A tale of two moduli spaces:

Hilbert schemes of singular curves and moduli of elliptic surfaces

by

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CHAPTER 1

Introduction

Algebraic geometry is the study of algebraic varieties, or more generally schemes, which are spaces given as the simulataneous solutions of a collection of algebraic equations. Algebraic varieties are more rigid than other classes of spaces (such as smooth manifolds). This means that we can study varieties through discrete algebraic data that is more tractable than the space itself. This often leads to the existence of finite dimensional parameter spaces for structures in algebraic geometry with fixed discrete data.

These parameter spaces, or moduli spaces, are the focus of this thesis. Broadly speaking, the study of moduli spaces of algebraic varieties or structures on them generally serves two (not mutually exclusive) purposes:

- cohomological computations on moduli spaces produce invariants of varieties – for example in curve counting theories such as Gromov-Witten theory and Donaldson-Thomas theory,

- computing the components of moduli spaces and the objects each component parametrizes gives a solution to the problem of classifying algebraic varieties
  – the components are often pinned down by fixing discrete data and each component will then parametrize all objects with given discrete data.

A general theme in both pursuits is the interplay between compactifications of moduli spaces and degenerations of the objects they classify.

In this thesis we will study two particular moduli spaces, each fulfilling one of the above purposes. The first is the Hilbert scheme of points on a singular curve which provides a rich invariant of the curve singularities. The second is the stable pair compactification of the moduli space of elliptic surfaces which classifies elliptic surfaces and their degenerations.

1.1 Hilbert schemes of points on singular curves

Let k be an algebraically closed field and $K_0(\mathrm{Var}_k)$ be the Grothendieck ring of varieties over k. Suppose X is a variety over k. The *motivic zeta function of* X is defined as the power series

$$Z_X^{\mathrm{mot}}(t) := \sum_{d \geq 0} [\mathrm{Sym}^d(X)] t^d \ \in \ K_0(\mathrm{Var}_k)[\![t]\!]$$

(where $[\mathrm{Sym}^0(X)] = 1$ by convention). For the remainder of this section, X will be a curve. In [Kap00], Kapranov observed that if X is a *smooth* curve, $Z_X^{\mathrm{mot}}(t)$ is a *rational* function in t. The zeta function is a rich invariant of X, and when X is defined over a finite field, specializes to the Weil zeta function via the point-counting measure. When the curve X is singular, $Z_X^{\mathrm{mot}}(t)$ is still a rational function in t, see for instance [Lit15, Corollary 30] for strong results in this direction. However, $Z_X^{\mathrm{mot}}(t)$

is not sensitive to the singularities of X. For example when X is a cuspidal cubic, $Z_X^{\mathrm{mot}}(t) = Z_{\mathbb{P}^1}^{\mathrm{mot}}(t)$.

An invariant that is more sensitive to the singularities of X is the *motivic Hilbert zeta function*:

$$Z_X^{\mathrm{Hilb}}(t) := \sum_{d \geq 0} [\mathrm{Hilb}^d(X)] t^d \ \in \ 1 + tK_0(\mathrm{Var}_k)[[t]].$$

Here, $\mathrm{Hilb}^d(X)$ is the Hilbert scheme parametrizing length d subschemes of X. When X is smooth, $\mathrm{Hilb}^d(X)$ coincides with $\mathrm{Sym}^d(X)$, so the Hilbert and usual motivic zeta functions coincide. However, when X is singular, $\mathrm{Hilb}^d(X)$ contains information about subschemes supported on the singularities. For instance, if $X = \mathrm{Spec}(k[x,y]/(xy))$ is a nodal curve with singular point $p \in X$, one can check that $\mathrm{Hilb}^2_p(X) \cong \mathbb{P}^1$ where $\mathrm{Hilb}^d_p(X) \subset \mathrm{Hilb}^d(X)$ is the locus parametrizing subschemes supported on the singularity (see Section 6.1). More generally, if $p \in X$ is a singular point, $\mathrm{Hilb}^2_p(X) \cong \mathbb{P}(T_pX)$ is the projectivization of the Zariski tangent space of the singularity. In particular, even $\mathrm{Hilb}^2(X)$ can be of arbitrarily large dimension. We prove the following:

Theorem 1.1.1. *Let X be a reduced curve over an algebraically closed field k. Then $Z_X^{\mathrm{Hilb}}(t)$ is a rational function of t, with constant term 1.*

As before, if X is defined over a finite field, by passing to point counting over $\mathbb{F}_q$ we obtain a generalization of the rationality part of the Weil conjectures for curves, to the case of Hilbert zeta functions. We note that even after passing to the Euler characteristic specialization, the result appears to be new.

Our approach to the proof of the main result is as follows. Given a singular point, we stratify the Hilbert scheme based on the lengths of the pullbacks of the universal subscheme to branches of the normalization. We then show that these lengths vary in a controlled manner, based only on the singularity. We use this to show that each of these strata stabilize, for large enough degree, inside an appropriate Grassmannian, mimicking the construction of the Hilbert scheme of points (see for instance [FGI$^+$05, Part 3]). In the last section we illustrate in an example how these methods may be used to compute the Hilbert zeta function explicitly.

We work over an algebraically closed field for simplicity. When k is not algebraically closed, X may fail to have a rational point and the standard argument for rationality [Kap00] is not sufficient, even when X is smooth. Nonetheless, even without a rational point, Litt [Lit15] has shown that the motivic zeta function is still rational. The present arguments generalize without substantial changes when the singular points of X are k-rational. When the singularities are not k-rational, the methods here may be adapted by applying the Cohen structure theorem at the singularities [Sta18, Tag 0323], with some additional careful bookkeeping.

In Section 3.5 we consider some special cases of rationality of the Hilbert zeta function for nonreduced curves.

Theorem 1.1.1 was known in special cases from work of others. When the singularities of X are planar, the result is implicit in work of Maulik and Yun [MY14] and for Gorenstein curves when $k = \mathbb{C}$, it is proved by Migliorini and Shende [MS13b, Proposition 16]. For curves that have unibranch singularities, one can deduce rationality from a result of Pfister and Steenbrink [PS92] that the punctual Hilbert schemes of a unibranch singularity become isomorphic as the number of points goes to infinity. We establish a similar stabilization for multibranch singularities.

When X is planar, the Hilbert zeta function is closely related to knot invariants, [OS$^+$12b]. One might hope that for non-planar curves X, the Hilbert zeta function is still related to such structures, for example the refined invariants defined by Aganagic and Shakirov [AS11]. To this end we propose the following expectation:

Expectation 1. *Let (X,p) be the germ of a reduced curve singularity over $\mathbb{C}$. The Euler characteristic Hilbert zeta function*

$$Z^{\mathrm{top}}_{X,p}(t) := \sum_{d\geq 0} \chi_{top}(\mathrm{Hilb}^d(X,p))t^d$$

depends only on the topology of X and combinatorics of an embedded resolution.

Here $\mathrm{Hilb}^d(X,p)$ is the reduced Hilbert scheme of d points supported on $p \in X$ which can be identified with the parameter space of ideals of codimension d in $\widehat{\mathcal{O}}_{X,p}$.

$$\mathrm{Hilb}^d(X,p) = \{I \subset \widehat{\mathcal{O}}_{X,p} \ : \ \dim_{\mathbb{C}} \widehat{\mathcal{O}}_{X,p}/I = d\}$$

Furthermore χ_{top} denotes the compactly supported topological Euler characteristic.

By a family of reduced curve singularities $(\mathcal{C} \to B, \sigma)$ we mean a flat family of reduced curves $\mathcal{C} \to B$ as well as a section $\sigma \colon B \to \mathcal{C}$ such that $\mathcal{C}_b \setminus \sigma(b)$ is nonsingular for all $b \in B$. The main result of Chapter 4 is the following evidence for Expectation 1.

Theorem 1.1.2. *Let $(\mathcal{C} \to B, \sigma)$ be a flat family of reduced curve singularities. Then*

$$b \mapsto Z^{\mathrm{top}}_{\mathcal{C}_b,\sigma(b)}(t)$$

is a constructible function $B \to \mathbb{Z}[\![t]\!]$.

Theorem 1.1.2 implies that the Hilbert zeta function $Z^{\mathrm{top}}_{C,p}(t)$ is a discrete invariant of the singularity (C,p). The main question then is exactly what type of discrete information about the singularity does the Hilbert zeta function encode?

For planar curves Maulik [Mau16], verifying a conjecture of Oblomkov-Shende [OS12a], proved $Z^{top}_{C,p}(t)$ is a topological invariant. An answer to the above question for planar curves has recently emerged due to a large body of work connecting $Z^{top}_{C,p}(t)$ to compactified Jacobians, knot invariants, string theory, enumerative geometry of Calabi-Yau threefolds, affine Springer theory, the Hitchin fibration, representation theory of Cherednik algebras, etc (e.g. [Kas15, OS12a, Mau16, MS13b, MSV15, MY14, DSV13, DHS12, GORS14, OY16, GN15, Ng6]). We hope that Theorem 1.1.2 as well as the rationality result of [BRV17] are the first steps in extending parts of this picture to non-planar curves.

1.2 Compact moduli spaces of elliptic surfaces

Elliptic fibrations are ubiquitous in mathematics, and the study of their moduli has been approached from many directions; e.g. via Hodge theory [HL02] and geometric invariant theory ([Mir81], [Mir80]). At the same time, moduli spaces often have many geometrically meaningful compactifications leading to different birational models. This leads to rich interactions between moduli theory and birational geometry.

The compact moduli space $\overline{\mathcal{M}}_g$ of genus g stable curves and its pointed analogue $\overline{\mathcal{M}}_{g,n}$ is exemplary. Studying the birational geometry of the moduli space of stable curves by varying the moduli problem has been a subject of active research over the past decade known as the *Hassett-Keel program* (see [FS13] for a survey). Our hope is to produce one of the first instances of this line of study for surfaces (see also [LO16] which initiates a similar line of study for quartic K3 surfaces). Building on previous work in [AB17b] and [AB16], we will continue a study of the birational geometry of the moduli space of *stable elliptic surfaces* initiated by La Nave [LN02].

One particular instance of the birational geometry of $\overline{\mathcal{M}}_{g,n}$ is developed by Hassett in [Has03], where various compactifications $\overline{\mathcal{M}}_{g,\mathcal{A}}$ of the moduli space of *weighted pointed curves* (see Section 8.1.1) are constructed. These compact moduli spaces parameterize degenerations of genus g curves with marked points weighted by a vector $\mathcal{A}$. A natural question is: what happens to the moduli spaces as one varies the weight vector? Among other things, Hassett proves that there are birational morphisms $\overline{\mathcal{M}}_{g,\mathcal{B}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ when $\mathcal{A} \leq \mathcal{B}$ (Theorem 8.1.1). Furthermore, there is a wall-and-chamber decomposition of the space of weight vectors $\mathcal{A}$ – inside a chamber the moduli spaces are isomorphic and there are explicit birational morphisms when crossing a wall.

Hassett's space is the one dimensional example of moduli spaces of *stable pairs*: pairs (X, D) of a variety along with a divisor having mild singularities and satisfying a positivity condition (see Definition 5.1.3). In this case, the variety is a curve C with at worst nodal singularities, the divisor is a weighted sum $D = \sum a_i p_i$ of smooth points on the curve, and one requires $\omega_C(D)$ to be an ample line bundle.

In this part of the thesis, we use stable pairs in higher dimensions to construct analogous compactifications of the moduli space of elliptic surfaces where the pair is given by an elliptic surface with section as well as $\mathcal{A}$-weighted marked fibers. The outcome is a picture for surface pairs which is more intricate, but analogous to that of $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

In general, the story of compactifications of moduli spaces in higher dimensions is quite subtle and relies on the full power of the minimal model program. Many fundamental constructions have been carried out over the past few decades (e.g. [KSB88], [Ale94], & [KP17]). Although stable pairs have been identified as the right analogue of stable curves in higher dimensions, it has proven difficult to find explicit examples of compactifications of moduli spaces in higher dimensions (see [Ale06] for some examples), and thus we take as one goal of thesis to establish a wealth of examples of compact moduli spaces of surfaces that illustrate both the difficulties, as well as methods necessary to overcome them.

More specifically, for admissible weights $\mathcal{A}$ (see Section 7.2.1), we construct and study $\mathcal{E}_{v,\mathcal{A}}$ (Definition 7.1.5): the compactification by stable pairs of the moduli space of $(f\colon X \to C, S + F_{\mathcal{A}})$, where $f\colon X \to C$ is an elliptic surface with chosen section S, marked fibers weighted by $\mathcal{A}$, and fixed volume v.

Theorem 1.2.1 (see Theorem 7.1.4)**.** *For admissible weights $\mathcal{A}$, there exists a moduli pseudofunctor of $\mathcal{A}$-weighted stable elliptic surfaces (see Definition 7.1.1) of volume v so that the main component $\mathcal{E}_{v,\mathcal{A}}$ is representable by a finite type separated Deligne-Mumford stack.*

To construct $\mathcal{E}_{v,\mathcal{A}}$ as an algebraic stack, we use the notion of a family of stable pairs given by Kovács-Patakfalvi in [KP17] and the construction of the moduli stack of stable pairs therein. However, representability of our functor does not follow immediately as we include the additional the data of the map $f\colon X \to C$ (see Section 7.1.1). Furthermore, the correct deformation theory for stable pairs has not yet been settled. As we are interested in the global geometry of the moduli space, we circumvent this issue by working exclusively with the normalization of the moduli stack. By the results of Appendix A, this amounts to only considering the functor on the subcategory of normal varieties.

Theorem 1.2.2 (see Theorem 7.2.7 and Theorem 7.2.9)**.** *The moduli space $\mathcal{E}_{v,\mathcal{A}}$ is proper. Its boundary parametrizes $\mathcal{A}$-broken elliptic surfaces (see Definition 7.1.9 and Figure 1.1).*

As with the previous theorem, it does not follow immediately from known results about stable pairs because of the data of the map $f\colon X \to C$. Rather, we prove

Theorem 1.2.2 by explicitly describing in Section 7.2 an algorithm for stable reduction that produces, as a limit, a stable pair as well as a map to a nodal curve. This is a generalization of the work of La Nave in [LN02]. The main input is the use of twisted stable maps of Abramovich-Vistoli to produce limits of fibered surface pairs as discussed in [AB16] as well as previous results in [AB17b] and [LN02], that describe the steps of the minimal model program on a one parameter family of elliptic surfaces. The final key input is a theorem of Inchiostro (Theorem 7.2.4) which guarantees these are the only steps that occur in the mmp.

Following Hassett, it is natural to ask how the moduli spaces change as we vary $\mathcal{A}$. The strategy in [Has03] is to understand how the objects themselves change as one varies $\mathcal{A}$, and then prove a strong vanishing theorem which guarantees that the formation of the relative log canonical sheaf commutes with base change. This ensures that the process of producing an $\mathcal{A}$-stable pointed curve from a $\mathcal{B}$-stable pointed curve with $\mathcal{A} \leq \mathcal{B}$ is functorial in families and leads to reduction morphisms on moduli spaces and universal families.

In [AB17b], we carried out a complete classification of the relative log canonical models of elliptic surface fibrations, and we extend this result here (see Section 6.1 and Theorem 6.1.10).

In Section 9, we prove an analogous base change theorem which implies that the steps of the minimal model program described in Section 6.1 are functorial in families of elliptic surfaces. The main technical tool is a vanishing theorem (Theorem 9.0.1) which relies on a careful analysis of the geometry of broken slc elliptic surfaces. We do not expect this vanishing theorem to hold in full generality for other classes of slc surfaces.

Theorem 1.2.3 (Invariance of log plurigenera, see Theorems 9.0.1 and 9.0.10)**.** *Let*

$\pi\colon (X \to C, S + F_{\mathcal{B}}) \to B$ be a family of $\mathcal{B}$-broken stable elliptic surfaces over a reduced base B. Let $0 \leq \mathcal{A} \leq \mathcal{B}$ be such that the divisor $K_{X/B} + S + F_{\mathcal{A}}$ is π-nef and $\mathbb{Q}$-Cartier. Then $\pi_*\mathcal{O}_X\Big(m(K_{X/B}+S+F_{\mathcal{A}})\Big)$ is a vector bundle on B whose formation is compatible with base change $B' \to B$ for $m \geq 2$ divisible enough.

The main difficulty in the above theorem, and in the study of stable pairs in general, is that smooth varieties will degenerate into non-normal varieties with several irreducible components. In dimension greater than 1 these *slc* varieties can become quite complicated: see Figure 1.1 for a $\mathcal{B}$-broken elliptic surface that appears in the limit of such a degeneration. Note in particular the map $f\colon X \to C$ is not equidimensional; there are irreducible components of X contracted to a point by f.

These components were first observed in the work of La Nave [LN02] and were coined *pseudoelliptic* surfaces. They are the result of contracting the section of an elliptic surface. In fact La Nave noticed in the study of stable reduction for elliptic surfaces with no marked fibers ($\mathcal{A} = 0$), that a component of the section of f is contracted by the minimal model program if and only if the corresponding component of the base nodal curve C needs to be contracted to obtain a stable curve. We generalize this (Proposition 7.1.13) to the case of marked fibers and as a result obtain a morphism to the corresponding Hassett space by forgetting the elliptic surface and remembering only the base weighted curve:

Theorem 1.2.4 (See Corollary 8.1.3)**.** *There are forgetful morphisms $\mathcal{E}_{\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$.*

Next we identify a wall and chamber decomposition of the space of admissible weights $\mathcal{A}$. In particular, we describe at which $\mathcal{A}$ a one parameter family of broken elliptic surfaces undergo birational transformations leading to different objects on the boundary of the moduli stack. In Section 7.2 we classify three types of birational transformations leading to three types of walls:

- there are $\mathrm{W_I}$ walls coming from the relative log minimal model program for the map $f\colon X \to C$ at which singular fibers change;
- there are $\mathrm{W_{II}}$ walls where a component of the section contracts to form a pseudoelliptic surface;
- there are $\mathrm{W_{III}}$ walls where an entire component of a broken elliptic surface may contract onto a curve or point.

Type $\mathrm{W_I}$ and $\mathrm{W_{III}}$ transformations result in divisorial contractions of the total space of a family of elliptic surfaces while type $\mathrm{W_{II}}$ result in small contractions which must then be resolved by a log flip. La Nave constructed this log flip explicitly and we show that this construction leads to a log flip of the universal family (see Section 8.2 and Figure 8.1). Putting this all together, our main theorem may be summarized as follows:

Theorem 1.2.5. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{Q}^r$ be weight vectors such that $0 \leq \mathcal{A} \leq \mathcal{B} \leq 1$. We have the following:*

(i) If $\mathcal{A}$ and $\mathcal{B}$ are in the same chamber, then the moduli spaces and universal families are isomorphic.

(ii) If $\mathcal{A} \leq \mathcal{B}$ then there are reduction morphisms $\mathcal{E}_{v,\mathcal{B}} \to \mathcal{E}_{v,\mathcal{A}}$ on moduli spaces which are compatible with the reduction morphisms on the Hassett spaces:

$$\begin{tikzcd} \mathcal{E}_{v,\mathcal{B}} \arrow[r] \arrow[d] & \mathcal{E}_{v,\mathcal{A}} \arrow[d] \\ \overline{\mathcal{M}}_{g,\mathcal{B}} \arrow[r] & \overline{\mathcal{M}}_{g,\mathcal{A}} \end{tikzcd}$$

(iii) The universal families are related by a sequence of explicit divisorial contractions and flips $\mathcal{U}_{v,\mathcal{B}} \dashrightarrow \mathcal{U}_{v,\mathcal{A}}$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{U}_{v,\mathcal{B}} \arrow[r, dashed] \arrow[d] & \mathcal{U}_{v,\mathcal{A}} \arrow[d] \\ \mathcal{E}_{v,\mathcal{B}} \arrow[r] & \mathcal{E}_{v,\mathcal{A}} \end{tikzcd}$$

More precisely, across $\mathrm{W_I}$ *and* $\mathrm{W_{III}}$ *walls there is a divisorial contraction of the universal family and across a* $\mathrm{W_{II}}$ *wall the universal family undergoes a log flip.*

The precise descriptions of the various wall crossing morphisms described above are given in Theorem 8.1.4, Corollary 7.2.10, Proposition 8.2.7, Theorem 8.2.1 and Proposition 8.2.4.

Now we will describe the objects that appear the boundary of $\mathcal{E}_{v,\mathcal{A}}$. While the minimal model program lends itself to an algorithmic approach towards finding minimal birational representatives of an equivalence class, it generally does *not* lead to an explicit stable reduction process as prevalent in $\overline{\mathcal{M}}_{g,n}$. However, using the minimal model program in addition to the theory of twisted stable maps developed by Abramovich-Vistoli [AV97], we are able to run an explicit stable reduction process, and classify precisely what objects live on the boundary of our moduli spaces. This is inspired by the work of [LN02].

The idea is that an elliptic fibration $f\colon X \to C$ with section S can be viewed as a rational map from the base curve to $\overline{\mathcal{M}}_{1,1}$, the stack of stable pointed genus one curves. One can use this to produce a birational model of f which can then be studied using twisted stable maps. The outcome is a compact moduli space of *twisted* fibered surface pairs studied in [AB16] which forms the starting point of our analysis of one parameter degenerations in $\mathcal{E}_{v,\mathcal{A}}$.

Combining these degenerations produced by twisted stable maps with the wall crossing transformations discussed above, in Section 7.2 we identify the boundary objects parametrized by $\mathcal{E}_{v,\mathcal{A}}$:

Theorem 1.2.6 (see Theorem 7.2.9)**.** *The boundary of the proper moduli space $\mathcal{E}_{v,\mathcal{A}}$ parametrizes $\mathcal{A}$-broken stable elliptic surfaces (see Definition 7.1.9) which are pairs $(f\colon X \to C, S + F_{\mathcal{A}})$ coming from a stable pair $(X, S + F_{\mathcal{A}})$ with a map to a nodal curve C such that:*

- *X is an slc union of elliptic surfaces with section S and marked fibers, as well as*
- *chains of pseudoelliptic surfaces of Type* I *and* II *(Definitions 7.1.7 and 7.1.8) contracted by f with marked pseudofibers (Definition 6.1.14).*

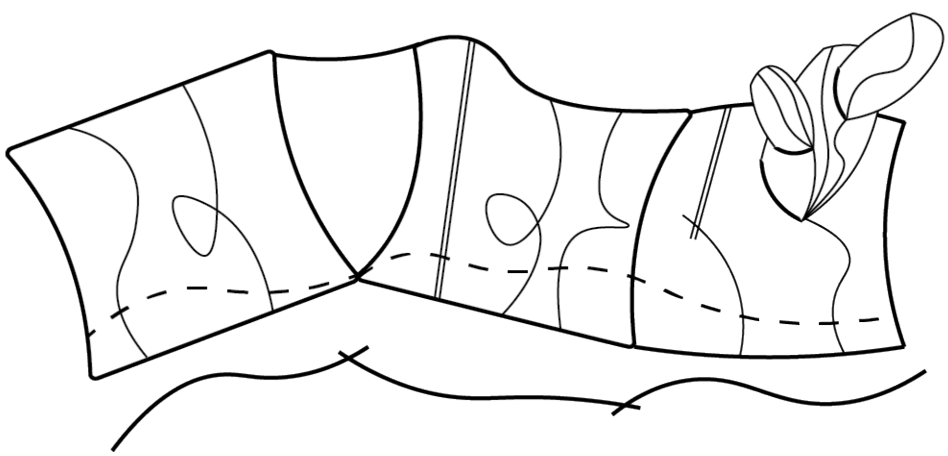


Figure 1.1: An $\mathcal{A}$-weighted broken elliptic surface.

Finally, in the appendix, we prove that in certain situations the normalization of an algebraic stack is uniquely determined by its values on normal base schemes (Proposition A.0.7) and that a morphism between normalizations of algebraic stacks can be constructed by specifying it on the category of normal schemes (Proposition

A.0.6). This material is probably well known but we include it here for lack of a suitable reference.

1.2.1 An example

We illustrate the main results in a specific example. Figure 1.2 depicts the central fiber of a particular one parameter stable degeneration of a rational elliptic surface with twelve marked nodal fibers, ten of which are marked with coefficient one, and the other two with coefficient α, as the coefficient α varies. The arrows depict the directions of the morphisms between the various models of the total space of the degeneration.

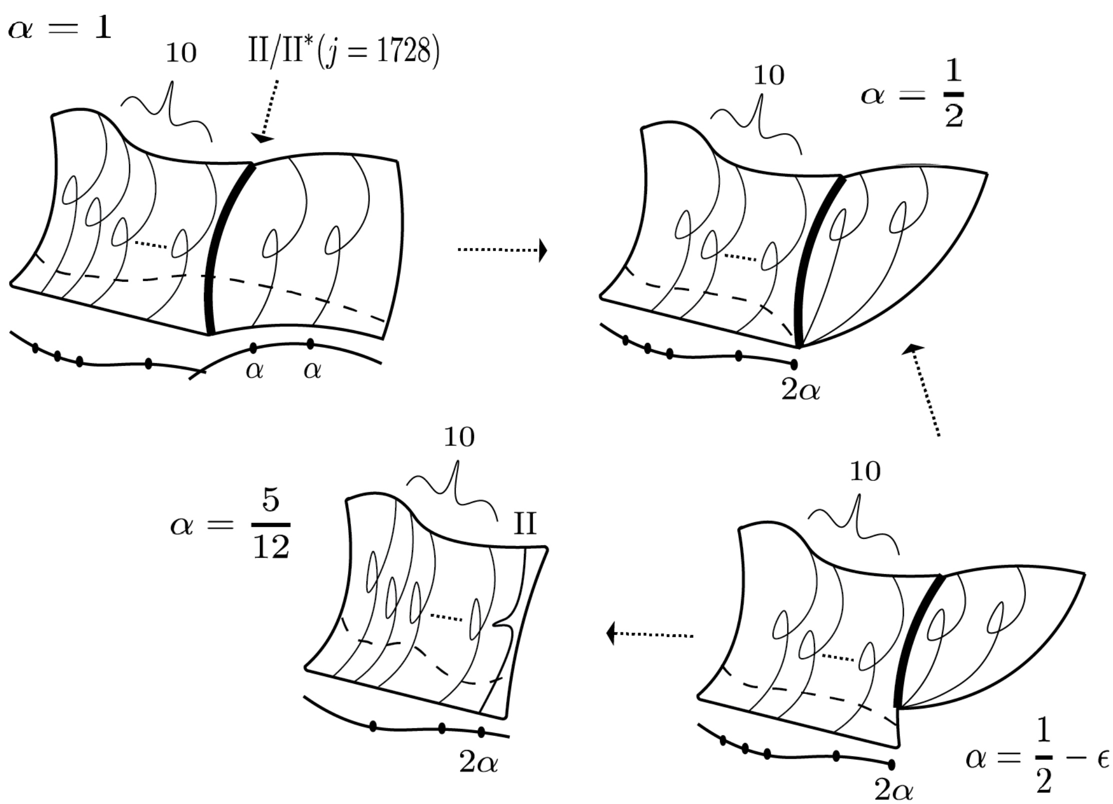


Figure 1.2: The wall crossing transformations on the central fiber of a stable degeneration of a rational elliptic surface.

The central fiber breaks up into a union of two components glued along twisted fibers of type II and II*, one containing 10 marked nodal fibers with coefficient one and a type II twisted fiber, and the other containing two nodal fibers marked with coefficient α and a type II* twisted fiber. At $\alpha = 1/2$ the section of the second component contracts to form a pseudoelliptic surface. At $\alpha = 1/2 - \epsilon$ for any small enough $\epsilon > 0$, this contraction of the section is a log flipping contraction of the total space of the degeneration and a flip results in a different stable model. Finally at $\alpha = 5/12$ the whole pseudoelliptic component contracts to a point yielding an elliptic surface with 10 nodal fibers marked with coefficient one and a type II Weierstrass fiber with coefficient 2α. Each surface maps to the corresponding Hassett stable base curve as depicted.

1.2.2 Applications and further work

A simple corollary of the preceding results is a classification of the singularities of stable degenerations of smooth elliptic surfaces. Combining Theorem 1.2.6 with the results of [AB17b] on singularities of log canonical models of elliptic surfaces (see also Section 6.1) as well as Proposition 6.1.20 we obtain the following:

Corollary 1.2.7. *Let $\mathscr{X}^0 \to \mathscr{C}^0 \to \Delta^0$ be a family of smooth relatively minimal elliptic surfaces over the punctured disc $\Delta^0 = \Delta \setminus \{0\}$ and with a fixed section and all singular fibers marked by a nonzero coefficient. Then after a base change of Δ^0, the family can be extended to $\mathscr{X} \to \mathscr{C} \to \Delta$ such that the central fiber $X \to C$ is a broken elliptic surface. Each component of X is an elliptic or pseudoelliptic surface with only quotient singularities and the singularities are all rational double points except along type* II, III *and* IV *fibers. In particular, the normalization X^ν has klt singularities.*

As another application, in [AB18] the results of this part of the thesis are used to construct a stable pairs compactification of the moduli space of anti-canonically polarized del Pezzo surfaces of degree one. By studying the wall-crossing morphisms we relate this compactification to the GIT compactification of the moduli space of rational elliptic surfaces of Miranda [Mir81]. In addition, we completely calculate all walls in the domain of admissible weights for the case of *rational* elliptic surfaces. Future work will expand upon these ideas, by comparing our compactifications to other compactifications of rational elliptic surfaces in the literature, e.g. the hodge theoretic approaches of Heckman-Looijenga [HL02]. As $\mathcal{E}_{v,\mathcal{A}}$ is modular, the explicit description of the boundary can be used to describe the boundaries of non-modular compactifications such as GIT models and compactifications of period domains.

Finally, we remark on our choice of boundary divisor. We fix the coefficient of the section to be one. This is the key reason that the base curve of a stable elliptic surface is a Hassett stable curve (see Proposition 7.1.13). On the other hand, it is the reason for the formation of pseudoelliptic surfaces which leads to interesting yet complicated behavior across type W_{II} and $\mathrm{W}_{\mathrm{III}}$ walls.

Our marked fibers consist of log canonical models of marked Weierstrass fibers which are classified in Theorem 6.1.10 and the preceding discussion. In particular, there are three types of fibers – Weierstrass fibers, twisted fibers obtained by stable reduction, and intermediate fibers which interpolate between them as the coefficient varies from zero to one. Since our fibers come as log canonical models of the Weierstrass fiber, they have to be marked with 1 on any exceptional divisor of the rational map to the Weierstrass model.

It would be interesting to extend our results to the case where the coefficient of these components and of S varies. When the coefficient of S is very small compared to the other numerical data, one expects to obtain a compactification of the moduli space of Weierstrass fibrations by equidimensional slc elliptic fibrations. This generalization is being carried out by Inchiostro in [Inc18b].

1.2.3 Previous results

La Nave [LN02] used twisted stable maps of Abramovich-Vistoli to prove properness of the moduli stack parameterizing elliptic surfaces in Weierstrass form via explicit stable reduction. He computes the stable models of one parameter families of elliptic surfaces in Weierstrass form. Roughly, given an elliptic surface $f\colon X \to C$ with section S, the Weierstrass form is obtained by contracting all components of the singular fibers of $f\colon X \to C$ that do not meet the section S. We will make repeated use of his work throughout. In our setting, this corresponds to the case of $\mathcal{E}_{\mathcal{A}}$ where $\mathcal{A} = 0$.

Brunyate [Bru15], described the KSBA stable pair limits of elliptic K3s with marked divisor $D = \delta S + \sum_{i=1}^{24} \epsilon F_i$, where $0 < \delta \ll \epsilon \ll 1$, the divisor S is a section, and the F_i are the 24 singular fibers.

In [Ale15], Alexeev provided another generalization of Hassett's picture for $\overline{\mathcal{M}}_{g,\mathcal{A}}$ to surfaces. He constructed reduction morphisms for the compact moduli spaces of *weighted hyperplane arrangements* – the moduli space parametrizing the union of hyperplanes in projective space.

Deopurkar in [Deo18] also suggested an alternate compactification of the moduli space of elliptic surfaces by admissible covers of the stacky curve $\overline{\mathcal{M}}_{1,1}$. It would be interesting to compare his space to those discussed here and in [AB16].

Part I

Hilbert schemes of points on singular curves

CHAPTER 2

Preliminaries

2.1 The Grothendieck ring of varieties

Let k be a field. The **Grothendieck ring of varieties** $K_0(\mathrm{Var}_k)$ of varieties of k is the ring generated by isomorphism classes $[X]$ of finite type schemes over k with the following relations:

- $[X] = [U] + [Z]$ whenever $Z \subset X$ is closed with open complement U,[1] and
- $[X \times Y] = [X][Y]$.

Note that these relations immediately imply that $[X] = [X^{red}]$.

We denote by $\mathbb{L} := \mathbb{A}^1$ the class of the affine line. $K_0(\mathrm{Var}_k)$ satisfies the following universal property. For any ring R and any function

$$\tilde{v} : \mathrm{Var}_k \to R$$

[1]This is called the scissor relation or cut-and-paste relation.

satisfying the relations

- $\tilde{v}(X) = \tilde{v}(X')$ whenever $X \cong X'$,
- $\tilde{v}(X) = \tilde{v}(U) + \tilde{v}(X \setminus U)$ for $U \hookrightarrow X$ an open immersion,
- $\tilde{v}(X \times Y) = \tilde{v}(X)\tilde{v}(Y)$,

there is a unique ring homomorphism $v : K_0(\mathrm{Var}_k) \to R$ such that the following diagram commutes.

$$\begin{tikzcd} K_0(\mathrm{Var}_k) \arrow[rr, "v"] & & R \\ & \mathrm{Var}_k \arrow[ul, "{[\ ]}"] \arrow[ur, "\tilde{v}"'] & \end{tikzcd}$$

Such homomorphism v are called *motivic measures*.

Example 2.1.1. (i) If $k = \mathbb{C}$, then the compactly supported euler characteristic

$$\chi_{top}(X) := \sum_i (-1)^i \dim_{\mathbb{Q}} H^i_c(X, \mathbb{Q}) \in \mathbb{Z}$$

is a motivic measure.

(ii) If $k = \mathbb{F}_q$, the point counting function $\#X(\mathbb{F}_q) \in \mathbb{Z}$ is a motivic measure.

(iii) Let k be a field of characteristic zero. Then the function which sends a smooth projective variety X to its Hodge polynomial

$$\sum_{p,q} \dim_k H^q(X, \Omega^p_X) u^p v^q \in \mathbb{Z}[u, v]$$

extends to a unique motivic measure. This can be proved over $\mathbb{C}$ using Deligne's mixed Hodge theory or over general k using Bittner's presentation of $K_0(\mathrm{Var}_k)$ and weak factorization of birational maps.

In particular, any result about classes in the Grothendieck ring of varieties immediately implies the same result about any of the above invariants.

In [GZLMH04], a power structure on the Grothendieck ring of varieties was constructed. This is an operation makes sense of expressions of the form

$$f(t)^M \in K_0(\mathrm{Var}_k)[\![t]\!]$$

for $f(t)$ a power series with coefficeints in $K_0(\mathrm{Var}_k)$ and constant term 1 and $M \in K_0(\mathrm{Var}_k)$. The power structure satisfies the following properties:

- $f(t)^0 = 1$;
- $f(t)^1 = 1$;
- $(f(t)g(t))^M = f(t)^M g(t)^M$;
- $f(t)^{M+N} = f(t)^M f(t)^N$;
- $f(t)^{MN} = (f(t)^M)^N$;
- $(1+t)^M = 1 + Mt +$ higher order terms;
- $f(t^k)^M = (f(t)^M)\Big|_{t \mapsto t^k}$;
- for X a quasiprojective variety,

$$\left(\frac{1}{1-t}\right)^{[X]} = \sum_{n \geq 0} [\mathrm{Sym}^n(X)] t^n$$

  is the Kapranov zeta function of X.

2.2 Hilbert schemes of points

For X a quasiprojective variety, the Hilbert scheme $\mathrm{Hilb}^d(X)$ is the moduli space for flat families of length d subschemes of X. Using the identification between length d subschemes $Z \subset X$ and ideal sheaves $\mathcal{J}$ with $\mathrm{colength}(\mathcal{J}) := \mathrm{length}(\mathcal{O}_X/\mathcal{J}) = d$, we will often represent the closed points of $\mathrm{Hilb}^d(X)$ by the corresponding ideals.

There is a well defined Hilbert-Chow morphism (see, for example, [FGI$^+$05, Chapter 7])

$$h\colon \mathrm{Hilb}^d(X) \to \mathrm{Sym}^d(X)$$

sending a subscheme to its support:

$$[\mathcal{J}] \mapsto \sum_{p \in \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})} \mathrm{length}(\mathcal{O}_{X,p}/\mathcal{J}_p)[p].$$

When X is a smooth curve, h is an isomorphism.

Let $Y \subset X$ be a closed k-subvariety. Then $\mathrm{Sym}^d(Y) \subset \mathrm{Sym}^d(X)$ is a closed subvariety and we define $\mathrm{Hilb}^d(X,Y)$ the *Hilbert scheme with support in* Y as the scheme theoretic preimage $h^{-1}(\mathrm{Sym}^d(Y))$ by the Hilbert-Chow morphism $h\colon \mathrm{Hilb}^d(X) \to \mathrm{Sym}^d(X)$. Set theoretically, $\mathrm{Hilb}^d(X,Y) \subset \mathrm{Hilb}^d(X)$ consists of length d subschemes $Z \subset X$ with support $\mathrm{supp}(\mathcal{O}_Z)$ contained in Y.

We define the *motivic Hilbert zeta function with support in* Y as

$$Z^{\mathrm{Hilb}}_{Y\subset X}(t) := \sum_{d\geq 0} [\mathrm{Hilb}^d(X,Y)] t^d \in 1 + tK_0(\mathrm{Var})[\![t]\!].$$

We write $Z^{\mathrm{Hilb}}_X(t)$ for the Hilbert zeta function of X where $Y = X$.

2.3 The geometry of singular curves

Consider a reduced curve singularity X with s branches. Let $\tilde{X} \to X$ be the normalization, opposite to the finite extension of rings

$$R \hookrightarrow \tilde{R} \cong \prod_{i=1}^{s} k[\![x_i]\!]$$

where R is the complete local ring of the singularity X. We will identify R with a subring of $\tilde{R}$, and write R_i for the coordinate ring of the branch i^{th} branch X_i. In other words, $R_i \subset k[\![x_i]\!]$ is a finite ring extension corresponding to the i^{th} branch $\varphi_i \colon B_i \to X_i \subset X$ of the normalization.

(1) Let

$$\delta := \dim_k \prod_{i=1}^{s} k[\![x_i]\!]/R$$

be the δ-*invariant* of X. Similarly, we denote by δ_i the δ-invariant $\dim_k k[\![x_i]\!]/R_i$ of the i^{th} branch.

