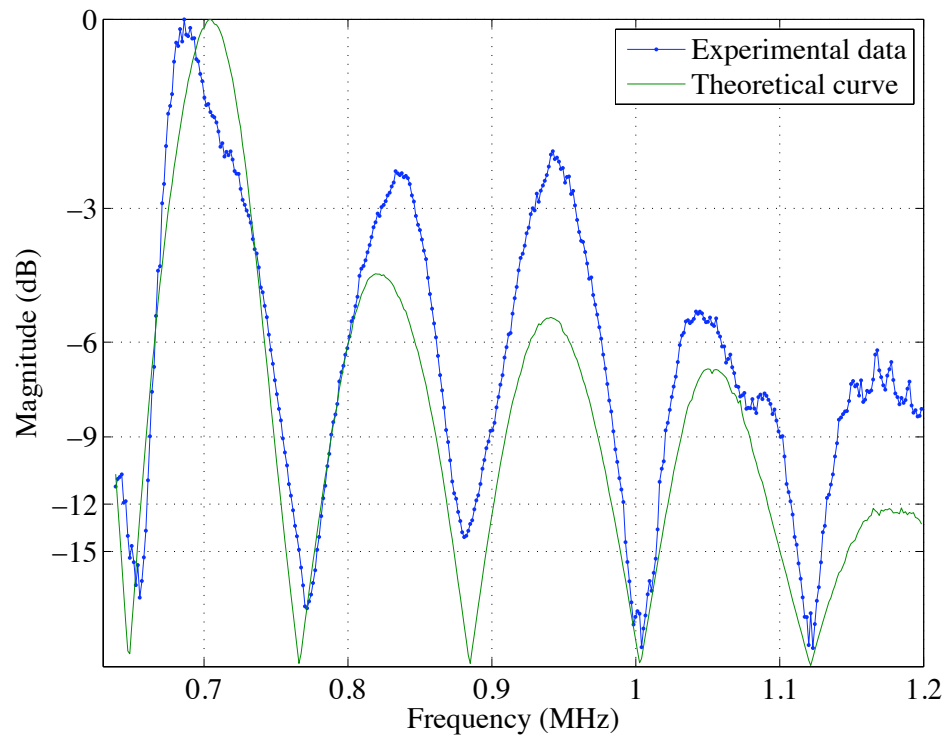


Figure 5.16: Experimental data (blue) and theoretical curve (green) for a colloidal sphere of diameter  $12.79 \pm 0.01$  mm. The nonlinear least squares fit to the data gave the diameter of the sphere to be  $12.52 \pm 0.02$  mm

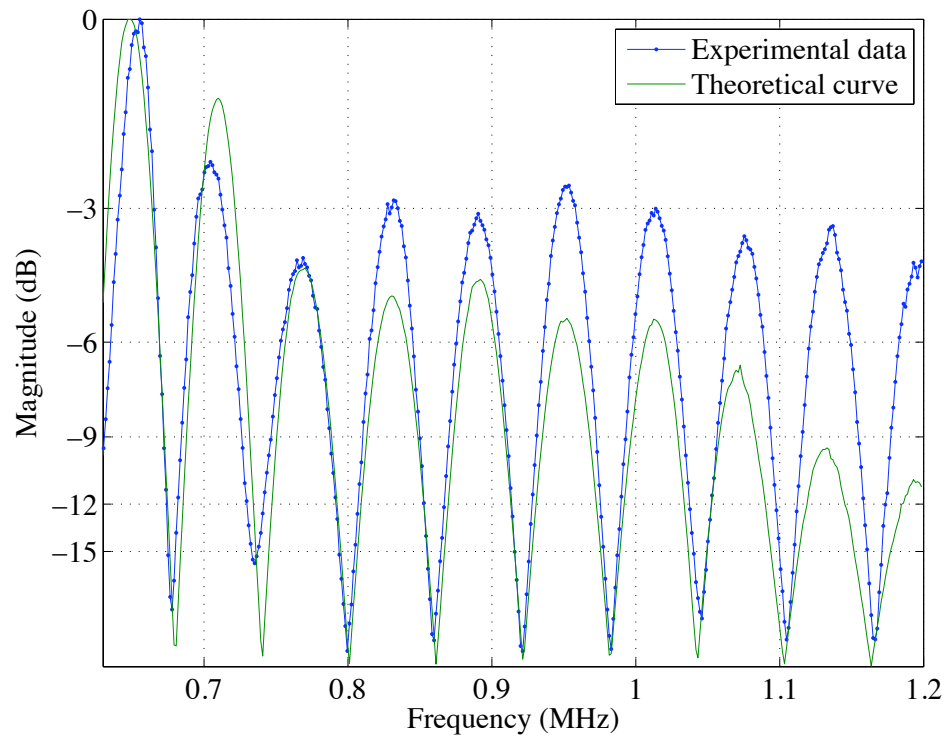


Figure 5.17: Experimental data (blue) and theoretical curve (green) for a colloidal cylinder of diameter  $24.88 \pm 0.01$  mm. The nonlinear least squares fit to the data gave the diameter of the cylinder to be  $24.49 \pm 0.02$  mm

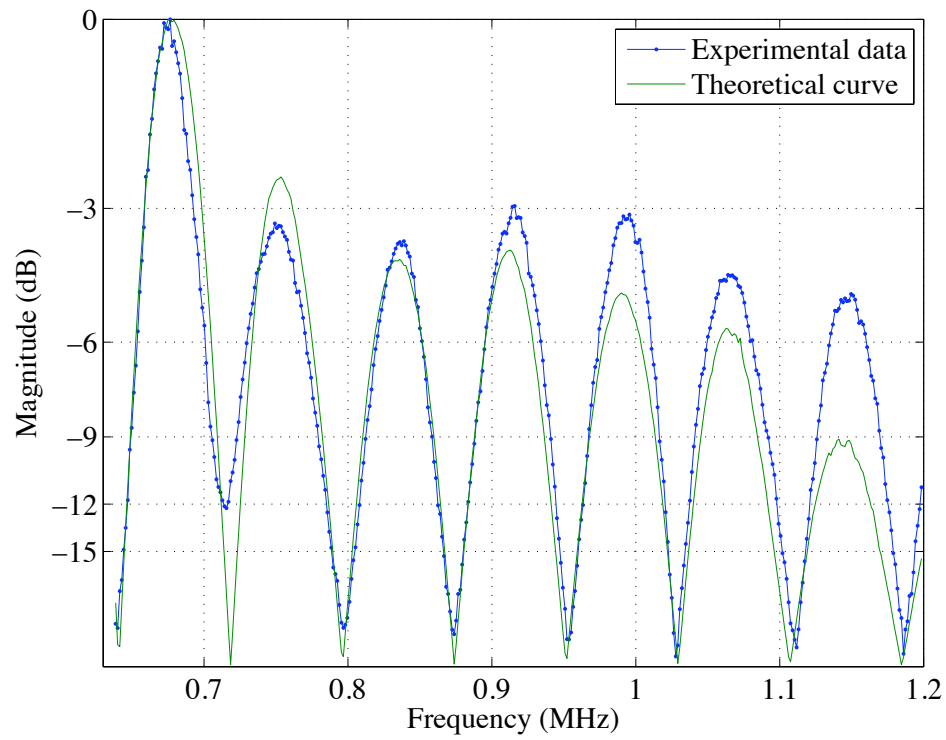


Figure 5.18: Experimental data (blue) and theoretical curve (green) for a colloidal cylinder of diameter  $19.08 \pm 0.01$  mm. The nonlinear least squares fit to the data gave the diameter of the cylinder to be  $19.04 \pm 0.02$  mm



