Boundary Layers for 2D Stationary Navier-Stokes Flows over a Moving Boundary

by

Sameer S. Iyer

Sc.B, Brown University; Providence, RI, 2012

A dissertation submitted in partial fulfillment of the

requirements for the degree of Doctor of Philosophy

in The Division of Applied Mathematics at Brown University

PROVIDENCE, RHODE ISLAND

May 2018

This dissertation by Sameer S. Iyer is accepted in its present form
by The Division of Applied Mathematics as satisfying the
dissertation requirement for the degree of Doctor of Philosophy.

Date____________________ ______________________________________

Yan Guo, Ph.D., Advisor

Recommended to the Graduate Council

Date____________________ ______________________________________

Govind Menon, Ph.D., Reader

Date____________________ ______________________________________

Benoit Pausader, Ph.D., Reader

Approved by the Graduate Council

Date____________________ ______________________________________

Andrew G. Campbell, Dean of the Graduate School

Vita

The author received his Sc. B. in Applied Mathematics from Brown University in May of 2012. In the Fall of 2013, he enrolled as a Ph.D. student in the Division of Applied Mathematics at Brown University. While at Brown, he has worked under the supervision of Professor Yan Guo. His preprints and publications include (Iye17a), (Iye16), (Iye17b), (IZ17), (IS16).

Preface and Acknowledgments

First and foremost, I would like to express my deepest gratitude to my advisor, Yan Guo. During the last four years, Yan has spent countless hours teaching me mathematics during which the subject of partial differential equations came alive and became ripe for investigation. Through his patience, dedication, generosity, and unbounded energy, Yan has taught me to strive for the highest ideals as a scientist and has instilled in me the notions of rigorous and relentless scientific pursuit. I am honored to be his student.

I would like to thank my thesis readers Govind Menon and Benoit Pausader. Govind taught my first ODE course (and several more along the way) and has been supportive ever since. Benoit has played a crucial role introducing me to exciting PDE topics which I foresee being a large part of my future career, and has continuously encouraged my pursuit of these areas. I am grateful to both for sharing, on many occasions, their perspectives on our discipline which has played a big role in shaping my own points of view.

I would like to thank Bjorn Sandstede for bringing me to the realm of applied analysis when I was an undergraduate, from whom I have learned many things, and with whom I have had a successful and exciting collaboration.

I would also like to thank the several other faculty at Brown from whom I have learned partial differential equations: Constantine Dafermos, Hongjie Dong, Justin Holmer, and Walter Strauss. Thanks to Basilis Gidas for always being available to share his unique perspectives on mathematics and for being supportive from the start. Thanks also to the many other teachers from my Brown undergraduate days who initially opened my eyes to higher mathematics, in particular Alex Kontorovich and Nimish Shah.

Thanks to my officemates Ian Alevy, Michael Burkhart, and Michael Snarski for valuable friendship during my time as a graduate student. Thanks also to the following people with whom I became friends while they were postdocs at Brown, and who have helped me along the way: Francesco di Plinio, Klaus Widmayer, and Sona Akopian. Thanks to the many people in the Division of Applied Mathematics at Brown who made the last five years a great time for me to develop as a scientist.

Finally, on a personal note, I would like to thank my Mom, Dad, and Briana for all the love and support. My dad was my very first teacher in mathematics who instilled in me a sense of curiosity, wonder, and marvel that has set the tone for my scientific pursuits. My mom has been a steadfast source of love and support from the beginning; a calm and reliable force in ever-changing circumstances and always cheering me on. Briana's love, companionship, and encouragement has been extraordinary, helping me always take fresh perspectives, to believe in myself, to pursue my passions, and to maintain strong positive energy throughout. Nothing would be possible without all of you!

Abstract of "Boundary Layers for 2D Stationary Navier-Stokes Flows over a Moving Boundary", by Sameer S. Iyer, Ph.D., Brown University, May 2018

In this thesis, we study Prandtl's boundary layer theory for 2D, stationary, incompressible Navier-Stokes flows posed on domains with boundaries. The boundary layer hypothesis posed by Prandtl in 1904 asserts that for flows with low viscosities, the effect of friction is prominent near physical boundaries and negligible in the bulk. Verifying this mathematically is, in general, an important open problem.

We establish the validity of Prandtl's hypothesis in three settings, all under the hypothesis of a translating boundary. The first setup we consider are flows on the exterior of a rotating disk. Here, the effect of the curved boundary poses the main obstacle to the analysis. Second, we study flows for which the tangential variable extends globally. Here, obtaining strong, integrable decay estimates is the main task. Third, we study boundary layers in the presence of a non-shear Euler flow. Here, a singularity of size $\varepsilon^{-1/2}$ is created at leading order, which we control by establishing estimates weighted from the far-field.

A fourth contribution of this thesis is the construction of a one-parameter family of stationary, 2D Navier-Stokes solutions which are arbitrarily close to the Couette flow.

Contents

List of Figures

Chapter One

Introduction

1.1 Boundary Layers and the Inviscid Limit:

In this thesis, I will study 2D, incompressible, stationary flows at low viscosities for which the Navier-Stokes equations read:

$$\left.\begin{aligned} &\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla P^\varepsilon = \varepsilon \Delta \mathbf{u}^\varepsilon \\ &\nabla \cdot \mathbf{u}^\varepsilon = 0 \\ &\mathbf{u}|_{\partial\Omega} = 0. \end{aligned}\right\} \tag{1.1.1}$$

Here, $\mathbf{u}^\varepsilon = (u^\varepsilon(x, Y), v^\varepsilon(x, Y)) : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field defined on an open set $\Omega \subset \mathbb{R}^2$. Describing the asymptotic behavior of (1.1.1) as the viscosity vanishes ($\varepsilon \downarrow 0$) in the presence of boundaries is a fundamental challenge in partial differential equations. This is due to the disparity between the no-slip boundary condition, $\mathbf{u}^\varepsilon|_{\partial\Omega} = 0$, for viscous flows and the no-penetration condition, $\mathbf{u}^0 \cdot \mathbf{n}|_{\partial\Omega} = 0$ for inviscid flows. Prandtl's boundary layer hypothesis, (Pra04), rectifies this boundary mismatch, and can be expressed mathematically as:

$$\mathbf{u}^\varepsilon(x, Y) = \mathbf{u}_e^0(x, Y) + \mathbf{u}_p^0(x, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon}\mathbf{u} \qquad \text{for } 0 < \varepsilon << 1, \tag{1.1.2}$$

where $\mathbf{u}_e^0 = (u_e^0, v_e^0)$ is a prescribed Euler flow, $\mathbf{u}_p^0 = (u_p^0, v_p^0)$ is a constructed boundary layer, and $\mathbf{u} = (u, v)$ is the remainder. For the purposes of being concrete, we will think of $Y = 0$ as the no-slip boundary, and thus the scaling $y = \frac{Y}{\sqrt{\varepsilon}}$ zooms in to a strip near the boundary. The remainder can be thought of as a function of either Y or the scaled variable $\frac{Y}{\sqrt{\varepsilon}}$. In this thesis, we will work in boundary layer coordinates, meaning we think of $\mathbf{u} = \mathbf{u}(x, y)$.

It is worthwhile to provide historical context regarding (1.1.2) (see (SG00) for more details). (1.1.2) was introduced by Prandtl in 1904, (Pra04). At the time, the Euler equations, which govern $\mathbf{u}_e^0$, were well known. Over the 1800's, the Eulerian theory was vigorously investigated by the likes of Kelvin, Helmholz, Rayleigh, Riemann, and many other influential mathematicians. Moreover, the Euler equations saw great agreement with certain physical experiments such as free surface ocean waves. However, the Euler theory saw great disparity with physical experiments which involved solid boundaries (such as pipe-flow). Because of this, experimentalists largely discarded the Euler theory.

The Navier-Stokes equations were also well known by the turn of the 1900's. However, they were also not being used in practice for two chief reasons. First, they were more complicated than the Euler equations from the point of view of special solutions. Second, the fluids that mattered the most were air and water, which have very low viscosity. Due to this, the general perception at the time was that the Navier-Stokes equations could not be very different from the Euler equations.

It was this that Prandtl disagreed with, and how he distinguished himself. He realized that, even for low viscosities, the Navier-Stokes and the Euler equations were very different in the presence of boundaries. He moreover characterized precisely how the difference should behave using the object $\mathbf{u}_p^0$ in (1.1.2). In so doing, he linked these equations, and also found great agreement to experiments. For this, he is credited as being the father of modern day aerodynamics.

An expansion of the form (1.1.2) is very natural from the point of view of *singularly perturbed systems*. For instance, one could consider the ODE $\varepsilon \partial_Y \varphi^{(\varepsilon)} + \varphi^{(\varepsilon)} = 1$ on the domain $Y > 0$, with boundary condition $\varphi^{(\varepsilon)}(0) = u_0 \neq 1$. This has explicit solution:

$$\varphi^{(\varepsilon)} = \underbrace{1}_{\text{Inviscid Profile}} + \underbrace{(u_0 - 1)e^{-\frac{Y}{\varepsilon}}}_{\text{Boundary Layer}}.$$

From this basic ODE example, we may gather several heuristics that will carry over the more complicated situation of Navier-Stokes and Euler. First, the boundary layer decays rapidly (exponentially) away from the boundary, $Y = 0$. Second, it decays in a new, scaled variable which in this case is $\frac{Y}{\varepsilon}$, whereas in the case of Navier-Stokes is $y = \frac{Y}{\sqrt{\varepsilon}}$ (as shown in (1.1.2)). A major implication of this scaling is that topology matters when speaking of inviscid limits. Indeed, the boundary layer profile, $\mathbf{u}_{BL}$ from (1.1.2) is size $\mathcal{O}(\varepsilon^{\frac{1}{2p}})$ in L^p, and so vanishes in the limit as $\varepsilon \to 0$ for all $p < \infty$. For this reason, we view the identity (1.1.2) as most important (and most challenging) in L^∞.

One of Prandtl's major contributions in his seminal work, (Pra04), was to obtain the proper scaling of the boundary layer variable $y = \frac{Y}{\sqrt{\varepsilon}}$. We briefly outline his argument. Suppose one has, to leading order, the expansion:

$$\begin{bmatrix} u^\varepsilon \\ v^\varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} u_p^0(x, \frac{Y}{\omega}) \\ v_p^0(x, \frac{Y}{\omega}) \end{bmatrix} \text{ for some } \omega > 0.$$

As the profiles u_p^0, v_p^0 are expected to decay rapidly in their variable, this expansion implies the thickness of the boundary layer is size ω. Thus, at $Y = 0$, one has the no-slip boundary condition $(u^\varepsilon, v^\varepsilon)|_{Y=0} = (0,0)$, whereas at $Y = \omega$ one has the "matching condition" of $(u^\varepsilon, v^\varepsilon)|_{Y=\omega} = (1,0)$. This implies that the relative orders of the gradients should be: $\partial_Y u^\varepsilon \sim o(\frac{1}{\omega})$, whereas $\partial_x u^\varepsilon \sim \partial_Y v^\varepsilon \sim o(1)$. This then indicates that one should scale out to the following normalized variables and unknowns:

$$y = \frac{Y}{\omega}, \quad U^\varepsilon(x,y) = u^\varepsilon(x,Y), \quad V^\varepsilon(x,y) = \frac{v^\varepsilon(x,Y)}{\omega}, \quad P^\varepsilon(x,y) = p^\varepsilon(x,Y).$$

Inserting the new scaled variables into the Navier-Stokes equations gives:

$$\begin{aligned}
&U^\varepsilon U_x^\varepsilon + V^\varepsilon U_y^\varepsilon + P_x^\varepsilon = \varepsilon U_{xx}^\varepsilon + \frac{\varepsilon}{\omega^2} U_{yy}^\varepsilon \\
&\omega\Big(U^\varepsilon V_x^\varepsilon + V^\varepsilon V_y^\varepsilon\Big) + \frac{P_y^\varepsilon}{\omega} = \varepsilon\omega V_{xx}^\varepsilon + \frac{\varepsilon}{\omega} V_{yy}^\varepsilon \\
&U_x^\varepsilon + V_y^\varepsilon = 0.
\end{aligned}$$

The final ingrediant is Prandtl's physical insight that within the boundary layer, the convective terms are of the same order as the viscosity in the normal, Y, direction. This is achieved if $\omega = \sqrt{\varepsilon}$. Subsequently retaining the leading order terms in ε yields the **Prandtl equations:**

$$
\begin{aligned}
&\bar{u}\bar{u}_x + \bar{v}\bar{u}_y - \bar{u}_{yy} = -\partial_x P_E \text{ on } \mathbb{R}_+ \times \mathbb{R}_+ \\
&\bar{u}_x + \bar{v}_y = 0, \\
&[\bar{u}, \bar{v}]|_{y=0} = [0,0], \quad \bar{u}|_{y\to\infty} \to u_e^0(0), \quad \bar{u}|_{x=0} = \bar{u}_0(y).
\end{aligned}
\tag{1.1.3}
$$

The unknowns $[\bar{u}, \bar{v}] = \mathbf{u_e^0}(0) + \mathbf{u_p^0}$ are expected to describe the leading order of the Navier-Stokes velocity field. A key observation to start with is that Equation (1.1.3) is a *parabolic* equation, with x occupying the time-like variable and y occupying the space-like variable. The boundary at $\{y = 0\}$ corresponds to a physical boundary past which the flow is occurring. On the other hand, the data prescribed at the $\{x = 0\}$ boundary corresponds to an in-flow profile.

Using $[\bar{u}, \bar{v}]$ in place of the Navier-Stokes velocity field has revolutionized many applied sciences, in particular aerodynamics. System (1.1.3) represents a significant simplification from (1.1.1) as parabolic initial-boundary value problems are in general computationally simpler than elliptic boundary value problems. Moreover, explicit solutions to (1.1.3) can be written down. This makes the computation of important physical quantities, such as drag over an airfoil, feasible and efficient. However, from the point-of-view of PDE, there are at least two important questions:

Q1. Establish well-posedness or ill-posedness of the Cauchy problem (1.1.3).

Q2. Establish the validity of expansion (1.1.2).

These two questions are obviously related. On the one hand, if one does not know the answer to Q2 in the affirmative, then a study of Q1 cannot indicate any useful information regarding the Navier-Stokes velocity field. On the other hand, one would, at the bare minimum, require well-posedness of system (1.1.3) and suitably strong estimates over $[\bar{u}, \bar{v}]$ if one hopes to answer Q2 in the affirmative. Oleinik's seminal work, (OS99), lists the major open problems in the boundary layer theory from a mathematical perspective, of which the first such problem is Q2:

> Is it possible to give a strict mathematical justification of [1.1.2] and find the limits of applicability of Prandtl's hypothesis.

1.2 Previous Works

Let us first mention that Prandtl's original work, (Pra04), was developed in the context of 2D, stationary flows. Oleinik's open problem regarding Q2 also concerns the 2D, stationary setting. Of course, one could also consider unsteady flows, for which many works exist regarding both Q1 and Q2. A discussion of these works would lead us astray, so we point the reader to (E00) for a comprehensive set of references regarding the unsteady theory. We will exclusively restrict our point of view to 2D stationary flows.

Let us now discuss work regarding Q1, system (1.1.3). Oleinik proved the following results in (OS99):

Theorem 1.2.1 (Oleinik, (OS99), Local Existence)**.** Suppose $\bar{u}_0(y) > 0$ for $y > 0$, $\bar{u}_0(0) = 0, \bar{u}_0'(0) > 0$, $\bar{u}_0$ is sufficiently regular, and appropriate compatibility conditions are satisfied at $x = 0, y = 0$. Then there exists a solution $[\bar{u}, \bar{v}]$ to (1.1.3) for $x \in (0, L)$ for some $L > 0$. On this interval, the following are satisfied: $\partial_y \bar{u}|_{y=0} > 0$, $\bar{u} > 0$, and $u, u_y, u_{yy}, u_x = v_y$ are all continuous and bounded.

Theorem 1.2.2 (Oleinik, (OS99), Global Existence)**.** Suppose, in addition to the hypotheses in Theorem 1.2.1, that $\partial_x P_E \leq 0$. Then L can be taken to be arbitrary.

The methods used to establish Theorems 1.2.1 - 1.2.2 rely on a nonlinear change of coordinates to $(x, \psi(x,y))$ where ψ is the stream function associated to $[\bar{u}, \bar{v}]$. Such a change of coordinates is clearly possible due to the positivity of $\bar{u}_0$ for $y > 0$. Under this change, letting $w = \bar{u}^2$, the following degenerate parabolic equation is derived: $w_x = \sqrt{w} w_{\psi\psi} - 2\partial_x P_E$. This procedure is known as the "von-Mises" transformation. Subsequently, Oleinik establishes maximum principles that apply to this transformed equation.

In the favorable pressure gradient case of $\partial_x P_E \le 0$, it is natural to investigate the asymptotic in x (or "downstream") behavior of $[\bar{u}, \bar{v}]$. This was investigated by Serrin in (Ser67). First, one seeks a self-similar solution, $f'(\frac{y}{\sqrt{x}})$, to the Prandtl equations, (1.1.3). Such a solution is known as the Falkner-Skan solution. These similarity solutions are unique once the outer streaming velocity $U(x) = \lim_{y\to\infty} \bar{u}$ is prescribed. In (1.1.3) we are taking $U(x) = 1$ for simplicity. Serrin then establishes convergence to these self-similar solutions (we state below a simplified version of his result):

Theorem 1.2.3 (Serrin, (Ser67))**.** Let $\partial_x P_E \le 0$. Let f' denote the unique Falkner-Skan self-similar solution specific to the boundary data provided in (1.1.3). Then the following asymptotics are valid $|\bar{u} - f'| = o(1)$ as $x \to \infty$.

Again, central to Serrin's approach is the von-Mises transformation and maximum principle arguments applied to the resulting quasilinear, degenerate, parabolic equation.

From the point of view of phenomena, the favorable pressure gradient condition in Theorems 1.2.2 and 1.2.3 prevents "boundary layer separation". Separation corresponds to a detachment of the boundary layer from the physical boundary. Figure 1.1 helps visualize such a process.

After the point x_0 in Figure 1.1, the velocity field points in the negative x-direction and $\partial_y \bar{u}|_{y=0}(x_0) = 0$. This subsequently creates a pinching off of the boundary layer. The natural notion of singularity for $[\bar{u}, \bar{v}]$ is thus when $\partial_y \bar{u}|_{y=0} \downarrow 0$. While Theorem 1.2.2 shows

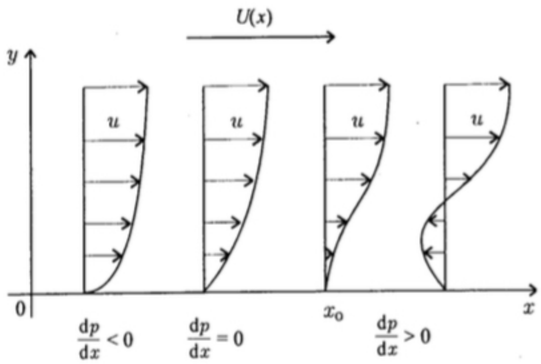

Figure 1.1: Boundary Layer Separation (Reprinted from (OS99), page 9).

that this cannot occur if $\partial_x P_E \le 0$, the following, very recent, result establishes the converse result:

Theorem 1.2.4 (Dalibard-Masmoudi, (DM18))**.** Let $\partial_x P_E = 1$. Then there exists an initial data $\bar{u}_0(y)$ satisfying the hypothesis of Theorem 1.2.1, and $0 < x_0 < \infty$ such that the solution $[\bar{u}, \bar{v}]$ exists on $(0, x_0)$, satisfies $\partial_y \bar{u}|_{y=0}(x) > 0$ on $0 < x < x_0$, and $\partial_y \bar{u}|_{y=0}(x) \downarrow 0$ as $x \to x_0$.

Although Theorems 1.2.1 - 1.2.4 paint a relatively well-developed picture of Q1, there have been very few approaches to answering Q2 in the stationary, 2D setting. From a PDE standpoint, the fundamental challenge to prove expansion (1.1.2) is to control the remainder, $\mathbf{u}$. Due to the multiple scales (∂_y of $\mathbf{u}_{BL}$ creates a singularity of $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$), there are no natural coercive quantities that yield estimates on $\mathbf{u}$ uniformly in ε. For 2D, stationary flows, the work (GN17) overcomes this difficulty, in the presence of a moving boundary.

In (GN17), the fluid domain, Ω, is $(0, L) \times \mathbb{R}_+$. The crucial no-slip condition is prescribed for the Navier-Stokes vector field at $\{y = 0\}$. In addition, this boundary is assumed to be translating to the right at velocity $u_b = 1 - \delta > 0$. Physically, one may imagine a fluid flowing over a plate, which in turn is translating. In contrast, the $\{x = 0\}$ and $\{x = L\}$ are in-flow and out-flow boundaries respectively. To visualize the parameter δ and the translating boundary, one should keep in mind Figure 1.2. The flow field of Figure 1.1 has

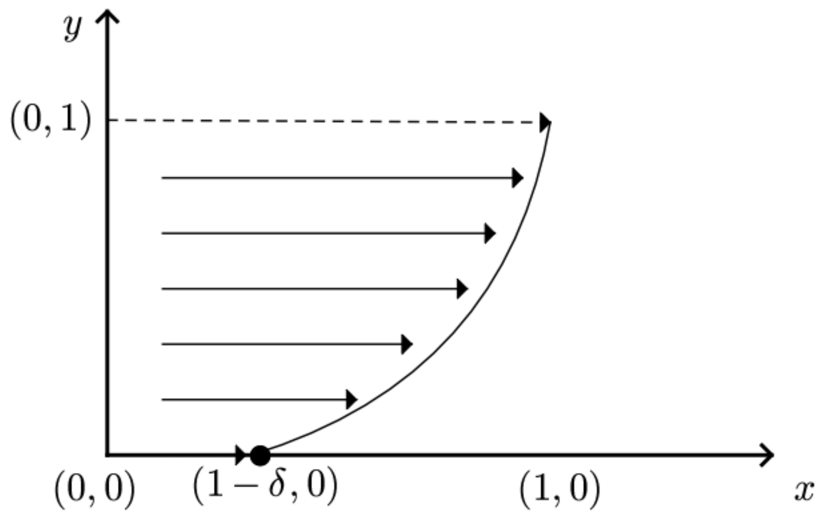

Figure 1.2: Flow over a Moving Boundary

thus been modified to include a transition from $(1-\delta)$ to 1 as opposed to from 0 to 1. This creates a boundary layer of size $\delta \in (0,1)$.

On this domain, Guo-Nguyen prove:

Theorem 1.2.5 (Guo-Nguyen, (GN17))**.** Let the parameter in the expansion (1.2.1) $\gamma \in (0, \frac{1}{4})$. Assume the boundary and initial data for all the profiles are given C^∞ functions that decay exponentially in their respective arguments at infinity. Without loss of generality assume $u_e(0) = 1$. Assume the boundary velocity $1 - \delta > 0$. Then (1.2.1) holds on the domain $[0, L] \times \mathbb{R}_+$, for $0 < L << 1$ sufficiently small relative to universal constants. Moreover, the remainder, $\mathbf{u}$, obeys the following estimates:

$$\|\nabla_\varepsilon \{u, v\}\|_2 + \varepsilon^{\frac{\gamma}{2}} \|\{u, \sqrt{\varepsilon} v\}\|_\infty \lesssim 1, \quad \nabla_\varepsilon := (\sqrt{\varepsilon}\partial_x, \partial_y).$$

Corollary 1.2.6 (Inviscid Convergence in L^∞)**.**

$$\|u^\varepsilon(x, Y) - u_e^0(Y) - u_p^0(x, \frac{Y}{\sqrt{\varepsilon}})\|_\infty \lesssim \sqrt{\varepsilon},$$

$$\|v^\varepsilon(x,Y) - \sqrt{\varepsilon} v_p^0(x, \frac{Y}{\sqrt{\varepsilon}}) - \sqrt{\varepsilon} v_e^1(x,Y)\|_\infty \lesssim \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}$$

The methods introduced in (GN17) are distinguished from those introduced in Theorems 1.2.1 - 1.2.2. Indeed, the central object of study in (GN17) is the remainder solution in 1.1.2, $\bar{u}$, which solves a linearized Navier-Stokes equation (specifically, see 1.2.6). This system is far more complicated than the Prandtl equation, and is not amenable to transformations such as the von-Mises transformation, nor to maximum principle methods. (GN17) thus introduce viable energy based methods to analyze $\bar{u}$. However, at the level of energy there is a competition between the viscosity term $\partial_{yy} u$ and the convection terms. Thus, delicate analysis is required to properly isolate coercive operators and prove their corresponding boundedness properties. This is manifested through the two-tiered structure that will be discussed in (1.2.13) - (1.2.14).

1.2.1 The Asymptotic Expansion:

(GN17) considers the following Ansatz (letting $\gamma \in (0, \frac{1}{4})$ be a free parameter):

$$\begin{aligned} U^\varepsilon &= u_e^0(Y) + u_p^0(x,y) + \sqrt{\varepsilon}\Big[u_e^1(x,Y) + u_p^1(x,y)\Big] + \varepsilon^{\frac{1}{2}+\gamma} u(x,y) \\ V^\varepsilon &= v_p^0(x,y) + v_e^1(x,Y) + \sqrt{\varepsilon} v_p^1(x,y) + \varepsilon^{\frac{1}{2}+\gamma} v(x,y) \\ P^\varepsilon &= \sqrt{\varepsilon} P_e^1(x,Y) + \sqrt{\varepsilon} P_p^1(x,y) + \varepsilon P_p^2(x,y) + \varepsilon^{\frac{1}{2}+\gamma} P(x,y) \end{aligned} \tag{1.2.1}$$

Let us discuss each of the terms in the above expansion, as many of the themes will carry over to the expansions we validate in this thesis. The profile $[u_e^0(Y), 0]$ is a given solution to the Euler equations, which one should think of heuristically as $[1, 0]$. Subsequently following the Euler profiles is an alternating sequence of boundary layer profiles, indexed by $[u_p^i, v_p^i]$ ("p" for Prandtl) and Eulerian profiles, indexed by u_e^i, ("e" for Euler). One may read-off the boundary conditions at $\{y = 0\}$ from the above expansions by enforcing the no-slip condition at each order. Thus, we take:

$$[u_e^0 + u_p^0]|_{y=0} = 1 - \delta, \quad [u_e^1 + u_p^1]|_{y=0} \quad u|_{y=0} = 0,$$
$$[v_p^0 + v_e^1]|_{y=0} = 0, \quad v_p^1|_{y=0} = 0, \quad v|_{y=0} = 0.$$

$[u_p^0, v_p^0]$ are Prandtl boundary layer profiles, which solve the nonlinear Prandtl equations:

$$\begin{aligned}
&(u_e^0(0) + u_p^0)\partial_x u_p^0 + (v_p^0 - v_p^0(0))\partial_y u_p^0 - \partial_{yy} u_p^0 = 0,\\
&\partial_x u_p^0 + \partial_y v_p^0 = 0, \qquad (1.2.2)\\
&u_p^0|_{x=0} = u_{0,p}(y), \quad u_p^0|_{y=0} = u_b - u_e^0(0), \quad \partial_y^k u_p^0|_{y\to\infty} = 0 \text{ for all } k \geq 0.
\end{aligned}$$

The profiles $[u_p^1, v_p^1]$ also satisfy a parabolic equation, the linearized Prandtl equation:

$$\begin{aligned}
&(u_e^0(0) + u_p^0)\partial_x u_p^1 + u_{px}^0 u_p^1 + (v_p^0 + v_e^1)\partial_y u_p^1\\
&+ (v_p^1 - v_p^1(0))\partial_y u_p^0 - \partial_{yy} u_p^1 + P_{px}^1 = f_1, \qquad (1.2.3)\\
&u_{px}^1 + v_{py}^1 = 0. \qquad (1.2.4)
\end{aligned}$$

In contrast, the profiles $[u_e^1, v_e^1]$ satisfy an elliptic boundary value problem in the variables (x, Y), which upon going to the vorticity formulation reads:

$$\begin{aligned}
&u_e^0 \Delta v_e^1 + u_{eYY}^0 v_e^1 = 0, \quad \partial_x u_e^1 + \partial_Y v_e^1 = 0\\
&v_e^1|_{x=0} = V_0(Y), \quad v_e^1|_{x=L} = V_L(Y), \quad v_e^1|_{Y=0} = -v_p^0|_{y=0}.
\end{aligned} \qquad (1.2.5)$$

1.2.2 The Remainder System

Let us briefly discuss the central ideas introduced in (GN17). The focus of our discussion will be entirely on the remainder solution $[\mathbf{u}, P]$, which satisfies the following system:

$$\begin{aligned}
&-\Delta_\varepsilon u + \partial_x P + S_u(u,v) = N_1(u,v) + f, \\
&-\Delta_\varepsilon v + \frac{\partial_y}{\varepsilon} P + S_v(u,v) = N_2(u,v) + g, \\
&u_x + v_y = 0,
\end{aligned} \tag{1.2.6}$$

together with boundary conditions:

$$\begin{aligned}
&[u,v]|_{x=0} = 0, \quad [u,v]|_{y=0} = 0, \quad [u,v]|_{y\to\infty} = 0, \\
&[u_y + \varepsilon v_x]|_{x=L} = 0, \quad [P - 2\varepsilon u_x]|_{x=L} = 0.
\end{aligned} \tag{1.2.7}$$

The boundary condition at $\{x = L\}$ is known as the "Stress-Free" boundary condition, which takes the place of Neumann boundary condition for fluid systems and corresponds to the vanishing of the Cauchy stress tensor. Let us now discuss the various quantities appearing in (1.2.6). The terms on the right-hand side, (f, g), are forcing terms that arise from previously constructed layers. The terms N_1, N_2 are quadratic interactions of $\mathbf{u}$ with itself:

$$N_1(u,v) := uu_x + vu_y, \quad N_2(u,v) := uv_x + vv_y. \tag{1.2.8}$$

The terms $S_u(u,v)$ and $S_v(u,v)$ contain linearizations around previously constructed layers:

$$S_u(u,v) := u_s u_x + u_{sx} u + v_s u_y + u_{sy} v, \tag{1.2.9}$$

$$S_v(u,v) := u_s v_x + v_{sx} u + v_s v_y + v_{sy} v, \tag{1.2.10}$$

$$u_s := u_e^0 + u_p^0 + \sqrt{\varepsilon}[u_e^1 + u_p^1], \qquad v_s := v_p^0 + v_e^1 + \sqrt{\varepsilon} v_p^1. \tag{1.2.11}$$

Let us also define the Rayleigh operator, which can be thought of as the "main" components of (S_u, S_v):

$$\mathbf{R}(u,v) := \begin{pmatrix} -u_s v_y + u_{sy} v \\ u_s v_x - u_{sx} v \end{pmatrix} \tag{1.2.12}$$

We will close by stating the basic structure of the linear estimates in (GN17):

Proposition 1.2.7 (Linear Estimates in (GN17))**.**

$$\|u_y\|_2^2 \lesssim \mathcal{O}(L)\|\nabla_\varepsilon v\|_2^2 + \text{Forcing and Nonlinear Terms} \tag{1.2.13}$$

$$\|\nabla_\varepsilon v\|_2^2 \lesssim \|u_y\|_2^2 + \text{Forcing and Nonlinear Terms.} \tag{1.2.14}$$

Estimate (1.2.13) is obtained via an energy estimate, whereas estimate (1.2.14) is obtained by identifying a cancellation enjoyed by the Rayleigh operator, (1.2.12). It is clear that by bringing L small, the sequence (1.2.13) and (1.2.14) can be combined to achieve linear control over $\|\nabla_\varepsilon \{u,v\}\|_2$. Let us briefly highlight the main mechanism behind the estimate (1.2.14). A natural way to generate positivity over $\|\nabla_\varepsilon v\|_2$ is to take the L^2 inner product of $\mathbf{R}(u,v)$ against $\nabla_\varepsilon^\perp v$. Doing this produces:

$$\int u_s v_y^2 + \frac{u_{syy}}{2} v^2 \not\gtrsim \int v_y^2.$$

Thus, Guo-Nguyen design a modified multiplier $\frac{v}{u_s}$ producing instead the quantity $\int v_y^2 + \frac{u_{syy}}{u_s} v^2$. This quantity is subsequently shown to be coercive over $\|v_y\|_2^2$ using a special cancellation identity.

1.3 Thesis Results

My thesis furthers the Prandtl theory in the following three ways: I proved the validity of (1.1.2) upon introducing curvature effects ((Iye17a)), infinite tangential length scales ((Iye16)), and non-shear effects ((Iye17b)), and characterized quantitatively how these flow features influence the expansion (1.1.2), all in the setting of a moving boundary. It is convenient to state a unifying "meta-theorem":

Meta-Theorem ((Iye17a), (Iye16), (Iye17b))**.** For each of the three setups in (Iye17a), (Iye16), (Iye17b) (described below), the expansion (1.1.2) holds, $\mathbf{u}_{BL}$ is smooth and rapidly decaying, and most importantly the remainder is controlled in a particular norm Z (that changes depending on the particular theorem, but controls L^∞ in all cases) $||\mathbf{u}||_Z \lesssim 1$. This therefore implies $||\mathbf{u}^\varepsilon - \mathbf{u}^0 - \mathbf{u}_{BL}||_\infty \lesssim \sqrt{\varepsilon}$.

The spirit of the above "meta-theorem" is as follows: the main challenge in the following analyses will be to control $\mathbf{u}$. This task is far from generic; rather the particulars of the model (be it the geometry of the domain, spatial length scales present, or the structure of the Euler flow) dictate the norm Z, which must be determined and controlled through careful analysis. The eventual form of $||\cdot||_Z$ tracks refined structural properties of $\mathbf{u}$ (e.g. growth/ decay rates in various spatial regimes) which change dramatically depending on the model. We now turn to specifics. For the sake of presentation, we will continue to refer to the model expansion (1.1.2) as opposed to more refined expansions such as (1.2.1).

1.3.1 Steady Prandtl Boundary Layer Expansions over a Rotating Disk

The first setup I considered, (Iye17a), was a shear Euler flow on the exterior of a rotating disk. The purpose is to investigate the influence of a curved boundary on (1.1.2). More precisely, the domain I consider is given in polar coordinates via $\Omega = (\omega, r) \in (0, \theta_0) \times$

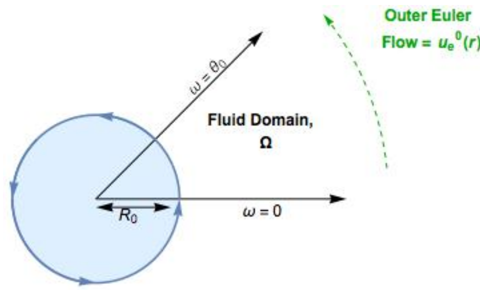

Figure 1.3: Flow past a rotating disk

(R_0, ∞). Moreover, the disk is assumed to be rotating at velocity $1 - \delta > 0$. This setup is displayed in Figure 1.3.

On this domain, the boundary layer scaling is defined in the following way:

$$R := R_0 + \frac{r - R_0}{\sqrt{\varepsilon}}, \qquad r = R_0 + \sqrt{\varepsilon}(R - R_0).$$

These scalings distinguish the curved boundary case from the Euclidean case because:

$$\int \int u^2 r \,\mathrm{d}r \,\mathrm{d}\omega = \sqrt{\epsilon} \int \int u^2(\omega, R) r \,\mathrm{d}R \,\mathrm{d}\omega \neq \sqrt{\epsilon} \int \int u^2(\omega, R) R \,\mathrm{d}R \,\mathrm{d}\omega. \tag{1.3.1}$$

Thus, Eulerian L^2 norms cannot be compared directly to Prandtl L^2 norms as in the Euclidean case, but rather to the measure $r(R)\,\mathrm{d}R\,\mathrm{d}\omega$. The Z-norm is designed to adapt to the scaling above by included specially selected weights r^σ, and is defined via:

Definition 1.3.1 (The norm Z)**.**

$$\begin{aligned}
||u, v||_Z^2 = &\int \int u_\omega^2 r^\sigma + u_R^2 r^{1+\sigma} d\omega dR + \int \int \epsilon v_\omega^2 r^\sigma + |\partial_R(rv)|^2 r^\sigma d\omega dR \\
&+ \epsilon^\gamma \left(\int \int u_R^{2q} r^{q+\alpha} d\omega dR \right)^{1/q} + \epsilon^\gamma \left(\int \int v_R^{2q} r^{q+\alpha} d\omega dR \right)^{1/q} \\
&+ \epsilon^\gamma \left(\int \int u_\omega^{2q} d\omega dR \right)^{1/q} + \epsilon^{\gamma+1} \left(\int \int v_\omega^{2q} r^{-\frac{q}{2p}} d\omega dR \right)^{1/q},
\end{aligned} \tag{1.3.2}$$

where $q = 1 + \delta'$, δ' arbitrarily small but positive, $\gamma \in (0, \frac{1}{4})$. Let p be the Holder conjugate of q, and $0 < \frac{q}{p} \le \alpha \le \frac{q\delta}{2}$. Most importantly, σ will be taken in the interval $1 - \frac{1}{2p} \le \sigma < 1$. The space Z depends on the weight σ, but we will refrain from depicting this explicitly.

The Z-norm should be thought of as a scaled, weighted H^1 norm, together with some H^{1+} quantities. These H^{1+} quantities are utilized to control nonlinear quantities: instead of estimated terms of the form $\|\mathbf{u}\nabla\mathbf{u}\|_2$ using $L^\infty \times L^2$ estimates, it turns out to be more useful to use $L^p \times L^{2+}$ estimates for p very large. This in turn is because we are able to control L^p with r-weights in a more refined manner (see for instance Lemma 1.3.3 below).

Theorem 1.3.2. Let $\delta \in (0, 1)$ from Figure 1.2, let $\varepsilon << \theta_0 << 1$ be sufficiently small relative to universal constants. Suppose appropriate boundary data are provided for the profiles appearing on the right-hand side of (1.1.2). Then there exists a unique solution, $[U^\varepsilon, V^\varepsilon, P^\varepsilon]$ to the Navier-Stokes equation, (1.1.1), which can be expressed in the expansion (1.1.2). Moreover, the following estimates are valid (consult (1.3.2) for the Z-norm specification):

$$\|u_p^0, v_p^0, u_e^1, v_e^1, u_p^1, v_p^1\|_\infty + \|u, v\|_Z \lesssim 1.$$

The main mathematical difficulties I encounter to control $\mathbf{u}$ is that the boundary layer scaling and curvature of the boundary interact in a nontrivial way and that exterior domains are not Poincare domains, which produces weak energy estimates. To handle these difficulties, I designed and controlled several radially weighted norms, established their embedding theorems, and used this calculus to control $\mathbf{u}$. Let us now introduce these norms. Let σ be arbitrary for now; it will be specifically based on the nonlinear iteration.

$$||u||^2_{L^2_{*,\sigma}} := \int\int u^2(\omega, R) r^\sigma \, dR d\omega,$$
$$||u||^2_A := \int\int \left(u_R^2 r^{1+\sigma} + \epsilon u_\omega^2 r^{-1+\sigma} + \epsilon u^2 r^{-1+\sigma}\right) dR d\omega,$$

$$||v||_B^2 := \int\int \left(r^\sigma |\partial_R(rv)|^2 + \epsilon v_\omega^2 r^\sigma + \epsilon v^2 r^\sigma \right) dR d\omega.$$

The essential property satisfied by norms A and B is loss of one weight of r when going from A to B. This property is independent of the choice of σ. The scheme of linear estimates reads for any $\sigma \in [-1, 1]$:

$$||u||_A^2 \lesssim \theta_0 ||v||_B^2 + \sigma\theta_0 ||P||^2_{L^2_{*,\sigma}} + ||f, \sqrt{\epsilon} g||^2_{L^2_{*,2+\sigma}}, \tag{1.3.3}$$

$$||v||_B^2 \lesssim ||u||_A^2 + (1-\sigma)^2 ||P||^2_{L^2_{*,\sigma}} + ||f, \sqrt{\epsilon} g||^2_{L^2_{*,2+\sigma}}, \tag{1.3.4}$$

$$||P||^2_{L^2_{*,\sigma}} \lesssim \mathcal{O}(\theta_0)||u||_A^2 + \mathcal{O}(\theta_0)||v||_B^2 + ||f, \sqrt{\epsilon} g||^2_{L^2_{*,2+\sigma}}. \tag{1.3.5}$$

Thus, in (2.2.31) there is a loss of one weight r, whereas in (1.3.4) we gain back this weight. This staggered structure is essentially due to the divergence free relation: $u_\omega \simeq r v_R$. At the culmination of (2.2.31) - (1.3.5), the choice of σ is still arbitrary. Let us now consider a sample trilinear term with $\sigma = -1$:

$$\begin{aligned} \Big| \int\int \Big(\epsilon^{\gamma+\frac{1}{2}} v u_R \Big) \cdot \partial_R(rv) \Big| &\le \epsilon^{\gamma+\frac{1}{2}} ||v r^{\frac{1}{2}}||_{L^\infty} ||u_R||_{L^2} ||r^{-\frac{1}{2}} \partial_R(rv)||_{L^2} \\ &\le \epsilon^{\gamma+\frac{1}{2}} ||v r^{\frac{1}{2}}||_{L^\infty} ||u, v||^2_{A\cup B}. \end{aligned} \tag{1.3.6}$$

However, $\|vr^{1/2}\|_\infty$ is far out of reach of $A \cup B$ with $\sigma = -1$. We must thus push σ much larger to estimate this trilinear quantity. However, $\sigma = 1$ faces a criticality created from the forcing term:

$$\|f\|^2_{L^2_{*,2+\sigma}} = \int \frac{u_e^2}{r^4} r^{2+\sigma} = \int \frac{1}{r} = \infty.$$

We must thus aim for selecting $\sigma = 1-$. For such a σ, we establish weighted $H^1 \hookrightarrow L^p$ embeddings of the type:

Lemma 1.3.3 (Embedding Theorems)**.**

$$\epsilon^{1/2}\left(\int\int v^p r^{\frac{p}{2}-1+\frac{\sigma p}{2}} dR d\omega\right)^{\frac{1}{p}} \le C_p ||v||_B \textit{ for } 2 \le p < \infty,$$
$$\left(\int\int u^p r^{\frac{\sigma p}{2}} r^{\frac{1}{2}}\right)^{\frac{1}{p}} \le C_p ||u||_{A\cup B} \textit{ for } 4 \le p < \infty,$$
$$\left(\int\int u^p r^{\frac{\sigma p}{2}}\right)^{\frac{1}{p}} \le C_p ||u||_{A\cup B} \textit{ for } 2 \le p < 4,$$

In the end, we find a small region of $\sigma \in (1-, 1)$ which is reflected in the Z-norm above for which all nonlinear quantities can be controlled.

1.3.2 Global Steady Prandtl Expansion Over a Moving Boundary

In my second work, (Iye16), I considered flows on the quadrant $[0,\infty) \times [0,\infty)$. The important feature here is that the x-coordinate is unbounded. Asymptotic in x behavior has long occupied a central role in the boundary layer theory due to the possibility of boundary layer separation, as has been discussed in Theorems 1.2.1 - 1.2.4. Theorem 1.3.4 is a global-in-x generalization of Theorem 1.2.5.

Theorem 1.3.4. Let the outer Euler flow be fixed at $[u_e^0, v_e^0] = [1, 0]$. Let $0 < \delta \le \delta_0 << 1$ from Figure 1.2, where δ_0 is sufficiently small relative to universal constants. Then the expansions (1.1.2) hold in the quadrant $[0,\infty) \times [0,\infty)$, and the following estimates are valid:

$$\|u^i, v_e^i, u_p^i, v_p^i\|_\infty + \|u, v\|_Z \lesssim 1.$$

Let us compare the above result with Theorem 1.2.5. There exists a scaling $L = L(\delta)$ present in Theorem 1.2.5 such that as $\delta \to 1$, $L(\delta) \downarrow 0$ and as $\delta \to 0$, $L(\delta) \uparrow 1$. This is analogous to a "large-data, local" result, where δ corresponds to the size of the data, L

plays the role of the time scale. This is substantially different than Theorem 1.3.4, which is analogous to a "small-data, global" result. Typically, in order to break scaling from a large-data local theorem, one needs to obtain sufficiently strong decay estimates of various quantities involved. This is encoded by the specification of the Z-norm, which we will now discuss.

Definition 1.3.5. The norm Z is defined through:

$$\begin{aligned} ||u,v||_Z :=& ||u,v||_{X_1 \cap X_2 \cap X_3} + \epsilon^{N_2} ||u,v||_{Y_2} + \epsilon^{N_3} ||u,v||_{Y_3} + \epsilon^{N_4} ||u x^{\frac{1}{4}}, \sqrt{\varepsilon} v x^{\frac{1}{2}}||_{L^\infty} \\ &+ \epsilon^{N_5} \sup_{x \ge 20} ||\sqrt{\varepsilon} v_x x^{\frac{3}{2}}, u_x x^{\frac{5}{4}}||_{L^\infty} + \varepsilon^{N_6} \sup_{x \ge 20} ||u_y x^{\frac{1}{2}}||_{L^2_y} \\ &+ \epsilon^{N_7} \Big[\int_{20}^\infty x^4 ||\sqrt{\varepsilon} v_{xx}||^2_{L^\infty_y} \, dx \Big]^{\frac{1}{2}}. \end{aligned} \tag{1.3.7}$$

Here, N_i, are large numbers which will be specified in (3.9.103) - (3.9.105). They depend only on universal constants. The parameter n from (3.2.14) - (3.2.15) will be taken much larger than any of the N_i. The norms $|| \cdot ||_{Y_i}$ are elliptic norms defined in (3.9.6) - (3.9.7). For the purposes of discussing the main result, we can refrain from being too specific with regards to the definitions of the Y_i norms. However, the norms $|| \cdot ||_{X_i}$ are important:

$$\begin{aligned} &\|u,v\|_{X_1} := \|u_y\|_2 + \|\nabla_\varepsilon v \cdot \sqrt{x}\|_2, \\ &\|u,v\|_{X_i} :\simeq \|(x \cdot \partial_x)^i \{u,v\}\|_{X_1} \text{ for } i = 2,3. \end{aligned}$$

The norm X_1 is estimated in a two-step procedure analogous to (1.2.13) - (1.2.14):

$$\begin{aligned} &\|u_y\|_2^2 \lesssim \mathcal{O}(\delta) \|\nabla_\varepsilon v \cdot \sqrt{x}\|_2^2, \\ &\|\nabla_\varepsilon v \cdot \sqrt{x}\|_2^2 \lesssim \|u_y\|_2^2. \end{aligned}$$

The first crucial point here is the loss of a factor of $\sqrt{x}$, which is due to the self-similarity of the Prandtl layers: $u_p^0 \sim \phi_*(\frac{y}{\sqrt{x}})$, where ϕ_* is a rapidly decaying profile in its argument. We establish this self-similarity by extracting an ODE governing ϕ_* in the self-similar variable, $z = \frac{y}{\sqrt{x}}$, and proving asymptotic convergence to this profile for general initial data. The second crucial point is the gain back of $\sqrt{x}$ weight in the second step. This in turn relies crucially on sharp, point-wise estimates on the Poisson kernel that we establish in order to conclude decay for the higher-order Euler layers: $|v_e^i| \lesssim x^{-1/2}$.

The norms X_2, X_3 establish X_1 control over anisotropic scaling vector-fields $x \cdot \partial_x$ applied to the solution $\{u, v\}$. In turn this relies on commutation properties of ∂_x with the linearized equation, (1.2.6).

This linearized strategy forces us to estimate trilinear quantities of the form $\int v u_y v_y \cdot x$, which in turn demands a sharp decay estimate for $\|v\sqrt{x}\|_\infty$. Obtaining the decay for v using the norms X_i is an extremely delicate matter, in which key quantities must overcome the critical Hardy inequality. To see this, we first use the $H^1_y(\mathbb{R}_+) \hookrightarrow L^\infty_y(\mathbb{R}_+)$ Sobolev embedding (ignoring factors of ε):

$$\sup_{x \geq 1} ||vx^{\frac{1}{2}}||_{L^\infty_y} \leq \sup_{x \geq 1} ||v||^{\frac{1}{2}}_{L^2_y} \cdot \sup_{x \geq 1} ||v_y x||^{\frac{1}{2}}_{L^2_y}. \tag{1.3.8}$$

For the first quantity on the right-hand side above, we write:

$$\partial_x \int v^2 \, \mathrm{d}y = \int 2 v v_x \, \mathrm{d}y. \tag{1.3.9}$$

Recall now the quantities, $||u_y, \sqrt{\varepsilon} v_x x^{\frac{1}{2}}, v_y x^{\frac{1}{2}}||^2_{L^2_{xy}}$, which are controlled on the left-hand sides (3.2.45), (3.2.46) and constitute the energy norm X_1. The right-hand side above fails to be x-integrable, precisely because of criticality of Hardy's inequality with power $x^{-\frac{1}{2}}$ in L^2:

$$|\int \int v v_x \, \mathrm{d}y \, \mathrm{d}x| \leq ||vx^{-\frac{1}{2}}||_{L^2_{xy}} ||v_x x^{\frac{1}{2}}||_{L^2_{xy}} \cancel{\lesssim} ||v_x x^{\frac{1}{2}}||^2_{L^2_{xy}}. \tag{1.3.10}$$

To avert this, we move to higher-order derivatives, which invokes the full strength of the norm Z. Indeed, suppose we knew $v_x \sim x^{-\frac{3}{2}}$, then coupled with the boundary condition $v \to 0$ as $x \to \infty$, this would immediately imply $v \sim x^{-\frac{1}{2}}$. Establishing the decay rate, $||v_x||_{L^\infty_y} \leq x^{-\frac{3}{2}}$, then becomes the goal, which requires us to use the full set of vector-fields X_2, X_3. Our main uniform estimates, given in Lemmas 3.9.15, 3.9.17 are given by the following sequence:

$$||vx^{\frac{1}{2}}||_{L^\infty_{xy}} \lesssim ||v_x x^{\frac{3}{2}}||_{L^\infty_{xy}} \lesssim \sup_{x\geq 1} ||v_x x||^{\frac{1}{2}}_{L^2_y} \cdot \sup_{x \geq 1} ||v_{xy}x^2||^{\frac{1}{2}}_{L^2_y} \lesssim ||u,v||_{X_1 \cap X_2 \cap X_3}. \tag{1.3.11}$$

The key point is that the quantities $||v_x x||_{L^2_y}$ and $||v_{xy}x^2||_{L^2_y}$ appearing above *do not* face issues of Hardy-criticality present in (1.3.10).

There are several other components to the norm Z, which I now discuss:

- Anisotropic Mixed Norms: Due to the anisotropy of the above procedure, some non-linear terms cannot be estimated through $L^\infty - L^2 - L^2$ type estimates. These must estimated using mixed norms, which we include in the above Z-norm and subsequently show can be controlled.
- Elliptic Norms: Due to the presence of boundaries, we must technically work with cut-off vector fields, that is $\chi(x \geq 10) \cdot x \cdot \partial_x$. In order to supplement the norm with ample control on $x \sim o(1)$ region, we must appeal to elliptic estimates available for the Stokes operator, which in turn scale poorly with ε.
- Null Forms: In order handle the large negative power of ε appearing in the elliptic norms, we must propagate the expansion of (1.1.2) up to high order, $\varepsilon^{\frac{n}{2}}$, all the while preserving the baseline decay estimates and self-similar estimates which are attained by propagating control of $\frac{y}{\sqrt{x}}$. The most dangerous contributions towards establishing this are Euler-Euler interactions, because weights of $\frac{y}{\sqrt{x}}$ are unfavorable for such interactions. The crucial observation is that these interactions enjoy a special

property, which is that they are of "gradient-type". That is, there exists a potential function whose gradient is exactly the vector of all Euler-Euler interactions. This is due to the special Cauchy-Riemann structure of the Euler flows. Such a potential can be exploited by adding it into the pressure to cancel out these interactions.

- Existence and Uniqueness: Existence in our space Z is achieved via compactness methods which rely on specially selected weights which we append to an approximating sequence of systems. These weights must be compatible with the scheme of estimates designed above. For uniqueness, we reapply the above estimates with weaker weights. It turns out only a small interval of weaker weights can work with the aforementioned scheme of estimates.

1.3.3 Steady Prandtl Expansions over a Moving Boundary: Nonshear Euler Flows

In the work (Iye17b), I considered again the domain $\Omega = (0, L) \times \mathbb{R}_+$, for $0 < L << 1$ small relative to universal constants. Let the boundary have velocity $0 < u_b$. In this case, suppose the prescribed Euler flow, $[u^0_e, v^0_e]$ is *not shear*. This means that $u^0_e = u^0_e(x, Y)$ is allowed to have x-dependence. I prove for "weakly nonshear" Euler flows (as specified by assumption 1.3.13), the following:

Theorem 1.3.6. Let the motion of the boundary equal $u_b > 0$. Consider an Euler flow $[u^0_e(x, Y), v^0_e(x, Y)]$ satisfying the following hypothesis:

$$0 < c_0 \le u^0_e \le C_0 < \infty, \tag{1.3.12}$$

$$||\frac{v^0_e}{Y}||_{L^\infty} << 1, \text{ and} \tag{1.3.13}$$

$$||Y^k \nabla^m v^0_e||_{L^\infty} < \infty \text{ for sufficiently large } k, m \ge 0, \tag{1.3.14}$$

$$||Y^k \nabla^m u^0_e||_{L^\infty} < \infty \text{ for sufficiently large } k \ge 0, m \ge 1. \tag{1.3.15}$$

Let the interval L be sufficiently small relative to universal constants. Suppose in addition that the boundary data described above are prescribed, assumed to be smooth and rapidly decaying in their arguments, and satisfy suitable compatibility conditions. Then the expansions (1.1.2) is valid, and the remainder solutions $[u, v, P]$ exist in the space Z and satisfy the estimate:

$$||u, v||_Z \lesssim 1. \tag{1.3.16}$$

Let us briefly highlight the physical importance of developing a method to handle non-shear Eulerian flows. A classical setup from fluid mechanics deals with horizontal flows past a rotating disk, see for instance (SG00). Such a flow is non-shear, as in the set-up considered here. In the simpler case when the flows are actually circular (and therefore shear), as opposed to horizontal, in the presence of a rotating disk, Theorem 1.3.2 develops machinery to handle the geometry of the boundary. The present article can be viewed as a first step in studying non-shear flows, without adding the complexities of a curved boundary.

Mathematically, the main difficulty of the analysis can be seen by explicitly writing the form of v_s, which presents a singularity of size $\frac{1}{\sqrt{\varepsilon}}$ at the leading order that must be addressed by our analysis:

$$v_s = \frac{v_e^0}{\sqrt{\varepsilon}} + v_p^0 + v_e^1 + \sqrt{\varepsilon} v_p^1. \tag{1.3.17}$$

In this case, we control this singularity by adding far-field controls over u in the Z-norm, which is defined via:

$$||u, v||_Z :\simeq ||\nabla_\varepsilon u \cdot y||_{L^2} + \|\nabla_\varepsilon v\|_{L^2} + ||\nabla_\varepsilon^2 u \cdot y||_{L^2} + \varepsilon^{\frac{\gamma}{2}} ||u, \sqrt{\varepsilon} v||_{L^\infty}.$$

Let us briefly demonstrate how this norm will be utilized and estimated. The scheme of linear estimates now read:

$$\|u_y\|_2^2 \lesssim \mathcal{O}(L)\|\nabla_\varepsilon v\|_2^2, \tag{1.3.18}$$

$$\|\nabla_\varepsilon v\|_2^2 \lesssim \Big\|\frac{v_e^0}{Y}\Big\|_\infty \|\nabla_\varepsilon u \cdot y\|_2^2 + \|u_y\|_2^2, \tag{1.3.19}$$

$$\|\nabla_\varepsilon u \cdot y\|_2^2 + \|\nabla_\varepsilon^2 u \cdot y\|_2^2 \lesssim \|u_y\|_2^2 + \|\nabla_\varepsilon v\|_2^2. \tag{1.3.20}$$

Let us highlight estimate (1.3.19) in which the following term creates a loss of one weight y:

$$|\int \frac{v_e^0}{\sqrt{\varepsilon}} u_y v_y| \le |\int \frac{v_e^0}{\sqrt{\varepsilon} y} u_y v_y y| \lesssim \Big\|\frac{v_e^0}{Y}\Big\|_\infty \|v_y\|_2 \|u_y y\|_2$$

Estimate (1.3.20) is the main contribution of this work. It is achieved by introducing a vector field: $G := y^2(1-x)\partial_y$. The factor of $1-x$ is sometimes known as a "Ghost Weight". The reason is because in absolute value, for x small, $1-x \sim 1$. However, at the level of one derivative, $\partial_x\{1-x\}$ is not similar at all to $\partial_x 1$. Thus, the inclusion of $(1-x)$ can amplify terms which have extra integration by parts available in the x direction. This is utilized in the present context to generate the following positive term:

$$\int G\{u_s \partial_x u\} \cdot u_y \simeq \int u_s u_{xy} u_y y^2 (1-x) \simeq \int u_y^2 y^2.$$

The remaining task in the estimate is to ensure that G interacts favorably with the remaining quantities in the equation (1.2.6).

1.3.4 Stationary Inviscid Limit to Shear Flows

We will now consider flows in a channel, that is $(x, Y) \in (0, L) \times (0, 2)$. In addition, assume the $\{Y = 2\}$ is translating rightward at velocity $u_b \geq 0$. We will consider Euler shear flows, $[u^0_e, v^0_e] = [\mu, 0]$ that themselves satisfy no-slip:

$$\mu(0) = 0, \;\; \mu(2) = u_b, \tag{1.3.21}$$

for which there is no leading order boundary layer. Denote now the asymptotic expansion:

$$\mathbf{u}^\varepsilon := \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix} = \begin{pmatrix} \mu + \varepsilon u^1_e + \varepsilon u^1_p + \varepsilon^{\frac{3}{2}} u^2_e + \varepsilon^{\frac{3}{2}} u^2_p + \varepsilon^{\frac{3}{2}+\gamma} u \\ \varepsilon v^1_e + \varepsilon^{\frac{3}{2}} v^1_p + \varepsilon^{\frac{3}{2}} v^2_e + \varepsilon^2 v^2_p + \varepsilon^{\frac{3}{2}+\gamma} v \end{pmatrix}. \tag{1.3.22}$$

Theorem 1.3.7 (joint with Chunhui Zhou)**.** Let $u_b \geq 0$ in (5.2.3). Let $\mu(y) \in C^\infty([0, 2])$ be a given function, satisfying the conditions:

$$\mu(0) = 0, \mu(2) = u_b, \tag{1.3.23}$$

$$\partial^j_y \mu(0) = \partial^j_y \mu(2) = 0 \text{ for } 2 \leq j \leq N_0, \tag{1.3.24}$$

$$\partial_y \mu(0) > 0, |\partial_y \mu(2)| > 0. \tag{1.3.25}$$

where $N_0 < \infty$ and large but unspecified. Let also standard compatibility conditions at the corners of Ω be prescribed for the layers in u_s. Then there exists a unique solution, $\mathbf{u}^\varepsilon$ satisfying the Navier-Stokes equations, (5.2.3), such that:

$$||u^\varepsilon - \mu||_\infty + ||v^\varepsilon||_\infty \leq c_0(\mu)\varepsilon. \tag{1.3.26}$$

The constant $c_0(\mu)$ satisfies:

$$c_0(\mu) \lesssim ||\frac{\mu'''}{\mu}||_{W^{100,\infty}}. \tag{1.3.27}$$

Our ultimate interest is motivated by Yudovich's ninth problem, (Yud03). Classical experiments starting with Reynolds have shown that unsteady flows in a 2D channel that start near Couette or Poiseulle flow do not converge to these flows. This indicates the existence of infinitely many stationary solutions to Navier-Stokes "near" Couette or Poiseulle. Establishing the existence of these solutions is an open problem. Our second result, Corollary 5.2.2, produces stationary solutions sufficiently close to Couette, assuming $x \in [0, L], L << 1$, and a moving boundary at $y = 2$. Our construction is local in x, and is a first step towards obtaining these global-in-x solutions.

Corollary 1.3.8. Let any $\alpha > 0$ be prescribed, which could depend on ε. Let $\tilde{\mu}$ be prescribed to satisfy the vanishing conditions: $\partial_y^k \tilde{\mu}|_{y=0} = \partial_y^k \tilde{\mu}|_{y=2} = 0$ for $0 \le k \le N_0$. There exists a unique solution, $\mathbf{u}^\varepsilon$ to (5.2.3) with $u_b = 2$ such that:

$$||u^\varepsilon - \Big(y + \alpha\tilde{\mu}(y)\Big)||_\infty + ||v^\varepsilon||_\infty \lesssim \alpha\varepsilon. \tag{1.3.28}$$

Proof. One can obtain this by applying Theorem 5.2.1 with $\mu(y) = y + \alpha\tilde{\mu}(y)$, where $\tilde{\mu}$ vanishes at high order near $y = 0, 2$. In this case, the constant $c_0(\mu) \lesssim \alpha$. □

Remark. The requirement of $u_b = 2$ is so that the no-slip condition is satisfied by the Couette flow. We do not use this motion of the boundary anywhere in the proof.

Chapter Two

Steady Prandtl Boundary Layer Expansions over a Rotating Disk

2.1 Abstract

This chapter concerns the validity of the Prandtl boundary layer theory for steady, incompressible Navier-Stokes flows over a rotating disk. We prove that the Navier-Stokes flows can be decomposed into Euler and Prandtl flows in the inviscid limit. In so doing, we develop a new set of function spaces and prove several embedding theorems which capture the interaction between the Prandtl scaling and the geometry of our domain.

2.2 Introduction

We consider the steady incompressible Navier-Stokes equations on the domain $\Omega = (0, \theta_0) \times (R_0, \infty)$ in polar coordinates. The boundary $\partial\Omega$ then consists of three components: $\{\omega = \theta_0\}, \{\omega = 0\}, \{r = R_0\}$. In cartesian coordinates, the equations read:

$$\left.\begin{aligned} &\bar{U}\bar{U}_x + \bar{V}\bar{U}_y + P_x = \epsilon\Delta\bar{U} \\ &\bar{U}\bar{V}_x + \bar{V}\bar{V}_y + P_y = \epsilon\Delta\bar{V} \\ &\bar{U}_x + \bar{V}_y = 0. \end{aligned}\right\} \quad \text{in } \Omega \tag{2.2.1}$$

In polar coordinates, the equations read (KC04, Page 739):

$$\left.\begin{aligned} &\frac{UU_\omega}{r} + VU_r + \frac{UV}{r} + \frac{P_\omega}{r} = \epsilon\left(U_{rr} + \frac{U_r}{r} + \frac{U_{\omega\omega}}{r^2} - \frac{U}{r^2} + \frac{2}{r^2}V_\omega\right) \\ &\frac{UV_\omega}{r} + VV_r - \frac{U^2}{r} + P_r = \epsilon\left(V_{rr} + \frac{1}{r}V_r + \frac{V_{\omega\omega}}{r^2} - \frac{V}{r^2} - \frac{2}{r^2}U_\omega\right) \\ &U_\omega + \partial_r(rV) = 0. \end{aligned}\right\} \quad \text{in } \Omega \tag{2.2.2}$$

Here, $\bar{U}$ and $\bar{V}$ represent the horizontal and vertical velocities of the flow, and U, V represent the angular and radial velocities of the flow. The Navier-Stokes equations are taken together with the no-slip boundary conditions on the boundary $\{r = R_0\}$. We suppose that the disk of radius R_0 is rotating counter-clockwise with a constant angular velocity of $u_b > 0$. The no slip boundary condition in our case is then

$$U|_{r=R_0} = u_b \text{ and } V|_{r=R_0} = 0. \tag{2.2.3}$$

The boundary conditions at $\{\omega = 0\}$ and $\{\omega = \theta_0\}$ will be prescribed in the text. We study the limit as $\epsilon \to 0$. Formally, one expects solutions to the above Navier-Stokes equations to converge to solutions of the Euler equations with $\epsilon = 0$, but this does not happen due the mismatch at the boundary between the no slip condition enforced for solutions to Navier Stokes equations and the no normal flow condition enforced for solutions to Euler equations.

To account for this mismatch, in 1904 Ludwig Prandtl proposed the formation of a boundary layer of size $\sqrt{\epsilon}$ near the boundary, such that the Navier-Stokes flow can be decomposed into the sum of the Euler flow and the boundary layer flow. This is regarded as one of the most important ideas in fluid mechanics in the last century, and the theory has led to astounding developments in the applied sciences. Indeed, many phenomena in fluids such as wake flows and plane jet flows are described by the Prandtl theory (SG00). Despite this, a rigorous mathematical justification of the boundary layer theory remains open in general.

For unsteady flows, there are several interesting results, see for instance (SC98a), (SC98b), (Mae14), (Asa91), (MT08). The work of Guo and Nguyen, (GN17), is the first result establishing validity of the boundary layer expansion for steady state flows in a rectangular domain over a moving plate. They do so using a combination of energy estimates, elliptic estimates, and a new positivity estimate obtained via the vorticity multiplier $\partial_y\left(\frac{v}{u_s}\right) - \epsilon\partial_x\left(\frac{v}{u_s}\right)$. The main goal of this paper is to generalize Guo and Nguyen's method in the presence of geometric curvature effects in order to establish the validity of the boundary layer theory for steady flows over a rotating disk.

2.2.1 Boundary Layer Expansion

We denote by u_e^0 to be an outer Euler shear flow which is radial:

$$u_e^0 = u_e^0(r). \tag{2.2.4}$$

Such a shear flow describes an Euler fluid which rotates counterclockwise. On the boundary $\{r = R_0\}$, we denote by $u_e = u_e^0(R_0)$ and assume that $u_e > 0$. We also suppose that the disk of radius R_0 is rotating at an angular velocity $u_b > 0$. We now scale to boundary layer variables in the following way:

Boundary Layer Scaling:	Euler Scaling:
$R = R(r) = R_0 + \frac{r-R_0}{\sqrt{\epsilon}}$,	$r = r(R) = R_0 + \sqrt{\epsilon}(R - R_0)$.

Note that $r, R \geq R_0 > 0$ and that $\partial_R r(R) = \sqrt{\epsilon}$, $\partial_r R(r) = 1/\sqrt{\epsilon}$. We scale to boundary layer velocities and pressure in the following way:

$$U^\epsilon(\omega, R) = U(\omega, r), \quad V^\epsilon(\omega, R) = \frac{1}{\sqrt{\epsilon}}V(\omega, r), \quad P^\epsilon(\omega, R) = P(\omega, r). \tag{2.2.5}$$

The boundary layer velocities and pressure satisfy the following scaled Navier-Stokes equations:

$$\frac{U^\epsilon U^\epsilon_\omega}{r} + V^\epsilon U^\epsilon_R + \frac{\sqrt{\epsilon}}{r}U^\epsilon V^\epsilon + \frac{1}{r}P^\epsilon_\omega = U^\epsilon_{RR} + \sqrt{\epsilon}\frac{U^\epsilon_R}{r} + \epsilon\frac{U^\epsilon_{\omega\omega}}{r^2} - \epsilon\frac{U^\epsilon}{r^2} + 2\epsilon^{3/2}\frac{V^\epsilon_\omega}{r^2}, \tag{2.2.6}$$

$$\frac{U^\epsilon V^\epsilon_\omega}{r} + V^\epsilon V^\epsilon_R - \frac{1}{\sqrt{\epsilon}}\frac{(U^\epsilon)^2}{r} + \frac{1}{\epsilon}P^\epsilon_R = V^\epsilon_{RR} + \sqrt{\epsilon}\frac{V^\epsilon_R}{r} + \epsilon\frac{V^\epsilon_{\omega\omega}}{r^2} - \epsilon\frac{V^\epsilon}{r} - 2\sqrt{\epsilon}\frac{U^\epsilon_\omega}{r^2},$$

$$U^\epsilon_\omega + \partial_R(rV^\epsilon) = 0.$$

We start with the following formal expansion:

$$U^\epsilon(\omega, R) = u^0_e(\omega, r) + u^0_p(\omega, R) + \sqrt{\epsilon}u^1_e(\omega, r) + \sqrt{\epsilon}u^1_p(\omega, R) + \epsilon^{\gamma+\frac{1}{2}}u^\epsilon(w, R), \tag{2.2.7}$$

$$V^\epsilon(\omega, R) = v^0_p(\omega, R) + v^1_e(w, r) + \sqrt{\epsilon}v^1_p(\omega, R) + \epsilon^{\gamma+1/2}v^\epsilon(\omega, R), \tag{2.2.8}$$

$$P^\epsilon(\omega, R) = P^0_e(r) + P^0_p(\omega, R) + \sqrt{\epsilon}P^1_e(\omega, r) + \sqrt{\epsilon}P^1_p(\omega, R) + \epsilon P^2_p(\omega, R) + \epsilon^{\frac{1}{2}+\gamma}P^\epsilon(\omega, R). \tag{2.2.9}$$

According to the expansions (2.2.7) - (2.2.8), the Prandtl decomposition up to leading order is then:

$$U(\omega, r) = U^\epsilon(\omega, R) \approx u^0_e(\omega, r) + u^0_p(\omega, R), \tag{2.2.10}$$

$$V(\omega, r) = \sqrt{\epsilon}V^\epsilon(\omega, R) \approx \sqrt{\epsilon}v^0_p(\omega, R) + \sqrt{\epsilon}v^1_e(\omega, r). \tag{2.2.11}$$

$[u^0_p, u^1_e, u^1_p]$ and $[v^0_p, v^1_e, v^1_p]$ are approximate boundary layers to be constructed, after which the remainders u^ϵ, v^ϵ must be constructed and controlled. We insert the expansions into the scaled Navier-Stokes equations, and obtain the different orders of the errors R^u and R^v, which are detailed in equations (2.4.1) - (2.4.19). The scaled divergence free condition is enforced at each stage of the expansion. So, for example, for the Prandtl-0 layer, we enforce $u^0_{p\omega} = -\partial_R(rv^0_p) = -\sqrt{\epsilon}v^0_p - rv^0_{pR}$, and for the Euler-1 layer, we enforce $u^1_{e\omega} = -\partial_r(rv^1_e) = -v^1_e - rv^1_{er}$. We define the following notation which will be in use throughout the paper:

Definition 2.2.1. $u_s = u^0_e + u^0_p + \sqrt{\epsilon}u^1_e$, $v_s = v^0_p + \sqrt{\epsilon}v^1_e$, and $u_{app} = u_s + u^1_p$, $v_{app} = v_s + v^1_p$.

Once the velocities of each layer has been constructed, the pressures are defined using the radial error contributions up to and including the ϵ^0 contributions. The ϵ^{-1} order equation, (2.4.7), for instance, dictates that the initial Prandtl pressure is constant in R. The $\epsilon^{-1/2}$ order error, (2.4.8), is the Euler-0 pressure as given by the Euler equation for the shear radial flow $u_e^0(r)$. We then estimate the error caused by this definition in the angular equations.

2.2.2 Boundary Data

The no-slip boundary conditions at $\{r = R_0\}$ must be enforced for each order of the expansion in (2.2.7, 2.2.8). Since the outer Euler flow u_e^0 is given, we have:

Boundary Conditions on $\{r = R_0\}$:

$$u_e^0(R_0) + u_p^0(\omega, R_0) = u_b, \qquad u_e^1(\omega, R_0) + u_p^1(\omega, R_0) = 0, \qquad u^\epsilon(\omega, R_0) = 0, \tag{2.2.12}$$

$$v_p^0(\omega, R_0) + v_e^1(\omega, R_0) = 0, \qquad v_p^1(\omega, R_0) = 0, \qquad v^\epsilon(\omega, R_0) = 0. \tag{2.2.13}$$

Boundary Conditions on $\{\omega = 0\}$:

$$u_p^0(0, R) = \bar{u}_0(R), \qquad u_p^1(0, R) = \bar{u}_1(R), \qquad u_e^1(0, r) = u_b^1(r), \tag{2.2.14}$$

$$v_e^1(0, r) = V_{b0}(r), \qquad u^\epsilon(0, R) = v^\epsilon(0, R) = 0. \tag{2.2.15}$$

Boundary Conditions on $\{\omega = \theta_0\}$:

$$v_e^1(\theta_0, r) = V_{b1}(r), \tag{2.2.16}$$

$$\epsilon v_\omega^\epsilon + r u_R^\epsilon = 0 \text{ and } P^\epsilon r = 2\epsilon u_\omega^\epsilon. \tag{2.2.17}$$

Boundary Conditions as $r \to \infty$:

$$u_p^0(\omega, R), u_p^1(\omega, R), [u_e^j, v_e^j](\omega, r) \to 0 \text{ as } r, R \to \infty. \tag{2.2.18}$$

We impose the following compatibility conditions for the Euler boundary conditions:

$$V_{b0}(R_0) = v_p^0(0, R_0), \qquad V_{b1}(R_0) = v_p^0(\theta_0, R_0). \tag{2.2.19}$$

The boundary conditions for the remainders (u^ϵ, v^ϵ) in (2.2.12) - (2.2.13) and (2.2.15) are the no-slip conditions, and the condition in (2.2.17) is the stress-free condition.

2.2.3 Main Result

In order to state our main result, we must first define the norm Z, which is the Prandtl layer norm in which we close our nonlinear analysis:

Definition 2.2.2.

$$\begin{aligned}||u,v||_Z^2 = &\int\int u^2 r^\delta + u_\omega^2 r^\delta + u_R^2 r^{1+\delta} d\omega dR + \int\int \epsilon v_\omega^2 r^\delta + |\partial_R(rv)|^2 r^\delta d\omega dR \\ &+ \epsilon^\gamma \left(\int\int u_R^{2q} r^{q+\alpha} d\omega dR\right)^{1/q} + \epsilon^\gamma \left(\int\int v_R^{2q} r^{q+\alpha} d\omega dR\right)^{1/q} \\ &+ \epsilon^\gamma \left(\int\int u_\omega^{2q} d\omega dR\right)^{1/q} + \epsilon^{\gamma+1} \left(\int\int v_\omega^{2q} r^{-\frac{q}{2p}} d\omega dR\right)^{1/q},\end{aligned} \tag{2.2.20}$$

where $q = 1 + \delta'$, δ' arbitrarily small but positive, $\gamma \in (0, \frac{1}{4})$. Let p be the Holder conjugate of q, and $0 < \frac{q}{p} \le \alpha \le \frac{q\delta}{2}$. Most importantly, δ will be taken in the interval $1 - \frac{1}{2p} \le \delta < 1$. The space Z depends on the weight δ, but we will refrain from depicting this explicitly.

Theorem 2.2.3. *Let $u_b > 0$ and $u_e^0(r)$ be a given Euler shear flow such that the derivatives $\partial_r^k u_e^0(r), k \ge 1$ decay exponentially. Suppose the boundary data in (2.2.12 - 2.2.18) are prescribed. Suppose that $\bar{u}_0$ and $\bar{u}_1$ decay exponentially fast in their arguments, that the compatibility conditions (2.2.19) are satisfied, and that $|V_{b0} - V_{b1}| \lesssim \theta_0$ for small θ_0. Suppose further that $\min\{u_b, u_e^0 + \bar{u}^0\} > 0$. There exists a positive angle θ_0 which depends on the* *prescribed data such that for* $\gamma \in (0, \frac{1}{4})$ *and* $\delta \in (0,1)$ *sufficiently close to* 1*, the asymptotic expansions given in equations (2.2.7 - 2.2.9) are valid. The approximate solutions appearing in the expansion are those constructed in Theorems 2.4.1, 2.4.5, and 2.4.12, and the Navier-Stokes remainder satisfies* $||u^\epsilon, v^\epsilon||_Z \le C_0$.

Corollary 2.2.4 (Inviscid L^p convergence)**.** *Under the assumptions of Theorem 2.2.3, we have the following inviscid* L^p *convergence:*

$$\left(\int\int |U(\omega,r) - u_e^0(r)|^p r^\delta dr d\omega\right)^{\frac{1}{p}} + \left(\int\int |V(\omega,r)|^p r^\delta dr d\omega\right)^{\frac{1}{p}} \le C\epsilon^{\frac{1}{2p}}$$

for $2 \le p < 4$,

and for δ *arbitrarily close to* 1*, and*

$$\left(\int\int |U(\omega,r) - u_e^0(r)|^p r dr d\omega\right)^{\frac{1}{p}} + \left(\int\int |V(\omega,r)|^p r dr d\omega\right)^{\frac{1}{p}} \le C\epsilon^{\frac{1}{2p}}$$

for $4 \le p < \infty$,

where U *and* V *are the original Navier-Stokes flows appearing in equation (2.2.2). In* L^∞ *we have the following convergence:*

$$\sup_{(\omega,r)\in\Omega} |U(\omega,r) - u_e^0(r) - u_p^0(\omega,R)| \lesssim \epsilon^{\frac{1}{4}+\frac{\gamma}{2}}, \tag{2.2.21}$$

$$\sup_{(\omega,r)\in\Omega} |V(\omega,r) - \sqrt{\epsilon}v_p^0(\omega,R) - \sqrt{\epsilon}v_e^1(\omega,r)| \lesssim \epsilon^{\frac{1}{4}+\frac{\gamma}{2}}, \tag{2.2.22}$$

Function Space Preliminaries

We briefly discuss the relevant function spaces in which we develop our analysis. Only basic definitions are given here because they are required to follow the steps of the outline below. The details of our functional analytic setup are presented in Section 2.3. The interaction between the Prandtl scaling and the geometry of our domain manifests itself in the functional framework of our analysis for the following reason:

Consider an L^2 function, $\bar{u}$, in Euler coordinates. By definition, this means $\int\int \bar{u}^2(\omega, r) r dr d\omega < \infty$. The corresponding scaled function in Prandtl coordinates is given by $u(\omega, R) = \bar{u}(\omega, r)$. The Eulerian L^2 norm scales down to:

$$||u||^2_{L^2(\text{Euler})} = \int\int \bar{u}^2 r dr d\omega = \sqrt{\epsilon} \int\int u^2(\omega, R) r dR d\omega \neq \sqrt{\epsilon} \int\int u^2(\omega, R) R dR d\omega.$$

Due to the mismatch between the scaled Euler L^2 norm and the actual Prandtl L^2 norm, we must work in a new set of function spaces (notationally depicted as $||\cdot||_*$ and variants thereof) which are natural to our problem, and build the corresponding analytic machinery we require to do our nonlinear analysis. Motivated by this, we define the following Prandtl-layer version of the L^2 norm.

Definition 2.2.5. $||u||^2_{L^2_*} := \int\int u^2 r dR d\omega.$

Applying this same analysis to the derivative operator, $\nabla = \left(\frac{\partial_\omega}{r}, \partial r\right)$, motivates the following definition:

Definition 2.2.6. $||u||_{H^1_*} := ||u||_{L^2_*} + ||\nabla_* u||_{L^2_*}$ where $\nabla_* = \left(\frac{\partial_\omega}{r}, \partial_R\right)$ and similarly for H^2_* where ∇^2_* has components $\left(\dfrac{\partial_{\omega\omega}}{r^2}, \dfrac{\partial_{\omega R}}{r}, \partial_{RR}\right)$. Occasionally we will refer to $\nabla_{*,\epsilon}$ which has components $\left(\frac{\sqrt{\epsilon}\partial_\omega}{r}, \partial_R\right)$. The corresponding weighted variants of these norms will be denoted with two subscripts: $||u||^2_{L^2_{*,\delta}} := \int\int u^2(\omega, R) r^\delta dR d\omega.$

Whenever we write L^p without any subscripts, this means the usual L^p in either the Prandtl layer or the Euler layer, which will be clear from context. The following are the norms in which energy estimates will be obtained:

Definition 2.2.7. $||u||^2_A := \int\int \left(u_R^2 r^{1+\delta} + \epsilon u_\omega^2 r^{-1+\delta} + \epsilon u^2 r^{-1+\delta}\right) dR d\omega.$

Definition 2.2.8. $||v||_B^2 := \int\int \left(r^\delta|\partial_R(rv)|^2 + \epsilon v_\omega^2 r^\delta + \epsilon v^2 r^\delta\right) dRd\omega.$

By the Fundamental Theorem of Calculus and Holder's inequality: $\int\int u^2 r^\delta dRd\omega \leq \theta_0^2 \int\int u_\omega^2 r^\delta dRd\omega$ if $u|_{\omega=0} = 0$. This paired with the divergence-free condition, $u_\omega = -\partial_R(rv)$, yields: $\int\int \left(u_\omega^2 + u^2\right) r^\delta dRd\omega \leq ||v||_B^2$. Motivated by this, we define the following norm:

Definition 2.2.9. $||u||_X^2 := \int\int u^2 r^\delta + u_\omega^2 r^\delta + u_R^2 r^{1+\delta}.$

With these definitions in hand, we detail the steps of our analysis.

Outline of Proof

Inserting the boundary layer expansions (2.2.7 - 2.2.9) into the scaled Navier-Stokes system (2.2.6), we obtain the following system for the Navier-Stokes remainders (for the remainder of this section, we replace $u^\epsilon, v^\epsilon, P^\epsilon$ by u, v, P for notational ease):

$$\begin{aligned} &\frac{1}{r}u_s u_\omega + \frac{1}{r}u_{s\omega}u + u_{sR}v + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v + \frac{1}{r}P_\omega && (2.2.23)\\ &\qquad - u_{RR} - \frac{\sqrt{\epsilon}}{r}u_r - \frac{\epsilon}{r^2}u_{\omega\omega} + \frac{\epsilon}{r^2}u - \frac{2}{r^2}\epsilon^{3/2}v_\omega = f, \\ &\frac{1}{r}u_s v_\omega + \frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u + \frac{1}{\epsilon}P_R && (2.2.24)\\ &\qquad - v_{RR} - \frac{\sqrt{\epsilon}}{r}v_R - \frac{\epsilon}{r^2}v_{\omega\omega} + \frac{2\sqrt{\epsilon}}{r^2}u_\omega + \frac{\epsilon}{r^2}v = g, \\ &\frac{1}{r}u_\omega + \sqrt{\epsilon}\frac{1}{r}v + v_R = 0. && (2.2.25) \end{aligned}$$

where

$$f(\omega, R) = -\epsilon^{-\gamma-\frac{1}{2}}R^u - \sqrt{\epsilon}R^{u,p} - \epsilon^{\gamma+\frac{1}{2}}\left(\frac{1}{r}uu_\omega + vu_R + \frac{\sqrt{\epsilon}}{r}uv\right), \tag{2.2.26}$$

$$g(\omega, R) = -\epsilon^{-\gamma-\frac{1}{2}}R^v - \sqrt{\epsilon}R^{v,p} - \epsilon^{\gamma+\frac{1}{2}}\left(\frac{1}{r}uv_\omega + vv_R - \frac{1}{\sqrt{\epsilon}}\frac{u^2}{r}\right). \tag{2.2.27}$$

Here, R^u, R^v are the remainders from the approximate solutions u_{app}, v_{app}, whose precise definitions are given in (2.4.1) - (2.4.19). $R^{u,p}, R^{v,p}$ are the linearizations of the Navier-Stokes remainders around the Prandtl-1 layer, which precisely are given by:

$$R^{u,p} = \frac{1}{r}u_p^1 u_\omega + \frac{1}{r}u_{p\omega}^1 u + u_{pR}^1 v + v_p^1 u_R + \frac{\sqrt{\epsilon}}{r}v_p^1 u + \frac{\sqrt{\epsilon}}{r}u_p^1 v, \tag{2.2.28}$$

$$R^{v,p} = \frac{1}{r}u_p^1 v_\omega + \frac{1}{r}v_{p\omega}^1 u + v_p^1 v_R + v_{pR}^1 v - \frac{2}{r\sqrt{\epsilon}}u_p^1 u. \tag{2.2.29}$$

The NS remainders u, v satisfy the following boundary conditions:

$$[u,v]|_{\omega=0} = [u,v]|_{R=R_0} = 0, \qquad \epsilon v_\omega + r u_R = 0 \text{ and } Pr = 2\epsilon u_\omega \text{ on } \{\omega = \theta_0\}. \tag{2.2.30}$$

Step I: Construction of Approximate Solutions

We first construct the approximate solutions u_{app}, v_{app} such that the resulting remainder terms R^u and R^v are higher order in ϵ. This involves three stages: constructing the Prandtl-0 layers (u_p^0, v_p^0) using equations (2.4.1, 2.4.7), the Euler-1 layers (u_e^1, v_e^1) using the equations (2.4.2, 2.4.10), and the Prandtl-1 layers (u_p^1, v_p^1) using the equation (2.4.3). The divergence free conditions are enforced at each stage, and the boundary conditions are given in (2.2.12 - 2.2.18).

The method of constructing the approximate solutions is as follows: the Prandtl-0 layer angular velocity, u_p^0, is constructed via a von-Mises transformation, for which the assumption $\min\{u_b, u_e^0 + \bar{u}^0\} > 0$ is crucial. The radial velocity v_p^0 is then obtained via the divergence free condition: $rv_p^0 = \int_R^\infty u_{p\omega}^0$. This choice creates rapid decay as $R \to \infty$ for the Prandtl-0 layers, but as a consequence a boundary condition for $v_p^0|_{R=R_0}$ cannot be enforced.

The second stage of the construction addresses the Euler-1 layer, (u_e^1, v_e^1) which is designed to correct for the normal boundary velocity of $v_p^0|_{R=R_0}$ by enforcing $v_p^0|_{R=R_0} +$

$v_e^1|_{R=R_0} = 0$. After passing to a vorticity formulation for v_e^1, this layer is obtained via standard methods from the second order elliptic theory.

The last stage of the construction addresses the Prandtl-1 layer, (u_p^1, v_p^1). The boundary conditions on $\{r = R_0\}$ are $u_p^1|_{R=R_0} = -u_e^1|_{R=R_0}$ and $v_p^1|_{R=R_0} = 0$. These are designed such that $u_{app}|_{R=R_0} = 0$ and $v_{app}|_{R=R_0} = 0$. This construction relies on the positivity estimate, which will be discussed in Step III.

After these three stages, we evaluate the remaining error, $\int\int \left(|R^u|^2 + \epsilon|R^v|^2\right) r^{2+\delta}$. The weight of $r^{2+\delta}$ must be included because R^u, R^v are contained in f, g in the equations (2.2.26 - 2.2.27), and $r^{2+\delta}$ accompanies f, g in the linear estimate (2.2.48). Interestingly, the angular error term arising from equation (2.4.4), $\int\int |u_e^0|^2 r^{\delta-2}$, is infinite in the critical case of $\delta = 1$, which is the reason the convergence in Corollary 2.2.4 cannot include $\delta = 1$ for $p < 4$, which in turn would correspond to the usual L^p convergence. Despite this, we make use of the delicate embedding theorems we prove in Section 2.3 in order to recover L^p convergence for $p \geq 4$.

The construction of the approximate solutions and evaluation of the resulting error culminates in the following:

Theorem 2.2.10. *Under the assumptions of Theorem 2.2.3, there exist approximate solutions such that* $\left(\int\int R_u^2 r^{2+\delta} dR d\omega\right)^{\frac{1}{2}} + \sqrt{\epsilon}\left(\int\int R_v^2 r^{2+\delta} dR d\omega\right)^{\frac{1}{2}} \lesssim \epsilon^{\frac{3}{4}-\kappa}$ *if* $0 \leq \delta < 1$ *and for* $\kappa > 0$ *but arbitrarily small. Moreover, the approximate solutions satisfy the various estimates which appear in Theorems 2.4.1, 2.4.5, and 2.4.12.*

Step II: Energy Estimate

In this step, we obtain the natural energy estimate associated to the linearized system (2.2.23) - (2.2.25).

Theorem 2.2.11.

$$||u||_A^2 \lesssim \theta_0 ||v||_B^2 + \delta\theta_0 ||P||^2_{L^2_{*,\delta}} + ||f||^2_{L^2_{*,2+\delta}} + \epsilon ||g||^2_{L^2_{*,2+\delta}} \textit{ for } \delta \in [0,1] \textit{ and } \epsilon << \theta_0. \tag{2.2.31}$$

This estimate is generated by applying the multiplier $(r^{1+\delta}u, \epsilon r^{1+\delta}v)$ to equations (2.2.23 - 2.2.24). Once the weight $r^{1+\delta}$ is fixed for the angular multiplier, the divergence free condition (2.2.25) forces a loss of one factor of r. We illustrate this by multiplying the right-hand side of (2.2.23) by $r^{1+\delta}u$, yielding:

$$\begin{aligned}\int\int f r^{1+\delta} u &\lesssim \int\int f^2 r^{2+\delta} + \int\int u^2 r^\delta \lesssim \int\int f^2 r^{2+\delta} + \theta_0^2 \int\int u_\omega^2 r^\delta \\ &\lesssim \int\int f^2 r^{2+\delta} + \theta_0^2 \int\int |\partial_R (rv)|^2 r^\delta \lesssim \int\int f^2 r^{2+\delta} + \theta_0^2 ||v||_B^2. \end{aligned} \tag{2.2.32}$$

Thus $||v||_B$ must appear in our estimate, which features an extra factor of r as compared to $||v||_A$ according to Definitions 2.2.7 and 2.2.8. The strongest weight for the radial multiplier which is then consistent with the presence of $||v||_B$ is $\epsilon r^{1+\delta} v$.

Step III: Positivity Estimate

In this crucial step, we estimate $||v||_B$ in terms of $||u||_A$. Such an estimate must overcome two difficulties. First and foremost, multiplying equation (2.2.24) by a multiplier which is $\mathcal{O}(\epsilon v)$, as in Step II, formally results in control over $\epsilon \int\int |\nabla_\epsilon v|^2$, which is too weak in the inviscid limit. This lack of a basic order-one estimate of v is the most fundamental difficulty in the boundary layer theory. In the case of a rectangular geometry, Guo and Nguyen overcame this difficulty by using the vorticity multiplier $\partial_y \left(\frac{v}{u_s}\right) - \epsilon \partial_x \left(\frac{v}{u_s}\right)$ (GN17). Second, since $||v||_B$, which appears on the right-hand side of estimate (2.2.31), contains an extra factor of r when compared to $||u||_A$, the positivity estimate must recover this factor. This difficulty is new to our problem due to the geometry of our domain Ω.

The starting point is the following calculation, which we use in Section 2.6 and throughout the construction of the approximate solutions. We temporarily ignore boundary contributions as we shall apply this calculation to functions v vanishing on relevant parts of the boundary.

Lemma 2.2.12 (Positivity Calculation)**.**

$$\int\int r^\delta |\partial_R(rv)|^2 \lesssim \int\int r^\delta |\partial_R(rv)|^2 + \int\int r^\delta \left(\frac{rv}{u_s}\right)^2 u_s u_{sRR} + \theta_0 ||v||_B^2. \tag{2.2.33}$$

Proof. For the sake of simplicity, we select the $\delta = 0$ case to showcase initially. The case for general δ which we shall need involves controlling a few more terms, and is proved rigorously in Section 2.6.

$$\begin{aligned}
&\int\int |\partial_R(rv)|^2 = \int\int |\partial_R(r\frac{v}{u_s}u_s)|^2 = \int\int |\partial_R(\frac{rv}{u_s})u_s + r\frac{v}{u_s}u_{sR}|^2 \\
&= \int\int |\partial_R(\frac{rv}{u_s})|^2|u_s|^2 + \int\int r^2\left(\frac{v}{u_s}\right)^2 u_{sR}^2 + 2\int\int \partial_R\left(\frac{rv}{u_s}\right)\frac{rv}{u_s}u_s u_{sR} \\
&= \int\int |\partial_R\left(\frac{rv}{u_s}\right)|^2 u_s^2 - \int\int \left(\frac{rv}{u_s}\right)^2 u_s u_{sRR},
\end{aligned} \tag{2.2.34}$$

$$\int\int |\partial_R(rv)|^2 = \int\int |\partial_R(r\frac{v}{u_s}u_s)|^2 \lesssim \int\int |\partial_R(\frac{rv}{u_s})|^2 u_s^2 + \int\int \frac{r^2v^2}{u_s^2}u_{sR}^2, \tag{2.2.35}$$

$$\begin{aligned}
\int\int \frac{r^2v^2}{u_s^2}u_{sR}^2 &= \int\int u_{sR}^2\left(\int_{R_0}^R \partial_R(\frac{rv}{u_s})\right)^2 \leq \int\int u_{sR}^2(R-R_0)\int_{R_0}^R |\partial_R(\frac{rv}{u_s})|^2 \\
&\lesssim \int\int |\partial_R(\frac{rv}{u_s})|^2.
\end{aligned} \tag{2.2.36}$$

Recalling that $\min u_s > 0$, and inserting (2.2.36) in (2.2.35) and then into (2.2.34) yields the desired estimate.

□

The key calculation is that in estimate (2.2.36), in which we've used the rapid decay of u_{sR} to conclude:

$$\sup_{\omega\in[0,\theta_0]}\int_{R_0}^{\infty} u_{sR}^2(R-R_0)dR < \infty \tag{2.2.37}$$

This will be proven rigorously in equation (2.5.27). Moreover, as in (GN17), our positivity estimate relies on the profile $u_s > 0$, which in turn relies upon our assumption that $u_b > 0$. This is the reason our analysis does not treat the case of a non-rotating boundary. This lemma is used to prove the following:

Theorem 2.2.13 (Positivity Estimate)**.** *For* $\delta \in [0,1]$,

$$||v||_B^2 + \int_{\omega=\theta_0} \epsilon u_\omega^2 r^{\delta-1} \lesssim ||u||_A^2 + (\theta_0+\sqrt{\epsilon})||v||_B^2 + (1-\delta)^2||P||^2_{L^2_{*,\delta}} + ||f||^2_{L^2_{*,2+\delta}} + \epsilon||g||^2_{L^2_{*,2+\delta}}. \tag{2.2.38}$$

In particular for θ_0 and ϵ small enough, this establishes control of $||v||_B$ in terms of $||u||_A$:

$$||v||_B^2 + \int_{\omega=\theta_0} \epsilon u_\omega^2 r^{\delta-1} \lesssim ||u||_A^2 + (1-\delta)^2||P||^2_{L^2_{*,\delta}} + ||f||^2_{L^2_{*,2+\delta}} + \epsilon||g||^2_{L^2_{*,2+\delta}}. \tag{2.2.39}$$

The essential mechanism behind the positivity estimate is to capitalize on the order 1 appearance of v_R in the positive profile term $\frac{u_s}{r}u_\omega$ in equation (2.2.23) through the divergence free condition. We apply the multiplier $(r^\delta \partial_R(\frac{r^2 v}{u_s}), -\epsilon\partial_\omega(\frac{r^{1+\delta}v}{u_s}))$ to equations (2.2.23 - 2.2.24), which is formally a weighted vorticity multiplier. The weights are designed carefully to capture the $||v||_B$ norm using the profile terms from (2.2.23 - 2.2.24). We highlight this using the three important profile terms below:

$$\int\int \frac{u_s}{r}u_\omega r^\delta \partial_R(\frac{r^2 v}{u_s}) \approx \int\int r^\delta u_\omega \partial_R(rv) + \int\int r^{1+\delta}\frac{u_{sR}}{u_s} v\partial_R(rv)$$

$$\approx -\int\int r^\delta |\partial_R(rv)|^2 + \frac{1}{2}\int\int r^\delta \frac{u_{sR}}{u_s}\partial_R((rv)^2), \tag{2.2.40}$$

$$-\epsilon\int\int u_s v_\omega r^\delta \partial_\omega\left(\frac{v}{u_s}\right) \approx -\epsilon\int\int r^\delta v_\omega^2, \tag{2.2.41}$$

$$\int\int v u_{sR} r^\delta \partial_R(\frac{r^2 v}{u_s}) \approx \frac{1}{2}\int\int r^\delta \frac{u_{sR}}{u_s}\partial_R((rv)^2) - \int\int r^{2+\delta}\frac{u_{sR}^2}{u_s^2}v^2. \tag{2.2.42}$$

Summing (2.2.40 - 2.2.42), integrating by parts, and using Lemma (2.2.12) yields:

$$(2.2.40 - 2.2.42) \approx -\int\int r^\delta|\partial_R(rv)|^2 - \epsilon\int\int r^\delta v_\omega^2 - \int\int r^{2+\delta}\frac{u_{sRR}}{u_s}v^2 \tag{2.2.43}$$

$$\lesssim -\int\int r^\delta|\partial_R(rv)|^2 - \epsilon\int\int r^\delta v_\omega^2. \tag{2.2.44}$$

Once the important quantities from $||v||_B$ have been extracted, the rest of the proof proceeds by estimating the remaining terms after the multiplier is applied to the equations (2.2.23) - (2.2.24).

Step IV: Pressure Estimate

The pressure term $||P||_{L^2_{*,\delta}}$ must be included in Theorems 2.2.11 and 2.2.13 due to geometric effects. The choice of $\delta = 0$ for the energy estimate multiplier in Theorem 2.2.11 forces the pressure term to drop out whereas the choice of $\delta = 1$ is required by Theorem 2.2.13 in order for this term to vanish. The lack of a consistent choice of δ which simultaneously forces the pressure to drop out of Theorems 2.2.11 and 2.2.13 requires us to estimate $||P||_{L^2_{*,\delta}}$ in the following:

Theorem 2.2.14. *For $\delta \in [0,1]$ and $\epsilon << \theta_0$,*

$$||P||^2_{L^2_{*,\delta}} \lesssim C_1(\theta_0,\epsilon)||u||_A^2 + C_2(\theta_0,\epsilon)||v||_B^2 + ||f, \sqrt{\epsilon}g||^2_{L^2_{*,\delta}}. \tag{2.2.45}$$

Here, $C_i(\theta_0,\epsilon) \to 0$ as either $\theta_0 \to 0$ or $\epsilon \to 0$.

We emphasize that this estimate is new in our analysis due to the presence of geometric effects, and therefore did not appear in (GN17). Moreover, it is surprising that the $||u||_A$ and $||v||_B$ terms appearing on the right-hand-side of Theorem 2.2.14 are accompanied by small parameters. The estimate relies on the existence of a vector field, $\mathbf{A_1} = (a(\omega, R), b(\omega, R))$ such that $\mathbf{div}(\mathbf{A_1}) \approx P$, and $||\mathbf{A_1}||_{H^1_*} \lesssim ||P||_{L^2_*}$, which is guaranteed to exist for $P \in L^2$ by (Orl98, Page 27) and the estimates we establish in Claims 5 - 8. Moreover, $\mathbf{A_1}$ can be selected to vanish on the Dirichlet portions of the boundary, $\{R = R_0\}$ and $\{\omega = 0\}$. Given this vector field, we apply the multiplier $(ar^\delta, \epsilon br^\delta)$ to (2.2.23 - 2.2.24). The weighted vector field is used as our multiplier in order to estimate the correct weight on the Pressure term:

$$\int\int \frac{P_\omega}{r} r^\delta a + P_R b r^\delta \approx -\int\int P(\frac{a_\omega}{r} + \sqrt{\epsilon}\frac{b}{r} + b_R) r^\delta \approx -\int\int P \mathrm{div}(A_1) r^\delta \approx -\int\int P^2 r^\delta. \tag{2.2.46}$$

Once $||P||_{L^2_{*,\delta}}$ has been extracted, the rest of the proof proceeds by controlling the terms arising from applying the multiplier $(ar^\delta, \epsilon br^\delta)$ to (2.2.23 - 2.2.24).

Summary of Linear Analysis in Steps II - IV:

Putting estimates (2.2.31), (4.4.5), (2.2.45) together yields the full energy estimate for the linearized system in (2.2.23 - 2.2.25):

$$||u||^2_A + ||v||^2_B + ||P||^2_{L^2_{*,\delta}} + \int_{\omega=\theta_0} \epsilon u^2_\omega r^{\delta-1} \lesssim ||f, \sqrt{\epsilon}g||^2_{L^2_{*,2+\delta}}. \tag{2.2.47}$$

When paired with the divergence-free condition, we can upgrade u, u_ω from order $\sqrt{\epsilon}$ to order 1:

$$||u||^2_X + ||v||^2_B + ||P||^2_{L^2_{*,\delta}} + \int_{\omega=\theta_0} \epsilon u^2_\omega r^{\delta-1} \lesssim ||f, \sqrt{\epsilon}g||^2_{L^2_{*,2+\delta}}. \tag{2.2.48}$$

The existence of a unique solution to the linear problem (2.2.23 - 2.2.25) is then given by an application of Schaefer's fixed point theorem:

Theorem 2.2.15. *Let u_s and v_s be the approximate solutions as defined in equations (2.2.7, 2.2.8). Then there exists a unique solution $[u, v, P]$ to the system in (2.2.23 - 2.2.25) on the domain Ω together with the boundary conditions (2.2.30) which satisfies estimate (2.2.48) uniformly in ϵ and small θ_0.*

Step V: High Regularity Estimates

In this step, we obtain higher regularity estimates for solutions to the problem (2.2.23 - 2.2.25). To do so, we rewrite the equations (2.2.23 - 2.2.25) by moving the profile-dependent terms to the right-hand-side:

$$-u_{RR} - \frac{\sqrt{\epsilon}}{r}u_r - \frac{\epsilon}{r^2}u_{\omega\omega} + \frac{\epsilon}{r^2}u - \frac{2}{r^2}\epsilon^{3/2}v_\omega + \frac{1}{r}P_\omega = \tilde{f}, \tag{2.2.49}$$

$$-v_{RR} - \frac{\sqrt{\epsilon}}{r}v_R - \frac{\epsilon}{r^2}v_{\omega\omega} + \frac{2\sqrt{\epsilon}}{r^2}u_\omega + \frac{\epsilon}{r^2}v + \frac{1}{\epsilon}P_R = \tilde{g}, \tag{2.2.50}$$

where

$$\tilde{f} = f - \left(\frac{1}{r}u_s u_\omega + \frac{1}{r}u_{s\omega}u + u_{sR}v + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v\right), \tag{2.2.51}$$

$$\tilde{g} = g - \left(\frac{1}{r}u_s v_\omega + \frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u\right). \tag{2.2.52}$$

From this point of view, we formally expect high regularity estimates using the standard theory of the Stokes equation: $||u, v||_{\dot{H}^2_*} \le \epsilon^{-M}||\tilde{f}, \sqrt{\epsilon}\tilde{g}||_{L^2_*}$ for some potentially large value M. This is only a formal estimate, however, because the corners of Ω, $(\omega = 0, R = R_0)$ and $(\omega = \theta_0, R = R_0)$, obstruct the H^2 regularity of the standard Stokes problem. To account for this, we use the results of (Orl98) to recover $H^{3/2}$ regularity for the solutions near the corners. Precisely, the main result of this section is:

Lemma 2.2.16. *The solutions u and v can be decomposed into $u = u_1 + u_2$ and $v = v_1 + v_2$, where u_2, v_2 are supported near the corners of the domain Ω in a region Ω_2 satisfying $(\omega, R) \in \Omega_2$ implies $R - R_0 \leq 1, r - R_0 \leq \sqrt{\epsilon}$. The decomposition obeys the following estimates:*

$$|u_1||_{\dot{H}^2_{*,2+\delta}} + ||v_1||_{\dot{H}^2_{*,2+\delta}} + ||u_2 r^m, v_2 r^m||_{H^{3/2}} \lesssim \epsilon^{-M} ||\tilde{f}, \sqrt{\epsilon}\tilde{g}||_{L^2_{*,2+\delta}} \tag{2.2.53}$$

for some possibly large value of M, where the constant is independent of θ_0, and for m arbitrarily large.

Step VI: Nonlinear Analysis

In the final step, we use a contraction mapping to obtain existence and uniqueness of the nonlinear problem (2.2.23) - (2.2.30). Consider a sample nonlinear term, $\epsilon^{\gamma+\frac{1}{2}} v u_R$, from equation (2.2.26). According to the right-hand-side of estimate (2.2.48), we must estimate the $|| \cdot ||_{L^2_{*,2+\delta}}$ norm of the nonlinearity:

$$||\epsilon^{\gamma+\frac{1}{2}} v u_R||^2_{L^2_{*,2+\delta}} = \int\int \left(\epsilon^{\gamma+\frac{1}{2}} v u_R\right)^2 r^{2+\delta} \leq \epsilon^{2\gamma+1} \left(\int\int v^{2p} r^{p-1+\delta p}\right)^{\frac{1}{p}} \left(\int\int u_R^{2q} r^{q+\alpha}\right)^{\frac{1}{q}}. \tag{2.2.54}$$

We have used Holder's inequality and we suppose for this discussion that the technical parameter α satisfies $\frac{\alpha}{q} \geq \frac{1}{p}$ in order to make the above inequality valid. We think of p as being very large and so $q = \frac{p}{p-1}$ is very close to 1. This calculation then motivates two features of our norm Z.

First, Z must control $\int\int u^{2p}$ and $\int\int (\sqrt{\epsilon} v)^{2p}$ for large p, together with the appropriate choice of weights r. Typically, the $H^1(\mathbb{R}^2) \hookrightarrow L^{2p}(\mathbb{R}^2)$ embedding yields the desired control, but cannot be applied in our setting for two reasons. First, the standard embeddings apply to integrals taken against the usual measure $RdRd\omega$ and for the usual gradient operator $\nabla = \left(\frac{\partial_\omega}{R}, \partial_R\right)$, whereas in our setting the measure is $r(R)dRd\omega$ and ∇ is replaced by ∇_*. Second, we must precisely determine the weight of $r(R)$ that can be controlled by our weighted energy norms, X and B. As such, we establish the required embeddings from scratch.

The second ingredient which is built into Z are high regularity quantities, because as seen in estimate (2.2.54), Z must control $\int\int |\nabla u, \nabla_\epsilon v|^{2q}$. In order to close a contraction mapping argument, we must in turn control these high regularity quantities (see Definition 2.2.2). The main result in this direction, which serves as the driving force behind the contraction mapping argument, is:

Theorem 2.2.17. *For u, v solutions to the system (2.2.23 - 2.2.25), there exists a $\delta' > 0$ such that if $q = 1 + \delta'$ and $p = \frac{q}{q-1}$, we have:*

$$\epsilon^{\frac{\gamma}{4}}||u_R||_{L^{2q}_{*,q+\alpha}} + \epsilon^{\frac{\gamma}{4}}||v_R||_{L^{2q}_{*,q+\alpha}} + \epsilon^{\frac{\gamma}{4}}||u_\omega||_{L^{2q}_{*,0}} + \epsilon^{\frac{\gamma}{4}+\frac{1}{2}}||v_\omega||_{L^{2q}_{*,-\beta}} \lesssim ||\tilde{f}, \sqrt{\epsilon}\tilde{g}||_{L^2_{*,2+\delta}} \quad (2.2.55)$$

for all δ such that $\max\{1 - \frac{\beta}{q}, \frac{1}{2}\} \le \delta \le 1$, where $\beta > 0$ and we can take $0 < \frac{q}{p} \le \alpha \le \frac{q\delta}{2}$.

The essence of the proof of Theorem 2.2.17 is as follows: since $2q$ is only slightly larger than 2, we interpolate between the H^1_* estimate in (2.2.48) which is uniform in ϵ and the high regularity estimates in Lemma 2.2.16, which scale poorly in ϵ. Again, these interpolations are highly sensitive to the weights r which can be controlled by $||\tilde{f}, \sqrt{\epsilon}\tilde{g}||_{L^2_{*,2+\delta}}$, and are also taking place in our $*-$spaces, and so must be developed from scratch. The required embedding theorems are proven in Section 2.3, and Theorem 2.2.17 is proved in Section 2.9. With Theorem 2.2.17 in hand, we are able to close a contraction mapping argument in the space Z, which we do in Section 2.10:

Theorem 2.2.18 (Nonlinear Existence and Uniqueness in Z)**.** *For $\delta \in (0,1)$ sufficiently close to 1 there exists unique Navier-Stokes remainders (u, v, P) to the system (2.2.23 - 2.2.29) such that $||u, v||_Z < \infty$.*

From here, the main result in Theorem 2.2.3 follows immediately.

2.3 Function Spaces and Embedding Theorems

2.3.1 Basic Properties of $*$-spaces

Here we establish a few basic facts which will be in use throughout our analysis:

Lemma 2.3.1. *Holder's Inequality for $L^p_{*,\delta}$: For p, q Holder conjugates, $||uv||_{L^1_{*,\delta}} \leq ||u||_{L^p_{*,\delta}}||v||_{L^q_{*,\delta}}$*

Proof.

$$\begin{aligned}
||uv||_{L^1_{*,\delta}} &= \int\int |uv|r^\delta = \int\int |uv|\frac{r^\delta}{R}RdRd\omega = \int\int |uv|\left(\frac{r^\delta}{R}\right)^{1/p}\left(\frac{r^\delta}{R}\right)^{1/q}RdRd\omega \\
&\leq \left(\int\int |u|^p\left(\left(\frac{r^\delta}{R}\right)^{1/p}\right)^p RdRd\omega\right)^{\frac{1}{p}}\left(\int\int |v|^q\left(\left(\frac{r^\delta}{R}\right)^{1/q}\right)^q Rd\omega dR\right)^{\frac{1}{q}} \\
&= ||u||_{L^p_{*,\delta}}||u||_{L^q_{*,\delta}}.
\end{aligned}$$

The third inequality above is the usual Holder inequality against the standard measure $RdRd\omega$.

□

Lemma 2.3.2. *The space $L^p_{*,\delta}(\Omega)$ endowed with the norm $||\cdot||_{L^p_{*,\delta}}$ is a Banach Space for $1 \leq p < \infty$.*

Proof. It is clear that $\displaystyle\int\int u^p r^\delta = 0 \iff u = 0$ and that $\displaystyle\int\int (cu)^p r^\delta = \int\int c^p u^p r^\delta = c^p \int\int u^p r^\delta$. We check the triangle inequality by using the corresponding triangle inequality for the usual L^p norm which corresponds to the weight $RdRd\omega$:

$$||u+v||_{L^p_{*,\delta}} = ||\frac{u+v}{R^{1/p}}r^{\delta/p}||_{L^p} \leq ||\frac{u}{R^{1/p}}r^{\delta/p}||_{L^p} + ||\frac{v}{R^{1/p}}r^{\delta/p}||_{L^p} = ||u||_{L^p_{*,\delta}} + ||v||_{L^p_{*,\delta}}.$$

We must argue that $L^p_{*,\delta}(\Omega)$ is complete under this norm. Suppose $\{u_n\}$ is a Cauchy sequence $\iff \{\frac{u_n}{R^{1/p}}r^{\delta/p}\}$ is Cauchy in the usual L^p norm, so there exists a limit function $\bar{u}$ such that $\frac{u_n}{R^{1/p}}r^{\delta/p} \to \bar{u}$ in L^p. Define $\bar{u} = u\frac{r^{\delta/p}}{R^{1/p}}$, so we have:

$$\int\int |u_n - u|^p r^\delta dRd\omega = \int\int \left|\frac{u_n}{R^{1/p}}r^{\delta/p} - u\frac{r^{\delta/p}}{R^{1/p}}\right|^p RdRd\omega \to 0.$$

$\square$

Lemma 2.3.3. *The space $L^2_{*,\delta}(\Omega)$ is a Hilbert space, endowed with the inner product $(u,v) = \int\int uvr^\delta dRd\omega$.*

Proof. By the Holder's inequality (established above), the inner product is well defined as a mapping $L^2_{*,\delta} \times L^2_{*,\delta} \to \mathbb{R}$. Moreover, it is easy to see linearity, symmetry, and non-degeneracy of the inner product. By the previous lemma, this inner product induces a norm, and the space is complete with respect to this norm. $\square$

In general, many properties of the usual L^p will be inherited by $L^p_{*,\delta}$ because the map $T: L^p_{*,\delta} \to L^p$ given by $T(f) = f\frac{r^{\delta/p}}{R^{1/p}}$ is a linear isometry. For instance, the characterization of the dual space to $L^p_{*,\delta}$ follows trivially from this observation:

Lemma 2.3.4. $\left(L^p_{*,\delta}\right)^* = L^q_{*,\delta}$ *where the superscript $*$ denotes (as always) the dual space.*

Proof. Given a bounded linear functional $I: L^p_{*,\delta} \to \mathbb{R}$, $I \circ T^{-1}$ is a bounded linear functional $L^p \to \mathbb{R}$, and is therefore given by $\bar{f} \to \int\int \bar{f}\bar{g}RdRd\omega$ for some $\bar{g} \in L^q$, where $\bar{f} =$

$f\frac{r^{\frac{\delta}{p}}}{R^{1/p}}$. Letting $g = \bar{g}R^{1/q}r^{-\frac{\delta}{q}}$, we readily check $\displaystyle\int\int fgr^{\delta}dRd\omega = \int\int \bar{f}\bar{g}RdRd\omega$ and that $||g||_{L^q_{*,\delta}} = ||\bar{g}||_{L^q}$. □

This immediately implies reflexivity for $1 < p < \infty$, and thus we will be able to obtain weak subsequential limits from sequences bounded uniformly in $_{*,\delta}$ spaces in the usual manner. We'll need a few more facts:

Lemma 2.3.5. *If $u_n \xrightarrow{L^p_{*,\kappa}} u$ for any $1 \leq p < \infty$ and any weight $\kappa \in \mathbb{R}$, a subsequence $u_{n_k} \xrightarrow{a.e.} u$.*

Proof. Define $\bar{u}_n = \dfrac{u_n}{R^{1/p}}r^{\frac{\kappa}{p}}$ and $u = \dfrac{u}{R^{1/p}}r^{\frac{\kappa}{p}}$. Then $\bar{u}_n \xrightarrow{L^p} \bar{u}$, so a subsequence $\bar{u}_{n_k} \xrightarrow{a.e.} \bar{u}$, which immediately implies $u_{n_k} \xrightarrow{a.e.} u$.

□

Lemma 2.3.6 (Density of C_c^∞ in $L^p_{*,\kappa}$)**.** *For $1 \leq p < \infty$ and any weight $\kappa \in \mathbb{R}$, we have that C_c^∞ is dense in $L^p_{*,\kappa}$.*

Proof. Given an $f \in L^p_{*,\kappa}$, define $\bar{f} = \dfrac{f}{R^{1/p}}r^{\frac{\kappa}{p}}$ which is now in the usual L^p. By density, there exists $\bar{\phi_n} \xrightarrow{L^p} \bar{f} \iff \int\int \left|\bar{f} - \bar{\phi_n}\right|^p RdRd\omega \to 0$. Now define $\dfrac{\phi_n}{R^{1/p}}r^{\kappa/p} = \bar{\phi_n}$, which immediately yields $\phi_n \xrightarrow{L^p_{*,\kappa}} f$ and moreover $\phi_n \in C_c^\infty(\Omega)$ because $R \geq R_0$.

□

2.3.2 Properties of Z, I

In this subsection, we prove the first basic property of the space Z:

Lemma 2.3.7. *The space Z together with the norm $||u, v||_Z$ defined above is a Banach space.*

Proof. Nondegeneracy and homogeneity of the $||\cdot||_Z$ follows from the definition. The triangle inequality follows from applying it separately to each component, showing $||\cdot||_Z$ is a norm. We must verify completeness. We first show that weak derivatives, when they are elements of the space $L^r_{*,\kappa}$ for any $r \geq 2$ and weight κ, are unique within this space.

Suppose we have two weak radial derivatives u^1_R and u^2_R in $L^r_{*,\kappa}$ of u. Then $\int\int \left(u^1_R - u^2_R\right)\phi d\omega dR = 0$ for all $\phi \in C^\infty_c(\Omega)$. Since the support of ϕ is compact, we also have $\int\int (u^1_R - u^2_R)\phi r^\kappa d\omega dR = 0$ for all $\phi \in C^\infty_c(\Omega)$. Let s be the Holder conjugate to r, and select an arbitrary $f \in L^s_{*,\kappa} = (L^r_{*,\kappa})^*$. By Lemma 2.3.6 approximate f by ϕ_n in the norm $L^s_{*,\kappa}$.

$$\int\int (u^1_R - u^2_R) f r^\kappa d\omega dR = \int\int (u^1_R - u^2_R)(L^s_{*,\kappa} \lim)\phi_n r^\kappa = \lim \int\int (u^1_R - u^2_R)\phi_n r^\kappa = 0.$$

We have exchanged the $L^s_{*,\kappa}$ lim and $\int\int$ by using Holder's inequality:

$$\left|\int\int \left(u^1_R - u^2_R\right)\left(\phi_n - f\right) r^\kappa d\omega dR\right| \leq \left(\int\int \left|u^1_R - u^2_R\right|^r r^\kappa\right)^{\frac{1}{r}} \left(\int\int \left|\phi_n - f\right|^s r^\kappa\right)^{\frac{1}{s}}$$

Since the right-hand side goes to zero in the above inequality, we are able to switch the limit and integral. Thus, $u^1_R - u^2_R$ has operator norm 0, and the only such element is the 0 element, showing radial derivatives are unique within the class L^r_κ for any r and κ. The choice of radial derivative as opposed to angular derivative was without loss of generality, so the above uniqueness result holds for angular derivative as well.

Suppose $\{u^n\}$ is Cauchy in Z. Then in particular $\{u^n\}$ is Cauchy in $L^2_{*,\delta}$, so there exists a limit u such that $u^n \xrightarrow{L^2_{*,\delta}} u$ by completeness of $L^2_{*,\delta}$. By the same argument, $u^n_\omega \xrightarrow{L^2_{*,\delta}} v$ and $u^n_R \xrightarrow{L^2_{*,\delta}} w$. We must verify $v = u_\omega$ and $w = u_R$. Let $\phi \in C^\infty_C(\Omega)$. Then:

$$\int\int w\phi dR d\omega = -\int\int \left(L^2_{*,\delta} \lim\right) u^n_R \phi = \lim \int\int u^n_R \phi = -\lim \int\int u^n \phi_R$$

$$= -\int\int \left(L^2_{*,\delta} \lim\right) u^n \phi_R = -\int\int u\phi_R.$$

We can exchange the $L^2_{*,\delta}$ limit and integral again by Holder's inequality. Since u_R is the unique element in $L^2_{*,\delta}$ satisfying the above equality, we have $u_R = w$. The identical argument shows $u_\omega = v$.

We now turn to the $||u_R||_{L^{2q}_{*,q+\alpha}}$ term. Again by completeness, there exists some limit function, w (we will abuse notation), such that $u^n_R \xrightarrow{L^{2q}_{*,q+\alpha}} w$. By passing to a subsequence, we can assume $u^n_R \xrightarrow{a.e.} w$. But we know from earlier that $u^n_R \xrightarrow{L^2_{*,\delta}} u_R$ so a further subsequence must converge almost everywhere to u_R. Since every subsequence of an a.e. converging sequence must also converge a.e. to the same limit function, we have $w = u_R$.

Since all of the weights above were done in full generality, we can repeat these arguments for all of the terms in the norm. This proves completeness since we have exhibited a single element u which serves as the $Z-$limit of the Cauchy sequence u^n.

□

Corollary 2.3.8. *The spaces $A, B,$ and X are individually Banach spaces.*

2.3.3 Properties of Z, II: Low Regularity Embeddings

We prove weighted embedding theorems which replace the usual $H^1 \hookrightarrow L^p$ Sobolev embedding in $\mathbb{R}^2$. This style of argument will be applied repeatedly in this paper. For this section, we suppose that u, v satisfy the boundary conditions displayed in (2.2.30) and satisfy the divergence free condition in (2.2.25).

Lemma 2.3.9. $\epsilon^{1/2}\left(\int\int v^p r^{p/2-1+\frac{\delta p}{2}} dR d\omega\right)^{\frac{1}{p}} \lesssim ||v||_B$ *for* $2 \le p < \infty$.

Proof. First consider the case $p = 2$. In this case the exponent on the weight r is δ, and so we have the result by definition of $||v||_B$.

Next, consider the case $p = 4$. We express:

$$\begin{aligned} v^4 r^{1+2\delta} = v^2 r^\delta v^2 r^{1+\delta} &= \int_0^\omega \partial_\omega (v^2) r^\delta \int_{R_0}^R \partial_y (v^2 r(y)^{1+\delta}) \\ &\approx \int_0^\omega v v_\omega r^\delta \int_{R_0}^R v v_R r(y)^{1+\delta} + \int_0^\omega v v_\omega r^\delta \int_{R_0}^R v^2 \sqrt{\epsilon} r(y)^\delta \\ &\lesssim \int_0^{\theta_0} |v v_\omega| r^\delta \int_{R_0}^\infty |v v_R| r^{1+\delta} + \sqrt{\epsilon} \int_0^{\theta_0} |v v_\omega| r^\delta \int_{R_0}^\infty v^2 r^\delta. \end{aligned} \tag{2.3.1}$$

Integrating both sides in $d\omega dR$ and applying Holder yields:

$$\begin{aligned} \int\int v^4 r^{1+2\delta} \lesssim & \int\int v^2 r^\delta \left(\int\int v_\omega^2 r^\delta \right)^{1/2} \left(\int\int v_R^2 r^{2+\delta} \right)^{1/2} \\ & + \sqrt{\epsilon} \left(\int\int v^2 r^\delta \right)^{\frac{3}{2}} \left(\int\int v_\omega^2 r^\delta \right)^{\frac{1}{2}}. \end{aligned} \tag{2.3.2}$$

We now have $\epsilon^{p/2} = \epsilon^2$ to distribute among the right hand side of the inequality, which yields the desired result. For $p \in (2, 4)$, we interpolate:

$$\begin{aligned} \left(\int\int v^p r^{\frac{p}{2} - 1 + \frac{\delta p}{2}} dR d\omega \right)^{\frac{1}{p}} &= \left(\int\int |v r^{\frac{1}{2} + \frac{\delta}{2}}|^p r^{-1} dR d\omega \right)^{\frac{1}{p}} \\ &\le \left(\int\int |v r^{\frac{1}{2} + \frac{\delta}{2}}|^2 r^{-1} dR d\omega \right)^{\frac{\theta}{2}} \left(\int\int |v r^{\frac{1}{2} + \frac{\delta}{2}}|^4 r^{-1} dR d\omega \right)^{\frac{1-\theta}{4}} \\ &\le ||v||_B. \end{aligned} \tag{2.3.3}$$

Now for $p \geq 4$ we can proceed inductively via the calculation:

$$v^p r^{\frac{p}{2} - 1 + \frac{\delta p}{2}} = v^{\frac{p}{2}} r^{\frac{p}{4} - 1 + \frac{\delta p}{4}} v^{\frac{p}{2}} r^{\frac{p}{4} + \frac{\delta p}{4}} = \int_0^\omega v^{\frac{p}{2} - 1} v_\omega r^{\frac{p}{4} - 1 + \frac{\delta p}{4}} \int_{R_0}^R v^{\frac{p}{2} - 1} v_R r(y)^{\frac{p}{4} + \frac{\delta p}{4}}$$

$$+\sqrt{\epsilon}\int_0^{\omega} v^{\frac{p}{2}-1}v_{\omega}r^{\frac{p}{4}-1+\frac{\delta p}{4}}\int_{R_0}^{R} v^{\frac{p}{2}}r(y)^{\frac{p}{4}+\frac{\delta p}{4}-1}. \tag{2.3.4}$$

Taking absolute value, integrating, and using Holder yields:

$$|\int\int v^p r^{\frac{p}{2}-1+\frac{\delta p}{2}}| \le \int\int v^{p-2}r^{\frac{p-2}{2}-1+\frac{\delta}{2}(p-2)}\left(\int\int v_{\omega}^2 r^{\delta}\right)^{1/2}\left(\int\int v_R^2 r^{2+\delta}\right)^{1/2}$$
$$+\sqrt{\epsilon}\left(\int\int v^{p-2}r^{\frac{p-2}{2}-1+\frac{\delta}{2}(p-2)}\right)\left(\int\int v_{\omega}^2 r^{\delta}\right)^{\frac{1}{2}}\left(\int\int v^{\frac{p}{2}}r^{\frac{p}{4}+\frac{\delta p}{4}-1}\right). \tag{2.3.5}$$

If p is an even integer, the absolute values on the left-hand side can be removed. We therefore establish the inequality for even integers successively starting at $p=6$ (since $p=4$ has been computed directly), and then interpolate in between.

□

Lemma 2.3.10. $\left(\int\int u^p r^{\frac{\delta p}{2}}r^{\frac{1}{2}}\right)^{\frac{1}{p}} \lesssim ||u||_X$ *for* $4 \le p < \infty$ *and* $\left(\int\int u^p r^{\frac{\delta p}{2}}\right)^{\frac{1}{p}} \lesssim ||u||_X$ *for* $2 \le p < 4$.

Proof. For $p=2$, we have $\int\int u^2 r^{\delta} \le ||u||_X^2$ by definition of the norm. We compute the case $p=4$:

$$u^4 r^{2\delta+\frac{1}{2}} = u^2 r^{\delta} u^2 r^{\frac{1}{2}+\delta} \approx \int_0^{\omega} u u_{\omega} r^{\delta}\left(\int_{R_0}^{R} u u_R r(y)^{\frac{1}{2}+\delta} + \sqrt{\epsilon}\int_{R_0}^{R} u^2 r(y)^{\delta-\frac{1}{2}}\right)$$
$$\le \int_0^{\theta_0} |u u_{\omega}| r^{\delta}\int_{R_0}^{\infty} |u u_R| r^{\frac{1}{2}+\delta} + \int_0^{\theta_0} |u u_{\omega}| r^{\delta}\int_{R_0}^{\infty} u^2 r^{\delta-\frac{1}{2}}. \tag{2.3.6}$$

Integrating both sides over $d\omega dR$ and applying Holder's inequality yields:

$$\int\int u^4 r^{2\delta+\frac{1}{2}} \lesssim \left(\int\int u^2 r^{\delta}\right)^{1/2}\left(\int\int u_{\omega}^2 r^{\delta}\right)^{1/2}\left(\int\int u^2 r^{\delta}\right)^{1/2}\left(\int\int u_R^2 r^{1+\delta}\right)^{1/2}$$

$$+\left(\int\int u^2 r^\delta\right)^{1/2}\left(\int\int u_\omega^2 r^\delta\right)^{1/2}\left(\int\int u^2 r^{\delta-\frac{1}{2}}\right)$$
$$\leq ||u||_X^4. \tag{2.3.7}$$

We now interpolate for $p \in (2,4)$:

$$\left(\int\int |ur^{\frac{\delta}{2}}|^p dRd\omega\right)^{\frac{1}{p}} \leq \left(\int\int u^2 r^\delta dRd\omega\right)^{\frac{\theta}{2}}\left(\int\int u^4 r^{2\delta} dRd\omega\right)^{\frac{1-\theta}{4}} \leq ||u||_X.$$

Once the above estimate for $p \in (2,4)$ has been established, the desired estimate can be inductively established via:

$$u^p r^{\frac{1}{2}+\frac{\delta p}{2}} = u^{\frac{p}{2}} r^{\frac{\delta p}{4}} u^{\frac{p}{2}} r^{\frac{\delta p}{4}+\frac{1}{2}} = \int_0^\omega u^{\frac{p}{2}-1} u_\omega r^{\frac{\delta p}{4}} \int_{R_0}^R \partial_R (u^{\frac{p}{2}} r(y)^{\frac{\delta p}{4}+\frac{1}{2}})$$
$$= \int_0^\omega u^{\frac{p}{2}-1} u_\omega r^{\frac{\delta p}{4}} \int_{R_0}^R u^{\frac{p}{2}-1} u_R r(y)^{\frac{\delta p}{4}+\frac{1}{2}} + \sqrt{\epsilon}\int_0^\omega u^{\frac{p}{2}-1} u_\omega r^{\frac{\delta p}{4}} \int_{R_0}^R u^{\frac{p}{2}} r(y)^{\frac{\delta p}{4}-\frac{1}{2}}. \tag{2.3.8}$$

Taking absolute values and applying Holder's inequality yields:

$$\int\int u^p r^{\frac{1}{2}+\frac{\delta p}{2}} \lesssim \int\int u^{p-2} r^{\frac{\delta}{2}(p-2)}\left(\int\int u_\omega^2 r^\delta\right)^{1/2}\left(\int\int u_R^2 r^{1+\delta}\right)^{1/2}$$
$$+\sqrt{\epsilon}\int\int u^{\frac{p}{2}} r^{\frac{\delta p}{4}-\frac{1}{2}}\left(\int\int u_\omega^2 r^\delta\right)^{1/2}\left(\int\int u^{p-2} r^{\frac{\delta}{2}(p-2)}\right)^{1/2}. \tag{2.3.9}$$

For $p \geq 4$ all of the quantities in the above estimate are inductively controlled by powers of $||u||_X$.

□

Remark. This argument is reminiscent of the proof of the classical Gagliardo-Nirenberg-Sobolev inequality. This method can be used to yield a direct proof of the standard $H^1 \hookrightarrow L^p$ embedding in $\mathbb{R}^2$ by replacing the weights r by R. The advantages of the direct approach above is the avoidance of defining the fractional Sobolev spaces H^s and consequently the avoidance of appealing to the Fourier Transform. Indeed, in the classical case, one must argue $H^1 \hookrightarrow H^s \hookrightarrow L^p$ for $0 \leq s < 1$ and $2 \leq p < \infty$ because the Sobolev exponent is critical. The drawbacks are that this method must take into account the behavior of u on the boundary $\partial\Omega$, and that it doesn't directly apply to more complex domains.

We now prove an embedding of the type $Z \hookrightarrow L^\infty$, from which Corollary 2.2.4 follows directly:

Lemma 2.3.11. *Given $\delta' > 0$, let $q = 1 + \delta'$. For $\frac{1}{1+\delta'} \leq \delta \leq 1$,*

$$\epsilon^{\frac{\gamma}{2}+\frac{1}{4q}}||u||_{L^\infty} + \epsilon^{\frac{\gamma}{2}+\frac{1}{4q}+\frac{1}{2}}||v||_{L^\infty} \lesssim ||u, v||_Z. \tag{2.3.10}$$

Proof. Since $2q = 2(1+\delta') > 2$, by Morrey's Inequality:

$$|u(\bar{\omega}, \bar{R})| \lesssim ||\nabla u||_{L^{2q}} + ||u||_{L^{2q}} = \left(\int\int |\nabla u|^{2q} RdRd\omega\right)^{\frac{1}{2q}} + \left(\int\int |u|^{2q} RdRd\omega\right)^{\frac{1}{2q}}. \tag{2.3.11}$$

Multiplying by $\epsilon^{\frac{1}{4q}+\frac{\gamma}{2}}$ yields:

$$\begin{aligned}\epsilon^{\frac{1}{4q}+\frac{\gamma}{2}}|u(\bar{\omega}, \bar{R})| &\lesssim \epsilon^{\frac{\gamma}{2}}\left(\int\int |\nabla u|^{2q} rdRd\omega\right)^{\frac{1}{2q}} + \epsilon^{\frac{\gamma}{2}}\left(\int\int |u|^{2q} rdRd\omega\right)^{\frac{1}{2q}} \\ &\lesssim \epsilon^{\frac{\gamma}{2}}\left(\int\int |\nabla_* u|^{2q} rdRd\omega\right)^{\frac{1}{2q}} + \epsilon^{\frac{\gamma}{2}}\left(\int\int |u|^{2q} rdRd\omega\right)^{\frac{1}{2q}} \lesssim ||u||_Z.\end{aligned} \tag{2.3.12}$$

where we have used $|\nabla u| = |\left(\frac{u_\omega}{R}, u_R\right)| \lesssim |\left(\frac{u_\omega}{r}, u_R\right)| = |\nabla_* u|$. The condition $\frac{1}{1+\delta'} \leq \delta$

ensures $||u||_{L^{2q}_{*,1}} \leq ||u||_{L^{2q}_{*,\delta q}} \leq ||u||_Z$ by Lemma 2.3.10. The proof for v works identically, where the extra factor of $\epsilon^{\frac{1}{2}}$ is required as the $||v||_Z$ contains $\epsilon^{\frac{1}{2}+\frac{\gamma}{2}} \left(\int\int v_\omega^{2q} r^{-\frac{q}{2p}}\right)^{\frac{1}{2q}}$.

□

Remark. Note that the condition in Definition 2.2.2, $1 - \frac{1}{2p} \leq \delta < 1$, implies the condition $\frac{1}{1+\delta'} \leq \delta \leq 1$.

2.3.4 Properties of Z, III: High Regularity Embeddings

In this subsection, we provide careful estimates which will yield control of the high regularity quantities appearing in $||\cdot||_Z$. Throughout this section, u, v are assumed to satisfy the boundary conditions displayed in (2.2.30) and the divergence free condition in (2.2.25).

Lemma 2.3.12. *Let $\delta \in [\frac{1}{2}, 1]$. There exists a $\delta' > 0$ such that if $q = 1 + \delta'$, then*

$$||u_R||_{L^4_{*,2+\frac{2\alpha}{q}}} \lesssim ||u||_X^{\frac{1}{2}} ||u||_{\dot{H}^2_{*,2+\delta}}^{\frac{1}{2}}; \text{ and } ||v_R||_{L^4_{*,2+\frac{2\alpha}{q}}} \lesssim ||v||_X^{\frac{1}{2}} ||v||_{\dot{H}^2_{*,2+\delta}}^{\frac{1}{2}} \tag{2.3.13}$$

for all $0 \leq \alpha \leq \frac{q\delta}{2}$. Moreover, α can be selected such that $\frac{q}{p} = \frac{1+\delta'}{p} \leq \alpha \leq \frac{q\delta}{2}$, where p is the Holder conjugate of q by taking δ' small enough.

Proof. After noticing that $v = v_R = 0$ on the boundary $\{\omega = 0\}$ and $v_R = \frac{-1}{r}(u_\omega + \sqrt{\epsilon} v) = 0$ on the boundary $\{R = R_0\}$, the u and v estimates follow in an identical manner, so we focus on u. We express:

$$u_R^4 r^{2+2\frac{\alpha}{q}} = u_R^4 r^{2+\alpha'} = u_R^2 r^{\frac{1}{2}+\alpha'} u_R^2 r^{\frac{3}{2}}. \tag{2.3.14}$$

Since $\delta \geq \frac{1}{2}$, the quantity $u_R^2 r^{\frac{3}{2}}$ is integrable and lies in the Sobolev space $W^{1,1}(\mathbb{R}_+)$ for each fixed ω by the definition of $\dot{H}^2_{*,2+\delta}$, implying that this quantity decays at ∞. This enables us to write:

$$u_R^2 r^{3/2} = -\int_R^\infty \partial_R(u_R^2(y) r^{3/2}) dy \leq \int_{R_0}^\infty |u_R u_{RR}| r^{3/2} dy + \sqrt{\epsilon} \int_{R_0}^\infty |u_R^2| r^{1/2} dy. \quad (2.3.15)$$

We also note that $u = u_R = 0$ on the boundary $\omega = 0$. Thus, we are able to write:

$$u_R^2 r^{\frac{1}{2}+\alpha'} = \int_0^\omega u_R u_{R\omega} r^{\frac{1}{2}+\alpha'} \leq \int_0^{\theta_0} |u_R u_{R\omega}| r^{1/2+\alpha'}. \quad (2.3.16)$$

Multiplying the previous two inequalities, integrating and applying Holder's inequality yields:

$$\begin{aligned} \int\int u_R^4 r^{2+\alpha'} &\lesssim \left(\int\int u_R^2 r^{1+\alpha'}\right)^{1/2} \left(\int\int u_{RR}^2 r^2\right)^{1/2} \left(\int\int u_{R\omega}^2 r^{\alpha'}\right)^{1/2} \left(\int\int u_R^2 r\right)^{1/2} \\ &\lesssim ||u||_X^2 ||u||_{\dot{H}^2_{*,2+\delta}}^2, \end{aligned} \quad (2.3.17)$$

where we use that $\alpha' = 2\frac{\alpha}{q} \leq \delta$. Taking fourth roots yields the result.

□

Lemma 2.3.13. *There exists a δ' such that for $q = 1 + \delta'$, $||u_\omega||_{L^4_{*,0}} \lesssim ||u||_X^{\frac{1}{2}} ||u||_{\dot{H}^2_{*,2+\delta}}^{\frac{1}{2}}$ for $\delta \in [\frac{1}{2}, 1]$.*

Proof. Writing $u_\omega^4 = u_\omega^2 u_\omega^2 = u_\omega^2 r^{-\delta} u_\omega^2 r^\delta$ and recalling that $u = u_\omega = 0$ on $R = R_0$, enables us to write:

$$u_\omega^2 r^\delta = \int_{R_0}^R \partial_R(u_\omega^2 r^\delta) dy = \int_{R_0}^R r^\delta u_\omega u_{\omega R} + \sqrt{\epsilon} \int_{R_0}^R u_\omega^2 r^{\delta-1}. \quad (2.3.18)$$

Using the divergence free condition, $u_\omega = -\partial_R(rv)$, we have $u_\omega = 0$ on $\omega = 0$. Therefore we can write:

$$u_\omega^2 r^{-\delta} = \int_0^\omega u_\omega u_{\omega\omega} r^{-\delta} = \int_0^\omega u_\omega r^{\frac{\delta}{2}} u_{\omega\omega} r^{-\frac{3\delta}{2}}.$$

Multiplying the two equalities above together, taking absolute values, and applying Holder yields:

$$\int\int u_\omega^4 \leq \left(\int\int u_\omega^2 r^\delta\right)\left(\int\int u_{\omega\omega}^2 r^{-3\delta}\right)^{1/2}\left(\int\int u_{\omega R}^2 r^\delta\right)^{1/2} \leq ||u||_X^2 ||u||_{\dot{H}^2_{*,2+\delta}}^2 \tag{2.3.19}$$

where we have used $\frac{1}{2} \leq \delta \leq 1 \Rightarrow -3\delta \leq -2+\delta$.

$\square$

Lemma 2.3.14. *There exists δ' such that for $q = 1+\delta'$, we have $\sqrt{\epsilon}||v_\omega||_{L^4_{*,-\frac{2\beta}{q}}} \lesssim ||v||_B^{\frac{1}{2}} ||v||_{\dot{H}^2_{*,2+\delta}}^{\frac{1}{2}}$ for any $\beta > 0$ and $1 - \frac{\beta}{q} \leq \delta \leq 1$.*

Proof. Temporarily writing $\beta' = \frac{2\beta}{q}$, we proceed to write: $v_\omega^4 r^{-\beta'} = v_\omega^2 r^{-\beta'} r^{-\delta} v_\omega^2 r^\delta$. Using that $v = v_\omega = 0$ on the boundary $R = R_0$, we can write

$$v_\omega^2 r^\delta = \int_{R_0}^R \partial_y(v_\omega^2 r^\delta) dy = \int_{R_0}^R v_\omega v_{\omega R} r^\delta + \sqrt{\epsilon}\int_{R_0}^R r^{-1+\delta} v_\omega^2.$$

Next, we recall the boundary conditions at $\omega = \theta_0$ are $v_\omega = -\frac{r}{\epsilon} u_R \rightarrow v_\omega^2(\theta_0, R) = \frac{r^2}{\epsilon^2} u_R(\theta_0, R)^2$. As such, we write:

$$v_\omega^2 r^{-\beta'} r^{-\delta} = \frac{r^{2-\beta'-\delta}}{\epsilon^2} u_R(\theta_0, R)^2 + \int_\omega^{\theta_0} v_\omega v_{\omega\omega} r^{-\delta-\beta'}.$$

Taking absolute values and multiplying the previous two equalities together yields:

$$\begin{aligned}v_\omega^4 r^{-\beta'} \lesssim \frac{r^{2-\beta'-\delta}}{\epsilon^2} u_R(\theta_0,R)^2 \int_{R_0}^{\infty} |v_\omega v_{\omega R}| r^\delta dy + \int_0^{\theta_0} |v_\omega v_{\omega\omega}| r^{-\delta-\beta'} d\omega \int_{R_0}^{\infty} |v_\omega v_{\omega R}| r^\delta dy \\ + \frac{r^{2-\beta'-\delta}}{\epsilon^2} u_R(\theta_0,R)^2 \sqrt{\epsilon} \int_{R_0}^{\infty} v_\omega^2 r^{-1+\delta} + \int_0^{\theta_0} |v_\omega v_{\omega\omega}| r^{-\delta-\beta'} d\omega \sqrt{\epsilon} \int_{R_0}^{\infty} v_\omega^2 r^{-1+\delta} dy.\end{aligned}$$

Integrating over $d\omega$ and dR yields:

$$\begin{aligned}\int\int v_\omega^4 r^{-\beta'} \lesssim & \frac{1}{\epsilon^2} \int_{R_0}^{\infty} r^{2-\beta'-\delta} u_R(\theta_0,R)^2 dR \left(\int\int v_\omega^2 r^\delta\right)^{1/2} \left(\int\int v_{\omega R}^2 r^\delta\right)^{1/2} \\ & + \left(\int\int v_\omega^2 r^\delta\right)^{1/2} \left(\int\int v_{\omega\omega} r^{-3\delta-2\beta'}\right)^{1/2} \left(\int\int v_\omega^2 r^\delta\right)^{1/2} \left(\int\int v_{\omega R}^2 r^\delta\right)^{1/2} \\ & + \frac{1}{\epsilon^{3/2}} \int_{R_0}^{\infty} r^{2-\beta'-\delta} u_R^2 \int\int v_\omega^2 r^{-1+\delta} \\ & + \sqrt{\epsilon} \left(\int\int v_\omega^2 r^\delta\right)^{\frac{1}{2}} \left(\int\int v_{\omega\omega}^2 r^{-3\delta-2\beta'}\right)^{\frac{1}{2}} \int\int v_\omega^2 r^{-1+\delta}. \end{aligned} \tag{2.3.20}$$

As $u = u_R$ on the boundary $\theta = 0$, we can write $u_R(\theta_0, R) = \int_0^{\theta_0} u_{R\omega} \Rightarrow u_R(\theta_0,R)^2 \lesssim \int_0^{\theta_0} u_{R\omega}^2$. Inserting this into (2.3.20):

$$\begin{aligned}\int\int v_\omega^4 r^{-\beta'} \lesssim & \frac{1}{\epsilon^2} \int_{R_0}^{\infty} \int_0^{\theta_0} r^{2-\beta'-\delta} u_{R\omega}^2 \left(\int\int v_\omega^2 r^\delta\right)^{1/2} \left(\int\int v_{\omega R}^2 r^\delta\right)^{1/2} \\ & + \left(\int\int v_\omega^2 r^\delta\right)^{1/2} \left(\int\int v_{\omega\omega} r^{-3\delta-2\beta'}\right)^{1/2} \left(\int\int v_\omega^2 r^\delta\right)^{1/2} \left(\int\int v_{\omega R}^2 r^\delta\right)^{1/2} \\ & + \frac{1}{\epsilon^{3/2}} \int\int r^{2-\beta'-\delta} u_{R\omega}^2 \int\int v_\omega^2 r^{-1+\delta} \\ & + \sqrt{\epsilon} \left(\int\int v_\omega^2 r^\delta\right)^{\frac{1}{2}} \left(\int\int v_{\omega\omega}^2 r^{-3\delta-2\beta'}\right)^{\frac{1}{2}} \int\int v_\omega^2 r^{-1+\delta}. \end{aligned} \tag{2.3.21}$$

We multiply estimate (2.3.21) by ϵ^2, and according to the weights we are able to estimate for the $\dot{H}^2_{*,2+\delta}$ terms, we require:

$$2-\beta'-\delta \le \delta \iff 2-\beta' \le 2\delta \iff 1-\frac{\beta'}{2} \le \delta, \tag{2.3.22}$$

and

$$-3\delta - 2\beta' \le \delta - 2 \iff 2-2\beta' \le 4\delta \iff \frac{1}{2}-\frac{\beta'}{2} \le \delta. \tag{2.3.23}$$

The lemma has been proved. □

Remark. The above estimate for v_ω is the most delicate of the high-order estimates as it relies on the stress-free boundary condition placed at the boundary $\{\omega = \theta_0\}$. For our purposes in Section 2.10, we will take $\beta = \frac{q}{2p}$ in which case the valid interval for δ is $\frac{1}{2p} \le \delta \le 1$.

2.4 Construction of Approximate Solutions

In this section, we construct the approximate solutions in the expansion (2.2.7, 2.2.8), and estimate the corresponding errors R^u and R^v. First, we record the errors R^u, R^v, which are obtained by inserting the expansion (2.2.7 - 2.2.9) into the scaled NS system (2.2.6):

Angular Error, R^u:

$$\epsilon^0 \text{ order error: } \frac{1}{r}(u_e^0+u_p^0)u_{p\omega}^0 + (v_p^0+v_e^1)u_{pR}^0 + \frac{1}{r}P_{p\omega}^0 - u_{pRR}^0; \tag{2.4.1}$$

$$\epsilon^{1/2}\text{order error, Euler: } \frac{u_{e\omega}^1 u_e^0}{r} + v_e^1 u_{er}^0 + \frac{u_e^0 v_e^1}{r} + \frac{P_{e\omega}^1}{r}; \tag{2.4.2}$$

$$\epsilon^{1/2}\text{order error, BL: } \frac{1}{r}(u_e^1+u_p^1)u_{p\omega}^0 + \frac{u_e^0 u_{p\omega}^1}{r} + \frac{u_p^0}{r}(u_{e\omega}^1+u_{p\omega}^1) + v_p^0(u_{er}^0+u_{pR}^1)$$
$$+ v_e^1 u_{pR}^1 + \sqrt{\epsilon}u_{er}^0 v_p^1 + v_p^1 u_{pR}^0 + \frac{u_e^0 v_p^0}{r} + \frac{1}{r}u_p^0(v_p^0+v_e^1) + \frac{P_{p\omega}^1}{r} - u_{pRR}^1 - \frac{1}{r}u_{pR}^0; \tag{2.4.3}$$

$$\epsilon^1 \text{ order error: } \frac{1}{r}(u_e^1+u_p^1)\partial_\omega(u_e^1+u_p^1)+(v_p^0+v_e^1)u_{er}^1+v_p^1u_{pR}^1 \tag{2.4.4}$$
$$+\frac{1}{r}(u_e^0+u_p^0)v_p^1+\frac{1}{r}(u_e^1+u_p^1)(v_p^0+v_e^1)+\frac{1}{r}P_{P\omega}^2-u_{err}^0-\frac{1}{r}(u_{er}^0+u_{pR}^1)$$
$$-\frac{1}{r^2}u_{p\omega\omega}^0+\frac{1}{r^2}(u_e^0+u_p^0);$$
$$\epsilon^{3/2} \text{ order error: } v_p^1u_{er}^1+\frac{1}{r}(u_e^1+u_p^1)v_p^1-u_{err}^1-\frac{1}{r}u_{er}^1-\frac{1}{r}(u_{e\omega\omega}^1+u_{p\omega\omega}^1) \tag{2.4.5}$$
$$+\frac{1}{r^2}(u_e^1+u_p^1)-\frac{2}{r^2}(v_{p\omega}^0+v_{e\omega}^1);$$
$$\epsilon^2 \text{ order error: } \frac{2}{r^2}v_{p\omega}^1. \tag{2.4.6}$$

After splitting into Euler and Boundary Layer variables, we have the following errors for the Radial component:

Radial Errors, R^v:

$$\epsilon^{-1} \text{ order error, BL: } P_{PR}^0; \tag{2.4.7}$$
$$\epsilon^{-1/2} \text{ order error, Euler: } P_{er}^0-\frac{(u_e^0)^2}{r}; \tag{2.4.8}$$
$$\epsilon^{-1/2} \text{ order error, BL: } P_{PR}^1-\frac{(u_p^0)^2}{r}-2\frac{u_e^0u_p^0}{r}; \tag{2.4.9}$$
$$\epsilon^0 \text{ order error, Euler: } P_{er}^1+\frac{u_e^0}{r}v_{e\omega}^1-2\frac{u_e^1u_e^0}{r}; \tag{2.4.10}$$
$$\epsilon^0 \text{ order error, BL: } P_{PR}^2+\frac{1}{r}\left(u_e^0v_{p\omega}^0+u_p^0\partial_\omega\left(v_p^0+v_e^1\right)\right)+(v_p^0+v_e^1)v_{pR}^0 \tag{2.4.11}$$
$$-\frac{2}{r}\left((u_e^1+u_p^1)u_p^0+u_p^1u_e^0\right)-v_{pRR}^0; \tag{2.4.12}$$
$$\epsilon^{1/2} \text{ order error, Euler: } \frac{1}{r}u_e^1v_{e\omega}^1+v_e^1v_{er}^1-\frac{1}{r}(u_e^1)^2; \tag{2.4.13}$$
$$\epsilon^{1/2} \text{ order error, BL: } \frac{1}{r}\left(u_e^1v_{p\omega}^0+u_p^1\partial_\omega(v_p^0+v_e^1)\right)+\frac{1}{r}((u_e^0+u_p^0)v_{p\omega}^1)+v_p^1v_{pR}^0 \tag{2.4.14}$$
$$+v_p^0(v_{er}^1+v_{pR}^1)+v_e^1(v_{pR}^1)-\frac{1}{r}(2u_e^1u_p^1+(u_p^1)^2)-\frac{1}{r}v_{pR}^0+\frac{2}{r^2}(u_{p\omega}^0)-v_{pRR}^1; \tag{2.4.15}$$
$$\epsilon^1 \text{ order error, Euler: } -v_{err}^1-\frac{1}{r}v_{er}^1-\frac{1}{r^2}v_{e\omega\omega}^1+\frac{2}{r^2}(u_{e\omega}^1)+\frac{v_e^1}{r^2}; \tag{2.4.16}$$
$$\epsilon^1 \text{ order error, BL: } \frac{1}{r}u_p^1v_{p\omega}^1+\frac{1}{r}u_e^1v_{p\omega}^1+v_p^1v_{er}^1+v_p^1v_{pR}^1-\frac{1}{r}v_{pR}^1-\frac{1}{r^2}v_{p\omega\omega}^0+\frac{2}{r^2}u_{p\omega}^1 \tag{2.4.17}$$
$$+\frac{1}{r^2}v_p^0; \tag{2.4.18}$$

$$\epsilon^{3/2} \text{ order error: } -\frac{1}{r^2} v^1_{p\omega\omega}. \tag{2.4.19}$$

2.4.1 Prandtl-0 Layer

We obtain the Prandtl-0 layer equations from the ϵ^0 order angular error, equation (2.4.1), and the ϵ^{-1} order radial error, equation (2.4.7) . We also enforce the divergence free condition. Thus, after dropping the subscripts, the Prandtl-0 layer equations are the following:

$$\begin{aligned} &(u^0_e + u)u_\omega + r(v^1_e + v)u_R + P_\omega = ru_{RR}, \\ &u_\omega + \sqrt{\epsilon}v + rv_R = 0, \\ &P_R = 0. \end{aligned} \tag{2.4.20}$$

The boundary conditions we take are: $u(\omega, R_0) = u_b - u_e$; $u(0, R) = \bar{u}_0(R)$; and $v(\omega, R_0) = -v^1_e(\omega, R_0)$. As will be shown rigorously in Theorem 2.4.1, the Prandtl-0 profiles u, v decay sufficiently rapidly, and so evaluating the equation above at $R = \infty$ yields $P_\omega = 0$. This implies the Prandtl pressure is constant when coupled with $P_R = 0$. We rewrite the first equation as:

$$\begin{aligned} &(u_e + u)\, u_\omega + \left(rv + R_0 v^1_e(\omega, R_0)\right) u_R = R_0 u_{RR} + e_0 + e_1 + e_2 \\ \Rightarrow\ &(u_e + u)\, u_\omega + (rv - R_0 v(\omega, R_0))\, u_R = R_0 u_{RR} + e_0 + e_1 + e_2. \end{aligned} \tag{2.4.21}$$

Here we have defined:

$$e_0 := \sqrt{\epsilon}(R - R_0)u^0_{er}(r)u_\omega + \sqrt{\epsilon}(R - R_0)\partial_r(rv^1_e(\omega, r))u_R, \tag{2.4.22}$$

$$e_1 := \sqrt{\epsilon}(R - R_0)u_{RR}, \tag{2.4.23}$$

$$\begin{aligned} e_2 \quad &:= u_\omega \sqrt{\epsilon} \int_{R_0}^{R} \left(u^0_{er}(r(\eta)) - u^0_{er}(r) \right) d\eta + u_R \sqrt{\epsilon} \int_{R_0}^{R} \partial_r (r(\theta) v^1_e(\omega, r(\theta))) - \partial_r (r v^1_e(\omega, r)) d\theta \\ &= \epsilon u_\omega \int_{R_0}^{R} \int_{R}^{\eta} u^0_{err}(r(\theta)) d\theta d\eta + \epsilon u_R \int_{R_0}^{R} \int_{R}^{\eta} \partial_r^2 (r(\theta) v^1_e(\omega, r(\theta))) d\theta d\eta. \end{aligned} \tag{2.4.24}$$

e_2 is high order in ϵ, as will be demonstrated in a later section. e_0 and e_1 will be put into the Prandtl-1 layer, and so we are left with solving $(u_e+u)u_\omega+(rv-R_0v(\omega,R_0))u_R = R_0u_{RR}$ together with the divergence free condition and the boundary conditions described above. To satisfy the divergence free condition, we take $rv(\omega, R) = -\int_R^\infty \partial_R(rv) = \int_R^\infty u_\omega$, which ensures the profile v decays at ∞. We have the following:

Theorem 2.4.1. *Suppose* $\min\{u_b, u_e + \bar{u}_0\} \geq c_0 > 0$. *Then for* θ_0 *sufficiently small, there exists a unique solution* $u^0_p(\omega, R)$ *such that:*

$$\sup_{[0,\theta_0]} ||R^{n/2} \partial_\omega^k u^0_p||_{L^2(\mathbb{R}_+)} + ||R^{n/2} \partial_\omega^k \partial_R u^0_p||_{L^2(0,\theta_0), L^2(\mathbb{R}_+)} \leq C. \tag{2.4.25}$$

Corollary 2.4.2.

$$\sup_{[0,\theta_0]} ||R^{n/2} \partial_\omega^k \partial_R^j [u^0_p, v^0_p]||_{L^2(\mathbb{R}_+)} \leq C. \tag{2.4.26}$$

Proof. The proof follows exactly as in (GN17), using the von-Mises transformation. Let $u_e := u^0_e(R_0)$, and define $\eta(\omega, R) = \int_{R_0}^R (u_e + u^0_p(\omega, y)) dy$, $\alpha(\omega, \eta) = u_e + u^0_p(\omega, R(\eta))$, where we've used that for each fixed ω, the transformation $(\omega, R) \to (\omega, \eta(\omega, R))$ is invertible by appealing to the maximum principle. We now compute:

$$\begin{aligned} &\eta_R = u_e + u^0_p(\omega, R) = \alpha, \\ &\eta_\omega = \int_{R_0}^R u^0_{p\omega} = -\int_{R_0}^R \partial_R (r(y) v) dy = R_0 v(\omega, R_0) - rv(\omega, R), \\ &u^0_{p\omega} = \alpha_\omega + \alpha_\eta \eta_\omega = \alpha_\omega + \alpha_\eta (R_0 v(\omega, R_0) - rv(\omega, R)), \\ &u^0_{pR} = \alpha_\eta \eta_R = \alpha \alpha_\eta. \end{aligned}$$

Inserting these identities into our equation:

$$\alpha\alpha_\omega + \alpha\alpha_\eta(R_0 v(\omega, R_0) - rv(\omega, R)) + (rv - R_0 v(\omega, R_0))\alpha\alpha_\eta = R_0\left(\alpha\alpha_\eta^2 + \alpha^2\alpha_{\eta\eta}\right) \quad (2.4.27)$$

$$\Rightarrow \alpha_\omega = R_0\left(\alpha\alpha_\eta\right)_\eta .$$

On the parabolic boundary of our domain, $\alpha(0, \eta) = u_e + u_p^0(0, R(\eta))$, and $\alpha(\omega, 0) = u_e + u_p^0(\omega, 0) = u_b$, both of which are strictly positive by assumption. Using the Parabolic maximum principle, $\alpha \geq C > 0$ for some constant C, and therefore our equation is non-degenerate. The equation (2.4.27) along with the boundary conditions is identical (apart from the constant R_0) to that in (GN17), and so the rest of the proof follows in the same manner.

□

e_0, e_1 will be solved for in the Prandtl-1 layer, and so the contribution to the angular error is e_2, given in (2.4.24).

2.4.2 Euler-1 Layer

The equations for the Euler-1 Layer arise from equations (2.4.2) and (2.4.10) together with the divergence free condition. We drop the subscript for u, v and P within this section, with the understanding that the unknowns appearing are that of the Euler-1 layer. The equations read:

$$\frac{u_e^0}{r} v_\omega - \frac{2}{r} u_e^0 u + P_r = 0, \qquad \frac{u_e^0}{r} u_\omega + u_{er}^0 v + \frac{u_e^0}{r} v + \frac{1}{r} P_\omega = 0, \qquad u_\omega + v + rv_r = 0. \quad (2.4.28)$$

As described in (2.2.12) - (2.2.18), the boundary conditions are as follows:

$$v(\omega, R_0) = -v_p^0(\omega, R_0), \qquad v(0,r) = V_{b0}(r), \qquad v(\theta_0, r) = V_{b1}(r). \tag{2.4.29}$$

We go to the vorticity formulation in order to eliminate the pressure term:

$$\begin{aligned}
0 &= \partial_r \left(u_e^0 u_\omega + r u_{er}^0 v + u_e^0 v + P_\omega\right) - \partial_\omega \left(\frac{u_e^0}{r} v_\omega - \frac{2}{r} u_e^0 u + P_r\right) \\
&= u_{er}^0 u_\omega + u_e^0 u_{\omega r} + u_{er}^0 v + r u_{err}^0 v + r u_{er}^0 v_r + u_e^0 v_r + u_{er}^0 v + P_{\omega r} \\
&- \frac{u_e^0}{r} v_{\omega\omega} + \frac{2}{r} u_e^0 u_\omega - P_{r\omega} \\
&= u_e^0 u_{\omega r} + r u_{err}^0 v + u_e^0 v_r + u_{er}^0 v - \frac{u_e^0}{r} v_{\omega\omega} + \frac{2}{r} u_e^0 u_\omega \\
&= u_e^0 \left(-2v_r - r v_{rr}\right) + r u_{err}^0 v + u_e^0 v_r + u_{er}^0 v - \frac{u_e^0}{r} v_{\omega\omega} + \frac{2}{r} u_e^0 u_\omega \\
&= -u_e^0 r \Delta v + \left(r u_{err}^0 + u_{er}^0\right) v + u_e^0 \left(-2v_r - \frac{2}{r} v\right) := u_e^0 L v,
\end{aligned} \tag{2.4.30}$$

where we have defined the linear operator L through equation (2.4.30) for ease of notation, and since $u_e^0 > 0$, $0 = Lv$ if and only if $0 = u_e^0 Lv$. Define the following boundary layer corrector:

$$\mathcal{B}(\omega, r) = \left(1 - \frac{\omega}{\theta_0}\right) \frac{v_p^0(\omega, 0)}{v_p^0(0,0)} V_{b0}(r) + \frac{\omega}{\theta_0} \frac{v_p^0(\omega, 0)}{v_p^0(\theta_0, 0)} V_{b1}(r). \tag{2.4.31}$$

Due to the compatibility conditions, $\mathcal{B}$ satisfies the same boundary conditions as v. Define

$$F(\omega, r) = -r\Delta\mathcal{B} + \left(\frac{r u_{err}^0}{u_e^0} + \frac{u_{er}^0}{u_e^0}\right)\mathcal{B} + \left(-2\mathcal{B}_r - \frac{2}{r}\mathcal{B}\right) = L\mathcal{B} \tag{2.4.32}$$

Since $\mathcal{B}$ and all of its derivatives decay exponentially fast, and since by assumption $|\partial_r(V_{b0} - V_{b1})| \lesssim \theta_0$ we have $||\langle r\rangle^k F||_{W^{k,p}} \leq C$ where C independent of θ_0. Next, for χ a cutoff function supported on $[0,1]$, define

$$E_b(\omega,r) = \chi(\frac{w}{\epsilon})F(0,r) + \chi(\frac{\theta_0-\omega}{\epsilon})F(\theta_0,r), \text{ for } \epsilon << \theta_0. \tag{2.4.33}$$

Since each differentiation of the cutoff function gives ϵ^{-1}, it is easy to see that $||\langle r\rangle^n \partial_\omega^k E_b||_{L^q} \leq C\epsilon^{-k+\frac{1}{q}}$. We also record for future use that $\partial_\omega^k E_b|_{\omega=0,\theta_0} = 0$. Consider $w = v - \mathcal{B}$, then $Lw = Lv - L\mathcal{B}$. In (2.4.67), v solves $Lv = E_b$ instead of $Lv = 0$ and the error made by this is accounted for in (2.4.72). Therefore

$$Lw = -r\Delta w + \left(r\frac{u^0_{err}}{u^0_e} + \frac{u^0_{er}}{u^0_e}\right)w + \left(-2w_r - \frac{2}{r}w\right) = E_b - F := f; \qquad w = 0 \text{ on } \partial\Omega \tag{2.4.34}$$

Since $\mathcal{B}$ is arbitrarily high regularity, obtaining estimates for w suffices to obtain estimates for v. $E_b = F$ on $\{\omega = 0, \theta_0\}$ implies $f|_{\omega=0,\theta_0} = 0$.

2.4.2.1 H^1 Estimates

Multiplying (2.4.34) by w and integrating by parts yields:

$$\begin{aligned}
&\int\int -r(w_{rr} - \frac{w_r}{r} - \frac{w_{\omega\omega}}{r^2})w - \int\int 2ww_r = \int\int fw + \frac{2}{r}w^2 - \left(\frac{ru^0_{err}}{u^0_e} + \frac{u^0_{er}}{u^0_e}\right)w^2 \\
&\Rightarrow \int\int rw_r^2 + \frac{w_\omega^2}{r} \leq N(\bar{\delta})\int\int f^2 r + \bar{\delta}\frac{w^2}{r} + ||u^0_{err} + u^0_{er}r^m||_\infty \int\int \frac{w^2}{r} \\
&\qquad \leq N(\bar{\delta})\int\int f^2 r + (\bar{\delta} + \theta_0^2)\int\int \frac{w_\omega^2}{r}.
\end{aligned} \tag{2.4.35}$$

We have used the rapid decay of the derivatives of u^0_e. For θ_0 sufficiently small, we obtain $\int\int rw_r^2 + \frac{w_\omega^2}{r} + \int\int \frac{w^2}{r} \lesssim \int\int f^2 r$, where the constant does not depend on small θ_0.

We obtain weighted estimates, $||r^n w||_{H^1}$, for $n \geq 1$ by testing the above equation against

$r^n w$:

$$\begin{aligned}\int\int r^{n+1}w_r^2 + r^{n-1}w_\omega^2 &= \int\int f w r^n - \int\int (\frac{r u^0_{err}}{u^0_e} + \frac{u^0_{er}}{u^0_e}) r^n w + 2\int\int w_r w r^n + 2\int\int w^2 r^{n-1} \\ &\lesssim \int\int f^2 r^{n+1} + \bar{\delta}\int\int w^2 r^{n-1} + \theta_0 \int\int w_\omega^2 r^{n-1} \Rightarrow \\ \int\int w_r^2 r^{n+1} + w_\omega^2 r^{n-1} + w^2 r^{n-1} &\lesssim \int\int f^2 r^{n+1}. \end{aligned} \tag{2.4.36}$$

With H^1 estimates in hand, we can establish existence and uniqueness. Again to ease notation, we define

$$\tilde{L}w := -r\Delta w = f - \left(r\frac{u^0_{err}}{u^0_e} + \frac{u^0_{er}}{u^0_e} \right) w + \left(\frac{2}{r} w \right) + 2w_r := g. \tag{2.4.37}$$

Lemma 2.4.3. *There exists a unique solution $w \in H^1$ to $Lw = f$ in Ω, $w|_{\partial\Omega} = 0$.*

Proof. First, consider the following problem posed on the bounded domain $\Omega_N = \{\omega \in (0,\theta_0), R \in (R_0, R_0+N)\}$:

$$\tilde{L}w^{(N)} = g \text{ on } \Omega_N, \qquad w^{(N)}|_{\partial\Omega_N} = 0, \tag{2.4.38}$$

where $\partial\Omega_N$ includes an additional boundary component, $D = \{R = R_0 + N\}$. It is clear that $w^{(N)}$ obeys H^1 estimates given above uniformly in N, so once each $w^{(N)}$ has been constructed, we can send $N \to \infty$. We first note the following version of the fundamental positivity estimate:

$$\begin{aligned}\int\int r w_r^2 &= \int\int r|\partial_r(\frac{w}{u^0_e}u^0_e)|^2 = \int\int r|\partial_r(\frac{w}{u^0_e})u^0_e + \frac{w}{u^0_e}u^0_{er}|^2 \\ &= \int\int r|\partial_r(\frac{w}{u^0_e})|^2 (u^0_e)^2 + \int\int r w^2 \frac{(u^0_{er})^2}{(u^0_e)^2} + \int\int r u^0_e u^0_{er} \partial_r \left((\frac{w}{u^0_e})^2 \right)\end{aligned}$$

$$= \int\int r|\partial_r(\frac{w}{u_e^0})|^2(u_e^0)^2 - \int\int \left(r\frac{u_{err}^0}{u_e^0} + \frac{u_{er}^0}{u_e^0} \right) w^2 \tag{2.4.39}$$

which, due to an identical calculation to (2.2.36), yields:

$$\int\int rw_r^2 \lesssim \int\int rw_r^2 + \int\int \left(r\frac{u_{err}^0}{u_e^0} + \frac{u_{er}^0}{u_e^0} \right) w^2. \tag{2.4.40}$$

Define the bilinear forms $\tilde{K}[w,\varphi] := \int\int \nabla w \cdot \nabla\varphi r dr d\omega + \int\int \left(r\frac{u_{err}^0}{u_e^0} + \frac{u_{er}^0}{u_e^0} \right) w\varphi$, and $K[w,\varphi] := \tilde{K}[w,\varphi] - \int\int \frac{2}{r} w\varphi$. Using the positivity estimate, it is clear that $\tilde{K}$ satisfies the hypothesis of Lax-Milgram, but due to lack of coercivity we cannot directly apply Lax-Milgram to K.

Note $K[w,\varphi] = \int\int f\varphi \iff \tilde{K}[w,\varphi] = \int\int f\varphi + \int\int \frac{2}{r} w\varphi \iff w = T^{-1}(f + \frac{2}{r}w)$ where T^{-1} is the solution operator to $\tilde{K} \iff w - T^{-1}(\frac{2w}{r}) = T^{-1}(f)$. T^{-1} is compact and self-adjoint, so the Fredholm alternative applied to the operator $I - T^{-1}(2\frac{\cdot}{r})$ enables us to conclude there either exists a unique solution to the original problem $K[w,\varphi] = \int\int f\varphi$ or there exists a nontrivial kernel. The latter option is ruled out by the H^1 estimates given above. $\square$

2.4.2.2 H^2 -H^4 Estimates

Our starting point is equation (2.4.37). The boundary layer corrector is defined such that:

$$w_{\omega\omega} = 0 \text{ on } \{\omega = 0, \theta_0\}. \tag{2.4.41}$$

Differentiating (2.4.37) in ω gives the third-order equation with Neumann boundary conditions:

$$-r\Delta w_\omega = g_\omega, \quad w_{\omega\omega}|_{\partial\Omega} = 0. \tag{2.4.42}$$

Applying the multiplier w_ω and noting that $g|_{\omega=0,\theta_0} = 0$ yields:

$$\int\int rw_{r\omega}^2 + \frac{w_{\omega\omega}^2}{r} \lesssim \int\int g_\omega w_\omega = -\int\int g w_{\omega\omega} \lesssim ||rw||_{H^1} + ||rf||_{L^2} + \bar{\delta}\int\int \frac{w_{\omega\omega}^2}{r}. \tag{2.4.43}$$

Applying the weighted multiplier $r^n w_\omega$ gives weighted estimates inductively. By using (2.4.37) to express w_{rr} in terms of the rest, we have the full H^2 estimate:

$$||r^n w||_{H^2} \leq C. \tag{2.4.44}$$

We differentiate the equation (2.4.42) again in ω, giving the Dirichlet problem for $w_{\omega\omega}$:

$$-r\Delta w_{\omega\omega} = g_{\omega\omega}, \quad w_{\omega\omega}|_{\partial\Omega} = 0. \tag{2.4.45}$$

Multiplying (2.4.45) by $w_{\omega\omega}$ gives:

$$\int\int rw_{\omega\omega r}^2 + \frac{w_{\omega\omega\omega}^2}{r} \leq ||w||_{H^2}^2 - \int\int g_\omega w_{\omega\omega\omega} \leq C + ||f_\omega r^n||_{L^2}^2 \lesssim \epsilon^{-1}. \tag{2.4.46}$$

Weighted estimates are obtained inductively via the multiplier $r^n w_{\omega\omega}$. The estimate for $w_{rr\omega}$ is obtained via equation (2.4.42), and the estimate for w_{rrr} is obtained by differentiating equation (2.4.37) in r to write w_{rrr} in terms of the other third order terms. This gives:

$$||r^n w||_{H^3} \lesssim \epsilon^{-1/2}. \tag{2.4.47}$$

Evaluating (2.4.45) at $\omega = 0, \theta_0$ gives the boundary condition

$$w_{\omega\omega\omega\omega} = rF_{\omega\omega} \text{ on } \{\omega = 0, \theta_0\}. \tag{2.4.48}$$

Differentiating (2.4.45) gives the fifth-order equation:

$$-r\Delta w_{\omega\omega\omega} = g_{\omega\omega\omega}, \tag{2.4.49}$$

to which we apply the multiplier $w_{\omega\omega\omega}$:

$$\int\int -rw_{rr\omega\omega\omega}w_{\omega\omega\omega} - \int\int \frac{w_{\omega\omega\omega\omega\omega}w_{\omega\omega\omega}}{r} = \int\int rw^2_{r\omega\omega\omega} + \int\int \frac{w^2_{\omega\omega\omega\omega}}{r} \tag{2.4.50}$$

$$- \int_{\omega=\theta_0} \frac{1}{r} w_{\omega\omega\omega}w_{\omega\omega\omega\omega}\Big|_{\omega=0}^{\omega=\theta_0} \tag{2.4.51}$$

For the boundary terms, we estimate:

$$\int \frac{1}{r} w_{\omega\omega\omega}w_{\omega\omega\omega\omega}dr\Big|_{\omega=0}^{\omega=\theta_0} = \int w_{\omega\omega\omega}F_{\omega\omega}dr\Big|_{\omega=0}^{\omega=\theta_0} = \int\int w_{\omega\omega\omega\omega}F_{\omega\omega} + w_{\omega\omega\omega}F_{\omega\omega\omega} \tag{2.4.52}$$

$$\lesssim \bar{\delta}||\frac{1}{r}w_{\omega\omega\omega\omega}||^2_{L^2} + ||w||_{H^3} + C \tag{2.4.53}$$

On the right-hand side of (2.4.49), we have up to harmless factors:

$$\int\int w_{\omega\omega\omega}\Big(f_{\omega\omega\omega} + w_{\omega\omega\omega} + w_{\omega\omega\omega r}\Big) \leq -\int\int w_{\omega\omega\omega\omega}f_{\omega\omega} + \int w_{\omega\omega\omega}F_{\omega\omega}dr\Big|_{\omega=0}^{\omega=\theta_0} \tag{2.4.54}$$

$$+ \bar{\delta}||w_{\omega\omega\omega r}||^2_{L^2} + ||w||^2_{H^3} \leq \bar{\delta}||w_{\omega\omega\omega r}||^2_{L^2} + \bar{\delta}||\frac{1}{r}w_{\omega\omega\omega\omega}||^2_{L^2} + \epsilon^{-3}. \tag{2.4.55}$$

Again, these calculations may be repeated using the weighted multiplier, $r^n w_{\omega\omega\omega}$, to obtain weighted estimates. Putting the above calculations together gives:

$$\int\int r^{n-1}w^2_{\omega\omega\omega\omega} + r^{n+1}w^2_{r\omega\omega\omega} \lesssim \epsilon^{-3}. \tag{2.4.56}$$

The estimate for $w_{rr\omega\omega}$ can be obtained from (2.4.45), the estimate for $w_{rrr\omega}$ may be obtained by differentiating (2.4.42) in r, and finally the estimate for w_{rrrr} may be obtained by differentiating (2.4.37) twice in r, ultimately yielding:

$$||r^n w||_{H^4} \lesssim \epsilon^{-3/2}. \tag{2.4.57}$$

2.4.2.3 $W^{k,q}$ Estimates

$W^{k,q}$ estimates are obtained using the framework of Agmon-Douglis-Nirenberg, (ADN59), where $k \leq 4$ and $q \in (1, \infty)$. To do so, cover the interior of the boundary $\{r = R_0\}$ using one open set, $\mathcal{U}^c$, the boundaries $\{\omega = 0, \theta_0\}$ using $\mathcal{U}^a$,$\mathcal{U}^b$ and the interior of the domain using $\mathcal{U}^d$. Let $\psi^{a,b,c,d}$ denote the partition of unity associated to this covering, and $w^{a,b,c,d} := \psi^{a,b,c,d} w$. $w^{a,b,c,d}$ satisfy the equation:

$$\begin{aligned} - r\Delta w^X &= -r w^X_{rr} - w^X_r - \frac{w^X_{\omega\omega}}{r} = \psi^X f + \left(r\frac{u^0_{err}}{u^0_e} + \frac{u^0_{er}}{u^0_e} \right) w^X + \left(2w^X_r + \frac{2}{r} w^X \right) \\ &\quad - 2r\psi^X_r w_r - r\psi^X_{rr} w - 3\psi^X_r w - \frac{2}{r}\psi^X_\omega w_\omega - \frac{\psi^X_{\omega\omega}}{r} w =: f^X, \quad X = a,b,c,d. \end{aligned} \tag{2.4.58}$$

The estimates for w^d follow from the standard interior $W^{k,q}$ estimates:

$$||w^d||_{W^{2,q}(\mathcal{U}^d)} \lesssim ||f||_{L^q(\Omega)} + ||w||_{W^{1,q}(\Omega)} \leq ||f||_{L^q(\Omega)} + ||w||_{H^2(\Omega)} \leq C(\theta_0), \tag{2.4.59}$$

$$||w^d||_{W^{3,q}(\mathcal{U}^d)} \lesssim ||f_\omega||_{L^q(\Omega)} + ||w||_{W^{2,q}(\Omega)} \leq C(\theta_0)\epsilon^{-1+\frac{1}{q}}, \tag{2.4.60}$$

$$||w^d||_{W^{4,q}(\mathcal{U}^d)} \lesssim ||f_{\omega\omega}||_{L^q(\Omega)} + ||w||_{W^{3,q}(\Omega)} \leq C(\theta_0)\epsilon^{-2+\frac{1}{q}}. \tag{2.4.61}$$

$C(\theta_0)$ is a constant that could depend poorly on θ_0. The weighted estimates, $||r^n w^d||_{W^{k,q}(\mathcal{U}^d)}$, are obtained by using the weighted H^k estimates. w^c is supported away from the corners of the domain, and so we can repeat a similar analysis as for w^c, remaining cognizant of the boundary condition $\partial_\omega^k w^c|_{r=R_0} = 0$ for $k \geq 0$. It remains to estimate $w^{a,b}$ which follow by taking odd angular extensions across the boundaries $\{\omega = 0\}$ and $\{\omega = \theta_0\}$. For concreteness, we proceed to treat the w^a case, with the w^b estimate being identical. Define:

$$\tilde{w}^a(\omega, r) = -w^a(-\omega, r) \text{ for } \omega \in (-\theta_0, 0), \qquad \tilde{w}^a = w^a \text{ for } \omega \in (0, \theta_0)$$

Applying a cutoff function:

$$\bar{w}^a = \chi(\omega)\tilde{w}^a, \quad \text{supp}(\chi) \subset (-\frac{\theta_0}{2}, \frac{\theta_0}{2}) \times (R_0, \infty) \tag{2.4.62}$$

ensures that $\bar{w}^a$ satisfies the boundary-value problem:

$$-r\Delta\bar{w}^a = -r\Delta\left(\chi(\omega)\tilde{w}^a\right) = \chi(\omega)\tilde{f}^a + \frac{\chi_{\omega\omega}}{r}\tilde{w}^a + 2\frac{\chi_\omega}{r}\tilde{w}^a_\omega, \tag{2.4.63}$$

$$\bar{w}^a|_{R=R_0} = \bar{w}^a|_{\omega=-\theta_0,\theta_0} = 0, \tag{2.4.64}$$

where $\tilde{f}^c$ is the odd angular extension of f^c. Moreover, $\bar{w}^a \in W^{4,q}$ whenever $w^a \in W^{4,q}$ by the condition $w = w_{\omega\omega} = 0$ at $\omega = 0$. Applying the standard $W^{2,q}$ estimates to the boundary-value problem in (2.4.63) gives:

$$||\bar{w}^a||_{W^{2,q}} \leq C(\theta_0). \tag{2.4.65}$$

We can differentiate the equation (2.4.63) in ω twice as $\bar{w}^a$ vanishes on a neighborhood of $\{\omega = -\theta_0, \theta_0\}$, and repeatedly apply the $W^{2,q}$ estimates. The full $W^{3,q}$ estimate is then recovered using the same procedure as in estimate (2.4.47), and the full $W^{4,q}$ estimate on w^c is recovered using the same procedure as in estimate (2.4.57). Combined with the estimates on $w^{c,d}$ we have established:

Lemma 2.4.4 ($W^{k,q}$ estimates, $q \in (1,\infty)$)**.**

$$||w||_{W^{2,q}} \lesssim C(\theta_0), \qquad ||w||_{W^{3,q}} \lesssim C(\theta_0)\epsilon^{-1+\frac{1}{q}}, \qquad ||w||_{W^{4,q}} \lesssim C(\theta_0)\epsilon^{-2+\frac{1}{q}}, \tag{2.4.66}$$

where $C(\theta_0)$ depends poorly on θ_0.