(2) Let

$$\mathfrak{c} := \mathrm{Ann}_R(\tilde{R}/R)$$

be the *conductor ideal*. This is an ideal of both $\tilde{R}$ and R. In particular $\mathfrak{c}$ is generated by a collection of monomials, say $\{x_i^{c_i}\}_{i=1}^{s}$, as an ideal of R. It's clear from the definition that c_i is the smallest positive integer such that for all $n \geq c_i$, $x_i^n \in R$. We will refer to c_i as the conductor of the i^{th} branch, denote by

$$C := \dim \tilde{R}/\mathfrak{c} = \sum_{i=1}^{s} c_i$$

the conductor of X, and by $\underline{c} = (c_1, \ldots, c_s)$ the conductor branch-length vector.

We will need the following result of Schwede:

Proposition 2.3.1. *(Schwede [Sch]) Let $R \subset \tilde{R}$ be the normalization of a reduced ring and let $\mathfrak{c}$ be the conductor ideal. Then*

$$\begin{tikzcd} \mathrm{Spec}\tilde{R}/\mathfrak{c} \arrow[r, hook] \arrow[d] & \mathrm{Spec}\tilde{R} \arrow[d] \\ \mathrm{Spec}R/\mathfrak{c} \arrow[r, hook] & \mathrm{Spec}R \end{tikzcd}$$

is a pushout diagram of schemes.

Proof. We want to show that $R \hookrightarrow \tilde{R}$ is the pullback of $R/\mathfrak{c} \hookrightarrow \tilde{R}/\mathfrak{c}$ along the quotient $\tilde{R} \to \tilde{R}/\mathfrak{c}$. Let A be this fiber product. There is a map $R \to A$ by universal property which is injective since the composition $R \to \tilde{R}$ is. Let $(x, \bar{y}) \in A$ so that $x \in \tilde{R}$, $\bar{y} \in \mathrm{Spec}R/\mathfrak{c}$ and $x + \mathfrak{c} = \bar{y}$. Then $x - y \in \mathfrak{c} \subset R \subset \tilde{R}$ where $y \in R$ is some lift of $\bar{y}$ so $x \in R$ and $R \to A$ is surjective.

$\square$

We use this to show that any reduced curve singularity appears as the unique singularity of a connected rational curve with all irreducible components unibranch.

Corollary 2.3.2. *Let R be the completed local ring of an s-branched curve singularity. There exists a connected affine curve Y with normalization $\bigsqcup_{i=1}^{s} \mathbb{A}^1$ and unique singular point $0 \in Y$ such that $\widehat{\mathcal{O}_{Y,0}} \cong R$ and the diagram*

$$\begin{tikzcd} \mathrm{Spec}\left(\prod_{i=1}^{s} k[\![x_i]\!]\right) \arrow[r] \arrow[d] & \bigsqcup_{i=1}^{s} \mathbb{A}^1 \arrow[d] \\ \mathrm{Spec}R \arrow[r] & Y \end{tikzcd}$$

commutes.

Proof. The composition $\prod_{i=1}^{s} k[x_i] \to \prod_{i=1}^{s} k[\![x_i]\!] \to \tilde{R}/\mathfrak{c}$ is evidently surjective and so induces a closed embedding $\mathrm{Spec}\tilde{R}/\mathfrak{c} \hookrightarrow \bigsqcup_{i=1}^{s} \mathbb{A}^1$. Now we define Y to be the pushout of the diagram

$$\begin{array}{ccc} \mathrm{Spec}\tilde{R}/\mathfrak{c} & \hookrightarrow & \bigsqcup_{i=1}^{s} \mathbb{A}^1 \\ \downarrow & & \\ \mathrm{Spec}R/\mathfrak{c} & & . \end{array}$$

which exists since everything is affine. By Proposition 2.3.1 and the universal property of pushouts, there exists a unique $\mathrm{Spec}R \to Y$ making the diagram in the statement commute. Finally, the induced map $\widehat{\mathcal{O}_{Y,0}} \to R$ is an isomorphism since completion of commutes with fiber products of rings.

□

2.3.1 The branch-length filtration and graded degenerations

Let $R \subset \tilde{R}$ as above be the completed local rings of an s-branched reduced curve singularity X and its normalization $\tilde{X}$. Let $A = \mathcal{O}_Y \subset R$ be the coordinate ring of a rational curve Y as constructed in Corollary 2.3.2 so that the normalization $\tilde{A} = \prod k[x_i] \subset \prod k[\![x_i]\!] = \tilde{R}$.

Denote by $v_i \colon \tilde{R} \to \mathbb{N}$ the composition of the projection onto $k[\![x_i]\!]$ with the valuation on $k[\![x_i]\!]$. This gives the order of vanishing of a function along the i^{th} branch of the normalization.

Definition 2.3.3. We define an $\mathbb{N}^s$-filtration $\tilde{F}^\bullet$ on $\tilde{R}$ by

$$\tilde{F}^{\underline{a}} := \{f \in \tilde{R} \mid v_i(f) \geq a_i \ i = 1, \ldots, s\} \subset \tilde{R}$$

for $\underline{a} \in \mathbb{N}^s$. The restriction to $R \subset \tilde{R}$ is denoted by

$$F^{\underline{a}} := \{f \in R \mid v_i(f) \geq a_i,\ i = 1, \ldots, s\} \subset R.$$

Equivalently, $\tilde{F}^{\underline{a}}$ is the ideal of $\tilde{R}$ generated by $x_i^{a_i}$ for $i = 1, \ldots, s$ and $F^{\underline{a}} = \tilde{F}^{\underline{a}} \cap R$ is the ideal of functions on X vanishing to order at least a_i along B_i. Note in particular that the conductor $\mathfrak{c} = F^{\underline{c}} = \tilde{F}^{\underline{c}}$ and $F^{\underline{a}} = \tilde{F}^{\underline{a}}$ if and only if $\tilde{F}^{\underline{a}} \subset \mathfrak{c}$.

Finally, note that $\tilde{F}^{\underline{\bullet}}$ restricts to filtrations on A and $\tilde{A}$ as well. We will abuse notation and also denote these by $\tilde{F}^{\bullet}$ and $F^{\bullet}$ respectively where the meaning will be clear from context.

Let $\underline{w} = (w_1, \ldots, w_s)$ be a vector of non-negative integers and denote by

$$\underline{w} \cdot \underline{a} = \sum w_i a_i$$

the usual inner product. Given such a weight vector $\underline{w}$, we obtain an $\mathbb{N}$-filtration $\tilde{F}^{\bullet}_{\underline{w}}$ on $\tilde{R}$ (resp. $\tilde{A}$) by

$$\tilde{F}^{n}_{\underline{w}} := \sum_{\underline{w} \cdot \underline{a} \geq n} \tilde{F}^{\underline{a}}.$$

Denote by $F^{\bullet}_{\underline{w}}$ the restriction of $\tilde{F}^{\bullet}_{\underline{w}}$ to R (resp. A).

Definition 2.3.4. The (extended) Rees algebra of an $\mathbb{N}$-filtered ring $(B, F^{\bullet})$ is

$$\mathcal{R}ees(B, F^{\bullet}) := \sum_{n \in \mathbb{Z}} F^n t^{-n} \subset B[t, t^{-1}]$$

where by convention, $F^n = B$ for $n \leq 0$.

The Rees algebra has the following useful properties:

- $\mathcal{R}ees(B, F^{\bullet})$ is flat over $k[t]$ by Lemma 2.3.7 below;

- $\mathcal{R}ees(B, F^\bullet)/(t-a)\mathcal{R}ees(B, F^\bullet) \cong B$ for $a \neq 0$;
- $\mathcal{R}ees(B, F^\bullet)/t\mathcal{R}ees(B, F^\bullet) \cong gr_{F^\bullet}B$ the associated graded ring.

These can be checked directly from the definition. We refer the reader to [Eis13, Chapter 6] for details on the Rees algebra.

We use this to construct an equinormalizable degeneration of a reduced curve singularity to a non-normal toric singularity. We take inspiration from Gröbner theory – by choosing a sufficiently generic weight vector $\underline{w}$, the Rees algebra construction allows us to construct a degeneration whose special fiber is a monomial subring generated by the "$\underline{w}$-leading terms", stated formally below. Let $Y = \mathrm{Spec}A$ be as in Corollary 2.3.2.

Theorem 2.3.5. *There exists a flat family of connected affine curves $\mathscr{Y} \to \mathbb{A}^1$ with a section $\sigma\colon \mathbb{A}^1 \to \mathscr{Y}$ such that $\mathscr{Y} \setminus \sigma$ is smooth over $\mathbb{A}^1$, the δ-invariant and number of branches of the singularity $(\mathscr{Y}_b, \sigma(b))$ are constant for all $b \in \mathbb{A}^1$, $\mathscr{Y}_b \cong Y$ for $b \neq 0$, and $\mathscr{Y}_0 = \mathrm{Spec}A_0$ where $A_0 \subset \tilde{A} = \prod_{i=1}^s k[x_i]$ is a monomial subring.*

Proof. Observe that the filtrations $\tilde{F}^{\underline{a}} = F^{\underline{a}}$ for all $a_i \geq c_i$, that is, A and $\tilde{A}$ agree in degrees above the conductors. In particular, there are only finitely many degrees a_{ij} such that $x_i^{a_{ij}} \in \tilde{A}$ but not A. Pick a positive integral weight vector such that $\underline{w}$ such that $\sum_i a_{ij}w_i$ are distinct integers for all choices of such j. That is, each monomial in low degree of $\tilde{A}$ is in a 1-dimensional graded piece of the split filtration $\tilde{F}^\bullet_{\underline{w}}$. [2]

Now let $\mathscr{Y} = \mathrm{Spec}\mathcal{R}ees(A, F^\bullet_{\underline{w}}) \to \mathbb{A}^1$. This is a flat family with all fibers away from zero isomorphic to Y and central fiber $\mathscr{Y}_0 = \mathrm{Spec}gr_{F^\bullet_{\underline{w}}}A =: \mathrm{Spec}A_0$. The inclusion $\mathcal{R}ees(A, F^\bullet_{\underline{w}}) \subset A[t, t^{-1}]$ induces a dominant morphism $Y \times \mathbb{G}_m \to \mathscr{Y}$. Let $\sigma \subset \mathscr{Y}$ be the smallest closed subscheme through which the the singular locus

[2] By a split filtration, we mean one induced by a direct sum decomposition.

$0 \times \mathbb{G}_m \subset Y \times \mathbb{G}_m$ factors – that is, the scheme theoretic image of the singular locus [Sta18, Tag 01R5]. σ is flat over $\mathbb{A}^1$ by Lemma 2.3.7 below and it has generic degree 1 so it is a section.

As $A \subset \tilde{A}$ is a finite ring extension of filtered rings, there is an induced finite ring extension $A_0 \subset gr_{\tilde{F}^\bullet_{\underline{w}}}\tilde{A}$. But $\tilde{A}$ is already graded so the latter is just $\tilde{A}$ and $A_0 \subset \tilde{A}$ is the normalization. By construction, $\sigma(0)$ is the vanishing locus of the ideal $\tilde{F}^{1,\dots,1} \cap A_0$ so the normalization is an isomorphism on the complement $\mathscr{Y}_0 \setminus \sigma(0)$. In particular, $\mathscr{Y} \setminus \sigma$ is smooth as required.

Furthermore, the normalization can be done in families. Indeed the isotrivial family

$$\mathrm{Spec}\mathcal{R}ees(\tilde{A}, \tilde{F}^\bullet_{\underline{w}}) \to \mathscr{Y}$$

is a simultaneous normalization. It follows that the number of branches and the δ-invariant of $(\mathscr{Y}, \sigma(b))$ is constant ([Tei77] [BG80, Theorem 5.2.2]).

Finally, $A_0 \subset \tilde{A}$ is a graded subalgebra and we chose the weight $\underline{w}$ so that the graded pieces in degrees smaller than the conductor are one dimensional spanned by monomials and A_0 and A agree in degree larger than the conductors so A_0 must be generated by monomials.

$\square$

Remark 2.3.6. A special case of Theorem 2.3.5 for planar unibranch curves is used by Goldin and Teissier (see [GT00, Proposition 3.1]) in order to study simultaneous resolution of a family curve singularities. Recently Kaveh and Murata [KM17] used Rees algebras to construct analagous toric degenerations of projective varieties.

Lemma 2.3.7. *Let $X \xrightarrow{f} Y \to \mathbb{A}^1$ be morphisms of schemes such that Y is the scheme theoretic image of f and $X \to \mathbb{A}^1$ is flat. Then $Y \to \mathbb{A}^1$ is flat. In particular, if $A \subset R$ is a $k[t]$-algebra extension and R is flat over $k[t]$, then so is A.*

Proof. The associated points of X map onto the associated points of its scheme theoretic image Y. A morphism to $\mathbb{A}^1$ is flat if and only if associated points all map to the generic point so the result follows. $\square$

Given an ideal $\mathcal{I} \subset A$, we can define an ideal sheaf $\mathscr{I}$ on $\mathscr{Y}$ by the intersection

$$\mathcal{I}A[t, t^{-1}] \cap \mathcal{R}ees(A, F^{\bullet}_{\underline{w}}).$$

It is evident that $\mathcal{I}_0 := \mathscr{I}/t\mathscr{I}$ is the associated graded ideal of A_0.

Corollary 2.3.8. *Suppose $Z \subset Y$ is a closed subscheme with ideal $\mathcal{I}$ and let $\mathscr{I}$ be as above. Then the closed subscheme $\mathscr{Z} \subset \mathscr{Y}$ cut out by $\mathscr{I}$ is flat over $\mathbb{A}^1$. Furthermore $\mathscr{Z}_b \cong Z$ for $b \neq 0$ and $\mathscr{Z}_0$ is a monomial subscheme.*

Proof. We need only check flatness as the rest follows from the definition of $\mathscr{I}$. By construction, $\mathscr{Z}$ is the scheme theoretic image of the constant family $Z \times \mathbb{G}_m$ under the dominant morphism $Y \times \mathbb{G}_m \to \mathscr{Y}$ so $\mathscr{Z}$ is flat over $\mathbb{A}^1$ by Lemma 2.3.7. $\square$

CHAPTER 3

The Hilbert zeta function is rational

The goal of this chapter is to prove the rationality of the Hilbert zeta function for a singular curve:

Theorem 3.0.1. *Let X be a reduced curve over an algebraically closed field k. Then $Z_X^{\mathrm{Hilb}}(t)$ is a rational function in t with constant term 1.*

This chapter is based on joint work with Ranganathan and Vakil [BRV17].

3.1 Reduction to a local calculation

We first reduce the proof of our main theorem to a local calculation at the singularities of the curve. The Hilbert zeta function respects the scissor relations on X in the following sense.

Lemma 3.1.1. *Let $Y \subset X$ a closed subset with open complement $U \subset X$. Then*

$$Z_X^{\mathrm{Hilb}}(t) = Z_U^{\mathrm{Hilb}}(t) \cdot Z_{Y \subset X}^{\mathrm{Hilb}}(t)$$

Proof. Stratify $\mathrm{Hilb}^d(X)$ into locally closed subsets

$$\mathrm{Hilb}^d(X) = \bigsqcup_{i+j=d} \mathrm{Hilb}^i(U) \times \mathrm{Hilb}^j(X, Y).$$

Here $\mathrm{Hilb}^i(U) \times \mathrm{Hilb}^j(X, Y)$ is the stratum consisting of subschemes of X of length d, such that the length of the subscheme supported on Y is exactly j. This implies that

$$[\mathrm{Hilb}^d(X)] = \sum_{i+j=d} [\mathrm{Hilb}^i(U)] \cdot [\mathrm{Hilb}^j(X, Y)]$$

in $K_0(\mathrm{Var})$ and the result follows. $\square$

Corollary 3.1.2. *Let X be a reduced curve over k with possibly singular points $p_1, \ldots, p_l$. Then*

$$Z_X^{\mathrm{Hilb}}(t) = Z_{X^{sm}}^{\mathrm{Hilb}}(t) \prod_{i=1}^{l} Z_{p_i \subset X}^{\mathrm{Hilb}}(t).$$

where X^{sm} denotes the smooth locus of X.

Upon applying Kapranov's theorem [Kap00, Theorem 1.1.9] in conjunction with the identification of $\mathrm{Hilb}^d(X)$ with $\mathrm{Sym}^d(X)$ for X a smooth curve, we see that $Z_{X^{sm}}^{\mathrm{Hilb}}(t)$ is a rational function. Thus, the proof of the main theorem will follow from the following result:

Theorem 3.1.3. *Let $(X, 0)$ be a reduced curve with singular point $0 \in X$. Then $Z_{0 \subset X}^{\mathrm{Hilb}}(t)$ is a rational function in t with denominator $(1-t)^s$ where s is the number of branches of the singularity $(X, 0)$.*

We will refer to the pair $(X, 0)$ as a curve singularity and $\mathrm{Hilb}^d(X, 0)$ as the punctual Hilbert scheme of $(X, 0)$. Note that $\mathrm{Hilb}^d(X, 0)$ depends only on the completed local ring $R = \widehat{\mathcal{O}_{X,0}}$. In fact there is a natural identification of the punctual Hilbert scheme

$$\mathrm{Hilb}^d(X,0) = \{[\mathcal{J}] \mid \mathcal{J} \subset R,\ \mathrm{colength}(\mathcal{J}) = d\}$$

as a parameter space for colength d ideals in R.

3.1.1 A stratification of the punctual Hilbert scheme

Let R be a reduced complete local ring of dimension 1 over k with residue field k, i.e., the completed local ring of the germ of a k-rational curve singularity. Let $\tilde{R}$ denote its normalization. If $X = \mathrm{Spec}R$ is an s-branched curve singularity, then we have an isomorphism

$$\tilde{X} := \mathrm{Spec}\tilde{R} \cong \mathrm{Spec}\left(k[\![x_1]\!] \times \cdots \times k[\![x_s]\!]\right).$$

Let $B_i := \mathrm{Spec}k[\![x_i]\!]$ be the i^{th} branch of $\tilde{X}$ and $\varphi_i : B_i \to \mathrm{Spec}(R)$ be the normalization map restricted to this branch.

Let $\mathrm{Hilb}^d(X,0)$ denote the punctual Hilbert scheme of points on X. Let

$$\underline{\boldsymbol{a}} = (a_1, \ldots, a_s) \in \mathbb{N}^s$$

be a vector of non-negative integers. For a length d subscheme defined by an ideal $\mathcal{I}$, let $[\mathcal{I}]$ denote the corresponding point in $\mathrm{Hilb}^d(X,0)$, over which the universal subscheme is $Z_{\mathcal{I}} := \mathrm{Spec}(R/\mathcal{I})$. Define the subset $\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}}(X,0)$ to be the locus

$$\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}}(X,0) := \{[\mathcal{I}] \in \mathrm{Hilb}^d(X,0) : \text{for all } i,\ \mathrm{length}(\varphi_i^\star(Z_{\mathcal{I}})) = a_i\}.$$

We will refer to the vector $\underline{\boldsymbol{a}}$ as the *branch-length vector* of the subscheme $Z_{\mathcal{I}}$.

Proposition 3.1.4. *Let d and $\underline{\boldsymbol{a}}$ be as above. The subset* $\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}}(X,0)$ *is a locally closed subscheme of* $\mathrm{Hilb}^d(X,0)$ *(possibly empty).*

Proof. It suffices to prove that for any single branch $\varphi\colon B \to X$,

$$\mathrm{Hilb}^{d,e}(X,0) := \{[\mathcal{I}] \in \mathrm{Hilb}^d(X,0) : \mathrm{length}(\varphi^*(Z_{\mathcal{I}})) = e\}$$

is locally closed. Consider the universal flat family

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d] & X \times \mathrm{Hilb}^d(X,0) \\ \mathrm{Hilb}^d(X,0) & \end{tikzcd}$$

of closed subschemes of X over $\mathrm{Hilb}^d(X,0)$. Pulling back this diagram along

$$(\varphi, \mathrm{id})\colon B \times \mathrm{Hilb}^d(X,0) \to X \times \mathrm{Hilb}^d(X,0)$$

gives us a diagram

$$\begin{tikzcd} \varphi^*\mathcal{Z} \arrow[r, hook] \arrow[d, "\pi"] & B \times \mathrm{Hilb}^d(X,0) \\ \mathrm{Hilb}^d(X,0) & \end{tikzcd}.$$

of closed subschemes of B over $\mathrm{Hilb}^d(X,0)$.

The morphism π is finite, so the function $[\mathcal{I}] \mapsto \mathrm{length}(\pi^{-1}[\mathcal{I}])$ is upper semicontinuous [Har77, Theorem III.2.8]. Thus we can stratify $\mathrm{Hilb}^d(X,0)$ into a disjoint union of locally closed $S_e \subset \mathrm{Hilb}^d(X,0)$ over which π is finite of constant degree e. It remains to check that $S_e = \mathrm{Hilb}^{d,e}(X,0)$. Indeed for any $[\mathcal{I}] \in \mathrm{Hilb}^d(X,0)$, $\pi^{-1}[\mathcal{I}] = \mathrm{Spec}(R/\mathcal{I} \otimes_R B) = \varphi^*(\mathrm{Spec}(R/\mathcal{I}))$, so that $\mathrm{Hilb}^{d,e}(X,0)$ is precisely the locus over which π has constant degree e.

$\square$

Remark

The stratification in the proof above is in fact the set-theoretic version of the flattening stratification for π. Observe that any finite morphism of constant degree is flat, so the restriction of π over each S_e is flat. On the other hand, finite flat morphisms have constant degree. In particular, it follows from the universal property of the flattening stratification that $\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}}(X,0)$ is a moduli space for length d subschemes $Z \subset X$ with $\mathrm{length}(\varphi_i^*(Z)) = a_i$. See [Sta18, Tag 052F] for details on this stratification.

3.2 Degree–branch-length bounds

Our proof proceeds in two main steps. In this section we show that the quantity $d - \sum a_i$ for which $\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}}(X,0)$ is nonempty is uniformly bounded by the invariants of the singularity $(X,0)$. In the next section, we use these bounds to embed $\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}}(X,0)$ into a Grassmannian and show that these locally closed subsets stabilize in the Grothendieck ring.

Lemma 3.2.1. *Let $\mathcal{J} \subset R$ be a finite colength ideal and suppose there exist $f_i \in \mathcal{J}$ for $i = 1, \ldots, s$ with $v_i(f_i) = l_i$. Then*

$$F^{l_1+c_1,\ldots,l_s+c_s} = \tilde{F}^{l_1+c_1,\ldots,l_s+c_s} \subset \mathcal{J}.$$

Proof. The equality between the two ideals is clear since they are both contained in the conductor. We claim that given $f_i \in \mathcal{J}$ with $v_i(f_i) = l_i$ then $x_i^{l_i+m} \in \mathcal{J}$ for all $m \geq c_i$. It suffices to check this one branch at a time so without loss of generality suppose $s = 1$.

Up to scaling, we may write $f = x^l + g(x)$ where $g(x)$ is higher order terms. We have $x^m f = x^{l+m} + x^m g(x) \in \mathcal{J}$ for any $m \geq 2c$. In particular, $x^{l+m} + h(x) \in \mathcal{J}$ for some $h(x)$ of arbitrarily large order. Since $\mathcal{J}$ is finite colength, $x^n \in \mathcal{J}$ for all n large enough so $x^{l+m} \in \mathcal{J}$.

$\square$

Proposition 3.2.2. *Let $\mathcal{I}$ be the ideal of a closed subscheme $Z \subset X$ having length d and branch-length vector $\underline{\boldsymbol{a}} = (a_1, \ldots, a_s)$. Then we have the inclusions*

$$F^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}} \subset \mathcal{I} \subset F^{\underline{\boldsymbol{a}}}.$$

Proof. There is a morphism of R-modules $\mathcal{I} \to \mathcal{I}\tilde{R}_i$ given by composing the inclusion $\mathcal{I} \subset R \subset \tilde{R}$ with the projection $\tilde{R} \to \tilde{R}_i$. The image $\operatorname{im}(\mathcal{I} \to \mathcal{I}\tilde{R}_i)$ generates $\mathcal{I}\tilde{R}_i$ as an R_i-module. Explicitly, this map is just $F \mapsto F \mod (x_1, \ldots, \hat{x}_i, \ldots x_s) \in k[\![x_i]\!]$.

Observe that the quotient $\tilde{R}_i/\mathcal{I}\tilde{R}_i$ has dimension a_i over k. Since the ideals of a power series ring are linearly ordered, it must be isomorphic to $k[\![x_i]\!]/(x_i^{a_i})$. We conclude that the monomial $x_i^{a_i}$ generates $\mathcal{I}\tilde{R}_i$ as an $\tilde{R}_i$-module. In particular, $x_i^{a_i} \in \operatorname{im}(\mathcal{I} \to \mathcal{I}\tilde{R}_i)$ so there exists an $F \in \mathcal{I}$ with

$$F \equiv x_i^{a_i} u(x_i) \mod (x_1, \ldots, \hat{x}_i, \ldots x_s).$$

where $u(x_i)$ is a unit in $k[\![x_i]\!]$. It follows that F can be written $F = x_i^{a_i} u(x_i) + G$ where $G \in \ker(\mathcal{I} \to \mathcal{I}\tilde{R}_i)$ and $x_i G = 0$. In particular, $v_i(F) = a_i$. As this holds for each i, we may apply Lemma 3.2.1 to obtain an inclusion

$$F^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}} = \tilde{F}^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}} \subset \mathcal{I}$$

as required.

On the other hand, since $\mathcal{I}$ has branch-length vector $(a_1, \ldots, a_s)$, then the order of vanishing of $f \in \mathcal{I}$ along the branch B_i cannot have valuation smaller than a_i. Applying this to each branch, we see that $\mathcal{I} \subset F^{\underline{\boldsymbol{a}}}$.

$\square$

Proposition 3.2.3. *Let $\mathcal{I}$ be the ideal of a closed subscheme $Z \subset X$ having length d and branch-length vector $\underline{\boldsymbol{a}} = (a_1, \ldots, a_s)$. Then we have*

$$-\delta \leq d - \sum_{i=1}^{s} a_i \leq C - \delta.$$

where the second inequality is strict if $(X, 0)$ is not smooth.

Proof. By Proposition 3.2.2, there are surjections

$$R/F^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}} \to R/\mathcal{I} \to R/F^{\underline{\boldsymbol{a}}}$$

which give us bounds

$$\dim_k R/F^{\underline{\boldsymbol{a}}} \leq d \leq \dim_k R/F^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}}.$$

For the upper bound, note that

$$\begin{aligned} d &\leq \dim_k R/F^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}} \\ &= \dim \tilde{R}/\tilde{F}^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}} - \dim \tilde{R}/R \\ &= \sum_{i=1}^{s} a_i + \sum_{i=1}^{s} c_i - \delta. \end{aligned}$$

where we have used that $F^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}} = \tilde{F}^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}}$. Note however, that we have equality if and only if $J = F^{\underline{\boldsymbol{a}}+\underline{\boldsymbol{c}}}$. If $(X,0)$ is not smooth, then this is impossible given that necessarily some $c_i > 0$ but J has length profile $\underline{\boldsymbol{a}}$. Thus $d \leq \sum a_i + C - \delta - 1$ when $(X,0)$ is not smooth.

The lower bound is more delicate as $\dim_k R/F^{\underline{\boldsymbol{a}}}$ depends on how the filtration on the normalization $\tilde{F}^{\underline{\boldsymbol{a}}}$ meets the singularity $R \subset \tilde{R}$, and thus could have complicated combinatorics. To overcome this difficulty, we use an equinormalizable degeneration of X to a toric singularity and observe that in the toric case, the filtration is controlled by monomials. The lower bound is more apparent in this case.

Choose $Y = \mathrm{Spec} A$ a rational curve as in Proposition 2.3.2 with normalization $\tilde{A}$ and let $\mathscr{Y} \to \mathbb{A}^1$ be an equinormalizable degeneration as in Theorem 2.3.5 to a monomial curve $\mathscr{Y}_0$. By Corollary 2.3.8, applied to the idea $F^{\underline{\boldsymbol{a}}} \subset A$, there is a flat family of subschemes $\mathscr{Z} \subset \mathscr{Y}$ of the total space whose nonzero fibers are each isomorphic to $\mathrm{Spec} A/F^{\underline{\boldsymbol{a}}}$. Furthermore, the special fiber $\mathscr{Z}_0$ is identified with $\mathrm{Spec} A_0/F_0$ where $F_0 = \tilde{F}^{\underline{\boldsymbol{a}}} \cap A_0$ is a monomial ideal of the monomial subring $A_0 \subset A$.

The algebra A_0/F_0 has a monomial basis, specifically consisting of those monomials $x_i^n \in \tilde{A}$ for $0 \leq n \leq a_i - 1$ that are contained in the toric singularity defined by A_0. The number of branches and δ-invariant of $\mathscr{Y}_0$ are the same as that of Y, which are in turn the same as that of $(X,0)$. By flatness of the degeneration (Corollary 2.3.8) we conclude that

$$\begin{aligned} d &\geq \dim_k A/F^{\underline{\boldsymbol{a}}} \\ &= \dim_k A_0/F_0 \\ &\geq \left(\sum_{i=1}^{s} a_i\right) - \delta, \end{aligned}$$

as desired. $\square$

3.3 Motivic stabilization for branch-length strata

In this section we prove the key stabilization result from which we deduce rationality. As a corollary we have that the dimensions of the punctual Hilbert schemes stabilize (Corollary 3.3.2). This is a generalization of [PS92, Theorem 3]. Following the ideas of [PS92], we use the uniform bounds on an ideal with fixed branch-length vector proved in the previous section to embed the strata $\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}}$ as subvarieties of a fixed Grassmannian of $\tilde{R}/N_0$ for an appropriate R-submodule $N_0 \subset \tilde{R}$. The image of this embedding lies inside a generalization of the Pfister–Steenbrink variety $\mathcal{M}$ defined in [PS92, Section 2].

We now argue that incrementing one entry in the branch-length vector stabilizes once the length on that branch is larger than the conductor. Note that for fixed d, the number of possible branch vectors of length d subschemes on a given singularity is finite. Fix an integer tuple $\underline{\boldsymbol{a}}' = (a_1, \ldots, a_{s-1})$ of length $s-1$. Let $\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}',e}(X,0)$ denote the stratum of $\mathrm{Hilb}^{d,a_1,\ldots,a_{s-1},e}$ with branch-length vector $(a_1, \ldots, a_{s-1}, e)$.

Theorem 3.3.1. *Let $(X,0)$ be a reduced curve singularity with s branches. Then for $e \geq c_s$, we have equalities*

$$[\mathrm{Hilb}^{d,\underline{\boldsymbol{a}}',e}(X,0)] \cong [\mathrm{Hilb}^{d+1,\underline{\boldsymbol{a}}',e+1}(X,0)].$$

in the Grothendieck ring.

Proof. We introduce the quantity

$$\alpha := \sum_{k=1}^{s-1} a_i.$$

There is an inclusion of R-modules

$$F^{a_1+c_1,\dots,a_{s-1}+c_{s-1},e+c_s} \subset \mathcal{I} \subset F^{a_1,\dots,a_{s-1},e} \subset \tilde{R}$$

for any $[\mathcal{I}] \in \mathrm{Hilb}^{d,\underline{\boldsymbol{a}}',e}$ by Proposition 3.2.2. Moreover, we have inequalities

$$d - \alpha + \delta - C \leq e \leq d - \alpha + \delta$$

by Proposition 3.2.3. Together, these produce the inclusions

$$N_1 := F^{a_1+c_1,\dots,a_{s-1}+c_{s-1},d-\alpha+\delta+c_s} \subset \mathcal{I} \subset F^{a_1,\dots,a_{s-1},d-\alpha+\delta-C}$$

Define

$$\epsilon_d := (x_1^{-a_1}, \dots, x_{s-1}^{-a_{s-1}}, x_s^{-d+\alpha-\delta+C}) \in \mathrm{Frac}(\tilde{R}) = \prod_{i=1}^{s} k(\!(x_i)\!)$$

where $\mathrm{Frac}(\tilde{R})$ is the total ring of fractions of $\tilde{R}$. Multiplication by ϵ_d is R-module automorphism of $\mathrm{Frac}(\tilde{R})$ and leads to an inclusion

$$N_0 := F^{c_1,\dots,c_{s-1},C+c_s} \subset \epsilon_d \mathcal{I} \subset \tilde{R}$$

of R-modules. Note that N_0 depends only on which entry of the branch-length vector is varying and not on the specific values of $d, \underline{\boldsymbol{a}}'$, or e.

We now compute the dimension

$$\dim_k(\epsilon_d \mathcal{I}/N_0) = \dim_k(\mathcal{I}/N_1).$$

By additivity of dimension, the right hand side is equal to

$$\dim_k(\tilde{R}/N_1) - \dim_k(\tilde{R}/R) - \dim_k(R/\mathcal{I}) = C.$$

Note that this dimension is independent of $d, \underline{\boldsymbol{a}}'$, and e. As a consequence we have a well defined map

$$\phi_{d,e}\colon \operatorname{Hilb}^{d,\underline{\boldsymbol{a}}',e}(X,0) \to \operatorname{Gr}(C, \tilde{R}/N_0)$$

given by

$$\mathcal{I} \mapsto \epsilon_d \mathcal{I}/N_0.$$

This is an embedding into the closed subvariety $\mathcal{M} \subset \operatorname{Gr}(C, \tilde{R}/N_0)$ consisting of those subspaces of $\tilde{R}/N_0$ that are R-submodules. To see this is a closed subvariety, apply the following observation to the generators of R: if V is a vector space and $f\colon V \to V$ is a linear map, then the set of f-stable subspaces of V is closed in the Grassmannian.

Suppose $e \geq c_s$ and let $[\epsilon_d \mathcal{I}/N_0] \in \operatorname{im}(\phi_{d,e})$. Consider the R-submodule $\xi_s \mathcal{I} \subset \tilde{R}$ where

$$\xi_s = (\underbrace{1, \ldots, 1}_{s-1 \text{ times}}, x_s)$$

Since $e \geq c_s$, the set $\xi_s \mathcal{I}$ is contained in R and so defines an ideal. Multiplication by ξ_s doesn't change the branch-length vector $\underline{\boldsymbol{a}}' = (a_1, \ldots, a_{s-1})$, but increases the length along the s^{th} branch/ It follows that

$$[\xi_s \mathcal{I}] \in \operatorname{Hilb}^{d+1,\underline{\boldsymbol{a}}',e+1}(X,0).$$

We have that $\epsilon_{d+1}\xi_s \mathcal{I}/N_0 = \epsilon_d \mathcal{I}/N_0$. It follows that $[\epsilon_d \mathcal{I}/N_0] \in \operatorname{im}(\phi_{d+1,e+1})$ and

$\mathrm{im}(\phi_{d,e})$ is contained in $\mathrm{im}(\phi_{d+1,e+1})$.

On the other hand, suppose $[\epsilon_{d+1}\mathcal{J}/N_0] \in \mathrm{im}(\phi_{d+1,e+1})$. By the same argument as above, we see that $\xi_s^{-1}\mathcal{J} \subset R$ is an ideal of length d with branch-length vector $(a_1, \ldots, a_{s-1}, e)$ so that $[\epsilon_{d+1}\mathcal{J}/N_0] = [\epsilon_d \xi_s^{-1}\mathcal{J}/N_0] \in \mathrm{im}(\phi_{d,e})$ and $\mathrm{im}(\phi_{d+1,e+1}) = \mathrm{im}(\phi_{d,e})$. The result follows since the maps $\phi_{d,e}$ are embeddings.

□

Corollary 3.3.2. *Let $(X, 0)$ be a reduced curve singularity. Then the dimension of $\mathrm{Hilb}^n(X, 0)$ stabilizes.*

3.3.1 Conclusion of the proof of the Main Theorem

Now we are equipped to conclude the proof of the Main Theorem. As observed in Section 3.1, this reduces to the proving Theorem 3.1.3.

Proof of Theorem 3.1.3. We compute $Z^{\mathrm{Hilb}}_{0\subset X}(t)$ by stratifying $\mathrm{Hilb}^d(X, 0)$ into branch-length strata $\mathrm{Hilb}^{d,\underline{\mathbf{a}}}(X, 0)$. By Proposition 3.2.3, for each d there are only finitely many branch-length vectors $\underline{\mathbf{a}}$ for which $\mathrm{Hilb}^{d,\underline{\mathbf{a}}}(X, 0)$ is non-empty. Thus we may compute as follows.

$$\begin{aligned} \sum_{d\geq 0}[\mathrm{Hilb}^d(X,0)]t^d &= \sum_{d\geq 0}\sum_{a_1,\ldots,a_s}[\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X,0)]t^d \\ &= \sum_{a_1,\ldots,a_s}\sum_{d\geq 0}[\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X,0)]t^d \end{aligned}$$

By Theorem 3.3.1, the Hilbert schemes stabilize under incrementing entries of the branch-length vector, once the lengths are beyond the conductor. Precisely, we have an equality

$$[\mathrm{Hilb}^{d+k,a_1,\ldots,c_i+k,\ldots a_s}(X,0)] = [\mathrm{Hilb}^{d,a_1,\ldots,c_i,\ldots a_s}(X,0)]$$

for any i and $k \geq 0$. Thus for fixed $a_1, \ldots, \hat{a}_i, \ldots, a_s$ we can sum over a_i to get

$$\sum_{a_i \geq 0} \sum_{d \geq 0} [\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X,0)]t^d = \sum_{a_i=0}^{c_i-1} \sum_{d \geq 0} [\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X,0)]t^d + \frac{1}{1-t} \sum_{d \geq 0} [\mathrm{Hilb}^{d,a_1,\ldots,c_i,\ldots,a_s}(X,0)]t^d.$$

By applying this to each branch and manipulating the summand, we calculate as follows.

$$\begin{aligned}
\sum_{a_1,\ldots,a_s} \sum_{d \geq 0} [\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X,0)]t^d = & \sum_{a_1=0}^{c_1-1} \cdots \sum_{a_s=0}^{c_s-1} \sum_{d \geq 0} [\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X,0)]t^d \\
& + \sum_{a_1=0}^{c_1-1} \cdots \sum_{a_{s-1}=0}^{c_{s-1}-1} \frac{1}{1-t} \sum_{d \geq 0} [\mathrm{Hilb}^{d,a_1,\ldots,c_s}(X,0)]t^d \\
& + \sum_{a_1=0}^{c_1-1} \cdots \sum_{a_{s-2}=0}^{c_{s-2}-1} \frac{1}{(1-t)^2} \sum_{d \geq 0} [\mathrm{Hilb}^{d,a_1,\ldots,c_{s-1},c_s}(X,0)]t^d \\
& \vdots \\
& + \frac{1}{(1-t)^s} \sum_{d \geq 0} [\mathrm{Hilb}^{d,c_1,\ldots,c_s}(X,0)]t^d
\end{aligned}$$

Finally, by Proposition 3.2.3, for each fixed branch-length vector $(a_1, \ldots, a_s)$, the value of d is bounded above and below. Thus the sums over d on the right hand side are all finite so we conclude that the left hand side $Z^{\mathrm{Hilb}}_{0 \subset X}(t)$ is a rational function with denominator $(1-t)^s$.