| <b>Sample</b> | $D_{\text{measured}}$ (mm) | $D_{\text{calculated}}$ (mm) | <b>Error (%)</b> |
|---------------|----------------------------|------------------------------|------------------|
| Sphere 1      | 12.79                      | 12.52                        | 2                |
| sphere 2      | 18.55                      | 18.49                        | 0.3              |
| sphere 3      | 22.20                      | 22.21                        | 0.05             |
| sphere 4      | 25.19                      | 25.12                        | 0.3              |
| sphere 5      | 31.30                      | 31.38                        | 0.3              |
| sphere 6      | 38.05                      | 38.12                        | 0.18             |
| cylinder 1    | 19.08                      | 19.04                        | 0.2              |
| cylinder 2    | 24.88                      | 24.49                        | 1.6              |

Table 5.1: Comparison between experimentally calculated and measured diameters of different colloidal spheres and cylinders.

## 5.4 Analysis and discussions

Comparison between experimental data and theoretical calculations shown in the former section show the excellent agreement between experimental data and theory for colloidal spheres and cylinders. This results encourage us to develop theory for more complicated objects as well as conduct experiments both to study and improve imaging based on the vibration potential.

However, there are certain limitations when conducting experiments with small spheres or cylinders. To main limitations will be discussed here. First, when the colloidal objects get smaller, the vibration signals produced are smaller and will become difficult to detect. In those cases, improvements in the signal processing system should be made to optimize the signal-to-noise ratio in the vibration potential signal.

Second, experimental data for smaller spheres or cylinders obtained with aforementioned transducers such as the current transducer, model E1410, might not be easily fit to the theoretical calculation, as the number zero points and peaks in the

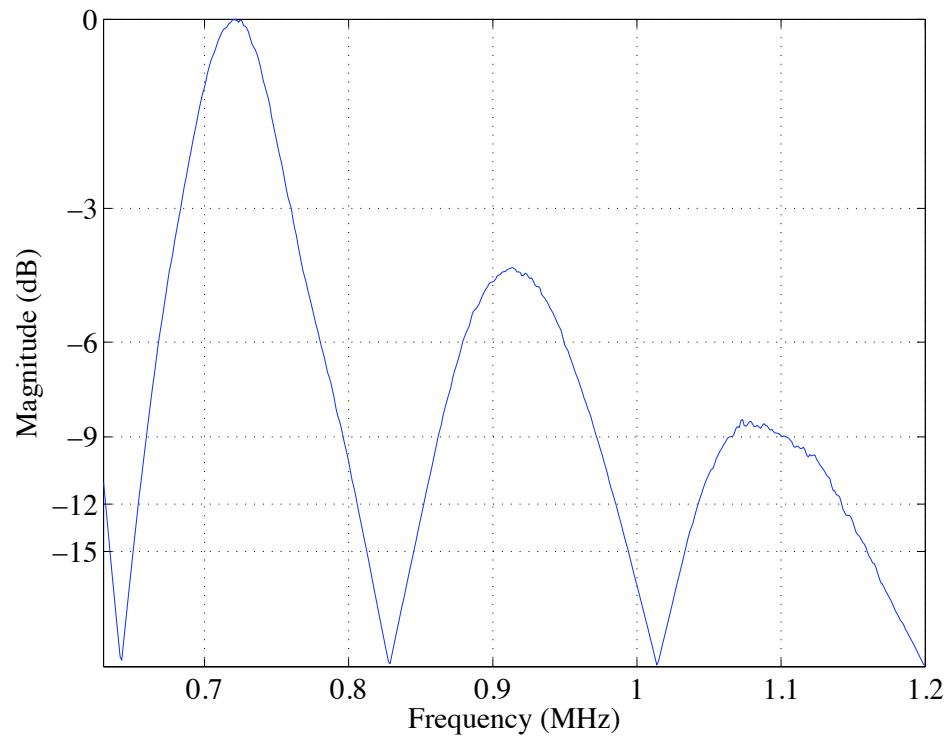


Figure 5.19: Theoretical signal versus frequency for a colloidal sphere of radius 4 mm.

signal in the operation of frequency range of the transducer becomes small (0.63 to 1.20 MHz). For example, Figure (5.19) shows the theoretical curve for a colloidal sphere of radius 4 mm. It is clear that there are only three zero points and three peaks in the operational bandwidth of transducer E1410. In order to have a good fit theory, a data over larger bandwidth should be obtained.

# Appendix A

## LabView Programs

### A.1 Gaussian burst generator

#### A.1.1 Program's interface

Figure (A.2) is a picture captured the interface of the LabView program called GaussianBurstToFG.vi used to control the function generator to generate the gaussian burst.

