Exact Smooth Piecewise Polynomial Sequences on Powell-Sabin and Worsey-Farin Splits

by

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Vita

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Publications

1. J. Guzmán, A. Lischke, and M. Neilan. Exact sequences on Worsey-Farin splits. *arXiv preprint arXiv:2008.05431*, 2020.

2. J. Guzmán, A. Lischke, and M. Neilan. Exact sequences on Powell-Sabin splits. *Calcolo* 57(2), 1–25, 2020.

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Chapter One

Introduction

Partial differential equations (PDEs) arise in a multitude of mathematics and engineering applications, hence the development of numerical methods for approximating the solutions of PDEs has inspired abundant and valuable research. In this thesis, we focus on the finite element method, which has become one of the most powerful tools in computational mathematics and engineering. In recent years, a framework known as *the finite element exterior calculus* (FEEC) emerged, catalyzing a new approach for analyzing numerical methods whose approximation spaces comprise differential complexes. The FEEC applies the calculus of differential forms, long studied in differential geometry, to Hilbert complexes, leading to a unification of many concepts in vector calculus as well as providing an elegant framework for proving the well-posedness of finite element methods.

The FEEC capitalizes on the ability of certain differential complexes to succinctly express structural properties of PDEs. These properties may then be preserved in their respective numerical approximations, yielding a sophisticated theory that ties together classical results in finite element methods through a deeper mathematical understanding. This powerful framework emerged in the study of PDEs for elasticity and electromagnetism [7, 8, 9, 18, 39], where certain differential complexes arose that had been well-studied in the homological algebra literature [48]. Perhaps the most significant work leading to the adoption of FEEC among a broader mathematical community is a 2006 publication by Arnold, Falk, and Winther [11], which led to an intensive effort to apply these tools to the analysis of numerical methods for PDEs. To fully introduce the FEEC requires a treatment of concepts from homological algebra and Hodge theory, but as the results of this thesis are mainly confined to Hilbert spaces of functions of two and three variables, and thus are described using vector calculus, we will restrict our discussion to a class of differential complexes used in our results, specifically *cochain complexes*, and their relevant properties. For a more thorough treatment of the FEEC, we refer the reader to [12, 6, 11].

Here, we introduce the *cochain complex*, which is a sequence of vector spaces V^k and linear maps d^k that map one vector space to the next, as in

$$\cdots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \longrightarrow \cdots$$

such that $d^k \circ d^{k-1} = 0$, and we denote this complex by (V, d). Each complex considered in this thesis is finite, where $V^k = 0$ if $k < 0$ or if k is large enough. An important example of a cochain complex is the *de Rham* complex. In the case where Ω is a domain in $\mathbb{R}^3$, the de Rham complex may be described using vector calculus:

$$\mathbb{R}^3 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \to 0. \tag{1.0.1}$$

This sequence indeed forms a complex due to the vector calculus identities $\operatorname{grad}\operatorname{curl} = 0$ and $\operatorname{curl}\operatorname{div} = 0$. It is also important to define the *Hilbert complex*, which consists of a sequence of Hilbert spaces V^k and closed, densely-defined linear operators d^k from V^k to V^{k+1} such that the range of d^k is a subset of the domain of d^{k+1}.

A *cochain map* is formed from one cochain complex (V, d) to a second cochain complex (W, d) by linear maps $\pi^k : V^k \to W^k$ that form a diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V^{k-1} & \xrightarrow{d^{k-1}} & V^k & \xrightarrow{d^k} & V^{k+1} & \xrightarrow{d^{k+1}} & \cdots \\
 & & \downarrow{\scriptstyle \pi^{k-1}} & & \downarrow{\scriptstyle \pi^k} & & \downarrow{\scriptstyle \pi^{k+1}} & & \\
\cdots & \longrightarrow & W^{k-1} & \xrightarrow{d^{k-1}} & W^k & \xrightarrow{d^k} & W^{k+1} & \xrightarrow{d^{k+1}} & \cdots .
\end{array}$$

A key property of the cochain map is that the above diagram commutes, i.e., $d^{k-1}\pi^{k-1}V^{k-1} = \pi^k d^{k-1} V^{k-1}$. Furthermore, the cochain map is said to be *bounded* if for each k, there exists a constant c such that for every $v \in V^k$, $\|\pi^k v\|_V^k \le c\|v\|_V^k$, where $\|\cdot\|_V$ is the norm associated with the spaces V^k. Moreover, when the W^k spaces represent finite element spaces, and π^k is the projection associated with a simplicial triangulation that maps from a Sobolev space into W^k, the boundedness of the cochain complex is important for the numerical stability of the finite element approximation [6].

We are especially interested in the case where the domain Ω in the de Rham complex (1.0.1) is given a simplicial triangulation Ω_h, and we develop projections π^k (i.e., degrees of freedom) mapping the smooth function spaces of (1.0.1) to a sequence of finite element spaces on Ω_h that preserve the cochain complex properties. It is in this sense that the finite element spaces of this thesis are *structure preserving*.

The study of exact sequences of finite element spaces was originally used to discretize the de Rham sequence [10]

$$\mathbb{R} \to H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}\,;\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div}\,;\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0.$$

The $H^1(\Omega)$-conforming finite elements were used to solve the Laplace equation [28]; the $H(\mathrm{curl}\,;\Omega)$-conforming finite element spaces were used to discretize Maxwell's equations [49]; and the $H(\mathrm{div}\,;\Omega)$ and $L^2(\Omega)$ finite element pairs we used to discretize Darcy flow [26]. We are interested in discretizing a sequence with smoother component spaces, so that the advantageous structure-preserving properties of exact sequences of finite element spaces can be used to find stable finite element spaces to discretize, for example, Stokes flow and the biharmonic equation.

1.1 Stokes complex

We consider the strong form of the Stokes equations, where the unknowns are the velocity, the vector-valued function u, and the pressure, the scalar function p:

$$\begin{aligned} -\mu\Delta u + \operatorname{grad} p = f, &\quad \text{in } \Omega, \\ \operatorname{div} u = 0, &\quad \text{in } \Omega, \end{aligned} \tag{1.1.1}$$

where Ω is a simply-connected domain in $\mathbb{R}^3$ and μ is a constant that represents viscosity. We impose no-slip (homogeneous Dirichlet) boundary conditions for simplicity. A finite element method for approximating (1.1.1) is based on its weak formulation, which is stated as follows: find the velocity $u \in \mathring{H}^1(\Omega, \mathbb{R}^3)$ and the pressure $p \in \mathring{L}^2(\Omega)$ such that

$$\begin{aligned} a(u,v) - (p, \operatorname{div} v) = (f, v), &\quad \forall v \in \mathring{H}^1(\Omega, \mathbb{R}^2), \\ (\operatorname{div} u, q) = 0, &\quad \forall q \in \mathring{L}^2(\Omega), \end{aligned} \tag{1.1.2}$$

where $(\cdot,\cdot)$ is the L^2-inner product, $\mathring{L}^2(\Omega) = \{p \in L^2(\Omega) : \int_\Omega p = 0\}$, and the bilinear form $a(u,v)$ is given by $a(u,v) = \mu(\operatorname{grad} u, \operatorname{grad} v)$.

In order to derive the *Galerkin method* for solving (1.1.2), one must select finite-dimensional normed spaces V_h and P_h associated with the triangulation Ω_h such that $V_h \subset \mathring{H}^1$ and $P_h \subset \mathring{L}^2$. Then the approximate solution is $(u_h, p_h) \in V_h \times P_h$, which must satisfy

$$\begin{aligned} a(u_h, v) - (p_h, \operatorname{div} v) = (f, v), &\quad \forall v \in V_h, \\ (\operatorname{div} u_h, q) = 0, &\quad \forall q \in P_h. \end{aligned}$$

One issue that may arise with this discretization is that the pressure p_h may not be unique.

Indeed, the space of functions $\{p_h \in P_h : (p_h, \operatorname{div} v) = 0 \; \forall v \in V_h\}$ may include spurious pressure modes. To resolve this issue, the function spaces V_h and P_h must satisfy the Babuska-Brezzi condition [14, 19, 36]:

$$\inf_{p \in P_h,\, p \neq 0} \sup_{v \in V_h,\, v \neq 0} \frac{(p, \operatorname{div} v)}{\|v\|_{H^1(\Omega)} \|p\|_{L^2(\Omega)}} = \beta_h > 0. \tag{1.1.3}$$

If (1.1.3) holds, then p_h is unique.

The physical condition of conservation of mass, imposed on the velocity in the strong form (1.1.1) as $\operatorname{div} u = 0$, may not be enforced in the discretization unless special care is taken in the choice of the spaces V_h and P_h. Classical Stokes element pairs such as the MINI elements, the P_2-P_0 elements, and the Taylor-Hood elements (see [16, Chapter 8]) do not enforce mass conservation, and the development of mass-conserving finite element pairs has been an active area of research. An equivalent way of enforcing conservation of mass is for the spaces V_h and P_h to satisfy the exactness property of an appropriate cochain complex. The Hilbert spaces containing the solutions u and p of the continuum equation (1.1.1) form a sequence within the *Stokes complex*,

$$0 \to H^2(\Omega) \xrightarrow{\text{grad}} H^1(\operatorname{curl}; \Omega) \xrightarrow{\text{curl}} H^1(\Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0, \tag{1.1.4}$$

which we have stated without boundary conditions for simplicity. The space $H^1(\operatorname{curl})$ represents the space of H^1 vector fields such that their curl is also in H^1. Suppose that the projections $\pi_V : H^1(\Omega) \to V_h$ and $\pi_S : L^2(\Omega) \to P_h$ are chosen such that $V_h \xrightarrow{\text{div}} P_h$ satisfies the subcomplex property, i.e., $\operatorname{div} V_h \subset P_h$. Since the Galerkin equation

$$(\operatorname{div} u_h, q) = 0$$

holds for all $q \in P_h$, and since the subcomplex property yields $\operatorname{div} u_h \in P_h$, it follows that $(\operatorname{div} u_h, \operatorname{div} u_h) = 0$, from which we infer that $\operatorname{div} u_h = 0$. Therefore the discretization enforces incompressibility of the numerical solution u_h for any mesh size h. Such projections π_V and π_P form a *commuting diagram*:

$$\begin{array}{ccc} H^1 & \xrightarrow{\operatorname{div}} & L^2 \\ \Big\downarrow \pi_V & & \Big\downarrow \pi_P \\ V_h & \xrightarrow{\operatorname{div}} & P_h. \end{array}$$

If the projections π_V and π_P are also bounded, the theory of FEEC can be used to show that such spaces V_h and P_h indeed satisfy the Brezzi condition (1.1.3). This example shows how a cochain complex and its associated commuting diagrams may be used to develop finite element methods that are consistent, stable, and mass conserving.

1.2 Geometrically refined meshes

Devising finite element methods for the Stokes equations on a general triangulation is an active area of research, as most natural choices of finite element pairs do not yield stable methods, e.g., the $\mathcal{P}^2 - \mathcal{P}^1$ finite elements. One approach to resolving this issue is to consider different types of mesh geometries, such that the stability and convergence properties of the finite element pairs may depend on the choice of the mesh family.

One such mesh geometry is known as the Alfeld refinement, which is obtained by connecting each vertex of each simplex with one interior "split" point. In two dimensions, this refinement is often called the "Clough-Tocher refinement", where each triangle is split into three sub-triangles, and in three dimensions, the Alfeld refinement splits a tetrahedron into four sub-tetrahedra. In 1992, Arnold and Qin [13] showed that the $\mathcal{P}^2 - \mathcal{P}^1$ finite elements are indeed stable if the mesh is an Alfeld refinement. Zhang [61] extended this work to three dimensions, and Guzmán and Neilan [38] extended this work to arbitrary dimensions using several different finite element pairs with any polynomial degree.

The Clough-Tocher finite elements were first introduced in 1965 as a way of reducing the polynomial order needed to construct a C^1 finite element space, and they were used by Clough and Tocher to analyze plate bending [25]. The C^1 interpolants could be constructed with cubic polynomials requiring only nine local degrees of freedom on each macro-element [51]. Zhang used the Alfeld refinement to solve the Stokes equations in three dimensions [61]. Alfeld extended this work to three dimensions in [2].

Peter Alfeld's work has been deeply influential within the spline and finite element communities. Although we do not attempt to thoroughly summarize the significance of Alfeld's work here, we wish to acknowledge those of his ideas upon which this thesis builds and extends. In particular, Alfeld introduced the first C^2 element based on a split of the triangle, where he used the double Clough-Tocher split [1]. Following this work, Alfeld and others, including [54], [41], [42], [43], [44], [45], [3], [4], and [46], introduced many macro-elements based on the Clough-Tocher and Powell-Sabin splits of a triangle. Furthermore, Alfeld's work [2] introduced the first C^1 three-dimensional macro-element based on what is now commonly known as the Alfeld split of a tetrahedron. An essential property of this macro-element, observed by Alfeld in [2], is that C^1 polynomial interpolants on the Alfeld split have intrinsic supersmoothness at the split point and at the vertices. In two dimensions, a C^1 piecewise polynomial on a Clough-Tocher split has two continuous derivatives at the split point (this holds for any choice of split point as long as it is strictly interior to the triangle). In three dimensions, a C^1 piecewise polynomial on the Alfeld split is C^3 at the split point and C^2 at the vertices. Alfeld and Schumaker defined the general notion of supersmoothness in [5], and Sorokina [57] characterized the supersmoothness for more general simplicial particians of polytopal domains in arbitrary dimensions.

Fu, Guzmán, and Neilan's work [33] showed that C^1 piecewise polynomials on an Alfeld split in any dimension $n \geq 2$ are connected to the Stokes finite element pairs via a de Rham sequence of piecewise polynomial spaces on a macro-element. They proved the exactness of these sequences on one macro-element, i.e., on a single Alfeld split of a simplex. Then they constructed degrees of freedom for three-dimensional finite element spaces that would induce global finite element spaces on the entire triangulation with the same exactness properties. In order to construct these degrees of freedom, they found it was necessary to use the intrinsic supersmoothness properties of piecewise polynomials on the Alfeld split geometry, and they needed to add some regularity at the vertices to the Stokes finite element pairs in the sequence. For example, as mentioned above, since Alfeld showed that C^1 piecewise polynomials on the Alfeld split are C^2 at the vertices [2], so their degrees of freedom for this space included data for the second derivatives on the vertices. In this sense, some of these degrees of freedom are not natural, and this issue motivates the work in this thesis. Our goal is to consider other types of splits for which the finite element spaces have degrees of freedom that do not rely on any supersmoothness properties, and instead only use regularity intrinsic to the PDE we aim to discretize. We considered the Powell-Sabin split in two dimensions and its three-dimensional analogue, the Worsey-Farin split, which turned out to be fruitful in this aspect. We are able to prove the exactness of sequences where the spaces have the same regularity properties as those in [33], and the degrees of freedom for these spaces require data with only as much regularity as the space.

A different approach was taken by Christensen and Hu [23], where they considered low-order approximations in any dimension while using different types of splits for each space in the de Rham sequence. For the first, smoothest space in the sequence, they used the split with the most refinement (which is the so-called Worsey-Piper split in three dimensions), and for the nth space in the sequence, they used the Alfeld split in the case where the pressure space was assumed to be continuous. If the pressures were allowed to be discontinuous, no splitting was used. In two dimensions, however, Christiansen and Hu were able to define a de Rham sequence with arbitrarily high polynomial order and with the same (Clough-Tocher) split for each space in the sequence.

In our work, we seek to avoid the seemingly unnatural reliance on supersmoothness by considering different types of splits. In particular, we develop finite element spaces on the Powell-Sabin split in two dimensions and the Worsey-Farin split in three dimensions that form exact sequences, and we are able to determine degrees of freedom for each space that are more natural in the sense that they do not make use of any supersmoothness properties and only rely on the smoothness intrinsic to the problem.

1.3 Finer splits

As discussed above, in order to define local exact sequences that would induce global spaces with the desired smoothness and without using any supersmoothness properties in the degrees of freedom, we considered Powell-Sabin splits in two dimensions and Worsey-Farin splits in three dimensions. We describe the Powell-Sabin split here. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and let $\mathcal{T}_h$ be the simplicial, shape-regular triangulation of Ω. Then the Powell-Sabin triangulation $\mathcal{T}_h^{\mathrm{ps}}$ is obtained as follows. We select an interior point of each triangle $T \in \mathcal{T}_h$ and adjoin this point with each vertex of T. Next, the interior points of each adjacent pair of triangles are connected with an edge. For any T that shares an edge with the boundary of Ω, an arbitrary point on the boundary edge is selected to connect with the interior point of T so that each $T \in \mathcal{T}_h$ is split into six triangles. See Figure 1.1. One common choice of interior points that produces a well-defined triangulation is the incenter of each $T \in \mathcal{T}_h$, i.e., the center point of the largest circle that fits within T

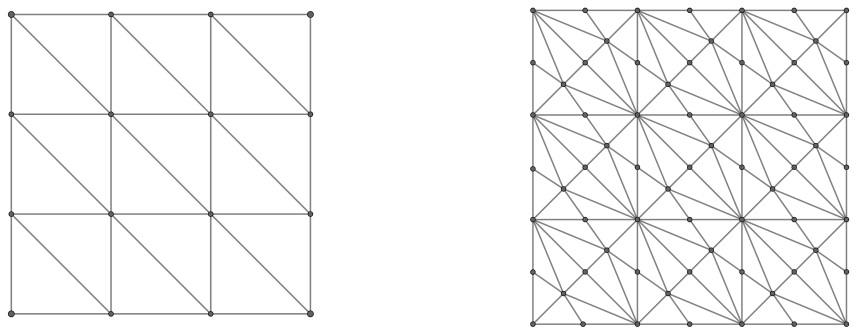

Figure 1.1: (*left*) A triangulation of the unit square, and (*right*) its Powell-Sabin refinement.

[47]. We define the set $\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ to be the points of intersection of the edges of $\mathcal{T}_h$ with the edges that adjoin interior points. An interesting fact about the meshes constructed is that the points in $\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ are singular vertices of the mesh $\mathcal{T}_h^{\mathrm{ps}}$; see [56]. In this thesis, we will construct finite element spaces on the Powell-Sabin split that form exact sequences such that the first space of the sequence is the space of C^1 piecewise polynomials, and the last two spaces are inf-sup stable Stokes finite element pairs. This work has been published in *Calcolo* Volume 57, Number 2; see [37].

Related to our work on the Powell-Sabin split are the papers [62, 63] by S. Zhang, where conforming finite element pairs are proposed and studied for the Stokes problem on Powell-Sabin meshes. Zhang showed that if the discrete velocity space is the linear Lagrange finite element space, and if the pressure space is the image of the divergence operator acting on the discrete velocity space, then the resulting pair is inf-sup stable. However, by design, the discrete pressure spaces in [62, 63], and correspondingly the range of the divergence operator, is not explicitly given. Practically, this issue is bypassed by using the iterative penalty method to solve the finite element method without explicitly constructing a basis of the discrete pressure space. In this thesis, we will explicitly construct the discrete pressure space and characterize the space of divergence-free functions for any polynomial degree.

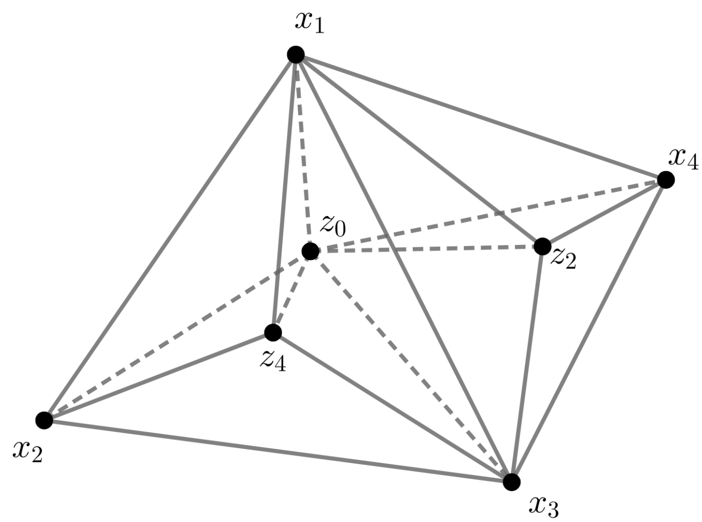

Figure 1.2: A representation of the Worsey-Farin split with two faces shown.

The Worsey-Farin split was first introduced by Worsey and Farin in 1987 [60] in order to construct a C^1 interpolant with data given only at the vertices and mid-edge points of an n-dimensional triangulation. We describe the Worsey-Farin split in $\mathbb{R}^3$ here. Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain, and let $\mathcal{T}_h$ be the simplicial, shape-regular triangulation of Ω. Then the Worsey-Farin triangulation $\mathcal{T}_h^{\mathrm{wf}}$ is obtained as follows. We select an interior point of T and adjoin this point with each vertex of T. The interior points of adjacent tetrahedra are then connected via an edge. The intersection of this edge with the shared face F of the two adjacent tetrahedra is added to the triangulation, and this point is then connected by three new edges with vertices of F. This intersection point always lies on the interior of the face F as long as the interior points of the tetrahedra $T \in \mathcal{T}_h$ are chosen as the incenters [47]. If T shares a face with the boundary of Ω, an arbitrary point on the boundary face is selected to split the face into three sub-triangles. See Figure 1.2. We will construct finite element spaces on the Worsey-Farin split that form exact sequences, generalizing the results on the Powell-Sabin split.

These types of triangulations have been of interest within the spline community for a long time. Sorokina and Worsey [58] developed C^1 piecewise quadratic splines on generalized Powell-Sabin splits for any dimension $\mathbb{R}^n$, which is equivalent to a Worsey-Farin split in $\mathbb{R}^3$. In an extension of this work, Floater and Hu [31] characterized the supersmoothness of C^r splines with $r \geq 1$ on different split geometries. Furthermore, Kolesnikov and Sorokina [40] used algebraic geometry techniques in addition to spline techniques to find the dimension of C^1 splines on the Alfeld split of any n-dimensional simplex. Foucart and Sorokina conjectured the dimension formula for general C^r splines on n-dimensional Alfeld splits [32], and this work was extended by Schenck to C^r splines in 2014 [55]. The book [47] by Lai and Schumaker contains an exhaustive study of smooth splines on many different types of splits in two and three dimensions. Lai and Schumaker also proved many geometrical results for triangulations based on these splits. For example, the Worsey-Piper split, which is a refinement where the faces of a tetrahedron are split by Powell-Sabin splits, does not induce a well-defined triangulation unless the original mesh satisfies some restrictive conditions on its geometry. In contrast, Lai and Schumaker showed that mesh refinements induced by the Worsey-Farin split are indeed well-defined when the split points are incenters. Many of these results from the spline literature informed, inspired, and enhanced our work.

Here, we outline the results contained in this thesis. In Chapter 2, we present useful definitions, vector calculus identities, and finite element spaces that are fundamental to the understanding of our results in the succeeding chapters. Chapter 2 also includes discussion on existing results on the Clough-Tocher, Powell-Sabin, and Worsey-Farin splits. Our work on exact sequences on Powell-Sabin splits is presented in Chapter 3, and Chapter 4 is the first part of our work on Worsey-Farin splits, where the exactness of sequences and the dimension counts of certain finite element spaces are proved. In Chapter 5, we develop unisolvent degrees of freedom that form commuting projections with the exact sequences of Chapter 4 in the lowest order case. We extend these results to degrees of freedom for any polynomial order in Chapter 6.

Chapter Two

Notation and Finite Element Spaces

2.1 Finite Elements

In this section, we develop some basic notation and terminology for describing the finite elements spaces used in this thesis.

Definition 2.1.1 (Mesh [30])**.** *Let* Ω *be a polygonal (resp., polyhedral) subset of* $\mathbb{R}^2$ *(resp.,* $\mathbb{R}^3$*). A mesh is a union of* N *compact, connected, disjoint, non-empty subsets* T_i *of* Ω *known as* cells *or* elements *such that* $\{T_i\}$ *forms a partition of* Ω*, i.e.,*

$$\overline{\Omega} = \bigcup_{i=1}^{N} T_i, \qquad \text{and} \qquad \mathring{T}_i \cap \mathring{T}_j = \emptyset \quad \text{for} \quad i \neq j.$$

We will focus on meshes where each element is a triangle (in $\mathbb{R}^2$) or a tetrahedron (in $\mathbb{R}^3$), otherwise known as *simplices*.

Definition 2.1.2 (Simplex, Simplicial triangulation [30])**.** *Let* $n \geq 1$*, and let* $\{x_0, \ldots, x_n\}$ *be a family of points in* $\mathbb{R}^n$ *such that the vectors* $\{x_1 - x_0, \ldots, x_n - x_0\}$ *are linearly independent. Then the convex hull* $\langle x_0, \ldots, x_d \rangle$ *of these points is called a* simplex. *A mesh* $\mathcal{T}_h$ *such that each cell* $T \in \mathcal{T}_h$ *is a simplex is called a* simplicial triangulation.

The meshes used in this thesis will be simplicial triangulations represented by $\mathcal{T}_h$, where h is a parameter that represents the level of refinement of the mesh. For each element $T \in \mathcal{T}_h$, the *diameter* h_T of T is defined $h_T = \text{diam}(T) = \max_{x_1, x_2 \in T} |x_1 - x_2|$, which is the largest Euclidean distance between two vertices of T. Then the mesh size h is defined as $\max_{T \in \mathcal{T}_h} h_T$. A family of meshes where h is decreasing and accumulates at zero will be denoted by $\{\mathcal{T}_h\}_{h>0}$. The set of vertices of a triangulation $\mathcal{T}_h$ is denoted by $\Delta_0(\mathcal{T}_h)$, edges are denoted by $\Delta_1(\mathcal{T}_h)$, triangles are denoted by $\Delta_2(\mathcal{T}_h)$, and tetrahedra are denoted by $\Delta_3(\mathcal{T}_h)$. All of these mesh entities are referred to as *facets* of the triangulation.

Definition 2.1.3 (Shape-regular [30])**.** *Given a cell* $T \in \mathcal{T}_h$*, let* ρ_T *represent the diameter* *of the largest ball that can fit within T. A family of meshes $\{\mathcal{T}_h\}_{h>0}$ is said to be shape-regular if there exists a positive constant β_0 such that for every h and for each $T \in \mathcal{T}_h$, $\beta_T = h_T/\rho_T \leq \beta_0$.*

If a family of triangulations is shape-regular, the triangles cannot become too "flat" as h goes to zero. This property is important in obtaining global error estimates and in proving that numerical solutions converge to the true solution.

Each finite element space in this thesis will be a space of piecewise polynomials on a simplicial triangulation. For $r \in \mathbb{N}$, let $\mathcal{P}_r(S)$ be the space of polynomials of degree less than or equal to r with domain S, where $\mathcal{P}_r(S) = \{0\}$ if $r < 0$. We represent piecewise polynomial functions on a triangulation $\mathcal{T}_h$ of $\Omega \subset \mathbb{R}^n$ as

$$\mathcal{P}_r(\mathcal{T}_h) = \{q \in L^2(\mathcal{T}_h) : q|_S \in \mathcal{P}_r(S),\ \forall S \in \Delta_n(\mathcal{T}_h)\}.$$

Now we are ready to describe some important finite element spaces, with the aim of describing well-known exact sequences and setting up the smoother extensions of these sequences developed in Chapters 3 - 6. Let S be a domain in $\mathbb{R}^d$, with $d = 2$ or 3, and let n_S be the outward unit normal of S on ∂S. Let $p \in \mathcal{P}_r(S)$ with $r \geq 0$, and let $q \in [\mathcal{P}_r(S)]^d$. We represent the Hilbert space of square-integrable functions by $L^2(S)$, and $\mathring{L}^2(S) = \{p \in L^2(S) : \int_S p = 0\}$. We will refer to the following Sobolev spaces throughout the thesis.

$$\begin{aligned} H^1(S) &= \{p \in L^2(S) : \operatorname{grad} p \in L^2(S)\}, \\ \mathring{H}^1(S) &= \{p \in H^1(S) : p = 0 \text{ on } \partial S\}, \\ H(\operatorname{div}; S) &= \{q \in [L^2(S)]^2 : \operatorname{div} q \in L^2(S)\}, \\ \mathring{H}(\operatorname{div}; S) &= \{q \in H(\operatorname{div}; S) : q \cdot n_S = 0 \text{ on } \partial S\}. \end{aligned}$$

Furthermore, we let $[x_1, x_2]^\top$ be a basis for $\mathbb{R}^2$, and we use the convention that the two-dimensional curl operator maps vector functions to scalar functions; specifically, given a vector function v, $\operatorname{curl} v = \partial_{x_1}(v \cdot x_2) - \partial_{x_2}(v \cdot x_1)$. The rot operator maps a scalar function u to a vector function and is defined rot $u = [\partial_{x_2} u, -\partial_{x_1} u]^\top$. In three dimensions, the curl of a vector function v has the usual definition $\operatorname{curl} v = \operatorname{grad} \times v$. Now, letting $d = 2$ or 3, we can define the useful Sobolev spaces

$$
\begin{aligned}
H(\operatorname{curl}; S) &= \{q \in [L^2(S)]^d : \operatorname{curl} q \in L^2(S)\},\\
\mathring{H}(\operatorname{curl}; S) &= \{q \in H(\operatorname{curl}; S) : q \times n_S = 0 \text{ on } \partial S\},\\
H(\text{rot}\,; S) &= \{p \in L^2(S) : \text{rot}\, p \in [L^2(S)]^2\},\\
\mathring{H}(\text{rot}\,; S) &= \{p \in H(\text{rot}\,; S) : p = 0 \text{ on } \partial S\},
\end{aligned}
$$

where the definition of curl should be understood from the dimension of the domain S, and the rot operator is only applied when the dimension $d = 2$.

Ciarlet defined a finite element as follows.

Definition 2.1.4 (Finite element [24])**.** *A* finite element *consists of a triplet* $\{T, V, \Sigma\}$ *such that*

(i) T *is a simplex of a triangulation* $\mathcal{T}_h$ *of a domain* $\Omega \subset \mathbb{R}^n$*,*

(ii) V *is a vector space of functions* $p : T \to \mathbb{R}^m$ *for some positive integer* $1 \leq m \leq n$*, and*

(iii) Σ *is a set of linear functionals* $\{\sigma_1, \ldots, \sigma_k\}$ *acting on the members of* V *such that the linear mapping* $p \in V$ *satisfies*

$$p \to (\sigma_1(p), \ldots, \sigma_k(p)) \in \mathbb{R}^k \tag{2.1.1}$$

is bijective. The linear functionals $\{\sigma_1, \ldots, \sigma_k\}$ *are called the* local degrees of freedom.

A consequence of the bijectivity of the mapping (2.1.1) is that there exists a basis $\{v_1, \ldots, v_k\}$ of V such that $\sigma_i(v_j) = \delta_{ij}$ for $1 \leq i, j \leq k$, where δ_{ij} is the Kronecker delta. This is often referred to as *unisolvence*, which is defined formally below.

Definition 2.1.5 (Unisolvence [30])**.** *The set* Σ *is unisolvent if and only if the following properties are satisfied.*

(i) $\dim V = |\Sigma| = k$,

(ii) for any $v \in V$*, if* $\sigma_j(v) = 0$ *for* $j = 1, \ldots, k$*, then* $v = 0$*.*

Here, we describe several particular finite element spaces that will be referenced throughout the thesis. The Lagrange finite elements are continuous piecewise polynomials used to discretize the H^1 space. The degrees of freedom for a Lagrange finite element on a triangulation consists of function evaluations at the nodes. In particular, if $\{T, V, \Sigma\}$ is a finite element where T is a simplex and $k = \dim V$, and if there is a set of points $\{x_1, \ldots x_k\}$ in T such that $\sigma_i(v) = v(x_i)$ for all $v \in V$ and for each $1 \leq i \leq k$, then $\{T, V, \Sigma\}$ is a Lagrange finite element.

Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3, and let $\mathcal{T}_h$ be a triangulation of Ω. Nédélec introduced four families of three-dimensional finite element spaces in two papers, [49] and [50], known as the $H(\mathrm{div}\,; \mathcal{T}_h)$- and $H(\mathrm{curl}\,; \mathcal{T}_h)$-conforming elements of the *first* and *second* kinds. Raviart and Thomas introduced an $H(\mathrm{div}\,; \mathcal{T}_h)$-conforming two-dimensional finite element space in [53], of which the $H(\mathrm{div}\,; \mathcal{T}_h)$-conforming Nédélec finite elements of the first kind are the three-dimensional extension. Hence these finite element spaces are often appropriately called the *Nédélec-Raviart-Thomas* finite element spaces. We will sometimes refer to these spaces simply as “Nédélec spaces”.

The Sobolev space $H(\operatorname{div}; \mathcal{T}_h)$ arises frequently in many problems in partial differential equations. Piecewise polynomials are $H(\operatorname{div}; \mathcal{T}_h)$-conforming if they have a continuous normal component on the facets of $\mathcal{T}_h$. Given a simplex T of a triangulation in $\mathbb{R}^d$ with $d = 2$ or 3, the Raviart-Thomas space RT_r^d of $\mathbb{R}^d$-valued polynomials is given by

$$RT_r^d = [\mathcal{P}_r(T)]^d \oplus x\tilde{\mathcal{P}}_r(T),$$

where $\tilde{\mathcal{P}}_r(T)$ are *homogeneous* polynomials [53]. Homogeneous polynomials have the form

$$\sum_{|\alpha|=r} c_\alpha x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$

for some constants c_α, and where α is a multi-index of degree r. Letting n represent the outward unit normal of T on ∂T, a function $q \in RT_r^d(T)$ is fully determined by the degrees of freedom

$$\int_f (q \cdot n) p \, ds, \qquad p \in \mathcal{P}_r(f), \ \forall f \in \Delta_{d-1}(T), \tag{2.1.2a}$$

$$\int_T q \cdot p \, dA, \qquad p \in [\mathcal{P}_{r-1}(T)]^d. \tag{2.1.2b}$$

Another divergence-conforming finite element space for two-dimensional triangulations was introduced by Brezzi, Douglas, and Marini in [21], and was extended to three dimensional simplices by Nédélec [50] and by Brezzi et al. [20]. Nédélec's definition is as follows. Given a triangle or tetrahedron T, the Brezzi-Douglas-Marini (or Nédélec) $H(\operatorname{div}; T)$-conforming finite element space is denoted by $V_r^2(T) = [\mathcal{P}_r(T)]^d$, with $r \geq 1$ and $d = 1$ or 2, where any function $v \in V_r^2(T)$ is fully determined by the degrees of freedom

$$\int_f (v \cdot n) p \, ds, \qquad p \in \mathcal{P}_r(f), \ \forall f \in \Delta_{d-1}(T), \tag{2.1.3a}$$

$$\int_T v \cdot p \, dx, \qquad p \in R_{r-1}(T) \text{ for } r \geq 2, \tag{2.1.3b}$$

where $R_r(T) = [\mathcal{P}_{r-1}(T)]^d \oplus S_r(T)$, and $S_r(T) = \{p \in \mathcal{P}_r(T) : (p \cdot x) = 0\}$.

For piecewise polynomials to be $H(\mathrm{curl}\,; \mathcal{T}_h)$-conforming, their tangential component must be continuous. In our work, we will only need to use the Nédélec elements of the second kind. Although Nédélec only considered three-dimensional $H(\mathrm{curl}\,; \mathcal{T}_h)$-conforming finite elements, we can also use the two-dimensional rotation operator to define $H(\mathrm{rot}\,; \mathcal{T}_h)$-conforming finite elements. If T is a triangle in $\mathcal{T}_h$, and $V_r = [\mathcal{P}_r(T)]^2$, then the Nédélec degrees of freedom are given by

$$\int_e (v \cdot t) p \, ds, \qquad \forall p \in \mathcal{P}_r(e), \ \forall e \in \Delta_1(T), \tag{2.1.4a}$$

$$\int_T v \cdot p \, dA, \qquad \forall p \in RT_{r-1}(T), \ r \geq 2. \tag{2.1.4b}$$

If T is a tetrahedron and $V = [\mathcal{P}_r(T)]^3$, then the degrees of freedom are

$$\int_e (v \cdot t) p \, ds, \qquad p \in \mathcal{P}_r(e), \ \forall e \in \Delta_1(T), \tag{2.1.5a}$$

$$\int_F v \cdot p \, dA, \qquad p \in RT^2_{r-1}(F), \ \forall F \in \Delta_2(T), \ r \geq 2, \tag{2.1.5b}$$

$$\int_T v \cdot p \, dx, \qquad p \in RT^3_{r-2}(T), \ r \geq 3, \tag{2.1.5c}$$

which are the Nédélec degrees of freedom for three dimensional $H(\mathrm{curl}\,; T)$-conforming finite elements.

The L^2-conforming finite elements approximate functions that are not necessarily continuous. These types of elements are frequently used in discretizations of the Poisson equations, Stokes equations, and elasticity. If T is either a triangle or a tetrahedron, and $V = \mathcal{P}_r(T)$, then a function $v \in V$ is fully determined by the degrees of freedom $\sigma_i(v) = v(x^i)$, where $\{x^i\}_{i=1}^k$ (with $k = \dim V$) is a set of points in T defined by

$$x = \begin{cases} (i/r, j/r), & 0 \leq i+j \leq r, \quad T \text{ is a triangle}, \\ (i/r, j/r, \ell/r), & 0 \leq i+j+\ell \leq r, \quad T \text{ is a tetrahedron}. \end{cases}$$

Now that we have described the local Nédélec-Raviart-Thomas and Lagrange finite element spaces and given their degrees of freedom, we are ready to formalize their notation on a triangulation. Let F_h represent the simplicial triangulation of a triangle F. Let $r \geq 0$. The Nédélec spaces of the second kind on F_h are denoted as

$$\begin{aligned} V_r^0(F_h) &= \mathcal{P}_r(F_h) \cap H^1(F), & \mathring{V}_r^0(F_h) &= V_r^0(F_h) \cap \mathring{H}^1(F), \\ V_{\mathrm{div},r}^1(F_h) &= [\mathcal{P}_r(F_h)]^2 \cap H(\mathrm{div}\,; F), & \mathring{V}_{\mathrm{div},r}^1(F_h) &= V_{\mathrm{div},r}^1(F_h) \cap \mathring{H}(\mathrm{div}\,; F), \\ V_{\mathrm{curl},r}^1(F_h) &= [\mathcal{P}_r(F_h)]^2 \cap H(\mathrm{curl}\,; F), & \mathring{V}_{\mathrm{curl},r}^1(F_h) &= V_{\mathrm{curl},r}^1(F_h) \cap \mathring{H}(\mathrm{curl}\,; F), \\ V_r^2(F_h) &= \mathcal{P}_r(F_h), & \mathring{V}_r^2(F_h) &= \mathcal{P}_r(F_h) \cap \mathring{L}^2(F). \end{aligned}$$

Next, let T_h represent the simplicial triangulation of a tethrahedron T. The Nédélec spaces of the second kind on T_h are denoted as

$$V_r^0(T_h) = \mathcal{P}_r(T_h) \cap H^1(T), \qquad \mathring{V}_r^0(T_h) = V_r^0(T_h) \cap \mathring{H}^1(T), \tag{2.1.6}$$

$$V_r^1(T_h) = [\mathcal{P}_r(T_h)]^3 \cap H(\mathrm{curl}\,; T), \qquad \mathring{V}_r^1(T_h) = V_r^1(T_h) \cap \mathring{H}(\mathrm{curl}\,; T), \tag{2.1.7}$$

$$V_r^2(T_h) = [\mathcal{P}_r(T_h)]^3 \cap H(\mathrm{div}\,; T), \qquad \mathring{V}_r^2(T_h) = V_r^2(T_h) \cap \mathring{H}(\mathrm{div}\,; T), \tag{2.1.8}$$

$$V_r^3(T_h) = \mathcal{P}_r(T_h), \qquad \mathring{V}_r^3(T_h) = \mathcal{P}_r(T_h) \cap \mathring{L}^2(F). \tag{2.1.9}$$

It is well known that the Nédélec spaces form an exact sequence [12, 11].

Lemma 2.1.6. *Let $r \geq 2$, and let F_h be a triangulation of a triangle F. The following sequences are exact, i.e., the range of each map is the kernel of the succeeding map.*

$$\mathbb{R} \longrightarrow V_r^0(F_h) \xrightarrow{\text{rot}} V^1_{\text{div},r-1}(F_h) \xrightarrow{\text{div}} V^2_{r-2}(F_h) \longrightarrow 0, \tag{2.1.10}$$

$$0 \longrightarrow \mathring{V}_r^0(F_h) \xrightarrow{\text{rot}} \mathring{V}^1_{\text{div},r-1}(F_h) \xrightarrow{\text{div}} \mathring{V}^2_{r-2}(F_h) \longrightarrow 0. \tag{2.1.11}$$

Now let $r \geq 3$, and let T_h be a triangulation of a tetrahedron T. The following sequences are exact.

$$\mathbb{R} \longrightarrow V_r^0(T_h) \xrightarrow{\text{grad}} V^1_{r-1}(T_h) \xrightarrow{\text{curl}} V^2_{r-2}(T_h) \xrightarrow{\text{div}} V^3_{r-3}(T_h) \longrightarrow 0, \tag{2.1.12}$$

$$0 \longrightarrow \mathring{V}_r^0(T_h) \xrightarrow{\text{grad}} \mathring{V}^1_{r-1}(T_h) \xrightarrow{\text{curl}} \mathring{V}^2_{r-2}(T_h) \xrightarrow{\text{div}} \mathring{V}^3_{r-3}(T_h) \longrightarrow 0. \tag{2.1.13}$$

We can also state an equivalent result of (2.1.14) - (2.1.15) where instead of rot and div, the two-dimensional operators grad and curl are used. The following result follows from Lemma 2.1.6 by rotating the coordinate axes. Let $r \geq 2$, then

$$\mathbb{R} \longrightarrow V_r^0(F_h) \xrightarrow{\text{grad}} V^1_{\text{curl},r-1}(F_h) \xrightarrow{\text{curl}} V^2_{r-2}(F_h) \longrightarrow 0, \tag{2.1.14}$$

$$0 \longrightarrow \mathring{V}_r^0(F_h) \xrightarrow{\text{grad}} \mathring{V}^1_{\text{curl},r-1}(F_h) \xrightarrow{\text{curl}} \mathring{V}^2_{r-2}(F_h) \longrightarrow 0. \tag{2.1.15}$$

The goal of our work is to extend these results to include the smoother finite element spaces defined below, which are of interest for solving the Stokes equations, Maxwell's equations, and the biharmonic equation.

The Lagrange spaces on F_h are denoted as

$$L_r^0(F_h) = \mathcal{P}_r(F_h) \cap C(F), \qquad \mathring{L}_r^0(F_h) = L_r^0(F_h) \cap \mathring{H}^1(F), \tag{2.1.16a}$$

$$L_r^1(F_h) = [L_r^0(F_h)]^2, \qquad \mathring{L}_r^1(F_h) = [\mathring{L}_r^0(F_h)]^2, \tag{2.1.16b}$$

$$L_r^2(F_h) = L_r^0(F_h), \qquad \mathring{L}_r^2(F_h) = \mathring{L}_r^0(F_h) \cap \mathring{V}_r^2(F_h). \tag{2.1.16c}$$

Notice that there is some redundancy in this notation: $L_r^0(F_h) = V_r^0(F_h)$, and $\mathring{L}_r^0(F_h) = \mathring{V}_r^0(F_h)$. The Lagrange spaces on T_h are denoted as

$$L_r^0(T_h) = \mathcal{P}_r(T_h) \cap C(T), \qquad \mathring{L}_r^0(T_h) = L_r^0(T_h) \cap \mathring{H}^1(T), \tag{2.1.17}$$

$$L_r^1(T_h) = [L_r^0(T_h)]^3, \qquad \mathring{L}_r^1(T_h) = [\mathring{L}_r^0(T_h)]^3, \tag{2.1.18}$$

$$L_r^2(T_h) = [L_r^0(T_h)]^3, \qquad \mathring{L}_r^2(T_h) = [\mathring{L}_r^0(T_h)]^3, \tag{2.1.19}$$

$$L_r^3(T_h) = L_r^0(T_h), \qquad \mathring{L}_r^3(T_h) = \mathring{L}_r^0(T_h) \cap \mathring{V}_r^3(T_h). \tag{2.1.20}$$

The same redundancy exists in the three-dimensional case as before, namely $L_r^0(T_h) = V_r^0(T_h)$ and $\mathring{L}_r^0(T_h) = \mathring{V}_r^0(T_h)$. Despite the overlap in notation for the two-dimensional and three-dimensional spaces above, the correct definitions are decipherable from the dimension of the triangulation of the underlying space (e.g., $V_r^2(F_h)$ versus $V_r^2(T_h)$).

Let grad_F and div_F represent the two-dimensional gradient and divergence operators on F, and let curl_F represent the two-dimensional scalar curl on F. The following "smooth spaces" on F_h are denoted with and without boundary conditions as

$$S_r^0(F_h) = \{v \in L_r^0(F_h) : \operatorname{grad}_F v \in [C(F)]^2\}, \tag{2.1.21a}$$

$$\mathring{S}_r^0(F_h) = \{v \in S_r^0(F_h) : v = 0 \text{ and } \operatorname{grad}_F v = 0 \text{ on } \partial F\}, \tag{2.1.21b}$$

$$S_{\mathrm{curl},r}^1(F_h) = \{v \in L_r^1(F_h) : \operatorname{curl}_F v \in C(F)\}, \tag{2.1.21c}$$

$$\mathring{S}_{\mathrm{curl},r}^1(F_h) = \{v \in S_r^1(F_h) : v = 0 \text{ and } \operatorname{curl}_F v = 0 \text{ on } \partial F\}, \tag{2.1.21d}$$

$$S^1_{\mathrm{div}\,,r}(F_h) = \{v \in L^1_r(F_h) : \mathrm{div}\,_F v \in C(F)\}, \tag{2.1.21e}$$

$$\mathring{S}^1_{\mathrm{div}\,,r}(F_h) = \{v \in S_{\mathrm{div}\,,r}(F_h) : v = 0 \text{ and } \mathrm{div}\,_F v = 0 \text{ on } \partial F\}, \tag{2.1.21f}$$

$$S^2_r(F_h) = L^2_r(F_h), \tag{2.1.21g}$$

$$\mathring{S}^2_r(F_h) = \mathring{L}^2_r(F_h). \tag{2.1.21h}$$

On the triangulation T_h of a tetrahedron T, the smooth spaces are denoted by

$$S^0_r(T_h) = \{v \in L^0_r(T_h) : \mathrm{grad}\, v \in [C(T)]^3\}, \tag{2.1.22a}$$

$$\mathring{S}^0_r(T_h) = \{v \in S^0_r(T_h) : v = 0 \text{ and } \mathrm{grad}\, v = 0 \text{ on } \partial T\}, \tag{2.1.22b}$$

$$S^1_r(T_h) = \{v \in L^1_r(T_h) : \mathrm{curl}\, v \in C(T)\}, \tag{2.1.22c}$$

$$\mathring{S}^1_r(T_h) = \{v \in S^1_r(T_h) : v = 0 \text{ and } \mathrm{curl}\, v = 0 \text{ on } \partial T\}, \tag{2.1.22d}$$

$$S^2_r(T_h) = \{v \in L^2_r(T_h) : \mathrm{div}\, v \in C(T)\}, \tag{2.1.22e}$$

$$\mathring{S}^2_r(T_h) = \{v \in S^2_r(T_h) : v = 0 \text{ and } \mathrm{div}\, v = 0 \text{ on } \partial T\}, \tag{2.1.22f}$$

$$S^3_r(T_h) = L^3_r(T_h), \tag{2.1.22g}$$

$$\mathring{S}^3_r(T_h) = \mathring{L}^3_r(T_h). \tag{2.1.22h}$$

We are interested in connecting these smooth spaces with Stokes element pairs via exact sequences. On general triangulations, it is not known how to form exact sequences involving these smooth spaces. Our approach is to consider different refinements of general triangulations on which we are able to formulate exact sequences using these spaces and to derive the appropriate commuting projections (i.e., degrees of freedom).

2.2 Vector calculus identities

In the course of proving the main results of this thesis, we will often invoke instances of the Stokes Theorem (i.e., integration by parts), which are stated in the identities below. Suppose that T is a tetrahedron and that F is a face in $\Delta_2(T)$. Let ∂T represent the boundary of T consisting of the faces $F \in \Delta_2(T)$, and let n represent the outward unit normal to T on ∂T. For functions $u, \psi \in \mathcal{P}_{r_1}(T)$ and $v, \phi \in [\mathcal{P}_{r_2}(T)]^3$ with $r_1, r_2 \geq 0$, we will need the following instances of integration by parts [22].

$$\int_T \operatorname{grad} u \cdot \phi \, dx = -\int_T u \operatorname{div} \phi \, dx + \int_{\partial T} n u \phi \, dA, \tag{2.2.1a}$$

$$\int_T \operatorname{curl} v \cdot \phi \, dx = \int_T v \cdot \operatorname{curl} \phi \, dx + \int_{\partial T} (n \times v) \cdot \phi \, dA, \tag{2.2.1b}$$

$$\int_T \operatorname{div} v \psi \, dx = -\int_T v \cdot \operatorname{grad} \psi \, dx + \int_{\partial T} (n \cdot v) \psi \, dA. \tag{2.2.1c}$$

Notice that the differential dx is used for three-dimensional integrands, and the differential dA is used for two-dimensional integrands.