$\square$

3.4 Extended example: the coordinate axes

The bounds in Proposition 3.2.3 and the stabilization in Theorem 3.3.1 yield an effective method for computing the Hilbert schemes of many curve singularities. We illustrate this in the following example.

Let $(X_N, 0)$ be the germ at the origin of the coordinate axes $V(x_ix_j = 0) \subset \mathbb{A}^N$. The normalization $\tilde{X}_N \to X_N$ consists of N branches mapping isomorphically to the branches of X_N. On coordinate rings, there is an inclusion

$$R = k[\![x_1, \dots, x_N]\!]/(\{x_ix_j\}_{i \neq j}) \subset \prod_{i=1}^{N} k[\![x_i]\!] = \tilde{R}.$$

We have $s = N$, $\delta = N - 1$, $\underline{c} = (1, \dots, 1)$ and $C = N$.

Let $\underline{a} = (a_1, \dots, a_s)$ be a branch-length vector and $[\mathcal{I}] \in \mathrm{Hilb}^{d,\underline{a}}(X_N, 0)$. By Proposition 3.2.3,

$$0 \leq \sum_{i=1}^{s} a_i - d \leq N - 1.$$

Since $c_i = 1$ for all i, it follows by Theorem 3.3.1 that the branch-length strata $\mathrm{Hilb}^{d,\underline{a}}(X_N, 0)$ stabilize at $a_i = 1$. In this case the above bounds become

$$0 \leq N - d \leq N - 1$$

or $1 \leq d \leq N$.

For each d, $\mathrm{Hilb}^{d,1,\dots,1}(X_N, 0)$ embeds into $Gr(N - d + 1, V)$ where

$$V = \langle x_1, \dots, x_N \rangle = F^{1,\dots,1}/F^{2,\dots,2}.$$

Explicitly, the embedding $\phi\colon \mathrm{Hilb}^{d,1,\dots,1}(X_N, 0) \subset Gr(N - d + 1, V)$ is given by

$$[\mathcal{I}] \mapsto [\mathcal{I}/F^{2,\dots,2}] \in Gr(N - d + 1, V)$$

where we have the containments $F^{2,\dots,2} \subset \mathcal{I} \subset F^{1,\dots,1}$ by Proposition 3.2.2.

As $\underline{c} = (1, \ldots, 1)$, multiplication by x_i acts by 0 on V so every subspace of V is an R-module. In particular, $\mathrm{im}(\phi)$ consists precisely of the locus of subspaces $W \subset V$ which have branch-length vector $(1, \ldots, 1)$. Equivalently, $W \subset V$ must not lie inside any coordinate hyperplane of V under the given coordinates. Denoting the open subset of the Grassmannian parametrizing such W by $Gr(N-d+1, V)^0$, we conclude that

$$[\mathrm{Hilb}^{d,1,\ldots,1}(X_N, 0)] = [Gr(N-d+1, V)^0].$$

Putting this all together, we obtain

Proposition 3.4.1. *Let $(X_N, 0)$ be the germ at the origin of the coordinate axes in $\mathbb{A}^N$. Then the Hilbert zeta function is the rational function*

$$Z^{\mathrm{Hilb}}_{0 \subset X_N}(t) = 1 + \frac{1}{(1-t)^N} \sum_{d=1}^{N} \left[Gr(N-d+1, V)^0 \right] t^d.$$

In particular, $[\mathrm{Hilb}^d(X_N, 0)]$ is a polynomial in $\mathbb{L}$ for all d and N.

Proof. This follows from the description of $[\mathrm{Hilb}^{d,1,\ldots,1}(X_N, 0)]$, Theorem 3.3.1 and the computation in Section 3.3.1. Finally, note that $Gr(k, V)^0$ is the complement of the union of $Gr(k, V_i) \subset Gr(k, V)$ where $V_i \subset V$ are the coordinate hyperplanes. By inclusion-exclusion it follows that $[Gr(k, V)^0]$ is a polynomial in $\mathbb{L}$. $\square$

Remark 3.4.2. Zheng [Zhe16, Section 2.3] has also performed the same computation for the coordinate axes using different methods.

3.4.1 The axes in three space

When $N = 3$ we get a particularly pleasant picture. In this case, V is 3-dimensional and we may compute

$$\begin{aligned}
\mathrm{Hilb}^{1,1,1,1}(X_3, 0) &= Gr(3, V)^0 = pt \\
\mathrm{Hilb}^{2,1,1,1}(X_3, 0) &= Gr(2, V)^0 = \mathbb{P}^2 \setminus \{P_1, P_2, P_3\} \\
\mathrm{Hilb}^{3,1,1,1}(X_3, 0) &= Gr(1, V)^0 = \mathbb{P}^2 \setminus (L_1 \cup L_2 \cup L_3)
\end{aligned}$$

where P_i are the distinguished points corresponding to the coordinate hyperplanes in V and L_i are the distinguished lines corresponding to the space of lines in the coordinate hyperplanes of V.

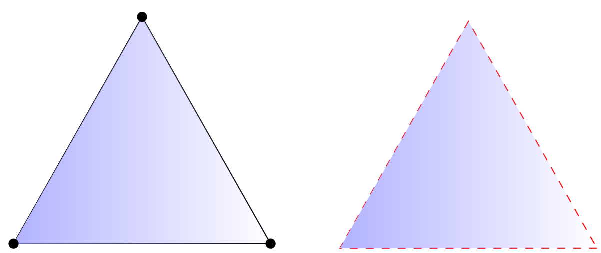


Figure 3.1: On the left, the stratum $\mathrm{Hilb}^{2,1,1,1}(X_3, 0)$ whose closure contains three zero dimensional strata corresponding to twisting $\mathrm{Hilb}^{1,1,1,1}(X_3, 0)$ along each of the three branches. On the right, the stratum $\mathrm{Hilb}^{3,1,1,1}(X_3, 0)$ whose closure contains coordinate lines inside of $\mathrm{Hilb}^{3,2,1,1}(X_3, 0)$ and its permutations.

The closure of $\mathrm{Hilb}^{2,1,1,1}(X_3, 0)$ contains three strata for branch-length $(2, 1, 1)$ and its permutations. These are simply the points P_i with the image of $\mathrm{Hilb}^{1,1,1,1}(X_3, 0)$ under the identification from Theorem 3.3.1. Concretely, P_1 corresponds to the ideal (x_1^2, x_2, x_3) and similarly for P_2 and P_3. These are all the possible strata for $d = 2$.

For $d = 3$ we have the new stratum $\mathrm{Hilb}^{3,1,1,1}(X_3, 0)$ as well as the strata coming from $\mathrm{Hilb}^2(X_3, 0)$ by twisting along the various branches as in Theorem 3.3.1. This corresponds to gluing in copies of $\mathbb{P}^2$ along each of the lines L_i which gives $\mathrm{Hilb}^3(X_3, 0)$

as a union of 4 copies of $\mathbb{P}^2$ glued along coordinate lines.

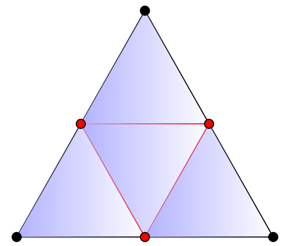


Figure 3.2: The reduced Hilbert scheme $\mathrm{Hilb}^3(X_3, 0)$ consists of 4 copies of $\mathbb{P}^2$ glued along coordinate lines as depicted. The solid shaded strata are $\mathrm{Hilb}^{3,2,1,1}(X_3, 0)$ and its permutations. The center stratum is $\mathrm{Hilb}^{3,1,1,1}(X_3, 0)$ and the vertices correspond to strata obtained by twisting $\mathrm{Hilb}^{1,1,1,1}(X_3, 0)$.

For larger d, the strata stabilize and are all obtained from twisting the strata for $d-1$ along each branch resulting in an arrangement of $\mathbb{P}^2$'s glued along coordinate lines with dual complex a regular subdivision of the triangle.

3.5 Nonreduced curves

It is natural to ask if the above rationality holds for nonreduced curves. For *generically reduced* curves one expects the answer to be yes:

Conjecture 3.5.1. *Let $(C, 0)$ be the germ of a generically reduced curve singularity. Then $Z^{Hilb}_{0\subset C}(t)$ is a rational function.*

For generically nonreduced curves, the picture seems more complicated. Here we compute the example of a "uniformly thickened" planar curve.

Proposition 3.5.2. *Let $C_n = \mathrm{Spec}k[x, y]/(y^n)$. Then the Hilbert zeta function of C_n is given by*

$$Z^{\mathrm{Hilb}}_{0\subset C_n}(t) := \sum_d [\mathrm{Hilb}^d(C_n, 0)]t^d = \prod_{m=1}^{n} \left(\frac{1}{1-\mathbb{L}^{m-1}t^m}\right).$$

Remark 3.5.3. Note that

$$\lim_{n\to\infty} Z^{\mathrm{Hilb}}_{0\subset C_n}(t) = \prod_{m=1}^{\infty}\left(\frac{1}{1-\mathbb{L}^{m-1}t^m}\right) = Z^{\mathrm{Hilb}}_{(0\subset\mathbb{A}^2)}(t)$$

as expected (see Proposition 3.5.4).

Before we prove the proposition, we give some background on $\mathrm{Hilb}^d(\mathbb{A}^2)$ following [Hai98, ES87].

3.5.1 Hilbert scheme of points on the plane

The action of $(\mathbb{C}^*)^2$ on $k[x,y]$ by $(t_1,t_2).(x,y) = (t_1x, t_2y)$ induces an action on $\mathrm{Hilb}^d(\mathbb{A}^2)$. The fixed points of the torus action are indexed by partitions $\lambda \vdash d$.

We denote the monomial ideal by I_λ and define

$$B_\lambda := \{x^iy^j \mid (i,j)\in\lambda\}$$

and

$$Z_\lambda := \mathrm{Spec} k[x,y]/I_\lambda.$$

The subset

$$U_\lambda := \{[Z]\in \mathrm{Hilb}^n(\mathbb{A}^2) \mid B_\lambda \text{ spans } \mathcal{O}_Z\}$$

is a maximal torus invariant open affine neighborhood of $[Z_\lambda]$. Coordinate functions on U_λ are given by $c^{r,s}_{i,j}$ satisfying

$$x^ry^s = \sum_{\lambda} c^{r,s}_{i,j}x^iy^j \mod I$$

for $[I]\in U_\lambda$.

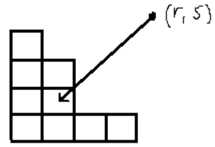


Figure 3.3: The function $c_{i,j}^{r,s}$ depicted as an arrow from box (r, s) to box (i, j).

We represent these as arrows starting at box (r, s) and ending at box $(i, j) \in \lambda$. Note that if $(r, s) \in \lambda$, then

$$c_{i,j}^{r,s} \equiv \begin{cases} 1 & (r, s) = (i, j) \\ 0 & \text{else} \end{cases}$$

Therefore, the nonconstant functions correspond to arrows that start at $(r, s) \in \mathbb{N}^2 \setminus \lambda$ and end in λ.

For each box $(i, j) \in \lambda$, there are two distinguished arrows $d_{i,j}$ and $u_{i,j}$ pointing southeast and northwest respectively as depicted:

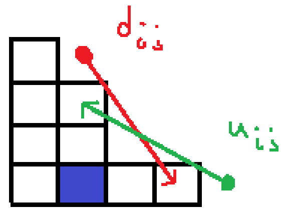


Figure 3.4: The distinguished arrows $d_{i,j}$ and $u_{i,j}$ associated to box (i, j) in blue.

The torus acts on $\mathcal{O}_{U_\lambda}$ by

$$(t_1, t_2) \cdot c_{i,j}^{r,s} = t_1^{r-i} t_2^{s-j} c_{i,j}^{r,s}$$

The cotangent space T_λ^* to the monomial subscheme $[Z_\lambda]$ in U_λ has basis given by the set distinguished arrows $d_{i,j}$ and $u_{i,j}$ as (i, j) runs through each box in λ.

Let $\sigma : \mathbb{C}^* \to (\mathbb{C}^*)^2$ be a 1-parameter subgroup $\sigma(t) = (t^p, t^q)$ for $q \gg p > 0$. This induces a Bialynicki-Birula decomposition of $\mathrm{Hilb}^d(\mathbb{P}^2)$ into affine cells. With these weights, a cell is either contained in $\mathrm{Hilb}^n(\mathbb{A}^2, 0)$ or is disjoint from it. Thus we get a Bialynicki-Birula decomposition of $\mathrm{Hilb}^d(\mathbb{A}^2, 0)$ into affine cells $D_\lambda \cong \mathbb{A}^{b(\lambda)}$ indexed by partitions.

The cell $D_\lambda \subset U_\lambda$ is the vanishing locus of all *positive weight* coordinate functions $c_{i,j}^{r,s}$. From the choice of weights, we see that $c_{i,j}^{r,s}$ is positive weight for σ if and only if $s > j$ (weakly south pointing arrows) or $s = j$ and $r > i$ (strictly west pointing arrows). In particular, the cotangent space to $[Z_\lambda]$ in D_λ is spanned by the set of $u_{i,j}$ that are not horizontal. Therefore

$$b(\lambda) = \dim D_\lambda = \#\{(i, j) \in \lambda \mid u_{i,j} \text{ is not horizontal } \}$$

Let $|\lambda|$ denote the number of boxes, $h(\lambda)$ the height (longest column) of the diagram and $l(\lambda)$ the length (longest row) of the diagram. Then a combinatorial argument shows that

$$b(\lambda) = |\lambda| - l(\lambda)$$

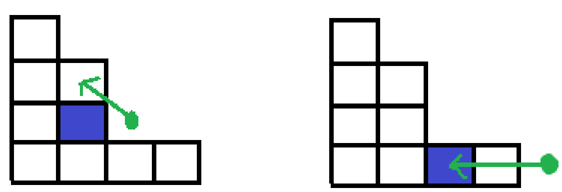


Figure 3.5: The arrow $u_{i,j}$ is not horizontal if and only if the box (i, j) is not top most in its column. Such boxes are clearly in bijection with boxes not in the first row.

Now we can compute $Z^{\mathrm{Hilb}}_{(0\subset\mathbb{A}^2)}(t)$.

Proposition 3.5.4.

$$Z^{\mathrm{Hilb}}_{(0\subset\mathbb{A}^2)}(t) = \prod_{m=1}^{\infty}\left(\frac{1}{1-\mathbb{L}^{m-1}t^m}\right)$$

Proof. Since $\mathrm{Hilb}^d(\mathbb{A}^2,0)$ is stratified by affine spaces, it suffices to compute the Betti number generating function. We see from above that

$$b_{2i}(\mathrm{Hilb}^d(\mathbb{A}^2,0)) = \#\{\lambda \vdash d \mid b(\lambda) = i\}.$$

Let

$$P(q,t) := \sum_{\lambda} q^{l(\lambda)}t^{|\lambda|}.$$

Then the generating function for the Betti numbers (up to a factor of 2) is

$$P(1/q,qt) = \sum_{\lambda} q^{|\lambda|-l(\lambda)}t^{|\lambda|}.$$

Since $l(\lambda)$ is the number of parts (columns) of the partition, $P(q,t)$ is just the generating function for the number of parts of a partition. This is

$$P(q,t) = \prod_{m\geq 1}\left(\frac{1}{1-qt^m}\right)$$

and we get the result by substituting $q \mapsto 1/\mathbb{L}$ and $t \mapsto \mathbb{L}t$. $\square$

3.5.2 Proof of Proposition 3.5.2

Let $C_n = \mathrm{Spec}k[x,y]/(y^n)$. An ideal I defines a subscheme of C_n if and only if $y^n \in I$. It follows that $\mathrm{Hilb}^d(C_n)$ is locally out of $\mathrm{Hilb}^d(\mathbb{A}^2)$ by the vanishing of the functions $c^{0,n}_{i,j}$ for all $(i,j) \in \lambda$ on the open set U_λ.

In particular, a monomial ideal I_λ defines a subscheme of C_n if and only if the height $h(\lambda)$ is bounded by n. That is, λ fits inside a horizontal height n strip. Equivalently, each part (column) of the partition is at most size n.

Therefore,

$$\mathrm{Hilb}^d(C_n) \subset \left(\bigcup_{h(\lambda)\leq n} U_\lambda \right) \setminus \left(\bigcup_{h(\lambda)>n} U_\lambda \right).$$

The affine cell D_λ is defined by the vanishing of positive weight arrows. If $h(\lambda) \leq n$, then $c_{i,j}^{0,n}$ has weight $q(n-j) - pi > 0$ since $j < n$. Therefore, $c_{i,j}^{0,n}$ is identically zero on D_λ. That is:

Lemma 3.5.5. *If $h(\lambda) \leq n$, $D_\lambda \subset \mathrm{Hilb}^d(C_n, 0)$ and otherwise $D_\lambda \cap \mathrm{Hilb}^d(C_n, 0) = \emptyset$. That is, $\mathrm{Hilb}^d(C_n, 0)$ admits an affine stratification by the cells D_λ for $h(\lambda) \leq n$.*

Proposition 3.5.6. *The Hilbert zeta function of C_n is given by*

$$Z^{\mathrm{Hilb}}_{0\subset C_n}(t) = \prod_{m=1}^{n} \left(\frac{1}{1-\mathbb{L}^{m-1}t^m} \right).$$

Proof. As before, it suffices to compute the Betti number generating function (up to a factor of 2) and $\dim D_\lambda = |\lambda| - l(\lambda)$. Letting

$$P(q,t) = \sum_{\lambda, h(\lambda)\leq n} q^{l(\lambda)} t^{|\lambda|},$$

the generating function is given by $P(1/q, qt)$. This is the generating function for partitions with parts bounded by n and statistic given by number of parts. As above this is given by

$$P(q,t) = \prod_{m=1}^{n} \left(\frac{1}{1-qt^m} \right)$$

and we obtain the Hilbert zeta function by $q \mapsto 1/\mathbb{L}$ and $t \mapsto \mathbb{L}t$.

$\square$

Remark 3.5.7. For a general monomial curve $C \subset \mathbb{A}^2$, one can run the same argument as to write the Hilbert zeta function as a sum over partitions of explicit powers of $\mathbb{L}$. However, these powers become more complicated to compute since Lemma 3.5.5 no longer holds. In this case $D_\lambda \cap \mathrm{Hilb}^d(C, 0)$ is an explicit affine subspace of D_λ given by the vanishing of certain coordinate functions $c_{i,j}^{r,s}$ depending on the monomials generating the ideal of C.

3.5.3 Locally planar uniformly thickened curves

Ribbons are uniform double structure on a smooth curve [BE95]. More generally we define a *uniformly n-fold thickened curve* to be a nonreduced curve X with $X^{red} = C$ smooth and such that the completed local ring at every point of X is isomorphic to the germ of $(C_n, 0)$.

Example 3.5.8. Let $C \subset S$ be asmooth curve inside a smooth surface with ideal I. Then the curve X with ideal I^n is a uniformly n-fold thickened curve.

Proposition 3.5.9. *Let X be a uniformly n-fold thickened curve with reduced subvariety C. Then*

$$Z_X^{Hilb}(t) = \prod_{m=1}^{n} Z_C(\mathbb{L}^{m-1}t).$$

In particular, it is a rational function.

Proof. There is a Hilbert-Chow morphism $h : \mathrm{Hilb}^d(X) \to \mathrm{Sym}^d(C)$ sending a subscheme of X to its support. We can stratify $\mathrm{Sym}^d(C)$ by partitions $d = \sum id_i$ where the d points have collided into d_i points of multiplicity i. Then over each stratum h is a Zariski locally trivial fibration with fiber

$$\prod \mathrm{Hilb}^i(C_n, 0)^{d_i}.$$

From the explicit form of the power structure on the Grothendieck ring of varieties (see also [GZLMH06]), wee see that

$$Z_X^{\mathrm{Hilb}}(t) = (Z_{0\subset C_n}^{\mathrm{Hilb}})^{[C]} = \prod_{m=1}^{n} \left(\frac{1}{1-\mathbb{L}^{m-1}t}\right)^{[C]}.$$

By [GZLMH04, Statement 2],

$$\left(\frac{1}{1-\mathbb{L}^{m-1}t}\right)^{[C]} = \left(\frac{1}{1-t}\right)^{[C]}\Big|_{t\mapsto\mathbb{L}^{m-1}t} = Z_C(\mathbb{L}^{m-1}t),$$

thus completing the proof. □

One expects that sometimes the moduli space of sheaves on a ribbon, or more generally a uniformly n-fold thickened curve X, should be related to the moduli space of rank n vector bundles on the underlying smooth curve (see for example [CK16]).

Question 1. *Is the expression above for $Z_X^{\mathrm{Hilb}}(t)$ related to motivic invariants of the moduli space of rank n vector bundles on the smooth curve X^{red}?*

CHAPTER 4

The Hilbert zeta function in families

The goal of this chapter is to prove Theorem 1.1.2:

Theorem 4.0.1. *Let $(\mathcal{C} \to B, \sigma)$ be a flat family of reduced curve singularities. Then*

$$b \mapsto Z^{\mathrm{top}}_{\sigma(b)\subset\mathcal{C}_b}(t)$$

is a constructible function $B \to \mathbb{Z}[\![t]\!]$.

4.1 Relative Hilbert schemes of points

Let $(f\colon \mathcal{C} \to B, \sigma)$ be a family of reduced curve singularities and $\mathfrak{m} \subset \mathcal{O}_{\mathcal{C}}$ the ideal of the section $S := \sigma(B) \subset \mathcal{C}$. The proof of the following closely follows [Ber12, Lemma 2.22]

Lemma 4.1.1. *Let $Z \subset \mathcal{C}$ be a subscheme flat and proper over B of degree d. Then $\mathfrak{m}^d \subset I(Z)$.*

Proof. We may suppose without loss of generality B is affine. Since our family embeds into $(\mathbb{C}^n \times B, 0 \times B)$ as the germ of some subvariety, $\mathfrak{m}$ is finitely generated. Consider $\bar{\mathfrak{m}} := \mathfrak{m}/I \subset \mathcal{O}_{\mathcal{C}}/I = \mathcal{O}_Z$ where $I = I(Z)$ is the ideal sheaf of Z. Note that $I \subset \mathfrak{m}$ and $\sqrt{I} = \sqrt{\mathfrak{m}}$ as Z is proper over B so it is necessarily supported on the section. Therefore every element of $\bar{\mathfrak{m}}$ is nilpotent and $\bar{\mathfrak{m}}$ is finitely generated so $\bar{\mathfrak{m}}^n = 0$ for large n.

On the other hand, $\bar{\mathfrak{m}}$ is the ideal of $\sigma(B)$ inside Z so it is contained in every maximal ideal of $\mathcal{O}_Z$. Therefore by Nakayama's lemma [AM69, Proposition 2.6] $\bar{\mathfrak{m}}^k = \bar{\mathfrak{m}}^{k+1}$ implies that $\bar{\mathfrak{m}} = 0$. In particular, $\bar{\mathfrak{m}}^n = 0$ if and only if $n \geq n_0 = \min\{k \ : \ \bar{\mathfrak{m}}^k = \bar{\mathfrak{m}}^{k+1}\}$. It follows that $\bar{\mathfrak{m}}^j/\bar{\mathfrak{m}}^{j-1} \neq 0$ for any $j < n_0$ and so for any $k \leq n_0$

$$\mathcal{O}_Z/\bar{\mathfrak{m}}^k$$

has rank at least k above some point $b \in B$. Since Z is finite of degree d we must have $k \leq d$. Therefore $n_0 \leq d$ and $\bar{\mathfrak{m}}^d = 0$.

□

Let $S_d = \mathrm{Spec}_B(\mathcal{O}_C/\mathfrak{m}^d)$ be the d^{th} formal neighborhood of the section in $\mathcal{C}$.

Lemma 4.1.2. *S_d is finite over B.*

Proof. $S_d \to B$ is quasi-finite and the induced morphism $(S_d)_{red} \to B_{red}$ is an isomorphism by existence of a section so $S_d \to B$ is proper. □

In particular, $S_d \to B$ is projective with relatively ample line bundle $\mathcal{O}_{S_d}$. By Lemma 4.1.1, every flat and proper subscheme $Z \subset \mathcal{C}$ of degree d over B is a subscheme of S_n for $n \geq d$.

Definition 4.1.3. We define the relative Hilbert scheme $\mathrm{Hilb}^d(\mathcal{C}/B, \sigma)$ of length d

subschemes supported on a family of curve singularities to be the Hilbert scheme $\mathrm{Hilb}^d(S_d/B)$.

Proposition 4.1.4. $\mathrm{Hilb}^d(\mathcal{C}/B, \sigma)$ *is a projective B-scheme and for each $b \in B$, we have an identification*

$$\mathrm{Hilb}^d(\mathcal{C}/B, \sigma) \times_B k(b) = \mathrm{Hilb}^d(\mathcal{C}_b, \sigma(b)).$$

Proof. Since $S_d \to B$ is a projective morphism and B is Noetherian, then $\mathrm{Hilb}^d(S_d/B)$ exists and is projective over B by a theorem of Grothendieck (e.g. [FGI+05, Theorem 5.14]). Furthermore, the formation of $\mathrm{Hilb}^d(S_d/B)$ is compatible with basechange [FGI+05, (5), page 114] so that

$$\mathrm{Hilb}^d(S_d/B) \times_B k(b) = \mathrm{Hilb}^d(\mathrm{Spec}(\mathcal{O}_{C_b}/\mathfrak{m}_b^d)).$$

By Lemma 4.1.1, every subscheme of $\mathcal{C}_b$ of length d supported on $\sigma(b)$ is a subscheme of $\mathrm{Spec}(\mathcal{O}_{C_b}/\mathfrak{m}_b^d)$ and so we may identify the right hand side with $\mathrm{Hilb}^d(\mathcal{C}_b, \sigma(b))$. $\square$

Remark 4.1.5. Note that $\mathrm{Hilb}^d(\mathcal{C}/B, \sigma)$ does not represent the functor for flat families of flat and proper subschemes of $\mathcal{C}$ of degree d over B. However, this is ok for our applications as the invariants we are interested in are insensitive to the scheme structure.

4.2 Singular curves and their deformations

In this section we will recall some facts about reduced curve singularities and their equisingular deformations including semicontinuity of δ and s. Furthermore, we show that the conductor c is also constructible, insuring the existince of a (δ, s, c)-constant stratification for any family of reduced curve singularities.

Let $(C, p) \subset (\mathbb{C}^N, 0)$ be the germ of a reduced curve singularity with s branches C_i and let $\mathcal{O}_C = \widehat{\mathcal{O}}_{C,p}$ denote the corresponding completed local ring. Let $n\colon \tilde{C} \to C$ be the normalization. By picking uniformizers for each branch, we identify $\mathcal{O}_{\widetilde{C}}$ with the ring $\prod_{i=1}^{s} \mathbb{C}[\![x_i]\!]$. The normalization induces a finite extension

$$\mathcal{O}_C \hookrightarrow \mathcal{O}_{\widetilde{C}} \cong \prod_{i=1}^{s} \mathbb{C}[\![x_i]\!]$$

of rings which factors through the inclusions $\mathcal{O}_{C_i} \subset \mathbb{C}[\![x_i]\!]$ corresponding to the i^{th} branch $n_i\colon \widetilde{C}_i \to C_i \subset C$ of the normalization.

(1) Let

$$\delta := \dim_{\mathbb{C}}(n_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C)$$

be the δ-*invariant* of C. Similarly, we denote by δ_i the δ-invariant $\dim_{\mathbb{C}} \mathbb{C}[\![x_i]\!]/\mathcal{O}_{C_i}$ of the i^{th} branch.

(2) Let

$$\mathfrak{c} := \mathrm{Ann}_{\mathcal{O}_C}(n_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C)$$

be the *conductor ideal*. This an ideal of both $\mathcal{O}_{\widetilde{C}}$ and $\mathcal{O}_C$. In particular $\mathfrak{c}$ is generated by monomials, say $x_i^{c_i}$, as an ideal of $\prod_{i=1}^{s} k[\![x_i]\!]$. It's clear from the definition that c_i is the smallest positive integer such that for all $n \geq c_i$, $x_i^n \in \mathcal{O}_C$. We will refer to c_i as the conductor of the i^{th} branch, denote by

$$c := \dim_{\mathbb{C}}(\mathcal{O}_{\widetilde{C}}/\mathfrak{c}) = \sum_{i=1}^{s} c_i$$

the conductor of C, and by $\underline{c} = (c_1, \ldots, c_s)$ the conductor branch-length vector.

More generally, for any finite homomorphism of rings $\varphi\colon R \to S$ the *conductor* of φ is defined as

$$\mathfrak{c}(\varphi) := \mathrm{Ann}_{\varphi(R)}(S/\varphi(R)).$$

Then it is clear that

$$c_i = \dim_{\mathbb{C}}(\mathcal{O}_{\widetilde{C}_i}/\mathfrak{c}(n_i)).$$

(3) The Milnor number $\mu(C)$ is defined as $\dim_{\mathbb{C}}(\omega_C/d\mathcal{O}_C)$ where $d\colon \mathcal{O}_C \to \omega_C$ is the differential composed with the canonical map $\Omega^1_C \to n_*\Omega^1_{\tilde{C}} \cong n_*\omega_{\tilde{C}} \to \omega_C$ to the dualizing sheaf of C. The Milnor number satisfies

$$\mu(C) = 2\delta(C) - s + 1$$

(see [BG80]).

4.2.1 Equisingular families

Let $(f\colon \mathcal{C} \to B, \sigma)$ be a flat family of germs of reduced curve singularities. Recall we will always assume that B is Noetherian and that there is an embedding of germs

$$(\mathcal{C}, \sigma) \subset (\mathbb{C}^N \times B, 0 \times B)$$

so that $(f\colon \mathcal{C} \to B, \sigma)$ is the germ of a family of reduced affine curves.

Definition 4.2.1. A morphism $\nu\colon \mathcal{C}' \to \mathcal{C}$ is a *simultaneous normalization* of f if for any $b \in B$, $\nu_b\colon \mathcal{C}'_b \to \mathcal{C}_b$ is the normalization. We say that f is *equinormalizable* if the normalization $\widetilde{\mathcal{C}} \to \mathcal{C}$ of the total space is a simultaneous normalization of f.

Theorem 4.2.2 (Tessier [Tei77], Reynaud, Chiang-Hsieh–Lipman [CHL06])**.** *Let*

$(f\colon \mathcal{C} \to B, \sigma)$ be a flat family of reduced curve singularities over a normal base B. Then f is equinormalizable if and only if $\delta(C_b, \sigma(b))$ is constant for $b \in B$.

Definition 4.2.3. Suppose B is connected, smooth and 1-dimensional with a basepoint $0 \in B$. We say that the family $(f\colon \mathcal{C} \to B, \sigma)$ is *equisingular*[1] if there is a homeomorphism

$$(\mathcal{C}, \sigma(B)) \cong_{top} (B \times \mathcal{C}_0, B \times \sigma(0))$$

compatible with the maps to B.

Theorem 4.2.4 (Buchweitz–Greuel [BG80, Theorems 5.2.2 and 6.1.7])**.** *Let $(f\colon \mathcal{C} \to B, \sigma)$ be a flat family of reduced curve singularities.*

(a) The function $\mu(\mathcal{C}_b, \sigma(b))$ for $b \in B$ is upper semicontinuous.

(b) Suppose B is a smooth, connected and 1-dimensional base. Then the following are equivalent:

(i) $(f\colon \mathcal{C} \to B, \sigma)$ is equisingular;

(ii) the Milnor number $\mu(\mathcal{C}_b, \sigma(b))$ is constant for $b \in B$;

(iii) $\delta(\mathcal{C}_b, \sigma(b))$ and the number of branches $s(\mathcal{C}_b, \sigma(b))$ are constant.

Corollary 4.2.5. *There exists a stratification $B = \bigsqcup B_i$ such that the pullback $f_i\colon \mathcal{C}_i \to B_i$ is a μ-constant family for each i. Furthermore, f_i is (δ, s)-constant and if B_i is normal then f_i is equinormalizable.*

We call such families (δ, s)-*constant* or *equisingular* families. If $(f : \mathcal{C} \to B, \sigma)$ is an equisingular family, then the normalization $\widetilde{f}\colon \widetilde{\mathcal{C}} \to B$ is a family of s germs of smooth curves with degree s multisection. That is, $\widetilde{\mathcal{C}}_b \cong \bigsqcup_{i=1}^{s} \widehat{\mathbb{A}}^1$ where $\widehat{\mathbb{A}}^1 = \mathrm{Spec}(\mathbb{C}[\![x]\!])$.

[1] There are several notions of equisingular deformations in the literature that are not always equivalent.

Proposition 4.2.6. *Let $(f\colon \mathcal{C} \to B, \sigma)$ be a (δ, s)-constant family of reduced curve singularities. Then the conductor c is constructible on B.*

Proof. Since the function $b \to c(\mathcal{C}_b, \sigma(b))$ depends only on the closed points of B, we may assume without loss of generality that B is normal. In this case f is equisingular and the normalization $n\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ is the simultaneous normalization. Consider the sequence

$$0 \to \mathcal{O}_C \to n_*\mathcal{O}_{\widetilde{C}} \to Q \to 0.$$

As f is equinormalizable, we have exactness of

$$0 \to \mathcal{O}_{C_b} \to n_*\mathcal{O}_{\widetilde{C}_b} \to Q_b \to 0$$

so that $\operatorname{length}(Q_b) = \delta$ is constant for all $b \in B$. Thus Q is finite of constant rank over B so it is flat.

Lemma 4.2.7. *Let $(f\colon \mathcal{C} \to B, \sigma)$ be a family of reduced curve singularities and let Q be a coherent sheaf on $\mathcal{C}$ that is flat and finite over B. Then*

$$b \to \operatorname{colength}_{\mathcal{O}_{C_b}}(\operatorname{Ann}_{\mathcal{O}_{C_b}}(Q_b))$$

is constructible.

Proof. Let d be the degree of Q over B and for any $k \leq d$ consider $\operatorname{Hilb}^k(\mathcal{C}/B, \sigma)$ with tautological subscheme $Z_k \subset \operatorname{Hilb}^k(\mathcal{C}/B, \sigma) \times_B \mathcal{C}$. Let Q_H the pullback of Q to $\operatorname{Hilb}^k(\mathcal{C}/B, \sigma) \times_B \mathcal{C}$ and Q_Z the pullback of Z_k. Then Q_H is flat over $\operatorname{Hilb}^k(\mathcal{C}/B, \sigma)$ of constant degree d over and Q_Z, as a quotient of Q_H, has degree at most d over $\operatorname{Hilb}^k(\mathcal{C}/B, \sigma)$.

Let $H_d^k \subset \mathrm{Hilb}^k(\mathcal{C}/B, \sigma)$ be the closed subset where Q_Z has degree exactly d, or equivalently the locus over which $Q_H \to Q_Z$ is an isomorphism. The image of H_d^k via $\mathrm{Hilb}^k(\mathcal{C}/B, \sigma) \to B$ is constructible in B and is by construction the locus over which Q is supported on a subscheme of length at most k. In particular, the image of H_d^d is all of B and the function

$$\varphi : b \to \min\{k \ : \in \mathrm{im}(H_k^d)\}$$

is constructible. On the other hand, since $V(\mathrm{Ann}_{\mathcal{O}_{C_b}}(Q_b)) = \mathrm{Supp}(Q_b)$ is the smallest subscheme on which Q_b is supported, then

$$\varphi(b) = \mathrm{colength}_{\mathcal{O}_{C_b}}(\mathrm{Ann}_{\mathcal{O}_{C_b}}(Q_b)).$$

$\square$

To complete the proof, note that δ is constant so

$$c(\mathcal{C}_b, \sigma(b)) = \delta + \mathrm{colength}_{\mathcal{O}_{C_b}}(\mathrm{Ann}_{\mathcal{O}_{C_b}}(Q_b))$$

is constructible by the lemma.

$\square$

Corollary 4.2.8. *For any (δ, s)-constant family, we may further stratify so that c is constant and $Z \to B$ is flat. We call such families* (δ, s, c)-constant *families.*

4.3 Proof of Theorem 1.1.2

By Theorem 1.1.1, we know that $Z^{\text{top}}_{p\subset C}(t)$ is a rational function of the form $P(t)/(1-t)^s$ where s is the number of branches. More precisely, there is an expansion of the following form.

$$\begin{aligned}
Z^{\text{top}}_{p\subset C}(t) &= \sum_{a_1=0}^{c_1-1}\cdots\sum_{a_s=0}^{c_s-1}\sum_{d\geq 0}\chi_{top}(\text{Hilb}^{d,a_1,\ldots,a_s}(C,p))t^d \\
&+\sum_{a_1=0}^{c_1-1}\cdots\sum_{a_{s-1}=0}^{c_{s-1}-1}\frac{1}{1-t}\sum_{d\geq 0}\chi_{top}(\text{Hilb}^{d,a_1,\ldots,c_s}(C,p))t^d \\
&+\sum_{a_1=0}^{c_1-1}\cdots\sum_{a_{s-2}=0}^{c_{s-2}-1}\frac{1}{(1-t)^2}\sum_{d\geq 0}\chi_{top}(\text{Hilb}^{d,a_1,\ldots,c_{s-1},c_s}(C,p))t^d \\
&\vdots \\
&+\frac{1}{(1-t)^s}\sum_{d\geq 0}\chi_{top}(\text{Hilb}^{d,c_1,\ldots,c_s}(C,p))t^d
\end{aligned}$$

Here $\text{Hilb}^{d,a_1,\ldots,a_s}(C,p)\subset\text{Hilb}^d(C,p)$ are certain subvarieties indexed by $a_i\in\mathbb{N}$ and c_i are the conductors of each branch (see Section 4.2).

Furthermore, there are uniform bounds

$$-\delta\leq d-\sum_{i=1}^{s}a_i\leq c-\delta.$$

where δ and $c=\sum c_i$ are the δ-invariant and total conductor (Section 4.2). In particular, $d\leq 2c-\delta$ in any of the terms $\text{Hilb}^{d,a_1,\ldots,a_s}(C,p)$ appearing in the expansion above. Multiplying the expression by $(1-t)^s$ we see that the degree of $P(t)$ is bounded above by $2c-\delta+s$. From the expression $Z^{\text{top}}_{p\subset C}(t)=P(t)/(1-t)^s$, we can then determine $P(t)$ from the first $2c-\delta+s$ coefficients of $Z^{\text{top}}_{p\subset C}(t)$.

Now let $(f\colon\mathcal{C}\to B,\sigma)$ be a family of reduced curve singularity over a base B

of finite type. We may suppose without loss of generality that B is normal. Then by Corollary 4.2.5 and Proposition 4.2.6 there is a finite stratification of the base over which δ, s and c are constant so we may suppose f is a (δ, s, c)-constant family. By Proposition 4.1.4 there exists a projective morphism $\pi_d \colon \operatorname{Hilb}^d(\mathcal{C}/B) \to B$ whose fiber over $b \in B$ is $\operatorname{Hilb}^d(\mathcal{C}_b, \sigma(b))$. For each d there exists a finite stratification of B so that over each stratum, the fibers of π_d have the same topological Euler characteristic. We may take the refinement of all these stratifications for $1 \leq d \leq 2c - \delta + s$. This produces a stratification such that $\chi_{top}(\operatorname{Hilb}^d(\mathcal{C}_b, \sigma(b))$ is constant on strata for all $1 \leq d \leq 2c - \delta + s$. As these coefficients suffice to determine the full zeta function $Z^{\text{top}}_{\sigma(b) \subset \mathcal{C}_b}(t)$, we are done.