2.4.2.4 Construction of Euler-1 Layers

The Euler-1 layers are defined to solve:

$$-u_e^0 r\Delta v_e^1 + \left(ru^0_{err} + u^0_{er}\right) v_e^1 + u_e^0\left(-2v_r - \frac{2}{r}v_e^1\right) = u_e^0 E_b, \tag{2.4.67}$$

$$u_e^1(\omega, r) = u_e^1(0,r) - \int_0^\omega \partial_r(rv_e^1)d\theta. \tag{2.4.68}$$

The pressure P_e^1 is defined to solve equation with P_r in (2.4.28) exactly: $P_e^1(\omega, r) = -\int_r^\infty \frac{2}{r}u_e^0 u - \frac{u_e^0}{r}v_\omega$. The error made in the P_ω equation in (2.4.28) is estimated as:

$$\begin{aligned} &u_e^0 u_\omega + ru^0_{er}v + u_e^0 v + P_\omega \\ &= u_e^0 u_\omega + ru^0_{er}v + u_e^0 v - \int_r^\infty \frac{2}{r}u_e^0 u_\omega + u_e^0 E_b + u_e^0(rv_{rr}) + 3u_e^0 v_r + \left(\frac{2u_e^0}{r} - ru^0_{err} - u^0_{er}\right)v. \end{aligned} \tag{2.4.69}$$

By direct computation:

$$\partial_r(u_e^0 u_\omega) = -u^0_{er}v - ru^0_{er}v_r - 2u_e^0 v_r - ru_e^0 v_{rr}, \tag{2.4.70}$$

$$\partial_r(ru^0_{er}v) = u^0_{er}v + ru^0_{err}v + ru^0_{er}v_r,$$

$$\partial_r(u_e^0 v) = u^0_{er}v + u_e^0 v_r.$$

Each of the three terms above is known to be in H^1, and therefore decay at infinity. We can write:

$$u_e^0 u_\omega + r u_{er}^0 v + u_e^0 v = -\int_r^\infty \partial_r (u_e^0 u_\omega + r u_{er}^0 v + u_e^0 v). \tag{2.4.71}$$

Matching these terms with those in the integral in equation (2.4.69), the only term remaining is:

$$\int_r^\infty u_e^0(\theta) E_b(\omega, \theta) d\theta. \tag{2.4.72}$$

This represents the second contribution to the angular error. The estimates for the Euler-1 layer are summarized:

Theorem 2.4.5 (Euler-1 Profile Estimates)**.**

$$||r^n v_e^1||_\infty + ||r^n v_e^1||_{H^2} \le C, \quad ||r^n v_e^1||_{H^3} \le C\epsilon^{-\frac{1}{2}}, \quad ||r^n v_e^1||_{H^4} \le C\epsilon^{-\frac{3}{2}}; \tag{2.4.73}$$

$$||r^n v_e^1||_{W^{2,q}} \le C(\theta_0), \quad ||r^n v_e^1||_{W^{3,q}} \le C(\theta_0)\epsilon^{-1+\frac{1}{q}}, \quad ||r^n v_e^1||_{W^{4,q}} \le C(\theta_0)\epsilon^{-2+\frac{1}{q}}, \tag{2.4.74}$$

where $C(\theta_0)$ could depend poorly on θ_0, for $q = (1, \infty)$. By definition of u_e^1, we have:

$$||r^n u_e^1||_{H^1} \le C, \quad ||r^n u_e^1||_\infty \le C. \tag{2.4.75}$$

Proof. Only the uniform bound on u_e^1 must be proven. To do so, we use the following:

$$\begin{aligned} &|u_e^1(\omega, r)|^2 = \left| \int_0^\omega \partial_r (rv)(\theta, r) d\theta \right|^2 \lesssim \int_0^{\theta_0} |\partial_r (rv)|^2 (\theta, r) d\theta \Rightarrow \\ &\sup_{[0,\theta_0]} \left| u_e^1(\omega, r) \right|^2 \le \int_0^{\theta_0} |\partial_r (rv)|^2 (\theta, r) d\theta \Rightarrow ||u_e^1||_\infty^2 \le \sup_{r \in [R_0, \infty)} \int_0^{\theta_0} |\partial_r (rv)|^2 (\theta, r) d\theta. \end{aligned} \tag{2.4.76}$$

Calling $\varphi(r) = \int_0^{\theta_0} |\partial_r(rv)|^2 \, d\theta$, the Sobolev embedding in $\mathbb{R}^1$ gives:

$$\sup_{r\in[R_0,\infty)} |\varphi(r)| \le ||\varphi||_{L^1} + ||\partial_r \varphi||_{L^1} \lesssim \int\int |\partial_r(rv)|^2 + \int\int |\partial_{rr}(rv)|^2 \lesssim ||r^n v_e^1||_{H^2} \le C. \tag{2.4.77}$$

We can proceed inductively to obtain weighted estimates on $||r^n u_e^1||_\infty$. These estimates are independent of small θ_0.

$\square$

2.4.3 Prandtl-1 Layer

2.4.3.1 Galerkin Formulation and a-Priori Estimates

In this subsection, we solve for the Prandtl-1 layers, u_p^1, v_p^1. For this subsection, we drop the subscripts on u_p^1, v_p^1. The starting point is a modification of equation (2.4.3):

$$\begin{aligned}(u_e^1 + u)u^0_{p\omega} + (u_e^0 + u_p^0)u_\omega + u_p^0 u^1_{e\omega} + rv_p^0 u^0_{er} + r(v_p^0 + v_e^1)u_R + ru^0_{pR}v + \\ (u_e^0 + u_p^0)v_p^0 + u_p^0 v_e^1 + P^1_{p\omega} + E_0 + E_1 + \sqrt{\epsilon} u^0_{er} v = ru_{RR} + u^0_{pR},\end{aligned} \tag{2.4.78}$$

where we have included E_0 and E_1, the contributions from the Prandtl-0 layer construction:

$$E_0 := (R - R_0)u^0_{er}(r)u^0_{p\omega} + (R - R_0)\partial_r(rv_e^1(\omega, r))u^0_{pR}, \quad E_1 := (R - R_0)u^0_{pRR}. \tag{2.4.79}$$

The boundary conditions are: $v_p^1(\omega, R_0) = 0; u_p^1(\omega, R_0) = -u_e^1(\omega, R_0); u_p^1(0, R) = \bar{u}_1(R)$. In order to solve the $\epsilon^{-1/2}$ order error in the radial equation, we take:

$$P_P^1 = \int_R^\infty \frac{(u_p^0)^2}{r(t)} + 2u_e^0 \frac{u_p^0}{r(t)} dt \Rightarrow P_{P\omega}^1 = \int_R^\infty 2\frac{u_p^0 u_{p\omega}^0}{r(t)} + 2u_e^0 \frac{u_{p\omega}^0}{r(t)} dt. \tag{2.4.80}$$

By the rapid decay of the u_p^0 terms, we have $|\partial^\alpha P_{p\omega}^1| \leq R^{-n}$ for arbitrarily large n and for any multi-index α. Let $u^0(\omega, R) = u_e^0(r) + u_p^0(w, R)$. We define

$$F = -u_e^1 u_{p\omega}^0 - u_p^0 u_{e\omega}^1 - r v_p^0 u_{er}^0 - u^0 v_p^0 - u_p^0 v_e^1 + u_R^0 - P_{P\omega}^1, \tag{2.4.81}$$

which gives:

$$u_\omega^0 u + u^0 u_\omega + r(v_p^0 + v_e^1) u_R + r u_R^0 v - r u_{RR} = F - E_1 - E_0. \tag{2.4.82}$$

We take ∂_R of the above equation, use the divergence-free condition, and divide by u^0 to obtain:

$$\begin{aligned} -\partial_{RR}(rv) + \frac{1}{u^0} r u_{RR}^0 v - \frac{1}{u^0}\partial_R(r u_{RR}) &= \frac{1}{u^0}\left(F_R - (E_0 + E_1)_R\right) \\ &\quad - \frac{1}{u^0}\left(u_{\omega R}^0 u + u_\omega^0 u_R + \partial_R\left(r(v_p^0 + v_e^1) u_R\right)\right) := G. \end{aligned} \tag{2.4.83}$$

Now we take ∂_ω of the above equation and again use the divergence free condition:

$$\begin{aligned} -\partial_{RR}(r v_\omega) + \frac{1}{u^0} r u_{RR}^0 v_\omega + \frac{1}{u^0}\partial_R\left(r\partial_{RRR}(rv)\right) v &= G_\omega - \partial_\omega\left(\frac{1}{u^0} r u_{RR}^0\right) v \\ &\quad + \partial_\omega\left(\frac{1}{u^0}\right)\partial_R\left(r u_{RR}\right). \end{aligned} \tag{2.4.84}$$

With an eye towards obtaining a weak formulation of the equation together with a-priori estimates, we rewrite the third term on the left-hand side as follows:

$$\frac{1}{u^0}\partial_R\left(r\partial_R^3(rv)\right) = r\partial_R^2(\frac{1}{u^0}\partial_R^2(rv)) + \sqrt{\epsilon}\frac{1}{u^0}\partial_R^3(rv) + r\partial_R^2(\frac{1}{u^0})\partial_R^2(rv) - 2r\partial_R(\frac{1}{u^0})\partial_R^3(rv). \tag{2.4.85}$$

Therefore, our equation now becomes:

$$\begin{aligned}&-\partial_{RR}(rv_\omega) + \frac{1}{u^0}ru^0_{RR}v_\omega + r\partial_R^2(\frac{1}{u^0}\partial_R^2(rv)) + \sqrt{\epsilon}\frac{1}{u^0}\partial_R^3(rv)\\ &- r\partial_R^2(\frac{1}{u^0})\partial_R^2(rv) - 2r\partial_R(\frac{1}{u^0})\partial_R^3(rv) = G_\omega - \partial_\omega(\frac{1}{u^0}ru^0_{RR})v\\ &+ \partial_\omega\left(\frac{1}{u^0}\right)\partial_R(ru_{RR}) = (2.4.86.1) - (2.4.86.9),\end{aligned} \tag{2.4.86}$$

and so we must obtain a-priori estimates for the equation:

$$-\partial_{RR}(rv_\omega) + \frac{1}{u^0}ru^0_{RR}v_\omega + r\partial_R^2(\frac{1}{u^0}\partial_R^2(rv)) = f_R + g. \tag{2.4.87}$$

Lemma 2.4.6. *There exists a unique solution v to Equation (2.4.87) on the domain $(0,\theta_0)\times(R_0, R_0+N)$ subject to the boundary conditions $v, v_R = 0$ at $\{R = R_0, R_0+N\}$ and the initial condition $v = \bar{v}_0$ at $\{\omega = 0\}$. This solution v satisfies the following estimate, uniform in N:*

$$\int\int|\partial_R(rv_\omega)|^2 + \sup\int r|\partial_{RR}(rv)|^2 \lesssim \int\int f^2 + \int\int g^2\langle R-R_0\rangle^3 + \int_{\omega=0} r|\partial_{RR}(rv)|^2. \tag{2.4.88}$$

Proof. Define an inner product by $[[v,w]] := \int_{R_0}^N \partial_R(rv)\partial_R(rw) + \int_{R_0}^N \frac{u^0_{RR}}{u^0}rvw$. All of the properties of inner-product follow from the properties of the integral, aside from non degeneracy. Supposing $[[v,v]] = 0$, by the positivity estimate, (2.2.33), $\int|\partial_R(rv)|^2 = 0 \Rightarrow |\partial_R(rv)| = 0$, coupled with the fact that $v = v_R = 0$ at $\{R = R_0, R_0+N\}$ implies $v = 0$.

Let e_j represent an orthonormal basis for H^1 with respect to this inner product. The weak formulation of (2.4.87) reads:

$$[[v_\omega, e_j]] + \int \frac{1}{u^0}\partial_{RR}(rv)\partial_{RR}(re_j) = -\int fe_{jR} + \int ge_j. \tag{2.4.89}$$

Define $v^k(\omega, R) = \sum_{i=1}^{k} a_{ik}(\omega)e_i(R)$. Inserting this into the weak formulation above enables us to solve the corresponding ODE for the coefficients a_{ik}. Multiplying the weak formulation by $\partial_\omega a_{ik}$ and summing over i yields:

$$[[v^k_\omega, v^k_\omega]] + \int \frac{1}{u^0}\partial_{RR}(rv^k_\omega)\partial_{RR}(rv^k) = -\int fv^k_{\omega R} + \int gv^k_\omega. \tag{2.4.90}$$

In order to pass to the limit as $k \to \infty$, we must obtain a-priori estimates for the above equation. We relabel v^k by v for this purpose. In order to obtain the desired a-priori estimate, we multiply equation (2.4.87) by rv_ω and integrate by parts:

$$\int -\partial_{RR}(rv_\omega)rv_\omega + \frac{1}{u^0}r^2u^0_{RR}v^2_\omega \geq \int |\partial_R(rv_\omega)|^2 \tag{2.4.91}$$

by the Positivity estimate, (2.2.33). Next, we have

$$\int \partial^2_R(\frac{1}{u^0}\partial^2_R(rv))r^2v_\omega = \partial_\omega \frac{1}{2}\int_{R_0}^\infty \frac{r}{u^0}|\partial_{RR}(rv)|^2 + 2\sqrt{\epsilon}\int_{R_0}^\infty \frac{1}{u^0}\partial_{RR}(rv)\partial_R(rv_\omega). \tag{2.4.92}$$

On the right-hand-side, we have:

$$\int f_R rv_\omega + \int grv_\omega \leq \int f^2 + \int g^2\langle R - R_0\rangle^3 + \delta\int |\partial_R(rv_\omega)|^2. \tag{2.4.93}$$

Using Gronwall's Inequality and integrating yields:

$$\int\int |\partial_R(rv_\omega)|^2 + \sup \int_{R_0}^{\infty} r|\partial_{RR}(rv)|^2 \lesssim \int\int f^2 + \int\int g^2 \langle R - R_0 \rangle^3 + \int_{\omega=0} r|\partial_{RR}(rv)|^2. \tag{2.4.94}$$

□

Next, the following high regularity estimate is established:

Lemma 2.4.7.

$$\begin{aligned}\int\int |\partial_R(rv_{\omega\omega})|^2 + \sup \int r|\partial_{RR}(rv_\omega)|^2 \lesssim \int\int f^2 + \int\int g^2\langle R - R_0\rangle^3 + \int_{R_0}^{\infty} r|\partial_{RR}(rv(0,\cdot))|^2 \\ + \int\int f_\omega^2 + \int\int g_\omega^2 \langle R - R_0\rangle^3 + \int_{R_0}^{\infty} r|\partial_{RR}(rv_\omega(0,\cdot))|^2.\end{aligned} \tag{2.4.95}$$

Proof. Differentiating the equation (2.4.87) once in ω yields:

$$\begin{aligned}-\partial_{RR}(rv_{\omega\omega}) + \left(\frac{u^0_{RR}}{u^0}\right)_\omega rv_\omega + \left(\frac{u^0_{RR}}{u^0}\right) rv_{\omega\omega} + r\partial_{RR}\left(\partial_\omega\left(\frac{1}{u^0}\right)\partial_{RR}(rv)\right) \\ + r\partial_{RR}\left(\frac{1}{u^0}\partial_{RR}(rv_\omega)\right) = f_{R\omega} + g_\omega.\end{aligned} \tag{2.4.96}$$

In order to obtain a-priori estimates of the equation (2.4.96), we multiply the above equation by $rv_{\omega\omega}$, remaining cognizant of the boundary condition that $v_R = 0 \Rightarrow v_{\omega\omega R} = v_{\omega R} = 0$ at $\{R = R_0, R_0 + N\}$:

$$\begin{aligned}\int \Bigg[- \partial_{RR}(rv_{\omega\omega}) + \left(\frac{u^0_{RR}}{u^0}\right)_\omega rv_\omega + \left(\frac{u^0_{RR}}{u^0}\right) rv_{\omega\omega} + r\partial_{RR}\left(\partial_\omega\left(\frac{1}{u^0}\right)\partial_{RR}(rv)\right) \\ + r\partial_{RR}\left(\frac{1}{u^0}\partial_{RR}(rv_\omega)\right)\Bigg] rv_{\omega\omega} = \int \Big[f_{R\omega} + g_\omega\Big] rv_{\omega\omega}.\end{aligned} \tag{2.4.97}$$

First, applying the positivity estimate (2.2.33),

$$-\int \partial_{RR}(rv_{\omega\omega})rv_{\omega\omega} + \left(\frac{u^0_{RR}}{u^0}\right) r^2 v^2_{\omega\omega} \geq \int |\partial_R(rv_{\omega\omega})|^2. \tag{2.4.98}$$

For the second term in (2.4.96), we have:

$$\begin{aligned}\int \left(\frac{u^0_{RR}}{u^0}\right)_\omega r^2 v_\omega v_{\omega\omega} &\leq \left(\int R^{-n} v^2_\omega\right)^{1/2} \left(\int R^{-n} v^2_{\omega\omega}\right)^{1/2} \leq \left(\int |\partial_R(v_\omega)|^2\right)^{1/2} \left(\int |\partial_R(v_{\omega\omega})|^2\right)^{1/2} \\ &\leq N(\bar{\delta}) \int |\partial_R(v_\omega)|^2 + \bar{\delta} \int |\partial_R(v_{\omega\omega})|^2.\end{aligned} \tag{2.4.99}$$

For the fourth term in (2.4.96), we have:

$$\begin{aligned}&\int r\partial_{RR}\left(\partial_\omega\left(\frac{1}{u^0}\right)\partial_{RR}(rv)\right) rv_{\omega\omega} = 2\sqrt{\epsilon}\int \left(\frac{1}{u^0}\right)_\omega \partial_{RR}(rv)\partial_R(rv_{\omega\omega}) \\ &+ \int r\left(\frac{1}{u^0}\right)_\omega \partial_{RR}(rv)\partial_{RR}(rv_{\omega\omega}) = (2.4.100.1) + (2.4.100.2).\end{aligned} \tag{2.4.100}$$

For the fifth term in (2.4.96), we have:

$$\begin{aligned}&\int \partial_{RR}\left(\frac{1}{u^0}\partial_{RR}(rv_\omega)\right) r^2 v_{\omega\omega} = \int \frac{1}{u^0}\partial_{RR}(rv_\omega)\partial_{RR}(r^2 v_{\omega\omega}) \\ &= \int \frac{1}{u^0}\partial_{RR}(rv_\omega)\left(2\sqrt{\epsilon}\partial_R(rv_{\omega\omega}) + r\partial_{RR}(rv_{\omega\omega})\right) \\ &= \partial_\omega \int \frac{1}{u^0}\left|\partial_{RR}(rv_\omega)\right|^2 - \int \left(\frac{1}{u^0}\right)_\omega |\partial_{RR}(rv_\omega)|^2 + \int \frac{1}{u^0}\partial_{RR}(rv_\omega)(2\sqrt{\epsilon}\partial_R(rv_{\omega\omega})\end{aligned} \tag{2.4.101}$$

$$= (2.4.101.1) + (2.4.101.2) + (2.4.101.3).$$

Finally, for the right-hand side,

$$\int f_{R\omega} rv_{\omega\omega} + \int g_\omega rv_{\omega\omega} \leq \int f^2_\omega + \int g^2_\omega \langle R - R_0\rangle^3 + \delta \int \partial_R(rv_{\omega\omega})^2. \tag{2.4.102}$$

(2.4.101.3) and (2.4.100.1) are estimated through Young's inequality. For (2.4.100.2) , we write:

$$\partial_{RR}(rv)\partial_{RR}(rv_{\omega\omega}) = \partial_\omega\left[\partial_{RR}(rv)\partial_{RR}(rv_\omega)\right] - \partial_{RR}(rv_\omega).$$

Applying Gronwall then yields:

$$\int\int |\partial_R(rv_{\omega\omega})|^2 + \sup\int r|\partial_{RR}(rv_\omega)|^2 \lesssim \int\int f^2 + \int\int g^2\langle R-R_0\rangle^3 + \int_{R_0}^\infty r|\partial_{RR}(rv(0,\cdot))|^2 + \int\int f_\omega^2 + \int\int g_\omega^2\langle R-R_0\rangle^3 + \int_{R_0}^\infty r|\partial_{RR}(rv_\omega(0,\cdot))|^2 + \int_{\omega=\theta_0}\partial_{RR}(rv)\partial_{RR}(rv_\omega) - \int_{\omega=0}\partial_{RR}(rv)\partial_{RR}(rv_\omega).$$

The final two terms are estimated through Young's inequality, where we recall Lemma 2.4.6 to estimate $N\int_{\omega=0,\theta_0}|\partial_{RR}(rv)|^2$.

□

We now obtain the weighted variants of the above two lemmas. Notationally, depict the weight by $p_m(R) = \langle R-R_0\rangle^m$.

Lemma 2.4.8.

$$\int\int p_m|\partial_R(rv_\omega)|^2 + \sup\int p_m r|\partial_{RR}(rv)|^2 \lesssim \int\int p_m f^2 + \int\int g^2 p_{m+3} + \int_{\omega=0} rp_m|\partial_{RR}(rv)|^2; \tag{2.4.103}$$

and

$$\int\int p_m|\partial_\omega\partial_R(rv_\omega)|^2 + \sup\int p_m r|\partial_{RR}(rv_\omega)|^2 \lesssim \int\int p_m f^2 + \int\int g^2 p_{m+3} + \int\int p_m f_\omega^2 + \int\int g_\omega^2 p_{m+3} + \int_{\omega=0} rp_m|\partial_{RR}(rv_\omega)|^2 + \int_{\omega=0} rp_m|\partial_{RR}(rv)|^2. \tag{2.4.104}$$

We will proceed in several steps to establish Lemma 2.4.8. As these are a-priori estimates and we eventually plan to send $N \to \infty$, we work in the domain $R \in (R_0, \infty)$. Define $w(R) = (R - R_0)^m$ on $[R_0, R_0 + M]$ and $w(R) = M^m$ for $R \geq M + R_0$. Note also that $w(R_0) = 0$, which eliminates boundary contributions from $\{r = R_0\}$.

Claim 1. $\sup \int w(R)|\partial_{RR}(rv)|^2 + \int\int rw(R)|\partial_R^3(rv)|^2 \lesssim \int_{\omega=0} p_m|\partial_{RR}(rv)|^2 + \int\int f^2 p_m + \int\int g^2 p_m$

Proof. Multiplying equation (2.4.87) by $-w(R)\partial_{RR}(rv)$ yields:

$$\int \left[\partial_{RR}(rv_\omega) - \frac{ru^0_{RR}}{u^0}v_\omega - r\partial_R^2(\frac{1}{u^0}\partial_R^2(rv))\right]w\partial_{RR}(rv) = \int (f_R + g)w\partial_{RR}(rv). \quad (2.4.105)$$

Integrating (2.4.105) by parts yields:

$$\partial_\omega \int w|\partial_{RR}(rv)|^2 + \int \frac{\partial_R(wr)}{u^0}\partial_R^3(rv)\partial_R^2(rv) + \int \frac{rw}{u^0}|\partial_R^3(rv)|^2 \leq \int (f^2 + g^2)w + J, \quad (2.4.106)$$

where J contains:

$$J = -\int \frac{ru^0_{RR}}{u^0}v_\omega w\partial_{RR}(rv) + \int R^{-N}|\partial_R^3(rv)||\partial_R^2(rv)| + \int R^{-n}|\partial_{RR}(rv)|^2. \quad (2.4.107)$$

Since $\partial_R(rw) = (m+1)\sqrt{\epsilon}w + R_0 m(R - R_0)^{m-1}$:

$$\int \frac{\partial_R(wr)}{u^0}\partial_R^3(rv)\partial_R^2(rv) = \int \frac{(m+1)\sqrt{\epsilon}w}{u^0}\partial_R^3(rv)\partial_R^2(rv) + mR_0\int \frac{(R-R_0)^{m-1}}{u^0}\partial_R^3(rv)\partial_R^2(rv)$$
$$\lesssim \sqrt{\epsilon}\int \frac{1}{u^0}|\partial_R^3(rv)|^2 + \sqrt{\epsilon}\int \frac{1}{u^0}|\partial_R^2(rv)|^2 + \int \frac{(R-R_0)^{m-1}}{u^0}|\partial_R^3(rv)|^2 + \int \frac{(R-R_0)^{m-1}}{u^0}|\partial_R^2(rv)|^2. \quad (2.4.108)$$

The first two terms above can be absorbed into (2.4.106), and the third and fourth terms above have been estimated inductively. Applying Gronwall and integrating yields the desired lemma.

□

Claim 2. $\sup\int w|\partial_{RR}(rv_\omega)|^2 + \int\int rw|\partial_R^3(rv_\omega)|^2 \lesssim \int_{\omega=0} p_m|\partial_{RR}(rv)|^2 + \int\int f^2 p_m + \int\int g^2 p_m + \int_{\omega=0} p_m|\partial_{RR}(rv_\omega)|^2 + \int\int f_\omega^2 p_m + \int\int g_\omega^2 p_m.$

Proof. We apply the multiplier $-w(R)\partial_{RR}(rv_\omega)$ to the differentiated equation (2.4.96):

$$\int (\text{Equation } 2.4.96) \times (-w(R)\partial_{RR}(rv_\omega)) \,. \tag{2.4.109}$$

On the left-hand-side, we have:

$$\partial_\omega \int w|\partial_{RR}(rv_\omega)|^2 + \int \frac{rw}{u^0}|\partial_R^3(rv_\omega)|^2 + \int \frac{\partial_R(rw)}{u^0}\partial_R^3(rv_\omega)\partial_R^2(rv_\omega) + J + I. \tag{2.4.110}$$

Here $J = \int \partial_R\left(\left(\frac{1}{u^0}\right)_\omega \partial_{RR}(rv)\right)\partial_R(rw\partial_{RR}(rv_\omega)) + \int\left(\frac{1}{u^0}\right)_R \partial_{RR}(rv_\omega)\partial_R(rw\partial_{RR}(rv_\omega))$ and $I \le \int v_{\omega\omega R}^2 + |\partial_{RR}(rv_\omega)|^2$. J and the third term of (2.4.110) can be treated through Young's inequality and induction after noticing that $\partial_R(rw) = (m+1)\sqrt{\epsilon}w + R_0 m(R-R_0)^{m-1}$ as in the previous lemma. The desired result now follows from Gronwall.

□

Claim 3. $\int\int w|\partial_R(rv_\omega)|^2 + \sup\int wr|\partial_R^2(rv)|^2 \lesssim \int_{\omega=0} rp_m|\partial_{RR}(rv)|^2 + \int\int f^2 p_m + \int\int g^2 p_{m+3}.$

Proof. By Claim 2, the quantity $\partial_R\left(\frac{r}{u^0}\partial_{RR}(rv)\right)$ is integrable and so integrating equation

(2.4.87) from R to ∞ to yield:

$$-\partial_R(rv_\omega) + \int_R^\infty \frac{ru^0_{RR}}{u^0} v_\omega + \partial_R \left(\frac{r}{u^0} \partial_R^2(rv) \right) - \sqrt{\epsilon} \frac{1}{u^0} \partial_R^2(rv) = f + \int_R^\infty g. \tag{2.4.111}$$

Multiplying the left-hand side of 2.4.111 by $-w(R)\partial_R(rv_\omega)$ gives:

$$\begin{aligned} &\int \left[\partial_R(rv_\omega) - \int_R^\infty \frac{ru^0_{RR}}{u^0} v_\omega - \partial_R \left(\frac{r}{u^0} \partial_R^2(rv) \right) + \sqrt{\epsilon} \frac{1}{u^0} \partial_R^2(rv) \right] w \partial_R(rv_\omega) \\ &= \int w |\partial_R(rv_\omega)|^2 + \int \frac{r}{u^0} \partial_R^2(rv) w(R) \partial_R^2(rv_\omega) + J, \end{aligned} \tag{2.4.112}$$

where

$$J = \int \frac{r}{u^0} \partial_R^2(rv) w'(R) \partial_R(rv_\omega) + \int \frac{\sqrt{\epsilon}}{u^0} \partial_{RR}(rv) w \partial_R(rv_\omega) - \int w \partial_R(rv_\omega) \int_R^\infty \frac{ru^0_{RR}}{u^0} v_\omega. \tag{2.4.113}$$

J can be controlled by Hardy's inequality, Young's inequality, and induction for the $rw' = \sqrt{\epsilon} m w + R_0 m (R - R_0)^{m-1}$ factor in the $\int \frac{r}{u^0} \partial_R^2(rv) w'(R) \partial_R(rv_\omega)$ term, as in the previous lemma. The result then follows from Gronwall.

□

Claim 4. $\int\int w(R)|\partial_R(rv_{\omega\omega})|^2 + \sup \int w(R) \partial_{RR}(rv_\omega) \lesssim \int_{\omega=0} p_m |\partial_{RR}(rv)|^2 + \int\int f^2 p_m + \int\int g^2 p_m + \int_{\omega=0} p_m |\partial_{RR}(rv_\omega)|^2 + \int\int f_\omega^2 p_m + \int\int g_\omega^2 p_{m+3}$

Proof. We differentiate equation (2.4.111) in ω to obtain:

$$-\partial_R(rv_{\omega\omega}) + \int_R^\infty \partial_\omega \left(\frac{ru^0_{RR}}{u^0} v_\omega \right) + \partial_{R\omega} \left(\frac{r}{u^0} \partial_R^2(rv) \right) - \sqrt{\epsilon} \partial_\omega \left(\frac{1}{u^0} \partial_R^2(rv) \right) \tag{2.4.114}$$

$$= f_\omega + \int_R^\infty g_\omega.$$

Multiplying (2.4.114) by $-w(R)\partial_R(rv_{\omega\omega})$ yields on the left-hand-side:

$$\int \left[\partial_R(rv_{\omega\omega}) - \int_R^\infty \partial_\omega\left(\frac{ru^0_{RR}}{u^0}v_\omega\right) - \partial_{R\omega}\left(\frac{r}{u^0}\partial_R^2(rv)\right) + \sqrt{\epsilon}\partial_\omega\left(\frac{1}{u^0}\partial_R^2(rv)\right)\right] w\partial_R(rv_{\omega\omega})$$
$$= \int w|\partial_R(rv_{\omega\omega})|^2 + \partial_\omega \int \frac{r}{u^0} w|\partial_{RR}(rv_\omega)|^2 + \int \frac{rw'(R)}{u^0}\partial_{RR}(rv_\omega)\partial_R(rv_{\omega\omega}) + J \quad (2.4.115)$$
$$= (2.4.115.1) + (2.4.115.2) + (2.4.115.3) + J,$$

where

$$J = \int \left(\frac{1}{u^0}\right)_\omega rw'(R)\partial_{RR}(rv)\partial_R(rv_{\omega\omega}) + \sqrt{\epsilon}\int \left(\frac{1}{u^0}\right)_\omega \partial_{RR}(rv)w\partial_R(rv_{\omega\omega})$$
$$+ \sqrt{\epsilon}\int \frac{w}{u^0}\partial_{RR}(rv_\omega)\partial_R(rv_{\omega\omega}) + \int w\partial_R(rv_{\omega\omega})\int_R^\infty \partial_\omega\left(\frac{ru^0_{RR}}{u^0}v_\omega\right)$$
$$- \int \partial_R\left(\left(\frac{1}{u^0}\right)_\omega rw(R)\partial_{RR}(rv)\right). \quad (2.4.116)$$

Lastly, we use that $rw'(R) = R_0 m(R-R_0)^{m-1} + \sqrt{\epsilon}mw(R)$ to estimate (2.4.115.3) inductively, as in the previous lemma. The result of the claim now follows from an application of Gronwall.

□

Proof of Lemma. The estimates in the claims above are uniform in M, enabling us to send $M \to \infty$ to obtain:

$$\int\int (R-R_0)^m|\partial_R(rv_\omega)|^2 + \sup\int (R-R_0)^m r|\partial_{RR}(rv)|^2 \lesssim \int\int p_m f^2 + \int\int g^2 p_{m+3}$$
$$+ \int_{\omega=0} rp_m|\partial_{RR}(rv)|^2, \quad (2.4.117)$$

and

$$\int\int (R-R_0)^m|\partial_\omega\partial_R(rv_\omega)|^2 + \sup\int (R-R_0)^m r|\partial_{RR}(rv_\omega)|^2 \lesssim \int\int p_m f^2 + \int\int g^2 p_{m+3} + \int\int p_m f_\omega^2 + \int\int g_\omega^2 p_{m+3} + \int_{\omega=0} rp_m|\partial_{RR}(rv_\omega)|^2 + \int_{\omega=0} rp_m|\partial_{RR}(rv)|^2. \tag{2.4.118}$$

Writing $p_m \lesssim 1+(R-R_0)^m$, it follows by pairing (2.4.117) - (2.4.118) with the unweighted estimates in (2.4.88) - (2.4.95), we can upgrade the weights on the left hand sides of (2.4.117) and (2.4.118) to $p_m = \langle R-R_0\rangle^m$, thereby establishing our lemma.

$\square$

2.4.3.2 Boundary Estimates

In the following theorem, we use the stream function formulation to give estimates of v and v_ω on the boundary $\{\omega = 0\}$.

Lemma 2.4.9. *$\int_{R_0}^\infty r^n(R-R_0)^m|\partial_R^{k+1}(rv(0,t))|^2 dt \le C + ||r^{n/2+1}(R-R_0)^{m/2}\bar{u}_1||^2_{H^{k+3}}$,*

and $\int_{R_0}^\infty r^n(R-R_0)^m|\partial_R^{k+1}(rv_\omega)|^2 \lesssim C + ||r^{n/2+1}(R-R_0)^{m/2}\bar{u}_1||^2_{H^{k+3}} + ||(R-R_0)^{-m}u^1_{e\omega\omega}(0,r(\cdot))||^2_{H^k}$.

Proof. Our starting point is equation (2.4.82). Define the stream function $\psi(\omega,R) = \int_{R_0}^R u(\omega,\theta)d\theta$, so that $\psi_\omega = \int_{R_0}^R u_\omega = -\int_{R_0}^R \partial_\theta(r(\theta)v)d\theta = -rv$, and $\psi_R = u$. Now define

$$w = -u^0_R\psi + u^0\psi_R, \tag{2.4.119}$$

so that:

$$w_\omega = -u^0_{\omega R}\psi - u^0_R\psi_\omega + u^0_\omega\psi_R + u^0\psi_{R\omega} = -u^0_{\omega R}\psi - u^0_R(-rv) + u^0_\omega u + u^0 u_\omega \qquad (2.4.120)$$
$$= -u^0_{\omega R}\psi + F - E_1 - E_0 + ru_{RR} - r(v^0_p + v^1_e)u_R.$$

According to the definitions of E_0, E_1, F in (2.4.79) and (2.4.81), $\int_{R_0}^{\infty} r^n R^m \partial_R^k F(0,t)^2 + E_0(0,t)^2 + E_1(0,t)^2 dt \le C$, where the constant C depends on the previously constructed profiles. Now we estimate the boundary data of w in terms of $\bar{u}_1$ using (2.4.119):

$$\int_{R_0}^{\infty} w(0,t)^2 dt \le \int_{R_0}^{\infty} (u^0_R)^2\psi^2 + \int_{R_0}^{\infty} (u^0)^2 \bar{u}_1^2 dt \lesssim \int_{R_0}^{\infty} \bar{u}_1^2 dt; \text{ and } \int_{R_0}^{\infty} \partial_t^k w(0,t)^2 \le ||\bar{u}_1||^2_{H^k}, \qquad (2.4.121)$$

where we have used the rapid decay of u^0_R for Hardy's inequality. We use (2.4.120) to do so similarly for w_ω:

$$\int_{R_0}^{\infty} w_\omega(0,t)^2 dt \le C + ||r\bar{u}_1||^2_{H^2}; \text{ and} \qquad (2.4.122)$$
$$\int_{R_0}^{\infty} r^n (R-R_0)^m \partial_t^k w_\omega(0,t)^2 dt \le C + ||r^{n/2+1}(R-R_0)^m \bar{u}_1||^2_{H^{k+2}}. \qquad (2.4.123)$$

Now we use (2.4.119) to express: $\psi = u^0 \int_{R_0}^{R} \frac{w(\omega,t)}{(u^0)^2} dt$. Differentiating, taking the L^2 norm of both sides, and using Hardy yields:

$$\int r^n (R-R_0)^m |\partial_R^{k+1}\partial_\omega \psi|^2 \le ||r^{n/2}(R-R_0)^{m/2} w(0,\cdot)||^2_{H^k(R_0,\infty)} + ||r^{n/2}(R-R_0)^{m/2} w_\omega(0,\cdot)||^2_{H^k(R_0,\infty)}.$$

This yields:

$$\int_{R_0}^{\infty} r^n (R-R_0)^m |\partial_R^{k+1}\left(rv(0,t)\right)|^2 dt = \int_{R_0}^{\infty} r^n (R-R_0)^m \partial_R^{k+1}\left(\psi_\omega(0,t)\right)^2 dt \qquad (2.4.124)$$

$$\leq ||r^{n/2}(R-R_0)^{m/2}w(0,\cdot)||^2_{H^k} + ||r^{n/2}(R-R_0)^{m/2}w_\omega(0,\cdot)||^2_{H^k} \leq C + ||r^{n/2+1}(R-R_0)^{m/2}\bar{u}_1||^2_{H^{k+3}}.$$

We now estimate the H^k of v_ω norm on the boundary $\{\omega = 0\}$. In particular, we start with:

$$rv_\omega = \psi_{\omega\omega} = \partial_{\omega\omega}\left(u^0\int_{R_0}^R \frac{w}{(u^0)^2}\right) \Rightarrow \partial_R(rv_\omega) = \partial_{\omega\omega}\left(u^0_R\int^R \frac{w}{(u^0)^2} + \frac{w}{u^0}\right) \Rightarrow$$
$$\int_{R_0}^\infty r^n(R-R_0)^m|\partial_R(rv_\omega)|^2 \lesssim \int w^2 + \int w_\omega^2 + \int r^n(R-R_0)^m w^2_{\omega\omega}$$
$$\lesssim C(1+||r\bar{u}_1||^2_{H^2}) + \int_{R_0}^\infty r^n(R-R_0)^m w^2_{\omega\omega}. \tag{2.4.125}$$

Differentiating (2.4.120):

$$w_{\omega\omega} = -u^0_{\omega\omega R}\psi - u^0_{\omega R}\psi_\omega + F_\omega - (E_0+E_1)_\omega + ru_{RR\omega} - r\partial_\omega\left(v^0_p + v^1_e\right)u_R - r(v^0_p+v^1_e)u_{R\omega} \tag{2.4.126}$$

$$= (2.4.126.1) + (2.4.126.2) + (2.4.126.3) + (2.4.126.4) + (2.4.126.5) + (2.4.126.6) + (2.4.126.7).$$

We estimate each of these terms:

$$(2.4.126.1) + (2.4.126.2) + (2.4.126.4) \leq \int \bar{u_1}^2 + \int (R-R_0)^{-n}\psi_\omega^2 + C(u^0_p, v^0_p);$$
$$(2.4.126.3) = \int F_\omega^2 \leq \int (u^1_e u^0_{p\omega})^2 + \int (u^1_e u^0_{p\omega\omega})^2 + \int (u^0_{p\omega}u^1_{e\omega})^2 + \int r^2(v^0_{p\omega}u^0_{er})^2$$
$$+ \int (u^0_\omega v^0_p)^2 + \int (u^0 v^0_{p\omega})^2 + \int (u^0_\omega v^0_p)^2 + \int (u^0 v^0_{p\omega})^2 + \int (u^0_{p\omega}v^1_e)^2 + \int (u^0_p v^1_{e\omega})^2 + \int (u^0_{R\omega})^2$$
$$+ \int (u^0_p)^2(u^1_{e\omega\omega})^2 + \int (P^1_{p\omega})^2;$$
$$(2.4.126.5) - (2.4.126.7) \leq \int r^{2+n}(R-R_0)^m u^2_{RR\omega} + \int r^{2+n}(R-R_0)^m u^2_R + r^{2+n}(R-R_0)^m u^2_{\omega R}.$$

All of the terms in (2.4.126.3) above are bounded by a constant C, using the H^2 estimate on v_e^1, aside from the $u^1_{e\omega\omega}$ term. For (2.4.126.5) - (2.4.126.7), we use the divergence free condition. This establishes the desired result.

$\square$

2.4.3.3 Construction of Prandtl Layer Solutions

We first define $\bar{v}(\omega,R) = v(\omega,R) - \frac{R-R_0}{R_0}\chi(R-R_0)u^1_{e\omega}(w,R_0)$. Here χ is a cutoff function near 0. Then $\bar{v}(\omega,R_0) = v(w,R_0) = 0$, and $\bar{v}_R(\omega,R_0) = v_R(\omega,R_0) - \frac{1}{R_0}u^1_{e\omega}(\omega,R_0) = 0$. Since the equation (2.4.87) above is linear, we easily find:

$$-\partial_{RR}(r\bar{v}_\omega) + \frac{1}{u^0}ru^0_{RR}\bar{v}_\omega + r\partial_{RR}\left(\frac{1}{u^0}\partial_{RR}\left(r\bar{v}\right)\right) = f_R + g, \tag{2.4.127}$$

where f and g are defined:

$$\begin{aligned} f &= \partial_\omega(\frac{1}{u^0})ru_{RR} - \sqrt{\epsilon}\frac{1}{u^0}\partial_R^2(rv) + 2r\partial_R^2(rv)\partial_R(\frac{1}{u^0}) + \frac{1}{u^0}\partial_\omega\left(r(v_p^0+v_e^1)u_R\right) \\ &+ \partial_\omega\left(\frac{1}{u^0}\right)u_Rr(v_p^0+v_e^1) = (f.1) - (f.5), \end{aligned} \tag{2.4.128}$$

$$\begin{aligned} g &= -\partial_\omega(\frac{1}{u^0}ru^0_{RR})v - \partial_{R\omega}(\frac{1}{u^0})ru_{RR} - \sqrt{\epsilon}\partial_R(\frac{1}{u^0})\partial_R^2(rv) - 2r\partial_R^2(rv)\partial_R^2(\frac{1}{u^0}) \\ &- \partial_R\left(\frac{1}{u^0}\right)\partial_\omega\left(r(v_p^0+v_e^1)u_R\right) - \partial_{\omega R}\left(\frac{1}{u^0}\right)\left(u_Rr(v_p^0+v_e^1)\right) \\ &+ \partial_\omega(\frac{1}{u^0})(F_R - E_{1R} - E_{0R}) + \frac{1}{u^0}\left(F_{R\omega} - E_{1R\omega} - E_{0R\omega}\right) - \partial_\omega(\frac{1}{u^0})\left(u^0_{\omega R}u + u^0_\omega u_R\right) \\ &- \frac{1}{u^0}\left(\partial_\omega\left(u^0_{\omega R}u + u^0_\omega u_R\right)\right) - \partial_{RR}(r\frac{R-R_0}{R_0}\chi(R-R_0)u^1_{e\omega\omega}(\omega,R_0)) \\ &+ \frac{1}{u^0}ru^0_{RR}\frac{R-R_0}{R_0}\chi u^1_{e\omega\omega} + r\partial_{RR}\left(\frac{1}{u^0}\partial_{RR}\left(r\frac{R-R_0}{R_0}\chi u^1_{e\omega}\right)\right). \end{aligned} \tag{2.4.129}$$

We define $|||\bar{v}|||^2 = \int\int |\partial_R(r\bar{v}_\omega)|^2 + \sup_{[0,\theta_0]} \int \frac{1}{u^0} r|\partial_{RR}(r\bar{v})|^2$. According to estimate (2.4.88):

$$|||\bar{v}|||^2 \leq \int\int f^2 + \int\int g^2 R^3 + \int_{R_0}^{\infty} \frac{r}{u^0(0,R)} |\partial_{RR}(r\bar{v})|^2. \tag{2.4.130}$$

Using the definition of $\bar{v}$, we record the following consequence of the divergence free condition:

$$u_\omega^2 \lesssim \epsilon \bar{v}^2 + \left(\frac{R-R_0}{R_0}\right)^2 \chi^2 (u_{e\omega}^1)^2 + r^2 \bar{v}_R^2 + r^2\chi^2(u_{e\omega}^1)^2 + \left(\frac{R-R_0}{R_0}\right)^2 (\chi')^2 (u_{e\omega}^1)^2, \tag{2.4.131}$$

$$u_{\omega R}^2 \leq \epsilon \bar{v}_R^2 + \partial_R \left(r\bar{v}_R\right)^2 + \partial_R \left(r\chi u_{e\omega}^1 + r\frac{R-R_0}{R_0}\chi' u_{e\omega}^1\right)^2 + \epsilon\partial_R \left(\frac{R-R_0}{R_0}\chi(R-R_0)u_{e\omega}^1(\omega,R_0)\right)^2. \tag{2.4.132}$$

We must now give estimates on the terms in f in terms of $|||\bar{v}|||$ in order to apply the contraction mapping principle. First, we relate u and v to $|||\bar{v}|||$:

$$\begin{aligned}\int\int v_{RR}^2 &\leq \int\int \bar{v}_{RR}^2 + \int\int (u_{e\omega}^1)^2 \left|\partial_{RR}\left(\frac{R-R_0}{R_0}\chi\right)\right|^2 \leq \theta_0 \sup \int |\bar{v}|^2 + ||u_{e\omega}^1||_{L^2(0,\theta_0)}^2 \\ &\leq \theta_0 |||\bar{v}|||^2 + ||u_{e\omega}^1||_{L^2}^2,\end{aligned} \tag{2.4.133}$$

$$\int\int \epsilon v^2 \leq \int\int \epsilon\bar{v}^2 + \epsilon||u_{e\omega}^1||_{L^2}^2 \leq \theta_0^2 \int\int \epsilon v_\omega^2 + \epsilon||u_{e\omega}^1||_{L^2}^2 \leq \theta_0^2|||\bar{v}|||^2 + \epsilon||u_{e\omega}^1||_{L^2}^2, \tag{2.4.134}$$

$$\int\int u_R^2 \leq \int\int \bar{u}_{1R}^2 + \theta_0 \int\int u_{\omega R}^2 \leq C + \int\int |\partial_{RR}(\bar{v})|^2 + ||u_{e\omega}^1||_{L^2}^2 \leq C + \theta_0|||\bar{v}|||^2 + ||u_{e\omega}^1||_{L^2}^2, \tag{2.4.135}$$

$$\begin{aligned}\int\int u_\omega^2 &\lesssim ||u_{e\omega}^1||_{L^2}^2 + \int\int |\partial_R(r\bar{v})|^2 \leq ||u_{e\omega}^1||_{L^2}^2 + \theta_0 \int_{R_0}^{\infty} |\partial_R(r\bar{v})(0,R)|^2 \\ &+ \theta_0^2 \int\int |\partial_R(r\bar{v}_\omega)|^2 \lesssim ||u_{e\omega}^1||_{L^2}^2 + \theta_0|||\bar{v}|||^2,\end{aligned} \tag{2.4.136}$$

$$\int\int u^2 \leq \int \bar{u}_1^2 + \theta_0 \int\int u_\omega^2 \text{ which has been estimated above;} \tag{2.4.137}$$

$$\int\int |v_{\omega R}|^2 \leq \int\int |\partial_R(r\bar{v}_\omega)|^2 + ||u_e^1||_{L^2}^2, \tag{2.4.138}$$

$$\int\int |u_{\omega\omega}|^2 = \int\int |\partial_R(rv_\omega)|^2 \leq ||u_{e\omega}^1||_{L^2}^2 + \int\int |\partial_R(r\bar{v}_\omega)|^2. \tag{2.4.139}$$

Now we turn to the terms in f, which are given in (2.4.128):

(f.1) For ru_{RR} we use equation (2.4.82):

$$\begin{aligned}
&\int\int |\partial_\omega(\frac{1}{u^0})|r^2|u_{RR}|^2 \\
&\leq \int\int |\frac{u_\omega^0}{|u^0|^2}|^2|u_\omega^0 u + u^0 u_\omega + \sqrt{\epsilon}u^0 v + r(v_p^0 + v_e^1)u_R + ru_R^0 v + E_1 + E_0 - F|^2 \\
&:= (f.1.1) - (f.1.8).
\end{aligned}$$

Each term in this equation is bounded by $C + C\theta_0|||\bar{v}|||$:

$$\begin{aligned}
\text{(f.1.1)} \;:\; &\int\int R^{-n}u^2 \leq \int\int R^{-n}\left(u(0,R)^2 + \theta_0\int u_\omega^2\right) \lesssim \theta_0\left(\int_{R_0}^\infty |\bar{u}_1|^2 + ||u_{e\omega}^1||_{L^2(0,\theta_0)}^2 + |||\bar{v}|||^2\right), \\
\text{(f.1.2)} \;:\; &\int\int R^{-n}u_\omega^2 \leq \int\int R^{-n}\left(\epsilon\bar{v}^2 + (u_{e\omega}^1)^2 + |\bar{v}_R|^2\right) \lesssim ||u_{e\omega}^1||_{L^2(0,\theta_0)}^2 + \int\int v_{RR}^2 \lesssim \\
&||u_{e\omega}^1||_{L^2(0,\theta_0)}^2 + \theta_0 \sup\int |v_{RR}|^2, \\
\text{(f.1.3)} \;:\; &\int\int R^{-n}\epsilon v^2 \leq \int\int R^{-n}\epsilon\bar{v}^2 + \int\int R^{-n}\epsilon|u_{e\omega}^1|^2, \\
\text{(f.1.4)} \;:\; &\int\int (v_p^0 + v_e^1)^2 u_R^2 \leq \int\int u_R^2 \leq \int\int \bar{u}_{1R}^2 + \theta_0\int\int u_{\omega R}^2 \\
&\leq C + \theta_0\left(\int\int |\partial_{RR}(r\bar{v})|^2 + \int |u_{e\omega}^1|^2\right), \\
\text{(f.1.5)} \;:\; &\int\int R^{-n}v^2 \leq \int\int R^{-n}\bar{v}^2 + \int |u_{e\omega}^1|^2, \\
\text{(f.1.6) - (f.1.8)} \;:\; &\text{The forcing terms decay rapidly and therefore} \leq \int\int R^{-n}|F - E_1 - E_0|^2 \leq C.
\end{aligned}$$

(f.2) $\epsilon\int\int |\frac{1}{u^0}|^2|\partial_R^2(rv)|^2 \leq \epsilon|||\bar{v}|||^2 + \epsilon\int\int |\partial_{RR}\left(\frac{R-R_0}{R_0}\chi u_{e\omega}^1\right)|^2,$

(f.3) $\int\int r^2|\partial_R^2(rv)|^2|\partial_R(\frac{1}{u^0})|^2 \leq \int\int R^{-n}|\partial_{RR}(r\bar{v})|^2 + \int\int R^{-n}|\partial_{RR}\left(\frac{R-R_0}{R_0}\chi u_{e\omega}^1\right)|^2,$

(f.4) $\int\int |\frac{1}{u^0}|^2|\partial_\omega\left(r(v_p^0 + v_e^1)u_R\right)|^2 \lesssim ||v_{e\omega}^1||_\infty \int\int u_R^2,$

(f.5) $\int\int |\partial_\omega(\frac{1}{u^0})|^2 u_R^2 r^2 (v_p^0 + v_e^1)^2 \lesssim \int\int R^{-n} u_R^2 |v_p^0|^2 + \int\int R^{-n} u_R^2 (v_e^1)^2 \lesssim \int\int u_R^2.$

Combining $(f.1)-(f.5)$ with (2.4.133) shows that $\int\int f^2 \lesssim C+\theta_0|||\bar{v}|||^2+||u_{e\omega}^1||_{L^2(r=R_0)}^2$. We now give estimates on $\int\int g^2 \langle R-R_0\rangle^3$, recalling the definition (2.4.129):

(g.1) $\int\int R^{-n}\left(v^2 + u_{RR}^2 + |\partial_{RR}(rv)|^2 + u_\omega^2 + u_R^2 + u^2 + u_{R\omega}^2\right) \le C+\theta_0|||\bar{v}|||^2+||u_{e\omega}^1||_{L^2(r=R_0)}^2$,

(g.2) $\int\int |\partial_R\left(\frac{1}{u^0}\right)|^2|\partial_\omega\left(r(v_p^0+v_e^1)u_R\right)|^2 + \int\int |\partial_{\omega R}(\frac{1}{u^0})u_R r(v_p^0+v_e^1)|^2 \le \int\int u_R^2$:
Here, we have used: $\int\int R^{-n}|v_{e\omega}^1|^2 u_R^2 \le ||R^{-n}v_{e\omega}^1||_\infty \int\int u_R^2$, and $||R^{-n}v_{e\omega}^1||_\infty \le ||R^{-n}v_e^1||_{H^3} \le \epsilon^n ||r^m v_e^1||_{H^3} \le C$.

(g.3) $\int\int |\partial_\omega(\frac{1}{u^0})|^2|F_R - E_{1R} - E_{0R}|^2 + \int\int |\frac{1}{u^0}|^2|F_{R\omega} - E_{1R\omega} - E_{0R\omega}|^2 \le C$ by Euler H^2 bounds,

(g.4) $\int\int |\partial_R(r\partial_R\left(\frac{R-R_0}{R_0}\chi(R-R_0)u_{e\omega\omega}^1(\omega,R_0)\right))|^2$:

$$\int u_{e\omega\omega}^1(\omega,R_0)^2 \le \int\int \partial_r(u_{e\omega\omega}^1)^2 = \int\int u_{e\omega\omega}^1 u_{e\omega\omega r}^1 = \int\int \left(v_{e\omega}^1 + r v_{er\omega}^1\right)\left(v_{e\omega r}^1 + \partial_r(r v_{er\omega}^1)\right)$$
$$\le ||r^n v_e^1||_{W^{2,p}}||r^n v_e^1||_{W^{3,q}} \lesssim \epsilon^{-\kappa} \text{ for } \kappa \text{ arbitrarily small.}$$

The boundary terms in (2.4.129) can be estimated by $\int_0^{\theta_0}|u_{e\omega}^1|^2 + \int_0^{\theta_0}|u_{e\omega\omega}^1|^2 \le C\epsilon^{-2\kappa}$.

Finally, it remains to give estimates on $\int_{R_0}^\infty r\frac{1}{u^0}\partial_R^2\left(r\bar{v}(0,R)\right)^2$ from (2.4.130):

$$\int_{R_0}^\infty r\frac{1}{u^0}\partial_R^2\left(r\bar{v}(0,R)\right)^2 \le \int_{R_0}^\infty r|\partial_{RR}(rv)|^2 + \int_{R_0}^\infty r|\partial_{RR}\left(\frac{R-R_0}{R_0}\chi\right)|^2|u_{e\omega}^1|^2$$
$$\le C + ||r^{3/2}\bar{u}_1||_{H^4}^2 + |u_{e\omega}^1(0,R_0)|^2 \le C + ||\bar{u}_1||_{H^4}.$$

Therefore, we can close the contraction mapping argument. Because the estimates are uniform in N, we can let $N\to\infty$ to obtain a global in R solution. We can repeat this same argument with the weights of R^m. Indeed, the only terms above sensitive to a weight are contained in f, and we treat them here:

$$\int\int R^m (v_p^0 + v_e^1)^2 u_R^2 \lesssim \int\int R^m u_R^2 \le C + \int\int R^m |\partial_{RR} r\bar{v}|^2 + \int |u^1_{e\omega}|^2, \tag{2.4.140}$$

$$\epsilon \int\int |\frac{1}{u^0}|^2 R^m |\partial_{RR}(rv)|^2 \le \epsilon \int\int R^m |\partial_{RR}(r\bar{v})|^2 + \epsilon \int |u^1_{e\omega}|^2, \tag{2.4.141}$$

$$\int\int |\frac{1}{u^0}|^2 R^m |\partial_\omega (r(v_p^0 + v_e^1) u_R)|^2 \lesssim \int\int R^m u_R^2 + \int\int R^m u_{R\omega}^2. \tag{2.4.142}$$

Again through contraction mapping, this establishes:

Lemma 2.4.10. $\int\int \langle R - R_0\rangle^m |\partial_R (r v_\omega)|^2 + \sup \int \langle R - R_0\rangle^m r |\partial_{RR}(rv)|^2 \le \epsilon^{-\kappa}$ *for* $\kappa > 0$, *arbitrarily small.*

Uniform estimates are obtained via the calculation:

$$\begin{aligned} u(\omega, R)^2 &= |\bar{u}_1(R)|^2 + \left(\int_0^\omega \partial_R (rv)\right)^2 \le |\bar{u}_1|^2 + \int_0^\theta |\partial_R(rv)|^2 \\ &\le C + \int_0^{\theta_0} \int_{R_0}^R \partial_{RR}(rv)\partial_R(rv) \\ &\le C + \int\int R^{-n} |\partial_R(rv)|^2 + \int\int R^n |\partial_{RR}(rv)|^2. \end{aligned} \tag{2.4.143}$$

Similarly, using the fact that $v(\omega, R_0) = 0$, we have

$$\begin{aligned} v(\omega, R)^2 &= \int_{R_0}^R v v_R \le \int_{R_0}^\infty R^{-m} |v|^2 dR + \int_{R_0}^\infty R^m |v_R|^2 \\ &\le \int_{R_0}^\infty R^{-m} R^3 ||v_{RR}||^2_{L^2} + C + \int_0^{\theta_0} \int_{R_0}^\infty |v_{R\omega}|^2. \end{aligned} \tag{2.4.144}$$

This yields: $||v||_\infty + ||u||_\infty \le C\epsilon^{-\kappa}$. We now obtain higher-regularity estimates. The starting point is (2.4.95), and as such we define

$$|||v|||^2 = \int\int |\partial_R(rv_{\omega\omega})|^2 + \sup \int r|\partial_{RR}(rv_\omega)|^2. \tag{2.4.145}$$

Since f and g have already been estimated, we compute f_ω and g_ω:

$$\begin{aligned} f_\omega &= \partial_{\omega\omega}(\frac{1}{u^0})ru_{RR} + \partial_\omega(\frac{1}{u^0})ru_{RR\omega} - \sqrt{\epsilon}\frac{1}{u^0}\partial_R^2(rv_\omega) + 2r\partial_R^2(rv_\omega)\partial_R(\frac{1}{u^0}) \\ &+ 2r\partial_R^2(rv)\partial_{\omega R}(\frac{1}{u^0}) + \frac{1}{u^0}\partial_{\omega\omega}\left(r(v_p^0+v_e^1)u_R\right) + \partial_{\omega\omega}\left(\frac{1}{u^0}\right)u_R r(v_p^0+v_e^1) \\ &+ \partial_\omega\left(\frac{1}{u^0}\right)u_{\omega R}r(v_p^0+v_e^1) + \partial_\omega\left(\frac{1}{u^0}\right)u_R r\partial_\omega(v_p^0+v_e^1) + \left(\frac{1}{u^0}\right)_\omega \partial_\omega(r(v_p^0+v_e^1)u_R). \end{aligned} \tag{2.4.146}$$

$$\begin{aligned} g_\omega &= -\partial_\omega^2(\frac{1}{u^0}ru_{RR}^0)v - \partial_\omega(\frac{1}{u^0}ru_{RR}^0)v_\omega - \partial_{R\omega}(\frac{1}{u^0})ru_{\omega RR} - \partial_{R\omega\omega}(\frac{1}{u^0})ru_{RR} \\ &- \sqrt{\epsilon}\partial_{\omega R}(\frac{1}{u^0})\partial_R^2(rv) - \sqrt{\epsilon}\partial_R(\frac{1}{u^0})\partial_R^2(rv_\omega) - 2r\partial_R^2(rv_\omega)\partial_R^2(\frac{1}{u^0}) \\ &- 2r\partial_R^2(rv)\partial_{RR\omega}(\frac{1}{u^0}) - \partial_{\omega R}\left(\frac{1}{u^0}\right)\partial_\omega\left(r(v_p^0+v_e^1)u_R\right) - \partial_R\left(\frac{1}{u^0}\right)\partial_{\omega\omega}\left(r(v_p^0+v_e^1)u_R\right) \\ &- \partial_{\omega\omega R}\left(\frac{1}{u^0}\right)\left(u_R r(v_p^0+v_e^1)\right) - \partial_{\omega R}\left(\frac{1}{u^0}\right)\partial_\omega\left(u_R r(v_p^0+v_e^1)\right) \\ &+ \partial_\omega\left(\partial_\omega(\frac{1}{u^0})(F_R - E_{1R} - E_{0R}) + \frac{1}{u^0}\left(F_{R\omega} - E_{1R\omega} - E_{0R\omega}\right)\right) \\ &- \partial_{\omega\omega}(\frac{1}{u^0})\left(u_{\omega R}^0 u + u_\omega^0 u_R\right) - 2\partial_\omega(\frac{1}{u^0})\partial_\omega\left(u_{\omega R}^0 u + u_\omega^0 u_R\right) - \frac{1}{u^0}\left(\partial_{\omega\omega}\left(u_{\omega R}^0 u + u_\omega^0 u_R\right)\right) \\ &- \partial_{RR}(r\frac{R-R_0}{R_0}\chi(R-R_0)u_{e\omega\omega\omega}^1(\omega,R_0)) + \partial_\omega\left(\frac{1}{u^0}ru_{RR}^0\right)\frac{R-R_0}{R_0}\chi u_{e\omega\omega}^1 \\ &+ \frac{1}{u^0}ru_{RR}^0\frac{R-R_0}{R_0}\chi u_{e\omega\omega\omega}^1 + r\partial_{\omega RR}\left(\frac{1}{u^0}\partial_{RR}\left(r\frac{R-R_0}{R_0}\chi u_{e\omega}^1\right)\right). \end{aligned} \tag{2.4.147}$$

We now treat the f_ω terms in (2.4.146)

$$\begin{aligned} \int\int f_\omega^2 \le \int\int R^{-n}\left(u_{RR}^2 + u_{RR\omega}^2 + |\partial_{RR}(rv_\omega|^2 + u_R^2 + u_{R\omega}^2\right) + \int\int |\frac{1}{u^0}|^2|\partial_{\omega\omega}(v_e^1 u_R)|^2 \\ + \epsilon\int\int |\frac{1}{u^0}|^2|\partial_{RR}(rv_\omega)|^2. \end{aligned} \tag{2.4.148}$$

All of these terms can be estimated in terms of the norm, with $u_{RR\omega}$ being estimated by taking ∂_ω of Equation (2.4.82). We now address g_ω, bearing in mind that all of these terms are accompanied by rapid decay:

$$\begin{aligned}\int\int g_\omega^2 \langle R-R_0\rangle^m \leq \int\int R^{-N}\left(v^2+v_\omega^2+u_{\omega RR}^2+u_{RR}^2+|\partial_{RR}(rv)|^2+|\partial_{RR}(rv_\omega)|^2+u_R^2\right)\\ +\int\int R^{-N}\left(|v_{\omega\omega}^1|^2u_R^2+\left|\partial_\omega\left(\partial_\omega(\frac{1}{u^0})(F_R-E_{1R}-E_{0R})+\frac{1}{u^0}\left(F_{R\omega}-E_{1R\omega}-E_{0R\omega}\right)\right)\right|^2+u^2\right).\end{aligned} \tag{2.4.149}$$

All of the above can be estimated in terms of $|||\bar{v}|||^2$. The most delicate boundary terms in g_ω is: $\int_{r=R_0}|u_{e\omega\omega\omega}^1|^2 \leq C(\theta_0)\epsilon^{-2}$, and the most delicate interior term is:

$$\begin{aligned}\int\int|\partial_R(\frac{1}{u^0})|^2|v_{e\omega\omega}^1|^2|u_R|^2 \leq ||R^{-n}v_{e\omega\omega}^1||_\infty^2\int\int R^{-n}u_R^2 \leq ||R^{-n}v_e^1||_{H^4}^2\int\int R^{-n}u_R^2\\ \leq \epsilon^n||r^m v_e^1||_{H^4}^2\int\int u_R^2.\end{aligned} \tag{2.4.150}$$

The latter boundary term in estimate (2.4.104) has been estimated before. Therefore, we must estimate the v_ω boundary term:

$$\begin{aligned}\int_{\omega=0} r\langle R-R_0\rangle^m|\partial_{RR}(r\bar{v}_\omega)|^2 &= \int_{\omega=0} r\langle R-R_0\rangle^m\left|\partial_{RR}\left(r\partial_\omega\left(v+\frac{R-R_0}{R_0}\chi u_{e\omega}^1(0,R_0)\right)\right)\right|^2\\ &\leq \int_{\omega=0} r\langle R-R_0\rangle^m|\partial_{RR}\left(rv_\omega\right)|^2+\int_{\omega=0} rR^m|\partial_{RR}(\frac{R-R_0}{R_0}\chi u_{e\omega\omega}^1(0,R_0))|^2\\ &\leq C+||r^{3/2}R^{m/2}\bar{u}_1||_{H^3}^2+||R^{-m}u_{e\omega\omega}^1(0,r(\cdot))||_{H^1}+|u_{e\omega\omega}^1(0,R_0)|^2\\ &\leq C(\theta_0)\epsilon^{-3/2},\end{aligned} \tag{2.4.151}$$

where for the final inequality we use the same calculation as in (GN17, page 25). Therefore, we can close the contraction mapping argument, and again let $N\to\infty$ yielding:

Lemma 2.4.11. $\int\int\langle R-R_0\rangle^m|\partial_R(rv_{\omega\omega})|^2+\sup\int\langle R-R_0\rangle^m r|\partial_{RR}(rv_\omega)|^2 \leq C(\theta_0)\epsilon^{-2}$

where the constant $C(\theta_0)$ could depend poorly on small θ_0.

2.4.3.4 Cutoff Prandtl Layers

We define our Prandtl-1 layers (u_p^1, v_p^1) by cutting off the previously constructed layers (u, v):

$$u_p^1(\omega, R) := \chi(\sqrt{\epsilon}(R - R_0))u + \sqrt{\epsilon}\chi'(\sqrt{\epsilon}(R - R_0)) \int_{R_0}^{R} u, \tag{2.4.152}$$
$$v_p^1(\omega, R) := \chi(\sqrt{\epsilon}(R - R_0))v.$$

Here, χ is a standard cutoff function which equals 1 on $[0, 1]$. (u_p^1, v_p^1) satisfy the divergence free condition: $\partial_\omega u_p^1 + \partial_R(rv_p^1) = 0$. We also have the following estimate for u_p^1:

$$|\sqrt{\epsilon}\chi'(\sqrt{\epsilon}(R - R_0)) \int_{R_0}^{R} u_p^1| \leq \sqrt{\epsilon}(R - R_0)\chi'||u_p^1||_\infty \leq C\epsilon^{-\kappa}. \tag{2.4.153}$$

We have the following lower-order estimates of u_p^1 and v_p^1:

$$\int\int r^n(v^1_{p\omega\omega})^2 \leq \epsilon^{-1/2} \int\int R^m(v^1_{p\omega\omega R})^2 \leq \epsilon^{-5/2}, \tag{2.4.154}$$
$$\int\int r^n(v^1_{p\omega})^2 \leq \epsilon^{-1/2} \int\int R^m(v^1_{p\omega R})^2 \leq \epsilon^{-1/2-\kappa}, \tag{2.4.155}$$
$$\int\int r^n(v^1_{pR})^2 \leq \epsilon^{-1/2} \int\int R^m(v^1_{pRR})^2 \leq \epsilon^{-1/2-\kappa}, \tag{2.4.156}$$
$$\begin{aligned}\int\int r^n|u^1_{p\omega}|^2 &\approx \int\int |\partial_R(rv_p^1)|^2 = \int\int\int_R^{\frac{1}{\sqrt{\epsilon}}} \partial_R(rv_p^1)\partial_{RR}(rv_p^1) \\ &\leq \epsilon^{-1/2}\int\int |\partial_R(rv)||\partial_{RR}(rv)| \\ &\leq \epsilon^{-1/2}\left(\int\int R^{-m}|\partial_R(rv)|^2 + \int\int R^m|\partial_{RR}(rv)|^2\right) \lesssim \epsilon^{-1/2-\kappa},\end{aligned} \tag{2.4.157}$$
$$\begin{aligned}\int\int r^n|u^1_{pR}|^2 &\approx \int\int |\bar{u}_R(R) + \int_0^\omega u^1_{p\omega R}(\theta, R)|^2 \leq C + \int\int |u^1_{p\omega R}|^2 = C + \int\int |\partial_{RR}(rv_p^1)|^2 \\ &\leq \epsilon^{-\kappa},\end{aligned} \tag{2.4.158}$$

$$\int\int r^n |u_p^1|^2 \approx \int |\bar{u}_1|^2 + \int\int |u_{p\omega}^1|^2 \le \epsilon^{-\frac{1}{2}-\kappa}, \tag{2.4.159}$$

$$\int\int r^n |v_p^1|^2 \approx \int\int_{R_0}^{R_0+\frac{1}{\sqrt{\epsilon}}} |v_p^1|^2 \lesssim ||v_p^1||_{L^\infty}^2 \int\int_{R_0}^{R_0+\frac{1}{\sqrt{\epsilon}}} dRd\omega \lesssim \epsilon^{-\frac{1}{2}-\kappa}. \tag{2.4.160}$$

The weight of r^n can be added in because on the support of v_p^1 and u_p^1, r is bounded uniformly. For the same reason, these weights can be added in for the uniform estimates. We summarize the results of the Prandtl-1 layer construction in the following:

Theorem 2.4.12. *Let u_p^1, v_p^1 denote the cutoff Prandtl layers described above. Then*

$$\int\int R^m |\partial_R(rv_{p\omega}^1)|^2 + \sup \int R^m r |\partial_{RR}(rv_p^1)|^2 \le C(\theta_0)\epsilon^{-\kappa} \text{ for } \kappa > 0 \text{ arbitrarily small,}$$

$$\int\int R^m |\partial_R(rv_{p\omega\omega}^1)|^2 + \sup \int R^m |\partial_{RR}(rv_{p\omega}^1)|^2 \le C(\theta_0)\epsilon^{-2}, \text{ and } ||r^n u_p^1, v_p^1||_\infty \le C(\theta_0)\epsilon^{-\kappa},$$

where $C(\theta_0)$ is a constant which could depend poorly on small θ_0.

2.4.4 Estimates on Profile Errors

2.4.4.1 Angular Errors, R^u:

There are three lower order error terms from the profile constructions, which come from (2.4.22) for the Prandtl-0 layer and (2.4.72) for the Euler-1 layer and for the Prandtl-1 layer by inserting (2.4.152) into the equation (2.4.78):

$$\epsilon^0(e_2) + \epsilon^{1/2}(\int_r^\infty E_b(\omega,\theta)d\theta) + \epsilon^{1/2}(\text{Prandtl-1 contribution}) = (2.4.161.1) + (2.4.161.2) + (2.4.161.3). \tag{2.4.161}$$

First, we have

$$(2.4.161.1) = ||r^m e_2||_{L^2} \leq ||r^m u^0_{pR} R^n||_{L^2} ||\partial_r^2 (r v^1_e)||_{L^2} + \epsilon ||r^m u^0_{p\omega} R^n||_{L^2} ||u^0_{err}||_\infty \leq C\epsilon. \tag{2.4.162}$$

Next, we have:

$$\int\int r(R)^m \Big(\int_{r(R)}^\infty E_b(\omega,\theta) d\theta \Big)^2 dR d\omega = \epsilon^{-\frac{1}{2}} \int\int r^m \Big(\int_r^\infty E_b(\omega,\theta) d\theta \Big)^2 dr d\omega$$
$$\lesssim \epsilon^{-\frac{1}{2}} \int \chi(\frac{\omega}{\epsilon})^2 d\omega \lesssim \epsilon^{\frac{1}{2}}. \tag{2.4.163}$$

To estimate (2.4.161.3), we write:

Prandtl-1 angular error contribution =:
$u^0_\omega \left(\chi u^1_p + \sqrt{\epsilon}\chi' \int^R u^1_p \right) + u^0 \left(\chi u^1_{p\omega} + \sqrt{\epsilon}\chi' \int^R u^1_{p\omega} \right) + r(v^0_p + v^1_e) \left(\chi u^1_{pR} + 2\sqrt{\epsilon}\chi' u^1_p + \epsilon\chi'' \int^R u^1_p \right) +$
$r u^0_R \chi v^1_p - r \left(\chi u^1_{pRR} + \epsilon\chi'' u^1_p + 2\sqrt{\epsilon}\chi' u^1_{pR} + \epsilon^{3/2}\chi''' \int^R u^1_p + 2\epsilon\chi'' + \sqrt{\epsilon}\chi' u^1_{pR} \right) - (F - E_1 -$
$E_0)$
$= (1-\chi)(F - E_0 - E_1) + \sqrt{\epsilon}\chi' \left(u^0_\omega \int^R u^1_p + u^0 \int^R u^1_{p\omega} + 2(v^0_p + v^1_e) u^1_p + 2r u^1_{pR} \right) +$
$\epsilon\chi'' \left((v^0_p + v^1_e) \int^R u^1_p + 3u^1_p \right) + \epsilon^{3/2}\chi''' \left(\int^R u^1_p \right).$

We give estimates on each of the terms in the above expression:

$$\int\int r^m (1-\chi)^2 |F - E_0 - E_1|^2 \leq \epsilon^n, \tag{2.4.164}$$

$$\epsilon \int\int r^m |\chi'(\sqrt{\epsilon}\cdot)|^2 |(v^0_p + v^1_e) u^1_p + u^1_{pR} + u^0_\omega \int^R u^1_p|^2 \leq \epsilon^{1-\kappa} \int\int |\chi'(\sqrt{\epsilon}\cdot)|^2 R^{-n} \leq \epsilon^{\frac{1}{2}-\kappa}, \tag{2.4.165}$$

$$\epsilon \int\int |\chi'(\sqrt{\epsilon}\cdot)|^2 |u^0|^2 |\int^R u^1_{p\omega}|^2 = \epsilon \int\int |\chi'(\sqrt{\epsilon}\cdot)|^2 |u^0|^2 |v^1_p|^2 \leq \epsilon^{1/2-\kappa}, \tag{2.4.166}$$

$$\int\int r^m \left[\epsilon\chi'' \left((v^0_p + v^1_e) \int^R u^1_p + 3u^1_p \right) + \epsilon^{3/2}\chi''' \left(\int^R u^1_p \right) \right]^2$$

$$\lesssim \epsilon^2 \int\int \left(\chi''(\sqrt{\epsilon}\cdot)\right)^2 \left(\int^R u_p^1\right)^2 r^m \lesssim \epsilon^{2-\kappa} \int\int \left(\chi''(\sqrt{\epsilon}\cdot)\right)^2 R^2 r^m$$
$$\lesssim \epsilon^{1-\kappa} \int\int \chi''(\sqrt{\epsilon}\cdot)^2 \le \epsilon^{1/2-\kappa}. \tag{2.4.167}$$

Now, we must estimate the higher order terms from the expansions (2.4.1) - (2.4.7) :

ϵ^1-order error:

$$\int\int r^m |u_e^1 + u_p^1|^2 |u_{e\omega}^1 + u_{p\omega}^1)|^2 \le ||u_e^1 + u_p^1||_\infty^2 \int\int |u_e^1 + u_{p\omega}^1|^2 \lesssim \epsilon^{-1/2-\kappa}, \tag{2.4.168}$$
$$\int\int r^m |v_p^0 + v_e^1|^2 |u_{er}^1|^2 \le ||r^m(v_p^0 + v_e^1)||_\infty^2 \int\int |u_{er}^1|^2 \lesssim \epsilon^{-1/2}, \tag{2.4.169}$$
$$\int\int r^m |v_p^1|^2 |u_{pR}^1|^2 \le ||v_p^1||_\infty^2 \int\int |u_{er}^0 + u_{pR}^1|^2 \le \epsilon^{-1/2-\kappa}, \tag{2.4.170}$$
$$\int\int r^m (u_e^0 + u_p^0)^2 |v_p^1|^2 \lesssim \epsilon^{-\kappa} \int\int^{\frac{1}{\sqrt{\epsilon}}} (u_e^0 + u_p^0)^2 \lesssim \epsilon^{-\kappa-1/2}, \tag{2.4.171}$$
$$\int\int r^m (u_e^1 + u_p^1)^2 (v_p^0 + v_e^1)^2 \lesssim \epsilon^{-1/2-\kappa}, \tag{2.4.172}$$
$$\int\int r^m |P_{p\omega}^2|^2 : \tag{2.4.173}$$

We define:

$$P_P^2 = \int_R^\infty \frac{1}{r(t)} \left(u_e^0 v_{p\omega}^0 + u_p^0 \partial_\omega (v_p^0 + v_e^1)\right) + (v_p^0 + v_e^1) v_{pR}^0 - \frac{2}{r(t)} (u_e^1 u_p^0) - v_{pRR}^0 dt$$
$$- \int_R^{R_0 + \frac{1}{\sqrt{\epsilon}}} \frac{2}{r(t)} u_p^1 u_p^0 dt - \int_R^{R_0 + \frac{1}{\sqrt{\epsilon}}} \frac{1}{r(t)} u_p^1 u_e^0 dt.$$

Using the rapid decay of Prandtl-0 profiles, after taking ∂_ω, the first integral is bounded by R^{-n} for arbitrarily large n. We must therefore treat the ∂_ω of second and third integrals:

$$\int_R^{R_0 + \frac{1}{\sqrt{\epsilon}}} \frac{2}{r(t)} u_{p\omega}^1 u_p^0 dt + \int_R^{R_0 + \frac{1}{\sqrt{\epsilon}}} \frac{2}{r(t)} u_p^1 u_{p\omega}^0 + \int_R^{R_0 + \frac{1}{\sqrt{\epsilon}}} \frac{1}{r(t)} u_{p\omega}^1 u_e^0.$$

The middle term above is bounded by $\epsilon^{-\kappa}R^{-n}$. For the first and third terms, we use the divergence free condition and integrate by parts, bearing in mind that $v_p^1|_{R_0+\frac{1}{\sqrt{\epsilon}}} = 0$ due to the cutoff:

$$
\begin{aligned}
|\int_R^{R_0+\frac{1}{\sqrt{\epsilon}}} \frac{2}{r(t)}\partial_t(r(t)v_p^1)u_p^0 dt| &= |-\int_R^{R_0+\frac{1}{\sqrt{\epsilon}}} r(t)v_p^1\partial_t(\frac{2}{r(t)}u_p^0)dt + v_p^1u_p^0| \\
&\le \epsilon^{-\kappa}|u_p^0| + \int_R^\infty |r(t)v_p^1|t^{-n}dt \lesssim \epsilon^{-\kappa}\left(|u_p^0| + R^{-n}\right),
\end{aligned}
$$

$$
|\int_R^{R_0+\frac{1}{\sqrt{\epsilon}}} \frac{2}{r(t)}u_{p\omega}^1 u_e^0(r(t))dt| = |v_p^1u_e^0(r) + \int_R^{R_0+\frac{1}{\sqrt{\epsilon}}} r(t)v_p^1\partial_t(\frac{u_e^0(r(t))}{r(t)})dt|.
$$

We place the terms above into integrals, the most delicate being:

$$
\begin{aligned}
&\int\int r^m \left(\int_R^{R_0+\frac{1}{\sqrt{\epsilon}}} r(t)v_p^1\partial_t(\frac{u_e^0}{r(t)})dt\right)^2 dRd\omega \\
&\lesssim \epsilon^{-\kappa}\int\int_{R_0}^{R_0+\frac{1}{\sqrt{\epsilon}}}\left(\int_R^{R_0+\frac{1}{\sqrt{\epsilon}}} |\partial_t(\frac{u_e^0(r(t))}{r(t)})|dt\right)^2 dRd\omega \\
&\lesssim \epsilon^{-\kappa}\int\int_{R_0}^{R_0+\frac{1}{\sqrt{\epsilon}}}\left(\int_{R_0}^{R_0+\frac{1}{\sqrt{\epsilon}}} \sqrt{\epsilon}|\frac{u_{er}^0(r(t))}{r(t)}| + \sqrt{\epsilon}|\frac{u_e^0(r(t))}{r(t)^2}|dt\right)^2 dRd\omega \\
&\lesssim \epsilon^{-\kappa}\int\int_{R_0}^{R_0+\frac{1}{\sqrt{\epsilon}}}\left(\int_{R_0}^{R_0+1} |u_{er}^0(r)| + |\frac{u_e^0}{r}|dr\right)^2 dRd\omega \lesssim \epsilon^{-\kappa-1/2}.
\end{aligned}
$$

$$
\int\int r^m|u_{err}^0|^2 + |u_{er}^0|^2 + |u_{pR}^1|^2 + |u_{p\omega\omega}^0|^2 + |u_p^0|^2 \lesssim \epsilon^{-1/2}, \tag{2.4.174}
$$

$$
\int\int \frac{(u_e^0)^2}{r^4}r^{2+\delta}dR \le ||u_e^0||_\infty^2 \int\int \frac{1}{r^{2-\delta}}dR \lesssim \epsilon^{-1/2}\int_{R_0}^\infty \frac{dr}{r^{2-\delta}} \le \epsilon^{-1/2} \text{ if } \delta < 1. \tag{2.4.175}
$$

We emphasize that estimate (2.4.175) is the most delicate and requires the weight parameter δ of $r^{2+\delta}$ to be strictly less than 1.

$\epsilon^{3/2}$-order error:

$$\int\int r^m |v_p^1|^2 |u_{er}^1 + u_e^1 + u_p^1|^2 + \int\int r^m |u_e^1 + u_p^1 + v_{p\omega}^0 + v_{e\omega}^1|^2 \lesssim \epsilon^{-\kappa-1/2}, \tag{2.4.176}$$

$$\int\int r^m |u_{err}^1 + u_{er}^1 + u_{e\omega\omega}^1|^2 \leq \epsilon^{-1/2} ||r^m u_e^1||_{H^2}^2 \leq \epsilon^{-1/2} ||r^m v_e^1||_{H^3}^2 \leq \epsilon^{-3/2}, \tag{2.4.177}$$

$$\int\int r^m |u_{p\omega\omega}^1|^2 \approx \int\int |\partial_R (r v_\omega)|^2 \leq \epsilon^{-\kappa}, \tag{2.4.178}$$

ϵ^2-order error:

$$\int\int r^m |v_{p\omega}^1|^2 \lesssim \epsilon^{-1/2-\kappa}. \tag{2.4.179}$$

2.4.4.2 Radial Errors, R^v:

The higher-order radial contributions must be estimated:

$\epsilon^{1/2}$ order error:

$$\int\int r^m |u_e^1 v_{e\omega}^1|^2 + \int\int r^m |v_e^1|^2 |v_{er}^1|^2 + \int\int r^m |u_e^1|^4$$

$$+ \int\int r^m \left(u_e^1 v_{p\omega}^0 + u_p^1 \partial_\omega (v_p^0 + v_e^1)\right)^2 + \int\int r^m |v_p^1|^2 |v_{pR}^0|^2 + \leq \epsilon^{-\kappa-1/2}, \tag{2.4.180}$$

$$\int\int r^m \left(|v_{p\omega}^1|^2 |u_e^0 + u_p^0|^2 + |v_p^0|^2 |v_{er}^1|^2 + |v_p^0|^2 |v_{pR}^1|^2 + |u_e^1|^2 |u_p^1|^2 + |u_p^1|^4 + |v_e^1|^2 |v_{pR}^1\right)^2$$

$$\leq \epsilon^{-\kappa-1/2}, \tag{2.4.181}$$

$$\int\int r^m \left(|v_{pR}^0|^2 + |u_{p\omega}^0|^2 + |v_{pRR}^1|^2\right) \leq \epsilon^{-\kappa}, \tag{2.4.182}$$

ϵ order error:

$$\int\int r^m |v_{err}^1 + v_{er}^1 + v_{e\omega\omega}^1 + u_{e\omega}^1 + v_e^1|^2 \leq \epsilon^{-1/2} ||v_e^1||_{H^2}^2, \tag{2.4.183}$$

$$\int\int r^m |u_p^1 v_{p\omega}^1 + u_e^1 v_{p\omega}^1 + v_p^1 v_{er}^1 + v_p^1 v_{pR}^1 + v_{pR}^1 + v_{p\omega\omega}^0 + u_{p\omega}^1 + v_p^0|^2 \leq \epsilon^{-\kappa-1/2}, \quad (2.4.184)$$

$\epsilon^{3/2}$ **order error:**

$$\int\int r^m |v_{p\omega\omega}^1|^2 \leq \epsilon^{-5/2}. \tag{2.4.185}$$

We have established the main result of this section, Theorem 2.2.10.

2.5 Energy Estimates

In this section, the energy estimates in Theorem 2.2.11 are proven. We will need to work with the restricted domain $\Omega_N = (0, \theta_0) \times (R_0, R_0 + N)$, and obtain estimates which are uniform in N. The boundary conditions $u = v = 0$ on $\{(\omega, R) : R = R_0 + N\}$ are enforced.

Step I: Laplacian and Lower Order Terms

By the divergence free condition:

$$\begin{aligned} 0 &= -\frac{\epsilon}{r^2}u_{\omega\omega} + \frac{\epsilon}{r^2}u_{\omega\omega} = -\frac{\epsilon}{r^2}u_{\omega\omega} + \frac{\epsilon}{r}\partial_\omega\left(-\frac{\sqrt{\epsilon}}{r}v - v_R\right) \\ &= -\frac{\epsilon}{r^2}u_{\omega\omega} - \frac{\epsilon^{3/2}}{r^2}v_\omega - \frac{\epsilon}{r}v_{\omega R}, \end{aligned} \tag{2.5.1}$$

$$\begin{aligned} 0 &= -v_{RR} + v_{RR} = -v_{RR} + \partial_R\left(-\frac{\sqrt{\epsilon}}{r}v - \frac{u_\omega}{r}\right) \\ &= -v_{RR} - \frac{\sqrt{\epsilon}}{r}v_R + \frac{\epsilon}{r^2}v - \frac{u_{\omega R}}{r} + \sqrt{\epsilon}\frac{u_\omega}{r^2}. \end{aligned} \tag{2.5.2}$$

In this step the terms in (2.2.23 - 2.2.24) which involve (u, v) but do not depend on the profiles, (u_s, v_s), are treated. After adding in (2.5.1) - (2.5.2) these terms are summarized:

$$-u_{RR} - \frac{\sqrt{\epsilon}}{r}u_R - \frac{2\epsilon}{r^2}u_{\omega\omega} + \frac{\epsilon}{r^2}u - \frac{3}{r^2}\epsilon^{3/2}v_\omega - \frac{\epsilon}{r}v_{\omega R}, \tag{2.5.3}$$

$$-2v_{RR} - \frac{2\sqrt{\epsilon}}{r}v_R - \frac{\epsilon}{r^2}v_{\omega\omega} + \frac{3\sqrt{\epsilon}}{r^2}u_\omega + 2\frac{\epsilon}{r^2}v - \frac{u_{\omega R}}{r}. \tag{2.5.4}$$

As described in the introduction, we proceed to multiply (2.5.3) by $r^{1+\delta}u$ and (2.5.4) by $\epsilon r^{1+\delta}v$:

$$\begin{aligned}&\int\int\left[-u_{RR} - \frac{\sqrt{\epsilon}}{r}u_R - \frac{2\epsilon}{r^2}u_{\omega\omega} + \frac{\epsilon}{r^2}u - \frac{3}{r^2}\epsilon^{3/2}v_\omega - \frac{\epsilon}{r}v_{\omega R}\right]\times r^{1+\delta}u \\ &+\int\int\left[-2v_{RR} - \frac{2\sqrt{\epsilon}}{r}v_R - \frac{\epsilon}{r^2}v_{\omega\omega} + \frac{3\sqrt{\epsilon}}{r^2}u_\omega + 2\frac{\epsilon}{r^2}v - \frac{u_{\omega R}}{r}\right]\times \epsilon r^{1+\delta}v.\end{aligned} \tag{2.5.5}$$

Integrating by parts the terms in (2.5.5) yields:

$$\begin{aligned}-\int\int(u_{RR} - \frac{\sqrt{\epsilon}}{r}u_R - \frac{2\epsilon}{r^2}u_{\omega\omega} + \frac{\epsilon u}{r^2})ur^{1+\delta} &= \int\int u_R^2 r^{1+\delta} + 2u_\omega^2 r^{-1+\delta} + \epsilon u r^{-1+\delta} \\ &- 2\epsilon\int_{\omega=\theta_0} r^{-1+\delta}uu_\omega + J_1,\end{aligned} \tag{2.5.6}$$

$$\begin{aligned}\int\int(-2v_{RR} - \frac{2\sqrt{\epsilon}v_R}{r} - \frac{\epsilon v_{\omega\omega}}{r^2} + 2\frac{\epsilon v}{r^2})\epsilon v r^{1+\delta} &= \int\int 2\epsilon v_R^2 r^{1+\delta} + \epsilon^2 v_\omega^2 r^{-1+\delta} \\ &+ 2\epsilon^2 v^2 r^{-1+\delta} - \int_{\omega=\theta_0}\epsilon^2 r^{-1+\delta}vv_\omega + J_2,\end{aligned} \tag{2.5.7}$$

$$\begin{aligned}-\epsilon\int\int u_{\omega R}vr^\delta - \epsilon\int\int v_{\omega R}ur^\delta &= 2\epsilon\int\int u_R v_\omega r^\delta + \delta\epsilon^{3/2}\int\int r^{-1+\delta}v_\omega u \\ &- \int_{\omega=\theta_0}\epsilon u_R v r^\delta = J_3 - \int_{\omega=\theta_0}\epsilon u_R v r^\delta,\end{aligned} \tag{2.5.8}$$

where $J_1 = \delta\sqrt{\epsilon}\int\int uu_R r^{\delta}$, $J_2 = C\epsilon^{3/2}\int\int vv_R r^{\delta}$, and $J_3 = 2\epsilon\int\int u_R v_\omega r^{\delta} + \delta\epsilon^{3/2}\int\int r^{-1+\delta}v_\omega u$. Putting the remaining terms into J_4 yields:

$$J_4 = -3\epsilon^{\frac{3}{2}}\int\int v_\omega u r^{-1+\delta} + 3\epsilon^{\frac{3}{2}}\int\int v u_\omega r^{-1+\delta}. \tag{2.5.9}$$

Letting $J = J_1 + J_2 + J_3 + J_4$, it is easy to see that:

$$|J| \leq C(\epsilon, \theta_0)||v||_B^2, \quad C(\theta_0, \epsilon) \sim \mathcal{O}(\theta_0, \sqrt{\epsilon}). \tag{2.5.10}$$

Using the stress-free boundary condition, (2.2.30), the boundary term from (2.5.7) cancels with that of (2.5.8):

$$-\int_{\omega=\theta_0}\epsilon^2 r^{-1+\delta}vv_\omega - \int_{\omega=\theta_0}\epsilon u_R v r^{\delta} = 0. \tag{2.5.11}$$

The only remaining boundary contribution is then that of (2.5.6):

$$\beta_1 = -2\epsilon\int_{\omega=\theta_0} uu_\omega r^{-1+\delta}. \tag{2.5.12}$$

The remaining interior terms from (2.5.6 - 2.5.7) are then:

$$I_1 = \int\int u_R^2 r^{1+\delta} + 2u_\omega^2 r^{-1+\delta} + \epsilon u r^{-1+\delta} + 2\epsilon v_R^2 r^{1+\delta} + \epsilon^2 v_\omega^2 r^{-1+\delta} + 2\epsilon^2 v^2 r^{-1+\delta}. \tag{2.5.13}$$

Summarizing the calculations from this step, we have:

$$\int\int r^{1+\delta}u \times (2.5.3) + \epsilon r^{1+\delta}v \times (2.5.4) = I_1 + \beta_1 + J, \tag{2.5.14}$$

where β_1 is defined in (2.5.12), J is estimated in (2.5.10), and the interior terms I_1 are defined in (2.5.13).

Step II: Profile Terms

In this step, we treat the terms from (2.2.23 - 2.2.25) which contain u_s, v_s. For clarity, we display these terms here:

$$\frac{1}{r}u_s u_\omega + \frac{1}{r}u_{s\omega}u + u_{sR}v + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v, \tag{2.5.15}$$

$$\frac{1}{r}u_s v_\omega + \frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u. \tag{2.5.16}$$

Applying our multiplier to (2.5.15 - 2.5.16) yields:

$$\begin{aligned}
&\int\int r^{1+\delta}u \times \left[\frac{1}{r}u_s u_\omega + \frac{1}{r}u_{s\omega}u + u_{sR}v + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v\right] \\
&= \int\int u_s u_\omega u r^\delta + u_{s\omega}u^2 r^\delta + u_{sR}uvr^{\delta+1} + v_s u_R u r^{\delta+1} + \sqrt{\epsilon}v_s r^\delta u^2 + \sqrt{\epsilon}u_s r^\delta uv \\
&\lesssim \theta_0||v||_B^2 + \theta_0||u||_A^2,
\end{aligned} \tag{2.5.17}$$

$$\begin{aligned}
&\int\int \epsilon r^{1+\delta}v \times \left[\frac{1}{r}u_s v_\omega + \frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u\right] \\
&= \int\int \epsilon u_s r^\delta v v_\omega + \epsilon v_{s\omega}r^\delta uv + \epsilon v_s v_R v r^{1+\delta} + \epsilon v_{sR}v^2 r^{1+\delta} - 2\sqrt{\epsilon}u_s uvr^\delta \\
&\lesssim C(\theta_0, \epsilon)||v||_B^2 \text{ where } C(\theta_0, \epsilon) \sim \mathcal{O}(\theta_0, \sqrt{\epsilon}).
\end{aligned} \tag{2.5.18}$$

We have used the following uniform bounds on the profiles, recalling that $u_s(\omega, R) = u_e^0(r) + u_p^0(\omega, R) + \sqrt{\epsilon}u_e^1(\omega, r)$, and $v_s(\omega, R) = v_p^0(\omega, R) + v_e^1(\omega, r)$.

$$||u_s||_\infty \leq C, \tag{2.5.19}$$

$$||v_s r^n||_\infty \leq ||v_P^0 r^n||_\infty + ||v_e^1 r^n||_\infty \leq C + ||v_e^1 r^n||_{H^2} \leq C, \tag{2.5.20}$$

$$\sup_{\omega \in [0,\theta_0]} \int_{R_0}^\infty u_{sR}^2 r^n (R - R_0) \leq C \text{ as shown below in (2.5.27)}, \tag{2.5.21}$$

$$||v_{sR} r^n||_\infty \leq ||r^n v_{pR}^0||_\infty + \sqrt{\epsilon}||r^n v_{er}^1||_\infty \leq C + \sqrt{\epsilon}||r^n v_e^1||_{H^3} \leq C, \tag{2.5.22}$$

$$||v_{s\omega} r^n||_\infty \leq C(\theta_0) \text{ by weighted Euler } W^{2,q} \text{ bounds in estimate (2.4.74)}, \tag{2.5.23}$$

$$||u_{s\omega} r^n||_\infty \leq ||u_{P\omega}^0 r^n||_\infty + \sqrt{\epsilon}||u_{e\omega}^1 r^n||_\infty \leq C + \sqrt{\epsilon}||r^n \partial_r (rv)||_\infty, \tag{2.5.24}$$

$$\leq C + \sqrt{\epsilon}||r^n v_e^1||_\infty + \sqrt{\epsilon}||r^n v_{er}^1||_\infty \leq C, \tag{2.5.25}$$

$$||u_{sR} r^n||_\infty \leq \sqrt{\epsilon}||r^n u_{er}^0||_\infty + ||u_p^0 r^n||_\infty + \epsilon||r^n u_{er}^1||_\infty \leq C + \epsilon||r^n v_e^1||_{H^3} \leq C. \tag{2.5.26}$$

Of these, $u_s(\omega, R) = u_e^0(r) + u_p^0(\omega, R) + \sqrt{\epsilon} u_e^1(\omega, r)$ is the term which cannot absorb factors of r due to the outer Euler flow u_e^0 being bounded below away from zero by assumption. For this reason the estimate for $\sqrt{\epsilon} \int \int u_s u v r^\delta$ in (2.5.17) is the most sensitive to the weight and forces the loss of one factor of r in the energy estimate.

The estimate $||u_{er}^1||_\infty \lesssim ||v_e^1||_{H^3}$ from (2.5.26) is given below:

$$\left(u_e^1(\omega, r)\right)^2 = \left(\int_0^\omega \partial_r (rv)(\theta, r) d\theta\right)^2 \Rightarrow \left(u_{er}^1(\omega, r)\right)^2 \leq \theta_0 \int_0^\omega |\partial_{rr}(rv)(\theta, r)|^2 d\theta \leq \theta_0 \int_0^{\theta_0} |\partial_{rr}(rv)|^2 d\theta,$$

$$\Rightarrow \sup_\omega |u_{er}^1(\omega, r)|^2 \leq \theta_0 \int_0^{\theta_0} |\partial_{rr}(rv)|^2(\theta, r) d\theta := \theta_0 \varphi(r) \Rightarrow ||u_{er}^1||_\infty \leq \theta_0 \sup_r |\varphi(r)|.$$

Since $\varphi : [R_0, \infty) \to \mathbb{R}$, according to the 1-dimensional Sobolev embedding:

$$\sup_r |\varphi| \leq ||\varphi||_{W^{1,1}} \leq \int_{R_0}^\infty |\varphi(r)| dr + \int_{R_0}^\infty |\partial_r \varphi(r)| dr \leq \int_{R_0}^\infty \int_0^{\theta_0} |\partial_{rr}(rv)|^2 d\theta dr + \int_{R_0}^\infty \int_0^{\theta_0} |\partial_{rrr}(rv)|^2 dr d\theta.$$

Combining these gives $||u_{er}^1||_\infty \lesssim ||r^n v_e^1||_{H^3}$ for a constant independent of small θ_0. We can repeat the argument for a weight of r^n.

The profile estimates in (2.5.19) - (2.5.26) are all independent of small θ_0, with the exception of $||v_{s\omega}||_\infty$. Any time $||v_{s\omega}||_\infty$ appears in our analysis, it must be accompanied by a power of ϵ to overcome the potentially poor dependence on θ_0. For the uniform bounds, this follows from the fact that the corresponding H^2 estimates were independent of small θ_0. For the $\sup_{\omega\in[0,\theta_0]} \int_{R_0}^\infty u_{sR}^2 r^n (R-R_0)$ estimate, we have:

$$\sup_{[0,\theta_0]} \int_{R_0}^\infty (R-R_0) r^n u_{sR}^2 \leq C(u_e^0, u_p^0) + \sup \int_{R_0}^\infty (R-R_0) r^n \epsilon^2 |u_{er}^1(r)|^2 dR$$
$$\leq C + \epsilon \sup \int_{R_0}^\infty r^{n+1} |u_e^1(r)|^2 dr \leq C + \epsilon\theta_0^{-1} ||r^{n+1} v_e^1||_{H^2}. \quad (2.5.27)$$

Step III: Pressure Term

Applying our multiplier to the two pressure terms in (2.2.23 - 2.2.25) yields:

$$\begin{aligned}
&\int\int P_\omega u r^\delta + \int\int P_R v r^{1+\delta} \\
&= -\int\int P u_\omega r^\delta - (1+\delta)\int\int \sqrt{\epsilon} v P r^\delta - \int\int r^{1+\delta} P v_R + \int_{\omega=\theta_0} P u r^\delta \\
&= -\delta \int\int P\sqrt{\epsilon} v r^\delta + \int_{\omega=\theta_0} P u r^\delta. \qquad (2.5.28)
\end{aligned}$$

Using the stress-free boundary condition at $\{\omega = \theta_0\}$, which is $Pr = 2\epsilon u_\omega$, this boundary contribution cancels the remaining boundary term β_1 from (2.5.12):

$$\int_{\omega=\theta_0} P u r^\delta - 2\epsilon \int_{\omega=\theta_0} u u_\omega r^{-1+\delta} = 0. \qquad (2.5.29)$$

The interior term is estimated as follows:

$$|\delta \int\int P\sqrt{\epsilon} v r^\delta| \leq \delta \left(\int\int P^2 r^\delta\right)^{\frac{1}{2}} \left(\int\int \epsilon v^2 r^\delta\right)^{\frac{1}{2}} \leq \delta\theta_0 ||v||_B ||P||_{L^2_{*,\delta}}. \qquad (2.5.30)$$

Step IV: Right-Hand Side

$$\int\int fr^{1+\delta}u \leq N(\bar{\delta})\int\int f^2r^{2+\delta} + \bar{\delta}\int\int u^2r^{\delta} \leq N(\bar{\delta})\int\int f^2r^{2+\delta} + \theta_0^2||v||_B^2 \quad (2.5.31)$$

$$\epsilon\int\int gr^{1+\delta}v \leq N(\bar{\delta})\epsilon\int\int g^2r^{2+\delta} + \bar{\delta}\int\int \epsilon v^2r^{\delta} \leq N\epsilon\int\int g^2r^{2+\delta} + \theta_0^2||v||_B^2. \quad (2.5.32)$$

Putting the calculations in (2.5.14), (2.5.17 - 2.5.18), (2.5.30), (2.5.31 - 2.5.32), together yields Theorem 2.2.11.

2.6 Positivity Estimate

In this section, we establish the positivity estimate given in Theorem 2.2.13. First, we must prove the positivity calculation in the generality we require, that is, using the weight of r^{δ}:

Proof of Lemma 2.2.12 for general δ.

$$\begin{aligned}\int\int r^{\delta}|\partial_R(rv)|^2 &= \int\int r^{\delta}|\partial_R(\frac{rv}{u_s}u_s)|^2 = \int\int r^{\delta}|\partial_R(\frac{rv}{u_s})u_s + u_{sR}(\frac{rv}{u_s})|^2 \quad (2.6.1)\\ &= \int\int r^{\delta}u_s^2|\partial_R(\frac{rv}{u_s})|^2 + \int\int r^{\delta}\left(\frac{rv}{u_s}\right)^2 u_{sR}^2 + \int\int r^{\delta}u_su_{sR}\partial_R((\frac{rv}{u_s})^2)\\ &= \int\int r^{\delta}|\partial_R(\frac{rv}{u_s})|^2u_s^2 - \int\int \delta r^{\delta-1}(\frac{rv}{u_s})^2\sqrt{\epsilon}u_su_{sR} - \int\int r^{\delta}\left(\frac{rv}{u_s}\right)^2 u_su_{sRR},\end{aligned}$$

$$\begin{aligned}\int\int r^{\delta}|\partial_R(rv\frac{u_s}{u_s})|^2 &= \int\int r^{\delta}|u_s\partial_R(\frac{rv}{u_s}) + u_{sR}(\frac{rv}{u_s}))|^2 \lesssim \int\int r^{\delta}|\partial_R(\frac{rv}{u_s})|^2u_s^2 + \int\int r^{\delta}(\frac{rv}{u_s})^2u_{sR}^2\\ &\lesssim \int\int r^{\delta}|\partial_R(\frac{rv}{u_s})|^2u_s^2, \quad (2.6.2)\end{aligned}$$

where we used the Fundamental Theorem of Calculus, the hypothesis that $\min u_s > 0$, and estimate (2.5.27) for the rapid decay of u_{sR}. Lastly, we deal with the term:

$$\delta \int\int \sqrt{\epsilon} r^{\delta-1} r^2 v^2 \frac{u_{sR}}{u_s} \lesssim \left(\int\int \epsilon v^2 r^\delta\right)^{1/2} \left(\int\int u_{sR}^2 v^2 r^{2+\delta}\right)^{1/2}$$
$$\lesssim \theta_0 ||v||_B \left(\sup \int u_{sR}^2 (R-R_0) r^n dR\right)^{1/2} \left(\int\int v_R^2\right)^{1/2} \lesssim \theta_0 ||v||_B^2. \tag{2.6.3}$$

Estimates (2.6.1) - (2.6.3) immediately imply the desired positivity.

□

Step I: Positive Profile Terms

As discussed in the introduction, we apply the multiplier $(r^\delta \partial_R(\frac{r^2 v}{u_s}), -\epsilon\partial_\omega(\frac{r^{1+\delta}v}{u_s}))$ to the system (2.2.23 - 2.2.25). Here we treat the three profile terms which enable us to obtain the necessary control of $||v||_B$. Explicitly, we are computing:

$$\int\int \left(\frac{u_s}{r} u_\omega + v u_{sR}\right) r^\delta \partial_R \left(\frac{r^2 v}{u_s}\right) - \int\int \epsilon \frac{u_s v_\omega}{r} r^{1+\delta} \partial_\omega \left(\frac{v}{u_s}\right). \tag{2.6.4}$$

We now compute each of the terms in (2.6.4) individually:

$$\int\int u_s r^{-1+\delta} u_\omega \partial_R(\frac{r^2 v}{u_s}) = \int\int u_s r^{-1+\delta}(-\partial_R(rv))\partial_R\left(\frac{r^2 v}{u_s}\right)$$
$$= -\int\int \sqrt{\epsilon} r^\delta v \partial_R(rv) - \int\int r^\delta |\partial_R(rv)|^2 + \int\int r^\delta \frac{u_{sR}}{u_s} rv\partial_R(rv)$$
$$= -\int\int \sqrt{\epsilon} r^\delta v\partial_R(rv) - \int\int r^\delta |\partial_R(rv)|^2 + \frac{1}{2}\int\int r^\delta \frac{u_{sR}}{u_s}\partial_R((rv)^2). \tag{2.6.5}$$

For the first term in (2.6.5), we estimate:

$$|\int\int \sqrt{\epsilon} r^\delta v \partial_R(rv)| \le \theta_0 \left(\int\int \epsilon v_\omega^2 r^\delta\right)^{\frac{1}{2}} \left(\int\int r^\delta |\partial_R(rv)|^2\right)^{\frac{1}{2}} \le \theta_0 ||v||_B^2. \tag{2.6.6}$$

The second term in (2.6.4) is:

$$\begin{aligned}\int\int v u_{sR} r^\delta \partial_R(\frac{r^2 v}{u_s}) &= \int\int v u_{sR} r^\delta \left(\sqrt{\epsilon}\frac{rv}{u_s} + r\frac{\partial_R(rv)}{u_s} - r^2 v \frac{u_{sR}}{u_s^2}\right) \\ &= \int\int v u_{sR} r^\delta \left(\sqrt{\epsilon}\frac{rv}{u_s} - r^2 v\frac{u_{sR}}{u_s^2}\right) + \frac{1}{2}\int\int r^\delta \frac{u_{sR}}{u_s}\partial_R\left((rv)^2\right).\end{aligned} \tag{2.6.7}$$

Combining (2.6.5) - (2.6.7), integrating by parts the final terms in both (2.6.5) and (2.6.7), and recalling estimate (2.6.3) yields:

$$\int\int \left(\frac{u_s}{r}u_\omega + v u_{sR}\right) r^\delta \partial_R\left(\frac{r^2 v}{u_s}\right) \le \theta_0 ||v||_B^2 - \int\int r^\delta |\partial_R(rv)|^2 - r^\delta \frac{u_{sRR}}{u_s}(rv)^2. \tag{2.6.8}$$

The third term in (2.6.4) is:

$$-\epsilon\int\int r^\delta u_s v_\omega \left(\partial_\omega(\frac{v}{u_s})\right) = -\epsilon\int\int r^\delta v_\omega^2 + \epsilon\int\int r^\delta \frac{u_{s\omega}}{u_s} v v_\omega = (2.6.9.1) + (2.6.9.2). \tag{2.6.9}$$

We estimate (2.6.9.2):

$$\epsilon\int\int r^\delta \frac{u_{s\omega}}{u_s} v v_\omega \le \theta_0 ||u_{s\omega}||_\infty \left(\int\int \epsilon v_\omega^2 r^\delta\right)^{1/2} \left(\int\int \epsilon v_\omega^2 r^\delta\right)^{1/2} \lesssim \theta_0 ||v||_B^2. \tag{2.6.10}$$

Putting (2.6.8 - 2.6.10) together and using the positivity estimate (2.2.33) yields:

$$\int\int r|\partial_R(rv)|^2 + \epsilon \int\int rv_\omega^2 \leq -(2.6.4) + \theta_0||v||_B^2. \tag{2.6.11}$$

Step II: Remaining Profile Terms

We now apply the multiplier to the remaining profile terms and provide bounds on each term, keeping in mind the estimates on the profiles we proved in (2.5.19 - 2.5.26). For reference, we include the specific profile terms from equations (2.2.23 - 2.2.25) that we treat in this step:

$$\frac{1}{r}u_{s\omega}u + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v, \text{ and} \tag{2.6.12}$$

$$\frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u. \tag{2.6.13}$$

Applying the multiplier to (2.6.12 - 2.6.13) yields:

$$\int\int \left(\frac{1}{r}u_{s\omega}u + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v\right) r^\delta \partial_R\left(\frac{r^2 v}{u_s}\right) \tag{2.6.14}$$

$$-\epsilon\int\int \left(\frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u\right)\partial_\omega\left(\frac{r^{1+\delta}v}{u_s}\right). \tag{2.6.15}$$

We estimate each of the eight terms appearing in (2.6.14) - (2.6.15) individually:

$$\begin{aligned}
&\int\int u_{s\omega}ur^{\delta-1}\partial_R\left(\frac{r^2v}{u_s}\right) = \int\int r^{-1+\delta}u_{s\omega}u\left(\sqrt{\epsilon}\frac{rv}{u_s} + r\frac{\partial_R(rv)}{u_s} + r^2 v\partial_R(\frac{1}{u_s})\right)\\
&\lesssim \theta_0\left(\int\int r^\delta u_\omega^2\right)^{1/2}\left(\left(\int\int r^\delta|\partial_R(rv)|^2\right)^{1/2} + \left(\int\int \epsilon r^\delta v^2\right)^{1/2}\right)\\
&\lesssim \theta_0||v||_B^2, \quad \text{where the constant } C = ||u_{s\omega}||_\infty + \sup\int u_{sR}^2(R-R_0)r^n.
\end{aligned} \tag{2.6.16}$$

$$\int\int v_s u_R r^\delta \partial_R\left(\frac{r^2 v}{u_s}\right) = \int\int v_s u_R r^\delta \left(\sqrt{\epsilon}\frac{rv}{u_s} + r\frac{\partial_R(rv)}{u_s} + r^2 v \partial_R(\frac{1}{u_s})\right)$$
$$\lesssim \left(\int\int r^{1+\delta}u_R^2\right)^{1/2}\left(\left(\int\int \epsilon v^2 r^\delta\right)^{1/2} + \left(\int\int r^\delta |\partial_R(rv)|^2\right)^{1/2}\right)$$
$$\lesssim \theta_0 ||v||_B ||u||_A, \quad \text{where the constant is: } C = ||v_s r^n||_\infty + \sup\int u_{sR}^2 r^n (R-R_0). \quad (2.6.17)$$

$$\int\int \sqrt{\epsilon} v_s u r^{\delta-1}\partial_R\left(\frac{r^2 v}{u_s}\right) = \int\int \sqrt{\epsilon} v_s r^{-1+\delta} u \left(\sqrt{\epsilon}\frac{rv}{u_s} + r\frac{\partial_R(rv)}{u_s} + r^2 v\partial_R(\frac{1}{u_s})\right)$$
$$\lesssim \left(\int\int \epsilon u^2 r^\delta\right)^{1/2}\left(\left(\int\int r^\delta|\partial_R(rv)|^2\right)^{1/2} + \left(\int\int r^\delta \epsilon v^2\right)^{1/2}\right)$$
$$\lesssim \sqrt{\epsilon}\theta_0 ||v||_B^2, \qquad \text{where the constant } C = ||v_s||_\infty + \sup\int u_{sR}^2 (R-R_0). \qquad (2.6.18)$$

$$\int\int \sqrt{\epsilon} u_s v r^{\delta-1}\partial_R\left(\frac{r^2 v}{u_s}\right) = \int\int \frac{\sqrt{\epsilon}}{r} u_s v r^\delta \left(\sqrt{\epsilon}\frac{rv}{u_s} + r\frac{\partial_R(rv)}{u_s} + r^2 v \partial_R(\frac{1}{u_s})\right) \quad (2.6.19)$$
$$= (2.6.19.1) + (2.6.19.2) + (2.6.19.3).$$

$$(2.6.19.1) = \int\int \epsilon r^\delta v^2 \leq \theta_0 \int\int \epsilon r^\delta v_\omega^2,$$
$$(2.6.19.2) = \int\int \sqrt{\epsilon} r^\delta v \partial_R(rv) \leq \theta_0 \left(\int\int \epsilon r^\delta v_\omega^2\right)^{1/2}\left(\int\int r^\delta |\partial_R(rv)|^2\right)^{1/2},$$
$$(2.6.19.3) = \int\int \sqrt{\epsilon} v^2 r^{1+\delta} u_s \partial_R(\frac{1}{u_s}) \leq \theta_0 \left(\int\int \epsilon r^\delta v_\omega^2\right)^{1/2}\left(\int\int r^n u_{sR}^2 rv\right)^{1/2}$$
$$\leq \theta_0 \left(\int\int \epsilon r^\delta v_\omega^2\right)^{1/2}\left(\sup\int r^n u_{sR}^2 (R-R_0)\right)\left(\int\int |\partial_R(rv)|^2\right)^{1/2}.$$

Thus, (2.6.19) $\lesssim \theta_0 ||v||_B^2$. We now individually compute the terms in (2.6.15):

$$\int\int -\epsilon r^\delta v_{s\omega} u \left(\frac{v_\omega}{u_s} - \frac{v u_{s\omega}}{u_s^2}\right) \leq \sqrt{\epsilon}\left(||v_{s\omega}||_\infty + ||u_{s\omega}||_\infty\right)\theta_0 ||v||_B^2. \qquad (2.6.20)$$

Here again $\sqrt{\epsilon}||v_{s\omega}||_\infty$ is a good term despite the potentially poor dependence of $||v_{s\omega}||_\infty$

on θ_0.

$$\int\int -\epsilon r^{1+\delta} v_s v_R \left(\frac{v_\omega}{u_s} - \frac{v u_{s\omega}}{u_s^2} \right) \le \sqrt{\epsilon} \left(||r v_s||_\infty + ||r u_{s\omega}||_\infty \right) ||v||_B^2, \tag{2.6.21}$$

$$\int\int -\epsilon r^{1+\delta} v_{sR} v \left(\frac{v_\omega}{u_s} - \frac{v u_{s\omega}}{u_s^2} \right) \le \left(||r v_{sR}||_\infty + ||r u_{s\omega}||_\infty \right) \theta_0 ||v||_B^2, \tag{2.6.22}$$

$$\begin{aligned}\int\int 2\sqrt{\epsilon} r^\delta u_s u \left(\frac{v_\omega}{u_s} - \frac{v u_{s\omega}}{u_s^2} \right) &\le \left(\int\int r^\delta u^2 \right)^{1/2} \left(\int\int \epsilon r^\delta v_\omega^2 \right)^{1/2} \\ &+ ||u_{s\omega}||_\infty \left(\int\int r^\delta u^2 \right)^{1/2} \left(\int\int \epsilon r^\delta v^2 \right)^{1/2} \lesssim \theta_0 ||v||_B^2. \end{aligned} \tag{2.6.23}$$

Terms (2.6.19) and (2.6.23) require precision with regards to the weight r^δ in our multipliers, as the profile u_s cannot absorb any factors of r. The results of this step are summarized:

$$(2.6.14) + (2.6.15) \lesssim C(\theta_0, \epsilon) ||v||_B^2 + N ||u||_A^2, \quad C(\theta_0, \epsilon) \sim \mathcal{O}(\theta_0, \sqrt{\epsilon}). \tag{2.6.24}$$

Step III: Laplacian and Lower Order Terms

In this step, we treat the terms which contain (u, v) and do not depend on the profiles u_s, v_s in equations (2.2.23) - (2.2.24). For clarity, these terms are summarized here:

$$- u_{RR} - \frac{\sqrt{\epsilon}}{r} u_r - \frac{\epsilon}{r^2} u_{\omega\omega} + \frac{\epsilon}{r^2} u - \frac{2}{r^2} \epsilon^{3/2} v_\omega, \text{ and} \tag{2.6.25}$$

$$- v_{RR} - \frac{\sqrt{\epsilon}}{r} v_R - \frac{\epsilon}{r^2} v_{\omega\omega} + \frac{2\sqrt{\epsilon}}{r^2} u_\omega + \frac{\epsilon}{r^2} v. \tag{2.6.26}$$

Applying the multiplier then yields:

$$\begin{aligned} &\int\int (\text{Equation } 2.6.25) \times r^\delta \partial_R \left(\frac{r^2 v}{u_s}\right) - \epsilon \int\int (\text{Equation } 2.6.27) \times \partial_\omega \left(\frac{r^{1+\delta} v}{u_s}\right) = \\ &\int\int \left(-u_{RR} - \frac{\sqrt{\epsilon}}{r} u_r - \frac{\epsilon}{r^2} u_{\omega\omega} + \frac{\epsilon}{r^2} u - \frac{2}{r^2} \epsilon^{3/2} v_\omega \right) r^\delta \partial_R \left(\frac{r^2 v}{u_s}\right) \end{aligned} \tag{2.6.27}$$

$$-\epsilon \int\int \left(-v_{RR} - \frac{\sqrt{\epsilon}}{r} v_R - \frac{\epsilon}{r^2} v_{\omega\omega} + \frac{2\sqrt{\epsilon}}{r^2} u_\omega + \frac{\epsilon}{r^2} v\right) \partial_\omega \left(\frac{r^{1+\delta} v}{u_s}\right). \tag{2.6.28}$$

We proceed to individually estimate all of the terms appearing in (2.6.27 - 2.6.28).

$$\int\int -u_{RR} r^\delta \partial_R(\frac{r^2 v}{u_s}) = -\int\int u_{RR} r^\delta \left(\sqrt{\epsilon}\frac{rv}{u_s} + \frac{r}{u_s}\partial_R(rv) + r^2 v \partial_R(\frac{1}{u_s})\right)$$
$$= (2.6.29.1) + (2.6.29.2) + (2.6.29.3). \tag{2.6.29}$$

$$(2.6.29.1) = \sqrt{\epsilon}\int\int u_R \partial_R(\frac{r^{1+\delta} v}{u_s}) = \epsilon \int\int u_R \frac{r^\delta v}{u_s} + \sqrt{\epsilon}\int\int u_R \frac{r^\delta}{u_s}\partial_R(rv)$$
$$+ \sqrt{\epsilon}\int\int u_R r^{1+\delta} v \partial_R(\frac{1}{u_s}) \le \sqrt{\epsilon}||u||_A ||v||_B,$$
$$(2.6.29.2) = \int\int u_R \partial_R\left(r^{1+\delta}\frac{\partial_R(rv)}{u_s}\right) = -\int\int u_R \partial_R\left(r^{1+\delta}\frac{u_\omega}{u_s}\right)$$
$$= -(1+\delta)\sqrt{\epsilon}\int\int u_R \frac{r^\delta u_\omega}{u_s} - \int\int u_R r^{1+\delta}\frac{u_{\omega R}}{u_s} - \int\int r^{1+\delta} u_R u_\omega \partial_R(\frac{1}{u_s})$$
$$= (2.6.29.2.1) + (2.6.29.2.2) + (2.6.29.2.3).$$
$$(2.6.29.2.1) \lesssim \sqrt{\epsilon}\left(\int\int r^\delta u_R^2\right)^{1/2}\left(\int\int r^\delta u_\omega^2\right)^{1/2} \lesssim \sqrt{\epsilon}||u||_A ||v||_B,$$
$$(2.6.29.2.2) = -\frac{1}{2}\int\int r^{1+\delta}\frac{1}{u_s}\partial_\omega\left(u_R^2\right) = +\frac{1}{2}\int\int r^{1+\delta}\partial_\omega\left(\frac{1}{u_s}\right)u_R^2 - \frac{1}{2}\int_{\omega=\theta_0} r^{1+\delta}\frac{1}{u_s}u_R^2,$$
$$(2.6.29.3) = \int\int u_R \partial_R(r^{2+\delta} v \partial_R(\frac{1}{u_s})) = (2+\delta)\sqrt{\epsilon}\int\int u_R r^{1+\delta} v \partial_R(\frac{1}{u_s}) + \int\int u_R v_R r^{2+\delta}\partial_R(\frac{1}{u_s}) +$$
$$\int\int r^{2+\delta} u_R v \partial_{RR}\left(\frac{1}{u_s}\right).$$

For the last term in (2.6.29.3), we use:

$$\sup \int_{R_0}^\infty r^m (R-R_0) u_{sRR}^2 \le C + \epsilon^3 \sup \int_{R_0}^\infty r^m (R-R_0)|u_{err}^1|^2 dR$$
$$\le C + \epsilon^2 \sup \int r^m |u_{err}^1|^2 dr$$
$$\le C + \epsilon^2 \theta_0^{-1} ||r^n v_e^1||_{H^3}^2 \le C + \epsilon\theta_0^{-1}, \tag{2.6.30}$$
$$\sup \int_{R_0}^\infty r^m (R-R_0)|u_{sR}|^4 \le C + \epsilon^4 \sup \int r^m (R-R_0)|u_e^1|^4 dR \le C + C(\theta_0)\epsilon^3. \tag{2.6.31}$$

Summarizing, we have shown (2.6.29) $\lesssim \bar{\delta}||v||_A^2 + N(\bar{\delta})||u||_A^2 - \frac{1}{2}\int_{\omega=\theta_0} r^{1+\delta}\frac{1}{u_s}u_R^2$. The next term in (2.6.25) is:

$$\int\int -\sqrt{\epsilon}u_R r^{-1+\delta}\left(\sqrt{\epsilon}\frac{rv}{u_s} + \frac{r}{u_s}\partial_R(rv) + r^2 v\partial_R(\frac{1}{u_s})\right)$$
$$\leq \sqrt{\epsilon}\left(\int\int r^{1+\delta}u_R^2\right)^{1/2}\left(\int\int \epsilon r^\delta v^2\right)^{1/2} + \sqrt{\epsilon}\left(\int\int r^{1+\delta}u_R^2\right)^{1/2}\left(\int\int |\partial_R(rv)|^2\right)^{1/2}$$
$$+ \theta_0||u_{sR}r^n||_\infty\left(\int\int u_R^2 r^{1+\delta}\right)^{1/2}\left(\int\int \epsilon v_\omega^2 r^\delta\right)^{1/2} \lesssim (\theta_0 + \sqrt{\epsilon})||u||_A||v||_B. \tag{2.6.32}$$

Next,

$$\int\int -\epsilon\frac{1}{r^{2-\delta}}u_{\omega\omega}\partial_R(\frac{r^2 v}{u_s}) = \int\int \frac{\epsilon}{r^{2-\delta}}u_\omega\partial_\omega\partial_R(\frac{r^2 v}{u_s}) - \epsilon\int_{\omega=\theta_0}\frac{u_\omega}{r^{2-\delta}}\partial_R(\frac{r^2 v}{u_s}). \tag{2.6.33}$$

We treat the interior term in (2.6.33), and place the boundary terms in the boundary contribution, β_Δ, which will be treated in the next subsection:

$$\int\int \frac{\epsilon}{r^{2-\delta}}u_\omega\partial_\omega\partial_R(\frac{r^2 v}{u_s}) = \int\int \frac{\epsilon}{r^{2-\delta}}u_\omega\partial_\omega\left(\sqrt{\epsilon}\frac{rv}{u_s} + r\frac{\partial_R(rv)}{u_s} + r^2 v\partial_R(\frac{1}{u_s})\right) \tag{2.6.34}$$
$$= (2.6.34.1) + (2.6.34.2) + (2.6.34.3),$$
$$(2.6.34.1) = \int\int \frac{\epsilon^{3/2}}{u_s}u_\omega v_\omega r^{\delta-1} + \epsilon^{\frac{3}{2}}\int\int r^{\delta-1}u_\omega v\partial_\omega\left(\frac{1}{u_s}\right) \lesssim \sqrt{\epsilon}||v||_B^2,$$
$$(2.6.34.2) = -\int\int \frac{\epsilon}{u_s}u_\omega u_{\omega\omega}r^{\delta-1} = -\frac{\epsilon}{2}\int\int \frac{1}{u_s}\partial_\omega\left(u_\omega^2\right)r^{\delta-1} = \frac{\epsilon}{2}\int\int u_\omega^2\partial_\omega\left(\frac{1}{u_s}\right)r^{\delta-1}$$
$$-\frac{\epsilon}{2}\int_{\omega=\theta_0}u_\omega^2\frac{1}{u_s}r^{\delta-1} \lesssim \epsilon||v||_B^2 - \frac{\epsilon}{2}\int_{\omega=\theta_0}u_\omega^2\frac{1}{u_s}r^{\delta-1},$$
$$(2.6.34.3) = \int\int \epsilon u_\omega r^\delta v_\omega\partial_R\left(\frac{1}{u_s}\right) + \epsilon\int\int r^\delta u_\omega v\partial_{\omega R}\left(\frac{1}{u_s}\right).$$

For the final term in (2.6.34.3) , we have used: $\epsilon\sup_{[0,\theta_0]}\int (R-R_0)|\partial_{\omega R}(\frac{1}{u_s})|^2 \lesssim C$, which can be estimated in a similar way as (2.6.30) - (2.6.31), to obtain:

$$\epsilon \int\int r^{\delta} u_\omega v \partial_{\omega R}\left(\frac{1}{u_s}\right) \le \sqrt{\epsilon}\left(\int\int u_\omega^2 r^{\delta}\right)^{1/2}\left(\int\int \epsilon|\partial_{\omega R}(\frac{1}{u_s})|^2 v^2 r^{\delta}\right)^{1/2} \lesssim \sqrt{\epsilon}||v||_B^2. \tag{2.6.35}$$

This establishes term (2.6.34) $\lesssim \sqrt{\epsilon}||v||_B^2 - \frac{\epsilon}{2}\int_{\omega=\theta_0} u_\omega^2 \frac{1}{u_s} r^{\delta-1} - \epsilon \int_{\omega=\theta_0} \frac{u_\omega}{r^{2-\delta}}\partial_R(\frac{r^2 v}{u_s})$. We now treat the second-order terms from (2.6.28):

$$\epsilon^2 \int\int r^{\delta-1} v_{\omega\omega}\left(\frac{v_\omega}{u_s} - v\frac{u_{s\omega}}{u_s^2}\right) = (2.6.36.1) + (2.6.36.2), \tag{2.6.36}$$

$$\begin{aligned}
(2.6.36.1) &= \epsilon^2 \int\int r^{\delta-1} v_{\omega\omega} v_\omega \frac{1}{u_s} = \frac{\epsilon^2}{2}\int\int r^{\delta-1}\partial_\omega\left(v_\omega^2\right)\frac{1}{u_s} = \\
&\quad -\frac{\epsilon^2}{2}\int\int r^{\delta-1} v_\omega^2 \partial_\omega\left(\frac{1}{u_s}\right) + \frac{\epsilon^2}{2}\int_{\omega=\theta_0} r^{\delta-1} v_\omega^2 \frac{1}{u_s} - \frac{\epsilon^2}{2}\int_{\omega=0} r^{\delta-1} v_\omega^2 \frac{1}{u_s}, \\
(2.6.36.2) &= \epsilon^2 \int\int r^{\delta-1} v_\omega \partial_\omega\left(v\frac{u_{s\omega}}{u_s^2}\right) - \epsilon^2 \int_{\omega=\theta_0} r^{\delta-1} v_\omega v \frac{u_{s\omega}}{u_s^2} \\
&= \epsilon^2 \int\int r^{\delta-1} v_\omega^2 \frac{u_{s\omega}}{u_s^2} + \epsilon^2 \int\int r^{\delta-1} v_\omega v \partial_{\omega\omega}\left(\frac{1}{u_s}\right) - \epsilon^2 \int_{\omega=\theta_0} r^{\delta-1} v_\omega v \frac{u_{s\omega}}{u_s^2}.
\end{aligned}$$

The middle term in (2.6.36.2) requires the following estimate on the profiles:

$$\begin{aligned}
\sup \int (R-R_0)|u_{s\omega\omega}|^2 dR &\le C + \epsilon \sup \int |u^1_{e\omega\omega}|^2(r)(R-R_0)dR \\
&\le C + \sup \int |u^1_{e\omega\omega}(r)|^2 r dr \le C + \theta_0^{-1}||v||_{H^3}^2 \le \theta_0^{-1}\epsilon^{-1},
\end{aligned} \tag{2.6.37}$$

$$\begin{aligned}
\sup \int |u_{s\omega}|^4 (R-R_0)dR &\le C + \epsilon^2 \sup \int |u^1_{e\omega}|^4(r)(R-R_0)dR \\
&\le C + \epsilon\theta_0^{-1}\left(\int\int r^n |u^1_{e\omega}|^4 + \int\int r^n |u^1_{e\omega\omega}|^4\right) \\
&\le C + \epsilon\theta_0^{-1}||v^1_e||_{W^{2,4}}.
\end{aligned} \tag{2.6.38}$$

Thus, we have term (2.6.36) $\lesssim \sqrt{\epsilon}||v||_B^2 + \frac{\epsilon^2}{2}\int_{\omega=\theta_0} r^{\delta-1}v_\omega^2\frac{1}{u_s} - \frac{\epsilon^2}{2}\int_{\omega=0} r^{\delta-1}v_\omega^2\frac{1}{u_s} - \epsilon^2\int_{\omega=\theta_0} r^{\delta-1}v_\omega v\frac{u_{s\omega}}{u_s^2}$. The final second-order term from (2.6.28) is:

$$\epsilon\int\int r^{1+\delta}v_{RR}\left(\frac{v_\omega}{u_s} - v\frac{u_{s\omega}}{u_s^2}\right) = (2.6.39.1) + (2.6.39.2), \tag{2.6.39}$$

$$(2.6.39.1) = -\epsilon\int\int v_R\partial_R\left(r^{1+\delta}\frac{v_\omega}{u_s}\right) = -\epsilon\int\int v_R\left(\frac{(1+\delta)\sqrt{\epsilon}r^\delta v_\omega}{u_s} + \frac{r^{1+\delta}v_{\omega R}}{u_s} + r^{1+\delta}\partial_R(\frac{1}{u_s})v_\omega\right)$$
$$\lesssim \sqrt{\epsilon}||v||_B^2 - \int_{\omega=\theta_0}\frac{\epsilon}{2}r^{1+\delta}v_R^2\frac{1}{u_s},$$
$$(2.6.39.2) = \epsilon\int\int v_R\partial_R\left(r^{1+\delta}v\frac{u_{s\omega}}{u_s^2}\right) = \epsilon\int\int r^{1+\delta}v_R^2\frac{u_{s\omega}}{u_s^2} + (1+\delta)\epsilon^{3/2}\int\int vv_Rr^\delta\frac{u_{s\omega}}{u_s^2}$$
$$+\epsilon\int\int r^{1+\delta}v_Rv\partial_{\omega R}\left(\frac{1}{u_s}\right).$$

For the final term in (2.6.39.2), we use that $\epsilon\sup_{[0,\theta_0]}\int r^m(R-R_0)|u_{s\omega R}|^2 \le C$. Thus, (2.6.39) $\lesssim \sqrt{\epsilon}||v||_B^2 - \int_{\omega=\theta_0}\frac{\epsilon}{2}r^{1+\delta}v_R^2\frac{1}{u_s}$. Finally, we have the low-order terms from (2.6.27) - (2.6.28):

$$\int\int\frac{\epsilon}{r^{2-\delta}}u\left(\sqrt{\epsilon}\frac{rv}{u_s} + \frac{r}{u_s}\partial_R(rv) + r^2v\partial_R(\frac{1}{u_s})\right) - \frac{2\epsilon^{3/2}}{r^{2-\delta}}v_\omega\left(\sqrt{\epsilon}\frac{rv}{u_s} + \frac{r}{u_s}\partial_R(rv) + r^2v\partial_R(\frac{1}{u_s})\right)$$
$$+\epsilon^{3/2}\int\int r^\delta v_R\left(\frac{v_\omega}{u_s} - v\frac{u_{s\omega}}{u_s^2}\right) - 2\int\int\epsilon^{3/2}r^{\delta-1}u_\omega\left(\frac{v_\omega}{u_s} - v\frac{u_{s\omega}}{u_s^2}\right)$$
$$-\int\int\epsilon^2r^{\delta-1}v\left(\frac{v_\omega}{u_s} - v\frac{u_{s\omega}}{u_s^2}\right) \lesssim C(\epsilon,\theta_0)||v||_B^2 + N||u||_A^2, \tag{2.6.40}$$

where $C(\epsilon,\theta_0)\to 0$ as either $\theta_0\to 0$ or $\epsilon\to 0$. Summarizing the results of this step:

$$(2.6.27) + (2.6.28) \le C(\epsilon,\theta_0)||v||_B^2 + N||u||_A^2 + \beta_\Delta, \quad C(\theta_0,\epsilon)\sim\mathcal{O}(\theta_0,\sqrt{\epsilon}), \tag{2.6.41}$$

where β_Δ contains the boundary terms from (2.6.29), (2.6.34), (2.6.36), (2.6.39):

$$\beta_\Delta = -\frac{1}{2}\int_{\omega=\theta_0} r^{1+\delta}\frac{1}{u_s}u_R^2 - \frac{\epsilon}{2}\int_{\omega=\theta_0}\frac{1}{u_s}u_\omega^2 r^{\delta-1} - \epsilon\int_{\omega=\theta_0} u_\omega r^{\delta-2}\partial_R(\frac{r^2 v}{u_s}) + \frac{\epsilon^2}{2}\int_{\omega=\theta_0}\frac{v_\omega^2}{u_s}r^{\delta-1}$$
$$- \epsilon^2\int_{\omega=\theta_0} v v_\omega \frac{u_{s\omega}}{u_s^2}r^{\delta-1} - \frac{\epsilon^2}{2}\int_{\omega=0} r^{\delta-1}v_\omega^2\frac{1}{u_s} - \frac{\epsilon}{2}\int_{\omega=\theta_0} r^{\delta+1}\frac{1}{u_s}v_R^2. \tag{2.6.42}$$

Step IV: Pressure Terms

In this step, we apply our multiplier to the pressure terms from (2.2.23 − 2.2.25), which immediately yield:

$$\int\int P_\omega r^{\delta-1}\partial_R\left(\frac{r^2 v}{u_s}\right) - \int\int P_R r^{1+\delta}\partial_\omega(\frac{v_\omega}{u_s}) = \int\int P_\omega r^{\delta-1}\partial_R(\frac{r^2 v}{u_s}) - \int\int P_R r^{\delta-1}\partial_\omega(\frac{r^2 v}{u_s})$$
$$= \int_{\omega=\theta_0} r^{\delta-1}P\partial_R(\frac{r^2 v}{u_s}) + (\delta-1)\int\int\sqrt{\epsilon}r^\delta P\left(\frac{v_\omega}{u_s} - \frac{u_{s\omega}}{u_s^2}v\right). \tag{2.6.43}$$

The interior term is estimated:

$$|\delta-1||\int\int\sqrt{\epsilon}r^\delta P\left(\frac{v_\omega}{u_s} - \frac{u_{s\omega}}{u_s^2}v\right)| \le |\delta-1|\left(\int\int r^\delta P^2\right)^{1/2}||v||_B. \tag{2.6.44}$$

We estimate the boundary term from (2.6.43) using the stress-free boundary condition in (2.2.30):

$$\beta_P := \int_{\omega=\theta_0} r^{\delta-1}P\partial_R(\frac{r^2 v}{u_s}) = -2\epsilon\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}u_\omega^2 + 2\epsilon^{3/2}\int_{\omega=\theta_0} r^{\delta-1}\frac{u_\omega v}{u_s} + 2\epsilon\int_{\omega=\theta_0} r^\delta u_\omega v\partial_R(\frac{1}{u_s})$$
$$\le -\epsilon\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}u_\omega^2 + N\int_{\omega=\theta_0}\epsilon v^2 r^{\delta-1}. \tag{2.6.45}$$

Step V: Boundary Terms

We rewrite the Boundary contributions of the Navier Stokes terms, starting with (2.6.42):

$$\beta_\Delta = -\frac{1}{2}\int_{\omega=\theta_0} r^{1+\delta}\frac{1}{u_s}u_R^2 - \frac{\epsilon}{2}\int_{\omega=\theta_0}\frac{1}{u_s}u_\omega^2 r^{\delta-1} - \epsilon\int_{\omega=\theta_0} u_\omega r^{\delta-2}\partial_R(\frac{r^2 v}{u_s}) + \frac{\epsilon^2}{2}\int_{\omega=\theta_0}\frac{v_\omega^2}{u_s}r^{\delta-1}$$
$$- \epsilon^2\int_{\omega=\theta_0} v v_\omega \frac{u_{s\omega}}{u_s^2}r^{\delta-1} - \frac{\epsilon^2}{2}\int_{\omega=0} r^{\delta-1}v_\omega^2\frac{1}{u_s} - \frac{\epsilon}{2}\int_{\omega=\theta_0} r^{\delta+1}\frac{1}{u_s}v_R^2$$
$$= -\frac{\epsilon}{2}\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}u_\omega^2 - \epsilon\int_{\omega=\theta_0} r^{\delta-2}u_\omega\partial_R(\frac{r^2 v}{u_s}) - \epsilon^2\int_{\omega=\theta_0} r^{\delta-1}v v_\omega\frac{u_{s\omega}}{u_s^2} - \frac{\epsilon^2}{2}\int_{\omega=0} v_\omega^2\frac{r^{\delta-1}}{u_s}$$
$$- \frac{\epsilon}{2}\int_{\omega=\theta_0}\frac{r^{\delta+1}}{u_s}v_R^2 \tag{2.6.46}$$
$$= \frac{\epsilon}{2}\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}u_\omega^2 + \epsilon^{3/2}\int_{\omega=\theta_0} r^{\delta-1}\frac{u_\omega v}{u_s} - \epsilon\int_{\omega=\theta_0} r^\delta u_\omega v\partial_R(\frac{1}{u_s}) - \epsilon^2\int_{\omega=\theta_0} r^{\delta-1}v v_\omega\frac{u_{s\omega}}{u_s^2}$$
$$- \frac{\epsilon}{2}\int_{\omega=\theta_0}\frac{r^{\delta+1}}{u_s}v_R^2 - \frac{\epsilon^2}{2}\int_{\omega=0} r^{\delta-1}\frac{v_\omega^2}{u_s}. \tag{2.6.47}$$

In the equality yielding (2.6.46), the stress-free condition from (2.2.30) was used. Using the divergence free condition: $u_\omega - \sqrt{\epsilon}v = rv_R \Rightarrow r^2 v_R^2 = u_\omega^2 + \epsilon v^2 - 2\sqrt{\epsilon}v u_\omega$, we can rewrite two of the terms in (2.6.47):

$$\frac{\epsilon}{2}\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}u_\omega^2 - \frac{\epsilon}{2}\int_{\omega=\theta_0}\frac{r^{\delta+1}}{u_s}v_R^2 = -\frac{\epsilon^2}{2}\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}v^2 + \epsilon^{\frac{3}{2}}\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}v u_\omega. \tag{2.6.48}$$

We also note that the $\int_{\omega=0}$ in (2.6.47) is of a beneficial sign, and so we only keep treating the $\int_{\omega=\theta_0}$ contributions. Summarizing, we have:

$$\beta_\Delta = \epsilon^{3/2}\int_{\omega=\theta_0} r^{\delta-1}\frac{u_\omega v}{u_s} - \epsilon\int_{\omega=\theta_0} r^\delta u_\omega v\partial_R(\frac{1}{u_s}) - \epsilon^2\int_{\omega=\theta_0} r^{\delta-1}v v_\omega\frac{u_{s\omega}}{u_s^2} - \frac{\epsilon^2}{2}\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}v^2$$
$$+ \epsilon^{\frac{3}{2}}\int_{\omega=\theta_0}\frac{r^{\delta-1}}{u_s}v u_\omega - \frac{\epsilon^2}{2}\int_{\omega=0} r^{\delta-1}\frac{v_\omega^2}{u_s}. \tag{2.6.49}$$

We now note that: $\displaystyle\int_{\omega=\theta_0}\epsilon v^2 r^{\delta-1} \le \theta_0^2\int\int \epsilon v_\omega^2 r^{\delta-1}$. Using Young's inequality we can absorb all of the terms in (2.6.49) and (2.6.45) into either $-\int_{\omega=\theta_0} \epsilon u_\omega^2 r^{\delta-1}$ or $\int_{\omega=\theta_0} \epsilon v^2 r^{\delta-1}$ terms except for the third term in (2.6.49), which we now estimate:

$$\begin{aligned}
-\epsilon^2 \int_{\omega=\theta_0} r^{\delta-1} v v_\omega \frac{u_{s\omega}}{u_s^2} &= \epsilon \int_{\omega=\theta_0} r^\delta u_R v \frac{u_{s\omega}}{u_s^2} = -\epsilon \int_{\omega=\theta_0} u \partial_R (r^\delta v \frac{u_{s\omega}}{u_s^2}) \\
&= -\epsilon \int_{\omega=\theta_0} u \delta r^{\delta-1} \sqrt{\epsilon} v \frac{u_{s\omega}}{u_s^2} - \epsilon \int_{\omega=\theta_0} r^\delta u v_R \frac{u_{s\omega}}{u_s^2} + \epsilon \int_{\omega=\theta_0} r^\delta u v \partial_{\omega R} \left(\frac{1}{u_s} \right) \qquad (2.6.50) \\
&= (2.6.50.1) + (2.6.50.2) + (2.6.50.3).
\end{aligned}$$

First, we have:

$$(2.6.50.1) \lesssim \epsilon^{\frac{3}{2}} \left(\int_{\omega=\theta_0} u^2 \right)^{\frac{1}{2}} \left(\int_{\omega=\theta_0} v^2 \right)^{\frac{1}{2}} \le \epsilon \theta_0^2 \left(\int \int u_\omega^2 \right)^{\frac{1}{2}} \left(\int \int \epsilon v_\omega^2 \right)^{\frac{1}{2}} \lesssim \epsilon \theta_0^2 ||v||_B^2.$$

For the second term, we use the divergence free condition $v_R = -\sqrt{\epsilon} \frac{1}{r} v - \frac{1}{r} u_\omega$, yielding:

$$\begin{aligned}
(2.6.50.2) &= \epsilon^{3/2} \int_{\omega=\theta_0} r^{\delta-1} \frac{u_{s\omega}}{u_s^2} u v + \epsilon \int_{\omega=\theta_0} r^{\delta-1} u u_\omega \frac{u_{s\omega}}{u_s^2} \\
&\lesssim \epsilon^{3/2} \left(\int_{\omega=\theta_0} u^2 \right)^{\frac{1}{2}} \left(\int_{\omega=\theta_0} v^2 \right)^{\frac{1}{2}} + \epsilon \left(\int_{\omega=\theta_0} u^2 \right)^{\frac{1}{2}} \left(\int_{\omega=\theta_0} u_\omega^2 \right)^{\frac{1}{2}} \\
&\lesssim \epsilon \theta_0^2 \left(\int \int u_\omega^2 \right)^{\frac{1}{2}} \left(\int \int \epsilon v_\omega^2 \right)^{\frac{1}{2}} + \epsilon \theta_0 \left(\int \int u_\omega^2 \right)^{\frac{1}{2}} \left(\int_{\omega=\theta_0} u_\omega^2 \right)^{\frac{1}{2}} \\
&\lesssim \epsilon \theta_0^2 ||v||_B^2 + \bar{\delta} \theta_0 \int_{\omega=\theta_0} \epsilon u_\omega^2 + N(\bar{\delta}) \theta_0 \epsilon \int \int u_\omega^2 \lesssim \bar{\delta} \theta_0 \int_{\omega=\theta_0} \epsilon u_\omega^2 + \epsilon \theta_0 ||v||_B^2.
\end{aligned}$$

For the third term, we have:

$$(2.6.50.3) \lesssim \sqrt{\epsilon} ||\partial_{\omega R} \left(\frac{1}{u_s} \right) ||_\infty \left(\int \int u_\omega^2 r^\delta \right)^{\frac{1}{2}} \theta_0 \left(\int \int \epsilon v_\omega^2 r^\delta \right)^{\frac{1}{2}} \lesssim \theta_0 ||v||_B^2. \qquad (2.6.51)$$

We have used that

$$||\partial_{\omega R} \left(\frac{1}{u_s} \right) ||_\infty \lesssim ||u_{s\omega R}||_\infty + ||u_{s\omega}||_\infty ||u_{sR}||_\infty \lesssim C + \epsilon ||v_e^1||_{H^4} \lesssim \epsilon^{-1/2}.$$

The results of this step may be summarized as:

$$|\beta_\Delta| + |\beta_P| \lesssim -\int_{\omega=\theta_0} \epsilon \frac{r^{\delta-1}}{u_s} u_\omega^2 + C(\theta_0, \epsilon)||v||_B^2, \quad C(\theta_0, \epsilon) \sim \mathcal{O}(\theta_0, \sqrt{\epsilon}). \tag{2.6.52}$$

Step VI: Right-Hand Side

$$\begin{aligned}\int\int f r^\delta \partial_R\Big(\frac{r^2 v}{u_s}\Big) &= \int\int f r^\delta \left(\sqrt{\epsilon}\frac{rv}{u_s} + r\partial_R(rv)\frac{1}{u_s} + r^2 v \partial_R\Big(\frac{1}{u_s}\Big)\right) \\ &\le N(\bar{\delta}) \int\int f^2 r^{2+\delta} + \bar{\delta}\int\int \epsilon v^2 r^\delta + \bar{\delta}\int\int r^\delta |\partial_R(rv)|^2,\end{aligned} \tag{2.6.53}$$

$$\epsilon \int\int g r^{1+\delta}\frac{v_\omega}{u_s} \le N(\bar{\delta})\epsilon \int\int g^2 r^{2+\delta} + \bar{\delta}\int\int \epsilon v_\omega^2 r^\delta, \tag{2.6.54}$$

$$\epsilon \int\int g r^{1+\delta} v \frac{u_{s\omega}}{u_s^2} \le N(\bar{\delta}) \int\int \epsilon g^2 r^{2+\delta} + \bar{\delta}\int\int \epsilon v^2 r^\delta. \tag{2.6.55}$$

Placing the above steps together finishes the proof of Theorem 2.2.13.

2.7 Pressure Estimate

Given $P(\omega, R) \in L^2(\Omega_N)$, there is a corresponding scaled $p(\omega, r) = P(\omega, R)$, whose domain is $\Omega_{\sqrt{\epsilon}N} = (0, \theta_0) \times (R_0, R_0 + \sqrt{\epsilon}N)$. Abusing notation, denote the Euclidean counterpart to $p(\omega, r)$ by $p(x, y)$.

Definition 2.7.1. L_0^2 denotes the mean-zero subspace of L^2: $q_0 \in L_0^2$ iff $\int\int q_0 dxdy = \int\int q_0 r dr d\omega = 0$.

Claim 5 (Mean-zero solvability of **div**). *Denote the annular domain* $\Omega_{\theta_0} = (0, \theta_0) \times$

$(R_0, R_0+\theta_0)$. *For each* $q_0 \in L_0^2(\Omega_{\theta_0})$, *there exists a vector field* $\mathbf{v_0} \in H_0^1(\Omega_{\theta_0})$ *such that* $\mathbf{div}(\mathbf{v_0}) = q_0$,$||\mathbf{v_0}||_{H_0^1(\Omega_{\theta_0})} \leq C_0||q_0||_{L^2(\Omega_{\theta_0})}$, *where* C_0 *is independent of small* θ_0.

Proof. The solvability of $\mathbf{div} : H_0^1 \to L_0^2$ is well known (see (Orl98, pp. 26-28)). The important point for our analysis is that the constant C_0 is independent of small θ_0. This is guaranteed by (Gal11, p. 162, estimate III.3.4), in which it is shown $C_0 \lesssim \left(\frac{\text{diam}(\Omega_{\theta_0})}{R}\right)^2$, where R is the radius of a ball $B_R \subset \Omega_{\theta_0}$ with respect to which Ω_{θ_0} is starlike. In our case, $R \approx \text{diam}(\Omega_{\theta_0})$. The claim is proven. $\square$

Claim 6. *For each* $q \in L^2(\Omega_{\theta_0})$, *there exists a vector field* $\mathbf{v}$ *such that* $\mathbf{v} = 0$ *on* $\{R = R_0, R_0+\theta_0\}, \{\omega = 0\}$, *and* $||\mathbf{v}||_{H^1(\Omega_{\theta_0})} \leq C||q||_{L^2(\Omega_{\theta_0})}$, *where* C *is independent of small* θ_0.

Proof. Similar to (Orl98, page 27), define the mean-zero function $q_0 = q - \left(\int\int_{\Omega_{\theta_0}} qrdrd\omega\right)\mathbf{div}(\mathbf{w})$, where $\mathbf{w} = \left(6\theta_0^{-4}(r-R_0)(\theta_0 - r + R_0)\omega, 0\right)$. By direct computation,

$$\int\int_{\Omega_{\theta_0}} \mathbf{div}(\mathbf{w})rdrd\omega = 1, \qquad ||\mathbf{div}(\mathbf{w})||_{L^2} \lesssim \theta_0^{-1}, \qquad ||\mathbf{w}||_{H^1} \lesssim \theta_0^{-1}. \tag{2.7.1}$$

Denoting by $\mathbf{v_0}$ the vector field guaranteed by Claim 5 for the function q_0,

$$||\mathbf{v_0}||_{H^1} \lesssim ||q_0||_{L^2} \lesssim ||q||_{L^2} + \left(\int\int |q|rdrd\omega\right)||\mathbf{div}(\mathbf{w})||_{L^2} \lesssim ||q||_{L^2} + ||q||_{L^2}\theta_0\theta_0^{-1} \lesssim ||q||_{L^2}. \tag{2.7.2}$$

The factor of θ in the final inequality in (2.7.2) arises from Holder's inequality:

$$\int\int_{\Omega_{\theta_0}} |q|rdrd\omega \leq ||q||_{L^2}\left(\int_0^{\theta_0}\int_{R_0}^{R_0+\theta_0} rdrd\omega\right)^{\frac{1}{2}} \lesssim ||q||_{L^2}\theta_0. \tag{2.7.3}$$

The desired vector field is now $\mathbf{v} = \mathbf{v}_0 + \left(\int\int qrdrd\omega\right)\mathbf{w}$. Clearly $\mathbf{v}$ vanishes on the required components of the boundary, and we have:

$$||\mathbf{v}||_{H^1} \leq ||\mathbf{v_0}||_{H^1} + \Big(\int\int |q|r\Big)||\mathbf{w}||_{H^1} \lesssim ||q||_{L^2} + ||q||_{L^2}\theta_0\theta_0^{-1} = ||q||_{L^2}. \tag{2.7.4}$$

The claim is proven. □

Claim 7. *There exists a vector field* $\mathbf{F}(x,y) = (f(x,y), g(x,y))$ *such that* $\mathbf{div}(\mathbf{F}) = p(x,y)$*, and:*

$$||\mathbf{F}||_{H^1(\Omega_{\sqrt{\epsilon}N})} \leq C||p||_{L^2(\Omega_{\sqrt{\epsilon}N})}, \tag{2.7.5}$$

where the constant C *is independent of* ϵ, N*, small* θ_0*, and where* $\mathbf{F}$ *vanishes on the Dirichlet portions of the boundary* $\{R = R_0\}, \{\omega = 0\}, \{R = R_0 + \sqrt{\epsilon}N\}$*.*

Proof. Divide the domain $\Omega_{\sqrt{\epsilon}N}$ into $\Omega^k = \{0, \theta_0\} \times \{R_0 + k\theta_0, R_0 + (k+1)\theta_0\}$. By Claim 6, there exists a vector field, $\mathbf{F}_k$, such that $\mathbf{div}(\mathbf{F_k}) = p$ on Ω^k, $\mathbf{F_k}(\omega, R_0 + \theta_0 k) = \mathbf{F_k}(\omega, R_0 + (k+1)\theta_0) = \mathbf{F_k}(0, r) = 0$, and $||\mathbf{F_k}||_{H^1(\Omega^k)} \leq C||p||_{L^2(\Omega^k)}$, where C does not depend on small θ_0. Define $\mathbf{F} = \sum_k \mathbf{F_k}$. Then $||\mathbf{F}||_{H^1(\Omega)} = \sum_k ||F_k||_{H^1(\Omega^k)} \leq C\sum_k ||p||_{L^2(\Omega^k)} = C||p||_{L^2}$, and $\mathbf{div}(\mathbf{F}) = p$. Finally $\mathbf{F}$ satisfies the required boundary conditions. The claim is proven. □

The vector field $\mathbf{F}$ can be expressed in the polar coordinate basis e_θ and e_r, and as functions of ω, r, in which case $\mathbf{div}(\mathbf{F}) = \frac{\phi_\omega}{r} + \frac{\psi}{r} + \psi_r = p(\omega, r)$. Converting estimate (2.7.5) to polar coordinates reads:

$$\int\int_{\Omega_{\sqrt{\epsilon}N}} \Big(\phi^2 + \psi^2 + \frac{\phi_\omega^2}{r^2} + \frac{\psi_\omega^2}{r^2} + \phi_r^2 + \psi_r^2\Big) r dr d\omega \lesssim \int\int_{\Omega_{\sqrt{\epsilon}N}} p^2(\omega, r) r dr d\omega. \tag{2.7.6}$$

Define $a(\omega, R) = \phi(\omega, r)$ and $\sqrt{\epsilon}b(\omega, R) = \psi(\omega, r)$, so $a_\omega(\omega, R) = \phi_\omega(\omega, r)$, $\frac{1}{\sqrt{\epsilon}}a_R(\omega, R) =$

$\phi_r(\omega, r)$. Also, $\sqrt{\epsilon} b_\omega(\omega, R) = \psi_\omega(\omega, r), b_R(\omega, R) = \psi_r(\omega, r)$. Note that this scaling is the same as the Prandtl scaling. Scaling all of the terms in (2.7.6) yields:

$$\int\int_{\Omega_N} \left(a^2 + \epsilon b^2 + \frac{a_\omega^2}{r^2} + \epsilon\frac{b_\omega^2}{r^2} + \frac{1}{\epsilon}a_R^2 + b_R^2\right) rdRd\omega \lesssim \int\int_{\Omega_N} P(\omega, R)^2 rdRd\omega, \quad (2.7.7)$$

and the scaled divergence equation:

$$\frac{a_\omega}{r} + \sqrt{\epsilon}\frac{b}{r} + b_R = P. \quad (2.7.8)$$

The admissible weights in (2.7.7) must be generalized to r^δ for $\delta \in [0, 1]$. The next claim shows this is possible as long as a small error is made in the scaled divergence equation (2.7.8).

Claim 8. *Given $P \in L^2(\Omega_N)$ there exists a vector field $\mathbf{A_1} = (a, b)$ such that*

$$\int\int_{\Omega_N} \left(a^2 + \epsilon b^2 + \frac{a_\omega^2}{r^2} + \epsilon\frac{b_\omega^2}{r^2} + \frac{1}{\epsilon}a_R^2 + b_R^2\right) r^\delta dRd\omega \lesssim \int\int_{\Omega_N} P(\omega, R)^2 r^\delta dRd\omega, \quad (2.7.9)$$

where the constant is independent of N, ϵ, and small θ_0. $\mathbf{A}_1$ vanishes on $\{R = R_0, R_0 + N\}$, $\{\omega = 0\}$, and satisfies the scaled divergence equation:

$$\frac{a_\omega}{r} + \sqrt{\epsilon}\frac{b}{r} + b_R = P + \left(\frac{1-\delta}{2}\right)\sqrt{\epsilon} b r^{-1}. \quad (2.7.10)$$

Proof. Given P, define $\bar{P} = P r^{\frac{\delta}{2} - \frac{1}{2}}$. Applying the procedure culminating in estimate (2.7.7) to $\bar{P}$ gives a vector field $(\bar{a}, \bar{b})$ satisfying:

$$\int\int_{\Omega_N} \left(\bar{a}^2 + \epsilon\bar{b}^2 + \frac{\bar{a}_\omega^2}{r^2} + \epsilon\frac{\bar{b}_\omega^2}{r^2} + \frac{1}{\epsilon}\bar{a}_R^2 + \bar{b}_R^2\right) rdRd\omega \lesssim \int\int_{\Omega_N} \bar{P}^2 r = \int\int_{\Omega_N} P^2 r^\delta, \quad (2.7.11)$$

and

$$\frac{\bar{a}_\omega}{r} + \sqrt{\epsilon}\frac{\bar{b}}{r} + \bar{b}_R = \bar{P} = Pr^{\frac{\delta}{2}-\frac{1}{2}}. \tag{2.7.12}$$

Define $\mathbf{A_1} = (a, b) = (\bar{a}r^{\frac{1}{2}-\frac{\delta}{2}}, \bar{b}r^{\frac{1}{2}-\frac{\delta}{2}})$. One readily computes the scaled divergence of $\mathbf{A_1}$ to check equation (2.7.10), as well as the desired estimates (2.7.9). It is also clear that $\mathbf{A_1}$ vanishes on $\{R = R_0, R_0 + N\}$, $\{\omega = 0\}$ because $(\bar{a}, \bar{b})$ vanishes on those components. The claim is proven. $\square$

We now test against our equation against the multiplier $(ar^\delta, \epsilon r^\delta b)$ in several steps.

Step I: Pressure Terms

Applying the the multiplier $(ar^\delta, \epsilon r^\delta b)$ to the terms in equation (2.2.23 - 2.2.25) containing the pressure, P, yields:

$$\int\int P_\omega a r^{\delta-1} + \int\int P_R b r^\delta = -\int\int P a_\omega r^{\delta-1} + \int_{\omega=\theta_0} P a r^{\delta-1} - \int\int P r^\delta b_R - \delta\int\int r^{\delta-1}\sqrt{\epsilon} b P$$
$$= \int\int -P^2 r^\delta + \frac{3}{2}(1-\delta)\int\int \sqrt{\epsilon} b r^{\delta-1} P + \int_{\omega=\theta_0} P a r^{\delta-1}. \tag{2.7.13}$$

We have used the relation (2.7.10). Using (2.7.9), the middle term above can be estimated:

$$|3\frac{1-\delta}{2}|\int\int |Pr^{\delta-1}\sqrt{\epsilon}b| \lesssim |1-\delta|\left(\int\int P^2 r^\delta\right)^{\frac{1}{2}}\left(\int\int \epsilon b^2 r^{\delta-2}\right)^{\frac{1}{2}}$$
$$\lesssim |1-\delta|\theta_0\left(\int\int P^2 r^\delta\right)^{\frac{1}{2}}\left(\int\int \epsilon b_\omega^2 r^{\delta-2}\right)^{\frac{1}{2}} \lesssim (1-\delta)\theta_0\int\int P^2 r^\delta. \tag{2.7.14}$$

Step II: Laplacian Terms and Lower Order Terms

In order to obtain the proper boundary cancellation, we use the representation of the Laplacian given in (2.5.3), (2.5.4). Applying our multiplier then yields:

$$\int\int\left[-u_{RR}-\frac{\sqrt{\epsilon}}{r}u_R-\frac{2\epsilon}{r^2}u_{\omega\omega}+\frac{\epsilon}{r^2}u-\frac{3}{r^2}\epsilon^{3/2}v_\omega-\frac{\epsilon}{r}v_{\omega R}\right]ar^\delta$$
$$+\int\int\left[-2v_{RR}-\frac{2\sqrt{\epsilon}}{r}v_R-\frac{\epsilon}{r^2}v_{\omega\omega}+\frac{3\sqrt{\epsilon}}{r^2}u_\omega+2\frac{\epsilon}{r^2}v-\frac{u_{\omega R}}{r}\right]\epsilon br^\delta. \tag{2.7.15}$$

We successively treat each term in (2.7.15), starting with the important, high order terms:

$$\begin{aligned}-\int\int u_{RR}r^\delta a &= \int\int r^\delta u_R a_R+\int\int \delta r^{\delta-1}\sqrt{\epsilon}au_R\\ &\le\left(\int\int r^\delta a_R^2\right)^{1/2}\left(\int\int r^\delta u_R^2\right)^{1/2}+\int\int\delta r^{\delta-1}\sqrt{\epsilon}au_R\\ &\le\sqrt{\epsilon}\left(\int\int P(\omega,R)^2r^\delta\right)^{1/2}\left(\int\int r^\delta u_R^2\right)^{1/2}+\int\int\delta r^{\delta-1}\sqrt{\epsilon}au_R\\ &\le\sqrt{\epsilon}||P||_{L^2_*,\delta}||u||_A,\end{aligned}\tag{2.7.16}$$

$$\begin{aligned}-2\int\int\epsilon r^{\delta-2}au_{\omega\omega} &= 2\int\int\epsilon r^{\delta-2}a_\omega u_\omega-2\epsilon\int_{\omega=\theta_0}r^{\delta-2}au_\omega\\ &\le\epsilon\left(\int\int a_\omega^2r^{\delta-2}\right)^{1/2}\left(\int\int u_\omega^2r^{\delta-2}\right)^{1/2}-2\epsilon\int_{\omega=\theta_0}r^{\delta-2}au_\omega\\ &\le\epsilon||P||_{L^2_*,\delta}||v||_B-2\epsilon\int_{\omega=\theta_0}r^{\delta-2}au_\omega,\end{aligned}\tag{2.7.17}$$

$$\begin{aligned}-2\int\int\epsilon v_{RR}r^\delta b &= 2\int\int\epsilon r^\delta v_Rb_R+2\epsilon^{3/2}\int\int\delta r^{\delta-1}bv_R\\ &\le\epsilon\left(\int\int r^\delta v_R^2\right)^{1/2}\left(\int\int r^\delta b_R^2\right)^{1/2}+2\epsilon^{3/2}\int\int\delta r^{\delta-1}bv_R\\ &\le\epsilon||v||_B||P||_{L^2_*,\delta},\end{aligned}\tag{2.7.18}$$

$$-\int\int \epsilon^2 v_{\omega\omega} b r^{\delta-2} = \epsilon^2 \int\int v_\omega b_\omega r^{\delta-2} - \int_{\omega=\theta_0} \epsilon^2 v_\omega b r^{\delta-2}$$
$$\leq \epsilon \left(\int\int \epsilon v_\omega^2 r^{\delta-2}\right)^{1/2} \left(\int\int \epsilon b_\omega^2 r^{\delta-2}\right)^{1/2} - \int_{\omega=\theta_0} \epsilon^2 r^{\delta-2} v_\omega b,$$
$$\leq \epsilon ||v||_B ||P||_{L^2_*,\delta} - \epsilon^2 \int_{\omega=\theta_0} r^{\delta-2} v_\omega b, \tag{2.7.19}$$

$$-\epsilon \int\int u_{\omega R} b r^{\delta-1} = \epsilon \int\int r^{\delta-1} u_R b_\omega - \epsilon \int_{\omega=\theta_0} r^{\delta-1} u_R b$$
$$\leq \sqrt{\epsilon} \left(\int\int r^\delta u_R^2\right)^{1/2} \left(\int\int \epsilon b_\omega^2 r^{\delta-2}\right)^{1/2} - \epsilon \int_{\omega=\theta_0} u_R b r^{\delta-1}$$
$$\leq \sqrt{\epsilon} ||u||_A ||P||_{L^2_*,\delta} - \epsilon \int_{\omega=\theta_0} u_R b r^{\delta-1}. \tag{2.7.20}$$

The remaining terms can be estimated through Young's inequality:

$$\int\int -\sqrt{\epsilon} u_R a r^{\delta-1} + \epsilon a u r^{\delta-2} - 3\epsilon^{3/2} r^{\delta-2} v_\omega a + \epsilon v_{\omega R} a r^{\delta-1} + 2\epsilon^{3/2} b v_R r^{\delta-1} + 3\epsilon^{3/2} b u_\omega r^{\delta-2} + \epsilon^2 r^{\delta-2} b v$$
$$\leq C(\epsilon, \theta_0) ||P||_{L^2_{*,\delta}} \left(||u||_A + ||v||_B\right), \tag{2.7.21}$$

where $C(\theta_0, \epsilon) \to 0$ as $\epsilon, \theta_0 \to 0$. Summarizing the results from this step,

$$\int\int (\text{Equation 2.5.3}) \times a r^\delta + (\text{Equation 2.5.4}) \times b r^\delta \leq C(\theta_0, \epsilon) ||P||_{L^2_{*,\delta}} \left(||u||_A + ||v||_B\right)$$
$$- 2\epsilon \int_{\omega=\theta_0} r^{\delta-2} a u_\omega - \epsilon^2 \int_{\omega=\theta_0} r^{\delta-2} v_\omega b - \epsilon \int_{\omega=\theta_0} u_R b r^{\delta-1}, \tag{2.7.22}$$

where $C(\theta_0, \epsilon) \to 0$ as either argument $\to 0$.

Step III: Boundary Contributions

We now treat the boundary contributions from the previous two steps. From (2.7.13) and (2.7.22), all of the boundary terms are:

$$-\epsilon\int_{\omega=\theta_0} u_R b r^{\delta-1} - \epsilon^2\int_{\omega=\theta_0} r^{\delta-2} v_\omega b - 2\epsilon\int_{\omega=\theta_0} r^{\delta-2} a u_\omega + \int_{\omega=\theta_0} P a r^{\delta-1} = 0. \quad (2.7.23)$$

We have used the stress free boundary conditions from (2.2.30).

Step IV: Profile Terms

In this step, we give estimates on the terms from (2.2.23) - (2.2.24) which depend on the profiles u_s, v_s. Precisely, we successively treat:

$$\int\int\left[\frac{1}{r}u_s u_\omega + \frac{1}{r}u_{s\omega}u + u_{sR}v + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v\right] a r^\delta \quad (2.7.24)$$

$$+\int\int\left[\frac{1}{r}u_s v_\omega + \frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u\right]\epsilon b r^\delta. \quad (2.7.25)$$

Throughout the following estimates, we use the bounds on the profiles u_s, v_s that were proven in (2.5.19) - (2.5.26).

$$\int\int u_s u_\omega a r^{\delta-1} \le \theta_0 ||u_s||_\infty \left(\int\int u_\omega^2 r^\delta\right)^{1/2}\left(\int\int a_\omega^2 r^{\delta-2}\right)^{1/2} \lesssim \theta_0 ||v||_B ||P||_{L^2_*,\delta}, \quad (2.7.26)$$

$$\int\int u_{s\omega} u a r^{\delta-1} \le ||u_{s\omega}||_\infty \left(\int\int u^2 r^\delta\right)^{1/2}\left(\int\int r^{\delta-2}a^2\right)^{1/2}$$

$$\le \theta_0^2 \left(\int\int u_\omega^2 r^\delta\right)^{1/2}\left(\int\int r^{\delta-2}a_\omega^2\right)^{1/2} \le \theta_0^2 ||v||_B ||P||_{L^2_*,\delta}, \quad (2.7.27)$$

$$\int\int u_{sR} v a r^\delta \le \theta_0 \left(\sup\int u_{sR}^2 r^n (R-R_0)\right)^{1/2}\left(\int\int r^\delta v_R^2\right)^{1/2}\left(\int\int a_\omega^2 r^{\delta-2}\right)^{1/2}$$

$$\lesssim \theta_0 ||v||_B ||P||_{L^2_*,\delta}, \tag{2.7.28}$$

$$\int\int v_s u_R r^\delta a \le ||v_s r||_\infty \left(\int\int r^\delta u_R^2\right)^{1/2} \left(\int\int a^2 r^{\delta-2}\right)^{1/2} \lesssim \theta_0 ||u||_A ||P||_{L^2_*,\delta}, \tag{2.7.29}$$

$$\int\int \sqrt{\epsilon} v_s u a r^{\delta-1} \lesssim \sqrt{\epsilon}\left(\int\int u^2 r^\delta\right)^{1/2} \left(\int\int a^2 r^{\delta-2}\right)^{1/2} \lesssim \theta_0\sqrt{\epsilon}||v||_B ||P||_{L^2_*,\delta}, \tag{2.7.30}$$

$$\int\int \sqrt{\epsilon} u_s v a r^{\delta-1} \lesssim \left(\int\int v^2 r^\delta\right)^{1/2} \left(\int\int a^2 r^{\delta-2}\right)^{1/2} \lesssim \theta_0 ||v||_B ||P||_{L^2_*,\delta}, \tag{2.7.31}$$

$$\epsilon\int\int u_s v_\omega b r^{\delta-1} = ||u_s||_\infty \left(\int\int \epsilon v_\omega^2 r^\delta\right)^{1/2} \left(\int\int \epsilon b^2 r^{\delta-2}\right)^{1/2} \lesssim \theta_0 ||v||_B ||P||_{L^2_*,\delta}, \tag{2.7.32}$$

$$\epsilon\int\int v_{s\omega} u b r^{\delta-1} \lesssim \sqrt{\epsilon}\left(\int\int u^2 r^\delta\right)^{1/2} \left(\int\int \epsilon b^2 r^{\delta-2}\right)^{1/2} \lesssim \sqrt{\epsilon}\theta_0^2 ||v||_B ||P||_{L^2_*,\delta}, \tag{2.7.33}$$

$$\epsilon\int\int v_s v_R b r^\delta \le \sqrt{\epsilon}\left(\int\int r^{2+\delta} v_R^2\right)^{1/2} \left(\int\int \epsilon b^2 r^{\delta-2}\right)^{1/2} \lesssim \sqrt{\epsilon}\theta_0 ||v||_B ||P||_{L^2_*,\delta}, \tag{2.7.34}$$

$$\epsilon\int\int v_{sR} v b r^\delta \le \theta_0 ||v||_B ||P||_{L^2_*,\delta}, \tag{2.7.35}$$

$$2\int\int \sqrt{\epsilon} u_s u b r^{\delta-1} \lesssim \left(\int\int u^2 r^\delta\right)^{1/2} \left(\int\int \epsilon b^2 r^{\delta-2}\right)^{1/2} \le \theta_0^2 ||v||_B ||P||_{L^2_*,\delta}. \tag{2.7.36}$$

We summarize the results of this step:

$$\int\int (\text{Equation 2.5.15}) \times a r^\delta + (\text{Equation 2.5.16}) \times \epsilon b r^\delta \lesssim C(\theta_0,\epsilon)||P||_{L^2_{*,\delta}} \left(||u||_A + ||v||_B\right), \tag{2.7.37}$$

where $C(\theta_0,\epsilon) \to 0$ as either $\theta_0, \epsilon \to 0$.