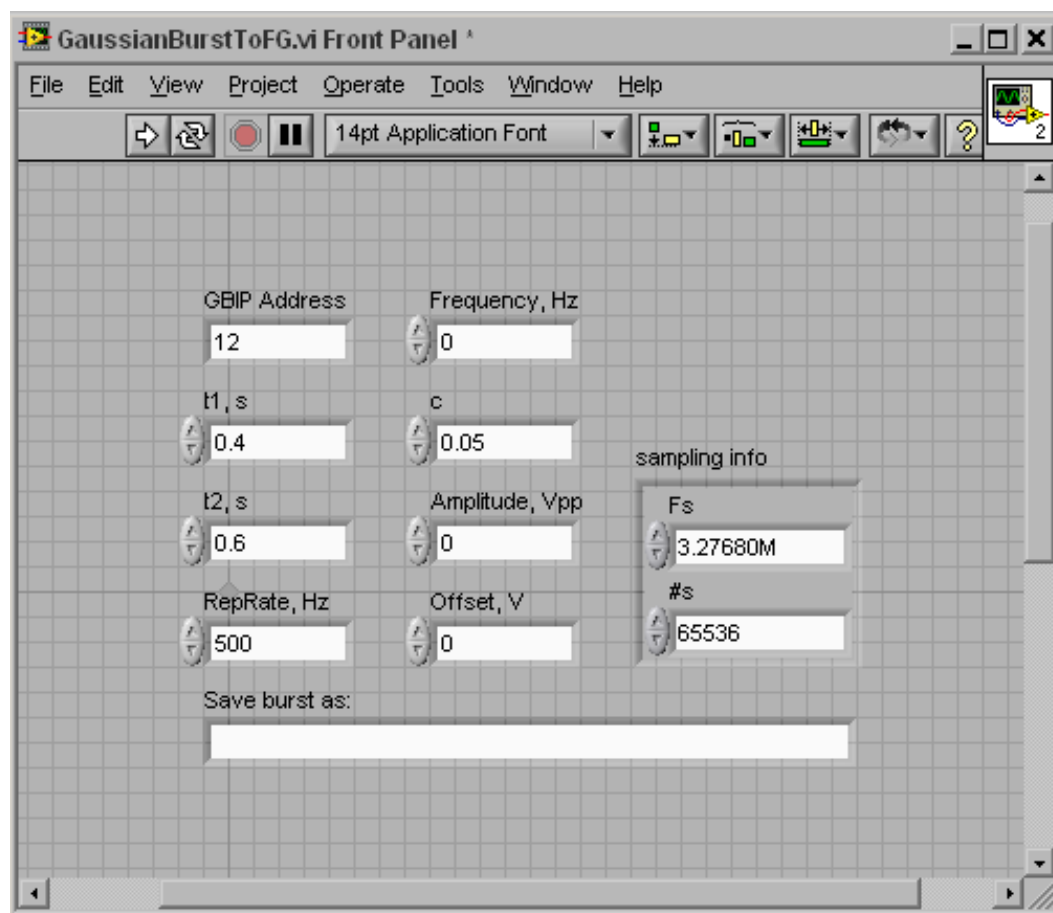


Figure A.1: Interface of a Gaussian burst generator written in LabView.

### A.1.2 Program's diagram

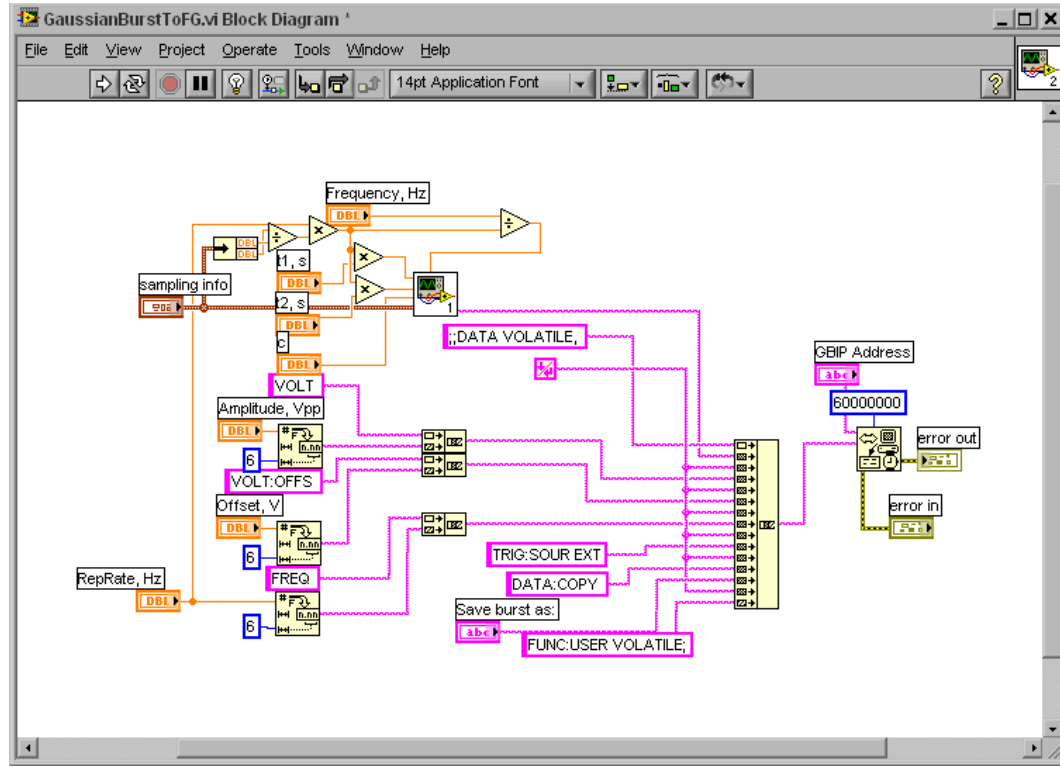


Figure A.2: Schematic diagram of a Gaussian burst generator written in LabView.

### A.1.3 Operational instruction

Enter necessary parameters in the program as follows:

- **GBIP Address:** GBIP address of the function generator used to generate Gaussian burst
- **Frequency, Hz:** frequency of the sinusoidal waveform, which is also the necessary frequency to drive the transducer, for example, 760000 Hz
- **t1, s:** beginning time of the burst, for example  $80 \times 10^{-6}$  s
- **t2, s:** ending time of the burst, for example  $80 \times 10^{-6}$  s
- **c** equal standard deviation constant

- **Amplitude, Vpp:** amplitude of the burst, for example 0.5 Vpp
- **RepRate, Hz:** repeat rate of the burst, for example 5000 Hz
- **Offset, V** offset value of the burst, normally is zero
- **sampling info:** information of sampling used to generate the burst
- **Save burst as:** name of the burst saved in Arbitrary shape mode of the function generator

## A.2 UVP Sweeper

### A.2.1 Program's interface

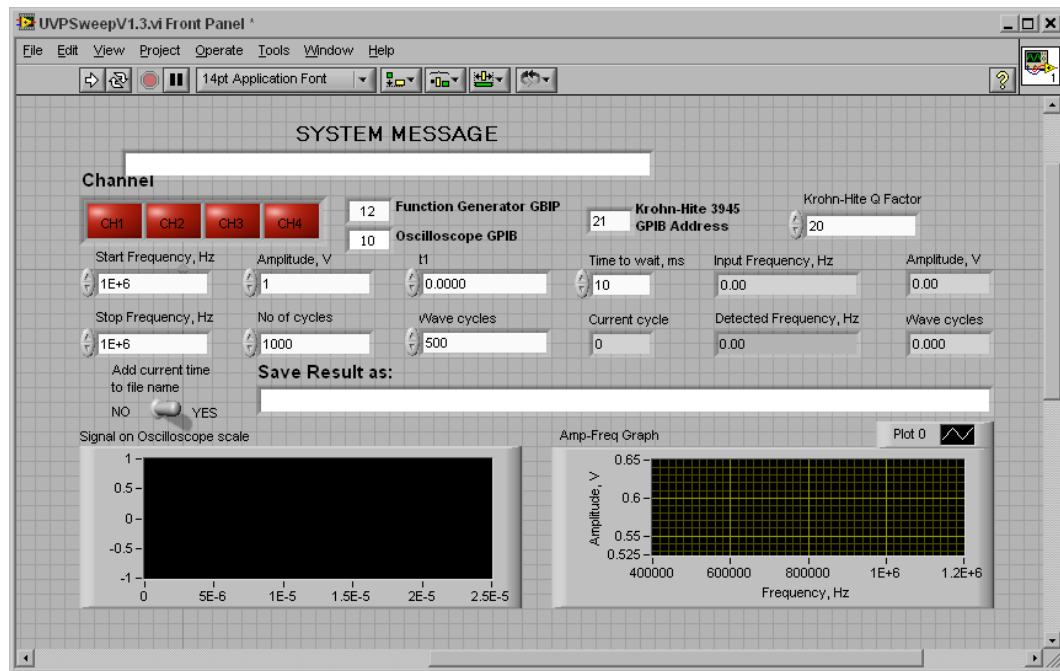


Figure A.3: Interface of the UVP Sweeper written in LabView.

