Investigations on the SYK Model and its Dual Gravity Theory

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1. "Space-Time in the SYK Model," arXiv:1712.02725 [hep-th], with S. R. Das, A. Ghosh and A. Jevicki.
2. "Three Dimensional View of Arbitrary q SYK models," JHEP **1802**, 162 (2018), [arXiv:1711.09839 [hep-th]], with S. R. Das, A. Ghosh and A. Jevicki.
3. "Three Dimensional View of the SYK/AdS Duality," JHEP **1709**, 017 (2017), [arXiv:1704.07208 [hep-th]], with S. R. Das and A. Jevicki.
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7. "Physical unitarity for a massive Yang-Mills theory without the Higgs field: A perturbative treatment," Phys. Rev. D **87**, no. 2, 025017 (2013), [arXiv:1209.3994 [hep-th]], with K. I. Kondo, H. Fukamachi, S. Nishino and T. Shinohara.

(ii). Conference Proceedings

1. "Finite Temperature Maps in Vector/Higher Spin Duality," Proceedings, International Workshop on Higher Spin Gauge Theories : Singapore, Singapore, November 4-6, 2015, with A. Jevicki and J. Yoon.

2. "Physical unitarity of a massive Yang-Mills theory without the Higgs field from a viewpoint of confinement," Proceedings, 10th Conference on Quark Confinement and the Hadron Spectrum (Confinement X) : Munich, Germany, October 8-12, 2012, [arXiv:1301.2480 [hep-th]], with K. I. Kondo, H. Fukamachi, S. Nishino and T. Shinohara.

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1 Introduction

The Sachdev-Ye-Kitaev (SYK) model was proposed as a simpler, yet non-trivial example of the AdS/CFT correspondence, which was based on the earlier model by Sachdev and Ye (SY) [1–5]. Detailed investigations of the SYK model [6–18] have shed light on interesting, highly non-trivial aspects of the AdS/CFT correspondence and provide a potential framework for quantum black holes. The model, which can be studied at Large N, features an emergent conformal reparametrization symmetry at the IR critical point, which triggers out-of-time-order correlators exhibiting quantum chaos, with a maximal Lyapunov exponent characteristic of quantum black holes [7], providing an example of the butterfly effect [19–25]. Connections of the model to Random Matrix Theory have further elaborated this quantum black hole interpretation of the model [26–37]. Related models have been studied also [38–43] as well as various generalizations [44–58]. The solution and properties in leading order in Large N are shared with tensor type models [59–85].

Despite great interest on the model, the precise bulk dual of the SYK model is still not understood. It has been conjectured in [86–89] that the gravity sector of this model is the Jackiw-Teitelboim model [90, 91] of dilaton-gravity with a negative cosmological constant studied in [92], while [93–97] provide strong evidences that it is actually Liouville theory. (Various other aspects of this dilaton-gravity sector have also been studied [98–105].) On the other hand, it is also known that the matter sector contains an infinite tower of particles [9–11]. Couplings of these particles have been computed by calculating higher-point functions in the SYK model [15, 16], and the spectrum of the matter sector can be understood from 3D gravity theory [106–108].

In this thesis, we describe the development of systematic Large N representation of the model given in [11, 12], through a nonlinear bi-local collective field theory. This representation systematically incorporates arbitrary n-point bi-local correlators through a set of $1/N$ vertices and propagator and as such gives the bridge to a dual description. It naturally provides a holographic interpretation along the lines proposed more generally in [109, 110], with the center of mass and relative coordinates of the two points in the bi-local fields. The Large

N SYK model represents a highly non-trivial non-linear system. At the IR critical point (strong coupling limit) there appears a zero mode problem which at the outset prevents a perturbative expansion. We treat this mode through introduction of collective "time" coordinate as a dynamical variable as in quantization of extended systems [111]. Its Faddeev-Popov quantization was seen to systematically project out the zero modes, providing for a well defined propagator and expansion around the IR point. What one has is a fully nonlinear interacting system of bi-local matter with a discrete gravitational degree of freedom governed by a Schwarzian action. We will demonstrate the non-linear derivation of the action to be exact at all orders [12], which leads to the so called "enhanced" contributions at the linearized quadratic level originally described in [10].

We will also describe the three dimensional interpretation of the bulk theory [106, 107]. The zero temperature SYK model with four point interactions corresponds to a background $\text{AdS}_2 \times I$, where $I = S^1/Z_2$ is a finite interval whose size needs to be suitably chosen. There is a single scalar field coupled to gravity, whose mass is equal to the Breitenlohner-Freedman bound [112] of AdS_2. The scalar field satisfies Dirichlet boundary conditions at the ends and feels an external delta function potential at the middle of the interval. The background can be thought of as coming from the near-horizon geometry of an extremal charged black hole which reduces the gravity sector to Jackiw-Teitelboim model with the metric in the third direction becoming the dilaton of the latter model [87]. The strong coupling limit of the SYK model corresponds to a trivial metric in the third direction, while at finite coupling this acquires a dependence on the AdS_2 spatial coordinate. With a suitable choice of the size of the interval L and the strength of the delta function potential V, we show that at strong coupling, (i) the spectrum of the Kaluza-Klein (KK) modes of the scalar is precisely the spectrum of the SYK model and (ii) the two point function with both points at the center of the interval is in precise agreement with the strong coupling bi-local propagator, using the simplest identification of the AdS coordinates proposed in [109]. For finite coupling, we adopt the proposal of [87, 88], and show that to order $1/J$, the poles of the propagator shift in a manner consistent with the explicit results in [10].

We will finally address the so called "i" problem discussed in [108]. If one considers the Euclidean partition function of the model, changing variables to the center of mass and relative coordinates of the bi-local, one reaches a solution (the propagator and quadratic fluctuations) with a Lorentzian signature, coming from the fact that the two points of the bi-local become coordinates of a Lorentzian signature. On the other hand, we expect that the dual theory of the Euclidean SYK model should live in Euclidean spacetime [10]. One issue which is detrimental to a potential Lorentzian identification associated with this data comes from the factors of " i " which inevitably appears in a Lorentzian dual theory, but absent in the SYK propagator. Secondly, the radial part of the AdS_2 wave functions which appear in the SYK eigenfunctions (whether or not we write this in the 3D language) are not the usual normalizable AdS wave functions, but satisfy different boundary conditions. These unusual wave functions are, however, required since these are the ones which diagonalize the SYK kernel [9, 11]. This suggests that they might be better thought of as dS_2 wave functions [10]

[1].

The resolution of both issues are given as follows. We will show that a non-local transform relates the bi-local field to a field whose underlying dynamics is in Euclidean AdS_2. We will arrive at this transform following the same principles underlying the derivation of the corresponding transform for the $O(N)$ model in $d = 3$ [114, 115]: the idea is to find a canonical transformation in the four dimensional phase space of the two points in the bi-local such that the symmetries of $EAdS_2$ are realized correctly. This suggests a simple transformation kernel for the momentum space fields. It turns out that the corresponding position space kernel is a H^2 Radon transform. Radon transforms have appeared (explicitly or implicitly) in discussions of AdS/CFT, most notably in [116–118] where this is used to go from the bulk to the kinematic space of the boundary field theory on a time slice. Indeed the space on which the bi-locals live is a version of kinematic space. However, unlike these papers we are not working on a time slice in the bulk - rather our transform takes unequal Euclidean time fields on $EAdS_2$ to bi-locals. Though mathematically identical, our transform is conceptually somewhat different. The necessity of a Radon transform in this context has been in fact mentioned in [10].

This transformation takes the particular combinations of Bessel functions which appear in the SYK propagators to the modified Bessel functions which appear in the standard $EAdS_2$ propagator. However there are additional factors which morally resemble the leg pole factors of the $c = 1$ matrix model necessary to relate the collective field [119] to the usual tachyon field of the dual 2D string theory and reproduce the S-Matrix [120–124] (for a recent improved understanding see [125]). In this latter case these factors are believed to arise from the discrete states of the 2D string. For our case it is tempting to speculate that the leg pole factors also arise from similar bulk degrees of freedom, which remain to be identified. In fact we find an intriguing analogy between the SYK propagator and the propagator of macroscopic loop operators [120].

2 The Model

2.1 Bi-local method

In this subsection, we will give a brief review of our formalism [11, 12]. The Sachdev-Ye-Kitaev model [7] is a quantum mechanical many body system with all-to-all random interactions on fermionic N sites ($N \gg 1$), represented by the Hamiltonian

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l\,, \tag{2.1}$$

where χ_i are Majorana fermions, which satisfy $\{\chi_i, \chi_j\} = \delta_{ij}$. The coupling constant J_{ijkl} are random with a Gaussian distribution. The original model is given by this four-point interaction; however, it is a simple generalization to analogous q-point interacting model [7, 10].

[1]This has been suggested by J. Maldacena [113].

In this thesis, we follow the more general q model, unless otherwise specified. Nevertheless, our main interest represents the original $q = 4$ model. After the disorder averaging for the random coupling J_{ijkl}, there is only one effective coupling J and the effective action is written as

$$S_q \;=\; -\frac{1}{2}\int dt \sum_{i=1}^{N}\sum_{a=1}^{n} \chi_i^a \partial_t \chi_i^a \;-\; \frac{J^2}{2qN^{q-1}}\int dt_1 dt_2 \sum_{a,b=1}^{n}\left(\sum_{i=1}^{N}\chi_i^a(t_1)\chi_i^b(t_2)\right)^q , \tag{2.2}$$

where a, b are the replica indexes. Throughout this thesis, we only consider this Euclidean time model. We do not expect a spin glass state in this model [8] and we can restrict to replica diagonal subspace [11]. Therefore, introducing a (replica diagonal) bi-local collective field:

$$\Psi(t_1, t_2) \;\equiv\; \frac{1}{N}\sum_{i=1}^{N}\chi_i(t_1)\chi_i(t_2)\,, \tag{2.3}$$

the model is described by a path-integral

$$Z \;=\; \int \prod_{t_1,t_2} \mathcal{D}\Psi(t_1,t_2)\; \mu[\Psi]\, e^{-S_{\rm col}[\Psi]}\,, \tag{2.4}$$

with an appropriate order $\mathcal{O}(N^0)$ measure μ and the collective action:

$$S_{\rm col}[\Psi] \;=\; \frac{N}{2}\int dt \left[\partial_t \Psi(t,t')\right]_{t'=t} \;+\; \frac{N}{2}\,{\rm Tr}\log\Psi \;-\; \frac{J^2 N}{2q}\int dt_1 dt_2\, \Psi^q(t_1,t_2)\,, \tag{2.5}$$

where the trace term comes from a Jacobian factor due to the change of path-integral variable, and the trace is taken over the bi-local time. This action being of order N gives a systematic $1/N$ expansion, while the measure μ found as in [126] begins to contribute at one loop level (in $1/N$). Here the first linear term represents a conformal breaking term, while the other terms respect conformal invariance. This naive expression of the breaking term represents a product at the same point, which will be receiving regularization in our perturbation. In the IR with the strong coupling limit $|t|J \gg 1$, the collective action is reduces to the critical action

$$S_{\rm c}[\Psi] \;=\; \frac{N}{2}\,{\rm Tr}\log\Psi \;-\; \frac{J^2 N}{2q}\int dt_1 dt_2\, \Psi^q(t_1,t_2)\,, \tag{2.6}$$

which exhibits the emergent conformal reparametrization symmetry

$$\Psi(t_1,t_2) \;\to\; \Psi_f(t_1,t_2) \;=\; \Big|f'(t_1)f'(t_2)\Big|^{\frac{1}{q}}\, \Psi(f(t_1),f(t_2))\,, \tag{2.7}$$

with an arbitrary function $f(t)$. The critical saddle-point solution is given by

$$\Psi_{0,f}(t_1,t_2) \;=\; b\left(\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right)^{\frac{2}{q}} , \tag{2.8}$$

where b is a time-independent constant. This symmetry is responsible for the appearance of zero modes in the strict IR critical theory. This problem was addressed in [11, 12] with analog of the quantization of extended systems with symmetry modes [111]. The above symmetry mode representing time reparametrization can be elevated to a dynamical variable through the Faddeev-Popov method which we summarize as follows: we insert into the partition function (2.4), the functional identity:

$$\int \prod_t \mathcal{D}f(t)\ \prod_t \delta\left(\int u \cdot \Psi_f\right) \left| \frac{\delta\left(\int u \cdot \Psi_f\right)}{\delta f} \right| = 1\,, \tag{2.9}$$

so that after an inverse change of the integration variable, it results in a combined representation

$$Z = \int \prod_t \mathcal{D}f(t) \prod_{t_1,t_2} \mathcal{D}\Psi(t_1,t_2)\ \mu(f,\Psi)\,\delta\left(\int u\cdot\Psi_f\right) e^{-S_{\rm col}[\Psi,f]}\,, \tag{2.10}$$

with an appropriate Jacobian. After separating the critical classical solution Ψ_0 from the bi-local field: $\Psi = \Psi_0 + \overline{\Psi}$, the total action is now given by

$$S_{\rm col}[\Psi,f] = S[f] + \frac{N}{2}\int \left[\overline{\Psi}_f\right]_s + S_{\rm c}[\Psi]\,. \tag{2.11}$$

Here $[\ ]_s$ represents a regularized expression for the breaking operator, that we will specify in Section 3.1. The action of the time collective coordinate is given by

$$S[f] = \frac{N}{2}\int \left[\Psi_{0,f}\right]_s. \tag{2.12}$$

We have in [11] given the explicit evaluation of the nonlinear action $S[f]$ for the case of $q=2$ demonstrating the Schwarzian form [11] conjectured by Kitaev and constructed at quadratic level by Maldacena and Stanford [10]. For general q, the naive form of the composite operator in (2.5) generates again a Schwarzian action, which we exhibited through an ε-expansion presented in Appendix A. Taking into account the regularized breaking term we confirm the Schwarzian form (in Appendix B)

$$S[f] = -\frac{N\alpha}{24\pi J}\int dt\left[\frac{f'''(t)}{f'(t)} - \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2\right], \tag{2.13}$$

with a coefficent

$$\alpha = -12\pi B_1\gamma\,, \tag{2.14}$$

where

$$\gamma(q) = -\frac{\tan(\frac{\pi}{q})}{12\pi b q}\left[\frac{2\pi(q-1)(q-2)}{q\sin(\frac{2\pi}{q})} - (q^2-6q+6)\right]. \tag{2.15}$$

and B_1 representing the coefficient of first order shift of the saddle-point solution which will be summarized in Section 3.1. All together our result for the prefactor of the Schwarzian action comes out in agreement with the value obtained first by Maldacena and Stanford through evaluation of zero mode dynamics [10].

Summarizing in the above construction we have an interacting picture of the emergent Schwarzian mode $f(t)$, and a bi-local matter field combined in the nonlinear collective action (2.11). It is important to emphasize that this action exhibits reparametrization symmetry both at and also away from the IR point. For this, the delta constraint condition projecting out the state associated with wave function $u(t_1, t_2)$ represents a gauge fixing condition with an corresponding Faddeev-Popov measure. This formulation then allows systematic perturbative calculations around the IR point.

2.2 Relation to Zero Mode Dynamics

Before we proceed with our perturbative calculations it is worth comparing the above exact treatment of the reparametrization mode (2.13) with a linearized determination of the zero mode dynamics, as considered in [10]. We will be able to see that the latter follows from the former.

Expanding the critical action around the critical saddle-point solution Ψ_0, we have the quadratic kernel (which defines the propagator) and a sequence of higher vertices and so on. This expansion is schematically written as

$$S_{\rm c}\Big[\Psi_0 + \sqrt{2/N}\,\eta\Big] \;=\; N\,S_{\rm c}[\Psi_0] \,+\, \frac{1}{2}\int \eta\cdot\mathcal{K}\cdot\eta \,+\, \frac{1}{\sqrt{N}}\int \mathcal{V}_{(3)}\cdot\eta\,\eta\,\eta \,+\, \cdots\,, \tag{2.16}$$

where the kernel is

$$\begin{aligned}\mathcal{K}(t_1,t_2;t_3,t_4) \;&=\; \frac{\delta^2 S_c[\Psi_0]}{\delta\Psi_0(t_1,t_2)\delta\Psi_0(t_3,t_4)}\\ &=\; \Psi_0^{-1}(t_1,t_3)\Psi_0^{-1}(t_2,t_4) \,+\, (q-1)J^2\,\delta(t_{13})\delta(t_{24})\,\Psi_0^{q-2}(t_1,t_2)\,,\end{aligned} \tag{2.17}$$

with $t_{ij} = t_i - t_j$. Then, the bi-local propagator $\mathcal{D}$ is determined as a solution of the following Green's equation:

$$\int dt_3 dt_4\,\mathcal{K}(t_1,t_2;t_3,t_4)\,\mathcal{D}(t_3,t_4;t_5,t_6) \;=\; \delta(t_{15})\delta(t_{26})\,. \tag{2.18}$$

In order to inverse the kernel $\mathcal{K}$ in the Green's equation (2.18) and determine the bi-local propagator, let us first consider an eigenvalue problem of the kernel $\mathcal{K}$:

$$\int dt_3 dt_4\,\mathcal{K}(t_1,t_2;t_3,t_4)\,u_{n,t}(t_3,t_4) \;=\; k_{n,t}\,u_{n,t}(t_1,t_2)\,, \tag{2.19}$$

where n and t are labels to distinguish the eigenfunctions. The zero mode, whose eigenvalue is $k_0 = 0$ is given by

$$u_{0,t}(t_1,t_2) \;=\; \frac{\delta\Psi_{0,f}(t_1,t_2)}{\delta f(t)}\bigg|_{f(t)=t}\,. \tag{2.20}$$

Now, we consider the zero mode quantum fluctuation around a shifted classical background

$$\Psi(t_1,t_2) \;=\; \Psi_{\rm cl}(t_1,t_2) \,+\, \int dt'\,\varepsilon(t')\,u_{0,t'}(t_1,t_2)\,, \tag{2.21}$$

with $\Psi_{\rm cl} = \Psi_0 + \Psi_1$ where Ψ_1 is a first $1/J$ shift of the classical field from the critical point. Then, the quadratic action of ε in the first order of the shift is given by expanding $S_{\rm c}[\Psi_{\rm cl} + \varepsilon \cdot u_0]$. This quadratic action can be written in terms of the shift of the kernel $\delta\mathcal{K}$ as

$$S_2[\varepsilon] \;=\; -\,\frac{N}{4}\int dtdt'\,\varepsilon(t)\,\varepsilon(t')\int dt_1dt_2dt_3dt_4\,u_{0,t}(t_1,t_2)\,\delta\mathcal{K}(t_1,t_2;t_3,t_4)\,u_{0,t'}(t_3,t_4)\,, \tag{2.22}$$

where

$$\delta\mathcal{K}(t_1,t_2;t_3,t_4) \;=\; \int dt_5dt_6\,\frac{\delta^3 S_c[\Psi_0]}{\delta\Psi_0(t_1,t_2)\delta\Psi_0(t_3,t_4)\delta\Psi_0(t_5,t_6)}\,\Psi_1(t_5,t_6)\,. \tag{2.23}$$

Let us formally denote the t_1 - t_4 integrals in Eq.(2.22) by

$$\delta k_t\,\delta(t-t') \;=\; \int dt_1dt_2dt_3dt_4\,u_{0,t}(t_1,t_2)\,\delta\mathcal{K}(t_1,t_2;t_3,t_4)\,u_{0,t'}(t_3,t_4)\,, \tag{2.24}$$

because this is related to the eigenvalue shift due to $\delta\mathcal{K}$ up to normalization. Then, we can write the quadratic action (2.22) as

$$S_2[\varepsilon] \;=\; -\,\frac{N}{4}\int dt\,\delta k_t\,\varepsilon^2(t)\,. \tag{2.25}$$

We now give a formal proof that the quadratic action (2.25) is equivalent to the quadratic action of Eq.(2.13). This statement can be easily seen from the following identity:

$$\begin{aligned}
&\int dt_1dt_2dt_3dt_4\;u_{0,t}(t_1,t_2)\,\frac{\delta^3 S_{\rm c}[\Psi_0]}{\delta\Psi_0(t_1,t_2)\delta\Psi_0(t_3,t_4)\delta\Psi_0(t_5,t_6)}\,u_{0,t'}(t_3,t_4)\\
&=\;-\int dt_3dt_4\;\mathcal{K}(t_3,t_4;t_5,t_6)\;\left.\frac{\delta^2\Psi_{0,f}(t_3,t_4)}{\delta f(t)\delta f(t')}\right|_{f(t)=t}\,.
\end{aligned} \tag{2.26}$$

This identity is derived as follows. In the zero mode equation $\int \mathcal{K}\cdot u_0 = 0$, rewriting the kernel as derivatives of $S_{\rm c}$ as in the first line of Eq.(2.17), and taking a derivative of this equation respect to $f(t')$, one finds

$$\begin{aligned}
0 \;=\; &\int dt_1dt_2dt_3dt_4\;\left.\frac{\delta\Psi_{0,f}(t_1,t_2)}{\delta f(t)}\right|_{f(t)=t}\cdot\frac{\delta^3 S_{\rm c}[\Psi_{0,f}]}{\delta\Psi_{0,f}(t_1,t_2)\delta\Psi_{0,f}(t_3,t_4)\delta\Psi_{0,f}(t_5,t_6)}\cdot\left.\frac{\delta\Psi_{0,f}(t_3,t_4)}{\delta f(t')}\right|_{f(t')=t'}\\
&+\int dt_3dt_4\;\frac{\delta^2 S_{\rm c}[\Psi_{0,f}]}{\delta\Psi_{0,f}(t_3,t_4)\delta\Psi_{0,f}(t_5,t_6)}\cdot\left.\frac{\delta^2\Psi_{0,f}(t_3,t_4)}{\delta f(t)\delta f(t')}\right|_{f(t)=t}\,,
\end{aligned} \tag{2.27}$$

where we used the zero mode expression (2.20). Since $S_{\rm c}$ is invariant under the reparametrization, we can change the argument of $S_{\rm c}$ from $\Psi_{0,f}$ to Ψ_0. Then, we get the identity (2.26).

We note that at next cubic level, one will have disagreement and the zero mode dynamics will not give the Schwarzian derivative. This follows from the further identity:

$$\begin{aligned}
&\int dt_5 dt_6\ \frac{\delta^2 S_{\rm c}[\Psi_0]}{\delta\Psi_0(t_5,t_6)\delta\Psi_0(t_7,t_8)}\cdot\frac{\delta^3\Psi_{0,f}(t_5,t_6)}{\delta f(t)\delta f(t')\delta f(t'')}\bigg|_{f(t)=t}\\
&= -\int dt_1 dt_2 dt_3 dt_4 dt_5 dt_6\ \frac{\delta^4 S_{\rm c}[\Psi_0]}{\delta\Psi_0(t_1,t_2)\delta\Psi_0(t_3,t_4)\delta\Psi_0(t_5,t_6)\delta\Psi_0(t_7,t_8)}\,u_{0,t}(t_1,t_2)\,u_{0,t'}(t_3,t_4)\,u_{0,t''}(t_5,t_6)\\
&-3\int dt_3 dt_4 dt_5 dt_6\ \frac{\delta^3 S_{\rm c}[\Psi_0]}{\delta\Psi_0(t_3,t_4)\delta\Psi_0(t_5,t_6)\delta\Psi_0(t_7,t_8)}\,\frac{\delta^2\Psi_{0,f}(t_3,t_4)}{\delta f(t)\delta f(t')}\bigg|_{f(t)=t}\,\frac{\delta\Psi_{0,f}(t_5,t_6)}{\delta f(t'')}\bigg|_{f(t)=t}\,,
\end{aligned}\tag{2.28}$$

where the second term in the right-hand side explains the expected discrepancy.