Remark. The choice of weight on the multiplier $(ar^\delta, \epsilon b r^\delta)$ is delicate in the sense that it is "critical" in several of the profile estimates given above. Specifically, in calculation (2.7.26), $||u_s||_{L^\infty}$ is unable to absorb any factors of r, and so the weight after applying the multiplier (which in this case is $u_s u_\omega a r^{\delta-1}$) must exactly match those of $||v||_B$ and $||P||_{L^2_{*,\delta}}$.

Step V: Right-Hand-Side

Finally, we apply our multiplier $(ar^\delta, \epsilon b r^\delta)$ to the right hand side of the system (2.2.23) - (2.2.25), which immediately yields:

$$\begin{aligned} \int\int far^\delta + \epsilon g r^\delta b &\lesssim \int\int f^2 r^\delta + \epsilon g^2 r^\delta + \bar{\delta}\int\int r^\delta \epsilon b^2 + \bar{\delta}\int\int r^\delta a^2 \\ &\lesssim \int\int f^2 r^\delta + \int\int \epsilon g^2 r^\delta + \bar{\delta}||P||^2_{L^2_*,\delta}. \end{aligned} \tag{2.7.38}$$

Putting the above steps together yields the desired Pressure estimate in Theorem 2.2.14.

2.8 Linearized Existence and Uniqueness for Navier-Stokes Remainders

In this section, we prove Theorem 2.2.15. The full estimate for the linear problem, given in (2.2.48) is uniform in N, ϵ, and small θ_0. It was established in Corollary 2.3.8 that the spaces X and B endowed with their respective norms are Banach spaces, and so we can establish existence and uniqueness of the linear problem by applying Schaefer's Fixed Point Theorem. We must apply the fixed point theorem for each fixed N, obtaining a solution u^N, v^N. We can subsequently send $N \to \infty$ as estimate (2.2.48) is uniform in N.

For the following discussion, we fix an ϵ and an $N < \infty$. Denote the Prandtl-layer version of the Stokes operator as S_*, so $S_*[u,v,P] = (f,g) \iff [u,v,P]$ satisfy the linear system:

$$u - u_{RR} - \frac{\sqrt{\epsilon}}{r}u_r - \frac{\epsilon}{r^2}u_{\omega\omega} + \frac{\epsilon}{r^2}u - \frac{2}{r^2}\epsilon^{3/2}v_\omega + \frac{1}{r}P_\omega = \tilde{f}, \tag{2.8.1}$$

$$\frac{v}{\epsilon} - v_{RR} - \frac{\sqrt{\epsilon}}{r}v_R - \frac{\epsilon}{r^2}v_{\omega\omega} + \frac{2\sqrt{\epsilon}}{r^2}u_\omega + \frac{\epsilon}{r^2}v + \frac{1}{\epsilon}P_R = \tilde{g}, \tag{2.8.2}$$

$$u_\omega + \partial_R(rv) = 0. \tag{2.8.3}$$

Claim 9. *$S_*^{-1} : L^2(\Omega_N) \to L^2(\Omega_N)$ is compact.*

Proof. The usual Stokes solution operator, S^{-1}, is compact on bounded domains from $L^2 \to L^2$. Suppose we take a sequence $\{f_n, g_n\} \in L^2_{*,\delta}(\Omega_N)$ which is uniformly bounded in the norm. Then define $[\bar{f}, \bar{g}]_n(\omega, r) = [f_n, g_n](\omega, R(r))$, which are uniformly bounded in $L^2(\Omega_{N'})$ where $N' = R_0 + \sqrt{\epsilon}N$. This implies $[\bar{u}_n, \bar{v}_n] = S(f_n, g_n)$ has a subsequence which converges in $L^2(\Omega_{N'})$. Define $[u_n(\omega, R), v_n(\omega, R)] = [u(\omega, r), \frac{1}{\sqrt{\epsilon}} v(\omega, r)]$. u_n, v_n solve the Stokes$-*$ operator, so $[u_n, v_n] = S_*^{-1}(f_n, g_n)$. Moreover, by scaling, u_n, v_n also have a convergent subsequence. Therefore S_*^{-1} is compact on $L^2(\Omega_N)$.

□

We will need a version of Korn's Inequality using the polar coordinate basis, for which we adapt the proof given in (Cia10).

Claim 10 (Lions' Lemma)**.** *Let U be a bounded, open set with Lipschitz boundary. Suppose a distribution $u \in \mathcal{D}'(U)$ has $\nabla u = \left(\frac{\partial_\omega}{r} u, \partial_r u\right) \in H^{-1}(U)$. Then $u \in L^2(U)$.*

Proof. Expressing $\frac{u_\omega}{r} = \cos\omega u_y - \sin\omega u_x$ and $u_r = \cos\omega u_x + \sin\omega u_y$, and the obvious inverse relationships, we have that $(u_x, u_y) \in H^{-1} \iff \left(\frac{\partial_\omega}{r} u, \partial_r u\right) \in H^{-1}$. From here, the claim follows from Lions' Lemma, found in (Cia10, Thm 1.1). □

Definition 2.8.1. Let $e(\bar{u}, \bar{v})$ be the symmetric gradient:

$$e(\bar{u}, \bar{v}) = \begin{pmatrix} \frac{\bar{u}_\omega}{r} & \frac{1}{2}\left(\frac{\bar{v}_\omega}{r} + \bar{u}_r\right) \\ \frac{1}{2}\left(\frac{\bar{v}_\omega}{r} + \bar{u}_r\right) & \bar{v}_r \end{pmatrix}. \tag{2.8.4}$$

Denote the norm $||\bar{u}, \bar{v}||_{E_\kappa} = ||\bar{u}, \bar{v}||_{L^2} + ||e(\bar{u}, \bar{v})||_{L^2} + \kappa||\frac{\bar{u}_\omega}{r^2}, \frac{\bar{v}_\omega}{r^2}||_{L^2}$.

Claim 11 (Korn-type Inequality)**.** *For the solutions (u, v), we have the variant of Korn's inequality:*

$$||\bar{u}, \bar{v}||_{H^1(U)} \lesssim ||\bar{u}, \bar{v}||_{E_\kappa}, \tag{2.8.5}$$

where the constant depends on the domain, U, but is independent of small κ.

Remark. As noted in (Cia10), these Korn-type inequalities do not make any restrictions on the behavior of (u, v) on the boundary ∂U.

Proof. Using standard arguments, E_κ is a Banach space. We now show that $E_\kappa(U)$ coincides with $H^1(U)$. Clearly, $H^1(U) \subset E_\kappa(U)$ continuously: $||\bar{u}, \bar{v}||_{E_\kappa} \leq C||\bar{u}, \bar{v}||_{H^1(U)}$ where the constant is independent of small κ. The reverse direction is delicate and requires a use of Lions' Lemma. We suppose $||\bar{u}, \bar{v}||_{E_\kappa} < \infty$, so $e_{ij} \in L^2 \Rightarrow \nabla e_{ij} \in H^{-1}$. The components of $\nabla^2(\bar{u}, \bar{v})$ can be expressed in terms of ∇e_{ij} via:

$$\frac{\bar{u}_{\omega\omega}}{r^2}, \bar{v}_{rr}, \frac{\bar{v}_{\omega r}}{r}, \frac{\bar{u}_{\omega r}}{r} = \partial_r(\frac{\bar{u}_\omega}{r}) + \frac{\bar{u}_\omega}{r^2}, \frac{\bar{v}_{\omega\omega}}{r^2} = 2\frac{\partial_\omega}{r} e_{12}(\bar{u}, \bar{v}) - \frac{\bar{u}_{r\omega}}{r}, \bar{u}_{rr} = 2\partial_r(e_{12}(\bar{u}, \bar{v})) + \frac{\bar{v}_\omega}{r^2} - \frac{\bar{v}_{\omega r}}{r^2}$$

Thus, $\nabla^2(\bar{u}, \bar{v}) \in H^{-1}$, so by Lions' Lemma, $\nabla(\bar{u}, \bar{v}) \in L^2$, so $(\bar{u}, \bar{v}) \in H^1$. The identity map $i : H^1(\Omega) \hookrightarrow E_\kappa$ is continuous and bijective, with bound independent of small κ, and therefore by Banach's inverse mapping the inverse map $i^{-1} : E_\kappa \hookrightarrow H^1$ is also bounded. By observing that $||1||_{E_\kappa} = ||1||_{H^1} = ||1||_{L^2(U)}$, we see that the operator norm $||i^{-1}||_{op}$ independent of small κ.

□

Claim 12.

$$\begin{aligned} &2\left[\int\int \epsilon u^2 r + \epsilon\frac{u_\omega^2}{r} + u_R^2 r + \int\int \epsilon^2 v^2 r + \epsilon^2\frac{v_\omega^2}{r} + \epsilon v_R^2 r\right] \\ &\quad \lesssim 2\int\int(\epsilon u^2 + \epsilon^2 v^2)r + \int\int 2\epsilon\frac{u_\omega^2}{r} + 2\epsilon v_R^2 r + u_R^2 r + \epsilon^2\frac{v_\omega^2}{r} + \int\int 2\epsilon u_R v_\omega. \end{aligned} \tag{2.8.6}$$

Proof. Expanding the definition of e_{ij} in (2.8.5), multiplying by 2 yields, and absorbing the $\kappa||\frac{\bar{u}_\omega}{r^2}, \frac{\bar{v}_\omega}{r^2}||_{L^2}$ term to the left-hand-side gives:

$$2||\bar{u}, \bar{v}||_{H^1}^2 \lesssim 2\int\int(\bar{u}^2+\bar{v}^2) + \int\int 2\frac{\bar{u}_\omega^2}{r} + 2\bar{v}_r^2 r + \frac{\bar{v}_\omega^2}{r} + \bar{u}_r^2 r + 2\int\int \bar{u}_r\bar{v}_\omega. \tag{2.8.7}$$

Given our solutions (u, v), we define

$$\bar{u}(\omega, r) = u(\omega, R), \qquad \bar{v}(\omega, r) = \sqrt{\epsilon}v(\omega, R). \tag{2.8.8}$$

Scaling back to (ω, R) coordinates yields the desired result:

$$\begin{aligned} &2\left[\int\int \epsilon u^2 r + \epsilon\frac{u_\omega^2}{r} + u_R^2 r + \int\int \epsilon^2 v^2 r + \epsilon^2\frac{v_\omega^2}{r} + \epsilon v_R^2 r\right] \\ &\quad\lesssim 2\int\int(\epsilon u^2 + \epsilon^2 v^2)r + \int\int 2\epsilon\frac{u_\omega^2}{r} + 2\epsilon v_R^2 r + u_R^2 r + \epsilon^2\frac{v_\omega^2}{r} + \int\int 2\epsilon u_R v_\omega. \end{aligned} \tag{2.8.9}$$

$\square$

Proof of Theorem 2.2.15. Consider the following map $T(u, v) = (T^1(u, v), T^2(u, v))$,

$$\begin{aligned} T^1(u,v) &:= f - [\frac{1}{r}u_s u_\omega + \frac{1}{r}u_{s\omega}u + u_{sR}v + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v] + u, \\ T^2(u,v) &:= g - [\frac{1}{r}u_s v_\omega + \frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u] + \frac{v}{\epsilon}. \end{aligned} \tag{2.8.10}$$

T is a bounded, affine map from $H^1_* \to L^2_*$. Our solution is a fixed point of $S_*^{-1}T$, which is a compact map from $H^1_* \to H^1_*$. Let u^λ, v^λ denote solutions to:

$$(1-\lambda)u^\lambda - u^\lambda_{RR} - \frac{\sqrt{\epsilon}}{r}u^\lambda_R - \frac{\epsilon}{r^2}u^\lambda_{\omega\omega} + \frac{\epsilon}{r^2}u^\lambda - \frac{2}{r^2}\epsilon^{3/2}v^\lambda_\omega + \frac{P^\lambda_\omega}{r}$$

$$= \lambda f - \lambda[\frac{1}{r}u_s u_\omega^\lambda + \frac{1}{r}u_{s\omega}u^\lambda + u_{sR}v^\lambda + v_s u_R^\lambda + \frac{\sqrt{\epsilon}}{r}v_s u^\lambda + \frac{\sqrt{\epsilon}}{r}u_s v^\lambda], \quad (2.8.11)$$

$$(1-\lambda)\frac{v^\lambda}{\epsilon} - v_{RR}^\lambda - \frac{\sqrt{\epsilon}}{r}v_R^\lambda - \frac{\epsilon}{r^2}v_{\omega\omega}^\lambda + \frac{2\sqrt{\epsilon}}{r^2}u_\omega^\lambda + \frac{\epsilon}{r^2}v^\lambda + \frac{P_R^\lambda}{\epsilon}$$

$$= \lambda g - \lambda[\frac{1}{r}u_s v_\omega^\lambda + \frac{1}{r}v_{s\omega}u^\lambda + v_s v_R^\lambda + v_{sR}v^\lambda - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u^\lambda]. \quad (2.8.12)$$

We obtain bounds uniform in λ in the following way. Select some $0 < \lambda_0 << \epsilon < 1$. For $\lambda_0 \leq \lambda \leq 1$, the energy, positivity, and pressure estimates in the previous sections can be repeated to obtain uniform bounds (where the size will depend on λ_0). For $0 \leq \lambda < \lambda_0 << \epsilon$, we must only perform an energy estimate by applying the multiplier $(ru, \epsilon rv)$ to (2.8.11 - 2.8.12). For the upcoming calculation we drop the superscript on u^λ, v^λ.

$$\int\int \left(-u_{RR} - \frac{\sqrt{\epsilon}}{r}u_R - 2\epsilon\frac{u_{\omega\omega}}{r^2}\right) ru = \int\int u_R^2 r + 2\epsilon\frac{u_\omega^2}{r} - \int_{\omega=\theta_0} 2\epsilon\frac{uu_\omega}{r}, \quad (2.8.13)$$

$$\int\int (-2v_{RR} - \frac{\sqrt{\epsilon}}{r}v_R - \epsilon\frac{v_{\omega\omega}}{r^2})\epsilon vr = 2\epsilon\int\int v_R^2 r + \epsilon^2\frac{v_\omega^2}{r} - \epsilon^2\int_{\omega=\theta_0}\frac{vv_\omega}{r}, \quad (2.8.14)$$

$$(1-\lambda)\int\int (u^2+v^2)r + \int\int \frac{2\epsilon^2 v^2 + \epsilon u^2}{r}, \quad (2.8.15)$$

$$-3\epsilon^{3/2}\int\int \frac{v_\omega u}{r} \leq \sqrt{\epsilon}\left(\int\int \epsilon^2\frac{v_\omega^2}{r}\right)^{\frac{1}{2}}\left(\int\int \frac{u^2}{r}\right)^{\frac{1}{2}}, \quad (2.8.16)$$

$$3\int\int \epsilon^{3/2}\frac{u_\omega v}{r} - \epsilon^{3/2}vv_R \leq \sqrt{\epsilon}\left(\int\int v^2 r\right)^{\frac{1}{2}}\left[\left(\int\int \epsilon\frac{u_\omega^2}{r}\right)^{\frac{1}{2}} + \left(\int\int \epsilon\frac{v_R^2}{r}\right)^{\frac{1}{2}}\right], \quad (2.8.17)$$

$$\int\int -\epsilon v_{\omega R}u - \epsilon u_{\omega R}v = 2\epsilon\int\int u_R v_\omega - \int_{\omega=\theta_0}\epsilon u_R v, \quad (2.8.18)$$

$$\int\int P_\omega u + P_R rv = \int_{\omega=\theta_0} Pu. \quad (2.8.19)$$

The boundary contributions from (2.8.13, 2.8.14, 2.8.18, 2.8.19) cancel, using the same calculation as (2.5.11) and (2.5.29):

$$\int_{\omega=\theta_0} Pu - \int_{\omega=\theta_0} \epsilon u_R v - \int_{\omega=\theta_0} 2\epsilon \frac{u u_\omega}{r} - \epsilon^2 \int_{\omega=\theta_0} \frac{v v_\omega}{r} = 0. \tag{2.8.20}$$

Summarizing the interior terms, and applying (2.8.6)

$$\begin{aligned} &\int\int u_R^2 r + 2\epsilon\frac{u_\omega^2}{r} + 2\epsilon v_R^2 r + \epsilon^2\frac{v_\omega^2}{r} + 2\epsilon u_R v_\omega + (1-\lambda)(u^2+v^2)r + \frac{2\epsilon^2 v^2 + \epsilon u^2}{r} + J \\ \geq\ & 2\Bigg[\int\int \epsilon u^2 r + \epsilon\frac{u_\omega^2}{r} + u_R^2 r + \int\int \epsilon^2 v^2 r + \epsilon^2\frac{v_\omega^2}{r} + \epsilon v_R^2 r\Bigg] + \frac{(1-\lambda)}{2}\int\int (u^2+v^2)r + J, \end{aligned} \tag{2.8.21}$$

where

$$J = 3\epsilon^{3/2}\int\int\Big(\frac{u_\omega v}{r} - \frac{v_\omega u}{r}\Big) \leq \sqrt{\epsilon}\Big(\int\int \epsilon^2\frac{v_\omega^2}{r}\Big)^{\frac{1}{2}}\Big(\int\int \frac{u^2}{r}\Big)^{\frac{1}{2}} + \epsilon\Big(\int\int \epsilon\frac{u_\omega^2}{r}\Big)^{\frac{1}{2}}\Big(\int\int \frac{v^2}{r}\Big)^{\frac{1}{2}}. \tag{2.8.22}$$

On the right-hand-side, we have

$$\begin{aligned} &\int\int \Bigg[\lambda f - \lambda[\frac{1}{r}u_s u_\omega + \frac{1}{r}u_{s\omega}u + u_{sR}v + v_s u_R + \frac{\sqrt{\epsilon}}{r}v_s u + \frac{\sqrt{\epsilon}}{r}u_s v]\Bigg] ur \\ &\qquad \lesssim \lambda\int\int f^2 r + \lambda\int\int u^2 r + \frac{u_\omega^2}{r} + u_R^2 r + v^2 r, \end{aligned} \tag{2.8.23}$$

$$\begin{aligned} &\int\int \Bigg[\lambda g - \lambda[\frac{1}{r}u_s v_\omega + \frac{1}{r}v_{s\omega}u + v_s v_R + v_{sR}v - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s u]\Bigg] \epsilon v r \\ &\qquad \lesssim \lambda\int\int g^2 r + \lambda\int\int \frac{v_\omega^2}{r} + u^2 r + v_R^2 r + v^2 r, \end{aligned} \tag{2.8.24}$$

and therefore since $0 \leq \lambda \leq \lambda_0 << \epsilon$, we have

$$\int\int (u^2+v^2)r + \int\int \Big(|\nabla_\epsilon u|^2 + \epsilon|\nabla_\epsilon v|^2\Big)r \lesssim \int\int \Big(f^2+\epsilon g^2\Big)r, \tag{2.8.25}$$

uniformly in λ. An application of Schaefer's fixed point theorem then shows there exists a solution u^N, v^N in the function space with the weak norm which we estimated (weak in the sense of weights and in ϵ), and by linearity of our equation, applying the estimate above implies uniqueness of this solution. With existence and uniqueness in the weak space in hand, we can bootstrap: each (u^N, v^N) is also an element of the space $X(\Omega_N), B(\Omega_N)$ because Ω_N is a bounded domain. Because this solution is an element of X, B, we can apply the strong estimate [2.2.48] which is uniform in ϵ, N, and small θ_0. We can therefore send $N \to \infty$ to obtain a global in R solution which also obeys estimate (2.2.48). This argument results in the proof of Theorem 2.2.15.

□

2.9 High Regularity Estimates

In this section we prove Lemma 2.2.16, the high regularity estimate for the solution u, v to the problem (2.2.49 - 2.2.52). Notationally, we will keep the $-M$, where $M > 0$, as a large negative exponent for ϵ, not taking care to rename different exponents as it is inconsequential to the estimate we are proving.

Proof of Lemma 2.2.16. We rescale via:

$$\bar{u}(\omega, r) = u(\theta_0\omega, \theta_0 R(r)); \bar{v}(\omega, r) = \sqrt{\epsilon} v(\theta_0\omega, \theta_0 R(r)); \bar{P}(\omega, r) = \frac{\theta_0}{\epsilon} P(\theta_0\omega, \theta_0 R(r)). \tag{2.9.1}$$

The equation satisfied by the normalized profiles is:

$$-\bar{u}_{rr} - \theta_0 \frac{\bar{u}}{r} - \frac{\bar{u}_{\omega\omega}}{r^2} + \theta_0^2 \frac{\bar{u}}{r^2} - \frac{2}{r^2}\theta_0 \bar{v}_\omega + \frac{P_\omega}{r} = \frac{\theta_0^2}{\epsilon}\tilde{f} := \bar{f}(\omega, r), \tag{2.9.2}$$

$$-\bar{v}_{rr} - \theta_0 \frac{\bar{v}}{r} - \frac{v_{\omega\omega}}{r^2} + \frac{2}{r^2}\theta_0 \bar{u}_\omega + \frac{\theta_0^2}{r^2}\bar{v} + \bar{P}_r = \frac{\theta_0^2}{\sqrt{\epsilon}}\tilde{g} := \bar{g}(\omega, r). \tag{2.9.3}$$

$\dot{H}_*^2$ Estimates

Using the standard regularity theory for Stokes equation, we can obtain $\dot{H}^2$ estimates for $\bar{u}$ and $\bar{v}$ away from the corners of Ω. Let $\chi(\omega, r)$ denote a cutoff function which is supported near the corners of the domain in such a way that $(\omega, r) \in \text{supp}(\chi) \Rightarrow r - R_0 \leq 1$. Define $\chi_1(\omega, r) = 1 - \chi(\omega, \frac{r-R_0}{\sqrt{\epsilon}})$. Then the equation for $\left(\bar{u}_1, \bar{v}_1, \bar{P}_1\right) := \chi_1 \cdot \left(\bar{u}, \bar{v}, \bar{P}\right)$ is given by:

$$\begin{aligned} -\Delta \bar{u}_1 - \frac{\bar{u}_1}{r^2} - \frac{2}{r^2}\bar{v}_{1\omega} + \frac{\bar{P}_{1\omega}}{r} &= \chi_1 \bar{f} - 2\nabla\chi_1 \cdot \nabla \bar{u} - \Delta\chi_1 \bar{u} - \frac{2}{r^2}\chi_{1\omega}\bar{v} + \frac{1}{r^2}\chi_{1\omega}\bar{P} \\ &\quad - (1-\theta_0^2)\frac{\bar{u}_1}{r^2} - \frac{2-2\theta_0}{r^2}\bar{v}_{1\omega}, \end{aligned} \tag{2.9.4}$$

$$\begin{aligned} -\Delta \bar{v}_1 + \frac{2}{r^2}\bar{u}_{1\omega} + \frac{1}{r^2}\bar{v}_1 + \bar{P}_{1r} &= \chi_1 \bar{g} - 2\nabla\chi_1 \cdot \nabla \bar{v} - \Delta\chi_1 \bar{v} + \partial_r(\chi_1)\bar{P} + \frac{2}{r^2}\theta_0 \partial_\omega(\chi_1)\bar{u}_1 \\ &\quad + (1-\theta_0^2)\frac{\bar{v}_1}{r^2} + (\frac{2-2\theta_0}{r^2})\bar{u}_{1\omega}. \end{aligned} \tag{2.9.5}$$

As $(\omega, r) \in \text{supp}\,\nabla\chi_1 \cup \text{supp}\,\Delta\chi_1$, we have $r - R_0 \leq \sqrt{\epsilon}$, and so by the standard Stokes estimate (with inhomogeneous divergence, see Remark 2.9):

$$||2\nabla\chi_1 \cdot \nabla\bar{u} - \Delta\chi_1\bar{u} - \frac{2}{r^2}\chi_{1\omega}\bar{v} + \frac{1}{r^2}\chi_{1\omega}\bar{P} - (1-\theta_0^2)\frac{\bar{u}_1}{r^2} - \frac{2-2\theta_0}{r^2}\bar{v}_{1\omega}||_{L^2} \lesssim \epsilon^{-1}||\bar{f}, \bar{g}||_{L^2}, \tag{2.9.6}$$

$$||-2\nabla\chi_1 \cdot \nabla\bar{v} - \Delta\chi_1\bar{v} + \partial_r(\chi_1)\bar{P} + \frac{2}{r^2}\theta_0\partial_\omega(\chi_1)\bar{u}_1 + (1-\theta_0^2)\frac{\bar{v}_1}{r^2} + (\frac{2-2\theta_0}{r^2})\bar{u}_{1\omega}||_{L^2} \lesssim \epsilon^{-1}||\bar{f}, \bar{g}||_{L^2}. \tag{2.9.7}$$

The ϵ^{-1} in estimate (2.9.6 - 2.9.7) arises from $\Delta\chi_1 = \Delta\left(\chi(\omega, \frac{r-R_0}{\sqrt{\epsilon}})\right)$. Thus using the standard $\dot{H}^2$ Stokes estimate:

$$||\bar{u}_1||_{\dot{H}^2} + ||\bar{v}_1||_{\dot{H}^2} + ||\bar{P}_1||_{\dot{H}^1} \lesssim \epsilon^{-1}\left(||\bar{f}||_{L^2} + ||\bar{g}||_{L^2}\right) \lesssim \epsilon^{-M}\left(||\tilde{f}||_{L^2_*} + \sqrt{\epsilon}||\tilde{g}||_{L^2_*}\right), \quad (2.9.8)$$

for some potentially large power M. We have used the calculation, according to the definition of $\bar{f}$ in (2.9.2)

$$||\bar{f}||^2_{L^2} = \int\int \bar{f}^2 r dr d\omega = \sqrt{\epsilon}\int\int \bar{f}^2 r dR d\omega = \theta_0^2\epsilon^{-3/2}\int\int \tilde{f}^2 r dR d\omega = \theta_0^2\epsilon^{-3/2}||\tilde{f}||^2_{L^2_*}, \quad (2.9.9)$$

and analogously for g. Defining the corresponding profiles in Prandtl variables by inverting (2.9.1) above: $u_1(\theta_0\omega, \theta_0 R) = \bar{u}_1(\omega, r), v_1(\theta_0\omega, \theta_0 R) = \frac{1}{\sqrt{\epsilon}}\bar{v}_1(\omega, r), P_1(\theta_0\omega, \theta_0 R) = \frac{\epsilon}{\theta_0}\bar{P}_1(\omega, r)$, we have:

$$||u_1, v_1||_{\dot{H}^2_*} \leq \epsilon^{-M}||\bar{u}_1, \bar{v}_1||_{\dot{H}^2} \lesssim \epsilon^{-M}||\tilde{f}, \sqrt{\epsilon}\tilde{g}||_{L^2_*}. \quad (2.9.10)$$

Since M can be arbitrarily large, the quantity appearing on the left of (2.9.10) above is independent of ϵ. We can also obtain weighted estimates by repeating the above analysis for the equation for $\bar{u}_1 r^{\frac{\delta}{2}+\frac{1}{2}}, \bar{v}_1 r^{\frac{\delta}{2}+\frac{1}{2}}$:

$$||r^{1/2+\delta/2}\left(u_1, v_1\right)||_{\dot{H}^2_*} \lesssim \epsilon^{-M}||r^{1/2+\delta/2}\left(\tilde{f}, \sqrt{\epsilon}\tilde{g}\right)||_{L^2_*}. \quad (2.9.11)$$

For the weighted estimate (2.9.11), we use that $r \leq C$ on $\operatorname{supp}\partial^k\chi_1, k \geq 1$. This establishes the desired estimate for u_1, v_1.

Remark. We apply the Stokes estimate with inhomogeneous divergence because the weighted, cutoff vector field $(\bar{u}_1 r^{\frac{\delta}{2}+\frac{1}{2}}, \bar{v}_1 r^{\frac{\delta}{2}+\frac{1}{2}})$ has divergence:

$$\frac{\partial_\omega}{r}(r^{\frac{1}{2}+\frac{\delta}{2}}\bar{u}_1) + \frac{r^{\frac{1}{2}+\frac{\delta}{2}}}{r}\bar{v}_1 + r^{\frac{\delta}{2}+\frac{1}{2}}\bar{v}_{1r} + r^{\frac{\delta}{2}-\frac{1}{2}}\bar{v}_1 = r^{\frac{1}{2}+\frac{\delta}{2}}\left(\chi_{1\omega}\frac{u}{r} + \chi_{1r}v\right) + r^{\frac{\delta}{2}-\frac{1}{2}}\bar{v}_1. \quad (2.9.12)$$

Therefore, a term $\epsilon^{-M}||u,v||_{L^2_{*,\delta}}$ appears on the right-hand side of the estimate (2.9.8), which is in turn controlled by $\epsilon^{-M}||\tilde{f},\sqrt{\epsilon}\tilde{g}||_{L^2_{*,2+\delta}}$ by our uniform energy estimates.

$H^{3/2}$ **Estimates:**

Let $\chi_2 = \chi(\omega, \frac{r-R_0}{\sqrt{\epsilon}})$, so

$$(\omega, r) \in \mathrm{supp}(\chi_2) \Rightarrow r - R_0 \leq \sqrt{\epsilon} \text{ and } R - R_0 \leq 1. \tag{2.9.13}$$

Define $(\bar{u}_2, \bar{v}_2, \bar{P}_2) = \chi_2 \cdot (u, v, P)$. By (OS95), the Stokes problem has an $H^{3/2}$ estimate:

$$||\bar{u}_2, \bar{v}_2||_{H^{3/2}} + ||\bar{P}_2||_{H^{1/2}} \lesssim \epsilon^{-M}\left(||\tilde{f}||_{L^2_*} + \sqrt{\epsilon}||\tilde{g}||_{L^2_*}\right). \tag{2.9.14}$$

We must relate the $H^{3/2}$ norm of $\bar{u}_2, \bar{v}_2$ to that of its scaled counterpart $[u_2, v_2](\omega, R)$, given again by inverting transformation in equation (2.9.1).

Claim 13. *Define the transformation $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ to be defined in polar coordinates via $\Psi(\omega, R) = (\omega, r)$. Denote by $u(\omega, R) = \bar{u}(\omega, r) = \bar{u} \circ \Psi$. Then $||u||_{H^{3/2}} = ||\bar{u} \circ \Psi||_{H^{3/2}} \lesssim ||\bar{u}||_{H^{3/2}}$. Here the constant depends on the derivatives of Ψ (which in turn depend on ϵ).*

Proof. The transformation $\Psi : (\omega, R) \to (\omega, r)$ is bijective and has derivatives which are bounded above and below as a map from $\Omega \subset \mathbb{R}^2 \to \Omega \subset \mathbb{R}^2$. Indeed, in the polar coordinate basis:

$$\nabla\Psi(\omega, R) = \begin{pmatrix} \frac{\Psi^1_\omega}{R} & \Psi^1_R \\ \frac{\Psi^2_\omega}{R} & \Psi^2_R \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\epsilon} \end{pmatrix}.$$

From here it is easy to see that Ψ is Bilipschitz, and $|\det\Psi|, |\det\Psi|^{-1}$ are bounded above and below, keeping in mind that the coordinate basis are functions of ω.

We may decompose $||u||_{H^{3/2}} = ||u||_{H^1} + ||u||_{H^{1/2}}$. Using the definition of weak derivative and an approximation argument, it is easy to see the usual chain rule holds, namely $\nabla u(x) = \nabla \bar{u}^T(\Psi(x))D\Psi(x)$. Moreover, since the derivatives of Ψ are bounded above and below, we have by the change of variables formula: $||u||_{H^1} = ||\bar{u} \circ \Psi||_{H^1} \lesssim ||\bar{u}||_{H^1}$.

We must now treat the $H^{1/2}$ portion. As shown in (NPV12), the $H^{1/2}$ norm of any function u is equivalent to the Gagliardo semi norm, which is defined as follows (where we calculate $n + sp = 2 + \frac{1}{2}2 = 3$):

$$[u]^2 = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^3} dxdy.$$

Therefore,

$$\begin{aligned}
[u]^2 = [\bar{u} \circ \Psi]^2 &= \int_\Omega \int_\Omega \frac{|\bar{u}(\Psi(x)) - \bar{u}(\Psi(y))|^2}{|x-y|^3} dxdy \\
&= \int_\Omega \int_\Omega \frac{|\bar{u}(x') - \bar{u}(y')|^2}{|\Psi^{-1}(x') - \Psi^{-1}(y')|^3} |\det D\Psi|^{-1} dx'dy' \\
&\leq ||\det D\Psi||_\infty \int_\Omega \int_\Omega \frac{|\bar{u}(x') - \bar{u}(y')|^2}{|\Psi^{-1}(x') - \Psi^{-1}(y')|^3} dx'dy' \\
&\lesssim \int_\Omega \int_\Omega \frac{|\bar{u}(x') - \bar{u}(y')|^2}{|x'-y'|^3} dx'dy' = [\bar{u}]^2.
\end{aligned}$$

We have used that $|\det D\Psi|$ is bounded above and below and that Ψ is Bilipschitz.

$\square$

By using Claim 13, we have:

$$||u_2, v_2||_{H^{3/2}} \lesssim \epsilon^{-M} ||\bar{u}_2, \bar{v}_2||_{H^{3/2}} \lesssim \epsilon^{-M} \left(||\tilde{f}, \sqrt{\epsilon}\tilde{g}||_{L^2_*} \right).$$

We can arbitrarily weight these norms due to (2.9.13). This concludes the proof of Lemma 2.2.16.

□

We are now able to control the high-regularity quantities appearing in $||\cdot||_Z$ given in equation (2.2.20):

Proof of Theorem 2.2.17. We first address the u_R term in term (2.2.20). Using interpolated Holder and the uniform energy estimates from estimate (2.2.48):

$$||u_R||_{L^{2q}_{*,q+\alpha}} \le ||u_R||^{\theta}_{L^2_{*,1+\frac{\alpha}{q}}} ||u_R||^{1-\theta}_{L^4_{*,2+\frac{2\alpha}{q}}} \lesssim ||\tilde{f}, \sqrt{\epsilon}\tilde{g}||^{\theta}_{L^2_{*,2+\delta}} ||u_R||^{1-\theta}_{L^4_{*,2+\frac{2\alpha}{q}}}. \tag{2.9.15}$$

We have used that $\alpha/q \le \delta/2$ in order to apply the uniform energy estimates. Here $\theta = \theta(\delta')$, where $\theta \to 1$ as $\delta' \to 0$. To estimate the L^4_* term (2.9.15), we decompose $u = u_1 + u_2$ as in Lemma 2.2.16:

$$||u_R||_{L^4_{*,2+\frac{2\alpha}{q}}} \le ||u_{1R}||_{L^4_{*,2+\frac{2\alpha}{q}}} + ||u_{2R}||_{L^4_{*,2+\frac{2\alpha}{q}}} \lesssim ||u_{1R}||_{L^4_{*,2+\frac{2\alpha}{q}}} + ||u_{2R}||_{L^4} \tag{2.9.16}$$

$$\le ||u_{1R}||_{L^4_{*,2+\frac{2\alpha}{q}}} + ||u_2||_{H^{3/2}} \le \epsilon^{-M} ||\tilde{f}, \sqrt{\epsilon} g||_{L^2_{*,2+\delta}}. \tag{2.9.17}$$

In the second inequality of (2.9.16), we have used that R is order 1 on the support of u_2 and so L^4 is equivalent to L^4_*. In the first inequality in (2.9.17), we have used the $H^{1/2} \hookrightarrow L^4$ embedding in $\mathbb{R}^2$. In the second inequality in (2.9.17), we used Lemma 2.2.16 to estimate $||u_2||_{H^{3/2}}$. By observing $u_1|_{r=R_0} = u_{1\omega}|_{r=R_0} = 0$, and $u_1|_{\omega=0} = u_{1R}|_{\omega=0} = 0$, we use estimate (2.3.13) for u_{1R} to yield:

$$||u_{1R}||_{L^4_{*,2+\frac{2\alpha}{q}}} \lesssim ||\chi_1 u||^{\frac{1}{2}}_{X} ||\chi_1 u||^{\frac{1}{2}}_{\dot{H}^2_{*,2+\delta}} \lesssim \epsilon^{-M} ||\tilde{f}, \sqrt{\epsilon} g||_{L^2_{*,2+\delta}}. \tag{2.9.18}$$

Inserting (2.9.17) into (2.9.15) and multiplying by $\epsilon^{\frac{\gamma}{4}}$:

$$\epsilon^{\frac{\gamma}{4}}||u_R||_{L^{2q}_{*,q+\alpha}} \leq \epsilon^{\frac{\gamma}{4}-M(1-\theta)}||\tilde{f},\sqrt{\epsilon}\tilde{g}||_{L^2_{*,2+\delta}}. \tag{2.9.19}$$

We can take δ' small enough such that $\frac{\gamma}{4} - M(1-\theta) > 0$. The u_ω and v_R terms in (2.2.20) are treated in an identical manner, after observing the relevant boundary conditions are respected by the cutoff quantity: $v_{1R}|_{r=R_0} = v_1|_{r=R_0} = 0$, $v_1|_{\omega=0} = v_{1R}|_{\omega=0} = 0$, $u_1|_{r=R_0} = u_{1\omega}|_{r=R_0} = 0$, and $u_1|_{\omega=0} = u_{1\omega}|_{\omega=0} = 0$. We now treat v_ω, which is slightly different from the previous terms. Recall from Theorem 2.2.17 that the parameter $\beta > 0$. Through interpolated Holder, we have:

$$||\sqrt{\epsilon}v_\omega||_{L^{2q}_{*,-\beta}} \leq ||\sqrt{\epsilon}v_\omega||^{\theta}_{L^2_{*,-\frac{\beta}{q}}}||\sqrt{\epsilon}v_\omega||^{1-\theta}_{L^4_{*,-\frac{2\beta}{q}}} \leq ||\tilde{f},\sqrt{\epsilon}\tilde{g}||^{\theta}_{L^2_{*,2+\delta}}||\sqrt{\epsilon}v_\omega||^{1-\theta}_{L^4_{*,-\frac{2\beta}{q}}}. \tag{2.9.20}$$

Again we decompose $v = v_1 + v_2$ where v_2 is supported near the corner of the domain as in Lemma 2.2.16, and so:

$$||\sqrt{\epsilon}v_\omega||_{L^4_{*,-\frac{2\beta}{q}}} \leq ||\sqrt{\epsilon}v_{1\omega}||_{L^4_{*,-\frac{2\beta}{q}}} + ||\sqrt{\epsilon}v_{2\omega}||_{L^4_{*,-\frac{2\beta}{q}}} \lesssim ||\sqrt{\epsilon}v_{1\omega}||_{L^4_{*,-\frac{2\beta}{q}}} + \epsilon^{-M}||\tilde{f},\sqrt{\epsilon}\tilde{g}||_{L^2_{*,2+\delta}}. \tag{2.9.21}$$

We cannot immediately apply Lemma 2.3.14 to the $||\sqrt{\epsilon}v_{1\omega}||_{L^4_{*,-\frac{2\beta}{q}}}$ term because $v_1 = \chi_1(\omega, R)v$ does not necessarily satisfy the stress-free boundary condition at $\{\omega = \theta_0\}$. However, $v_{1\omega}(\theta_0, R) = \chi_1(\theta_0, R)v_\omega(\theta_0, R) + \chi_{1\omega}(\theta_0, R)v(\theta_0, R)$, and so a trivial modification of the proof of Lemma 2.3.14 yields the required result.

$\square$

2.10 Nonlinear Existence and Uniqueness for Navier-Stokes Remainders

We now apply contraction mapping on the space Z. For this section, call L the linear operator in the linearized problem appearing in equation (2.2.23).

Theorem 2.10.1. *Suppose $L\bar{u}, \bar{v} = f(u,v), g(u,v)$. Select the $\delta' > 0$ guaranteed by Theorem [2.2.17], and let $p = \frac{1+\delta'}{\delta'}$, the Holder conjugate to $q = 1+\delta'$. Then for $\delta \in (0,1)$ sufficiently close to 1, and for $2\gamma + \kappa < \frac{1}{2}$, we have*

$$||\bar{u}, \bar{v}||_Z \leq C(u_s, v_s)\Big[1 + \epsilon^{\frac{1}{2}-\frac{\gamma}{2}-\kappa}||u,v||_Z + \epsilon^{\frac{\gamma}{2}}||u,v||_Z^2\Big]. \tag{2.10.1}$$

Thus, the solution operator to the nonlinear problem (2.2.23) - (2.2.30) maps the ball of radius $2C(u_s, v_s)$ in Z to itself.

Proof. We apply Theorem 2.2.17, after choosing the parameter $\beta = \frac{q}{2p}$ and subsequently δ in the interval $1 - \frac{1}{2p} \leq \delta < 1$, which yields $||\bar{u}, \bar{v}||_Z \lesssim ||\tilde{f}, \sqrt{\epsilon}\tilde{g}||_{L^2_{*,2+\delta}}$. $\tilde{f}, \tilde{g}$ are given by:

$$\tilde{f} = f - \epsilon^{\frac{\gamma}{4}}\left(\frac{1}{r}u_s\bar{u}_\omega + \frac{1}{r}u_{s\omega}\bar{u} + u_{sR}\bar{v} + v_s\bar{u}_R + \frac{\sqrt{\epsilon}}{r}v_s\bar{u} + \frac{\sqrt{\epsilon}}{r}u_s\bar{v}\right), \tag{2.10.2}$$

$$\tilde{g} = g - \epsilon^{\frac{\gamma}{4}}\left(\frac{1}{r}u_s\bar{v}_\omega + \frac{1}{r}v_{s\omega}\bar{u} + v_s\bar{v}_R + v_{sR}\bar{v} - \frac{2}{r}\frac{1}{\sqrt{\epsilon}}u_s\bar{u}\right), \tag{2.10.3}$$

where f, g are defined in (2.2.26) - (2.2.27). f, g are used to estimate the $||u||_X, ||v||_B$ components of $||u,v||_Z$ according to the linear estimate (2.2.48), and the profile terms in (2.10.2) - (2.10.3) are required to estimate the high-regularity components of $||u,v||_Z$, according to Theorem 2.2.17. The factor of $\epsilon^{\frac{\gamma}{4}}$ accompanies these profile terms because $\epsilon^{\frac{\gamma}{2}}$ was used in the definition of the norm $||\cdot||_Z$, while only a factor of $\epsilon^{\frac{\gamma}{4}}$ was required in Theorem 2.2.17. We now proceed to estimate $||\tilde{f}, \sqrt{\epsilon}\tilde{g}||_{L^2_{*,2+\delta}}$ in terms of $||u,v||_Z$.

From Theorem 2.2.10, we have:

$$\epsilon^{-2\gamma-1}\left(\int\int R_u^2 r^{2+\delta} dR d\omega + \epsilon \int\int R_v^2 r^{2+\delta} dR d\omega\right) \leq \epsilon^{-2\gamma-1}\epsilon^{3/2-\kappa} = \epsilon^{\frac{1}{2}-2\gamma-\kappa}. \quad (2.10.4)$$

Here we use that $2\gamma+\kappa<\frac{1}{2}$. Next we have $\sqrt{\epsilon}R^{u,p}$, as defined in (2.2.28):

$$\epsilon\int\int r^\delta (u_p^1)^2 u_\omega^2 \leq ||u_p^1||_{L^\infty}^2 \epsilon \int\int u_\omega^2 r^\delta \leq C(u_p^1)\epsilon^{1-\kappa}|||u,v|||_Z^2, \quad (2.10.5)$$

$$\epsilon\int\int r^\delta (u_{p\omega}^1)^2 u^2 \lesssim \epsilon\left(\sup\int (u_{p\omega}^1)^2 (R-R_0)\right)\int\int u_R^2 \leq \epsilon^{1-\kappa}|||u,v|||_Z^2, \quad (2.10.6)$$

$$\epsilon\int\int r^{2+\delta}(u_{pR}^1)^2 v^2 \lesssim \left(\sup\int (R-R_0)(u_{pR}^1)^2\right)\int\int \epsilon v_R^2 \leq \epsilon^{1-\kappa}|||u,v|||_Z^2, \quad (2.10.7)$$

$$\epsilon\int\int r^{2+\delta}(v_p^1)^2 u_R^2 \lesssim \epsilon||v_p^1||_\infty^2 \int\int u_R^2 \leq \epsilon^{1-\kappa}|||u,v|||_Z^2, \quad (2.10.8)$$

$$\epsilon^2\int\int r^\delta \left(v_p^1\right)^2 u^2 \lesssim \epsilon^2 ||v_p^1||_{L^\infty}^2 \int\int u^2 \leq \epsilon^{2-\kappa}|||u,v|||_Z^2, \quad (2.10.9)$$

$$\epsilon^2\int\int r^\delta \left(u_p^1\right)^2 v^2 \lesssim \epsilon||u_p^1||_\infty^2 \int\int \epsilon v^2 \leq \epsilon^{1-\kappa}|||u,v|||_Z^2. \quad (2.10.10)$$

We now estimate $\sqrt{\epsilon}R^{v,p}$, as defined in (2.2.29):

$$\epsilon\int\int r^\delta (u_p^1)^2 v_\omega^2 \lesssim ||u_p^1||_\infty^2 \int\int \epsilon v_\omega^2 \leq \epsilon^{-\kappa}|||u,v|||_Z^2, \quad (2.10.11)$$

$$\epsilon\int\int r^\delta (v_{p\omega}^1)^2 u^2 \leq \epsilon||u||_{L^\infty}^2 ||v_{p\omega}^1||_{L^2}^2 \leq \epsilon^{-\gamma-\kappa}|||u,v|||_Z^2, \quad (2.10.12)$$

$$\epsilon\int\int r^{2+\delta}(v_p^1)^2 v_R^2 \lesssim \epsilon||v_p^1||_\infty^2 \int\int v_R^2 \leq \epsilon^{1-\kappa}|||u,v|||_Z^2, \quad (2.10.13)$$

$$\int\int r^\delta (u_p^1)^2 u^2 \lesssim ||u_p^1||_\infty^2 \int\int u^2 \leq \epsilon^{-\kappa}|||u,v|||_Z^2, \quad (2.10.14)$$

$$\epsilon\int\int r^{2+\delta}(v_{pR}^1)^2 v^2 \lesssim \epsilon\left(\sup\int (R-R_0)(v_{pR}^1)^2\right)\int\int v_R^2 \leq \epsilon^{-\kappa}|||u,v|||_Z^2. \quad (2.10.15)$$

For (2.10.12) we use that $||v_{p\omega}^1||_{L^2}^2 \leq \epsilon^{-\frac{1}{2}-\kappa}$ and $||u||_{L^\infty}^2 \lesssim \epsilon^{-\gamma-\frac{1}{2}}||u||_Z^2$, according to

(2.3.10). For (2.10.15) we have used the bound:

$$\epsilon \sup \int (R - R_0)(v^1_{pR})^2 \le \sqrt{\epsilon} \sup \int \left(v^1_{pR}\right)^2 \le \frac{\sqrt{\epsilon}}{\theta_0} \left(\int\int (v^1_{pR})^2 + \int\int (v^1_{pR\omega})^2 \right) \le \frac{\epsilon^{-\kappa}}{\theta_0}.$$

Summarizing the linear components of f, g,

$$\epsilon^{-\gamma-\frac{1}{2}} ||R_u, \sqrt{\epsilon} R_v||_{L^2_{*,2+\delta}} + \sqrt{\epsilon} ||R^{u,p}, \sqrt{\epsilon} R^{v,p}||_{L^2_{*,2+\delta}} \lesssim C(u_s, v_s) + \epsilon^{\frac{1}{2} - \frac{\gamma}{2} - \kappa} ||u, v||_Z. \quad (2.10.16)$$

For the nonlinear terms we recall the definition of $|| \cdot ||_Z$ in (2.2.20), where $q = 1 + \delta'$, $p = \frac{q}{q-1}$, and $0 < \frac{q}{p} \le \alpha \le \frac{q\delta}{2}$. We also recall the low regularity embeddings in Lemmas 2.3.9 and 2.3.10.

$$\epsilon^{2\gamma+1} \int\int r^\delta u^2 u_\omega^2 \le \epsilon^{2\gamma+1} \left(\int\int u_\omega^{2q} \right)^{\frac{1}{q}} \left(\int\int u^{2p} r^{\delta p} \right)^{\frac{1}{p}} \lesssim \epsilon^{\gamma+1} ||u||_Z^4, \quad (2.10.17)$$

$$\epsilon^{2\gamma+1} \int\int r^{2+\delta} v^2 u_R^2 \le \epsilon^{2\gamma+1} \left(\int\int v^{2p} r^{p-1} r^{\delta p} \right)^{\frac{1}{p}} \left(\int\int u_R^{2q} r^{q+\alpha} \right)^{\frac{1}{q}} \lesssim \epsilon^{\gamma} ||u, v||_Z^4. \quad (2.10.18)$$

Here the inequality holds because we have selected $\delta' > 0$ so that $\frac{\alpha}{q} \ge \frac{1}{p}$.

$$\epsilon^{2\gamma+2} \int\int r^\delta u^2 v^2 \le \epsilon^{2\gamma+2} \left(\int\int r^{2\delta} v^4 \right)^{\frac{1}{2}} \left(\int\int u^4 \right)^{\frac{1}{2}} \le \epsilon^{2\gamma+1} ||u, v||_Z^4. \quad (2.10.19)$$

Nonlinear terms in g:

$$\epsilon^{2\gamma+1} \int\int r^\delta u^2 v_\omega^2 \le \epsilon^{2\gamma+1} \left(\int\int u^{2p} r^{\delta p} r^{\frac{1}{2}} \right)^{\frac{1}{p}} \left(\int\int r^{-\frac{q}{2p}} v_\omega^{2q} \right)^{\frac{1}{q}} \le \epsilon^{\gamma} ||u, v||_Z^4, \quad (2.10.20)$$

$$\epsilon^{2\gamma+1}\int\int r^{2+\delta}v^2 v_R^2 \le \epsilon^{2\gamma+1}\left(\int\int v^{2p}r^{p-1}r^{\delta p}\right)^{\frac{1}{p}}\left(\int\int v_R^{2q}r^{q+\alpha}\right)^{\frac{1}{q}} \lesssim \epsilon^{\gamma}||u,v||_Z^4, \tag{2.10.21}$$

$$\epsilon^{2\gamma}\int\int u^4 r^\delta \lesssim \epsilon^{2\gamma}||u||_Z^4. \tag{2.10.22}$$

Combined with (2.10.16), we now have:

$$||f,\sqrt{\epsilon}g||_{L^2_{*,2+\delta}} \lesssim C(u_s,v_s) + \epsilon^{\frac{1}{2}-\frac{\gamma}{2}-\kappa}||u,v||_Z + \epsilon^{\frac{\gamma}{2}}||u,v||_Z^2. \tag{2.10.23}$$

We now provide estimates for the profile terms in (2.10.2) - (2.10.3).

$$\epsilon^{\frac{\gamma}{2}}\int\int r^\delta u_s \bar{u}_\omega^2 \le ||u_s||_\infty \epsilon^{\frac{\gamma}{2}}\int\int \bar{u}_\omega^2 r^\delta, \tag{2.10.24}$$

$$\epsilon^{\frac{\gamma}{2}}\int\int u_{s\omega}^2 \bar{u}^2 r^\delta \le ||u_{s\omega}||_\infty^2 \epsilon^{\frac{\gamma}{2}}\int\int \bar{u}^2 r^\delta, \tag{2.10.25}$$

$$\epsilon^{\frac{\gamma}{2}}\int\int r^{2+\delta}v_s^2 \bar{u}_R^2 \le \epsilon^{\frac{\gamma}{2}}||v_s r||_\infty^2 \int\int \bar{u}_R^2 r^\delta, \tag{2.10.26}$$

$$\epsilon^{1+\frac{\gamma}{2}}\int\int r^\delta v_s^2 \bar{u}^2 \le \epsilon^{1+\frac{\gamma}{2}}||v_s||_\infty^2 \int\int \bar{u}^2 r^\delta, \tag{2.10.27}$$

$$\epsilon^{1+\frac{\gamma}{2}}\int\int r^\delta u_s^2 \bar{v}^2 \le \epsilon^{\frac{\gamma}{2}}||u_s||_\infty \int\int \epsilon \bar{v}^2 r^\delta, \tag{2.10.28}$$

$$\epsilon^{\frac{\gamma}{2}}\int\int u_{sR}^2 r^{2+\delta}\bar{v}^2 \le \epsilon^{\frac{\gamma}{2}}\left(\sup\int u_{sR}^2 r^2 (R-R_0)\right)\int\int \bar{v}_R^2, \tag{2.10.29}$$

$$\epsilon^{1+\frac{\gamma}{2}}\int\int r^\delta u_s^2 \bar{v}_\omega^2 \le \epsilon^{\frac{\gamma}{2}}||u_s||_\infty^2 \int\int \epsilon r^\delta \bar{v}_\omega^2, \tag{2.10.30}$$

$$\epsilon^{1+\frac{\gamma}{2}}\int\int r^{2+\delta}v_s^2 \bar{v}_R^2 \le \epsilon^{1+\frac{\gamma}{2}}||v_s||_\infty^2 \int\int r^{2+\delta}\bar{v}_R^2, \tag{2.10.31}$$

$$\epsilon^{1+\frac{\gamma}{2}}\int\int r^{2+\delta}v_{sR}^2 \bar{v}^2 \le ||v_{sR}r||_\infty^2 \epsilon^{\frac{\gamma}{2}}\int\int \epsilon r^\delta \bar{v}^2, \tag{2.10.32}$$

$$\epsilon^{\frac{\gamma}{2}}\int\int r^\delta u_s^2 \bar{u}^2 \le ||u_s||_\infty^2 \epsilon^{\frac{\gamma}{2}}\int\int r^\delta \bar{u}^2, \tag{2.10.33}$$

$$\epsilon^{1+\frac{\gamma}{2}}\int\int r^\delta v_{s\omega}^2 \bar{u}^2 \le ||v_{s\omega}||_\infty^2 \epsilon^{1+\frac{\gamma}{2}}\int\int r^\delta \bar{u}^2. \tag{2.10.34}$$

Thus, (2.10.24) + ... + (2.10.34) $\lesssim \epsilon^{\frac{\gamma}{2}-\kappa}||f,\sqrt{\epsilon}g||_{L^2_{*,2+\delta}}^2$. This concludes the proof. □

Corollary 2.10.2. *For δ sufficiently close to 1, the solution operator of the nonlinear equation is a contraction map on the space Z, satisfying:*

$$\begin{aligned}||\bar{u}^1 - \bar{u}^2, \bar{v}^1 - \bar{v}^2||_Z \leq & C(u_s, v_s)\Big[\epsilon^{\frac{\gamma}{2}} \left(||u^1, v^1||_Z + ||u^2, v^2||_Z\right) ||u^1 - u^2, v^1 - v^2||_Z \\ & + \epsilon^{\frac{1}{2} - \frac{\gamma}{2} - \kappa} ||u^1 - u^2, v^1 - v^2||_Z\Big]. \end{aligned} \tag{2.10.35}$$

By applying the contraction mapping theorem, we have proven Theorem 2.2.18 and therefore the main result, Theorem 2.2.3.

Chapter Three

Global-in-x Prandtl Layers over a Moving Boundary

3.1 Abstract

In this three-part monograph, we prove that steady, incompressible Navier-Stokes flows posed over the moving boundary, $y = 0$, can be decomposed into Euler and Prandtl flows in the inviscid limit globally in $[1, \infty) \times [0, \infty)$, assuming a sufficiently small velocity mismatch. Sharp decay rates and self-similar asymptotics are extracted for both Prandtl and Eulerian layers. We then develop a functional framework to capture precise decay rates of the remainders, and prove the corresponding embedding theorems by establishing weighted estimates for their higher order tangential derivatives. These tools are then used in conjunction with a third order energy analysis, which in particular enables us to control the nonlinearity vu_y globally.

3.2 Introduction

We consider the steady, incompressible Navier-Stokes equations in two dimensions:

$$U^{NS}U^{NS}_X + V^{NS}U^{NS}_Y + P^{NS}_X = \epsilon\Delta U^{NS}, \tag{3.2.1}$$

$$U^{NS}V^{NS}_X + V^{NS}V^{NS}_Y + P^{NS}_Y = \epsilon\Delta V^{NS}, \tag{3.2.2}$$

$$U^{NS}_X + V^{NS}_Y = 0, \tag{3.2.3}$$

in the domain,

$$\Omega = [1, \infty) \times \mathbb{R}_+. \tag{3.2.4}$$

The boundary $Y = 0$ is moving with velocity $u_b > 0$. The no-slip boundary conditions are placed on this portion of the boundary:

$$U^{NS}(X, 0) = u_b = 1 - \delta, \quad V^{NS}(X, 0) = 0. \tag{3.2.5}$$

The boundary conditions at $X = 1$ will be prescribed explicitly in the text. We take $X = 1$ for convenience (this enables us to replace weights of $(1+x)^k$ with x^k). Throughout this paper, we assume that prescribed Euler flow is the shear flow:

$$(U^E, V^E) = (1, 0). \tag{3.2.6}$$

We are interested in the limit as $\epsilon \to 0$. Formally, one expects that the solutions to Navier-Stokes equations in (3.2.1) - (3.2.3) converges to the Euler shear flow in (3.2.6). This does not happen, however, due to the mismatch at the boundary $Y = 0$, between the no-slip condition enforced for Navier-Stokes, (3.2.5), and $U^E(X, Y = 0) = 1$.

To account for the mismatch at the boundary, Prandtl in 1904 proposed a thin fluid boundary layer which connects the velocity of u_b to the Euler velocity of 1. The Prandtl hypothesis is that the Navier-Stokes solutions can be decomposed, up to leading order in ϵ, as the sum of the prescribed Euler flow and a boundary layer, the latter of which corrects the disparity at the boundary between Euler and Navier-Stokes:

$$U^{NS} = 1 + u_p^0 + h.o.t(\epsilon), \quad V^{NS} = 0 + \sqrt{\epsilon} v_p^0 + \sqrt{\epsilon} v_e^1 + h.o.t(\epsilon).^{1} \tag{3.2.7}$$

The contribution of this paper is to validate the boundary layer theory, equations (3.2.7), in the domain Ω*, which in particular implies that the tangential variable can be taken in* $[1, \infty)$ *if the mismatch is sufficiently small*:

$$U^E - U^{NS}|_{Y=0} = 1 - u_b = 1 - (1 - \delta) = \delta << 1. \tag{3.2.8}$$

[1] Here, "h.o.t" is an acronym for "higher order terms."

Boundary Layer Expansion

We will work with scaled, boundary layer variables:

$$x = X, \quad y = \frac{Y}{\sqrt{\epsilon}}. \tag{3.2.9}$$

The scaled Navier-Stokes unknowns are then given by:

$$U^\epsilon(x,y) = U^{NS}(X,Y), \quad V^\epsilon(x,y) = \frac{V^{NS}(X,Y)}{\sqrt{\epsilon}}, \quad P^\epsilon(x,y) = P^{NS}(X,Y). \tag{3.2.10}$$

These unknowns satisfy the following system:

$$U^\epsilon U^\epsilon_x + V^\epsilon U^\epsilon_y + P^\epsilon_x = U^\epsilon_{yy} + \epsilon U^\epsilon_{xx}, \tag{3.2.11}$$

$$U^\epsilon V^\epsilon_x + V^\epsilon V^\epsilon_y + \frac{P^\epsilon_y}{\epsilon} = V^\epsilon_{yy} + \epsilon V^\epsilon_{xx}, \tag{3.2.12}$$

$$U^\epsilon_x + V^\epsilon_y = 0. \tag{3.2.13}$$

Note that the steady Prandtl system is obtained by considering the leading order in ε of the above system (3.2.11) - (3.2.13). We start with the following asymptotic expansion:

$$U^\epsilon(x,y) = 1 + u^0_p + \sum_{i=1}^{n} \epsilon^{\frac{i}{2}} u^i_e + \epsilon^{\frac{i}{2}} u^i_p + \epsilon^{\frac{n}{2}+\gamma} u(x,y), \tag{3.2.14}$$

$$V^\epsilon(x,y) = \sum_{i=0}^{n-1} \epsilon^{\frac{i}{2}} v^i_p + \epsilon^{\frac{i}{2}} v^{i+1}_e + \epsilon^{\frac{n}{2}} v^n_p + \epsilon^{\frac{n}{2}+\gamma} v(x,y), \tag{3.2.15}$$

$$P^\epsilon(x,y) = \sum_{i=1}^{n} \epsilon^{\frac{i}{2}} P^i_e + \epsilon^{\frac{i}{2}} P^i_p + \epsilon^{i} P^{i,a}_e + \epsilon^{\frac{i+1}{2}} P^{i,a}_p + \epsilon^{\frac{n}{2}+\gamma} P(x,y). \tag{3.2.16}$$

Here, $\gamma \in [0, \frac{1}{4})$. We will use the word "profiles" to refer to the terms which appear in the expansions (3.2.14) - (3.2.15), excluding the remainders, $[u, v, P]$. All of the profiles with subscript-e are functions of Eulerian variables, (x, Y), whereas all terms with subscript-p are functions of boundary layer variables, (x, y). Here, $[u^i_p, v^i_p]$ are boundary layers to be constructed. The number of intermediate layers, n, is dependent on universal constants. The pressures P^i_e, P^i_p are the pressures associated with the $i'th$ Euler and Prandtl layers, respectively. We will show that $P^i_p = 0$, that is the leading-order pressure in the boundary layers is zero. The pressures $P^{i,a}_P, P^{1,a}_e$ are *auxiliary* pressures, which are higher-order, whose purpose is to capitalize on the gradient structure of our problem (see 3.7.33). After these layers are constructed, the Navier-Stokes remainders $[u, v, P]$ are then constructed. Let us now designate names for the partial expansions:

$$u^{(i)}_s := 1 + \sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} u^j_p + \sum_{j=1}^{i} \epsilon^{\frac{j}{2}} u^j_e, \quad \bar{u}^{(i)}_s := u^{(i)}_s + \epsilon^{\frac{i}{2}} u^i_p = 1 + \sum_{j=0}^{i} \epsilon^{\frac{j}{2}} u^j_p + \sum_{j=1}^{i} \epsilon^{\frac{j}{2}} u^j_e \quad (3.2.17)$$

$$v^{(i)}_s := \sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} v^j_p + \sum_{j=1}^{i} \epsilon^{\frac{j}{2}-\frac{1}{2}} v^j_e, \quad \bar{v}^{(i)}_s := v^{(i)}_s + \epsilon^{\frac{i}{2}} v^i_p = \sum_{j=0}^{i} \epsilon^{\frac{j}{2}} v^j_p + \sum_{j=1}^{i} \epsilon^{\frac{j}{2}-\frac{1}{2}} v^j_e, \quad (3.2.18)$$

For the Pressure expansion:

$$P^{(i)}_s := \sum_{j=1}^{i-1} \epsilon^{\frac{j}{2}} P^j_p + \sum_{j=1}^{i-1} \epsilon^{\frac{j+1}{2}} P^{j,a}_p + \sum_{j=1}^{i} \epsilon^{\frac{j}{2}} P^j_e + \sum_{j=1}^{i} \epsilon^j P^{j,a}_e. \quad (3.2.19)$$

$$\bar{P}^{(i)}_s := \sum_{j=1}^{i} \epsilon^{\frac{j}{2}} P^j_p + \sum_{j=1}^{i} \epsilon^{\frac{j}{2}} P^j_e + \sum_{j=1}^{i} \epsilon^j P^{j,a}_e + \sum_{j=1}^{i} \epsilon^{\frac{j+1}{2}} P^{j,a}_p. \quad (3.2.20)$$

We insert the expansions (3.2.14) - (3.2.16) into (3.2.11) - (3.2.13) and collect a heirarchy of equations in powers of ϵ. Doing so yields the linearized Prandtl-equations:

$$(1 + u^0_p) u^i_{px} + u^{(i)}_{sx} u^i_p + v^{(i)}_s u^i_{py} + u^0_{py} \Big(v^i_p - v^i_p(x, 0) \Big) + P^i_{px} = u^i_{pyy} + f^{(i)}, \quad (3.2.21)$$

$$u^i_p(x, 0) = -u^i_e(x, 0), \quad \lim_{y \to \infty} u^i_p(x, y) = 0, \quad u^i_p(1, y) = U_i(y). \quad (3.2.22)$$

and the Euler equations:

$$u^i_{ex} + P^i_{ex} = 0, \quad v^i_{ex} + P^i_{eY} = 0, \quad u^i_{ex} + v^i_{eY} = 0. \tag{3.2.23}$$

The forcing term $f^{(i)}$ will be defined precisely in (3.7.47). These equations are derived rigorously in the analysis leading up to equations (3.5.5), (3.6.21), (3.7.19), and (3.7.48).

Boundary Data:

The no-slip boundary condition at the boundary $\{y = 0\}$ is the most important, and must be enforced at each order in ϵ, which gives:

$$u^0_p(x, 0) = -\delta, \quad u^i_e(x, 0) + u^i_p(x, 0) = 0 \text{ for } i \geq 1, \quad u(x, 0) = 0, \tag{3.2.24}$$

$$v^{i-1}_p(x, 0) + v^i_e(x, 0) = 0 \text{ for } i \geq 1, \quad v^n_p(x, 0) = 0, \quad v(x, 0) = 0. \tag{3.2.25}$$

The in-flow $(x = 1)$ boundary conditions for the leading order boundary layer, u^0_p, is:

$$u^0_p(1, y) = U_0(y), \quad U_0(0) = -\delta, \quad \lim_{y\to\infty} U_0(y) = 0. \tag{3.2.26}$$

We assume the rapid decay of the profile:

$$||\langle y\rangle^m \partial^j_y U_0(y)||_{L^\infty} \leq C(m, j), \text{ for any } m, j \geq 0. \tag{3.2.27}$$

We will in addition assume the following smallness condition:

$$||\langle y\rangle^m \partial^j_y U_0(y)||_{L^\infty} \leq \mathcal{O}(\delta; j, m) \text{ for any } m \geq 0, j = 0, 1, 2. \tag{3.2.28}$$

The in-flow ($x = 1$) boundary conditions for the boundary-layer profiles are:

$$u_p^i(1, y) = U_i(y), \quad u(1, y) = 0, \quad v(1, y) = 0, \text{ for } 1 \leq i \leq n - 1. \tag{3.2.29}$$

Here, $U_i(y)$, for $i \leq 1 \leq n - 1$, will be prescribed to be rapidly decaying in y, so:

$$||\langle y \rangle^m u_p^i(1, y)||_{L^\infty} = ||\langle y \rangle^m U_i(y)||_{L^\infty} \leq C(i, m) \text{ for } 1 \leq i \leq n - 1. \tag{3.2.30}$$

For the final Prandtl layer, u_p^n, which occurs at order $\varepsilon^{\frac{n}{2}}$, the in-flow data is determined through the analysis and is not explicitly prescribed. This is due to retaining that $[u_p^n, v_p^n]$ are divergence free, while cutting off v_p^n for large values of y. The reader is invited to turn to equation (3.7.142) and corresponding discussion for details regarding this matter. We will enforce $||\langle y \rangle^m U_n(y)||_{L^\infty} \leq C(m)$. However U_n is an auxiliary in-flow, which is used to construct $[u_p^n, v_p^n]$. That is: $u_p^n(1, y) \neq U_n(y)$, and the in-flow velocity, $u_p^n(1, y)$, is given by an implicit, bounded profile which decays as $y \to \infty$. We will need several compatibility conditions on the in-flow data for the Prandtl layers. The first of these is:

$$U_0(0) = -\delta, \quad \partial_{yy} U_0(y) = 0, \quad U_i(0) = -u_e^i(1, 0). \tag{3.2.31}$$

However, we shall also need higher-order compatibility conditions on the $U_i(y)$ at $y = 0$, (see for instance Remarks 3.4.2, 3.6.2) which we refrain from depicting explicitly here. Finally, the boundary conditions of the boundary-layer profiles as $y \to \infty$ are:

$$\lim_{y \to \infty} [u_p^i(x), v_p^i(x)] = \lim_{y \to \infty} [u(x), v(x)] = 0 \text{ for all } x \geq 1, \text{ and all } 0 \leq i \leq n. \tag{3.2.32}$$

These boundary conditions are known as the "matching condition", and physically correspond to the Navier-Stokes flow matching the outer Euler flow away from the boundary,

$y = 0$. According to our construction, we will have rapid matching up to order $\varepsilon^{\frac{n-1}{2}}$:

$$\lim_{y\to\infty} [\langle y\rangle^N u_p^i(x), \langle y\rangle^N v_p^i(x)] = 0 \text{ for all } x \geq 1, \text{ for } 1 \leq i \leq n-1. \tag{3.2.33}$$

At the highest-order in ε, we enforce the matching condition:

$$\lim_{y\to\infty} \varepsilon^{\frac{n}{2}} [u_p^n(x,y), v_p^n(x,y)] = \lim_{y\to\infty} \varepsilon^{\frac{n}{2}+\gamma}[u(x,y), v(x,y)] = 0 \text{ for all } x \geq 1. \tag{3.2.34}$$

Let us now turn to the Euler flows. At leading order, we have the prescription: $[u_e^0, v_e^0] = [1, 0]$. The higher-order Euler flows will be described starting in Section 3.5 and Subsection 3.7.1. The higher-order Euler flows are obtained as suitable Poisson extensions of the $Y = 0$ boundary data, $v_p^{i-1}(x, 0)$ (see (3.2.24)), which depend on the constructed Prandtl layers. For these higher-order Euler flows, we do not prescribe the in-flow data, $u_e^i(1, Y)$. Rather, the in-flow conditions are obtained through the analysis, so we state:

$$\text{Eulerian In-Flow} = 1 + \sum_{i=1}^{n} \epsilon^{\frac{i}{2}} u_e^i(1, \cdot). \tag{3.2.35}$$

Main Result:

In order to state our main result, we need to introduce the norm Z in which we control the remainder solutions, $[u, v]$:

Definition 3.2.1. The norm Z is defined through:

$$\begin{aligned} ||u, v||_Z :=& ||u, v||_{X_1 \cap X_2 \cap X_3} + \epsilon^{N_2} ||u, v||_{Y_2} + \epsilon^{N_3} ||u, v||_{Y_3} + \epsilon^{N_4} ||u x^{\frac{1}{4}}, \sqrt{\varepsilon} v x^{\frac{1}{2}}||_{L^\infty} \\ &+ \epsilon^{N_5} \sup_{x \geq 20} ||\sqrt{\varepsilon} v_x x^{\frac{3}{2}}, u_x x^{\frac{5}{4}}||_{L^\infty} + \varepsilon^{N_6} \sup_{x\geq 20} ||u_y x^{\frac{1}{2}}||_{L^2_y} \\ &+ \epsilon^{N_7} \Big[\int_{20}^\infty x^4 ||\sqrt{\varepsilon} v_{xx}||^2_{L^\infty_y} dx \Big]^{\frac{1}{2}}. \end{aligned} \tag{3.2.36}$$

Here, N_i, are large numbers which will be specified in (3.9.103) - (3.9.105). They depend only on universal constants. The parameter n from (3.2.14) - (3.2.15) will be taken much larger than any of the N_i. The norms $||\cdot||_{X_i}$ are energy norms defined in (3.9.3) - (3.9.5). The norms $||\cdot||_{Y_i}$ are elliptic norms defined in (3.9.6) - (3.9.7). For the purposes of stating the main result, we can refrain from being too specific with regards to the definitions of these norms. The essential point that we will record concerns the uniform component:

$$\epsilon^{N_4}||ux^{\frac{1}{4}}, \sqrt{\varepsilon}vx^{\frac{1}{2}}||_{L^\infty} \le ||u,v||_Z, \tag{3.2.37}$$

The main result of this paper is:

Theorem 3.2.2. *Suppose the the outer Euler flow is prescribed with $u_e^0 = 1$. Suppose the boundary and in-flow data are specified satisfying the conditions outlined in (3.2.24) - (3.2.34). Then there exists an n depending on only universal constants such that the asymptotic expansions in (3.2.14) - (3.2.16) are valid globally on the domain Ω, for $0 \le \gamma < \frac{1}{4}$, so long as the mismatch between the Eulerian boundary trace and the motion of the boundary, δ, and the viscosity, ε, are taken sufficiently small relative to universal constants, and $\varepsilon << \delta$. The remainders, $[u,v]$, in the expansions (3.2.14) - (3.2.15) are uniquely determined in the space Z:*

$$||u,v||_Z \lesssim \varepsilon^{\frac{1}{4}-\gamma-\kappa}, \tag{3.2.38}$$

where κ is any fixed constant such that $\gamma + \kappa < \frac{1}{4}$.

Because $\frac{n}{2}$ is large relative to N_4 in (3.2.37), we immediately find:

Corollary 3.2.3 (Inviscid L^∞ Convergence)**.** *Under the hypothesis of Theorem 3.2.2, there* *exists a unique Navier-Stokes solution* $[U^{NS}, V^{NS}, P^{NS}]$ *on* Ω *such that:*

$$\sup_{(X,Y)\in\Omega} \Big| U^{NS}(X,Y) - 1 - u_p^0(X,y) \Big| X^{\frac{1}{4}} \lesssim \epsilon^{\frac{1}{2}}, \tag{3.2.39}$$

$$\sup_{(X,Y)\in\Omega} \Big| V^{NS}(X,Y) - \sqrt{\epsilon} v_p^0(X,y) - \sqrt{\epsilon} v_e^1(X,Y) \Big| X^{\frac{1}{2}} \lesssim \epsilon. \tag{3.2.40}$$

Existing Literature:

Let us first discuss the issue of establishing wellposedness of the Prandtl equation, which becomes an issue in the unsteady setting (in contrast to the steady setting of the present paper). This program was initiated in the classic works (OS99), (Ole67), in which, under the monotonicity assumption $U_y^\varepsilon(t=0) > 0$, globally regular solutions are constructed on the $[0, L] \times \mathbb{R}_+$, where L is sufficiently small, and local solutions are constructed for arbitrary, but finite L. This was extended in (XZ04), in which global weak solutions were constructed for arbitrary L, under both monotonicity and favorable outer-Euler pressure ($\partial_x P^E(t,x) \leq 0$ for $t \geq 0$) assumptions.

From a physical standpoint, the monotonicity and favorable pressure assumptions mentioned above are stabilizing and in particular prevent boundary layer separation. This phenomena was known to Prandtl, see Figure 2 in (Pra04). More recently, it was announced in (DM18) that a proof of boundary layer separation in the steady setting has been obtained.

The main tool used both in (Ole67) and (XZ04) is the Crocco transform. Still under monotonicity hypothesis, local wellposedness was obtained in (AYXY15) and (MW15), neither works using the Crocco transform. (AYXY15) use energy methods coupled with a Nash-Moser iteration, and (MW15) use energy methods applied to a good unknown which enjoys crucial cancellation properties. Generalizing to multiple monotonicity regions, (KMVW14) have shown the Prandtl equation is locally well-posed, if an analyticity assumption is made on the complement of the monotonicity regions.

Indeed, when the assumption of monotonicity is removed, the wellposedness results are largely in the analytic or Gevrey setting. The reader should consult (SC98a) - (SC98b), (KMVW13), (LCS03), (IV16), and (GVM13) for some results in this direction. In the Sobolev setting without monotonicity, the equations are linearly and nonlinearly ill-posed (see (GVD10) and (GVN12)). A finite-time blowup result was obtained in (EE97) when the outer Euler flow is taken to be zero, in (KVW15) for a particular, periodic outer Euler flow, and in (HH03) for both the inviscid and viscous Prandtl equations. The above discussion is not comprehensive: we refer the reader to the review articles, (E00), (GJT16) and references therein for a more thorough review of the wellposedness theory.

The question with which we are concerned is the validity of the asymptotic expansion (3.2.14) - (3.2.16) in the inviscid limit. Let us first discuss unsteady flows. Local-in-time convergence is established in (SC98a), (SC98b) in the analyticity framework, in (GVMM16) in the Gevrey setting, and in (Mae14) when the initial vorticity distribution is supported away from the boundary. The reader should see also (Asa91), (MT08) for related results. Despite the boundary layer classically having thickness $\sqrt{\varepsilon}$, an interesting criteria was given in (Kat84) which points to phenomena occurring in a sub-layer of size ϵ. There are also several linear and nonlinear instability results (for instance, (Gre00), (GGN16b), (GGN16a), (GGN15), (GN11)) which show the invalidity of Prandtl's expansion generically in Sobolev spaces in the unsteady setting.

For steady flows, there are very few validity results. (GN17) is the first result in this direction, establishing validity of the boundary layer expansion for steady state flows in a rectangular domain over a moving boundary. Geometric effects of the boundary were subsequently considered in (Iye17a). The crucial idea in (GN17) was the use of a positivity estimate, which is coupled with energy estimates and elliptic estimates. Both of these results are *local* in the tangential variable. In the present work, we prove validity of the boundary layer expansion *globally* in the tangential variable, x, in the setting of small data.

One preliminary piece of our analysis is to obtain the asymptotics of the Prandtl layer, u_p^0.

This has nontrivial dynamics due to the mismatched boundary conditions, $u^0_p(y=0) = -\delta$, while $\lim_{y\to\infty} u^0_p(y) = 0$. These asymptotics were first studied in (Ser67, pg. 493, Inequality 5) using maximum principle techniques and are valid for large data. The result in (Ser67) gives that the difference between u^0_p and a Gaussian"front" solution to the heat equation (call it w) is $o(1)$ in x, uniformly in y. Under the hypothesis of small data, we sharpen these asymptotics in the following sense: first, w is shown to belong to a higher-order Sobolev space $H^k(m)$, where this weight is in the self-similar variable $z = \frac{y}{\sqrt{x}}$. Second, we obtain rates of decay of w in x in various norms.

We now detail the main difficulties and ideas behind our analysis.

Sharp Decay of Profiles (Chapter I):

The key issue that we must capture in our analysis is the decay as $x \to \infty$ of various quantities. More specifically, a central difficulty is to control contributions from the nonlinearity $V^\epsilon U^\epsilon_y$. Let us now introduce the equations for the remainders, $[u, v, P]$:

$$-\Delta_\epsilon u + S_u + P_x = f, \quad -\Delta_\epsilon v + S_v + \frac{P_y}{\epsilon} = g, \quad u_x + v_y = 0. \tag{3.2.41}$$

Here, the terms S_u, S_v contain the linearizations of $[u, v]$ around the previously constructed profiles, $[\bar{u}^{(n)}_s, \bar{v}^{(n)}_s]$. These terms, together with f, g are specifically defined in (3.8.5) - (3.8.6). To organize this discussion, let us record the heuristic:

$$\text{Difficult Contributions from } V^\varepsilon U^\varepsilon_y = \bar{u}^{(n)}_{sy} v + \bar{v}^{(n)}_s u_y + \varepsilon^{\frac{n}{2}+\gamma} v u_y. \tag{3.2.42}$$

First, let us discuss $\bar{u}^{(n)}_{sy} v$ from (3.2.42), which will motivate the crucial decay rates appearing in (3.2.48). Applying the scaled multiplier of $(u, \epsilon v)$ to the system (3.2.41) requires controlling the large convective term, $\int\int u^0_{py} uv$. We do not have the ability to create a derivative through the Poincare inequality, and so we trade factors of x and y in the following manner:

$$\Big| \int\int u^0_{py} uv \Big| \le ||u^0_{py} y^2 x^{-\frac{1}{2}}||^2_{L^\infty} ||\frac{u}{y}||_{L^2} ||\frac{vx^{\frac{1}{2}}}{y}||_{L^2} \le \mathcal{O}(\delta) ||u_y||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2}. \tag{3.2.43}$$

The first crucial observation we make is the identification of a self-similar front, $\phi_*(\frac{y}{\sqrt{x}})$, which bridges the boundary conditions: $u^0_p(x,0) = -\delta$, $u^0_p(x,\infty) = 0$. Then, temporarily identifying $u^0_p \approx \phi_*(\frac{y}{\sqrt{x}})$, (3.2.43) will be satisfied:

Requirement 1 (Self-Similarity of Prandtl profiles)**.**

$$||y^2 x^{-\frac{1}{2}} u^0_{py}||_{L^\infty} = ||y^2 x^{-1} \phi'_* \Big(\frac{y}{\sqrt{x}}\Big)||_{L^\infty} = ||z^2 \phi'_*(z)||_{L^\infty} \le \mathcal{O}(\delta). \tag{3.2.44}$$

Summarizing the energy estimate that we obtain:

$$||u_y||^2_{L^2} \le \mathcal{O}(\delta) ||\{\sqrt{\varepsilon} v_x, v_y\} x^{\frac{1}{2}}||^2_{L^2} + \text{ Forcing Terms.} \tag{3.2.45}$$

Above, the key point is the loss of weight, $x^{\frac{1}{2}}$. The next ingredient is recovering this weight in the Positivity estimate, (see Proposition 3.10.2), which is summarized:

$$||\{\sqrt{\varepsilon} v_x, v_y\} x^{\frac{1}{2}}||^2_{L^2} \lesssim ||u_y||^2_{L^2} + \text{ Forcing Terms.} \tag{3.2.46}$$

In order to prove (3.2.46), we must apply the weighted multilplier $v_y x$. Referring to the final two terms in (3.2.42), this gives (temporarily ignoring factors of ϵ):

$$\Big| \int\int \bar{v}^{(n)}_s u_y \cdot v_y x \Big| + \Big| \int\int v u_y \cdot v_y x \Big| \le ||\{\bar{v}^{(n)}_s, v\} x^{\frac{1}{2}}||_{L^\infty} ||u_y||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2}. \tag{3.2.47}$$

The latter two L^2 quantities are controlled by the left-hand sides of (3.2.46) - (3.2.47).

From this, we obtain the requirements:

Requirement 2 (Uniform Decay)**.**

$$\bar{v}_s^{(n)} + v \sim x^{-\frac{1}{2}}, \;\; as \; x \to \infty. \tag{3.2.48}$$

We emphasize that this requirement is inflexible, and we cannot sacrifice even a logarithmic factor of x here. The main contribution of Chapter I is the construction of profiles $u_p^i, v_p^i, u_e^i, v_e^i$ which satisfy the requirement of $v_s^{(n)}$ in (3.2.48). The most difficult task is to obtain the estimate (3.2.48) for the Eulerian profiles, v_e^1, as these are solutions to elliptic boundary value problems in which the boundary condition exhibits exactly the required decay rate, $|v_e^1(x,0)| \le x^{-\frac{1}{2}}$. We refer the reader to the crucial Proposition 3.5.2 in which we introduce novel techniques centered around the explicit integral representation of v_e^1, enabling us to prove the required decay, $|v_e^1| \lesssim x^{-\frac{1}{2}}$.

Note that we construct the expansions, (3.2.14) - (3.2.16) for any $n \in \mathbb{N}$, which is required as discussed in the paragraph following (3.2.52). Our ability to do this relies on the Cauchy-Riemann structure of the Eulerian profiles, which we use in Lemma 3.7.33.

The Norm Z (Chapter II):

Let us now turn to the v term in Requirement (3.2.48): the main contribution of Chapter II is to prove the required decay estimate $|v| \lesssim x^{-\frac{1}{2}}$ by using the crucial norm Z (see Lemmas 3.9.15, 3.9.17). The challenge is to extract this precise uniform decay information from the energy norms that are controlled. Bearing in mind the inflexibility of (3.2.48), obtaining the decay for v using the norms X_i is an extremely delicate matter, in which key quantities must overcome the critical Hardy inequality. To see this, we first use the $H^1_y(\mathbb{R}_+) \hookrightarrow L^\infty_y(\mathbb{R}_+)$

Sobolev embedding (ignoring factors of ε):

$$\sup_{x \geq 1} ||vx^{\frac{1}{2}}||_{L^\infty_y} \leq \sup_{x \geq 1} ||v||^{\frac{1}{2}}_{L^2_y} \cdot \sup_{x \geq 1} ||v_y x||^{\frac{1}{2}}_{L^2_y}. \tag{3.2.49}$$

For the first quantity on the right-hand side above, we write:

$$\partial_x \int v^2 \, \mathrm{d}y = \int 2vv_x \, \mathrm{d}y. \tag{3.2.50}$$

Recall now the quantities, $||u_y, \sqrt{\varepsilon} v_x x^{\frac{1}{2}}, v_y x^{\frac{1}{2}}||^2_{L^2_{xy}}$, which are controlled on the left-hand sides (3.2.45), (3.2.46) and constitute the energy norm X_1. The right-hand side above fails to be x-integrable, precisely because of criticality of Hardy's inequality with power $x^{-\frac{1}{2}}$ in L^2:

$$|\int \int vv_x \, \mathrm{d}y \, \mathrm{d}x| \leq ||vx^{-\frac{1}{2}}||_{L^2_{xy}} ||v_x x^{\frac{1}{2}}||_{L^2_{xy}} \not\lesssim ||v_x x^{\frac{1}{2}}||^2_{L^2_{xy}}. \tag{3.2.51}$$

To avert this, we move to higher-order derivatives, which invokes the full strength of the norm Z. Indeed, suppose we knew $v_x \sim x^{-\frac{3}{2}}$, then coupled with the boundary condition $v \to 0$ as $x \to \infty$, this would immediately imply $v \sim x^{-\frac{1}{2}}$. Establishing the decay rate, $||v_x||_{L^\infty_y} \leq x^{-\frac{3}{2}}$, then becomes the goal, which requires us to go to third-order energy estimates (thus explaining the presence of X_1, X_2, X_3 in the norm Z). Our main uniform estimates, given in Lemmas 3.9.15, 3.9.17 are given by the following sequence:

$$||vx^{\frac{1}{2}}||_{L^\infty_{xy}} \lesssim ||v_x x^{\frac{3}{2}}||_{L^\infty_{xy}} \lesssim \sup_{x \geq 1} ||v_x x||^{\frac{1}{2}}_{L^2_y} \cdot \sup_{x \geq 1} ||v_{xy} x^2||^{\frac{1}{2}}_{L^2_y} \lesssim ||u, v||_{X_1 \cap X_2 \cap X_3}. \tag{3.2.52}$$

The key point is that the quantities $||v_x x||_{L^2_y}$ and $||v_{xy} x^2||_{L^2_y}$ appearing above *do not* face issues of Hardy-criticality present in (3.2.51), which can be seen in Lemma 3.9.15.

Applying ∂_x^k to the system creates singularities near the corner at $(1, 0)$. To handle this we cutoff near the boundary, $x = 1$, when performing higher order energy estimates. Cutoff functions interact poorly with nonlinearities, and so we need to supplement energy estimates with elliptic estimates which retain additional control of $[u, v]$ near $x = 1$ (though not all the way up to the boundary, $x = 1$). These are characterized by the norms $||\cdot||_{Y_i}$ in (3.2.36). These Y_i norms are controlled by invoking the elliptic theory, which in turn requires sacrificing factors of ε. For this, we require a high power of ε^n to accompany nonlinear terms, which in turn requires us to go to high order expansions in (3.2.14) - (3.2.16).

Existence and Uniqueness (Chapter III):

Upon proving our main *a-priori* estimate, Theorem 3.8.1, we prove existence and uniqueness of a solution in Z. As in (3.2.43), applying the multiplier u produces the nonlinearity $vu_y \cdot u$, which is a perfect derivative and therefore vanishes. This cancellation property is destroyed upon taking differences, and so we cannot rely on a standard application of the contraction mapping theorem. A sequence of auxiliary, approximate systems are then carefully designed to produce enough compactness enabling us to show existence of a solution in the space Z.

The uniqueness in Z is a more delicate matter, again due to a lack of the perfect derivative structure. For this, we take the difference of two solutions in Z and repeat the energy analysis with weaker weights. The reader is referred to Lemma 3.17.1, in which the weights must be selected carefully in a small interval below those weights appearing in the energy analysis, for instance in (3.2.45), (3.2.46). These methods are carried out in Chapter III.

Notation and Important Parameters

There are three important parameters in this paper: ε, δ, and n (see (3.2.14) - (3.2.16)). The notation $A \lesssim B$ means $A \leq CB$, where C is some constant which is independent of small

δ, ε, and large n. Constants denoted by $\mathcal{O}(\delta)$ or $\mathcal{O}(\varepsilon)$ satisfy $\mathcal{O}(\delta), \mathcal{O}(\varepsilon) \to 0$ as $\delta, \varepsilon \to 0$, respectively. Given any parameter, say p, constants denoted by $C(p)$ mean those constants which depend (perhaps poorly) on large values of p. Given two parameters, δ and σ, for instance, we will write $\mathcal{O}(\delta;\sigma)$ to denote a constant which depends on σ and δ, but such that for fixed σ, δ can be made small to make the constant small, for instance $\delta \times \sigma$. We define $||f||^p_{L^p_y} := \int f(x,y)^p dy$. When unspecified, $||\cdot||_{L_p}$ means the L^p norm of two-variables. We define here the differential operators:

$$\Delta_\varepsilon u := (\partial_{xx} + \varepsilon \partial_{yy})u \text{ for any profile } u; \qquad \Delta[u^i_e, v^i_e] := (\partial_{xx} + \partial_{YY})[u^i_e, v^i_e]. \tag{3.2.53}$$

The variable z will denote a self-similar variable, so typically $z = \frac{y}{\sqrt{x}}$ or $z = \frac{\eta}{\sqrt{x}}$. Finally, the word "profiles" refers to terms in the expansion (3.2.14) - (3.2.15), excluding the remainders $[u, v]$, and "profile terms" refers to the linearizations in S_u, S_v, defined in (3.8.6).

165

Part I: Construction of Profiles

3.3 Overview of Profile Constructions

The purpose of this chapter is to construct each of the profiles appearing in the expansions (3.2.14) - (3.2.16), with the exception of the final terms, $[u, v, P]$. This results of this chapter are used extensively in Chapter II. Let us first introduce the following notation for $\bar{u}_s^{(n)}, \bar{v}_s^{(n)}$:

$$u_R^P := \sum_{j=0}^n \epsilon^{\frac{j}{2}} u_p^j, \quad u_R^E := 1 + \sum_{j=1}^n \epsilon^{\frac{j}{2}} u_e^j, \quad u_R := \bar{u}_s^{(n)} = u_R^P + u_R^E, \tag{3.3.1}$$

$$v_R^P := \sum_{j=0}^n \epsilon^{\frac{j}{2}} v_p^j, \quad v_R^E := \sum_{j=1}^n \epsilon^{\frac{j}{2}-\frac{1}{2}} v_e^j, \quad v_R := \bar{v}_s^{(n)} = v_R^P + v_R^E. \tag{3.3.2}$$

We shall also have occasion to further split u_R^P to distinguish the final layer via:

$$u_R^{P,n-1} = \sum_{j=0}^n \epsilon^{\frac{j}{2}} u_p^j, \text{ so that } u_R^P = u_R^{P,n-1} + \epsilon^{\frac{n}{2}} u_p^n. \tag{3.3.3}$$

Inserting the expansion (3.2.14) - (3.2.16) into the scaled NS equations, (3.2.11) - (3.2.13), motivates the following definition:

Definition 3.3.1. The n'th remainder is denoted by:

$$R^{u,n} := -\Delta_\epsilon \bar{u}_s^{(n)} + \bar{u}_s^{(n)} \bar{u}_{sx}^{(n)} + \bar{v}_s^{(n)} \bar{u}_{sy}^{(n)} + \bar{P}_{sx}^{(n)}, \tag{3.3.4}$$

$$R^{v,n} := -\Delta_\epsilon \bar{v}_s^{(n)} + \bar{u}_s^{(n)} \bar{v}_{sx}^{(n)} + \bar{v}_s^{(n)} \bar{v}_{sy}^{(n)} + \frac{\partial_y}{\epsilon} \bar{P}_s^{(n)}. \tag{3.3.5}$$

The main result of this chapter is:

Theorem 3.3.2. *Let $n \geq 2 \in \mathbb{N}$. Let δ, ε be sufficiently small relative to universal constants, and $\varepsilon << \delta$. Let the boundary and in-flow data from (3.2.24) - (3.2.35) be prescribed. Then there exist Prandtl profiles $[u_p^j, v_p^j, P_p^j]$ for $j = 1, ..., n$, Euler profiles $[u_e^j, v_e^j, P_e^j]$ for $j = 1, ..., n$, and auxiliary pressures $[P_p^{j,a}, P_e^{j,a}]$ for $j = 1, ..., n$ such that for $R^{u,n}, R^{v,n}$ as defined in (3.3.4) - (3.3.5), and for any $\gamma \in [0, \frac{1}{4})$, $n \geq 2$, and for $\sigma_n = \frac{1}{10{,}000}$, $\kappa > 0$*

arbitrarily small, the following remainder estimate holds for any $k \geq 0$:

$$\epsilon^{-\frac{n}{2}-\gamma}\Big|\partial_x^k R^{u,n} + \sqrt{\epsilon}\partial_x^k R^{v,n}\Big| \leq C(n,\kappa)\epsilon^{\frac{1}{4}-\gamma-\kappa}x^{-k-\frac{3}{2}+2\sigma_n}, \tag{3.3.6}$$

$$\epsilon^{-\frac{n}{2}-\gamma}||\sqrt{\epsilon}\partial_x^k R^{u,n}, \sqrt{\epsilon}\partial_x^k R^{v,n}||_{L^2_y} \leq C(n,\kappa)\epsilon^{\frac{1}{4}-\gamma-\kappa}x^{-k-\frac{5}{4}+2\sigma_n+\kappa}. \tag{3.3.7}$$

The following bounds hold on $[u_R, v_R]$ by construction, for any $[k,j,m] \geq 0$, so long as n is sufficiently large relative to m.

$$||\partial_x^k \partial_y^j v_R^P z^m x^{k+\frac{j}{2}+\frac{1}{2}}||_{L^\infty} \leq C(k,j,m) \text{ if } k \geq 1, \tag{3.3.8}$$

$$||\partial_y^j v_R^P z^m x^{\frac{j}{2}+\frac{1}{2}}||_{L^\infty} \leq C(j,m) \text{ if } j \geq 2, \tag{3.3.9}$$

$$||\partial_y^j v_R^P z^m x^{\frac{j}{2}+\frac{1}{2}}||_{L^\infty} \leq \mathcal{O}(\delta; m, j) \text{ if } j = 0, 1. \tag{3.3.10}$$

$$||\partial_x^k \partial_y^j u_R^P z^m x^{k+\frac{j}{2}}||_{L^\infty} \leq C(k,j,m) \text{ for } k > 1, j \geq 0 \tag{3.3.11}$$

$$||\partial_x u_R^P z^m x||_{L^\infty} \leq \mathcal{O}(\delta; m), \tag{3.3.12}$$

$$||\partial_x \partial_y^j u_R^P z^m x||_{L^\infty} \leq C(m,j) \text{ for } j \geq 1 \tag{3.3.13}$$

$$||\partial_y^j u_R^{P,n-1} y^j z^m||_{L^\infty} \leq \mathcal{O}(\delta; m, j), \text{ for } 0 \leq j \leq 2, \tag{3.3.14}$$

$$||\partial_y^j u_R^{P,n-1} y^j z^m||_{L^\infty} \leq C(m,j), \text{ for } j > 2, \tag{3.3.15}$$

$$||\partial_y^j u_p^n y^j x^{\frac{1}{2}-\sigma_n}||_{L^\infty} \leq C(n,j) \text{ for all } j \geq 0, \tag{3.3.16}$$

$$||\partial_x^k \partial_Y^j v_R^E x^{k+j+\frac{1}{2}}||_{L^\infty} \leq C(k,j) \text{ for } k+j > 0, \tag{3.3.17}$$

$$||\partial_x^k \partial_Y^j u_R^E x^{k+j+\frac{1}{2}}||_{L^\infty} \leq \sqrt{\varepsilon}C(k,j) \text{ for } k+j > 0 \tag{3.3.18}$$

$$||\partial_x^k v_R^E x^{k-\frac{1}{2}} Y||_{L^\infty} \leq C(k,j) \text{ for } k \geq 1, \tag{3.3.19}$$

$$||\{u_R^E - 1, v_R^E\} x^{\frac{1}{2}}, v_{RY}^E x^{\frac{3}{2}}||_{L^\infty} \leq \mathcal{O}(\delta). \tag{3.3.20}$$

The profiles u_R, v_R from (3.3.1) - (3.3.2) arise as coefficients in the linearized problem for the Navier-Stokes remainders $[u, v, P]$, which is to be analyzed in Chapter II. The estimates obtained in (3.3.8) - (3.3.18) are therefore essential to the analysis of Chapter II. In particular, we invite the reader to compare the requirements discussed in (3.2.44) and

(3.2.48) with the estimates we prove in (3.3.8) - (3.3.20).

The structure of this chapter is as follows:

(Step 1) Construction of the zeroeth-order Prandtl layers, $[u_p^0, v_p^0]$ (Section 3.4): The distinguishing feature of u_p^0 are the mismatched boundary conditions, as seen from (3.4.5). As shown in Proposition 3.4.2, this contributes a "front"-profile which looks similar to e_δ, defined in (3.4.11). The decay rates for Prandtl profiles from estimates (3.3.8) - (3.3.18) are dictated by this front profile. For the zeroeth layer, this is formalized in Proposition 3.4.7, Corollaries 3.4.8 and 3.4.9. It is essential that we obtain estimates weighted in the self-similar variable, $z = \frac{y}{\sqrt{x}}$, as can be seen from (3.3.8) - (3.3.20) above.

(Step 2) Construction of Euler-1 layers, $[u_e^1, v_e^1]$ (Section 3.5): Given the boundary conditions, which are known to satisfy the progressive estimate: $|\partial_x^k v_e^1(x,0)| \lesssim x^{-\frac{1}{2}-k}$, we must obtain the sharp uniform estimate $|\partial_x^k v_e^1(x,Y)| \lesssim x^{-\frac{1}{2}-k}$. This is a delicate matter, as these profiles are elliptic, and as indicated in (3.2.48), the required decay rates must be obtained *exactly*. We introduce a method to obtain the required pointwise estimates using directly the Poisson integral formulation, which is carried out in Proposition 3.5.2. This section is a major contribution of Chapter I.

(Step 3) Construction of Prandtl-1 layer, $[u_p^1, v_p^1]$ (Section 3.6): This profile is controlled using coupled energy and positivity estimates, given in Lemmas 3.6.4 and Lemma 3.6.5.

(Step 4) Construction of Intermediate Euler and Prandtl layers (Section 3.7):. The essential mechanism here is as follows: as one consider linearizations for $[u_p^j, v_p^j]$ for $j > 1$, one encounters terms which scale poorly in $z = \frac{y}{\sqrt{x}}$, due to Euler-Euler interactions. However, due to the Cauchy-Riemann structure present in the Euler profiles (see (3.5.7)), we may introduce auxiliary pressures $P_p^{i,a}, P_E^{i,a}$ which creates cancellations of all terms which are "purely-Eulerian". This is seen in (3.7.32) - (3.7.33).

(Step 5) Construction of Final Prandtl layer, $[u_p^n, v_p^n]$ (Subsection 3.7.3): The final Prandtl layer satisfies the boundary condition $v_p^n(x,0) = 0$, and so has a contribution as $y \uparrow \infty$. We cut-off this layer in the region $z \leq \frac{1}{\sqrt{\varepsilon}}$, which honors the parabolic scaling of the Prandtl layers. This is a generalization of the cut-off used in (GN17). It is delicate to ensure these cutoff layers obey desirable estimates, which is done in Lemma 3.7.23. It is also delicate to ensure that this process contributes an error that satisfies estimates (3.3.6) - (3.3.7) above. This is proven in Lemma 3.7.22.

3.4 Asymptotics of Prandtl Layer, u_p^0:

Our starting point is the leading order terms from (3.2.21), which yields the following system for the Prandtl layer, $[u_p^0, v_p^0]$:

$$\Big(1 + u_p^0\Big)u_{px}^0 + \Big(v_p^0 + v_e^1(x,0)\Big)u_{py}^0 = u_{pyy}^0, \qquad u_{px}^0 + v_{py}^0 = 0, \tag{3.4.1}$$

$$u_p^0(x,0) = -\delta, \quad v_p^0(x,0) = -v_e^1(x,0), \quad u_p^0(1,y) = U_0(y). \tag{3.4.2}$$

As shown in (GN17, pg. 9), taking $[u_p^0, v_p^0]$ to solve the system (3.4.1) - (3.4.2) creates an error:

$$R^{u,0} := \epsilon u_{pxx}^0 + \sqrt{\epsilon} y v_{eY}^1 u_{py}^0 + \epsilon u_{py}^0 \int_0^y \int_{y'}^y v_{eYY}^1 \, dy'' dy' \tag{3.4.3}$$

This $R^{u,0}$ contribution is higher-order in ε, and so will be accounted for as a forcing term in the construction of the next Prandtl layer, $[u_p^1, v_p^1]$ (see (3.6.23)). After introducing the von-Mises coordinates,

$$\eta = \int_0^y \Big(1 + u_p^0\Big) dy', \tag{3.4.4}$$

the equation for u_p^0 becomes parabolic, with x being the time-like variable:

$$u_{px}^0 = \partial_\eta((1+u_p^0)u_{p\eta}^0), \quad u_p^0(x,0) = -\delta, \quad u_p^0(x,\infty) = 0, \quad u_p^0(1,\eta) = U_0, \tag{3.4.5}$$

Via the maximum principle, as in (GN17),

$$1 + u_p^0 \geq 1 - \delta. \tag{3.4.6}$$

For the analysis in Section 3.4, it is convenient to introduce the shifted unknown

$$q = u_p^0 + \delta. \tag{3.4.7}$$

The shifted unknown then satisfies the IBVP:

$$q_x = \partial_\eta\Big((1-\delta+q)q_\eta\Big), \quad q(x,0) = 0, \quad q(x,\infty) = \delta, \quad q(1,\eta) = u_p^0(1,\eta) + \delta. \tag{3.4.8}$$

3.4.1 Existence of Front Profile

The dynamics of the solution to equations (3.4.1) - (3.4.2), or equivalently, (3.4.8), are governed by a self-similar "front", in the sense of (BKL94). This is due to the mismatch in boundary conditions at $y = 0$ and $y = \infty$, as seen from (3.4.8). Inserting a self-similar anzatz $\phi_*(z) = \phi_*(\frac{\eta}{\sqrt{x}})$ into equation (3.4.8) gives the following ODE:

$$(1-\delta+\phi_*)\phi_*'' + \Big|\phi_*'\Big|^2 + \frac{z}{2}\phi_*' = 0, \quad \phi_*(0) = q(x,0) = 0, \quad \phi_*(\infty) = q(x,\infty) = \delta. \tag{3.4.9}$$

Here the $'$ denotes ∂_z, where z is the self-similar variable for ϕ_*. As the profile ϕ_* that we seek is a nonlinear variant of the Gaussian error function, we study:

$$\psi = \phi_* - e_\delta, \tag{3.4.10}$$

where e_δ is the Gaussian front profile with value δ at $+\infty$:

$$e_\delta(z) = \frac{\delta}{\sqrt{\pi}} \int_0^z e^{-\frac{t^2}{4}}\, dt. \tag{3.4.11}$$

Let us record the following:

Lemma 3.4.1. *With e_δ being defined as (3.4.11), $e_\delta(\frac{\eta}{\sqrt{x}})$ is an explicit solution of the heat equation, which bridges two distinct boundary conditions at $y = 0$ and $y = \infty$:*

$$\Big(\partial_x - \partial_{\eta\eta}\Big) e_\delta(\frac{\eta}{\sqrt{x}}) = 0, \quad e_\delta(0) = 0, \quad e_\delta(\infty) = \delta. \tag{3.4.12}$$

Proof. We first record the identities:

$$\frac{\partial z}{\partial \eta} = \frac{1}{\sqrt{x}}, \quad \frac{\partial z}{\partial x} = -\frac{z}{2x}. \tag{3.4.13}$$

Differentiating (3.4.11) gives the identities:

$$e'_\delta(z) = \frac{\delta}{\sqrt{\pi}} e^{-\frac{z^2}{4}}, \quad e''_\delta(z) = -\frac{z}{2x} \frac{\delta}{\sqrt{\pi}} e^{-\frac{z^2}{4}} = -\frac{z}{2x} e'_\delta(z). \tag{3.4.14}$$

One then checks that: $\partial_x e_\delta(\frac{\eta}{\sqrt{x}}) = \partial_{\eta\eta} e_\delta(\frac{\eta}{\sqrt{x}})$ is equivalent to (3.4.14). The boundary condition at 0 is trivial from (3.4.11), and the boundary condition at ∞ arises from: $e_\delta(\infty) = \frac{\delta}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4}}\, dt = \delta$. The lemma is proven.

$\square$

Using the heat equation for e_δ, coupled with (3.4.10) we obtain:

$$\Big(1-\delta+\psi\Big)\psi''+\Big|\psi'\Big|^2+\frac{z}{2}\psi'=-2\psi'e_\delta'-|e_\delta'|^2-\psi e_\delta''-e_\delta\psi''-e_\delta e_\delta'',$$
$$\psi(0)=\psi(\infty)=0. \tag{3.4.15}$$

The first task is to obtain existence of a solution, ψ, to the above boundary value problem in a suitable Sobolev space.[2]

Proposition 3.4.2. *For δ sufficiently small, there exists a unique solution to the equation (3.4.15) in $H^2_w(\mathbb{R}_+)$ satisfying $||\psi||_{H^2_w}\lesssim\delta$, where H^2_w is a weighted variant of H^2 which is formally defined in (3.4.24)*

Proof. For this argument, fix $r>0$. We will eventually let $r\to\infty$. Our starting point is to establish existence of solutions to the linear operator

$$\Big(-\partial_z^2-\frac{z}{2}\partial_z\Big). \tag{3.4.16}$$

We recall the identity given in (BKL94):

$$e^{\frac{z^2}{8}}\Big(-\partial_z^2-\frac{z}{2}\partial_z\Big)e^{-\frac{z^2}{8}}\psi=\Big(-\partial_z^2+\frac{z^2}{16}+\frac{1}{4}\Big)\psi, \tag{3.4.17}$$

which in turn implies

$$\Big(-\partial_z^2-\frac{z}{2}\partial_z\Big)\psi=f,\quad \psi(0)=\psi(r)=0 \tag{3.4.18}$$

[2] It is clear by rescaling $z\to(1-\delta)^{\frac{1}{2}}z$, we can replace the factor of $1-\delta$ in front of ψ'' by simply 1. This rescaling would change the main linear operator, $(1-\delta)\psi''+\frac{z}{2}\psi'$ to $\psi''+\frac{z}{2}\psi'$. For notational ease, then, we work simply with the ψ'' instead of $(1-\delta)\psi''$. The actual self-similar variable, then, is really $(1-\delta)\frac{\eta}{\sqrt{x}}$, but as $(1-\delta)$ is near 1, this causes no confusion in the analysis to follow.

if and only if

$$\Big(-\partial_z^2+\frac{z^2}{16}+\frac{1}{4}\Big)\tilde{\psi}_{(r)}=\tilde{f},\quad \tilde{\psi}_{(r)}(0)=\tilde{\psi}_{(r)}(r)=0,\quad \tilde{\psi}_{(r)}=e^{\frac{z^2}{8}}\psi_{(r)},\quad \tilde{f}=e^{\frac{z^2}{8}}f. \tag{3.4.19}$$

The subscripts are included to emphasize the domain, $(0,r)\subset\mathbb{R}$. Consider:

$$B[\tilde{\psi}_{(r)},v]:=\int_0^r \partial_z\tilde{\psi}_{(r)}\cdot\partial_z v+\int_0^r\Big(\frac{z^2}{16}+\frac{1}{4}\Big)\tilde{\psi}_{(r)}v,\quad H_0^1(0,r)\times H_0^1(0,r)\to\mathbb{R}. \tag{3.4.20}$$

B is clearly bounded and coercive on $H_0^1(0,r)$. One obtains the existence of a unique H^1 weak solution to the system (3.4.19), and correspondingly to (3.4.18) from the Lax-Milgram Lemma. Via elliptic regularity, this implies H^2 regularity which, in original unknowns, translates to the existence of a unique $\psi_{(r)}\in H^2(0,r)$ for each $f\in L^2(0,r)$ such that:

$$||\psi_{(r)}||_{H^2(0,r)}\leq C(r)||f||_{L^2(0,r)},\ \text{and}\ \psi_{(r)}(0)=\psi_{(r)}(r)=0. \tag{3.4.21}$$

Let us rewrite the nonlinear problem (3.4.15) as a fixed point to:

$$-\psi''_{(r)}-\frac{z}{2}\psi'_{(r)}=f(\overline{\psi}_{(r)})+(e_\delta+\delta)\overline{\psi}''_{(r)},\quad \psi_{(r)}(0)=\psi_{(r)}(r)=0, \tag{3.4.22}$$

where

$$f(\overline{\psi}_{(r)})=\overline{\psi}_{(r)}\overline{\psi}''_{(r)}+\Big|\overline{\psi}'_{(r)}\Big|^2+2e_\delta\overline{\psi}'_{(r)}+\Big|e'_\delta\Big|^2+e''_\delta\overline{\psi}_{(r)}+e_\delta e''_\delta. \tag{3.4.23}$$

Define now the norm:

$$||\psi_{(r)}||^2_{H^2_w}:=\int|\psi''_{(r)}|^2+\int(1+z^2)\Big|\psi_{(r)},\psi'_{(r)}\Big|^2, \tag{3.4.24}$$

and the parameter:

$$R(\delta) = \max\Big\{||e_\delta||_{L^\infty}, ||e'_\delta||_{L^\infty}, ||e''_\delta||_{L^\infty}, ||(1+z^2)^{\frac{1}{2}}e''_\delta||_{L^2}, ||e'_\delta(1+z^2)^{\frac{1}{2}}||_{L^2}\Big\} \qquad (3.4.25)$$

Via a standard integration by parts argument applied to (3.4.22), we obtain the stabilizing estimate:

$$||\psi_{(r)}||^2_{H^2_w} \leq R(\delta)^4 + R(\delta)^2||\overline{\psi}_{(r)}||^2_{H^2_w} + ||\overline{\psi}_{(r)}||^4_{H^2_w}, \qquad (3.4.26)$$

which proves that if $||\overline{\psi}_{(r)}||_{H^2_w} \leq R(\delta)$, then

$$||\psi_{(r)}||^2_{H^2_w} \leq R(\delta)^4 + 2R(\delta)^2||\overline{\psi}_{(r)}||^2_{H^2_w} \qquad (3.4.27)$$

Selecting δ small enough then ensures $3R(\delta)^4 < R(\delta)^2$, ensuring that

$$\psi_{(r)} = N(\overline{\psi_{(r)}}) \in B_{R(\delta)} \subset H^2_w \text{ whenever } \overline{\psi_{(r)}} \in B_{R(\delta)} \subset H^2_w. \qquad (3.4.28)$$

The second step is to prove the nonlinear map N is a contraction map on $B_{R(\delta)} \subset H^2_w$. As such, label the pairs:

$$-\psi''_{(i,r)} - \frac{z}{2}\psi'_{(i,r)} = f(\overline{\psi_{(i,r)}}) + \Big(e_\delta + \delta\Big)\overline{\psi}''_{(r)}, \quad \psi_{(i,r)}(0) = \psi_{(i,r)}(r) = 0, \quad i = 1, 2. \quad (3.4.29)$$

Taking differences yields and performing a standard integration by parts argument gives:

$$||\psi_{1,r} - \psi_{2,r}||^2_{H^2_w} \leq R(\delta)^2||\overline{\psi}_{1,r} - \overline{\psi}_{2,r}||^2_{H^2_w}, \qquad (3.4.30)$$

By the contraction mapping theorem there exists a unique solution to the nonlinear problem (3.4.22) - (3.4.23). Moreover, as this solution lies in the ball $B_{R(\delta)}$, it obeys the estimate

$$||\psi_{(r)}||_{H^2_w(0,r)} \leq R(\delta), \tag{3.4.31}$$

uniformly in r. Therefore, we may let $r \to \infty$ to obtain a solution to the problem (3.4.15).

$\square$

We may repeat the procedure in the above theorem with any weight, giving:

$$||\langle z\rangle^M \psi||_{H^2} \leq \mathcal{O}(\delta; M). \tag{3.4.32}$$

By further differentiating equation (3.4.15), we can obtain

$$||\langle z\rangle^M \psi||_{H^k} \leq \mathcal{O}(\delta; M, k). \tag{3.4.33}$$

Our front from here on will be denoted as $\phi^* = \psi + e_\delta$. We will abuse notation and depict

$$\phi^*(z) = \phi^*(x, \eta). \tag{3.4.34}$$

We have the following corollary to the preceding analysis:

Corollary 3.4.3. *The front ϕ^* obeys the following bounds, for any $m, k, l \geq 0$:*

$$||z^m \partial_\eta^l \partial_x^k \Big(\phi^* - \delta\Big)||_{L^p_\eta} \leq \mathcal{O}(\delta) x^{-k-\frac{l}{2}+\frac{1}{2p}} \tag{3.4.35}$$

Proof. This follows immediately from writing $\phi^* - \delta = \psi + (e_\delta - \delta)$, the definition of e_δ

in (3.4.11), the bounds in (3.4.33), and then the chain rule together with the identities (3.4.13). $\square$

Let us now record the following fact:

Corollary 3.4.4. *A solution ϕ_* to the system (3.4.9) must satisfy $\phi_*'(0) > 0$.*

Proof. Define $u_f = \phi_*, v_f = \phi_*'$. By (3.4.35), $|u_f| \le c_0\delta$ for some constant c_0. The system (3.4.9) is then: $u_f' = v_f$, $(1-\delta+u_f)v_f' + |v_f|^2 + \frac{z}{2}v_f = 0$. The invariant set $\Gamma = \{u_f = C, v_f = 0\}$, for $-c_0\delta \le C \le c_0\delta$ contains equilibria. The solution ϕ_* corresponds to an orbit with initial condition $(u_f(0) = 0, v_f(0))$ and final condition $(u_f(\infty) = \delta, v_f(\infty))$. By ODE uniqueness, the trajectory cannot cross the set Γ unless at $(\delta, 0)$. Thus, if $v_f(0) \le 0$, $v_f(z) \le 0$ for all z, which violates $u_f(\infty) = \delta$. Thus, we must have $v_f(0) > 0$. $\square$

3.4.2 Zeroeth Prandtl Layer, u_p^0

We define the remainder, w, via:

$$w = q - \phi^*(\cdot), \quad g := w(1,\cdot) = q(1,\cdot) - \phi^*(\cdot) = U_0(1,\cdot) + \delta - \phi^*(\cdot). \tag{3.4.36}$$

The initial data in (3.4.36) are clearly rapidly decaying after recalling (3.2.26) - (3.2.28), via the relation: $g = U_0 - (e_\delta - \delta) - \psi$. Moreover, the following smallness is obtained, again by recalling the assumptions (3.2.26) - (3.2.28):

$$||\langle y\rangle^m \partial_y^j g||_{L^\infty} \le \mathcal{O}(\delta; j, m) \text{ for any } m, \text{ and } j = 0, 1, 2. \tag{3.4.37}$$

The equation satisfied by w is the following (after substituting $q = w + \phi^*$):

$$w_x = (1-\delta)w_{\eta\eta} + \partial_\eta\Big(qw_\eta + w\partial_\eta\phi^*\Big) = (1-\delta)w_{\eta\eta} + K, \tag{3.4.38}$$

$$w(x,0) = 0, \quad w(x,\infty) = 0, \quad w(1,\eta) = g(\eta), \tag{3.4.39}$$

where

$$K = w_\eta^2 + 2\phi^*_\eta w_\eta + ww_{\eta\eta} + \phi^* w_{\eta\eta} + w\phi^*_{\eta\eta}. \tag{3.4.40}$$

Let us first make the following basic observation regarding our initial data:

Lemma 3.4.5 (Compatibility of Initial Data)**.** *Suppose the compatibility conditions in (3.2.31) are enforced. Then the initial data of w_x respects the boundary condition $w_x(1,0) = 0$.*

Proof. This follows upon evaluating (3.4.1) at $y = 0$ and (3.4.8) at $\eta = 0, x = 1$. ☐

Remark (Higher Order Compatibility)*.* This same calculation for higher-order compatibility conditions at $x = 1, y = 0$ can be made. We omit displaying them explicitly, but will feel free to assume higher-order compatibility of the initial data at $y = 0$ as needed.

Let us now define a series of norms in which we control the remainder, w:

$$||w||^2_{Q(0,0)} := \sup_{x\ge 1} \int w^2, w_\eta^2 x, w_{\eta\eta}^2 x^2 + \int_1^\infty \int w_\eta^2, w_{\eta\eta}^2 x, w_{\eta\eta\eta}^2 x^2. \tag{3.4.41}$$

We shall need the general, differentiated and weighted variant of the above norms:

$$\begin{aligned} ||w||^2_{Q(\sigma_0,k)} := \sup_{x\ge 1} \int \Big|\partial_x^k w\Big|^2 x^{2k-2\sigma_0}, \Big|\partial_x^k w_\eta\Big|^2 x^{2k+1-2\sigma_0}, \Big|\partial_x^k w_{\eta\eta}\Big|^2 x^{2k+2-2\sigma_0} \\ + \int_1^\infty \int \Big|\partial_x^k w_\eta\Big|^2 x^{2k-2\sigma_0}, \Big|\partial_x^k w_{\eta\eta}\Big|^2 x^{2k+1-2\sigma_0}, \Big|\partial_x^k w_{\eta\eta\eta}\Big|^2 x^{2k+2-2\sigma_0}. \end{aligned} \tag{3.4.42}$$

Remark. One should take note of several points. First, each application of ∂_x to the system (3.4.38) adds enhanced decay of x^{-1}, which is reflected in the norms above. This is because of the behavior of the profiles ϕ^* under the application of ∂_x, see estimate (3.4.35).

Second, upon controlling w in the Q-norms in (3.4.41), the dynamics of $q = w + \phi^*$ will be dominated by those of the front ϕ^* (so long as the parameter $\sigma_0 < \frac{1}{4}$, which we will enforce). The small parameter σ_0 arises for technical reasons when performing weighted estimates, but should be ignored for the unweighted estimates.

The final point is that upon heuristically identifying $w_x \approx w_{\eta\eta}$ through the equation (3.4.38), one sees that the norms $Q(\sigma_0, k)$ have an iterative structure:

$$\int |\partial_x^{k+1} w|^2 x^{2(k+1)} \, \mathrm{d}\eta \approx \int |\partial_x^k w_{\eta\eta}|^2 x^{2k+2} \, \mathrm{d}\eta, \tag{3.4.43}$$

the quantity on the left-hand side above being in $||w||_{Q(0,k+1)}$ and the quantity on the right-hand side above being in $||w||_{Q(0,k)}$ (upon taking sup in x). The reason for this structure is that the equation (3.4.38) is quasilinear.

Through standard Sobolev interpolation, it is clear that:

Lemma 3.4.6. *For any $\sigma_0 \geq 0$, and $k \geq 0$,*

$$\sup_x ||\partial_x^k w x^{\frac{1}{4} - \sigma_0 + k}||_{L^\infty_\eta} + \sup_x ||\partial_x^k w_\eta x^{\frac{3}{4} - \sigma_0 + k}||_{L^\infty_\eta} \lesssim ||w||_{Q(\sigma_0, k)}. \tag{3.4.44}$$

Proof. As the profile w decays at $\eta \to \infty$ for each fixed x, we have:

$$\begin{aligned} |\partial_x^k w|^2 x^{\frac{1}{2} - 2\sigma_0 + 2k} = \Big| 2 \int_\eta^\infty \partial_x^k w \partial_x^k w_\eta x^{\frac{1}{2} - 2\sigma_0 + 2k} d\eta' \Big| &\lesssim ||w x^{k-\sigma_0}||_{L^2_\eta} ||w_\eta x^{k - \sigma_0 + \frac{1}{2}}||_{L^2_\eta} \\ &\lesssim ||w||^2_{Q(\sigma_0, k)}. \end{aligned} \tag{3.4.45}$$

A similar computation works for the $\partial_x^k w_\eta$ term in (3.4.44).

$\square$

We now give the energy estimates for w:

Proposition 3.4.7. *For any $\sigma_0 > 0$, and $m \geq 0$, w satisfies the following estimate:*

$$||z^m w||_{Q(\sigma_0,0)} \leq \mathcal{O}(\delta; m, \sigma_0), \tag{3.4.46}$$

$$||z^m w||_{Q(\sigma_0,k)} \leq C(k, m, \sigma_0), \text{ for } k > 0. \tag{3.4.47}$$

Remark. Notice that the weights, $z = \frac{\eta}{\sqrt{x}}$, honor the parabolic scaling of (3.4.38), and are propagated by the linear flow.

Proof. The proof proceeds in several steps.

Step 1: Multiplier $M = w$:

Applying the multiplier $M = w$ to the system (3.4.38) generates the following positive terms:

$$\int \Big(w_x - (1-\delta) w_{\eta\eta} \Big) \cdot w = \frac{\partial_x}{2} \int w^2 + (1-\delta) \int w_\eta^2. \tag{3.4.48}$$

Next, we must treat the nonlinear and linearized terms in K. The boundary condition $w(x,0) = 0$ enables us to integrate by parts the quasilinear term:

$$\Big| \int w_\eta^2 \cdot w + w w_{\eta\eta} \cdot w \Big| = \Big| \int w w_\eta^2 \Big| \lesssim ||w||_{L^\infty_\eta} \Big(\int w_\eta^2 \Big), \tag{3.4.49}$$

$$\begin{aligned} \Big| \int \phi^* w_{\eta\eta} \cdot w + \int \phi^*_{\eta\eta} w \cdot w + \int 2\phi^*_\eta w_\eta \cdot w \Big| &= \Big| \int \phi^*_{\eta\eta} w^2 + \int \phi^* w_\eta^2 \Big| \\ \leq ||\phi^*_{\eta\eta} \eta^2||_{L^\infty_\eta} \int \frac{w^2}{\eta^2} + ||\phi^*||_{L^\infty} ||w_\eta||^2_{L^2_\eta} &\leq \mathcal{O}(\delta) \int w_\eta^2. \end{aligned} \tag{3.4.50}$$

Above, we have used Corollary 3.4.3 to absorb two factors of η to into $\phi^*_{\eta\eta}$. We have also used the Hardy inequality, which is available as $w(x,0)=0$. Summarizing, we have:

$$\partial_x \int w^2 + \int w_\eta^2 \lesssim \Big[\mathcal{O}(\delta) + ||w||_{L^\infty_\eta}\Big] \int w_\eta^2. \tag{3.4.51}$$

Step 2: Multiplier $M = -w_{\eta\eta}x$

Next, we will apply the multiplier $M = -w_{\eta\eta}x$. This generates the positive terms:

$$-\int \Big(\partial_x - (1-\delta)\partial_{\eta\eta}\Big) w \cdot w_{\eta\eta}x = \frac{\partial_x}{2}\int w_\eta^2 x - \int w_\eta^2 + (1-\delta)\int w_{\eta\eta}^2 x. \tag{3.4.52}$$

Let us now turn to the terms in K:

$$\Big|\int w_\eta^2 w_{\eta\eta}x + \int w w_{\eta\eta}^2 x\Big| \le ||w, w_\eta x^{\frac{1}{2}}||_{L^\infty_\eta}\Big(||w_\eta||^2_{L^2_\eta} + ||x^{\frac{1}{2}} w_{\eta\eta}||^2_{L^2_\eta}\Big), \tag{3.4.53}$$

$$\begin{aligned} &\Big|\int \phi^*_\eta w_\eta w_{\eta\eta}x + \int \phi^* w_{\eta\eta}^2 x + \int \phi^*_{\eta\eta} w w_{\eta\eta} x\Big| \\ &\qquad \le ||\phi^*, \eta x^{\frac{1}{2}}\phi^*_{\eta\eta}, x^{\frac{1}{2}}\phi^*_\eta||_{L^\infty}\Big(||w_\eta||^2_{L^2_\eta} + ||w_{\eta\eta}x^{\frac{1}{2}}||^2_{L^2_\eta}\Big) \\ &\qquad \le \mathcal{O}(\delta)\Big(||w_\eta||^2_{L^2_\eta} + ||w_{\eta\eta}x^{\frac{1}{2}}||^2_{L^2_\eta}\Big). \end{aligned} \tag{3.4.54}$$

Summarizing, we have:

$$\partial_x \int w_\eta^2 x + \int w_{\eta\eta}^2 x \lesssim \Big(1 + ||w_\eta x^{\frac{1}{2}}||_{L^\infty_\eta}\Big)\int w_\eta^2 + ||w, w_\eta x^{\frac{1}{2}}||_{L^\infty_\eta}\int w_{\eta\eta}^2 x. \tag{3.4.55}$$

Step 3: Multiplier $w_x x^2$

The third step is to take ∂_x of the equation (3.4.38). Doing so gives the system:

$$\Big(\partial_x - (1-\delta)\partial_{\eta\eta}\Big) w_x = \partial_x K, \quad w_x(x,0) = 0, \tag{3.4.56}$$

$$w_x(1,\cdot) = (1-\delta) g_{\eta\eta} + g_\eta^2 + 2\phi^*_\eta g_\eta + g g_{\eta\eta} + \phi^* g_{\eta\eta} + \phi^*_{\eta\eta} g. \tag{3.4.57}$$

First, we will record that $||\langle \eta \rangle^m w_x(1,\eta)||_{L^\infty} \lesssim \mathcal{O}(\delta; m)$, for any $m \geq 0$, by (3.4.57) and (3.2.26) - (3.2.28). Let us now apply the multiplier $M = w_x x^2$ to the system (3.4.56). Again, we will remain cognizant of the boundary condition $w_x(x,0) = 0$ when integrating by parts. Doing so gives the positive terms:

$$\int \Big(\partial_x - (1-\delta)\partial_{\eta\eta}\Big) w_x \cdot w_x x^2 = \partial_x \int w_x^2 x^2 - \int w_x^2 x + (1-\delta) \int w_{\eta x}^2 x^2. \tag{3.4.58}$$

Next, we come to the nonlinearity in K_x, where integrating by parts as necessary:

$$\Big| \int w_\eta w_{\eta x} w_x x^2 + w_x w_{\eta\eta} w_x x^2 + w w_{\eta\eta x} w_x x^2 \Big|$$

$$\leq ||w_\eta x^{\frac{1}{2}}||_{L^\infty_\eta} ||w_{\eta x} x||_{L^2_\eta} ||w_x x^{\frac{1}{2}}||_{L^2_\eta} + ||w||_{L^\infty} ||w_{\eta x} x||^2_{L^2_\eta}, \tag{3.4.59}$$

$$\Big| \int \phi^*_{\eta x} w_\eta w_x x^2 + \int \phi^*_\eta w_{\eta x} w_x x^2 \Big|$$

$$\leq ||\phi^*_{\eta x} x \eta||_{L^\infty_\eta} ||w_\eta||_{L^2_\eta} ||w_{x\eta} x||_{L^2_\eta} + ||\phi^*_\eta \eta||_{L^\infty_\eta} ||w_{\eta x} x||^2_{L^2_\eta},$$

$$\leq \mathcal{O}(\delta) ||w_\eta||_{L^2_\eta} ||w_{x\eta} x||_{L^2_\eta} + \mathcal{O}(\delta) ||w_{\eta x} x||^2_{L^2_\eta} \tag{3.4.60}$$

$$\Big| \int \phi^*_x w_{\eta\eta} w_x x^2 + \int \phi^* w_{\eta\eta x} w_x x^2 \Big|$$

$$\leq ||\phi^*_x x||_{L^\infty_\eta} ||w_{\eta\eta} x^{\frac{1}{2}}||_{L^2_\eta} ||w_x x^{\frac{1}{2}}||_{L^2_\eta} + ||\phi^*, \phi^*_\eta \eta||_{L^\infty_\eta} ||w_{\eta x} x||^2_{L^2_\eta},$$

$$\leq \mathcal{O}(\delta) ||w_{\eta\eta} x^{\frac{1}{2}}||_{L^2_\eta} ||w_x x^{\frac{1}{2}}||_{L^2_\eta} + \mathcal{O}(\delta) ||w_{\eta x} x||^2_{L^2_\eta} \tag{3.4.61}$$

$$\Big| \int \phi^*_{\eta\eta x} w w_x x^2 + \int \phi^*_{\eta\eta} w_x^2 x^2 \Big|$$

$$\leq ||\phi^*_{\eta\eta x} \eta x^{\frac{3}{2}}||_{L^\infty_\eta} ||w_x x^{\frac{1}{2}}||_{L^2_\eta} ||w_\eta||_{L^2_\eta} + ||\phi^*_{\eta\eta} x||_{L^\infty} ||w_x x^{\frac{1}{2}}||^2_{L^2}$$

$$\leq \mathcal{O}(\delta)||w_x x^{\frac{1}{2}}||_{L^2_\eta}||w_\eta||_{L^2_\eta} + \mathcal{O}(\delta)||w_x x^{\frac{1}{2}}||^2_{L^2_\eta}. \tag{3.4.62}$$

Summarizing this piece:

$$\begin{aligned}\partial_x \int w_x^2 x^2 + \int w_{\eta x}^2 x^2 &\lesssim \Big[\mathcal{O}(\delta) + ||w, w_\eta x^{\frac{1}{2}}||_{L^\infty_\eta}\Big] \int w_{\eta x}^2 x^2 \\ &+ \Big[1 + \mathcal{O}(\delta) + ||w, w_\eta x^{\frac{1}{2}}||_{L^\infty_\eta}\Big] \int w_x^2 x + \mathcal{O}(\delta) \int w_\eta^2, w_{\eta\eta}^2 x.\end{aligned} \tag{3.4.63}$$

With these estimates in hand, we may now apply a standard continuous induction argument to conclude the global existence of w satisfying (3.4.46) with $k = m = \sigma_0 = 0$.

Step 4: Weighted Estimates

We now apply the weighted multiplier $M = x^{-2\sigma_0} z^m w$ to the system (3.4.38). First, we have the following positive terms:

$$\begin{aligned}\int \Big(w_x - w_{\eta\eta}\Big) \cdot w z^{2m} x^{-2\sigma_0} &= \frac{\partial_x}{2} \int w^2 z^{2m} x^{-2\sigma_0} + \Big(\frac{m}{2} + \sigma_0\Big) \int w^2 z^{2m} x^{-1-2\sigma_0} \\ &+ \int w_\eta^2 z^{2m} x^{-2\sigma_0} - m(2m-1) \int w^2 z^{2m-2} x^{-1-2\sigma_0}.\end{aligned} \tag{3.4.64}$$

The last term above has been estimated inductively at the $m-1$'th iterate. Next, we address the terms in K:

$$\Big| \int \Big(w_\eta^2 + w w_{\eta\eta}\Big) \cdot w z^{2m} x^{-\sigma_0} \Big| \lesssim ||w||_{L^\infty_\eta} ||w_\eta z^m x^{-\sigma_0}||^2_{L^2_\eta}, \tag{3.4.65}$$

$$\Big| \int \Big(2\phi^*_\eta w_\eta + \phi^* w_{\eta\eta} + w \phi^*_{\eta\eta}\Big) \cdot z^{2m} x^{-2\sigma_0} w \Big| \lesssim ||\eta \phi^*_\eta, \eta^2 \phi^*_{\eta\eta}||_{L^\infty_\eta} ||w_\eta z^m x^{-\sigma_0}||^2_{L^2_\eta}. \tag{3.4.66}$$

Summarizing:

$$\partial_x \int w^2 z^{2m} x^{-2\sigma_0} + \int w_\eta^2 z^{2m} x^{-2\sigma_0} + \int w^2 z^{2m} x^{-1-2\sigma_0} \lesssim \int w^2 z^{2m-2} x^{-1-2\sigma_0}. \quad (3.4.67)$$

By integrating in x:

$$\begin{aligned} \sup_{x\geq 1} \int w^2 z^{2m} x^{-2\sigma_0} + \int\int w_\eta^2 z^{2m} x^{-2\sigma_0} + \int\int w^2 z^{2m} x^{-1-2\sigma_0} \\ \lesssim \int_{x=1} w^2 y^{2m} + \int\int w^2 z^{2m-2} x^{-1-2\sigma_0} \leq \mathcal{O}(\delta, m, \sigma_0). \end{aligned} \quad (3.4.68)$$

The next step is to apply the multiplier $M = -w_{\eta\eta} x^{1-2\sigma_0} z^{2m}$. First, this gives:

$$\begin{aligned} -\int \Big(\partial_x - \partial_{\eta\eta}\Big) w \cdot w_{\eta\eta} z^{2m} x^{1-2\sigma_0} = \\ \frac{\partial_x}{2} \int w_\eta^2 z^{2m} x^{1-2\sigma_0} + \int w_{\eta\eta}^2 z^{2m} x^{1-2\sigma_0} + \frac{m}{2} \int w_\eta^2 z^{2m} x^{-2\sigma_0} \\ - \frac{1-2\sigma_0}{2} \int w_\eta^2 z^{2m} x^{-2\sigma_0} + 2m \int w_\eta w_x z^{2m-1} x^{\frac{1}{2}-2\sigma_0}. \end{aligned} \quad (3.4.69)$$

Next, let us turn to the nonlinear terms in K:

$$\begin{aligned} \Big| \int \Big(w_\eta^2 + w w_{\eta\eta} \Big) \cdot w_{\eta\eta} z^{2m} x^{1-2\sigma_0} \Big| \leq ||w, w_\eta x^{\frac{1}{2}}||_{L^\infty_\eta} \Big(||w_\eta z^m x^{-\sigma_0}||^2_{L^2_\eta} \times \\ ||w_{\eta\eta} z^m x^{\frac{1}{2}-\sigma_0}||^2_{L^2_\eta} \Big), \end{aligned} \quad (3.4.70)$$

and the linearized terms:

$$\begin{aligned} \Big| \int \Big(\phi^*_\eta w_\eta + \phi^* w_{\eta\eta} + \phi^*_{\eta\eta} w \Big) \cdot w_{\eta\eta} z^{2m} x^{1-2\sigma} \Big| \\ \leq ||x^{\frac{1}{2}} \phi^*_\eta, x^{\frac{1}{2}} \eta \phi^*_{\eta\eta}, \phi^*||_{L^\infty} ||w_\eta x^{-\sigma_0} z^m||_{L^2_\eta} ||w_{\eta\eta} z^m x^{\frac{1}{2}-\sigma_0}||_{L^2_\eta}. \end{aligned} \quad (3.4.71)$$

Summarizing,

$$\partial_x \int w_\eta^2 z^{2m} x^{1-2\sigma_0} + \int w_{\eta\eta}^2 z^{2m} x^{1-2\sigma_0} + \int w_\eta^2 z^{2m} x^{-2\sigma_0} \lesssim \int w_\eta w_x z^{2m-1} x^{\frac{1}{2}-2\sigma_0} \tag{3.4.72}$$

$$\lesssim \frac{1}{10,000} \int w_\eta^2 z^{2m} x^{-2\sigma_0} + C \int w_x^2 x^{1-2\sigma_0} z^{2m-2}. \tag{3.4.73}$$

Integrating in x:

$$\sup_{x\geq 1} \int w_\eta^2 z^{2m} x^{1-2\sigma_0} + \int_1^\infty \int w_{\eta\eta}^2 z^{2m} x^{1-2\sigma_0} + \int_1^\infty \int w_\eta^2 z^{2m} x^{-2\sigma_0}$$
$$\lesssim \int_{x=1} w_\eta^2 y^{2m} + \int \int w_x^2 x^{1-2\sigma_0} z^{2m-2} \leq \mathcal{O}(\delta; \sigma_0, m). \tag{3.4.74}$$

The final step is to apply the multiplier $M = w_x x^{2-2\sigma_0} z^{2m}$ to the system (3.4.56). This generates the following terms:

$$\int \Big(\partial_x - \partial_{\eta\eta}\Big) w_x \cdot w_x x^{2-2\sigma_0} z^{2m} = \frac{\partial_x}{2} \int w_x^2 x^{2-2\sigma_0} z^{2m} + \int w_{\eta x}^2 x^{2-2\sigma_0} z^{2m}$$
$$+ \Big(\frac{m}{2} - 1 + \sigma_0\Big) \int w_x^2 x^{1-2\sigma_0} z^{2m} - m(2m-1) \int w_x^2 x^{1-2\sigma_0} z^{2m-2}. \tag{3.4.75}$$

We turn to the nonlinearities contained in K_x, for which we integrate by parts as needed:

$$\Big| \int \Big(w_\eta w_{\eta x} + w_x^2 w_{\eta\eta} + w w_{\eta\eta x} w \Big) \cdot w_x x^{2-2\sigma_0} z^{2m} \Big| \tag{3.4.76}$$
$$\lesssim ||w, w_\eta x^{\frac{1}{2}}||_{L^\infty} \Big(||w_x x^{\frac{1}{2}-\sigma_0} z^m||_{L^2_\eta}^2 + ||w_x x^{\frac{1}{2}-\sigma_0} z^{m-1}||_{L^2_\eta}^2 + ||w_x x^{\frac{1}{2}-\sigma_0} z^m||_{L^2_\eta}^2 \Big).$$

Next, we come to the linearizations around the front profile, ϕ^*:

$$\Big| \int \phi^*_{\eta x} w_\eta w_x x^{2-2\sigma_0} z^{2m} + \int \phi^*_\eta w_{\eta x} w_x x^{2-2\sigma_0} z^{2m} \Big|$$
$$\leq ||\phi^*_{\eta x} \eta x z^{2m}, \phi^*_\eta \eta z^{2m}||_{L^\infty} \Big(||w_\eta||_{L^2_\eta}^2 + ||w_{x\eta} x||_{L^2_\eta}^2 \Big), \tag{3.4.77}$$

$$\Big| \int \phi^*_x w_{\eta\eta} w_x x^{2-2\sigma_0} z^{2m} \Big| \le ||z^{2m} x \phi^*_x||_{L^\infty} \Big(||w_{\eta\eta} x^{\frac{1}{2}-\sigma_0}||^2_{L^2_\eta} + ||w_x x^{\frac{1}{2}-\sigma_0}||^2_{L^2_\eta} \Big), \tag{3.4.78}$$

$$\Big| \int \phi^* w_{x\eta\eta} w_x x^{2-2\sigma_0} z^{2m} \Big| = \Big| \int \phi^*_\eta w_x w_{x\eta} x^{2-2\sigma_0} z^{2m} + 2m \int \phi^* w_x w_{x\eta} x^{\frac{3}{2}-2\sigma_0} z^{2m-1} \Big|$$
$$\lesssim ||\eta \phi^*_\eta z^{2m}, \phi^*||_{L^\infty} \Big(||x w_{x\eta}||^2_{L^2_\eta} + ||x^{1-\sigma_0} w_{x\eta} z^m||^2_{L^2_\eta} + ||z^{m-1} w_x x^{\frac{1}{2}-\sigma_0}||^2_{L^2_\eta} \Big), \tag{3.4.79}$$

$$\Big| \int \phi^*_{\eta\eta x} w w_x x^{2-2\sigma_0} z^{2m} + \int \phi^*_{\eta\eta} w_x^2 x^{2-2\sigma_0} z^{2m} \Big|$$
$$\lesssim ||z^{2m} x \eta^2 \phi^*_{\eta\eta x}, z^{2m} \eta^2 \phi^*_{\eta\eta}||_{L^\infty} \Big(||w_\eta||^2_{L^2_\eta} + ||w_{x\eta} x||^2_{L^2_\eta} \Big). \tag{3.4.80}$$

Summarizing the results of this multiplier:

$$\frac{\partial_x}{2} \int w_x^2 x^{2-2\sigma_0} z^{2m} + \int w_{\eta x}^2 x^{2-2\sigma_0} z^{2m} \lesssim \int w_x^2 x^{1-2\sigma_0} z^{2m}$$
$$+ \int w_x^2 x^{1-2\sigma_0} z^{2m-2} + \mathcal{O}(\delta) \int w_\eta^2, w_{\eta\eta}^2 x, w_{x\eta}^2 x^2, \tag{3.4.81}$$

Again, we may integrate in x to obtain:

$$\sup_{x\ge 1} \int w_x^2 x^{2-2\sigma_0} z^{2m} + \int_1^\infty \int w_{\eta x}^2 x^{2-2\sigma_0} z^{2m}$$
$$\lesssim \int_{x=1} w_x(1)^2 y^{2m} + \int_1^\infty \int w_x^2 x^{1-2\sigma_0} z^{2m} + \int_1^\infty \int w_x^2 x^{1-2\sigma_0} z^{2m-2}$$
$$+ \mathcal{O}(\delta) \int_1^\infty \int w_\eta^2, w_{\eta\eta}^2 x^{\frac{1}{2}}, w_{x\eta}^2 x. \tag{3.4.82}$$

The third and fourth terms on the right-hand side are controlled by $||w||^2_Q$, and the first term is controlled by the initial data. For the second term, we must use the equation (3.4.38):

$$||w_x x^{\frac{1}{2}-\sigma_0} z^m||_{L^2} \le ||w_{\eta\eta} x^{\frac{1}{2}-\sigma_0} z^m||_{L^2} + ||\Big(w_\eta^2 + 2\phi^*_\eta w_\eta + w w_{\eta\eta} + \phi^* w_{\eta\eta} + w \phi^*_{\eta\eta} \Big) x^{\frac{1}{2}-\sigma_0} z^m||_{L^2}$$
$$\le ||w_{\eta\eta} x^{\frac{1}{2}-\sigma_0} z^m||_{L^2} + ||w_\eta x^{\frac{1}{2}}, \phi^*_\eta x^{\frac{1}{2}}, w, \phi^*, \phi^*_{\eta\eta} \eta x^{\frac{1}{2}}||_{L^\infty} ||w_\eta x^{-\sigma_0} z^m, w_{\eta\eta} x^{\frac{1}{2}-\sigma_0} z^m||_{L^2}$$
$$\lesssim ||w_{\eta\eta} x^{\frac{1}{2}-\sigma_0} z^m, w_\eta x^{-\sigma_0} z^m||_{L^2}. \tag{3.4.83}$$

As the majorizing terms above have been controlled, we can conclude:

$$\sup_{x \geq 1} \int w_x^2 x^{2-2\sigma_0} z^{2m} + \int_1^\infty \int w_{\eta x}^2 x^{2-2\sigma_0} z^{2m} \leq \mathcal{O}(\delta). \tag{3.4.84}$$

By subsequently relating w_x to $w_{\eta\eta}$ and $w_{\eta x}$ to $w_{\eta\eta\eta}$, we have shown that $||z^m w||_{Q(\sigma_0,0)} \leq \mathcal{O}(\delta, m, \sigma_0)$, which is the $k = 0$ case of (3.4.46). We may upgrade the previous set of estimates to higher order x derivatives by iterating Steps 2, 3, and 4. The mechanism for doing so is that each application of ∂_x to the profile terms ϕ^* produces an extra factor of x^{-1}, and similarly each time ∂_x hits z, one extra factor of x^{-1} is produced by (3.4.13). As this is similar to the previous steps, we omit the details. We remark that controlling higher order derivatives does not require smallness of the initial datum (hence, the smallness assumption (3.2.28) is only for $j = 0, 1, 2$).

□

We now give the following corollaries:

Corollary 3.4.8 (u_p^0 Asymptotics)**.** *For any $m \geq 0$, $2 \leq p \leq \infty$,*

$$||z^m \partial_y^l \partial_x^k u_p^0||_{L_y^p} \leq C(m,l,k) x^{-k-\frac{l}{2}+\frac{1}{2p}} \text{ for } 2k + l > 2, \tag{3.4.85}$$

$$||z^m \partial_y^l \partial_x^k u_p^0||_{L_y^p} \leq \mathcal{O}(\delta; m,l,k) x^{-k-\frac{l}{2}+\frac{1}{2p}} \text{ for } 2k + l \leq 2. \tag{3.4.86}$$

Proof. We start with the representation of u_p^0 via (3.4.36):

$$||z^m \partial_\eta^l \partial_x^k u_p^0||_{L_\eta^p} = ||z^m \partial_\eta^l \partial_x^k \Big(\psi + e_\delta - \delta\Big)||_{L_\eta^p} + ||z^m \partial_\eta^l \partial_x^k w||_{L_\eta^p}. \tag{3.4.87}$$

The first quantity on the right-hand side above is controlled by Corollary 3.4.3. For the second quantity on the right-hand side, we simply use (3.4.46) with $0 < \sigma_0 < \frac{1}{4}$, coupled with the standard Sobolev interpolation. Recall now: $\eta = \int_0^y 1 + u_p^0(x,\theta) d\theta$, from which we have the equivalence: $(1-\delta)y \le \eta \le (1+\delta)y$. Moreover, we have the Jacobians of the change of coordinates are given by: $\eta'(y) = 1 + u_p^0, \quad y'(\eta) = \frac{1}{1+u_p^0}$, both of which are bounded and bounded away from zero. Therefore, we can transfer (3.4.87) to (x, y) coordinates.

$\square$

We may give bounds for v_p^0:

Corollary 3.4.9 (Estimates for v_p^0)**.** *We have the following bounds for v, for any $m \ge 0$:*

$$||z^m \partial_x^k v_p^0(x,\cdot)||_{L_y^2} \le C(k,m) x^{-k-\frac{1}{4}} \text{ for } k \ge 1, \tag{3.4.88}$$

$$||z^m v_p^0(x,\cdot)||_{L_y^2} \le \mathcal{O}(\delta; m) x^{-\frac{1}{4}}, \tag{3.4.89}$$

$$||z^m \partial_x^k v_p^0(x,\cdot)||_{L_y^\infty} \le C(k,m) x^{-k-\frac{1}{2}} \text{ for } k \ge 1, \tag{3.4.90}$$

$$||z^m v_p^0(x,\cdot)||_{L_y^\infty} \le \mathcal{O}(\delta; m) x^{-\frac{1}{2}}. \tag{3.4.91}$$

Proof. First, we give L^2 via the Hardy inequality, which is available for $m \ge 0$ as $\partial_x^k v(x, y = \infty) = 0$:

$$||z^m \partial_x^k v_p^0(x,\cdot)||_{L_y^2} \le ||z^m y \partial_x^k v_{py}^0(x,\cdot)||_{L_y^2} = ||z^m y \partial_x^k u_{px}^0(x,\cdot)||_{L_y^2} \lesssim x^{-k-\frac{1}{4}}. \tag{3.4.92}$$

We have used estimate (3.4.85) for $k > 0$, and one obtains the smallness of $\mathcal{O}(\delta)$ for $k = 0$ by applying (3.4.86). The L^∞ estimate then follows via standard Sobolev interpolation. Indeed, as $\lim_{y\to\infty} v_p^0(x, y) = 0$, with rapid decay for fixed x, one writes:

$$\begin{aligned} |y^m \partial_x^k v_p^0(x,y)|^2 &= \int_y^\infty 2(y^m \partial_x^k v_p^0) \partial_y (y^m \partial_x^k v_p^0) \, \mathrm{d}y' \\ &= \int_y^\infty 2y^m \partial_x^k v_p^0 m y^{m-1} \partial_x^k v_p^0 \, \mathrm{d}y' + \int_y^\infty 2(y^m \partial_x^k v_p^0) y^m \partial_x^k v_{py}^0 \, \mathrm{d}y'. \end{aligned} \tag{3.4.93}$$

Dividing both sides by x^m then gives:

$$||z^m \partial_x^k v_p^0(x,\cdot)||_{L_y^\infty} \lesssim ||z^m \partial_x^k v_p^0(x)||_{L_y^2}^{\frac{1}{2}} ||z^m \partial_x^k v_{py}^0(x)||_{L_y^2}^{\frac{1}{2}} + x^{-\frac{1}{4}} ||z^{m-\frac{1}{2}} \partial_x^k v_p^0||_{L_y^2}, \tag{3.4.94}$$

from which the desired results follow upon consultation with (3.4.92) and (3.4.85) - (3.4.86). □

3.5 Euler-1 Layer

3.5.1 Derivation of Equations

In this section we construct the next order in the expansion, $[u_e^1, v_e^1, P_e^1]$. The analysis will be taking place in Euler coordinates, that is (x, Y), where $Y = \sqrt{\epsilon} y$ as in (3.2.9). We make the notational convention that differential operators applied to $[u_e^1, v_e^1, P_e^1]$ will always be in (x, Y) coordinates, so for example $\Delta u_e^1 := \partial_x^2 u_e^1 + \partial_Y^2 u_e^1$. Let us now expand the partial expansion in order to find the lowest-order terms which we then take to solve the Euler-1 equation. The first equation is:

$$\begin{aligned} -\Delta_\epsilon u_s^{(1)} + u_s^{(1)} u_{sx}^{(1)} + v_s^{(1)} u_{sy}^{(1)} + P_{sx}^{(1)} = -\Delta_\epsilon u_p^0 - \epsilon^{\frac{3}{2}} \Delta u_e^1 + \Big(\bar{u}_s^0 + \sqrt{\epsilon} u_e^1\Big)\Big(u_{px}^0 + \sqrt{\epsilon} u_{ex}^1\Big) \\ + \Big(v_p^0 + v_e^1\Big)\Big(u_{py}^0 + \epsilon u_{eY}^1\Big) + \sqrt{\epsilon} P_{ex}^1 + \epsilon P_{ex}^{1,a}. \end{aligned} \tag{3.5.1}$$

Removing now the terms found in equation (3.4.1), and retaining the lowest-order purely Euler terms gives:

$$\sqrt{\epsilon}\Big[u_{ex}^1 + P_{ex}^1\Big] = 0. \tag{3.5.2}$$

The remaining terms from the above expansion are placed into the next order, which is discussed in detail in equations (3.6.1) - (3.6.8). Let us now turn to the second equation:

$$-\Delta_\epsilon v_s^{(1)} + u_s^{(1)} v_{sx}^{(1)} + v_s^{(1)} v_{sy}^{(1)} + \frac{\partial_y}{\epsilon} P_s^{(1)} = -\Delta_\epsilon v_p^0 - \epsilon \Delta v_e^1 + \left(\bar{u}_s^0 + \sqrt{\epsilon} u_e^1\right)\left(v_{px}^0 + v_{ex}^1\right)$$
$$+ \left(v_p^0 + v_e^1\right)\left(v_{py}^0 + \sqrt{\epsilon} v_{eY}^1\right) + P_{eY}^1 + \sqrt{\epsilon} P_{eY}^{1,a}. \tag{3.5.3}$$

Taking the order-1 purely Eulerian terms gives:

$$v_{ex}^1 + P_{eY}^1 = 0. \tag{3.5.4}$$

The remaining terms are contributed to the next-order error in (3.6.1) - (3.6.8). Putting (3.5.2) - (3.5.4) together with the divergence-free condition yields the following system for the Euler-1 layer:

$$u_{ex}^1 + P_{ex}^1 = 0, \quad v_{ex}^1 + P_{eY}^1 = 0, \quad u_{ex}^1 + v_{eY}^1 = 0, \text{ in } \Omega. \tag{3.5.5}$$

with prescribed boundary data

$$v_e^1(x,0) = -v_p^0(x,0). \tag{3.5.6}$$

Without loss of generality, taking $P_e^1 = -u_e^1$, gives the div-curl system

$$v_{ex}^1 - u_{eY}^1 = 0, \quad u_{ex}^1 + v_{eY}^1 = 0. \tag{3.5.7}$$

Notice that these equations are the Cauchy-Riemann equations for $v_e^1 = Re(f)$, $u_e^1 =$

$Im(f)$, f holomorphic on Ω. Then by taking the curl of the equation,

$$\Delta v_e^1 = 0 \text{ in } \Omega, \quad v_e^1(x,0) = -v_p^0(x,0). \tag{3.5.8}$$

The following boundary decay rates follow from (3.4.90) - (3.4.91):

$$\left|\partial_x^k v_p^0(x,0)\right| \le C(k) x^{-\frac{1}{2}-k}, \quad \left|v_p^0(x,0)\right| \le \mathcal{O}(\delta) x^{-\frac{1}{2}} \tag{3.5.9}$$

3.5.2 Uniform Decay Estimates

Motivated by (3.5.9), let us define for this section $f := \chi_{x \ge -10} E^{(m)}_{\mathbb{R}_+} v_p^0$, where $E^{(m)}_{\mathbb{R}_+}$ is the m'th order Sobolev extension operator on the half-line $\mathbb{R}_+$ (see (AF03, Chapter 5)). Here, m is selected as large as required by the remainder of our analysis. Then, f agrees with v_p^0 on $[0,\infty)$, and is cut-off past $x \le -10$. Then by standard Sobolev extension theory, $||f||_{H^m} \le C(m) ||v_p^0||_{H^m}$. Our setup for this subsection is:

$$\Delta v = 0 \text{ on } \mathbb{H}, \qquad v(x,0) = f(x), \quad \left|\partial_x^k f(x)\right| \le x^{-k-\frac{1}{2}}, \quad \left|f(x)\right| \le \mathcal{O}(\delta) x^{-\frac{1}{2}}. \tag{3.5.10}$$

Using the Poisson Kernel, we have:

$$v(x,Y) = P_Y * f = \int_{-\infty}^{\infty} P_Y(x-t) f(t) dt = \int_{-\infty}^{\infty} \frac{Y}{Y^2 + (x-t)^2} f(t) dt. \tag{3.5.11}$$

We will need the following pointwise estimates on the Poisson Kernel:

Lemma 3.5.1. *For $x > 0$, $k \ge 0$, and $0 \le s \le k$, we have:*

$$|\partial_x^k P_Y(x)| \lesssim x^{-s-1} Y^{-(k-s)}. \tag{3.5.12}$$

Proof. First, let us address the $k = 0$ case. Indeed,

$$xP_Y(x) = \frac{Yx}{Y^2 + x^2} \leq \frac{Y^2 + x^2}{Y^2 + x^2} \leq 1. \tag{3.5.13}$$

For $k \geq 1$, we record the basic identities:

$$\partial_x^k P_Y(x) = \sum_{j=0}^{\frac{k}{2}} c_{j,k} Y x^{2j} (Y^2 + x^2)^{-(\frac{k}{2}+1)-j} \text{ if } k \text{ even}, \tag{3.5.14}$$

$$\partial_x^k P_Y(x) = \sum_{j=0}^{\frac{k-1}{2}} c_{j,k} Y x^{2j+1} (Y^2 + x^2)^{-\frac{k+1}{2}-1-j} \text{ if } k \text{ odd}. \tag{3.5.15}$$

For the k even case, we multiply (3.5.14) by x^{k+1}, use the binomial formula, and then apply Young's inequality for products in the following manner:

$$\begin{aligned}
|x^{s+1} Y^{k-s} \partial_x^k P_Y(x)| &= |\sum_{j=0}^{\frac{k}{2}} c_{j,k} Y^{1+k-s} x^{2j+s+1} (Y^2 + x^2)^{-(\frac{k}{2}+1)-j}| \\
&\lesssim \sum_{j=0}^{\frac{k}{2}} c_{j,k} \frac{Y^{1+k-s} x^{2j+s+1}}{Y^{k+2+2j} + x^{k+2+2j}} \\
&\lesssim \sum_{j=0}^{\frac{k}{2}} \frac{Y^{(1+k-s)(\frac{k+2+2j}{1+k-s})} + x^{(s+1+2j)(\frac{k+2+2j}{s+1+2j})}}{Y^{k+2+2j} + x^{k+2+2j}} \lesssim 1,
\end{aligned} \tag{3.5.16}$$

where the Young's conjugates are:

$$1 = \frac{1 + k - s}{k + 2 + 2j} + \frac{s + 1 + 2j}{k + 2 + 2j}. \tag{3.5.17}$$

Performing the same calculation using (3.5.15), we have:

$$|x^{s+1} Y^{k-s} \partial_x^k P_Y(x)| = |\sum_{j=0}^{\frac{k-1}{2}} c_{j,k} Y^{1+k-s} x^{s+2j+2} (Y^2 + x^2)^{-\frac{k+1}{2}-1-j}|$$

$$\lesssim \sum_{j=0}^{\frac{k-1}{2}} c_{j,k} \frac{Y^{1+k-s} x^{s+2j+2}}{Y^{k+3+2j} + x^{k+3+2j}}$$
$$\lesssim \sum_{j=0}^{\frac{k-1}{2}} c_{j,k} \frac{Y^{(1+k-s)(\frac{k+2j+3}{1+k-s})} + x^{(s+2j+2)(\frac{k+2j+3}{s+2j+2})}}{Y^{k+3+2j} + x^{k+3+2j}} \lesssim 1. \tag{3.5.18}$$

Here, the Young's conjugates are:

$$1 = \frac{1+k-s}{k+2j+3} + \frac{s+2j+2}{k+2j+3}. \tag{3.5.19}$$

This proves (3.5.12).

□

We now prove the following uniform estimates for the $v_e^1(x,Y)$, which now drop superscripts and denote by v for notational ease:

Proposition 3.5.2. *Let v be the Poisson extension defined via (3.5.11). Then v satisfies the following bounds:*

$$\sup_{Y\geq 0} \Big|\partial_x^k v(x,Y)\Big| \leq C(k) x^{-k-\frac{1}{2}}, \quad \sup_{Y\geq 0} \Big|v(x,Y)\Big| \leq \mathcal{O}(\delta) x^{-\frac{1}{2}}. \tag{3.5.20}$$

Proof. The $Y = 0$ case is clear by (3.5.9), and so we must treat the case when $Y > 0$. In this regime, several *qualitative* facts are available for use. In particular, P_Y is smooth, has unit mass, and is positive everywhere. As is standard, we shall exploit these qualitative facts in order to extract quantitative bounds independent of $Y > 0$. Fixing any $\alpha > 0$, we shall split the integral (3.5.11) into three pieces:

$$\begin{aligned} v(x,Y) &= \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t) f(t) dt + \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} P_Y(x-t) f(t) dt + \int_{\frac{x}{1+\alpha}}^{\infty} P_Y(x-t) f(t) dt \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.5.21}$$

For the sake of concreteness, fix: $\alpha = \frac{1}{10,000}$. δ and ε will be taken small relative to this universal constant, α. Let us first turn to I_3:

$$
\begin{aligned}
x^{\frac{1}{2}}\Big|I_3\Big| &= \Big|\int_{\frac{x}{1+\alpha}}^{\infty} P_Y(x-t)x^{\frac{1}{2}}f(t)dt\Big| && (3.5.22)\\
&\leq \Big|\int_{\frac{x}{1+\alpha}}^{\infty} P_Y(x-t)\{x^{\frac{1}{2}} - |t|^{\frac{1}{2}}\}f(t)dt\Big| + \Big|\int_{\frac{x}{1+\alpha}}^{\infty} P_Y(x-t)|t|^{\frac{1}{2}}f(t)dt\Big| && (3.5.23)\\
&\leq \int_{\frac{x}{1+\alpha}}^{\infty} P_Y(x-t)\{\Big(\frac{x}{|t|}\Big)^{\frac{1}{2}} - 1\}|t|^{\frac{1}{2}}f(t)dt + ||fx^{\frac{1}{2}}||_{L^\infty}\int_{-\infty}^{\infty} P_Y(x-t)dt && (3.5.24)\\
&\leq \sup_{t\geq\frac{x}{1+\alpha}} \Big|\Big(\frac{x}{|t|}\Big)^{\frac{1}{2}} - 1\Big| ||fx^{\frac{1}{2}}||_{L^\infty}\int_{-\infty}^{\infty} P_Y(x-t)dt + \mathcal{O}(\delta) \leq \mathcal{O}(\delta). && (3.5.25)
\end{aligned}
$$

where we have used for each fixed $Y > 0$, $P_Y \geq 0$ and $\int P_Y = 1$. By symmetry, I_1 works the same way. Therefore, we are left with I_2, where the main mechanism are pointwise estimates on P_Y. According to (3.5.12), with $k = s = 0$,

$$
\sup_{-\frac{x}{1+\alpha}\leq t\leq\frac{x}{1+\alpha}} P_Y(x-t) \lesssim \frac{1}{x}, \tag{3.5.26}
$$

as $x - t > 0$ int the region $-\frac{x}{1+\alpha} \leq t \leq \frac{x}{1+\alpha}$. Thus, we have:

$$
\begin{aligned}
\Big|I_2\Big| &\leq \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} P_Y(x-t)|f(t)|dt \lesssim \frac{1}{x}\int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} |f(t)|dt \leq \frac{\mathcal{O}(\delta)}{x}\int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} |1+t|^{-\frac{1}{2}}dt\\
&\leq \mathcal{O}(\delta)x^{-\frac{1}{2}}. && (3.5.27)
\end{aligned}
$$

Combining (3.5.25) - (3.5.27), we may conclude that

$$
\Big|x^{\frac{1}{2}}v\Big| \leq \mathcal{O}(\delta). \tag{3.5.28}
$$

We now need to establish this procedure for higher-order x derivatives for v. We continue with the I_1, I_2, I_3 splitting used above. Upon differentiating (3.5.21) with respect to x, let us treat each term I_i individually.

$$\begin{aligned}\partial_x I_1 &= \frac{-1}{1+\alpha}P_Y(x+\frac{x}{1+\alpha})f(-\frac{x}{1+\alpha}) + \int_{-\infty}^{-\frac{x}{1+\alpha}} \partial_x P_Y(x-t)f(t)dt \\ &= \frac{-1}{1+\alpha}P_Y(x+\frac{x}{1+\alpha})f(-\frac{x}{1+\alpha}) - \int_{-\infty}^{-\frac{x}{1+\alpha}} \partial_t P_Y(x-t)f(t)dt \\ &= B_1 + \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)f'(t)dt. \end{aligned} \tag{3.5.29}$$

Here,

$$B_1 = \frac{-1}{1+\alpha}P_Y(x+\frac{x}{1+\alpha})f(-\frac{x}{1+\alpha}) - P_Y(x+\frac{x}{1+\alpha})f(\frac{-x}{1+\alpha}). \tag{3.5.30}$$

Note now in a similar manner to (3.5.12),

$$\Big|B_1\Big| \lesssim P_Y(\frac{2+\alpha}{1+\alpha}x)|f(-\frac{x}{1+\alpha})| \lesssim \frac{1}{x}\mathcal{O}(\delta)x^{-\frac{1}{2}} \lesssim x^{-\frac{3}{2}}. \tag{3.5.31}$$

Using the same method as evaluating (3.5.25):

$$\begin{aligned} &x^{\frac{3}{2}}\Big|\int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)f'(t)dt\Big| \\ &\leq \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)|t|^{\frac{3}{2}}\Big|f'(t)\Big|dt + \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)\Big|\{x^{\frac{3}{2}} - |t|^{\frac{3}{2}}\}\Big|\Big|f'(t)\Big|dt \\ &\leq \int_{-\infty}^{\infty} P_Y(x-t)|t|^{\frac{3}{2}}\Big|f'(t)\Big|dt + \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)\Big|\{x^{\frac{3}{2}} - |t|^{\frac{3}{2}}\}\Big|\Big|f'(t)\Big|dt \\ &\leq ||x^{\frac{3}{2}}f'||_{L^\infty} + \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)\Big|\{\Big(\frac{x}{|t|}\Big)^{\frac{3}{2}} - 1\}\Big||t|^{\frac{3}{2}}\Big|f'(t)\Big|dt \\ &\lesssim ||x^{\frac{3}{2}}f'||_{L^\infty} + \sup_{t\leq -\frac{x}{1+\alpha}}\Big|\{\Big(\frac{x}{|t|}\Big)^{\frac{3}{2}} - 1\}\Big|||x^{\frac{3}{2}}f'||_{L^\infty}\int_{-\infty}^{\infty} P_Y(x-t)dt \leq C. \end{aligned} \tag{3.5.32}$$

Next, we turn to I_2, which after differentiating gives:

$$\partial_x I_2 = \frac{1}{1+\alpha} P_Y(x - \frac{x}{1+\alpha}) f(\frac{x}{1+\alpha}) + \frac{1}{1+\alpha} P_Y(x + \frac{x}{1+\alpha}) f(-\frac{x}{1+\alpha}) \\ + \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} \partial_x P_Y(x-t) f(t) dt. \tag{3.5.33}$$

Via (3.5.12) with $k = 1, s = 0$,

$$\sup_{Y>0, t \in [-\frac{x}{1+\alpha}, \frac{x}{1+\alpha}]} \Big| \partial_x P_Y(x-t) \Big| \le \sup_{Y>0, t \in [-\frac{x}{1+\alpha}, \frac{x}{1+\alpha}]} \frac{1}{(x-t)^2} \lesssim \frac{1}{x^2}. \tag{3.5.34}$$

For I_2, this then facilitates the bound:

$$\Big| \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} \partial_x P_Y(x-t) f(t) dt \Big| \lesssim \frac{1}{x^2} \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} (1+t)^{-\frac{1}{2}} dt \lesssim x^{-\frac{3}{2}}. \tag{3.5.35}$$

For the $\partial_x I_3$ term, we must integrate by parts in the following manner:

$$\partial_x I_3 = \frac{1}{1+\alpha} P(x - \frac{x}{1+\alpha}) f(\frac{x}{1+\alpha}) + \int_{\frac{x}{1+\alpha}}^{\infty} \partial_x P_Y(x-t) f(t) dt \\ = B_3 + \int_{\frac{x}{1+\alpha}}^{\infty} P_Y(x-t) f'(t) dt. \tag{3.5.36}$$

Here, B_3 contains the boundary terms:

$$B_3 = \frac{1}{1+\alpha} P_Y(x - \frac{x}{1+\alpha}) f(\frac{x}{1+\alpha}) - P_Y(x - \frac{x}{1+\alpha}) f(\frac{x}{1+\alpha}) \\ = \frac{1}{1+\alpha} P_Y(\frac{\alpha}{1+\alpha} x) f(\frac{x}{1+\alpha}) - P_Y(\frac{\alpha}{1+\alpha} x) f(\frac{x}{1+\alpha}). \tag{3.5.37}$$

Again, via direct calculation using the definition of P_Y, it is clear that:

$$\Big| B_3 \Big| \lesssim x^{-\frac{3}{2}}. \tag{3.5.38}$$

We may estimate the integral in (3.5.36) in identical fashion to (3.5.32). This procedure can be iterated for higher-order derivatives. To see this, let us start with the splitting given in (3.5.21). We will apply ∂_x^k, where $k \geq 2$:

$$\partial_x^k v(x,Y) = \partial_x^k I_1 + \partial_x^k I_2 + \partial_x^k I_3. \tag{3.5.39}$$

Our starting point is the expression (3.5.29), from which it becomes clear that applying ∂_x^k to I_1, for generic constants $c_\alpha > 0$ (which can change from term to term), we have:

$$\partial_x^k I_1 = \sum_{j=0}^{k-1} c_j \partial_x^j P_Y(c_\alpha x)\partial_x^{k-1-j} f(c_\alpha x) + \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)\partial_t^k f(t)dt. \tag{3.5.40}$$

Using (3.5.12), we estimate:

$$|\partial_x^j P_Y(c_\alpha x)\partial_x^{k-1-j} f(c_\alpha x)| \leq c_\alpha x^{-j-1} x^{-(k-1-j)-\frac{1}{2}} \lesssim x^{-k-\frac{1}{2}}. \tag{3.5.41}$$

For the integration term in (3.5.40), we follow (3.5.32) to give:

$$\begin{aligned}
&x^{k+\frac{1}{2}}\Big|\int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)\partial_t^k f(t)dt\Big| && (3.5.42)\\
&\leq \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)|t|^{k+\frac{1}{2}}\Big|\partial_t^k f(t)\Big|dt + \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)\Big|\{x^{k+\frac{1}{2}} - |t|^{k+\frac{1}{2}}\}\Big|\Big|\partial_t^k f(t)\Big|dt\\
&\leq \int_{-\infty}^{\infty} P_Y(x-t)|t|^{k+\frac{1}{2}}\Big|\partial_t^k f(t)\Big|dt + \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)\Big|\{x^{k+\frac{1}{2}} - |t|^{k+\frac{1}{2}}\}\Big|\Big|\partial_t^k f(t)\Big|dt\\
&\leq ||x^{k+\frac{1}{2}}\partial_x^k f||_{L^\infty} + \int_{-\infty}^{-\frac{x}{1+\alpha}} P_Y(x-t)\Big|\{\Big(\frac{x}{|t|}\Big)^{k+\frac{1}{2}} - 1\}\Big||t|^{k+\frac{1}{2}}\Big|\partial_t^k f(t)\Big|dt\\
&\lesssim ||x^{k+\frac{1}{2}}\partial_x^k f||_{L^\infty} + \sup_{t\leq -\frac{x}{1+\alpha}}\Big|\{\Big(\frac{x}{|t|}\Big)^{k+\frac{1}{2}} - 1\}\Big|||x^{k+\frac{1}{2}}\partial_x^k f||_{L^\infty}\int_{-\infty}^{\infty} P_Y(x-t)dt \leq C.
\end{aligned} \tag{3.5.43}$$

By symmetry, the same estimate applies to I_3, and so we turn to I_2. Referring to (3.5.33), one sees the following expression:

$$\partial_x^k I_2 = \sum_{j=0}^{k-1} c_j \partial_x^j P_Y(c_\alpha x)\partial_x^{k-1-j} f(c_\alpha x) + \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} \partial_x^k P_Y(x-t) f(t)\,\mathrm{d}t, \tag{3.5.44}$$

where $c_\alpha > 0$ denotes generic constants which depend on α, not on δ, ε, and are strictly positive. Referring to (3.5.41), it remains to estimate the integral term above:

$$\begin{aligned}|\int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} \partial_x^k P_Y(x-t) f(t)\,\mathrm{d}t| &\leq \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} |\partial_x^k P_Y(x-t)||f(t)|\,\mathrm{d}t \\ &\lesssim \sup_{-\frac{x}{1+\alpha}\leq t\leq \frac{x}{1+\alpha}} |\partial_x^k P_Y(x-t)| \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} |(1+t)|^{\frac{1}{2}}\,\mathrm{d}t \\ &\lesssim x^{-k-1} x^{\frac{1}{2}} = x^{-k-\frac{1}{2}}. \end{aligned} \tag{3.5.45}$$

This then proves the desired result.

□

By repeating the above proof, we actually have the following:

Corollary 3.5.3. *Consider boundary values $g(x)$ satisfying:*

$$\left|\partial_x^k g(x)\right| \leq x^{-k-w}, \text{ where } w \in (0,1). \tag{3.5.46}$$

Then the Poisson extension satisfies:

$$\sup_Y \left|\partial_x^k \{P_Y * g\}\right| \lesssim x^{-k-w}. \tag{3.5.47}$$

We now give an estimate more suitable to obtaining L^2 in Y estimates.

Lemma 3.5.4. *For any $k \geq 1$, and $0 \leq k - s \leq 1$, we have:*

$$\left|\partial_x^k v(x, Y)\right| \lesssim x^{-s-\frac{1}{2}} Y^{-(k-s)}. \tag{3.5.48}$$

Proof. By Lemma 3.5.2, the claim is only meaningful for $Y \geq x$, so make this assumption to begin. Again, we start with the splitting (3.5.21). By symmetry it suffices to consider I_2, I_3. Applying ∂_x^k, we have the expression (for generic constants $c_\alpha > 0$):

$$\partial_x^k I_2 = \sum_{j=0}^{k-1} c_{j,k,\alpha} \partial_x^j P_Y(c_\alpha x) \partial_x^{k-1-j} f(c_\alpha x) + \int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} \partial_x^k P_Y(x-t) f(t) dt, \tag{3.5.49}$$

First,

$$\begin{aligned} |\partial_x^j P_Y(c_\alpha x) \partial_x^{k-1-j} f(c_\alpha x)| &\lesssim Y^{-(k-s)} x^{-j-1+(k-s)} x^{-(k-1-j)-\frac{1}{2}} \\ &\lesssim Y^{-(k-s)} x^{-s-\frac{1}{2}}. \end{aligned} \tag{3.5.50}$$

Above, we have used (3.5.12) for the case $j \geq 1$. For the $j = 0$ case we use the assumption that $Y \geq x$, and the following estimate on P_Y, which is valid so long as $k - s \leq 1$:

$$P_Y(x-t) = \frac{Y}{Y^2 + (x-t)^2} \leq \frac{Y}{Y^2} = \frac{1}{Y} \leq Y^{-(k-s)} x^{-1+(k-s)}. \tag{3.5.51}$$

For the integral term in (3.5.49), via (3.5.12),

$$\sup_{\substack{Y \geq 0 \\ t \in [-\frac{x}{1+\alpha}, \frac{x}{1+\alpha}]}} \left|\partial_x^k P_Y(x-t)\right| \lesssim Y^{-(k-s)} x^{-s-1}. \tag{3.5.52}$$

From here, the estimate on the integral in (3.5.49) follows:

$$\Big|\int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}} \partial_x P_Y(x-t)f(t)dt\Big| \lesssim Y^{-(k-s)}x^{-s-1}\int_{-\frac{x}{1+\alpha}}^{\frac{x}{1+\alpha}}(1+t)^{-\frac{1}{2}}dt \lesssim Y^{-(k-s)}x^{-s-\frac{1}{2}}. \quad (3.5.53)$$

For the I_3 contribution, we apply ∂_x^k and integrate by parts in t, reads:

$$\partial_x I_3 = \sum_{j=0}^{k-1} c_{j,k,\alpha}\partial_x^j P_Y(c_\alpha x)\partial_x^{k-1-j}f(c_\alpha x) + \int_{\frac{x}{1+\alpha}}^{\infty} P_Y(x-t)\partial_t^k f(t)dt. \quad (3.5.54)$$

First, as shown in (3.5.50)

$$|\partial_x^j P_Y(c_\alpha x)\partial_x^{k-1-j}f(c_\alpha x)| \lesssim Y^{-(k-s)}x^{-s-\frac{1}{2}}. \quad (3.5.55)$$

Turning to the integration in (3.5.54), our point of view on the kernel:

$$P_Y(x-t) = \frac{Y}{Y^2+(x-t)^2} \le \frac{Y}{Y^2} = \frac{1}{Y}. \quad (3.5.56)$$

Then in the regime $Y \ge x$, we have (since $k-s \le 1$):

$$\begin{aligned}\Big|\int_{\frac{x}{1+\alpha}}^{\infty} P_Y(x-t)\partial_t^k f(t)dt\Big| &\le \frac{1}{Y}\int_{\frac{x}{1+\alpha}}^{\infty} t^{-k-\frac{1}{2}}dt \le \frac{1}{Y}x^{-k+\frac{1}{2}} \\ &\le \frac{1}{Y}x^{-k+\frac{1}{2}}\Big(\frac{Y}{x}\Big)^{1-(k-s)} \le Y^{-(k-s)}x^{-s-\frac{1}{2}}. \quad (3.5.57)\end{aligned}$$

This establishes the desired result. □

Lemma 3.5.5 (Interpolation Estimates)**.** *We have the following decay estimates:*

$$x^{k+\frac{1}{2}}||\partial_Y^k v(x,\cdot)||_{L_Y^\infty} \le C, \quad x^{\frac{3}{2}}||\partial_Y v(x,\cdot)||_{L_Y^\infty} \le \mathcal{O}(\delta). \quad (3.5.58)$$

Proof. We treat the case when $k = 1$, with the extension for higher k values being obvious. This follows via standard interpolation arguments: fix an x, and consider $v_Y(x, \cdot)$ as a function of the Y variable, defined on the half-space $Y \geq 0$. Then via Gagliardo-Nirenberg interpolation estimates, and via harmonicity $v_{xx} = -v_{YY}$,

$$x^{\frac{3}{2}} ||v_Y(x, \cdot)||_{L^\infty_Y} \lesssim \Big(x^{\frac{1}{2}} ||v(x, \cdot)||_{L^\infty_Y} \Big)^{\frac{1}{2}} \Big(x^{\frac{5}{2}} ||v_{xx}(x, \cdot)||_{L^\infty_Y} \Big)^{\frac{1}{2}}. \tag{3.5.59}$$

Importantly, via Remark 4, Page 125 in (Nir59), we need not include a lower-order term on the right-hand side of (3.5.59), which is due to the fact that our domain, for each fixed x, is the half-space $Y \geq 0$. Taking the supremum over x and applying (3.5.20) yields the desired result. The smallness of $\mathcal{O}(\delta)$ for $k = 1$ case is guaranteed due to the v term in (3.5.59).