Let n_F represent the outward unit normal to T on a face $F \in \Delta_2(T)$. Then we denote the tangential components $n_F \times v \times n_F$ on F by v_F, and we denote the restriction of u to F as u_F. Let $[e_1, e_2]^\top$ be an orthonormal basis spanning the plane containing F. Then we will define the following surface operators on F.

$$\begin{aligned} \operatorname{grad}_F u_F &= n_F \times \operatorname{grad} u \times n_F, \\ \operatorname{curl}_F v_F &= \operatorname{curl} v \cdot n_F, \\ \operatorname{div}_F v_F &= \partial_{e_1} (v \cdot e_1)_F + \partial_{e_2} (v \cdot e_2)_F. \end{aligned} \tag{2.2.2}$$

We also use the two-dimensional rotated gradient $\operatorname{rot}_F u_F = \operatorname{grad}_F u_F \times n_F$.

We can now state the Stokes Theorem for the surface faces $F \in \Delta_2(T)$. Let u, v, ϕ, and ψ be defined as before, and let $n_{\partial F}$ represent the outward unit normal to F on ∂F (so $n_{\partial F}$ is tangent to the face F). Then we have

$$\int_F \operatorname{grad}_F u_F \cdot \phi_F \, dA = -\int_F u_F \operatorname{div}_F \phi_F \, dA + \int_{\partial F} n_{\partial F} u_F \phi_F \, ds, \tag{2.2.3a}$$

$$\int_F \operatorname{curl}_F v_F \, \psi_F \, dA = \int_F v_F \cdot \operatorname{curl}(\psi_F n_F) \, dA + \int_{\partial F} (n_{\partial F} \times v_F) \cdot (\psi_F n_F) \, ds, \tag{2.2.3b}$$

$$\int_F \operatorname{div}_F v_F \, \psi_F \, dA = -\int_F v_F \cdot \operatorname{grad}_F \psi_F \, dA + \int_{\partial F} (n_{\partial F} \cdot v_F) \psi_F \, ds. \tag{2.2.3c}$$

We have used dx for two-dimensional integrands and ds for one-dimensional integrands.

2.3 The Clough-Tocher split

Let us describe the Clough-Tocher split. Let $\Omega \subset \mathbb{R}^2$ be a polyhedral domain, and let $\mathcal{T}_h$ be a simplicial, shape-regular triangulation of Ω. Then the Clough-Tocher triangulation $\mathcal{T}_h^{\text{ct}}$ is obtained as follows. We select an interior point of each triangle $F \in \mathcal{T}_h$ and adjoin this point with each vertex of F, so that each $F \in \mathcal{T}_h$ is split into three triangles. See Figure 2.1. We denote the Clough-Tocher split of F by F^{ct}.

Given an orthonormal basis $[e_1, e_2]^\top$ of $\mathbb{R}^2$, we use the convention that a vector-valued function v has $\operatorname{curl} v = -\partial_{x_2}(v \cdot e_1) + \partial_{x_1}(v \cdot e_2)$, and $\operatorname{grad} v = \partial_{x_1}(v \cdot e_1)e_1 + \partial_{x_2}(v \cdot e_2)e_2$. It should be understood from context whether we refer to the two-dimensional scalar curl or to the three-dimensional vector curl.

The dimensions of the Nédélec and Lagrange spaces on the Clough-Tocher split were given in [6] and [33], respectively.

$$\dim V_r^0(F^{\text{ct}}) = \frac{3}{2}r^2 + \frac{3}{2}r + 1, \qquad \dim \mathring{V}_r^0(F^{\text{ct}}) = \frac{3}{2}r^2 - \frac{3}{2}r + 1, \tag{2.3.1}$$

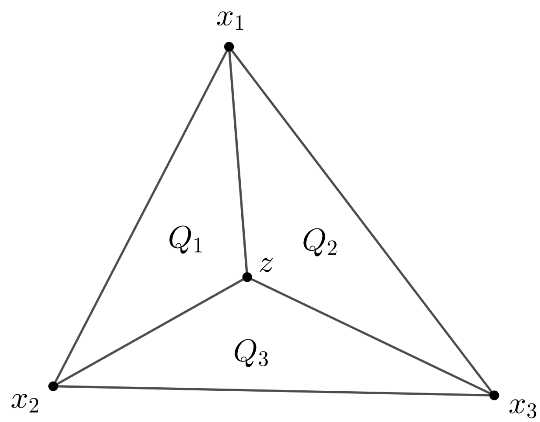

Figure 2.1: Representation of the Clough-Tocher split of a triangle.

$$\dim V^1_{\mathrm{div},r}(F^{\mathrm{ct}}) = 3(r+1)^2, \qquad \dim \mathring{V}^1_{\mathrm{div},r}(F^{\mathrm{ct}}) = 3r(r+1), \tag{2.3.2}$$

$$\dim V^2_r(F^{\mathrm{ct}}) = \frac{3}{2}(r+1)(r+2), \qquad \dim \mathring{V}^2_r(F^{\mathrm{ct}}) = \frac{3}{2}(r+1)(r+2) - 1, \tag{2.3.3}$$

$$\dim L^0_r(F^{\mathrm{ct}}) = \frac{1}{2}(3r^2+3r+2), \qquad \dim \mathring{L}^0_r(F^{\mathrm{ct}}) = \frac{1}{2}(3r^2-3r+2), \tag{2.3.4}$$

$$\dim L^1_r(F^{\mathrm{ct}}) = 3r^2+3r+2, \qquad \dim \mathring{L}^1_r(F^{\mathrm{ct}}) = 3r^2-3r+2, \tag{2.3.5}$$

$$\dim L^2_r(F^{\mathrm{ct}}) = \frac{1}{2}(3r^2+3r+2), \qquad \dim \mathring{L}^2_r(F^{\mathrm{ct}}) = \frac{3}{2}r(r-1). \tag{2.3.6}$$

We note that the dimension of $V^1_{\mathrm{curl},r}(F^{\mathrm{ct}})$ (resp., $\mathring{V}^1_{\mathrm{curl},r}(F^{\mathrm{ct}})$) is equal to the dimension of $V^1_{\mathrm{div},r}(F^{\mathrm{ct}})$ (resp., $\mathring{V}^1_{\mathrm{div},r}(F^{\mathrm{ct}})$).

The spaces $S^k_r(F^{\mathrm{ct}})$ and $\mathring{S}^k_r(F^{\mathrm{ct}})$ have the dimension counts [33]:

$$\dim S^0_r(F^{\mathrm{ct}}) = \frac{3}{2}(r^2-r+2), \qquad \dim \mathring{S}^0_r(F^{\mathrm{ct}}) = \frac{3}{2}(r^2-5r+6), \tag{2.3.7}$$

$$\dim S^1_r(F^{\mathrm{ct}}) = 3r^2+3, \qquad \dim \mathring{S}^1_r(F^{\mathrm{ct}}) = 3r^2-9r+6, \tag{2.3.8}$$

$$\dim S^2_r(F^{\mathrm{ct}}) = \frac{1}{2}(3r^2+3r+2), \qquad \dim \mathring{S}^2_r(F^{\mathrm{ct}}) = \frac{3}{2}r(r-1). \tag{2.3.9}$$

We will use the following intermediate spaces when developing commuting projections on the Worsey-Farin split.

Lemma 2.3.1. *We define the spaces* $\mathcal{R}^0_r(F^{\rm ct}) := \{v \in S^0_r(F^{\rm ct}) : v|_{\partial F} = 0\}$, $\mathcal{R}^1_r(F^{\rm ct}) := \{v \in S^1_{{\rm div}\,,r}(F^{\rm ct}) : v|_{\partial F} = 0\}$. *Then, we have*

$$\dim \mathcal{R}^0_r(F^{\rm ct}) = \frac{3}{2}(r-1)(r-2), \qquad \dim \mathcal{R}^1_r(F^{\rm ct}) = 3(r-1)^2.$$

Proof. Let $v \in \mathcal{R}^k_r(F^{\rm ct})$. To calculate the dimension of $\mathcal{R}^k_r(F^{\rm ct})$, we must count the number of constraints imposed by setting $v|_{\partial F} = 0$.

(i) *Case* $k = 0$. In this case, v is a scalar function, so on each (external) edge $e \in \Delta_1(F^{\rm ct})$, setting $v|_e = 0$ requires $r+1$ constraints. As v is continuous, the values of v at the vertices $\Delta_0(F^{\rm ct})$ need only be counted once. Hence the number of constraints imposed is $3(r+1) - 3 = 3r$. Therefore $\dim \mathcal{R}^0_r(F^{\rm ct}) = \dim S^0_r(F^{\rm ct}) - 3r = \frac{3}{2}(r^2 - 3r + 2) = \frac{3}{2}(r-1)(r-2)$.

(ii) *Case* $k = 1$. Now v is a two-dimensional vector function, hence the argument above must be applied in both components of v. Therefore setting $v|_{\partial F} = 0$ imposes $6r$ constraints, so $\dim \mathcal{R}^1_r(F^{\rm ct}) = \dim S^1_{{\rm div}\,,r}(F^{\rm ct}) - 6r = 3r^2 + 3 - 6r = 3(r-1)^2$. $\square$

Now we are ready to state the results of [33] on the Clough-Tocher split.

Theorem 2.3.2. *Let* $r \geq 3$. *The following sequences are exact [33].*

$$\mathbb{R} \longrightarrow L^0_r(F^{\rm ct}) \xrightarrow{\rm grad} V^1_{{\rm curl}\,,r-1}(F^{\rm ct}) \xrightarrow{\rm curl} V^2_{r-2}(F^{\rm ct}) \longrightarrow 0, \tag{2.3.10a}$$

$$\mathbb{R} \longrightarrow S^0_r(F^{\rm ct}) \xrightarrow{\rm grad} L^1_{r-1}(F^{\rm ct}) \xrightarrow{\rm curl} V^2_{r-2}(F^{\rm ct}) \longrightarrow 0, \tag{2.3.10b}$$

$$\mathbb{R} \longrightarrow S^0_r(F^{\rm ct}) \xrightarrow{\rm grad} S^1_{r-1}(F^{\rm ct}) \xrightarrow{\rm curl} L^2_{r-2}(F^{\rm ct}) \longrightarrow 0, \tag{2.3.10c}$$

$$0 \longrightarrow \mathring{L}^0_r(F^{\rm ct}) \xrightarrow{\rm grad} \mathring{V}^1_{{\rm curl}\,,r-1}(F^{\rm ct}) \xrightarrow{\rm curl} \mathring{V}^2_{r-2}(F^{\rm ct}) \longrightarrow 0, \tag{2.3.10d}$$

$$0 \longrightarrow \mathring{S}_r^0(F^{\mathrm{ct}}) \xrightarrow{\mathrm{grad}} \mathring{L}_{r-1}^1(F^{\mathrm{ct}}) \xrightarrow{\mathrm{curl}} \mathring{V}_{r-2}^2(F^{\mathrm{ct}}) \longrightarrow 0, \tag{2.3.10e}$$

$$0 \longrightarrow \mathring{S}_r^0(F^{\mathrm{ct}}) \xrightarrow{\mathrm{grad}} \mathring{S}_{r-1}^1(F^{\mathrm{ct}}) \xrightarrow{\mathrm{curl}} \mathring{L}_{r-2}^2(F^{\mathrm{ct}}) \longrightarrow 0. \tag{2.3.10f}$$

Theorems 2.3.2 has an alternate form that follows from a rotation of the coordinate axes, where the operators grad and curl are replaced by rot and div, respectively.

Corollary 2.3.3. *Let $r \geq 3$. The following sequences are exact [33].*

$$\mathbb{R} \longrightarrow L_r^0(F^{\mathrm{ct}}) \xrightarrow{\mathrm{rot}} V_{\mathrm{div},r-1}^1(F^{\mathrm{ct}}) \xrightarrow{\mathrm{div}} V_{r-2}^2(F^{\mathrm{ct}}) \longrightarrow 0, \tag{2.3.11a}$$

$$\mathbb{R} \longrightarrow S_r^0(F^{\mathrm{ct}}) \xrightarrow{\mathrm{rot}} L_{r-1}^1(F^{\mathrm{ct}}) \xrightarrow{\mathrm{div}} V_{r-2}^2(F^{\mathrm{ct}}) \longrightarrow 0, \tag{2.3.11b}$$

$$\mathbb{R} \longrightarrow S_r^0(F^{\mathrm{ct}}) \xrightarrow{\mathrm{rot}} S_{r-1}^1(F^{\mathrm{ct}}) \xrightarrow{\mathrm{div}} L_{r-2}^2(F^{\mathrm{ct}}) \longrightarrow 0, \tag{2.3.11c}$$

$$0 \longrightarrow \mathring{L}_r^0(F^{\mathrm{ct}}) \xrightarrow{\mathrm{rot}} \mathring{V}_{\mathrm{div},r-1}^1(F^{\mathrm{ct}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-2}^2(F^{\mathrm{ct}}) \longrightarrow 0, \tag{2.3.11d}$$

$$0 \longrightarrow \mathring{S}_r^0(F^{\mathrm{ct}}) \xrightarrow{\mathrm{rot}} \mathring{L}_{r-1}^1(F^{\mathrm{ct}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-2}^2(F^{\mathrm{ct}}) \longrightarrow 0, \tag{2.3.11e}$$

$$0 \longrightarrow \mathring{S}_r^0(F^{\mathrm{ct}}) \xrightarrow{\mathrm{rot}} \mathring{S}_{r-1}^1(F^{\mathrm{ct}}) \xrightarrow{\mathrm{div}} \mathring{L}_{r-2}^2(F^{\mathrm{ct}}) \longrightarrow 0. \tag{2.3.11f}$$

2.4 The Alfeld split

Next, we describe the Alfeld split. Let T be a tetrahedron with vertices $\{x_1, \ldots, x_4\}$, and let z_0 be an interior point of T. The Alfeld split $T^{\mathrm{a}} = \langle x_1, \ldots, x_4, z_0 \rangle$ of T is constructed by connecting each vertex x_i with the interior point z_0 by an edge, resulting in a triangulation with 4 tetrahedra, 10 edges, and 5 vertices. The Alfeld refinement $\mathcal{T}_h^a$ of a general triangulation is achieved by constructing an Alfeld split on each triangle $T \in \mathcal{T}_h$, and such a refinement is always a well-defined triangulation as long as each split point is chosen to be strictly interior to the original simplex.

Fu, Guzmán, and Neilan extended the results of Lemma 2.1.6 to smoother sequences on the Alfeld split, as detailed in the following theorem.

Theorem 2.4.1. *The following sequences are exact.*

$$\mathbb{R} \longrightarrow L_r^0(T^{\mathrm{a}}) \xrightarrow{\text{grad}} V_{r-1}^1(T^{\mathrm{a}}) \xrightarrow{\text{curl}} V_{r-2}^2(T^{\mathrm{a}}) \xrightarrow{\text{div}} V_{r-3}^3(T^{\mathrm{a}}) \longrightarrow 0, \tag{2.4.1a}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{a}}) \xrightarrow{\text{grad}} L_{r-1}^1(T^{\mathrm{a}}) \xrightarrow{\text{curl}} V_{r-2}^2(T^{\mathrm{a}}) \xrightarrow{\text{div}} V_{r-3}^3(T^{\mathrm{a}}) \longrightarrow 0, \tag{2.4.1b}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{a}}) \xrightarrow{\text{grad}} S_{r-1}^1(T^{\mathrm{a}}) \xrightarrow{\text{curl}} L_{r-2}^2(T^{\mathrm{a}}) \xrightarrow{\text{div}} V_{r-3}^3(T^{\mathrm{a}}) \longrightarrow 0, \tag{2.4.1c}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{a}}) \xrightarrow{\text{grad}} S_{r-1}^1(T^{\mathrm{a}}) \xrightarrow{\text{curl}} S_{r-2}^2(T^{\mathrm{a}}) \xrightarrow{\text{div}} L_{r-3}^3(T^{\mathrm{a}}) \longrightarrow 0. \tag{2.4.1d}$$

Furthermore, the following sequences with boundary conditions are exact.

$$0 \longrightarrow \mathring{L}_r^0(T^{\mathrm{a}}) \xrightarrow{\text{grad}} \mathring{V}_{r-1}^1(T^{\mathrm{a}}) \xrightarrow{\text{curl}} \mathring{V}_{r-2}^2(T^{\mathrm{a}}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{\mathrm{a}}) \longrightarrow 0, \tag{2.4.2a}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{a}}) \xrightarrow{\text{grad}} \mathring{L}_{r-1}^1(T^{\mathrm{a}}) \xrightarrow{\text{curl}} \mathring{V}_{r-2}^2(T^{\mathrm{a}}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{\mathrm{a}}) \longrightarrow 0, \tag{2.4.2b}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{a}}) \xrightarrow{\text{grad}} \mathring{S}_{r-1}^1(T^{\mathrm{a}}) \xrightarrow{\text{curl}} \mathring{L}_{r-2}^2(T^{\mathrm{a}}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{\mathrm{a}}) \longrightarrow 0, \tag{2.4.2c}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{a}}) \xrightarrow{\text{grad}} \mathring{S}_{r-1}^1(T^{\mathrm{a}}) \xrightarrow{\text{curl}} \mathring{S}_{r-2}^2(T^{\mathrm{a}}) \xrightarrow{\text{div}} \mathring{L}_{r-3}^3(T^{\mathrm{a}}) \longrightarrow 0. \tag{2.4.2d}$$

It is important to note that the exactness of these sequences is proved on a single macroelement, and that constructing global spaces (i.e., finite element spaces on the entire triangulation) with the desired exactness and smoothness properties requires particular attention. The approach to constructing such global spaces begins with the degrees of freedom for the local spaces, where the idea is to show that a function p defined on $\mathcal{T}_h$ by the degrees of freedom $\Sigma(T)$ for each $T \in \mathcal{T}_h$ has the desired properties on all of $\mathcal{T}_h$. For example, in the case of the Lagrange finite element spaces, one must show that the local degrees of freedom on each T induce a global function that is continuous on all of $\mathcal{T}_h$.

In order to develop commuting projections for their sequences on the Alfeld split, Fu, Guzmán, and Neilan found that they needed to consider finite element spaces with some extra smoothness at the vertices. In particular, they considered subspaces of $S^k_{r-k}(T^{\mathrm{a}})$, $L^k_{r-k}(T^{\mathrm{a}})$, and $V^k_{r-k}(T^{\mathrm{a}})$ that have C^{2-k} continuity on the vertices of T in the cases $0 \leq k \leq 2$, and degrees of freedom associated with these derivatives at the vertices were included. It turns out that functions in $S^0_r(T^{\mathrm{a}})$ are intrinsically C^2 at the vertices of T. Fu, Guzmán, and Neilan introduced the following spaces with added continuity at the vertices.

$$
\begin{aligned}
L^1_{c,r}(T^{\mathrm{a}}) &= \{v \in L^1_r(T^{\mathrm{a}}) : v \text{ is } C^1 \text{ on } \Delta_0(T)\},\\
V^2_{c,r}(T^{\mathrm{a}}) &= \{v \in V^2_r(T^{\mathrm{a}}) : v \text{ is } C^0 \text{ on } \Delta_0(T)\},\\
\mathring{L}^1_{c,r}(T^{\mathrm{a}}) &= L^1_{c,r}(T^{\mathrm{a}}) \cap \mathring{L}^1_r(T^{\mathrm{a}}),\\
\mathring{V}^2_{c,r}(T^{\mathrm{a}}) &= V^2_{c,r}(T^{\mathrm{a}}) \cap \mathring{V}^2_r(T^{\mathrm{a}}).
\end{aligned}
$$

Next, they proved that the following sequence that includes these modified spaces is exact.

$$
\mathbb{R} \longrightarrow \mathring{S}^0_r(T^{\mathrm{a}}) \xrightarrow{\text{grad}} \mathring{L}^1_{c,r-1}(T^{\mathrm{a}}) \xrightarrow{\text{curl}} \mathring{V}^2_{c,r-2}(T^{\mathrm{a}}) \xrightarrow{\text{div}} \mathring{V}^3_{r-3}(T^{\mathrm{a}}) \longrightarrow 0. \quad (2.4.3a)
$$

For all other sequences, any additional smoothness required for their degrees of freedom turned out to be inherent properties of the component spaces due to supersmoothness. Fu, Guzmán, and Neilan went on to develop degrees of freedom on the three-dimensional Alfeld split that induce global spaces that form exact sequences analogous with the local sequences presented above. Since the exactness of the sequences proved in [33] applies more generally to these spaces in any dimension, we first aimed to construct similarly general commuting projections that would yield exact global sequences for any dimension.

To this end, we considered the space $S^0_r(T^{\mathrm{a}})$ on the Alfeld split in four dimensions, with the goal of extending the commuting projections of [33] $\mathbb{R}^4$ first. In the course of devising a set of degrees of freedom for $S_r^0(T^{\mathrm{a}})$, we discovered that up to eighth-order derivatives on the vertices of T would be needed, many of which would be multi-valued. These constraints are arduous and, in a sense, unnatural, so this study led us to conclude that it would be more practical to consider other types of splits. We present the details of this study here.

Let the four-dimensional simplex T have five vertices $\{x_i\}$ with $1 \leq i \leq 5$, and let T^{a} represent the Alfeld split of T, which is formed by adding an interior point z_0 that is connected to the five vertices of T via five interior edges. Then T^{a} has five 4-dimensional facets, which we call 4-simplices, as well as 10 tetrahedra, 10 triangles, and 5 edges interior to T^{a}. We label these five 4-simplices $Q_i = \langle x_1, \ldots, \hat{x}_i, \ldots, x_6 \rangle$, where the notation $\hat{x}_i$ means that vertex x_i is not in the set, and $\langle \cdot \rangle$ represents the convex hull. Then any vertex x_i of T is contained in four of these 4-simplices. Recall that $\Delta_s(T)$ denote the set of facets of T of dimension s. The critical idea of the following discussion is that along an interface between two facets, a derivative of a continuous function in the direction tangent to the interface is continuous along that interface.

A special property of the Alfeld refinement of a simplex is that of *supersmoothness*, which is when a piecewise polynomial with a prescribed smoothness inherits additional regularity due to the geometry of the triangulation. We will make use of the result that a C^1 polynomial on a three-dimensional Alfeld split is C^2 at the vertices and C^3 at the split point. This result was proven in Alfeld's 1984 paper [2] and, using a different approach, in the 2010 paper of Sorokina [57].

We will use the three-dimensional supersmoothness properties to show that a piecewise C^1 polynomial on a four-dimensional Alfeld split must be C^3 at the vertices. Let q be a C^1 piecewise polynomial on the four-dimensional Alfeld split. Consider a vertex of

T, say x_1, and let y_0 be an arbitrary point on the interior edge e_{16}. Let P_{y_0} be a hyperplane that contains y_0 but does contain the edge e_{16}. Then the intersection of the hyperplane P_{y_0} with T is a three-dimensional simplex that has y_0 as an interior vertex, and the boundary vertices occur at the intersections:

$$y_1 = P_{y_0} \cap \langle x_1, x_2, x_6 \rangle, \qquad y_2 = P_{y_0} \cap \langle x_1, x_3, x_6 \rangle,$$
$$y_3 = P_{y_0} \cap \langle x_1, x_4, x_6 \rangle, \qquad y_4 = P_{y_0} \cap \langle x_1, x_5, x_6 \rangle.$$

Then the points y_1, y_2, y_3, y_4 are vertices of the resulting three-dimensional triangulation, which is an Alfed split, and we denote it by T^y. Since q is a C^1 function on T, it is C^1 on T^y. Then by Sorokina's and Alfeld's supersmoothness result mentioned above, it follows that q is C^3 on y_0. Since y_0 and P_{y_0} are chosen arbitrarily, it follows that q is C^3 at any point along the interior edge e_{16}. In particular, q is C^3 at the vertex x_1. Hence $q \in C^3(S)$ for any $S \in \Delta_0(T)$.

Now, we will show how many 4th-order derivatives of q are continuous at x_i. Since some of the 4th-order derivatives of q may not be continuous at each vertex $x_i \in \Delta_0(T)$, we will also determine how many values the fourth derivatives of q may take at each x_i.

Suppose that t_{i6} is the unit vector tangent to edge $e_{i6} = \langle x_i, z_0 \rangle$, where $1 \le i \le 5$, and let t_{i6} be oriented such that it points away from the interior point z_0. Since T is non-degenerate, it follows that any 4 of these vectors t_{i6} will form a spanning set of $\mathbb{R}^4$. Hence with an abuse of notation, we can write each partial derivative of q with respect to a selection of four of these edges.

The vertex x_1 lies in the intersection of $Q_2 \cap Q_3 \cap Q_4 \cap Q_5$. A fourth-order derivative of q is continuous at x_1 if it is continuous across all interfaces between these four Q_i's at x_1. Notice that interior edge $e_{16} = Q_2 \cap Q_3 \cap Q_4 \cap Q_5$, which means that t_{16}, the unit vector tangent to e_{16}, is tangent to all four $\{Q_i\}_{i=2}^4$, hence it is tangent to all interfaces between them. So any order derivative of q with respect to e_{16} is continuous at x_1, such as $\partial^4 q/\partial e_{16}^4$. We will choose the four directions tangent to the edge set $E_1 = \{e_{16}, e_{26}, e_{36}, e_{46}\}$ as a basis to represent derivatives of q in the following discussion. It is convenient to choose e_{16} to belong to this set while considering smoothness at x_1, but the other three edges in E_1 may be chosen arbitrarily from the remaining four interior edges.

Now we will prove that 32 fourth-order partial derivatives of q are continuous at x_1. This leaves 3 fourth-order derivatives to be multi-valued at x_1, since the number of fourth-order partial derivatives in $\mathbb{R}^4$ is $1+3\binom{4}{3}+\binom{4}{2}+4\binom{4}{1}=35$. First, consider derivatives of the form

$$\frac{\partial}{\partial e_{16}} \frac{\partial^3 q}{\partial e_{26}^i \partial e_{36}^j \partial e_{46}^k}, \quad i+j+k=3. \tag{2.4.4}$$

There are ten of these derivatives. Since $q \in C^3(x_1)$, $\frac{\partial^3 q}{\partial e_{26}^i \partial e_{36}^j \partial e_{46}^k}$ is continuous at x_1. Since t_{16} is tangent to every 4-simplex,

$$\frac{\partial}{\partial e_{16}} \frac{\partial^3 q}{\partial e_{26}^i \partial e_{36}^j \partial e_{46}^k}$$

is a tangential derivative of a continuous function at x_1, so (2.4.4) is continuous at x_1. The same logic yields that the 6 derivatives of the form

$$\frac{\partial^2}{\partial e_{16}^2} \frac{\partial^2 q}{\partial e_{26}^i \partial e_{36}^j \partial e_{46}^k}, \quad i+j+k=2$$

are continuous at x_1, and the 3 derivatives of the form

$$\frac{\partial^3}{\partial e_{16}^3} \frac{\partial^3 q}{\partial e_{26}^i \partial e_{36}^j \partial e_{46}^k}, \quad i+j+k=1$$

are continuous at x_1. So far, we have identified 20 fourth-order partial derivatives of q that are continuous at x_1, leaving 15 total four-order derivatives remaining. Now, consider the 15 derivatives of the form

$$\frac{\partial^4 q}{\partial e_{26}^i \partial e_{36}^j \partial e_{46}^k}, \quad i+j+k=4. \tag{2.4.5}$$

Case 1. Suppose $i, j, k < 4$, i.e., we exclude the derivatives of the form $\partial^4 q/\partial e_{\ell 6}^4$ with $\ell = 2, 3, 4$. Without loss of generality, suppose $i, j > 0$. Using the fact that q is C^3 at x_1,

$$\frac{\partial^3 q}{\partial e_{26}^{i-1} \partial e_{36}^j \partial e_{46}^k}$$

is continuous at x_1. The vector t_{26} is tangent to the 4-simplices Q_3, Q_4, and Q_5, hence it is tangent to the interfaces at the pairwise intersections of these three 4-simplices. So

$$\frac{\partial}{\partial e_{26}} \frac{\partial^3 q}{\partial e_{26}^{i-1} \partial e_{36}^j \partial e_{46}^k}\bigg|_{Q_3}(x_1) = \frac{\partial}{\partial e_{26}} \frac{\partial^3 q}{\partial e_{26}^{i-1} \partial e_{36}^j \partial e_{46}^k}\bigg|_{Q_4}(x_1) = \frac{\partial}{\partial e_{26}} \frac{\partial^3 q}{\partial e_{26}^{i-1} \partial e_{36}^j \partial e_{46}^k}\bigg|_{Q_5}(x_1).$$

Following similar logic,

$$\frac{\partial^3 q}{\partial e_{26}^{i} \partial e_{36}^{j-1} \partial e_{46}^k}$$

is continuous at x_1, and the partial derivative $\partial/\partial e_{36}$ is tangential to Q_2, Q_4, and Q_5, hence it is tangential to their pairwise intersections. This yields

$$\frac{\partial}{\partial e_{36}} \frac{\partial^3 q}{\partial e_{26}^{i} \partial e_{36}^{j-1} \partial e_{46}^k}\bigg|_{Q_2}(x_1) = \frac{\partial}{\partial e_{36}} \frac{\partial^3 q}{\partial e_{26}^{i} \partial e_{36}^{j-1} \partial e_{46}^k}\bigg|_{Q_4}(x_1) = \frac{\partial}{\partial e_{36}} \frac{\partial^3 q}{\partial e_{26}^{i} \partial e_{36}^{j-1} \partial e_{46}^k}\bigg|_{Q_5}(x_1), \tag{2.4.6}$$

and we know from the previous argument and by commuting the partials that the second and third values in (2.4.6) are equal to

$$\frac{\partial}{\partial e_{36}} \frac{\partial^3 q}{\partial e_{26}^{i} \partial e_{36}^{j-1} \partial e_{46}^k}\bigg|_{Q_3}(x_1).$$

Therefore,

$$\left.\frac{\partial^4 q}{\partial e_{26}^i \partial e_{36}^j \partial e_{46}^k}\right|_{Q_2}(x_1) = \left.\frac{\partial^4 q}{\partial e_{26}^i \partial e_{36}^j \partial e_{46}^k}\right|_{Q_3}(x_1),$$

so this derivative is continuous at x_1. The same logic holds in the cases $i, k > 0$ and $j, k > 0$, so any derivative of the form (2.4.5) where $i, j, k < 4$ is continuous at x_1. In other words, derivatives that do not have any partials with respect to edge e_{16} are continuous as long as partials with respect to at least two different edges appear in the derivative.

Case 2. Without loss of generality, suppose $i = 4$, so $j = k = 0$. Then $\partial^4 q/\partial e_{26}^4$ is tangential to 4-simplices Q_3, Q_4, and Q_5. Therefore, the derivative is continuous across the interfaces $Q_3 \cap Q_4$, $Q_3 \cap Q_5$, and $Q_4 \cap Q_5$, and

$$\left.\frac{\partial^4 q}{\partial e_{26}^4}\right|_{Q_3}(x_1) = \left.\frac{\partial^4 q}{\partial e_{26}^4}\right|_{Q_4}(x_1) = \left.\frac{\partial^4 q}{\partial e_{26}^4}\right|_{Q_5}(x_1).$$

However, e_{26} is not tangential to Q_2, so it is not necessarily continuous across the interfaces $Q_2 \cap Q_3$, $Q_2 \cap Q_4$, and $Q_2 \cap Q_5$. Hence

$$\left.\frac{\partial^4 q}{\partial e_{26}^4}\right|_{Q_2}(x_1)$$

may take on a different value at x_1 than on the other 4-simplices, so this derivative has 2 values at x_1. The same argument holds for the cases $j = 4$ and $k = 4$, yielding three 4th-order partial derivatives of q that take two values at x_1.

The same argument holds on the other vertices, where the important edge direction e_{16} is replaced by e_{i6} in the proof above for each vertex x_i. Hence in applying degrees of freedom $D^4 q(x_i)$ requires 35 + 3 constraints on each of the five vertices of $\Delta_0(T)$.

Let $\mathcal{P}_r^{c1}(T^{\mathrm{a}})$ represent the space of C^1 piecewise polynomials on T^{a}, and let $\mathring{\mathcal{P}}_r^{c1}(T^{\mathrm{a}})$ represent the space of piecewise polynomials in $\mathcal{P}_r^{c1}(T^{\mathrm{a}})$ that are equal to zero on ∂T.

The number of exterior DOFs should be

$$\dim \mathcal{P}_r^{c1}(T^{\mathrm{a}}) - \dim \mathring{\mathcal{P}}_r^{c1}(T^{\mathrm{a}}) = \frac{5}{3}r^3 - \frac{15}{2}r^2 + \frac{155}{6}r - 20.$$

For a facet $S \in \Delta_d(T^{\mathrm{a}})$, $0 \leq d \leq 4$, let $\hat{\mathcal{P}}_r(S)$ represent the space of polynomials that vanish at the vertices of the simplex S. For $S \in \Delta_d(T)$, let $b_S \in \mathcal{P}_{d+1}(S)$ denote the corresponding bubble function. There holds $D^\alpha b_S|_F = 0$ for all $F \in \Delta_m(S)$ and $|\alpha| \leq d - m - 1$ for $m = 0, 1, \ldots, d-1$. For example, consider the case $d = 3$. Then we have that the bubble function $b_S \in \mathcal{P}_4(S)$ with $S \in \Delta_3(T)$ satisfy

$$\operatorname{grad} b_S|_F = 0 \;\forall F \in \Delta_1(S), \qquad D^2 b_S|_F = 0 \;\forall F \in \Delta_0(S).$$

Furthermore, we have that $D^3 b_S|_F \neq 0$ for $F \in \Delta_0(S)$. Now we can state the degrees of freedom and their dimension for the exterior facets of T in the following theorem.

Theorem 2.4.2. *Let $r \geq 9$. A function $v \in S_r^0(T^{\mathrm{a}})$ may be uniquely determined on ∂T by the following degrees of freedom.*

$$
\begin{array}{llrr}
 & & \textit{No. of DOFs} & \\
D^\alpha v(S), & S \in \Delta_0(T),\ |\alpha| \leq 4 & 365, & \text{(2.4.7a)} \\
\displaystyle\int_S v\kappa\, ds, & S \in \Delta_1(T),\ \kappa \in \mathcal{P}_{r-10}(S), & 10(r-9), & \text{(2.4.7b)} \\
\displaystyle\int_S \frac{\partial v}{\partial n_i}\kappa\, ds, & S \in \Delta_1(T),\ \kappa \in \mathcal{P}_{r-9}(S), & 30(r-8), & \text{(2.4.7c)} \\
\displaystyle\int_S \frac{\partial^2 v}{\partial n_i \partial n_j}\kappa\, ds, & S \in \Delta_1(T),\ \kappa \in \mathcal{P}_{r-8}(S), & 60(r-7), & \text{(2.4.7d)} \\
\displaystyle\int_S v\kappa\, dA, & S \in \Delta_2(T),\ \kappa \in \mathcal{P}_{r-9}(S), & 5(r-7)(r-8), & \text{(2.4.7e)} \\
\displaystyle\int_S \frac{\partial v}{\partial n_i}\kappa\, dA, & S \in \Delta_2(T),\ \kappa \in \mathcal{P}_{r-7}(S), & 10(r-5)(r-6), & \text{(2.4.7f)} \\
\displaystyle\int_S v\kappa\, dx, & S \in \Delta_3(T),\ \kappa \in \mathcal{P}_{r-8}(S), & \frac{5}{6}(r-5)(r-6)(r-7), & \text{(2.4.7g)}
\end{array}
$$

$$\int_S \frac{\partial v}{\partial n_S}\kappa\,dx, \qquad S\in\Delta_3(T),\ \kappa\in\hat{\mathcal{P}}_{r-5}(S), \quad \frac{5}{6}(r-2)(r-3)(r-4)-20. \quad \text{(2.4.7h)}$$

Proof. The number of DOFs given above is $\frac{5}{3}r^3-\frac{15}{2}r^2+\frac{155}{6}r-20$, which is exactly the desired amount.

The conditions (2.4.7a) - (2.4.7d) yield $D^\alpha v|_S=0$ for all $S\in\Delta_1(T)$ and $|\alpha|\le 2$. Thus, on $S\in\Delta_2(T)$, we have $v=b_S^3 q$ and $\partial v/\partial n_S^{(i)}=b_S^2 p_i$ for some $q\in\mathcal{P}_{r-9}(S)$ and $p_i\in\mathcal{P}_{r-7}(S)$. Thus the conditions (2.4.7e) - (2.4.7f) imply $v=0$ and $\operatorname{grad} v=0$ on all $S\in\Delta_2(T)$.

Now let $S\in\Delta_3(T)$. Since $v|_{\partial S}=0$ and $\operatorname{grad} v|_{\partial S}=0$, we have $v=b_S^2 q$ and $\partial v/\partial n_S=b_S p$ with $b_S\in\mathcal{P}_4(S)$, $q\in\mathcal{P}_{r-8}(S)$, and $p\in\mathcal{P}_{r-5}(S)$. The DOFs (2.4.7g) then imply that $v|_S=0$.

Note that $D^\alpha(\partial n/\partial n_S)=D^\alpha(b_S p)=0$ on $\Delta_0(S)$ and $|\alpha|\le 3$. But on $\Delta_0(S)$, we have $D^3(b_S p)=D^3 b_S p$ because $D^2 b_S=0$ and $\operatorname{grad} b_S=0$ on $\Delta_0(S)$. Since $D^3 b_S\neq 0$, on $\Delta_0(S)$, we must have that p vanishes on $\Delta_0(S)$, i.e., $p\in\hat{\mathcal{P}}_{r-5}(S)$. Thus, the DOFs (2.4.7h) imply $\partial v/\partial n_S=0$ for all $S\in\Delta_3(T)$. Therefore $v|_{\partial T}=0$ and $\operatorname{grad} v|_{\partial T}=0$, which is the desired result. $\square$

To achieve the full unisolvent set of degrees of freedom for $S_r^0(T^{\mathrm{a}})$, one needs to add the DOFs $\int_T \operatorname{grad}(v)\cdot p\,dx$ for all $p\in\operatorname{grad}\mathring{S}_r^0(T^{\mathrm{a}})$, where $\mathring{S}_r^0(T^{\mathrm{a}})=\{v\in S_r^0(T^{\mathrm{a}}):v|_{\partial T}=0 \text{ and } \operatorname{grad} v|_{\partial T}=0\}$.

In summary, we have seen that fourth-order derivative data is sufficient on the vertices for the degrees of freedom, although the supersmoothness of C^1 polynomials on Alfeld splits only yields smooth third-order derivatives. Hence the fourth-order derivatives are multi-valued, further complicating the degrees of freedom. Since the degrees of freedom on the four-dimensional Alfeld split seem to be arduous to the point of being impractical, we wanted to see whether we could devise degrees of freedom for the smoother spaces that would only rely on the smoothness inherent in the problem. This goal led us to consider other types of splits with more facets: the Powell-Sabin split in two dimensions and its three-dimensional analogue, the Worsey-Farin split. These splits and their relevant properties are discussed in the next sections.

2.5 The Powell-Sabin split

Let us describe the Powell-Sabin split. Let T be a triangle with vertices $z_1, z_2,$ and $z_3,$ labeled counter–clockwise, and let z_0 be an interior point of T. Denote the edges of T by $\{e_i\}_{i=1}^3$, labeled such that z_i is not a vertex of e_i, i.e., $e_i = [z_{i+1}, z_{i+2}]$. We denote the outward unit normal of ∂T restricted to e_i as n_i and the tangent vector by t_i. Let z_{3+i} be an interior point of edge e_i. We then construct the triangulation $T^{\mathrm{ps}} = \{T_1, \ldots, T_6\}$ by connecting each z_i to z_0 for $1 \le i \le 6$; see Figure 2.2. We let $\mathcal{E}^b(T^{\mathrm{ps}})$ be the set containing the six boundary edges of T^{ps}. We also let $\mathcal{M}(T^{\mathrm{ps}}) = \{z_4, z_5, z_6\}$ and use the notation for $z \in \mathcal{M}(T^{\mathrm{ps}})$, $\mathcal{T}(z) = \{K_1, K_2\}$, where each $K_i \in T^{\mathrm{ps}}$ have z as a vertex. We also set $T(z) = K_1 \cup K_2$. Let $z \in \mathcal{M}(T^{\mathrm{ps}})$, then we define the jump as follows

$$[\![p]\!](z) = p_1(z)m_1 + p_2(z)m_2,$$

where $p_i = p|_{K_i}$ and m_i is the outward pointing normal to K_i perpendicular to e. We see then that $[\![p]\!](z) = (p_1(z) - p_2(z))m_1 = -(p_1(z) - p_2(z))m_2$.

Let μ be the unique piecewise linear function on the mesh T^{ps} such that $\mu(z_0) = 1$ and

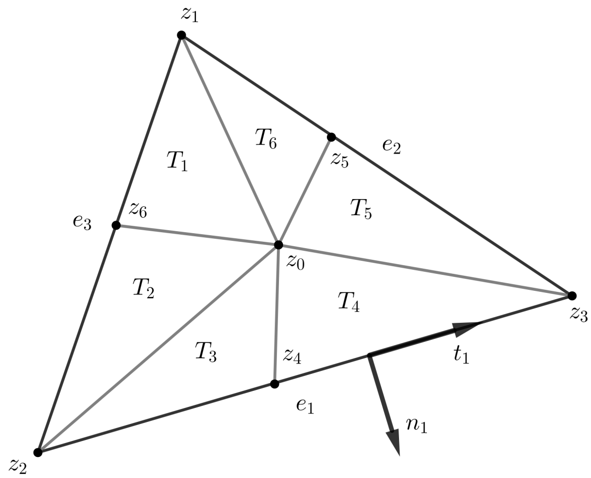

Figure 2.2: A pictorial description of a Powell-Sabin split of a triangle.

$\mu = 0$ on ∂T. We use the notation $\nabla \mu_i := \nabla \mu|_{e_i} = \nabla \mu|_{T(z_{3+i})}$ and note that

$$\frac{1}{|\nabla \mu_i|} \nabla \mu_i = -n_i \quad (i = 1, 2, 3), \tag{2.5.1}$$

and hence

$$\nabla \mu_i \cdot t_i = 0 \quad (i = 1, 2, 3). \tag{2.5.2}$$

One main result of Chapter 4 is to show that sequences with these smoother component spaces are exact. An integral component of this result is a characterization of the range of the divergence operator acting on the (vector-valued) Lagrange space. For example, it is known [56, Proposition 2.1] that if $v \in \mathring{L}^1_r(T^{\mathrm{ps}})$ then $\operatorname{div} v$ is continuous at the vertices z_4, z_5, z_6. In particular, this is because each of these vertices is a *singular vertex*, i.e., the edges meeting at the vertex fall on exactly two straight lines. Hence, in order to extend Lemma 2.1.6 and to characterize the range of $\operatorname{div} \mathring{L}^1_r(T^{\mathrm{ps}})$, we will consider the spaces

$$\begin{aligned} \mathcal{V}^2_r(T^{\mathrm{ps}}) &= \{q \in V^2_r(T^{\mathrm{ps}}) : q \text{ is continuous at } z_4, z_5, z_6\}, \\ \mathring{\mathcal{V}}^2_r(T^{\mathrm{ps}}) &= \mathcal{V}^2_r(T^{\mathrm{ps}}) \cap \mathring{L}^2(T). \end{aligned} \tag{2.5.3}$$

Now we can write the sequences for which we will prove exactness in Chapter 4.

$$\mathbb{R} \longrightarrow L_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} V_{\mathrm{div}\,,r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} V_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0, \tag{2.5.4a}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} L_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} V_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0, \tag{2.5.4b}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} S_{\mathrm{div}\,,r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} L_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0, \tag{2.5.4c}$$

$$0 \longrightarrow \mathring{L}_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} \mathring{V}_{\mathrm{div}\,,r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0, \tag{2.5.4d}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} \mathring{L}_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0, \tag{2.5.4e}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} \mathring{S}_{\mathrm{div}\,,r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathring{L}_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0. \tag{2.5.4f}$$

2.6 The Worsey-Farin split

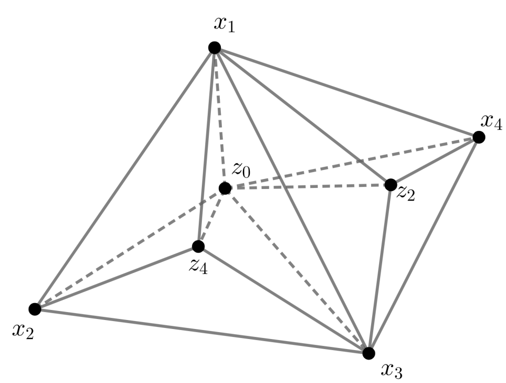

Figure 2.3: Representation of a Worsey-Farin split (with two faces shown).

Here, we describe the Worsey-Farin split. Let T be a tetrahedron with vertices $\{x_1, x_2, x_3, x_4\}$, and let z_0 be an interior point of T. Denote the faces of T by $F_i = \langle x_1, \ldots, \hat{x}_i, \ldots x_4 \rangle$, with $1 \leq i \leq 4$, where the notation $\hat{x}_i$ indicates that x_i is not in F_i. Let z_i be an interior point of face F_i. Then the triangulation T^{wf}, consisting of the tetrahedra $K_i^{\mathrm{wf}} = \langle z_0, z_i, x_1, \ldots, \hat{x}_i, \ldots, x_4 \rangle$, with $1 \leq i \leq 4$, is constructed by connecting each z_i to the vertices of F_i and to the interior point z_0 via an edge. The resulting split T^{wf} has 12 tetrahedra, 30 triangles, 26 edges, and 9 vertices; see Figure 2.3. Recall that

$\Delta_d(S)$ represents the set of d-dimensional facets of a simplicial triangulation S, and we let $\Delta_d^I(S)$ represent the set of d-dimensional facets that are interior to S. We let K_i be the tetrahedron $\langle x_1, x_2, \ldots, \hat{x}_i, \ldots, x_4, z_0\rangle$. Since the triangulation T^{wf} is a refinement of the Alfeld split T^{a} of T using the same interior split point z_0, we can denote the set of tetrahedra $\{K_i\}$ by $\Delta_3(T^{\mathrm{a}})$. The triangulation K_i^{wf} of $K_i \in \Delta_3(T^{\mathrm{a}})$ is constructed by splitting the face of K_i that lies on the boundary of T by a Clough-Tocher split with the split point z_i as in Figure 2.4. The outward unit normal of T on a face $F \in \Delta_2(T)$ is denoted by n_F, and the outward unit normal of a face F on ∂F is denoted $n_{\partial F}$, which is tangent to the plane containing F. Furthermore, each interior edge e of the triangulation F^{ct} of a face $F \in \Delta_2(T)$ is associated with two unit vectors that are both tangent to F, which we write as $[t, s]^\top$, where t is the unit vector tangent to e and s is normal to t; see Figure 2.4. Then $[t, s, n_F]^\top$ forms a basis for $\mathbb{R}^3$.

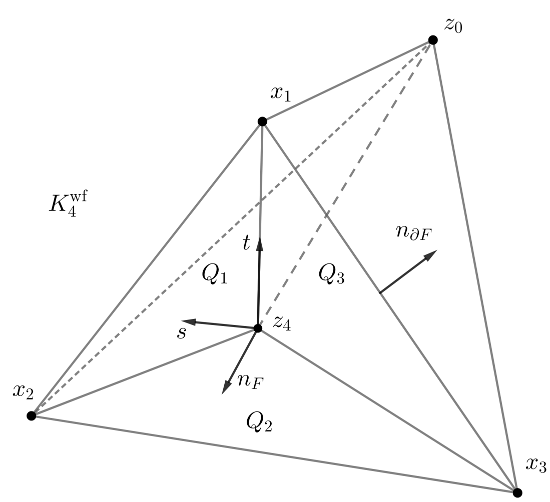

Figure 2.4: Representation of the triangulation K_4^{wf} where $K \in \Delta_3(T^{\mathrm{a}})$.