3 Shift of the Classical Solution

In large N limit, the exact classical solution $\Psi_{\rm cl}$ is given by the solution of the saddle-point equation of the collective action (2.5). This classical solution corresponds to the one-point function:

$$\big\langle\Psi(t_1,t_2)\big\rangle\;=\;\Psi_{\rm cl}(t_1,t_2)\,.\tag{3.1}$$

At the strict strong coupling limit, the classical solution is given by the critical solution Ψ_0, which is a solution of the saddle-point equation of the critical action (2.6). One can then develop a perturbative $1/J$ expansion for the full solution .

3.1 Evaluation of Ψ_1

Let us consider the first order shift Ψ_1 of the classical solution from the critical solution induced by the breaking term. We start with the naive delta function breaking term of the action $S_{\rm col}$ (2.5). Substitution of $\Psi_{\rm cl}=\Psi_0+\Psi_1$ gives

$$\int dt_3 dt_4\,\mathcal{K}(t_1,t_2;t_3,t_4)\Psi_1(t_3,t_4)\;=\;\partial_1\delta(t_{12})\,,\tag{3.2}$$

where the kernel is given in Eq.(2.17).

It is useful to separate the J dependence from the bi-local field by

$$\Psi_{\rm cl}(t_1,t_2)\;=\;J^{-\frac{2}{q}}\,\Psi_0(t_1,t_2)\;+\;\cdots\,,\tag{3.3}$$

so that the critical solution Ψ_0, now reads

$$\Psi_0(t_1,t_2)\;=\;b\,\frac{\text{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}}}\,,\tag{3.4}$$

with

$$b\;=\;-\left[\frac{\tan\left(\frac{\pi}{q}\right)}{2\pi}\left(1-\frac{2}{q}\right)\right]^{\frac{1}{q}}\,.\tag{3.5}$$

Now the kernel (2.17) does not have the explicit J^2 factor in the second term, and such rescaled kernel denoted by $\mathcal{K}$ will be used in the rest of the thesis. Since

$$\Psi_0^{-1}(t_1,t_2) \;=\; -\,b^{q-1}\;\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2-\frac{2}{q}}}\,, \tag{3.6}$$

and the kernel has dimension $\mathcal{K}\sim|t|^{-4+4/q}$, from dimension analysis Ψ_1 would need to be the form of

$$\Psi_1(t_1,t_2) \;=\; A\;\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{4}{q}}}\,, \tag{3.7}$$

where A is a t-independent coefficient. In checking this ansatz we have the following integral in the first term of the LHS of Eq.(3.2)

$$Ab^{2q-2}\int dt_3dt_4\;\frac{\mathrm{sgn}(t_{13})\,\mathrm{sgn}(t_{24})\,\mathrm{sgn}(t_{34})}{|t_{13}|^{2-\frac{2}{q}}\,|t_{24}|^{2-\frac{2}{q}}\,|t_{34}|^{\frac{4}{q}}}\,. \tag{3.8}$$

This type of integral is already evaluated in Appendix A of [9]. In general, the result is

$$\begin{aligned}\int dt_3dt_4\;\frac{\mathrm{sgn}(t_{13})\,\mathrm{sgn}(t_{24})\,\mathrm{sgn}(t_{34})}{|t_{13}|^{2\Delta}\,|t_{24}|^{2\Delta}\,|t_{34}|^{2\alpha}} &= -\,\pi^2\left[\frac{\sin(2\pi\alpha)\;+\;2\sin(2\pi(\alpha+\Delta))\;+\;\sin(2\pi(\alpha+2\Delta))}{\sin(2\pi\alpha)\sin(2\pi\Delta)\sin(2\pi(\alpha+\Delta))\sin(2\pi(\alpha+2\Delta))}\right]\\ &\times\frac{\Big[\sin(2\pi\Delta)+\sin(2\pi(\alpha+\Delta))\Big]\Gamma(1-2\Delta)}{\Gamma(2\alpha)\Gamma(2\Delta)\Gamma(3-2\alpha-4\Delta)}\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2\alpha+4\Delta-2}}\,.\end{aligned} \tag{3.9}$$

Our interest is $\Delta=1-1/q$. For this case, the result is inversely proportional to $\Gamma(4/q-2\alpha-1)$. If we plug $\alpha=2/q$ into this equation, we can see that the Gamma function in the denominator gives infinity: $\Gamma(4/q-2\alpha-1)=\Gamma(-1)=\infty$, while other part is finite. Therefore, the first term of the LHS of Eq.(3.2) vanishes. The second term is trivial to evaluate; however the resulting form does not agree with the naive δ-function source in RHS. Hence, we conclude that the δ'-source is only matched in the non-perturbative solution level, where all the $1/J$ corrections are summed over.

To proceed, consider a more general ansatz for Ψ_1:

$$\Psi_1(t_1,t_2) \;=\; B_1\;\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+2s}}\,, \tag{3.10}$$

where B_1 is a t-independent coefficient. The parameter s has to be $s>0$, because the dimension of Ψ_1 needs to be less than the scaling dimension of Ψ_0. Now using this ansatz, we are going to evaluate Eq.(3.2). The integral of the first term of LHS of Eq.(3.2) is evaluated from Eq.(3.9) with $\Delta=1-1/q$ and $\alpha=s+1/q$ as

$$\frac{B_1\,b^{2q-2}\,\pi^2\cot\left(\frac{\pi}{q}\right)\Gamma\left(\frac{2}{q}-1\right)}{\sin\left(\pi\left(\frac{1}{q}+s\right)\right)\cos\left(\pi\left(s-\frac{1}{q}\right)\right)\Gamma\left(\frac{2}{q}+2s\right)\Gamma\left(2-\frac{2}{q}\right)\Gamma\left(\frac{2}{q}-2s-1\right)}\;\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2-\frac{2}{q}+2s}}\,. \tag{3.11}$$

Hence, after a slight manipulation the LHS of Eq.(3.2) becomes

$$\int dt_3 dt_4\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_1(t_3,t_4) \;=\; (q-1)B_1 b^{q-2}\gamma(s,q)\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2-\frac{2}{q}+2s}}\,, \tag{3.12}$$

where we used Eq.(3.5) and we defined

$$\gamma(s,q) \;=\; 1-\frac{\pi\,\Gamma\left(\frac{2}{q}\right)}{q\sin\left(\pi\left(\frac{1}{q}+s\right)\right)\cos\left(\pi\left(s-\frac{1}{q}\right)\right)\Gamma\left(\frac{2}{q}+2s\right)\Gamma\left(3-\frac{2}{q}\right)\Gamma\left(\frac{2}{q}-2s-1\right)}\,. \tag{3.13}$$

Now we note that for $s=1/2$, $\gamma(1/2,q)=0$, so that the ansatz (3.10) would be the homogeneous equation associated with Eq.(3.2). This limit $s\to 1/2$ therefore leads to the following first order shift of the background:

$$\Psi_{\mathrm{cl}}(t_1,t_2) \;=\; J^{-\frac{2}{q}}\Big[\,\Psi_0(t_1,t_2)\;+\;J^{-1}\,\Psi_1(t_1,t_2)\;+\;\cdots\Big]\,, \tag{3.14}$$

with

$$\Psi_0(t_1,t_2) \;=\; b\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}}}\,,\qquad\qquad \Psi_1(t_1,t_2) \;=\; B_1\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+1}}\,. \tag{3.15}$$

We will however keep the parameter s infinitesimally away from $1/2$ as a regularization. Then,

$$\gamma(s,q) \;=\; \frac{6\,q\,\gamma}{(q-1)\,b^{q-1}}\big(s-\tfrac{1}{2}\big)\;+\;\mathcal{O}\Big(\big(s-\tfrac{1}{2}\big)^2\Big)\,, \tag{3.16}$$

where γ is defined in Eq.(2.15), and the RHS in Eq.(3.12) can be interpreted as a regularized non-zero source term of the form

$$Q_s(t_1,t_2) \;\equiv\; \big(s-\tfrac{1}{2}\big)\,6qB_1b^{-1}\gamma\;\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2-\frac{2}{q}+2s}}\;+\;\mathcal{O}\big(\big(s-\tfrac{1}{2}\big)^2\big)\,. \tag{3.17}$$

The γ is obtained by expanding $\gamma(s,q)$ (3.13) around $s=1/2$ so that

$$\gamma \;=\; \frac{(q-1)b^{q-1}}{6q}\;\gamma'(s=\tfrac{1}{2},q)\,. \tag{3.18}$$

Here, the prime denotes a derivative respect to s. We use this regularized source to define the regularized breaking term by

$$\int\;\big[\Psi_f\big]_s\;\equiv\;-\lim_{s\to\frac{1}{2}}\int dt_1dt_2\,\Psi_f(t_1,t_2)\,Q_s(t_1,t_2)\,. \tag{3.19}$$

Finally, the coefficient B_1 can be deduced from the numerical result found in [10]. Comparison of the two results gives the relation:

$$\frac{B_1}{bJ}\;=\;\frac{\alpha_G}{\mathcal{J}}\,, \tag{3.20}$$

with the numerical approximated value of α_G established in [10]

$$\alpha_G \approx \frac{2(q-2)}{16/\pi + 6.18(q-2) + (q-2)^2}\,, \tag{3.21}$$

and $\mathcal{J} = \frac{\sqrt{q}}{2^{\frac{q-1}{2}}} J$.

3.2 Evaluation of Ψ_2

Now we would like to go further higher order term in the expansion of the classical solution. This term is given by

$$\Psi_{\rm cl}(t_1,t_2) \;=\; J^{-\frac{2}{q}}\Big[\,\Psi_0(t_1,t_2) \,+\, J^{-1}\,\Psi_1(t_1,t_2) \,+\, J^{-2}\,\Psi_2(t_1,t_2) \,+\, \cdots\Big]\,, \tag{3.22}$$

with

$$\Psi_2(t_1,t_2) \;=\; B_2\;\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+2}}\,, \tag{3.23}$$

where B_2 is a t-independent coefficient. The dimension of Ψ_2 is already fixed by Ψ_1, so what we need to do is just to fix the coefficient B_2. Substituting the above expansion of the classical field into the critical action $S_{\rm c}$ (2.6) and expanding it, one finds that the equation determining Ψ_2 is given by

$$\begin{aligned} &\int dt_3 dt_4\,\mathcal{K}(t_1,t_2;t_3,t_4)\Psi_2(t_3,t_4) \\ &= \;-\;[\Psi_0^{-1}\star\Psi_1\star\Psi_0^{-1}\star\Psi_1\star\Psi_0^{-1}](t_1,t_2) \;-\; \frac{(q-1)(q-2)}{2}\;\Psi_0^{q-3}(t_1,t_2)\,\Psi_1^2(t_1,t_2)\,, \end{aligned} \tag{3.24}$$

where the star product is defined by $[A\star B](t_1,t_2) \equiv \int dt_3\,A(t_1,t_3)B(t_3,t_2)$. Now, we are going to evaluate each term of this equation. For the first term in the LHS is again given by Eq.(3.9) with $\Delta = 1-1/q$ and $\alpha = 1/q+1$ as

$$(\text{LHS 1st}) \;=\; 2\pi\,B_2\,b^{2q-2}\;\frac{q(q-1)(3q-2)}{(q^2-4)\tan(\frac{\pi}{q})}\;\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{4-\frac{2}{q}}}\,. \tag{3.25}$$

For the first term of the RHS, we need to use Eq.(3.9) twice. First for the middle of the term: $\Psi_1\star\Psi_0^{-1}\star\Psi_1$, and then for the result sandwiched by the remaining Ψ_0^{-1}'s. Then, we have

$$(\text{RHS 1st}) \;=\; -\,B_1^2\,b^{3(q-1)}\,\frac{2\pi^2 q^2(q-1)(3q-2)}{(q-2)^2}\;\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{4-\frac{2}{q}}}\,. \tag{3.26}$$

The second terms in the LHS and RHS are trivially evaluated. Therefore, now one can see that all terms have the same t_{12} dependence. Then, comparing their coefficients, we finally fix B_2 as

$$B_2 \;=\; -\,\frac{B_1^2}{b}\left(\frac{q+2}{8q}\right)\left[(q-2)+(3q-2)\tan^2\left(\frac{\pi}{q}\right)\right]\,. \tag{3.27}$$

3.3 All Order Evaluation in $q > 2$

In this subsection, we extend our previous perturbative expansion of the classical solution to all order contributions in the $1/J$ expansion. Because of the dimension of Ψ_1 (3.15), the time-dependence is already fixed for all order as in Eq.(3.32). Therefore, we only need to determine the coefficient B_n, and in this subsection we will give a recursion relation which fixes the coefficients. However, we will not use this subsection's result in the rest of the thesis, so readers who are interested only in the first few terms in the $1/J$ expansion (3.22) may skip this subsection and move on to Section 4. As we saw in Section 3.1, the structure of the classical solution in $q = 2$ model is different from $q > 2$ case. In this subsection, we focus on $q > 2$ case.

We generalize the expansion (3.22) to all order by

$$\Psi_{\mathrm{cl}}(t_1, t_2) \;=\; J^{-\frac{2}{q}} \sum_{m=0}^{\infty} J^{-m}\, \Psi_m(t_1, t_2)\,. \tag{3.28}$$

Now, we substitute this expansion into the critical action S_{c} (2.6). As we saw before, the kinetic term does not contribute to the perturbative analysis when $q > 2$; therefore, we discard the kinetic term here. The contribution of the kinetic term will be recovered in the full classical solution with correct UV boundary conditions. Hence, the saddle-point equation is now formally written as

$$0 \;=\; \left[\sum_{m=0}^{\infty} J^{-m}\, \Psi_m(t_1, t_2)\right]^{-1} \;+\; \left[\sum_{m=0}^{\infty} J^{-m}\, \Psi_m(t_1, t_2)\right]^{q-1} . \tag{3.29}$$

Using the multinomial theorem, each term can be reduced to polynomials of Ψ_m's. Substituting these results into Eq.(3.29) leads the saddle-point equation written in terms polynomials with all order of $1/J$ expansion. From this equation, one can further pick up order $\mathcal{O}(J^{-n})$ terms. For $n = 0$, it is the equation of Ψ_0. Therefore, we consider $n \geq 1$ case, which is given by

$$\begin{aligned}0 \;=&\; \sum_{k_1+2k_2+\cdots=n} (-1)^{k_1+k_2+\cdots}\, \frac{(k_1+k_2+\cdots)!}{k_1!k_2!k_3!\cdots} \times \left[\Psi_0^{-1} \star \left(\Psi_1 \star \Psi_0^{-1}\right)^{k_1} \star \left(\Psi_2 \star \Psi_0^{-1}\right)^{k_2} \star \cdots\right](t_1, t_2)\\ &+ \sum_{k_1+2k_2+\cdots=n} \frac{(q-1)!}{k_0!k_1!k_2!\cdots} \times \Psi_0^{k_0}(t_1, t_2)\, \Psi_1^{k_1}(t_1, t_2)\, \Psi_2^{k_2}(t_1, t_2) \cdots\,,\end{aligned} \tag{3.30}$$

with $k_0 = q-(1+k_1+\cdots+k_{n-1})$. Let us consider this order $\mathcal{O}(J^{-n})$ equation more. Because of the constraint $k_1 + 2k_2 + \cdots = n$, we know that $k_{n+1} = k_{n+2} = \cdots = 0$. Also the same constraint implies that $k_n = 0$ or 1, and when $k_n = 1$, then $k_1 = k_2 = \cdots = k_{n-1} = 0$. Therefore, it is useful to separate $k_n = 1$ terms from $k_n = 0$ ones. After this separation, the order $\mathcal{O}(J^{-n})$ equation is reduced to a more familiar form:

$$\begin{aligned}
&\int dt_3 dt_4\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_n(t_3,t_4) \\
&= -\sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n} (-1)^{k_1+\cdots+k_{n-1}}\, \frac{(k_1+\cdots+k_{n-1})!}{k_1!\cdots k_{n-1}!} \\
&\qquad\qquad \times \left[\Psi_0^{-1}\star\left(\Psi_1\star\Psi_0^{-1}\right)^{k_1}\star\cdots\star\left(\Psi_{n-1}\star\Psi_0^{-1}\right)^{k_{n-1}}\right](t_1,t_2) \\
&\quad - \sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n} \frac{(q-1)!}{k_0!k_1!\cdots k_{n-1}!} \times \Psi_0^{k_0}(t_1,t_2)\,\Psi_1^{k_1}(t_1,t_2)\,\cdots\,\Psi_{n-1}^{k_{n-1}}(t_1,t_2)\,,
\end{aligned} \tag{3.31}$$

where $k_0 = q-(1+k_1+\cdots+k_{n-1})$. This is the equation which determines Ψ_n from $\{\Psi_0,\Psi_1,\cdots,\Psi_{n-1}\}$ sources. However, we already know the t_{12} dependence of $\Psi_n(t_1,t_2)$. Namely,

$$\Psi_n(t_1,t_2) \,=\, B_n\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+n}}\,. \tag{3.32}$$

Therefore, we only need to determine the coefficient B_n. Probably it is hard to evaluate the star products in the RHS of Eq.(3.31) by direct integrations of t's, and it is better to use momentum space representations.

$$\Psi_m(t_1,t_2) \,=\, B_m\,\int\frac{d\omega}{2\pi}\,e^{-i\omega t_{12}}\,\Psi_m(\omega)\,, \tag{3.33}$$

where we excluded the coefficient B_m from $\Psi_m(\omega)$ for later convenience, and $\Psi_m(\omega) = C_m\,|\omega|^{\frac{2}{q}+m-1}\,\mathrm{sgn}(\omega)$, with

$$C_m \,\equiv\, i\,2^{1-m-\frac{2}{q}}\sqrt{\pi}\,\frac{\Gamma(1-\frac{1}{q}-\frac{m}{2})}{\Gamma(\frac{1}{q}+\frac{m}{2}+\frac{1}{2})}\,. \tag{3.34}$$

With this definition of C_m, we can write the inverse of the critical solution as

$$\Psi_0^{-1}(t_1,t_2) \,=\, \int\frac{d\omega}{2\pi}\,e^{-i\omega t_{12}}\,\Psi_0^{-1}(\omega) \,=\, -\,b^{q-1}\,C_{2-\frac{4}{q}}\int\frac{d\omega}{2\pi}\,e^{-i\omega t_{12}}\,|\omega|^{1-\frac{2}{q}}\,\mathrm{sgn}(\omega)\,. \tag{3.35}$$

Now, we can evaluate each term in Eq.(3.31) using these Fourier transforms. Then, every term has the same ω integral; therefore, comparing the coefficients, one obtains

$$\begin{aligned}
&b^{q-2}\Big[\,(q-1)\,C_{2+n-\frac{4}{q}}\,-\,b^q\,C^2_{2-\frac{4}{q}}\,C_n\,\Big]B_n \\
&= -\sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n} (-1)^{k_1+\cdots+k_{n-1}}\,\frac{(k_1+\cdots+k_{n-1})!}{k_1!\cdots k_{n-1}!} \\
&\qquad\qquad \times\Big(-\,b^{q-1}\,C_{2-\frac{4}{q}}\Big)^{k_1+\cdots+k_{n-1}+1}\Big(B_1\,C_1\Big)^{k_1}\cdots\Big(B_{n-1}\,C_{n-1}\Big)^{k_{n-1}} \\
&\quad - \sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n} \frac{(q-1)!}{k_0!k_1!\cdots k_{n-1}!}\,\times\,b^{k_0}B_1^{k_1}\cdots B_{n-1}^{k_{n-1}}\,C_{2+n-\frac{4}{q}}\,,
\end{aligned} \tag{3.36}$$

with $k_0 = q - (1 + k_1 + \cdots + k_{n-1})$. This is the recursion relation which determines B_n from $\{B_1, B_2, \cdots, B_{n-1}\}$. Note that C_m's are a priori known numbers as defined in Eq.(3.34).

4 Finite Temperature

Up to here, we have been considering only zero-temperature solutions in the SYK model. In this section, we will consider the finite-temperature solutions $\Psi_{1,\beta}$ and $\Psi_{2,\beta}$ and the tree-level free energy in the low temperature region.

4.1 Classical Solutions

As we saw in Section 3, the $1/J$ expansion of the classical solution in the strongly coupling region is given by

$$\Psi_{\rm cl}(t_1,t_2) \;=\; J^{-\frac{2}{q}}\Big[\,\Psi_0(t_1,t_2) \;+\; J^{-1}\,\Psi_1(t_1,t_2) \;+\; J^{-2}\,\Psi_2(t_1,t_2) \;+\; \cdots\Big]\,, \tag{4.1}$$

where

$$\Psi_0(t_1,t_2) \;=\; b\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}}}\,,\quad \Psi_1(t_1,t_2) \;=\; B_1\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+1}}\,,\quad \Psi_2(t_1,t_2) \;=\; B_2\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+2}}\,. \tag{4.2}$$

In order to evaluate tree-level free energy, we first need finite-temperature versions of these classical solutions. Ψ_0 is the solution of the strict strong coupling limit, where the model exhibits an emergent conformal reparametrization symmetry: $t \to f(t)$ with the Ψ_0 transformation (2.7). Therefore, to obtain the finite-temperature version of Ψ_0, we just need to use $f(t) = \frac{\beta}{\pi}\tan(\frac{\pi t}{\beta})$ with the above transformation [7]. This map maps the infinitely long zero-temperature time to periodic thermal circle. Thus, this gives us

$$\Psi_{0,\beta}(t_1,t_2) \;=\; b\,\left[\frac{\pi}{\beta\sin(\frac{\pi t_{12}}{\beta})}\right]^{\frac{2}{q}}\,\mathrm{sgn}(t_{12})\,. \tag{4.3}$$

Since Ψ_1 and Ψ_2 are the shifts of the classical solution from the strict IR limit, they do not enjoy the reparametrization symmetry. Therefore, we cannot use the above method to get their finite-temperature counterparts. However, we can approximate finite-temperature solutions by mapping the zero-temperature solutions onto a thermal circle and summing over all image charges:

$$\Psi_\beta(t_{12}) \;=\; \sum_{m=-\infty}^{\infty}(-1)^m\,\Psi_{\beta=\infty}(t_{12}+\beta m)\,. \tag{4.4}$$

In this approximation, the finite-temperature solutions (two-point function in terms of the fundamental fermions) trivially satisfy the KMS condition. This approximation also works order by order in the $1/J$ expansion. Therefore, after separating positive m and negative m and changing the labeling, one finds

$$\Psi_{1,\beta}(t_{12}) \;=\; B_1\left[\,\sum_{m=0}^{\infty}\frac{(-1)^m}{(\beta m+t_{12})^{\frac{2}{q}+1}} \;-\; \sum_{m=1}^{\infty}\frac{(-1)^m}{(\beta m-t_{12})^{\frac{2}{q}+1}}\,\right]. \tag{4.5}$$

The summations of m can be evaluated to give the Hurwitz zeta functions. In the same way, we can approximate Ψ_2 in terms the Hurwitz zeta functions.