□

Proposition 3.5.6 (Euler Correctors)**.** *Suppose $[u^1_e, v^1_e]$ solves the boundary value problem:*

$$\Delta v^1_e = 0, \quad v^1_e(x, 0) = -v^0_p(x, 0), \quad u^1_e = \int_x^\infty v^1_{eY}(x', Y) dx', \text{ for } x \geq 1. \tag{3.5.60}$$

Then the following bounds holds for any $k, j \geq 0$:

$$\sup_{Y \geq 0} \Big| \partial_x^k \partial_Y^j v^1_e \Big| x^{k+j+\frac{1}{2}} \leq C, \qquad \sup_{Y \geq 0} \Big[\Big| u^1_e, v^1_e \Big| x^{\frac{1}{2}} + \Big| u^1_{ex}, v^1_{eY} \Big| x^{\frac{3}{2}} \Big] \leq \mathcal{O}(\delta). \tag{3.5.61}$$

Proof. We shall take $v^1_e = v$ from the above lemmas. For the u^1_e profile, we define for $x \geq 1$:

$$u^1_e(x, Y) := \int_x^\infty v^1_{eY}(x', Y) dx'. \tag{3.5.62}$$

From here it is clear that the Cauchy-Riemann equations hold:

$$u^1_{ex} = -v^1_{eY}, \quad u^1_{eY} = \int_x^\infty v^1_{eYY} = -\int_x^\infty v^1_{exx} = v^1_{ex}. \tag{3.5.63}$$

We have used that $v^1_{ex}(x, Y) \to 0$ as $x \to \infty$. From Lemma 3.5.5, it is then clear that:

$$\left|u^1_e\right| = \left|\int_x^\infty u^1_{ex} dx'\right| = \left|\int_x^\infty v^1_{eY} dx'\right| \leq \mathcal{O}(\delta) \int_x^\infty |x'|^{-\frac{3}{2}} dx' \leq \mathcal{O}(\delta) x^{-\frac{1}{2}}. \tag{3.5.64}$$

□

Remark (In-flow Conditions). Note that we do not solve for the Euler correctors given an arbitrary in-flow condition at $x = 1$. Rather, we take $[u^1_e, v^1_e, P^1_e]$ to be the explicit profiles obtained by applying the Poisson extension to f. In this sense, the set-up we consider here is distinct from the construction of Euler profiles in (GN17).

3.6 Prandtl Layer 1

3.6.1 Derivation of Linearized Prandtl Equations:

In this section, we will construct the Prandtl correctors, $[u^1_p, v^1_p, P^1_p]$. Let us now obtain the equations that they will satisfy. We do this in a manner which can be easily generalized in the next section. We expand the nonlinear terms:

$$\begin{aligned}
\bar{u}^{(1)}_s \bar{u}^{(1)}_{sx} &= \Big(\bar{u}^{(0)}_s + \sqrt{\epsilon} u^1_e + \sqrt{\epsilon} u^1_p\Big)\Big(\bar{u}^{(0)}_{sx} + \sqrt{\epsilon} u^1_{ex} + \sqrt{\epsilon} u^1_{px}\Big) \\
&= \bar{u}^{(0)}_s \bar{u}^{(0)}_{sx} + \sqrt{\epsilon} u^{(1)}_{sx} u^1_p + \sqrt{\epsilon} u^{(1)}_s u^1_{px} + \epsilon u^1_p u^1_{px} + \sqrt{\epsilon} u^1_{ex} u^0_p + \sqrt{\epsilon} u^1_e u^0_{px} \\
&+ \sqrt{\epsilon} u^1_{ex} + \epsilon u^1_e u^1_{ex}, \qquad (3.6.1) \\
\bar{v}^{(1)}_s \bar{u}^{(1)}_{sy} &= \Big(v^0_p + v^1_e + \sqrt{\epsilon} v^1_p\Big)\Big(u^0_{py} + \epsilon u^1_{eY} + \sqrt{\epsilon} u^1_{py}\Big)
\end{aligned}$$

$$= v_s^{(1)} u_{py}^0 + \sqrt{\epsilon} u_{sy}^{(1)} v_p^1 + \sqrt{\epsilon} v_s^{(1)} u_{py}^1 + \epsilon v_p^1 u_{py}^1 + \epsilon v_p^0 u_{eY}^1 + \epsilon v_e^1 u_{eY}^1, \tag{3.6.2}$$

$$\begin{aligned}
\bar{u}_s^{(1)} \bar{v}_{sx}^{(1)} &= \Big(1 + u_p^0 + \sqrt{\epsilon} u_e^1 + \sqrt{\epsilon} u_p^1\Big)\Big(v_{px}^0 + v_{ex}^1 + \sqrt{\epsilon} v_{px}^1\Big) \\
&= \bar{u}_s^{(0)} v_{px}^0 + \sqrt{\epsilon} v_{px}^1 u_s^{(1)} + \sqrt{\epsilon} v_{sx}^{(1)} u_p^1 + \epsilon u_p^1 v_{px}^1 + u_p^0 v_{ex}^1 + \sqrt{\epsilon} u_e^1 v_{px}^0 \\
&\quad + v_{ex}^1 + \sqrt{\epsilon} u_e^1 v_{ex}^1,
\end{aligned} \tag{3.6.3}$$

$$\begin{aligned}
\bar{v}_s^{(1)} \bar{v}_{sy}^{(1)} &= \Big(v_p^0 + v_e^1 + \sqrt{\epsilon} v_p^1\Big)\Big(v_{py}^0 + \sqrt{\epsilon} v_{eY}^1 + \sqrt{\epsilon} v_{py}^1\Big) \\
&= v_p^0 v_{py}^0 + \sqrt{\epsilon} v_{py}^1 v_s^{(1)} + \sqrt{\epsilon} v_{sy}^{(1)} v_p^1 + \epsilon v_p^1 v_{py}^1 + \sqrt{\epsilon} v_p^0 v_{eY}^1 + v_e^1 v_{py}^0 + \sqrt{\epsilon} v_e^1 v_{eY}^1.
\end{aligned} \tag{3.6.4}$$

Let us now denote:

$$R_E^{u,1} := v_e^1 u_{py}^0 + \sqrt{\epsilon} u_{ex}^1 u_p^0 + \sqrt{\epsilon} u_e^1 u_{px}^0 + \epsilon v_p^0 u_{eY}^1, \quad E_E^{u,1} := \epsilon v_e^1 u_{eY}^1 + \epsilon u_e^1 u_{ex}^1, \tag{3.6.5}$$

$$R_E^{v,1} = u_p^0 v_{ex}^1 + \sqrt{\epsilon} u_e^1 v_{px}^0 + \sqrt{\epsilon} v_p^0 v_{eY}^1 + v_e^1 v_{py}^0, \quad E_E^{v,1} := \sqrt{\epsilon} u_e^1 v_{ex}^1 + \sqrt{\epsilon} v_e^1 v_{eY}^1. \tag{3.6.6}$$

The purpose of the terms above is to separate the Euler-Prandtl terms and the pure-Euler terms. The pure-Euler terms are harmful to our analysis, because they scale differently than the Euler-Prandtl terms. From a practical point of view, their presence prevents the application of weights of the form $z = \frac{y}{\sqrt{x}}$, and therefore obstructs self-similarity. It turns out that all pure-Euler terms are of "gradient-type", see (3.7.32). Thus, by introducing appropriate potential functions in the pressure expansion, we can force these terms to vanish identically. With this notation, we may write the Navier-Stokes expansion as:

$$\begin{aligned}
-\Delta_\epsilon \bar{u}_s^{(1)} + \bar{u}_s^{(1)} \bar{u}_{sx}^{(1)} + \bar{v}_s^{(1)} \bar{u}_{sy}^{(1)} + \bar{P}_x^{(1)} &= -\Delta_\epsilon \bar{u}_s^{(0)} + \bar{u}_s^{(0)} \bar{u}_{sx}^{(0)} + \bar{v}_s^{(0)} \bar{u}_{sy}^{(0)} + \bar{P}_x^{(0)} \\
&\quad + \sqrt{\epsilon}\Big[-\Delta_\epsilon u_p^1 + u_s^{(1)} u_{px}^1 + u_{sx}^{(1)} u_p^1 + u_{sy}^{(1)} v_p^1 + v_s^{(1)} u_{py}^1 + \sqrt{\epsilon} u_p^1 u_{px}^1 \\
&\quad + \sqrt{\epsilon} v_p^1 u_{py}^1 + P_{px}^1 \Big] + \epsilon P_{px}^{1,a} + R_E^{u,1} + E_E^{u,1} - \epsilon^{\frac{3}{2}} \Delta u_e^1 + \epsilon P_{Ex}^{1,a} + \sqrt{\epsilon}(u_{ex}^1 + P_{ex}^1).
\end{aligned} \tag{3.6.7}$$

The normal equation is expanded via:

$$
\begin{aligned}
-\Delta_\epsilon \bar{v}_s^{(1)} + \bar{u}_s^{(1)}\bar{v}_{sx}^{(1)} + \bar{v}_s^{(1)}\bar{v}_{sy}^{(1)} + \frac{\bar{P}_y^{(1)}}{\epsilon} &= -\Delta_\epsilon \bar{v}_s^{(0)} + \bar{u}_s^{(0)}\bar{v}_{sx}^{(0)} + \bar{v}_s^{(0)}\bar{v}_{sy}^{(0)} + \frac{\bar{P}_y^{(0)}}{\epsilon} \\
&+ \sqrt{\epsilon}\Big[-\Delta_\epsilon v_p^1 + u_s^{(1)} v_{px}^1 + v_{sx}^{(1)} u_p^1 + v_s^{(1)} v_{py}^1 + v_{sy}^{(1)} v_p^1 + \sqrt{\epsilon} u_p^1 v_{px}^1 \\
&+ \sqrt{\epsilon} v_p^1 v_{py}^1 + \frac{P_{py}^1}{\epsilon}\Big] + \frac{\partial_y}{\epsilon}\epsilon P_p^{1,a} + R_E^{v,1} + E_E^{v,1} - \epsilon\Delta v_e^1 + \frac{\partial_y}{\epsilon}\epsilon P_E^{1,a} + (\frac{\partial_y}{\epsilon}\sqrt{\epsilon} P_e^1 + v_{ex}^1).
\end{aligned}
\tag{3.6.8}
$$

In the above expressions, the $[\bar{u}_s^{(0)}, \bar{v}_s^{(0)}]$, $[u_s^{(1)}, v_s^{(1)}]$ are known at this stage, and the u_p^1, v_p^1 are unknowns, to be constructed in this step. Similarly for the pressures, the the P_e^1 is known, but the auxiliary pressures $P_E^{1,a}, P_p^{1,a}$ are to be defined in this section. Finally, as can be seen from the definitions in (3.6.5), (3.6.6), the $R_E^{u,1}, R_E^{v,1}, E_E^{u,1}, E_E^{v,1}$ are all knowns. Let us now simplify the above expressions. First, via the construction of $[u_p^0, v_p^0]$ we may write:

$$
\begin{aligned}
R^{u,0} &:= -\Delta_\epsilon \bar{u}_s^{(0)} + \bar{u}_s^{(0)}\bar{u}_{sx}^{(0)} + \bar{v}_s^{(0)}\bar{u}_{sy}^{(0)} + \bar{P}_x^{(0)} + v_e^1 u_{py}^0 \\
&= -\epsilon u_{pxx}^0 + \sqrt{\epsilon} y v_{eY}^1 u_{py}^0 + \epsilon u_{py}^0 \int_0^y \int_y^{y'} v_{eYY}^1 \, dy'' dy'.
\end{aligned}
\tag{3.6.9}
$$

This then accounts for the lowest order term from $R_E^{u,1}$ in (3.6.5), allowing us to redefine:

$$
R_E^{u,1} = \sqrt{\epsilon} u_{ex}^1 u_p^0 + \sqrt{\epsilon} u_e^1 u_{px}^0 + \epsilon v_p^0 u_{eY}^1. \tag{3.6.10}
$$

Let us now define the auxiliary Euler pressure:

$$
P_E^{1,a} := -\frac{1}{2}\Big(\Big|u_e^1\Big|^2 + \Big|v_e^1\Big|^2 \Big), \tag{3.6.11}
$$

so that, combined with the equations we have taken for $[u_e^1, v_e^1]$, we have:

Lemma 3.6.1. *With $P^{1,a}_E$ as in (3.6.11), and $E^{u,1}_E, E^{v,1}_E$ as in (3.6.5) - (3.6.6),*

$$\partial_x \epsilon P^{1,a}_E + E^{u,1}_E = \frac{\partial_y}{\epsilon}\epsilon P^{1,a}_E + E^{v,1}_E = 0. \tag{3.6.12}$$

Remark. Denoting by $\nabla_\epsilon := (\partial_x, \frac{\partial_Y}{\sqrt{\varepsilon}})$, the above lemma reads:

$$\nabla_\varepsilon \varepsilon P^{1,a}_E + (E^{u,1}_E, E^{v,1}_E) = 0. \tag{3.6.13}$$

Thus, the purely Eulerian terms are of gradient structure, which is then exploited with the introduction of our auxiliary pressure.

Proof. The proof follows by direct calculation and an appeal to (3.5.5):

$$\varepsilon \partial_x P^{1,a}_E = -\varepsilon u^1_e u^1_{ex} - \varepsilon v^1_e v^1_{ex} = -\varepsilon u^1_e u^1_{ex} - \varepsilon v^1_e u^1_{eY} = -E^{u,1}_E. \tag{3.6.14}$$

Similarly,

$$\frac{\partial_Y}{\sqrt{\varepsilon}}\varepsilon P^{1,a}_E = -\sqrt{\varepsilon} u^1_e u^1_{eY} - \varepsilon v^1_e v^1_{eY} = -\sqrt{\varepsilon} u^1_e v^1_{ex} - \varepsilon v^1_e v^1_{eY} = -E^{v,1}_E. \tag{3.6.15}$$

The claim has been proven.

∎

Corollary 3.6.2. *All "purely Eulerian" terms from the expansions (3.6.7), (3.6.8) vanish identically. That is:*

$$E^{u,1}_E - \epsilon^{\frac{3}{2}}\Delta u^1_e + \epsilon P^{1,a}_{Ex} + \sqrt{\epsilon}(u^1_{ex} + P^1_{ex}) = 0, \tag{3.6.16}$$

$$E^{v,1}_E - \epsilon \Delta v^1_e + \frac{\partial_y}{\epsilon}\epsilon P^{1,a}_E + (\frac{\partial_y}{\epsilon}\sqrt{\epsilon} P^1_e + v^1_{ex}) = 0. \tag{3.6.17}$$

Proof. First, by harmonicity of $[u^1_e, v^1_e]$, we have:

$$\epsilon^{\frac{3}{2}} \Delta u^1_e = \epsilon \Delta v^1_e = 0. \tag{3.6.18}$$

Next, regarding the final terms in both (3.6.7) and (3.6.8), by the equations (3.5.2) and (3.5.4), we see that these terms drop out:

$$u^1_{ex} + P^1_{ex} = 0, \quad v^1_{ex} + P^1_{eY} = 0. \tag{3.6.19}$$

Coupled with (3.6.12), this establishes the desired claim.

$\square$

In the above calculation, we are using crucially the Cauchy-Riemann structure of $[u^1_e, v^1_e]$ in 3.5.7; harmonicity alone does not suffice here. Next, define the auxiliary Prandtl pressure:

$$P^{1,a}_p := \int_y^\infty \Big[- \Delta_\epsilon \bar{v}^{(0)}_s + \bar{u}^{(0)}_s \bar{v}^{(0)}_{sx} + \bar{v}^{(0)}_s \bar{v}^{(0)}_{sy} \Big] + \int_y^\infty R^{v,1}_E. \tag{3.6.20}$$

Combining all this, we take our Prandtl-1 equation to be:

$$-u^1_{pyy} + (1 + u^0_p) u^1_{px} + u^{(1)}_{sx} u^1_p + u^0_{py} \Big(v^1_p - v^1_p(x,0) \Big) + v^{(1)}_s u^1_{py} + P^1_{px} = f^{(1)}, \tag{3.6.21}$$

together with the divergence free condition $u^1_{px} + v^1_{py} = 0$, and the boundary conditions:

$$u^1_p(x,0) = 0, \quad \lim_{y \to \infty} [u^1_p(x,y), v^1_p(x,y)] = 0, \quad v^1_p(x,0) = -v^2_e(x,0), \quad u^1_p(1,y) = U_1(y). \tag{3.6.22}$$

Here, the forcing term is defined as:

$$f^{(1)} := \epsilon^{-\frac{1}{2}}\Big[R^{u,0} + R^{u,1}_E + \epsilon P^{1,a}_{px}\Big]. \tag{3.6.23}$$

The boundary contribution of $v^1_p(x,0)$ in (3.6.21) again arises from the calculation:

$$\begin{aligned} v^2_e(x,Y)u^0_{py} &= v^2_e(x,0)u^0_{py} + \sqrt{\epsilon}yu^0_{py}v^2_{eY} + \epsilon u^0_{py}\int_0^y\int_y^{y'} v^2_{eYY} \\ &= -v^1_p(x,0)u^0_{py} + \sqrt{\epsilon}yu^0_{py}v^2_{eY} + \epsilon u^0_{py}\int_0^y\int_y^{y'} v^2_{eYY}. \end{aligned} \tag{3.6.24}$$

Consolidating (3.6.18), (3.6.19), (3.6.12), (3.6.20), (3.6.21), (3.6.24) with the expressions (3.6.7) and (3.6.8) then shows that the following remainder is contributed:

$$\begin{aligned} R^{u,1} = \sqrt{\epsilon}\Big[&- \epsilon u^1_{pxx} + \sqrt{\epsilon}yv^2_{eY}u^0_{py} + \epsilon u^0_{py}\int_0^y\int_y^{y'} v^2_{eYY} + (\epsilon u^1_{eY} + \sqrt{\epsilon}u^1_{py})v^1_p \\ &+ \sqrt{\epsilon}(u^1_e + u^1_p)u^1_{px}\Big], \end{aligned} \tag{3.6.25}$$

$$R^{v,1} = \sqrt{\epsilon}\Big[- \Delta_\epsilon v^1_p + u^{(1)}_s v^1_{px} + v^{(1)}_{sx}u^1_p + v^{(1)}_s v^1_{py} + v^{(1)}_{sy}v^1_p + \sqrt{\epsilon}u^1_p v^1_{px} + \sqrt{\epsilon}v^1_p v^1_{py}\Big]. \tag{3.6.26}$$

Let us emphasize the boundary condition at $y \to \infty$ for v^1_p means that we will define:

$$v^1_p(x,y) = \int_y^\infty u^1_{px}(x,y')dy'. \tag{3.6.27}$$

We may evaluate the equation (3.6.21) at $y = \infty$ to see that the pressure term drops out. That is the pressure in the Prandtl layer is constant, so we may WLOG take $P^1_p = 0$.

3.6.2 Global in x Existence and Decay:

The first step is to homogenize the boundary conditions by introducing the new unknowns:

$$u = u_p^1 + \chi(y)u_e^1(x,0), \quad v = v_p^1 - v_p^1(x,0) + u_{ex}^1(x,0)I_\chi(y), \quad I_\chi(y) = \int_y^\infty \chi(\theta)d\theta, \tag{3.6.28}$$

where $\chi(y)$ is a cutoff function satisfying:

$$\chi(0) = 1, \quad \partial_y^k \chi(0) = 0, \quad \int_0^\infty \chi(y)dy = 0. \tag{3.6.29}$$

The mean-zero condition is meant to ensure that $v(0) = 0$. The homogenized profiles now satisfy the following system:

$$(1 + u_p^0)u_x - u_{yy} + \mathcal{P}(u,v) = f^{(1)} + \mathcal{J}; \quad u(x,0) = v(x,0) = 0, \quad \lim_{y\to\infty} u(x,y) = 0. \tag{3.6.30}$$

along with the divergence free condition $u_x + v_y = 0$. Here:

$$\mathcal{P} := u_{sx}^{(1)}u + u_{py}^0 v + v_s^{(1)}u_y, \tag{3.6.31}$$

$$\begin{aligned}\mathcal{J} := {} & u_s^{(1)}\chi(y)u_{ex}^1(x,0) - \chi'' u_e^1(x,0) - \chi(y)u_{sx}^{(1)}u_e^1(x,0) - u_{py}^0 I_\chi(y)u_{ex}^1(x,0) \\ & + v_s^{(1)}\chi'(y)u_e^1(x,0). \end{aligned}\tag{3.6.32}$$

We note that $v(x,y)$ does not vanish as $y \to \infty$ due to the definition in (4.9.80). The essential feature of $v(x,y)$ that will be in use is that $v(x,0) = 0$. Examining (3.6.31), one observes that v is always accompanied by u_p^0, which decays rapidly in y for each fixed x.

Let us now define the norm in which we shall control the Prandtl solutions:

$$||u||^2_{P(X_1,\sigma)} := \sup_{1 \le x \le X_1} \{\int u^2 x^{-2\sigma} + u_y^2 x^{1-2\sigma}\} + \int_1^{X_1} \int \{u^2 x^{-1-2\sigma} + u_y^2 x^{-2\sigma} + u_x^2 x^{1-2\sigma}\}. \tag{3.6.33}$$

We shall also need the following, differentiated version, of the above Prandtl-layer norm:

$$\begin{aligned} ||u||^2_{P_k(X_1,\sigma)} := &\sup_{1 \le x \le X_1} \{\int |\partial_x^k u|^2 x^{2k-2\sigma} + |\partial_x^k u_y|^2 x^{2k+1-2\sigma}\} \\ &+ \int_1^{X_1} \int \{|\partial_x^k u|^2 x^{2k-1-2\sigma} + |\partial_x^k u_y|^2 x^{2k-2\sigma} + |\partial_x^{k+1} u|^2 x^{2k+1-2\sigma}\}. \end{aligned} \tag{3.6.34}$$

Through the $H^1_y \hookrightarrow L^\infty_y$ embedding, it is clear that:

$$\sup_{1 \le x \le X_1} x^{k+\frac{1}{4}-\sigma} ||\partial_x^k u||_{L^\infty} \le ||u||_{P_k(X_1,\sigma)}. \tag{3.6.35}$$

Finally, we shall write the global norms as:

$$||u||_{P(\sigma)} := ||u||_{P(\infty,\sigma)}, \quad ||u||_{P_k(\sigma)} := ||u||_{P_k(\infty,\sigma)}. \tag{3.6.36}$$

Remark. A comparison of the norms P_k with the estimates valid for $[u^0_p, v^0_p]$ in (3.4.86), (3.4.91) show that u^1_p is roughly "$x^{-\frac{1}{4}}$-better" than u^0_p. There is no front-profile as in the case of $[u^0_p, v^0_p]$, which is because the present boundary condition, $u^1_p(x,0) = -u^1_e(x,0)$ decays as $x \to \infty$, due to (3.5.64), in contrast with the boundary condition for $u^0_p(x,0) = -\delta$. The secondary reason is because $f^{(1)}$ in (3.6.23) contains derivative and nonlinear terms from the previous layers, which enhances the decay in x.

The first step is to give the following estimates on the forcing terms, which capitalize on the structure of these terms either having many derivatives (thereby enhancing decay in x) or are of product form (also enhancing decay in x):

Lemma 3.6.3 (Forcing Estimates)**.** *For any $k, m \geq 0$, and arbitrary $N > 0$*

$$\Big| z^m \partial_x^k \mathcal{J} \Big| \leq C(k,m) \langle y \rangle^{-N} x^{-k-\frac{1}{2}}, \tag{3.6.37}$$

$$||z^m \partial_x^k f^{(1)}||_{L^2_y} \leq C(k,m) \langle x \rangle^{-k-\frac{5}{4}}. \tag{3.6.38}$$

Proof. The estimate for the $\mathcal{J}$ terms is direct, thanks to Proposition 3.5.6, estimate (3.5.61). For $f^{(1)}$, in consultation with the definition (3.6.23), we start with the $R^{u,0}$, defined in (3.6.9):

$$\sqrt{\epsilon} ||u^0_{pxx}||_{L^2_y} \lesssim \sqrt{\epsilon} x^{-\frac{7}{4}}, \tag{3.6.39}$$

$$||y u^0_{py} v^1_{eY}||_{L^2_y} \leq ||x^{-\frac{1}{4}} y u^0_{py}||_{L^2_y} \sup_y \Big| v^1_{eY} \Big| x^{\frac{1}{4}} \lesssim x^{-\frac{5}{4}}, \tag{3.6.40}$$

$$\sqrt{\epsilon} ||u^0_{py} \int_0^y \int_y^{y'} v^1_{exx} dy'' dy' ||_{L^2_y} \leq \sqrt{\epsilon} ||y^2 u^0_{py}||_{L^2_y} ||v^1_{exx}||_{L^\infty_y} \lesssim \sqrt{\epsilon} x^{-\frac{7}{4}}. \tag{3.6.41}$$

Second, we control $R^{u,1}_E$ via:

$$\begin{aligned} \epsilon^{-\frac{1}{2}} ||R^{u,1}_E||_{L^2_y} &\leq ||u^1_{ex}||_{L^\infty_y} ||u^0_p||_{L^2_y} + ||u^1_e||_{L^\infty_y} ||u^0_{px}||_{L^2_y} + \sqrt{\epsilon} ||v^0_p||_{L^2_y} ||u^1_{eY}||_{L^\infty_y} \\ &\lesssim x^{-\frac{5}{4}}. \end{aligned} \tag{3.6.42}$$

Next, according to (3.6.20), we have:

$$||P^{1,a}_{px}||_{L^2_y} \leq ||\langle y \rangle^{\frac{3}{2}+\kappa} \partial_x \Big(- \Delta_\epsilon \bar{v}^{(0)}_s + \bar{u}^{(0)}_s \bar{v}^{(0)}_{sx} + \bar{v}^{(0)}_s \bar{v}^{(0)}_{sy} \Big) ||_{L^\infty_y} + ||\langle y \rangle^{\frac{3}{2}+\kappa} R^{v,1}_{Ex}||_{L^\infty_y}. \tag{3.6.43}$$

For each of these terms, the ability to trade $x^{\frac{3}{4}+\frac{\kappa}{2}}$ for $y^{\frac{3}{2}+\kappa}$ is in constant use:

$$\begin{aligned} &||\langle y \rangle^{\frac{3}{2}+\kappa} \Delta_\epsilon v^0_{px}||_{L^\infty_y} + ||\langle y \rangle^{\frac{3}{2}+\kappa} \partial_x \{ \bar{u}^{(0)}_s \bar{v}^{(0)}_{sx} \} ||_{L^\infty_y} + ||\langle y \rangle^{\frac{3}{2}+\kappa} \partial_x \{ \bar{v}^{(0)}_s \bar{v}^{(0)}_{sy} \} ||_{L^\infty_y} \\ &+ ||\langle y \rangle^{\frac{3}{2}+\kappa} \partial_x \{ u^0_p v^1_{ex} \} ||_{L^\infty_y} + ||\langle y \rangle^{\frac{3}{2}+\kappa} \partial_x \{ \sqrt{\epsilon} u^1_e v^0_{px} \} ||_{L^\infty_y} + ||\langle y \rangle^{\frac{3}{2}+\kappa} \partial_x \{ \sqrt{\epsilon} v^0_p v^1_{eY} \} ||_{L^\infty_y} \end{aligned}$$

$$+ ||\langle y\rangle^{\frac{3}{2}+\kappa}\partial_x\{v_e^1 v_{py}^0\}||_{L_y^\infty} \lesssim x^{-\frac{3}{2}}. \tag{3.6.44}$$

Above, we are using the established estimates in (3.4.86), (3.4.91) for the Prandtl-0 profiles, and (3.5.61) for the Euler-1 profiles. The desired estimates are proven for $m = 0$. For general m the estimates work in an identical manner, after noticing that powers of z play no role when accompanied by $[u_p^0, v_p^0]$, which appear in every term above.

□

The above lemma relies crucially on Corollary 3.6.2 in order to apply the weight z^m. We now give the following energy estimate:

Lemma 3.6.4. *Let $\sigma > 0$ and fix any $X_1 > 1$. Then:*

$$\sup_{x\in[1,X_1]}\int x^{-2\sigma}u^2 + \int_1^{X_1}\int x^{-1-2\sigma}u^2 + \int_1^{X_1}\int x^{-2\sigma}u_y^2$$
$$\lesssim \mathcal{O}(\delta;\sigma)\int_1^{X_1}\int v_y^2 x^{1-2\sigma} + C(\sigma). \tag{3.6.45}$$

The constant above depends poorly on small σ.

Remark. The need for this $\sigma > 0$ is to avoid certain x-integrations being critical. We make the notational convention that we will not rename different values for σ (for instance 2σ), as we can always redefine σ to be smaller. However, *within* a single calculation (for instance the upcoming proof) we fix a $\sigma > 0$.

Proof. Applying the multiplier $M = ux^{-2\sigma}$ to the system (3.6.30) gives the following terms:

$$\int\Big((1+u_p^0)\partial_x-\partial_{yy}\Big)u\cdot ux^{-2\sigma} \gtrsim \partial_x\int(1+u_p^0)x^{-2\sigma}u^2+\int x^{-1-2\sigma}u^2+\int x^{-2\sigma}u_y^2-\int u_{px}^0 u^2 x^{-2\sigma}. \tag{3.6.46}$$

The constant in the above estimate depends poorly on σ, as there is a factor of σ accompanying the $\int x^{-1-2\sigma}u^2$ term. The final term above then gets placed into the contributions from $\mathcal{P}$, to which we now turn (see the definition in (3.6.31)):

$$\begin{aligned}\Big|\int \mathcal{P}\cdot ux^{-2\sigma}\Big| &\le ||xu^{(1)}_{sx}, yu^0_{py}, v^{(1)}_{sy}x||_\infty\Big(\int u^2x^{-1-2\sigma}+\int v_y^2x^{1-2\sigma}\Big)\\ &\le \mathcal{O}(\delta)\Big(\int u^2x^{-1-2\sigma}+\int v_y^2x^{1-2\sigma}\Big). \end{aligned} \tag{3.6.47}$$

Above, we have used the Prandtl-0 bounds in (3.4.86), which crucially provides the smallness of $\{u^0_{px}, u^0_{py}\}$ in terms of $\mathcal{O}(\delta)$. No smallness of Eulerian profiles is required due to the extra factor of $\sqrt{\varepsilon}$. The key estimate which forces a loss of ∂_x derivative is the following:

$$\Big|\int u^0_{py}v\cdot ux^{-2\sigma}\Big| \le ||u^0_{py}y||_{L^\infty}\int\Big|\frac{v}{y}ux^{-2\sigma}\Big| \le \mathcal{O}(\delta)||v_yx^{\frac12-\sigma}||_{L^2_y}||ux^{-\frac12-\sigma}||_{L^2_y}. \tag{3.6.48}$$

The structure of the key estimate above, (3.6.48), is omnipresent in our analysis: we use the y-absorption of u^0_{py} to produce a v_y term via Hardy's inequality (which is valid as $v(x,0)=0$). This then forces a loss of $x^{\frac12-\sigma}$ in the decay, which must then be regained in the next lemma. Again, the estimate $||yu^0_{py}||_{L^\infty}\le\mathcal{O}(\delta)$ arises from (3.4.86). Next, we arrive at the forcing terms. First, via (3.6.37):

$$\begin{aligned}\Big|\int \mathcal{J}\cdot ux^{-2\sigma}\Big| &\lesssim \int\langle y\rangle^{-N}x^{-\frac12}|u|x^{-2\sigma} \lesssim ||x^{-\frac12-\sigma}\langle y\rangle^{-\frac{N}{2}}||_{L^2_y}||x^{-\sigma}\frac{u}{\langle y\rangle^{\frac{N}{2}}}||_{L^2_y}\\ &\le \frac{1}{100{,}000}||u_yx^{-\sigma}||^2_{L^2_y}+Cx^{-1-2\sigma}. \end{aligned} \tag{3.6.49}$$

Upon taking an integration in dx, the majorizing terms above are finite. Next, according to (3.6.38), via Young's inequality:

$$\Big|\int f^{(1)}\cdot ux^{-2\sigma}\Big| \le ||f^{(1)}x^{\frac12-\sigma}||_{L^2_y}||ux^{-\frac12-\sigma}||_{L^2_y}\le Cx^{-\frac32}+\frac{1}{100{,}000}\int u^2x^{-1-2\sigma}dy.$$

Placing these estimates together:

$$\partial_x \int x^{-2\sigma} u^2 + \int x^{-1-2\sigma} u^2 + \int x^{-2\sigma} u_y^2 \lesssim \mathcal{O}(\delta) \int u_x^2 x^{1-2\sigma} + x^{-1-2\sigma}.$$

Integrating above from $x = 1$ to $x = X_1$ yields:

$$\int X_1^{-2\sigma} u^2(X_1) dy + \int_1^{X_1} \int x^{-1-2\sigma} u^2 + \int_1^{X_1} \int x^{-2\sigma} u_y^2 \lesssim \mathcal{O}(\delta) \int_1^{X_1} \int v_y^2 x^{1-2\sigma} + C. \tag{3.6.50}$$

Finally, we reason as follows: the second and third terms on the left-hand side above are positive, which then gives for this X_1:

$$\int X_1^{-2\sigma} u^2(X_1) dy \lesssim \mathcal{O}(\delta) \int_1^{X_1} \int v_y^2 x^{1-2\sigma} + C. \tag{3.6.51}$$

For any $X_2 \in [1, X_1]$, the same estimate holds, namely:

$$\int X_2^{-2\sigma} u^2(X_2) dy \lesssim \mathcal{O}(\delta) \int_1^{X_2} \int v_y^2 x^{1-2\sigma} + C \lesssim \mathcal{O}(\delta) \int_1^{X_1} \int v_y^2 x^{1-2\sigma} + C. \tag{3.6.52}$$

Above, we have used the monotonicity of the right-hand side as $X_2 < X_1$. This allows us to replace in (3.6.50) the first term on the left-hand side with $\sup_{1 \le x \le X_1} \int x^{-2\sigma} u^2 dy$. This then gives the desired estimate in (3.6.45).

$\square$

We now recover the v_y term on the right-hand side of (3.6.45) via:

Lemma 3.6.5. *Let $\sigma > 0$, and fix any $X_1 > 1$. Then:*

$$\sup_{1 \le x \le X_1} \int u_y^2 x^{1-2\sigma} + \int_1^{X_1} \int u_x^2 x^{1-2\sigma} \lesssim C + \mathcal{O}(\delta) \int_1^{X_1} \int u^2 x^{-1-2\sigma} + \int_1^{X_1} \int u_y^2 x^{-2\sigma}. \tag{3.6.53}$$

Proof. We now apply the multiplier $M = u_x x^{1-2\sigma}$ to the system in (3.6.30). Doing so yields the following positive terms:

$$\int \Big((1 + u_p^0)\partial_x - \partial_{yy} \Big) u \cdot u_x x^{1-2\sigma} \gtrsim \partial_x \int u_y^2 x^{1-2\sigma} - \int u_y^2 x^{-2\sigma} + \int u_x^2 x^{1-2\sigma}. \tag{3.6.54}$$

Note crucially that the middle term in the above estimate, $||u_y x^{-\sigma}||^2_{L^2_y}$, has been estimated in (3.6.45) upon taking an x-integration. To close this sequence of estimates, therefore, it is crucial that this small parameter $\mathcal{O}(\delta)$ is attached to the $||v_y x^{\frac{1}{2}-\sigma}||^2_{L^2}$ term in (3.6.45). Next, we come to the profile terms, $\mathcal{P}$ (see (3.6.31) for the definition):

$$\begin{aligned} \Big| \int \mathcal{P} \cdot u_x x^{1-2\sigma} \Big| &\le ||x u_{sx}^{(1)}, y u_{py}^0, v_s^{(1)} x^{\frac{1}{2}}||_{L^\infty} \Big(||u x^{-\frac{1}{2}-\sigma}||^2_{L^2_y} + ||u_x x^{\frac{1}{2}-\sigma}||^2_{L^2_y} \Big) \\ &\le \mathcal{O}(\delta) \Big(||u x^{-\frac{1}{2}-\sigma}||^2_{L^2_y} + ||u_x x^{\frac{1}{2}-\sigma}||^2_{L^2_y} \Big). \end{aligned} \tag{3.6.55}$$

We have used estimates (3.4.86), (3.4.91), and (3.5.61), which provide the smallness of $\mathcal{O}(\delta)$. Next, we come to $f^{(1)}$, for which we use the estimate in (3.6.38) (with $k = 0$):

$$\Big| \int f^{(1)} u_x x^{1-2\sigma} \Big| \le ||x^{\frac{1}{2}-\sigma} f^{(1)}||_{L^2_y} ||u_x x^{\frac{1}{2}-\sigma}||_{L^2_y} \le C x^{-\frac{3}{2}} + \frac{1}{100{,}000} ||u_x x^{\frac{1}{2}-\sigma}||^2_{L^2_y}. \tag{3.6.56}$$

Piecing the above estimates together gives:

$$\partial_x \int u_y^2 x^{1-2\sigma} + \int u_x^2 x^{1-2\sigma}$$

$$\leq \int u_y^2 x^{-2\sigma} + \mathcal{O}(\delta)\int u^2 x^{-1-2\sigma} + Cx^{-1-\sigma} + \int \mathcal{J}\cdot u_x x^{1-2\sigma}. \tag{3.6.57}$$

Now, we take an integration from $x=1$ to $x=X_1$:

$$\begin{aligned}\int u_y^2(X_1)X_1^{1-2\sigma} + \int_1^{X_1}\int u_x^2 x^{1-2\sigma} \lesssim \int_1^{X_1}\int u_y^2 x^{-2\sigma} \\ + \mathcal{O}(\delta)\int_1^{X_1}\int u^2 x^{-1-2\sigma} + C + \int_1^{X_1}\int \mathcal{J}\cdot u_x x^{1-2\sigma}.\end{aligned} \tag{3.6.58}$$

The last step is to come to the terms in $\mathcal{J}$, from (3.6.32). The most delicate term here requires successive integration by parts:

$$\int_1^{X_1}\int \chi''(y)x^{1-2\sigma}u_e^1(x,0)u_x \tag{3.6.59}$$

$$\begin{aligned}= -\int_1^{X_1}\int \chi''(y)u\partial_x\{u_e^1(x,0)x^{1-2\sigma}\} - \int_{x=1} u_e^1(1,0)u\chi''(y)dy \\ + \int_{x=X_1} X_1^{1-2\sigma}u_e^1(X_1,0)\chi''(y)u(X_1,y)dy\end{aligned} \tag{3.6.60}$$

$$\begin{aligned}= -\int_1^{X_1}\int \chi''(y)u\partial_x\{u_e^1(x,0)x^{1-2\sigma}\} + C \\ - \int_{x=X_1} X_1^{1-2\sigma}u_e^1(X_1,0)\chi'(y)u_y(X_1,y)dy\end{aligned} \tag{3.6.61}$$

$$\leq \int_1^{X_1}\int \chi'(y)u_y\partial_x\{u_e^1(x,0)x^{1-2\sigma}\} + C + \frac{1}{100,000}\int_{x=X_1} X_1^{1-2\sigma}u_y^2(X_1) \tag{3.6.62}$$

$$\lesssim ||u_y x^{-\sigma}||_{L^2}||\chi'(y)x^{-\frac{1}{2}-\sigma}||_{L^2} + C \leq C + \frac{1}{100,000}||u_y x^{-\sigma}||_{L^2}^2. \tag{3.6.63}$$

Going from (3.6.61) to (3.6.62), we have used $|u_e^1(X_1,0)| \lesssim X_1^{-\frac{1}{2}}$, according to (3.5.61). Going from (3.6.62) to (3.6.63), we have used $|u_{ex}^1(x,0)| \lesssim x^{\frac{3}{2}}$, also according to (3.5.61). Finally, we have absorbed the boundary contribution at $x=X_1$, $\int u_y^2 X_1^{1-2\sigma}$ into the left-hand side of (3.6.58). A consultation with (3.6.32) shows that we can perform a similar calculation with the remaining terms in $\mathcal{J}$ because these terms either have one extra x-derivative, or are accompanied by $v_s^{(1)}$, which contributes additional decay of $x^{-\frac{1}{2}}$. This then gives:

$$\int u_y^2(X_1)X_1^{1-2\sigma} + \int_1^{X_1}\int u_x^2 x^{1-2\sigma} \lesssim \int_1^{X_1}\int u_y^2 x^{-2\sigma} + \mathcal{O}(\delta)\int_1^{X_1}\int u^2 x^{-1-2\sigma} + C.$$

Using a similar line of reasoning as in the Lemma 3.6.4, we can replace the $\int u_y^2(X_1)X_1^{1-2\sigma}$ with the supremum over all $x \in [1, X_1]$, thereby yielding the desired result.

$\square$

Consolidating the results of the previous two lemmas and applying contraction mapping:

Corollary 3.6.6. *For δ, ϵ sufficiently small relative to σ, a solution to the Prandtl system in (3.6.30) satisfies the following a-priori estimate in the space P:*

$$||u||^2_{P(X_1,\sigma)} \lesssim C(\sigma). \tag{3.6.64}$$

For δ, ϵ sufficiently small, there exists a unique solution to (3.6.30), satisfying $||u||_{P(X_1)} \leq C(\sigma)$. The constant above in (3.6.64) is independent of X_1, and so we can immediately send $X_1 \to \infty$, thereby yielding a global solution on $x \in [1, \infty)$ satisfying: $||u||_P \leq C(\sigma)$.

It is possible to successively differentiate the system in ∂_x, and re-apply the previous estimates, noting that the added $x-$derivative adds a factor of x^{-1} to each term above, enabling us to enhance the multiplier to $[x^{k-2\sigma}\partial_x^k u, x^{1+k-2\sigma}\partial_x^{k+1}u]$. This is the reason for the differentiated version of the Prandtl-norm in (3.6.34). To do so, we simply need:

Lemma 3.6.7 (Initial Conditions)**.** *For each $k \geq 0$, the initial data $\partial_x^k u(1, y)$ is of order δ and decays rapidly in y.*

Proof. This follows from using the equation (3.6.21) to write:

$$\begin{aligned} u^1_{px}(1,y) &= U_{1yy}(y) + u^{(1)}_{sx}(1,y)U_1(y) + yu^0_{py}(1,y)\frac{v^1_p(1,y) - v^1_p(1,0)}{y} \\ &+ v^{(1)}_s(1,y)U_{1y}(y) + f^{(1)}(1,y). \end{aligned} \tag{3.6.65}$$

We may now multiply by $(1+y)^n$, use the inequality $||\frac{v(1,y)}{y}||_{L^\infty} \lesssim ||v_y(1,y)||_{L^\infty}$ for functions v satisfying $v(x,0) = 0$ (here we take $v = v^1_p(1,y) - v^1_p(1,0)$), and use $||\langle y\rangle^n U_1(y) u^0_{px}(1,y)||_{L^\infty} \le \mathcal{O}(\delta)||u^0_{px}(1,y)||_{L^\infty}$, to obtain: $||\langle y\rangle^n u^1_{px}(1,y)||_{L^\infty_y} \le C$. It is clear that the same procedure may be applied to higher order x-derivatives.

□

Remark. [Higher Order Compatibility] In order to apply the procedure described, we require high-order compatibility condition such that the initial data of $u^1_{px}(1,0) = 0$, thereby honoring the boundary condition. These condition can be ascertained inductively from (3.6.65). For instance:

$$0 = u^1_{px}(1,0) = U_{1yy}(0) + u^{(1)}_{sx}(1,0)U_1(0) + v^{(1)}_s U_{1y}(0) + f^{(1)}(1,0). \tag{3.6.66}$$

We suppose these compatibility conditions for large k.

Repeating the previous set of estimates after applying the self-similar weight z^M, and upon applying ∂^k_x to the system, we arrive at the following:

Lemma 3.6.8. *Given any $\sigma > 0$, let δ, ϵ be sufficiently small relative to σ. Consider the system given in (3.6.21), together with the boundary conditions (3.6.22). Let all derivatives of the prescribed data $U_1(y)$ be exponentially decaying in its argument. Then there exists a unique, global in x solution $[u_p, v_p]$, satisfying:*

$$||z^M \partial^k_x u||_{P_k(\sigma)} \le C(M,k,\sigma). \tag{3.6.67}$$

It is now our task to extract similar estimates for the profiles u^1_p, v^1_p from (3.6.67). First,

Lemma 3.6.9. *For any $m, k \ge 0, \sigma > 0$,*

$$\sup_{x \ge 1} \int z^{2m} |\partial_x^k u^1_p|^2 x^{2k-2\sigma} \le C(M, m, k, \sigma). \tag{3.6.68}$$

Proof. This follows by writing $u^1_p = u - \chi(y) u^1_e(x, 0)$, and using $\left| \partial_x^k u^1_e(x, 0) \right| \lesssim x^{-k-\frac{1}{2}}$. □

Next, we may give the following uniform decay estimate for v^1_p:

Corollary 3.6.10 (Uniform Estimates for v^1_p)**.** *For any $m \ge 0$,*

$$||z^m \partial_x^k v^1_p||_{L^\infty_y} \lesssim x^{-k-\frac{3}{4}+\sigma} C(\sigma, m, k). \tag{3.6.69}$$

Proof. First, according to (3.6.27), we have the rapid decay $v^1_p \to 0$ as $y \to \infty$. By the divergence-free condition, and the trace inequality, and for any small $\kappa > 0$,

$$\left| v^1_p \right|^2 \lesssim \left| \int_y^\infty v^1_p v_{py} \right| \le \int_0^\infty \left| \frac{v^1_p}{y^{\frac{1}{2}-\kappa}} y^{\frac{1}{2}-\kappa} v^1_{py} \right| \le || \frac{v^1_p}{y^{\frac{1}{2}-\kappa}} ||_{L^2_y} || y^{\frac{1}{2}-\kappa} v^1_{py} ||_{L^2_y} \tag{3.6.70}$$

$$\le || y^{\frac{1}{2}+\kappa} v^1_{py} ||_{L^2_y} || y^{\frac{1}{2}-\kappa} v^1_{py} ||_{L^2_y} = || x^{\frac{1}{4}+\kappa} z^{\frac{1}{2}+\kappa} v^1_{py} ||_{L^2_y} || x^{\frac{1}{4}-\kappa} z^{\frac{1}{2}-\kappa} v^1_{py} ||_{L^2_y} \tag{3.6.71}$$

$$\le x^{\frac{1}{2}} x^{-1+\sigma} x^{-1+\sigma} = x^{-\frac{3}{2}+2\sigma}. \tag{3.6.72}$$

We have used the $\kappa > 0$ to avoid the critical Hardy inequality. The Hardy inequality we have used (with power $y^{-\frac{1}{2}+\kappa} v^1_p$) relies on the vanishing of v^1_p at $y = \infty$. From here the desired bound follows for $m = 0, k = 0$, and $k, m \ge 1$ works analogously.

□

The final ingredient we will need is to understand the connection between the norms we have controlled, P_k, and the quantities u^1_{py}, u^1_{pyy}. This is the content of the following:

Lemma 3.6.11.

$$x^{\frac{3}{4}-\sigma}||z^m u^1_{py}||_{L^\infty_y} + x^{\frac{1}{2}-\sigma}||z^m u^1_{py}||_{L^2_y} + x^{1-\sigma}||z^m u^1_{pyy}||_{L^2_y} \le C < \infty. \tag{3.6.73}$$

Proof. First, let us record:

$$\begin{aligned} x^{1-2\sigma}\int z^{2m}\Big|u^1_{py}\Big|^2 &\le x^{1-2\sigma}\int z^{2m}u_y^2 + x^{1-2\sigma}\int z^{2m}\chi^2|u^1_e|^2(x,0) \\ &\le ||z^m u||_{P(\sigma)} + C. \end{aligned} \tag{3.6.74}$$

Via the equation (3.6.21), we have:

$$\begin{aligned} x^{1-\sigma}||u^1_{pyy}||_{L^2_y} &\le x^{1-\sigma}||u^1_{px}||_{L^2_y} + x^{1-\sigma}||\mathcal{P}||_{L^2_y} + x^{1-\sigma}||f^{(1)}||_{L^2_y} \\ &\lesssim ||u||_{P_1(\sigma)} + ||u||_{P(\sigma)} + C. \end{aligned} \tag{3.6.75}$$

We are using the decay rates established for u^0_p in (3.4.86), and the pointwise decay of the Euler profiles established in (3.5.48) and the relations in (3.6.68). Let us give $\mathcal{P}$ in detail,

$$x^{1-\sigma}||u^{(1)}_{sx}u^1_p||_{L^2_y} \le ||xu^{(1)}_{sx}||_{L^\infty}x^{-\sigma}||u^1_p||_{L^2_y} \le \mathcal{O}(\delta)||u||_P + \mathcal{O}(\delta), \tag{3.6.76}$$

$$\begin{aligned} x^{1-\sigma}||u^0_{py}\Big(v^1_p(x,y) - v^1_p(x,0)\Big)||_{L^2_y} &\le x^{1-\sigma}||yu^0_{py}||_{L^\infty}||\frac{v^1_p(x,y)-v^1_p(x,0)}{y}||_{L^2_y} \\ &\le \mathcal{O}(\delta)x^{1-\sigma}||u^1_{px}||_{L^2_y} \le \mathcal{O}(\delta)||u||_{P_1}, \end{aligned} \tag{3.6.77}$$

$$x^{1-\sigma}||v^{(1)}_s u^1_{py}||_{L^2_y} \le ||x^{\frac{1}{2}}v^{(1)}_s||_{L^\infty}x^{\frac{1}{2}-\sigma}||u^1_{py}||_{L^2_y} \le \mathcal{O}(\delta)||u||_P + \mathcal{O}(\delta). \tag{3.6.78}$$

For the final line we have used (3.6.74). Next, via Sobolev interpolation in the y direction,

$$x^{\frac{3}{4}-\sigma}||u^1_{py}||_{L^\infty_y} \le \Big(x^{1-\sigma}||u^1_{pyy}||_{L^2_y}\Big)^{\frac{1}{2}}\Big(x^{\frac{1}{2}-\sigma}||u^1_{py}||_{L^2_y}\Big)^{\frac{1}{2}} \le C||u||_P + \frac{1}{100}x^{1-\sigma}||u^1_{pyy}||_{L^2_y}. \tag{3.6.79}$$

Coupled with the u^1_{pyy} estimate in (3.6.75), this then establishes the desired bounds. The weighted estimate in z follows analogously. □

We will select now,

$$\sigma = \sigma_1 = \frac{3}{3^n \times 10,000}. \tag{3.6.80}$$

Summarizing the results of this section:

Proposition 3.6.12 (Prandtl-1 Layer Bounds)**.** *Given any $n \in \mathbb{N}$, let σ_1 be as in (3.6.80), and let δ, ϵ be sufficiently small relative to n and σ_1. Consider the system given in (3.6.21), together with the boundary conditions (3.6.22). Let all derivatives of the prescribed data $U_0(y)$ be exponentially decaying in its argument. Then there exists a unique, global in x solution $[u^1_p, v^1_p]$, satisfying:*

$$||z^M \partial_x^k u^1_p||_{P_k(\sigma_1)} + x^{k+\frac{3}{4}-\sigma_1}||z^m \partial_x^k v^1_p||_{L^\infty_y} \le C(M,k,n), \text{ for any } k, m \ge 0. \tag{3.6.81}$$

3.7 Intermediate Layers

We now construct intermediate layers $i = 2$ through $i = n-1$. This is achieved inductively, starting with the construction of the Euler Layer, u^i_e, v^i_e. Let us fix the parameters:

$$\sigma_i = \frac{3^i}{3^n \times 10,000} \text{ for } i = 2, ..., n. \tag{3.7.1}$$

The reason for this selection will be seen in (3.11.55).

3.7.1 Construction of Euler Layer, $[u_e^i, v_e^i]$

For this step in the construction, we suppose that $[\bar{u}_s^{(i-1)}, \bar{v}_s^{(i-1)}]$ have already been constructed. The inductive hypothesis on the $1, ..., i-1$ Prandtl profiles are as follows:

$$||z^m \partial_x^k u_p^j||_{P_k(\sigma_j)} \le C(k, m, n), \text{ for } j = 1, ..., i-1, \tag{3.7.2}$$

where $[u_p^j, v_p^j]$ satisfy the system given in (3.7.48) for $2 \le j \le i-1$, and for $i = 2$, that $[u_p^1, v_p^1]$ satisfy (3.6.21). In order to obtain the equations for $[u_e^i, v_e^i]$, we expand the nonlinear terms including the new Euler terms:

$$\begin{aligned} u_s^{(i)} u_{sx}^{(i)} &= \left(\bar{u}_s^{(i-1)} + \epsilon^{\frac{i}{2}} u_e^i\right)\left(\bar{u}_{sx}^{(i-1)} + \epsilon^{\frac{i}{2}} u_{ex}^i\right) \\ &= \bar{u}_s^{(i-1)} \bar{u}_{sx}^{(i-1)} + \epsilon^{\frac{i}{2}} \bar{u}_{sx}^{(i-1)} u_e^i + \epsilon^{\frac{i}{2}} \bar{u}_s^{(i-1)} u_{ex}^i + \epsilon^i u_e^i u_{ex}^i, \end{aligned} \tag{3.7.3}$$

$$\begin{aligned} v_s^{(i)} u_{sy}^{(i)} &= \left(\bar{v}_s^{(i-1)} + \epsilon^{\frac{i-1}{2}} v_e^i\right)\left(\bar{u}_{sy}^{(i-1)} + \sqrt{\epsilon}\epsilon^{\frac{i}{2}} u_{eY}^i\right) \\ &= \bar{v}_s^{(i-1)} \bar{u}_{sy}^{(i-1)} + \epsilon^{\frac{i-1}{2}} \bar{u}_{sy}^{(i-1)} v_e^i + \sqrt{\epsilon}\epsilon^{\frac{i}{2}} \bar{v}_s^{(i-1)} u_{eY}^i + \epsilon^i v_e^i u_{eY}^i. \end{aligned} \tag{3.7.4}$$

$$\begin{aligned} u_s^{(i)} v_{sx}^{(i)} &= \left(\bar{u}_s^{(i-1)} + \epsilon^{\frac{i}{2}} u_e^i\right)\left(\bar{v}_{sx}^{(i)} + \epsilon^{\frac{i-1}{2}} v_{ex}^i\right) \\ &= \bar{u}_s^{(i-1)} \bar{v}_{sx}^{(i-1)} + \epsilon^{\frac{i}{2}} \bar{v}_{sx}^{(i-1)} u_e^i + \epsilon^{\frac{i-1}{2}} \bar{u}_s^{(i-1)} v_{ex}^i + \epsilon^{i-\frac{1}{2}} u_e^i v_{ex}^i, \end{aligned} \tag{3.7.5}$$

$$\begin{aligned} v_s^{(i)} v_{sy}^{(i)} &= \left(\bar{v}_s^{(i-1)} + \epsilon^{\frac{i-1}{2}} v_e^i\right)\left(\bar{v}_{sy}^{(i-1)} + \epsilon^{\frac{i}{2}} v_{eY}^i\right) \\ &= \bar{v}_s^{(i-1)} \bar{v}_{sy}^{(i-1)} + \epsilon^{\frac{i-1}{2}} \bar{v}_{sy}^{(i-1)} v_e^i + \epsilon^{\frac{i}{2}} \bar{v}_s^{(i-1)} v_{eY}^i + \epsilon^{i-\frac{1}{2}} v_e^i v_{eY}^i. \end{aligned} \tag{3.7.6}$$

We will now define several terms:

Definition 3.7.1. The $i-1$'th remainder is denoted by:

$$R^{u,i-1} := -\Delta_\epsilon \bar{u}_s^{(i-1)} + \bar{u}_s^{(i-1)} \bar{u}_{sx}^{(i-1)} + \bar{v}_s^{(i-1)} \bar{u}_{sy}^{(i-1)} + \bar{P}_{sx}^{(i-1)} + \varepsilon^{\frac{i-1}{2}} v_e^i u_{py}^0, \tag{3.7.7}$$

$$R^{v,i-1} := -\Delta_\epsilon \bar{v}_s^{(i-1)} + \bar{u}_s^{(i-1)} \bar{v}_{sx}^{(i-1)} + \bar{v}_s^{(i-1)} \bar{v}_{sy}^{(i-1)} + \frac{\partial_y}{\epsilon} \bar{P}_s^{(i-1)}. \tag{3.7.8}$$

We will also split the Euler-Euler interaction terms and the Euler-Prandtl terms via:

Definition 3.7.2.

$$R_E^{u,i} := \epsilon^{\frac{i}{2}} u_{ex}^i \Big[\sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} u_p^j\Big] + \epsilon^{\frac{i}{2}} u_e^i \Big[\sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} u_{px}^j\Big]$$
$$+ \epsilon^{\frac{i}{2}} \sqrt{\epsilon} u_{eY}^i \Big[\sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} v_p^j\Big] + \epsilon^{\frac{i-1}{2}} v_e^i \Big[\sum_{j=1}^{i-1} \epsilon^{\frac{j}{2}} u_{py}^j\Big], \tag{3.7.9}$$

$$R_E^{v,i} := \epsilon^{\frac{i-1}{2}} v_{ex}^i \sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} u_p^j + \epsilon^{\frac{i}{2}} u_e^i \sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} v_{px}^j$$
$$+ \epsilon^{\frac{i-1}{2}} v_e^i \sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} v_{py}^j + \sqrt{\epsilon} \epsilon^{\frac{i-1}{2}} v_{eY}^i \sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} v_p^j, \tag{3.7.10}$$

$$E_e^{u,i} := \epsilon^i \Big[u_e^i u_{ex}^i + v_e^i u_{eY}^i\Big] + \epsilon^{\frac{i}{2}} u_{ex}^i \sum_{j=1}^{i-1} \epsilon^{\frac{j}{2}} u_e^j + \epsilon^{\frac{i}{2}} u_e^i \sum_{j=1}^{i-1} \epsilon^{\frac{j}{2}} u_{ex}^j$$
$$+ \sqrt{\epsilon} \epsilon^{\frac{i}{2}} u_{eY}^i \Big[\sum_{j=1}^{i-1} \epsilon^{\frac{j-1}{2}} v_e^j\Big] + \epsilon^{\frac{i-1}{2}} v_e^i \Big[\sum_{j=1}^{i-1} \epsilon^{\frac{1}{2}} \epsilon^{\frac{j}{2}} u_{eY}^j\Big], \tag{3.7.11}$$

$$E_e^{v,i} := \epsilon^{i-\frac{1}{2}} \Big[u_e^i v_{ex}^i + v_e^i v_{eY}^i\Big] + \epsilon^{\frac{i}{2}} u_e^i \Big[\sum_{j=1}^{i-1} \epsilon^{\frac{j-1}{2}} v_{ex}^j\Big] + \epsilon^{\frac{i-1}{2}} v_{ex}^i \Big[\sum_{j=1}^{i-1} \epsilon^{\frac{j}{2}} u_e^j\Big]$$
$$+ \sqrt{\epsilon} \epsilon^{\frac{i-1}{2}} v_e^i \sum_{j=1}^{i-1} \epsilon^{\frac{j-1}{2}} v_{eY}^j + \sqrt{\epsilon} \epsilon^{\frac{i-1}{2}} v_{eY}^i \sum_{j=1}^{i-1} \epsilon^{\frac{j-1}{2}} v_e^j. \tag{3.7.12}$$

Remark. The natural definition of the remainder term, $R^{u,i-1}$ should be:

$$R^{u,i-1} \text{``} = \text{''} -\Delta_\epsilon \bar{u}_s^{(i-1)} + \bar{u}_s^{(i-1)} \bar{u}_{sx}^{(i-1)} + \bar{v}_s^{(i-1)} \bar{u}_{sy}^{(i-1)} + \bar{P}_{sx}^{(i-1)}, \tag{3.7.13}$$

and the natural definition of $R_E^{u,i}$ would contain the lowest-order term, $\varepsilon^{\frac{i-1}{2}} v_e^i u_{py}^0$. However, for $i < n$, the quantity of interest in (3.7.22) is the sum, $R^{u,i-1} + R_E^{u,i}$, that is contributed to the next order (see the definition of the forcing in (3.7.31)). Thus, for convenience (see calculation 3.7.56 and (3.7.58)), we add and subtract one factor $\varepsilon^{\frac{i-1}{2}} v_e^i u_{py}^0$, which explains the definitions of (3.7.7) and (3.7.9). This, however, will not be done for $i = n$.

The Navier-Stokes expansion reads:

$$\begin{aligned}
-\Delta_\epsilon u_s^{(i)} + u_s^{(i)} u_{sx}^{(i)} + v_s^{(i)} u_{sy}^{(i)} + P_{sx}^{(i)} &= -\Delta_\epsilon \bar{u}_s^{(i-1)} + \bar{u}_s^{(i-1)} \bar{u}_{sx}^{(i-1)} + \bar{v}_s^{(i-1)} \bar{u}_{sy}^{(i-1)} + \bar{P}_{sx}^{(i-1)} \\
&- \epsilon^{\frac{i}{2}+1} \Delta u_e^i + \epsilon^{\frac{i}{2}} \bar{u}_{sx}^{(i-1)} u_e^i + \epsilon^{\frac{i}{2}} \bar{u}_s^{(i-1)} u_{ex}^i + \epsilon^i u_e^i u_{ex}^i + \epsilon^{\frac{i-1}{2}} \bar{u}_{sy}^{(i-1)} v_e^i \\
&+ \sqrt{\epsilon} \epsilon^{\frac{i}{2}} \bar{v}_s^{(i-1)} u_{eY}^i + \epsilon^i v_e^i u_{eY}^i + \epsilon^{\frac{i}{2}} P_{ex}^i + \epsilon^i P_{ex}^{1,a} \qquad (3.7.14) \\
&= \varepsilon^{\frac{i}{2}} \Big[u_{ex}^i + P_{ex}^i \Big] + \varepsilon^{\frac{i}{2}+1} \Delta u_e^i + R^{u,i-1} + R_E^{u,i} + E_e^{u,i} + \varepsilon^i P_{ex}^{i,a}. \qquad (3.7.15)
\end{aligned}$$

For the normal equation, the expansions read:

$$\begin{aligned}
-\Delta_\epsilon v_s^{(i)} + u_s^{(i)} v_{sx}^{(i)} + v_s^{(i)} v_{sy}^{(i)} + \frac{\partial_y}{\epsilon} P_s^{(i)} &= -\Delta_\epsilon \bar{v}_s^{(i-1)} + \bar{u}_s^{(i-1)} \bar{v}_{sx}^{(i-1)} + \bar{v}_s^{(i-1)} \bar{v}_{sy}^{(i-1)} + \frac{\partial_y}{\epsilon} \bar{P}_s^{(i-1)} \\
&- \epsilon^{\frac{i+1}{2}} \Delta v_e^i + \epsilon^{\frac{i}{2}} \bar{v}_{sx}^{(i-1)} u_e^i + \epsilon^{\frac{i-1}{2}} \bar{u}_s^{(i-1)} v_{ex}^i + \epsilon^{i-\frac{1}{2}} u_e^i v_{ex}^i + \epsilon^{\frac{i-1}{2}} \bar{v}_{sy}^{(i-1)} v_e^i \\
&+ \epsilon^{\frac{i}{2}} \bar{v}_s^{(i-1)} v_{eY}^i + \epsilon^{i-\frac{1}{2}} v_e^i v_{eY}^i + \epsilon^{\frac{i-1}{2}} P_{eY}^i + \epsilon^{i-\frac{1}{2}} P_{eY}^{1,a} \qquad (3.7.16) \\
&= \varepsilon^{\frac{i-1}{2}} \Big[v_{ex}^i + P_{eY}^i \Big] + \varepsilon^{\frac{i+1}{2}} \Delta v_e^i + R^{v,i-1} + R_E^{v,i} + E_e^{v,i} + \varepsilon^{i-\frac{1}{2}} P_{eY}^i. \qquad (3.7.17)
\end{aligned}$$

From here, we simply read off the highest order terms that are "purely-Eulerian". All of the remaining terms will be treated in the next subsection. In (3.7.14), these are at order $\epsilon^{\frac{i}{2}}$, and in (3.7.16), these are at order $\epsilon^{\frac{i-1}{2}}$:

$$\epsilon^{\frac{i}{2}} \Big[u_{ex}^i + P_{ex}^i \Big] = 0, \quad \epsilon^{\frac{i-1}{2}} \Big[v_{ex}^i + P_{eY}^i \Big] = 0 \qquad (3.7.18)$$

When paired with the divergence-free condition, we arrive at the equations that are taken for the $[u_e^i, v_e^i]$, which are the Cauchy-Riemann equations:

$$u_{ex}^i + P_{ex}^i = 0, \quad v_{ex}^i + P_{eY}^i = 0, \quad u_{ex}^i + v_{eY}^i = 0. \qquad (3.7.19)$$

The boundary conditions for the $i'th$ Euler layer is $v_e^i(x,0) = -v_p^{i-1}(x,0)$. According to the inductive hypothesis, the decay rate of this boundary condition is:

$$\left|v_e^i(x,0)\right| = \left|v_p^{i-1}(x,0)\right| \le ||v_p^{i-1}(x,y)||_{L_y^2}^{\frac{1}{2}} ||v_{py}^{i-1}(x,y)||_{L_y^2}^{\frac{1}{2}} \le ||yv_{py}^{i-1}(x,y)||_{L_y^2}^{\frac{1}{2}} ||v_{py}^{i-1}(x,y)||_{L_y^2}^{\frac{1}{2}}$$
$$\le C(\sigma,n)x^{-\frac{3}{4}+\sigma_{i-1}}, \text{ for } i \ge 2. \tag{3.7.20}$$

A comparison of (3.7.20) to (3.5.9) shows that the decay rate of the boundary condition has improved, enabling us to improve the Euler decay rates. Indeed, as the system (3.7.19) is the identical system to the first Euler layer (and is in particular the Cauchy-Riemann equations), we may simply repeat the analysis given there to conclude:

Proposition 3.7.3 (Euler-i Layer)**.** *Let $i \ge 2$. The i'th Euler layer, defined by the Cauchy-Riemann equations (3.7.19) taken with boundary conditions (3.7.20) satisfy the enhanced decay rates:*

$$x^{k+m+\frac{3}{4}-\sigma_{i-1}}\left|\partial_x^k \partial_Y^m v_e^i\right| + x^{\frac{3}{4}-\sigma_{i-1}}\left|u_e^i\right| \le C(k,m,n). \tag{3.7.21}$$

Proof. This follows by repeating the arguments in Section 3.5 with the enhanced boundary condition (3.7.20). □

Remark. It is not possible to enhance the rates (3.7.21) much more (for instance, past x^{-1} for $k = m = 0$. This is due to the restriction of $w \in (0,1)$ in Corollary 3.5.3.

We have the simplified expression for (3.7.15), (3.7.17)

Lemma 3.7.4. *According to definitions in (3.7.7) - (3.7.12), one has*

$$-\Delta_\epsilon u_s^{(i)} + u_s^{(i)} u_{sx}^{(i)} + v_s^{(i)} u_{sy}^{(i)} + P_{sx}^{(i)} = R^{u,i-1} + R_E^{u,i} + E_e^{u,i} + \varepsilon^i P_{ex}^{i,a}, \tag{3.7.22}$$

$$-\Delta_\epsilon v_s^{(i)} + u_s^{(i)} v_{sx}^{(i)} + v_s^{(i)} v_{sy}^{(i)} + \frac{\partial_y}{\epsilon} P_s^{(i)} = R^{v,i-1} + R_E^{v,i} + E_e^{v,i} + \varepsilon^{i-\frac{1}{2}} P_{eY}^i. \tag{3.7.23}$$

Proof. By construction in Proposition 3.7.3, the u_e^i, v_e^i are harmonic, and so the $\Delta u_e^i, \Delta v_e^i$ terms vanish identically. This then accounts for all of the terms in (3.7.14) - (3.7.16). □

3.7.2 Construction of Prandtl Layer, $[u_p^i, v_p^i]$

For this step in the construction, we suppose that $[u_s^{(i)}, v_s^{(i)}]$ have been constructed. The inductive hypothesis on these profiles are that the following remainders (according to the definitions in (3.7.7) for $R^{u,i-1}$ and (3.7.8) for $R^{v,i-1}$) have been accumulated:

$$R^{u,i-1} = \epsilon^{\frac{i-1}{2}}\Big[\epsilon u_{pxx}^{i-1} + \sqrt{\epsilon} y u_{py}^0 v_{eY}^i + u_{px}^{i-1}\sum_{j=1}^{i-1}\varepsilon^{\frac{j}{2}}\{u_e^j + u_p^j\}$$
$$+ \epsilon u_{py}^0 \int_0^y \int_y^{y'} v_{eYY}^i dy'' dy' + v_p^{i-1}\Big(\sum_{j=1}^{i-1}\epsilon^{\frac{j}{2}}\{u_{py}^j + \sqrt{\epsilon} u_{eY}^j\}\Big)\Big], \qquad (3.7.24)$$
$$R^{v,i-1} = \epsilon^{\frac{i-1}{2}}\Big[- \Delta_\epsilon v_p^{i-1} + u_s^{(i-1)} v_{px}^{i-1} + v_{sx}^{(i-1)} u_p^{i-1} + v_s^{(i-1)} v_{py}^{i-1} + v_{sy}^{(i-1)} v_p^{i-1}$$
$$+ \epsilon^{\frac{i-1}{2}} u_p^{i-1} v_{px}^{i-1} + \epsilon^{\frac{i-1}{2}} v_p^{i-1} v_{py}^{i-1}\Big]. \qquad (3.7.25)$$

The induction will start at $i = 2$, and so (3.7.24) - (3.7.25) should be compared to (3.6.26) for this case and to (3.7.50) for the general i case. The relevant profile estimates, according to Propositions 3.5.6, 3.7.3 and 3.6.12 which hold inductively are:

$$x^{k+m+\frac{1}{2}}|\partial_x^k \partial_Y^m v_e^1| + x^{\frac{1}{2}}|u_e^1| \le C(k,m,n), \qquad (3.7.26)$$
$$x^{k+m+\frac{3}{4}-\sigma_{j-1}}\left|\partial_x^k \partial_Y^m v_e^j\right| + x^{\frac{3}{4}-\sigma_{j-1}}\left|u_e^j\right| \le C(k,m,n) \text{ for } j = 2,..,i, \qquad (3.7.27)$$
$$||z^m \partial_x^k u_p^j||_{P_k(\sigma_j)} \le C(k,m,n) \text{ for } j = 1,...,i-1. \qquad (3.7.28)$$

By writing $\bar{u}_s^{(i)} = u_s^{(i)} + \epsilon^{\frac{i}{2}} u_p^{(i)}$, $\bar{v}_s^{(i)} = v_s^{(i)} + \epsilon^{\frac{i-1}{2}} v_p^{(i)}$, and expanding the Navier-Stokes equations, we obtain the two expansions:

$$-\Delta_\varepsilon \bar{u}_s^{(i)} + \bar{u}_s^{(i)}\bar{u}_{sx}^{(i)} + \bar{v}_s^{(i)}\bar{u}_{sy}^{(i)} + \bar{P}_{sx}^{(i)} = -\Delta_\epsilon u_s^{(i)} + u_s^{(i)} u_{sx}^{(i)} + v_s^{(i)} u_{sy}^{(i)} + P_{sx}^{(i)} + \epsilon^{\frac{i+1}{2}} P_{px}^{i,a}$$
$$+ \epsilon^{\frac{i}{2}}\Big[- \Delta_\varepsilon u_p^i + u_{sx}^{(i)} u_p^i + u_s^{(i)} u_{px}^i + u_{sy}^{(i)} v_p^i + v_s^{(i)} u_{py}^i + \epsilon^{\frac{i}{2}} u_p^i u_{px}^i + \epsilon^{\frac{i}{2}} v_p^i u_{py}^i + P_{px}^i\Big], \qquad (3.7.29)$$

and:

$$-\Delta_\epsilon \bar{v}_s^{(1)} + \bar{u}_s^{(i)}\bar{v}_{sx}^{(i)} + \bar{v}_s^{(i)}\bar{v}_{sy}^{(i)} + \frac{\partial_y}{\epsilon}\bar{P}_s^{(i)} = -\Delta_\epsilon v_s^{(i)} + u_s^{(i)}v_{sx}^{(i)} + v_s^{(i)}v_{sy}^{(i)} + \frac{P_{sy}^{(i)}}{\epsilon} + \epsilon^{\frac{i-1}{2}}P_{py}^{i,a}$$
$$+ \epsilon^{\frac{i}{2}}\Big[-\Delta_\varepsilon v_p^i + u_s^{(1)}v_{px}^i + v_{sx}^{(i)}u_p^i + v_{sy}^{(i)}v_p^i + v_s^{(i)}v_{py}^i + \epsilon^{\frac{i}{2}}u_p^i v_{px}^i + \epsilon^{\frac{i}{2}}v_p^i v_{py}^i + \frac{P_{py}^i}{\epsilon}\Big]. \tag{3.7.30}$$

Define the forcing term to be those terms from (3.7.29) which do not appear in the bracket:

$$\begin{aligned} -\epsilon^{\frac{i}{2}}f^{(i)} &:= -\Delta_\epsilon u_s^{(i)} + u_s^{(i)}u_{sx}^{(i)} + v_s^{(i)}u_{sy}^{(i)} + P_{sx}^{(i)} + \epsilon^{\frac{i+1}{2}}P_{px}^{i,a} \\ &= R^{u,i-1} + R_E^{u,i} + E_e^{u,i} + \epsilon^i P_{ex}^{1,a} + \epsilon^{\frac{i+1}{2}}P_{px}^{i,a}, \end{aligned} \tag{3.7.31}$$

where we have used (3.7.22) to simplify the expression. The first step, here, is to introduce a potential Pressure which eliminates the "purely" Eulerian terms from above:

Definition 3.7.5. The i'th auxiliary Euler pressure, $P_e^{1,a}$, is defined by:

$$P_e^{1,a} := -\sum_{j=1}^{i-1}\epsilon^{\frac{j-i}{2}}v_e^i v_e^j - \sum_{j=1}^{i-1}\epsilon^{\frac{j-i}{2}}u_e^i u_e^j - \frac{1}{2}\Big|v_e^i\Big|^2 - \frac{1}{2}\Big|u_e^i\Big|^2. \tag{3.7.32}$$

We may now check that:

Lemma 3.7.6. *With the definition above (3.7.32), $P_e^{1,a}$ serves as a gradient potential to eliminate the purely-Eulerian terms, $[E_e^{u,i}, E_e^{v,i}]$ from the expansion*

$$\Big(\partial_x, \frac{\partial_y}{\epsilon}\Big)\epsilon^i P_e^{1,a} + \Big(E_e^{u,i}, E_e^{v,i}\Big) = 0. \tag{3.7.33}$$

Proof. By scaling, we will write:

$$\Big(\partial_x, \frac{\partial_y}{\epsilon}\Big)\epsilon^i P_e^{1,a} = \Big(\partial_x, \frac{\partial_Y}{\sqrt{\epsilon}}\Big)\epsilon^i P_e^{1,a}. \tag{3.7.34}$$

We will go term by term through the definition in (3.7.32), starting with:

$$\varepsilon^i \partial_x \Big(-\sum_{j=1}^{i-1} \varepsilon^{\frac{j-i}{2}} v_e^i v_e^j \Big) = -\sum_{j=1}^{i-1} \varepsilon^{\frac{j+i}{2}} \Big(v_{ex}^i v_e^j + v_e^i v_{ex}^j \Big) = -\sum_{j=1}^{i-1} \varepsilon^{\frac{j+i}{2}} \Big(u_{eY}^i v_e^j + v_e^i u_{eY}^j \Big). \tag{3.7.35}$$

Next,

$$\varepsilon^i \partial_x \Big(-\sum_{j=1}^{i-1} \varepsilon^{\frac{j-i}{2}} u_e^i u_e^j \Big) = -\sum_{j=1}^{i-1} \varepsilon^{\frac{j+i}{2}} \Big(u_{ex}^i u_e^j + u_e^i u_{ex}^j \Big). \tag{3.7.36}$$

Third,

$$\varepsilon^i \partial_x \Big(-\frac{1}{2}|v_e^i|^2 - \frac{1}{2}|u_e^i|^2 \Big) = -\varepsilon^i v_e^i v_{ex}^i - \varepsilon^i u_e^i u_{ex}^i = -\varepsilon^i v_e^i u_{eY}^i - \varepsilon^i u_e^i u_{ex}^i. \tag{3.7.37}$$

Comparing these expressions, (3.7.35) - (3.7.37) to the expression (3.7.11), one observes the exact cancellation:

$$\partial_x \varepsilon^i P_e^{1,a} + E_E^{u,i} = 0. \tag{3.7.38}$$

Next, we will move to the $\frac{\partial_Y}{\sqrt{\varepsilon}}$ terms:

$$-\frac{\partial_Y}{\sqrt{\varepsilon}} \varepsilon^i \Big(\sum_{j=1}^{i-1} \varepsilon^{\frac{j-i}{2}} v_e^i v_e^j \Big) = -\sum_{j=1}^{i-1} \varepsilon^{\frac{i+j}{2} - \frac{1}{2}} \Big(v_{eY}^i v_e^j + v_e^i v_{eY}^j \Big). \tag{3.7.39}$$

Next,

$$-\frac{\partial_Y}{\sqrt{\varepsilon}} \varepsilon^i \Big(\sum_{j=1}^{i-1} \varepsilon^{\frac{j-i}{2}} u_e^i u_e^j \Big) = -\sum_{j=1}^{i-1} \varepsilon^{\frac{i+j}{2} - \frac{1}{2}} \Big(u_{eY}^i u_e^j + u_e^i u_{eY}^j \Big)$$

$$= -\sum_{j=1}^{i-1} \varepsilon^{\frac{i+j}{2}-\frac{1}{2}} \Big(v^i_{ex} u^j_e + u^i_e v^j_{ex} \Big). \tag{3.7.40}$$

Third,

$$\frac{\partial_Y}{\sqrt{\varepsilon}} \varepsilon^i \Big(-|v^i_e|^2 - |u^i_e|^2 \Big) = -\varepsilon^{i-\frac{1}{2}} v^i_e v^i_{eY} - \varepsilon^{i-\frac{1}{2}} u^i_e u^i_{eY} = -\varepsilon^{i-\frac{1}{2}} v^i_e v^i_{eY} - \varepsilon^{i-\frac{1}{2}} u^i_e v^i_{ex}. \tag{3.7.41}$$

Comparing these expressions, (3.7.39) - (3.7.41) to the expression (3.7.11), one observes the exact cancellation:

$$\frac{\partial_Y}{\sqrt{\varepsilon}} \varepsilon^i P^{1,a}_e + E^{v,i}_E = 0. \tag{3.7.42}$$

This establishes the desired result, (3.7.33).

□

Remark. One should notice the essential role played by the Cauchy-Riemann equations, $u^j_{eY} = v^j_{ex}$ in the equalities above.

Let us now turn to the terms outside of the bracket in (3.7.30), which we also simplify via (3.7.23) and subsequently via (3.7.33):

$$\begin{aligned} -\Delta_\epsilon v^{(i)}_s + u^{(i)}_s v^{(i)}_{sx} + v^{(i)}_s v^{(i)}_{sy} + \frac{P^{(i)}_{sy}}{\epsilon} + \epsilon^{\frac{i-1}{2}} P^{i,a}_{py} & \\ = R^{v,i-1} + R^{v,i}_E + E^{v,i}_e + \epsilon^{i-\frac{1}{2}} P^{1,a}_{eY} + \epsilon^{\frac{i-1}{2}} P^{i,a}_{py} & \\ = R^{v,i-1} + R^{v,i}_E + \epsilon^{\frac{i-1}{2}} P^{i,a}_{py}. & \end{aligned} \tag{3.7.43}$$

Motivated by this, define the auxiliary Pressure via:

Definition 3.7.7. The i'th auxiliary Prandtl pressure, $P_P^{i,a}$ is defined via:

$$\epsilon^{\frac{i+1}{2}} P_P^{i,a} := \epsilon \int_y^\infty R^{v,i-1} + R_E^{v,i}. \tag{3.7.44}$$

Immediately from this definition, we have:

Lemma 3.7.8. *According to the Definition 3.7.44, the following identity holds:*

$$-\Delta_\epsilon v_s^{(i)} + u_s^{(i)} v_{sx}^{(i)} + v_s^{(i)} v_{sy}^{(i)} + \frac{P_{sy}^{(i)}}{\epsilon} + \epsilon^{\frac{i-1}{2}} P_{py}^{i,a} = 0. \tag{3.7.45}$$

Proof. By direct calculation from (3.7.44),

$$\varepsilon^{\frac{i-1}{2}} P_P^{i,a} = -R^{v,i-1} - R_E^{v,i}. \tag{3.7.46}$$

Combined with (3.7.43) then implies the desired result. □

We are now able to rewrite the forcing for our equation, which was defined in (3.7.31):

$$f^{(i)} = -\epsilon^{-\frac{i}{2}} \Big[R^{u,i-1} + R_E^{u,i} + \epsilon^{\frac{i+1}{2}} P_{px}^{i,a} \Big]. \tag{3.7.47}$$

The $\epsilon^{-\frac{i}{2}}$ factor arises as we are in the $i'th$ order of the construction. Let us briefly comment on the orders of the three-terms on the right-hand side of (3.7.47). According to (3.7.24), it is evident that $R^{u,i-1}$ is order $\epsilon^{\frac{i}{2}}$. According to (3.7.9), it is clear that $R_E^{u,i}$ is order $\epsilon^{\frac{i}{2}}$. According to (3.7.44) coupled with (3.7.25) and (3.7.10), it is clear that $P_P^{i,a}$ is order 1. Thus, all of the terms in (3.7.47) are order 1 or higher. Reading off from (3.7.29) - (3.7.30), the system we will now be considering is:

$$(1 + u_p^0) u_{px}^i + u_{sx}^{(i)} u_p^i + v_s^{(i)} u_{py}^i + u_{py}^0 \Big(v_p^i - v_p^i(x, 0) \Big) + P_{px}^i = u_{pyy}^i + f^{(i)}, \tag{3.7.48}$$

$$u_p^i(x,0) = -u_e^i(x,0), \quad \lim_{y\to\infty} u_p^i(x,y) = 0, \quad u_p^i(1,y) = U_i(y), \quad P_{py}^i = 0. \tag{3.7.49}$$

Again, by evaluating the equation at $y = \infty$ gives $P_{px}^i = 0$. Coupled with the normal equation $P_{py}^i = 0$ then shows that $P_p^i = 0$. After this construction, $R^{u,i}, R^{v,i}$ contain the terms from (3.7.29) - (3.7.30) which were omitted in the construction of $[u_p^i, v_p^i]$:

Lemma 3.7.9. *For each i, with $[R^{u,i}, R^{v,i}]$ defined as in (3.7.7) - (3.7.8), with $[u_p^i, v_p^i]$ taken to solve the system (3.7.48), the following identities hold:*

$$\begin{aligned} R^{u,i} = \epsilon^{\frac{i}{2}}\Big[&\epsilon u_{pxx}^i + \sqrt{\epsilon} y u_{py}^0 v_{eY}^{i+1} + \epsilon u_{py}^0 \int_0^y \int_y^{y'} v_{eYY}^{i+1} dy'' dy' \\ &+ v_p^i \sum_{j=1}^i \epsilon^{\frac{j}{2}} \{u_{py}^j + \sqrt{\epsilon} u_{eY}^j\} + u_{px}^i \sum_{j=1}^i \varepsilon^{\frac{j}{2}} \{u_e^j + u_p^j\}\Big], \end{aligned} \tag{3.7.50}$$

$$R^{v,i} = \epsilon^{\frac{i}{2}}\Big[-\Delta_\epsilon v_p^i + u_s^{(i)} v_{px}^i + v_{sx}^{(i)} u_p^i + v_s^{(i)} v_{py}^i + v_{sy}^{(i)} v_p^{(i)} + \epsilon^{\frac{i}{2}} u_p^i v_{px}^i + \epsilon^{\frac{i}{2}} v_p^i v_{py}^i\Big]. \tag{3.7.51}$$

Proof. Starting with the definition in equation (3.7.7), we have:

$$R^{u,i} := -\Delta_\epsilon \bar{u}_s^{(i)} + \bar{u}_s^{(i)} \bar{u}_{sx}^{(i)} + \bar{v}_s^{(i)} \bar{u}_{sy}^{(i)} + \bar{P}_{sx}^{(i)} + \varepsilon^{\frac{i}{2}} v_e^{i+1} u_{py}^0 \tag{3.7.52}$$

$$\begin{aligned} &= -\Delta_\epsilon u_s^{(i)} + u_s^{(i)} u_{sx}^{(i)} + v_s^{(i)} u_{sy}^{(i)} + P_{sx}^{(i)} + \epsilon^{\frac{i+1}{2}} P_{px}^{i,a} \\ &\qquad + \epsilon^{\frac{i}{2}}\Big[-\Delta_\varepsilon u_p^i + u_{sx}^{(i)} u_p^i + u_s^{(i)} u_{px}^i + u_{sy}^{(i)} v_p^i + v_s^{(i)} u_{py}^i \\ &\qquad + \epsilon^{\frac{i}{2}} u_p^i u_{px}^i + \epsilon^{\frac{i}{2}} v_p^i u_{py}^i + P_{px}^i\Big] + \varepsilon^{\frac{i}{2}} v_e^{i+1} u_{py}^0 \end{aligned} \tag{3.7.53}$$

$$\begin{aligned} &= -\varepsilon^{\frac{i}{2}} f^{(i)} + \epsilon^{\frac{i}{2}}\Big[-\Delta_\varepsilon u_p^i + u_{sx}^{(i)} u_p^i + u_s^{(i)} u_{px}^i + u_{sy}^{(i)} v_p^i + v_s^{(i)} u_{py}^i \\ &\qquad + \epsilon^{\frac{i}{2}} u_p^i u_{px}^i + \epsilon^{\frac{i}{2}} v_p^i u_{py}^i + P_{px}^i\Big] + -\varepsilon^{\frac{i}{2}} v_e^{i+1} u_{py}^0 \end{aligned} \tag{3.7.54}$$

$$\begin{aligned} &= \varepsilon^{\frac{i}{2}} f^{(i)} + \epsilon^{\frac{i}{2}}\Big[-u_{pyy}^i + u_{sx}^{(i)} u_p^i + (1+u_p^0) u_{px}^i + u_{py}^0 v_p^i + v_s^{(i)} u_{py}^i \\ &\qquad + \epsilon^{\frac{i}{2}} u_p^i u_{px}^i + \epsilon^{\frac{i}{2}} v_p^i u_{py}^i + P_{px}^i - \varepsilon u_{pxx}^i + \sum_{j=1}^i \varepsilon^{\frac{j}{2}} \{u_e^j + u_p^j\} u_{px}^i \\ &\qquad + \sum_{j=1}^i \varepsilon^{\frac{j}{2}} \{u_{py}^j + \sqrt{\varepsilon} u_{eY}^j\} v_p^i\Big] + \varepsilon^{\frac{i}{2}} v_e^{i+1} u_{py}^0 \end{aligned} \tag{3.7.55}$$

$$= -\varepsilon^{\frac{i}{2}} f^{(i)} + \epsilon^{\frac{i}{2}}\Big[-u_{pyy}^i + u_{sx}^{(i)} u_p^i + (1+u_p^0) u_{px}^i + u_{py}^0 v_p^i + v_s^{(i)} u_{py}^i$$

$$+ \epsilon^{\frac{i}{2}} u_p^i u_{px}^i + \epsilon^{\frac{i}{2}} v_p^i u_{py}^i + P_{px}^i - \varepsilon u_{pxx}^i + \sum_{j=1}^{i} \varepsilon^{\frac{j}{2}} \{u_e^j + u_p^j\} u_{px}^i$$

$$+ \sum_{j=1}^{i} \varepsilon^{\frac{j}{2}} \{u_{py}^j + \sqrt{\varepsilon} u_{eY}^j\} v_p^i \Big] + \varepsilon^{\frac{i}{2}} \Big[- v_p^i(x,0) u_{py}^0 + \sqrt{\varepsilon} y u_{py}^0 v_{eY}^{i+1}$$

$$+ \varepsilon u_{py}^0 \int_0^y \int_y^{y'} v_{eYY}^{i+1} \, \mathrm{d}y'' \, \mathrm{d}y' \Big] \tag{3.7.56}$$

$$= \varepsilon^{\frac{i}{2}} \Big[- \varepsilon u_{pxx}^i + \sum_{j=1}^{i} \varepsilon^{\frac{j}{2}} \{u_e^j + u_p^j\} u_{px}^i + \sum_{j=1}^{i} \varepsilon^{\frac{j}{2}} \{u_{py}^j + \sqrt{\varepsilon} u_{eY}^j\} v_p^i + \sqrt{\varepsilon} y u_{py}^0 v_{eY}^{i+1}$$

$$+ \varepsilon u_{py}^0 \int_0^y \int_y^{y'} v_{eYY}^{i+1} \, \mathrm{d}y'' \, \mathrm{d}y' \Big]. \tag{3.7.57}$$

For the calculation in (3.7.53), we have used the expansion (3.7.29). For the calculation in (3.7.54), we have used (3.7.31). For the calculation in (3.7.55), we have simply rearranged terms. For the calculation in (3.7.56), we have used:

$$\varepsilon^{\frac{i}{2}} v_e^{i+1} u_{py}^0 = \varepsilon^{\frac{i}{2}} \Big[- v_p^i(x,0) u_{py}^0 + \sqrt{\varepsilon} y u_{py}^0 v_{eY}^{i+1} + \varepsilon u_{py}^0 \int_0^y \int_y^{y'} v_{eYY}^{i+1} \, \mathrm{d}y'' \, \mathrm{d}y' \Big]. \tag{3.7.58}$$

Finally, for the calculation in (3.7.57), we used (3.7.48). The identity (3.7.50) then follows. For the $R^{v,i}$ contribution, we combine (3.7.45) with the expression (3.7.30), and finally with the condition that $P_p^i = 0$ from (3.7.49). □

A comparison with (3.7.24) - (3.7.25) shows that this closes the construction. We now give estimates on the forcing term based on the inductively assumed decay rates and regularity in (3.7.2), together with the $[u_p^0, v_p^0]$ bounds in (3.4.86) and the $[u_e^1, v_e^1]$ bounds in (3.5.61).

Lemma 3.7.10 (Forcing Estimates)**.** *Let $i \geq 2$. For any $m, k \geq 0$, with $f^{(i)}$ as in (3.7.47),*

$$||z^m \partial_x^k f^{(i)}||_{L^2_y} \leq C(k,m,\sigma,n) x^{-k-\frac{5}{4}+2\sigma_{i-1}}. \tag{3.7.59}$$

Proof. We start with the $R^{u,i}_E$ terms, which are defined in (3.7.9):

$$\sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} ||u^i_{ex} u^j_p||_{L^2_y} \le \sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} ||u^i_{ex} x^{\frac{3}{2}}||_{L^\infty} x^{-\frac{3}{2}} ||u^j_p||_{L^2_y} \lesssim x^{-\frac{5}{4}}, \tag{3.7.60}$$

$$\sum_{j=0}^{i-1} ||u^i_e \epsilon^{\frac{j}{2}} u^j_{px}||_{L^2_y} \lesssim \sum_{j=0}^{i-1} \epsilon^{\frac{j}{2}} ||u^i_e x^{\frac{1}{2}}||_{L^\infty} ||x^{-\frac{1}{2}} u^j_{px}||_{L^2_y} \le x^{-\frac{5}{4}}, \tag{3.7.61}$$

$$\sum_{j=0}^{i-1} \epsilon^{\frac{j+1}{2}} ||u^i_{eY} v^j_p||_{L^2_y} \lesssim \sum_{j=0}^{i-1} \epsilon^{\frac{j+1}{2}} ||u^i_{eY} x^{\frac{3}{2}}||_{L^\infty} x^{-\frac{3}{2}} ||z^m v^j_p||_{L^2_y} \lesssim x^{-\frac{7}{4}}, \tag{3.7.62}$$

$$\epsilon^{\frac{i-1}{2}} \sum_{j=1}^{i-1} \epsilon^{\frac{j}{2}} ||v^i_e u^j_{py}||_{L^2_y} \lesssim \sum_{j=1}^{i-1} ||v^i_e x^{\frac{3}{4}-\sigma_{i-1}}||_{L^\infty} ||x^{-\frac{3}{4}+\sigma_{i-1}} u^j_{py}||_{L^2_y} \le x^{-\frac{5}{4}+2\sigma_{i-1}}. \tag{3.7.63}$$

In (3.7.63), we have used the enhanced Eulerian decay rate in (3.7.21). Next, we have the Prandtl contributions from $R^{u,i-1}$, according to (3.7.24):

$$\sqrt{\epsilon} ||u^{i-1}_{pxx}||_{L^2_y} \le \sqrt{\epsilon} x^{-\frac{7}{4}+\sigma_{i-1}}, \tag{3.7.64}$$

$$||y u^0_{py} v^i_{eY}||_{L^2_y} \le ||x^{-\frac{1}{4}} y u^0_{py}||_{L^2_y} x^{\frac{1}{4}} ||v^i_{eY}||_{L^\infty_y} \lesssim x^{-\frac{5}{4}}, \tag{3.7.65}$$

$$\sqrt{\epsilon} ||u^0_{py} \int_0^y \int_y^{y'} v^i_{eYY} dy' dy''||_{L^2_y} \le \sqrt{\epsilon} ||y^2 x^{-\frac{3}{4}} u^0_{py}||_{L^2_y} x^{\frac{3}{4}} ||v^i_{eYY}||_{L^\infty_y} \le \sqrt{\epsilon} x^{-\frac{7}{4}}, \tag{3.7.66}$$

$$\epsilon^{\frac{j-1}{2}} ||v^{i-1}_p u^j_{py}||_{L^2_y} \le \epsilon^{\frac{j-1}{2}} x^{-\frac{5}{4}+2\sigma_{i-1}}, j \ge 1 \tag{3.7.67}$$

$$\epsilon^{\frac{j}{2}} ||v^{i-1}_p u^j_{eY}||_{L^2_y} \le \epsilon^{\frac{j}{2}} x^{-2+2\sigma_{i-1}}, j \ge 1, \tag{3.7.68}$$

$$\varepsilon^{\frac{j}{2}} ||u^{i-1}_{px} \{u^j_e + u^j_p\}||_{L^2_y} \lesssim \sqrt{\varepsilon} x^{-\frac{5}{4}+2\sigma_{i-1}}, j \ge 1. \tag{3.7.69}$$

We now move to the term $P^{i,a}_{px}$. For this we note:

$$\epsilon^{-\frac{i}{2}} \epsilon^{\frac{i+1}{2}} ||P^{i,a}_{px}||_{L^2_y} \le \sqrt{\epsilon} \epsilon^{-\frac{i-1}{2}} ||\langle y \rangle^{\frac{3}{2}+\kappa} \partial_x \Big(R^{v,i-1} + R^{v,i}_E \Big)||_{L^\infty_y}. \tag{3.7.70}$$

We now turn to evaluating the right-hand side of (3.7.70). Let us comment that it is essential at this stage that the pressure $P^{1,a}_e$ was introduced to eliminate all of the purely Eulerian contributions, as those terms cannot handle the weight of $y^{\frac{3}{2}+\kappa}$. Each term below has at least one Prandtl ($[u^i_p, v^i_p]$) factor, and so we may exchange the weight of $y^{\frac{3}{2}+\kappa}$ for decay, $x^{\frac{3}{4}+\frac{\kappa}{2}}$. For the calculations below, we simply pair this with the decay rates in (3.7.2):

$$||y^{\frac{3}{2}+\kappa}\partial_x\Big(v^i_{ex}u^j_p\Big)||_{L^\infty_y} + ||y^{\frac{3}{2}+\kappa}\partial_x\Big(u^i_e v^j_{px}\Big)||_{L^\infty_y} \lesssim x^{-\frac{3}{2}+2\sigma_{i-1}}, \tag{3.7.71}$$

$$||y^{\frac{3}{2}+\kappa}\partial_x\Big(v^i_e v^j_{py}\Big)||_{L^\infty_y} \lesssim x^{-\frac{3}{2}+2\sigma_{i-1}} \tag{3.7.72}$$

$$||y^{\frac{3}{2}+\kappa}\partial_x\Big(v^i_{eY}v^j_p\Big)||_{L^\infty_y} \lesssim x^{-2+2\sigma_{i-1}}, \tag{3.7.73}$$

$$||y^{\frac{3}{2}+\kappa}\partial_x\Big(\Delta_\epsilon v^{i-1}_p\Big)\Big)||_{L^\infty_y} \lesssim x^{-\frac{5}{4}+2\sigma_{i-1}}, \tag{3.7.74}$$

$$||y^{\frac{3}{2}+\kappa}\partial_x\Big(u^{(i-1)}_s v^{i-1}_{px}\Big)||_{L^\infty_y} + ||y^{\frac{3}{2}+\kappa}\partial_x\Big(v^{(i-1)}_{sx}u^{i-1}_p\Big)||_{L^\infty_y} \lesssim x^{-\frac{3}{2}+2\sigma_{i-1}}, \tag{3.7.75}$$

$$||y^{\frac{3}{2}+\kappa}\partial_x\Big(v^{(i-1)}_s v^{i-1}_{py}\Big)||_{L^\infty_y} + ||y^{\frac{3}{2}+\kappa}\partial_x\Big(v^{(i-1)}_{sy}v^{i-1}_p\Big)||_{L^\infty_y} \lesssim x^{-\frac{3}{2}+2\sigma_{i-1}}, \tag{3.7.76}$$

$$||y^{\frac{3}{2}+\kappa}\partial_x\Big(\epsilon^{\frac{i-1}{2}}u^{i-1}_p v^{i-1}_{px} + v^{i-1}_p v^{i-1}_{py}\Big)||_{L^\infty_y} \lesssim \epsilon^{\frac{i-1}{2}}x^{-\frac{5}{4}+2\sigma_{i-1}}. \tag{3.7.77}$$

This proves the claim for $k=0, m=0$. The general case can be obtained in an identical fashion, upon noticing that powers of z will not influence the estimates when accompanied by Prandtl profiles, which are present in each term above.

□

Indeed, the boundary condition in (3.7.48) has improved, referring to (3.7.21):

$$\Big|u^i_e(x,0)\Big| \le x^{-\frac{3}{4}+\sigma_{i-1}}, \tag{3.7.78}$$

according to (3.7.21). We can expect enhanced decay due to the enhanced decay of the boundary data. This is the content of the following:

Proposition 3.7.11 (Construction of $i'th$ Prandtl Layer)**.** *For $i \ge 2$, there exists a solution $[u^i_p, v^i_p]$ to the system (3.7.48), satisfying for any $m, k \ge 0$:*

$$||z^m x^{\frac{1}{4}}\partial^k_x u^i_p||_{P_k(\sigma_i)} + x^{k+1-\sigma_i}||z^m\partial^k_x v^i_p||_{L^\infty_y} \le C(k,m,n). \tag{3.7.79}$$

This will be proven in several steps. We homogenize the boundary conditions by defining:

$$u = u_p + \chi(y) u^i_e(x,0), \quad v = v_p - v_p(x,0) + u^n_{ex}(x,0) I_\chi(y), \quad I_\chi(y) = \int_y^\infty \chi(\theta) d\theta,$$

where χ is a localized cut-off function selected to have mean-zero. These profiles satisfy:

$$(1 + u^0_p) u_x - u_{yy} + \mathcal{P}(u,v) = f^{(i)} + \mathcal{J}, \tag{3.7.80}$$

where

$$\begin{aligned} \mathcal{J} := u^{(i)}_s \chi(y) u^i_{ex}(x,0) - \chi'' u^i_e(x,0) - \chi(y) u^{(i)}_{sx} u^n_e(x,0) \\ - u^0_{py} I_\chi(y) u^i_{ex}(x,0) + v^{(i)}_s \chi'(y) u^i_e(x,0). \end{aligned} \tag{3.7.81}$$

We will record the estimate on $\mathcal{J}$, which follows directly from (3.7.78):

$$|\mathcal{J}| \lesssim \langle y \rangle^{-N} x^{-\frac{3}{4} + \sigma_{i-1}}. \tag{3.7.82}$$

We introduce the stream function,

$$\psi(x,y) := \int_y^\infty u(x,y') dy', \quad \psi_y = -u, \quad \psi_x = v - v(x,\infty). \tag{3.7.83}$$

The stream function satisfies the following system:

$$\Big(\partial_x - \partial_{yy}\Big)\psi = \int_y^\infty u^0_p u_x + \int_y^\infty \mathcal{P} + \int_y^\infty f^{(n)} + \int_y^\infty \mathcal{J}, \tag{3.7.84}$$

$$\psi_y(x,0) = u(x,0) = 0, \quad \psi(x,\infty) = 0, \quad \psi(1,y) = \int_y^\infty U_n(y') dy'. \tag{3.7.85}$$

For technical reasons (due to certain integrals being critical), it is necessary to start with the stream-function formulation in order to obtain the desired, enhanced decay.

Lemma 3.7.12. *The stream function ψ solving the system (3.7.84) satisfies:*

$$\sup_x \int x^{-\frac{1}{2}-2\sigma_i}\psi^2 + \int\int x^{-\frac{3}{2}-2\sigma_i}\psi^2 + \int\int x^{-\frac{1}{2}-2\sigma_i}u^2 \lesssim C + \mathcal{O}(\delta)\int\int v_y^2 x^{\frac{3}{2}-2\sigma_i}. \tag{3.7.86}$$

Proof. Applying the multiplier $M = x^{-\frac{1}{2}-2\sigma_i}\psi$ to the above system yields the following terms on the left-hand side:

$$\partial_x \int x^{-\frac{1}{2}-2\sigma_i}\psi^2 + \int x^{-\frac{3}{2}-2\sigma_i}\psi^2 + \int x^{-\frac{1}{2}-2\sigma_i}\psi_y^2 \lesssim \int \Big(\partial_x - \partial_{yy}\Big)\psi \cdot \psi x^{-\frac{1}{2}-2\sigma_i}. \tag{3.7.87}$$

For the first term on the right-hand side of (3.7.84), we will now give the bound via Hardy's inequality, and (3.4.86):

$$\begin{aligned}\int x^{-\frac{1}{2}-2\sigma_i}\psi \int_y^\infty u_p^0 u_x dy' dy &\leq ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y}||x^{\frac{1}{4}}\int_y^\infty u_p^0 u_x dy'||_{L^2_y} \\ &\leq ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y}||x^{\frac{1}{4}-\sigma_i} y u_p^0 u_x||_{L^2_y} \leq \mathcal{O}(\delta)||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y}||x^{\frac{3}{4}-\sigma_i}u_x||_{L^2_y}.\end{aligned} \tag{3.7.88}$$

Next, let us address the profile terms in $\mathcal{P}$. For the first term from $\mathcal{P}$, we will split:

$$\begin{aligned}&\int x^{-\frac{1}{2}-2\sigma_i}\psi \int_y^\infty u_{sx}^{(i)} u \,\mathrm{d}y'\,\mathrm{d}y \\ &\quad = \sum_{j=0}^{i-1}\int x^{-\frac{1}{2}-2\sigma_i}\psi\int_y^\infty \varepsilon^{\frac{j}{2}}u_{px}^j u\,\mathrm{d}y'\,\mathrm{d}y + \sum_{j=1}^{i}\int x^{-\frac{1}{2}-2\sigma_i}\psi\int_y^\infty \varepsilon^{\frac{j}{2}}u_{ex}^j u\,\mathrm{d}y'\,\mathrm{d}y\end{aligned} \tag{3.7.89}$$

For the first term above, we apply the Hardy inequality in y (it suffices to consider

$j = 0$):

$$\Big| \int x^{-\frac{1}{2}-2\sigma_i} \psi \int_y^\infty u^0_{px} u \, \mathrm{d}y' \, \mathrm{d}y \Big| \le ||x^{-\frac{3}{4}-\sigma_i} \psi||_{L^2_y} ||x^{\frac{1}{4}-\sigma_i} \int_y^\infty u^0_{px} u \, \mathrm{d}y'||_{L^2_y}$$
$$\le ||x^{-\frac{3}{4}-\sigma_i} \psi||_{L^2_y} ||x^{\frac{1}{2}} y u^0_{px}||_{L^\infty} ||x^{-\frac{1}{4}-\sigma_i} u||_{L^2_y}$$
$$\le \mathcal{O}(\delta) ||x^{-\frac{3}{4}-\sigma_i} \psi||^2_{L^2_y} + \mathcal{O}(\delta) ||x^{-\frac{1}{4}-\sigma_i} u||^2_{L^2_y}. \tag{3.7.90}$$

We have used the smallness which is guaranteed by (3.4.86). For the second term in (3.7.89), we will integrate by parts in y and recall $\psi \xrightarrow{y \to \infty} 0$, (again it suffices to consider $j = 1$):

$$\int x^{-\frac{1}{2}-2\sigma_i} \psi \int_y^\infty \sqrt{\varepsilon} u^1_{ex} \partial_y \psi = \int x^{-\frac{1}{2}-2\sigma_i} \sqrt{\varepsilon} u^1_{ex} \psi^2 + \int x^{-\frac{1}{2}-2\sigma_i} \psi \int_y^\infty \varepsilon u^1_{exY} \psi. \tag{3.7.91}$$

The first term in (3.7.91):

$$\Big| \int x^{-\frac{1}{2}-2\sigma_i} \sqrt{\varepsilon} u^1_{ex} \psi^2 \Big| \le \sqrt{\varepsilon} ||u^1_{ex} x||_{L^\infty} ||x^{-\frac{3}{4}-\sigma_i} \psi||^2_{L^2_y} \lesssim \sqrt{\varepsilon} ||x^{-\frac{3}{4}-\sigma_i} \psi||^2_{L^2_y}. \tag{3.7.92}$$

For the second term in (3.7.91), we apply Hardy in y and appeal to estimate (3.5.48) with $k = 2, s = 1$:

$$\Big| \int x^{-\frac{1}{2}-2\sigma_i} \psi \int_y^\infty \varepsilon u^1_{exY} \psi \Big| \le \sqrt{\varepsilon} ||u^1_{exY} Y x||_{L^\infty} ||x^{-\frac{3}{4}-\sigma_i} \psi||^2_{L^2_y}$$
$$= \sqrt{\varepsilon} ||v^1_{exx} Y x||_{L^\infty} ||x^{-\frac{3}{4}-\sigma_i} \psi||^2_{L^2_y} \lesssim \sqrt{\varepsilon} ||x^{-\frac{3}{4}-\sigma_i} \psi||^2_{L^2_y}. \tag{3.7.93}$$

Let us now move to the second term in $\mathcal{P}$. One integration by parts in y produces:

$$\int x^{-\frac{1}{2}-2\sigma_i} \psi \int_y^\infty v^{(i)}_s u_y dy' dy = \int x^{-\frac{1}{2}-2\sigma_i} v^{(i)}_s \psi \psi_y + \int x^{-\frac{1}{2}-2\sigma_i} \psi \int_y^\infty u^{(i)}_{sx} u dy' dy. \tag{3.7.94}$$

For the second term in (3.7.94), one notices this is exactly the term on the left-hand side of (3.7.89). For the first term in (3.7.94), we estimate:

$$\begin{aligned}\Big|\int x^{-\frac{1}{2}-2\sigma_i} v_s^{(i)} \psi \psi_y\Big| &\le ||v_s^{(i)} x^{\frac{1}{2}}||_{L^\infty} \Big(||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y}^2 + ||x^{-\frac{1}{4}-\sigma_i}\psi_y||_{L^2_y}^2 \Big) \\ &\le \mathcal{O}(\delta)\Big(||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y}^2 + ||x^{-\frac{1}{4}-\sigma_i}\psi_y||_{L^2_y}^2 \Big). \end{aligned} \tag{3.7.95}$$

We have used the smallness guaranteed by (3.4.91). The final term from $\mathcal{P}$ is:

$$\begin{aligned}\Big|\int x^{-\frac{1}{2}-2\sigma_i} \psi \int_y^\infty u^0_{py} v dy' dy\Big| &\le ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y} ||x^{\frac{1}{4}-\sigma_i} \int_y^\infty u^0_{py} v||_{L^2_y} \\ &\le ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y} x^{\frac{1}{4}-\sigma_i} ||y u^0_{py} v||_{L^2_y} \\ &\le ||y^2 x^{-\frac{1}{2}} u^0_{py}||_{L^\infty} ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y} ||v_y x^{\frac{3}{4}-\sigma_i}||_{L^2_y} \\ &\le \mathcal{O}(\delta) ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y} ||v_y x^{\frac{3}{4}-\sigma_i}||_{L^2_y}. \end{aligned} \tag{3.7.96}$$

Consolidating the previous estimates, taking δ small enough relative to σ_i (and consequently relative to large n, according to (3.7.1)) and applying Young's inequality gives:

$$\begin{aligned} &\partial_x \int x^{-\frac{1}{2}-2\sigma_i}\psi^2 + \int x^{-\frac{3}{2}-2\sigma_i}\psi^2 + \int x^{-\frac{1}{2}-2\sigma_i}\psi_y^2 \\ &\qquad \lesssim \mathcal{O}(\delta) \int v_y^2 x^{\frac{3}{2}-2\sigma_i} + \int x^{-\frac{1}{2}-2\sigma_i} \psi \int_y^\infty \Big[f^{(i)} + \mathcal{J} \Big]. \end{aligned} \tag{3.7.97}$$

For the forcing terms, we appeal to the bounds in (3.7.59), coupled with (3.7.82):

$$||x^{-\frac{1}{2}-2\sigma_i}\psi \int_y^\infty f^{(i)} \,\mathrm{d}y'||_{L^2_y} \le ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y} ||x^{\frac{1}{4}-\sigma_i} \int_y^\infty f^{(i)}\,\mathrm{d}y'||_{L^2_y} \tag{3.7.98}$$

$$\le ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y} ||x^{\frac{1}{4}-\sigma_i} y f^{(i)}||_{L^2_y} \tag{3.7.99}$$

$$\le ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y} ||x^{\frac{3}{4}-\sigma_i} z f^{(i)}||_{L^2_y} \tag{3.7.100}$$

$$\le ||x^{-\frac{3}{4}-\sigma_i}\psi||_{L^2_y} x^{-\frac{1}{2}-\sigma_i} \tag{3.7.101}$$

$$\leq \frac{1}{100,000}||x^{-\frac{3}{4}-\sigma_i}\psi||_{L_y^2}^2 + Cx^{-1-\sigma'}, \tag{3.7.102}$$

where $\sigma' > 0$. We have used (3.7.1) via: $\sigma_i - 2\sigma_{i-1} = 3\sigma_{i-1} - 2\sigma_{i-1} = \sigma_{i-1}$, to calculate:

$$||x^{\frac{3}{4}-\sigma_i} z f^{(i)}||_{L_y^2} \lesssim x^{\frac{3}{4}-\sigma_i} x^{-\frac{5}{4}+2\sigma_{i-1}} = Cx^{-\frac{1}{2}-\sigma_i+2\sigma_{i-1}} = Cx^{-\frac{1}{2}-\sigma_{i-1}}. \tag{3.7.103}$$

Next, through Young's inequality

$$|\int x^{-\frac{1}{2}-2\sigma_i}\psi \int_y^\infty \mathcal{J}\, \mathrm{d}y'| \lesssim \int x^{-\frac{1}{2}-2\sigma_i}\psi\langle y\rangle^{-N} x^{-\frac{3}{4}+\sigma_{i-1}}$$
$$\lesssim \frac{1}{100,000}||x^{-\frac{3}{4}-\sigma_i}\psi||_{L_y^2}^2 + Cx^{-1-\sigma'}, \tag{3.7.104}$$

where $\sigma' > 0$. Inserting these into (3.7.97), relabeling $\psi_y = u$, and integrating in x, one obtains the desired result in (3.7.86).

□

From here, we may repeat the calculations in Lemmas 3.6.4 - 3.6.5:

Lemma 3.7.13. *The solutions $[u, v]$ to the system (3.7.48) satisfy the following inequality:*

$$\sup_x \int u^2 x^{\frac{1}{2}-2\sigma_i} + \int\int u_y^2 x^{\frac{1}{2}-2\sigma_i} \lesssim C + \mathcal{O}(\delta)\int\int v_y^2 x^{\frac{3}{2}-2\sigma_i} + \int\int u^2 x^{-\frac{1}{2}-2\sigma_i}. \tag{3.7.105}$$

Proof. We multiply both sides of equation (3.7.80) by $ux^{\frac{1}{2}-2\sigma_i}$ and integrate by parts. This gives on the left-hand side:

$$\int \Big((1+u_p^0)\partial_x - \partial_{yy}\Big)u \cdot ux^{\frac{1}{2}-2\sigma_i} \gtrsim \partial_x \int (1+u_p^0)u^2 x^{\frac{1}{2}-2\sigma_i} + \int u_y^2 x^{\frac{1}{2}-2\sigma_i}$$
$$- \int u_{px}^0 u^2 x^{\frac{1}{2}-2\sigma_i} - \int (1+u_p^0)u^2 x^{-\frac{1}{2}-2\sigma_i}. \tag{3.7.106}$$

One now sees that it is crucial to have controlled the final term in (3.7.106), which is the purpose of (3.7.97). Moving next to the sequence of terms in $\mathcal{P}(u,v)$:

$$|\int u^{(i)}_{sx} u^2 x^{\frac{1}{2}-2\sigma_i}| \le ||u^{(i)}_{sx} x||_{L^\infty} \int u^2 x^{-\frac{1}{2}-2\sigma_i} \le \mathcal{O}(\delta) ||u x^{-\frac{1}{4}-\sigma_i}||^2_{L^2_y}, \tag{3.7.107}$$

$$\begin{aligned} |\int v^{(i)}_s u_y u x^{\frac{1}{2}-2\sigma_i}| &= |\int \frac{v^{(i)}_{sy}}{2} u^2 x^{\frac{1}{2}-2\sigma_i}| \\ &\lesssim ||u^{(i)}_{sx} x||_{L^\infty} ||u x^{-\frac{1}{4}-\sigma_i}||^2_{L^2_y} \le \mathcal{O}(\delta) ||u x^{-\frac{1}{4}-\sigma_i}||^2_{L^2_y}, \end{aligned} \tag{3.7.108}$$

$$\begin{aligned} |\int u^0_{py} u v x^{\frac{1}{2}-2\sigma_i}| &\lesssim ||u^0_{py} y||_{L^\infty} ||\frac{v}{y} x^{\frac{3}{4}-\sigma_i}||_{L^2_y} ||u x^{-\frac{1}{4}-\sigma_i}||_{L^2_y} \\ &\lesssim \mathcal{O}(\delta) ||v_y x^{\frac{3}{4}-\sigma_i}||_{L^2_y} ||u x^{-\frac{1}{4}-\sigma_i}||_{L^2_y}. \end{aligned} \tag{3.7.109}$$

We can summarize the above terms by writing:

$$|\int \mathcal{P} \cdot u x^{\frac{1}{2}-\sigma_i}| \le \mathcal{O}(\delta) ||v_y x^{\frac{3}{4}-\sigma_i}||^2_{L^2_y} + \mathcal{O}(\delta) ||u x^{-\frac{1}{4}-\sigma_i}||^2_{L^2_y}. \tag{3.7.110}$$

Note that the smallness in the above estimates is guaranteed by (3.4.86), (3.4.91), and (3.5.61). We now move to the forcing terms, f and $\mathcal{J}$:

$$\begin{aligned} |\int f^{(i)} \cdot u x^{\frac{1}{2}-2\sigma_i}| &\le ||f^{(i)} x^{\frac{3}{4}-\sigma_i}||_{L^2_y} ||u x^{-\frac{1}{4}-\sigma_i}||_{L^2_y} \lesssim x^{-\frac{1}{2}-\sigma'} ||u x^{-\frac{1}{4}-\sigma_i}||_{L^2_y} \\ &\le C x^{-1-\sigma'} + \frac{1}{100,000} ||u x^{-\frac{1}{4}-\sigma_i}||^2_{L^2_y}. \end{aligned} \tag{3.7.111}$$

Next, we use the Hardy inequality:

$$\begin{aligned} |\int \mathcal{J} \cdot u x^{\frac{1}{2}-2\sigma_i}| &\lesssim \int \langle y \rangle^{-N} x^{-\frac{3}{4}+\sigma_{i-1}} |u| x^{\frac{1}{2}-2\sigma_i} \\ &\lesssim ||\langle y \rangle^{-N+1}||_{L^2_y} x^{-\frac{1}{2}-\sigma'} ||\frac{u}{y}||_{L^2_y} x^{\frac{1}{4}-\sigma_i} \\ &\le C x^{-1-\sigma'} + \frac{1}{100,000} ||u_y x^{\frac{1}{4}-\sigma_i}||^2_{L^2_y}. \end{aligned} \tag{3.7.112}$$

Combining the above estimates together, one obtains, for some $\sigma' > 0$ small,

$$\partial_x \int (1+u_p^0)u^2 x^{\frac{1}{2}-2\sigma_i} + \int u_y^2 x^{\frac{1}{2}-2\sigma_i} \lesssim Cx^{-1-\sigma'} + \mathcal{O}(\delta)\int v_y^2 x^{\frac{3}{2}-2\sigma_i} + \int u^2 x^{-\frac{1}{2}-2\sigma_i}, \tag{3.7.113}$$

and so integrating in x gives the desired result.

□

The third step in establishing the desired bounds is:

Lemma 3.7.14. *For solutions $[u,v]$ to the system in (3.7.80) one has the following positivity estimate:*

$$\sup_x \int u_y^2 x^{\frac{3}{2}-2\sigma_i} + \int\int u_x^2 x^{\frac{3}{2}-2\sigma_i} \lesssim C + \int\int u_y^2 x^{\frac{1}{2}-2\sigma_i} + \mathcal{O}(\delta)\int\int u^2 x^{-\frac{1}{2}-2\sigma_i}. \tag{3.7.114}$$

Proof. We apply the multiplier $u_x x^{\frac{3}{2}-2\sigma_i}$ to the system (3.7.80). This gives on the left-hand side:

$$\int \Big((1+u_p^0)\partial_x - \partial_{yy}\Big)u \cdot u_x x^{\frac{3}{2}-2\sigma_i} \gtrsim \int (1+u_p^0)u_x^2 x^{\frac{3}{2}-2\sigma_i} + \partial_x \int u_y^2 x^{\frac{3}{2}-2\sigma_i} - \int u_y^2 x^{\frac{1}{2}-2\sigma_i} \tag{3.7.115}$$

Let us now move to the terms in $\mathcal{P}$:

$$\begin{aligned} |\int u_{sx}^{(i)} u u_x x^{\frac{3}{2}-2\sigma_i}| &\leq ||u_{sx}^{(i)} x||_{L^\infty} ||u x^{-\frac{1}{4}-\sigma_i}||_{L^2_y} ||u_x x^{\frac{3}{4}-\sigma_i}||_{L^2_y} \\ &\lesssim \mathcal{O}(\delta) ||u x^{-\frac{1}{4}-\sigma_i}||_{L^2_y} ||u_x x^{\frac{3}{4}-\sigma_i}||_{L^2_y}, \end{aligned} \tag{3.7.116}$$

$$\begin{aligned} |\int v_s^{(i)} u_y u_x x^{\frac{3}{2}-2\sigma_i}| &\leq ||v_s^{(i)} x^{\frac{1}{2}}||_{L^\infty} ||u_y x^{\frac{1}{4}-\sigma}||_{L^2_y} ||u_x x^{\frac{3}{4}-\sigma_i}||_{L^2_y} \\ &\leq \mathcal{O}(\delta) ||u_y x^{\frac{1}{4}-\sigma_i}||_{L^2_y} ||u_x x^{\frac{3}{4}-\sigma_i}||_{L^2_y}, \end{aligned} \tag{3.7.117}$$

$$|\int u^0_{py} u_x v x^{\frac{3}{2}-2\sigma_i}| \lesssim ||u^0_{py} y||_{L^\infty} ||\frac{v}{y} x^{\frac{3}{4}-\sigma_i}||_{L^2_y} ||u_x x^{\frac{3}{4}-\sigma_i}||_{L^2_y}$$
$$\lesssim \mathcal{O}(\delta) ||v_y x^{\frac{3}{4}-\sigma_i}||^2_{L^2_y}. \tag{3.7.118}$$

We can summarize this contribution via:

$$|\int \mathcal{P} \cdot u_x x^{\frac{3}{2}-2\sigma_i}| \le \mathcal{O}(\delta)||u_x x^{\frac{3}{4}-\sigma_i}||^2_{L^2_y} + \mathcal{O}(\delta)||u_y x^{\frac{1}{4}-\sigma_i}||^2_{L^2_y} + \mathcal{O}(\delta)||u x^{-\frac{1}{4}-\sigma_i}||^2_{L^2_y}. \tag{3.7.119}$$

Again, the smallness is guaranteed by (3.4.86), (3.4.91), and (3.5.61). Next, the forcing is given in the same way, due to the choice of σ_i relative to σ_{i-1}:

$$|\int f^{(i)} \cdot u_x x^{\frac{3}{2}-2\sigma_i}| \le ||f^{(i)} x^{\frac{3}{4}-\sigma_i}||_{L^2_y} ||u_x x^{\frac{3}{4}-\sigma_i}||_{L^2_y}$$
$$\lesssim x^{-\frac{1}{2}-\sigma'} ||u_x x^{\frac{3}{4}-\sigma_i}||_{L^2_y} \lesssim x^{-1-\sigma'} + \frac{1}{100,000} ||u_x x^{\frac{3}{4}-\sigma_i}||^2_{L^2_y}. \tag{3.7.120}$$

We now integrate up in x up to some fixed point X_1:

$$\int u_y^2(X_1) X_1^{\frac{3}{2}-2\sigma_i} + \int_1^{X_1} \int u_x^2 x^{\frac{3}{2}-2\sigma_i} \tag{3.7.121}$$
$$\lesssim C + \int_1^{X_1} \int u_y^2 x^{\frac{1}{2}-2\sigma_i} + \mathcal{O}(\delta) \int_1^{X_1} \int u^2 x^{-\frac{1}{2}-2\sigma_i} + \int_1^{X_1} \int \mathcal{J} \cdot u_x x^{\frac{3}{2}-2\sigma_i}.$$

The last part is to control the $\mathcal{J}$ term above:

$$\int_1^{X_1} \int \chi''(y) x^{\frac{3}{2}-2\sigma_i} u^i_e(x,0) u_x \tag{3.7.122}$$
$$= -\int_1^{X_1} \int \chi''(y) u \partial_x \{ u^i_e(x,0) x^{\frac{3}{2}-2\sigma_i} \} - \int_{x=1} u^i_e(0,0) u \chi''(y) dy$$
$$+ \int_{x=X_1} X_1^{\frac{3}{2}-2\sigma_i} u^i_e(X_1,0) \chi''(y) u(X_1,y) dy \tag{3.7.123}$$

$$= -\int_1^{X_1}\int \chi''(y)u\partial_x\{u_e^i(x,0)x^{\frac{3}{2}-2\sigma_i}\} + C$$

$$- C\int_{x=X_1} X_1^{\frac{3}{2}-2\sigma_i}u_e^i(X_1,0)\chi'(y)u_y(X_1,y)dy \tag{3.7.124}$$

$$\leq \int_1^{X_1}\int \chi'(y)u_y\partial_x\{u_e^i(x,0)x^{\frac{3}{2}-2\sigma_i}\} + C + \frac{1}{100,000}\int_{x=X_1} X_1^{\frac{3}{2}-2\sigma_i}u_y^2(X_1) \tag{3.7.125}$$

$$\leq C||u_y x^{\frac{1}{4}-\sigma_i}||_{L^2}||\chi'(y)x^{-\frac{1}{2}-\sigma_i}||_{L^2} + C \leq C + \frac{1}{100,000}||u_y x^{\frac{1}{4}-\sigma_i}||_{L^2}^2. \tag{3.7.126}$$

We have absorbed the x boundary contribution, $\int u_y^2 X_1^{\frac{3}{2}-2\sigma_i}$ into the left-hand side of (3.7.121). For the other terms in $\mathcal{J}$, we can perform a similar calculation. As X_1 is arbitrary, this then implies the desired result, (3.7.114).

□

Proof of Proposition 3.7.11. Consolidating estimates the three estimates (3.7.86), (3.7.105), and (3.7.114), and subsequently taking δ small enough, one obtains:

$$\sup_x \int \psi^2 x^{-\frac{1}{2}-2\sigma_i} + u^2 x^{\frac{1}{2}-2\sigma_i} + u_y^2 x^{\frac{3}{2}-2\sigma_i}$$

$$+ \int\int \psi^2 x^{-\frac{3}{2}-2\sigma_i} + u^2 x^{-\frac{1}{2}-2\sigma_i} + u_y^2 x^{\frac{1}{2}-2\sigma_i} + u_x^2 x^{\frac{3}{2}-2\sigma_i} \lesssim \mathcal{O}(\delta; n). \tag{3.7.127}$$

It is a standard matter now to obtain weighted in z estimates, and to also successively differentiate the system in x and repeat the previously established bounds. This procedure establishes the desired result fpr u_p^i in (3.7.79). To estimate v_p^i, we take:

$$v_p^i = \int_y^\infty u_{px}^i \, \mathrm{d}y', \tag{3.7.128}$$

and repeat the procedure as in estimate (3.6.69).

□

3.7.3 Final Prandtl Layer

We now construct the final Prandtl layer, $[u_p^n, v_p^n]$. This final, n'th layer is slightly different because v_p^n will be taken to satisfy the boundary condition: $v_p^n(x,0) = 0$. According to (3.7.48), the system for the n'th layer is:

$$(1+u_p^0)u_{px} + u_{sx}^{(n)}u_p + v_s^{(n)}u_{py} + u_{py}^0 v_p + P_{px}^n = u_{pyy} + f^{(n)}, \tag{3.7.129}$$

$$[u_p(x,0), v_p(x,0)] = [-u_e^n(x,0), 0], \quad \lim_{y\to\infty} u_p(x,y) = 0, \quad u_p(1,y) = U_n(y), \tag{3.7.130}$$

$$P_{py}^n = 0, \quad u_{px} + v_{py} = 0. \tag{3.7.131}$$

As usual from the Prandtl layers, by evaluating the equation at $y = \infty$, it is clear that the leading order Prandtl pressure is constant, that is $P_p^n = 0$. Once u_p, v_p are constructed to solve (3.7.129), we shall cut them off, thereby defining $[u_p^n, v_p^n]$. The relevant remainders are given in (3.3.4) - (3.3.5), which we recall here for convenience:

Definition 3.7.15. The n'th remainder is denoted by:

$$R^{u,n} := -\Delta_\epsilon \bar{u}_s^{(n)} + \bar{u}_s^{(n)}\bar{u}_{sx}^{(n)} + \bar{v}_s^{(n)}\bar{u}_{sy}^{(n)} + \bar{P}_{sx}^{(n)}, \tag{3.7.132}$$

$$R^{v,n} := -\Delta_\epsilon \bar{v}_s^{(n)} + \bar{u}_s^{(n)}\bar{v}_{sx}^{(n)} + \bar{v}_s^{(n)}\bar{v}_{sy}^{(n)} + \frac{\partial_y}{\epsilon}\bar{P}_s^{(n)}. \tag{3.7.133}$$

Remark. We refer the reader to Remark 3.7.1. In this case, there is no $n+1$'th Euler construction, which explains the definition in (3.7.132). This is seen as the remainder which is contributed to the next order in the specification of system (3.8.1) - (3.8.7).

Using similar arguments to (3.7.50) - (3.7.51), we have the following errors:

$$R^{u,n} = \epsilon^{\frac{n}{2}} \Big[\epsilon u^n_{pxx} + v^n_p \Big(\sum_{j=1}^n \epsilon^{\frac{j}{2}} \{ u^j_{py} + \sqrt{\epsilon} u^j_{eY} \} \Big) + u^n_{px} \sum_{j=1}^n \epsilon^{\frac{j}{2}} \{ u^j_e + u^j_p \} + \mathcal{E}^{(n)} \Big], \quad (3.7.134)$$

$$R^{v,n} = \epsilon^{\frac{n}{2}} \Big[- \Delta_\epsilon v^n_p + u^{(n)}_s v^n_{px} + v^{(n)}_{sx} u^n_p + v^{(n)}_s v^n_{py} + v^{(n)}_{sy} v^n_p + \epsilon^{\frac{n}{2}} u^n_p v^n_{px} + \epsilon^{\frac{n}{2}} v^n_p v^n_{py} \Big], \quad (3.7.135)$$

Here $\mathcal{E}^{(n)} = \mathcal{E}^{(n)}(u_p, v_p)$ is an error term created by cutting off these layers, which will be defined in (3.7.155). One can repeat the procedure used to construct the previous Prandtl layers to conclude:

Proposition 3.7.16. *Let σ_n be defined according to (3.7.1). The Prandtl layer, u_p constructed to satisfy the equation (3.7.129) together with the boundary conditions in (3.7.130), satisfies:*

$$||x^{\frac{1}{4}} z^m \partial_x^k u_p||_{P_k(\sigma_n)} \le C(k,m,n). \quad (3.7.136)$$

Let us summarize the decay rates which result from this construction, by recalling the definition of our Prandtl norms, P_k, given in (3.6.33) - (3.6.34):

Corollary 3.7.17. *For any $k, j, M \ge 0$, the profiles u_p satisfy the following decay rates:*

$$x^{k+\frac{j}{2}+\frac{1}{2}-\sigma_n} ||z^M \partial_x^k \partial_y^j u_p||_{L^\infty_y} + x^{k+\frac{j}{2}+\frac{1}{4}-\sigma_n} ||z^M \partial_x^k \partial_y^j u_p||_{L^2_y} \le C(k,j,M,n). \quad (3.7.137)$$

In order to satisfy $v_p(x,0) = 0$, v_p is obtained from u_p via: $v_p(x,y) = -\int_0^y u_{px}(x,y')\,\mathrm{d}y'$. This is distinct from (3.6.27) and (3.7.128), and distinguishes the final, $n-$th Prandtl layer from the previous layers. Then,

Corollary 3.7.18. *v_p obeys the following uniform decay estimate:*

$$x^{k+1-\sigma_n} ||\partial_x^k v_p||_{L^\infty_y} \le C(k,n). \quad (3.7.138)$$

Proof. Using the boundary condition $v_p(x,0)=0$, one can write:

$$\Big|v_p\Big|^2 \lesssim \Big|\int_0^y v_p v_{py}\Big| \le \int_0^\infty \Big|\frac{v_p}{y^{\frac{1}{2}+\kappa}} y^{\frac{1}{2}+\kappa} v_{py}\Big| \le ||\frac{v_p}{y^{\frac{1}{2}+\kappa}}||_{L^2_y} ||y^{\frac{1}{2}+\kappa} v_{py}||_{L^2_y} \tag{3.7.139}$$

$$\le ||y^{\frac{1}{2}-\kappa} v_{py}||_{L^2_y} ||y^{\frac{1}{2}+\kappa} v_{py}||_{L^2_y} = ||x^{\frac{1}{4}-\kappa} z^{\frac{1}{2}-\kappa} v_{py}||_{L^2_y} ||x^{\frac{1}{4}+\kappa} z^{\frac{1}{2}+\kappa} v_{py}||_{L^2_y} \tag{3.7.140}$$

$$\le x^{\frac{1}{2}} x^{-1+\sigma_n} x^{-1+\sigma_n} = x^{-\frac{3}{2}+2\sigma_n}. \tag{3.7.141}$$

We have used the $\kappa>0$ to avoid the critical Hardy inequality. The Hardy inequality we have used (with power $y^{-\frac{1}{2}-\kappa} v^1_p$) relies on the vanishing of v^n_p at $y=0$. ☐

Next, we introduce a cutoff function which honors the scaling, $z=\frac{y}{\sqrt{x}}$:

$$v^n_p := \chi(\sqrt{\epsilon} z) v_p, \quad u^n_p := \int_x^\infty \partial_y \Big[\chi\Big(\sqrt{\epsilon}\frac{y}{\sqrt{x'}}\Big) v_p(x',y)\Big)\Big] dx'. \tag{3.7.142}$$

Here, we make the notational convention for integrands: $z'=\frac{y}{\sqrt{x'}}$ or $z'=\frac{y'}{\sqrt{x}}$ where $'$ denotes the integration variable. It is clear, then, that $\partial_x u^n_p + \partial_y v^n_p = 0$. The following boundary conditions are also clear:

$$[u^n_p, v^n_p] \to 0 \text{ as } y\to\infty, \quad [u^n_p, v^n_p]|_{y=0} = [u_p, v_p]|_{y=0}. \tag{3.7.143}$$

Remark. This cut-off honors the parabolic scaling of the Prandtl layers for all $x>0$. The cut-off introduced in (GN17) was in the region $y \ge \frac{1}{\sqrt{\varepsilon}}$. Locally in x, z is equivalent to y, so these cut-offs agree locally in x.

We must now record two properties about the cut-off layers. First, the uniform estimates of the cut-off layers remain unchanged:

Lemma 3.7.19. *The following pointwise decay bounds hold for the cut-off layers, for any*

$k \geq 0$:

$$\left|\partial_x^k v_p^n\right| \lesssim C(k,n)x^{-k-1+\sigma_n}, \tag{3.7.144}$$

$$\left|\partial_x^k v_{py}^n\right| \lesssim C(k,n)x^{-k-\frac{3}{2}+\sigma_n}, \tag{3.7.145}$$

$$\left|\partial_x^k u_{py}^n\right| \lesssim C(k,n)x^{-k-1+\sigma_n}, \tag{3.7.146}$$

$$\left|\partial_x^k u_p^n\right| \lesssim C(k,n)x^{-k-\frac{1}{2}+\sigma_n}. \tag{3.7.147}$$

Proof. First,

$$|v_{py}^n| \leq \frac{\sqrt{\epsilon}}{\sqrt{x}}\chi'|v_p| + \chi|v_{py}| \lesssim x^{-\frac{3}{2}+\sigma_n}. \tag{3.7.148}$$

Next,

$$\begin{aligned}\left|u_{py}^n\right| = \Big|\int_x^\infty \partial_{yy}\Big[\chi(\sqrt{\epsilon}z')v_p\Big]dx'\Big| \leq \Big|\int_x^\infty \frac{\epsilon}{x'}\chi''v_p\Big| + \Big|2\int_x^\infty \frac{\sqrt{\epsilon}}{\sqrt{x'}}v_{py}\Big| + \Big|\int_x^\infty \chi v_{pyy}\Big| \\ \lesssim x^{-1+\sigma_n}.\end{aligned} \tag{3.7.149}$$

Finally, for u_p^n,

$$\left|u_p^n\right| \leq \int_x^\infty \sqrt{\epsilon}\frac{1}{\sqrt{x'}}\chi'|v_p| + \int_x^\infty |v_{py}| \leq x^{-\frac{1}{2}+\sigma_n}. \tag{3.7.150}$$

By differentiating successively in x, the result for $k \geq 1$ follows in the same manner. $\square$

Lemma 3.7.20 (L^2_y Estimates)**.** *The following L^2_y decay bounds hold for the cut-off layers, for any $k \geq 0$:*

$$||\partial_x^k \partial_y^j u_p^n||_{L^2_y} \lesssim C(k,\sigma,n)x^{-k-\frac{j}{2}-\frac{1}{4}+\sigma_n}. \tag{3.7.151}$$

Proof. In order to compute the L^2 norms, we must trade in the following way:

$$o(\chi)\Big|\varepsilon^{\frac{1}{4}+\frac{\kappa}{2}}\langle y\rangle^{\frac{1}{2}+\kappa}x^{-\frac{1}{4}-\frac{\kappa}{2}}\Big| \lesssim 1, \tag{3.7.152}$$

where $o(\chi)$ means χ or any derivative of χ, the essential feature being that $z \le \frac{1}{\sqrt{\varepsilon}}$. Using this:

$$||u_p^n||_{L_y^2} \le \int_x^\infty ||\frac{\sqrt{\varepsilon}}{\sqrt{x'}}\chi' v_p||_{L_y^2} + \int_x^\infty ||\chi v_{py}||_{L_y^2} \lesssim x^{-\frac{1}{4}+\sigma_n}. \tag{3.7.153}$$

For u_{py}^n, again according to the expression in (3.7.149), and the decay rates in (3.7.137):

$$||u_{py}^n||_{L_y^2} \lesssim \int_x^\infty ||\frac{\epsilon\chi''}{x'}v_p||_{L_y^2} + \int_x^\infty ||\frac{\sqrt{\varepsilon}}{\sqrt{x'}}v_{py}||_{L_y^2} + \int_x^\infty ||v_{pyy}||_{L_y^2} \lesssim x^{-\frac{3}{4}+\sigma_n}. \tag{3.7.154}$$

It is now clear we can repeat these calculations for higher x and y derivatives, thereby obtaining the desired result.

□

The next task is to control the error made by cutting off the layers. This error is obtained by inserting the new, cut-off layers into the equation (3.7.129):

$$\mathcal{E}^{(n)} := (1+u_p^0)u_{px}^n + u_{sx}^{(n)}u_p^n + v_s^{(n)}u_{py}^n + u_{py}^0 v_p^n - u_{pyy}^n - f^{(n)}. \tag{3.7.155}$$

We will proceed to expand (3.7.155), term-by-term:

$$(1+u_p^0)u_{px}^n = \chi(1+u_p^0)u_{px} - (1+u_p^0)\frac{\sqrt{\epsilon}}{\sqrt{x}}\chi' v_p, \tag{3.7.156}$$

$$u_{sx}^{(n)}u_p^n = \chi u_{sx}^{(n)}u_p + u_{sx}^{(n)}\int_x^\infty \frac{\sqrt{\epsilon}}{\sqrt{x'}}\chi' v_p + \chi'\frac{\sqrt{\epsilon}z}{x'}u_p, \tag{3.7.157}$$

$$v_s^{(n)} u_{py}^n = \chi v_s^{(n)} u_{py} + v_s^{(n)} \int_x^\infty \frac{\epsilon}{x'} \chi'' v_p + 2 \frac{\sqrt{\epsilon}}{\sqrt{x'}} \chi' v_{py} - \frac{\sqrt{\epsilon}}{\sqrt{x'}} \chi' z u_{py}, \tag{3.7.158}$$

$$u_{py}^0 v_p^n = \chi u_{py}^0 v_p, \tag{3.7.159}$$

$$u_{pyy}^n = \chi u_{pyy} - \int_x^\infty \frac{\sqrt{\epsilon}}{x'} \chi' \frac{z}{2} u_{pyy} + C_1 \frac{\epsilon}{x'} \chi'' v_{py} + C_2 \chi' \frac{\sqrt{\epsilon}}{\sqrt{x'}} v_{pyy}$$
$$+ \frac{\epsilon^{\frac{3}{2}}}{(x')^{\frac{3}{2}}} \chi''' v_p. \tag{3.7.160}$$

Let us provide justification to the above expressions, first turning to (3.7.157). Applying the definition in (3.7.142) yields:

$$\begin{aligned} u_{sx}^{(n)} u_p^n &= u_{sx}^{(n)} \int_x^\infty \frac{\sqrt{\epsilon}}{\sqrt{x'}} \chi' v_p + u_{sx}^{(n)} \int_x^\infty \chi v_{py} \\ &= u_{sx}^{(n)} \int_x^\infty \frac{\sqrt{\epsilon}}{\sqrt{x'}} \chi' v_p - u_{sx}^{(n)} \int_x^\infty \chi u_{px} dx' \\ &= u_{sx}^{(n)} \int_x^\infty \Big[\frac{\sqrt{\epsilon}}{\sqrt{x'}} \chi' v_p + u_{sx}^{(n)} \frac{z\sqrt{\epsilon}}{x'} \chi' u_p \Big] dx' + \chi u_{sx}^{(n)} u_p, \end{aligned} \tag{3.7.161}$$

the final equality following from an integration by parts in x. Similarly, for (3.7.158):

$$\begin{aligned} v_s^{(n)} u_{py}^n &= v_s^{(n)} \int_x^\infty \partial_{yy} \Big(\chi v_p \Big) dx' \\ &= v_s^{(n)} \int_x^\infty \Big(\frac{\epsilon}{x'} \chi'' v_p + 2\chi' \frac{\sqrt{\epsilon}}{\sqrt{x'}} v_{py} + \chi v_{pyy} \Big) \mathrm{d}x' \\ &= v_s^{(n)} \int_x^\infty \frac{\epsilon}{x'} \chi'' v_p + 2\chi' \frac{\sqrt{\epsilon}}{x'} v_{py} - v_s^{(n)} \int_x^\infty \chi u_{pxy} dx' \\ &= v_s^{(n)} \int_x^\infty \Big[\frac{\epsilon}{x'} \chi'' v_p + 2\chi' \frac{\sqrt{\epsilon}}{x'} v_{py} - \sqrt{\epsilon} \frac{\chi'}{x'} z u_{py} \Big] \mathrm{d}x' + v_s^{(n)} \chi u_{py}. \end{aligned} \tag{3.7.162}$$

This same computation is performed for (3.7.160):

$$\begin{aligned} u_{pyy}^n &= \int_x^\infty \partial_y^3 \Big(\chi v_p \Big) dx' \\ &= \int_x^\infty \Big[\frac{\varepsilon^{\frac{3}{2}}}{(x')^{\frac{3}{2}}} \chi''' v_p + C_1 \frac{\varepsilon}{x'} \chi'' v_{py} + C_2 \frac{\sqrt{\varepsilon}}{\sqrt{x'}} v_{pyy} \chi' \Big] + \int_x^\infty \chi v_{pyyy} \end{aligned}$$

$$= \int_x^\infty \Big[\frac{\varepsilon^{\frac{3}{2}}}{(x')^{\frac{3}{2}}}\chi'''v_p + C_1\frac{\varepsilon}{x'}\chi''v_{py} + C_2\frac{\sqrt{\varepsilon}}{\sqrt{x'}}v_{pyy}\chi'\Big] - \int_x^\infty \chi u_{pxyy}$$

$$= \int_x^\infty \Big[\frac{\varepsilon^{\frac{3}{2}}}{(x')^{\frac{3}{2}}}\chi'''v_p + C_1\frac{\varepsilon}{x'}\chi''v_{py} + C_2\frac{\sqrt{\varepsilon}}{\sqrt{x'}}v_{pyy}\chi'\Big] - \int_x^\infty \Big[\chi' u_{pyy}\frac{z}{2x'}\sqrt{\varepsilon}\Big]\,\mathrm{d}x'$$
$$+ \chi u_{pyy}. \tag{3.7.163}$$

Summing the above terms together, we arrive at our expression:

$$\mathcal{E}^{(n)} = -(1+u_p^0)\frac{\sqrt{\epsilon}}{\sqrt{x}}\chi' v_p + u_{sx}^{(n)}\int_x^\infty \frac{\sqrt{\epsilon}}{\sqrt{x'}}\chi' v_p + u_{sx}^{(n)}\int_x^\infty \chi'\frac{\sqrt{\epsilon}z}{x'}u_p \tag{3.7.164}$$
$$+ v_s^{(n)}\int_x^\infty \frac{\epsilon}{x'}\chi'' v_p + 2\frac{\sqrt{\epsilon}}{\sqrt{x'}}\chi' v_{py} - \frac{\sqrt{\epsilon}}{x'}\chi' z u_{py} - \int_x^\infty \frac{\sqrt{\epsilon}}{x'}\chi'\frac{z}{2}u_{pyy} \tag{3.7.165}$$
$$+ \int_x^\infty \Big[C_2\chi'\frac{\sqrt{\epsilon}}{\sqrt{x'}}v_{pyy} + C_1\frac{\epsilon}{x'}\chi''v_{py} + \frac{\epsilon^{\frac{3}{2}}}{(x')^{\frac{3}{2}}}\chi'''v_p\Big] + (1-\chi)f^{(n)}. \tag{3.7.166}$$

Lemma 3.7.21. *For $\kappa > 0$ arbitrarily small, the error, $\mathcal{E}^{(n)}$ obeys the following bound:*

$$\Big|\mathcal{E}^{(n)}\Big| \le C(n,\kappa)\epsilon^{\frac{1}{4}-\kappa}x^{-\frac{3}{2}+\sigma_n+\kappa}, \quad ||\mathcal{E}^{(n)}||_{L^2_y} \le C(n,\kappa)\epsilon^{\frac{1}{4}-\kappa}x^{-\frac{5}{4}+\sigma_n+\kappa}. \tag{3.7.167}$$

Proof. We will proceed in order from (3.7.164) - (3.7.166), and we will focus on the L^2_y estimates, as the uniform estimates are straight-forward. First,

$$||(1+u_p^0)\frac{\sqrt{\epsilon}}{\sqrt{x}}\chi'(z)v_p||_{L^2_y} \lesssim \sqrt{\epsilon}||\langle y\rangle^{\frac{1}{2}+\kappa}\langle y\rangle^{-\frac{1}{2}-\kappa}\chi' v_p\frac{1}{\sqrt{x}}||_{L^2_y}$$
$$\lesssim \sqrt{\epsilon}||\langle y\rangle^{\frac{1}{2}+\kappa}\chi' x^{-\frac{1}{2}}v_p||_{L^\infty_y} \lesssim \sqrt{\epsilon}||z^{\frac{1}{2}+\kappa}\chi' x^{-\frac{1}{4}+\frac{\kappa}{2}}v_p||_{L^\infty_y}$$
$$\lesssim \epsilon^{\frac{1}{4}-\kappa}x^{-\frac{5}{4}+\frac{\kappa}{2}+\sigma_n}. \tag{3.7.168}$$

The essential characteristic in the above calculation is the ability to pay a factor of $\epsilon^{\frac{1}{4}+\frac{\kappa}{2}}x^{-\frac{1}{4}-\frac{\kappa}{2}}$ in order to obtain an L^2_y quantity due to the presence of the cut-off function χ' (see 3.7.152). In similar manner, we have:

$$||u_{sx}^{(n)}\int_x^\infty \frac{\sqrt{\epsilon}}{\sqrt{x'}}\chi' v_p dx'||_{L^2_y} \le ||u_{sx}^{(n)}||_{L^\infty}\int_x^\infty ||\frac{\sqrt{\epsilon}}{\sqrt{x'}}\chi' v_p||_{L^\infty_y}||\langle y\rangle^{-\frac{1}{2}-\kappa}||_{L^2_y}dx'$$

$$\lesssim x^{-1}\int_x^\infty (x')^{-1+\sigma_n}\epsilon^{\frac14-\frac{\kappa}{2}}x^{-\frac14+\frac{\kappa}{2}}||\epsilon^{\frac14+\frac{\kappa}{2}}x^{-\frac14-\frac{\kappa}{2}}\langle y\rangle^{\frac12+\kappa}\chi'||_{L^\infty}$$
$$\lesssim \epsilon^{\frac14+\kappa}x^{-\frac54+\sigma_n+\frac{\kappa}{2}}. \tag{3.7.169}$$

The third term on line (3.7.164) can be estimated analogously. Coming now to the first term in (3.7.165),

$$||v_s^{(n)}\int_x^\infty \frac{\epsilon}{x}\chi'' v_p||_{L^2_y} \le ||v_s^{(n)}||_{L^\infty}\int_x^\infty \epsilon^{\frac34-\frac{\kappa}{2}}(x')^{-\frac34+\frac{\kappa}{2}}(x')^{-1+\sigma_n}||\epsilon^{\frac14+\frac{\kappa}{2}}x^{-\frac14-\frac{\kappa}{2}}\langle y\rangle^{\frac12+\kappa}\chi''||_{L^\infty}$$
$$\lesssim \epsilon^{\frac34-\frac{\kappa}{2}}x^{-\frac54+\frac{\kappa}{2}+\sigma_n}. \tag{3.7.170}$$

The second term in (3.7.165):

$$||v_s^{(n)}\int_x^\infty \frac{\sqrt{\epsilon}}{\sqrt{x'}}\chi' v_{py}||_{L^2_y} \le ||v_s^{(n)}||_{L^\infty}\int_x^\infty \epsilon^{\frac14-\frac{\kappa}{2}}(x')^{-\frac14+\frac{\kappa}{2}}(x')^{-\frac32+\sigma_n}dx'$$
$$\lesssim \epsilon^{\frac14-\frac{\kappa}{2}}x^{-\frac54+\sigma_n+\frac{\kappa}{2}}. \tag{3.7.171}$$

Moving to the third, which is slightly different due to the weight z, but this causes no harm as zu_{py} is known to be in L^∞_y according to (3.7.136):

$$||v_s^{(n)}\int_x^\infty \frac{\sqrt{\epsilon}}{x'}\chi' z u_{py}||_{L^2_y} \lesssim x^{-\frac12}\int_x^\infty \frac{\sqrt{\epsilon}}{x'}(x')^{-\frac34+\sigma_n}dx' \lesssim \sqrt{\epsilon}x^{-\frac54-\sigma_n}. \tag{3.7.172}$$

Next, we move to the u^n_{pyy} contributions:

$$\int_x^\infty ||C_1\frac{\varepsilon}{x'}\chi'' v_{py} + C_2\frac{\sqrt{\varepsilon}}{\sqrt{x'}}v_{pyy}\chi' - \chi' u_{pyy}\frac{z}{2x'}\sqrt{\varepsilon}||_{L^2_y} \lesssim \sqrt{\varepsilon}x^{-\frac54+\sigma_n}, \tag{3.7.173}$$

according to the rates, $||u_{pyy}zx^{\frac54-\sigma_n}, v_{pyy}x^{\frac74-\sigma_n}, v_{py}x^{\frac54-\sigma_n}||_{L^2_y} \le C$ from (3.7.137). Finally, for the v_p term in (3.7.166), one must use the tradeoff in (3.7.152), to obtain:

$$\int_x^\infty ||\frac{\varepsilon^{\frac{3}{2}}}{(x')^{\frac{3}{2}}}\chi''' v_p||_{L^2_y} dx' \lesssim \varepsilon^{\frac{5}{4}-\frac{\kappa}{2}} \int_x^\infty (x')^{\frac{1}{4}+\frac{\kappa}{2}}(x')^{-\frac{3}{2}}||v_p||_{L^\infty_y}$$
$$\lesssim \varepsilon^{\frac{5}{4}-\frac{\kappa}{2}} \int_x^\infty (x')^{\frac{1}{4}+\frac{\kappa}{2}}(x')^{-\frac{3}{2}}(x')^{-1+\sigma_n} \lesssim \varepsilon^{\frac{5}{4}-\frac{\kappa}{2}} x^{-\frac{5}{4}+\frac{\kappa}{2}+\sigma_n}. \tag{3.7.174}$$

The $f^{(n)}$ term can then be estimated upon observing that $f^{(n)}$ is exponentially small in the support of $1-\chi$:

$$||(1-\chi)f^{(n)}||_{L^2_y} = ||(1-\chi)z^N z^{-N} f^{(n)}||_{L^2_y}$$
$$\lesssim \epsilon^{\frac{N}{2}}||z^N f^{(n)}||_{L^2_y} \lesssim \epsilon^{\frac{N}{2}} x^{-\frac{5}{4}+\sigma_{n-1}} \lesssim \epsilon^{\frac{N}{2}} x^{-\frac{5}{4}+\sigma_n}, \tag{3.7.175}$$

for any N large, according to estimate (3.7.59). This now concludes the proof of (3.7.167). □

By construction, one has the following:

Lemma 3.7.22 (Remainder Estimates)**.** *For $R^{u,n}, R^{v,n}$ as defined in (3.7.135), and for any $\gamma \in [0, \frac{1}{4})$, $n \ge 2$, and for σ_n as in (3.7.1), $\kappa > 0$ arbitrarily small,*

$$\epsilon^{-\frac{n}{2}-\gamma}\Big|\partial_x^k R^{u,n} + \sqrt{\epsilon}\partial_x^k R^{v,n}\Big| \lesssim C(n,\kappa)\epsilon^{\frac{1}{4}-\gamma-\kappa} x^{-k-\frac{3}{2}+2\sigma_n}, \tag{3.7.176}$$
$$\epsilon^{-\frac{n}{2}-\gamma}||\sqrt{\epsilon}\partial_x^k R^{u,n}, \sqrt{\epsilon}\partial_x^k R^{v,n}||_{L^2_y} \lesssim C(n,\kappa)\epsilon^{\frac{1}{4}-\gamma-\kappa} x^{-k-\frac{5}{4}+2\sigma_n+\kappa}. \tag{3.7.177}$$

Proof. This follows from (3.7.167) and those bounds established in (3.7.136). We shall proceed term by term from (3.7.135). First,

$$\varepsilon^{1-\gamma}||u^n_{pxx}||_{L^2_y} = \varepsilon^{1-\gamma}||\partial_x\Big(\chi v_p\Big)_y||_{L^2_y} = \varepsilon^{1-\gamma}||\partial_x\Big(\frac{\chi'}{\sqrt{x}}\sqrt{\varepsilon}v_p + \chi v_{yp}\Big)||_{L^2_y}$$
$$= \varepsilon^{1-\gamma}||\varepsilon\frac{\chi''}{x^{\frac{3}{2}}}zv_p + \frac{\chi'\sqrt{\varepsilon}}{x^{\frac{3}{2}}}v_p + \frac{\chi'}{\sqrt{x}}\sqrt{\varepsilon}v_{px} + \sqrt{\varepsilon}\frac{\chi'}{x}zv_{py} + \chi v_{pxy}||_{L^2_y} \tag{3.7.178}$$

$$= (3.7.178.1) + ...(3.7.178.5).$$

First,

$$\begin{aligned}|(3.7.178.1)| &\lesssim \varepsilon^{1-\gamma}||\sqrt{\varepsilon}z\cdot\Big(\langle y\rangle^{\frac{1}{2}+\kappa}x^{-\frac{1}{4}-\frac{\kappa}{2}}\varepsilon^{\frac{1}{4}+\frac{\kappa}{2}}\Big)\chi''||_{L^\infty_y}\varepsilon^{\frac{1}{4}-\frac{\kappa}{2}}||v_p||_{L^\infty}x^{-\frac{5}{4}+\frac{\kappa}{2}} \\ &\lesssim \varepsilon^{\frac{5}{4}-\gamma-\frac{\kappa}{2}}x^{-\frac{9}{4}+\frac{\kappa}{2}+\sigma_n}. \end{aligned} \tag{3.7.179}$$

It is clear that (3.7.178.1), (3.7.178.2), and (3.7.178.3) are identical, and so we move to:

$$|(3.7.178.4)| \lesssim \varepsilon^{1-\gamma}\frac{\sqrt{\varepsilon}}{x}||zv_{py}||_{L^2_y} \lesssim \varepsilon^{1-\gamma}\frac{\sqrt{\varepsilon}}{x}x^{-\frac{5}{4}+\sigma_n}, \tag{3.7.180}$$

and finally:

$$|(3.7.178.5)| = \varepsilon^{1-\gamma}||\chi v_{pxy}||_{L^2_y} \lesssim \varepsilon^{1-\gamma}x^{-\frac{9}{4}+\sigma_n}. \tag{3.7.181}$$

We next move to the second term in $R^{u,n}$, for which we immediately use (3.7.152)

$$\begin{aligned}\varepsilon^{\frac{n}{2}-\gamma}||v_p^n u_{py}^n||_{L^2_y} &\le \varepsilon^{\frac{n}{2}-\gamma-\frac{1}{4}-\frac{\kappa}{2}}x^{\frac{1}{4}+\frac{\kappa}{2}}||v_p^n||_{L^\infty_y}||u_{py}^n||_{L^\infty_y} \\ &\lesssim \varepsilon^{\frac{n}{2}-\gamma-\frac{1}{4}-\frac{\kappa}{2}}x^{\frac{1}{4}+\frac{\kappa}{2}}x^{-2+2\sigma_n} = \varepsilon^{\frac{n}{2}-\gamma-\frac{1}{4}-\frac{\kappa}{2}}x^{-\frac{7}{4}+\frac{\kappa}{2}+2\sigma_n}. \end{aligned} \tag{3.7.182}$$

where we have used (3.7.144), (3.7.146). Similarly, for $j \ge 1$, one has:

$$\varepsilon^{\frac{1}{2}-\gamma}||v_p^n u_{py}^j||_{L^2_y} \le \varepsilon^{\frac{1}{2}-\gamma}||v_p^n||_{L^\infty}||u_{py}^j||_{L^2_y} \lesssim \varepsilon^{\frac{1}{2}-\gamma}x^{-1+\sigma_n}x^{-\frac{1}{2}+\sigma_n} \lesssim \varepsilon^{\frac{1}{2}-\gamma}x^{-\frac{3}{2}+2\sigma_n} \tag{3.7.183}$$

where we have used the specification of the norm $||\cdot||_P$ given in (3.6.33). For the term

$v_p^n u_{eY}^j$, we calculate, for $\kappa > 0$:

$$\epsilon^{\frac{1}{2}+\frac{j}{2}-\gamma}||v_p^n u_{eY}^j||_{L_y^2} \le \epsilon^{\frac{1}{4}+\frac{j}{2}-\gamma}||v_p^n||_{L_y^\infty}||u_{eY}^j||_{L_Y^2} \lesssim C(n)\epsilon^{\frac{1}{4}+\frac{j}{2}-\gamma}x^{-1+\sigma_n}x^{-1+\kappa}. \quad (3.7.184)$$

The last estimate follows from applying L_Y^2 to the Eulerian self-similar estimate, (3.5.48):

$$\left|u_{eY}^1\right| = \left|v_{ex}^1\right| \le C(\kappa)x^{-1+\kappa}Y^{-\frac{1}{2}-\kappa}. \quad (3.7.185)$$

The next term in $R^{u,n}$ is, for $j \ge 1$, for which we use (3.6.81) for the u_p^j term, (3.7.21) for the u_e^j term, and (3.7.151) for the $u_{px}^{(n)}$ term in L^2:

$$||\varepsilon^{\frac{j}{2}}u_{px}^{(n)}(u_e^j + u_p^j)||_{L_y^2} \lesssim \varepsilon^{\frac{j}{2}}x^{-\frac{1}{4}+\sigma_n}||u_{px}^n||_{L_y^2} \lesssim \varepsilon^{\frac{1}{4}-\frac{\kappa}{2}}x^{-\frac{1}{4}+\sigma_n}x^{-\frac{5}{4}+\sigma_n}. \quad (3.7.186)$$

The final term in $R^{u,n}$ is the error term, $\mathcal{E}^{(n)}$, which has been controlled in (3.7.167). We may now move to $R^{v,n}$. In so doing, we first note that εv_{pxx}^n can be estimated identically to (3.7.187). Second, using (3.7.151), we have:

$$\varepsilon^{\frac{1}{2}-\gamma}||v_{pyy}^n||_{L_y^2} \lesssim \varepsilon^{\frac{1}{2}-\gamma}x^{-\frac{7}{4}+\sigma_n}.$$

For the L^2 bound on the v_{px}^n term in (3.7.135), we employ (3.7.152):

$$\begin{aligned}
\epsilon^{\frac{1}{2}-\gamma}||u_s^{(n)}v_{px}^n||_{L_y^2} &\lesssim \epsilon^{\frac{1}{2}-\gamma}||v_{px}^n||_{L_y^2} \le \epsilon^{\frac{1}{2}-\gamma}||1_{z\le\frac{1}{\sqrt{\epsilon}}}v_{px}^n||_{L_y^2} \\
&\le \epsilon^{\frac{1}{2}-\gamma}||1_{z\le\frac{1}{\sqrt{\epsilon}}}x^{-2+\sigma}||_{L_y^2} = \epsilon^{\frac{1}{2}-\gamma}||1_{z\le\frac{1}{\sqrt{\epsilon}}}x^{-2+\sigma_n}(1+y)^{\frac{1}{2}+\kappa}(1+y)^{-\frac{1}{2}-\kappa}||_{L_y^2} \\
&\le \epsilon^{\frac{1}{2}-\gamma}||1_{z\le\frac{1}{\sqrt{\epsilon}}}(1+y)^{\frac{1}{2}+\kappa}x^{-\frac{1}{4}-\frac{\kappa}{2}}||_{L_y^\infty}x^{-\frac{7}{4}+\sigma_n+\frac{\kappa}{2}}||(1+y)^{-\frac{1}{2}-\kappa}||_{L_y^2} \\
&\le \epsilon^{\frac{1}{2}-\gamma}||1_{z\le\frac{1}{\sqrt{\epsilon}}}(1+z)^{\frac{1}{2}+\kappa}||_{L_y^\infty}x^{-\frac{7}{4}+\sigma_n+\frac{\kappa}{2}} \le \epsilon^{\frac{1}{4}-\gamma-\kappa}x^{-\frac{7}{4}+\sigma_n+\frac{\kappa}{2}}. \quad (3.7.187)
\end{aligned}$$

The next two terms:

$$\varepsilon^{\frac{1}{2}-\gamma}||v_{sx}^{(n)}u_p^n||_{L_y^2} \le \sqrt{\varepsilon}||v_{sx}^{(n)}||_{L^\infty}||u_p^n||_{L_y^2} \lesssim \sqrt{\varepsilon}x^{-\frac{3}{2}}x^{-\frac{1}{4}}, \tag{3.7.188}$$

$$\varepsilon^{\frac{1}{2}-\gamma}||v_s^{(n)}v_{py}^n||_{L_y^2} \le \varepsilon^{\frac{1}{2}-\gamma}x^{-\frac{7}{4}+2\sigma_n}. \tag{3.7.189}$$

Now, by using the trade-off in (3.7.152):

$$\varepsilon^{\frac{1}{2}-\gamma}||v_{sy}^{(n)}v_p^n||_{L_y^2} \le \varepsilon^{\frac{1}{4}-\gamma-\kappa}||v_{sy}^{(n)}||_{L_y^\infty}||v_p^n||_{L_y^\infty}x^{\frac{1}{4}+\frac{\kappa}{2}} \lesssim \varepsilon^{\frac{1}{4}-\gamma-\kappa}x^{-\frac{7}{4}+2\sigma_n+\frac{\kappa}{2}}. \tag{3.7.190}$$

We will now move to:

$$\varepsilon^{\frac{n}{2}-\gamma}\sqrt{\varepsilon}||u_p^n v_{px}^n||_{L_y^2} \lesssim \varepsilon^{\frac{n}{2}-\gamma+\frac{1}{2}}||u_p^n||_{L_y^2}||v_{px}^n||_{L_y^\infty} \lesssim \varepsilon^{\frac{n}{2}-\gamma+\frac{1}{2}}x^{-\frac{9}{4}+2\sigma_n}, \tag{3.7.191}$$

$$\varepsilon^{\frac{n}{2}-\gamma}\sqrt{\varepsilon}||v_p^n v_{py}^n||_{L_y^2} \lesssim \varepsilon^{\frac{n}{2}-\gamma+\frac{1}{2}}x^{-\frac{9}{4}+2\sigma_n}. \tag{3.7.192}$$

where we have used (3.7.144) and (3.7.151). This completes all of the terms in (3.7.135), thereby establishing (3.7.177). The uniform estimates follow in an analogous manner.

□

For the energy estimates in Section 3.10, we will need to retain the self-similarity of u_p^n (the ability to absorb factors of z). As the profiles u_p^n, v_p^n are higher order in ϵ, we will be happy to pay factors of ϵ in order to retain this ability:

Lemma 3.7.23. *The following point-wise decay estimates holds for any $m \ge 0$,*

$$\left|z^m\partial_x^k v_p^n\right| \lesssim \epsilon^{-\frac{m}{2}}C(k,n,m)x^{-k-1+\sigma_n}, \tag{3.7.193}$$

$$\left|z^m\partial_x^k v_{py}^n\right| \lesssim \epsilon^{-\frac{m}{2}}C(k,n,m)x^{-k-\frac{3}{2}+\sigma_n}, \tag{3.7.194}$$

$$\left|y^j\partial_y^j u_p^n\right| \lesssim C(k,n,m)x^{-\frac{1}{2}+\sigma_n}, \tag{3.7.195}$$

$$\left| z^m \partial_x^k u^n_{py} \right| \lesssim \epsilon^{-\frac{m}{2}} C(k,n,m) x^{-1+\sigma_n - k}, \; \textit{for } k \geq 1. \tag{3.7.196}$$

Proof. Via the definition:

$$z^m |v^n_p| \leq z^m |\chi(\sqrt{\epsilon} z) v_p| \leq \epsilon^{-\frac{m}{2}} |v_p| \lesssim \epsilon^{-\frac{m}{2}} x^{-1+\sigma_n}. \tag{3.7.197}$$

Next, applying ∂_y yields:

$$z^m |v^n_{py}| = z^m \sqrt{\epsilon} |\chi' \frac{v_p}{\sqrt{x}}| + \chi z^m |v_{py}| \lesssim \epsilon^{-\frac{m}{2}} x^{-\frac{3}{2}+\sigma_n}. \tag{3.7.198}$$

The estimate for u^n_{py} is more complicated. The $m = 0$ case was treated in (3.7.147). Suppose $m = 1$. Then,

$$yu^n_{py} = y \int_x^\infty \partial_{yy}(\chi v_p) dx' = \int_x^\infty \frac{\epsilon}{x'} \chi'' y v_p + \int_x \frac{\sqrt{\epsilon}}{\sqrt{x'}} \chi' y v_{py} + \int_x^\infty \chi y v_{pyy} \tag{3.7.199}$$

$$= I_1 + I_2 + I_3. \tag{3.7.200}$$

We have:

$$|I_1| \leq ||\chi'' z||_{L^\infty} \int_x^\infty \epsilon \frac{|v_p|}{\sqrt{x'}} dx' \leq \sqrt{\epsilon} \int_x^\infty (x')^{-\frac{3}{2}+\sigma} dx' \lesssim \sqrt{\epsilon} x^{-\frac{1}{2}+\sigma_n}. \tag{3.7.201}$$

Next,

$$|I_2| \leq \int_x^\infty \sqrt{\epsilon} \chi' |z v_{py}| dx' \lesssim \int_x^\infty \sqrt{\epsilon} (x')^{-\frac{3}{2}} dx' \lesssim \sqrt{\epsilon} x^{-\frac{1}{2}+\sigma_n}. \tag{3.7.202}$$

Lastly,

$$\int_x^\infty |y\chi v_{pyy}|dx' \leq \int_x^\infty (x')^{\frac{1}{2}} z v_{pyy} dx' \lesssim x^{-\frac{1}{2}+\sigma_n}. \tag{3.7.203}$$

Piecing these estimates together gives:

$$y|u^n_{py}| \lesssim x^{-\frac{1}{2}+\sigma_n}, \tag{3.7.204}$$

as desired. Higher y-derivatives of u^n_p follow an identical calculation. Let us now move to x-derivatives of u^n_{py}:

$$u^n_{pyx} = \partial_{yy}(\chi v_p) = \frac{\epsilon}{x}\chi'' v_p + \frac{\sqrt{\epsilon}}{\sqrt{x}}\chi' v_{py} + \chi v_{pyy}. \tag{3.7.205}$$

From here, once can estimate:

$$\begin{aligned} z^m|u^n_{pxy}| &\lesssim \frac{\epsilon}{x}|\chi''|z^m|v_p| + \frac{\sqrt{\epsilon}}{\sqrt{x}}|\chi'|z^m|v_{py}| + \chi z^m|v_{pyy}| \\ &\leq \epsilon^{-\frac{m}{2}} x^{-2+\sigma_n}. \end{aligned} \tag{3.7.206}$$

□

We are now ready to conclude Chapter I:

Proof of Theorem 3.3.2. Consolidating Corollaries 3.4.8 - 3.4.9, Propositions 3.5.6, 3.6.12, 3.7.3, 3.7.11, Lemma 3.7.22, and Lemma 3.7.23 gives Theorem 3.3.2. □

Part II: a-Priori Estimates in $\| \cdot \|_Z$

3.8 Overview of a-priori Z-Norm Estimates

We will now consider the system satisfied by the remainders, $[u, v, P]$ as defined in (3.2.14) - (3.2.16). Expanding the Navier-Stokes equations in (3.2.11) - (3.2.13), one obtains:

$$-\Delta_\epsilon u + S_u(u,v) + P_x = f(u,v), \tag{3.8.1}$$

$$-\Delta_\epsilon v + S_v(u,v) + \frac{P_y}{\epsilon} = g(u,v), \tag{3.8.2}$$

$$u_x + v_y = 0, \tag{3.8.3}$$

which are taken together with the boundary conditions:

$$[u,v]|_{\{y=0\}} = [u,v]|_{\{x=1\}} = \lim_{y\to\infty}[u,v] = \lim_{x\to\infty}[u,v] = 0. \tag{3.8.4}$$

The terms in equations (3.8.1) - (3.8.2) are defined:

$$f(u,v) := \epsilon^{-\frac{n}{2}-\gamma} R^{u,n} + \mathcal{N}^u(u,v), \quad g(u,v) := \epsilon^{-\frac{n}{2}-\gamma} R^{v,n} + \mathcal{N}^v(u,v), \tag{3.8.5}$$

$$S_u(u,v) := u_R u_x + u_{Rx} u + v_R u_y + u_{Ry} v, \quad S_v(u,v) := u_R v_x + v_{Rx} u + v_R v_y + v_{Ry} v, \tag{3.8.6}$$

$$\mathcal{N}^u(u,v) := \epsilon^{\frac{n}{2}+\gamma} u u_x + \epsilon^{\frac{n}{2}+\gamma} v u_y, \quad \mathcal{N}^v(u,v) := \epsilon^{\frac{n}{2}+\gamma} u v_x + \epsilon^{\frac{n}{2}+\gamma} v v_y. \tag{3.8.7}$$

The forcing terms, $R^{u,v}, R^{v,n}$ are given by the expressions in (3.7.134) - (3.7.135), and have been controlled in Theorem 3.3.2. The coefficients u_R, v_R in $S_u(u,v), S_v(u,v)$ given in (3.8.6) have been defined in (3.3.1) - (3.3.2), and controlled according to the estimates in Theorem 3.3.2. Due to the eventual need to perform a fixed-point argument, we will actually consider the slightly generalized system, where f, g above are replaced by:

$$f(u,\bar{u},\bar{v}) = \epsilon^{-\frac{n}{2}-\gamma} R^{u,n} + \epsilon^{\frac{n}{2}+\gamma} \bar{u}\bar{u}_x + \epsilon^{\frac{n}{2}+\gamma} \bar{v} u_y, \tag{3.8.8}$$

$$g(\bar{u}, \bar{v}) = \epsilon^{-\frac{n}{2}-\gamma} R^{v,n} + \epsilon^{\frac{n}{2}+\gamma} \bar{u}\bar{v}_x + \epsilon^{\frac{n}{2}+\gamma} \bar{v}\bar{v}_y. \tag{3.8.9}$$

When $u = \bar{u}, v = \bar{v}$ (which corresponds to a fixed point of an appropriately defined map), one obtains the actual system of interest, with f as in (3.8.5). For technical purposes, in this chapter we will consider the system (3.8.1) - (3.8.3) on the domain:

$$\Omega^N := \{(x, y) : x > 0, 0 < y < N\}. \tag{3.8.10}$$

All estimates will be made independent of N, allowing us to eventually send $N \to \infty$. When working on this domain, we will use the boundary conditions:

$$[u, v]|_{\{y=0\}} = [u, v]|_{\{x=1\}} = [u, v]|_{\{y=N\}} = \lim_{x\to\infty} [u, v] = 0. \tag{3.8.11}$$

The norms which we will work in are defined in the next section, Section 3.9, starting with (3.9.3) - (3.9.8). We invite the reader to read these definitions at this point. The main result of this chapter is then:

Theorem 3.8.1 (Complete Z Estimate)**.** *Fix any $N > 0$. Let δ, ϵ be sufficiently small based on universal constants, with $\epsilon << \delta$, and $n \in \mathbb{N}$ sufficiently large relative to universal constants. Let N_i be parameters in the definition of Z, given in (3.9.8). Let $\gamma \in (0, \frac{1}{4})$, and parameter $\kappa > 0$ arbitrarily small. Suppose $||\bar{u}, \bar{v}||_{Z(\Omega^N)} \le 1$. Then $[u, v] \in Z(\Omega^N)$, solutions to the system (3.8.1) - (3.8.3), (3.8.5) - (3.8.9), with boundary conditions (3.8.11), obey the following a-priori estimate:*

$$||u, v||^2_{Z(\Omega^N)} \lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa} + \epsilon^{\frac{n}{2}-\omega(N_i)} \Big(||\bar{u}, \bar{v}||^4_{Z(\Omega^N)} \Big), \tag{3.8.12}$$

for some fixed function $\omega(N_i)$, where the constants above are independent of N.