The Worsey-Farin refinement of a triangulation admits a special structure where any two macroelements attach. Let $\mathcal{T}_h$ be a triangulation of a domain $\Omega \subset \mathbb{R}^3$, and let T_1 and T_2 be adjacent tetrahedra in $\mathcal{T}_h$ that share a face $F = \langle x_1, x_2, x_3\rangle$ as in Figure 2.5. Then the construction of the Worsey-Farin refinement $\mathcal{T}_h^{\mathrm{wf}}$ proceeds by adding the incenters z_0^1 and z_0^2 of T_1 and T_2, respectively, as well as a new edge $\langle z_0^1, z_0^2\rangle$, that intersects the interior

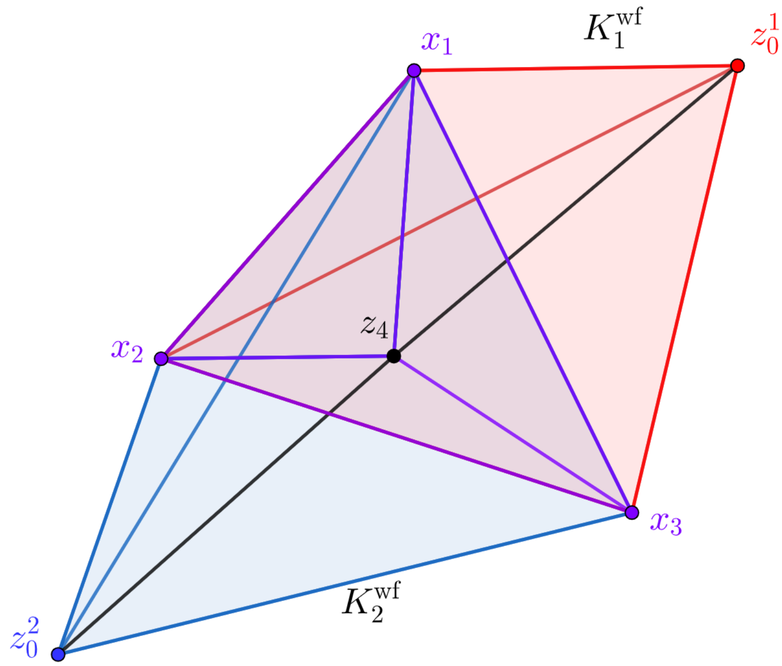

Figure 2.5: The interface F^{ct} of adjacent triangulations K_1^{wf} and K_2^{wf} within a Worsey-Farin refinement.

of F. Such an intersection point always exists when the interior points z_0^1 and z_0^2 are chosen to be the incenters of the tetrahedra T_1 and T_2, respectively [47]. This intersection point is labeled z_4, and three interior edges of F, $\{\langle z_4, x_i\rangle\}_{i=1}^3$, are added. Figure 2.5 is a representation of the resulting triangulation. Furthermore, the tetrahedron $\langle z_0^1, x_1, x_2, x_3\rangle$ is labeled K_1, and $\langle z_0^2, x_1, x_2, x_3\rangle$ is labeled K_2. These tetrahedra are each split into three subtetrahedra in the course of the Worsey-Farin refinement, and the triangulations of K_1 and K_2 are represented by K_1^{wf} and K_2^{wf}.

Since the edges $\langle z_0^1, z_4\rangle$ and $\langle z_0^2, z_4\rangle$ are colinear, the triangles $\langle z_0^1, z_4, x_1\rangle$ and $\langle z_0^2, z_4, x_1\rangle$ are coplanar. Similarly, triangles $\langle z_0^1, z_4, x_2\rangle$ and $\langle z_0^2, z_4, x_2\rangle$ are coplanar. Thus the 3-dimensional facet $\langle z_0^1, x_1, z_0^2, x_2\rangle$ forms a tetrahedron, as shown in Figure 2.6, where the face $\langle z_0^1, x_i, z_0^2\rangle$ is the union of the two coplanar triangles $\langle z_0^1, z_4, x_i\rangle$ and $\langle z_0^2, z_4, x_i\rangle$ for $i = 1, 2$. In the same way, $\langle z_0^1, x_1, z_0^2, x_3\rangle$ and $\langle z_0^1, x_2, z_0^2, x_3\rangle$ form tetrahedra.

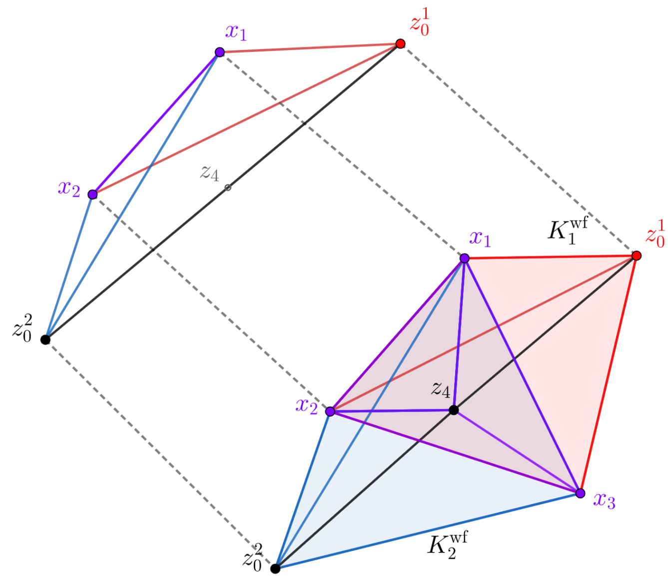

Figure 2.6: Representation of one of three tetrahedra formed between two adjacent Worsey-Farin splits by the colinearity of points $\{z_0^1, z_4, z_0^2\}$.

Remark 2.6.1. *The importance of this structure is that the natural extension of a piecewise polynomial from K_1^{wf} to all of $K_1^{\mathrm{wf}} \cup K_2^{\mathrm{wf}}$ maintains its original smoothness properties across the interior faces of K_2^{wf}, since all the faces of a given subtetrahedron in K_1^{wf} are coplanar to the faces of the adjacent subtetrahedron of K_2^{wf}.*

Next, we discuss the formation of singular edges in the Worsey-Farin refinement of a triangulation.

Definition 2.6.2. *An edge e is a singular edge if the faces of the triangulation that meet at edge e fall on exactly two planes.*

Each of the interior edges of F_i is a singular edge since the interior triangles of each $T^{\mathrm{wf}} \in \mathcal{T}_h^{\mathrm{wf}}$ meeting at these edges are coplanar. A singular edge occurs when each triangle intersecting at that edge lies in one of two planes, i.e., every point along a singular edge is a singular point. This property leads to some additional continuity of functions in the ranges of the Lagrange spaces, $\operatorname{curl} L_r^1(\mathcal{T}_h^{\mathrm{wf}})$ and $\operatorname{div} L_r^2(\mathcal{T}_h^{\mathrm{wf}})$, so we need to define some new spaces that incorporate this additional continuity. Let $\Delta_d(S)$ represent the set of subsimplices of S that have dimension d.

$$\mathcal{V}_r^2(T^{\mathrm{wf}}) = \{v \in V_r^2(T^{\mathrm{wf}}) : v \times n_F|_F \text{ is continuous on each } F \in \Delta_2(T)\}, \tag{2.6.1a}$$

$$\mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}}) = \{v \in \mathcal{V}_r^2(T^{\mathrm{wf}}) : v \cdot n_F|_F = 0 \text{ on each } F \in \Delta_2(T)\}, \tag{2.6.1b}$$

$$\mathcal{V}_r^3(T^{\mathrm{wf}}) = \{q \in V_r^3(T^{\mathrm{wf}}) : q|_F \text{ is continuous on each } F \in \Delta_2(T)\}, \tag{2.6.1c}$$

$$\mathring{\mathcal{V}}_r^3(T^{\mathrm{wf}}) = \{q \in \mathcal{V}_r^3(T^{\mathrm{wf}}) : \int_T q = 0\}, \tag{2.6.1d}$$

where $q|_F$ is the restriction of q to a face F of T. We will prove in Chapter 4 that $\mathrm{div} : \mathring{L}_r^2(T^{\mathrm{wf}}) \to \mathring{\mathcal{V}}_r^3(T^{\mathrm{wf}})$ is surjective, and therefore $\mathrm{div}\, \mathring{L}_r^2(T^{\mathrm{wf}}) = \mathring{\mathcal{V}}_{r-1}^3(T^{\mathrm{wf}})$.

Now we can state the sequences that will be shown to be exact in Chapter 5.

$$\mathbb{R} \longrightarrow L_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} V_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} V_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} V_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{2.6.2a}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} L_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} V_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} V_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{2.6.2b}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} S_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} L_{r-2}^2(F^{\mathrm{ct}}) \xrightarrow{\mathrm{div}} V_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{2.6.2c}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} S_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} S_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} L_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0. \tag{2.6.2d}$$

We will also show that the following sequences with boundary conditions are exact.

$$0 \longrightarrow \mathring{L}_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} \mathring{V}_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} \mathring{V}_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{2.6.3a}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} \mathring{L}_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{2.6.3b}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} \mathring{S}_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} \mathring{L}_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathring{\mathcal{V}}_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{2.6.3c}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} \mathring{S}_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} \mathring{S}_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathring{L}_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0. \tag{2.6.3d}$$

It is useful to have the dimension counts of each of the spaces that make up (2.6.2) and (2.6.3) in developing their degrees of freedom. Here, we state some known dimension counts of the Nédélec and Lagrange spaces on the Worsey-Farin split.

Lemma 2.6.3. *Let $r \geq 0$. The following dimension counts may be found in [6], pp. 86–87.*

$$
\begin{aligned}
\dim V_r^0(T^{\mathrm{wf}}) &= (2r+1)(r^2+r+1), \\
\dim V_{r-1}^1(T^{\mathrm{wf}}) &= 6r^3+2r, \\
\dim V_{r-2}^2(T^{\mathrm{wf}}) &= \max\{3r(2r-1)(r-1), 0\}, \\
\dim V_{r-3}^3(T^{\mathrm{wf}}) &= \max\{2r(r-1)(r-2), 0\}.
\end{aligned}
$$

Lemma 2.6.4. *The following dimension counts may be found in [47] and [6].*

$$
\begin{aligned}
\dim L_r^0(T^{\mathrm{wf}}) &= (2r+1)(r^2+r+1), \\
\dim L_{r-1}^1(T^{\mathrm{wf}}) &= \max\{3(2r-1)(r^2-r+1), 0\}, \\
\dim L_{r-2}^2(T^{\mathrm{wf}}) &= \max\{3(2r-3)(r^2-3r+3), 0\}, \\
\dim L_{r-3}^3(T^{\mathrm{wf}}) &= \max\{(2r-5)(r^2-5r+7), 0\}.
\end{aligned}
$$

We prove new formulae for the dimension of the smooth spaces $S_r^k(T^{\mathrm{wf}})$ in Chapter 5. In the next chapter, we prove the exactness of sequences on the Powell-Sabin split as stated in Section 2.5 and derive commuting projections for these sequences that induce the appropriate global spaces.

CHAPTER THREE

Exact Sequences on Powell-Sabin Splits

3.1 Exact sequences on a macro triangle

The goal of this section is to extend Lemma 2.1.6, the exact sequences formed by Nédélec spaces on general triangulations, to incorporate the smoother spaces defined in Section 2.1. Throughout this chapter, will use the form (2.5.4) of the exact sequences using the two-dimensional rot and div operators, so for ease of notation, we will simply represent $V^1_{\mathrm{div}\,,r}(T^{\mathrm{ps}})$ by $V^1_r(T^{\mathrm{ps}})$ and $S^1_{\mathrm{div}\,,r}(T^{\mathrm{ps}})$ by $S^1_r(T^{\mathrm{ps}})$.

First, we show that $\mathrm{div}\,\mathring{L}^1_r(T^{\mathrm{ps}}) = \ker \mathring{\mathcal{V}}^2_{r-1}(T^{\mathrm{ps}})$, where the spaces $\mathcal{V}^2_{r-1}(T^{\mathrm{ps}})$ and $\mathring{\mathcal{V}}^2_{r-1}(T^{\mathrm{ps}})$ are defined as in (2.5.3). Notice that $\ker \mathring{\mathcal{V}}^2_{r-1}(T^{\mathrm{ps}}) = \mathring{\mathcal{V}}^2_{r-1}(T^{\mathrm{ps}})$, since the entire space is mapped to zero in the sequence (2.5.4e). In the following lemma, we show that the mapping $\mathrm{div}\, : \mathring{L}^1_{r+1}(T^{\mathrm{ps}}) \to \mathring{\mathcal{V}}^2_{r-1}(T^{\mathrm{ps}})$ is injective.

Lemma 3.1.1. *For any* $r \geq 0$, $\mathring{\mathcal{V}}^2_r(T^{\mathrm{ps}}) \subseteq \mathrm{div}\,\mathring{L}^1_{r+1}(T^{\mathrm{ps}})$.

Proof. Let $v \in \mathring{L}^1_{r+1}(T^{\mathrm{ps}})$, so by the Stokes theorem (2.2.3c), we have

$$\int_T \mathrm{div}\, v\, dx = \int_{\partial T} n_i \cdot v\, dx = 0,$$

since $v|_{\partial T} = 0$. Let $e_i \in \Delta_1(T)$, and we can see that $\mathrm{div}\, v$ is continuous along e_i. Letting z_{i+3} be the split point of e_i as in Figure 2.2, and letting the unit vector t_i be tangent to e_i, we have that $\partial_{t_i}(v \cdot t_i) = 0$ along e_i since $v \cdot t_i$ is zero on e_i. Let s_i be the unit vector tangent to edge $\langle z_0, z_{i+3}\rangle$ that intersects edge e_i at z_{i+3}. Then the derivative $\partial_{s_i}(v \cdot s_i)$ is continuous at z_{i+3} because v is continuous along $\langle z_0, z_{i+3}\rangle$. Since t_i and n_i form a basis for $\mathbb{R}^2$, we can write

$$s_i = (s_i \cdot t_i)t_i + (s_i \cdot n_i)n_i,$$

and solving for n_i yields

$$n_i = \frac{s_i - (s_i \cdot t_i)t_i}{s_i \cdot n_i}.$$

It follows that, along e_i,

$$\begin{aligned}
\operatorname{div} v &= t_i \cdot \operatorname{grad}(v \cdot t_i) + n_i \cdot \operatorname{grad}(v \cdot n_i) \\
&= 0 + \frac{s_i}{s_i \cdot n_i} \cdot \operatorname{grad}\left(v \cdot \left(\frac{s_i - (s_i \cdot t_i)t_i}{s_i \cdot n_i}\right)\right) - \\
&\quad \frac{(s_i \cdot t_i)t_i}{s_i \cdot n_i} \cdot \operatorname{grad}\left(v \cdot \left(\frac{s_i - (s_i \cdot t_i)t_i}{s_i \cdot n_i}\right)\right) \\
&= \frac{s_i}{(s_i \cdot n_i)^2} \cdot \operatorname{grad}(v \cdot s_i) - \frac{(s_i \cdot t_i)s_i}{(s_i \cdot n_i)^2} \cdot \operatorname{grad}(v \cdot t_i) - \\
&\quad \frac{(s_i \cdot t_i)t_i}{(s_i \cdot n_i)^2} \cdot \operatorname{grad}(v \cdot s_i) + \frac{(s_i \cdot t_i)^2 t_i}{(s_i \cdot n_i)^2} \cdot \operatorname{grad}(v \cdot t_i).
\end{aligned} \tag{3.1.1}$$

The first term on the right hand side of (3.1.1) is continuous because $v \cdot s_i$ is continuous on T, therefore $s_i \cdot \operatorname{grad}(v \cdot s_i)$ is continuous along $\langle z_0, z_{i+3}\rangle$, which includes the point z_{i+3}. Hence $s_i \cdot \operatorname{grad}(v \cdot s_i)$ is continuous along e_i. By the same logic, $s_i \cdot \operatorname{grad}(v \cdot t_i)$ is continuous along e_i since $v \cdot t_i$ is continuous on T. The third term is continuous because $v \cdot s_i$ is zero along e_i, hence $t_i \cdot \operatorname{grad}(v \cdot s_i)$ is zero on e_i as well. By the same logic, $t_i \cdot \operatorname{grad}(v \cdot t_i)$ is also zero along e_i. Therefore $\operatorname{div} v$ is continuous along e_i and has average zero in T, so $\operatorname{div} v \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{ps}})$. $\square$

It remains to show that $\operatorname{div} \mathring{L}_{r+1}^1(T^{\mathrm{ps}}) \subseteq \mathring{\mathcal{V}}_r^2(T^{\mathrm{ps}})$. We proceed using multiple steps proved in the following lemmas, where our goal is ultimately to construct a function $v \in \mathring{L}_{r+1}^1(T^{\mathrm{ps}})$ for any given $q \in \mathcal{V}_r^2(T^{\mathrm{ps}})$ such that $\operatorname{div} v = q$. First, we will need the following remark.

Remark 3.1.2. *For any function $v \in [\mathcal{P}_r(T^{\mathrm{ps}})]^d$, with $d = 1, 2$, such that $v|_{\partial T} = 0$, there exists a function $w \in [\mathcal{P}_{r-1}(T^{\mathrm{ps}})]^d$ such that $v = \mu w$.*

Lemma 3.1.3. *Let $q \in \mathcal{V}_r^2(T^{\mathrm{ps}})$ and $r \geq 1$, then there exists $w \in L_r^1(T^{\mathrm{ps}})$ and $g \in V_{r-1}^2(T^{\mathrm{ps}})$ such that $\mu^s q = \operatorname{div}(\mu^{s+1} w) + \mu^{s+1} g$ for any $s \geq 0$.*

Proof. Let $b_i \in \mathcal{P}_1(e_i)$ be the linear function such that $q|_{e_i} - b_i$ vanishes at the end points of e_i. Because $q - b_i$ vanishes at the endpoints and q is continuous at z_{3+i}, there exists $a_i \in L_r^0(T^{\mathrm{ps}})$ such that $a_i|_{e_i} = (q - b_i)|_{e_i}$ and $\operatorname{supp} a_i \in T(z_{3+i})$. Note that $a_i|_{e_j} = 0$ for $i \neq j$.

Next, using (2.5.1) and the Nédélec degrees of freedom (2.1.3), we construct a unique function $w_1 \in [\mathcal{P}_1(T)]^2$ such that

$$(s+1) w_1 \cdot \nabla \mu_i = b_i \quad \text{on } e_i, \quad i = 1, 2, 3.$$

We set $\ell_i = \frac{\nabla \mu_i}{|\nabla \mu_i|^2}$,

$$w_2 = \frac{1}{s+1}(a_1 \ell_1 + a_2 \ell_2 + a_3 \ell_3), \quad \text{and} \quad w = w_1 + w_2.$$

We then see that, on e_i,

$$(s+1) w \cdot \nabla \mu_i = (s+1) w_1 \cdot \nabla \mu_i + (s+1) w_2 \cdot \nabla \mu_i = b_i + a_i = q.$$

Therefore the function $(s+1) w \cdot \nabla \mu - q$ vanishes on ∂T, which implies that $\mu v = (s+1) w \cdot \nabla \mu - q$ for some $v \in V_{r-1}^2(T^{\mathrm{ps}})$; see Remark 3.1.2.

Finally we compute

$$\mu^s q = \mu^s q + \operatorname{div}(\mu^{s+1} w) - \mu^{s+1} \operatorname{div}(w) - \mu^s (s+1) w \cdot \nabla \mu = \operatorname{div}(\mu^{s+1} w) - \mu^{s+1}(\operatorname{div}(w) + v).$$

The proof is complete upon setting $g = -(\operatorname{div} w + v)$. $\square$

Lemma 3.1.4. *For any $\theta \in V_r^2(T^{\rm ps})$ with $r \geq 0$, there exists $\psi \in L_1^1(T^{\rm ps})$ and $\gamma \in \mathcal{V}_r^2(T^{\rm ps})$ such that*

$$\mu^s\theta = \operatorname{div}(\mu^s\psi) + \mu^s\gamma \quad \textit{for any } s \geq 0. \tag{3.1.2}$$

Proof. Given $\theta \in V_r^2(T^{\rm ps})$, we define $a_i \in L_1^0(T^{\rm ps})$ uniquely by the conditions

$$a_i(z_j) = 0, \;\; j = 0, 1, 2, 3, \qquad a_i(z_{3+j}) = 0, \;\; j \neq i, \qquad [\![\nabla a_i \cdot t_i]\!](z_{3+i}) = [\![\theta]\!](z_{3+i}).$$

We clearly have supp $a_i \in T(z_{3+i})$. Setting $\psi = a_1t_1 + a_2t_2 + a_3t_3$ we have

$$\operatorname{div}\psi|_{e_i} = \nabla a_i \cdot t_i,$$

and therefore, by the construction of a_i, $\gamma := \theta - \operatorname{div}\psi \in \mathcal{V}_r^2(T^{\rm ps})$. Furthermore, we have $\psi \cdot \nabla\mu|_{T(z_{3+i})} = a_it_i \cdot \nabla\mu|_{T(z_{3+i})} = 0$ for $i = 1, 2, 3$ by (2.5.2), and so $\psi \cdot \nabla\mu = 0$ in T. It then follows that

$$\mu^s\theta - \operatorname{div}(\mu^s\psi) = \mu^s(\theta - \operatorname{div}\psi) - s\mu^{s-1}\nabla\mu \cdot \psi = \mu^s\gamma.$$

$\square$

We combine the previous two lemmas to obtain the following.

Lemma 3.1.5. *Let $q \in \mathcal{V}_r^2(T^{\rm ps})$ and $r \geq 1$. Then there exists $v \in L_r^1(T^{\rm ps})$ and $Q \in \mathcal{V}_{r-1}^2(T^{\rm ps})$ such that $\mu^s q = \operatorname{div}(\mu^{s+1}v) + \mu^{s+1}Q$ for any $s \geq 0$.*

The last lemma handles the lowest order case which follows from [33, Lemma 3.11].

Lemma 3.1.6. *Let $q \in \mathcal{V}_0^2(T^{\rm ps})$ with $\int_T \mu^s q = 0$. Then there exists $w \in L_0^1(T^{\rm ps})$ such that $\mu^s q = \operatorname{div}(\mu^{s+1}w)$ for any $s \geq 0$.*

We can now state and prove the main result.

Theorem 3.1.7. *For each* $p \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{ps}})$*, with* $r \geq 0$*, there exists a* $v \in \mathring{L}_{r+1}^1(T^{\mathrm{ps}})$ *such that* $\operatorname{div} v = p$*.*

Proof. We adopt similar arguments to those given in [38]. Let $p_r = p$ and suppose we have found $w_{r-j} \in L_{r-j}^1(T^{\mathrm{ps}})$ for $0 \leq j \leq \ell - 1$ and $p_{r-j} \in \mathcal{V}_{r-j}^2(T^{\mathrm{ps}})$ for $0 \leq j \leq \ell$ such that

$$\operatorname{div}(\mu^{j+1} w_{r-j}) = \mu^j p_{r-j} - \mu^{j+1} p_{r-(j+1)} \qquad \text{for all } 0 \leq j \leq \ell - 1. \tag{3.1.3}$$

We can then apply Lemma 3.1.5 to find $w_{r-\ell} \in L_{r-\ell}^1(T^{\mathrm{ps}})$ and $p_{r-(\ell+1)} \in \mathcal{V}_{r-(\ell+1)}^2(T^{\mathrm{ps}})$ such that

$$\operatorname{div}(\mu^{\ell+1} w_{r-\ell}) = \mu^\ell p_{r-\ell} - \mu^{\ell+1} p_{r-(\ell+1)}. \tag{3.1.4}$$

Hence, by induction we can find $w_{r-j} \in L_{r-j}^1(T^{\mathrm{ps}})$ for $0 \leq j \leq r-1$ and $p_{r-j} \in \mathcal{V}_{r-j}^2(T^{\mathrm{ps}})$ for $0 \leq j \leq r$ such that (3.1.4) holds. Therefore,

$$\operatorname{div}(\mu w_r + \mu^2 w_{r-1} + \cdots + \mu^r w_1) = p - \mu^r p_0.$$

We have that $\int_T \mu^r p_0 = 0$ and hence by Lemma 4.1.9 we can find $w_0 \in L_0^1(T^{\mathrm{ps}})$ such that $\operatorname{div}(\mu^{r+1} w_0) = \mu^r p_0$. The result follows after setting $v = \mu w_r + \mu^2 w_{r-1} + \cdots \mu^r w_1 + \mu^{r+1} w_0$. $\square$

We have several corollaries that follow from Theorem 3.1.7. First we show that the analogous result without boundary conditions is satisfied.

Corollary 3.1.8. *For each* $p \in V_r^2(T^{\mathrm{ps}})$ *there exists a* $v \in L_{r+1}^1(T^{\mathrm{ps}})$ *such that* $\operatorname{div} v = p$*.*

Proof. Let $p \in V_r^2(T^{\mathrm{ps}})$. By Lemma 4.1.6 there exists $w \in L_1^1(T^{\mathrm{ps}})$ and $g \in \mathcal{V}_r^2(T^{\mathrm{ps}})$ with

$$p = \operatorname{div} w + g.$$

We let $\psi = (\frac{1}{|T|}\int_T g)\frac{1}{2}x \in L_1^1(T^{\mathrm{ps}})$ and hence $\int_T \operatorname{div}\psi = \int_T g$. We then have

$$p = \operatorname{div}(w+\psi) + (g - \operatorname{div}\psi).$$

By Theorem 3.1.7 there exists a $\theta \in \mathring{L}_{r+1}^1(T^{\mathrm{ps}})$ such that $\operatorname{div}\theta = g - \operatorname{div}\psi$. Therefore, we have

$$p = \operatorname{div}(w+\psi+\theta).$$

The proof is complete after we set $v = w + \psi + \theta$. □

Corollary 3.1.9. *For each* $p \in \mathring{L}_r^2(T^{\mathrm{ps}})$ *(resp.,* $p \in L_r^2(T^{\mathrm{ps}})$*) there exists a* $v \in \mathring{S}_{r+1}^1(T^{\mathrm{ps}})$ *(resp.,* $v \in S_{r+1}^1(T^{\mathrm{ps}})$*) such that* $\operatorname{div} v = p$*. Likewise for each* $v \in \mathring{L}_r^1(T^{\mathrm{ps}})$ *(resp.,* $v \in L_r^1(T^{\mathrm{ps}})$*) with* $\operatorname{div} v = 0$ *there exists a* $z \in \mathring{S}_{r+1}^0(T^{\mathrm{ps}})$ *(resp.,* $z \in S_{r+1}^0(T^{\mathrm{ps}})$*) such that* $\operatorname{rot} z = v$.

Proof. Let $p \in \mathring{L}_r^2(T^{\mathrm{ps}}) \subset \mathring{\mathcal{V}}_r^2(T^{\mathrm{ps}})$ and we can apply Theorem 3.1.7 to find $v \in \mathring{L}_{r+1}^1(T^{\mathrm{ps}})$ such that $\operatorname{div} v = p$. However, clearly $v \in \mathring{S}_{r+1}^1(T^{\mathrm{ps}})$.

Next, let $v \in \mathring{L}_r^1(T^{\mathrm{ps}}) \subset V_r^1(T^{\mathrm{ps}})$ be divergence–free. Lemma 2.1.6 shows that there exists $z \in \mathring{V}_r^0(T^{\mathrm{ps}})$ such that $\operatorname{rot} z = v$. Since v is continuous and vanishes on the boundary, we have $\operatorname{rot} z \in [C(T)]^2$ and $z|_{\partial T} = 0$, $\operatorname{rot} z|_{\partial T} = 0$. Thus $z \in \mathring{S}_r^0(T^{\mathrm{ps}})$ by definition.

This proof applies *mutatis mutandis* to the statements without boundary conditions. □

Remark 3.1.10. *To summarize, Lemma 2.1.6, Theorem 3.1.7, and Corollaries 3.1.8 and*

3.1.9 show that the following two sets of sequences are exact:

$$
\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & L_r^0(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & V_{r-1}^1(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & V_{r-2}^2(T^{\mathrm{ps}}) & \longrightarrow & 0, \\
\mathbb{R} & \longrightarrow & S_r^0(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & L_{r-1}^1(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & V_{r-2}^2(T^{\mathrm{ps}}) & \longrightarrow & 0, \\
\mathbb{R} & \longrightarrow & S_r^0(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & S_{r-1}^1(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & L_{r-2}^2(T^{\mathrm{ps}}) & \longrightarrow & 0,
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathring{L}_r^0(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & \mathring{V}_{r-1}^1(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & \mathring{V}_{r-2}^2(T^{\mathrm{ps}}) & \longrightarrow & 0, \\
0 & \longrightarrow & \mathring{S}_r^0(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & \mathring{L}_{r-1}^1(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{ps}}) & \longrightarrow & 0, \\
0 & \longrightarrow & \mathring{S}_r^0(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & \mathring{S}_{r-1}^1(T^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & \mathring{L}_{r-2}^2(T^{\mathrm{ps}}) & \longrightarrow & 0.
\end{array}
$$

3.2 Dimension counts

We can easily count the dimensions of the smooth spaces $S_r^k(T^{\mathrm{ps}})$ via the rank–nullity theorem and the exactness of sequences ($k = 0, 1$):

$$
\begin{aligned}
\dim S_r^k(T^{\mathrm{ps}}) &= \dim \operatorname{range} S_r^k(T^{\mathrm{ps}}) + \dim \ker S_r^k(T^{\mathrm{ps}}) \\
&= \dim \ker L_{r-1}^{k+1}(T^{\mathrm{ps}}) + \dim \ker L_r^k(T^{\mathrm{ps}}) \\
&= \dim L_{r-1}^{k+1}(T^{\mathrm{ps}}) - \dim \operatorname{range} L_{r-1}^{k+1}(T^{\mathrm{ps}}) + \dim L_r^k(T^{\mathrm{ps}}) - \operatorname{range} L_r^k(T^{\mathrm{ps}}) \\
&= \dim L_{r-1}^{k+1}(T^{\mathrm{ps}}) + \dim L_r^k(T^{\mathrm{ps}}) - \dim \ker V_{r-2}^{k+2}(T^{\mathrm{ps}}) - \dim \ker V_{r-1}^{k+1}(T^{\mathrm{ps}}) \\
&= \dim L_{r-1}^{k+1}(T^{\mathrm{ps}}) + \dim L_r^k(T^{\mathrm{ps}}) - \dim V_{r-1}^{k+1}(T^{\mathrm{ps}}).
\end{aligned}
$$

Now we easily find

$$\dim L_r^k(T^{\mathrm{ps}}) = \binom{2}{k}\big[3r^2+3r+1\big], \quad \dim V_r^k(T^{\mathrm{ps}}) = \begin{cases} 3r^2+3r+1 & k=0, \\ 6r^2+12r+6 & k=1, \\ 3r^2+9r+6 & k=2. \end{cases}$$

Thus, we have

$$\dim S_r^k(T^{\mathrm{ps}}) = \begin{cases} 3r^2-3r+3 & k=0, \\ 6r^2+3 & k=1, \\ 3r^2+3r+1 & k=2. \end{cases}$$

Similar calculations also show that

$$\dim \mathring{S}_r^k(T^{\mathrm{ps}}) = \begin{cases} 3(r-2)(r-3) & k=0, \\ 6(r-1)(r-2) & k=1, \\ 3r(r-1) & k=2. \end{cases}$$

3.3 Commuting projections on a macro triangle

In this section we define commuting projections. In order to do so, we give the degrees of freedom for C^1 polynomials on a line segment. Let $a < m < b$, and define the space

$$W_r(\{a,m,b\}) = \{v \in C^1([a,b]) : v|_{[a,m]} \in \mathcal{P}_r([a,m]) \text{ on } v|_{[m,b]} \in \mathcal{P}_r([m,b])\}.$$

The classical degrees of freedom for $W_r(\{a,m,b\})$ is given in the next result.

Lemma 3.3.1. *Let $r \geq 1$. A function $z \in W_r(\{a,m,b\})$ is uniquely determined by the* *following degrees of freedom.*

$$
\begin{aligned}
z(a), z(b) & \\
z'(a), z'(b) & \quad \textit{if } r \geq 2, \\
z(m), z'(m) & \quad \textit{if } r \geq 3, \\
\int_a^m z(x)q(x) & \quad \textit{for all } q \in \mathcal{P}_{r-4}([a,m]), \\
\int_m^b z(x)q(x) & \quad \textit{for all } q \in \mathcal{P}_{r-4}([m,b]).
\end{aligned}
$$

Other degrees of freedom are given in the next lemma. Its proof is found in the appendix.

Lemma 3.3.2. *Let $r \geq 1$. A function $z \in W_r(\{a,m,b\})$ is uniquely determined by the following degrees of freedom.*

$$z(a), z(b) \tag{3.3.1a}$$

$$\int_a^m z(x)q(x) \quad \textit{for all } q \in \mathcal{P}_{r-2}([a,m]), \tag{3.3.1b}$$

$$\int_m^b z(x)q(x) \quad \textit{for all } q \in \mathcal{P}_{r-2}([m,b]). \tag{3.3.1c}$$

Lemma 3.3.3. *Suppose that $q \in S_r^0(T^{\mathrm{ps}})$ with $q|_{e_i} = 0$ for some $i \in \{1,2,3\}$. Then $q|_{T(z_{3+i})} = \mu p|_{T(z_{3+i})}$ for some $p \in L_{r-1}^0(T^{\mathrm{ps}})\big|_{T(z_{3+i})}$, and $p \in C^1(T(z_{3+i}))$. In particular, if $q|_{\partial T} = 0$, then $q = \mu p$ for some $p \in L_{r-1}^0(T^{\mathrm{ps}})$ and $p|_{T(z_{3+i})} \in C^1(T(z_{3+i}))$ for $i = 1,2,3$.*

Proof. The statement $q|_{T(z_{3+i})} = \mu p|_{T(z_{3+i})}$ is a consequence of Remark 3.1.2. Because q and μ are continuous, it follows that p is continuous, i.e., $p \in L_{r-1}^0(T^{\mathrm{ps}})\big|_{T(z_{3+i})}$. We also have $\nabla q = \mu \nabla p + p \nabla \mu$, and therefore

$$\mu \nabla p|_{T(z_{3+i})} = \big(\nabla q - p \nabla \mu\big)|_{T(z_{3+i})}.$$

Since $\nabla \mu$ is constant on $T(z_{3+i})$, we find that $\mu \nabla p|_{T(z_{3+i})}$ is continuous. Because μ is positive in the interior of $T(z_{3+i})$, we conclude that ∇p is continuous on $T(z_{3+i})$. $\square$

We are now ready to give degrees of freedom (DOFs) for functions in $S_r^0(T^{\mathrm{ps}})$.

Lemma 3.3.4. *A function $q \in S_r^0(T^{\mathrm{ps}})$, with $r \geq 2$, is uniquely determined by*

$$q(z_i), \nabla q(z_i) \qquad 1 \leq i \leq 3, \qquad \textit{(9 DOFs)} \tag{3.3.2a}$$

$$q(z_{3+i}), \partial_t q(z_{3+i}) \qquad 1 \leq i \leq 3, \ \textit{if } r \geq 3, \qquad \textit{(6 DOFs)} \tag{3.3.2b}$$

$$\int_e \partial_n q \, p \qquad \forall p \in \mathcal{P}_{r-3}(e), \ e \in \mathcal{E}^b(T^{\mathrm{ps}}), \qquad (6(r-2) \textit{ DOFs}) \tag{3.3.2c}$$

$$\int_e qp \qquad \forall p \in \mathcal{P}_{r-4}(e), \ e \in \mathcal{E}^b(T^{\mathrm{ps}}), \qquad (6(r-3) \textit{ DOFs}) \tag{3.3.2d}$$

$$\int_T \mathrm{rot}\, q \cdot \mathrm{rot}\, p \qquad \forall p \in \mathring{S}_r^0(T^{\mathrm{ps}}), \qquad (3(r-2)(r-3) \textit{ DOFs}) \tag{3.3.2e}$$

Proof. The number of DOFs given is $3r^2 - 3r + 3 = \dim S_r^0(T^{\mathrm{ps}})$. We will show that the only function q for which (3.3.2a)–(3.3.2e) are equal to zero must be zero on T. Suppose that q vanishes on (3.3.2a)–(3.3.2d) restricted to a single edge e_i. Then q satisfies all conditions of Lemma 3.3.1 on each edge of T, so $q \equiv 0$ on e_i. It then follows from Lemma 3.3.3 that $q|_{T(z_{3+i})} = \mu p|_{T(z_{3+i})}$, where $p \in C^1(T(z_{3+i}))$ is a piecewise polynomial of degree $(r-1)$. We then have $\nabla q|_{e_i} = p \nabla \mu_i|_{e_i}$, and so by (3.3.2a), $p = 0$ on the endpoints of e_i. Also (3.3.2c) yields $\int_e p w \partial_n \mu = 0$ for all $w \in \mathcal{P}_{r-3}(e)$ and for all $e \in \mathcal{E}^b(T^{\mathrm{ps}})$ with $e \subset e_i$. Since $\partial_n \mu$ is constant on each edge $e \in \mathcal{E}^b(T^{\mathrm{ps}})$, we have $\int_e pw = 0$ for all $w \in \mathcal{P}_{r-3}(e)$ and $e \subset e_i$. Using Lemma 3.3.2, it follows that $p \equiv 0$ on e_i. Thus $\nabla q|_{e_i} = 0$.

We conclude that if q vanishes on (3.3.2), then $q \in \mathring{S}_r^0(T^{\mathrm{ps}})$. Finally, condition (3.3.2e) yields rot $q = 0$ on T, and hence $q \equiv 0$ on T.

□

Lemma 3.3.5. *A function $v \in L_r^1(T^{\rm ps})$ is uniquely determined by*

		No. of DOFs	
$v(z_i)$,	$1 \le i \le 3$,	6,	(3.3.3a)
$\int_{e_i} (v \cdot n_i)\, ds$,	*if* $r = 1$,		(3.3.3b)
$[\![\mathrm{div}\, v]\!](z_{3+i})$,	$1 \le i \le 3$,	3,	(3.3.3c)
$v(z_{3+i}) \cdot n_i$,	$1 \le i \le 3$, *if* $r \ge 2$,	3,	(3.3.3d)
$\int_e v \cdot w\, ds$,	$\forall w \in [\mathcal{P}_{r-2}(e)]^2,\ \forall e \in \mathcal{E}^b(T^{\rm ps})$,	$12(r-1)$,	(3.3.3e)
$\int_T v \cdot \mathrm{rot}\, w\, dx$,	$\forall w \in \mathring{S}_{r+1}^0(T^{\rm ps})$,	$3(r-1)(r-2)$,	(3.3.3f)
$\int_T \mathrm{div}\, v\, w\, dx$,	$\forall w \in \mathring{\mathcal{V}}_{r-1}^2(T^{\rm ps})$,	$3r(r+1) - 4$.	(3.3.3g)

Proof. The number of degrees of freedom given is $6r^2+6r+2$ which equals the dimension of $L_r^1(T^{\rm ps})$. We show that if $v \in L_r^1(T^{\rm ps})$ vanishes on (3.3.3), then v is identically zero.

Suppose that v vanishes on (3.3.3a)–(3.3.3e) restricted to a single edge e_i. Recall that $T(z_{3+i}) = T_{2i+1} \cup T_{2i+2}$ is the union of two triangles that have z_{3+i} as a vertex, and n_i and t_i are, respectively, the outward normal and unit tangent vectors of the edge $e_i = \partial T \cap \partial T(z_{3+i})$. Let s_i be a unit vector that is tangent to the interior edge $[z_0, z_{3+i}]$, which is necessarily linearly independent of t_i. Thus we may write

$$v|_{T(z_{3+i})} = a_i t_i + b_i s_i$$

for some $a_i, b_i \in L_r^0(T^{\rm ps})|_{T(z_{3+i})}$. We then see that

$$\operatorname{div} v|_{T(z_{i+3})} = \partial_{t_i} a_i + \partial_{s_i} b_i.$$

Because b_i is continuous on $T(z_{3+i})$ we have that $[\![\partial_{s_i} b_i]\!](z_{i+3}) = 0$ and hence $0 = [\![\operatorname{div} v]\!](z_{3+i}) = [\partial_{t_i} a_i](z_{3+i})$. Therefore $a_i|_{e_i}$ is C^1 on e_i. To continue, we split the proof into two step.

Case $r = 1$*:*

By the first set of DOFs (3.3.3a), there holds $a_i(z_j) = b_i(z_j) = 0$ for $j \in \{1, 2, 3\} \backslash \{i\}$. Because $a_i|_{e_i}$ is piecewise linear and C^1, we conclude that $a_i \equiv 0$ on e_i. Next, using (3.3.3b) yields

$$\int_{e_i} b_i (s_i \cdot n_i) = 0.$$

Because $s_i \cdot n_i \neq 0$, we conclude that $\int_{e_i} b_i = 0$. Since b_i vanishes at the endpoints of e_i, and since b_i is piecewise linear on e_i, we conclude that $b_i = 0$ on e_i, and therefore $v|_{e_i} = 0$.

Case $r \geq 2$*:*

Again, there holds $a_i(z_j) = b_i(z_j) = 0$ by the first set of DOFs (3.3.3a). Combining Lemma 3.3.2 with the DOFs (3.3.3e), noting that a_i is C^1 on e_i, then yields $a_i = 0$ on e_i. Likewise the DOFs (3.3.3a), (3.3.3e), and (3.3.3d) show that $b_i = 0$ on e_i. We conclude that $v|_{e_i} = 0$.

Thus, if v vanishes on (3.3.3) then $v \in \mathring{L}_1^1(T^{\rm ps})$. The DOFs (3.3.3g) then show that $\operatorname{div} v = 0$, and therefore, by Corollary 3.1.9, $v = \operatorname{rot} z$ for some $z \in \mathring{S}_{r+1}(T^{\rm ps})$. Finally, by (3.3.3f), we conclude that $v \equiv 0$.

$\square$

Lemma 3.3.6. *A function $q \in V_r^2(T^{\mathrm{ps}})$ is uniquely determined by*

		No. of DOFs	
$[\![q]\!](z_{3+i}),$	$1 \le i \le 3,$	$3,$	(3.3.4a)
$\int_T q\,dx,$		$1,$	(3.3.4b)
$\int_T qp\,dx,$	$\forall p \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{ps}}),$	$3(r+1)(r+2)-4.$	(3.3.4c)

Proof. If $q \in V_r^2(T^{\mathrm{ps}})$ is such that (3.3.4a) are zero then q is continuous at z_{3+i} for $1 \le i \le 3$. Then (3.3.4b) yields that $q \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{ps}})$, and it follows from (3.3.4c) that $q \equiv 0$ on T. □

Lemma 3.3.7. *A function $v \in S_r^1(T^{\mathrm{ps}})$ is uniquely determined by the following degrees of freedom.*

		No. of DOFs	
$v(z_i),\ \operatorname{div} v(z_i),$	$1 \le i \le 3,$	$9,$	(3.3.5a)
$\int_{e_i} v \cdot n_i\,ds,$	$1 \le i \le 3,$ *if* $r = 1,$		(3.3.5b)
$v(z_{3+i}) \cdot n,\ \operatorname{div} v(z_{3+i}),$	$1 \le i \le 3,$ *if* $r \ge 2,$	$6,$	(3.3.5c)
$\int_e v \cdot w\,ds,$	$\forall w \in [\mathcal{P}_{r-2}(e)]^2,\ \forall e \in \mathcal{E}^b(T^{\mathrm{ps}}),$	$12(r-1),$	(3.3.5d)
$\int_e (\operatorname{div} v) q\,ds,$	$\forall q \in \mathcal{P}_{r-3}(e),\ \forall e \in \mathcal{E}^b(T^{\mathrm{ps}}),$	$6(r-2),$	(3.3.5e)
$\int_T v \cdot \operatorname{rot} q\,dx,$	$\forall q \in \mathring{S}_{r+1}^0(T^{\mathrm{ps}}),$	$3(r-1)(r-2),$	(3.3.5f)
$\int_T (\operatorname{div} v) q\,dx,$	$\forall q \in \mathring{L}_{r-1}^2(T^{\mathrm{ps}}),$	$3(r-1)(r-2).$	(3.3.5g)

Proof. If v vanishes at the DOFs, then $v \in S_r^1(T^{\mathrm{ps}}) \subset L_r^1(T^{\mathrm{ps}})$ vanishes on (3.3.3a)–(3.3.3e). The proof of Lemma 3.3.5 then shows that $v|_{\partial T} = 0$, and therefore $\int_T \operatorname{div} v = 0$. Using (3.3.5a),(3.3.5c), and (3.3.5e), we also find that $\operatorname{div} v|_{\partial T} = 0$, i.e., $\operatorname{div} v \in$

$\mathring{L}^2_{r-1}(T^{\rm ps})$. The DOFs (3.3.5g) yield $\operatorname{div} v = 0$ in T, and therefore $v = \operatorname{rot} q$ for some $q \in \mathring{S}^0_{r+1}(T^{\rm ps})$ by Corollary 3.1.9. Finally (3.3.5f) gives $v \equiv 0$. Noting that the number of DOFs is $6r^2 + 3$, the dimension of $S^1_r(T^{\rm ps})$, we conclude that (3.3.5) form a unisolvent set over $S^1_r(T^{\rm ps})$. $\square$

Lemma 3.3.8. *Let $q \in L^2_r(T^{\rm ps})$ with $r \geq 1$. Then q is uniquely determined by the following degrees of freedom.*

		No. of DOFs	
$q(z_i),$	$1 \leq i \leq 3,$	$3,$	(3.3.6a)
$q(z_{3+i}),$	$1 \leq i \leq 3,$	$3,$	(3.3.6b)
$\int_e qp\, ds,$	$\forall p \in \mathcal{P}_{r-2}(e),\ \forall e \in \mathcal{E}^b(T^{\rm ps}),$	$6(r-1),$	(3.3.6c)
$\int_T q\, dx,$		$1,$	(3.3.6d)
$\int_T qp\, dx,$	$\forall p \in \mathring{L}^2_r(T^{\rm ps}),$	$3r(r-1).$	(3.3.6e)

Proof. Let $q \in L^2_r(T^{\rm ps})$ such that all DOFs (3.3.6) are equal to zero. The conditions (3.3.6a)–(3.3.6c) yield that $q \equiv 0$ on ∂T. Therefore, using (3.3.6d), $q \in \mathring{L}^2_r(T^{\rm ps})$, and by (3.3.6e), $q \equiv 0$ on T. $\square$

The next two theorems show that projections induced by the degrees of freedom given in Lemmas 3.3.4–3.3.8 commute.

Theorem 3.3.9. *Let $\Pi^r_0 : C^\infty(T) \to S^0_r(T^{\rm ps})$ be the projection induced by the DOFs* (3.3.2)*, that is,*

$$\phi(\Pi^r_0 p) = \phi(p), \quad \forall \phi \in \textit{DOFs in } (3.3.2).$$

Likewise, let $\Pi^{r-1}_1 : [C^\infty(T)]^2 \to L^1_{r-1}(T^{\rm ps})$ be the projection induced by the DOFs

(3.3.3), *and let* $\Pi_2^{r-2} : C^\infty(T) \to V_{r-2}^2(T^{\mathrm{ps}})$ *be the projection induced by the DOFs* (3.3.4). *Then for* $r \geq 2$*, the following diagram commutes*

$$\begin{array}{ccccccccc} \mathbb{R} & \longrightarrow & C^\infty(T) & \xrightarrow{\text{rot}} & [C^\infty(T)]^2 & \xrightarrow{\text{div}} & C^\infty(T) & \longrightarrow & 0 \\ & & \Big\downarrow \Pi_0^r & & \Big\downarrow \Pi_1^{r-1} & & \Big\downarrow \Pi_2^{r-2} & & \\ \mathbb{R} & \longrightarrow & S_r^0(T^{\mathrm{ps}}) & \xrightarrow{\text{rot}} & L_{r-1}^1(T^{\mathrm{ps}}) & \xrightarrow{\text{div}} & V_{r-2}^2(T^{\mathrm{ps}}) & \longrightarrow & 0. \end{array}$$

In other words, we have for $r \geq 2$

$$\operatorname{div} \Pi_1^{r-1} v = \Pi_2^{r-2} \operatorname{div} v, \quad \forall v \in [C^\infty(T)]^2, \tag{3.3.7a}$$

$$\operatorname{rot} \Pi_0^r p = \Pi_1^{r-1} \operatorname{rot} p, \quad \forall p \in C^\infty(T). \tag{3.3.7b}$$

Proof. (i) *Proof of* (3.3.7a). We take $v \in [C^\infty(T)]^2$. Since $\rho := \operatorname{div} \Pi_1^{r-1} v - \Pi_2^{r-2} \operatorname{div} v \in V_{r-2}^2(T^{\mathrm{ps}})$, we only need to prove that ρ vanishes at the DOFs (3.3.4). For the jump condition at points z_{3+i} for $1 \leq i \leq 3$, we have

$$[\![\rho]\!](z_{3+i}) = [\![\operatorname{div} \Pi_1^{r-1} v - \Pi_2^{r-2} \operatorname{div} v]\!](z_{3+i}) = [\![\operatorname{div} \Pi_1^{r-1} v - \operatorname{div} v]\!](z_{3+i}) = 0,$$

where we have used the definitions of Π_2^{r-2} and Π_1^{r-1} along with the DOFs (3.3.4a) and (3.3.3c).

For the interior DOFs, we have,

$$\int_T \rho = \int_T \left(\operatorname{div} \Pi_1^{r-1} v - \operatorname{div} v\right) = \int_{\partial T} \left(\Pi_1^{r-1} v - v\right) \cdot n = 0,$$

where we have used the definitions of Π_1^{r-1} and Π_2^{r-2} and DOFs (3.3.4b) and either (3.3.3b) if $r = 2$ or (3.3.3e) if $r \geq 3$. Finally, for any $p \in \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{ps}})$,

$$\int_T \rho p = \int_T \left(\operatorname{div} \Pi_1^{r-1} v - \Pi_2^{r-1} \operatorname{div} v\right) p = 0$$

by the definitions of Π_1^{r-1} and Π_2^{r-2} along with DOFs (3.3.4c) and (3.3.3g). By Lemma 3.3.6, ρ is exactly zero on T, and the projections in (3.3.7a) commute.

(ii) *Proof of* (3.3.7b). Let $p \in C^\infty(T)$ and set $\rho := \text{rot}\, \Pi_0^r p - \Pi_1^{r-1}\text{rot}\, p \in L_{r-1}^1(T^{\text{ps}})$. We will show that ρ vanishes for all DOFs (3.3.3).