## A.2.2 Program's diagram

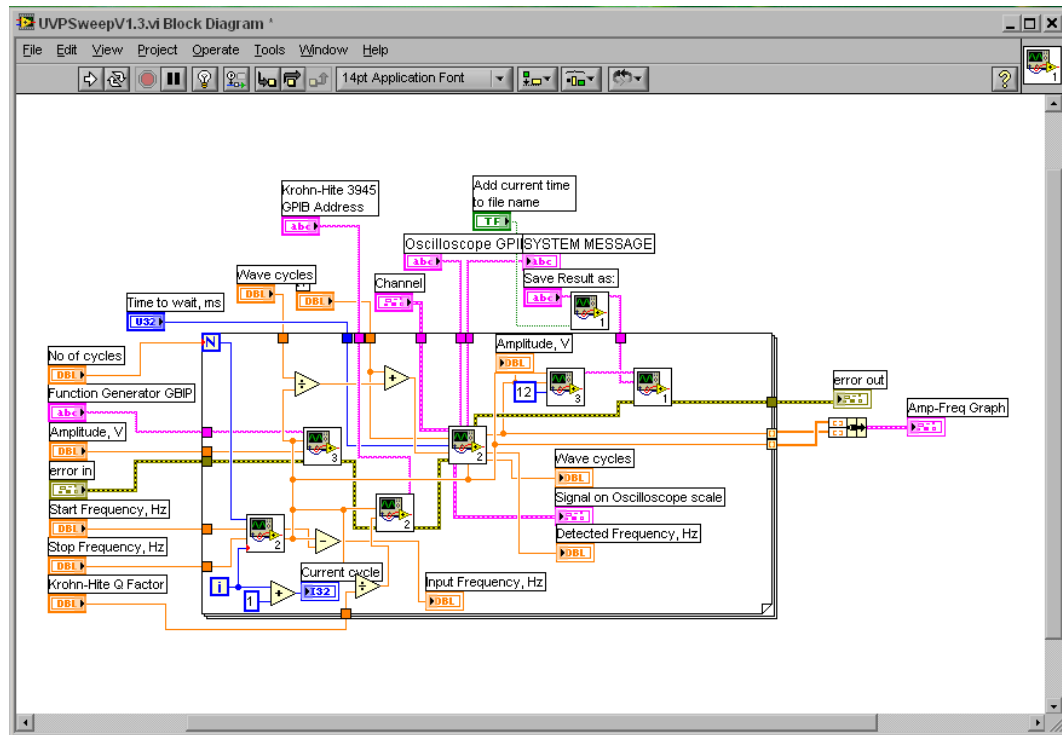


Figure A.4: Schematic diagram of the UVP Sweeper written in LabView.

## A.2.3 Instruction and explanation

Parameters on the program are:

- **SYSTEM MESSAGE:** message which will be displayed there indicating the status of the system
- **Channel:** channel on the oscilloscope used to display the waveform
- **Function Generator GBIP:** GBIP address of the function generator
- **Oscilloscope GBIP:** GBIP address of the oscilloscope
- **Krohn-Hite 3945 GBIP Address:** address of the filter model 3945 from Krohn-Hite

- **Krohn-Hite Q Factor:** qualification factor of the filter
- **Start Frequency, Hz:** starting frequency of the sweep, in Hertz
- **Stop Frequency, Hz:** ending frequency of the sweep, in Hertz
- **Amplitude, V:** amplitude of the function generator to be fed to the power amplifier
- **No of cycles:** number of running times of the sweep
- **t1:** starting of the burst to be captured by the program, in seconds
- **Wave cycles:** number of wave cycles to be captured by the program
- **Time to wait, ms:** time, in milliseconds, that the program should wait before each running time
- **current cycle:** display the current running cycle of the program
- **Input Frequency, Hz:** display the input frequency of the burst
- **Detected Frequency, Hz:** display the detected frequency of the burst
- **Amplitude, V:** display the amplitude of the signal
- **Wave cycles:**
- **Add current time to file name:** adding the current time to the file name or not
- **Save result as:** name of the text file which will be saved in the computer



## A.3 UVP Scanner

### A.3.1 Program's interface

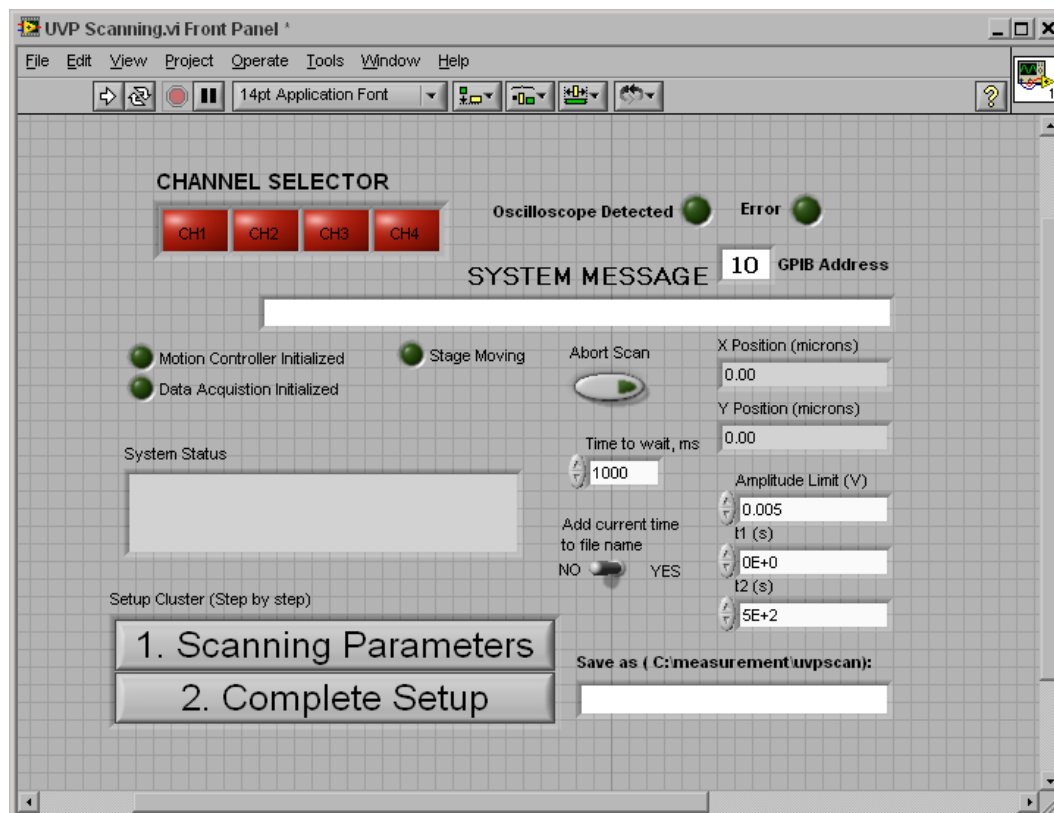


Figure A.5: Interface of the UVP Scanner written in LabView.