Remark 4.3.1. Note in fact that the proof of Theorem 1.1.2 applies verbatim with χ_{top} replaced by any invariant χ satisfying the following two properties: (1) χ factors through the Grothendieck ring of varieties, (2) χ is constructible in families of varieties. The theory of characteristic classes of mixed Hodge modules produces many such invariants [MS13a]. The methods above can be used to generalize rationality of the Hilbert zeta function to the generating series for mixed Hodge modules associated to the relative Hilbert schemes of a family $(f \colon \mathcal{C} \to B, \sigma)$ of curve singularities. One can view this as a motivic version of constructibility for the Hilbert zeta function. This will be pursued in the future.

Part II

Compact moduli spaces of elliptic surfaces

CHAPTER 5

Preliminaries

Part II is based on joint work with Ascher and Inchiostro [AB17a, Inc18a].

We work throughout over the complex numbers for simplicity.

5.1 The minimal model program and moduli of stable pairs

We work with $\mathbb{Q}$-divisors. Whenever we write equality for divisors, e.g. $K_X = \Delta$, unless otherwise noted, we mean $\mathbb{Q}$-linear equivalence.

5.1.1 Semi-log canonical pairs

To compactify the moduli space of pairs of log general type, one needs to introduce pairs on the boundary which have *semi-log canonical (slc) singularities.* We begin with their definition.

Definition 5.1.1. Let $(X, D = \sum d_i D_i)$ be a pair of a normal variety and a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Suppose that there is a log resolution $f\colon Y \to X$ such that

$$K_Y + \sum a_E E = f^*(K_X + D),$$

where the sum goes over all irreducible divisors on Y. We say that the pair (X, D) has **log canonical singularities** (or is lc) if all $a_E \leq 1$.

Definition 5.1.2. Let (X, D) be a pair of a reduced variety and a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. The pair (X, D) has **semi-log canonical singularities** (or is an slc pair) if:

- The variety X is S2,
- X has only double normal crossings in codimension 1, and
- If $\nu\colon X^\nu \to X$ is the normalization, then the pair $(X^\nu, \nu_*^{-1}D + D^\nu)$ is log canonical, where D^ν denotes the preimage of the double locus on X^ν.

Definition 5.1.3. A pair (X, D) of a projective variety and $\mathbb{Q}$-divisor is a **stable pair** if:

(i) (X, D) is an slc pair, and

(ii) $\omega_X(D)$ is ample.

Definition 5.1.4. Let (X, D) be an (s)lc pair and let $f\colon X \to B$ be a projective morphism. The **(semi-)log canonical model** of $f\colon (X, D) \to B$, if it exists, is the unique (s)lc pair $(Y, \mu_* D)$ given by

$$Y := \operatorname{Proj}_B \left(\bigoplus_m f_* \mathcal{O}_X(m(K_X + D)) \right) \to B$$

and $\mu\colon X \dashrightarrow Y$. When B is a point, $(Y, \mu_* D)$ is a stable pair.

We will make repeated use of abundance for slc surface pairs in computing log canonical models of slc surface pairs.

Proposition 5.1.5 (Abundance for slc surfaces, see [AFKM02] and [Kaw92])**.** *Let (X, D) be an slc surface pair and $f\colon X \to B$ a projective morphism. If $K_X + D$ is f-nef, then it is f-semiample.*

The following results are standard (see for example [AB17b, Section 3]).

Lemma 5.1.6. *Let X be seminormal and $\mu\colon Y \to X$ a projective morphism with connected fibers. Then for any coherent sheaf $\mathcal{F}$ on X, we have that $\mu_*\mu^*\mathcal{F} = \mathcal{F}$.*

Proposition 5.1.7. *Let (X, Δ) be an slc pair and $\mu\colon Y \to X$ a (partial) semi-resolution. Write*

$$K_Y + \mu_*^{-1}\Delta + \Gamma = \mu^*(K_X + \Delta) + B$$

where $\Gamma = \sum_i E_i$ is the exceptional divisor of μ and B is effective and exceptional. Then

$$\mu_*\mathcal{O}_Y\Big(m(K_Y + \mu_*^{-1}\Delta + \Gamma)\Big) \cong \mathcal{O}_X\Big(m(K_X + \Delta)\Big).$$

Corollary 5.1.8. *Notation as above; the morphism μ induces an isomorphism of global sections*

$$H^0\Big(X, \mathcal{O}_X\Big(m(K_X + \Delta)\Big)\Big) \cong H^0\Big(Y, \mathcal{O}_Y\Big(m(K_Y + \mu_*^{-1}\Delta + \Gamma)\Big)\Big).$$

In particular, $K_X + \Delta$ is big if and only if $K_Y + \mu_*^{-1}\Delta + \Gamma$ is big.

Proof. The first part is the definition of pushforwards. The second statement follows since $\dim Y = \dim X$. $\square$

Corollary 5.1.9. *Notation as above; the morphism μ induces an injection*

$$H^1\Big(X, \mathcal{O}_X\Big(m(K_X + \Delta)\Big)\Big) \hookrightarrow H^1\Big(Y, \mathcal{O}_Y\Big(m(K_Y + \mu_*^{-1}\Delta + \Gamma)\Big)\Big).$$

Proof. This follows from the five-term exact sequence of the Leray spectral sequence for μ applied to $\mathcal{O}_Y\Big(m(K_Y + \mu_*^{-1}\Delta + \Gamma)\Big)$. □

Let (X, D) be a pair consisting of a normal variety X and a divisor D such that the rounding up $\lceil D \rceil$ is a reduced divisor. We *do not* assume that (X, D) is log canonical.

Definition 5.1.10. The **log canonical model** of a pair (X, D) as above is the log canonical model of the lc pair $(Y, \mu_*^{-1}D + \Gamma)$ where $\mu \colon Y \to X$ is a log resolution of (X, D) and Γ is the exceptional divisor.

Remark 5.1.11. By Proposition 5.1.7 and its corollaries, the log canonical model of (X, D) is independent of choice of log resolution and therefore is well defined.

Lemma 5.1.12. *[KM98, 2.35] If $(X, D + D')$ is an lc pair, and D' is an effective $\mathbb{Q}$-Cartier divisor, then (X, D) is also an lc pair.*

Definition 5.1.13. Let (X, D) be be a pair with (semi-)log canonical singularities and $A \subset X$ a divisor. The **(semi-)log canonical threshold** $\operatorname{lct}(X, D, A)$ is

$$\operatorname{lct}(X, D, A) := \max\{a \mid (X, D + aA) \text{ has (semi-)log canonical singularities } \}.$$

5.1.2 Moduli spaces of stable pairs

The curve case

First we review Hassett's weighted stable curves, as these will be used extensively, and they illuminate some of the basic geometric concepts.

Definition 5.1.14. Let $\mathcal{A} = (a_1, \ldots, a_r)$ for $0 < a_i \leq 1$. An **$\mathcal{A}$-stable curve** is a pair $(C, D = \sum a_i p_i)$, of a reduced connected projective curve X together with a divisor D consisting of n weighted marked points p_i on C such that:

- C has at worst nodal singularities, the points p_i lie in the smooth locus of C, and for any subset $\{p_1, \cdots, p_s\}$ with nonempty intersection we have $a_1 + \cdots + a_s \leq 1$;
- $\omega_C(D)$ is ample.

In particular, if $\mathcal{A} = (1, \ldots, 1)$, then one obtains an r-pointed stable curve [Knu83].

Theorem 5.1.15. *[Has03] Let $\mathcal{A} = (a_1, \ldots, a_r)$ be a weight vector $0 < a_i \leq 1$ and fix an integer $g \geq 0$. Then there is a smooth Deligne-Mumford stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$ with projective coarse moduli space $\overline{M}_{g,\mathcal{A}}$ parametrizing $\mathcal{A}$-stable curves.*

Moreover, if one considers the domain of admissible weights, there is a *wall and chamber* decomposition – we say that $(a'_1, \ldots, a'_r) \leq (a_1, \ldots, a_r)$ if $a'_i \leq a_i$ for all i. Hassett proved the following theorem.

Theorem 5.1.16. *[Has03] There is a wall and chamber decomposition of the domain of admissible weights such that:*

(i) If $\mathcal{A}$ and $\mathcal{A}'$ are in the same chamber, then the moduli stacks and universal families are isomorphic.

(ii) If $\mathcal{A}' \leq \mathcal{A}$, then there is a reduction morphism $\overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}'}$ and a compatible contraction morphism on universal families.

Higher dimensions

In full generality, it has been difficult to construct a proper moduli space parametrizing stable pairs (X, D) with suitable numerical data. An example due to Hassett (see Section 1.2 in [KP17]), shows that when the coefficients of D are *not* all $> 1/2$, the divisor D might not deform as expected in a flat family of pure codimension 1 subvarieties of X – the limit of the divisor D may acquire an embedded point. However, we first make the following remarks:

Remark 5.1.17.

- Hassett and Alexeev (see [Has01] and [Ale08]) have demonstrated properness when all coefficients of D are equal to 1.
- It is well known that by results of Kollár, the moduli space exists and is proper when the coefficients are all $> 1/2$ (see e.g. [Kol18a, Sec 4.2] and [Kol18b]).

While it is clear what the objects are (see Definition 5.1.3), it is not clear what the proper definition for families are, and thus it is unclear what exactly the moduli functor should be. Many functors have been suggested, but no functor seems to be "better" than any other. That being said, the projectivity results of [KP17], namely Theorem 1.1 in *loc. cit.*, is independent of the choice of functor, and applies to any moduli functor whose objects are stable pairs. We do remark that Kovács-Patakfalvi demonstrate their results using a proposed functor of Kollár (see Section 5 in [KP17]). We also note that it *is* clear what the definition of a stable family (i.e. a family of stable pairs) is over a *normal* base:

Definition 5.1.18. [KP17, Definition 2.11] A **family of stable pairs** of dimension n and volume v over a normal variety Y consists of a pair (X, D) and a flat proper surjective morphism $f\colon X \to Y$ such that

(i) D avoids the generic and codimension 1 singular points of every fiber,

(ii) $K_{X/Y} + D$ is $\mathbb{Q}$-Cartier,

(iii) (X_y, D_y) is a connected n-dimensional stable pair for all $y \in Y$, and

(iv) $(K_{X_y} + D_y)^n = v$ for $y \in Y$.

We denote a family of stable pairs by $f\colon (X, D) \to Y$.

If we are satisfied working only over normal bases, then this definition of a family of stable pairs suffices. In fact, any moduli functor $\mathcal{M}$ with

$$\mathcal{M}(Y) = \left\{ \begin{array}{l} \text{families of stable pairs } f\colon (X,D) \to Y \text{ of} \\ \text{dimension } n \text{ and volume } v \text{ as in Definition} \\ \text{5.1.18} \end{array} \right\}$$

for Y normal has the same normalization by Proposition A.0.7 (see also Definition 5.2 and Remark 5.15 of [KP17]). Therefore for many questions about moduli of stable pairs, one needs only consider families over a normal base.

5.1.3 Vanishing theorems

The existence of reduction morphisms between the moduli spaces will rely on the proof of a vanishing theorem for higher cohomologies which implies invariance of log plurigenera for a family of $\mathcal{A}$-weighted broken elliptic surfaces (see Section 9). There are various preliminary vanishing results we will use along the way that we record here for convenience.

The first is a version of Grauert-Riemenschneider vanishing theorem for surfaces. The proof is analagous to the proof of [Kol13, Theorem 10.4].

Proposition 5.1.19. *(Grauert-Riemenschneider vanishing) Let X be an slc surface and $f\colon X \to Y$ a proper, generically finite morphism with exceptional curves C_i such that $E = \bigcup_i C_i$ is a connected curve with arithmetic genus 0. Let L be a line bundle on X. Suppose*

(i) C_i *is a $\mathbb{Q}$-Cartier divisor for all i;*

(ii) $C_i.E \leq 0$ *for all i; and*

(iii) $\deg(L|_{C_i}) = 0$ *for all i.*

Then $R^1 f_* L = 0$.

Proof. Let $Z = \sum_{i=1}^{s} r_i C_i$ be an effective integral cycle. Then we prove using induction that the stalk $(R^1 f_* L)_{Y,p} = \varprojlim_Z H^1(Z, L|_Z) = 0$. As f is finite away from $p = f(E)$, this gives $R^1 f_* L = 0$.

Let C_i be an irreducible curve contained in $\operatorname{Supp}(Z)$, and let $Z_i = Z - C_i$. Consider the short exact sequence:

$$0 \to \mathcal{O}_{C_i} \otimes \mathcal{O}_X(-Z_i) \to \mathcal{O}_Z \to \mathcal{O}_{Z_i} \to 0.$$

Tensoring with L we obtain:

$$0 \to \mathcal{O}_{C_i} \otimes L(-Z_i) \to L \otimes \mathcal{O}_Z \to L \otimes \mathcal{O}_{Z_i} \to 0.$$

By induction on $\sum r_i$, we know that $H^1(Z_i, L|_{Z_i}) = 0$. Therefore, it suffices to show that $H^1(C_i, \mathcal{O}_{C_i} \otimes L(-Z_i)) = 0$ for some i. Moreover, by Serre duality it suffices to show that

$$L \cdot C_i - Z_i \cdot C_i > \deg \omega_{C_i} = -2.$$

By assumption, $L \cdot C_i = 0$, so it suffices to show that $-Z_i \cdot C_i > -2$, or equivalently that $Z_i \cdot C_i < 2$. This follows from Artin's results on intersection theory of exceptional curves for rational surface singularities [Art66] applied to the normalization of X, as C_i and E are rational exceptional curves. $\square$

Next we will use Fujino's generalization of the Kawamata-Viehweg vanishing theorem for slc pairs. Before stating the result, we will need to make a preliminary definition.

Definition 5.1.20. Let (X, Δ) be a semi-log canonical pair and let $\nu \colon X^\nu \to X$ be the normalization. Let Θ be a divisor on X^ν, so that $(K_{X^\nu} + \Theta) = \nu^*(K_X + \Delta)$. A subvariety $W \subset X$ is called an **slc center** of (X, Δ) if there exists a resolution of singularities $f \colon Y \to X^\nu$ and a prime divisor E on Y such that the discrepancies $a(E, X^\nu, \Theta) = -1$ and $\nu \circ f(E) = W$. A subvariety $W \subset X$ is called an **slc stratum** if W is an slc center, or an irreducible component.

Now we state Fujino's theorem.

Theorem 5.1.21. *[Fuj14, Theorem 1.10] Let (X, Δ) be a projective semi-log canonical pair, L a $\mathbb{Q}$-Cartier divisor whose support does not contain any irreducible components of the conductor, and $f \colon X \to S$ a projective morphism. Suppose $L - (K_X + \Delta)$ is f-nef and additionally is f-big over each slc stratum of (X, Δ). Then $R^i f_* \mathcal{O}_X(L) = 0$ for $i > 0$.*

5.2 Elliptic surfaces

5.2.1 Standard elliptic surfaces

We point the reader to [Mir89] for a detailed exposition on the theory of elliptic surfaces.

Definition 5.2.1. An irreducible **elliptic surface with section** $(f\colon X \to C, S)$ is an irreducible surface X together with a surjective proper flat morphism $f\colon X \to C$ to a proper smooth curve and a section S such that:

(i) the generic fiber of f is a stable elliptic curve, and

(ii) the generic point of the section is contained in the smooth locus of f.

We say $(f\colon X \to C, S)$ is **standard** if all of S is contained in the smooth locus of f.

This definition differs from the usual definition of an elliptic surface in that we only require the generic fiber to be a *stable* elliptic curve.

Definition 5.2.2. A **Weierstrass fibration** $(f\colon X \to C, S)$ is an elliptic surface with section as above, such that the fibers are reduced and irreducible.

Definition 5.2.3. A surface is **semi-smooth** if it only has the following singularities:

(i) 2-fold normal crossings (locally $x^2 = y^2$), or

(ii) pinch points (locally $x^2 = zy^2$).

Definition 5.2.4. A **semi-resolution** of a surface X is a proper map $g\colon Y \to X$ such that Y is semi-smooth and g is an isomorphism over the semi-smooth locus of X.

Definition 5.2.5. An elliptic surface is called **relatively minimal** if it is semi-smooth and there is no (-1)-curve in any fiber.

Note that a relatively minimal elliptic surface with section is standard. If $(f\colon X \to C, S)$ is a standard elliptic surface then there are finitely many fiber components not intersecting the section. We can contract these to obtain an elliptic surface with all fibers reduced and irreducible:

Definition 5.2.6. If $(f\colon X \to C, S)$ is a standard elliptic surface then the Weierstrass fibration $f'\colon X' \to C$ with section S' obtained by contracting any fiber components not intersecting S is the **Weierstrass model** of $(f\colon X \to C, S)$. If $(f\colon X \to C, S)$ is relatively minimal, then we refer to $f'\colon X' \to C$ as the **minimal Weierstrass model**.

Definition 5.2.7. The **fundamental line bundle** of a standard elliptic surface $(f\colon X \to C, S)$ is $\mathscr{L} := (f_* N_{S/X})^{-1}$ where $N_{S/X}$ denotes the normal bundle of S in X. For $(f\colon X \to C, S)$ an arbitrary elliptic surface, we define $\mathscr{L} := (f'_* N_{S'/X'})^{-1}$ where $(f'\colon X' \to C, S')$ is a minimal semi-resolution.

Since $N_{S/X}$ only depends on a neighborhood of S in X, the line bundle $\mathscr{L}$ is invariant under taking a semi-resolution or the Weierstrass model of a standard elliptic surface. Therefore $\mathscr{L}$ is well defined and equal to $(f'_* N_{S'/X'})^{-1}$ for $(f'\colon X' \to C, S')$ a minimal semi-resolution of $(f\colon X \to C, S)$.

The fundamental line bundle greatly influences the geometry of a minimal Weierstrass fibration. The line bundle $\mathscr{L}$ has non-negative degree on C and is independent of choice of section S [Mir89]. Furthermore, $\mathscr{L}$ determines the canonical bundle of X:

Proposition 5.2.8. *[Mir89, Proposition III.1.1] Let $(f\colon X \to C, S)$ be either (i) a*

Weierstrass fibration, or (ii) a relatively minimal smooth elliptic surface. Then $\omega_X = f^*(\omega_C \otimes \mathscr{L})$.

We prove a more general canonical bundle formula in [AB17b] (see Proposition 6.1.13).

Definition 5.2.9. We say that $f \colon X \to C$ is **properly elliptic** if $\deg(\omega_C \otimes \mathscr{L}) > 0$.

We note that X is properly elliptic if and only if the Kodaira dimension $\kappa(X) = 1$.

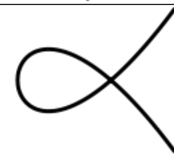
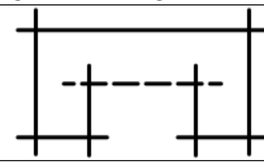
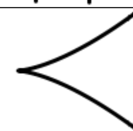
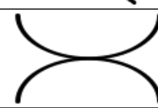
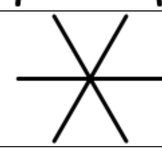
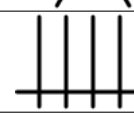
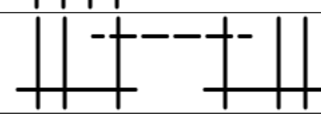
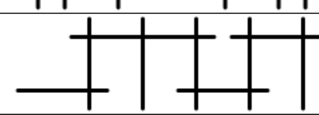
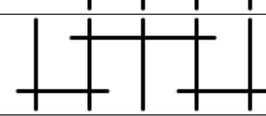
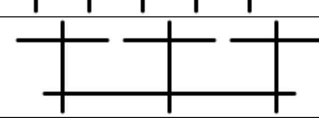
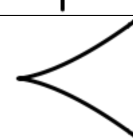
5.2.2 Singular fibers

When $(f \colon X \to C, S)$ is a smooth relatively minimal elliptic surface, then f has finitely many singular fibers. These are unions of rational curves with possibly non-reduced components whose dual graphs are ADE Dynkin diagrams. The possible singular fibers were classified independently by Kodaira and Nerón.

Table 5.1 gives the full classification in Kodaira's notation for the fiber. Fiber types I_n for $n \geq 1$ are reduced and normal crossings, fibers of type $\mathrm{I}_n^*, \mathrm{II}^*, \mathrm{III}^*$, and IV^* are normal crossings but nonreduced, and fibers of type $\mathrm{II}, \mathrm{III}$ and IV are reduced but not normal crossings.

For $f \colon X \to C$ isotrivial with $j = \infty$, La Nave classified the Weierstrass models with log canonical singularities in [LN02, Lemma 3.2.2] (see also [AB17b, Section 5]). They have equation $y^2 = x^2(x - t^k)$ for $k = 0, 1$ and 2 and we call these N_0, N_1 and N_2 fibers respectively.

Table 5.1: Singular fibers of a smooth minimal elliptic surface

Kodaira Type	# of components	Fiber
I_0	1	
I_1	1	
I_2	2	
I_n, $n \geq 2$	n (nodal cycle)	
II	1 (cusp)	
III	2 (tangent)	
IV	3 (meet at 1 pt)	
I_0^*	5	
I_n^*, $n \geq 1$	$5 + n$	
II^*	9	
III^*	8	
IV^*	7	
N_I	1	

CHAPTER 6

Log canonical models of elliptic surfaces

6.1 Log canonical models of $\mathcal{A}$-weighted elliptic surfaces

Let $\mathcal{A} = (a_1, \ldots, a_r) \in (\mathbb{Q} \cap [0,1])^r$ be a rational *weight vector* with $0 \leq a_i \leq 1$. We will consider *Weierstrass* elliptic surfaces marked by an $\mathcal{A}$-weighted sum $F_{\mathcal{A}} = \sum_{i=1}^r a_i F_i$ where F_i are fibers of the Weierstrass surface. Note that the weights come with a natural partial ordering. We say that

$$\mathcal{A}' = (a'_1, \ldots, a'_r) < \mathcal{A}$$

if $a'_i \leq a_i$ for all i, and if the inequality is strict for at least one i. If $s \in \mathbb{Q}$ is a rational number, we write $\mathcal{A} \leq s$ ($\mathcal{A} \geq s$) if $a_i \leq s$ ($a_i \geq s$) for all i. Our goal is to compare stable pair compactifications of the moduli space of $\mathcal{A}$-weighted elliptic surface pairs for various weight vectors $\mathcal{A}$.

As a first step, we need to understand the log canonical models of Weierstrass elliptic surface pairs and how they depend on the weights $\mathcal{A}$. That is, given a Weierstrass elliptic surface pair $(g\colon Y \to C, S)$ and an $\mathcal{A} = (a_1, \ldots, a_n)$-weighted sum of marked fibers

$$F_{\mathcal{A}} = \sum_i a_i F_i,$$

we need to compute the log canonical model for all weights $\mathcal{A}$. This is based on the computations in [AB17b].

Our study of log canonical models of an elliptic surface pair $(f\colon X \to C, S + F_{\mathcal{A}})$ proceeds in two steps: first we compute the relative canonical model of $(X, S + F_{\mathcal{A}})$ over the curve C and then contract the section or whole components if necessary according to the log minimal model program.

6.1.1 Relative log canonical models

Let $(g\colon Y \to C, S + F_{\mathcal{A}})$ be an $\mathcal{A}$-weighted Weierstrass elliptic fibration over a smooth curve. We want to compute the relative log canonical model of the pair $(Y, S + F_{\mathcal{A}})$ relative to the fibration g. That is, we wish to take a suitable log resolution $\mu\colon Y' \to Y$ and compute the log canonical model of $(Y', \mu_*^{-1}S + \mu_*^{-1}(F_{\mathcal{A}}) + \mathrm{Exc}(\mu))$ relative to $g \circ \mu\colon Y' \to C$. In what follows, unless otherwise specified, by relative log canonical model we mean relative to the base curve C.

This computation is local on the base so for the rest of this subsection, we assume that $C = \mathrm{Spec}(R)$ is a spectrum of a DVR with closed point s and generic point η. We then consider the log pair $(Y, S + aF)$ where $F = g^*(s)$ and $0 \leq a \leq 1$.

Definition 6.1.1. [LN02, Definition 3.2.3] A normal Weierstrass elliptic fibration

$(g\colon Y \to C, S)$ over the spectrum of a DVR with Weierstrass equation $y^2 = x^3 + ax + b$ is called a **standard Weierstrass model** if $\min(\mathrm{val}(3n), \mathrm{val}(2m)) \leq 12$. A non-normal Weierstrass fibration with equation $y^2 = x^2(x - at^k)$ is called a **standard Weierstrass model** if $k \leq 2$.

Proposition 6.1.2. *[LN02, Corollary 3.2.4] A Weierstrass elliptic fibration $Y \to C$ over the spectrum of a DVR is (semi-)log canonical if and only if it is a standard Weierstrass model.*

Definition 6.1.3. Let $(g\colon Y \to C, S' + aF')$ be a Weierstrass elliptic surface pair over the spectrum of a DVR and let $(f\colon X \to C, S + F_a)$ be its relative log canonical model. We say that X has a:

(i) **twisted fiber** if the special fiber $f^*(s)$ is irreducible and $(X, S + E)$ has (semi-)log canonical singularities where $E = f^*(s)^{red}$;

(ii) **intermediate fiber** if $f^*(s)$ is a nodal union of an arithmetic genus zero component A and a possibly non-reduced arithmetic genus one component supported on a curve E such that the section meets A along the smooth locus of $f^*(s)$ and the pair $(X, S + A + E)$ has (semi-)log canonical singularities.

(iii) **standard (resp. minimal) intermediate fiber** if $(g\colon Y \to C, S)$ is a standard (resp. minimal) Weierstrass model.

Let X be the relative log canonical model of $(Y, S' + aF') \to C$, and let $\mu\colon X \dashrightarrow Y$ be the birational map to Y. Then the divisor E in both the twisted and intermediate cases is an exceptional divisor for μ. Therefore $\mathrm{lct}(X, 0, E) = 1$ and E appears with coefficient one in the log canonical pair $(X, \mu_*^{-1}(S' + aF') + \mathrm{Exc}(\mu))$. In particular, $F_a = \mu_*^{-1}(aF') + \mathrm{Exc}(\mu)$ contains E with coefficient one.

Lemma 6.1.4. *Let $(f\colon X \to C, S + F_a)$ be the relative log canonical model of a Weierstrass model and suppose that $f\colon X \to C$ has twisted central fiber. Then $F_a = E$.*

Proof. The boundary divisor of the log canonical model is given by

$$\mu_*^{-1}(S' + aF') + \mathrm{Exc}(\mu)$$

where $\mu\colon X \dashrightarrow Y$ is the natural birational map. Then $F_a = \mu_*^{-1}(aF') + \mathrm{Exc}(\mu)$ is a divisor supported on the fiber of f and contains E with coefficient one. Since the fiber is twisted then $F_a = E$. □

The terminology for a twisted fiber comes from the fact that these fibers are exactly those that appear in the coarse space of a flat family $\mathcal{X} \to \mathcal{C}$ of *stable* elliptic curves over a *orbifold* base curve $\mathcal{C}$. Equivalently, a twisted fiber is obtained by taking the quotient of a family of stable curves over the spectrum of a DVR by a subgroup of the automorphism group of the central fiber. This notion is introduced in [AV97] for the purpose of obtaining fibered surfaces from twisted stable maps. In [AB16, Proposition 4.12] it is proved that any twisted fiber pair $(f\colon X \to C, S + E)$ as in the conclusion of Lemma 6.1.4 is obtained as the coarse space of family of stable curves over an orbifold curve. Moreover, twisted models exist and are unique.

Lemma 6.1.5. *Let $(g\colon Y \to C, S')$ be a Weierstrass elliptic surface over the spectrum of a DVR and suppose that there exists a birational model of Y with an intermediate fiber. Then the following birational models are isomorphic:*

(i) the twisted model $(f_1\colon X_1 \to C, S_1 + E_1)$,

(ii) the log canonical model of the intermediate model $(f\colon X \to C, S + A + E)$ with coefficient one,

(iii) the log canonical model of the Weierstrass fiber $(g\colon Y \to C, S' + F')$ *with coefficient one.*

Moreover, in this case there is a morphism $X \to X_1$ *contracting the component* A *to a point.*

Proof. The contraction of $\mu\colon X \to Y$ to its Weierstrass model provides a log canonical partial resolution of $(Y, S' + F')$ with boundary divisor $\mu_*^{-1}(S' + F') = S + A + E$ and so (2) and (3) agree by definition of log canonical model. Now the pair $(X, S + A + E)$ has log canonical singularities and so we may run an mmp and use abundance to compute its relative log canonical model $\mu_0\colon X \to X_0$. Note μ_0 is a morphism since X is a surface. Then $(X_0, \mu_{0_*}(S + A + E))$ is a relative log canonical model with fiber marked with coefficient one and so it must be the twisted model by [AB16, Proposition 4.12]. $\square$

Lemma 6.1.6. *Let* $(f\colon X \to C, S)$ *be a standard intermediate model. Then the relative log canonical model of* $f\colon (X, S + aA + E) \to C$ *is the contraction* $\mu\colon X \to X'$ *of* E *to a point with Weierstrass fiber* $A' = \mu_* A$ *for any* $0 \le a \le \mathrm{lct}(Y, S', F')$ *where* $(Y \to C, S')$ *is the corresponding standard Weierstrass model.*

Proof. By construction a standard intermediate model maps onto the corresponding standard Weierstrass model – call this map $\mu\colon X \to Y$. Then $\mu_* A = F'$ and $\mu_* E = 0$. In particular, $\mu\colon (X, S + aA + E) \to (Y, S' + aF')$ is a log resolution of $(Y, S' + aF')$ for any a. If $a \le \mathrm{lct}(Y, S', F')$ then $(Y, S' + aF')$ is log canonical and μ is the relative log canonical model of $(X, S + aA + E) \to C$. As (Y, S') is log canonical, then $\mathrm{lct}(Y, S', F') \ge 0$. $\square$

Proposition 6.1.7. *Let* $(g\colon Y \to C, S')$ *be a standard Weierstrass model with central fiber* F'*. There exists a number* b_0 *such that* $\mathrm{lct}(Y, S', F') < b_0 \le 1$ *and the relative* *log canonical model of* $(Y, S' + aF') \to C$ *is*

(i) a standard intermediate fiber for $\mathrm{lct}(Y, S', F') < a < b_0$*;*

(ii) a twisted fiber for $b_0 \leq a \leq 1$*.*

Proof. Standard intermediate models for a standard Weierstrass model are computed to exist (by taking a log resolution and blowing down extra components) in [AB17b]. Furthermore, there it is shown that a standard intermediate model $(X \to C, S)$ has a contraction $p\colon X \to X'$ contracting the A component onto a twisted model with central fiber $E' = p_*E$. Now $(X', S + E')$ has log canonical singularities and p is a partial log resolution so (X', E') is the relative log canonical model of $(X, S + \mathrm{Exc}(p) + E) = (X, S + A + E)$.

On the other hand, $\mu\colon X \to Y$ is a log resolution of the pair $(Y, S' + aF')$ with boundary $\mu_*^{-1}(S' + aF') + \mathrm{Exc}(\mu) = S + aA + E$. Thus when $a = 1$, the relative log canonical model of $(Y, S' + F')$, which is equal to the relative log canonical model of the log resolution $(X, S + A + E)$, is the twisted fiber.

Furthermore, A is an exceptional divisor of the log resolution $p\colon X \to X'$ and so the intersection number $A.(K_X + S + A + E) \leq 0$ and similarly E is exceptional for $\mu\colon X \to Y$ and $E.(K_X + S + a_0A + E) = 0$ for $a_0 = \mathrm{lct}(Y, S', F')$. Thus by linearity of intersection numbers, there is a b_0 such that $a_0 \leq b_0 \leq 1$ and

$$A.(K_X + S + aA + E) > 0$$
$$E.(K_X + S + aA + E) > 0$$

for any $a_0 < a < b_0$. □

Proposition 6.1.8. *Let* $(g\colon Y \to C, S')$ *be a non-standard Weierstrass model for* *which there exists an intermediate model. Then there is a number* $0 \leq b_0 \leq 1$ *such that the relative log canonical model of* $(Y, S' + aF')$ *is*

(i) a standard intermediate fiber for $0 \leq a < b_0$*;*

(ii) a twisted fiber for $b_0 \leq a \leq 1$*.*

Proof. By Lemma 6.1.5, the log canonical model is the twisted model for $a = 1$. Now consider the contraction $\mu\colon X \to Y$ from the intermediate to the Weierstrass model. The pair $(X, S + aA + E)$ is a log canonical resolution of (Y, S') so we may compute the relative log canonical model (Y, S') by computing that of $(X, S + aA + E)$.

Y is Gorenstein since it is cut out by a single Weierstrass equation and so K_Y is Cartier. Furthermore, since g is a genus one fibration, K_Y must be supported on fiber components. We may write

$$\mu^*(K_Y) = K_X + \alpha E$$

where $\alpha > 1$ since the singularities of $(Y, 0)$ are *not* log canonical. It follows that

$$(K_X + S + E).E = (1 - \alpha)E^2 > 0$$

so that the first step of the log MMP does not contract E. As a increases, this intersection number also increases and so E is never contracted in the first step of the log MMP.

If $(K_X + S + E).A = 1 + (1 - \alpha)A.E \leq 0$, then A is contracted by either the log MMP or the log canonical linear series for all $a \geq 0$ as increasing a decreases this intersection number. Thus the log canonical model is the twisted model for all a and $b_0 = 0$.

Otherwise if $(K_X + S + E).A > 0$, then the $K_X + S + E$ is already ample and so the intermediate fiber is the log canonical model for $a = 0$. By linearity of intersection numbers there is a unique b_0 such that $(K_X + S + b_0A + E).A = 0$ and this b_0 has the required property. $\square$

To summarize, given a standard Weierstrass model $(g\colon Y \to C, S' + aF')$ over the spectrum of a DVR, there is a standard intermediate model $g\colon X \to C$ which maps to the Weierstrass model by contracting the component E, and maps to the twisted model by contracting the component A. Thus the intermediate fiber can be seen as interpolating between the relative log canonical model being Weierstrass and twisted as the coefficient a varies (see Figure 6.1). For a non-standard Weierstrass model, there is a similar picture except the Weierstrass model is never log canonical and so there is only a single transition from intermediate to twisted.

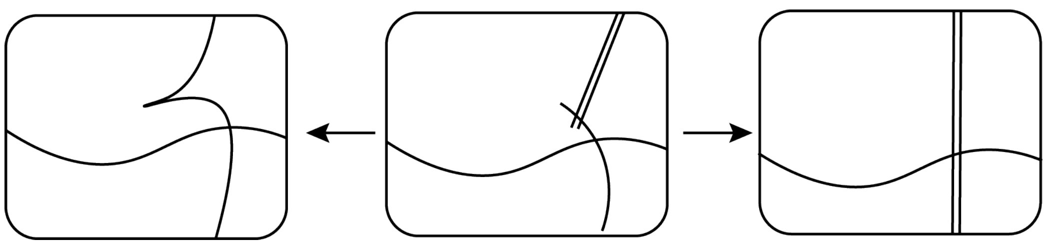


Figure 6.1: Here we illustrate the relative log canonical models and morphisms between them. From left to right: **standard Weierstrass** model $(0 \leq a \leq a_0)$ – a single reduced and irreducible component meeting the section, **standard intermediate** model $(a_0 < a < b_0 \leq 1)$ – a nodal union of a reduced component meeting the section and a nonreduced component, and **twisted** model $(b_0 \leq a \leq 1)$ – a single possibly nonreduced component meeting the section in a singular point of the surface.

Remark 6.1.9. Note since E is a log canonical center of the intermediate fiber pair $(X, S + aA + E)$, then $(E, (S + aA + E)|_E)$ is itself a log canonical pair. In particular, E must be at worst nodal. Since E is irreducible then either the intermediate fiber is reduced and E is a stable elliptic curve, or E supports a nonreduced arithmetic genus one component so E is a smooth rational curve.

The calculations in [AB17b] allow us to make precise the coefficients $a_0 = \mathrm{lct}(Y, S', F')$ and b_0 where the transitions from the various fiber models occur for the minimal Weierstrass models (see Table 5.1). We summarize the calculations here and direct the reader to [AB17b] for more details.

Table 6.1: Intersection pairings in a standard intermediate fiber

Singular fiber	A^2	E^2	$A.E$	Mult. of E in $f^{-1}(p)$
I_n^*	-2	$-1/2$	1	2
II	-6	$-1/6$	1	6
III	-4	$-1/4$	1	4
IV	-3	$-1/3$	1	3
II^*	$-6/5$	$-1/30$	$1/5$	6
III^*	$-4/3$	$-1/12$	$1/3$	4
IV^*	$-3/2$	$-1/6$	$1/2$	3

Theorem 6.1.10. *Let $(g\colon Y \to C, S' + aF')$ be a standard Weierstrass model over the spectrum of a DVR, and let $(f\colon X \to C, S + F_a)$ be the relative log canonical model. Suppose the special fiber F' of g is either either (a) one of the Kodaira singular fiber types, or (b) g is isotrivial with constant j-invariant ∞ and F' is an N_0 or N_1 fiber.*

(i) If F is a type I_n or N_0 fiber, then the relative log canonical model is the Weierstrass model for all $0 \leq a \leq 1$.

(ii) For any other fiber type, there is an a_0 such that the relative log canonical model is

(i) the Weierstrass *model for any $0 \leq a \leq a_0$,*

(ii) a twisted fiber *consisting of a single non-reduced component supported on a smooth rational curve when $a = 1$, and*

(iii) *a* standard intermediate fiber *with* E *a smooth rational curve for any* $a_0 < a < 1$.

The constant a_0 *is as follows for the other fiber types:*

$$a_0 = \begin{cases} 5/6 & \mathrm{II} \\ 3/4 & \mathrm{III} \\ 2/3 & \mathrm{IV} \\ 1/2 & \mathrm{N}_1 \end{cases} \quad a_0 = \begin{cases} 1/6 & \mathrm{II}^* \\ 1/4 & \mathrm{III}^* \\ 1/3 & \mathrm{IV}^* \\ 1/2 & \mathrm{I}_n^* \end{cases}$$

Remark 6.1.11. The difference between Theorem 6.1.10 and the corresponding theorem in [AB17b], is that here we are marking the E component of the intermediate fiber with coefficient one rather than marking it by a. By the above discussion, this is equivalent to taking the log canonical model of the Weierstrass pair rather than taking the log canonical models of the minimal resolution as in [AB17b]. With this in mind, the result above for types $\mathrm{II}, \mathrm{III}, \mathrm{IV}$ and N_1 fibers are unchanged as for these types of fibers, the coefficient of E was already one in [AB17b] and the results for I_n and N_0 are unchanged as these fibers are already log canonical models regardless of coefficient. The reason for this change in convention is to avoid an unwanted flip (Theorem 7.2.4).