First, for each vertex z_i with $1 \le i \le 3$,

$$\rho(z_i) = \text{rot}\, \Pi_0^r p(z_i) - \Pi_1^{r-1}\text{rot}\, p(z_i) = \text{rot}\, p(z_i) - \Pi_1^{r-1}\text{rot}\, p(z_i) = 0, \tag{3.3.8}$$

by (3.3.2a) and (3.3.3a). Furthermore, at nodes z_{3+i}, we have by (3.3.3c)

$$\begin{aligned} [\![\text{div}\, \rho]\!](z_{3+i}) &= [\![\text{div}\,\text{rot}\, \Pi_0^r p - \text{div}\, \Pi_1^{r-1}\text{rot}\, p]\!](z_{3+i}) \\ &= -[\![\text{div}\, \Pi_1^{r-1}\text{rot}\, p]\!](z_{3+i}) \\ &= -[\![\text{div}\,\text{rot}\, p]\!](z_{3+i}) = 0, \end{aligned}$$

For the DOFs on each edge $e \in \mathcal{E}^b(T^{\text{ps}})$, we will use that rot $\varphi \cdot n = \partial_t \varphi$ and rot $\varphi \cdot t = -\partial_n \varphi$. Then we have, for $r \ge 3$,

$$\begin{aligned} \rho(z_{3+i}) \cdot n_i &= (\text{rot}\, \Pi_0^r p(z_{3+i})) \cdot n_i - \left(\Pi_1^{r-1}\text{rot}\, p(z_{3+i})\right) \cdot n_i \\ &= \partial_t p(z_{3+i}) - \left(\Pi_1^{r-1}\text{rot}\, p(z_{3+i})\right) \cdot n_i \\ &= \partial_t p(z_{3+i}) - \text{rot}\, p(z_{3+i}) \cdot n_i = 0 \end{aligned} \tag{3.3.9}$$

by (3.3.2b) and (3.3.3d). If $r = 2$ (so that $\rho \in L_1^1(T^{\text{ps}})$),

$$\int_{e_i} \rho \cdot n_i = \int_{e_i} \left(\text{rot}\, \Pi_r^0 p - \Pi_{r-1}^1 \text{rot}\, p\right) \cdot n_i = \int_{e_i} \partial_{t_i} \left(\Pi_r^0 p - p\right) = 0$$

by (3.3.3b) and (3.3.2a), so (3.3.7b) is proved.

Now let $r \geq 3$. We have, for all $q \in \mathcal{P}_{r-3}(e)$ and for all $e \in \mathcal{E}^b(T^{\mathrm{ps}})$,

$$\begin{aligned}\int_e (\rho \cdot n) q &= \int_e \left(\text{rot } \left(\Pi_0^r p - p\right) \cdot n\right) q \\ &= \int_e \partial_t \left(\Pi_0^r p - p\right) q \\ &= -\int_e \left(\Pi_0^r p - p\right) \partial_t q = 0,\end{aligned}$$

by (3.3.3e), (3.3.2b) and (3.3.2d). Likewise, for $q \in \mathcal{P}_{r-3}(e)$,

$$\begin{aligned}\int_e (\rho \cdot t) q &= \int_e \left(\left(\text{rot } \Pi_0^r p - \Pi_1^{r-1} \text{rot } p\right) \cdot t\right) q \\ &= \int_e \left(\text{rot } \left(\Pi_0^r p - p\right) \cdot t\right) q \\ &= \int_e -\partial_n \left(\Pi_0^r p - p\right) q = 0\end{aligned}$$

by (3.3.3e) and (3.3.2c). For the interior DOFs, for any $w \in \mathring{S}_{r-1}^0(T^{\mathrm{ps}})$, we have

$$\int_T \rho \cdot \text{rot } w = \int_T \left(\text{rot } \Pi_0^r p - \Pi_1^{r-1} \text{rot } p\right) \cdot \text{rot } w = 0$$

by (3.3.2e) and (3.3.3f). Finally, for any $w \in \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{ps}})$,

$$\begin{aligned}\int_T \text{div } \rho \, w &= \int_T \text{div } \left(\text{rot } \Pi_0^r p - \Pi_1^{r-1} \text{rot } p\right) w \\ &= \int_T -\text{div } (\text{rot } p) w = 0\end{aligned}$$

where we used the DOF (3.3.3g). Therefore ρ is equal to zero on T, and the identity (3.3.7b) is proved. $\square$

Theorem 3.3.10. *Let* $\Pi_0^r : C^\infty(T) \to S_r^0(T^{\mathrm{ps}})$ *be the projection induced by the DOFs* (3.3.2)*, that is,*

$$\phi(\Pi_0^r p) = \phi(p), \quad \forall \phi \in \textit{DOFs in } (3.3.2).$$

Likewise, let $\varpi_1^{r-1} : [C^\infty(T)]^2 \to S_{r-1}^1(T^{\mathrm{ps}})$ *be the projection induced by the DOFs* (3.3.5)*, and let* $\varpi_2^{r-2} : C^\infty(T) \to L_{r-2}^2(T^{\mathrm{ps}})$ *be the projection induced by the DOFs* (3.3.6)*. Then for* $r \geq 2$*, the following diagram commutes*

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{rot}"] \arrow[d, "\Pi_0^r"] & {[C^\infty(T)]^2} \arrow[r, "\mathrm{div}"] \arrow[d, "\varpi_1^{r-1}"] & C^\infty(T) \arrow[r] \arrow[d, "\varpi_2^{r-2}"] & 0 \\
\mathbb{R} \arrow[r] & S_r^0(T^{\mathrm{ps}}) \arrow[r, "\mathrm{rot}"] & S_{r-1}^1(T^{\mathrm{ps}}) \arrow[r, "\mathrm{div}"] & L_{r-2}^2(T^{\mathrm{ps}}) \arrow[r] & 0.
\end{tikzcd}
$$

In other words, we have for $r \geq 2$

$$
\begin{aligned}
\mathrm{rot}\, \Pi_0^r p =& \varpi_1^{r-1} \mathrm{rot}\, p, \quad \forall p \in C^\infty(T), && \text{(3.3.10a)} \\
\mathrm{div}\, \varpi_1^{r-1} v =& \varpi_2^{r-2} \mathrm{div}\, v, \quad \forall v \in [C^\infty(T)]^2. && \text{(3.3.10b)}
\end{aligned}
$$

Proof. (i) *Proof of* (3.3.10a). Let $p \in C^\infty(T)$ and $\rho := \mathrm{rot}\, \Pi_0^r p - \varpi_1^{r-1} \mathrm{rot}\, p \in S_{r-1}^1(T^{\mathrm{ps}})$. We show that ρ vanishes on (3.3.5).

First,

$$
\begin{aligned}
\rho(z_i) &= \mathrm{rot}\, \Pi_0^r p(z_i) - \varpi_1^{r-1} \mathrm{rot}\, p(z_i) = 0, \\
\mathrm{div}\, \rho(z_i) &= -\mathrm{div}\, \varphi_1^{r-1} \mathrm{rot}\, p(z_i) = -\mathrm{div}\, \mathrm{rot}\, p(z_i) = 0,
\end{aligned}
$$

by the definitions of Π_0^r and ϖ_1^{r-1} along with DOFs (3.3.2a) and (3.3.5a).

Next, if $r = 2$,

$$
\begin{aligned}
\int_{e_i} \rho \cdot n_i &= \int_{e_i} \left(\mathrm{rot}\, \Pi_0^r p - \varpi_1^{r-1} \mathrm{rot}\, p\right) \cdot n_i \\
&= \int_{e_i} \left(\mathrm{rot}\, \Pi_0^r p - \Pi_1^{r-1} \mathrm{rot}\, p\right) \cdot n_i = 0,
\end{aligned}
$$

using (3.3.5b), (3.3.3b) and (3.3.7b). Similar arguments show that, for $r \geq 3$,

$$\begin{aligned} \rho(z_{3+i}) \cdot n_i &= (\text{rot}\, \Pi_0^r p(z_{3+i}) - \Pi_1^{r-1} \text{rot}\, p(z_{3+i})) \cdot n_i = 0, \\ \int_e \rho \cdot w &= \int_e (\text{rot}\, \Pi_0^r p - \varpi_1^{r-1} \text{rot}\, p) \cdot w = \int_e (\text{rot}\, \Pi_0^r p - \Pi_1^{r-1} \text{rot}\, p) \cdot w = 0, \end{aligned}$$

and

$$\int_T \rho \cdot \text{rot}\, w = \int_T (\text{rot}\, \Pi_0^r p - \Pi_1^{r-1} \text{rot}\, p) \cdot w = 0.$$

Next using (3.3.5c) gives

$$\text{div}\, \rho(z_{3+i}) = -\text{div}\, \varpi_1^{r-1} \text{rot}\, p(z_{3+i}) = -\text{div}\, \text{rot}\, p(z_{3+i}) = 0,$$

and (3.3.5e) yields

$$\int_e (\text{div}\, \rho) q = -\int_e (\text{div}\, \varpi_1^{r-1} \text{rot}\, p) q = -\int_e (\text{div}\, \text{rot}\, p) q = 0$$

for all $q \in \mathcal{P}_{r-4}(e)$ and $e \in \mathcal{E}^b(T^{\text{ps}})$. The same arguments, but using (3.3.5g), gives

$$\int_T (\text{div}\, \rho) q = 0 \qquad \forall q \in \mathring{L}^2_{r-1}(T^{\text{ps}}).$$

Applying Lemma 3.3.7 shows that $\rho \equiv 0$, and so (3.3.10a) holds.

(ii) *Proof of* (3.3.10b). For some $v \in [C^\infty(T)]^2$, we define $\rho := \text{div}\, \varpi_1^{r-1} v - \varpi_2^{r-2} \text{div}\, v \in L^2_{r-2}(T^{\text{ps}})$. Then we need only show that ρ is zero for all DOFs in (3.3.6). For the vertex DOFs, we have for each z_i,

$$\rho(z_i) = \text{div}\, \varpi_1^{r-1} v(z_i) - \varpi_2^{r-2} \text{div}\, v(z_i) = 0,$$

by (3.3.5a) and (3.3.6a). Next, for each $i = 1, 2, 3$,

$$\rho(z_{3+i}) = \operatorname{div} \varpi_1^{r-1} v(z_{3+i}) - \varpi_2^{r-2} \operatorname{div} v(z_{3+i}) = 0,$$

where we have used (3.3.5a) and (3.3.6b). Similar arguments show that

$$\int_e \rho q = 0 \qquad \forall q \in \mathcal{P}_{r-4}(e),\ e \in \mathcal{E}^b(T^{\mathrm{ps}}),$$

by (3.3.5e) and (3.3.6c), and that

$$\int_T \rho q = 0 \qquad \forall q \in \mathring{L}^2_{r-2}(T^{\mathrm{ps}})$$

by (3.3.5g) and (3.3.6e). Using (3.3.6d) and (3.3.5b) if $r = 2$ or (3.3.5d) if $r > 2$,

$$\int_T \rho = \int_T \operatorname{div} \varpi_1^{r-1} v - \varpi_2^{r-2} \operatorname{div} v = \int_T \operatorname{div} (\varpi_1^{r-1} v - v) = \int_{\partial T} (\varpi_1^{r-1} v - v) \cdot n = 0.$$

Therefore, $\rho \equiv 0$ on T by Lemma 3.3.8, and (3.3.10b) is proved. $\square$

3.4 Global spaces on Powell-Sabin refinements

In this section, we study the global finite element spaces induced by the degrees of freedom in Section 3.3. We let $\mathcal{T}_h$ represent the simplicial triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$, and $\mathcal{T}_h^{\mathrm{ps}}$ represent the Powell-Sabin refinement of $\mathcal{T}_h$, as discussed in the introduction. We define the set $\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ to be the points of intersection of the edges of $\mathcal{T}_h$ with the edges that adjoin interior points. We also let $\mathcal{E}^b(\mathcal{T}_h^{\mathrm{ps}})$ be the collection of all the new edges of $\mathcal{T}_h^{\mathrm{ps}}$ that were obtained by sub-dividing edges of $\mathcal{T}_h$. We let $\mathcal{E}(\mathcal{T}_h)$ be the edges of $\mathcal{T}_h$. By the construction of $\mathcal{T}_h^{\text{ps}}$ every $x \in \mathcal{M}(\mathcal{T}_h^{\text{ps}})$ belongs to edges that lie on two straight lines. Therefore, these vertices are singular vertices [56]. It is important to note that to make our global spaces to have the correct continuity it is essential to construct the meshes in such a way [47, 52]. Furthermore, as previously mentioned, the divergence of continuous, piecewise polynomials have a weak continuity property at singular vertices, i.e., at the vertices in $\mathcal{M}(\mathcal{T}_h^{\text{ps}})$. In detail, let $z \in \mathcal{M}(\mathcal{T}_h^{\text{ps}})$ and suppose that z is an interior vertex. Then it is a vertex of four triangles $K_1, \ldots, K_4 \in \mathcal{T}_h^{\text{ps}}$. For a function q we define

$$\theta_z(q) := |q|_{K_1}(z) - q|_{K_2}(z) + q|_{K_3}(z) - q|_{K_4}(z)|.$$

Then, if v is a continuous piecewise polynomial with respect to $\mathcal{T}_h^{\text{ps}}$, there holds $\theta_z(\operatorname{div} v) = 0$ [56].

The degrees of freedom stated in Lemmas 3.3.4–3.3.8 induce the following spaces

$$\begin{aligned}
S_r^0(\mathcal{T}_h^{\text{ps}}) =& \{q \in C^1(\Omega) : q|_T \in S_r^0(T^{\text{ps}}) \, \forall T \in \mathcal{T}_h\}, \\
S_r^1(\mathcal{T}_h^{\text{ps}}) =& \{v \in [C(\Omega)]^2 : \operatorname{div} v \in C(\Omega), v|_T \in S_r^1(T^{\text{ps}}) \, \forall T \in \mathcal{T}_h\}, \\
L_r^1(\mathcal{T}_h^{\text{ps}}) =& \{v \in [C(\Omega)]^2 : v|_T \in L_r^1(T^{\text{ps}}) \, \forall T \in \mathcal{T}_h\}, \\
L_r^2(\mathcal{T}_h^{\text{ps}}) =& \{p \in C(\Omega) : p|_T \in L_r^2(T^{\text{ps}}) \, \forall T \in \mathcal{T}_h\}, \\
\mathcal{V}_r^2(\mathcal{T}_h^{\text{ps}}) =& \big\{p \in L^2(\Omega) : p|_T \in V_r^2(T^{\text{ps}}) \, \forall T \in \mathcal{T}_h, \, \theta_z(p) = 0, \, \forall z \in \mathcal{M}(\mathcal{T}_h^{\text{ps}}) \text{ and} \\
& \qquad z \text{ an interior node}\big\}.
\end{aligned}$$

Remark 3.4.1. *Let $z \in \mathcal{M}(\mathcal{T}_h^{ps})$ be an interior vertex and $T_1, T_2 \in \mathcal{T}_h$ share a common edge where z lies. Then $\theta_z(q) = 0$ if and only if $[\![q_1]\!](z) = [\![q_2]\!](z)$ where $q_i = q|_{T_i}$. Therefore, the local degrees of freedom for $V_r^2(T^{\text{ps}})$ with the jump condition* (3.3.4a) *do indeed induce the global space $\mathcal{V}_r^2(\mathcal{T}_h^{ps})$ above.*

We list the degrees of freedom of these spaces. The global DOF come directly from the local DOF. We list them here to be precise.

It follows from Lemma 3.3.4 that a function $q \in S_r^0(\mathcal{T}_h^{\text{ps}})$, with $r \geq 2$, is uniquely determined by

$$
\begin{aligned}
q(z),\ \nabla q(z) \qquad & \text{for every vertex } z \text{ of } \mathcal{T}_h,\\
q(z), \partial_t q(z) \qquad & \forall z \in \mathcal{M}(\mathcal{T}_h^{\text{PS}}),\ \text{if } r \geq 3,\\
\int_e \partial_n q\, p \qquad & \forall p \in \mathcal{P}_{r-3}(e),\ \text{for all } e \in \mathcal{E}^b(\mathcal{T}_h^{\text{PS}})\\
\int_e qp \qquad & \forall p \in \mathcal{P}_{r-4}(e),\ \text{for all } e \in \mathcal{E}^b(\mathcal{T}_h^{\text{PS}}),\\
\int_T \operatorname{rot} q \cdot \operatorname{rot} p \qquad & \forall p \in \mathring{S}_r^0(T^{\text{ps}}),\ \text{for all } T \in \mathcal{T}_h.
\end{aligned}
$$

Remark 3.4.2. *The degrees of freedom for $r = 2$ coincide with the known degrees of freedom of Powell-Sabin [52, 47]. Recently, results for polynomial degrees $r = 3, 4$ have appeared [34, 35].*

Lemma 3.3.5 shows that a function $v \in L_r^1(\mathcal{T}_h^{\text{ps}})$ is uniquely determined by the values

$$
\begin{aligned}
v(z), \qquad & \text{for every vertex } z \text{ of } \mathcal{T}_h,\\
\int_e (v \cdot n), \qquad & \forall e \in \mathcal{E}(\mathcal{T}_h),\ \text{if } r = 1,\\
[\![\operatorname{div} v]\!](z), \qquad & \forall z \in \mathcal{M}(\mathcal{T}_h^{\text{ps}}),\\
v(z) \cdot n, \qquad & \forall z \in \mathcal{M}(\mathcal{T}_h^{\text{ps}}),\ \text{if } r \geq 2,\\
\int_e v \cdot w, \qquad & \forall w \in [\mathcal{P}_{r-2}(e)]^2,\ \forall e \in \mathcal{E}^b(\mathcal{T}_h^{\text{ps}}),\\
\int_T v \cdot \operatorname{rot} w, \qquad & \forall w \in \mathring{S}_{r+1}^0(T^{\text{ps}}), \forall T \in \mathcal{T}_h,\\
\int_T \operatorname{div} v\, w, \qquad & \forall w \in \mathring{\mathcal{V}}_{r-1}^2(T^{\text{ps}}), \forall T \in \mathcal{T}_h.
\end{aligned}
$$

A function $q \in \mathcal{V}_r^2(\mathcal{T}_h^{\text{ps}})$, for $r \geq 0$, is uniquely determined by

$$
\begin{aligned}
[\![q]\!](z), &\qquad \forall z \in \mathcal{M}(\mathcal{T}_h^{\text{ps}}),\\
\int_T q = 0, &\qquad \forall T \in \mathcal{T}_h,\\
\int_T qp &\qquad \forall p \in \mathring{\mathcal{V}}_r^2(\mathcal{T}_h^{\text{ps}}), \forall T \in \mathcal{T}_h.
\end{aligned}
$$

A function $v \in S_r^1(\mathcal{T}_h^{\text{ps}})$ is determined by the following degrees of freedom.

$$
\begin{aligned}
v(z),\ \operatorname{div} v(z) &\qquad \text{for every vertex } z \text{ of } \mathcal{T}_h,\\
\int_e (v \cdot n_i), &\qquad \forall e \in \mathcal{E}(\mathcal{T}_h), \text{if } r = 1,\\
v(z) \cdot n,\ \operatorname{div} v(z) &\qquad \forall z \in \mathcal{M}(\mathcal{T}_h^{\text{ps}}),\ \text{if } r \geq 2,\\
\int_e v \cdot w &\qquad \forall w \in [\mathcal{P}_{r-2}(e)]^2,\ e \in \mathcal{E}^b(\mathcal{T}_h^{\text{ps}}),\\
\int_e (\operatorname{div} v) q &\qquad \forall q \in \mathcal{P}_{r-3}(e),\ e \in \mathcal{E}^b(\mathcal{T}_h^{\text{ps}}),\\
\int_T v \cdot \text{rot } w &\qquad \forall w \in \mathring{S}_{r+1}^0(T^{\text{ps}}) \text{ for all } T \in \mathcal{T}_h,\\
\int_T \operatorname{div} v\, w &\qquad \forall w \in \mathring{L}_{r-1}^2(T^{\text{ps}}) \text{ for all } T \in \mathcal{T}_h.
\end{aligned}
$$

A function $q \in L_r^2(\mathcal{T}_h^{\text{ps}})$, if $r \geq 1$, is determined by the degrees of freedom

$$
\begin{aligned}
q(z) &\qquad 1 \leq i \leq 3, \quad \text{for every vertex } z \text{ of } \mathcal{T}_h,\\
q(z) &\qquad 1 \leq i \leq 3, \quad \forall z \in \mathcal{M}(\mathcal{T}_h^{\text{ps}}),\\
\int_e qp &\qquad \forall p \in \mathcal{P}_{r-2}(e),\ \forall e \in \mathcal{E}^b(T^{\text{ps}}),\\
\int_T q &\\
\int_T qp &\qquad \forall p \in \mathring{L}_r^2(\mathcal{T}_h^{\text{ps}}).
\end{aligned}
$$

Each of the following sequences of spaces forms a complex.

$$\mathbb{R} \longrightarrow S_r^0(\mathcal{T}_h^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} L_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathcal{V}_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) \longrightarrow 0, \quad r \geq 2, \tag{3.4.2a}$$

$$\mathbb{R} \longrightarrow S_r^0(\mathcal{T}_h^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} S_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} L_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) \longrightarrow 0, \quad r \geq 3. \tag{3.4.2b}$$

Remark 3.4.3. *The spaces $L_1^1(\mathcal{T}_h^{ps})$ and $\operatorname{div} L_1^1(\mathcal{T}_h^{ps})$ were considered by Zhang [62] for approximating incompressible flows. In particular, he proved inf-sup stability of this pair. However, he does not explicitly write the relationship $\mathcal{V}_{r-2}^2(\mathcal{T}_h^{ps}) = \operatorname{div} L_{r-1}^1(\mathcal{T}_h^{ps})$, which we know holds.*

Additionally, we can define commuting projections. For example, for the sequences (3.4.2a) and (3.4.2b), we define π_i^r such that, for $0 \leq i \leq 2$, $\pi_i^r v|_T = \Pi_i^r(v|_T)$ for all $T \in \mathcal{T}_h$. By using Theorem 3.3.9, we find that following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(S) & \xrightarrow{\mathrm{rot}} & [C^\infty(S)]^2 & \xrightarrow{\mathrm{div}} & C^\infty(S) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_0^r} & & \downarrow{\scriptstyle \pi_1^{r-1}} & & \downarrow{\scriptstyle \pi_2^{r-2}} & & \\
\mathbb{R} & \longrightarrow & S_r^0(\mathcal{T}_h^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & L_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & \mathcal{V}_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) & \longrightarrow & 0.
\end{array}$$

Similarly, defining the projections $\chi_i^r v|_T = \varpi_i^r(v|_T)$ for $i = 1, 2$, it follows from Theorem 3.3.10 that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(S) & \xrightarrow{\mathrm{rot}} & [C^\infty(S)]^2 & \xrightarrow{\mathrm{div}} & C^\infty(S) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_0^r} & & \downarrow{\scriptstyle \chi_1^{r-1}} & & \downarrow{\scriptstyle \chi_2^{r-2}} & & \\
\mathbb{R} & \longrightarrow & S_r^0(\mathcal{T}_h^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & S_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & L_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) & \longrightarrow & 0.
\end{array}$$

The proofs that these projections commute are similar to the local cases. The top sequences (the non-discrete spaces) are exact if S is simply connected [27]. In the next result, we will show that the bottom sequences (the discrete spaces) are also exact on simply connected domains.

Theorem 3.4.4. *Suppose that Ω is simply connected. Then the sequence* (3.4.2a) *is exact for $r \geq 2$, and the sequence* (3.4.2b) *is exact for $r \geq 3$.*

Proof. Suppose that $v \in L^1_{r-1}(\mathcal{T}_h^{\text{ps}})$ satisfies $\operatorname{div} v = 0$. Using the inclusion $S^1_{r-1}(\mathcal{T}_h^{\text{ps}}) \subset H(\operatorname{div};\Omega)$ and standard results, there exists $q \in H(\operatorname{rot};\Omega)$ such that $v = \text{rot}\, q$. Because v is a piecewise polynomial of degree $r-1$, it follows that q is a piecewise polynomial of degree r. Moreover, v is continuous and therefore $q \in C^1(S)$. Thus it follows that $q \in S^0_r(\mathcal{T}_h^{\text{ps}})$. Note that this result shows that if $v \in L^1_{r-1}(\mathcal{T}_h^{\text{ps}})$ satisfies $\operatorname{div} v = 0$, then $v = \text{rot}\, q$ for some $q \in S^0_r(\mathcal{T}_h^{\text{ps}})$.

Thus to prove the result, it suffices to show that the mappings $\operatorname{div} : L^1_{r-1}(\mathcal{T}_h^{\text{ps}}) \to V^2_{r-2}(\mathcal{T}_h^{\text{ps}})$ and $\operatorname{div} : S^1_{r-1}(\mathcal{T}_h^{\text{ps}}) \to L^2_{r-2}(\mathcal{T}_h^{\text{ps}})$ are surjections. This will be accomplished by showing that $\dim(\operatorname{div} L^1_{r-1}(\mathcal{T}_h^{\text{ps}})) = \dim V^2_{r-2}(\mathcal{T}_h^{\text{ps}})$ and $\dim(\operatorname{div} S^1_{r-1}(\mathcal{T}_h^{\text{ps}})) = \dim L^2_{r-2}(\mathcal{T}_h^{\text{ps}})$.

Denote by $\mathbb{V}$, $\mathbb{E}$, and $\mathbb{T}$ the number of vertices, edges, and triangles in $\mathcal{T}_h$, respectively. The degrees of freedom given above show that, for $r \geq 2$,

$$\begin{aligned}
\dim S^0_r(\mathcal{T}_h^{\text{ps}}) &= 3\mathbb{V} + (4r-8)\mathbb{E} + 3(r-2)(r-3)\mathbb{T}, \\
\dim L^1_{r-1}(\mathcal{T}_h^{\text{ps}}) &= 2\mathbb{V} + (4r-6)\mathbb{E} + 3(r-2)(r-3)\mathbb{T} + \big(3(r-1)r-4\big)\mathbb{T}, \\
\dim V^2_{r-2}(\mathcal{T}_h^{\text{ps}}) &= \mathbb{E} + \mathbb{T} + \big(3(r-1)r-4\big)\mathbb{T}.
\end{aligned}$$

We then find, by the rank–nullity theorem and the Euler relation $\mathbb{V} - \mathbb{E} + \mathbb{T} = 1$ that

$$\begin{aligned}
\dim(\operatorname{div} L^1_{r-1}(\mathcal{T}_h^{\text{ps}})) &= \dim L^1_{r-1}(\mathcal{T}_h^{\text{ps}}) - \dim(\text{rot}\, S^0_r(\mathcal{T}_h^{\text{ps}})) \\
&= \dim L^1_{r-1}(\mathcal{T}_h^{\text{ps}}) - \dim S^0_r(\mathcal{T}_h^{\text{ps}}) + 1 \\
&= \dim L^1_{r-1}(\mathcal{T}_h^{\text{ps}}) - \dim S^0_r(\mathcal{T}_h^{\text{ps}}) + (\mathbb{V} - \mathbb{E} + \mathbb{T})
\end{aligned}$$

$$
\begin{aligned}
&= 2\mathbb{V} + (4r-6)\mathbb{E} + 3(r-2)(r-3)\mathbb{T} + \big(3(r-1)r-4\big)\mathbb{T} \\
&\qquad - \big(3\mathbb{V} + (4r-8)\mathbb{E} + 3(r-2)(r-3)\mathbb{T}\big) + (\mathbb{V} - \mathbb{E} + \mathbb{T}) \\
&= \mathbb{E} + \mathbb{T} + \big(3(r-1)r-4\big)\mathbb{T} = \dim V_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}).
\end{aligned}
$$

Likewise, we have for $r \geq 3$,

$$
\begin{aligned}
\dim S_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) &= 3\mathbb{V} + (6r-12)\mathbb{E} + 3(r-2)(r-3)\mathbb{T} + 3(r-2)(r-3)\mathbb{T}, \\
\dim L_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) &= \mathbb{V} + (2r-5)\mathbb{E} + \mathbb{T} + 3(r-2)(r-3)\mathbb{T},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\dim(\operatorname{div} S_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}})) &= \dim S_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) - \dim S_r^0(\mathcal{T}_h^{\mathrm{ps}}) + (\mathbb{V} - \mathbb{E} + \mathbb{T}) \\
&= 3\mathbb{V} + (6r-12)\mathbb{E} + 6(r-2)(r-3)\mathbb{T} \\
&\qquad - \big(3\mathbb{V} + (4r-8)\mathbb{E} + 3(r-2)(r-3)\mathbb{T}\big) + (\mathbb{V} - \mathbb{E} + \mathbb{T}) \\
&= \mathbb{V} + (2r-5)\mathbb{E} + 3(r-2)(r-3)\mathbb{T} + \mathbb{T} = L_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}).
\end{aligned}
$$

$\square$

We have developed smooth finite element spaces on Powell-Sabin splits that form exact sequences in two dimensions. In the following sections, we extend this work to higher dimensions using the Worsey-Farin split.

3.5 An Application of Powell-Sabin finite elements

As discussed in the introduction, the finite elements developed in this thesis have applications to fluid flow problems. Here, we discuss another application: eigenvalue problems in electro-magnetics. The results in this section are not ours; they have appeared in [17]. With permission from the authors, we will present their findings here to exhibit an important application of the Powell-Sabin finite elements and exact sequences developed in this chapter.

It is well known that using Lagrange finite elements to solve the eigenvalue problem of electro-magnetics normally leads to spurious eigenvalues. However, interestingly, Wong and Cendes [59] numerically found that if one uses the Powell-Sabin split with linear Lagrange elements, the numerical eigenvalues seem to converge to the correct ones. Theoretical justification of this fact remained open until recently [17]. The key to the analysis in [17] is the use of the exact sequence properties developed earlier in this chapter. Let us describe the problem. For an overview of finite elements for eigenvalue problems we refer the reader to [15].

Let $\Omega \subset \mathbb{R}^2$ be a contractible polygonal domain and consider the eigenvalue problem

$$(\text{rot } u, \text{rot } v) = \eta^2(u, v) \quad \forall v \in \mathring{H}(\text{rot}; \Omega).$$

Given a finite element space $\mathring{V}_h \subset \mathring{H}(\text{rot}; \Omega)$, a finite element method seeks $u_h \in \mathring{V}_h \backslash \{0\}$ and $\eta_h \in \mathbb{R}$ satisfying

$$(\text{rot } u_h, \text{rot } v_h) = \eta_h^2(u_h, v_h) \quad \forall v_h \in V_h.$$

It is well known that the Nédélec finite elements do well for this problem and that Lagrange elements generally do not do well. For example, if one uses a generic Delaunay triangulation (see Figure 3.1) and quartic finite elements, Table 3.2 (which is from [17]) shows that the first twenty eigenvalues do not convergence.

On the other hand, using linear Lagrange elements with Powell-Sabin triangulations,

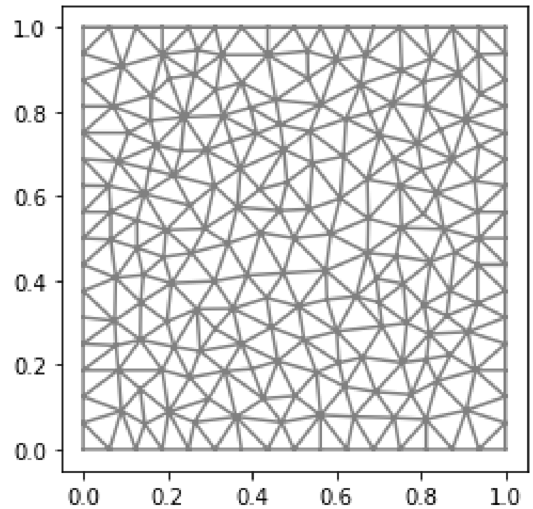

Figure 3.1: Unstructured mesh with $h \approx 1/10$

one can prove that the eigenvalues converge [17]. In Table 3.1 (which is from [17]), one sees convergence to the first eigenvalue. In fact, we see that the first eigenvalue converges like h^2.

h	error of first eigenvalue	rate
2^{-3}	1.084194558097806E-1	
2^{-4}	3.835460507298371E-2	1.8228
2^{-5}	2.952141736802360E-3	1.8768
2^{-6}	7.488421347368046E-4	1.9790

Table 3.1: The rate of convergence with respect to h of first non-zero eigenvalue using for Powell–Sabin split and the linear Lagrange finite element space.

h	error of first twenty eigenvalues	rate
2^{-2}	8.38611345105E-03	
2^{-3}	5.61831120933E-05	7.2217
2^{-4}	59.2176263988	-20.008
2^{-5}	59.2176264065	0.000

Table 3.2: Maximum error of the first 20 eigenvalues on (non-perturbed) Delaunay triangulations using quartic Lagrange elements
.

CHAPTER FOUR

Exact Sequences on Worsey-Farin Splits

4.1 Local Exact Sequences

A crucial result is to prove the local sequences are exact. The first sequences are the ones with homogeneous boundary conditions.

$$0 \longrightarrow \mathring{V}_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} \mathring{V}_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} \mathring{V}_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{4.1.1a}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} \mathring{\mathsf{L}}_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{4.1.1b}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} \mathring{S}_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} \mathring{\mathsf{L}}_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathring{\mathcal{V}}_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0, \tag{4.1.1c}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} \mathring{S}_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} \mathring{S}_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathring{\mathsf{L}}_{r-3}^3(T^{\mathrm{wf}}) \longrightarrow 0. \tag{4.1.1d}$$

The second set of sequences do not have boundary conditions.

$$\mathbb{R} \to V_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} V_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} V_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} V_{r-3}^3(T^{\mathrm{wf}}) \to 0, \tag{4.1.2a}$$

$$\mathbb{R} \to S_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} L_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} V_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} V_{r-3}^3(T^{\mathrm{wf}}) \to 0, \tag{4.1.2b}$$

$$\mathbb{R} \to S_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} S_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} L_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} V_{r-3}^3(T^{\mathrm{wf}}) \to 0, \tag{4.1.2c}$$

$$\mathbb{R} \to S_r^0(T^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} S_{r-1}^1(T^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} S_{r-2}^2(T^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} L_{r-3}^3(T^{\mathrm{wf}}) \to 0. \tag{4.1.2d}$$

We know that the first sequences (4.1.1a) and (4.1.2a) are exact by Nédélec [50]. The major result of this section is the following results.

Theorem 4.1.1. *Let* $r \geq 3$*. Then the sequences* (4.1.1) *are exact.*

Proof. Again, we already know that (4.1.1a) is exact. The exactness of the rest of the sequences follow from the following results that are found below. Corollaries 4.1.17, 4.1.18, Theorem 4.1.5, and Theorem 4.1.11. $\square$

Similarly we can prove.

Theorem 4.1.2. *Let $r \geq 3$. Then the sequences* (4.1.2) *are exact.*

Then we will use the following result which goes back to Arnold and Qin [13].

Lemma 4.1.3. *Let $F \in \Delta_2(T)$, then for $\omega \in \mathring{V}_r^2(F^{\mathrm{ct}})$. There exists a $\rho \in \mathring{\mathsf{L}}_{r+1}^1(F^{\mathrm{ct}})$ such that $\operatorname{div}_F \rho = \omega$ on F. Similarly, there exists $\eta \in \mathring{\mathsf{L}}_{r+1}^1(F^{\mathrm{ct}})$ such that $\operatorname{curl}_F \eta = \omega$.*

We also need the well known result that follows from a simple argument.

Lemma 4.1.4. *Let $F \in \Delta_2(T)$, then for $\omega \in \mathring{V}_r^1(F^{\mathrm{ct}})$ and $\operatorname{div}_F \rho = 0$. There exists a $\rho \in \mathring{\mathsf{L}}_{r+1}^0(F^{\mathrm{ct}})$ such that $\operatorname{rot}_F \rho = \omega$ on F.*

4.1.1 Surjectivity of the divergence operator on discrete local spaces

The goal of this section is to prove the following results.

Theorem 4.1.5. *Let $r \geq 0$. Then:*

(i) for each $p \in \mathring{\mathcal{V}}_r^3(T^{\mathrm{wf}})$, there exists a $v \in \mathring{\mathsf{L}}_{r+1}^2(T^{\mathrm{wf}})$ such that $\operatorname{div} v = p$.

(ii) for each $p \in \mathring{V}_r^3(T^{\mathrm{wf}})$, there exists a $v \in \mathsf{L}_{r+1}^2(T^{\mathrm{wf}}) \cap \mathring{V}_{r+1}^2(T^{\mathrm{wf}})$ such that $\operatorname{div} v = p$.

(iii) for each $p \in V_r^3(T^{\mathrm{wf}})$, there exists a $v \in \mathsf{L}_{r+1}^2(T^{\mathrm{wf}})$ such that $\operatorname{div} v = p$.

(iv) for each $p \in \mathring{\mathsf{L}}^3_r(T^{\mathrm{wf}})$ *(resp.,* $p \in \mathsf{L}^3_r(T^{\mathrm{wf}})$*), there exists a* $v \in \mathring{S}^2_{r+1}(T^{\mathrm{wf}})$ *(resp.,* $v \in S^2_{r+1}(T^{\mathrm{wf}})$*) such that* $\operatorname{div} v = p$*.*

The proofs of Theorems 4.1.5 parts i and ii depend on four preliminary lemmas.

Lemma 4.1.6. *Let* $r \geq 1$ *and* $s \geq 0$ *be integers. Then for any* $q \in \mathcal{V}^3_r(T^{\mathrm{wf}})$*, there exists* $w \in \mathsf{L}^2_r(T^{\mathrm{wf}})$ *and* $g \in V^3_{r-1}(T^{\mathrm{wf}})$*, such that* $\mu^s q = \operatorname{div}(\mu^{s+1} w) + \mu^{s+1} g$*.*

Proof. Let $q \in \mathcal{V}^3_r(T^{\mathrm{wf}})$ and $s \geq 0$. Because $q|_{F_i}$ is continuous on each $F_i \in \Delta_2(T)$, there exists $b_i \in \mathcal{P}_r(F_i)$ such that $b_i = q|_{F_i}$ on ∂F_i. Thus $q - b_i$ is continuous on F_i and vanishes on ∂F_i. Consequently, there exists $a_i \in \mathsf{L}^2_r(T^{\mathrm{wf}})$ such that $a_i = (q - b_i)$ on F_i and supp$(a_i) \subseteq K_i$. Using the divergence-conforming Nédélec degrees of freedom of the second kind [50] , and the fact that $\operatorname{grad} \mu_i$ is parallel to the outward unit normal of F_i, there exists $w_1 \in [\mathcal{P}_r(T)]^3$ such that

$$(s+1) w_1 \cdot \operatorname{grad} \mu_i = b_i \quad \text{on } F_i.$$

We also define $w_2 \in \mathsf{L}^2_r(T^{\mathrm{wf}})$

$$w_2 := \frac{1}{s+1} \sum_{i=1}^{4} a_i \ell_i,$$

where $\ell_i := \frac{\operatorname{grad} \mu_i}{|\operatorname{grad} \mu_i|^2}$. Finally, we set $w := w_1 + w_2 \in \mathsf{L}^2_r(T^{\mathrm{wf}})$. We then see that

$$q - (s+1) w \cdot \operatorname{grad} \mu = 0 \quad \text{on } \partial T,$$

and, hence, there exists $p \in V^3_{r-1}(T^{\mathrm{wf}})$ such that

$$q = (s+1) w \cdot \operatorname{grad} \mu + \mu p \quad \text{on } T.$$

Setting $g := \operatorname{div} w - p \in V_{r-1}^3(T^{\mathrm{wf}})$ we have

$$\mu^s q = (s+1)\mu^s w \cdot \operatorname{grad} \mu + \mu^{s+1} p = \operatorname{div}(\mu^s w) + \mu^{s+1} g.$$

$\square$

We will need the following result in several occasions.

Lemma 4.1.7. *Let $F \in \Delta_2(T)$ and $K \in T^{\mathrm{a}}$ be such that $F \subset \partial K$. If $p \in \mathring{\mathsf{L}}_r^1(F^{\mathrm{ct}})$ then there exists $q \in \mathsf{L}_r^1(T^{\mathrm{wf}})$ such that $q|_F = p$, supp$(q) \subset K$ and $q \cdot n_F = 0$ on K.*

Proof. Let t, s, n_F be an orthonormal set with t and s parallel to F. Then $p = at + bs$ for some $a, b \in \mathring{\mathsf{L}}_r^0(F^{\mathrm{ct}})$. We extend a and b to all of K, which we denote by $\tilde{a}, \tilde{b} \in \mathring{\mathsf{L}}_r^0(K^{\mathrm{wf}})$, by setting all the other Lagrange degrees of freedom to be zero. In particular $\tilde{a}$ and $\tilde{b}$ vanish on $\partial K \backslash F$. Hence, we can further extend them by zero to all of T and $\tilde{a}, \tilde{b} \in \mathring{\mathsf{L}}_r^0(T^{\mathrm{wf}})$. We set $q = \tilde{a}t + \tilde{b}s$. $\square$

Lemma 4.1.8. *For any $\theta \in V_r^3(T^{\mathrm{wf}})$, with $r \geq 0$, there exists $\psi \in \mathsf{L}_{r+1}^2(T^{\mathrm{wf}}) \cap \mathring{V}_{r+1}^2(T^{\mathrm{wf}})$ and $\gamma \in \mathcal{V}_r^3(T^{\mathrm{wf}})$ such that*

$$\mu^s \theta = \operatorname{div}(\mu^s \psi) + \mu^s \gamma \qquad \forall s \geq 0. \tag{4.1.3}$$

Proof. Let $K_i \in \Delta_3(T^{\mathrm{a}})$ be the tetrahedron containing the face $F_i \in \Delta_2(T)$, and let $\kappa_i \in V_0^3(T^{\mathrm{wf}})$ be defined on K_i as $\kappa_i = \frac{1}{|F_i|}\int_{F_i} \theta$. Then on F_i,

$$\int_{F_i} (\theta - \kappa_i) = 0,$$

so $(\theta - \kappa_i)_{F_i} \in \mathring{V}_r^2(F_i^{\mathrm{ct}})$ by definition. Hence, by Lemma 4.1.3, there exists a function

$\rho_i \in \mathring{L}^1_{r+1}(F_i^{\text{ct}})$ such that

$$\operatorname{div}_{F_i} \rho_i = (\theta - \kappa_i) \quad \text{ on } F_i. \tag{4.1.4}$$

Since ρ_i vanishes on ∂F_i, by Lemma 4.1.7, there exists an extension $\psi_i \in \mathsf{L}^3_{r+1}(T^{\text{wf}})$ such that $(\psi_i)_{F_i} = \rho_i$, supp$(\psi_i) \subseteq K_i$, and $\psi_i \cdot n_{F_i} = 0$ on K_i. We then define $\psi = \sum_{i=0}^{3} \psi_i \in \mathsf{L}^3_{r+1}(T^{\text{wf}}) \cap \mathring{V}^2_{r+1}(T^{\text{wf}})$. The construction of ψ, and using (4.1.4), yields the identities

$$\psi \cdot n_{F_i} = 0 \qquad \text{on } K_i, \tag{4.1.5}$$

$$\operatorname{div} \psi = \operatorname{div}_{F_i} \rho_i \qquad \text{on } F_i. \tag{4.1.6}$$

Now set $\gamma := \theta - \operatorname{div} \psi$, so that $\gamma = \kappa_i$ on F_i by (4.1.6) and (4.1.4). Since κ_i is continuous on F_i, it follows that $\gamma \in \mathcal{V}^3_r(T^{\text{wf}})$. Rearranging yields $\theta = \operatorname{div} \psi + \gamma$, which proves the result in the case $s = 0$. Furthermore, since $\operatorname{grad} \mu$ is parallel to n_{F_i} on each K_i, we have by (4.1.5),

$$\mu^s \theta - \operatorname{div}(\mu^s \psi) = \mu^s \theta - \mu^s \operatorname{div} \psi - s\mu^{s-1} \psi \cdot \operatorname{grad} \mu = \mu^s \gamma,$$

which is the desired result. $\square$

Lemma 4.1.9. *Let $q \in \mathcal{V}^3_r(T^{\text{wf}})$ with $r \geq 1$, and $s \geq 0$. Then there exists $v \in \mathsf{L}^2_r(T^{\text{wf}})$ and $Q \in \mathcal{V}^3_{r-1}(T^{\text{wf}})$ such that $\mu^s q = \operatorname{div}(\mu^{s+1} v) + \mu^{s+1} Q$.*

Proof. By Lemma 4.1.6, there exist $w \in \mathsf{L}^2_r(T^{\text{wf}})$ and $g \in V^3_{r-1}(T^{\text{wf}})$ such that

$$\mu^s q = \operatorname{div}(\mu^{s+1} w) + \mu^{s+1} g.$$

Since $g \in V^3_{r-1}(T^{\mathrm{wf}})$, Lemma 4.1.8 yields the existence of $\psi \in \mathsf{L}^2_r(T^{\mathrm{wf}})$ and $Q \in \mathcal{V}^3_{r-1}(T^{\mathrm{wf}})$ such that

$$\mu^{s+1}g = \operatorname{div}\,(\mu^{s+1}\psi) + \mu^{s+1}Q.$$

Therefore, $\mu^s q = \operatorname{div}\,(\mu^{s+1}(w+\psi)) + \mu^{s+1}Q$. Setting $v = w + \psi$ achieves the desired result. □

The final preliminary lemma follows from a result shown in [33].

Lemma 4.1.10. *Let $s \geq 0$, and let $q \in \mathcal{V}^3_0(T^{\mathrm{wf}})$ with $\int_T \mu^s q = 0$. Then there exists $w \in \mathsf{L}^2_0(T^{\mathrm{wf}})$ such that $\mu^s q = \operatorname{div}\,(\mu^{s+1}w)$.*

Proof. Since $q \in \mathcal{V}^3_0(T^{\mathrm{wf}})$ it is easy to see that $q \in V^3_0(T^{\mathrm{a}})$. From [33, Lemma 3.11], there exists $w \in [\mathcal{P}_0(T)]^3 \subset \mathsf{L}^2_0(T^{\mathrm{wf}})$ such that $\operatorname{div}\,(\mu^{s+1}w) = q$. □

We can now prove Theorem 4.1.5 Parts i and ii.

proof of Theorem 4.1.5, Part i. Let $1 \leq \ell \leq r-1$ and $p_r = p$. Suppose we have constructed $w_{r-j} \in \mathsf{L}^2_{r-j}(T^{\mathrm{wf}})$ with $0 \leq j \leq \ell - 1$ and $p_{r-j} \in \mathcal{V}^3_{r-j}(T^{\mathrm{wf}})$ with $0 \leq j \leq \ell$ such that

$$\operatorname{div}\,(\mu^{j+1}w_{r-j}) = \mu^j p_{r-j} - \mu^{j+1}p_{r-(j+1)}, \qquad 0 \leq j \leq \ell - 1.$$

We apply Lemma 4.1.9 to find $w_{r-\ell} \in \mathsf{L}^2_{r-\ell}(T^{\mathrm{wf}})$ and $p_{r-(\ell+1)} \in \mathcal{V}^3_{r-(\ell+1)}(T^{\mathrm{wf}})$ such that

$$\operatorname{div}\,(\mu^{\ell+1}w_{r-\ell}) = \mu^\ell p_{r-\ell} - \mu^{\ell+1}p_{r-(\ell+1)}. \tag{4.1.7}$$

By induction, there exists $w_{r-\ell} \in \mathsf{L}^2_{r-\ell}(T^{\mathrm{wf}})$ for $0 \le \ell \le r-1$ and $p_{r-\ell} \in \mathcal{V}^3_{r-\ell}(T^{\mathrm{wf}})$ for $0 \le \ell \le r$ such that (4.1.7) holds. Therefore,

$$\operatorname{div}(\mu w_r + \mu^2 w_{r-1} + \cdots + \mu^r w_1) = p - \mu^r p_0.$$

By the hypothesis $\int_T p = 0$, there holds $\int_T \mu^r p_0 = 0$. By Lemma 4.1.10, there exists $w_0 \in \mathsf{L}^2_0(T^{\mathrm{wf}})$ such that $\operatorname{div}(\mu^{r+1} w_0) = \mu^r p_0$. The result follows by setting $v = \mu w_r + \mu^2 w_{r-1} + \cdots + \mu^r w_1 + \mu^{r+1} w_0$. □

proof of Theorem 4.1.5, Part ii. By Lemma 4.1.8 (with $s = 0$), there exists $\psi \in \mathsf{L}^2_{r+1}(T^{\mathrm{wf}}) \cap \mathring{V}^2_{r+1}(T^{\mathrm{wf}})$ and $\gamma \in \mathcal{V}^3_r(T^{\mathrm{wf}})$ satisfying

$$p = \operatorname{div} \psi + \gamma.$$

Note that $\int_T p = 0$, and $\int_T \operatorname{div} \psi = \int_{\partial T} \psi \cdot n = 0$ since $\psi \cdot n = 0$ on ∂T. Thus, we have that $\int_T \gamma = 0$ which implies $\gamma \in \mathring{\mathcal{V}}^3_r(T^{\mathrm{wf}})$. Therefore, we apply Part i of Theorem 4.1.5 to find $g \in \mathring{\mathsf{L}}_{r+1}(T^{\mathrm{wf}})$ such that $\operatorname{div} g = \gamma$. The result follows by setting $v = \psi + g$. □

We now prove Parts iii and iv of Theorem 4.1.5, which are corollaries to Parts i and ii of Theorem 4.1.5.

proof of Part iii. We decompose $p = (p - \overline{p}) + \overline{p}$ where $\overline{p} := \frac{1}{|T|} \int_T p$. There exists $w \in [\mathcal{P}_1(T)]^3$ such that $\operatorname{div} w = \overline{p}$, and by Part ii of Theorem 4.1.5 we have $\psi \in \mathsf{L}^2_{r+1}(T^{\mathrm{wf}}) \cap \mathring{V}^2_{r+1}(T^{\mathrm{wf}})$ such that $\operatorname{div} \psi = p - \overline{p}$. Thus, setting $v := \psi + w$ completes the proof. □

proof of Part iv. Let $p \in \mathring{\mathsf{L}}^3_r(T^{\mathrm{wf}}) \subset \mathring{\mathcal{V}}^3_r(T^{\mathrm{wf}})$. Applying Part i of Theorem 4.1.5, we find $v \in \mathring{\mathsf{L}}^2_{r+1}(T^{\mathrm{wf}})$ such that $\operatorname{div} v = p$. But clearly $v \in \mathring{S}^2_{r+1}(T^{\mathrm{wf}})$, since $\operatorname{div} v$ is continuous and has average zero by definition of $\mathring{\mathsf{L}}^3_r(T^{\mathrm{wf}})$. □

4.1.2 Surjectivity of the curl operator on discrete local spaces

The main goal of this section is to derive the analagous results of Section 4.1.1, but for the curl operator; that is, we show the curl operator acting on piecewise polynomial spaces with respect to the Worsey–Farin split is surjective onto spaces of divergence–free functions. Before making this precise and to prove the result, we first state a simple result that will be used many times in the arguments below.