In [10], Maldacena and Stanford obtained a first order shift of the classical solution in finite-temperature through a numerical solution of the exact Schwinger-Dyson equation. The estimation of $\Psi_{1,\beta}$ by the above "image charge" method can be seen to agree well with the numerical ansatz. The solution of [10] is shown in their Eq.(3.122) reading:

$$\frac{\delta G(t_1,t_2)}{G_c(t_1,t_2)} \;=\; -\frac{\alpha_G}{\beta\mathcal{J}}\,f_0(t_{12})\,, \qquad f_0(t_{12}) \;=\; 2 \,+\, \frac{\pi - \frac{2\pi|t_{12}|}{\beta}}{\tan|\frac{\pi t_{12}}{\beta}|}\,. \tag{4.6}$$

with the notation, $G_c = \Psi_{0,\beta}$ and $\delta G = \Psi_{1,\beta}$. We can see in Figure 1 that our approximated solution for $\Psi_{1,\beta}$ is pretty close to this solution. It is more convenient to introduce a new variable

$$y \;\equiv\; \frac{|t_{12}|}{\beta} - \frac{1}{2}\,. \qquad \left(-\frac{1}{2} \le y \le \frac{1}{2}\right) \tag{4.7}$$

Then, we have $f_0(y) = 2 + 2\pi y\tan(\pi y)$. On the other hand for the figure, we rewrite our approximated solution by

$$\frac{\Psi_{1,\beta}(t_{12})}{\Psi_{0,\beta}(t_{12})} \;=\; \frac{B_1}{2(2\pi)^{\frac{2}{q}}b\beta}\left[\zeta\left(\tfrac{2}{q}+1,\tfrac{1}{4}\right) \,-\, \zeta\left(\tfrac{2}{q}+1,\tfrac{3}{4}\right)\right] \times F_0(y,q)\,, \tag{4.8}$$

where

$$F_0(y,q) \;\equiv\; (\cos\pi y)^{\frac{2}{q}}\left[\frac{\zeta\left(\frac{2}{q}+1,\frac{1}{4}+\frac{y}{2}\right) + \zeta\left(\frac{2}{q}+1,\frac{1}{4}-\frac{y}{2}\right) - \zeta\left(\frac{2}{q}+1,\frac{3}{4}+\frac{y}{2}\right) - \zeta\left(\frac{2}{q}+1,\frac{3}{4}-\frac{y}{2}\right)}{\zeta\left(\frac{2}{q}+1,\frac{1}{4}\right) - \zeta\left(\frac{2}{q}+1,\frac{3}{4}\right)}\right]. \tag{4.9}$$

Here, we adjusted the normalization of F_0 so that $F_0(y=0,q) = 2 = f_0(y=0)$. In Figure 1, we plotted $f_0(y)$ and $F_0(y,q)$ with $q = 2, 4, 1000$. We can see that for any value of q, F_0 is pretty close to f_0 in all range of y.

We will now develop a small temperature expansion which will give further useful information about the finite temperature solution and also the free energy. For this one expands the equation iteratively starting from $\Psi_{0,\beta}$ as sources. We develop this method for $\Psi_{1,\beta}$ in the rest of this subsection. The expansion of $\Psi_{0,\beta}$ solution (4.3) in the small temperature region is given by

$$\Psi_{0,\beta}(t_{12}) \;=\; b\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}}}\left[1 + \frac{\pi^2}{3q}\left|\frac{t_{12}}{\beta}\right|^2 + \frac{(q+5)\pi^4}{90q^2}\left|\frac{t_{12}}{\beta}\right|^4 + \cdots\right]. \tag{4.10}$$

We then expand the finite-temperature solution $\Psi_{1,\beta}$ by

$$\Psi_{1,\beta}(t_1,t_2) \;=\; B_1\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+1}}\left[1 + c_{1,1}\left|\frac{t_{12}}{\beta}\right| + c_{1,2}\left|\frac{t_{12}}{\beta}\right|^2 + c_{1,3}\left|\frac{t_{12}}{\beta}\right|^3 + \cdots\right], \tag{4.11}$$

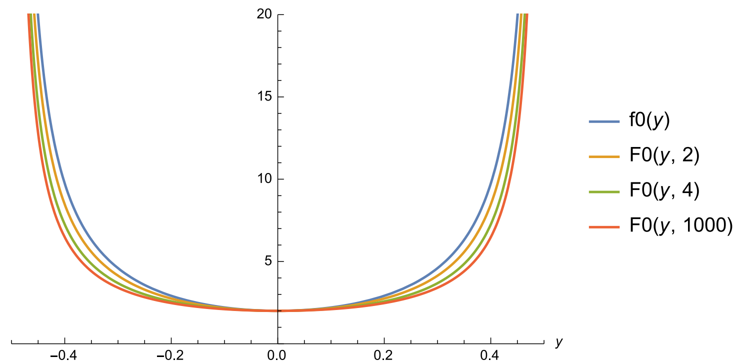

Figure 1. $f_0(y)$ and $F_0(y,q)$ with $q = 2, 4, 1000$ in the range of $-\frac{1}{2} \leq y \leq \frac{1}{2}$.

and then, using the equation of motion for Ψ_1 (3.2) we iteratively determine the coefficients $c_{1,i}$ starting from the lower order ones. As we will see in the next subsection, to evaluate its free energy contribution, we need $a_1 \equiv c_{1,3}$. First we consider $\mathcal{O}(\beta^{-1})$ order. The equation in this order reads

$$B_1 c_{1,1} \beta^{-1} \int dt_3 dt_4 \, \mathcal{K}(t_1, t_2; t_3, t_4) \frac{\text{sgn}(t_{34})}{|t_{34}|^{\frac{2}{q}}} = 0 \,, \tag{4.12}$$

where $\mathcal{K}$ denotes the zero temperature kernel. Using the formula in Eq.(3.9), one can evaluate the left-hand side integrals. In general, the integral does not vanish. Therefore, to satisfy the equation, we need $c_{1,1} = 0$. Next for $\mathcal{O}(\beta^{-2})$ order, we have an equation

$$\begin{aligned} &B_1 c_{1,2} \beta^{-2} \int dt_3 dt_4 \, \mathcal{K}(t_1, t_2; t_3, t_4) \frac{\text{sgn}(t_{34})}{|t_{34}|^{\frac{2}{q}-1}} \\ &= -\frac{\pi^2 (q-1) B_1 b^{q-2}}{3q\beta^2} \int dt_3 dt_4 \left[b^q \left(\frac{\text{sgn}(t_{13})\text{sgn}(t_{24})}{|t_{13}|^{-\frac{2}{q}} |t_{24}|^{2-\frac{2}{q}}} + \frac{\text{sgn}(t_{13})\text{sgn}(t_{24})}{|t_{13}|^{2-\frac{2}{q}} |t_{24}|^{-\frac{2}{q}}} \right) + (q-2) \frac{\delta(t_{13})\delta(t_{24})}{|t_{12}|^{-\frac{4}{q}}} \right] \frac{\text{sgn}(t_{34})}{|t_{34}|^{\frac{2}{q}+1}} \,. \end{aligned} \tag{4.13}$$

Again one can evaluate the integrals and find $c_{1,2} = -(q-1)\pi^2/3q$. Finally we consider $\mathcal{O}(\beta^{-3})$ order. The equation of this order reads

$$B_1 c_{1,3} \beta^{-3} \int dt_3 dt_4 \, \mathcal{K}(t_1, t_2; t_3, t_4) \frac{\text{sgn}(t_{34})}{|t_{34}|^{\frac{2}{q}-2}} = 0 \,. \tag{4.14}$$

The LHS integral identically vanishes. Hence, we cannot determine the coefficient $c_{1,3}$ from this equation. Nevertheless, this iterative method precisely recovers the expansion of (4.6) up to the third order:

$$\delta G(t_1, t_2) = -B_1 \frac{\text{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+1}} \left[1 - \frac{(q-1)\pi^2}{3q} \left| \frac{t_{12}}{\beta} \right|^2 + \frac{2\pi^2}{3} \left| \frac{t_{12}}{\beta} \right|^3 - \frac{(2q-1)(q+5)\pi^4}{90q^2} \left| \frac{t_{12}}{\beta} \right|^4 + \cdots \right] . \tag{4.15}$$

where we used the relation (3.20). Using this $\Psi_{1,\beta}$ expansion as source together with $\Psi_{0,\beta}$, we can also apply this method to determine low temperature expansion of $\Psi_{2,\beta}$.

4.2 Tree-Level Free Energy

Now we evaluate the tree-level free energy through the regularized breaking term. The order $(\beta J)^0$ contribution to the tree-level free energy, which comes from $S_{\rm c}[\Psi_{0,\beta}]$, was already evaluated in [7, 10, 86]. Therefore in this section, we will evaluate higher order contributions of the $1/\beta J$ expansion to the tree-level free energy.

The action of the collective time coordinate was evaluated in Appendix B from the regularized breaking term, which leads to the Schwarzian action given in Eq.(2.13). Now, we use the classical solution: $f(t) = \frac{\beta}{\pi}\tan(\frac{\pi t}{\beta})$. Then, the integral can be evaluated to give $2\pi^2/\beta$. Therefore, the $S[f]$ contribution to the tree-level free energy is

$$\beta F_0 \;=\; \frac{NB_1\gamma\pi^2}{\beta J}\,. \tag{4.16}$$

This contribution can actually be evaluated directly from the regularized breaking term by

$$\beta F_0 \;=\; -\,\frac{N}{2}\lim_{s\to\frac{1}{2}}\int dt_1 dt_2\,\Psi_{0,\beta}(t_1,t_2)\,Q_s(t_1,t_2)\,, \tag{4.17}$$

where the finite temperature critical solution $\Psi_{0,\beta}$ and the regularized source Q_s are given in Eq.(4.3) and (3.17), respectively. Since the regularized source Q_s has a factor $(s-1/2)$, in order to obtain non-vanishing contribution after the limit, we only need to extract a single pole $(s-1/2)^{-1}$ term from the integral. For this purpose, we expand the finite temperature critical solution $\Psi_{0,\beta}$ by power series of $|t_{12}|$ up to $|t|^{2-\frac{2}{q}}$ order, which is responsible for a single pole term. This leads to

$$\beta F_0 \;=\; -\,\frac{NB_1\pi^2}{6q\beta J}\,(q-1)b^{q-1}\left[\gamma(s,q)\int\frac{dt}{|t|^{2s}}\right]_{s\to\frac{1}{2}}\,. \tag{4.18}$$

Hence, using the expansion of $\gamma(s,q)$ in Eq.(3.16) and taking the limit $s\to 1/2$, we obtain the final result. This result agrees with the result found in Eq.(4.16) from the Schwarzian action.

Now we consider the next $(\beta J)^{-2}$ order contribution. The contribution from the breaking term to such order is given by

$$\beta F_1 \;=\; -\,\frac{N}{2}\lim_{s\to\frac{1}{2}}\int dt_1 dt_2\,\Psi_{1,\beta}(t_1,t_2)\,Q_s(t_1,t_2)\,. \tag{4.19}$$

Again to compute this free energy, we only need to extract the $|t|^{2-\frac{2}{q}}$ order term from $\Psi_{1,\beta}$. From the expansion in Eq.(4.15), one can read off the $|t|^{2-\frac{2}{q}}$ order term as

$$\Psi_{1,\beta}(t_1,t_2) \;=\; -\,\frac{2B_1\pi^2}{3\beta^3 J^{1+\frac{2}{q}}}\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}-2}} + \cdots\,. \tag{4.20}$$

Following the same process as in the previous subsection, one can evaluate the contribution from the breaking term to this order free energy. However, this is not the all contributions to this $(\beta J)^{-2}$ order free energy. The critical action $S_{\rm c}$ part also gives a contribution to this $(\beta J)^{-2}$ order, which is half of the breaking term contribution with opposite sign. Therefore, combining these two contributions, the final answer for the $(\beta J)^{-2}$ order free ernrgy is given by

$$\beta F_1 \;=\; -\,\pi^2 q\,\frac{NB_1^2\gamma}{b(\beta J)^2}\,. \tag{4.21}$$

In the following, we discuss the general $(\beta J)^{-n}$ order contribution of the tree-level free energy. For this purpose, let us first look at the collective action (2.11). After rescaling the bi-local field by $\Psi \to J^{-2/q}\Psi$, one sees the explicit J-dependence appearing only in the breaking term. Hence, from the J-derivative trick, the tree-level free energy is solely determined by the breaking term by

$$J\frac{\partial}{\partial J}\big(\beta F_n\big) \;=\; \frac{N}{2J}\lim_{s\to\frac{1}{2}}\int dt_1 dt_2\,\Psi_{n,\beta}(t_1,t_2)Q_s(t_1,t_2)\,. \tag{4.22}$$

We know that any order of $1/J$ correction for the zero temperature classical solution is given by Eq.(3.32). Even though we don't know exact finite-temperature version of these corrections, we nevertheless expect the finite-temperature solution can be expanded in low temperature region as

$$\Psi_{n,\beta}(t_1,t_2) \;=\; \frac{B_n}{J^n}\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+n}}\left[1+\cdots+a_n\left|\frac{t_{12}}{\beta}\right|^{n+2}+\cdots\right], \tag{4.23}$$

where a_n is a q-dependent constant, but independent of J, β or t. As we saw in the previous sections, the $|t|^{2-\frac{2}{q}}$ order term is only needed to extract the $(s-1/2)^{-1}$ poles. Hence, substituting this order term into Eq.(4.22), one can perform the integrals and the limit together with Eq.(3.17). This result is given by

$$J\frac{\partial}{\partial J}\big(\beta F_n\big) \;=\; -\,\frac{3qa_n}{b}\,\frac{NB_1B_n\gamma}{(\beta J)^{n+1}}\,. \tag{4.24}$$

After the integration of J, the free energy is given by

$$\beta F_n \;=\; \frac{3qa_n}{(n+1)b}\,\frac{NB_1B_n\gamma}{(\beta J)^{n+1}}\,. \tag{4.25}$$

We can check the consistency of this formula with previous results. For $n=0$, we have $B_0=b$ and $a_0=\pi^2/(3q)$. Then the formula gives the result we found above. For $n=1$, we have $a_1=-2\pi^2/3$, and then the formula again leads to the result found above. For general order we only need to determine a_n to evaluate the free energy. We note that in principle the coefficient of the zero temperature solution B_n can be determined from the recursion relation (3.36).

In summary, we have obtained the following $1/J$ corrections to the tree-level free energy

$$\frac{\beta F}{N} = \frac{B_1 \gamma \pi^2}{\beta J} - \frac{B_1^2 \gamma \pi^2 q}{b(\beta J)^2} - \frac{3 q a_2 B_1^3 \gamma}{b^2 (\beta J)^3} \left(\frac{q+2}{8q}\right) \left[(q-2) + (3q-2)\tan^2\left(\frac{\pi}{q}\right)\right] + \cdots . \quad (4.26)$$

For $q = 4$, we can compute the coefficients as

$$\frac{\beta F}{N} = -0.197\,(\beta J)^{-1} + 0.208\,(\beta J)^{-2} + 0.038 \times a_2\,(\beta J)^{-3} + \cdots , \quad (4.27)$$

with a_2 to be determined. These results agree with the recent numerical results of [27, 28].

5 Bi-local Propagator and Spectrum

In this section, we consider the bi-local two-point function:

$$\Big\langle \Psi(t_1, t_2) \Psi(t_3, t_4) \Big\rangle , \quad (5.1)$$

where the expectation value is evaluated by the path integral (2.4). After the Faddeev-Popov procedure and changing the integration variable as we discussed in Section 1, this two-point function becomes

$$\Big\langle \Psi_f(t_1, t_2) \Psi_f(t_3, t_4) \Big\rangle , \quad (5.2)$$

where now the expectation value is evaluated by the gauged path integral (2.10).

Now, we expand the bi-local field around a classical (large N) background solution Ψ_{cl}. Namely,

$$\Psi(t_1, t_2) = \Psi_{\text{cl}}(t_1, t_2) + \sqrt{\frac{2}{N}}\, \overline{\eta}(t_1, t_2) , \quad (5.3)$$

where $\overline{\eta}$ are quantum fluctuations, but the zero mode is eliminated from its Hilbert space. We will discuss the zero mode contribution in the following subsection. Therefore, the two-point function is now decomposed as

$$\Big\langle \Psi_f(t_1, t_2) \Psi_f(t_3, t_4) \Big\rangle = \Big\langle \Psi_{\text{cl},f}(t_1, t_2) \Psi_{\text{cl},f}(t_3, t_4) \Big\rangle + \frac{2}{N} \Big\langle \overline{\eta}(t_1, t_2) \overline{\eta}(t_3, t_4) \Big\rangle . \quad (5.4)$$

The second term in the RHS is the bi-local propagator $\mathcal{D}$ determined by Eq.(2.18), which was evaluated in [11] for $q = 4$ (and also in [9, 10]) as

$$\begin{aligned} \mathcal{D}(t_1, t_2; t_3, t_4) = &- \operatorname{sgn}(t_- t'_-)\, \frac{8}{J\sqrt{\pi}} \sum_{m=1}^{\infty} \int d\omega\, \frac{e^{-i\omega(t_+ - t'_+)}}{\sin(\pi p_m)} \frac{p_m^2}{p_m^2 + (3/2)^2} \\ &\times \left[J_{-p_m}(|\omega t_-|) + \frac{p_m + \frac{3}{2}}{p_m - \frac{3}{2}} J_{p_m}(|\omega t_-|) \right] J_{p_m}(|\omega t'_-|) , \end{aligned} \quad (5.5)$$

where p_m are the solutions of $2p_m/3 = -\tan(\pi p_m/2)$, and $t_\pm = (t_1 \pm t_2)/2$ and $t'_\pm = (t_3 \pm t_4)/2$.

Let us now describe the derivation of this bi-local propagator $\mathcal{D}$ with $q = 4$ for notational simplicity, but everything can be generalized to any q by small modifications. Fluctuations around the critical IR background can be studied by expanding the bi-local field as in Eq.(5.3) with

$$\Psi_{\rm cl}(t_1, t_2) \;=\; \Psi_0(t_1, t_2)\,, \tag{5.6}$$

where the critical IR background solution is given by

$$\Psi_0(t_1, t_2) \;=\; \left(\frac{1}{4\pi J^2}\right)^{\frac{1}{4}} \frac{\mathrm{sgn}(t_{12})}{\sqrt{|t_{12}|}}\,. \tag{5.7}$$

At the quadratic level, we have the quadratic kernel $\mathcal{K}$. The diagonalization of this quadratic kernel is done by the eigenfunction $u_{\nu,\omega}$ and the eigenvalue $\widetilde{g}(\nu)$ as

$$\int dt_1' dt_2'\, \mathcal{K}(t_1, t_2; t_1', t_2')\, u_{\nu,\omega}(t_1', t_2') \;=\; \widetilde{g}(\nu)\, u_{\nu,\omega}(t_1, t_2)\,. \tag{5.8}$$

The quadratic kernel $\mathcal{K}$ is in fact a function of the bi-local $SL(2,\mathcal{R})$ Casimir

$$\begin{aligned} C_{1+2} \;&=\; \big(\hat{D}_1 + \hat{D}_2\big)^2 \;-\; \frac{1}{2}\big(\hat{P}_1 + \hat{P}_2\big)\big(\hat{K}_1 + \hat{K}_2\big) \;-\; \frac{1}{2}\big(\hat{K}_1 + \hat{K}_2\big)\big(\hat{P}_1 + \hat{P}_2\big) \\ &=\; -\,(t_1 - t_2)^2\, \partial_1 \partial_2\,, \end{aligned} \tag{5.9}$$

with the $SL(2,\mathcal{R})$ generators $\hat{D} = -t\partial_t$, $\hat{P} = \partial_t$, and $\hat{K} = t^2 \partial_t$. The common eigenfunctions of the bi-local $SL(2,\mathcal{R})$ Casimir (5.9) are, due to the properties of the conformal block, given by the three-point function of the form

$$\Big\langle \mathcal{O}_h(t_0)\, \mathcal{O}_\Delta(t_1)\, \mathcal{O}_\Delta(t_2) \Big\rangle \;=\; \frac{\mathrm{sgn}(t_{12})}{|t_{10}|^h |t_{20}|^h |t_{12}|^{2\Delta - h}}\,. \tag{5.10}$$

Since the SYK quadratic kernel $\mathcal{K}$ is a function of this bi-local $SL(2,\mathcal{R})$ Casimir, this three-point function is also the eigenfunction of the SYK quadratic kernel. For the investigation of dual gravity theory, it is more useful to Fourier transform from t_0 to ω by

$$\begin{aligned} \Big\langle \widetilde{\mathcal{O}_h}(\omega)\, \mathcal{O}_\Delta(t_1)\, \mathcal{O}_\Delta(t_2) \Big\rangle \;&\equiv\; \int dt_0\, e^{i\omega t_0} \Big\langle \mathcal{O}_h(t_0)\, \mathcal{O}_\Delta(t_1)\, \mathcal{O}_\Delta(t_2) \Big\rangle \\ &=\; -\sqrt{\pi}\, \cot(\pi\nu)\, \Gamma(\tfrac{1}{2} - \nu)\, |\omega|^\nu\, \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2\Delta - \frac{1}{2}}}\, e^{i\omega(\frac{t_1 + t_2}{2})} Z_\nu(|\tfrac{\omega t_{12}}{2}|)\,, \end{aligned} \tag{5.11}$$

where we used $h = \nu + 1/2$ and defined

$$Z_\nu(x) \;=\; J_\nu(x) \;+\; \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu \;=\; \frac{\tan(\pi\nu/2) + 1}{\tan(\pi\nu/2) - 1}\,. \tag{5.12}$$

The t_0 integral in the Fourier transform can be performed by decomposing the integration region into three pieces. The complete set of ν can be understood from the representation theory of the conformal group, as discussed recently in [132]. We have the discrete modes $\nu = 2n + 3/2$ with $(n = 0, 1, 2, \cdots)$ and the continuous modes $\nu = ir$ with $(0 < r < \infty)$. Adjusting the normalization, we define our eigenfunctions by

$$u_{\nu,\omega}(t, z) \;\equiv\; \mathrm{sgn}(z)\, z^{\frac{1}{2}}\, e^{i\omega t}\, Z_\nu(|\omega z|)\,, \tag{5.13}$$

which have normalization condition

$$\int_{-\infty}^{\infty} \frac{dt}{2\pi} \int_0^{\infty} \frac{dz}{z^2}\; u^*_{\nu,\omega}(t, z)\, u_{\nu',\omega'}(t, z) \;=\; N_\nu\, \delta(\nu - \nu')\delta(\omega - \omega')\,, \tag{5.14}$$

with

$$N_\nu \;=\; \begin{cases} (2\nu)^{-1} & \text{for } \nu = 3/2 + 2n \\ 2\nu^{-1} \sin \pi\nu & \text{for } \nu = ir\,. \end{cases} \tag{5.15}$$

Here we used the change of the coordinates by

$$t \;\equiv\; \frac{t_1 + t_2}{2}\,, \qquad z \;\equiv\; \frac{t_1 - t_2}{2}\,, \tag{5.16}$$

and then the bi-local field $\eta(t_1, t_2)$

$$\eta(t_1, t_2) \;\equiv\; \Phi(t, z)\,, \tag{5.17}$$

can be then considered as a field in two dimensions (t, z). Expand the fluctuation field as

$$\Phi(t, z) \;=\; \sum_{\nu,\omega} \tilde{\Phi}_{\nu,\omega} u_{\nu,\omega}(t, z)\,, \tag{5.18}$$

the quadratic action can be written as

$$S_{(2)} \;=\; \frac{3J}{32\sqrt{\pi}} \sum_{\nu,\omega} N_\nu\, \tilde{\Phi}_{\nu,\omega} \big(\tilde{g}(\nu) - 1\big) \tilde{\Phi}_{\nu,\omega}\,, \tag{5.19}$$

where the kernel is given by

$$\tilde{g}(\nu) \;=\; -\,\frac{2\nu}{3}\, \cot\left(\frac{\pi\nu}{2}\right)\,. \tag{5.20}$$

Now this leads to the bi-local propagator

$$\begin{aligned} \mathcal{D}(t, z; t', z') \;=&\; \frac{16\sqrt{\pi}}{3J} \int_0^{\infty} \frac{dr}{N_{ir}} \int dw\; \frac{u^*_{ir,w}(t, z)\, u_{ir,w}(t', z')}{\widetilde{g}(ir) - 1} \\ &+\; \frac{16\sqrt{\pi}}{3J} \sum_{n=0}^{\infty} \frac{1}{N_{\nu_n}} \int dw\; \frac{u^*_{\nu_n w}(t, z)\, u_{\nu_n w}(t', z')}{\widetilde{g}(\nu_n) - 1}\bigg|_{\nu_n = 2n + \frac{3}{2}}\,. \end{aligned} \tag{5.21}$$

The r-integral is evaluated as explained in Appendix E which picks up poles determined by solutions of $\tilde{g}(\nu) = 1$, they represent a sequence denoted by p_m as

$$\frac{2p_m}{3} \;=\; -\,\tan\left(\frac{\pi p_m}{2}\right)\,, \qquad 2m + 1 < p_m < 2m + 2 \qquad (m = 0, 1, 2, \cdots) \tag{5.22}$$

Therefore, the bi-local propagator is written as residues of $\nu = p_m$ poles as

$$\mathcal{D}(t,z;t',z') \;=\; -\frac{32\pi^{\frac{3}{2}}}{3J}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')}\sum_{m=1}^{\infty} R(p_m)\,\frac{Z_{-p_m}(|\omega|z^{>})J_{p_m}(|\omega|z^{<})}{N_{p_m}}\,, \tag{5.23}$$

where $z^{>}(z^{<})$ is the greater (smaller) number among z and z'. The residue function is defined by

$$R(p_m) \;\equiv\; \text{Res}\left(\frac{1}{\tilde{g}(\nu)-1}\right)\bigg|_{\nu=p_m} \;=\; \frac{3p_m^2}{[p_m^2+(3/2)^2][\pi p_m - \sin(\pi p_m)]}\,. \tag{5.24}$$

Since that p_m are zeros of $\widetilde{g}(\nu)-1$, near each pole p_m, we can approximate as

$$\widetilde{g}(\nu) - 1 \;\approx\; \left[\nu^2 - (p_m)^2\right] f_m\,, \tag{5.25}$$

where f_m can be determined from residue of $1/(\widetilde{g}(\nu)-1)$ at $\nu = p_m$. Explicitly evaluating these residues, the inverse kernel is written as an exact expansion

$$\frac{1}{\tilde{g}(\nu)-1} \;=\; \sum_{m=1}^{\infty}\frac{6\,p_m^3}{[p_m^2+(3/2)^2][\pi p_m - \sin(\pi p_m)]}\left(\frac{1}{\nu^2 - p_m^2}\right)\,. \tag{5.26}$$

The effective action near a pole labelled by m is that of a scalar field with mass, $M_m^2 = p_m^2 - \frac{1}{4}$, $(m > 0)$ in AdS_2:

$$S_m^{\text{eff}} \;=\; \frac{1}{2}\int \sqrt{-g}\, d^2x\left[-g^{\mu\nu}\partial_\mu\phi_m\partial_\nu\phi_m - \left(p_m^2 - \frac{1}{4}\right)\phi_m^2\right]\,, \tag{5.27}$$

where the metric $g_{\mu\nu}$ is given by $g_{\mu\nu} = \text{diag}(-1/z^2, 1/z^2)$. It is clear from the above analysis that a spectrum of a sequence of 2D scalars, with growing conformal dimensions is being packed into a single bi-local field. In other words the bi-local representation effectively packs an infinite product of AdS Laplacians with growing masses. It is this feature which leads to the suggestion that the theory should be represented by an enlarged number of fields, or equivalently by an extra Kaluza-Klein dimension, which we will explain in the next section.