### A.3.2 Program's diagram

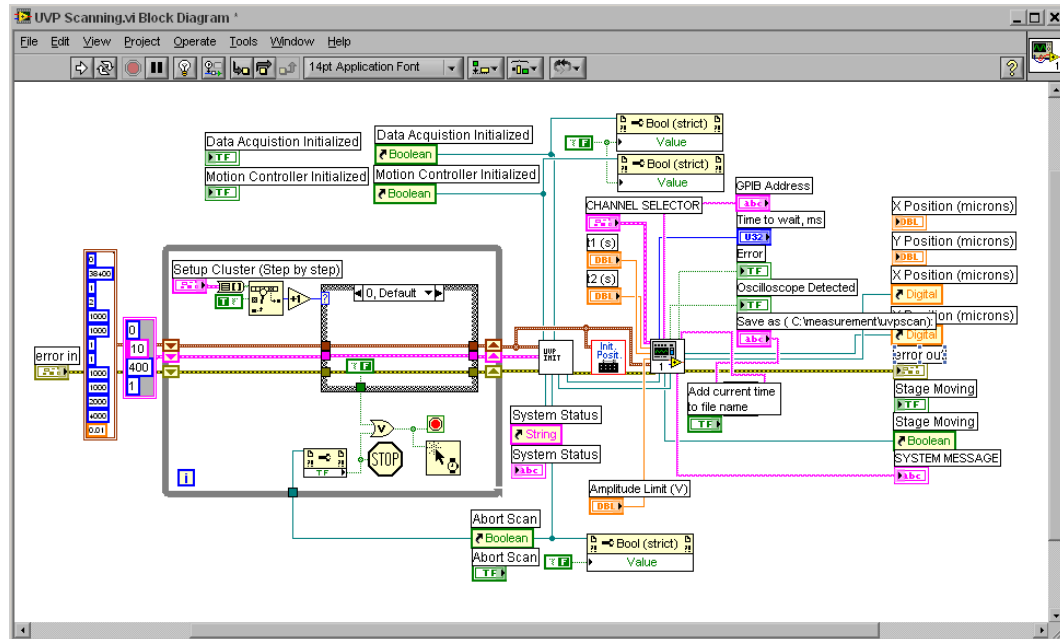


Figure A.6: Schematic diagram of the UVP Scanner written in LabView.

### A.3.3 Instruction and explanation

Parameters on the program are:

- **SYSTEM MESSAGE:** message which will be displayed there indicating the status of the system
- **Channel selector:** channel on the oscilloscope used to display the waveform
- **GBIP address:** GBIP address of the oscilloscope
- **Oscilloscope Detected:** on if the oscilloscope is detected by the program
- **Error:** on if error is detected when program runs
- **Motion Controller Initialized:** on if the motion controller is initialized
- **Data Acquisition Initialized:** on if the acquisition of data is currently active

- **Stage Moving:** on if the stage motion is moving
- **Abort Scan:** abort the scan or not
- **System status:** display the message what the program is doing during running
- **Time to wait, ms:** time, in milliseconds, that the program should wait for each running time
- **Add current time to file name:** adding the current time to the file name or not
- **X Position (microns):** display the current position of the x coordinate of the motion controller, in micrometers
- **Y Position (microns):** display the current position of the y coordinate of the motion controller, in micrometers
- **Amplitude Limit (V):** the smallest value of signal that the program can detect, otherwise the program will set the amplitude to be zero
- **t1 (s):** starting time of the burst in seconds
- **t2 (s):** ending time of the burst in seconds
- **Save as:** name of the text file which will be saved in the computer

# Appendix B

## Matlab programs

### B.1 UVP distribution in 2D with FEM method

#### B.1.1 Finite layer with width of 2 cm and height of 2 cm

```
1 clear all;
2 % Parameters used in the program
3 freq = 75e3;           % frequency of sound, kHz;
4 c = 1500;              % sound speed – consider travelling in water, 1500 m/s;
5 k = 2*pi*freq/c;       % wave vector of sound
6 % MAIN PROGRAM
7 % FEM to solve equation Laplace(U) = Delta(f)/dy
8 % Number of points in x and y axis
9 n=99; m=100;
10 % x vector
11 xx = linspace(0,0.1,n);
12 % y vector
13 yy = linspace(0,0.1,m);
14 Dy = linspace(0,0.1,m+1);
15 % Creat a grid of [0;0.1]x[0,0.1] m^2
16 [x,y]=meshgrid(xx,yy);
17 [dx,dy]=meshgrid(xx,Dy);
18 % making triangles grid
19 % matrix listing x,y coordinates of all nodes
20 xy2no=[x(:),y(:)];
21 % number of nodes
```

```

22 noNodes=size(xy2no,1);
23 % 3 node indeces of two triangles in first square
24 tglesIdx=[1,2,m+2; 1,m+2,m+1];
25 % matrix listing 3 nodes indeces of triangles
26 tglesIdx=kron(tglesIdx,ones(m-1,1))+kron(ones(size(tglesIdx)),(0:m-2)');
27 tglesIdx=kron(tglesIdx,ones(n-1,1))+kron(ones(size(tglesIdx)),(0:n-2)'*m);
28 % number of triangles
29 noTgles=size(tglesIdx,1);
30 % position of UVP sample
31 y1 = 0.03;y2 = 0.05;      % Sample locates from 4.5 cm to 5.5 cm, sound moves along↔
    y axis
32 x1 = 0.03;x2 = 0.05;      % Sample locates from 4 cm to 6 cm
33 % looking indeces of of xi and yi on grid
34 [x1m, x1Idx]=min(abs(xx-x1));
35 [x2m, x2Idx]=min(abs(xx-x2));
36 [y1m, y1Idx]=min(abs(yy-y1));
37 [y2m, y2Idx]=min(abs(yy-y2));
38 % Chosing \alpha0=1, function f=(dp/dy)*(heaviside(y-y1)-heaviside(y-y2)), where p ↔
    =
39 % p0*exp(-i*k*y) for a plane sound wave moving along y axis; hence f =
40 % -i*k*exp(-i*k*y) in sample and 0 out sample. I take the real part of f,
41 % and let p0 = 1. There are some more coefficients but I do not count on
42 % them here.
43 f = -i.*k.*exp(-i*k*dy).*( dx>=x1 & dx<=x2 & dy>=y1 & dy<=y2 );
44 % Delta (f)
45 df = -diff(f).*diff(dy);
46 % make vector F from df
47 F = reshape(df,noNodes,1);
48 % creating zero matrix
49 A=sparse(noNodes,noNodes);
50
51 for e=1:noTgles;
52     nodes=tglesIdx(e,:);
53     % 3 by 3 matrix with rows=[1 xcorner ycorner]
54     Pe=[ones(3,1),xy2no(nodes,:) ];
55     % area of each triangle
56     Atot=abs(det(Pe))/2;
57     % compute the gradient of the vectors
58     grad_phie=inv(Pe);
59     grad_phi=grad_phie(2:3,:);
60     % compute element matrix Ae(i,j) and add the contribution to the global