The main results of this section are the following.

Theorem 4.1.11. *Let $r \geq 0$. Then:*

(i) for any $v \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$ satisfying $\operatorname{div} v = 0$ there exists $w \in \mathring{\mathsf{L}}_{r+1}^1(T^{\mathrm{wf}})$ satisfying $\operatorname{curl} w = v$.

(ii) let $v \in V_r^2(T^{\mathrm{wf}})$ with $\operatorname{div} v = 0$. Then there exists $w \in \mathsf{L}_{r+1}^1(T^{\mathrm{wf}})$ such that $\operatorname{curl} w = v$.

(iii) for each $v \in \mathring{\mathsf{L}}_r^2(T^{\mathrm{wf}})$ (resp., $v \in \mathsf{L}_r^2(T^{\mathrm{wf}})$) where $\operatorname{div} v = 0$, there exists a $w \in \mathring{S}_{r+1}^1(T^{\mathrm{wf}})$ (resp., $w \in S_{r+1}^1(T^{\mathrm{wf}})$) such that $\operatorname{curl} w = v$.

(iv) for each $v \in \mathring{S}_r^2(T^{\mathrm{wf}})$ (resp., $v \in S_r^2(T^{\mathrm{wf}})$) where $\operatorname{div} v = 0$, there exists $w \in \mathring{S}_{r+1}^1(T^{\mathrm{wf}})$ (resp., $w \in S_{r+1}^1(T^{\mathrm{wf}})$ such that $\operatorname{curl} w = v$.

We omit the proofs of Parts iii and iv of Theorem 4.1.11 since they follow easily from Parts i and ii of the same theorem.

Before we prove Parts i and ii of Theorem 4.1.11, we first establish several lemmas.

Lemma 4.1.12. *Let* $r \geq 0$ *and let* $v \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$. *Then there exist functions* $z \in [\mathcal{P}_r(T^{\mathrm{wf}})]^3$ *and* $\gamma \in [\mathcal{P}_{r-1}(T^{\mathrm{wf}})]^3$ *such that*

$$v = \operatorname{grad}\mu \times z + \mu\gamma, \tag{4.1.8}$$

and so $\operatorname{grad}\mu \times z$ *is continuous on* F *for each* $F \in \Delta_2(T)$. *Moreover,* $z \cdot t$ *is single-valued for all* $e \in \Delta_1(T)$, *where* t *is a unit tangent vector to* e.

Proof. By [33, Lemma 4.1], there exists $z \in [\mathcal{P}_r(T^{\mathrm{wf}})]^3$ and $\gamma \in [\mathcal{P}_{r-1}(T^{\mathrm{wf}})]^3$ such that (4.1.8) holds. For each $F \in \Delta_2(T)$, there holds $v = \operatorname{grad}\mu \times z$ on F, and hence $\operatorname{grad}\mu \times z$ is continuous on F. Following exactly the proof of [33, Lemma 4.2], we see that $z \cdot t$ is single-valued for all $e \in \Delta_1(T)$. $\square$

Lemma 4.1.13. *For any* $v \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$, *with* $r \geq 1$, *and any integer* $s \geq 0$, *there exists* $w \in \mathsf{L}_r^1(T^{\mathrm{wf}})$ *and* $g \in V_{r-1}^2(T^{\mathrm{wf}})$ *such that*

$$\mu^s v = \operatorname{curl}(\mu^{s+1} w) + \mu^{s+1} g. \tag{4.1.9}$$

Proof. From Lemma 4.1.12, there exists $z \in [\mathcal{P}_r(T^{\mathrm{wf}})]^3$ and $\gamma \in [\mathcal{P}_{r-1}(T^{\mathrm{wf}})]^3$ satisfying (4.1.8) with $z \times \operatorname{grad}\mu$ continuous on F for each $F \in \Delta_2(T)$ and $z \cdot t$ is single-valued for all $e \in \Delta_1(T)$. Let $\{F_i\}_{i=0}^3$ be the four faces of T. For each i we choose $b_i \in [\mathcal{P}_r(F_i)]^2$ so that $b_i = \operatorname{grad}\mu \times z$ on ∂F_i, which we are allowed to do since $z \times \operatorname{grad} u$ continuous on F_i. Since $z \cdot t$ is single valued for all $e \in \Delta_1(T)$ we have that $b_i \cdot t = b_j \cdot t$ if $e = F_i \cap F_j$. Hence, using the curl-conforming Nédélec degrees of freedom of the second kind [50], there exists $w_1 \in [\mathcal{P}_r(T)]^3$ such that

$$\operatorname{grad}\mu \times w_1 = b_i \qquad \text{on } F_i,\ 0 \leq i \leq 3.$$

Since $\operatorname{grad}\mu\times z|_{F_i}-b_i\in \mathring{\mathsf{L}}_r^1(F_i^{\mathrm{ct}})$, according to Lemma 4.1.7, there exists $a_i\in \mathsf{L}_r^1(T^{\mathrm{wf}})$ such that supp $(a_i)\subset K_i$ and $\operatorname{grad}\mu\times a_i=\operatorname{grad}\mu\times z-b_i$ on F_i. We set $w_2:=\sum_{i=0}^3 a_i$ and finally $w:=\frac{1}{s+1}(w_1+w_2)\in \mathsf{L}_r^1(T^{\mathrm{wf}})$. Hence,

$$
\begin{aligned}
(s+1)\operatorname{grad}\mu\times w =&\operatorname{grad}\mu\times w_1+\operatorname{grad}\mu\times w_2\\
=&b_i+\operatorname{grad}\mu\times a_i\\
=&\operatorname{grad}\mu\times z \qquad\qquad\qquad \text{on } F_i,\ 0\le i\le 3.
\end{aligned}
$$

Thus, we have $\phi\in[\mathcal{P}_{r-1}(T^{\mathrm{wf}})]^3$ such that

$$
(s+1)\operatorname{grad}\mu\times w=\operatorname{grad}\mu\times z+\mu\phi=v+\mu(\theta-\gamma)\quad \text{on } T. \tag{4.1.10}
$$

We write $\operatorname{curl}(\mu^{s+1}w)=(s+1)\mu^s\operatorname{grad}\mu\times w+\mu^{s+1}\operatorname{curl}w=\mu^s v+\mu^{s+1}(\operatorname{curl}w-\gamma+\phi)$. Setting $g:=-(\operatorname{curl}w-\gamma+\phi)$, we have that (4.1.9) holds. Finally, since $\mu^s v\cdot n$ and $\operatorname{curl}(\mu^{s+1}w)\cdot n$ are single-valued on interior faces, $\mu^{s+1}g\cdot n$ is single-valued. Because μ is continuous and strictly positive in the interior of T, this implies $g\cdot n$ is single-valued on interior faces, and thus $g\in V_{r-1}^2(F^{\mathrm{ct}})$. $\square$

We will use the following Lemma repeatedly.

Lemma 4.1.14. *For any $g\in \mathring{V}_r^2(T^{\mathrm{wf}})$ we have that $g_F\in H(\operatorname{div}_F;F)$ for $F\in\Delta_2(T)$.*

Proof. Let $e\in F^{\mathrm{ct}}$, and let f be the corresponding an internal face of T^{wf} that has e as an edge. We let t be a unit vector parallel to e and set $s=t\times n_F$. Note that $\{n_F,s,t\}$ forms an orthonormal basis of $\mathbb{R}^3$. To prove $g_F\in H(\operatorname{div}_F;F)$, it suffices to show $g_F\cdot s$ is single-valued on e.

Let n_f be a unit-normal to f. Since $n_f\cdot t=0$, we have that $n_f=(n_f\cdot s)s+(n_f\cdot n_F)n_F$

and thus, $g \cdot n_f = g \cdot s(n_f \cdot s) + g \cdot n_F(n_f \cdot n_F)$ on e. However, $g \cdot n_F = 0$ on F by definition of $\mathring{V}_r^2(T^{\mathrm{wf}})$, and so $g \cdot n_f = g \cdot s(n_f \cdot s)$ on e. Since $g \cdot n_f$ is single valued on e (since $e \subset \partial f$ and $g \in V_r^2(T^{\mathrm{wf}})$) we have that $g \cdot s$ is single valued on e. Finally, since $g_F \cdot s = g \cdot s$ we conclude $g_F \in H(\mathrm{div}_F; F)$. □

Lemma 4.1.15. *Let $r \geq 0$ and $s \geq 0$. For any $g \in \mathring{V}_r^2(T^{\mathrm{wf}})$ there exists $\psi \in \mathsf{L}_{r+1}^1(T^{\mathrm{wf}})$ and $\gamma \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$ such that*

$$\mu^s g = \mathrm{curl}\,(\mu^s \psi) + \mu^s \gamma.$$

Proof. By Lemma 4.1.14, $g_F \in H(\mathrm{div}\,_F; F)$ for $F \in \Delta_2(T)$. Next, let $\{F_i\}_{i=0}^3$ be the four faces of T. We use the (two-dimensional) divergence-conforming Nédélec degrees of freedom to construct $p_i \in [\mathcal{P}_r(F_i)]^2$ so that for $r \geq 1$,

$$p_i \cdot (n_F \times t) = g_{F_i} \cdot (n_F \times t) \qquad \text{on } e, \forall e \in \Delta_1(F_i),$$

where t is tangent to the edge e. If $r = 0$, we can satisfy the above equation for two of the three edges, however, on the third edge the equation will be automatically be satisfied since $\mathrm{div}_{F_i}\,(g_{F_i} - p_i) = 0$.

Using $g_{F_i} - p_i \in \mathring{V}_{\mathrm{div},r}^1(F_i^{\mathrm{ct}})$ and Stokes theorem, there holds $\int_{F_i} \mathrm{div}\,(g_{F_i} - p_i) = 0$ and, hence, $\mathrm{div}\,(g_{F_i} - p_i) \in \mathring{V}_{r-1}^2(F_i^{\mathrm{ct}})$. By Lemma 4.1.3, there exists $m_i \in \mathring{\mathsf{L}}_r^1(F_i^{\mathrm{ct}})$ so that $\mathrm{div}_{F_i} m_i = \mathrm{div}_{F_i}\,(g_{F_i} - p_i)$ on F_i. Thus, if we let $\theta_i := p_i + m_i$ we have $\theta_i \in \mathsf{L}_r^1(F_i^{\mathrm{ct}})$ and $g_{F_i} - \theta_i \in \mathring{V}_{\mathrm{div},r}^1(F_i^{\mathrm{ct}})$ with $\mathrm{div}_{F_i}\,(g_{F_i} - \theta_i) = 0$. By Lemma 4.1.4, there exists $\kappa_i \in \mathring{\mathsf{L}}_{r+1}^0(F_i^{\mathrm{ct}})$ such that $\mathrm{rot}_{F_i}\kappa_i = g_{F_i} - \theta_i$. Since κ_i vanishes on ∂F_i there exists $\beta_i \in \mathring{\mathsf{L}}_{r+1}^0(T^{\mathrm{wf}})$ with $\mathrm{supp}\,(\beta_i) \subset K_i$ such that $\beta_i = \kappa_i$ on F_i. We let $\psi = \sum_{i=0}^3 \beta_i n_{F_i} \in \mathring{\mathsf{L}}_{r+1}^1(T^{\mathrm{wf}})$. Note that this immediately implies that $\mathrm{grad}\,\mu \times \psi \equiv 0$ on T. Also, we have that

$$\mathrm{curl}\,\psi = \mathrm{grad}\,\beta_i \times n_{F_i} = \mathrm{rot}\,_{F_i}(\kappa_i) = g_{F_i} - \theta_i \qquad \text{on } F_i.$$

Setting $\gamma = g - \mathrm{curl}\,\psi$ we see that $\gamma \in \mathring{V}_r^2(T^{\mathrm{wf}})$. Moreover, noting, in addition, to the above equation, that $\mathrm{curl}\,\psi|_{F_i} = (\mathrm{curl}\,\psi)_{F_i}$ since $\mathrm{curl}\,\psi \cdot n_{F_i} = 0$ on F_i, we see that $\gamma_{F_i} = \theta_i \in \mathsf{L}_r^1(F_i^{\mathrm{ct}})$ and, hence, $\gamma \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$. Finally, since $\mathrm{grad}\,\mu \times \psi \equiv 0$ we have $\mathrm{curl}\,(\mu^s\psi) = \mu^s \mathrm{curl}\,\psi = \mu^s(g-\gamma)$. □

Lemma 4.1.16. *Let* $r \geq 1, s \geq 0$ *then for any* $v \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$ *such that* $\mathrm{div}\,(\mu^s v) = 0$ *on* T *there exists* $w \in L_r^1(T^{\mathrm{wf}})$ *and* $g \in \mathring{\mathcal{V}}_{r-1}^2(T^{\mathrm{wf}})$ *satisfying* $\mu^s v = \mathrm{curl}\,(\mu^{s+1}w) + \mu^{s+1}g$.

Proof. By (4.1.9) we have $w_1 \in \mathsf{L}_r^1(T^{\mathrm{wf}})$ and $q \in V_{r-1}^2(T^{\mathrm{wf}})$ satisfying

$$\mu^s v = \mathrm{curl}\,(\mu^{s+1}w_1) + \mu^{s+1}g_1 \tag{4.1.11}$$

By our hypothesis we have $0 = \mathrm{div}\,(\mu^{s+1}g_1) = \mu^s((s+1)\mathrm{grad}\,\mu \cdot g_1 + \mu \mathrm{div}\,g_1)$. Hence, $(s+1)\mathrm{grad}\,\mu \cdot g_1 + \mu\mathrm{div}\,g_1 = 0$ on T which implies $(\mathrm{grad}\,\mu) \cdot g_1 = 0$ on ∂T. In other words, we have $g_1 \in \mathring{V}_{r-1}^2(T^{\mathrm{wf}})$. We then apply Lemma 4.1.15 to write $\mu^{s+1}g_1 = \mathrm{div}\,(\mu^{s+1}w_2) + \mu^{s+1}g_2$ where $w_2 \in \mathsf{L}_r^1(T^{\mathrm{wf}})$ and $g_2 \in \mathring{\mathcal{V}}_{r-1}^2(T^{\mathrm{wf}})$. The proof is complete if we set $w := w_1 + w_2$ and $g = g_2$. □

Now we can prove Parts i and ii of Theorem 4.1.11.

proof of Part i of Theorem 4.1.11. Assume that we have found $w_r, \ldots, w_{r-j}$ with $w_\ell \in L_r^1(T^{\mathrm{wf}})$ and $g_{r-(j+1)} \in \mathring{\mathcal{V}}_{r-(j+1)}^2(T^{\mathrm{wf}})$ such that

$$v = \mathrm{curl}\,(\mu w_r + \mu^2 w_{r-1} + \cdots + \mu^{j+1}w_{r-j}) + \mu^{j+1}g_{r-(j+1)}.$$

Since $\mathrm{div}\,(\mu^{j+1}g_{r-(j+1)}) = 0$ on T, we apply apply Lemma 4.1.16 to get

$$\mu^{j+1}g_{r-(j+1)} = \mathrm{curl}\,(\mu^{j+2}w_{r-(j+1)}) + \mu^{j+2}g_{r-(j+2)},$$

where $w_{r-(j+1)} \in L^1_{r-(j+1)}(T^{\text{wf}})$ and $g_{r-(j+2)} \in \mathring{\mathcal{V}}^2_{r-(j+2)}(T^{\text{wf}})$. It follows that

$$v = \operatorname{curl}\left(\mu w_r + \mu^2 w_{r-1} + \cdots + \mu^{j+1} w_{r-j} + \mu^{j+2} w_{r-(j+1)}\right) + \mu^{j+2} g_{r-(j+2)}.$$

Continuing by induction, we have

$$v = \operatorname{curl}\left(\mu w_r + \mu^2 w_{r-1} + \cdots + \mu^r w_1\right) + \mu^r g_0, \qquad \text{with } g_0 \in \mathring{\mathcal{V}}^2_0(T^{\text{wf}}).$$

It is easy to see that $g_0 \in \mathring{V}^2_0(T^{\text{a}})$. Hence by Lemma 4.3 in [33] there exists $w_0 \in [\mathcal{P}_0(T)]^3 \subset \mathsf{L}^1_0(T^{\text{wf}})$ such that $\operatorname{curl}\left(\mu^{r+1} w_0\right) = \mu^r g_0$. Setting $w := \mu w_r + \mu^2 w_{r-1} + \cdots + \mu^{r+1} w_0$ completes the proof. $\square$

proof of Part ii of Theorem 4.1.11. Set $\phi = v - \Pi_0^{\text{RT}} v$, where $\Pi_0^{\text{RT}} v$ is the lowest-order Raviart-Thomas projection of v on T. Then $\int_{F_i} \phi \cdot n_{F_i} = 0$ for each $F_i \in \Delta_2(T)$. Applying Lemma 4.1.3, there exists a $\rho_i \in \mathring{\mathsf{L}}^1_{r+1}(F_i^{\text{ct}})$ such that $\operatorname{curl}_{F_i} \rho_i = \phi \cdot n_{F_i}$ on F_i. By Lemma 4.1.7 we can extend ρ_i to a function $p_i \in L^1_{r+1}(T^{\text{wf}})$ with support only on K_i, such that $n \times p_i \times n = \rho_i$ on F_i. We let $p = \sum_{i=0}^3 p_i \in \mathsf{L}^1_{r+1}(T^{\text{wf}})$. Hence, $\operatorname{curl} p \cdot n_{F_i} = \phi \cdot n_{F_i}$ on F_i. Furthermore, there exists $s \in [\mathcal{P}_1(T)]^3$ such that $\operatorname{curl} s = \Pi_0^{\text{RT}} v$. We set $\psi := s + p \in \mathsf{L}^1_{r+1}(T^{\text{wf}})$, then

$$v \cdot n_{F_i} = (\phi + \Pi_0^{\text{RT}} v) \cdot n_{F_i} = (\operatorname{curl} \psi) \cdot n_{F_i} \qquad \text{on } F_i.$$

Hence, we see that $v - \operatorname{curl} \psi \in \mathring{V}^2_r(T^{\text{wf}})$. By Lemma 4.1.15 we have $v - \operatorname{curl} \psi = \operatorname{curl} m + \gamma$ where $m \in \mathsf{L}^1_{r+1}(T^{\text{wf}})$ and $\gamma \in \mathring{\mathcal{V}}^2_r(T^{\text{wf}})$. By Part i of Theorem 4.1.11, there exists $z \in \mathring{\mathsf{L}}^1_{r+1}(T^{\text{wf}})$ such that $\operatorname{curl} z = \gamma$. Setting $w = \psi + m + z$ completes the proof. $\square$

4.1.3 Surjectivity of the gradient operator on discrete local spaces

Corollary 4.1.17. *For each* $v \in \mathring{\mathsf{L}}_r^1(T^{\mathrm{wf}})$ *(resp.,* $v \in \mathsf{L}_r^1(T^{\mathrm{wf}})$*) with* $\operatorname{curl} v = 0$*, there exists a* $w \in \mathring{S}_{r+1}^0(T^{\mathrm{wf}})$ *(resp.,* $w \in S_{r+1}^0(T^{\mathrm{wf}})$*) such that* $\operatorname{grad} w = v$*.*

Proof. Let $v \in \mathring{L}_r^1(T^{\mathrm{wf}}) \subset \mathring{V}_r^1(T^{\mathrm{wf}})$ such that $\operatorname{curl} v = 0$. Then there exists $w \in \mathring{V}_{r+1}^0(T^{\mathrm{wf}})$ such that $\operatorname{grad} w = v$. However, clearly $w \in \mathring{S}_{r+1}^0(T^{\mathrm{wf}})$. □

Another simple corollary is the following result:

Corollary 4.1.18. *For each* $v \in \mathring{S}_r^1(T^{\mathrm{wf}})$ *(resp.,* $v \in S_r^1(T^{\mathrm{wf}})$*) where* $\operatorname{curl} v = 0$*, there exists a* $w \in \mathring{S}_{r+1}^0(T^{\mathrm{wf}})$ *(resp.,* $w \in S_{r+1}^0(T^{\mathrm{wf}})$*) such that* $\operatorname{grad} w = v$*.*

4.2 Dimension Counts

Here, we give dimension counts for the spaces that will be used in the next section where we give degrees of freedom. We start by listing the dimension counts of the the Nédélec and Lagrange spaces. These counts follow from well-known dimension formulas of these spaces and the fact that T^{wf} contains 9 vertices, 26 edges, 30 faces, and 12 tetrahedra.

$$\dim V_r^0(T^{\mathrm{wf}}) = (2r+1)(r^2+r+1), \qquad \dim V_r^1(T^{\mathrm{wf}}) = 2(r+1)(3r^2+6r+4) \tag{4.2.1a}$$

$$\dim V_r^2(T^{\mathrm{wf}}) = 3(r+1)(r+2)(2r+3), \quad \dim V_r^3(T^{\mathrm{wf}}) = 2(1+r)(2+r)(3+r), \tag{4.2.1b}$$

$$\dim \mathsf{L}_r^0(T^{\mathrm{wf}}) = (2r+1)(r^2+r+1), \qquad \dim \mathsf{L}_r^1(T^{\mathrm{wf}}) = 3\dim L_r^0(T^{\mathrm{wf}}). \tag{4.2.1c}$$

and of course and $\dim \mathsf{L}_r^2(T^{\text{wf}}) = \dim \mathsf{L}_r^1(T^{\text{wf}})$.

With homogenous boundary conditions, we use that T^{wf} contains 1 internal vertex, 8 internal edges, and 18 internal faces to conclude

$$\dim \mathring{V}_r^0(T^{\text{wf}}) = (2r-1)(r^2-r+1), \qquad \dim \mathring{V}_r^1(T^{\text{wf}}) = 2(r+1)(3r^2+1), \tag{4.2.2a}$$

$$\dim \mathring{V}_r^2(T^{\text{wf}}) = 3(1+r)(2+r)(1+2r), \quad \dim \mathring{V}_r^3(T^{\text{wf}}) = 2r^3+12r^2+22r+11, \tag{4.2.2b}$$

$$\dim \mathring{\mathsf{L}}_r^0(T^{\text{wf}}) = (2r-1)(r^2-r+1), \qquad \dim \mathring{\mathsf{L}}_r^1(T^{\text{wf}}) = 3(2r-1)(r^2-r+1), \tag{4.2.2c}$$

$$\dim \mathring{\mathsf{L}}_r^2(T^{\text{wf}}) = 3(2r-1)(r^2-r+1), \qquad \dim \mathring{\mathsf{L}}_r^3(T^{\text{wf}}) = (r-1)(2r^2-r+2). \tag{4.2.2d}$$

In order to calculate the dimension count of the rest of the spaces, we need the dimension counts of $\mathring{\mathcal{V}}_r^2(T^{\text{wf}})$ and $\mathring{\mathcal{V}}_r^3(T^{\text{wf}})$. For each $F \in \Delta_2(T)$, let $e_F \in \Delta_1^I(F^{\text{ct}})$ be an arbitrary, but fixed, internal edge of F^{ct}.

Lemma 4.2.1. *Let $p \in V_{\text{div},r}^1(F^{\text{ct}})$ and suppose that*

$$\int_{e_F} [\![p \cdot t]\!] m = 0 \quad \textit{for all } m \in \mathcal{P}_r(e_F) \tag{4.2.3a}$$

$$\int_{e} [\![p \cdot t]\!] m = 0 \quad \textit{for all } m \in \mathcal{P}_{r-1}(e), \forall e \in \Delta_1^I(F^{\text{ct}}) \backslash \{e_F\}, \tag{4.2.3b}$$

where t is the unit vector tangent to an edge e. Then, $p \in \mathsf{L}_r^1(F^{\text{ct}})$.

Proof. Let $e \in \Delta_1^I(F^{\text{ct}})$, and let s be a vector parallel to F that is perpendicular to the edge e. Then since $p \in V_{\text{div},r}^1(F^{\text{ct}})$, $[\![p \cdot s]\!] = 0$. In order to show that $p \in \mathsf{L}_r^1(F^{\text{ct}})$ we need to show that $[\![p \cdot t]\!] = 0$ for all internal edges $e \in \Delta_1^I(F^{\text{ct}})$. By (4.2.3a) this is certainly true for $e = e_F$. In fact, this shows that p is continuous accross e_F. Since $[\![p \cdot s]\!] = 0$ on the two remaining edges this show that p is continuous on the interior vertex z. In particular, $[\![p \cdot t]\!](z)$ vanishes on the two remaining edges. Hence, using (4.2.3b) shows that $[\![p \cdot t]\!] = 0$. $\square$

Corollary 4.2.2. *Let $v \in \mathring{V}_r^2(T^{\mathrm{wf}})$ and suppose that for all $F \in \Delta_2(T)$, the following holds*

$$
\begin{aligned}
\int_{e_F} [\![v_F \cdot t]\!] m = 0 \quad & \textit{for } m \in \mathcal{P}_r(e_F), \\
\int_{e} [\![v_F \cdot t]\!] m = 0 \quad & \textit{for } m \in \mathcal{P}_{r-1}(e), \forall e \in \Delta_1^I(F^{\mathrm{ct}}) \backslash \{e_F\}.
\end{aligned}
$$

Then, $v \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$.

Proof. By Lemma 4.1.14 we have $v_F \in V^1_{\mathrm{div}\,,r}(F^{\mathrm{ct}})$ for all $F \in \Delta_2(T)$. The result now follows by applying Lemma 4.2.1. $\square$

We see that the number of constraints in Corollary 4.2.2 is $4(3r+1)$. We use this result to determine the dimension of the space $\mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$.

Lemma 4.2.3. *Let $v \in \mathring{\mathcal{V}}_r^2(T^{\mathrm{wf}})$ with $r \geq 1$. Then v is fully determined by the following degrees of freedom.*

$$v|_f \cdot n_f(a), \qquad \forall a \in \Delta_0(T), \qquad \forall f \in \Delta_2^I(T^{\mathrm{wf}}), a \subset \overline{f}, \tag{4.2.4a}$$

$$\int_e (v|_f \cdot n_f)\kappa \, ds, \quad \forall \kappa \in \mathcal{P}_{r-2}(e), \quad \forall e \in \Delta_1(T), \forall f \in \Delta_2(T^{\mathrm{wf}}), \subset e \subset \overline{f}, \tag{4.2.4b}$$

$$\int_e (v_F \cdot t)\kappa \, ds, \quad \forall \kappa \in \mathcal{P}_{r-2}(e), \quad \forall e \in \Delta_1(F^{\mathrm{ct}}) \backslash \Delta_1^I(F^{\mathrm{ct}}), \forall F \in \Delta_2(T), \tag{4.2.4c}$$

$$\int_F v_F \cdot \kappa \, dx, \qquad \forall \kappa \in \mathring{\mathsf{L}}_r^1(F^{\mathrm{ct}}), \qquad \forall F \in \Delta_2(T), \tag{4.2.4d}$$

$$\int_T v \cdot \kappa \, dx, \qquad \forall \kappa \in V_{r-1}^2(T^{\mathrm{wf}}). \tag{4.2.4e}$$

Here t *is tangent to* e*. Furthermore,* $\dim \mathring{\mathcal{V}}_r^2(T^{\text{wf}}) = 6r^3 + 21r^2 + 9r + 2.$

Proof. From Corollary 4.2.2 we have

$$\dim \mathring{\mathcal{V}}_r^2(T^{\text{wf}}) \geq \dim \mathring{V}_r^2(T^{\text{wf}}) - 4(3r+1) = 6r^3 + 21r^2 + 9r + 2. \tag{4.2.5}$$

We see that the number of DOFs from (4.2.4a) are $12 = 4 \cdot 3$. There are $6(r-1)$ DOFs for (4.2.4b) and $12(r-1)$ DOFs for (4.2.4c). We have $4(3(r-1)(r-2) + 3(r-1) + 2)$ DOFs from (4.2.4d), and finally $3r(2r+1)(r+1)$ for (4.2.4d). Hence, the total number of DOFs (4.2.4) is

$$3r(2r+1)(r+1) + 12(r-1)(r-2) + 42(r-1) + 20 = 6r^3 + 21r^2 + 9r + 2.$$

Hence, we will prove that $\dim \mathring{\mathcal{V}}_r^2(T^{\text{wf}}) = 6r^3 + 21r^2 + 9r + 2$ if we show the constraints (4.2.4) determine a function $v \in \mathring{\mathcal{V}}_r^2(T^{\text{wf}})$. To this end, suppose that the DOFs (4.2.4) vanish. The DOFs (4.2.4a) shows that v vanishes $\forall a \in \Delta_0(T)$. The DOFs (4.2.4b) and (4.2.4b) show that v vanishes $\forall e \in \Delta_1(T)$. Also, the DOFs (4.2.4d) show that v_F vanishes $\forall F \in \Delta_2(T)$. Thus, $v = 0$ on ∂T and so $v = \mu w$ where $w \in V_{r-1}^2(T^{\text{wf}})$. Finally, (4.2.4e) shows that w vanishes. Thus, $v \equiv 0$. $\square$

In a similar fashion, but significantly easier way we can show that.

$$\dim \mathcal{V}_r^3(T^{\text{wf}}) \geq \dim V_r^3(T^{\text{wf}}) - 4(2(r+1) + r) = 2(r^3 + 6r^2 + 5r + 2). \tag{4.2.6}$$

Lemma 4.2.4. *It holds* $\dim \mathcal{V}_r^3(T^{\text{wf}})$ *has dimension* $2(r^3 + 6r^2 + 5r + 2)$.

Proof. We can easily show that the following DOFs determine $q \in \mathcal{V}_r^3(T^{\mathrm{wf}})$

$$\int_F qp\, dA, \qquad \forall p \in L_r^2(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \tag{4.2.7a}$$

$$\int_T qp\, dx, \qquad \forall p \in V_{r-1}^3(T^{\mathrm{wf}}). \tag{4.2.7b}$$

The number of DOFs are $2(r^3 + 6r^2 + 5r + 2)$ which are exactly the number given by (4.2.6). $\square$

Theorem 4.2.5. *For* $r \geq 1$*, it holds*

$$\dim \mathring{S}_r^0(T^{\mathrm{wf}}) = \max\{2(r-2)(r-3)(r-4), 0\},$$

$$\dim \mathring{S}_r^1(T^{\mathrm{wf}}) = \max\{3(2r-3)(r-2)(r-3), 0\},$$

$$\dim \mathring{S}_r^2(T^{\mathrm{wf}}) = \max\{2(r-2)(3r^2-6r+4), 0\},$$

$$\dim \mathring{S}_r^3(T^{\mathrm{wf}}) = (r-1)(2r^2-r+2).$$

Proof. Using the exactness of the sequences (4.1.1) we have

$$\dim \mathring{S}_r^0(T^{\mathrm{wf}}) - \dim \mathring{\mathsf{L}}_{r-1}^1(T^{\mathrm{wf}}) + \dim \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{wf}}) - \dim \mathring{V}_{r-3}^3(T^{\mathrm{wf}}) = 0,$$

$$\dim \mathring{S}_r^0(T^{\mathrm{wf}}) - \dim \mathring{S}_{r-1}^1(T^{\mathrm{wf}}) + \dim \mathring{\mathsf{L}}_{r-2}^2(T^{\mathrm{wf}}) - \dim \mathring{\mathcal{V}}_{r-3}^3(T^{\mathrm{wf}}) = 0,$$

$$\dim \mathring{S}_r^0(T^{\mathrm{wf}}) - \dim \mathring{S}_{r-1}^1(T^{\mathrm{wf}}) + \dim \mathring{S}_{r-2}^2(T^{\mathrm{wf}}) - \dim \mathring{\mathsf{L}}_{r-3}^3(T^{\mathrm{wf}}) = 0.$$

This along with (4.2.2) and Lemmas 4.2.3–4.2.4 give the result. $\square$

Theorem 4.2.6. *For* $r \geq 1$*, it holds:*

$$\dim S_r^0(T^{\mathrm{wf}}) = 2r^3 - 6r^2 + 10r - 2, \quad \dim S_r^1(T^{\mathrm{wf}}) = 3r(2r^2 - 3r + 5),$$

$$\dim S_r^2(T^{\mathrm{wf}}) = 6r^3 + 8r + 2, \qquad \dim S_r^3(T^{\mathrm{wf}}) = \big(2r+1\big)(r^2+r+1).$$

Proof. Using the exactenss of the sequences (4.1.2) we have, for $r \geq 3$,

$$\dim S_r^0(T^{\mathrm{wf}}) - \dim \mathsf{L}_{r-1}^1(T^{\mathrm{wf}}) + \dim V_{r-2}^2(T^{\mathrm{wf}}) - \dim V_{r-3}^3(T^{\mathrm{wf}}) = 1,$$
$$\dim S_r^0(T^{\mathrm{wf}}) - \dim S_{r-1}^1(T^{\mathrm{wf}}) + \dim \mathsf{L}_{r-2}^2(T^{\mathrm{wf}}) - \dim V_{r-3}^3(T^{\mathrm{wf}}) = 1,$$
$$\dim S_r^0(T^{\mathrm{wf}}) - \dim S_{r-1}^1(T^{\mathrm{wf}}) + \dim S_{r-2}^2(T^{\mathrm{wf}}) - \dim \mathsf{L}_{r-3}^3(T^{\mathrm{wf}}) = 1.$$

Using this with (4.2.2) give the result. □

For small r, some of these spaces are trivialized. In particular, when $r = 1, 2$, $S_r^0(T^{\mathrm{wf}}) = \mathcal{P}_r(T)$, and $S_1^1(T^{\mathrm{wf}}) = [\mathcal{P}_1(T)]^3$.

CHAPTER FIVE

Commuting Projections on Worsey-Farin Splits: Lowest Polynomial Order

In this chapter, we develop unisolvent sets of degrees of freedom that define commuting projections for the complexes considered in Chapter 4. We restrict our study to the case of the lowest non-trivial polynomial order, $r = 3$. We present this case separately for readability, as even the lowest order case requires significant effort. The commuting projections for general polynomial orders, which generalize the results of this chapter, are presented in Chapter 6.

Before we discuss commuting projections on the Worsey-Farin split, we will need some new lemmas relating to piecewise polynomials on the Clough-Tocher split. In the following lemma, we provide degrees of freedom for space $S_3^0(F^{\mathrm{ct}})$ that will be used in the projection for $S_3^0(T^{\mathrm{wf}})$.

Lemma 5.0.1. *A function $q \in S_3^0(F^{\mathrm{ct}})$ is fully determined by the degrees of freedom*

$$q(a),\ \nabla q(a), \qquad \forall a \in \Delta_0(F), \tag{5.0.1a}$$

$$\int_e \frac{\partial q}{\partial n_e}\, ds, \qquad \forall e \in \Delta_1(F), \tag{5.0.1b}$$

where n_e represents the outward unit normal vector to edge e.

Proof. Let $q \in S_0^3(F^{\mathrm{ct}})$ such that q vanishes on (5.0.1). Since q is cubic, it follows from the DOFs (5.0.1a) that $q|_e = 0$ for each edge $e \in \Delta_1(F)$. Then if t is the unit vector tangential to an edge $e \in \Delta_1(F)$, $\partial q/\partial t$ is also zero along e. Using the fact $\nabla q(a) = 0$ for all $a \in \Delta_0(F)$ and DOFs (5.0.1b), it follows that $\nabla q|_e = 0$ as well. Then $q \in \mathring{S}_3^0(F^{\mathrm{ct}}) = \{0\}$, since $\dim \mathring{S}_3^0(F^{\mathrm{ct}}) = 0$. $\square$

Next, we show that the space $S_2^0(T^{\mathrm{wf}})$ reduces to $S_2^0(T^{\mathrm{a}})$ on the Alfeld split of T. We will also use this result in proving unisolvency of the degrees of freedom for $S_3^0(T^{\mathrm{wf}})$.

In defining commuting projections, we will make use of the following definitions.

Definition 5.0.2. *Given a face $F \in \Delta_1^I(F^{\mathrm{ct}})$, each edge $e \in \Delta_1^I(F^{\mathrm{ct}})$ is associated with two orthonormal vectors, $[t, s]^\top$. The unit vector t is tangent to the edge e and points outward from the split point z of F^{ct}. The unit vector s is orthogonal to t and tangent to the face F, oriented such that $s \times n_F = t$. Furthermore, let r be the unit vector orthogonal to t and s that is tangent to the interior face $f \in \Delta_2^I(T^{\mathrm{wf}})$ that contains edge e.*

In the remainder of this thesis, the edge associated with vectors t, s, or r should be inferred from the context. For example, in the expression $\int_e v \cdot t\, ds$, the unit vector t should be understood to be the tangent vector of the edge e. For any face F in the Worsey Farin split we designate an edge $e_F \in \Delta_1^I(F^{\mathrm{ct}})$. We will also make use of the notion of a "jump" of a function across an edge, as defined below.

Definition 5.0.3. *Suppsoe that $F \in \Delta_2(T)$, $e \in \Delta_1^I(F^{\mathrm{ct}})$ and $f \in \Delta_2(T^{\mathrm{wf}})$ with $e \subset \overline{f}$. Furthermore, let $T_1, T_2 \in \Delta_2(T^{\mathrm{wf}})$ with $f = \overline{T_1} \cap \overline{T_2}$ and let s_i be proportional to $n_F \times t$ with s_i pointing out of T_i (here t is a tangent vector to e). We define the jump as*

$$[\![p]\!]_e = p|_{T_1} s_1 + p|_{T_2} s_2 \qquad \textit{on } e.$$

5.1 SLVV degrees of freedom

We will first consider the sequence (4.1.2b), which we denote as the "SLVV" sequence.

Proposition 5.1.1. *We can construct the projections:*

$$\Pi_3^0 : C^\infty(T) \to S_3^0(T^{\mathrm{wf}}),$$

$$\Pi_2^1 : [C^\infty(T)]^3 \to L_2^1(T^{\mathrm{wf}}),$$

$$\Pi_1^2 : [C^\infty(T)]^3 \to V_1^2(T^{\mathrm{wf}}),$$

$$\Pi_0^3 : C^\infty(T) \to V_0^3(T^{\mathrm{wf}}),$$

such that the following diagram commutes.

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(T) & \xrightarrow{\mathrm{grad}} & [C^\infty(T)]^3 & \xrightarrow{\mathrm{curl}} & [C^\infty(T)]^3 & \xrightarrow{\mathrm{div}} & C^\infty(T) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Pi_3^0} & & \downarrow{\scriptstyle \Pi_2^1} & & \downarrow{\scriptstyle \Pi_1^2} & & \downarrow{\scriptstyle \Pi_0^3} & & \\
\mathbb{R} & \longrightarrow & S_3^0(T^{\mathrm{wf}}) & \xrightarrow{\mathrm{grad}} & L_2^1(T^{\mathrm{wf}}) & \xrightarrow{\mathrm{curl}} & V_1^2(T^{\mathrm{wf}}) & \xrightarrow{\mathrm{div}} & V_0^3(T^{\mathrm{wf}}) & \longrightarrow & 0.
\end{array}$$

In other words, we have

$$\begin{aligned}
\mathrm{grad}\,\Pi_3^0 q &= \Pi_2^1 \mathrm{grad}\, q, \quad \forall q \in C^\infty(T), \\
\mathrm{curl}\,\Pi_2^1 v &= \Pi_1^2 \mathrm{curl}\, v, \quad \forall v \in [C^\infty(T)]^3, \\
\mathrm{div}\,\Pi_1^2 w &= \Pi_0^3 \mathrm{div}\, w, \quad \forall w \in [C^\infty(T)]^3.
\end{aligned}$$

This proposition will be proved using the following Lemmas.

Lemma 5.1.1. *A function $q \in S_3^0(T^{\mathrm{wf}})$ is fully determined by the degrees of freedom*

		No. of DOFs	
$q(a)$,	$\forall a \in \Delta_0(T)$,	4,	(5.1.1a)
$\mathrm{grad}\, q(a)$,	$\forall a \in \Delta_0(T)$,	12,	(5.1.1b)
$\displaystyle\int_e \frac{\partial q}{\partial n_e^\pm}\, ds$,	$\forall e \in \Delta_1(T)$,	12,	(5.1.1c)

where $\frac{\partial}{\partial n_e^\pm}$ represents two normal derivatives to edge e, so that n_e^+, n_e^- and t, the unit vector tangent to e, form a basis of $\mathbb{R}^3$. Then the DOFs (5.1.1) *define the projection $\Pi_3^0 : C^\infty(T) \to S_3^0(T^{\mathrm{wf}})$.*

Proof. The number of degrees of freedom in (5.1.1) is 28, which is consistent with the dimension count of $S_3^0(T^{\mathrm{wf}})$ in (4.2.6).

Let $q \in S_0^3(T^{\mathrm{wf}})$ such that q vanishes on the DOFs (5.1.1a) - (5.1.1c). On each face F^{ct}, $q|_F$ is zero by Lemma 5.0.1. Then we write $q = \mu p$ for some piecewise quadratic function p on T^{wf}. Let $F \in \Delta_2(T)$ and let $K \in T^{\mathrm{a}}$ contain F. Then since μ is linear on K, $p \in C^1$ on K. Moreover, $\operatorname{grad} q = p \operatorname{grad} \mu$ on F and so p vanishes on ∂F. Hence, we have that $p \in \mathcal{R}_2^0(F^{\mathrm{ct}}) = \{0\}$. Thus $\operatorname{grad} q = 0$ on ∂T, or $q \in \mathring{S}_3^0(T^{\mathrm{wf}}) = \{0\}$.

$\square$

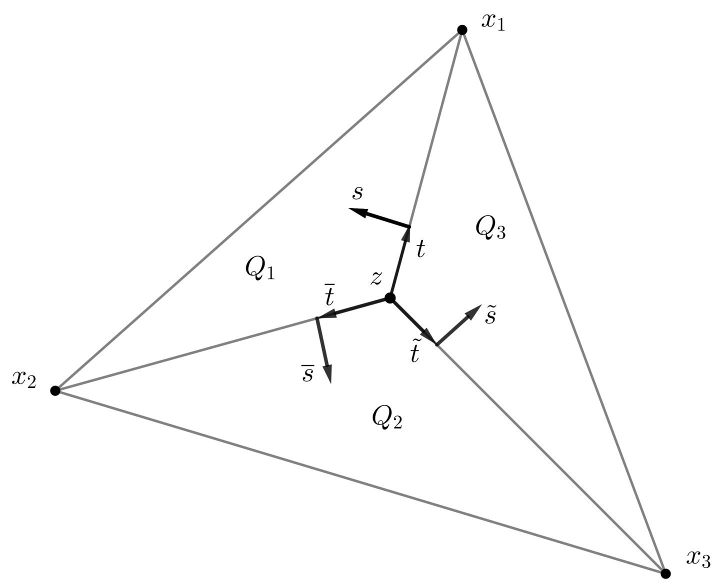

Figure 5.1: Representation of the Clough-Tocher split with associated interior edge vectors.

Lemma 5.1.2. *Given a triangulation F^{ct} of a face $F \in \Delta_2(T)$, the spaces $S_2^0(F^{\mathrm{ct}})$ and $\mathcal{P}_2(F^{\mathrm{ct}})$ are equivalent.*

Proof. Notice that $S_2^0(F^{\mathrm{ct}}) \subseteq \mathcal{P}_2(F^{\mathrm{ct}}) = 6$, hence $\dim S_2^0(F^{\mathrm{ct}}) \leq \dim \mathcal{P}_2(F^{\mathrm{ct}})$. However, from the formula for the dimension of $S_2^0(F^{\mathrm{ct}})$ in (2.3.7), we see that $\dim S_2^0(F^{\mathrm{ct}}) = 6$. It follows that these two spaces must be equivalent.

□

We will use Corollary 4.2.2 and Lemma 5.1.2 to determine a unisolvent set of degrees of freedom for $L_2^1(T^{\mathrm{wf}})$.

Lemma 5.1.3. *A function $v \in L_2^1(T^{\mathrm{wf}})$ is fully determined by the following degrees of freedom.*

$$
\begin{array}{llrr}
 & & \textit{No. of DOFs} & \\
v(a), & \forall a \in \Delta_0(T), & 12, & (5.1.2a) \\
\int_e v\, ds, & \forall e \in \Delta_1(T), & 18, & (5.1.2b) \\
\int_F \operatorname{curl}_F v_F\, p\, dA, & \forall p \in \mathring{V}_1^2(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), & 32, & (5.1.2c) \\
\int_e [\![\operatorname{curl} v \cdot t]\!]_e\, ds, & \forall e \in \Delta_1^I(F^{\mathrm{ct}})\backslash\{e_F\},\ \forall F \in \Delta_2(T), & 8, & (5.1.2d) \\
\int_{e_F} [\![\operatorname{curl} v \cdot t]\!]_{e_F} q\, ds, & \forall q \in \mathcal{P}_1(e), \forall F \in \Delta_2(T), & 8, & (5.1.2e) \\
\int_T \operatorname{curl} v \cdot q\, dx, & \forall q \in \operatorname{curl} \mathring{L}_2^1(T^{\mathrm{wf}}), & 27. & (5.1.2f)
\end{array}
$$

Then the DOFs (5.1.2) *define the projection* $\Pi_2^1 : [C^\infty(T)]^3 \to L_2^1(T^{\mathrm{wf}})$.

Proof. The total number of degrees of freedom is 105, which matches the dimension count for $L_2^1(T^{\mathrm{wf}})$ in Lemma 2.6.4.

Let $v \in L_2^1(T^{\mathrm{wf}})$ such that v vanishes on the DOFs (5.1.2). On a face $F \in \Delta_2(T)$, $v|_{\partial F} = 0$ due to DOFs (5.1.2a) - (5.1.2b). Since $v|_{\partial F} = 0$, we have $v_F \in \mathring{L}_2^1(F^{\mathrm{ct}})$, so using sequence (2.3.10b) on the Clough-Tocher split F^{ct} yields $\operatorname{curl}_F v_F \in \mathring{V}_1^2(F^{\mathrm{ct}})$. Then $\operatorname{curl}_F v_F = 0$ by DOF (5.1.2c). Using that $v_F \in \mathring{L}_2^1(F^{\mathrm{ct}})$, and by the exactness of (2.3.10b), there exists a function $p \in \mathring{S}_3^0(F^{\mathrm{ct}})$ such that $\operatorname{grad}_F p = v_F$. But $\mathring{S}_3^0(F^{\mathrm{ct}}) = \{0\}$, therefore $p = 0$ and $v_F = 0$ on F.