Let us give a brief overview of the steps to prove Theorem 3.8.1.

(Step 1) The Space Z (Section 3.9): We introduce the various components of the crucial norm Z. The energy norms, X_1, X_2, X_3 are to be controlled using energy and positivity estimates (in the next steps). The elliptic norms, Y_2, Y_3, provide additional controls near the boundary, $x = 1$, and are related to the X_i norms in subsection 3.9.1. Most importantly, there are uniform-type norms, defined in (3.9.16), of which the $||vx^{\frac{1}{2}}||_{L^\infty}$ is the most crucial ingredient. These are controlled in Lemmas 3.9.15, 3.9.17. The relation between all of the norms is summarized in Theorem 3.9.20. This analysis of Z is the main novelty of Chapter II.

(Step 2) Energy Estimates: The lowest-order energy estimates are performed in Proposition 3.10.1, for which the sharp profile estimates from (3.3.8) - (3.3.20) are essential. An examination of Proposition 3.10.1 shows that the energy estimate loses a factor of $x^{\frac{1}{2}}$ which must be recovered in the next step.

(Step 3) Positivity Estimates: The lowest-order positivity estimates are performed in Proposition 3.10.2. Here again, the estimates from (3.3.8) - (3.3.18) are used in an essential way. In particular, the estimate (3.10.66) forces the requirement shown in (3.2.48).

(Step 4) Higher-Order Energy/ Positivity Estimates: In Propositions 3.10.4, 3.10.6, 3.10.7, 3.10.8, we control the higher-order energy norms X_2, X_3 from definitions (3.9.4) - (3.9.5). This is achieved by applying appropriate iterations of ∂_x to the system (3.8.1) - (3.8.3).

(Step 5) Nonlinear Estimates: The estimates of the nonlinearities present in (f, g) from (3.8.8) - (3.8.9) is performed in Section 3.11, in particular in Lemma 3.11.1. Here, there are several delicate matters. First, the sharp decay of $||x^{\frac{1}{2}}v||_{L^\infty}$, which is present in (3.9.8) due to Lemma 3.9.17, is essential in order to control the term $\bar{v}\bar{u}_y$. This estimate is (3.11.8). Second, estimate (3.11.5) capitalizes on a cancellation structure which enables us to control a term that would be otherwise out of reach. Third, the top-order nonlinear terms in estimates (3.11.20) and (3.11.24) are controlled by using mixed-norms.

3.9 The Function Space Z

In this section, we define and analyze the high-order, weighted norm, Z, on which we will close our nonlinear analysis. The functional framework of this section is required in order to follow the calculations in the upcoming sections. First, we need to define the energy norms in which we obtain energy estimates, and also several auxiliary norms that will supplement the energy norms. To define these, first define the cut-off functions:

$$\zeta_3(x) = \begin{cases} 0 \text{ for } 1 \le x \le \frac{3}{2}, \\ 1 \text{ for } x \ge 2. \end{cases} \tag{3.9.1}$$

$$\rho_k(x) = \begin{cases} 0 \text{ for } 1 \le x \le 50 + 50(k-2), \\ 1 \text{ for } x \ge 60 + 50(k-2). \end{cases} \tag{3.9.2}$$

The energy norms are defined as follows:

$$||u,v||_{X_1}^2 := ||u_y||_{L^2}^2 + ||\{\sqrt{\epsilon} v_x, v_y\} x^{\frac{1}{2}}||_{L^2}^2 \tag{3.9.3}$$

$$||u,v||_{X_2}^2 := ||u_{xy} \cdot \rho_2 x||_{L^2}^2 + ||\{\sqrt{\epsilon} v_{xx}, v_{xy}\} \cdot (\rho_2 x)^{\frac{3}{2}}||_{L^2}^2, \tag{3.9.4}$$

$$||u,v||_{X_3}^2 := ||u_{xxy} \cdot (\rho_3 x)^2||_{L^2}^2 + ||\{\sqrt{\epsilon} v_{xxx}, v_{xxy}\} \cdot (\rho_3 x)^{\frac{5}{2}}||_{L^2}^2. \tag{3.9.5}$$

Definition 3.9.1. The norms Y_2, Y_3 are strengthenings of X_2, X_3 near the boundary, $x = 1$, and defined through:

$$||u,v||_{Y_2}^2 := ||u_{xy} x||_{L^2}^2 + ||\{\sqrt{\epsilon} v_{xx}, v_{xy}\} x^{\frac{3}{2}}||_{L^2}^2 + ||u_{yy}||_{L^2(x \le 2000)}, \tag{3.9.6}$$

$$||u,v||_{Y_3}^2 := ||u_{xxy} \cdot \zeta_3 x^2||_{L^2}^2 + ||\{\sqrt{\epsilon} v_{xxx}, v_{xxy}\} \cdot \zeta_3 x^{\frac{5}{2}}||_{L^2}^2. \tag{3.9.7}$$

Definition 3.9.2. The norm Z is defined through:

$$\begin{aligned}||u,v||_Z :=& ||u,v||_{X_1\cap X_2\cap X_3} + \epsilon^{N_2}||u,v||_{Y_2} + \epsilon^{N_3}||u,v||_{Y_3} + \epsilon^{N_4}||ux^{\frac{1}{4}}, \sqrt{\varepsilon}vx^{\frac{1}{2}}||_{L^\infty} \\ &+ \epsilon^{N_5}\sup_{x\ge 20}||\sqrt{\varepsilon}v_x x^{\frac{3}{2}}, u_x x^{\frac{5}{4}}||_{L^\infty} + \varepsilon^{N_6}\sup_{x\ge 20}||u_y x^{\frac{1}{2}}||_{L^2_y} \\ &+ \epsilon^{N_7}\Big[\int_{20}^\infty x^4||\sqrt{\varepsilon}v_{xx}||^2_{L^\infty_y}\,dx\Big]^{\frac{1}{2}}. \end{aligned} \tag{3.9.8}$$

Here, N_i, are some large numbers which will be specified in (3.9.103) - (3.9.105). They depend only on universal constants. It will be understood that N_i are much larger than any of the quantities M_i appearing in the forthcoming lemmas, and that n is much larger than any of the N_i, M_i.

Now that the norms we will be working in have been specified, we will define corresponding spaces. First, some basic notations:

Definition 3.9.3. For any open set $U \subset \Omega$, the space $C^\infty_0(U)$ denotes the space of smooth functions with compact support in U. We will also use $C^\infty_0(U)$ to denote the space of smooth vector fields with compact support in U, which will be clear from context. The space $C^\infty_{0,D}(U)$ denotes the space of smooth, divergence-free vector fields with compact support in U. When the set U is clear from context, we shall suppress it.

Definition 3.9.4. Given any open set $U \subset \Omega$, the space $Z(U)$ is defined to be space of those divergence-free vector fields which lie in the closure of $C^\infty_{0,D}(U)$ under the norm X_1, such that $||u,v||_Z < \infty$.

Definition 3.9.5. Given any open set $U \subset \Omega$, the space $(X_1 \cap X_2 \cap X_3)(U)$ is defined to be space of those divergence-free vector fields which lie in the closure of $C^\infty_{0,D}(U)$ under the norm X_1, such that $||u,v||_{X_1\cap X_2\cap X_3} < \infty$.

Due to the weights in the norm Z and the energy norms $X_1\cap X_2\cap X_3$, there is no "$H = W$ Theorem" generically available, which would equate the density of smooth functions with compact support with the class of functions satisfying $||u,v||_Z < \infty$. This is the reason must specify that:

$$Z(U) \subset \overline{C_{0,D}^{\infty}}^{||\cdot||_{X_1}}(U). \tag{3.9.9}$$

We will first record the boundary behavior of $Z(\Omega)$-vector fields:

Lemma 3.9.6. *For $[u,v] \in Z(\Omega)$, the following boundary conditions are satisfied:*

$$[u,v]|_{x=1} = [u,v]|_{y=0} = \lim_{x\to\infty}[u,v] = \lim_{y\to\infty}[u,v] = 0. \tag{3.9.10}$$

Proof. The boundary conditions at $x = 1$ and $y = 0$ are satisfied by the density specification (3.9.9). The boundary conditions as $x \to \infty$ is satisfied by the definition of norm Z, (3.9.8) because the decay is encoded (with rates) in the norm. The boundary conditions as $y \to \infty$ is enforced because $[u,v] \in Z(\Omega)$ implies that $[u,v] \in H^1_{(x\leq A)}$, and candidacy in such a Sobolev space automatically encodes decay as $y \to \infty$. $\square$

The above lemma shows that $Z(\Omega)$ is a suitable space to obtain solutions to our boundary-value problem, after a consultation with (3.8.4). That $X_1 \cap X_2 \cap X_3$ also encodes the boundary conditions from (3.9.10) is less obvious, and is proven in Lemma 3.9.16.

Lemma 3.9.7. *For any open set $U \subset \Omega$, $Z(U)$ and $(X_1 \cap X_2 \cap X_3)(U)$ are Banach spaces.*

Proof. This follows from standard arguments. $\square$

For the energy norm, $X_1 \cap X_2 \cap X_3$, we have Hilbertian structure. Given two divergence-free vector-fields:

$$\mathbf{u} := (u,v), \quad \mathbf{a} := (a,b), \tag{3.9.11}$$

define the following inner-products:

$$\langle \mathbf{u}, \mathbf{a} \rangle_{X_1} := \int\int u_y a_y + \int\int u_x a_x x + \int\int \varepsilon v_x b_x x, \tag{3.9.12}$$

$$\langle \mathbf{u}, \mathbf{a} \rangle_{X_2} := \int\int u_{xy} a_{xy} (\rho_2 x)^2 + \int\int u_{xx} a_{xx} (\rho_2 x)^3 + \int\int \varepsilon v_{xx} b_{xx} (\rho_2 x)^3, \tag{3.9.13}$$

$$\langle \mathbf{u}, \mathbf{a} \rangle_{X_3} := \int\int u_{xxy} a_{xxy} (\rho_3 x)^4 + \int\int u_{xxx} a_{xxx} (\rho_3 x)^5 + \int\int \varepsilon v_{xxx} b_{xxx} (\rho_3 x)^5, \tag{3.9.14}$$

$$\langle \mathbf{u}, \mathbf{a} \rangle_{X_1 \cap X_2 \cap X_3} := \langle \mathbf{u}, \mathbf{a} \rangle_{X_1} + \langle \mathbf{u}, \mathbf{a} \rangle_{X_2} + \langle \mathbf{u}, \mathbf{a} \rangle_{X_3}. \tag{3.9.15}$$

Temporarily accepting Lemma 3.9.16, we have:

Lemma 3.9.8. *With the inner product (3.9.15), $(X_1 \cap X_2 \cap X_3)(\Omega)$ is a Hilbert space.*

Proof. The inner-product (3.9.15) is non-degenerate due to the boundary conditions $[u, v]|_{y=0}$. The remaining inner-product axioms are straightforward to check, which coupled with the completeness in Lemma 3.9.7 gives the result.

□

Notationally, it sometimes is convenient to refer at once to all of the "uniform-type" quantities in the norm Z, so we designate the following notation:

Definition 3.9.9. The norm $\mathcal{U}$ is defined by:

$$\begin{aligned} ||u, v||_{\mathcal{U}} := \epsilon^{N_4} ||u x^{\frac{1}{4}}, \sqrt{\varepsilon} v x^{\frac{1}{2}}||_{L^\infty} + \epsilon^{N_5} \sup_{x \geq 20} ||\sqrt{\varepsilon} v_x x^{\frac{3}{2}}, u_x x^{\frac{5}{4}}||_{L^\infty} \\ + \varepsilon^{N_6} \sup_{x \geq 20} ||u_y x^{\frac{1}{2}}||_{L^2_y} + \epsilon^{N_7} \Big[\int_{20}^\infty x^4 ||\sqrt{\varepsilon} v_{xx}||^2_{L^\infty_y} dx \Big]^{\frac{1}{2}}. \end{aligned} \tag{3.9.16}$$

3.9.1 Elliptic Estimates and the Spaces Y_i

We will first utilize elliptic theory in order to obtain basic H^k estimates for our solution, which hold generically for Stokes-type equations. These elliptic estimates are meant to supplement the energy estimates that we will perform in Section 3.10 which are meant to control the energy norms, $X_1 \cap X_2 \cap X_3$. In particular, the estimates from this section cannot *replace* the energy estimates for two reasons:

(1) They scale poorly in ϵ, and

(2) One cannot extract sharp enough global-in-x information using this procedure.

Their purpose is to provide more controls near the boundary $x = 1$, which is reflected in the norms Y_2, Y_3. To see this one should compare the support of the cut-off functions in our energy norms, ρ_k, with the support of ζ_3 in Y_3. For the set of calculations in this subsection, there are many different cutoff functions which arise in addition to the important cutoffs, ρ_k, ζ_k which have already been defined.

Remark (Notational Convention). To simplify notations, given a cut-off function χ, we introduce the notation $o(\chi)$ to mean either χ or any of its derivatives, or any positive powers of χ or any of its derivatives. Roughly speaking, any "variant" of χ is denoted by $o(\chi)$, the essential feature being $o(\chi)$ is supported in the same (or similar) region.

The first step will be to provide some controls of $||u, v||_{\dot{H}^2}$.

Lemma 3.9.10 (H^2 Regularity)**.** *Let $[u, v], [\bar{u}, \bar{v}] \in X_1$ be solutions to (3.8.1) - (3.8.3), with f, g as in (3.8.8) - (3.8.9). For some M_2, perhaps large, dependent only on universal constants:*

$$\sup_{x \le 2000} ||u, v||_{L^\infty_y} + ||u, v||_{\dot{H}^2(x \le 2000)} \lesssim \epsilon^{-M_2} \Big[C + \epsilon^{\frac{n}{2}+\gamma} ||\bar{u}, \bar{v}||_{L^\infty} \Big(||\bar{u}, \bar{v}||_{X_1} + ||u, v||_{X_1} \Big) + ||u, v||_{X_1} \Big]. \quad (3.9.17)$$

Proof. Notationally, we will not take care to rename generic constants M_2 within this proof. We rescale the system back to Eulerian coordinates via:

$$\tilde{u}(x,Y) := u(x,y), \quad \tilde{v}(x,Y) := \sqrt{\epsilon}v(x,y), \quad \tilde{P}(x,Y) = \frac{1}{\epsilon}P(x,y), \tag{3.9.18}$$

which, by rescaling (3.8.1) - (3.8.3), yields the Stokes-sytem, for some M_2 perhaps large:

$$\begin{aligned} & -\Delta\tilde{u} + \tilde{P}_x = \epsilon^{-M_2}\Big[\tilde{f} + \tilde{S}_u\Big], \quad -\Delta\tilde{v} + \tilde{P}_Y = \epsilon^{-M_2}\Big[\tilde{g} + \tilde{S}_v\Big], \\ & \tilde{u}_x + \tilde{v}_Y = 0. \end{aligned} \tag{3.9.19}$$

Above, $\tilde{f}, \tilde{g}, \tilde{S}_u, \tilde{S}_v$ have been also rescaled to Eulerian coordinates. By considering the stream function $\psi = \int_0^Y \tilde{u}$, we obtain the following biharmonic problem:

$$\Delta^2\psi = F := \epsilon^{-M_2}\{\partial_Y\Big[\tilde{f} + \tilde{S}_u\Big] - \partial_x\Big[\tilde{g} + \tilde{S}_v\Big]\}, \tag{3.9.20}$$

$$\psi(x,0) = \psi_Y(x,0) = 0, \quad \psi(1,Y) = \psi_x(1,Y) = 0. \tag{3.9.21}$$

We now introduce the partition of unity, $\{\chi_m\}_{m\geq 1}$ of the region $[1,2000] \times [0,\infty)$. The specifications of these cut-off functions are as follows. First, define:

$$\chi^{(a)}(x) = \begin{cases} 1 \text{ for } 1 \le x \le 2000, \\ 0 \text{ for } x \ge 4000, \end{cases} \qquad \chi^{(b)}_m(Y) = \begin{cases} 1 \text{ for } m-1 \le Y \le m+1, \\ 0 \text{ for } Y \ge m+2, \\ 0 \text{ for } Y \le m-2. \end{cases} \tag{3.9.22}$$

for all $m \geq 1$. Then define:

$$\chi_m(x,Y) = \chi^{(a)}(x)\chi^{(b)}_m(Y). \tag{3.9.23}$$

The purpose of selecting such a partition is to localize near $x = 1$, in such a way that the region between 1 and the support of ρ_2 is captured (see the definition in (3.9.2)). Denote by $\bar{\psi}_m = \chi_m \psi$ Then:

$$\Delta^2 \bar{\psi}_m = \chi_m F + [\Delta^2, \chi_m]\psi, \quad \bar{\psi}_m(x, 0) = \partial_Y \bar{\psi}_m(x, 0) = \bar{\psi}_m(1, Y) = \partial_x \bar{\psi}_m(1, Y) = 0. \tag{3.9.24}$$

Here the commutator is defined as:

$$\begin{aligned}
[\Delta^2, \chi]\psi :=& \chi_{YYYY}\psi + 6\chi_{YY}\psi_{YY} + 4\chi_{YYY}\psi_Y + 4\chi_Y\psi_{YYY} + 6\chi_{xx}\psi_{xx} + 4\chi_{xxx}\psi_x \\
&+ 4\chi_x\psi_{xxx} + \chi_{xxxx}\psi + \chi_{xxYY}\psi + 2\chi_{YY}\psi_{xx} + 2\chi_{YYx}\psi_x + 2\chi_x\psi_{xYY} \\
&+ 2\chi_{xxY}\psi_y + 2\chi_y\psi_{xxY} + 4\chi_{xY}\psi_{xY}.
\end{aligned} \tag{3.9.25}$$

According to (BR80), Theorems 1 and 2, with $k = 1$, coupled with Figure 2, P. 562 in (BR80) with "C/C" boundary conditions, we have the following H^3 estimate for ψ:

$$||\bar{\psi}_m||_{H^3} \lesssim ||\chi_m F||_{H^{-1}} + ||[\Delta^2, \chi_m]\psi||_{H^{-1}}. \tag{3.9.26}$$

We will first address the commutator terms in (3.9.26). First, consider the terms in (3.9.25) which have three derivatives on ψ, which are denoted by $o(\partial^3\psi)$. Using the definition of H^{-1}, for any compactly supported test function $\alpha(x, Y) \in H^1_0$,

$$\begin{aligned}
\left|\langle o(\chi_m)o(\partial^3\psi), \alpha(x, y)\rangle_{H^{-1}, H^1_0}\right| &= \left|\langle o(\partial\chi_m)o(\partial^2\psi), \alpha\rangle\right| + \left|\langle o(\chi_m)o(\partial^2\psi), \partial\alpha\rangle\right| \\
&\leq ||o(\chi_m)o(\partial^2\psi)||_{L^2}||\alpha||_{H^1_0}.
\end{aligned} \tag{3.9.27}$$

Then taking the sup over all $||\alpha||_{H^1_0} = 1$, we obtain:

$$||o(\chi_m)o(\partial^3\psi)||_{H^{-1}} \lesssim ||o(\chi_m)o(\partial^2\psi)||_{L^2} \lesssim ||o(\chi_m)o(\partial)[\tilde{u},\tilde{v}]||_{L^2} \lesssim \epsilon^{-M_2}||o(\chi_m)]\{u,v\}||_{X_1}. \tag{3.9.28}$$

Next, turn to the terms in (3.9.25) which has two derivatives on ψ:

$$||o(\chi_m)o(\partial^2\psi)||_{H^{-1}} \le ||o(\chi_m)o(\partial^2\psi)||_{L^2} \le \varepsilon^{-M_2}||o(\chi_m)]\{u,v\}||_{X_1}. \tag{3.9.29}$$

Next, we must address those terms in (3.9.25) which has one derivatives on ψ. For this, we use the Poincare inequality in x direction:

$$\begin{aligned} ||o(\chi_m)\partial\psi||_{H^{-1}} \le ||o(\chi_m)\partial\psi||_{L^2} \le ||o(\chi_m)\{u,v\}||_{L^2} \le ||u_x, v_x||_{L^2(x\le 2000)} \\ \le ||o(\chi_m)\{u_x,v_x\}||_{L^2} \le \varepsilon^{-M_2}||o(\chi_m)\{u,v\}||_{X_1}. \end{aligned} \tag{3.9.30}$$

For the zeroeth order terms in ψ, we must argue as follows:

$$\begin{aligned} ||o(\chi_m)\psi||_{H^{-1}} \lesssim ||o(\chi_m)\psi||_{L^2} = ||o(\chi_m)\int_0^x v||_{L^2} \lesssim ||o(\chi_m)\frac{1}{x}\int_0^x v||_{L^2} \\ \lesssim \varepsilon^{-1}||o(\chi_m)\{u,v\}||_{X_1}. \end{aligned} \tag{3.9.31}$$

Summarizing, then,

$$||[\Delta^2,\chi_m]\psi||_{H^{-1}} \lesssim \epsilon^{-M_2}||o(\chi_m)\{u,v\}||_{X_1}, \tag{3.9.32}$$

It remains, then, to analyze the majorizing terms in F in estimate (3.9.26). Up to redefining M_2 in (3.9.20), we may scale back to Prandtl coordinates (x,y). By integrating by parts against compactly supported test functions, we have:

$$||\chi_m \cdot [\partial_y(f + S_u) - \partial_x(g + S_v)]||_{H^{-1}} \lesssim ||o(\chi_m)\Big[f + S_u + g + S_v\Big]||_{L^2}. \tag{3.9.33}$$

First, we'll start with:

$$\begin{aligned} ||o(\chi_m) \cdot S_u||_{L^2} &= ||o(\chi_m)\Big[u_R u_x + u_{Rx} u + v_R u_y + u_{Ry} v\Big]||_{L^2} \\ &\le ||u_R, x u_{Rx}, v_R, u^P_{Ry} y, u^E_{RY} x^{\frac{3}{2}}||_{L^\infty} ||o(\chi_m)\{u_x, u_y, v_y, v_x\}||_{L^2} \\ &\lesssim ||o(\chi_m)\{u, v\}||_{X_1}. \end{aligned} \tag{3.9.34}$$

Similarly,

$$\begin{aligned} ||o(\chi_m) \cdot S_v||_{L^2} &= ||o(\chi_m)\Big[u_R v_x + v_{Rx} u + v_R v_y + v_{Ry} v\Big]||_{L^2} \\ &\le ||u_R, x v_{Rx}, v_R, v_{Ry} y||_{L^\infty} ||o(\chi_m)\{v_x, u_x, v_y\}||_{L^2} \lesssim ||o(\chi_m)\{u, v\}||_{X_1}. \end{aligned} \tag{3.9.35}$$

Next, we come to the nonlinear terms:

$$\begin{aligned} &\epsilon^{\frac{n}{2}+\gamma} ||o(\chi_m) \cdot \Big[\bar{u}\bar{u}_x + \bar{v} u_y + \bar{u}\bar{v}_x + \bar{v}\bar{v}_y\Big]||_{L^2} \\ &\quad \le \epsilon^{\frac{n}{2}+\gamma} ||o(\chi_m)\{\bar{u}, \bar{v}\}||_{L^\infty} ||o(\chi_m)\{\bar{u}_x, \bar{v}_x, u_y\}||_{L^2} \\ &\quad \lesssim \epsilon^{\frac{n}{2}+\gamma} ||o(\chi_m)\{\bar{u}, \bar{v}\}||_{L^\infty} \Big(||o(\chi_m)\{\bar{u}, \bar{v}\}||_{X_1} + ||o(\chi_m)\{u, v\}||_{X_1} \Big). \end{aligned} \tag{3.9.36}$$

Finally, we have the forcing terms in f, g for which we cite Lemma 3.7.22,

$$\sum_{m=1}^{\infty} ||o(\chi_m)\{R^{u,n}, R^{v,n}\}||_{L^2} \le C. \tag{3.9.37}$$

Upon taking summation in m:

$$\begin{aligned}||\psi||_{H^3(x\le 2000)} &\le ||\sum_m \chi_m\psi||_{H^3} \le \sum_m ||\chi_m\psi||_{H^3} \le \sum_m ||\chi_m F||_{H^{-1}} + \sum_m ||[\Delta^2,\chi_m]\psi||_{H^{-1}} \\ &\lesssim \varepsilon^{-M_2}\Big[\sum_m ||o(\chi_m)\{R^{u,n},R^{v,n}\}||_{L^2} + \sum_m ||o(\chi_m)\{u,v\}||_{X_1} \\ &\qquad + \varepsilon^{\frac{n}{2}+\gamma}||o(\chi_m)\{||\bar{u},\bar{v}\}||_{L^\infty}\Big(||o(\chi_m)\{\bar{u},\bar{v}\}||_{X_1} + ||o(\chi_m)\{u,v\}||_{X_1}\Big)\Big] \\ &\lesssim \varepsilon^{-M_2}\Big[C + ||\{u,v\}||_{X_1} + \varepsilon^{\frac{n}{2}+\gamma}||\bar{u},\bar{v}||_{L^\infty}\Big(||\{\bar{u},\bar{v}\}||_{X_1} + ||\{u,v\}||_{X_1}\Big)\Big].\end{aligned}$$

For the L^∞ component of our desired claim, we simply use the standard H^2 embedding.

$\square$

Corollary 3.9.11. *Suppose $||\bar{u},\bar{v}||_Z \le 1$. There exists a universal constant M_2 such that for any selection of N_2, N_4, n, γ, the following estimate holds:*

$$\begin{aligned}\epsilon^{N_2}||u,v||_{Y_2} &\lesssim \epsilon^{N_2-M_2} + \epsilon^{\frac{n}{2}+\gamma+N_2-M_2-N_4}||\bar{u},\bar{v}||_Z^2 \\ &\quad + \Big(\epsilon^{N_2-M_2} + \varepsilon^{\frac{n}{2}+\gamma+N_2-M_2-N_4}\Big)||u,v||_{X_1} + \epsilon^{N_2}||u,v||_{X_2}. \end{aligned} \tag{3.9.38}$$

We will now bootstrap the above elliptic regularity, away from the boundary $x = 1$ (thereby avoiding the corners of our domain).

Lemma 3.9.12 (Third-Order Elliptic Regularity)**.** *For some $M_3 \ge 0$ perhaps large, but independent of n, and supposing $||\bar{u},\bar{v}||_Z \le 1$, we have:*

$$\begin{aligned}||\{u_x,\sqrt{\epsilon}v_x\}\cdot\zeta_3||_{\dot{H}^2(x\le 1000)} &\lesssim \epsilon^{-M_3}\Big[C + ||u,v||_{X_1\cap Y_2} + \varepsilon^{\frac{n}{2}+\gamma-N_4-N_2}||u,v||_{X_1\cap Y_2}\Big] \\ &\quad + \epsilon^{-M_3-2N_2-2N_4+\frac{n}{2}+\gamma}||\bar{u},\bar{v}||_Z^2. \end{aligned} \tag{3.9.39}$$

Proof. Start with the system in (3.9.20), and differentiate once in x:

$$\Delta^2\psi_x = F_x := \varepsilon^{-M_2}\{\partial_Y\Big[\tilde{f}_x + \partial_x\tilde{S}_u\Big] - \partial_x\Big[\tilde{g}_x + \partial_x\tilde{S}_v\Big]\},$$
$$\partial_x\psi(x,0) = \partial_x\psi_Y(x,0) = 0. \tag{3.9.40}$$

We shall now define a new cut-off function, via:

$$\chi^{(2,a)}(x) = \begin{cases} 0 \text{ for } 1 \le x \le \frac{3}{2}, \\ 1 \text{ for } 2 \le x \le 1000, \\ 0 \text{ for } 2000 \le x. \end{cases} \tag{3.9.41}$$

Then, referring back to (3.9.22),

$$\chi_m^{(2)}(x,Y) := \chi^{(2,a)}(x)\chi_m^{(b)}(Y) \tag{3.9.42}$$

The collection $\{\chi_m^{(2)}\}$ is meant to fill the gap between ζ_3 and ρ_3 (see the definitions in (3.9.1), (3.9.2)). We will consider the unknown $\chi_m^{(2)}\psi_x$, which satisfies the system:

$$\Delta^2\Big(\chi_m^{(2)}\psi_x\Big) = \chi_m^{(2)}F_x + [\Delta^2, \chi_m^{(2)}]\psi_x, \tag{3.9.43}$$

where the commutator expression is given by (3.9.25). Then via the standard, local in x, H^3 estimate, one has:

$$||\chi_m^{(2)}\psi_x||_{H^3} \lesssim ||\chi_m^{(2)}F_x||_{H^{-1}} + ||[\Delta^2, \chi_m^{(2)}]\psi_x||_{H^{-1}}. \tag{3.9.44}$$

Again, as $[\Delta^2, \chi_m^{(2)}]$ is localized in x and contains three-derivatives of ψ_x, we easily obtain:

$$||[\Delta^2, \chi_m^{(2)}]\psi_x||_{H^{-1}} \lesssim \varepsilon^{-M_3}||o(\chi_m^{(2)})\{u,v\}||_{Y_2\cap X_1}. \tag{3.9.45}$$

We will now evaluate the F_x term above. By selecting the exponent M_3 large enough, one can rescale $\tilde{f}_x, \tilde{g}_x$ to f_x, g_x and $\partial_x\tilde{S}_u, \partial_x\tilde{S}_v$ to S_u, S_v, which we automatically do. Then,

$$\begin{aligned} ||\chi_m^{(2)}F_x||_{H^{-1}} &\leq \varepsilon^{-M_3}\Big[||\chi_m^{(2)}f_{xy}||_{H^{-1}} + ||\chi_m^{(2)}g_{xx}||_{H^{-1}}\Big] \\ &\leq \varepsilon^{-M_3}\Big[||o(\chi_m^{(2)})f_x||_{L^2} + ||o(\chi_m^{(2)})g_x||_{L^2}\Big]. \end{aligned} \tag{3.9.46}$$

Then, according to the definition (3.8.8) - (3.8.9), one has for the nonlinear terms:

$$\begin{aligned} ||o(\chi_m^{(2)})\cdot\mathcal{N}_x^u||_{L^2} &= \epsilon^{\frac{n}{2}+\gamma}||o(\chi_m^{(2)})[\bar{u}\bar{u}_{xx} + \bar{u}_x^2 + \bar{v}_x u_y + \bar{v}u_{xy}]||_{L^2} \\ &\leq \epsilon^{\frac{n}{2}+\gamma}\Big[||o(\chi_m^{(2)})\bar{u}||_{L^\infty}||o(\chi_m^{(2)})\bar{u}_{xx}||_{L^2} + ||o(\chi_m^{(2)})\bar{u}_x||_{L^4}^2 \\ &\qquad + ||o(\chi_m^{(2)})\bar{v}_x||_{L^4}||o(\chi_m^{(2)})u_y||_{L^4} + ||o(\chi_m^{(2)})\bar{v}||_{L^\infty}||o(\chi_m^{(2)})u_{xy}||_{L^2}\Big] \\ &\leq \epsilon^{\frac{n}{2}+\gamma}\Big[||o(\chi_m^{(2)})\bar{u}||_{L^\infty}||o(\chi_m^{(2)})\bar{u}_{xx}||_{L^2} + ||o(\chi_m^{(2)})\bar{u}_x||_{L^2}||o(\chi_m^{(2)})\bar{u}_x||_{\dot{H}^1} \\ &\qquad + ||o(\chi_m^{(2)})\bar{v}_x||_{L^2}^{\frac{1}{2}}||o(\chi_m^{(2)})\bar{v}_x||_{\dot{H}^1}^{\frac{1}{2}}||o(\chi_m^{(2)})u_y||_{L^2}^{\frac{1}{2}}||o(\chi_m^{(2)})u_y||_{\dot{H}^1}^{\frac{1}{2}} \\ &\qquad + ||o(\chi_m^{(2)})\bar{v}||_{L^\infty}||o(\chi_m^{(2)})u_{xy}||_{L^2}\Big] \\ &\lesssim \epsilon^{\frac{n}{2}+\gamma}\Big[\varepsilon^{-2N_2-2N_4}||o(\chi_m^{(2)})\{\bar{u},\bar{v}\}||_Z^2 + \varepsilon^{-2N_2-2N_4}||o(\chi_m^{(2)})\{u,v\}||_{X_1\cap Y_2}\Big]. \end{aligned} \tag{3.9.47}$$

Above, we have used the assumption that $||\bar{u},\bar{v}||_Z \leq 1$ in the term:

$$\begin{aligned} &||o(\chi_m^{(2)})\bar{v}_x||_{L^2}^{\frac{1}{2}}||o(\chi_m^{(2)})\bar{v}_x||_{\dot{H}^1}^{\frac{1}{2}}||o(\chi_m^{(2)})u_y||_{L^2}^{\frac{1}{2}}||o(\chi_m^{(2)})u_y||_{\dot{H}^1}^{\frac{1}{2}} \\ &\qquad \leq \varepsilon^{-N_2}||\bar{v}||_Z||o(\chi_m^{(2)})u_y||_{L^2}^{\frac{1}{2}}||o(\chi_m^{(2)})u_y||_{\dot{H}^1}^{\frac{1}{2}} \\ &\qquad \leq \varepsilon^{-N_2}||o(\chi_m^{(2)})u_y||_{L^2}^{\frac{1}{2}}||o(\chi_m^{(2)})u_y||_{\dot{H}^1}^{\frac{1}{2}} \end{aligned}$$

$$\leq \varepsilon^{-N_2}||o(\chi_m^{(2)})u_y||_{X_1}^{\frac{1}{2}}\varepsilon^{-N_2}||o(\chi_m^{(2)})u||_{Y_2}^{\frac{1}{2}}. \tag{3.9.48}$$

In the final estimate, we have used that $\chi^{(2)}$ is supported on a strictly smaller region in x than the region over which we have controlled u_{yy} (compare (3.9.23) to the u_{yy} estimate in (3.9.17)). Similarly,

$$\begin{aligned}
||o(\chi_m^{(2)})\mathcal{N}_x^v||_{L^2} &= \epsilon^{\frac{n}{2}+\gamma}||o(\chi_m^{(2)})[\bar{u}\bar{v}_{xx} + \bar{u}_x\bar{v}_x + \bar{v}_x\bar{v}_y + \bar{v}\bar{v}_{xy}]||_{L^2} \\
&\leq \epsilon^{\frac{n}{2}+\gamma}\Big[||o(\chi_m^{(2)})\bar{u}||_{L^\infty}||o(\chi_m^{(2)})\bar{v}_{xx}||_{L^2} + ||o(\chi_m^{(2)})\bar{u}_x||_{L^4}||o(\chi_m^{(2)})\bar{v}_x||_{L^4} \\
&+ ||o(\chi_m^{(2)})\bar{v}_x||_{L^4}||o(\chi_m^{(2)})\bar{v}_y||_{L^4} + ||o(\chi_m^{(2)})\bar{v}||_{L^\infty}||o(\chi_m^{(2)})\bar{v}_{xy}||_{L^2}\Big] \\
&\leq \epsilon^{\frac{n}{2}+\gamma}\Big[\varepsilon^{-2N_2-2N_4}||o(\chi_m^{(2)})\{\bar{u},\bar{v}\}||_Z^2\Big].
\end{aligned} \tag{3.9.49}$$

Next, according to estimates (3.7.177), one has:

$$\sum_{m\geq 1}||o(\chi_m^{(2)})R_x^{u,n}, o(\chi_m^{(2)})R_x^{v,n}||_{L^2} \leq C. \tag{3.9.50}$$

Taking summation in m and scaling back to Prandtl coordinates gives the desired result upon comparing the supports of χ_2 and ρ_2:

$$\begin{aligned}
||\chi^{(2)}\psi||_{H^3} &\leq \sum_m ||\chi_m^{(2)}\psi||_{H^3} \\
&\lesssim \varepsilon^{-M_3}\sum_m||o(\chi_m^{(2)})u,v||_{Y_2\cap X_1} + \epsilon^{\frac{n}{2}+\gamma}\sum_m\varepsilon^{-2N_2-2N_4}||o(\chi_m^{(2)})\{\bar{u},\bar{v}\}||_Z^2 \\
&+ \epsilon^{\frac{n}{2}+\gamma}\varepsilon^{-N_2-N_4}\sum_m||o(\chi_m^{(2)})\{u,v\}||_{X_1\cap Y_2} + \sum_m||o(\chi_m^{(2)})R_x^{u,n}, o(\chi_m^{(2)})R_x^{v,n}||_{L^2} \\
&\lesssim \varepsilon^{-M_3}\Big[C + ||u,v||_{Y_2\cap X_1} + \varepsilon^{\frac{n}{2}+\gamma-2N_2-2N_4}||\bar{u},\bar{v}||_Z^2 + \varepsilon^{\frac{n}{2}+\gamma-N_2-N_4}||u,v||_{X_1\cap Y_2}\Big].
\end{aligned} \tag{3.9.51}$$

□

Corollary 3.9.13. *There exists a universal constant M_3 such that for any selection of N_2, N_3, N_4, n, γ, so long as $||u,v||_Z \le 1$, we have:*

$$\epsilon^{N_3}||u,v||_{Y_3} \lesssim \epsilon^{N_3-M_3}\Big[C + ||u,v||_{X_1\cap Y_2}\Big] + \varepsilon^{N_3}||u,v||_{X_3} + \varepsilon^{\frac{n}{2}+\gamma-N_2-N_4-M_3+N_3}||u,v||_{X_1\cap Y_2} + \epsilon^{N_3-M_3-2N_2-2N_4+\frac{n}{2}+\gamma}||\bar{u},\bar{v}||_Z^2. \quad (3.9.52)$$

We now upgrade the previous estimate to a fourth order, weighted estimate. This fourth order estimate is not included in our norm, (3.9.8). This will simply be used in order to justify rigorously one of the integrations by parts in the energy estimates (in particular, calculation (3.10.95)). This is the reason our right-hand side below in (3.9.53) need not be depicted explicitly; we only need the qualitative information that this quantity is finite.

Lemma 3.9.14 (Fourth-Order Elliptic Regularity)**.** *Suppose $||\bar{u},\bar{v}||_Z \le 1$. Solutions $[u,v,P] \in Z$ to the system (3.8.1) - (3.8.3), with forcing terms as in (3.8.8) - (3.8.9) satisfy the following fourth-order estimate:*

$$||\{u_{xx}, \sqrt{\varepsilon}v_{xx}\} \cdot x||_{\dot{H}^2(x\ge 20)} < \infty. \quad (3.9.53)$$

Proof. Taking two derivatives of (3.9.20), one has $\Delta^2(\psi_{xx}) = F_{xx}$. Define a partition of unity of $\{x \ge 20\}$, $\{\chi_m\}_{m\ge 0}$, such that χ_0 is supported on $Y \ge 1$, and all of the $\chi_m, m \ge 1$ are localized near the boundary $Y = 0$ in the region $Y \in [0,2]$ and near $x = m$. Then:

$$\Delta^2(\chi_m x\psi_{xx}) = [\Delta^2, \chi_m x]\psi_{xx} + x\chi_m F_{xx}, \quad (3.9.54)$$

$$\Delta^2(\chi_0 x\psi_{xx}) = [\Delta^2, \chi_0 x]\psi_{xx} + x\chi_0 F_{xx}, \quad (3.9.55)$$

where we refer the reader to the commutator expression in (3.9.25). For equation (3.9.55), one uses the standard, interior $\dot{H}^3$ estimate for the Bi-Laplacian:

$$||\chi_0 x\psi_{xx}||_{\dot{H}^3} \lesssim ||[\Delta^2, \chi_0 x]\psi_{xx}||_{H^{-1}} + ||x\chi_0 F_{xx}||_{H^{-1}}. \quad (3.9.56)$$

For equation (3.9.54), one uses the boundary H^3 estimate which gives:

$$||\chi_m x\psi_{xx}||_{H^3} \lesssim ||[\Delta^2, \chi_m x]\psi_{xx}||_{H^{-1}} + ||x\chi_m F_{xx}||_{H^{-1}}. \tag{3.9.57}$$

Let us write the expression for F_{xx} that we will read from, referring to (3.9.20) (we will rename the power of ε)

$$F_{xx} = \varepsilon^{-M_4}\Big[\partial_{xxY}\tilde{f} + \partial_{xxY}\tilde{S}_u - \partial_{xxx}\tilde{g} - \partial_{xxx}\tilde{S}_v\Big]. \tag{3.9.58}$$

As usual, by sacrificing powers of ε, it suffices to evaluate the H^{-1} norm of the above expression in Prandtl coordinates. We will start with S_u:

$$||\{\chi_m, \chi_0\}x\partial_{xxY}\tilde{S}_u||_{H^{-1}} \lesssim ||\{\chi_m, \chi_0\}x\partial_{xx}\tilde{S}_u||_{L^2} + ||\{o(\partial_y\chi_m, \partial_y\chi_0)x\partial_{xx}\tilde{S}_u\}||_{L^2}. \tag{3.9.59}$$

Similarly for S_v, one obtains:

$$\begin{aligned} ||o(\chi_m, \chi_0)\partial_{xxx}xS_v||_{H^{-1}} \le{}& ||o(\chi_m, \chi_0)\partial_{xx}S_v||_{L^2} + ||o(\partial_x\chi_m, \partial_x\chi_0)\partial_{xx}xS_v||_{L^2} \\ &+ ||o(\chi_m, \chi_0)x\partial_{xx}S_v||_{L^2}. \end{aligned} \tag{3.9.60}$$

Computing two derivatives of S_u, S_v, one obtains:

$$\begin{aligned} \partial_{xx}S_u ={}& u_{Rxxx}u + 2u_{Rxx}u_x + u_{Rx}u_{xx} + u_{Rxx}u_x + 2u_{Rx}u_{xx} \\ &+ u_R u_{xxx} + u_{Ryxx}v + 2u_{Ryx}v_x + u_{Ry}v_{xx} + v_{Rxx}u_y + 2v_{Rx}u_{xy} + v_R u_{xxy}, \end{aligned} \tag{3.9.61}$$

$$\begin{aligned} \partial_{xx}S_v ={}& u_{Rxx}v_x + 2u_{Rx}v_{xx} + u_R v_{xxx} + v_{Rxxx}u + 2v_{Rxx}u_x \\ &+ v_{Rx}u_{xx} + v_{Rxx}v_y + 2v_{Rx}v_{xy} + v_R v_{xxy} + v_{Rxxy}v + 2v_{Rxy}v_x + v_{Ry}v_{xx}. \end{aligned} \tag{3.9.62}$$

From here, given χ_0, χ_m are supported on $x \geq 20$, it is easy to see that:

$$||o(\chi_m, \chi_0)\Big[x\partial_{xx}S_u, x\partial_{xx}S_v\Big]||_{L^2(x\geq 20)} \lesssim ||o(\chi_m, \chi_0)\{u, v\}||_Z. \tag{3.9.63}$$

Next, turning to the expressions in f, g:

$$f_{xx} = \varepsilon^{-\frac{n}{2}-\gamma}R^{u,n}_{xx} + \bar{u}\bar{u}_{xxx} + 3\bar{u}_x\bar{u}_{xx} + \bar{v}_{xx}u_y + 2\bar{v}_x u_{xy} + \bar{v}u_{xxy}, \tag{3.9.64}$$

$$g_{xx} = \varepsilon^{-\frac{n}{2}-\gamma}R^{v,n}_{xx} + \bar{u}_{xx}\bar{v}_x + \bar{u}\bar{v}_{xxx} + 2\bar{u}_x\bar{v}_{xx} + \bar{v}_{xx}\bar{v}_y + \bar{v}\bar{v}_{xxy} + 2\bar{v}_x\bar{v}_{xy}. \tag{3.9.65}$$

From here, using (3.7.177), and the definition the norm Z in (3.9.8), one observes:

$$||o(\chi_m, \chi_0)\Big[x\partial_{xx}f, x\partial_{xx}g\Big]||_{L^2(x\geq 20)} \lesssim ||o(\chi_m, \chi_0)\{u, v\}||_Z^2 + ||o(\chi_m, \chi_0)\{\bar{u}, \bar{v}\}||_Z^2. \tag{3.9.66}$$

Summarizing this:

$$\begin{aligned} \sum_{m\geq 0} ||x\chi_m F_{xx}||_{H^{-1}} \lesssim & \sum_{m\geq 0} ||o(\chi_m)\{u, v\}||_Z + \sum_{m\geq 0} ||o(\chi_m)\{u, v\}||_Z^2 \\ & + \sum_{m\geq 0} ||o(\chi_m)\{\bar{u}, \bar{v}\}||_Z < \infty. \end{aligned} \tag{3.9.67}$$

We now move to the commutator terms from (3.9.56) - (3.9.57), which we write out, denoting by χ generically either χ_m or χ_0:

$$\begin{aligned} [\Delta^2, x\chi]v_x = & \partial_{xyy}(x\chi)v_{xx} + \partial_{xxy}(x\chi)v_{xy} + \partial_x(x\chi)\partial_{xyy}v_x + x\partial_{yy}\chi v_{xxx} \\ & + \partial_y(x\chi)\partial_{xxy}v_x + \partial_{xx}(x\chi)v_{xyy} + \partial_{xy}(x\chi)v_{xxy} \\ & + \sum_{k=1}^{4}\partial_x^k(x\chi)\partial_x^{4-k}v_x + \sum_{k=1}^{4}x\partial_y^k\chi\partial_y^{4-k}v_x. \end{aligned} \tag{3.9.68}$$

Examining the commutator terms, one observes that the worst term is when all derivatives fall on the cut-off, and none on either x or v_x. Such a term arises, for instance, when $k = 4$ in the final summation above in (3.9.68). In this case, one uses that the partition of unity is selected such that $\partial_y \chi_0, \chi_m$ is a bounded region in y:

$$||\partial_y^4 \{\chi_m, \chi_0\} \cdot x v_x||_{H^{-1}} \le ||\partial_y^4 \{\chi_m, \chi_0\} \cdot x v_x||_{L^2} \lesssim ||o(\chi_m), o(\partial \chi_0) x v_{xy}||_{L^2}. \tag{3.9.69}$$

It is straightforward to check that all terms in the commutator can be estimated either in this manner or directly using the definitions of Z in (3.9.8). Thus, taking summation over the partition of unity again gives:

$$\sum_{m \ge 0} ||[\Delta^2, \chi_m x]\psi_{xx}||_{H^{-1}} \lesssim ||u, v||_Z < \infty. \tag{3.9.70}$$

Combining (3.9.56), (3.9.57), (3.9.67), and (3.9.70), the lemma is proven.

□

3.9.2 Embedding Theorems for the Space Z

The next task is to pinpoint the interplay between the uniform quantities in the norm Z and the norms Y_i. Let us start with:

Lemma 3.9.15. *For $\sigma > 0$ arbitrarily small,*

$$\sup_{x \ge 1} \Big[||\sqrt{\varepsilon} \psi x^{-1-\sigma}||_{L^2_y}^2 + ||u x^{-\sigma}||_{L^2_y}^2 + ||\sqrt{\varepsilon} v x^{-\sigma}||_{L^2_y}^2 \Big] \lesssim C(\sigma) ||u, v||_{X_1}^2, \tag{3.9.71}$$

$$\sup_{x \ge 1} \Big[||u_y x^{\frac{1}{2}}||_{L^2_y}^2 + ||u_x x||_{L^2_y}^2 \Big] + \sup_{x \ge 20} ||\sqrt{\epsilon} v_x x||_{L^2_y}^2 \lesssim ||u, v||_{X_1 \cap Y_2}^2, \tag{3.9.72}$$

$$\sup_{x \ge 20} \Big[||u_{xy} x^{\frac{3}{2}}||_{L^2_y}^2 + ||\{v_{xy}, \sqrt{\varepsilon} v_{xx}\} x^2||_{L^2_y}^2 \Big] \lesssim ||u, v||_{Y_2 \cap Y_3}^2. \tag{3.9.73}$$

The constant $C(\sigma) \uparrow \infty$ as $\sigma \downarrow 0$. Finally, for $[u, v] \in Z$, we have the following property:

$$\sup_{x \geq 20} ||\{v_{xxx}, u_{xxy}\}x||_{L^2_y} < \infty, \tag{3.9.74}$$

Remark. The estimate (3.9.74) is required in order to rigorously justify one integration by parts in our energy estimates, in particular calculation (3.10.100), and is not required as part of the norm Z, which is why we do not characterize the right-hand side. The most important parts of this lemma are the first three estimates, (3.9.71) - (3.9.73).

Proof. First, take a differentiation of:

$$\partial_x \int u^2 x^{-2\sigma} \, dy = 2 \int u u_x x^{-2\sigma} \, dy - 2\sigma \int u^2 x^{-1-2\sigma} \, dy, \tag{3.9.75}$$

which upon an integration in x, and recalling $u(1, y) = 0$, yields:

$$\int u^2 x_1^{-2\sigma} \, dy = 2 \int_1^{x_1} \int u u_x x^{-2\sigma} \, dx \, dy - 2\sigma \int_1^{x_1} \int u^2 x^{-1-2\sigma} \, dx \, dy. \tag{3.9.76}$$

Taking absolute values, applying Holder's inequality, and taking the supremum in x_1:

$$\sup_{x \geq 1} \int u^2 x^{-2\sigma} \lesssim ||u x^{-\sigma - \frac{1}{2}}||_{L^2} ||u_x x^{\frac{1}{2}}||_{L^2} + ||u x^{-\frac{1}{2} - \sigma}||^2_{L^2} \lesssim ||u_x x^{\frac{1}{2}}||^2_{L^2} \lesssim ||u, v||^2_{X_1}. \tag{3.9.77}$$

Here the factor of $\sigma > 0$ is required to avert the critical Hardy inequality, which occurs with weight $x^{-\frac{1}{2}}$ in L^2. The stream function estimate follows similarly:

$$\partial_x \int \psi^2 x^{-2-\sigma} = 2 \int \psi v x^{-2-\sigma} - (2 - \sigma) \int \psi^2 x^{-3-\sigma},$$

and so integration gives:

$$\sup_{x \geq 1} \int \psi^2 x^{-2-\sigma} \leq ||\psi v x^{-2-\sigma}||_{L^1} + ||\psi x^{-\frac{3}{2}-\frac{\sigma}{2}}||_{L^2}^2 \lesssim ||v x^{-\frac{1}{2}-\frac{\sigma}{2}}||_{L^2}^2 \lesssim ||v_x x^{\frac{1}{2}-\frac{\sigma}{2}}||_{L^2}^2$$
$$\lesssim ||u, v||_{X_1}^2. \tag{3.9.78}$$

The estimate for v in (3.9.71) works in a similar fashion. We will now move to first-order estimates in (3.9.72). The u_y estimate follows easily after a differentiation:

$$\partial_x \int u_y^2 x = 2 \int u_y u_{xy} x + \int u_y^2. \tag{3.9.79}$$

Taking an integration in x, and recalling that $u_y(1, y) = 0$, then gives:

$$\sup_{x \geq 1} \int u_y^2 x \lesssim ||u_y||_{L^2} ||u_{xy} x||_{L^2} + ||u_y||_{L^2}^2 \lesssim ||u, v||_{X_1 \cap Y_2}^2. \tag{3.9.80}$$

Next,

$$\partial_x \int u_x^2 x^2 = 2 \int u_x u_{xx} x^2 + 2 \int u_x^2 x, \tag{3.9.81}$$

so taking an x-integration and using that $u_x(1, y) = 0$, we have:

$$\sup_{x \geq 1} \int u_x^2 x^2 \lesssim ||u_x x^{\frac{1}{2}}||_{L^2} ||u_{xx} x^{\frac{3}{2}}||_{L^2} + ||u_x x^{\frac{1}{2}}||_{L^2}^2 \lesssim ||u||_{X_1 \cap Y_2}^2.$$

Let us now introduce new cut-off functions, for $k = 2, 3$:

$$\eta_k(x) = \begin{cases} 0 \text{ for } 1 \leq x \leq 3 + 6(k-2), \\ 1 \text{ for } x \geq 6 + 6(k-2). \end{cases} \tag{3.9.82}$$

Then η_2, η_3 are "in-between" ζ_3 and ρ_2, ρ_3. Consider now the quantity $\int \epsilon v_x^2 x^2 \eta_2(x)$:

$$\partial_x \int \epsilon v_x^2 x^2 \eta_2(x) = 2\int \epsilon v_x^2 x \eta_2(x) + 2\int \varepsilon v_x v_{xx} x^2 \eta_2 + \int \varepsilon v_x^2 x^2 \eta_2'(x). \tag{3.9.83}$$

As η_2 vanishes on $[1,3]$, we can take the integration up from $x = 1$:

$$\sup_{x \ge 20} \int \varepsilon v_x^2 x^2 \le \sup_{x \ge 1} \int \epsilon v_x^2 x^2 \eta_2(x) \lesssim ||\sqrt{\varepsilon} v_x x^{\frac{1}{2}}||_{L^2}^2 + ||\sqrt{\varepsilon} v_{xx} x^{\frac{3}{2}}||_{L^2}^2, \tag{3.9.84}$$

where we have used that $|\eta_2'(x) x^2| \lesssim 1$. We now move to the second-order estimates in (3.9.73), starting with:

$$\partial_x \int u_{xy}^2 x^3 \eta_3 = 2\int u_{xy} u_{xxy} x^3 \eta_3 + \int u_{xy}^2 3x^2 \eta_3 + \int u_{xy}^2 x^3 \eta_3'. \tag{3.9.85}$$

Using that $\eta_3 u_{xy}(1, y) = 0$, we have:

$$\sup_{x \ge 1} \int u_{xy}^2 x^3 \eta_3 \lesssim ||u_{xy} x||_{L^2}^2 + ||u_{xy} x||_{L^2} ||u_{xxy} x^2 \eta_3||_{L^2} \lesssim ||u, v||_{Y_2 \cap Y_3}^2. \tag{3.9.86}$$

Above, we have used that $\eta_3 \le \zeta_3$. The final calculation is:

$$\partial_x \int v_{xy}^2 x^4 \eta_3 = C\int v_{xy} v_{xxy} x^4 \eta_3 + C\int v_{xy}^2 x^3 \eta_3 + C\int v_{xy}^2 x^2 \eta_3'. \tag{3.9.87}$$

Integrating up from $x = 1$,

$$\sup_{x \ge 1} \int v_{xy}^2 x^4 \eta_3 \lesssim ||v_{xy} x^{\frac{3}{2}}||_{L^2}^2 + ||v_{xxy} x^{\frac{5}{2}} \eta_3||_{L^2}^2 \lesssim ||u, v||_{Y_2 \cap Y_3}^2. \tag{3.9.88}$$

It is clear that $\sqrt{\varepsilon} v_{xx}$ works in an identical manner to (3.9.87), and also in an identical manner, one obtains (3.9.74) by pairing with (3.9.53).

□

We will now record the following about the $x \to \infty$ behavior of elements in $(X_1 \cap X_2 \cap X_3)(\Omega)$:

Lemma 3.9.16. *Suppose $[u, v] \in X_1 \cap X_2 \cap X_3(\Omega)$. Then the following boundary conditions are automatically enforced:*

$$[u, v]|_{x=1} = [u, v]|_{y=0} = \lim_{x\to\infty} [u, v] = \lim_{y\to\infty} [u, v] = 0 \tag{3.9.89}$$

Proof. All follow as in Lemma 3.9.6 aside from the condition at $x \to \infty$. For this, we use:

$$||ux^{\frac{1}{4}-\sigma}||_{L^\infty_y} \le ||ux^{-2\sigma}||^{\frac{1}{2}}_{L^2_y} ||u_y x^{\frac{1}{2}}||^{\frac{1}{2}}_{L^2_y} < \infty, \tag{3.9.90}$$

$$||vx^{\frac{1}{2}-\sigma}||_{L^\infty_y} \le ||vx^{-2\sigma}||^{\frac{1}{2}}_{L^2_y} ||v_y x||^{\frac{1}{2}}_{L^2_y} < \infty, \tag{3.9.91}$$

according to the estimates in (3.9.71). Thus, we can conclude that $[u, v] \to 0$ as $x \to \infty$.

□

The estimates (3.9.90) - (3.9.91) yield uniform decay of $[u, v]$ as $x \to \infty$. However, the factor of $x^{-\sigma}$ is too weak to close our nonlinear analysis (see estimate (3.11.8)). This in turn is caused by Hardy's inequality being critical (see (3.9.77)). In the next lemma, we use higher-order decay estimates to avoid this criticality:

Lemma 3.9.17 (Uniform Embeddings)**.** *For $[u, v] \in X_1 \cap X_2 \cap X_3(\Omega)$, we have:*

$$\sup_{x\ge 1} ||ux^{\frac{1}{4}}, \sqrt{\varepsilon} v x^{\frac{1}{2}}||_{L^\infty_y} \lesssim ||u, v||_{X_1 \cap Y_2 \cap Y_3}, \tag{3.9.92}$$

$$\sup_{x\ge 20} ||u_x x^{\frac{5}{4}}, \sqrt{\varepsilon} v_x x^{\frac{3}{2}}||_{L^\infty_y} \lesssim ||u, v||_{X_1 \cap Y_2 \cap Y_3}. \tag{3.9.93}$$

Proof. We will first turn to the first-order estimates in (3.9.93). These follow from the evolution estimates in the previous lemma via standard Sobolev interpolation:

$$||u_x x^{\frac{5}{4}}||_{L^\infty_y} \le ||u_x x||_{L^2_y}^{\frac{1}{2}} ||u_{xy} x^{\frac{3}{2}}||_{L^2_y}^{\frac{1}{2}}. \tag{3.9.94}$$

Similarly,

$$||v_x x^{\frac{3}{2}}||_{L^\infty_y} \le ||v_x x||_{L^2_y}^{\frac{1}{2}} ||v_{xy} x^2||_{L^2_y}^{\frac{1}{2}}. \tag{3.9.95}$$

The result now follows after taking the supremum in $x \ge 20$ and appealing to the evolution estimates above. Next, we address (3.9.92). For $x \le 20$, we may simply appeal to the uniform estimates in estimate (3.9.17). For v, the result follows immediately from:

$$\Big|v(x,y)\Big| = \Big|\int_x^\infty v_x(x',y)dx'\Big| \le ||v_x x^{\frac{3}{2}}||_{L^\infty_y} \int_x^\infty (x')^{-\frac{3}{2}} dx' = \Big[\sup_{x\ge 20} ||v_x x^{\frac{3}{2}}||_{L^\infty_y}\Big] x^{-\frac{1}{2}}. \tag{3.9.96}$$

We have used the qualitative boundary condition that $v \to 0$ as $x \to \infty$, which is available due to Lemma 3.9.16. For the u estimate follows in the same manner:

$$\Big|u(x,y)\Big| \le ||u_x x^{\frac{5}{4}}||_{L^\infty_y} \int_x^\infty (x')^{-\frac{5}{4}} dx' \lesssim \Big[\sup_{x\ge 20} ||u_x x^{\frac{5}{4}}||_{L^\infty_y}\Big] x^{-\frac{1}{4}}. \tag{3.9.97}$$

Now taking the sup in x, y of both of the above inequalities yields the desired result.

□

We now turn to controlling the top-order uniform-type norm:

Lemma 3.9.18 (Top Order Uniform Embedding)**.** *For $[u,v] \in X_1 \cap Y_2 \cap Y_3$, one has the* *following mixed-norm estimate:*

$$\int_{20}^{\infty} x^4 ||\sqrt{\varepsilon} v_{xx}||_{L_y^\infty}^2 \, dx + \int_{20}^{\infty} x^{\frac{7}{2}} ||u_{xx}||_{L_y^\infty}^2 \, dx \lesssim ||u, v||_{X_1 \cap Y_2 \cap Y_3}^2. \tag{3.9.98}$$

Proof. For $x \geq 20$, we start with:

$$x^2 ||\sqrt{\varepsilon} v_{xx}||_{L_y^\infty} \leq ||x^{\frac{3}{2}} \sqrt{\varepsilon} v_{xx}||_{L_y^2}^{\frac{1}{2}} ||x^{\frac{5}{2}} \sqrt{\varepsilon} v_{xxy}||_{L_y^2}^{\frac{1}{2}}.$$

Taking square on both sides:

$$x^4 ||\sqrt{\varepsilon} v_x||_{L_y^\infty}^2 \leq ||x^{\frac{3}{2}} \sqrt{\varepsilon} v_{xx}||_{L_y^2}^2 + ||x^{\frac{5}{2}} \sqrt{\varepsilon} v_{xxy}||_{L_y^2}^2, \tag{3.9.99}$$

so integrating

$$\int_{20}^{\infty} x^4 ||\sqrt{\varepsilon} v_{xx}||_{L_y^\infty}^2 \, dx \leq ||u, v||_{X_1 \cap Y_2 \cap Y_3}^2. \tag{3.9.100}$$

For the u_{xx} estimate, we have:

$$x^{\frac{7}{4}} ||u_{xx}||_{L_y^\infty} \leq ||x^{\frac{3}{2}} u_{xx}||_{L_y^2}^{\frac{1}{2}} ||x^2 u_{xxy}||_{L_y^2}^{\frac{1}{2}}, \tag{3.9.101}$$

Squaring both sides and taking an x-integration gives:

$$\begin{aligned} \int_{20}^{\infty} x^{\frac{7}{2}} ||u_{xx}||_{L_y^\infty}^2 &\leq \int_{20}^{\infty} ||x^{\frac{3}{2}} u_{xx}||_{L_y^2}^2 + \int_{20}^{\infty} ||x^2 u_{xxy}||_{L_y^2}^2 \\ &\lesssim ||u, v||_{X_1 \cap Y_2 \cap Y_3}^2. \end{aligned} \tag{3.9.102}$$

□

We shall make the following selections, given arbitrary constants $M_2, M_3 \geq 0$:

$$N_2 \text{ is selected such that } N_2 - M_2 = 100; \tag{3.9.103}$$

$$N_3 \text{ is selected such that } N_3 - M_3 = 2N_2; \tag{3.9.104}$$

$$N_k = N_3 \text{ for } k = 4, 5, 6, 7, \tag{3.9.105}$$

$$n \text{ sufficiently large relative to } N_i, M_i. \tag{3.9.106}$$

We now consolidate the above series of lemmas, coupled with (3.9.38) and (3.9.52).

Corollary 3.9.19. *With $N_4, ..., N_7$ as in (3.9.105), we have:*

$$||u, v||_{\mathcal{U}} \lesssim \varepsilon^{\max\{N_2, N_3\}} ||u, v||_{X_1 \cap Y_2 \cap Y_3}. \tag{3.9.107}$$

Theorem 3.9.20 (Z embedding)**.** *Suppose $||\bar{u}, \bar{v}||_Z \leq 1$. For appropriate choices of $N_2, ...N_7$, based only on universal constants, there exists a universal constant $\omega(N_i)$ such that:*

$$||u, v||_Z \lesssim \epsilon^{100} + ||u, v||_{X_1 \cap X_2 \cap X_3} + \varepsilon^{\frac{n}{2} + \gamma - \omega(N_i)} ||\bar{u}, \bar{v}||_Z^2. \tag{3.9.108}$$

Proof. According to (3.9.107), it suffices to treat the Y_2, Y_3 terms in $|| \cdot ||_Z$. For this we simply use the selections in (3.9.103) - (3.9.106) to rewrite (3.9.52)

$$\varepsilon^{N_3} ||u, v||_{Y_3} \lesssim \varepsilon^{100} + \varepsilon^{100} ||u, v||_{X_1 \cap X_3} + \varepsilon^{N_2} ||u, v||_{Y_2} + \varepsilon^{\frac{n}{2} + \gamma - \omega(N_i)} ||\bar{u}, \bar{v}||_Z^2. \tag{3.9.109}$$

The condition (3.9.106) allows us to control, in estimate (3.9.52):

$$\varepsilon^{\frac{n}{2} + \gamma - N_2 - N_4 - M_3 + N_3} ||u, v||_{X_1 \cap Y_2} \leq \varepsilon^{N_2 + 100} ||u, v||_{X_1 \cap Y_2}, \tag{3.9.110}$$

for instance, so long as: $\frac{n}{2} + \gamma - N_2 - N_4 - M_3 + N_3 > N_2 + 100$. There are many such criteria which arise over the course of our analysis, and so we retain the generality as stated in (3.9.106). Next, rewriting (3.9.38) in a similar fashion, one has:

$$\varepsilon^{N_2}||u,v||_{Y_2} \lesssim \varepsilon^{100} + \varepsilon^{100}||u,v||_{X_1 \cap X_2} + \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2. \tag{3.9.111}$$

Summing these two estimates yields the desired result.

□

3.9.3 Function Space, $Z(\Omega^N)$

We will have occasion to consider, $Z(\Omega^N)$, where Ω^N is defined in (3.8.10), and N is some large but finite, fixed number. Due to the boundedness in the y-direction of Ω^N:

Lemma 3.9.21. *For* $[u,v] \in Z(\Omega^N)$*,*

$$||\psi||_{L^2(\Omega^N)} + ||\{\sqrt{\epsilon}v, v_y\}x||_{L^2(\Omega^N)} \le C(N)||u,v||_{Z(\Omega^N)}, \tag{3.9.112}$$

where $C(N)$ *depends poorly on large* N*.*

Proof. This follows from the Poincare inequality, as both $\psi = v = v_y = 0$ on $y = 0$:

$$||vx||_{L^2} \le C(N)||v_y x||_{L^2} \le C(N)||v_{yy}x||_{L^2} = ||u_{xy}x||_{L^2_y} \lesssim C(N)||u,v||_{Z(\Omega^N)}. \tag{3.9.113}$$

Similarly,

$$||\psi||_{L^2} \le C(N)||\psi_y||_{L^2} = ||u||_{L^2} \le C(N)||u_y||_{L^2} \le C(N)||u||_{Z(\Omega^N)}. \tag{3.9.114}$$

□

Lemma 3.9.22. *For* $[u, v] \in Z(\Omega^N)$,

$$\sup_{x \geq 20} \Big[x^{\frac{3}{2}} ||v, v_y||_{L^2_y} + x^{\frac{1}{2}} ||\psi||_{L^2_y} + x^{\frac{1}{2}} ||u||_{L^2_y} + x^2 ||v_x||_{L^2_y} \Big] \leq C(N) ||u, v||_{Z(\Omega^N)}, \quad (3.9.115)$$

where $C(N)$ *depends poorly on large* N.

Proof. By applying the Poincare inequality twice in the y-direction,

$$||vx^{\frac{3}{2}}||_{L^2_y} \leq C(N) ||v_y x^{\frac{3}{2}}||_{L^2_y} \leq C(N) ||u_{xy} x^{\frac{3}{2}}||_{L^2_y} \lesssim ||u, v||_Z. \quad (3.9.116)$$

The final inequality following from (3.9.73). Similarly,

$$||v_x x^2||_{L^2_y} \leq C(N) ||v_{xy} x^2||_{L^2_y} \lesssim ||u, v||_Z. \quad (3.9.117)$$

□

Finally, by repeating all of the calculations which culminated in Theorem 3.9.20 on the domain Ω^N, the following estimate holds:

Theorem 3.9.23 ($Z(\Omega^N)$ embedding)**.** *Fix any* $N > 0$, *large. Suppose* $||\bar{u}, \bar{v}||_{Z(\Omega^N)} \leq 1$. *For appropriate choices of* $N_2, ... N_7$, *based only on universal constants, there exists a universal constant* $\omega(N_i)$ *such that we have:*

$$||u, v||_{Z(\Omega^N)} \lesssim \epsilon^{100} + ||u, v||_{X_1 \cap X_2 \cap X_3(\Omega^N)} + \varepsilon^{\frac{n}{2} + \gamma - \omega(N_i)} ||\bar{u}, \bar{v}||^2_{Z(\Omega^N)}. \quad (3.9.118)$$

The constants in the above estimate are independent of N.

3.10 Navier-Stokes Remainders: Energy Estimates

In this section, we shall obtain a family of energy and positivity estimates for the system in (3.8.1) - (3.8.3). As mentioned in the prior section, we seek a solution $[u, v] \in Z(\Omega)$, where $Z(\Omega)$ is defined precisely in equation (3.9.4). Our point of view for this section, then, is to obtain *a-priori* estimates under the assumption that $[u, v] \in Z$. Such a solution necessarily encodes decay rates of the solutions and their derivatives (see, for instance, (3.9.71) - (3.9.73), (3.9.92), and (3.9.93)). We shall give *a-priori* estimates on the domain Ω^N, as defined in (3.8.10), which are independent of N, allowing us to take $N \to \infty$. For this purpose, we take the boundary conditions shown in (3.8.11).

Remark (Notational Convention). For Section 3.10, all integrations $\int\int$ and norms, $||\cdot||$, without further specification of domains are over Ω^N.

Going to the vorticity formulation of (3.8.1) - (3.8.3):

$$\partial_y\Big(-\Delta_\epsilon u + P_x + S_u\Big) - \epsilon\partial_x\Big(-\Delta_\epsilon v + \frac{P_y}{\epsilon} + S_v\Big) =$$
$$\partial_y\Big(-\Delta_\epsilon u + S_u\Big) - \epsilon\partial_x\Big(-\Delta_\epsilon v + S_v\Big) = f_y - \epsilon g_x. \tag{3.10.1}$$

Define the stream function through

$$\psi(x,y) = -\int_0^y u(x,y')dy', \quad \psi_x = v, \quad \psi_y = -u, \tag{3.10.2}$$

and note via the boundary conditions (3.8.4) and (3.8.11),

$$\psi|_{y=0,y=N} = \psi|_{x=1} = \psi_x|_{x=1} = \psi_y|_{y=0,y=N} = 0. \tag{3.10.3}$$

To see that $\psi|_{y=N} = 0$, we can write:

$$\partial_x \psi = -\int_0^N u_x(x,y')dy' = \int_0^N v_y(x,y')dy' = v(x,N) - v(x,0) = 0. \tag{3.10.4}$$

Next, the boundary condition $\psi(1,y) = 0$ enables us to evaluate ψ the corner: $\psi(1,N) = 0$. Thus, coupling these two facts yields $\psi(x,N) = 0$. Next, we record the observation:

$$\psi(x,y) = \psi(x,y) - \psi(1,y) = \int_1^x \partial_x \psi(x',y)dx' = \int_1^x v(x',y)dx', \tag{3.10.5}$$

and so taking absolute values, and supremum in y yields:

$$||\psi(x')||_{L^\infty_y} \le \int_1^x ||v(x')||_{L^\infty_y} dx' \le \int_1^x (x')^{-\frac{1}{2}} dx' \lesssim x^{\frac{1}{2}}. \tag{3.10.6}$$

We also will have occassional need for the auxiliary domain:

$$\Omega^N_M := \{0 < x < M, 0 < y < N\}. \tag{3.10.7}$$

3.10.1 Energy Estimates

We now give the energy estimates on $[u,v]$. Let us first introduce the notation:

$$\mathcal{W}_{1,E} = |\int\int f \cdot u| + \int\int \varepsilon |g||v|, \tag{3.10.8}$$

$$\mathcal{W}_{1,P} = \int\int |f||v_y|x + \int\int \varepsilon |g||v_x|x, \tag{3.10.9}$$

$$\mathcal{W}_1 = \mathcal{W}_{1,E} + \mathcal{W}_{1,P}. \tag{3.10.10}$$

Proposition 3.10.1. *Let $\epsilon << \delta$ and δ, ε be sufficiently small relative to universal constants. Let $[u,v] \in Z(\Omega^N)$ be solutions to the system (3.8.1) - (3.8.3) on the domain Ω^N. Then* *these solutions satisfy the a-priori energy estimate:*

$$||\sqrt{\epsilon}u_x, u_y||_{L^2}^2 \le \mathcal{O}(\delta)||\sqrt{\epsilon}v_x x^{\frac{1}{4}}, v_y x^{\frac{1}{2}}||_{L^2}^2 + \mathcal{W}_{1,E}. \tag{3.10.11}$$

The constant in the above estimate is independent of N.

Remark. Note carefully in (3.10.8) that we do not bring the absolute value inside of the integration for the $\int\int fu$ term, which is important to treat term (3.11.5). For the remaining terms above in $\mathcal{W}_1$, we place the absolute values inside the integration for convenience (in terms of comparison with other terms that arise, see for instance (3.10.160) - (3.10.161)).

Proof. The vorticity equation (3.10.1) is multiplied by the stream function ψ and we proceed to integrate by parts. According to the definition of the norm Z in (3.9.8) and the estimate (3.9.112), all integrands appearing in this estimate will be $L^1(\Omega^N)$, and so all applications of Fubini are justified. First, we will treat the highest order terms:

$$\int\int \partial_y\Big(- \Delta_\epsilon u\Big)\psi + \epsilon\partial_x\Delta_\epsilon v\psi = \int\int \Delta_\epsilon u\psi_y - \epsilon\Delta_\epsilon v\psi_x \tag{3.10.12}$$

$$= -\int\int \Delta_\epsilon uu - \epsilon\int\int \Delta_\epsilon vv \tag{3.10.13}$$

$$= \int\int |\nabla_\epsilon u|^2 + \epsilon\int\int |\nabla_\epsilon v|^2 - \lim_{M\to\infty}\Big[\int_{x=M} \varepsilon u_x u + \varepsilon^2 v_x v\Big] \tag{3.10.14}$$

$$= \int\int |\nabla_\epsilon u|^2 + \epsilon\int\int |\nabla_\epsilon v|^2. \tag{3.10.15}$$

For the limiting integrals over $x = M$ above, we have used the bounds from (3.9.112) to conclude:

$$|\int_{x=M} \varepsilon uu_x| \le \varepsilon||ux^{\frac{1}{2}}||_{L^2_y}||u_x x^{\frac{3}{2}}||_{L^2_y}M^{-2} \xrightarrow{M\to\infty} 0, \tag{3.10.16}$$

$$|\int_{x=M} \varepsilon^2 vv_x| \le \varepsilon^2||vx^{\frac{3}{2}}||_{L^2_y}||v_x x^2||_{L^2_y}M^{-\frac{7}{2}} \xrightarrow{M\to\infty} 0. \tag{3.10.17}$$

Let us justify rigorously the integration by parts found in line (3.10.12). To isolate the corners, define $C_r^{1,2}$ to be solid balls of radius r centered at the two corners, $(1,0)$, and $(1,N)$. Then,

$$\int\int \partial_y\Big(-\Delta_\epsilon u\Big)\psi = \int\int_{\Omega^N - \cup_{i=1}^2 C_r^i} \partial_y\Big(-\Delta_\epsilon u\Big)\psi + \sum_{i=1}^2 \int\int_{C_r^i} \partial_y\Big(-\Delta_\epsilon u\Big)\psi \tag{3.10.18}$$

$$= \int\int_{\Omega^N - \cup_{i=1}^2 C_r^i} -\Delta_\epsilon u u - \int_{\partial C_r^i} \Delta_\epsilon u \psi dS + \int\int_{C_r^i} \partial_y\Big(-\Delta_\epsilon u\Big)\psi. \tag{3.10.19}$$

First, as we know $-\Delta_\epsilon u u \in L^1(\Omega^N)$, we have:

$$\lim_{r\to 0} \int\int_{\Omega^N - \cup_{i=1}^2 C_r^i} -\Delta_\epsilon u u = \int\int -\Delta_\epsilon u u. \tag{3.10.20}$$

Next, we appeal to the classical expansion in the vicinity of a corner point in (BR80), Page 57, equation (5.5), from which it follows that:

$$|u,v| \lesssim r, \quad |\psi| \lesssim r, \quad |\nabla^2 u, \nabla^2 v| \lesssim r^{-1} \text{ in } C_r^i. \tag{3.10.21}$$

Then the boundary integral from (3.10.19) can be controlled via:

$$\Big|\int_{\partial C_r^i} \Delta_\epsilon u \psi\Big| \le \int_{\partial C_r^i} \Big|\Delta_\epsilon u \psi\Big| \le \int_{\partial C_r^i} r r^{-1} dS \le r \xrightarrow{r\to 0} 0. \tag{3.10.22}$$

Finally, we arrive at the interior term from the corners in (3.10.19). For this, the expansion in (BR80), Page 57, equation (5.5) implies that:

$$|\nabla^3[u,v]| + |\nabla^3\psi| \lesssim r^{-2} + \tilde{u}, \text{ where } \tilde{u} \in L^2. \tag{3.10.23}$$

Using this gives:

$$\Big|\int\int_{C_r^i}\partial_y\Big(-\Delta_\epsilon u\Big)\psi dxdy\Big| \lesssim \int\int_{C_r^i}\Big|r^{-2}r\Big|dxdy + \int\int_{C_r^i}|\tilde{u}|rdxdy$$
$$\le \int\int_{C_r^i} r^{-2}rrdrd\omega + ||\tilde{u}||_{L^2(C_r^i)}||r||_{L^2(C_r^i)} \xrightarrow{r\to 0} 0. \tag{3.10.24}$$

Summarizing, we have shown the validity of the integration by parts

$$\int\int \partial_y(-\Delta_\epsilon u)\psi = \int\int -\Delta_\epsilon uu. \tag{3.10.25}$$

This calculation works generically at the corners (it was not specific to the particular derivatives involved, just the order of them), and so we will avoid repeating it each time. We now turn to the v-terms in (3.10.12), for which we write:

$$\begin{aligned}\int\int_{\Omega^N - C_r^i}\partial_x\Delta_\epsilon v\psi &= \lim_{M\to\infty}\int\int_{\Omega^N_M - C_r^i}\partial_x\Delta_\epsilon v\psi \\ &= -\lim_{M\to\infty}\Big[\int\int_{\Omega^N_M - C_r^i}\Delta_\epsilon vv + \int_{x=M}\Delta_\epsilon v\psi dy - \int_{\partial C_r^i}\Delta_\epsilon v\psi dy\Big] \\ &= -\int\int_{\Omega^N - C_r^i}\Delta_\epsilon vv + \lim_{M\to\infty}\int_{x=M}\Delta_\epsilon v\psi dy - \int_{\partial C_r^i}\Delta_\epsilon v\psi dy. \end{aligned}\tag{3.10.26}$$

For the $x = M$ boundary term in (3.10.26), we have:

$$\Big|\int_{x=M}\Delta_\epsilon v\psi\Big| \le ||\Delta_\epsilon v||_{L^2_y}||\psi||_{L^2_y} \lesssim ||\psi x^{\frac{1}{2}}||_{L^2_y}||\Delta_\varepsilon v x^{\frac{3}{2}}||_{L^2_y}M^{-2} \xrightarrow{M\to\infty} 0, \tag{3.10.27}$$

according to (3.9.71), (3.9.73), and (3.9.115). Subsequently, sending $r \to 0$ as above gives the desired identity:

$$\int\int \partial_x\Delta_\epsilon v\psi = -\int\int \Delta_\epsilon vv. \tag{3.10.28}$$

Next, we come to the profile terms, S_u, from the equation (3.8.1). We refer the reader to the definition of S_u, which is in (3.8.6).

$$\int\int \partial_y\Big(S_u\Big)\psi = -\int\int S_u\psi_y = \int\int S_u u$$
$$= \int\int \Big[u_R u_x + u_{Rx}u + v_R u_y + u_{Ry}v\Big]u. \tag{3.10.29}$$

The first three of these terms in $\int\int S_u u$ are handled through an integration by parts:

$$\int\int u_R u_x u + u_{Rx}u^2 + v_R u u_y = \int\int u_{Rx}u^2 + \lim_{M\to\infty}\int_{x=M}\frac{u_R}{2}u^2 = -\int\int v_{Ry}u^2. \tag{3.10.30}$$

Above, we have used the estimate for $||ux^{\frac{1}{2}}||_{L^2_y}$ in (3.9.115). The term on the right-hand side of (3.10.30) is handled via:

$$\int\int v_{Ry}u^2 = \int\int \Big(v^P_{Ry} + \sqrt{\epsilon}v^E_{RY}\Big)u^2$$
$$\leq ||v^P_{Ry}y^2||_{L^\infty}||\frac{u}{y}||^2_{L^2} + \sqrt{\epsilon}||v^E_{RY}x^{\frac{3}{2}}||_{L^\infty}||\frac{u}{x^{\frac{3}{4}}}||^2_{L^2}$$
$$\leq \mathcal{O}(\delta)||u_y||^2_{L^2} + \mathcal{O}(\delta)||u_x x^{\frac{1}{4}}||^2_{L^2}. \tag{3.10.31}$$

In (3.10.31), we have first used the profile estimates in (3.3.10) and (3.3.20), and subsequently the Hardy inequality which is available (for exponents of x which are not equal to $\frac{1}{2}$) as $u(1,y) = u(x,0) = 0$. The large convective term in $\int\int S_u u$, (3.10.29), is given by:

$$\int\int u_{Ry}uv = \int\int \{u^{P,n-1}_{Ry} + \epsilon^{\frac{n}{2}}u^n_p + \sqrt{\epsilon}u^E_{RY}\}uv. \tag{3.10.32}$$

First, by estimate (3.3.14), with $j = 1, m = 1$, we have:

$$\Big| \int\int u^{P,n-1}_{Ry} uv \Big| \le ||y^2 x^{-\frac{1}{2}} u^{P,n-1}_{Ry}||_{L^\infty} ||\frac{u}{y}||_{L^2} ||\frac{vx^{\frac{1}{2}}}{y}||_{L^2} \le \mathcal{O}(\delta)||u_y||_{L^2}||v_y x^{\frac{1}{2}}||_{L^2}. \quad (3.10.33)$$

Second, according to (3.3.16) with $j = 1$:

$$\begin{aligned} \Big| \int\int \epsilon^{\frac{n}{2}} u^n_{py} uv \Big| &\le \epsilon^{\frac{n}{2}} ||u^n_{py} y x^{\frac{1}{2}-\sigma_n}||_{L^\infty} ||ux^{-1+\sigma_n}||_{L^2} ||\frac{vx^{\frac{1}{2}}}{y}||_{L^2} \\ &\le C(n)\epsilon^{\frac{n}{2}} ||u_x x^{\sigma_n}||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2} \lesssim C(n)\epsilon^{\frac{n}{2}} ||u_x x^{\frac{1}{2}}||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2}. \quad (3.10.34) \end{aligned}$$

Third, according to (3.3.18):

$$\begin{aligned} \Big| \int\int \sqrt{\epsilon} u^E_{RY} uv \Big| &\le ||u^E_{RY} x^{\frac{3}{2}}||_{L^\infty} ||\frac{u}{x^{\frac{3}{4}}}||_{L^2} ||\sqrt{\epsilon}\frac{v}{x^{\frac{3}{4}}}||_{L^2} \\ &\le \sqrt{\epsilon}||u_x x^{\frac{1}{4}}||_{L^2} ||\sqrt{\epsilon} v_x x^{\frac{1}{4}}||_{L^2}. \quad (3.10.35) \end{aligned}$$

In (3.10.33), we have used the Hardy inequality in the y direction:

$$||\frac{v}{y} x^{\frac{1}{2}}||_{L^2} = ||||\frac{v}{y} x^{\frac{1}{2}}||_{L^2_y}||_{L^2_x} \le ||v_y x^{\frac{1}{2}}||_{L^2}, \quad (3.10.36)$$

which is available as $v|_{y=0} = 0$. We have also used the Hardy inequality in x direction, which is available as $v|_{x=1} = 0$. Rigorously, turning to (3.10.34):

$$\begin{aligned} ||\frac{u}{x^{1-\sigma}}||_{L^2} &= ||||\frac{u}{x^{1-\sigma}}||_{L^2_x}||_{L^2_y} \le ||||\frac{u}{(x-1)^{1-\sigma}}||_{L^2_x}||_{L^2_y} \lesssim ||||u_x (x-1)^\sigma||_{L^2_x}||_{L^2_y} \quad (3.10.37) \\ &\lesssim ||||u_x x^\sigma||_{L^2_x}||_{L^2_y} = ||u_x x^\sigma||_{L^2}. \quad (3.10.38) \end{aligned}$$

Summarizing,

$$\Big| \int \int S_u u \Big| \le \mathcal{O}(\delta) ||u_y||_{L^2}^2 + \mathcal{O}(\delta) ||\sqrt{\epsilon} v_x x^{\frac{1}{4}}, v_y x^{\frac{1}{2}}||_{L^2}^2. \tag{3.10.39}$$

The important mechanism in controlling (3.10.33) is the ability to trade a factor of $y^2 x^{-\frac{1}{2}}$ which is absorbed by the Prandtl profiles, $u_{Ry}^{P,n-1}$, according to (3.3.16) with $j = 1, m = 2$. This creates two y derivatives, u_y and $v_y x^{\frac{1}{2}}$, both of which are order 1. The next step is to control the profile terms S_v:

$$-\epsilon \int \int \partial_x S_v \psi = \epsilon \int \int S_v v - \varepsilon \lim_{M \to \infty} \int_{x=M} S_v \psi = \epsilon \int \int S_v v. \tag{3.10.40}$$

By inspecting (3.9.115), one sees easily that the above limit vanishes. We now treat the interior terms from (3.10.40), which we expand for convenience:

$$\int \int S_v \epsilon v = \int \int \Big(u_R v_x + v_{Rx} u + v_R v_y + v_{Ry} v \Big) \epsilon v. \tag{3.10.41}$$

The first, third, and fourth terms above in (3.10.41) are given via the following calculations:

$$\begin{aligned} \int \int \epsilon u_R v_x v + \epsilon v_R v v_y + \epsilon v_{Ry} v^2 &= \epsilon \int \int v_{Ry} v^2 + \lim_{M \to \infty} \int_{x=M} \frac{\varepsilon}{2} u_R v^2 \\ &= \epsilon \int \int v_{Ry} v^2 && (3.10.42) \\ &\le \epsilon ||v_{Ry}^P y^2||_{L^\infty} ||v_y||_{L^2}^2 + \sqrt{\epsilon} ||v_{RY}^E x^{\frac{3}{2}}||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{4}}||_{L^2}^2 && (3.10.43) \\ &\lesssim \varepsilon ||v_y||_{L^2}^2 + \sqrt{\varepsilon} ||\sqrt{\varepsilon} v_x x^{\frac{1}{4}}||_{L^2}^2. && (3.10.44) \end{aligned}$$

The $M-$limit vanishes by the estimate for $||v x^{\frac{3}{2}}||_{L^2_y}$ in (3.9.115). We have also used estimates (3.3.10) and (3.3.17) for the profiles. Next,

$$\Big| \int \int \epsilon v_{Rx} u v \Big| \le \sqrt{\epsilon} ||x^{\frac{3}{2}} v_{Rx}||_{L^\infty} ||u_x x^{\frac{1}{4}}||_{L^2} ||\sqrt{\epsilon} v_x x^{\frac{1}{4}}||_{L^2}. \tag{3.10.45}$$

Again, we have used (3.3.8) and (3.3.17) for the profiles. Thus,

$$\Big| \int \int S_v \epsilon v \Big| \lesssim \sqrt{\epsilon} ||\sqrt{\varepsilon} v_x x^{\frac{1}{4}}, v_y x^{\frac{1}{4}}||_{L^2}^2. \tag{3.10.46}$$

On the right-hand side of equation (3.10.1), we have:

$$\int \int \Big(f_y - \epsilon g_x \Big) \psi = - \int \int f \psi_y + \int \int \epsilon g \psi_x = \int \int f u + \epsilon g v. \tag{3.10.47}$$

First, we will note that each term in the above integration by parts is in $L^1(\Omega^N)$, which follows from the definitions in (3.8.5), and a consultation with the definition of Z in (3.9.8):

$$f_y = \epsilon^{-\frac{n}{2}-\gamma} R_y^{u,n} + \epsilon^{\frac{n}{2}+\gamma} \Big(u_y u_x + u u_{xy} + v_y u_y + v u_{yy} \Big), \tag{3.10.48}$$

$$g_x = \epsilon^{-\frac{n}{2}-\gamma} R_x^{v,n} + \epsilon^{\frac{n}{2}+\gamma} \Big(u_x v_x + u v_{xx} + v_x v_y + v v_{xy} \Big). \tag{3.10.49}$$

It remains to justify the boundary terms resulting from the x-integration by parts, at $x \to \infty$, in (3.10.47). This, however, follows just as in (3.10.40) by inspecting the decay rates in (3.9.115). A comparison with (3.10.8) then gives the desired result. Combining all of the previous estimates proves (3.10.11). $\square$

3.10.2 Positivity Estimate

We now give the following Positivity estimate:

Proposition 3.10.2. *Let $\epsilon << \delta$ and δ, ε be sufficiently small relative to universal constants. Then $[u, v] \in Z(\Omega^N)$ solutions to the system (3.8.1) - (3.8.3) satisfy the following estimate:*

$$\epsilon^2 \int_{x=1} v_x^2 \, \mathrm{d}y + \lim_{M \to \infty} \int_{x=M} u_y^2 x \, \mathrm{d}y + ||\Big(\sqrt{\epsilon} v_x, v_y\Big) x^{\frac{1}{2}}||_{L^2}^2 \lesssim ||u_y||_{L^2}^2 + \mathcal{W}_1. \tag{3.10.50}$$

Remark (Selection of Multiplier). There is a distinction between the Positivity estimate from (GN17), Page 31, and the present case. In the case of (GN17), the profiles, u_R, were not assumed small, and so they required the normalized multiplier $\frac{v_y}{u_R} - \varepsilon \frac{v_x}{u_R}$. In our case as the profiles are assumed size δ, we need not normalize by a factor of u_R. On the other hand, we need to capture precise behavior at $x \to \infty$, which is the reason our multiplier is $(v_y - \varepsilon v_x) \cdot x$, or equivalently in the vorticity formulation, $v \cdot x$.

Proof. We apply the multiplier $x\psi_x = xv$ to the equation (3.10.1), and we will subsequently take the following integration:

$$\lim_{M \to \infty} \int \int_{\Omega_M^N} \text{Equation (3.10.1)} \cdot xv \, \mathrm{d}y \, \mathrm{d}x. \tag{3.10.51}$$

The purpose of specifying the order of integration is for the terms in (3.10.54), which are not automatically in $L^1(\Omega^N)$ prior to integrating by parts in y. All other terms are in $L^1(\Omega^N)$ according to the norm Z, (3.9.8), and the estimate (3.9.112). Thus, with the exception of the term in (3.10.54), the limiting procedure above can (and will) be omitted, and Fubini can be justified.

Second Order Terms

First, we will treat the highest order terms. Let us begin with:

$$-\int \int \partial_y \Delta_\epsilon u v x = \int \int (\epsilon u_{xx} + u_{yy}) v_y x. \tag{3.10.52}$$

We will note that the u_{xx} term on the right-hand side above is in $L^1(\Omega^N)$, away from the corners. This follows from the definition of Z, and (3.9.112). Thus, the forthcoming applications of Fubini are justified:

$$\begin{aligned}\int\int \epsilon u_{xx} v_y x &= -\int\int \epsilon v_{xy} v_y x = -\frac{\epsilon}{2}\int\int \partial_x\Big(v_y^2\Big)x \\ &= \frac{\epsilon}{2}\int\int v_y^2 - \lim_{M\to\infty}\frac{\epsilon}{2}\int_{x=M} v_y^2 x = \frac{\epsilon}{2}\int\int v_y^2.\end{aligned} \tag{3.10.53}$$

The above limit vanishes according to (3.9.115). For the u_{yy} term in (3.10.52), we can integrate by parts again in y, due to (3.10.51), to obtain:

$$\int\int_{\Omega_M^N} u_{yy} v_y x \,\mathrm{d}y\,\mathrm{d}x = -\int\int_{\Omega_M^N} u_y v_{yy} x \,\mathrm{d}y\,\mathrm{d}x. \tag{3.10.54}$$

Now, the right-hand side of (3.10.54) is in $L^1(\Omega^N)$ according to our norm Z. Interchanging the order of integration:

$$\int\int_{\Omega_M^N} u_{yy} v_y x \,\mathrm{d}y\,\mathrm{d}x = \int\int_{\Omega_M^N} \partial_x\Big(\frac{u_y^2}{2}\Big) x \,\mathrm{d}x\,\mathrm{d}y = -\int\int_{\Omega_M^N} \frac{u_y^2}{2}\,\mathrm{d}x\,\mathrm{d}y + \int_{x=M}\frac{u_y^2}{2}x. \tag{3.10.55}$$

As the solid integrals on the right-hand side of (3.10.55) is known to be in $L^1(\Omega^N)$, we can pass to the limit $M\to\infty$ (and also drop the notation $\mathrm{d}x\,\mathrm{d}y$ when the order no longer matters):

$$\lim_{M\to\infty} -\int\int_{\Omega_M^N} \partial_y u_{yy}\cdot v x \,\mathrm{d}y\,\mathrm{d}x = -\int\int\frac{u_y^2}{2} + \lim_{M\to\infty}\int_{x=M} u_y^2 x. \tag{3.10.56}$$

The limit above appears with a good sign, and therefore contributes to the left-hand side of the desired estimate in (3.10.50). We have omitted the delicate limiting process near the corners in this calculation, as this process is identical to that of (3.10.19). We shall now examine the term:

$$\int\int \epsilon \partial_x \Delta_\epsilon v v x \, \mathrm{d}y \, \mathrm{d}x = \int\int \epsilon \Big(\epsilon v_{xxx} + v_{xyy} \Big) v x \, \mathrm{d}y \, \mathrm{d}x \tag{3.10.57}$$

A direct computation then gives:

$$\int\int \epsilon^2 v_{xxx} v x = -\int\int \epsilon^2 v_{xx} \partial_x (vx) + \epsilon^2 \lim_{M \to \infty} \int_{x=M} v_{xx} v x dy \tag{3.10.58}$$

$$= -\int\int \epsilon^2 v_{xx} \partial_x (vx) \tag{3.10.59}$$

The $x = M$ term in (4.10.19) is controlled by using (3.9.73) and (3.9.115):

$$\Big| \int_{x=M} v_{xx} v x \Big| \le ||v_{xx} x^{\frac{3}{2}}||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-3} \to 0. \tag{3.10.60}$$

Again, we omit displaying the limiting process which handles the corners of the domain, because this is identical to (3.10.19). We now turn to the next integration by parts for the interior term in (4.10.19). The first observation is that both $v_{xx} v_x x$ and $v_{xx} v$ are in $L^1(\Omega^N)$ by inspection of the norm Z. Therefore, we are justified in the integration by parts:

$$\begin{aligned} -\epsilon^2 \int\int v_{xx} v_x x - \epsilon^2 \int\int v_{xx} v \\ &= \frac{3\epsilon^2}{2} \int\int v_x^2 + \frac{\epsilon^2}{2} \int_{x=1} v_x^2 - \lim_{M \to \infty} \Big[\frac{\epsilon^2}{2} \int_{x=M} v_x^2 x dy + \frac{\epsilon^2}{2} \int_{x=M} v v_x dy \Big] \\ &= \frac{3\epsilon^2}{2} \int\int v_x^2 + \frac{\epsilon^2}{2} \int_{x=1} v_x^2, \end{aligned} \tag{3.10.61}$$

where we have appealed to estimates (3.9.115) to show the limits above vanish. We must now treat the v_{xyy} term from (3.10.57):

$$\int\int \epsilon v_{xyy} v x = -\epsilon \int\int v_{yx} v_y x = -\epsilon \int\int \partial_x \Big(\frac{v_y^2}{2} \Big) x$$

$$= \frac{\epsilon}{2}\int\int v_y^2 - \lim_{M\to\infty}\frac{\epsilon}{2}\int_{x=M} v_y^2 x = \frac{\epsilon}{2}\int\int v_y^2. \tag{3.10.62}$$

Again, we appeal to (3.9.115) to show the limit vanishes. Summarizing, then:

$$\int\int \epsilon\partial_x\Delta_\epsilon v x = \frac{3\epsilon^2}{2}\int\int v_x^2 + \int\int \frac{\epsilon}{2}v_y^2 + \frac{\epsilon^2}{2}\int_{x=1} v_x^2. \tag{3.10.63}$$

Profile Terms, S_u:

Next, we treat the S_u profile terms, for which we refer the reader to the expressions in (3.8.6). Integrating by parts in y:

$$\int\int \partial_y S_u v x = -\int\int S_u v_y x = -\int\int u_R u_x v_y x + \mathcal{R}_3 = \int\int u_R v_y^2 x + \mathcal{R}_3. \tag{3.10.64}$$

We give estimates on $\mathcal{R}_3$, starting with:

$$\begin{aligned}\int\int u_{Rx} u x v_y &= \int\int \{u^P_{Rx} + u^E_{Rx}\} u x v_y \\ &\le ||y x^{\frac{1}{2}} u^P_{Rx}||_{L^\infty} ||u_y||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2} + ||u^E_{Rx} x^{\frac{3}{2}}||_{L^\infty} ||u_x||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2} \\ &\le \mathcal{O}(\delta)||u_y||^2_{L^2} + \sqrt{\varepsilon}||v_y x^{\frac{1}{2}}||^2_{L^2}. \end{aligned} \tag{3.10.65}$$

Above, we have used estimate (3.3.12) with $m = 1$, and (3.3.18). Next, by using (3.3.10), we have:

$$\Big|\int\int v_R u_y v_y x\Big| \le ||v_R x^{\frac{1}{2}}||_{L^\infty} ||u_y||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2} \le \mathcal{O}(\delta)||u_y||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2}. \tag{3.10.66}$$

The estimate (3.10.66) is significant in that it essentially determines the rate of decay, $x^{\frac{1}{2}}$, that must be satisfied exactly by the profiles, v_R. The next profile term, according to

(3.3.14) with $j = 1, m = 0$, estimates (3.3.16), and (3.3.18), is:

$$\begin{aligned}\int\int u_{Ry} v v_y x &= \int\int \{u^P_{Ry} + \sqrt{\epsilon} u^E_{RY}\} v v_y x \\ &\le ||y u^P_{Ry}||_{L^\infty} ||v_y x^{\frac{1}{2}}||^2_{L^2} + ||u^E_{RY} x^{\frac{3}{2}}||_{L^\infty} ||v_y x^{\frac{1}{2}}||_{L^2} ||\sqrt{\epsilon} v_x||_{L^2} \\ &\le \mathcal{O}(\delta) ||v_y x^{\frac{1}{2}}||^2_{L^2} + \sqrt{\epsilon} ||\sqrt{\epsilon} v_x||^2_{L^2}. \end{aligned} \tag{3.10.67}$$

Summarizing, we have:

$$\int\int \partial_y S_u v x = \int\int u_R v_y^2 x + \mathcal{R}_3, \quad \Big|\mathcal{R}_3\Big| \le \mathcal{O}(\delta)||u_y||^2_{L^2} + \mathcal{O}(\delta)||v_y x^{\frac{1}{2}}||^2_{L^2} + \sqrt{\epsilon}||\sqrt{\epsilon} v_x||^2_{L^2}. \tag{3.10.68}$$

Profile Terms, S_v:

Next, we treat the S_v profile terms, for which we refer the reader to (3.8.6). The first step is to integrate by parts in x:

$$\begin{aligned} -\int\int \epsilon \partial_x S_v v x &= \epsilon \int\int S_v x v_x + \epsilon \int\int S_v v - \lim_{M\to\infty} \int_{x=M} \varepsilon S_v v x \\ &= \epsilon \int\int S_v x v_x + \epsilon \int\int S_v v. \end{aligned} \tag{3.10.69}$$

The limit above is easily seen to vanish using (3.9.115). We now treat the first term on the right-hand side of (3.10.69). We start with the following term which enables control over $\sqrt{\epsilon} v_x$:

$$\int\int \epsilon u_R v_x^2 x \ge \min u_R \int\int \epsilon v_x^2 x, \tag{3.10.70}$$

The second term from S_v can be controlled according to estimate (3.3.8) and (3.3.17), via:

$$\epsilon \int\int v_{Rx} u v_x x \leq \sqrt{\epsilon} ||x^{\frac{3}{2}} v_{Rx}||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} ||u_x||_{L^2}$$
$$\leq \sqrt{\epsilon} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} ||u_x||_{L^2} \tag{3.10.71}$$

Next, we come to the third term in S_v, where again we use (3.3.10) and (3.3.17):

$$\epsilon \Big| \int\int v_R v_y v_x x \Big| \leq \sqrt{\epsilon} ||v_R||_{L^\infty} ||v_y x^{\frac{1}{2}}||_{L^2} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2}$$
$$\leq \sqrt{\epsilon} \mathcal{O}(\delta) ||v_y x^{\frac{1}{2}}||_{L^2} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} \tag{3.10.72}$$

The fourth profile term in S_v is controlled according to estimates (3.3.10) and (3.3.20) by:

$$\epsilon \Big| \int\int v_{Ry} v v_x x \Big| \leq \sqrt{\epsilon} ||v^P_{Ry} y||_{L^\infty} ||v_y x^{\frac{1}{2}}||_{L^2} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} + \sqrt{\epsilon} ||v^E_{RY} x^{\frac{3}{2}}||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||^2_{L^2}$$
$$\leq \sqrt{\epsilon} \mathcal{O}(\delta) ||v_y x^{\frac{1}{2}}||^2_{L^2} + \sqrt{\varepsilon} \mathcal{O}(\delta) ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||^2_{L^2} \tag{3.10.73}$$

Next, we note that the second interior term on the right-hand side of (3.10.69) is exactly that contained in (3.10.46), which yields:

$$-\int\int \epsilon \partial_x S_v v x = \int\int \epsilon u_R v_x^2 x + \mathcal{R}_4,$$
$$\Big|\mathcal{R}_4\Big| \leq \mathcal{O}(\delta) ||\{\sqrt{\epsilon} v_x, v_y\} x^{\frac{1}{2}}||^2_{L^2} + \mathcal{O}(\delta) ||u_y||^2_{L^2}. \tag{3.10.74}$$

Right-Hand Side

On the right-hand side, we have

$$\int\int\Big(f_y - \epsilon g_x\Big) vx = -\int\int f v_y x + \epsilon \int\int g\Big(v_x x + v\Big). \tag{3.10.75}$$

We now justify both the integration by parts above. First, let us turn to the f_y term, which, according to (3.8.5) contains the forcing terms $R^{u,n}$ and the nonlinearity $\mathcal{N}^u$:

$$\begin{aligned} f_y vx &= \Big(\epsilon^{-\frac{n}{2}-\gamma}\partial_y R^{u,n} + \partial_y\{uu_x + vu_y\}\Big)vx \\ &= \Big(\epsilon^{-\frac{n}{2}-\gamma}\partial_y R^{u,n} + u_y u_x + uu_{xy} + v_y u_y + vu_{yy}\Big)vx. \end{aligned} \tag{3.10.76}$$

From here it is easy to see that $[f_y vx, \epsilon g_x vx] \in L^1(\Omega^N)$. Therefore, it remains to treat the x-integration by parts boundary terms at $x = \infty$, for which we simply appeal to (3.9.115) in an identical fashion to (3.10.69). Combining the previous estimates proves (3.10.50).

□

3.10.3 Second Order Bounds

In this part, we obtain second order control of the solution to the system (3.8.1) - (3.8.3). To do so, we consider the differentiated system in vorticity form:

$$\begin{aligned} &\partial_{xy}\Big(-\Delta_\epsilon u + P_x + S_u\Big) - \epsilon\partial_{xx}\Big(-\Delta_\epsilon v + \frac{P_y}{\epsilon} + S_v\Big) = \\ &\qquad \partial_{xy}\Big(-\Delta_\epsilon u + S_u\Big) - \epsilon\partial_{xx}\Big(-\Delta_\epsilon v + S_v\Big) = f_{xy} - \epsilon g_{xx}. \end{aligned} \tag{3.10.77}$$

We will now repeat the Energy and Positivity estimates from the previous section, with higher order multipliers. One should briefly recall the definition of the cut-off function ρ_2 from (3.9.2). Define our weight via:

$$w_2 = \rho_2 x. \tag{3.10.78}$$

The essential property of this weight is that:

Lemma 3.10.3 (Almost Linear Property)**.**

$$|\partial_x^k w_2| \le \partial_x^k x \text{ for } k = 0, 1, \text{ and } |\partial_x^k w_2| \lesssim x^{-M}, \text{ for } k \ge 2, \text{ for any } M. \tag{3.10.79}$$

Remark. This property of the weight distinguishes it from a generic weight approximating the function x in that all of the nonlinear fluctuations are in an order-1 region around $x = 1$. This structure is needed in (3.10.138), and distinguishes the second order energy estimates from the third-order estimates.

Define:

$$\mathcal{W}_{2,E} = \int\int |f_x||u_x||\rho_2^2 x^2| + \epsilon |g_x||v_x||\rho_2^2 x^2|. \tag{3.10.80}$$

$$\mathcal{W}_{2,P} = \int\int |f_x||v_{xy}||\rho_2^3 x^3| + \epsilon |g_x||v_{xx}||\rho_2^3 x^3|\}, \tag{3.10.81}$$

$$\mathcal{W}_2 = \mathcal{W}_{2,E} + \mathcal{W}_{2,P}. \tag{3.10.82}$$

Proposition 3.10.4 (Second-Order Energy Estimate)**.** *Let $\epsilon << \delta$, and δ, ε be sufficiently small relative to universal constants. Then solutions $[u, v] \in Z(\Omega^N)$ to the system (3.8.1) - (3.8.3) satisfy the following energy estimate:*

$$||u_{xy} w_2||_{L^2}^2 + \epsilon ||\{v_{xy}, \sqrt{\epsilon} v_{xx}\} w_2||_{L^2}^2 \le \mathcal{O}(\delta) ||\{\sqrt{\epsilon} v_{xx}, v_{xy}\} w_2^{\frac{3}{2}}||_{L^2}^2 + ||u, v||_{X_1}^2 + \mathcal{W}_1 + \mathcal{W}_{2,E}. \tag{3.10.83}$$

Remark. Note the presence of absolute values inside the integration in the definition of $\mathcal{W}_2$ for the f term, unlike in $\mathcal{W}_1$. This will be important for calculation (3.11.5).

Remark (Degenerate Weights near $x = 1$). Due to our weight, w_2, degenerating near the boundary $x = 1$, we cannot say that $w_2^2 \lesssim w_2^3$. It is imperative that we retain control of the non-degenerate weight of w on the left-hand side of (3.10.83) for the terms $\{v_{xy}, \sqrt{\epsilon} v_{xx}\}$.

Remark. We will continue to justify rigorously each integration by parts, as we have not cut-off as $x \to \infty$. Starting with the second-order positivity estimate (see (3.10.173)), it becomes possible to work with cut-offs in x, and so the rigorous justifications of Fubini and vanishing boundary contributions at $x = \infty$ become automatic. However, for the present calculation, as in the first-order positivity estimate, we take integrations in the order:

$$\lim_{M \to \infty} \int \int_{\Omega_M^N} \cdot \, \mathrm{d}y \, \mathrm{d}x. \tag{3.10.84}$$

We also refer the reader to the results on the auxiliary space $Z(\Omega^N)$ in Subsection 3.9.3, which will be cited in the forthcoming calculation, for some formal justifications.

Proof of Proposition. We apply the multiplier vw_2^2 to the system (3.10.77). We shall drop the subscript-2 from w_2, for the proof, with the understanding that $w = w_2$ for this calculation. Integration by parts several times gives the highest order terms:

$$\begin{aligned} &\int \int \{\partial_{xy}\Big(- \Delta_\epsilon u\Big) - \epsilon \partial_{xx}\Big(- \Delta_\epsilon v\Big)\} \cdot vw^2 \\ &= \int \int u_{xy}^2 w^2 + \int \int \epsilon u_{xx}^2 w^2 + \int \int \epsilon v_{xy}^2 w^2 + \int \int \epsilon^2 v_{xx}^2 w^2 + J_0, \end{aligned} \tag{3.10.85}$$

where $|J_0| \lesssim \epsilon ||u, v||_{X_1}^2$. To see this, let us first start with the $\Delta_\epsilon u$ terms:

$$\begin{aligned} - \int \int \partial_{xy}(\Delta_\epsilon u) v w^2 &= - \int \int \Delta_\epsilon u_x u_x w^2 \\ &= \int \int \epsilon u_{xx} \partial_x \Big(u_x w^2 \Big) + \int \int u_{xy}^2 w^2 &(3.10.86) \\ &= \int \int (\epsilon u_{xx}^2 + u_{yx}^2) w^2 + 2 \int \int \epsilon u_{xx} u_x w \partial_x w &(3.10.87) \\ &= \int \int (\epsilon u_{xx}^2 + u_{yx}^2) w^2 - \int \int \epsilon u_x^2 \partial_x^2 w^2. &(3.10.88) \end{aligned}$$

We will now examine the weight in (3.10.88). Indeed,

$$|\partial_x^2 w^2| = |2w\partial_x^2 w + 2(\partial_x w)^2| \lesssim 1, \tag{3.10.89}$$

according to (3.10.79). Therefore,

$$\Big| \int\int \epsilon u_x^2 \partial_x^2 w^2 \Big| \lesssim \int\int \epsilon u_x^2 \le \epsilon ||u||_{X_1}^2. \tag{3.10.90}$$

Due to the cutoff function, ρ_2, in w, there are no boundary terms at $x = 1$ when integrating by parts in the x direction. We shall now provide some formalities. First, due to the order of integration in (3.10.84), one can integrate by parts twice in y for the following term:

$$\begin{aligned} -\int\int_{\Omega_N^M} \partial_x \partial_y u_{yy} \cdot v w^2 \,\mathrm{d}y\,\mathrm{d}x = \int\int_{\Omega_N^M} \partial_x u_{yy} v_y w^2 \,\mathrm{d}y\,\mathrm{d}x = -\int\int_{\Omega_N^M} u_{xy} v_{yy} w^2 \\ = \int\int_{\Omega_N^M} u_{xy}^2 w^2. \end{aligned} \tag{3.10.91}$$

The final quantity above is known to be in $L^1(\Omega^N)$ according to our norm Z, and therefore we can take the limit:

$$\lim_{M\to\infty} \int\int_{\Omega_N^M} u_{xy}^2 w^2 = \int\int u_{xy}^2 w^2. \tag{3.10.92}$$

We next turn to the integration in x in (3.10.86). One easily checks that the integrand $\varepsilon u_{xxx} u_x w^2 \in L^1(\Omega^N)$ for $u \in Z$, and so the following calculation is justified:

$$\int\int \varepsilon u_{xxx} u_x w^2 = -\int\int \varepsilon u_{xx} \partial_x (u_x w^2) + \lim_{M\to\infty} \int_{x=M} \varepsilon u_{xx} u_x w^2. \tag{3.10.93}$$

For the limit, we use (3.9.72) - (3.9.73):

$$|\int_{x=M} u_{xx}u_x w^2| \lesssim ||u_{xx}x||_{L^2_y}||u_x x||_{L^2_y} \lesssim M^{-1} \xrightarrow{M\to\infty} 0. \tag{3.10.94}$$

The x-integration in (3.10.87) works in an identical manner. Let us turn to the $\Delta_\epsilon v$ term:

$$\int\int \epsilon\partial_{xx}(\Delta_\epsilon v)vw^2 = -\int\int \epsilon\partial_x(\Delta_\epsilon v)\partial_x(vw^2) \tag{3.10.95}$$
$$= \int\int (\epsilon^2 v_{xx}^2 + \epsilon v_{xy}^2)w^2 + \int\int \epsilon^2 v^2\partial_x^4 w^2 - \epsilon^2 v_x^2\partial_x^2 w^2 - \epsilon v_y^2\partial_x^2 w^2.$$

The final three integrations above are estimated as:

$$\Big|\int\int \epsilon^2 v_x^2\partial_x^2 w^2\Big| \le ||\partial_x^2 w^2||_{L^\infty}\int\int \epsilon^2 v_x^2 \lesssim \epsilon||v||_{X_1}^2, \tag{3.10.96}$$
$$\Big|\int\int \epsilon^2 v^2\partial_x^4 w^2\Big| \le ||x^2\partial_x^4 w^2||_{L^\infty}\int\int \epsilon^2\frac{v^2}{x^2} \lesssim \int\int \epsilon^2 v_x^2 \lesssim \epsilon||v||_{X_1}^2, \tag{3.10.97}$$
$$\Big|\int\int \epsilon v_y^2\partial_x^2 w^2\Big| \lesssim ||\partial_x^2 w^2||_{L^\infty}\int\int \epsilon v_y^2 \lesssim \epsilon||v||_{X_1}^2. \tag{3.10.98}$$

Let us now give formal justifications for (3.10.95). First, we note the following bound:

$$|\int\int \varepsilon\partial_{xx}(\varepsilon v_{xx} + v_{yy})vw^2| \lesssim ||(v_{xxxx} + v_{xxyy})w||_{L^2}||vw||_{L^2}$$
$$\lesssim ||\{u_{xx}, v_{xx}\}\zeta_4 x||_{\dot{H}^2}||vw||_{L^2} < \infty. \tag{3.10.99}$$

For the final estimate, we have used the elliptic regularity established in (3.9.53) and the estimate found in (3.9.112). This then justifies:

$$\int\int \varepsilon\partial_{xx}(\Delta_\varepsilon v)vw^2 = -\int\int \varepsilon\partial_x(\Delta_\varepsilon v)\partial_x(vw^2) + \lim_{M\to\infty}\int \varepsilon\partial_x\Delta_\varepsilon v v w^2 \tag{3.10.100}$$
$$= \int\int \varepsilon\partial_x(\Delta_\varepsilon v)\partial_x(vw^2). \tag{3.10.101}$$

For the above limit, one must use (3.9.115) together with the estimate (3.9.74). The remaining integrations in (3.10.95) are justified in the standard way, as in (3.10.93) - (3.10.94) with the aid of (3.9.112) - (3.9.115). In so doing, one computes limits of the following types:

$$|\int_{x=M} v_{xx} v \partial_x w^2| \le ||v_{xx} x^2||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-\frac{7}{2}} M^2, \tag{3.10.102}$$

$$|\int_{x=M} v_{xx} v_x w^2| \le ||v_{xx} x^2||_{L^2_y} ||v_x x^2||_{L^2_y} M^{-4} M^2, \tag{3.10.103}$$

$$|\int_{x=M} v_x^2 w \le ||v_x x^2||^2_{L^2_y} M^{-4} M, \tag{3.10.104}$$

$$|\int_{x=M} v_x v \partial_x^2 w^2| \le ||v_x x^2||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-\frac{7}{2}}, \tag{3.10.105}$$

$$|\int_{x=M} v^2 \partial_x^3 w^2| \le ||v x^{\frac{3}{2}}||^2_{L^2_y} M^{-3}. \tag{3.10.106}$$

All of these terms vanish upon taking $M \to \infty$. This completes the formal justification of all of the integrations thus far. For the profile terms, calculations which are analogous to the lowest-order case yield:

$$\Big| \int\int \{\partial_{yx} S_u - \epsilon \partial_{xx} S_v\} \cdot v w^2 \Big| \lesssim ||u,v||^2_{X_1} + \mathcal{O}(\delta) ||\{\sqrt{\epsilon} v_{xx}, v_{xy}\} w^{\frac{3}{2}}||^2_{L^2}.$$

For completeness, we include all details here. We will now be working through the following set of terms, referring to the definition in (3.8.6):

$$\begin{aligned} \int\int \partial_{yx} S_u \cdot v w^2 &= \int\int \partial_x S_u \cdot u_x w^2 \\ &= \int\int \partial_x \Big[u_R u_x + u_{Rx} u + u_{Ry} v + v_R u_y \Big] \cdot u_x w^2. \end{aligned} \tag{3.10.107}$$

We begin with:

$$\begin{aligned} |\int\int \partial_x (u_R u_x) u_x w^2| &= |\int\int u_{Rx} u_x^2 w^2 + \int\int u_R u_{xx} u_x w^2| \\ &= |\int\int \frac{u_{Rx}}{2} u_x^2 w^2 - \int\int \frac{u_R}{2} u_x^2 \partial_x w^2 + \lim_{M\to\infty} \int_{x=M} \frac{u_R}{2} u_x^2 w^2| \end{aligned}$$

$$\leq ||u_{Rx}x, u_R||_{L^\infty}||\partial_x w||_{L^\infty}||u_x x^{\frac{1}{2}}||_{L^2}^2 \lesssim ||u||_{X_1}^2. \tag{3.10.108}$$

Above, we have used the estimate (3.3.12) and (3.3.18) for the u^E_{Rx} term. The limit above has vanished according to (3.9.115). Next,

$$\Big|\int\int \partial_x(u_{Rx}u)u_x w^2\Big| = \Big|\int\int u_{Rxx}uu_x w^2 + u_{Rx}u_x^2 w^2\Big| \tag{3.10.109}$$

We must break up the profile term, u_{Rxx}, into Euler and Prandtl. For the u^P_{Rxx} term we use (3.3.11) with $k = 2, j = 0, m = 1$, and subsequently the Hardy inequality in y:

$$\begin{aligned}|\int\int u^P_{Rxx}uu_x w^2| &\leq ||u^P_{Rxx}yx^{\frac{3}{2}}||_{L^\infty}||\frac{u}{y}||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \\ &\lesssim ||u_y||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \lesssim ||u||_{X_1}^2. \end{aligned} \tag{3.10.110}$$

For the u^E_{Rxx} term, we use (3.3.18), followed by the Hardy inequality in x:

$$|\int\int u^E_{Rxx}uu_x w^2|| \leq ||u^E_{Rxx}x^{\frac{5}{2}}||_{L^\infty}||\frac{u}{x}||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \lesssim \sqrt{\varepsilon}||u||_{X_1}^2, \tag{3.10.111}$$

$$|\int\int u_{Rx}u_x^2 w^2| \leq ||u_{Rx}x||_{L^\infty}||u_x x^{\frac{1}{2}}||_{L^2}^2 \leq \mathcal{O}(\delta)||u||_{X_1}^2. \tag{3.10.112}$$

The next profile term from (3.10.107) is the most delicate convective term:

$$\int\int \partial_x(u_{Ry}v)\cdot u_x w^2 = \int\int u_{Rxy}vu_x w^2 + \int\int u_{Ry}v_x u_x w^2 \tag{3.10.113}$$

For the first term in (3.10.113), we bound, according to (3.3.11) for the Prandtl contribution, coupled with the Hardy inequality in y:

$$|\int\int u^P_{Rxy}vu_x w^2| \leq ||u^P_{Rxy}xy||_{L^\infty}||\frac{v}{y}x^{\frac{1}{2}}||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2}$$

$$\leq ||u^P_{Rxy}xy||_{L^\infty}||v_y x^{\frac{1}{2}}||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \lesssim ||u,v||^2_{X_1}, \tag{3.10.114}$$

and (3.3.18) for the Euler contribution, followed by the Hardy inequality in x direction:

$$\begin{aligned}|\int\int \sqrt{\epsilon}u^E_{RxY}vu_x w^2| &\leq \sqrt{\epsilon}||u^E_{RxY}x^{\frac{5}{2}}||_{L^\infty}||\frac{v}{x}||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \\ &\lesssim \sqrt{\epsilon}||\sqrt{\epsilon}v_x||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \lesssim \sqrt{\epsilon}||u,v||^2_{X_1}.\end{aligned} \tag{3.10.115}$$

For the second term in (3.10.113), we bound by using profile estimates (3.3.14), (3.3.16), (3.3.18):

$$\begin{aligned}\Big|\int\int u_{Ry}v_x u_x w^2\Big| &= \Big|\int\int \Big(u^P_{Ry} + \sqrt{\epsilon}u^E_{RY}\Big)v_x u_x w^2\Big| \\ &\leq ||u^P_{Ry}y||_{L^\infty}||\frac{v_x}{y}w^{\frac{3}{2}}||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} + ||u^E_{RY}x^{\frac{3}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \\ &\leq \mathcal{O}(\delta)||v_{xy}w^{\frac{3}{2}}||_{L^2}||u||_{X_1} + \sqrt{\varepsilon}||u,v||^2_{X_1}.\end{aligned} \tag{3.10.116}$$

Implicit in the above calculation is the fact that both the profile terms u_{Ry} and the terms from X_1 are controlled on the full domain, Ω^N, and therefore do not demand any of the "nondegeneracy" of the cut-off weight w near $x = 1$. The next profile term from S_u is:

$$\int\int \partial_x\Big(v_R u_y\Big)\cdot u_x w^2 = \int\int v_{Rx}u_y u_x w^2 + \int\int v_R u_{xy}u_x w^2 \tag{3.10.117}$$

We estimate by using (3.3.8) - (3.3.10), and the Eulerian estimates in (3.3.17) and (3.3.20):

$$\Big|\int\int v_{Rx}u_y u_x w^2\Big| \leq ||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||u_y||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \lesssim ||u,v||^2_{X_1}, \tag{3.10.118}$$

$$\Big|\int\int v_R u_{xy}u_x w^2\Big| \leq ||v_R x^{\frac{1}{2}}||_{L^\infty}||u_{xy}w||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2}$$

$$\lesssim ||u, v||_{X_1}^2 + \mathcal{O}(\delta)||u_{xy}w||_{L^2}^2. \tag{3.10.119}$$

Let us now summarize the S_u contribution:

$$\Big| \int \int \partial_x \Big(S_u \Big) \cdot u_x w^2 \Big| \lesssim ||u, v||_{X_1}^2 + \mathcal{O}(\delta)||v_{xy} w^{\frac{3}{2}}||_{L^2}^2 + \mathcal{O}(\delta)||u_{xy}w||_{L^2}^2. \tag{3.10.120}$$

The u_{xy} term in the above estimate is absorbed to the left-hand side of (3.10.88), by taking δ sufficiently small. The next task is to move to the four profile terms in S_v, which we now do, and recall the terms for convenience:

$$\int \int -\epsilon \partial_{xx} \{ u_R v_x + v_{Rx} u + v_R v_y + v_{Ry} v \} v w^2. \tag{3.10.121}$$

Let us begin by giving some formal justification to the initial x-integration by parts which will be required to treat the above set of terms. First, one observes using (3.9.112) and the definition of Z in (3.9.8) that all terms are in $L^1(\Omega^N)$. Therefore, an integration by parts in x would contribute the following integration in the limit:

$$\begin{aligned} &\lim_{M \to \infty} \int -\epsilon \partial_x \Big[u_R v_x + v_{Rx} u + v_R v_y + v_{Ry} u \Big] v w^2 \, \mathrm{d}y \\ &= \lim_{M \to \infty} \int -\epsilon \Big[u_{Rx} v_x + u_R v_{xx} + v_{Rxx} u + v_{Rx} u_x + v_{Rx} v_y \\ &\qquad\qquad + v_R v_{xy} + v_{Rxy} u + v_{Ry} u_x \Big] v w^2 \, \mathrm{d}y. \end{aligned} \tag{3.10.122}$$

We shall estimate each term above, with the aid of the profile estimates in (3.3.10) - (3.3.17), and also the $Z(\Omega^N)$ estimates in Subsection 3.9.3, (3.9.115).

$$| \int u_{Rx} v_x v | \le ||u_{Rx} x||_{L^\infty} ||v_x x^2||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-\frac{9}{2}}, \tag{3.10.123}$$

$$| \int u_R v_{xx} v | \le ||v_{xx} x^2||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-\frac{7}{2}}, \tag{3.10.124}$$

$$|\int v_{Rxx}uv| \leq ||v_{Rxx}x^{\frac{5}{2}}||_{L^\infty}||ux^{\frac{1}{2}}||_{L^2_y}||vx^{\frac{3}{2}}||_{L^2_y}M^{-\frac{9}{2}}, \tag{3.10.125}$$

$$|\int v_{Rx}u_xv| \leq ||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||u_xx^{\frac{3}{2}}||_{L^2_y}||vx^{\frac{3}{2}}||_{L^2}M^{-\frac{9}{2}}, \tag{3.10.126}$$

$$|\int v_Rv_{xy}v| \leq ||v_Rx^{\frac{1}{2}}||_{L^\infty}||v_{xy}x^2||_{L^2_y}||vx^{\frac{3}{2}}||_{L^2_y}M^{-4}, \tag{3.10.127}$$

$$|\int v_{Rxy}uv| \leq ||v_{Rxy}x^2||_{L^\infty}||ux^{\frac{1}{2}}||_{L^2_y}||vx^{\frac{3}{2}}||_{L^2_y}M^{-4}, \tag{3.10.128}$$

$$|\int v_{Ry}u_xv| \leq ||v_{Ry}x||_{L^\infty}||u_xx^{\frac{3}{2}}||_{L^2_y}||vx^{\frac{3}{2}}||_{L^2_y}M^{-4}. \tag{3.10.129}$$

From here, it is clear that the limit above in (3.10.122) is zero. With this formal justification in hand, we continue with the *a-priori* estimate. For the first term from (3.10.121),

$$\begin{aligned}\Big|\int\int -\epsilon\partial_{xx}(u_Rv_x)\cdot vw^2\Big| &= \Big|\int\int \epsilon\partial_x(u_Rv_x)\cdot\{v_xw^2+2vww'\}\Big| \\ &= \Big|\int\int \epsilon\{u_{Rx}v_x+u_Rv_{xx}\}\cdot\{v_xw^2+2vww'\}\Big|.\end{aligned} \tag{3.10.130}$$

Let us individually treat each term in (3.10.130). First, by combining (3.3.12) and (3.3.18):

$$\Big|\int\int \epsilon u_{Rx}v_x^2w^2\Big| \lesssim ||u_{Rx}x||_{L^\infty}||\sqrt{\epsilon}v_xx^{\frac{1}{2}}||_{L^2}^2 \leq \mathcal{O}(\delta)||v||_{X_1}^2. \tag{3.10.131}$$

Next,

$$\begin{aligned}\Big|\int\int \epsilon u_Rv_{xx}v_xw^2\Big| &= \Big|\int\int \frac{\epsilon}{2}v_x^2\partial_x(u_Rw^2) + \lim_{M\to\infty}\int_{x=M}\frac{\varepsilon}{2}u_Rv_x^2w^2\Big| \\ &= \Big|\int\int \frac{\epsilon}{2}v_x^2u_{Rx}w^2\Big| + \Big|\int\int \epsilon v_x^2u_Rww'\Big| \\ &\lesssim ||u_{Rx}w, w'||_{L^\infty}||\sqrt{\epsilon}v_xx^{\frac{1}{2}}||_{L^2}^2 \lesssim ||v||_{X_1}^2.\end{aligned} \tag{3.10.132}$$

The limit above vanishes according to (3.9.115). Next, we shall split $u_{Rx} = u^P_{Rx} + u^E_{Rx}$:

$$
\begin{aligned}
\Big| \int\int \epsilon u_{Rx} v_x v w w' \Big| &\le \Big| \int\int \epsilon u^P_{Rx} v_x v w w' \Big| + \Big| \int\int \epsilon u^E_{Rx} v_x v w w' \Big| \\
&\le \sqrt{\epsilon} ||u^P_{Rx} y x^{\frac{1}{2}}||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} ||\frac{v}{y}||_{L^2} \\
&\qquad + ||u^E_{Rx} x^{\frac{3}{2}}||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} ||\sqrt{\epsilon}\frac{v}{x}||_{L^2} \\
&\le \sqrt{\epsilon} ||u^P_{Rx} y x^{\frac{1}{2}}||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} ||v_y||_{L^2} \\
&\qquad + ||u^E_{Rx} x^{\frac{3}{2}}||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} ||\sqrt{\epsilon} v_x||_{L^2} \\
&\le \mathcal{O}(\delta) ||v||^2_{X_1}. \qquad (3.10.133)
\end{aligned}
$$

Above, we have used (3.3.12) (with $m = 1$), coupled with Hardy inequalities in y and x, and (3.3.18) for the Euler term. The fourth term in (3.10.130) is by far the most delicate:

$$
\begin{aligned}
\int\int \epsilon u_R v_{xx} v \partial_x w^2 &= -\int\int \epsilon v_x \partial_x \Big(u_R v \partial_x w^2 \Big) + \lim_{M\to\infty} \int_{x=M} \varepsilon u_R v_x v \partial_x w^2 \\
&= -\int\int \epsilon v_x \{ u_{Rx} v \partial_x w^2 + u_R v_x \partial_x w^2 + u_R v \partial_x^2 w^2 \} \\
&= I_1 + I_2 + I_3. \qquad (3.10.134)
\end{aligned}
$$

We justify the above integration. It is clear, according to (3.9.112), that $v_{xx} v \partial_x w^2 \in L^1(\Omega^N)$, and so the boundary contribution at $x = \infty$ is:

$$
|\int_{x=M} \varepsilon u_R v_x v \partial_x w^2| \le ||v_x x^2||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-\frac{7}{2}} M \xrightarrow{M\to\infty} 0. \qquad (3.10.135)
$$

I_1 follows similarly to (3.10.133). For I_2, we have:

$$
|I_2| = |\int\int \varepsilon u_R v_x^2 \partial_x w^2| \lesssim ||\partial_x w||_{L^\infty} ||u_R||_{L^\infty} ||\sqrt{\varepsilon} v_x x^{\frac{1}{2}}||^2_{L^2} \lesssim ||v||^2_{X_1}. \qquad (3.10.136)
$$

We will estimate I_3. For this, we note that due to the almost-linear structure of our weight,

$$|\partial_x^3 w^2| \lesssim x^{-K}, \text{ for any } K, \tag{3.10.137}$$

and so (again with the aid of (3.9.115)):

$$\begin{aligned}\Big|\int\int \epsilon u_R v_x v \partial_x^2 w^2\Big| &= \Big|\int\int \frac{\epsilon}{2} v^2 \Big(u_{Rx}\partial_x^2 w^2 + u_R \partial_x^3 w^2\Big) + \lim_{M\to\infty}\int_{x=M} \varepsilon u_R v^2 \partial_x^2 w^2\Big| \\ &= \Big|\int\int \frac{\epsilon}{2} v^2 \Big(u_{Rx}\partial_x^2 w^2 + u_R \partial_x^3 w^2\Big)\Big|.\end{aligned} \tag{3.10.138}$$

For the u_{Rx} term above in (3.10.138), we shall spit into Euler and Prandtl components, and use estimates (3.3.12) (with $m = 2$), (3.3.18):

$$|\int\int \epsilon v^2 u^P_{Rx}\partial_x^2 w^2| \lesssim \epsilon ||u^P_{Rx} y^2||_{L^\infty} ||\frac{v}{y}||^2_{L^2} \le \epsilon ||u^P_{Rx} y^2||_{L^\infty} ||v_y||^2_{L^2} \lesssim \epsilon \mathcal{O}(\delta) ||v||^2_{X_1}, \tag{3.10.139}$$

$$|\int\int \epsilon v^2 u^E_{Rx}\partial_x^2 w^2| \lesssim ||u^E_{Rx} x^{\frac{3}{2}}||_{L^\infty} ||\sqrt{\epsilon}\frac{v}{x^{\frac{3}{4}}}||^2_{L^2} \lesssim \sqrt{\varepsilon} ||\sqrt{\epsilon} v_x x^{\frac{1}{4}}||^2_{L^2} \lesssim \mathcal{O}(\delta) ||v||^2_{X_1}. \tag{3.10.140}$$

For the u_R term in (3.10.138) above, the structure of our weight, (3.10.137) is important:

$$|\int\int \epsilon v^2 \partial_x^3 w^2| \lesssim ||\sqrt{\epsilon} v_x||^2_{L^2} \lesssim ||v||^2_{X_1}. \tag{3.10.141}$$

For the second term from (3.10.121), we integrate by parts again to arrive at:

$$\begin{aligned}&\Big|\int\int \epsilon \partial_x (v_{Rx} u) \cdot \{v_x w^2 + 2 v w w'\}\Big| \\ &\qquad = \Big|\int\int \epsilon \{v_{Rxx} u + v_{Rx} u_x\} \cdot \{v_x w^2 + 2 v w w'\}\Big|\end{aligned} \tag{3.10.142}$$

Of these, we estimate, according to (3.3.8) - (3.3.10) and (3.3.17):

$$\epsilon\Big|\int\int v_{Rxx}uvww'\Big| \le \epsilon||v_{Rxx}x^{\frac{5}{2}}||_{L^\infty}||\frac{u}{x^{\frac{3}{4}}}||_{L^2}||\frac{v}{x^{\frac{3}{4}}}||_{L^2} \lesssim \sqrt{\epsilon}||u_x x^{\frac{1}{4}}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{4}}||_{L^2}, \tag{3.10.143}$$

$$\epsilon\Big|\int\int v_{Rxx}uv_xw^2\Big| \le \sqrt{\epsilon}||v_{Rxx}x^{\frac{5}{2}}||_{L^\infty}||\frac{u}{x}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2} \lesssim \sqrt{\varepsilon}||u,v||^2_{X_1}, \tag{3.10.144}$$

$$\epsilon\Big|\int\int v_{Rx}u_xv_xw^2\Big| \le \sqrt{\varepsilon}||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||u_x||_{L^2}||\sqrt{\varepsilon}v_x x^{\frac{1}{2}}||_{L^2} \lesssim \sqrt{\varepsilon}||u,v||^2_{X_1}, \tag{3.10.145}$$

$$\epsilon\Big|\int\int v_{Rx}u_xvww'\Big| \le \epsilon||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||u_x x^{\frac{1}{2}}||_{L^2}||\frac{v}{x}||_{L^2} \lesssim \sqrt{\epsilon}||u_x x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_x||_{L^2}. \tag{3.10.146}$$

The third term from (3.10.121) is given by:

$$\begin{aligned} -\epsilon\int\int\partial_{xx}(v_Rv_y)vw^2 &= \int\int\epsilon\partial_x(v_Rv_y)\{v_xw^2+2ww'v\} \\ &= \int\int\epsilon\{v_{Rx}v_y+v_Rv_{xy}\}\{v_xw^2+2ww'v\}. \end{aligned} \tag{3.10.147}$$

We estimate term by term, appealing to estimate (3.3.8) and (3.3.17):

$$\Big|\int\int\epsilon v_{Rx}v_yv_xw^2\Big| \lesssim \sqrt{\epsilon}||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||v_y||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2} \lesssim \sqrt{\epsilon}||v||^2_{X_1}, \tag{3.10.148}$$

$$\begin{aligned} \Big|\int\int\epsilon v_{Rx}v_yvww'\Big| &\lesssim \epsilon||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||v_y x^{\frac{1}{2}}||_{L^2}||\frac{v}{x}||_{L^2} \lesssim \sqrt{\epsilon}||v_y x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_x||_{L^2} \\ &\lesssim \sqrt{\epsilon}||v||^2_{X_1}. \end{aligned} \tag{3.10.149}$$

Next, appealing to estimate (3.3.10) (with $k=0, j=0,1, m=0,1$), and the Euler estimate in (3.3.20), coupled with the Hardy inequality in x and in y directions:

$$\begin{aligned} \Big|\int\int\epsilon v_Rv_{xy}vww'\Big| &= \Big|\int\int\epsilon v_x\{v_Rv_y+v_{Ry}v\}ww'\Big| \\ &\lesssim ||v_Rx^{\frac{1}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}||v_y||_{L^2} + \sqrt{\epsilon}||v^P_{Ry}yx^{\frac{1}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}||\frac{v}{y}||_{L^2} \\ &\quad + \sqrt{\varepsilon}||v^E_{RY}x^{\frac{3}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}\frac{v}{x}||_{L^2} \end{aligned}$$

$$\lesssim ||v_R x^{\frac{1}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}||v_y||_{L^2} + \sqrt{\epsilon}||v^P_{Ry} y x^{\frac{1}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}||v_y||_{L^2}$$
$$+ \sqrt{\varepsilon}||v^E_{RY} x^{\frac{3}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_x||_{L^2}$$
$$\lesssim \mathcal{O}(\delta)||u,v||^2_{X_1}. \tag{3.10.150}$$

Upon integrating by parts and appealing to (3.3.10) (with $k = 0, j = 1, m = 0$) and (3.3.20):

$$\Big|\int\int \epsilon v_R v_{xy} v_x w^2\Big| = \Big|\int\int \frac{\epsilon}{2} v_{Ry} v_x^2 w^2\Big| \lesssim \sqrt{\epsilon}||x v_{Ry}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||^2_{L^2} \le \mathcal{O}(\delta)||v||^2_{X_1}. \tag{3.10.151}$$

The final contribution from S_v in (3.10.121) is:

$$\epsilon\int\int \partial_{xx}(v_{Ry}v)\cdot vw^2 = \epsilon\int\int \{v_{Rxxy}v + 2v_{Rxy}v_x + v_{Ry}v_{xx}\}vw^2 \tag{3.10.152}$$

For the first term, we split:

$$\Big|\int\int \epsilon v^P_{Rxxy} v^2 w^2\Big| \le ||v^P_{Rxxy} x^2 y^2||_{L^\infty}||\frac{v}{y}||^2_{L^2} \lesssim ||v_y||^2_{L^2} \lesssim ||u,v||^2_{X_1}, \tag{3.10.153}$$
$$\Big|\int\int \epsilon v^E_{Rxxy} v^2 w^2\Big| \le \epsilon^{\frac{3}{2}}||v^E_{RxxY} x^{\frac{7}{2}}||_{L^\infty}||\frac{v}{x^{\frac{3}{4}}}||^2_{L^2} \lesssim \sqrt{\epsilon}||\sqrt{\varepsilon}v_x x^{\frac{1}{4}}||^2_{L^2} \lesssim \sqrt{\epsilon}||u,v||^2_{X_1}. \tag{3.10.154}$$

For (3.10.153), we appeal to estimate (3.3.8) (with $k = 2, j = 1, m = 2$), and for (3.10.154), we appeal to estimate (3.3.17). For the next term in (3.10.152), again with the splitting $v_R = v^P_R + v^E_R$, by using estimate (3.3.8), we have:

$$|\int\int \varepsilon v^P_{Rxy} v_x v w^2| \le \sqrt{\varepsilon}||v^P_{Rxy} y x^{\frac{3}{2}}||_{L^\infty}||\frac{v}{y} x^{\frac{1}{2}}||_{L^2}||\sqrt{\varepsilon}v_x x^{\frac{1}{2}}||_{L^2}$$
$$\le \sqrt{\varepsilon}||v^P_{Rxy} y x^{\frac{3}{2}}||_{L^\infty}||v_y x^{\frac{1}{2}}||_{L^2}||\sqrt{\varepsilon}v_x x^{\frac{1}{2}}||_{L^2}$$

$$\le \sqrt{\varepsilon}||u,v||_{X_1}^2. \tag{3.10.155}$$

For the Euler contribution, by (3.3.17), we have:

$$\begin{aligned}|\int\int \varepsilon^{\frac{3}{2}} v^E_{RxY} v v_x w^2| &\le \sqrt{\varepsilon}||v^E_{RxY} x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\varepsilon}\frac{v}{x}||_{L^2}||\sqrt{\varepsilon} v_x x^{\frac{1}{2}}||_{L^2} \\ &\le \sqrt{\varepsilon}||v^E_{RxY} x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\varepsilon} v_x||_{L^2}||\sqrt{\varepsilon} v_x x^{\frac{1}{2}}||_{L^2} \\ &\le \sqrt{\varepsilon}||v||_{X_1}^2. \end{aligned} \tag{3.10.156}$$

The third term in (3.10.152) we split into Euler and Prandtl components, and use (3.3.10) (with $j = 1, m = 1$), and (3.3.17):

$$\begin{aligned}\Big|\int\int \epsilon v_{Ry} v_{xx} v w^2\Big| &\le \Big|\int\int \epsilon v^P_{Ry} v_{xx} v w^2\Big| + \Big|\int\int \epsilon v^E_{Ry} v_{xx} v w^2\Big| \\ &\le \sqrt{\epsilon}||v^P_{Ry} y x^{\frac{1}{2}}||_{L^\infty}||\frac{v}{y} x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon} v_{xx} w^{\frac{3}{2}}||_{L^2} \\ &\qquad + \sqrt{\varepsilon}||v^E_{RY} x^{\frac{3}{2}}||_{L^\infty}||\sqrt{\epsilon} v_{xx} w^{\frac{3}{2}}||_{L^2}||\sqrt{\epsilon}\frac{v}{x}||_{L^2} \\ &\le \sqrt{\varepsilon}||\{v_y, \sqrt{\epsilon} v_x\} x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon} v_{xx} w^{\frac{3}{2}}||_{L^2} \\ &\le \sqrt{\varepsilon}||v||_{X_1}||\sqrt{\epsilon} v_{xx} w^{\frac{3}{2}}||_{L^2}. \end{aligned} \tag{3.10.157}$$

Summarizing the contributions from S_v:

$$\Big|\int\int -\epsilon\partial_{xx}\{S_v\} v w^2\Big| \lesssim ||u,v||_{X_1}^2 + \mathcal{O}(\delta)||\sqrt{\epsilon} v_{xx} w^{\frac{3}{2}}||_{L^2}^2 \tag{3.10.158}$$

On the right-hand side, we have:

$$\begin{aligned}\int\int \partial_{xy} f \cdot v w^2 &= -\int\int f_x \cdot v_y w^2, \qquad (3.10.159)\\ -\epsilon\int\int \partial_{xx} g \cdot v w^2 &= \epsilon\int\int g_x \cdot \{v_x w^2 + 2 v w w'\}\end{aligned}$$

$$= \epsilon \int\int g_x v_x w^2 - \epsilon \int\int g\{v_x w w' + v\partial_{xx}(w^2)\}. \tag{3.10.160}$$

We estimate the g term:

$$\begin{aligned} |\int\int \varepsilon g v_x w w' + g v \partial_{xx}(w^2)| &\le \varepsilon \int\int |g||v_x||ww'| + \varepsilon \int\int |g||v||\partial_{xx}(w^2)| \\ &\le \int\int \varepsilon |g||v_x||w| + \varepsilon |g||v| \le \mathcal{W}_1. \end{aligned} \tag{3.10.161}$$

A formal point: one can easily check that $\partial_{xx} g \cdot v w^2$ and $g_x\{v_x w^2 + v\partial_x w^2\} \in L^1(\Omega^N)$, and so the term at $x = \infty$ which is contributed as a result of integrating by parts in x is:

$$\begin{aligned} &\lim_{M\to\infty} |\int_{x=M} \varepsilon \partial_x g v w^2| \\ &= \lim_{M\to\infty} |\int_{x=M} \varepsilon \Big(\varepsilon^{-\frac{n}{2}-\gamma} R^{v,n}_x + \varepsilon^{\frac{n}{2}+\gamma}(u_x v_x + u v_{xx} + v_x v_y + v v_{xy}) \Big) v w^2|. \end{aligned} \tag{3.10.162}$$

We can estimate each term, using (3.7.177), (3.9.115), for arbitrarily small constants $\kappa > 0$, and σ_n as in (3.7.1)

$$|\int_{x=M} R^{v,n}_x \cdot v w^2| \lesssim ||R^{v,n}_x x^{\frac{9}{4}-2\sigma_n-\kappa}||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-\frac{15}{4}+2\sigma_n+\kappa} M^2, \tag{3.10.163}$$

$$|\int_{x=M} u_x v_x v w^2| \lesssim ||v_x x^{\frac{3}{2}}||_{L^\infty} ||u_x x^{\frac{3}{2}}||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} \lesssim M^{-\frac{9}{2}} M^2, \tag{3.10.164}$$

$$|\int_{x=M} u v_{xx} v w^2| \lesssim ||u||_{L^\infty} ||v_{xx} x^2||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-\frac{7}{2}} M^2, \tag{3.10.165}$$

$$|\int_{x=M} v_x v_y v w^2| \le ||v_x x^{\frac{3}{2}}||_{L^\infty} ||v_y x^{\frac{3}{2}}||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} M^{-\frac{9}{2}} M^2, \tag{3.10.166}$$

$$|\int_{x=M} v v_{xy} v w^2| \le ||v x^{\frac{1}{2}}||_{L^\infty} ||v_{xy} x^2||_{L^2_y} ||v x^{\frac{3}{2}}||_{L^2_y} \le M^{-4} M^2. \tag{3.10.167}$$

It is clear that all of these vanish as $M \to \infty$. This justifies the integration by parts in (3.10.160). The second terms in (3.10.160) are part of (3.10.10). The desired estimate is established.

□

Next, we come to the second-order positivity estimate. It is necessary to apply approximations to the actual weights we would like to control in order to avoid boundary contributions from $x = \infty$. To this end, let us define $\phi(x)$ to be the standard mollifier, with the usual properties of unit mass, positivity, and support in $B(0,1)$. Next, let $\chi(x)$ be a standard cutoff function, equal to 1 inside $[1,2]$, and equal to zero on the interval $[3,\infty)$. Also, recall the definition of ρ_2 provided in equation (3.9.2). Then let us define

$$a_L(x) := \min\{x, L\}, \quad \phi_L(x) := \frac{1}{(L/2)}\phi\Big(\frac{x}{(L/2)}\Big), \tag{3.10.168}$$

$$w_{2,L}(x) := \Big(a_L * \phi_L\Big)\chi\Big(\frac{x}{10L}\Big)\rho_2(x). \tag{3.10.169}$$

The relevant properties of this weight are summarized:

Lemma 3.10.5. *The weight w_L satisfies:*

$$w_{2,L}(x) = x, \text{ for } 60 \le x \le \frac{L}{2}, \quad w_{2,L}(x) = L \text{ for } \frac{3L}{2} \le x \le 10L, \tag{3.10.170}$$

$$w_{2,L}(x) = 0 \text{ for } x \ge 30L \text{ and } 1 \le x \le 50, \quad \Big|x^{k-1}\partial_x^k w_{2,L}\Big| \le C \text{ independent of } L. \tag{3.10.171}$$

Proof. All are clear by basic properties of convolutions. □

Heuristically, the property (3.10.171) ensures that the weight $w_{2,L}(x)$ behaves like x in the sense that each additional derivative eliminates one factor of the weight. We record:

$$\lim_{L\to\infty} w_{2,L}(x) = x\rho_2(x) = w_2(x), \text{ for all } x \ge 1. \tag{3.10.172}$$

Proposition 3.10.6 (Second-Order Positivity)**.** *For δ, ϵ sufficiently small relative to uni-*

versal constants and $\varepsilon << \delta$, solutions $[u,v] \in Z(\Omega^N)$ to the system (3.8.1) - (3.8.3) satisfy:

$$||\{v_{xy}, \sqrt{\epsilon}v_{xx}\}w_2^{\frac{3}{2}}||_{L^2}^2 \lesssim ||u_{xy}w_2||_{L^2}^2 + \epsilon||\{v_{xy}, \sqrt{\epsilon}v_{xx}\}w_2||_{L^2}^2 + ||u,v||_{X_1}^2 + \mathcal{W}_1 + \mathcal{W}_2. \tag{3.10.173}$$

Proof. We apply the multiplier $v_x w_{2,L}^3$. We will drop the subscript-2 from $w_{2,L}$ and simply use w_L for this calculation. Upon integrating by parts, the highest-order terms are:

$$\int\int \partial_{xy}\Big(- u_{yy}\Big) \cdot v_x w_L^3 = -\frac{3}{2}\int\int u_{xy}^2 w_L^2 w_L', \tag{3.10.174}$$

$$\int\int \partial_{xy}\Big(- \epsilon u_{xx}\Big) \cdot v_x w_L^3 = \frac{3\epsilon}{2}\int\int u_{xx}^2 w_L^2 w_L', \tag{3.10.175}$$

$$\int\int \epsilon^2 \partial_x^4 v \cdot v_x w_L^3 = C\int\int \epsilon^2 v_{xx}^2 w_L^2 w_L' + \int\int \epsilon v_x^2 \partial_x^3\{w_L^3\}, \tag{3.10.176}$$

$$\int\int \epsilon \partial_{xxyy} v \cdot v_x w_L^3 = C\int\int \epsilon v_{xy}^2 w_L^2 w_L'. \tag{3.10.177}$$

Above, we have used the property (3.10.171):

$$\Big|\int\int \epsilon v_x^2 \partial_x^3\{w_L^3\}\Big| \le ||\partial_x^3 w_L^3||_{L^\infty}\int\int \epsilon v_x^2 \lesssim ||v||_{X_1}^2. \tag{3.10.178}$$

Similarly, we have:

$$\Big|\int\int \epsilon^2 v_{xx}^2 w_L^2 w_L' + \epsilon v_{xy}^2 w_L^2 w_L'\Big| \le ||w_L'||_{L^\infty}\epsilon||\{\sqrt{\epsilon}v_{xx}, v_{xy}\}w_L||_{L^2}^2. \tag{3.10.179}$$

Thus, all of the terms on the right-hand sides of (3.10.174) - (3.10.177) appear on the right-hand side of (3.10.173), which in turn are controlled by the Energy Estimate, see (3.10.83). We will now work through the profile terms, contained in S_u, which we write below upon using definition (3.8.6):

$$\Big|\int\int \partial_{xy}\{u_R u_x + u_{Rx}u + u_{Ry}v + v_R u_y\} \cdot v_x w_L^3\Big| \tag{3.10.180}$$

First, we have the main profile terms:

$$\int\int \partial_{xy}\Big(u_R u_x\Big) \cdot v_x w_L^3 = -\int\int \partial_x(u_R u_x) v_{xy} w_L^3$$
$$= \int\int u_{Rx} u_x v_{xy} w_L^3 + u_R u_{xx}^2 w_L^3. \tag{3.10.181}$$

As usual, this term retains control of the main term:

$$\int\int u_R u_{xx}^2 w_L^3 \gtrsim \min|u_R| \int\int u_{xx}^2 w_L^3. \tag{3.10.182}$$

The other term in (3.10.181) may be estimated by recalling (3.3.12) and (3.3.18), via:

$$\Big|\int\int u_{Rx} u_x v_{xy} w_L^3\Big| \leq ||u_{Rx} x||_{L^\infty} ||u_x x^{\frac{1}{2}}||_{L^2} ||v_{xy} w_L^{\frac{3}{2}}||_{L^2}$$
$$\leq \mathcal{O}(\delta)||u,v||_{X_1}^2 + \mathcal{O}(\delta)||v_{xy} w_L^{\frac{3}{2}}||_{L^2}^2. \tag{3.10.183}$$

The second term on the right-hand side above, in (3.10.183), can be absorbed by the main positive term, (3.10.182) by taking δ small enough. Next, we have the remaining profile terms from S_u:

$$\Big|\int\int \partial_{xy}\{u_{Rx} u + u_{Ry} v + v_R u_y\} \cdot v_x w_L^3\Big| \tag{3.10.184}$$

For the first term in (3.10.184), we will integrate by parts in y, expand the product, and use Young's inequality:

$$\Big|\int\int \partial_{xy}\{u_{Rx} u\} \cdot v_x w_L^3\Big| = \Big| -\int\int \{u_{Rxx} u + u_{Rx} u_x\} v_{xy} w_L^3\Big|$$
$$\leq ||u_{Rxx}^P y x^{\frac{3}{2}}||_{L^\infty} ||u_y||_{L^2} ||v_{xy} w_L^{\frac{3}{2}}||_{L^2} + ||x^{\frac{5}{2}} u_{Rxx}^E||_{L^\infty} ||u_x||_{L^2} ||v_{xy} w_L^{\frac{3}{2}}||_{L^2}$$
$$+ ||u_{Rx} x||_{L^\infty} ||u_x x^{\frac{1}{2}}||_{L^2} ||v_{xy} w_L^{\frac{3}{2}}||_{L^2}$$

$$\lesssim ||u,v||_{X_1}||v_{xy}w_L^{\frac{3}{2}}||_{L^2} \le \frac{1}{100,000}||v_{xy}w_L^{\frac{3}{2}}||_{L^2}^2 + C||u,v||_{X_1}^2. \tag{3.10.185}$$

We have used (3.3.11), (3.3.12), and the Euler estimates from (3.3.18). The next term from (3.10.184) is the convection term, which we start by integrating by parts:

$$\begin{aligned}
\int\int \partial_{xy}(u_{Ry}v)v_x w_L^3 &= -\int\int \partial_x(u_{Ry}v)v_{xy}w_L^3 \\
&= -\int\int u_{Rxy}vv_{xy}w_L^3 - \int\int u_{Ry}v_x v_{xy}w_L^3 \\
&= -\int\int u_{Rxy}vv_{xy}w_L^3 + \frac{1}{2}\int\int u_{Ryy}v_x^2 w_L^3 \qquad (3.10.186) \\
&= (3.10.186.1) + (3.10.186.2).
\end{aligned}$$

First, by applying (3.3.11) (with $k = j = m = 1$) and the Euler estimate in (3.3.18), and subsequently the Hardy inequality in both the y and x directions,

$$\begin{aligned}
\Big|(3.10.186.1)\Big| &\le ||u^P_{Rxy}xy||_{L^\infty}||\frac{v}{y}x^{\frac{1}{2}}||_{L^2}||v_{xy}w_L^{\frac{3}{2}}||_{L^2} + ||u^E_{RxY}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}\frac{v}{x}||_{L^2}||v_{xy}w_L^{\frac{3}{2}}||_{L^2} \\
&\le ||u^P_{Rxy}xy||_{L^\infty}||v_y x^{\frac{1}{2}}||_{L^2}||v_{xy}w_L^{\frac{3}{2}}||_{L^2} + ||u^E_{RxY}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x||_{L^2}||v_{xy}w_L^{\frac{3}{2}}||_{L^2} \\
&\lesssim ||v||_{X_1}||v_{xy}w_L^{\frac{3}{2}}||_{L^2} \le \frac{1}{100,000}||v_{xy}w_L^{\frac{3}{2}}||_{L^2}^2 + C||v||_{X_1}^2. \qquad (3.10.187)
\end{aligned}$$

Next, by applying (3.3.14), and (3.3.16) with $j = 2$ and (3.3.18), we have:

$$\begin{aligned}
\Big|(3.10.186.2)\Big| &\le ||y^2 u^P_{Ryy}||_{L^\infty}||\frac{v_x}{y}w_L^{\frac{3}{2}}||_{L^2}^2 + ||u^E_{RYY}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}^2 \\
&\le ||y^2 u^P_{Ryy}||_{L^\infty}||v_{xy}w_L^{\frac{3}{2}}||_{L^2}^2 + ||u^E_{RYY}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}^2 \\
&\le \mathcal{O}(\delta)||v_{xy}w_L^{\frac{3}{2}}||_{L^2}^2 + \sqrt{\varepsilon}||u,v||_{X_1}^2. \qquad (3.10.188)
\end{aligned}$$

The final term from (3.10.184), upon integrating by parts once in y, is:

$$\begin{aligned}\Big|\int\int \partial_x\Big(v_R u_y\Big)\cdot v_{xy}w_L^3\Big| &\le \Big|\int\int v_{Rx}u_y v_{xy}w_L^3\Big| + \Big|\int\int v_R u_{xy}v_{xy}w_L^3\Big| \\ &\lesssim ||v_R x^{\frac{1}{2}}||_{L^\infty}\Big(||u_{xy}w_L||_{L^2}^2 + ||v_{xy}w_L^{\frac{3}{2}}||_{L^2}^2\Big) + ||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||u_y||_{L^2}||v_{xy}w_L^{\frac{3}{2}}||_{L^2} \\ &\lesssim \mathcal{O}(\delta)||u_{xy}w_L||_{L^2}^2 + \Big(\mathcal{O}(\delta) + \frac{1}{100,000}\Big)||v_{xy}w_L^{\frac{3}{2}}||_{L^2}^2 + C||u,v||_{X_1}.\end{aligned} \tag{3.10.189}$$

We have used (3.3.10) and (3.3.20) to estimate v_R, in which we crucially retain the smallness from $\mathcal{O}(\delta)$. We have used (3.3.8), (3.3.17), followed by Young's inequality for the v_{Rx} term. Summarizing:

$$\begin{aligned}&\Big|\int\int \partial_{xy}\{u_{Rx}u + u_{Ry}v + v_R u_y\}\cdot v_x w_L^3\Big| \\ &\quad\lesssim \mathcal{O}(\delta)||u,v||_{X_1}^2 + \Big(\mathcal{O}(\delta) + \frac{1}{100,000}\Big)\Big(||v_{xy}w_L^{\frac{3}{2}}||_{L^2}^2\Big) + \mathcal{O}(\delta)||u_{xy}w_L||_{L^2}^2.\end{aligned} \tag{3.10.190}$$

The middle term on the right-hand side above gets absorbed into (3.10.182). We will now come to the profile terms from the normal equation, S_v. For convenience, we display the terms we will be reading from here, according to the definition in (3.8.6):

$$-\epsilon\int\int \partial_{xx}\Big[u_R v_x + v_{Rx}u + v_{Ry}v + v_R v_y\Big]\cdot v_x w_L^3. \tag{3.10.191}$$

The main term upon integrating by parts once in x and expanding the product is:

$$-\int\int \epsilon\partial_{xx}(u_R v_x)\cdot v_x w_L^3 = \int\int \epsilon\{u_R v_{xx} + u_{Rx}v_x\}\cdot\{v_{xx}w_L^3 + 3v_x w_L^2 w_L'\}. \tag{3.10.192}$$

The first term above yields the desired positivity, namely:

$$\int\int \epsilon u_R v_{xx}^2 w_L^3 \gtrsim \min |u_R| \int\int \epsilon v_{xx}^2 w_L^3. \tag{3.10.193}$$

Let us now turn to the remaining terms above in (3.10.192). First an integration by parts gives:

$$\begin{aligned} \Big| \int\int \epsilon u_R v_{xx} v_x \partial_x w_L^3 \Big| &= \Big| - \int\int \epsilon v_x^2 \partial_x \{ u_R \partial_x w_L^3 \} \Big| \\ &= \Big| - \int\int \epsilon v_x^2 \{ u_{Rx} \partial_x w_L^3 + u_R \partial_x^2 w_L^3 \} \Big| \\ &\lesssim ||u_{Rx} x||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2}^2 \lesssim \mathcal{O}(\delta) ||v||_{X_1}^2, \end{aligned} \tag{3.10.194}$$

where we have used the estimate $|\partial_x^2 w_L^3| \lesssim w_L \lesssim x$, and also (3.3.12) and (3.3.18), both of which guarantee the smallness of $\mathcal{O}(\delta)$. Still from (3.10.192), by using (3.3.11) - (3.3.12) and (3.3.18), we have:

$$\begin{aligned} \Big| \int\int \epsilon u_{Rx} v_x v_{xx} w_L^3 \Big| &= \Big| - \int\int \frac{\epsilon}{2} v_x^2 \partial_x \{ u_{Rx} w_L^3 \} \Big| \\ &= \Big| - \int\int \frac{\epsilon}{2} v_x^2 u_{Rxx} w_L^3 - \int\int \frac{\epsilon}{2} v_x^2 u_{Rx} \partial_x w_L^3 \Big| \\ &\lesssim ||u_{Rxx} x^2, u_{Rx} x||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2}^2 \lesssim ||v||_{X_1}^2. \end{aligned} \tag{3.10.195}$$

Last from (3.10.192),

$$\Big| \int\int \epsilon v_x^2 u_{Rx} w_L^2 w_L' \Big| \lesssim ||u_{Rx} x||_{L^\infty} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2}^2 \lesssim \mathcal{O}(\delta) ||v||_{X_1}^2. \tag{3.10.196}$$

Above, we have used (3.3.12) and (3.3.18). We now move to the second term in S_v, for which we integrate by parts once in x and expand the resulting product:

$$\begin{aligned}\int\int \epsilon\partial_{xx}(v_{Rx}u)\cdot v_x w_L^3 &= -\int\int \epsilon\partial_x(v_{Rx}u)\partial_x\{v_x w_L^3\} \\ &= \int\int -\epsilon v_{Rxx}uv_{xx}w_L^3 - \epsilon v_{Rx}u_x v_{xx}w_L^3 - \epsilon v_{Rxx}uv_x\partial_x w_L^3 - \epsilon v_{Rx}u_x v_x \partial_x w_L^3 \\ &= (3.10.197.1) + ... + (3.10.197.4). \end{aligned} \tag{3.10.197}$$

First, by (3.3.8) and (3.3.17):

$$\begin{aligned}\Big|(3.10.197.1)\Big| &\le \sqrt{\epsilon}||v_{Rxx}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}||\frac{u}{x}||_{L^2} \\ &\lesssim \sqrt{\epsilon}||v_{Rxx}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}||u_x||_{L^2} \\ &\lesssim \sqrt{\epsilon}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}||u||_{X_1}. \end{aligned} \tag{3.10.198}$$

Second,

$$\Big|(3.10.197.2)\Big| \lesssim \sqrt{\epsilon}||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||u_x||_{L^2}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2} \lesssim \sqrt{\epsilon}||u||_{X_1}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}. \tag{3.10.199}$$

Third,

$$\Big|(3.10.197.3)\Big| \lesssim \sqrt{\epsilon}||v_{Rxx}x^{\frac{5}{2}}||_{L^\infty}||\frac{u}{x}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2} \lesssim \sqrt{\epsilon}||u,v||_{X_1}^2. \tag{3.10.200}$$

Fourth,

$$\Big|(3.10.197.4)\Big| \lesssim \sqrt{\epsilon}||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||u_x x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_x||_{L^2} \lesssim \sqrt{\epsilon}||u,v||_{X_1}^2. \tag{3.10.201}$$

We have used estimate (3.3.8) and (3.3.17) in the above calculations. Let us now move to the third term in S_v for which we integrate by parts once in x and expand the product:

$$\begin{aligned}\int\int \epsilon\partial_{xx}(v_{Ry}v)v_x w_L^3 &= -\epsilon\int\int \partial_x(v_{Ry}v)\partial_x(v_x w_L^3)\\ &= \int\int -\epsilon v_{Rxy}v v_{xx}w_L^3 - \epsilon v_{Ry}v_x v_{xx}w_L^3 - 3\epsilon v_{Rxy}v v_x w_L^2 w_L'\\ &\qquad - 3\epsilon v_{Ry}v_x^2 w_L^2 w_L' \qquad (3.10.202)\\ &= (3.10.202).1 + ... + (3.10.202).4.\end{aligned}$$

We shall treat term by term above, starting with:

$$\begin{aligned}|(3.10.202).1| &= |\int\int -\epsilon v^P_{Rxy}v v_{xx}w_L^3 - \epsilon^{\frac{3}{2}}v^E_{RxY}v v_{xx}w_L^3|\\ &\leq \sqrt{\epsilon}||v^P_{Rxy}yx^{\frac{3}{2}}||_{L^\infty}||\frac{v}{y}x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}\\ &\qquad + \sqrt{\epsilon}||v^E_{RxY}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}||\sqrt{\epsilon}\frac{v}{x}||_{L^2}\\ &\leq \sqrt{\epsilon}||v^P_{Rxy}yx^{\frac{3}{2}}||_{L^\infty}||v_y x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}\\ &\qquad + \sqrt{\epsilon}||v^E_{RxY}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}||\sqrt{\epsilon}v_x||_{L^2}\\ &\lesssim \sqrt{\epsilon}||v||_{X_1}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}. \qquad (3.10.203)\end{aligned}$$

Above, we have used (3.3.8) with $k = 1, j = 1, m = 1$, and the Euler bounds in (3.3.17). Next, according to estimate (3.3.10), (3.3.17):

$$|(3.10.202).2| \leq ||v_{Ry}x||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2} \lesssim \mathcal{O}(\delta)||v||_{X_1}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}. \qquad (3.10.204)$$

Third, also using the estimates from (3.3.8) and (3.3.17),

$$\begin{aligned}|(3.10.202).3| &\leq \sqrt{\epsilon}||v^P_{Rxy}yx^{\frac{3}{2}}||_{L^\infty}||\frac{v}{y}x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2} + \sqrt{\epsilon}||v^E_{RxY}x^{\frac{5}{2}}||_{L^\infty}||\sqrt{\epsilon}\frac{v}{x}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}\\ &\leq \sqrt{\varepsilon}||v_y x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2} + \sqrt{\epsilon}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||^2_{L^2} \lesssim \sqrt{\varepsilon}||v||^2_{X_1}. \qquad (3.10.205)\end{aligned}$$

Fourth, again by estimate (3.3.10) and (3.3.17):

$$|(3.10.202).4| \lesssim ||v_{Ry}x||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}^2 \le \mathcal{O}(\delta)||v||_{X_1}^2. \tag{3.10.206}$$

We now move to the fourth term in (3.10.191), for which we integrate by parts once and distribute the product:

$$\begin{aligned}\int\int \epsilon\partial_{xx}(v_R v_y)v_x w_L^3 &= -\int\int \epsilon\partial_x(v_R v_y)\partial_x(v_x w_L^3)\\ &= \int\int -\epsilon v_{Rx}v_y v_{xx}w_L^3 - \epsilon v_R v_{xy}v_{xx}w_L^3 - 3\epsilon v_{Rx}v_y v_x w_L^2 w_L'\\ &\qquad - 3\epsilon v_R v_{xy}v_x w_L^2 w_L' \qquad (3.10.207)\\ &= (3.10.207).1 + ... + (3.10.207).4.\end{aligned}$$

First, by (3.3.8), (3.3.10) and (3.3.17):

$$|(3.10.207).1| \le \sqrt{\epsilon}||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||v_y x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}, \tag{3.10.208}$$

$$|(3.10.207).2| \le \sqrt{\epsilon}||v_R x^{\frac{1}{2}}||_{L^\infty}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}||v_{xy}w_L^{\frac{3}{2}}||_{L^2}, \tag{3.10.209}$$

$$|(3.10.207).3| \le \sqrt{\epsilon}||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||v_y x^{\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}. \tag{3.10.210}$$

For the fourth term, we integrate by parts in y and again appeal to (3.3.10) and (3.3.17):

$$|(3.10.207).4| = |\int\int \frac{3\epsilon}{2}v_{Ry}v_x^2 w_L^2 w_L'| \lesssim ||v_{Ry}x||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}^2. \tag{3.10.211}$$

Summarizing the last three terms from the S_v contribution:

$$\begin{aligned}&\Big|\int\int -\epsilon\partial_{xx}\{v_{Rx}u + v_R v_y + v_{Ry}v\}\cdot v_x w_L^3\Big|\\ &\qquad \le \mathcal{O}(\delta)||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}^2 + \mathcal{O}(\delta)||v_{xy}w_L^{\frac{3}{2}}||_{L^2}^2 + \mathcal{O}(\delta)||u,v||_{X_1}^2. \qquad (3.10.212)\end{aligned}$$

Finally, on the right-hand side, we have:

$$\int\int \partial_{xy} f \cdot v_x w_L^3 = \int\int f_x u_{xx} w_L^3, \tag{3.10.213}$$
$$\int\int \epsilon \partial_{xx} g \cdot v_x w_L^3 = \int\int \epsilon g_x \{v_{xx} w_L^3 + 3 v_x w_L^2 w_L'\}.$$

We estimate the term:

$$|\int\int \varepsilon g_x v_x w_L^2 w_L'| \le \int\int \varepsilon |g_x||v_x|(\rho_2 x)^2 \le \mathcal{W}_2. \tag{3.10.214}$$

Taking the limit as $L \to \infty$, and appealing to the Monotone Convergence Theorem then completes the calculation.

□

3.10.4 Third Order Bounds

In this step, we obtain third-order bounds for our solution. We must repeat the above calculations to the twice-differentiated system:

$$\partial_y \partial_x^2 \Big(- \Delta_\epsilon u + P_x + S_u \Big) - \epsilon \partial_x \partial_x^2 \Big(- \Delta_\epsilon v + \frac{P_y}{\epsilon} + S_v \Big) =$$
$$\partial_y \partial_x^2 \Big(- \Delta_\epsilon u + S_u \Big) - \epsilon \partial_x \partial_x^2 \Big(- \Delta_\epsilon v + S_v \Big) = \partial_y \partial_x^2 f - \epsilon \partial_x \partial_x^2 g, \tag{3.10.215}$$

To state our energy estimate, we will recall the definition of ρ_3 from (3.9.2) and the definitions of a_L, ϕ_L given in (3.10.168). We will then define the weights:

$$w_3 = \rho_3(x) x, \quad w_{3,L}(x) := \Big(a_L * \phi_L \Big) \chi\Big(\frac{x}{10L}\Big) \rho_3(x). \tag{3.10.216}$$

Remark (Selection of cut-offs, ρ_k)*.* As k increases from 2 to 3, the supports of ρ_k shift away from $x = 1$. The purpose is so that when obtaining the third-order estimate, the second-order terms should have a non-degenerate estimate in the support of ρ_3, which is achieved so long as ρ_3 is supported sufficiently far to the right of $x = 1$ as compared to ρ_2.

Remark. It is worth emphasizing again the distinguishing feature of the second-order estimate, which vanishes for the third-order estimate. This is estimate (3.10.138) where factors of v (no derivative) appear. This subsequently forces the weight w to be "almost-linear" in the sense of (3.10.137). As soon as the order is upgraded to third-order, this problem vanishes, enabling us to apply the weights $w_{3,L}$ which vanish at $x = \infty$. See (3.10.245) in the forthcoming estimate to contrast with (3.10.138).