Convention 6.1.12. Let $(f\colon X \to C, S + F_a)$ be an elliptic fibration over the spectrum of a DVR with section S, central fiber F, and boundary divisor F_a supported on F. From now on we will say this pair is a relative log canonical if it is the relative log canonical model of a Weierstrass model. That is, either

(i) F is a Weierstrass fiber and $F_a = aF$ for $a \leq \mathrm{lct}(X, S, F)$,

(ii) F is a standard intermediate fiber with $F_a = aA + E$ and $\mathrm{lct}(Y, S', F') < a < 1$, or

(iii) F is a twisted fiber with $F_a = E$ and $a = 1$.

More generally, we will say $(f\colon X \to C, S + F)$ over a smooth curve is a **relative log canonical model** or **relatively stable** if it is the relative log canonical model of its Weierstrass model so that the restriction of $(X, S + F)$ to the local ring of each point in C is a relative log canonical model of Weierstrass, intermediate or twisted type. We will call it **standard (resp. minimal)** if each of the fibers are log canonical models of standard (resp. minimal) Weierstrass models.

6.1.2 Canonical bundle formula

In [AB17b], we computed a formula for the canonical bundle of relative log canonical model.

Theorem 6.1.13. *[AB17b, Theorem 1.2] Let $(f\colon X \to C, S + F_{\mathcal{A}})$ be a relative log canonical model where $f\colon X \to C$ is a minimal irreducible elliptic surface with section S, and let $F_{\mathcal{A}} = F_{a_i}$ is a sum of marked fibers as in 6.1.12 with $0 \leq a_i \leq 1$. Then*

$$\omega_X = f^*(\omega_C \otimes \mathscr{L}) \otimes \mathcal{O}_X(\Delta).$$

where Δ is effective and supported on fibers of type $\mathrm{II}, \mathrm{III}$, and IV contained in $\mathrm{Supp}(F)$. The contribution of a type II, III or IV fiber to Δ is given by αE where E

supports the unique nonreduced component of the fiber and

$$\alpha = \begin{cases} 4 & \mathrm{II} \\ 2 & \mathrm{III} \\ 1 & \mathrm{IV} \end{cases}$$

It is important to emphasize here that only type II, III or IV fibers that are *not* in Weierstrass form affect the canonical bundle. If all of the type II, III and IV fibers of $f\colon X \to C$ are Weierstrass, then the usual canonical bundle formula $\omega_X = f^*(\omega_C \otimes \mathscr{L})$ holds.

6.1.3 Pseudoelliptic contractions

In [LN02], La Nave studied compactifications of the moduli space of Weierstrass fibrations by stable elliptic surface pairs $(f\colon Y \to C, S)$ – i.e. where $\mathcal{A} = 0$. There it was shown that the section of some irreducible components of a reducible elliptic surface may be contracted by the log MMP, inspiring the following.

Definition 6.1.14. A **pseudoelliptic surface** is a surface X obtained by contracting the section of an irreducible elliptic surface pair $(f\colon Y \to C, S)$. For any fiber of f, we call its pushforward via $\mu\colon Y \to X$ a **pseudofiber** of X. We call $(f\colon Y \to C, S)$ the **associated elliptic surface** to X. If $(f\colon Y \to C, S + F'_{\mathcal{A}})$ is an $\mathcal{A}$-weighted relative log canonical model then we call $(X, F_{\mathcal{A}})$ a **pseudoelliptic pair** where $F_{\mathcal{A}} = \mu_* F'_{\mathcal{A}}$.

In the next section we will discuss when a pseudoelliptic surface forms. That is, for which $\mathcal{A}$ does the minimal model program necessitate that the section of a relative log canonical model $(f\colon Y \to C, S + F_{\mathcal{A}})$ contracts to form a pseudoelliptic surface?

In this section we are tasked with understanding when the log canonical contraction of pseudoelliptic pair $(X, F_\mathcal{A})$ corresponding to a relatively stable elliptic surface contracts X to a lower dimensional variety.

Proposition 6.1.15. *[AB17b, Proposition 7.1] Let* $f\colon Y \to C$ *be an irreducible properly elliptic surface with section* S*. Then* $K_Y + S$ *is big.*

Corollary 6.1.16. *If* X *is a log canonical pseudoelliptic surface such that the associated elliptic surface* $\mu\colon Y \to X$ *is properly elliptic, then* K_X *is big.*

Proof. $\mu\colon Y \to X$ is a partial log resolution so $K_Y + \mathrm{Exc}(\mu) = K_Y + S$ is big if and only if K_X is big by Corollary 5.1.8. □

Definition 6.1.17. The **fundamental line bundle** $\mathscr{L}$ of a pseudoelliptic surface X is the fundamental line bundle (see Definition 5.2.7) for $(f\colon Y \to C, S)$ the corresponding elliptic surface.

Proposition 6.1.18. *[AB17b, Proposition 7.4] Let* $(X, F_\mathcal{A})$ *be an* $\mathcal{A}$*-weighted slc pseudoelliptic surface pair corresponding to an elliptic surface* $(f\colon Y \to C, S)$ *over a rational curve* $C \cong \mathbb{P}^1$*. Denote by* $\mu\colon Y \to X$ *the contraction of the section. Suppose* $\deg \mathscr{L} = 1$ *and* $0 \leq \mathcal{A} \leq 1$ *such that* $K_X + F_\mathcal{A}$ *is a nef and* $\mathbb{Q}$*-Cartier. Then either*

i) $K_X + F_\mathcal{A}$ *is big and the log canonical model is an elliptic or pseudoelliptic surface;*

ii) $K_X + F_\mathcal{A} \sim_\mathbb{Q} \mu_*\Sigma$ *where* Σ *is a multisection of* Y *and the log canonical contraction maps* X *onto a rational curve; or*

iii) $K_X + F_\mathcal{A} \sim_\mathbb{Q} 0$ *and the log canonical map contracts* X *to a point.*

The cases above correspond to $K_X + F_\mathcal{A}$ *having Iitaka dimension* $2, 1$ *and* 0 *respectively.*

Remark 6.1.19. The proof of [AB17b, Proposition 7.4] actually gives a method for determining which situation of $(i), (ii)$, and (iii) we are in. Indeed since $K_X + F_{\mathcal{A}}$ is nef, it is big if and only if $(K_X + F_{\mathcal{A}})^2 > 0$. Furthermore, $K_X + F_{\mathcal{A}} \sim_{\mathbb{Q}} 0$ if and only if $t = 0$ where

$$K_Y + tS + \tilde{F}_{\mathcal{A}} = \mu^*(K_X + F_{\mathcal{A}}).$$

So if $K_X + F_{\mathcal{A}}$ is not big, it suffices to compute whether $t > 0$ or $t = 0$ to decide if the log canonical map contracts the pseudoelliptic to a curve or to a point.

Proposition 6.1.20. *Let $(f\colon X \to C, S + F_{\mathcal{B}})$ be an irreducible elliptic surface over a rational curve $C \cong \mathbb{P}^1$ such that $(X, S + F_{\mathcal{B}})$ is a stable pair and $\deg \mathscr{L} = 1$. Suppose $\mathcal{B} = (1, b_2, \ldots, b_s)$, $0 < \mathcal{A} \leq \mathcal{B}$ such that $a_1 = b_1 = 1$, and F_1 is a type I_n fiber. Then $K_X + S + F_{\mathcal{A}}$ is big.*

Proof. By the canonical bundle formula, $K_X = -G + \Delta$ for G a general fiber and Δ effective. All fibers are linearly equivalent as C is rational, and type I_n fibers are reduced so that $F_1 \sim_{\mathbb{Q}} G$. Thus $K_X + F_1 = \Delta$ is effective and

$$K_X + S + F_{\mathcal{B}} = \Delta + S + \sum_{i=2}^{s} F_{b_i},$$

$$K_X + S + F_{\mathcal{A}} = \Delta + S + \sum_{i=2}^{s} F_{a_i}$$

with $0 < a_i \leq b_i$ with

$$F_a = \begin{cases} aF \\ aA + E \\ E \end{cases}$$

depending on whether F is a Weierstrass, intermediate or twisted fiber. Furthermore $K_X + S + F_{\mathcal{B}}$ ample. Since $a_i > 0$, for m large enough we can write

$m(K_X+S+F_{\mathcal{A}})-(K_X+S+F_{\mathcal{B}})=D$ where D is effective. Therefore $K_X+S+F_{\mathcal{A}}$ is big by Kodaira's lemma. $\square$

Proposition 6.1.21. *[AB17b, Proposition 7.3] Let $(f\colon X \to C, S+F_{\mathcal{A}})$ be an irreducible elliptic surface over a rational curve $C \cong \mathbb{P}^1$ such that $(X, S+F_{\mathcal{A}})$ is a relative log canonical model and suppose that* $\deg \mathscr{L} = 2$.

(a) If $\mathcal{A} > 0$, then $K_X+S+F_{\mathcal{A}}$ is big and the log canonical model is either the relative log canonical model, or the pseudoelliptic obtained by contracting the section of the relative log canonical model.

(b) If $\mathcal{A} = 0$, then the minimal model program results in a pseudoelliptic surface with a log canonical contraction that contracts this surface to a point.

Proposition 6.1.22. *Let (X, G_1+G_2) be an slc pseudoelliptic surface pair with pseudofibers G_1 and G_2 marked with coefficient one. Then $K_X+G_1+G_2$ is big.*

Proof. Consider the blowup $\mu\colon Y \to X$, where $(f\colon Y' \to \mathbb{P}^1, S'+G_1''+G_2'')$ is the corresponding elliptic surface. Taking the relative log canonical model, we obtain a pair $(f\colon Y \to \mathbb{P}^1, S+G_1'+G_2')$, where by construction $K_Y+S+G_1'+G_2'$ is relatively ample. Note that $(K_Y+S+G_1'+G_2').S = 0$ by Proposition 7.1.13 and $K_Y+S+G_1'+G_2'$ has positive degree on all other curve classes as it is f'-ample. Therefore $K_Y+S+G_1'+G_2'$ is actually nef, and thus semiample by Proposition 5.1.5. Therefore the only curve contracted by $|m(K_Y+S+G_1'+G_2')|$ is the section S and the log canonical model of (X, G_1+G_2) is the corresponding pseudoelliptic surface of $(f\colon Y \to \mathbb{P}^1, S+G_1'+G_2')$. Therefore, (X, G_1+G_2) is log general type and $K_X+G_1+G_2$ must be big. $\square$

CHAPTER 7

Moduli spaces of weighted stable elliptic surfaces

7.1 Weighted stable elliptic surfaces

In this section we will construct a compactification of the moduli space of log canonical models $(f\colon X \to C, S+F_{\mathcal{A}})$ of $\mathcal{A}$-weighted Weierstrass elliptic surface pairs by allowing our surface pairs to degenerate to *semi-log canonical (slc)* pairs (see Definition 5.1.2). As such our surfaces can acquire non-normal singularities and break up into multiple components.

The first definition we give, inspired by the minimal model program, yields a finite type and separated algebraic stack (see Theorem 7.1.4) with possibly too many components. In Definition 7.1.9, we will give a more refined definition of the objects that appear on the boundary of the compactified moduli stack when one runs stable reduction (see Theorem 7.2.9).

Definition 7.1.1. An **$\mathcal{A}$-weighted slc elliptic surface with section**

$$(f\colon X \to C, S + F_{\mathcal{A}})$$

, (see Figure 7.1) is an slc surface pair $(X, S + F_{\mathcal{A}})$ and a proper surjective morphism with connected fibers $f\colon X \to C$ to a projective nodal curve such that:

(a) S is a section with generic points contained in the smooth locus of f, and $F_{\mathcal{A}}$ is an $(\mathcal{A} \sqcup 1)$-weighted sum of reduced divisors contracted by f;

(b) every component of $Z \subset X$ is either an elliptic surface with fibration $f|_Z$ and section $S|_Z$, or a surface contracted to a point by f;

(c) for each elliptic component Z, the restriction $(F_{\mathcal{A}})|_Z$ makes the pair

$$(f|_Z\colon Z \to C, S|_Z + (F_{\mathcal{A}})|_Z)$$

into a $\mathcal{A}$-weighted relative log canonical model such that all the marked fibers lie over smooth points of C.

We say that $(f\colon X \to C, S + F_{\mathcal{A}})$ is an **$\mathcal{A}$-stable elliptic surface** if the $\mathbb{Q}$-Cartier divisor $K_X + S + F_{\mathcal{A}}$ is ample, that is, if $(X, S + F_{\mathcal{A}})$ is a stable pair.

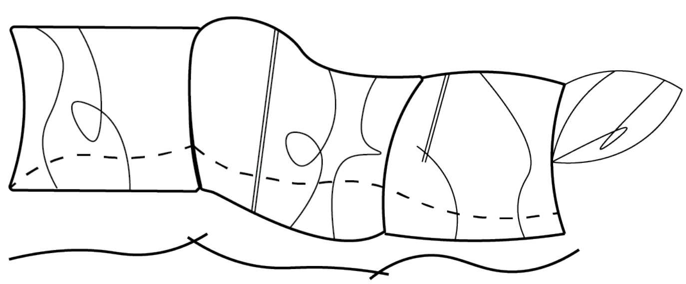


Figure 7.1: An $\mathcal{A}$-weighted slc elliptic surface.

We will elaborate on parts (b) and (c) of the above Definition 7.1.1. The components in condition (b) contracted to a point by f were the *pseudoelliptic components* (see Definition 6.1.14). We will study them further in Section 7.1.3. The condition (c) ensures that the restriction of $F_{\mathcal{A}}$ to any elliptic component consists of a-weighted twisted, intermediate or Weierstrass fibers F_a marked as in Definition 6.1.12.

7.1.1 Moduli functor for elliptic surfaces

Following [KP17], we introduce the following notion of a pseudofunctor (following Definition 5.2 of [KP17]) of stable elliptic surfaces:

Definition 7.1.2. Fix $v \in \mathbb{Q}_{>0}$. A pseudofunctor $\mathcal{E} \colon \mathfrak{Sch}_k \to \mathfrak{Grp}$ from the category of k-schemes to groupoids is a **moduli pseudofunctor for $\mathcal{A}$-stable elliptic surfaces of volume** v if for any normal variety T,

$$\mathcal{E}(T) = \left\{ \left(\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & C \arrow[dl, "h"] \\ & T & \end{tikzcd} , S + F_{\mathcal{A}} \right) \middle| \begin{array}{ll} \text{(i)} & g \colon (X, S + F_{\mathcal{A}}) \to T \text{ is a flat family of stable pairs of dimension 2 and volume } v \text{ as in Definition 5.1.18;} \\ \text{(ii)} & h \text{ is a flat family of connected nodal curves;} \\ \text{(iii)} & f \text{ is a morphism over } T\text{; and} \\ \text{(iv)} & \text{for each } t \in T \text{, the fiber } f_t \colon (X_t, S_t + (F_{\mathcal{A}})_t) \to C_t \text{ is an } \mathcal{A}\text{-weighted slc elliptic surface with section and marked fibers.} \end{array} \right\}.$$

Let $\mathcal{E}^{\circ}$ be the subfunctor consisting of families with $(f_t \colon X_t \to C_t, S_t)$ a *minimal relative log canonical model* with section over a smooth curve as in Definition 5.2.1.

The **main component** $\mathcal{E}^m$ will denote the closure $\overline{\mathcal{E}^\circ}$ in $\mathcal{E}$.

Remark 7.1.3. Despite the terminology, it is not true in general that $\mathcal{E}^m$ is irreducible. Rather, it has components labeled by the configurations of singular fibers on the irreducible elliptic surfaces.

Theorem 7.1.4. *There exists a moduli pseudofunctor of $\mathcal{A}$-stable elliptic surfaces of volume v such that the main component $\mathcal{E}^m$ is a separated Deligne-Mumford stack of finite type.*

Proof. In [KP17], a suitable pseudofunctor $\mathcal{M}_{v,I,n}$ for stable pairs (X, D) with volume v, coefficient set I and index n is defined. Here n is a fixed integer such that $n(K_X+D)$ is required to be Cartier. Furthermore, $\mathcal{M}_{v,I,n}$ is a finite type Deligne-Mumford stack with projective coarse space (see Proposition 5.11 and Corollary 6.3 in [KP17]). Take I to be the additively closed set generated by the weight vector $\mathcal{A}$. By boundedness for surface pairs (see Theorem 9.2. in [Ale94]), there exists an index n such that $n(K_X + S + F_{\mathcal{A}})$ is a very ample Cartier divisor for all $\mathcal{A}$-stable elliptic surfaces of volume v.

Consider the stack of stable pairs $\mathcal{M}_{v,I,n}$ and denote $\mathcal{M} := \mathcal{M}_{v,I,n}$ for convenience. Let $\mathcal{X} \to \mathcal{M}$ be the the universal family. Furthermore, let $\mathfrak{M}_g$ be the algebraic stack of prestable curves with universal family $\mathfrak{C}_g \to \mathfrak{M}_g$. Consider the Hom-stack

$$\mathscr{H}om_{\mathcal{M}\times\mathfrak{M}_g}(\mathcal{X} \times \mathfrak{M}_g, \mathcal{M} \times \mathfrak{C}_g).$$

This is a quasi-separated algebraic stack locally of finite presentation with affine stabilizers by Theorem 1.2 in [HR14]. Now we consider the pseudofunctor given by

$$\mathcal{E}_{v,\mathcal{A},n}\colon B \mapsto \left\{ \begin{tikzcd} (X, S+F_{\mathcal{A}}) \arrow[rr, "f"] \arrow[dr] & & C \arrow[dl] \\ & B & \end{tikzcd} \right\}$$

where $(X, S+F_{\mathcal{A}}) \to B$ is a flat family of stable pairs in the sense of [KP17], $C \to B$ is a flat family of pre-stable curves, and $(f_b\colon X_b \to C_b, S_b + (F_{\mathcal{A}})_b)$ is an $\mathcal{A}$-stable elliptic surface with volume v for each $b \in B$.

It is clear that $\mathcal{E}_{v,\mathcal{A},n}$ is a substack of the Hom-stack $\mathscr{H}om_{\mathcal{M}\times\mathfrak{M}_g}(\mathcal{X}\times\mathfrak{M}_g, \mathcal{M}\times\mathfrak{C}_g)$. The substack $\mathcal{E}^{\circ}_{v,\mathcal{A},n}$ parametrizing irreducible minimal log canonical models of elliptic surfaces over base curves is an algebraic substack of the Hom-stack, as flatness, irreducibility and smoothness are algebraic conditions. Thus the closure $\mathcal{E}^{m}_{v,\mathcal{A},n}$ in the Hom-stack is a quasi-separated algebraic stack locally of finite presentation with affine stabilizers, and is a pseudofunctor for $\mathcal{A}$-stable elliptic surfaces of volume v.

To prove that $\mathcal{E}^{m}_{v,\mathcal{A},n}$ is separated, let B be a smooth curve and let

$$(X^0, S^0 + F^0_{\mathcal{A}}) \xrightarrow{f^0} C^0 \longrightarrow B^0 = B \setminus p$$

be a flat family of $\mathcal{A}$-stable elliptic surfaces over the complement of a point $p \in B$. Suppose

$$(X, S + F_{\mathcal{A}}) \xrightarrow{f} C \longrightarrow B$$
$$(X', S' + F'_{\mathcal{A}}) \xrightarrow{f'} C' \longrightarrow B$$

are two extensions to B.

Then $(X, S + F_{\mathcal{A}}) \to B$ and $(X', S' + F'_{\mathcal{A}}) \to B$ are two families of stable pairs over B with isomorphic restrictions to B^0. Since log canonical models are unique, $(X', S' + F'_{\mathcal{A}}) = (X, S + F_{\mathcal{A}})$ over B. Furthermore, the compositions $S \to C$ and $S' \to C'$ are isomorphisms so $C \cong C'$ over B. Therefore, we have $f, f' \colon X \to C \to B$ with $f|_{X^0} = f'|_{X^0}$. Since $X \to B$ is flat, X^0 is dense in X, therefore $f = f'$ since C is separated. Thus an extension to B is unique and so $\mathcal{E}^m_{v,\mathcal{A},n}$ is separated.

Finally, we show that the stack is Deligne-Mumford, by showing that the objects have finitely many automorphisms. An automorphism of $(X, S + F_{\mathcal{A}}) \to C$ is an automorphism σ of the elliptic surface pair $(X, S + F_{\mathcal{A}})$, as well as an automorphism τ of C such that the autormophisms commute. Since the autormophism σ fixes the fibers $F_{\mathcal{A}}$, the compatibility of the automorphisms implies that τ actually fixes the marked points $D_{\mathcal{A}}$ on the base curve C (see Definition 7.1.12). We will show in Corollary 7.1.15 that the base curve is actually a weighted stable pointed curve in the sense of Hassett, and thus has finitely automorphisms. Moreover, there are finitely automorphisms of the stable surface pair (see e.g. [Iit82, 11.12]). □

As it is not clear how to define families of stable pairs over a general base (see Remark 5.1.17), from now on we restrict to only considering families of elliptic surfaces over a normal base.

Definition 7.1.5. Define

$$\mathcal{E}_{v,\mathcal{A}} := \left(\mathcal{E}^m_{v,\mathcal{A},n}\right)^\nu$$

to be the normalization of the stack constructed in Theorem 7.1.4 (see Appendix A for a discussion on normalizations) and $\mathcal{U}_{v,\mathcal{A}} \to \mathcal{E}_{v,\mathcal{A}}$ the pullback of the universal family.

$\mathcal{E}_{v,\mathcal{A}}$ is a separated algebraic stack locally of finite type with affine stabilizers. By

Proposition A.0.7, the stack $\mathcal{E}_{v,\mathcal{A}}$ is independent of n for n large enough, and more generally independent of the choice of pseudofunctor $\mathcal{E}$ as in Definition 7.1.2.

7.1.2 Broken elliptic surfaces

In this section we refine the definition of an $\mathcal{A}$-weighted stable elliptic surface pair to more accurately reflect the type of surfaces that will appear as a result of stable reduction. Our strategy for this, inspired by [LN02], is to compute a limit in the twisted stable maps moduli space [AV97, AV02, AB16], replace this family with its $\mathcal{A}$-weighted relative log canonical model, and then run the minimal model program to produce a limit of stable pairs.

To this end, let $(f\colon X \to C, S + F_{\mathcal{A}})$ be an $\mathcal{A}$-weighted slc elliptic surface. We want to perform a sequence of extremal and log canonical contractions over C to make $K_X + S + F_{\mathcal{A}}$ an f-ample divisor.

Let $\nu\colon C' \to C$ be the normalization and let X' be the pullback:

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ C' \arrow[r, "\nu"] & C \end{tikzcd} .$$

Then $\varphi^*(K_X + S + F) = K_{X'} + G + S' + F'$ is f'-ample if and only if $K_X + S + F$ is f-ample. Here $\varphi^* S = S'$ is a section of f' and $F' = \varphi^* F$. The divisor G is the reduced divisor above the points of C' lying over the nodes of C. In particular, to compute the relative canonical model over C starting with a log smooth model, it suffices to assume C is smooth and $f\colon X \to C$ is an irreducible elliptic surface and so the computation of relative log canonical models reduces to that in Chapter 6.

We now move on to the question of what sorts of pseudoelliptic components appear and how are they attached? There are two *types* of pseudoelliptic components that will appear as irreducible components of a stable limit of elliptic surfaces.

Definition 7.1.6. Let $(T, 0)$ be a rooted tree with root vertex $0 \in V(T)$. We make $V(T)$ into a poset by declaring that $\alpha \leq \beta$ if vertex α lies on the unique minimal length path from vertex β to the root 0. We denote by $T[i]$ the set of vertices of distance i from the root so that $T[0] = \{0\}$. Finally, if $\alpha \in T[i]$, we denote by $\alpha[1]$ the set of vertices $\beta \in T[i+1]$ with $\alpha \leq \beta$.

Definition 7.1.7. Let $(T, 0)$ be a rooted tree. A **pseudoelliptic tree** $(Y, F_{\mathcal{A}})$ with dual graph $(T, 0)$ is an slc pair consisting of the union of pseudoelliptic components Y_α with dual graph T constructed inductively: every component Y_β for $\beta \in \alpha[1]$ is attached to Y_α by gluing a twisted pseudofiber G_β of Y_β to the arithmetic genus one component E_α of an intermediate pseudofiber with reduced component A_α of Y_α. The $\mathcal{A}$-weighted marked fibers $F_{\mathcal{A}}$ satisfy

$$\mathrm{Coeff}(A_\alpha, F_{\mathcal{A}}) = \sum_{\beta \in \alpha[1]} \sum_{D \in \mathrm{Supp}(F_{\mathcal{A}}|_{Y_\beta})} \mathrm{Coeff}(D, F_{\mathcal{A}}). \tag{7.1}$$

A component $(Y_\alpha, F_{\mathcal{A}}|_{Y_\alpha})$ is a **Type I** pseudoelliptic (See Figure 7.2).

Definition 7.1.8. A pseudoelliptic surface of **Type II** (see Figure 7.3) is formed by the log canonical contraction of a section of an elliptic component attached along twisted or stable fibers.

One important fact is that the section S is often contracted even for $\mathcal{A}$-weighted elliptic surfaces with small but nonzero weights. In fact, we will see (Section 7.1.3) that contracting the section of a component to form a pseudoelliptic corresponds to

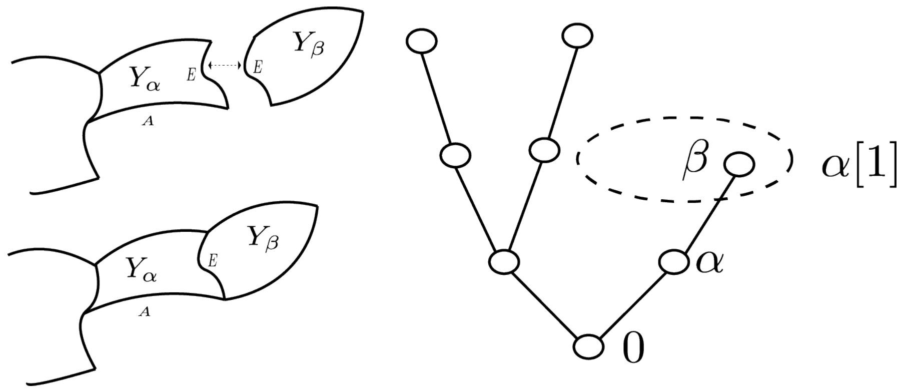


Figure 7.2: A pseudoelliptic tree of is constructed inductively by attaching a *Type I* pseudoelliptic surface Y_β to Y_α for each $\beta \in \alpha[1]$ as pictured. The component A on Y_α is marked by the sum of the weights of the markings on Y_β.

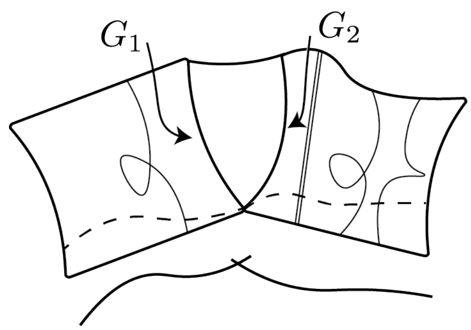


Figure 7.3: A pseudoelliptic surface of *Type II* attached along twisted fibers G_1 and G_2.

stabilizing the base curve as an $\mathcal{A}$-stable curve in the sense of Hassett (see Section 5.1.2).

We can now define the particular $\mathcal{A}$-weighted stable elliptic surfaces that will appear on the boundary of the main components of the moduli space (see Figure 7.4).

Definition 7.1.9. An **$\mathcal{A}$-broken elliptic surface** is an $\mathcal{A}$-weighted slc elliptic surface pair

$(f\colon X \to C, S + F_{\mathcal{A}})$ such that (see Figure 7.4)

(a) each component of X contracted by f is a type I or type II pseudoelliptic surface with marked pseudofibers;

(b) the elliptic components and type II pseudoelliptics are attached along twisted fibers;

(c) the type I pseudoelliptics appear in pseudoelliptic trees attached by gluing a twisted pseudofiber G_0 on the root to an arithmetic genus one component E of an intermediate (pseudo)fiber of an elliptic (type II pseudoelliptic) component;

(d) all marked intermediate (pseudo)fibers are minimal.

We say $(f\colon X \to C, S + F_{\mathcal{A}})$ is an **$\mathcal{A}$-broken stable elliptic surface** if $(X, S + F_{\mathcal{A}})$ is a stable pair.

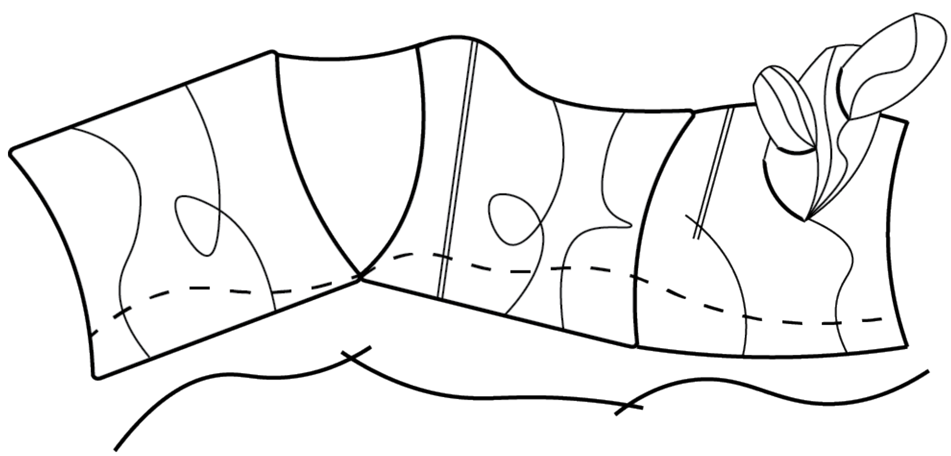


Figure 7.4: An $\mathcal{A}$-weighted broken elliptic surface.

Remark 7.1.10. Note this definition allows for non-minimal Weierstrass cusps and also non-minimal intermediate fibers contained in the double locus.

Remark 7.1.11. For each pseudoelliptic component $X_0 \subset X$ with associated elliptic surface $f_0\colon Y_0 \to C_0$ and morphism $\mu_0\colon Y_0 \to X_0$ contracting the section, there is a unique slc elliptic surface $f'\colon X' \to C'$ with $Y_0 \subset X'$ and $f'|_{Y_0} = f_0$. There is a morphism $\mu\colon X' \to X$ contracting the section of Y_0. Thus one can think of a broken elliptic surface as one obtained by contracting the sections and corresponding components in the base curve of some irreducible components of an $\mathcal{A}$-weighted slc elliptic pair $(f\colon X' \to C', S' + F'_{\mathcal{A}})$ where *every component* is a relative log canonical model of an elliptic surface and in particular only has twisted, intermediate or Weierstrass fibers marked appropriately.

7.1.3 Formation of pseudoelliptic components

In this subsection, we record various statements about the formation of pseudoelliptic components.

The following describes how the log canonical divisor class intersects the section (see also Proposition 6.5 in [AB17b] and Proposition 4.3.2 in [LN02]). This determines when the section of a component contracts to form a pseudoelliptic surface.

Given an $\mathcal{A}$-broken elliptic surface $(f\colon X \to C, S + F_{\mathcal{A}})$, we obtain an $\mathcal{A}$-weighted pointed curve $(C, D_{\mathcal{A}})$ as follows

Definition 7.1.12. Let $(f\colon X \to C, S + F_{\mathcal{A}})$ be a $\mathcal{A}$-broken elliptic surface. Define $D_{\mathcal{A}} = \sum a_i p_i$ where $p_i = f(F_{a_i})$ is the image of the i^{th} marked fiber and a_i its coefficient.

We form the dual graph of C by assigning a vertex to each irreducible component $C_\alpha \subset C$ and an edge for each node. Let v_α be the valence of C_α in the dual graph and $g(C_\alpha)$ the geometric genus of C_α.

Proposition 7.1.13. *[AB16, Proposition 5.3] Let $(f\colon X \to C, S + F_{\mathcal{A}})$ be an $\mathcal{A}$-broken elliptic surface with section S. Let $(C, D_{\mathcal{A}})$ be the $\mathcal{A}$-weighted pointed curve and $C \subset C_\alpha$ an irreducible component. Then for the component S_α of the section* *lying above C_α, we have*

$$\begin{aligned}(K_X + S + F).S_\alpha &= 2g(C_\alpha) - 2 + v_\alpha + \deg\left(D_{\mathcal{A}}|_{C_\alpha}\right) \\ &= \deg\left(\omega_C(D_{\mathcal{A}})|_{C_\alpha}\right).\end{aligned}$$

Proof. The case where $\mathcal{A} = 1$ is precisely Proposition 5.3 of [AB16] (see also Proposition 6.5 in [AB17b] and Proposition 4.3.2 in [LN02]). This more general case follows from the adjunction formula, as the section passes through the smooth locus of the surface in a neighborhood of any fiber that is not marked with coefficient $a_i = 1$. □

Remark 7.1.14. Indeed this computation really shows that $D_{\mathcal{A}}$ is the unique divisor on C such that

$$\sigma^*(K_X + S + F_{\mathcal{A}}) = K_C + D_{\mathcal{A}}$$

and $(C, D_{\mathcal{A}})$ is an slc pair where $\sigma \colon C \to X$ is the map identifying C with the section S. That is, $D_{\mathcal{A}}$ is the *different* $\mathrm{Diff}_S(F_{\mathcal{A}})$ (see [Kol13, Section 4.2] and [Pat16, Corollary 2.11]).

Corollary 7.1.15. *[AB17b, Corollary 6.7 & 6.8] Let $(f \colon X \to C, S + F_{\mathcal{A}})$ be an $\mathcal{A}$-broken elliptic surface such that each component is elliptically fibered. Then $(K_X + S + F_{\mathcal{A}}).S_\alpha > 0$ for every component S_α of S if and only if $(C, D_{\mathcal{A}})$ is an $\mathcal{A}$-pointed stable curve in the sense of Hassett. In this case, the relative log canonical model over C is stable.*

Corollary 7.1.16. *[AB17b, Corollary 6.9] The log minimal model program contracts the section of an elliptic component $X_\alpha \to C_\alpha$ of $(f \colon X \to C, S + F_{\mathcal{A}})$ to produce a pseudoelliptic if and only if either:*

(a) $C \cong \mathbb{P}^1$ *and* $\sum a_i \leq 2$*, or*

(b) C *is a genus one curve and* $a_i = 0$ *for all* i*.*

7.2 Stable reduction

The goal of this section is to prove a stable reduction theorem for $\mathcal{A}$-broken elliptic surfaces in the spirit of La Nave [LN02]. As a result we obtain properness of the moduli spaces $\mathcal{E}_{v,\mathcal{A}}$ and give a description of the surfaces that appear in the boundary.

Our strategy for stable reduction is to first compute stable limits of a family of irreducible elliptic surfaces with large coefficients. To this end, in [AB16] we use the theory of *twisted stable maps* to compute stable limits in the case when all singular fibers are marked with coefficient $b_i = 1$. We then run the minimal model program while reducing the coefficients to compute the stable limit for weights $\mathcal{A}$ using the classification of log canonical models of elliptic surfaces as well Theorem 9.0.10.

7.2.1 Wall and chamber structure

Let $\mathcal{D} \subset (\mathbb{Q} \cap [0,1])^n$ be the set of *admissible weights*: weight vectors $\mathcal{A}$ such that $K_X + S + F_{\mathcal{A}}$ is pseudoeffective. A *wall and chamber* decomposition of $\mathcal{D}$ is a finite collection $\mathcal{W}$ of real codimension one hyperplanes (the *walls*), and the *chambers* are the connected components of the complement of $\mathcal{W}$ in $\mathcal{D}$.

First we describe a wall and chamber decomposition of $\mathcal{D}$ defined by where the log canonical model of an $\mathcal{A}$-slc elliptic surface changes as $\mathcal{A}$ varies. The collection of walls $\mathcal{W}$ corresponds to the steps in the MMP required to produce a stable limit of a family of elliptic surfaces over a smooth curve.

Definition 7.2.1. The collection $\mathcal{W}$ consists of the following types of walls:

I A wall of *Type* $\mathrm{W_I}$ is a wall arising from the log canonical transformations seen in Section 6.1 – that is, the walls where the fibers of the relative log canonical model transition from twisted, to intermediate, to Weierstrass fibers.

II A wall of *Type* $\mathrm{W_{II}}$ is a wall at which the log canonical morphism induced by the log canonical contracts the section of some components. By Corollary 7.1.15 these are the same as the walls for Hassett space $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

III A wall of *Type* $\mathrm{W_{III}}$ is a wall where the morphism induced by the log canonical contracts a rational pseudoelliptic component. These are determined by Proposition 6.1.18 and Remark 6.1.19

There are also *boundary walls* given by $a_i = 0, 1$ at the boundary of $\mathcal{D}$. These can be of any of the types above.

Theorem 7.2.2. *The non-boundary walls of each type are described as follows:*

(a) Type $\mathrm{W_I}$ *walls are defined by the equations*

$$a_i = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}.$$

(b) Type $\mathrm{W_{II}}$ *walls are defined by equations*

$$\sum_{j=1}^{k} a_{i_j} = 1.$$

where $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$. *When the base curve is rational there is another* $\mathrm{W_{II}}$ *wall at*

$$\sum_{i=1}^{r} a_i = 2.$$

(c) Type $\mathrm{W_{III}}$ *walls where a rational pseudoelliptic component contracts to a point are given by*

$$\sum_{j=1}^{k} a_i = c$$

where $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ *and* $c = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$ *are the log canonical thresholds of minimal Weierstrass fibers.*

(d) Finitely many Type $\mathrm{W_{III}}$ *walls where an isotrivial rational pseudoelliptic component contracts onto the* E *component of a pseudoelliptic surface it is attached to.*

In particular, there are only finitely many walls and chambers.

Proof. Part (a) follows from the results of Section 6.1 since these are exactly the coefficients at which minimal Weierstrass cusps transition from Weierstrass models to intermediate models.

Part (b) follows from Proposition 7.1.13 since $(K_X + S + F_{\mathcal{A}}).S > 0$ if and only if the base curve is a weighted stable pointed curve. When $\sum a_{i_j} = 1$, the section of any component fibered over a rational curve, which is attached to the other components of the surface along one attaching fiber, and contains marked fibers $i_1, \ldots, i_k$ gets contracted. When the base curve is $\mathbb{P}^1$ and $\sum a_i = 2$, the section of *every* elliptic surface gets contracted so that all $\mathcal{A}$-slc elliptic surfaces have *only* pseudoelliptic components.