Next, using (5.1.2d) - (5.1.2e) and Corollary 4.2.2, we have that $(\operatorname{curl} v)_F$ is continuous on F. By Lemma 6.1.3 we have that $\operatorname{grad}_F(v \cdot n_F)$ is continuous on F. Thus $v \cdot n_F|_F \in \mathcal{R}_2^0(F^{\mathrm{ct}}) = \{0\}$. Hence $v|_{\partial T} = 0$, so $v \in \mathring{L}_2^1(T^{\mathrm{wf}})$. By DOF (5.1.2f), $\operatorname{curl} v = 0$ on T. By exactness of the sequence (4.1.2c), there exists a function $q \in \mathring{S}_3^0(T^{\mathrm{wf}})$ such that $\operatorname{grad} q = v$. However, $\mathring{S}_3^0(T^{\mathrm{wf}}) = \{0\}$, therefore $v = 0$. $\square$

Lemma 5.1.4. *A function $w \in V_1^2(T^{\mathrm{wf}})$ is fully determined by the following degrees of freedom.*

$$
\begin{array}{lllr}
 & & \textit{No. of DOFs} & \\
\displaystyle\int_F w \cdot n_F \, dA, & \forall F \in \Delta_2(T), & 4, & (5.1.3a) \\
\displaystyle\int_F (w \cdot n_F) p \, dA, & \forall p \in \mathring{V}_1^2(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), & 32, & (5.1.3b) \\
\displaystyle\int_e [\![w \cdot t]\!]_e \, ds, & \forall e \in \Delta_1^I(F^{\mathrm{ct}}) \backslash \{e_F\},\ \forall F \in \Delta_2(T), & 8, & (5.1.3c) \\
\displaystyle\int_{e_F} [\![w \cdot t]\!]_{e_F} q \, ds, & \forall q \in \mathcal{P}_1(e), \forall F \in \Delta_2(T), & 8, & (5.1.3d) \\
\displaystyle\int_T (\operatorname{div} w) q \, dx, & \forall q \in \mathring{V}_0^3(T^{\mathrm{wf}}), & 11, & (5.1.3e) \\
\displaystyle\int_T w \cdot q \, dx, & \forall q \in \operatorname{curl} \mathring{L}_2^1(T^{\mathrm{wf}}), & 27. & (5.1.3f)
\end{array}
$$

Then the DOFs (5.1.3) *define the projection* $\Pi_1^2 : [C^\infty(T)]^3 \to V_1^2(T^{\mathrm{wf}})$.

Proof. The total number of DOFs in (5.1.3) is 90, which is consistent with our dimension count for $V_1^2(T^{\mathrm{wf}})$ from Lemma 2.6.3.

Let $w \in V_1^2(T^{\mathrm{wf}})$ such that w vanishes on the DOFs (5.1.3). Then the triangulation F^{ct} of a face $F \in \Delta_2(T)$, it follows from (5.1.3a) that $w \cdot n_F|_F \in \mathring{V}_1^2(F^{\mathrm{ct}})$. Then by (5.1.3b), $w \cdot n_F|_F = 0$. Using (5.1.3c) - (5.1.3d) and Corollary 4.2.2 $w \in \mathring{\mathcal{V}}_1^2(T^{\mathrm{wf}})$, so $\operatorname{div} w \in \mathring{V}_0^3(T^{\mathrm{wf}})$ by the sequence (4.1.1b). Therefore (5.1.3e) yields $\operatorname{div} w = 0$. By the exactness of sequence (4.1.1b), there exists $v \in \mathring{L}_2^1(T^{\mathrm{wf}})$ such that $\operatorname{curl} v = w$. Then

(5.1.3f) yields $w = 0$. $\square$

Lemma 5.1.5. *A function $p \in V_0^3(T^{\mathrm{wf}})$ is fully determined by the following degrees of freedom.*

$$\int_T p \, dx, \qquad \textit{(1 DOF)} \tag{5.1.4a}$$

$$\int_T pq \, dx, \qquad \forall q \in \mathring{V}_0^3(T^{\mathrm{wf}}). \qquad \textit{(11 DOFs)} \tag{5.1.4b}$$

Then the DOFs (5.1.4) *define the projection $\Pi_0^3 : C^\infty(T) \to V_0^3(T^{\mathrm{wf}})$.*

Proof. Let $p \in V_0^3(T^{\mathrm{wf}})$ such that p vanishes on the DOFs (5.1.4). Then the average of p on T is zero by (5.1.4a), so $p \in \mathring{V}_0^3(T^{\mathrm{wf}})$. Therefore by (5.1.4b), $p = 0$. $\square$

5.2 SLVV commuting diagram

Theorem 5.2.1. *Given the definitions of the projections $\Pi_3^0, \Pi_2^1, \Pi_1^2$, and Π_0^3 in Lemmas 5.1.1 - 5.1.5, the diagram* (5.1.1) *of Proposition 5.1.1 commutes, i.e.,*

$$\operatorname{grad} \Pi_3^0 q = \Pi_2^1 \operatorname{grad} q, \quad \forall q \in C^\infty(T), \tag{5.2.1a}$$

$$\operatorname{curl} \Pi_2^1 v = \Pi_1^2 \operatorname{curl} v, \quad \forall v \in [C^\infty(T)]^3, \tag{5.2.1b}$$

$$\operatorname{div} \Pi_1^2 w = \Pi_0^3 \operatorname{div} w, \quad \forall w \in [C^\infty(T)]^3. \tag{5.2.1c}$$

Proof. (a) *Proof of* (5.2.1a). Given $q \in C^\infty(T)$, let $\rho = \operatorname{grad} \Pi_3^0 q - \Pi_2^1 \operatorname{grad} q$, and we aim to show $\rho = 0$. Then $\rho \in L_2^1(T^{\mathrm{wf}})$, so it is sufficient to show that ρ vanishes on the DOFs of Lemma 5.1.3. From (5.1.2a), we have $\rho(a) = \operatorname{grad} \Pi_3^0 q(a) - \Pi_2^1 \operatorname{grad} q(a) = 0$ for each $a \in \Delta_1(T)$ by (5.1.1a) and (5.1.2a). Using (5.1.1a), (5.1.1b), and (5.1.2b), for each $e \in \Delta_1(T)$,

$$\begin{aligned}
\int_e \rho \, ds &= \int_e \operatorname{grad} \Pi_3^0 q - \Pi_2^1 \operatorname{grad} q \, ds \\
&= \int_e \operatorname{grad} (\Pi_3^0 q - q) \, ds \\
&= \int_e \frac{\partial}{\partial n_e^+} (\Pi_3^0 q - q) n_e^+ + \frac{\partial}{\partial n_e^-} (\Pi_3^0 q - q) n_e^- + \frac{\partial}{\partial t} (\Pi_3^0 q - q) t \, ds \\
&= \int_e \frac{\partial}{\partial t} (\Pi_3^0 q - q) t \, ds \\
&= (\Pi_3^0 q(a_2) - q(a_2)) - (\Pi_3^0 q(a_1) - q(a_1)) = 0,
\end{aligned}$$

where a_1 and a_2 are the vertices of edge e.

Next, by (5.1.2c), for all $p \in \mathring{V}_1^2(F^{\text{ct}})$ and for each $F \in \Delta_2(T)$,

$$\begin{aligned}
\int_F \operatorname{curl}_F \rho_F \, p \, dA &= \int_F \operatorname{curl}_F (\operatorname{grad} \Pi_3^0 q - \Pi_2^1 \operatorname{grad} q) p \, dA \\
&= \int_F \operatorname{curl}_F (\operatorname{grad} (\Pi_3^0 q - q) p \, dA = 0,
\end{aligned}$$

since the curl of a gradient is zero. By (5.1.2d), for each $e \in \Delta_1^I(F^{\text{ct}})$ and for every $F \in \Delta_2(T)$,

$$\begin{aligned}
\int_e [\![\operatorname{curl} (\rho) \cdot t]\!]_e \, ds &= \int_e [\![\operatorname{curl} (\operatorname{grad} \Pi_3^0 q - \Pi_2^1 \operatorname{grad} q) \cdot t]\!]_e \, ds \\
&= \int_e [\![\operatorname{curl} (\operatorname{grad} (\Pi_3^0 q - q)) \cdot t]\!]_e \, ds = 0.
\end{aligned}$$

Similarly, from (5.1.2e), we have

$$\begin{aligned}
[\![\operatorname{curl} (\rho) \cdot t]\!]_e(z) &= [\![\operatorname{curl} (\operatorname{grad} \Pi_3^0 q - \Pi_2^1 \operatorname{grad} q) \cdot t]\!]_e(z) \\
&= [\![\operatorname{curl} (\operatorname{grad} (\Pi_3^0 q - q)) \cdot t]\!]_e(z) = 0.
\end{aligned}$$

Finally, using (5.1.2f), for every $p \in \mathring{L}_2^1(T^{\mathrm{wf}})$,

$$\begin{aligned}\int_T \operatorname{curl}\rho \cdot \operatorname{curl} p\, dx &= \int_T \operatorname{curl}(\operatorname{grad}\Pi_3^0 q - \Pi_2^1 \operatorname{grad} q)\cdot \operatorname{curl} p\, dx \\ &= \int_T \operatorname{curl}(\operatorname{grad}(\Pi_3^0 q - q))\cdot \operatorname{curl} p\, dx = 0.\end{aligned}$$

Then by Lemma 5.1.3, it follows that $\rho = 0$, therefore $\operatorname{grad}\Pi_3^0 q = \Pi_2^1 \operatorname{grad} q$.

(b) *Proof of* (5.2.1b). Given $v \in [C^\infty(T)]^3$, let $\rho = \operatorname{curl}(\Pi_2^1 v) - \Pi_1^2 \operatorname{curl} v$. Then $\rho \in V_1^2(T^{\mathrm{wf}})$, so we need only show that ρ vanishes on the DOFs (5.1.3). We apply the Stokes Theorem of Equation (2.2.3b) as well as (5.1.2b) and (5.1.3a) to get

$$\begin{aligned}\int_F \rho\cdot n_F\, dA &= \int_F (\operatorname{curl}(\Pi_2^1 v) - \Pi_1^2 \operatorname{curl} v)\cdot n_F\, dA \\ &= \int_F \operatorname{curl}_F((\Pi_2^1 v - v)_F)\, dA \\ &= 0.\end{aligned}$$

Next, for all $p \in \mathring{V}_1^2(F^{\mathrm{ct}})$, we have

$$\begin{aligned}\int_F (\rho\cdot n_F) p\, dA &= \int_F ((\operatorname{curl}(\Pi_2^1 v) - \Pi_1^2 \operatorname{curl} v)\cdot n_F) p\, dA \\ &= \int_F (\operatorname{curl}(\Pi_2^1 v - v)\cdot n_F)\, p\, dA = 0,\end{aligned}$$

where we used (5.1.2c) and (5.1.3b). Using (5.1.3c), for every $e \in \Delta_1^I(F^{\mathrm{ct}})\backslash\{e_F\}$ and for each $F \in \Delta_2(T^{\mathrm{wf}})$,

$$\int_e [\![\rho\cdot t]\!]_e\, ds = \int_e [\![\operatorname{curl}(\Pi_2^1 v - v)\cdot t]\!]_e\, ds = 0$$

by (5.1.2d). Similarly, we can show that ρ vanishes on the DOFs (5.1.3d).

Next, using (5.1.3e), for all $q \in \mathring{V}_0^3(T^{\mathrm{wf}})$,

$$\int_T (\operatorname{div} \rho) q \, dx = \int_T (\operatorname{div} \operatorname{curl} (\Pi_2^1 v - v)) q \, dx = 0.$$

Finally, using (5.1.3f), for any $q \in \operatorname{curl} \mathring{L}_2^1(T^{\mathrm{wf}})$,

$$\int_T \rho \cdot q \, dx = \int_T \operatorname{curl} (\Pi_2^1 v - v) \cdot q \, dx = 0$$

by (5.1.2f). Therefore $\rho = 0$, and $\operatorname{curl} (\Pi_2^1 v) = \Pi_1^2 \operatorname{curl} v$.

(c) *Proof of* (5.2.1c). Set $\rho = \operatorname{div} \Pi_1^2 w - \Pi_0^3 \operatorname{div} w$, where $w \in [C^\infty(T)]^3$, so that $\rho \in V_0^3(T^{\mathrm{wf}})$. We will show that ρ vanishes on (5.1.4). By the Stokes Theorem from Equation (2.2.1c) as well as (5.1.4a) and (5.1.3a), we have

$$\begin{aligned}
\int_T \rho \, dx &= \int_T \operatorname{div} \Pi_1^2 w - \Pi_0^3 \operatorname{div} w \, dx \\
&= \int_T \operatorname{div} (\Pi_1^2 w - w) \, dx \\
&= \int_{\partial T} (\Pi_1^2 w - w) \cdot n \, dA \\
&= 0.
\end{aligned}$$

For all $q \in \mathring{V}_0^3(T^{\mathrm{wf}})$, by (5.1.4b) and (5.1.3e),

$$\int_T \rho q \, dx = \int_T (\operatorname{div} \Pi_1^2 w - \Pi_0^3 \operatorname{div} w) q \, dx = \int_T \operatorname{div} (\Pi_1^2 w - w) q \, dx = 0.$$

Therefore by Lemma 5.1.5, $\rho = 0$. $\square$

5.3 SSLV degrees of freedom

Now we will develop commuting projections for the sequence (4.1.2c) with $r = 3$.

Proposition 5.3.1. *Let $\Pi_0^3 : C^\infty(T) \to S_3^0(T^{\mathrm{wf}})$ be the projection defined in Lemma 5.1.1. We can construct projections*

$$\pi_2^1 : [C^\infty(T)]^3 \to S_2^1(T^{\mathrm{wf}}),$$
$$\pi_1^2 : [C^\infty(T)]^3 \to L_1^2(T^{\mathrm{wf}}),$$
$$\pi_0^3 : C^\infty(T) \to V_0^3(T^{\mathrm{wf}})$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi_3^0"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{curl}"] \arrow[d, "\pi_1^2"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{div}"] \arrow[d, "\pi_2^1"] & C^\infty(T) \arrow[r] \arrow[d, "\pi_3^0"] & 0 \\
\mathbb{R} \arrow[r] & S_3^0(T^{\mathrm{wf}}) \arrow[r, "\mathrm{grad}"] & S_2^1(T^{\mathrm{wf}}) \arrow[r, "\mathrm{curl}"] & L_1^2(T^{\mathrm{wf}}) \arrow[r, "\mathrm{div}"] & V_0^3(T^{\mathrm{wf}}) \arrow[r] & 0.
\end{tikzcd}$$

In other words, we have

$$\mathrm{grad}\,\Pi_0^3 q = \pi_1^2 \mathrm{grad}\, q, \quad \forall q \in C^\infty(T),$$
$$\mathrm{curl}\,\pi_1^2 v = \pi_2^1 \mathrm{curl}\, v, \quad \forall v \in [C^\infty(T)]^3,$$
$$\mathrm{div}\,\pi_2^1 w = \pi_3^0 \mathrm{div}\, w, \quad \forall w \in [C^\infty(T)]^3.$$

We will make use of the following Lemma in determining the degrees of freedom for $S_2^1(T^{\mathrm{wf}})$.

Lemma 5.3.1. *A function $v \in S_2^1(T^{\mathrm{wf}})$ is fully determined by the following degrees of freedom.*

$$\begin{array}{llrr} & & \textit{No. of DOFs} & \\ v(a), & \forall a \in \Delta_0(T), & 12, & \text{(5.3.1a)} \\ \operatorname{curl} v(a), & \forall a \in \Delta_0(T), & 12, & \text{(5.3.1b)} \\ \int_e v\, ds, & \forall e \in \Delta_1(T), & 18. & \text{(5.3.1c)} \end{array}$$

Then the DOFs (5.3.1) *define the projection* $\pi_2^1 : [C^\infty(T)]^3 \to S_2^1(T^{\mathrm{wf}})$.

Proof. The number of degrees of freedom in (5.3.1) is 42, which matches the dimension count of $S_2^1(T^{\mathrm{wf}})$ from the formula in Theorem 4.2.6.

Let $v \in S_2^1(T^{\mathrm{wf}})$ such that v vanishes on the DOFs (5.3.1), and let $F \in \Delta_2(T)$. The DOFs (5.3.1a) and (5.3.1c) yield that $v|_e = 0$ for each $e \in \Delta_1(T)$. Furthermore, since $\operatorname{curl} v$ is linear, it follows from (5.3.1b) that $\operatorname{curl} v|_e = 0$ as well. Hence $v_F \in \mathring{S}_2^1(F^{\mathrm{ct}}) = \{0\}$ by the dimension count given in (2.3.8).

Also, since $\operatorname{curl} v$ is continuous on F and $v_F = 0$ it follows from Lemma 6.1.3 that $\operatorname{grad}_F(v \cdot n_F)|_F$ is continuous. Therefore $v \cdot n_F|_F \in \mathcal{R}_2^0(F^{\mathrm{ct}}) = \{0\}$. It follows that $v|_{\partial T} = 0$.

Since $\operatorname{curl} v \in \mathring{V}_1^1(T^{\mathrm{wf}})$ we can apply Lemma 4.1.14 to deduce that $(\operatorname{curl} v)_F \in \mathcal{R}_1^1(F^{\mathrm{ct}}) = \{0\}$, where we also used that $(\operatorname{curl} v)_F = 0$ on ∂F. We already had that $\operatorname{curl} v \cdot n_F = 0$, so $\operatorname{curl} v|_F = 0$ on each face $F \in \Delta_2(T)$.

Now we have $v \in \mathring{S}_2^1(T^{\mathrm{wf}})$, but $\dim \mathring{S}_2^1(T^{\mathrm{wf}}) = 0$ by Theorem 4.2.5, therefore $v = 0$. $\square$

Lemma 5.3.2. *A function $w \in L_1^2(T^{\mathrm{wf}})$ is fully determined by the following degrees of freedom.*

$$\begin{array}{lllr}
 & & \textit{No. of DOFS} & \\
w(a), & \forall a \in \Delta_0(T), & 12, & \text{(5.3.2a)}\\
\displaystyle\int_F w \cdot n_F \, dA, & \forall F \in \Delta_2(T), & 4, & \text{(5.3.2b)}\\
\displaystyle\int_e [\![\operatorname{div} w]\!]_e \, ds, & \forall e \in \Delta_1^I(F^{\mathrm{ct}})\backslash\{e_F\},\ \forall F \in \Delta_2(T), & 8, & \text{(5.3.2c)}\\
\displaystyle\int_T \operatorname{div} w \, v \, dx, & \forall v \in \operatorname{div} \mathring{L}_1^2(T^{\mathrm{wf}}), & 3. & \text{(5.3.2d)}
\end{array}$$

Then the DOFs (5.3.2) *define the projection* $\pi_1^2 : [C^\infty(T)]^3 \to L_1^2(T^{\mathrm{wf}})$.

Proof. The number of degrees of freedom in (5.3.2) is 27, which matches the dimension count of $L_1^2(T^{\mathrm{wf}})$ from Lemma 2.6.4.

Let $w \in L_1^2(T^{\mathrm{wf}})$ such that w vanishes on the DOFs (5.3.2). Then by (5.3.2a), $w|_e = 0$ for every $e \in \Delta_1(T)$. Then, by (5.3.2b), we have $w \cdot n_F|_F \in \mathring{L}_1^2(F^{\mathrm{ct}})$, and since $\dim \mathring{L}_1^2(F^{\mathrm{ct}}) = 0$ by (2.3.6), it follows that $w \cdot n_F = 0$ on F. Let $K \in \Delta_3(T^{\mathrm{a}})$ with $F \in \Delta_2(K)$. Thus, we can write $w \cdot n_F = \mu\psi$ for some $\psi \in \mathcal{P}_0(T^{\mathrm{wf}})$. However, since $w \cdot n_F$ is continuous on K and μ is linear on positive on K it must be that ψ is continuous on K. Moreover, since $n_F \cdot \operatorname{grad}(w \cdot n_F) = \psi \operatorname{grad} \mu \cdot n_F$ on F which implies that $n_F \cdot \operatorname{grad}(w \cdot n_F)$ is continuous on F.

Using DOFs (5.3.2c) - (5.3.2d) and Lemma 6.3.2, we have that $\operatorname{div} w|_F \in \mathring{L}_0^2(F^{\mathrm{ct}})$ for each $F \in \Delta_2(T)$. We can write $\operatorname{div}_F w_F = \operatorname{div} w|_F - n_F \cdot \operatorname{grad}(w \cdot n_F)$ and, hence, $\operatorname{div}_F w_F$ is continuous which implies that $w_F \in \mathcal{R}_1^1(F^{\mathrm{ct}}) = \{0\}$.

Now we have $w \in \mathring{L}_1^2(T^{\mathrm{wf}})$, so by (5.3.2d), $\operatorname{div} w = 0$. Then we can use the exactness of the sequence (4.1.1c) to see that there exists a function $q \in \mathring{S}_2^1(T^{\mathrm{wf}})$ such that $\operatorname{curl} q = w$. But $\dim \mathring{S}_2^1(T^{\mathrm{wf}}) = 0$ by Theorem 4.2.5, therefore $q = 0$, so $w = 0$. $\square$

In order for the projections of Proposition 5.3.1 to commute, we must use a new set of degrees of freedom for the space $V_0^3(T^{\mathrm{wf}})$.

Lemma 5.3.3. *A function $p \in V_0^3(T^{\mathrm{wf}})$ is fully determined by the following degrees of freedom.*

No. of DOFs

$$\int_e [\![p]\!]_e \, ds, \qquad \forall e \in \Delta_1^I(F^{\mathrm{ct}}) \backslash \{e_F\},\ \forall F \in \Delta_2(T), \qquad 8, \tag{5.3.3a}$$

$$\int_T p \, dx, \qquad 1, \tag{5.3.3b}$$

$$\int_T pq \, dx, \qquad \forall q \in \mathring{\mathcal{V}}_0^3(T^{\mathrm{wf}}), \qquad 3. \tag{5.3.3c}$$

Then the DOFs (5.3.3) *define the projection $\pi_0^3 : C^\infty(T) \to V_0^3(T^{\mathrm{wf}})$.*

Proof. The number of DOFs in (5.3.3) is equal to 12, which is the dimension of $V_0^3(T^{\mathrm{wf}})$ given in Lemma 2.6.3.

Let $p \in V_0^3(T^{\mathrm{wf}})$ such that the DOFs (5.3.3) are zero. Since p is piecewise constant, the DOFs (5.3.3a) yield that p is continuous on each $F \in \Delta_2(T^{\mathrm{wf}})$. Furthermore, it follows from (5.3.3b) that $p \in \mathring{\mathcal{V}}_0^3(T^{\mathrm{wf}})$, and using (5.3.3c), $p = 0$. □

5.4 SSLV commuting diagram

In this section, we prove that the degrees of freedom presented in Section 5.3 yield commuting projections for the sequence (4.1.2c).

Theorem 5.4.1. *Given the definitions of the projections π_2^1, π_1^2, and π_0^3 from Lemmas 5.3.1*

- 5.3.3 as well as Π_3^0 *from Lemma 5.1.1, the diagram in Proposition 5.3.1 commutes, i.e.,*

$$\operatorname{grad}\Pi_0^3 q = \pi_2^1 \operatorname{grad} q, \quad \forall q \in C^\infty(T), \tag{5.4.1a}$$

$$\operatorname{curl}\pi_2^1 v = \pi_1^2 \operatorname{curl} v, \quad \forall v \in [C^\infty(T)]^3, \tag{5.4.1b}$$

$$\operatorname{div}\pi_1^2 w = \pi_3^0 \operatorname{div} w, \quad \forall w \in [C^\infty(T)]^3. \tag{5.4.1c}$$

Proof. (a) *Proof of* (5.4.1a). Set $\rho = \operatorname{grad}\Pi_0^3 q - \pi_1^2 \operatorname{grad} q \in S_2^1(T^{\mathrm{wf}})$. We show that ρ vanishes on (5.3.1).

We have $\rho(a) = \operatorname{grad}(\Pi_0^3 q(a) - q(a)) = 0$ using (5.3.1a) and (5.1.1a). Using (5.3.1b), $\operatorname{curl}\rho(a) = \operatorname{curl}(\operatorname{grad}(\Pi_0^3 q(a) - q(a))) = 0$ since the curl of a gradient is always zero. Then we use (5.3.1c) and (5.1.1a) so that

$$\begin{aligned}
\int_e \rho\, ds &= \int_e \operatorname{grad}(\Pi_0^3 q - q)\, ds \\
&= \int_e \frac{\partial}{\partial n_e^+}(\Pi_3^0 q - q) n_e^+ + \frac{\partial}{\partial n_e^-}(\Pi_3^0 q - q) n_e^- + \frac{\partial}{\partial t}(\Pi_3^0 q - q) t\, ds \\
&= \int_e \frac{\partial}{\partial t}(\Pi_3^0 q - q) t\, ds \\
&= (\Pi_3^0 q(a_2) - q(a_2)) - (\Pi_3^0 q(a_1) - q(a_1)) = 0,
\end{aligned}$$

where a_1 and a_2 are the vertices of edge e. Thus, ρ vanishes on the DOFs (5.3.1), so $\rho = 0$ by Lemma 5.3.1, and the identity (5.4.1a) holds.

(b) *Proof of* (5.4.1b). Set $\rho = \operatorname{curl}\pi_2^1 v - \pi_1^2 \operatorname{curl} v \in L_1^2(T^{\mathrm{wf}})$. We show that ρ vanishes on (5.3.2). By (5.3.2a) and (5.3.1a), we have $\rho(a) = \operatorname{curl}\pi_2^1 v(a) - \pi_1^2 \operatorname{curl} v(a) = 0$. Then using the Stokes Theorem of Equation (2.2.3b) as well as (5.3.2b) and (5.3.1c),

$$\begin{aligned}
\int_F \rho \cdot n_F\, dA &= \int_F \operatorname{curl}_F(\pi_2^1 v - v)_F\, dA \\
&= \int_{\partial F} ((\pi_2^1 v - v)_F \times n_F) \cdot \mathbf{1}\, ds = 0,
\end{aligned}$$

since $(\pi_2^1 v - v)_F$ is orthogonal to n_F. By (5.3.2c), for any $e \in \Delta_1^I(F^{\rm ct})\backslash\{e_F\}$ of each $F \in \Delta_2(T)$,

$$\begin{aligned}\int_e [\![\operatorname{div}\rho]\!]_e \, ds &= \int_e [\![\operatorname{div}(\operatorname{curl}\pi_1^2 v - \pi_2^1 \operatorname{curl} v)]\!]_e \, ds \\ &= \int_e [\![\operatorname{div}(\operatorname{curl}(\pi_1^2 v - v))_F]\!]_e \, ds = 0.\end{aligned}$$

Then by (5.3.2d), for every $\kappa \in \operatorname{div} \mathring{L}_1^2(T^{\rm wf})$

$$\begin{aligned}\int_T \operatorname{div}\rho \, \kappa \, dx &= \int_T \operatorname{div}(\operatorname{curl}\pi_2^1 v - \pi_1^2 \operatorname{curl} v)\kappa \, dx \\ &= \int_T \operatorname{div}(\operatorname{curl}(\pi_2^1 v - v))\kappa \, dx = 0.\end{aligned}$$

Therefore ρ vanishes on the DOFs (5.3.2), so $\rho = 0$ by Lemma 5.3.2. Thus the identity (5.4.1b) holds.

(c) *Proof of* (5.4.1c). Set $\rho = \operatorname{div}\pi_1^2 w - \pi_3^0 \operatorname{div} w$, where $w \in [C^\infty(T)]^3$. We will show that ρ vanishes on (5.3.3). Using (5.3.3a) and (5.3.2c), we have, for any $e \in \Delta_1^I(F^{\rm ct})\backslash\{e_F\}$ in $F \in \Delta_2(T)$,

$$\int_e [\![\rho]\!]_e \, ds = \int_e [\![\operatorname{div}(\pi_1^2 w - w)]\!]_e \, ds = 0.$$

Next, using Stokes Theorem of Equation (2.2.1c) as well as (5.3.3b) and (5.3.2b),

$$\int_T \rho \, dx = \int_T \operatorname{div}(\pi_1^2 w - w) \, dx = \int_{\partial T} (\pi_2^1 w - w) \cdot n \, dA = 0.$$

Finally, using the Stokes Theorem (2.2.1c) again, in addition to (5.3.2d) and (5.3.3c), for all $q \in \mathring{\mathcal{V}}_0^3(T^{\rm wf})$,

$$\int_T \rho q \, dx = \int_T \operatorname{div}(\pi_1^2 w - w) q \, dx = 0,$$

since $\operatorname{div} \mathring{L}_1^2(T^{\mathrm{wf}}) = \mathring{\mathcal{V}}_0^3(T^{\mathrm{wf}})$ by the sequence (4.1.2c). Thus ρ vanishes on (5.3.3), so $\rho = 0$ by Lemma 5.3.3. Therefore the identity (5.4.1c) holds. □

5.5 SSSL degrees of freedom

Next, we will determine degrees of freedom for the spaces $S_1^2(T^{\mathrm{wf}})$ and $L_0^3(T^{\mathrm{wf}})$ of the sequence (4.1.2d) such that the following proposition holds.

Proposition 5.5.1. *We can construct projections $\varpi_1^2 : [C^\infty(T)]^3 \to S_1^2(T^{\mathrm{wf}})$ and $\varpi_0^3 : C^\infty(T) \to L_0^3(T^{\mathrm{wf}})$ such that the following diagram commutes.*

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(T) & \xrightarrow{\text{grad}} & [C^\infty(T)]^3 & \xrightarrow{\text{curl}} & [C^\infty(T)]^3 & \xrightarrow{\text{div}} & C^\infty(T) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Pi_0^3} & & \downarrow{\scriptstyle \pi_1^2} & & \downarrow{\scriptstyle \varpi_2^1} & & \downarrow{\scriptstyle \varpi_3^0} & & \\
\mathbb{R} & \longrightarrow & S_3^0(T^{\mathrm{wf}}) & \xrightarrow{\text{grad}} & S_2^1(T^{\mathrm{wf}}) & \xrightarrow{\text{curl}} & S_1^2(T^{\mathrm{wf}}) & \xrightarrow{\text{div}} & L_0^3(T^{\mathrm{wf}}) & \longrightarrow & 0.
\end{array}$$

In other words, we have

$$\operatorname{grad} \Pi_0^3 q = \pi_2^1 \operatorname{grad} q, \quad \forall q \in C^\infty(T), \tag{5.5.1a}$$

$$\operatorname{curl} \pi_2^1 v = \varpi_1^2 \operatorname{curl} v, \quad \forall v \in [C^\infty(T)]^3, \tag{5.5.1b}$$

$$\operatorname{div} \varpi_1^2 w = \varpi_0^3 \operatorname{div} w, \quad \forall w \in [C^\infty(T)]^3. \tag{5.5.1c}$$

Lemma 5.5.1. *A function $w \in S_1^2(T^{\mathrm{wf}})$ is fully determined by the degrees of freedom*

$$w(a), \qquad \forall a \in \Delta_0(T), \quad \textit{(12 DOFs)} \tag{5.5.2a}$$

$$\int_F w \cdot n_F \, dA, \qquad \forall F \in \Delta_2(T). \quad \textit{(4 DOFs)} \tag{5.5.2b}$$

Then the DOFs (5.5.2) *define the projection $\varpi_1^2 : [C^\infty(T)]^3 \to S_1^2(T^{\mathrm{wf}})$.*

Proof. The number of degrees of freedom is 16, which matches the dimension of $S_1^2(T^{\mathrm{wf}})$ given in Theorem 4.2.6.

Let $w \in S_1^2(T^{\mathrm{wf}})$ such that w vanishes on the DOFs (5.5.2). Then by (5.5.2a), $w|_e = 0$ for all $e \in \Delta_1(T)$. Then $w_F \in \mathcal{R}_1^1(F^{\mathrm{ct}})$. But $\dim \mathcal{R}_1^1(F^{\mathrm{ct}}) = 0$ by Lemma 2.3.1, so $w_F = 0$ on F. Furthermore, by (5.5.2b), we have $w \cdot n_F \in \mathring{L}_1^2(F^{\mathrm{ct}})$, which has dimension equal to 0 by (2.3.6), hence $w \cdot n_F = 0$ on F. Then there exists a constant vector $c \in \mathbb{R}^3$ such that $w = c\mu$. Hence $\operatorname{div} w = c\operatorname{div} \mu$, which is only continuous if $c = 0$. Since $\operatorname{div} w$ is continuous by definition of the space $S_1^2(T^{\mathrm{wf}})$, it follows that $w = 0$. □

Lemma 5.5.2. *A function $p \in L_0^3(T^{\mathrm{wf}})$ is fully determined by the degree of freedom $\int_T p\, dx$. This DOF defines the projection $\varpi_0^3 : C^\infty(T) \to L_0^3(T^{\mathrm{wf}})$.*

Proof. Let $p \in L_0^3(T^{\mathrm{wf}})$ such that $\int_T p\, dx = 0$. Since p is a constant on T with average 0, it follows that $p = 0$. This is the correct number of DOFs, as the dimension of the space $L_0^3(T^{\mathrm{wf}})$ is 1 by Lemma 2.6.4. □

5.6 SSSL commuting diagram

Theorem 5.6.1. *Given the definitions of projections ϖ_1^2 and ϖ_0^3 in Lemmas 5.5.1 - 5.5.2 as well as the projections Π_3^0 and π_2^1 from Lemmas 5.1.1 and 5.3.1, respectively, the diagram of Proposition 5.5.1 commutes, i.e.,*

$$\operatorname{grad} \Pi_3^0 q = \pi_2^1 \operatorname{grad} q, \quad \forall q \in C^\infty(T), \tag{5.6.1a}$$

$$\operatorname{curl} \pi_2^1 v = \varpi_1^2 \operatorname{curl} v, \quad \forall v \in [C^\infty(T)]^3, \tag{5.6.1b}$$

$$\operatorname{div} \varpi_1^2 w = \varpi_0^3 \operatorname{div} w, \quad \forall w \in [C^\infty(T)]^3. \tag{5.6.1c}$$

Proof. (a) The identity (5.6.1a) holds by Theorem 6.4.1.

(b) *Proof of* (5.6.1b). Let $v \in [C^\infty(T)]^3$ and set $\rho = \operatorname{curl} \pi_2^1 v - \varpi_1^2 \operatorname{curl} v \in S_1^2(T^{\mathrm{wf}})$. We will show that ρ vanishes on the DOFs (5.5.2). First, $\rho(a) = \operatorname{curl} \pi_2^1 v(a) - \varpi_1^2 \operatorname{curl} v(a) = 0$ by (5.5.2a) and (5.3.1a). Then using the Stokes Theorem of Equation (2.2.3b), as well as (5.5.2b) and (5.3.1c), we have

$$\int_F \rho \cdot n_F \, dA = \int_F \operatorname{curl}_F (\pi_2^1 v - v)_F \, dA = \int_{\partial F} ((\pi_2^1 v - v)_F \times n_F) \cdot \mathbf{1} \, ds = 0,$$

since $(\pi_2^1 v - v)_F$ is orthogonal to n_F. Thus ρ vanishes on (5.5.2), so $\rho = 0$ by Lemma 5.5.1. Hence the identity (5.6.1b) holds.

(c) *Proof of* (5.6.1c). Let $w \in [C^\infty(T)]^3$, and set $\rho = \operatorname{div} \varpi_1^2 w - \varpi_0^3 \operatorname{div} w \in L_0^3(T^{\mathrm{wf}})$. The only DOF is $\int_T \rho \, dx$, so

$$\int_T \rho \, dx = \int_T \operatorname{div} (\varpi_1^2 w - w) \, dx = \int_{\partial T} (\varpi_1^2 w - w) \cdot n \, dA = 0,$$

by Stokes Theorem of Equation (2.2.1c) and (5.5.2b). Thus $\rho = 0$ by Lemma 5.5.2, so the identity (5.6.1c) holds. $\square$

5.7 Global sequences and commuting diagrams

In this section, we discuss the global finite element spaces induced by the degrees of freedom from Sections 5.1, 5.3, and 5.5. Let $\mathcal{T}_h$ be the triangulation of the polygonal domain $\Omega \subset \mathbb{R}^3$, and let $\mathcal{T}_h^{\mathrm{wf}}$ be the Worsey-Farin refinement of $\mathcal{T}_h$. Before we discuss global finite element spaces, we must first revisit the discussion of singular edges of Section 2.6.

By construction, every edge connecting a vertex of $\Delta_0(\mathcal{T}_h)$ with a split point of a face in $\Delta_2(\mathcal{T}_h)$ is a "singular edge", meaning the edge is in the intersection of four triangles that together lie in two planes (see Definition 2.6.2). In other words, the interior edges of the Clough-Tocher splits F^{ct} of $\mathcal{T}_h^{\mathrm{wf}}$ are singular edges. In order for the global spaces to have the correct continuity across adjacent macro-elements, we must define an operator $\theta_e(\cdot)$ in terms of the singular edges of $\mathcal{T}_h^{\mathrm{wf}}$ that places a condition on the multiple values a piecewise polynomial may take at these edges.

Definition 5.7.1. *We define the set $\mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})$ as the collection of edges that are internal to a Clough-Tocher split of a face $F \in \Delta_2(\mathcal{T}_h)$, i.e., $\mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})$ is the set of singular edges of the triangulation $\mathcal{T}_h^{\mathrm{wf}}$.*

Definition 5.7.2. *Let $e \subset \Delta_1^I(F^{\mathrm{ct}})$ be an internal edge. Let $T_1, T_2 \in \mathcal{T}_h$ be such that $F = \overline{T_1} \cap \overline{T_2}$. Furthemore, let $K_i^1, K_i^2 \in \Delta_3(T_i^{\mathrm{wf}})$, $1 \le i \le 2$ be such that $e \subset \overline{K_i^j}, 1 \le i \le 2, 1 \le j \le 2$ and K_1^2 shares a face with K_2^1 then we define*

$$\theta_e(p) = |p_1^1 - p_1^2 + p_2^1 - p_2^2| \qquad \textit{on } e,$$

where $p_i^j = p|_{K_i^j}$.

Note that if $\theta_e(p) = 0$ if and only if $[\![p_1]\!]_e = [\![p_2]\!]_e$ where $p_i = p|_{T_i}$.

Now, we are ready to consider the following global finite element spaces on $\mathcal{T}_h^{\mathrm{wf}}$.

$$\mathcal{S}_3^0(\mathcal{T}_h^{\mathrm{wf}}) = \{q \in C^1(\Omega) : q|_T \in S_3^0(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\}, \tag{5.7.1a}$$

$$\mathcal{S}_2^1(\mathcal{T}_h^{\mathrm{wf}}) = \{v \in [C(\Omega)]^3 : \operatorname{curl} v \in [C(\Omega)]^3,\ v|_T \in S_2^1(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\}, \tag{5.7.1b}$$

$$\mathcal{S}_1^2(\mathcal{T}_h^{\mathrm{wf}}) = \{w \in [C(\Omega)]^3 : \operatorname{div} w \in C(\Omega), w|_T \in S_1^2(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\}, \tag{5.7.1c}$$

$$\mathcal{L}_2^1(\mathcal{T}_h^{\mathrm{wf}}) = \{v \in [C(\Omega)]^3 : v|_T \in L_2^1(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\}, \tag{5.7.1d}$$

$$\mathcal{L}_1^2(\mathcal{T}_h^{\mathrm{wf}}) = \{w \in [C(\Omega)]^3 : w|_T \in L_1^2(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\}, \tag{5.7.1e}$$

$$\mathscr{V}_1^2(\mathcal{T}_h^{\mathrm{wf}}) = \big\{w \in H(\mathrm{div}\,;\Omega) : w|_T \in V_1^2(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h,\, \theta_e(w \cdot t) = 0,$$
$$\forall e \in \mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})\big\}, \tag{5.7.1f}$$
$$\mathscr{V}_0^3(\mathcal{T}_h^{\mathrm{wf}}) = \{p \in L^2(\Omega) : p|_T \in V_0^3(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h,\, \theta_e(p) = 0\,\forall e \in \mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})\}, \tag{5.7.1g}$$
$$V_0^3(\mathcal{T}_h^{\mathrm{wf}}) = \mathcal{P}_0(\mathcal{T}_h^{\mathrm{wf}}). \tag{5.7.1h}$$

We will refer to the spaces (5.7.1) as *global spaces*, since they are defined for the entire triangulation $\mathcal{T}_h^{\mathrm{wf}}$ of the domain Ω. The ranges of the Lagrange finite element spaces are particularly affected by the presence of singular edges, which are reflected in the definitions of the spaces $\mathscr{V}_r^k(\mathcal{T}_h^{\mathrm{wf}})$ above by the presence of the condition involving θ_e. The following lemma describes an intrinsic property of the curl of functions belonging to $L_2^1(\mathcal{T}_h^{\mathrm{wf}})$ on singular edges in $\mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})$.

Lemma 5.7.3. *Let the function $v \in \mathcal{L}_2^1(\mathcal{T}_h^{\mathrm{wf}})$, and let e be a singular edge in $\mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})$. Then $\theta_e(\mathrm{curl}\, v \cdot t) = 0$.*

Proof. We use the same notation as Lemma 5.7.2. We have that

$$\begin{aligned}
&\theta_e(\mathrm{curl}\, v \cdot t)\\
=&|\mathrm{curl}\, v_1^1 \cdot t - \mathrm{curl}\, v_1^2 \cdot t + \mathrm{curl}\, v_2^1 \cdot t + \mathrm{curl}\, v_2^2 \cdot t|\\
=&|(\mathrm{grad}\,(v_1^1 \cdot s) \cdot n - \mathrm{grad}\,(v_1^1 \cdot n) \cdot s) - (\mathrm{grad}\,(v_1^2 \cdot s) \cdot n - \mathrm{grad}\,(v_1^2 \cdot n) \cdot s)\\
&+ (\mathrm{grad}\,(v_2^1 \cdot s) \cdot n - \mathrm{grad}\,(v_2^1 \cdot n) \cdot s) - (\mathrm{grad}\,(v_2^2 \cdot s) \cdot n - \mathrm{grad}\,(v_2^2 \cdot n) \cdot s)|
\end{aligned}$$

Here we used that $\mathrm{curl}\, w \cdot t = \mathrm{grad}\,(w \cdot s) \cdot n - \mathrm{grad}\,(w \cdot n) \cdot s$.

We know that the following vanish since s is tangent to F and v is continuous.

$$\begin{aligned}
\mathrm{grad}\,(v_1^1 \cdot n) \cdot s - \mathrm{grad}\,(v_2^2 \cdot n) \cdot s =&0 \qquad \text{on } e,\\
\mathrm{grad}\,(v_1^2 \cdot n) \cdot s - \mathrm{grad}\,(v_2^1 \cdot n) \cdot s =&0 \qquad \text{on } e.
\end{aligned}$$

Let $f_i \in \Delta_2(T_i^{\mathrm{wf}})$ be the faces such that $e \subset \overline{f_i}$. We know that f_i belong to the same plane, which we call f, and we let r be a vector tangent to both f and perpendicular to t. We can write

$$n = ar + bs,$$

where $a := \frac{1}{(r \cdot n)}$ and $b := -\frac{(r \cdot s)}{(r \cdot n)}$. Hence, on e

$$\begin{aligned}
&\operatorname{grad}(v_1^1 \cdot s) \cdot n - \operatorname{grad}(v_1^2 \cdot s) \cdot n + \operatorname{grad}(v_2^1 \cdot s) \cdot n - \operatorname{grad}(v_2^2 \cdot s) \cdot n \\
=& a\operatorname{grad}((v_1^1 - v_1^2) \cdot s) \cdot r + a\operatorname{grad}((v_2^1 - v_2^2) \cdot s) \cdot r \\
&+ b\operatorname{grad}((v_1^1 - v_2^2) \cdot s) \cdot s + b\operatorname{grad}((v_2^1 - v_1^2) \cdot s) \cdot s \\
=& 0.
\end{aligned} \tag{5.7.2}$$

□

Next, we describe an intrinsic property of the divergence of functions belonging to $\mathcal{L}_1^2(\mathcal{T}_h^{\mathrm{wf}})$ on singular edges $\mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})$.

Lemma 5.7.4. *Let $w \in \mathcal{L}_1^2(\mathcal{T}_h^{\mathrm{wf}})$, and let e be a singular edge in $\mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})$. Then $\theta_e(\operatorname{div} w) = 0$.*

Proof. We use the same notation as Lemma 5.7.2. We have that

$$\begin{aligned}
&\theta_e(\operatorname{div} w) \\
=& |\operatorname{div} w_1^1 - \operatorname{div} w_1^2 + \operatorname{div} w_2^1 + \operatorname{div} w_2^2| \\
=& |\big(\operatorname{grad}(w_1^1 \cdot n) \cdot n + \operatorname{grad}(w_1^1 \cdot s) \cdot s + \operatorname{grad}(w_1^1 \cdot t) \cdot t\big) \\
&- \big(\operatorname{grad}(w_1^2 \cdot n) \cdot n + \operatorname{grad}(w_1^2 \cdot s) \cdot s + \operatorname{grad}(w_1^2 \cdot t) \cdot t\big) \\
&+ \big(\operatorname{grad}(w_2^1 \cdot n) \cdot n + \operatorname{grad}(w_2^1 \cdot s) \cdot s + \operatorname{grad}(w_2^1 \cdot t) \cdot t\big)
\end{aligned}$$

$$
\begin{aligned}
&- \big(\operatorname{grad}(w_2^2 \cdot n)\cdot n + \operatorname{grad}(w_2^2 \cdot s)\cdot s + \operatorname{grad}(w_2^2 \cdot t)\cdot t\big)| \\
=& |\operatorname{grad}(w_1^1 \cdot n)\cdot n - \operatorname{grad}(w_1^2 \cdot n)\cdot n + \operatorname{grad}(w_2^1 \cdot n)\cdot n - \operatorname{grad}(w_2^2 \cdot n)\cdot n|.
\end{aligned}
$$

Here we used that since t, s are tangent to F and w is continuous we obtain

$$
\begin{aligned}
\operatorname{grad}(w_1^1 \cdot t)\cdot t - \operatorname{grad}(w_2^2 \cdot t)\cdot t =& 0 \qquad \text{on } e,\\
\operatorname{grad}(w_1^2 \cdot t)\cdot t - \operatorname{grad}(w_2^1 \cdot t)\cdot t =& 0 \qquad \text{on } e,\\
\operatorname{grad}(w_1^1 \cdot s)\cdot s - \operatorname{grad}(w_2^2 \cdot s)\cdot s =& 0 \qquad \text{on } e,\\
\operatorname{grad}(w_1^2 \cdot s)\cdot s - \operatorname{grad}(w_2^1 \cdot s)\cdot s =& 0 \qquad \text{on } e.
\end{aligned}
$$

As we did in (5.7.2) we can show that

$$
\operatorname{grad}((w_1^1 - w_1^2 + w_2^1 - w_2^2)\cdot n)\cdot n = 0 \qquad \text{on } e.
$$

$\square$

Let us now describe what it means for a set of degrees of freedom to "induce" a global space, using the $\mathcal{S}_3^0(\mathcal{T}_h^{\mathrm{wf}})$ space as an example. Let T_1 and T_2 be two tetrahedra in $\mathcal{T}_h$ that share a face F. Suppose we have the locally-defined functions $q_1 \in S_3^0(T_1^{\mathrm{wf}})$ and $q_2 \in S_3^0(T_2^{\mathrm{wf}})$ such that the degrees of freedom for q_1 and q_2 are set equal to each other on the face F. Specifically, using Lemma 5.1.1, suppose that we have

$$
\begin{aligned}
& q_1(a) = q_2(a) && \forall a \in \Delta_0(F),\\
& \operatorname{grad} q_1(a) = \operatorname{grad} q_2(a) && \forall a \in \Delta_0(F),\\
& \int_e \frac{\partial q_1}{\partial n_e^\pm}\, ds = \int_e \frac{\partial q_2}{\partial n_e^\pm}\, ds && \forall e \in \Delta_1(F).
\end{aligned}
$$

Let $\chi(S)$ be the characteristic function on a simplex S. If it follows from these degrees of freedom that the function $q_1\chi(T_1)+q_2\chi(T_2)$ is C^1 across the face F, then the function must also be C^1 on all of $T_1 \cup T_2$. Then we can infer that the local degrees of freedom of Lemma 5.1.1 induce the global space (5.7.1a). In this section, we will show that the degrees of freedom presented in Sections 5.1, 5.3, and 5.5 induce the global spaces of (5.7.1). From this result, the exactness of sequences made up of these global spaces follows from the fact that the local spaces that induce these global spaces form exact sequences within each macroelement.

Let us define some notation used in this section. We let T_1 and T_2 be adjacent tetrahedra in $\mathcal{T}_h$ that share a face F, as before. Let K_1 and K_2 be tetrahedra in $\Delta_3(T_1^{\mathrm{a}})$ and $\Delta_3(T_2^{\mathrm{a}})$, respectively, such that K_1 and K_2 share the face F. Let F^{ct} represent the triangulation of F in $\mathcal{T}_h$, and let K_i^{wf} be the triangulation of K_i, where $1 \le i \le 2$. Without loss of generality, we choose $n_F = n_1$, the outward normal to T_1 on F.

Lemma 5.7.5. *The local degrees of freedom stated in Lemma 5.1.1 induce the global space $\mathcal{S}_3^0(\mathcal{T}_h^{\mathrm{wf}})$.*

Proof. Let $q_1 \in S_3^0(T_1^{\mathrm{wf}})$ and $q_2 \in S_3^0(T_2^{\mathrm{wf}})$ such that the DOFs (5.1.1) for q_1 and q_2 are equal on F. Then we extend q to K_2 as in Remark 2.6.1, and we define $p = q_1 - q_2$. Using Lemma 5.1.1, we see that p and $\operatorname{grad} p$ must be zero on F since the DOFs (5.1.1) applied to p are zero on F. Therefore $q_1 = q_2$ and $\operatorname{grad} q_1 = \operatorname{grad} q_2$ on F, so $q_1\chi(T_1) + q_2\chi(T_2)$ is C^1 on $T_1 \cup T_2$. It follows that the DOFs (5.1.1) of Lemma 5.1.1 induce the global space $\mathcal{S}_3^0(\mathcal{T}_h^{\mathrm{wf}})$. $\square$

Next, we consider the global Lagrange space $\mathcal{L}_2^1(\mathcal{T}_h^{\mathrm{wf}})$, where we must show that the intrinsic property describe in Lemma 5.1.3 is induced by the local DOFs (5.1.2) in addition to continuity between macroelements.