5.1 Zero Mode Contribution

In the above discussion, we have excluded the zero mode ($m = 0$) contribution which corresponds to the pole $p_0 = 3/2$. If we had this mode in Eq.(5.23), indeed $Z_{-p_0} = Z_{-3/2}$ leads to a divergence of the propagator because of $\xi_{-3/2} = -\infty$. This divergence can be treated by shifting the classical solution slightly away from the critical IR fixed point as first discussed by [10] in a $1/J$ expansion.

Therefore, for the first term in the RHS of Eq.(5.4), expanding the classical field up to the second order [2]

$$\Psi_{\text{cl}}(t_1,t_2) \;=\; \Psi_0(t_1,t_2) \,+\, \frac{1}{J}\,\Psi_1(t_1,t_2)\,, \tag{5.28}$$

[2] Here we have rescaled the entire field by $J^{2/q}$ to separate out all J dependencies from Ψ.

one has

$$\begin{aligned}&\Big\langle \Psi_{\text{cl},f}(t_1,t_2)\Psi_{\text{cl},f}(t_3,t_4)\Big\rangle\\&=\Big\langle \Psi_{0,f}(t_1,t_2)\Psi_{0,f}(t_3,t_4)\Big\rangle+\frac{1}{J}\left[\Big\langle \Psi_{0,f}(t_1,t_2)\Psi_{1,f}(t_3,t_4)\Big\rangle+\begin{pmatrix}t_1\leftrightarrow t_3\\t_2\leftrightarrow t_4\end{pmatrix}\right]+\cdots,\end{aligned}\tag{5.29}$$

where

$$\begin{aligned}\Psi_{0,f}(t_1,t_2)&=\Big|f'(t_1)f'(t_2)\Big|^{\frac{1}{q}}\;\Psi_0(f(t_1),f(t_2))\,,\\\Psi_{1,f}(t_1,t_2)&=\Big|f'(t_1)f'(t_2)\Big|^{\frac{1}{q}+\frac{1}{2}}\;\Psi_1(f(t_1),f(t_2))\,.\end{aligned}\tag{5.30}$$

Now, we consider an infinitesimal reparametrization $f(t)=t+\varepsilon(t)$. Then, the classical fields are expanded as

$$\begin{aligned}\Psi_{0,f}(t_1,t_2)&=\Psi_0(t_1,t_2)+\int dt\,\varepsilon(t)\,u_{0,t}(t_1,t_2)+\cdots,\\\Psi_{1,f}(t_1,t_2)&=\Psi_1(t_1,t_2)+\int dt\,\varepsilon(t)\,u_{1,t}(t_1,t_2)+\cdots,\end{aligned}\tag{5.31}$$

where

$$u_{0,t}(t_1,t_2)\equiv\frac{\partial\Psi_{0,f}(t_1,t_2)}{\partial f(t)}\bigg|_{f(t)=t},\qquad u_{1,t}(t_1,t_2)\equiv\frac{\partial\Psi_{1,f}(t_1,t_2)}{\partial f(t)}\bigg|_{f(t)=t}.\tag{5.32}$$

Therefore, in the quadratic order of ε, the classical field two-point function is now written in term of the two-point function of ε. For later convenience, it is better to write down this as momentum space integral as

$$\begin{aligned}&\Big\langle \Psi_{\text{cl},f}(t_1,t_2)\Psi_{\text{cl},f}(t_3,t_4)\Big\rangle\\&=\int\frac{d\omega}{2\pi}\,\langle\varepsilon(\omega)\varepsilon(-\omega)\rangle\left[u^*_{0,\omega}(t_1,t_2)u_{0,\omega}(t_3,t_4)+\frac{1}{J}\left(u^*_{0,\omega}(t_1,t_2)u_{1,\omega}(t_3,t_4)+\begin{pmatrix}t_1\leftrightarrow t_3\\t_2\leftrightarrow t_4\end{pmatrix}\right)+\cdots\right].\end{aligned}\tag{5.33}$$

Let us first evaluate the ε two-point function. The collective coordinate action is given in Eq.(2.13). Expanding $f(t)=t+\varepsilon(t)$, the quadratic action of ε can be obtained from this action. Hence, the two-point function in momentum space is

$$\langle\varepsilon(\omega)\varepsilon(-\omega)\rangle=\frac{24\pi J}{\alpha N}\frac{1}{\omega^4}\,.\tag{5.34}$$

One can also Fourier transform back to the time representation to get

$$\langle\varepsilon(t_1)\varepsilon(t_2)\rangle=\frac{2\pi J}{\alpha N}\,|t_{12}|^3\,.\tag{5.35}$$

Next, we evaluate u_0 and u_1. Taking the derivative respect to $f(t)$, one obtains

$$\begin{aligned}
u_{0,t}(t_1,t_2) &= \frac{1}{q}\left[\delta'(t_1-t) + \delta'(t_2-t) - 2\left(\frac{\delta(t_1-t)-\delta(t_2-t)}{t_1-t_2}\right)\right]\Psi_0(t_1,t_2)\,, \\
u_{1,t}(t_1,t_2) &= \frac{2+q}{2q}\left[\delta'(t_1-t) + \delta'(t_2-t) - 2\left(\frac{\delta(t_1-t)-\delta(t_2-t)}{t_1-t_2}\right)\right]\Psi_1(t_1,t_2) \\
&= \frac{(2+q)B_1}{2b}\,\frac{u_{0,t}(t_1,t_2)}{|t_{12}|}\,.
\end{aligned} \tag{5.36}$$

After some manipulation, one can show that the momentum space expressions are given by

$$\begin{aligned}
u_{0,\omega}(t_1,t_2) &= -\frac{ib\sqrt{\pi}}{q}\,\frac{|\omega|^{\frac{3}{2}}\,\mathrm{sgn}(\omega t_-)}{|2\,t_-|^{\frac{2}{q}-\frac{1}{2}}}\;e^{i\omega t_+}\,J_{\frac{3}{2}}(|\omega t_-|)\,, \\
u_{1,\omega}(t_1,t_2) &= \frac{(2+q)B_1}{4b}\,\frac{u_{0,\omega}(t_1,t_2)}{|t_-|}\,.
\end{aligned} \tag{5.37}$$

Using the two-point function of ε and above u_0 and u_1 expressions, finally the two-point function (5.4) up to order J^0 is given by

$$\begin{aligned}
\Big\langle \Psi_f(t_1,t_2)\Psi_f(t_3,t_4)\Big\rangle &= \frac{12}{\alpha N}\left[J + \frac{(2+q)B_1}{4b}\left(\frac{1}{|t_-|}+\frac{1}{|t'_-|}\right)\right]\int\frac{d\omega}{\omega^4}\,u^*_{0,\omega}(t_1,t_2)u_{0,\omega}(t_3,t_4) \\
&\quad + \mathcal{D}(t_1,t_2;t_3,t_4)\,.
\end{aligned} \tag{5.38}$$

What we have established therefore is the following. What one has is first the leading "classical" contribution to the bi-local two-point function which usually factorizes, due to the dynamics of the reparametrization symmetry mode. It now represents the leading "big" contribution, as in [10], and a sub-leading one.

6 3D Interpretation

In this section, we describe the spectrum of matter fields predicted by the $q = 4$ SYK bi-local propagator (5.22) can be understood as a Kaluza-Klein of a single scalar field in 3-dimensional space-time. A generalization to arbitrary even integer q is also constructed in [107], but here we restrict ourselves to $q = 4$ case just for notational simplicity.

According to [87, 88], the bulk dual of the SYK model involves Jackiw-Teitelboim theory of two dimensional dilaton gravity, whose action is given by (up to usual boundary terms)

$$S_{JT} = -\frac{1}{16\pi G}\int\sqrt{-g}\Big[\phi(R+2) - 2\phi_0\Big]\,, \tag{6.1}$$

where ϕ_0 is a constant, and ϕ is a dilaton field. The zero temperature background is given by AdS_2 with a metric

$$ds^2 = \frac{-dt^2+dz^2}{z^2}\,, \tag{6.2}$$

and a dilaton

$$\phi(z) \;=\; \phi_0 \,+\, \frac{a}{z} \,+\, \cdots\,, \tag{6.3}$$

where a is a parameter which scales as $1/J$ and the ellipsis denotes higher order corrections. In the following we will choose, without loss of generality, $\phi_0 = 1$.

This action can be thought as arising from a higher dimensional system which has extremal black holes, and the AdS_2 is the near horizon geometry [87]. The three dimensional metric, with the dilaton being the third direction, is given by

$$ds^2 = \frac{1}{z^2}\big[-dt^2 + dz^2\big] \,+\, \Big(1+\frac{a}{z}\Big)^2 dy^2\,. \tag{6.4}$$

This is in fact the near-horizon geometry of a charged extremal BTZ black hole.

6.1 Kaluza-Klein Decomposition

We will now show that the infinite sequence of poles in the previous section from the Kaluza-Klein tower of a single scalar in a three dimensional metric (6.4) where the direction y is an interval $-L < y < L$. The action of the scalar is

$$S \;=\; \frac{1}{2}\int d^3x\sqrt{-g}\Big[-g^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi \;-\; m_0^2\,\Phi^2 \;-\; V(y)\Phi^2\Big]\,, \tag{6.5}$$

where $V(y) = V\delta(y)$, with the constant V and the size L to be determined. This is similar to Horava-Witten compactification on S^1/Z_2 [127] with an additional delta function potential. [3] The scalar satisfies Dirichlet boundary conditions at the ends of the interval.

We now proceed to decompose the 3D theory into 2 dimensional modes. Using Fourier transform for the t coordinate:

$$\Phi(t,z,y) \;=\; \int\frac{d\omega}{2\pi}\,e^{-i\omega t}\,\chi_\omega(z,y)\,, \tag{6.6}$$

one can rewrite the action (6.5) in the form of

$$S \;=\; \frac{1}{2}\int dzdy\int\frac{d\omega}{2\pi}\;\chi_{-\omega}\,(\mathcal{D}_0+\mathcal{D}_1)\,\chi_\omega\,, \tag{6.7}$$

where $\mathcal{D}_0$ is the a-independent part and $\mathcal{D}_1$ is linear in a:

$$\begin{aligned}\mathcal{D}_0 \;&=\; \partial_z^2 \,+\, \omega^2 \,-\, \frac{m_0^2}{z^2} \,+\, \frac{1}{z^2}\Big(\partial_y^2 - V(y)\Big)\,,\\ \mathcal{D}_1 \;&=\; \frac{a}{z}\left[\partial_z^2 \,-\, \frac{1}{z}\,\partial_z \,+\, \omega^2 \,-\, \frac{m_0^2}{z^2} \,-\, \frac{1}{z^2}\Big(\partial_y^2 + V(y)\Big)\right]\,.\end{aligned} \tag{6.8}$$

Here, we neglected higher order contributions of a. The eigenfunctions of $\mathcal{D}_0$ can be clearly written in the form

$$\chi_\omega(z,y) \;=\; \chi_\omega(z)\,f_k(y)\,. \tag{6.9}$$

[3] See also [128, 129]. We are grateful to Cheng Peng for bringing this to our attention.

Then $f_k(y)$ is an eigenfunction of the Schrödinger operator $-\partial_y^2 + V(y)$ with eigenvalue k^2. This is a well known Schrodinger problem: the eigenfunctions and the eigenvalues are presented in detail in Appendix C.

After solving this part, the kernels are reduced to

$$\mathcal{D}_0 \,=\, \partial_z^2 \,+\, \omega^2 \,-\, \left(\frac{m_0^2 + p_m^2}{z^2}\right)\,, \qquad \mathcal{D}_1 \,=\, \frac{a}{z}\left[\partial_z^2 \,-\, \frac{1}{z}\,\partial_z \,+\, \omega^2 \,-\, \left(\frac{m_0^2 - q_m^2}{z^2}\right)\right]\,, \tag{6.10}$$

where p_m are the solutions of

$$-\,(2/V)k = \tan(kL) \tag{6.11}$$

while q_m are the expectation values of $-\partial_y^2 - V(y)$ operator respect to f_{p_m}. If we choose $V = 3$ and $L = \frac{\pi}{2}$ the solutions of (6.11) agree precisely with the strong coupling spectrum of the SYK model given by $\tilde{g}(\nu) = 1$, as is clear from (5.19) and (5.20). This is our main observation.

For these values of V, L, the propagator G is determined by the Green's equation of $\mathcal{D}$. We now use the perturbation theory to evaluate it. This will then be compared with the corresponding propagator of the bi-local SYK theory.

6.2 Evaluation of $G^{(0)}$

We start by determining the leading, zero-th order $G^{(0)}$ propagator obeying

$$\mathcal{D}_0\, G^{(0)}_{\omega,\omega'}(z,y;z',y') \,=\, -\,\delta(z-z')\delta(y-y')\delta(\omega+\omega')\,. \tag{6.12}$$

We first separate the scaling part of the propagator by $G^{(0)} = \sqrt{z}\,\widetilde{G}^{(0)}$ and multiplying z^2. Expanding in a basis of eigenfunctions $f_k(y)$,

$$\widetilde{G}^{(0)}(z,y,\omega;z',y',\omega') = \sum_{k,k'} f_k(y) f_{k'}(y') \widetilde{G}^{(0)}_{\omega,k;\omega',k'}(z;z') \tag{6.13}$$

The Green's function $\widetilde{G}^{(0)}_{\omega,k;\omega',k'}(z,z')$ is clearly proportional to $\delta(k-k')$ and satisfies the equation

$$\left[z^2\,\partial_z^2 \,+\, z\,\partial_z \,+\, \omega^2\,z^2 \,-\, \nu_0^2\right]\widetilde{G}^{(0)}_{\omega,k;\omega',k'}(z;z') \,=\, -\,z^{\frac{3}{2}}\,\delta(z-z')\delta(\omega+\omega')\delta(k-k')\,. \tag{6.14}$$

where we have defined

$$\nu_0^2 \equiv k^2 + m_0^2 + 1/4. \tag{6.15}$$

The operator which appears in (6.14) is the Bessel operator. Thus the Green's function can be expanded in the complete orthonormal basis. For this, we use the same basis form Z_ν as in the SYK evaluation [4]:

$$\widetilde{G}^{(0)}_{\omega,k;-\omega,k}(z;z') \,=\, \int d\nu\, \widetilde{g}^{(0)}_\nu(z')\, Z_\nu(|\omega z|)\,. \tag{6.16}$$

[4] This represents a modified set of wave functions with boundary conditions at $z \to \infty$ in contrast to the standard AdS wave functions. Some basic aspects of Euclidean AdS scalar propagators are summarized in Appendix F

Then, substituting this expansion into the Green's equation (6.12) and using Eqs.(D.6) and (D.2), one can fix the coefficient $\widetilde{g}_\nu^{(0)}$. Finally, the ν-integral form of the propagator is given by

$$G^{(0)}_{\omega,k;-\omega,k}(z;z') \;=\; -\,|zz'|^{\frac{1}{2}} \int \frac{d\nu}{N_\nu}\, \frac{Z^*_\nu(|\omega z|)\, Z_\nu(|\omega z'|)}{\nu^2-\nu_0^2}\,. \tag{6.17}$$

We now note that if we choose $m_0^2 = -1/4$, which is the BF bound of AdS_2, we have $\nu_0^2 = p_m^2$, and the equation which determine p_m, (6.11) is precisely the equation which determines the spectrum of the SYK theory found in [9, 11]. With this choice, the real space zeroth order propagator in three dimensions is

$$G^{(0)}(t,z,y;t',z',y') \;=\; -\,|zz'|^{\frac{1}{2}} \sum_{m=0}^{\infty} f_{p_m}(y) f_{p_m}(y') \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \int \frac{d\nu}{N_\nu}\, \frac{Z^*_\nu(|\omega z|)\, Z_\nu(|\omega z'|)}{\nu^2-p_m^2}\,. \tag{6.18}$$

We now show that the above propagator with $y = y' = 0$ is in exact agreement with the bi-local propagator of the SYK model. The Green's function with these end points is

$$G^{(0)}(t,z,0;t',z',0) \;=\; -\,|zz'|^{\frac{1}{2}} \sum_{m=0}^{\infty} C(p_m) \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \int \frac{d\nu}{N_\nu}\, \frac{Z^*_\nu(|\omega z|)\, Z_\nu(|\omega z'|)}{\nu^2-p_m^2}\,, \tag{6.19}$$

where we have defined

$$C(p_m) \;\equiv\; f_{p_m}(0) f_{p_m}(0) \;=\; B_m^2\, \frac{p_m^2}{p_m^2+(3/2)^2} \;=\; \frac{2p_m^3}{[p_m^2+(3/2)^2][\pi p_m - \sin(\pi p_m)]}\,. \tag{6.20}$$

Now we note that Kaluza-Klein wave function coefficient coincides in detail with the SYK one, namely:

$$C(p_m) \;=\; \frac{2p_m}{3}\, R(p_m)\,, \tag{6.21}$$

where $R(p_m)$ was given in Eq.(5.24).

As in Eq.(D.6), the integration of ν is a short-hand notation which denotes a summation of $\nu = 3/2 + 2n$, $(n = 0, 1, 2\cdots)$ and an integral of $\nu = ir$, $(0 < r < \infty)$. The sum over these discrete values of ν and the integral over the continuous values can be now performed exactly as in the calculation of the SYK bi-local propagator [11]. Closing the contour for the continuous integral in $\mathrm{Re}(\nu)\to\infty$, one finds that there are two types of poles inside of this contour. (1): $\nu = 2n + 3/2$, $(n = 0, 1, 2, \cdots)$, and (2): $\nu = p_m$, $(m = 0, 1, 2, \cdots)$. The contributions of the former type of poles precisely cancel with the contribution from the discrete sum over n. Details of the evaluation which explicitly shows the cancelation are presented in Appendix E. Therefore, the final remaining contribution is just written as residues of $\nu = p_m$ poles as

$$G^{(0)}(t,z,0;t',z',0) \;=\; \frac{1}{3}\, |zz'|^{\frac{1}{2}} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')}\, R(p_m)\, \frac{Z_{-p_m}(|\omega|z^>) J_{p_m}(|\omega|z^<)}{N_{p_m}}\,. \tag{6.22}$$

Altogether we have shown that $y = 0$ mode 3D propagator is in precise agreement with the $q = 4$ SYK bi-local propagator at large J given in Eq.(5.23). The propagator is a sum of non-standard propagators in AdS_2. While it vanishes on the boundary, the boundary conditions at the horizon are different from that of the standard propagator in AdS.

6.3 First Order Eigenvalue Shift

In this section, we study the first order eigenvalue shift due to $\mathcal{D}_1$ by treating this operator as a perturbation onto the $\mathcal{D}_0$ operator. The result will confirm the duality $a = 1/J$, where a is defined in the dilaton background (6.3) and J is the coupling constant in the SYK model.