Let us now define the third-order forcing term:

$$\mathcal{W}_{3,E} = \int\int |f_{xx}||u_{xx}|w_3^4 + \epsilon|g_{xx}||v_{xx}|w_3^4, \tag{3.10.217}$$

$$\mathcal{W}_{3,P} = \int\int |f_{xx}||u_{xxx}|w_3^5 + \epsilon|g_{xx}||v_{xxx}|w_3^5, \tag{3.10.218}$$

$$\mathcal{W}_3 = \mathcal{W}_{3,E} + \mathcal{W}_{E,P}. \tag{3.10.219}$$

We are now ready to state our energy estimate:

Proposition 3.10.7 (Third-Order Energy Estimate)**.** *Let δ, ϵ be sufficiently small relative to universal constants, and $\varepsilon << \delta$. Then solutions $[u, v] \in Z(\Omega^N)$ to the system (3.8.1) - (3.8.3) satisfy:*

$$\begin{aligned} &||\partial_x^2 u_y w_3^2||_{L^2}^2 + \epsilon||\partial_x^2\{\sqrt{\epsilon}v_x, v_y\}w_3^2||_{L^2}^2 \\ &\qquad \lesssim \mathcal{O}(\delta)||\partial_x^2\{\sqrt{\epsilon}v_x, v_y\}w_3^{\frac{5}{2}}||_{L^2}^2 + ||u, v||_{X_1\cap X_2}^2 + \sum_{j=1}^{2}\mathcal{W}_j + \mathcal{W}_{3,E}. \end{aligned} \tag{3.10.220}$$

Proof. We shall apply the multiplier $v_x w_{3,L}^4$ to the system (3.10.215). Notationally, we drop the subscript-3 from the weight, and simply call w_L the weight appearing in (3.10.216) for this calculation. We start with the highest-order terms, first from $\Delta_\epsilon u$:

$$\int\int \partial_y \partial_x^2 \Big(-u_{yy} - \epsilon u_{xx}\Big) \cdot v_x w_L^4 = -\int\int \Delta_\epsilon u_{xx} \cdot u_{xx} w_L^4$$
$$= \int\int \Big(u_{xxy}^2 + \epsilon u_{xxx}^2\Big) w_L^4 - \frac{1}{2}\int\int \epsilon u_{xx}^2 \partial_x^2 w_L^4. \tag{3.10.221}$$

Next, the terms from $\Delta_\epsilon v$:

$$\int\int \epsilon^2 \partial_x \partial_x^2 v_{xx} v_x w_L^4$$
$$= -\int\int \epsilon^2 \partial_x^2 v_{xx} v_{xx} w_L^4 - \int\int \epsilon^2 \partial_x^2 v_{xx} v_x \partial_x w_L^4$$
$$= \int\int \epsilon^2 v_{xxx}^2 w_L^4 - \frac{1}{2}\epsilon^2 v_{xx}^2 \partial_x^2 w_L^4 + C\epsilon^2 v_x^2 \partial_x^4 w_L^4. \tag{3.10.222}$$

Finally,

$$\int\int \epsilon \partial_x \partial_x^2 v_{yy} \cdot v_x w_L^4$$
$$= -\int\int \epsilon \partial_x^2 v_{yy} v_{xx} w_L^4 - c\epsilon \partial_x^2 v_{yy} v_x \partial_x w_L^4$$
$$= \int\int \epsilon v_{xxy}^2 w_L^4 - \epsilon v_{xy}^2 \partial_x^2 w_L^4. \tag{3.10.223}$$

For the final integrations from (3.10.221) - (3.10.223), we use the inequalities

$$|\partial_x^2 w_L^4| \lesssim w_L^2, \quad |\partial_x^4 w_L^4| \lesssim C, \tag{3.10.224}$$

to estimate:

$$|\int\int \epsilon u_{xx}^2 \partial_x^2 w_L^4| \lesssim ||\sqrt{\epsilon} u_{xx} w_L||_{L^2}^2 \lesssim \epsilon ||u||_{X_2}^2, \tag{3.10.225}$$
$$|\int\int \epsilon^2 v_{xx}^2 \partial_x^2 w_L^4| \lesssim \epsilon ||\sqrt{\epsilon} v_{xx} w_L||_{L^2}^2 \lesssim \epsilon ||v||_{X_2}^2, \tag{3.10.226}$$

$$|\int\int \epsilon^2 v_x^2 \partial_x^4 w_L^4| \lesssim \epsilon ||\sqrt{\epsilon} v_x||_{L^2}^2 \lesssim \epsilon ||v||_{X_1}^2, \tag{3.10.227}$$

$$|\int\int \epsilon v_{xy}^2 \partial_x^2 w_L^4| \lesssim \epsilon ||u_{xx} w_L||_{L^2}^2 \leq \epsilon ||u||_{X_2}^2. \tag{3.10.228}$$

This then leaves from (3.10.221) - (3.10.223) the three terms on the left-hand side of (3.10.220). We now move to the profile terms contained in S_u, which we will display here for convenience upon integrating by parts in y and using the divergence-free condition:

$$\begin{aligned} &\int\int \partial_y \partial_{xx} \Big[u_R u_x + u_{Rx} u + v_R u_y + u_{Ry} v \Big] \cdot v_x w_L^4 \\ &\qquad = \int\int \partial_{xx} \Big[u_R u_x + u_{Rx} u + v_R u_y + u_{Ry} v \Big] \cdot u_{xx} w_L^4. \end{aligned} \tag{3.10.229}$$

First, we will expand via the product rule:

$$\int\int \partial_{xx}(u_R u_x) u_{xx} w_L^4 = \sum_{k=0}^{2} c_k \int\int \partial_x^k u_R \partial_x^{2-k} u_x \cdot u_{xx} w_L^4. \tag{3.10.230}$$

For $k = 0$, we integrate by parts and appeal to (3.3.12), (3.3.17):

$$\begin{aligned} |\int\int u_R \partial_x \Big(|u_{xx}|^2 \Big) w_L^4| &= |-\int\int |u_{xx}|^2 \partial_x (u_R w_L^4)| \\ &= |\int\int |u_{xx}|^2 \Big(u_{Rx} w_L^4 + c u_R \partial_x w_L^4 \Big)| \\ &\lesssim ||u_{Rx} x, u_R||_{L^\infty} ||u_{xx} w_L^{\frac{3}{2}}||_{L^2}^2 \lesssim ||u||_{X_2}^2. \end{aligned} \tag{3.10.231}$$

For $k > 0$, we directly estimate using (3.3.11) and (3.3.17):

$$\begin{aligned} &|\int\int \partial_x^k u_R \partial_x^{2-k} u_x \cdot u_{xx} w_L^4| \\ &\qquad \lesssim ||\partial_x^k u_R x^k||_{L^\infty} ||\partial_x^{2-k} u_x w_L^{\frac{5}{2}-k}||_{L^2} ||u_{xx} x^{\frac{3}{2}}||_{L^2} \\ &\qquad \lesssim ||u||_{X_1 \cap X_2}^2. \end{aligned} \tag{3.10.232}$$

Let us now move to the second term from (3.10.229):

$$\int\int \partial_x^2(u_{Rx}u)\cdot u_{xx}w_L^4 = \sum_{k=0}^{2} c_k \int\int \partial_x^k u_{Rx}\partial_x^{2-k}u\cdot u_{xx}w_L^4 \tag{3.10.233}$$

For the $k=2$ case, that is when all derivatives avoid the u term, we have to employ the splitting $\partial_x^3 u_R = \partial_x^3 u_R^P + \partial_x^3 u_R^E$, and treat each piece separately. First, by (3.3.11):

$$\begin{aligned} |\int\int \partial_x^3 u_R^P u\cdot u_{xx}w_L^4| &\leq ||\partial_x^3 u_R^P y x^{\frac{5}{2}}||_{L^\infty}||\frac{u}{y}||_{L^2}||u_{xx}w_L^{\frac{3}{2}}||_{L^2} \\ &\lesssim ||u_y||_{L^2}||u_{xx}w_L^{\frac{3}{2}}||_{L^2} \lesssim ||u||_{X_1}||u||_{X_2}. \end{aligned} \tag{3.10.234}$$

Next, for the u_R^E contribution, by (3.3.18):

$$\begin{aligned} |\int\int \partial_x^3 u_R^E u\cdot u_{xx}w_L^4| &\leq ||\partial_x^3 u_R^E x^{\frac{7}{2}}||_{L^2}||\frac{u}{x}||_{L^2}||u_{xx}w_L^{\frac{3}{2}}||_{L^2} \\ &\lesssim ||u_x||_{L^2}||u_{xx}w_L^{\frac{3}{2}}||_{L^2} \lesssim ||u||_{X_1}||u||_{X_2}. \end{aligned} \tag{3.10.235}$$

For (3.10.233) when $k<2$, we have by (3.3.11), (3.3.18):

$$\begin{aligned} &|\int\int \partial_x^k u_{Rx}\partial_x^{2-k}u\cdot u_{xx}w_L^4| \\ &\qquad \leq ||\partial_x^k u_{Rx}x^{1+k}||_{L^\infty}||\partial_x^{2-k}u x^{\frac{3}{2}-k}||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\ &\qquad \leq ||u||_{X_{2-k}}||u||_{X_2}. \end{aligned} \tag{3.10.236}$$

We will now move to the third term from (3.10.229):

$$\begin{aligned} \int\int \partial_{xx}(v_R u_y)\cdot u_{xx}w_L^4 &= \sum_{k=0}^{2} c_k\int\int \partial_x^k v_R \partial_x^{2-k}u_y u_{xx}w_L^4 \\ &\leq \sum_{k=0}^{2} ||\partial_x^k v_R x^{k+\frac{1}{2}}||_{L^\infty}||\partial_x^{2-k}u_y w_L^{2-k}||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \end{aligned}$$

$$\lesssim \mathcal{O}(\delta)||\partial_x^2 u_y w_L^2||_{L^2}||u,v||_{X_2} + \sum_{k=1}^{2} ||u||_{X_{3-k}}||u||_{X_2}. \tag{3.10.237}$$

Above we have used estimate (3.3.10) and (3.3.20) to obtain the smallness of $\mathcal{O}(\delta)$ for the v_R term, and estimate (3.3.8) for the remaining terms. We must take δ small enough to absorb the top order term above into (3.10.221). Finally, the fourth convective term from (3.10.229) is expanded into,

$$\int\int \partial_x^2(u_{Ry}v)\cdot u_{xx}w_L^4 = \sum_{k=0}^{2} c_k \int\int \partial_x^k u_{Ry}\partial_x^{2-k}v\cdot u_{xx}w_L^4 \tag{3.10.238}$$

We now split $u_R = u_R^P + u_R^E$. First, according to (3.3.11), (3.3.13), (3.3.14), and (3.3.16):

$$\begin{aligned} &|\int\int \partial_x^k u_{Ry}^P \partial_x^{2-k}v\cdot u_{xx}w_L^4| \\ &\qquad \le ||\partial_x^k u_{Ry}^P y x^k||_{L^\infty}||\frac{\partial_x^{2-k}v}{y}x^{\frac{5}{2}-k}||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\ &\qquad \lesssim ||\partial_x^k u_{Ry}^P y x^k||_{L^\infty}||\partial_x^{2-k}v_y x^{\frac{5}{2}-k}||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\ &\qquad \lesssim \mathcal{O}(\delta)||\partial_x^2 v_y x^{\frac{5}{2}}||_{L^2}||u||_{X_2} + \sum_{k=1}^{2}||u,v||_{X_{3-k}}||u||_{X_2}. \end{aligned} \tag{3.10.239}$$

Let us remark that term (3.10.239), when $k=0$, requires the top order norm, $||v||_{X_3}$ in order to control, and so the smallness of $\mathcal{O}(\delta)$ is essential above. This smallness is obtained via (3.3.14) and (3.3.16), the latter of which is accompanied by $\varepsilon^{\frac{n}{2}}$. Next, we treat the Eulerian contribution via estimate (3.3.18):

$$\begin{aligned} &|\sqrt{\epsilon}\int\int \partial_x^k u_{RY}^E \partial_x^{2-k}v\cdot u_{xx}w_L^4| \\ &\qquad \le ||\partial_x^k u_{RY}^E x^{k+\frac{3}{2}}||_{L^\infty}||\sqrt{\epsilon}\partial_x^{2-k}v x^{1-k}||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\ &\qquad \lesssim \sqrt{\varepsilon}||u,v||_{X_{2-k}}||u||_{X_2}. \end{aligned} \tag{3.10.240}$$

For the previous calculation, in the event that $k = 2$, we have used the Hardy inequality:

$$||\sqrt{\epsilon}\frac{v}{x}||_{L^2} \lesssim ||\sqrt{\epsilon}v_x||_{L^2} \lesssim ||v||_{X_1}. \tag{3.10.241}$$

We are now ready to move to the terms from S_v, which we depict here for convenience:

$$\int\int -\epsilon\partial_x\partial_x^2\Big[u_R v_x + v_{Rx}u + v_R v_y + v_{Ry}v\Big] \cdot v_x w_L^4 \tag{3.10.242}$$

We will now work through the first term:

$$\begin{aligned}\int\int -\epsilon\partial_x\partial_{xx}(u_R v_x)v_x w_L^4 \\ &= \int\int \epsilon\partial_{xx}(u_R v_x)\Big(v_{xx}w_L^4 + v_x\partial_x w_L^4\Big) \\ &= \sum_{k=0}^{2}\sum_{i=1}^{2} c_{k,i}\int\int \epsilon\partial_x^k u_R\partial_x^{2-k}v_x \cdot \partial_x^i v\partial_x^{2-i}w_L^4. \end{aligned}\tag{3.10.243}$$

First, we will treat the $k = 0$ terms from (3.10.243), using estimates (3.3.12) and (3.3.18):

$$\begin{aligned}\int\int \epsilon u_R v_{xxx} \cdot v_{xx}w_L^4 = -\frac{1}{2}\int\int \epsilon v_{xx}^2\partial_x(u_R w_L^4) = C\int\int \epsilon v_{xx}^2\Big(u_{Rx}w_L^4 + u_R\partial_x w_L^4\Big) \\ \lesssim ||u_R, u_{Rx}x||_{L^\infty}||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}^2 \lesssim ||v||_{X_2}^2. \end{aligned}\tag{3.10.244}$$

The second $k = 0$ term, again using (3.3.12) and (3.3.18):

$$\begin{aligned}\int\int \epsilon u_R v_{xxx}v_x\partial_x w_L^4 &= -\int\int \epsilon v_{xx} \cdot \partial_x\{u_R v_x\partial_x w_L^4\} \\ &= -\int\int \epsilon v_{xx}u_{Rx}v_x\partial_x w_L^4 - \epsilon v_{xx}^2 u_R\partial_x w_L^4 - \epsilon v_{xx}u_R v_x\partial_x^2 w_L^4 \end{aligned}\tag{3.10.245}$$

$$\lesssim ||u_R, u_{Rx}x||_{L^\infty}\Big(||\sqrt{\epsilon}v_{xx}w_L^{\frac{3}{2}}||_{L^2}^2 + ||\sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}^2\Big) \tag{3.10.246}$$

$$\lesssim ||v||^2_{X_1 \cap X_2}. \tag{3.10.247}$$

Remark. The third term in line (3.10.245) is the term which has improved for the higher-order energy estimates from the second order energy estimate, allowing us to use the cut-off weight w_L. Indeed, reading this term with $k = 2$ shows that a factor of v appears, which cannot be controlled in L^2.

Next, we move to the $k > 0$ terms from (3.10.243), for which (3.3.11), (3.3.12), and (3.3.18) are in constant use:

$$\begin{aligned}
&|\int\int \epsilon \partial_x^k u_R \partial_x^{2-k} v_x \cdot v_{xx} w_L^4| \\
&\qquad \lesssim ||\partial_x^k u_R x^k||_{L^\infty} ||\sqrt{\epsilon}\partial_x^{3-k} v x^{\frac{5}{2}-k}||_{L^2} ||\sqrt{\epsilon} v_{xx} x^{\frac{3}{2}}||_{L^2} \\
&\qquad \lesssim ||v||_{X_{3-k}} ||v||_{X_2}, \qquad (3.10.248) \\
&|\int\int \epsilon \partial_x^k u_R \partial_x^{2-k} v_x \cdot v_x \partial_x w_L^4| \\
&\qquad \lesssim ||\partial_x^j u_R x^j||_{L^\infty} ||\sqrt{\epsilon}\partial_x^{3-k} v x^{\frac{5}{2}-k}||_{L^2} ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} \\
&\qquad \lesssim ||v||_{X_{3-k}} ||v||_{X_1}. \qquad (3.10.249)
\end{aligned}$$

We will now approach the second term from (3.10.242), for which we use (3.3.8) and (3.3.17):

$$\begin{aligned}
&\int\int -\epsilon \partial_x \partial_x^2 (v_{Rx} u) \partial_x v w_L^4 \\
&\qquad = \sum_{k=0}^{2} \int\int \epsilon \partial_x^k v_{Rx} \partial_x^{2-k} u \cdot v_{xx} w_L^4 - \epsilon \partial_x^k v_{Rx} \partial_x^{2-k} u \cdot v_x \partial_x w_L^4 \\
&\qquad \le \sum_{k=0}^{2} \sqrt{\epsilon} ||\partial_x^k v_{Rx} x^{k+\frac{3}{2}}||_{L^\infty} ||\partial_x^{2-k} u x^{1-k}||_{L^2} \Big(||\sqrt{\epsilon} v_{xx} w_L^{\frac{3}{2}}||_{L^2} \\
&\qquad\qquad + ||\sqrt{\epsilon} v_x w_L^{\frac{1}{2}}||_{L^2} \Big) \\
&\qquad \le ||u, v||^2_{X_1 \cap X_2}. \qquad (3.10.250)
\end{aligned}$$

Above, in the event when $k = 2$, we have used the Hardy inequality to obtain control $||\frac{u}{x}||_{L^2} \lesssim ||u_x||_{L^2}$. We will now approach the third term from (3.10.242), where again we use (3.3.8), (3.3.10), and (3.3.17):

$$
\begin{aligned}
&|\int\int \epsilon \partial_x \partial_x^2 (v_R v_y) \partial_x v w_L^4| \\
&\quad = |\int\int -\epsilon \partial_x^2 (v_R v_y) v_{xx} w_L^4 + \partial_x^2 (v_R v_y) v_x \partial_x w_L^4| \\
&\quad = |\sum_{k=0}^{2} c_k \int\int \epsilon \partial_x^k v_R \partial_x^{2-k} v_y \Big(v_{xx} w_L^4 + v_x \partial_x w_L^4 \Big)| \\
&\quad \lesssim \sum_{k=0}^{2} \sqrt{\epsilon} ||\partial_x^k v_R x^{\frac{1}{2}+k}||_{L^\infty} ||\partial_x^{2-k} v_y w_L^{\frac{5}{2}-k}||_{L^2} \Big(||\sqrt{\epsilon} v_{xx} w_L^{\frac{3}{2}}||_{L^2} + ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}||_{L^2} \Big) \\
&\quad \lesssim ||v||_{X_1 \cap X_2}^2 + \varepsilon ||\partial_x^2 v_y w_L^{\frac{5}{2}}||_{L^2}^2. \qquad (3.10.251)
\end{aligned}
$$

We move to the fourth term from (3.10.242):

$$
\begin{aligned}
&\int\int \epsilon \partial_x \partial_x^2 (v_{Ry} v) v_x w_L^4 \\
&\qquad = \int\int -\epsilon \partial_x^2 (v_{Ry} v) \Big(v_{xx} w_L^4 + C v_x \partial_x w_L^4 \Big) \\
&\qquad = \sum_{k=0}^{2} c_k \int\int \epsilon \partial_x^k v_{Ry} \partial_x^{2-k} v \Big(v_{xx} w_L^4 + C v_x \partial_x w_L^4 \Big). \qquad (3.10.252)
\end{aligned}
$$

We must split $v_R = v_R^P + v_R^E$. First, we treat the Prandtl component by using estimates (3.3.8) - (3.3.10):

$$
\begin{aligned}
&|\int\int \epsilon \partial_x^k v_{Ry}^P \partial_x^{2-k} v \Big(v_{xx} w_L^4 + C v_x \partial_x w_L^4 \Big)| \\
&\qquad \le \sqrt{\epsilon} ||\partial_x^k v_{Ry}^P y x^{\frac{1}{2}+k}||_{L^\infty} ||\partial_x^{2-k} \frac{v}{y} w_L^{\frac{5}{2}-k}||_{L^2} ||\sqrt{\epsilon} v_{xx} w_L^{\frac{3}{2}}, \sqrt{\epsilon} v_x x^{\frac{3}{2}}||_{L^2} \\
&\qquad \le \sqrt{\epsilon} ||\partial_x^k v_{Ry}^P y x^{\frac{1}{2}+k}||_{L^\infty} ||\partial_x^{2-k} v_y w_L^{\frac{5}{2}-k}||_{L^2} ||\sqrt{\epsilon} v_{xx} w_L^{\frac{3}{2}}, \sqrt{\epsilon} v_x x^{\frac{3}{2}}||_{L^2} \\
&\qquad \le \sqrt{\epsilon} ||u, v||_{X_1 \cap X_2}^2 + \sqrt{\varepsilon} ||\partial_x^2 v_y w_L^{\frac{5}{2}}||_{L^2} ||u, v||_{X_1 \cap X_2}. \qquad (3.10.253)
\end{aligned}
$$

Now, fix $k < 2$. The Eulerian contribution is controlled via (3.3.17):

$$\sqrt{\epsilon}\int\int \epsilon\partial_x^k v_{RY}^E \partial_x^{2-k} v\Big(v_{xx}w_L^4 + Cv_x\partial_x w_L^4\Big)$$
$$\leq \sqrt{\epsilon}||\partial_x^k v_{RY}^E x^{\frac{3}{2}+k}||_{L^\infty}||\sqrt{\epsilon}\partial_x^{2-k} v x^{1-k}||_{L^2}||\sqrt{\epsilon}v_{xx}x^{\frac{3}{2}}, \sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}$$
$$\lesssim \sqrt{\epsilon}||v||_{X_{2-k}}||v||_{X_1\cap X_2}. \tag{3.10.254}$$

For $k = 2$, we must employ the Hardy inequality in addition to (3.3.17) to conclude:

$$\sqrt{\epsilon}\int\int \epsilon\partial_x^2 v_{RY}^E v\Big(v_{xx}w_L^4 + Cv_x\partial_x w_L^4\Big)$$
$$\leq \sqrt{\epsilon}||\partial_x^2 v_{RY}^E x^{\frac{3}{2}+2}||_{L^\infty}||\sqrt{\epsilon}v x^{-1}||_{L^2}||\sqrt{\epsilon}v_{xx}x^{\frac{3}{2}}, \sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}$$
$$\leq \sqrt{\epsilon}||\partial_x^2 v_{RY}^E x^{\frac{3}{2}+2}||_{L^\infty}||\sqrt{\epsilon}v_x||_{L^2}||\sqrt{\epsilon}v_{xx}x^{\frac{3}{2}}, \sqrt{\epsilon}v_x x^{\frac{1}{2}}||_{L^2}$$
$$\leq \sqrt{\epsilon}||\partial_x^2 v_{RY}^E x^{\frac{3}{2}+2}||_{L^\infty}||v||_{X_1}||v||_{X_1\cap X_2}. \tag{3.10.255}$$

The final task is to address the right-hand side:

$$\int\int \partial_{xxy} f\cdot v_x w_L^4 = \int\int f_{xx}u_{xx}w_L^4, \tag{3.10.256}$$
$$-\epsilon\int\int \partial_{xxx} g\cdot v_x w_L^4 = \int\int \epsilon g_{xx}\{v_{xx}w_L^4 + v_x 4 w_L^3 w_L'\}$$
$$= \int\int \epsilon g_{xx}v_{xx}w_L^4 + C\epsilon g_x\{v_{xx}w_L^3 w_L' + v_x\partial_{xx}\{w_L^4\}\}. \tag{3.10.257}$$

We estimate the terms:

$$|\int\int \varepsilon g_x\{v_{xx}w_L^3 w_L' + v_x\partial_{xx}\{w_L^4\}\}| \lesssim \int\int \varepsilon|g_x|\{|v_{xx}|w_L^3 + |v_x|w_L^2\}$$
$$\leq \int\int \varepsilon|g_x|\{|v_{xx}|w_3^3 + |v_x|w_3^2\}$$
$$\leq \int\int \varepsilon|g_x|\{|v_{xx}|w_2^3 + |v_x|w_2^2\} \leq \mathcal{W}_2. \tag{3.10.258}$$

A comparison now with the definitions in (3.10.219) gives the desired result upon taking $L \to \infty$.

$\square$

We now give the third-order positivity estimate, which is the final estimate for our linear analysis:

Proposition 3.10.8 (Third Order Positivity Estimate). *Let δ, ϵ be sufficiently small relative to universal constants. Then solutions $[u, v] \in Z(\Omega^N)$ to the system (3.8.1) - (3.8.3) satisfy:*

$$||\{\sqrt{\epsilon}v_{xxx}, v_{xxy}\}w_3^{\frac{5}{2}}||_{L^2}^2 \lesssim ||\{u_{xxy}, \sqrt{\epsilon}u_{xxx}, \epsilon v_{xxx}\}w_3^2||_{L^2}^2 + ||u, v||_{X_1 \cap X_2}^2 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3. \tag{3.10.259}$$

Proof. We apply the multiplier $v_{xx}w_{3,L}^5$. As usual, we shall drop the subscript-3 for this calculation, with the understanding that $w_{3,L} = w_L$.The highest-order terms are:

$$\begin{aligned}\int\int \{\partial_{yxx}(-\Delta_\epsilon u) + \epsilon\partial_{xxx}(\Delta_\epsilon v)\} \cdot v_{xx}w_L^5 \\ &= -\int\int u_{yxx}\partial_x w_L^5 + C\epsilon\int\int u_{xxx}^2\partial_x w_L^5 + C\epsilon^2 v_{xxx}^2\partial_x w_L^5 \\ &+ C\epsilon^2\int\int v_{xx}^2\partial_x^3\{w_L^5\}.\end{aligned} \tag{3.10.260}$$

Next, we have the main profile term from S_u, which we will list here for convenience:

$$\begin{aligned}\int\int \partial_y\partial_{xx}\Big[u_R u_x + u_{Rx}u + v_R u_y + u_{Ry}v\Big] \cdot v_{xx}w_L^5 \\ = \int\int \partial_{xx}\Big[u_R u_x + u_{Rx}u + v_R u_y + u_{Ry}v\Big] \cdot u_{xxx}w_L^5.\end{aligned} \tag{3.10.261}$$

We will now treat the first term from (3.10.261):

$$\int\int \partial_{xx}(u_R u_x)\cdot u_{xxx}w_L^5 = \int\int \Big(u_R u_{xxx} + 2u_{Rx}u_{xx} + u_{Rxx}u_x\Big)\cdot u_{xxx}w_L^5$$
$$\gtrsim \min u_R \int\int u_{xxx}^2 w_L^5 + \int\int u_{Rx}u_{xx}u_{xxx}w_L^5 + u_{Rxx}u_x u_{xxx}w_L^5. \tag{3.10.262}$$

Bounds on the final two integrals on the right-hand side above follow from (3.3.11), (3.3.18):

$$|\int\int u_{Rx}u_{xx}u_{xxx}w_L^5| \le ||u_{Rx}x||_{L^\infty}||u_{xx}w_L^{\frac{3}{2}}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\le C||u,v||_{X_2}^2 + \frac{1}{100,000}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}, \tag{3.10.263}$$
$$|\int\int u_{Rxx}u_x u_{xxx}w_L^5| \le ||u_{Rxx}x^2||_{L^\infty}||u_x x^{\frac{1}{2}}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\le C||u,v||_{X_1}^2 + \frac{1}{100,000}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}^2. \tag{3.10.264}$$

The u_{xxx} terms on the right-hand side above can be absorbed by the first term in (3.10.262). Here, we use that in the support of ρ_3, $C_1 x \le \rho_2(x) \le C_2 x$ so that u_{xx} from (3.10.262), for instance, does not require any of the degeneration from w_L. The next term from (3.10.261) is:

$$\int\int \partial_{xx}(u_{Rx}u)\cdot u_{xxx}w_L^5| = |\int\int \Big(u_{Rxxx}u + 2u_{Rxx}u_x + u_{Rx}u_{xx}\Big)\cdot u_{xxx}w_L^5.$$

We shall now split $u_R = u_R^P + u_R^E$. First, the u_R^P contribution, via estimate (3.3.11):

$$|\int\int \partial_x^k u_R^P \partial_x^i \cdot u_{xxx}w_L^5| \le ||\partial_x^k u_R^P x^{k-\frac{1}{2}}y||_{L^\infty}||\frac{\partial_x^{3-k}u}{y}x^{3-k}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\le ||\partial_x^k u_R^P x^{k-\frac{1}{2}}y||_{L^\infty}||\partial_x^{3-k}u_y x^{3-k}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\lesssim ||u||_{X_{4-k}}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2} \le C||u,v||_{X_1\cap X_2}^2 + \frac{1}{100,000}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}^2, \tag{3.10.265}$$

for $k = 3, 2$. It is worth distinguishing the $k = 1$ case, although the calculation is identical, by appealing to estimate (3.3.12):

$$|\int\int u^P_{Rx} u_{xx} u_{xxx} w_L^5| \le ||u^P_{Rx} y x^{\frac{1}{2}}||_{L^\infty} ||u_{xxy} w_L^2||_{L^2} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2}$$
$$\le \mathcal{O}(\delta) ||u_{xxy} w_L^2||_{L^2} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2}, \tag{3.10.266}$$

because now both majorizers are part of the X_3 norm: u_{xxy} has been estimated in the energy estimate, and u_{xxx} appears in (3.10.262). We may then absorb the u_{xxx} term from above into (3.10.262) by taking δ sufficiently small. Next, the Eulerian contribution, for which we use (3.3.18), first with $k = 1, 2$:

$$|\int\int \partial_x^k u^E_R \partial_x^{3-k} u \cdot u_{xxx} w_L^5| \le ||\partial_x^k u^E_R x^{k+\frac{1}{2}}||_{L^\infty} ||\partial_x^{3-k} u x^{3-k-\frac{1}{2}}||_{L^2} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2}$$
$$\le \sqrt{\varepsilon} ||u||_{X_{3-k}} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2}. \tag{3.10.267}$$

For the $k = 3$ case, we must add an extra step via Hardy's inequality:

$$|\int\int u^E_{Rxxx} u u_{xxx} w_L^5| \le ||u^E_{Rxxx} x^{\frac{7}{2}}||_{L^\infty} ||\frac{u}{x}||_{L^2} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2}$$
$$\lesssim \sqrt{\varepsilon} ||u_x||_{L^2} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2} \lesssim \sqrt{\varepsilon} ||u||_{X_1} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2}. \tag{3.10.268}$$

Next, we have the convection term (we set $k = 0, 1, 2$ below):

$$\int\int \partial_{xx}(u_{Ry} v) \cdot u_{xxx} w_L^5 = \sum_{k=0}^2 \int\int \partial_x^k u_{Ry} \partial_x^{2-k} v \cdot u_{xxx} w_L^5$$
$$= \sum_{k=0}^2 \int\int \partial_x^k \Big(u^P_{Ry} + \sqrt{\epsilon} u^E_{RY} \Big) \partial_x^{2-k} v \cdot u_{xxx} w_L^5. \tag{3.10.269}$$

First, the Prandtl contributions, using estimates (3.3.13):

$$|\int\int \partial_x^k u^P_{Ry}\partial_x^{2-k}v\cdot u_{xxx}w_L^5| \lesssim ||\partial_x^k u^P_{Ry} y x^k||_{L^\infty}||\frac{\partial_x^{2-k}v}{y}w_L^{3-k-\frac{1}{2}}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\lesssim ||\partial_x^k u^P_{Ry} y x^k||_{L^\infty}||\partial_x^{2-k}v_y w_L^{3-k-\frac{1}{2}}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}.$$

Again, we distinguish the $k=0$ case above, both majorizing terms are in X_3, and so we must use the smallness of $\mathcal{O}(\delta)$ to absorb into the (3.10.262) positive term. In particular, appealing to estimate (3.3.14), (3.3.16), we have:

$$||u^P_{Ry}y||_{L^\infty}||\partial_x^2 v_y w_L^{\frac{5}{2}}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2} \le \mathcal{O}(\delta)||\partial_x^2 v_y w_L^{\frac{5}{2}}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}. \tag{3.10.270}$$

For $k>0$, we have by using (3.3.11) - (3.3.13) and then Young's inequality:

$$||\partial_x^k u^P_{Ry} y x^k||_{L^\infty}||\partial_x^{2-k}v_y w_L^{3-k-\frac{1}{2}}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2} \lesssim ||u,v||_{X_1\cap X_2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\le C||u,v||^2_{X_1\cap X_2} + \frac{1}{100,000}||u_{xxx}w_L^{\frac{5}{2}}||^2_{L^2}. \tag{3.10.271}$$

For the Euler contributions u^E_R, we estimate for the $k=0,1$ cases, using (3.3.18):

$$|\int\int \sqrt{\epsilon}\partial_x^k u^E_{RY}\partial_x^{2-k}v u_{xxx}w_L^5| \lesssim ||\partial_x^k u^E_{RY}x^{k+1+\frac{1}{2}}||_{L^\infty}||\sqrt{\epsilon}\partial_x^{2-k}v w_L^{2-k-\frac{1}{2}}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\lesssim \sqrt{\varepsilon}||v||_{X_{2-k}}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}. \tag{3.10.272}$$

For the $k=2$ case, we must additionally use the Hardy inequality:

$$|\int\int \sqrt{\epsilon}u^E_{RYxx}v u_{xxx}w_L^5| \lesssim ||u^E_{RYxx}x^{\frac{7}{2}}||_{L^\infty}||\sqrt{\epsilon}\frac{v}{x}||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\lesssim ||u^E_{RYxx}x^{\frac{7}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x||_{L^2}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}$$
$$\lesssim \sqrt{\varepsilon}||v||_{X_1}||u_{xxx}w_L^{\frac{5}{2}}||_{L^2}. \tag{3.10.273}$$

The final term in S_u is easily estimated directly, where $k = 0, 1, 2$, by applying estimates (3.3.8) - (3.3.10), and (3.3.17), (3.3.20), crucially obtaining the factor $\mathcal{O}(\delta)$ when $k = 0$:

$$\Big| \int\int \partial_{xx}(v_R u_y) \cdot u_{xxx} w_L^5 \Big| \lesssim ||\partial_x^k v_R x^{k+\frac{1}{2}}||_{L^\infty} ||\partial_x^{2-k} u_y w_L^{2-k}||_{L^2} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2} \quad (3.10.274)$$
$$\lesssim \mathcal{O}(\delta) ||\partial_x^2 u_y w_L^2||_{L^2} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2} + \frac{1}{100,000} ||u_{xxx} w_L^{\frac{5}{2}}||_{L^2}^2 + C||u, v||_{X_1 \cap X_2}^2.$$

We now address the profile terms from S_v. We shall record these below for convenience:

$$\int\int -\epsilon \partial_x \partial_{xx} \Big[u_R v_x + v_{Rx} u + v_R v_y + v_{Ry} v \Big] \cdot v_{xx} w_L^5$$
$$= \int\int \epsilon \partial_{xx} \Big[u_R v_x + v_{Rx} u + v_R v_y + v_{Ry} v \Big] \cdot \Big[v_{xxx} w_L^5 + v_{xx} \partial_x w_L^5 \Big]. \quad (3.10.275)$$

Let us start with the first term above, which yields the desired positivity:

$$\int\int \epsilon \partial_{xx}(u_R v_x) \cdot (v_{xxx} w_L^5 + v_{xx} \partial_x w_L^5)$$
$$= \int\int \epsilon u_R v_{xxx}^2 w_L^5 + \sum_{k=1}^{2} \sum_{i=2}^{3} \epsilon \partial_x^k u_R \partial_x^{3-k} v \cdot \partial_x^i v \partial_x^{3-i} w_L^5$$
$$\geq \min u_R \int\int \epsilon v_{xxx}^2 w_L^5 + \sum_{k=1}^{2} \sum_{i=2}^{3} \epsilon \partial_x^k u_R \partial_x^{3-k} v \cdot \partial_x^i v \partial_x^{3-i} w_L^5. \quad (3.10.276)$$

We estimate the summation on the right-hand side above, by using (3.3.11) - (3.3.12), (3.3.18)

$$| \int\int \epsilon \partial_x^k u_R \partial_x^{3-k} v \cdot \partial_x^i v \partial_x^{3-i} w_L^5 ||$$
$$\leq ||\partial_x^k u_R x^k||_{L^\infty} ||\sqrt{\epsilon} \partial_x^{3-k} v x^{3-k-\frac{1}{2}}||_{L^2} ||\sqrt{\epsilon} \partial_x^i v w_L^{i-\frac{1}{2}}||_{L^2}$$
$$\lesssim ||v||_{X_{3-k}} ||v||_{X_i} \leq C||u, v||_{X_1 \cap X_2}^2 + \frac{1}{100,000} ||\sqrt{\varepsilon} v_{xxx} w_L^{\frac{5}{2}}||_{L^2}^2 \quad (3.10.277)$$

When $i = 3$, one obtains the top norm X_3 above, and must be absorbed into the positive term from (3.10.276), which is done via Young's inequality. The next profile term follows by using the bounds in (3.3.8) - (3.3.10) for v_R^P and (3.3.17) for v_R^E:

$$
\begin{aligned}
&|\int\int \epsilon\partial_{xx}(v_{Rx}u)\cdot(v_{xxx}w_L^5 + v_{xx}\partial_x w_L^5)| \\
&\qquad \le \sum_{k=0}^{2}\sum_{i=2}^{3}|\int\int \epsilon\partial_x^{k+1}v_R\partial_x^{2-k}u\partial_x^i v\partial_x^{3-i}w_L^5| \\
&\qquad \lesssim \sqrt{\epsilon}\sum_{k=0}^{2}\sum_{i=2}^{3}||\partial_x^{k+1}v_R x^{k+\frac{3}{2}}||_{L^\infty}||\partial_x^{2-k}u x^{1-k}||_{L^2}||\sqrt{\epsilon}\partial_x^i v\cdot w_L^{i-\frac{1}{2}}||_{L^2} \\
&\qquad \lesssim \sqrt{\epsilon}||u,v||^2_{X_1\cap X_2} + \sqrt{\varepsilon}||\sqrt{\varepsilon}\partial_x^3 v w_L^{\frac{5}{2}}||^2_{L^2}. \qquad (3.10.278)
\end{aligned}
$$

Above, we have used the Hardy inequality in the case when $k = 2$, for the term:

$$
||\frac{u}{x}||_{L^2} \lesssim ||u_x||_{L^2} \lesssim ||u||_{X_1}. \qquad (3.10.279)
$$

The third profile term requires a splitting into Euler and Prandtl components:

$$
\begin{aligned}
&\int\int \epsilon\partial_{xx}(v_{Ry}v)\cdot(v_{xxx}w_L^5 + v_{xx}\partial_x w_L^5) \\
&\qquad = \sum_{k=0}^{2}\sum_{i=2}^{3}\int\int \epsilon\partial_x^k v_{Ry}\partial_x^{2-k}v\cdot\partial_x^i v\partial_x^{3-i}w_L^5 \\
&\qquad = \sum_{k=0}^{2}\sum_{i=2}^{3}\int\int \epsilon\partial_x^k\Big(v_{Ry}^P + \sqrt{\epsilon}v_{RY}^E\Big)\partial_x^{2-k}v\cdot\partial_x^i v\partial_x^{3-i}w_L^5 \qquad (3.10.280)
\end{aligned}
$$

First, according to (3.3.8) - (3.3.10),

$$
\begin{aligned}
&|\int\int \epsilon\partial_x^k v_{Ry}^P\partial_x^{2-k}v\partial_x^i v\partial_x^{3-i}w_L^5| \\
&\qquad \le \sqrt{\epsilon}||\partial_x^k v_{Ry}^P y x^{k+\frac{1}{2}}||_{L^\infty}||\frac{\partial_x^{2-k}v}{y}w_L^{3-k-\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}\partial_x^i v w_L^{i-\frac{1}{2}}||_{L^2} \\
&\qquad \lesssim \sqrt{\epsilon}||\partial_x^k v_{Ry}^P y x^{k+\frac{1}{2}}||_{L^\infty}||\partial_x^{2-k}v_y w_L^{3-k-\frac{1}{2}}||_{L^2}||\sqrt{\epsilon}\partial_x^i v w_L^{i-\frac{1}{2}}||_{L^2}
\end{aligned}
$$

$$\lesssim \sqrt{\epsilon}||u,v||^2_{X_1\cap X_2} + \sqrt{\varepsilon}||\sqrt{\varepsilon}\partial_x^3 v w_L^{\frac{5}{2}}||^2_{L^2} + \sqrt{\varepsilon}||\partial_x^2 v_y w_L^{\frac{5}{2}}||^2_{L^2}. \quad (3.10.281)$$

Next, according to (3.3.17),

$$\begin{aligned} &|\int\int \epsilon^{\frac{3}{2}}\partial_x^k v^E_{RY}\partial_x^{2-k}v\partial_x^i v\partial_x^{3-i}w_L^5| \\ &\qquad \le \sqrt{\epsilon}||\partial_x^k v^E_{RY}x^{k+\frac{3}{2}}||_{L^\infty}||\sqrt{\epsilon}\partial_x^{2-k}vw_L^{1-k}||_{L^2}||\sqrt{\epsilon}\partial_x^i v w_L^{i-\frac{1}{2}}||_{L^2} \\ &\qquad \lesssim \sqrt{\epsilon}||u,v||^2_{X_1\cap X_2} + \sqrt{\varepsilon}||\sqrt{\varepsilon}\partial_x^3 v w_L^{\frac{5}{2}}||^2_{L^2}. \end{aligned} \quad (3.10.282)$$

Above, we have used the Hardy inequality in the case when $k=2$ for the term:

$$||\sqrt{\epsilon}\frac{v}{x}||_{L^2} \lesssim ||\sqrt{\epsilon}v_x||_{L^2} \lesssim ||v||_{X_1}. \quad (3.10.283)$$

The final profile term is handled by appealing to estimates (3.3.8) - (3.3.10) and (3.3.17):

$$\begin{aligned} &|\int\int \epsilon\partial_{xx}(v_R v_y)\cdot\Big(v_{xxx}w_L^5 + v_{xx}\partial_x w_L^5\Big)| \\ &\qquad \le \varepsilon\sum_{k=0}^{2}\sum_{i=2}^{3}|\int\int \epsilon\partial_x^k v_R\partial_x^{2-k}v_y\partial_x^i v\partial_x^{3-i}w_L| \\ &\qquad \le \varepsilon\sum_{k=0}^{2}\sum_{i=2}^{3}||\partial_x^k v_R x^{k+\frac{1}{2}}||_{L^\infty}||\partial_x^{2-k}v_y w_L^{3-k-\frac{1}{2}}||_{L^2}||\partial_x^i v w_L^{i-\frac{1}{2}}||_{L^2} \\ &\qquad \lesssim ||u,v||^2_{X_1\cap X_2} + \sqrt{\varepsilon}||\partial_x^2 v_y w_L^{\frac{5}{2}}||^2_{L^2} + \sqrt{\varepsilon}||\sqrt{\varepsilon}\partial_x^3 v w_L^{\frac{5}{2}}||^2_{L^2}. \end{aligned} \quad (3.10.284)$$

Finally, we have the right-hand side:

$$\int\int \partial_{xxy}f\cdot v_{xx}w_L^5 = -\int\int f_{xx}\cdot v_{xxy}w_L^5, \quad (3.10.285)$$

$$-\int\int \epsilon\partial_{xxx}g\cdot v_{xx}w_L^5 = \int\int \epsilon g_{xx}\{v_{xxx}w_L^5 + 5v_{xx}w_L^4 w_L'\}. \quad (3.10.286)$$

We estimate:

$$|\int\int \varepsilon g_{xx} v_{xx} w_L^4 w_L'| \le \int\int \varepsilon |g_{xx}||v_{xx}|w_L^4 \le \int\int \varepsilon |g_{xx}||v_{xx}|w_3^4 \le \mathcal{W}_3. \tag{3.10.287}$$

We end by taking $L \to \infty$. A comparison with (3.10.219) shows that our claim is proven.

□

Summarizing, then, the conclusions of the linear analysis:

Theorem 3.10.9 (Linear Estimates)**.** *Let ε, δ be sufficiently small relative to universal constants, and let $\varepsilon << \delta$. Then $[u, v] \in Z$, solutions to the system (3.8.1) - (3.8.3), (3.8.5) - (3.8.9), with boundary conditions (3.8.11), satisfy the following a-priori estimate:*

$$||u, v||^2_{X_1 \cap X_2 \cap X_3} \lesssim \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3. \tag{3.10.288}$$

3.11 Nonlinear Analysis

3.11.1 a-priori Estimate of Nonlinearities

In this subsection, we exhibit control of the right-hand side of (3.10.288). We continue to consider the system (in a manner independent of N):

$$\left.\begin{aligned} -\Delta_\varepsilon u + S_u + P_x &= \varepsilon^{-\frac{n}{2}-\gamma} R^{u,n} + \varepsilon^{\frac{n}{2}+\gamma}\Big[\bar{u}\bar{u}_x + \bar{v}u_y\Big] \\ -\Delta_\varepsilon v + S_v + \frac{P_y}{\varepsilon} &= \varepsilon^{-\frac{n}{2}-\gamma} R^{v,n} + \varepsilon^{\frac{n}{2}+\gamma}\Big[\bar{u}\bar{v}_x + \bar{v}\bar{v}_y\Big], \\ u_x + v_y &= 0. \end{aligned}\right\} \quad \text{in } \Omega^N \tag{3.11.1}$$

That is, $f = f(u, \bar{u}, \bar{v})$ and $g = g(\bar{u}, \bar{v})$, as in (3.8.8), (3.8.9). Notationally, we continue to depict integration over Ω^N by $\int\int$ and similarly, norms without further specification are taken over Ω^N. For the forthcoming calculation, we refer the reader to the definitions of $\mathcal{W}^i$, in equations (3.10.10), (3.10.82), (3.10.219).

Lemma 3.11.1. *Suppose $||\bar{u},\bar{v}||_Z \le 1$. For $0 \le \gamma < \frac{1}{4}$, and fixed parameters $\kappa > 0$ arbitrarily small, for δ, ϵ sufficiently small, n sufficiently large:*

$$\Big|\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3\Big| \lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa} + \epsilon^{\frac{1}{4}-\gamma-\kappa}||u,v||^2_{X_1\cap X_2\cap X_3} + \epsilon^{\frac{n}{2}-\omega(N_i)}||u,v||^2_Z + \epsilon^{\frac{n}{2}-\omega(N_i)}||\bar{u},\bar{v}||^4_Z, \tag{3.11.2}$$

where $\omega(N_i)$ is a function which depends only on universal constants and N_i in the definition of norm Z.

Proof. For clarity of exposition, the order in which we treat the terms are as follows: we will first treat the nonlinear terms, $\mathcal{N}^u$, arising from f in $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$, then those nonlinear terms, $\mathcal{N}^v$, arising from g, and finally the forcing terms, $R^{u,n}, R^{v,n}$, in both f and g. Turning first to $\mathcal{W}_1$, equation (3.10.10), we start with the term:

$$\int\int \mathcal{N}^u \cdot u = \int\int \varepsilon^{\frac{n}{2}+\gamma}\Big(\bar{u}\bar{u}_x + \bar{v}u_y\Big)\cdot u. \tag{3.11.3}$$

First,

$$\begin{aligned} \Big|\int\int \varepsilon^{\frac{n}{2}+\gamma}\bar{u}\bar{u}_x \cdot u\Big| &\lesssim \varepsilon^{\frac{n}{2}+\gamma}||\bar{u}x^{\frac{1}{4}}||_{L^\infty}||\bar{u}_x x^{\frac{1}{2}}||_{L^2}||\frac{u}{x^{\frac{3}{4}}}||_{L^2} \\ &\lesssim \varepsilon^{\frac{n}{2}+\gamma}||\bar{u}x^{\frac{1}{4}}||_{L^\infty}||\bar{u}_x x^{\frac{1}{2}}||_{L^2}||u_x x^{\frac{1}{4}}||_{L^2} \\ &\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||^2_Z||u,v||_Z. \end{aligned} \tag{3.11.4}$$

For the next term in (3.11.3), we must use the structure of the nonlinearity by integrating by parts once in y:

$$\Big| \int\int \varepsilon^{\frac{n}{2}+\gamma} \bar{v} u_y \cdot u \Big| = \Big| \int\int \varepsilon^{\frac{n}{2}+\gamma} \bar{v}_y \frac{u^2}{2} \Big| \tag{3.11.5}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma} ||ux^{\frac{1}{4}}||_{L^\infty} ||\bar{v}_y x^{\frac{1}{2}}||_{L^2} ||ux^{-\frac{3}{4}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma} ||ux^{\frac{1}{4}}||_{L^\infty} ||u_x x^{\frac{1}{4}}||_{L^2} ||\bar{v}_y x^{\frac{1}{2}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||u,v||_Z^2 ||\bar{u},\bar{v}||_Z.$$

Note carefully that no absolute values were included on this term in the definition of $\mathcal{W}_1$, equation (3.10.10). We then move to the next term from (3.10.10), which we display below:

$$\int\int |\mathcal{N}^u||v_y|x \le \int\int \varepsilon^{\frac{n}{2}+\gamma} |\bar{u}\bar{u}_x + \bar{v}u_y||v_y|x. \tag{3.11.6}$$

The nonlinear terms here are treated via:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma} |\bar{u}\bar{u}_x v_y x| \le \varepsilon^{\frac{n}{2}+\gamma} ||\bar{u}||_{L^\infty} ||\bar{u}_x x^{\frac{1}{2}}||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2},$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^2 ||u,v||_Z, \tag{3.11.7}$$
$$\int\int \varepsilon^{\frac{n}{2}+\gamma} |\bar{v}u_y v_y x| \le \varepsilon^{\frac{n}{2}+\gamma} ||\bar{v}x^{\frac{1}{2}}||_{L^\infty} ||u_y||_{L^2} ||v_y x^{\frac{1}{2}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z ||u,v||_Z^2. \tag{3.11.8}$$

Staying with $\mathcal{N}^u$, we will move to $\mathcal{W}_2$ in (3.10.82):

$$\int\int |\partial_x \mathcal{N}^u| \Big[|u_x|\rho_2^2 x^2 + |u_{xx}|\rho_2^3 x^3 \Big]$$
$$= \int\int \varepsilon^{\frac{n}{2}+\gamma} |\partial_x(\bar{u}\bar{u}_x + \bar{v}u_y)| \Big[|u_x|\rho_2^2 x^2 + |u_{xx}|\rho_2^3 x^3 \Big]. \tag{3.11.9}$$

First, we will expand:

$$\begin{aligned}
\int\int \varepsilon^{\frac{n}{2}+\gamma}|\partial_x\Big(\bar{u}\bar{u}_x\Big)||u_x|\rho_2^2x^2& \\
&= \int\int \varepsilon^{\frac{n}{2}+\gamma}|\bar{u}_x^2||u_x|\rho_2^2x^2 + \int\int \varepsilon^{\frac{n}{2}+\gamma}|\bar{u}\bar{u}_{xx}||u_x|\rho_2^2x^2 \\
&\le \varepsilon^{\frac{n}{2}+\gamma}||u_x x^{\frac{5}{4}}||_{L^\infty(x\ge 20)}||\bar{u}_x x^{\frac{1}{2}}||_{L^2}^2 + \varepsilon^{\frac{n}{2}+\gamma}||\bar{u}||_{L^\infty}||u_x x^{\frac{1}{2}}||_{L^2}||\bar{u}_{xx}x^{\frac{3}{2}}||_{L^2} \\
&\le \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||u,v||_Z||\bar{u},\bar{v}||_Z^2. && (3.11.10)
\end{aligned}$$

Above, we have used that the support of ρ_2 is when $x \ge 50$. Next,

$$\begin{aligned}
\int\int \varepsilon^{\frac{n}{2}+\gamma}|\partial_x\Big(\bar{u}\bar{u}_x\Big)||u_{xx}|\rho_2^3x^3 &= \int\int \varepsilon^{\frac{n}{2}+\gamma}|\{\bar{u}_x^2+\bar{u}\bar{u}_{xx}\}|\cdot|u_{xx}|\rho_2^3x^3 \\
&\le ||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x\ge 20)}||\bar{u}_x x^{\frac{1}{2}}||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} + ||\bar{u}x^{\frac{1}{4}}||_{L^\infty}||\bar{u}_{xx}x^{\frac{3}{2}}||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||u,v||_Z||\bar{u},\bar{v}||_Z^2. && (3.11.11)
\end{aligned}$$

Next, the second nonlinearity in (3.11.9):

$$\begin{aligned}
\int\int \varepsilon^{\frac{n}{2}+\gamma}|\partial_x\Big(\bar{v}u_y\Big)||u_x|\rho_2^2x^2 &= \int\int \varepsilon^{\frac{n}{2}+\gamma}|\{\bar{v}_x u_y+\bar{v}u_{xy}\}||u_x|\rho_2^2x^2 \\
&\le \varepsilon^{\frac{n}{2}+\gamma}||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x\ge 20)}||u_y||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} + \varepsilon^{\frac{n}{2}+\gamma}||\bar{v}x^{\frac{1}{2}}||_{L^\infty}||u_{xy}x||_{L^2}||u_x x^{\frac{1}{2}}||_{L^2} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||u,v||_Z^2. && (3.11.12)
\end{aligned}$$

For this same nonlinearity:

$$\begin{aligned}
\int\int \varepsilon^{\frac{n}{2}+\gamma}|\{\bar{v}u_{xy}+\bar{v}_x u_y\}||u_{xx}|\rho_2^3x^3& \\
&\le \varepsilon^{\frac{n}{2}+\gamma}||\bar{v}x^{\frac{1}{2}}||_{L^\infty}||u_{xy}x||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} + \varepsilon^{\frac{n}{2}+\gamma}||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x\ge 20)}||u_y||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||u,v||_Z^2. && (3.11.13)
\end{aligned}$$

We'll now move to the $\mathcal{N}^u$ terms in $\mathcal{W}_3$ (see (3.10.219)), which are the most delicate. These terms are:

$$\int\int \partial_{xx}\mathcal{N}^u\Big[|u_{xx}\rho_3^4 x^4 + |u_{xxx}|^2\rho_3^5 x^5\Big] \\ = \int\int \varepsilon^{\frac{n}{2}+\gamma}\partial_{xx}\Big(\bar{u}\bar{u}_x + \bar{v}u_y\Big)\cdot\Big[|u_{xx}|\rho_3^4 x^4 + |u_{xxx}|^2\rho_3^5 x^5\Big]. \tag{3.11.14}$$

First,

$$\begin{aligned}\int\int \varepsilon^{\frac{n}{2}+\gamma}|\partial_{xx}\Big(\bar{u}\bar{u}_x\Big)||u_{xx}|\rho_3^4 x^4 &= \int\int \varepsilon^{\frac{n}{2}+\gamma}|\{\bar{u}\bar{u}_{xxx} + 3\bar{u}_x\bar{u}_{xx}\}||u_{xx}|\rho_3^4 x^4 \\ &\leq \varepsilon^{\frac{n}{2}+\gamma}||\bar{u}||_{L^\infty}||\bar{u}_{xxx}x^{\frac{5}{2}}||_{L^2(x\geq 20)}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\ &\quad + \varepsilon^{\frac{n}{2}+\gamma}||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x\geq 20)}||\bar{u}_{xx}x^{\frac{3}{2}}||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\ &\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \end{aligned} \tag{3.11.15}$$

Above, we have used that ρ_3 is supported in a strict subset of $\{x \geq 20\}$, which in turn is supported in a strict subset of ζ_3, which appears in the norm Z (see (3.9.1) - (3.9.2)). Referring back to (3.11.14), for this same nonlinear term:

$$\begin{aligned}&\int\int \varepsilon^{\frac{n}{2}+\gamma}|\{\bar{u}\bar{u}_{xxx} + 3\bar{u}_x\bar{u}_{xx}\}||u_{xxx}|\rho_3^5 x^5 \\ &\leq \varepsilon^{\frac{n}{2}+\gamma}||\bar{u}||_{L^\infty}||\bar{u}_{xxx}x^{\frac{5}{2}}||_{L^2(x\geq 20)}||u_{xxx}x^{\frac{5}{2}}||_{L^2(x\geq 20)} \\ &\quad + \varepsilon^{\frac{n}{2}+\gamma}||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x\geq 20)}||\bar{u}_{xx}x^{\frac{3}{2}}||_{L^2}||u_{xxx}x^{\frac{5}{2}}||_{L^2(x\geq 20)} \\ &\leq \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \end{aligned} \tag{3.11.16}$$

We now move to the final nonlinearity contributed by $\mathcal{N}^u$, which is the $\partial_{xx}\{vu_y\}$ term from (3.11.14). This term is the most delicate to control. First, expanding yields:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma}|\partial_{xx}\Big(\bar{v}u_y\Big)||u_{xx}|\rho_3^4 x^4 = \int\int \varepsilon^{\frac{n}{2}+\gamma}|\{\bar{v}u_{xxy} + 2\bar{v}_x u_{xy} + \bar{v}_{xx}u_y\}||u_{xx}|\rho_3^4 x^4 \tag{3.11.17}$$

First,

$$\begin{aligned}\int\int \varepsilon^{\frac{n}{2}+\gamma}|\bar{v}u_{xxy}||u_{xx}|\rho_3^4 x^4 &\leq \varepsilon^{\frac{n}{2}+\gamma}||\bar{v}x^{\frac{1}{2}}||_{L^\infty}||u_{xxy}x^2||_{L^2(x\geq 20)}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\ &\leq \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||u,v||_Z^2.\end{aligned} \tag{3.11.18}$$

Second,

$$\begin{aligned}\int\int \varepsilon^{\frac{n}{2}+\gamma}|\bar{v}_x u_{xy}||u_{xx}|\rho_3^4 x^4 &\leq \varepsilon^{\frac{n}{2}+\gamma}||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x\geq 20)}||u_{xy}x||_{L^2}||u_{xx}x^{\frac{3}{2}}||_{L^2} \\ &\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||u,v||_Z^2||\bar{u},\bar{v}||_Z.\end{aligned} \tag{3.11.19}$$

Finally,

$$\begin{aligned}\int\int \varepsilon^{\frac{n}{2}+\gamma}|\bar{v}_{xx}u_y||u_{xx}|\rho_3^4 x^4 \leq& \varepsilon^{\frac{n}{2}+\gamma}\Big[\sup_{x\geq 20}||u_y x^{\frac{1}{2}}||_{L^2_y}\Big]||u_{xx}x^{\frac{3}{2}}||_{L^2}\Big[\int_{x=20} x^4||\bar{v}_{xx}||^2_{L^\infty_y}\Big]^{\frac{1}{2}} \\ &\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||u,v||_Z^2.\end{aligned} \tag{3.11.20}$$

Turning back to (3.11.14), for this same nonlinearity, it remains to treat:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma}|\{\bar{v}u_{xxy} + 2\bar{v}_x u_{xy} + \bar{v}_{xx}u_y\}||u_{xxx}|\rho_3^5 x^5. \tag{3.11.21}$$

First,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma}|\bar{v}u_{xxy}||u_{xxx}|\rho_3^5 x^5 \leq \varepsilon^{\frac{n}{2}+\gamma}||\bar{v}x^{\frac{1}{2}}||_{L^\infty}||u_{xxy}x^2||_{L^2(x\geq 20)}||u_{xxx}x^{\frac{5}{2}}||_{L^2(x\geq 20)}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||u,v||_Z^2. \tag{3.11.22}$$

Second,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma}|\bar{v}_x u_{xy}||u_{xxx}|\rho_3^5 x^5 \leq \varepsilon^{\frac{n}{2}+\gamma}||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x\geq 20)}||u_{xy}x||_{L^2}||u_{xxx}x^{\frac{5}{2}}||_{L^2(x\geq 20)}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||u,v||_Z^2. \tag{3.11.23}$$

Last,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma}|\bar{v}_{xx}u_y||u_{xxx}|\rho_3^5 x^5$$
$$\leq \varepsilon^{\frac{n}{2}+\gamma}\sup_{x\geq 20}||u_y x^{\frac{1}{2}}||_{L^2_y}||u_{xxx}x^{\frac{5}{2}}||_{L^2(x\geq 20)}\Big[\int_{x=20}^\infty x^4||\bar{v}_{xx}||_{L^\infty}^2\,\mathrm{d}x\Big]^{\frac{1}{2}}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||u,v||_Z^2. \tag{3.11.24}$$

We now move to the nonlinear terms, $\mathcal{N}^v$, from g, defined in (3.8.9), starting with $\mathcal{W}_1$, equation (3.10.10):

$$\int\int \varepsilon|\mathcal{N}^v|\Big[|v|+|v_x|x\Big] = \int\int \varepsilon\Big[\varepsilon^{\frac{n}{2}+\gamma}|\bar{u}\bar{v}_x|+|\bar{v}\bar{v}_y|\Big]\Big[|v|+|v_x|x\Big]. \tag{3.11.25}$$

First,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{u}\bar{v}_x|\cdot|v| \lesssim \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{u}x^{\frac{1}{4}}||_{L^\infty}||\bar{v}_x x^{\frac{1}{2}}||_{L^2}||x^{-\frac{3}{4}}v||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma}||\bar{u}x^{\frac{1}{4}}||_{L^\infty}||\sqrt{\varepsilon}\bar{v}_x x^{\frac{1}{2}}||_{L^2}||\sqrt{\varepsilon}v_x x^{\frac{1}{4}}||_{L^2},$$

$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z, \tag{3.11.26}$$

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{v}\bar{v}_y|\cdot|v| \lesssim \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{v}x^{\frac{1}{2}}||_{L^\infty}||\bar{v}_y x^{\frac{1}{2}}||_{L^2}||v_x||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.27}$$

Next,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{u}\bar{v}_x||v_x|x \leq \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{u}||_{L^\infty}||\bar{v}_x x^{\frac{1}{2}}||_{L^2}||v_x x^{\frac{1}{2}}||_{L^2},$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z, \tag{3.11.28}$$
$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{v}\bar{v}_y||v_x|x \leq \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{v}x^{\frac{1}{2}}||_{L^\infty}||\bar{v}_y x^{\frac{1}{2}}||_{L^2}||v_x x^{\frac{1}{2}}||_{L^2},$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||u,v||_Z||\bar{u},\bar{v}||_Z^2. \tag{3.11.29}$$

We will now move to the nonlinear terms, $\mathcal{N}^v$, arising from $\mathcal{W}_2$, (see equation (3.10.82)), which are summarized here:

$$\int\int \varepsilon|\partial_x\mathcal{N}^v|\Big[|v_x|\rho_2^2x^2+|v_{xx}|\rho_2^3x^3\Big]$$
$$=\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\partial_x\Big[\bar{u}\bar{v}_x+\bar{v}\bar{v}_y\Big]|\Big[|v_x|\rho_2^2x^2+|v_{xx}|\rho_2^3x^3\Big] \tag{3.11.30}$$

We will go through (3.11.30) term by term, starting with:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\partial_x\Big(\bar{u}\bar{v}_x\Big)||v_x|\rho_2^2x^2 = \int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\{\bar{u}_x\bar{v}_x+\bar{u}\bar{v}_{xx}\}|\cdot|v_x|\rho_2^2x^2$$
$$\leq \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x\geq 20)}||\bar{v}_x x^{\frac{1}{2}}||_{L^2}||v_x x^{\frac{1}{2}}||_{L^2}$$
$$+\varepsilon^{\frac{n}{2}+\gamma+1}||\bar{u}||_{L^\infty}||\bar{v}_{xx}x^{\frac{3}{2}}||_{L^2}||v_x x^{\frac{1}{2}}||_{L^2},$$
$$\leq \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.31}$$

Staying with this nonlinearity,

$$\begin{aligned}
\int\int \varepsilon^{\frac{n}{2}+\gamma+1} &|\{\bar{u}_x \bar{v}_x + \bar{u}\bar{v}_{xx}\}| \cdot |v_{xx}| \rho_2^3 x^3 \\
&\le \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x \ge 20)} ||\bar{v}_x x^{\frac{1}{2}}||_{L^2} ||v_{xx} x^{\frac{3}{2}}||_{L^2} \\
&\quad + \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{u}||_{L^\infty} ||\bar{v}_{xx} x^{\frac{3}{2}}||_{L^2} ||v_{xx} x^{\frac{3}{2}}||_{L^2} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^2 ||u,v||_Z. \qquad (3.11.32)
\end{aligned}$$

We now move to the vv_y nonlinearity, still in term (3.11.30), which we expand:

$$\begin{aligned}
\int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\partial_x \Big(\bar{v}\bar{v}_y\Big)| |v_x| \rho_2^2 x^2 &= \int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\{\bar{v}_x \bar{v}_y + \bar{v}\bar{v}_{xy}\}| |v_x| \rho_2^2 x^2 \\
&\le \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x \ge 20)} ||\bar{v}_y x^{\frac{1}{2}}||v_x x^{\frac{1}{2}}||_{L^2} \\
&\quad + \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{v} x^{\frac{1}{2}}||_{L^\infty} ||v_x x^{\frac{1}{2}}||_{L^2} ||\bar{v}_{xy} x^{\frac{3}{2}}||_{L^2} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^2 ||u,v||_Z. \qquad (3.11.33)
\end{aligned}$$

For this same nonlinear term:

$$\begin{aligned}
\int\int \varepsilon^{\frac{n}{2}+\gamma+1} &|\{\bar{v}_x \bar{v}_y + \bar{v}\bar{v}_{xy}\}| \cdot |v_{xx}| \rho_2^3 x^3 \\
&\le \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x \ge 20)} ||\bar{v}_y x^{\frac{1}{2}}||_{L^2} ||v_{xx} x^{\frac{3}{2}}||_{L^2} \\
&\quad + \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{v} x^{\frac{1}{2}}||_{L^\infty} ||\bar{v}_{xy} x^{\frac{3}{2}}||_{L^2} ||v_{xx} x^{\frac{3}{2}}||_{L^2} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||u,v||_Z ||\bar{u},\bar{v}||_Z^2. \qquad (3.11.34)
\end{aligned}$$

We now move to the highest-order terms, which we read from (3.10.219), and summarize here:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\partial_{xx} \mathcal{N}^v| \Big[|v_{xx}| \rho_3^4 x^4 + |v_{xxx}| \rho_3^5 x^5 \Big]$$

$$= \int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\partial_{xx}\Big[\bar{u}\bar{v}_x + \bar{v}\bar{v}_y\Big]| \Big[|v_{xx}|\rho_3^4 x^4 + |v_{xxx}|\rho_3^5 x^5\Big]. \tag{3.11.35}$$

We shall expand:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\partial_{xx}\Big(\bar{u}\bar{v}_x\Big)| \cdot |v_{xx}|\rho_3^4 x^4$$
$$= \int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\{\bar{u}\bar{v}_{xxx} + 2\bar{u}_x\bar{v}_{xx} + \bar{u}_{xx}\bar{v}_x\}| \cdot |v_{xx}|\rho_3^4 x^4. \tag{3.11.36}$$

First,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\bar{u}\bar{v}_{xxx}| \cdot |v_{xx}|\rho_3^4 x^4 \le \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{u}||_{L^\infty} ||\bar{v}_{xxx} x^{\frac{5}{2}}||_{L^2(x\ge 20)} ||v_{xx} x^{\frac{3}{2}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^2 ||u,v||_Z. \tag{3.11.37}$$

Next,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\bar{u}_x\bar{v}_{xx}| \cdot |v_{xx}|\rho_3^4 x^4 \le \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x\ge 20)} ||\bar{v}_{xx} x^{\frac{3}{2}}||_{L^2} ||v_{xx} x^{\frac{3}{2}}||_{L^2}$$
$$\le \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^2 ||u,v||_Z. \tag{3.11.38}$$

Third,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\bar{v}_x\bar{u}_{xx}||v_{xx}|\rho_3^4 x^4 \le \varepsilon^{\frac{n}{2}+\gamma+1} ||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x\ge 20)} ||\bar{u}_{xx} x^{\frac{3}{2}}||_{L^2} ||\bar{v}_{xx} x^{\frac{3}{2}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^2 ||u,v||_Z. \tag{3.11.39}$$

Turning back to (3.11.35), for this same nonlinearity, we will now treat:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1} |\{\bar{u}\bar{v}_{xxx} + 2\bar{u}_x\bar{v}_{xx} + \bar{u}_{xx}\bar{v}_x\}||v_{xxx}|\rho_3^5 x^5 \tag{3.11.40}$$

First,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{u}\bar{v}_{xxx}||v_{xxx}|\rho_3^5 x^5 \le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{u}||_{L^\infty}||\bar{v}_{xxx}x^{\frac{5}{2}}||_{L^2(x\ge 20)}||v_{xxx}x^{\frac{5}{2}}||_{L^2(x\ge 20)}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.41}$$

Second,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{u}_x\bar{v}_{xx}||v_{xxx}|\rho_3^5 x^5 \le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x\ge 20)}||\bar{v}_{xx}x^{\frac{3}{2}}||_{L^2}||v_{xxx}x^{\frac{5}{2}}||_{L^2(x\ge 20)}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.42}$$

Third,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{u}_{xx}\bar{v}_x||v_{xxx}|\rho_3^5 x^5 \le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x\ge 20)}||\bar{u}_{xx}x^{\frac{3}{2}}||_{L^2}||v_{xxx}x^{\frac{5}{2}}||_{L^2(x\ge 20)}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.43}$$

Turning to (3.11.35), we now approach the final nonlinear term in g (see definition (3.8.9)), which is the $\bar{v}\bar{v}_y$ term. First, we will expand:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\partial_{xx}\Big(\bar{v}\bar{v}_y\Big)|\cdot|v_{xx}|\rho_3^4 x^4$$
$$= \int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\{\bar{v}_{xx}\bar{v}_y+\bar{v}_x\bar{v}_{xy}+\bar{v}\bar{v}_{xxy}\}|\cdot|v_{xx}|\rho_3^4 x^4. \tag{3.11.44}$$

First,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{v}_{xx}\bar{v}_y|\cdot|v_{xx}|\rho_3^4 x^4 \le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x\ge 20)}||\bar{v}_{xx}x^{\frac{3}{2}}||_{L^2}||v_{xx}x^{\frac{3}{2}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.45}$$

Second,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{v}_x\bar{v}_{xy}||v_{xx}|\rho_3^4 x^4 \le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x\ge 20)}||\bar{v}_{xy}x^{\frac{3}{2}}||_{L^2}||\bar{v}_{xx}x^{\frac{3}{2}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.46}$$

Third,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{v}\bar{v}_{xxy}||v_{xx}|\rho_3^4 x^4 \le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{v}x^{\frac{1}{2}}||_{L^\infty}||\bar{v}_{xxy}x^{\frac{5}{2}}||_{L^2(x\ge 20)}||v_{xx}x^{\frac{3}{2}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.47}$$

Again turning to (3.11.35), for this same nonlinearity, we must also treat:

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\{\bar{v}_{xx}\bar{v}_y + \bar{v}_x\bar{v}_{xy} + \bar{v}\bar{v}_{xxy}\}|\cdot|v_{xxx}|\rho_3^5 x^5. \tag{3.11.48}$$

First,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{v}_{xx}\bar{v}_y||v_{xxx}|\rho_3^5 x^5$$
$$\le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{u}_x x^{\frac{5}{4}}||_{L^\infty(x\ge 20)}||\bar{v}_{xx}x^{\frac{3}{2}}||_{L^2}||v_{xxx}x^{\frac{5}{2}}||_{L^2(x\ge 20)}$$
$$\le \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.49}$$

Second,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{v}_x\bar{v}_{xy}||v_{xxx}|\rho_3^5 x^5$$
$$\le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{v}_x x^{\frac{3}{2}}||_{L^\infty(x\ge 20)}||\bar{v}_{xy}x^{\frac{3}{2}}||_{L^2}||v_{xxx}x^{\frac{5}{2}}||_{L^2(x\ge 20)}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.50}$$

Finally,

$$\int\int \varepsilon^{\frac{n}{2}+\gamma+1}|\bar{v}\bar{v}_{xxy}||v_{xxx}|\rho_3^5 x^5$$
$$\le \varepsilon^{\frac{n}{2}+\gamma+1}||\bar{v}x^{\frac{1}{2}}||_{L^\infty}||\bar{v}_{xxy}x^{\frac{5}{2}}||_{L^2(x\ge 20)}||v_{xxx}x^{\frac{5}{2}}||_{L^2(x\ge 20)}$$
$$\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z^2||u,v||_Z. \tag{3.11.51}$$

This now concludes all of the nonlinear terms in $\mathcal{W}_i$. The final task is to control the $R^{u,n}, R^{v,n}$ terms in f, g. Via Lemma 3.7.22, for κ arbitrarily small and σ_n as in (3.7.1), we choose now $\sigma' = \frac{1}{10{,}000}$. Then, we have:

$$\Big|\int\int \epsilon^{-\frac{n}{2}-\gamma}R^{u,n}u + \int\int \epsilon^{-\frac{n}{2}-\gamma}R^{u,n}v_y x\Big| \lesssim \epsilon^{-\frac{n}{2}-\gamma}||R^{u,n}x^{\frac{1}{2}+\sigma'}||_{L^2}||x^{-\frac{1}{2}-\sigma'}u, x^{\frac{1}{2}}v_y||_{L^2}$$
$$\le C(n)\epsilon^{-\frac{n}{2}-\gamma}||||R^{u,n}||_{L^2_y}x^{\frac{1}{2}+\sigma'}||_{L^2_x}||u,v||_{X_1} \tag{3.11.52}$$
$$\le C(n)\epsilon^{\frac{1}{4}-\gamma-\kappa}||x^{-\frac{3}{4}+2\sigma_n+\sigma'+\kappa}||_{L^2_x}||u,v||_{X_1} \tag{3.11.53}$$
$$\le C(n)\epsilon^{\frac{1}{4}-\gamma-\kappa}||u,v||_{X_1}. \tag{3.11.54}$$

In (3.11.52), we have used the Hardy inequality with power $x^{-\frac{1}{2}-\sigma}$, which is admissible as $u(1,y)=0$. Upon citing (3.7.1), one has:

$$||x^{-\frac{3}{4}+2\sigma_n+\sigma'+\kappa}||_{L^2_x}|| = ||x^{-\frac{3}{4}+\frac{2}{10{,}000}+\frac{1}{10{,}000}+\kappa}||_{L^2_x} < \infty. \tag{3.11.55}$$

Next, again via Lemma 3.7.22, we have:

$$\int\int \epsilon^{1-\frac{n}{2}-\gamma}|R^{v,n}|\Big(|v|+|v_x|x\Big) \le \epsilon^{-\frac{n}{2}-\gamma}||\sqrt{\epsilon}R^{v,n}x^{\frac{1}{2}+\sigma'}||_{L^2}||x^{-\frac{1}{2}-\sigma'}\sqrt{\epsilon}v, x^{\frac{1}{2}}\sqrt{\epsilon}v_x||_{L^2}$$
$$\lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa}||u,v||_{X_1}. \tag{3.11.56}$$

Summarizing these forcing terms:

$$\epsilon^{-\frac{n}{2}-\gamma}\Big|\int\int R^{u,n}u+\epsilon R^{v,n}v+R^{u,n}v_yx+\epsilon R^{v,n}v_xx\Big|$$
$$\le \epsilon^{\frac{1}{4}-\gamma-\kappa}+\epsilon^{\frac{1}{4}-\gamma-\kappa}||u,v||^2_{X_1}. \tag{3.11.57}$$

Similarly, for higher order terms,

$$\int\int \epsilon^{-\frac{n}{2}-\gamma}|R^{u,n}_x|\{|u_x|\rho_2^2x^2+|u_{xx}|\rho_2^3x^3\}$$
$$\lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa}||x^{-1-\frac{5}{4}+\sigma'+\kappa_0}x^{\frac{3}{2}}||_{L^2_x}||u_xx^{\frac{1}{2}},u_{xx}\rho_2^{\frac{3}{2}}x^{\frac{3}{2}}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{1}{4}-\gamma-\kappa}||u,v||_{X_1\cap X_2}. \tag{3.11.58}$$

and

$$\int\int \epsilon^{-\frac{n}{2}-\gamma}|R^{u,n}_{xx}|\{|u_{xx}|\rho_3^4x^4+|u_{xxx}|\rho_3^5x^5\}$$
$$\lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa}||x^{-2-\frac{5}{4}+\sigma'+\kappa}x^{\frac{5}{2}}||_{L^2_x}||u_{xx}\rho_3^{\frac{3}{2}}x^{\frac{3}{2}},u_{xxx}\rho_3^{\frac{5}{2}}x^{\frac{5}{2}}||_{L^2}$$
$$\lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa}||u,v||_{X_1\cap X_2\cap X_3}. \tag{3.11.59}$$

For the terms from g:

$$\epsilon^{-\frac{n}{2}-\gamma}\int\int \epsilon|R^{v,n}_x|\{|v_x|\rho_2^2x^2+|v_{xx}|\rho_2^3x^3\}$$
$$\lesssim\epsilon^{\frac{1}{4}-\gamma-\kappa}||x^{-1-\frac{5}{4}+\sigma'+\kappa}x^{\frac{3}{2}}||_{L^2_x}||\sqrt{\epsilon}v_xx^{\frac{1}{2}},\sqrt{\epsilon}v_{xx}\rho_2^{\frac{3}{2}}x^{\frac{3}{2}}||_{L^2}$$
$$\lesssim\varepsilon^{\frac{1}{4}-\gamma-\kappa}||u,v||_{X_1\cap X_2}. \tag{3.11.60}$$

and

$$\epsilon^{-\frac{n}{2}-\gamma} \int\int \epsilon |R^{v,n}_{xx}| \{ |v_{xx}| \rho_3^4 x^4 + |v_{xxx}| \rho_3^5 x^5 \}$$
$$\lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa} || x^{-2-\frac{5}{4}+\sigma'+\kappa} x^{\frac{5}{2}} ||_{L^2_x} || \sqrt{\epsilon} v_{xx} \rho_3^{\frac{3}{2}} x^{\frac{3}{2}}, \sqrt{\epsilon} v_{xxx} \rho_3^{\frac{5}{2}} x^{\frac{5}{2}} ||_{L^2}$$
$$\lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa} ||u,v||_{X_1 \cap X_2 \cap X_3}. \tag{3.11.61}$$

Combining all of the estimates we have established, we have proven Lemma 3.11.1.

□

We may now prove the main result, Theorem 3.8.1:

Proof of Theorem 3.8.1. The starting point is estimate (3.9.118):

$$||u,v||_Z^2 \lesssim \epsilon^{100} + \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^4 + ||u,v||^2_{X_1\cap X_2 \cap X_3} \tag{3.11.62}$$
$$\lesssim \epsilon^{100} + \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^4 + \Big| \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 \Big| \tag{3.11.63}$$
$$\lesssim \epsilon^{100} + \epsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\bar{u},\bar{v}||_Z^4 + \epsilon^{\frac{1}{4}-\gamma-\kappa} + \epsilon^{\frac{1}{4}-\gamma-\kappa} ||u,v||^2_{X_1\cap X_2\cap X_3}$$
$$+ \epsilon^{\frac{n}{2}-\omega(N_i)} ||u,v||_Z^2 + \epsilon^{\frac{n}{2}-\omega(N_i)} ||\bar{u},\bar{v}||_Z^4, \tag{3.11.64}$$

so absorbing the $||u,v||$ terms to the left-hand side gives:

$$||u,v||_Z^2 \lesssim \epsilon^{\frac{1}{4}-\gamma-\kappa} + \epsilon^{\frac{n}{2}-\omega(N_i)} \Big(||\bar{u},\bar{v}||_Z^4 \Big). \tag{3.11.65}$$

□

Part III: Existence and Uniqueness

3.12 Overview of Existence and Uniqueness

With the main estimate, (3.11.65), in hand, we will prove existence and uniqueness of solutions to the nonlinear system specified in (3.8.1) - (3.8.3), which is defined on the domain Ω, with boundary conditions given in (3.8.4), and f, g as in (3.8.5). The main result of this chapter is:

Theorem 3.12.1. *For ε, δ sufficiently small, $\varepsilon << \delta$, $\kappa > 0$ small, and $0 \le \gamma < \frac{1}{4}$, there exists a unique solution $[u, v] \in Z(\Omega)$ to the system (3.8.1) - (3.8.3), (3.8.4), (3.8.5) satisfying the bound:*

$$||u, v||_{Z(\Omega)} \lesssim C(u_R, v_R)\varepsilon^{\frac{1}{4}-\gamma-\kappa}. \tag{3.12.1}$$

The main result, Theorem 3.2.2 follows immediately from Theorem 3.12.1. The proof of this theorem proceeds in several steps, which we now outline:

(Step 1) Linear existence of solutions to weighted Stokes system, defined as follows:

$$\Delta_\epsilon^2 \psi + \alpha A(\psi) = F_y - \epsilon G_x \text{ on } \Omega^N, \quad F, G \in L^2(\Omega^N), \tag{3.12.2}$$

$$\psi|_{y=0,N} = \psi_y|_{y=0,N} = 0, \text{ and } \psi|_{x=1} = \psi_x|_{x=1} = 0, \tag{3.12.3}$$

$$\lim_{x\to\infty} [\psi_x, \psi_y] = 0. \tag{3.12.4}$$

where $\alpha > 0$, and

$$A(\psi) = \Big[\psi x^{2m} - \psi_{yy}x^{2m+2} - \partial_x(\psi_x x^{2m+2}) + \psi_{yyyy}x^{2m+4} + \partial_x\Big(\psi_{yyx}x^{2m+4}\Big) + \partial_{xx}\Big(\psi_{xx}x^{2m+4}\Big)\Big]. \tag{3.12.5}$$

Here, $m > 0$ is sufficiently large, and can remain temporarily unspecified. The scaled Bilaplacian is defined as $\Delta_\epsilon^2 := \partial_y^4 + \epsilon\partial_y^2\partial_x^2 + \epsilon^2\partial_x^4$. The right-hand sides, F, G, should be thought of as generic elements satisfying $F_y - \varepsilon G_x \in H^{-1}$. Upon introducing appropriate function spaces, we define the weak formulation of (3.12.2) - (3.12.3) in (3.13.6). Depicting the weak-solution operator to the above system by S_α^{-1} (see 3.13.12 for a precise definition), Step 1 amounts to studying the solvability of $S_\alpha \psi = F_y - \varepsilon G_x$. The boundary conditions as $x \to \infty$ in (3.12.4) are selected in order to be consistent with (3.8.4). However, due to the terms in $A(\psi)$, the weak solution, $[\psi, u, v]$ exhibits rapid decay as $x \to \infty$.

(Step 2) Linear existence of compact perturbations to S_α. Define the maps:

$$T[\psi] := \partial_y \Big[- u_R \psi_{xy} - u_{Rx}\psi_y - (v_R + \varepsilon^{\frac{n}{2}+\gamma}\bar{v})\psi_{yy} + u_{Ry}\psi_x \Big] - \varepsilon \partial_x \Big[u_R \psi_{xx} - v_{Ry}\psi_y + v_R \psi_{xy} + v_{Ry}\psi_x \Big], \quad (3.12.6)$$

$$T_0[\psi] := T[\psi] + \varepsilon^{\frac{n}{2}+\gamma}\partial_y[\bar{v}\psi_{yy}], \quad (3.12.7)$$

$$T_a[\psi] := \Big[- u_R \psi_{xy} - u_{Rx}\psi_y - v_R \psi_{yy} + u_{Ry}\psi_x \Big], \quad (3.12.8)$$

$$T_b[\psi] := \Big[u_R \psi_{xx} - v_{Ry}\psi_y + v_R \psi_{xy} + v_{Ry}\psi_x \Big]. \quad (3.12.9)$$

T has a dependence on $\bar{v}$, so to be precise we will sometimes write $T[\psi; \bar{v}]$. When there is no danger of confusion, we simply write $T[\psi]$. The map $T_0[\psi]$ is defined to match the profile terms, $S_u(u, v), S_v(u, v)$ (see the definition in (3.8.6)), when they are written in terms of the stream function, ψ. We have defined the notation T_a, T_b so that we can write $T_0 = \partial_y T_a - \varepsilon \partial_x T_b$. In this step, we are interested in establishing solvability of the system:

$$S_\alpha \psi + T[\psi] = F_y - \varepsilon G_x \text{ on } \Omega^N, \quad (3.12.10)$$

$$[\psi = \psi_x]|_{x=1} = [\psi = \psi_y]|_{y=0} = [\psi = \psi_y]|_{y=N} = \lim_{x\to\infty} [\psi_x, \psi_y] = 0. \quad (3.12.11)$$

The essence of the arguments in this step is that upon applying S_α^{-1} to both sides above, $S_\alpha^{-1} T$ is seen as a compact perturbation of the identity. Despite Ω^N being unbounded in the x-direction, the required compactness arises from the weights, w, present in $A(\psi)$ above in (3.12.5). The solution of (3.12.10) is known to decay rapidly as $x \to \infty$, due to the presence of $A(\psi)$. This is captured in estimate (3.14.67).

(Step 3) Nonlinear existence of auxiliary system: we first invite the reader to refer back to (3.8.5) and (3.8.8) - (3.8.9) for the definitions of f and g. Given this and the definition of T in (3.12.6), we define:

$$\tilde{f}(\bar{u}, \bar{v}) := \varepsilon^{-\frac{n}{2}-\gamma} R^{u,n} + \varepsilon^{\frac{n}{2}+\gamma} \bar{u}\bar{u}_x, \quad \text{so that } f(u, \bar{u}, \bar{v}) = \tilde{f}(\bar{u}, \bar{v}) + \varepsilon^{\frac{n}{2}+\gamma} \bar{v} u_y. \tag{3.12.12}$$

The aim of this step is to obtain existence of solutions (which we now index by α and N for clarity) to the nonlinear system:

$$S_\alpha \psi^{\alpha,N} + T[\psi^{\alpha,N}; v^{\alpha,N}] = \tilde{f}_y(u^{\alpha,N}, v^{\alpha,N}) + \varepsilon g_x(u^{\alpha,N}, v^{\alpha,N}) \text{ on } \Omega^N. \tag{3.12.13}$$

This existence is obtained in the unit ball of $Z(\Omega^N)$ via Schaefer's fixed point theorem.

(Step 4) Nonlinear existence of solutions to the system (3.8.1) - (3.8.3), with f, g as in (3.8.5): By re-applying the analyses in Sections 3.9 - 3.10 and in Lemma 3.11.1, one obtains the uniform-in-(α, N) estimate: $||u^{\alpha,N}, v^{\alpha,N}||_{Z(\Omega^N)} \lesssim \mathcal{O}(\delta)\varepsilon^{\frac{1}{4}-\gamma-\kappa}$, which then enables the passage to weak limits in the space $X_1 \cap X_2 \cap X_3$. The weak limit is denoted by $[u, v]$, and is demonstrated to satisfy a weak formulation of system (3.8.1) - (3.8.3), see (3.16.10) for this formulation. Moreover, $[u, v] \in X_1 \cap X_2 \cap X_3$, gives enough regularity to upgrade immediately to a strong solution of (3.8.1) - (3.8.3).

Remark. To establish existence, we rely on compactness methods as opposed to applying a contraction mapping. The essential reason for this is seen by examining calculation (3.11.5), in which the structure is not preserved under taking differences.

Remark. It is important to establish nonlinear existence of the auxiliary system before establishing nonlinear existence of the system (3.8.1) - (3.8.3), as opposed to jumping from linear existence of (3.8.1) - (3.8.3) to nonlinear existence because the compactness methods we rely on require the weights from $\alpha A(\psi)$.

(Step 5) Nonlinear uniqueness for solutions to the system (3.8.1) - (3.8.3), with f, g as in (3.8.5): In order to prove uniqueness, we re-apply the estimates in Sections 3.9 - 3.10 with weights that are weaker by x^{-b}, where $b < 1$, but is arbitrarily close to 1. This step is necessary (with the weaker weight) due again to the calculation in (3.11.5), whose structure is destroyed upon considering differences.

3.13 Step 1: Invertibility of Weighted Stokes Operator, S_α

In this step, we study the system (3.12.2) - (3.12.3). We remind the reader that still, all integrations and all norms are taken over Ω^N unless otherwise specified. There is an abuse of notation here; ψ should be indexed by α and N, but this will not cause any confusion for this step, as we view both α and N as fixed. Our intention of this subsection is to exhibit solvability of the system (3.12.2) in the space $Z(\Omega^N)$. Denote by $\chi_1(x)$ a cut-off function satisfying (refer to (3.9.1) for the definition of ζ_3):

$$\chi_1 = 1 \text{ on } x \geq \frac{12}{10}, \quad \chi_1 = 0 \text{ for } 1 \leq x \leq \frac{11}{10}. \tag{3.13.1}$$

We define higher-order cut-offs similar to (3.13.1), satisfying the following property: support $\chi_k \subset \{\chi_{k-1} = 1\}$. Define the following auxiliary norms via:

$$||\psi||^2_{H^2_w} := \int\int \psi^2 x^{2m} + |\nabla\psi|^2 x^{2m+2} + |\nabla^2\psi|^2 x^{2m+4} \tag{3.13.2}$$

$$||\psi||^2_{H^3_w} := ||\psi||_{H^2_w} + \int\int \Big|\nabla^2\psi_x\Big|^2 x^{2m+4}, \tag{3.13.3}$$

$$||\psi||^2_{G^k_{w,B}} := ||\psi||^2_{H^k_w} + \int\int_B \Big|\partial_y^k\psi\Big|^2, \text{ for any bounded subset } B \subset \Omega^N, k = 0, ...3, \tag{3.13.4}$$

$$||\psi||^2_{H^k_w} := ||\psi||^2_{H^3_w} + \int\int \chi_k^2 \Big|\nabla^2\partial_x^{k-2}\psi\Big|^2 x^{2m+4}, \text{ for } k \geq 4. \tag{3.13.5}$$

We will also call $G^k_{w,loc}(\Omega^N)$ the space such that $||\psi||_{G^k_{w,B}} \le C(B)$ for all compact subsets B. Define the weak formulation of (3.12.2) to be:

$$\int\int \nabla^2_\epsilon \psi : \nabla^2_\epsilon \phi + \alpha\Big[\int\int \psi\phi x^{2m} + \int\int \nabla\psi\cdot\nabla\phi x^{2m+2} + \int\int \nabla^2\psi : \nabla^2\phi x^{2m+4}\Big]$$
$$= \langle F_y - \epsilon G_x, \phi\rangle_{H^{-1},H^1} \text{ for all } \phi \in C^\infty_0(\Omega^N), \text{ where } \psi \in H^2_w(\Omega^N). \tag{3.13.6}$$

Above, ∇^2 is the Hessian matrix, and the inner product between two matrices is given by $A : B = \text{trace}(AB)$. We will need one more norm:

$$||(F,G)||_{H^{-1}_k} := \sum_{j=0}^{k} ||\partial^j_x\{F_y - \varepsilon G_x\}||_{H^{-1}}. \tag{3.13.7}$$

Relevant spaces are defined here:

Definition 3.13.1. $H^2_w(\Omega^N)$ is defined to be the closure of $C^\infty_0(\Omega^N)$ under the norm $||\cdot||_{H^2_w}$. $H^k_w(\Omega^N)$ for $k \ge 3$ consists of the subspace of $H^2_w(\Omega^N)$ whose $H^k_w(\Omega^N)$ norm is finite. Note that $H^3_w(\Omega^N)$ does not contain all of the third derivatives of ψ; it is missing $\partial^3_y\psi$, which is the reason for the norm, $||\cdot||_{G_{w,B}}$.

Remark. There is a distinction between $H^2_w(\Omega^N)$, and $H^k_w(\Omega^N)$ in that:

$$H^2_w(\Omega^N) = \overline{C^\infty_0(\Omega^N)}^{||\cdot||_{H^2_w}} \text{ but for } k \ge 3,\ H^3_w(\Omega^N) \cancel{=} \overline{C^\infty_0(\Omega^N)}^{||\cdot||_{H^3_w}}. \tag{3.13.8}$$

Due to the weights, there is no "$H = W$" theorem generically for $H^k_w(\Omega^N)$.

Lemma 3.13.2. *For $\psi \in H^2_w(\Omega^N)$, the following boundary conditions are satisfied:*

$$\psi|_{y=0,N} = \psi_y|_{y=0,N} = \psi|_{x=1} = \psi_x|_{x=1} = 0 \tag{3.13.9}$$

Proof. If $\psi \in H^2_w(\Omega^N)$, obtain a sequence $\phi^{(n)}$ such that $||\phi^{(n)} - \psi||_{H^2_w} \to 0$. The claim now follows by the standard boundedness properties of the trace operator. $\square$

Lemma 3.13.3. *$H^2_w(\Omega^N)$ as defined in Definition 3.13.1 is a Banach space.*

Proof. Consider the auxiliary space:

$$H^2_{0,w}(\Omega^N) = \Big\{\psi : \nabla\psi, \nabla^2\psi \text{ exist in the weak sense, and } ||\psi||_{H^2_w(\Omega^N)} < \infty\Big\}. \tag{3.13.10}$$

Through standard arguments, $H^2_{0,w}(\Omega^N)$ is a Banach space. Suppose $\{\psi^{(n)}\}$ is a Cauchy sequence in $H^2_w(\Omega^N)$. Then $\{\psi^{(n)}\}$ is Cauchy in $H^2_{0,w}(\Omega^N)$, and so there exists a limit point ψ such that: $||\psi - \psi^{(n)}||_{H^2_w} \xrightarrow{n\to\infty} 0$. As $\psi^{(n)} \in H^2_w(\Omega^N)$, we may find a sequence $\{\phi^{(n)}_m\}_{m\geq 1}$ such that $||\phi^{(n)}_m - \psi^{(n)}||_{H^2_w} \xrightarrow{m\to\infty} 0$, where $\phi^{(n)}_m \in C^\infty_0(\Omega^N)$. In particular, define, for each n, by selecting m large enough: $||\phi^{(n)} - \psi^{(n)}||_{H^2_w} < 2^{-n}$. Thus, $||\phi^{(n)} - \psi||_{H^2_w} \xrightarrow{n\to\infty} 0$, proving that $\psi \in \overline{C^\infty_0}^{||\cdot||_{H^2_w}}$. This establishes the desired result.

□

Lemma 3.13.4. *Endowed with the inner product,*

$$\langle \psi, \varphi \rangle_{H^2_w} := \int\int \psi\varphi x^{2m} + \nabla\psi \cdot \nabla\varphi x^{2m+2} + \nabla^2\psi : \nabla^2\varphi x^{2m+4}, \tag{3.13.11}$$

H^2_w is a Hilbert Space. The inner product in (3.13.11) induces the norm defined in (3.13.2).

Proof. One easily verifies the standard axioms of an inner-product for (3.13.11). Non-degeneracy of (3.13.11) is obtained via the boundary conditions in (3.12.3). Completeness is then obtained via Lemma 3.13.3 □

Definition 3.13.5. The $\alpha-$Stokes operator is defined through:

$$S_\alpha\psi = F_y - \varepsilon G_x \text{ for } \psi \in H^2_w(\Omega^N), F_y - \varepsilon G_x \in H^{-1}(\Omega^N), \text{ if and only if (3.13.6) holds.} \tag{3.13.12}$$

It is our aim to study the invertibility of S_α:

Lemma 3.13.6. *Given $F_y - \varepsilon G_x \in H^{-1}(\Omega^N)$, there exists a unique weak solution $\psi \in H^2_w(\Omega^N)$ satisfying (3.13.6). Such a weak solution satisfies the energy inequality:*

$$||\psi||^2_{H^2_w} \lesssim \frac{1}{\alpha}||F_y - \epsilon G_x||^2_{H^{-1}} = \frac{1}{\alpha}||S_\alpha \psi||^2_{H^{-1}}. \tag{3.13.13}$$

Proof. Define:

$$B[\psi,\phi] := \int\int \nabla^2_\epsilon \psi : \nabla^2_\epsilon \phi + \alpha\Big[\int\int \psi\phi x^{2m} + \int\int \nabla\psi\cdot\nabla\phi x^{2m+2} + \int\int \nabla^2\psi : \nabla^2\phi x^{2m+4}\Big]. \tag{3.13.14}$$

It is immediate to see that B is bilinear, bounded, and coercive on $H^2_w(\Omega^N)$. Next, $F_y - \varepsilon G_x$ act as bounded linear functionals on $H^2_w(\Omega^N)$ through the pairing: $\langle F_y - \epsilon G_x, \phi\rangle_{H^{-2}_w, H^2_w} := \langle F_y - \epsilon G_x, \phi\rangle_{H^{-1},H^1}$. This follows from: $\Big|\langle F_y - \epsilon G_x, \phi\rangle_{H^{-1},H^1}\Big| \le ||F_y - \epsilon G_x||_{H^{-1}}||\phi||_{H^2_w}$. The existence of $\psi \in H^2_w(\Omega^N)$ a solution to (3.13.6) is then a standard application of the Lax-Milgram Lemma to the Hilbert Space $H^2_w(\Omega^N)$. The energy identity above follows from density of $C^\infty_0(\Omega^N)$ in $H^2_w(\Omega^N)$, which enables us to replace ϕ with ψ in (3.13.6). $\square$

The above lemma then says that $S_\alpha^{-1} : H^{-1}(\Omega^N) \to H^2_w(\Omega^N)$ is well-defined. Our intention now is to upgrade regularity.

Lemma 3.13.7. *Given $F_y - \varepsilon G_x \in H^{-1}(\Omega)$, the unique weak solution in $H^2_w(\Omega^N)$ guaranteed by Lemma 3.13.6 is in $H^3_w(\Omega^N)$ and satisfies:*

$$||\psi||^2_{H^3_w} \lesssim \frac{1}{\alpha}||F_y - \epsilon G_x||^2_{H^{-1}} = \frac{1}{\alpha}||S_\alpha \psi||^2_{H^{-1}}. \tag{3.13.15}$$

Proof. As our weak solutions are only in $H^2_w(\Omega^N)$, we must formally use difference quotients within the weak formulation (3.13.6) to upgrade to $H^3_w(\Omega^N)$. However, we will generate the

H^3_w estimate via differentiating (3.12.2), with the understanding that everything that is done can be formalized through the use of difference quotients in the standard manner. As such, we take ∂_x of the system (3.12.2), which gives:

$$\Delta_\varepsilon^2 \psi_x + \alpha A(\psi_x) + [\partial_x, \alpha A]\psi = \partial_x(F_y - \epsilon G_x), \tag{3.13.16}$$

where

$$\begin{aligned}[\partial_x, \alpha A]\psi = \alpha\Big[&2mx^{2m-1}\psi - (2m+2)x^{2m+1}\psi_{yy} - (2m+2)\partial_x(\psi_x x^{2m+1}) \\ &+ (2m+4)x^{2m+3}\psi_{yyyy} + (2m+4)\partial_x\Big(\psi_{yyx}x^{2m+3}\Big) \\ &+ (2m+4)\partial_{xx}\Big(\psi_{xx}x^{2m+3)}\Big)\Big].\end{aligned} \tag{3.13.17}$$

Let χ_1 be as above in (3.13.1). Define the quantities:

$$\tilde{\chi}_1 = 1 - \chi_1, \qquad \rho_M(x) = \chi_1(x)\tilde{\chi}_1(\frac{x}{M}), \text{ which implies } \Big|x^k\partial_x^k\rho_M(x)\Big| \le 2. \tag{3.13.18}$$

We now test the above equation, (3.13.16), against the multiplier $\rho_M\psi_x$. Doing so first gives from the Bilaplacian:

$$\begin{aligned}\int\int \Delta_\varepsilon^2\psi_x \cdot \rho_M\psi_x = &\int\int \rho_M\Big[\varepsilon|\psi_{xxy}|^2 + \varepsilon^2|\psi_{xxx}|^2 + |\psi_{xyy}|^2\Big] \\ &+ c_0\int\int \rho_M''\Big[\varepsilon|\psi_{xy}|^2 + \varepsilon^2|\psi_{xx}|^2\Big] + c_1\int\int \partial_x^4\rho_M \cdot |\psi_x|^2,\end{aligned} \tag{3.13.19}$$

for constants c_0, c_1. Next, we have the terms coming from A:

$$\begin{aligned}&\int\int \alpha A(\psi_x)\cdot\rho_M\psi_x \\ &\qquad \gtrsim \alpha\int\int[\psi_x^2 x^{2m} + \psi_{xy}^2 x^{2m+2} + \psi_{xx}^2 x^{2m+4} + \psi_{xyy}^2 x^{2m+4}\end{aligned}$$

$$+\psi_{xxy}^2 x^{2m+4} + \psi_{xxx}^2 x^{2m+4}]\rho_M - ||\psi||_{H_w^2}^2. \tag{3.13.20}$$

Through a direct integration by parts, the commutator contains lower order terms:

$$\Big| \int\int [\partial_x, \alpha A]\psi \cdot \psi_x \rho_M \Big| \lesssim ||\psi||_{H_w^2}^2 \lesssim \frac{1}{\alpha} ||F_y - \epsilon G_x||_{H^{-1}}^2. \tag{3.13.21}$$

For detailed proofs of calculations (3.13.20) and (3.13.21), we refer the reader to (3.14.99) - (3.14.106). Finally, on the right-hand side of (3.13.16), we have:

$$\Big| \langle \partial_x (F_y - \epsilon G_x), \psi_x \rho_M \rangle_{H^{-2}, H^2} \Big| \leq ||F_y - \epsilon G_x||_{H^{-1}} ||\rho_M \psi_{xx}||_{H^1}. \tag{3.13.22}$$

We can send $M \to \infty$ so that the weight $\rho_M \uparrow \chi_1$, resulting in

$$\int\int \chi_1 \Big| \nabla^2 \psi_x \Big|^2 x^{2m+4} \lesssim \frac{1}{\alpha} ||F_y - \epsilon G_x||_{H^{-1}}^2. \tag{3.13.23}$$

For the region $1 \leq x \leq 20$, and $0 \leq y \leq N$, we apply the standard $\dot{H}^2(\Omega^N)$ estimate for solutions, u^α, v^α Stokes' equation near corners (see (BR80), Theorems 1 and 2, and Figure 2, P. 562 also in (BR80) with "C/C" boundary conditions). Formally, fix another cut-off function, $\chi_2(x, y)$ localized near the corner $(1, 0)$ (the identical argument can be given for the other corner, $(1, N)$). First, by calculation, we have:

$$\Delta_\epsilon^2 \Big(\chi_2 \psi \Big) = \chi_2 \Delta_\epsilon^2 \psi + [\Delta_\epsilon^2, \chi]\psi, \tag{3.13.24}$$

where the expression for the commutator is given explicitly:

$$[\Delta_\epsilon, \chi_2]\psi = 4\partial_y \chi_2 \partial_y^3 \psi + 4\partial_y^3 \chi_2 \partial_y \psi + 6\partial_y^2 \chi_2 \partial_y^2 \psi + 2\epsilon \partial_x^2 \partial_y^2 \chi_2 \psi + 2\epsilon \partial^2 \chi_2 \partial_x^2 \psi$$

$$+ 4\epsilon\partial_x\partial_y^2\chi_2\partial_x\psi + 2\epsilon\partial_x^2\chi_2\partial_y^2\psi + 4\epsilon\partial_x\chi_2\partial_x\partial_y^2\psi + 4\epsilon\partial_x^2\partial_y\chi_2\partial_y\psi$$
$$+ 4\epsilon\partial_y\chi_2\partial_x^2\partial_y\psi + 8\epsilon\partial_{xy}\chi_2\partial_{xy}\psi + 6\epsilon^2\partial_x^2\chi_2\partial_x^2\psi + \epsilon^2\partial_x^4\chi_2\psi + 4\epsilon^2\partial_x^3\chi_2\partial_x\psi$$
$$+ 4\epsilon^2\partial_x\chi_2\partial_x^3\psi. \tag{3.13.25}$$

The salient feature of (3.13.25) will be:

$$[\Delta_\epsilon, \chi_2]\psi = o(\chi_2\partial^3\psi), \tag{3.13.26}$$

where this is short-hand notation for containing up to three ψ-derivatives, and localized by χ_2 (or any derivative of χ_2 which is also localized). Localizing (3.12.2) using χ_2:

$$||\chi_2\psi||_{H^3} \lesssim ||\chi_2(F_y - \epsilon G_x)||_{H^{-1}} + ||o(\chi_2\partial^3\psi)||_{H^{-1}} \lesssim ||\chi_2(F_y - \epsilon G_x)||_{H^{-1}}. \tag{3.13.27}$$

Combining (3.13.27) and (3.13.23) gives the desired result.

□

Lemma 3.13.8. *Fix any bounded set $B \subset \Omega^N$. Then we have:*

$$||\psi||^2_{G^3_{w,B}} \lesssim C(B)\frac{1}{\alpha}||F_y - \epsilon G_x||^2_{H^{-1}}, \tag{3.13.28}$$

where the constant $C(B)$ depends on B.

Proof. This argument proceeds identically to the calculation from the previous lemma which resulted in (3.13.27) by simply replacing χ_2 with cut-off functions localized to each interval $x \in [M, M+1]$. The dependence on B in the constant in (3.13.28) arises from the weights $x^{2m}, x^{2m+2}, x^{2m+4}$ appearing in the equation (3.12.2) through $A(\psi)$. □

The above lemmas roughly show that S_α^{-1} gains four derivatives. By repeating this procedure for higher-order x-derivatives, we can upgrade to higher-regularity:

Lemma 3.13.9. *Given $(F,G) \in H_2^{-1}$, the unique weak solution guaranteed by Lemma 3.13.6 satisfies:*

$$||\psi|^2_{H^5_w} \lesssim \frac{1}{\alpha}||(F,G)||^2_{H_2^{-1}} \tag{3.13.29}$$

For $(F,G) \in H_2^{-1}$, we can upgrade weak solutions to strong solutions:

Lemma 3.13.10. *Given $(F,G) \in H_2^{-1}$, the unique weak solution guaranteed by Lemma 3.13.6 is a strong solution of (3.12.2).*

Proof. An integration by parts of the weak formulation (3.13.6), justified according to the previous lemma, is equivalent to the equation (3.12.2) being satisfied pointwise on Ω^N. The boundary conditions at $x = 1, y = 0, y = N$ are satisfied by Lemma 3.13.2. The boundary condition at $x \to \infty$ comes from the norms, (3.13.2), which when applying with $k = 5$, imply that up to four derivatives of ψ vanish rapidly at $x \to \infty$. $\square$

3.14 Step 2: Compact Perturbations, $S_\alpha \psi + T[\psi]$

For this step, we invite the reader to refer back to the specification of $T[\psi]$, given in (3.12.6), and the system that we will focus on, given in (3.12.10). Note that $T[\psi]$ contains a loss of three-derivatives for ψ. Note also the presence of the term $\varepsilon^{\frac{n}{2}+\gamma}\bar{v}$. We will now need some compactness lemmas.

Lemma 3.14.1. *Fix two weights, $w_1 = x^{m_1}$, and $w_2 = x^{m_2}$, where $m_2 > m_1 \geq 0$. Then, one has the following compact embedding:*

$$H^1_{loc}(\Omega^N) \cap L^2_{w_2}(\Omega^N) \hookrightarrow\hookrightarrow L^2_{w_1}(\Omega^N). \tag{3.14.1}$$

Proof. Consider a family of functions $\{f^n\}$ defined on Ω^N such that:

$$\sup_n \int\int f_n^2 w_2^2 < \infty, \tag{3.14.2}$$

and such that $f_n \in H^1_{loc}(\Omega^N)$, uniformly in n. By taking Sobolev extensions across $\partial\Omega^N$, and subsequently cutting off in the y and negative x directions, we can assume $\{f_n\}$ are defined on $\mathbb{R}^2$, compactly supported in the y direction and negative x direction. Fix any $\delta' > 0$. Since $m_2 > m_1$, there exists a compact set $K = K(\delta')$ such that:

$$\sup_n ||f_n||_{L^2_{w_1}(K^c)} \leq \frac{\delta'}{2}. \tag{3.14.3}$$

On K, by Rellich compactness, there exists a subsequence (depending on δ') such that

$$\limsup_{j,k\to\infty} ||f_{n_j} - f_{n_k}||_{L^2(K)} \leq \frac{\delta'}{2 \times diam(K)^{m_1}}. \tag{3.14.4}$$

Then,

$$\limsup_{j,k\to\infty} ||f_{n_j} - f_{n_k}||_{L^2_{w_1}(K)} \leq \frac{\delta'}{2}. \tag{3.14.5}$$

Combining the above two estimates,

$$\limsup_{j,k\to\infty} ||f_{n_j} - f_{n_k}||_{L^2_{w_1}} \leq \delta'. \tag{3.14.6}$$

Taking successively $\delta' = 2^{-n}$ and applying a diagonalization argument gives the result.

□

Lemma 3.14.2. *Let the weight, x^{2m}, in the expression for $A(\psi)$, equation (3.12.5), be* *selected for any $m > 0$. Then the map $S_\alpha^{-1}T$ is well-defined and compact $H^2(\Omega^N) \to H^2(\Omega^N)$.*

Proof. According to (3.13.28), this follows from the compactness of $G^3_{w,loc}(\Omega^N) \hookrightarrow\hookrightarrow H^2(\Omega^N)$, which in turn follows from (3.14.1). The lemma is proven. $\square$

We are now ready to study system (3.12.10). The first task is to obtain an energy estimate to the inhomogeneous problem:

Lemma 3.14.3. *Suppose $\psi \in H^2(\Omega^N)$ is a solution to (3.12.10), where $(F,G) \in H_2^{-1}$, and $||\bar{u},\bar{v}||_Z \le 1$. Then ψ obeys the following energy estimate:*

$$||u_y||^2_{L^2} + \alpha||\psi||^2_{H^2_w} \lesssim \mathcal{O}(\delta)||\sqrt{\epsilon}v_x x^{\frac{1}{2}}, v_y x^{\frac{1}{2}}||^2_{L^2} + \int\int Fu + \varepsilon|G||v|. \tag{3.14.7}$$

Proof. Supposing there existed such a ψ, we would have $T[\psi] \in H^{-1}(\Omega^N)$, and so by (3.13.15), we know $\psi \in H^3_w(\Omega^N)$. By bootstrapping this regularity, we obtain that:

$$\psi \in H^5_w(\Omega^N). \tag{3.14.8}$$

We would like to apply the multiplier ψ to the equation (3.12.10) in order to repeat the energy estimate from Proposition 3.10.1. Select test functions, $\phi^{(n)} \in C_0^\infty(\Omega^N)$, which satisfy:

$$||\phi^{(n)} - \psi||_{H^2_w} \to 0. \tag{3.14.9}$$

This is possible according to the density of $C_0^\infty(\Omega^N)$ in H^2_w in Definition 3.13.1. Multiplying (3.12.10) by $\phi^{(n)}$, then gives on the left-hand side:

$$\int\int \Big(\Delta_\varepsilon^2\psi + T\psi\Big)\cdot\phi^{(n)} + \alpha\int\int A(\psi)\phi^{(n)}. \tag{3.14.10}$$

First, we shall use (3.12.7) to write:

$$\int\int\Big(\Delta_\varepsilon^2\psi + T[\psi]\Big)\phi^{(n)} = \int\int\Big(\Delta_\varepsilon^2\psi + T_0[\psi]\Big)\phi^{(n)} - \int\int \varepsilon^{\frac{n}{2}+\gamma}\partial_y(\bar{v}u_y)\cdot\phi^{(n)}$$
$$= \int\int\Big(\Delta_\varepsilon^2\psi + T_0[\psi]\Big)\phi^{(n)} + \int\int \varepsilon^{\frac{n}{2}+\gamma}(\bar{v}u_y)\cdot\phi_y^{(n)}. \quad (3.14.11)$$

According to (3.14.9), we pass to limits in the following terms:

$$\int\int(\Delta_\varepsilon^2\psi + T_0[\psi])\phi^{(n)} = \int\int \nabla_\varepsilon^2\psi : \nabla_\varepsilon^2\phi^{(n)} + \int\int T_0[\psi]\phi^{(n)}$$
$$\xrightarrow{n\to\infty} \int\int |\nabla_\varepsilon^2\psi|^2 + \int\int T_0[\psi]\psi. \quad (3.14.12)$$

We have used:

$$|\int\int T_0[\psi]\phi^{(n)} - T_0[\psi]\psi| = \int\int(\partial_y T_a[\psi] - \partial_x T_b[\psi])\cdot(\phi^{(n)}-\psi)$$
$$\leq ||T_a[\psi], T_b[\psi]||_{L^2}||\phi^{(n)}-\psi||_{H^1}$$
$$\leq ||\psi||_{H^5_w}||\phi^{(n)}-\psi||_{H^1} \xrightarrow{n\to\infty} 0, \quad (3.14.13)$$

according to (3.14.8) and the definition in equation (3.12.7). The integration on the right-hand side of (3.14.12) arises exactly from the energy estimates, Proposition 3.10.1, in particular, terms (3.10.12), (3.10.29), (3.10.41), and so we may write:

$$|\lim_{n\to\infty}\int\int\Big(\Delta_\varepsilon^2\psi + T_0[\psi]\Big)\phi^{(n)}| \gtrsim ||u_y||_{L^2}^2 - \mathcal{O}(\delta)||\sqrt{\epsilon}v_x x^{\frac{1}{2}}, v_y x^{\frac{1}{2}}||_{L^2}^2. \quad (3.14.14)$$

We may pass to the limit in the final term of (3.14.11) due to the calculation:

$$|\int\int \bar{v}u_y(\phi_y^{(n)} - \psi_y)| \leq ||\bar{v}||_{L^\infty}||u_y||_{L^2}||\phi_y^{(n)}-\psi_y||_{L^2}$$
$$\leq \varepsilon^{-N_4}||\bar{v}||_Z||u_y||_{L^2}||\phi_y^{(n)}-\psi_y||_{L^2} \xrightarrow{n\to\infty} 0. \quad (3.14.15)$$

Upon passing to the limit, we integrate by parts:

$$-\int\int \varepsilon^{\frac{n}{2}+\gamma}(\bar{v}u_y)\cdot\phi^{(n)}_y \xrightarrow{n\to\infty} \int\int \varepsilon^{\frac{n}{2}+\gamma}(\bar{v}u_y)\cdot u = -\int\int \frac{\varepsilon^{\frac{n}{2}+\gamma}}{2}\bar{v}_y u^2 \tag{3.14.16}$$

From here, we estimate identically as in (3.11.5):

$$|\int\int \frac{\varepsilon^{\frac{n}{2}+\gamma}}{2}\bar{v}_y u^2| \le \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||v_y x^{\frac{1}{2}}||^2_{L^2} \le \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||v_y x^{\frac{1}{2}}||^2_{L^2}. \tag{3.14.17}$$

It remains to treat (3.14.10), for which we use the compact support of $\phi^{(n)}$ to justify the integration by parts:

$$\int\int \alpha A(\psi)\cdot\phi^{(n)} = \int\int \psi\phi^{(n)}x^{2m} + \nabla\psi\cdot\nabla\phi^{(n)}x^{2m+2} + \nabla^2\psi : \nabla^2\phi^{(n)}x^{2m+4}. \tag{3.14.18}$$

Passing to the limit, according to (3.14.9):

$$\lim_{n\to\infty}\int\int \alpha A(\psi)\phi^{(n)} = \alpha\int\int \psi^2 x^{2m} + |\nabla\psi|^2 x^{2m+2} + |\nabla^2\psi|^2 x^{2m+4}, \tag{3.14.19}$$

On the right-hand side, we have:

$$\int\int F_y\cdot\phi^{(n)} = -\int\int F\phi^{(n)}_y \xrightarrow{n\to\infty} \int\int Fu, \tag{3.14.20}$$

$$\int\int -\varepsilon G_x\cdot\phi^{(n)} = \int\int \varepsilon G\cdot\phi^{(n)}_x \xrightarrow{n\to\infty} \int\int \varepsilon G v. \tag{3.14.21}$$

Consolidating the previous estimates gives the desired estimate, (3.14.7).

□

The task now is to estimate the right-hand side of (3.14.7) in terms of the left-hand side using the smallness of $\mathcal{O}(\delta)$. We refer the reader to Proposition 3.10.2, whose proof we follow closely. We will point out the subtle differences:

Lemma 3.14.4. *Suppose $\psi \in H^2(\Omega^N)$ is a solution to (3.12.10), where $(F, G) \in H_2^{-1}$, and $||\bar{u}, \bar{v}||_Z \leq 1$. Suppose the weight $w = x^{2m}$ from equation (3.12.5) is selected such that m is sufficiently large relative to universal constants. Then ψ obeys:*

$$||\{\sqrt{\epsilon} v_x, v_y\} x^{\frac{1}{2}}||_{L^2}^2 \lesssim ||u_y||_{L^2}^2 + \alpha ||\psi||_{H_w^2}^2 + \int\int |F||u_x|x + \varepsilon |G||v| + \varepsilon |G||v_x|x. \quad (3.14.22)$$

Proof. We will repeat the positivity estimate of Proposition 3.10.2. To do so, we apply the multiplier $\psi_x x \chi_{L,\alpha}^2$ to (3.12.10). Here, χ is a normalized cut-off function equal to 1 on $[1, 2]$ and 0 on $[3, \infty)$, and

$$\chi_{L,\alpha}(x) := \chi(\frac{\alpha}{L} x), \text{ so that } \partial_x^k \chi_{L,\alpha} = \frac{\alpha^k}{L^k} \chi^{(k)}. \quad (3.14.23)$$

Such a cut-off function was not present in Proposition 3.10.2. The necessity of it is due to the terms arising from $A(\psi)$. The presence of this cut-off function enables us to justify all integrations by parts in the x-direction. For our fixed $\alpha > 0$, we will eventually send $L \to \infty$. Applying the multiplier $\psi_x x \chi_{L,\alpha}^2$ to (3.12.10), gives on the left-hand side:

$$\begin{aligned} &\int\int \Big(T[\psi] + \Delta_\varepsilon^2 \psi \Big) \cdot \psi_x x \chi_{L,\alpha}^2 + \alpha \int\int A(\psi) \cdot \psi_x x \chi_{L,\alpha}^2 \quad (3.14.24) \\ &\quad = \int\int \Big(T_0[\psi] + \Delta_\varepsilon^2 \psi + \varepsilon^{\frac{n}{2}+\gamma} \partial_y [\bar{v} u_y] \Big) \cdot \psi_x x \chi_{L,\alpha}^2 + \alpha \int\int A(\psi) \cdot \psi_x x \chi_{L,\alpha}^2. \end{aligned}$$

We will first focus on the first two integrands above in (3.14.24), which appeared in Proposition 3.10.2. The obstacle to repeating the calculations exactly as in Proposition 3.10.2 is the presence of the cut-off function, $\chi_{L,\alpha}^2$, in the multiplier. The essential idea is this: when no derivative falls on $\chi_{L,\alpha}^2$, the estimate will be the same as the corresponding term in Proposition 3.10.2. When at least one derivative falls on $\chi_{L,\alpha}^2$, we may use the factors of α obtained from the scaling in (3.14.23) to absorb the new terms into the left-hand side of (3.14.7). Let us start with the profile terms from $T_0[\psi]$, for which we refer the reader to estimates (3.10.64) - (3.10.68). We will transfer all of the terms to velocity formulation so as to remain consistent with estimates (3.10.64) - (3.10.68).

$$\begin{aligned}\int\int \partial_y S_u \cdot vx\chi_{L,\alpha}^2 &= \int\int S_u u_x x\chi_{L,\alpha}^2 \\ &= \int\int [u_R u_x + u_{Rx}u + v_R u_y + u_{Ry}v]u_x x\chi_{L,\alpha}^2 \\ &\gtrsim \int\int u_x^2 x\chi_{L,\alpha}^2 - |\int\int [u_{Rx}u + v_R u_y + u_{Ry}v]u_x x\chi_{L,\alpha}^2|. \end{aligned} \tag{3.14.25}$$

We will treat the three terms on the right-hand side above, using (3.3.12) and (3.3.18) starting with:

$$\begin{aligned}\int\int u_{Rx}uxu_x\chi_{L,\alpha}^2 &= \int\int \{u_{Rx}^P + u_{Rx}^E\}uxu_x\chi_{L,\alpha}^2 \\ &\le ||yx^{\frac{1}{2}}u_{Rx}^P||_{L^\infty}||u_y||_{L^2}||v_y x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2} \\ &\qquad + ||u_{Rx}^E x^{\frac{3}{2}}||_{L^\infty}||\frac{u}{x}\chi_{L,\alpha}||_{L^2}||u_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2} \\ &\le \mathcal{O}(\delta)||u_y||_{L^2}^2 + \mathcal{O}(\delta)||u_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}^2 + \frac{\alpha}{L}||u||_{L^2}^2 \\ &\le \mathcal{O}(\delta)||u_y||_{L^2}^2 + \mathcal{O}(\delta)||u_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H_w^2}^2. \end{aligned} \tag{3.14.26}$$

Above, we have used the Hardy inequality:

$$\begin{aligned}||\frac{u}{x}\chi_{L,\alpha}||_{L^2} &\lesssim ||\partial_x(u\chi_{L,\alpha})||_{L^2} \le ||u_x\chi_{L,\alpha}||_{L^2} + \frac{\alpha}{L}||u\chi_{L,\alpha}'||_{L^2} \\ &\lesssim ||u_x\chi_{L,\alpha}||_{L^2} + \frac{\alpha}{L}||\psi||_{H_w^2}. \end{aligned} \tag{3.14.27}$$

Next, by (3.3.10), (3.3.20), we have:

$$\begin{aligned}\Big|\int\int v_R u_y v_y x\chi_{L,\alpha}^2\Big| &\le ||v_R x^{\frac{1}{2}}||_{L^\infty}||u_y||_{L^2}||v_y x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2} \\ &\le \mathcal{O}(\delta)||u_y||_{L^2}||v_y x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}. \end{aligned} \tag{3.14.28}$$

Next, by (3.3.14), (3.3.16), (3.3.18) we have:

$$\begin{aligned}
\int\int u_{Ry} v u_x x \chi_{L,\alpha}^2 &= \int\int \{u_{Ry}^P + \sqrt{\epsilon} u_{RY}^E\} v v_y x \chi_{L,\alpha}^2 \\
&\le ||y u_{Ry}^P||_{L^\infty} ||v_y x^{\frac{1}{2}} \chi_{L,\alpha}||_{L^2}^2 \\
&\qquad + \sqrt{\epsilon} ||u_{RY}^E x^{\frac{3}{2}}||_{L^\infty} ||v_y x^{\frac{1}{2}} \chi_{L,\alpha}||_{L^2} ||\sqrt{\epsilon} \frac{v}{x} \chi_{L,\alpha}||_{L^2} \\
&\le ||y u_{Ry}^P||_{L^\infty} ||v_y x^{\frac{1}{2}} \chi_{L,\alpha}||_{L^2}^2 \\
&\qquad + \sqrt{\epsilon} ||u_{RY}^E x^{\frac{3}{2}}||_{L^\infty} ||v_y x^{\frac{1}{2}} \chi_{L,\alpha}||_{L^2} ||\sqrt{\epsilon} v_x \chi_{L,\alpha}||_{L^2} \\
&\qquad + \frac{\alpha}{L} \sqrt{\epsilon} ||u_{RY}^E x^{\frac{3}{2}}||_{L^\infty} ||v_y x^{\frac{1}{2}} \chi_{L,\alpha}||_{L^2} ||\sqrt{\epsilon} v \chi_{L,\alpha}'||_{L^2} \\
&\le \mathcal{O}(\delta) ||v_y x^{\frac{1}{2}} \chi_{L,\alpha}||_{L^2}^2 + \sqrt{\epsilon} \mathcal{O}(\delta) ||\sqrt{\epsilon} v_x \chi_{L,\alpha}||_{L^2}^2 + \mathcal{O}(\delta) \frac{\alpha}{L} ||\psi|_{H_w^2}^2.
\end{aligned} \tag{3.14.29}$$

Summarizing the previous four terms:

$$\begin{aligned}
\int\int \partial_y S_u \cdot v x \chi_{L,\alpha}^2 \gtrsim{}& ||v_y x^{\frac{1}{2}} \chi_{L,\alpha}||_{L^2}^2 - \frac{\alpha}{L} ||\psi||_{H_w^2}^2 \\
&- \mathcal{O}(\delta) ||u_y||_{L^2}^2 - \mathcal{O}(\delta) \sqrt{\varepsilon} ||\sqrt{\varepsilon} v_x \chi_{L,\alpha}||_{L^2}^2.
\end{aligned} \tag{3.14.30}$$

We will now move to the profile terms from S_v, which are located starting from estimate (3.10.69). First,

$$\int\int -\varepsilon \partial_x S_v \cdot v x \chi_{L,\alpha}^2 = \int\int \varepsilon S_v \cdot [v_x x \chi_{L,\alpha}^2 + v \chi_{L,\alpha}^2 + v x \frac{2\alpha}{L} \chi_{L,\alpha} \chi_{L,\alpha}']. \tag{3.14.31}$$

Referring to definition (3.8.5), consider the term $u_R v_x$ in S_v, which is the most delicate profile term:

$$\int\int \varepsilon u_R v_x^2 x \chi_{L,\alpha}^2 + \int\int \varepsilon u_R v_x v [\chi_{L,\alpha} + 2x \frac{\alpha}{L} \chi_{L,\alpha} \chi_{L,\alpha}']. \tag{3.14.32}$$

The first term above in (3.14.32) gives positivity:

$$\int\int \varepsilon u_R v_x^2 x\chi_{L,\alpha}^2 \gtrsim \int\int \varepsilon v_x^2 x\chi_{L,\alpha}^2. \tag{3.14.33}$$

We will treat the second term on the right-hand side of (3.14.32):

$$\int\int \varepsilon u_R v v_x \chi_{L,\alpha}^2 = -\int\int \varepsilon \frac{v^2}{2}[u_{Rx}\chi_{L,\alpha}^2 + 2u_R\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha}], \tag{3.14.34}$$

$$\begin{aligned}\int\int \varepsilon u_R v v_x x\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha} = -\int\int \varepsilon \frac{v^2}{2}[&u_{Rx}x\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha} \\ &+ u_R\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha} + u_R x\frac{\alpha^2}{L^2}\chi_{L,\alpha}\chi''_{L,\alpha}].\end{aligned} \tag{3.14.35}$$

The first term on the right-hand side of (3.14.34) yields:

$$|\int\int \varepsilon v^2 u_{Rx}\chi_{L,\alpha}^2| \lesssim \sqrt{\varepsilon}||\chi_{L,\alpha}\{\sqrt{\varepsilon}v_x, v_y\}x^{\frac{1}{2}}||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H^2_w}^2, \tag{3.14.36}$$

in nearly an identical manner to estimate (3.10.43). We now estimate:

$$|\int\int \varepsilon \frac{u_R}{2}v^2\frac{\alpha}{L}\chi'_{L,\alpha}| \lesssim \int\int \varepsilon v^2\frac{\alpha}{L} \lesssim \frac{\alpha}{L}||\psi||_{H^2_w}^2. \tag{3.14.37}$$

This same estimate can be performed for all the terms in (3.14.35). Consolidating these bounds:

$$\int\int \varepsilon v_x^2 x\chi_{L,\alpha}^2 \lesssim (3.14.32) + \sqrt{\varepsilon}||\chi_{L,\alpha}\{\sqrt{\varepsilon}v_x, v_y\}x^{\frac{1}{2}}||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H^2_w}. \tag{3.14.38}$$

It remains now to treat the remaining three terms in S_v. The second, third, and fourth terms from S_v can be controlled in the same manner as in (3.10.71) - (3.10.73):

$$\epsilon|\int\int v_{Rx}uv_x x\chi_{L,\alpha}^2| \le \sqrt{\epsilon}||x^{\frac{3}{2}}v_{Rx}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}||u_x\chi_{L,\alpha}||_{L^2} + \frac{\alpha}{L}||\psi||_{H^2_w}^2. \tag{3.14.39}$$

$$\epsilon\Big|\int\int v_R v_y v_x x\chi_{L,\alpha}^2\Big| \le \sqrt{\epsilon}||v_R||_{L^\infty}||v_y x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2} + \frac{\alpha}{L}||\psi||_{H^2_w}^2. \tag{3.14.40}$$

$$\begin{aligned}\epsilon\Big|\int\int v_{Ry}vv_x x\chi_{L,\alpha}^2\Big| \le{}& \sqrt{\epsilon}||v_{Ry}^P y||_{L^\infty}||v_y x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2} \\ &+ \sqrt{\epsilon}||v_{RY}^E x^{\frac{3}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H^2_w}^2.\end{aligned} \tag{3.14.41}$$

We now turn back to (3.14.31), addressing the second term in the bracket for the final three profile terms from S_v:

$$\int\int[v_{Rx}u + v_R v_y + v_{Ry}v]\cdot\varepsilon v\chi_{L,\alpha}^2. \tag{3.14.42}$$

First, through the Hardy inequality and (3.3.8), (3.3.17):

$$\begin{aligned}|\int\int\varepsilon v_{Rx}uv\chi_{L,\alpha}^2| &\le \sqrt{\varepsilon}||x^{\frac{3}{2}}v_{Rx}||_{L^\infty}||\frac{u}{x^{\frac{3}{4}}}\chi_{L,\alpha}||_{L^2}||\sqrt{\varepsilon}\frac{v}{x^{\frac{3}{4}}}\chi_{L,\alpha}||_{L^2} \\ &\le \sqrt{\varepsilon}\Big[||u_x x^{\frac{1}{4}}\chi_{L,\alpha}||_{L^2}^2 + ||\sqrt{\varepsilon}v_x x^{\frac{1}{4}}\chi_{L,\alpha}||_{L^2}^2 + \frac{\alpha}{L}||\{u,\sqrt{\varepsilon}v x^{\frac{1}{4}}\chi_{L,\alpha}'||_{L^2}^2\Big] \\ &\le \sqrt{\varepsilon}\Big[||u_x x^{\frac{1}{4}}\chi_{L,\alpha}||_{L^2}^2 + ||\sqrt{\varepsilon}v_x x^{\frac{1}{4}}\chi_{L,\alpha}||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H^2_w}^2\Big],\end{aligned} \tag{3.14.43}$$

so long as $w = x^m$ is selected larger than $x^{\frac{1}{4}}$, which is true by the assumption of this lemma. Next, through an integration by parts and (3.3.10), (3.3.20):

$$\begin{aligned}\int\int[v_R v_y + v_{Ry}v]\varepsilon v\chi_{L,\alpha}^2 = \frac{1}{2}\int\int v_{Ry}v^2\varepsilon\chi_{L,\alpha}^2 &\le \epsilon||v_{Ry}^P y^2||_{L^\infty}||v_y\chi_{L,\alpha}||_{L^2}^2 \\ &+ \sqrt{\epsilon}||v_{RY}^E x^{\frac{3}{2}}||_{L^\infty}||\sqrt{\epsilon}v_x x^{\frac{1}{4}}\chi_{L,\alpha}||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H^2_w}^2.\end{aligned} \tag{3.14.44}$$

The final task for the S_v profile contributions is the third term from (3.14.31):

$$|\int\int \varepsilon[v_{Rx}u + v_R v_y + v_{Ry}v] \cdot vx\frac{2\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha}| \leq \frac{\alpha}{L}||\psi||^2_{H^2_w}, \tag{3.14.45}$$

so long as $w = x^m$ is selected larger than x, which is true by assumption of this lemma. Let us consolidate all of the calculations from S_v:

$$\int\int -\varepsilon\partial_x S_v \cdot vx\chi^2_{L,\alpha} \gtrsim \int\int \varepsilon v_x^2 x\chi^2_{L,\alpha} - \sqrt{\varepsilon}||\chi_{L,\alpha}\{\sqrt{\varepsilon}v_x, v_y\}x^{\frac{1}{2}}||^2_{L^2} - \frac{\alpha}{L}||\psi||^2_{H^2_w}. \tag{3.14.46}$$

It now remains to come to those terms contributed by $\Delta^2_\varepsilon\psi$ into (3.14.24). We will follow closely the calculations from (3.10.52) - (3.10.63) in Proposition 3.10.2. We will again omit the justifications near the corners of our domain as these are identical to Proposition 3.10.2. Again, we will write these terms in the velocity form, to remain consistent with the calculations in (3.10.52) - (3.10.63). First,

$$|\int\int -u_{yy}u_x x\chi^2_{L,\alpha}| = |-\int\int \frac{u_y^2}{2}[\chi^2_{L,\alpha} + 2x\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha}]| \lesssim ||u_y||^2_{L^2}. \tag{3.14.47}$$

Next,

$$\int\int -\varepsilon u_{xx}u_x x\chi^2_{L,\alpha}| = |\int\int \varepsilon\frac{u_x^2}{2}[\chi^2_{L,\alpha} + 2x\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha}] \lesssim \varepsilon||u_x\chi_{L,\alpha}||^2_{L^2} + \varepsilon\frac{\alpha}{L}||\psi||^2_{H^2_w}. \tag{3.14.48}$$

We now move to the terms from $\Delta_\varepsilon v$, starting with:

$$\begin{aligned}|\int\int \varepsilon^2 v_{xxx} \cdot vx\chi^2_{L,\alpha}| &= |\int\int \varepsilon^2 v_{xx} \cdot [v_x x\chi^2_{L,\alpha} + v\chi^2_{L,\alpha} + 2vx\frac{\alpha}{L}\chi_{L,\alpha}\chi'_{L,\alpha}]| \\ &\leq \varepsilon||\sqrt{\varepsilon}v_x\chi_{L,\alpha}||^2_{L^2} + \frac{\alpha}{L}||\psi||^2_{H^2_w}.\end{aligned} \tag{3.14.49}$$

Finally, we have:

$$
\begin{aligned}
|\int\int -\varepsilon v_{xyy} v x \chi_{L,\alpha}^2| &= |\int\int \varepsilon v_{xy} v_y x \chi_{L,\alpha}^2| \\
&= |\int\int \varepsilon v_y^2 [\chi_{L,\alpha}^2 + 2x\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}]| \\
&\lesssim \varepsilon ||v_y \chi_{L,\alpha}||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H_w^2}^2. \qquad (3.14.50)
\end{aligned}
$$

By combining calculations (3.14.30), (3.14.46), (3.14.47) - (3.14.50), and absorbing relevant terms to the left-hand side below, we have:

$$
\begin{aligned}
||\{\sqrt{\epsilon} v_x, v_y\} x^{\frac{1}{2}} \chi_{L,\alpha}||_{L^2}^2 \lesssim\ & ||u_y||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H_w^2}^2 \\
& + \alpha \int\int A(\psi)\cdot \psi_x x \chi_{L,\alpha}^2 + \int\int \varepsilon^{\frac{n}{2}+\gamma} \bar{v} u_y u_x x \chi_{L,\alpha}^2 \\
& + \int\int F_y \cdot v x \chi_{L,\alpha}^2 - \varepsilon G_x \cdot v x \chi_{L,\alpha}^2. \qquad (3.14.51)
\end{aligned}
$$

Via direct integration by parts, which is justified due to the presence of the cut-off function in x, we compute:

$$
\Big|\alpha \int\int A(\psi)\psi_x x \chi_{L,\alpha}^2\Big| \lesssim \alpha ||\psi||_{H_w^2}^2. \qquad (3.14.52)
$$

Let us compute each term in $A(\psi)$ to verify (3.14.52), referring to the definition in (3.12.5), starting with:

$$
\begin{aligned}
|\int\int \alpha\psi x^{2m} v x \chi_{L,\alpha}^2| &= |-\frac{\alpha}{2}\int\int \psi^2 \partial_x[x^{2m+1}\chi_{L,\alpha}^2]| \\
&= |-\frac{\alpha}{2}\int\int \psi^2 [C x^{2m}\chi_{L,\alpha}^2 + 2x^{2m+1}\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}']| \qquad (3.14.53) \\
&\leq \alpha ||\psi||_{H_w^2}^2. \qquad (3.14.54)
\end{aligned}
$$

For the second term in (3.14.53), we have used: $|\frac{\alpha}{L} x \chi_{L,\alpha}| \lesssim 1$. Next, let us turn to:

$$
\begin{aligned}
\alpha \int\int (-\psi_{yy} x^{2m+2} + \psi_{yyyy} x^{2m+4}) v x \chi_{L,\alpha}^2 & \\
&= \alpha \int\int u_y v x^{2m+3} \chi_{L,\alpha}^2 - \alpha u_y u_{xy} x^{2m+5} \chi_{L,\alpha}^2 \\
&= \alpha \int\int u u_x x^{2m+3} \chi_{L,\alpha}^2 + \alpha u_y^2 \partial_x (x^{2m+5} \chi_{L,\alpha}^2) \\
&\lesssim \alpha \int\int u^2 x^{2m+2} + u_y^2 x^{2m+4} \lesssim \alpha ||\psi||_{H^2_w}^2. \qquad (3.14.55)
\end{aligned}
$$

Next,

$$
\begin{aligned}
\alpha \int\int \partial_x (\psi_x x^{2m+2}) v x \chi_{L,\alpha}^2 &= \alpha \int\int v x^{2m+2} \partial_x (v x \chi_{L,\alpha}^2) \\
&= \alpha \int\int v x^{2m+2} [v_x x \chi_{L,\alpha}^2 + v \chi_{L,\alpha}^2 + 2 v x \frac{\alpha}{L} \chi_{L,\alpha} \chi_{L,\alpha}'] \\
&\lesssim \alpha \int\int v^2 x^{2m+2} \lesssim \alpha ||\psi||_{H^2_w}^2. \qquad (3.14.56)
\end{aligned}
$$

Next,

$$
\begin{aligned}
\alpha \int\int \partial_x (\psi_{yyx} x^{2m+4}) v x \chi_{L,\alpha}^2 &= \alpha \int\int \partial_x (\psi_{xy} x^{2m+4}) v_y x \chi_{L,\alpha}^2 \\
&= \alpha \int\int \partial_x (u_x x^{2m+4}) u_x x \chi_{L,\alpha}^2 \lesssim \alpha \int\int u_x^2 x^{2m+4} \lesssim \alpha ||\psi||_{H^2_w}^2.
\end{aligned}
\qquad (3.14.57)
$$

The final term in $A(\psi)$ is:

$$
\begin{aligned}
\alpha \int\int \partial_{xx} (\psi_{xx} x^{2m+4}) v x \chi_{L,\alpha}^2 &= \alpha \int\int \psi_{xx} x^{2m+4} \partial_{xx} [v x \chi_{L,\alpha}^2] \\
&= \alpha \int\int v_x x^{2m+4} [v_{xx} x \chi_{L,\alpha}^2 + 2 v_x \partial_x (x \chi_{L,\alpha}) + v \partial_{xx} (x \chi_{L,\alpha})] \\
&\lesssim \alpha ||\psi||_{H^2_w}^2. \qquad (3.14.58)
\end{aligned}
$$

This concludes all the terms in $A(\psi)$, according to (3.12.5). Estimating the next term in (3.14.51) exactly as in (3.11.8) yields:

$$\begin{aligned}|\int\int \varepsilon^{\frac{n}{2}+\gamma}\bar{v}u_y u_x x\chi_{L,\alpha}^2| &\le \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u},\bar{v}||_Z||u_y||_{L^2}||u_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2} \\ &\le \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||u_x x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}^2 + \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||u_y||_{L^2}^2. \quad (3.14.59)\end{aligned}$$

Finally, we come to the right-hand side:

$$\begin{aligned}\int\int [F_y - \varepsilon G_x]\cdot vx\chi_{L,\alpha} &= \int\int Fu_x x\chi_{L,\alpha} + \varepsilon G\partial_x[vx\chi_{L,\alpha}] \\ &\le \int\int |F||u_x|x + |\int\int \varepsilon G[v_x x\chi_{L,\alpha} + v\chi_{L,\alpha} + vx(\frac{\alpha}{L})\chi'_{L,\alpha}]| \\ &\le \int\int |F||u_x|x + \varepsilon|G||v_x|x + \varepsilon|G||v|., \quad (3.14.60)\end{aligned}$$

Inserting the previous few calculations into estimate (3.14.51) gives:

$$\begin{aligned}||\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}}\chi_{L,\alpha}||_{L^2}^2 &\lesssim ||u_y||_{L^2}^2 + \frac{\alpha}{L}||\psi||_{H_w^2}^2 \\ &+ \alpha||\psi||_{H_w^2}^2 + \int\int |F||u_x|x + \varepsilon|G|[|v_x|x + |v|]. \quad (3.14.61)\end{aligned}$$

We now send $L \to \infty$, and appeal to Monotone Convergence Theorem, as $\chi_{L,\alpha} \uparrow 1$ to establish the desired result.

□

Having understood the inhomogeneous problem:

Lemma 3.14.5. *For $(F, G) \in H_2^{-1}$, and $||\bar{u},\bar{v}||_Z \le 1$, there exists a unique weak solution $\psi \in H_w^2(\Omega^N)$ to the system (3.12.10).*

Proof. We apply S_α^{-1} to both sides of (3.12.10), which is valid as the right-hand side and therefore the left-hand side is assumed to be in at least $H^{-1}(\Omega^N)$, thereby yielding:

$$\psi + S_\alpha^{-1} T\psi = S_\alpha^{-1}\Big(F_y - \varepsilon G_x\Big). \tag{3.14.62}$$

We will study the equation (3.14.62) as an equality in the space $H^2(\Omega^N)$. According to the Fredholm alternative, which is available according to Lemma 3.14.2, there either exists a unique solution $\psi \in H^2(\Omega^N)$ to the system (3.14.62), or a non-trivial solution $\psi \in H^2(\Omega^N)$ to:

$$\psi + S_\alpha^{-1} T\psi = 0 \iff S_\alpha \psi = -T\psi. \tag{3.14.63}$$

Therefore, coupling (3.14.22) with (3.14.7), taking $F = G = 0$, we have:

$$||\sqrt{\epsilon} u_x, u_y||_{L^2}^2 + \alpha ||\psi||_{H_w^2}^2 + ||\sqrt{\epsilon} v_x x^{\frac{1}{2}}, v_y x^{\frac{1}{2}}||_{L^2}^2 \leq 0, \tag{3.14.64}$$

implying $\psi, u, v = 0$. Thus, by the Fredholm alternative, there exists a unique solution $\psi \in H^2(\Omega^N)$ to (3.14.62). Rearranging (3.14.62):

$$\psi = S_\alpha^{-1}\Big(F_y - \varepsilon G_x - T\psi\Big), \tag{3.14.65}$$

where $F_y - \varepsilon G_x - T\psi \in H^{-1}(\Omega^N)$, and so an application of (3.13.13) shows that $\psi \in H_w^2(\Omega^N)$. This concludes the proof.

$\square$

Lemma 3.14.6. *Let ψ be the unique $H_w^2(\Omega^N)$ weak solution from Lemma 3.14.5. Then for $(F, G) \in H_2^{-1}$, $\psi \in H_w^5(\Omega^N)$.*

Proof. $T\psi \in H^{-1}(\Omega^N)$, and so $S_\alpha \psi = -T\psi + F_y - \varepsilon G_x \in H^{-1}(\Omega^N)$, which implies that $\psi \in H^3_w(\Omega^N)$ according to (3.13.15). Iterating this regularity then gives $\psi \in H^5_w(\Omega^N)$. □

We now introduce more notation, which is more suitable for the velocities:

$$L_{\alpha,\bar{v}}[u,v] = (\tilde{f}, g) \iff S_\alpha \psi + T[\psi; \bar{v}] = \tilde{f}_y - \varepsilon g_x. \tag{3.14.66}$$

Summarizing the established results, we have:

Corollary 3.14.7. *For $\tilde{f}, g \in H_2^{-1}(\Omega^N)$, $\alpha > 0$, and $||\bar{u},\bar{v}||_{Z(\Omega^N)} \le 1$, the map $L_{\alpha,\bar{v}}[u,v]$ is invertible, where*

$$L^{-1}_{\alpha,\bar{v}} : (\tilde{f}, g) \in H_2^{-1}(\Omega^N) \to [u,v] \in H^4_w(\Omega^N). \tag{3.14.67}$$

Moreover, the boundary conditions (3.12.3) are satisfied by $[u,v] = L^{-1}_{\alpha,\bar{v}}[\tilde{f}, g]$.

It is now our intention to repeat the second and third order energy and positivity estimates from Section 3.10, with our new system (3.12.10). For this, we will need to understand several calculations. First, we introduce some norms:

$$||\psi||^2_{J^2} := ||\psi||^2_{H^2_w}, \tag{3.14.68}$$

$$||\psi||^2_{J^{k+2}} := \int\int |\partial_x^k \psi|^2 (\rho_{k+1})^{2k} x^{2m+2k} + |\nabla \partial_x^k \psi|^2 \rho_{k+1}^{2k} x^{2m+2k+2} \tag{3.14.69}$$

$$+ |\nabla^2 \partial_x^k \psi|^2 \rho_{k+1}^{2k} x^{2m+2k+4} \text{ for } k \ge 1. \tag{3.14.70}$$

The reader is referred to the definitions of ρ_k provided in (3.9.2). The essential difference between these J^k-norms and the H^k_w norms introduced in (3.13.2) are the growing weights of x which each application of ∂_x, which mimics the structure of the energy norms, X_k, in (3.9.3).

Lemma 3.14.8.

$$\int\int A(\partial_x^k\psi)\cdot\partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k} \gtrsim ||\chi_{L,\alpha}\psi||_{J^{k+2}}^2 - \sum_{i=0}^{k-1}||\psi||_{J^{i+2}}^2 \tag{3.14.71}$$

Proof. Referring to (3.12.5), the first term is:

$$\int\int|\partial_x^k\psi|^2x^{2m}x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}. \tag{3.14.72}$$

The next terms, via an integration by parts in y:

$$\int\int -\partial_x^k\psi_{yy}x^{2m+2}\cdot\partial_x^k\psi x^{2k}\rho_{k+1}^{2k}\chi_{L,\alpha}^2 = \int\int|\partial_x^k\psi_y|^2x^{2m+2k+2}\rho_{k+1}^{2k}\chi_{L,\alpha}^2, \tag{3.14.73}$$

$$\int\int \partial_x^k\psi_{yyyy}x^{2m+4}\cdot\partial_x^k\psi x^{2k}\rho_{k+1}^{2k}\chi_{L,\alpha}^2 = \int\int|\partial_x^k\psi_{yy}|^2x^{2m+2k+4}\rho_{k+1}^{2k}\chi_{L,\alpha}^2. \tag{3.14.74}$$

Next,

$$\int\int\partial_x[(\partial_x^k\psi)_{yyx}x^{2m+4}]\cdot\partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}$$

$$= \int\int\partial_x^{k+1}\psi_yx^{2m+4}\cdot\partial_x[\partial_x^k\psi_yx^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}] \tag{3.14.75}$$

$$\gtrsim \int\int|\partial_x^{k+1}\psi_y|^2x^{2m+2k+4}\chi_{L,\alpha}^2\rho_{k+1}^{2k} - \int\int|\partial_x^k\psi_y|^2x^{2m+2k+2}\rho_k^{2(k-1)} \tag{3.14.76}$$

$$\gtrsim \int\int|\partial_x^{k+1}\psi_y|^2x^{2m+2k+4}\chi_{L,\alpha}^2\rho_{k+1}^{2k} - \sum_{i=0}^{k-1}||\psi||_{J^{i+2}}. \tag{3.14.77}$$

Above, we have used the calculation:

$$\partial_x[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}] = 2kx^{2k-1}\chi_{L,\alpha}^2\rho_{k+1}^{2k} + x^{2k}\frac{2\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\rho_{k+1}^{2k} + x^{2k}\chi_{L,\alpha}^2 2k\rho_{k+1}^{2k-1}\rho_{k+1}'. \tag{3.14.78}$$

For the second term on the right-hand side of (3.14.78), we estimate: $\frac{2\alpha}{L} x \chi_{L,\alpha} \lesssim 1$. For the third term on the right-hand side, we use that the support of ρ'_{k+1} is localized in x. We also use that: support$(\rho_k) \subset \{\rho_{k-1} = 1\}$. Next, we integrate by parts twice in x to obtain:

$$\begin{aligned}
\int\int \partial_{xx}[(\partial_x^k \psi)_{xx} x^{2m+4}] \cdot \partial_x^k \psi x^{2k} \chi_{L,\alpha}^2 \rho_{k+1}^{2k} \\
&= \int\int \partial_x^{k+2} \psi x^{2m+4} \partial_{xx}[\partial_x^k \psi x^{2k} \chi_{L,\alpha}^2 \rho_{k+1}^{2k}] \qquad (3.14.79) \\
&= \int\int |\partial_x^{k+2}\psi|^2 x^{2m+2k+4} \chi_{L,\alpha}^2 \rho_{k+1}^{2k} + \int\int \partial_x^{k+2}\psi x^{2m+4} \partial_x^{k+1}\psi \partial_x [x^{2k} \chi_{L,\alpha}^2 \rho_{k+1}^{2k}] \\
&+ \int\int \partial_x^{k+2}\psi x^{2m+4} \partial_x^k \psi \partial_{xx}[x^{2k}\chi_{L,\alpha}^2 \rho_{k+1}^{2k}]. \qquad (3.14.80)
\end{aligned}$$

The final two terms on the right-hand side of (3.14.80) are estimated through further integrations by parts:

$$\begin{aligned}
|\int\int \partial_x^{k+2}\psi x^{2m+4} \partial_x^{k+1}\psi \partial_x[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}] + \int\int \partial_x^{k+2}\psi x^{2m+4}\partial_x^k\psi \partial_{xx}[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}]| \\
\lesssim ||\psi||_{J^{k+1}}^2. \qquad (3.14.81)
\end{aligned}$$

Finally,

$$\begin{aligned}
-\int\int \partial_x[(\partial_x^k\psi)_x x^{2m+2}] \cdot \partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k} = \int\int \partial_x^{k+1}\psi x^{2m+2}\partial_x[\partial_x^k\psi x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}] \\
= \int\int |\partial_x^{k+1}\psi|^2 x^{2m+2+2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k} \qquad (3.14.82) \\
+ \int\int \partial_x^{k+1}\psi x^{2m+2}\partial_x^k\psi\partial_x[x^{2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k}] \qquad (3.14.83) \\
\gtrsim \int\int |\partial_x^{k+1}\psi|^2 x^{2m+2+2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k} - ||\psi||_{J^{k+1}}^2. \qquad (3.14.84)
\end{aligned}$$

Piecing all of the above estimates together yields the desired bound.

□

Lemma 3.14.9.

$$|\int\int A(\partial_x^k\psi)\cdot\partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}|\lesssim\sum_{i=0}^{k}||\psi||_{J^{i+2}}^2. \tag{3.14.85}$$

Proof. Again, referring to definition (3.12.5), we will proceed term by term, starting with the following, for which we integrate by parts once:

$$|\int\int\partial_x^k\psi x^{2m}\cdot\partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}|=-\frac{1}{2}\int\int|\partial_x^k\psi|^2\partial_x[x^{2m+2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}] \tag{3.14.86}$$

Let us expand the product rule above:

$$\begin{aligned}\partial_x[x^{2m+2k+1}\chi_{L,\alpha}^2\rho_k^{2k+1}]&=Cx^{2m+2k}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}+Cx^{2m+2k+1}\frac{\alpha}{L}\chi_{L,\alpha}\chi_{L,\alpha}'\rho_{k+1}^{2k+1}\\&+x^{2m+2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k}\rho_{k+1}'\lesssim x^{2m+2k}\rho_{k+1}^{2k}.\end{aligned} \tag{3.14.87}$$

This, the term (3.14.86) can be controlled via:

$$|(3.14.86)|\lesssim\int\int|\partial_x^k\psi|^2x^{2m+2k}\rho_{k+1}^{2k}\lesssim\sum_{i=0}^{k}||\psi||_{J^{i+2}}^2. \tag{3.14.88}$$

The second term in (3.12.5) is treated via:

$$\begin{aligned}&\int\int-\partial_x(\partial_x^{k+1}\psi x^{2m+2})\cdot\partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}\\&\quad=\int\int[-\partial_x^{k+2}\psi x^{2m+2}-C\partial_x^{k+1}\psi x^{2m+1}]\cdot\partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}\\&\quad=\int\int|\partial_x^{k+1}\psi|^2\partial_x[x^{2m+2k+3}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}]-C|\partial_x^{k+1}\psi|^2x^{2m+2k+2}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}\end{aligned} \tag{3.14.89}$$

$$\lesssim\int\int|\partial_x^{k+1}\psi|^2x^{2m+2k+2}\rho_{k+1}^{2k}\lesssim\sum_{i=0}^{k}||\psi||_{J^{i+2}} \tag{3.14.90}$$

We have expanded the product in the first term on the right-hand side of (3.14.89):

$$\partial_x[x^{2m+2k+3}\chi^2_{L,\alpha}\rho^{2k+1}_{k+1}] = Cx^{2m+2k+2}\chi^2_{L,\alpha}\rho^{2k+1}_{k+1} + Cx^{2m+2k+3}(\frac{\alpha}{L})\chi_{L,\alpha}\chi'_{L,\alpha}\rho^{2k+1}_{k+1}$$
$$+ Cx^{2m+2k+3}\chi^2_{L,\alpha}\chi^{2k+1}_{k+1}\rho^{2k}_{k+1}\rho'_{k+1}$$
$$\lesssim x^{2m+2k+2}\rho^{2k}_{k+1}. \tag{3.14.91}$$

Next, we have:

$$\int\int -\partial_x^k\psi_{yy}x^{2m+2}\cdot\partial_x^{k+1}\psi x^{2k+1}\chi^2_{L,\alpha}\rho^{2k+1}_{k+1} = \int\int|\partial_x^k\psi_y|^2\partial_x[x^{2m+2k+3}\chi^2_{L,\alpha}\rho^{2k+1}_{k+1}] \tag{3.14.92}$$

We will expand the product rule above:

$$\partial_x[x^{2m+2k+3}\chi^2_{L,\alpha}\rho^{2k+1}_{k+1}] = Cx^{2m+2k+2}\chi^2_{L,\alpha}\rho^{2k+1}_{k+1} + C\frac{\alpha}{L}x^{2m+2k+3}\chi_{L,\alpha}\chi'_{L,\alpha}\rho^{2k+1}_{k+1}$$
$$+ Cx^{2m+2k+3}\chi^2_{L,\alpha}\rho^{2k}_{k+1}\rho'_k \lesssim x^{2k+2m+2}\rho^{2k}_{k+1}. \tag{3.14.93}$$

Inserting this above yields: $|(3.14.92)| \lesssim \sum_{i=0}^k ||\psi||^2_{J^{i+2}}$. Next, after two integrations by parts in y, and one in x:

$$\int\int \partial_x^k\psi_{yyyy}x^{2m+4}\cdot\partial_x^{k+1}\psi x^{2k+1}\rho^{2k+1}_{k+1}\chi^2_{L,\alpha} = \int\int|\partial_x^k\psi_{yy}|^2\partial_x[x^{2m+2k+5}\rho^{2k+1}_{k+1}\chi^2_{L,\alpha}] \tag{3.14.94}$$

Expanding the product rule above yields:

$$\partial_x[x^{2m+2k+5}\rho^{2k+1}_{k+1}\chi^2_{L,\alpha}] = Cx^{2m+2k+4}\rho^{2k+1}_{k+1}\chi^2_{L,\alpha} + Cx^{2m+2k+5}\rho^{2k}_{k+1}\rho'_{k+1}\chi^2_{L,\alpha}$$
$$+ Cx^{2m+2k+5}\rho^{2k+1}_{k+1}\chi_{L,\alpha}\chi'_{L,\alpha}\frac{\alpha}{L} \lesssim x^{2m+2k+4}\rho^{2k}_{k+1} \tag{3.14.95}$$

Inserting above yields: $|(3.14.94)| \lesssim \sum_{i=0}^{k} ||\psi||^2_{J^{i+2}}$. The next term from $A(\psi)$ in definition (3.12.5) is:

$$
\begin{aligned}
&\int\int \partial_x[(\partial_x^k \psi)_{yyx} x^{2m+4}] \cdot \partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2 \\
&\quad = -\int\int \partial_x^{k+1}\psi_y x^{2m+4} \cdot \partial_x[\partial_x^{k+1}\psi_y x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2] \\
&\quad = -\int\int \partial_x^{k+1}\psi_y x^{2m+4} \cdot \partial_x^{k+2}\psi_y x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2 \\
&\qquad\qquad - \int\int |\partial_x^{k+1}\psi_y|^2 x^{2m+4}\partial_x[x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2] \\
&\quad = \int\int |\partial_x^{k+1}\psi_y|^2 \partial_x[x^{2m+2k+5}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2] \\
&\qquad\qquad - \int\int |\partial_x^{k+1}\psi_y|^2 x^{2m+4}\partial_x[x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2] \\
&\quad \lesssim \int\int |\partial_x^{k+1}\psi_y|^2 x^{2m+2k+4}\rho_{k+1}^{2k} \lesssim \sum_{i=0}^{k} ||\psi||^2_{J^{i+2}}. \qquad (3.14.96)
\end{aligned}
$$

The final term from $A(\psi)$ in definition (3.12.5) is:

$$
\begin{aligned}
&\int\int \partial_{xx}(\partial_x^{k+2}\psi x^{2m+4}) \cdot \partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2 \qquad (3.14.97) \\
&\quad = -\int\int \partial_x^{k+2}\psi x^{2m+4} \cdot \partial_{xx}[\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2] \\
&\quad = -\int\int \partial_x^{k+2}\psi x^{2m+4} \cdot \partial_x^{k+3}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2 \\
&\qquad\qquad - \int\int |\partial_x^{k+2}\psi|^2 x^{2m+4} \cdot \partial_x[x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2] \\
&\qquad\qquad - \int\int |\partial_x^{k+1}\psi|^2 \partial_{xxx}[x^{2m+2k+5}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2] \\
&\quad \lesssim \sum_{i=0}^{k} ||\psi||^2_{J^{i+2}}. \qquad (3.14.98)
\end{aligned}
$$

This concludes the proof of the desired estimate, (3.14.85).

□

Lemma 3.14.10.

$$|\int\int [\partial_x^k, A]\psi \cdot \partial_x^k \psi x^{2k} \chi_{L,\alpha}^2 \rho_{k+1}^{2k}| \lesssim \sum_{i=0}^{k-1} ||\psi||_{J^{i+2}}^2. \tag{3.14.99}$$

Proof. To keep notations simple, we will prove the $k = 1$ case, with the $k \geq 2$ cases following identically. We will proceed term by term from the commutator expression in (3.13.17). First,

$$\begin{aligned} \int\int \psi x^{2m-1} \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 &= -\frac{1}{2} \int\int |\psi|^2 \partial_x [x^{2m+1} \chi_{L,\alpha}^2 \rho_2^2] \\ &\lesssim \int\int \psi^2 x^{2m} \lesssim ||\psi||_{J^2}^2. \end{aligned} \tag{3.14.100}$$

Next,

$$\begin{aligned} \int\int -\psi_{yy} x^{2m+1} \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 &= \int\int \psi_y^2 \partial_x [x^{2m+3} \chi_{L,\alpha}^2 \rho_2^2] \\ &\lesssim \int\int \psi_y^2 x^{2m+2} \lesssim ||\psi||_{J^2}^2. \end{aligned} \tag{3.14.101}$$

Next,

$$\int\int -\partial_x(\psi_x x^{2m+1}) \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 \lesssim \int\int \psi_x^2 x^{2m+2} \lesssim ||\psi||_{J^2}^2. \tag{3.14.102}$$

We will now move to the high order terms, starting with:

$$\begin{aligned} \int\int \psi_{yyyy} x^{2m+3} \cdot \psi_x x^2 \chi_{L,\alpha}^2 \rho_2^2 &= \int\int \psi_{yy} \psi_{yyx} x^{2m+5} \chi_{L,\alpha}^2 \rho_2^2 \\ &= -\frac{1}{2} \int\int |\psi_{yy}|^2 \partial_x [x^{2m+5} \chi_{L,\alpha}^2 \rho_2^2] \lesssim ||\psi||_{J^2}^2. \end{aligned} \tag{3.14.103}$$

Next, again integrating by parts several times:

$$\int\int \partial_x[\psi_{yyx}x^{2m+3}]\cdot\psi_x x^2\chi_{L,\alpha}^2\rho_2^2$$
$$= \int\int \psi_{xy}^2[x^{2m+3}\partial_x[x^2\chi_{L,\alpha}^2\rho_2^2] - \partial_x[x^{2m+5}\chi_{L,\alpha}^2\rho_2^2]] \lesssim ||\psi||_{J^2}^2. \tag{3.14.104}$$

The final term from $A(\psi)$, which after integrating by parts several times in the same way as above,

$$\int\int \partial_{xx}[\psi_{xx}x^{2m+3}]\cdot\psi_x x^2\chi_{L,\alpha}^2\rho_2^2 \lesssim \int\int ||\psi||_{J^2}^2. \tag{3.14.105}$$

This concludes the proof of (3.14.99).

□

Lemma 3.14.11.

$$|\int\int[\partial_x^k, A]\psi\cdot\partial_x^{k+1}\psi x^{2k+1}\chi_{L,\alpha}^2\rho_{k+1}^{2k+1}| \lesssim \sum_{i=0}^{k}||\psi||_{J^{i+2}}. \tag{3.14.106}$$

Proof. This estimate proceeds in the same manner as those from (3.14.99), with the adjustment that the extra derivative in the multiplier from (3.14.106) is accounted for by the increment in order on the right-hand sides of (3.14.106) versus (3.14.99). Indeed, let us take the highest order term from the commutator, $[\partial_x, A]\psi$:

$$|\int\int \partial_{xx}(\psi_{xx}x^{2m+3})\cdot\psi_{xx}x^3\chi_{L,\alpha}^2\rho_2^3| \lesssim \int\int|\psi_{xxx}|^2x^{2m+6}\chi_{L,\alpha}^2\rho_2^3 + ||\psi||_{J^2}^2. \tag{3.14.107}$$

The first term on the right-hand side above can be controlled by $||\psi||_{J^3}^2$, as can be seen from a comparison to (3.14.70) with $k=1$. The remaining terms work identically.

□

Using the above calculations, we may repeat the energy and positivity estimates, for $k \geq 1$:

Lemma 3.14.12 ($k+1$'th order Auxiliary Energy Estimate)**.** *Let* $k = 1, 2$. *Then,*

$$||\partial_x^k u_y \cdot (\rho_{k+1} x)^k||_{L^2}^2 + \alpha ||\psi||_{J^{k+2}}^2 \lesssim \alpha \sum_{i=0}^{k-1} ||\psi||_{J^{i+2}}^2 + \mathcal{O}(\delta) ||\partial_x^k \{\sqrt{\varepsilon} v_x, v_y\} x^{k+\frac{1}{2}} \rho_{k+1}^{k+\frac{1}{2}} ||_{L^2}^2 + \mathcal{W}_1 + \sum_{i=1}^{k} \mathcal{W}_{i+1}. \tag{3.14.108}$$

Proof. We apply the operator ∂_x^k to the system (3.12.10):

$$\Delta_\varepsilon \partial_x^k \psi + \partial_x^k T[\psi] + \alpha A(\partial_x^k \psi) + \alpha [\partial_x^k, A]\psi = \partial_x^k \{F_y - \varepsilon G_x\}. \tag{3.14.109}$$

We subsequently apply the multiplier $\partial_x^k \psi x^{2k} \rho_{k+1}^{2k} \chi_{L,\alpha}^2$:

$$\int \int [\Delta_\varepsilon \partial_x^k \psi + \partial_x^k T[\psi] + \alpha A(\partial_x^k \psi) + \alpha [\partial_x^k, A]\psi] \cdot \partial_x^k \psi x^{2k} \rho_{k+1}^{2k} \chi_{L,\alpha}^2 = \int \int [\partial_x^k \{F_y - \varepsilon G_x\}] \cdot \partial_x^k \psi x^{2k} \rho_{k+1}^{2k} \chi_{L,\alpha}^2. \tag{3.14.110}$$

The desired estimate now follows using similar calculations as in Lemma 3.14.3.

□

Lemma 3.14.13 ($k+1$'th order Auxiliary Positivity Estimate)**.**

$$||\partial_x^k \{\sqrt{\varepsilon} v_x, v_y\} x^{k+\frac{1}{2}} \rho_{k+1}^{k+\frac{1}{2}} ||_{L^2}^2 \lesssim ||\partial_x^k u_y \cdot (\rho_{k+1} x)^k||_{L^2}^2 + \alpha \sum_{i=0}^{k} ||\psi||_{J^{i+2}}^2 + \mathcal{W}_1 + \sum_{i=1}^{k} \mathcal{W}_{i+1}. \tag{3.14.111}$$

Proof. We apply the multiplier $\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2$ to the system (3.14.109):

$$\int\int[\Delta_\varepsilon\partial_x^k\psi+\partial_x^kT[\psi]+\alpha A(\partial_x^k\psi)+\alpha[\partial_x^k,A]\psi]\cdot\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2$$
$$=\int\int[\partial_x^k\{F_y-\varepsilon G_x\}]\cdot\partial_x^{k+1}\psi x^{2k+1}\rho_{k+1}^{2k+1}\chi_{L,\alpha}^2. \tag{3.14.112}$$

The desired estimate now follows using similar calculations as in Lemma 3.14.4.

□

3.15 Step 3: Nonlinear Existence of Auxiliary Systems

For this subsection, it is necessary to be more precise with notation; we will index solutions by (α, N) and also specify domains over which norms are being taken. We shall also transition our right-hand sides from being generic (F, G) to being the particular right-hand sides of interest, $(\tilde{f}, g)$ as defined in (3.12.12). Our intention now is to study the map, M^α:

$$\begin{aligned}&M^\alpha[\bar{u}^{\alpha,N},\bar{v}^{\alpha,N}]=[u^{\alpha.N},v^{\alpha,N}]\\&\quad\iff L_{\alpha,\bar{v}^{\alpha,N}}[u^{\alpha,N},v^{\alpha,N}]=\tilde{f}_y(\bar{u}^{\alpha,N},\bar{v}^{\alpha,N})-\varepsilon g_x(\bar{u}^{\alpha,N},\bar{v}^{\alpha,N})\\&\quad\iff [u^{\alpha,N},v^{\alpha,N}]=L^{-1}_{\alpha,\bar{v}^{\alpha,N}}\{\tilde{f}_y(\bar{u}^{\alpha,N},\bar{v}^{\alpha,N})-\varepsilon g_x(\bar{u}^{\alpha,N},\bar{v}^{\alpha,N})\}.\end{aligned} \tag{3.15.1}$$

which corresponds to the system written in vorticity form:

$$\Delta_\varepsilon^2\psi^{\alpha,N}+\alpha A(\psi^{\alpha,N})+T(\psi^{\alpha,N};\bar{v}^{\alpha,N})=\tilde{f}_y(\bar{u}^{\alpha,N},\bar{v}^{\alpha,N})-\varepsilon g_x(\bar{u}^{\alpha,N},\bar{v}^{\alpha,N})\quad\text{on }\Omega^N. \tag{3.15.2}$$

A fixed point of (3.15.2) corresponds to the desired solution of (3.12.13). By repeating the analysis in Section 3.9,the energy and positivity estimates in Section 3.10, and finally the estimates on $\mathcal{W}_i$ in Lemma 3.11.1 for the system, one obtains

Lemma 3.15.1. *Suppose* $||\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}||_{Z(\Omega^N)} \leq 1$*. Fix any open set* $B \subset \Omega^N$*. Let* $\alpha > 0$ *and* $N >> 1$*. Solutions* $\psi^{\alpha,N}$*, or equivalently* $[u^{\alpha,N}, v^{\alpha,N}]$*, to the system (3.15.2) satisfy the following estimates, independent of* N*, where* $\omega(N_i)$ *is based on universal constants:*

$$\begin{aligned} &\varepsilon^{N_0} C(B)||\psi^{\alpha,N}||_{H^5(B)} + ||u^{\alpha,N}, v^{\alpha,N}||_{Z(\Omega^N)} \\ &\qquad \lesssim \epsilon^{100} + ||u^{\alpha,N}, v^{\alpha,N}||_{X_1 \cap X_2 \cap X_3(\Omega^N)} + \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}||^2_{Z(\Omega^N)}. \end{aligned} \tag{3.15.3}$$

The following energy and positivity estimates hold:

$$\alpha||\psi^{\alpha,N}||^2_{H^4_w(\Omega^N)} + ||u^{\alpha,N}, v^{\alpha,N}||^2_{X_1 \cap X_2 \cap X_3(\Omega^N)} \lesssim \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3, \tag{3.15.4}$$

Finally, one has:

$$\begin{aligned} &\alpha||\psi^{\alpha,N}||^2_{H^4_w(\Omega^N)} + \varepsilon^{N_0} C(B)||\psi^{\alpha,N}||^2_{H^5(B)} + ||u^{\alpha,N}, v^{\alpha,N}||^2_{Z(\Omega^N)} \\ &\qquad \lesssim \varepsilon^{\frac{1}{4}-\gamma-\kappa} + \varepsilon^{\frac{n}{2}-\omega(N_i)}||\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}||^4_{Z(\Omega^N)}. \end{aligned} \tag{3.15.5}$$

All constants appearing in the above estimates are independent of (α, N)*.*

Proof of Estimate (3.15.3). This follows by repeating the proofs of elliptic regularity in Subsection 3.9.1, namely Lemmas 3.9.10 and 3.9.12, to the new system, (3.15.2). The only new term in (3.15.2) as compared to 3.8.1 - (3.8.3), (3.8.8) - (3.8.9) are $\alpha A(\psi)$. The proof of Lemmas 3.9.10 and 3.9.12 then follows identically, as these are local-in-x estimates, which are unaffected by the weights in $A(\psi)$. At this point, one repeats the estimates in Subsection 3.9.2, which hold independent of any equation.

□

Proof of Estimate 3.15.4. This follows from Lemmas 3.14.12 - 3.14.13, and subsequently comparing $||\cdot||_{J^k}$ with $||\cdot||_{H^k_w}$. □

Proof of Estimate 3.15.5. This follows by repeating the proof of Lemma 3.11.1.

□

Motivated by (3.15.5), we define the notation:

$$||u^{\alpha,N},v^{\alpha,N}||_{\mathcal{F}(\Omega^N)} := \alpha||\psi^{\alpha,N}||^2_{H^4_w(\Omega^N)} + ||u^{\alpha,N},v^{\alpha,N}||^2_{Z(\Omega^N)}. \tag{3.15.6}$$

Lemma 3.15.2 (Properties of M^α)**.** *Fix any $\alpha > 0$ and any $N > 0$, and $\gamma, \kappa > 0$ arbitrarily small.*

(1) *$M^\alpha : B_Z(1) \subset Z(\Omega^N) \to B_Z(1) \subset Z(\Omega^N)$, where $B_Z(1)$ is the unit ball in $Z(\Omega^N)$;*

(2) *M^α is continuous and compact as an operator on $B_Z(1)$.*

(3) *There exists a fixed point, $[u^{\alpha,N},v^{\alpha,N}] = M^\alpha[u^{\alpha,N},v^{\alpha,N}]$ in $B_Z(1)$.*

(4) *The fixed point satisfies, $||u^{\alpha,N},v^{\alpha,N}||_{Z(\Omega^N)} \lesssim \varepsilon^{\frac{1}{4}-\gamma-\kappa}$, independent of α, N.*

(3.15.7)

Proof. The outline of this proof is as follows. The map M^α is shown to be well-defined in the appropriate domains and codomains, according to (1) above. Continuity of M^α is investigated by considering differences, and compactness of M^α is obtained using our compactness lemmas above. One then applies a fixed point argument to prove (3) and (4).

(1) Suppose $[\bar{u},\bar{v}] \in Z(\Omega^N)$. This implies that $(\tilde{f}, g) \in H^{-1}_2$, so by (3.14.67), the map M^α is well-defined on $Z(\Omega^N)$. The property (3.9.9) is verified according to Lemma

3.13.6, and the definition of $H^2_w(\Omega^N)$, Definition 3.13.1, which ensures that $[u^\alpha, v^\alpha]$ are contained in $\overline{C^\infty_{0,D}}^{||\cdot||_{X_1}}$. Supposing the pre-images are contained in the unit ball of $Z(\Omega^N)$, $||\bar{u}^{\alpha,N}, \bar{v}^{\alpha,N}||_{Z(\Omega^N)} \leq 1$, one has estimate (3.15.5), which implies that $M^\alpha(\bar{u}, \bar{v}) \in B_Z(1)$.

(2) To check continuity of the map M^α on $B_Z(1)$, suppose:

$$M^\alpha[\bar{u}^{\alpha,N}_i, \bar{v}^{\alpha,N}_i] = [u^{\alpha,N}_i, v^{\alpha,N}_i] \text{ for } i = 1, 2, \tag{3.15.8}$$

where

$$||\bar{u}^{\alpha,N}_i, \bar{v}^{\alpha,N}_i||_Z \leq 1. \tag{3.15.9}$$

Define the notation for the differences,

$$[\hat{\bar{\psi}}, \hat{\bar{u}}, \hat{\bar{v}}] = [\bar{\psi}^{\alpha,N}_2 - \bar{\psi}^{\alpha,N}_1, \bar{u}^{\alpha,N}_2 - \bar{u}^{\alpha,N}_1, \bar{v}^{\alpha,N}_2 - \bar{v}^{\alpha,N}_1], \tag{3.15.10}$$

$$[\hat{\psi}, \hat{u}, \hat{v}] = [\psi^{\alpha,N}_2 - \psi^{\alpha,N}_1, u^{\alpha,N}_2 - u^{\alpha,N}_1, v^{\alpha,N}_2 - v^{\alpha,N}_1]. \tag{3.15.11}$$

By consulting (3.15.2), one then obtains the following system satisfied by the differences:

$$\begin{aligned}\Delta^2_\varepsilon \hat{\psi} + \alpha A(\hat{\psi}) + T(\hat{\psi}) &= \tilde{f}_y(u^{\alpha,N}_2, \bar{u}^{\alpha,N}_2, \bar{v}^{\alpha,N}_2) - \tilde{f}_y(u^{\alpha,N}_1, \bar{u}^{\alpha,N}_1, \bar{v}^{\alpha,N}_1) \\ &\quad - \varepsilon g_x(\bar{u}^{\alpha,N}_2, \bar{v}^{\alpha,N}_2) + \varepsilon g_x(\bar{u}^{\alpha,N}_1, \bar{v}^{\alpha,N}_1). \end{aligned} \tag{3.15.12}$$

We may then repeat the estimates which resulted in (3.15.3) - (3.15.5) to obtain:

$$||\hat{u}, \hat{v}||^2_{Z(\Omega^N)} \lesssim \frac{1}{\alpha^2} ||\hat{\bar{u}}, \hat{\bar{v}}||^2_{Z(\Omega^N)}. \tag{3.15.13}$$

The only non-trivial calculation when repeating the estimates which resulted in (3.15.3) - (3.15.5) is to handle the nonlinearity (3.11.5) under taking differences. For this, we first write:

$$\begin{aligned}
&\bar{v}_2^{\alpha,N} u_{2y}^{\alpha,N} - \bar{v}_1^{\alpha,N} u_{1y}^{\alpha,N} \\
&\qquad = \bar{v}_2^{\alpha,N} u_{2y}^{\alpha,N} - \bar{v}_2^{\alpha,N} u_{1y}^{\alpha,N} + \bar{v}_2^{\alpha,N} u_{1y}^{\alpha,N} - \bar{v}_1^{\alpha,N} u_{1y}^{\alpha,N} \\
&\qquad = \bar{v}_2^{\alpha,N} \hat{u}_y + \hat{\bar{v}} u_{1,y}^{\alpha,N}. \qquad (3.15.14)
\end{aligned}$$

Repeating calculation (3.11.5) then yields:

$$\begin{aligned}
\int\int \varepsilon^{\frac{n}{2}+\gamma} [\bar{v}_2^{\alpha,N} u_{2y}^{\alpha,N} - \bar{v}_1^{\alpha,N} u_{1y}^{\alpha,N}] \cdot \hat{u} &= \int\int \varepsilon^{\frac{n}{2}+\gamma} [\bar{v}_2^{\alpha,N} \hat{u}_y + \hat{\bar{v}} u_{1,y}^{\alpha,N}] \cdot \hat{u} \\
&= -\int\int \frac{\varepsilon^{\frac{n}{2}+\gamma}}{2} \hat{u}^2 \bar{v}_{2y}^{\alpha,N} + \int\int \varepsilon^{\frac{n}{2}+\gamma} \hat{\bar{v}} u_{1,y}^{\alpha,N} \hat{u}
\end{aligned} \qquad (3.15.15)$$

For the first term in the right-hand side above, we give the same estimate as in (3.11.5), which shows:

$$|\int\int \frac{\varepsilon^{\frac{n}{2}+\gamma}}{2} \hat{u}^2 \bar{v}_{2y}^{\alpha,N}| \lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\hat{u}||^2_{Z(\Omega^N)} ||\bar{u}_2^{\alpha,N}, \bar{v}_2^{\alpha,N}||_{Z(\Omega^N)} \lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\hat{u}||^2_{Z(\Omega^N)}. \qquad (3.15.16)$$

This then gets absorbed into the left-hand side of (3.15.13). For the second term on the right-hand side above, we estimate:

$$\begin{aligned}
|\int\int \varepsilon^{\frac{n}{2}+\gamma} \hat{\bar{v}} u_{1,y}^{\alpha,N} \hat{u}| &\le \varepsilon^{\frac{n}{2}+\gamma} ||\hat{\bar{v}} x^{\frac{1}{2}}||_{L^\infty} ||u_{1,y}^{\alpha,N} x^m||_{L^2} ||\hat{u} x^{-m-\frac{1}{2}}||_{L^2} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\hat{\bar{v}}||_{Z(\Omega^N)} ||u_1^{\alpha,N}||_{H^2_w(\Omega^N)} ||\hat{u}||_{Z(\Omega^N)} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\hat{\bar{v}}||_{Z(\Omega^N)} \frac{1}{\alpha} ||u_1^{\alpha,N}||_{\mathcal{F}(\Omega^N)} ||\hat{u}||_{Z(\Omega^N)} \\
&\lesssim \varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)} ||\hat{\bar{v}}||_{Z(\Omega^N)} \frac{1}{\alpha} ||\hat{u}||_{Z(\Omega^N)} \\
&\lesssim \varepsilon^{2(\frac{n}{2}+\gamma-\omega(N_i))} ||\hat{u}||^2_{Z(\Omega^N)} + \frac{1}{\alpha^2} ||\hat{\bar{v}}||^2_{Z(\Omega^N)}, \qquad (3.15.17)
\end{aligned}$$

where we have used (3.15.9) coupled with (3.15.5) to conclude that: $||u_1^{\alpha,N}||_{\mathcal{F}(\Omega^N)} \lesssim \varepsilon^{\frac{1}{4}-\gamma-\kappa}$. The weight, x^m, arises from the definition (3.12.5), and consequently in (3.13.2). The first term on the right-hand side of (3.15.17) is absorbed into the left-hand side of (3.15.13), whereas the second term contributes to the right-hand side of (3.15.13). All of the remaining calculations which produced (3.15.5) can be repeated in a similar fashion. Estimate (3.15.13) then implies the continuity of M^α on $B_Z(1)$. The modulus of continuity of M^α is $\frac{1}{\alpha^2}$, which prevents M^α from being a contraction map. Nevertheless, continuity is retained for all $\alpha > 0$.