For type $\mathrm{W_{III}}$ walls, note that by the results of Section 6.1.3, if $K_X + S + F_{\mathcal{A}}$ is *not big* on a pseudoelliptic component Y, then the component is rational. Suppose Y is attached to a component E of an intermediate (pseudo)fiber $A \cup E$ on X'.

By Proposition 6.1.18, if $(K_X + S + F_{\mathcal{A}})|_Y$ is not big then either the log canonical linear series contracts Y to a point or contracts Y along a morphism $Y \to E$.

In particular, Y contracts to a point if and only if E contracts to a point. We can do the computation by restricting $(K_X + S + F_{\mathcal{A}})$ to X' first. In this case we have an intermediate marked (pseudo)fiber $F_a = aA + E$ where

$$a = \sum_{a_i \,:\, F_{a_i} \text{ lies on } Y} a_i$$

is a sum of markings on the pseudoelliptic Y by Equation 7.1. Then by Proposition 6.1.7 the fiber F_a contracts onto its Weierstrass model if and only if $a \leq c$ where c is the log canonical threshold of the Weierstrass fiber. By Theorem 6.1.10 the nonzero log canonical thresholds are exactly the ones written above.

On the other hand, suppose that Y contracts along a morphism $Y \to E$. By Proposition 6.1.20, the curve E is notfiber of type I_n, so it has to support a nonreduced twisted fiber. In particular $E \cong \mathbb{P}^1$ is a rational curve. Let $\mu \colon Y' \to Y$ be the associated elliptic surface. Then there is a morphism $Y' \to E$ by composition which is induced by the linear series $\mu^*((K_X + S + F_{\mathcal{A}})|_Y) = K_{Y'} + \alpha S' + \mu_*^{-1}(F_{\mathcal{A}})|_Y$. By Proposition 6.1.18, the coefficient $\alpha > 0$, and the generic fiber of $Y' \to E$ is a generic multisection M of the elliptic fibration $Y' \to C$ that is disjoint to S'. In particular, $M.S' = 0$.

Let $p \colon Y' \to Y_0$ be the contraction of the rational components of each intermediate fiber of $Y' \to C$ and let $S_0 = p_*(S'), F_0 = p_*(\mu_*^{-1}(F_{\mathcal{A}})|_Y)$ and $M_0 = p_*(M)$. We claim that $Y' \to E$ factors through $Y' \to Y_0$. That is, $M.A = 0$ for A the genus zero component of each intermediate fiber. By Lemma 7.2.5 we have the fact that $S_0^2 \leq 0$

and the following inequalities:

$$0 = (K_{Y'} + \alpha S' + \mu_*^{-1}(F_{\mathcal{A}})|_Y).M = (K_{Y'} + \mu_*^{-1}(F_{\mathcal{A}})|_Y).M \geq (K_{Y_0} + F_0).M_0$$
$$= m(K_{Y_0} + F_0).S_0 \geq m(K_{Y_0} + \alpha S_0 + F_0).S_0 \geq m(K_{Y'} + \alpha S' + \mu_*^{-1}(F_{\mathcal{A}})|_Y).S' = 0.$$

In particular, all the inequalities are equalities. Now $\alpha > 0$ by Proposition 6.1.18 and by the first inequality on the second line, $S_0^2 = 0$ on the twisted model. Now we conclude by the following lemma.

□

Lemma 7.2.3. *Suppose $(f\colon X \to C, S)$ is an irreducible relative log canonical model with only twisted fibers. Suppose further that $S^2 = 0$. Then X is the quotient of a trivial fibration $B \times E \to B$.*

Proof. By [AB16, Proposition 4.12], the pair $f\colon X \to C, S$ is the coarse space of a family of stable curves over a stable denoted by curve $\mathcal{X} \to \mathcal{C}$. Pick a projective curve with a finite cover $B \to \mathcal{C}$ and consider the pullback $Y \to B$ of $\mathcal{X} \to \mathcal{C}$. Then $Y \to B$ is a family of stable curves over B with section T pulled back from S. On the other hand, by the projection formula, $S^2 = 0$ implies that $T^2 = 0$. However, a Weierstrass elliptic fibration with $T^2 = -\deg \mathscr{L} = 0$ is trivial by [Mir89, III.1.4].

□

7.2.2 The birational transformations across each wall

We wish to describe the birational transformations that a family of $\mathcal{A}$-broken stable elliptic surfaces undergoes as $\mathcal{A}$ crosses a wall. Let $(f\colon X \to C, S + F_{\mathcal{A}}) \to B$ be a one parameter family of broken elliptic surfaces with normal generic fiber and special fiber $f'\colon X' \to C'$.

Type $\mathrm{W_I}$

If F is a minimal intermediate fiber, then at the wall at coefficients $a_i = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $\frac{3}{4}$ or $\frac{5}{6}$ (depending on the Kodaira fiber type of F), X undergoes a divisorial contraction where F transforms from an intermediate to Weierstrass model. Similarly, at the boundary wall $a_i = 1$, the surface X undergoes a divisorial contraction where F transforms from an intermediate into a twisted fiber.

Type $\mathrm{W_{II}}$

Let $\mathcal{A}_0$ be a weight on the non-boundary wall defined by $\sum a_{i_j} = 1$ for $\{i_1, \ldots, i_k\}$. Let $\mathcal{A}_\pm$ be in the adjacent chambers with $\sum a_{i_j} = 1 \pm \epsilon$ for ϵ very small. $\mathcal{A}_0$ is on a wall for the Hassett space where a leaf component C'_0 of the central fiber C' is contracted.

La Nave studied this situation in [LN02, Section 4.3]. At A_0, the section S'_0 of an elliptic component X'_0 lying over C' in the central fiber $X' \to C'$ of the $\mathcal{A}_+$ stable family $X \to C$ must contract by Proposition 7.1.13. This is a log canonical contraction of the pair $(X, S + F_{\mathcal{A}_0})$, but it is an extremal contraction of the pair $(X, S + F_{\mathcal{A}_-})$.

Since the total space X is a threefold and S'_0 is a curve, this is a small contraction so we must perform a flip to compute the $\mathcal{A}_-$ stable model. La Nave computes this flip explicitly using a local toric model around S'_0 inside the total space X [LN02, Theorem 7.1.2]. This leads to the formation of a type I pseudoelliptic surface Z in the central fiber attached to the component E of an intermediate (pseudo)fiber $E \cup A$ where A is the flipped curve, as depicted in Figure 7.5:

At a boundary type $\mathrm{W_{II}}$ wall, a rational component C'_0 of C' which is not a leaf may contract. The contraction of the corresponding section component S'_0 in

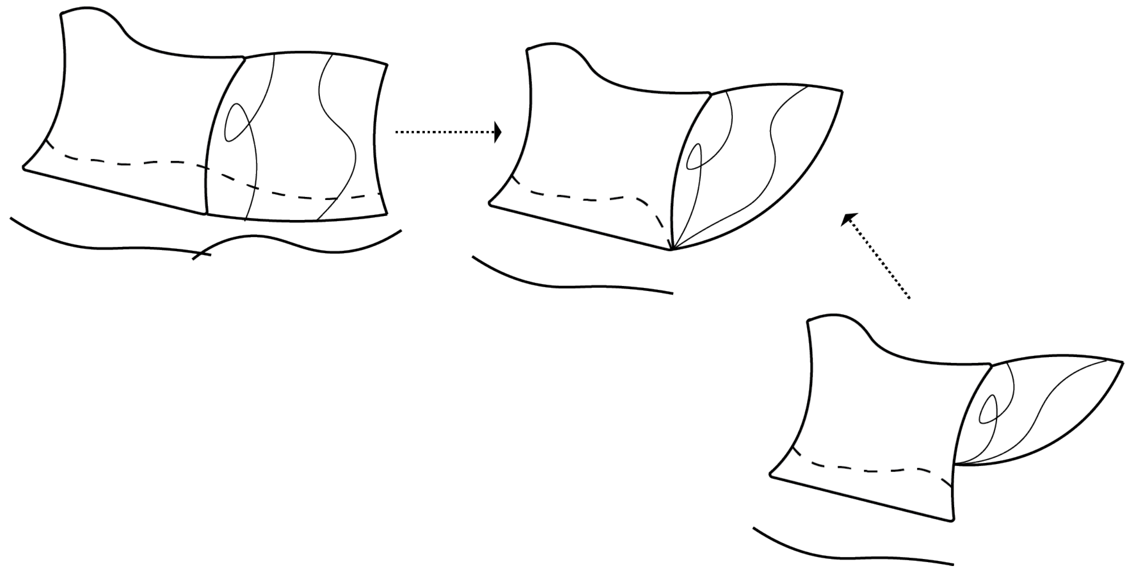


Figure 7.5: This depicts, from left to right, the central fiber of the $\mathcal{A}_+$, $\mathcal{A}_0$ and $\mathcal{A}_-$ stable families where $\mathcal{A}_0$ is a type W_{II} wall.

the central fiber X' of $X \to B$ is a log canonical contraction which forms a type II pseudoelliptic surface.

Finally when the genus of the base curve is 0, we must consider the wall defined by $\sum a_i = 2$. In this case the base curve is contracted to a point and so the section of the total family $X \to C \to B$ is contracted by a divisorial log canonical contraction. This produces a one parameter family of pseudoelliptic surfaces $Z \to B$ with normal generic fiber and special fiber consisting of an $\mathcal{A}$-broken pseudoelliptic surface.

Type $\mathrm{W}_{\mathrm{III}}$

At $\mathcal{A}_0$, there is a pseudoelliptic component Z in the central fiber of X' for which $K_X + S + F_{\mathcal{A}_0}$ is nef but not big. Then the total space $(X, S + F_{\mathcal{A}_0})$ undergoes a divisorial log canonical contraction $X \to Y$ which contracts Z onto either a point or a curve as determined by Remark 6.1.19.

When Z contracts to a point, this results in a cuspidal cubic fiber on the central fiber $Y' \to C'$ of Y at the point that Z contracted to. When $\mathcal{A}_0$ is not on a boundary wall, then the surface Y' has at worst a rational singularity at this point by Proposition 6.1.20. At a boundary, the contraction of Z may produce an elliptic singularity at the cusp.

Multiple walls

Figures 7.6, 7.7 and 7.8 illustrate some of the multi-step transformations the central fiber can undergo due to the birational transformations of X when crossing several walls at once.

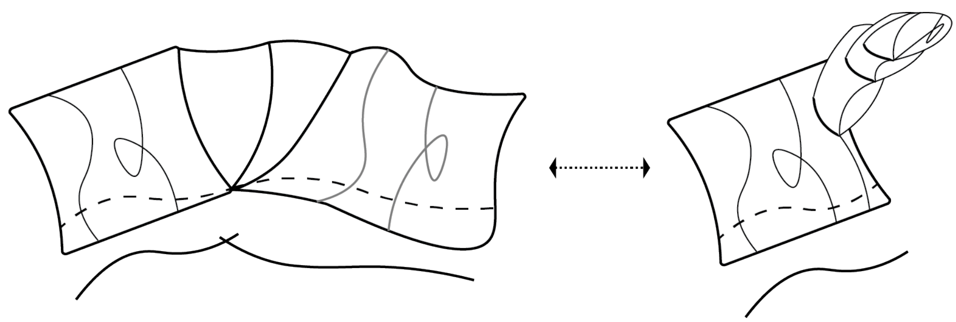


Figure 7.6: Here a type $\mathrm{W_{II}}$ wall is crossed which causes the right most component to transform into a type I pseudoelliptic. However, that then causes the type II pseudoelliptics to also become type I since they have no marked fibers.

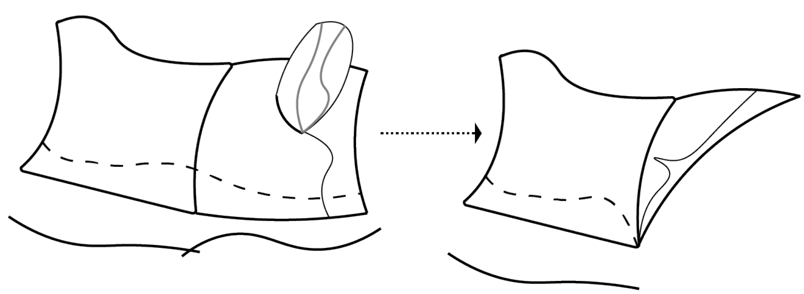


Figure 7.7: This is a simultaneous $\mathrm{W_{II}}$ and $\mathrm{W_{III}}$ wall where the type I pseudoelliptic component contracts onto a point and the right most elliptic component becomes a pseudoelliptic.

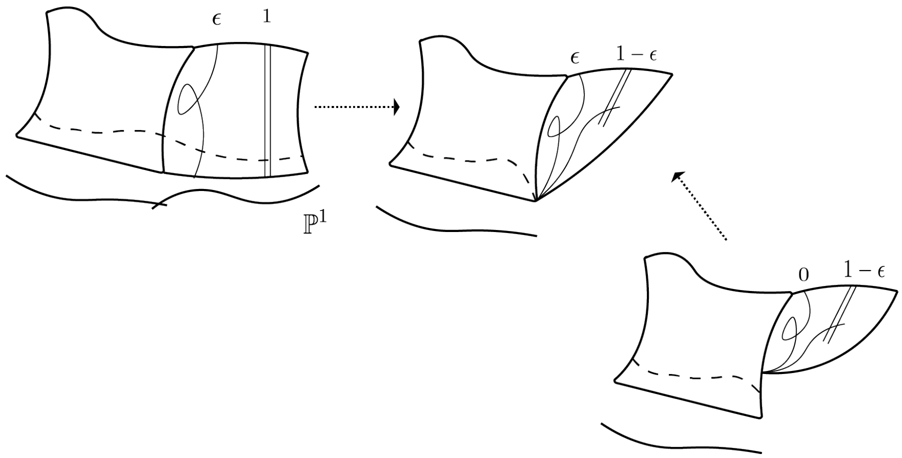


Figure 7.8: This is a simultaneous $\mathrm{W_I}$ and $\mathrm{W_{II}}$ where the twisted fiber becomes an intermediate fiber and a type I pseudoelliptic forms.

7.2.3 No other flipping contractions

In this section we record some results of Inchiostro [Inc18a] that ensure that there are no flips other than the one occuring at a type $\mathrm{W_{II}}$ wall introduced by La Nave [LN02].

Theorem 7.2.4 (Inchiostro)**.** *Let $(f\colon X \to C, S + F_{\mathcal{B}}) \to B$ be a family of $\mathcal{A}$-stable broken elliptic surfaces over a smooth curve B. Let $\mathcal{A} \le \mathcal{B}$ so that $K_X + S + F_{\mathcal{A}}$ is nef and $\mathbb{Q}$-Cartier. Then the only codimension two exceptional locus of the log canonical contraction consists of section components of the central fiber. In particular, the only flips that appear in the minimal model program for reducing coefficients on a one parameter family of broken elliptic surfaces are La Nave's.*

The main input into the above result is to show that any time the log canonical contraction contracts either (i) a multisection, or (ii) the A component of an intermediate fiber on a pseudoelliptic component, then in fact it the entire component. This shows such contractions are not flipping contractions. The following lemmas about intersection products on broken elliptic surfaces are a key input which we also need:

Lemma 7.2.5. *Let $(f\colon X \to C, S + F_{\mathcal{A}})$ be an irreducible $\mathcal{A}$-broken elliptic surface and M an irreducible multisection disjoint from S. Let $p\colon X \to X_0$ be the contraction of all intermediate components to their twisted models with $S_0 = p_*(S)$, $F_0 = p_*(F_{\mathcal{A}})$, and $M_0 = p_*(M)$.*

a) We have $S^2 \leq 0$ and if $S^2 = 0$, then X has no Weierstrass or intermediate fibers.

b) We have the following inequalities:

$$(K_X + \alpha S + F_{\mathcal{A}}).S \leq (K_{X_0} + \alpha S_0 + F_0).S_0,$$

$$(K_X + \alpha S + F_{\mathcal{A}}).M \geq (K_{X_0} + \alpha S_0 + F_0).M_0.$$

7.2.4 Explicit stable reduction

Recall the following definition (see Definition 4.9 [AB16]):

Definition 7.2.6. An $\mathcal{A}$-broken elliptic surface $(f\colon X \to C, S + F_{\mathcal{A}})$ is **twisted** if $a_i = 1$ for all i, there are no pseudoelliptic components, and the support of every non-reduced fiber is contained in $\mathrm{Supp}(F_{\mathcal{A}})$.

In [AB16], we used the Abramovich-Vistoli moduli space of twisted stable maps [AV02] to construct a proper moduli space of twisted elliptic surfaces analogous to the moduli spaces of fibered surfaces considered in [AV97]. In particular, in [AB16, Proposition 4.12] we proved that any surface with only twisted fibers is the coarse space of a stacky family of stable curves. This is the starting point for computing the stable limits in $\mathcal{E}_{v,\mathcal{A}}$ for any $\mathcal{A}$.

Given a family of $\mathcal{A}$-stable irreducible elliptic surface $(X \to C, S + F_{\mathcal{A}}) \to U$ over a punctured curve U, the idea is to

(i) increase the coefficients so that $a_i = 1$ for all i, and

(ii) add the supports of any unstable fibers to the boundary divisor.

Then the stable model of this new pair will be a family of twisted elliptic surfaces. By the results of [AB16], this family extends uniquely after a base change $U' \to U$. Finally, we can run the log MMP to compute the stable model as we decrease coefficients again. This is analogous to the approach used by La Nave [LN02] to compute stable limits of stable Weierstrass fibrations, i.e. when $\mathcal{A} = 0$.

Theorem 7.2.7. *The moduli stack $\mathcal{E}_{v,\mathcal{A}}$ is proper.*

Proof. Consider a family of normal $\mathcal{A}$-stable elliptic surfaces $(X^0, S^0 + F^0_{\mathcal{A}}) \to C^0 \to U$ over $U = B \setminus p$, a smooth curve minus one point. Let $\mathcal{B}_1 = (1, \ldots, 1)$ be the constant weight 1 vector and let $G^0 = G^0_{r+1} + \ldots + G^0_s$ be the reduced divisor whose support consists of the singular fibers not contained in $\mathrm{Supp}(F_{\mathcal{A}})$. Define $D^0_{\mathcal{B}} = F^0_{\mathcal{B}_1} + G^0$ so that $(X^0, S^0 + D^0_{\mathcal{B}}) \to C^0 \to U$ is a family of pairs with all non-stable fibers marked and all fibers marked with coefficient one. We index the weight vector $\mathcal{B} = (b_1, \ldots, b_r, b_{r+1}, \ldots, b_s)$ such that b_i for $i = 1, \ldots, r$ are the coefficients of the original marked fibers F_i.

After performing a log resolution, we can take the log canonical model of this pair to obtain a family of slc elliptic surfaces $(X^1, S^1 + D^1_{\mathcal{B}}) \to C^0 \to U$, such that all fibers are either stable or twisted, and all fibers that are not of type I_n are contained in either the double locus of X or in $D^1_{\mathcal{B}}$. By [AB16, Corollary 5.10], there is a map $C^0 \to \overline{M}_{1,1}$ making $(X^1, S^1 + D^1_{\mathcal{B}}) \to \overline{M}_{1,1}$ an Alexeev stable map from a twisted elliptic surface (see Section 5 and Proposition 5.2 of [AB16]).

By [AB16, Proposition 5.2], the moduli space of Alexeev stable maps from a twisted elliptic surface is proper. Therefore, after a finite base change $B' \to B$, this family extends uniquely to a family $(Z_1, S_1 + D_{\mathcal{B}}) \to C_1 \to B'$ of twisted elliptic surfaces over B' with a well defined j-invariant map $C_1 \to \overline{M}_{1,1}$. Furthermore the central fiber consists of only elliptic components fibered over a possibly reducible nodal curve.

Now consider the line segment $\mathcal{A}(t) := t\mathcal{B} + (1-t)\mathcal{A}_0$ for $t \in [0,1]$ where $\mathcal{A}_\delta = (a_1, \dots, a_r, 0, \dots, 0)$. By Theorem 7.2.2, there are finitely many $t_0 = 0, t_1, \dots,$ $t_{n-1}, t_n = 1$ so that $\mathcal{A}(t_k)$ are on walls.

By invariance of log plurigenera (Theorem 9.0.10), we can compute the stable model of

$$\pi\colon (Z_1, S_1 + D_{\mathcal{B}(t)}) \to C_1 \to B'$$

as we decrease t from $t = 1$ by taking the stable model of each fiber as long as $K_\pi + S_1 + D_{\mathcal{B}(t)}$ remains π-nef, and $\mathbb{Q}$-Cartier. First we need that each wall-crossing preserves the structure of a fibered surface:

Lemma 7.2.8. *Let $(f\colon X \to C, S + F_{\mathcal{A}})$ be a $\mathcal{B}$-broken stable elliptic surface. Let $\mathcal{A} \leq \mathcal{B}$ and denote by X' and C' the stabilizations of X and C with respect to $\mathcal{A}$ respectively. Then there exists a commutative diagram as follows.*

$$\begin{tikzcd} X \arrow[r] \arrow[d, dashed] & C \arrow[d] \\ X' \arrow[r] & C' \end{tikzcd}$$

Proof. Let $\phi\colon X \dashrightarrow X'$ be the log canonical birational map induced by $m(K_X+S+F_{\mathcal{A}})$. We can factor ϕ into a sequence of type W_{I}, W_{II} and W_{III} birational transformations described in Section 7.2. We reduce to checking that for each of these birational transformations, there is a compatible factorization of $X' \to C'$.

I. If ϕ is a $\mathrm{W_I}$ type transformation, that is, a transition between twisted, intermediate and Weierstrass fibers, then ϕ is a composition of blowups and blowdowns of fiber components so there is a factorization $X' \to C$.

II. If ϕ is a $\mathrm{W_{II}}$ type transformation, then there is a diagram

$$\begin{tikzcd} X_- \arrow[dr] & & X_+ \arrow[dl] \\ & X_0 & \end{tikzcd}$$

where $X_+ \to X_0$ is the contraction of a section component $X_- \to X_0$ is birational on every component and ϕ is either $X_+ \to X_0$ or $X_+ \dashrightarrow X_-$ (see Section 8.2 for details). By Proposition 7.1.13, the map $X_+ \to X_0$ contracts a section component if and only if that component of the base curve is contracted by the morphism $C \to C'$. Therefore there is a unique factorization $X_0 \to C'$ also inducing a unique map $X_- \to C'$ by composition.

III. If ϕ is a type $\mathrm{W_{III}}$ transformation, then it contracts components of X which are contracted to a point by $f\colon X \to C$. Therefore there is a unique factorization $X' \to C$.

$\square$

Now for $0 < \epsilon \ll 1$, there exists a unique family of $\mathcal{A}(1-\epsilon)$-weighted stable elliptic surfaces $(Z_{1-\epsilon}, S_{1-\epsilon} + D_{\mathcal{A}(1-\epsilon)}) \to C_1 \to B'$ obtained by the blowup from twisted to intermediate models of the marked fibers. Then one performs the following whenever $\mathcal{A}(t)$ crosses a wall:

- Each time t crosses a type $\mathrm{W_I}$ or $\mathrm{W_{III}}$ wall t_k, the family undergoes a divisorial contraction as described in Sections 7.2.2 and 7.2.2. In this case one obtains a $\mathcal{A}(t_k)$-weighted stable family $(Z_{t_k}, S_{t_k} + D_{\mathcal{A}(t_k)}) \to C_{t_k} \to B'$;

- Across a type $\mathrm{W_{II}}$ wall t_l, the map $X_t \to X_{t_l}$ is a flipping contraction of a section of a component of the central fiber. As described in Section 7.2.2 there is a unique flip $X_{t'} \to X_{t_l}$ constructed by La Nave in [LN02] giving an $\mathcal{A}(t')$-weighted stable family

$$(Z_{t'}, S_{t'} + D_{\mathcal{A}(t')}) \to C_{t'} \to B'.$$

Since there are only finitely many walls crossed, we eventually obtain an $\mathcal{A}(0) = \mathcal{A}_0$-weighted stable family $\pi\colon (Z_\delta, S_0 + D_{\mathcal{A}_0}) \to C_0 \to B'$. Forgetting about the auxillary divisors G now marked with 0, this is in fact a $\mathcal{A}$-stable family. $\square$

Theorem 7.2.9. *The stable limit of a family of irreducible $\mathcal{A}$-stable elliptic surfaces is an $\mathcal{A}$-broken stable elliptic surface. In particular, the compact moduli stack $\mathcal{E}_{v,\mathcal{A}}$ parametrizes $\mathcal{A}$-broken stable elliptic surfaces.*

Proof. Every step of the proof of Theorem 7.2.7 produces a central fiber which is a broken elliptic surface. $\square$

Corollary 7.2.10. *For any $\mathcal{A}$ and $\mathcal{B}$ within the same chamber, $\mathcal{E}_{v,\mathcal{A}} \cong \mathcal{E}_{v,\mathcal{B}}$.*

Proof. The walls of type $\mathrm{W_I}$, $\mathrm{W_{II}}$ and $\mathrm{W_{III}}$ describe precisely when the log canonical divisor is nef rather than ample. Within a chamber $K_X + S + F_{\mathcal{A}}$ is ample if and only if $K_X + S + F_{\mathcal{B}}$ is ample and the log canonical models are the same. It follows that $\mathcal{E}^m_{v,\mathcal{A},n}(T) = \mathcal{E}^m_{v,\mathcal{B},n}(T)$ for any normal base T and so $\mathcal{E}_{v,\mathcal{A}} \cong \mathcal{E}_{v,\mathcal{B}}$ by Proposition A.0.7. $\square$

CHAPTER 8

Wall-crossing morphisms for $\mathcal{E}_{v,\mathcal{A}}$

8.1 Reduction morphisms

We begin by reviewing the notion of reduction morphisms present in the work of [Has03].

8.1.1 Hassett's moduli space

Recall the moduli spaces $\overline{\mathcal{M}}_{g,\mathcal{A}}$, parametrizing genus g curves with r marked points weighted by a weight vector $\mathcal{A} = (a_1, \ldots, a_r)$ were defined in [Has03]. Hassett studied what happened as one lowers the weight vector $\mathcal{A}$. Namely, the following theorem guarantees the existence of reduction morphisms on the level of moduli spaces.

Theorem 8.1.1. *[Has03, Theorem 4.1] Fix g and n and let $\mathcal{A} = (a_1, \ldots, a_r)$ and $\mathcal{B} = (b_1, \ldots, b_r)$ two collections of admissible weights and suppose that $\mathcal{A} \leq \mathcal{B}$. Then* *there exists a natural birational reduction morphism*

$$\overline{\mathcal{M}}_{g,\mathcal{B}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}.$$

Given an element $(C, p_1, \ldots, p_r) \in \overline{\mathcal{M}}_{g,\mathcal{B}}$*, the image in* $\overline{\mathcal{M}}_{g,\mathcal{A}}$ *is obtained by collapsing components of* C *along which* $K_C + a_1 p_1 + \cdots + a_r p_r$ *fails to be ample.*

We will construct analagous reduction morphisms on the moduli spaces $\mathcal{E}_{v,\mathcal{A}}$ and their universal families which are compatible with the reduction morphisms of Hassett in the following way. The image curve $(C, D_\mathcal{A})$ (recall Definition 7.1.12) is naturally an $\mathcal{A}$-weighted curve in the sense of Hassett. We obtain a natural forgetful morphism from $\mathcal{E}_{v,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ for all $0 \leq a \leq 1$ (see Corollary 8.1.3) and the reduction morphisms (Theorem 8.1.4) will commute with Hassett's reduction morphisms above.

8.1.2 Preliminary results

Let $(f\colon X \to C, S + F_\mathcal{A})$ be an $\mathcal{A}$-broken elliptic surface. Denote by $D_\mathcal{A}$ the weighted divisor on C corresponding to the weighted marked fibers of $f\colon X \to C$ (Definition 7.1.12). Then $(C, D_\mathcal{A})$ is a weighted pointed nodal curve in the sense of Hassett.

Lemma 8.1.2. *Let* $(X, S + F_\mathcal{B}) \xrightarrow{f} C \xrightarrow{q} B$ *be a flat family of* $\mathcal{B}$*-stable elliptic surfaces over a base* B*. Denoting the composition* $p = q \circ f$*, then the formation of* $p_*(f^*\omega_q(D_\mathcal{A})^{[m]})$ *commutes with base change for any* $\mathcal{A} \leq \mathcal{B}$ *and* $m \geq 1$*.*

Proof. By Lemma 5.1.6, $p_*(f^*\omega_q(D_\mathcal{B})^{[m]}) = q_* f_* f^* \omega_q(D_\mathcal{B})^{[m]} = q_*\omega_q(D_\mathcal{B})^{[m]}$, and the latter commutes with base change by Proposition 3.3 of [Has03]. □

First, we show the base curve of our weighted elliptic surface pairs are weighted stable curves in the sense of Hassett, so we can use these spaces to gain understanding of $\mathcal{E}_{v,\mathcal{A}}$.

Corollary 8.1.3. *There is a natural forgetful morphism $\mathcal{E}_{v,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ given by sending a family of $\mathcal{A}$-broken stable elliptic surfaces $p\colon (f\colon X \to C, S + F_{\mathcal{A}}) \to B$ to the family of $\mathcal{A}$-weighted stable curves $q\colon (C, D_{\mathcal{A}}) \to B$.*

Proof. By Lemma 8.1.2, the formation of $p_*(f^*\omega_q(D_{\mathcal{A}})^{[m]}) = q_*\omega_q(D_{\mathcal{A}})^{[m]}$ commutes with base change. Therefore it suffices to check that $(C_b, (D_{\mathcal{A}})_b)$ is an $\mathcal{A}$-stable curve for each $b \in B$ and this is Corollary 7.1.15. $\square$

8.1.3 Reduction morphisms

We are now ready to state and prove our main theorem on reduction morphisms for moduli of elliptic surfaces analagous to [Has03, Theorem 4.1].

Theorem 8.1.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be rational tuples such that $\mathcal{A} \leq \mathcal{B}$. Suppose that that $\mathcal{A}(t)$ never lies on a Type W_{II} wall for $t > 0$ (see Remark 1). Then there exists a reduction morphisms $\rho_{\mathcal{A},\mathcal{B}}\colon \mathcal{E}_{v,\mathcal{B}} \to \mathcal{E}_{v,\mathcal{A}}$. If we further suppose that $\lfloor \mathcal{A} \rfloor = \lfloor \mathcal{B} \rfloor$, then there exists a compatible $\widetilde{\rho}_{\mathcal{A},\mathcal{B}}\colon \mathcal{U}_{v,\mathcal{B}} \to \mathcal{U}_{v,\mathcal{A}}$ making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{U}_{v,\mathcal{B}} & \xrightarrow{\widetilde{\rho}_{\mathcal{A},\mathcal{B}}} & \mathcal{U}_{v,\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathcal{E}_{v,\mathcal{B}} & \xrightarrow{\rho_{\mathcal{A},\mathcal{B}}} & \mathcal{E}_{v,\mathcal{A}} \end{array} .$$

All of the above reduction morphisms commute via the forgetful morphism of Corollary 8.1.3 with the reduction morphisms for Hassett space.

Remark 8.1.5. The condition $\lfloor \mathcal{A} \rfloor = \lfloor \mathcal{B} \rfloor$ just means that $a_i = 1$ if and only if $b_i = 1$. We consider the case when $a_i < b_i = 1$ in Proposition 8.2.7.

Proof. The proof that $\rho_{\mathcal{A},\mathcal{B}}$ is a morphism is modeled off of the proof of [Has03, Theorem 4.1]. Let $\mathcal{A} = (a_1, \ldots, a_r)$ and $\mathcal{B} = (b_1, \ldots, b_r)$ be so that $\mathcal{A} \leq \mathcal{B}$. Denote

$\mathcal{A}(t) := (1-t)\mathcal{A} + t\mathcal{B}$, where $t \in \mathbb{Q}$ and $0 \le t \le 1$.

With notation from the proof of Theorem 7.1.4, we define a natural transformation of pseudofunctors

$$\mathcal{E}^m_{v,\mathcal{B},n}(B) \to \mathcal{E}^m_{v,\mathcal{A},n}(B)$$

for a normal base scheme B that is compatible with base change. This will lead to a morphism of moduli spaces $\rho_{\mathcal{A},\mathcal{B}} \colon \mathcal{E}_{v,\mathcal{B}} \to \mathcal{E}_{v,\mathcal{A}}$ by Proposition A.0.6. There are necessarily finitely many $t_0 = 0, t_1, \ldots, t_{k-1}, t_k = 1$ so that $\mathcal{A}(t_j)$ lie on walls and for any $t \neq t_j$, weights $\mathcal{A}(t)$ are not on any wall. It is clear that the weight vectors $\mathcal{A}(t_j) \le \mathcal{A}(t_{j+1})$ satisfy the hypothesis of the theorem so it suffices to construct reduction morphisms $\rho_{\mathcal{A}(t_j),\mathcal{A}(t_{j+1})}$ so that

$$\rho_{\mathcal{A},\mathcal{B}} = \rho_{\mathcal{A}(t_0),\mathcal{A}(t_1)} \circ \ldots \circ \rho_{\mathcal{A}(t_{k-1}),\mathcal{A}(t_k)}$$

Therefore we may assume that $\mathcal{A}(t)$ does not lie on a wall for any $t \neq 0, 1$, and that $\mathcal{A}$ is either in a chamber or on a wall of type W_{I} or $\mathrm{W}_{\mathrm{III}}$. Writing $\mathcal{A}(t) = (a_1(t), \ldots, a_r(t))$ this means explicitly that for all $0 < t < 1$,

i) $a_j(t) \neq \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$ (there are no type W_{I} walls);

ii) there is no subset $\{i_1, .., i_k\} \subset \{1, \ldots, r\}$ such that $a_{\mathrm{I}_1}(t) + \cdots + a_{i_k}(t) = 1$ (there are no type W_{II} walls);

iii) $K_X + S + F_{\mathcal{A}(t)}$ is big on every irreducible component of every $\mathcal{B}$-broken stable elliptic surface $(X \to C, S + F_{\mathcal{B}})$ (there are no type $\mathrm{W}_{\mathrm{III}}$ walls).

Let $\pi \colon (f \colon X \to C, S + F_{\mathcal{B}}) \to B$ be a family of $\mathcal{B}$-broken elliptic surfaces over a normal base B. By our above assumption, $K_X + S + F_{\mathcal{A}(t)}$ is ample for $t > 0$ and $K_X + S + F_{\mathcal{A}(0)} = K_{X/B} + S + F_{\mathcal{A}}$ is π-nef and $\mathbb{Q}$-Cartier. By Proposition 5.1.5,

$K_{X/B} + S + F_{\mathcal{A}}$ is π-semiample. Then we can write

$$C' = \operatorname{Proj}_B \left(\bigoplus_m \pi_* f^* \omega_{C/B}(mnD_{\mathcal{A}}) \right)$$
$$X' = \operatorname{Proj}_B \left(\bigoplus_m \pi_* \mathcal{O}_X(mn(K_{X/B} + S + F_{\mathcal{A}})) \right)$$

where n is a large enough integer such that $nD_{\mathcal{A}}$ and $n(K_{X/B} + S + F_{\mathcal{A}})$ are Cartier.

There are canonical maps $C \to C'$ and $X \dashrightarrow X'$. It follows from the basechange results Theorem 9.0.10 and Lemma 8.1.2 that X' and C' are families of $\mathcal{A}$-broken stable elliptic surfaces and $\mathcal{A}$-weighted pointed stable curves respectively. By Lemma 7.2.8, there is a map $f' \colon X' \to C'$ making $\pi' \colon (f' \colon X' \to C', S + F_{\mathcal{B}}) \to B$ a a family of $\mathcal{A}$-broken stable elliptic surfaces over B. Since the construction of $\pi_* f^* \omega_{C/B}(mnD_{\mathcal{A}})$ and $\pi_* \mathcal{O}_X(mn(K_{X/B} + S + F_{\mathcal{A}})$ commute with basechange by Theorem 9.0.10 and Lemma 8.1.2, it follows that the construction of the family of $\mathcal{A}$-broken stable elliptic surfaces is functorial in B. Furthermore, it is clear that the map on closed points, $\mathcal{E}^m_{v,\mathcal{B},n}(k) \to \mathcal{E}^m_{v,\mathcal{A},n}(k)$ is dominant on each component by observing that it is dominant on the locus of irreducible elliptic surfaces. This induces the required morphism

$$\rho_{\mathcal{A},\mathcal{B}} \colon \mathcal{E}_{v,\mathcal{B}} \to \mathcal{E}_{v,\mathcal{A}}.$$

Next, we show existence of the morphism $\widetilde{\rho}_{\mathcal{A},\mathcal{B}}$ on the level of universal families under the assumption $a_i = 1$ if and only if $b_i = 1$. In this case, there are no type W_{I} transformations from twisted to intermediate fibers so the rational map $X \dashrightarrow X'$ is actually a morphism $X \to X'$. The universal family $\mathcal{U}_{v,\mathcal{B}} \to \mathcal{E}_{v,\mathcal{B}}$ is itself a family of $\mathcal{B}$-weighted stable elliptic surfaces. Therefore applying the above construction gives a family $\mathcal{Y} \to \mathcal{E}_{v,\mathcal{B}}$ of $\mathcal{A}$-stable elliptic surfaces with a morphism $\mathcal{U}_{v,\mathcal{B}} \to \mathcal{Y}$ over $\mathcal{E}_{v,\mathcal{B}}$.

This induces the morphism $\rho_{\mathcal{A},\mathcal{B}}$ so that

$$\mathcal{Y} = \rho^*_{\mathcal{A},\mathcal{B}}\, \mathcal{U}_{v,\mathcal{A}}.$$

The composition $\mathcal{U}_{v,\mathcal{B}} \to \mathcal{Y} \to \mathcal{U}_{v,\mathcal{A}}$ gives the required $\widetilde{\rho}_{\mathcal{A},\mathcal{B}}$.

The fact that these morphisms commute with the reduction morphisms for Hassett space is immediate since the forgetful map to the base curve is a morphism, and the family of base curves is stabalized by the linear series $\omega_{C/B}(nD_{\mathcal{A}})$ by Proposition 7.1.13 and Lemma 8.1.2. □

Corollary 8.1.6. *The reduction morphisms $\rho_{\mathcal{A},\mathcal{B}}$ are surjective.*

Proof. This follows since $\rho_{\mathcal{A},\mathcal{B}}$ is a dominant morphism of proper stacks. □

8.2 Flipping walls

Theorem 8.1.4, shows that there are reduction morphisms

$$\rho_{\mathcal{A},\mathcal{B}} \colon \mathcal{E}_{v,\mathcal{B}} \to \mathcal{E}_{v,\mathcal{A}}$$

whenever $\mathcal{A}(t) := (1-t)\mathcal{A} + t\mathcal{B}$ never crosses a type W_{II} wall for $t \in (0,1]$. The key point is that if $\mathcal{A}(t_0)$ is a type W_{II} wall for $t_0 \in (0,1]$ and $t_\pm := t_0 \pm \epsilon$ for $0 < \epsilon \ll 1$, then

$$K_X + S + F_{\mathcal{A}(t_-)}$$

is *not necessarily* $\mathbb{Q}$-Cartier where $(f \colon X \to C, S + F_{\mathcal{A}(t_0)})$ is an $\mathcal{A}(t_0)$-stable elliptic surface. Therefore it no longer makes sense to construct the $\mathcal{A}(t_-)$-stable model as a Proj of the section ring.