Since the t and y directions are trivial, let us start with the kernels already solved for these two directions given in Eq.(6.10). The eigenfunction of $\mathcal{D}_0$ operator is

$$|z|^{\frac{1}{2}}\, Z_\nu(|\omega z|)\,, \tag{6.23}$$

and using the orthogonality condition (D.3), its matrix element in the ν space is found as

$$N_\nu \Big[\nu^2 - (m_0^2 + p_m^2 + \tfrac{1}{4})\Big]\, \delta_{\nu,\nu'}\,. \tag{6.24}$$

Now following the first order perturbation theory, we are going to determine the first order eigenvalue shift. Using the Bessel equation, the action of $\mathcal{D}_1$ on the $\mathcal{D}_0$ eigenfunction (6.23) is found as

$$\mathcal{D}_1\, |z|^{\frac{1}{2}}\, Z_\nu(|\omega z|) \;=\; \frac{a}{|z|^{\frac{1}{2}}}\left[\frac{\partial_z}{z} \;-\; \left(\frac{m_0^2 - q_m^2 + \frac{3}{4}}{z^2}\right)\right] Z_\nu(|\omega z|)\,. \tag{6.25}$$

For the derivative term, we use the Bessel function identity (for example, see 8.472 of [130])

$$\partial_x J_\nu(x) \;=\; \pm\, J_{\nu\mp 1}(x) \,\mp\, \frac{\nu}{x}\, J_\nu(x)\,, \tag{6.26}$$

to obtain

$$\partial_z\, Z_\nu(|\omega z|) \;=\; \frac{\nu}{|z|}\, Z_\nu(|\omega z|) \;-\; |\omega|\Big[J_{\nu+1}(|\omega z|) \;-\; \xi_\nu\, J_{-\nu-1}(|\omega z|)\Big]\,. \tag{6.27}$$

Therefore, now the matrix element is determined by integrals

$$\begin{aligned}\int_0^\infty dz\, |z|^{\frac{1}{2}}\, Z^*_{\nu'}(|\omega z|)\mathcal{D}_1\, |z|^{\frac{1}{2}}\, Z_\nu(|\omega z|) \;=\; & a\Big[\nu - \big(m_0^2 - q_m^2 + \tfrac{3}{4}\big)\Big]\int_0^\infty dz\, \frac{Z^*_{\nu'}(|\omega z|)Z_\nu(|\omega z|)}{z^2} \\ & -\, a|\omega|\int_0^\infty dz\, \frac{Z^*_{\nu'}(|\omega z|)}{z}\Big[J_{\nu+1}(|\omega z|) \;-\; \xi_\nu\, J_{-\nu-1}(|\omega z|)\Big]\,.\end{aligned} \tag{6.28}$$

For the continuous mode ($\nu = ir$), the integrals might be hard to evaluate. In the following, we restrict ourself to the real discrete mode $\nu = 3/2 + 2n$. In such case, $\xi_\nu = 0$. Therefore, the linear combination of the Bessel function is reduced to a single Bessel function as $Z_\nu(x) = J_\nu(x)$. Since

$$\begin{aligned}\int_0^\infty dx\, \frac{J_\alpha(x)J_\beta(x)}{x} &= \frac{2}{\pi}\, \frac{\sin\left[\frac{\pi}{2}(\alpha-\beta)\right]}{\alpha^2-\beta^2}\,, & [\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0] \\ \int_0^\infty dx\, \frac{J_\alpha(x)J_\beta(x)}{x^2} &= \frac{4}{\pi}\, \frac{\sin\left[\frac{\pi}{2}(\alpha-\beta-1)\right]}{\left[(\alpha+\beta)^2-1\right]\left[(\alpha-\beta)^2-1\right]}\,, & [\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 1]\end{aligned} \tag{6.29}$$

we have now found the matrix element for the discrete mode is given by

$$\frac{2a|\omega|}{\pi}\, \frac{\sin\left[\frac{\pi}{2}(\nu-\nu'-1)\right]}{(\nu+1)^2-\nu'^2} \left[\frac{2\left[\nu-(m_0^2-q_m^2+\frac{3}{4})\right]}{(\nu-1)^2-\nu'^2} - 1\right]. \tag{6.30}$$

Next, let us focus on the zero mode ($\nu = \nu' = 3/2$) eigenvalue. In the above formula, taking the bare mass to the BF bound: $m_0^2 = -1/4$ as before, the zero mode first order eigenvalue shift is found as

$$\frac{a|\omega|}{2\pi}\,(2+q_0^2)\,. \tag{6.31}$$

Now, we compare this result with the $1/J$ first order eigenvalue shift of the SYK model, which is for the zero mode found in [10] as

$$k(2,\omega) = 1 - \frac{\alpha_K|\omega|}{2\pi\mathcal{J}} + \cdots\,, \qquad \text{(zero temperature)} \tag{6.32}$$

where $\alpha_K \approx 2.852$ for $q = 4$. The ω-dependence of our result (6.31) thus agrees with that of the SYK model. Furthermore, this comparison confirms the duality $a = 1/J$.

Finally, we can now complete our comparison by showing agreement for the $m = 0$ mode contribution to the propagator. We include the first $\mathcal{O}(a)$ order shift for the pole as

$$\nu = \frac{3}{2} + \frac{a|\omega|}{6\pi}\left(2+q_0^2\right) + \mathcal{O}(a^2)\,. \tag{6.33}$$

For the zero mode part ($m = 0$) of the on-shell propagator in Eq.(E.6), the leading order is $\mathcal{O}(1/a)$. This contribution comes from the coefficient factor of the Bessel function, which was responsible for the double pole at $\nu = 3/2$. For other p_0 setting them to $3/2$, we obtain the leading order contribution from the zero mode as

$$G^{(0)}_{\rm zero-mode}(t,z,0;t',z',0) = -\frac{9\pi}{4a}\,\frac{B_0^2}{(2+q_0^2)}\,|zz'|^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{d\omega}{|\omega|}\, e^{-i\omega(t-t')} J_{\frac{3}{2}}(|\omega z|) J_{\frac{3}{2}}(|\omega z'|)\,. \tag{6.34}$$

This agrees with the order $\mathcal{O}(J)$ contribution of the SYK bi-local propagator of Maldacena/Stanford [10].

7 Question of Dual Spacetime

7.1 "i" Problem

In this section, we clarify the question regarding the signature of the SYK dual gravity theory.

The bi-local $SL(2,\mathcal{R})$ Casimir (5.9) can be seen to take the form of a Laplacian of Lorentzian two dimensional Anti de-Sitter or de-Sitter space-time (in this two dimensional case they are characterized by the same isometry group SO(2,1) or SO(1,2)). Under the canonical identification with AdS

$$ds^2 = \frac{-dt^2 + d\hat{z}^2}{\hat{z}^2}, \tag{7.1}$$

it equals

$$C_{1+2} = z^2(-\partial_t^2 + \partial_{\hat{z}}^2). \tag{7.2}$$

Consequently the SYK eigenfunctions should be compared with known AdS_2 or dS_2 basis wave functions.

Note that the Bessel function Z_ν (5.12) are not the standard normalizable modes used in quantization of scalar fields in AdS_2: in particular they have rather different boundary conditions at the Poincare horizon. Another important property of this basis is that when viewed as a Schrodinger problem as in [9] it has a set of bound states, in addition to the scattering states. This will be discussed in detail in Section 7.2 (see the left picture of FIG. 2).

This leads one to try an identification with de-Sitter basis functions [5]. In fact as we will argue, the bi-local SYK wave functions can be realized as a particular α-vacuum of Lorentzian dS_2 with a choice of $\alpha = i\pi h = i\pi(\nu + 1/2)$. This is seen as follows. We consider the dS_2 background with a metric given by

$$ds^2 = \frac{-d\eta^2 + dt^2}{\eta^2}. \tag{7.3}$$

This can be obtained by the coordinate change (5.16) by replacing $z \to \eta$. The Euclidean (Bunch-Davies [133]) wave function of a massive scalar field is given by

$$\phi^E_\omega(\eta)\, e^{i\omega t}, \tag{7.4}$$

with

$$\phi^E_\omega(\eta) = \eta^{\frac{1}{2}} H^{(2)}_\nu(|\omega|\eta), \qquad \nu = \sqrt{\frac{1}{4} - m^2}, \tag{7.5}$$

where $H^{(2)}_\nu$ is the Hankel function of the second kind. Since the t-dependence is always like $e^{i\omega t}$, in the following we will focus only on the η dependence. The α-vacuum wave function

[5] This possibility has been emphasized to us by J. Maldacena [113].

is defined by Bogoliubov transformation from this Euclidean wave function [134, 135] as

$$\begin{aligned}\phi_\omega^\alpha(\eta) &\equiv N_\alpha\Big[\phi_\omega^E(\eta) + e^\alpha\phi_\omega^{E*}(\eta)\Big]\\ &= N_\alpha\,\eta^{\frac{1}{2}}\Big[H_\nu^{(2)}(|\omega|\eta) + e^\alpha H_\nu^{(1)}(|\omega|\eta)\Big]\,,\end{aligned} \tag{7.6}$$

where

$$N_\alpha = \frac{1}{\sqrt{1-e^{\alpha+\alpha^*}}}\,, \tag{7.7}$$

and α is a complex parameter. Now let us consider a possibility of α-vacuum with

$$\alpha = i\pi\left(\nu+\frac{1}{2}\right) = i\pi h\,. \tag{7.8}$$

With this choice of α, using the definition of the Hankel functions

$$H_\nu^{(1)}(x) = \frac{J_{-\nu}(x) - e^{-i\pi\nu}J_\nu(x)}{i\sin(\pi\nu)}\,, \qquad H_\nu^{(2)}(x) = \frac{J_{-\nu}(x) - e^{i\pi\nu}J_\nu(x)}{-i\sin(\pi\nu)}\,, \tag{7.9}$$

one can rewrite the α-vacuum wave function as

$$\phi_\omega^\alpha(\eta) = \left(\frac{2\,\eta^{\frac{1}{2}}}{1+\xi_\nu\,e^{-i\pi\nu}}\right) Z_\nu(|\omega|\eta)\,, \tag{7.10}$$

where Z_ν is defined in Eq.(5.12). After excluding the η-independent part of the wave function, we can write the η-dependent part as

$$\phi_\omega^\alpha(\eta) = \eta^{\frac{1}{2}}\,Z_\nu(|\omega|\eta)\,. \tag{7.11}$$

This wave function agrees with the eigenfunction of the SYK quadratic kernel (5.13) after the identifications of $\eta = (t_1-t_2)/2$ and $t = (t_1+t_2)/2$.

Due to this observation, one might attempt to claim that the dual gravity theory of the SYK model is given by Lorentzian dS_2 space-time. However, there is a critical issue in this claim. Apart from the Lorentzian signature in this metric (7.3), we still have a discrepancy in the exponent of the partition function (2.4) with a factor of "i". Namely, if the dual gravity theory (higher spin gravity or string theory) is Lorentzian dS_2, it must have

$$Z = \int \mathcal{D}h_n\,\mathcal{D}\Phi_m\,\exp\Big[i\Big(S_{\text{grav}}[h,\Phi] + S_{\text{matter}}[h,\Phi]\Big)\Big]\,, \tag{7.12}$$

where we collectively denote the graviton and other "higher spin" gauge fields by h_n and the dilaton and other matter fields by Φ_m. Hence the agreement of the SYK bi-local propagator

$$\mathcal{D}_{\text{SYK}}(t_1,t_2;t_1',t_2') = \Big\langle\overline{\Psi}(t_1,t_2)\overline{\Psi}(t_1',t_2')\Big\rangle = \sum_{m=0}^{\infty} G_{p_m}(t_1,t_2;t_1',t_2')\,, \tag{7.13}$$

with a dS_2 propagator

$$\mathcal{D}_{\text{dS}}(\eta,t;\eta',t') = \frac{1}{i}\sum_{m=0}^{\infty}\Big\langle\Phi_m(\eta,t)\Phi_m(\eta',t)\Big\rangle = \frac{1}{i}\sum_{m=0}^{\infty} G_m(\eta,t;\eta',t')\,, \tag{7.14}$$

is only up to the factor i. Namely, even if we have a complete agreement of G_{p_m} with G_m by identifying the coordinates by (5.16) (with a replacement of $z \to \eta$), there is a problem with the signature (i.e. the discrepancy of the factor i). For higher point functions, the same i-problem proceeds due to the i factors coming from the propagator and each vertex.

To conclude, for the Euclidean SYK model under consideration, one needs a dual gravity theory to be in the hyperbolic plane H_2 (i.e. Euclidean AdS_2) for the matching of n-point functions. We will set the basis for the $EAdS_2$ realization in the next subsection.

7.2 Transformations and Leg Factors

As we have commented in the Introduction in order to identify an Euclidean bulk dual description (rather than a Lorentzian), we will need a transformation which brings the SYK eigenfunctions (as given on bi-local space-time) to the standard eigenfunctions of the $EAdS_2$ Laplacian. We will arrive at this transformation by considering the bi-local map described in [114, 115] for higher dimensional case. In our current $d = 1$ case, the map is even simpler. It will be seen to take the form of a H^2 Radon transform (a related suggestion was made in [10]). The need for a non-local transform on external legs appears to be characteristic of collective theory (which as a rule contains a minimal set of physical degrees of freedom). The first appearance of Radon type transforms in identifying holographic space-time was seen in the $c = 1$ / $D = 2$ string correspondence. This is seen precisely in the form of what is known as the regular Radon transform.

Let us describe procedure formulated in [114, 115] for constructing the bi-local to space-time map. The method is based on construction of canonical transformations in phase space: bi-local (t_1, p_1), (t_2, p_2) and $EAdS_2$ (τ, p_τ), (z, p_z). We consider the Poincare coordinates for the Euclidean AdS_2 space

$$ds^2 \,=\, \frac{d\tau^2 + dz^2}{z^2} \,. \tag{7.15}$$

One way to obtain the bi-local map is to equate the $SL(2, \mathcal{R})$ generators.

$$\hat{J}_{1+2} \,=\, \hat{J}_{\text{EAdS}} \,. \tag{7.16}$$

The one-dimensional bi-local conformal generators are

$$\hat{D}_{1+2} \,=\, t_1\, p_1 + t_2\, p_2 \,, \qquad \hat{P}_{1+2} \,=\, -p_1 - p_2 \,, \qquad \hat{K}_{1+2} \,=\, -\, t_1^2\, p_1 \,-\, t_2^2\, p_2 \,, \tag{7.17}$$

and the $EAdS_2$ generators are given by

$$\hat{D}_{\text{EAdS}} \,=\, \tau\, p_\tau + z\, p_z \,, \qquad \hat{P}_{\text{EAdS}} \,=\, -p_\tau \,, \qquad \hat{K}_{\text{EAdS}} \,=\, (z^2 - \tau^2)\, p_\tau \,-\, 2\tau z\, p_z \,, \tag{7.18}$$

where we defined $p_1 \equiv -\partial_{t_1}$, $p_2 \equiv -\partial_{t_2}$, $p_\tau \equiv -\partial_\tau$, $p_z \equiv -\partial_z$. Equating the generators, we can determine the map. From the $\hat{P}$ generators, we have $p_\tau = p_1 + p_2$. Using this result for the other generators, we get two equations to solve:

$$\begin{aligned} z\, p_z \,&=\, (t_1 - \tau) p_1 \,+\, (t_2 - \tau) p_2 \\ -\, z^2\, p_\tau \,&=\, (t_1 - \tau)^2 p_1 \,+\, (t_2 - \tau)^2 p_2 \,. \end{aligned} \tag{7.19}$$

These are solved by

$$\tau = \frac{t_1 p_1 - t_2 p_2}{p_1 - p_2}, \quad p_\tau = p_1 + p_2, \quad z^2 = -\left(\frac{t_1 - t_2}{p_1 - p_2}\right)^2 p_1 p_2, \quad p_z^2 = -4 p_1 p_2. \tag{7.20}$$

One can see that the canonical commutators are preserved under the transform (at least classically, i.e. in terms of the Poisson bracket). Namely, $[\tau, p_\tau] = [z, p_z] = 1$ and others vanish provided that $[t_i, p_j] = \delta_{ij}$, with $(i, j = 1, 2)$. Hence, we conclude the map is canonical transformation, which is also a point transformation in momentum space. For the kernel which implements this momentum space correspondence we can take

$$\mathcal{R}(p_1, p_2; p_\tau, p_z) = \frac{\delta(p_\tau - (p_1 + p_2))}{\sqrt{p_z^2 + 4 p_1 p_2}}. \tag{7.21}$$

Through Fourier transforming all momenta to corresponding coordinates, the associated coordinate space kernel becomes [6]

$$\mathcal{R}(t_1, t_2; \tau, z) = \delta(\eta^2 - (\tau - t)^2 - z^2). \tag{7.22}$$

With a multiplicative factor of additional of η this is known as the Circular Radon transform (7.25) which has a simple relationship to Radon transform on H^2.

There is another construction of the Radon transform which is used in [116–118] and is based on integration over geodesics. For the Euclidean AdS_2 space-time (7.15), a geodesic is given by a semicircle

$$(\tau - \tau_0)^2 + z^2 = \frac{1}{E^2}, \tag{7.23}$$

where $\tau = \tau_0$ is the center of the semicircle and $1/E$ is the radius. The Radon transform of a function of the bulk coordinates $f(\tau, z)$ is a function of the parameters of a geodesic (E, τ_0) defined by

$$\big[\mathcal{R}f\big](E, \tau_0) \equiv \int_\gamma ds\, f(\tau, z(\tau)), \tag{7.24}$$

where the integral is over the geodesic. From the geodesic equation (7.23), this transform is explicitly written as

$$\big[\mathcal{R}f\big](\eta, t) = 2\eta \int_{t-\eta}^{t+\eta} d\tau \int_0^\infty \frac{dz}{z}\, \delta\Big(\eta^2 - (\tau - t)^2 - z^2\Big)\, f(\tau, z), \tag{7.25}$$

where we have used the identifications $1/E = \eta$ and $\tau_0 = t$; the resulting function $[\mathcal{R}f](\eta, t)$ is understood as a function on the Lorentzian dS_2 (7.3).

We will now explicitly evaluate the Radon transformation of (unit-normalized) EAdS_2 wave functions (see Appendix G)

$$\overline{\phi}_{\mathrm{EAdS}_2}(\tau, z) = \alpha_\nu\, z^{\frac{1}{2}}\, e^{-i\omega\tau}\, K_\nu(|\omega| z) \tag{7.26}$$

[6] Here, we have ignored possible issues related to the range of variables.

From the above formula of the Radon transform (7.25), we get

$$\left[\mathcal{R}\overline{\phi}_{\mathrm{EAdS}_2}\right](\eta,t) \;=\; \alpha_\nu\,\eta\int_{t-\eta}^{t+\eta}\frac{d\tau}{\eta^2-(\tau-t)^2}\;(\eta^2-(\tau-t)^2)^{\frac14}\,e^{-i\omega\tau}\,K_\nu(|\omega|\sqrt{\eta^2-(\tau-t)^2})\,. \tag{7.27}$$

Now shifting the integral variable $\tau\to\tau+t$ and using the symmetry of the integrand, one can rewrite this integral as

$$\left[\mathcal{R}\overline{\phi}_{\mathrm{EAdS}_2}\right](\eta,t) \;=\; 2\,\alpha_\nu\,\eta\,e^{-i\omega t}\int_0^\eta d\tau\,\left(\frac{1}{\eta^2-\tau^2}\right)^{\frac34}\cos(\omega\tau)\,K_\nu(|\omega|\sqrt{\eta^2-\tau^2})\,. \tag{7.28}$$

Further rewriting the $\cos(\omega\tau)$ in terms of $J_{-1/2}(\omega\tau)$ and changing the integration variable to $\tau=\eta\sin\theta$, we find

$$\begin{aligned}\left[\mathcal{R}\overline{\phi}_{\mathrm{EAdS}_2}\right](\eta,t) \;&=\; \sqrt{\frac{\pi^3}{2}}\,\frac{\alpha_\nu|\omega|^{\frac12}\eta}{\sin(\pi\nu)}\,e^{-i\omega t}\\ &\times\int_0^{\frac{\pi}{2}}d\theta\,(\tan\theta)^{\frac12}\,J_{-\frac12}(|\omega|\eta\sin\theta)\left[I_{-\nu}(|\omega|\eta\cos\theta)\;-\;I_\nu(|\omega|\eta\cos\theta)\right],\end{aligned} \tag{7.29}$$

where we decomposed the modified Bessel function of the second kind into two first kinds. This θ integral is indeed given in Eq.(4) of $12\cdot 11$ of [131], which leads to

$$\left[\mathcal{R}\overline{\phi}_{\mathrm{EAdS}_2}\right](\eta,t) \;=\; -2i\sqrt{\pi}\,\frac{\Gamma(\frac14+\frac{\nu}{2})}{\Gamma(\frac34+\frac{\nu}{2})}\,\beta_\nu\,\eta^{\frac12}\,e^{-i\omega t}\left[J_\nu(|\omega|\eta)\;+\;\frac{\tan\frac{\pi\nu}{2}+1}{\tan\frac{\pi\nu}{2}-1}\,J_{-\nu}(|\omega|\eta)\right], \tag{7.30}$$

where we also used Eq.(G.7). The inside of the square bracket precisely agrees with the particular combination of Bessel functions, $Z_\nu(|\omega|\eta)$ function defined in Eq.(5.12).

When $\nu_n=3/2+2n$ the second term in this square bracket vanishes. As will be clear soon, we need the radon transform of the modified Bessel function I_{ν_n} with. This can be likewise evaluated to yield

$$\mathcal{R}[\alpha'_{\nu_n}z^{1/2}e^{-ik\tau}I_{\nu_n}(|k|z)] = (2\nu_n\eta)^{1/2}e^{-ikx}J_{\nu_n}(|k|\eta) \tag{7.31}$$

where

$$\alpha'_{\nu_n} = \left(\frac{2\nu_n}{\pi}\right)^{\frac12}\frac{\Gamma(\frac34+\frac{\nu_n}{2})}{\Gamma(\frac14+\frac{\nu_n}{2})} \tag{7.32}$$

The extra ν-dependent factor in (7.30) which appears in front of the unit-normalized dS_2 wave function described in Appendix G should be understood as a leg factor (7.34). As we will see later, this is analogous to what happens in the $c=1$ matrix model [120–124].

In summary, we have the Radon transform

$$\mathcal{R}\,\overline{\phi}^{(\mathrm{EAdS}_2)}_{\omega,\nu}(\tau,z) \;=\; L(\nu)\,\overline{\psi}^{(\mathrm{dS}_2)}_{\omega,\nu}(\eta,t)\,, \tag{7.33}$$

where $\overline{\phi}_{\mathrm{EAdS}_2}$ and $\overline{\psi}_{\mathrm{dS}_2}$ are the unit-normlized wave functions defined in Eq.(G.1) and Eq.(G.4), respectively, while the leg factor is defined by

$$L(\nu) \;\equiv\; (\text{Leg Factor}) \;=\; -2i\sqrt{\pi}\,\frac{\Gamma(\frac14+\frac{\nu}{2})}{\Gamma(\frac34+\frac{\nu}{2})}\,. \tag{7.34}$$

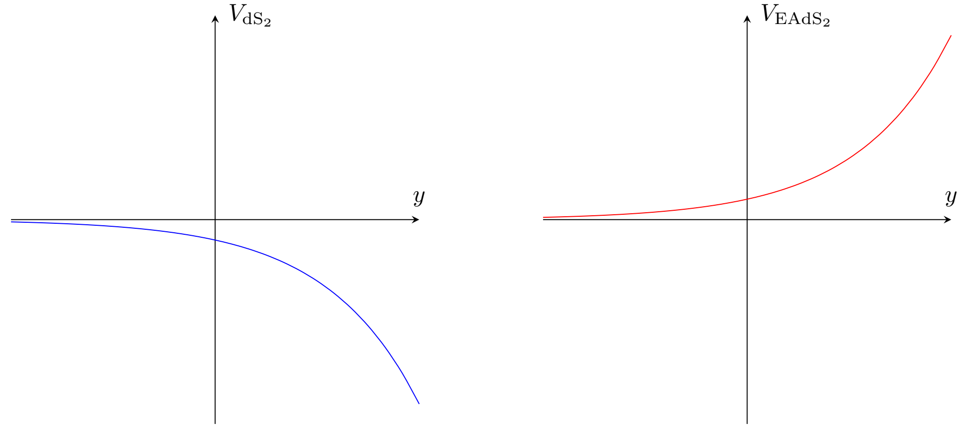

Figure 2. The de Sitter potential $V_{\rm dS_2}$ has bound states and scattering states. On the other hand, the Euclidean AdS potential $V_{\rm AdS_2}$ has only scattering modes.

The inverse transformations are

$$\mathcal{R}^{-1}\,\overline{\psi}^{(\rm dS_2)}_{\omega,\nu}(\eta,t) \;=\; L^{-1}(\nu)\,\overline{\phi}^{(\rm EAdS_2)}_{\omega,\nu}(\tau,z)\,. \tag{7.35}$$

for $\nu \neq 3/2+2n$, while for $\nu = 3/2+2n$ we have instead

$$\mathcal{R}^{-1}\,\overline{\psi}^{(\rm dS_2)}_{\omega,\nu_n}(\eta,t) \;=\; \alpha'_{\nu_n} z^{1/2} e^{-ik\tau} I_{\nu_n}(|k|z) \tag{7.36}$$

Under the Radon transform $\mathcal{R}$, the Laplacian of Lorentzian $\rm dS_2$ is transformed into that of Euclidean $\rm AdS_2$:

$$\Box_{\rm ds_2}\,\psi_{\rm dS_2}(\eta,t) \;=\; -\,\mathcal{R}\,\Box_{\rm EAdS_2}\,\phi_{\rm EAdS_2}(\tau,z)\,, \tag{7.37}$$

with

$$\Box_{\rm ds_2} \;=\; \eta^2(-\partial_\eta^2+\partial_t^2)\,, \qquad \Box_{\rm EAdS_2} \;=\; z^2(\partial_\tau^2+\partial_z^2)\,. \tag{7.38}$$

Here, $\psi_{\rm dS_2} = \mathcal{R}\phi_{\rm EAdS_2}$ and wave functions are not normalized.