```

```

61     % matrix
62     Ae=Atot*grad_phi'*grad_phi;
63     A(nodes,nodes)=A(nodes,nodes)+Ae;
64     Fe=Atot/3;
65     F(nodes)=F(nodes)+Fe;
66 end
67
68 % Boundaries indeces
69 %bdry=[1:m,m+1:m:m*n,2*m:m:m*n,m*n-m+2:m*n-1];
70 bdry0=[1:m:m*n-m+1,m:m:m*n];
71 %bdry1=[1:m,m*n-m+1:m*n];
72 % applying boundary conditions; the left and right boundary are approximately zero
73 % when the area of calculating of sample are small
74 %A(bdry,:)=0; A(:,bdry)=0; F(bdry)=0;
75 A(bdry0,:)=0;A(:,bdry0)=0;F(bdry0)=0;
76 %A(1:m,:)=A(m*n-m+1:m*n,:);A(:,1:m)=A(:,m*n-m+1:m*n);
77 %A(bdry,bdry)=speye(length(bdry),length(bdry));
78 A(bdry0,bdry0)=speye(length(bdry0),length(bdry0));
79 Ab=A; Fb=F;
80 % solving final potential u
81 U=Ab\Fb;
82 u = reshape(real(U),m,n);
83 % plotting result
84 figure;
85 mesh(x,y,u);
86 xlabel('x (m)');ylabel('y (m)');zlabel('U (AU)');
87 [Ex,Ey] = gradient(-u,xx(2)-xx(1),yy(2)-yy(1));
88 figure;
89 plot([x1 x2 x2 x1 x1],[y1 y1 y2 y2 y1],'r—'); hold on;
90 streamslice(x,y,Ex,Ey,20); axis equal;
91 axis([0 0.1 0 0.1]);
92 xlabel('x (m)');ylabel('y (m)');

```

## B.1.2 Cylindrical object of radius 2 cm

```

1 clear all;
2 % Parameters used in the program
3 freq = 150e3; % frequency of sound, 400 kHz;
4 c = 1500; % sound speed — consider travelling in water, 1500 m/s;

```

```

5 k = 2*pi*freq/c; % wave vector of sound
6 % MAIN PROGRAM
7 % FEM to solve equation Laplace(U) = Delta(f)/dy
8 % Number of points in x and y axis
9 n=99; m=98;
10 % x vector
11 xx = linspace(0,0.1,n);
12 % y vector
13 yy = linspace(0,0.1,m);
14 % Creat a grid of [0;0.1]x[0,0.1] m^2
15 [x,y]=meshgrid(xx,yy);
16 % making triangles grid
17 % matrix listing x,y coordinates of all nodes
18 xy2no=[x(:),y(:)];
19 % number of nodes
20 noNodes=size(xy2no,1);
21 % 3 node indeces of two triangles in first square
22 tglesIdx=[1,2,m+2; 1,m+2,m+1];
23 % matrix listing 3 nodes indeces of triangles
24 tglesIdx=kron(tglesIdx,ones(m-1,1))+kron(ones(size(tglesIdx)),(0:m-2)');
25 tglesIdx=kron(tglesIdx,ones(n-1,1))+kron(ones(size(tglesIdx)),(0:n-2)*m);
26 % number of triangles
27 noTgles=size(tglesIdx,1);
28 % position of UVP sample
29 x0 = 0.05; y0 = 0.05;
30 R = 0.02;
31 % function f=(dp/dy)*(heaviside(y-y1)-heaviside(y-y2)), where p =
32 % p0*exp(-i*k*y) for a plane sound wave moving along y axis; hence f =
33 % -i*k*exp(-i*k*y) in sample and 0 out sample. I take the real part of f,
34 % and let p0 = 1. There are some more coefficients but I do not count on
35 % them here.
36 f = zeros(m+1,n);
37 for xIdx=1:n
38     for yIdx=1:m
39         if ((xx(xIdx)-x0)^2+(yy(yIdx)-y0)^2<=R^2)
40             %f(yIdx,xIdx)=k*sin(k*y(yIdx,xIdx));
41             f(yIdx,xIdx)=-i*k*exp(-i*k*y(yIdx,xIdx));
42         end
43     end
44 end
45 % Delta (f)

```

```

46 df = diff(f);
47 % make vector F from df
48 F = reshape(df,noNodes,1);
49 % creating zero matrix
50 A=sparse(noNodes,noNodes);
51
52 for e=1:noTgles;
53     nodes=tglesIdx(e,:);
54     % 3 by 3 matrix with rows=[1 xcorner ycorner]
55     Pe=[ones(3,1),xy2no(nodes,:)];
56     % area of each triangle
57     Atot=abs(det(Pe))/2;
58     % compute the gradient of the vectors
59     grad_phie=inv(Pe);
60     grad_phi=grad_phie(2:3,:);
61     % compute element matrix Ae(i,j) and add the contribution to the global
62     % matrix
63     Ae=Atot*grad_phi'*grad_phi;
64     A(nodes,nodes)=A(nodes,nodes)+Ae;
65     Fe=Atot/3;
66     F(nodes)=F(nodes)+Fe;
67 end
68
69 % Boundaries indeces
70 bdry=[1:m,m+1:m*m*n,2*m:m*m*n,m*n-m+2:m*n-1];
71 % applying boundary conditions; the left and right boundary are approximately zero
72 % when the area of calculating of sample are small
73 A(bdry,:)=0; A(:,bdry)=0; F(bdry)=0;
74 A(bdry,bdry)=speye(length(bdry),length(bdry));
75 Ab=A; Fb=F;
76 % solving final potential u
77 U=Ab\Fb;
78 u = reshape(real(U),m,n);
79 % plotting result
80 figure;
81 mesh(x,y,u);
82 xlabel('x (m)'); ylabel('y (m)'); zlabel('U (AU)');
83 [Ex,Ey] = gradient(-u,xx(2)-xx(1),yy(2)-yy(1));
84 figure;
85 phi = linspace(0, 2*pi, 100);
86 plot(R*cos(phi)+x0,R*sin(phi)+y0,'r—'); hold on;

```



```

87 streamslice(x,y,Ex,Ey,10);
88 axis equal;
89 axis([0 0.1 0 0.1]);
90 xlabel('x (m)');ylabel('y (m)');