We now turn to compactness. According to Lemma 3.14.2, (3.15.5) shows that $M^\alpha(B_Z(1))$ is compactly embedded in $B_Z(1)$ so long as m is sufficiently large.

(3 and 4) Consider the family of solutions:

$$[u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}] = \lambda M^\alpha[u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}], \qquad \text{for } 0 \le \lambda \le 1. \tag{3.15.18}$$

By (3.15.1) and linearity of L_α^{-1}, this occurs if and only if

$$[u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}] = L_\alpha^{-1}\{\lambda \tilde{f}_y(u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}) - \varepsilon\lambda g_x(u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N})\}. \tag{3.15.19}$$

By repeating the estimates which culminated in (3.15.5), one sees the uniform in λ bound:

$$||u_\lambda^{\alpha,N}, v_\lambda^{\alpha,N}||_{Z(\Omega^N)}^2 \lesssim \varepsilon^{\frac{1}{4}-\gamma-\kappa}. \tag{3.15.20}$$

Thus, Schaefer's fixed point theorem applied to the convex subset $B_Z(1) \subset Z(\Omega^N)$ produces a fixed point, $[u^{\alpha,N}, v^{\alpha,N}] \in B_Z(1)$. The estimate it obeys follows from (3.15.5). □

3.16 Step 4: Nonlinear Existence

We now need to pass to the limit as $\alpha \to 0$ and as $N \to \infty$. The fixed point of the system (3.15.2), from Lemma 3.15.2 satisfies the following integral identity for any $\phi \in C_0^\infty(\Omega^N)$:

$$\begin{aligned}\int\int_{\Omega^N} \nabla_\epsilon^2 \psi^{N,\alpha} : \nabla_\epsilon^2 \phi + \alpha \Big[\int\int_{\Omega^N} \psi^{N,\alpha} \phi x^{2m} + \int\int_{\Omega^N} \nabla \psi^{N,\alpha} \cdot \nabla \phi x^{2m+2} \\ + \int\int_{\Omega^N} \nabla^2 \psi^{N,\alpha} : \nabla^2 \phi x^{2m+4} \Big] + \int\int_{\Omega^N} -S_u \cdot \phi_y + \varepsilon S_v \cdot \phi_x \\ + \int\int_{\Omega^N} \varepsilon^{-\frac{n}{2}-\gamma} \Big[R^{u,n} \cdot \phi_y - \varepsilon R^{v,n} \cdot \phi_x \Big] \\ = \int\int_{\Omega^N} \varepsilon^{\frac{n}{2}+\gamma} \Big[-u^{N,\alpha} u_x^{N,\alpha} \phi_y - v^{N,\alpha} u_y^{N,\alpha} \phi_y + \varepsilon u^{N,\alpha} v_x^{N,\alpha} \phi_x + \varepsilon v^{N,\alpha} v_y^{N,\alpha} \phi_x \Big]. \quad (3.16.1)\end{aligned}$$

First, we shall pass to the limit as $\alpha \to 0$, fixing an N. To do so, we first use (3.15.7) to obtain a weak subsequential limit point:

$$u^{N,\alpha} \rightharpoonup u^N, \text{ weakly in } (X_1 \cap X_2 \cap X_3)(\Omega^N). \quad (3.16.2)$$

It is now our task to pass to the limit in the equation, (3.16.1), along the subsequence $\alpha \to 0$. Given a test-function, denote by U_ϕ to be the support of ϕ. As U_ϕ is bounded, we have Poincare inequalities available:

$$\begin{aligned}\alpha \Big| \Big[\int\int_{\Omega^N} \psi^{N,\alpha} \phi x^{2m} + \int\int_{\Omega^N} \nabla \psi^{N,\alpha} \cdot \nabla \phi x^{2m+2} + \int\int_{\Omega^N} \nabla^2 \psi^{N,\alpha} : \nabla^2 \phi x^{2m+4} \Big] \Big| \\ \le C(\phi) \alpha \Big[||\psi^{N,\alpha}||_{L^2(U_\phi)} + ||\nabla \psi^{N,\alpha}||_{L^2(U_\phi)} + ||\nabla^2 \psi^{N,\alpha}||_{L^2(U_\phi)} \Big] \\ \le C(\phi) \alpha ||\nabla u^{N,\alpha}, \nabla v^{N,\alpha}||_{L^2(U_\phi)} \le C(\phi) \alpha ||u^{N,\alpha}, v^{N,\alpha}||_{Z(\Omega^N)} \xrightarrow{\alpha \to 0} 0. \quad (3.16.3)\end{aligned}$$

For all of the linear terms, we use the weak convergence in $(X_1 \cap X_2 \cap X_3)(\Omega^N)$:

$$\lim_{\alpha \to 0} \int\int_{\Omega^N} \nabla_\varepsilon^2 \psi^{\alpha,N} : \nabla_\varepsilon^2 \phi - \int\int_{\Omega^N} S_u(u^{N,\alpha}, v^{N,\alpha}) \phi_y + \varepsilon S_v(u^{N,\alpha}, v^{N,\alpha}) \cdot \phi_x$$

$$= \int\int_{\Omega^N} \nabla_\varepsilon^2 \psi^N : \nabla_\varepsilon^2 \phi - \int\int_{\Omega^N} S_u(u^N, v^N)\phi_y + \varepsilon S_v(u^N, v^N) \cdot \phi_x. \tag{3.16.4}$$

Finally, we turn to the nonlinear terms for which we integrate by parts:

$$\int\int_{\Omega^N} u^{N,\alpha} u_x^{N,\alpha} \phi_y + v^{N,\alpha} u_y^{N,\alpha} \phi_y = \int\int_{\Omega^N} -|u^{N,\alpha}|^2 \phi_{xy} - u^{N,\alpha} v^{N,\alpha} \phi_{yy}, \tag{3.16.5}$$

$$\int\int_{\Omega^N} u^{N,\alpha} v_x^{N,\alpha} \phi_x + v^{N,\alpha} v_y^{N,\alpha} \phi_x = \int\int_{\Omega^N} -|v^{N,\alpha}|^2 \phi_{xy} - u^{N,\alpha} v^{N,\alpha} \phi_{xx}. \tag{3.16.6}$$

Fixing a compactly supported ϕ, we can localize the integrations above to U_ϕ. On this set, the weak convergence of $u^{N,\alpha} \rightharpoonup u^N$ implies strong convergence in L^2. Thus,

$$\begin{aligned} &| \int\int_{U_\phi} \Big[|u^{N,\alpha}|^2 - u^{N,\alpha} u^N + u^{N,\alpha} u^N - |u^N|^2 \Big] \phi_{xy} \\ &\qquad \lesssim ||u^{N,\alpha} - u^N||_{L^2(U_\phi)} ||u^{N,\alpha}||_{L^2(U_\phi)} + ||u^N||_{L^2(U_\phi)} ||u^{N,\alpha} - u^N||_{L^2(U_\phi)}. \end{aligned} \tag{3.16.7}$$

The right-hand side converges to zero. The same bound works for all of the other nonlinear terms. Thus, the weak limit $[u^N, v^N]$ or equivalently ψ^N satisfies the weak formulation:

$$\begin{aligned} &\int\int_{\Omega^N} \nabla_\epsilon^2 \psi^N : \nabla_\epsilon^2 \phi - \int\int_{\Omega^N} S_u(u^N, v^N) \cdot \phi_y + \varepsilon S_v(u^N, v^N) \cdot \phi_x \\ &\qquad\qquad + \int\int_{\Omega^N} \varepsilon^{-\frac{n}{2} - \gamma} \Big[R^{u,n} \cdot \phi_y - \varepsilon R^{v,n} \cdot \phi_x \Big] \\ &= \int\int_{\Omega^N} \varepsilon^{\frac{n}{2} + \gamma} \Big[- u^N u_x^N \phi_y - v^N u_y^N \phi_y + \varepsilon u^N v_x^N \phi_x + \varepsilon v^N v_y^N \phi_x \Big]. \end{aligned} \tag{3.16.8}$$

The weak limit $[u^N, v^N]$ must satisfy the bound:

$$||u^N, v^N||_{(X_1 \cap X_2 \cap X_3)(\Omega^N)} \lesssim C(u_R, v_R) \varepsilon^{\frac{1}{4} - \gamma - \kappa}, \tag{3.16.9}$$

independent of N. We may now repeat this exact procedure with the subsequential N

limit: denote by $[u,v]$ and ψ the subsequential $(X_1 \cap X_2 \cap X_3)(\Omega)$-weak limit as $N \to \infty$, guaranteed by (3.16.9). One then passes to the limit in the equation (3.16.8) to obtain:

$$\begin{aligned}\int\int_\Omega \nabla_\epsilon^2 \psi : \nabla_\epsilon^2 \phi - \int\int_\Omega S_u(u,v)\cdot\phi_y + \varepsilon S_v(u,v)\cdot\phi_x \\ + \int\int_\Omega \varepsilon^{-\frac{n}{2}-\gamma}\Big[R^{u,n}\cdot\phi_y - \varepsilon R^{v,n}\cdot\phi_x\Big] \\ = \int\int_\Omega \varepsilon^{\frac{n}{2}+\gamma}\Big[-uu_x\phi_y - vu_y\phi_y + \varepsilon uv_x\phi_x + \varepsilon vv_y\phi_x\Big],\end{aligned} \tag{3.16.10}$$

with the limit satisfying:

$$||u,v||_{(X_1\cap X_2\cap X_3)(\Omega)} \lesssim \varepsilon^{\frac{1}{4}-\gamma-\kappa}. \tag{3.16.11}$$

We now state the main existence result:

Theorem 3.16.1. *For ε, δ sufficiently small, $\kappa > 0$ small, and $0 \le \gamma < \frac{1}{4}$, there exists a solution to the system (3.8.1) - (3.8.3), (3.8.4), (3.8.5) satisfying:*

$$||u,v||_{Z(\Omega)} \lesssim C(u_R, v_R)\varepsilon^{\frac{1}{4}-\gamma-\kappa}. \tag{3.16.12}$$

Proof. Estimate (3.16.11) implies enough regularity to integrate by parts identity (3.16.1) to:

$$\int\int_\Omega \Big[\Delta_\varepsilon^2\psi + \partial_y S_u - \varepsilon\partial_x S_v - \partial_y f + \varepsilon\partial_x g\Big]\cdot\phi = 0, \tag{3.16.13}$$

which then implies that the PDE is satisfied pointwise in Ω. The boundary conditions (3.8.4) are satisfied by elements in $(X_1 \cap X_2 \cap X_3)(\Omega)$, according to Lemma 3.9.16. From here, one repeats the embedding theorems in Section 3.9 which give estimate (3.16.12). That this is possible for those embeddings in Subsection 3.9.2 is straightforward to see, as these did not require $[u,v]$ to satisfy any equations. Let us then turn to Subsection 3.9.1.

We must repeat the proofs of Lemmas 3.9.10 and 3.9.12 to the *nonlinear* system, (3.8.1) - (3.8.3), with f, g as in (3.8.5). This amounts to replacing $[\bar{u}, \bar{v}]$ with $[u, v]$ in Lemmas 3.9.10 and 3.9.12, and foregoing the assumption that $||\bar{u}, \bar{v}||_Z \le 1$. A nearly identical proof to Lemma 3.9.10 then yields:

$$\sup_{x \le 2000} ||u, v||_{L^\infty_y} + ||u, v||_{\dot{H}^2(x \le 2000)} \lesssim \varepsilon^{-M_2}. \tag{3.16.14}$$

One now bootstraps the estimate in Lemma 3.9.12 in the identical manner. This gives estimate (3.16.12). We have verified that $[u, v] \in Z(\Omega)$ satisfies (3.8.1) - (3.8.3), (3.8.4), (3.8.5). $\square$

3.17 Step 5: Uniqueness

In this final subsection, we prove uniqueness of the solution $[u, v]$ from Theorem 3.16.1. Suppose there existed two solutions, $[u_1, v_1]$ and $[u_2, v_2]$ to the system in (3.8.1) - (3.8.3), (3.8.4), (3.8.5). Define:

$$\hat{u} = u_1 - u_2, \quad \hat{v} = v_1 - v_2, \quad \hat{P} = P_1 - P_2. \tag{3.17.1}$$

Then the new unknowns satisfy:

$$-\Delta_\epsilon \hat{u} + S_u(\hat{u}, \hat{v}) + \hat{P}_x = \hat{f} := \varepsilon^{\frac{n}{2}+\gamma}\Big[u_1 u_{1x} - u_2 u_{2x} + v_1 u_{1y} - v_2 u_{2y}\Big], \tag{3.17.2}$$

$$-\Delta_\epsilon \hat{v} + S_v(\hat{u}, \hat{v}) + \frac{\hat{P}_y}{\epsilon} = \hat{g} := \varepsilon^{\frac{n}{2}+\gamma}\Big[u_1 v_{1x} - u_2 v_{2x} + v_1 v_{1y} - v_2 v_{2y}\Big], \tag{3.17.3}$$

together with the divergence-free condition, $\hat{u}_x + \hat{v}_y = 0$, and also satisfy the boundary conditions:

$$\{\hat{u}, \hat{v}\}|_{\{y=0\}} = \{\hat{u}, \hat{v}\}|_{\{x=1\}} = 0. \tag{3.17.4}$$

Going to vorticity,

$$\partial_y \Big[-\Delta_\varepsilon \hat{u} + S_u(\hat{u}, \hat{v}) \Big] - \varepsilon \partial_x \Big[-\Delta_\varepsilon v + S_v(\hat{u}, \hat{v}) \Big] = \varepsilon^{\frac{n}{2}} \Big\{ \partial_y \Big[u_1 u_{1x} - u_2 u_{2x} + v_1 u_{1y} - v_2 u_{2y} \Big] - \varepsilon \partial_x \Big[u_1 v_{1x} - u_2 v_{2x} + v_1 v_{1y} - v_2 v_{2y} \Big] \Big\}. \tag{3.17.5}$$

We shall repeat the basic energy and positivity estimates using a slightly weaker weight. It is convenient to work with the weak formulation, which is given in (3.16.10). Then, $\hat{u}, \hat{v}$ satisfy the following:

$$\int\int \nabla_\epsilon^2 \hat{\psi} : \nabla_\epsilon^2 \phi + \int\int \varepsilon S_v(\hat{u}, \hat{v}) \cdot \phi_x - S_u(\hat{u}, \hat{v}) \cdot \phi_y = \int\int \Big[-\hat{f}\phi_y + \varepsilon \hat{g} \phi_x \Big], \tag{3.17.6}$$

for all $\phi \in C_0^\infty(\Omega)$. We make the notational convention that

$$\int\int := \int\int_\Omega . \tag{3.17.7}$$

Lemma 3.17.1. *There exists a $0 < b < 1$, sufficiently close to 0, depending only on universal constants, such that for δ, ε sufficiently small and $\varepsilon << \delta << b$, the solutions $[\hat{u}, \hat{v}] \in Z$ to the system (3.17.2) - (3.17.3) with boundary conditions (3.17.4) satisfy the following estimate:*

$$b||\{\hat{u}, \sqrt{\varepsilon}\hat{v}\} x^{-b-\frac{1}{2}}||_{L^2}^2 + ||\hat{u}_y x^{-b}||_{L^2}^2 \lesssim \mathcal{O}(\delta) ||\{\sqrt{\varepsilon}\hat{v}_x, \hat{v}_y\} x^{\frac{1}{2}-b}||_{L^2}^2 + \mathcal{W}_{1,E,b}, \tag{3.17.8}$$

where

$$\mathcal{W}_{1,E,b} := \int\int \hat{f}\hat{u}x^{-2b} + \varepsilon\hat{g}\hat{v}x^{-2b} - 2b\varepsilon\hat{g}\hat{\psi}x^{-2b-1}, \tag{3.17.9}$$

$$\mathcal{W}_{1,P,b} := \int\int \hat{f}\hat{u}_x x^{1-2b} + \varepsilon\hat{g}\hat{v}_x x^{1-2b}, \tag{3.17.10}$$

$$\mathcal{W}_{1,b} = \mathcal{W}_{1,E,b} + \mathcal{W}_{1,P,b}. \tag{3.17.11}$$

Proof. The estimate will follow upon applying the multiplier $\hat{\psi} \cdot x^{-2b}$ to the system in (3.17.5). To work rigorously, we will apply approximate multipliers, and work with the weak formulation given in (3.17.6). Fix $[\hat{u}^n, \hat{v}^{(n)}, \hat{\psi}^{(n)}] \in C_0^\infty(\Omega)$, such that:

$$[\hat{u}^{(n)}, \hat{v}^{(n)}] \xrightarrow{X_1} [\hat{u}, \hat{v}], \tag{3.17.12}$$

where X_1 is defined in (3.9.3). Within the notation of (3.17.6), $\phi = \hat{\psi}^{(n)}x^{-2b}$. The existence of the sequence specified in (3.17.12) is guaranteed by $[\hat{u}, \hat{v}] \in Z(\Omega)$. That ϕ is compactly supported in (x, y) follows from the representations:

$$\hat{\psi}^{(n)} = -\int_0^y \hat{u}^{(n)} = \int_0^x \hat{v}^{(n)}. \tag{3.17.13}$$

Let us first treat the second-order terms:

$$\begin{aligned}\int\int \nabla_\varepsilon^2\hat{\psi} : \nabla_\varepsilon^2\phi &= \int\int \nabla_\varepsilon^2\hat{\psi} : \nabla_\varepsilon^2(\hat{\psi}^{(n)}x^{-2b}) \\ &= \int\int \hat{\psi}_{yy}\hat{\psi}^{(n)}_{yy}x^{-2b} + 2\varepsilon\hat{\psi}_{xy}\partial_x\Big(\hat{\psi}^{(n)}_y x^{-2b}\Big) + \varepsilon^2\hat{\psi}_{xx}\partial_{xx}\Big(\hat{\psi}^{(n)}x^{-2b}\Big).\end{aligned} \tag{3.17.14}$$

The first two terms from (3.17.14) above are:

$$\int\int \hat{\psi}_{yy}\hat{\psi}^{(n)}_{yy}x^{-2b} + 2\varepsilon\hat{\psi}_{xy}\hat{\psi}^{(n)}_{xy}x^{-2b} + 2\varepsilon\hat{\psi}_{xy}\hat{\psi}^{(n)}_y\partial_x x^{-2b}$$

$$= \int\int \hat{u}_y \hat{u}_y^{(n)} x^{-2b} + 2\varepsilon \hat{u}_x \hat{u}_x^{(n)} x^{-2b} - 2\varepsilon \hat{u}_x \hat{u}^{(n)} \partial_x x^{-2b}. \tag{3.17.15}$$

We shall take the limit as $n \to \infty$ above. According to the definition (3.9.3), the convergence in (3.17.12) implies:

$$\begin{aligned} &|\int\int \hat{u}_y(\hat{u}_y^{(n)} - \hat{u}_y)x^{-2b}| + |\int\int \hat{u}_x(\hat{u}_x^{(n)} - \hat{u}_x)x^{-2b}| \\ &\qquad + |\int\int \hat{u}_x(\hat{u}^{(n)} - \hat{u})\partial_x x^{-2b}| \xrightarrow{n\to\infty} 0. \end{aligned} \tag{3.17.16}$$

Expanding the third term from (3.17.14),

$$\int\int \varepsilon^2 \hat{\psi}_{xx} \partial_{xx}\Big(\hat{\psi}^{(n)} x^{-2b}\Big) = \int\int \varepsilon^2 \hat{v}_x \cdot \Big[\hat{v}_x^{(n)} x^{-2b} + 2\hat{v}^{(n)} \partial_x x^{-2b} + \hat{\psi}^{(n)} \partial_{xx} x^{-2b}\Big]. \tag{3.17.17}$$

By referring to the definition of X_1 in (3.9.3) and (3.17.12), we may pass to the limit:

$$\begin{aligned} \text{Equation (3.17.15)} &\xrightarrow{n\to\infty} \int\int [\hat{u}_y^2 + 2\varepsilon \hat{u}_x^2] x^{-2b} - \int\int \varepsilon \hat{u}_x \hat{u} \partial_x x^{-2b} \\ &= \int\int [\hat{u}_y^2 + 2\varepsilon \hat{u}_x^2] x^{-2b} + b(2b+1)\varepsilon \hat{u}^2 x^{-2-2b} + \varepsilon b \lim_{M\to\infty} \int_{x=M} \hat{u}^2 x^{-1-2b} \\ &= \int\int [\hat{u}_y^2 + 2\varepsilon \hat{u}_x^2] x^{-2b} + b(2b+1)\varepsilon \hat{u}^2 x^{-2-2b}, \end{aligned} \tag{3.17.18}$$

and:

$$\text{Equation (3.17.17)} \xrightarrow{n\to\infty} \int\int \varepsilon^2 \hat{v}_x^2 x^{-2b} - 4b\varepsilon^2 \hat{v}_x \hat{v} x^{-1-2b} + 2b(2b+1)\varepsilon^2 \hat{v}_x \hat{\psi} x^{-2-2b}. \tag{3.17.19}$$

Integrating by parts the final two terms above in (3.17.19), and referring to estimate

(3.9.71),

$$-4b\int\int \varepsilon^2 \hat{v}_x \hat{v} x^{-1-2b} = \int\int 2b\varepsilon^2 \hat{v}^2 \partial_x x^{-1-2b} - 2b \lim_{M\to\infty}\int_{x=M} \varepsilon^2 \hat{v}^2 x^{-1-2b}$$
$$= -2b(1+2b)\int\int \varepsilon^2 \hat{v}^2 x^{-2-2b}, \tag{3.17.20}$$

and similarly, to treat the final term in (3.17.19), we appeal to the estimates in (3.9.71):

$$\int\int \varepsilon^2 \hat{v}_x \hat{\psi} x^{-2-2b} = -\int\int \varepsilon^2 \hat{v}^2 x^{-2-2b} - \int\int \varepsilon^2 \hat{v}\hat{\psi}\partial_x x^{-2-2b}$$
$$+ \lim_{M\to\infty}\int_{x=M} \varepsilon^2 \hat{v}\hat{\psi} x^{-2-2b} \tag{3.17.21}$$
$$= -\int\int \varepsilon^2 \hat{v}^2 x^{-2-2b} + \int\int \frac{(2b+3)(2b+2)}{2}\varepsilon^2 \hat{\psi}^2 x^{-4-2b}$$
$$+ \lim_{M\to\infty}\int_{x=M} \frac{2b+2}{2}\varepsilon^2 \hat{\psi}^2 x^{-3-2b} \tag{3.17.22}$$
$$= -\int\int \varepsilon^2 \hat{v}^2 x^{-2-2b} + \int\int \frac{(2b+3)(2b+2)}{2}\varepsilon^2 \hat{\psi}^2 x^{-4-2b}. \tag{3.17.23}$$

Therefore, summarizing the highest order calculation:

$$\int\int \nabla_\varepsilon^2 \hat{\psi} : \nabla_\varepsilon^2 (\hat{\psi}^{(n)} x^{-2b}) \gtrsim \int\int [\hat{u}_y^2 x^{-2b} + 2\varepsilon \hat{u}_x^2 + \varepsilon^2 \hat{v}_x^2] x^{-2b}$$
$$- \int\int [\varepsilon^2 \hat{v}^2 + \varepsilon \hat{u}^2] x^{-2-2b} + \varepsilon^2 \hat{\psi}^2 x^{-4-2b} \tag{3.17.24}$$
$$\gtrsim \int\int \hat{u}_y^2 x^{-2b} - C\int\int \varepsilon^2 \hat{v}_x^2 x^{-2b} - C\int\int \varepsilon \hat{u}_x^2 x^{-2b}. \tag{3.17.25}$$

To go from (3.17.24) to (3.17.25), we have used the Hardy inequality in the x-direction. We will now address the profile terms arising from $S_u(\hat{u},\hat{v})$ in the weak formulation (3.17.6), whose definition has been given in (3.8.5):

$$-\int\int \Big[u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y\Big]\cdot \partial_y \phi$$
$$= -\int\int \Big[u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y\Big]\cdot \partial_y \hat{\psi}^{(n)} x^{-2b}$$

$$= \int\int \Big[u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y\Big] \cdot \hat{u}^{(n)} x^{-2b}. \tag{3.17.26}$$

We will first pass to the limit in (3.17.26), using the definition of X_1 in (3.9.3), which gives:

$$(3.17.26) \xrightarrow{n\to\infty} \int\int \Big[u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y\Big] \cdot \hat{u} x^{-2b}. \tag{3.17.27}$$

We proceed to treat each term in (3.17.27), starting with:

$$\begin{aligned}\int\int u_R \hat{u}_x \hat{u} x^{-2b} &= -\int\int \hat{u}^2 \frac{\partial_x}{2}\Big(u_R x^{-2b}\Big) + \lim_{M\to\infty}\int_{x=M} \hat{u}^2 x^{-2b} \\ &= -\int\int \hat{u}^2 \Big(u_{Rx} x^{-2b} - 2b u_R x^{-2b-1}\Big) \\ &\gtrsim -||u_{Rx} x||_{L^\infty} \int\int \hat{u}^2 x^{-2b-1} + 2b \min u_R \int\int \hat{u}^2 x^{-2b-1} \\ &\gtrsim b\int\int \hat{u}^2 x^{-2b-1}, \end{aligned} \tag{3.17.28}$$

according to estimates (3.3.12), (3.3.17), so long as δ is taken small relative to b. For the M-limit above, we have used estimate (3.9.71), which is valid so long as $b > 0$. For the second term in (3.17.27), we again appeal to estimates (3.3.12), (3.3.18):

$$|\int u_{Rx}\hat{u}^2 x^{-2b}| \lesssim ||u_{Rx} x||_{L^\infty} ||\hat{u} x^{-2b-\frac{1}{2}}||_{L^2}^2 \lesssim \mathcal{O}(\delta) ||\hat{u} x^{-b-\frac{1}{2}}||_{L^2}^2.$$

For the third term, we shall split $u_R = u_R^{n-1,p} + \varepsilon^{\frac{n}{2}} u_{pR}^n + u_R^E$. First, we apply estimate (3.3.14):

$$\begin{aligned}|\int\int u_{Ry}^{P,n-1} \hat{v}\hat{u} x^{-2b}| &\leq ||y^2 x^{-\frac{1}{2}} u_{Ry}^{P,n-1}||_{L^\infty} ||\frac{\hat{u}}{y} x^{-b}||_{L^2} ||\frac{\hat{v}}{y} x^{\frac{1}{2}-b}||_{L^2} \\ &\leq \mathcal{O}(\delta) ||\hat{u}_y x^{-b}||_{L^2} ||\hat{v}_y x^{\frac{1}{2}-b}||_{L^2}. \end{aligned} \tag{3.17.29}$$

Next, for σ_n as in (3.7.1), according to estimate (3.3.16),

$$|\int\int u^{P,n}_{Ry}\hat{v}\hat{u}x^{-2b}| \le \varepsilon^{\frac{n}{2}}||u^n_{py}yx^{\frac{1}{2}-\sigma_n}||_{L^\infty}||\hat{u}x^{-1+\sigma_n-b}||_{L^2}||\frac{\hat{v}}{y}x^{\frac{1}{2}-b}||_{L^2}$$
$$\lesssim \varepsilon^{\frac{n}{2}}||\hat{u}_x x^{\sigma_n-b}||_{L^2}||\hat{v}_y x^{\frac{1}{2}-b}||_{L^2} \lesssim \varepsilon^{\frac{n}{2}}\mathcal{O}(\delta)||\hat{v}_y x^{\frac{1}{2}-b}||^2_{L^2}. \tag{3.17.30}$$

Finally, the Eulerian contribution is handled by an application of (3.3.18):

$$|\int\int \sqrt{\varepsilon}u^E_{RY}\hat{u}\hat{v}x^{-2b}| \le \sqrt{\varepsilon}||u^E_{RY}x^{\frac{3}{2}}||_{L^\infty}||\frac{\hat{u}}{x^{\frac{3}{4}+b}}||_{L^2}||\frac{\hat{v}}{x^{\frac{3}{4}+b}}||_{L^2}$$
$$\lesssim \sqrt{\varepsilon}||\hat{u}_x x^{\frac{1}{4}-b}||_{L^2}||\sqrt{\varepsilon}\hat{v}_x x^{\frac{1}{4}-b}||_{L^2}. \tag{3.17.31}$$

The fourth term from (3.17.26), upon using estimate (3.3.12) and (3.3.18), reads:

$$|\int\int v_R\hat{u}_y\hat{u}x^{-2b}| = |\int\int \frac{v_{Ry}}{2}\hat{u}^2x^{-2b}| \lesssim ||u_{Rx}x||_{L^\infty}||\hat{u}x^{-\frac{1}{2}-b}||^2_{L^2} \lesssim \mathcal{O}(\delta)||\hat{u}x^{-\frac{1}{2}-b}||^2_{L^2}. \tag{3.17.32}$$

Summarizing these calculations,

$$|(3.17.27)| \gtrsim b||\hat{u}x^{-\frac{1}{2}-b}||^2_{L^2} - \mathcal{O}(\delta)||\hat{u}x^{-\frac{1}{2}-b}||^2_{L^2} - \mathcal{O}(\delta)||\hat{u}_y x^{-b}||^2_{L^2}$$
$$- \mathcal{O}(\delta)||\{\sqrt{\varepsilon}v_x\hat{v}_y\}x^{\frac{1}{2}-b}||^2_{L^2}$$
$$\gtrsim b||\hat{u}x^{-\frac{1}{2}-b}||^2_{L^2} - \mathcal{O}(\delta)||\{\sqrt{\varepsilon}v_x, v_y\}x^{\frac{1}{2}-b}||^2_{L^2}. \tag{3.17.33}$$

We have absorbed the $\hat{u}_y$ terms into (3.17.24), and taken δ sufficiently small relative to b. We shall now address the profile terms from S_v:

$$\int\int \varepsilon S_v(\hat{u},\hat{v})\cdot\hat{\phi}_x = \int\int \varepsilon\Big[u_R\hat{v}_x + v_{Rx}\hat{u} + v_R\hat{v}_y + v_{Ry}\hat{v}\Big]\times$$
$$\Big[\hat{v}^{(n)}x^{-2b} - 2b\hat{\psi}^{(n)}x^{-2b-1}\Big]. \tag{3.17.34}$$

We may take $n \to \infty$ above due to the definition of X_1 from (3.9.3) and (3.17.12):

$$(3.17.34) \xrightarrow{n\to\infty} \int\int \varepsilon\Big[u_R\hat{v}_x + v_{Rx}\hat{u} + v_R\hat{v}_y + v_{Ry}\hat{v}\Big]\cdot\Big[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\Big]. \tag{3.17.35}$$

We will now proceed to treat each term in (3.17.35). The first profile term, $u_R v_x$ is the most delicate:

$$\int\int \varepsilon u_R\hat{v}_x[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}]. \tag{3.17.36}$$

First,

$$\begin{aligned}\int\int \varepsilon u_R\hat{v}_x\hat{v}x^{-2b} &= -\int\int \varepsilon\hat{v}^2\frac{\partial_x}{2}\Big(u_Rx^{-2b}\Big) + \lim_{M\to\infty}\int_{x=M}\frac{\varepsilon u_R}{2}\hat{v}^2x^{-2b} \\ &= -\int\int \varepsilon\hat{v}^2\frac{u_{Rx}}{2}x^{-2b} + \int\int b\varepsilon u_R\hat{v}^2x^{-2b-1}. \end{aligned}\tag{3.17.37}$$

The M-limit above vanishes due to (3.9.71). Staying with the term (3.17.36):

$$\begin{aligned}-2b\int\int \varepsilon u_R\hat{v}_x\hat{\psi}x^{-2b-1} &= 2b\int\int \varepsilon\hat{v}\partial_x\Big(u_R\hat{\psi}x^{-2b-1}\Big) \\ &= \int\int 2b\varepsilon u_{Rx}\hat{v}\hat{\psi}x^{-2b-1} + \int\int 2b\varepsilon u_R\hat{v}^2x^{-2b-1} \\ &\qquad - \int\int 2b(2b+1)\varepsilon u_R\hat{\psi}\hat{v}x^{-2b-2} \qquad (3.17.38) \\ &= \int\int 2b\varepsilon u_{Rx}\hat{v}\hat{\psi}x^{-2b-1} + \int\int 2b\varepsilon u_R\hat{v}^2x^{-2b-1} \\ &\qquad + \int\int b(2b+1)\varepsilon\hat{\psi}^2u_{Rx}x^{-2b-2} \\ &\qquad - \int\int b(2b+1)(2b+2)\varepsilon u_R\hat{\psi}^2x^{-2b-3}. \qquad (3.17.39)\end{aligned}$$

Combining the positive terms in (3.17.39) and (3.17.37), the total positive contribution is $\int\int 3b\varepsilon u_R \hat{v}^2 x^{-2b-1}$. For the final term in (3.17.39), we will now give the estimate:

$$\begin{aligned}\int\int u_R\hat{\psi}^2 x^{-2b-3} &= \int\int u_R\hat{\psi}^2 \frac{-\partial_x}{2b+2} x^{-2b-2} \\ &= \int\int \frac{2}{2b+2} u_R\hat{\psi}\hat{v}x^{-2b-2} + \int\int \frac{u_{Rx}}{2b+2}\hat{\psi}^2 x^{-2b-2} \\ &\le \Big[\frac{1}{2}||u_R^{\frac{1}{2}}\hat{\psi}x^{-b-\frac{3}{2}}||_{L^2}^2 + \frac{1}{2}\frac{4}{(2b+2)^2}||u_R^{\frac{1}{2}}\hat{v}x^{-b-\frac{1}{2}}||_{L^2}^2\Big] \\ &\qquad + \frac{||u_{Rx}x||_{L^\infty}}{2b+2}\frac{\sup|u_R|}{\inf|u_R|}\int\int u_R\hat{\psi}^2 x^{-2b-3}. \qquad (3.17.40)\end{aligned}$$

By collecting terms and rearranging, we obtain:

$$\Big[1-\frac{1}{2}-\frac{||u_{Rx}x||_{L^\infty}}{2b+2}\frac{\sup|u_R|}{\inf|u_R|}\Big]||u_R^{\frac{1}{2}}\hat{\psi}x^{-b-\frac{3}{2}}||_{L^2}^2 \le \frac{2}{(2b+2)^2}||u_R^{\frac{1}{2}}\hat{v}x^{-b-\frac{1}{2}}||_{L^2}^2. \qquad (3.17.41)$$

This then implies:

$$||u_R^{\frac{1}{2}}\hat{\psi}x^{-b-\frac{3}{2}}||_{L^2}^2 \le \frac{1}{1-\mathcal{O}(\delta)}\frac{4}{(2b+2)^2}||u_R^{\frac{1}{2}}\hat{v}x^{-b-\frac{1}{2}}||_{L^2}^2. \qquad (3.17.42)$$

Inserting this into (3.17.39), one arrives at:

$$\begin{aligned}&|\int\int b(2b+1)(2b+2)\varepsilon u_R\hat{\psi}^2 x^{-2b-3}| \\ &\qquad \le \frac{1}{1-\mathcal{O}(\delta)}\frac{4b(2b+1)(2b+2)}{(2b+2)^2}\int\int \varepsilon u_R\hat{v}^2 x^{-1-2b} \\ &\qquad \le \int\int \frac{5b}{2}u_R\varepsilon\hat{v}^2 x^{-1-2b}, \qquad (3.17.43)\end{aligned}$$

so long as b is sufficiently close to 0, by the following calculation:

$$\lim_{b\to 0}\frac{(2b+1)(2b+2)}{(2b+2)^2} = \frac{1}{2}. \qquad (3.17.44)$$

Thus, taking b sufficiently small, and recalling the positive contributions from (3.17.39) and (3.17.37), we have:

$$3b\int\int \varepsilon u_R \hat{v}^2 x^{-2b-1} - \frac{5b}{2}\int\int \varepsilon u_R \hat{v}^2 x^{-2b-1} = \frac{b}{2}\int\int \varepsilon u_R \hat{v}^2 x^{-2b-1}. \tag{3.17.45}$$

The remaining terms from (3.17.37) and (3.17.39) are then estimated in terms of (3.17.45) using the smallness of $\mathcal{O}(\delta)$. Summarizing, we have established control over:

$$\int\int \varepsilon u_R \hat{v}_x \cdot \Big[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\Big] \gtrsim \int\int b\varepsilon \hat{v}^2 x^{-1-2b}, \tag{3.17.46}$$

for a constant independent of small δ and b. We will now move to the second term from (3.17.34), for which we recall estimates (3.3.8) and (3.3.17):

$$\begin{aligned} |\int\int \varepsilon v_{Rx}\hat{u}\cdot\Big[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\Big]| &\le \sqrt{\varepsilon}||v_{Rx}x^{\frac{3}{2}}||_{L^\infty}||\frac{\hat{u}}{x^{\frac{3}{4}-b}}||_{L^2}||\sqrt{\varepsilon}\frac{\hat{v}}{x^{\frac{3}{4}-b}}||_{L^2} \\ &\le \sqrt{\varepsilon}||\hat{u}_x x^{\frac{1}{4}-b}||_{L^2}||\sqrt{\varepsilon}\hat{v}_x x^{\frac{1}{4}-b}||_{L^2}. \end{aligned} \tag{3.17.47}$$

For the third term from (3.17.34), we use Young's inequality and estimates (3.3.10), (3.3.20):

$$\begin{aligned} &|\int\int \varepsilon v_R \hat{v}_y\Big[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\Big]| \\ &\qquad \le ||v_R x^{\frac{1}{2}}||_{L^\infty}\Big[||\hat{v}_y x^{\frac{1}{2}-b}||_{L^2}^2 + ||\sqrt{\varepsilon}\hat{v}x^{-b-\frac{1}{2}}||_{L^2}^2 + ||\sqrt{\varepsilon}\hat{\psi}x^{-b-\frac{3}{2}}||_{L^2}^2\Big] \\ &\qquad \le \mathcal{O}(\delta)\Big[||\hat{v}_y x^{\frac{1}{2}-b}||_{L^2}^2 + ||\sqrt{\varepsilon}\hat{v}x^{-b-\frac{1}{2}}||_{L^2}^2 + ||\sqrt{\varepsilon}\hat{\psi}x^{-b-\frac{3}{2}}||_{L^2}^2\Big]. \end{aligned} \tag{3.17.48}$$

For the final term from (3.17.34), we use Young's inequality and estimates (3.3.10), (3.3.20):

$$|\int\int \varepsilon v_{Ry}\hat{v}\cdot\Big[\hat{v}x^{-2b} - 2b\hat{\psi}x^{-2b-1}\Big]| \tag{3.17.49}$$
$$\lesssim ||v_{Ry}x||_{L^\infty}\Big[||\sqrt{\varepsilon}\hat{v}x^{-b-\frac{1}{2}}||^2_{L^2} + b||\sqrt{\varepsilon}\hat{\psi}x^{-b-\frac{3}{2}}||^2_{L^2}\Big].$$

Summarizing these last few terms, we obtain:

$$|(3.17.35)| \gtrsim \int\int b\varepsilon\hat{v}x^{-1-2b} + \mathcal{O}(\delta)\Big[||\{\hat{u},\sqrt{\varepsilon}v\}x^{\frac{1}{2}-b}||^2_{L^2} + ||\hat{v}_y x^{\frac{1}{2}-b}||^2_{L^2}\Big]. \tag{3.17.50}$$

The final task is to turn to the right-hand side. Reading from (3.17.6), and (3.17.2) - (3.17.3):

$$\int\int \hat{f}\cdot\phi_y + \varepsilon\hat{g}\cdot\phi_x = \int\int \hat{f}\cdot\hat{u}^{(n)}x^{-2b} + \varepsilon\hat{g}\cdot[\hat{v}^{(n)}x^{-2b} + \hat{\psi}^{(n)}\partial_x x^{-2b}]$$
$$\xrightarrow{n\to\infty} \int\int \hat{f}\cdot\hat{u}^{(n)}x^{-2b} + \varepsilon\hat{g}\cdot[\hat{v}x^{-2b} + \hat{\psi}\partial_x x^{-2b}], \tag{3.17.51}$$

where we have passed to the limit using again the definition of X_1 from (3.9.3). Combining (3.17.24), (3.17.33), (3.17.50), and (3.17.51), one obtains the desired result, estimate (3.17.8).

□

We now repeat the positivity estimate, with a correspondingly weaker weight in order to close the above energy estimate. We refer the reader to Proposition 3.10.2 for a comparison.

Lemma 3.17.2. *Fix any $0 < b < 1$. Let δ, ε be sufficiently small relative to universal constants, and $\varepsilon << \delta$. Then for $[\hat{u},\hat{v}] \in Z$ solutions to (3.17.2) - (3.17.3) with boundary* *conditions (3.17.4) satisfy the following estimate:*

$$||\{\hat{u}_x, \sqrt{\varepsilon}\hat{v}_x\}x^{\frac{1}{2}-b}||_{L^2}^2 \lesssim ||\hat{u}_y x^{-b}||_{L^2}^2 + ||\{\sqrt{\varepsilon}\hat{v}, \hat{u}\}x^{-\frac{1}{2}-b}||_{L^2}^2 + \mathcal{W}_{1,P,b}. \tag{3.17.52}$$

Proof. The estimate will follow upon applying the multiplier $\hat{v}x^{1-2b}$ to the system (3.17.5). In order to proceed formally, we must start with the weak formulation given in (3.17.6), and select the test function:

$$\phi = \hat{v}^{(n)}x^{1-2b}, \quad [\hat{u}^{(n)}, \hat{v}^{(n)}] \xrightarrow{X_1} [\hat{u}, \hat{v}], \tag{3.17.53}$$

where X_1 is defined in (3.9.3). Turning to the weak formulation in (3.17.6), we will first expand the second-order terms:

$$\begin{aligned}\int\int \nabla_\varepsilon^2 \hat{\psi} : \nabla_\varepsilon^2 \phi &= \int\int \nabla_\varepsilon^2 \hat{\psi} : \nabla_\varepsilon^2 \hat{v}^{(n)} x^{1-2b} \\ &= \int\int \hat{\psi}_{yy}\hat{v}_{yy}^{(n)}x^{1-2b} + 2\varepsilon\hat{\psi}_{xy}\partial_x\Big(\hat{v}_y^{(n)}x^{1-2b}\Big) + \varepsilon^2\hat{\psi}_{xx}\partial_{xx}\Big(\hat{v}^{(n)}x^{1-2b}\Big) \\ &= \int\int -\hat{u}_y\hat{v}_{yy}^{(n)}x^{1-2b} + 2\varepsilon\hat{v}_y\partial_x\Big(\hat{v}_y^{(n)}x^{1-2b}\Big) + \varepsilon^2\hat{v}_x\partial_{xx}\Big(\hat{v}^{(n)}x^{1-2b}\Big)\end{aligned} \tag{3.17.54}$$

We first arrive at the first two terms from (3.17.54):

$$\begin{aligned}&\int\int -\hat{u}_y\hat{v}_{yy}^{(n)}x^{1-2b} - 2\varepsilon\hat{u}_x\hat{v}_{xy}^{(n)}x^{1-2b} - 2\varepsilon\hat{u}_x\hat{v}_y^{(n)}x^{-2b} \\ &\qquad = \int\int -\hat{u}_y^{(n)}\partial_x[\hat{u}_y x^{1-2b}] - 2\varepsilon\hat{u}_x^{(n)}\partial_x[\hat{u}_x x^{1-2b}] - 2\varepsilon\hat{u}_x\hat{v}_y^{(n)}x^{-2b}.\end{aligned} \tag{3.17.55}$$

Referring to the definition of X_1 in (3.9.3), according to (3.17.53), we may pass to the limit as $n \to \infty$, and appeal to the estimates in (3.9.71) and (3.9.72), to obtain:

$$(3.17.55) \xrightarrow{n\to\infty} \int\int -\hat{u}_y\partial_x[\hat{u}_y x^{1-2b}] - 2\varepsilon\hat{u}_x\partial_x[\hat{u}_x x^{1-2b}] - 2\varepsilon\hat{u}_x\hat{v}_y x^{-2b}$$

$$= \int\int -\frac{(1-2b)}{2}\hat{u}_y^2 x^{-2b} + (1+2b)\varepsilon\hat{u}_x^2 x^{-2b} + \lim_{M\to\infty}\int_{x=M}\left[\frac{1}{2}\hat{u}_y^2 x^{1-2b} - \varepsilon\hat{u}_x^2 x^{1-2b}\right]$$
$$= \int\int -\frac{(1-2b)}{2}\hat{u}_y^2 x^{-2b} + (1+2b)\varepsilon\hat{u}_x^2 x^{-2b}. \tag{3.17.56}$$

Again referring to the definition in (3.9.3), the third term from (3.17.54) is treated by:

$$\int\int \varepsilon^2\hat{v}_x\partial_{xx}\left(\hat{v}^{(n)}x^{1-2b}\right) = -\int\int \varepsilon^2\hat{v}_{xx}\partial_x\left(\hat{v}^{(n)}x^{1-2b}\right)$$
$$\xrightarrow{n\to\infty} -\int\int \varepsilon^2\hat{v}_{xx}\partial_x\left(\hat{v}x^{1-2b}\right) = \int\int -\varepsilon^2\hat{v}_{xx}\hat{v}_x x^{1-2b} - \int\int \varepsilon^2\hat{v}_{xx}\hat{v}(1-2b)x^{-2b}. \tag{3.17.57}$$

Integrating by parts the first term on the right-hand side of (3.17.57), and appealing to estimate (3.9.72):

$$\int\int -\varepsilon^2\hat{v}_{xx}\hat{v}_x x^{1-2b} = \int\int \varepsilon^2\frac{1-2b}{2}|\hat{v}_x|^2 x^{-2b} - \lim_{M\to\infty}\int_{x=M}\frac{\varepsilon^2}{2}\hat{v}_x^2 x^{1-2b}$$
$$= \int\int \varepsilon^2\frac{1-2b}{2}|\hat{v}_x|^2 x^{-2b}. \tag{3.17.58}$$

Integrating by parts the second term on the right-hand side of (3.17.57), and again appealing to estimates (3.9.71) - (3.9.73) for the M-limit below:

$$-\int\int \varepsilon^2\hat{v}_{xx}\hat{v}(1-2b)x^{-2b} = \int\int \varepsilon^2(1-2b)\hat{v}_x\partial_x\left[\hat{v}x^{-2b}\right] + \lim_{M\to\infty}\int_{x=M}\varepsilon^2(1-2b)\hat{v}_x\hat{v}x^{-2b}$$
$$= \int\int \varepsilon^2(1-2b)\hat{v}_x^2 x^{-2b} - \int\int \varepsilon^2 2b(1-2b)\hat{v}_x\hat{v}x^{-2b-1}$$
$$= \int\int \varepsilon^2(1-2b)\hat{v}_x^2 x^{-2b} + \int\int \varepsilon^2 b(1-2b)\hat{v}^2\partial_x x^{-2b-1}$$
$$- \lim_{M\to\infty}\varepsilon^2 b(1-2b)\hat{v}^2 x^{-2b-1}$$
$$= \int\int \varepsilon^2(1-2b)\hat{v}_x^2 x^{-2b} - \int\int \varepsilon^2 b(1-2b)(2b+1)\hat{v}^2 x^{-2b-2}. \tag{3.17.59}$$

Combining the above estimates:

$$(3.17.57) = \int\int \frac{3}{2}(1-2b)\varepsilon^2 \hat{v}_x^2 x^{-2b} - b(2b+1)(1-2b)\varepsilon^2 \hat{v}^2 x^{-2b-2}. \tag{3.17.60}$$

Hence, summarizing (3.17.55) - (3.17.59):

$$|\lim_{n\to\infty} \int\int \nabla_\varepsilon^2 \hat{\psi} : \nabla_\varepsilon^2 \phi| = |\int\int \nabla_\varepsilon^2 \hat{\psi} : \nabla_\varepsilon^2 (\hat{v} x^{1-2b})| \lesssim \int\int [\varepsilon^2 \hat{v}_x^2 + \varepsilon \hat{u}_x^2 + \hat{u}_y^2] x^{-2b}. \tag{3.17.61}$$

We will now turn to the profile terms from S_u, which upon consultation with (3.17.6), the definition in (3.9.3), and (3.17.53), read:

$$\begin{aligned} &\int\int \Big[u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y\Big] \cdot \hat{u}_x^{(n)} x^{1-2b} \\ &\qquad \xrightarrow{n\to\infty} \int\int \Big[u_R \hat{u}_x + u_{Rx}\hat{u} + u_{Ry}\hat{v} + v_R \hat{u}_y\Big] \cdot \hat{u}_x x^{1-2b}. \end{aligned} \tag{3.17.62}$$

We now turn our attention to (3.17.62). The first term yields the desired positivity:

$$\int\int u_R \hat{u}_x^2 x^{1-2b} \gtrsim \min u_R \int\int \hat{u}_x^2 x^{1-2b}. \tag{3.17.63}$$

Next, by (3.3.12), (3.3.18):

$$\begin{aligned} |\int\int u_{Rx}\hat{u}\hat{u}_x x^{1-2b}| &\leq ||u_{Rx} x||_{L^\infty} ||\hat{u}_x x^{\frac{1}{2}-b}||_{L^2} ||\hat{u} x^{-\frac{1}{2}-b}||_{L^2} \\ &\leq \mathcal{O}(\delta) ||\hat{u}_x x^{\frac{1}{2}-b}||_{L^2}^2. \end{aligned} \tag{3.17.64}$$

Next, we shall split $u_R = u_R^P + u_R^E$, and use estimate (3.3.14) and (3.3.16) for (3.17.65) below and (3.3.18) for (3.17.66) below:

$$|\int\int u^P_{Ry}\hat{v}\hat{u}_x x^{1-2b}| \le ||yu^P_{Ry}||_{L^\infty}||\hat{v}_y x^{\frac{1}{2}-b}||^2_{L^2}, \tag{3.17.65}$$

$$\begin{aligned}|\int\int \sqrt{\varepsilon}u^E_{RY}\hat{v}\hat{u}_x x^{1-2b}| &\le ||u^E_{RY}x^{\frac{3}{2}}||_{L^\infty}||\sqrt{\varepsilon}\hat{v}x^{-\frac{1}{2}-b}||_{L^2}||\hat{u}_x x^{\frac{1}{2}-b}||_{L^2} \\ &\lesssim \sqrt{\varepsilon}\Big[||\sqrt{\varepsilon}\hat{v}x^{-\frac{1}{2}-b}||^2_{L^2} + ||\hat{u}_x x^{\frac{1}{2}-b}||^2_{L^2}\Big].\end{aligned} \tag{3.17.66}$$

For the fourth term from (3.17.62), by estimates (3.3.10) and (3.3.20):

$$\begin{aligned}|\int\int v_R\hat{u}_y\hat{u}_x x^{1-2b}| &\le ||v_R x^{\frac{1}{2}}||_{L^\infty}||\hat{u}_y x^{-b}||_{L^2}||\hat{u}_x x^{\frac{1}{2}-b}||_{L^2} \\ &\le \mathcal{O}(\delta)||\hat{u}_y x^{-b}||_{L^2}||\hat{u}_x x^{\frac{1}{2}-b}||_{L^2}.\end{aligned} \tag{3.17.67}$$

Summarizing the last four calculations:

$$\begin{aligned}|(3.17.62)| \gtrsim \int\int \hat{u}_x^2 x^{1-2b} - \mathcal{O}(\delta)\Big[&||\hat{u}x^{-\frac{1}{2}-b}||^2_{L^2} \\ &+ ||\hat{u}_y x^{-b}||^2_{L^2} + ||\sqrt{\varepsilon}\hat{v}x^{-\frac{1}{2}-b}||^2_{L^2}\Big].\end{aligned} \tag{3.17.68}$$

The final three terms appearing on the right-hand side above all appear on the right-hand side of estimate (3.17.52). Turning now to the profile terms, from S_v, for which we read (3.17.6) with $\phi = \hat{v}^{(n)}x^{1-2b}$, appeal to (3.9.3) and (3.17.53), giving ultimately:

$$\begin{aligned}&\int\int \varepsilon\Big[u_R\hat{v}_x + v_{Rx}\hat{u} + v_R\hat{v}_y + v_{Ry}\hat{v}\Big]\cdot\partial_x[\hat{v}^{(n)}x^{1-2b}] \\ &\xrightarrow{n\to\infty} \int\int \varepsilon\Big[u_R\hat{v}_x + v_{Rx}\hat{u} + v_R\hat{v}_y + v_{Ry}\hat{v}\Big]\cdot\partial_x[\hat{v}x^{1-2b}].\end{aligned} \tag{3.17.69}$$

We will treat each term in (3.17.69). For the first term from (3.17.69):

$$\int\int \varepsilon u_R\hat{v}_x\Big(\hat{v}_x x^{1-2b} + (1-2b)\hat{v}x^{-2b}\Big)$$

$$= \int\int \varepsilon u_R \hat{v}_x^2 x^{1-2b} + b(1-2b) u_R \varepsilon \hat{v}^2 x^{-1-2b} - \int\int \frac{1-2b}{2} \varepsilon u_{Rx} \hat{v}^2 x^{-2b}$$
$$\gtrsim \int\int \varepsilon \hat{v}_x^2 x^{1-2b} + b \int\int \varepsilon \hat{v}^2 x^{-1-2b}. \tag{3.17.70}$$

Above we have used (3.3.12) and (3.3.18). For the second term, we integrate by parts:

$$\begin{aligned}\int\int \varepsilon v_{Rx} \hat{u} \partial_x [\hat{v} x^{1-2b}] &= \int\int -\varepsilon \partial_x \Big(v_{Rx} \hat{u} \Big) \cdot \hat{v} x^{1-2b} + \lim_{M \to \infty} \int_{x=M} v_{Rx} \hat{u} \hat{v} x^{1-2b} \\ &= \int\int -\varepsilon v_{Rxx} \hat{u} \hat{v} x^{1-2b} - \varepsilon v_{Rx} \hat{v} \hat{u}_x x^{1-2b} \\ &\leq \sqrt{\varepsilon} ||v_{Rx} x^{\frac{3}{2}}, v_{Rxx} x^{\frac{5}{2}}||_{L^\infty} \Big[||\hat{u} x^{-\frac{1}{2}-b}||_{L^2}^2 \\ &\qquad + ||\hat{u}_x x^{\frac{1}{2}-b}||_{L^2}^2 + ||\sqrt{\varepsilon} \hat{v} x^{-\frac{1}{2}-b}||_{L^2}^2 \Big].\end{aligned} \tag{3.17.71}$$

The above $M-$limit vanishes according to estimates (3.9.71), and we have used estimates (3.3.8) and (3.3.17). For the third term, we recall estimates (3.3.10), (3.3.20):

$$\begin{aligned}&\int\int \varepsilon v_R \hat{v}_y \hat{v}_x x^{1-2b} + c_0 \varepsilon v_R \hat{v}_y \hat{v} x^{-2b} \\ &\qquad \leq \sqrt{\varepsilon} ||v_R x^{\frac{1}{2}}||_{L^\infty} \Big[||\hat{v}_y x^{\frac{1}{2}-b}||_{L^2}^2 + ||\sqrt{\varepsilon} \hat{v}_x x^{\frac{1}{2}-b}||_{L^2}^2 + ||\sqrt{\varepsilon} \hat{v} x^{-\frac{1}{2}-b}||_{L^2}^2 \Big].\end{aligned} \tag{3.17.72}$$

For the fourth term, we integrate by parts and appeal to (3.9.71), (3.3.8) - (3.3.10), and (3.3.17):

$$\begin{aligned}\int\int \varepsilon v_{Ry} \hat{v} \cdot \partial_x [\hat{v} x^{1-2b}] &= -\int\int \varepsilon \partial_x [v_{Ry} \hat{v}] \cdot \hat{v} x^{1-2b} + \lim_{M \to \infty} \int_{x=M} \varepsilon v_{Ry} \hat{v}^2 x^{1-2b} \\ &= \int\int -\varepsilon v_{Rxy} \hat{v}^2 x^{1-2b} - \varepsilon v_{Ry} \hat{v}_x \hat{v} x^{1-2b} \\ &\leq ||v_{Ry} x, v_{Rxy} x^2||_{L^\infty} \Big[||\sqrt{\varepsilon} \hat{v}_x x^{\frac{1}{2}-b}||_{L^2}^2 + ||\sqrt{\varepsilon} \hat{v} x^{-\frac{1}{2}-b}||_{L^2}^2 \Big].\end{aligned} \tag{3.17.73}$$

Summarizing these four terms,

$$|(3.17.69)| \gtrsim \int\int \varepsilon \hat{v}_x^2 x^{1-2b} + b\int\int \varepsilon \hat{v}^2 x^{-1-2b} - \mathcal{O}(\delta)\Big[||\hat{u}x^{-\frac{1}{2}-b}, \hat{u}_x x^{\frac{1}{2}-b}||_{L^2}^2 + ||\sqrt{\varepsilon}\hat{v}x^{-\frac{1}{2}-b}, \sqrt{\varepsilon}\hat{v}_x x^{\frac{1}{2}-b}||_{L^2}^2\Big]. \tag{3.17.74}$$

On the right-hand side, appealing again to (3.9.3), (3.17.53), and the definitions of $\hat{f}, \hat{g}$ in (3.17.2) - (3.17.3), one obtains:

$$\int\int \hat{f}\hat{u}_x^{(n)} x^{1-2b} + \hat{g}\Big[\hat{v}_x^{(n)} x^{1-2b} + (1-2b)\hat{v}^{(n)} x^{-2b}\Big] \xrightarrow{n\to\infty} \int\int \hat{f}\hat{u}_x x^{1-2b} + \hat{g}\Big[\hat{v}_x x^{1-2b} + (1-2b)\hat{v}x^{-2b}\Big], \tag{3.17.75}$$

Placing the above estimates together yields the estimate (3.17.52).

□

We will now introduce some notation, which is a natural adaptation of what is found in Section 3.9 to the weaker weight of x^{-b}. The reader should recall the definitions of the cutoff functions introduced in (3.9.1) - (3.9.2). The energy norms are defined as follows:

$$||u,v||_{X_{1,b}}^2 := ||u_y x^{-b}||_{L^2}^2 + ||\{\sqrt{\epsilon}v_x, v_y\}x^{\frac{1}{2}-b}||_{L^2}^2 \tag{3.17.76}$$

$$||u,v||_{X_{2,b}}^2 := ||u_{xy}\cdot\rho_2 x^{1-b}||_{L^2}^2 + ||\{\sqrt{\epsilon}v_{xx}, v_{xy}\}\cdot\rho_2^{\frac{3}{2}} x^{\frac{3}{2}-b}||_{L^2}^2, \tag{3.17.77}$$

$$||u,v||_{X_{3,b}}^2 := ||u_{xxy}\cdot\rho_3^2 x^{2-b}||_{L^2}^2 + ||\{\sqrt{\epsilon}v_{xxx}, v_{xxy}\}\cdot\rho_3^{\frac{5}{2}} x^{\frac{5}{2}-b}||_{L^2}^2. \tag{3.17.78}$$

Definition 3.17.3. The norms $Y_{2,b}, Y_{3,b}$ are strengthenings of $X_{2,b}, X_{3,b}$ near the boundary, $x = 1$, and defined through:

$$||u,v||_{Y_{2,b}}^2 := ||u_{xy}x^{1-b}||_{L^2}^2 + ||\{\sqrt{\epsilon}v_{xx}, v_{xy}\}x^{\frac{3}{2}-b}||_{L^2}^2 + ||u_{yy}||_{L^2(x\le 2000)}, \tag{3.17.79}$$

$$||u,v||_{Y_{3,b}}^2 := ||u_{xxy}\cdot\zeta_3 x^{2-b}||_{L^2}^2 + ||\{\sqrt{\epsilon}v_{xxx}, v_{xxy}\}\cdot\zeta_3 x^{\frac{5}{2}-b}||_{L^2}^2. \tag{3.17.80}$$

Definition 3.17.4. The norm Z_b is defined through:

$$\begin{aligned}||u,v||_{Z_b} :=& ||u,v||_{X_{1,b}\cap X_{2,b}\cap X_{3,b}} + \epsilon^{N_2}||u,v||_{Y_{2,b}} + \epsilon^{N_3}||u,v||_{Y_{3,b}} \\ &+ \epsilon^{N_4}||ux^{\frac{1}{4}-b}, \sqrt{\varepsilon} v x^{\frac{1}{2}-b}||_{L^\infty} + \epsilon^{N_5} \sup_{x\geq 20} ||\sqrt{\varepsilon} v_x x^{\frac{3}{2}-b}, u_x x^{\frac{5}{4}-b}||_{L^\infty} \\ &+ \varepsilon^{N_6} \sup_{x\geq 20} ||u_y x^{\frac{1}{2}-b}||_{L^2_y} + \epsilon^{N_7}\Big[\int_{20}^\infty x^{4-b}||\sqrt{\varepsilon} v_{xx}||^2_{L^\infty_y} dx\Big]^{\frac{1}{2}}. \end{aligned} \tag{3.17.81}$$

Next, we record the second and third order versions of the energy and positivity estimates, which mimic Propositions 3.10.4, 3.10.6, 3.10.7, 3.10.8. We will omit most details, and record only those differences which arise.

Lemma 3.17.5 (Second-Order Energy Estimate)**.** *Fix any $0 < b < 1$. Let δ, ε be sufficiently small relative to universal constants, and $\varepsilon << \delta$. Then for $[\hat{u}, \hat{v}] \in Z$ solutions to (3.17.2) - (3.17.3):*

$$||\hat{u}_{xy}\rho_2 x^{1-b}||^2_{L^2} \lesssim \mathcal{O}(\delta)||\{\sqrt{\varepsilon}\hat{v}_{xx}, \hat{v}_{xy}\}\rho_2^{\frac{3}{2}} x^{\frac{3}{2}-b}||^2_{L^2} + ||\hat{u},\hat{v}||^2_{X_{1,b}} + \mathcal{W}_{1,b} + \mathcal{W}_{2,E,b}, \tag{3.17.82}$$

where (recall the definition of ρ_2 from (3.9.2)):

$$\mathcal{W}_{2,E,b} := \int\int \hat{f}_x \hat{u}_x \rho_2^2 x^{2-2b} + \int\int \varepsilon \hat{g}_x \hat{v}_x \rho_2^2 x^{2-2b}, \tag{3.17.83}$$

$$\mathcal{W}_{2,P,b} = \int\int \hat{f}_x \hat{u}_{xx} \rho_2^3 x^{3-2b} + \int\int \varepsilon \hat{g}_x \hat{v}_{xx} \rho_2^3 x^{3-2b}, \tag{3.17.84}$$

$$\mathcal{W}_{2,b} := \mathcal{W}_{2,E,b} + \mathcal{W}_{2,P,b}. \tag{3.17.85}$$

Proof. Differentiating the weak formulation gives:

$$\begin{aligned}\int\int \nabla_\epsilon^2 \hat{\psi}_x : \nabla^2_\epsilon \phi - \int\int \partial_x S_u(\hat{u},\hat{v})\cdot \phi_y + \varepsilon \partial_x S_v(\hat{u},\hat{v})\cdot\phi_x \\ = \int\int \varepsilon^{\frac{n}{2}+\gamma}\Big[-\partial_x \hat{f}\phi_y + \varepsilon\partial_x\hat{g}\phi_x\Big],\end{aligned} \tag{3.17.86}$$

For the second-order energy estimate, we select $\phi = \rho_2^2 \hat{v}^{(n)} x^{2-2b}$, where:

$$[\hat{u}^{(n)}, \hat{v}^{(n)}] \in C^\infty_{0,D}, \quad [\hat{u}^{(n)}, \hat{v}^{(n)}] \xrightarrow{X_1} [\hat{u}, \hat{v}]. \tag{3.17.87}$$

Let us turn to the highest-order terms:

$$\begin{aligned}
\int\int \nabla_\varepsilon^2 \hat{\psi}_x : \nabla_\varepsilon^2 \phi &= \int\int \nabla_\varepsilon^2 \hat{v} : \nabla_\varepsilon^2 \rho_2^2 \hat{v}^{(n)} x^{2-2b} \\
&= \int\int \hat{v}_{yy} \rho_2^2 \hat{v}^{(n)}_{yy} x^{2-2b} + 2\varepsilon \hat{v}_{xy} \partial_x [\hat{v}^{(n)}_y \rho_2^2 x^{2-2b}] + \varepsilon^2 \hat{v}_{xx} \partial_{xx} [\rho_2^2 \hat{v}^{(n)} x^{2-2b}] \\
&= \int\int \hat{u}_{xy} \rho_2^2 \hat{u}^{(n)}_{xy} x^{2-2b} + 2\varepsilon \hat{v}_{xy} \partial_x [\hat{v}^{(n)}_y \rho_2^2 x^{2-2b}] + \varepsilon^2 \hat{v}_{xx} \partial_{xx} [\rho_2^2 \hat{v}^{(n)} x^{2-2b}] \\
&= \int\int -\partial_x [\hat{u}_{xy} \rho_2^2 x^{2-2b}] \hat{u}^{(n)}_y - 2\varepsilon \hat{v}_{xxy} \hat{v}^{(n)}_y \rho_2^2 x^{2-2b} - \varepsilon^2 \hat{v}_{xxx} \partial_x [\rho_2^2 \hat{v}^{(n)} x^{2-2b}]
\end{aligned} \tag{3.17.88}$$

One now checks according to the definition (3.9.3), that (3.17.87) suffices to pass to the limit in the above identity, which upon integrating by parts in x yields:

$$\begin{aligned}
(3.17.88) \xrightarrow{n \to \infty} &\int\int -\partial_x [\hat{u}_{xy} \rho_2^2 x^{2-2b}] \hat{u}_y - 2\varepsilon \hat{v}_{xxy} \hat{v}_y \rho_2^2 x^{2-2b} - \varepsilon^2 \hat{v}_{xxx} \partial_x [\rho_2^2 \hat{v} x^{2-2b}] \\
&= \int\int [\hat{u}_{xy}^2 + \varepsilon \hat{u}_{xx}^2 + \varepsilon^2 \hat{v}_{xx}^2] \rho_2^2 x^{2-2b} + J.
\end{aligned}$$

where $|J| = |c_0 \varepsilon^2 \hat{v}_x^2 \partial_{xx} (\rho_2^2 x^{2-2b}) + c_1 \varepsilon^2 \hat{v}^2 \partial_x^4 (\rho_2^2 x^{2-2b})| \lesssim ||u, v||^2_{X_{1,b}}$. From here, repeating the calculations in Proposition 3.10.4 gives the desired result, where the required integrations by parts are justified upon using that $b > 0$, combined with the estimates in (3.9.71) - (3.9.73). These justifications are analogous to those in Lemma 3.17.1, and so we omit the details.

□

Lemma 3.17.6 (Second-Order Positivity Estimate)**.** *Fix any $0 < b < 1$. Let δ, ε be suffi-*

ciently small relative to universal constants, and $\varepsilon << \delta$. Then for $[\hat{u}, \hat{v}] \in Z$ solutions to (3.17.2) - (3.17.3):

$$||\{\sqrt{\varepsilon}\hat{v}_{xx}, \hat{v}_{xy}\}\rho_2^{\frac{3}{2}} x^{\frac{3}{2}-b}||_{L^2}^2 \lesssim ||\hat{u}_{xy}\rho_2 x^{1-b}||_{L^2}^2 + ||\hat{u}, \hat{v}||_{X_{1,b}}^2 + \mathcal{W}_{1,b} + \mathcal{W}_{2,b}. \tag{3.17.89}$$

Proof. We start again with the weak formulation in (3.17.86). Fix a large $0 < L < \infty$. We then make the selection: $\phi = \hat{v}_x^{(n)} \cdot \rho_2^3 x_L^{3-2b}$, where, referring to (3.10.168), the weight x_L is defined via: $x_L := \Big(a_L * \phi_L\Big)\chi\Big(\frac{x}{10L}\Big)$. Define the domain: $\Omega_L := \{x : 3 < x < 50L + 100\}$, so that $\hat{v}_x \cdot \rho_2^3 x_L^{3-2b} = 0$ on Ω_L^C. The sequence $\hat{v}^{(n)}$ is selected according to:

$$[\hat{u}_x^{(n)}, \hat{v}_x^{(n)}] \in C_{0,D}^\infty(\Omega_L), \qquad [\hat{u}_x^{(n)}, \hat{v}_x^{(n)}] \xrightarrow{H^1(\Omega_L)} [\hat{u}_x, \hat{v}_x]. \tag{3.17.90}$$

The existence of such a sequence is guaranteed due to the standard Sobolev space theory, because we are now in the un-weighted setting. It is now straightforward to repeat all estimates in Proposition 3.10.6 using the test function ϕ. Upon doing so, we pass to the limit first as $n \to \infty$, and then as $L \to \infty$ to obtain the desired estimate.

$\square$

Lemma 3.17.7 (Third-Order Energy Estimate)**.** *Fix any $0 < b < 1$. Let δ, ε be sufficiently small relative to universal constants, and $\varepsilon << \delta$. Then for $[\hat{u}, \hat{v}] \in Z$ solutions to (3.17.2) - (3.17.3):*

$$||\hat{u}_{xxy}\rho_3^2 x^{2-b}||_{L^2}^2 \lesssim \mathcal{O}(\delta)||\{\sqrt{\varepsilon}\hat{v}_{xxx}, \hat{v}_{xxy}\}\rho_3^{\frac{5}{2}} x^{\frac{5}{2}-b}||_{L^2}^2 + ||\hat{u}, \hat{v}||_{X_{1,b}\cap X_{2,b}}^2 + \sum_{i=1}^2 \mathcal{W}_{i,b} + W_{3,E,b}, \tag{3.17.91}$$

where

$$\mathcal{W}_{3,E,b} := \int\int \hat{f}_{xx}\hat{u}_{xx}\rho_3^4 x^{4-2b} + \int\int \varepsilon\hat{g}_{xx}\hat{v}_{xx}\rho_3^4 x^{4-2b}, \tag{3.17.92}$$

$$\mathcal{W}_{3,P,b} := \int\int \hat{f}_{xx}\hat{u}_{xxx}\rho_3^5 x^{5-2b} + \int\int \varepsilon\hat{g}_{xx}\hat{v}_{xxx}\rho_3^5 x^{5-2b}, \tag{3.17.93}$$

$$\mathcal{W}_{3,b} := W_{3,E,b} + W_{3,P,b}. \tag{3.17.94}$$

Proof. The first step is to differentiate the weak formulation (3.17.86) yet again, which formally takes place using difference quotients, yielding:

$$\int\int \nabla_\epsilon^2 \hat{\psi_{xx}} : \nabla_\epsilon^2 \phi - \int\int \partial_{xx} S_u(\hat{u},\hat{v}) \cdot \phi_y + \varepsilon \partial_{xx} S_v(\hat{u},\hat{v}) \cdot \phi_x$$
$$= \int\int \varepsilon^{\frac{n}{2}+\gamma}\Big[- \partial_{xx}\hat{f}\phi_y + \varepsilon\partial_{xx}\hat{g}\phi_x\Big], \tag{3.17.95}$$

Fix any L large, finite. The selection of test function is now $\phi := \hat{v}_x^{(n)}\rho_3^4 x_L^{4-2b}$, where the sequence:

$$[\hat{u}_{xx}^{(n)}, \hat{v}_{xx}^{(n)}] \in C_{0,D}^\infty(\Omega_L), \qquad [\hat{u}_{xx}^{(n)}, \hat{v}_{xx}^{(n)}] \xrightarrow{H^1(\Omega_L)} [\hat{u}_{xx}, \hat{v}_{xx}]. \tag{3.17.96}$$

From here, repeating the estimates given in Proposition 3.10.7, and sending $n \to \infty$ and then $L \to \infty$ gives the desired result.

$\square$

Lemma 3.17.8 (Third-Order Positivity Estimate)**.** *Fix any $0 < b < 1$. Let δ, ε be sufficiently small relative to universal constants, and $\varepsilon << \delta$. Then for $[\hat{u},\hat{v}] \in Z$ solutions to (3.17.2) - (3.17.3):*

$$||\{\sqrt{\varepsilon}\hat{v}_{xxx}, \hat{v}_{xxy}\}\rho_3^{\frac{5}{2}} x^{\frac{5}{2}-b}||_{L^2}^2 \lesssim ||\hat{u}_{xxy}\rho_3^2 x^{2-b}||_{L^2}^2 + ||\hat{u},\hat{v}||_{X_{1,b}\cap X_{2,b}}^2 + \sum_{i=1}^3 \mathcal{W}_{i,b}. \tag{3.17.97}$$

Proof. Again, fix any L large, finite. The selection of the test function is now $\phi :=$

$\hat{v}_{xx}^{(n)} \rho_3^5 x_L^{5-2b}$, where the sequence $[\hat{u}^{(n)}, \hat{v}^{(n)}]$ is selected according to:

$$[\hat{u}_{xx}^{(n)}, \hat{v}_{xx}^{(n)}] \in C_{0,D}^{\infty}(\Omega_L), \qquad [\hat{u}_{xx}^{(n)}, \hat{v}_{xx}^{(n)}] \xrightarrow{H^1(\Omega_L)} [\hat{u}_{xx}, \hat{v}_{xx}]. \tag{3.17.98}$$

From here, repeating the estimates in Proposition 3.10.8, and sending $n \to \infty$ and then $L \to \infty$ gives the desired result.

□

Piecing together the above set of estimates,

Proposition 3.17.9. *Let δ, ε be sufficiently small relative to universal constants, and $\varepsilon << \delta << b$. Then for $[\hat{u}, \hat{v}] \in Z$ solutions to (3.17.2) - (3.17.3):*

$$||\hat{u}, \hat{v}||^2_{X_{1,b} \cap X_{2,b} \cap X_{3,b}} \lesssim \mathcal{W}_{1,b} + \mathcal{W}_{2,b} + \mathcal{W}_{3,b}, \tag{3.17.99}$$

where $\mathcal{W}_{i,b}$ have been defined in (3.17.11), (3.17.85), (3.17.94).

By repeating the analysis in Section 3.9, one has:

Lemma 3.17.10. *Let δ, ε be sufficiently small relative to universal constants, and $\varepsilon << \delta << b$. Then for $[\hat{u}, \hat{v}] \in Z$ solutions to (3.17.2) - (3.17.3):*

$$||\hat{u}, \hat{v}||^2_{Z_b} \lesssim \varepsilon^{\frac{n}{2} + \gamma - \omega(N_i)} ||\hat{u}, \hat{v}||^4_{Z_b} + ||\hat{u}, \hat{v}||^2_{X_{1,b} \cap X_{2,b} \cap X_{3,b}}. \tag{3.17.100}$$

Due to (3.17.99), we will now turn to estimating $\mathcal{W}_{i,b}$

Lemma 3.17.11. *Let $\mathcal{W}_{1,b}, \mathcal{W}_{2,b}, \mathcal{W}_{3,b}$ be as in (3.10.10), (3.10.82), (3.10.219). Then:*

$$|\mathcal{W}_{1,b} + \mathcal{W}_{2,b} + \mathcal{W}_{3,b}| \lesssim C(b) \varepsilon^{\frac{n}{2} + \gamma - \omega(N_i)} ||\hat{u}, \hat{v}||^2_{Z_b}, \tag{3.17.101}$$

where $C(b) \uparrow \infty$ *as* $b \downarrow 0$.

Proof. We will work with the expression:

$$\begin{aligned}
\hat{f} &= \varepsilon^{\frac{n}{2}+\gamma}\Big[u^{(1)}u^{(1)}_x - u^{(2)}u^{(2)}_x + v^{(1)}u^{(1)}_y - v^{(2)}u^{(2)}_y\Big] \\
&= \varepsilon^{\frac{n}{2}+\gamma}\Big[\hat{u}u^{(1)}_x + u^{(2)}\hat{u}_x + \hat{v}u^{(1)}_y + v^{(2)}\hat{u}_y\Big], \qquad (3.17.102)\\
\hat{g} &= \varepsilon^{\frac{n}{2}+\gamma}\Big[u^{(1)}v^{(1)}_x - u^{(2)}v^{(2)}_x + v^{(1)}v^{(1)}_y - v^{(2)}v^{(2)}_y\Big] \\
&= \varepsilon^{\frac{n}{2}+\gamma}\Big[\hat{u}v^{(1)}_x + u^{(2)}\hat{v}_x + \hat{v}v^{(1)}_y + v^{(2)}\hat{v}_y\Big]. \qquad (3.17.103)
\end{aligned}$$

Concerning $\mathcal{W}_{1,b}$, let us bring particular attention to the following term from $\int\int |\hat{f}| \cdot |\hat{u}|x^{-2b}$:

$$\begin{aligned}
\int\int & \varepsilon^{\frac{n}{2}+\gamma}[\hat{v}u^{(1)}_y + v^{(2)}\hat{u}_y] \cdot |\hat{u}|x^{-2b} \\
&\le \varepsilon^{\frac{n}{2}+\gamma}||\hat{v}x^{\frac{1}{2}-b}||_{L^\infty}||u^{(1)}_y||_{L^2}||\hat{u}x^{-\frac{1}{2}-b}||_{L^2} \\
&\qquad + \varepsilon^{\frac{n}{2}+\gamma}||v^{(2)}x^{\frac{1}{2}}||_{L^\infty}||\hat{u}_y x^{-b}||_{L^2}||\hat{u}x^{-\frac{1}{2}-b}||_{L^2} \\
&\le \varepsilon^{\frac{n}{2}+\gamma}\Big[||\hat{v}x^{\frac{1}{2}-b}||_{L^\infty}||u^{(1)}_y||_{L^2}||\hat{u}_x x^{\frac{1}{2}-b}||_{L^2} \\
&\qquad + ||v^{(2)}x^{\frac{1}{2}}||_{L^\infty}||\hat{u}_y x^{-b}||_{L^2}||\hat{u}_x x^{\frac{1}{2}-b}||_{L^2}\Big] \\
&\le C(b)\varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||u^{(i)}, v^{(i)}||_Z||\hat{u}, \hat{v}||^2_{Z_b}. \qquad (3.17.104)
\end{aligned}$$

The above term requires the weight of $x^{-2b}, b > 0$, in order to apply the Hardy inequality. Indeed, this was not required for the existence proof (see calculation (3.11.5)), because the structure of $vu_y \cdot u$ enabled us to integrate by parts, unlike in the present situation. The remaining terms in $\mathcal{W}_{1,b}$, and all terms in $\mathcal{W}_{2,b}, \mathcal{W}_{3,b}$ are treated nearly identically to the Lemma 3.11.1, and so we omit repeating those calculations.

□

Corollary 3.17.12. *Fix $0 < b < 1$ sufficiently small, relative to universal constants. Suppose ε, δ are sufficiently small, such that $\varepsilon << \delta << b$. Then $\hat{u}, \hat{v} = 0$.*

Proof. Combining estimate (3.17.101) and (3.17.100) with estimate (3.16.12) yields:

$$||\hat{u}, \hat{v}||^2_{Z_b} \lesssim C(b)\varepsilon^{\frac{n}{2}+\gamma-\omega(N_i)}||\hat{u}, \hat{v}||^2_{Z_b}. \tag{3.17.105}$$

For ε sufficiently small, this then implies $||\hat{u}, \hat{v}||_{Z_b} = 0$. Upon consultation with the norm Z_b, and (3.17.4), this implies that $\hat{u}, \hat{v} = 0$. □

Remark. We have controlled the second and third order energy norms, (3.17.77) - (3.17.78) in order to treat the term $\int\int \hat{v}u_y^{(1)}|\hat{u}|x^{-2b}$, which appears in (3.17.104). This term forces us to control $||\hat{v}x^{\frac{1}{2}-b}||_{L^\infty}$. One cannot get around placing this term in L^∞ (for instance by integrating by parts from $u_y^{(1)}$) because this produces suboptimal decay rates, according to (3.9.92) - (3.9.93).

This then establishes Theorem 3.12.1, and controlling $[u, v] \in Z$ then immediately establishes the main result, Theorem 3.2.2.

Chapter Four

Steady Prandtl Layers over a Moving Boundary: Nonshear Flows

4.1 Abstract

In this article we establish the validity of Prandtl layer expansions around Euler flows which are not shear. The presence of non-shear flows at the leading order creates a singularity of $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$. A new y-weighted positivity estimate is developed to control this leading-order growth at the far field.

4.2 Introduction

We consider the steady, incompressible Navier-Stokes equations on the domain $\Omega = (0, L) \times (0, \infty)$. The boundary consists of three components, $Y = 0, x = 0$, and $x = L$. The system reads:

$$\left.\begin{aligned} &U^{NS}U^{NS}_x + V^{NS}U^{NS}_Y + P^{NS}_x = \epsilon\Delta U^{NS} \\ &U^{NS}V^{NS}_x + V^{NS}V^{NS}_Y + P^{NS}_Y = \epsilon\Delta V^{NS} \\ &U^{NS}_x + V^{NS}_Y = 0. \end{aligned}\right\} \quad \text{in } \Omega \tag{4.2.1}$$

The system above is taken together with the no-slip boundary condition on $Y = 0$, which in addition is assumed to be moving with velocity $u_b > 0$. The boundary conditions at $x = 0, L$ are inflow and outflow conditions, to be prescribed specifically in the article.

We are interested in the asymptotic behavior of solutions to (4.2.1) as $\varepsilon \to 0$. Such asymptotics must capture the formation of boundary layers, which we now describe in generality. Suppose an outer Euler flow is prescribed:

$$[u^0_e(x, Y), v^0_e(x, Y), P^0_e(x, Y)], \tag{4.2.2}$$

satisfying the Euler equations:

$$\left.\begin{aligned} &u_e^0 u_{ex}^0 + v_e^0 u_{eY}^0 + P_{ex}^0 = 0 \\ &u_e^0 v_{ex}^0 + v_e^0 v_{eY}^0 + P_{eY}^0 = 0 \\ &u_{ex}^0 + v_{eY}^0 = 0, \end{aligned}\right\} \qquad \text{in } \Omega \tag{4.2.3}$$

together with the no penetration boundary conditions at $Y = 0, Y \to \infty$:

$$v_e^0|_{Y=0} = v_e^0|_{Y\to\infty} = 0. \tag{4.2.4}$$

Generically there is a mismatch between the boundary velocity $u_e^0(x, 0)$ and u_b, indicating that one should not expect solutions of (4.2.1) to converge to $[u_e^0, v_e^0]$ in the L^∞ norm. Rather, it was proposed in 1904 by Ludwig Prandtl that one should expect the formation of boundary layers, which can be expressed mathematically as an asymptotic expansion:

$$\begin{aligned} &U^{NS}(x, Y) = u_e^0(x, Y) + u_p^0(x, \frac{Y}{\sqrt{\varepsilon}}) + \mathcal{O}(\sqrt{\varepsilon}), \qquad (4.2.5) \\ &V^{NS}(x, Y) = v_e^0(x, Y) + \sqrt{\varepsilon} v_p^0(x, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon} v_e^1(x, Y) + \mathcal{O}(\varepsilon), \\ &P^{NS}(x, Y) = P_e^0(x, Y) + P_p^0(x, \frac{Y}{\sqrt{\varepsilon}}) + \mathcal{O}(\sqrt{\varepsilon}). \end{aligned}$$

The flows considered under the present setup are elliptic. Thus, a mathematical formulation of validating the expansion (4.2.5) is to assume boundary data are prescribed so that the expansions (4.2.5) are valid at the boundaries, $x = 0, L$, and to then prove that they must be valid in the interior of the domain, Ω.

Under the setup described above, (4.2.5) has been justified rigorously for shear flows in (GN17). Our aim in this article is to generalize the results to non-shear flows that are "sufficiently close to shear", to be made rigorous by assumption (4.2.25) in our main result.

As is evident from (4.2.5), such a generalization is a *leading order effect*, which when scaled to Prandtl variables creates a singularity of $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$; this is evident in the specification of (4.2.14) below.

Let us briefly highlight the physical importance of developing a method to handle non-shear Eulerian flows. A classical setup from fluid mechanics deals with horizontal flows past a rotating disk, see for instance (SG00). Such a flow is non-shear, as in the set-up considered here. In the simpler case when the flows are actually circular (and therefore shear), as opposed to horizontal, in the presence of a rotating disk, the article of (Iye17a) develops machinery to handle the geometry of the boundary. The present article can be viewed as a first step in studying non-shear flows, without adding the complexities of a curved boundary.

Boundary Layer Expansions

We will work with scaled, boundary layer variables $y = \frac{Y}{\sqrt{\varepsilon}}$, and consider the scaled Navier-Stokes unknowns:

$$U^\varepsilon(x,y) = U^{NS}(x,Y), \quad V^\varepsilon(x,y) = \frac{V^{NS}(x,Y)}{\sqrt{\varepsilon}} \quad P^\varepsilon(x,y) = P^{NS}(x,Y). \tag{4.2.6}$$

In the new unknowns, the system (4.2.1) becomes:

$$U^\varepsilon U^\varepsilon_x + V^\varepsilon U^\varepsilon_y + P^\varepsilon_x = U^\varepsilon_{yy} + \varepsilon U^\varepsilon_{xx}, \tag{4.2.7}$$

$$U^\varepsilon V^\varepsilon_x + V^\varepsilon V^\varepsilon_y + \frac{P^\varepsilon_y}{\varepsilon} = V^\varepsilon_{yy} + \varepsilon V^\varepsilon_{xx} \tag{4.2.8}$$

$$U^\varepsilon_x + V^\varepsilon_y. \tag{4.2.9}$$

We start with the following expansions:

$$U^\varepsilon = u_e^0 + u_p^0 + \sqrt{\varepsilon} u_e^1 + \sqrt{\varepsilon} u_p^1 + \varepsilon^{\frac{1}{2}+\gamma} u := u_s + \varepsilon^{\frac{1}{2}+\gamma} u, \tag{4.2.10}$$

$$V^\varepsilon = \frac{v_e^0}{\sqrt{\varepsilon}} + v_p^0 + v_e^1 + \sqrt{\varepsilon} v_p^1 + \varepsilon^{\frac{1}{2}+\gamma} v = v_s + \varepsilon^{\frac{1}{2}+\gamma} v, \tag{4.2.11}$$

$$P^\varepsilon = P_e^0 + P_p^0 + \sqrt{\varepsilon} P_e^1 + \sqrt{\varepsilon} P_p^1 + \varepsilon P_p^2 + \varepsilon^{\frac{1}{2}+\gamma} P = P_s + \varepsilon^{\frac{1}{2}+\gamma} P. \tag{4.2.12}$$

We are prescribed the Euler flow:

$$[u_e^0, v_e^0, P_e^0]. \tag{4.2.13}$$

Importantly, the fact that u_e^0 is not shear means that it can have an x-dependence. This in turn implies that v_e^0 and P_e^0 are nonzero. Our analysis does not assume a sign condition for $\partial_x P_e^0$. Due to the x-dependence of u_e^0, it is natural that in the scaled, Prandtl variable, there is a singularity of $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$, (see below, equation (4.2.14)).

We will construct the remaining terms in $[u_s, v_s, P_s]$, as defined by (4.2.10) - (4.2.12), in Appendix 4.9. We will specify the particular equations satisfied by each of the terms in $[u_s, v_s]$ in Appendix 4.9. Let us explicitly write the form of v_s:

$$v_s = \frac{v_e^0}{\sqrt{\varepsilon}} + v_p^0 + v_e^1 + \sqrt{\varepsilon} v_p^1. \tag{4.2.14}$$

As can be seen from above, the presence of nonzero v_e^0 creates a leading order singularity of $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$, which is the main difficulty that must be addressed by our analysis.

The main part of the article will be to construct and control the final term in the expansion, $[u, v, P]$, which we term the "remainders". The equations satisfied by the remainders $[u, v, P]$ are specified in (4.2.30) - (4.2.32).

We now discuss the boundary data of each term above. The key point is that the no slip condition on $Y = 0$ must be enforced at each order in the expansion:

$$u_e^0(x,0) + u_p^0(x,0) = u_b, \quad u_p^1(x,0) = -u_e^1(x,0), \quad u(x,0) = 0 \tag{4.2.15}$$

$$v_e^0(x,0) = 0, \quad v_e^1(x,0) = -v_p^0(x,0), \quad v_p^1(x,0) = 0, \quad v(x,0) = 0. \tag{4.2.16}$$

The boundary data at $x = 0$ must be specified for the Prandtl layers as follows:

$$u_p^0(x,0) = u_{p0}^0(y), \quad u_p^1(x,0) = u_{p0}^1(y). \tag{4.2.17}$$

The equations for u_p^i are diffusion equations, and so need to only be prescribed initial data at $x = 0$. v_p^i are then recovered via the divergence free condition, and therefore do not need in-flow boundary conditions. We will assume that u_{p0}^i are smooth and exponentially decaying.

In contrast, the Euler layers, $[u_e^1, v_e^1]$ satisfy an elliptic system, and we must prescribe boundary data at both $x = 0, L$. We do so at the level of the stream function, where $\nabla^\perp \phi^1 = [u_e^1, v_e^1]$:

$$\phi^1(0,Y) = \phi_0^1(Y), \quad \phi^1(L,Y) = \phi_L^1(Y). \tag{4.2.18}$$

These are also assumed smooth and rapidly decaying, and in addition must satisfy a compatibility condition which we call "well-prepared" boundary data defined in Definition 4.9.6.

Finally, we can describe the boundary data for the remainders, $[u, v, P]$:

$$[u,v]|_{x=0} = [a_0(y), b_0(y)], \quad [u,v]|_{y=0} = [u,v]|_{y\to\infty} = 0, \tag{4.2.19}$$

$$P - 2\varepsilon u_x|_{x=L} = a_L(y), \quad u_y + \varepsilon v_x|_{x=L} = b_L(y). \tag{4.2.20}$$

The boundary condition at $x = 0$ allows the prescription of in-flow data. The boundary conditions at $x = L$ in (4.2.20) is known as the (inhomogeneous) stress-free boundary condition, and corresponds to evaluating the Cauchy stress tensor at the boundary $x = L$. We will provide assumptions on the boundary data:

$$|\partial_y^k a_L| \lesssim \sqrt{\varepsilon}\langle y\rangle^{-N}, \quad |\partial_y^k\{a_0, b_0, b_L\}| \lesssim \langle y\rangle^{-N}, \quad \text{supp}\{a_0, b_0, a_L, b_L\} \subset \{y \geq 1\}. \tag{4.2.21}$$

for sufficiently large k, N.

Main Theorem

In order to state our result, we must introduce the norm in which will control the solution. Define our $\mathcal{X}$ norm to be:

$$\begin{aligned} ||u, v||_{\mathcal{X}} := {} & ||u_y \cdot y||_{L^2} + ||\sqrt{\varepsilon}u_x \cdot y||_{L^2} + ||v_y, \sqrt{\varepsilon}v_x||_{L^2} \\ & + ||\Big\{u_{yy}, \sqrt{\varepsilon}u_{xy}, \varepsilon u_{xx}\Big\} \cdot y||_{L^2} + \varepsilon^{\frac{\gamma}{2}}||u, \sqrt{\varepsilon}v||_{L^\infty} \\ & + ||u, v||_B, \end{aligned} \tag{4.2.22}$$

where the boundary norm is given by:

$$||u, v||_B := ||u_y \cdot y, \sqrt{\varepsilon}u_x \cdot y||_{L^2(x=L)} + ||\sqrt{\varepsilon}u_x||_{L^2(x=L)}. \tag{4.2.23}$$

We will also have to define the space, $\mathcal{X}$, for which we refer the reader to Appendix B, equation 4.10.9.

Theorem 4.2.1. *Consider an Euler flow $[u^0_e(x,Y), v^0_e(x,Y)]$ satisfying the following hypothesis:*

$$0 < c_0 \le u^0_e \le C_0 < \infty, \tag{4.2.24}$$

$$||\frac{v^0_e}{Y}||_{L^\infty} << 1, \text{ and} \tag{4.2.25}$$

$$||Y^k \nabla^m v^0_e||_{L^\infty} < \infty \text{ for sufficiently large } k, m \ge 0, \tag{4.2.26}$$

$$||Y^k \nabla^m u^0_e||_{L^\infty} < \infty \text{ for sufficiently large } k \ge 0, m \ge 1. \tag{4.2.27}$$

Let the interval L be sufficiently small relative to universal constants. Suppose in addition that the boundary data described above are prescribed, assumed to be smooth and rapidly decaying in their arguments, satisfy the assumptions (4.2.21), and satisfy the compatibility conditions given in Definition 4.9.6. Then the remainder solutions $[u, v, P]$ exist in the space $\mathcal{X}$ and satisfy the estimate:

$$||u, v||_{\mathcal{X}} \lesssim 1. \tag{4.2.28}$$

Corollary 4.2.2. *In the inviscid limit, we have the convergence:*

$$||U^{NS} - u^0_e - u^0_p||_{L^\infty} + ||V^{NS} - v^0_e||_{L^\infty} \le \sqrt{\varepsilon}. \tag{4.2.29}$$

Remark. The Euler flows which satisfy the assumptions of (4.2.24) - (4.2.27) are plentiful, see Proposition 4.9.1 in Appendix 4.9.

Let us place this result in the context of recent developments in the boundary layer theory. We will restrict to stationary, two dimensional flows. A central task in this setting is to establish validity of an expansion of the type (4.2.5), and this is considered to be one of the most challenging open problems in fluid mechanics. It has been achieved in the setting of a moving boundary in (GN17), (Iye17a), (Iye16). The method introduced by (GN17) relies on establishing a crucial positivity estimate which gives $o(1)$ control over the remainder quantity $||v_y, \sqrt{\varepsilon} v_x||_{L^2}$. The flows considered in those works were all shear flows, and the aim of the present result is to generalize (in particular the result of (GN17)) to the case of non-shear flows. As can be seen in the expansion (4.2.11), this is a leading order effect, and therefore requires a new y-weighted estimate.

Within the stationary, two dimensional setting, the recent work of (DM18) addresses the related question of blowup of the Prandtl equation in the presence of an unfavorable pressure gradient. For unsteady flows, the validity of an asymptotic expansion of the form (4.2.5) has been established in the analyticity framework, (Asa91), (SC98a), (SC98b), in the Gevrey setting in (GVMM16), for initial vorticity bounded away from the origin in (Mae14), and for special flows in (MT08). Giving a more exhaustive survey of results in the unsteady setting would lead us astray, and so we refer the reader to the review articles of (E00), (GJT16), and (MM17) and the references therein.

Overview of Proof

Let us introduce the system satisfied by the remainders. For our discussion, we will consider the linearized version of system (4.9.105) - (4.9.107) and regard f, g and generic elements of L^2.

$$-\Delta_\varepsilon u + S_u + P_x = f, \tag{4.2.30}$$

$$-\Delta_\varepsilon v + S_v + \frac{P_y}{\varepsilon} = g, \tag{4.2.31}$$

$$u_x + v_y = 0, \tag{4.2.32}$$

together with the inhomogeneous boundary conditions:

$$[u, v]|_{y=0} = [u, v]|_{x=0} = [u, v]|_{y \to \infty} = 0, \tag{4.2.33}$$

$$P - 2\varepsilon u_x|_{x=L} = a_L(y), \quad \{u_y + \varepsilon v_x\}|_{x=L} = b_L(y). \tag{4.2.34}$$

In actuality, f, g contain the nonlinear components. Also note that we can, up to redefining a_L, b_L, reduce the boundary data from (4.2.19) - (4.2.20) to the homogenized boundary data (4.2.33) - (4.2.34). This is proven in Lemma 4.9.14. We provide relevant definitions below:

$$S^u = u_s u_x + u_{sx} u + v_s u_y + u_{sy} v, \quad S^v = u_s v_x + v_{sx} u + v_s v_y + v_{sy} v, \tag{4.2.35}$$

$$N^u(u,v) = \varepsilon^{\frac{1}{2}+\gamma}\Big[u u_x + v u_y\Big], \quad N^v(u,v) = \varepsilon^{\frac{1}{2}+\gamma}\Big[u v_x + v v_y\Big], \tag{4.2.36}$$

$$f = \varepsilon^{-\frac{1}{2}-\gamma} R^{u,1} + N^u + L_1^b, \quad g = \varepsilon^{-\frac{1}{2}-\gamma} R^{v,1} + N^v + L_2^b. \tag{4.2.37}$$

Here, R^u, R^v are high order profile remainders which are defined specifically in (4.9.15), (4.9.24) and estimated in (4.9.123), and L_1^b, L_2^b arise from homogenizing the boundary data, are defined in (4.9.117) and estimated in (4.9.118). The important consideration for the purposes of this discussion is the rough specification of $[u_s, v_s]$, which we can write:

$$u_s \approx u_e^0 + u_p^0 + \mathcal{O}(\sqrt{\varepsilon}), \tag{4.2.38}$$

$$v_s \approx \frac{v_e^0}{\sqrt{\varepsilon}} + \mathcal{O}(1). \tag{4.2.39}$$

Here, the Prandtl layer, u_p^0 is rapidly decaying in the Prandtl variable, y. The main idea is to close a y-weighted estimate which can control the $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$ contribution from v_s. The first estimate in our scheme is the basic energy estimate:

$$||u_y||_{L^2}^2 \lesssim \mathcal{O}(L)||v_y, \sqrt{\varepsilon} v_x||_{L^2}^2 + ||f, \sqrt{\varepsilon} g||_{L^2}^2 + C(a_0, b_0, a_L, b_L). \tag{4.2.40}$$

This estimate is standard, and is obtained by applying $(u, \varepsilon v)$ to the system (4.2.30) - (4.2.31). The main coercive term is the Δ_ε, which yields control over $||u_y, \sqrt{\varepsilon} v_y, \varepsilon v_x||_{L^2}^2$. The $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$ singular term from v_s does not play a role at the level of energy estimates because a factor of ∂_y hits each instance of v_s upon applying the multiplier $(u, \varepsilon v)$. Nevertheless, the energy estimate is too weak to close as a standalone estimate due to large convective terms. For instance, one considers:

$$|\int u_{sy}uv| = |\int \Big(\sqrt{\varepsilon}u^0_{eY} + u^0_{py} + \mathcal{O}(\sqrt{\varepsilon})\Big)uv| \le \mathcal{O}(L)||u_x||_{L^2}||v_y||_{L^2}. \tag{4.2.41}$$

Thus, it is required that v_y be controlled at $\mathcal{O}(1)$, which is a famous difficulty in the boundary layer theory. This is the content of the next step, which generates the following positivity estimate:

$$\begin{aligned}||v_y, \sqrt{\varepsilon}v_x||^2_{L^2} + ||\sqrt{\varepsilon}u_x||_{L^2(x=L)} \lesssim &||u_y||^2_{L^2} + ||\frac{v^0_e}{Y}||_{L^\infty}||u_y \cdot y, \sqrt{\varepsilon}v_y \cdot y||^2_{L^2} \\ &+ ||f, \sqrt{\varepsilon}g||^2_{L^2} + C(a_0, b_0, a_L, b_L).\end{aligned} \tag{4.2.42}$$

The above estimate is generating by applying $[\partial_y \frac{v}{u_s}, -\varepsilon\partial_x \frac{v}{u_s}]$ to the system (4.2.30) - (4.2.32). Here, $\mathcal{O}(v^0_e)$ is a constant that can be made small according to the assumption in (4.2.25). This estimate was introduced in the context of shear flows by (GN17), and crucially utilizes the multiplier $\frac{v}{u_s}$ which is able to generate coercivity over $||v_y, \sqrt{\varepsilon}v_x||^2_{L^2}$. We refer the reader to the article of (GN17) for more details, but emphasize that the significant difference when addressing non-shear flows is the term $||\frac{v^0_e}{Y}||_{L^\infty}||u_y \cdot y, \sqrt{\varepsilon}v_y \cdot y||^2_{L^2}$. *The key difficulty* for our analysis is the loss of one y-weight on the right-hand side due to this term, which is the leading order effect of the non-shear flow. Specifically, consider the term $v_s u_y$ appearing in S^u, as seen from definition (4.2.35), and recall that according to (4.2.39) the leading order of $v_s \approx \frac{v^0_e}{\sqrt{\varepsilon}}$. The outcome then becomes:

$$\begin{aligned}\int v_y^2 \lesssim \int \frac{v^0_e}{\sqrt{\varepsilon}}u_y v_y = \int \frac{v^0_e}{\sqrt{\varepsilon}y} y u_y v_y \\ \le ||\frac{v^0_e}{Y}||_{L^\infty}||u_y \cdot y||_{L^2}||v_y||_{L^2}.\end{aligned} \tag{4.2.43}$$

Our first main contribution of this paper is to develop the following y-weighted estimate which controls the term $||u_y \cdot y||_{L^2}$ term from (4.2.42):

$$||\{u_{yy}, \sqrt{\varepsilon}u_{xy}, \varepsilon u_{xx}\} \cdot y||_{L^2}^2 + ||\{u_y, \sqrt{\varepsilon}u_x\} \cdot y||_{L^2}^2 + ||\{u_y, \sqrt{\varepsilon}u_x\} \cdot y||_{L^2(x=L)}$$
$$\lesssim ||\sqrt{\varepsilon}u_x||_{L^2(x=L)} + ||u_y||_{L^2}^2 + ||v_y, \sqrt{\varepsilon}v_x||_{L^2}^2$$
$$+ \text{Forcing Terms } + C(a_0, b_0, a_L, b_L). \tag{4.2.44}$$

The key idea is to first apply ∂_y to the system (4.2.35). Let us extract the main terms coming from S^u:

$$\partial_y S^u = u_s u_{xy} + v_s u_{yy} + u_{syy} v. \tag{4.2.45}$$

We now introduce a mixed weight multiplier $u_y y^2 \cdot 1 - x$, where the $1 - x$ can take advantage of ∂_x integrating by parts:

$$\int u_s u_{xy} \cdot u_y y^2 (1-x) = -\int \frac{u_{sx}}{2} u_y^2 y^2 (1-x) + \int \frac{u_s}{2} u_y^2 y^2$$
$$+ \int_{x=L} \frac{u_s}{2} u_y^2 y^2 (1-L), \tag{4.2.46}$$
$$\int v_s u_{yy} \cdot u_y y^2 (1-x) = -\int \frac{v_{sy}}{2} u_y^2 y^2 (1-x) - \int v_s y u_y^2 (1-x). \tag{4.2.47}$$

Summing (4.2.46) - (4.2.47), using $u_{sx} + v_{sy} = 0$, and the smallness given in (4.2.25), we have:

$$(4.2.46) + (4.2.47) \gtrsim \int \frac{u_s}{2} u_y^2 y^2 - \int v_s y u_y^2 (1-x) + \int_{x=L} \frac{u_s}{2} u_y^2 y^2 (1-L)$$
$$\gtrsim \int \frac{u_s}{2} u_y^2 y^2 + \int_{x=L} \frac{u_s}{2} u_y^2 y^2 (1-L). \tag{4.2.48}$$

At the level of the convection, the additional ∂_y is necessary to generate an additional factor of $\sqrt{\varepsilon}$:

$$\begin{aligned}\int u_{syy} v \cdot u_y y^2 (1-x) &\approx \int \varepsilon u^0_{eYY} v \cdot u_y y^2 (1-x) \\ &\le \|Y^2 u^0_{eYY}\|_{L^\infty} \|\frac{v}{y}\|_{L^2} \|u_y \cdot y\|_{L^2} \lesssim \|v_y\|_{L^2} \|u_y y\|_{L^2}.\end{aligned} \tag{4.2.49}$$

The above series of estimates closes by using the smallness of L and v^0_e. Let us make a few remarks. Although the purpose of the weighted estimate, (4.2.44), is to capture behavior for large y, we cannot introduce a cut-off function that avoids the $y = 0$ boundary into the multiplier for instance by selecting $u_y y^2 (1-x)\chi(y)$. This is because the higher-order terms arising from $\partial_y \Delta_\varepsilon$ will generate local terms which cannot be controlled.

Apart from from the weighted estimate, (4.2.44), a second novelty of our analysis is that we treat a large class of inhomogeneous boundary data at $x = 0, x = L$, as is shown in (4.2.19) - (4.2.20). This level of generality is important and very physical: it corresponds to taking measurements of the fluid at the inflow and outflow edges, $x = 0$ and $x = L$, and taking these values as inputs. The technique for treating these boundary conditions is based on Lemma 4.9.14, proved in Appendix 4.9. We first construct an auxiliary divergence free vector field which attains the boundary data from (4.2.19) at $\{x = 0\}$. Using this auxiliary vector field to homogenize then creates a boundary contribution at $x = L$, which cannot be removed by a further homogenization due to the need to preserve the divergence-free condition. This has the effect of contributing several new boundary terms from $x = L$ into the positivity estimate, (4.2.42), which must then be controlled.

A third novelty of our analysis is to develop a scaled, weighted version of Korn's inequality to close the above scheme of estimates. Such an estimate is needed due to the higher order contributions which are created in order to perform estimate (4.2.44). In particular, the estimate we prove is a coercivity estimate of the form:

$$\int \Big[u^2_{yy} + 4\varepsilon u^2_{xy} + \varepsilon^2 u^2_{xx} - 2\varepsilon u_{yy} u_{xx}\Big] y^2 \cdot (1-x) \tag{4.2.50}$$

$$\gtrsim \int \Big[u_{yy}^2 + \varepsilon u_{xy}^2 + \varepsilon^2 u_{xx}^2\Big] y^2 \cdot (1-x) - \text{Acceptable Contributions.}$$

Notation

Within lemmas, we will use $X \sim \mathcal{O}(\text{LHS})$ and $X \sim \mathcal{O}(\text{RHS})$ to mean X can be controlled, up to a universal constant, by the left-hand side (or right-hand side, respectively) of the lemma we are proving. Quantities denoted by $\mathcal{O}(L)$ refer to those which can be made small by making L small, and quantities denoted by $\mathcal{O}(v_e^0)$ refer to those which can be made small according to the smallness assumptions in (4.2.25).

4.3 Energy Estimate

We will now give the basic energy estimate. The reader should recall the properties of the profiles, given in Appendix 4.9, in particular Lemma 4.9.16, and the space $\mathcal{X}$, as defined by the norm introduced in (4.2.22), and the definition in (4.10.9).

Proposition 4.3.1. *Solutions $[u, v, P] \in \mathcal{X}$, as defined by (4.10.9), to the system (4.2.30) - (4.2.32), with the boundary conditions (4.2.33) - (4.2.34), satisfy the following estimate:*

$$||u_y||_{L^2}^2 + \int_{x=L} \frac{u_s}{2}\Big(u^2 + \varepsilon v^2\Big) \lesssim \mathcal{O}(L)||v_y, \sqrt{\varepsilon} v_x||_{L^2}^2 + \mathcal{R}_1 + ||a_L, b_L||_{L^2}^2, \tag{4.3.1}$$

where:

$$\mathcal{R}_1 := \int f \cdot u + \varepsilon g \cdot v. \tag{4.3.2}$$

Proof. This follows upon applying $(u, \varepsilon v)$ to the system (4.2.30) - (4.2.32). First, we will write the Δ_ε terms in the following way:

$$\Delta_\varepsilon u = u_{yy} + 2\varepsilon u_{xx} + \varepsilon v_{xy}, \qquad \Delta_\varepsilon v = 2v_{yy} + \varepsilon \partial_x\{u_y + \varepsilon v_x\}. \tag{4.3.3}$$

Using the above representation, we now integrate by parts:

$$-\int u_{yy} \cdot u - \int 2\varepsilon u_{xx} \cdot u - \int \varepsilon v_{xy} u$$
$$= \int u_y^2 + \int 2\varepsilon u_x^2 + \int \varepsilon v_x u_y - \int_{x=L} 2\varepsilon u_x u, \tag{4.3.4}$$
$$-\int 2v_{yy} \cdot \varepsilon v - \int \varepsilon \partial_x\{u_y + \varepsilon v_x\} \cdot v$$
$$= + \int 2\varepsilon v_y^2 + \int \varepsilon^2 v_x^2 + \int \varepsilon u_y v_x - \int_{x=L} \varepsilon v b_L(y), \tag{4.3.5}$$
$$\int P_x \cdot u + \int P_y v = - \int_{x=L} Pu, \tag{4.3.6}$$

For (4.3.4) and (4.3.5) we have used the boundary conditions from (4.2.34). First, we will estimate the interior term from (4.3.5):

$$|\int \varepsilon u_y v_x| \le \sqrt{\varepsilon}||\sqrt{\varepsilon} v_x||_{L^2}||u_y||_{L^2} \le \sqrt{\varepsilon}\Big[||u_y||_{L^2}^2 + ||\sqrt{\varepsilon} v_x||_{L^2}^2\Big]. \tag{4.3.7}$$

Next, the boundary term from (4.3.5):

$$|\int_{x=L} \varepsilon v b_L| \le ||\varepsilon v||_{L^2(x=L)}||b_L||_{L^2(x=L)} \le \mathcal{O}(L)\varepsilon||\sqrt{\varepsilon} v_x||_{L^2}^2 + ||b_L||_{L^2}^2. \tag{4.3.8}$$

We combine the boundary term from (4.3.4) and (4.3.6) by invoking the stress free boundary condition, in (4.2.34):

$$-\int_{x=L} \{P - 2\varepsilon u_x\} \cdot u = -\int_{x=L} a_L(y) \cdot u$$
$$\le ||a_L||_{L^2(x=L)}||u||_{L^2(x=L)}$$

$$\leq ||a_L||^2_{L^2(x=L)} + \mathcal{O}(L)||u_x||^2_{L^2}. \tag{4.3.9}$$

We will now move to the terms from S^u, as defined in (4.2.35):

$$\int S^u \cdot u = \int \Big[u_s u_x + u_{sx} u + v_s u_y + u_{sy} v\Big] \cdot u \tag{4.3.10}$$

The most difficult convective term from S^u is:

$$|\int u_{sy} uv| \leq \mathcal{O}(L)||u_{sy} \cdot y||_{L^\infty} ||v_y, \sqrt{\varepsilon} v_x||^2_{L^2}.$$

We have used the estimate (4.9.124) with $k = 1$. The remaining profile terms:

$$\begin{aligned}\int \{u_s u_x + u_{sx} u + v_s u_y\} u &= \int_{x=L} \frac{u_s}{2} u^2 + \int u_{sx} u^2 \\ &\gtrsim \int_{x=L} \frac{u_s}{2} u^2 - \mathcal{O}(L)||u_{sx}||_{L^\infty} ||u_x||^2_{L^2}.\end{aligned} \tag{4.3.11}$$

Notice that crucially, the $v_s \sim \mathcal{O}(\frac{1}{\sqrt{\varepsilon}}) v^0_e$ singular term is accompanied by a factor of ∂_y which cancels the singularity. We now move to the profile terms from S^v, as defined in (4.2.35):

$$\begin{aligned}&\int \{u_s v_x + v_{sx} u + v_s v_y + v_{sy} v\} \varepsilon v \\ &\qquad = + \int_{x=L} \frac{1}{2} u_s v^2 + \int \varepsilon v_{sx} uv + \int \varepsilon v_{sy} v^2 \\ &\qquad \gtrsim + \int_{x=L} \frac{1}{2} u_s v^2 - ||\sqrt{\varepsilon} v_{sx}||_{L^\infty} \mathcal{O}(L) ||u_x||_2 ||\sqrt{\varepsilon} v_x||_2 \\ &\qquad + ||v_{sy}||_{L^\infty} \mathcal{O}(L) ||\sqrt{\varepsilon} v_x||^2_{L^2}.\end{aligned} \tag{4.3.12}$$

Above, we have again used that $\partial_y v_s \mathcal{O}(1)$, according to (4.9.124) with $k = 1$. This concludes the proof.

□

4.4 Positivity Estimate

For the positivity estimate, we must work with the new unknown:

$$\beta = \frac{v}{u_s}. \tag{4.4.1}$$

Note that this quantity is well-defined because $u_s > 0$, according to (4.9.125). We first establish the equivalence:

Lemma 4.4.1. *For any function v satisfying $v|_{y=0} = 0 = v|_{x=0} = 0$, and β defined through (4.4.1), the following estimate is valid:*

$$||v_y, \sqrt{\varepsilon} v_x||_{L^2}^2 \lesssim ||\beta_y, \sqrt{\varepsilon}\beta_x||_{L^2}^2 \tag{4.4.2}$$

Proof. The proof forwards directly from:

$$\begin{aligned}\int v_y^2 = \int |\partial_y \{u_s \beta\}|^2 = \int \Big(u_{sy}\beta + u_s \beta_y \Big)^2 \\ \lesssim \int u_{sy}^2 \beta^2 + \int u_s^2 \beta_y^2 \lesssim ||y u_{sy}||_{L^\infty}^2 \int u_s^2 \beta_y^2.\end{aligned} \tag{4.4.3}$$

We have used above that $\beta|_{y=0} = 0$, according to the assumptions of the lemma. We have also used estimate (4.9.124) with $k = 1$. Next,

$$\int v_x^2 = \int |\partial_x \{u_s \beta\}|^2 = \int \Big(u_{sx}\beta + u_s \beta_x \Big)^2$$

$$\lesssim \int u_{sx}^2 \beta^2 + \int u_s^2 \beta_x^2 \lesssim \int \beta_x^2. \tag{4.4.4}$$

Above, we have used that $\beta|_{x=0} = 0$, according to the assumptions of the lemma. This concludes the proof.

□

According to the above lemma, it suffices to control $||\nabla_\varepsilon \beta||_{L^2}$, to which we now turn:

Proposition 4.4.2 (Positivity Estimate)**.** *Solutions $[u, v, P] \in \mathcal{X}$, defined in (4.2.22), (4.10.9), to the system (4.2.30) - (4.2.32), with the boundary conditions (4.2.33) - (4.2.34) satisfy:*

$$||\beta_y, \sqrt{\varepsilon}\beta_x||_{L^2}^2 + \int_{x=L} \frac{1}{u_s} \varepsilon v_y^2 \lesssim ||u_y||_{L^2}^2 + \mathcal{O}(v_e^0)||u_y \cdot y, \sqrt{\varepsilon} v_y \cdot y||_{L^2}^2 \\ + \mathcal{R}_2 + ||b_L, \partial_y b_L, \frac{a_L}{\sqrt{\varepsilon}}||_{L^2(x=L)}^2. \tag{4.4.5}$$

where:

$$\mathcal{R}_2 := -\int f \cdot \beta_y + \varepsilon g \cdot \beta_x. \tag{4.4.6}$$

Proof. We will apply to the system (4.2.30) - (4.2.32) the multiplier:

$$[-\beta_y, +\varepsilon \beta_x]. \tag{4.4.7}$$

Our analysis consists of a series of steps, which we now detail:

Step 1: S_u Profile Terms

Referring to the definition of S^u in (4.2.35), we have via the divergence-free condition:

$$S^u = -u_s v_y + u_{sy} v + v_s u_y + u_{sx} u = -u_s^2 \beta_y + v_s u_y + u_{sx} u. \tag{4.4.8}$$

We gain:

$$\int -u_s^2 \beta_y \cdot -\beta_y = \int u_s^2 \beta_y^2. \tag{4.4.9}$$

Referring to the definition of v_s in (4.2.11), the main convective term is:

$$-\int v_s u_y \cdot \beta_y = \int \Big(\frac{v_e^0}{\sqrt{\varepsilon}}\Big) u_y \beta_y + \int \{v_p^0 + v_e^1 + \sqrt{\varepsilon} v_p^1\} u_y \beta_y. \tag{4.4.10}$$

The lowest order term is the most dangerous:

$$\begin{aligned} |\int \frac{v_e^0}{\sqrt{\varepsilon} y} y u_y \beta_y| &\le ||\frac{v_e^0}{Y}||_{L^\infty} ||u_y \cdot y||_{L^2} ||\beta_y||_{L^2} \\ &\le \mathcal{O}(v_e^0) \Big[||u_y \cdot y||_{L^2}^2 + ||\beta_y||_{L^2}^2 \Big]. \end{aligned} \tag{4.4.11}$$

Here we need the small parameter $||\frac{v_e^0}{Y}||_{L^\infty}$. For the higher-order contributions:

$$\begin{aligned} |\int \Big(v_s - \frac{v_e^0}{\sqrt{\varepsilon}}\Big) u_y \beta_y| &\le ||v_s - \frac{v_e^0}{\sqrt{\varepsilon}}||_{L^\infty} ||u_y||_{L^2} ||\beta_y||_{L^2} \\ &\le \delta ||\beta_y||_{L^2}^2 + N_\delta ||u_y||_{L^2}^2. \end{aligned} \tag{4.4.12}$$

Finally, the last term from (4.4.8)

$$|\int u_{sx} u \cdot -\beta_y| \le \mathcal{O}(L) ||u_{sx}||_{L^\infty} ||u_x||_{L^2} ||\beta_y||_{L^2} \lesssim \mathcal{O}(L) ||\beta_y||_{L^2}^2. \tag{4.4.13}$$

Step 2: S_v Profile Terms

Referring to the definition of S^v in (4.2.35), here we will be treating:

$$\int S_v \cdot -\varepsilon\partial_x\{uy^2w\} = \int \Big(u_s v_x + v_{sx}u + v_s v_y + v_{sy}v\Big) \cdot -\varepsilon\partial_x\{uy^2w\} \tag{4.4.14}$$

First:

$$\begin{aligned}\int u_s v_x \cdot \varepsilon\beta_x &= \int \varepsilon u_s^2\beta_x^2 + \int \varepsilon u_s u_{sx}\beta\beta_x \\ &\gtrsim \int \varepsilon u_s^2\beta_x^2 - \mathcal{O}(L)\int \varepsilon u_s^2\beta_x^2 \gtrsim \int \varepsilon u_s^2\beta_x^2,\end{aligned} \tag{4.4.15}$$

$$|\int v_{sx}u \cdot \varepsilon\beta_x| \le \mathcal{O}(L)||\sqrt{\varepsilon}v_{sx}||_{L^\infty}||u_x||_{L^2}||\sqrt{\varepsilon}\beta_x||_{L^2}. \tag{4.4.16}$$

Next, we will use the smallness of $||\frac{v_e^0}{Y}||_{L^\infty}$:

$$\begin{aligned}|\int \frac{v_e^0}{\sqrt{\varepsilon}}v_y \cdot \varepsilon\beta_x| &\le ||\frac{v_e^0}{\sqrt{\varepsilon}y}||_{L^\infty}||\sqrt{\varepsilon}v_y \cdot y||_{L^2}||\sqrt{\varepsilon}\beta_x||_{L^2} \\ &\le \mathcal{O}(v_e^0)\Big[||\sqrt{\varepsilon}v_y \cdot y||_{L^2}^2 + \mathcal{O}(\text{LHS})\Big],\end{aligned} \tag{4.4.17}$$

$$\begin{aligned}|\int \{v_s - \frac{v_e^0}{\sqrt{\varepsilon}}\}v_y \cdot \varepsilon\beta_x| &\le \sqrt{\varepsilon}||v_s - \frac{v_e^0}{\sqrt{\varepsilon}}||_{L^\infty}||v_y||_{L^2}||\sqrt{\varepsilon}\beta_x||_{L^2} \\ &\le \sqrt{\varepsilon}\mathcal{O}(\text{LHS}),\end{aligned} \tag{4.4.18}$$

$$|\int v_{sy}v \cdot \varepsilon\beta_x| \le \mathcal{O}(L)||v_{sy}||_{L^\infty}||\sqrt{\varepsilon}v_x||_{L^2}||\sqrt{\varepsilon}\beta_x||_{L^2}. \tag{4.4.19}$$

Step 3: Pressure Terms

$$\begin{aligned}\int P_x \cdot -\beta_y + \int P_y \cdot \beta_x &= -\int_{x=L} P\beta_y = -\int_{x=L} 2\varepsilon u_x\beta_y - \int_{x=L} a_L\beta_y \\ &= +\int_{x=L} 2\varepsilon\frac{v_y^2}{u_s} - \int_{x=L} 2\varepsilon u_x v\partial_y\{\frac{1}{u_s}\} - \int_{x=L} a_L\beta_y \\ &= +\int_{x=L} 2\varepsilon\frac{v_y^2}{u_s} - \mathcal{O}(L)||u_{sy}||_{L^\infty}||\sqrt{\varepsilon}v_x||_{L^2}||\sqrt{\varepsilon}v_y||_{L^2(x=L)}\end{aligned}$$

$$- \int_{x=L} a_L \beta_y. \tag{4.4.20}$$

The above term crucially yields control over the boundary term appearing in (4.4.5). We must estimate the contribution:

$$\begin{aligned} | \int_{x=L} a_L \beta_y | &\le || \frac{a_L}{\sqrt{\varepsilon}} ||_{L^2(x=L)} || \sqrt{\varepsilon} \beta_y ||_{L^2(x=L)} \\ &\lesssim N_\delta || \frac{a_L}{\sqrt{\varepsilon}} ||^2_{L^2(x=L)} + \delta || \sqrt{\varepsilon} \beta_y ||^2_{L^2(x=L)}, \end{aligned} \tag{4.4.21}$$

the latter of which can be absorbed into (4.4.20).

Step 4: Vorticity Terms

We will now move to the vorticity terms from (4.2.30) - (4.2.32), where the stress-free boundary condition shown in (4.2.34) will be used repeatedly.

$$\begin{aligned} + \int u_{yy} \beta_y &= \int u_{yy} \cdot \frac{v_y}{u_s} - u_{yy} v \frac{u_{sy}}{u_s^2} \\ &= - \int u_y \partial_y \{ \frac{v_y}{u_s} \} + u_y \partial_y \{ v \frac{u_{sy}}{u_s^2} \} \\ &= - \int u_y \frac{v_{yy}}{u_s} - \int 2 u_y v_y \partial_y \{ \frac{1}{u_s} \} + \int u_y v \partial_y \frac{u_{sy}}{u_s^2} \\ &= - \int \frac{u_y^2}{2} \partial_x \frac{1}{u_s} + \int_{x=L} \frac{u_y^2}{2u_s} - \int 2 u_y v_y \partial_y \{ \frac{1}{u_s} \} \\ &+ \int u_y v \partial_y \{ \frac{u_{sy}}{u_s^2} \} \\ &\gtrsim \int_{x=L} \frac{u_y^2}{2u_s} - || u_{sx}, u_{sy}, y u_{sy}^2, y u_{syy} ||_{L^\infty} \Big[N_\delta || u_y ||^2_{L^2} + \delta || v_y ||^2_{L^2} \Big]. \end{aligned} \tag{4.4.22}$$

The boundary term above, as with all boundary terms from this set of calculations, will be put into (4.4.28), and subsequently estimated. Next:

$$
\begin{aligned}
+\int \varepsilon u_{xx}\beta_y &= -\int \varepsilon u_x \beta_{xy} + \int_{x=L} \varepsilon u_x \beta_y \\
&= -\int \varepsilon u_x \partial_x \{ \frac{v_y}{u_s} - v\frac{u_{sy}}{u_s^2} \} + \int_{x=L} \varepsilon u_x \beta_y \\
&= -\int \frac{\varepsilon}{2} u_x^2 \partial_x \{ \frac{1}{u_s} \} + \int_{x=L} \frac{\varepsilon}{2u_s} u_x^2 - \int \varepsilon u_x v_y \partial_x \frac{1}{u_s} \\
&\quad + \int \varepsilon u_x v_x \frac{u_{sy}}{u_s^2} + \int \varepsilon u_x v \partial_x \{ \frac{u_{sy}}{u_s^2} \} + \int_{x=L} \varepsilon u_x \beta_y. \qquad (4.4.23)
\end{aligned}
$$

The boundary terms are estimated as in:

$$
\begin{aligned}
&+\int_{x=L} \frac{\varepsilon}{2u_s} u_x^2 + \int_{x=L} \varepsilon u_x \beta_y \\
&\qquad = -\int_{x=L} \varepsilon u_x^2 \frac{1}{2u_s} + \int_{x=L} \varepsilon u_x v \frac{u_{sy}}{u_s^2} \\
&\qquad \leq -\int_{x=L} \varepsilon u_x^2 \frac{1}{2u_s} + \mathcal{O}(L) ||u_{sy}||_{L^\infty} ||\sqrt{\varepsilon} v_x||_{L^2} ||\sqrt{\varepsilon} u_x||_{L^2(x=L)}, \qquad (4.4.24)
\end{aligned}
$$

the final term above being absorbed into (4.4.20) using the smallness of L. The bulk terms are estimated via:

$$
\begin{aligned}
&|\int \frac{\varepsilon}{2} u_x^2 \partial_x \{ \frac{1}{u_s} \}| + |\int \varepsilon u_x v_y \partial_x \frac{1}{u_s}| + |\int \varepsilon u_x v_x \frac{u_{sy}}{u_s^2}| + |\int \varepsilon u_x v \partial_x \frac{u_{sy}}{u_s^2}| \\
&\qquad \leq \sqrt{\varepsilon} ||u_{sx}, u_{sy}, u_{sxy}||_{L^\infty} \Big[||u_x||_{L^2}^2 + ||\sqrt{\varepsilon} v_x||_{L^2}^2 \Big]. \qquad (4.4.25)
\end{aligned}
$$

Next:

$$
\begin{aligned}
-\int \varepsilon v_{yy} \beta_x &= +\int \varepsilon v_y \beta_{xy} \\
&= +\int \varepsilon v_y \partial_y \{ \frac{v_x}{u_s} - \frac{u_{sx}}{u_s^2} v \} \\
&= +\int \varepsilon v_y \Big(\frac{v_{xy}}{u_s} - v_x \frac{u_{sy}}{u_s^2} - \partial_y \{ \frac{u_{sx}}{u_s^2} \} v - \frac{u_{sx}}{u_s^2} v_y \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= + \int \frac{\varepsilon}{2}\frac{u_{sx}}{u_s^2} v_y^2 + \int_{x=L} \frac{\varepsilon}{2u_s} v_y^2 - \int \varepsilon v_x v_y \frac{u_{sy}}{u_s^2} \\
&- \int \varepsilon v v_y \partial_y \{\frac{u_{sx}}{u_s^2}\} - \int \varepsilon v_y^2 \frac{u_{sx}}{u_s^2} \\
&\gtrsim \int_{x=L} \frac{\varepsilon}{2u_s} v_y^2 - ||u_{sx}, u_{sy}, u_{sxy}||_{L^\infty} | \Big[\mathcal{O}(L) ||\sqrt{\varepsilon} v_x||_{L^2}^2 + \sqrt{\varepsilon} ||v_y||_{L^2}^2 . \Big] \qquad (4.4.26)
\end{aligned}
$$

Finally:

$$
\begin{aligned}
- \int \varepsilon^2 v_{xx} \beta_x &= - \int \varepsilon^2 v_{xx} \frac{v_x}{u_s} - \int \varepsilon^2 v_{xx} v \partial_x \frac{1}{u_s} \\
&= + \int \frac{\varepsilon^2}{2} \partial_x \frac{1}{u_s} v_x^2 - \int_{x=L} \frac{\varepsilon^2}{2u_s} v_x^2 \\
&+ \int_{x=0} \frac{\varepsilon^2}{2u_s} v_x^2 + \int \varepsilon^2 v_x^2 \partial_x \frac{1}{u_s} \\
&+ \int \varepsilon^2 v v_x \partial_{xx} \{\frac{1}{u_s}\} - \int_{x=L} \varepsilon^2 v v_x \partial_x \{\frac{1}{u_s}\} \\
&\gtrsim \int_{x=0} \frac{\varepsilon^2}{2u_s} v_x^2 - \int_{x=L} \frac{\varepsilon^2}{2u_s} v_x^2 - \int_{x=L} \varepsilon^2 v v_x \partial_x \{\frac{1}{u_s}\} \\
&- \varepsilon ||u_{sx}, u_{sxx}||_{L^\infty} \mathcal{O}(L) ||\sqrt{\varepsilon} v_x||_{L^2}^2 . \qquad (4.4.27)
\end{aligned}
$$

Collecting the highest order $x = L$ boundary contributions from (4.4.22), (4.4.24), (4.4.26), (4.4.27):

$$
\begin{aligned}
&+ \int_{x=L} \frac{u_y^2}{2u_s} - \int_{x=L} \frac{\varepsilon}{2u_s} u_x^2 + \int_{x=L} \frac{\varepsilon}{2u_s} v_y^2 - \int_{x=L} \frac{\varepsilon^2}{2u_s} v_x^2 \\
&= + \int_{x=L} \frac{u_y^2}{2u_s} - \int_{x=L} \frac{\varepsilon^2}{2u_s} v_x^2 \\
&= + \int_{x=L} \frac{u_y^2}{2u_s} - \int_{x=L} \frac{(u_y + \varepsilon v_x - u_y)^2}{2u_s} \\
&= + \int_{x=L} \frac{u_y^2}{2u_s} - \int_{x=L} \frac{(b_L - u_y)^2}{2u_s} \\
&= + \int_{x=L} \frac{u_y^2}{2u_s} - \int_{x=L} \frac{b_L^2}{2u_s} - \int_{x=L} \frac{u_y^2}{2u_s} + \int_{x=L} \frac{1}{u_s} b_L u_y \\
&= - \int_{x=L} \frac{b_L^2}{2u_s} - \int_{x=L} u \partial_y \{\frac{b_L}{u_s}\} \\
&\lesssim ||b_L, \partial_y b_L||_{L^2(x=L)}^2 + ||u||_{L^2(x=L)}^2
\end{aligned}
$$

$$\lesssim ||b_L, \partial_y b_L||^2_{L^2(x=L)} + \mathcal{O}(L)||u_x||^2_{L^2}. \tag{4.4.28}$$

The final boundary term from (4.4.27) can be estimated via:

$$\begin{aligned}
|\int_{x=L} \varepsilon^2 v v_x \partial_x\{\frac{1}{u_s}\}| &= |\int \varepsilon v u_y \partial_x \{\frac{1}{u_s}\}| \\
&= |\int \varepsilon v_y u \partial_x\{\frac{1}{u_s}\}| + |\int \varepsilon v u \partial_{xy}\{\frac{1}{u_s}\}| \\
&\leq \sqrt{\varepsilon}||u_{sx}, y\partial_{xy}\{\frac{1}{u_s}\}||_{L^\infty}||\sqrt{\varepsilon}v_y||_{L^2(x=L)}\mathcal{O}(L)||u_x||_{L^2} \qquad (4.4.29)
\end{aligned}$$

The boundary contribution from (4.4.29) can be absorbed into (4.4.20). This concludes the proof.

□

4.5 Weighted Estimates

In this section, we will bootstrap to the weighted estimates described in (4.2.44). By differentiating the system (4.2.30) - (4.2.32), we have:

$$-\Delta_\varepsilon u_y + P_{xy} + \partial_y S_u = \partial_y f \tag{4.5.1}$$

$$-\Delta_\varepsilon v_y + \frac{P_{yy}}{\varepsilon} + \partial_y S_v = \partial_y g, \tag{4.5.2}$$

where:

$$\partial_y S_u = u_s u_{xy} + v_s u_{yy} + u_{syy} v + u_{sxy} u \tag{4.5.3}$$

$$\partial_y S_v = u_s v_{xy} + v_s v_{yy} + u_{sy} v_x + v_{sxy} u + v_{sx} u_y + 2 v_{sy} v_y + v_{syy} v. \tag{4.5.4}$$

We will now prove the main weighted estimate. The reader should keep in mind Lemma 4.9.16 which will be in constant use.

Proposition 4.5.1. *Consider $[u, v, P] \in \mathcal{X}$ solutions to (4.2.30) - (4.2.32), with the boundary conditions (4.2.33) - (4.2.34). Such a solution satisfies the following estimate:*

$$\begin{aligned} &||\Big\{u_{yy}, \sqrt{\varepsilon}u_{xy}, \varepsilon u_{xx}\Big\} \cdot y||_{L^2}^2 + ||\Big\{u_y, \sqrt{\varepsilon}u_x\Big\} \cdot y||_{L^2}^2 \\ &\quad + ||\Big\{u_y, \sqrt{\varepsilon}u_x\Big\} \cdot y||_{L^2(x=L)}^2 \lesssim ||u_y||_{L^2}^2 + ||v_y, \sqrt{\varepsilon}v_x||_{L^2}^2 \\ &\quad + ||\sqrt{\varepsilon}u_x||_{L^2(x=L)}^2 + ||\{a_L, \partial_y a_L, b_L, \partial_y b_L\}\langle y\rangle^2||_{L^2(x=L)}^2 + \mathcal{R}_3, \end{aligned} \tag{4.5.5}$$

where:

$$\mathcal{R}_3 := \int \partial_y f \cdot \partial_y\{uy^2 w\} - \varepsilon \partial_y g \cdot \partial_x\{uy^2 w\}. \tag{4.5.6}$$

Proof. We will apply the weighted multiplier:

$$\Big[\partial_y\{uw(x)y^2\}, -\varepsilon\partial_x\{uw(x)y^2\}\Big], \tag{4.5.7}$$

where $w(x) = 1 - x$. The analysis proceeds in several steps which we will now detail.

Step 1: Positive Profile Terms

We will now generate the positive quantities on the left-hand side of (4.5.5), by considering from (4.5.3) - (4.5.4) the following terms:

$$\int \Big(u_s u_{xy} + v_s u_{yy}\Big) \cdot \partial_y\{uwy^2\} - \varepsilon \int \Big(u_s v_{xy} + v_s v_{yy}\Big) \cdot \partial_x\{uwy^2\}. \tag{4.5.8}$$

First from (4.5.8):

$$\begin{aligned}\int u_s u_{xy} \cdot \partial_y\{uy^2 w\} &= \int u_s u_{xy} u_y y^2 w + \int 2u_s u_{xy} uyw \\ &= -\int \frac{u_{sx}}{2} u_y^2 y^2 w + \int \frac{u_s}{2} u_y^2 y^2 + \int_{x=L} \frac{u_s}{2} u_y^2 y^2 w \\ &\qquad - \int 2u_x \partial_y\{u_s uy\} w \\ &= -\int \frac{u_{sx}}{2} u_y^2 y^2 w + \int \frac{u_s}{2} u_y^2 y^2 + \int_{x=L} \frac{u_s}{2} u_y^2 y^2 w \\ &\quad - \int 2u_x u_s u_y yw - \int 2u_x u_{sy} uyw - \int 2u_x u_s uw. \end{aligned} \tag{4.5.9}$$

The final three terms above are estimated:

$$|\int 2u_s u_x u_y yw| \le \delta ||u_y y||_{L^2}^2 + N_\delta ||u_x||_{L^2}^2, \tag{4.5.10}$$

$$|\int 2u_{sy} u_x uyw| \le \mathcal{O}(L) ||u_{sy} y||_{L^\infty} ||u_x||_{L^2}^2, \tag{4.5.11}$$

$$|\int 2u_s u u_x w| \le \mathcal{O}(L) ||u_x||_{L^2}^2. \tag{4.5.12}$$

Next from (4.5.8):

$$\begin{aligned}\int v_s u_{yy} \partial_y\{uy^2 w\} &= \int v_s u_{yy} u_y y^2 w + \int 2v_s u_{yy} uwy \\ &= -\int \frac{v_{sy}}{2} u_y^2 y^2 w - \int v_s y u_y^2 w - \int 2v_{sy} u u_y yw \\ &\quad - \int 2wv_s u u_y - \int 2v_s y u_y^2 w \\ &= -\int \frac{v_{sy}}{2} u_y^2 y^2 w - \int 3v_s y u_y^2 w \\ &\quad - \int 2v_{sy} y u u_y w + \int w v_{sy} u^2. \end{aligned} \tag{4.5.13}$$

The final two terms above are estimated:

$$|\int 2v_{sy}yuu_yw| \le \mathcal{O}(L)||v_{sy}||_{L^\infty}||yu_y||_{L^2}||u_x||_{L^2}, \tag{4.5.14}$$

$$|\int v_{sy}u^2w| \le \mathcal{O}(L)||v_{sy}||_{L^\infty}||u_x||_{L^2}^2. \tag{4.5.15}$$

Summing (4.5.9) - (4.5.13):

$$(4.5.9) + (4.5.13) \gtrsim \int \{\frac{u_s}{2}y^2 - 3v_syw\}u_y^2 + \int_{x=L} \frac{u_s}{2}u_y^2y^2w - \mathcal{O}(\text{RHS}). \tag{4.5.16}$$

We will consider the v_s term above. At leading order:

$$-\int \frac{v_e^0}{\sqrt{\varepsilon}}ywu_y^2 = -\int \frac{v_e^0}{Y}y^2wu_y^2 \le_{|\cdot|} ||\frac{v_e^0}{Y}||_{L^\infty}||u_yy||_{L^2}^2. \tag{4.5.17}$$

Here we use that $||\frac{v_e^0}{Y}||_{L^\infty}$ is taken sufficiently small by assumption (4.2.25) to absorb into the positive contribution from (4.5.16.) The higher order contributions can be estimated:

$$\begin{aligned} |\int \{v_s - \frac{v_e^0}{\sqrt{\varepsilon}}\}yu_y^2w| &\le ||v_s - \frac{v_e^0}{\sqrt{\varepsilon}}||_{L^\infty}||u_y \cdot y||_{L^2}||u_y||_{L^2} \\ &\lesssim \delta||u_y \cdot y||_{L^2}^2 + N_\delta||u_y||_{L^2}^2. \end{aligned} \tag{4.5.18}$$

Ultimately this yields:

$$(4.5.9) + (4.5.13) \gtrsim \int \frac{u_s}{2}y^2u_y^2 + \int_{x=L} \frac{u_s}{2}u_y^2y^2 - \mathcal{O}(\text{RHS}). \tag{4.5.19}$$

We now move to the positive terms from ∂_yS_v:

$$-\int \varepsilon u_sv_{xy} \cdot \partial_x\{uy^2w\} = +\int \varepsilon u_sv_{xy}v_yy^2w + \int \varepsilon u_sv_{xy}uy^2$$

$$
\begin{aligned}
&= -\int \frac{u_{sx}}{2} v_y^2 y^2 \varepsilon w + \int \varepsilon u_s \frac{v_y^2}{2} y^2 + \int_{x=L} \varepsilon \frac{u_s}{2} v_y^2 y^2 w \\
&- \int \varepsilon v_y \partial_x \{u_s u y^2\} + \int_{x=L} u_s v_y u y^2 \varepsilon \\
&= -\int \frac{u_{sx}}{2} v_y^2 y^2 \varepsilon w + \int \varepsilon u_s \frac{v_y^2}{2} y^2 + \int_{x=L} \varepsilon \frac{u_s}{2} v_y^2 y^2 w \\
&- \int \varepsilon v_y u_{sx} u y^2 - \int \varepsilon v_y u_s u_x y^2 + \int_{x=L} u_s v_y u y^2 \varepsilon \\
&= -\int \frac{u_{sx}}{2} v_y^2 y^2 \varepsilon w + \int \varepsilon \frac{3}{2} u_s v_y^2 y^2 + \int_{x=L} \varepsilon \frac{u_s}{2} v_y^2 y^2 w \\
&- \int \varepsilon v_y u_{sx} u y^2 + \int_{x=L} u_s v_y u y^2 \varepsilon. \qquad (4.5.20)
\end{aligned}
$$

We will estimate the final two terms from (4.5.20):

$$
|\int \varepsilon u_{sx} v_y u y^2| \le \mathcal{O}(L) ||u_{sx}||_{L^\infty} ||\sqrt{\varepsilon} u_x y||_{L^2}^2, \qquad (4.5.21)
$$

$$
|\int_{x=L} \varepsilon u_s v_y u y^2| \le ||\sqrt{\varepsilon} v_y y||_{L^2(x=L)} ||\sqrt{\varepsilon} u y||_{L^2(x=L)}. \qquad (4.5.22)
$$

Finally, from above:

$$
||\sqrt{\varepsilon} u y||_{L^2(x=L)}^2 = \int_{x=L} \varepsilon u^2 y^2 \le \mathcal{O}(L) ||\sqrt{\varepsilon} u_x y||_{L^2}^2. \qquad (4.5.23)
$$

Next:

$$
\begin{aligned}
-\int \varepsilon v_s v_{yy} \partial_x \{u w y^2\} &= +\int \varepsilon v_s v_{yy} v_y w y^2 + \int \varepsilon v_s v_{yy} u y^2 \\
&= -\int \varepsilon \frac{v_y^2}{2} \partial_y \{v_s w y^2\} - \int \varepsilon v_y \partial_y \{v_s u y^2\} \\
&= -\int \varepsilon \frac{v_{sy}}{2} v_y^2 y^2 w - \int \varepsilon v_s y w v_y^2 - \int \varepsilon v_{sy} v_y u y^2 \\
&- \int \varepsilon v_s v_y u_y y^2 - \int 2\varepsilon v_s v_y u y. \qquad (4.5.24)
\end{aligned}
$$

We will estimate three of the terms above:

$$|\int \varepsilon v_{sy} v_y u y^2| \le \mathcal{O}(L)||v_{sy}||_{L^\infty}||\sqrt{\varepsilon} v_y y||_{L^2}^2, \tag{4.5.25}$$

$$|\int 2\varepsilon v_s v_y u y| \le \mathcal{O}(L)||\sqrt{\varepsilon} v_s||_{L^\infty}||\sqrt{\varepsilon} v_y y||_{L^2}||u_x||_{L^2}, \tag{4.5.26}$$

$$\begin{aligned}|\int \varepsilon v_s v_y u_y y^2| &\le |\int \varepsilon \frac{v_e^0}{\sqrt{\varepsilon}} v_y u_y y^2| + |\int \varepsilon \{v_s - \frac{v_e^0}{\sqrt{\varepsilon}}\} v_y u_y y^2| \qquad (4.5.27)\\ &\le ||v_e^0 \sqrt{\varepsilon} y||_{L^\infty}||v_y||_{L^2}||u_y \cdot y||_{L^2} \\ &\quad + \sqrt{\varepsilon}||v_s - \frac{v_e^0}{\sqrt{\varepsilon}}||_{L^\infty}||\sqrt{\varepsilon} v_y y||_{L^2}||u_y y||_{L^2} \\ &\lesssim ||v_e^0 \cdot Y||_{L^\infty}||v_y||_{L^2}||u_y y||_{L^2} + \sqrt{\varepsilon}\mathcal{O}(\text{LHS}) \\ &\lesssim \delta||u_y \cdot y||_{L^2}^2 + N_\delta ||v_y||_{L^2}^2 + \sqrt{\varepsilon}\mathcal{O}(\text{LHS}). \qquad (4.5.28)\end{aligned}$$

The $||u_y y||_{L^2}^2$ term can be absorbed into the positive contribution from (4.5.20), whereas the $||v_y||_{L^2}^2$ term is $\mathcal{O}(\text{RHS})$. Thus, summing (4.5.24) and (4.5.20) yields:

$$(4.5.24) + (4.5.20) \gtrsim \int \Big(\frac{3}{2} u_s y^2 - v_s y w\Big) \varepsilon v_y^2 - \mathcal{O}(\text{RHS}). \tag{4.5.29}$$

We must now examine the v_s term above:

$$\int \{v_p^0 + \sqrt{\varepsilon} v_p^1\} y w \varepsilon v_y^2 \le ||v_p^0 y, \sqrt{\varepsilon} v_p^1 y||_{L^\infty}||\sqrt{\varepsilon} v_y||_{L^2}^2, \tag{4.5.30}$$

$$|\int \varepsilon v_e^1 y w v_y^2| \le ||v_e^1 Y||_\infty \sqrt{\varepsilon}||v_y||_{L^2}^2, \tag{4.5.31}$$

$$|\int \sqrt{\varepsilon} v_e^0 y w v_y^2| \le ||v_e^0 \cdot Y||_{L^\infty}||v_y||_{L^2}^2, \tag{4.5.32}$$

all of which are acceptable contributions according to the right-hand side of (4.5.5). Summarizing this set of calculations:

$$(4.5.8) \gtrsim \int \frac{u_s}{2} y^2 \Big(u_y^2 + \varepsilon v_y^2\Big) + \int_{x=L} \frac{u_s}{2}\Big(u_y^2 y^2 + \varepsilon v_y^2\Big) - \mathcal{O}(\text{RHS}). \tag{4.5.33}$$

Step 2: Remaining Profile Terms

We now extract the remaining terms from (4.5.3) - (4.5.4):

$$\begin{aligned}
\int u_{syy} v \cdot \partial_y \{uy^2 w\} &= \int u_{syy} v u_y y^2 w + \int u_{syy} v u 2yw \\
&\le ||u_{syy} y^2||_{L^\infty} ||\frac{v}{y}||_{L^2} ||u_y y||_{L^2} \\
&\qquad + \mathcal{O}(L) ||u_{syy} y^2||_{L^\infty} ||\frac{v}{y}||_{L^2} ||u_x||_{L^2} \\
&\le ||u_{syy} y^2||_{L^\infty} ||v_y||_{L^2} ||u_y y||_{L^2} \\
&\qquad + \mathcal{O}(L) ||u_{syy} y^2||_{L^\infty} ||v_y||_{L^2} ||u_x||_{L^2}, \\
&\le ||u_{syy} y^2||_{L^\infty} \Big[N_\delta ||v_y||^2_{L^2} + \delta ||u_y y||_{L^2} \Big] \\
&\qquad + \mathcal{O}(L) ||u_{syy} y^2||_{L^\infty} ||v_y||_{L^2} ||u_x||_{L^2},
\end{aligned} \tag{4.5.34}$$

all of which are acceptable contributions. Note that we have used estimate (4.9.124) to absorb y^2 into u_{syy}. Next:

$$\begin{aligned}
\int u_{sxy} u \cdot \partial_y \{uy^2 w\} &= \int u_{sxy} u u_y y^2 w + \int u_{sxy} u^2 2yw \\
&\le_{|\cdot|} ||u_{sxy} y||_{L^\infty} \mathcal{O}(L) \Big[||u_x||^2_{L^2} + ||u_y y||_{L^2} \Big],
\end{aligned} \tag{4.5.35}$$

which is an acceptable contribution by taking $L << 1$. Next, we move to the terms from $\partial_y S_v$ according to (4.5.4), starting with:

$$\begin{aligned}
-\int \varepsilon u_{sy} v_x \partial_x \{uwy^2\} &= -\int \varepsilon u_{sy} v_x u_x w y^2 + \int \varepsilon u_{sy} v_x u y^2 \\
&\le ||u_{sy} y||_{L^\infty} ||\sqrt{\varepsilon} u_x y||_{L^2} ||\sqrt{\varepsilon} v_x||_{L^2} \\
&\le ||u_{sy} y||_{L^\infty} \Big[\delta ||\sqrt{\varepsilon} u_x y||^2_{L^2} + N_\delta ||\sqrt{\varepsilon} v_x||^2_{L^2} \Big],
\end{aligned} \tag{4.5.36}$$

which is seen to be an acceptable contribution according to (4.5.33). Next:

$$-\int \varepsilon v_{sxy} u\Big[u_x w y^2 - u y^2\Big] \le \mathcal{O}(L)||v_{sxy}||_{L^\infty}||\sqrt{\varepsilon} u_x y||_{L^2}^2, \tag{4.5.37}$$

$$-\int \varepsilon v_{sx} u_y \Big[u_x w y^2 - u y^2\Big] \le ||\sqrt{\varepsilon} v_{sx} Y||_{L^\infty}||u_y y||_{L^2}||u_x||_{L^2}, \tag{4.5.38}$$

$$\lesssim ||\sqrt{\varepsilon} v_{sx} Y||_{L^\infty}\Big[\delta ||u_y y||_{L^2}^2 + N_\delta ||u_x||_{L^2}^2\Big],$$

$$-\int 2\varepsilon v_{sy} v_y \Big[u_x w y^2 - u y^2\Big] \lesssim ||v_{sy} Y^2||_{L^\infty}||u_x||_{L^2}^2, \tag{4.5.39}$$

$$-\int \varepsilon v_{syy} v \Big[u_x w y^2 - u y^2\Big] \le \mathcal{O}(L)||v_{syy} y||_{L^\infty}\Big[||\sqrt{\varepsilon} v_x||_{L^2}^2 + ||\sqrt{\varepsilon} u_x y||_{L^2}^2\Big]. \tag{4.5.40}$$

Step 3: Vorticity Terms

We record the following identities:

$$-\Delta_\varepsilon u_y = -u_{yyy} - 2\varepsilon u_{xxy} - \varepsilon v_{xyy}, \tag{4.5.41}$$

$$-\Delta_\varepsilon v_y = -2v_{yyy} - \varepsilon \partial_x \{u_{yy} + \varepsilon v_{xy}\}. \tag{4.5.42}$$

In the forthcoming calculations, we provide estimates on the vorticity terms:

$$-\int \Delta_\varepsilon u_y \cdot \partial_y\{u y^2 w\} = \int \Big\{ - u_{yyy} - 2\varepsilon u_{xxy} - \varepsilon v_{xyy}\Big\} \cdot \partial_y\{u y^2 w\}, \tag{4.5.43}$$

$$+\int \Delta_\varepsilon v_y \cdot \varepsilon\partial_x\{u y^2 w\} = \int \Big\{2v_{yyy} + \varepsilon\partial_x\{u_{yy} + \varepsilon v_{xy}\}\Big\} \cdot \varepsilon\partial_x\{u y^2 w\}. \tag{4.5.44}$$

Starting with the first term from (4.5.43):

$$\begin{aligned} -\int u_{yyy} \cdot \partial_y\{u w y^2\} &= +\int u_{yy}\partial_y^2\{u y^2 w\} \\ &= \int u_{yy}^2 y^2 w + \int 4 u_{yy} u_y y w + \int 2 u_{yy} u w \\ &= +\int u_{yy}^2 y^2 w - 4\int u_y^2 w \end{aligned} \tag{4.5.45}$$

$$\gtrsim + \int u_{yy}^2 y^2 - \mathcal{O}(\text{RHS}). \tag{4.5.46}$$

We must provide the rigorous justification of the integration found in (4.5.45). The delicate calculation occurs near $x = L, y = 0$ corner, for which we use the regularity theory in (OS95), which yields the asymptotic behavior: [1]

$$|u| \lesssim r^{\frac{1}{2}}, \quad |u_x, u_y| \lesssim r^{-\frac{1}{2}}, \quad |D^2 u| \lesssim r^{-\frac{3}{2}}, \tag{4.5.47}$$

where r is the distance to the corner. Defining C_r to be a solid ball of radius r around the corner, we have:

$$-\int u_{yyy} \cdot \partial_y \{uwy^2\} = -\int_{\Omega - C_r} u_{yyy} \cdot \partial_y \{uwy^2\} - \int_{C_r} u_{yyy} \cdot \partial_y \{uwy^2\}. \tag{4.5.48}$$

First, the expansions in (OS95) show that $u_{yyy} r^{\frac{3}{2}} \in L^2$. Therefore, taking limit as $r \to 0$, the latter term in (4.5.48) vanishes, and it remains to treat the former term:

$$-\int_{\Omega - C_r} u_{yyy} \cdot \partial_y \{uwy^2\} = +\int_{\Omega - C_r} u_{yy} \cdot \partial_{yy} \{uwy^2\} - \int_{\partial C_r} u_{yy} \cdot \partial_y \{uwy^2\} \,\mathrm{d}S. \tag{4.5.49}$$

For the surface integral, we use the expansions from (4.5.47), and that $y \le r$:

$$-\int_{\partial C_r} u_{yy} \partial_y \{uwy^2\} \,\mathrm{d}S \le \int_{\partial C_r} r^{-\frac{3}{2}} r^{-\frac{1}{2}} y^2 \le \int_{\partial C_r} 1 \to 0. \tag{4.5.50}$$

[1] One applies Theorem 4.1 in (OS95) with $\beta = 1 + \delta, q = q_1 = 2, h = -\delta$ and $h_1 = \frac{1}{2}+$ to obtain $\beta_1 = \frac{1}{2}-$. Theorem 4.1 gives $||r^{-\frac{3}{2}} u, r^{-\frac{1}{2}} Du, r^{\frac{1}{2}} D^2 u||_{L^2} < \infty$. One can then bootstrap this regularity to obtain $||r^{\frac{1}{2}+k} D^{2+k} u||_{L^2} < \infty$. Standard Sobolev embedding arguments give the pointwise asymptotics in (4.5.47).

We now move to the second term from (4.5.43):

$$
\begin{aligned}
-\int 2\varepsilon u_{xxy} \cdot \partial_y \{uy^2 w\} &= + \int 2\varepsilon u_{xy} \partial_{xy} \{uy^2 w\} - \int_{x=L} 2\varepsilon u_{xy} \partial_y \{uy^2 w\} \\
&= + \int 2\varepsilon u_{xy}^2 y^2 + \int 4\varepsilon u_{xy} u_x y w \\
&\quad - \int 2\varepsilon u_{xy} u_y y^2 - \int 4\varepsilon u_{xy} u y - \int_{x=L} 2\varepsilon u_{xy} \partial_y \{uy^2 w\} \\
&\gtrsim \int 2\varepsilon u_{xy}^2 y^2 - \int_{x=L} 2\varepsilon u_{xy} \partial_y \{uy^2 w\} \\
&\qquad - \mathcal{O}(\text{RHS}) - \varepsilon \mathcal{O}(\text{LHS}),
\end{aligned} \tag{4.5.51}
$$

where we have used the following estimates:

$$
\int 4\varepsilon u_{xy} u_x y w = - \int 2\varepsilon u_x^2 w, \tag{4.5.52}
$$

$$
- \int 2\varepsilon u_{xy} u_y y^2 = - \int_{x=L} \varepsilon u_y^2 y^2 \leq \varepsilon \cdot (4.5.33) \leq \varepsilon \mathcal{O}(\text{LHS}), \tag{4.5.53}
$$

$$
- \int 4\varepsilon u_{xy} u y = + \int 4\varepsilon u_x u + \int 4\varepsilon u_x u_y y \leq \varepsilon \Big[||u_x||_{L^2}^2 + ||u_y y||_{L^2}^2 \Big]. \tag{4.5.54}
$$

Next, the third term from (4.5.43):

$$
\begin{aligned}
-\int \varepsilon v_{xyy} \partial_y \{uy^2 w\} &= + \int \varepsilon v_{xy} \partial_{yy} \{uy^2 w\} \\
&= + \int \varepsilon v_{xy} \Big[u_{yy} y^2 w + 2uw + 4u_y y w \Big] \\
&= - \int \varepsilon u_{xx} u_{yy} y^2 w - \int 2\varepsilon v_x u_y w - \int 4\varepsilon u_{xx} u_y y w \\
&= - \int \varepsilon u_{xx} u_{yy} y^2 w - \int 2\varepsilon v_x u_y w + \int 4\varepsilon u_x u_{xy} y w \\
&\quad - \int_{x=L} 4\varepsilon u_x u_y y w \\
&= - \int \varepsilon u_{xx} u_{yy} y^2 w + \mathcal{O}(\text{RHS}) + \varepsilon \mathcal{O}(\text{LHS}).
\end{aligned} \tag{4.5.55}
$$

where we have estimated:

$$|\int_{x=L} 4\varepsilon u_x u_y y w| \lesssim \sqrt{\varepsilon}||\sqrt{\varepsilon}u_x||_{L^2(x=L)}||u_y y||_{L^2(x=L)}, \tag{4.5.56}$$

$$|\int 2\varepsilon v_x u_y w| \le \sqrt{\varepsilon}||u_y||_{L^2}||\sqrt{\varepsilon}v_x||_{L^2}, \tag{4.5.57}$$

$$\int 4\varepsilon u_{xy}u_x y w = -\int 2\varepsilon u_x^2 w. \tag{4.5.58}$$

We now come to the first term from (4.5.44):

$$\begin{aligned}
&\int \varepsilon \partial_x \{u_{yy} + \varepsilon v_{xy}\} \cdot \partial_x \{u y^2 w\} \\
&\quad = -\int \{u_{yy} + \varepsilon v_{xy}\} \cdot \partial_{xx}\{u y^2 w\} + \int_{x=L} \varepsilon \{u_{yy} + \varepsilon v_{xy}\} \cdot \partial_x \{u y^2 w\} \\
&\quad = -\int \{u_{yy} + \varepsilon v_{xy}\} \cdot \Big[u_{xx} y^2 w - 2u_x y^2\Big] + \int_{x=L} \varepsilon \partial_y b_L \cdot \partial_x \{u y^2 w\} \\
&\quad = -\int \varepsilon u_{yy} u_{xx} w y^2 + \int \varepsilon^2 u_{xx}^2 y^2 w + \int 2\varepsilon u_{yy} u_x y^2 + \int 2\varepsilon^2 v_{xy} u_x y^2 \\
&\qquad\qquad + \int_{x=L} \varepsilon \partial_y b_L \cdot \partial_x \{u y^2 w\} \\
&\quad = -\int \varepsilon u_{yy} u_{xx} w y^2 + \int \varepsilon^2 u_{xx}^2 y^2 w + \varepsilon \mathcal{O}(\text{RHS}) + \varepsilon \mathcal{O}(\text{LHS}) \\
&\qquad\qquad + ||\partial_y b_L \langle y \rangle^2||_{L^2(x=L)}^2 + \varepsilon ||u, \sqrt{\varepsilon} u_x||_{L^2(x=L)}^2.
\end{aligned} \tag{4.5.59}$$

We have estimated:

$$\begin{aligned}
+ \int 2\varepsilon u_{yy} u_x y^2 &= -\int 2\varepsilon u_y u_{xy} y^2 - \int 4\varepsilon u_y u_x y \\
&= -\int_{x=L} \varepsilon u_y^2 y^2 - \int 4\varepsilon u_y u_x y, \\
&\le \varepsilon ||u_y y||_{L^2(x=L)}^2 + \varepsilon ||u_y y||_{L^2} ||u_x||_{L^2} \\
&\lesssim \varepsilon \Big[\mathcal{O}(\text{LHS}) + \mathcal{O}(\text{RHS})\Big],
\end{aligned} \tag{4.5.60}$$

$$+ \int 2\varepsilon^2 v_{xy} u_x y^2 = -\int 2\varepsilon^2 u_{xx} u_x y^2 = -\int_{x=L} \varepsilon^2 u_x^2 y^2 \le \varepsilon \mathcal{O}(\text{LHS}). \tag{4.5.61}$$

Next from (4.5.42):

$$
\begin{aligned}
+\int 2v_{yyy}\cdot\varepsilon\partial_x\{uwy^2\} &= -\int 2\varepsilon v_{yyy}v_y wy^2 - \int 2\varepsilon v_{yyy}uy^2 \\
&= +\int 2\varepsilon v_{yy}^2 y^2 + \int 4\varepsilon v_{yy}v_y wy + \int 2\varepsilon v_{yy}u_y y^2 + \int 4\varepsilon v_{yy}uy \\
&= \int 2\varepsilon v_{yy}^2 y^2 w - \int 2\varepsilon v_y^2 w - \int 2\varepsilon u_{xy}u_y y^2 \\
&\qquad - \int 4\varepsilon v_y u_y y - \int 4\varepsilon v_y u \\
&\gtrsim \int 2\varepsilon v_{yy}^2 y^2 w - \varepsilon\Big[\mathcal{O}(\text{LHS}) + \mathcal{O}(\text{RHS})\Big], \qquad (4.5.62)
\end{aligned}
$$

where we have estimated the following terms:

$$
-\int 2\varepsilon u_{xy}u_y y^2 = -\int_{x=L}\varepsilon u_y^2 y^2, \qquad (4.5.63)
$$

$$
|\int 4\varepsilon v_y u_y y| \le \varepsilon||u_y y||_{L^2}||v_y||_{L^2}, \qquad (4.5.64)
$$

$$
|\int 4\varepsilon v_y u| \le \varepsilon\mathcal{O}(L)||u_x||_{L^2}^2. \qquad (4.5.65)
$$

We can now collect the estimates from (4.5.46), (4.5.51), (4.5.55), (4.5.59), (4.5.62) to get:

$$
\begin{aligned}
&-\int \Delta_\varepsilon u_y\cdot\partial_y\{uy^2w\} + \int \varepsilon v_y\cdot\partial_x\{uy^2w\} \\
&\qquad \gtrsim \int\Big[u_{yy}^2 + 4\varepsilon u_{xy}^2 + \varepsilon^2 u_{xx}^2 - 2\varepsilon u_{xx}u_{yy}\Big]y^2w - \int_{x=L} 2\varepsilon u_{xy}\partial_y\{uy^2w\} \\
&\qquad - \mathcal{O}(\text{RHS}) - \varepsilon\mathcal{O}(\text{LHS}). \qquad (4.5.66)
\end{aligned}
$$

We now have the Pressure contributions:

$$
\int P_{yx}\cdot\partial_y\{uy^2w\} + \int P_{yy}\cdot\partial_x\{uy^2w\} = \int_{x=L} P_y\cdot\partial_y\{uy^2w\}. \qquad (4.5.67)
$$

Using $P_y - 2\varepsilon u_{xy} = a_L(y)$ on $x = L$, the boundary term above can be combined with that in (4.5.66) yielding:

$$\int_{x=L} \Big(P_y - 2\varepsilon u_{xy}\Big) \cdot \partial_y\{uy^2 w\} = \int_{x=L} \partial_y a_L \cdot \partial_y\{uy^2 w\}$$
$$\leq ||\partial_y a_L \cdot \langle y\rangle^2||_{L^2(x=L)} ||u_y y, u||_{L^2(x=L)} \tag{4.5.68}$$
$$\lesssim N_\delta ||\partial_y a_L \cdot \langle y\rangle^2||^2_{L^2(x=L)} + \delta ||u_y y, u||^2_{L^2(x=L)}, \tag{4.5.69}$$

the latter of which can be absorbed into the left-hand side of our estimate, specifically the positive contribution of (4.5.33). Thus, summing (4.5.67) with (4.5.66) yields:

$$- \int \Delta_\varepsilon u_y \cdot \partial_y\{uy^2 w\} + \int \varepsilon v_y \cdot \partial_x\{uy^2 w\}$$
$$+ \int P_{yx} \cdot \partial_y\{uy^2 w\} + \int P_{yy} \cdot \partial_x\{uy^2 w\}$$
$$\gtrsim \int \Big\{u_{yy}^2 + 4\varepsilon u_{xy}^2 + \varepsilon^2 u_{xx}^2 - 2\varepsilon u_{xx} u_{yy}\Big\} y^2 w - \mathcal{O}(\text{RHS}) - \varepsilon \mathcal{O}(\text{LHS})$$
$$\gtrsim \int \Big\{u_{yy}^2 + \varepsilon u_{xy}^2 + \varepsilon^2 u_{xx}^2\Big\} y^2, \tag{4.5.70}$$

where the final inequality follows from (4.5.71). This concludes the proof.

□

4.5.1 The Korn's Inequality

Lemma 4.5.2. *For any functions $[u, v] \in \mathcal{X}$, the following estimate is valid:*

$$\int \Big[u_{yy}^2 + 4\varepsilon u_{xy}^2 + \varepsilon^2 u_{xx}^2 - 2\varepsilon u_{yy} u_{xx}\Big] y^2 w(x) \,\mathrm{d}x \,\mathrm{d}y$$
$$\gtrsim \int \Big[u_{yy}^2 + \varepsilon u_{xy}^2 + \varepsilon^2 u_{xx}^2\Big] y^2 w(x) \,\mathrm{d}x \,\mathrm{d}y$$
$$- \varepsilon ||u_y y, \sqrt{\varepsilon} u_x y||^2_{L^2} - ||u_y, u_x||^2_{L^2}. \tag{4.5.71}$$

Proof. We would like to apply the Korn inequality to generate positive terms:

$$\int \Big[u_{yy}^2 + 4\varepsilon u_{xy}^2 + \varepsilon^2 u_{xx}^2 - 2\varepsilon u_{yy}u_{xx}\Big] y^2 w(x)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.5.72}$$

We will first rescale to original Eulerian coordinates, so as to ensure all estimates are independent of L:

$$X = \frac{x}{L}, \quad Y = \frac{\sqrt{\varepsilon}}{L}y, \quad U(X,Y) = u(x,y), \quad V(X,Y) = \sqrt{\varepsilon}v(x,y). \tag{4.5.73}$$

Define also $w_L(X) = 1 - LX$. This gives the following relations:

$$U_X = Lu_x, U_Y = \frac{L}{\sqrt{\varepsilon}}u_y, U_{YY} = \frac{L^2}{\varepsilon}u_{yy}, \tag{4.5.74}$$

$$V_X = \sqrt{\varepsilon}Lv_x, V_Y = Lv_y, V_{XX} = L^2\sqrt{\varepsilon}v_{xx}, V_{YY} = \frac{L^2}{\sqrt{\varepsilon}}v_{yy}. \tag{4.5.75}$$

It is clear that:

$$(4.5.72) = \sqrt{\varepsilon}\int \Big[U_{YY}^2 + 4U_{XY}^2 + U_{XX}^2 - 2U_{YY}U_{XX}\Big] Y^2 w_L(X)\,\mathrm{d}X\,\mathrm{d}Y. \tag{4.5.76}$$

We will define:

$$U^{(1)} := U_Y Y\sqrt{w_L}, \qquad V^{(1)} := V_Y Y\sqrt{w_L}. \tag{4.5.77}$$

$$U^{(1)}_Y = U_{YY}Y\sqrt{w_L} + U_Y\sqrt{w_L}, \tag{4.5.78}$$

$$U^{(1)}_X = U_{XY}Y\sqrt{w_L} + U_Y Y\frac{1}{2\sqrt{w_L}}L, \tag{4.5.79}$$

$$V^{(1)}_Y = V_{YY}Y\sqrt{w_L} + V_Y\sqrt{w_L} \tag{4.5.80}$$

$$V_X^{(1)} = V_{XY} Y \sqrt{w_L} + V_Y Y \frac{1}{2\sqrt{w_L}} L. \tag{4.5.81}$$

We will now calculate:

$$\sqrt{\varepsilon} \int U_Y^2 w_L \,\mathrm{d}X \,\mathrm{d}Y = \int u_y^2 w \,\mathrm{d}x \,\mathrm{d}y, \tag{4.5.82}$$

$$\sqrt{\varepsilon} \int U_Y^2 Y^2 \frac{L^2}{w_L} \,\mathrm{d}X \,\mathrm{d}Y = \varepsilon \int u_y^2 y^2 \,\mathrm{d}x \,\mathrm{d}y, \tag{4.5.83}$$

$$\sqrt{\varepsilon} \int V_Y^2 w_L \,\mathrm{d}X \,\mathrm{d}Y = \int \varepsilon v_y^2 w \,\mathrm{d}x \,\mathrm{d}y, \tag{4.5.84}$$

$$\sqrt{\varepsilon} \int V_Y^2 Y^2 \frac{L^2}{w_L} \,\mathrm{d}X \,\mathrm{d}Y = \varepsilon^2 \int v_y^2 y^2 \,\mathrm{d}x \,\mathrm{d}y, \tag{4.5.85}$$

$$\sqrt{\varepsilon} \int |U^{(1)}|^2 \,\mathrm{d}X \,\mathrm{d}Y = \frac{\varepsilon}{L^2} \int u_y^2 y^2 w \,\mathrm{d}x \,\mathrm{d}y, \tag{4.5.86}$$

$$\sqrt{\varepsilon} \int |V^{(1)}|^2 \,\mathrm{d}X \,\mathrm{d}Y = \frac{\varepsilon^2}{L^2} \int v_y^2 y^2 w \,\mathrm{d}x \,\mathrm{d}y, \tag{4.5.87}$$

Thus:

$$\begin{aligned}
\sqrt{\varepsilon} \int U_{YY}^2 Y^2 w_L &= \sqrt{\varepsilon} \int |U_Y^{(1)}|^2 + \mathcal{C}, \\
\sqrt{\varepsilon} \int U_{XY}^2 Y^2 w_L &= \sqrt{\varepsilon} \int 4 |U_X^{(1)}|^2 + \mathcal{C}, \\
\sqrt{\varepsilon} \int U_{XX}^2 Y^2 w_L &= \sqrt{\varepsilon} \int |V_X^{(1)}|^2 + \mathcal{C}, \\
-2\sqrt{\varepsilon} \int U_{YY} U_{XX} Y^2 w_L &= -\sqrt{\varepsilon} \int 2 U_Y^{(1)} V_X^{(1)} + \mathcal{C},
\end{aligned}$$

where:

$$\begin{aligned}
|\mathcal{C}| \lesssim & N_\delta \cdot ||u_y, u_x||_{L^2} + \varepsilon \cdot ||u_y y, \sqrt{\varepsilon} u_x y||_{L^2}^2 \\
& + \delta \sqrt{\varepsilon} \int |U_Y^{(1)}|^2 + |V_X^{(1)}|^2 + |U_X^{(1)}|^2.
\end{aligned} \tag{4.5.88}$$

According to this, we can write:

$$(4.5.72) \gtrsim \sqrt{\varepsilon}\Big[\int |U_Y^{(1)}|^2 + 4|U_X^{(1)}|^2 + |V_X^{(1)}|^2 - 2U_Y^{(1)}V_X^{(1)}\Big] - |\mathcal{C}|. \tag{4.5.89}$$

By adding and subtracting (4.5.86) - (4.5.87) and up to redefining $\mathcal{C}$, we have:

$$\begin{aligned}(4.5.72) \gtrsim &\sqrt{\varepsilon}\int \Big[|U_Y^{(1)}|^2 + 4|U_X^{(1)}|^2 + |V_X^{(1)}|^2 - 2U_Y^{(1)}V_X^{(1)}\Big]\,\mathrm{d}X\,\mathrm{d}Y \\ &+ \sqrt{\varepsilon}\int \Big[|U^{(1)}|^2 + |V^{(1)}|^2\Big]\,\mathrm{d}X\,\mathrm{d}Y - |\mathcal{C}|.\end{aligned}$$

An application of Korn's inequality yields:

$$\begin{aligned}(4.5.72) &\gtrsim \sqrt{\varepsilon}\int |U_Y^{(1)}|^2 + |U_X^{(1)}|^2 + |V_X^{(1)}|^2 \\ &\qquad\qquad - ||u_y, u_x||_{L^2}^2 + \varepsilon||u_y y, \sqrt{\varepsilon}u_x y||_{L^2}^2 \\ &\gtrsim \int \Big\{u_{yy}^2 + \varepsilon u_{xy}^2 + \varepsilon^2 u_{xx}^2\Big\} y^2 w(x)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad\qquad - ||u_y, u_x||_{L^2}^2 + \varepsilon||u_y y, \sqrt{\varepsilon}u_x y||_{L^2}^2.\end{aligned} \tag{4.5.90}$$

This concludes the proof.

$\square$

4.5.2 Summary of L^2 Estimates:

Let us now consolidate the L^2-based estimates, by combining (4.3.1), (4.4.5), (4.5.5). First, we will define the following L^2 based norm:

$$||u, v||_{X_1} := ||u_y \cdot y||_{L^2} + ||\sqrt{\varepsilon}u_x \cdot y||_{L^2} + ||v_y, \sqrt{\varepsilon}v_x||_{L^2}$$

$$+ ||\Big\{u_{yy}, \sqrt{\varepsilon}u_{xy}, \varepsilon u_{xx}\Big\} \cdot y||_{L^2}. \tag{4.5.91}$$

Recalling the boundary norm given in (4.2.23), accumulating estimates (4.5.5), (4.3.1), and (4.4.5), and taking $0 < L << ||\frac{v_e^0}{Y}||_{L^\infty} << 1$ gives:

$$||u, v||_{X_1}^2 + ||u, v||_B^2 \lesssim \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3. \tag{4.5.92}$$

4.6 Uniform Estimates

We will now obtain L^∞ estimates for solutions $[u, v]$ to the system (4.2.30) - (4.2.32), which are based on bootstrapping estimates that are valid for the Stokes operator.

Lemma 4.6.1. *Solutions $[u, v] \in \mathcal{X}$ to the system (4.2.30) - (4.2.32), with the boundary conditions (4.2.33) - (4.2.34) satisfy the following uniform estimate:*

$$\begin{aligned}\varepsilon^{\frac{\gamma}{4}}||u, \sqrt{\varepsilon}v||_{L^\infty(\bar{\Omega})} \lesssim & C(\gamma, L)\Big\{||u, \sqrt{\varepsilon}v||_{H^1} + C(a_L, b_L) \\ & + ||S_u, \sqrt{\varepsilon}S_v||_{L^2} + ||f, \sqrt{\varepsilon}g||_{L^2}\Big\}.\end{aligned} \tag{4.6.1}$$

Proof. The proof follows from (GN17), Lemma 4.1. Note that the estimate up to the boundary, $L^\infty(\bar{\Omega})$, is guaranteed according to (AF03), P. 98, Equation 9, as our domain Ω satisfies the strong local Lipschitz property, as defined by (AF03), P. 66.

□

We emphasize that for our analysis, it is important to obtain the uniform control on the boundary $x = L$, due for instance, to the nonlinear contributions from (4.7.6). We now relate the right-hand side above to our norms.