In the rest of this section, we will show that the Radon transformation flips the sign of the potential appearing in the equivalent Schrodinger problem as formulated in [9]. We start from the Radon transformation (7.37). Expanding the wave functions by

$$\begin{aligned}\psi_{\rm dS_2}(\eta,t) \;&=\; \eta^{\frac12}\sum_\omega e^{-i\omega t}\,\widetilde{\psi}_{\rm dS_2}(\eta;k)\,,\\ \phi_{\rm EAdS_2}(\tau,z) \;&=\; z^{\frac12}\sum_\omega e^{-i\omega\tau}\,\widetilde{\phi}_{\rm EAdS_2}(\omega;z)\,,\end{aligned} \tag{7.39}$$

we have corresponding Bessel equations for $\widetilde{\psi}_{\rm dS_2}$ and $\widetilde{\phi}_{\rm EAdS_2}$. By changing the coordinates by $y \equiv \log(\omega\eta)$ or $y \equiv \log(\omega z)$, these Bessel equations are reduced to the Schrodinger equations as

$$\Big(-\partial_y^2 - e^y \Big)\, \widetilde{\psi}_{\rm dS_2} \;=\; -\nu^2\, \widetilde{\psi}_{\rm dS_2}\,,$$
$$\Big(-\partial_y^2 + e^y \Big)\, \widetilde{\phi}_{\rm EAdS_2} \;=\; -\nu^2\, \widetilde{\phi}_{\rm EAdS_2}\,. \tag{7.40}$$

Therefore, the Radon transform flips the sign of the corresponding Schrodinger potential (see FIG. 2). The de Sitter potential $V_{\rm dS_2} = -e^y$ has bound states as well as scattering states. On the other hand, the Euclidean AdS potential $V_{\rm AdS_2} = e^y$ has only scattering modes.

7.3 Green's Functions and Leg Factors

In this subsection, we start from the SYK bi-local propagator (5.21). Using the inverse Radon transformation (7.35), we will show that the resulting propagator can be written in terms of functions which appear in $EAdS_2$, except for the leg-factors.

The SYK bi-local propagator is given by

$$G(t_1,t_2;t_1',t_2') \;\propto\; J^{-1} \int_{-\infty}^{\infty} d\omega \sum_{\nu} \frac{u^*_{\nu,\omega}(t_1,t_2)u_{\nu,\omega}(t_1',t_2')}{N_\nu[\widetilde{g}(\nu)-1]}\,, \tag{7.41}$$

where $u_{\nu,\omega}$ are the eigenfunctions defined in Eq.(5.13). Here the summation over ν is a short-hand notation denotes the discrete mode sum and the continuous mode sum. Now, we identify $\eta = (t_1 - t_2)/2$ and $t = (t_1 + t_2)/2$. Then, the propagator can be written in terms of the dS$_2$ wave functions as

$$G(\eta,t;\eta',t') \;=\; 2\pi J^{-1} \int_{-\infty}^{\infty} d\omega \Bigg\{ \sum_{n=0}^{\infty} \frac{4\sin\pi\nu_n}{\widetilde{g}(\nu_n)-1}\; \overline{\psi}^{\,*}_{\omega,\nu_n}(\eta,t)\, \overline{\psi}_{\omega,\nu_n}(\eta',t') + \int_0^\infty dr\; \frac{\overline{\psi}^{\,*}_{\omega,\nu}(\eta,t)\, \overline{\psi}_{\omega,\nu}(\eta',t')}{\widetilde{g}(\nu)-1}\bigg|_{\nu=ir} \Bigg\}, \tag{7.42}$$

where $\nu_n = 2n + 3/2$. Next, we use the inverse Radon transform (7.35) to bring the dS wave functions into the EAdS wave functions.

$$G(\tau,z;\tau',z') \;=\; 2\pi J^{-1} \int_{-\infty}^{\infty} d\omega \Bigg\{ \sum_{n=0}^{\infty} \frac{4\sin\pi\nu_n}{\widetilde{g}(\nu_n)-1}\; |L^{-1}(\nu_n)|^2\; \overline{\phi}^{\,*}_{\omega,\nu_n}(\tau,z)\, \overline{\phi}_{\omega,\nu_n}(\tau',z') + \int_0^\infty dr\; |L^{-1}(\nu)|^2\; \frac{\overline{\phi}^{\,*}_{\omega,\nu}(\tau,z)\, \overline{\phi}_{\omega,\nu}(\tau',z')}{\widetilde{g}(\nu)-1}\bigg|_{\nu=ir} \Bigg\}. \tag{7.43}$$

Here we have *defined* $\overline{\phi}_{\omega,\nu_n}(\tau,z)$ by

$$\overline{\phi}_{\omega,\nu_n}(\tau,z) = \alpha'_{\nu_n} z^{1/2} e^{-ik\tau} I_{\nu_n}(|k|z) \tag{7.44}$$

Note that $\overline{\phi}_{\omega,\nu_n}(\tau,z)$ is not really a $EAdS$ wavefunction.

We can directly evaluate the continuous mode summation for the full Green's function including the leg factors in the integrand. However, we postpone this for a while and we first demonstrate how to extract the leg factors from the integrand and evaluate the integral without the leg factors. Writing the ν's appearing in the leg factors in terms of the Bessel differential operators, we can pull out the leg factors from the summation over m or the integral of r as

$$G(\tau,z;\tau',z') \;=\; 2\pi J^{-1}\left|L^{-1}(\hat{p}_{\mathrm{EAdS}_2})\right|^2 \int_{-\infty}^{\infty} d\omega \Bigg\{ \sum_{n=0}^{\infty} \frac{4\sin\pi\nu_n}{\widetilde{g}(\nu_n)-1}\, \overline{\phi}^{\,*}_{\omega,\nu_n}(\tau,z)\,\overline{\phi}_{\omega,\nu_n}(\tau',z') + \int_0^\infty dr\, \frac{\overline{\phi}^{\,*}_{\omega,\nu}(\tau,z)\,\overline{\phi}_{\omega,\nu}(\tau',z')}{\widetilde{g}(\nu)-1}\bigg|_{\nu=ir} \Bigg\}. \tag{7.45}$$

with

$$\hat{p}_{\mathrm{EAdS}_2} \;\equiv\; \sqrt{\Box_{\mathrm{EAdS}_2}+\frac{1}{4}}\,, \tag{7.46}$$

where the Laplacian of EAdS_2 is defined in Eq.(7.38). The above expression for the leg factor differential operators is slightly ambiguous. What we mean is that one of the leg factor differential operator is acting on (τ, z) and the other leg factor operator is acting on (τ', z'). To make this more explicit, we can introduce two delta functions $\delta(z-z_1)\delta(z'-z_2)$ and integrals over z_1 and z_2, where one of the leg factor is acting on $\delta(z-z_1)$ and the other is acting on $\delta(z'-z_2)$. Furthermore, we can rewrite the delta functions in terms of the completeness of the Bessel function (H.1). Therefore, now the propagator is written as

$$G(\tau,z;\tau',z') \;=\; \int_0^\infty \frac{dz_1}{z_1}\int_0^\infty \frac{dz_2}{z_2}\, L^*(z;z_1)\, G_{\mathrm{EAdS}_2}(\tau,z_1;\tau',z_2)\, L(z_2;z')\,, \tag{7.47}$$

with

$$G_{\mathrm{EAdS}_2}(\tau,z;\tau',z') \;=\; 2\pi J^{-1}\int_{-\infty}^{\infty} d\omega \Bigg\{ \sum_{n=0}^{\infty} \frac{4\sin\pi\nu_n}{\widetilde{g}(\nu_n)-1}\, \overline{\phi}^{\,*}_{\omega,\nu_n}(\tau,z)\,\overline{\phi}_{\omega,\nu_n}(\tau',z') + \int_0^\infty dr\, \frac{\overline{\phi}^{\,*}_{\omega,\nu}(\tau,z)\,\overline{\phi}_{\omega,\nu}(\tau',z')}{\widetilde{g}(\nu)-1}\bigg|_{\nu=ir} \Bigg\}, \tag{7.48}$$

and we defined the leg factor integral kernel as

$$L(z_1;z_2) \;\equiv\; \frac{i}{\pi^{5/2}}\int_0^\infty dr\, r\sinh(\pi r)\, \frac{\Gamma(\frac{3}{4}+\frac{ir}{2})}{\Gamma(\frac{1}{4}+\frac{ir}{2})}\, K_{ir}(z_1)\, K_{ir}(z_2)\,. \tag{7.49}$$

Let us now evaluate the continuous mode summation in the EAdS_2 propagator (7.48). Evaluating this integral as a contour integral as in [11, 106], there is only one set of poles coming from $\nu = p_m$, with $(m = 0, 1, 2, \cdots)$ which are the solutions of $\widetilde{g}(p_m) = 1$. Therefore, after this integral the EAdS_2 propagator (7.48) becomes

$$G_{\mathrm{EAdS}_2}(\tau,z;\tau',z') \;=\; \frac{2}{J}\,|zz'|^{\frac{1}{2}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(\tau-\tau')} \Bigg\{ \sum_{m=0}^{\infty} \frac{p_m}{\widetilde{g}'(p_m)}\, K_{p_m}(|\omega|z^{>})\, I_{p_m}(|\omega|z^{<}) + \frac{4}{\pi^2}\sum_{n} \frac{\nu_n}{\widetilde{g}(\nu_n)-1}\, K_{\nu_n}(|\omega|z)\, K_{\nu_n}(|\omega|z') \Bigg\}. \tag{7.50}$$

As we will see in the next section this form of the propagator is analogous to the Wilson loop (or macroscopic loop) operator propagators in the $c = 1$ matrix model (7.61).

Now we go back to the off-shell expression of the propagator (7.43) and evaluate the continuous mode summation for the full Green's function with including the leg factors in the integrand:

$$\mathcal{I}_{\rm cont} \equiv \int_0^\infty dr\, |L^{-1}(\nu)|^2 \, \frac{\overline{\phi}^*_{\omega,\nu}(\tau,z)\,\overline{\phi}_{\omega,\nu}(\tau',z')}{\widetilde{g}(\nu)-1}\bigg|_{\nu=ir} . \tag{7.51}$$

We evaluate this integral as a contour integral as before. We note that since the modified Bessel function K_ν is regular on the entire ν-complex plane, we have two sets of poles: (i). $\nu = p_m$, with $(m = 0, 1, 2, \cdots)$. (ii). $\nu = \nu_n = 2n + 3/2$, with $(n = 0, 1, 2, \cdots)$ where $\Gamma(\frac{3}{4} - \frac{\nu}{2}) = \infty$. After evaluating the residues at these poles, we find the integral as

$$\begin{aligned}\mathcal{I}_{\rm cont} = \frac{|zz'|^{\frac{1}{2}}}{4\pi^2}\, e^{-i\omega(\tau-\tau')} \Bigg\{ &\sum_{m=0}^\infty \frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})}\, \frac{p_m}{\widetilde{g}'(p_m)}\, K_{p_m}(|\omega|z^>)I_{p_m}(|\omega|z^<) \\ &+ \frac{2}{\pi}\sum_{n=0}^\infty \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1}\right) K_{\nu_n}(|\omega|z^>)I_{\nu_n}(|\omega|z^<)\Bigg\} . \end{aligned}\tag{7.52}$$

The second line in the RHS looks similar to the discrete mode contribution to the propagator (7.43). However, these two contributions do not cancel each other. Hence there are two types of the contributions to the final result as

$$\begin{aligned} &G(\tau,z;\tau',z') \\ &= \frac{|zz'|^{\frac{1}{2}}}{2\pi J}\int_{-\infty}^\infty d\omega\, e^{-i\omega(\tau-\tau')} \Bigg\{ \sum_{m=0}^\infty \frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})}\, \frac{p_m}{\widetilde{g}'(p_m)}\, K_{p_m}(|\omega|z^>)I_{p_m}(|\omega|z^<) \\ &\quad + \sum_{n=0}^\infty \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1}\right) I_{\nu_n}(|\omega|z^<)\Big[2I_{\nu_n}(|\omega|z^>) - I_{-\nu_n}(|\omega|z^>)\Big]\Bigg\} . \end{aligned}\tag{7.53}$$

Of course, here we still have the zero mode ($p_0 = 3/2$) problem coming from $\Gamma(\frac{3}{4} - \frac{p_0}{2}) = \infty$. In this expression, the Bessel function part of the first contribution in the RHS is the standard form for EAdS propagator, while the extra factor coming from the leg-factors can be possibly understood as a contribution from the naively pure gauge degrees of freedom as in the $c = 1$ model (c.f. [121–124]), in which case the second contribution in RHS represents the contribution from these modes as in [120].

In section 6, we presented the 3D picture of the SYK theory, based on the fact that the non-trivial spectrum predicted by the model, which are solutions of $\widetilde{g}(p_m) = 1$ with $(m = 0, 1, 2, \cdots)$ can be reproduced through Kaluza-Klein mechanism in one higher dimension. This picture is more natural in the AdS_2 interpretation of the bi-local space. Now, we will point out a similarity between the 3D picture of the SYK model [106, 107] and the $c = 1$ Liouville theory (2D string theory) [119–124].

In the 3D description we have a scalar field Φ

$$S_{\rm 3D} = \frac{1}{2}\int dx^3 \sqrt{-g}\Big[- g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m_0^2\Phi^2 - V(y)\Phi^2\Big] , \tag{7.54}$$

with a background metric

$$ds^2 \;=\; \frac{-dt^2 + d\hat{z}^2}{\hat{z}^2} \;+\; \left(1 + \frac{a}{\hat{z}}\right)^2 dy^2\,, \tag{7.55}$$

where $a \sim J^{-1}$, but here we only consider the leading in $1/J$ contribution and suppress the subleading contributions coming from the yy-component of the metric. The detail of the potential $V(y)$ depends on q and for that readers should refer to [106, 107]. The propagator for the scalar field in this background in the leading order of $1/J$ is given by

$$G^{(0)}(\hat{z}, t, y; \hat{z}', t', y') \;=\; |\hat{z}\hat{z}'|^{\frac{1}{2}} \sum_k f_k(y) f_k(y') \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \int \frac{d\nu}{N_\nu}\, \frac{Z^*_\nu(|\omega\hat{z}|)\, Z_\nu(|\omega\hat{z}'|)}{\nu^2 - k^2}\,, \tag{7.56}$$

where $f_k(y)$ is the wave function along the third direction y with momentum k. This is simply a rewriting the propagator (7.42) by treating the non-local kernel (eigenvalue) by an extra dimension. The identical procedure leads to the leg-factors. After the (inverse) Radon transform and the contour integral for the continuous mode sum, the propagator is reduced to

$$\begin{aligned} &G^{(0)}_{\omega;-\omega}(z, y; z', y') \\ &= \frac{|zz'|^{\frac{1}{2}}}{4\pi} \sum_k f_k(y) f_k(y') \left\{ \frac{\Gamma(\frac{3}{4} + \frac{k}{2})\Gamma(\frac{3}{4} - \frac{k}{2})}{\Gamma(\frac{1}{4} + \frac{k}{2})\Gamma(\frac{1}{4} - \frac{k}{2})}\, K_k(|\omega| z^>) I_k(|\omega| z^<) \right. \\ &\qquad \left. +\; 2\sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4} + \frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4} + \frac{\nu_n}{2})} \left(\frac{\nu_n}{\nu_n^2 - k^2}\right) I_{\nu_n}(|\omega| z^<) \Big[2 I_{\nu_n}(|\omega| z^>) - I_{-\nu_n}(|\omega| z^>)\Big] \right\}. \end{aligned} \tag{7.57}$$

On the other hand, for the $c = 1$ matrix model / 2D string duality, the Wilson loop operator is related to the matrix eigenvalue density field ϕ by

$$W(t, \ell) \;\equiv\; \mathrm{Tr}\Big(e^{-\ell M(t)}\Big) \;=\; \int_0^\infty dx\, e^{-\ell x}\, \phi(t, x)\,. \tag{7.58}$$

The corresponding propagator was found by Moore and Seiberg [120] as

$$\Big\langle w(t, \varphi) w(t', \varphi') \Big\rangle \;=\; \int_{-\infty}^{\infty} dE \int_0^\infty dp\; \frac{p}{\sinh \pi p}\, \frac{\phi^*_{E,p}(t, \varphi) \phi_{E,p}(t', \varphi')}{E^2 - p^2}\,, \tag{7.59}$$

with $\ell = e^{-\varphi}$ and the normalized wave function

$$\phi_{E,p}(t, \varphi) \;=\; \sqrt{p \sinh \pi p}\, e^{-iEt}\, K_{ip}(\sqrt{\mu} e^{-\varphi})\,. \tag{7.60}$$

After evaluating the p-integral as a contour integral, we obtain the propagator as

$$\begin{aligned} \Big\langle w(t, \varphi) w(t', \varphi') \Big\rangle \;=\; -\pi \int_{-\infty}^{\infty} dE\, e^{-iE(t-t')} &\left\{ \frac{\pi E}{2 \sinh \pi E}\, K_{iE}(\sqrt{\mu} e^{-\varphi^<})\, I_{iE}(\sqrt{\mu} e^{-\varphi^>}) \right. \\ &\left. +\; \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{E^2 + n^2}\, K_n(\sqrt{\mu} e^{-\varphi^<})\, I_n(\sqrt{\mu} e^{-\varphi^>}) \right\}. \end{aligned} \tag{7.61}$$

The point we want to make here is that this 3D picture is completely parallel to the $c = 1$ Liouville theory (2D string theory) [119–124]. Namely, if we make a change of coordinate by $z = e^{-\varphi}$, then the φ-direction becomes the Liouville direction, while the y-direction (at least in the leading order of $1/J$) can be understood as the $c = 1$ matter direction. In this comparison, the τ-direction serves as an extra direction. Finally, the ν appearing in the SYK model is realized as a momentum k along the y-direction in the 3D picture (7.55). Therefore, we have the following correspondence between the $c = 1$ Liouville theory and the 3D picture of the SYK model.

$c = 1$	3D SYK
$ie^{-\varphi}$	z
$-it$	y
ip	ν
iE	k
$\sqrt{\mu}$	$\lvert\omega\rvert$

8 Conclusion

In this thesis we described the bi-local formulation of the SYK model by defining a reparametrization invariant collective theory at the IR point and away from it. A regularized action representing an interacting theory between a Schwarzian coordinate and bi-local matter is specified. We presented the non-linear derivation of the Schwarzian action, which is exact at all orders. It generates perturbative calculations in the SYK model around the conformal IR point which are systematic in the inverse of the strong coupling J. We gave the evaluation of the tree level free energy in this expansion. Even though, the present calculations are done at tree level in $1/N$, the formalism given allows for loop level calculations with no difficulty: by projection of the zero mode the perturbation expansion is well defined, while the Jacobian(s) of the changes of variables provide exact counter terms which are expected to cancel infinities appearing in loop diagrams.

Next, we have presented a three dimensional perspective of the bulk dual of the SYK model. At strong coupling we showed that the spectrum and the propagator of the bi-local field can be exactly reproduced by that of a scalar field living in $AdS_2 \times S^1/Z_2$ with a delta function potential at the center. The metric on the interval in the third direction is the dilaton of Jackiw-Teitelboim theory, which is a constant at strong coupling. We also calculated the leading $1/J$ correction to the propagator which comes from the corresponding term in the metric in the third direction, and showed that form of the poles of the propagator are consistent with the results of the SYK model [10].

Finally we addressed the question of what represents the bulk dual space-time of the SYK model. At the outset, the question seems simple since the small fluctuations of the (Euclidean) SYK model are completely given by a set of Lorentzian wave functions associated with the $SL(2, \mathcal{R})$ isometry group. With a simple identification of space-time, these are seen to associated with eigenfunctions in de-Sitter (or Anti de-Sitter) space-time (as was discussed in [7, 9–11]). Likewise the propagator and higher point n-functions continue to feature this Lorentzian space-time structure.

Even though this Lorentzian bulk dual interpretation seems to be straightforwardly associated with the SYK bi-local data, we have stressed that there is a problem with this interpretation. Concentrating on the effective bi-local Large N version of the model, we have provided a resolution, which follows from a further non-local redefinition of space-time. This comes in terms of Leg transformations of Green's functions which places the theory in Euclidean AdS dual setting. Such transformations are actually characteristic of collective field representations of Large N theories. The leg transformations that we explicitly implement (apart from providing the EAdS_2 space-time setting) also bring out the couplings of additional "discrete" states.

We expect that all features of the SYK model we presented here will play a central role in full identification of the bulk dual for the SYK model.

A ϵ-Expansion

In this appendix, we will exemplify how to obtain the non-linear Schwarzian action (2.13) associated with the naive form of the breaking term in Eq.(2.5). This is done by using ε-expansion with $q = 2/(1-\varepsilon)$ and treating ε as a small parameter. We note that for any q in the range of $2 \leq q \leq \infty$, the value of ε is $0 \leq \varepsilon \leq 1$. Therefore, the convergence of this ϵ-expansion is guaranteed. Even though we use the ε-expansion, we can nevertheless calculate all order contributions of ε as we will see below. We first rewrite the critical solution in the following way:

$$\begin{aligned}\Psi_{0,f}(t_1,t_2) \;=\; & -\frac{1}{\pi J}\left(\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right) \\ & \times\left[1-\varepsilon\,\log\left(\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right)+\frac{\varepsilon^2}{2}\left(\log\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right)^2+\cdots\right],\end{aligned} \tag{A.1}$$

where the first term is the contribution from $q = 2$ case, which leads to the result Eq.(2.13) with $\alpha = 1$. To evaluate higher order ε contributions, we use the following expansions of the logarithm in the $t_1 \to t_2$ limit:

$$\log\left(\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right) = -\log|t_1-t_2| - \frac{1}{8}\frac{|f''(t_2)|^2}{|f'(t_2)|^2}|t_1-t_2|^2 + \frac{1}{12}\frac{|f'''(t_2)|}{|f'(t_2)|}|t_1-t_2|^2 + \cdots . \tag{A.2}$$

The first log term gives an f-independent divergent term which we will eliminate in the following. One also expands the factor representing $q = 2$ reparametrized critical solution and then one finds $\mathcal{O}(\varepsilon) = 0$. For order $\mathcal{O}(\varepsilon^2)$ contribution, from Eq.(A.2), one can find

$$\begin{aligned}\mathcal{O}(\varepsilon^2) \;&=\; -\frac{N\varepsilon^2}{4\pi J}\int dt_1\,\partial_1\left[\left(\frac{1}{4}\frac{|f''(t_2)|^2}{|f'(t_2)|^2} \;-\; \frac{1}{6}\frac{|f'''(t_2)|}{|f'(t_2)|}\right)|t_1-t_2|\,\log|t_1-t_2|\right]_{t_2\to t_1}\\ &=\; \frac{N\varepsilon^2}{24\pi J}\int dt_1\left[\frac{f'''(t_1)}{f'(t_1)} \;-\; \frac{3}{2}\left(\frac{f''(t_1)}{f'(t_1)}\right)^2\right], \end{aligned} \tag{A.3}$$

where we again eliminated the divergence term and used integration by parts. Hence, the total contribution up to $\mathcal{O}(\varepsilon^2)$ for $q = 2/(1-\varepsilon)$ action is given by

$$S[f] \;=\; -\frac{N\alpha}{24\pi J}\int dt\left[\frac{f'''(t)}{f'(t)} \;-\; \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2\right], \tag{A.4}$$

where

$$\alpha(\varepsilon) \;=\; 1 \;-\; \varepsilon^2 \;+\; \mathcal{O}(\varepsilon^3)\,. \tag{A.5}$$

In fact, there is no higher order contributions from $\mathcal{O}(\varepsilon^3)$. This can be seen from an expansion

$$\begin{aligned}&\left(\log\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right)^n\\ =\;&\Big(-\log|t_1-t_2|\Big)^n \;-\; n\left(\frac{1}{8}\frac{|f''(t_2)|^2}{|f'(t_2)|^2} \;-\; \frac{1}{12}\frac{|f'''(t_2)|}{|f'(t_2)|}\right)|t_1-t_2|^2\Big(-\log|t_1-t_2|\Big)^{n-1} \;+\;\cdots\,. \end{aligned} \tag{A.6}$$

This expansion together with the expansion of $q = 2$ reparametrized critical solution does not give any non-zero finite contribution to the action after the limit when $n \geq 3$. Namely, the $(\log|t_1-t_2|)^n$ factor gives a strong divergence when n is large. However, if one wants to lower the power of this logarithm, then one gets a higher power of $|t_1-t_2|^n$, which strongly vanishes after setting $t_2 = t_1$. This naive form of α will turn out to be renormalized with our regularization of the breaking term. We evaluate this renormalized coefficient in the next appendix.