```

## B.2 UVP distribution in 2D with analytical method

### B.2.1 Triangle wedge

#### Main program

```

1 clear all;
2 % Parameters used in the program
3 f = 150e3; % frequency of sound, kHz;
4 c = 1500; % sound speed – consider travelling in water, 1500 m/s;
5 z1 = 0.04;z2 = 0.06; % Sample locates from 4.5 cm to 5.5 cm, sound moves ←
    along y axis
6 x1 = 0.03;x2 = 0.07; % Sample locates from 4 cm to 6 cm
7 % MAIN PROGRAM
8 % Theory Solution of equation Laplace(U) = Delta(f)/dy
9 % Number of points in x and y axis. n should be even to prevent x = 0
10 n=99; m=99;
11 % x vector
12 xx = linspace(0,0.1,n);
13 % y vector
14 zz = linspace(0,0.1,m);
15 % Creat a grid of [0;0.1]x[0,0.1]
16 [x,z]=meshgrid(xx,zz);
17 % prism function uvp2DprismTM(freq,c,x1,x2,z1,z2,x,z)
18 u = real(uvp2DprismTM(f,c,x1,x2,z1,z2,xx,zz,z));
19 figure;
20 mesh(x,z,u);
21 xlabel('x (m)');ylabel('z (m)');zlabel('U (AU)');
22 title('Infinity prism');
23 [Ex,Ez] = gradient(-u,xx(2)-xx(1),zz(2)-zz(1));
24 figure;
25 hs = plot([x1 x2 x1 x1],[z1 z1 z2 z1],'r—'); hold on;
26 set(hs,'LineWidth',2);

```

```

27 he = plot([xx(1) xx(n)], [zz(1) zz(1)], 'r', [xx(1) xx(n)], [zz(m) zz(m)], 'r'); hold on;
28 set(he, 'LineWidth', 5);
29 streamslice(x,z,Ex,Ez,10); axis equal;
30 axis([xx(1) xx(n) zz(1) zz(m)]);
31 xlabel('x (m)'); ylabel('z (m)');
32 title('Infinity prism');

```

## Sub-function

```

1 function u = uvp2DprismTM(freq,c,x1,x2,z1,z2,xx,zz,z)
2 % uvp2DcylinderTM stands for uvp 2D case – sphere – Theory Method
3 % uvp's potential calculation in 2D case based on the theory and notes
4 % parameters.
5 % freq          % frequency of sound, kHz;
6 % c             % sound speed – consider travelling in water, 1500 m/s;
7 k = 2*pi*freq/c;          % wave vector of sound
8 h = 0.1;                  % distance between two electrodes
9 a0 = 1;                   % alpha0 coefficient (acoustic gradient factor)
10 p0 = 1;                   % pressure amplitude p = p0*exp(-i*k*y) transverses ←
    along y
11 % temporary negnect factor (e-i*sigma/omega). This factor seems to be
12 % constant in the range of considered
13 % sample is a prism lies at (x1,z1), (x2,z1) and (x1,z2)
14 U0 = -i*k*a0*p0/h;
15 % U1 contains the sum from 1 to infinity
16 U1 = 0;
17 n = 1;
18 nxp = 2001;
19 xp = linspace(x1,x2,nxp);
20 while n <= 200
21     kn = n*pi/h;
22     U11 = sin(kn*z);
23     U1 = U1 + U11.*uvp2DprismTMB(k,kn,x1,x2,z1,z2,xx,zz,xp);
24     n = n + 1;
25 end
26 u = U0*U1;
27
28 function b = uvp2DprismTMB(k,kn,x1,x2,z1,z2,xx,zz,xp)
29 %xx = reshape(x, size(x,1)*size(x,2),1);
30 %zl = z1 + (z2-z1)*(x2-xp)/(x2-x1);

```

```

31 %if k==kn
32     %fl = i*exp(-2*i*k*z1)/4/k+z1/2 - i*exp(-2*i*k*z1)/4/k-z1/2;
33 %else
34     %fl = exp(-i*k*z1).*(i*k*cos(kn*z1)-kn*sin(kn*z1))/(k^2-kn^2)...
35     %    - exp(-i*k*z1).*(i*k*cos(kn*z1)-kn*sin(kn*z1))/(k^2-kn^2);
36 %end
37 bvec = xx*0;
38 for j=1:length(xx)
39     frho = exp(-kn*abs(xp-xx(j))).*((i*exp(-2*i*k*(z1 + (z2-z1)*(x2-xp)/(x2-x1)))↵
        /4/k+(z1 + (z2-z1)*(x2-xp)/(x2-x1))/2 - i*exp(-2*i*k*z1)/4/k-z1/2)*(k==kn)↵
        +...
40     (exp(-i*k*(z1 + (z2-z1)*(x2-xp)/(x2-x1))).*(i*k*cos(kn*(z1 + (z2-z1)*(x2-xp)↵
        )/(x2-x1))-kn*sin(kn*(z1 + (z2-z1)*(x2-xp)/(x2-x1))))/(k^2-kn^2)...
41     - exp(-i*k*z1).*(i*k*cos(kn*z1)-kn*sin(kn*z1))/(k^2-kn^2))*(k~=kn));
42     %bvec(j) = quad(frho,x1,x2,1e-6);
43     %frho = exp(-kn*abs(xp-xx(j))).* fl;
44     bvec(j) = splquad2(xp,frho,x1,x2,2001);
45     %bveccum = cumsimpson(xp,frho);
46     %bvec(j) = bveccum(length(bveccum));
47     %bvec(j) = simps(xp,frho);
48 end
49 b=meshgrid(bvec,zz);

```

## B.2.2 Spherical object of radius 2 cm

### Main program

```

1 clear all;
2 % Parameters used in the program
3 c = 1500; % sound speed - consider travelling in water, 1500 m/s;
4 R = 0.02;
5 f = 3*(1500/0.02); % frequency of sound, kHz;
6 x0 = 0.05;
7 z0 = 0.05; % x position of center of the sphere
8 % MAIN PROGRAM
9 % Theory Solution of equation Laplace(U) = Delta(f)/dy
10 % Number of points in x and y axis. n should be even to prevent x = 0
11 n=100; m=99;
12 % x vector

```

```

13 xx = linspace(0,0.1,n);
14 % y vector
15 zz = linspace(0,0.1,m);
16 % Creat a grid of [0;0.1]x[0,0.1]
17 [x,z]=meshgrid(xx,zz);
18 % sphere function uvp2DsphereTM(freq,c,y0,R,x,y)
19 u = real(uvp2DsphereTM(f,c,x0,z0,R,x,z,xx,zz));
20
21 figure('color','w');
22 mesh(x,z,u);
23 xlabel('x (m)','FontName','Arial','FontSize',15);
24 ylabel('z (m)','FontName','Arial','FontSize',15);
25 zlabel('\phi_R (AU)','FontName','Arial','FontSize',15);
26 set(gca,'FontName','Arial','FontSize',15);
27 %title('Sphere');
28
29 [Ex,Ez] = gradient(-u,xx(2)-xx(1),zz(2)-zz(1));
30 figure('color','w');
31 phi = linspace(0, 2*pi, 100);
32 hs = plot(R*cos(phi)+x0,R*sin(phi)+z0,'r'); hold on;
33 set(hs,'LineWidth',2);
34 he = plot([xx(1) xx(n)],[zz(1) zz(1)],'r',[xx(1) xx(n)],[zz(m) zz(m)],'r');hold on;
35 set(he,'LineWidth',5);
36 streamslice(x,z,Ex,Ez,10); axis equal;
37 axis([xx(1) xx(n) zz(1) zz(m)]);
38 xlabel('x (m)','FontName','Arial','FontSize',15);
39 ylabel('z (m)','FontName','Arial','FontSize',15);
40 set(gca,'FontName','Arial','FontSize',15);
41 %title('Sphere');