Lemma 5.7.6. *The local degrees of freedom stated in Lemma 5.1.3 induce the global space* $\mathcal{L}_2^1(\mathcal{T}_h^{\mathrm{wf}})$.

Proof. We let $v_1 \in L_2^1(T_1^{\mathrm{wf}})$ and $v_2 \in L_2^1(T_2^{\mathrm{wf}})$ such $v_1 - v_2$ vanishes on the DOFs (5.1.2) associated with the triangulation F^{ct} of the face F, and we extend v_1 to K_2 as in Remark 2.6.1. Then we define $w = v_1 - v_2$, and following the proof of Lemma 5.1.3, we see that $w = 0$ on F since the DOFs (5.1.2) applied to w are equal to zero on F. Hence, $v := v_1\chi(T_1) + v_2\chi(T_2)$ is continuous on all of $T_1 \cup T_2$. It follows that the local DOFs (5.1.2) induce the global space $\mathcal{L}_2^1(\mathcal{T}_h^{\mathrm{wf}})$. □

Lemma 5.7.7. *The local degrees of freedom of Lemma 5.1.4 induce the global space* $\mathscr{V}_1^2(\mathcal{T}_h^{\mathrm{wf}})$.

Proof. Let $w_1 \in V_1^2(T_1^{\mathrm{wf}})$ and $w_2 \in V_1^2(T_2^{\mathrm{wf}})$ such that $w_1 - w_2$ vanishes on the DOFs (5.1.3a) - (5.1.3d) associated with the triangulation F^{ct} of F. Then we extend w_1 to K_2 in T_2^{wf} as in Remark (2.6.1), and we define $v = w_1 - w_2$. Then by Lemma 5.1.4, v vanishes on F. In particular, $v \cdot n_F = 0$ on F, and it follows that $w := w_1\chi(T_1) + w_2\chi(T_2)$ is divergence-conforming across F. For each $e \in \Delta_1^I(F^{\mathrm{ct}})$, we also have that $0 = [\![w_1 \cdot t]\!]_e - [\![w_2 \cdot t]\!]_e$ by (5.1.3c) - (5.1.3d) and Corollary 4.2.2, which implies that $\theta_e(w \cdot t) = 0$. Therefore, the DOFs (5.1.3) induce the global space $\mathscr{V}_1^2(\mathcal{T}_h^{\mathrm{wf}})$. □

Lemma 5.7.8. *The local degrees of freedom in Lemma 5.1.5 induce the global space* $V_0^3(\mathcal{T}_h^{\mathrm{wf}})$.

Proof. Let $p_1 \in V_0^3(T_1^{\mathrm{wf}})$ and $p_2 \in V_0^3(T_2^{\mathrm{wf}})$, such that $\int_F p_1\, dA = \int_F p_2\, dA$, as in the DOFs (5.1.4a). The DOFs (5.1.4) yield that $p_1\chi(T_1) + p_2\chi(T_2)$ is piecewise constant on $K_1^{\mathrm{wf}} \cup K_2^{\mathrm{wf}}$, therefore these DOFs induce the global space $V_0^3(\mathcal{T}_h^{\mathrm{wf}})$. □

Now we can see that the sequence

$$\mathbb{R} \longrightarrow \mathcal{S}_3^0(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\text{grad}} \mathcal{L}_2^1(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\text{curl}} \mathscr{V}_1^2(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\text{div}} V_0^3(\mathcal{T}_h^{\mathrm{wf}}) \longrightarrow 0 \tag{5.7.3a}$$

forms a complex, which follows from Theorem 4.1.2 and Lemmas 5.7.5 - 5.7.8. Furthermore, we can define commuting projections $\tilde{\Pi}_i^j$ by $\tilde{\Pi}_i^j v|_T = \Pi_i^j(v|_T)$ for all $T \in \mathcal{T}_h$, with $0 \le i, j \le 3$. Using Theorem 6.4.1, we get the following commuting diagram for sequence (5.7.3a).

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(\Omega) & \xrightarrow{\text{grad}} & [C^\infty(\Omega)]^3 & \xrightarrow{\text{curl}} & [C^\infty(\Omega)]^3 & \xrightarrow{\text{div}} & C^\infty(\Omega) & \longrightarrow & 0 \\
 & & \Big\downarrow \tilde{\Pi}_3^0 & & \Big\downarrow \tilde{\Pi}_2^1 & & \Big\downarrow \tilde{\Pi}_1^2 & & \Big\downarrow \tilde{\Pi}_0^3 & & \\
\mathbb{R} & \longrightarrow & \mathcal{S}_3^0(\mathcal{T}_h^{\mathrm{wf}}) & \xrightarrow{\text{grad}} & \mathcal{L}_2^1(\mathcal{T}_h^{\mathrm{wf}}) & \xrightarrow{\text{curl}} & \mathscr{V}_1^2(\mathcal{T}_h^{\mathrm{wf}}) & \xrightarrow{\text{div}} & V_0^3(\mathcal{T}_h^{\mathrm{wf}}) & \longrightarrow & 0.
\end{array}$$

Next, we will show that the DOFs for the local spaces $S_2^1(T^{\mathrm{wf}})$, $L_1^2(T^{\mathrm{wf}})$, and $V_0^3(T^{\mathrm{wf}})$ of sequence (4.1.2c) with $r = 3$ induce the global spaces $\mathcal{S}_2^1(\mathcal{T}_h^{\mathrm{wf}})$, $\mathcal{L}_1^2(\mathcal{T}_h^{\mathrm{wf}})$, and $\mathscr{V}_0^3(\mathcal{T}_h^{\mathrm{wf}})$, respectively.

Lemma 5.7.9. *The local degrees of freedom in Lemma 5.3.1 induce the global space $\mathcal{S}_2^1(\mathcal{T}_h^{\mathrm{wf}})$.*

Proof. Let $v_1 \in S_2^1(T_1^{\mathrm{wf}})$ and $v_2 \in S_2^1(T_2^{\mathrm{wf}})$ such that $v_1 - v_2$ vanishes on the DOFs (5.3.1) associated with the triangulation F^{ct} of the face F. We extend v_1 to K_2 as in Remark 2.6.1, and we define $w = v_1 - v_2$. Then by Lemma 5.3.1, w and $\operatorname{curl} w$ vanish on F. Hence $v_1\chi(T_1) + v_2\chi(T_2)$ is continuous with continuous curl on F^{ct}, so the DOFs (5.3.1) induce the global space $\mathcal{S}_2^1(\mathcal{T}_h^{\mathrm{wf}})$. $\square$

Lemma 5.7.10. *The local degrees of freedom stated in Lemma 5.3.2 induce the global space $\mathcal{L}_1^2(\mathcal{T}_h^{\mathrm{wf}})$.*

Proof. Let $w_1 \in L_1^2(T_1^{\mathrm{wf}})$ and $w_2 \in L_1^2(T_2^{\mathrm{wf}})$ such that $w_1 - w_2$ vanishes on the DOFs (5.3.2a) - (5.3.2c) associated with the triangulation F^{ct} of the face F. We extend w_1 to K_2 as in Remark 2.6.1, and set $v = w_1 - w_2$. By following the proof of Lemma 5.3.2 we can show that $v = 0$ on F. Then we see that $w := w_1\chi(T_1) + w_2\chi(T_2)$ is continuous accross F. $\square$

Lemma 5.7.11. *The local degrees of freedom stated in Lemma 5.3.3 induce the global space* $\mathscr{V}_0^3(\mathcal{T}_h^{\mathrm{wf}})$.

Proof. Let $p_1 \in V_0^3(T_1^{\mathrm{wf}})$ and $p_2 \in V_0^3(T_2^{\mathrm{wf}})$ such that $p_1 - p_2$ vanishes on the DOFs (5.3.3a) associated with the triangulation F^{ct} of the face F. We extend p_1 to K_2 as in Remark 2.6.1.

Given an edge $e \in \Delta_1^I(F^{\mathrm{ct}})$, and using DOF (5.3.3a), we have that $[\![p_1]\!]_e = [\![p_2]\!]_e$, which implies that $\theta_e(p) = 0$, where $p = p_1\chi(K_1) + p_2\chi(K_2)$. $\square$

Now we can see that the sequence

$$\mathbb{R} \longrightarrow S_3^0(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} S_2^1(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} L_1^2(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathcal{V}_0^3(\mathcal{T}_h^{\mathrm{wf}}) \longrightarrow 0$$

forms a complex. Furthermore, we can define commuting projections $\tilde{\pi}_i^j$ such that $\tilde{\pi}_i^j v|_T = \pi_i^j(v|_T)$ for all $T \in \mathcal{T}_h$. Then by using Theorem 6.4.1, we get the following commuting diagram.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{grad}"] \arrow[d, "\tilde{\Pi}_3^0"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{curl}"] \arrow[d, "\tilde{\pi}_2^1"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{div}"] \arrow[d, "\tilde{\pi}_1^2"] & C^\infty(T) \arrow[r] \arrow[d, "\tilde{\pi}_0^3"] & 0 \\
\mathbb{R} \arrow[r] & \mathcal{S}_3^0(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{grad}"] & \mathcal{S}_2^1(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{curl}"] & \mathcal{L}_1^2(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{div}"] & \mathscr{V}_0^3(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r] & 0.
\end{tikzcd}$$

From Lemma 5.5.2, we see that the dimension of the final space of sequence (4.1.2d),

$L_0^3(T^{\mathrm{wf}})$, is 1, i.e., $L_0^3(T^{\mathrm{wf}})$ is the space of constant scalar functions on T. However, in the global setting, it is not possible for a locally constant function to be globally continuous unless it is a global constant. Therefore, in the case $r = 3$, the sequence (4.1.2d) is only conforming on the global Worsey-Farin refinement $\mathcal{T}_h^{\mathrm{wf}}$ if it consists of trivial global polynomial spaces. In the next chapter, we develop commuting projections for general polynomial order r, and we will show that the sequence (4.1.2d) is non-trivial in the global setting when $r \geq 4$.

Chapter Six

Commuting Projections on Worsey-Farin Splits: General Polynomial Order

In this chapter, we extend the results of Chapter 5 to general polynomial orders. In particular, we develop projections for the local, three-dimensional finite element spaces described in Section 2.1 such that the diagrams associated with the exact sequences (4.1.2a) - (4.1.2d) commute. Furthermore, we ensure that, in the lowest order case, these projections recover those defined in Chapter 5, where we set the polynomial order $r = 3$.

6.1 SLVV degrees of freedom

First, we give degrees of freedom for the local finite element spaces in the sequence (4.1.2b) such that the following proposition holds.

Proposition 6.1.1. *Let $r \geq 3$. There exists projections*

$$\begin{aligned}
\Pi_r^0 &: C^\infty(T) \to S_r^0(T^{\mathrm{wf}}), \\
\Pi_{r-1}^1 &: [C^\infty(T)]^3 \to L_{r-1}^1(T^{\mathrm{wf}}), \\
\Pi_{r-2}^2 &: [C^\infty(T)]^3 \to V_{r-2}^2(T^{\mathrm{wf}}), \\
\Pi_{r-3}^3 &: C^\infty(T) \to V_{r-3}^3(T^{\mathrm{wf}})
\end{aligned}$$

such that the following diagram commutes.

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(T) & \xrightarrow{\mathrm{grad}} & [C^\infty(T)]^3 & \xrightarrow{\mathrm{curl}} & [C^\infty(T)]^3 & \xrightarrow{\mathrm{div}} & C^\infty(T) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Pi_r^0} & & \downarrow{\scriptstyle \Pi_{r-1}^1} & & \downarrow{\scriptstyle \Pi_{r-2}^2} & & \downarrow{\scriptstyle \Pi_{r-3}^3} & & \\
\mathbb{R} & \longrightarrow & S_r^0(T^{\mathrm{wf}}) & \xrightarrow{\mathrm{grad}} & L_{r-1}^1(T^{\mathrm{wf}}) & \xrightarrow{\mathrm{curl}} & V_{r-2}^2(T^{\mathrm{wf}}) & \xrightarrow{\mathrm{div}} & V_{r-3}^3(T^{\mathrm{wf}}) & \longrightarrow & 0.
\end{array}$$

In other words, the following identities hold.

$$
\begin{aligned}
\operatorname{grad}\Pi_r^0 q &= \Pi_{r-1}^1 \operatorname{grad} q, \quad \forall q \in C^\infty(T),\\
\operatorname{curl}\Pi_{r-1}^1 v &= \Pi_{r-2}^2 \operatorname{curl} v, \quad \forall v \in [C^\infty(T)]^3,\\
\operatorname{div}\Pi_{r-2}^2 w &= \Pi_{r-3}^3 \operatorname{div} w, \quad \forall w \in [C^\infty(T)]^3.
\end{aligned} \tag{6.1.1}
$$

The degrees of freedom for each of the spaces $S_r^0(T^{\mathrm{wf}}), L_{r-1}^1(T^{\mathrm{wf}}), V_{r-2}^2(T^{\mathrm{wf}})$, and $V_{r-3}^3(T^{\mathrm{wf}})$ will define the projections $\Pi_r^0, \Pi_{r-1}^1, \Pi_{r-2}^2$, and Π_{r-3}^3, respectively.

Now, we give degrees of freedom for $S_r^0(T^{\mathrm{wf}})$ for $r \geq 3$. When $r < 3$, this space reduces to $\mathcal{P}_r(T)$.

Lemma 6.1.1. *A function $q \in S_r^0(T^{\mathrm{wf}})$, with $r \geq 3$, is fully determined by the following degrees of freedom.*

No. of DOFs

$$
\begin{aligned}
& q(a), && \forall a \in \Delta_0(T), && 4, && \text{(6.1.2a)}\\
& \operatorname{grad} q(a), && \forall a \in \Delta_0(T), && 12, && \text{(6.1.2b)}\\
& \int_e q\kappa\, ds, && \forall \kappa \in \mathcal{P}_{r-4}(e),\ \forall e \in \Delta_1(T), && 6(r-3), && \text{(6.1.2c)}\\
& \int_e \frac{\partial q}{\partial n_e^\pm}\kappa\, ds, && \forall \kappa \in \mathcal{P}_{r-3}(e),\ \forall e \in \Delta_1(T), && 12(r-2), && \text{(6.1.2d)}\\
& \int_F \operatorname{grad}_F q \cdot \kappa\, dA, && \forall \kappa \in \operatorname{grad}_F \mathring{S}_r^0(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), && 6(r-2)(r-3), && \text{(6.1.2e)}\\
& \int_F (n_F \cdot \operatorname{grad} q)\kappa\, dA, && \forall \kappa \in \mathcal{R}_{r-1}^0(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), && 6(r-2)(r-3), && \text{(6.1.2f)}\\
& \int_T \operatorname{grad} q \cdot \kappa\, dx, && \forall \kappa \in \operatorname{grad} \mathring{S}_r^0(T^{\mathrm{wf}}), && 2(r-2)(r-3)(r-4), && \text{(6.1.2g)}
\end{aligned}
$$

where $\frac{\partial}{\partial n_e^\pm}$ represents two normal derivatives to edge e, so that n_e^+, n_e^- and t, the unit vector tangent to e, form a basis of $\mathbb{R}^3$. Then the DOFs (6.1.2) *define the projection $\Pi_r^0 : C^\infty(T) \to S_r^0(T^{\mathrm{wf}})$.*

Proof. The dimension of $S_r^0(T^{\mathrm{wf}})$ is $2r^3 - 6r^2 + 10r - 2$, which is equal to the sum of the number of the given DOFs.

Let $q \in S_r^0(T^{\mathrm{wf}})$ such that q vanishes on the DOFs (6.1.2). On each edge $e \in \Delta_1(T)$, $q|_e = 0$ by DOFs (6.1.2a) - (6.1.2c). Furthermore, $\operatorname{grad} q|_e = 0$ by DOFs (6.1.2b) and (6.1.2d). Then $q|_F \in \mathring{S}_r^0(F^{\mathrm{ct}})$ for each $F \in \Delta_2(T)$, and (6.1.2e) yields $\operatorname{grad}_F q|_F = 0$. Hence $q|_F$ is constant, and since $q|_{\partial F} = 0$, it follows that $q|_F = 0$ for each $F \in \Delta_2(T)$.

Now we can write $q = \mu p$, where $p \in L_{r-1}^0(T^{\mathrm{wf}})$. Since μ is linear on each $K \in \Delta_3(T^{\mathrm{a}})$, and $q|_K \in S_r^0(K^{\mathrm{wf}})$, it follows that $p \in {S_{r-1}}^0(K^{\mathrm{wf}})$, hence $p|_F \in {S_{r-1}}^0(F^{\mathrm{ct}})$. We have that $\operatorname{grad} q = \mu \operatorname{grad} p + p \operatorname{grad} \mu$, hence on F, $n_F \cdot \operatorname{grad} q|_F = p(n_F \cdot \operatorname{grad} \mu)|_F$. Since $\operatorname{grad} q|_{\partial F} = 0$, it follows that $p|_{\partial F} = 0$. Therefore $p \in \mathcal{R}_{r-1}^0(F^{\mathrm{ct}})$, so $p|_F = 0$ by (6.1.2f). Now $\operatorname{grad} q|_{\partial T} = 0$, hence $q \in \mathring{S}_r^0(T^{\mathrm{wf}})$, and by (6.1.2g), we have $\operatorname{grad} q = 0$. Therefore $q = 0$, which is the desired result. $\square$

Remark 6.1.2. *In two dimensions, the work of [29] provided nodal degrees of freedom for the space $S_r^0(F^{\mathrm{ct}})$ with $r \geq 3$.*

Lemma 6.1.3. *Let e be an internal edge of F^{ct}, and let t and s be unit vectors tangent and orthogonal to e, respectively, as in Definition 5.0.2. Let $v \in L_k^1(T^{\mathrm{wf}})$ for some $k \geq 0$. If $v \times n_F = 0$ on F, then $[\![\operatorname{curl} v \cdot t]\!]_e = [\![\operatorname{grad}(v \cdot n_F) \cdot s]\!]_e$.*

Proof. Since $[t, s, n_F]^\top$ forms an orthonormal basis of $\mathbb{R}^3$, we write $v = a_t t + a_s s + a_n n_F$, where $a_t = v \cdot t$, $a_s = v \cdot s$, and $a_n = v \cdot n_F$. Since $v \times n_F = 0$ on F, we have $a_t = a_s = 0$ on F. Then, on F,

$$\operatorname{grad}_F(a_t) = \operatorname{grad}_F(a_s) = 0. \tag{6.1.3}$$

Since $\operatorname{curl} v$ can be written as $\operatorname{grad} a_t \times t + \operatorname{grad} a_s \times s + \operatorname{grad} a_n \times n_F$, we have

$$\operatorname{curl} v \cdot t = (\operatorname{grad} a_s \times s + \operatorname{grad} a_n \times n_F) \cdot t. \tag{6.1.4}$$

We can also write $\operatorname{grad} a_s$ as

$$\operatorname{grad} a_s = (t \cdot \operatorname{grad} a_s)t + (s \cdot \operatorname{grad} a_s)s + (n_F \cdot \operatorname{grad} a_n)n_F,$$

hence

$$(\operatorname{grad} a_s \times s) \cdot t = (n_F \cdot \operatorname{grad} a_s)(n_F \times s) \cdot t, \tag{6.1.5}$$

since $(t \times s) \cdot t = 0$ and $(s \times s) \cdot t = 0$.

Let f be the interior face of T^{wf} that contains e, and let r be the unit vector tangent to f and orthogonal to t. Then r may be written $r = (r \cdot s)s + (r \cdot n_F)n_F$, therefore

$$n_F = \frac{r - (r \cdot s)s}{r \cdot n_F}.$$

Then by (6.1.3), on F we have

$$\begin{aligned} n_F \cdot \operatorname{grad} a_s &= \frac{1}{r \cdot n_F}(r - (r \cdot s)s) \cdot \operatorname{grad} a_s \\ &= \frac{1}{r \cdot n_F}(r \cdot \operatorname{grad} a_s - s \cdot \operatorname{grad} a_s) \\ &= \frac{1}{r \cdot n_F}(r \cdot \operatorname{grad} a_s). \end{aligned} \tag{6.1.6}$$

Since r is tangent to f and a_s is continuous, we have $[\![r \cdot \operatorname{grad} a_s]\!]_e = 0$, which yields $[\![n_F \cdot \operatorname{grad} a_s]\!]_e = 0$ and in turn implies $[\![(\operatorname{grad} a_s \times s) \cdot t]\!]_e = 0$ by (6.1.5). It follows that $[\![\operatorname{curl} v \cdot t]\!]_e = [\![(\operatorname{grad} a_n \times n_F) \cdot t]\!]_e$.

We expand $\operatorname{grad} a_n$ in terms of $[t, s, n_F]^\top$ as

$$\operatorname{grad} a_n = (t \cdot \operatorname{grad} a_n \cdot)t + (s \cdot \operatorname{grad} a_n)s + (n_F \cdot \operatorname{grad} a_n)n_F.$$

So $(\operatorname{grad} a_n \times n_F) \cdot t = (s \cdot \operatorname{grad} a_n (s \times n_F)) \cdot t$, since $(t \times n_F) \cdot t = 0$ and $(n_F \times n_F) \cdot t = 0$. Because $(s \times n_F) \cdot t = 1$, it follows that $(\operatorname{grad} a_n \times n_F) \cdot t = s \cdot \operatorname{grad} a_n$. Therefore $[\![\operatorname{curl} v \cdot t]\!]_e = [\![s \cdot \operatorname{grad} a_n]\!]_e = [\![s \cdot \operatorname{grad} (v \cdot n_F)]\!]_e$, which is the desired result. $\square$

We remind the reader that the notation $\ker \mathring{L}^1_{r-1}(F^{\mathrm{ct}})$ represents the space $\{v \in \mathring{L}^1_{r-1}(F^{\mathrm{ct}}) : \operatorname{curl}_F v = 0\}$, which is equal to the space $\operatorname{grad}_F \mathring{S}^0_r(F^{\mathrm{ct}})$. Now we are ready to give the degrees of freedom for $L^1_{r-1}(T^{\mathrm{wf}})$.

Lemma 6.1.4. *A function $v \in L^1_{r-1}(T^{\mathrm{wf}})$, with $r \geq 3$, is fully determined by the following degrees of freedom.*

		No. of DOFs	
$v(a)$,		12,	(6.1.7a)
$\int_e v \cdot \kappa \, ds$,	$\forall \kappa \in [\mathcal{P}_{r-3}(e)]^3, \ \forall e \in \Delta_1(T)$,	$18(r-2)$,	(6.1.7b)
$\int_e [\![\operatorname{curl} v \cdot t]\!]_e \kappa \, ds$,	$\forall \kappa \in \mathcal{P}_{r-3}(e), \ \forall e \in \Delta_1^I(F^{\mathrm{ct}}) \backslash \{e_F\}$, $\forall F \in \Delta_2(T)$,	$8(r-2)$,	(6.1.7c)
$\int_{e_F} [\![\operatorname{curl} v \cdot t]\!]_{e_F} \kappa \, ds$,	$\forall \kappa \in \mathcal{P}_{r-2}(e_F), \ \forall F \in \Delta_2(T)$,	$4(r-1)$,	(6.1.7d)
$\int_F (v \cdot n_F) \kappa \, dA$,	$\forall \kappa \in \mathcal{R}^0_{r-1}(F^{\mathrm{ct}}), \forall F \in \Delta_2(T)$,	$6(r-2)(r-3)$,	(6.1.7e)
$\int_F \operatorname{curl}_F v_F \kappa \, dA$,	$\forall \kappa \in \mathring{V}^2_{r-2}(F^{\mathrm{ct}}), \forall F \in \Delta_2(T)$,	$6r^2 - 6r - 4$,	(6.1.7f)
$\int_F v_F \cdot \kappa \, dA$,	$\forall \kappa \in \ker \mathring{L}^1_{r-1}(F^{\mathrm{ct}}), \forall F \in \Delta_2(T)$,	$6(r-2)(r-3)$,	(6.1.7g)
$\int_T \operatorname{curl} v \cdot \kappa \, dx$,	$\forall \kappa \in \operatorname{curl} \mathring{L}^1_{r-1}(T^{\mathrm{wf}})$,	$4r^3 - 9r^2 - 7r + 21$,	(6.1.7h)
$\int_T v \cdot \kappa \, dx$,	$\forall \kappa \in \operatorname{grad} \mathring{S}^0_r(T^{\mathrm{wf}})$,	$2(r-2)(r-3)(r-4)$.	(6.1.7i)

Then the DOFs (6.1.7) *define the projection* $\Pi^1_{r-1} : [C^\infty(T)]^3 \to L^1_{r-1}(T^{\mathrm{wf}})$.

Proof. The dimension of $L^1_{r-1}(T^{\mathrm{wf}})$ is $6r^3 - 9r^2 + 9r - 3$, which is equal to the number of DOFs in (6.1.7). Let $v \in L^1_{r-1}(T^{\mathrm{wf}})$ such that v vanishes on the DOFs (6.1.7). Then $v|_e = 0$ for each edge $e \in \Delta_1(T)$ by (6.1.7a) - (6.1.7b), so $v_F \in \mathring{L}^1_{r-1}(F^{\mathrm{ct}})$ on each $F \in \Delta_2(T)$. From (2.3.10d), we can see that $\operatorname{curl}_F v_F \in \mathring{V}^2_{r-2}(F^{\mathrm{ct}})$. Then (6.1.7f) yields $\operatorname{curl}_F v_F = 0$ and by the exactness of the sequence (2.3.10d), we have $v_F \in \ker \mathring{L}^1_{r-1}(F^{\mathrm{ct}})$, so $v_F = 0$ by (6.1.7g).

Since $\operatorname{curl} v \cdot n_F = 0$ on F it follows from Corollary 4.2.2, DOFs (6.1.7c) - (6.1.7d) that $[\![\operatorname{curl} v \cdot t]\!]_e = 0$ for each $e \in \Delta_1^I(F^{\mathrm{ct}})$. Hence, by Lemma 6.1.3, $v \cdot n_F \in S^0_{r-1}(F^{\mathrm{ct}})$, and since $v \cdot n_F|_{\partial F} = 0$, we have $v \cdot n_F|_F \in \mathcal{R}^0_{r-1}(F^{\mathrm{ct}})$. Then $v \cdot n_F|_F = 0$ by (6.1.7e). We, therefore, conclude that $v|_{\partial T} = 0$.

Now $v \in \mathring{L}^1_{r-1}(T^{\mathrm{wf}})$, so $\operatorname{curl} v = 0$ by (6.1.7h). Using the exactness of sequence (4.1.1b), there exists a $p \in \mathring{S}^0_r(T^{\mathrm{wf}})$ such that $\operatorname{grad} p = v$. So by (6.1.7i), $v = 0$, which is the desired result. □

Next, we can write the degrees of freedom for $V^2_{r-2}(T^{\mathrm{wf}})$.

Lemma 6.1.5. *A function* $w \in V^2_{r-2}(T^{\mathrm{wf}})$*, with* $r \geq 3$*, is fully determined by the following degrees of freedom.*

		No. of DOFs	
$\int_e [\![w \cdot t]\!]_e q\, ds,$	$\forall q \in \mathcal{P}_{r-3}(e),\ \forall e \in \Delta_1^I(F^{\mathrm{ct}}) \backslash \{e_F\},$		
	$\forall F \in \Delta_2(T),$	$8(r-2),$	(6.1.8a)
$\int_{e_F} [\![w \cdot t]\!]_{e_F} q\, ds,$	$\forall q \in \mathcal{P}_{r-2}(e_F),\ \forall F \in \Delta_2(T),$	$4(r-1),$	(6.1.8b)

$$\int_F w \cdot n_F q\, dA, \qquad \forall q \in V_{r-2}^2(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \qquad 6r(r-1), \tag{6.1.8c}$$

$$\int_T (\operatorname{div} w) q\, dx, \qquad \forall q \in \mathring{V}_{r-3}^3(T^{\mathrm{wf}}), \qquad 2r^3 - 6r^2 + 4r - 1, \tag{6.1.8d}$$

$$\int_T w \cdot q\, dx, \qquad \forall q \in \operatorname{curl} \mathring{L}_{r-1}^1(T^{\mathrm{wf}}), \qquad 4r^3 - 9r^2 - 7r + 21. \tag{6.1.8e}$$

Then the DOFs (6.1.8) *define the projection* $\Pi_{r-2}^2 : [C^\infty(T)]^3 \to V_{r-2}^2(T^{\mathrm{wf}})$.

Proof. The dimension of $V_{r-2}^2(T^{\mathrm{wf}})$ is $6r^3 - 9r^2 + 3r + 12$, which is the number of DOFs in (6.1.8). Let $w \in V_{r-2}^2(T^{\mathrm{wf}})$ such that w vanishes on (6.1.8). By DOF (6.1.8c), we have $w \cdot n_F = 0$ on each $F \in \Delta_2(T)$. By DOFs (6.1.8a) - (6.1.8b), and Corollary 4.2.2 we have that $w \in \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{wf}})$, so $\operatorname{div} w = 0$ by (6.1.8d). By the exactness of sequence (4.1.1b), there exists a $v \in \mathring{L}_{r-1}^1(T^{\mathrm{wf}})$ such that $\operatorname{curl} v = w$. Therefore $w = 0$ by (6.1.8e), which is the desired result. $\square$

Lemma 6.1.6. *A function* $p \in V_{r-3}^3(T^{\mathrm{wf}})$*, with* $r \geq 3$*, is fully determined by the following degrees of freedom.*

No. of DOFs

$$\int_T p\, dx, \qquad 1, \tag{6.1.9a}$$

$$\int_T pq\, dx, \qquad \forall q \in \mathring{V}_{r-3}^3(T^{\mathrm{wf}}), \qquad 2r(r-1)(r-2) - 1. \tag{6.1.9b}$$

Then the DOFs (6.1.9) *define the projection* $\Pi_{r-3}^3 : C^\infty(T) \to V_{r-3}^3(T^{\mathrm{wf}})$.

Proof. The dimension of $V_{r-3}^3(T^{\mathrm{wf}})$ is $2r(r-1)(r-2)$, which is the number of DOFs in (6.1.9).

Let $p \in V_{r-3}^3(T^{\mathrm{wf}})$ such that p vanishes on (6.1.9). From (6.1.9a), we have that $p \in \mathring{V}_{r-3}^3(T^{\mathrm{wf}})$. Hence by (6.1.9b), $p = 0$, which is the desired result. $\square$

6.2 SLVV commuting diagram

Theorem 6.2.1. *Let $r \geq 3$. Given the definitions of the projections $\Pi_r^0, \Pi_{r-1}^1, \Pi_{r-2}^2$, and Π_{r-3}^3 in Lemmas 6.1.1 - 6.1.6, the diagram of Proposition 6.1.1 commutes, i.e.,*

$$\operatorname{grad} \Pi_r^0 q = \Pi_{r-1}^1 \operatorname{grad} q, \quad \forall q \in C^\infty(T), \tag{6.2.1a}$$

$$\operatorname{curl} \Pi_{r-1}^1 v = \Pi_{r-2}^2 \operatorname{curl} v, \quad \forall v \in [C^\infty(T)]^3, \tag{6.2.1b}$$

$$\operatorname{div} \Pi_{r-2}^2 w = \Pi_{r-3}^3 \operatorname{div} w, \quad \forall w \in [C^\infty(T)]^3. \tag{6.2.1c}$$

Proof. (i) *Proof of* (6.2.1a). Given $q \in C^\infty(T)$, let $\rho = \operatorname{grad} \Pi_r^0 q - \Pi_{r-1}^1 \operatorname{grad} q \in L_{r-1}^1(T^{\mathrm{wf}})$. Then to show (6.2.1a) holds, it is sufficient to show that ρ vanishes on the DOFs (6.1.7) of Lemma 6.1.4.

Using (6.1.7a) and (6.1.2b), we have $\rho(a) = \operatorname{grad} \Pi_r^0 q(a) - \Pi_{r-1}^1 \operatorname{grad} q(a) = 0$ for each $a \in \Delta_0(T)$. Using (6.1.7b) and (6.1.2d), for each $e \in \Delta_1(T)$ and for any $\kappa \in [\mathcal{P}_{r-3}(e)]^3$, where $e \in \Delta_1(T)$,

$$\begin{aligned}
\int_e \rho \cdot \kappa \, ds &= \int_e \operatorname{grad} (\Pi_r^0 q - q) \cdot \kappa \, ds \\
&= \int_e \big(\frac{\partial}{\partial n_e^+} (\Pi_r^0 q - q) n_e^+ + \frac{\partial}{\partial n_e^-} (\Pi_r^0 q - q) n_e^- + \frac{\partial}{\partial t} (\Pi_r^0 q - q) t \big) \cdot \kappa \, ds \\
&= \int_e \frac{\partial}{\partial t} (\Pi_r^0 q - q) t \cdot \kappa \, ds \\
&= 0,
\end{aligned}$$

where the last line follows from (6.1.2a). Using (6.1.7c), for each $e \in \Delta_1^I(F^{\mathrm{ct}}) \backslash \{e_F\}$, for all $F \in \Delta_2(T)$, and for any $\kappa \in \mathcal{P}_{r-3}(e)$,

$$\int_e [\![\operatorname{curl} \rho \cdot t]\!]_e \kappa \, ds = \int_e [\![\operatorname{curl} \operatorname{grad} (\Pi_r^0 q - q) \cdot t]\!]_e \kappa \, ds = 0,$$

since the curl of the gradient is zero. By the same reasoning, the DOFS (6.1.7d) of ρ vanish. By (6.1.7e), for any $\kappa \in \mathcal{R}^0_{r-1}(F^{\mathrm{ct}})$,

$$\int_F (\rho \cdot n_F)\kappa \, dA = \int_F (\operatorname{grad}(\Pi^0_r q - q) \cdot n_F)\kappa \, dA = 0,$$

by (6.1.2f).

Similarly, using (6.1.7f), $\int_F \operatorname{curl}_F \rho_F \, \kappa \, dA = 0$ for every $\kappa \in \mathring{V}^2_{r-2}(F^{\mathrm{ct}})$. Next, for $\kappa \in \ker \mathring{L}^1_{r-1}(F^{\mathrm{ct}})$,

$$\int_F \rho_F \cdot \kappa \, dA = \int_F \operatorname{grad}_F(\Pi^0_r q - q) \cdot \kappa \, dA = 0$$

using (6.1.2e) and (6.1.7g).

On the macro-elements, we use (6.1.7h) so that for all $\kappa \in \operatorname{curl} \mathring{L}^1_{r-1}(T^{\mathrm{wf}})$,

$$\int_T \operatorname{curl} \rho \cdot \kappa \, dx = \int_T \operatorname{curl} \operatorname{grad}(\Pi^0_r q - q) \cdot \kappa \, dx = 0.$$

Finally, we use (6.1.7i) to see that for all $\kappa \in \operatorname{grad} \mathring{S}^0_r(T^{\mathrm{wf}})$,

$$\int_T \rho \cdot \kappa \, dx = \int_T \operatorname{grad}(\Pi^0_r q - q) \cdot \kappa \, dx = 0,$$

by (6.1.2g). Hence by Lemma 6.1.4, $\rho = 0$, and the identity (6.2.1a) is proved.

(ii) *Proof of* (6.2.1b). Given $v \in [C^\infty(T)]^3$, let $\rho = \operatorname{curl} \Pi^1_{r-1} v - \Pi^2_{r-2} \operatorname{curl} v \in V^2_{r-2}(T^{\mathrm{wf}})$. To prove that (6.2.1b) holds, we will show that ρ vanishes on the DOFs (6.1.8) of Lemma 6.1.5.

On the interior edges $e \in \Delta^I_1(F^{\mathrm{ct}}) \backslash \{e_F\}$ of each face $F \in \Delta_2(T)$, and for all $q \in$

$\mathcal{P}_{r-3}(e)$, we have

$$\int_e [\![\rho \cdot t]\!]_e \, q \, ds = \int_e [\![\mathrm{curl}\,(\Pi^1_{r-1} v - v) \cdot t]\!]_e \, q \, ds = 0,$$

using (6.1.7c) and (6.1.8a). Similarly, using the DOFs (6.1.8c) of ρ vanish.

To show that the DOFs (6.1.8c) of ρ vanish we consider first constant functions and then functiosn orthognal to constants. To this end, we use (6.1.8c), (6.1.7b) and the Stokes Theorem of Equation (2.2.3b), so that

$$\int_F \rho \cdot n_F \, dA = \int_F \mathrm{curl}\,_F (\Pi^1_{r-1} v - v)_F \, dA = 0,$$

Here we used that $r \geq 3$. Moreover, for any $p \in \mathring{V}^2_{r-2}(F^{\mathrm{ct}})$, from (6.1.8c), we have

$$\int_F \rho \cdot n_F \, p \, dA = \int_F \mathrm{curl}\,_F (\Pi^1_{r-1} v - v)_F p \, dA = 0.$$

On the macro-elements, it follows from (6.1.8d) that for all $p \in \mathring{V}^3_{r-3}(T^{\mathrm{wf}})$

$$\int_T (\mathrm{div}\, \rho) p \, dx = \int_T \mathrm{div}\,\mathrm{curl}\,(\Pi^1_{r-1} v - v) p \, dx = 0.$$

Finally, for all $p \in \mathrm{curl}\, \mathring{L}^1_{r-1}(T^{\mathrm{wf}})$, it follows from (6.1.7h) and (6.1.8e)

$$\int_T \rho \cdot p \, dx = \int_T \mathrm{curl}\,(\Pi^1_{r-1} v - v) \cdot p \, dx = 0.$$

Hence by Lemma 6.1.5, $\rho = 0$, and the identity (6.2.1b) is proved.

(iii) *Proof of* (6.2.1c). Given $w \in [C^\infty(T)]^3$, let $\rho = \mathrm{div}\, \Pi^2_{r-2} w - \Pi^3_{r-3} \mathrm{div}\, w \in V^3_{r-3}(T^{\mathrm{wf}})$. We will show that ρ vanishes on the DOFs (6.1.9), so that $\rho = 0$ and identity (6.2.1c) holds.

First, by (6.1.8c), (6.1.9a), and the Stokes Theorem of Equation (2.2.1c), we have

$$\int_T \rho\, dx = \int_T \operatorname{div}(\Pi^2_{r-2} w - w)\, dx = \int_{\partial T} (\Pi^2_{r-2} w - w) \cdot n\, dx = 0.$$

Next, using (6.1.8d) and (6.1.9b), for any $q \in \mathring{V}^3_{r-3}(T^{\mathrm{wf}})$,

$$\int_T \rho q\, dx = \int_T \operatorname{div}(\Pi^2_{r-2} w - w) q\, dx = 0,$$

since $\mathring{V}^3_{r-3}(T^{\mathrm{wf}}) = \operatorname{div} \mathring{\mathcal{V}}^2_{r-2}(T^{\mathrm{wf}})$. Then by Lemma 6.1.6, $\rho = 0$, and the identity (6.2.1c) is proved.

□

6.3 SSLV degrees of freedom

In this section we consider the sequence that will allow Lagrange elements for the third space. The third and last space will be especially well suited for fluid flow problems as the introducton describes.

Proposition 6.3.1. *Let $r \geq 3$. There exists projections*

$$\pi^1_{r-1} : [C^\infty(T)]^3 \to S^1_{r-1}(T^{\mathrm{wf}}),$$
$$\pi^2_{r-2} : [C^\infty(T)]^3 \to L^2_{r-2}(T^{\mathrm{wf}}),$$
$$\pi^3_{r-3} : C^\infty(T) \to V^3_{r-3}(T^{\mathrm{wf}})$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi_r^0"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{curl}"] \arrow[d, "\pi_{r-1}^1"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{div}"] \arrow[d, "\pi_{r-2}^2"] & C^\infty(T) \arrow[r] \arrow[d, "\pi_{r-3}^3"] & 0 \\
\mathbb{R} \arrow[r] & S_r^0(T^{\mathrm{wf}}) \arrow[r, "\mathrm{grad}"] & S_{r-1}^1(T^{\mathrm{wf}}) \arrow[r, "\mathrm{curl}"] & L_{r-2}^2(T^{\mathrm{wf}}) \arrow[r, "\mathrm{div}"] & V_{r-3}^3(T^{\mathrm{wf}}) \arrow[r] & 0.
\end{tikzcd}$$

In other words, the following identities hold.

$$\begin{aligned}
\operatorname{grad} \Pi_3^0 q &= \pi_2^1 \operatorname{grad} q, \quad \forall q \in C^\infty(T), \\
\operatorname{curl} \pi_2^1 v &= \pi_1^2 \operatorname{curl} v, \quad \forall v \in [C^\infty(T)]^3, \\
\operatorname{div} \pi_1^2 w &= \pi_0^3 \operatorname{div} w, \quad \forall w \in [C^\infty(T)]^3.
\end{aligned} \tag{6.3.1}$$

We will define degrees of freedom for each of the spaces $S_{r-1}^1(T^{\mathrm{wf}})$, $L_{r-2}^2(T^{\mathrm{wf}})$, and $V_{r-3}^3(T^{\mathrm{wf}})$ that determine the projections π_{r-1}^1, π_{r-2}^2, and π_{r-3}^3, respectively, such that the identities (6.3.1) hold.

Lemma 6.3.1. *A function $v \in S_{r-1}^1(T^{\mathrm{wf}})$, with $r \geq 2$, is fully determined by the following degrees of freedom.*

		No. of DOFs	
$v(a),$	$\forall a \in \Delta_0(T),$	$12,$	(6.3.2a)
$\operatorname{curl} v(a),$	$\forall a \in \Delta_0(T),$	$12,$	(6.3.2b)
$\int_e v \cdot q \, ds,$	$\forall q \in [\mathcal{P}_{r-3}(e)]^3,\ \forall e \in \Delta_1(T),$	$18(r-2),$	(6.3.2c)
$\int_e \operatorname{curl} v \cdot q \, ds,$	$\forall q \in [\mathcal{P}_{r-4}(e)]^3,\ \forall e \in \Delta_1(T),$	$18(r-3),$	(6.3.2d)
$\int_F \operatorname{curl}_F v_F \, q \, dA,$	$\forall q \in L_{r-3}^0(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T^{\mathrm{wf}}),$	$6r^2 - 30r + 40,$	(6.3.2e)
$\int_F (v \cdot n_F) q \, dA,$	$\forall q \in \mathcal{R}_{r-1}^0(F^{\mathrm{ct}}), \forall F \in \Delta_2(T^{\mathrm{wf}}),$	$6(r-2)(r-3),$	(6.3.2f)
$\int_F v_F \cdot q \, dA,$	$\forall q \in \ker \mathring{S}_{r-1}^1(F^{\mathrm{ct}}), \forall F \in \Delta_2(T^{\mathrm{wf}}),$	$6r^2 - 30r + 32$	(6.3.2g)
$\int_F (\operatorname{curl} v)_F \cdot q \, dA,$	$\forall q \in \mathcal{R}_{r-2}^1(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T^{\mathrm{wf}}),$	$12(r-3)^2,$	(6.3.2h)
$\int_T \operatorname{curl} v \cdot q \, dx,$	$\forall q \in \operatorname{curl} \mathring{S}_{r-1}^1(T^{\mathrm{wf}}),$	$(4r-11)(r-3)(r-4),$	(6.3.2i)

$$\int_T v \cdot q\, dx, \qquad \forall q \in \operatorname{grad} \mathring{S}_r^0(T^{\mathrm{wf}}), \qquad 2(r-2)(r-3)(r-4). \tag{6.3.2j}$$

Then the DOFs (6.3.2) *define the projection* $\pi_{r-1}^1 : [C^\infty(T)]^3 \to S_{r-1}^1(T^{\mathrm{wf}})$.

Proof. The dimension of $S_{r-1}^1(T^{\mathrm{wf}})$ is $6r^3-27r^2+51r-30$, which is equal to the number of DOFs in (6.3.2).

Let $v \in S_{r-1}^1(T^{\mathrm{wf}})$ such that v vanishes on (6.3.2). Then DOFs (6.3.2a) and (6.3.2c) yield that $v|_e = 0$ for every $e \in \Delta_1(T)$. Furthermore, it follows from DOFs (6.3.2b) and (6.3.2d) that $\operatorname{curl} v|_e = 0$ for each $e \in \Delta_1(T)$.

Since $\operatorname{curl}_F v_F \in \mathring{L}_{r-2}^0(F^{\mathrm{ct}})$, there exists a function $\beta \in L_{r-3}^0(F^{\mathrm{ct}})$ such that $\operatorname{curl}_F v_F = \lambda_F \beta$, where λ_F is the continuous linear function on F such that $\lambda_F(z) = 1$ at the split point z and $\lambda_F|_{\partial F} = 0$. Thus we have $\operatorname{curl}_F v_F = 0$ by (6.3.2e). From the exactness of sequence (2.3.10e), it follows that $v_F \in \ker \mathring{S}_{r-1}^1(F^{\mathrm{ct}})$, so $v_F = 0$ by (6.3.2g). Since $\operatorname{curl} v$ is continuous and $v_F = 0$, by Lemma 6.1.3 we have that $\operatorname{grad}(v \cdot n_F)|_F$ is continuous. Therefore, $v \cdot n_F|_F \in \mathcal{R}_{r-1}^0(F^{\mathrm{ct}})$, so $v \cdot n_F|_F = 0$ by (6.3.2f).

Since $\operatorname{curl} v \in \mathring{V}_{r-2}^1(T^{\mathrm{wf}})$ we can apply Lemma 4.1.14 to deduce that $(\operatorname{curl} v)_F \in \mathcal{R}_{r-2}^1(F^{\mathrm{ct}})$, where we also used that $(\operatorname{curl} v)_F = 0$ on ∂F. Hence, by (6.3.2h), we have that $(\operatorname{curl} v)_F = 0$. We already had that $\operatorname{curl} v \cdot n_F = 0$, so $\operatorname{curl} v|_F = 0$ on each face $F \in \Delta_2(T)$.

On the macro-elements, we use (6.3.2i) to see that $\operatorname{curl} v = 0$. By the exactness of sequence (4.1.1b), there exists a $p \in \mathring{S}_r^0(T^{\mathrm{wf}})$ such that $\operatorname{grad} p = v$. Hence by (6.3.2j), $v = 0$, which is the desired result. $\square$

Lemma 6.3.2. *Let $p \in V_r^3(T^{\mathrm{wf}})$ and $r \geq 0$. For $F \in \Delta_2(T^{\mathrm{wf}})$, if*

$$\int_e [\![p]\!]_e q\, ds = 0 \qquad \forall q \in \mathcal{P}_r(e) \qquad e \in \Delta_1^I(F^{\mathrm{ct}})\backslash\{e_F\}, \quad \textit{and} \tag{6.3.3a}$$

$$\int_{e_F} [\![p]\!]_{e_F}\, ds = 0 \qquad \forall q \in \mathcal{P}_{r-1}(e_F) \qquad e_F \in \Delta_1^I(F^{\mathrm{ct}}), \tag{6.3.3b}$$

then $p|_F$ is continuous.

Proof. We label the three triangles in $\Delta_2(F^{\mathrm{ct}})$ as Q_1, Q_2, and Q_3 such that $e_F = Q_1 \cap Q_2$. We let $p_i = p|_{Q_i}$ and let $z \in \Delta_0^I(F^{\mathrm{ct}})$. Since $p \in V_r^3(T^{\mathrm{wf}})$, condition (6.3.3a) yields that $[\![p]\!]_e = 0$ for both interior edges $e \in \Delta_1^I(F^{\mathrm{ct}})\backslash\{e_F\}$. It follows that $p_1(z) = p_2(z)$ and $p_2(z) = p_3(z)$, therefore p is continuous at z. Hence $[\![p]\!]_{e_F}(z) = 0$. Then, (6.3.3b) shows that $[\![p]\!]_{e_F} = 0$. $\square$

Lemma 6.3.3. *A function $w \in L_{r-2}^2(T^{\mathrm{wf}})$, with $r \geq 3$, is fully determined by the following degrees of freedom.*

No. of DOFs

$$w(a), \qquad \forall a \in \Delta_0(T), \qquad 12, \tag{6.3.4a}$$

$$\int_e w \cdot q\, ds, \qquad \forall q \in [\mathcal{P}_{r-4}(e)]^3,\ \forall e \in \Delta_1(T), \qquad 18(r-3), \tag{6.3.4b}$$

$$\int_F (w \cdot n_F) q\, dA, \qquad \forall q \in L_{r-3}^0(F^{\mathrm{ct}}), \quad \forall F \in \Delta_2(T), \qquad 6(r-2)(r-3)+4, \tag{6.3.4c}$$

$$\int_e [\![\mathrm{div}\, w]\!]_e q\, ds, \qquad \forall q \in \mathcal{P}_{r-3}(e),\ e \in \Delta_1^I(F^{\mathrm{ct}})\backslash\{e_F\}, \quad \forall F \in \Delta_2(T), \qquad 8(r-2), \tag{6.3.4d}$$

$$\int_{e_F} [\![\mathrm{div}\, w]\!]_{e_F} q\, ds, \qquad \forall q \in \mathcal{P}_{r-4}(e_F),\ e_F \in \Delta_1^I(F^{\mathrm{ct}}), \quad \forall F \in \Delta_2(T), \qquad 4(r-3), \tag{6.3.4e}$$

$$\int_F w_F \cdot q\, dA, \qquad \forall q \in \mathcal{R}_{r-2}^1(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \qquad 12(r-3)^2, \tag{6.3.4f}$$

$$\int_T \operatorname{div} w\, q\, dx, \qquad \forall q \in \operatorname{div} \mathring{L}^2_{r-2}(T^{\mathrm{wf}}), \qquad 2(r-3)(r-2)(r+2)+3, \quad (6.3.4g)$$
$$\int_T w \cdot q\, dx, \qquad \forall q \in \operatorname{curl} \mathring{S}^1_{r-1}(T^{\mathrm{wf}}), \qquad (4r-11)(r-3)(r-4). \quad (6.3.4h)$$

The the DOFs (6.3.4) *define the projection* $\pi^2_{r-2} : [C^\infty(T)]^3 \to L^2_{r-2}(T^{\mathrm{wf}})$.