B s-Regularization and Schwarzian Action

In this appendix, we will directly evaluate the collective coordinate action with the regularized breaking term:

$$S[f] \;=\; \frac{N}{2}\int\,\big[\Psi_{0,f}\big]_s \;=\; -\frac{N}{2}\lim_{s\to\frac{1}{2}}\int dt_1dt_2\,\Psi_{0,f}(t_1,t_2)\,Q_s(t_1,t_2)\,, \tag{B.1}$$

and confirm that the result is given by

$$S[f] \;=\; -\frac{N\alpha}{24\pi J}\int dt\left[\frac{f'''(t)}{f'(t)} \;-\; \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2\right], \tag{B.2}$$

with a coefficient

$$\alpha \,=\, -12\pi B_1 \gamma\,. \tag{B.3}$$

For this purpose, we expand the reparametrized critical solution with $f(t) = t + \varepsilon(t)$ as

$$\begin{aligned}\Psi_{0,f}(t_1,t_2) \,=\, \Psi_0(t_{12}) \,+\, \int dt_a\, \varepsilon(t_a)\, u_{0,t_a}(t_1,t_2) \,+\, \frac{1}{2}\int dt_a dt_b\, \varepsilon(t_a)\varepsilon(t_b)\, u_{(1),t_a,t_b}(t_1,t_2) \\ +\, \frac{1}{6}\int dt_a dt_b dt_c\, \varepsilon(t_a)\varepsilon(t_b)\varepsilon(t_c)\, u_{(2),t_a,t_b,t_c}(t_1,t_2) \,+\, \cdots\,,\end{aligned} \tag{B.4}$$

where we defined

$$\begin{aligned} u_{0,t_a}(t_1,t_2) \,&\equiv\, \left.\frac{\partial \Psi_{0,f}(t_{12})}{\partial f(t_a)}\right|_{f(t)=t}\,, \\ u_{(2),t_a,t_b}(t_1,t_2) \,&\equiv\, \left.\frac{\partial^2 \Psi_{0,f}(t_{12})}{\partial f(t_a)\partial f(t_b)}\right|_{f(t)=t}\,, \\ u_{(3),t_a,t_b,t_c}(t_1,t_2) \,&\equiv\, \left.\frac{\partial^3 \Psi_{0,f}(t_{12})}{\partial f(t_a)\partial f(t_b)\partial f(t_c)}\right|_{f(t)=t}\,, \\ u_{(n),\{t_a,\cdots,t_n\}}(t_1,t_2) \,&\equiv\, \left.\frac{\partial^n \Psi_{0,f}(t_{12})}{\partial f(t_a)\cdots\partial f(t_n)}\right|_{f(t)=t}\,. \end{aligned} \tag{B.5}$$

Let us first consider the quadratic and cubic order contributions. Taking derivatives and expressing in the momentum space, the quadratic and cubic coefficients are given by

$$\begin{aligned} u_{(2),\omega_a,\omega_b}(t_1,t_2) =\, &\frac{2}{q}\, e^{i(\omega_a+\omega_b)t_+} \left[|\omega_a\omega_b| \cos((\omega_a+\omega_b)t_-) \,-\, \frac{1}{|t_-|^2}\sin|\omega_a t_-|\sin|\omega_b t_-|\right]\, \Psi_0(t_{12}) \\ -\, &\frac{2}{\pi q^2}\, e^{i(\omega_a+\omega_b)t_+}\, |\omega_a\omega_b|^{\frac{1}{2}} |t_-|\, J_{\frac{3}{2}}(|\omega_a t_-|)\, J_{\frac{3}{2}}(|\omega_b t_-|)\, \Psi_0(t_{12})\,, \end{aligned} \tag{B.6}$$

and

$$\begin{aligned} &u_{(3),\omega_a,\omega_b,\omega_c}(t_1,t_2) \\ &= -\frac{4i}{q}\, e^{i(\omega_a+\omega_b+\omega_c)t_+} \left[|\omega_a\omega_b\omega_c| \cos((\omega_a+\omega_b+\omega_c)t_-) \,-\, \frac{1}{|t_-|^3}\sin|\omega_a t_-|\sin|\omega_b t_-|\sin|\omega_c t_-|\right]\, \Psi_0(t_{12}) \\ &- \frac{4i}{q^2}\sqrt{\frac{\pi|\omega_c^3 t_-|}{2}} e^{i(\omega_a+\omega_b+\omega_c)t_+} \left[|\omega_a\omega_b| \cos((\omega_a+\omega_b)t_-) \,-\, \frac{1}{|t_-|^2}\sin|\omega_a t_-|\sin|\omega_b t_-|\right]\, J_{\frac{3}{2}}(|\omega_c t_-|)\, \Psi_0(t_{12}) \\ &+ \frac{8i}{q^3}\left|\frac{\pi t_-}{2}\right|^{\frac{3}{2}} e^{i(\omega_a+\omega_b+\omega_c)t_+}\, |\omega_a\omega_b\omega_c|^{\frac{3}{2}}\, J_{\frac{3}{2}}(|\omega_a t_-|)\, J_{\frac{3}{2}}(|\omega_b t_-|)\, J_{\frac{3}{2}}(|\omega_c t_-|)\, \Psi_0(t_{12})\,. \end{aligned} \tag{B.7}$$

In fact, there are two more terms in the second line of RHS in $u_{(3)}$ obtained by permutations of $(\omega_a,\omega_b,\omega_c)$, but we omitted these terms in the above expression. Substituting these expressions into the action (B.1) and performing the t_1, t_2 integrals, one finds single poles

$(s-1/2)^{-1}$ coming from the double sine term in $u_{(2)}$ and from the triple sine term in $u_{(3)}$. Namely for the quadratic contribution, we have

$$\begin{aligned} &- \int dt_1 dt_2 dt_3 dt_4 \, u_{(2),\omega_a,\omega_b}(t_1,t_2)\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_1(t_3,t_4) \\ &= 2^{2-2s}\pi \left(\frac{q-1}{q}\right) B_1 b^{q-1} \gamma(s,q)\, \delta(\omega_a+\omega_b) \int_0^\infty \frac{dt_-}{|t_-|^{4+2s}} \sin^2|\omega_a t_-| \,, \end{aligned} \tag{B.8}$$

with

$$\int_0^\infty dx \, \frac{\sin^2 x}{x^{4+2s}} = \frac{1}{6}\frac{1}{s-\frac{1}{2}} + \mathcal{O}((s-\tfrac{1}{2})^0)\,. \tag{B.9}$$

Also for the cubic contribution

$$\begin{aligned} &- \int dt_1 dt_2 dt_3 dt_4 \, u_{(3),\omega_a,\omega_b,\omega_c}(t_1,t_2)\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_1(t_3,t_4) \\ &= -2^{3-2s}\pi i \left(\frac{q-1}{q}\right) B_1 b^{q-1} \gamma(s,q)\, \delta(\omega_a+\omega_b+\omega_c) \int_0^\infty \frac{dt_-}{|t_-|^{5+2s}} \sin|\omega_a t_-| \, \sin|\omega_b t_-| \, \sin|\omega_c t_-| \,, \end{aligned} \tag{B.10}$$

with

$$\begin{aligned} &\int_0^\infty \frac{dt_-}{|t_-|^{5+2s}} \sin|\omega_a t_-| \, \sin|\omega_b t_-| \, \sin|\omega_c t_-| \\ &= |\omega_a\,\omega_b\,\omega_c| \left(|\omega_a|^2+|\omega_b|^2+|\omega_c|^2\right) \frac{1}{12(s-\frac{1}{2})} + \mathcal{O}((s-\tfrac{1}{2})^0)\,. \end{aligned} \tag{B.11}$$

There are no other terms giving such $(s-1/2)^{-1}$ pole. Such single pole factor $(s-1/2)^{-1}$ cancels with the $(s-1/2)$ factor in the regularized source Q_s (3.17), and lead to

$$- \int dt_1 dt_2 dt_3 dt_4 \, u_{(2),t_a,t_b}(t_1,t_2)\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_1(t_3,t_4) = B_1\,\gamma\, \partial_{t_a}^2 \partial_{t_b}^2 \delta(t_{ab})\,, \tag{B.12}$$

and

$$\begin{aligned} &- \int dt_1 dt_2 dt_3 dt_4 \, u_{(3),t_a,t_b,t_c}(t_1,t_2)\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_1(t_3,t_4) \\ &= B_1\,\gamma\, \partial_{t_a}\partial_{t_b}\partial_{t_c}\left(\partial_{t_a}^2+\partial_{t_b}^2+\partial_{t_c}^2\right) \delta(t_{ac})\delta(t_{bc})\,. \end{aligned} \tag{B.13}$$

With the experience of quadratic and cubic order computations, now we would like to evaluate all order contributions. As we saw above the poles associated to the limit $s\to 1/2$ only come from the double and triple sine terms. Therefore, we expect this structure is also true for any higher order contributions. Taking derivatives of the reparameterized critical solution, we find such term in n-th order is given by

$$u_{(n),\{\omega_a,\cdots,\omega_n\}}(t_1,t_2) = \frac{2}{q}(-i)^n (n-1)!\, e^{i(\omega_a+\cdots+\omega_n)t_+} \left(\prod_{i=a}^{n} \sin|\omega_i t_-|\right) \frac{\Psi_0(t_{12})}{|t_-|^n} + \cdots\,, \tag{B.14}$$

where the ellipsis denotes non-singular terms in the limit $s \to 1/2$. Now, using the result (3.17), one obtains the contribution from the n-th order as

$$
\begin{aligned}
&- \int dt_1 dt_2 dt_3 dt_4 \, u_{(n),\{\omega_a,\cdots,\omega_n\}}(t_1,t_2)\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_1(t_3,t_4) \\
&= -12\pi(-i)^n (n-1)!\,\gamma B_1\,(s-\tfrac{1}{2})\delta(\omega_a+\cdots+\omega_n) \int_0^\infty \frac{dt_-}{|t_-|^{2+2s+n}} \left(\prod_{i=a}^n \sin|\omega_i t_-|\right) + \mathcal{O}(s-\tfrac{1}{2})\,.
\end{aligned}
\tag{B.15}
$$

The t_--integral is given by

$$
\int_0^\infty \frac{dt_-}{|t_-|^{2+2s+n}} \prod_{i=1}^n \sin|\omega_i t_-| \;=\; \left(\prod_{i=1}^n |\omega_i|\right)\left(\sum_{i=1}^n |\omega_i|^2\right) \frac{1}{12(s-\frac{1}{2})} + \mathcal{O}((s-\tfrac{1}{2})^0)\,. \tag{B.16}
$$

Now, using this result and Fourier transforming back to $\{t_a,\cdots,t_n\}$ from $\{\omega_a,\cdots,\omega_n\}$, we get

$$
\begin{aligned}
&- \int dt_1 dt_2 dt_3 dt_4 \, u_{(n),\{t_a,\cdots,t_n\}}(t_1,t_2)\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_1(t_3,t_4) \\
&= \frac{(n-1)!}{2} B_1\gamma \left(\prod_{i=a}^n \partial_{t_i}\right)\left(\sum_{i=a}^n \partial_{t_i}^2\right) \delta(t_{an})\cdots\delta(t_{n-1,n})\,,
\end{aligned}
\tag{B.17}
$$

where we have already taken $s \to 1/2$ limit.

Finally, together with the expansion (B.4), one can see that the n-th order contribution to the collective coordinate action (2.13) is given by

$$
\begin{aligned}
S[f] &= \frac{NB_1\gamma}{4nJ} \int dt_1\cdots dt_n\, \varepsilon(t_1)\cdots\varepsilon(t_n) \left(\prod_{i=1}^n \partial_{t_i}\right)\left(\sum_{i=1}^n \partial_{t_i}^2\right) \delta(t_{1n})\cdots\delta(t_{n-1,n}) \\
&= \frac{NB_1\gamma}{2J} \int dt\, \frac{(-1)^n}{2} \varepsilon'''(t)\Big(\varepsilon'(t)\Big)^{n-1}\,.
\end{aligned}
\tag{B.18}
$$

This result can be summed over for all order to get

$$
S[f] \;=\; \frac{NB_1\gamma}{2J} \int dt\, \mathrm{Sch}(f;t)\,, \tag{B.19}
$$

where

$$
\mathrm{Sch}(f;t) \;=\; \frac{f'''(t)}{f'(t)} - \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2\,. \tag{B.20}
$$

To see this correspondence, one first rewrites the Schwarzian derivative by integration by parts as

$$
\int dt\, \mathrm{Sch}(f;t) \;=\; -\frac{1}{2}\frac{f'''(t)}{f'(t)}\,. \tag{B.21}
$$

Then, we use $f(t) = t + \varepsilon(t)$ and expand the Schwarzian derivative by powers of ε as

$$
\int dt\, \mathrm{Sch}(f;t) \;=\; \int dt \sum_{n=1}^\infty \frac{(-1)^n}{2}\, \varepsilon'''(t)\Big(\varepsilon'(t)\Big)^{n-1}\,. \tag{B.22}
$$

This expansion completely agrees with the result found in Eq.(B.18).

Finally as a reference, we give a relation of our coefficients to the coefficients α_S and α_G defined in [10]:

$$- \frac{\alpha}{12\pi} = B_1\gamma = -2\alpha_S \left(\frac{J}{\mathcal{J}}\right) = b\,\gamma\,\alpha_G \left(\frac{J}{\mathcal{J}}\right) , \tag{B.23}$$

where $\mathcal{J} = \frac{\sqrt{q}}{2^{\frac{q-1}{2}}} J$.

C Schrodinger Equation

In this appendix, we consider the equation of $f(y)$, which is the Schrödinger equation:

$$\Big[-\partial_y^2 \pm V\delta(y)\Big] f(y) = E\, f(y) , \tag{C.1}$$

where E is an eigenvalue of the equation. Since we confined the field in $-L < y < L$, we have boundary conditions: $f(\pm L) = 0$. The continuation conditions at $y = 0$ are $f(+0) = f(-0)$ and the other can be derived by integrating the Schrödinger equation (C.1) over $(-\varepsilon, \varepsilon)$ and taking limit $\varepsilon \to 0$ as

$$f'(+0) - f'(-0) = \pm V\, f(0) . \tag{C.2}$$

Since the potential of the Schrödinger equation is even function, the wave function is either odd or even function of y.

(i) odd: For odd parity case, a solution satisfying the boundary conditions at $y = \pm L$ is given by

$$f(y) = \begin{cases} A\sin(k(y-L)) & (0 < y < L) \\ A\sin(k(y+L)) & (-L < y < 0) \end{cases} \tag{C.3}$$

where $k^2 = E$. For odd parity solution, to satisfy the boundary condition $f(+0) = f(-0)$, we need $f(\pm 0) = 0$. This implies that

$$k = \frac{\pi n}{L} , \qquad (n = 1, 2, 3, \cdots) \tag{C.4}$$

Then, the continuity condition (C.2) is automatically satisfied. The normalization constant is fixed as $A = 1/\sqrt{L}$.

(ii) even: For even parity case, a solution satisfying the boundary conditions at $y = \pm L$ is given by

$$f(y) = \begin{cases} B\sin(k(y-L)) & (0 < y < L) \\ -B\sin(k(y+L)) & (-L < y < 0) \end{cases} \tag{C.5}$$

where $k^2 = E$. The evenness of the parity guarantees $f(-0) = f(+0)$. So, we only need to impose the condition (C.2) on this solution. This condition gives an equation

$$\mp \frac{2}{V} k = \tan(kL) . \tag{C.6}$$

Now we set $L = \pi/2$ and $V = 3$, then we have $\mp(2/3)k = \tan(\pi k/2)$, which is precisely the same transcendental equation determining poles of the $q = 4$ SYK bi-local propagator (5.22). We denote the solutions of $-(2/3)k = \tan(\pi k/2)$ by p_m, $(2m-1 < p_m < 2m)$, $(m = 1, 2, 3, \cdots)$, and the solutions of $(2/3)k = \tan(\pi k/2)$ by q_m, $(2m-2 < q_m < 2m-1)$, $(m = 1, 2, 3, \cdots)$. The normalization constant is fixed as

$$B \;=\; \sqrt{\frac{2k}{2kL - \sin(2kL)}}\,. \tag{C.7}$$

Finally, let us prove the orthogonality of the parity even wave function (C.5):

$$\int_{-L}^{L} dy\, f_m(y) f_{m'}(y) \;=\; \delta_{m,m'}\,. \tag{C.8}$$

Using the solution (C.5) and evaluating the integral in the left-hand side, one obtains

$$B^2 \left[\frac{\sin(L(k-k'))}{k-k'} \;-\; \frac{\sin(L(k+k'))}{k+k'} \right]. \tag{C.9}$$

Now let's assume $k \neq k'$. Then, the integral result can be rearranged to the form of

$$\frac{B^2}{k^2 - k'^2}\, \cos(Lk) \cos(Lk') \Big[k' \tan(Lk) \;-\; k \tan(Lk') \Big] \;=\; 0\,, \tag{C.10}$$

where the final equality is due to the relation $\tan(LK) = -2k/3$. Next, we consider $k = k'$ case. In this case, due to the delta function identity, the result (C.9) is reduced to

$$B^2 \left[L \;-\; \frac{\sin(2Lk)}{2k} \right] \delta_{k,k'} \;=\; \delta_{k,k'}\,, \tag{C.11}$$

where for the equality we used Eq.(C.7). Therefore, now we have proven the orthogonality (C.8).

D Completeness Condition of Z_ν

In this appendix, we give a derivation of the completeness condition (D.6), which is used to determine the zero-th order propagator (6.18). The linear combination of the Bessel functions is defined by [9]

$$Z_\nu(x) \;=\; J_\nu(x) \,+\, \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu \;=\; \frac{\tan(\pi\nu/2) + 1}{\tan(\pi\nu/2) - 1}\,, \tag{D.1}$$

which satisfies the Bessel equation

$$\left[\, z^2 \,\partial_z^2 \,+\, z\,\partial_z \,+\, \omega^2\, z^2 \,\right] Z_\nu(|\omega z|) \;=\; \nu^2\, Z_\nu(|\omega z|)\,. \tag{D.2}$$

In [9], the orthogonality condition of the linear combination of the Bessel function Z_ν (D.1) is given by

$$\int_0^\infty \frac{dx}{x}\, Z_\nu^*(x)\, Z_{\nu'}(x) \;=\; N_\nu\, \delta(\nu-\nu')\,, \tag{D.3}$$

where N_ν is defined in (5.15)

Since Z_ν is complete, one can expand any function on the basis of Z_ν. In particular, we are interested in a delta function

$$\delta(x-x') \;=\; \int d\nu\, \gamma_\nu(x)\, Z_\nu(|x'|)\,, \tag{D.4}$$

where $\gamma_\nu(x)$ is the coefficient of the expansion and the integral symbol of ν is a short-hand notation of a combination of summation over $\nu = 3/2 + 2n$, $(n = 0, 1, , 2\cdots)$ and integration of $\nu = ir$, $(r > 0)$. One can fix the coefficient $\gamma_\nu(x)$ by multiplying $Z_{\nu'}^*(|x'|)/x'$ to Eq.(D.4) and integrating over x' with Eq.(D.3) as

$$\gamma_\nu(x) \;=\; \frac{Z_\nu^*(|x|)}{N_\nu x}\,. \tag{D.5}$$

Therefore, finally we find

$$\int \frac{d\nu}{N_\nu}\, Z_\nu^*(|x|)\, Z_\nu(|x'|) \;=\; x\,\delta(x-x')\,. \tag{D.6}$$

E Evaluation of the Contour Integral

In this appendix, we give a detail evaluation of the continuous and the discrete sums appearing in Eq.(6.19). As we defined before, the integral symbol $d\nu$ is a short-hand notation of a combination of summation over $\nu = 3/2 + 2n$, $(n = 0, 1, 2, \cdots)$ and integration of $\nu = ir$, $(r > 0)$. Namely,

$$\int \frac{d\nu}{N_\nu}\, \frac{Z_\nu^*(|\omega z|)\, Z_\nu(|\omega z'|)}{\nu^2 - p_m^2} \;=\; I_1 \,+\, I_2\,, \tag{E.1}$$

with

$$\begin{aligned} I_1 \;&\equiv\; \sum_{n=0}^{\infty} \frac{2\nu}{\nu^2-p_m^2}\, J_\nu(|\omega z|)\, J_\nu(|\omega z'|)\Big|_{\nu=\frac{3}{2}+2n}\,, \\ I_2 \;&\equiv\; -\int_0^\infty \frac{dr}{2\sinh(\pi r)}\, \frac{r}{r^2+p_m^2}\, Z_{ir}^*(|\omega z|)\, Z_{ir}(|\omega z'|)\,. \end{aligned} \tag{E.2}$$

Let us evaluate the continuous sum I_2 first. Using the symmetry of the integrand, one can rewrite the integral as

$$I_2 \;=\; -\frac{i}{2}\int_{-i\infty}^{i\infty} \frac{d\nu}{\sin(\pi\nu)}\, \frac{\nu}{\nu^2-p_m^2}\, \Big[J_{-\nu}(|\omega z|) \,+\, \xi_{-\nu}\, J_\nu(|\omega z|)\Big]\, J_\nu(|\omega z'|)\,. \tag{E.3}$$

We evaluate this integral by a contour integral on the complex ν plane by closing the contour in the Re(ν)> 0 half of the complex plane if $z > z'$. Inside of this contour, we have two types of the poles. (i) at $\nu = p_m$ coming from the coefficient factor. (ii) at $\nu = 3/2 + 2n$, $(n = 0, 1, 2, \cdots)$ coming from $\xi_{-\nu}$, where $\xi_{-\nu} = \infty$. After evaluating residues at these poles, one obtains

$$I_2 = -\frac{\pi}{2\sin(\pi p_m)}\Big[J_{-p_m}(|\omega z|) + \xi_{-p_m} J_{p_m}(|\omega z|)\Big] J_{p_m}(|\omega z'|) - \sum_{n=0}^{\infty} \frac{2\nu}{\nu^2 - p_m^2} J_\nu(|\omega z|) J_\nu(|\omega z'|)\Big|_{\nu = \frac{3}{2}+2n} . \tag{E.4}$$

Now, one can notice that the second term exactly cancels with the contribution from I_1. One can also repeat the above discussion for $z' > z$ case. Therefore, combining these two cases the total contribution is now

$$I_1 + I_2 = -\frac{\pi}{2\sin(\pi p_m)}\left[J_{-p_m}(|\omega| z^>) + \left(\frac{p_m + \frac{3}{2}}{p_m - \frac{3}{2}}\right) J_{p_m}(|\omega| z^>)\right] J_{p_m}(|\omega| z^<) , \tag{E.5}$$

where $z^> (z^<)$ is the greater (smaller) number among z and z'. Then, the propagator is reduced to

$$G^{(0)}(t, z, 0; t', z', 0) = \frac{1}{4} |zz'|^{\frac{1}{2}} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')} \frac{B_m^2}{\sin(\pi p_m)} \frac{p_m^2}{p_m^2 + (3/2)^2} \times \left[J_{-p_m}(|\omega| z^>) + \left(\frac{p_m + \frac{3}{2}}{p_m - \frac{3}{2}}\right) J_{p_m}(|\omega| z^>)\right] J_{p_m}(|\omega| z^<) . \tag{E.6}$$

This agrees with the result given in Eq.(6.22).