Rather, to construct the $\mathcal{A}(t_-)$-stable model from $(X, S + F_{\mathcal{A}}(t_0))$, we need to first perform a log resolution to make the log canonical divisor $\mathbb{Q}$-Cartier before running the steps of the minimal model program. Therefore, across a type W_{II} wall, we obtain a morphism resembling a *flip* (see Figure 8.1).

We fix some notation. Let $t_0, t_\pm$ be as above where $\mathcal{A}_0 := \mathcal{A}(t_0)$ is on a wall of type W_{II} and $\mathcal{A}_- := \mathcal{A}(t_-) < \mathcal{A}_0 < \mathcal{A}(t_+) =: \mathcal{A}_+$ so that $\mathcal{A}_\pm$ are in the interiors of chambers. We will use $(X_0, S_0 + F_{\mathcal{A}_0})$ and $(X_\pm, S_\pm + F_{\mathcal{A}_\pm})$ to denote $\mathcal{A}_0$-stable and $\mathcal{A}_\pm$-stable elliptic surfaces respectively.

Theorem 8.2.1. *There exist morphisms $\tilde{\epsilon}^-_{\mathcal{A}_0}, \epsilon^-_{\mathcal{A}_0}$ making the following diagram commute:*

$$\begin{tikzcd}
\mathcal{U}_{v,\mathcal{A}_-} \arrow[r, "\tilde{\epsilon}^-_{\mathcal{A}_0}"] \arrow[d] & \mathcal{U}_{v,\mathcal{A}_0} \arrow[d] \\
\mathcal{E}_{v,\mathcal{A}_-} \arrow[r, "\epsilon^-_{\mathcal{A}_0}"] & \mathcal{E}_{v,\mathcal{A}_0}
\end{tikzcd} .$$

Proof. This proof is analogous to the proof of Theorem 8.1.4. Under these assumptions $K_{X_-} + S_- + F_{\mathcal{A}_0}$ is a semiample $\mathbb{Q}$-Cartier divisor and the $\mathcal{A}_0$-stable model is

$$\mathrm{Proj}\left(\bigoplus_{k \geq 0} H^0(X_-, \mathcal{O}_X(km_0(K_{X_-} + S_- + F_{\mathcal{A}_0})))\right)$$

where m_0 is the index. Thus it suffices to prove a vanishing result analogous to Theorem 9.0.1.

Lemma 8.2.2. *In this situation, $H^i(\mathcal{O}_X(m(K_{X_-} + S_- + F_{\mathcal{A}_0}))) = 0$ for $i > 0$ and m large and divisible.*

Proof. We consider the irreducible components of X_-. There are three types of components:

(a) a pseudoelliptic whose section was contracted at the wall $\mathcal{A}_0$;

(b) a component along which a pseudoelliptic from case (a) is attached;

(c) a component not in either of the above cases.

The pair $(X_-, S_- + F_{\mathcal{A}_0})$ is slc and the linear series $|K_{X_-} + S_- + F_{\mathcal{A}_0}|$ is semi-ample by Proposition 5.1.5. It induces a morphism $g \colon X_- \to X_0$ which is necessarily an isomorphism on the components in cases (a) and (c) above.

Suppose X' is a component in case (b). Then it is attached to a pseudoelliptic Z in case (a) along a fiber component G. As explained in La Nave (see Section 4.3 and Theorem 7.1.2 in [LN02]), the contraction of the section of a component to form Z at the wall $\mathcal{A}_0$ may be a log flipping contraction inside of the total space of a one parameter degeneration with central fiber X_-. In this case, Z is a type I pseudoelliptic attached along an irreducible pseudofiber G to an intermediate (pseudo)fiber $G \cup A$ on X' (see Figure 8.1). The coefficient $\mathrm{Coeff}(A, F_{\mathcal{A}})$ given by the sum of weights of fibers on Z as can be seen from La Nave's local toric model and the morphism $g \colon X_- \to X_0$ contracts A. In particular $\mathrm{Coeff}(A, F_{\mathcal{A}_0}) = 1$.

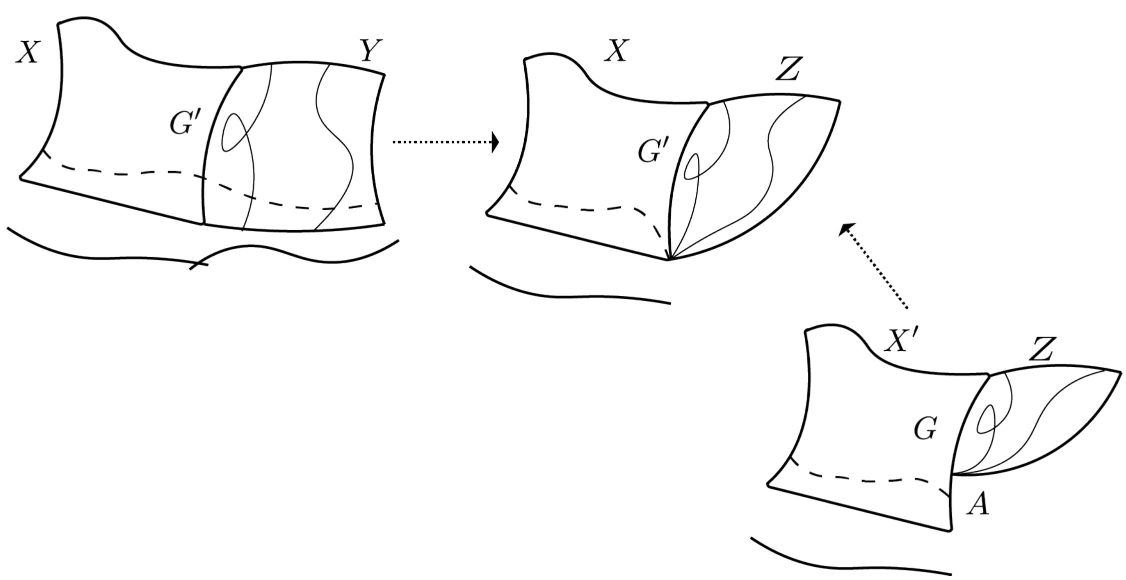


Figure 8.1: From left to right, the $\mathcal{A}_+$, $\mathcal{A}_0$ and $\mathcal{A}_-$ stable models. The sum of the weights of the marked pesudofibers on Z is equal to the coefficient of A in $F_{\mathcal{A}}$.

Thus $g\colon X_- \to X_0$ is precisely the contraction of these rational curves A produced by La Nave's flips. Denote $S_- + F_{\mathcal{A}_0} = \Delta$. Then by Proposition 5.1.19,

$$R^1 g_* \mathcal{O}_{X_-}(m(K_{X_-} + \Delta)) = 0.$$

On the other hand, $g_*\mathcal{O}_{X_-}(m(K_{X_-} + \Delta)) = \mathcal{O}_{X_0}(m(K_{X_0} + g_*\Delta))$ by Proposition 5.1.7, since $g_*^{-1} g_*\Delta + \operatorname{Exc}(g) = \Delta$ as each curve A appears with coefficient 1. Therefore

$$H^1(X_-, \mathcal{O}_{X_-}(m(K_{X_-} + \Delta))) = H^1(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + g_*\Delta))) = 0$$

since $(X_0, g_*\Delta) = (X_0, S_0 + F_{\mathcal{A}_0})$ is the $\mathcal{A}_0$-stable model. $\square$

Now we can proceed as in the proof of Theorem 8.1.4: let

$$\pi\colon (X_- \to C, S_- + F_{\mathcal{A}_-}) \to B$$

be an $\mathcal{A}_-$-weighted stable family of elliptic surfaces over a normal base B. Then the construction of

$$\operatorname{Proj}_B \left(\bigoplus_k \pi_* \mathcal{O}_{X_-}(k m_0 (K_{X_-} + S_- + F_{\mathcal{A}_-}) \right)$$

commutes with base change and produces a family $\pi_0\colon (X_0 \to C, S_0 + F_{\mathcal{A}_0})$ of $\mathcal{A}_0$-stable elliptic surfaces and realizes the morphism $\epsilon^-_{\mathcal{A}_0}$. Applying this construction to $B = \mathcal{E}_{v,A_-}$ with the universal family yields $\tilde{\epsilon}^-_{\mathcal{A}_0}$. $\square$

Remark 8.2.3. Note that in the above construction, the $\mathcal{A}_0$-stable family

$$(X_0 \to C, S_0 + F_{\mathcal{A}_0})$$

associated to the $\mathcal{A}_-$-stable family $(X_- \to C, S_- + F_{\mathcal{A}_-})$ has the same base curve C.

This is because a marked curve is $\mathcal{A}_0$-stable if and only if it is $\mathcal{A}_-$-stable where $\mathcal{A}_0$ is one of the walls for the space of weighted stable curves. That is, the reduction morphism $\mathcal{M}_{g,\mathcal{A}_0} \to \mathcal{M}_{g,\mathcal{A}_-}$ is a canonical isomorphism. In particular there is a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_{v,\mathcal{A}_-} \arrow[r, "\epsilon^-_{\mathcal{A}_0}"] \arrow[d] & \mathcal{E}^m_{v,\mathcal{A}_0} \arrow[d] & \mathcal{E}_{v,\mathcal{A}_+} \arrow[l, "\epsilon^+_{\mathcal{A}_0}"'] \arrow[d] \\
\mathcal{M}_{g,\mathcal{A}_-} \arrow[r, equal] & \mathcal{M}_{g,\mathcal{A}_0} & \mathcal{M}_{g,\mathcal{A}_-} \arrow[l]
\end{tikzcd}$$

showing compatibility with the reduction morphisms on Hassett spaces.

Let $\tilde{\epsilon}^+_{\mathcal{A}_0} := \widetilde{\rho}_{\mathcal{A}_+,\mathcal{A}}$ and $\epsilon^+_{\mathcal{A}_0} := \rho_{\mathcal{A}_+,\mathcal{A}}$ be the reduction morphisms of the previous section. Then we have a commuting diagram

$$\begin{tikzcd}
\mathcal{U}_{v,\mathcal{A}_-} \arrow[dr, "\tilde{\epsilon}^-_{\mathcal{A}_0}"'] \arrow[dd] & & \mathcal{U}_{v,\mathcal{A}_+} \arrow[dl, "\tilde{\epsilon}^+_{\mathcal{A}_0}"] \arrow[dd] \\
& \mathcal{U}_{v,\mathcal{A}_0} \arrow[dd] & \\
\mathcal{E}_{v,\mathcal{A}_-} \arrow[dr, "\epsilon^-_{\mathcal{A}_0}"'] & & \mathcal{E}_{v,\mathcal{A}_+} \arrow[dl, "\epsilon^+_{\mathcal{A}_0}"] \\
& \mathcal{E}_{v,\mathcal{A}_0} &
\end{tikzcd} .$$

We want to compare $\mathcal{A}_+$-, $\mathcal{A}_0$-, and $\mathcal{A}_-$-stable families of elliptic surfaces over the same base B. To do this, it is natural to consider the fiber product

$$\mathcal{E}_{v,\mathcal{A}_-} \times_{\mathcal{E}_{v,\mathcal{A}_0}} \mathcal{E}_{v,\mathcal{A}_+} =: \mathfrak{F}.$$

Pulling back the universal families gives a commutative diagram

$$\begin{tikzcd} \mathfrak{U}_- \arrow[dr] \arrow[ddr] & & \mathfrak{U}_+ \arrow[dl] \arrow[ddl] \\ & \mathfrak{U}_0 \arrow[d] & \\ & \mathfrak{F} & \end{tikzcd}$$

Then a map $B \to \mathfrak{F}$ is equivalent to a commutative diagram

$$\begin{tikzcd} X_- \arrow[dr] \arrow[ddr] & & X_+ \arrow[dl] \arrow[ddl] \\ & X_0 \arrow[d] & \\ & B & \end{tikzcd}$$

of compatible families $X_0, X_\pm \to B$ of $\mathcal{A}_0$-stable (resp. $\mathcal{A}_\pm$-) stable elliptic surfaces.

We show that the diagram

$$\begin{tikzcd} \mathfrak{U}_- \arrow[dr] & & \mathfrak{U}_+ \arrow[dl] \\ & \mathfrak{U}_0 & \end{tikzcd}$$

is a *universal flip* in the following sense:

Proposition 8.2.4. *For any normal and irreducible base B and map $B \to \mathfrak{F}$ with generic point mapping to the interior of the moduli space, the induced diagram*

$$\begin{tikzcd} X_- \arrow[dr, "g_-"'] & & X_+ \arrow[ll, dashed, "\varphi"'] \arrow[dl, "g^+"] \\ & X_0 & \end{tikzcd}$$

is a $(K_{X_+} + S_+ + F_{\mathcal{A}_-})$*-flip of the total spaces.*

Proof. Let $V \subset B$ be the open locus over which the elliptic surfaces are irreducible and let $Z = B \setminus V$. By assumption V is nonempty and the morphisms $X_- \to X_0$ and $X_+ \to X_0$ are isomorphisms over V. Thus the exceptional locus $\mathrm{Exc}(\varphi)$ lies over Z. On each fiber over Z, the map $X_+ \to X_0$ contracts the section of a pseudoelliptic component and $X_- \to X_0$ contracts a curve in an attaching fiber. Therefore the $\mathrm{Exc}(\varphi)$ is codimension at least 2.

We need to show that $-(K_{X_+} + S_+ + F_{\mathcal{A}_-})$ is g_+-ample and $K_{X_-} + \varphi_*(S_+ + F_{\mathcal{A}_-})$ is g_+-ample. Note that $\varphi_*(S_+ + F_{\mathcal{A}_-}) = S_- + F_{\mathcal{A}_-}$, where by abuse of notation, we write $F_{\mathcal{A}}$ for $\mathcal{A}$-weighted fibers on any of the birational models. Since g_- and g_+ are proper, relative ampleness is a fiberwise condition. Thus it suffices to check this after pulling back to a smooth curve $B' \to B$ so without loss of generality, we may take B to be an irreducible smooth curve so that $V = B \setminus \{p\}$.

In this case $X_+ \to X_0$ is the contraction of the section in a component of the central fiber $(X_+)_p$. It is then proven in [LN02, Theorem 7.1.2] that $X_+ \to X_0$ is a flipping contraction induced by $K_{X_+} + S_+ + F_{\mathcal{A}_-}$ with log flip $X_- \to X_0$. □

Corollary 8.2.5. *The morphism* $\epsilon^-_{\mathcal{A}_0}$ *is an isomorphism.*

Proof. $\epsilon^-_{\mathcal{A}_0}$ is a proper bijection and our moduli spaces are normal. □

Remark 8.2.6. Since we normalize the moduli spaces, we make no claims about the infinitessimal structure of $\epsilon^-_{\mathcal{A}_0}$. Indeed the deformation theories of $\mathcal{A}_0$ and $\mathcal{A}_-$ broken elliptic surfaces may be very different.

8.2.1 The "wall" at $a = 1$

In this section we discuss an analogous behavior to the flipping morphism

$$\epsilon^-_{\mathcal{A}_0} : \mathcal{E}^m_{v,A_-} \to \mathcal{E}^m_{v,\mathcal{A}_0}$$

that occurs in the limit as a coefficient $a \to 1$.

Indeed if we take $X_- = X' \cup Z$ as in the proof of of Theorem 8.2.1 so that X' is an elliptic component, Z is a pseudoelliptic component of type I attached to X' along an intermediate fiber $G \cup A$, then we saw that the morphism $\epsilon^-_{\mathcal{A}_0}$ contracts the fiber component A. Locally on X' around the fiber $G \cup A$, this contraction of A is the transition from an intermediate to a twisted fiber Section 6.1. In both cases, this contraction occurs when the intermediate fiber components G and E are both marked with coefficient $a = 1$ and in both cases, this induces a morphism on moduli spaces:

Proposition 8.2.7. *Let $\mathcal{B} = (b_1, \ldots, b_r)$ and fix j such that $b_j = 1$. Let $\mathcal{A} < \mathcal{B}$ be a weight vector with $a_i = b_i$ for $i \neq j$ and $a_j = 1 - \epsilon$ where $0 < \epsilon \ll 1$. Then there are morphisms $\theta_j \colon \mathcal{E}_{v,\mathcal{A}} \to \mathcal{E}_{v,\mathcal{B}}$ and $\tilde{\theta}_j \colon \mathcal{U}_{v,\mathcal{A}} \to \mathcal{U}_{v,\mathcal{B}}$ making the following diagram commute:*

$$\begin{tikzcd}
\mathcal{U}_{v,\mathcal{A}} \arrow[r, "\tilde{\theta}_j"] \arrow[d] & \mathcal{U}_{v,\mathcal{B}} \arrow[d] \\
\mathcal{E}_{v,\mathcal{A}} \arrow[r, "\theta_j"] & \mathcal{E}_{v,\mathcal{B}}
\end{tikzcd}$$

Proof. Since we are taking $\epsilon \ll 1$, then $K_X + S + F_{\mathcal{B}}$ is a nef $\mathbb{Q}$-Cartier divisor on a $\mathcal{A}$-stable elliptic surface $(X \to C, S + F_{\mathcal{A}})$. Therefore $K_X + S + F_{\mathcal{B}}$ is semiample and by the results of Section 6.1, there are two possibilities for the Iitaka map $\varphi := \varphi_{m(K_X+S+F_{\mathcal{B}})} \colon X \to X'$ depending on the fiber F_j whose coefficient is changing:

- the fiber F_j is a smooth or stable fiber (type I_n) so that the birational model does *not* change when $b_j = 1$ and φ is the identity;

- the fiber F_j is not type I_n so that it is an intermediate fiber given by a union $A \cup E$ of a reduced component A and a nonreduced component E. The Iitaka map φ is the contraction of A to produce a twisted fiber.

In the first case there is nothing to prove. In the second,

$$R^1\varphi_*(\mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = 0$$

by Proposition 5.1.19 and $\varphi_*(\mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = \mathcal{O}_{X'}(m(K_{X'} + \varphi_*(S + F_{\mathcal{A}})))$ by Proposition 5.1.7. It follows that $H^1(X, \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = 0$ by the Leray spectral sequence.

Now if $\pi\colon (X \to C, S + F_{\mathcal{A}}) \to B$ is a family of $\mathcal{A}$-stable elliptic surfaces, then as in the construction of reduction morphisms and flipping morphisms,

$$\mathrm{Proj}_B\left(\bigoplus_k \pi_*\mathcal{O}_X(km_0(K_X + S + F_{\mathcal{B}}))\right)$$

gives a family $\mathcal{B}$-stable elliptic surfaces over B. This construction is compatible with base change by the above vanishing and induces the required morphism θ_j.

The morphism $\tilde{\theta}_j$ is induced by applying the above to the universal family $\mathcal{U}_{v,\mathcal{A}} \to \mathcal{E}_{v,\mathcal{A}}$. □

Corollary 8.2.8. *In the situation above, the morphism θ_j is inverse to the reduction morphism $\rho_{\mathcal{B},\mathcal{A}}$. In particular, $\mathcal{E}_{v,\mathcal{A}} \cong \mathcal{E}_{v,\mathcal{B}}$.*

Remark 8.2.9. As in Remark 8.2.6, the validity of the above corollary hinges on the fact that we are defining our moduli spaces to be the normalizations of the appropriate pseudofunctors. In general the deformation theories of $\mathcal{A}$-stable and $\mathcal{B}$-stable models might differ depending on the choice of functor of stable pairs and we can only hope for θ_j to be some type of partial normalization.

CHAPTER 9

Invariance of log plurigenera for broken elliptic surfaces

In [AB16], we investigated the stable pairs compactification of the space of *twisted elliptic surfaces* using the theory of twisted stable maps. A twisted elliptic surface is an irreducible $\mathcal{A}$-stable elliptic surface $(f\colon X \to C, S + F_{\mathcal{A}})$ for the constant weight vector $\mathcal{A} = (1, \ldots, 1)$ satisfying the property that the support of every non-stable fiber is contained in $\mathrm{Supp}(F_{\mathcal{A}})$. In particular, the compactification of the space of twisted elliptic surfaces is a component of the space $\mathcal{E}_{v,\mathcal{A}}$ we denote $\mathcal{F}_v(1,1)$.

The main result of [AB16] regarding the space $\mathcal{F}_v(1,1)$ is the characterization of the boundary components as consisting of broken elliptic surfaces (see Theorem 1.4 [AB16]). Our goal is to generalize this result to $\mathcal{A}$-stable elliptic surfaces for arbitrary weights and use it to construct various morphisms between the moduli spaces for different weights analagous to the reduction morphisms of Hassett spaces (see Theorem 5.1.16).

To show the existence of such morphisms on the level of moduli spaces and universal families, we demonstrate that the pushforwards of the pluri-log canonical sheaves of a family are compatible with base change so that the construction of log canonical models is functorial in families. The main technical step is the following vanishing theorem for the pluri-log canonical divisor which we prove in this section:

Theorem 9.0.1. *Let $(f\colon X \to C, S + F_{\mathcal{B}})$ be a $\mathcal{B}$-broken stable elliptic surface with section S and marked fibers $F_{\mathcal{B}}$. Let $0 \leq \mathcal{A} \leq \mathcal{B}$ such that $K_X + S + F_{\mathcal{A}}$ is nef and $\mathbb{Q}$-Cartier.*

(a) If $\mathcal{A}$ is not identically zero. Then

$$H^i\Big(X, \mathcal{O}_X\Big(m(K_X + S + F_{\mathcal{A}})\Big)\Big) = 0$$

for $i > 0$ and $m \geq 2$ divisible enough.

(b) If $\mathcal{A}$ is identically zero, suppose further that i) $p_g(C) \neq 1$, and ii) if $p_g(C) = 0$ then $\deg \mathscr{L} \neq 2$. Then

$$H^i\Big(X, \mathcal{O}_X\Big(m(K_X + S + F_{\mathcal{A}})\Big)\Big) = 0$$

for $i > 0$ and $m \geq 2$ divisible enough.

Remark 9.0.2.

(i) Let $\mathcal{A}_t = t\mathcal{B} + (1-t)\mathcal{A}$. Since the nef cone is the closure of the ample cone, the divisor $K_X + S + F_{\mathcal{A}_t}$ is also ample for $t > 0$. That is, $K_X + S + F_{\mathcal{A}}$ is the first time we drop from ample to nef along the segment connecting $\mathcal{B}$ to $\mathcal{A}$.

(ii) We consider the case $p_g(C) = 1$ and $\mathcal{A} = 0$ and the case $p_g(C) = 0, \mathcal{A} = 0$ and $\deg \mathscr{L} = 2$ in Theorem 9.0.10.

Proof. We will prove both cases at once, pointing out where in the argument the hypothesis in case (b) are necessary if $\mathcal{A} = 0$. For convenience we sometimes denote the $\mathbb{Q}$-line bundle $L^{[m]} := \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))$. The proof will proceed through several steps.

Step 1: First we carefully break X up into several components.

Let $Y \subset X$ be a union of irreducible components and let X' be a union of the complementary irreducible components. Then there is an exact sequence

$$0 \to L^{[m]}|_{X'}(-M) \to L^{[m]} \to L^{[m]}|_Y \to 0$$

where $M = \sum_{j=1}^{s} M_j$ is the sum of fiber components along which X' and Y are attached to obtain X (see the proof of [KK02, Corollary 10.34]). Since

$$\mathcal{O}_{X'}(K_X|_{X'}) = \mathcal{O}_{X'}(K_{X'} + M)$$

and $\mathcal{O}_Y(K_X|_Y) = \mathcal{O}_Y(K_Y + M)$, we see that

$$\begin{aligned} L^{[m]}|_Y &= \mathcal{O}_Y\Big(m\Big(K_Y + S|_Y + F_{\mathcal{A}}|_Y + M\Big)\Big) \\ L^{[m]}|_{X'}(-M) &= \mathcal{O}_{X'}\Big(m\Big(K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + \frac{m-1}{m}M\Big)\Big). \end{aligned}$$

By the long exact sequence of cohomology, it suffices to prove vanishing for the divisor $L^{[m]}|_{X'}(-M)$ on X' and $L^{[m]}|_Y$ on Y. To do this, we need to guarantee some positivity for $L^{[m]}|_{X'}(-M)$, namely that it is nef. This is not immediate due to the twisting by $-M$, and therefore we need to pick X' and Y judiciously to ensure that twisting by $-M$ still yields a nef divisor. Note on the other hand that $L^{[m]}|_Y$ is automatically nef.

Let Y be a pseudoelliptic tree (see Definition 7.1.7) indexed by the rooted tree $(T,0)$ with root component Y_0. Suppose that Y is attached to X' by gluing a twisted pseudofiber of Y_0 to the arithmetic genus 1 component of an intermediate fiber on X'. In this case M is an irreducible curve. Let A denote the rational component of the intermediate fiber of X'. Suppose finally that $\mathrm{Coeff}(A, F_{\mathcal{A}}) < \mathrm{Coeff}(A, F_{\mathcal{B}})$.

Lemma 9.0.3. *In the situation above, $L^{[m]}|_{X'}(-M)$ and $L^{[m]}|_Y$ are nef and $\mathbb{Q}$-Cartier.*

Proof. $L^{[m]}|_Y$ is nef and $\mathbb{Q}$-Cartier as it is the restriction of a nef and $\mathbb{Q}$-Cartier divisor. On the other hand, we need to check that

$$L^{[m]}|_{X'}(-M) = \mathcal{O}_{X'}\left(m(K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + \frac{m-1}{m}M)\right)$$

is nef and $\mathbb{Q}$-Cartier on X'. For $\mathbb{Q}$-Cartier, it suffices to note that X' has quotient singularities in a neighborhood of M (see Section 6.2 of [AB16]). To see that it is nef, note that we only need to check

$$\begin{aligned}\left(K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + \frac{m-1}{m}M\right).M \geq 0\\ \left(K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + \frac{m-1}{m}M\right).A \geq 0\end{aligned}$$

since $K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + M$ is nef and reducing the coefficient of M does not affect the degree on the other components of the marked divisor. Furthermore, the intersections we are computing are all on the single component of X' containing A, so we may suppose X' is irreducible.

The first inequality is clear – recall that $M^2 < 0$, so reducing its coefficient *increases* the intersection with M. For the second inequality, we take a log resolution $\mu\colon X_0 \to X'$ if necessary, so that we can assume that A lies on an elliptic component $f_0\colon X_0 \to C_0$ with section S_0. Using the fact that the $\mathcal{B}$-weighted divisor $K_X+S+F_{\mathcal{B}}$ is ample, we see that $K_{X_0}+S_0+F_{\mathcal{B}}|_{X_0}+M$ is f_0-ample. Furthermore A is disjoint from the other marked fibers and $A^2<0$, so that decreasing the coefficient of A *increases* the degree on A. That is,

$$(K_{X_0}+S_0+F_{\mathcal{A}}|_{X_0}+M).A>0$$

so for large enough m,

$$\left(K_{X_0}+S_0+F_{\mathcal{A}}|_{X_0}+\frac{m-1}{m}M\right).A>0.$$

In particular, $K_{X_0}+S_0+F_{\mathcal{A}}|_{X_0}+\frac{m-1}{m}M$ is f_0-nef. Thus, after possibly contracting the section if necessary, we obtain a log minimal model $\left(X',\mu_*\left(S_0+F_{\mathcal{A}}|_{X_0}+\frac{m-1}{m}M\right)\right)$. In particular, $K_{X'}+(S+F_{\mathcal{A}})|_{X'}+\frac{m-1}{m}M$ is nef. $\square$

Now we check that the condition $\mathrm{Coeff}(A,F_{\mathcal{A}})<\mathrm{Coeff}(A,F_{\mathcal{B}})$ is satisfied whenever Y is a pseudoelliptic tree which contains at least one marked divisor whose coefficient is lowered. Indeed, if Y_α is a component and A_α is the reduced component of an intermediate fiber where another pseudoelliptic Y_β with $\beta\geq\alpha$ is attached, then

$$\mathrm{Coeff}(A_\alpha,F_{\mathcal{A}})=\sum_{D\subset\mathrm{Supp}(F_{\mathcal{A}}|_{Y_\beta})}\mathrm{Coeff}(D,F_{\mathcal{A}})$$

is a sum of the coefficients of marked fibers on Y_β. In particular, if A as above is the reduced component of an intermediate fiber on X' where the root component Y_0 of Y is attached, then $\mathrm{Coeff}(A,F_{\mathcal{A}})<\mathrm{Coeff}(A,F_{\mathcal{B}})$ since there is some D on some Y_β

with $\mathrm{Coeff}(D, F_{\mathcal{A}}) < \mathrm{Coeff}(D, F_{\mathcal{B}})$.

Now by induction on the number of pseudoelliptic trees where we have reduced coefficients, we use the long exact sequence on cohomology associated to

$$0 \to L^{[m]}|_{X'}(-M) \to L^{[m]} \to L^{[m]}|_Y \to 0$$

and reduce to proving vanishing for the following two cases:

(i) $(X, S + F_{\mathcal{A}})$ is an slc $\mathcal{A}$-broken elliptic surface such that $F_{\mathcal{A}}|_Y = F_{\mathcal{B}}|_Y$ for any pseudoelliptic tree, or

(ii) $(X, F_{\mathcal{A}})$ is an slc pseudoelliptic tree.

We will denote this pair (X, Δ) and take care to note which case we are in if necessary.

Step 2: We consider Case 1. Here we show that we may assume that $K_X + S + F_{\mathcal{A}}$ is big on every component of X. Indeed $K_X + S + F_{\mathcal{A}}$ is *ample* on every pseudoelliptic of Type I by assumption. By Proposition 6.1.22, it is big on every pseudoelliptic of Type II (see Definition 7.1.8) and every elliptic component with $\deg \mathscr{L} > 0$.

We are left to consider a component $X_1 \cong E \times C_1$ isomorphic to a product with section S_1. By Proposition 7.1.13, if $(K_X + S + F_{\mathcal{A}})|_{X_1}$ is nef but not big, then C_1 is rational and $(K_X + S + F_{\mathcal{A}}).S_1 = 0$. In this case, the log canonical morphism factors through a morphism $\mu \colon X \to Z$ which contracts the component X_1 onto E and is an isomorphism away from X_1.

Now $(Z, \mu_*(S + F_{\mathcal{A}}))$ is an $\mathcal{A}$-broken elliptic surface and

$$\mu_*\mathcal{O}_X\Big(m(K_X + S + F_{\mathcal{A}})\Big) = \mathcal{O}_Z\Big(m\Big(K_Z + \mu_*(S + F_{\mathcal{A}})\Big)\Big).$$

Therefore we want to show $R^i\mu_*L^{[m]} = 0$ for $i > 0$ so that

$$H^j\Big(X, \mathcal{O}_X\Big(m(K_X + S + F_{\mathcal{A}})\Big)\Big) = H^j\left(Z, \mathcal{O}_Z\Big(m\Big(K_Z + \mu_*(S + F_{\mathcal{A}})\Big)\Big)\right).$$

By the Theorem on Formal Functions, it suffices to show that

$$H^i(X_n, L^{[m]}|_{X_n}) = 0$$

for all $i > 0$ and n, where X_n is the n^{th} formal neighborhood of X_1 in X. The fibration $X_1 \to C_1$ extends to a fibration $X_n \to C_n$ with all fibers isomorphic to E, where C_n is isomorphic to the n^{th} formal neighborhood of the component C_1 in C. That is, C_n is a rational curve with two embedded points of length n, and is locally isomorphic to $k[x, y]/(xy, y^n)$ around these points. Furthermore, $L|_{X_n} \cong \mathcal{O}_{X_n}(S_n)$, where S_n is a formal neighborhood of the section.

Lemma 9.0.4. *Let $f_n \colon X_n \to C_n$ be an elliptic fibration with all fibers isomorphic to E over a rational curve C_n with finitely many embedded points locally isomorphic to $k[x, y]/(xy, y^n)$. Let S_n be a section. Then $H^i\Big(X_n, \mathcal{O}_{X_n}(mS_n)\Big) = 0$ for any $m, n \geq 1$ and $i > 0$.*

Proof. A direct computation on E shows that

$$H^i(E, mP) = 0$$

for $i > 0$ and $m \geq 1$ where $P = (S_n)|_E$ is a point. Therefore $R^i(f_n)_*\Big(\mathcal{O}_{X_n}(mS_n)\Big) = 0$ for $i > 0$. Similarly,

$$h^0(E, mP) = m$$

so $R_{m,n} := (f_n)_*\Big(\mathcal{O}_{X_n}(mS_n)\Big)$ is a rank m vector bundle.

For $m, n = 1$, the pushforward $(f_1)_*\Big(\mathcal{O}_X(S)\Big)$ is a line bundle on $C_1 \cong \mathbb{P}^1$ with a section coming from pushing forward the section $\mathcal{O}_{X_1} \to \mathcal{O}_{X_1}(S_1)$. Therefore $H^i(C_1, R_{1,1}) = 0$ for $i > 0$. Pushing forward the exact sequence

$$0 \to \mathcal{O}_{X_1}\Big((m-1)S_1\Big) \to \mathcal{O}_{X_1}(mS_1) \to \mathcal{O}_{S_1}(mS_1|_{S_1}) \to 0$$

and noticing that $S_1|_{S_1} = 0$, we get

$$0 \to R_{m-1,1} \to R_{m,1} \to \mathcal{O}_{C_1} \to 0.$$

Since $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for $i > 0$, then $H^i(C_1, R_{m,1}) = 0$ for $i > 0$ by induction on m.

Now consider the ideal sequence

$$0 \to I_n \to \mathcal{O}_{C_n} \to \mathcal{O}_{C_{n-1}} \to 0$$

where I_n is torsion supported on finitely many points. Applying $(-) \otimes_{C_n} R_{m,n}$ and using base change for the Cartesian square

$$\begin{tikzcd} E \times C_{n-1} \arrow[r, "j"] \arrow[d, "f_{n-1}"'] & E \times C_n \arrow[d, "f_n"] \\ C_{n-1} \arrow[r, "i"'] & C_n \end{tikzcd}$$

gives an exact sequence

$$0 \to K_{m,n} \to R_{m,n} \to R_{m,n-1} \to 0$$

where $K_{m,n}$ is torsion supported on finitely many points. Now by induction on n

and the previous vanishing for $R_{m,1}$, we obtain $H^i(C_n, R_{m,n}) = 0$ for all $i > 0$. The required vanishing then follows from the Leray spectral sequence. $\square$

This shows it suffices to prove vanishing in Case 1 for the $\mathcal{A}$-broken elliptic surface pair $(Z, \mu_*(S + F_{\mathcal{A}}))$ after contracting the component X_1. Applying this inductively, we can assume that in Case 1, the divisor $K_X + S + F_{\mathcal{A}}$ is big on every component.

Step 3: Next we reduce to the case when $K_X + S + F_{\mathcal{A}}$ has positive degree on every component of the section. Let $(X_0 \to C_0, S_0)$ be an elliptically fibered component such that $(K_X + S + F_{\mathcal{A}}).S_0 = 0$.

Let $\mu \colon X \to Z$ be the morphism contracting S_0. Then $\Big(Z, \mu_*(S + F_{\mathcal{A}})\Big)$ is an $\mathcal{A}$-broken elliptic surface pair and

$$\mu_*\mathcal{O}_X\Big(m(K_X + S + F_{\mathcal{A}})\Big) = \mathcal{O}_Z\Big(m\Big(K_Z + \mu_*(S + F_{\mathcal{A}})\Big)\Big)$$

by Proposition 5.1.7. We want to show that

$$R^i\mu_*\mathcal{O}_X\Big(m(K_X + S + F_{\mathcal{A}})\Big) = 0$$

for $i > 0$. This follows by Proposition 5.1.19, since the exceptional locus of μ is a rational curve $S_0 \cong \mathbb{P}^1$ with $S_0^2 < 0$ and $(K_X + S + F_{\mathcal{A}}).S_0 = 0$. Here we have used the hypothesis that if $\mathcal{A} = 0$, then the genus of the base curve is not 1 so that S_0 is necessarily a rational curve.

Step 4: We complete the proof in Case 1, under the assumption that $K_X + S + F_{\mathcal{A}}$ is big on every irreducible component of X, and has positive degree on every component of the section.

Proposition 9.0.5. *Let $(f\colon X \to C, S + F_{\mathcal{B}})$ be a $\mathcal{B}$-broken stable elliptic surface. Let $L^{[m]}$ denote the divisor $m(K_X + S + F_{\mathcal{A}})$ for $m \geq 2$, where $0 \leq \mathcal{A} \leq \mathcal{B}$. Suppose that $K_X + S + F_{\mathcal{A}}$ is nef, $\mathbb{Q}$-Cartier and big on every irreducible component of X and that $(K_X + S + F_{\mathcal{A}}).S_0 > 0$ for every component S_0 of S. Suppose that $F_{\mathcal{A}}|_Y = F_{\mathcal{B}}|_Y$ for every pseudoelliptic tree $Y \subset X$. Finally suppose either (a) $p_g(C) \neq 1$, or (b) $\mathcal{A}$ is not identically zero. Then $H^i(X, L^{[m]}) = 0$ for all $i > 0$.*

Proof. We will apply Fujino's Theorem 5.1.21 to $L^{[m]}$. We have that

$$L^{[m]}\Big(-(K_X + S + F_{\mathcal{A}})\Big) = \mathcal{O}_X\Big((m-1)(K_X + S + F_{\mathcal{A}})\Big)$$

is big and nef on every irreducible component of X by assumption. Therefore, to apply the theorem, it suffices to prove that $K_X + S + F_{\mathcal{A}}$ is big on every slc center of $(X, S + F_{\mathcal{A}})$. This is clear for zero dimensional slc centers.

The one dimensional slc centers of $(X, S + F_{\mathcal{A}})$ are (a) the components of the section S, (b) the twisted fibers F_j, (c) E components of marked intermediate fibers, (d) and the components of the double locus D, Now $K_X + S + F_{\mathcal{A}}$ is big on every component of the section by assumption, and

$$(K_X + S + F_{\mathcal{A}}).F_j = 1/d$$

where F_i supports a possibly nonreduced fiber of multiplicity d. Here we have used the fact that a twisted fiber is irreducible so that $F_{\mathcal{A}}.F_i = 0$.