Proof. The dimension of $L^2_{r-2}(T^{\mathrm{wf}})$ is $3(2r-3)(r^2-3r+3)$, which is equal to the number of DOFs in (6.3.4).

Let $w \in L^2_{r-2}(T^{\mathrm{wf}})$ such that w vanishes on the DOFs (6.3.4). Using DOFs (6.3.4a) and (6.3.4b), we have that $w|_e = 0$ for every $e \in \Delta_1(T)$, hence $w \cdot n_F|_F \in \mathring{L}^0_{r-2}(F^{\mathrm{ct}})$. Then there exists a function $\beta \in L^0_{r-3}(F^{\mathrm{ct}})$ such that $w \cdot n_F|_F = \lambda\beta$. Hence by (6.3.4c), $\beta = 0$, so $w \cdot n_F|_F = 0$. Let $K \in \Delta_3(T^{\mathrm{a}})$ with $F \in \Delta_2(K)$. Thus, we can write $w \cdot n_F = \mu\psi$ for some $\psi \in \mathcal{P}_{r-3}(T^{\mathrm{wf}})$. However, since $w \cdot n_F$ is continuous on K and μ is linear on positive on K it must be that ψ is continuous on K. Moreover, since $n_F \cdot \operatorname{grad}(w \cdot n_F) = \psi \operatorname{grad} \mu \cdot n_F$ on F which implies that $n_F \cdot \operatorname{grad}(w \cdot n_F)$ is continuous on F.

Using DOFs (6.3.4d) - (6.3.4e) and Lemma 6.3.2, we have that $\operatorname{div} w|_F \in \mathring{L}^2_{r-3}(F^{\mathrm{ct}})$ for each $F \in \Delta_2(T)$. We can write $\operatorname{div}_F w_F = \operatorname{div} w|_F - n_F \cdot \operatorname{grad}(w \cdot n_F)$ and, hence, $\operatorname{div}_F w_F$ is continuous which implies that $w_F \in \mathcal{R}^1_{r-2}(F^{\mathrm{ct}})$. By (6.3.4f), it follows that $w_F = 0$ on F.

Now we have that $w \in \mathring{L}^2_{r-2}(T^{\mathrm{wf}})$. On the macro-element, $\operatorname{div} w \in \mathring{L}^2_{r-3}(T^{\mathrm{wf}})$, so by (6.3.4g), $\operatorname{div} w = 0$. Using the exactness property of sequence (4.1.1b), there exists a function $p \in \mathring{S}^1_{r-1}(T^{\mathrm{wf}})$ such that $\operatorname{curl} p = w$. Then by (6.3.4h), $w = 0$, which is the desired result. $\square$

Lemma 6.3.4. *A function $p \in V^3_{r-3}(T^{\mathrm{wf}})$, with $r \geq 3$, is fully determined by the following* *degrees of freedom.*

No. of DOFs

$$\int_e [\![p]\!]_e q\, ds, \qquad \forall q \in \mathcal{P}_{r-3}(e),\ e \in \Delta_1^I(F^{\mathrm{ct}})\backslash\{e_F\},$$

$$\forall F \in \Delta_2(T), \qquad 8(r-2), \tag{6.3.5a}$$

$$\int_{e_F} [\![p]\!]_{e_F} q\, ds, \quad \forall q \in \mathcal{P}_{r-4}(e_F),\ e_F \in \Delta_1^I(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \qquad 4(r-3), \tag{6.3.5b}$$

$$\int_T p\, dx, \qquad 1, \tag{6.3.5c}$$

$$\int_T pq\, dx, \qquad \forall q \in \mathring{\mathcal{V}}_{r-3}^3(T^{\mathrm{wf}}), \qquad 2r^3 - 6r^2 - 8r + 27. \tag{6.3.5d}$$

Then the DOFs (6.3.5) *define the projection* $\pi_{r-3}^3 : C^\infty(T) \to V_{r-3}^3(T^{\mathrm{wf}})$.

Proof. The dimension of $V_{r-3}^3(T^{\mathrm{wf}})$ is $2r(r-1)(r-2)$, which is equal to the number of DOFs in (6.3.5).

Let $p \in V_{r-3}^3(T^{\mathrm{wf}})$ such that p vanishes on the DOFs (6.3.5). Then by (6.3.5a) - (6.3.5b), $[\![p]\!]_e = 0$ for every $e \in \Delta_1^I(F^{\mathrm{ct}})$ for each $F \in \Delta_2(T)$. Combined with (6.3.5c), it follows that $p \in \mathring{\mathcal{V}}_{r-3}^3(T^{\mathrm{wf}})$. So by (6.3.5d), $p = 0$. $\square$

6.4 SSLV commuting diagram

Theorem 6.4.1. *Let $r \geq 3$, and let $\Pi_r^0 : C^\infty(T) \to S_r^0(T^{\mathrm{wf}})$ be the projection defined in Lemma 6.1.1, let $\pi_{r-1}^1 : [C^\infty(T)]^3 \to S_{r-1}^1(T^{\mathrm{wf}})$ be the projection defined in Lemma 6.3.1, let $\pi_{r-2}^2 : [C^\infty(T)]^3 \to L_{r-2}^2(T^{\mathrm{wf}})$ be the projection defined in Lemma 6.3.3, and let $\pi_{r-3}^3 : C^\infty(T) \to V_{r-3}^3(T^{\mathrm{wf}})$ be the projection defined in Lemma 6.3.4. Then the following diagram commutes.*

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(T) & \xrightarrow{\text{grad}} & [C^\infty(T)]^3 & \xrightarrow{\text{curl}} & [C^\infty(T)]^3 & \xrightarrow{\text{div}} & C^\infty(T) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \Pi_r^0} & & \Big\downarrow{\scriptstyle \pi_{r-1}^1} & & \Big\downarrow{\scriptstyle \pi_{r-2}^2} & & \Big\downarrow{\scriptstyle \pi_{r-3}^3} & & \\
\mathbb{R} & \longrightarrow & S_r^0(T^{\mathrm{wf}}) & \xrightarrow{\text{grad}} & S_{r-1}^1(T^{\mathrm{wf}}) & \xrightarrow{\text{curl}} & L_{r-2}^2(T^{\mathrm{wf}}) & \xrightarrow{\text{div}} & V_{r-3}^3(T^{\mathrm{wf}}) & \longrightarrow & 0.
\end{array}$$

In other words, we have

$$\operatorname{grad} \Pi_r^0 q = \pi_{r-1}^1 \operatorname{grad} q, \quad \forall q \in C^\infty(T), \tag{6.4.1a}$$

$$\operatorname{curl} \pi_{r-1}^1 v = \pi_{r-2}^2 \operatorname{curl} v, \quad \forall v \in [C^\infty(T)]^3, \tag{6.4.1b}$$

$$\operatorname{div} \pi_{r-2}^2 w = \pi_{r-3}^3 \operatorname{div} w, \quad \forall w \in [C^\infty(T)]^3. \tag{6.4.1c}$$

Proof. (i) *Proof of* (6.4.1a). Let $q \in C^\infty(T)$, and set $\rho = \operatorname{grad} \Pi_r^0 q - \Pi_{r-1}^1 \operatorname{grad} q$. Then $\rho \in S_{r-1}^1(T^{\mathrm{wf}})$, so we must show that ρ vanishes on the DOFs (6.3.2).

For each $a \in \Delta_0(T)$, $\rho(a) = \operatorname{grad} \Pi_r^0 q(a) - \Pi_{r-1}^1 \operatorname{grad} q(a) = 0$ by (6.1.2a) and (6.3.2a). Then, using (6.3.2b), $\operatorname{curl} \rho(a) = \operatorname{curl} (\operatorname{grad} (\Pi_r^0 q - q)) = 0$. By (6.3.2c), we have, for all $p \in [\mathcal{P}_{r-3}(e)]^3$ on each $e \in \Delta_1(T)$,

$$\begin{aligned}
\int_e \rho \cdot p \, ds &= \int_e \operatorname{grad} (\Pi_r^0 q - q) \cdot p \, ds \\
&= \int_e \left(\frac{\partial}{\partial n_e^+} (\Pi_r^0 q - q) n_e^+ + \frac{\partial}{\partial n_e^-} (\Pi_r^0 q - q) n_e^- + \frac{\partial}{\partial t} (\Pi_r^0 q - q) t \right) \cdot p \, ds \\
&= \int_e \frac{\partial}{\partial t} (\Pi_r^0 q - q)(p \cdot t) \, ds && \text{by (6.1.2d)} \\
&= - \int_e (\Pi_r^0 q - q) \frac{\partial}{\partial t} (p \cdot t) \, ds && \text{by (6.1.2a)} \\
&= 0. && \text{by (6.1.2c)}
\end{aligned}$$

Next, using (6.3.2d), for all $p \in [\mathcal{P}_{r-4}(e)]^3$,

$$\int_e \operatorname{curl} \rho \cdot p \, ds = \int_e \operatorname{curl} \operatorname{grad} (\Pi_r^0 q - q) \cdot p \, ds = 0.$$

On the faces, from (6.3.2e), we have for all $p \in L^0_{r-3}(F^{\mathrm{ct}})$,

$$\int_F \operatorname{curl}_F \rho_F \, p \, dA = \int_F \operatorname{curl}_F \operatorname{grad}_F (\Pi^0_r q - q) p \, dA = 0.$$

Using (6.1.2f) and (6.3.2f), for all $p \in \mathcal{R}^0_{r-1}(F^{\mathrm{ct}})$,

$$\int_F (\rho \cdot n_F) p \, dA = \int_F (n_F \cdot \operatorname{grad} (\Pi^0_r q - q)) \, p \, dA = 0.$$

Next, using (6.3.2g), we have for all $p \in \ker \mathring{S}^1_{r-1}(F^{\mathrm{ct}})$,

$$\int_F \rho_F \cdot p \, dA = \int_F (\operatorname{grad}_F (\Pi^0_r q - q)_F) \cdot p \, dA = 0,$$

where we have used (6.1.2e) and the result $\operatorname{grad}_F \mathring{S}^0_r(F^{\mathrm{ct}}) = \ker \mathring{S}^1_{r-1}(F^{\mathrm{ct}})$ due to the exactness of sequence (2.3.10f). Then we use (6.3.2h) and (6.1.2e), so that for all $p \in \mathcal{R}^1_{r-2}(F^{\mathrm{ct}})$,

$$\int_F (\operatorname{curl} \rho)_F \cdot p \, dA = \int_F (\operatorname{curl} (\operatorname{grad} (\Pi^0_r q - q)))_F \cdot p \, dA = 0.$$

On the macro-elements, we use (6.3.2i) so that, for all $p \in \operatorname{curl} \mathring{S}^1_{r-1}(T^{\mathrm{wf}})$,

$$\int_T \operatorname{curl} \rho \cdot p \, dx = \int_T \operatorname{curl} \operatorname{grad} (\Pi^0_r q - q) \cdot p \, dx = 0.$$

Lastly, we use (6.1.2g) and (6.3.2j) to see that, for every $p \in \operatorname{grad} \mathring{S}^0_r(T^{\mathrm{wf}})$,

$$\int_T \rho \cdot p \, dx = \int_T \operatorname{grad} (\Pi^0_r q - q) \cdot p \, dx = 0.$$

Therefore, by Lemma 6.3.1, $\rho = 0$, and the identity (6.4.1a) is proved.

(ii) *Proof of* (6.4.1b). Let $v \in [C^\infty(T)]^3$, and set $\rho = \operatorname{curl} \pi^1_{r-1} v - \pi^2_{r-2} \operatorname{curl} v$. Then

$\rho \in L^2_{r-2}(T^{\mathrm{wf}})$, so we must show that ρ vanishes on the DOFs (6.3.4). By (6.3.2b) and (6.3.4a), $\rho(a) = \operatorname{curl} \pi^1_{r-1} v(a) - \pi^2_{r-2} \operatorname{curl} v(a) = 0$. By (6.3.2d) and (6.3.4b), for all $p \in [\mathcal{P}_{r-4}(e)]^3$ where $e \in \Delta_1(T)$,

$$\int_e \rho \cdot p \, ds = \int_e \operatorname{curl} (\pi^1_{r-1} v - v) \cdot p \, ds = 0.$$

By (6.3.4c) and (6.3.2e), for every $p \in L^0_{r-3}(F^{\mathrm{ct}})$,

$$\int_F (\rho \cdot n_F) p \, dA = \int_F \operatorname{curl}_F ((\pi^1_{r-1} v)_F - v_F) p \, dA = 0.$$

Using (6.3.4d), for all $p \in \mathcal{P}_{r-3}(e)$, $e \in \Delta^I_1(F^{\mathrm{ct}}) \backslash \{e_F\}$ and $F \in \Delta_2(T)$, we have

$$\int_e [\![\operatorname{div} \rho]\!]_e p \, ds = \int_e [\![\operatorname{div} \operatorname{curl} (\pi^1_{r-1} v - v)]\!]_e p \, ds = 0.$$

Similarly, (6.3.4e) yields that $\int_{e_F} [\![\operatorname{div} \rho]\!]_{e_F} p \, ds = 0$ for $p \in \mathcal{P}_{r-2}(e)$. Next, using (6.3.4f), for any $p \in \mathcal{R}^1_{r-2}(F^{\mathrm{ct}})$, we have

$$\int_F \rho_F \cdot p \, dA = \int_F (\operatorname{curl} (\pi^1_{r-1} v - v))_F \cdot p \, dA = 0$$

by (6.3.2h).

By (6.3.4g) and for any $p \in \operatorname{div} \mathring{L}^2_{r-2}(T^{\mathrm{wf}})$,

$$\int_T \operatorname{div} \rho \, p \, dx = \int_T \operatorname{div} \operatorname{curl} (\pi^1_{r-1} v - v) p \, dx = 0.$$

Finally, by (6.3.2i), (6.3.4h), and for any $p \in \operatorname{curl} \mathring{S}^1_{r-1}(T^{\mathrm{wf}})$,

$$\int_T \rho \cdot p \, dx = \int_T \operatorname{curl} (\pi^1_{r-1} v - v) \cdot p \, dx = 0.$$

Therefore, $\rho = 0$ by Lemma 6.3.3, which is the desired result.

(iii) *Proof of* (6.4.1c). Let $w \in [C^\infty(T)]^3$, and set $\rho = \mathrm{div}\, \pi^2_{r-2} w - \pi^3_{r-3} \mathrm{div}\, w$. Then $\rho \in V^3_{r-3}(T^{\mathrm{wf}})$, so we must show that ρ vanishes on the DOFs (6.3.5).

First, we see from (6.3.4d) and (6.3.5a) that for any $p \in \mathcal{P}_{r-3}(e)$, $e\Delta^I_1(F^{\mathrm{ct}})\backslash\{e_F\}$ and $F \in \Delta_2(T)$, we have

$$\int_e [\![\rho]\!]_e p\, ds = \int_e [\![\mathrm{div}\, (\pi^2_{r-2} w - w)]\!]_e p\, ds = 0.$$

Similarly, we can show that ρ vanish on the DOFs of (6.3.5b).

On the macro-elements, we use (6.3.5c), (6.3.4c), and the Stokes Theorem of Equation (2.2.1c) to see that

$$\int_T \rho\, dx = \int_T \mathrm{div}\, (\pi^2_{r-2} w - w)\, dx = \int_{\partial T} (\pi^2_{r-2} w - w) \cdot n_F\, dA = 0.$$

Lastly, for any $p \in \mathring{\mathcal{V}}^3_{r-3}(T^{\mathrm{wf}})$,

$$\int_T \rho\, p\, dx = \int_T (\pi^2_{r-2} w - w) p\, dx = 0$$

by (6.3.4g) and (6.3.5d) and using the fact $\mathrm{div}\, \mathring{L}^2_{r-2}(T^{\mathrm{wf}}) = \mathring{\mathcal{V}}^3_{r-3}(T^{\mathrm{wf}})$ (from the exactness of sequence (4.1.1c)). Therefore $\rho = 0$ by Lemma 6.3.4, which is the desired result. $\square$

6.5 SSSL degrees of freedom

Proposition 6.5.1. *Let $r \geq 3$. We can construct the projections*

$$\varpi^2_{r-2} : [C^\infty(T)]^3 \to V^2_{r-2}(T^{\mathrm{wf}}),$$
$$\varpi^3_{r-3} : C^\infty(T) \to V^3_{r-3}(T^{\mathrm{wf}})$$

such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_r"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{curl}"] \arrow[d, "\pi^1_{r-1}"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{div}"] \arrow[d, "\varpi^2_{r-2}"] & C^\infty(T) \arrow[r] \arrow[d, "\varpi^3_{r-3}"] & 0 \\
\mathbb{R} \arrow[r] & S^0_r(T^{\mathrm{wf}}) \arrow[r, "\mathrm{grad}"] & S^1_{r-1}(T^{\mathrm{wf}}) \arrow[r, "\mathrm{curl}"] & S^2_{r-2}(T^{\mathrm{wf}}) \arrow[r, "\mathrm{div}"] & L^3_{r-3}(T^{\mathrm{wf}}) \arrow[r] & 0.
\end{tikzcd}$$

In other words, the following identities hold.

$$\begin{aligned}
\mathrm{curl}\,\pi^1_2 v &= \varpi^2_1 \mathrm{curl}\, v, \quad \forall v \in [C^\infty(T)]^3, \\
\mathrm{div}\,\varpi^2_1 w &= \varpi^3_0 \mathrm{div}\, w, \quad \forall w \in [C^\infty(T)]^3.
\end{aligned} \tag{6.5.1}$$

We will define degrees of freedom for each of the spaces $S^2_{r-2}(T^{\mathrm{wf}})$ and $L^3_{r-3}(T^{\mathrm{wf}})$ that determine the projections ϖ^2_{r-2} and ϖ^3_{r-3}, respectively, such that the identities (6.5.1) hold.

Lemma 6.5.1. *A function $w \in S^2_{r-2}(T^{\mathrm{wf}})$, with $r \geq 3$, is fully determined by the following degrees of freedom.*

	No. of DOFs	
$w(a)$,	12,	(6.5.2a)
$\mathrm{div}\, w(a)$,	4,	(6.5.2b)

$$\int_e w \cdot q\, ds, \qquad \forall q \in [\mathcal{P}_{r-4}(e)]^3,\ \forall e \in \Delta_1(T), \qquad 18(r-3), \tag{6.5.2c}$$

$$\int_e \operatorname{div} wq\, ds, \qquad \forall q \in \mathcal{P}_{r-5}(e),\ \forall e \in \Delta_1(T), \qquad 6(r-4), \tag{6.5.2d}$$

$$\int_F (w \cdot n_F) q\, dA, \qquad \forall q \in L^0_{r-3}(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \qquad 6(r-2)(r-3)+4, \tag{6.5.2e}$$

$$\int_F w_F \cdot q\, dA, \qquad \forall q \in \mathcal{R}^1_{r-2}(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \qquad 12(r-3)^2, \tag{6.5.2f}$$

$$\int_F \operatorname{div} w\, q\, dA, \qquad \forall q \in L^2_{r-4}(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \qquad 6(r-3)(r-4)+4, \tag{6.5.2g}$$

$$\int_T \operatorname{div} wq\, dx, \qquad \forall q \in \mathring{L}^3_{r-3}(T^{\mathrm{wf}}), \qquad (r-4)(2r^2-13r+23), \tag{6.5.2h}$$

$$\int_T w \cdot q\, dx, \qquad \forall q \in \operatorname{curl} \mathring{S}^1_{r-1}(T^{\mathrm{wf}}), \qquad (4r-11)(r-3)(r-4). \tag{6.5.2i}$$

Then the DOFs 6.5.2 define the projection $\varpi^2_{r-2}(T^{\mathrm{wf}}) : [C^\infty(T)]^3 \to S^2_{r-2}(T^{\mathrm{wf}})$.

Proof. The dimension of $S^2_{r-2}(T^{\mathrm{wf}})$ is $6r^3-36r^2+80r-62$, which is equal to the number of DOFs in (6.5.2).

Let $w \in S^2_{r-2}(T^{\mathrm{wf}})$ such that w vanishes on the DOFs (6.5.2). Then from (6.5.2a) and (6.5.2c), $w|_e = 0$ for each $e \in \Delta_1(T)$, and $\operatorname{div} w|_e = 0$ by (6.5.2b) and (6.5.2d).

On each $F \in \Delta_2(T)$, $w \cdot n_F|_F \in \mathring{L}^0_{r-2}(F^{\mathrm{ct}})$, hence there exists a function $\beta \in L^0_{r-3}(F^{\mathrm{ct}})$ such that $w \cdot n_F|_F = \lambda\beta$. Then $w \cdot n_F|_F = \lambda\beta = 0$ by (6.5.2e). As we did in the proof of Lemma 6.3.3, since $w, \operatorname{div} w$ are continuous and $w \cdot n_F$ vanishes on F, we have that $\operatorname{div}_F w_F$ is continuous. Hence, $w_F \in \mathcal{R}^1_{r-2}(F^{\mathrm{ct}})$. Therefore (6.5.2f) yields $w_F = 0$. Now we have $\operatorname{div} w \in L^2_{r-3}(F^{\mathrm{ct}})$ and $\operatorname{div} w$ vanishes on ∂F. So, $\operatorname{div} w|_F = 0$ by (6.5.2g), therefore $w \in \mathring{S}^2_{r-2}(T^{\mathrm{wf}})$.

On the macro-element, we have that $\operatorname{div} w = 0$ by (6.5.2h). So by the exactness of sequence (4.1.1c), there exists a $v \in \mathring{S}^1_{r-1}(T^{\mathrm{wf}})$ such that $\operatorname{curl} v = w$. Hence by (6.5.2i), $w = 0$, which is the desired result. □

Lemma 6.5.2. *A function $p \in L^3_{r-3}(T^{\mathrm{wf}})$, with $r \geq 3$, is fully determined by the following degrees of freedom.*

$$
\begin{array}{llr}
 & & \textit{No. of DOFs} \\
p(a), & & 4, \quad (6.5.3a) \\
\int_e pq\, ds, & \forall q \in \mathcal{P}_{r-5}(e), \forall e \in \Delta_1(T), & 6(r-4), \quad (6.5.3b) \\
\int_F pq\, dA, & \forall q \in L^2_{r-4}(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), & 6(r-3)(r-4)+4, \quad (6.5.3c) \\
\int_T p\, dx, & & 1, \quad (6.5.3d) \\
\int_T pq\, dx, & \forall q \in \mathring{L}^3_{r-3}(T^{\mathrm{wf}}), & (r-4)(2r^2-13r+23). \quad (6.5.3e)
\end{array}
$$

Then the DOFs (6.5.3) *define the projection $\varpi^3_{r-3}(T^{\mathrm{wf}}) : C^\infty(T) \to L^3_{r-3}(T^{\mathrm{wf}})$.*

Proof. The dimension of $L^3_{r-3}(T^{\mathrm{wf}})$ is $(2r-5)(r^2-5r+7)$, which matches the number of DOFs in (6.5.3).

Let $p \in L^3_{r-3}(T^{\mathrm{wf}})$ such that p vanishes on the DOFs (6.5.3). Then by (6.5.3a) and (6.5.3b), $p|_e = 0$ for every $e \in \Delta_1(T)$. For each $F \in \Delta_2(T)$, we have that $p|_F \in \mathring{L}^2_{r-3}(F^{\mathrm{ct}})$, so $p|_F = 0$ by (6.5.3c). Then by (6.5.3d), we have $p \in \mathring{L}^3_{r-3}(T^{\mathrm{wf}})$, and by (6.5.3e), $p = 0$. $\square$

6.6 SSSL commuting diagram

Theorem 6.6.1. *Recall that $\Pi^0_r : C^\infty(T) \to S^0_r(T^{\mathrm{wf}})$ is the projection defined in Lemma 6.1.1, $\pi^1_{r-1} : [C^\infty(T)]^3 \to S^1_{r-1}(T^{\mathrm{wf}})$ is the projection defined in Lemma 6.3.1, $\varpi^2_{r-2} : [C^\infty(T)]^3 \to L^2_{r-2}(T^{\mathrm{wf}})$ is the projection defined in Lemma 6.5.1, and*

$\varpi^3_{r-3} : C^\infty(T) \to V^3_{r-3}(T^{\mathrm{wf}})$ *is the projection defined in Lemma 6.3.4. Then the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^0_r"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{curl}"] \arrow[d, "\pi^1_{r-1}"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{div}"] \arrow[d, "\varpi^2_{r-2}"] & C^\infty(T) \arrow[r] \arrow[d, "\varpi^3_{r-3}"] & 0 \\
\mathbb{R} \arrow[r] & S^0_r(T^{\mathrm{wf}}) \arrow[r, "\mathrm{grad}"] & S^1_{r-1}(T^{\mathrm{wf}}) \arrow[r, "\mathrm{curl}"] & S^2_{r-2}(T^{\mathrm{wf}}) \arrow[r, "\mathrm{div}"] & L^3_{r-3}(T^{\mathrm{wf}}) \arrow[r] & 0.
\end{tikzcd}$$

In other words, we have

$$\mathrm{grad}\,\Pi^0_r q = \pi^1_{r-1}\mathrm{grad}\,q, \quad \forall q \in C^\infty(T), \tag{6.6.1a}$$

$$\mathrm{curl}\,\pi^1_{r-1} v = \varpi^2_{r-2}\mathrm{curl}\,v, \quad \forall v \in [C^\infty(T)]^3, \tag{6.6.1b}$$

$$\mathrm{div}\,\varpi^2_{r-2} w = \varpi^3_{r-3}\mathrm{div}\,w, \quad \forall w \in [C^\infty(T)]^3. \tag{6.6.1c}$$

Proof. (i) *Proof of* (6.6.1a). The identity (6.6.1a) holds by Theorem 6.4.1.

(ii) *Proof of* (6.6.1b). Let $v \in [C^\infty(T)]^3$, and set $\rho = \mathrm{curl}\,\pi^1_{r-1}v - \varpi^2_{r-2}\mathrm{curl}\,v$. Then $\rho \in S^2_{r-2}(T^{\mathrm{wf}})$, so we must show that ρ vanishes on the DOFs (6.5.2). By (6.3.2b) and (6.5.2a), $\rho(a) = \mathrm{curl}\,\pi^1_{r-1}v(a) - \mathrm{curl}\,v(a) = 0$, and by (6.5.2b), $\mathrm{div}\,\rho(a) = \mathrm{div}\,\mathrm{curl}\,\pi^1_{r-1}v(a) - \mathrm{div}\,\mathrm{curl}\,v(a) = 0$ for each $a \in \Delta_0(T)$.

For all $e \in \Delta_1(T)$ and for all $p \in [\mathcal{P}_{r-4}(e)]^3$,

$$\int_e \rho \cdot p\,ds = \int_e \mathrm{curl}\,(\pi^1_{r-1}v - v) \cdot p\,ds = 0$$

by (6.3.2d) and (6.5.2c). Using (6.5.2d), for all $p \in \mathcal{P}_{r-5}(e)$, we have

$$\int_e \mathrm{div}\,\rho\,p\,ds = \int_e \mathrm{div}\,\mathrm{curl}\,(\pi^1_{r-1}v - v)p\,ds = 0.$$

On each face $F \in \Delta_2(T)$, for every $p \in L^0_{r-3}(F^{\mathrm{ct}})$,

$$\int_F (\rho \cdot n_F) p \, dA = \int_F \operatorname{curl}_F((\pi^1_{r-1} v)_F - v_F) p \, dA = 0$$

by (6.3.2e) and (6.5.2e). For each $p \in \mathcal{R}^1_{r-2}(F^{\mathrm{ct}})$, we have

$$\int_F \rho_F \cdot p \, dA = \int_F \operatorname{curl}(\pi^1_{r-1} v - v)_F \cdot p \, dA = 0,$$

where we used (6.3.2h) and (6.5.2f). Next, for every $p \in L^2_{r-4}(F^{\mathrm{ct}})$, (6.5.2g) yields

$$\int_F \operatorname{div} \rho \, p \, dA = \int_F \operatorname{div} \operatorname{curl}(\pi^1_{r-1} v - v) \, p \, dA = 0.$$

On the macro-element T^{wf}, for each $p \in \mathring{L}^3_{r-3}(T^{\mathrm{wf}})$, we use (6.5.2h) so that

$$\int_T \operatorname{div} \rho \, p \, dx = \int_T \operatorname{div} \operatorname{curl}(\pi^1_{r-1} v - v) p \, dx = 0.$$

Finally, for all $p \in \operatorname{curl} \mathring{S}^1_{r-1}(T^{\mathrm{wf}})$,

$$\int_T \rho \cdot p \, dx = \int_T \operatorname{curl}(\pi^1_{r-1} v - v) \cdot p \, dx = 0,$$

by (6.3.2i) and (6.5.2i). Therefore, by Lemma 6.5.1, $\rho = 0$, and the identity (6.6.1b) is proved.

(iii) *Proof of* (6.6.1c). Let $w \in [C^\infty(T)]^3$, and set $\rho = \operatorname{div} \varpi^2_{r-2} w - \varpi^3_{r-3} \operatorname{div} w$. Then $\rho \in L^3_{r-3}(T^{\mathrm{wf}})$, so we must show that ρ vanishes on the DOFs (6.5.3).

For all $a \in \Delta_0(T)$, $\rho(a) = \operatorname{div} \varpi^2_{r-2} w(a) - \varpi^3_{r-3} \operatorname{div} w(a) = 0$ by (6.5.2b) and

(6.5.3a). On each edge $e \in \Delta_1(T)$ and for all $p \in \mathcal{P}_{r-5}(e)$,

$$\int_e \rho\, p\, ds = \int_e \operatorname{div}(\varpi_{r-2}^2 w - w) p\, ds = 0$$

by (6.5.2d) and (6.5.3b).

On each face $F \in \Delta_2(T)$, using (6.5.3c), we have, for all $p \in L_{r-4}^2(F^{\mathrm{ct}})$,

$$\int_F \rho\, p\, dA = \int_F \operatorname{div}(\varpi_{r-2}^2 w - w)\, p\, ds = 0,$$

by (6.5.2g).

Now we use (6.5.3d) and the Stokes Theorem of Equation (2.2.1c) to see that

$$\int_T \rho\, dx = \int_T \operatorname{div}(\varpi_{r-2}^2 w - w)\, dx = \int_{\partial T} (\varpi_{r-2}^2 w - w) \cdot n\, dA = 0$$

by (6.5.2e). Then by (6.5.2h) and (6.5.3e), for any $p \in \mathring{L}_{r-3}^3(T^{\mathrm{wf}})$,

$$\int_T \rho\, p\, dx = \int_T \operatorname{div}(\varpi_{r-2}^2 w - w) p\, dx = 0.$$

Hence $\rho = 0$ by Lemma 6.5.2, and the identity (6.6.1c) is proved. $\square$

6.7 Global spaces and commuting diagrams

In this section, we discuss the global finite element spaces of any polynomial order induced by the degrees of freedom of Sections 6.1, 6.3, and 6.5, thereby extending the results of Section 5.7. We let $\mathcal{T}_h$ be the triangulation of the polygonal domain $\Omega \subset \mathbb{R}^3$, and we let $\mathcal{T}_h^{\mathrm{wf}}$ be the Worsey-Farin refinement of $\mathcal{T}_h$. Recall the operator $\theta_e(\cdot)$ and the set $\mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})$

given in Definitions 5.7.2 and 5.7.1, respectively. We will show that the projections defined in Sections 6.1, 6.3, and 6.5 induce the following global spaces.

$$
\begin{aligned}
\mathcal{S}_r^0(\mathcal{T}_h^{\mathrm{wf}}) &= \{q \in C^1(\Omega) : q|_T \in S_r^0(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\},\\
\mathcal{S}_{r-1}^1(\mathcal{T}_h^{\mathrm{wf}}) &= \{v \in [C(\Omega)]^3 : \operatorname{curl} v \in [C(\Omega)]^3,\, v|_T \in S_{r-1}^1(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\},\\
\mathcal{S}_{r-2}^2(\mathcal{T}_h^{\mathrm{wf}}) &= \{w \in [C(\Omega)]^3 : \operatorname{div} w \in C(\Omega), w|_T \in S_{r-2}^2(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\},\\
\mathcal{L}_{r-1}^1(\mathcal{T}_h^{\mathrm{wf}}) &= \{v \in [C(\Omega)]^3 : v|_T \in L_{r-1}^1(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\},\\
\mathcal{L}_{r-2}^2(\mathcal{T}_h^{\mathrm{wf}}) &= \{w \in [C(\Omega)]^3 : w|_T \in L_{r-2}^2(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h\},\\
\mathscr{V}_{r-2}^2(\mathcal{T}_h^{\mathrm{wf}}) &= \{w \in H(\operatorname{div};\Omega) : w|_T \in V_{r-2}^2(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h,\\
&\qquad\qquad \theta_e(w \cdot t) = 0\,\forall e \in \mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})\},\\
\mathscr{V}_{r-3}^3(\mathcal{T}_h^{\mathrm{wf}}) &= \{p \in L^2(\Omega) : p|_T \in V_{r-3}^3(T^{\mathrm{wf}})\,\forall T \in \mathcal{T}_h,\, \theta_e(p) = 0\,\forall e \in \mathcal{E}(\mathcal{T}_h^{\mathrm{wf}})\},\\
V_{r-3}^3(\mathcal{T}_h^{\mathrm{wf}}) &= \mathcal{P}_{r-3}(\mathcal{T}_h^{\mathrm{wf}}).
\end{aligned}
$$

These spaces generalize those defined in Chapter 5 to arbitrary polynomial order. Let T_1 and T_2 be adjacent tetrahedra in $\mathcal{T}_h$ that share a face F. Let K_1 and K_2 be tetrahedra in T_1^{a} and T_2^{a}, respectively, such that K_1 and K_2 share the face F. Let F^{ct} represent the triangulation of F^{ct} in $\mathcal{T}_h^{\mathrm{wf}}$, and let K_i^{wf} be the triangulation of K_i in $\mathcal{T}_h^{\mathrm{wf}}$, where $1 \le i \le 2$. Given a simplex $S \in \Delta_s(\mathcal{T}_h^{\mathrm{wf}})$, with $0 \le s \le 3$, let $\chi(S)$ represent that characteristic function that equals 1 on S and 0 otherwise. Without loss of generality, we choose $n_F = n_1$, the outward normal to T_1 on F.

Remark 6.7.1. *Due to the singular edges formed through a Worsey-Farin refinement of a triangulation, the global space $\mathcal{L}_r^1(\mathcal{T}_h^{\mathrm{wf}})$ with $r \ge 1$ has the same inherent property described in Lemma 5.7.3 for the case $r = 1$. Let $T \in \mathcal{T}_h$, $F \in \Delta_2(T)$, and $e \in \Delta_1^I(F^{\mathrm{ct}})$. Then for any $w \in \mathcal{L}_r^1(\mathcal{T}_h^{\mathrm{wf}})$,*

$$\theta_e(\operatorname{curl} w \cdot t) = 0, \tag{6.7.1}$$

where t is the unit tangent vector to edge e. The proof of this result is exactly the same as the proof of Lemma 5.7.3. Similarly, any function $v \in \mathcal{L}_r^2(\mathcal{T}_h^{\mathrm{wf}})$ satisfies

$$\theta_e(\operatorname{div} v) = 0. \tag{6.7.2}$$

This result follows using the same proof as Lemma 5.7.4.

Now we will show that the local DOFs of Section 6.1 induce the associated global spaces. As these proofs are quite similar to those in Section 5.7 in the lowest order case, we only include a sketch of the proofs of the general results.

Lemma 6.7.2. *The local degrees of freedom stated in Lemma 6.1.1 induce the global space $\mathcal{S}_r^0(\mathcal{T}_h^{\mathrm{wf}})$.*

Proof. Let $q_1 \in S_r^0(T_1^{\mathrm{wf}})$ and $q_2 \in S_r^0(T_2^{\mathrm{wf}})$ such that $q_1 - q_2$ vanishes on the DOFs (6.1.2a) - (6.1.2f) associated with the triangulation F^{ct}. We extend q_1 to K_2 according to Remark 2.6.1, and we set $p = q_1 - q_2$. Then by Lemma 6.1.1, we have $p = 0$ and $\operatorname{grad} p = 0$ on F, therefore the function $q_1\chi(K_1) + q_2\chi(K_2)$ is C^1 across F. Therefore the DOFs (6.1.2) induce the global space $\mathcal{S}_r^0(\mathcal{T}_h^{\mathrm{wf}})$. $\square$

Lemma 6.7.3. *The local degrees of freedom stated in Lemma 6.1.4 induce the global space $\mathcal{L}_{r-1}^1(\mathcal{T}_h^{\mathrm{wf}})$.*

Proof. Let $v_1 \in L_{r-1}^1(T_1^{\mathrm{wf}})$ and $v_2 \in L_{r-1}^1(T_2^{\mathrm{wf}})$ such that $v_1 - v_2$ vanishes on the DOFs (6.1.7a) - (6.1.7g) associated with the triangulation F^{ct} of the face F, and we extend v_1 to K_2 as in Remark 2.6.1. Then we set $w = v_1 - v_2$, and following the proof of Lemma 6.1.4, $w = 0$ on F, since w vanishes on the DOFs (6.1.7) on F. Hence, $v := v_1\chi(T_1) + v_2\chi(T_2)$ is continuous on all of $T_1 \cup T_2$. It follows that the local DOFs (6.1.7) induce the global space $\mathcal{L}_{r-1}^1(\mathcal{T}_h^{\mathrm{wf}})$. $\square$

Lemma 6.7.4. *The local degrees of freedom stated in Lemma 6.1.5 induce the global space* $\mathscr{V}^2_{r-2}(\mathcal{T}_h^{\mathrm{wf}})$.

Proof. Let $w_1 \in V^2_{r-2}(T_1^{\mathrm{wf}})$ and $w_2 \in V^2_{r-2}(T_2^{\mathrm{wf}})$ such that $w_1 - w_2$ vanishes on the DOFs (6.1.8a) - (6.1.8d) associated with the triangulation F^{ct} of the face F. We extend w_1 to K_2 as in Remark 2.6.1 and set $v = w_1 - w_2$. Then by Lemma 6.1.5, $v = 0$ on F. Therefore $w := w_1\chi(K_1) + w_2\chi(K_2)$ is divergence-conforming across F. For each $e \in \Delta_1^I(F^{\mathrm{ct}})$, we also have $[\![w_1 \cdot t]\!]_e - [\![w_2 \cdot t]\!]_e = 0$, and it follows that $\theta_e(w \cdot t) = 0$. Therefore, the DOFs (6.1.8) induce the global space $\mathscr{V}^2_{r-2}(\mathcal{T}_h^{\mathrm{wf}})$. $\square$

Lemma 6.7.5. *The local degrees of freedom stated in Lemma 6.1.6 induce the global space* $V^3_{r-3}(\mathcal{T}_h^{\mathrm{wf}})$.

Proof. The DOFs (6.1.9) simply determine the piecewise polynomials $\mathcal{P}_{r-3}(T^{\mathrm{wf}})$. Hence these DOFs naturally induce the global piecewise polynomial space $\mathcal{P}_{r-3}(\mathcal{T}_h^{\mathrm{wf}})$. $\square$

Now we can see that the following sequence forms a complex by Theorem 6.2.1 for $r \geq 3$.

$$\mathbb{R} \longrightarrow S^0_r(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\text{grad}} L^1_{r-1}(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\text{curl}} \mathcal{V}^2_{r-2}(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\text{div}} V^3_{r-3}(\mathcal{T}_h^{\mathrm{wf}}) \longrightarrow 0.$$

Furthermore, for $0 \leq k \leq 3$ and $r \geq 3$ we have commuting projections $\tilde{\Pi}^k_{r-k}$ such that $\tilde{\Pi}^k_{r-k} v|_T = \Pi^k_{r-k}(v|_T)$ for all $T \in \mathcal{T}_h$. Then by Theorem 6.2.1, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\text{grad}"] \arrow[d, "\tilde{\Pi}^r_3"] & {[C^\infty(T)]^3} \arrow[r, "\text{curl}"] \arrow[d, "\tilde{\Pi}^1_{r-1}"] & {[C^\infty(T)]^3} \arrow[r, "\text{div}"] \arrow[d, "\tilde{\Pi}^2_{r-2}"] & C^\infty(T) \arrow[r] \arrow[d, "\tilde{\Pi}^3_{r-3}"] & 0 \\
\mathbb{R} \arrow[r] & \mathcal{S}^0_r(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\text{grad}"] & \mathcal{L}^1_{r-1}(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\text{curl}"] & \mathscr{V}^2_{r-2}(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\text{div}"] & V^3_{r-3}(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r] & 0.
\end{tikzcd}$$

Next, we will show that the global analogue of sequence (4.1.2c) is induced by the local DOFs of Section 6.3.

Lemma 6.7.6. *The local degrees of freedom stated in Lemma 6.3.1 induce the global space $\mathcal{S}^1_{r-1}(\mathcal{T}_h^{\mathrm{wf}})$.*

Proof. Let $v_1 \in S^1_{r-1}(T_1^{\mathrm{wf}})$ and $v_2 \in S^1_{r-1}(T_2^{\mathrm{wf}})$ such that $v_1 - v_2$ vanishes on the DOFs (6.3.2a) - (6.3.2h) associated with the triangulation F^{ct}. We extend v_1 to K_2 as in Remark 2.6.1, and we set $w = v_1 - v_2$. Then by Lemma 6.3.1, $w = 0$ and $\operatorname{curl} w = 0$ on F, therefore the DOFs (6.3.2) induce the global space $\mathcal{S}^1_{r-1}(\mathcal{T}_h^{\mathrm{wf}})$. □

Lemma 6.7.7. *The local degrees of freedom stated in Lemma 6.3.3 induce the global space $\mathcal{L}^2_{r-2}(\mathcal{T}_h^{\mathrm{wf}})$.*

Proof. Let $w_1 \in L^2_{r-2}(T_1^{\mathrm{wf}})$ and $w_2 \in L^2_{r-2}(T_2^{\mathrm{wf}})$ such that $w_1 - w_2$ vanishes on the DOFs (6.3.4a) - (6.3.4f) associated with the triangulation F^{ct} of the face F. We extend w_1 to K_2 as in Remark 2.6.1, and set $v = w_1 - w_2$. Since v vanishes on the DOFs (6.3.4a) - (6.3.4f), it follows from Lemma 6.3.3 that $v = 0$ on F. Then we see that $w := w_1\chi(T_1) + w_2\chi(T_2)$ is continuous across F, hence the DOFs of Lemma 6.3.3 induce the global space $\mathcal{L}^2_{r-2}(\mathcal{T}_h^{\mathrm{wf}})$. □

Lemma 6.7.8. *The local degrees of freedom stated in Lemma 6.3.4 induce the global space $\mathscr{V}^3_{r-3}(\mathcal{T}_h^{\mathrm{wf}})$.*

Proof. Let $q_1 \in \mathcal{V}^3_{r-3}(T_1^{\mathrm{wf}})$ and $q_2 \in \mathcal{V}^3_{r-3}(T_2^{\mathrm{wf}})$ such that $q_1 - q_2$ vanishes on the DOFs (6.3.5a) - (6.3.5b) associated with the triangulation F^{ct} of the face F. We extend q_1 to K_2 as in Remark 2.6.1. Given an edge $e \in \Delta_1^I(F^{\mathrm{ct}})$, and using DOFs (6.3.5a) - (6.3.5b), we have that $[\![q_1]\!]_e = [\![q_2]\!]_e$, which implies $\theta_e(q) = 0$, where $q = q_1\chi(K_1) + q_2\chi(K_2)$. □

Now we can see that the following sequence forms a complex by Theorem 6.4.1 for $r \geq 3$.

$$\mathbb{R} \longrightarrow S_r^0(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} S_{r-1}^1(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} L_{r-2}^2(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} \mathcal{V}_{r-3}^3(\mathcal{T}_h^{\mathrm{wf}}) \longrightarrow 0.$$

Furthermore, for $1 \leq k \leq 3$ and $r \geq 3$ we have commuting projections $\tilde{\pi}_{r-k}^k$ such that $\tilde{\pi}_{r-k}^k v|_T = \pi_{r-k}^k(v|_T)$ for all $T \in \mathcal{T}_h$, and by Theorem 6.4.1, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{grad}"] \arrow[d, "\tilde{\Pi}_3^r"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{curl}"] \arrow[d, "\tilde{\pi}_{r-1}^1"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{div}"] \arrow[d, "\tilde{\pi}_{r-2}^2"] & C^\infty(T) \arrow[r] \arrow[d, "\tilde{\pi}_{r-3}^3"] & 0 \\
\mathbb{R} \arrow[r] & \mathcal{S}_r^0(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{grad}"] & \mathcal{S}_{r-1}^1(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{curl}"] & \mathcal{L}_{r-2}^2(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{div}"] & \mathscr{V}_{r-3}^3(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r] & 0.
\end{tikzcd}$$

Lastly, we will show that the global analogue of sequence (4.1.2d) is induced by the local DOFs of Section 6.5.

Lemma 6.7.9. *The local degrees of freedom stated in Lemma 6.5.1 induce the global space* $\mathcal{S}_{r-2}^2(\mathcal{T}_h^{\mathrm{wf}})$.

Proof. Let $w_1 \in S_{r-2}^2(T_1^{\mathrm{wf}})$ and $w_2 \in S_{r-2}^2(T_2^{\mathrm{wf}})$ such that $w_1 - w_2$ vanishes on the DOFs (6.5.2a) - (6.5.2g) associated with the triangulation F^{ct}. We extend w_1 to K_2 as in Remark 2.6.1, and we set $v = w_1 - w_2$. Then by Lemma 6.5.1, $v = 0$ and $\operatorname{div} v = 0$ on F. Therefore, the local DOFs (6.5.2) induce the global space $\mathcal{S}_{r-2}^2(\mathcal{T}_h^{\mathrm{wf}})$. $\square$

Lemma 6.7.10. *The local degrees of freedom stated in Lemma 6.5.2 induce the global space* $\mathcal{L}_{r-3}^3(\mathcal{T}_h^{\mathrm{wf}})$.

Proof. Let $q_1 \in L_{r-3}^3(T_1^{\mathrm{wf}})$ and $q_2 \in L_{r-3}^3(T_2^{\mathrm{wf}})$ such that $q_1 - q_2$ vanishes on the DOFs (6.5.3a) - (6.5.3d) associated with the triangulation F^{ct} of the face F. We extend q_1 to K_2

as in Remark 2.6.1, and we set $p = q_1 - q_2$. It follows from Lemma 6.5.2 that $p = 0$ on F, which means $q := q_1\chi(K_1) + q_2\chi(K_2)$ is continuous across F. Therefore the local DOFs (6.5.3) induce the global space $\mathcal{L}^3_{r-3}(\mathcal{T}_h^{\mathrm{wf}})$. $\square$

Now we can see that the following sequence forms a complex by Theorem 6.6.1 for $r \geq 3$.

$$\mathbb{R} \longrightarrow S^0_r(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{grad}} S^1_{r-1}(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{curl}} S^2_{r-2}(\mathcal{T}_h^{\mathrm{wf}}) \xrightarrow{\mathrm{div}} L^3_{r-3}(\mathcal{T}_h^{\mathrm{wf}}) \longrightarrow 0.$$

For $2 \leq k \leq 3$ and $r \geq 3$, we have commuting projections $\tilde{\varpi}^k_{r-k}$ such that $\tilde{\varpi}^k_{r-k} v|_T = \varpi^k_{r-k}(v|_T)$ for all $T \in \mathcal{T}_h$, and by Theorem 6.6.1, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{grad}"] \arrow[d, "\tilde{\Pi}^r_3"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{curl}"] \arrow[d, "\tilde{\pi}^1_{r-1}"] & {[C^\infty(T)]^3} \arrow[r, "\mathrm{div}"] \arrow[d, "\tilde{\varpi}^2_{r-2}"] & C^\infty(T) \arrow[r] \arrow[d, "\tilde{\varpi}^3_{r-3}"] & 0 \\
\mathbb{R} \arrow[r] & \mathcal{S}^0_r(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{grad}"] & \mathcal{S}^1_{r-1}(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{curl}"] & \mathcal{S}^2_{r-2}(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r, "\mathrm{div}"] & \mathcal{L}^3_{r-3}(\mathcal{T}_h^{\mathrm{wf}}) \arrow[r] & 0.
\end{tikzcd}$$

We have developed smooth finite element spaces on Worsey-Farin splits that for exact sequences in three dimensions, and we constructed commuting projections for these spaces. In the following section, we will use numerical experiments to verify our two-dimensional results on Powell-Sabin splits.

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