Lemma 4.6.2. *For any functions, $[u, v] \in \mathcal{X}$, the following estimate holds:*

$$||u, \sqrt{\varepsilon} v||_{H^1} + ||S_u, \sqrt{\varepsilon} S_v||_{L^2} + ||\varepsilon^{-\frac{1}{2}-\gamma}\{R^u, \sqrt{\varepsilon} R^v\}||_{L^2} + ||L_1^b, \sqrt{\varepsilon} L_2^b||_{L^2}$$
$$+ ||N^u(\bar{u}, \bar{v}), \sqrt{\varepsilon} N^v(\bar{u}, \bar{v})||_{L^2} \lesssim 1 + ||u, v||_{X_1} + ||\bar{u}, \bar{v}||_{\mathcal{X}}^2. \tag{4.6.2}$$

Proof. The estimates on $||u, \sqrt{\varepsilon} v||_{H^1}, ||\varepsilon^{-\frac{1}{2}-\gamma}\{R^u, \sqrt{\varepsilon} R^v\}||_{L^2}$ follow trivially, the latter from (4.9.123). The estimates on $||L_1^b, \sqrt{\varepsilon} L_2^b||_{L^2}$, as defined in (4.9.116) - (4.9.117), follow from (4.9.118). Next, referring to the definition of S_u in (4.2.35), and the estimates in (4.9.124),

$$||S_u||_{L^2} = ||u_s u_x + u_{sx} u + v_s u_y + u_{sy} v||_{L^2}$$
$$\leq ||u_s, u_{sx}, \frac{v_e^0}{Y}, v_s - \frac{v_e^0}{\sqrt{\varepsilon}}, u_{sy} y||_{L^2} ||u_x, u_y y||_{L^2}. \tag{4.6.3}$$

Similarly, referring to the definition of S_v given in (4.2.35) and the estimates (4.9.124):

$$||\sqrt{\varepsilon} S_v||_{L^2} \leq ||\sqrt{\varepsilon}\Big(u_s v_x + v_{sx} u + v_s v_y + v_{sy} v\Big)||_{L^2}$$
$$\leq ||u_s, \sqrt{\varepsilon} v_{sx}, \sqrt{\varepsilon} v_s, v_{sy}||_{L^2} ||\sqrt{\varepsilon} v_x||_{L^2}. \tag{4.6.4}$$

Referring to the definitions of the nonlinearities given in (4.2.36):

$$||N^u(\bar{u}, \bar{v})||_{L^2} = \varepsilon^{\frac{1}{2}+\gamma} ||\bar{u}\bar{u}_x + \bar{v}\bar{u}_y||_{L^2}$$
$$\leq \varepsilon^{\frac{\gamma}{2}} ||\varepsilon^{\frac{\gamma}{2}} \Big\{\bar{u}, \sqrt{\varepsilon}\bar{v}\Big\}||_{L^\infty} ||\bar{u}_x, \bar{u}_y||_{L^2}, \tag{4.6.5}$$

$$||\sqrt{\varepsilon} N^v(\bar{u}, \bar{v})||_{L^2} \leq ||\varepsilon^{1+\gamma}\Big(\bar{u}\bar{v}_x + \bar{v}\bar{v}_y\Big)||_{L^2}$$
$$\leq \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} ||\varepsilon^{\frac{\gamma}{2}} \bar{u}||_{L^\infty} ||\sqrt{\varepsilon}\bar{v}_x||_{L^2} + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\bar{v}_y||_{L^2}. \tag{4.6.6}$$

The above estimates imply the result. □

Combining (4.6.1), (4.6.2) with the definition of (f,g) given in (4.2.36) - (4.2.37), together with relevant definitions in (4.9.15), (4.9.24), and (4.9.117) gives the following:

Corollary 4.6.3. *Solutions $[u,v] \in \mathcal{X}$ to the system (4.2.30) - (4.2.32), with the boundary conditions (4.2.33) - (4.2.34) satisfy the following uniform estimate:*

$$\varepsilon^{\frac{\gamma}{2}} ||u, \sqrt{\varepsilon} v||_{L^\infty} \lesssim \varepsilon^{\frac{\gamma}{4}} + \varepsilon^{\frac{\gamma}{4}} \Big[||u,v||_{X_1} + ||\bar{u}, \bar{v}||_{\mathcal{X}}^2 \Big]. \tag{4.6.7}$$

Combining with (4.5.92), we have now controlled the full $\mathcal{X}$ norm:

Corollary 4.6.4. *Solutions $[u,v] \in \mathcal{X}$ to the system (4.2.30) - (4.2.32), with the boundary conditions (4.2.33) - (4.2.34) satisfy the following estimate:*

$$||u,v||_{\mathcal{X}}^2 \lesssim \varepsilon^{\frac{\gamma}{2}} + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \varepsilon^{\frac{\gamma}{2}} ||\bar{u}, \bar{v}||_{\mathcal{X}}^4. \tag{4.6.8}$$

It remains to control $\mathcal{R}_i$, which we now expand by recalling (4.3.2), (4.4.6), (4.5.6), and (4.2.37):

$$\begin{aligned} \mathcal{R}_1 = & \int \Big[\varepsilon^{-\frac{1}{2}-\gamma} R^{u,1} + N^u + L_1^b \Big] \cdot u \\ & + \int \varepsilon \Big[\varepsilon^{-\frac{1}{2}-\gamma} R^{v,1} + N^v + L_2^b \Big] v, && (4.6.9) \\ \mathcal{R}_2 = & - \int \Big[\varepsilon^{-\frac{1}{2}-\gamma} R^{u,1} + N^u + L_1^b \Big] \cdot \beta_y \\ & + \int \varepsilon \Big[\varepsilon^{-\frac{1}{2}-\gamma} R^{v,1} + N^v + L_2^b \Big] \cdot \beta_x && (4.6.10) \\ \mathcal{R}_3 = & \int \Big[\varepsilon^{-\frac{1}{2}-\gamma} \partial_y R^{u,1} + \partial_y N^u + \partial_y L_1^b \Big] \cdot \partial_y \{ u y^2 w \} \\ & - \int \varepsilon \Big[\varepsilon^{-\frac{1}{2}-\gamma} \partial_y R^{v,1} + \partial_y N^v + \partial_y L_2^b \Big] \cdot \partial_x \{ u y^2 w \}. && (4.6.11) \end{aligned}$$

We now turn to controlling these quantities.

4.7 Nonlinearities

We now provide estimates on $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, as displayed in (4.6.9) - (4.6.11). We will first estimate the nonlinear terms, N^u, N^v, which are in turn defined in (4.2.36). Because we will eventually perform a contraction mapping argument, we will consider $N^u(\bar{u},\bar{v}), N^v(\bar{u},\bar{v})$, where $[\bar{u},\bar{v}] \in \mathcal{X}$. We have:

$$\partial_y N^u(\bar{u},\bar{v}) = \varepsilon^{\frac{1}{2}+\gamma}\Big\{\bar{u}\bar{u}_{xy} + \bar{v}\bar{u}_{yy}\Big\}, \tag{4.7.1}$$

$$\partial_y N^v(\bar{u},\bar{v}) = \varepsilon^{\frac{1}{2}+\gamma}\Big\{\bar{u}_y\bar{v}_x + \bar{u}\bar{v}_{xy} + \bar{v}_y^2 + \bar{v}\bar{v}_{yy}\Big\}. \tag{4.7.2}$$

The first step is to provide estimates on the nonlinear contributions from $\mathcal{R}_3$, as defined in (4.5.6). For this we have:

Lemma 4.7.1. *For any vector fields $[u,v], [\bar{u},\bar{v}] \in \mathcal{X}$, the following estimate holds:*

$$|\int \partial_y N^u(\bar{u},\bar{v}) \cdot \partial_y\{uwy^2\}| + |\int \partial_y N^v(\bar{u},\bar{v}) \cdot \varepsilon\partial_x\{uwy^2\}| \lesssim \varepsilon^{\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^2||u,v||_{\mathcal{X}}. \tag{4.7.3}$$

Proof. Turning to the first term from (4.7.1), we will expand via the product rule:

$$\begin{aligned}
&\int \varepsilon^{\frac{1}{2}+\gamma}\bar{u}\bar{u}_{xy} \cdot \partial_y\{uy^2w\} \\
&\quad = \int \varepsilon^{\frac{1}{2}+\gamma}\bar{u}\bar{u}_{xy} \cdot u_y y^2 w + \int \varepsilon^{\frac{1}{2}+\gamma}\bar{u}\bar{u}_{xy} \cdot u2yw \\
&\quad = \int \varepsilon^{\frac{1}{2}+\gamma}\bar{u}\bar{u}_{xy}u_y y^2 w - \int \varepsilon^{\frac{1}{2}+\gamma}\bar{u}_y\bar{u}_x u2yw \\
&\qquad\qquad - \int \varepsilon^{\frac{1}{2}+\gamma}\bar{u}\bar{u}_x u_y 2yw - \int \varepsilon^{\frac{1}{2}+\gamma}\bar{u}u\bar{u}_x 2w \\
&\quad \le \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}\bar{u}||_{L^\infty}||\sqrt{\varepsilon}\bar{u}_{xy}y||_{L^2}||u_y y||_{L^2} \\
&\qquad\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}u||_{L^\infty}||\bar{u}_y y||_{L^2}||\bar{u}_x||_{L^2} \\
&\qquad\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}\bar{u}||_{L^\infty}||u_y y||_{L^2}||\bar{u}_x||_{L^2} \\
&\qquad\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}u||_{L^\infty}||\bar{u}_x||_{L^2}^2.
\end{aligned} \tag{4.7.4}$$

Turning to the second term from (4.7.1):

$$
\begin{aligned}
&\int \varepsilon^{\frac{1}{2}+\gamma} \bar{v}\bar{u}_{yy} \cdot \partial_y \{uy^2 w\} \\
&\qquad = \int \varepsilon^{\frac{1}{2}+\gamma} \bar{v}\bar{u}_{yy} u_y y^2 w + \int \varepsilon^{\frac{1}{2}+\gamma} \bar{v}\bar{u}_{yy} u 2yw \\
&\qquad \le \varepsilon^{\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\bar{u}_{yy} y||_{L^2} \Big[||u_y y||_{L^2} + \mathcal{O}(L) ||u_x||_{L^2} \Big].
\end{aligned} \tag{4.7.5}
$$

We will now turn to the first term from (4.7.2), which is the most delicate because $\bar{v}_x$ cannot accept any weights of y, according to our norm $\mathcal{X}$, (4.2.22). As a result, we must rely on an integration by parts in x:

$$
\begin{aligned}
\int \varepsilon^{\frac{3}{2}+\gamma} \bar{u}_y \bar{v}_x \partial_x \{uy^2 w\} &= \int \varepsilon^{\frac{3}{2}+\gamma} \bar{u}_y \bar{v}_x u_x y^2 w - \int \varepsilon^{\frac{3}{2}+\gamma} \bar{u}_y \bar{v}_x u y^2 \\
&= - \int \varepsilon^{\frac{3}{2}+\gamma} \bar{v}\bar{u}_{xy} u_x y^2 w - \int \varepsilon^{\frac{3}{2}+\gamma} \bar{v}\bar{u}_y u_{xx} y^2 w \\
&\qquad - \int \varepsilon^{\frac{3}{2}+\gamma} \bar{v}\bar{u}_y u_x y^2 - \int \varepsilon^{\frac{3}{2}+\gamma} \bar{u}_y \bar{v}_x u y^2 \\
&\qquad + \int_{x=L} \varepsilon^{\frac{3}{2}+\gamma} \bar{v}\bar{u}_y u_x y^2 \\
&= - \int \varepsilon^{\frac{3}{2}+\gamma} \bar{v}\bar{u}_{xy} u_x y^2 w - \int \varepsilon^{\frac{3}{2}+\gamma} \bar{v}\bar{u}_y u_{xx} y^2 w \\
&\qquad - \int \varepsilon^{\frac{3}{2}+\gamma} \bar{v}\bar{u}_y u_x y^2 + \int \varepsilon^{\frac{3}{2}+\gamma} \bar{v} u_x \bar{u}_y y^2 + \int \varepsilon^{\frac{3}{2}+\gamma} \bar{v} u \bar{u}_{xy} y^2 \\
&\qquad - \int_{x=L} \varepsilon^{\frac{3}{2}+\gamma} \bar{v} u \bar{u}_y y^2 + \int_{x=L} \varepsilon^{\frac{3}{2}+\gamma} \bar{v}\bar{u}_y u_x y^2 \\
&\le \varepsilon^{\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\sqrt{\varepsilon} u_{xy} y||_{L^2} ||\sqrt{\varepsilon} u_x y||_{L^2} \\
&\qquad + \varepsilon^{\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\bar{u}_y y||_{L^2} ||\varepsilon u_{xx} y||_{L^2} \\
&\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\bar{u}_y y||_{L^2} ||\sqrt{\varepsilon} u_x y||_{L^2} \\
&\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\sqrt{\varepsilon} u_x y||_{L^2} ||\bar{u}_y y||_{L^2} \\
&\qquad + \varepsilon^{\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\sqrt{\varepsilon} u_x y||_{L^2} ||\sqrt{\varepsilon} u_{xy} y||_{L^2} \\
&\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\bar{u}_y y||_{L^2(x=L)} ||\sqrt{\varepsilon} u y||_{L^2(x=L)} \\
&\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} ||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}} \bar{v}||_{L^\infty} ||\bar{u}_y y||_{L^2(x=L)} ||\sqrt{\varepsilon} u_x y||_{L^2(x=L)}.
\end{aligned} \tag{4.7.6}
$$

Note that for the above term, (4.7.6), it is imperative to obtain control of v on the boundary $x = L$, as shown in estimate (4.6.1). We now move to the second term from (4.7.2):

$$\begin{aligned}\int \varepsilon^{\frac{3}{2}+\gamma}\bar{u}\bar{v}_{xy}\partial_x\{uy^2w\} &= \int \varepsilon^{\frac{3}{2}+\gamma}\bar{u}\bar{v}_{xy}u_xy^2w - \int \varepsilon^{\frac{3}{2}+\gamma}\bar{u}\bar{v}_{xy}uy^2 \\ &\leq \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}\bar{u}||_{L^\infty}||\varepsilon\bar{u}_{xx}y||_{L^2}||\sqrt{\varepsilon}u_xy||_{L^2} \\ &\qquad + \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}u||_{L^\infty}||\varepsilon u_{xx}y||_{L^2}||\sqrt{\varepsilon}u_xy||_{L^2}. \end{aligned} \tag{4.7.7}$$

Now we turn to the third term from (4.7.2):

$$\begin{aligned}\int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}_y^2\partial_x\{uy^2w\} &= \int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}_y^2u_xy^2w - \int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}_y^2uy^2 \\ &= -\int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}\bar{v}_{yy}u_xy^2w - \int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}\bar{v}_yu_{xy}y^2w \\ &\qquad - \int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}\bar{v}_yu_x2yw - \int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}_y^2uy^2 \\ &\leq \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}\bar{v}||_{L^\infty}||\sqrt{\varepsilon}\bar{v}_{yy}y||_{L^2}||\sqrt{\varepsilon}u_xy||_{L^2} \\ &\qquad + \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}\bar{v}||_{L^\infty}||\sqrt{\varepsilon}\bar{u}_xy||_{L^2}||\sqrt{\varepsilon}u_{xy}y||_{L^2} \\ &\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}\bar{v}||_{L^\infty}||\sqrt{\varepsilon}\bar{u}_xy||_{L^2}||u_x||_{L^2} \\ &\qquad + \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}u||_{L^\infty}||\sqrt{\varepsilon}\bar{u}_xy||_{L^2}^2. \end{aligned} \tag{4.7.8}$$

Now we turn to the fourth, final term from (4.7.2):

$$\begin{aligned}\int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}\bar{v}_{yy}\partial_x\{uwy^2\} &= \int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}\bar{v}_{yy}u_xwy^2 - \int \varepsilon^{\frac{3}{2}+\gamma}\bar{v}\bar{v}_{yy}uy^2 \\ &\lesssim \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}\bar{v}||_{L^\infty}||\sqrt{\varepsilon}\bar{u}_{xy}y||_{L^2}||\sqrt{\varepsilon}u_xy||_{L^2}. \end{aligned} \tag{4.7.9}$$

These estimates conclude the proof of the desired result, estimate (4.7.3). $\square$

We will now come to the nonlinear contributions to the energy estimates, which are contained in (4.6.9):

Lemma 4.7.2. *For any vector fields* $[u,v],[\bar{u},\bar{v}] \in \mathcal{X}$*, the following estimate holds:*

$$|\int N^u(\bar{u},\bar{v})\cdot u + \varepsilon N^v(\bar{u},\bar{v})\cdot v| \le \varepsilon^{\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^2||u,v||_{\mathcal{X}}. \tag{4.7.10}$$

Proof. We turn to the definitions of N^u, N^v which are given in (4.2.36). From there, the following calculations follow:

$$\varepsilon^{\frac{1}{2}+\gamma}|\int \bar{u}\bar{u}_x\cdot u| \le \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}\bar{u}||_{L^\infty}||\bar{u}_x||_{L^2}^2 \lesssim \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^2||u,v||_{\mathcal{X}}, \tag{4.7.11}$$

$$\varepsilon^{\frac{1}{2}+\gamma}|\int \bar{v}\bar{u}_y\cdot u| \le \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}\sqrt{\varepsilon}\bar{v}||_{L^\infty}||\bar{u}_x||_{L^2}||u_y||_{L^2} \lesssim \varepsilon^{\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^2||u,v||_{\mathcal{X}}, \tag{4.7.12}$$

$$\varepsilon^{\frac{1}{2}+\gamma}|\int \bar{u}\bar{v}_x\cdot\varepsilon v| \le \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}\bar{u}||_{L^\infty}||\sqrt{\varepsilon}\bar{v}_x||_{L^2}||\sqrt{\varepsilon}v_x||_{L^2} \lesssim \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^2||u,v||_{\mathcal{X}}, \tag{4.7.13}$$

$$\varepsilon^{\frac{1}{2}+\gamma}|\int \bar{v}\bar{v}_y\cdot\varepsilon v| \le \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}\bar{v}||_{L^\infty}||\sqrt{\varepsilon}v_x||_{L^2}||\bar{v}_y||_{L^2} \lesssim \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^2||u,v||_{\mathcal{X}}. \tag{4.7.14}$$

The desired result follows from these calculations.

□

We will now provide nonlinear estimates arising from the positivity estimate, in particular we must evaluate the contributions of the nonlinearity in (4.6.10):

Lemma 4.7.3. *For any vector fields* $[u,v],[\bar{u},\bar{v}] \in \mathcal{X}$*, the following estimate holds:*

$$|\int N^u(\bar{u},\bar{v})\cdot -\beta_y| + |\int N^v(\bar{u},\bar{v})\cdot\varepsilon\beta_x| \le \varepsilon^{\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^2||u,v||_{\mathcal{X}}. \tag{4.7.15}$$

Proof. We again turn to the definitions of N^u, N^v from (4.2.36):

$$\varepsilon^{\frac{1}{2}+\gamma}|\int \bar{u}\bar{u}_x\cdot\beta_y| \le \varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}\bar{u}||_{L^\infty}||\bar{u}_x||_{L^2}||\beta_y||_{L^2}, \tag{4.7.16}$$

$$\varepsilon^{\frac{1}{2}+\gamma}|\int \bar{v}\bar{u}_y \cdot \beta_y| \le \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}\bar{v}||_{L^\infty}||\bar{u}_y||_{L^2}||\beta_y||_{L^2}, \tag{4.7.17}$$

$$\varepsilon^{\frac{1}{2}+\gamma}|\int \bar{u}\bar{v}_x \cdot \varepsilon\beta_x| \le \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{\gamma}{2}}\bar{u}||_{L^\infty}||\sqrt{\varepsilon}\bar{v}_x||_{L^2}||\sqrt{\varepsilon}\beta_x||_{L^2}, \tag{4.7.18}$$

$$\varepsilon^{\frac{1}{2}+\gamma}|\int \bar{v}\bar{v}_y \cdot \varepsilon\beta_x| \le \varepsilon^{\frac{\gamma}{2}}||\varepsilon^{\frac{1}{2}+\frac{\gamma}{2}}\bar{v}||_{L^\infty}||\bar{v}_y||_{L^2}||\sqrt{\varepsilon}\beta_x||_{L^2}. \tag{4.7.19}$$

This concludes the proof. □

4.8 Forcing

Recall the definitions given in (4.9.15) and (4.9.24), and the definitions given in (4.9.116) - (4.9.117). The purpose of the following estimates is to estimate the contributions of the forcing terms $R^{u,1}, L_1^b, R^{v,1}, L_2^b$ into $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, as shown in (4.6.9) - (4.6.11) Thus, we will analyze the forcing contributions:

Lemma 4.8.1. *For any vector fields $[u, v] \in \mathcal{X}$, the following estimates hold:*

$$\begin{aligned}
&|\int \varepsilon^{-\frac{1}{2}-\gamma}\Big\{R^{u,1}\cdot u + \varepsilon R^{v,1}\cdot v\Big\}| + |\int \varepsilon^{-\frac{1}{2}-\gamma}\Big\{R^{u,1}\cdot -\beta_y + \varepsilon R^{v,1}\beta_x\Big\}| \\
&\quad + |\int \varepsilon^{-\frac{1}{2}-\gamma}\partial_y R^{u,1}\cdot \partial_y(uy^2 w)| + |\int \varepsilon^{-\frac{1}{2}-\gamma}\varepsilon\partial_y R^{v,1}\partial_x\{uy^2w\}| \\
&\quad + |\int L_1^b \cdot u + \varepsilon L_2^b v| + |\int L_1^b\cdot -\beta_y + \int \varepsilon L_2^b\beta_x| \\
&\quad + |\int \partial_y L_1^b \cdot \partial_y\{uy^2w\}| + |\int \varepsilon\partial_y L_2^b\cdot\partial_x\{uy^2w\}| \\
&\quad \lesssim (C(a_0, b_0, a_L, b_L) + \varepsilon^{\frac{1}{4}-\gamma})||u, v||_{\mathcal{X}}.
\end{aligned} \tag{4.8.1}$$

Proof. We recall estimate (4.9.123) from the Appendix, which we then directly use. First, we start with the contributions to $\mathcal{R}_1$, shown in (4.6.9):

$$\begin{aligned}
\int \varepsilon^{-\frac{1}{2}-\gamma}\Big\{R^{u,1}\cdot u + \varepsilon R^{v,1}\cdot v\Big\} &\le \varepsilon^{-\frac{1}{2}-\gamma}||R^{u,1}, \sqrt{\varepsilon}R^{v,1}||_{L^2}||u, \sqrt{\varepsilon}v||_{L^2} \\
&\le \varepsilon^{-\frac{1}{2}-\gamma}\varepsilon^{\frac{3}{4}}\mathcal{O}(L)||u_x, \sqrt{\varepsilon}v_x||_{L^2}.
\end{aligned} \tag{4.8.2}$$

We now move the contributions from $\mathcal{R}_2$, shown in (4.6.10):

$$\int \varepsilon^{-\frac{1}{2}-\gamma}\Big\{R^{u,1}\cdot -\beta_y + \varepsilon R^{v,1}\beta_x\Big\} \le \varepsilon^{-\frac{1}{2}-\gamma}\varepsilon^{\frac{3}{4}}||\beta_y, \sqrt{\varepsilon}\beta_x||_{L^2}. \tag{4.8.3}$$

Next, we move to the higher order quantities from $\mathcal{R}_3$, shown in (4.6.11):

$$\begin{aligned}
\int \varepsilon^{-\frac{1}{2}-\gamma}\partial_y R^{u,1}\cdot\partial_y(uy^2w) &= \int \varepsilon^{-\frac{1}{2}-\gamma}\partial_y R^{u,1}\cdot u_y y^2 w + \int \varepsilon^{-\frac{1}{2}-\gamma}\partial_y R^{u,1}\cdot u2yw \\
&\le \varepsilon^{-\frac{1}{2}-\gamma}||\partial_y R^{u,1}y||_{L^2}\Big[||u_y y||_{L^2} + \mathcal{O}(L)||u_x||_{L^2}\Big] \\
&\le \varepsilon^{-\frac{1}{2}-\gamma}\varepsilon^{\frac{3}{4}}\Big[||u_y y||_{L^2} + \mathcal{O}(L)||u_x||_{L^2}\Big], \qquad (4.8.4)\\
\int \varepsilon^{-\frac{1}{2}-\gamma}\varepsilon\partial_y R^{v,1}\partial_x\{uy^2w\} &= \int \varepsilon^{-\frac{1}{2}-\gamma}\varepsilon\partial_y R^{v,1}\Big[u_x y^2 w - uy^2\Big] \\
&\le \varepsilon^{-\frac{1}{2}-\gamma}||\sqrt{\varepsilon}\partial_y R^{v,1}y||_{L^2}||\sqrt{\varepsilon}u_x y||_{L^2} \\
&\le \varepsilon^{-\frac{1}{2}-\gamma}\varepsilon^{\frac{3}{4}}||\sqrt{\varepsilon}u_x y||_{L^2}. \qquad (4.8.5)
\end{aligned}$$

The estimates on L_1^b, L_2^b contributions follow directly from estimate (4.9.118). This concludes the proof.

□

Combining (4.7.3), (4.7.10), (4.7.15), and (4.8.1):

Corollary 4.8.2. *For $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ defined as in (4.4.6), (4.5.6), (4.3.2), we have:*

$$|\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3| \lesssim \Big[C(a_0, b_0, a_L, b_L) + \varepsilon^{\frac{1}{4}-\gamma}\Big]||u,v||_{\mathcal{X}} + \varepsilon^{\frac{\gamma}{2}}||u,v||_{\mathcal{X}}^2 + \varepsilon^{\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^4. \tag{4.8.6}$$

Combining the above estimate with (4.5.92) and (4.6.7), and performing Young's inequality for the product $C(a_0, b_0, a_L, b_L)||u,v||_{\mathcal{X}}$ above to absorb $||u,v||_{\mathcal{X}}^2$ to the left-hand side of (4.8.7), we have now established the main *a-priori* estimate:

Theorem 4.8.3 ($\mathcal{X}$-Estimate)**.** *Solutions $[u,v] \in \mathcal{X}$ to the system (4.2.30) - (4.2.32), with the boundary conditions (4.2.33) - (4.2.34) satisfy the following estimate:*

$$||u,v||_{\mathcal{X}}^2 \lesssim C(a_0,b_0,a_L,b_L) + \varepsilon^{\frac{1}{4}-\gamma} + \varepsilon^{\frac{\gamma}{2}}||\bar{u},\bar{v}||_{\mathcal{X}}^4. \tag{4.8.7}$$

With the main *a-priori* estimate in hand, we give the formal arguments leading to existence of a solution in $\mathcal{X}$ in Appendix 4.10. In particular, Theorem 4.8.3 coupled with Proposition 4.10.2 gives the main result, Theorem 4.2.1.

4.9 Construction of Profiles

4.9.1 Specification of R^u

Define:

$$R^u := U^\varepsilon \partial_x U^\varepsilon + V^\varepsilon \partial_y U^\varepsilon + \partial_x P^\varepsilon - \partial_{yy} U^\varepsilon - \varepsilon \partial_{xx} U^\varepsilon, \tag{4.9.1}$$

$$R^v := U^\varepsilon \partial_x V^\varepsilon + V^\varepsilon \partial_y V^\varepsilon + \frac{\partial_y}{\varepsilon} P^\varepsilon - \partial_{yy} V^\varepsilon - \varepsilon \partial_{xx} V^\varepsilon. \tag{4.9.2}$$

In this subsection, we will specify the equations we shall take for R^u. We will first expand the nonlinear terms in the following manner:

$$\begin{aligned} U^\varepsilon \partial_x U^\varepsilon =& \Big(u_e^0 + u_p^0 + \sqrt{\varepsilon} u_e^1 + \sqrt{\varepsilon} u_p^1 + \varepsilon^{\frac{1}{2}+\gamma} u\Big) \times \\ & \qquad \Big(u_{ex}^0 + u_{px}^0 + \sqrt{\varepsilon} u_{ex}^1 + \sqrt{\varepsilon} u_{px}^1 + \varepsilon^{\frac{1}{2}+\gamma} u_x\Big) \\ &= \{u_e^0(x,0) + u_p^0\} u_{px}^0 + u_{ex}^0(x,0) u_p^0 \\ & \qquad + \{u_e^0 - u_e^0(x,0)\} u_{px}^0 + \{u_{ex}^0 - u_{ex}^0(x,0)\} u_p^0 \\ & \qquad + \sqrt{\varepsilon}\Big[u_p^0 u_{ex}^1 + u_e^1 u_{px}^0\Big] + \sqrt{\varepsilon}\Big[\{u_e^0 - u_e^0(x,0)\} u_{px}^1 \end{aligned}$$

$$
\begin{aligned}
&+\sqrt{\varepsilon}\Big[\{u_e^0(x,0)+u_p^0\}u_{px}^1+\{u_{ex}^0(x,0)+u_{px}^0\}u_p^1\Big]\\
&+\{u_{ex}^0-u_{ex}^0(x,0)\}u_p^1\Big]+\varepsilon\Big[(u_e^1+u_p^1)u_{px}^1\\
&+u_{ex}^1u_p^1\Big]+\Big[u_e^0u_{ex}^0+\sqrt{\varepsilon}\Big(u_{ex}^1u_e^0+u_e^1u_{ex}^0\Big)+\varepsilon u_e^1u_{ex}^1\Big]\\
&+\varepsilon^{\frac{1}{2}+\gamma}\{u_su_x+u_{sx}u\}+\varepsilon^{1+2\gamma}uu_x.
\end{aligned}
\tag{4.9.3}
$$

$$
\begin{aligned}
V^\varepsilon\partial_yU^\varepsilon=&\Big(\frac{v_e^0}{\sqrt{\varepsilon}}+v_p^0+v_e^1+\sqrt{\varepsilon}v_p^1+\varepsilon^{\frac{1}{2}+\gamma}v\Big)\\
&\times\Big(\sqrt{\varepsilon}u_{eY}^0+u_{py}^0+\varepsilon u_{eY}^1+\sqrt{\varepsilon}u_{py}^1+\varepsilon^{\frac{1}{2}+\gamma}u_y\Big)\\
=&\Big(yv_{eY}^0(x,0)+v_p^0+v_e^1(x,0)\Big)u_{py}^0\\
&+\sqrt{\varepsilon}\Big[\{v_p^0+yv_{eY}^0(x,0)+v_e^1(x,0)\}u_{py}^1+u_{py}^0v_p^1\Big]\\
&+\sqrt{\varepsilon}\Big[v_p^0(u_{eY}^0+\sqrt{\varepsilon}u_{eY}^1)\Big]\\
&+\Big[\frac{v_e^0}{\sqrt{\varepsilon}}-yv_{eY}^0(x,0)\Big]u_{py}^0+\varepsilon v_p^1\Big(u_{eY}^0+\sqrt{\varepsilon}u_{eY}^1\Big)+\varepsilon v_p^1u_{py}^1\\
&+\Big[v_e^0-Yv_{eY}^0(x,0)\Big]u_{py}^1+\sqrt{\varepsilon}\Big[v_e^1-v_e^1(x,0)\Big]u_{py}^1\\
&+\Big[v_e^1-v_e^1(x,0)-Yv_{eY}^1\Big]u_{py}^0+\sqrt{\varepsilon}yv_{eY}^1u_{py}^0\\
&+\Big[v_e^0u_{eY}^0+\sqrt{\varepsilon}\Big(v_e^0u_{eY}^1+v_e^1u_{eY}^0\Big)+\varepsilon v_e^1u_{eY}^1\Big]\\
&+\varepsilon^{\frac{1}{2}+\gamma}\{u_{sy}v+v_su_y\}+\varepsilon^{1+2\gamma}vu_y.
\end{aligned}
\tag{4.9.4}
$$

Inserting into the system (4.9.1) gives the following expansion:

$$
\begin{aligned}
R^u=&\Big\{\{u_e^0(x,0)+u_p^0\}u_{px}^0+u_{ex}^0(x,0)u_p^0+\{yv_{eY}^0(x,0)+v_p^0+v_e^1(x,0)\}u_{py}^0\\
&+P_{px}^0-u_{pyy}^0\Big\}
\end{aligned}
\tag{4.9.5}
$$

$$
\begin{aligned}
&+\sqrt{\varepsilon}\Big\{\{u_e^0(x,0)+u_p^0\}u_{px}^1+\{u_{ex}^0(x,0)+u_{px}^0\}u_p^1\\
&\quad+\{yv_{eY}^0(x,0)+v_p^0+v_e^1(x,0)\}u_{py}^1+u_{py}^0v_p^1-u_{pyy}^1+P_{px}^1-F_1\Big\}
\end{aligned}
\tag{4.9.6}
$$

$$+\sqrt{\varepsilon}\Big[u_e^0 u_{ex}^1 + u_{ex}^0 u_e^1 + v_e^0 u_{eY}^1 + v_e^1 u_{eY}^0 + P_{ex}^1\Big] \tag{4.9.7}$$

$$+R^{\tilde{u},1} + \varepsilon P_{px}^2 + \varepsilon^{\frac{1}{2}+\gamma}\Big[-\Delta_\varepsilon u + S^u(u,v) + P_x + N^u(u,v)\Big]. \tag{4.9.8}$$

We will define:

$$\begin{aligned}
F_1 :=& v_p^0\{u_{eY}^0 + \sqrt{\varepsilon}u_{eY}^1\} + \frac{1}{\sqrt{\varepsilon}}\Big[\frac{v_e^0}{\sqrt{\varepsilon}} - y v_{eY}^0(x,0)\Big]u_{py}^0 \\
&+ y v_{eY}^1 u_{py}^0 + \frac{1}{\sqrt{\varepsilon}}\Big[\{u_e^0 - u_e^0(x,0)\}u_{px}^0 + \{u_{ex}^0 - u_{ex}^0(x,0)\}u_p^0\Big] \\
&+ u_p^0 u_{ex}^1 + u_e^1 u_{px}^0, \qquad (4.9.9) \\
R^{\tilde{u},1} :=& \varepsilon^{\frac{3}{2}} v_p^1 u_{eY}^1 + \varepsilon v_p^1 u_{py}^1 + \Big(v_e^0 - Y v_{eY}^0(x,0)\Big)u_{py}^1 \\
&+ \Big(v_e^1 - v_e^1(x,0) - Y v_{eY}^1(x,0)\Big)u_{py}^0 + \sqrt{\varepsilon}\Big[v_e^1 - v_e^1(x,0)\Big]u_{py}^1 \\
&+ \sqrt{\varepsilon}\{u_e^0 - u_e^0(x,0)\}u_{px}^1 + \sqrt{\varepsilon}u_p^1\{u_{ex}^0 - u_{ex}^0(x,0)\} \\
&+ \varepsilon\Big[(u_e^1 + u_p^1)u_{px}^1 + u_{ex}^1 u_p^1 + u_{eY}^0 v_p^1\Big] \\
&+ \varepsilon u_{pxx}^0 + \varepsilon^{\frac{3}{2}} u_{pxx}^1 + \varepsilon\Big[u_e^1 u_{ex}^1 + v_e^1 u_{eY}^1 + \Delta u_e^0 + \sqrt{\varepsilon}\Delta u_e^1\Big], \qquad (4.9.10) \\
N^u(u,v) :=& \varepsilon^{\frac{1}{2}+\gamma}\Big(u u_x + v u_y\Big), \qquad (4.9.11) \\
N^v(u,v) :=& \varepsilon^{\frac{1}{2}+\gamma}\Big(u v_x + v v_y\Big), \qquad (4.9.12) \\
S^u(u,v) :=& u_s u_x + u_{sx} u + v_s u_y + u_{sy} v, \qquad (4.9.13) \\
S^v(u,v) :=& u_s v_x + v_{sx} u + v_s v_y + v_{sy} v. \qquad (4.9.14)
\end{aligned}$$

Equations (4.9.5) - (4.9.7) define the equations for our approximate layers, as seen in (4.9.53), (4.9.59), and (4.9.74), thereby contributing the final line, (4.9.8) into the remainder equation, (4.9.105). We must actually modify $\tilde{R}^{u,1}$ to $R^{u,1}$, which accounts for the fact that the layers $[u_p^1, v_p^1]$ are cutoff at $y \to \infty$:

$$R^{u,1} := \tilde{R}^{u,1} + \sqrt{\varepsilon}R_p^u + \varepsilon P_{px}^2, \tag{4.9.15}$$

where:

$$R_p^u := \{u_e^0(x,0) + u_p^0\}u_{px}^1 + \{u_{ex}^0(x,0) + u_{px}^0\}u_p^1 + \{yv_{eY}^0(x,0) + v_p^0\}u_{py}^1$$
$$+ u_{py}^0 v_p^1 - u_{pyy}^1 + P_{px}^1 - F_1. \tag{4.9.16}$$

We are then left with:

$$\varepsilon^{\frac{1}{2}+\gamma}\Big[- \Delta_\varepsilon u + S^u(u,v) + P_x + N^u(u,v)\Big] = R^{u,1}. \tag{4.9.17}$$

4.9.2 Specification of R^v

We turn now to the simplification of (4.9.2).

$$R^v = \frac{1}{\sqrt{\varepsilon}}\Big[u_e^0 v_{ex}^0 + v_e^0 v_{eY}^0 + P_{eY}^0\Big] + \frac{P_{py}^0}{\varepsilon} + \frac{P_{py}^1}{\sqrt{\varepsilon}} \tag{4.9.18}$$
$$+ \Big[u_e^0 v_{ex}^1 + u_e^1 v_{ex}^0 + v_e^0 v_{eY}^1 + v_{eY}^0 v_e^1 + P_{eY}^1\Big] \tag{4.9.19}$$
$$+ v_{px}^0\Big(u_e^0 + u_p^0 + \sqrt{\varepsilon}u_e^1 + \sqrt{\varepsilon}u_p^1\Big)$$
$$+ \frac{v_{ex}^0}{\sqrt{\varepsilon}}\Big(u_p^0 + \sqrt{\varepsilon}u_p^1\Big) + v_{ex}^1\Big(u_p^0 + \sqrt{\varepsilon}u_p^1\Big)$$
$$+ \frac{v_e^0}{\sqrt{\varepsilon}}v_{py}^0 + v_e^0 v_{py}^1 + v_p^0\Big(v_{eY}^0 + v_{py}^0 + \sqrt{\varepsilon}v_{eY}^1 + \sqrt{\varepsilon}v_{py}^1\Big)$$
$$+ v_e^1(v_{py}^0 + \sqrt{\varepsilon}v_{py}^1) + \Delta_\varepsilon v_p^0 + P_{py}^2 \tag{4.9.20}$$
$$+ \sqrt{\varepsilon}v_{px}^1\Big(u_e^0 + u_p^0 + \sqrt{\varepsilon}u_e^1 + \sqrt{\varepsilon}u_p^1\Big) + \Delta_\varepsilon v_p^1$$
$$+ \sqrt{\varepsilon}v_p^1\Big(v_{eY}^0 + v_{py}^0 + \sqrt{\varepsilon}v_{eY}^1 + \sqrt{\varepsilon}v_{py}^1\Big)$$
$$+ \Big[\sqrt{\varepsilon}\Delta v_e^0 + \varepsilon\Delta v_e^1 + \sqrt{\varepsilon}u_e^1 v_{ex}^1 + \sqrt{\varepsilon}v_e^1 v_{eY}^1\Big]$$
$$+ \varepsilon^{\frac{1}{2}+\gamma}\Big[- \Delta_\varepsilon v + S^v(u,v) + \frac{P_y}{\varepsilon} + N^v(u,v)\Big] \tag{4.9.21}$$

We shall make the identifications so that (4.9.18) and (4.9.19) vanish by using these equations to define the construction of the approximate layers in (4.9.53), (4.9.60), and (4.9.74). We then define P_p^2 via (4.9.20):

$$\begin{aligned} P_p^2 = &- \int_y^\infty v^0_{px}\Big(u^0_e + u^0_p + \sqrt{\varepsilon}u^1_e + \sqrt{\varepsilon}u^1_p\Big) + \frac{v^0_{ex}}{\sqrt{\varepsilon}}\Big(u^0_p + \sqrt{\varepsilon}u^1_p\Big) \\ &+ v^1_{ex}\Big(u^0_p + \sqrt{\varepsilon}u^1_p\Big) + \frac{v^0_e}{\sqrt{\varepsilon}}v^0_{py} + v^0_e v^1_{py} + v^0_p\Big(v^0_{eY} + v^0_{py} + \sqrt{\varepsilon}v^1_{eY} + \sqrt{\varepsilon}v^1_{py}\Big) \\ &+ v^1_e(v^0_{py} + \sqrt{\varepsilon}v^1_{py}) + \Delta_\varepsilon v^0_p. \end{aligned} \tag{4.9.22}$$

This choice enforces the vanishing of line (4.9.20). We are then left with:

$$\varepsilon^{\frac{1}{2}+\gamma}\Big[- \Delta_\varepsilon v + S^v(u,v) + \frac{P_y}{\varepsilon} + N^v(u,v)\Big] = R^{v,1}, \tag{4.9.23}$$

where

$$\begin{aligned} R^{v,1} :=& \sqrt{\varepsilon}v^1_{px}\Big(u^0_e + u^0_p + \sqrt{\varepsilon}u^1_e + \sqrt{\varepsilon}u^1_p\Big) + \Delta_\varepsilon v^1_p \\ &+ \sqrt{\varepsilon}v^1_p\Big(v^0_{eY} + v^0_{py} + \sqrt{\varepsilon}v^1_{eY} + \sqrt{\varepsilon}v^1_{py}\Big) \\ &+ \Big[\sqrt{\varepsilon}\Delta v^0_e + \varepsilon\Delta v^1_e + \sqrt{\varepsilon}u^1_e v^1_{ex} + \sqrt{\varepsilon}v^1_e v^1_{eY}\Big]. \end{aligned} \tag{4.9.24}$$

This defines the second equation for the remainder, as seen in (4.9.106).

4.9.3 Construction of Layers

We are prescribed the Euler flow $[u^0_e, v^0_e, P^0_e]$. The first task is to verify that there exists Euler flows satisfying assumptions (4.2.24) - (4.2.27):

Proposition 4.9.1. *There exists a nontrivial set of Euler flows, $[u^0_e, v^0_e, P^0_e]$ satisfying assumptions (4.2.24) - (4.2.27).*

We will start with the shear flow $U_0(Y)$, satisfying the following hypothesis:

$$c_0 \le U_0 \le C_0, \tag{4.9.25}$$

$$U_0 \text{ smooth, with rapidly decaying derivatives,} \tag{4.9.26}$$

$$\partial_Y U_0 \ge 0, \tag{4.9.27}$$

$$U_0 = 1 \text{ in a neighborhood of } 0. \tag{4.9.28}$$

Such a shear has stream function $\phi_0(Y) = \int_0^Y U_0$. Such a stream function has the following asymptotics:

$$\phi_0|_{Y=0} = 0, \quad \phi_0|_{x=0} = \phi_0|_{x=L} = \phi_0(Y), \quad \lim_{Y\to\infty} \frac{\phi_0}{Y} = U_\infty \in (c_0, C_0). \tag{4.9.29}$$

Note that assumption (4.9.25) implies $c_0 Y \le \phi_0 \le C_0 Y$. To define our final Euler flow, we must first solve for an perturbative stream function, ψ, using the following elliptic equation:

$$\begin{aligned} -\Delta\psi = \partial_Y U_0 + f_e(\phi_0 + \psi), \quad \psi|_{x=0} = A_0(Y), \quad \psi_{x=L} = A_L(Y), \\ \psi|_{Y=0} = 0, \qquad \psi|_{Y\to\infty} = 0. \end{aligned} \tag{4.9.30}$$

We will assume the following conditions on f_e and the boundary data $A_{0,L}$:

$$0 \le f_e \le \delta << 1, \tag{4.9.31}$$

$$|\partial^k f_e(x+a)| \lesssim |\partial^k f_e(x)| \text{ for } a \ge 0, \tag{4.9.32}$$

$$f_e \in C^\infty(\mathbb{R}), \text{ rapidly decaying in it's argument,} \tag{4.9.33}$$

$$f_e \text{ supported in a neighborhood away from } 0\ , \tag{4.9.34}$$

$$0 \le A_0, A_L \le \delta \times L^{10}, \tag{4.9.35}$$

$$|\partial_Y^k \{A_0, A_L\}| \le \delta \times L^{10} \tag{4.9.36}$$

$$A_0, A_L \in C^\infty(\mathbb{R}_+), \text{ rapidly decaying in it's argument,} \tag{4.9.37}$$

$$A_0, A_L \text{ supported in a neighborhood away from } 0. \tag{4.9.38}$$

$$A_0 \neq A_L. \tag{4.9.39}$$

It is straightforward to see that the set of admissible f_e, A_0, A_L is nonempty. First, via hypothesis (4.9.27) and (4.9.31), we have $\Delta\psi \le 0$, so that via the maximum principle and assumption on the boundary data (4.9.35):

$$\psi \ge 0. \tag{4.9.40}$$

Lemma 4.9.2. *Assume (4.9.25) - (4.9.28) and the assumptions (4.9.31) - (4.9.39) are satisfied. For $0 < L << \delta << 1$, the following energy estimate holds:*

$$||Y^k \psi||_{H^1} \le C_k \mathcal{O}(\delta). \tag{4.9.41}$$

Proof. Define:

$$B(x, Y) = \frac{L - x}{L} A_0(Y) + \frac{x}{L} A_L(Y). \tag{4.9.42}$$

B is smooth and all derivatives are order δ by the assumptions (4.9.36) on $A_{0,L}$. Define now $\bar{\psi} = \psi - B$, which satisfies:

$$-\Delta\bar{\psi} = \Delta B + \partial_Y U_0 + f_e(\phi_0 + \psi), \qquad \bar{\psi}|_{\partial\Omega} = 0. \tag{4.9.43}$$

An energy estimate coupled with Poincare's inequality gives:

$$\int |\nabla \bar{\psi}|^2 = \int \Big(\Delta B + \partial_Y U_0 \Big) \cdot \bar{\psi} + \int f_e(\phi_0 + \psi) \cdot \bar{\psi}$$
$$\leq \mathcal{O}(\delta, L) ||\bar{\psi}_x||_{L^2} + ||f_e(\phi_0 + \psi)||_{L^2} \mathcal{O}(L) ||\bar{\psi}_x||_{L^2}. \tag{4.9.44}$$

We now use (4.9.40) together with assumptions (4.9.32) and (4.9.25) to estimate:

$$||f_e(\phi_0 + \psi)||_{L^2} \leq ||f_e(\phi_0)||_{L^2} \leq ||f_e(c_0 Y)||_{L^2} \leq \mathcal{O}(\delta). \tag{4.9.45}$$

This concludes the proof. □

We now upgrade to weighted estimates, and higher regularity:

Lemma 4.9.3. *Assume (4.9.25) - (4.9.28) and the assumptions (4.9.31) - (4.9.39) are satisfied. For $0 < L << \delta << 1$, the following energy estimate holds:*

$$||Y^m \partial_x^j \partial_Y^k \psi||_{L^2} \leq C_{m,k,j} \delta \textit{ for any } k, m, j \geq 0. \tag{4.9.46}$$

Proof. The first step is to differentiate (4.9.30) in Y. Defining $\psi^{(1)} := \partial_Y \psi$, this produces:

$$-\Delta \psi^{(1)} = \partial_Y^2 U_0 + f_e'(\phi_0 + \psi)(\partial_Y \phi_0 + \psi^{(1)}),$$
$$\psi^{(1)}_Y|_{Y=0} = 0, \quad \psi^{(1)}|_{x=0,L} = \partial_Y A_{0,L}, \tag{4.9.47}$$

where we have evaluated (4.9.30) using the condition (4.9.28), (4.9.34), and (4.9.38) to obtain the Neumann boundary condition above. A homogenization procedure and energy estimate nearly identical to (4.9.43) - (4.9.44) produces:

$$\int |\psi_{xY}|^2 + |\psi_{YY}|^2 \lesssim \mathcal{O}(\delta). \tag{4.9.48}$$

By using now the equation(4.9.30), we also obtain ψ_{xx} in L^2. Note crucially that $||\partial_Y U_0||_{L^2} \leq \mathcal{O}(L)$ due to the integration in the x-direction, which prevents us from requiring a smallness condition on $\partial_Y U_0$. One can iterate this procedure for higher derivatives. It is also straightforward to obtain weighted in Y estimates, using hypothesis (4.9.26), (4.9.33), and (4.9.37) to absorb weights of Y. This concludes the proof of (4.9.46).

□

Proof of Proposition 4.9.1. If we define $\phi^E := \phi_0 + \psi$, then ϕ^E solves:

$$-\Delta\phi^E = f_e(\phi^E), \quad \phi^E(0,Y) = \phi_0 + A_0(Y), \quad \phi^E(L,Y) = \phi_0 + A_L(Y),$$
$$\phi^E(x,0) = 0, \quad \frac{\phi^E(x,Y)}{Y} \xrightarrow{Y\to\infty} U_\infty. \tag{4.9.49}$$

Solutions to such elliptic equations solve the 2D Euler equations (see (CS12)) by setting:

$$u_e^0 = \partial_Y\phi^E, \quad v_e^0 = -\partial_x\phi^E = -\partial_x\psi, \quad P_e^0 = -\frac{1}{2}|\nabla\phi^E|^2 + F_e(\phi^E), \quad F_e' = f_e. \tag{4.9.50}$$

We view ψ as a $\mathcal{O}(\delta)$-perturbation to the shear flow $(U_0(Y), 0)$ for which $\phi^E = \phi_0$, which is therefore achieved by setting $f_e = A_0 = A_L = 0$. Note that the property (4.9.39) creates the x-dependence, for if $A_0 = A_L$, one could solve (4.9.30) for ψ^1 as just a function of Y, creating another shear flow. All properties (4.2.24) - (4.2.27) are easily verified, where the crucial smallness is obtained through the use of (4.9.46):

$$||\frac{v_E^0}{Y}||_{L^\infty} \leq ||v_{eY}^0||_{L^\infty} = ||\psi_{xY}||_{L^\infty} \leq ||\psi_{xY}||_{H^2} \leq \mathcal{O}(\delta). \tag{4.9.51}$$

This concludes the proof of the proposition. □

We will now abandon the particular construction of Proposition 4.9.1, and consider *any* flow satisfying the assumptions of the paper, namely (4.2.24) - (4.2.27). Similar to the above considerations, there exists a function f_e such that:

$$u^0_{eY} - v^0_{ex} = w^0_e = -\Delta\phi^0 = f_e(\phi^0), P^0_e = -\frac{1}{2}|\nabla\phi^0|^2 + F_e(\phi^0), \quad F'_e = f_e. \tag{4.9.52}$$

Our assumptions (4.2.24) - (4.2.27) guarantee the following:

$$c_0 \le u^0_e \le C_0 \Rightarrow \phi^0 = \int_0^Y u^0_e \sim Y,$$

coupled with w^0_e is bounded and decaying in Y implies that f_e together with derivatives are bounded and decaying, which we state now as a lemma:

Lemma 4.9.4. *Define f_e to satisfy the equalities in (4.9.52). The assumptions on $[u^0_e, v^0_e]$ stated in (4.2.24) - (4.2.27) imply that f_e together with sufficiently many derivatives is bounded and decaying in its argument.*

The above lemma is in spirit a converse to Proposition 4.9.1, which will be convenient for later constructions (see specifically equation (4.9.62)).

In accordance with (4.9.5) and (4.9.18), we will take the following system for the leading order Prandtl layer:

$$\{u^0_e(x,0) + u^0_p\}u^0_{px} + u^0_{ex}(x,0)u^0_p + \{yv^0_{eY}(x,0) + v^0_p + v^1_e(x,0)\}u^0_{py}$$
$$+ P^0_{px} - u^0_{pyy}, \qquad P^0_{py} = 0, \tag{4.9.53}$$

$$u^0_p(x,0) = u_b - u^0_e(x,0), \qquad u^0_p(0,y) = u^0_{p,0}(y), \qquad v^0_p(x,0) = -v^1_e(x,0). \tag{4.9.54}$$

Remark. By rewriting the system (4.9.53) for the unknowns:

$$\bar{u} := u_e^0(x,0) + u_(^0 x,y), \qquad \bar{v} = yv_{eY}^0(x,0) + v_p^0(x,y) + v_e^1(x,0), \tag{4.9.55}$$

we obtain:

$$\begin{aligned} &\bar{u}\bar{u}_x + \bar{v}\bar{u}_y - \bar{u}_{yy} = u_e^0(x,0)u_{ex}^0(x,0), \qquad \bar{v} = -\int_y^\infty \bar{u}_x, \\ &\bar{u}|_{y=0} = u_b, \qquad \bar{u}|_{y=\infty} = u_e^0(x,0). \end{aligned} \tag{4.9.56}$$

By evaluating equation (4.2.3) at $Y = 0$, we see that $u_e^0 u_{ex}^0|_{Y=0} = -P_{ex}^0|_{Y=0}$. Note that we do not demand any sign condition on this forcing term.

For the system (4.9.53), we have:

Proposition 4.9.5. *There exists a unique solution, $[u_p^0, v_p^0]$, to the system (4.9.53), satisfying the following:*

$$\sup_x ||y^M \partial_x^j \partial_y^k \{u_p^0, v_p^0\}||_{L_y^2} \lesssim C(M,k,j). \tag{4.9.57}$$

Moreover, the following profile is strictly positive:

$$u_p^0 + u_e^0 \gtrsim 1. \tag{4.9.58}$$

Proof. The proof follows via an appropriate von-Mises transformation, an application of the standard parabolic maximum principle, and energy estimates in a very similar manner to (GN17). We therefore omit the proof.

∎

We will next move to the first Euler layer, which in accordance to (4.9.7) and (4.9.19) is obtained via the following system:

$$u_e^0 u_{ex}^1 + u_{ex}^0 u_e^1 + v_e^0 u_{eY}^1 + v_e^1 u_{eY}^0 + P_{ex}^1 = 0, \tag{4.9.59}$$

$$u_e^0 v_{ex}^1 + u_e^1 v_{ex}^0 + v_e^0 v_{eY}^1 + v_{eY}^0 v_e^1 + P_{eY}^1 = 0, \tag{4.9.60}$$

$$u_{ex}^1 + v_{eY}^1 = 0. \tag{4.9.61}$$

By going to the stream function formulation, where $\nabla^\perp \phi^1 = [u_e^1, v_e^1]$, we have:

$$-\Delta \phi^1 = f_e'(\phi^0)\phi^1, \tag{4.9.62}$$

$$\phi_x^1(x,0) = -v_e^1(x,0) = v_p^0(x,0), \Rightarrow \phi^1(x,0) = 1 + \int_0^x v_p^0(x',0)\,\mathrm{d}x', \tag{4.9.63}$$

$$\phi^1(0,y) = \phi_0^1(y), \qquad \phi^1(L,y) = \phi_L^1(y). \tag{4.9.64}$$

We assume the data in (4.9.63), (4.9.64) are well-prepared in the following sense:

Definition 4.9.6 (Well Prepared Boundary Data)**.** There exists a value of $\phi_{YY}^1|_{Y=0}$ which is given by evaluating equation (4.9.62) on $Y = 0$ and using (4.9.63): $\phi_{YY}^1(x,0) = -\phi_{xx}^1(x,0) - f_e'(\phi^0)\phi^1(x,0)$. The value of $\phi_{YY}^1(x,0)|_{x=0}$ should equal $\partial_{YY}\phi_0^1|_{Y=0}$. Similarly, $\phi_{YY}^1(x,0)|_{x=0} = \partial_{YY}\phi_L^1|_{Y=0}$. If this is the case, we say the boundary data are well-prepared up to order 2. The generalization to order k is obtained by repeating the above procedure.

By standard elliptic regularity, one has:

Lemma 4.9.7. *Assuming well-prepared boundary data, there exists a solution ϕ^1 to the system (4.9.62) - (4.9.64), satisfying the following estimate:*

$$||Y^m \phi^1||_{H^k} \lesssim_{k,m} 1. \tag{4.9.65}$$

Proof. Introduce the corrector:

$$B(x,Y) = (1-\frac{x}{L})\phi_0^1(Y)\phi^1(x,0) + \frac{x}{L}\frac{\phi_L^1(Y)}{\phi^1(L,0)}\phi^1(x,0). \tag{4.9.66}$$

By definition, B is regular and decays exponentially fast in Y. Homogenizing:

$$\bar{\phi} = \phi^1 - B, \tag{4.9.67}$$

we have:

$$-\Delta\bar{\phi} - f'(\phi^0)\bar{\phi} = -\Delta B - f'(\phi^0)B, \qquad \bar{\phi}|_{\partial\Omega} = 0. \tag{4.9.68}$$

As our boundary data are well-prepared according to Definition 4.9.6, we may take ∂_Y^2 of the system and repeat the procedure. In particular:

$$-\Delta\partial_Y^2\phi^1 - f_e'(\phi^0)\partial_Y^2\phi^1 = 2f''(\phi^0)\phi_y^0\phi_Y^1 + f'''(\phi^0)|\phi_Y^0|^2\phi^1 + f''(\phi^0)\phi_{YY}^0\phi^1. \tag{4.9.69}$$

One may define the new corrector B analogously to (4.9.66) and perform standard elliptic estimates to conclude that:

$$||Y^m\{\phi_{YYY}^1, \phi_{YYX}^1, \phi_{YY}^1\}||_{L^2} \lesssim 1. \tag{4.9.70}$$

By Hardy inequality, as all derivatives of ϕ^1 decay as $Y \to \infty$, we can conclude:

$$||Y^m\phi_{xY}^1||_{L^2} \lesssim 1. \tag{4.9.71}$$

From equation (4.9.62), it is clear that:

$$||\phi^1_{xx} Y^m||_{L^2} \lesssim 1. \tag{4.9.72}$$

We have thus obtained all H^2 quantities. Taking ∂_Y of (4.9.62) enables us to estimate ϕ^1_{xxY} and taking ∂_x of (4.9.62) enables us to estimate ϕ_{xxx}, giving the full H^3 estimate. Next, we can conclude that:

$$||\phi^1_Y||_{L^\infty([0,\infty])} \leq ||\phi^1_Y||_{H^1((0,L))} \leq ||\phi^1||_{H^3} \lesssim 1. \tag{4.9.73}$$

This enables us to iterate the procedure.

□

In accordance to the (4.9.6) and (4.9.18), we will take the following system for the Prandtl-1 layer:

$$u^0 u_{px} + u^0_x u_p + v^0 u_{py} + u^0_y v_p - u_{pyy} = F_1, \qquad P^1_{py} = 0, \tag{4.9.74}$$

$$u_{px} + v_{py} = 0, \qquad u^1_p(x,0) = -u^1_e(x,0), \qquad u^1_p(0,y) = u^1_{p0}(y), \tag{4.9.75}$$

$$u^0 := u^0_e(x,0) + u^0_p, \qquad v^0 := y v^0_{eY}(x,0) + v^0_p + v^1_e(x,0). \tag{4.9.76}$$

Here, v_p will be recovered via:

$$v_p = \int_0^y u^1_{px} \, \mathrm{d}y'. \tag{4.9.77}$$

We will homogenize in the following manner: define χ such that:

$$\chi(0) = 1, \qquad \partial_y^k \chi(0) = 0 \text{ for } k \geq 1, \qquad \int_0^\infty \chi \, \mathrm{d}y = 0. \tag{4.9.78}$$

Then define:

$$u = u_p^1 + \chi(y) u_e^1(x,0), \qquad v = v_p^1 + u_{ex}^1(x,0) I_\chi(y), \qquad I_\chi(y) = \int_y^\infty \chi. \tag{4.9.79}$$

The new unknowns, $[u, v]$ satisfy the following system:

$$u^0 u_x + u_x^0 u + v^0 u_y + u_y^0 v - u_{yy} = F_1 + H_1, \tag{4.9.80}$$

$$\begin{aligned} H_1 &:= u^0 \chi u_{ex}^1(x,0) + u_x^0 \chi u_e^1(x,0) \\ &\qquad + v^0 \chi' u_e^1(x,0) + u_y^0 I_\chi u_{ex}^1(x,0) - \chi'' u_e^1(x,0). \end{aligned} \tag{4.9.81}$$

We recall the definition of F_1 given in (4.9.9). Furthermore, we have the following estimate on the forcing:

Lemma 4.9.8. *For any $m, k, j \geq 0$, the following estimate for F_1 holds:*

$$||\langle y \rangle^m \partial_y^k \partial_x^j F_1||_{L^2}^2 \lesssim_{m,k,j} 1. \tag{4.9.82}$$

Proof. The proof follows directly due to the smoothness and rapid decay properties of $[u_p^0, v_p^0]$. $\square$

Lemma 4.9.9. *Solutions $[u, v]$ as defined in (4.9.79) to the problem (4.9.80) satisfy the following estimate:*

$$\sup_{x \in [0,L]} ||u||_{L_y^2}^2 + ||u_y||_{L^2}^2 \leq C(u_{p0}^1) + \mathcal{O}(L) ||u_x||_{L^2}^2. \tag{4.9.83}$$

Proof. One applies u to the above system, (4.9.80), and integrates:

$$\int \Big(u^0 u_x + u^0_x u + v^0 u_y + u^0_y v\Big) \cdot u = \int \{F_1 + H_1\} \cdot u. \tag{4.9.84}$$

The result follows upon integrating in x and estimating:

$$\begin{aligned} &|\int\int u^0_y uv| + |\int\int u^0_x u^2| + |\int\int \{F_1 + H_1\}u| \\ &\qquad \le C(u^1_{p0}) + \mathcal{O}(L)||u^0_y \cdot y, u^0_x||_{L^\infty}\Big[||u_x||^2_{L^2} + ||F_1, H_1||^2_{L^2}\Big]. \end{aligned} \tag{4.9.85}$$

□

Lemma 4.9.10. *Solutions $[u,v]$ as defined in (4.9.79) to the problem (4.9.80) satisfy the following estimate:*

$$||u_x||^2_{L^2} + \sup ||u_y||^2_{L^2_y} \lesssim C(u^1_{p0}) + ||u_y||^2_{L^2} + C(v^0_e)||yu_y||^2_{L^2}. \tag{4.9.86}$$

Proof. Introduce $v = \beta u^0$. Then the system becomes:

$$-u^0 v_y + u^0_y v + u^0_x u + v^0 u_y - u_{yy} = -|u^0|^2\beta_y + u^0_x u + v^0 u_y - u_{yy}.$$

Multiplying by $-\beta_y$ and integrating in y yields:

$$\begin{aligned} \int |u^0|^2\beta_y^2 + \int u_{yy}\beta_y &= \int |u^0|^2\beta_y^2 - \int u_y\beta_{yy} \\ &= \int |u^0|^2\beta^2_y + \int u_y \frac{u_{xy}}{u^0} - \int 2u_y v_y \partial_y \frac{1}{u^0} - \int u_y v \partial_y^2 \frac{1}{u^0} \\ &= \int |u^0|^2 \beta^2_y + \frac{\partial_x}{2}\int \frac{1}{u^0}u^2_y - \int u^2_y \partial_x\{\frac{1}{u^0}\} - 2\int u_y v_y \partial_y \frac{1}{u^0} \\ &- \int u_y v \partial^2_y \frac{1}{u^0}. \end{aligned} \tag{4.9.87}$$

Upon integrating further in x, the final three terms above are estimated:

$$|-\int\int u_y^2\partial_x\{\frac{1}{u^0}\}-2\int\int u_y v_y\partial_y\frac{1}{u^0}-\int\int u_y v\partial_y^2\frac{1}{u^0}| \lesssim ||u_y||_{L^2}^2+\delta||v_y||_{L^2}^2. \tag{4.9.88}$$

The remaining terms, upon integrating in x:

$$|\int\int u_x^0 u\cdot\beta_y| \le \mathcal{O}(L)||\beta_y||_{L^2}^2, \tag{4.9.89}$$

$$|\int\int v^0 u_y\cdot\beta_y| \le C(v_e^0)||yu_y||_{L^2}||\beta_y||_{L^2}. \tag{4.9.90}$$

The right-hand side is estimated simply using Holder's inequality.

□

Lemma 4.9.11 (Weighted Estimates)**.** *Solutions* $[u,v]$ *as defined in (4.9.79) to the problem (4.9.80) satisfy the following estimate:*

$$||\{u_y,u_{yy}\}\cdot y\chi(y)||_{L^2}^2 \lesssim 1+||u_x||_{L^2}^2+||u_y||_{L^2}^2. \tag{4.9.91}$$

Proof. Applying ∂_y to the system gives:

$$u^0u_{xy}+u_{xy}^0u+v^0u_{yy}+u_{yy}^0v-u_{yyy}=\partial_y\{F_1+H_1\}. \tag{4.9.92}$$

We apply the multiplier $u_y\chi y^2\cdot(1-x)$. The main positive terms are:

$$\int u^0u_{xy}\cdot u_y\chi y^2(1-x)+\int v^0u_{yy}\cdot u_y\chi y^2(1-x) = \frac{\partial_x}{2}\int u^0u_y^2y^2\chi(1-x)+\int\frac{u^0}{2}u_y^2y^2\chi-\int\frac{u_x^0}{2}u_y^2y^2\chi(1-x)$$

$$-\int \frac{v_y^0}{2} u_y^2 \chi y^2 (1-x) - \int u_y^2 v^0 y(1-x) - \int \frac{u_y^2}{2} y^2 (1-x)\chi'$$
$$= \frac{\partial_x}{2} \int u^0 u_y^2 y^2 \chi (1-x) + \int \frac{u^0}{2} u_y^2 y^2 \chi - \int u_y^2 v^0 y(1-x)$$
$$- \int \frac{u_y^2}{2} y^2 (1-x)\chi'. \tag{4.9.93}$$

Upon taking integration in x from 0 to X_*, we obtain using the smallness of $v^0_{eY}(x,0)$:

$$\int_0^{X_*} (4.9.93) \gtrsim \int_{x=X_*} u^0 u_y^2 y^2 \chi (1-X_*) + ||u_y y \chi||_{L^2}^2 - ||u_y||_{L^2}^2. \tag{4.9.94}$$

The remaining terms, upon integrating in x from $[0, X_*]$:

$$|\int u^0_{xy} u \cdot u_y \chi y^2 (1-x)| \le ||u^0_{xy} y^2||_{L^\infty} ||u_y||_{L^2}^2, \tag{4.9.95}$$

$$|\int u^0_{yy} v \cdot u_y \chi y^2 (1-x)| \le ||u^0_{yy} y^2||_{L^\infty} ||u_y||_{L^2} ||v_y||_{L^2}, \tag{4.9.96}$$

$$-\int u_{yyy} u_y \chi y^2 (1-x) \gtrsim ||u_{yy} y \chi||_{L^2}^2 - ||u_y||_{L^2}^2, \tag{4.9.97}$$

$$|\int \partial_y \{F_1 + H_1\} \cdot u_y \chi y^2 (1-x)| \lesssim 1 + ||u_y||_{L^2}^2. \tag{4.9.98}$$

□

Placing the above series of estimates together closes the basic estimate for u^1_p. It is possible to take ∂_x^k and repeat with weights y^m. We omit these details. Summarizing:

Lemma 4.9.12. *For any $k, m \ge 0$, solutions $[u,v]$ as defined in (4.9.79) to the problem (4.9.80) satisfy the following estimate:*

$$\sup_x ||y^m \partial_x^k u_p||_{L^2_y} + ||y^m \partial_x^k u_{py}||_{L^2} + ||v_p||_{L^\infty} \lesssim C(k,m). \tag{4.9.99}$$

Proof. Only the v_p estimate remains to be proven, for which we appeal to Hardy (as $v_p|_{y=0} =$

0):

$$v_p^2 = \int_0^y v_p v_{py} \le ||v_{py}||_{L^2_y} ||y v_{py}||_{L^2} = ||u_{px}||_{L^2_y} ||y u_{px}||_{L^2} \lesssim 1, \tag{4.9.100}$$

the final estimate following from the u_p estimates in (4.9.99). □

The final task is to cut-off the Prandtl-1 layer:

$$u_p^1 = \chi(\sqrt{\varepsilon} y) u_p - \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon} y) \int_y^\infty u_p(x,s)\,\mathrm{d}s, \qquad v_p^1 = \chi(\sqrt{\varepsilon} y) v_p \tag{4.9.101}$$

It is clear that the divergence free structure is preserved and the same estimates from (4.9.99) hold. The error created by such a cut-off layer is:

$$\begin{aligned} R_p^u = (1-\chi)F_1 &+ \sqrt{\varepsilon} u^0 \chi' v_p - \sqrt{\varepsilon} u^0_x \chi' \int_y^\infty u_p \\ &+ 2\sqrt{\varepsilon} v^0 \chi' u_p - \varepsilon v^0 \chi'' \int_y^\infty u_p + 3\sqrt{\varepsilon}\chi' u_{py} \\ &+ 3\varepsilon \chi'' u_p - \varepsilon^{\frac{3}{2}} \chi''' \int_y^\infty u_p. \end{aligned} \tag{4.9.102}$$

R_p^u then contributes into $R^{u,1}$, according to (4.9.15).

Lemma 4.9.13. *The remainder R_p^u defined in (4.9.102) satisfies the following estimate:*

$$||R_p^u||_{L^2} + ||y \partial_y R_p^u||_{L^2} \lesssim \varepsilon^{\frac{1}{4}}. \tag{4.9.103}$$

Proof. All follow via the estimates in (4.9.99) aside from F_1, for which we must use the rapid decay and that support of $1-\chi$ is on $y \ge \frac{1}{\sqrt{\varepsilon}}$. Next, the term with v_p, we must use:

$$||\sqrt{\varepsilon} u^0 \chi' v_p||_{L^2} \le \sqrt{\varepsilon} ||v_p||_{L^\infty} ||1||_{L^2(y \le \frac{1}{\sqrt{\varepsilon}})} \le \varepsilon^{\frac{1}{4}}. \tag{4.9.104}$$

Upon applying $y\partial_y$, an identical calculation yields the desired result. $\square$

4.9.4 Remainder System

Collecting the constructions above, according to (4.9.17) and (4.9.23), the remainders $[u, v, P]$ are to satisfy the following system:

$$-\Delta_\varepsilon u + S^u(u, v) + P_x = N^u(u, v) + \varepsilon^{-\frac{1}{2}-\gamma} R^{u,1} := f_0, \tag{4.9.105}$$

$$-\Delta_\varepsilon v + S^v(u, v) + \frac{P_y}{\varepsilon} = N^v(u, v) + \varepsilon^{-\frac{1}{2}-\gamma} R^{v,1} := g_0. \tag{4.9.106}$$

$$u_x + v_y = 0, \tag{4.9.107}$$

together with the boundary conditions:

$$[u, v]|_{x=0} = [a_0(y), b_0(y)], [u, v]|_{y=0} = [u, v]|_{y\to\infty} = 0, \tag{4.9.108}$$

$$\{u_y + \varepsilon v_x\}|_{x=L} = b_L(y), \quad \{P - 2\varepsilon u_x\}|_{x=L} = a_L(y). \tag{4.9.109}$$

The definitions of S_u, S_v, N^u, N^v are given in (4.9.11) - (4.9.14). The assumptions on $[a_0, b_0, a_L, b_L]$ are given in (4.2.21). Up to renaming a_L, b_L, it is possible to, without loss of generality, consider the following simplification:

$$-\Delta_\varepsilon \bar{u} + S^u(\bar{u}, \bar{v}) + P_x = N^u(\bar{u}, \bar{v}) + \varepsilon^{-\frac{1}{2}-\gamma} R^{u,1} := f, \tag{4.9.110}$$

$$-\Delta_\varepsilon \bar{v} + S^v(\bar{u}, \bar{v}) + \frac{P_y}{\varepsilon} = N^v(\bar{u}, \bar{v}) + \varepsilon^{-\frac{1}{2}-\gamma} R^{v,1} := g. \tag{4.9.111}$$

$$\bar{u}_x + \bar{v}_y = 0, \tag{4.9.112}$$

together with homogenized boundary conditions:

$$[\bar{u}, \bar{v}]|_{x=0} = [0,0], [\bar{u}, \bar{v}]|_{y=0} = [\bar{u}, \bar{v}]|_{y\to\infty} = 0, \tag{4.9.113}$$

$$\{\bar{u}_y + \varepsilon \bar{v}_x\}|_{x=L} = \bar{b}_L(y), \quad \{P - 2\varepsilon \bar{u}_x\}|_{x=L} = \bar{a}_L(y). \tag{4.9.114}$$

The reason is:

Lemma 4.9.14. *If the assumptions in (4.2.21) are satisfied, $[u,v]$ solves the system (4.9.105) - (4.9.107) with boundary conditions (4.9.108) - (4.9.109) if and only if $[\bar{u}, \bar{v}] = [u - u_0, v - v_0]$ solves (4.9.105) - (4.9.107) with (4.9.113) - (4.9.114), and with modifying $[f_0, g_0]$ to $[f, g]$ as defined by:*

$$f := f_0 + L_1^b, \quad g := g_0 + L_2^b, \tag{4.9.115}$$

where:

$$\begin{aligned} L_1^b :=& u_s u_{0x} + u_{sx} u_0 + v_s u_{0y} + u_{sy} v_0 \\ &+ \varepsilon^{\frac{1}{2}+\gamma}\Big(u_0 u_x + u u_{0x} + v_0 u_y + u_{0y} v\Big), \\ &+ \varepsilon^{\frac{1}{2}+\gamma}\Big(u_0 u_{0x} + v_0 u_{0y}\Big) \end{aligned} \tag{4.9.116}$$

$$\begin{aligned} L_2^b :=& u_s v_{0x} + v_{sx} u_0 + v_s v_{0y} + v_{sy} v_0 \\ &+ \varepsilon^{\frac{1}{2}+\gamma}\Big(u_0 v_x + u v_{0x} + v_0 v_y + v_{0y} v\Big) \\ &+ \varepsilon^{\frac{1}{2}+\gamma}\Big(u_0 v_{0x} + v_0 v_{0y}\Big). \end{aligned} \tag{4.9.117}$$

where $[u_0, v_0]$ are defined below in (4.9.119). Finally, we have the following estimate:

$$||\langle y \rangle^N \Big\{ L_1^b, \sqrt{\varepsilon} L_2^b, \partial_y L_1^b, \sqrt{\varepsilon}\partial_y L_2^b \Big\}||_{L^2} \lesssim 1. \tag{4.9.118}$$

Proof. We will define the following auxiliary profiles:

$$u_0 = a_0(y) - x\partial_y b_0(y), \qquad v_0 = b_0(y). \tag{4.9.119}$$

It is clear that $[u_0, v_0]$ is a divergence free vector field, that achieves the boundary conditions at $x = 0, y = 0, y \to \infty$. It is also clear that $[u_0, v_0]$ are order-1, and decay rapidly in y. Consider now the difference:

$$\bar{u} = u - u_0, \quad \bar{v} = v - v_0. \tag{4.9.120}$$

At $x = L$, the following boundary conditions are satisfied:

$$\bar{a}_L := P - 2\varepsilon\bar{u}_x|_{x=L} = P - 2\varepsilon u_x - 2\varepsilon\partial_y b_0(y) = a_L(y) - 2\varepsilon\partial_y b_0(y), \tag{4.9.121}$$

$$\bar{b}_L := \bar{u}_y + \varepsilon\bar{v}_x|_{x=L} = \Big(u_y + \varepsilon v_x\Big)|_{x=L} - \partial_y u_0 = b_L - \partial_y u_0. \tag{4.9.122}$$

It is clear that $[\bar{u}, \bar{v}]$ will achieve the boundary conditions in (4.9.109), with $[a_L, b_L]$ replaced by $[\bar{a}_L, \bar{b}_L]$, and that $[\bar{a}_L, \bar{b}_L]$ satisfy the required assumptions, (4.2.21). Finally, the new profiles $[\bar{u}, \bar{v}]$ satisfy the new system (4.9.105) - (4.9.107) with f, g defined in (4.9.115) according to a standard linearization. The estimate in (4.9.118) follows from the definitions (4.9.116) - (4.9.117), together with (4.9.119). □

Remark (Notation). Due to this lemma, we can restrict to considering (4.9.110) - (4.9.114), and we will rename $[\bar{u}, \bar{v}]$ to $[u, v]$ to help simplify notation.

Proposition 4.9.15. *For $[R^{u,1}, R^{v,1}]$ defined as in (4.9.15), (4.9.24), we have:*

$$||R^{u,1}, \sqrt{\varepsilon}R^{v,1}||_{L^2} + ||\langle y\rangle\partial_y\{R^{u,1}, \sqrt{\varepsilon}R^{v,1}\}||_{L^2} \le \varepsilon^{\frac{3}{4}}. \tag{4.9.123}$$

Proof. We will start with $R^{u,1}$, as defined in (4.9.15). The estimate on R^u_p follows from recalling the prefactor of $\sqrt{\varepsilon}$ given in (4.9.15) coupled with (4.9.103). The estimate on εP^2_{px} follows upon noticing that each term in the definition (4.9.22) exhibits rapid decay, and therefore $|P^2_p| \le \langle y \rangle^{-M}$. We can thus move to the terms from $\tilde{R}^{u,1}$, as defined in (4.9.10), and $R^{v,1}$ as defined in (4.9.24). Combining estimates (4.9.57), (4.9.65), and (4.9.99) immediately implies the desired result. $\square$

We will also record here the following, which will be in constant use throughout the paper:

Lemma 4.9.16 (Uniform Estimates of Profiles). *With $[u_s, v_s]$ defined as in (4.2.10) - (4.2.11), for any $k, j \ge 0$, we have:*

$$||y^k \partial_y^k \partial_x^j u_s, y^k \partial_y^{k+1} \partial_x^j v_s||_{L^\infty} \lesssim 1. \tag{4.9.124}$$

Moreover, we have the strict positivity:

$$u_s \gtrsim 1. \tag{4.9.125}$$

Proof. Using the definitions provided in (4.2.10), we see that:

$$\partial_y^k u_s = \partial_y^k \{u_e^0 + u_p^0 + \sqrt{\varepsilon} u_e^1 + \sqrt{\varepsilon} u_p^1\} \tag{4.9.126}$$

$$= \varepsilon^{\frac{k}{2}} \{\partial_Y^k u_e^0, \sqrt{\varepsilon} \partial_Y^k u_e^1\} + \partial_y^k \{u_p^0, \sqrt{\varepsilon} u_p^1\}. \tag{4.9.127}$$

Multiplying by y^k and using $\sqrt{\varepsilon}^k y^k = Y^k$ gives the desired result. An analogous computation can be made using the definition (4.2.11), and finally iterates of ∂_x do not contribute factors of $\sqrt{\varepsilon}$, which is why they do not enhance the weight of y.

Finally, the positivity in (4.9.125) follows from (4.9.58) and the uniform estimates on u_e^1, u_p^1 found in (4.9.65), (4.9.99).

□

4.10 Existence and Uniqueness of Remainder

The main result of this appendix is the following:

Proposition 4.10.1 (Linear Existence). *Given $(f, g) \in L^2$, and given (a_L, b_L) satisfying the assumptions (4.2.21), for L sufficiently small, there exists a unique solution to the linear problem (4.2.30) - (4.2.32), together with boundary conditions (4.2.33) -(4.2.34).*

Proposition 4.10.2 (Nonlinear Existence). *Given boundary data satisfying assumptions (4.2.21), there exists a unique solution $[u, v] \in \mathcal{X}$ to the full nonlinear problem (4.9.110) - (4.9.114).*

We will define the operator:

$$\begin{aligned} S_{\alpha,m}[u,v,P] :=& -\Delta_\varepsilon u + P_x - 10\alpha\partial_y\{\langle y\rangle^{2m}u_y\chi_1(y)\} \\ & -\Delta_\varepsilon v + \frac{P_y}{\varepsilon} - 2\alpha\partial_y\{\langle y\rangle^{2m}v_y\chi_1(y)\} \\ & -\alpha\partial_x\{\langle y\rangle^{2m}\{u_y + \varepsilon v_x\}\chi_1(y)\}. \end{aligned} \tag{4.10.1}$$

defined always on divergence free vector fields, together with the boundary conditions:

$$\begin{aligned} &[u,v]|_{x=0} = [u,v]|_{y=0} = [u,v]|_{y\to\infty} = 0, \\ &P - 2\varepsilon u_x|_{x=L} = a_L(y), \qquad u_y + \varepsilon v_x|_{x=L} = b_L(y). \end{aligned} \tag{4.10.2}$$

Here χ_1 a cutoff function which is equal to 0 on $[0, 1)$ and 1 on $(2, \infty)$. Strictly, $S_{\alpha,m}$ must return a four-tuple, with the first two components being (4.10.1), and the final two components including (a_L, b_L). Fix another cut-off function $\chi_2(y) = 0$ on $[0, 10)$ and $\chi_2 = 1$

on $(20, \infty)$. Define now the norms:

$$||u, v||^2_{H^1_m} := ||\Big\{u_y, \sqrt{\varepsilon} v_y, \varepsilon v_x\Big\} \langle y \rangle^m ||^2_{L^2}, \tag{4.10.3}$$

$$||u, v||^2_{H^2_m} := ||\Big\{u_{yy}, \sqrt{\varepsilon} v_{yy}, \varepsilon v_{xx}\Big\} \langle y \rangle^m \chi_2 ||^2_{L^2}. \tag{4.10.4}$$

Notationally, we will refer to the $m = 0$ norm as simply H^1. Define now the space:

$$C_{0,S} := \Big\{(\varphi, \phi) \in C^\infty : \text{compactly supported in } y, \tag{4.10.5}$$
$$\text{supported away from } x = 0, \text{ and } \partial_x \varphi + \partial_y \phi = 0\Big\}.$$

We will define:

$$H^1_m := \overline{C_{0,S}}^{||\cdot||_{H^1_m}}, \tag{4.10.6}$$

and the scaled, symmetric gradient via:

$$D_\varepsilon = \begin{pmatrix} \sqrt{\varepsilon} u_x & u_y + \varepsilon v_x \\ u_y + \varepsilon v_x & \sqrt{\varepsilon} v_y \end{pmatrix}. \tag{4.10.7}$$

Recalling the definition of $|| \cdot ||_{\mathcal{X}}$ from (4.2.22), our ultimate space $\mathcal{X}$ is defined via:

$$\mathcal{X}_0 := \overline{C_{0,s}}^{||\cdot||_{H^1}}, \tag{4.10.8}$$

$$\mathcal{X} := \{[u, v] \in \mathcal{X}_0 : ||u, v||_{\mathcal{X}} < \infty\}. \tag{4.10.9}$$

Define the weak formulation of (4.10.1) to be:

$$\int D_\varepsilon u \cdot D_\varepsilon \varphi + \int \varepsilon D_\varepsilon v \cdot D_\varepsilon \phi + \int_{x=L} a_L \varphi - \int_{x=L} \varepsilon b_L \phi$$
$$+ \alpha \int \Big(10 u_y \cdot \varphi_y + 2\varepsilon v_y \cdot \phi_y + \{\varepsilon^2 v_x + \varepsilon u_y\} \cdot \phi_x \Big) \chi_1(y) \langle y \rangle^{2m}$$
$$- \alpha \int_{x=L} \varepsilon y^{2m} \chi_1 b_L(y) \phi = \int f \cdot \varphi + \varepsilon g \cdot \phi, \tag{4.10.10}$$

for all $(\varphi, \phi) \in C_{0,S}$.

Lemma 4.10.3. *Given $(f,g) \in L^2$, and boundary values (a_L, b_L) satisfying the assumptions (4.2.21), there exists a weak solution $[u,v,P] \in H^1_m$ satisfying the estimate:*

$$\alpha ||u,v||^2_{H^1_m} \lesssim ||f, \sqrt{\varepsilon} g||^2_{L^2} + ||\Big\{ \frac{a_L}{\sqrt{\varepsilon}}, b_L \Big\} y^{2m} ||^2_{L^2(x=L)}. \tag{4.10.11}$$

Proof. The existence of solutions follows directly from Lax-Milgram. We must verify the Bilinear form in (4.10.10) is coercive:

$$B[(u,v),(\varphi,\phi)] := \int D_\varepsilon u \cdot D_\varepsilon \varphi + \int \varepsilon D_\varepsilon v \cdot D_\varepsilon \phi \tag{4.10.12}$$
$$+ \alpha \int \Big(10 u_y \cdot \varphi_y + 2\varepsilon v_y \cdot \phi_y + \{\varepsilon^2 v_x + \varepsilon u_y\} \cdot \phi_x \Big) \chi_1(y) \langle y \rangle^{2m}.$$

This is immediate, apart from the cross term, to which we first appeal to the density:

$$\int \varepsilon u_y \phi_x^{(n)} \chi_1 \langle y \rangle^{2m} \xrightarrow{n \to \infty} \int \varepsilon u_y v_x \chi_1 \langle y \rangle^{2m}$$
$$\le_{|\cdot|} \frac{1}{2} \int \varepsilon^2 v_x^2 \chi_1 \langle y \rangle^{2m} + \frac{1}{2} \int u_y^2 \chi_1 \langle y \rangle^{2m}, \tag{4.10.13}$$

which explains the constants of 10 appearing in (4.10.1). We view the terms:

$$\int f \cdot \varphi + \varepsilon g \cdot \phi - \int_{x=L} a_L \varphi + \int_{x=L} \varepsilon b_L \phi + \alpha \int_{x=L} \varepsilon y^{2m} \chi_1 b_L \phi, \tag{4.10.14}$$

as a functional on H^{-1}. We must thus estimate the following boundary term:

$$\begin{aligned}|\int_{x=L}\alpha\varepsilon y^{2m}\chi_1 b_L\phi| &\le \alpha||b_L\langle y\rangle^{2m}||_{L^2(x=L)}||\varepsilon\phi||_{L^2(x=L)}\\ &\lesssim ||b_L\langle y\rangle^{2m}||^2_{L^2(x=L)} + \alpha^2\mathcal{O}(L)||\varepsilon\phi_x||^2_{L^2}, \end{aligned} \tag{4.10.15}$$

the latter term being absorbed into the positive contributions from $\int|D_\varepsilon v|^2$ using the smallness of L and α. Next, we must estimate the boundary terms:

$$\begin{aligned}|\int_{x=L}a_L\varphi| &\le ||\frac{a_L}{\sqrt{\varepsilon}}||_{L^2(x=L)}||\sqrt{\varepsilon}\varphi||_{L^2(x=L)}\\ &\lesssim ||\frac{a_L}{\sqrt{\varepsilon}}||^2_{L^2(x=L)} + \mathcal{O}(L)||\sqrt{\varepsilon}\varphi_x||^2_{L^2}, \end{aligned} \tag{4.10.16}$$

$$\begin{aligned}|\int_{x=L}\varepsilon b_L\phi| &\le ||b_L||_{L^2(x=L)}||\varepsilon\phi||_{L^2(x=L)}\\ &\le ||b_L||^2_{L^2(x=L)} + \mathcal{O}(L)||\varepsilon\phi_x||^2_{L^2}. \end{aligned} \tag{4.10.17}$$

the latter terms in both of the above calculations can be absorbed into the positive contributions from $\int|D_\varepsilon v|^2$. □

It is clear that each solution $[u,v]\in H^1_m$ is automatically in $\mathcal{X}_0$. We will now bootstrap to H^2_m solutions.

Lemma 4.10.4. *Solutions $[u,v]\in H^1_m$ to the system (4.10.1) satisfy:*

$$\alpha||u,v||_{H^2_m} \lesssim ||f,\sqrt{\varepsilon}g||_{L^2} + ||\Big\{\frac{a_L}{\sqrt{\varepsilon}}, b_L\Big\}y^{2m}||^2_{L^2(x=L)}. \tag{4.10.18}$$

Moreover, such solutions are strong solutions, which satisfy the boundary conditions of (4.10.2).

Proof. This follows formally from differentiating (4.10.1) in y and applying the multiplier u_y, with the help of the cut-off function χ_2 in (4.10.4) to avoid the corners. Rigorously, one needs to work with difference quotients within the weak formulation (4.10.10). We demonstrate this now for the main weighted term. Given $[u, v] \in H^1_m$, there exists a sequence φ^n, ϕ^n such that: $[\varphi^n, \phi^n] \xrightarrow{H^1_m} [u, v]$ by the density (4.10.6). Denote by D^h the difference quotient in the y-direction: $D^h u(x, y) = \frac{u(x,y+h)-u(x,y)}{h}$. We will select the multiplier $D^{-h}D^h\varphi$ to apply the weak formulation (4.10.10):

$$\int y^m \chi_1(y) u_y D^{-h} D^h \varphi_y^{(n)} = \int D^h \Big\{ y^m \chi_1(y) u_y \Big\} D^h \varphi_y^{(n)}. \tag{4.10.19}$$

By definition of difference quotient, for each fixed h, $\langle y \rangle^m D^h \varphi_y^{(n)} \xrightarrow{L^2} \langle y \rangle^m D^h u_y$. Similarly, for each fixed h, $\langle y \rangle^m D^h u_y \in L^2$. Thus, for each fixed h, we can take $n \to \infty$:

$$(4.10.19) \xrightarrow{n \to \infty} \int D^h \Big\{ y^m \chi_1(y) u_y \Big\} D^h u_y. \tag{4.10.20}$$

Next taking limits in h gives:

$$(4.10.20) \xrightarrow{h \to 0} \int \partial_y \{ y^m \chi_1(y) u_y \} \cdot u_{yy}. \tag{4.10.21}$$

Performing similar calculations for each of the terms yields the desired result. The boundary conditions (4.10.2) are satisfied by integrating by parts (4.10.1) against a test function, justified as $[u, v]$ are strong solutions, and comparing the boundary terms with (4.10.10).

$\square$

Near the boundary $y = 0$, standard Stokes theory (applicable due to cutoff $\chi_1(y)$) implies:

Lemma 4.10.5. *Solutions $[u, v]$ to the system (4.10.1) satisfy the following estimate:*

$$||u, v||^2_{H^{\frac{3}{2}}_{loc}} \lesssim ||f, \sqrt{\varepsilon} g||^2_{L^2} + ||\Big\{ \frac{a_L}{\sqrt{\varepsilon}}, b_L \Big\} y^{2m}||^2_{L^2(x=L)}. \tag{4.10.22}$$

To summarize, we have established that:

Corollary 4.10.6. *For $m, \alpha > 0$, the map $S^{-1}_{\alpha,m} : [L^2]^{\times 2} \times [L^2(x = L)]^{\times 2} \to \Big[H^2_m \cap H^{\frac{3}{2}}_{loc}\Big]^2$ is well defined, and returns a solution to the system (4.10.1) which satisfies the boundary conditions specified by the third and fourth inputs of $S_{\alpha,m}$.*

We now define:

$$\begin{aligned} T[u, v] =& u_s u_x + u_{sx} u + v_s u_y + u_{sy} v \\ & u_s v_x + v_{sx} u + v_s v_y + v_{sy} v. \end{aligned} \tag{4.10.23}$$

We will study:

$$\begin{aligned} & S_{\alpha,m}[u, v] + T[u, v] = (f, g) \Rightarrow \\ & [u, v] + S^{-1}_{\alpha,m}\Big[T[u, v], a_L, b_L\Big] = S^{-1}_{\alpha,m}\Big[f, g, a_L, b_L\Big] \end{aligned} \tag{4.10.24}$$

as an equality in $H^1 \times H^1$.

Lemma 4.10.7. *For $m > 0$, we have the following compact embedding:*

$$H^2_m \cap H^{\frac{3}{2}}_{loc} \subset\subset H^1. \tag{4.10.25}$$

Proof. The proof follows from a standard argument, see for instance (Iye17a, P. 145, Lemma 13.1). □

As a direct consequence, $S_{\alpha,m}^{-1}T$ is a compact operator on H^1. An application of the Fredholm Alternative shows that to produce an H^1 solution of (4.10.24), we must rule out nontrivial solutions to the homogeneous problem, which occurs when $f = g = a_L = b_L = 0$. For this purpose, we give a-priori estimates of the problem (4.10.24), under the hypothesis that $[u, v] \in H^1$. For such functions, we automatically know that $[u, v] \in H^2_m$ due to (4.10.18).

Lemma 4.10.8 (Energy Estimates)**.** *Solutions $[u, v] \in H^2_m$ to the system (4.10.1) satisfy the following energy estimate:*

$$\begin{aligned} &||u_y, \sqrt{\varepsilon} u_x, \varepsilon v_x||^2_{L^2} + \alpha ||\{u_y, \sqrt{\varepsilon} v_y, \varepsilon v_x\} \cdot \chi y^m||^2_{L^2} \\ &\quad \lesssim \mathcal{O}(L) ||u_x, \sqrt{\varepsilon} v_x||^2_{L^2} + \mathcal{R}_1 + ||\Big\{\frac{a_L}{\sqrt{\varepsilon}}, b_L\Big\} y^{2m}||^2_{L^2(x=L)}. \end{aligned} \tag{4.10.26}$$

Proof. This follows upon testing the system (4.10.24) against $[\varphi^{(n)}, \phi^{(n)}]$, where the sequence $[\varphi^{(n)}, \phi^{(n)}] \xrightarrow{H^1_m} [u, v]$, and repeating the energy estimate in Proposition 4.3.1.

□

Lemma 4.10.9 (Positivity Estimates)**.** *Let $m = 1$. Then solutions $[u, v] \in H^2_m$ to the system (4.10.1) satisfy the following estimate:*

$$\begin{aligned} ||v_y, \sqrt{\varepsilon} v_x||^2_{L^2} + ||\sqrt{\varepsilon} u_x||^2_{L^2(x=L)} \lesssim ||u_y||^2_{L^2} &+ \mathcal{O}(v_e^0) ||u_y \cdot y, \sqrt{\varepsilon} v_y y||^2_{L^2} \\ &+ \alpha ||\{u_y, \sqrt{\varepsilon} v_y, \varepsilon v_x\} \cdot \chi y^m||^2_{L^2} + \mathcal{R}_2 \\ &+ ||\Big\{\frac{a_L}{\sqrt{\varepsilon}}, b_L, \partial_y b_L\Big\} y^{2m}||^2_{L^2(x=L)}. \end{aligned} \tag{4.10.27}$$

Proof. We must perform estimates on the new, weighted quantities appearing from (4.10.1). Temporarily omitting the prefactor of 10, we have:

$$\begin{aligned} +\alpha \int \partial_y \{u_y y^{2m}\} \cdot \partial_y \frac{v}{u_s} &= -\alpha \int u_y y^{2m} \partial_y \Big[\frac{v_y}{u_s} - v \frac{u_{sy}}{u_s^2}\Big] \\ &= -\alpha \int u_y y^{2m} \frac{v_{yy}}{u_s} + \alpha \int u_y y^{2m} v_y \frac{u_{sy}}{u_s^2} \end{aligned} \tag{4.10.28}$$

$$
\begin{aligned}
&- \alpha \int u_y y^{2m} v \partial_y \{ \frac{u_{sy}}{u_s^2} \} \\
&= +\alpha \int u_y y^{2m} \frac{u_{xy}}{u_s} + \alpha \int u_y y^{2m} v_y \frac{u_{sy}}{u_s^2} \\
&- \alpha \int u_y y^{2m} v \partial_y \{ \frac{u_{sy}}{u_s^2} \} \\
&= -\alpha \int u_y^2 y^{2m} \partial_x \{ \frac{1}{u_s} \} + \int_{x=L} \alpha u_y^2 y^{2m} \frac{1}{2u_s} \\
&+ \alpha \int u_y y^{2m} v_y \frac{u_{sy}}{u_s^2} - \alpha \int u_y y^{2m} v \partial_y \{ \frac{u_{sy}}{u_s^2} \}.
\end{aligned} \tag{4.10.29}
$$

The boundary contribution above is positive, whereas the other terms can all be estimated by the α term in (4.10.27). We need to justify the integration by parts leading to the equality in (4.10.28). For this we notice that our solution is in H^2_m, and so both the left and right-hand sides of (4.10.28) are in L^1. This then justifies the following limit:

$$
\begin{aligned}
\int \partial_y \{ u_y y^{2m} \} \cdot \partial_y \frac{v}{u_s} &= \lim_{M \to \infty} \int_{y=0}^{M} \partial_y \{ u_y y^{2m} \} \cdot \partial_y \frac{v}{u_s} \\
&= \lim_{M \to \infty} \Big[- \int_0^M u_y y^{2m} \partial_{yy} \frac{v}{u_s} + \int_{y=M} u_y y^{2m} \partial_y \frac{v}{u_s} \Big] \\
&= - \int u_y y^{2m} \partial_{yy} \frac{v}{u_s},
\end{aligned} \tag{4.10.30}
$$

where the limit of the boundary contribution vanishes as $||u_y y^m||^2_{L^2_x}$ and $||v_y y^m||_{L^2_x}$ are H^1_y functions, according to the definition of H^2_m. We similarly have:

$$
\begin{aligned}
&\int -2\alpha \partial_y \{ \chi_1 y^{2m} v_y \} \cdot \varepsilon \partial_x \{ \frac{v}{u_s} \} \\
&\quad = \int 2\alpha \chi y^{2m} \varepsilon v_y \Big(\frac{v_{xy}}{u_s} + v_y \partial_x \frac{1}{u_s} + v_x \partial_y \frac{1}{u_s} + v \partial_{xy} \frac{1}{u_s} \Big).
\end{aligned} \tag{4.10.31}
$$

Finally:

$$
- \int \alpha \varepsilon \partial_x \{ \chi y^{2m} \{ u_y + \varepsilon v_x \} \} \cdot \partial_x \{ \frac{v}{u_s} \}
$$

$$= -\int \alpha\varepsilon\chi y^{2m} u_{xy}\partial_x\{\frac{v}{u_s}\} - \int \alpha\varepsilon^2 v_{xx}v_x\chi y^{2m}\frac{1}{u_s}$$
$$- \int \varepsilon^2\alpha v_{xx}v\partial_x\frac{1}{u_s}\chi y^{2m}. \tag{4.10.32}$$

The latter two terms in (4.10.32) are estimated according to standard calculations. For the first term:

$$-\int \alpha\varepsilon\chi y^{2m}u_{xy}\partial_x\{\frac{v}{u_s}\} = \int \alpha\varepsilon u_x\partial_y\{\chi y^{2m}\partial_x\{\frac{v}{u_s}\}\}$$
$$= \int \varepsilon\alpha u_x\chi y^{2m}\Big[\frac{v_{xy}}{u_s} + v_x\partial_y\frac{1}{u_s} + v_y\partial_x\{\frac{1}{u_s}\} + v\partial_{xy}\frac{1}{u_s}\Big]$$
$$+ \int \varepsilon\alpha u_x\partial_x\frac{v}{u_s}\partial_y\{\chi y^{2m}\}. \tag{4.10.33}$$

For the final term above, we must use that $m = 1$:

$$\int \varepsilon\alpha u_x\partial_x\frac{v}{u_s}\partial_y\{\chi y^{2m}\} \le \alpha||\sqrt{\varepsilon}u_x y||_{L^2}||\sqrt{\varepsilon}v_x||_{L^2}. \tag{4.10.34}$$

The remaining terms can all be estimated similarly to estimate (4.4.5).

$\square$

We first recall the definition of $\mathcal{R}_1$ given in (4.3.2).

Lemma 4.10.10 (Weighted Estimate)**.** *Solutions $[u, v] \in H^2_m$ to the system (4.10.1) satisfy the following estimate:*

$$||\Big\{u_{yy}, \sqrt{\varepsilon}u_{xy}, \varepsilon u_{xx}\Big\}\cdot y||^2_{L^2} + ||\{u_y, \sqrt{\varepsilon}u_x\}y||^2_{L^2} + ||\{u_y, \sqrt{\varepsilon}u_x\}y||^2_{L^2(x=L)}$$
$$+ \alpha||\Big\{u_{yy}, \sqrt{\varepsilon}u_{xy}, \varepsilon u_{xx}\Big\}\cdot y^{m+1}||^2_{L^2} \lesssim \alpha||\{u_y, \sqrt{\varepsilon}v_y, \varepsilon v_x\}\cdot\chi y^m||^2_{L^2}$$
$$+ ||v_y, \sqrt{\varepsilon}v_x||^2_{L^2} + ||u_y||^2_{L^2} + ||\{a_L, \partial_y a_L, b_L, \partial_y b_L\}\langle y\rangle^2||^2_{L^2(x=L)}$$
$$+ ||\sqrt{\varepsilon}u_x||^2_{L^2(x=L)} + \mathcal{R}_1. \tag{4.10.35}$$

Proof. For this step, we can apply a cut-off $\chi_N(y) = \chi(\frac{y}{N})$, and take $N \to \infty$. Due to the cut-off, there is no need to justify contributions from $y = \infty$. Consider the new term:

$$\begin{aligned} -\alpha \int \partial_{yy}\{y^{2m}\chi(y)u_y\} \cdot \partial_y\{uy^2 w(x)\}\chi_N(y) \\ = +\alpha \int \partial_y\{y^{2m}\chi(y)u_y\} \cdot \partial_y\{uy^2 w\}\frac{1}{N}\chi_N'(y) \\ = +\alpha \int \chi\chi_N y^{2m+2}u_{yy}^2 + \alpha\mathcal{O}(||u_y||^2_{H^1_m}). \end{aligned} \tag{4.10.36}$$

Analogous calculations can be performed for the remaining $\alpha-$ terms from (4.10.1). For the remaining terms from (4.10.24), one can repeat the proof of Proposition 4.5.1 with the additional cutoff term $\chi_N(y)$. We omit repeating those details.

□

Putting the above estimates, (4.10.26), (4.10.27), (4.10.35) together gives the following uniform in α estimate:

$$\begin{aligned} ||u,v||^2_{\mathcal{X}} + \alpha||\Big\{u_{yy}, \sqrt{\varepsilon}u_{xy}, \varepsilon u_{xx}\Big\} \cdot y^{m+1}||^2_{L^2} + \alpha||\{u_y, \sqrt{\varepsilon}v_y, \varepsilon v_x\} \cdot \chi y^m||^2_{L^2} \\ \lesssim \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + C(a_L, b_L). \end{aligned} \tag{4.10.37}$$

Taking the forcing $f = g = a_L = b_L = 0$ (thus $\mathcal{R}_i = 0$), we can apply the Fredholm Alternative to conclude that there exists an H^1 solution $[u, v]$ to the problem (4.10.24). Such a solution is automatically H^2 by (4.10.18), and so is a strong solution. The final task is to establish a solution to our original system (4.9.105) - (4.9.107), which can be achieved as a weak limit in $\mathcal{X}$ as $\alpha \to 0$ using the uniform in α estimate (4.10.37). This then proves Proposition 4.10.1. Proposition 4.10.2 then follows upon applying the Contraction Mapping Theorem when coupled with the main $\mathcal{X}$-estimate in Theorem 4.8.3.

Chapter Five

Stationary Inviscid Limit to Shear Flows

5.1 Abstract

This is based on a joint work with Chunhui Zhou (see (IZ17)). We establish a density result for certain stationary shear flows, $\mu(y)$, that vanish at the boundaries of a horizontal channel. We construct stationary solutions to 2D Navier-Stokes that are ε-close in L^∞ to the given shear flow. Our construction is based on a coercivity estimate for the Rayleigh operator, $R[v]$, which is based on a decomposition made possible by the vanishing of μ at the boundaries.

5.2 Introduction

We are considering 2D, stationary flows on the strip:

$$\Omega = (0, L) \times (0, 2). \tag{5.2.1}$$

We consider an Euler shear flow:

$$\mathbf{u}^0 = (\mu(y), 0). \tag{5.2.2}$$

Let $\mathbf{u}^\varepsilon$ solve the Navier-Stokes equations:

$$\left.\begin{aligned} &\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla P^\varepsilon = \varepsilon \Delta \mathbf{u}^\varepsilon \\ &\nabla \cdot \mathbf{u}^\varepsilon = 0 \\ &\mathbf{u}^\varepsilon|_{y=0} = 0, \mathbf{u}^\varepsilon|_{y=2} = u_b \end{aligned}\right\}. \tag{5.2.3}$$

Here $u_b \geq 0$ denotes the velocity of the boundary at $\{y = 2\}$. Our main result, Theorem

5.2.1 treats the non-moving case of $u_b = 0$. We are interested in the asymptotic behavior of $\mathbf{u}^\varepsilon$ as $\varepsilon \to 0$. In the presence of boundaries, the vanishing viscosity asymptotics are a major open problem in fluids made challenging due to the mismatch between the no-slip condition $\mathbf{u}^\varepsilon|_{\partial\Omega} = 0$ and the no penetration condition typically satisfied by Euler flows: $\mathbf{u}^0 \cdot \mathbf{n} = 0$. This mismatch is typically rectified by the presence of Prandtl's boundary layer (see (GN17), (Iye17a), (Iye16), (Iye17b) for relevant results in the 2D stationary setting). In this article, we will consider Euler flows that themselves satisfy no-slip:

$$\mu(0) = 0, \tag{5.2.4}$$

for which there is no leading order boundary layer. Denote now the asymptotic expansion:

$$\mathbf{u}^\varepsilon := \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix} = \begin{pmatrix} \mu + \varepsilon u_e^1 + \varepsilon u_p^1 + \varepsilon^{\frac{3}{2}} u_e^2 + \varepsilon^{\frac{3}{2}} u_p^2 + \varepsilon^{\frac{3}{2}+\gamma} u \\ \varepsilon v_e^1 + \varepsilon^{\frac{3}{2}} v_p^1 + \varepsilon^{\frac{3}{2}} v_e^2 + \varepsilon^2 v_p^2 + \varepsilon^{\frac{3}{2}+\gamma} v \end{pmatrix}. \tag{5.2.5}$$

We denote:

$$u_s := \mu + \varepsilon u_e^1 + \varepsilon u_p^1 + \varepsilon^{\frac{3}{2}} u_e^2 + \varepsilon^{\frac{3}{2}} u_p^2, \tag{5.2.6}$$

$$v_s := \varepsilon v_e^1 + \varepsilon^{\frac{3}{2}} v_p^1 + \varepsilon^{\frac{3}{2}} v_e^2 + \varepsilon^2 v_p^2. \tag{5.2.7}$$

We impose the boundary conditions:

$$[u, v]|_{x=0} = [u, v]|_{y=0} = [u, v]|_{y=2} = 0, \tag{5.2.8}$$

$$\partial_y u + \partial_x v = 0, \quad P = 2\varepsilon \partial_x u. \tag{5.2.9}$$

The system satisfied by $[u, v]$ is:

$$\left.\begin{aligned} & -\varepsilon\Delta u + S_u + \partial_x P = f := \mathcal{N}_1(u,v) + \mathcal{F}_u \\ & -\varepsilon\Delta v + S_v + \partial_y P = g := \mathcal{N}_2(u,v) + \mathcal{F}_v \\ & \partial_x u + \partial_y v = 0 \end{aligned}\right\} \text{ in } \Omega. \tag{5.2.10}$$

We have defined:

$$S_u := u_s u_x + u_{sx} u + u_{sy} v + v_s u_y, \quad S_v := u_s v_x + v_{sx} u + v_s v_y + v v_{sy}, \tag{5.2.11}$$

$$\mathcal{N}_1 := -\varepsilon^{\frac{3}{2}+\gamma}\Big(u\partial_x u + v\partial_y u\Big), \quad \mathcal{N}_2 := -\varepsilon^{\frac{3}{2}+\gamma}\Big(u\partial_x v + v\partial_y v\Big), \tag{5.2.12}$$

and $\mathcal{F}_u, \mathcal{F}_v$ are defined in (5.5.55). Let us now define several norms in which we will control the solution:

$$||u,v||_{\mathcal{E}} := ||\sqrt{\varepsilon}\nabla u||_{L^2} + ||\sqrt{\varepsilon}\nabla v||_{L^2}, \tag{5.2.13}$$

$$||u,v||_{\mathcal{P}} := ||\sqrt{u_s}\nabla v||_{L^2}, \tag{5.2.14}$$

$$||u,v||_X := ||u,v||_{\mathcal{E}} + \varepsilon^{\frac{\gamma}{2}}||\sqrt{\varepsilon}\{u,v\}||_\infty \tag{5.2.15}$$

We introduce here the notation:

$$\tilde{y} = y \cdot (2-y). \tag{5.2.16}$$

The main theorems we prove are the following:

Theorem 5.2.1. *Let $u_b \geq 0$ in (5.2.3). Let $\mu(y) \in C^\infty([0,2])$ be a given function, satisfying the conditions:*

$$\mu(0) = 0, \mu(2) = u_b, \tag{5.2.17}$$

$$\partial_y^j \mu(0) = \partial_y^j \mu(2) = 0 \text{ for } 2 \leq j \leq N_0, \tag{5.2.18}$$

$$\partial_y \mu(0) > 0, |\partial_y \mu(2)| > 0. \tag{5.2.19}$$

where $N_0 < \infty$ and large but unspecified.[1] *Let also standard compatibility conditions at the corners of Ω be prescribed for the layers in u_s.*[2] *Then there exists a unique solution, $\mathbf{u}^\varepsilon$ satisfying the Navier-Stokes equations, (5.2.3), such that:*

$$||u^\varepsilon - \mu||_\infty + ||v^\varepsilon||_\infty \leq c_0(\mu)\varepsilon. \tag{5.2.20}$$

The constant $c_0(\mu)$ satisfies:

$$c_0(\mu) \lesssim ||\frac{\mu'''}{\mu}||_{W^{100,\infty}}. \tag{5.2.21}$$

Our ultimate interest is motivated by Yudovich's ninth problem, (Yud03). Classical experiments starting with Reynolds have shown that unsteady flows in a 2D channel that start near Couette or Poiseulle flow do not converge to these flows. This indicates the existence of infinitely many stationary solutions to Navier-Stokes "near" Couette or Poiseulle. Establishing the existence of these solutions is an open problem. Our second result, Corollary 5.2.2, produces stationary solutions sufficiently close to Couette, assuming $x \in [0, L], L << 1$, and a moving boundary at $y = 2$.

Corollary 5.2.2. *Let any $\alpha > 0$ be prescribed, which could depend on ε. Let $\tilde{\mu}$ be prescribed to satisfy the vanishing conditions: $\partial_y^k \tilde{\mu}|_{y=0} = \partial_y^k \tilde{\mu}|_{y=2} = 0$ for $0 \leq k \leq N_0$. There exists a unique solution, $\mathbf{u}^\varepsilon$ to (5.2.3) with $u_b = 2$ such that:*

$$||u^\varepsilon - \Big(y + \alpha\tilde{\mu}(y)\Big)||_\infty + ||v^\varepsilon||_\infty \lesssim \alpha\varepsilon. \tag{5.2.22}$$

[1] We have selected not to optimize N_0. The optimal N_0 is likely between 4 and 10.

[2] We omit stating the precise form of these compatibility conditions here. They can be found in (5.5.34), (5.5.39).

Proof. One can obtain this by applying Theorem 5.2.1 with $\mu(y) = y + \alpha\tilde{\mu}(y)$, where $\tilde{\mu}$ vanishes at high order near $y = 0, 2$. In this case, the constant $c_0(\mu) \lesssim \alpha$. $\square$

Remark. The requirement of $u_b = 2$ is so that the no-slip condition is satisfied by the Couette flow. We do not use this motion of the boundary anywhere in the proof.

The present article is structured as follows: the construction of the approximate layers, u_s, v_s, in the expansion (5.2.5) is performed in the Appendix. The main analysis in Sections 5.3, 5.4 is centered around the system (5.2.10).

5.3 Linear Estimates

We will analyze the system (5.2.10). The reader is urged to consult Lemma 5.5.3 for relevant properties of the linearizations, u_s, and the forcing terms, f, g.

5.3.1 Energy Estimate

Proposition 5.3.1. *For any $\theta > 0$, solutions $[u, v]$ to (5.2.10) satisfy:*

$$||u, v||_{\mathcal{E}}^2 + ||\sqrt{u_s}\{u, v\}||_{L^2(x=L)}^2 \lesssim C(\theta)\varepsilon^{-\theta}||u, v||_{\mathcal{P}}^2 + \mathcal{R}_1, \tag{5.3.1}$$

where:

$$\mathcal{R}_1 := \int f \cdot u + \int \varepsilon g \cdot v. \tag{5.3.2}$$

Proof. Apply $[u, v]$ to (5.2.10). The coercive quantities are:

$$\begin{aligned}
\int -\varepsilon\Delta u \times u - \int \varepsilon\Delta v \times v + \int \nabla P \cdot \mathbf{u} \\
&= \int \varepsilon\Big[-\partial_{yy}u - 2\partial_{xx}u - \partial_{xy}v\Big] \times u + \int \partial_x P u \\
&\quad + \int \varepsilon\Big[-2\partial_{yy}v - \partial_x\{\partial_y u + \partial_x v\}\Big] \times v + \int \partial_y P v \\
&= \int \varepsilon\Big[|\partial_y u^2 + |\partial_x v|^2 + 4|\partial_y v|^2 + 2\partial_x v\partial_y u\Big] \\
&\gtrsim \int \varepsilon\Big[|\nabla u|^2 + |\nabla v|^2\Big].
\end{aligned} \tag{5.3.3}$$

Above, we have used the stress-free boundary condition in (5.2.9). We now have the convection terms:

$$\int [u_s u_x + u_{sx} u + v_s u_y] \cdot u = \int u_{sx} u^2 + \frac{1}{2} \int_{x=L} u_s u^2, \tag{5.3.4}$$

$$\int [u_s v_x + v_s v_y + v_{sy} v] \cdot v = \int v_{sy} v^2 + \frac{1}{2} \int_{x=L} u_s v^2. \tag{5.3.5}$$

We estimate the two bulk terms above using (5.5.59) - (5.5.61):

$$|\int u_{sx} u^2| + |\int v_{sy} v^2| \le |\int \sqrt{\varepsilon} \tilde{y} [u^2 + v^2]| \le \sqrt{\varepsilon} \mathcal{O}(L) ||\sqrt{u_s} \nabla v||_2^2. \tag{5.3.6}$$

We now move to:

$$|\int v_{sx} uv| \le \varepsilon ||\sqrt{\tilde{y}} u_x||_2 ||\sqrt{\tilde{y}} v_x||_2, \tag{5.3.7}$$

again by using (5.5.61). For the $u_{sy} v$ convection term, we first handle the leading order contribution from μ, and we must take care to avoid the critical Hardy inequality:

$$|\int \mu' uv| = |\int \mu' uv \Big[\chi(y \le \frac{1}{10}) + \chi(\frac{1}{10} \le y \le \frac{19}{10}) + \chi(y \ge \frac{19}{10}) \Big]|.$$

For the interior contributions:

$$|\int \mu' uv \chi(\frac{1}{10} \le y \le \frac{19}{10})| \le \mathcal{O}(L) ||\sqrt{u_s} \partial_x u||_2 ||\sqrt{u_s} \partial_x v||_2 \tag{5.3.8}$$

The $y \le \frac{1}{10}$ contribution is exactly analogous to the $y \ge \frac{1}{10}$, and so we treat the former. Let $\tilde{\chi}$ denote a fattened relative to $\chi(y \le \frac{1}{10})$. Fix an $\omega > 0$ small.

$$|\int \mu' uv \chi(y \le \frac{1}{10})| \le ||\mu'||_\infty ||y^{-(\frac{1}{2} - \frac{\omega}{2})} v \tilde{\chi}||_2 ||y^{\frac{1}{2} - \frac{\omega}{2}} u \tilde{\chi}||_2 \tag{5.3.9}$$

We estimate each L^2 term above individually.

$$\begin{aligned}\int y^{-1+\omega} v^2 \tilde{\chi} &= \int \frac{\partial_y}{\omega}\{y^\omega\} v^2 \tilde{\chi} \\ &= -\int \frac{y^\omega}{\omega} 2v\partial_y v \tilde{\chi} - \int \frac{y^\omega}{\omega} v^2 \tilde{\chi}' \\ &\le \frac{1}{\omega}||y^{-(\frac{1}{2}-\frac{\omega}{2})} v\tilde{\chi}||_2 ||\sqrt{u_s}\partial_y v||_2 + \frac{1}{\omega}\mathcal{O}(L)||\sqrt{u_s}\partial_x v||_2^2. \end{aligned} \tag{5.3.10}$$

Next from (5.3.9):

$$\begin{aligned} ||y^{\frac{1}{2}-\frac{\omega}{2}} u\tilde{\chi}||_2 &\lesssim ||y^{\frac{1}{2}} u\tilde{\chi}||_2^{1-\theta(\omega)} ||\frac{u}{y}\tilde{\chi}||_2^{\theta(\omega)} \\ &\lesssim \varepsilon^{-\theta(\omega)} ||\sqrt{u_s}\partial_x u||_2^{1-\theta(\omega)} \Big[||\sqrt{\varepsilon}\partial_y u\tilde{\chi}||_2 + ||\sqrt{u_s}\partial_x u\tilde{\chi}||_2 \Big]^{\theta(\omega)} \end{aligned} \tag{5.3.11}$$

Inserting (5.3.10) and (5.3.11) into (5.3.9), one obtains for small $\kappa > 0$

$$|(5.3.9)| \le \kappa ||u,v||_{\mathcal{E}}^2 + N_\kappa \varepsilon^{0-} ||u,v||_{\mathcal{P}}^2. \tag{5.3.12}$$

For the higher-order contributions, we use the estimate (5.5.60), and subsequently split:

$$|\int [u_{sy} - \mu'] uv| \le \int \sqrt{\varepsilon}|u||v|[\chi_\delta^- + \chi_\delta^c + \chi_\delta^+] = (5.3.13.1) + (5.3.13.2) + (5.3.13.3). \tag{5.3.13}$$

Here, $\chi_\delta^+(y) = \chi_\delta^-(2-y)$ and $\chi_\delta^c = 1 - \chi_\delta^+ - \chi_\delta^-$, where:

$$\chi_\delta^-(y) = \begin{cases} 1 \text{ on } y \le \delta \\ 0 \text{ on } y \ge 2\delta \end{cases} \tag{5.3.14}$$

Terms (5.3.13.1) and (5.3.13.3) are identical. We estimate:

$$|(5.3.13.1)| \lesssim \sqrt{\delta}\mathcal{O}(L)||\sqrt{\varepsilon}\frac{u}{y}\sqrt{\tilde{\chi}_\delta}||_2||\sqrt{u_s}u_x||_2, \tag{5.3.15}$$

$$|(5.3.13.2)| \lesssim \frac{\sqrt{\varepsilon}}{\delta}||\sqrt{u_s}\nabla v||_2^2. \tag{5.3.16}$$

We now estimate:

$$\begin{aligned}||\sqrt{\varepsilon}\frac{u}{y}\sqrt{\tilde{\chi}_\delta}||_2^2 &= \int \varepsilon\frac{\partial_y}{-1}\{y^{-1}\}u^2\tilde{\chi}_\delta \\ &= \int \varepsilon y^{-1}2u\partial_y u\tilde{\chi}_\delta + \int \varepsilon y^{-1}u^2\frac{\tilde{\chi}'_\delta}{\delta} \\ &\leq ||\sqrt{\varepsilon}\frac{u}{y}\sqrt{\tilde{\chi}_\delta}||_2||\sqrt{\varepsilon}\partial_y u||_2 + \frac{\varepsilon}{\delta^3}||\sqrt{u_s}\partial_x u||_2^2. \end{aligned} \tag{5.3.17}$$

We may thus take $\delta = \varepsilon^{\frac{1}{4}}$ and insert (5.3.17) into (5.3.15) to conclude.

□

5.3.2 Positivity Estimate

Proposition 5.3.2. *Solutions* $[u,v]$ *to (5.2.10) satisfy, for any* $\kappa > 0$*:*

$$||u,v||_{\mathcal{P}}^2 + ||\sqrt{\varepsilon}\partial_x v||_{L^2(x=0)}^2 + ||\sqrt{\varepsilon}\partial_x u||_{L^2(x=L)}^2 \lesssim \varepsilon^{1-\kappa}||u,v||_{\mathcal{E}}^2 + \mathcal{R}_2, \tag{5.3.18}$$

where:

$$\mathcal{R}_2 := \int f\cdot -\partial_y v + \int g\cdot\partial_x v. \tag{5.3.19}$$

Proof. We will apply the multiplier $\mathbf{M} := (-\partial_y v, \partial_x v)$ to the system (5.2.10). This gives:

$$\begin{aligned}
\int \Big(- u_s \partial_y v + v \partial_y u_s \Big) \cdot - \partial_y v + \int u_s \partial_x v \cdot \partial_x v \\
= \int u_s \Big[|\partial_y v|^2 + |\partial_x v|^2 \Big] + \int \frac{u_{syy}}{2} v^2 \\
\geq \int u_s |\nabla v|^2 - || \frac{u_{syy}}{2} ||_\infty \int v^2 \\
\gtrsim \int u_s |\nabla v|^2.
\end{aligned} \tag{5.3.20}$$

We have used the splitting:

$$\begin{aligned}
\int v^2 &= \int v^2 [\chi_\delta + \chi_\delta^c] \\
&\leq \frac{L^2}{\delta} \int |\partial_x v|^2 + | \int \partial_y \{y\} v^2 \chi_\delta | \\
&\leq \frac{L^2}{\delta} \int u_s |\partial_x v|^2 + | \int y 2 v \partial_y v \chi_\delta | + | \int y v^2 \frac{\chi_\delta'}{\delta} | \\
&\lesssim \frac{L^2}{\delta} \int u_s |\partial_x v|^2 + \sqrt{\delta} || \sqrt{u_s} \partial_y v ||_2 \times \sqrt{\text{LHS}} \\
&\overset{\delta = L}{=} L \int u_s |\partial_x v|^2 + \sqrt{L} || \sqrt{u_s} \partial_y v ||_2 \times \sqrt{\text{LHS}}.
\end{aligned} \tag{5.3.21}$$

For the vorticity terms, repeated integration by parts gives:

$$\begin{aligned}
\int -\varepsilon \Delta u \cdot \partial_y v - \int \varepsilon \Delta v \cdot \partial_x v + \int \nabla P \cdot \mathbf{M} \\
= \frac{\varepsilon}{2} \int_{x=0} |\partial_x v|^2 + \frac{\varepsilon}{2} \Big[\int_{x=L} \Big(|\partial_y u|^2 - |\partial_x u|^2 + |\partial_y v|^2 \\
- |\partial_x v|^2 \Big) \Big] + \int_{x=L} P \partial_x u \\
= \frac{\varepsilon}{2} \int_{x=0} |\partial_x v|^2 + \int_{x=L} 2\varepsilon |\partial_x u|^2,
\end{aligned} \tag{5.3.22}$$

where we have used the Stress-Free boundary condition from (5.2.9). We now come to the remaining linearized terms from (5.2.10):

$$|\int \Big(u\partial_x u_s + v_s\partial_y u\Big)\cdot -\partial_y v| + |\int \Big(u\partial_x v_s + v_s\partial_y v + v\partial_y v_s\Big)\cdot \partial_x v|$$
$$\lesssim \varepsilon^{\kappa} \times \text{LHS of (5.3.18)} + \varepsilon^{1-\kappa}||u,v||_{\mathcal{E}}^2,$$

where we have used the Poincare inequality and the estimates in (5.5.59) - (5.5.61). Finally, the right-hand side of (5.3.18) follows from the definition of $\mathcal{R}_2$.

□

Lemma 5.3.3. *For any* $\theta > 0$,

$$\varepsilon^{\theta}||\sqrt{\varepsilon}u, \sqrt{\varepsilon}v||_{\infty} \le C_{\theta}\Big[||u,v||_{\mathcal{E}} + ||f,g||_2\Big]. \tag{5.3.23}$$

Proof. We omit the proof, this is found in (GN17) using interpolation arguments and estimates for the Stokes operator on domains with corners. □

As a direct corollary to (5.3.1), (5.3.18), and taking $\theta = \frac{\gamma}{4}$ in (5.3.23):

Corollary 5.3.4.

$$||u,v||_X^2 \lesssim \mathcal{R}_1 + \mathcal{R}_2 + \varepsilon^{\frac{\gamma}{2}}||f,g||_2^2. \tag{5.3.24}$$

5.4 Evaluation of Right-Hand Sides

We first provide the nonlinear estimates:

Lemma 5.4.1. *With $\mathcal{N}_1, \mathcal{N}_2$ defined as in (5.2.12):*

$$
\begin{aligned}
&|\int \varepsilon^{\frac{3}{2}+\gamma}\mathcal{N}_1 \cdot [u+\partial_x u]| + |\int \varepsilon^{\frac{3}{2}+\gamma}\mathcal{N}_2 \cdot [v+\partial_x v]| \\
&\qquad + \varepsilon^{\frac{\gamma}{4}}||\mathcal{N}_1, \mathcal{N}_2||_2 \le \varepsilon^{\frac{\gamma}{2}}\Big[||u,v||_X^3 + ||u,v||_X^2\Big].
\end{aligned} \tag{5.4.1}
$$

Proof. We compute directly:

$$
\begin{aligned}
&|\int \varepsilon^{\frac{3}{2}+\gamma}[u\partial_x u + v\partial_y u] \cdot [u+\partial_x u]| \\
&\qquad \le \varepsilon^{\frac{\gamma}{2}}||\sqrt{\varepsilon}\varepsilon^{\frac{\gamma}{2}}\{u,v\}||_\infty ||\sqrt{\varepsilon}\nabla\{u,v\}||_2^2 \lesssim \varepsilon^{\frac{\gamma}{2}}||u,v||_X^3.
\end{aligned} \tag{5.4.2}
$$

Similarly:

$$
\begin{aligned}
&|\int \varepsilon^{\frac{3}{2}+\gamma}[u\partial_x v + v\partial_y v] \cdot [v+\partial_x v]| \\
&\qquad \le \varepsilon^{\frac{\gamma}{2}}||\sqrt{\varepsilon}\varepsilon^{\frac{\gamma}{2}}\{u,v\}||_\infty ||\sqrt{\varepsilon}\nabla\{u,v\}||_2^2 \lesssim \varepsilon^{\frac{\gamma}{2}}||u,v||_X^3.
\end{aligned} \tag{5.4.3}
$$

Finally:

$$
||\mathcal{N}_1, \mathcal{N}_2||_2 \le \varepsilon^{\frac{3}{2}+\gamma}\Big[||u\{\partial_x u, \partial_x v\}||_2 + ||v\{\partial_y u, \partial_y v\}||_2\Big]. \tag{5.4.4}
$$

□

Lemma 5.4.2. *With $\mathcal{F}_u, \mathcal{F}_v$ defined as in (5.5.55), for any $\delta > 0$:*

$$
\Big|\int \varepsilon^{-\frac{3}{2}-\gamma}\mathcal{F}_u \cdot [u+\partial_x u] + \int \varepsilon^{-\frac{3}{2}-\gamma}\mathcal{F}_v \cdot [v+\partial_x v]\Big|
$$

$$\leq \delta ||u,v||_X^2 + C(u_s, v_s) c_0 (\frac{\mu'''}{\mu}),$$

$$\varepsilon^{\frac{\gamma}{4}} ||\varepsilon^{-\frac{3}{2}-\gamma} \{\mathcal{F}_u, \mathcal{F}_v\}||_2 \lesssim \varepsilon^{\frac{1}{2}-\frac{3\gamma}{4}} c_0 (\frac{\mu'''}{\mu}). \tag{5.4.5}$$

Proof. First recall the decomposition of $\mathcal{F}_u, \mathcal{F}_v$ given in (5.5.56). We first estimate:

$$\int \varepsilon^{\frac{1}{2}-\gamma} T_1 \cdot u = \int \varepsilon^{\frac{1}{2}-\gamma} T_1 \cdot u [\chi_\delta + \chi_\delta^c]. \tag{5.4.6}$$

For the nonlocal part, we use estimate (5.5.62):

$$|\int \varepsilon^{\frac{1}{2}-\gamma} T_1 u \chi_\delta^c| \leq \delta^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}-\gamma} ||T_1||_2 ||\sqrt{u_s} \partial_x u||_2. \tag{5.4.7}$$

For the local component, we integrate by parts in y:

$$\begin{aligned}
|\int \varepsilon^{\frac{1}{2}-\gamma} T_1 u \chi_\delta(y)| &= \varepsilon^{\frac{1}{2}-\gamma} |\int y \partial_y u T_1 \chi_\delta + y u \partial_y T_1 \chi_\delta + y u T_1 \frac{\chi_\delta'}{\delta}| \\
&\leq \delta \varepsilon^{-\gamma} ||\sqrt{\varepsilon} \partial_y u||_2 ||T_1||_2 + \varepsilon^{\frac{1}{2}-\gamma} \sqrt{\delta} ||\partial_y T_1||_2 ||\sqrt{u_s} \partial_x u||_2 \\
&\quad + \delta^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}-\gamma} ||T_1||_2 ||\sqrt{u_s} \partial_x u||_2.
\end{aligned} \tag{5.4.8}$$

The same estimates can be used for $\varepsilon^{\frac{1}{2}-\gamma} T_2 \cdot v$. We now come to the higher order terms, in which the non-local contributions are estimated via:

$$|\int \varepsilon^{\frac{1}{2}-\gamma} T_1 \cdot \partial_x u \chi_\delta^c + \int \varepsilon^{\frac{1}{2}-\gamma} T_2 \cdot \partial_x v \chi_\delta^c| \leq \delta^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}-\gamma} ||T_1||_2 ||\sqrt{u_s} \nabla v||_2. \tag{5.4.9}$$

We now focus on the T_1 localized contributions individually. First:

$$\varepsilon^{\frac{1}{2}-\gamma} |\int \mu' v_p^2 \partial_y v \chi_\delta| \leq \varepsilon^{-\gamma} ||\mu', \frac{v_p^2}{Y}||_\infty \sqrt{\delta} ||\sqrt{u_s} \partial_y v||_2, \tag{5.4.10}$$

$$\varepsilon^{\frac{1}{2}-\gamma} |\int \partial_Y u_p^2 v_e^1 \partial_y v \chi_\delta| \leq \varepsilon^{\frac{1}{2}-\gamma} ||\partial_Y u_p^2||_\infty || \frac{v_e^1}{y} ||_2 ||\sqrt{u_s} \partial_y v||_2, \tag{5.4.11}$$

$$\varepsilon^{\frac{1}{2}-\gamma}|\int \Big[\mu\chi' v_p^{2,0} + 3\chi'\partial_Y u_p^{2,0}\Big]\partial_y v| \le \varepsilon^{\frac{1}{2}-\gamma}||\mu\chi' v_p^{2,0} + 3\chi'\partial_Y u_p^{2,0}||_\infty||\sqrt{u_s}\partial_y v||_2. \quad (5.4.12)$$

The remaining terms in T_1 are handled by integrating by parts in y and proceeding as in (5.4.8):

$$\begin{aligned}
&\varepsilon^{\frac{1}{2}-\gamma}\int \Big[u_e^1\partial_x u_e^1 + v_e^1\partial_y u_e^1 - \Delta u_e^1\Big]\cdot \partial_y v\chi_\delta \\
&\qquad = -\int \varepsilon^{\frac{1}{2}-\gamma}\underbrace{\partial_y\Big[u_e^1\partial_x u_e^1 + v_e^1\partial_y u_e^1 - \Delta u_e^1\Big]}_{m_1}\cdot v\chi_\delta \\
&\qquad\quad - \int \varepsilon^{\frac{1}{2}-\gamma}\Big[u_e^1\partial_x u_e^1 + v_e^1\partial_y u_e^1 - \Delta u_e^1\Big]\cdot v\frac{\chi_\delta'}{\delta} \\
&\qquad = (5.4.13.1) + (5.4.13.2).
\end{aligned} \quad (5.4.13)$$

First:

$$|(5.4.13.1)| = |\int \varepsilon^{\frac{1}{2}-\gamma}m_1 v| \le \varepsilon^{\frac{1}{2}-\gamma}||m_1||_2||v||_2 \le \varepsilon^{\frac{1}{2}-\gamma}||m_1||_2||\sqrt{u_s}\nabla v||_2. \quad (5.4.14)$$

Second:

$$|(5.4.13.2)| \le \varepsilon^{\frac{1}{2}-\gamma}\delta^{-\frac{3}{2}}||u_e^1\partial_x u_e^1 + v_e^1\partial_y u_e^1 - \Delta u_e^1||_2||\sqrt{u_s}\partial_x v||_2 \quad (5.4.15)$$

We now consider the localized contributions from T_2, for which we apply estimate (5.5.62):

$$\varepsilon^{\frac{1}{2}-\gamma}|\int T_2\cdot\partial_x v| \le ||\frac{T_2}{\tilde{y}}||_2||\sqrt{u_s}\partial_x v||_2. \quad (5.4.16)$$

We now make the selection of $\delta = \varepsilon^{10\gamma}$, and $\gamma << 1$ sufficiently small, which closes all of the above estimates. Finally, the $\mathcal{O}(\varepsilon^{\frac{5}{2}})$ are handled easily via:

$$\begin{aligned}
\varepsilon^{-\frac{3}{2}-\gamma}| \int [\mathcal{F}_u - T_1] \cdot [u + \partial_x u] + \int [\mathcal{F}_v - T_2] \cdot [v + \partial_x v]| \\
\lesssim \int \varepsilon^{1-\gamma} \cdot [u + \partial_x u + v + \partial_x v]| \\
\lesssim \varepsilon^{\frac{1}{2}-\gamma} ||\sqrt{\varepsilon} \nabla v||_2.
\end{aligned} \tag{5.4.17}$$

□

We now obtain our complete nonlinear estimate:

Corollary 5.4.3. *Solutions* $[u, v]$ *to the system (5.2.10) satisfy:*

$$||u, v||_X^2 \lesssim C(u_s, v_s) c_0(\frac{\mu'''}{\mu}) + \varepsilon^{\frac{\gamma}{2}} ||u, v||_X^3. \tag{5.4.18}$$

From here, the main result, Theorem 5.2.1 follows from a straightforward application of the contraction mapping theorem.

5.5 Construction of Layers

We start with the asymptotic expansions:

$$u^\varepsilon := \mu + \varepsilon u_e^1 + \varepsilon u_p^1 + \varepsilon^{\frac{3}{2}} u_e^2 + \varepsilon^{\frac{3}{2}} u_p^2 + \varepsilon^{\frac{3}{2}+\gamma} u, \tag{5.5.1}$$

$$v^\varepsilon := \varepsilon v_e^1 + \varepsilon^{\frac{3}{2}} v_p^1 + \varepsilon^{\frac{3}{2}} v_e^2 + \varepsilon^2 v_p^2 + \varepsilon^{\frac{3}{2}+\gamma} v, \tag{5.5.2}$$

$$P^\varepsilon := \varepsilon P_e^1 + \varepsilon^{\frac{3}{2}} P_e^2 + \varepsilon[P_p^1 + \varepsilon P_p^{1,a}] + \varepsilon^{\frac{3}{2}} P_p^2 + \varepsilon^{\frac{3}{2}+\gamma} P \tag{5.5.3}$$

5.5.1 Formal Asymptotic Expansion

Here the Eulerian profiles are functions of (x, y), whereas the boundary layer profiles are functions of (x, Y), where:

$$Y = \begin{cases} Y_+ := \dfrac{2-y}{\sqrt{\varepsilon}} \text{ if } 1 \le y \le 2, \\ Y_- := \dfrac{y}{\sqrt{\varepsilon}} \text{ if } 0 \le y \le 1. \end{cases} \tag{5.5.4}$$

Due to this, we break up the boundary layer profiles into two components, one supported near $y = 0$ and one supported near $y = 2$:

$$u_p^i(x, Y) = u_p^{i,-}(x, Y_-) + u_p^{i,+}(x, Y_+). \tag{5.5.5}$$

As a <u>notational convention</u>, we use:

$$\partial_Y u_p^i := \partial_{Y_-} u_p^{i,-} - \partial_{Y_+} u_p^{i,+}. \tag{5.5.6}$$

The purpose of such a convention is to obtain the chain rule:

$$\partial_y u_p^i = \frac{1}{\sqrt{\varepsilon}}\partial_Y u_p^i. \tag{5.5.7}$$

Let us set the following notations:

$$u_E^\varepsilon := \mu + \varepsilon u_e^1 + \varepsilon^{\frac{3}{2}} u_e^2, \quad v_E^\varepsilon := \varepsilon v_e^1 + \varepsilon^{\frac{3}{2}} v_e^2, \tag{5.5.8}$$

$$u_s^{(2)} := \mu + \varepsilon u_e^1 + \varepsilon u_p^1 + \varepsilon^{\frac{3}{2}} u_e^2 + \varepsilon^{\frac{3}{2}} u_p^2, \tag{5.5.9}$$

$$v_s^{(2)} := \varepsilon v_e^1 + \varepsilon^{\frac{3}{2}} v_p^1 + \varepsilon^{\frac{3}{2}} v_e^2 + \varepsilon^2 v_p^2, \tag{5.5.10}$$

$$P_s^{(2)} := P^\varepsilon := \varepsilon P_e^1 + \varepsilon^{\frac{3}{2}} P_e^2 + \varepsilon[P_p^1 + \varepsilon P_p^{1,a}] + \varepsilon^{\frac{3}{2}} P_p^2. \tag{5.5.11}$$

Using the expansions (5.5.1) - (5.5.3), we will first expand out the purely Euler terms:

$$\begin{aligned} u_E^\varepsilon \partial_x u_E^\varepsilon &= \Big[\mu + \varepsilon u_e^1 + \varepsilon^{\frac{3}{2}} u_e^2\Big] \cdot \Big[\varepsilon u_{ex}^1 + \varepsilon^{\frac{3}{2}} u_{ex}^2\Big] \\ &= \varepsilon \mu u_{ex}^1 + \varepsilon^2 u_e^1 u_{ex}^1 + \varepsilon^{\frac{5}{2}} u_e^2 u_{ex}^1 + \varepsilon^{\frac{3}{2}} \mu u_{ex}^2 + \varepsilon^{\frac{5}{2}} u_e^1 u_{ex}^2 + \varepsilon^3 u_e^2 u_{ex}^2 \end{aligned} \tag{5.5.12}$$

$$\begin{aligned} v_E^\varepsilon \partial_y u_E^\varepsilon &= \Big[\varepsilon v_e^1 + \varepsilon^{\frac{3}{2}} v_e^2\Big] \cdot \Big[\mu' + \varepsilon u_{ey}^1 + \varepsilon^{\frac{3}{2}} u_{ey}^2\Big] \\ &= \varepsilon \mu' v_e^1 + \varepsilon^2 v_e^1 u_{ey}^1 + \varepsilon^{\frac{5}{2}} v_e^1 u_{ey}^2 + \varepsilon^{\frac{3}{2}} \mu' v_e^2 + \varepsilon^{\frac{5}{2}} v_e^2 u_{ey}^1 + \varepsilon^3 v_e^2 u_{ey}^2 \end{aligned} \tag{5.5.13}$$

$$\begin{aligned} u_E^\varepsilon \partial_x v_E^\varepsilon &= \Big[\mu + \varepsilon u_e^1 + \varepsilon^{\frac{3}{2}} u_e^2\Big] \cdot \Big[\varepsilon v_{ex}^1 + \varepsilon^{\frac{3}{2}} v_{ex}^2\Big] \\ &= \varepsilon \mu v_{ex}^1 + \mu \varepsilon^{\frac{3}{2}} v_{ex}^2 + \varepsilon^2 u_e^1 v_{ex}^1 + \varepsilon^{\frac{5}{2}} u_e^1 v_{ex}^2 + \varepsilon^{\frac{5}{2}} u_e^2 v_{ex}^1 + \varepsilon^3 u_e^2 v_{ex}^2, \end{aligned} \tag{5.5.14}$$

$$v^\varepsilon \partial_y v_E^\varepsilon = \Big[\varepsilon v_e^1 + \varepsilon^{\frac{3}{2}} v_e^2\Big] \cdot \Big[\varepsilon v_{ey}^1 + \varepsilon^{\frac{3}{2}} v_{ey}^2\Big]$$

$$= \varepsilon^2 v_e^1 v_{ey}^1 + \varepsilon^{\frac{5}{2}} v_e^1 v_{ey}^2 + \varepsilon^{\frac{5}{2}} v_e^2 v_{ey}^1 + \varepsilon^3 v_e^2 v_{ey}^2. \tag{5.5.15}$$

$$\partial_x P_E^\varepsilon = \varepsilon P_{ex}^1 + \varepsilon^{\frac{3}{2}} P_{ex}^2, \tag{5.5.16}$$

$$\partial_y P_E^\varepsilon = \varepsilon P_{ey}^1 + \varepsilon^{\frac{3}{2}} P_{ey}^2 \tag{5.5.17}$$

$$\varepsilon \Delta u_E^\varepsilon = \varepsilon \mu''(y) + \varepsilon^2 \Delta u_e^1 + \varepsilon^{\frac{5}{2}} \Delta u_e^2, \tag{5.5.18}$$

$$\varepsilon \Delta v_E^\varepsilon = \varepsilon^2 \Delta v_e^1 + \varepsilon^{\frac{5}{2}} \Delta v_e^2. \tag{5.5.19}$$

We now expand:

$$\begin{aligned} u_s^{(2)} \partial_x u_s^{(2)} =& u_E^\varepsilon \partial_x u_E^\varepsilon + \varepsilon^2 u_p^1 u_{ex}^1 + \varepsilon^2 u_p^1 u_{px}^1 + \varepsilon^{\frac{5}{2}} u_p^1 u_{ex}^2 \\ &+ \varepsilon \mu u_{px}^1 + \varepsilon^2 u_e^1 u_{px}^1 + \varepsilon^{\frac{5}{2}} u_e^2 u_{px}^1 + \varepsilon^{\frac{5}{2}} u_p^2 u_{ex}^1 + \varepsilon^{\frac{5}{2}} u_p^2 u_{px}^1 \\ &+ \varepsilon^3 u_p^2 u_{ex}^2 + \varepsilon^3 u_p^2 u_{px}^2 + \varepsilon^{\frac{3}{2}} \mu u_{px}^2 + \varepsilon^{\frac{5}{2}} u_e^1 u_{px}^2 \\ &+ \varepsilon^{\frac{5}{2}} u_p^1 u_{px}^2 + \varepsilon^3 u_e^2 u_{px}^2. \end{aligned} \tag{5.5.20}$$

$$\begin{aligned} v_s^{(2)} \partial_y u_s^{(2)} =& v_E^\varepsilon \partial_y u_E^\varepsilon + \varepsilon^{\frac{3}{2}} \mu' v_p^1 + \varepsilon^{\frac{5}{2}} v_p^1 u_{ey}^1 + \varepsilon^2 v_p^1 u_{pY}^1 + \varepsilon^3 v_p^1 u_{ey}^2 \\ &+ \varepsilon^{\frac{3}{2}} v_e^1 u_{pY}^1 + \varepsilon^2 v_e^2 u_{pY}^1 + \varepsilon^2 \mu' v_p^2 + \varepsilon^3 u_{ey}^1 v_p^2 + \varepsilon^{\frac{5}{2}} v_p^2 u_{pY}^1 \\ &+ \varepsilon^{\frac{3}{2}} v_p^2 u_{ey}^2 + \varepsilon^3 v_p^2 u_{pY}^2 + \varepsilon^{\frac{5}{2}} v_e^2 u_{pY}^2 + \varepsilon^{\frac{5}{2}} v_p^1 u_{pY}^2 + \varepsilon^2 v_e^1 u_{pY}^2. \end{aligned} \tag{5.5.21}$$

$$\begin{aligned} u_s^{(2)} \partial_x v_s^{(2)} =& u_E^\varepsilon \partial_x v_E^\varepsilon + \varepsilon^2 u_p^1 v_{ex}^1 + \varepsilon^{\frac{5}{2}} u_p^1 v_{px}^1 + \varepsilon^{\frac{5}{2}} u_p^1 v_{ex}^2 + \varepsilon^{\frac{3}{2}} \mu v_{px}^1 \\ &+ \varepsilon^{\frac{5}{2}} u_e^1 v_{px}^1 + \varepsilon^3 u_e^2 v_{px}^1 + \varepsilon^2 \mu v_{px}^2 + \varepsilon^3 u_e^1 v_{px}^2 \end{aligned}$$

$$+\varepsilon^3 u_p^1 v_{px}^2 + \varepsilon^{\frac{5}{2}} u_e^2 v_{px}^2 + \varepsilon^{\frac{7}{2}} u_p^2 v_{px}^2 + \varepsilon^3 u_p^2 v_{ex}^2$$
$$+\varepsilon^3 u_p^2 v_{px}^1 + \varepsilon^{\frac{5}{2}} v_{ex}^1 u_p^2. \tag{5.5.22}$$

$$\begin{aligned} v_s^{(2)}\partial_y v_s^{(2)} =& v_E^\varepsilon \partial_y v_E^\varepsilon + \varepsilon^{\frac{5}{2}} v_p^1 v_{ey}^1 + \varepsilon^{\frac{5}{2}} v_p^1 v_{pY}^1 + \varepsilon^3 v_p^1 v_{ey}^2 + \varepsilon^2 v_e^1 v_{pY}^1 \\ &+ \varepsilon^{\frac{5}{2}} v_e^2 v_{pY}^1 + \varepsilon^3 v_{ey}^1 v_p^2 + \varepsilon^3 v_{pY}^1 v_p^2 + \varepsilon^{\frac{7}{2}} v_p^2 v_{ey}^2 + \varepsilon^{\frac{7}{2}} v_p^2 v_{pY}^2 \\ &+ \varepsilon^{\frac{5}{2}} v_e^1 v_{pY}^2 + \varepsilon^3 v_p^1 v_{pY}^2 + \varepsilon^3 v_e^2 v_{pY}^2. \end{aligned} \tag{5.5.23}$$

Finally, we have the linear terms:

$$\partial_x P_s = \partial_x P_E^\varepsilon + \varepsilon^{\frac{3}{2}} P_{px}^2 + \varepsilon P_{px}^1 + \varepsilon^2 P_{px}^{1,a}, \tag{5.5.24}$$

$$\partial_y P_s = \partial_y P_E^\varepsilon + \varepsilon P_{pY}^2 + \sqrt{\varepsilon} P_{pY}^1 + \varepsilon^{\frac{3}{2}} P_{pY}^{1,a} \tag{5.5.25}$$

$$\varepsilon \Delta u^\varepsilon = \varepsilon \Delta u_E^\varepsilon + \varepsilon^2 u_{pxx}^1 + \varepsilon^{\frac{5}{2}} u_{pxx}^2 + \varepsilon u_{pYY}^1 + \varepsilon^{\frac{3}{2}} u_{pYY}^2, \tag{5.5.26}$$

$$\varepsilon \Delta v^\varepsilon = \varepsilon \Delta v_E^\varepsilon + \varepsilon^{\frac{5}{2}} v_{pxx}^1 + \varepsilon^{\frac{3}{2}} v_{pYY}^1 + \varepsilon^3 v_{pxx}^2 + \varepsilon^2 v_{pYY}^2. \tag{5.5.27}$$

5.5.2 Euler Equations

The equations satisfied by the Euler layers are obtained by collecting the $\mathcal{O}(\varepsilon)$ order terms from (5.5.12) - (5.5.19), and is now shown:

$$\left.\begin{aligned} &\mu \partial_x u_e^1 + \mu' v_e^1 + \partial_x P_e^1 = \mu''(y) \\ &\mu \partial_x v_e^1 + \partial_y P_e^1 = 0, \\ &\partial_x u_e^1 + \partial_y v_e^1 = 0, \\ &v_e^1|_{x=0} = v_e^1|_{y=0} = v_e^1|_{y=2} = v_e^1|_{x=L} = 0. \end{aligned}\right\} \tag{5.5.28}$$

By going to the vorticity formulation, we arrive at the following problem:

$$-\mu\Delta v_e^1 + \mu'' v_e^1 = \mu'''(y), \quad v_e^1|_{\partial\Omega} = 0, \quad u_e^1 := \int_0^x v_{ey}^1. \tag{5.5.29}$$

We will make the assumptions that:

$$\frac{\mu''}{\mu}, \frac{\mu'''}{\mu} \text{ vanish at high order at } y = 0, 2. \tag{5.5.30}$$

According to (5.5.30), we divide (5.5.29) by μ to obtain:

$$-\Delta v_e^1 + \frac{\mu''}{\mu} v_e^1 = \frac{\mu'''}{\mu}, \qquad v_e^1|_{\partial\Omega} = 0. \tag{5.5.31}$$

By evaluating (5.5.31) at $y = 0, 2$ and recalling (5.5.30), it is clear that $\partial_{yy} v_e^1|_{y=0,2} = 0$. The system satisfied by the second Euler layer is obtained by collecting the $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ terms from (5.5.12) - (5.5.19), and is shown here:

$$\left.\begin{aligned}
&\mu\partial_x u_e^2 + \mu' v_e^2 + \partial_x P_e^2 = 0 \\
&\mu\partial_x v_e^2 + \partial_y P_e^2 = 0, \\
&\partial_x u_e^2 + \partial_y v_e^2 = 0, \\
&v_e^2|_{x=0} = v_e^2|_{x=L} = v_e^2|_{y=2} = 0, \quad v_e^2|_{y=0} = -v_p^1|_{Y=0}.
\end{aligned}\right\} \tag{5.5.32}$$

Going to vorticity produces the system:

$$-\mu\Delta v_e^2 + \mu'' v_e^2 = 0, \quad v_e^2|_{y=0,2} = -v_p^1|_{y=0,2}. \tag{5.5.33}$$

We will assume high-order compatibility conditions on the data $v_e^2|_{x=0,L}$ with $v_e^2|_{y=0,2}$ at the four corners of the domain, Ω. The first of these conditions at the corner $x = 0, y = 0$

is as follows:

$$\partial_{yy} v_e^1|_{x=0}(0) = \partial_{yy} v_e^1|_{y=0}(0) = -\frac{\mu''}{\mu} v_p^1|_{y=0}. \tag{5.5.34}$$

The remaining compatibility conditions may be derived in the same manner. These will contribute higher order terms, which are the $\mathcal{O}(\varepsilon^2)$ terms from (5.5.12) - (5.5.19):

$$\begin{aligned}
\mathcal{C}_{1,u} :=& \varepsilon^2 [u_e^1 \partial_x u_e^1 + \sqrt{\varepsilon} u_e^2 \partial_x u_e^1 + \sqrt{\varepsilon} u_e^1 \partial_x u_e^2 + \varepsilon u_e^2 \partial_x u_e^2] + \\
& \varepsilon^2 \Big[v_e^1 \partial_y u_e^1 + \sqrt{\varepsilon} v_e^2 \partial_y u_e^1 + \sqrt{\varepsilon} v_e^1 \partial_y u_e^2 + \varepsilon v_e^2 \partial_y u_e^2 \Big] \\
& - \varepsilon^2 \Delta u_e^1 - \varepsilon^{\frac{5}{2}} \Delta u_e^2, && (5.5.35) \\
\mathcal{C}_{1,v} :=& \varepsilon^2 u_e^1 \partial_x v_e^1 + \varepsilon^{\frac{5}{2}} u_e^1 \partial_x v_e^2 + \varepsilon^{\frac{5}{2}} u_e^2 \partial_x v_e^1 + \varepsilon^3 u_e^2 \partial_x v_e^2 \\
& + \varepsilon^2 v_e^1 \partial_y v_e^1 + \varepsilon^{\frac{5}{2}} v_e^2 \partial_y v_e^1 + \varepsilon^{\frac{5}{2}} v_e^1 \partial_y v_e^2 + \varepsilon^3 v_e^2 \partial_y v_e^2 \\
& - \varepsilon^2 \Delta v_e^1 - \varepsilon^{\frac{5}{2}} \Delta v_e^2. && (5.5.36)
\end{aligned}$$

The following follow from standard elliptic theory:

Lemma 5.5.1. *Assuming (5.5.30) and compatibility conditions for both v_e^1, v_e^2 for arbitrary order as in (5.5.34), there exist unique solutions, v_e^1, v_e^2 to (5.5.28) and (5.5.32) that are regular:*

$$|\partial_x^l \partial_y^m \{u_e^i, v_e^i\}| \lesssim c_0(\frac{\mu'''}{\mu}) \times C_{l,k} \text{ for } i = 1, 2. \tag{5.5.37}$$

5.5.3 Boundary Layer Equations

Collecting the $\mathcal{O}(\varepsilon)$ terms from (5.5.20)- (5.5.27):

$$\left.\begin{aligned} &\mu\partial_x u_p^{1,0,-} - \partial_{Y_-Y_-} u_p^{1,0,-} = 0, \quad \partial_{Y_-} P_p^{1,0,-} = 0, \\ &u_p^{1,0,-}|_{x=0} =, \quad u_p^{1,0,-}|_{Y_-=0} = -u_e^1|_{y=0}, u_p^{1,0,-}|_{Y_-\to\infty} = 0 \\ &v_p^{1,0,-} = \int_{Y_-}^{\infty} \partial_x u_p^{1,0,-}. \end{aligned}\right\} \tag{5.5.38}$$

Here we must assume the compatibility condition:

$$u_p^{1,0,-}(0,Y_-)|_{Y_-=0} = -u_e^1|_{y=0}, \quad \partial_{Y_-}^2 u_p^{1,0,-}(0,Y)|_{Y_-=0} = 0. \tag{5.5.39}$$

We will also assume higher order compatibility conditions that can be obtained by differentiating the above system and reading the resulting equalities. Note that we construct $u_p^{1,0,-}, v_p^{1,0,-}$ on $(0,L)\times(0,\infty)$. We now cut-off these layers and make a $\mathcal{O}(\sqrt{\varepsilon})$-order error:

$$u_p^{1,-} = \chi(\frac{\sqrt{\varepsilon}Y}{100})u_p^{1,0,-} - \frac{\sqrt{\varepsilon}}{100}\chi'(\frac{\sqrt{\varepsilon}Y}{100})\int_0^x v^{1,0,-}, \quad v_p^{1,-} := \chi(\frac{\sqrt{\varepsilon}Y}{100})v_p^{1,0,-} \tag{5.5.40}$$

$u^{1,0,+}, v^{1,0,+}, u^{1,+}, v^{1,+}$ are defined analogously, and we omit these details. We then define:

$$u_p^{1,0} := u_p^{1,0,-} + u_p^{1,0,+}, \qquad v_p^1 := v_p^{1,0,-} + v_p^{1,0,+}, \tag{5.5.41}$$

$$u_p^1 := u_p^{1,-} + u_p^{1,+}, \qquad v_p^1 := v_p^{1,-} + v_p^{1,+}. \tag{5.5.42}$$

Note that due to the cut-off in (5.5.40), u_p^1, v_p^1 is smooth. The contributions to the next layer are:

$$\begin{aligned}
\mathcal{C}_{2,u} :=& \varepsilon^2 \partial_x P_p^{1,a} + \varepsilon^2 u_e^1 \partial_x u_p^1 + \varepsilon^2 u_p^1 \partial_x u_e^1 + \varepsilon^2 u_p^1 \partial_x u_p^1 \\
&+ \varepsilon^{\frac{3}{2}} v_e^1 \partial_Y u_p^1 + \varepsilon^{\frac{3}{2}} v_p^1 \mu' + \varepsilon^{\frac{5}{2}} v_p^1 \partial_y u_e^1 + \varepsilon^2 v_p^1 \partial_Y u_p^1 \\
&- \varepsilon^2 \partial_{xx} u_p^1 + \left[\varepsilon^{\frac{5}{2}} \left(u_e^2 u_{px}^1 + u_p^1 u_{ex}^2 \right) + \varepsilon^2 v_e^2 u_{pY}^1 + \varepsilon^3 v_p^1 u_{ey}^2 \right] + \mathcal{C}_{cut}^1.
\end{aligned} \tag{5.5.43}$$

Here $\mathcal{C}_{cut}^1$ is the error introduced by the cut-off functions in (5.5.40):

$$\begin{aligned}
\mathcal{C}_{cut}^1 :=& \sqrt{\varepsilon} \mu \chi' v_p^{1,0} + 3\sqrt{\varepsilon} \chi' \partial_Y u_p^{2,0} \\
&+ 3\varepsilon \chi'' u_p^{2,0} - \varepsilon^{\frac{3}{2}} \chi''' \int_Y^\infty u_p^{2,0},
\end{aligned} \tag{5.5.44}$$

Define the auxiliary pressure via:

$$\begin{aligned}
P_p^{1,a} := - \int_Y^1 \Big[& \mu \partial_x v_p^1 + \varepsilon u_e^1 \partial_x v_p^1 + \sqrt{\varepsilon} u_p^1 \partial_x v_e^1 + \varepsilon u_p^1 \partial_x v_p^1 \\
&+ \sqrt{\varepsilon} v_e^1 \partial_Y v_p^1 + \varepsilon v_p^1 \partial_y v_e^1 + \varepsilon v_p^1 \partial_Y v_p^1 - \partial_{YY} v_p^1 - \varepsilon \partial_{xx} v_p^1 \\
&+ \varepsilon^{-\frac{3}{2}} \left(\varepsilon^3 u_e^2 v_{px}^1 + \varepsilon^{\frac{5}{2}} u_p^1 v_{ex}^2 + \varepsilon^{\frac{5}{2}} v_e^2 v_{py}^1 + \varepsilon^3 v_p^1 v_{ey}^2 \right) \Big].
\end{aligned} \tag{5.5.45}$$

With such a choice,

$$\mathcal{C}_{2,v} := 0. \tag{5.5.46}$$

Collecting the $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ terms from (5.5.20) - (5.5.27), the system satisfied by the second boundary layers is:

$$\left.\begin{aligned}
&\mu\partial_x u_p^{2,0} - \partial_{YY} u_p^{2,0} = f_2 := \varepsilon^{-\frac{3}{2}}\mathcal{C}_{2,u}, \quad \partial_Y P_p^2 = 0, \quad \partial_x u_p^{2,0} + \partial_Y v_p^{2,0} = 0,\\
&u_p^{2,0}|_{x=0} =, \quad u_p^{2,0,-}|_{Y=0} = -u_e^2|_{y=0}, \quad u_p^{2,0,+}|_{y=2} = -u_e^2|_{y=2},\\
&u_p^{2,0,-}|_{Y\to\infty} = 0, \quad u_p^{2,0,+}|_{Y\to-\infty} = 0\\
&v_p^{2,0,-} = -\int_0^{Y_-} \partial_x u_p^{2,0,-}, \quad v_p^{2,0,+} = -\int_2^{Y_+} \partial_x u_p^{2,0,+}.
\end{aligned}\right\} \tag{5.5.47}$$

Note that in the same manner as in u_p^1, v_p^1, we have two boundary layer variables, Y_-, Y_+. We compactify the notation in (5.5.47) to simultaneously address both. Define the cut-off layer via:

$$u_p^2 := \chi(\frac{\sqrt{\varepsilon}Y}{100})u_p^{2,0} - \frac{\sqrt{\varepsilon}}{100}\chi'(\frac{\sqrt{\varepsilon}Y}{100})\int_0^x v_p^2, \quad v_p^2 := \chi(\frac{\sqrt{\varepsilon}Y}{100})v_p^{2,0}. \tag{5.5.48}$$

Lemma 5.5.2. *Assume high order compatibility conditions in the sense of (5.5.39) for both u_p^1, u_p^2. There exist unique solutions to (5.5.38) and (5.5.47) that are regular and satisfy the following estimates:*

$$|Y^m\partial_x^k\partial_Y^l\{u_p^1, v_p^1\}| \le c_0(\frac{\mu'''}{\mu}) \times C_{m,k,l} \text{ for any } k,l,m \ge 0, \tag{5.5.49}$$

$$|Y^m\partial_x^k\partial_Y^l u_p^2| \le c_0(\frac{\mu'''}{\mu}) \times C_{m,k,l} \text{ for any } k,l,m \ge 0, \tag{5.5.50}$$

$$|\partial_x^k v_p^2| \le c_0(\frac{\mu'''}{\mu}) \times C_k \text{ for any } k \ge 0. \tag{5.5.51}$$

Proof. These follow from standard heat equation estimates. $\square$

The following are the errors contributed to the next layer:

$$\begin{aligned}
\mathcal{C}_{3,u} :=& \varepsilon^{\frac{5}{2}}[u_e^1 + u_p^1 + \sqrt{\varepsilon}u_p^2 + \sqrt{\varepsilon}u_e^2]\partial_x u_p^2 + \varepsilon^{\frac{3}{2}}u_p^2[\varepsilon\partial_x u_e^1 + \varepsilon\partial_x u_p^1\\
&+ \varepsilon^{\frac{3}{2}}\partial_x u_p^2 + \varepsilon^{\frac{3}{2}}\partial_x u_e^2] + \varepsilon\partial_Y u_p^2\Big[\varepsilon v_e^1 + \varepsilon^{\frac{3}{2}}v_p^1 + \varepsilon^{\frac{3}{2}}v_e^2 + \varepsilon^2 v_p^2\Big]
\end{aligned}$$

$$+\varepsilon^2 v_p^2\Big[\mu' + \varepsilon\partial_y u_e^1 + \sqrt{\varepsilon}\partial_Y u_p^1 + \varepsilon^{\frac{3}{2}}\partial_y u_e^2\Big] - \varepsilon^{\frac{5}{2}}\partial_{xx} u_p^2 + \varepsilon^{\frac{3}{2}}\mathcal{C}_{cut}, \tag{5.5.52}$$

$$\begin{aligned}\mathcal{C}_{3,v} :=&\varepsilon^2\partial_x v_p^2\Big[\mu + \varepsilon u_e^1 + \varepsilon u_p^1 + \varepsilon^{\frac{3}{2}} u_e^2\Big] + \varepsilon^{\frac{3}{2}} u_p^2\Big[\varepsilon\partial_x v_e^1 + \varepsilon^{\frac{3}{2}}\partial_x v_p^1 + \varepsilon^{\frac{3}{2}}\partial_x v_e^2\Big]\\ &+ \varepsilon^{\frac{7}{2}} u_p^2\partial_x v_p^2 + \varepsilon^2 v_p^2\Big[\varepsilon\partial_y v_e^1 + \varepsilon\partial_y v_p^1 + \varepsilon^{\frac{3}{2}}\partial_y v_e^2\Big]\\ &+ \varepsilon^{\frac{3}{2}}\partial_Y v_p^2\Big[\varepsilon v_e^1 + \varepsilon^{\frac{3}{2}} v_p^1 + \varepsilon^{\frac{3}{2}} v_e^2\Big] + \varepsilon^{\frac{7}{2}} v_p^2\partial_y v_p^2.\end{aligned} \tag{5.5.53}$$

Here $\mathcal{C}_{cut}$ is the error contributed by cutting off the layers:

$$\begin{aligned}\mathcal{C}_{cut} :=&(1-\chi)f_2 + \sqrt{\varepsilon}\mu\chi' v_p^{2,0} + 3\sqrt{\varepsilon}\chi'\partial_Y u_p^{2,0}\\ &+ 3\varepsilon\chi'' u_p^{2,0} - \varepsilon^{\frac{3}{2}}\chi'''\int_Y^\infty u_p^{2,0}.\end{aligned} \tag{5.5.54}$$

The total contributions to the remainder forcing is:

$$\mathcal{F}_u := \mathcal{C}_{1,u} + \mathcal{C}_{3,u}, \quad \mathcal{F}_v := \mathcal{C}_{1,v} + \mathcal{C}_{3,v}. \tag{5.5.55}$$

We can break up the forcing contribution into:

$$\mathcal{F}_u = \mathcal{T}_{u,\varepsilon^2} + \mathcal{O}(\varepsilon^{\frac{5}{2}}), \qquad \mathcal{F}_v = \mathcal{T}_{v,\varepsilon^2} + \mathcal{O}(\varepsilon^{\frac{5}{2}}), \tag{5.5.56}$$

where the terms at $\mathcal{O}(\varepsilon^2)$ are the following:

$$\mathcal{T}_{u,\varepsilon^2} := \varepsilon^2\underbrace{\Big[u_e^1\partial_x u_e^1 + v_e^1\partial_y u_e^1 - \Delta u_e^1 + \partial_Y u_p^2 v_e^1 + \mu' v_p^2 + \mu\chi' v_p^{2,0} + 3\chi'\partial_Y u_p^{2,0}\Big]}_{T_1} \tag{5.5.57}$$

$$\mathcal{T}_{v,\varepsilon^2} := \varepsilon^2\underbrace{\Big[u_e^1\partial_x v_e^1 + v_e^1\partial_y v_e^1 - \Delta v_e^1 + \mu\partial_x v_p^2\Big]}_{T_2}. \tag{5.5.58}$$

Summarizing the above constructions:

Lemma 5.5.3. *The following estimates are satisfied by u_s, v_s:*

$$|\partial_x u_s| + |\partial_y v_s| + |u_s - \mu| \le \min\{\mathcal{O}(\sqrt{\varepsilon})\tilde{y}, \mathcal{O}(\varepsilon)\}, \tag{5.5.59}$$

$$|\partial_y u_s - \mu'| \lesssim \sqrt{\varepsilon}, \tag{5.5.60}$$

$$|\partial_x^l v_s| \lesssim \varepsilon \tilde{y} \textit{ for } l \ge 0, \tag{5.5.61}$$

The following are satisfied by $T_{u,\varepsilon^2}, T_{v,\varepsilon^2}$:

$$||T_1, \frac{T_2}{\tilde{y}}||_2 \lesssim c_0(\frac{\mu'''}{\mu}), \tag{5.5.62}$$

$$||\partial_y T_1, \partial_y T_2||_2 \lesssim c_0(\frac{\mu'''}{\mu}) \times \varepsilon^{-\frac{1}{4}}. \tag{5.5.63}$$

Proof. Only $\frac{T_2}{y}$ is non-trivial, and it follows by examining that all terms in T_2 satisfy $T_2|_{y=0} = 0 = T_2|_{y=2}$. Note that we have used (5.5.29) and (5.5.30) to conclude that $\Delta v_e^1|_{y=0,2} = 0$. $\square$

Bibliography

[ADN59] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I. *Comm. Pure Appl. Math.*, (12):623–727, 1959.

[AF03] R. Adams and J. Fourier. *Sobolev Spaces.* Elsevier Science Ltd,, 2 edition, 2003.

[Asa91] A. Asano. Zero viscosity limit of incompressible Navier-Stokes equations. *Conference at the Fourth Workshop on Mathematical Aspects of Fluid and Plasma Dynamics*, 1991.

[AYXY15] R. Alexandre, Y.G.Wang, C.J. Xu, and T. Yang. Well-posedness of the Prandtl equation in Sobolev spaces. *Journal of the American Mathematical Society*, (3):745–784, 2015.

[BKL94] J. Bricmont, A. Kupiainen, and G. Lin. Renormalization Group and Asymptotics of Solutions of Nonlinear Parabolic Equations. *Comm. Pure. Appl. Math.*, (47):893–922, 1994.

[BR80] H. Blum and R. Rannacher. On the Boundary Value Problem of the Biharmonic Operator on Domains with Angular Corners. *Math. Meth. in the Appl. Sci.*, pages 556–581, 1980.

[Cia10] P. Ciarlet. On Korn's Inequality. *Chinese Annals of Mathematics, Series B*, pages 607–618, 2010.

[CS12] A. Choffrut and V. Sverak. Local structure of the set of steady-state solutions to the 2d incompressible Euler equations. *Geometric and Functional Analysis*, 22(1):136–201, 2012.

[DM18] A.L. Dalibard and N. Masmoudi. Separation for the stationary Prandtl equation. *arXiv preprint*, (1802.04039), 2018.

[E00] W. E. Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta Math. Sin. (Engl. Ser.)*, 2(16):207–218, 2000.

[EE97] W. E and B. Engquist. Blowup of solutions of the unsteady Prandtl's equation. *Comm. Pure Appl. Math.*, 12(50):1287–1293, 1997.

[Gal11] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations.* Springer,, 2 edition, 2011.

[GGN15] E. Grenier, Y. Guo, and T. Nguyen. Spectral instability of Prandtl boundary layers: an overview. *Analysis (Berlin)*, 35(4):343–355, 2015.

[GGN16a] E. Grenier, Y. Guo, and T. Nguyen. Spectral instability of characteristic boundary layer flows. *Duke Math J.*, (165):3085–3146, 2016.

[GGN16b] E. Grenier, Y. Guo, and T. Nguyen. Spectral instability of symmetric shear flows in a two-dimensional channel. *Adv. Math.*, (292):52–110, 2016.

[GJT16] G-M. Gie, C-Y. Jung, and R. Temam. Recent Progresses in the Boundary Layer Theory. *Disc. Cont. Dyn. Syst. A*, 5(36):2521–2583, 2016.

[GN11] Y. Guo and T. Nguyen. A note on the Prandtl boundary layers. *Comm. Pure Appl. Math.*, 64(10):1416–1438, 2011.

[GN17] Y. Guo and T. Nguyen. Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. *Annals of PDE*, 2017.

[Gre00] E. Grenier. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.*, 9(53):1067–1091, 2000.

[GVD10] D. Gerard-Varet and E. Dormy. On the ill-posedness of the Prandtl equation. *Journal of the American Mathematical Society*, 23(2):591–609, 2010.

[GVM13] D. Gerard-Varet and N. Masmoudi. Well-posedness for the Prandtl system without analyticity or monotonicity. *arXiv preprint*, (1305.0221v1), 2013.

[GVMM16] D. Gerard-Varet, Y. Maekawa, and N. Masmoudi. Gevrey stability of Prandtl expansions for 2D Navier-Stokes flows. *arXiv preprint*, (1607.06434v1), 2016.

[GVN12] D. Gerard-Varet and T. Nguyen. Remarks on the ill-posedness of the Prandtl equation. *Asymptotic Analysis*, 77(1-2):71–88, 2012.

[HH03] L. Hong and J. Hunter. Singularity formation and instability in the unsteady inviscid and viscous Prandtl equations. *Comm. Math. Sci.*, 1(2):293–316, 2003.

[IS16] S. Iyer and B. Sandstede. Mixing in Reaction-Diffusion Systems: Large Phase Offsets. *arXiv preprint*, (1610.06527), 2016.

[IV16] M. Ignatova and V. Vicol. Almost Global Existence for the Prandtl Boundary Layer Equations. *Arch. Rational Mech. Anal.*, 220(2):809–848, 2016.

[Iye16] S. Iyer. Global Steady Prandtl Expansion Over a Moving Boundary (3 Part Monograph). *arXiv preprint*, (1609.05397), 2016.

[Iye17a] S. Iyer. Steady Prandtl Boundary Layer Expansion of Navier-Stokes Flows over a Rotating Disk. *Arch. Rational Mech. Anal.*, 224(2):421–469, 2017.

[Iye17b] S. Iyer. Steady Prandtl Layers over a Moving Boundary: Non-shear Euler Flows. *arXiv preprint*, (1705.05936v2), 2017.

[IZ17] S. Iyer and C. Zhou. Steady Inviscid Limit to Shear Flows. *arXiv preprint*, (1711.05664), 2017.

[Kat84] T. Kato. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. *Math. Sci. Res. Inst. Publ.*, (2):85–98, 1984.

[KC04] P. Kundu and I. Cohen. *Fluid Mechanics*. Elsevier Academic Press, 3 edition, 2004.

[KMVW13] I. Kukavica, N. Masmoudi, V. Vicol, and T.K. Wong. On the local existence of analytic solutions to the Prandtl boundary layer equations. *Commun. Math. Sci.*, 1(11):269–292, 2013.

[KMVW14] I. Kukavica, N. Masmoudi, V. Vicol, and T.K. Wong. On the local well-posedness of the Prandtl and Hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.*, 46(6):3865–3890, 2014.

[KVW15] I. Kukavica, V. Vicol, and F. Wang. The van Dommelen and Shen singularity in the Prandtl equations. *arXiv preprint*, (1512.07358v1):269–292, 2015.

[LCS03] M. C. Lombardo, M. Cannone, and M. Sammartino. Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.*, 35(4):987–1004, 2003.

[Mae14] Y. Maekawa. On the inviscid problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure. Appl. Math.*, 67:1045–1128, 2014.

[MM17] Y. Maekawa and A. Mazzucato. The inviscid limit and boundary layers for Navier-Stokes flows. *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, pages 1–48, 2017.

[MT08] A. Mazzucato and M. Taylor. Vanishing viscosity plane parallel channel flow and related singular perturbation problems. *Anal. PDE*, 1(1):35–93, 2008.

[MW15] N. Masmoudi and T.K. Wong. Local in time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure. Appl. Math.*, 68:1683–1741, 2015.

[Nir59] L. Nirenberg. On elliptic partial differential equations. *Annali Della Scoula Normale Superiore Di Pisa.*, 1959.

[NPV12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to fractional Sobolev spaces. *Bulletin des Sciences Mathematiques*, (136), 2012.

[Ole67] O.A. Oleinik. On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid. *J. Appl. Math. Mech.*, 30:951–974, 1967.

[Orl98] M. Orlt. Regularity for Navier-Stokes in domains with corners. *Ph.D. Thesis*, 1998.

[OS95] M. Orlt and A.M. Sandig. Regularity of viscous Navier-Stokes flows in non smooth domains. *Lecture Notes in Pure and Appl. Math.*, 167, 1995.

[OS99] O.A. Oleinik and V.A. Samokhin. *Mathematical models in boundary layer theory.* Applied Mathematics and Mathematical Computation, 15. Champan and Hall/ CRC, 1999.

[Pra04] L. Prandtl. Uber flussigkeits-bewegung bei sehr kleiner reibung. *Verhandlungen des III Internationalen Mathematiker-Kongresses, Heidelberg*, pages 484–491, 1904.

[SC98a] M. Sammartino and R. Caflisch. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.*, 192(2):433–461, 1998.

[SC98b] M. Sammartino and R. Caflisch. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.*, 192(2):463–491, 1998.

[Ser67] J. Serrin. Asymptotic behaviour of velocity profiles in the Prandtl boundary layer theory. *Proc. R. Soc. Lond. A*, 299:491–507, 1967.

[SG00] H. Schlichting and K. Gersten. *Boundary Layer Theory.* Springer-Verlag, 8 edition, 2000.

[XZ04] Z. Xin and L. Zhang. On the global existence of solutions to the Prandtl's system. *Adv. Math.*, 181(1):88–133, 2004.

[Yud03] V.I. Yudovich. Eleven great problems of hydrodynamics. *Mosc. Math. J.*, 3(2):711–737, 2003.