Next we need to consider the E components of marked intermediate fibers. Since $K_X + S + F_{\mathcal{A}}$ is nef, we have

$$(K_X + S + F_{\mathcal{A}}).E \geq 0.$$

If this is positive then the restriction $(K_X + S + F_{\mathcal{A}})|_E$ is big. If this intersection is 0 then the log canonical linear series factors through the contraction of E to a minimal Weierstrass cusp. Let $\mu\colon X \to Z$ be this contraction. Then $(Z, \mu_*(S + F_{\mathcal{A}}))$ is log canonical and

$$\mu_*\mathcal{O}_X(m(K_X + S + F_{\mathcal{A}})) = \mathcal{O}_Z(m(K_Z + \mu_*(S + F_{\mathcal{A}})))$$

by Proposition 5.1.7. We want to show that

$$R^i\mu_*\mathcal{O}_X(m(K_X + S + F_{\mathcal{A}})) = 0$$

for $i > 0$. This follows by Proposition 5.1.19 since E is a rational curve as the marked intermediates are all minimal, $E^2 = 0$ and $(K_X + S + F_{\mathcal{A}}).E = 0$. Therefore it suffices to compute cohomology vanishing for the pair $(Z, \mu_*(S + F_{\mathcal{A}}))$ and by induction we can assume that $K_X + S + F_{\mathcal{A}}$ is big on all E components of marked intermediate fibers.

This leaves case (d), the double locus D, which consists of three types of irreducible components:

(i) For a stable or twisted (pseudo)fiber F along which an elliptic or type II pseudoelliptic is glued to the rest of X, we have

$$(K_X + S + F_{\mathcal{A}}).F = 1/d > 0;$$

(ii) For every isotrivial component Z with $j = \infty$, there is the self intersection locus B. If Z is a pseudoelliptic component, then the morphism $Z' \to Z$ contracting the section of the associated elliptic component is an isomorphism in a neighborhood of B so we may suppose that Z is elliptic. In this case B is a section of Z disjoint from S and

$$(K_X + S + F_{\mathcal{A}}).B > 0.$$

(iii) For every pseudoelliptic tree Y, there is the component M along which the root component Y_0 is attached to the rest of X. By the assumption

$$(K_X + S + F_{\mathcal{A}})|_Y = (K_X + S + F_{\mathcal{B}})|_Y$$

is ample on Y. In particular, $(K_X + S + F_{\mathcal{A}})|_Y.M > 0$.

Therefore $K_X + S + F_{\mathcal{A}}$ is big and nef on each slc stratum of $(X, F_{\mathcal{A}})$. Applying Theorem 5.1.21 we have the required vanishing

$$H^i\Big(X, \mathcal{O}_X\Big(m(K_X + F_{\mathcal{A}})\Big)\Big) = 0, \quad i > 0.$$

□

Step 5: Now we consider Case 2 of a pseudoelliptic tree Y indexed by a rooted tree $(T, 0)$. If $(Y, F_{\mathcal{A}})$ is already a stable pair, then we are done. Otherwise, there is some Y_α where the coefficients have been reduced. This implies the coefficients have been reduced on Y_β for any $\beta \leq \alpha$ as well.

Suppose Y_α is a leaf of the tree and that Y' is the union of Y_β for $\beta \neq \alpha$, i.e. the pseudoelliptic tree with dual graph $(T \setminus \alpha, 0)$. Suppose Y_α is attached to Y' along M a component of an intermediate fiber on Y' with genus 0 component A. Since the coefficients of Y_α have been reduced, then $L^{[m]}|_{Y'}(-M)$ and $L^{[m]}|_{Y_\alpha}$ are nef and

$\mathbb{Q}$-Cartier by Lemma 9.0.3. By the attaching sequence, it suffices to show that

$$H^i(Y_\alpha, L^{[m]}|_{Y_\alpha}) = H^i(Y', L^{[m]}|_{Y'}(-M)) = 0.$$

By induction on the number of leaves of T, it suffices to prove that

$$H^i(Y, L^{[m]}|_Y(-M)) = 0$$

where $(Y, S + F_{\mathcal{A}})$ is a pseudoelliptic tree, M is a sum of the supports of finitely many arithmetic genus 1 components of intermediate pseudofibers of Y, and either

(i) Y is irreducible, or

(ii) for each leaf $\alpha \in T$, $F_{\mathcal{A}}|_{Y_\alpha} = F_{\mathcal{B}}|_{Y_\alpha}$.

That is, we have separated of all of the leaves on which coefficients have been decreased. Therefore, we have reduced to proving vanishing on the leaves themselves, as well as on a pseudoelliptic tree for which the coefficients of all emanating leaves have *not* been decreased.

Step 6: Let $(Y, F_{\mathcal{A}})$ be a pseudoelliptic tree with dual graph $(T, 0)$ and suppose that $F_{\mathcal{A}}|_{Y_\beta} = F_{\mathcal{B}}|_{Y_\beta}$ for each leaf β, that is, we are in case (2) of Step 5 above. If $F_\beta|_Y = F_\alpha|_Y$ then $(K_X + S + F_\alpha)|_Y$ is ample so were done. Thus suppose that there exists a component Y_α with $F_{\mathcal{A}}|_{Y_\alpha} < F_{\mathcal{B}}|_{Y_\alpha}$. We may take α to be maximal so that $F_{\mathcal{A}}|_{Y_\beta} = F_{\mathcal{B}}|_{Y_\beta}$ for all $\beta > \alpha$.

Let $\beta \in \alpha[1]$ (Definition 7.1.6) and $T_{\geq\beta} = \{\gamma \in V(T) \colon \gamma \geq \beta\}$ the subtree of T with root β (see Figure 9.1). Then $Y_{\geq\beta} = \bigcup_{\gamma \in T_{\geq\beta}} Y_\gamma$ is a pseudoelliptic subtree of Y with root component Y_β. Denoting by Y' the union of components of Y not in

$Y_{\geq\beta}$, then $Y_{\geq\beta}$ is attached to Y' by gluing a twisted pseudofiber M on $Y_{\geq\beta}$ to the arithmetic genus 1 component of an intermediate pseudofiber $M \cup A$ on $Y_\alpha \subset Y'$.

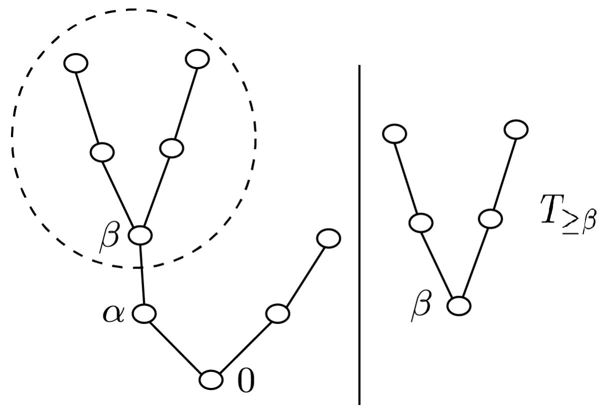


Figure 9.1: The rooted subtree $(T_{\geq\beta}, \beta)$ corresponds to the pseudoelliptic tree obtained by separating Y_β from Y_α.

We consider the gluing sequence

$$0 \to L^{[m]}|_{Y_{\geq\beta}}(-M) \to L^{[m]} \to L^{[m]}|_{Y'} \to 0.$$

Lemma 9.0.6. *$L^{[m]}|_{Y_{\geq\beta}}(-M)$ is ample on $Y_{\geq\beta}$ and $L^{[m]}|_{Y'}$ is nef with positive degree on M.*

Proof. Since no coefficients have been reduced on $Y_{\geq\beta}$, then $L|_{Y_{\geq\beta}}$ is ample so $L^{[m]}|_{Y_{\geq\beta}}(-M)$ is ample for m large enough. By assumption $\mathrm{Coeff}(A, F_{\mathcal{A}}) = \mathrm{Coeff}(A, F_{\mathcal{B}})$ so that in particular $L.M > 0$ since $(K_X + S + F_{\mathcal{A}})|_{Y'}$ is ample. $\square$

Thus, we have that

$$H^i(Y_{\geq\beta}, L^{[m]}|_{Y_{\geq\beta}}(-M)) = 0$$

for $i > 0$ so $H^i(Y, L^{[m]}) = H^i(Y', L^{[m]}|_{Y'})$. Therefore it suffices to prove vanishing for $L^{[m]}|_{Y'} = \mathcal{O}_Y(m(K_Y + M + F_{\mathcal{A}}|_{Y'}))$, where $(Y', M + F_{\mathcal{A}}|_{Y'})$ is a pseudoelliptic tree with a leaf Y_α such that coefficients on Y_α have been reduced. By induction on the number of leaves, we may suppose $(Y', F_{\mathcal{A}} + M)$ is a pseudoelliptic tree that $F_{\mathcal{A}}|_{Y_\alpha} < F_{\mathcal{B}}|_{Y_\alpha}$ for every leaf α and M is a sum of reduced arithmetic genus 1 components of intermediate pseudofibers on the leaf components. Furthermore, by the above Lemma, $(K_Y + F_{\mathcal{A}} + M).M_0 > 0$ for each component M_0 of M.

Since $F_{\mathcal{A}}|_{Y_\alpha} < F_{\mathcal{B}}|_{Y_\alpha}$ for every leaf α, we can apply Step 5 to the pseudoelliptic tree $(Y, F_{\mathcal{A}} + M)$. That is, we can separate the irreducible components of Y. This reduces to proving

$$H^i(Y, L^{[m]}|_Y(-M')) = 0,$$

for $i > 0$ where Y is an irreducible pseudoelliptic surface, M' is a union of the supports of arithmetic genus 1 components of intermediate fibers, and we can write

$$L^{[m]}|_Y(-M') = \mathcal{O}_Y\left(m\left(K_Y + F_{\mathcal{A}}|_Y + G + M + \frac{m-1}{m}M'\right)\right),$$

where M is a union of components of intermediate fibers with $L.M > 0$ and G is a twisted fiber. Denoting $\Delta = F_{\mathcal{A}}|_Y + G + M + \frac{m-1}{m}M'$, we are then left to consider an irreducible pseudoelliptic pair (Y, Δ).

Step 7: Let (Y, Δ) be an irreducible pseudoelliptic pair as in Step 5 case (1) or the conclusion of Step 6 above and suppose $K_Y + \Delta$ is big. Now we may take the partial log semi-resolution $\mu\colon X \to Y$ by the associated elliptic surface $(X \to C, S)$ over a necessarily rational curve.

We may write

$$K_X + S + F = \mu^*(K_Y + \Delta) + tS$$

for $0 \leq t \leq 1$ where $F = \mu_*^{-1}\Delta$ is a union of (not necessarily reduced) fiber components.

By Proposition 5.1.7 we have

$$\mu_* \mathcal{O}_X\Big(m(K_X + S + F)\Big) = \mathcal{O}_Y\Big(m(K_Y + \Delta)\Big).$$

Proceeding as in Proposition 9.0.5, we aim to apply Fujino's Theorem 5.1.21. That is, we need to check that $K_Y + \Delta$ is big on each of the slc strata of (Y, Δ). The divisor $K_Y + \Delta$ is big on Y by assumption, and it is trivially big on the zero dimensional log canonical centers. This leaves the one dimensional log canonical centers of (Y, Δ). These are exactly the images of the log canonical centers of $(X, S + F_{\mathcal{A}})$, noting that the image of S is a point so we need not consider it. Now $(K_Y + \Delta).M > 0$ for any log canonical center supported on an intermediate pseudofiber where a pseudoelliptic tree was attached by Lemma 9.0.6. Using the projection formula we may proceed to check the other log canonical centers as in the proof of Proposition 9.0.5 (where as in *loc. cit.* we might need to first contract necessarily rational E components of marked intermediate pseudofibers). Therefore

$$H^i\Big(Y, \mathcal{O}_Y\Big(m(K_Y + \Delta)\Big)\Big) = 0,$$

for all $i > 0$ by Theorem 5.1.21.

In particular, this finishes the proof for the following cases, where we know that $K_Y + \Delta$ is big:

- Y is an irreducible pseudoelliptic with $\deg \mathscr{L} \geq 3$ (Proposition 6.1.15),
- Y is an irreducible pseudoelliptic with with $\deg \mathscr{L} = 2$ with $\Delta \not\sim_{\mathbb{Q}} 0$ (Proposition 6.1.21).

We are left to deal with irreducible pseudoelliptics (Y, Δ) with $K_Y + \Delta$ *not* big.

Step 8: Let (Y, Δ) be a pseudoelliptic with $\deg \mathscr{L} = 1$ and Iitaka dimension $\kappa(K_Y + \Delta) = 1$. By Proposition 6.1.18, $K_Y + \Delta \sim_{\mathbb{Q}} \mu_*\Sigma$, where $\mu\colon Z \to Y$ is the contraction of the section of the associated elliptic surface $f\colon Z \to C$ and Σ is a rational multisection of f disjoint from S. Since Σ is in the locus where μ is an isomorphism, it suffices to prove that $H^i(Z, \mathcal{O}_Z(m\Sigma)) = 0$.

By Lemma 9.0.7 below, $f_*\mathcal{O}_Z(m\Sigma)$ is a semipositive vcector bundle on $\mathbb{P}^1$. In particular,

$$H^i(\mathbb{P}^1, f_*\mathcal{O}_Z(m\Sigma)) = 0.$$

Furthermore, $G.\Sigma > 0$ for any irreducible fiber G, since Σ is an effective multisection. In particular, $m\Sigma - (K_Z + S + F_{\mathcal{A}})$ is f-nef and f-big over each slc stratum of Y for $m \gg 1$. Therefore $R^i f_*\mathcal{O}_Z(m\Sigma) = 0$ by Fujino's Theorem 5.1.21 for $i > 0$ and $m \gg 0$, and so $H^i(Z, \mathcal{O}_Z(m\Sigma)) = 0$ by the Leray spectral sequence.

Lemma 9.0.7. *Let $f\colon Y \to C$ be a fibration from an irreducible slc surface to a reduced curve and let Σ be a multisection with $|\Sigma|$ basepoint free. Then $f_*\mathcal{O}_Y(m\Sigma)$ is a semipositive vector bundle on C for $m \gg 0$.*

Proof. Note that for a finite morphism $\varphi\colon B \to C$ and a vector bundle $\mathcal{E}$ on C, the vector bundle $\mathcal{E}$ is semipositive on C if and only if $\varphi^*\mathcal{E}$ is semipositive on B.

Since $R^i f_*\mathcal{O}_Y(m\Sigma) = 0$ for $i > 0$ and $m \gg 0$, we may apply cohomology and base change to conclude that $f_*\mathcal{O}_Y(m\Sigma)$ is a vector bundle and its formation commutes with basechange. Let $\nu\colon \widetilde{C} \to C$ be the normalization. Consider the base change

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\mu"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ \widetilde{C} \arrow[r, "\nu"] & C \end{tikzcd}$$

Since ν is finite, it suffices to prove that $\nu^* f_* \mathcal{O}_Y(m\Sigma) \cong g_* \mathcal{O}_Y(m\mu^*\Sigma)$ is semipositive. Since $\mu^*\Sigma$ is a multisection of g, we may assume without loss of generality that C is smooth.

Since $|m\Sigma|$ is basepoint free, there exists a general member D that is a reduced divisor, i.e. $D = \sum_{i=1}^{s} C_i$ where each C_i is a distinct multisection. There exists a finite base change

$$\begin{tikzcd} Y' \arrow[r, "\phi"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ C' \arrow[r, "\varphi"] & C \end{tikzcd}$$

such that $\phi^* C_i \sim_{\mathbb{Q}} \sum_j S_{ij}$ is a sum of distinct sections. Therefore we may assume that

$$m\Sigma \sim_{\mathbb{Q}} D = \sum_{i=1}^{s} C_i,$$

is a finite sum of distinct sections.

Now we use induction on the number of sections. Let $D_k = \sum_{i=1}^{k} C_i$. Then for $k = 1$, D_1 is a single section and $f_* \mathcal{O}_Y(D_1)$ is a line bundle with a section induced by pushing forward the section $\mathcal{O}_Y \to \mathcal{O}_Y(D_1)$. Therefore $f_* \mathcal{O}_Y(D_1)$ is semipositive.

Now let $k \geq 2$. We consider the exact sequence

$$0 \to \mathcal{O}_Y(D_{k-1}) \to \mathcal{O}_Y(D_k) \to \mathcal{O}_{C_k}(D_{k-1}|_{C_k}) \to 0$$

induced by adding C_k. Since the sections are all distinct, then $D_{k-1}.C_k \geq 0$ so that $\mathcal{O}_{C_k}(D_{k-1}|_{C_k})$ is a semipositive line bundle. Pushing forward, and noting that $R^1 f_* \mathcal{O}_Y(D_{k-1}) = 0$, then the sequence

$$0 \to f_* \mathcal{O}_Y(D_{k-1}) \to f_* \mathcal{O}_Y(D_k) \to f_* \mathcal{O}_{C_k}(D_{k-1}|_{C_k}) \to 0$$

is exact. The first term is semipositive by the inductive hypothesis, and the last term is semipositive since f is an isomorphism on C_k. Therefore the middle term is semipositive. $\square$

Step 9: Finally, we are left with the case of an irreducible pseudoelliptic (X, Δ) with Iitaka dimension $\kappa(K_X + \Delta) = 0$, which occurs for $\deg \mathscr{L} = 1$ and $\deg \mathscr{L} = 2$ as in Propositions 6.1.18 and 6.1.21. We have $K_X + \Delta \sim_{\mathbb{Q}} 0$ so for m large and divisible enough, the sheaf $\mathcal{O}_X\Big(m(K_X + \Delta)\Big) = \mathcal{O}_X$. Thus we need to compute cohomology of the structure sheaf on an irreducible pseudoelliptic surface with associated elliptic surface $(f \colon Y \to \mathbb{P}^1, S)$ and contraction $p \colon Y \to X$.

By Proposition 5.1.19, $R^i p_* \mathcal{O}_Y = 0$ for $i > 0$ and $p_* \mathcal{O}_Y = \mathcal{O}_X$ so

$$H^i(X, \mathcal{O}_X) = H^i(Y, \mathcal{O}_Y)$$

. Similarly, if Y has any nonreduced twisted fiber (in fact it can have at most one such fiber otherwise the section would not contract to form a pseudoelliptic by Proposition 7.1.13), let $\mu \colon Y' \to Y$ be the partial resolution blowing up the twisted fibers to their intermediate models. Then $\mu \colon Y' \to Y$ contracts the genus 0 component A of each such intermediate fiber. Again by Proposition 5.1.19, $R^i \mu_* \mathcal{O}_{Y'} = 0$ for $i > 0$ so we may suppose without loss of generality that Y has no nonreduced twisted fibers. In particular, we may assume the section S of Y passes through the smooth locus of $f \colon Y \to \mathbb{P}^1$, that is, $(f \colon Y \to \mathbb{P}^1, S)$ is standard.

Lemma 9.0.8. *Let $(f \colon Y \to \mathbb{P}^1, S)$ be a standard elliptic surface. Then $H^1(Y, \mathcal{O}_Y) = 0$ and $H^2(Y, \mathcal{O}_Y) = \deg \mathscr{L} - 1$.*

Proof. For a standard elliptic surface, we have $R^1 f_* \mathcal{O}_Y = \mathscr{L}^{-1}$ (see [Mir89, II.3.6]). Since $\mathscr{L}$ is effective, then $H^0(\mathbb{P}^1, \mathscr{L}^{-1}) = 0$ and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ so the Leray spectral sequence for f implies that $H^1(Y, \mathcal{O}_Y) = 0$ and

$$H^2(Y, \mathcal{O}_Y) = H^1(\mathbb{P}^1, \mathscr{L}^{-1}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\deg \mathscr{L} - 2))$$

by Serre duality. $\square$

Now we apply the lemma to the case at hand. If $\mathcal{A}$ is not zero, then by Propositions 6.1.18 and 6.1.21, we must have $\deg \mathscr{L} = 1$ and so $H^i(Y, \mathcal{O}_Y) = 0$. If $\mathcal{A} = 0$, then we are assuming that if the *original* broken elliptic surface from Step 1 is fibered over a rational curve, then the total degree of $\mathscr{L} \neq 2$ so any (pseudo)elliptic component with $\deg \mathscr{L} = 2$ must be attached. In particular the log canonical must be big on it by Proposition 6.1.21.

This concludes the proof of Theorem 9.0.1. $\square$

Remark 9.0.9. Note that the only place we required that if $\mathcal{A} = 0$ and $p_g(C) = 0$ then $\deg \mathscr{L} \neq 2$ is in Lemma 9.0.8. However, this is a completely trivial edge case as $L^{[m]} = \mathcal{O}_X$, so while $H^2(X, L^{[m]}) \neq 0$, the formation of $L^{[m]}$ still commutes with base change in families.

Theorem 9.0.10. *(Invariance of log plurigenera) Let $\pi \colon (X \to C, S + F_{\mathcal{B}}) \to B$ be a family of $\mathcal{B}$-stable broken elliptic surfaces over a reduced base B. Let $0 \leq \mathcal{A} \leq \mathcal{B}$ such that $K_{X/B} + S + F_{\mathcal{A}}$ is a π-nef and $\mathbb{Q}$-Cartier divisor. Then $\pi_* \mathcal{O}_X\big(m(K_{X/B} + S + F_{\mathcal{A}})\big)$ is a vector bundle on B whose formation is compatible with base change $B' \to B$ for $m \geq 2$ divisible enough.*

Proof. If either (a) $\mathcal{A}$ is not identically zero, or (b) $\mathcal{A} = 0$ but $p_g(C) \neq 1$ and if $p_g(C) = 0$ then $\deg \mathscr{L} \neq 2$, then we may apply Theorem 9.0.5 to see that

$H^i(X_b, m(K_{X/B} + S + F_{\mathcal{A}})|_{X_b}) = 0$ for $i > 0$ and for all closed points $b \in B$ so the result follows by the proper base change theorem.

Suppose $p_g(C_b) = 1$ and $\mathcal{A} = (0, \ldots, 0)$ is identically zero. We may suppose that $\mathcal{B} = (a, 0, \ldots, 0)$ has exactly one nonzero entry by applying the above result and first decreasing all but one coefficient to 0. Then (C_b, ap) is a one pointed stable genus 1 curve with $a < 1$. In particular, it is irreducible. Therefore X_b contains a single elliptically fibered component $X_0 \to C_b$ with a marked divisor $(F_a)_b$ lying over p. There are three cases to consider:

(i) X_0 is properly elliptic and $F_a = aF$ is a Weierstrass fiber,

(ii) X_0 is properly elliptic and there is a pseudoelliptic tree $(Y_b, (F_a)_b|_{Y_b})$ attached to an intermediate fiber $E \cup A$ above p and $(F_a)_b|_{X_0} = aA$, or

(iii) $\deg \mathscr{L} = 0$ and $X_b = X_0 = C_b \times E_b$ is a product.

In either of case (i) and (ii) there may be unmarked type I or II pseudoelliptics attached elsewhere to X_0.

Let us denote $L^{[m]} := \mathcal{O}_X(m(K_{X/B} + S))$. The linear series $|L_b^{[m]}|$ is semi-ample by Proposition 5.1.5 and $L_b^{[m]}.S_b = 0$. Thus the linear series factors through the contraction of S_b which gives a morphism $\mu \colon X_b \to Z_b$. In case (i) and (ii) this maps onto an slc broken pseudoelliptic surface with an elliptic singularity at $\mu(S_b)$ and $\mu_* L_b^{[m]} = \mathcal{O}_{Z_b}(m(K_{Z_b}))$.

In case (i), the pair (X_0, S_b) is log general type by Proposition 6.1.15 and for every other component $W \subset X_b$, the line bundle $L_b^{[m]}|_W = \mathcal{O}_{X_b}(m(K_{X_b} + S_b + (F_a)_b))|_W$ is still ample on W. It follows that K_{Z_b} is big and nef on each slc stratum of $(Z_b, 0)$ so

$H^1(Z_b, \mu_* L_b^{[m]}) = 0$ by Theorem 5.1.21.

In case (ii) we consider the attaching sequence

$$0 \to \mu_* L_b^{[m]}|_{Z'_b}(-M) \to \mu_* L_b^{[m]} \to L_b^{[m]}|_{Y_b} \to 0$$

where $M = \mu_* E$ is the curve along which Y_b is attached to $Z_0 = \mu(X_0)$ and Z'_b is the union of components of Z_b not contained in Y_b. Now $L_b|_{Y_b} = K_{Y_b} + E$ and (Y_b, E) is a broken pseudoelliptic tree and we can apply Theorem 9.0.1 to conclude $H^1(Y_b, L_b^{[m]}|_{Y_b}) = 0$. On the other hand, $K_{Z_b}|_{Z'_b} = K_{Z'_b} + M$ so

$$\mu_* L_b^{[m]}|_{Z'_b}(-M) = \mathcal{O}_{Z'_b}\left(m(K_{Z'_b} + \frac{m-1}{m}M)\right).$$

As in case (i), the divisor $K_{Z'_b} + \frac{m-1}{m}M$ is big and nef on every slc stratum of $(Z'_b, \frac{m-1}{m}M)$ so

$$H^1(Z'_b, \mu_* L_b^{[m]}|_{Z'_b}(-M)) = 0$$

by Theorem 5.1.21 and we conclude that $H^1(Z_b, \mu_* L_b^{[m]}) = 0$.

In either case (i) or (ii), it follows that $H^1(X_b, L_b^{[m]}) = H^0(Z_b, R^1\mu_* L_b^{[m]})$. Now

$$(K_{X_b} + S_b)|_{S_b} \sim_{\mathbb{Q}} 0$$

so $L_b^{[m]}|_{S_b} = \mathcal{O}_{S_b}$ for m divisible enough. On the other hand S_b is an irreducible nodal arithmetic genus 1 curve so by the theorem on formal functions, $R^1\mu_* L_b^{[m]}$ is a skyscraper sheaf supported on $\mu(S_b)$ with 1 dimensional fiber and $h^1(X_b, L_b^{[m]}) = 1$.

In case (iii), consider the trivial fibration $f\colon X_b \to C_b$ with section S_b and $g(C_b) = 1$. Then $K_{X_b} = 0$ and $L_b^{[m]} = \mathcal{O}_{X_b}(mS_b)$. Furthermore, $R^1 f_* \mathcal{O}_{X_b}(mS_b) = 0$

and $f_*\mathcal{O}_{X_b}(mS_b) = \mathcal{O}_{C_b}^{\oplus m}$ for $m \geq 1$ by [Mir89, II.3.5 and II.4.3]. It follows that $h^1(X_b, L_b^{[m]}) = h^1(C_b, \mathcal{O}_{C_b}^{\oplus m}) = m$.

In each case $h^1\Big(X_b, \mathcal{O}_X(m(K_{X/B}+S))|_{X_b}\Big)$ is constant and since the base is reduced, it follows from cohomology and base change over a reduced base scheme (see e.g. [Oss, Theorem 1.2]) that formation of

$$\pi_*\mathcal{O}_X\Big(m(K_{X/B}+S)\Big)$$

is compatible with base change.

Finally, when $\mathcal{A} = 0$, $p_g(C) = 0$ and $\deg \mathscr{L} = 2$, we have that

$$\mathcal{O}_{X_b}(m(K_{X_b}+S_b)) = \mathcal{O}_{X_b}$$

(Remark 9.0.9) and $h^0(X_b, \mathcal{O}_{X_b}) = 1$ is constant for $b \in B$ since X_b is connected and reduced so formation of $\pi_*L^{[m]}$ commutes with base change. $\square$

Remark 9.0.11. Note that for the first part of Theorem 9.0.10, we do *not* need to assume that B is reduced. Indeed, whenever we can apply the vanishing theorem 9.0.5, a strong form of proper base change ensures that the formation of $\pi_*\mathcal{O}_X(m(K_{X/B}+S+F_{\mathcal{A}}))$ commutes with arbitrary base change for any base B. It is only in the second case when the higher cohomology *does not* vanish that we need to assume B is reduced to apply cohomology and base change. This will not matter in the sequel as we restrict to normal base schemes.

The above Theorem 9.0.10 allows us to compute the $\mathcal{A}$-stable model of a $\mathcal{B}$-stable family by working fiber by fiber. This is used in the next section to explicitly describe the steps of the log MMP to compute the stable limits of a 1-parameter family. Then in Section 8.1, we use Theorem 9.0.10 to show that performing the steps of the log MMP on a family of elliptic surfaces is functorial. This leads to the existence of reduction morphisms between moduli spaces of elliptic surfaces for weights $0 \leq \mathcal{A} \leq \mathcal{B}$ as above.

APPENDIX A

Normalizations of algebraic stacks

In this appendix, we justify the fact that we only work with normal base schemes throughout Part II of the thesis. Specifically, the goal is to prove that in certain situations, the normalization of an algebraic stack is uniquely determined by its values on normal base schemes (Proposition A.0.7) and that a morphism between normalizations of algebraic stacks can be constructed by specifying it on the category of normal schemes (Proposition A.0.6). This material is probably well known but we include it here for lack of a suitable reference.

If X is a locally Noetherian scheme, the normalization $\nu\colon X^\nu \to X$ is defined as the normalization of X in its total ring of fractions. We denote by $|X|$ (resp. $|\mathcal{X}|$) the underlying topological space of points of a scheme (resp. algebraic stack). We begin with some facts about normalizations of schemes.

Lemma A.0.1. *[Sta18, Tag 035Q] Let X be a locally Noetherian scheme;*

(i) the normalization $X^\nu \to X$ is integral, surjective and induces a bijection on

irreducible components;

(ii) *for any normal scheme Z and morphism $Z \to X$ such that each irreducible component of Z dominates an irreducible component of X, there exists a unique factorization $Z \to X^\nu \to X$.*

Lemma A.0.2. *[Sta18, Tag 07TD] Let $X \to Y$ be a smooth morphism of locally Noetherian schemes. Let $Y^\nu \to Y$ be the normalization of Y. Then $X \times_Y Y^\nu \to X$ is the normalization of X.*

This motivates the following definitions:

Definition A.0.3. Let $\mathcal{X}$ be a locally Noetherian algebraic stack. We say that $\mathcal{X}$ is **normal** if there is a smooth surjection $U \to \mathcal{X}$ where U is a normal scheme. A **normalization** of $\mathcal{X}$ is a representable morphism

$$\nu\colon \mathcal{X}^\nu \to \mathcal{X}$$

from an algebraic stack $\mathcal{X}^\nu$ such that for any scheme U and any smooth morphism $U \to \mathcal{X}$, the pullback $\mathcal{X}^\nu \times_{\mathcal{X}} U \to U$ is the normalization of U.

Lemma A.0.4. *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then a normalization $\nu\colon \mathcal{X}^\nu \to \mathcal{X}$ exists and it is unique up to unique isomorphism.*

Proof. The proof closely follows [Sta18, Tag 07U4] which proves the claim for algebraic spaces. Indeed let $R \rightrightarrows U$ be a smooth groupoid presentation for $\mathcal{X}$. Then by Lemma A.0.2 one sees that the pullback of R to U^ν under both morphisms is isomorphic to R^ν. One can then check as in *loc. cit.* that $R^\nu \rightrightarrows U^\nu$ is a smooth groupoid and define $\mathcal{X}^\nu = [U^\nu/R^\nu]$ with morphism to $\mathcal{X}$ induced by $U^\nu \to U$ and $R^\nu \to R$.

Normality is local on the base in the smooth topology [Sta18, Tag 034F] so that for any scheme T and smooth morphism $T \to \mathcal{X}$, we can check normality of $T \times_{\mathcal{X}} \mathcal{X}^{\nu}$ by pulling back to the smooth cover $U \to \mathcal{X}$. Here the result follows from Lemma A.0.2. Finally uniqueness is clear from the construction. □

Lemma A.0.5. *Let $\mathcal{X}$ be a locally Noetherian algebraic stack, then;*

(i) $\mathcal{X}^{\nu}$ is normal;

(ii) $\mathcal{X}^{\nu} \to \mathcal{X}$ is an integral surjection that induces a bijection on irreducible components;

(iii) for any normal algebraic stack $\mathcal{Z}$ and a morphism $\mathcal{Z} \to \mathcal{X}$ such that every irreducible component of $\mathcal{Z}$ dominates an irreducible component of $\mathcal{X}$, there exists a unique factorization $\mathcal{Z} \to \mathcal{X}^{\nu} \to \mathcal{X}$.

Proof. The proof follows the analagous result [Sta18, Tag 0BB4] for algebraic spaces. (1) is clear and (2) follows from Lemma A.0.1 and descent.

For (3) let $U \to \mathcal{X}$ be a smooth surjection and $R = U \times_{\mathcal{X}} U \rightrightarrows U$. Pulling back to $\mathcal{Z}$ gives a smooth morphism $\mathcal{Y} := U \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}$. Let $U' \to \mathcal{Y}$ be a smooth cover of $\mathcal{Y}$ by a scheme and U'. The composition $U' \to \mathcal{Z}$ is a smooth cover with groupoid presentation $R' \colon U' \times_{\mathcal{Z}} U' \rightrightarrows U'$ and a commutative square

$$\begin{tikzcd} R' \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & R \arrow[d, shift left] \arrow[d, shift right] \\ U' \arrow[r] & U \end{tikzcd}.$$

The conditions on $\mathcal{Z} \to \mathcal{X}$ imply that we can apply Lemma A.0.1 to obtain unique factorizations $R' \to R^{\nu}$ and $U' \to U^{\nu}$. By uniqueness, these morphisms are compatible with the groupoid data so that we get a unique factorization $\mathcal{Z} \to \mathcal{X}^\nu$ by descent. $\square$

Now we are ready for the main results of this appendix.

Proposition A.0.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally Noetherian algebraic stacks. Suppose that for each normal scheme T, there exist functors*

$$f_T \colon \mathcal{X}(T) \to \mathcal{Y}(T)$$

compatible with base change and such that the morphism on points $|f| \colon |\mathcal{X}| \to |\mathcal{Y}|$ is dominant on irreducible components. Then f_T induces a unique representable morphism

$$f^\nu \colon \mathcal{X}^\nu \to \mathcal{Y}^\nu.$$

Proof. Let $U \to \mathcal{X}$ be a smooth surjection from a scheme U and let $U^\nu \to U$ be the normalization. Then $U^\nu \to \mathcal{X}$ is an integral surjection that induces a bijection on irreducible components by Lemma A.0.5 (2). Let $\xi \in \mathcal{X}(U^\nu)$ be the object inducing this morphism. Then we have an object $f_T(\xi) \in \mathcal{Y}(U^\nu)$ inducing a morphism $U^\nu \to \mathcal{Y}$. By assumption this is compatible with the pullbacks to $R^\nu = U^\nu \times_{\mathcal{X}^\nu} \times U^\nu$ and thus induces a morphism $g \colon \mathcal{X}^\nu \to \mathcal{Y}$.

The map $|g| \colon |\mathcal{X}| \to |\mathcal{Y}|$ factors as

$$\begin{tikzcd} {|\mathcal{X}^\nu|} \arrow[r, "|\nu|"] \arrow[dr, "|g|"'] & {|\mathcal{X}|} \arrow[d, "|f|"] \\ & {|\mathcal{Y}|} \end{tikzcd} .$$

By Lemma A.0.5 (2) and the assumptions on $|f|$, $|g|$ is dominant on irreducible components. Therefore there is a unique factorization $f^\nu \colon \mathcal{X}^\nu \to \mathcal{Y}^\nu$ by Lemma A.0.5

(3). $\square$

Proposition A.0.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be separated locally Noetherian algebraic stacks. Suppose that for each normal scheme T, there is an isomorphism $f_T\colon \mathcal{X}(T) \cong \mathcal{Y}(T)$ compatible with base change. Then there is an isomorphism $f\colon \mathcal{X}^\nu \to \mathcal{Y}^\nu$.*

Proof. First let $\mathcal{T}$ be a normal algebraic stack. Then there is a smooth cover $U \to \mathcal{T}$ where U is normal giving a groupoid presentation $R \rightrightarrows U$ of $\mathcal{T}$. Since normality is local in the smooth topology [Sta18, Tag 034F], R is normal and we have equivalences $\mathcal{X}(R) \cong \mathcal{Y}(R)$ and $\mathcal{X}(U) \cong \mathcal{Y}(U)$ compatible with base change by the two morphisms $R \rightrightarrows U$. By descent, this induces an equivalence $f_{\mathcal{T}}\colon \mathcal{X}(\mathcal{T}) \cong \mathcal{Y}(\mathcal{T})$ compatible with base change by a normal algebraic stack. Denote the inverse by $g_{\mathcal{T}}$.

By Proposition A.0.6 there exist morphisms $f\colon \mathcal{X}^\nu \to \mathcal{Y}^\nu$ and $g\colon \mathcal{Y}^\nu \to \mathcal{X}^\nu$ induced by f_T and its inverse. The map $\mathcal{X}^\nu \to \mathcal{X}$ is induced by an object $\xi \in \mathcal{X}(\mathcal{X}^\nu)$ and under the equivalence described in the preceding paragraph, $f_{\mathcal{X}^\nu}(\xi) \in \mathcal{Y}(\mathcal{X}^\nu)$ corresponds to the composition $\mathcal{X}^\nu \to \mathcal{Y}^\nu \to \mathcal{Y}$. Similarly, if $\xi' \in \mathcal{Y}(\mathcal{Y}^\nu)$ is the object inducing the normalization $\mathcal{Y}^\nu \to \mathcal{Y}$, then $g_{\mathcal{Y}^\nu}(\xi') \in \mathcal{X}(\mathcal{Y}^\nu)$ corresponds to the composition $\mathcal{Y}^\nu \to \mathcal{X}^\nu \to \mathcal{X}$.

By compatibility of the equivalences with pullbacks, we get that $g^*\xi = g_{\mathcal{Y}^\nu}(\xi')$ so that

$$\xi' = f_{\mathcal{Y}^\nu} g^*\xi = g^* f_{\mathcal{X}^\nu}\xi \in \mathcal{Y}(\mathcal{Y}^\nu)$$

. But the latter is the object corresponding to the composition

$$\mathcal{Y}^\nu \to \mathcal{X}^\nu \to \mathcal{Y}^\nu \to \mathcal{Y}.$$

Therefore $\nu \circ f \circ g = \nu$, i.e. the morphism $fg\colon \mathcal{Y}^\nu \to \mathcal{Y}^\nu$ commutes with the normalization $\mathcal{Y}^\nu \to \mathcal{Y}$.

Since the normalization factors uniquely through $\mathcal{Y}^{red}$, we may suppose that $\mathcal{Y}$ is reduced. Then ν is an isomorphism over a dense open subset of each irreducible component of $\mathcal{Y}$. Therefore fg must agree with the identity over this dense open subset so $fg = \mathrm{id}_{\mathcal{Y}^\nu}$, since $\mathcal{Y}^\nu$ is separated. Applying the same argument to $\mathcal{X}^\nu$ yields that $gf = \mathrm{id}_{\mathcal{X}^\nu}$. $\square$

Remark A.0.8. Note that $\mathcal{X}^\nu(T)$ is *not* necessarily equal to $\mathcal{X}(T)$ for T normal even though $\mathcal{X}^\nu$ is uniquely determined by the values of $\mathcal{X}(T)$ for T normal. Indeed this fails even for schemes. For example the inclusion of the node of nodal curve has multiple lifts to the normalization. It is an interesting question to determine a functorial way to define the normalization of $\mathcal{X}$ directly as a category fibered in groupoids over schemes without knowing a priori that $\mathcal{X}$ is algebraic.

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