F EAdS Scalar Propagators

In this Appendix, we summarize basic aspects of Euclidean AdS scalar propagators.

Let us consider a scalar field

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right] , \tag{F.1}$$

in Euclidean AdS_{d+1} Poincare coordinates

$$ds^2 = \frac{dx_i^2 + dz^2}{z^2} . \tag{F.2}$$

A propagator satisfies the scalar Green's equation

$$(\Delta + m^2)\, G(x_i, z\,; x_i', z') = \frac{1}{\sqrt{g}} \delta^d(x_i - x_i')\, \delta(z - z') , \tag{F.3}$$

where

$$\Delta \equiv -\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, g^{\mu\nu}\, \partial_\nu) . \tag{F.4}$$

More explicitly, one can write down the equation as

$$\Big[- z^2\,\partial_z^2 \,+\, (d-1)z\,\partial_z - z^2\,\partial_i^2 \,+\, m^2 \Big]\, G(x_i, z\,; x_i', z') \;=\; z^{d+1}\,\delta^d(x_i - x_i')\,\delta(z - z')\,. \tag{F.5}$$

Excluding the scaling behavior of the Green's function by $G(x_i, z\,; x_i', z') = z^{\frac{d}{2}} g(x_i, z\,; x_i', z')$, the equation is reduced to

$$\Big[z^2\,\partial_z^2 \,+\, z\,\partial_z \,+\, z^2\,\partial_i^2 \,-\, \nu^2 \Big]\, g(x_i, z\,; x_i', z') \;=\; -z^{\frac{d+2}{2}}\,\delta^d(x_i - x_i')\,\delta(z - z')\,, \tag{F.6}$$

with $\nu^2 \equiv m^2 + d^2/4$. In the following, we will give two different expressions of the off-shell scalar propagators and show that these two form of propagators are indeed equivalent.

F.1 p-Integral Form

In this subsection, following the discussion in Appendix A of [136], we consider p-integral form of the propagator. First we rewrite the Green's equation (F.6) as

$$\Big[\Delta_\nu \,+\, \partial_i^2 \Big]\, g(x_i, z\,; x_i', z') \;=\; -z^{\frac{d-2}{2}}\,\delta^d(x_i - x_i')\,\delta(z - z')\,. \tag{F.7}$$

where

$$\Delta_\nu \;\equiv\; \partial_z^2 \,+\, \frac{1}{z}\partial_z \,-\, \frac{\nu^2}{z^2}\,. \tag{F.8}$$

The differential operator Δ_ν has eigenvalues

$$\Delta_\nu\, J_\nu(pz) \;=\; -\,p^2\, J_\nu(pz)\,. \tag{F.9}$$

The Bessel function completeness condition is given by

$$\delta(z - z') \;=\; z' \int_0^\infty dp\, p\, J_\nu(pz)\, J_\nu(pz')\,. \tag{F.10}$$

Now, we expand the rescaled Green's function on the Bessel function basis with coefficients $\tilde{g}_{p,\vec{k}}$ as

$$g(x_i, z\,; x_i', z') \;=\; \int \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot\vec{x}} \int_0^\infty dp\, p\, J_\nu(pz)\, \tilde{g}_{p,\vec{k}}(x_i', z')\,. \tag{F.11}$$

Plugging this expression into the Green's equation (F.7), one can fix the coefficients as

$$\tilde{g}_{p,\vec{k}}(x_i, z) \;=\; \frac{z^{\frac{d}{2}}}{p^2 + (\vec{k})^2}\; e^{-i\vec{k}\cdot\vec{x}'}\; J_\nu(pz)\,. \tag{F.12}$$

Hence the off-shell form of the Green's function is given by

$$G(x_i, z\,; x_i', z') \;=\; (zz')^{\frac{d}{2}} \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int_0^\infty dp\, \frac{p}{p^2 + (\vec{k})^2}\, J_\nu(pz)\, J_\nu(pz')\,. \tag{F.13}$$

The p integral is evaluated in Appendix F.3. The result is given by

$$\int_0^\infty dp\, \frac{p}{p^2+(\vec{k})^2}\, J_\nu(pz)\, J_\nu(pz') \;=\; K_\nu(kz^>)\, I_\nu(kz^<)\,, \tag{F.14}$$

where $k \equiv \sqrt{\vec{k}^2}$ and $z^>(z^<)$ is the greater (smaller) number among z and z'. Therefore, the Green's function is now

$$G(x_i, z\,; x_i', z') \;=\; (zz')^{\frac{d}{2}} \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, K_\nu(kz^>)\, I_\nu(kz^<)\,. \tag{F.15}$$

F.2 ν-Integral Form

In this subsection, we express the Green's equation (F.6) as

$$\Big[\Delta_k - \nu^2\Big]\, g(x_i, z\,; x_i', z') \;=\; -z^{\frac{d+2}{2}}\, \delta^d(x_i - x_i')\, \delta(z-z')\,. \tag{F.16}$$

where

$$\Delta_k \;\equiv\; z^2\,\partial_z^2 \,+\, z\,\partial_z \,-\, z^2\,k^2\,. \tag{F.17}$$

The differential operator Δ_k has eigenvalues

$$\Delta_k\, K_{i\bar\nu}(kz) \;=\; -\,\bar\nu^2\, K_{i\bar\nu}(kz)\,. \tag{F.18}$$

The modified Bessel function completeness condition is given by

$$\int_0^\infty d\nu\, \nu \sinh(\pi\nu)\, K_{i\nu}(x)\, K_{-i\nu}(y) \;=\; \frac{\pi^2}{2}\, x\, \delta(x-y)\,. \tag{F.19}$$

Now, we expand the rescaled Green's function on the basis of the modified Bessel functions with coefficients $\tilde{g}_{\bar\nu,\vec{k}}$ as

$$g(x_i, z\,; x_i', z') \;=\; \int \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot\vec{x}} \int_0^\infty d\bar\nu\;\bar\nu\,\sinh(\pi\bar\nu)\, K_{i\bar\nu}(kz)\, \tilde{g}_{\bar\nu,\vec{k}}(x_i', z')\,. \tag{F.20}$$

Plugging this expression back into the Green's equation (F.16), one can fix the coefficients as

$$\tilde{g}_{\bar\nu,\vec{k}}(x_i', z') \;=\; \frac{2}{\pi^2}\, \frac{z'^{\frac{d}{2}}}{\bar\nu^2+\nu^2}\, e^{-i\vec{k}\cdot\vec{x}'}\, K_{-i\bar\nu}(kz')\,. \tag{F.21}$$

Hence the off-shell form of the Green's function is given by

$$G(x_i, z\,; x_i', z') \;=\; \frac{2}{\pi^2}\,(zz')^{\frac{d}{2}} \int \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int_0^\infty d\bar\nu\, \frac{\bar\nu}{\bar\nu^2+\nu^2}\, \sinh(\pi\bar\nu)\, K_{i\bar\nu}(kz)\, K_{-i\bar\nu}(kz')\,. \tag{F.22}$$

The $\bar\nu$ integral is evaluated in Appendix F.4. The result is

$$\int_0^\infty d\bar\nu\, \frac{\bar\nu}{\bar\nu^2+\nu^2}\, \sinh(\pi\bar\nu)\, K_{i\bar\nu}(kz)\, K_{-i\bar\nu}(kz') \;=\; \frac{\pi^2}{2}\, K_\nu(kz^>)\, I_\nu(kz^<)\,. \tag{F.23}$$

Therefore, the Green's function is reduced to

$$G(x_i, z\,; x_i', z') \;=\; (zz')^{\frac{d}{2}} \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, K_\nu(kz^>)\, I_\nu(kz^<)\,. \tag{F.24}$$

This agrees with the result of the previous subsection.

F.3 p-Integral

In this subsection, we evaluate the p integral (F.14)

$$\mathcal{I} \equiv \int_0^\infty dp \, \frac{p}{p^2 + (\vec{k})^2} \, J_\nu(pz) \, J_\nu(pz') \,. \tag{F.25}$$

In order to use the contour integral, it is convenient to extend the integration region to

$$\mathcal{I} \,=\, \frac{1}{2} \int_{-\infty}^\infty dp \, \frac{|p|}{p^2 + (\vec{k})^2} \, J_\nu(|p|z) \, J_\nu(|p|z') \,. \tag{F.26}$$

Here, the absolute value is understood in the sense of real variables, not in the sense of complex variables (i.e. $|p| = p \, \text{sgn}(\text{Re}(p))$). Since the large argument behavior ($|z| \gg 1$) of the Bessel function is

$$J_\nu(z) \,=\, \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \,, \tag{F.27}$$

we cannot close the contour neither upper nor lower half of the complex p plane. To manage this, we decompose the Bessel function into the Hankel functions:

$$J_\nu(z) \,=\, \frac{1}{2}\Big(H_\nu^{(1)}(z) \,+\, H_\nu^{(2)}(z)\Big) \,. \tag{F.28}$$

This is because the Hankel functions have nice asymptotic behaviors in $|z| \gg 1$

$$\begin{aligned} H_\nu^{(1)}(z) \,&=\, \sqrt{\frac{2}{\pi z}} \exp\left[i\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right] \,, \\ H_\nu^{(2)}(z) \,&=\, \sqrt{\frac{2}{\pi z}} \exp\left[-i\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right] \,. \end{aligned} \tag{F.29}$$

Now, we have four terms in the p integral

$$\begin{aligned} \mathcal{I} \,=\, \frac{1}{8} \int_{-\infty}^\infty dp \, \frac{|p|}{p^2 + (\vec{k})^2} \Big[& H_\nu^{(1)}(|p|z) \, H_\nu^{(1)}(|p|z') \,+\, H_\nu^{(1)}(|p|z) \, H_\nu^{(2)}(|p|z') \\ & + \, H_\nu^{(2)}(|p|z) \, H_\nu^{(1)}(|p|z') \,+\, H_\nu^{(2)}(|p|z) \, H_\nu^{(2)}(|p|z') \Big] \,. \end{aligned} \tag{F.30}$$

We label each term by $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4\}$ in the order of appearing in (F.30). In order to pick up a pole at $p = \pm ik + 0^+$, where $k \equiv \sqrt{\vec{k}^2}$, the contour of the complex p integral is closed in upper or lower half of the complex plane for each term as

$$\mathcal{I}_1 \,=\, \text{upper} \qquad \mathcal{I}_2 \,=\, \begin{cases} \text{upper} & (z > z') \\ \text{lower} & (z < z') \end{cases} \qquad \mathcal{I}_3 \,=\, \begin{cases} \text{lower} & (z > z') \\ \text{upper} & (z < z') \end{cases} \qquad \mathcal{I}_4 \,=\, \text{lower} \,. \tag{F.31}$$

Evaluating the contour integral by residue theorem, each term gives

$$
\begin{aligned}
\mathcal{I}_1 &= \frac{\pi i}{8}\, H_\nu^{(1)}(ikz)\, H_\nu^{(1)}(ikz')\,, \\
\mathcal{I}_2 &= \begin{cases} +\frac{\pi i}{8}\, H_\nu^{(1)}(ikz)\, H_\nu^{(2)}(ikz') & (z > z') \\ -\frac{\pi i}{8}\, H_\nu^{(1)}(-ikz)\, H_\nu^{(2)}(-ikz') & (z < z') \end{cases} \\
\mathcal{I}_3 &= \begin{cases} -\frac{\pi i}{8}\, H_\nu^{(2)}(-ikz)\, H_\nu^{(1)}(-ikz') & (z > z') \\ +\frac{\pi i}{8}\, H_\nu^{(2)}(ikz)\, H_\nu^{(1)}(ikz') & (z < z') \end{cases} \\
\mathcal{I}_4 &= -\frac{\pi i}{8}\, H_\nu^{(2)}(-ikz)\, H_\nu^{(2)}(-ikz')\,.
\end{aligned} \tag{F.32}
$$

We note that we can combine $\mathcal{I}_2$ and $\mathcal{I}_3$ into more compact notation as

$$
\mathcal{I}_2 + \mathcal{I}_3 = \frac{\pi i}{8}\Big[H_\nu^{(1)}(ikz^>)\, H_\nu^{(2)}(ikz^<) - H_\nu^{(1)}(-ikz^<)\, H_\nu^{(2)}(-ikz^>)\Big]\,, \tag{F.33}
$$

where $z^>(z^<)$ is the greater (smaller) number among z and z'. Therefore, now the entire p integral is written as

$$
\mathcal{I} = \frac{\pi i}{4}\Big[H_\nu^{(1)}(ikz^>)\, J_\nu(ikz^<) - H_\nu^{(2)}(-ikz^>)\, J_\nu(-ikz^<)\Big]\,. \tag{F.34}
$$

Finally, using

$$
I_\nu(z) = i^{-\nu} J_\nu(iz) = i^{\nu} J_\nu(iz)\,, \tag{F.35}
$$

and

$$
K_\nu(z) = \frac{\pi}{2}\, i^{\nu+1} H_\nu^{(1)}(iz) = \frac{\pi}{2}\, i^{-\nu-1} H_\nu^{(2)}(-iz)\,, \tag{F.36}
$$

one obtains

$$
\mathcal{I} = K_\nu(kz^>)\, I_\nu(kz^<)\,. \tag{F.37}
$$

F.4 ν-Integral

In this Appendix, we evaluate the $\bar\nu$ integral (F.23)

$$
\mathcal{I} \equiv \int_0^\infty d\bar\nu\, \frac{\bar\nu}{\bar\nu^2 + \nu^2}\, \sinh(\pi\bar\nu)\, K_{i\bar\nu}(kz)\, K_{-i\bar\nu}(kz')\,. \tag{F.38}
$$

We use the same latter $\mathcal{I}$ as in previous section to denote different integrals, but it is obvious that $\mathcal{I}$ in Appendix F.3 defined in (F.25), while $\mathcal{I}$ in this section is defined in (F.38).

We first consider $z > z'$ case. It is convenient to decompose the second kind of modified Bessel functions into the first kind of modified Bessel functions by

$$
K_{i\bar\nu}(kz) = \frac{\pi i}{2}\, \frac{I_{i\bar\nu}(kz) - I_{-i\bar\nu}(kz)}{\sinh(\pi\bar\nu)}\,. \tag{F.39}
$$

Note that even though $\sinh(\pi\bar{\nu})$ in the denominator becomes zero at $\bar{\nu} = in$, $(n \in \mathbb{Z})$, these points are not poles of $K_{i\bar{\nu}}$, because $I_{-n}(x) = I_n(x)$ for $n \in \mathbb{Z}$. Now, the $\bar{\nu}$ integral is

$$\begin{aligned}
\mathcal{I} \;&=\; -\frac{\pi^2}{8}\int_{-\infty}^{\infty} d\bar{\nu}\,\frac{\bar{\nu}}{\bar{\nu}^2+\nu^2}\,\frac{1}{\sinh(\pi\bar{\nu})}\Big[I_{i\bar{\nu}}(kz) - I_{-i\bar{\nu}}(kz)\Big]\Big[I_{i\bar{\nu}}(kz') - I_{-i\bar{\nu}}(kz')\Big] \\
&=\; -\frac{\pi^2}{4}\int_{-\infty}^{\infty} d\bar{\nu}\,\frac{\bar{\nu}}{\bar{\nu}^2+\nu^2}\,\frac{1}{\sinh(\pi\bar{\nu})}\Big[I_{i\bar{\nu}}(kz) - I_{-i\bar{\nu}}(kz)\Big]\,I_{i\bar{\nu}}(kz') \\
&=\; \frac{\pi i}{2}\int_{-\infty}^{\infty} d\bar{\nu}\,\frac{\bar{\nu}}{\bar{\nu}^2+\nu^2}\,K_{i\bar{\nu}}(kz)\,I_{i\bar{\nu}}(kz')\,.
\end{aligned} \tag{F.40}$$

We note that in this combination of $K_{i\bar{\nu}}I_{i\bar{\nu}}$ is well defined in both $z \to \infty$ and $z' \to 0$ limits. This can be seen from the asymptotic behaviors of the Bessel functions

$$K_{i\bar{\nu}}(kz) \;\sim\; \sqrt{\frac{\pi}{2kz}}\,e^{-kz}\,, \qquad (z \to \infty) \tag{F.41}$$

$$I_{i\bar{\nu}}(kz') \;\sim\; \frac{1}{\Gamma(1+i\bar{\nu})}\left(\frac{kz'}{2}\right)^{i\bar{\nu}}\,, \qquad (z' \to 0) \tag{F.42}$$

The opposite combination is ill-defined because the limit $I_{i\bar{\nu}}(z \to \infty)$ leads to divergence. In order to evaluate the $\bar{\nu}$ integral (F.40) as a contour integral on the complex $\bar{\nu}$ plane, we need to decide whether we will close the contour in the upper half or lower half of the complex plane. This can be determined from (F.42). The limit $z' \to 0$ is converged only if we close the contour in the lower half of the complex $\bar{\nu}$ plane. Hence now the residue theorem at the pole $\bar{\nu} = -i\nu$ gives

$$\mathcal{I} \;=\; \frac{\pi^2}{2}\,K_\nu(kz)\,I_\nu(kz')\,. \tag{F.43}$$

This can be generalized to include $z' > z$ case also as

$$\mathcal{I} \;=\; \frac{\pi^2}{2}\,K_\nu(kz^>)\,I_\nu(kz^<)\,. \tag{F.44}$$

where $z^>(z^<)$ is the greater (smaller) number among z and z'.

G Unit Normalized EAdS/dS Wave Functions

The unit-normalized Euclidean AdS_2 wave function is given by

$$\overline{\phi}_{\mathrm{EAdS}_2}(\tau, z) \;=\; \alpha_\nu\, z^{\frac{1}{2}}\, e^{-i\omega\tau}\, K_\nu(|\omega| z)\,, \tag{G.1}$$

where the normalization factor can be chosen as

$$\alpha_\nu \;=\; i\sqrt{\frac{\nu\sin(\pi\nu)}{\pi^3}}\,. \tag{G.2}$$

Then from the Bessel K_ν orthogonality condition (H.2), the wave function is unit normalized:

$$\int_{-\infty}^{\infty} d\tau \int_0^{\infty} \frac{dz}{z^2}\, \overline{\phi}^*_{\omega,\nu}(\tau,z)\, \overline{\phi}_{\omega',\nu'}(\tau,z) \;=\; \delta(\omega-\omega')\delta(\nu-\nu')\,. \tag{G.3}$$

The Lorentzian dS_2 wave function is given by

$$\overline{\psi}_{dS_2}(\eta,t) \;=\; \beta_\nu\, \eta^{\frac{1}{2}}\, e^{-i\omega t}\, Z_\nu(|\omega|\eta)\,. \tag{G.4}$$

Here, let us only consider the continuous modes ($\nu = ir$). Now choosing the normalization factor as

$$\beta_\nu \;=\; \sqrt{\frac{\nu}{4\pi\sin(\pi\nu)}}\,, \tag{G.5}$$

then the wave function is unit normalized for the continuous modes as

$$\int_{-\infty}^{\infty} dt \int_0^{\infty} \frac{d\eta}{\eta^2}\, \overline{\psi}^*_{\omega,\nu}(\eta,t)\, \overline{\psi}_{\omega',\nu'}(\eta,t) \;=\; \delta(\omega-\omega')\delta(\nu-\nu')\,, \qquad \text{for } (\nu = ir) \tag{G.6}$$

We note that

$$\alpha_\nu \;=\; 2i\,\frac{\sin\pi\nu}{\pi}\,\beta_\nu\,. \tag{G.7}$$

H Completeness and Orthogonality of $K_{i\nu}$

In this appendix, we derive the completeness condition of modified Bessel function of the second kind

$$\int_0^{\infty} d\nu\, \nu\sinh(\pi\nu)\, K_{i\nu}(x)\, K_{i\nu}(y) \;=\; \frac{\pi^2}{2}\, x\,\delta(x-y)\,, \tag{H.1}$$

and the orthogonality condition

$$\int_0^{\infty} \frac{dx}{x}\, K_{i\nu}(x)\, K_{i\nu'}(x) \;=\; \frac{\pi^2}{2}\,\frac{\delta(\nu-\nu')}{\nu\sinh(\pi\nu)}\,, \tag{H.2}$$

where $x, y > 0$. We start with defining

$$\mathcal{I} \;\equiv\; \int_0^{\infty} d\nu\, \nu\sinh(\pi\nu)\, K_{i\nu}(x)\, K_{i\nu}(y)\,. \tag{H.3}$$

Using the integral representations of the modified Bessel function

$$\begin{aligned} K_{i\nu}(x) \;&=\; \frac{1}{\sinh(\frac{\pi\nu}{2})}\int_0^{\infty} ds\, \sin(x\sinh s)\, \sin(\nu s)\,, \\ K_{i\nu}(y) \;&=\; \frac{1}{\cosh(\frac{\pi\nu}{2})}\int_0^{\infty} dt\, \cos(y\sinh t)\, \cos(\nu t)\,, \end{aligned} \tag{H.4}$$

one can rewrite $\mathcal{I}$ as

$$\mathcal{I} \;=\; \frac{1}{4}\int_{-\infty}^{\infty} d\nu\,\nu \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\; \sin(\nu s)\, \cos(\nu t)\, \sin(x\sinh s)\, \cos(y\sinh t)\,, \tag{H.5}$$

where using the symmetry of ν, s and t, we extended the integration regions to $(-\infty,\infty)$. The ν integral is evaluated as

$$\int_{-\infty}^{\infty} d\nu\, \nu\, \sin(\nu s)\, \cos(\nu t) \;=\; -\pi\, \frac{\partial}{\partial s}\Big(\delta(s+t) \,+\, \delta(s-t)\Big)\,. \tag{H.6}$$

Therefore, now

$$\begin{aligned}
\mathcal{I} \;&=\; -\frac{\pi}{4}\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\; \frac{\partial}{\partial s}\Big(\delta(s+t) \,+\, \delta(s-t)\Big)\, \sin(x\sinh s)\, \cos(y\sinh t) \\
&=\; \frac{\pi}{4}\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\; \Big(\delta(s+t) \,+\, \delta(s-t)\Big)\, \frac{\partial}{\partial s}\sin(x\sinh s)\, \cos(y\sinh t) \\
&=\; \frac{\pi x}{2}\int_{-\infty}^{\infty} ds\;\; \cosh s\, \cos(x\sinh s)\, \cos(y\sinh s)\,.
\end{aligned} \tag{H.7}$$

Changing the integration variable to $u = \sinh s$,

$$\begin{aligned}
\mathcal{I} \;&=\; \frac{\pi x}{2}\int_{-\infty}^{\infty} du\, \cos(xu)\, \cos(yu) \\
&=\; \frac{\pi^2}{2}\, x\, \Big[\delta(x+y) \,+\, \delta(x-y)\Big]\,.
\end{aligned} \tag{H.8}$$

Since $x, y > 0$, we keep only $\delta(x-y)$ term and obtain the completeness condition Eq.(H.1).

The normalization of the orthogonality condition can be fixed by the completeness. Since $K_{i\nu}$ is complete, we can expand any analytical function on this basis. In particular,

$$\delta(\nu-\nu') \;=\; \int_0^{\infty} dx\, \gamma_{\nu'}(x)\, K_{i\nu}(x)\,, \tag{H.9}$$

with some coefficient $\gamma_{\nu'}$. Using the completeness (H.1), one can fix the coefficient as

$$\gamma_\nu(x) \;=\; \frac{2}{\pi^2 x}\; \nu \sinh(\pi\nu)\, K_{i\nu}(x)\,. \tag{H.10}$$

This leads to the orthogonality condition (H.2).

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