```

## Sub-function

```

1 function u = uvp2DsphereTM(freq,c,x0,z0,R,x,z,xx,zz)
2 % uvp2DcylinderTM stands for uvp 2D case – sphere – Theory Method
3 % uvp's potential calculation in 2D case based on the theory and notes
4 % parameters.
5 %freq = 100e3;           % frequency of sound, kHz;
6 %c = 1500;               % sound speed – consider travelling in water, 1500 m/s↔
7 %                        ;
7 k = 2*pi*freq/c;        % wave vector of sound

```

```

8 h = 0.1; % distance between two electrodes
9 a0 = 1; % alpha0 coefficient (acoustic gradient factor)
10 p0 = 1; % pressure amplitude p = p0*exp(-i*k*y) transverses ←
    along y
11 % temporary negnect factor (e-i*sigma/omega). This factor seems to be
12 % constant in the range of considered
13 % sample is a sphere lyes at (x0,y0,z0) and radius R
14 U0 = -2*i*k*a0*p0/h;
15 % U1 contains the sum from 1 to infinity
16 U1 = 0;
17 n = 1;
18 nxp = 2001;
19 xp = linspace(-1+eps,1-eps,nxp);
20 while n <= 100
21     Kn = n*pi/h;
22     U11 = sin(Kn*z);
23     U1 = U1 + U11.*R^2.*uvp2DsphereTMBrho(Kn,R,x0,z0,k,xx,zz,xp);
24     n = n + 1;
25 end
26 u = U0*U1;
27
28 function b = uvp2DsphereTMBrho(kn,R,x0,z0,k,xx,zz,xp)
29 % subfunction of B(rho).
30 %frho = zeros(size(x,1)*size(x,2),nz);
31 xx = xx-x0;bvec=xx*0;
32 for j=1:length(xx)
33     frho = exp(-i*k*(R*xp+z0)).*cos(kn*R*(xp+z0/R)).*(1/R/kn-sqrt(1-xp.^2)).*...
34         besselk(1,kn*R*sqrt(1-xp.^2)).*besseli(0,kn*abs(xx(j))).*...
35         ( abs(xx(j))<R*sqrt(1-xp.^2) ) + exp(-i*k*(R*xp+z0)).*cos(kn*R*(xp+z0/R))←
36         ...
37         .*sqrt(1-xp.^2).*besseli(1,kn*R*sqrt(1-xp.^2)).*besselk(0,kn*abs(xx(j)))←
38         .*...
39         ( abs(xx(j))>=R*sqrt(1-xp.^2) );
40     bvec(j) = splquad2(xp,frho,-1,1,2001);
41 end
42 b = meshgrid(bvec,zz);

```

## B.3 Least square regression to find diameter of the sphere

### B.3.1 Main program

```

1 % A clean start
2 clear all;
3 % Raw data loading
4 SilicaAndTransResData = load(' ../../ experiment-data/frequency-curve↵
    /20080201-113659-ElectrodeRes-maybeTransducerRes-31mmSphere-0.5M-1.2M.txt ');
5 SiAndTransRes = SilicaAndTransResData(:,2);
6 TransducerResData = load(' ../../ experiment-data/frequency-curve↵
    /20080124-175728-0.76MHzFlatTransducerHomeMadeHydrophoneFilterCurrentAmp↵
    FrequencyResponse-01.txt ');
7 TransducerRes = TransducerResData(:,2);
8 SiRes = SiAndTransRes./TransducerRes;
9 SiResMax = max(SiRes);
10 SphereData = load(' ../../ experiment-data/frequency-curve/selected↵
    /20080130-165034-12mmSphere-0.5M-1.2M.txt ');
11 Freq = SphereData(:,1);
12 RawSig = SphereData(:,2);
13 ResData = load(' ../../ experiment-data/frequency-curve/20080130-151328-↵
    SilicaAgarseFreqResponse-0.5M-1.2M-incomingburst.txt ');
14 Res = ResData(:,2);
15
16 figure;
17 CorrectedSig = RawSig./Res;
18 CorrectedSigMax = max(CorrectedSig);
19 format long
20 Nm = 100;
21 NM = 500;
22 [result,Resid,Jacob,Covb,Mse] = nlinfit(Freq(Nm:NM),CorrectedSig(Nm:NM),↵
    @UVPCurrentSphere,[1 0.012/2]);
23 error = nlparci(result,Resid,'jacobian',Jacob);
24 dR = max((result(2)-error(2,1)),(-result(2)+error(2,2)));
25
26 % Print out constant A and diameter of the sphere
27 disp(['Gain factor A = ',num2str(result(1))]);

```

```

28 disp(['Radius of the sphere R = ',num2str(1000*result(2)), ' + ',num2str(1000*dR), ' ←
      mm or ',num2str(dR/result(2)*100), '%']);
29 disp(['Diameter of the sphere D = ',num2str(2000*result(2)), ' + ',num2str(2000*dR) ←
      , ' mm or ',num2str(dR/result(2)*100), '%']);
30 TheoryCurve = UVPCurrentSphere([result(1) result(2)],Freq(Nm:NM)).*SiRes(Nm:NM);
31 TheoryCurveMax = max(TheoryCurve);
32
33 % Plot experimental and theoretical curves
34 plot(Freq(Nm:NM)/1e6,CorrectedSig(Nm:NM)/CorrectedSigMax,'.-',...
35      Freq(Nm:NM)/1e6,TheoryCurve/TheoryCurveMax);
36 grid on;
37 axis([0.63,1.2,0,1]);
38 xlabel('Frequency (MHz)','FontName','Times New Roman','FontSize',15);
39 ylabel('Magnitude (dB)','FontName','Times New Roman','FontSize',15);
40 set(gca,'YTick',[10^(-15/20) 10^(-12/20) 10^(-9/20) 10^(-6/20) 10^(-3/20) 1],...
41      'YTickLabel',{'-15','-12','-9','-6','-3','0'},'FontSize',15,'FontName','Times ←
      New Roman');
42 text(0.45,0.015,'-\infty','FontSize',15,'FontName','Times New Roman');
43 legend('Experimental data','Theoretical curve');

```

### B.3.2 Sub-function used to fit the data

```

1 function u = UVPCurrentSphere(b,f)
2 % Assumptions
3 a0 = 1;
4 p0 = 1;
5 % Distance between two electrodes
6 h = 0.2;
7 % Sound speed
8 c = 1480;
9 % Angular frequency
10 w = 2*pi*f;
11 % Wavevector
12 k = w/c;
13 % Current density constant
14 I0 = w*a0*p0/h;
15 % Two unknown inputs
16 A = b(1);
17 R = b(2);

```

```
18 % Main function
19 u = abs(A.*4*pi.*R^2.*IO./(k.*R).*(cos(k.*R)-sin(k.*R)./(k.*R)));
```



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